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GKM-theory for torus actions on cyclic quiver Grassmannians

Martina Lanini and Alexander Pütz

We define and investigate algebraic torus actions on quiver Grassmannians for nilpotent representations of the equioriented cycle. Examples of such varieties are type `A` flag varieties, their linear degenerations and finite-dimensional approximations of both the affine flag variety and affine Grassmannian for GL_n. We show that these quiver Grassmannians equipped with our specific torus action are GKM-varieties and that their moment graph admits a combinatorial description in terms of the coefficient quiver of the underlying quiver representations. By adapting to our setting results by Gonzales, we are able to prove that moment graph techniques can be applied to construct module bases for the equivariant cohomology of the quiver Grassmannians listed above.

Introduction

GKM-theory is named after the seminal paper by Goresky, Kottwitz and MacPherson [Goresky et al. 1998], where the authors establish several localisation results in the derived category setting. In the present article, we do not make use of the full strength of [Goresky et al. 1998], as we only deal with equivariant cohomology.

Let X be a complex projective algebraic variety equipped with an action of an algebraic torus T. For instance, consider the projective plane $X = \mathbb{P}^2(\mathbb{C})$ equipped with the following action of $T = \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*$:

$$(\gamma_1, \gamma_2, \gamma_3) \cdot [x_1 : x_2 : x_3] = [\gamma_1 x_1 : \gamma_2 x_2 : \gamma_3 x_3],$$

for $[x_1 : x_2 : x_3] \in X$ and $(\gamma_1, \gamma_2, \gamma_3) \in T$.

GKM-theory aims to identify the equivariant cohomology ring with the image of the pullback $H_T^\bullet(X) \to H_T^\bullet(X^T)$ and to describe this image in terms of the corresponding moment graph. This is the one-skeleton of the T-action on X (that is, the set of fixed points and one-dimensional orbits) plus some extra information coming from the torus action on the one-dimensional orbits. In the case of the projective plane equipped with the 3-dimensional torus action above, the T-fixed points are

$$P_1 = [1 : 0 : 0], \quad P_2 = [0 : 1 : 0] \quad \text{and} \quad P_3 = [0 : 0 : 1],$$

and there are three one-dimensional T-orbits, say O_i, for $i = 1, 2, 3$, each given by the vanishing of the i-th coordinate, and each containing in its closure the pair of fixed points P_j, P_k, with $i \neq j, k$. The

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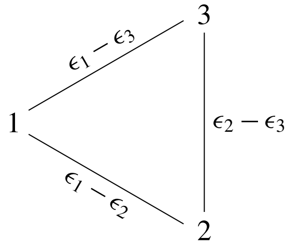
one-skeleton of this torus action is hence a triangle. This is not the desired moment graph yet, since we need to keep track of the torus action on the one-dimensional orbits. For the moment, let us say that this is equivalent to put on any edge a degree one homogeneous polynomial from $S = \mathbb{Q}[\epsilon_1, \epsilon_2, \epsilon_3]$ by following a specific recipe (see Section 1C). In fact, it will be useful to equip the above graph with an orientation, but for now we can ignore this. All in all, the (unoriented) moment graph of our example is



Once the above moment graph is obtained, GKM-theory reduces the determination of the equivariant cohomology to a problem of commutative algebra. In our example, $H_T^\bullet(X)$ can be identified with the following module over S:

$$\big\{(f_1, f_2, f_3) \in S \oplus S \oplus S \mid f_i - f_j \equiv 0 \bmod \epsilon_i - \epsilon_j\big\},$$

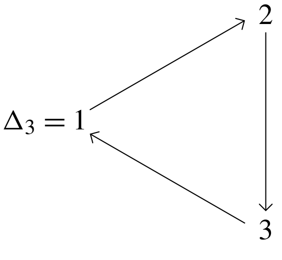
which can be read off from the graph: an element of $H_T^\bullet(X)$ can be realised as a tuple of polynomials, one for any vertex of the moment graph, chosen in such a way that if two vertices are related by an edge, then the corresponding polynomials have to agree modulo the label of such an edge. Observe that the module we have described is free over S, and that the following is an S-basis:

$$(1, 1, 1), \quad (0, \epsilon_1 - \epsilon_2, \epsilon_1 - \epsilon_3), \quad \big(0, 0, (\epsilon_1 - \epsilon_3)(\epsilon_2 - \epsilon_3)\big).$$

Goresky, Kottwitz and MacPherson studied a big class of varieties acted upon by a torus whose equivariant cohomology can be read off from the corresponding moment graph, as in our example. We refer to them as GKM-varieties (see Definition 1.4). Examples of GKM-varieties are flag varieties and their Schubert varieties (see, for example, [Carrell 2002]), as well as rationally smooth standard embeddings of reductive groups [Gonzales 2011].

The aim of this paper is to apply GKM-theory to certain varieties coming from quiver representation theory.

A quiver Q is a finite oriented graph, for instance



We refer to this as the equioriented cycle of length three. A representation M of a quiver Q is a configuration of finite-dimensional vector spaces $M^{(i)}$ (one for each vertex) and linear maps $M_{i\to j} : M^{(i)} \to M^{(j)}$ among them (one for each arrow). For example,

$$\begin{tikzcd}
 & M^{(2)} = \mathbb{C}^3 \arrow[dd, "M_{2\to 3}"] \\
M := M^{(1)} = \mathbb{C}^3 \arrow[ur, "M_{1\to 2}"] & \\
 & M^{(3)} = \mathbb{C}^3 \arrow[ul, "M_{3\to 1}"]
\end{tikzcd}$$

where for the standard basis of $\mathbb{C}^3$, the linear maps have the following matrix presentation:

$$M_{1\to 2} = M_{2\to 3} = M_{3\to 1} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Note that if we keep composing the linear maps of the above representation following the orientation of the edges, we always end up with the zero homomorphism:

$$M_{3\to 1} \circ M_{2\to 3} \circ M_{1\to 2} = M_{1\to 2} \circ M_{3\to 1} \circ M_{2\to 3} = M_{2\to 3} \circ M_{1\to 2} \circ M_{3\to 1} = 0.$$

This is an example of a nilpotent representation. For a collection $\boldsymbol{e}$ of nonnegative integers $e_i \leq \dim M^{(i)}$, the quiver Grassmannian $\mathrm{Gr}_{\boldsymbol{e}}(M)$ is the variety of configurations of vector spaces $U^{(i)}$ of dimensions prescribed by $\boldsymbol{e}$ which are compatible with the maps $M_{i\to j}$. For the collection $\boldsymbol{e} = (1, 1, 1)$ and M as above, the quiver Grassmannian $\mathrm{Gr}_{\boldsymbol{e}}(M)$ contains the point U, with

$$U^{(1)} := \mathrm{span}(e_1), \quad U^{(2)} := \mathrm{span}(e_2) \quad \text{and} \quad U^{(3)} := \mathrm{span}(e_3),$$

where $\{e_1, e_2, e_3\}$ denotes the standard basis of $\mathbb{C}^3$. Note that if the quiver is just the graph with one vertex and no arrows, a representation is just a finite-dimensional vector space, and quiver Grassmannians are classical Grassmann varieties.

In this work, we will focus on the special case in which Q is the equioriented cycle of length n and M is a nilpotent representation. Our primary goal is to equip these class of quiver Grassmannians with a torus action which provides them with a GKM-variety structure.

This is not the first time that GKM-theory meets representation theory of quivers: in [Cerulli Irelli et al. 2013a], the moment graph of a torus action on a quiver Grassmannian for a very special representation of the equioriented quiver of type A is described (see Section 7A2 for more details). In [Weist 2013], a torus action on quiver moduli is introduced with localisation results in mind. Observe that quiver Grassmannians for a fixed quiver are quiver moduli for the one-point extension of the same quiver. Both articles work with one explicit torus depending on the representation. The results in [Cerulli Irelli et al. 2013a] are limited to this special torus, whereas the results from [Weist 2013] can be generalised as described in [Boos and Franzen 2022, Remark 3.2]. Also, the action as introduced in [Weist 2013] has been applied recently, for example, in [Franzen 2020; Boos and Franzen 2022].

Unluckily, Weist's torus action does not equip the corresponding quiver moduli with the structure of a GKM-variety in general (some of the obstructions are explained in the introduction of [Franzen 2020]). A known class where this works requires strong restrictions, among them acyclicity of the quiver. Our torus action, instead, turns every quiver Grassmannian for a nilpotent representation of the equioriented cycle into a GKM-variety, with no further restrictions.

We hope that this paper will motivate both the reader familiar with GKM-theory, as well as the reader familiar with quiver representations, to (further) apply moment graph techniques to quiver Grassmannians.

In order to reach both communities, we have decided to spend some time recalling the basic definitions and results of both theories. To help the reader navigate the paper, we now describe the content of the various sections.

In Section 1, we deal with varieties equipped with a torus action, and describe the properties we want them to satisfy (equivariant formality, T-skeletality, BB-filterability). We also state the GKM-version of the localisation theorem for equivariant cohomology in Theorem 1.20.

The primary goal of Section 2 is to produce a cohomology module basis (under GKM-localisation). We adapt Gonzales' work [2014] and show that the normality assumption in his article can be dropped if the variety is BB-filterable. This is needed since quiver Grassmannians are not normal in general. The main result of this section is Theorem 2.12, which provides existence and uniqueness of an equivariant basis with certain suitable properties. The basis we propose generalises the equivariant Schubert cycle basis for the cohomology of the flag variety. Following Gonzales' recipe, the definition of the basis relies on the concept of local indices and equivariant Euler classes (see Section 1C).

In Section 3, we provide some background material on quiver representations and quiver Grassmannians. In particular, we recall the definition of the coefficient quiver of a quiver representation in Definition 3.8, a combinatorial gadget encoding all information about the given representation and one particular chosen basis for the representation. This object will play a central role in the rest of paper.

From Section 4 on, we restrict our attention to the equioriented cycle with n vertices (denoted by Δ_n). We show that in this case, any nilpotent representation admits a basis, whose corresponding coefficient quiver behaves in a particularly convenient way (see Section 4A).

In Section 5, we use this good combinatorial behaviour to define torus actions on quiver Grassmannians for nilpotent representations of Δ_n. We start by defining a $\mathbb{C}^*$-action, which induces a cellular decomposition of the variety (see Theorem 5.7). Then, in Section 5B, we define an action of a larger-rank torus T and show that the previously defined $\mathbb{C}^*$-action corresponds to a generic cocharacter of the larger torus. We conclude the section by showing that the quiver Grassmannian equipped with the T-action is a BB-filterable variety in Corollary 5.15.

Finally, we describe the moment graph for the T-action on the quiver Grassmannian in Section 6. More precisely, we show that this oriented graph with labelled edges has a combinatorial description: the vertices of the graph are given by successor closed subquivers (see Definition 6.7) of the coefficient quiver and the edges by fundamental mutations (see Definition 6.9). The precise statement, which also explains how to label the edges of the graph via torus characters, is Theorem 6.15.

Section 7 deals with some special cases. We start by focusing on quiver Grassmannians for the equioriented type $\mathtt{A_n}$ Dynkin quiver. Our results apply, since any of its representations can be trivially extended to a nilpotent representation of Δ_n. We, hence, show that in the case of the variety of complete flags and Feigin's degeneration of it, Theorem 6.15 allows us to recover known moment graphs: the Bruhat graph and the graph described in [Cerulli Irelli et al. 2013a], respectively. Our constructions also apply to certain finite-dimensional approximations of the affine flag variety and affine Grassmannian for GL_n, as defined in [Pütz 2022] (see Lemma 7.6). For one example of such degenerations, we draw its moment graph, determine the module basis from Theorem 2.12 and describe the ring structure of the equivariant cohomology.

In the Appendix, we explain how to construct equivariant resolutions of singularities in the explicit example from Section 7. This allows us to compute the equivariant Euler classes of T-varieties (at singular points).

1. Torus actions, cellular decompositions and GKM-theory

1A. ***GKM-varieties.*** Throughout this section, X will denote a complex projective algebraic variety. We say that X is a T-variety if it is acted upon by an algebraic torus $T \cong (\mathbb{C}^*)^r$. If X is a T-variety, we denote by $H_T^\bullet(X)$ the T-equivariant cohomology of X with rational coefficients.

We are interested in a class of T-varieties with a particularly nice T-action.

Definition 1.1. A T-variety X is *equivariantly formal* if one of the following equivalent conditions is satisfied:

(1) the Serre spectral sequence degenerates at E_2,

(2) the ordinary rational cohomology can be recovered by extension of scalars:

$$H^\bullet(X) \cong H_T^\bullet(X) \otimes_{H_T^\bullet(\mathrm{pt})} \mathbb{Q},$$

(3) $H_T^\bullet(X)$ is a free $H_T^\bullet(\mathrm{pt})$-module.

Condition (1) of the above definition is discussed in detail in [Borel 1960, Section XII]. A proof that the other conditions are equivalent can be found in [Goresky et al. 1998, Theorem 1.6.2] or [Brion 2000, Lemma 1.2], where the following lemma is also proven:

Lemma 1.2. *The T-variety X is equivariantly formal if the rational cohomology of X vanishes in odd degrees. Both conditions are equivalent if X has finitely many T-fixed points.*

Since the variety X is equivariantly formal with respect to the T-action, we will often denote an equivariantly formal variety by (X, T). In order to apply localisation techniques, we require more than equivariant formality.

Definition 1.3. We say that the T-action on X is

(1) *skeletal* if the number of T-fixed points and one-dimensional T-orbits in X is finite,

(2) *locally linearisable* if for each one-dimensional orbit E in X there is a linear action of T on $\mathbb{CP}^1$ and a T-equivariant isomorphism $h : \bar{E} \to \mathbb{CP}^1$.

Definition 1.4. We say that X, or (X, T), is a *GKM-variety* if it is equivariantly formal and the T-action is skeletal.

Remark 1.5. Recall that, for us, X is always a projective variety. Then, by [Goresky et al. 1998, Equation (1.2)], the T-action is locally linearisable, as soon as (X, T) is a GKM-variety.

Remark 1.6. Our definition of GKM-variety differs from the definition by Gonzales [2011, Definition 1.4.13], as we do not assume normality. This is central for us, since the varieties we want to deal with fail to be normal in general [Cerulli Irelli et al. 2017, Theorem 13]. By [Sumihiro 1974, Corollary 2], the T-action on normal varieties is locally linearisable.

The above definition of GKM-variety is based on the assumptions by Goresky, Kottwitz and MacPherson [Goresky et al. 1998, Section 7.1].

Example 1.7. Examples of GKM-varieties are (finite-dimensional) Schubert varieties (of flag varieties for a Kac–Moody group) [Carrell 2002], toric varieties [Brion 1998] and rationally smooth embeddings of reductive groups [Gonzales 2011].

1B. ***BB-filterable varieties.*** Assume that X is equipped with a $\mathbb{C}^*$-action, and denote by $X_1, \dots, X_m$ the connected components of the fixed point set of X, which we denote by $X^{\mathbb{C}^*}$. This induces a decomposition

$$X = \bigcup_{i \in [m]} W_i, \quad \text{with } W_i := \{x \in X \mid \lim_{z \to 0} z.x \in X_i\}, \tag{1-8}$$

for $[m] := \{1, \dots, m\}$. We call this a *BB-decomposition* since decompositions of this type were first studied by Białynicki-Birula [1973].

Definition 1.9. We say that W_i from (1-8) is a *rational cell* if it is rationally smooth at all $w \in W_i$. This in turn holds if

$$H^{2 \dim_{\mathbb{C}}(W_i)}(W_i, W_i \setminus \{w\}) \simeq \mathbb{Q} \quad \text{and} \quad H^m(W_i, W_i \setminus \{w\}) = 0$$

for any $m \neq 2 \dim_{\mathbb{C}}(W_i)$ (see [Gonzales 2014, p. 292, Definition 3.4]).

Remark 1.10. These W_i are called *attractive sets* and are isomorphic to affine spaces in the original BB-decomposition. Requiring the attractive sets to be affine spaces is a strong restriction, so that usually the BB-decomposition does not have to be a cellular decomposition. Nevertheless, the notion of rational cells provides a reasonable replacement of this condition for the study of topological properties in the case of singular varieties, see [Gonzales 2014].

Remark 1.11. We will show in Theorem 5.7 that it is possible to obtain attractive sets which are in fact affine spaces for the class of varieties we are interested in. We decided, nevertheless, to deal with rational cells in this section, as the results we achieved are intended to.

Let $\mathfrak{X}_*(T)$ be the cocharacter lattice of an algebraic torus T. If X is a T-variety, then every $\chi \in \mathfrak{X}_*(T)$ determines a $\mathbb{C}^*$-action on X.

Definition 1.12. A cocharacter χ is *generic* (for T acting on X) if $X^{\chi(\mathbb{C}^*)} = X^T$.

Remark 1.13. Recall that X always denotes a complex projective variety. Under such an assumption, it is enough to have $|X^{\chi(\mathbb{C}^*)}| < \infty$ to conclude that the cocharacter χ is generic. Indeed, since $\chi(\mathbb{C}^*)$ is a subgroup of T, then $X^T \subseteq X^{\chi(\mathbb{C}^*)}$. It is a known fact that the Euler characteristic of X agrees with the number of fixed points of any algebraic torus action on X, as soon as the latter number is finite. It follows that the two fixed point sets have the same cardinality and, hence, have to coincide.

Definition 1.14. A projective T-variety X is *BB-filterable* if:

(BB1) the fixed point set X^T is finite,

(BB2) there exists a generic cocharacter $\chi : \mathbb{C}^* \to T$, i.e., $X^{\chi(\mathbb{C}^*)} = X^T$, such that the associated BB-decomposition consists of rational cells.

The above definition is very much inspired by Gonzales' definition [2014, Definition 4.6] of $\mathbb{Q}$-filterable variety. Here we relax the assumptions in [Gonzales 2014] and do not require that X is normal. The following theorem extends [Gonzales 2014, Theorem 4.7] to the class of BB-filterable varieties. Its proof is based on Gonzales' idea, but has to be adapted to the setting of BB-filterable varieties.

Theorem 1.15. *Let X be a BB-filterable projective T-variety. Then*:

(1) *X admits a filtration into T-stable closed subvarieties Z_i such that*

$$\varnothing = Z_0 \subset Z_1 \subset \cdots \subset Z_{m-1} \subset Z_m = X.$$

(2) *Each $W_i = Z_i \setminus Z_{i-1}$ is a rational cell, for all $i \in [m]$.*

(3) *The singular rational cohomology of Z_i vanishes in odd degrees, for $i \in [m]$. In other words, each Z_i is equivariantly formal.*

(4) *If, additionally, the T-action on X is skeletal, each Z_i is a GKM-variety.*

Proof. X is BB-filterable, which by Definition 1.14 implies that the attractive loci of the BB-decomposition are rational cells. These cells are T-stable since $\mathbb{C}^*$ acts via some generic cocharacter $\chi \in \mathfrak{X}_*(T)$. By [Carrell 2002, Lemma 4.12], there exists a total order of the fixed points such that, if we define the subvarieties Z_i inductively by removing the rational cell W_{i+1}, they are nested and closed in X.

We apply [Gonzales 2014, Lemma 4.4] inductively to the filtered BB-decomposition and get that the Z_i have no odd cohomology. Lemma 1.2 implies that they are equivariantly formal. Hence, a T-skeletal action implies that the Z_i are GKM-varieties. □

Remark 1.16. In particular, we obtain that the Z_i are GKM-varieties, as soon as we have finitely many one-dimensional T-orbits. This suffices since $|X^T| < \infty$ holds by the definition of BB-filterable varieties.

Remark 1.17. By Remark 1.5, Theorem 1.15 implies that the T-action on BB-filterable projective T-varieties is locally linearisable.

Remark 1.18. If $\{Z_0, Z_1, \dots, Z_m\}$ and $\{W_1, \dots, W_m\}$ are as in Theorem 1.15, then for any i, we have that W_i is open in Z_i and $Z_i \setminus W_i$ is a (closed) T-stable subvariety of Z_i.

1C. ***Equivariant localisation after Goresky, Kottwitz, and MacPherson.*** The equivariant cohomology of a GKM-variety (X, T) can be described by looking at the one-skeleton of the T-action. The idea of extracting all needed data from the zero- and one-dimensional T-orbits is actually due to Chang and Skjelbred [1974], but such an approach is nowadays known as GKM-theory after the paper [Goresky et al. 1998].

Functoriality of equivariant cohomology implies that there is a $\mathbb{N}$-graded algebra homomorphism

$$\psi : H_T^\bullet(X) \to H_T^\bullet\Big(\bigcup_{i=1}^{m} X_i\Big) \simeq \bigoplus_{i\in[m]} H_T^\bullet(X_i),$$

where $X_1, \dots, X_m$ are the connected components of the fixed point set as in (1-8). In particular, if X has a finite number of (isolated) T-fixed points, we can identify $H_T^\bullet(X^T)$ with $\bigoplus_{x\in X^T} H_T^\bullet(\mathrm{pt})$. From now on, we use $S := H_T^\bullet(\mathrm{pt})$ as shorthand notation. Then S can be identified with the symmetric algebra of the $\mathbb{Q}$-vector space over the torus character lattice $\mathfrak{X}^*(T) \otimes_{\mathbb{Z}} \mathbb{Q}$.

If the T-action on X is locally linearisable, any one-dimensional orbit E contains exactly two fixed points in its closure, say x_E and y_E. Clearly, the torus acts on E via a character (uniquely defined up to a sign, depending on the isomorphism $\bar{E} \simeq \mathbb{P}^1$). Since the sign choice does not play any role in the following theorem, we just pick a torus character, denoted by α_E, for each one-dimensional orbit E.

The above data concerning T-fixed points, one-dimensional orbits and their closure is encoded in an oriented graph whose edges are labelled by torus characters.

Definition 1.19. Let (G, T) be a GKM-variety, and let $\chi \in \mathfrak{X}_*(T)$ be a generic cocharacter. The corresponding *moment graph* $\mathcal{G} = \mathcal{G}(X, T, \chi)$ of a GKM-variety is given by the following data:

(MG0) the T-fixed points as vertices, i.e., $\mathcal{G}_0 = X^T$,

(MG1) the closures of one-dimensional T-orbits $\bar{E} = E \cup \{x, y\}$ as edges in $\mathcal{G}_1$, oriented from x to y if $\lim_{\lambda\to 0} \chi(\lambda).p = x$ for $p \in E$,

(MG2) every $\bar{E}$ is labelled by a character $\alpha_E \in \mathfrak{X}^*(T)$ describing the T-action on E.

If the choice of the cocharacter χ is clear from the context or the orientation is not relevant, we sometimes drop it from the notation for the moment graph.

Theorem 1.20 [Goresky et al. 1998, Theorem 1.2.2]. *Let (X, T) be a GKM-variety. Then ψ is injective and its image is*

$$\mathrm{Im}(\psi) = \Big\{(f_x) \in \bigoplus_{x\in\mathcal{G}(X,T)_0} S \;\Big|\; f_{x_E} - f_{y_E} \in \alpha_E S \textit{ for any } \bar{E} = E \cup \{x_E, y_E\} \in \mathcal{G}(X, T)_1\Big\}.$$

Remark 1.21. Since the appearance of [Goresky et al. 1998], moment graph techniques have been extensively — and successfully — applied to the study of equivariant cohomology of Schubert varieties (in Kac–Moody flag varieties) [Carrell 2002], Hessenberg varieties, standard group embeddings, and more. For more examples see the excellent survey article [Tymoczko 2005]. The aim of our paper is to further expand the class of varieties whose equivariant cohomology ring can be investigated by looking at their moment graphs.

2. Construction of cohomology module bases

By definition, the equivariant cohomology of an equivariantly formal space X is a free module over S. It is, hence, natural to look for an S-basis of $H_T^\bullet(X)$. In this section, we address this question in the generality of GKM-varieties.

2A. *Equivariant Euler classes.*

To construct our basis, we will use the same recipe as Gonzales [2014], and hence need equivariant Euler classes and local indices. For a T-variety Y and a fixed point $y \in Y^T$, we denote by $\mathrm{Eu}_T(y, Y)$ the equivariant Euler class of y in Y. This is an element of the fraction field Q of S, whose inverse (up to a sign) is obtained by localising the fundamental class in Borel–Moore homology. We refer the reader to [Arabia 1998, Section 2.2.1] for the precise definition, and limit ourselves to three properties, which very often are enough to determine the equivariant Euler classes.

Lemma 2.1 (cf. [Brion 1998, Corollary 15, Lemma 16, Theorem 18]). *Let Y be a T-variety and $y \in Y^T$.*

1. *If Y is smooth at y, then* $\mathrm{Eu}_T(y, Y) = (-1)^{\dim(Y)} \det T_y Y$, *where* $\det T_y Y$ *is the product of the characters by which T acts on the tangent space $T_y Y$.*
2. *If Y is rationally smooth at y, then* $\mathrm{Eu}_T(y, Y) = z \cdot \det T_y Y$, *for some* $z \in \mathbb{Q} \setminus \{0\}$.
3. *If $\pi : Y \to X$ is a T-equivariant resolution of singularities and* $|Y^T| < \infty$, *then*

$$\mathrm{Eu}_T(x, X)^{-1} = \sum_{\substack{y \in Y^T \\ \pi(y) = x}} \mathrm{Eu}_T(y, Y)^{-1}.$$

Remark 2.2. Actually, Brion [1998] studied equivariant multiplicities rather than Euler classes; they are inverse to each other (up to some sign which has been taken care of in the statement of Lemma 2.1).

Remark 2.3. By using the properties in the previous lemma, Arabia [1998, Section 2.7 (27)] determined (the inverse) equivariant Euler classes of Schubert varieties by looking at Bott–Samelson resolutions. The above lemma also allows us to determine equivariant Euler classes, and hence the desired module basis for the equivariant cohomology, by constructing desingularisations of the quiver Grassmannians, we are looking at (see the Appendix).

In the following, thanks to Theorem 1.20, we identify $H^\bullet(X)$ with $\mathrm{Im}(\psi)$, so that $f \in H_T^\bullet(X)$ will be given by a collection $(f_x) \in \bigoplus S$, satisfying the conditions given by the edge labels of the moment graph $\mathcal{G}(X, T)$.

Lemma 2.4. *Let (X, T) be a BB-filterable GKM-variety with filtration*

$$\varnothing = Z_0 \subset Z_1 \subset \cdots \subset Z_m = X,$$

as in Theorem 1.15. Let $X^T = \{x_1, \dots, x_m\}$, *with* $x_i \in W_i = Z_i \setminus Z_{i-1}$. *For* $i \in [m]$, *define*

$$\tau_{x_j}^{(i)} := \begin{cases} 0 & \text{if } j \neq i, \\ \mathrm{Eu}_T(x_i, Z_i) & \text{if } j = i, \end{cases} \qquad \text{where } j \in [m].$$

Then, $\tau^{(i)} := (\tau_{x_j}^{(i)})_{j \in [m]} \in H_T^\bullet(Z_i)$.

Proof. By Theorem 1.15 (4), (Z_i, T) is itself a GKM-variety for any $i \in [m]$. Therefore, by Theorem 1.20, $\tau^{(i)} \in H_T^\bullet(Z_i)$ if and only if all relations coming from the edges are verified. Since all but one entry of $\tau^{(i)}$ vanish, we only have to check that

$$\mathrm{Eu}_T(x_i, Z_i) \equiv 0 \mod \alpha_E$$

for any $\overline{E} \in \mathcal{G}(Z_i, T)_1$ adjacent to x_i. To obtain this, we just notice that the proof of [Gonzales 2014, Lemma 6.4] also works under our assumptions. Indeed, by [Gonzales 2014, Corollary 5.6], there exists a nonzero $z \in \mathbb{Q}$ such that

$$Eu_T(x_i, W_i) = z \cdot \alpha_{E_1} \cdots \alpha_{E_r},$$

where $E_1 \dots, E_r$ are the 1-dimensional T-orbits lying in W_i and whose closure contains x_i. Recall that W_i is open in Z_i and its complement is T-stable. Hence, thanks to local linearisability, we can apply the proof of [Gonzales 2014, Lemma 6.3] to deduce that all one-dimensional T-orbits, lying in Z_i and containing x_i in their closures, are actually contained in W_i. We conclude that the edges in $\mathcal{G}(Z_i, T)_1$ that are adjacent to x_i are exactly $\{\overline{E_1}, \dots, \overline{E_r}\}$ and the product of their labels is a nonzero multiple of $\mathrm{Eu}_T(x_i, Z_i)$. □

The following theorem is due to Gonzales, and our only contribution is to notice that, once again, his proof works also under our hypotheses:

Theorem 2.5. *Let (X, T) be a BB-filterable GKM-variety with filtration*

$$\varnothing = Z_0 \subset Z_1 \subset \cdots \subset Z_m = X,$$

as in Theorem 1.15. Let $X^T = \{x_1, \dots, x_m\}$ with $x_i \in W_i = Z_i \setminus Z_{i-1}$. There exists a basis $\{\varphi^{(i)}\}_{i\in[m]}$ of $H_T^\bullet(X)$ as a free S-module satisfying the following two properties:

(1) $\varphi^{(i)}_{x_j} = 0$ *for any $j < i$*,

(2) $\varphi^{(i)}_{x_i} = \mathrm{Eu}_T(x_i, Z_i)$.

Proof. The proof is by induction on the length m of the filtration. If $m = 1$, then X is a point and the statement is trivial. Since (Z_{m-1}, T) is a BB-filterable GKM-variety, we get cohomology generators $\{\tilde{\varphi}^{(i)}\}$ of $H_T^\bullet(Z_{m-1})$ satisfying (1) and (2). These elements, can be lifted to $H_T^\bullet(Z_m) = H_T^\bullet(X)$, in a way which is compatible with the localisation map ψ, thanks to the commutative diagram [Gonzales 2014, Equation (1)]. At this point, we have $m-1$ elements $\varphi^{(1)}, \dots, \varphi^{(m-1)}$ satisfying the desired properties (1) and (2). For the missing generator, we set $\varphi^{(m)} := \tau^{(m)}$, where $\tau^{(m)}$ is the one from Lemma 2.4.

Standard arguments imply that a set of elements satisfying properties (1) and (2) is linearly independent and generates $H_T^\bullet(X)$ (cf. [Gonzales 2014, Lemma 6.2]). □

Remark 2.6. Notice that Theorem 2.5 gives existence, but not uniqueness, of the basis. Indeed, the induction step of the proof consists in lifting classes from $H_T^\bullet(Z_{m-1})$ to $H_T^\bullet(X)$, and in general, this lift does not need to be unique. It is, hence, natural to ask whether there is a preferred basis, among the ones which satisfy properties (1) and (2) of Theorem 2.5, and if so, how to choose it.

Remark 2.7. Observe that $\tau^{(i)}$ is a special element of the T-equivariant cohomology of the i-th piece in the filtration of X, whereas $\varphi^{(i)}$ denotes the i-th element in the ordered basis for the T-equivariant cohomology of X. For the running example from the introduction we have $\tau^{(2)} = (0, \epsilon_1 - \epsilon_2, 0)$ and $\varphi^{(2)} = (0, \epsilon_1 - \epsilon_2, \epsilon_1 - \epsilon_3)$.

2B. ***Local indices and a special module basis.*** Before constructing the desired basis, we need to introduce another ingredient: the local index of a cohomology class at a fixed point. We then show that Theorem 2.5 produces only one basis that also satisfies a particular condition with respect to the local index.

The local index of $f \in H_T^\bullet(X)$ at $x_i \in X^T$ is defined in terms of what is called integration map introduced in [Arabia 1998, Section 1.4]. Instead of the original definition, we will define it via an explicit formula (under the localisation map ψ):

Definition 2.8 (see [Gonzales 2014, Lemma 6.7]). Let $X^T = \{x_1, \dots, x_m\}$. For $i \in [m]$, the local index of $f \in H_T^\bullet(X)$ at $x_i \in X^T$ is

$$I_i(f) = \sum_{\substack{j \in [m] \\ x_j \in Z_i}} \frac{f_{x_j}}{\mathrm{Eu}_T(x_j, Z_i)}. \tag{2-9}$$

Example 2.10. Consider the action of $T = (\mathbb{C}^*)^3$ on $X = \mathbb{P}^2(\mathbb{C})$ as studied in the introduction. The local index of $f = (0, \epsilon_1 - \epsilon_2, \epsilon_1 - \epsilon_3)$ at 2 is

$$I_2(f) = \frac{f_1}{\mathrm{Eu}_T(x_1, Z_2)} + \frac{f_2}{\mathrm{Eu}_T(x_2, Z_2)} = 0 + \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 - \epsilon_2} = 1.$$

Remark 2.11. The above definition is useful for computations, but has the disadvantage that by the formula one cannot tell that $I_i(f)$ is actually polynomial. Luckily, this is the case, and it is immediate by the definition in terms of the integration map.

The following theorem provides us with a preferred choice among the bases from Theorem 2.5. Since everything depends on the order of enumeration of the fixed points (and hence of the filtration), which is not unique, we refrain from referring to this basis as canonical.

Theorem 2.12. *Let (X, T) be a BB-filterable GKM-variety with filtration*

$$\varnothing = Z_0 \subset Z_1 \subset \cdots \subset Z_m = X,$$

as in Theorem 1.15. Let $X^T = \{x_1, \dots, x_m\}$, with $x_i \in W_i = Z_i \setminus Z_{i-1}$. There exists a unique basis $\{\theta^{(i)}\}_{i \in [m]}$ of $H_T^\bullet(X)$ as an S-free module, such that for any $i \in [m]$ the following properties hold:

(1) $I_i(\theta^{(i)}) = 1$,

(2) $I_j(\theta^{(i)}) = 0$ *for all* $j \neq i$,

(3) $\theta^{(i)}_{x_j} = 0$ *for all* $j < i$,

(4) $\theta^{(i)}_{x_i} = \mathrm{Eu}_T(x_i, Z_i)$.

Proof. As for previous results, Gonzales' proof [2014] of Theorem 6.9 goes through, and hence we limit ourselves to give only a sketch.

Firstly, we show existence. Let $i \in [m]$, and consider $\widetilde{\theta^{(i)}} := z^{-1} \cdot \varphi^{(i)}$, where $\varphi^{(i)}$ is any element of $H_T^\bullet(X)$ satisfying (1) and (2) from Theorem 2.5, and $z \in \mathbb{Q}$ is such that $\varphi^{(i)}_{x_i} = z \cdot \mathrm{Eu}_T(x_i, Z_i)$. Thanks to (2-9), it is easy to check that (1), (3) and (4) hold. If (2) holds too, we are done; otherwise, we inductively modify $\widetilde{\theta^{(i)}}$ as follows: let $k_0 := \min\{j > i \mid I_j(\widetilde{\theta^{(i)}}) \neq 0\}$ and replace $\widetilde{\theta^{(i)}}$ by $\widetilde{\theta^{(i)}} - I_{k_0}(\widetilde{\theta^{(i)}})\widetilde{\theta^{(k_0)}}$. It is again an easy check to see that the local index of this new element vanishes at any point x_j, with $j \leq k_0$ and $j \neq i$, and that (1), (3) and (4) still hold. At the end of this process, we get an element of $H_T^\bullet(X)$ that we denote by $\theta^{(i)}$ and that satisfies (1), (2), (3) and (4).

Secondly, they freely generate $H_T^\bullet(X)$ by standard arguments (cf. proof of [Gonzales 2014, Lemma 6.2]).

Finally, the uniqueness is shown by contradiction. Assume that we can find $\theta^{(i)}$ and $\psi^{(i)}$ both satisfying (1)–(4) and such that $\theta^{(i)} \neq \psi^{(i)}$. As they are distinct, we can find $k_0 := \min\{j \mid \theta^{(i)}_{x_j} - \psi^{(i)}_{x_j} \neq 0\}$. Since they both satisfy (4), $k_0 \neq i$, and we have that $I_{k_0}(\theta^{(i)} - \psi^{(i)}) = 0$. But from (2-9), we get

$$0 \neq \theta^{(i)}_{x_{k_0}} - \psi^{(i)}_{x_{k_0}} = \underbrace{I_{k_0}(\theta^{(i)} - \psi^{(i)})}_{=0} \cdot \mathrm{Eu}_T(x_{k_0}, Z_{k_0}),$$

which gives us the desired contradiction. □

Remark 2.13. If $G \supset P \supset T$ are, respectively, a complex linear reductive algebraic group, a parabolic subgroup and a maximal torus, then the above basis of $H_T^\bullet(G/P)$ coincides with the one given by equivariant Schubert classes.

The rest of this article is devoted to providing a class of applications for this result. Namely, we want to introduce certain quiver Grassmannians and show that they are projective BB-filterable GKM-varieties.

3. Generalities on quiver Grassmannians

We recall here some definitions concerning quivers, their representations and quiver Grassmannians which are required later. For more details we refer the reader to the articles by Cerulli Irelli [2011; 2016] and the book by Schiffler [2014].

Definition 3.1. A (finite) *quiver* $Q = (Q_0, Q_1)$ is an ordered pair, where

- Q_0 is a finite set of vertices,
- Q_1 is a finite set of oriented edges.

For an edge $a \in Q_1$, we denote the source of a by s_a and the target by t_a.

Definition 3.2. Let Q be a quiver.

(1) A (finite-dimensional) *Q-representation M* over the field $\Bbbk$ is given by $((M^{(i)})_{i \in Q_0}, (M_a)_{a \in Q_1})$, where
  - $M^{(i)}$ is a (finite-dimensional) $\Bbbk$-vector space for any $i \in Q_0$,
  - $M_a : M^{(s_a)} \to M^{(t_a)}$ is a $\Bbbk$-linear map for any $a \in Q_1$.

(2) The *dimension vector* of a finite-dimensional Q-representation M is

$$\mathbf{dim}\, M := (\dim_{\Bbbk} M^{(i)})_{i \in Q_0} \in \mathbb{Z}_{\geq 0}^{Q_0}.$$

(3) A *morphism* between two Q-representations M and N is a collection $(\psi_i : M^{(i)} \to N^{(i)})_{i \in Q_0}$ of $\Bbbk$-linear maps such that the following diagram commutes:

$$\begin{tikzcd}
M^{(s_a)} \arrow[r, "\psi_{s_a}"] \arrow[d, "M_a"'] \arrow[dr, phantom, "///"] & N^{(s_a)} \arrow[d, "N_a"] \\
M^{(t_a)} \arrow[r, "\psi_{t_a}"'] & N^{(t_a)}
\end{tikzcd}$$

Observe that parts (1) and (3) of the above definition work with finite and infinite-dimensional vector spaces. The above-defined Q-representations together with the morphisms among them form a category, which is denoted by $\mathrm{Rep}_{\Bbbk}(Q)$. By $\mathrm{rep}_{\Bbbk}(Q)$, we denote the full subcategory whose objects are the finite-dimensional Q-representations. The following theorem tells us that $\mathrm{rep}_{\Bbbk}(Q)$ is Krull–Schmidt:

Theorem 3.3 (see [Kirillov 2016, Theorem 1.11]). *Every object of* $\mathrm{rep}_{\Bbbk}(Q)$ *is isomorphic to a direct sum of indecomposable objects, and this decomposition is unique up to reordering.*

Definition 3.4. Let Q be a quiver and M an object of $\mathrm{rep}_{\Bbbk}(Q)$.

(1) A subobject of M is called a *subrepresentation.*

(2) For any $\boldsymbol{e} \in \mathbb{Z}_{\geq 0}^{Q_0}$, the *quiver Grassmannian* $\mathrm{Gr}_{\boldsymbol{e}}(M)$ is the variety that parametrises all $\boldsymbol{e}$-dimensional subrepresentations of M.

Remark 3.5. It is immediate to see that if there is an $i \in Q_0$ such that $\boldsymbol{e}_i > \dim_{\Bbbk} M^{(i)}$, then $\mathrm{Gr}_{\boldsymbol{e}}(M)$ is empty. We will, therefore, only consider $\boldsymbol{e}$ such that $e_i \leq \dim_{\Bbbk} M^{(i)}$ for all $i \in Q_0$. We will denote this relation between dimension vectors by $\boldsymbol{e} \leq \mathbf{dim}\, M$.

Remark 3.6. The algebraic variety structure of the quiver Grassmannian is obtained by embedding it into the classical Grassmannian of $\sum_i e_i$-dimensional subspaces of $V = \bigoplus_{i \in Q_0} M^{(i)}$, therefore it does not depend on the choice of bases for the $M^{(i)}$'s.

Example 3.7. Let Q be the type $\mathtt{A_n}$ equioriented Dynkin quiver, that is, the quiver with $Q_0 = \{1, 2, \ldots, n\}$ and $Q_1 = \{i \to i+1 \mid i = 1 \ldots n-1\}$.

Consider the complex Q-representation M given by $M^{(i)} = \mathbb{C}^{n+1}$, for any $i \in Q_0$, and $M_a = \mathrm{id}_{\mathbb{C}^{n+1}}$, for any $a \in Q_1$. Then $\mathrm{Gr}_{(1,2,\ldots,n)}(M)$ is isomorphic to the variety $\mathcal{F}l_{n+1}$ of complete flags in $\mathbb{C}^{n+1}$.

If we relax the map conditions, that is, if we consider any complex Q-representation N with $N^{(i)} = \mathbb{C}^{n+1}$ for any $i \in Q_0$, then $\mathrm{Gr}_{(1,2,\ldots,n)}(N)$ is a linear degeneration of $\mathcal{F}l_{n+1}$, see [Cerulli Irelli et al. 2017].

For any Q-representation M and any collection of bases for the vector spaces $M^{(i)}$, with $i \in Q_0$, it is possible to define a new quiver. This will help to provide a combinatorial description of the moment graph of a torus action on the class of quiver Grassmannians, which we will be interested in later.

Definition 3.8. Let Q be a quiver, and let M be an object of $\mathrm{rep}_{\Bbbk}(Q)$. For $i \in Q_0$, let $B^{(i)} := \{v_k^{(i)}\}$ be a basis of $M^{(i)}$, and let $B := \bigcup_{i\in Q_0} B^{(i)}$. The *coefficient quiver* $Q(M, B)$ is given by

- $Q(M, B)_0 := B$,
- $v_k^{(i)} \to v_\ell^{(j)} \in Q(M, B)_1$ if and only if there exists an $a \in Q_1$ such that $s_a = i$, $t_a = j$ and the coefficient of $v_\ell^{(j)}$ in $M_a(v_k^{(i)})$ is nonzero.

Remark 3.9. By [Kirillov 2016, Theorem 1.11], see Theorem 3.3, every quiver representation is isomorphic to a direct sum of indecomposable quiver representations, which is unique up to the order of the summands. This isomorphism translates to a base change of the representation M and implies that there exists a basis B such that the connected components of the coefficient quiver $Q(M, B)$ are in bijection with the indecomposable summands in the decomposition of M. From now on, we always work with bases satisfying this property.

We conclude this subsection by introducing the notion of attractive grading on the vertex set of the coefficient quiver. This is a crucial tool to study cellular decompositions of quiver Grassmannians.

Definition 3.10. Let M and B be as in Definition 3.8, and let $Q(M, B)$ be the corresponding coefficient quiver.

(1) A *grading* on $Q(M, B)_0$ is a tuple $\mathbf{wt} = (\mathrm{wt}(v_k^{(i)})) \in \mathbb{Z}^B$.

(2) A grading $\mathbf{wt}$ on $Q(M, B)_0$ is *attractive* if

(AG1) for any $i \in Q_0$, it holds that $\mathrm{wt}(v_k^{(i)}) > \mathrm{wt}(v_\ell^{(i)})$ whenever $k > \ell$,

(AG2) for any $a \in Q_1$, there exists a weight $d(a) \in \mathbb{Z}$ such that

$$\mathrm{wt}(v_\ell^{(t_a)}) = \mathrm{wt}(v_k^{(s_a)}) + d(a)$$

whenever $v_k^{(s_a)} \to v_\ell^{(t_a)} \in Q(M, B)_1$.

Remark 3.11. For a special class of quiver representations, we describe an approach to construct attractive gradings of their coefficient quivers in Proposition 5.1.

Remark 3.12. The above definition is inspired by [Cerulli Irelli 2011, Theorem 1], where a grading on $Q(M, B)_0$ with property (AG2) and (AG1) with "$\neq$" instead of "$>$" is used to define a $\mathbb{C}^*$-action on $\mathrm{Gr}_{\boldsymbol{e}}(M)$ [Cerulli Irelli 2011, Lemma 1.1] via

$$z \cdot b := z^{\mathrm{wt}(b)} b \quad \text{for } z \in \mathbb{C}^*,\ b \in B. \tag{3-13}$$

Looking at the fixed point set of such an action allowed Cerulli Irelli [2011, Theorem 1] to compute the Euler characteristic of $\mathrm{Gr}_{\boldsymbol{e}}(M)$. His construction was generalised by Haupt [2012, Theorem 1.2].

Remark 3.14. A different approach to compute cellular decompositions of quiver Grassmannians, which does not rely on a $\mathbb{C}^*$-action, is presented in [Cerulli Irelli et al. 2021]. Since we are interested in cellular decompositions which are stable under the action of some larger-rank torus, it is convenient for us to start from a $\mathbb{C}^*$-action on our varieties.

4. Nilpotent representations of the equioriented cycle

Let Δ_n denote the equioriented cycle on n vertices with arrows $i \to i+1$ for $i \in 1, \dots, n-1$ and $n \to 1$. The set of vertices and the set of arrows in Δ_n are in bijection with $\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$.

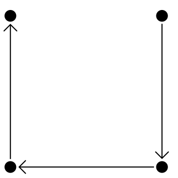
Definition 4.1. An object M of $\operatorname{rep}_{\Bbbk}(\Delta_n)$ is called *nilpotent* if there exists an $N \in \mathbb{Z}_{\geq 0}$ such that $M_{a+N} \circ M_{a+N-1} \circ \cdots \circ M_a = 0$, for any $a \in Q_1 = \mathbb{Z}/n\mathbb{Z}$. The minimal N such that this is satisfied is called the *nilpotence parameter* of M.

Remark 4.2. Notice that a representation $M = ((M^{(i)})_{i \in \mathbb{Z}/n\mathbb{Z}}, (M_a)_{a \in \mathbb{Z}/n\mathbb{Z}})$ of Δ_n, is the same as a $\mathbb{Z}/n\mathbb{Z}$-graded $\Bbbk$-vector space $V = \bigoplus_{i \in \mathbb{Z}/n\mathbb{Z}} M^{(i)}$, together with a $\Bbbk$-linear operator $A \in \operatorname{End}(V)$ such that $AM^{(i)} = M_a(M^{(i)}) \subseteq M^{(i+1)}$, for any $i \in \mathbb{Z}/n\mathbb{Z}$ with $a \in Q_1$ such that $i = s_a$. Then M is nilpotent if and only if A is a nilpotent endomorphism. From now on, we write M_i for the map along the arrow a with $i = s_a$.

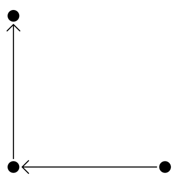
Example 4.3. Let $i \in \mathbb{Z}/n\mathbb{Z}$, and let $\ell \in \mathbb{Z}_{\geq 1}$. Consider the $\Bbbk$-vector space V with basis $W = \{w_1, \dots, w_\ell\}$ equipped with the $\mathbb{Z}/n\mathbb{Z}$-grading given by $\deg(w_k) = i + k - \ell \in \mathbb{Z}/n\mathbb{Z}$. Consider, moreover, the operator $A \in \operatorname{End}(V)$ uniquely determined by setting $Aw_k = w_{k+1}$ for any $k < \ell$ and $Aw_\ell = 0$. The corresponding Δ_n-representation is immediately seen to be nilpotent. We denote this representation by $U(i; \ell)$. For $n = 4$, we draw Δ_4 and the coefficient quivers of $U(1; 4)$ and $U(2; 3)$:

$$\Delta_4 = \begin{tikzcd} 4 \arrow[r] & 1 \arrow[d] \\ 3 \arrow[u] & 2 \arrow[l] \end{tikzcd}$$

$Q(U(1; 4), W) =$



$Q(U(2; 3), W) =$



The following theorem tells us that any indecomposable nilpotent representation of the cycle is isomorphic to some $U(i; \ell)$:

Theorem 4.4 (see [Kirillov 2016, Theorem 7.6]). (1) *The representation $U(i; \ell)$ defined in Example 4.3 is indecomposable.*

(2) *Let M be an indecomposable nilpotent representation of Δ_n. Then there exist $i \in \mathbb{Z}/n\mathbb{Z}$ and $\ell \in \mathbb{Z}_{>0}$ such that $M \simeq U(i; \ell)$.*

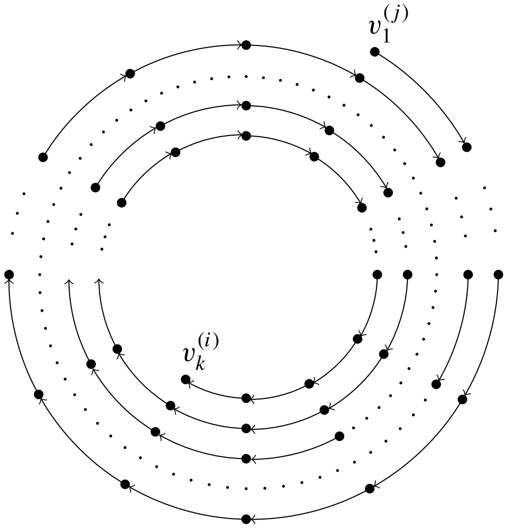
By the above theorem, together with Theorem 3.3, we deduce that if M is a nilpotent representation of Δ_n, then there exists a nilpotence parameter $N \in \mathbb{N}$ such that

$$M \cong U(\boldsymbol{d}) := \bigoplus_{i \in \mathbb{Z}_n} \bigoplus_{\ell=1}^{N} U(i; \ell) \otimes \mathbb{C}^{d_{i,\ell}}, \tag{4-5}$$

with $d_{i,\ell} \in \mathbb{Z}_{\geq 0}$. The investigation of torus actions on quiver Grassmannians for nilpotent representations of Δ_n is the main purpose of the rest of this paper.

4A. ***Coefficient quivers for nilpotent representations of Δ_n.*** Observe that in Example 4.3 we defined the representation $U(i;\ell)$ by choosing a basis $B=\{w_1,\dots,w_\ell\}$ of the underlying $\mathbb{Z}/n\mathbb{Z}$-graded vector space. This can be obviously rearranged into the union of ordered bases $B^{(j)}$, for $j\in\mathbb{Z}/n\mathbb{Z}$. We fix these bases once and for all, and therefore, we write $Q(U(i;\ell))$ for $Q(U(i;\ell),B)$. Notice that $Q(U(i;\ell))$ is a segment on ℓ points, which starts at vertex $v_1^{(j)}$ (for $j=i-\ell+1 \mod n$) and ends at the vertex $v_k^{(i)}$ (for $k=1+\lfloor(\ell-1)/n\rfloor$).

Example 4.6. With basis as above, the coefficient quiver of $U(i;\ell)$ has the form



Let M be a nilpotent representation of Δ_n. By the above discussion, Remark 3.9 and Theorem 4.4, we deduce that there exists a basis B such that the connected components of the coefficient quiver are segments, parametrised by a terminal vertex i and a length parameter ℓ. Now, we want to rearrange these segments in a particular way, which allows us to prove the existence of attractive gradings on the coefficient quiver (see Proposition 5.1). We use these gradings to compute a cellular decomposition of $\mathrm{Gr}_{\boldsymbol{e}}(M)$ as in [Pütz 2022, Theorem 4.13]. This new arrangement corresponds to a base change for the representation M, and hence does not affect the geometry of the quiver Grassmannian (see Remark 3.6).

Definition 4.7. A nilpotent Δ_n-representation M is *alignable* if there exists a basis B, such that for $Q(M,B)$ the following holds over each $i\in\mathbb{Z}_n$:

- (QM0) B is the union of standard basis of the indecomposable direct summands of M.
- (QM1) The end points of segments have larger indices than every point with outgoing arrows: if $M_i v_j^{(i)}=0$ and $M_i v_k^{(i)}\neq 0$, then $j>k$.
- (QM2) The outgoing arrows are order preserving: if $M_i v_j^{(i)}=v_{j'}^{(i+1)}$ and $M_i v_k^{(i)}=v_{k'}^{(i+1)}$ with $j>k$, then $j'>k'$.

Proposition 4.8. *Every nilpotent Δ_n-representation M is alignable.*

Proof. All nilpotent Δ_n-representations decompose as in (4-5). For $i \in \mathbb{Z}/n\mathbb{Z}$, set

$$d_i := \sum_{\ell=1}^{N} d_{i,\ell}, \quad r_i := d_i - d_{i,1}, \quad q_i := \dim_{\Bbbk} M^{(i)} - d_i,$$

so that in the coefficient quiver of M there will be d_i segments ending in vertices corresponding to basis elements in $B^{(i)}$. We construct the coefficient quiver inductively, by truncating and then extending the various segments step-by-step.

In Step 1, we draw the coefficient quiver of

$$M^1 = \bigoplus_{i \in \mathbb{Z}_n} \bigoplus_{\ell=1}^{N} U(i;1) \otimes \mathbb{C}^{d_{i,\ell}},$$

which has no edges, and the vertices are $\{v^{(1)}_{q_1+1}, \ldots, v^{(1)}_{q_1+d_1}, \ldots, v^{(n)}_{q_n+1}, \ldots, v^{(n)}_{q_n+d_n}\}$.

In Step 2, we extend the segments of the $U(i;\ell)$ with $\ell \geq 2$, to get the coefficient quiver of

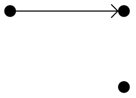
$$M^2 = \bigoplus_{i \in \mathbb{Z}_n} \Bigl(U(i;1) \otimes \mathbb{C}^{d_{i,1}} \oplus \bigoplus_{\ell=2}^{N} U(i;2) \otimes \mathbb{C}^{d_{i,\ell}} \Bigr),$$

so we do not touch the vertices $\{v^{(i)}_{q_i+r_i+1}, \ldots, v^{(i)}_{q_i+d_i} \mid i \in \mathbb{Z}_n\}$ corresponding to the $U(i;1)$-segments.

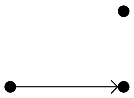
For $i \in \mathbb{Z}_n$ and $k_i \in [r_i]$, each of the $v^{(i)}_{q_i+k_i}$ is connected via an edge to $v^{(i-1)}_{q_{i-1}+k_i-r_i}$ in $B^{(i-1)}$. This procedure is continued until all segments are fully rearranged. In the k-th step we modify segments corresponding to $U(i;\ell)$ with $\ell \geq k$, while the shorter ones are already complete and remain unchanged. Hence, the procedure ends after N steps. ☐

Remark 4.9. Given a nilpotent representation M as before, the aligned coefficient quiver we have obtained is uniquely determined by the decomposition (4-5) of M, up to the order of segments of the same length, but this does not change the isomorphism type of the graph. Without any ambiguity, we denote it by $Q(M)$.

Remark 4.10. Observe that this is not the only way to obtain an aligned coefficient quiver of M. For example,



and



are aligned coefficient quivers of the Δ_2-representation $U(2;2) \oplus U(2;1)$. The role of different alignments will be discussed in Example 6.14. The explicit alignment as in Proposition 4.8 allows us to prove the existence of attractive gradings by constructing one specific attractive grading of $Q(M)$.

We hope that an example will make the above construction clear.

Example 4.11. Let $n = 4$ and

$$M = U(1;4) \oplus U(1;2) \oplus U(2;3) \oplus U(2;2) \oplus U(2;1) \oplus U(4;6).$$

We compute $d_1 = 2$, $d_2 = 3$, $d_3 = 0$ and $d_4 = 1$. Following the procedure described above, the coefficient quiver $Q(M)$ is constructed as shown in Figure 1.

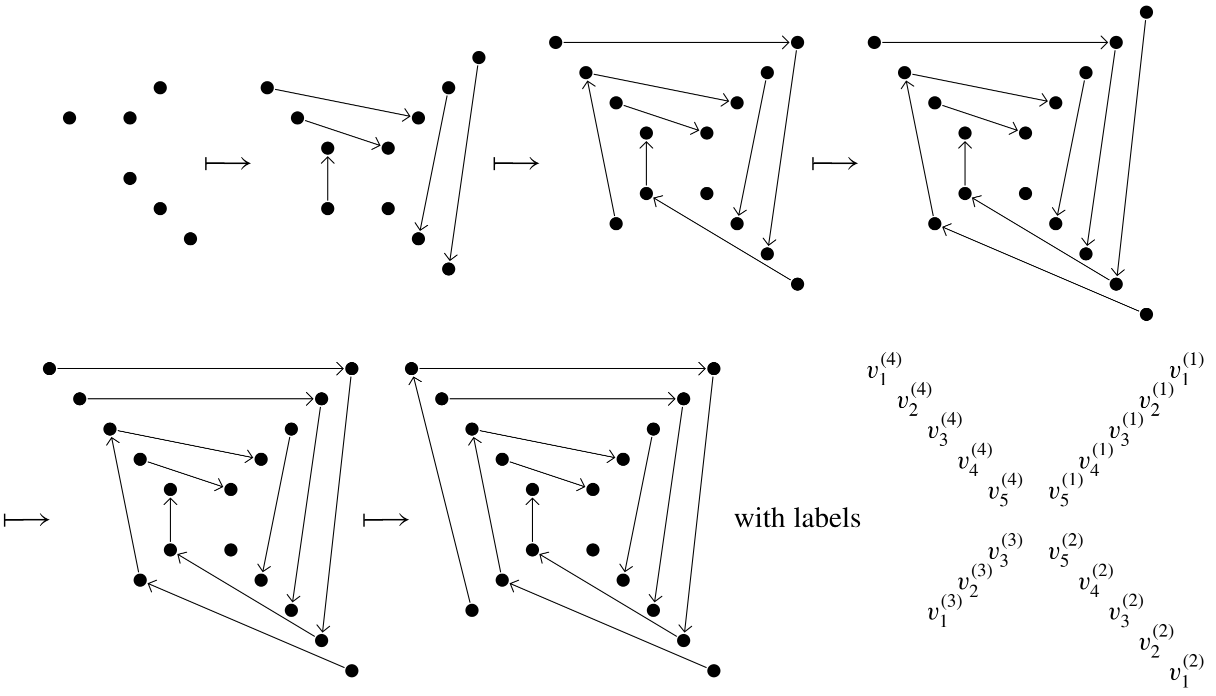


Figure 1. Construction of the coefficient quiver $Q(M)$.

5. Torus actions

5A. ***$\mathbb{C}^*$-action and cellular decomposition.*** Now, we describe one explicit attractive grading of the coefficient quiver of any nilpotent representation of the cycle.

Proposition 5.1. *Let M be a nilpotent representation of Δ_n with decomposition as in* (4-5). *There exists an attractive grading of $Q(M)$ with* (*constant*) *weight function on the edges given by*

$$d(a) := D := \max\{d_i - d_{i,1} \mid i \in \mathbb{Z}_n\} \quad \textit{for all edges } a \in \mathbb{Z}/n\mathbb{Z}.$$

Proof. Take $i_0 \in \mathbb{Z}_n$ so that $d_{i_0,N} \geq d_{i,N}$ for all $i \in \mathbb{Z}_n$. This is equivalent to picking a vertex of Δ_n such that the number of segments of length N ending on that vertex is maximal. This choice is not unique and, indeed, the grading depends on it.

Set $j_0 := i_0 - N + 1 \bmod n$, and define $\operatorname{wt}(v_1^{(j_0)}) := 1$. Condition (AG2), with $d(a)$ as in the statement of the proposition, uniquely determines the weights on any vertex belonging to the (length N) segment starting in $v_1^{(j_0)}$. Notice that such a segment ends in $v_{q_{i_0}+1}^{(i_0)}$ (see (QM1)). Observe that (AG2) implies in particular $w := \operatorname{wt}(v_{q_{i_0}+1}^{(i_0)}) = 1 + D(N-1)$. Next, we let $k := w + d_{i_0} - d_{i_0,1}$ and, for any $i \in \mathbb{Z}/n\mathbb{Z}$, set

$$\operatorname{wt}(v_{q_i+p}^{(i)}) := k + p - 1 + d_{i,1} - d_i, \quad \text{for all } p \in [d_i].$$

Observe that if $i = i_0$ and $p = 1$, we obtain the already defined weight of $v_{q_{i_0}+1}^{(i_0)}$. This formula allows to compute the weight of the end point of any segment. The remaining weights are determined by imposing (AG2).

To conclude, we have to show the attractiveness of the above defined grading. Observe that by definition of the grading, (AG2) is automatically satisfied. Therefore, we only have to make sure that also (AG1) holds, that is $\mathrm{wt}(v_{h+1}^{(i)}) > \mathrm{wt}(v_h^{(i)})$ for any $i \in \mathbb{Z}/nZ$ and any $h < \dim_{\mathbb{k}} M^{(i)}$. We prove this by induction on the length s of the segments in $Q(M)$, just as we did in the construction of $Q(M)$. If we restrict to the vertices belonging to $B^{(i)}$ for some $i \in \mathbb{Z}/n\mathbb{Z}$, it is clear for the end points that all weights are distinct and strictly increasing with the indices of the basis vectors. We, hence, assume that if we consider the truncated representation M^{s-1} for $s > 1$ and restrict the grading to its coefficient quiver $Q(M^{s-1})$, we obtain an attractive grading.

Recall that to get $Q(M^s)$, we have to add an arrow and its starting point to all segments corresponding to isotypical components $U(i;\ell)$ with $\ell \geq s$. Consider $v_{h+1}^{(i)}, v_h^{(i)} \in B^{(i)}$ such that they both are vertices of $Q(M^s)$ and $v_h^{(i)} \notin Q(M^{s-1})_0$ (otherwise, the claim follows immediately by induction). If $v_{h+1}^{(i)}$ is not an end point, then $v_h^{(i)}$ and $v_{h+1}^{(i)}$ are sources of two arrows, say a_1 and a_2, respectively, whose targets lie in $B^{(i+1)}$ and by induction $\mathrm{wt}(t_{a_1}) < \mathrm{wt}(t_{a_2})$. Thus, by (AG2),

$$\mathrm{wt}(v_{h+1}^{(i)}) = \mathrm{wt}(t_{a_2}) - D > \mathrm{wt}(t_{a_1}) - D = \mathrm{wt}(v_h^{(i)}).$$

Assume now that $v_{h+1}^{(i)}$ is an end point. In this case, as $v_h^{(i)}$ is not an end point, we have $h = q_i$, and hence, $\mathrm{wt}(v_{h+1}^{(i)}) = k + d_{i,1} - d_i$. Now recall that $v_h^{(i)}$ belongs to a segment whose end point is $v_{q_j+p}^{(j)} \in B^{(j)}$, with $j = i + s - 1 \mod n$ and $p \in [d_j - d_{j,1}]$, so that

$$\mathrm{wt}(v_h^{(i)}) = \mathrm{wt}(v_{q_j+p}^{(j)}) - D(s-1) = k + p - 1 + d_{j,1} - d_j - D(s-1) \leq k - 1 - D(s-1).$$

The claim now follows from the fact that $D \geq d_i - d_{i,1}$ and $s - 1 \geq 1$. □

Example 5.2. We compute the weights for the vertices in the coefficient quiver $Q(M)$ of the representation M from Example 4.11. The edge weight is $D = 2$. We now determine the attractive grading, following the procedure described in the proof of Proposition 5.1. There is a unique segment of length $N = 6$, which corresponds to the subrepresentation $U(4;6)$, and hence we take $i_0 = 4$ and compute $d_{i_0} - d_{i_{0,1}} = 1$. This procedure is shown in Figure 2.

Let M be a nilpotent representation of Δ_n, and let $(\mathrm{wt}(b))_{b\in B}$ be an attractive grading on an aligned $Q(M, B)$. Define a $\mathbb{C}^*$-action on M by (3-13). It is immediate to check that all hypotheses of [Cerulli Irelli 2011, Lemma 1.1] are satisfied, hence the $\mathbb{C}^*$-action extends to the quiver Grassmannian.

Lemma 5.3. *Let M be a nilpotent representation of Δ_n, and let $(\mathrm{wt}(b))_{b\in B}$ be an attractive grading on $Q(M, B)_0$. Then for any $U \in \mathrm{Gr}_{\boldsymbol{e}}(M)$ and any $z \in \mathbb{C}^*$, also $z \cdot U \in \mathrm{Gr}_{\boldsymbol{e}}(M)$.*

It is also possible to describe the fixed points of the introduced $\mathbb{C}^*$-action. As usual,

$$B^{(i)} = \{v_k^{(i)} \mid k \in [m_i]\}$$

denotes the basis of $M^{(i)}$, which we use to construct the aligned $Q(M, B)$. The following lemma is just a special case of [Cerulli Irelli 2011, Theorem 1]:

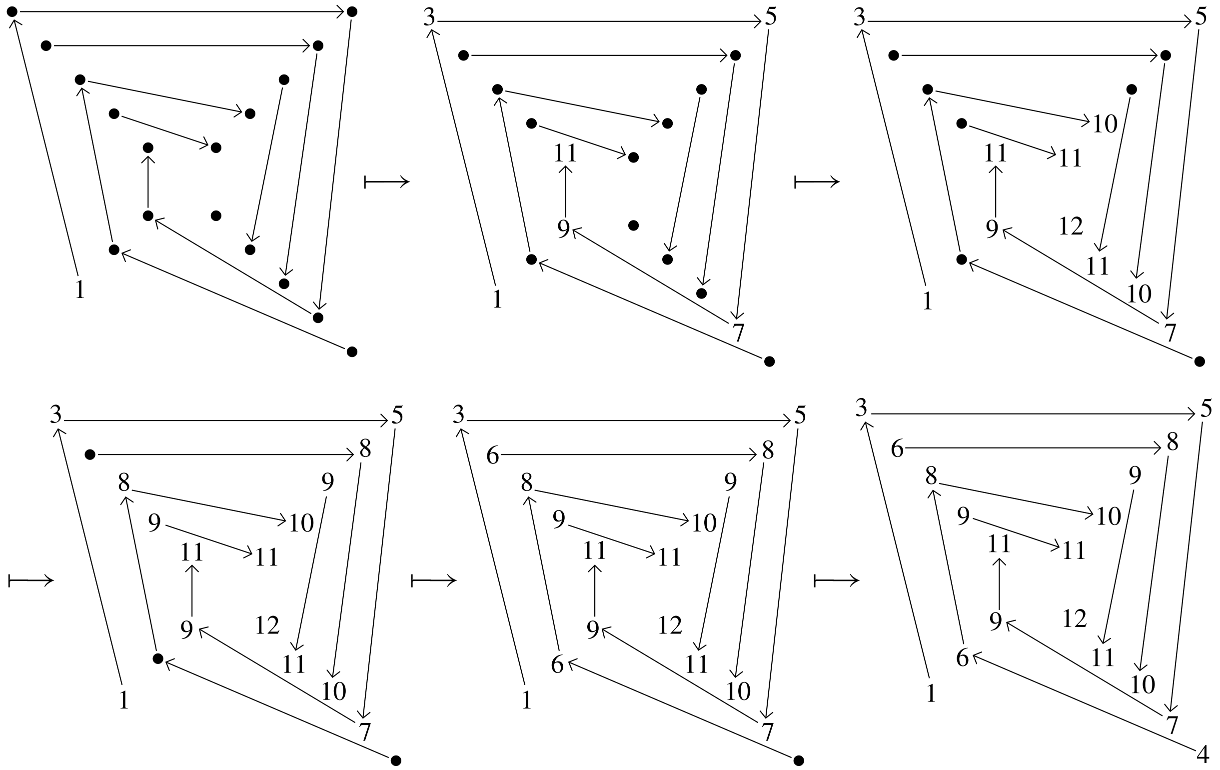
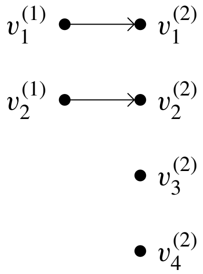


Figure 2. Determination of the attractive grading.

Lemma 5.4. *Let M be a nilpotent Δ_n-representation, and let $(\mathrm{wt}(b))_{b\in B}$ be an attractive grading on $Q(M, B)$. Then, the fixed point set of the above defined $\mathbb{C}^*$-action is*

$$\Big\{L \in \mathrm{Gr}_{\boldsymbol{e}}(M) \;\Big|\; \textit{for all } i \in \mathbb{Z}_n \textit{ there are } K_i \in \binom{[m_i]}{e_i} \textit{ such that } L^{(i)} = \langle v_k^{(i)} | k \in K_i\rangle\Big\}.$$

Example 5.5. Let $M = U(2; 2) \otimes \mathbb{C}^2 \oplus U(2; 1) \otimes \mathbb{C}^2$ be a representation of Δ_2, and set $\boldsymbol{e} := (1, 2)$. Following the construction in the proof of Proposition 4.8, the aligned coefficient quiver of M together with the corresponding basis vectors is given as



The pair of vector spaces

$$U = \big(U^{(1)} = \langle v_1^{(1)} + av_2^{(1)}\rangle,\, U^{(2)} = \langle v_1^{(2)} + av_2^{(2)},\, v_3^{(2)} + bv_4^{(2)}\rangle\big),$$

with $a, b \in \mathbb{C}$ describes a point in the quiver Grassmannian $\mathrm{Gr}_{\boldsymbol{e}}(M)$, since

$$M_1 U^{(1)} = \langle v_1^{(2)} + av_2^{(2)}\rangle \subset U^{(2)},$$

$\dim_{\mathbb{C}} U^{(1)} = 1$ and $\dim_{\mathbb{C}} U^{(2)} = 2$. The grading $\mathrm{wt}(v_j^{(i)})$ is attractive and for the induced $\mathbb{C}^*$-action, we compute

$$\begin{aligned} z.U &= \big(U^{(1)} = \langle z^1 v_1^{(1)} + az^2 v_2^{(1)}\rangle,\, U^{(2)} = \langle z^1 v_1^{(2)} + az^2 v_2^{(2)},\, z^3 v_3^{(2)} + bz^4 v_4^{(2)}\rangle\big) \\ &= \big(U^{(1)} = \langle v_1^{(1)} + azv_2^{(1)}\rangle,\, U^{(2)} = \langle v_1^{(2)} + azv_2^{(2)},\, v_3^{(2)} + bzv_4^{(2)}\rangle\big), \end{aligned}$$

which is contained in $\mathrm{Gr}_{\boldsymbol{e}}(M)$ by Lemma 5.3.

Following Lemma 5.4, the fixed points of the $\mathbb{C}^*$-action, as defined above, are

$$\begin{aligned} L_1 &= \big(L_1^{(1)} = \langle v_1^{(1)}\rangle,\, L_1^{(2)} = \langle v_1^{(2)}, v_2^{(2)}\rangle\big), \\ L_2 &= \big(L_2^{(1)} = \langle v_1^{(1)}\rangle,\, L_2^{(2)} = \langle v_1^{(2)}, v_3^{(2)}\rangle\big), \\ L_3 &= \big(L_3^{(1)} = \langle v_1^{(1)}\rangle,\, L_3^{(2)} = \langle v_1^{(2)}, v_4^{(2)}\rangle\big), \\ L_4 &= \big(L_4^{(1)} = \langle v_2^{(1)}\rangle,\, L_4^{(2)} = \langle v_1^{(2)}, v_2^{(2)}\rangle\big), \\ L_5 &= \big(L_5^{(1)} = \langle v_2^{(1)}\rangle,\, L_5^{(2)} = \langle v_2^{(2)}, v_3^{(2)}\rangle\big), \\ L_6 &= \big(L_6^{(1)} = \langle v_2^{(1)}\rangle,\, L_6^{(2)} = \langle v_2^{(2)}, v_4^{(2)}\rangle\big). \end{aligned}$$

By Lemma 5.4, we have a finite number of fixed points, and thus, we can consider the corresponding attractive loci to get the decomposition (1-8). Since we have not fixed an order on the fixed point set, we will use the notation

$$W_L := \big\{V \in \mathrm{Gr}_{\boldsymbol{e}}(M) \mid \lim_{z\to 0} z.V = L\big\}, \quad \text{where } L \in \mathrm{Gr}_{\boldsymbol{e}}(M)^{\mathbb{C}^*}. \tag{5-6}$$

Theorem 5.7. *Let M be a nilpotent representation of Δ_n, and consider the $\mathbb{C}^*$-action on $\mathrm{Gr}_{\boldsymbol{e}}(M)$ corresponding to an attractive grading on an aligned $Q(M, B)$. Then, for every $L \in \mathrm{Gr}_{\boldsymbol{e}}(M)^{\mathbb{C}^*}$, the subset W_L is an affine space, and hence, the quiver Grassmannian admits a cellular decomposition*

$$\mathrm{Gr}_{\boldsymbol{e}}(M) = \coprod_{L \in \mathrm{Gr}_{\boldsymbol{e}}(M)^{\mathbb{C}^*}} W_L.$$

Proof. In the case of the specific attractive grading of the aligned coefficient quiver as described in the proof of Proposition 4.8, the statement is [Pütz 2022, Theorem 4.13]. It is immediate to see that the proof can be extended, because it only relies on the attractiveness of the grading if the underlying coefficient quiver is aligned. For the convenience of the reader we summarise the main steps from the proof of [Pütz 2022, Theorem 4.13] and highlight the where the generalisation takes place.

The proof has two main steps. First we show that the BB-decomposition is an α-partition, i.e., there exists a total order of the fixed points $\mathrm{Gr}_{\boldsymbol{e}}^{\Delta_n}(M)^{\mathbb{C}^*} = \{L_1, \ldots, L_r\}$ such that $\bigsqcup_{j=1}^{s} \mathcal{C}(L_j)$ is closed in $\mathrm{Gr}_{\boldsymbol{e}}(M)$ for all $s \in [r]$. This follows from [Carrell 2002, Lemma 4.12] and does not depend on the attractive grading.

It remains to show that the $\mathcal{C}(L)$ are isomorphic to affine spaces. From the definition of quiver Grassmannians we know that

$$\mathrm{Gr}_{\boldsymbol{e}}(M) = \Big\{ (V^{(i)})_{i \in \mathbb{Z}_n} \in \prod_{i \in \mathbb{Z}_n} \mathrm{Gr}_{e_i}(m_i) \;\Big|\; M_i V^{(i)} \subseteq V^{(i+1)} \text{ for all } i \in \mathbb{Z}_n \Big\},$$

where $\mathrm{Gr}_e(m)$ is the Grassmannian of e-dimensional subspaces in $\mathbb{C}^m$. We start by showing that the attractive sets $\mathcal{C}(L^{(i)}) := \mathcal{C}(L) \cap \mathrm{Gr}_{e_i}(m_i)$ are isomorphic to affine spaces.

By Lemma 5.4 there exists an index set $K_i \in \binom{[m_i]}{e_i}$ such that $L^{(i)}$ is the span of the $v_k^{(i)}$ for $k \in K_i$. To apply this result, we only need $\neq$ in (AG1) of Definition 3.10 (see Remark 3.12). From properties (AG1) and (AG2), we deduce that a point $V^{(i)}$ in the attracting set $\mathcal{C}(L^{(i)})$ has generators

$$w_k^{(i)} = v_k^{(i)} + \sum_{\substack{j \in [m_i] \setminus [k] \\ j \notin K_i}} u_{j,k}^{(i)} v_j^{(i)}, \qquad \text{with } u_{j,k}^{(i)} \in \mathbb{C}, \tag{5-8}$$

for $k \in K_i$ and $\mu_{\ell,k}^{(i)} \in \mathbb{C}$. Hence, $\mathcal{C}(L^{(i)})$ is an affine space. It is the key observation of this proof that the above description of the generators is not only valid for the specific attractive grading from the proof of Proposition 4.8, but also for all other attractive gradings. The following computation is exactly the same as in the proof of [Pütz 2022, Theorem 4.13].

Observe that for a representation V in an attracting set of a $\mathbb{C}^*$-fixed point L, it holds that

$$V \in \mathcal{C}(L) \iff V \in \mathrm{Gr}_{\boldsymbol{e}}^{\Delta_n}(M) \cap \prod_{i \in \mathbb{Z}_n} \mathcal{C}(L^{(i)}).$$

Now we describe the equations arising from the condition $M_i V^{(i)} \subseteq V^{(i+1)}$. By the arrangement of the segments in $Q(M)$, it follows that $M_i w_k^{(i)} = 0$ if $M_i v_k^{(i)} = 0$. In this case, there are no relations. Assume $M_i v_k^{(i)} \neq 0$ and let $k' \in K_{i+1}$ be such that $M_i v_k^{(i)} = v_{k'}^{(i+1)}$. Analogously, we define the index set $K_i' \subseteq [m_{i+1}]$. If $M_i v_k^{(i)} \neq 0$, the coefficients are subject to the conditions

$$u_{j,k}^{(i)} = u_{j',k'}^{(i+1)} + \sum_{\substack{\ell \in [j-1] \setminus [k] \\ M_i v_\ell^{(i)} \neq 0 \\ \ell' \in K_{i+1} \setminus K_i'}} u_{\ell,k}^{(i)} u_{j',\ell'}^{(i+1)} \quad \text{if } j \in [m_i] \setminus [k] \text{ with } M_i v_j^{(i)} \neq 0,\ j' \notin K_{i+1}, \tag{5-9}$$

$$0 = u_{h,k'}^{(i+1)} + \sum_{\substack{\ell \in [m_i] \setminus [k] \\ M_i v_\ell^{(i)} \neq 0,\, \ell' < h \\ \ell' \in K_{i+1} \setminus K_i'}} u_{\ell,k}^{(i)} u_{h,\ell'}^{(i+1)} \quad \text{if } h \in [m_{i+1}] \setminus [k'] \text{ with } h \notin K_{i+1} \text{ and } M_i v_\ell^{(i)} \neq v_h^{(i+1)} \text{ for all } \ell \in [m_i] \setminus [k]. \tag{5-10}$$

Finally, it is shown as in [Pütz 2022, Theorem 4.13] that these equations parametrise an affine subspace in the product of Grassmannians $\mathrm{Gr}_{e_i}(m_i)$. ☐

Example 5.11. For the fixed point L_2 from Example 5.5, the vector spaces of the points in the cell $\mathcal{C}(L_2)$ have the generators

$$w_1^{(1)} = v_1^{(1)} + u_{2,1}^{(1)} v_2^{(1)}, \quad w_1^{(2)} = v_1^{(2)} + u_{2,1}^{(2)} v_2^{(2)} + u_{4,1}^{(2)} v_4^{(2)} \quad \text{and} \quad w_3^{(2)} = v_3^{(2)} + u_{4,3}^{(2)} v_4^{(2)}$$

by (5-8) in the proof of the above theorem. Following (5-9) and (5-10) these generators are subject to

$$u_{2,1}^{(1)} = u_{2,1}^{(2)} \quad \text{and} \quad u_{4,1}^{(2)} = 0.$$

Hence, we obtain that the cell $\mathcal{C}(L_2)$ is two-dimensional.

5B. ***Action of a bigger torus.*** In this section we introduce an action of a bigger torus T on $\mathrm{Gr}_{\boldsymbol{e}}(M)$ and we show that the $\mathbb{C}^*$-action, coming from an attractive grading as in Proposition 5.1, corresponds to a (generic) cocharacter of T.

Let $d_0 := \sum_{i \in \mathbb{Z}_0} d_i$ be the number of indecomposable summands of M. We fix once and for all an enumeration $U(i_1; \ell_1), \ldots, U(i_{d_0}; \ell_{d_0})$ of the segments of $Q(M, B)$. Each point in the coefficient quiver and, hence, each basis vector $b \in B$ is uniquely determined by the index $j \in [d_0]$ of the segment it belongs to, and its position $p \in \{0, \ldots, \ell_j - 1\}$ on the segment itself. Here, we declare that the position of a starting point is $p = 0$. We will denote by $b_{j,p}$ such a basis vector.

Let $T := (\mathbb{C}^*)^{d_0+1}$. For any $\gamma := (\gamma_0, (\gamma_j)_{j \in [d_0]}) \in T$, we set

$$\gamma . b_{j,p} := \gamma_0^p \gamma_j \cdot b_{j,p}.$$

With T', we denote the subtorus obtained from T by setting $\gamma_0 = 1$. By extending linearly, we get an action on the graded vector space $\bigoplus_{i \in \mathbb{Z}/n\mathbb{Z}} M_i$, which preserves each graded piece.

Lemma 5.12. *Let M be a representation of Δ_n, and let T act on $\bigoplus M^{(i)}$ as above. Then for any $U \in \mathrm{Gr}_{\boldsymbol{e}}(M)$ and any $\gamma \in T$, we have $\gamma \cdot U \in \mathrm{Gr}_{\boldsymbol{e}}(U)$.*

Proof. Let $M = \big(\bigoplus_{i \in \mathbb{Z}/n\mathbb{Z}} M^{(i)}, A\big)$ as in Remark 4.2. For $\gamma \in T$, we denote by γ also the corresponding automorphism of $\bigoplus_{i \in \mathbb{Z}/n\mathbb{Z}} M^{(i)}$. It is easy to verify that, up to a nonzero scalar, γ commutes with A. Indeed, it is enough to check this statement on the basis vectors. For $b_{j,p} \in B$ with $p = \ell_j - 1$, we have that $Ab_{j,p} = 0$ holds and the statement is trivial. If $p \neq \ell_j - 1$, then $Ab_{j,p} = b_{j,p+1}$ and hence,

$$(A \circ \gamma)(b_{j,p}) = \gamma_0^p \gamma_j (A(b_{j,p})) = \gamma_0^p \gamma_j b_{j,p+1} = \gamma_0^{-1}\big(\gamma.(b_{j,p+1})\big) = \gamma_0^{-1}(\gamma \circ A)(b_{j,p}).$$

Let $U = \big(\bigoplus_{i \in \mathbb{Z}/n\mathbb{Z}} U^{(i)}, \bar{A}\big) \in \mathrm{Gr}_{\boldsymbol{e}}(M)$, where $\bar{A} = A_{|\bigoplus U^{(i)}}$. Since γ is an automorphism of $\bigoplus M^{(i)}$, which preserves the $\mathbb{Z}/n\mathbb{Z}$-grading, it preserves inclusions and dimensions of graded subspaces, i.e., $\dim_{\Bbbk} U^{(i)} = \dim_{\Bbbk} \gamma . U^{(i)}$. Moreover, by the previous computation, we obtain

$$A(\gamma . U_i) = \gamma_0^{-1}(\gamma . A(U^{(i)})) \subseteq \gamma_0^{-1}(\gamma . U^{(i+1)}) = \gamma . U^{(i+1)} \quad \text{for any } i \in \mathbb{Z}/n\mathbb{Z}. \qquad \square$$

Remark 5.13. In particular, we obtain that T' commutes with A. Hence, T' is a subgroup in the automorphism group $\mathrm{Aut}_{\Delta_n}(M)$ of the Δ_n-representation M, whereas T has no embedding into $\mathrm{Aut}_{\Delta_n}(M)$. If the support of M is acyclic (i.e., at least one of the maps M_a is zero), it is possible to show that it is sufficient to work with the T'-action, in order to obtain the structure of a GKM-variety. For the special case of the Feigin degeneration of the flag variety of type A, the T'-action is studied in [Cerulli Irelli et al. 2013a]. The following computations for T can be specialised to the T'-action on M with acyclic support:

Theorem 5.14. *Let M be a nilpotent representation of Δ_n, and let $(\mathrm{wt}(b))_{b\in B}$ be an attractive grading on $Q(M, B)$ with $d(a) = D$ for all $a \in \mathbb{Z}_n$. Then*

$$\chi : \mathbb{C}^* \to T = (\mathbb{C}^*)^{d_0+1}, \quad z \mapsto \bigl(z^D, (z^{wt(b_{j,1})})_{j\in[d_0]}\bigr),$$

is a generic cocharacter for the above described T-action on $\mathrm{Gr}_{\boldsymbol{e}}(M)$.

Proof. For any $z \in \mathbb{C}^*$ and any $b_{j,p} \in B$, we have

$$\chi(z).b_{j,p} = (\chi(z)_0^p \chi(z)_j) \cdot b_{j,p} = z^{Dp+\mathrm{wt}(b_{j,1})} b_{j,p} = z^{\mathrm{wt}(b_{j,p})} b_{j,p}.$$

Thus the $\mathbb{C}^*$-action on $\mathrm{Gr}_{\boldsymbol{e}}(M)$, induced by the cocharacter χ, coincides with the $\mathbb{C}^*$-action in (3-13), coming from the attractive grading, and we can apply Lemma 5.4 to deduce that $|\mathrm{Gr}_{\boldsymbol{e}}(M)^{\chi(\mathbb{C}^*)}| < \infty$. The proof is concluded by Remark 1.13. ☐

Corollary 5.15. *Let M be a nilpotent representation of Δ_n. Then, the T-variety $\mathrm{Gr}_{\boldsymbol{e}}(M)$ is BB-filterable.*

Proof. From the proof of the previous theorem,

$$|\mathrm{Gr}_{\boldsymbol{e}}(M)^T| = |\mathrm{Gr}_{\boldsymbol{e}}(M)^{\chi(\mathbb{C}^*)}| < \infty,$$

which implies Property (BB1) of Definition 1.14.

Let us take the generic cocharacter $\chi \in \mathfrak{X}_*(T)$ as in Theorem 5.14. From the proof of the latter result, we know that the $\mathbb{C}^*$-action induced by χ coincides with the one coming from the attractive grading. We can then apply Theorem 5.7, to deduce that W_L is an affine space for any $L \in \mathrm{Gr}_{\boldsymbol{e}}(M)$. So it is smooth (and hence, rationally smooth). This gives us (BB2). ☐

Recall that any point of W_L, corresponding to the collection $(K_i)_{i\in[n]} \in \prod \binom{[m_i]}{e_i}$, can be described by a collection of tuples

$$\bigl((u^{(i)}_{j,k} \mid k \in K_i,\ j \notin K_i,\ j \in [m_i] \setminus [k])\bigr)_{i\in[n]}$$

as in (5-8). In order to describe the T-action on W_L, in terms of such a description, we introduce some notation: for a basis vector $v^{(i)}_k$, we denote by $s^{(i)}_k$ the segment on which it lies on, and by $p^{(i)}_k$ its position. Then it is immediate to see that

$$\gamma.\bigl((u^{(i)}_{j,k})\bigr) = \bigl((u^{(i)}_{j,k} \gamma_0^{p^{(i)}_j - p^{(i)}_k} \gamma_{s^{(i)}_j} \gamma^{-1}_{s^{(i)}_k})\bigr). \tag{5-16}$$

Example 5.17. The points in the cell $\mathcal{C}(L_2)$ from Example 5.11 are described by the following collection:

$$\bigl((u^{(1)}_{2,1}),\ (u^{(2)}_{2,1}, u^{(2)}_{4,1}, u^{(2)}_{4,3})\bigr),$$

with relations as computed in that example. The coefficient quiver of the representation M of this running example is shown in Example 5.5 and has four segments. Hence, $T = (\mathbb{C}^*)^{4+1}$ acts on the quiver Grassmannian $\mathrm{Gr}_{\boldsymbol{e}}(M)$. It follows from the position of the vertices in the coefficient quiver that

$$\begin{aligned}\gamma.u^{(1)}_{2,1} &= u^{(1)}_{2,1}\gamma_0^0\gamma_2\gamma_1^{-1},\\ \gamma.u^{(2)}_{2,1} &= u^{(2)}_{2,1}\gamma_0^0\gamma_2\gamma_1^{-1},\\ \gamma.u^{(2)}_{4,1} &= u^{(2)}_{4,1}\gamma_0^1\gamma_4\gamma_1^{-1},\\ \gamma.u^{(2)}_{4,3} &= u^{(2)}_{4,3}\gamma_0^0\gamma_4\gamma_3^{-1},\end{aligned}$$

because

$$s^{(i)}_k = k, \quad p^{(1)}_1 = p^{(1)}_2 = 1, \quad p^{(2)}_1 = p^{(2)}_2 = 2 \quad \text{and} \quad p^{(2)}_3 = p^{(2)}_4 = 1$$

hold in our running example.

6. GKM-variety structure

6A. *One-dimensional torus orbits.* We deal now with the one-dimensional torus orbits on $\mathrm{Gr}_{\boldsymbol{e}}(M)$, where, as in the previous sections, M is a nilpotent representation of Δ_n. We recall that for a basis vector $v^{(i)}_j \in B^{(i)}$ such that $M_i v^{(i)}_j \neq 0$, we have denoted by j' the unique element in $[m_{i+1}]$ such that $M_i v^{(i)}_j = v^{(i+1)}_{j'}$.

Before proving that T acts with finitely many one-dimensional orbits, we give a definition.

Definition 6.1. Let W_L be the cell corresponding to the collection $(K_i)_{i\in[n]} \in \prod \binom{[m_i]}{e_i}$. The triple (i, j, k), with $i \in [n]$, $k \in K_i$ and $j \in [m_i] \setminus ([k] \cup K_i)$, is said to be *terminal* for W_L if either $M_i v^{(i)}_j = 0$ or $j' \in K_{i+1}$ and for all $v^{(i')}_\ell$ with $s^{(i')}_\ell = s^{(i)}_k$, $p^{(i')}_\ell < p^{(i)}_k$ and $\ell \in K_{i'}$ there exists a $v^{(i')}_q$ with $s^{(i')}_q = s^{(i)}_j$ and $p^{(i)}_k - p^{(i')}_\ell = p^{(i)}_j - p^{(i')}_q$.

Here, we used the same notation as in the proof of Corollary 5.15.

Remark 6.2. Notice that, once the collection $(K_i)_{i\in[n]}$ is fixed, for any $k \in K_i$, there is at most one vertex $v^{(i)}_j$ on each segment such that (i, j, k) is terminal. The only case in which a segment has no such vertex is when the starting point of the segment is contained in one of the K_j's (and hence, the whole segment is contained in $\bigcup_{i\in[n]} K_i$).

Example 6.3. The terminal triples for the cell of L_2 from Example 5.5 are $(2, 2, 1)$ and $(2, 4, 3)$. The corresponding terminal vertices are $v^{(2)}_2$ and $v^{(2)}_4$.

Proposition 6.4. *Let M be a nilpotent representation of Δ_n with d_0 indecomposable direct summands, and let $\boldsymbol{e} \leq \mathbf{dim}\, M$ be such that $\mathrm{Gr}_{\boldsymbol{e}}(M)$ is nonempty. Let $T = (\mathbb{C}^*)^{d_0+1}$ act as in Section 5B. Then,*

(1) *the number of one-dimensional T-orbits on $\mathrm{Gr}_{\boldsymbol{e}}(M)$ is finite,*

(2) *the one-dimensional orbits contained in the cell W_L are parametrised by the terminal triples for W_L.*

Proof. Since any cell W_L is T-stable, any T-orbit is contained in a unique cell, and we can therefore apply the coordinate description from (5-8), to analyse the T-orbit of a point. Assume to have a point in the cell corresponding to the collection $(K_i)_{i\in[n]}$ and consider its coordinate description $((u^{(i)}_{j,k}))$.

By (5-16),

$$T.((u^{(i)}_{j,k})) = \Big\{\big((u^{(i)}_{j,k}\gamma_0^{p_j^{(i)}-p_k^{(i)}}\gamma_{s_j^{(i)}}\gamma_{s_k^{(i)}}^{-1})_{j,k}\big)_i \mid \gamma_0, \gamma_1, \dots, \gamma_{d_0} \in \mathbb{C}^*\Big\}.$$

Observe that $((u^{(i)}_{j,k}))$ consists of all 0's if and only if the corresponding point is the fixed point of W_L. Since we are interested in one-dimensional orbits, we can assume that there is at least one nonzero entry, say $u^{(i)}_{j,k}$. We see immediately that if $u^{(i)}_{j,k} \neq 0$, then the orbit is one-dimensional only if $u^{(r)}_{h,\ell} = 0$ unless $s^{(r)}_h = s^{(i)}_j$, $s^{(r)}_\ell = s^{(i)}_k$, and $p^{(r)}_h - p^{(r)}_\ell = p^{(i)}_j - p^{(i)}_k$.

Since we have assumed that $u^{(i)}_{j,k} \neq 0$, there exists a terminal triple (r, h, ℓ) such that $s^{(r)}_\ell = s^{(i)}_k$, $s^{(r)}_h = s^{(i)}_j$ and $p^{(r)}_h - p^{(i)}_j \geq 0$. We show now by induction on $p^{(r)}_h - p^{(i)}_j$ that $u^{(i)}_{j,k} = u^{(r)}_{h,\ell}$. If $p^{(r)}_h - p^{(i)}_j = 0$, then $(i, j, k) = (r, h, \ell)$ and the statement is trivial. Otherwise, $M_i v^{(i)}_j \neq 0$, and by induction, $u^{(i+1)}_{j',k'} = u^{(r)}_{h,\ell}$. Observe that if $q \in K_{i+1} \setminus K'_i$, then $s^{(i+1)}_q \neq s^{(i)}_k$, and hence, $u^{(i+1)}_{j',q} = 0$. Therefore, by (5-9), we conclude that $u^{(i)}_{j,k} = u^{(r)}_{h,\ell}$ as desired. This also implies $p^{(r)}_h - p^{(r)}_\ell = p^{(i)}_j - p^{(i)}_k$.

From what we have just discussed, we conclude that there is at most one orbit of dimension one for any terminal triple. Since there are only finitely many terminal triples, we deduce that T acts on $\mathrm{Gr}_{\boldsymbol{e}}(M)$ with a finite number of one-dimensional orbits.

To prove that there is a one-dimensional orbit for any terminal triple (r, h, ℓ), we only have to observe that the tuple $(u^{(i)}_{j,k})$ given by

$$\mu^{(i)}_{j,k} = \begin{cases} 1, & \text{if } s^{(i)}_j = s^{(r)}_h,\ s^{(i)}_k = s^{(r)}_\ell \text{ and } p^{(r)}_h - p^{(i)}_j = p^{(r)}_\ell - p^{(i)}_k \geq 0, \\ 0, & \text{otherwise,} \end{cases}$$

satisfies both (5-9) and (5-10), and its orbit is one-dimensional by the above considerations. ☐

Corollary 6.5. *Let M be a nilpotent representation of Δ_n with d_0 indecomposable direct summands, and let $\boldsymbol{e} \leq \mathbf{dim}\, M$ be such that $\mathrm{Gr}_{\boldsymbol{e}}(M)$ is nonempty. Then for any $L \in \mathrm{Gr}_{\boldsymbol{e}}(M)^T$,*

$$\dim W_L = \#\{(i, j, k) \mid (i, j, k) \textit{ is a terminal triple for } W_L\}.$$

Proof. By the previous result, all one-dimensional T-orbits in W_L, which contain the fixed point L in their closures, are parametrised by the set of terminal triples for W_L. By [Brion 1999, Section 1.4, Corollary 2], the number of closed curves through L (which is finite) coincides with the dimension of W_L, since L is the unique isolated fixed point of the rationally smooth T-variety W_L. ☐

Theorem 6.6. *Let M be a nilpotent representation of Δ_n with d_0 indecomposable direct summands, and let $\boldsymbol{e} \leq \mathbf{dim}\, M$ be such that $\mathrm{Gr}_{\boldsymbol{e}}(M)$ is nonempty. Let $T := (\mathbb{C}^*)^{d_0+1}$ act on $\mathrm{Gr}_{\boldsymbol{e}}(M)$ as in Section 5B. Then $(\mathrm{Gr}_{\boldsymbol{e}}(M), T)$ is a projective BB-filterable GKM-variety.*

Proof. By Corollary 5.15, $\mathrm{Gr}_{\boldsymbol{e}}(M)$ is a BB-filterable projective T-variety. By Proposition 6.4, the number of one-dimensional T-orbits is finite. Hence, the T-action on $\mathrm{Gr}_{\boldsymbol{e}}(M)$ is skeletal, since the number of one-dimensional T-orbits is finite by Theorem 5.14. Then Theorem 1.15 implies that $(\mathrm{Gr}_{\boldsymbol{e}}(M), T)$ is also a GKM-variety. ☐

6B. ***Combinatorial description of the moment graph.*** We have just proven that any nonempty quiver Grassmannian for a nilpotent representation of the quiver Δ_n, admits the structure of a GKM-variety. In order to be able to apply the techniques presented in Section 1, first of all we need to describe the moment graph arising from the torus action.

Definition 6.7. Let Q be a quiver, let M be an object of $\mathrm{rep}_{\mathbb{k}}(Q)$ and let $\boldsymbol{e} \leq \mathbf{dim}\, M$ be such that $\mathrm{Gr}_{\boldsymbol{e}}(M)$ is nonempty. A subquiver Q' of $Q(M, B)$ is said to be *successor closed* with dimension vector $\boldsymbol{e}$ if $\# \, Q'_0 \cap B^{(i)} = e_i$ for any $i \in [n]$, and if for all $a \in Q(M, B)_1$ with $s_a \in Q'_0$, then also $t_a \in Q'_0$.

Denote by $\mathrm{SC}^{Q}_{\boldsymbol{e}}(M)$ the set of successor closed subquivers of $Q(M, B)$ with dimension vector $\boldsymbol{e}$. Notice that each $S \in \mathrm{SC}^{Q}_{\boldsymbol{e}}(M)$ is a collection of (successor closed) subsegments of the segments of $Q(M, B)$. For the rest of this section, we restrict to the case where $Q = \Delta_n$ and a basis B such that $Q(M, B)$ is aligned.

Definition 6.8. Let $S \in \mathrm{SC}^{\Delta_n}_{\boldsymbol{e}}(M)$. A connected subquiver of a segment of S is called a *movable part* of S if it has the same starting point as the segment.

For $S \in \mathrm{SC}^{\Delta_n}_{\boldsymbol{e}}(M)$, we denote by $\mathrm{MP}(S)$ the set of movable parts of S. Notice that all the segments of S are contained in $\mathrm{MP}(S)$.

Definition 6.9. For $S, H \in \mathrm{SC}^{\Delta_n}_{\boldsymbol{e}}(M)$, we say that H is obtained from S by a *fundamental mutation* if we obtain H from S by moving down exactly one movable part of S.

The definition of successor closed subquiver is known and, for example, is used in [Cerulli Irelli 2011], whereas movable parts and fundamental mutations are new to the current paper.

Remark 6.10. Whenever we speak about subquivers of $Q(M, B)$, we mean, by abuse of terminology, full subquivers, so that they are uniquely determined by their set of vertices. In the above definition, the quiver H is the full subquiver of $Q(M, B)$, whose set of vertices is obtained by removing from S_0 the set of vertices belonging to the movable part and adding the set of vertices corresponding to the target vertices of the mutation.

Remark 6.11. In Definition 6.9, downwards means that the operation is index increasing in our preferred basis B (such that $Q(M, B)$ is aligned): if $v^{(i)}_k$ is the starting point of the movable part of S, then it can only be moved to some $v^{(i)}_j$ with $j > k$ (and $v^{(i)}_j \notin S_0$). The condition $S, H \in \mathrm{SC}^{\Delta_n}_{\boldsymbol{e}}(M)$ implies that the target $v^{(r)}_h$ of the end point $v^{(r)}_\ell$ of the moved part is

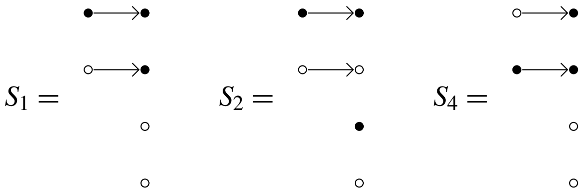
(1) either the end point of the segment $s^{(r)}_h$ of $Q(M, B)$ (i.e., $M_i v^{(r)}_h = 0$),

(2) or the predecessor (in $Q(M, B)$) of the starting point of the segment $s^{(r)}_h \cap H$ (i.e., $v^{(r+1)}_{h'} \in H_0$).

Notice that a fundamental mutation of $S \in \mathrm{SC}^{\Delta_n}_{\boldsymbol{e}}(M)$ is uniquely determined by the vertices $v^{(r)}_\ell$ and $v^{(r)}_h$ above. For convenience, we will denote such a fundamental mutation by $\mu_{(s^{(r)}_\ell, p^{(r)}_\ell), (s^{(r)}_h, p^{(r)}_h)}$.

Example 6.12. Consider M and $\boldsymbol{e}$ as in Example 5.5. To describe an element S of $\mathrm{SC}^{\Delta_2}_{\boldsymbol{e}}(M)$, we take $Q(M, B)$ and fill in white the vertices which are not contained in S_0. The fixed points L_1, L_2 and L_4

have the following coefficient quivers:



The mutation $\mu_{(2,1),(3,0)} : S_1 \to S_2$ is subject to Condition (1) of Remark 6.11, whereas the mutation $\mu_{(1,0),(2,0)} : S_1 \to S_4$ is subject to Condition (2) of Remark 6.11.

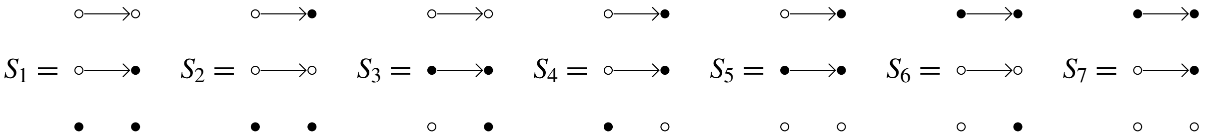
Remark 6.13. There are no restrictions on the difference $j-k$ in Remark 6.11, so that it can happen that a fundamental mutation is the concatenation of two (or more) other fundamental mutations.

Example 6.14. Let $Q = \Delta_2$, and take the Δ_2-representation

$$M = U(1;1) \oplus U(2;2) \oplus U(2;2) \oplus U(2;1)$$

and $\boldsymbol{e} := (1,2)$. We now apply the above constructions in this setting.

The quiver Grassmannian $\mathrm{Gr}_{\boldsymbol{e}}(M)$ is isomorphic to the Feigin degeneration of the classical flag variety $\mathcal{F}l_3$, as explained in [Cerulli Irelli et al. 2012, Proposition 2.7]. With the algorithm as described in the proof of Proposition 4.8, we obtain the following coefficient quiver for M: The successor closed subquivers in the set $\mathrm{SC}^{\Delta_2}_{\boldsymbol{e}}(M)$ are



Observe that the basis of M, as obtained with the algorithm from Proposition 4.8, is a permutation of the basis described in [Cerulli Irelli et al. 2017, Section 6.4] and used in [Cerulli Irelli et al. 2013a, Remark 3.14]. Here, we order the segments ending over a fixed vertex from long to short. For their computations shorter segments are above the longer ones if they end over the same vertex, see [Cerulli Irelli et al. 2017, Example 4]. For the Feigin-degenerate flag variety, their order of segments is described in Section 7A2.

To determine which fundamental mutation we are applying, we enumerated the four segments of $Q(M,B)$, increasingly from the top to the bottom and from left to right. From S_4, we can obtain S_1 via the mutation $\mu_{(1,1),(4,0)}$, or by first applying $\mu_{(2,1),(4,0)}$ which gives us S_2 and then applying $\mu_{(1,1),(2,1)}$.

The combinatorics of such moves on coefficient quivers can be used to describe the moment graph associated to the T-action on $\mathrm{Gr}_{\boldsymbol{e}}(M)$.

For $(\gamma_0, \dots, \gamma_{d_0}) \in T$, we define

$$\epsilon_i : T \to \mathbb{C}^*, \quad (\gamma_0, \gamma_1, \dots, \gamma_{d_0}) \mapsto \gamma_i, \quad \text{where } i \geq 1,$$

and

$$\delta : T \to \mathbb{C}^*, \quad (\gamma_0, \gamma_1, \dots, \gamma_{d_0}) \mapsto \gamma_0.$$

Theorem 6.15. *Let M be a nilpotent Δ_n-representation. The vertices of the moment graph $\mathcal{G}_{\boldsymbol{e}}(M)$ are in bijection with the set of successor closed subquivers $SC_{\boldsymbol{e}}^{\Delta_n}(M)$. For $S, H \in SC_{\boldsymbol{e}}^{\Delta_n}(M)$, there exists an arrow from S to H in the moment graph if and only if there exists a fundamental mutation $\mu_{(i,p),(j,q)}(S) = H$. If this is the case, the label of such an edge is given by $\epsilon_j - \epsilon_i + (q-p)\delta$.*

Proof. The successor closed subquivers in the set $SC_{\boldsymbol{e}}^{\Delta_n}(M)$ are in bijection with the $\mathbb{C}^*$-fixed points of $\mathrm{Gr}_{\boldsymbol{e}}(M)$ [Cerulli Irelli 2011, Proposition 1]. By Theorem 5.14, the $\mathbb{C}^*$-fixed points are exactly the T-fixed points and, hence, the vertices of the moment graph.

Let $S \in SC_{\boldsymbol{e}}^{\Delta_n}(M)$. Under the bijection from [Cerulli Irelli 2011, Proposition 1], we have that the corresponding collection $(K_i)_{i\in[n]}$ is given by $K_i := S_0 \cap B^{(i)}$. Let L be the corresponding quiver representation. It follows from Remark 6.11 that there is a bijection

$$\left\{(r,h,\ell) \;\middle|\; \begin{matrix}(r,h,\ell) \text{ is a terminal}\\ \text{triple for } W_L\end{matrix}\right\} \leftrightarrow \left\{\mu_{(i,p)(j,q)} \;\middle|\; \begin{matrix}\text{there exists } H \in SC_{\boldsymbol{e}}^{\Delta_n}(M)\\ \text{such that } H = \mu_{(i,p)(j,q)}S\end{matrix}\right\}$$

which sends (r,h,ℓ) to the fundamental mutation $\mu_{(s_\ell^{(r)},p_\ell^{(r)}),(s_h^{(r)},p_h^{(r)})}$. The edge label is a direct consequence of the description of the T-action in the proof of Proposition 6.4. □

Remark 6.16. The orientation of the moment graph depends on the choice of the basis B for the Q-representation M and on the attractive grading on $Q(M,B)_0$, which determines the cocharacter $\chi : \mathbb{C}^* \to T$ as in Theorem 5.14. But the vertices and unoriented edges only depend on the T-action, which is independent of the order of the segments in the coefficient quiver. Hence, the unoriented graph for Example 6.14 coincides with the one given in Section 7A2.

Remark 6.17. We can equip the set $SC_{\boldsymbol{e}}^{\Delta_n}(M)$ with a partial order, given by the transitive closure of the relation $H \le S$ if $H = \mu_{(h,p),(\ell,q)}(S)$. It is then possible to refine this to a total order, which is compatible with Theorem 1.15. This is used to determine the order of the fixed points in Example 6.14 and Example 7.7.

7. Special cases

7A. ***Quiver Grassmannians for equioriented quivers of type A.*** The constructions, as introduced in Section 4 and Section 5, also apply to equioriented quivers of type A. The following result is a special case of Theorem 6.6:

Corollary 7.1. *Let Q be an equioriented quiver of type A on n vertices. Let M be a representation of Q with d_0 indecomposable direct summands. Take a dimension vector $\boldsymbol{e} \le \mathbf{dim}\, M$ such that $\mathrm{Gr}_{\boldsymbol{e}}(M)$ is nonempty. With respect to the action of the torus $T := (\mathbb{C}^*)^{d_0+1}$, this quiver Grassmannian is a projective BB-filterable GKM-variety.*

Proof. All indecomposable representations of Q are indecomposable nilpotent representations of Δ_n. □

7A1. *Recovering the Bruhat graph.* Let $\mathcal{F}l_{n+1}$ denote the variety of complete flags of subspaces in $\mathbb{C}^{n+1}$ as in Example 3.7. This variety can be obtained as $\mathrm{Gr}_{\boldsymbol{e}}(M)$ for $M = U_{1,n} \otimes \mathbb{C}^{n+1}$ and $\boldsymbol{e} = (1, 2, \dots, n)$. The coefficient quiver consists of $n+1$ segments, all of length n, starting in 1. A subquiver S of $Q(M,B)$

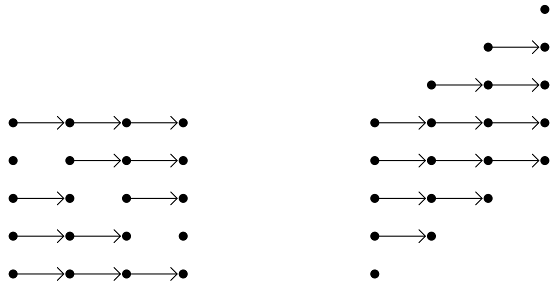


Figure 3. For $n = 4$, the coefficient quiver of this representation (left) and after reordering the segments (right).

is successor closed if and only if $K_i := S_0 \cap B^{(i)} \subset K_{i+1} := S_0 \cap B^{(i+1)}$ for any $i \in [n-1]$. Therefore, $SC_{\boldsymbol{e}}^{\Delta_n}(M)$ is in bijection with the set S_{n+1} of permutations of $[n+1]$: if $S \in SC_{\boldsymbol{e}}^{\Delta_n}(M)$, then the corresponding permutation σ_S sends $i \in [n+1]$ to the unique element, contained in $K_i \setminus K_{i-1}$, where, by convention, $K_0 = \varnothing$ and $K_{n+1} = [n+1]$.

Let $v_k^{(i)}$ be the starting point of a segment in $S \in SC_{\boldsymbol{e}}^{\Delta_n}(M)$, and assume that $j > k$ is such that $v_j^{(i)} \notin K_i$, then there is exactly one movable part of the i-th segment which can be moved to the segment j. We observe that, in permutation terms, this is equivalent to left multiplying σ_S by (i, j). Since the end point $v_k^{(r)}$ of the movable part lies on the k-th segment in position r, and the vertex $v_j^{(r)}$ lies on the j-th segment, also in position r, we deduce that the corresponding edge is labelled by the torus character $\epsilon_j - \epsilon_i$. Thus, the 0-th coordinate of any element $(\gamma_0, \gamma_1, \dots, \gamma_{n+1}) \in T = \mathbb{C}^* \times (\mathbb{C}^*)^{n+1} = \mathbb{C}^* \times T'$ acts trivially, and the action of T' on $\mathcal{F}l_{n+1}$ coincides with the action of the maximal torus of diagonal matrices in GL_{n+1} (induced by the natural action of GL_{n+1} on $\mathbb{C}^{n+1}$).

Remark 7.2. The moment graph we have just described is the so-called Bruhat graph. The partial order, obtained as in Remark 6.17, is in this case nothing but the (opposite) Bruhat order.

7A2. *Feigin degeneration of* $\mathcal{F}l_{n+1}$. Replacing the identity maps of the quiver representation M from the previous subsection by arbitrary linear maps, we obtain the linear degenerations of the flag variety introduced in [Cerulli Irelli et al. 2017]. It is, therefore, possible to apply Theorem 6.15 and Theorem 2.12 to this class of varieties.

In particular, we can recover the moment graph, for the Feigin degeneration of the flag variety (denoted by $\mathcal{F}l_{n+1}^a$), as constructed in [Cerulli Irelli et al. 2013a, Section 3.2]. This is a special degeneration where we replace the identity map along the i-th arrow by the projection pr_{i+1}.

For $n = 4$, the coefficient quiver of this representation is displayed on the left in Figure 3 and after reordering the segments, we arrive at the right picture in Figure 3. The basis corresponding to the right picture satisfies the assumptions of an aligned coefficient quiver and is the same as described in [Cerulli Irelli et al. 2013a, Remark 3.14]. We obtain an attractive grading if we use the row indices as weights for the corresponding basis vectors. Hence, we can apply the results from Section 6 in this setting.

By [Cerulli Irelli et al. 2013a, Equation (2.1), Remark 3.14], the admissible collections $\boldsymbol{S}$, which parametrise the vertices of the moment graph, are a special case of the tuple of index sets $(K_i)_{i \in [n]}$, as in Lemma 5.4. The one-dimensional orbits, starting at the vertex parametrised by $\boldsymbol{S}$, are parametrised by $\boldsymbol{S}$-effective pairs [Cerulli Irelli et al. 2013a, Definition 3.5]. Their geometric interpretation in [Cerulli Irelli et al. 2013a, Remark 3.6] coincides with the description of fundamental mutations of the coefficient quiver, corresponding to the admissible collection $\boldsymbol{S}$. Hence, the indices of the $\boldsymbol{S}$-effective pairs and the leading indices of the fundamental mutations coincide, if we use the same order of the indecomposable summands of the quiver representation.

The structure of the unoriented moment graph is independent of the chosen basis, as long as the basis yields an aligned coefficient quiver. But to recover the edge labels as in [Cerulli Irelli et al. 2013a, Theorem 3.18], we have to use the basis described above instead of the one from Example 6.14. Nevertheless, the edge labels are slightly different, since our torus has one additional parameter, which is required to obtain finitely many one-dimensional orbits for general quiver Grassmannians for the cyclic quiver.

For the equioriented quiver of type A, the parameter γ_0 is not necessary to obtain the structure of a GKM-variety. Instead we can work with the subtorus T' as defined in Section 5B, where we set γ_0 equal to one. In the following example, we exhibit the different edge labels for $n = 3$:

Example 7.3. The T- (and T'-)fixed points in $\mathcal{F}l_3^a$ are given by

$$\begin{tikzcd}[column sep=small]
 & & \circ & & & \bullet & & & \circ & & & \bullet & & & \circ & & & \bullet & & & \bullet \\
p_1 = & \circ \arrow[r] & \bullet & p_2 = & \circ \arrow[r] & \circ & p_3 = & \circ \arrow[r] & \bullet & p_4 = & \circ \arrow[r] & \bullet & p_5 = & \bullet \arrow[r] & \bullet & p_6 = & \circ \arrow[r] & \circ & p_7 = & \bullet \arrow[r] & \bullet \\
 & \circ \arrow[r] & \bullet & & \circ \arrow[r] & \bullet & & \bullet \arrow[r] & \bullet & & \circ \arrow[r] & \circ & & \circ \arrow[r] & \bullet & & \bullet \arrow[r] & \bullet & & \circ \arrow[r] & \circ \\
 & \bullet & & & \bullet & & & \circ & & & \bullet & & & \circ & & & \circ & & & \circ &
\end{tikzcd}$$

In Figure 4 (left), we have the moment graph for the T-action on $\mathcal{F}l_3^a$ and in Figure 4 (right), we have the moment graph for the T'-action, which coincides with the results in [Cerulli Irelli et al. 2013a, Example 3.17, Theorem 3.18].

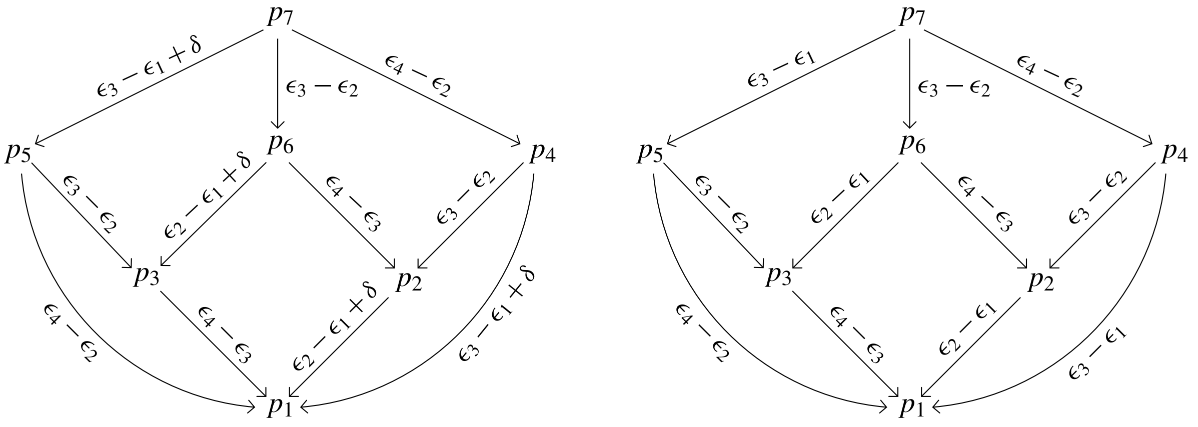


Figure 4. The moment graph for the T-action on $\mathcal{F}l_3^a$ (left), and the moment graph for the T'-action (right).

7B. ***Linear degenerations of the affine Grassmannian and the affine flag variety of type A.*** We briefly recall the definition of the affine Grassmannian, its linear degenerations and their finite-dimensional approximations (see [Feigin et al. 2017] and [Pütz 2022, Section 3 and 6] for more details). Let $P \subset \widehat{\mathfrak{gl}}_n$ be the maximal parahoric subgroup. The *affine Grassmannian* of type $\mathfrak{gl}_n$ is defined as $\mathrm{Gr}(\widehat{\mathfrak{gl}}_n) := \widehat{\mathfrak{gl}}_n/P$. For $\ell \in \mathbb{Z}$, define

$$V_\ell := \mathrm{span}(v_\ell, v_{\ell-1}, v_{\ell-2}, \dots)$$

as the subspace of the infinite-dimensional vector space V with basis vectors v_i for all $i \in \mathbb{Z}$. The *Sato Grassmannian* SGr_k for $k \in \mathbb{Z}$ is defined as

$$\mathrm{SGr}_k := \big\{U \subset V \mid \text{there exists an } \ell < k \text{ s.t. } V_\ell \subset U \text{ and } \dim U/V_\ell = k - \ell\big\}.$$

Proposition 7.4 (cf. [Feigin et al. 2017, Section 1.3]). *The affine Grassmannian* $\mathrm{Gr}(\widehat{\mathfrak{gl}}_n)$ *as a subset in the Sato Grassmannian is parametrised as*

$$\mathrm{Gr}(\widehat{\mathfrak{gl}}_n) \cong \{U \in \mathrm{SGr}_0 \mid U \subseteq s_n U\},$$

where $s_n : V \to V$ *maps* v_i *to* v_{i+n} *for all* $i \in \mathbb{Z}$.

It is possible to define linear degenerations in the same way as for the affine flag variety (see [Pütz 2022, Definition 6.2]). For a linear map $f : V \to V$, the *f-linear degenerate affine Grassmannian* is defined as

$$\mathrm{Gr}^f(\widehat{\mathfrak{gl}}_n) := \{U \in \mathrm{SGr}_0 \mid fU \subseteq U\}.$$

The degeneration

$$\mathrm{Gr}^a(\widehat{\mathfrak{gl}}_n) := \{U \in \mathrm{SGr}_0 \mid s_{-n} \circ \mathrm{pr}_1 \circ \mathrm{pr}_2 \circ \cdots \circ \mathrm{pr}_n U \subseteq U\}$$

was already studied in [Feigin et al. 2017]. For a positive integer $N \in \mathbb{N}$, the finite approximation of the f-linear degenerate affine Grassmannian is defined as

$$\mathrm{Gr}^f_N(\widehat{\mathfrak{gl}}_n) := \{U \in \mathrm{Gr}^f(\widehat{\mathfrak{gl}}_n) \mid V_{-N} \subseteq U \subseteq V_N\}.$$

Utilising the finite-dimensional vector space

$$V_{(N)} := \mathrm{span}(v_N, v_{N-1}, \dots, v_{-N+2}, v_{-N+1}),$$

we can identify each finite-dimensional approximation with a quiver Grassmannian (see [Pütz 2022, Theorem 6.3]):

$$\mathrm{Gr}^f_N(\widehat{\mathfrak{gl}}_n) \cong \mathrm{Gr}_N(M^f_N),$$

where $M^f_N := (V_{(N)}, f|_{V_{(N)}})$. A linear degeneration is called nilpotent if $f|_{V_{(N)}}$ is nilpotent for every $N \in \mathbb{N}$. In this case, M^f_N is a nilpotent representation of the loop quiver (see [Pütz 2022, Remark 6.5]). The isomorphism classes of nilpotent degenerations are parametrised by the corank of f [Pütz 2022, Section 6.4]. We call $\mathrm{Gr}^f_N(\widehat{\mathfrak{gl}}_n)$ a partial degeneration if the corank of f is between the corank of s_{-n} and the corank of $s_{-n} \circ \mathrm{pr}_1 \circ \cdots \circ \mathrm{pr}_n$. The corresponding isomorphism classes of nilpotent linear degenerations between

$\mathrm{Gr}(\widehat{\mathfrak{gl}}_n)$ and $\mathrm{Gr}^a(\widehat{\mathfrak{gl}}_n)$ are labelled by the integers $k \in \{0, 1, \dots, n\}$, where zero corresponds to the affine Grassmannian and n to $\mathrm{Gr}^a(\widehat{\mathfrak{gl}}_n)$. Each isomorphism class has the representative

$$\mathrm{Gr}^k(\widehat{\mathfrak{gl}}_n) := \{U \in \mathrm{SGr}_0 \mid s_{-n} \circ \mathrm{pr}_1 \circ \mathrm{pr}_2 \circ \cdots \circ \mathrm{pr}_k \, U \subseteq U\}.$$

Proposition 7.5. *For $k \in \{0, 1, \dots, n\}$, the finite approximations of nilpotent partial degenerations can be realised as quiver Grassmannians in the following way*:

$$\mathrm{Gr}^k_N(\widehat{\mathfrak{gl}}_n) \cong \mathrm{Gr}_{nN}(A_{2N} \otimes \mathbb{C}^{n-k} \oplus A_N \otimes \mathbb{C}^{2k}),$$

where $A_N \cong \mathbb{C}[t]/(t^N)$.

This is a special case of the construction in [Pütz 2022, Theorem 3.7, Lemma 4.14].

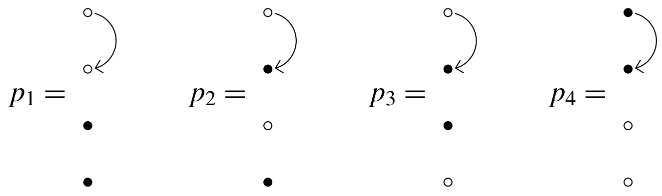
Lemma 7.6. *The quiver Grassmannians providing finite approximations for nilpotent linear degenerations of affine Grassmannians and affine flag varieties are BB-filterable GKM-varieties.*

Proof. The quiver representations as in Proposition 7.5 and [Pütz 2022, Lemma 4.14], which are used to define the approximations, satisfy the assumptions of Theorem 6.6. □

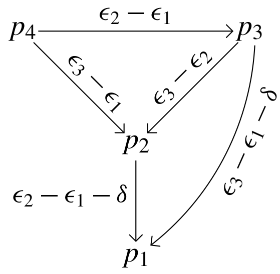
These quiver Grassmannians are, in general, not normal, which implies that it is not possible to apply [Gonzales 2014, Theorem 6.9], in order to compute the S-module basis of the T-equivariant cohomology. Nevertheless, by Theorem 2.12, Gonzales' recipe also works in our setting.

Example 7.7. For $n = 2$, there are three isomorphism classes of linear degenerations of the affine Grassmannian. We want to consider the representative $\mathrm{Gr}^k(\widehat{\mathfrak{gl}}_n)$ for $k = 1$. This corresponds to the intermediate degeneration between the (nondegenerate) affine Grassmannian and its degeneration $\mathrm{Gr}^a(\widehat{\mathfrak{gl}}_2)$.

For $N = 1$, its approximation is isomorphic to the quiver Grassmannian $\mathrm{Gr}_2(M)$ for the representation $M = A_2 \oplus A_1 \oplus A_1$. By Lemma 7.6, we know that this quiver Grassmannian admits a T-action such that it becomes a BB-filterable GKM-variety. We can also apply Theorem 6.15 to compute its moment graph. There are four torus fixed points, corresponding to the following successor closed subquivers:



If we number the segments in the coefficient quiver from top to bottom, we obtain the moment graph



$$\theta_1 = \begin{pmatrix} 1 & & 1 \\ & 1 & \\ & 1 & \end{pmatrix},$$

$$\theta_2 = \begin{pmatrix} 2\epsilon_1 - \epsilon_2 - \epsilon_3 + \delta & & \epsilon_1 - \epsilon_3 + \delta \\ & \epsilon_1 - \epsilon_2 + \delta & \\ & 0 & \end{pmatrix},$$

$$\theta_3 = \begin{pmatrix} (\epsilon_3 - \epsilon_1)(\epsilon_3 - \epsilon_2 - \delta) & & (\epsilon_3 - \epsilon_2)(\epsilon_3 - \epsilon_1 - \delta) \\ & 0 & \\ & 0 & \end{pmatrix},$$

$$\theta_4 = \begin{pmatrix} (\epsilon_3 - \epsilon_1)(\epsilon_2 - \epsilon_1) & & 0 \\ & 0 & \\ & 0 & \end{pmatrix}$$

Figure 5. S-module basis of the T-equivariant cohomology

In particular, we can apply Theorem 2.12 in order to compute the S-module basis for the T-equivariant cohomology. The results of this computation are presented in Figure 5. For the filtered T-stable subvarieties corresponding to the fixed points, we use the same notation as in Theorem 1.15. Observe that Z_4 is not smooth in p_2 and p_3, We apply Lemma 2.1 (3) to compute the equivariant Euler classes $\mathrm{Eu}_T(p_2, Z_4)$ and $\mathrm{Eu}_T(p_3, Z_4)$. These computations are described in the Appendix. For all other equivariant Euler classes we can apply Lemma 2.1 (1).

We conclude this example by describing the ring structure of $H_T^\bullet(\mathrm{Gr}_2(M))$. Under localisation, the addition and multiplication laws, are defined componentwise. It is, hence, immediate to see that θ_1 is the unit of the ring. It is also easy to check that

$$\begin{aligned} \theta_2^2 &= (\epsilon_1 - \epsilon_2 + \delta)\theta_2 + \theta_3 + 2\theta_4, \\ \theta_2\theta_3 = \theta_3\theta_2 &= (\epsilon_1 - \epsilon_3 + \delta)\theta_3 + (\epsilon_2 - \epsilon_3 + \delta)\theta_4, \\ \theta_2\theta_4 = \theta_4\theta_2 &= (2\epsilon_1 - \epsilon_2 - \epsilon_3 + \delta)\theta_4, \\ \theta_3^2 &= (\epsilon_3 - \epsilon_2)(\epsilon_3 - \epsilon_1 - \delta)\theta_3 + (\epsilon_1 - \epsilon_2)(\epsilon_3 - \epsilon_2 - \delta)\theta_4, \\ \theta_3\theta_4 = \theta_4\theta_3 &= (\epsilon_3 - \epsilon_1)(\epsilon_3 - \epsilon_2 - \delta)\theta_4, \\ \theta_4^2 &= (\epsilon_3 - \epsilon_1)(\epsilon_2 - \epsilon_1)\theta_4. \end{aligned}$$

This completely determines the ring structure of $H_T^\bullet(\mathrm{Gr}_2(M))$. We observe that the equivariant cohomology is an S-algebra and that S acts diagonally under localisation. By looking at the above multiplication table, we immediately see that there are two possible subsets of our module basis which generate $H_T^\bullet(\mathrm{Gr}_2(M))$ as an S-algebra: $\{\theta_2, \theta_3\}$ and $\{\theta_2, \theta_4\}$. Recall that we have been focusing on cohomology with rational coefficients. Actually, our basis $\{\theta_1, \dots, \theta_4\}$ generates the equivariant cohomology with coefficients in any field. On the other hand, if the field has characteristic 2, only $\{\theta_2, \theta_4\}$ generates $H_T^\bullet(\mathrm{Gr}_2(M))$ as an algebra.

8. Open problems and further research directions

Constructions and results presented in this paper are a first step towards the application of moment graph techniques to the investigation of quiver Grassmannians, related combinatorics and representation theory. We believe that this is only the tip of the iceberg. We list here some of the questions which remain to be addressed.

8A. ***Explicit formulae for the θ-basis.*** In the classical setting of flag varieties, it is possible to give an explicit formula for any entry of the GKM-presentation of an equivariant Schubert class. This formula is known as the Billey formula because of the article [Billey 1999], but had already been noticed by Andersen, Jantzen and Soergel [Andersen et al. 1994, Appendix D]. Explicit, positive Billey formulae for varieties other than flag varieties were asked for by Tymoczko [2016, Question 15].

It would be hence interesting to provide such formulae at least for some special class of nilpotent Δ_n-representations.

Recall that the determination of the θ basis relies on the computation of equivariant Euler classes, so that to get an analogue of Billey formula it is first necessary to have an explicit formula for such classes. In the flag variety case this is achieved by Arabia [1998, Section 2.7, Equation (27)] by exploiting Bott–Samelson resolutions. Therefore, a strategy to obtain the desired formulae is to first find an equivariant desingularisation of the quiver Grassmannian of interest, then compute the equivariant Euler classes and finally use them to determine the formula for the θ-basis, as we do in Example 7.7.

After this paper was written, (equivariant) resolutions for a (very special) class of nilpotent representations for the equioriented cycle have been constructed in [Feigin et al. 2023]. This construction has not been applied to calculate equivariant Euler classes yet.

8B. ***Extension to a broader class of quiver Grassmannians.*** For string representations, $\mathbb{C}^*$-actions on quiver Grassmannians have been studied by Cerulli Irelli [2011]. These are representations such that there exists a basis for which the coefficient quiver of the representation consists only of orientations of Dynkin diagrams of type $\mathtt{A}$. He gives a combinatorial description of the fixed points in terms of successor closed subquivers in the coefficient quiver. In the case of nilpotent representations of the equioriented cycle, the same description is valid for the fixed points of the higher rank torus as introduced above (see Theorem 6.15). The results of Cerulli Irelli [2011] were generalised to the setting of tree and band representations by Haupt [2012]. These are representations with coefficient quivers consisting of oriented trees and bands.

The $\mathbb{C}^*$-action, as introduced in [Cerulli Irelli 2011], is subject to slightly less restrictive conditions than our assumptions in the definition of attractive gradings (see Definition 3.10). Hence, we believe that it is possible to generalise our results (concerning the cellular decomposition, torus action and the moment graph structure) from the present paper to the setting of string, band and tree representations which admit an attractive grading and some sort of aligned coefficient quiver (see Remark 4.9). This combination was important to prove existence of the cellular decomposition.

Thus, the first step towards a generalisation would be to find the class of quiver representations which admit attractive gradings and aligned coefficient quivers. Then one has to check if these assumptions are sufficient to obtain a cellular decomposition of the corresponding quiver Grassmannians. Next, one has to adapt the construction for the action of the larger torus to this setting and check if it has the desired properties.

8C. ***Applications to geometric representation theory.*** Geometric representation theory exploits geometric tools to investigate representations of groups or algebras. If on one hand the geometric realisation of a representation allows one to apply geometric methods, on the other hand it is often not very explicit. Tymoczko [2008b] studied Weyl group representations on cohomology rings of Schubert varieties via GKM-theory, obtaining hence an explicit description of the space and at the same time the desired geometric construction. In the survey paper [Tymoczko 2008a], she proposes the challenge to find other spaces whose (equivariant) cohomology rings are endowed with group actions and which can be described via GKM-theory.

After the first version of this paper was written, we managed in [Lanini and Pütz 2023] to extend Tymoczko's work [2008b] and equip, under some technical assumptions, the equivariant cohomology of quiver Grassmannians for nilpotent representations of the equioriented cycle with the action of appropriate products of symmetric groups. These representations were studied and decomposed into irreducible representations in [Lanini and Pütz 2023]. Moreover, under the same technical assumptions, the action of a certain Nil Hecke ring on the equivariant cohomology was obtained (see [Lanini and Pütz 2023, Theorem 6.8]), but not investigated. While in the classical setting of the flag variety the equivariant cohomology is a cyclic module for the action of the Nil Hecke ring, in the quiver Grassmannian case this does not hold any more. Thus, it would be interesting to further investigate this Nil Hecke algebra module structure. For instance, it would be fascinating to determine when the equivariant cohomology is a cyclic module for the Nil Hecke algebra.

8D. ***Sheaves on moment graphs for quiver Grassmannians.*** Given an oriented graph without oriented cycles whose edges are labelled by elements of a $\mathbb{Z}$-module, it is possible to define the corresponding category of sheaves on it (see, for example, [Fiebig 2008]).

If the graph is the moment graph of a torus action on a complex projective GKM-algebraic variety equipped with a T-stable (Whitney) stratification, as in [Braden and MacPherson 2001, Section 1.1], then an appropriate class of sheaves on this moment graph (the so-called BMP-sheaves) allows one to compute T-equivariant intersection cohomology. In general, the cellularisations of the quiver Grassmannians that we obtain in this article are not stratifications. Nevertheless, it would be interesting to find some classes of quiver Grassmannians of nilpotent representations of the equioriented cycle such that we do get a stratification. For example, in the special case treated in [Feigin et al. 2023], a stratification is obtained, and it is, hence, possible to apply Braden–MacPherson theory to determine the equivariant intersection cohomology. It would be interesting to see whether the Poincaré polynomials of the local intersection cohomology groups have interesting combinatorial features, as it happens in the flag variety case, where one gets Kazhdan–Lusztig polynomials.

The study of categories of sheaves on moment graphs coming form the GKM-variety structure on quiver Grassmannians for nilpotent quiver representations has not yet be initiated, but we expect it to be fruitful.

If the moment graph is the Bruhat graph of some Coxeter group (or a parabolic analogue), the full subcategory of BMP sheaves produces a (weak) categorification of the Hecke algebra (or a certain parabolic module over it, see [Lanini 2014]) of the underlying Coxeter group. By [Fiebig 2008], it is in fact a moment graph realisation of the famous category of Soergel bimodules. This fact led to interesting categorical lifting of properties of Kazhdan–Lusztig polynomials (see, e.g., [Lanini 2012; 2015]).

We believe that the investigation of the category $\mathcal{B}$ of BMP-sheaves at the very least for the quiver Grassmannians appearing in [Feigin et al. 2023] is worth to be pursued. In this case, the Grothendieck group of $\mathcal{B}$ has a basis indexed by a combinatorially interesting set, that is the set of Grassmann necklaces, or of juggling patterns.

8E. ***Combinatorics of moment graphs coming from quiver Grassmannians.*** A combinatorial study of the moment graphs obtained by our construction might produce interesting algebro-combinatorial results, as well as have geometrical consequences.

If the underlying GKM variety is coming from a (full) flag variety of an algebraic group (acted upon by a maximal torus of such a group), the obtained moment graph is called Bruhat graph and was firstly considered by Dyer [1991]. There is a vast literature on Bruhat graph combinatorics and applications, and we would be surprised if the combinatorics of our moment graphs were not of some interest itself. For example, in the case of the quiver Grassmannians studied in [Feigin et al. 2023], the resulting moment graphs can be described in terms of Grassmann necklace combinatorics (see [Feigin et al. 2023, Proposition 6.3]).

As for the geometric applications of a combinatorial study of the moment graphs, we only mention the possibility of reading off from the graphs the rational smoothness of the variety at a given fixed point (see, for example, [Brion 1999]). In the case of a Bruhat graph, this led to the so–called Carrel–Peterson sufficient and necessary criterion for a KL-polynomial to be equal to 1. It would be certainly relevant to have a purely combinatorial criterion for a successor closed subquiver to index a rationally smooth point of a quiver Grassmannian for a nilpotent Δ_n-representation.

Appendix: Equivariant desingularisations of quiver Grassmannians

By Lemma 2.1 (3), equivariant desingularisations can be used to compute the equivariant Euler classes at singular points. In [Cerulli Irelli et al. 2013a], desingularisations of quiver Grassmannians for Dynkin quivers are provided. More general constructions for quiver Grassmannians can be found in [Keller and Scherotzke 2014; Scherotzke 2017]. The explicit nature of [Cerulli Irelli et al. 2013a] seemed, to us, to be better suited for our purposes and to the approach of the present article, allowing us to work with a coordinate description, to define torus actions and hence to obtain the needed equivariant resolutions.

We expect that the construction in [Cerulli Irelli et al. 2013a] can be generalised to the equioriented cycle, where some of their key assumptions are not satisfied. For the rest of this appendix, we restrict us to the special case of the quiver Grassmannian from Example 7.7. In this special case, we construct an equivariant resolution of singularities, by adapting methods from [Cerulli Irelli et al. 2013b].

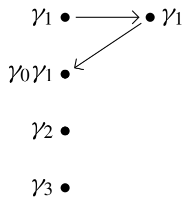
Following the procedure from [Cerulli Irelli et al. 2013b, Section 8.1], we obtain

$$\hat{Q} = \begin{tikzcd} \bullet \arrow[r, bend left, "a"] & \bullet \arrow[l, bend left, "b"] \end{tikzcd} \quad \text{and} \quad \hat{M} = \begin{tikzcd} \mathbb{C}^4 \arrow[r, bend left, "\hat{M}_a"] & \mathbb{C} \arrow[l, bend left, "\hat{M}_b"] \end{tikzcd} \quad \text{with } {}^t\hat{M}_a = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \text{ and } \hat{M}_b = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

Remark A.1. In the notation of the previous sections, we have $\hat{Q} = \Delta_2$ and $\hat{M} = U(1;3) \oplus U(1;1) \otimes \mathbb{C}^2$. Hence, $\hat{M}$ is also a nilpotent representation of a quiver of affine type A. The shape of $\hat{Q}$ depends on the structure of M, and it is, in general, not of the same type as Q.

Observe that $\hat{Q}$ and $\hat{M}$ fail to satisfy the assumptions in [Cerulli Irelli et al. 2013b, Proposition 7.1]. Nevertheless, we obtain similar results about the desingularisation as in [Cerulli Irelli et al. 2013b, Section 7].

Recall that each dot in the coefficient quiver $Q(\hat{M})$ stands for one basis vector of the vector spaces in the representation. We define a T-action on the vector space $\hat{M}_1 \oplus \hat{M}_2$ by declaring that the element $(\gamma_0, \gamma_1, \gamma_2, \gamma_3) \in T$ acts on the basis vectors as follows:



We also consider the $\mathbb{C}^*$-action on $\hat{M}_1 \oplus \hat{M}_2$ induced by the generic cocharacter $\mathbb{C}^* \to T$ given by

$$z \mapsto (z, z, z^3, z^4).$$

It is immediate to see that the above defined T- and $\mathbb{C}^*$-actions on $\hat{M}_1 \oplus \hat{M}_2$ extend to actions on the (nonempty) quiver Grassmannians for $\hat{M}$. Since it corresponds to an attractive grading, the $\mathbb{C}^*$-action induces cellularisations of these quiver Grassmannians by Theorem 5.7.

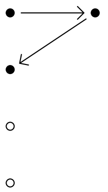
For $U \in \mathrm{Gr}_{\boldsymbol{e}}(M)$, we denote by $\mathcal{S}_U$ the subvariety of $\mathrm{Gr}_{\boldsymbol{e}}(M)$ which consists of subrepresentations in $\mathrm{Gr}_{\boldsymbol{e}}(M)$, which are isomorphic to U (as quiver representations). In $\mathrm{Gr}_2(M)$, there are two isomorphism classes of subrepresentations with representatives $U_1 = A_2$ and $U_2 = A_1 \oplus A_1$. Thus, we have $\mathrm{Gr}_2(M) = \mathcal{S}_{U_1} \sqcup \mathcal{S}_{U_2}$. For U_1 and U_2, we compute the $\hat{Q}$-representations, in the same way as done for M, and obtain $\hat{U}_1 = U(1;3)$ and $\hat{U}_2 = U(1;1) \oplus U(1;1)$. Their dimension vectors are $\mathbf{dim}\,\hat{U}_1 = (2,1)$ and $\mathbf{dim}\,\hat{U}_2 = (2,0)$, and it follows from the definition of $\hat{M}$ that

$$\mathrm{Gr}_{(2,1)}(\hat{M}) \cong \mathrm{Gr}_1(\mathbb{C}^3) \quad \text{and} \quad \mathrm{Gr}_{(2,0)}(\hat{M}) \cong \mathrm{Gr}_2(\mathbb{C}^3). \tag{A-2}$$

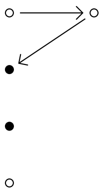
Therefore, $\mathrm{Gr}_{(2,1)}(\hat{M})$ and $\mathrm{Gr}_{(2,0)}(\hat{M})$ are irreducible and smooth.

The top-dimensional cells in both Grassmannians are attractive loci of the fixed points $\hat{q}_1$ and $\hat{q}_2$, which are isomorphic to $\hat{U}_1 \in \mathrm{Gr}_{(2,1)}(\hat{M})$ and $\hat{U}_2 \in \mathrm{Gr}_{(2,0)}(\hat{M})$, respectively, and correspond to the following successor closed subquivers of $Q(\hat{M})$:

$\hat{q}_1 =$



and $\hat{q}_2 =$



The T-action equips both Grassmannians with the structure of a GKM-variety. Starting from these fixed points, we compute the moment graphs of both Grassmannians analogously to Theorem 6.15. Observe that the labels are different from the ones we would get from Theorem 6.15, as we do not use the T-action as defined in Section 5B, but rather an action which is compatible with the T-action on $\mathrm{Gr}_2(M)$. More precisely, the moment graph corresponding to $(\mathrm{Gr}_{(2,1)}(\hat{M}), T)$, respectively to $(\mathrm{Gr}_{(2,0)}(\hat{M}), T)$, is the full (labelled) subgraph of the moment graph in Example 7.7 whose vertex set is $\{p_2, p_3, p_4\}$, respectively $\{p_1, p_2, p_3\}$.

As in [Cerulli Irelli et al. 2013b, Section 7], for $U \in \{U_1, U_2\}$, we define the map

$$\pi_{[U]} : \mathrm{Gr}_{\mathbf{dim}\,\hat{U}}(\hat{M}) \to \mathrm{Gr}_2(M).$$

In our case, it has the explicit form $V = (V_1, V_2) \mapsto V_1$, so that $\pi_{[U_1]}(\hat{q}_1) = p_4$ and $\pi_{[U_2]}(\hat{q}_2) = p_3$. As before, p_3 and p_4 are the fixed points in $\mathrm{Gr}_2(M)$ as computed in Example 7.7.

For our example, the same conclusions as in [Cerulli Irelli et al. 2013b, Theorem 7.5] hold true.

Proposition A.3. *With the same notation as before, for $U \in \{U_1, U_2\}$, we have*

(1) *$Gr_{\mathbf{dim}\,\hat{U}}(\hat{M})$ is smooth and irreducible,*

(2) *the map $\pi_{[U]}$ is projective and T-equivariant,*

(3) *the image of $\pi_{[U]}$ is closed in $\mathrm{Gr}_2(M)$ and contains $\overline{\mathcal{S}_U}$,*

(4) *the map $\pi_{[U]}$ is one-to-one over $\mathcal{S}_U$.*

Proof. Part (1) follows from (A-2), as already noticed. Since both Grassmannians of subspaces are projective, the projectivity of $\pi_{[U]}$ is clear.

The T-equivariance follows immediately from the coordinate description of the maps $\pi_{[U]}$, induced by the cellular decompositions of the involved quiver Grassmannians. The image of $\pi_{[U]}$ is closed, since projective morphisms are closed. By construction $\pi_{[U]}$ is one-to-one even over $\overline{\mathcal{S}_U}$, and hence its image contains $\overline{\mathcal{S}_U}$. □

Corollary A.4. *With the same notation as before, for $U \in \{U_1, U_2\}$, the map*

$$\pi = \coprod_{U \in \{U_1, U_2\}} \pi_{[U]} : \coprod_{U \in \{U_1, U_2\}} \mathrm{Gr}_{\mathbf{dim}\,\hat{U}}(\hat{M}) \to \mathrm{Gr}_2(M)$$

is a T-equivariant desingularisation of $\mathrm{Gr}_2(M)$.

Proof. By Proposition A.3, π is T-equivariant, and the closure of $\mathcal{S}_{\hat{U}}$ is the whole quiver Grassmannian $\mathrm{Gr}_{\mathbf{dim}\,\hat{U}}(\hat{M})$. Hence, it is smooth and of the same dimension as the quiver Grassmannian. The rest of the proof is analogous to the proof of [Cerulli Irelli et al. 2013b, Corollary 7.7]. □

By Corollary A.4, we can apply Lemma 2.1 (3) to π and compute

$$\begin{aligned}\mathrm{Eu}_T(p_2, Z_4)^{-1} &= \frac{\epsilon_2 - \epsilon_3 - \delta}{(\epsilon_3 - \epsilon_1)(\epsilon_3 - \epsilon_2)(\epsilon_2 - \epsilon_1 - \delta)},\\ \mathrm{Eu}_T(p_3, Z_4)^{-1} &= \frac{\epsilon_2 - \epsilon_3 + \delta}{(\epsilon_3 - \epsilon_2)(\epsilon_2 - \epsilon_1)(\epsilon_3 - \epsilon_1 - \delta)}.\end{aligned}$$

Remark A.5. In particular, this example is compatible with the irreducibility conjecture for the resolving quiver Grassmannians $Gr_{\mathbf{dim}\,\hat{U}}(\hat{M})$, as stated in [Cerulli Irelli et al. 2013b, Remark 7.8], whereas in general quiver Grassmannians for the cycle are not irreducible.

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The de Rham–Fargues–Fontaine cohomology

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We show how to attach to any rigid analytic variety V over a perfectoid space P a rigid analytic motive over the Fargues–Fontaine curve $\mathcal{X}(P)$ functorially in V and P. We combine this construction with the overconvergent relative de Rham cohomology to produce a complex of solid quasicoherent sheaves over $\mathcal{X}(P)$, and we show that its cohomology groups are vector bundles if V is smooth and proper over P or if V is quasicompact and P is a perfectoid field, thus proving and generalizing a conjecture of Scholze. The main ingredients of the proofs are explicit $\mathbb{B}^1$-homotopies, the motivic proper base change and the formalism of solid quasicoherent sheaves.

1. Introduction

The aim of this article is twofold. On the one hand, we define a *relative* version of the overconvergent de Rham cohomology for rigid analytic varieties over an (admissible) adic space S in characteristic zero, generalizing the work of Große-Klönne [2000; 2002; 2004] for rigid varieties over a field. We prove that this cohomology theory can be canonically defined for any variety X locally of finite type over S, takes values in the infinity-category of solid quasicoherent $\mathcal{O}_S$-modules, in the sense of Clausen and Scholze [2020], is functorial, has étale descent and is $\mathbb{B}^1$-invariant. In particular, we deduce that it is *motivic*, i.e., it can be defined as a contravariant realization functor

$$\mathrm{dR}_S : \mathrm{RigDA}(S) \to \mathrm{QCoh}(S)^{\mathrm{op}}$$

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on the (unbounded, derived, stable, étale) category $\mathrm{RigDA}(S)$ of rigid analytic motives over S with values in the infinity-category of solid quasicoherent $\mathcal{O}_S$-modules. As a matter of fact, in order to prove the properties above we make extensive use of the theory of motives, and more specifically of their six-functor formalism [Ayoub et al. 2022] and of a homotopy-based relative version of Artin's approximation lemma (Theorem 3.9) inspired by the absolute motivic proofs given in [Vezzani 2018]. If X is a proper smooth rigid variety over S, $\mathrm{dR}_S(X)$ is a perfect complex, whose cohomology groups are vector bundles. To prove this finiteness result, we combine the characterization of dualizable objects in $\mathrm{QCoh}(S)$ due to Andreychev [2021] (see also [Scholze 2020]), the motivic proper base change and the "continuity" property for rigid analytic motives (see [Ayoub et al. 2022]). The latter result, which is based on the use of explicit rigid homotopies, states that whenever one has a weak limit of adic spaces (in the sense of Huber) $X \sim \varprojlim X_i$, any compact motive over X has a model over some X_i. We apply this fact to reduce ourselves to the case $S = \operatorname{Spa} A$ with A being a classical Tate algebra, and eventually to the case of a field $S = \operatorname{Spa}(K, K^\circ)$, by considering the limit $x \sim \varprojlim_{x \in U} U$ whenever x is a closed point (a technique that was already exploited in [Scholze 2012]).

On the other hand, in the second part of this paper, we define a motivic version of a pullback functor along the relative Fargues–Fontaine curve that works for smooth rigid analytic *varieties* over a perfectoid space P in positive characteristic. More specifically, we define a monoidal functor $\mathcal{D}$ from rigid analytic motives over P to the category of rigid analytic motives over the relative Fargues–Fontaine curve $\mathcal{X}(P)$. This lets us associate to an adic space V which is locally of finite type over P the motive of a *rigid analytic variety* over $\mathcal{X}(P)$ (and not a relatively perfectoid space!). Let us sketch the simple idea of the construction in the case where $P = \operatorname{Spa}(C, C^\circ)$, with C a complete algebraically closed nonarchimedean field of characteristic p. The adic space $\mathcal{Y}_{[0,\infty)}(C)$, as defined by Fargues and Fontaine, is equipped with an action of Frobenius φ such that, for any quasicompact neighborhood U of the point C, one has $U \subset \varphi(U)$. By motivic continuity applied to $\operatorname{Spa} C \sim \varprojlim_{\varphi_*} U$ we can extend any motive V over C to some motive $U(V)$ defined on U. We may also extend the (motivically invertible!) geometric Frobenius map $\varphi^* V \cong V$ to some gluing datum $U(V) \cong (\varphi_* U(V))|_U$ enabling us to stretch $U(V)$ to $\mathcal{Y}_{[0,\infty)}(C)$ and eventually to $\mathcal{X}(C)$.

This motivic take on Dwork's trick (see, for example, [de Jong 1998; Kedlaya 2005]) admits an explicit description when applied to varieties with good reduction and, in general, gives a "globalization" of the motivic tilting equivalence $\mathrm{RigDA}(C) \cong \mathrm{RigDA}(C^\sharp)$ of [Vezzani 2019a] at the level of each classical point $C^\sharp$ of $\mathcal{X}(C)$. The functor $\mathcal{D}$ above can be considered as being the avatar of the pullback p^* along the map $p : \mathcal{Y}_{(0,\infty)}(C) \to C$ as if it existed in adic spaces (and not just diamonds).

Putting the two main results above together, we are led to consider the composition

$$\mathrm{RigDA}(P) \xrightarrow{\mathcal{D}} \mathrm{RigDA}(\mathcal{X}(P)) \xrightarrow{\mathrm{dR}_{\mathcal{X}(P)}} \mathrm{QCoh}(\mathcal{X}(P))^{\mathrm{op}}$$

giving rise to a functorial cohomology theory for adic spaces which are locally of finite type over a perfectoid space P *in positive characteristic* that takes values in the category of solid quasicoherent sheaves on the relative Fargues–Fontaine curve $\mathcal{X}(P)$. When P is a geometric point, this is closely related to a conjecture which was formulated in [Fargues 2018, Conjecture 1.13] and in [Scholze 2018, Conjecture 6.4] that we prove below; but the construction makes good sense for any P. More precisely (see Theorem 6.3):

Theorem. *Let P be an admissible perfectoid space of characteristic p. There is a functor*

$$\mathrm{RigDA}(P) \to \mathrm{QCoh}(\mathcal{X}(P)), \quad M \mapsto \mathrm{dR}^{\mathrm{FF}}_P(M),$$

where $\mathrm{QCoh}(\mathcal{X}(P))$ is the category of solid quasicoherent sheaves over the relative Fargues–Fontaine curve $\mathcal{X}(P)$ with the following properties:

(1) *It satisfies étale descent, $\mathbb{B}^1$-invariance and a Künneth formula.*

(2) *For any untilt $P^\sharp$ of P, the pullback of $\mathrm{dR}^{\mathrm{FF}}_P(M)$ along $P^\sharp \to \mathcal{X}(P)$ is isomorphic to the overconvergent de Rham cohomology $\mathrm{dR}_{P^\sharp}(M^\sharp)$ of the motive $M^\sharp$ corresponding to M via the motivic equivalence $\mathrm{RigDA}(P) \cong \mathrm{RigDA}(P^\sharp)$.*

(3) *The object $\mathrm{dR}^{\mathrm{FF}}_P(M)$ is a perfect complex of $\mathcal{O}_{\mathcal{X}(P)}$-modules whose cohomology sheaves are vector bundles, whenever M is (the motive of) a smooth proper variety over P or whenever M is compact and P is a perfectoid field.*

Examples of admissible perfectoid spaces include those which are pro-étale over rigid analytic varieties, and examples of compact motives over a field include motives of quasicompact smooth varieties or analytifications of algebraic varieties. The cohomology theory induced by $\mathrm{dR}^{\mathrm{FF}}_P$ will be called the *de Rham–Fargues–Fontaine* cohomology. Its construction is purely made at the level of the generic fibers, makes no use of log-geometry and requires weak hypotheses on the base P. It is expected to enhance the de Rham and the de Rham–Fargues–Fontaine realizations with coefficients, in a compatible way with the motivic six-functor formalism.

One may precompose this realization functor with the motivic tilting equivalence

$$\mathrm{RigDA}(P) \cong \mathrm{RigDA}(P^\flat)$$

allowing P to be a perfectoid space in characteristic zero as well (in this case, the target category would be obviously $\mathrm{QCoh}(\mathcal{X}(P^\flat))$ or with the analytification functor. On the other hand, if P is a characteristic p perfectoid space, one can postcompose it with specialization along a chosen untilt $P^\sharp \to \mathcal{X}(P)$ and get a perfect complex of $\mathcal{O}_{P^\sharp}$-modules. By doing so when $P = C$ is an algebraically closed perfectoid field of characteristic p, we recover a construction from [Vezzani 2019b] and also Bhatt, Morrow and Scholze's $B^+_{\mathrm{dR}}(C^\sharp)$-cohomology [Bhatt et al. 2018, Section 13] for each untilt $C^\sharp$ of C. This proves that $\mathrm{dR}^{\mathrm{FF}}$ satisfies all the requirements of conjecture 6.4 in [Scholze 2018]. There is also a connection to rigid cohomology that we sketch at the end of the article.

In Section 2 we begin by recalling the properties of rigid analytic motives and we give a proof of their pro-étale descent. This allows us to define motives over any (admissible) diamond. In Section 3 we give a definition of relative dagger varieties (or relative varieties with an overconvergent structure) and we show that up to homotopy, any smooth relative variety can be equipped with such a structure. In Section 4 we introduce the de Rham complex of a relative dagger space and prove that it gives rise to a motivic realization with values in solid modules, or even perfect complexes, under suitable hypotheses.

In the second part, we build the motivic rigid-analytic version of the relative Fargues–Fontaine curve and we compare it to the usual construction in Section 5. Finally, in Section 6 we put together the ingredients of the previous sections introducing the de Rham–Fargues–Fontaine cohomology and its properties, including its relation to the cohomology theories mentioned above.

2. Adic étale motives

We start by laying down the main definitions and properties of the type of adic spaces we consider and the homotopy theory associated to them.

Definitions and formal properties. Our conventions and notation are mostly taken from [Ayoub 2015; Ayoub et al. 2022] even if we typically omit any visual reference to the étale topology and the ring of coefficients in what follows.

Definition 2.1. We say that a Tate Huber pair (A, A^+) over $\mathbb{Z}_p$ is *stably strongly uniform* if for any $n \in \mathbb{N}$ and any map $(A\langle T_1, \dots, T_n\rangle, A^+\langle T_1, \dots, T_n\rangle) \to (B, B^+)$ obtained as a composition of rational localizations and finite étale maps (as defined in Definition 7.1(i) of [Scholze 2012]), the space $\mathrm{Spa}(B, B^+)$ is uniform, that is, the ring B^+ is (open and) bounded. An adic space is *stably strongly uniform* if it is locally the spectrum of a stably strongly uniform pair. Examples of stably strongly uniform spaces include diamantine spaces [Hansen and Kedlaya 2020, Theorem 11.14], sous-perfectoid spaces (such as perfectoid spaces) [Scholze and Weinstein 2020, Proposition 6.3.3], and reduced rigid analytic varieties over nonarchimedean fields [Bosch et al. 1984, Theorem 6.2.4/1]. We let Adic be the full subcategory of quasiseparated adic spaces over $\mathbb{Z}_p$ which consists of stably strongly uniform spaces having a cover of affinoid open spaces with finite (topological) Krull dimension (see, for example, [Stacks 2018, Section 0054]). Its objects will be sometimes referred to as *admissible adic spaces*. For any full subcategory C of Adic we let $\mathrm{C}^{\mathrm{qcqs}}$ be the subcategory of C of quasicompact quasiseparated morphisms (referred to as qcqs from now on). We let $\mathbb{B}^n$ and $\mathbb{T}^n$ be the adic spaces

$$\mathbb{B}^n = \mathrm{Spa}(\mathbb{Z}_p\langle T_1, \dots, T_n\rangle, \mathbb{Z}_p\langle T_1, \dots, T_n\rangle) \quad \text{and} \quad \mathbb{T}^n = \mathrm{Spa}(\mathbb{Z}_p\langle T_1^{\pm 1}, \dots, T_n^{\pm 1}, \rangle, \mathbb{Z}_p\langle T_1^{\pm 1}, , \dots, T_n^{\pm 1}\rangle).$$

We remark that $\mathbb{B}^n_S = S \times_{\mathbb{Z}_p} \mathbb{B}^n$ and $\mathbb{T}^n_S = S \times_{\mathbb{Z}_p} \mathbb{T}^n$ lie in Adic for any $S \in$ Adic and any $n \in \mathbb{N}$.

Remark 2.2. We point out that reduced rigid analytic varieties over a nonarchimedean field K are admissible. Also their perfection (assuming K has characteristic p) is an admissible perfectoid space, and as we will remark later (Remark 5.2) the Fargues–Fontaine curves associated to such perfectoid spaces are admissible too. As a matter of fact, in all that follows one can replace the category Adic with any subcategory of adic spaces over $\mathbb{Z}_p$ which are locally of finite Krull dimension that is stable under open immersions, finite étale extensions as well as relative discs, and that contains reduced rigid analytic varieties and relative Fargues–Fontaine curves. Alternatively, one may consider the (larger) category of rigid spaces as defined by [Fujiwara and Kato 2018] and considered in [Ayoub et al. 2022]. In this article, we stick to an adic perspective and we leave it to the reader to extend the statements and definitions of the present article to any more general setting.

Definition 2.3. Let $f : X \to S$ be a morphism in Adic.

- We say that f is *étale* if it is, locally on X and S, the composition of an open immersion and a finite étale morphism. A collection of étale maps $\{X_i \to S\}$ is an *étale cover* if it is jointly surjective on the underlying topological spaces.
- We say f is *smooth* (or even, by abuse of notation, that X is a *smooth rigid analytic variety over* S) if it is, locally on X, the composition of an étale map $X \to \mathbb{B}^N_S$ and the canonical projection $\mathbb{B}^N_S \to S$ for some N. The category of smooth rigid analytic varieties over S is denoted by Sm/S.

We point out that if S is in Adic and f is smooth (using the above definition) then X lies in Adic as well. Also, we remark that pullbacks of smooth (resp. étale) maps exist in Adic and they are again smooth (resp. étale).

Definition 2.4. Let S be in Adic.

- For any $X \in \mathrm{Sm}/S$ we let $\mathbb{Q}_S(X)$ be the (free) presheaf of $\mathbb{Q}$-modules represented by X. That is $\Gamma(Y, \mathbb{Q}_S(X)) = \mathbb{Q}[\mathrm{Hom}_S(Y, X)]$.

- We let $\mathrm{Psh}(\mathrm{Sm}/S, \mathbb{Q})$ be the infinity-category of presheaves on the category Sm/S taking values on the derived infinity-category of $\mathbb{Q}$-modules, and we let $\mathrm{RigDA}^{\mathrm{eff}}(S, \mathbb{Q})$ be its full stable infinity-subcategory spanned by those objects $\mathcal{F}$ such that:

(1) For any $X \in \mathrm{Sm}/S$ the canonical map $\mathcal{F}(X \times_S \mathbb{B}^1_S) \to \mathcal{F}(X)$ is an equivalence (we refer to this property as $\mathbb{B}^1$-*invariance*).

(2) For any Cech étale hypercover $\mathcal{U} \to X$ in Sm/S the canonical map $\mathcal{F}(X) \to \operatorname{holim} \mathcal{F}(\mathcal{U})$ is an equivalence (we refer to this property as *étale descent*).

We will typically omit $\mathbb{Q}$ in the notation. The category $\mathrm{RigDA}^{\mathrm{eff}}(S)$ is equipped with the structure of a symmetric monoidal infinity-category and a localization functor

$$L : \mathrm{Psh}(\mathrm{Sm}/S, \mathbb{Q}) \to \mathrm{RigDA}^{\mathrm{eff}}(S)$$

which is symmetric monoidal and left adjoint to the canonical inclusion.

- For any $X \in \mathrm{Sm}/S$ we use the notation $\mathbb{Q}_S(X)$ also to refer to the object $L\mathbb{Q}_S(X)$ in $\mathrm{RigDA}^{\mathrm{eff}}(S)$. There is a symmetric monoidal structure on $\mathrm{RigDA}^{\mathrm{eff}}(S)$ which is such that $\mathbb{Q}_S(X) \otimes \mathbb{Q}_S(Y) \cong \mathbb{Q}_S(X \times_S Y)$.

- We let T_S be the object of $\mathrm{Psh}(S, \mathbb{Q})$ which is the split cofiber of the morphism $\mathbb{Q}_S(S) \to \mathbb{Q}_S(\mathbb{T}^1_S)$ induced by 1 and we set $\mathrm{RigDA}(S, \mathbb{Q}) = \mathrm{RigDA}^{\mathrm{eff}}(S, \mathbb{Q})[T_S^{-1}]$ in Pr^{L} (see [Robalo 2015, Definition 2.6]). We will typically omit $\mathbb{Q}$ in the notation. The (extension of the) endofunctor $M \mapsto M \otimes T_S^{\otimes n}$ in $\mathrm{RigDA}(S)$ will be denoted by $M \mapsto M(n)$ and its quasi-inverse by $M \mapsto M(-n)$. We still denote by $\mathbb{Q}_S(X)$ the images of these objects by the natural functor $\mathrm{RigDA}^{\mathrm{eff}}(S) \to \mathrm{RigDA}(S)$.

- When we write $\mathrm{RigDA}^{(\mathrm{eff})}(S)$ in a statement, we mean that the statement holds both for $\mathrm{RigDA}^{\mathrm{eff}}(S)$ (sometimes called the category of *effective motives*) and for $\mathrm{RigDA}(S)$.

Remark 2.5. In [Ayoub et al. 2022], the category $\mathrm{RigDA}^{(\mathrm{eff})}(S)$ is denoted by $\mathrm{RigSH}^{(\mathrm{eff})}(S, \mathbb{Q})$. We use the notation DA which is more customary in the case of sheaves of Λ-modules for a ring Λ. All adic spaces in Adic are rigid analytic spaces in the sense of [Ayoub et al. 2022, Notation 1.1.8] by [Ayoub et al. 2022, Corollary 1.2.7]. Contrary to [Ayoub et al. 2022], we use the notation $\mathrm{RigDA}^{(\mathrm{eff})}(S)$ to refer both to the presentable category in Pr^{L} as well as to the structure $\mathrm{RigDA}^{(\mathrm{eff})}(S)^{\otimes}$ of symmetric monoidal category in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ it is equipped with.

Remark 2.6. We now give a triangulated, more down-to-earth definition of $\mathrm{RigDA}^{\mathrm{eff}}(S)$. One can consider the derived category of étale sheaves on Sm/S with values in $\mathbb{Q}$-modules. Its full subcategory given by complexes of sheaves $\mathcal{F}$ such that $\mathbb{R}\Gamma(X, \mathcal{F}) \cong \mathbb{R}\Gamma(\mathbb{B}^1_X, \mathcal{F})$ is (the triangulated category underlying) $\mathrm{RigDA}^{\mathrm{eff}}(S)$. We remark that there is a left adjoint to the canonical inclusion, and that these categories are actually DG-categories. Similarly, we can give a more down-to-earth definition of $\mathrm{RigDA}(S)$: its objects are collections $\{\mathcal{F}_i\}_{i\in\mathbb{N}}$ of complexes of sheaves in $\mathrm{RigDA}^{\mathrm{eff}}(S)$ together with quasi-isomorphisms $\mathcal{F}_i \to \underline{\mathrm{Hom}}(T_S, \mathcal{F}_{i+1})$.

Remark 2.7. We now give a more blue-sky definition of $\mathrm{RigDA}^{\mathrm{eff}}(S)$. By [Lurie 2017, Proposition 4.8.1.17] one can consider the (presentable) infinity-category $\mathrm{Sh}_{\text{ét}}(\mathrm{Sm}/S)$ of simplicial étale sheaves on Sm/S as well as its tensor product $\mathrm{Sh}_{\text{ét}}(\mathrm{Sm}/S) \otimes \mathrm{Ch}\,\mathbb{Q}$ with the derived infinity category of (chain complexes of) $\mathbb{Q}$-modules and let $\mathrm{RigDA}^{\mathrm{eff}}(S)$ be its full infinity-subcategory of $\mathbb{B}^1_S$-invariant objects (one may equivalently consider étale *hypersheaves* by [Ayoub et al. 2022, Corollary 2.4.19]). We can also define $\mathrm{RigDA}(S)$ as the homotopy colimit $\varinjlim \mathrm{RigDA}^{\mathrm{eff}}(S)$ following the functor $\mathcal{F} \mapsto \mathcal{F} \otimes T_S$, computed in the category of presentable infinity-categories and left adjoint functors Pr^{L}. Equivalently, it is the homotopy limit $\varprojlim \mathrm{RigDA}^{\mathrm{eff}}(S)$ following the functor $\mathcal{F} \mapsto \underline{\mathrm{Hom}}(T_S, \mathcal{F})$, computed in the category of presentable infinity-categories and right adjoint functors Pr^{R} (or computed in infinity-categories) by [Robalo 2015, Corollary 2.22].

Remark 2.8. By definition, (a suitable localization of) the projective model structure on presheaves makes the natural functor $\mathrm{Sm}/S \to \mathrm{RigDA}(S)$ universal among functors $R : \mathrm{Sm}/S \to \mathbb{Q}$-enriched model categories M satisfying the requirements

(i) $R(X) \cong \operatorname{holim} R(\mathcal{U})$ for any Cech étale hypercover $\mathcal{U} \to X$;

(ii) the maps $R(\mathbb{B}^1_X) \to R(X)$ are invertible in the homotopy category;

(iii) $R(M) \mapsto R(T^1 \otimes M)$ is an automorphism on the homotopy category.

The same is true by replacing M with an arbitrary infinity-category with small colimits (see [Robalo 2015, Theorem 2.30]). We remark that, as we take coefficients in $\mathbb{Q}$, the condition on Cech hypercovers extends automatically to arbitrary étale hypercovers (see [Ayoub et al. 2022, Proposition 2.4.19]).

Remark 2.9. We use the fact that coefficients are in $\mathbb{Q}$ already in Theorems 2.10 and 2.15. Nonetheless, for most of the results in this article, it is possible to replace $\mathbb{Q}$ with $\mathbb{Z}[1/p]$ or even more general ring spectra, by eventually restricting the category Adic to its full subcategory of objects having a suitably bounded pointwise cohomological dimension (see, for example, [Ayoub et al. 2022, Proposition 2.4.22]). As we are mostly interested in a rational cohomology theory here, we leave this task to the reader.

The following statement follows from the results of [Ayoub et al. 2022]. For the definition of the category of (symmetric monoidal) presentable infinity-categories and (symmetric monoidal) left adjoint functors Pr^{L} (resp. $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$), as well as the definition of compactly generated (symmetric monoidal) presentable categories and (symmetric monoidal) compact-preserving left adjoint functors $\mathrm{Pr}^{\mathrm{L}}_{\omega}$ (resp. $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\omega})$) we refer to Definitions 5.5.3.1 and 5.5.7.5 in [Lurie 2009] (resp. to Proposition 4.8.1.15 and Lemma 5.3.2.11(2) in [Lurie 2017]).

Theorem 2.10. (1) *For any $S \in \mathrm{Adic}$ the category $\mathrm{RigDA}^{(\mathrm{eff})}(S)$ is a compactly generated stable symmetric monoidal category, in which a set of compact generators is given by $\mathbb{Q}_S(X)(n)$ with $X \in \mathrm{Sm}/S$ affinoid and $n \in \mathbb{Z}$. Also, $\mathbb{Q}_S(X)(n) \otimes \mathbb{Q}_S(X')(n') \cong \mathbb{Q}_S(X \times_S X')(n+n')$.*

(2) *For any morphism $f : S' \to S$ in Adic the pullback functor $X \mapsto X \times_S S'$ induces a symmetric monoidal left (Quillen) adjoint functor $f^* : \mathrm{RigDA}^{(\mathrm{eff})}(S) \to \mathrm{RigDA}^{(\mathrm{eff})}(S')$ whose right adjoint will be denoted by f_*. If f is quasicompact and quasiseparated, then f^* is compact-preserving.*

(3) *One can define contravariant functors $\mathrm{RigDA}^{(\mathrm{eff})*}$ from Adic to the infinity-category $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ of symmetric monoidal, presentable infinity-categories and left adjoint symmetric monoidal functors, sending S to $\mathrm{RigDA}^{(\mathrm{eff})}(S)$ and a morphism f to f^*. Their restrictions to $\mathrm{Adic}^{\mathrm{qcqs}}$ take values in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\omega})$.*

(4) *For any smooth morphism $f : S' \to S$ in Adic the "forgetful" functor $(X \to S') \mapsto (X \to S' \to S)$ induces a compact-preserving left (Quillen) adjoint functor $f_\sharp : \mathrm{RigDA}^{(\mathrm{eff})}(S') \to \mathrm{RigDA}^{(\mathrm{eff})}(S)$ whose right adjoint coincides with f^*.*

(5) *The functors $\mathrm{RigDA}^{(\mathrm{eff})*}$ satisfy étale hyperdescent. This means that for any étale hypercover $\mathcal{U} \to S$ in Adic which is levelwise representable, one has the following equivalence in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$:*

$$\mathrm{RigDA}^{(\mathrm{eff})}(S) \cong \lim \mathrm{RigDA}^{(\mathrm{eff})}(\mathcal{U}).$$

Proof. As S is locally of finite Krull dimension by hypothesis, it is $(\mathbb{Q}, \text{ét})$-admissible in the sense of [Ayoub et al. 2022, Definition 2.4.14]. Points (1)–(3) follow then from [Ayoub et al. 2022, Propositions 2.1.21 and 2.4.22], Point (4) can be deduced from (1) and [Ayoub et al. 2022, Proposition 2.2.1] while point (5) is proved in [Ayoub et al. 2022, Theorem 2.3.4]. □

Remark 2.11. The formal properties above hold true already for the infinity categories of hypersheaves $\mathrm{Sh}_{\text{ét}}(\mathrm{Sm}/S)$ and are easily inherited by $\mathrm{RigDA}^{\mathrm{eff}}(S)$ and its stabilization $\mathrm{RigDA}(S)$. Homotopies play therefore no special role in their proofs.

Continuity and pro-étale descent. We now list further properties which are satisfied by rigid motives. In all that follows, the role of homotopies over $\mathbb{B}^1$ is crucial, and the analogous statements for the categories of (hyper)sheaves are not expected to hold in general. We start by a "spreading out" result.

Theorem 2.12 [Ayoub et al. 2022, Theorem 2.8.14 and Remark 2.3.5]. *Let $\{S_i\}$ be a cofiltered diagram in* Adic *with quasicompact and quasiseparated transition maps, and let $S \in \mathrm{Adic}$ be such that $S \sim \varprojlim S_i$ in the sense of Huber* (*see* [Huber 1996, Definition 2.4.2] *and* [Ayoub et al. 2022, Definition 2.8.9]). *The pullback functors induce an equivalence in* $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$:

$$\varinjlim \mathrm{RigDA}^{(\mathrm{eff})}(S_i) \cong \mathrm{RigDA}^{(\mathrm{eff})}(S).$$

Remark 2.13. If the maps $S \to S_i$ are also quasicompact and quasiseparated, then the equivalence holds true in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\omega})$, as colimits in $\mathrm{Pr}^{\mathrm{L}}_{\omega}$ can be computed in Pr^{L} by [Lurie 2017, Lemma 5.3.2.9].

Remark 2.14. The algebraic analog of the spreading out result above is also true, and it is much more straightforward as it holds at the level of sheaves, without the need of using $\mathbb{A}^1$-homotopies (see, for example, [Ayoub et al. 2022, Proposition 2.5.11]). In the adic setting, this is no longer true: even if $S \sim \varprojlim S_i$, the (big) étale topos $\mathrm{Sh}_{\text{ét}}(\mathrm{Sm}/S)$ may not be equivalent to $\mathrm{Sh}_{\text{ét}}(\varinjlim \mathrm{Sm}/S_i)$. The main difference is that here a *completion* of the underlying topological rings is performed.

The continuity property above strongly suggests that the étale sheaf RigDA is also a pro-étale sheaf. This is indeed the case, and is the content of the next theorem. We remark nonetheless that its proof is more complicated than the analogous statement for sheaves of sets or groups (see, for example, [Scholze 2017, Proposition 8.5]) as RigDA takes values in the infinity-category Pr^{L} in which the cosimplicial Cech diagrams appearing in the descent criterion cannot be truncated on the right.[1] In the proof, we will use crucially some results on pro-étale sheaves from [Scholze 2017, Section 14].

Theorem 2.15. *The functors* $\mathrm{RigDA}^{(\mathrm{eff})*} : \mathrm{Adic}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ *satisfy pro-étale descent. This means that for any bounded pro-étale hypercover $\mathcal{U} \to S$ in* Adic, *one has the following equivalence in* $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$:

$$\mathrm{RigDA}^{(\mathrm{eff})}(S) \cong \lim \mathrm{RigDA}^{(\mathrm{eff})}(\mathcal{U}).$$

Proof. The proof will be split into some intermediate steps. In what follows, whenever $(\mathcal{C}, \tau)$ is a site, we will use the symbol $\mathcal{D}_\tau(\mathcal{C})$ to refer to the derived infinity-category of τ-sheaves of $\mathbb{Q}$-vector spaces, for brevity.

Step 1: Since the functor $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}) \to \mathrm{Pr}^{\mathrm{L}}$ is limit-preserving and conservative (see [Lurie 2017, Corollary 3.2.2.5 and Lemma 3.2.2.6]) we might as well prove the statement for $\mathrm{RigDA}^{(\mathrm{eff})}$ as functors with values in Pr^{L}. We first consider the case of $\mathrm{RigDA}^{\mathrm{eff}}$.

Step 2: As we already know that $\mathrm{RigDA}^{\mathrm{eff}}$ is an étale hypersheaf, we may prove the claim for its restriction to the subcategory Aff of Adic made of affinoid spaces. It suffices to show then that if $p : P \sim \varprojlim_{i \in I} P_i \to X$

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[1] The same proof shows that an étale sheaf with a "spreading out" property, taking values in an n-category with $n < \infty$ in which filtered colimits commute with finite limits, has pro-étale descent.

is a pro-étale affinoid cover of an affinoid X with $p_i : P_i \to X$ étale surjective, then

$$\mathrm{RigDA}^{\mathrm{eff}}(X) \cong \lim\Big(\mathrm{RigDA}^{\mathrm{eff}}(P) \rightrightarrows \mathrm{RigDA}^{\mathrm{eff}}(P \times_X P) \mathrel{\substack{\rightarrow\\ \rightarrow\\ \rightarrow}} \cdots\Big). \tag{$\star$}$$

Step 3: From now on we consider the category $\mathrm{Pro}_{\text{ét}} \operatorname{Aff} \mathrm{Sm}/X$ of pro-objects in affinoid smooth varieties over X with étale transition maps with a quasicompact weak limit. We will use the letter $\tilde{P}$ to refer to the object $\varprojlim P_i$ in this category. We say that a map in $\mathrm{Pro}_{\text{ét}} \operatorname{Aff} \mathrm{Sm}/X$ is smooth (resp. étale) if it is of the form $\varprojlim T_0 \times_{S_0} S_i \to \varprojlim S_i$ for some smooth (resp. étale) map $T_0 \to S_0$, we say it is pro-étale if it has a strictification which is levelwise étale, and pro-smooth if it is a composition of a pro-étale map, followed by a smooth map. We say it is a cover if the map on the underlying topological spaces $\varprojlim |T_i| \to \varprojlim |S_i|$ is surjective. In particular, we may consider the full subcategory $\mathrm{Pro}\,\mathrm{Sm}/\tilde{P}$ whose objects are pro-smooth maps over $\tilde{P}$, and equip it with the pro-étale topology. We remark that $\tilde{P} \to X$ is a cover by assumption, and that there are continuous equivalences $(\mathrm{Pro}\,\mathrm{Sm}/X)/\tilde{P} \cong \mathrm{Pro}\,\mathrm{Sm}/\tilde{P}$ giving rise to the following diagram (see [Ayoub et al. 2022, Proposition 2.3.7] which is essentially [Lurie 2009, Proposition 6.3.5.14]):

$$\mathcal{D}_{\text{proét}}(\mathrm{Pro}\,\mathrm{Sm}/X) \cong \lim\Big(\mathcal{D}_{\text{proét}}(\mathrm{Pro}\,\mathrm{Sm}/\tilde{P}) \rightrightarrows \mathcal{D}_{\text{proét}}(\mathrm{Pro}\,\mathrm{Sm}/\tilde{P} \times_X \tilde{P}) \mathrel{\substack{\rightarrow\\ \rightarrow\\ \rightarrow}} \cdots\Big).$$

Step 4: By definition, the étale topos on $\mathrm{Sm}/\tilde{P}$ is equivalent to the one on $\varinjlim \mathrm{Sm}/P_i$ (these toposes are *not* equivalent to the one on Sm/P!). By the proof of [Ayoub et al. 2022, Proposition 2.5.8] we deduce that $\mathcal{D}_{\text{ét}}(\mathrm{Sm}/\tilde{P}) \cong \varinjlim \mathcal{D}_{\text{ét}}(\mathrm{Sm}/P_i)$ and that $\mathrm{RigDA}^{\mathrm{eff}}(P) \cong \mathrm{RigDA}^{\mathrm{eff}}(\tilde{P}) \cong \varinjlim \mathrm{RigDA}^{\mathrm{eff}}(P_i)$ (using Theorem 2.12 for the first equivalence) where the colimits are taken in Pr^{L}. Note that the map of sites $\nu : (\mathrm{Pro}\,\mathrm{Sm}/\tilde{P}, \text{proét}) \to (\mathrm{Sm}/\tilde{P}, \text{ét})$ induces a functor $\nu^* : \mathrm{Sh}_{\text{ét}}(\mathrm{Sm}/\tilde{P}, \mathbb{Q}) \to \mathrm{Sh}_{\text{proét}}(\mathrm{Pro}\,\mathrm{Sm}/\tilde{P}, \mathbb{Q})$. By adapting the proof of [Scholze 2017, Proposition 14.10] this functor can be described explicitly as

$$\nu^*\mathcal{F}(\varprojlim Q_i) = \varinjlim \mathcal{F}(Q_i \times_{P_i} \tilde{P})$$

and induces a fully faithful inclusion $\nu^* : \mathcal{D}^+_{\text{ét}}(\mathrm{Sm}/\tilde{P}) \to \mathcal{D}^+_{\text{proét}}(\mathrm{Pro}\,\mathrm{Sm}/\tilde{P})$. We may extend this inclusion by left-completion (we are using that any object has a finite rational étale cohomological dimension; see [Ayoub et al. 2022, Corollary 2.4.13]) to a fully faithful inclusion $\nu^* : \mathcal{D}_{\text{ét}}(\mathrm{Sm}/\tilde{P}) \to \mathcal{D}_{\text{proét}}(\mathrm{Pro}\,\mathrm{Sm}/\tilde{P})$.

Step 5: We claim that $\mathcal{D}_{\text{ét}}(\mathrm{Sm}/X)$ fits in the pullback square

$$\begin{tikzcd}
\mathcal{D}_{\text{ét}}(\mathrm{Sm}/X) \arrow[r, "p^*"] \arrow[d, hook, "\nu^*"] & \mathcal{D}_{\text{ét}}(\mathrm{Sm}/\tilde{P}) \arrow[d, hook, "\nu^*"] \\
\mathcal{D}_{\text{proét}}(\mathrm{Pro}\,\mathrm{Sm}/X) \arrow[r, "p^*"] & \mathcal{D}_{\text{proét}}(\mathrm{Pro}\,\mathrm{Sm}/\tilde{P})
\end{tikzcd}$$

i.e., we claim that for any $\mathcal{F}$ in $\mathcal{D}_{\text{proét}}(\mathrm{Pro}\,\mathrm{Sm}/X)$, one has $\mathcal{F} \cong \nu^*\nu_*\mathcal{F}$ provided that $p^*\mathcal{F} \cong \nu^*\nu_*p^*\mathcal{F}$. Note that the analogous claim for the *small* (pro-)étale sites holds [Scholze 2017, Proposition 14.10] and we now show that we can reduce to it. As any object in $\mathrm{Pro}\,\mathrm{Sm}/X$ is locally pro-étale over some affinoid variety Y in Sm/X, we may prove the equivalence $\mathcal{F} \cong \nu^*\nu_*\mathcal{F}$ by restricting to each one of the small sites $\mathrm{Pro}\,\mathrm{Et}/Y$ with Y as before. In other words, it suffices to check that $\iota_*\mathcal{F} \cong \iota_*\nu^*\nu_*\mathcal{F}$ with ι being the natural map of sites $\operatorname{Pro}\mathrm{Sm}/X \to \operatorname{Pro}\mathrm{Et}/Y$. By construction, we have $\iota'_* p^* \cong p'^* \iota_*$, $\iota_*\nu_* \cong \nu'_*\iota'_*$ and $\iota_*\nu^* \cong \nu'^*\iota'_*$ with p' being $\tilde{P} \times_X Y \to Y$ and ν' (resp. ι') being the map of sites $\nu' : \operatorname{Pro}\mathrm{Et}/Y \to \mathrm{Et}/Y$ (resp. $\iota' : \mathrm{Sm}/X \to \mathrm{Et}/Y$). In particular, we can deduce the claim from the analogous claim on the small (pro-)étale sites as claimed. We can reproduce this proof also for each one of the pro-étale maps of pro-objects $\delta : \tilde{P}^{\times_X n+1} \to \tilde{P}^{\times_X n}$. This also proves the equivalence

$$\mathcal{D}_{\text{ét}}(\mathrm{Sm}/X) \cong \lim\Big(\mathcal{D}_{\text{ét}}(\mathrm{Sm}/\tilde{P}) \rightrightarrows \mathcal{D}_{\text{ét}}(\mathrm{Sm}/\tilde{P} \times_X \tilde{P}) \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \cdots\Big)$$

and implies in particular that the map $p^* : \mathcal{D}_{\text{ét}}(\mathrm{Sm}/X) \to \mathcal{D}_{\text{ét}}(\mathrm{Sm}/\tilde{P})$ is conservative.

Step 6: We show that the functor $p^* : \mathcal{D}_{\text{ét}}(\mathrm{Sm}/X) \to \mathcal{D}_{\text{ét}}(\mathrm{Sm}/\tilde{P})$ sends a class of compact generators to a class of compact generators. As we have $\mathcal{D}_{\text{ét}}(\mathrm{Sm}/\tilde{P}) = \varinjlim \mathcal{D}_{\text{ét}}(\mathrm{Sm}/P_i)$, it suffices to show that the functors p_i^* send compact generators to compact generators. In other words (see [Ayoub et al. 2022, Lemma 2.8.3]) we need to show that the functor e_* is conservative whenever $e : Y \to X$ is an étale map of affinoid varieties. The statement is étale-local on X so we may assume e is given by a trivial finite étale cover $Y = X \sqcup X \to X$ and e_* is thus the functor $\mathcal{D}_{\text{ét}}(\mathrm{Sm}/Y) \cong \mathcal{D}_{\text{ét}}(\mathrm{Sm}/X) \times \mathcal{D}_{\text{ét}}(\mathrm{Sm}/X) \to \mathcal{D}_{\text{ét}}(\mathrm{Sm}/X)$, $(\mathcal{F}, \mathcal{F}') \mapsto \mathcal{F} \oplus \mathcal{F}'$, which is obviously conservative. The same proof shows also that $p^* : \mathrm{RigDA}^{\mathrm{eff}}(X) \to \mathrm{RigDA}^{\mathrm{eff}}(P)$ sends a class of compact generators to a class of compact generators.

Step 7: We now claim that $\mathrm{RigDA}^{\mathrm{eff}}(X)$ fits in the pullback square

$$\begin{tikzcd}
\mathrm{RigDA}^{\mathrm{eff}}(X) \arrow[r] \arrow[d, hook] & \mathrm{RigDA}^{\mathrm{eff}}(P) \cong \varinjlim \mathrm{RigDA}^{\mathrm{eff}}(P_i) \arrow[d, hook] \\
\mathcal{D}_{\text{ét}}(\mathrm{Sm}/X) \arrow[r] & \mathcal{D}_{\text{ét}}(\mathrm{Sm}/\tilde{P})
\end{tikzcd}$$

This amounts to saying that an object $\mathcal{F}$ in $\mathcal{D}_{\text{ét}}(\mathrm{Sm}/X)$ is $\mathbb{B}^1$-invariant if and only if it is so after applying the pullback functor p^*, i.e., we claim that $\mathcal{F} \cong \pi_*\pi^*\mathcal{F}$ provided that $p^*\mathcal{F} \cong \pi_*\pi^*p^*\mathcal{F}$ where π denotes the natural projection $\mathbb{B}^1_X \to X$ (as well as its pullback over $\tilde{P}$). From step 5 we already know that the functor $p^* : \mathcal{D}_{\text{ét}}(\mathrm{Sm}/X) \to \mathcal{D}_{\text{ét}}(\mathrm{Sm}/\tilde{P})$ is conservative, so it suffices to show that it commutes with π^* (which is obvious) and with π_*. To this aim, by step 6, we fix a compact object M in $\mathrm{RigDA}^{\mathrm{eff}}(X)$ and we prove that $\mathrm{Map}(p^*M, p^*\pi_*\mathcal{F}) \cong \mathrm{Map}(p^*M, \pi_*p^*\mathcal{F})$ for any $\mathcal{F}$ in $\mathcal{D}_{\text{ét}}(\mathrm{Sm}/\mathbb{B}^1_{\tilde{P}}) \cong \varinjlim \mathcal{D}_{\text{ét}}(\mathrm{Sm}/\mathbb{B}^1_{P_i})$. This follows from the sequence of equivalences

$$\begin{aligned}
\mathrm{Map}(p^*M, p^*\pi_*\mathcal{F}) &\cong \varinjlim \mathrm{Map}(p_i^*M, p_i^*\pi_*\mathcal{F}) \\
&\cong \varinjlim \mathrm{Map}(p_i^*M, \pi_*p_i^*\mathcal{F}) \\
&\cong \varinjlim \mathrm{Map}(p_i^*\pi^*M, p_i^*\mathcal{F}) \\
&\cong \mathrm{Map}(p^*\pi^*M, p^*\mathcal{F}) \\
&\cong \mathrm{Map}(p^*M, \pi_*p^*\mathcal{F}),
\end{aligned} \tag{$\star\star$}$$

where we used the obvious commutation $\pi^*p^* \cong p^*\pi^*$ and the commutation $\pi_*p_i^* \cong p_i^*\pi_*$ which follows from the natural equivalence $\pi^*p_{i\sharp} \cong p_{i\sharp}\pi^*$ (see [Ayoub et al. 2022, Proposition 2.2.1]). The same proof shows more generally that $\mathrm{RigDA}^{\mathrm{eff}}(P^{\times x^n})$ is the pullback of $\mathrm{RigDA}^{\mathrm{eff}}(P^{\times x^{n+1}})$ along $\delta^* : \mathcal{D}_{\text{ét}}(\mathrm{Sm}/\tilde{P}^{\times x^n}) \to \mathcal{D}_{\text{ét}}(\mathrm{Sm}/\tilde{P}^{\times x^{n+1}})$. We have then finally deduced ($\star$), i.e., descent for effective motives $\mathrm{RigDA}^{\mathrm{eff}}$.

Step 8: We now move to proving the statement for RigDA. Just like in the proof of [Ayoub et al. 2022, Theorem 2.3.4] this follows formally from the commutation $\underline{\mathrm{Hom}}(T, -) \circ p^* \cong p^* \circ \underline{\mathrm{Hom}}(T, -)$ which can be deduced from the commutation $\underline{\mathrm{Hom}}(T, -) \circ p_i^* \cong p_i^* \circ \underline{\mathrm{Hom}}(T, -)$ using a similar argument to the one used in step 7 for the sequence ($\star\star$). ☐

Pro-étale descent implies the possibility to extend motives to diamonds (provided that we impose the same conditions on their Krull dimension as in Definition 2.1).

Definition 2.16. We say a diamond is *admissible* if it is pro-étale locally a perfectoid space in Adic (i.e., locally of finite Krull dimension).

Corollary 2.17. *Consider the restrictions of the functors* $\mathrm{RigDA}^{(\mathrm{eff})}$ *to the category* $\mathrm{Adic}_{/\mathbb{F}_p}$. *They can be extended uniquely as pro-étale sheaves to the category of admissible diamonds.*

Proof. This follows (see [Lurie 2009, Lemma 6.4.5.6] or [Ayoub et al. 2022, Lemma 2.1.4]) from pro-étale descent and the equivalence between the pro-étale toposes on perfectoid spaces over $\mathbb{F}_p$ and on diamonds. ☐

Remark 2.18. At this stage, we can't say that the construction of RigDA is compatible with the "diamondification" functor from adic spaces to diamonds. In other words, it is not yet clear that $\mathrm{RigDA}(S) \cong \mathrm{RigDA}(S^\diamond)$ if S is an adic space in $\mathrm{Adic}_{/\mathbb{Q}_p}$. We will show this only in Theorem 5.13.

Frobenius-invariance and perfectoid motives. We continue to inspect the formal properties of RigDA which depend on homotopies, now focusing on the behavior of the functor RigDA under the action of Frobenius which is studied in [Ayoub et al. 2022, Section 2.9].

Theorem 2.19. *Let* $S' \to S$ *be a universal homeomorphism in* Adic. *The pullback functor induces an equivalence* $\mathrm{RigDA}^{(\mathrm{eff})}(S) \cong \mathrm{RigDA}^{(\mathrm{eff})}(S')$. *In particular, if* S *is in* $\mathrm{Adic}_{/\mathbb{F}_p}$ *then the pullback along* $S^{\mathrm{Perf}} \to S$ *induces an equivalence in* $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\omega})$:

$$\mathrm{RigDA}^{(\mathrm{eff})}(S) \cong \mathrm{RigDA}^{(\mathrm{eff})}(S^{\mathrm{Perf}})$$

which is compatible with the functors f^*.

Proof. By [Ayoub et al. 2022, Corollary 2.9.10] only the last sentence needs to be proved, and that follows from Theorem 2.12. ☐

Remark 2.20. The same is true for algebraic motives, provided that we consider their stable version. On the other hand, there is no need for any hypothesis on the Krull dimension of the base scheme; see [Ayoub et al. 2022, Theorem 2.9.7; Ayoub 2014, Théorème 3.9; Elmanto and Khan 2020].

Corollary 2.21. *Let S be in* Adic *and let $f : X' \to X$ be a universal homeomorphism in* RigSm/S. *The induced map of motives $\mathbb{Q}_S(X') \to \mathbb{Q}_S(X)$ is invertible in* $\mathrm{RigDA}^{(\mathrm{eff})}(S)$.

Proof. Let p and p' be the structural smooth morphisms $X \to S$ and $X' \to S$, respectively. The map of motives in the statement can be written as $(p'_\sharp \circ f^*)(\mathbb{Q}_X) \to p_\sharp \mathbb{Q}_X$. But $p'_\sharp \circ f^*$ is canonically equivalent to $p_\sharp$ as they are both left adjoint functors to p^* by Theorem 2.19. □

Corollary 2.22. *Let S be a perfectoid space over a perfectoid field K of characteristic p. The base change along Frobenius defines an endofunctor* $\varphi^* : \mathrm{RigDA}^{(\mathrm{eff})}(S) \to \mathrm{RigDA}^{(\mathrm{eff})}(S)$ *and the relative Frobenius morphisms $X \to X^{(1)} := X \times_{S,\mathrm{Frob}} S$ induce a natural transformation* $\mathrm{id} \Rightarrow \varphi^*$ *which is an equivalence.*

Proof. We are left to prove that the transformation is pointwise invertible (in the homotopy category). It suffices to show this for the generators of the form $\mathbb{Q}_S(X)(n)$ with $p : X \to S$ in Sm/S and this follows from Corollary 2.21. □

Definition 2.23. Let $\mathcal{C}$ be a presentable infinity-category and $F : \mathcal{C} \to \mathcal{C}$ an endofunctor with a right adjoint.

(1) The category of homotopically stable F-objects $\mathcal{C}^{hF}$ is the pullback

$$\begin{tikzcd}
\mathcal{C}^{hF} \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "\Gamma_F"] \\
\mathcal{C} \arrow[r, "\Delta"] & \mathcal{C} \times \mathcal{C}
\end{tikzcd}$$

More concretely, its objects are given by pairs (X, α) with X in $\mathcal{C}$ and α an equivalence $X \xrightarrow{\sim} FX$ (or, equivalently, an equivalence $FX \xrightarrow{\sim} X$).

(2) Suppose that $\mathcal{C}$ is compactly generated and that F preserves compact objects. The category $\mathcal{C}^{hF}_\omega$ is the pullback of the diagram above, computed in the category $\mathrm{Pr}^{\mathrm{L}}_\omega$.

(3) By means of [Lurie 2017, Corollary 3.2.2.5] we may use the same notation when $\mathcal{C}$ is a (compactly generated) symmetric monoidal presentable category, F is also symmetric monoidal and the pullback is computed in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ (resp. in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_\omega)$).

Remark 2.24. Our notation is justified by the following remark: $\mathcal{C}^{hF}$ is the category of homotopically fixed points $\mathcal{C}^{h\mathbb{N}}$ by letting the monoid $\mathbb{N}$ act on $\mathcal{C}$ via F.

Remark 2.25. Even if $\mathcal{C}$ is compactly generated and F preserves compact object, it may not be true that $\mathcal{C}^{hF}$ is compactly generated. Nonetheless, by [Lurie 2009, Lemma 5.4.5.7(2)] its full subcategory generated (under filtered colimits) by compact objects is $\mathcal{C}^{hF}_\omega$. In particular, whenever $\mathcal{C}^{hF}$ is compactly generated, the natural functor $\mathcal{C}^{hF}_\omega \subset \mathcal{C}^{hF}$ in Pr^{L} is an equivalence.

Corollary 2.26. *Let S be a perfectoid space in* $\mathrm{Adic}_{/\mathbb{F}_p}$ *and φ^* be the automorphism of* $\mathrm{RigDA}^{(\mathrm{eff})}(S)$ *induced by pullback along Frobenius. There is a natural functor*

$$\mathrm{RigDA}^{(\mathrm{eff})}(S) \to \mathrm{RigDA}^{(\mathrm{eff})}(S)^{h\varphi^*}_\omega \cong \mathrm{RigDA}^{(\mathrm{eff})}(S)^{h\varphi^{-1*}}_\omega$$

sending each motive M to the datum $M \xrightarrow{\sim} \varphi^* M$ given by the relative Frobenius functor. This gives rise to a natural transformation of étale hypersheaves with values in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$:

$$\mathrm{RigDA}^{(\mathrm{eff})*} \to (\mathrm{RigDA}^{(\mathrm{eff})*})^{h\varphi^*}_{\omega}$$

defined on the category of perfectoid spaces over $\mathbb{F}_p$.

Proof. For the first claim, it suffices to consider the diagram

$$\begin{tikzcd}
\mathrm{RigDA}(S) \arrow[r, equal] \arrow[d, equal] & \mathrm{RigDA}(S) \arrow[d, "\Gamma_{\varphi^*}"] \\
\mathrm{RigDA}(S) \arrow[r, "\Delta"'] \arrow[ur, Rightarrow, "\sim"] & \mathrm{RigDA}(S) \times \mathrm{RigDA}(S)
\end{tikzcd}$$

where the natural transformation is defined by the relative Frobenius functor (see Corollary 2.22).

In order to prove functoriality with respect to S, we fix a morphism $f : S' \to S$ and denote by φ_S and $\varphi_{S'}$ the relative Frobenius functor over S and S', respectively. We first remark that the canonical natural transformation $\varphi^*_{S'} f^* \Rightarrow f^* \varphi^*_S$ is an equivalence: when tested on compact generators of the form $\mathbb{Q}_S(X)(n)$ with X/S smooth, it corresponds to a universal homeomorphism; hence it is invertible by means of Theorem 2.19. With this remark, it is possible to define a lax functor from $\mathrm{Adic}^{\mathrm{op}}_{/\mathbb{F}_p} \times \boldsymbol{B}\mathbb{N}$ to relative categories which, by usual strictification techniques (see, for example, [May 1980, Theorem 3.4]) induces a functor from $\mathrm{Adic}^{\mathrm{op}}_{/\mathbb{F}_p} \times \boldsymbol{B}\mathbb{N}$ to relative categories, and hence to infinity-categories (see [Barwick and Kan 2012]). This promotes φ^*_S into an automorphism of the functors $\mathrm{RigDA}^{(\mathrm{eff})*}$ and the natural transformation $\mathrm{id} \Rightarrow \varphi^*_S$ into a map between automorphisms of these functors, concluding the claim. Alternatively, to prove the functoriality of $\mathrm{RigDA}(-)^{h\varphi^*}$ one may use the explicit model-theoretical description of such categories given in [Bergner 2011]. □

Perfectoid motives over a perfectoid field were introduced in [Vezzani 2019a]. We now easily extend their definitions and some properties to the relative setting.

Definition 2.27. We let Perf be the full subcategory of Adic made of perfectoid spaces over some perfectoid field, and we let S be in Perf. We let PerfSm/S be the full subcategory of Adic/S whose objects are locally étale over $\widehat{\mathbb{B}}^n_S := S \times_{\mathbb{Z}_p} \mathrm{Spa}\,\mathbb{Z}_p\langle T_1^{1/p^\infty}, \dots, T_n^{1/p^\infty}\rangle$ (sometimes called *geometrically smooth* perfectoid spaces over S). We let $\widehat{\mathbb{T}}^n_S$ be $S \times_{\mathbb{Z}_p} \mathrm{Spa}\,\mathbb{Z}_p\langle T_1^{\pm 1/p^\infty}, \dots, T_n^{\pm 1/p^\infty}\rangle$ and $\widehat{T}_S$ be the cokernel of the split inclusion of presheaves $\mathbb{Q}_S(S) \to \mathbb{Q}_S(\widehat{\mathbb{T}}^1_S)$ induced by the unit. We let $\mathrm{Psh}(\mathrm{PerfSm}/S, \mathbb{Q})$ be the infinity-category of presheaves on the category PerfSm/S taking values on the derived infinity-category of $\mathbb{Q}$-modules, and we let $\mathrm{PerfDA}^{\mathrm{eff}}(S)$ be its full stable infinity-subcategory spanned by those objects $\mathcal{F}$ which are $\widehat{\mathbb{B}}^1$-invariant and with ét-descent. Finally, we set $\mathrm{PerfDA}(S, \mathbb{Q}) = \mathrm{PerfDA}^{\mathrm{eff}}(S, \mathbb{Q})[\widehat{T}_S^{-1}]$ in Pr^{L} (see [Robalo 2015, Definition 2.6]). These categories are endowed with a symmetric monoidal structure for which $\mathbb{Q}_S(X) \otimes \mathbb{Q}_S(Y) \cong \mathbb{Q}_S(X \times_S Y)$.

Remark 2.28. The Krull dimension of an adic space X (which is a spectral space) can be computed by the maximal height of the valuations at each point x of X. As such (see, for example, [Ayoub et al. 2022, Definition 2.8.10 and Example 2.8.11] or [Scholze and Weinstein 2013, Proposition 2.4.2]) pro-étale maps can only decrease the topological Krull dimension and therefore any perfectoid space that is locally pro-étale above a rigid analytic variety lies in Perf.

Proposition 2.29. *One can define contravariant functors* $\mathrm{PerfDA}^{(\mathrm{eff})*}$ *on* Perf *with values in* $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ *such that any morphism* $f : S' \to S$ *in* Perf *is mapped to the functor* $\mathrm{PerfDA}^{(\mathrm{eff})}(S) \to \mathrm{PerfDA}^{(\mathrm{eff})}(S')$ *induced by pullback along* f. *They satisfy étale hyperdescent and their restrictions to* $\mathrm{Perf}^{\mathrm{qcqs}}$ *take values in* $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\omega})$.

Proof. The proofs of [Ayoub et al. 2022, Proposition 2.1.21, Theorem 2.3.4 and Proposition 2.4.22] can be easily adapted to the perfectoid context. ☐

Remark 2.30. It is clear that $\mathrm{PerfDA}^{(\mathrm{eff})}(P) \cong \mathrm{PerfDA}^{(\mathrm{eff})}(P^\flat)$ for any perfectoid space P, functorially in P, by [Scholze 2012].

Theorem 2.31. *Let S be an object of* $\mathrm{Perf}_{/\mathbb{F}_p}$. *The functor induced by relative perfection* $\mathrm{Perf} : \mathrm{RigSm}/S \to \mathrm{PerfSm}/S$ *gives an equivalence*

$$\mathrm{Perf}^* : \mathrm{RigDA}^{(\mathrm{eff})}(S) \xrightarrow{\sim} \mathrm{PerfDA}^{(\mathrm{eff})}(S).$$

More generally, the relative perfection induces an equivalence of presheaves $\mathrm{RigDA}^{(\mathrm{eff})*} \cong \mathrm{PerfDA}^{(\mathrm{eff})*}$ *on* $\mathrm{Perf}_{/\mathbb{F}_p}$ *with values in* $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$.

Proof. The natural transformation of functors can be defined as in [Robalo 2015]. By étale hyperdescent, it suffices to prove Perf^* is an equivalence whenever S is an affinoid perfectoid. The case $S = \mathrm{Spa}(K, K^\circ)$ has been proved in [Vezzani 2019a] and the same proof works for any affinoid base; see [Vezzani 2022]. ☐

Corollary 2.32. *Let* $f : S' \to S$ *be a map of admissible diamonds that, pro-étale locally on* S, *lies in* PerfSm/S. *Then the functor* $f^* : \mathrm{RigDA}^{(\mathrm{eff})}(S) \to \mathrm{RigDA}^{(\mathrm{eff})}(S')$ *has a left adjoint given by*

$$\mathrm{RigDA}^{(\mathrm{eff})}(S') \cong \mathrm{PerfDA}^{(\mathrm{eff})}(S') \xrightarrow{f_\sharp} \mathrm{PerfDA}^{(\mathrm{eff})}(S) \cong \mathrm{RigDA}^{(\mathrm{eff})}(S)$$

with $f_\sharp$ *defined as the functor induced by*

$$\mathrm{PerfSm}/S' \to \mathrm{PerfSm}/S, \quad (X \to S') \mapsto (X \to S' \to S).$$

Proof. If S is itself a perfectoid space, the proof is straightforward and similar to Theorem 2.10(4). We remark that in this case, by construction, if one has a cartesian diagram of perfectoid spaces

$$\begin{tikzcd} T' \arrow[r, "g'"] \arrow[d, "f'"] & S' \arrow[d, "f"] \\ T \arrow[r, "g"] & S \end{tikzcd}$$

with $f \in \mathrm{PerfSm}/S$, then $g^* f_\sharp \cong f'_\sharp g'^*$.

Let $\mathcal{P} \to S$ be a perfectoid pro-étale hypercover and $\mathcal{P}' \to S'$ be the hypercover of S induced by base change. By the previous part of the proof, there are functors of diagrams $\mathrm{RigDA}^{(\mathrm{eff})}(\mathcal{P}') \to \mathrm{RigDA}^{(\mathrm{eff})}(\mathcal{P})$ which are levelwise left adjoint to the base-change functors. They then induce a functor $f_\sharp$ between the two homotopy limits (computed by pro-étale descent; see Theorem 2.15) $\mathrm{RigDA}^{(\mathrm{eff})}(S') \to \mathrm{RigDA}^{(\mathrm{eff})}(S)$ which is a left adjoint to the base-change functor (see [Lurie 2017, Proposition 4.7.4.19]) as wanted. □

Definition 2.33. We may and do extend the functor $\mathrm{PerfDA}^{(\mathrm{eff})}(-)$ from $\mathrm{Perf}_{/\mathbb{F}_p}$ to diamonds, by considering its pro-étale sheafification. For any $S \in \mathrm{Adic}$ we write $\mathrm{PerfDA}^{(\mathrm{eff})}(S)$ for the category $\mathrm{PerfDA}^{(\mathrm{eff})}(S^\diamond)$. By Theorem 2.31, it is canonically equivalent to $\mathrm{RigDA}^{(\mathrm{eff})}(S^\diamond)$.

Remark 2.34. There is an alternative "naive" definition of $\mathrm{PerfDA}^{(\mathrm{eff})}(S)$ in the case $S \in \mathrm{Adic}$ is not necessarily perfectoid: we may consider the category PerfSm_n/S (n standing for naive) as being the full subcategory of $\mathrm{Adic}_{/S}$ which are locally étale over some space $\widehat{\mathbb{B}}^N \times S$, equip it with the étale topology and consider the induced category of (effective) motives $\mathrm{PerfDA}_n^{(\mathrm{eff})}(S)$. This construction defines functors $\mathrm{PerfDA}_n^{(\mathrm{eff})}$ with values in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ which are equipped with natural transformations $\sigma : \mathrm{PerfDA}_n^{(\mathrm{eff})} \to \mathrm{PerfDA}^{(\mathrm{eff})} \cong \mathrm{RigDA}^{(\mathrm{eff})}$. We note that σ is invertible when restricted to the category of perfectoid spaces and it therefore exhibits PerfDA as the pro-étale sheaf associated to $\mathrm{PerfDA}_n^{(\mathrm{eff})}$.

3. Relative overconvergent varieties and motives

We now introduce the category of overconvergent motives, generalizing the situation of [Vezzani 2018]. To this aim, we first define the category of *smooth dagger rigid analytic varieties* $\mathrm{Sm}^\dagger/S$ (or *smooth varieties with an overconvergent structure*) over a base S which is in $\mathrm{Adic}_{/\mathbb{Q}_p}$.

Relative overconvergent rigid varieties. Our definition is based on the absolute notion introduced in [Große-Klönne 2000] (see also [Vezzani 2018, Appendix A] for an adic perspective). We remark that we do not put any overconvergent structure on the base S, so that $\mathrm{Et}^\dagger/S = \mathrm{Et}/S$ and that for any open U of S we have $\mathrm{Sm}^\dagger/U = (\mathrm{Sm}^\dagger/S)/U$.

Definition 3.1. Let $U \to S$ be a morphism in Adic which is locally qcqs and topologically of finite type, and let $U \subset V$ be an open inclusion. We write $U \Subset_S V$ if the morphism $U \subset V$ extends to a morphism of adic spaces $U^{/S} \subset V$ where $U^{/S}$ is the universal compactification of U/S (see [Huber 1996, Theorem 5.1.5]). In the affinoid setting, say for a map $f : (R, R^+) \to (R', R'^+)$ over (A, A^+) this means that $f(R^+)$ is included in the algebraic closure of $A^+ + R'^{\circ\circ}$ in R'.

Remark 3.2. Even though in [Huber 1996] every adic space is assumed to be noetherian (in order to ensure the sheafyness property), this hypothesis is not used in the proof of [Huber 1996, Theorem 5.1.5].

Definition 3.3. Let S be in $\mathrm{Adic}_{/\mathbb{Q}_p}$. We let $\mathrm{Sm}^\dagger/S$ be the subcategory of $(\mathrm{Sm}/S) \times \mathrm{Pro}(\mathrm{Sm}/S)$ whose objects are given by pairs $(\widehat{X}, \{X_h\})$ with $\widehat{X} \in \mathrm{Sm}/S$ and $\{X_h\}$ is a cofiltered system of open inclusions $\widehat{X} \Subset_V X_h \subset X_{h'}$ in Sm/S such that $\widehat{X}^{/V} \sim \varprojlim X_h$, where we let V be the open subvariety of S given by $\mathrm{Im}(\widehat{X} \to S)$. Morphisms are defined levelwise and required to be compatible with the inclusions $\widehat{X} \subset X_h$. For an object $X = (\widehat{X}, \{X_h\})$ in $\mathrm{Sm}^\dagger/S$ we let $\mathcal{O}^\dagger(X)$ be $\varinjlim_h \mathcal{O}(X_h)$ and $\mathcal{O}^{+\dagger}(X)$ be $\varinjlim_h \mathcal{O}^+(X_h)$.

Fix a map $(\widehat{X}, \{X_h\}) \to (\widehat{Y}, \{Y_h\})$ in $\mathrm{Sm}^\dagger/S$. We say it is an *open immersion* (resp. *étale*) if the map of pro-objects has a strictification which is made of morphisms $X_h \to Y_h$ that are open immersions (resp. étale). We remark that under these hypotheses, the map $\widehat{X} \to \widehat{Y}$ is automatically an open immersion (resp. étale). A collection of morphisms $\{(\widehat{U}_i, \{U_{h_i}\}) \to (\widehat{X}, \{X_h\})\}$ is a *cover* if for every $x \in \widehat{X}^{/V}$ there is some i for which x lies in the image of each U_{h_i}.

Remark 3.4. A choice of a strict inclusion $\widehat{X} \Subset_V X_0$ of smooth rigid analytic varieties over S with $V = \mathsf{Im}(\widehat{X} \to S)$ defines an object of $\mathrm{Sm}^\dagger/S$ by taking the filtered diagram of open subsets of X_0 containing the closure of $\widehat{X}$. Any morphism, open immersion, étale map of strict inclusions $(\widehat{X} \to X_0) \to (\widehat{Y} \to Y_0)$ induces a morphism, open immersion, étale map in $\mathrm{Sm}^\dagger/S$, respectively. Up to replacing X_0 with $X_0 \times_S V$ one may assume that $V = \mathsf{Im}(X_0 \to S)$. We can actually define $\mathrm{Sm}^\dagger/S$ to be the category of such strict inclusions, up to refinement, where maps are morphisms $\widehat{X} \to \widehat{Y}$ extending to $X_h \to Y_0$ for some strict neighborhood X_h of $\widehat{X}$ in X_0 (i.e., containing its closure).

Remark 3.5. By [Huber 1996, Proposition 2.4.4] (which holds even without the noetherianity hypothesis imposed in [Huber 1996]; see, for example, [Ayoub et al. 2022, Corollary 1.4.20]) if $\widehat{X}$ is qcqs, any étale cover of $(\widehat{X}, \{X_h\})$ consisting of a finite number of étale maps can be refined by one of the form $\{(\widehat{U}_i, \{U_{ih}\})\}_{i=1,\dots,N}$ such that all indices h vary in the same category, that we can suppose to be directed, and each map of pro-objects comes from a map of diagrams, with each $\{U_{ih} \to X_h\}$ being an étale cover.

Proposition 3.6. *The big étale site on the category* $\mathrm{Sm}^\dagger/S$ *is equivalent to the site whose objects are pairs* $X = (\widehat{X}, \mathcal{O}^\dagger(X))$ *with* $\widehat{X}$ *a smooth variety over S of the form*

$$\mathrm{Spa}\big(\mathcal{O}(V)\langle \underline{x}, \underline{y}\rangle/(p_1, \dots, p_m), \mathcal{O}(V)\langle \underline{x}, \underline{y}\rangle/(p_1, \dots, p_m)^+\big)$$

with V being an affinoid subset of S which is the image of $\widehat{X}$, $\underline{x}$ *and* $\underline{y}$ *some sets of variables* $\underline{x} = (x_1, \dots, x_n)$, $\underline{y} = (y_1, \dots, y_m)$, p_i *are in* $\mathcal{O}(V)[\underline{x}, \underline{y}]$ *such that* $\det(\partial p_i/\partial y_j)$ *is invertible in* $\mathcal{O}(\widehat{X})$ *and* $\mathcal{O}^\dagger(X)$ *is a subring of* $\mathcal{O}(\widehat{X})$ *of the form*

$$\mathcal{O}^\dagger(X) = \varinjlim \mathcal{O}(V)\langle \pi^{1/h}\underline{x}, \pi^{1/h}\underline{y}\rangle/(p_1, \dots, p_m).$$

Morphisms $X \to X'$ *are defined as being the maps* $\widehat{X} \to \widehat{X}'$ *sending* $\mathcal{O}^\dagger(X')$ *to* $\mathcal{O}^\dagger(X)$ *and étale covers are families* $\{X_i \to X\}$ *such that the maps* $\widehat{X}_i \to \widehat{X}$ *are étale and jointly surjective.*

Proof. We first prove that the category above is a full subcategory of $\mathrm{Sm}^\dagger/S$. Let $X = (\widehat{X}, \mathcal{O}^\dagger(X))$ as in the statement. We remark that since $d := \det(\partial p_i/\partial y_j) \in \mathcal{O}^\dagger(X)$ is invertible in $\mathcal{O}(\widehat{X})$ in which $\mathcal{O}^\dagger(X)$ is dense, and $\widehat{X}$ is quasicompact, we have d is invertible in some ring $R_h := \mathcal{O}(V)\langle \pi^{1/h}\underline{x}, \pi^{1/h}\underline{y}\rangle/(p_1, \dots, p_m)$ and therefore $\widehat{X} \Subset_V \mathrm{Spa}\, R_h =: X_h$ defines an object of $\mathrm{Sm}^\dagger/S$ (see Remark 3.4).

We now show that morphisms $X \to Y$ computed in $\mathrm{Sm}^\dagger/S$ amount to morphisms $\widehat{X} \to \widehat{Y}$ such that the images $\underline{s}, \underline{t}$ of $\underline{x}, \underline{y}$ lie in $\mathcal{O}^\dagger(X) \cap \mathcal{O}^+(\widehat{X})$. It suffices to show that an (R, R^+)-morphism from $X^\dagger$ to $\mathbb{B}^{1\dagger}_{\mathrm{Spa}(R,R^+)} = (\mathbb{B}^1_{\mathrm{Spa}(R,R^+)}, R\langle x\rangle^\dagger)$ amounts to a choice of an element in $\mathcal{O}^+(\widehat{X}) \cap \mathcal{O}^\dagger(X)$. Fix such an element s. We may suppose that it lies in $\mathcal{O}(X_0)$. But then we have $\widehat{X} \subset U(s/1) \Subset_{X_0} U(\pi s/1)$ which implies that $X_h \subset U(\pi s/1)$ for $h \gg 0$ so that $\pi s \in \mathcal{O}^{+\dagger}(X)$ showing that the map $\widehat{X} \to \mathbb{B}^1$ extends to some map $X_h \to \mathrm{Spa} R\langle \pi x\rangle$ as wanted. Conversely, if the map $\widehat{X} \to \mathbb{B}^1_{(R,R^+)}$ defined by $s \in \mathcal{O}^+(\widehat{X})$ extends to $X_h \to \mathrm{Spa}\, R\langle \pi x\rangle$ then $\pi s \in \mathcal{O}^+(X_h)$ so that $s \in \mathcal{O}^\dagger(X) \cap \mathcal{O}^+(\widehat{X})$.

We now show that the subcategory of the statement is dense in $\mathrm{Sm}^\dagger/S$. This is analogous to [Vezzani 2018, Corollary 3.4]. Indeed, locally with respect to the analytic topology, any object $X = (\widehat{X} \Subset X_0)$ is such that $\widehat{X}$ is of the form prescribed. We now show that there is an automorphism of $\widehat{X}$ identifying the two (dense) subrings $\varinjlim \mathcal{O}(X_h)$ and $\mathcal{O}^\dagger(X)$ of the statement. By [Vezzani 2019a, Corollary A.2] we can find some power series F in $\mathcal{O}(\widehat{X})[\![\underline{\sigma} - \underline{x}]\!]$ ($\underline{\sigma}$ being some variable as in [Vezzani 2018, Corollary 3.4]) with a positive radius of convergence such that $(x, y) \mapsto (\tilde{s}, F(\tilde{s}))$ defines an endomorphism of $\widehat{X}$ for every $\tilde{s}$ sufficiently close to x. By density, we may take $\tilde{s}$ in $\varinjlim \mathcal{O}(X_h) \cap \mathcal{O}^+(\widehat{X})$. We remark that under this hypothesis, $F(\tilde{s})$ lies in $\varinjlim \mathcal{O}(X_h) \cap \mathcal{O}^+(\widehat{X})$. This follows from the equivalence $\mathrm{Et}/\widehat{X}^{/V} \cong \varprojlim \mathrm{Et}/X_h$ of [Huber 1996, Proposition 2.4.4] by considering the étale morphism $\mathrm{Spa}\,\mathcal{O}(X_h)\langle \underline{\tau}\rangle/(p(\tilde{s}, \tau)) \to X_h$ ($\underline{\tau}$ being some variable) that splits above $\widehat{X}^{/V}$. This shows that there is an endomorphism ψ of $\widehat{X}$ which is close to the identity (in the sense that $\|\psi(f) - f\| \leq |\pi^2|$ whenever $\|f\| \leq 1$ with respect to some Banach norm $\|\cdot\|$ of $\mathcal{O}(\widehat{X})$) mapping $\mathcal{O}^\dagger(\widehat{X})$ to $\varinjlim \mathcal{O}(X_h)$. Any endomorphism which is close to the identity is invertible; hence the claim.

We are left to prove that the small étale site over $X^\dagger = (\widehat{X} \Subset_V X_0)$ is equivalent to the small étale site on $\widehat{X}$ via the functor mapping $(\widehat{U} \Subset_{V_U} U_0)$ to $\widehat{U}$. Indeed, if $\widehat{U} \subset \widehat{X}$ is a rational open, we may lift it to $U = (\widehat{U} \Subset_{V_U} X_0)$, and if $\widehat{E} \to \widehat{X}$ is finite étale between affinoids, we may extend it to a finite étale map $\widehat{E}^{/V} \to \widehat{X}^{/V}$ and hence to some finite étale map $E_h \to X_h$ with $\widehat{E} \Subset_V E_h$. This shows that any étale dagger space over $\widehat{X}$ has a cover made of objects descending to $X^\dagger$. Since $\left(\bigcup \widehat{U}_i\right)^{/V} = \bigcup(\widehat{U}_i^{/V_i})$ we also deduce that a family $\{\widehat{U}_i \Subset_{V_i} U_i\}$ of étale maps over $X^\dagger$ is a cover if and only if the family $\{\widehat{U}_i\}$ covers $\widehat{X}$, proving the claim. □

Relative overconvergent motives. It is straightforward to generalize the definition of motives to the dagger setting.

Definition 3.7. Let S be an object of $\mathrm{Adic}_{/\mathbb{Q}_p}$. We let $\mathbb{B}^{1\dagger}_S$ (resp. $\mathbb{T}^{1\dagger}_S$) be the object of $\mathrm{Sm}^\dagger/S$ induced by the inclusions $\mathbb{B}^1_S \Subset_S \mathbb{P}^1_S$ (resp. $\mathbb{T}^1_S \Subset_S \mathbb{P}^1_S$) and $T^\dagger_S$ be the quotient of the split inclusion $\mathbb{Q}_S(S) \to \mathbb{Q}_S(\mathbb{T}^{1\dagger}_S)$ in $\mathrm{Psh}(\mathrm{Sm}^\dagger/S, \mathbb{Q})$. We let $\mathrm{Psh}(\mathrm{Sm}^\dagger/S, \mathbb{Q})$ be the infinity-category of presheaves on the category Sm/S taking values on the derived infinity-category of $\mathbb{Q}$-modules, and we let $\mathrm{RigDA}^{\mathrm{eff}\,\dagger}(S)$ be its full stable infinity-subcategory spanned by those objects $\mathcal{F}$ which are $\mathbb{B}^{1\dagger}$-invariant and with ét-descent. Finally, we set $\mathrm{RigDA}^\dagger(S, \mathbb{Q}) = \mathrm{RigDA}^{\mathrm{eff}\,\dagger}(S, \mathbb{Q})[T^{\dagger\,-1}_S]$ in Pr^{L} (see [Robalo 2015, Definition 2.6]).

The following result is essentially formal; see Theorem 2.10.

Proposition 3.8. *There are contravariant functors* $\mathrm{RigDA}^{(\mathrm{eff})\dagger *}$ *defined on* $\mathrm{Adic}_{/\mathbb{Q}_p}$ *with values in* $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ *such that any* $f : S' \to S$ *in* $\mathrm{Adic}_{/\mathbb{Q}_p}$ *is sent to the functor* $f^* : \mathrm{RigDA}^{(\mathrm{eff})\dagger}(S) \to \mathrm{RigDA}^{(\mathrm{eff})\dagger}(S')$ *induced by pullback along* f*. They satisfy étale hyperdescent and their restrictions to* $\mathrm{Adic}^{\mathrm{qcqs}}_{/\mathbb{Q}_p}$ *take values in* $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_\omega)$.

Proof. One can adapt the proofs of [Ayoub et al. 2022, Propositions 2.1.21 and 2.4.22, Theorem 2.3.4 and Remark 2.3.5] to the dagger setting. ☐

The following theorem allows one to equip any motive with an overconvergent structure, if needed. It is a generalization of [Vezzani 2018] to a base S with no overconvergent structure. Once again, we crucially use some explicit homotopies in the proof of the statement.

Theorem 3.9. *Let S be in $\mathrm{Adic}_{/\mathbb{Q}_p}$. The functor $l : X \mapsto \widehat{X}$ induces an equivalence*

$$l^* : \mathrm{RigDA}^{\dagger(\mathrm{eff})}(S) \cong \mathrm{RigDA}^{(\mathrm{eff})}(S).$$

Proof. The proof will be divided into several steps, most of which follow closely the proof of [Vezzani 2019a, Proposition 4.5].

Step 1: It suffices to prove the claim for effective motives. By Proposition 3.6 we may and do use as models for $\mathrm{RigDA}^{\dagger\,\mathrm{eff}}(S)$ (resp. $\mathrm{RigDA}^{\mathrm{eff}}(S)$) the category of spectra on the (ét, $\mathbb{B}^1$)-localization of complexes of étale presheaves on $\mathcal{C}^\dagger$ (resp. $\mathcal{C}$) which is the (dense) subcategory of $\mathrm{RigSm}^\dagger/S$ (resp. RigSm/S) whose objects are of the form $X = (\widehat{X}, \mathcal{O}^\dagger(X))$ (resp. l^*X) described in Proposition 3.6. The functor l induces a Quillen pair (l^*, l_*) between these two model categories; hence a pair of (derived) functors $(\mathbb{L}l^*, \mathbb{R}l_*)$ between the associated infinity-categories. Moreover, $\mathbb{R}l_* = l_*$ is exact as it commutes with étale sheafification and preserves $\mathbb{B}^1$-weak equivalences. We then remark that it suffices to prove that the functor $\mathbb{L}l^*$ between the $\mathbb{B}^1$-localizations $\mathrm{Ch}_{\mathbb{B}^{1\dagger}_S} \mathrm{Psh}(\mathcal{C}^\dagger, \mathbb{Q})$ and $\mathrm{Ch}_{\mathbb{B}^1_S} \mathrm{Psh}(\mathcal{C}, \mathbb{Q})$ is an equivalence. Since it sends a class of compact generators to a class of compact generators, we are left to prove it is fully faithful.

Step 2: We show the following claim. Fix varieties $X = (\mathrm{Spa}(R, R^+), R^\dagger)$ and $X' = (\mathrm{Spa}(R', R'^+), R'^\dagger)$ in $\mathcal{C}^\dagger$ and a morphism $f : \widehat{X}' = \mathrm{Spa}(R', R'^+) \to \widehat{X} = \mathrm{Spa}(R, R^+)$ over S. Then there exists a map $H : \mathbb{B}^1_{\widehat{X}'} \cong \mathrm{Spa}(R'\langle\chi\rangle, R'^+\langle\chi\rangle) \to \widehat{X}$ such that $H \circ i_0 = f$ and $H \circ i_1$ lies in $\mathrm{Hom}(X, X')$. Explicitly, if f is induced by the map $\sigma \mapsto s$, $\tau \mapsto t$, the map H can be defined via

$$(\sigma, \tau) \mapsto \big(s + (\tilde{s} - s)\chi,\, F(s + (\tilde{s} - s)\chi)\big),$$

where F is the unique array of formal power series (implicit functions) with positive radius of convergence in $R'[\![\sigma - s]\!]$ associated by [Vezzani 2019a, Corollary A.2] to the polynomials $p(\sigma, \tau)$ which are such that $F(s) = t$ and $p(\sigma, F(\sigma)) = 0$, and $\tilde{s}$ are elements in $R'^\dagger$ such that the radius of convergence of F is larger than $\|\tilde{s} - s\|$ and $F(\tilde{s})$ lies in R^+. As $R'^\dagger$ is dense in R'^+ we can find elements $\tilde{s}_i \in R'_0 \cap R'^+$ such that $\|\tilde{s} - s\|$ is smaller than the convergence radius of F. As F is continuous and R'^+ is open, we can also assume that the elements $\tilde{t}_j := F_j(\tilde{s})$ lie in R'^+. We are left to prove that they actually lie in $R'^\dagger$. We consider the R'_0-algebra E defined as $E = R'_0\langle\tau\rangle/(p(\tilde{s}, \tau))$ which is étale over R'_h, and over which the map $R'_0 \to R'$ factors. In particular, the étale morphism $\mathrm{Spa}(E, E^+) \times_{X'_0} \widehat{X}' \to \widehat{X}'$ splits. In light of the equivalence between the étale toposes on $\widehat{X}'$ and on X' (see the end of the proof of Proposition 3.6), if we let Y be the étale map in $\mathcal{C}^\dagger_{/X'}$ induced by (E, E^+), Yoneda ensures that $Y \to X'$ splits as well, proving that $\tilde{t}_j$ lies in R'_h as wanted.

Step 3: We show the following claim. For a given finite set of maps $\{f_1, \dots, f_N\}$ in $\operatorname{Hom}_S(\widehat{X}' \times_S \mathbb{B}^n_S, \widehat{X})$ we can find corresponding maps $\{H_1, \dots, H_N\}$ in $\operatorname{Hom}_S(\widehat{X}' \times_S \mathbb{B}^n_S \times_S \mathbb{B}^1_S, \widehat{X})$ such that

1. for all $1 \leq k \leq N$ we have $i_0^* H_k = f_k$ and $i_1^* H_k$ has a model in $\operatorname{Hom}(X' \times_S \mathbb{B}^n_S, X)$;
2. if $f_k \circ d_{r,\epsilon} = f_{k'} \circ d_{r,\epsilon}$ for some $1 \leq k, k' \leq N$ and some $(r, \epsilon) \in \{1, \dots, n\} \times \{0, 1\}$ then $H_k \circ d_{r,\epsilon} = H_{k'} \circ d_{r,\epsilon}$;
3. if for some $1 \leq k \leq N$ the map $f_k \circ d_{1,1} \in \operatorname{Hom}(\widehat{X}' \times_S \mathbb{B}^{n-1}_S, \widehat{X})$ has a model in $\operatorname{Hom}(X' \times_S \mathbb{B}^{(n-1)\dagger}_S, X)$ then the element $H_k \circ d_{1,1}$ of $\operatorname{Hom}_S(\widehat{X}' \times_S \mathbb{B}^{n-1}_S \times_S \mathbb{B}^1_S, \widehat{X})$ is constant on $\mathbb{B}^1_S$ equal to $f_k \circ d_{1,1}$;

where we denote by $d_{r,\epsilon}$ the morphisms $\mathbb{B}^{n-1} \to \mathbb{B}^n$ induced by the evaluation of the r-th coordinate of $\mathbb{B}^n$ at ϵ. We may suppose that each f_k is induced by maps $(\sigma, \tau) \mapsto (s_k, t_k)$ from R to $R'\langle \theta_1 \dots, \theta_n \rangle$ for some m-tuples s_k and n-tuples t_k in $R'\langle \underline{\theta} \rangle$. Moreover, by step 2 there exists a sequence of power series $F_k = (F_{k1}, \dots, F_{km})$ associated to each f_k such that

$$(\sigma, \tau) \mapsto \big(s_k + (\tilde{s}_k - s_k)\chi,\ F_k(s_k + (\tilde{s}_k - s_k)\chi)\big) \in R'\langle \underline{\theta}, \chi \rangle$$

defines a map H_k satisfying the first claim, for any choice of $\tilde{s}_k \in R'\langle \underline{\theta} \rangle^\dagger$ such that $\tilde{s}_k$ is in the convergence radius of F_k and $F_k(\tilde{s}_k)$ is in $R'\langle \underline{\theta} \rangle^+$. Let ϵ be a positive real number, smaller than all radii of convergence of the series F_{kj} and such that $F(a) \in R'\langle \underline{\theta} \rangle^+$ for all $|a - s| < \epsilon$. Denote by $\tilde{s}_{ki}$ the elements associated to s_{ki} by applying [Vezzani 2019a, Proposition A.5] with respect to the chosen ϵ. In particular, they induce a well-defined map H_k and the elements $\tilde{s}_{ki}$ lie in $R'\langle \underline{\theta} \rangle_{\bar{h}}$ for some index $\bar{h}$. We show that the maps H_k induced by this choice also satisfy the second and third claims of the proposition. Suppose that $f_k \circ d_{r,\epsilon} = f_{k'} \circ d_{r,\epsilon}$ for some $r \in \{1, \dots, n\}$ and $\epsilon \in \{0, 1\}$. This means that $\bar{s} := s_k|_{\theta_r = \epsilon} = s_{k'}|_{\theta_r = \epsilon}$ and $\bar{t} := t_k|_{\theta_r = \epsilon} = t_{k'}|_{\theta_r = \epsilon}$. This implies that both $F_k|_{\theta_r = \epsilon}$ and $F_{k'}|_{\theta_r = \epsilon}$ are two m-tuples of formal power series $\bar{F}$ with coefficients in $\mathcal{O}(\widehat{X}' \times \mathbb{B}^{n-1})$ converging around $\bar{s}$ and such that $p(\sigma, \bar{F}(\sigma)) = 0$, $\bar{F}(\bar{s}) = \bar{t}$. By the uniqueness of such power series stated in [Vezzani 2019a, Corollary A.2], we conclude that they coincide. Moreover, by our choice of the elements $\tilde{s}_k$ it follows that $\bar{\tilde{s}} := \tilde{s}_k|_{\theta_r = \epsilon} = \tilde{s}_{k'}|_{\theta_r = \epsilon}$. In particular one has

$$F_k((\tilde{s}_k - s_k)\chi)|_{\theta_r = \epsilon} = \bar{F}((\bar{\tilde{s}} - \bar{s})\chi) = F_{k'}((\tilde{s}_{k'} - s_{k'})\chi)|_{\theta_r = \epsilon}$$

and therefore $H_k \circ d_{r,\epsilon} = H_{k'} \circ d_{r,\epsilon}$ proving the second claim. The third claim follows immediately since the elements $\tilde{s}_{ki}$ satisfy the condition (iv) of [Vezzani 2019a, Proposition A.5].

Step 4: We remark that (see [Vezzani 2018, Proposition 4.22] or [Vezzani 2019a, Proposition 4.5]) the claim proved in step 3 admits the following interpretation: the natural map

$$\phi : (\operatorname{Sing}^{\mathbb{B}^{1\dagger}_S} \mathbb{Q}(X))(X') \to (\operatorname{Sing}^{\mathbb{B}^1_S} \mathbb{Q}_S(\widehat{X}))(\widehat{X}')$$

is a quasi-isomorphism, where for any complex of presheaves $\mathcal{F}$ we let $\operatorname{Sing}^{\mathbb{B}^{1\dagger}_S} \mathcal{F}$ be the singular complex associated to the cocubical complex $\underline{\operatorname{Hom}}(\mathbb{Q}_S(\mathbb{B}^{\bullet\dagger}_S), \mathcal{F})$ which is $\mathbb{B}^{1\dagger}$-equivalent to $\mathcal{F}$. Indeed, the lifting property of step 3 allows one to prove directly that the homology groups of the normalized complexes associated to the cocubical complexes above are isomorphic; we refer to the proof of [Vezzani 2018, Proposition 4.22] for details. This implies that, considering the Quillen adjunction

$$\mathbb{L}l^* : \mathrm{Ch}_{\mathbb{B}^{1\dagger}_S} \mathrm{Psh}(\mathcal{C}^\dagger, \mathbb{Q}) \rightleftarrows \mathrm{Ch}_{\mathbb{B}^1_S} \mathrm{Psh}(\mathcal{C}, \mathbb{Q}) : \mathbb{R}l_* = l_*,$$

we have

$$\mathbb{R}l_* \mathbb{L}l^* \mathbb{Q}_S(X) = l_* \operatorname{Sing}^{\mathbb{B}^1_S} \mathbb{Q}_S(\widehat{X}) \cong \operatorname{Sing}^{\mathbb{B}^1_S} \mathbb{Q}_S(X).$$

Since $\operatorname{Sing}^{\mathbb{B}^1_S} \mathbb{Q}_S(X) \cong \mathbb{Q}_S(X)$ (see, for example, [Vezzani 2018, Proposition 4.10]) this proves that $\mathbb{L}l^*$ is fully faithful; hence the claim by step 1. ☐

4. The relative overconvergent de Rham cohomology

The aim of this section is to define the analog of the overconvergent de Rham cohomology in the relative setting. One of the main problems of its "naive" definition is that a nice category of quasicoherent sheaves over an adic space wasn't available until very recently. Clausen and Scholze's formalism of condensed mathematics [Scholze 2019; 2020] allows one to define such a category with a symmetric monoidal structure. Although this category is big, its dualizable objects are nothing but (classical) perfect complexes, as proved in [Andreychev 2021] for the case of interest to us. By upgrading the relative de Rham cohomology to the condensed level, we are then able to formulate and prove a base change formula and the Künneth formula for it. Combined with the above characterization of dualizable objects, this produces some finiteness statements for relative de Rham cohomology.

The relative de Rham complex. We initially give the definition of the module of differentials of a smooth map in Adic, and prove its basic properties. As far as we know, the current literature treats mainly the case of a noetherian base (see [Huber 1996], for example) and we make here some straightforward extensions of this case.

Definition 4.1. Let $f : X \to S$ be a smooth morphism in Adic. Let $\mathcal{I}_{X/S} \subset \mathcal{O}_{X \times_S X}$ be the ideal sheaf of the diagonal $\Delta_f : X \to X \times_S X$. The *sheaf of differentials of X over S* is

$$\Omega^1_{X/S} := \mathcal{I}_{X/S}/\mathcal{I}^2_{X/S},$$

seen as an $\mathcal{O}_X$-module through the identification $\mathcal{O}_X \simeq \mathcal{O}_{X \times_S X}/\mathcal{I}_{X/S}$.

Note that by construction, $\Omega^1_{X/S}$ comes with an $\mathcal{O}_S$-linear derivation $d : \mathcal{O}_X \to \Omega^1_{X/S}$, sending a section s to $1 \otimes s - s \otimes 1$.

Definition 4.2. Let $d \geq 0$. Let $f : X \to S$ be a smooth morphism in Adic. We say that f is of *dimension d* if locally on X and S the morphism factors as the composition of an étale morphism $X \to \mathbb{B}^d_S$ with the projection $\mathbb{B}^d_S \to S$.

Since the dimension of a smooth morphism $f : X \to S$ is locally constant on X, it is no loss of generality in practice to assume that f is of fixed dimension.

The following statement is proved in [Fargues and Scholze 2021]. We recall how the argument goes, in order to fix some notation.

Proposition 4.3. *Let $f : X \to S$ be a smooth morphism in* Adic. *The $\mathcal{O}_X$-module $\Omega^1_{X/S}$ is a vector bundle. If f is of dimension d, it is of constant rank d.*

Proof. Since this is a local assertion, we can assume that f is the composite of an étale morphism $g : X \to \mathbb{B}^d_S$ with the projection $h : \mathbb{B}^d_S \to S$. We can also assume that $S = \mathrm{Spa}(A, A^+)$ and $X = \mathrm{Spa}(B, B^+)$ are both affinoid. In this case, we will prove that $\Omega^1_{X/S}$ is in fact a free $\mathcal{O}_X$-module of rank d. For brevity, write $Y := \mathbb{B}^d_S$. The diagonal map $\Delta_f : X \to X \times_S X$ can be decomposed as the composition of

$$X \xrightarrow{\Delta_g} X \times_Y X = Y \times_{Y \times_S Y} (X \times_S X) \to X \times_S X,$$

where the second map is obtained by base changing $\Delta_h : Y \to Y \times_S Y$ along $X \times_S X \to Y \times_S Y$. Since g is étale, the map Δ_g is an open immersion. Therefore, the $\mathcal{O}_{X \times_S X}$-module $\mathcal{I}_{X/S}$ is the pullback of the $\mathcal{O}_{Y \times_S Y}$-module $\mathcal{I}_{Y/S}$ along the map $X \times_S X \to Y \times_S Y$.

The map $Y \to Y \times_S Y$ is of the form

$$\mathrm{Spa}(A\langle \underline{T} \rangle, A^+\langle \underline{T} \rangle) \to \mathrm{Spa}(A\langle \underline{T}, \underline{T}' \rangle, A^+\langle \underline{T}, \underline{T}' \rangle)$$

for some sets of variables $\underline{T} = (T_1, \dots, T_d)$ and $\underline{T}' = (T'_1, \dots, T'_d)$, and $\mathcal{I}_{Y/S}$ is the ideal sheaf given by the ideal $(T_1 - T'_1, \dots, T_d - T'_d)$. To conclude the proof, it suffices to check that $T_1 - T'_1, \dots, T_N - T'_N$ define a regular sequence in $B \widehat{\otimes}_A B$ and that the ideal $(T_1 - T'_1, \dots, T_d - T'_d) \cdot B \widehat{\otimes}_A B$ is closed in $B \widehat{\otimes}_A B$. This is the content of [Fargues and Scholze 2021, Proposition IV.4.12]. ☐

Definition 4.4. Let $f : A \to B$ a be morphism of complete Huber rings. A *universal A-derivation of B* is a continuous A-derivation $d_{B/A} : B \to \Omega_{B/A}$ such that for any continuous A-derivation $d : B \to M$ from B to a complete topological B-module M, there is a unique continuous B-linear map $g : \Omega_{B/A} \to M$ such that $d = g \circ d_{B/A}$.

Proposition 4.5. *Let $f : X \to S$ be a smooth morphism in* Adic. *Locally on X, $X = \mathrm{Spa}(B, B^+)$, $S = \mathrm{Spa}(S, S^+)$ and $\Omega^1_{X/S}$ is the $\mathcal{O}_X$-module attached to the finite projective B-module $\Omega_{B/A} := I/I^2$, where I is the kernel of the multiplication map $B \widehat{\otimes}_A B \to B$. The map $d_{B/A} : B \to \Omega_{B/A}$, induced by the map $b \mapsto 1 \otimes b - b \otimes 1$, is a universal A-derivation of B.*

Proof. The first part follows from the proof of Proposition 4.3. Moreover, this proof shows that the ideal I is closed and finitely generated, therefore a complete B-module of finite type. Choose a finite subset N of B such that the subring $A[N]$ is dense in B. The proof of [Huber 1996, Proposition 1.6.2(ii)] shows that the ideal J generated by the elements $1 \otimes n - n \otimes 1$, $n \in N$, is dense in I. Thus, by [Bhatt et al. 2019, Lemma 1.1.13], we must have $J = I$ (note that the topology on I induced by the topology on B is necessarily the natural topology, by [Bhatt et al. 2019, Corollary 1.1.12]). From there, the same proof as the usual algebraic proof shows that $\Omega_{B/A}$ is a universal A-derivation of B. ☐

This allows us to check that $\Omega^1_{X/S}$ has the expected properties listed in the following proposition.

Proposition 4.6. *Let $f : X \to S$ be a smooth morphism in* Adic.

(1) *Let $g : S' \to S$ be a map in* Adic, *and let $f' : X' := X \times_S S' \to S'$ be the base change of f, which is again smooth. Then $\Omega^1_{X'/S'}$ is the pullback of $\Omega^1_{X/S}$ along $g' : X' \to X$.*

(2) *Let $g : Y \to X$ be a smooth morphism. Then one has a short exact sequence*

$$0 \to g^*\Omega^1_{X/S} \to \Omega^1_{Y/S} \to \Omega^1_{Y/X}.$$

(3) *Let $g : Y \to S$ be a smooth morphism. There is a natural isomorphism*

$$\Omega^1_{(X\times_S Y)/S} \cong g'^*\Omega^1_{X/S} \oplus f'^*\Omega^1_{Y/S},$$

where $g' : X \times_S Y \to X$, $f' : X \times_S Y \to Y$ denote the two projections.

Proof. The proofs of (1) and (2) are the same as in the algebraic case, using the universal property, given Proposition 4.5. The assertion (3) follows from (1) and (2). ☐

Definition 4.7. Let $f : X \to S$ be a smooth morphism in Adic of dimension d. For each $i \geq 1$, write $\Omega^i_{X/S} = \bigwedge^i \Omega^1_{X/S}$. The derivation $d : \mathcal{O}_X \to \Omega^1_{X/S}$ extends naturally to a complex of sheaves of $\mathcal{O}_S$-modules on X:

$$\mathcal{O}_X \xrightarrow{d} \Omega^1_{X/S} \xrightarrow{d} \cdots \xrightarrow{d} \Omega^d_{X/S},$$

(with $\mathcal{O}_X$ sitting in degree 0) called *the de Rham complex of X over S* and denoted by $\Omega^\bullet_{X/S}$.

Recollection on solid quasicoherent sheaves. Clausen and Scholze have developed a formalism allowing one to attach to any analytic adic space X an infinity-category $\mathrm{QCoh}(X)$ of *solid quasicoherent sheaves* on X, serving the same purposes as the category of quasicoherent sheaves in algebraic category (and even more, since it allows one to build a full 6-functor formalism; see [Scholze 2019]). If $f : X \to S$ is a smooth (dagger) morphism in Adic, the (overconvergent) de Rham complex naturally defines an object of $\mathrm{QCoh}(S)$ and it will be important for us to adopt this point of view in the following. This is what we explain in this subsection. We start by recalling several properties of analytic rings attached to complete Huber pairs that we gather essentially from [Scholze 2020; Andreychev 2021] and that we summarize here for the convenience of the reader.

Definition 4.8. For the basic notation on condensed abelian groups we refer to [Scholze 2019]. We will typically consider them as abelian sheaves on the site of extremally disconnected sets with covers given by finite collections of jointly surjective maps (see [Scholze 2019, Proposition 2.7]).

(1) If A is a topological abelian group we denote by $\underline{A}$ the condensed abelian group defined by $\underline{A}(S) = \mathrm{Hom}(S, A)$ (the group of continuous maps) for any extremally disconnected set S. If A has a topological ring structure, then $\underline{A}$ is a condensed ring.

(2) If R is a condensed ring (for example, $R = \underline{A}$ for some topological ring A) and S is an extremally disconnected set, we denote by $R[S]$ the condensed R-module representing the functor $M \mapsto M(S)$ on condensed R-modules.

(3) An *analytic ring* is given by a condensed ring R, a functor M_R taking an extremally disconnected set S to some R-module $M_R[S]$ in condensed abelian groups, and a natural transformation $R[S] \to M_R[S]$ satisfying some extra properties (see [Scholze 2020, Definition 6.12]). The category of (R, M_R)*-modules* M_R -Mod is the full abelian subcategory with products and sums inside condensed R-modules generated by the objects $M_R[S]$. The natural transformation which is part of the definition gives rise to a localization functor R -Mod $\to$ M_R -Mod that is denoted by $M \mapsto M \otimes_R (R, M_R)$ and is the unique colimit-preserving extension of the functor $R[S] \to M_R[S]$. More generally, any map of analytic rings (defined as in [Scholze 2019, Lecture VII]) $f : (A, M_A) \to (B, M_B)$ induces a base-change functor $f^* : M_A$ -Mod $\to$ M_B -Mod, $M \mapsto M \otimes_{(A, M_A)} (B, M_B)$, which is a left adjoint to the "forgetful" functor f_*. If R is commutative, the category M_R -Mod is endowed with a symmetric monoidal tensor product $\otimes_{(R, M_R)}$ making the functor $M \mapsto M \otimes_R (R, M_R)$ symmetric monoidal. One says (R, M_R) is *complete* or *normalized* (see [Scholze 2020, Definition 12.9]) if $M_R[*] \cong R$.

(4) We recall that an *animated analytic ring* is given by a condensed animated ring $\mathcal{R}$, a functor $\mathcal{M}_{\mathcal{R}}$ taking an extremally disconnected set S to some $\mathcal{R}$-module $\mathcal{M}_{\mathcal{R}}[S]$ in condensed animated abelian groups, and a natural transformation $\mathcal{R}[S] \to \mathcal{M}_{\mathcal{R}}[S]$ satisfying some extra properties (see [Scholze 2020, Definition 12.1]). The category $\mathcal{D}(\mathcal{R}, \mathcal{M}_{\mathcal{R}})$ is the stable infinity-category generated under sifted colimits by the shifts of $\mathcal{M}_{\mathcal{R}}[S]$ in (unbounded) derived condensed $\mathcal{R}$-modules (see [Scholze 2020, Definition 12.3 and Remark 12.5]). The natural transformation which is part of the definition gives rise to a localization functor $\mathcal{D}(\mathcal{R}) \to \mathcal{D}(\mathcal{M}_{\mathcal{R}})$ that is denoted by $M \mapsto M \otimes_{\mathcal{R}} (\mathcal{R}, \mathcal{M}_{\mathcal{R}})$. More generally, any map of analytic rings (defined as in [Scholze 2020, Lecture XII]) $f : (\mathcal{A}, \mathcal{M}_A) \to (\mathcal{B}, \mathcal{M}_B)$ induces a base-change functor $f^* : \mathcal{D}(\mathcal{M}_{\mathcal{A}}) \to \mathcal{D}(\mathcal{M}_{\mathcal{B}})$, $M \mapsto M \otimes_{(\mathcal{A}, \mathcal{M}_A)} (\mathcal{B}, \mathcal{M}_{\mathcal{B}})$, which is a left adjoint to the "forgetful" functor f_*. If $\mathcal{R}$ is a condensed animated commutative ring, there is a unique symmetric monoidal structure $\otimes_{(\mathcal{R}, \mathcal{M}_{\mathcal{R}})}$, making the functor $- \otimes_{\mathcal{R}} (\mathcal{R}, \mathcal{M}_{\mathcal{R}})$ symmetric monoidal. Any analytic ring structure (R, M_R) can be seen as an animated ring structure $\mathcal{M}_R$ on $R[0]$.

Remark 4.9. In [Andreychev 2021] the adjective *animated* is often dropped. What we call here *analytic rings* are there called 0*-truncated* (animated) analytic rings.

Remark 4.10. Beware that the functor $- \otimes_{R[0]} (R[0], \mathcal{M}_{\mathcal{R}})$ may not be the left derived functor of the functor $- \otimes_R (R, M_R)$ (see [Scholze 2019, Warning 7.6]) but it is so in all the examples we are interested in (see Proposition 4.12 below).

Example 4.11.

- If $\mathcal{R}$ is a condensed animated ring, the functor $S \mapsto \mathcal{R}[S]$ defines a ("trivial") analytic ring structure on $\mathcal{R}$, which we denote by $\mathcal{R}_{\text{triv}}$.
- The pair $(\underline{\mathbb{Z}}, \mathbb{Z}_\blacksquare)$ with $\mathbb{Z}_\blacksquare[\varprojlim S_i] := \varprojlim \mathbb{Z}[S_i]$ defines an analytic ring structure on the condensed discrete ring $\underline{\mathbb{Z}}$ (see [Scholze 2019, Theorem 5.8]). Similarly, if R is a finitely generated discrete ring, the datum $(\underline{R}, R_\blacksquare)$ with $R_\blacksquare[S] := \varprojlim R[S_i]$ defines an analytic ring structure on $\underline{R}$ (see [Scholze 2019, Theorem 8.1]). More generally, if R is a (discrete, 0-truncated) ring, the functor

$S \mapsto R_\blacksquare[S] := \varinjlim_{R'} R'_\blacksquare[S]$, as R' runs among finitely generated subrings of R, is an analytic ring structure on $\underline{R}$. From now on, the analytic ring structure $(\underline{R}, R_\blacksquare)$ will simply be denoted by $R_\blacksquare$.

All the analytic rings that we will consider lie above $\mathbb{Z}_\blacksquare$. The following fact is therefore particularly convenient for us.

Proposition 4.12 [Andreychev 2021, Proposition 2.11 and Corollary 2.11.2]. *If (R, M_R) is an analytic ring over $\mathbb{Z}_\blacksquare$ then $M_R[S] \otimes^{\mathbb{L}}_{(R,M_R)} M_R[T]$ is concentrated in degree zero for any pair of extremally disconnected sets (S, T). In particular, the tensor product in $\mathcal{D}(\mathcal{M}_\mathcal{R})$ coincides with the derived tensor product of M_R* -Mod.

There is a convenient way to produce animated analytic ring structures given in [Scholze 2020].

Proposition 4.13 [Scholze 2020, Proposition 12.8]. *Let $(\mathcal{R}, \mathcal{M}_\mathcal{R})$ be an animated analytic ring and $\mathcal{R} \to \mathcal{R}'$ a map of condensed animated rings. The functor*

$$S \mapsto \mathcal{R}'[S] \otimes_\mathcal{R} (\mathcal{R}, \mathcal{M}_\mathcal{R})$$

defines an animated analytic ring structure on $\mathcal{R}'$, which is the pushout $(\mathcal{R}, \mathcal{M}_\mathcal{R}) \otimes_{\mathcal{R}_{\mathrm{triv}}} \mathcal{R}'_{\mathrm{triv}}$ in animated analytic rings.

Under suitable hypotheses, the recipe above is internal to normalized analytic rings. The proof of the following fact is immediate.

Proposition 4.14 [Andreychev 2021, Proposition 2.16]. *Let (R, M_R) be a normalized analytic ring. Let $R \to R'$ be a map of condensed rings such that R' is an (R, M_R)-module and such that $R'[S] \otimes^{\mathbb{L}}_R (R, M_R)$ lies in degree zero for any extremally disconnected set S. The functor*

$$S \mapsto R'[S] \otimes_R (R, M_R)$$

defines a structure of a normalized analytic ring on R' above (R, M_R) whose associated animated analytic ring structure is $R'[0]_{\mathrm{triv}} \otimes_{R[0]_{\mathrm{triv}}} (R[0], \mathcal{M}_R)$.

We shall refer to the (animated) analytic structure introduced in the previous propositions as the one *induced* by $\mathcal{M}_\mathcal{R}$ and the map $\mathcal{R} \to \mathcal{R}'$.

Example 4.15. The analytic ring structure induced by $\mathbb{Z}_\blacksquare$ and the map (of discrete rings) $\mathbb{Z} \to \mathbb{Z}[T]$ will be denoted by $(\mathbb{Z}[T], \mathbb{Z})_\blacksquare$.

Another example of this situation, which is crucial to our setting, has been studied by [Andreychev 2021]. Let (A, A^+) be a complete Huber pair. Recall that the discrete ring A^+_{disc} (the ring A^+ endowed with the discrete topology) is equipped with a (normalized) analytic ring structure denoted by $(A^+_{\mathrm{disc}})_\blacksquare$ (see Example 4.11).

Definition 4.16. Let (A, A^+) be a complete Huber pair. We define $(A, A^+)_\blacksquare$ as the animated ring structure given by $\underline{A}[0]_{\mathrm{triv}} \otimes_{\underline{A}^+_{\mathrm{disc}}[0]_{\mathrm{triv}}} (A^+_{\mathrm{disc}})_\blacksquare$.

Proposition 4.17 [Andreychev 2021, Lemmas 3.24 and 3.25]. *The map* $\underline{A}^{+}_{\mathrm{disc}} \to \underline{A}$ *satisfies the hypotheses of Proposition 4.14. In particular, there is an analytic ring structure on* $\underline{A}$ *associated to* $(A, A^{+})_{\blacksquare}$.

We will use the same notation $(A, A^{+})_{\blacksquare}$ to refer both to the analytic ring structure on A and the animated one. The $(A, A^{+})_{\blacksquare}$-modules are also called *solid* (A, A^{+})-modules. We note that in particular one has, for any complete Huber pair (A, A^{+}), an infinity-category

$$\mathrm{QCoh}(\mathrm{Spa}(A, A^{+})) := \mathcal{D}((A, A^{+})_{\blacksquare}),$$

which is the infinity-category of (unbounded derived) solid (A, A^{+})-modules. Whenever we write $\otimes_{(A,A^{+})_{\blacksquare}}$ or f^{*}, for a morphism $f : (A, A^{+}) \to (B, B^{+})$ of complete Huber pairs, we will always mean it in the animated sense.

One of the main results of Andreychev is the following theorem.

Theorem 4.18 [Andreychev 2021, Theorem 4.1]. *Let X be an analytic adic space. The functor* $U \mapsto \mathrm{QCoh}(U)$ *from rational open subsets of X to infinity-categories has rational descent.*

Definition 4.19. For any $X \in \mathrm{Adic}$ we will denote by $\mathrm{QCoh}(X)$ the infinity-category obtained by rational descent from the functor QCoh defined on affinoid subspaces $U \subset X$. It is endowed with a symmetric monoidal structure $\otimes_{\mathrm{QCoh}(X)}$.

Remark 4.20. There is a natural t-structure on $\mathrm{QCoh}(X)$ when $X = \mathrm{Spa}(A, A^{+})$, whose heart is the abelian category of solid (A, A^{+})-modules, but there is no canonical t-structure on $\mathrm{QCoh}(X)$ in general.

Some pushouts in normalized animated analytic rings were introduced in Proposition 4.13 but actually, general pushouts in the category of normalized (animated) analytic rings exist, even though they are defined rather unexplicitly (see [Scholze 2020, Proposition 12.12]). However, there is a condition that turns them into something more tractable: we recall that a map of normalized analytic rings $f : (\mathcal{A}, \mathcal{M}_{\mathcal{A}}) \to (\mathcal{B}, \mathcal{M}_{\mathcal{B}})$ is *steady* (see [Scholze 2020, Definition 12.13]) if for any other map $g : (\mathcal{A}, \mathcal{M}_{\mathcal{A}}) \to (\mathcal{C}, \mathcal{M}_{\mathcal{C}})$ of normalized analytic rings, the pushout $(\mathcal{B}, \mathcal{M}_{\mathcal{B}}) \otimes_{(\mathcal{A},\mathcal{M}_{\mathcal{A}})} (\mathcal{C}, \mathcal{M}_{\mathcal{C}})$ is given by the functor

$$\mathcal{M}_{\mathcal{E}}[S] = \mathcal{M}_{\mathcal{C}}[S] \otimes_{(\mathcal{A},\mathcal{M}_{\mathcal{A}})} (\mathcal{B}, \mathcal{M}_{\mathcal{B}})$$

defining an analytic ring structure on the normalization $\mathcal{E}$ of $\mathcal{B} \otimes_{\mathcal{A}} \mathcal{C}$.

The following fact is essentially proved in [Scholze 2020].

Lemma 4.21. *Let* $(A, A^{+}) \to (B, B^{+})$ *be an adic map of Huber pairs. The induced map of analytic rings* $(A, A^{+})_{\blacksquare} \to (B, B^{+})_{\blacksquare}$ *is steady.*

Proof. We may decompose the map into two maps

$$(A, A^{+})_{\blacksquare} \to (B, B^{+}_{A})_{\blacksquare} \to (B, B^{+})_{\blacksquare}$$

with B^{+}_{A} being the smallest ring of integers for B containing the image of A^{+}. We remark that $(B, B^{+}_{A})_{\blacksquare} = (B, A^{+})_{\blacksquare}$, i.e., the analytic ring structure is the one induced by $(A, A^{+})_{\blacksquare}$ and the map $A \to B$. Since $A \to$

B is adic, we deduce that the map $(A, A^+)_■ \to (B, B^+_A)_■$ is steady by [Scholze 2020, Proposition 13.14 and page 102].

The map $(B, B^+_A)_■ \to (B, B^+)_■$ is an ind-steady open immersion defined by putting $|f| \leq 1$ for all $f \in B^+$ and as such (see [Scholze 2020, Proposition 12.15 and Example 13.15(3)]) it is steady.

We can then conclude the lemma, as compositions of steady maps are steady by [Scholze 2020, Proposition 12.15]. □

The following proposition will be used freely in what follows, and shows some compatibility between base change maps of adic spaces, and base change maps of their relative analytic spaces. It relies on results in [Andreychev 2021]. We say that a rational open immersion $U \subset \operatorname{Spa}(A, A^+)$ is *Laurent* if it is of the form $U = U(1/f)$ or $U = U(f/1)$ for some $f \in A$. We recall that any rational open immersion $U = U((f_1, \dots, f_n)/g) \subset \operatorname{Spa}(A, A^+)$ of Tate algebras is a composition of Laurent open immersions (see, for example, [Scholze 2012, Remark 2.8]).

Proposition 4.22. *Let*

$$f : X = \operatorname{Spa}(B, B^+) \to S = \operatorname{Spa}(A, A^+) \quad \textit{and} \quad g : Y = \operatorname{Spa}(C, C^+) \to S = \operatorname{Spa}(A, A^+)$$

be maps in Adic *such that f is smooth and can be written as a composition of rational open immersions, finite étale maps and projections of the form $\mathbb{B}^d_T \to T$. The pushout of (animated) analytic rings $(B, B^+)_■ \otimes_{(A,A^+)_■} (C, C^+)_■$ coincides with the analytic ring structure $(B \widehat{\otimes}_A C, B^+ \widehat{\otimes}_{A^+} C^+)_■$ on the completed tensor product of Huber pairs.*

Proof. We may and do consider separately the cases in which f is a Laurent rational open immersion, f is the projection of the unit disc and f is finite étale. In the first case, the result follows from the compatibility of (steady) localizations with base change [Scholze 2020, Proposition 12.18]. More explicitly, if $B = A\langle a/1 \rangle$ for some $a \in A$ then by [Andreychev 2021, Proposition 4.11] and Lemma 4.21 we can write

$$(A\langle a/1 \rangle, A\langle a/1 \rangle^+)_■ \cong (A, A^+)_■ \otimes_{(\mathbb{Z}[T],\mathbb{Z})_■} \mathbb{Z}[T]_■,$$

where the map $(\mathbb{Z}[T], \mathbb{Z})_■ \to (A, A^+)$ is the one induced by $T \mapsto a$. We then deduce

$$\begin{aligned} (C\langle a/1 \rangle, C\langle a/1 \rangle^+) &\cong (C, C^+)_■ \otimes_{(\mathbb{Z}[T],\mathbb{Z})_■} \mathbb{Z}[T]_■ \\ &\cong (C, C^+)_■ \otimes_{(A,A^+)_■} ((A, A^+)_■ \otimes_{(\mathbb{Z}[T],\mathbb{Z})_■} \mathbb{Z}[T]_■) \\ &\cong (C, C^+)_■ \otimes_{(A,A^+)_■} (A\langle a/1 \rangle, A\langle a/1 \rangle^+). \end{aligned}$$

The case $B = A\langle 1/a \rangle$ is dealt with similarly, by writing

$$(A\langle 1/a \rangle, A\langle 1/a \rangle^+)_■ \cong (A, A^+)_■ \otimes_{(\mathbb{Z}[T],\mathbb{Z})_■} (\mathbb{Z}[T^{\pm 1}], \mathbb{Z}[T^{-1}])_■.$$

Suppose f is the projection $\mathbb{B}^1_S \to S$. By [Andreychev 2021, Lemma 4.7] we have that $(A\langle T \rangle, A^+\langle T \rangle)_■$ coincides with the (steady) rational localization at $|T| \leq 1$ (see Proposition 4.14) of the analytic structure

$(\underline{A}[T]\otimes_{\underline{A}}(A,A^+)_\blacksquare)$ induced by the map of rings $A\to A[T]$ which is $(A,A^+)_\blacksquare\otimes_{\mathbb{Z}_\blacksquare}(\mathbb{Z}[T],\mathbb{Z})_\blacksquare$. By what was shown in the first part, we deduce that

$$\begin{aligned}(C\langle T\rangle, C^+\langle T\rangle) &\cong (C,C^+)_\blacksquare\otimes_{\mathbb{Z}_\blacksquare}\mathbb{Z}[T]_\blacksquare\\ &\cong (C,C^+)_\blacksquare\otimes_{(A,A^+)_\blacksquare}((A,A^+)_\blacksquare\otimes_{\mathbb{Z}_\blacksquare}\mathbb{Z}[T]_\blacksquare)\\ &\cong (C,C^+)_\blacksquare\otimes_{(A,A^+)_\blacksquare}(A\langle T\rangle, A^+\langle T\rangle)\end{aligned}$$

as wanted. The case in which f is finite étale is immediate, as in this case $(B,B^+)_\blacksquare$ is again induced by some (finite) map $A\to B$. ☐

An important consequence of the previous fact is the following base change result.

Corollary 4.23. *Under the hypotheses of Proposition 4.22, we let $f':X\times_S Y\to Y$, $g':X\times_S Y\to X$ be the base change of the maps f and g in* Adic. *For any object M of* $\mathrm{QCoh}(X)$ *the base change map*

$$g^*f_*M\to f'_*g'^*M$$

is an isomorphism in $\mathrm{QCoh}(Y)$.

Proof. The morphism g is adic; hence steady by Lemma 4.21. Therefore, by [Scholze 2020, Proposition 12.14], we know that

$$(M_{|A})\otimes_{(A,A^+)_\blacksquare}(C,C^+)_\blacksquare\cong(M\otimes_{(B,B^+)_\blacksquare}((B,B^+)_\blacksquare\otimes_{(A,A^+)_\blacksquare}(C,C^+)_\blacksquare))_{|C},$$

where on the right-hand side, $(B,B^+)_\blacksquare\otimes_{(A,A^+)_\blacksquare}(C,C^+)_\blacksquare$ denotes the analytic ring structure obtained by pushout. But for f satisfying the geometric hypotheses of the proposition, we know by Proposition 4.22 that this pushout is the same as $(B\widehat{\otimes}_A C, E^+)_\blacksquare$ with E^+ being the smallest ring of integers containing $B^+\widehat{\otimes}_{A^+}C^+$, whence the claim. ☐

Let us spell out a corollary of this, which will be useful later.

Corollary 4.24. *Under the hypotheses on Proposition 4.22, the modules $\underline{B}$ and $\underline{C}$ are solid (A,A^+)-modules, and $\underline{B}\otimes_{(A,A^+)_\blacksquare}\underline{C}$ is isomorphic to $(\underline{B\widehat{\otimes}_A C})[0]$ in* $\mathrm{QCoh}(S)$. ☐

Proof. We may harmlessly replace (C,C^+) with the Huber pair (C,C^+_A) where C^+_A denotes the smallest ring of integral elements containing A^+. In this case, the analytic structure $(C,C^+_A)_\blacksquare$ coincides with $(A,A^+)_\blacksquare\otimes_{\underline{A}}\underline{C}$, i.e., to the one induced by $(A,A^+)_\blacksquare$ and the continuous ring map $A\to C$. In particular, the base change functor g^* is given by the functor $M\mapsto M\otimes_{(A,A^+)_\blacksquare}\underline{C}$.

We may then rewrite the module $\underline{B}\otimes_{(A,A^+)_\blacksquare}\underline{C}$ as $g^*f_*\underline{B}$ which by Corollary 4.23 is canonically isomorphic to $f'_*g'^*\underline{B}=\underline{B\widehat{\otimes}_A C}$ as claimed. ☐

Remark 4.25. From Corollary 4.24 we obtain in particular that the complex $\underline{B}\otimes_{(A,A^+)_\blacksquare}\underline{C}$ is concentrated in degree zero and as such, it coincides with the *underived* tensor product $\underline{B}\otimes^{\mathrm{un}}_{(A,A^+)_\blacksquare}\underline{C}$ in solid (A,A^+)-modules (see Proposition 4.12).

The relative de Rham complex in the solid world. We would like to upgrade the de Rham cohomology complex to a complex of solid quasicoherent sheaves. In fact, we will strictly speaking do so only when everything in sight is affinoid and then glue using analytic descent. For most of this section we will then restrict to the following special smooth maps.

Definition 4.26. Let $S = \mathrm{Spa}(A, A^+)$ be an affinoid space in Adic. We say that a smooth map $X \to S$ is *smooth with good coordinates* if $X \to S$ can be factored into $X \xrightarrow{f} \mathbb{B}^d_S \xrightarrow{p} S$ with $d \in \mathbb{N}$, f being a composition of rational open immersions and finite étale maps, and with p being the natural projection. We remark that in this case $\Omega^1_{X/S}$ is free. We denote by $\mathrm{Sm}^{\mathrm{gc}}/S$ the full subcategory of Sm/S whose objects are smooth with good coordinates.

Locally on X, any smooth map has good coordinates so that the analytic étale topos on $\mathrm{Sm}^{\mathrm{gc}}/S$ is equivalent to the one on Sm/S.

Definition 4.27. Let $S = \mathrm{Spa}(A, A^+)$ be affinoid and $X \to S$ be smooth with good coordinates. We let $\underline{\Omega}^\bullet(X/S)$ be the complex of solid (A, A^+)-modules obtained by levelwise underlining the complex of Banach A-modules given by global sections of the complex $\Omega^\bullet_{X/S}$ of Definition 4.7 (note that since $\Omega^1_{X/S}$ is a finite free $\mathcal{O}_X$-module, $\Omega^i_{X/S}(X)$ has a natural structure of a Banach A-module for each i). We denote by $R\Gamma_{\mathrm{dR}}(X/S)_{\blacksquare}$ the object of $\mathcal{D}((A, A^+)_{\blacksquare}) = \mathrm{QCoh}(S)$ attached to the complex $\underline{\Omega}^\bullet(X/S)$.

The notation $\mathrm{R}\Gamma_{\mathrm{dR}}(X/S)_{\blacksquare}$ could a priori be confusing, as it may suggest that alternatively we see $\Omega^\bullet_{X/S}$ as a complex of sheaves valued in $\mathcal{D}((A, A^+)_{\blacksquare})$ (say, defined on $\mathrm{Sm}^{\mathrm{gc}}/S$) and compute its (hyper)cohomology on X. The following proposition shows that these two definitions agree, as a basic consequence of Tate's acyclicity.

Proposition 4.28. *Let* $S = \mathrm{Spa}(A, A^+)$ *be in* Adic. *The functor*

$$\mathrm{R}\Gamma_{\mathrm{dR}}(-/S)_{\blacksquare} : U \mapsto \mathrm{R}\Gamma_{\mathrm{dR}}(U/S)_{\blacksquare}$$

from $(\mathrm{Sm}^{\mathrm{gc}}/S)$ *to* $\mathrm{QCoh}(S)$ *has étale descent. That is, if* $\mathcal{U} \to X$ *is an étale Cech-hypercover in* $\mathrm{Sm}^{\mathrm{gc}}/S$,

$$\mathrm{R}\Gamma_{\mathrm{dR}}(X/S)_{\blacksquare} \cong \lim \mathrm{R}\Gamma_{\mathrm{dR}}(\mathcal{U}/S)_{\blacksquare}$$

in $\mathrm{QCoh}(S)$.

Proof. We shall prove that the statement follows from Tate's acyclicity. The proof will be divided into some intermediate steps.

Step 1: For any Cech hypercover $\mathcal{U} \to X$ in $\mathrm{Sm}^{\mathrm{gc}}/S$, the map $\mathrm{colim}\, \mathbb{Z}(\mathcal{U}) \to \mathbb{Z}(X)$ is an ét-local equivalence in $\mathcal{D}(\mathrm{Psh}(\mathrm{Sm}^{\mathrm{gc}}/S), \mathbb{Z})$ (see, for example, [SGA 4_2 1972, Théorème V.7.3.2]); hence so is the analogous map between the two induced free presheaves of solid (A, A^+)-modules. It therefore suffices to show that $\mathrm{R}\Gamma_{\mathrm{dR}}(-/S)_{\blacksquare}$ is ét-local in the category $\mathcal{D}\big(\mathrm{Psh}(\mathrm{Sm}^{\mathrm{gc}}/S, \mathrm{QCoh}(S))\big)$, that is, the homology groups $H^i\Gamma(X, \mathrm{R}\Gamma_{\mathrm{dR}}(-/S)_{\blacksquare})$ coincide with the hypercohomology groups $\mathbb{H}^i_{\mathrm{ét}}(X, \mathrm{R}\Gamma_{\mathrm{dR}}(-/S)_{\blacksquare})$. To this aim, we may show that $\mathrm{R}\Gamma_{\mathrm{dR}}(-/S)_{\blacksquare}$ is a bounded complex of Cech-acyclic sheaves (of solid (A, A^+)-modules), that is, each $\underline{\Omega}^i_{-/S}$ is a Cech-acyclic sheaf.

Step 2: Since $\Omega^1_{X/S}$ is free for any $X \in \mathrm{Sm}^{\mathrm{gc}}/S$ and $\underline{\mathcal{O}}(U)$ is a solid (A, A^+)-module, it suffices to show that $\underline{\mathcal{O}}$ is a Cech-acyclic étale sheaf of condensed $\mathcal{O}(S)$-modules in $\mathrm{Sm}^{\mathrm{gc}}/S$. We fix an étale cover $\mathcal{U} = \{U_i \to X\}_{i=1,\dots,n}$ in this site. We are left to show that the (bounded) complex

$$0 \to \underline{\mathcal{O}}(X) \to \bigoplus \underline{\mathcal{O}}(U_i) \to \bigoplus \underline{\mathcal{O}}(U_{ij}) \to \cdots$$

is exact. By the classical Tate acyclicity theorem and the Banach open mapping theorem, we know that the sequence

$$0 \to \mathcal{O}(X) \to \bigoplus \mathcal{O}(U_i) \to \bigoplus \mathcal{O}(U_{ij}) \to \cdots$$

is a strict exact complex of Banach A-modules, so the claim follows from Lemma 4.29. □

We learnt the following fact, which was used in the previous proof, from Guido Bosco.

Lemma 4.29. *Let $S = \mathrm{Spa}(A, A^+)$ be in* Adic. *The functor $M \mapsto \underline{M}$ from the* (*exact*) *category of Banach A-modules and continuous maps to the category of condensed A-modules is exact.*

Proof. Since the “underlining” functor is left exact, it is enough to prove that if $f : M' \to M$ is a surjective map between two Banach A-modules, the map $\underline{f} : \underline{M'} \to \underline{M}$ remains surjective; in other words, that whenever S is an extremally disconnected set and $g : S \to M$ is a continuous map, there is a continuous map $g' : S \to M'$ lifting g. But the image $g(S)$ is compact, and thus by [Trèves 1967, Lemma 45.1] (which we can apply, thanks to [Bhatt et al. 2019, Theorem 1.1.9]) it is the image $f(K)$ of a compact subset K of M'. This concludes the claim, since extremally disconnected sets are projective objects in the category of compact Hausdorff spaces [Gleason 1958, Theorem 2.5]. □

Proposition 4.30. *Let $f : X \to S = \mathrm{Spa}(A, A^+)$ be a smooth map with good coordinates and let $g : Y = \mathrm{Spa}(C, C^+) \to S$ be a map in* Adic.

1. *There is a canonical equivalence $g^*\mathrm{R}\Gamma_{\mathrm{dR}}(X/S)_{\blacksquare} \cong \mathrm{R}\Gamma_{\mathrm{dR}}(X \times_S Y/Y)_{\blacksquare}$.*
2. *Suppose that g is also smooth with good coordinates. Then there is a canonical equivalence $\mathrm{R}\Gamma_{\mathrm{dR}}(X/S)_{\blacksquare} \otimes_{(A,A^+)_{\blacksquare}} \mathrm{R}\Gamma_{\mathrm{dR}}(Y/S)_{\blacksquare} \cong \mathrm{R}\Gamma_{\mathrm{dR}}(X \times_S Y/S)_{\blacksquare}$.*

Proof. We consider the first statement. We let f' (resp. g') be the map $X \times_S Y \to Y$ (resp. $X \times_S Y \to X$) obtained by pullback. It suffices to prove that the levelwise one satisfies $g^* f_* \underline{\Omega}^d_{X/S} \cong f'_* \underline{\Omega}^d_{X \times_S Y/Y}$. This follows from Corollary 4.23 together with Proposition 4.6(1).

Now we move to the second statement. By Proposition 4.6(3), we deduce the equivalence of complexes of topological A-modules

$$\Gamma(X \times_S Y, \Omega^\bullet_{X \times_S Y/S}) \cong \mathrm{Tot}\big((\Gamma(X, \Omega^\bullet_{X/S}) \otimes_B (B \widehat{\otimes}_A C)) \otimes_{B \widehat{\otimes}_A C} ((B \widehat{\otimes}_A C) \otimes_C \Gamma(Y, \Omega^\bullet_{Y/S}))\big).$$

The right-hand side can be simplified and we get

$$\Gamma(X \times_S Y, \Omega^\bullet_{X \times_S Y/S}) \cong \mathrm{Tot}(\Gamma(X, \Omega^\bullet_{X/S}) \widehat{\otimes}_A \Gamma(Y, \Omega^\bullet_{Y/S})).$$

Underlining both sides, we deduce (using the notation of Definition 4.27)

$$\underline{\Omega}^\bullet(X \times_S Y/S) \cong \mathrm{Tot}(\underline{\Gamma(X, \Omega^\bullet_{X/S})\widehat{\otimes}_A \Gamma(Y, \Omega^\bullet_{Y/S})}).$$

Since the terms of the complexes $\underline{\Omega}^\bullet(X/S) = \underline{\Gamma(X, \Omega^\bullet_{X/S})}$ and $\underline{\Omega}^\bullet(Y/S) = \underline{\Gamma(Y, \Omega^\bullet_{Y/S})}$ are finite locally free B-modules and finite locally free C-modules, respectively, we deduce from Corollary 4.24 (see also Remark 4.25) that

$$\mathrm{Tot}(\underline{\Gamma(X, \Omega^\bullet_{X/S})\widehat{\otimes}_A \Gamma(Y, \Omega^\bullet_{Y/S})}) \cong \mathrm{Tot}(\underline{\Omega}^\bullet(X/S) \otimes^{\mathrm{un}}_{(A,A^+)_\blacksquare} \underline{\Omega}^\bullet(Y/S)),$$

where the tensor product on the right is the *underived* tensor product of solid (A, A^+)-modules. Moreover, (see [EGA III_2 1963, Proposition 6.3.2])

$$\mathrm{Tot}(\underline{\Omega}^\bullet(X/S) \otimes^{\mathrm{un}}_{(A,A^+)_\blacksquare} \underline{\Omega}^\bullet(Y/S)) \cong \mathrm{R}\Gamma_{\mathrm{dR}}(X/S)_\blacksquare \otimes_{(A,A^+)_\blacksquare} \mathrm{R}\Gamma_{\mathrm{dR}}(Y/S)_\blacksquare,$$

proving the claim. ☐

The results above allow us to extend the definition of $\mathrm{R}\Gamma_{\mathrm{dR}}(X/S)_\blacksquare$ to arbitrary smooth maps $X \to S$.

Definition 4.31. Let $X \to S$ be a smooth map in Adic.

(1) Let S be affinoid. We define $\mathrm{R}\Gamma_{\mathrm{dR}}(X/S)_\blacksquare$ to be the object in $\mathrm{QCoh}(S)$ defined by rational descent (see Proposition 4.28) of the functor $\mathrm{R}\Gamma_{\mathrm{dR}}(-/S)_\blacksquare : (\mathrm{Sm}^{\mathrm{gc}}/S)_{/X} \to \mathrm{QCoh}(S)^{\mathrm{op}}$.

(2) In the general case, we can define $\mathrm{R}\Gamma_{\mathrm{dR}}(X/S)_\blacksquare$ by rational descent of the category $\mathrm{QCoh}(S)$, i.e., we may chose an affinoid rational hypercover $S_\bullet \to S$, and let $\mathrm{R}\Gamma_{\mathrm{dR}}(X/S)_\blacksquare$ be the object of $\mathrm{QCoh}(S) \cong \lim \mathrm{QCoh}(S_\bullet)$ induced by the objects $\mathrm{R}\Gamma_{\mathrm{dR}}(X_n/S_n)_\blacksquare$. The compatibility is ensured by Proposition 4.30.

Remark 4.32. Infinity-categorically, one may rephrase the definition above as follows. If S is affinoid, by rational descent of $\mathrm{R}\Gamma_{\mathrm{dR}}(-/S)_\blacksquare$, we can extend it to a functor of infinity-categories $\mathcal{D}_{\mathrm{an}}(\mathrm{Sm}/S) \cong \mathcal{D}_{\mathrm{an}}(\mathrm{Sm}^{\mathrm{gc}}/S) \to \mathrm{QCoh}(S)^{\mathrm{op}}$. By letting S vary, the compatibility with pullbacks along open immersions translates into a natural transformation between analytic sheaves of infinity-categories (see [Ayoub et al. 2022, Proposition 2.3.7] and Theorem 4.18) $\mathcal{D}_{\mathrm{an}}(\mathrm{Sm}/-) \to \mathrm{QCoh}(-)$ on affinoid spaces open in S that can then be extended to S.

We deduce formally from Proposition 4.30 the following extension.

Corollary 4.33. *Let $f : X \to S$, $g : S' \to S$ be maps in* Adic *with f smooth.*

(1) *Let $\mathcal{U} \to X$ be an étale Cech hypercover. Then* $\mathrm{R}\Gamma_{\mathrm{dR}}(X/S)_\blacksquare \cong \lim \mathrm{R}\Gamma_{\mathrm{dR}}(\mathcal{U}/S)_\blacksquare$.

(2) *If g is an open immersion, there is a canonical equivalence* $g^*\mathrm{R}\Gamma_{\mathrm{dR}}(X/S)_\blacksquare \cong \mathrm{R}\Gamma_{\mathrm{dR}}(X'/S')_\blacksquare$ *where $X' = X \times_S S'$.*

(3) *If f is qcqs, there is a canonical equivalence* $g^*\mathrm{R}\Gamma_{\mathrm{dR}}(X/S)_\blacksquare \cong \mathrm{R}\Gamma_{\mathrm{dR}}(X'/S')_\blacksquare$ *where $X' = X \times_S S'$.*

(4) *Suppose that f, g are both smooth and qcqs. Then*

$$\mathrm{R}\Gamma_{\mathrm{dR}}(X/S)_\blacksquare \otimes_{\mathrm{QCoh}(S)} \mathrm{R}\Gamma_{\mathrm{dR}}(S'/S)_\blacksquare \cong \mathrm{R}\Gamma_{\mathrm{dR}}(X \times_S S'/S)_\blacksquare.$$

Proof. The first point comes directly from the definition. All points are local on S so we can assume that S is affinoid. By (1), if f is qcqs we can write $\mathrm{R}\Gamma_{\mathrm{dR}}(X/S)_{\blacksquare}$ as a finite limit of objects $\mathrm{R}\Gamma_{\mathrm{dR}}(U/S)_{\blacksquare}$ with U affinoid. We then deduce (3) and (4) from the affinoid case treated in Proposition 4.30, and the commutation of g^* and $\otimes$ with finite limits. In the case g is an open immersion, we claim that g^* commutes with arbitrary limits, which will give us the compatibility with pullbacks along open immersions in full generality. To justify this, we note that using [Andreychev 2021, Propositions 4.11 and 4.12(ii)] (and the fact that forgetful functors are conservative and commute with limits) the claim can be deduced from the commutation with limits of the functor j^*, where j is a localization of analytic rings which is either $j : (\mathbb{Z}[T], \mathbb{Z})_{\blacksquare} \to \mathbb{Z}[T]_{\blacksquare}$ or $j : (\mathbb{Z}[T], \mathbb{Z})_{\blacksquare} \to (\mathbb{Z}[T^{\pm 1}], \mathbb{Z}[T^{-1}])_{\blacksquare}$.

Assume first that j is $(\mathbb{Z}[T], \mathbb{Z})_{\blacksquare} \to \mathbb{Z}[T]_{\blacksquare}$. In [Scholze 2019, Theorem 8.1] a left adjoint $j_!$ to j^* is constructed. In particular, j^* commutes with limits. Next, assume that j is $(\mathbb{Z}[T], \mathbb{Z})_{\blacksquare} \to (\mathbb{Z}[T^{\pm 1}], \mathbb{Z}[T^{-1}])_{\blacksquare}$. We decompose j into

$$(\mathbb{Z}[T], \mathbb{Z}) \xrightarrow{\alpha} (\mathbb{Z}[T, U], \mathbb{Z}[U]) \xrightarrow{\iota} (\mathbb{Z}[T, U]/(TU - 1), \mathbb{Z}[U]).$$

To keep notation simple, we will write $A = \mathbb{Z}[U]$, $B = \mathbb{Z}[T, U]$, $C = \mathbb{Z}[T, U]/(TU - 1)$ in what follows. Then $j^* = \iota^* \circ \alpha^* = \iota^*[-1] \circ \alpha^*[1]$, and the statement will be proved if we can prove that both $\alpha^*[1]$ and $\iota^*[-1]$ commute with limits. For ι, note that the forgetful functor ι_* commutes with colimits and hence has a right adjoint which by the Hom-tensor adjunction is given by $R\underline{\mathrm{Hom}}_B(C, -)$ (which is solid). We claim that the natural map

$$R\underline{\mathrm{Hom}}_B(C, B) \otimes_{(C,A)_{\blacksquare}} \iota^*(-) \to R\underline{\mathrm{Hom}}_B(C, -)$$

is an equivalence. We may and do check this in the category $\mathrm{QCoh}((B, A)_{\blacksquare})$. Using that $C \cong (B \xrightarrow{TU-1} B)$ we deduce

$$\begin{aligned} R\underline{\mathrm{Hom}}_B(C, B) \otimes_{(C,A)_{\blacksquare}} \iota^*(-) &\cong C[-1] \otimes_{(C,A)_{\blacksquare}} (C, A)_{\blacksquare} \otimes_{(B,A)_{\blacksquare}} (-) \\ &\cong C[-1] \otimes_{(B,A)_{\blacksquare}} (-) \\ &\cong R\underline{\mathrm{Hom}}_B(C, -), \end{aligned}$$

whence our claim. Therefore, we see that $\iota^*[-1]$ agrees with the right adjoint of ι_*, and thus commutes with limits.

Finally, we turn to α. The map α is the base change along $\mathbb{Z}_{\blacksquare} \to (\mathbb{Z}[T], \mathbb{Z})_{\blacksquare}$ of the map $\alpha' : \mathbb{Z}_{\blacksquare} \to \mathbb{Z}[U]_{\blacksquare}$. Using base change as above (which holds here: to see it, argue as in the proof of Corollary 4.23 using that α' is steady and that we can compute the pushout of analytic rings by Proposition 4.22, since α' is smooth), we reduce to showing that $(\alpha')^*[1]$ commutes with limits. But [Scholze 2019, Pages 57–58] shows that $(\alpha')^*[1]$ has a left adjoint $\alpha_!$ defined there, and thus commutes with limits, as desired. □

Overconvergent version and extension to rigid-analytic motives. It is straightforward now to give an overconvergent version of $\mathrm{R}\Gamma_{\mathrm{dR}}(X/S)_{\blacksquare}$ for dagger varieties over S in $\mathrm{Adic}_{/\mathbb{Q}_p}$.

Definition 4.34. Let S be affinoid in $\mathrm{Adic}_{/\mathbb{Q}_p}$. We let $\mathrm{Sm}^{\mathrm{gc}\,\dagger}/S$ be the full subcategory of $\mathrm{Sm}^{\dagger}/S$ of those objects $(\widehat{X}, X_h)$ with $\widehat{X}, X_h$ in $\mathrm{Sm}^{\mathrm{gc}}/S$. For any $X = (\widehat{X}, X_h)$ in $\mathrm{Aff}\,\mathrm{Sm}^{\dagger}/S$, we let $\mathrm{R}\Gamma^{\dagger}_{\mathrm{dR}}(X/S)_{\blacksquare}$ be the object of $\mathrm{QCoh}(S)$ defined as $\operatorname{colim} \mathrm{R}\Gamma_{\mathrm{dR}}(X_h/S)_{\blacksquare}$.

Remark 4.35. Filtered colimits of solid modules are solid, and filtered colimits are exact in condensed $\mathcal{O}(S)$-modules. Therefore $\mathrm{R}\Gamma^{\dagger}_{\mathrm{dR}}(X/S)_{\blacksquare}$ is a bounded complex whose terms are $\varinjlim f_{h*}\underline{\Omega}^{d}_{X_h/S}$ (f_h being the smooth map $X_h \to S$).

Proposition 4.36. *Let S be affinoid in $\mathrm{Adic}_{/\mathbb{Q}_p}$ and X be in $\mathrm{Sm}^{\mathrm{gc}\,\dagger}/S$.*

(1) *Let $\mathcal{U} \to X$ be an étale Cech hypercover in $\mathrm{Aff}\,\mathrm{Sm}^{\dagger}/S$. Then $\mathrm{R}\Gamma^{\dagger}_{\mathrm{dR}}(X/S)_{\blacksquare} \cong \lim \mathrm{R}\Gamma^{\dagger}_{\mathrm{dR}}(\mathcal{U}/S)_{\blacksquare}$.*

(2) *Let $g : S' \to S$ be a map of affinoid spaces in Adic. There is a canonical equivalence $g^{*}\mathrm{R}\Gamma^{\dagger}_{\mathrm{dR}}(X/S)_{\blacksquare} \cong \mathrm{R}\Gamma^{\dagger}_{\mathrm{dR}}(X'/S')_{\blacksquare}$ where $X' = X \times_S S'$.*

(3) *Let $g : Y \to S$ be another object of $\mathrm{Sm}^{\mathrm{gc}\,\dagger}/S$. Then*

$$\mathrm{R}\Gamma^{\dagger}_{\mathrm{dR}}(X/S)_{\blacksquare} \otimes_{\mathrm{QCoh}(S)} \mathrm{R}\Gamma^{\dagger}_{\mathrm{dR}}(Y/S)_{\blacksquare} \cong \mathrm{R}\Gamma^{\dagger}_{\mathrm{dR}}(X \times_S Y/S)_{\blacksquare}.$$

Proof. Just like in the proof of Proposition 4.28, it suffices to show that the sheaf of solid modules $\underline{\Omega}^{i\dagger}$ is Cech-acyclic. We let $\mathcal{U}$ be a Cech étale hypercover of X that we may assume to be arising from an étale cover of X_0. We let $\mathcal{U}_h$ be the corresponding Cech hypercover on each X_h. But then $\Gamma(\mathcal{U}, \underline{\Omega}^{\dagger i}) \cong \varinjlim \Gamma(\mathcal{U}_h, \underline{\Omega}^{i})$. As filtered colimits commute with finite limits in $\mathrm{QCoh}(S)$, the claim follows from the acyclicity of $\underline{\Omega}^{i}$. Properties (2) and (3) follow from Proposition 4.30 and the commutation of filtered colimits with tensor products and base change functors. ☐

Corollary 4.37. *The functor $X \mapsto \mathrm{R}\Gamma^{\dagger}_{\mathrm{dR}}(X/S)_{\blacksquare}$ can be uniquely extended into a functor $\mathrm{R}\Gamma^{\dagger}_{\mathrm{dR}}(-/S)_{\blacksquare}$ from $\mathrm{RigSm}^{\dagger}/S$ to $\mathrm{QCoh}(S)$ for any $S \in \mathrm{Adic}_{/\mathbb{Q}_p}$ such that:*

(1) *For any $\mathcal{U} \to X$ étale Cech hypercover in $\mathrm{Aff}\,\mathrm{Sm}^{\dagger}/S$ one has $\mathrm{R}\Gamma^{\dagger}_{\mathrm{dR}}(X/S)_{\blacksquare} \cong \lim \mathrm{R}\Gamma_{\mathrm{dR}}(\mathcal{U}/S)_{\blacksquare}$.*

(2) *For any open immersion $j : U \to S$ in Adic there is a canonical equivalence $j^{*}\mathrm{R}\Gamma^{\dagger}_{\mathrm{dR}}(X/S)_{\blacksquare} \cong \mathrm{R}\Gamma_{\mathrm{dR}}(X \times_S U/U)^{\dagger}_{\blacksquare}$.*

It satisfies the following properties.

(3) *If X is qcqs in $\mathrm{RigSm}^{\dagger}/S$ and if $g : S' \to S$ is a map in Adic, then $g^{*}\mathrm{R}\Gamma^{\dagger}_{\mathrm{dR}}(X/S)_{\blacksquare} \cong \mathrm{R}\Gamma_{\mathrm{dR}}(X'/S')^{\dagger}_{\blacksquare}$ where $X' = X \times_S S'$.*

(4) *If $f : X \to S$ and $g : Y \to S$ are qcqs in $\mathrm{Sm}^{\dagger}/S$ then*

$$\mathrm{R}\Gamma^{\dagger}_{\mathrm{dR}}(X/S)_{\blacksquare} \otimes_{\mathrm{QCoh}(S)} \mathrm{R}\Gamma^{\dagger}_{\mathrm{dR}}(Y/S)_{\blacksquare} \cong \mathrm{R}\Gamma^{\dagger}_{\mathrm{dR}}(X \times_S Y/S)_{\blacksquare}.$$

(5) *The natural projection induces an equivalence $\mathrm{R}\Gamma^{\dagger}_{\mathrm{dR}}(\mathbb{B}^{1\dagger}_{X}/S)_{\blacksquare} \cong \mathrm{R}\Gamma^{\dagger}_{\mathrm{dR}}(X/S)_{\blacksquare}$.*

(6) *One has $\mathrm{R}\Gamma^{\dagger}_{\mathrm{dR}}(\mathbb{T}^{1\dagger}_{S}/S)_{\blacksquare} \cong 1 \oplus 1[-1]$ where 1 is the unit of the monoidal structure on $\mathrm{QCoh}(S)$.*

Proof. As any smooth dagger space over S is locally in $\mathrm{Sm}^{\mathrm{gc}\,\dagger}/S$, the first four claims follow formally from Proposition 4.36 as in the proof of Corollary 4.33. We now move to the last two. Using (2)–(3), it is enough to compute $\mathrm{R}\Gamma^{\dagger}_{\mathrm{dR}}(X/S)_{\blacksquare}$ when $S=\mathrm{Spa}(\mathbb{Q}_p)$ and $X=\mathbb{B}^{1\dagger}_{\mathbb{Q}_p}$ (resp. $X=\mathbb{T}^{1\dagger}_{\mathbb{Q}_p}$). We note that the classical computations show that the underlying $\mathbb{Q}_p$-vector spaces are the expected ones, and we now have to promote these computations to solid $\mathbb{Q}_p$-vector spaces. To this aim, we will use once again Lemma 4.29.

By cofinality, we may rewrite the complex $\mathrm{R}\Gamma^{\dagger}_{\mathrm{dR}}(X/S)_{\blacksquare}$ as

$$\varinjlim \underline{\mathcal{O}(X^{\circ}_{\epsilon})} \to \varinjlim \underline{\mathcal{O}(X^{\circ}_{\epsilon})dT},$$

where $\mathcal{O}(X^{\circ}_{\epsilon})$ is the Fréchet algebra of functions on the open disc (resp. annulus) of radius $1+\epsilon$ (and $1-\epsilon$) with $\sqrt{|\mathbb{Q}_p|}\ni\epsilon\to 0$ inside $\mathrm{Spa}\,\mathbb{Q}_p\langle pT\rangle$. We need to show that its cohomology in degree one is trivial (resp. isomorphic to $\underline{\mathbb{Q}}_p$). We show that the H^1 of each complex $\underline{\mathcal{O}(X^{\circ}_{\epsilon})}\to\underline{\mathcal{O}(X^{\circ}_{\epsilon})dT}$ is trivial (resp. $\underline{\mathbb{Q}_p}$).

Noting that Lemma 4.29 also holds for Fréchet spaces (since the open mapping theorem holds for them as well; see [Schneider 2002, Proposition 8.6]) and that the differential map is strict[2] (it is so for any smooth Stein space over a finite extension of $\mathbb{Q}_p$; see [Große-Klönne 2000, Lemma 4.7]) we conclude that the solid vector space H^1 coincides with $\underline{\mathcal{O}(X^{\circ}_{\epsilon})dT/d\mathcal{O}(X^{\circ}_{\epsilon})}$ which is zero (resp. $\underline{\mathbb{Q}_p}$) by the standard computations of the (overconvergent) de Rham cohomology of such Stein spaces [Monsky and Washnitzer 1968; Große-Klönne 2004]. ☐

Definition 4.38. We let $\mathrm{RigDA}(S)^{\mathrm{ct}}$ (ct standing for *constructible*) be the full idempotent complete subcategory of $\mathrm{RigDA}(S)$ stable under shifts and finite colimits generated by the objects $\mathbb{Q}_S(X)(n)$ with $X\to S$ smooth and qcqs, and $n\in\mathbb{Z}$. It coincides with the category of compact objects $\mathrm{RigDA}(S)^{\omega}$ if S is itself quasicompact and quasiseparated (see Theorem 2.10(1)) and it is stable under tensor products and pullbacks.

The infinity-categorical translation of the corollary above is the following (compare with Remark 4.32).

Corollary 4.39. *Let S be in $\mathrm{Adic}_{/\mathbb{Q}_p}$.*

(1) *There is a unique functor*

$$\mathrm{dR}_S:\mathrm{RigDA}(S)\cong\mathrm{RigDA}^{\dagger}(S)\to\mathrm{QCoh}(S)^{\mathrm{op}}$$

associating to each motive $\mathbb{Q}_S(X)$ with $X\in\mathrm{RigSm}^{\dagger}/S$ the complex $\mathrm{R}\Gamma^{\dagger}_{\mathrm{dR}}(X/S)_{\blacksquare}$.

(2) *The functor above is compatible with j^* for any open immersion $j:U\to S$.*

(3) *The restriction to constructible objects*

$$\mathrm{RigDA}(S)^{\mathrm{ct}}\to\mathrm{QCoh}(S)^{\mathrm{op}}$$

is symmetric monoidal and compatible with f^* for any morphism $f:S'\to S$, giving rise to a natural transformation

$$\mathrm{dR}:\mathrm{RigDA}(-)^{\mathrm{ct}}\to\mathrm{QCoh}(-)^{\mathrm{op}}$$

between contravariant functors from $\mathrm{Adic}_{/\mathbb{Q}_p}$ with values in symmetric monoidal infinity-categories.

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[2] Recall that a morphism $f:V\to W$ of topological vector spaces is *strict* if the quotient topology on $\mathrm{im}(f)$ induced from V coincides with the subspace topology induced from W.

Proof. For the first point, in light of Theorem 3.9, by the universal property of $\mathrm{RigDA}^{\dagger}(S)$ (see Remark 2.8) it suffices to prove that the functor $\mathbb{Q}_S(X) \mapsto \mathrm{R}\Gamma^{\dagger}_{\mathrm{dR}}(X/S)_{\blacksquare}$ is $\mathbb{B}^{1\dagger}_S$-invariant, has étale descent and sends the motive $T^{\dagger}_S$ to an invertible one. All these properties were proved in Corollary 4.37. Corollary 4.37 also implies that dR_S is symmetric monoidal and compatible with pullbacks on the full pseudoabelian stable subcategory of $\mathrm{RigDA}(S)$ generated under finite colimits by the objects $\mathbb{Q}(X)(d)$ with X affinoid and $d \in \mathbb{Z}$, which is precisely $\mathrm{RigDA}(S)^{\mathrm{ct}}$. □

Definition 4.40. Under the hypotheses of Corollary 4.39 we call the functor

$$\mathrm{dR}_S : \mathrm{RigDA}(S) \to \mathrm{QCoh}(S)^{\mathrm{op}}$$

the (*relative*) *overconvergent de Rham realization*. When M is the motive $M = \mathbb{Q}_S(X)$ of a smooth variety X over S, or more generally if $M = p_! p^! \mathbb{Q}_S$ for some map $p : X \to S$ which is locally of finite type (see [Ayoub et al. 2022, Corollary 4.3.18]) we will often write $\mathrm{dR}_S(X)$ instead of $\mathrm{dR}_S(M)$.

Remark 4.41. We point out that the equivalence $\mathrm{RigDA}(S) \cong \mathrm{RigDA}^{\dagger}(S)$ and the fact that dR_S is motivic imply in particular that the overconvergent de Rham complex $\mathrm{R}\Gamma^{\dagger}_{\mathrm{dR}}(X/S)_{\blacksquare}$ doesn't depend on the choice of a dagger structure on X.

Remark 4.42. In the case S is affinoid, we may take the cohomology groups $H^i_{\mathrm{dR}}(M/S)^{\dagger} := H^i(\mathrm{dR}_S(M))$ with respect to the t-structure of Remark 4.20 and call them the *i-th overconvergent de Rham cohomology group of M over S*. In the case $M = p_! p^! \mathbb{Q}_S$ for a map $p : X \to S$ which is locally of finite type, we may abbreviate them as $H^i_{\mathrm{dR}}(X/S)^{\dagger}$.

Just like in the absolute case, there is no need of an overconvergent structure for smooth proper varieties.

Proposition 4.43. *Let $X \to S$ be a smooth proper map in $\mathrm{Adic}_{/\mathbb{Q}_p}$. The complex $\mathrm{R}\Gamma^{\dagger}_{\mathrm{dR}}(X/S)_{\blacksquare}$ is equivalent to the complex $\mathrm{R}\Gamma_{\mathrm{dR}}(X/S)_{\blacksquare}$.*

Proof. We may and do assume S is affinoid. Let $\{U_0, \dots, U_N\}$ be a finite open cover of X made of objects in $\mathrm{Sm}^{\mathrm{gc}}/S$. The inclusions $U_i \Subset_S X$ induce overconvergent structures $V_i = (U_i, U_{ih})$ which are such that $\{U_{1h}, \dots, U_{Nh}\}$ is again an open cover of X. But then we get

$$\begin{aligned}\mathrm{R}\Gamma^{\dagger}_{\mathrm{dR}}(X/S)_{\blacksquare} &\cong \lim \mathrm{R}\Gamma^{\dagger}_{\mathrm{dR}}(V_{\bullet}/S)_{\blacksquare} \\ &\cong \lim \varinjlim_h \mathrm{R}\Gamma_{\mathrm{dR}}(U_{\bullet h}/S)_{\blacksquare} \\ &\cong \varinjlim_h \lim \mathrm{R}\Gamma_{\mathrm{dR}}(U_{\bullet h}/S)_{\blacksquare} \\ &\cong \mathrm{R}\Gamma_{\mathrm{dR}}(X/S)_{\blacksquare},\end{aligned}$$

where we used the commutation of filtered colimits with finite limits and descent of $\mathrm{R}\Gamma_{\mathrm{dR}}(-/S)_{\blacksquare}$ (see Corollary 4.33). □

Remark 4.44. Even if the overconvergent setting is "superfluous" when dealing with smooth proper maps X/S, we stress that it is crucial in order to have a realization dR_S on motives $\mathrm{RigDA}(S)$ (and not just "pure" ones). This allows one to use the motivic six-functor formalism and its consequences, which give nontrivial results even when applied to "pure" motives (see, for example, Corollary 4.47).

Finiteness. We would like to conclude the same finiteness results for the relative rigid de Rham cohomology as the relative *algebraic* de Rham cohomology (see, for example, [Hartshorne 1975]), that is, the fact that it defines vector bundles on the base in the case X/S is proper and smooth or whenever S is a field.

Definition 4.45. Let $\mathcal{C}$ be a symmetric monoidal infinity-category. We denote by $\mathcal{C}^{\mathrm{fd}}$ the full subcategory of $\mathcal{C}$ whose objects are (fully) dualizable in the sense of [Lurie 2017, Definition 4.6.1.7].

We now prove the main theorem of this section.

Theorem 4.46. *Let S be an adic space in $\mathrm{Adic}_{/\mathbb{Q}_p}$. The relative overconvergent de Rham realization*

$$\mathrm{dR}_S : \mathrm{RigDA}(S) \to \mathrm{QCoh}(S)^{\mathrm{op}}$$

sends dualizable motives to split perfect complexes. In particular, if M is a dualizable motive, then the cohomology groups of $\mathrm{dR}_S(M)$ (for the t-structure on the derived category of perfect complexes induced by the natural t-structure on the derived category of $\mathcal{O}_S$-modules) are vector bundles on S and equal to 0 *if $|i| \gg 0$.*

Proof. We may and do assume that S is affinoid. We divide the proof into various steps.

Step 1: As the unit object in $\mathrm{RigDA}(S)$ is compact, any dualizable object is compact. As the functor dR_S is symmetric monoidal when restricted to compact objects by Corollary 4.39(3), it sends dualizable objects to dualizable objects. Since dualizable objects in $\mathrm{QCoh}(S)$ are perfect complexes by [Andreychev 2021, Theorem 5.9 and Corollary 5.51.1], we deduce that dR restricts to a functor $\mathrm{RigDA}(S)^{\mathrm{fd}} \to \mathcal{P}(S)^{\mathrm{op}}$ where we let $\mathcal{P}(S)$ be the full subcategory of perfect complexes in $\mathrm{QCoh}(S)$.

Step 2: Let $f : S \to T$ be a morphism of affinoid spaces in $\mathrm{Adic}_{/\mathbb{Q}_p}$ and suppose that a dualizable motive $M \in \mathrm{RigDA}(S)$ has a dualizable model $N \in \mathrm{RigDA}(T)$ (N is dualizable and $f^*N \cong M$). We then deduce from Corollary 4.39 the commutative diagram

$$\begin{tikzcd}
\mathrm{RigDA}(T)^{\mathrm{fd}} \arrow[r] \arrow[d] & \mathcal{P}(\mathcal{O}(T))^{\mathrm{op}} \arrow[d] \\
\mathrm{RigDA}(S)^{\mathrm{fd}} \arrow[r] & \mathcal{P}(\mathcal{O}(S))^{\mathrm{op}}
\end{tikzcd}$$

and hence that $\mathrm{dR}_S(M) \cong f^* \mathrm{dR}_T(N)$. As split perfect complexes are stable under base change, if we know the statement holds for N, we can deduce it for M as well.

Step 3: Since $\mathcal{O}(S)$ is a uniform Tate–Huber ring, $\mathcal{O}(S)^+$ is a ring of definition and has the p-adic topology. Write $\mathcal{O}(S)^+$ as the union of its finitely generated $\mathbb{Z}_p$-subalgebras R. Since $\mathcal{O}(S)^+$ is p-adically complete, we therefore get a presentation of $(\mathcal{O}(S), \mathcal{O}(S)^+)$ as the filtered colimit of the complete affinoid rings $(\widehat{R}[1/p], \widehat{R})$ for R as before. Applying [Scholze and Weinstein 2013, Proposition 2.4.2] (with ideals of definition generated by p), we deduce that $S \sim \varprojlim \mathrm{Spa}(A, A^+)$, with $A = \widehat{R}[1/p]$ being a Tate algebra of topologically finite type over $\mathbb{Q}_p$. By Theorem 2.12 we deduce that $\mathrm{RigDA}(S) \cong \varinjlim \mathrm{RigDA}(\mathrm{Spa}(A, A^+))$ so that any dualizable motive M has a model $N_A \in \mathrm{RigDA}(\mathrm{Spa}(A, A^+))^{\mathrm{fd}}$ for some A. By step 2, it suffices to prove the statement in the case $S = \mathrm{Spa}(A, A^+)$ with A an affinoid Tate algebra of topologically finite type over a finite extension K of $\mathbb{Q}_p$.

Step 4: Any perfect complex of A-modules with projective cohomology groups is split. As $\mathrm{dR}_S(M)$ is a perfect complex, and each cohomology group $H^i\mathrm{dR}_S(M)$ is a finite type module over A, we are left to prove that they are free after base change to each stalk $\mathcal{O}_{\mathrm{Spec}(A),s}$ with s being a closed point of $\mathrm{Spec}(A)$, corresponding to a maximal ideal $\mathfrak{m}$ of A. Fix such an s. Since $\mathcal{O}_{\mathrm{Spec}(A),s}$ is noetherian, it suffices in fact to do so after base change to the $\mathfrak{m}$-adic completion $\widehat{\mathcal{O}}_{\mathrm{Spec}(A),s}$ of $\mathcal{O}_{\mathrm{Spec}(A),s}$, as the map $\mathcal{O}_{\mathrm{Spec}(A),s} \to \widehat{\mathcal{O}}_{\mathrm{Spec}(A),s}$ is faithfully flat. The completion $\widehat{\mathcal{O}}_{\mathrm{Spec}(A),s}$ agrees with the completion of the local ring $\mathcal{O}_{S,s}$ of the adic space S at s (now seen as a point of S). In particular, it suffices to show that for each integer i, there exists some rational domain U over s such that $H^i\,\mathrm{dR}_S(M) \otimes_A \mathcal{O}(U)$ is projective. Since A is an affinoid algebra of finite type, the natural map $A \to \mathcal{O}(U)$ is flat for any such U, and therefore $H^i\,\mathrm{dR}_S(M) \otimes_A \mathcal{O}(U)$ is nothing but $H^i\,\mathrm{dR}(M_U)$. Up to taking a finite étale cover of $\mathrm{Spa}\,A$ and enlarging K we may assume that $k(s) = K$. By means of Theorem 2.12 we have $\varinjlim_{s \in U} \mathrm{RigDA}(U) \cong \mathrm{RigDA}(K)$ where U runs among affinoid neighborhood of x. We remark that in this case, the functor from right to left is induced by pullback Π^* over the structure morphisms $\Pi : U \to \mathrm{Spa}\,K$. We deduce that for some open neighborhood U of s, the motive M_U is isomorphic to $\Pi^* M_s$ with M_s in $\mathrm{RigDA}(K)$, which implies by step 2 that the complex $\mathrm{dR}_S(M) \otimes_A \mathcal{O}(U) \cong \mathrm{dR}_U(M_U)$ is quasi-isomorphic to $\mathrm{dR}_s(M_s) \otimes_K \mathcal{O}(U)$, which is split, proving the claim. ☐

It is well known that the relative de Rham cohomology groups $H^i_{\mathrm{dR}}(X/S)$ of a map $f : X \to S$ of algebraic varieties in characteristic zero are vector bundles on the base, whenever f is smooth and proper. We can prove the analogous statement for the overconvergent de Rham cohomology of adic spaces.

Corollary 4.47. *Let $f : X \to S$ be a smooth and proper map in $\mathrm{Adic}_{/\mathbb{Q}_p}$. Then $\mathrm{dR}_S(X)$ is a perfect complex and its cohomology groups* (*see Theorem 4.46*) *are vector bundles on S, and equal to zero if $i \gg 0$.*

Proof. By the six-functor formalism, the motive $f_! f^! \mathbb{Q} = \mathbb{Q}_S(X)$ is dualizable in $\mathrm{RigDA}(S)$ with dual $f_* f^* \mathbb{Q}$ as shown in [Ayoub et al. 2022, Corollary 4.1.8]. ☐

Remark 4.48. We also remark that Theorem 4.46 generalizes [Vezzani 2018] as any compact motive in $\mathrm{RigDA}(K)$ with K a complete nonarchimedean field is dualizable: this can be seen by [Ayoub 2020, Proposition 2.31; Riou 2005].

Remark 4.49. We point out that Theorem 4.46 and Corollary 4.47 hold for any motivic realization which is compatible with tensor products and pullbacks, taking values in solid quasicoherent sheaves.

5. A rigid analytic Fargues–Fontaine construction

In this section we construct a functorial motivic realization from *rigid analytic* motives over a base in characteristic p with values in motives over the corresponding adic Fargues–Fontaine curve (in characteristic zero). This is akin to the usual *perfectoid* constructions of Fargues and Fontaine and of Scholze, that we de-perfectoidify using homotopies via the motivic results shown in Section 2.

Motives on Fargues–Fontaine curves. We first apply the formalism of motives for a special kind of adic space, namely Fargues–Fontaine curves associated to perfectoid spaces. We briefly recall how they are constructed.

Definition 5.1. Let S be a perfectoid space in characteristic p with some pseudouniformizer $\pi \in \mathcal{O}^\times(S)$. We let $\mathcal{Y}_{[0,\infty)}(S)$ (resp. $\mathcal{Y}_{(0,\infty)}(S)$) be the adic space $S \overset{\bullet}{\times} \operatorname{Spa} \mathbb{Z}_p$ (resp. $S \overset{\bullet}{\times} \operatorname{Spa} \mathbb{Q}_p$) using the notation of [Scholze and Weinstein 2020, Section 11.2]. In the case S is affinoid and $S = \operatorname{Spa}(R, R^+)$, it coincides with the open locus $\{|\pi| \neq 0\}$ (resp. $\{|p\pi| \neq 0\}$) in the spectrum $\operatorname{Spa}(W(R^+), W(R^+))$ and is obtained by gluing along affinoids in the general case. For any $r = (a/b) \in \mathbb{Q}_{>0}$ we also let $\mathbb{B}_{[0,r]}(S)$ (resp. $\mathbb{B}_{(0,r]}(S)$) be the open locus of $\mathcal{Y}_{[0,\infty)}(S)$ (resp. of $\mathcal{Y}_{(0,\infty)}(S)$) defined by $|p|^b \leq |\pi|^a$ (resp. $0 < |p|^b \leq |\pi|^a$).

The (invertible) Frobenius endomorphism $\mathcal{O}_S^+ \to \mathcal{O}_S^+$ induces an automorphism

$$\varphi : \mathcal{Y}_{[0,\infty)}(S) \xrightarrow{\sim} \mathcal{Y}_{[0,\infty)}(S)$$

which restricts to the Frobenius automorphism on the φ-stable closed subspace $S \cong \{p = 0\} \subset \mathcal{Y}_{[0,\infty)}(S)$. One has $\varphi(\mathbb{B}_{[0,r]}(S)) = \mathbb{B}_{[0,pr]}(S)$ (see, for example, [Scholze and Weinstein 2020, Page 136]) so that the action on $\mathcal{Y}_{(0,\infty)}(S)$ is properly discontinuous; hence it makes sense to define the quotient adic space $\mathcal{X}(S) := \mathcal{Y}_{(0,\infty)}(S)/\varphi^{\mathbb{Z}}$ which is *the relative Fargues–Fontaine curve over* S.

Remark 5.2. If S lies in Adic (i.e., it is admissible) then also the spaces $\mathcal{Y}_{[0,\infty)}(S)$, $\mathcal{Y}_{(0,\infty)}(S)$, $\mathcal{X}(S)$ are admissible. Indeed, they are stably strongly uniform, as they are sous-perfectoid (see the proof of [Scholze and Weinstein 2020, Proposition 11.2.1]). We are left to prove the condition on the Krull dimension. To this aim, we may suppose that S has global Krull dimension d and show that the Krull dimension of $\mathcal{Y}_{[0,\infty)}(S)$ is bounded. As this condition translates into a condition on the maximal height of the valuations at the residue fields, we may consider separately the closed space S (of dimension d) and its open complementary $\mathcal{Y}_{(0,\infty)}(S)$. For the latter, we can replace it by a pro-étale cover, since this does not alter the Krull dimension, and consider $\mathcal{Y}_{(0,\infty)}(S) \times_{\operatorname{Spa}(\mathbb{Q}_p)} \operatorname{Spa}(\mathbb{Q}_p^{\mathrm{cyc}})$. This is a perfectoid space, and its tilt is isomorphic to the perfectoid punctured open unit disk over S. Since tilting and perfection do not change the (topological!) Krull dimension, this space has the same dimension as the open disk over S, which is finite by assumption on S.

We let U be an open neighborhood of S in $\mathcal{Y}_{[0,\infty)}(S)$ of the form $U = \mathbb{B}_{[0,r]}(S)$ with $r \in \mathbb{Z}[1/p]_{>0}$. The natural inclusion $j : U \subset \varphi(U)$ and the map $\varphi : U \xrightarrow{\sim} \varphi(U)$ induce a triple of endofunctors (see Theorem 2.10) $j_\sharp$, j^*, j_* on $\mathrm{RigDA}_{\text{ét}}(U, \mathbb{Q})$ defined as follows:

$$j_\sharp : \mathrm{RigDA}^{(\mathrm{eff})}(U) \xrightarrow{j_\sharp} \mathrm{RigDA}^{(\mathrm{eff})}(\varphi(U)) \underset{\sim}{\xrightarrow{\varphi^*}} \mathrm{RigDA}^{(\mathrm{eff})}(U),$$

$$j^* : \mathrm{RigDA}^{(\mathrm{eff})}(U) \xrightarrow{j^*} \mathrm{RigDA}^{(\mathrm{eff})}(\varphi^{-1}(U)) \underset{\sim}{\xrightarrow{\varphi^{-1*}}} \mathrm{RigDA}^{(\mathrm{eff})}(U),$$

$$j_* : \mathrm{RigDA}^{(\mathrm{eff})}(U) \xrightarrow{j_*} \mathrm{RigDA}^{(\mathrm{eff})}(\varphi(U)) \underset{\sim}{\xrightarrow{\varphi^*}} \mathrm{RigDA}^{(\mathrm{eff})}(U),$$

and from the canonical equivalence $\varphi^* j^* \cong j^* \varphi^*$ we deduce that they form a triple of adjoint functors $(j_\sharp, j^*, j_*)$ such that $j^* j_\sharp \cong \mathrm{id}$ and $j^* j_* \cong \mathrm{id}$.

In the following proposition, we specialize some of the general motivic results of Section 2 to the setting of the subspaces of the relative Fargues–Fontaine curves introduced above.

Proposition 5.3. *Let S be a perfectoid space in $\mathrm{Adic}_{/\mathbb{F}_p}$ and let U be an open neighborhood of S in $\mathcal{Y}_{[0,\infty)}(S)$ of the form $U = \mathbb{B}_{[0,r]}(S)$ for some $r \in \mathbb{Z}[1/p]_{>0}$.*

(1) *The pullback to S induces an equivalence in* $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\omega})$:

$$\varinjlim_{j^*} \mathrm{RigDA}^{(\mathrm{eff})}(U) \cong \mathrm{RigDA}^{(\mathrm{eff})}(S).$$

Under the equivalence above, the endofunctor j^* on the left-hand side corresponds to the endofunctor φ^{-1*} on the right-hand side.

(2) *The pullbacks induce an equivalence in* $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$:

$$\varprojlim_{j^*} \mathrm{RigDA}^{(\mathrm{eff})}(U) \cong \mathrm{RigDA}^{(\mathrm{eff})}(\mathcal{Y}_{[0,\infty)}(S)).$$

Under the equivalence above, the endofunctor j^* on the left-hand side corresponds to the endofunctor φ^{-1*} on the right-hand side.

(3) *The canonical functors induce the following equivalences in* $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$:

$$\mathrm{RigDA}^{(\mathrm{eff})}(S)^{h\varphi^*}_{\omega} \cong (\varinjlim_{j^*} \mathrm{RigDA}^{(\mathrm{eff})}(U))^{hj^*}_{\omega} \cong \mathrm{RigDA}^{(\mathrm{eff})}(U)^{hj^*}_{\omega},$$
$$\mathrm{RigDA}^{(\mathrm{eff})}(\mathcal{Y}_{[0,\infty)}(S))^{h\varphi^*} \cong (\varprojlim_{j^*} \mathrm{RigDA}^{(\mathrm{eff})}(U))^{hj^*} \cong \mathrm{RigDA}^{(\mathrm{eff})}(U)^{hj^*}.$$

(4) *If we let ι be the closed inclusion $S \subset \mathcal{Y}_{[0,\infty)}(S)$, the functor ι^* induces an equivalence in* $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\omega})$:

$$\mathrm{RigDA}^{(\mathrm{eff})}(\mathcal{Y}_{[0,\infty)}(S))^{h\varphi^*}_{\omega} \cong \mathrm{RigDA}^{(\mathrm{eff})}(S)^{h\varphi^*}_{\omega}.$$

(5) *The pullback functor defines the following equivalences in* $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$:

$$\mathrm{RigDA}^{(\mathrm{eff})}(\mathcal{X}(S)) \cong \mathrm{RigDA}^{(\mathrm{eff})}(\mathcal{Y}_{(0,\infty)}(S))^{h\varphi^*} \cong \mathrm{RigDA}^{(\mathrm{eff})}(\mathcal{Y}_{(0,\infty)}(S))^{h\varphi^*}_{\omega}.$$

Proof. The forgetful functors $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}) \to \mathrm{Pr}^{\mathrm{L}}$, $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\omega}) \to \mathrm{Pr}^{\mathrm{L}}_{\omega}$ (see [Lurie 2017, Lemma 3.2.26]) are conservative and detect filtered colimits and limits (see [Lurie 2017, Corollaries 3.2.2.5 and 3.2.3.2]). Hence, as all the functors involved are monoidal, we may prove all statements by ignoring the monoidal structure. We first prove (1). The diagram

$$\mathrm{RigDA}^{(\mathrm{eff})}(U) \xrightarrow{j^*} \mathrm{RigDA}^{(\mathrm{eff})}(U) \xrightarrow{j^*} \mathrm{RigDA}^{(\mathrm{eff})}(U) \xrightarrow{j^*} \cdots$$

is equivalent to the diagram

$$\mathrm{RigDA}^{(\mathrm{eff})}(U) \xrightarrow{j^*} \mathrm{RigDA}^{(\mathrm{eff})}(\varphi^{-1}(U)) \xrightarrow{j^*} \mathrm{RigDA}^{(\mathrm{eff})}(\varphi^{-2}(U)) \xrightarrow{j^*} \cdots .$$

Since $|S| = \bigcap |U_{[0,r/p^n]}|$ the first claim follows from Theorem 2.12 and Remark 2.13. The second claim follows from the definition and the fact that φ on $\mathcal{Y}(S)$ restricts to φ on S.

We also remark that, dually, the diagram

$$\mathrm{RigDA}^{(\mathrm{eff})}(U) \xrightarrow{j_\sharp} \mathrm{RigDA}^{(\mathrm{eff})}(U) \xrightarrow{j_\sharp} \mathrm{RigDA}^{(\mathrm{eff})}(U) \xrightarrow{j_\sharp} \cdots$$

is equivalent to the diagram of inclusions of full subcategories of $\mathrm{RigDA}^{(\mathrm{eff})}(\mathcal{Y}_{[0,\infty)}(Y))$:

$$\mathrm{RigDA}^{(\mathrm{eff})}(U) \xrightarrow{j_\sharp} \mathrm{RigDA}^{(\mathrm{eff})}(\varphi(U)) \xrightarrow{j_\sharp} \mathrm{RigDA}^{(\mathrm{eff})}(\varphi^2(U)) \xrightarrow{j_\sharp} \cdots .$$

We point out that its union contains a set of compact generators of $\mathrm{RigDA}^{(\mathrm{eff})}(\mathcal{Y}_{[0,\infty)}(Y))$ since $\mathcal{Y}_{[0,\infty)} = \bigcup \varphi^n(U)$. We then deduce $\varinjlim_{j_\sharp} \mathrm{RigDA}^{(\mathrm{eff})}(U) \cong \mathrm{RigDA}^{(\mathrm{eff})}(\mathcal{Y}_{[0,\infty)}(Y))$ in Pr^{L}. On the other hand, since $j_\sharp$ is the left adjoint to j^* and limits in Pr^{L} as well as in Pr^{R} are computed in infinity-categories (see [Lurie 2009, Proposition 5.5.3.13 and Theorem 5.5.3.18]) we may rewrite $\varprojlim_{j^*} \mathrm{RigDA}^{(\mathrm{eff})}(U)$ as $\varinjlim_{j_\sharp} \mathrm{RigDA}^{(\mathrm{eff})}(U)$ in Pr^{L}. The latter is a colimit of fully faithful inclusions (since $j^* j_\sharp \cong \mathrm{id}$) which is $\mathrm{RigDA}^{(\mathrm{eff})}(\mathcal{Y}_{[0,\infty)}(S))$ as, indeed, any compact object here is defined over some $\varphi^n(U)$. We can then deduce the equivalence in (2). By definition, the functor $j_\sharp$ corresponds to φ^*; hence the final claim.

We now move to (3) and we start by the first row. We remark that the functors involved are monoidal, so it suffices to prove the statement in Pr^{L}, and that colimits computed in Pr^{L} coincide with those computed in $\mathrm{Pr}^{\mathrm{L}}_{\omega}$ by [Lurie 2017, Lemma 5.3.2.9]. The first equivalence follows immediately from (1). As $\mathrm{Pr}^{\mathrm{L}}_{\omega}$ is compactly generated (for a proof of this folklore fact, see, e.g., [Ayoub et al. 2022, Proposition 2.8.4]) finite limits commute with filtered colimits (since it is the case for spaces). We deduce

$$(\varinjlim_{j^*} \mathrm{RigDA}^{(\mathrm{eff})}(U))^{hj^*}_{\omega} \cong \varinjlim_{j^*} (\mathrm{RigDA}^{(\mathrm{eff})}(U)^{hj^*}_{\omega}) \cong \mathrm{RigDA}^{(\mathrm{eff})}(U)^{hj^*}_{\omega},$$

where the last equivalence follows from the fact that the extension of j^* to $\mathrm{RigDA}^{\mathrm{eff}}(U)^{hj^*}$ is an equivalence.

Similarly, for the second row, we point out that the first equivalence follows from (2) and for the second we may use the commutation of limits in Pr^{L} and conclude

$$(\varprojlim_{j^*} \mathrm{RigDA}^{(\mathrm{eff})}(U))^{hj^*} \cong \varprojlim_{j^*} (\mathrm{RigDA}^{(\mathrm{eff})}(U)^{hj^*}) \cong \mathrm{RigDA}^{(\mathrm{eff})}(U)^{hj^*}.$$

By Remark 2.25, the category $\mathrm{RigDA}^{(\mathrm{eff})}(U)^{hj^*}_{\omega}$ is the presentable subcategory of $\mathrm{RigDA}^{(\mathrm{eff})}(U)^{hj^*}$ generated by compact objects. Using (3) we then deduce that $\mathrm{RigDA}^{(\mathrm{eff})}(S)^{h\varphi^*}_{\omega}$ is equivalent to the presentable subcategory of $\mathrm{RigDA}^{(\mathrm{eff})}(\mathcal{Y}_{[0,\infty)}(S))^{h\varphi^*}$ generated by compact objects, which in turn coincides with $\mathrm{RigDA}^{(\mathrm{eff})}(\mathcal{Y}_{[0,\infty)}(S))^{h\varphi^*}_{\omega}$ (using Remark 2.25 once again) and this proves (4).

Next, we prove (5). By étale descent for RigDA applied to the cover $\mathcal{Y}_{(0,\infty)}(S) \to \mathcal{X}(S) = \mathcal{Y}_{(0,\infty)}(S)/\varphi^{\mathbb{Z}}$ we deduce (we denote here $\mathcal{Y}_{(0,\infty)}(S)$ by $\mathcal{Y}$, for brevity)

$$\mathrm{RigDA}(\mathcal{X}(S)) \cong \lim\Bigl(\mathrm{RigDA}(\mathcal{Y}) \rightrightarrows \mathrm{RigDA}(\mathcal{Y}) \times \mathbb{Z} \mathrel{\substack{\rightarrow\\\rightarrow\\\rightarrow}} \mathrm{RigDA}(\mathcal{Y}) \times \mathbb{Z}^2 \mathrel{\substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow}} \cdots\Bigr),$$

which computes $\mathrm{RigDA}(\mathcal{Y}_{(0,\infty)}(S))^{h\mathbb{Z}}$. This category, using Remarks 2.24 and 2.25, coincides with $\mathrm{RigDA}(\mathcal{Y}_{(0,\infty)}(S))^{h\varphi^*}_{\omega}$. ☐

Remark 5.4. The homotopy limit appearing in (2) coincides with the homotopy limit of the Cech hypercover generated by the cover $\{\varphi^N(U)\}$ of $\mathcal{Y}_{[0,\infty)}(S)$. In particular, (2) is also a special instance of analytic descent.

A motivic Dwork's trick. We now give another interpretation of Proposition 5.3 giving rise to a method to associate a motive over S to a motive over the (relative) Fargues–Fontaine curve $\mathcal{X}(S)$. This is reminiscent of the so-called Dwork's trick and produces a "universal" way to transform a rigid space in equicharacteristic p to a mixed characteristic space (up to homotopy). We now give the formal, precise definition of the functor $\mathcal{D}$ already mentioned in the introduction.

Corollary 5.5. *Let S be in $\mathrm{Adic}_{/\mathbb{F}_p}$. There is a functor*

$$\mathcal{D}(S) : \mathrm{RigDA}^{(\mathrm{eff})}(S) \to \mathrm{RigDA}^{(\mathrm{eff})}(\mathcal{X}(S^{\mathrm{Perf}}))$$

defined as follows:

$$\begin{tikzcd}
\mathrm{RigDA}^{(\mathrm{eff})}(S) \arrow[r, equal, "\sim"] & \mathrm{RigDA}^{(\mathrm{eff})}(S^{\mathrm{Perf}}) \arrow[d] & & \\
& \mathrm{RigDA}^{(\mathrm{eff})}(S^{\mathrm{Perf}})^{h\varphi^*}_{\omega} \arrow[r, equal, "\sim"] & \mathrm{RigDA}^{(\mathrm{eff})}(\mathcal{Y}_{[0,\infty)}(S^{\mathrm{Perf}}))^{h\varphi^*}_{\omega} \arrow[d, hook] & \\
& & \mathrm{RigDA}^{(\mathrm{eff})}(\mathcal{Y}_{[0,\infty)}(S^{\mathrm{Perf}}))^{h\varphi^*} \arrow[d, "j^*"] & \\
& & \mathrm{RigDA}^{(\mathrm{eff})}(\mathcal{Y}_{(0,\infty)}(S^{\mathrm{Perf}}))^{h\varphi^*} \arrow[r, equal, "\sim"] & \mathrm{RigDA}^{(\mathrm{eff})}(\mathcal{X}(S^{\mathrm{Perf}})).
\end{tikzcd}$$

It is compatible with tensor products and pullbacks, *inducing a functor*

$$\mathcal{D} : \mathrm{RigDA}^{(\mathrm{eff})} \to \mathrm{RigDA}^{(\mathrm{eff})}(\mathcal{X}(-))$$

between étale hypersheaves on $\mathrm{Perf}_{/\mathbb{F}_p}$ *with values in* $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$.

Proof. We can define a functor $\mathrm{RigDA}^{(\mathrm{eff})}(S) \to \mathrm{RigDA}^{(\mathrm{eff})}(\mathcal{X}(S^{\mathrm{Perf}}))$ as in the statement, where the first equivalence follows from Theorem 2.19, the first vertical map is defined in Corollary 2.26, the second equivalence follows from Proposition 5.3(4), the second vertical map is the natural inclusion (see Remark 2.25), and the third is simply given by j^* with $j : \mathcal{Y}_{(0,\infty)}(S^{\mathrm{Perf}}) \subset \mathcal{Y}_{[0,\infty)}(S^{\mathrm{Perf}})$ being the φ-equivariant open inclusion, while the last equivalence follows from Proposition 5.3(5). All these maps are monoidal. Compatibility with pullbacks follows from Corollary 2.26 and the commutativity of j^* with pullbacks. ☐

Remark 5.6. The recipe sketched above uses the specific formal properties of the categories of (adic) motives in various instances. It is impossible to follow a similar strategy directly on the category of smooth spaces over S in general (even the first step would not hold; see [Le Bras 2018]). As a consequence, even when the motive $\bar{M}$ is the motive of a smooth rigid variety over S, we cannot claim the motive $M_{\mathcal{X}}$ to be attached to a smooth rigid variety over $\mathcal{X}(S)$ in general (but see Proposition 5.11).

Remark 5.7. Consider now a Tate curve $E = \mathbb{G}_m^{\mathrm{an}}/\varphi$ over a nonarchimedean field K with φ being the automorphism $x \mapsto q \cdot x$ of $\mathbb{A}^1_K$ with $0 \neq q \in K^{\circ\circ}$. Following the proof of the previous corollary, one can also construct a functor

$$\mathrm{RigDA}^{(\mathrm{eff})}(K) \to \mathrm{RigDA}^{(\mathrm{eff})}(K)^{h\,\mathrm{id}} \cong \mathrm{RigDA}^{(\mathrm{eff})}(\mathbb{A}_K^{1\mathrm{an}})^{h\varphi^*} \to \mathrm{RigDA}^{(\mathrm{eff})}(E).$$

In this situation, this composition coincides with the pullback p^* along the projection $p : E \to \operatorname{Spa} K$ since $\iota^* p^* = \mathrm{id}$. We may then interpret the functor $\mathcal{D}(S)$ as playing the same role as the functor p^* with p being the (nonexistent) map $p : \mathcal{X}(S) \dashrightarrow S$. We will make this more precise in Proposition 5.17.

Remark 5.8. There is a perfectoid version of the previous constructions. We remark that in this case, the functor obtained by Dwork's trick

$$\mathrm{PerfDA}(P) \xrightarrow{\mathcal{D}(P)} \mathrm{PerfDA}(\mathcal{X}(P)) \cong \mathrm{RigDA}(\mathcal{X}(P)^{\diamond})$$

(the category on the right is defined by pro-étale descent; see Corollary 2.17) coincides canonically with the functor induced by the relative Fargues–Fontaine curve construction $X \mapsto \mathcal{X}(X)$. This can be seen from the fact that $\mathbb{Q}_S(\mathcal{X}(X))$ is naturally an object on $\mathrm{PerfDA}_n(\mathcal{X}(S))$ (see Remark 2.34) using [Kedlaya and Liu 2015, Lemma 8.7.15] and that $X \mapsto \mathcal{Y}_{[0,\infty)}(X)$ defines an inverse to ι^*. This is compatible with the idea that $\mathcal{D}(S)$ must be seen as a rigid-analytic model of the relative Fargues–Fontaine construction, as we will prove in Proposition 5.17.

Remark 5.9. There is a more direct way to define a map from $\mathrm{RigDA}(S)$ to $\mathrm{RigDA}(\mathcal{Y}_{[0,\infty)})^{h\varphi^*}$, namely, by using the functor ι_* (the right adjoint to the pullback functor). Nonetheless, we remark that the composition

$$\mathrm{RigDA}(S)^{h\varphi^*} \xrightarrow{\iota_*} \mathrm{RigDA}(\mathcal{Y}_{[0,\infty)}(S))^{h\varphi^*} \xrightarrow{j^*} \mathrm{RigDA}(\mathcal{Y}_{(0,\infty)}(S))^{h\varphi^*} \cong \mathrm{RigDA}(\mathcal{X}(S))$$

is trivial, since the objects $\iota_* M$ are concentrated on S and hence are in the kernel of j^*. The functor $\mathcal{D}(S)$ defined above is far from being trivial. Indeed, as it is a monoidal functor, it sends $1 = \mathbb{Q}_S(S)$ to $1 = \mathbb{Q}_{\mathcal{X}(S)}(\mathcal{X}(S))$.

We can even be more precise by computing the image under $\mathcal{D}$ of motives of "good reduction". We recall some basic facts on formal motives.

Definition 5.10. As in [Ayoub et al. 2022, Remark 3.1.5(2)], whenever $\mathfrak{S}$ is a formal scheme, we denote by $\mathrm{FDA}(\mathfrak{S}, \mathbb{Q}) = \mathrm{FDA}(\mathfrak{S})$ the infinity-category of (unbounded, derived, $\mathbb{Q}$-linear, étale) *formal motives* over $\mathfrak{S}$, i.e., the infinity-category arising as in Definition 2.4 from the étale site on smooth formal schemes over $\mathfrak{S}$ with coefficients in the ring $\mathbb{Q}$ (typically omitted) by imposing homotopy invariance, and invertibility of the Tate twist. Suppose now that $\mathfrak{S}_\eta$ is an adic space.

The special fiber functor $\mathfrak{X} \mapsto \mathfrak{X}_\sigma$ (resp. the generic fiber functor $\mathfrak{X} \mapsto \mathfrak{X}_\eta$) (see [Ayoub et al. 2022, Notation 1.1.6 and 1.1.8]) induces a natural map $\sigma^* : \mathrm{FDA}(\mathfrak{S}) \to \mathrm{DA}(\mathfrak{S}_\sigma)$ (resp. $\eta^* : \mathrm{FDA}(\mathfrak{S}) \to \mathrm{RigDA}(\mathfrak{S}_\eta)$) and the former is even an equivalence (see [Ayoub et al. 2022, Theorem 3.1.10]).

In particular, whenever $S = \mathrm{Spa}(R, R^+)$ is a perfectoid affinoid in $\mathrm{Perf}/\mathbb{F}_p$ with pseudouniformizer π, we have $\mathrm{FDA}(\mathrm{Spf}\, W(R^+)) \cong \mathrm{FDA}(\mathrm{Spf}\, R^+) \cong \mathrm{DA}(\mathrm{Spec}\, R^+/\pi)$. By Remark 2.20, the Frobenius endomorphism φ defines an invertible automorphism of $\mathrm{FDA}(\mathrm{Spf}\, W(R^+))$ and, arguing as in Corollary 2.26, we obtain a functor $\mathrm{FDA}(\mathrm{Spf}\, W(R^+)) \to \mathrm{FDA}(\mathrm{Spf}\, W(R^+))^{h\varphi^*}$ that we can compose with η^* and the pullback along the inclusion $\mathcal{Y}_{(0,\infty)}(S) \subset \mathcal{Y}_{[0,\infty]}(S) = \mathrm{Spf}\, W(R^+)_\eta$ getting the composition (one may temporarily lift any condition on Krull dimensions, as we do not use compact generators in this construction)

$$\begin{tikzcd}
\mathrm{FDA}(R^+) \arrow[d, equal, "\sim"] & & & \mathrm{RigDA}(\mathcal{X}(S)) \arrow[d, equal, "\sim"'] \\
\mathrm{FDA}(W(R^+)) \arrow[r] & \mathrm{FDA}(W(R^+))^{h\varphi^*} \arrow[r, "\eta^*"] & \mathrm{RigDA}(W(R^+)_\eta)^{h\varphi^*} \arrow[r, "j^*"] & \mathrm{RigDA}(\mathcal{Y}_{(0,\infty)}(S))^{h\varphi^*}
\end{tikzcd}$$

This produces a functor $\widetilde{\mathcal{D}}(R^+) : \mathrm{FDA}(R^+) \to \mathrm{RigDA}(\mathcal{X}(S))$.

Proposition 5.11. *Let* $S = \mathrm{Spa}(R, R^+)$ *be a perfectoid affinoid in* $\mathrm{Perf}_{/\mathbb{F}_p}$ *and let* M *be a motive of* $\mathrm{FDA}(R^+)$. *Then* M *can be defined over* $W(R^+)$ *and the image of* M_η *in* $\mathrm{RigDA}(\mathcal{Y}_{(0,\infty)}(S))$ *via* $\mathcal{D}(S)$ *is canonically isomorphic to* $M \times_{W(R^+)} \mathcal{Y}_{(0,\infty)}$.

More precisely, the following diagram commutes up to a natural invertible transformation:

$$\begin{tikzcd}
\mathrm{FDA}(R^+) \arrow[d, "\eta^*"'] \arrow[dr, "\widetilde{\mathcal{D}}(R^+)"] & \\
\mathrm{RigDA}(S) \arrow[r, "\mathcal{D}(S)"] & \mathrm{RigDA}(\mathcal{X}(S))
\end{tikzcd}$$

Proof. From the equivalence $\mathrm{FDA}(R^+) \cong \mathrm{FDA}(W(R^+))$ we know that M has a model over $W(R^+)$. In order to prove the final claim, it suffices to prove the commutativity of the diagram

$$\begin{tikzcd}
\mathrm{FDA}(W(R^+)) \arrow[r] \arrow[d] & \mathrm{FDA}(W(R^+))^{h\varphi^*}_\omega \arrow[d] \arrow[dr] & \\
\mathrm{RigDA}(S) \arrow[r] & \mathrm{RigDA}(S)^{h\varphi^*}_\omega \arrow[r, equal, "\sim"] & \mathrm{RigDA}(U)^{hj^*}_\omega
\end{tikzcd}$$

which in turn follows from the commutativity of the φ^*-equivariant, compact-preserving diagram, whose sides are all defined by pullback

$$\begin{tikzcd}
\mathrm{FDA}(W(R^+)) \arrow[d] \arrow[dr] & \\
\mathrm{RigDA}(U) \arrow[r] & \mathrm{RigDA}(S)
\end{tikzcd}$$

which is straightforward. ☐

Remark 5.12. We recall that $\mathrm{RigDA}(S)$ is generated by motives which are of good reduction over some étale extension $S' \to S$ by [Ayoub et al. 2022, Corollary 3.7.19]. Proposition 5.11 allows us then to have an explicit description of $\mathcal{D}(S)(M)$ for any compact motive $M \in \mathrm{RigDA}(S)$ up to some étale extension of the base.

(De)perfectoidification and rigid-analytic tilting. We now quickly show that the construction of the functor $\mathcal{D}(S)$ given above allows one to "globalize" the motivic rigid-analytic tilting equivalence given in [Vezzani 2019a], that is, to prove that $\mathrm{RigDA}(S) \cong \mathrm{RigDA}(S^{\diamond})$ for any space $S \in \mathrm{Adic}_{/\mathbb{Q}_p}$. This allows one to give, a posteriori, another construction of $\mathcal{D}$ in terms of the relative Fargues–Fontaine curve, paired up with motivic (de)perfectoidification.

Theorem 5.13. *There are equivalences of presheaves on* $\mathrm{Adic}_{/\mathbb{Q}_p}$ *with values in* $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$:

$$\mathrm{RigDA}(-) \cong \mathrm{RigDA}((-)^{\diamond}) \cong \mathrm{PerfDA}((-)^{\diamond}) \cong \mathrm{PerfDA}(-).$$

Proof. The proof is divided into various steps.

Step 1: By Theorem 2.31 it suffices to produce the first equivalence. By pro-étale descent we may restrict to $\mathrm{Perf}^{\mathrm{qcqs}}_{/\mathbb{C}_p}$ and show $\mathrm{RigDA}(P) \cong \mathrm{RigDA}(P^{\flat})$ in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\omega})$ functorially on P. We can produce a natural transformation between the functors $\mathrm{RigDA}(-)$ and $\mathrm{RigDA}(-^{\flat})$ by means of the composition

$$F : \mathrm{RigDA}(P^{\flat}) \xrightarrow{\mathcal{D}(P^{\flat})} \mathrm{RigDA}(\mathcal{X}(P^{\flat})) \xrightarrow{\infty^*} \mathrm{RigDA}(P).$$

We now restrict the two functors on the affinoid analytic site of P where they are analytic (hyper)sheaves with values in $\mathrm{Pr}^{\mathrm{L}}_{\omega}$ (see Theorem 2.10). To show they are equivalent, it suffices to show that F is invertible on analytic stalks (see [Ayoub et al. 2022, Lemma 2.8.4]), that is, on a fixed perfectoid space of the form $P = \mathrm{Spa}(K, K^+)$ with K a complete field (by Theorem 2.12; see also [Ayoub et al. 2022, Theorem 2.8.5]). By pro-étale descent, we may then actually suppose that K is algebraically closed. We remark that we are almost in the same setting as in [Vezzani 2019a], with the difference that K^+ may not be equal to K°. In particular, we can't use duality as it is done in [Vezzani 2019a, Theorem 7.11]. We will replace this ingredient with [Ayoub et al. 2022, Theorem 3.7.21].

Step 2: We consider the adjoint pairs

$$\xi : \mathrm{FDA}(K^+) \rightleftarrows \mathrm{RigDA}(\mathrm{Spa}(K, K^+)) : \eta, \quad \xi^{\flat} : \mathrm{FDA}(K^+) \rightleftarrows \mathrm{RigDA}(\mathrm{Spa}(K^{\flat}, K^{\flat +})) : \eta^{\flat}.$$

We remark that, by means of Proposition 5.11, we have $F\xi \cong \xi^{\flat}$. Using [Ayoub et al. 2022, Theorem 3.7.21] we may replace the categories $\mathrm{RigDA}(\mathrm{Spa}(K, K^+))$ and $\mathrm{RigDA}(\mathrm{Spa}(K^{\flat}, K^{\flat +}))$ with $\mathrm{FDA}(\mathrm{Spf}\, K^+, \chi 1)$ and $\mathrm{FDA}(\mathrm{Spf}\, K^+, \chi^{\flat} 1)$, respectively, which denote the categories of modules in formal motives over the commutative algebra object $\chi 1$ and $\chi^{\flat} 1$, respectively (see [Ayoub et al. 2022, Section 3.4]). Accordingly, we may replace the functor F with the base change along the map $\chi^{\flat} 1 \to \chi 1$ which is induced by $F\xi \cong \xi^{\flat}$. The fact that this morphism is invertible can be deduced if we prove $G1 \cong 1$, where we denote by G the right adjoint to F. Equivalently, we are left to prove that for any compact $M \in \mathrm{FDA}(W(K^+))$, there is a canonical equivalence $\mathrm{Map}_{\mathrm{RigDA}(K,K^+)}(M_{(K,K^+)}, 1) \cong \mathrm{Map}_{\mathrm{RigDA}(K,K^+)}(M_{(K^{\flat},K^{\flat +})}, 1)$. From the equivalence $\mathrm{Map}_{\mathrm{RigDA}(K,K^+)}(M, 1) \cong \varinjlim \mathrm{Map}_{\mathrm{RigDA}^{\mathrm{eff}}(K,K^+)}(M(n), 1(n))$ and since $\mathbb{Q}(1)$ is a direct summand of $\mathbb{Q}(\mathbb{T}^1)$, it suffices to show an equivalence

$$\mathrm{Map}_{\mathrm{RigDA}^{\mathrm{eff}}(K,K^+)}(M_{(K,K^+)}, \mathbb{Q}(\mathbb{T}^n)) \cong \mathrm{Map}_{\mathrm{RigDA}^{\mathrm{eff}}(K,K^+)}(M_{(K^\flat,K^{\flat+})}, \mathbb{Q}(\mathbb{T}^n))$$

for any M ranging among a class of compact generators of $\mathrm{FDA}^{\mathrm{eff}}(W(K^+))$. Since universal homeomorphisms become invertible in $\mathrm{FDA}(W(K^+))$ (see [Ayoub et al. 2022, Theorems 2.9.7 and 3.1.10]) and hence in $\mathrm{RigDA}(K, K^+)$, we may and do invert formally on $\mathrm{RigDA}^{\mathrm{eff}}(K, K^+)$ universal homeomorphisms of formal schemes over K^+ without changing the stable category $\mathrm{RigDA}(K, K^+)$.

Step 3: We can now use the results of [Vezzani 2019a] which do not use the hypothesis $K^+ = K^\circ$ to conclude. Assume M to be the motive of a variety X which is étale over the some affine space over $W(K^+)$. We may use these coordinates to define a perfectoid pro-étale cover $\widehat{X}_{(K,K^+)} \sim \varprojlim X_h$ of $X_{(K,K^+)}$ and a perfectoid pro-étale cover $\widehat{X}_{(K^\flat,K^{\flat+})}$ of $X_{(K^\flat,K^{\flat+})}$ which coincides with its perfection. By [Vezzani 2019a, Proposition 4.5] we have $\mathrm{Map}(\mathbb{Q}(\widehat{X}_{(K,K^+)}), \mathbb{Q}(\mathbb{T}^n)) \cong \varinjlim_h \mathrm{Map}(\mathbb{Q}(X_h), \mathbb{Q}(\mathbb{T}^n))$. As the maps $X_h \to X_{(K,K^+)}$ are invertible in $\mathrm{RigDA}^{\mathrm{eff}}(K, K^+)$ by construction, we deduce that $\mathrm{Map}(\mathbb{Q}(X), \mathbb{Q}(\mathbb{T}^n)) \cong \mathrm{Map}(\mathbb{Q}(\widehat{X}_{(K,K^+)}), \mathbb{Q}(\mathbb{T}^n))$. On the other hand, by Theorem 2.31 we have

$$\mathrm{Map}(\mathbb{Q}(X_{(K^\flat,K^{\flat+})}), \mathbb{Q}(\mathbb{T}^n)) \cong \mathrm{Map}(\mathbb{Q}(\widehat{X}_{(K^\flat,K^{\flat+})}), \mathbb{Q}(\widehat{\mathbb{T}}^n)) = \mathrm{Map}(\mathbb{Q}(\widehat{X}_{(K,K^+)}, \mathbb{Q}(\widehat{\mathbb{T}}^n)).$$

The equivalence $\mathrm{Map}(\mathbb{Q}(\widehat{X}_{(K,K^+)}), \mathbb{Q}(\mathbb{T}^n)) \cong \mathrm{Map}(\mathbb{Q}(\widehat{X}_{(K,K^+)}, \mathbb{Q}(\widehat{\mathbb{T}}^n))$ proved in [Vezzani 2019a, Propositions 7.5–7.6] then gives the desired equivalence. □

Remark 5.14. One could replace step 3 of the previous proof with the explicit description of the algebras $\chi 1$ and $\chi^\flat 1$ given in [Ayoub et al. 2022, Section 3.8]: when evaluated on each point v of $\mathrm{Spf}\,\mathcal{O}_C$ (corresponding to some valuation ring K_v^+ containing K^+) they can be shown to be both isomorphic to $(1 \oplus 1(-1)[-1])^{\otimes n}$ with n being the rank over $\mathbb{Q}$ of the valuation group Γ_v of the valuation (K, K_v^+) (resp. $(K^\flat, K_v^{\flat+})$) via a map induced by the choice of some generators $|\varpi_1|, \dots, |\varpi_n|$ of Γ. The morphism $\chi^\flat 1 \to \chi 1$ corresponds to the one induced by $\varpi \mapsto \varpi^\sharp$ which fixes the $\mathbb{Q}$-basis $|\varpi_i|$ and is then invertible.

Remark 5.15. The result above is stated only for stable motives (as seen in the proof we made use of this hypothesis). On the other hand, over points of the form (K, K°) it holds even for effective motives, using [Vezzani 2019a, Theorem 7.10] together with [Ayoub et al. 2022, Remark 2.9.12].

The proof of Theorem 5.13 also shows the following.

Corollary 5.16. *Let K be a perfectoid field of characteristic p and P be in $\mathrm{Perf}_{/K}$. For any closed point $x^\sharp$ of $\mathcal{X}(K)$ associated to an untilt $K^\sharp$ of K the composition*

$$\mathrm{RigDA}(P) \xrightarrow{\mathcal{D}(P)} \mathrm{RigDA}(\mathcal{X}(P)) \xrightarrow{x^{\sharp *}} \mathrm{RigDA}(P^\sharp)$$

is an equivalence and recovers the equivalence of [Vezzani 2019a] *in the case $P = \mathrm{Spa}(K)$.* □

We end this section by linking the functor $\mathcal{D}$ to the base change along $\mathcal{X}(S)^\diamond \to S^\diamond$.

Proposition 5.17. *Let P be a perfectoid space in $\mathrm{Perf}_{/\mathbb{F}_p}$.*

(1) *The relative Fargues–Fontaine curve functor* $X \in \mathrm{PerfSm}\,/\,P \mapsto \mathcal{X}(X)$ *induces a functor*

$$\mathcal{X} : \mathrm{PerfDA}(P) \to \mathrm{PerfDA}(\mathcal{X}(P))$$

and the following diagram, with vertical maps given by Theorem 5.13, is commutative (*up to a canonical invertible transformation*):

$$\begin{tikzcd}
\mathrm{RigDA}(P) \arrow[r, "\mathcal{D}(P)"] \arrow[d, "\sim"] & \mathrm{RigDA}(\mathcal{X}(P)) \arrow[d, "\sim"] \\
\mathrm{PerfDA}(P) \arrow[r, "\mathcal{X}"] & \mathrm{PerfDA}(\mathcal{X}(P))
\end{tikzcd}$$

In particular, one can define $\mathcal{D}(P)$ *as the functor induced by the relative Fargues–Fontaine curve construction and motivic* (*de*)*perfectoidification.*

(2) *The pullback along* $\Pi : \mathcal{Y}_{(0,\infty)}(P)^{\diamond} \to P^{\diamond}$ *induces a functor*

$$\Pi^* : \mathrm{RigDA}(P^{\diamond}) \to \mathrm{RigDA}(\mathcal{Y}_{(0,\infty)}(P)^{\diamond})$$

and the following diagram, with vertical maps given by Theorem 5.13, is commutative (*up to a canonical invertible transformation*):

$$\begin{tikzcd}
\mathrm{RigDA}(P) \arrow[r, "\mathcal{D}(P)"] \arrow[d, "\sim"] & \mathrm{RigDA}(\mathcal{X}(P)) \arrow[r] & \mathrm{RigDA}(\mathcal{Y}_{(0,\infty)}(P)) \arrow[d, "\sim"] \\
\mathrm{RigDA}(P^{\diamond}) \arrow[rr, "\Pi^*"] & & \mathrm{RigDA}(\mathcal{Y}_{(0,\infty)}(P)^{\diamond})
\end{tikzcd}$$

In particular, one can define the functor $\mathcal{D}(P)$ *by means of the pullback along the diamond map* $\mathcal{Y}_{(0,\infty)}(P)^{\diamond} \to P^{\diamond}$ *and motivic* (*de*)*diamondification.*

Proof. Since the functor $\Pi^* : \mathrm{PerfDA}(P) \to \mathrm{PerfDA}(\mathcal{Y}_{(0,\infty)}(P))$ obtained by pullback coincides with the one induced by $X \mapsto \mathcal{Y}_{(0,\infty)}(X)$, we easily see that the two claims are actually equivalent. We recall that, if we put $Q := \mathcal{Y}_{(0,\infty)}(P)_{\mathbb{C}_p}$, the map $e : Q \to \mathcal{Y}_{(0,\infty)}(P)$ is a pro-étale perfectoid cover and hence, by pro-étale descent, it suffices to construct a Galois-equivariant invertible natural transformation between the functors $e^* \circ \widetilde{\mathcal{D}} : \mathrm{RigDA}(P) \to \mathrm{RigDA}(Q)$ and $\widetilde{\Pi} : \mathrm{RigDA}(P) \to \mathrm{RigDA}(Q^{\flat})$ where we put $\widetilde{\mathcal{D}}$ to be the composition of $\mathcal{D}$ with $(\mathcal{Y}_{(0,\infty)}(P) \to \mathcal{X}(P))^*$ and $\widetilde{\Pi}$ to be $Q^{\diamond} \to P$.

This follows from the functoriality of $\mathcal{D}$ and the construction of the equivalence $\mathrm{RigDA}(Q) \cong \mathrm{RigDA}(Q^{\flat})$ showed in Theorem 5.13, which give the commutative diagram

$$\begin{tikzcd}
\mathrm{RigDA}(P) \arrow[r, "\widetilde{\Pi}^*"] \arrow[d, "\widetilde{\mathcal{D}}"] & \mathrm{RigDA}(Q^{\flat}) \arrow[d, "\widetilde{\mathcal{D}}"] \arrow[dr, bend left, "\sim"] & \\
\mathrm{RigDA}(\mathcal{Y}_{(0,\infty)}(P)) \arrow[r, "\mathcal{Y}(\widetilde{\Pi})^*"] \arrow[rr, bend right, "e^*"'] & \mathrm{RigDA}(\mathcal{Y}_{(0,\infty)}(Q^{\flat})) \arrow[r, "\infty_{\mathbb{C}_p}^*"] & \mathrm{RigDA}(Q)
\end{tikzcd}$$

This proves the statement (the commutativity of the lower part of the diagram is simply expressing the adjunction between Witt vectors and tilting). For the final claim, we remark that one could then define $\mathcal{D}$ using the composition

$$\mathrm{RigDA}(P) \to \mathrm{RigDA}(P)^{h\varphi^*} \xrightarrow{\Pi^*} \mathrm{RigDA}(\mathcal{Y}_{(0,\infty)}(P)^{\diamond})^{h\varphi^*} \cong \mathrm{RigDA}(\mathcal{X}(P)^{\diamond}) \cong \mathrm{RigDA}(\mathcal{X}(P)),$$

where the first map is induced by Corollary 2.26. ☐

6. The de Rham–Fargues–Fontaine cohomology

In this final section, we combine the results above, by merging the Fargues–Fontaine realization $\mathcal{D}$ with the overconvergent de Rham realization, giving rise to a de Rham-like cohomology theory for analytic spaces in positive characteristic with values in modules over the associated Fargues–Fontaine curves.

Definition and properties. We can juxtapose Corollary 4.39 and Corollary 5.5 as follows.

Definition 6.1. Let S be an adic space in $\mathrm{Adic}_{/\mathbb{F}_p}$. The composition of the functors

$$\mathrm{dR}_S^{\mathrm{FF}} : \mathrm{RigDA}(S) \xrightarrow{\mathcal{D}(S^{\mathrm{Perf}})} \mathrm{RigDA}(\mathcal{X}(S^{\mathrm{Perf}})) \xrightarrow{\mathrm{dR}_{\mathcal{X}(S^{\mathrm{Perf}})}} \mathrm{QCoh}(\mathcal{X}(S^{\mathrm{Perf}}))^{\mathrm{op}}$$

will be called the *de Rham–Fargues–Fontaine realization*.

In the case $M = \mathbb{Q}_S(X)$ for some smooth map $X \to S$, or more generally if $M = p_! p^! \mathbb{Q}_S$ for some map $p : X \to S$ which is locally of finite type (see [Ayoub et al. 2022, Corollary 4.3.18]), we alternatively write $\mathrm{dR}_S^{\mathrm{FF}}(X)$ instead of $\mathrm{dR}_S^{\mathrm{FF}}(M)$.

Remark 6.2. In the case S is affinoid, we may define the cohomology groups $H^i_{\mathrm{FF}}(M/\mathcal{X}(S)) := H^i(\mathrm{dR}_S^{FF}(M))$ with respect to the t-structure of Remark 4.20 and call them the *i-th de Rham–Fargues–Fontaine cohomology group of M over $\mathcal{X}(S)$*. In the case $M = p_! p^! \mathbb{Q}_S$ for a map $p : X \to S$ which is locally of finite type, we may even use the symbol $H^i_{\mathrm{FF}}(X/\mathcal{X}(S))$.

We recall that we denote by $\mathrm{RigDA}(S)^{\mathrm{fd}}$ the full subcategory of dualizable motives (see Definition 4.45) and by $\mathcal{P}(S)$ the full subcategory of perfect complexes in $\mathrm{QCoh}(S)$.

Theorem 6.3. *Let S be in $\mathrm{Adic}_{/\mathbb{F}_p}$. The de Rham–Fargues–Fontaine realization $\mathrm{dR}_S^{\mathrm{FF}}$ restricts to a symmetric monoidal functor compatible with pullbacks*:

$$\mathrm{dR}_S^{\mathrm{FF}} : \mathrm{RigDA}(S)^{\mathrm{fd}} \to \mathcal{P}(\mathcal{X}(S^{\mathrm{Perf}}))^{\mathrm{op}}.$$

For any M in $\mathrm{RigDA}(S)^{\mathrm{fd}}$, $\mathrm{dR}_S^{\mathrm{FF}}(M)$ is a split perfect complex of $\mathcal{O}_{\mathcal{X}(S^{\mathrm{Perf}})}$-modules over the relative Fargues–Fontaine curve $\mathcal{X}(S^{\mathrm{Perf}})$. In particular, its cohomology groups are vector bundles on S and equal to 0 if $|i| \gg 0$.

Proof. The functor $\mathcal{D}(S)$, being monoidal, preserves dualizable objects. The claim then follows from Theorem 4.46. ☐

One of the key features of the relative de Rham cohomology for algebraic varieties is that it defines a vector bundle on the base whenever the map $f : X \to S$ is proper and smooth. The analogous statement holds for the de Rham–Fargues–Fontaine cohomology:

Corollary 6.4. *If* $X \to S$ *is a smooth proper morphism in* $\mathrm{Adic}_{/\mathbb{F}_p}$, *then* $\mathrm{dR}^{\mathrm{FF}}_S(X)$ *is a split perfect complex of* $\mathcal{O}_{\mathcal{X}(S^{\mathrm{Perf}})}$*-modules over the relative Fargues–Fontaine curve* $\mathcal{X}(S^{\mathrm{Perf}})$. *In particular, its cohomology groups are vector bundles on* S *and equal to* 0 *if* $|i| \gg 0$.

Proof. It suffices to point out that the motive $\mathbb{Q}_S(X)$ is dualizable, and this follows from [Ayoub et al. 2022, Corollary 4.1.8]. ☐

It is also well known that the absolute de Rham cohomology for algebraic varieties over a field (of characteristic zero) is finite, for any sort of variety X. Once again, the same result holds for the de Rham–Fargues–Fontaine cohomology, as the next corollary shows.

Corollary 6.5. *Let* K *be a perfectoid field of characteristic* p. *If* M *is a compact motive* (*e.g.*, *the motive attached to a smooth quasicompact rigid variety over* K) *in* $\mathrm{RigDA}(K)$, *then* $\mathrm{dR}^{\mathrm{FF}}_K(X)$ *is a split perfect complex of* $\mathcal{O}_{\mathcal{X}(K)}$*-modules over the relative Fargues–Fontaine curve* $\mathcal{X}(K)$.

Proof. Whenever the base is the spectrum of a field K, any compact motive in $\mathrm{DA}(K)$ is dualizable, as proved in [Riou 2005] (we use the fact that we have rational coefficients). Since the image of the (monoidal) functor $\mathrm{DA}(K) \to \mathrm{RigDA}(K)$ induced by analytification generates the target category (again, since we have rational coefficients; see [Ayoub 2020, Proposition 2.31]) we deduce that also in $\mathrm{RigDA}(K)$ any compact motive is dualizable. ☐

Remark 6.6. We stress that there is no "smoothness" nor "properness" condition on the motive M above: for example, any (eventually singular or nonproper) algebraic variety $p : X \to K$ has an attached (homological) motive $p_! p^! \mathbb{Q}(K)$ which is dualizable in $\mathrm{DA}(K)$ (by [Ayoub 2014, Théorème 8.10]) and hence in $\mathrm{RigDA}(K)$, after analytification. It coincides with the homological motive of the analytified variety by [Ayoub 2015, Théorème 1.4.40].

Remark 6.7. By precomposing $\mathcal{D}$ with other symmetric monoidal functors, we can deduce further cohomology theories. For example, if $S = \mathrm{Spa}(A, A^+)$ is affinoid, we may consider the analytification functor (see [Ayoub et al. 2022, Proposition 2.2.13])

$$\mathrm{An}^* : \mathrm{DA}(\mathrm{Spec}\, A) \to \mathrm{RigDA}(S),$$

getting a de Rham–Fargues–Fontaine realization for *algebraic* varieties over A.

Comparison with the B^+_{dR}-cohomology of [Bhatt et al. 2018]. To conclude this text, we would like to briefly discuss the relation between the de Rham–Fargues–Fontaine realization and some other cohomology theories.

Let K be a perfectoid field of characteristic p. From Corollary 5.16 one deduces that, under the hypotheses of Corollary 6.5, the specialization of $\mathrm{dR}^{\mathrm{FF}}_K(M)$ at some untilt $K^\sharp$ of K is isomorphic to the $K^\sharp$-overconvergent de Rham cohomology $R\Gamma_{\mathrm{dR}}(M, K^\sharp)$ defined in [Vezzani 2019b, Definition 4.2]. Therefore, $\mathrm{dR}^{\mathrm{FF}}_K(M)$ is a perfect complex on the Fargues–Fontaine curve interpolating between the overconvergent de Rham cohomologies of M at various untilts of K, which are parametrized by rigid points of the curve.

Remark 6.8. Using the above notations, if X is the analytification of a smooth algebraic qcqs variety over $K^\sharp$ (resp. a smooth proper rigid analytic variety over $K^\sharp$), the $K^\sharp$-overconvergent de Rham cohomology $R\Gamma_{\mathrm{dR}}(\mathbb{Q}_{K^\sharp}(X), K^\sharp)$ coincides with the algebraic de Rham cohomology over $K^\sharp$ (resp. with the analytic de Rham cohomology of X over $K^\sharp$); see [Vezzani 2018, Proposition 5.12]. However, we stress that the Hodge filtration on the latter is not expected to be recovered by this rigid-analytic motivic construction.

Suppose now that C is a perfectoid field of characteristic zero (or, more generally, an admissible perfectoid space over it). We notice that the overconvergent de Rham cohomology over C extends to a cohomology with values over $\mathrm{QCoh}(\mathcal{X}(C))$ via the composition

$$\mathrm{RigDA}(C) \cong \mathrm{RigDA}(C^\flat) \xrightarrow{\mathrm{dR}^{\mathrm{FF}}} \mathrm{QCoh}(\mathcal{X}(C^\flat))^{\mathrm{op}}.$$

We now consider the particular case where C is algebraically closed. Let k be its residue field and B^+_{dR} be Fontaine's pro-infinitesimal thickening

$$B^+_{\mathrm{dR}} := W(\mathcal{O}_C^\flat)[1/p]^{\wedge_\xi} \xrightarrow{\theta} C$$

with ξ denoting a generator of the kernel of the map $\theta : W(\mathcal{O}_C^\flat) \to \mathcal{O}_C$. We also pick a section of $\mathcal{O}_C/p \to k$ giving rise to a splitting $k \to \mathcal{O}_{C^\flat}$. The overconvergent de Rham cohomology over C can be extended over B^+_{dR} as follows:

$$\mathrm{RigDA}(C)^{\mathrm{fd}} \cong \mathrm{RigDA}(C^\flat)^{\mathrm{fd}} \xrightarrow{\mathrm{dR}^{\mathrm{FF}}} \mathcal{P}(\mathcal{X}(C^\flat))^{\mathrm{op}} \to \mathcal{P}(B^+_{\mathrm{dR}})^{\mathrm{op}},$$

where the last arrow is induced by the section at ∞ of the Fargues–Fontaine curve and the identification $\widehat{\mathcal{O}}_{\mathcal{X}(C^\flat),\infty} \cong B^+_{\mathrm{dR}}$. We note that by Corollary 5.16, this is equivalent to considering a spreading out from C to its open neighborhoods on the curve as follows:

$$\mathrm{RigDA}(C)^{\mathrm{fd}} \cong \varinjlim_{\infty\in U} \mathrm{RigDA}(\mathcal{O}(U))^{\mathrm{fd}} \xrightarrow{\mathrm{dR}} \varinjlim_{\infty\in U} \mathcal{P}(\mathcal{O}(U))^{\mathrm{op}} \to \mathcal{P}(B^+_{\mathrm{dR}})^{\mathrm{op}}. \tag{+}$$

In [Bhatt et al. 2018, Section 13] Bhatt, Morrow and Scholze also constructed, for proper smooth rigid varieties over C, a deformation of de Rham cohomology along B^+_{dR} using a different spreading out argument that we now recall in order to set some notation. By de Jong's theorem (see the proof of [Bhatt et al. 2018, Lemma 13.7]) we have $\mathrm{Spa}(C) \sim \varprojlim_{S,\eta} S$ where S runs among affinoid spaces that are *smooth* over the discrete valued field $K := W(k)[1/p]$ equipped with a C-rational point $\eta : \mathrm{Spa}\, C \to S$. By eventually taking an open neighborhood of η, we may also assume that $S \to \mathrm{Spa}\, K$ factors as $S \xrightarrow{e} \mathbb{B}^N_K \to \mathrm{Spa}\, K$ for some $N \in \mathbb{N}$ and some étale map e. If we let A be $\mathcal{O}(S)$, we remark that

$\eta : A \to C$ has a (nonunique) lift $\ell : A \to B^+_{\mathrm{dR}}$ over C, by the smoothness of A/K. More precisely, we have the following.

Proposition 6.9. *With the notation above, there is an affinoid open neighborhood U of ∞ and a map $f : U \to S$ such that η factors as* $\operatorname{Spa} C \xrightarrow{\infty} U \xrightarrow{f} S$.

Proof. Choose a lift $\alpha : U \to \mathbb{B}^N_K$ of the map $e \circ \eta$ and consider the étale map $e_U : S \times_{\mathbb{B}^N_K} U \to U$. We note that η defines a section of the map $e_C : S \times_{\mathbb{B}^N_K} \operatorname{Spa} C \to \operatorname{Spa} C$. Since $\infty \sim \varprojlim_{\infty \in U} U$ we deduce that, up to shrinking U, there is also a section η_U to the map e_U and hence a map $f : U \to S$ with the required property. ☐

Let X/C be a smooth and proper variety. By [Bhatt et al. 2018, Corollary 13.16] there exists (S, η) as above and a smooth and proper variety $\widetilde{X}/S$ such that $\widetilde{X} \times_{S,\eta} C \cong X$. The B^+_{dR}-cohomology is then given by

$$\mathrm{R}\Gamma_{\mathrm{crys}}(X/B^+_{\mathrm{dR}}) := \mathrm{R}\Gamma_{\mathrm{dR}}(\widetilde{X}/S) \otimes_{A,\ell} B^+_{\mathrm{dR}},$$

and it can be made independent on the various choices made, as shown in [Bhatt et al. 2018, Section 13.1 and Theorem 13.19]. We also note that, by Proposition 4.43, the functor $\widetilde{X} \mapsto \mathrm{R}\Gamma_{\mathrm{crys}}(X/B^+_{\mathrm{dR}})$ is easily seen to be extended by the composition

$$\mathrm{RigDA}(S)^{\mathrm{fd}} \xrightarrow{\mathrm{dR}} \mathcal{P}(A)^{\mathrm{op}} \xrightarrow{\ell^*} \mathcal{P}(B^+_{\mathrm{dR}})^{\mathrm{op}}. \tag{++}$$

Remark 6.10. In [Bhatt et al. 2018], the B^+_{dR}-cohomology is defined for arbitrary smooth varieties over C, but it is not $\mathbb{B}^1$-invariant. We may interpret (++) as being an *overconvergent* version of their construction.

Theorem 6.11. *Let X be a smooth and proper variety over C. Then $\mathrm{R}\Gamma_{\mathrm{crys}}(X/B^+_{\mathrm{dR}})$ is canonically equivalent to $\mathrm{dR}^{\mathrm{FF}}_{C^\flat}(M_C(X)^\flat) \otimes_{\mathcal{O}_{\mathcal{X}(C^\flat)}} B^+_{\mathrm{dR}}$. In particular the de Rham–Fargues–Fontaine cohomology over a complete algebraically closed field C is compatible with (an overconvergent version of) the B^+_{dR}-cohomology of* [Bhatt et al. 2018].

Proof. By $\mathrm{RigDA}(C) \cong \varinjlim \mathrm{RigDA}_{S,\eta}(S)$ we fix a (S, η) as above and show that for a given $\ell : A \to B^+_{\mathrm{dR}}$, the functor (++) coincides with

$$\mathrm{RigDA}^{\mathrm{fd}}(S) \to \mathrm{RigDA}^{\mathrm{fd}}(C) \xrightarrow{(+)} \mathcal{P}(B^+_{\mathrm{dR}})^{\mathrm{op}}.$$

To this aim, it suffices to choose a lift $\tilde{\ell} : U \to S$ as in Proposition 6.9 and put $\ell : A \to B^+_{\mathrm{dR}}$ to be the one induced by $A \xrightarrow{\tilde{\ell}} \mathcal{O}(U) \to B^+_{\mathrm{dR}}$. The claim then follows from the commutative diagram below (which also reproves that (++) is independent on the choice of ℓ):

$$\begin{tikzcd}
\mathrm{RigDA}(S) \arrow[r, "\tilde{\ell}^*"'] \arrow[d, "\mathrm{dR}"] \arrow[rrr, bend left, "\eta^*"] & \mathrm{RigDA}(U) \arrow[r] \arrow[d, "\mathrm{dR}"] & \varinjlim \mathrm{RigDA}(U) \arrow[r, dash, "\sim"'] \arrow[d, "\mathrm{dR}"] & \mathrm{RigDA}(C) \arrow[d, "(+)"] \\
\mathcal{P}(A)^{\mathrm{op}} \arrow[r, "\tilde{\ell}^*"] \arrow[rrr, bend right, "\ell^*"'] & \mathcal{P}(\mathcal{O}(U))^{\mathrm{op}} \arrow[r] & \varinjlim \mathcal{P}(\mathcal{O}(U))^{\mathrm{op}} \arrow[r] & \mathcal{P}(B^+_{\mathrm{dR}})^{\mathrm{op}}
\end{tikzcd}$$

☐

Remark 6.12. This completes our proof that $R\Gamma_{\mathrm{FF}_C}(-) := \mathrm{dR}^{\mathrm{FF}}_{C^\flat}(-^\flat)$ satisfies all the requirements of [Scholze 2018, Conjecture 6.4]. Notice that the description given in (++) shows that its completion at ∞ is an overconvergent version of $R\Gamma_{\mathrm{crys}}(-/B^+_{\mathrm{dR}})$ as defined in [Bhatt et al. 2018, Section 13].

Remark 6.13. de Jong's theorem allows one to write $\operatorname{Spa} C \sim \varprojlim_{(S,\eta)} S$ with S being smooth over $\mathbb{Q}_p$. By motivic continuity we deduce $\mathrm{RigDA}(C)^{\mathrm{fd}} \cong \varinjlim \mathrm{RigDA}(S)^{\mathrm{fd}}$ so that one can spread out a compact motive over C to some dualizable motive defined over $\operatorname{Spa}(A)$ with A smooth over $\mathbb{Q}_p$. This is the motivic version of the spreading out arguments of Conrad and Gabber mentioned in [Bhatt et al. 2018, Remark 13.17].

Comparison with rigid cohomology. We first describe the de Rham–Fargues–Fontaine realization on objects with good reduction. Let us do it in the affinoid case, for simplicity. Let $S = \operatorname{Spa}(R, R^+) \in \mathrm{Perf}_{/\mathbb{F}_p}$. As an immediate consequence of Proposition 5.11, we see, using the notation introduced there, the composition

$$\mathrm{FDA}(\operatorname{Spf}(R^+)) \xrightarrow{\eta^*} \mathrm{RigDA}(S) \xrightarrow{\mathrm{dR}^{\mathrm{FF}}_S} \mathrm{QCoh}(\mathcal{X}(S))^{\mathrm{op}}$$

is simply given by composing $\widetilde{\mathcal{D}}(R^+)$ with $\mathrm{dR}_{\mathcal{X}(S)}$. Informally speaking, formal motives over R^+ uniquely lift to the Witt vectors of R^+, and the de Rham–Fargues–Fontaine realization of their generic fiber can be deduced from the overconvergent de Rham cohomology of this lift after inverting p.

Here is a variant without topology, i.e., on *discrete* rings. Let A be a perfect $\mathbb{F}_p$-algebra and $S = \operatorname{Spa}(R, R^+) \in \mathrm{Aff\,Perf}_{/A}$, that is, an affinoid perfectoid space with a map $f : S \to \operatorname{Spa}(A)$ (A is endowed with the discrete topology). The composition

$$\mathrm{DA}(\operatorname{Spec}(A)) \cong \mathrm{FDA}(\operatorname{Spf}(A)) \xrightarrow{f^*} \mathrm{FDA}(\operatorname{Spf}(R^+)) \xrightarrow{\eta^*} \mathrm{RigDA}(S) \xrightarrow{\mathrm{dR}^{\mathrm{FF}}_S} \mathrm{QCoh}(\mathcal{X}(S))^{\mathrm{op}}$$

defines a functor

$$\mathrm{Rig}^{\mathrm{FF}}_{A,S} : \mathrm{DA}(\operatorname{Spec}(A)) \to \mathrm{QCoh}(\mathcal{X}(S))^{\mathrm{op}}$$

which is compatible with pullbacks along maps $g : S' \to S$ in $\mathrm{Aff\,Perf}_{/A}$. By Theorem 6.3, the restriction of the functor above to fully dualizable objects takes values in the infinity-subcategory $\mathcal{P}(\mathcal{X}(S))$ made of perfect complexes on $\mathcal{X}(S)$. In particular, we obtain for each $S \in \mathrm{Aff\,Perf}_{/A}$ a functor

$$\mathrm{Rig}^{\mathrm{FF}}_{A,S} : \mathrm{DA}(\operatorname{Spec}(A))^{\mathrm{fd}} \to \mathcal{P}(\mathcal{X}(S))^{\mathrm{op}}$$

which is compatible with base change in S. The category $\mathcal{P}(\mathcal{X}(S))$ satisfies v-descent with respect to S (see [Anschütz and Le Bras 2021, Proposition 2.4]). We may then introduce the following.

Definition 6.14. We denote by $\mathcal{P}(\mathcal{X}(\operatorname{Spa}(A)))$ the category

$$\lim_{S \in \mathrm{AffPerf}_{/A}} \mathcal{P}(\mathcal{X}(S)),$$

that is, the category of global sections of the v-stack $\mathcal{P}(\mathcal{X}(-))$ restricted to $\mathrm{AffPerf}_{/A}$.

One may think of $\mathcal{P}(\mathcal{X}(\mathrm{Spa}(A)))$ as the category of perfect complexes over the nonexisting $\mathcal{X}(\mathrm{Spa}(A))$. This category is a priori inexplicit, but receives a functor from a more familiar category, as we now explain.

Definition 6.15. Set $Y_A := \mathrm{Spa}(W(A)[1/p], W(A))$. It is a sheafy adic space ([Scholze and Weinstein 2020, Remark 13.1.2]), endowed with a Frobenius endomorphism φ. We let Isoc_A be the category $(\mathcal{P}(Y_A))^{h\varphi}$ of φ-equivariant perfect complexes on Y_A.

When $A = k$ is a perfect field of characteristic p, objects of Isoc_A are bounded complexes of isocrystals over k, whence the notation. We have for each $S = \mathrm{Spa}(R, R^+) \in \mathrm{Aff\,Perf}_{/A}$ a functor

$$\mathcal{E}_{A,S} : \mathrm{Isoc}_A \to \mathcal{P}(\mathcal{X}(S))$$

induced by the pullback functor on solid quasicoherent sheaves along the (φ-equivariant) map $W(A) \to W(R^+)$. It is functorial in $S \in \mathrm{Aff\,Perf}_{/A}$. Taking the limit over S, we deduce a functor

$$\mathcal{E}_A : \mathrm{Isoc}_A \to \mathcal{P}(\mathcal{X}(\mathrm{Spa}(A))).$$

Remark 6.16. In the case $A = \overline{\mathbb{F}}_p$, the functor $\mathcal{E}_{\overline{\mathbb{F}}_p}$ is an equivalence, as proved in [Anschütz 2023, Theorem 3.5].

Definition 6.17. We let $\mathrm{Rig}_A^{\mathrm{FF}}$ be the functor

$$\mathrm{Rig}_A^{\mathrm{FF}} : \mathrm{DA}(\mathrm{Spec}(A))^{\mathrm{fd}} \to \mathcal{P}(\mathcal{X}(\mathrm{Spa}(A)))^{\mathrm{op}}$$

obtained by taking the limit of the functors $\mathrm{Rig}_{A,S}^{\mathrm{FF}}$ for $S \in \mathrm{Aff\,Perf}_{/A}$.

The functor $\mathrm{Rig}_A^{\mathrm{FF}}$ is nothing surprising: it is simply rigid cohomology in disguise. To make this precise, let us recall the definition of the latter.

Definition 6.18. Let A be a perfect $\mathbb{F}_p$-algebra. The functor

$$\mathrm{DA}(\mathrm{Spec}(A))^{\mathrm{fd}} \to \mathrm{Isoc}_A^{\mathrm{op}}$$

obtained as the restriction to fully dualizable objects of the composition of the Monsky–Washnitzer-type functor

$$\mathrm{DA}(\mathrm{Spec}(A)) \overset{\sigma^*}{\cong} \mathrm{FDA}(\mathrm{Spf}(W(A))) \to \mathrm{FDA}(\mathrm{Spf}(W(A)))^{h\varphi^*} \xrightarrow{\eta^*} \mathrm{RigDA}(Y_A)^{h\varphi^*}$$

with

$$\mathrm{dR}_{X_A}^{h\varphi^*} : \mathrm{RigDA}(Y_A)^{h\varphi^*} \to \mathrm{Isoc}_A^{\mathrm{op}}$$

is called *rigid cohomology* and denoted by $\mathrm{R}\Gamma_R^{\mathrm{rig}}$.

Rigid cohomology of the motive of a proper smooth variety over R is simply crystalline cohomology of its special fiber by Berthelot's comparison result between crystalline cohomology and de Rham cohomology of a lift (see [Bhatt and de Jong 2011, Corollary 3.8] for a short proof).

Again as an immediate consequence of the definitions and of Proposition 5.11, we get:

Proposition 6.19. *Let A be a perfect $\mathbb{F}_p$-algebra. We have a natural isomorphism*

$$\mathcal{E}_A \circ \mathrm{R}\Gamma_A^{\mathrm{rig}} \cong \mathrm{Rig}_A^{\mathrm{FF}}$$

of functors from $\mathrm{DA}(\mathrm{Spec}(A))^{\mathrm{fd}}$ *to* $\mathcal{P}(\mathcal{X}(\mathrm{Spa}(A)))^{\mathrm{op}}$. ☐

In particular, when $A = \overline{\mathbb{F}}_p$, by the equivalence of Remark 6.16, the functor $\mathrm{Rig}_A^{\mathrm{FF}}$ is literally just rigid cohomology.

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On the variation of Frobenius eigenvalues in a skew-abelian Iwasawa tower

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We study towers of varieties over a finite field such as $y^2 = f(x^{\ell^n})$ and prove that the characteristic polynomials of the Frobenius on the étale cohomology show a surprising ℓ-adic convergence. We prove this by proving a more general statement about the convergence of certain invariants related to a skew-abelian cohomology group. The key ingredient is a generalization of Fermat's little theorem to matrices. Along the way, we will prove that many natural sequences of polynomials $(p_n(x))_{n\geq 1} \in \mathbb{Z}_\ell[x]^{\mathbb{N}}$ converge ℓ-adically and give explicit rates of convergence.

Notation 1. We will work throughout over a fixed finite field $\mathbb{F}_q$. A curve C over $\mathbb{F}_q$ refers to a smooth, projective, geometrically connected scheme of dimension 1. The base change to the algebraic closure $\bar{\mathbb{F}}_q$ is denoted by $\overline{C}$. We denote its étale cohomology with $\mathbb{Z}_\ell$ coefficients by $H^1_{\text{ét}}(\overline{C}, \mathbb{Z}_\ell)$. By standard functoriality arguments, it comes endowed with a linear action of the geometric Frobenius σ_q. We fix an auxiliary prime ℓ throughout and for simplicity assume that $\ell > 2$ and $q \equiv 1 \pmod{\ell}$.[1]

1. Introduction

The eigenvalues of the Frobenius on the étale cohomology of a smooth, projective variety over a finite field carry significant arithmetic information. By the Weil conjectures, these eigenvalues are algebraic integers and their absolute values under any complex embedding are understood.

We draw inspiration from Iwasawa theory to study the asymptotic behavior of these eigenvalues in an "Iwasawa tower" and in particular, we show that there is a strong ℓ-adic convergence statement to be made in many natural examples. The Iwasawa algebras arising in this study are noncommutative due

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[1] As usual, the theorems go through if $\ell = 2$ with appropriately stronger hypothesis. For instance, if $\ell = 2$ then we need $q \equiv 1 \pmod{\ell^2}$.

to the nontrivial action of the Frobenius on this monodromy group and we hope that this perspective is interesting too. Let us begin with an example.

Example 2. Consider the smooth projective curves C_n corresponding to the equations

$$Y^2 = X^{2^n} + 1 \text{ over } \mathbb{F}_5.$$

They define a tower $\cdots \to C_2 \to C_1$ with maps $C_{n+1} \to C_n$ defined by $(X, Y) \to (X^2, Y)$. The characteristic polynomial of σ_5 on $H^1_{\text{ét}}(\bar{C}_n, \mathbb{Z}_\ell)$ is

$$f_n(x) := \det(1 - \sigma_2 x \mid H^1_{\text{ét}}(\bar{C}_n, \mathbb{Z}_\ell)) = (1 - 2x + 5x^2) \prod_{i=1}^{n-2} (1 + x^{2^i} 5^{2^{i-1}})^2.$$

Note that $f_{n-1}(x)$ divides $f_n(x)$ and the inverse of the roots of $g_n(x) = f_n(x)/f_{n-1}(x)$ are of the form $\sqrt{5}\zeta$ for ζ a root of unity of order 2^{n-1} for $n \geq 3$. In Section 4, we show that for n sufficiently large, the normalized (by $\alpha \to \alpha/|\alpha|$ so that the complex absolute value is 1) roots of $g_{n+1}(x)$ are exactly all possible ℓ-th roots of the normalized roots of $g_n(x)$.[2]

In fact, we prove the same statement for towers of Fermat curves (from which the above follows) and Artin–Schreier curves. The proof of this statement follows from realizing the roots of $g_n(x)$ as Jacobi sums and using results of Coleman [1987] on identities for Gauss sums (coming from the Gross–Kubota p-adic Gamma function [Gross and Koblitz 1979]).

1.1. ***A congruence on characteristic polynomials.*** This prompts the question of what happens in a more general context. For instance, we could take a map $f : C \to \mathbb{P}^1$ or $f : C \to A$ for A an abelian variety of dimension d and pull back by the following diagrams:

$$\begin{tikzcd} C_n \arrow[r, "f_n"] \arrow[d, "\pi_n"] & \mathbb{P}^1 \arrow[d, "t \to t^{\ell^n}"] \\ C \arrow[r, "f"] & \mathbb{P}^1 \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} C_n \arrow[r, "f_n"] \arrow[d, "\pi_n"] & A \arrow[d, "{[\ell^n]}"] \\ C \arrow[r, "f"] & A \end{tikzcd}$$

We denote the first family of examples by Case A and the second family by Case B. Note that in both the families, the $C_n \to C$ are geometrically (branched) Galois extensions with an abelian Galois group $G_n \cong (\mathbb{Z}/\ell^n\mathbb{Z})^b$ for $b = 1$ or $2d$ in Cases A and B respectively. Note that the G_n themselves have an action of σ_q and this will be crucial.

We define $M_n = H^1_{\text{ét}}(\bar{C}_n, \mathbb{Z}_\ell)/H^1_{\text{ét}}(\bar{C}, \mathbb{Z}_\ell)$, $f_n(x)$ to be the characteristic polynomial of σ_q on M_n and g_n to be the characteristic polynomial $\det(1 - \sigma_q x \mid M_n/M_{n-1})$. It does not seem to be true that g_n determines g_{n+1} as in Example 2. Nonetheless, the following weaker convergence statement is true.

Let k_n be the order of σ_q acting on $\mu_{\ell^n} = \mathbb{G}_m[\ell^n]$ in Case A while in Case B, k_n is a close relative of the order of σ_q acting on $A[\ell^n]$. In particular, it is independent of C and can be made completely explicit. In either case k_n is of the form $\max\{1, \ell^{n-n_0}\}$ with n_0 depending on which case we are considering.

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[2] we note that the complex norm $|\alpha|$ is independent of the embedding to $\mathbb{C}$ by the Weil conjectures

Theorem (Theorem 19). *In the above set up* (*with some mild assumptions on f and q*):

(1) *We have a factorization into monic polynomials*

$$f_m(x) = \prod_{n \le m} g_n(x)$$

where the g_n are independent of m.

(2) *There exist polynomials $h_n(y), \tilde{h}_n(y) \in \mathbb{Z}[y]$ such that, **in Case A***

$$g_n(x) = h_n(x^{k_n}).$$

*while in **Case B***

$$g_n(x) = \tilde{h}_n(x^{k_n}).$$

(3) ***In Case A**, for n sufficiently large so that $k_{n+1} = \ell k_n$* (*Lemma 21*), *we have the ℓ-adic convergence*

$$h_{n+1}(y) \equiv h_n(y) \pmod{\ell^n}.$$

In particular, the following ℓ-adic limit exists in $\mathbb{Z}_\ell[y]$:

$$h_\infty(y) = \lim_{n\to\infty} h_n(y).$$

***In Case B**, for $n \ge n_0$ sufficiently large so that $k_{n+1} = \ell k_n$, we have the congruence*

$$\tilde{h}_{n+1}(y) \equiv \tilde{h}_n^{\ell^{(b-1)}}(y) \pmod{\ell^n}.$$

In particular, the following ℓ-adic limit exists in $\mathbb{Z}_\ell[y]$ with exp, log *defined formally as power series*:

$$\tilde{h}_\infty(y) = \exp\left(\lim_{n\to\infty} \frac{1}{\ell^{(n-n_0)(b-1)}} \log(\tilde{h}_n(y))\right).$$

The first two properties of the theorem are fairly standard and follow from understanding the structure of M_n as a module over $\mathbb{Z}_\ell[G_n, \sigma_q]$ and in particular, depends on σ_q having "large" orbits when acting upon the characters of G_n.[3] The main body of the paper proves a more abstract statement (Theorem 26) about the convergence of certain invariants of a nonabelian cohomology group which implies the third part of the above theorem on the towers of curves.

We note that this more abstract statement can be applied to many more geometric contexts than just our two examples of towers of curves above although we do not pursue this in our paper. It applies to any tower of varieties with an action of an abelian group such that the Frobenius action on the cohomology has a "large" orbit. For instance, we could take hypersurfaces of the form

$$f(x_0^{\ell^n}, \dots, x_n^{\ell^n}) = 0 \subset \mathbb{P}^n_{\mathbb{F}_q}.$$

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[3] As a reviewer pointed out, part 2 has been "known for a long time and rediscovered several times", for instance see [Gordon 1979, Lemma 1.1]. For completeness, we give our own proof too.

All the interesting cohomology is concentrated in the middle dimensional cohomology and the above theorem holds for the characteristic polynomial of the Frobenius action on this middle dimensional cohomology group.

We will see in Section 3 that $M_\infty = \varprojlim_n M_n$ is a free module for a certain skew-abelian Iwasawa algebra and in particular, the characteristic polynomials we study are all determined by a Galois cohomology class with coefficients in matrices over a ring of power series. The bulk of this paper consists in studying the ℓ-adic properties of these power series.

A key role in the study of the study of these algebraic properties is played by the following generalization of Fermat's little theorem (conjectured by Arnold [2006] and proven by Zarelua [2008]):

Theorem (Arnold and Zarelua, Theorem 29). *Let A be a $r \times r$ matrix over $\mathbb{Z}_\ell$. Then, the congruence*

$$\operatorname{tr}(A^{\ell^{n+1}}) \equiv \operatorname{tr}(A^{\ell^n}) \pmod{\ell^{n+1}}$$

holds for any prime ℓ and any $n \in \mathbb{N}$.

Arnold's conjecture goes back to before Arnold (Jänichen [1921] and Schur [1937]). For a more recent expository survey and applications to topology and dynamics; see Zarelua [2008]. Arnold's conjecture has since been proven many times in the literature; for instance, see [Mazur and Petrenko 2010]. We give a new proof of a slightly refined statement since we will use a similar technique in proving our main theorem.[4]

To keep notation simple, we state a special (yet nontrivial) case of our general ℓ-adic convergence theorem.

Theorem (Theorems 23 and 26). *Let $F(t)$ be a $r \times r$ matrix with entries in $\mathbb{Z}_\ell[t]$. Suppose that q is a prime such that $q-1$ is divisible by ℓ but not ℓ^2. For each $n \geq 1$, we define the matrix*

$$A_n = \prod_{i=1}^{\ell^{n-1}} F(\zeta_{\ell^n}^{q^i})$$

with characteristic polynomial $p_n(x)$. Then, the limit $p_\infty(x) = \lim_n p_n(x)$ exists and we have the congruence

$$p_{n+1}(x) \equiv p_n(x) \pmod{\ell^n}.$$

We note that even in the simplest case where $r = 1$, the above theorem is not obvious.

1.2. *Some questions for future work.*

We pose a few questions suggested by this work.

Question 3. Our main theorem establishes the existence of ℓ-adic limits $h_\infty(x)$, $\tilde{h}_\infty(x)$ in the two cases. In some simple cases, the $h_n(x)$ are independent of n for n large enough and by the proof of the Weil conjectures, are known to in fact be polynomials over $\mathbb{Z}$ while a-priori $h_\infty(x)$ is only defined over $\mathbb{Z}_\ell$.

Are the roots of $h_\infty(x)$ always transcendental numbers (except in the cases where h_n is eventually constant)?

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[4] In the course of writing this paper, we found essentially the same proof by Qiaochu Yuan in a blog post from 2009.

Question 4. Even if the roots of $h_\infty(x)$ are transcendental, is it possible to describe them using simple ℓ-adic transcendental functions?

Question 5. What information about the original morphism $f : C \to \mathbb{P}^1$ does $h_\infty(x)$ remember? In the classical Iwasawa theory set up, the limiting characteristic polynomials turn out to be equal to various ℓ-adic L-functions up to a unit (by the main conjecture of Iwasawa theory), can we hope for something similar in this case?

Question 6. Let (Λ, σ_q) be as in Section 3 and M a finite, free Λ module with a σ_q semilinear endomorphism $\Phi : M \to M$. It might be possible and interesting to completely classify such endomorphisms Φ in the hope of a more conceptual proof of the main results. This question is reminiscent of Manin's classification of Dieudonne modules [Manin 1963]. Indeed, the $(M \otimes_\Lambda \Lambda_n(v), \Phi)$ form a "compatible" system of an "ℓ-adic analogue of Dieudonne modules" over the "compatible" system of cyclotomic local rings with endomorphism $(\Lambda_n(v), \sigma_q)$ as n varies - this final sentence is purely impressionistic!

Question 7. Let $Q : \mathbb{Z}_\ell^n \to \mathbb{Z}_\ell^n$ be a linear automorphism and for $v \in \mathbb{Z}_\ell^n$, let $k_n(v)$ be the smallest positive integer so that $Q^{k_n(v)} v \equiv v \pmod{\ell^n}$. Let $\lambda : \mathbb{Z}_\ell^n \to \mathbb{Z}_\ell$ be an arbitrary linear form. Does the sequence

$$S_n(\lambda, v) := \sum_{j=1}^{k_n(v)} \zeta_{\ell^n}^{\lambda(Q^{-j} v)}$$

defined in Remark 32 converge to 0 as $n \to \infty$? If so, what is the rate of convergence and is it uniform as v ranges over primitive vectors?

Outline of the paper. For expository reasons, the paper is not presented in strictly logical order. Section 3 is independent of the rest of the paper and its main results (Theorems 23 and 26) are used in proving our main geometric result (Theorem 19). The reader interested in the geometry and willing to take the ℓ-adic analysis on faith can skip Section 3. The reader interested only in the ℓ-adic convergence results can skip Section 2.

2. On the cohomology of a tower of curves

In this section, we reduce Theorem 19 to an abstract statement about the convergence of characteristic polynomials of a sequence of matrices.

We fix an odd prime ℓ and a finite field $\mathbb{F}_q$ with q large enough to be specified soon. For a variety $X/\mathbb{F}_q$, the notation $H^i_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell)$ denotes as usual the étale cohomology of the variety $X \times_{\mathbb{F}_q} \bar{\mathbb{F}}_q$ with $\mathbb{Z}_\ell$ coefficients. The Frobenius σ_q acts on it through a linear automorphism.

2.1. *Two families of Iwasawa towers.*

Definition 8. Let $C/\mathbb{F}_q$ be a curve. We will be interested in the following two classes of towers:

<u>Case A:</u> Given a nonconstant map $f : C \to \mathbb{P}^1$, we can construct extensions $\pi_n : C_n \to C$ by the pull back diagram:

$$\begin{tikzcd}
C_n \arrow[r, "f_n"] \arrow[d, "\pi_n"] & \mathbb{P}^1 \arrow[d, "t \to t^{\ell^n}"] \\
C \arrow[r, "f"] & \mathbb{P}^1
\end{tikzcd}$$

The C_n form an inverse system with an action by the group

$$\Gamma_n = \mathbb{Z}/\ell^n\mathbb{Z} \rtimes \mathbb{Z}$$

where we denote a generator for the first factor by θ (corresponding to $\theta(t) = \zeta_{\ell^n} t$) and a generator for the second factor by σ_q corresponding to the Frobenius operation. They satisfy the commutation identity

$$\sigma_q \theta = \theta^q \sigma_q.$$

We require the $\pi_n : C_n \to C$ to be totally ramified over the preimage $f^{-1}(\{0, \infty\})$ — for instance, this is satisfied if f is unramified over $0, \infty$ or more generally, if the ramification indices of f over $0, \infty$ are coprime to ℓ. This guarantees that the C_n are geometrically irreducible.

<u>Case B:</u> Given an abelian variety $A/\mathbb{F}_q$ of dimension d and a map $f : C \to A$, we construct $\pi_n : C_n \to C$ by the pullback diagram:

$$\begin{tikzcd}
C_n \arrow[r, "f_n"] \arrow[d, "\pi_n"] & A \arrow[d, "{[\ell^n]}"] \\
C \arrow[r, "f"] & A
\end{tikzcd}$$

We require the C_n to be geometrically irreducible, this is achieved for instance if the induced map $\pi_1^{\text{ét}}(f) : \pi_1^{\text{ét}}(C) \to \pi_1^{\text{ét}}(A)$ on the étale fundamental groups is surjective. In this case, the C_n are acted upon by (with $b = 2d$)

$$\Gamma_n = (\mathbb{Z}/\ell^n\mathbb{Z})^b \rtimes \mathbb{Z}.$$

The first factor can be identified with $A[\ell^n]$ and we denote a basis of it by $\alpha_1, \dots, \alpha_d, \beta_1, \dots, \beta_d$ so that the Frobenius σ_q (corresponding to the second factor) acts by a $b \times b$ matrix Q as

$$\sigma_q v = Qv$$

for $v \in A[\ell^n]$. The congruence $Q \equiv \mathrm{I} \pmod{\ell}$ is equivalent to σ_q acting as the identity on $A[\ell]$. This can always be achieved by a finite extension of the base field $\mathbb{F}_q$ and we suppose that q is large enough so that $Q \equiv \mathrm{I} \pmod{\ell}$.[5] Note that 1 is not an eigenvalue of Q since $\sigma_q - 1 : A \to A$ has finite degree equal to $A(\mathbb{F}_q)$.

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[5] If $\ell = 2$, we would need $Q \equiv \mathrm{I} \pmod{\ell^2}$.

Remark 9. These aren't the only cases our main theorem applies to and in fact, we can even generalize to higher dimensions. What is important is that our tower of varieties has an action by a pro-ℓ abelian group as above and that the growth in cohomology is "regular" in the tower so that as a module over the group algebra, the rank of the cohomology groups are constant. For example, we could take Fermat hypersurfaces of the form

$$X_n : \sum_{j=0}^{d} x_j^{\ell^n} = 0 \subset \mathbb{P}^{b+1}$$

with action by $G_n = (\mathbb{Z}/\ell^n\mathbb{Z})^b$. The only interesting cohomology group is in degree $i = b$, in which case it is a rank 1 module over $\mathbb{Z}_\ell[G_n]$ ([Anderson 1987, Theorem 6] for instance) and a straightforward variant of Theorem 14 shows that growth in cohomology is regular.

Remark 10. Note that the automorphism groups Γ_n aren't abelian but they are very close to being abelian, being the extension of an abelian group by the Frobenius action. Therefore one could view this as an example of skew-abelian Iwasawa theory.

In the remainder of this subsection we prove some basic results about the cohomology of these towers (ignoring the Frobenius action initially).

Lemma 11. *For a finite extension of curves $f : X \to Y$, $H^1_{\text{ét}}(\overline{Y}, \mathbb{Z}_\ell)$ is a direct factor of $H^1_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell)$.*

Proof. Since $H^1_{\text{ét}}(\overline{Y}, \mathbb{Z}_\ell)$ is dual to the Tate module $T_\ell(Y)$, it suffices to show the corresponding fact for the Tate modules of X and Y, i.e., we need to show that the natural map $f : T_\ell(X) \to T_\ell(Y)$ is surjective and that the kernel is torsion free.

One easily checks the following composite map

$$\text{Jac}(Y) \xrightarrow{f^*} \text{Jac}(X) \xrightarrow{f_*} \text{Jac}(Y)$$

is simply multiplication by the degree of f, for instance by using an isomorphism $\text{Jac}(X) \cong \text{Pic}(X)$ and computing the map explicitly in terms of divisors supported away from the ramification locus. This shows that the second map is surjective which in turn implies that the map on Tate modules $T_\ell(f) : T_\ell(X) \to T_\ell(Y)$ is surjective.

Moreover, the kernel of $T_\ell(f)$ is torsion free since if $[P_n]_{n\geq 1} \in T_\ell(X)$ mapped to zero, then $P_n \in \ker(f)$ which would imply that $\deg(f) \geq \ell^n$ for all n which is a contradiction. □

When the extension is generically Galois, we can say more.

Lemma 12. *Suppose $f : X \to Y$ is a generically Galois (branched) extension of (smooth, proper) curves with Galois group G. Then, $H^1_{\text{ét}}(\overline{Y}, \mathbb{Z}_\ell)$ is exactly the submodule of $H^1_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell)$ fixed by the G action.*

Proof. Let us first suppose that X, Y are not necessarily proper but that $f : X \to Y$ is unramified. By the Hochschild–Serre spectral sequence,

$$H^r(G, H^s_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell)) \Rightarrow H^{r+s}_{\text{ét}}(\overline{Y}, \mathbb{Z}_\ell).$$

If we want to let $r+s=1$, then we have either $r=0, s=1$ or $r=1, s=0$. But, $H^0_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell) = \mathbb{Z}_\ell$ with the trivial G action and therefore

$$H^1(G, H^0_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell)) = \operatorname{Hom}(G, \mathbb{Z}_\ell) = 0$$

since G is torsion and $\mathbb{Z}_\ell$ is torsion free. This causes the spectral sequence to degenerate at the (1, 1) term and we have the required isomorphism

$$H^1_{\text{ét}}(\overline{Y}, \mathbb{Z}_\ell) \cong H^0(G, H^1_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell)).$$

Now, for a general (branched) $f : X \to Y$, let $T \subset Y$ be the ramification divisor on Y and $f^{-1}(T) = S \subset X$ its preimage in X with $U = X - S$, $V = Y - T$. With this set-up, we have following commutative diagram:

$$\begin{tikzcd}
H^1_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell) \arrow[r, hook] & H^s_{\text{ét}}(\overline{U}, \mathbb{Z}_\ell) \\
H^1_{\text{ét}}(\overline{Y}, \mathbb{Z}_\ell) \arrow[r, hook] \arrow[u, hook] & H^1_{\text{ét}}(\overline{V}, \mathbb{Z}_\ell) \arrow[u, hook]
\end{tikzcd}$$

Note that the cokernels along the horizontal rows have weight 2 (i.e., σ_q acts by q on the cokernel) as can be seen either from the excision long exact sequence or from the Lefschetz fixed point theorem for compactly supported cohomology along with Poincaré duality. On the other hand $H^1_{\text{ét}}(\overline{Y}, \mathbb{Z}_\ell)$, $H^1_{\text{ét}}(\overline{Y}, \mathbb{Z}_\ell)$ are both of weight 1 (by the Weil conjectures, for instance).

The above diagram is G-equivariant since S, T are. Therefore, the G-invariants of $H^1_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell)$ are contained in $H^1_{\text{ét}}(\overline{V}, \mathbb{Z}_\ell)$ but the above weight argument shows that it is in fact contained in $H^1_{\text{ét}}(\overline{Y}, \mathbb{Z}_\ell)$ as required. ☐

Let us return to our specific towers above.

Definition 13. In *Case A*, let $G_n = \mathbb{Z}/\ell^n\mathbb{Z}$ with generator θ while in *Case B*, let $G_n = (\mathbb{Z}/\ell^n\mathbb{Z})^b$ with generators α_i, β_j as discussed before. We also define the group algebra $R_n = \mathbb{Z}_\ell[G_n]$.

By Lemma 11 and 12, $M_n = H^1_{\text{ét}}(\overline{C}_n, \mathbb{Z}_\ell)/H^1_{\text{ét}}(\overline{C}, \mathbb{Z}_\ell)$ is a free $\mathbb{Z}_\ell$ module with an action of R_n described by the following theorem with g_0 the genus of C.

Theorem 14. *Let us define*

$$r = \begin{cases} 2g_0 + s - 2 & \text{in Case } A, \\ 2g_0 - 2 & \text{in Case } B, \end{cases}$$

where in Case A, s is the number of preimages of $0, \infty$ *for the defining map* $f : C \to \mathbb{P}^1$.

As R_n modules, we have an exact sequence

$$0 \to \mathbb{Z}_\ell^r \to R_n^r \to M_n \to 0,$$

where G_n acts trivially on the first term.

As a preliminary to the above theorem, we use Riemann–Hurwitz to compute the dimensions of M_n.

Lemma 15. *Let g_n be the genus of C_n. M_n is a free $\mathbb{Z}_\ell$ module of rank $2(g_n - g_0)$ and in Case A, we have*

$$\dim_{\mathbb{Z}_\ell} M_n = (\ell^n - 1)(2g_0 + s - 2)$$

while in Case B, we have

$$\dim_{\mathbb{Z}_\ell} M_n = (\ell^{bn} - 1)(2g_0 - 2).$$

Proof. By Lemmas 11 and 12, M_n is a free $\mathbb{Z}_\ell$ module. It remains to compute its $\mathbb{Z}_\ell$ rank ($= 2(g_n - g_0)$). In *Case A*, let $S_0, S_\infty \subset C(\bar{\mathbb{F}}_q)$ be the preimages of $0, \infty$ under f so that $s = |S_0| + |S_\infty|$. Note that π_n is only ramified over S_0, S_∞ and by assumption, it is totally ramified to order ℓ^n over these points. By Riemann–Hurwitz, we then have

$$2g_n - 2 = \ell^n(2g_0 - 2) + s(\ell^n - 1) \implies 2(g_n - g_0) = (\ell^n - 1)(2g_0 + s - 2).$$

In *Case B*, π_n is unramified and of degree ℓ^{bn} and therefore, we simply have

$$2g_n - 2 = \ell^{bn}(2g_0 - 2) \implies 2(g_n - g_0) = (\ell^{bn} - 1)(2g_0 - 2). \qquad \square$$

We finish the proof of Theorem 14 by using the Lefschetz fixed point theorem to compute the character of M in terms of fixed points.

Proof. In *Case A*, let $g \in G_n$ be nontrivial. Since g is not the identity, the only points it fixes on C_n are the points lying over $0, \infty$ under the map $C_n \to \mathbb{P}^1$. In the notation of the previous lemma, there are s such points in total and the local index at each point is $+1$. Moreover, g acts trivially on the degree 0, 2 cohomology groups. Therefore, by the Lefschetz fixed point formula

$$\operatorname{tr}(g \mid H^1_{\text{ét}}(\overline{C}_n, \mathbb{Z}_\ell)) = 2 - s$$

and since G acts trivially on C_0,

$$\operatorname{tr}(g \mid H^1_{\text{ét}}(\overline{C}_n, \mathbb{Z}_\ell)) - \operatorname{tr}(g \mid H^1_{\text{ét}}(\overline{C}, \mathbb{Z}_\ell)) = -(2g_0 + s - 2) = -r.$$

On the other hand, the identity $\mathrm{id} \in G_n$ of course acts trivially so that

$$\operatorname{tr}(\mathrm{id} \mid H^1_{\text{ét}}(\overline{C}_n, \mathbb{Z}_\ell)) - \operatorname{tr}(\mathrm{id} \mid H^1_{\text{ét}}(\overline{C}, \mathbb{Z}_\ell)) = 2(g_n - g_0) = r(\ell^n - 1)$$

where the final equality is by the previous lemma.

In *Case B*, any $g \neq \mathrm{id} \in G_n$ acts on the abelian variety A by a nontrivial translation and hence has no fixed points on either C_n or A. As before, by the Lefschetz fixed point theorem

$$\operatorname{tr}(g \mid H^1_{\text{ét}}(\overline{C}_n, \mathbb{Z}_\ell)) - \operatorname{tr}(g \mid H^1_{\text{ét}}(\overline{C}, \mathbb{Z}_\ell)) = 2 - 2g_0 = -r.$$

The identity element has trace equal to

$$\dim H^1_{\text{ét}}(\overline{C}_n, \mathbb{Z}_\ell) - \dim H^1_{\text{ét}}(\overline{C}, \mathbb{Z}_\ell) = 2(g_n - g_0) = r(\ell^{bn} - 1).$$

If we then examine the exact sequence

$$0 \to \mathbb{Z}_\ell^r \to R_n^r \to X \to 0,$$

we see that X has the character we computed above in both cases for M_n proving that $X \cong M_n$ as G_n representations. $\square$

Remark 16. As an immediate corollary of the above theorem, we notice that in both *Case A* and *B*, for every nontrivial character $\chi : G_n \to \bar{\mathbb{Z}}^\times$, the corresponding eigenspace $M_n(\chi)$ of $M_n \otimes \mathbb{Z}_\ell[\zeta_{\ell^n}]$ is of dimension $2g_0 + s - 2$ and $2g_0 - 2$ in the two cases respectively. In particular it is independent of n and we call the characters appearing in $P_n = M_n/M_{n-1}$ "new" or "primitive" characters of level n.

We fix a set of generators $t_1, \dots, t_b$ for $G_n \cong (\mathbb{Z}/\ell^n\mathbb{Z})^b$ and identify characters χ of G_n by vectors $v = (v_1, \dots, v_b) \in (\mathbb{Z}/\ell^n\mathbb{Z})^b$ by defining $\chi_v(t_i) = t_i^{v_i}$. Under this identification, primitive characters correspond exactly to primitive vectors as defined below in Definition 18. We denote the eigenspace of χ_v by $M_n(v)$.

The exact sequence in the above theorem implies that M_n is not a free R_n module but nevertheless, the inverse limit $M_\infty := \lim_n M_n$ is a free module over $\Lambda = \mathbb{Z}_\ell[[T_1, \dots, T_b]] = \lim_n R_n$.

Lemma 17. *Let $\theta_1, \dots, \theta_b$ be the generators of $G_n = (\mathbb{Z}/\ell^n\mathbb{Z})^b$ as above. Then the projective limit $M_\infty := \varprojlim_n M_n$ is a free module of rank r over $\Lambda = \mathbb{Z}_\ell[[T_1, \dots, T_b]]$. The Frobenius σ_q acts* ***semi****linearly on M_∞, i.e., σ_q is $\mathbb{Z}_\ell$ linear and satisfies*

$$\sigma_q \circ (1 + T_i) = \sigma_q(1 + T_i) \circ \sigma_q$$

where we identify $1 + T_i$ with θ_i so that σ_q acts on $1 + T_i$ through its action on $\varprojlim_n G_n$.

Proof. By the above theorem, we have the following identification as $\mathbb{Z}_\ell[G_n]$-modules

$$M_n \cong \left(\frac{\mathbb{Z}_\ell[\theta_1, \dots, \theta_b]}{\theta_1^{\ell^n} = 1, \dots, \theta_b^{\ell^n} = 1, \prod_{i=1}^b (\sum_{j=0}^{\ell^n - 1} \theta_i^j)} \right)^r$$

since $\prod_{i=1}^b (\sum_{j=0}^{\ell^n - 1} \theta_i^j)$ generates the unique 1-dimensional $\mathbb{Z}_\ell$ submodule of $\mathbb{Z}_\ell[G_n]$ with trivial G_n action. Using this explicit presentation, we define a map

$$\Lambda^r = (\mathbb{Z}_\ell[[T_1, \dots, T_b]])^r \to M_\infty$$

by mapping, for each factor, the $T_i \to \theta_i - 1$ in each term in the projective limit. We will prove that this map is an isomorphism. Since the map is defined on each factor, we can assume henceforth that $r = 1$. The kernel of the induced map to M_n is generated by the elements

$$(1 + T_i)^{\ell^n} - 1 = \sum_{j=1}^{\ell^n} \binom{\ell^n}{j} T_i^j \quad \text{for } i = 1, \dots, b$$

and

$$\prod_{i=1}^{b}\Big(\frac{(1+T_i)^{\ell^n}-1}{T_i}\Big)=\prod_{i=1}^{b}\Big(\sum_{j=1}^{\ell^n}\binom{\ell^n}{j}T_i^j\Big).$$

As $n \to \infty$, these elements tend to 0 in the $(\ell, T_1, \dots, T_b)$-adic topology of Λ so that the map $\Lambda \to M_\infty$ is injective. On the other hand, surjectivity is also clear since the θ_i generate G_n, and consequently the $\theta_i - 1$ generate $\mathbb{Z}[G_n]$. The Frobenius action is induced through this morphism, thus completing the proof. ☐

2.2. ***On the distribution of Frobenius eigenvalues in towers of curves.*** In this subsection, we prove that the characteristic polynomials $f_n(x)$ of σ_q on $H^1_{\text{ét}}(\overline{C}_n, \mathbb{Z}_\ell)$ in our two cases satisfy some striking congruences. We will treat the cases uniformly by letting $Q = q$, $b = 1$ in Case A.

Definition 18. For R a discretely valued ring (DVR) or a quotient of a DVR, we call $v \in R^b$ primitive if at least one of its coordinates is a unit. We denote the space of primitive vectors by $\mathcal{P}(R^b)$.

For a primitive vector $v \in H^1_{\text{ét}}(\overline{C}_n, \mathbb{Z}_\ell)$, we define $k_n(v)$ to be the smallest positive integer such that $Q^{k_n(v)}v \equiv v \pmod{\ell^n}$. We define k_n to be the minimum of $k_n(v)$ as v ranges over primitive vectors. Lemma 21 shows the existence of a positive integer β_v such that $k_n(v) = \ell^{n-\beta_v}$ for $n \geq \beta_v$. Moreover, $n_0 = \max_{v \text{ primitive}} \beta_v$ is finite so that $k_n = \ell^{n-n_0}$ for $n \geq n_0$.

Theorem 19. *Let C_n be as in Case A or B of Definition 8 and*

$$f_n(x) = \det(1 - \sigma_q x \mid M_n)$$

be the characteristic polynomial of the Frobenius σ_q acting on $M_n = H^1_{\text{ét}}(\overline{C}_n, \mathbb{Z}_\ell)/H^1_{\text{ét}}(\overline{C}, \mathbb{Z}_\ell)$. It satisfies the following properties:

(1) *We have a factorization into monic polynomials*

$$f_m(x) = \prod_{n\leq m} g_n(x) \tag{1}$$

where the g_n are independent of m.

(2) *There exist polynomials $h_n(y)$, $\tilde{h}_n(y)$ such that,* ***in Case A***

$$g_n(x) = h_n(x^{k_n}). \tag{2}$$

While ***in Case B***

$$g_n(x) = \tilde{h}_n(x^{k_n}). \tag{3}$$

(3) ***In Case A****, for $n \geq n_0$ (Lemma 21), we have the ℓ-adic convergence*

$$h_{n+1}(y) \equiv h_n(y) \pmod{\ell^n}. \tag{4}$$

In particular, the following ℓ-adic limit exists in $\mathbb{Z}_\ell[y]$:

$$h_\infty(y) = \lim_{n\to\infty} h_n(y).$$

In Case B, *for $n \geq n_0$, we have the congruence*

$$\tilde{h}_{n+1}(y) \equiv (\tilde{h}_n(y))^{\ell^{(b-1)}} \pmod{\ell^n}. \tag{5}$$

In particular, the following ℓ-adic limit exists in $\mathbb{Z}_\ell[y]$ with exp, log *defined formally as power series*:

$$\tilde{h}_\infty(y) = \exp\left(\lim_{n\to\infty} \frac{1}{\ell^{(n-n_0)(b-1)}} \log(\tilde{h}_n(y))\right).$$

Remark 20. The Frobenius σ_q is known to act semisimply on the étale cohomology of a curve and conjectured to act semisimply with rational coefficients on any variety over $\mathbb{F}_q$.[6] While the following proof simplifies slightly if we use the semisimplicity of σ_q on M_n, we do not assume this so that the following proof can be adapted more easily to cases where semisimplicity is not known.

Proof. Part 1, i.e., equation (1) is an immediate consequence of Lemma 11 once we define $g_n(x)$ to be the characteristic polynomial of σ_q on $P_n = M_n/M_{n-1}$.

To prove Part 2, i.e., equations (2) and (3), we treat the two cases simultaneously by taking $b = 1$, $Q = q$ in Case A. Recall the notation that, for $v \in \mathbb{Z}_\ell^b$, $M_n(v)$ is the eigenspace of G_n for the character $\chi_v(t_i) = t_i^{v_i}$. The eigenspaces $M_n(v)$ get permuted by σ_q in the following manner:

$$\sigma_q : M_n(v) \to M_n(Q^{-1}v)$$

and therefore $\sigma_q^{k_n(v)}$ is an automorphism of $M_n(v)$. We will prove that a Jordan block of $\sigma_q^{k_n(v)}$ acting on $M_n(v) \subset P_n$ (with eigenvalue $\lambda \neq 0$) corresponds to $k_n(v)$ distinct Jordan blocks of σ_q acting on P_n (with eigenvalues $\mu^{1/k_n(v)}$). Since this claim is independent of passing to an extension, we replace P_n by $P_n \otimes_{\mathbb{Z}_\ell} \overline{Q}_\ell$.

To that end, let $m_1, \dots, m_s \in M_n(v)$ be some generalized eigenvectors of $\sigma_q^{k_n}$ corresponding to a pure Jordan block of eigenvalue λ (possibly defined over an extension $\mathbb{Z}_\ell$) so that

$$\sigma_q^{k_n}(m_{i+1}) = \lambda m_{i+1} + m_i$$

(with the convention that $m_0 = 0$). We will first show that the eigenvector m_1 for $\sigma_q^{k_n(v)}$ corresponds to $k_n(v)$ distinct eigenvectors for σ_q. For $m_i \in M_n(v)$, let $m_{i,j} = \sigma_q^{j-1}(m_i)$ for $j = 1, \dots, k_n(v)$. Note that

$$\sigma_q(m_{i+1,k_n(v)}) = \sigma_q^{k_n(v)}(m_{i+1}) = \lambda m_{i+1,1} + m_{i,1}.$$

For each μ a k_n root of λ, $n_\mu = \sum_{j=1}^{k_n(v)} \mu^{-j} m_{1,j}$ is an eigenvector of σ_q. Indeed, we have

$$\sigma_q(n_\mu) = \sum_{j=1}^{k_n(v)-1} \mu^{-j} m_{1,j+1} + \lambda\mu^{-k_n(v)} m_{1,1} = \mu n_\mu.$$

---

[6] Semisimplicity for abelian varieties.

Therefore, the n_μ are each an eigenvector of σ_q and the subspace $N = \mathrm{span}(n_\mu : \mu^{k_n(v)} = \lambda)$ is stable under σ_q and contains

$$m_{1,j} = \frac{1}{k_n(v)} \sum_{\mu^{k_n(v)}=\lambda} \mu^j n_\mu \quad \text{for } j = 1, \dots, k_n(v).$$

Passing to the quotient P_n/N therefore corresponds to replacing the $m_1, \dots, m_s$ by $m_2, \dots, m_s$ (with m_2 now an eigenvalue of $\sigma_q^{k_n(v)}$) and we continue inductively to show that each m_i corresponds to $k_n(v)$ distinct generalized eigenvectors $n_{i,\mu}$ with eigenvalue μ.

Let $g_{n,v}(x) = \det(\mathrm{I} - \sigma_q x)$ be the characteristic polynomial of σ_q on $N_n(v) = \bigoplus_{i=0}^{k_n(v)-1} M_n(Q^i(v))$. This module has dimension exactly $k_n(v)$ times the dimension of $M_n(v)$ and since for each generalized eigenvector m_i of $M_n(v)$, we have constructed $k_n(v)$ distinct generalized eigenvectors $n_{i,\mu}$ of $N_n(v)$ corresponding to the $k_n(v)$ distinct roots of λ, the $n_{i,\mu}$ together in fact span $N_n(v)$.

The identity

$$\prod_{j=1}^{k_n(v)} (1 - x\mu\zeta_{k_n(v)}^j) = 1 - x^{k_n(v)}\mu^{k_n(v)}$$

then shows that $g_{n,v}(x) = h_{n,v}(x^{k_n(v)})$ for some polynomial $h_{n,v}(y)$ with roots $y = \lambda = \mu^{k_n(v)}$. We note that the above proof in fact computes the $h_{n,v}(x)$ to be exactly the characteristic polynomial of $\sigma_q^{k_n(v)}$ on $M_n(v)$. Since

$$g_n(x) = \prod_{v \in \mathcal{P}(\mathbb{Z}/\ell^n\mathbb{Z})/\sim} g_{n,v}(x)$$

where the product is over a set of representatives for the σ_q action on primitive vectors, the proof of part (2) in Case A is completed by defining

$$h_n(y) = \prod_{v \in \mathcal{P}(\mathbb{Z}/\ell^n\mathbb{Z})/\sim} h_{n,v}(y)$$

and setting $y = x^{k_n}$.

For Case B, we define (again as a product over a similar set of representatives for the σ_q action on primitive vectors)

$$\tilde{h}_n(y) = \prod_{v \in \mathcal{P}(\mathbb{Z}/\ell^n\mathbb{Z})^b/\sim} h_{n,v}(y^{k_n(v)/k_n})$$

so that (with $y = x^{k_n}$)

$$g_n(x) = \prod_{v \in \mathcal{P}(\mathbb{Z}/\ell^n\mathbb{Z})^b/\sim} g_{n,v}(x) = \prod_{v \in \mathcal{P}(\mathbb{Z}/\ell^n\mathbb{Z})^b/\sim} h_{n,v}(x^{k_n(v)}) = \tilde{h}_n(x^{k_n}).$$

Finally, we prove Part (3), i.e., equations (4) and (5). Let us fix a generating set $m_1, \dots, m_r$ for M_n over $\mathbb{Z}_\ell[G_n] \cong \mathbb{Z}_\ell[t_1, \dots, t_b]/(t_i^{\ell^n} - 1 : i = 1, \dots, b)$. Since M_n is not a free $\mathbb{Z}_\ell[G_n]$ module, it might not be completely clear what a generating set should mean. For our purposes, it suffices to choose $m_1, \dots, m_r$ so that under any specialization that maps the t_i to ℓ^n roots of unity, the m_i specialize to a genuine basis over the induced specialization of M_n. That this is indeed possible follows from the explicit description of the M_n as G_n modules in Lemma 17. Such a specialization corresponds to a representation $\chi_v : G_n \to \overline{\mathbb{Q}}$ for $v \in \mathbb{Z}_\ell^b$ and we denote the induced specialization also by $\chi_v : M_n \to M_n(v)$

In terms of the $m_1, \dots, m_r$, σ_q acting on M_n can be represented by some invertible matrix $F(t_1, \dots, t_b)$. From this point on, *we will be concerned only with this matrix* $F(t_1, \dots, t_b)$. Since σ_q skew commutes with the t_i, we have

$$\sigma_q^{k_n(v)} = \prod_{i=1}^{k_n(v)} F(t^{Q^{k_n(v)-i}v}).$$

Therefore, with respect to the basis $\chi_v(m_1), \dots, \chi_v(m_r)$ of $M_n(v)$, the action of $\sigma_q^{k_n(v)}$ corresponds to evaluating the above product using the character χ_v and is represented by the matrix

$$A_n(v) = \prod_{i=1}^{k_n(v)} F(\zeta_{\ell^n}^{Q^{k_n(v)-i}v})$$

of Section 3 (and we note that $A_n(v)$ is independent of our choice of F or the $m_1, \dots, m_r$). As noted above, the $h_{n,v}(y)$ are the characteristic polynomials of $\sigma_q^{k_n(v)}$ on $M_n(v)$ and therefore, correspond to the $p_{n,v}(y)$ in Section 3. We further see that the $\tilde{h}_n(y)$ correspond to the polynomials $r_n(y)$ of Theorem 26 and by this theorem, we have the required congruence:

$$\tilde{h}_{n+1}(y) \equiv \tilde{h}_n(y) \pmod{\ell^n}. \qquad \square$$

3. On the convergence of a skew-abelian Iwasawa theoretic invariant

In this section, we prove a general, abstract result about the convergence of a certain cohomological invariant defined for a skew commutative Iwasawa algebra. The set up is as follows.

We fix an odd prime ℓ and positive integers b, r throughout this section.[7] All cohomology groups in this section represent group cohomology unless indicated otherwise. All congruences in this paper are in $\mathbb{Z}_\ell$ (and hence only concerned with the ℓ-adic valuation) unless explicitly mentioned otherwise.

Let $\Lambda = \mathbb{Z}_\ell[\![T_1, \dots, T_b]\!]$ be the b dimensional Iwasawa algebra and set $t_i = 1 + T_i$. It is a local ring with maximal ideal $\mathfrak{m} = (\ell, T_1, \dots, T_b)$. Note that for $\lambda \in \mathbb{Z}_\ell$, the expression

$$t_i^\lambda = (1 + T_i)^\lambda = \sum_{k \geq 0} \binom{\lambda}{k} T_i^k$$

converges in Λ. For $v = (v_1, \dots, v_b) \in (\mathbb{Z}_\ell)^b$, we define $t^v = (t_1^{v_1}, \dots, t_b^{v_b})$. We suppose that Λ has an endomorphism σ_q acting through a matrix $Q = Q_{ij} \in \mathrm{GL}_b(\mathbb{Z}_\ell)$ in the following way:

$$\sigma_q(t^v) = t^{Qv} \iff \sigma_q(T_i) = \left[\prod_j (1 + T_j)^{Q_{ji}}\right] - 1 \text{ for all } i.$$

---

[7] As usual, the arguments of this paper go through if $\ell = 2$ with minor, standard modifications.

We note that the action is well defined since $\sigma_q(T_i) \in \mathfrak{m}$. For $v \in \mathbb{Z}_\ell^b$, we denote the size of the orbit of v under Q in $(\mathbb{Z}_\ell/\ell^n\mathbb{Z}_\ell)^b$ by $k_n(v)$.[8] We also define

$$k_n = \min_{v \text{ primitive}} k_n(v).$$

Assumption. We suppose henceforth that $Q \equiv \mathrm{I} \pmod{\ell}$ and that Q fixes no vectors.[9]

Lemma 21. *Let v be a primitive vector. Then there exist integers $\alpha \geq 1$, $\beta_v \geq 0$ so that*

$$k_n(v) = \begin{cases} \ell^{n-\alpha-\beta_v} & \textit{if } n \geq \alpha + \beta_v, \\ 1 & \textit{otherwise.} \end{cases}$$

Moreover, there is some (minimal) β_0 such that $\beta_v \leq \beta_0$ for all primitive v.

In particular, we have

$$k_n = \begin{cases} \ell^{n-\alpha-\beta_0} & \textit{if } n \geq n_0 := \alpha + \beta_0, \\ 1 & \textit{otherwise.} \end{cases}$$

Proof. Since $Q \equiv \mathrm{I} \pmod{\ell}$, we have $\log Q = \ell^\alpha X$ for $\alpha \geq 1$ with $X \in M_b(\mathbb{Z}_\ell)$ not divisible by ℓ. Since $\ell \geq 3$,

$$(Q^m - \mathrm{I})v = \exp(m \log Q)v - v = m\ell^\alpha X v + \frac{(m\ell^\alpha)^2}{2} X^2 v + \cdots .$$

Since Q does not fix any vectors, $Xv \neq 0$ so let β_v be the largest value such that $Xv \equiv 0 \pmod{\ell^{\beta_v}}$. We see that $k_n(v)$ is the smallest m so that $(Q^m - \mathrm{I})v$ is divisible by ℓ^n. Since $X^k v \equiv 0 \pmod{\ell^{\beta_v}}$ for any $k \geq 1$ too, the ℓ-adic valuation of $(Q^m - \mathrm{I})v$ is determined by the leading term $m\ell^\alpha X v$ so that

$$k_n(v) = \begin{cases} \ell^{n-\alpha-\beta_v} & \text{if } n \geq \alpha + \beta_v, \\ 1 & \text{otherwise.} \end{cases}$$

It remains to show that there is a uniform upper bound on β_v.

Let $\pi : \mathbb{Z}_\ell^b \to \mathbb{F}_\ell^b$ be the reduction map. The primitive vectors correspond to the subspace $\mathcal{P} = \pi^{-1}(\mathbb{F}_\ell^b - \{0\})$ which is a closed (and open) subset of $\mathbb{Z}_\ell^b$. Therefore $\mathcal{P}$ is compact and by continuity of multiplication by X,

$$X\mathcal{P} = \{Xv : v \in \mathcal{P}\} \subset \mathbb{Z}_\ell^b$$

is compact and closed too. By assumption on Q, $X\mathcal{P}$ does not contain 0 (since this would correspond to a fixed point of Q). This implies that $X\mathcal{P}$ is in fact bounded away from 0, i.e, there is some minimal β_0 so that the image of $X\mathcal{P}$ in $(\mathbb{Z}/\ell^{\beta_0+1}\mathbb{Z}_\ell)^b$ does not contain 0 so that $\beta_v \leq \beta_0$ for every primitive v (and $\beta_0 = \beta_v$ for some primitive v). □

Remark 22. It is easy to see why we need to restrict to v primitive and to Q not having any fixed vectors. If $Qv = v$, then $k_n(v) = 1$ and if $v = \ell^s v_0$, then $k_n(v) = 1$ for $n \leq s$ which is an obstruction to a uniform bound on n.

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[8] i.e., $Q^{k_n(v)} v \equiv v \pmod{\ell^n}$ and $k_n(v)$ is the least such positive integer.

[9] If $\ell = 2$, then we would need to assume that $Q \equiv 1 \pmod{4}$.

3.1. ***A cohomological interpretation.*** Let M be a free Λ module of rank r with a Λ-linear endomorphism $\Phi : M \to M$. Upon picking a basis $m_1, \dots, m_r$ for M, we express Φ as a matrix $F(T_1, \dots, T_b)$ with entries in Λ. We suppose that Φ skew commutes with σ_q in the following sense:

$$\sigma_q \circ F = F(\sigma_q(T_1), \dots, \sigma_q(T_b)) \circ \sigma_q.$$

Note that σ_q acts on $\mathrm{GL}_r(\Lambda)$ through its action on Λ. This data of M and the endomorphism Φ as above gives rise to an element η in the nonabelian cohomology group $H^1(\mathbb{Z}\sigma_q, \mathrm{GL}_r(\Lambda))$ in the following way:

Given a F as above, we can define a cocycle representative by $\eta(\sigma_q) = F \in \mathrm{GL}_r(\Lambda)$. A change of basis by a matrix $P \in \mathrm{GL}_r(\Lambda)$ corresponds to $F \to P(\sigma_q(T_1), \dots, \sigma_q(T_b))FP^{-1}$ which is exactly the boundary action. Therefore, the cohomology class $\eta \in H^1(\mathbb{Z}\sigma_q, \mathrm{GL}_r(\Lambda))$ depends only on (M, Φ).

For a positive integer n and $v = (v_1, \dots, v_b) \in \mathbb{Z}_\ell^b$, note that since $T_i = \zeta_{\ell^n}^{v_i} - 1$ is in the maximal ideal of $\mathbb{Z}_\ell[\zeta_{\ell^n}]$, we can define the quotient

$$\Lambda_n(v) = \frac{\mathbb{Z}_\ell[\![T_1, \dots, T_b]\!]}{(t_1 = \zeta_{\ell^n}^{v_1}, \dots, t_b = \zeta_{\ell^n}^{v_b})}.$$

We note that $\sigma_q^{k_n(v)}$ fixes the ideal $(t_1 - \zeta_{\ell^n}^{v_1}, \dots, t_b - \zeta_{\ell^n}^{v_b}) \subset \mathbb{Z}_\ell[\![T_1, \dots, T_b]\!]$ and thus descends to an endomorphism of $\Lambda_n(v)$.

Henceforth, we fix $\eta \in H^1(\mathbb{Z}\sigma_q, \mathrm{GL}_r(\Lambda))$, $v \in \mathbb{Z}_\ell^b$ and define the following sequence of invariants (implicitly depending on η) taking values in polynomials in one variable:

$$p_{n,-}(y) : v \in H^1(\mathbb{Z}\sigma_q, \mathrm{GL}_r(\Lambda)) \xrightarrow{\text{restriction}} H^1(\mathbb{Z}\sigma_q^{k_n(v)}, \mathrm{GL}_r(\Lambda_n(v))) \xrightarrow{\text{char poly}} \Lambda_n(v)[y] \ni p_{n,v}(y)$$

where for the first map, we restrict along $\mathbb{Z}\sigma_q^{k_n(v)} \subset \mathbb{Z}\sigma_q$ and push forward along the quotient $\mathrm{GL}_r(\Lambda) \to \mathrm{GL}_r(\Lambda_n(v))$ and for the second map, since $\sigma_q^{k_n(v)}$ acts trivially on $\mathrm{GL}_r(\Lambda_n(v))$, we have

$$H^1(\mathbb{Z}\sigma_q^{k_n(v)}, \mathrm{GL}_r(\Lambda_n(v))) = \mathrm{Hom}(\mathbb{Z}\sigma_q^{k_n(v)}, \mathrm{GL}_r(\Lambda_n(v)))/\text{conjugacy} = \mathrm{GL}_r(\Lambda_n(v))/\text{conjugacy}$$

which shows that the characteristic polynomial is a well defined invariant. Tracing through the definition in terms of the value of $F = \eta(\sigma_q)$ for $\eta \in H^1(\mathbb{Z}\sigma_q, \mathrm{GL}_r(\Lambda))$, $p_{n,v}(y)$ has the following explicit formula. For $v \in \mathbb{Z}_\ell^b$, we denote $F(t_1 = \zeta_{\ell^n}^{v_1}, \dots, t_b = \zeta_{\ell^n}^{v_b})$ by $F(\zeta_{\ell^n}^v)$ and define

$$A_n(v) := F(\zeta_{\ell^n}^{Q^{k_n(v)-1}v}) \cdots F(\zeta_{\ell^n}^v) = \prod_{i=1}^{k_n} F(\zeta_{\ell^n}^{Q^{-i}v}) \tag{6}$$

where we implicitly use that $Q^{k_n(v)}v \equiv v \pmod{\ell^n}$ for the second equality. The characteristic polynomial of $A_n(v)$ is exactly

$$p_{n,v}(y) = \det(\mathrm{I} - yA_n(v)).$$

Equivalently, it is the characteristic polynomial of $\sigma_q^{k_n(v)}$ acting on $\Lambda_n(v)$.

As the main results of this section, we will prove two ℓ-adic convergence results regarding the sequence of polynomials $p_{n,v}(y)$ as $n \to \infty$.

Theorem 23. *Suppose that $Q = q\,\mathrm{I}$ is a scalar matrix. For n sufficiently large so that $k_{n+1} = \ell k_n$, the characteristic polynomials satisfy the congruence*

$$p_{n+1,v}(y) \equiv p_{n,v}(y) \pmod{k_{n+1}}.$$

Remark 24. Unfortunately, this strong congruence is not true in general if Q is not scalar (even if $r = 1$) as the following example shows. Take $\ell = 5$, $q_1 = 6$, $q_2 = 11$ and let Q be the diagonal matrix with entries q_1, q_2. Take $F(t_1, t_2) = 1 + t_1^3 t_2$ and $v = (1, 1) \in \mathbb{Z}_\ell^2$. Computation shows that $A_3(v) = 49$, $A_2(v) = 7$ so that the difference is only divisible by 7 and not $k_3 = 49$ as the above theorem would suggest. Nevertheless, the computational evidence also suggests that the $A_n(v)$ still converge, just with a slower rate of convergence. As we will see in Remark 32, this will be related to the vanishing of certain sums of roots of unity.

For our geometric applications, the following statement is sufficient. Recall that $\mathcal{P}((\mathbb{Z}/\ell^n\mathbb{Z})^b)$ denotes the space of primitive vectors. It is acted upon by Q and we denote a set of representatives for the orbits of $Q^{\mathbb{Z}}$ acting on $\mathcal{P}((\mathbb{Z}/\ell^n\mathbb{Z})^b)$ by $\mathcal{P}((\mathbb{Z}/\ell^n\mathbb{Z})^b)/\sim$. For $v' = Qv$, we note that $p_{n,v} \equiv p_{n,v'}$ so that $p_{n,v}$ is independent of the choice of representative. The following polynomial depends only on the class η.

Definition 25. With $A_n(v)$ and $p_{n,v}$ as before, define

$$r_n(y) := \prod_{v \in \mathcal{P}((\mathbb{Z}/\ell^n\mathbb{Z})^b)/\sim} p_{n,v}(y^{k_n(v)/k_n}).$$

Theorem 26. *Let Q be any matrix in the kernel of $\mathrm{GL}_b(\mathbb{Z}_\ell) \to \mathrm{GL}_b(\mathbb{F}_\ell)$. For $n \geq n_0$ so that $k_{n+1} = \ell k_n$, We have*

$$r_{n+1}(y) \equiv r_n^{\ell^{b-1}}(y) \pmod{\ell^n}.$$

If $Q = q\,\mathrm{I}$, we have the stronger congruence

$$r_{n+1}(y) \equiv r_n^{\ell^{b-1}}(y) \pmod{\ell^{nb}}.$$

Remark 27. When $b = 1$, the two bounds agree since all matrices are scalar! Note that Theorem 23 only implies the following weaker congruence for $b = 1$:

$$r_{n+1}(y) \equiv r_n(y) \pmod{k_{n+1}}.$$

Remark 28. Numerical evidence shows that these congruences are in fact sharp and the bounds in Theorems 23 and 26 are realized in most cases (but not always!). For instance, with $r = 1, b = 2, \ell = 3$ and $Q = (1 + \ell^2)\,\mathrm{I}$ a scalar matrix, the computation

$$A_3\left(\frac{1}{1-\ell}, \frac{1}{1-\ell}\right) - A_2\left(\frac{1}{1-\ell}, \frac{1}{1-\ell}\right) = 70\ell$$

shows the sharpness of Theorem 23. The same example also shows the sharpness of part 2 of Theorem 26. Let $d \geq 1$ and $\tau_3, \tau_2 \in \mathbb{Z}_\ell$ so that $r_3(y) = 1 - \tau_3 y + \cdots$ and $r_2^\ell(y) = 1 - \tau_2 y + \ldots$. Then

$$\tau_3 - \ell\tau_2 = 560\ell^4.$$

Both the theorems above will depend on the following generalization of Fermat's little theorem to matrices to deal with the case when $r \geq 1$. This generalization of Fermat's little theorem can be seen as the degenerate case of Theorem 23 when $F(T_1, \ldots, T_b) = F_0$ is constant in the T_i.

3.2. ***A generalization of Fermat's little theorem to matrices.*** In this subsection we state and prove a generalization of Fermat's little theorem to the case of matrices. As noted in the introduction, this generalization was conjectured by Arnold [2006] and proved by Zarelua [2008] (and many other following works). Our proof is short and apparently new and therefore we present it here.[10]

Theorem 29 (Arnold and Zarelua). *Let $A \in M_r(\mathbb{Z}_\ell)$. Then*

$$\operatorname{tr} A^{\ell^{n+1}} \equiv \operatorname{tr} A^{\ell^n} \pmod{\ell^{n+1}}.$$

In fact, we also have

$$\det(1 - xA^{\ell^{n+1}}) \equiv \det(1 - xA^{\ell^n}) \pmod{\ell^{n+1}}.$$

Proof. We fix a n. Since we are proving a congruence modulo ℓ^{n+1}, we can replace A by a $r \times r$ matrix with nonnegative integer entries. Let G be the directed multigraph with adjacency matrix A, i.e it has r vertices labeled from 1 to r and there are a_{ij} many edges from i to j.

A closed path of length n on the graph corresponds to a sequence of edges $e_1, \ldots, e_{n-1}$ such that the in-vertex of e_{i+1} is the out-vertex of e_i and the path starts and ends at the same vertex. The quantity $\operatorname{tr} A^n$ has the graph theoretic interpretation of being the number of closed paths of length n on G.

Now, consider a closed path P of length ℓ^{n+1}. The cyclic group of order ℓ^{n+1} acts on the path by permuting

$$(e_1, \ldots, e_{n-1}) \to (e_2, \ldots, e_{n-1}, e_1).$$

Since we are working modulo ℓ^{n+1}, we can ignore those paths P where the orbit by this action has size ℓ^{n+1}. The remaining paths P are exactly those which are concatenations of ℓ copies of a path of length ℓ^n. These are exactly counted by $\operatorname{tr}(A^{\ell^n})$ and therefore we have shown the required congruence

$$\operatorname{tr}(A^{\ell^{n+1}}) \equiv \operatorname{tr}(A^{\ell^n}) \pmod{\ell^{n+1}}.$$

To prove the corresponding congruence for characteristic polynomials, we use the well known determinant to trace exponential identity (as formal power series in x)

$$\det(1 - xB) = \exp\left(-\sum_{d \geq 1} \frac{\operatorname{tr}(B^d)x^d}{d}\right). \tag{7}$$

Let $d = d_0 \ell^e$ for d_0 coprime to ℓ. The congruence above on powers of A^{d_0} then implies that

$$\operatorname{tr}(A^{d\ell^{n+1}}) \equiv \operatorname{tr}(A^{d\ell^n}) \pmod{d\ell^{n+1}}.$$

Since $\ell > 2$, $\alpha \equiv \beta \pmod{\ell^n}$ for $n \geq 1$ implies that $\exp(\alpha) \equiv \exp(\beta) \pmod{\ell^n}$.

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[10] In the course of writing this paper, we found essentially the same proof by Qiaochu Yuan in a blog post from 2009.

To see this, let $t \in \mathbb{Z}_\ell$ such that $\ell^n \mid t$. We will show that $e^t \equiv 1 \pmod{\ell^n}$. Supposing this, we see that

$$\alpha \equiv \beta \pmod{\ell^n} \implies e^{\alpha-\beta} \equiv 1 \pmod{\ell^n} \implies e^\alpha \equiv e^\beta \pmod{\ell^n}$$

since $e^\beta \in \mathbb{Z}_\ell[\![x]\!]$ in our case.

To show that $e^t \equiv 1 \pmod{\ell^n}$, we argue by cases. The terms appearing in the Taylor expansion of $\exp(t)$ are of the form $t^r/r!$. If $r = 1$, then $\ell^n \mid t$. In general, Legendre's formula shows that t^r/r is divisible by $\ell^{\delta_{n,r}}$ for $\delta_{n,r} := nr - r/(\ell-1)$. For $r \geq 2$, note that

$$\delta_{n,r} \geq n \impliedby nr - \frac{r}{2} - n \geq 0 \iff 2n \geq \frac{r}{r-1} \text{ which is always true for } n \geq 1.$$

We finish our proof now by noting that the congruences on the traces implies (by the exponential identity)

$$\det(1 - xA^{\ell^{n+1}}) \equiv \det(1 - xA^{\ell^n}) \pmod{\ell^{n+1}}. \qquad \square$$

3.3. ***A proof of the main congruences.*** In this subsection, we prove Theorems 23 and 26. It will help to set up some notation and make some easy reductions first.

Recall that $F(T_1, \ldots, T_b)$ is a power series in the T_i and to define $A_{n+1}(v)$, we are required to evaluate F at $T_i = \zeta_{\ell^{n+1}}^{v_i} - 1$ (for $i = 1, \ldots, b$) which is in the maximal ideal for the local ring $\mathbb{Z}_\ell[\zeta_{\ell^{n+1}}]$. Since we are interested in a congruence modulo $k_{n+1}(v)$ (or k_{n+1}), we can truncate the F at some finite degree d so that $(\zeta_{\ell^{n+1}} - 1)^d \equiv 0 \pmod{k_{n+1}(v)}$ and suppose that it is a polynomial in the $t_i = T_i + 1$ of the form

$$F = \sum_{I \in \mathbb{N}^b} F_I t_1^{i_1} \cdots t_b^{t_b}$$

where the F_I are $r \times r$ matrices over $\mathbb{Z}_\ell$.

Let $\rho \geq 1$ and for a tuple $J = (I_1, \ldots, I_\rho) \in (N^b)^\rho$, we define $F_J = \prod_{j=1}^{\rho} F_{I_j}$. Using the standard notation $\langle\,\cdot\,,\cdot\,\rangle$ for inner products (and considering $\mathbb{N}^b \subset \mathbb{Z}_\ell^b$), we also define the linear form

$$\lambda_J(v) = \sum_{j=1}^{\rho} \langle I_j, Q^{-j} v \rangle.$$

In terms of this notation, we see that

$$A_{n+1}^d(v) = \sum_{J \in (N^b)^{dk_{n+1}(v)}} F_J \zeta_{\ell^{n+1}}^{\lambda_J(v)}$$

where we have implicitly used that $Q^{k_{n+1}(v)} v \equiv v \pmod{\ell^{n+1}}$. We denote cyclic permutations by

$$\tau(J) = (I_2, I_3, \ldots, I_\rho, I_1)$$

and if $k_n(v) \mid \rho$, we note that

$$\lambda_{\tau J}(v) \equiv \lambda_J(Qv) \pmod{\ell^n}. \tag{8}$$

Notation 30. We will argue by considering each tuple along with its cyclic permutations. To that end, we fix some notation that we will use repeatedly. Let $K = (I_1, \dots, I_\rho)$ be a tuple of length ρ such that it is nonperiodic.[11] For any $\delta = r\rho \in \mathbb{N}$, we define $J_K(\delta) = (I_1, \dots, I_\delta) := (K, \dots, K)$ to be the tuple of length δ where K is concatenated to itself r times. We suppose that $r = r_0\ell^s$ with r_0 coprime to ℓ.

We need one more lemma (which will in fact control the rate of congruence) before the proof of Theorem 23.

Lemma 31. *For $n \geq 0$, suppose ρ is an integer multiple of k_n. For any $w \in \mathbb{Z}_\ell$,*

$$S_{\rho,n}(w) := \sum_{i=1}^{\rho} \zeta_{\ell^n}^{q^i w} \equiv 0 \pmod{\rho}.$$

Proof. Let $w = \ell^m w_0$ with w_0 a unit. Since $q^{k_n} \equiv 1 \pmod{\ell^n}$ and $\rho/k_n \in \mathbb{Z}$, we see that

$$S_{\rho,n}(w) = \sum_{i=1}^{\rho} \zeta_{\ell^{n-m}}^{q^i w_0} = \frac{\rho}{k_{n-m}} S_{k_{n-m},n-m}(w_0)$$

where we use the convention that $\zeta_{-m} = 1$ if $m \geq 0$. Therefore, we can suppose that w is a unit and $\rho = k_n$ without loss of generality. Let $\log q = \ell^\alpha x$ with x a unit so that $q^i - 1 = i\ell^\alpha x \pmod{\ell^{\alpha+1}}$. We now have two cases to consider. Either $\alpha \geq n$ in which case $\zeta_{\ell^n}^{q^i w} = \zeta_{\ell^n}^{w}$ and

$$S_{\rho,n}(w) = \rho\zeta_{\ell^n}^{w} \equiv 0 \pmod{\rho}$$

or $\alpha < n$. In this second case, note that the $\zeta_{\ell^n}^{q^i w}$ are all pairwise distinct for $i \leq k_n = \ell^{n-\alpha}$.

If $1 \leq j < i \leq \ell^{n-\alpha}$, then

$$i - j < \ell^{n-\alpha} \implies \zeta_{\ell^n}^{(q^i - q^j)w} = \zeta_{\ell^n}^{(i-j)w\ell^\alpha x + \cdots} \neq 1.$$

In fact, the $\zeta_{\ell^n}^{q^i w}$ are a complete set of roots for the polynomial $z^{\ell^{n-\alpha}} = \zeta_{\ell^\alpha}^{w}$ and $S_{\rho,n}(w)$ is equal to the linear term of this polynomial which is 0 thus completing the proof. □

Proof of Theorem 23. For this proof, recall that $Q = q\,\mathrm{I}$ is a scalar matrix so that $k_n(v) = k_n$ for all primitive v. We reduce the congruence on the characteristic polynomials $p_{n,v}$ to a congruence on traces using the exponential identity (7)

$$p_{n,v}(y) = \exp\left(-\sum_{d \geq 0} \operatorname{tr}(A_n^d(v))\frac{y^d}{d}\right)$$

as in the proof of Theorem 29. Upon fixing n such that $k_{n+1} = \ell k_n$, it suffices to show the congruence

$$t_n := \operatorname{tr}(A_{n+1}^d(v)) - \operatorname{tr}(A_n^d(v)) \equiv 0 \pmod{dk_{n+1}}.$$

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[11] i.e., the tuples $\tau^i J$ are pairwise distinct for $1 \leq i < \rho$.

We will consider the contributions to t_n from each tuple and its cyclic permutations. In the notation of Notation 30, we take $\delta = dk_{n+1}$ and $J = J_K(dk_{n+1})$ and if $\ell \mid r$, $J_0 = J_K(dk_n)$. Note that

$$\lambda_J(v) = \sum_{i=1}^{\rho} \langle I_i, q^{-i}(1 + q^{-\rho} + \cdots + q^{-(r-1)\rho})v\rangle = \frac{q^{-r\rho} - 1}{q^{-\rho} - 1}\lambda_K(v).$$

Since $q^i - 1 = i\log(q) + \frac{1}{2}(i\log(q))^2 + \cdots$ is exactly divisible by $i\log(q)$, there exists some $w \in \mathbb{Z}_\ell$ so that

$$\frac{q^{-r\rho} - 1}{q^{-\rho} - 1} = \frac{\ell^s\rho}{\rho}w = \ell^s w \Longrightarrow \lambda_J(v) = \ell^s w\lambda_K(v). \tag{9}$$

Moreover, there exists some $y \in \mathbb{Z}_\ell$ so that $q^{-dk_n} \equiv 1 + \ell^n y \pmod{\ell^{n+1}}$ and therefore

$$\sum_{i=1}^{\ell-1} q^{-idk_n} \equiv \ell + \ell^n y \sum_{i=0}^{\ell-1} i \equiv \ell + \ell^{n+1} y \frac{\ell-1}{2} \pmod{\ell^{n+1}} \Longrightarrow \lambda_J(v) \equiv \ell\lambda_{J_0}(v) \pmod{\ell^{n+1}}. \tag{10}$$

We now have to consider two cases:

First, suppose $s = 0$. In this case, the only contributions from tuples that are repetitions of K and its cyclic permutations comes from $\operatorname{tr}(A_{n+1}^d(v))$ and is of the form

$$\sum_{i=1}^{\rho} \operatorname{tr}(F_{\tau^i J})\zeta_{\ell^{n+1}}^{\lambda_{\tau^i J}(v)} = \operatorname{tr}(F_J)\sum_{i=1}^{\rho} \zeta_{\ell^{n+1}}^{\lambda_J(q^i v)} = \operatorname{tr}(F_J)\sum_{i=1}^{\rho} \zeta_{\ell^{n+1}}^{q^i w\lambda_K(v)}$$

where for the first equality, we use that the trace is invariant under cyclic permutations and (8) while for the second equality, we use (9) above and that $s = 0$ by assumption. Now, $r\rho = dk_{n+1}$ and since r is coprime to ℓ, k_{n+1} being a ℓ-power necessarily divides ρ. In fact, ρ and dk_{n+1} have the same ℓ-adic valuation. Thus, we can apply Lemma 31 to conclude

$$\sum_{i=1}^{\rho} \operatorname{tr}(F_{\tau^i J})\zeta_{\ell^{n+1}}^{\lambda_{\tau^i J}(v)} \equiv 0 \pmod{\rho} \Longleftrightarrow \sum_{i=1}^{\rho} \operatorname{tr}(F_{\tau^i J})\zeta_{\ell^{n+1}}^{\lambda_{\tau^i J}(v)} \equiv 0 \pmod{dk_{n+1}}.$$

Next, suppose $s > 0$. In this case, we will have contributions from both $\operatorname{tr}(A_{n+1}^d(v))$ and $\operatorname{tr}(A_n^d(v))$ and they are of the form

$$\begin{aligned}
\sum_{i=1}^{\rho} \operatorname{tr}(F_{\tau^i J})\zeta_{\ell^{n+1}}^{\lambda_{\tau^i J}(v)} - \sum_{i=1}^{\rho} \operatorname{tr}(F_{\tau^i J_0})\zeta_{\ell^n}^{\lambda_{\tau^i J_0}(v)} &= (\operatorname{tr}(F_K^r) - \operatorname{tr}(F_K^{r/\ell}))\sum_{i=1}^{\rho} \zeta_{\ell^n}^{\lambda_J(q^i v)} \\
&= (\operatorname{tr}(F_K^r) - \operatorname{tr}(F_K^{r/\ell}))\sum_{i=1}^{\rho} \zeta_{\ell^{n+1}}^{q^i \ell^s w\lambda_K(v)} \\
&\equiv 0 \pmod{r\rho = dk_{n+1}}
\end{aligned}$$

where the first equality follows from invariance of trace under cyclic permutations and (10) while the second equation follows from (9). For the last congruence, Theorem 29 implies that

$$\operatorname{tr}(F_K^r) - \operatorname{tr}(F_K^{r/\ell}) \equiv 0 \pmod{r}.$$

Moreover, since $dk_{n+1} = r\rho$, we see that ρ is divisible by $dk_{n+1}\ell^{-s}$ and in particular by k_{n+1-s}. Therefore, we can apply Lemma 31 to conclude

$$\sum_{i=1}^{\rho} \zeta_{\ell^{n+1-s}}^{q^i w \lambda_K(v)} \equiv 0 \pmod{\rho}. \qquad \square$$

Remark 32. We remark that the failure of this proof for the general case (see Remark 24) happens exactly at Lemma 31. If Q is not scalar, it is no longer true that $\lambda_J(Qv) = Q\lambda_J(v)$ and consequently, there exist examples (with λ a linear form) such that

$$S_n(\lambda; v) := \sum_{j=1}^{k_n(v)} \zeta_{\ell^n}^{\lambda(Q^{-j}v)} \not\equiv 0 \pmod{k_n(v)}.$$

Nevertheless, the above proof shows that if the $S_n(\lambda_J; v) \to 0$ as $n \to \infty$, then the characteristic polynomials $p_{n,v}(y)$ will also converge as $n \to \infty$. If $\lambda_J(\log(Q)v) \neq 0$, a variation of Lemma 31 still applies to $S_n(\lambda_J; v)$. In fact, numerical evidence supports the vanishing of the limit (for λ_J an arbitrary linear form) but we do not know how to prove it.

From now on, we again let $Q \equiv \mathrm{I} \pmod{\ell}$ be a general matrix. We recall some notation before the proof of Theorem 26. We let $V = \mathbb{Z}_\ell^b$ be a free $\mathbb{Z}_\ell$ module, $V_n = V/\ell^n V$, $\mathcal{P}(V_n)$ to be the primitive vectors in V_n and $\mathcal{P}(V_n)/\sim$ to be a set of representatives under the action by Q. The characteristic polynomials we are interested in are

$$r_n(y) = \prod_{v \in \mathcal{P}((\mathbb{Z}/\ell^n\mathbb{Z})^b)/\sim} p_{n,v}(y^{k_n(v)/k_n}).$$

We also fix n sufficiently large and define (in the notation of Lemma 21)

$$V_e = \left\{ v \in V : \frac{k_n(v)}{k_n} \,\middle|\, \ell^e \iff \beta_v \geq \beta_0 - e \iff Xv \equiv 0 \pmod{\ell^{\beta_0 - e}} \right\} \subset V.$$

By the last equivalent condition, we see that V_e is a (nonempty) submodule of V. Since Q commutes with $\log(Q)$ and hence also X, we see that Q preserves V_e. When $Q = q\,\mathrm{I}$, $V_e = V$ since $\beta_v = \beta_0$ for all primitive v. Also define $V_{e,n}$ to be the image of V_e in $V/\ell^n V$ under the reduction map. Note that, in general, $V_e \not\cong (\mathbb{Z}/\ell^n\mathbb{Z})^c$ for some c and is only a-priori a finite $\mathbb{Z}/\ell^n\mathbb{Z}$ module.

So, let M be an arbitrary finite $\mathbb{Z}_\ell$ module and $n \geq 0$ be the smallest value such that $\ell^n M = 0$. An element $v \in M$ is said to be primitive (generalizing our usual notion) when $\ell^{n-1}v \neq 0$ and the set of primitive elements is denoted $\mathcal{P}(M)$. Our two definitions of primitive are compatible in the sense that

$$\mathcal{P}(V_{e,n}) = \mathcal{P}(V/\ell^n V) \cap V_{e,n}.$$

We need one more lemma (analogous to Lemma 31 and also the determining factor for the rate of convergence) before the proof of Theorem 26.

Lemma 33. *Let M be as above with*

$$\chi : M \to \bar{\mathbb{Z}}_\ell^\times$$

a character. Then, we have the congruence

$$\sum_{v\in\mathcal{P}(M)} \chi(v) \equiv 0 \pmod{\ell^{n-1}}.$$

If $M = (\mathbb{Z}_\ell/\ell^n\mathbb{Z}_\ell)^b$, we have the stronger congruence

$$\sum_{v\in\mathcal{P}(M)} \chi(v) \equiv 0 \pmod{\ell^{(n-1)b}}.$$

Proof. Let $|M| = m$, note that $S_M := \sum_{v\in M} \chi(v) \equiv 0 \pmod{m}$. There are two cases to consider: First, if χ is the trivial character, then $S_M = m$ and the congruence is clear. Second, if χ is not trivial, we can find some $m_0 \in M$ so that $\chi(m_0) \neq 1$ and $S_M = \chi(m_0)S_M \implies S_M = 0 \equiv 0 \pmod{m}$.

Define

$$N = \{v \in M : \ell^{n-1}v = 0\} \subset M$$

so that $\mathcal{P}(M) = M - N$. The module M has size at least ℓ^n and the module N has size at least ℓ^{n-1}. Therefore, we have

$$\sum_{v\in\mathcal{P}(M)} \chi(v) = \sum_{v\in M} \chi(v) - \sum_{w\in N} \chi(w) = S_M - S_N \equiv 0 \pmod{\ell^{n-1}}$$

since $|M| \equiv |N| \equiv 0 \pmod{\ell^{n-1}}$.

If $M = (\mathbb{Z}_\ell/\ell^n\mathbb{Z}_\ell)^b$ so that $N = (\mathbb{Z}_\ell/\ell^{n-1}\mathbb{Z}_\ell)^b$, then the above argument shows the stronger congruence

$$\sum_{v\in\mathcal{P}(M)} \chi(v) = \sum_{v\in M} \chi(v) - \sum_{w\in N} \chi(w) = S_M - S_N \equiv 0 \pmod{\ell^{(n-1)b}}. \qquad \square$$

We now prove Theorem 26, along the same general lines as the proof of Theorem 23.

Proof of Theorem 26. By the exponential identity (7), we have

$$r_n(y) = \exp\biggl(-\sum_{v\in\mathcal{P}(V/\ell^n V)/\sim}\ \sum_{f\geq 0} \frac{\operatorname{tr} A_n^f(v)}{f} y^{fk_n(v)/k_n}\biggr).$$

Let us fix some $d = d_0\ell^e$ (with d_0 coprime to ℓ) and collect the terms corresponding to y^d so that with

$$C_{d,n} = \sum_{v\in\mathcal{P}(V_{e,n})/\sim} \frac{k_n(v)}{dk_n} \operatorname{tr} A_n^{dk_n/k_n(v)}(v), \quad \text{we have } r_n(y) = \exp\biggl(-\sum_{d\geq 0} C_{d,n} y^d\biggr).$$

As in the proof of Theorem 29, the congruence

$$r_{n+1}(y) \equiv r_n^{\ell^{b-1}}(y) \pmod{\ell^n},$$

is reduced to the congruence

$$C_{d,n+1} \equiv \ell^{b-1} C_{d,n} \pmod{\ell^n}.$$

Since a representative $v \in \mathcal{P}(V_{e,n})/\sim$ represents $k_n(v)$ many vectors in $\mathcal{P}(V_{e,n})$ and $A_n(Qv)$ is conjugate to $A_n(v)$ so that their powers have the same traces, we can express $C_{d,n}$ as a sum over *all* primitive vectors by

$$C_{d,n} = \sum_{v \in \mathcal{P}(V_{e,n})} \frac{1}{dk_n} \operatorname{tr} A_n^{dk_n/k_n(v)}(v).$$

Therefore we are reduced to proving the congruence

$$t_n := \sum_{v \in \mathcal{P}(V_{e,n+1})} \frac{1}{dk_{n+1}} \operatorname{tr} A_{n+1}^{dk_{n+1}/k_{n+1}(v)}(v) - \sum_{v \in \mathcal{P}(V_{e,n})} \frac{\ell^b}{dk_{n+1}} \operatorname{tr} A_n^{dk_n/k_n(v)}(v) \equiv 0 \pmod{\ell^n}$$

where we have implicitly used the assumption that n is sufficiently large so that $k_{n+1} = \ell k_n$. Since every vector in $\mathcal{P}(V_{e,n})$ has ℓ^b many lifts to $\mathcal{P}(V_{e,n+1})$, we also have

$$t_n = \frac{1}{dk_{n+1}} \sum_{v \in \mathcal{P}(V_{e,n+1})} \left(\operatorname{tr} A_{n+1}^{dk_{n+1}/k_{n+1}(v)}(v) - \operatorname{tr} A_n^{dk_n/k_n(v)}(v) \right).$$

Note that in the expansion

$$\operatorname{tr} A_n^{dk_n/k_n(v)}(v) = \sum_{J \in (\mathbb{N}^b)^{dk_n}} \operatorname{tr}(F_J) \zeta_{\ell^n}^{\lambda_J(v)},$$

the tuples all have size dk_n independent of v. As before, we will argue by fixing a tuple K and considering the contributions from tuples that are multiples of K and their cyclic permutations. In the notation of Notation 30, let $J = J_K(dk_{n+1})$ and when $\ell \mid r$, $J_0 = J_K(dk_n)$.

First, we suppose that $s = 0$. In this case, the only contribution to t_n from K will be through J and will be of the form

$$\frac{1}{dk_{n+1}} \sum_{v \in \mathcal{P}(V_{e,n+1})} \operatorname{tr}(F_J) \zeta_{\ell^{n+1}}^{\lambda_J(v)}.$$

We note that $\zeta_{\ell^{n+1}}^{\lambda_J(v)}$ is a character on $V_{e,n+1}$ and therefore, by Lemma 33, there exists some $T_{\lambda_J} \in \mathbb{Z}_\ell$ such that

$$\frac{1}{dk_{n+1}} \sum_{v \in \mathcal{P}(V_{e,n+1})} \operatorname{tr}(F_J) \zeta_{\ell^{n+1}}^{\lambda_J(v)} = \frac{\ell^n}{dk_{n+1}} T_{\lambda_J}.$$

Moreover, for any cyclic permutation $\tau^i J$ of J, the corresponding contribution is of the same form as before since Q^i permutes $\mathcal{P}(V_{e,n+1})$

$$\frac{1}{dk_{n+1}} \sum_{v \in \mathcal{P}(V_{e,n+1})} \operatorname{tr}(F_{\tau^i J}) \zeta_{\ell^{n+1}}^{\lambda_J(Q^i v)} = \frac{1}{dk_{n+1}} \sum_{v \in \mathcal{P}(V_{e,n+1})} \operatorname{tr}(F_J) \zeta_{\ell^{n+1}}^{\lambda_J(v)} = \frac{\ell^n}{dk_{n+1}} T_{\lambda_J}.$$

Therefore, the contribution from all the cyclic permutations of J is together equal to

$$\frac{\rho \ell^n}{dk_{n+1}} \operatorname{tr}(F_J) T_{\lambda_J} \equiv 0 \pmod{\ell^n}$$

since the ℓ-adic valuation of ρ is equal to the ℓ-adic valuation of dk_{n+1}.

Next, *suppose* $s > 0$. In this case, the contribution from K will be through J and J_0. Since $v \in V_e$, dk_{n+1} is divisible by $k_{n+1}(v)$ so that $Q^{-idk_n}v = v + i\ell^n Y v$ for some $Y \in M_b(\mathbb{Z}_\ell)$ and

$$(\mathrm{I}+Q^{-dk_n}+\cdots+Q^{-(\ell-1)dk_n})v = \ell v + i\ell^n \sum_{i=0}^{\ell-1} Y v = \ell v + \ell^{n+1}\frac{\ell-1}{2} Y v \equiv \ell v \pmod{\ell^{n+1}}.$$

This implies that

$$\lambda_J(v) = \sum_{i=1}^{dk_n} \langle I_i, q^{-i}(1+Q^{-dk_n}+\cdots+Q^{-(\ell-1)dk_n})v\rangle \equiv \ell\lambda_{J_0}(v) \pmod{\ell^{n+1}}$$

which is equivalent to $\zeta_{\ell^{n+1}}^{\lambda_J(v)} = \zeta_{\ell^n}^{\lambda_{J_0}(v)}$. Therefore, the contribution from J, J_0 in t_n is of the form

$$\frac{1}{dk_{n+1}}(\mathrm{tr}(F_k^r)-\mathrm{tr}(F_k^{r/\ell})) \sum_{v\in\mathcal{P}(V_{e,n+1})} \zeta_{\ell^{n+1}}^{\lambda_J(v)} = \frac{\ell^n}{dk_{n+1}}(\mathrm{tr}(F_k^r)-\mathrm{tr}(F_k^{r/\ell}))T_{\lambda_J}.$$

As above, the cyclic permutations of K give rise to exactly the same contribution so that the total contribution from all cyclic permutations of K is

$$\frac{\rho\ell^n}{dk_{n+1}}(\mathrm{tr}(F_k^r)-\mathrm{tr}(F_k^{r/\ell}))T_{\lambda_J} \equiv 0 \pmod{\ell^n}$$

since $(\mathrm{tr}(F_k^r)-\mathrm{tr}(F_k^{r/\ell}))$ is divisible by r by Theorem 29 and $r\rho = dk_{n+1}$.

When $Q = q\,\mathrm{I}$, the proof is exactly the same as above except that we have the stronger congruence

$$\sum_{v\in\mathcal{P}(V_{e,n+1})} \mathrm{tr}(F_J)\zeta_{\ell^{n+1}}^{\lambda_J(v)} \equiv 0 \pmod{\ell^{nb}}.$$

This follows from the second part of Lemma 33 since $V_e = V = \mathbb{Z}_\ell^b$ in this case and $V_{e,n+1} = (\mathbb{Z}_\ell/\ell^{n+1}\mathbb{Z}_\ell)^b$. □

Remark 34. As one sees from the proof, the modulus of the congruence in Theorem 26 depends on the structure of $V_{e,n+1}$.

4. Explicit examples

In this section, we prove that the normalized eigenvalues of the characteristic polynomials $h_{n,v}(x)$ defined in the proof of Theorem 19 are independent of n for n sufficiently large in the following two examples:

- *Fermat Curves*: This is the family of curves defined by the equation

$$C_n : x^{\ell^n} + y^{\ell^n} + z^{\ell^n} = 0 \subset \mathbb{P}^2.$$

We have maps

$$\cdots \to C_n \to C_{n-1} \to \cdots \to C_1 \cong \mathbb{P}^1$$

with $G_n = \mathrm{Aut}(C_n/C_1) = (\mu_{\ell^n})^2$ and the element (ζ_1, ζ_2) acts by $[x : y : z] \to [x\zeta_1 : y\zeta_2 : z]$.

- *Artin–Schreier Curves*: This is the family of curves defined by the projective closure of the equation
$$C_n : y^q - y = x^{\ell^n} \subset \mathbb{P}^2/\mathbb{F}_q.$$
The automorphism group in this case is $G_n = \mathbb{F}_q \times \mu_{\ell^n}$. An element (a, ζ) in this group acts on the curve by $(x, y) \to (\zeta x, y + a)$.

Remark 35. The results of this section work in somewhat greater generality, for instance we don't need to restrict to Fermat or Artin–Schreier curves of degree a power of ℓ. The results also work for various quotients of these curves such as the superelliptic curves $y^m = x^{\ell^n} + a$.

Since the computations in other cases are exactly analogous, we only deal with the above two cases.

Throughout this section, we identify characters $\chi : \mu_{\ell^n} \to \bar{\mathbb{Z}}_\ell$ with vectors $v \in \mathbb{Z}_\ell$ by $\chi(v) : \zeta_{\ell^n} \to \zeta_{\ell^n}^v$. We also fix a compatible family of additive characters $\psi_n : \mathbb{F}_{q^n} \to \bar{\mathbb{Z}}_\ell$ that satisfy $\psi_n = \mathrm{tr}(\mathbb{F}_{q^n}/\mathbb{F}_q) \circ \psi_1$.

In both of the above families of curves, we can decompose $M_n = H^1_{\text{ét}}(\bar{C}_n, \mathbb{Z}_\ell)$ into one dimensional eigenspaces $M_n(\chi)$ indexed by characters χ of G_n. In the Fermat curve case, the characters are naturally indexed by $v \in (\mathbb{Z}/\ell^n\mathbb{Z})^2$ while in the second case, the characters are indexed by (ψ, v) where ψ is an additive character of $\mathbb{F}_q$ and $v \in \mathbb{Z}/\ell^n\mathbb{Z}$.

Given a character $\chi : \mu_{\ell^n} \to \bar{\mathbb{Z}}_\ell$ and $q \equiv 1 \pmod{\ell^n}$, we can define a multiplicative character of $\mathbb{F}_q^\times$ since the map $x \to x^{(q-1)/\ell^n}$ induces a surjection
$$\mathbb{F}_q^\times \to \mu_{\ell^n}(\mathbb{F}_q) \cong \mu_{\ell^n}$$
and we compose this surjection with χ. By a slight abuse of notation, we also denote this character by χ.

The following well-known theorem [Katz 1981, Corollary 2.2 and Lemma 2.3] identifies the eigenvalues of the Frobenius σ_q on $M(\chi)$ with Gauss and Jacobi sums respectively.

Theorem 36. *We assume that* $q \equiv 1 \pmod{\ell^n}$:

- *For the Fermat curves C_n, let $\eta = (\chi, \chi_2)$ be a character of $G_n = (\mu_{\ell^n})^2$. The eigenvalues of σ_q on the eigenspace $M_n(\eta)$ are given by the Jacobi sum*
$$-J_q(\chi_1, \chi_2) = -\sum_{x \in \mathbb{F}_q} \chi_1(x)\chi_2(1-x).$$
- *For the Artin–Schreier curves, let $\eta = (\psi, \chi)$ be a character of $G_n = \mathbb{F}_q \times \mu_{\ell^n}$. The eigenvalues of σ_q on the eigenspace $M_n(\eta)$ are given by the Gauss sums*
$$-g_q(\psi, \chi) = -\sum_{x \in \mathbb{F}_q} \psi(x)\chi(x).$$

Proof. We sketch the proof for completeness. In the case of Fermat curves, we would like to count points on the affine curve $x^{\ell^n} + y^{\ell^n} = -1$ while in the case of Artin–Schreier curves, we would like to count points on $y^q - y = x^{\ell^n}$.

We have the identities

$$\sum_{\chi:\mathbb{F}_q^\times\to\mu_{\ell^n}} \chi(x) = \begin{cases} \ell^n & \text{if } x = y^{\ell^n}, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\sum_{\psi:\mathbb{F}_q\to\mu_q} \psi(x) = \begin{cases} q & \text{if } x = y^q - y, \\ 0 & \text{otherwise.} \end{cases}$$

We can use these identities to test if an element $x \in \mathbb{F}_q$ is a ℓ^n-th power or of the form $y^q - y$ and therefore use it to count points.

For the Fermat curve, we have

$$C_n(F_q) = \sum_{z+w=-1} \sum_{\chi_1,\chi_2:\mathbb{F}_q^\times\to\mu_{\ell^n}} \chi_1(x)\chi_2(y)$$

while for Artin–Schreier curves

$$C_n(\mathbb{F}_q) = \sum_{z\in\mathbb{F}_q} \sum_{\psi,\chi} \psi(z)\chi(z).$$

Exchanging the summation, this shows that the point counts on the two curves can be expressed in terms of Jacobi and Gauss sums respectively. Finally, we use the Weil-conjectures to identify eigenvalues of the Frobenius action with Jacobi/Gauss sums by varying over all powers of q. □

Let us return to the set-up of Theorem 19. The roots of the characteristic polynomial $h_{n,v}(x)$ therefore correspond to $(-J_q(\chi_1, \chi_2))^{k_n} = -J_{q^{k_n}}(\chi_1, \chi_2)$ with v corresponding to the character χ_1, χ_2 and similarly for the Gauss sum in the two cases we are interested in. Put another way, we choose the minimal q so that $q - 1$ is exactly divisible by ℓ^n and we are looking for a relation between these values for varying n.

Luckily, the exact statement we need is a result of Coleman [1987] proved using the p-adic Gamma function of Gross and Koblitz [1979]. Stated in our notation and specialized to our needs, [Coleman 1987, Theorem 11] takes the following form:

Theorem 37 (Coleman). *Let $v \in \mathbb{Z}_\ell$, $q = p^f$ be such that ℓ^n exactly divides $q - 1$. In the notation of the previous theorem, we have*

$$g_{q^\ell}(\psi, \chi_{q^\ell}(v)) = g_q(\psi, \chi_q(v))\chi_q(v)(\ell)c_q$$

for $c_q = c_p^f$ and $c_p = (-1)^r p^{(\ell-1)/2}$ where r depends only on ℓ.

Proof. In Theorem 11 of [loc. cit.], take $b = v/\ell^{n+1}$, $d = \ell$. Note that there is exactly one orbit of size ℓ and $c = (\sqrt{-p}^{\,\ell-1}\phi_d(0))^f$, $r = r_\ell + (\ell - 1)/2$ in the notation of that paper. □

The following theorem is an immediate consequence of Coleman's theorem and is the required relation.

Theorem 38. *Suppose that q is such that ℓ^n exactly divides $q-1$. Let $v_1, v_2 \in \mathbb{Z}_\ell$, $\chi_{q^m}(v_i)$ multiplicative characters of $\mu_{\ell^\infty}(\mathbb{F}_{q^m})$ corresponding to v_i and $\psi_n : \mathbb{F}_{q^n} \to \bar{\mathbb{Z}}_\ell$ a compatible series of additive characters as above.*

Then, we have the following identities:

$$\frac{J_q(\chi_q(v_1), \chi_q(v_2))}{q^{1/2}} = \frac{J_{q^\ell}(\chi_{q^\ell}(v_1), \chi_{q^\ell}(v_2))}{q^{\ell/2}} \tag{11}$$

and

$$\frac{g_q(\psi, \chi_q(v))\chi_q(v)(\ell)}{q^{1/2}} = \frac{g_{q^\ell}(\psi, \chi_{q^\ell}(v))}{q^{\ell/2}}. \tag{12}$$

Proof. We first prove (11). We have the well known identity

$$J_q(\chi_1, \chi_2)g_q(\psi, \chi_1\chi_2) = g_q(\psi, \chi_1)g_q(\psi, \chi_2).$$

By Theorem 37, we then have

$$\begin{aligned} J_{q^\ell}(\chi_{q^\ell}(v_1), \chi_{q^\ell}(v_2)) &= \frac{g_{q^\ell}(\psi, \chi_{q^\ell}(v_1))g_{q^\ell}(\psi, \chi_{q^\ell}(v_2))}{g_{q^\ell}(\psi, \chi_1\chi_2)} \\ &= \frac{g_q(\psi, \chi_q(v_1))g_q(\psi, \chi_q(v_2))c_q}{g_q(\psi, \chi_1\chi_2)} \\ &= J_q(\chi_q(v_1), \chi_q(v_2))c_q \end{aligned}$$

where $q = p^f$. Since $c_q = \pm q^{(\ell-1)/2}$, we recover (11) *up to a sign* by dividing by $q^{\ell/2}$. Finally, upon reducing Theorem 23 $\pmod{\ell}$, we note that the normalized eigenvalues are all congruent $\pmod{\ell}$ and therefore the sign has to be $+1$.

Equation (12) follows in exactly the same manner from Theorem 37. ☐

Remark 39. We note that the above theorem is in exact accord with Case A, Theorem 19 since in the notation of that theorem, it shows that the roots of $h_{n+1}(y)$ are equal to the roots of $h_n(y)$. In other words, we not only have a congruence $h_{n+1}(y) \equiv h_n(y) \pmod{\ell^n}$, we have an equality $h_{n+1}(y) = h_n(y)$ in the two cases considered in this section.

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Limit multiplicity for unitary groups and the stable trace formula

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We give upper bounds on limit multiplicities of certain nontempered representations of unitary groups $U(a,b)$, conditionally on the endoscopic classification of representations. Our result applies to some cohomological representations, and we give applications to the growth of cohomology of cocompact arithmetic subgroups of unitary groups. The representations considered are transfers of products of characters and discrete series on endoscopic groups, and the bounds are obtained using Arthur's stabilization of the trace formula and the classification established by Mok, and Kaletha, Minguez, Shin and White.

1. Introduction

Let G be a semisimple Lie group and let $\Gamma \subset G$ be an arithmetic lattice. Such a group is an analogue of the "$\mathbb{Z}$-points of G": is realized as the intersection $\mathcal{G}(\mathbb{Q}) \cap K \subset \mathcal{G}(\mathbb{A}_f)$ for a choice of algebraic group $\mathcal{G}/\mathbb{Q}$ such that $\mathcal{G}(\mathbb{R}) = G$ and K a compact-open subgroups of the finite adelic points of G. A natural invariant to study is the group cohomology $H^i(\Gamma, \mathbb{C})$. Yet beyond some low-rank examples, the dimension of this cohomology has only been computed explicitly for specific instances of Γ; for example [11; 21]. A variant of the problem is to study this question *in towers*: one studies the asymptotic properties of $\dim H^i(\Gamma_n, \mathbb{C})$ for sequences Γ_n of nested subgroups as $n \to \infty$; see for example [10; 12; 35].

A central family of such sequences Γ_n are congruence towers $\Gamma(p^n)$. These are obtained by fixing a suitable prime p and considering sequences of subgroups $K(p^n) = K^p K_p(p^n)$. The group K^p is a fixed compact-open subgroup of $G(\mathbb{A}_f^p)$, the finite adelic points away from p, and

$$K_p(p^n) = \mathcal{G}(\mathbb{Q}_p) \cap \{g \in \mathrm{GL}_n(\mathbb{Z}_p) \mid g \equiv I \mod p_n\}$$

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for a choice of embedding $\mathcal{G}(\mathbb{Q}_p) \hookrightarrow \mathrm{GL}_n(\mathbb{Q}_p)$. The resulting nested sequence of subgroups $\Gamma(p^n) = \mathcal{G}(\mathbb{Q}) \cap K(p^n)$ are referred to as principal p-power congruence towers.

This article is motivated by the study of rates of growth of $\dim H^i(\Gamma(p^n), \mathbb{C})$ as n grows, for $\Gamma(p^n)$ cocompact.

These dimensions can be expressed representation-theoretically by using Matsushima's formula [36]:

$$H^i(\Gamma(p^n), \mathbb{C}) = \bigoplus_{\pi} m(\pi, p^n) H^i(\mathfrak{g}, K; \pi).$$

Here the sum is taken over isomorphism classes of unitary representations of G, the number $m(\pi, p^n)$ is the multiplicity of π in the regular representation of G on $L^2(\Gamma(p^n)\backslash G)$, and $H^i(\mathfrak{g}, K; \pi)$ is the $(\mathfrak{g}, K)$-cohomology of π. Following the work of Vogan and Zuckerman [49], the finitely many representations contributing nontrivially to the above sum are well-understood. Thus the question is reduced to the growth of multiplicities $m(\pi, p^n)$ of cohomological representations.

Multiplicity growth rates are best understood for discrete series representations, which contribute to cohomology only in the middle degree. In that case, DeGeorge and Wallach [19] and later Savin [43] have shown that $m(\pi, p^n)$ grows proportionally to the index $[\Gamma(1) : \Gamma(p^n)]$. This leaves open the question of multiplicity growth for cohomological representations in lower degrees. In general, these are nontempered, and DeGeorge and Wallach show that their multiplicities $m(\pi, p^n)$ satisfy

$$m(\pi, p^n)/[\Gamma(1) : \Gamma(p^n)] \xrightarrow[n\to\infty]{} 0.$$

Sarnak and Xue [42] have predicted upper bounds on growth, interpolating between the rate for discrete series and the constant multiplicity of the trivial representation. Here "$f(n) \ll g(n)$" means that for n large enough, $f(n)$ is bounded by a constant multiple of $g(n)$, and $\ll_\epsilon$ indicates that the implied constants depends on ϵ.

Conjecture 1 (Sarnak and Xue). *Let π be a unitary representation of G and let*

$$p(\pi) = \inf\{p \geq 2 \mid \text{ the } K\text{-finite matrix coefficients of } \pi \text{ are in } L^p(G)\}.$$

Then

$$m(\pi, p^n) \ll_\epsilon [\Gamma(1) : \Gamma(p^n)]^{(2/p(\pi))+\epsilon}.$$

By definition, the representation π is tempered if $p(\pi) = 2$. Thus Sarnak and Xue expect the failure of temperedness to dictate the rate of growth of $m(\pi, p^n)$.

1.1. ***Main Theorem.*** In this article, we give upper bounds on the multiplicity growth of certain cohomological representations of unitary groups. The results are conditional on the endoscopic classification of representations, as discussed in Section 1.2. Let E/F be a CM extension of number fields with $F \neq \mathbb{Q}$, and $\mathfrak{p}$ a prime of F such that the cardinality $\mathrm{Nm}(\mathfrak{p})$ of the residue field is large enough, see Section 4.2. Let $a \leq N/2$ and let G be a unitary group defined from a Hermitian form of signature $((a, N-a), (N, 0), \ldots, (N, 0))$ relative to E/F. Finally, let $\Gamma(\mathfrak{p}^n)$ be a sequence of principal level cocompact lattices in G, defined in Section 4. Denote by $\nu(n)$ the unique n-dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{C})$. Our main theorem concerns cohomological representations π of G satisfying the following two conditions:

(i) π belongs to a cohomological Arthur packet associated to a parameter ψ with $\psi\,|_{\mathrm{SL}_2(\mathbb{C})} = \nu(2k) \oplus \nu(1)^{N-2k}$.

(ii) π does not appear in any other Arthur packet.

Such π are endoscopic transfers of products of discrete series with characters, from endoscopic groups of G of the form $U(2k) \times U(N-2k)$. They exist if $a+k \geq N/2$. For example, they include a family π_k described in Section 5 and which contributes to cohomology in degrees

$$i = i(N, a, k) = \begin{cases} ((N-2)/2)^2 + 2a - k^2 & N \text{ even,} \\ ((N-1)/2)^2 + a - k^2 & N \text{ odd.} \end{cases}$$

We recall that $f(n) \ll g(n)$ means that for n large enough, f is bounded by a constant multiple of g.

Theorem 2. *Assume the endoscopic classification of representations for unitary groups stated in* [26]*. Let* $\Gamma(\mathfrak{p}^n)$ *be a tower of principal level cocompact lattices in* $G = U(a, N-a)$*, such that the size* $\mathrm{Nm}(\mathfrak{p})$ *of the residue field is large enough. Let* $N/2 > k \geq N/2 - a$*, and let* π *be a cohomological representation of* G *satisfying properties* (i) *and* (ii) *above. Then*

$$m(\pi, \mathfrak{p}^n) \ll \mathrm{Nm}(\mathfrak{p}^n)^{N(N-2k)}.$$

In particular, Conjecture 1 holds for π.

Our method of proof leads us to believe that these bounds are sharp, in the sense that one should be able to achieve them for a suitable choice of lattices. Indeed, our strategy is to decompose the multiplicity count and show that the leading term comes from a smaller group for which exact asymptotics are known. We expect that the other terms can be made to oscillate and not contribute in the limit. For $G = U(2, 1)$, this type of method was carried out successfully by Simon Marshall [34].

Our representations do not account for all of the cohomology, but in some low degrees, we expect them to do so asymptotically. For example, the smallest nonzero degree i for which $H^i(\Gamma(\mathfrak{p}^n), \mathbb{C})$ is nontrivial is $i = a$. When N is odd, the representations associated to $k = (N-1)/2$ contribute asymptotically all the cohomology in degree a, yielding the following bounds.

Corollary 3. *Under the assumptions of Theorem 2, assume additionally that* N *is odd. Then*

$$\dim H^a(\Gamma(\mathfrak{p}^n), \mathbb{C}) \ll \mathrm{Nm}(\mathfrak{p}^n)^N.$$

In order to describe the more general range of degrees in which we predict that the representations we can control contribute all the cohomology, we state our main technical theorem. It concerns bounds on limit multiplicity for representations belonging to a prescribed archimedean Arthur packet. This result does not require that the representations be cohomological, and our most general limit multiplicity result is the following.

Theorem 4. *Under the assumptions of Theorem 2, let ψ_∞ be an Arthur parameter with regular infinitesimal character and such that $\psi_\infty\,|_{\mathrm{SL}_2(\mathbb{C})}= \nu(2k) \oplus \nu(1)^{N-2k}$. Let $\pi_\infty \in \Pi_{\psi_\infty}$, and let $\Psi_{\psi_\infty,1}$ be the set of Arthur parameters for G whose specialization at infinity is ψ_∞, and associated to representations with trivial central character. Then*

$$\sum_{\psi\in\Psi_{\psi_\infty,1}} \sum_{\pi=\pi_\infty\otimes\pi_f\in\Pi_\psi} m(\pi)\dim \pi_f^{K(\mathfrak{p}^n)} \ll \mathrm{Nm}(\mathfrak{p}^n)^{N(N-2k)}, \tag{1}$$

where $m(\pi)$ denotes the multiplicity of π in $L^2_{\mathrm{disc}}(G(F)\backslash G(\mathbb{A}_F), 1)$.

These types of Arthur parameters seem to control the growth of certain degrees of cohomology. The combinatorics of intersections between various Arthur packets rapidly get complicated, but here is a sample of behavior we expect.

Conjecture 5. *Let $G = U(N-a, a)$ be as above. Then:*

(i) *The representations belonging to Arthur packets attached to parameters ψ with $\psi\,|_{\mathrm{SL}_2}= \nu(N-\ell) \oplus \nu(1)^\ell$ contribute asymptotically all the cohomology in degrees $a\cdot\ell$ for $0 \le \ell \le N-2(a-1)$.*

(ii) *For these degrees,*

$$\dim H^{a\cdot\ell}(\Gamma(\mathfrak{p}^n), \mathbb{C}) \ll \mathrm{Nm}(\mathfrak{p}^n)^{N\cdot\ell}.$$

The range of degrees to which the conjecture applies is larger for smaller values of a, i.e., when G is farther from being quasisplit. For $a = 1$, Marshall and Shin [35] proved (ii) under some assumptions on $\mathfrak{p}$, and conjectured (i).

1.1.1. *Outline of the proof.* The results are proved in the framework of endoscopy, Arthur parameters, and the stable trace formula. The theorem is a consequence of the endoscopic classification of representations for unitary groups. The classification is a result of Mok [38] if the group G is quasisplit, and of Kaletha, Minguez, Shin and White [26] for inner forms, building on the seminal work of Arthur [5]. It gives a decomposition of the regular representation of $G(\mathbb{A}_F)$ on the discrete spectrum

$$L^2_{\mathrm{disc}}(G(F)\backslash G(\mathbb{A}_F)) \simeq \bigoplus_{\psi} \bigoplus_{\pi\in\Pi_\psi} m(\pi)\pi$$

where the irreducible summands $\pi = \otimes'_v \pi_v$ are automorphic representations; they appear in the discrete spectrum with multiplicity $m(\pi)$. This decomposition is given in terms of Arthur packets Π_ψ indexed by Arthur parameters ψ. These parameters are formal objects

$$\psi = \boxplus_i(\mu_i \boxtimes \nu(m_i))$$

where each μ_i is a cuspidal automorphic representation of GL_{n_i} and $\nu(m_i)$ is the unique irreducible m_i-dimensional representation of $\mathrm{SL}_2(\mathbb{C})$. Such a ψ is associated to a packet of representations of a unitary group of rank N if $\sum_i n_i m_i = N$ and if ψ is self-dual in a suitable sense. The parameters stand in for homomorphisms

$$\psi : L_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G$$

where ${}^LG$ is the L-group of G and L_F is the Langlands group of F, an object whose existence is at the present moment only hypothetical. Despite Arthur parameters being purely formal objects, one can consider the restriction $\psi\,|_{\mathrm{SL}_2(\mathbb{C})} := \oplus_i \nu(m_i)^{n_i}$ which is an actual finite-dimensional representation. The classification of parameters in terms of this restriction plays a central role in our argument, and we refer to the group $\mathrm{SL}_2(\mathbb{C})$ used to build Arthur parameters as the "Arthur SL_2".

Endoscopy is a specific instance of the principle of functoriality in the Langlands program. It concerns certain groups H, the so-called endoscopic groups of G, and states that if ψ factors through an embedding ${}^LH \hookrightarrow {}^LG$, then there must be trace identities between the characters of the representations $\pi \in \Pi_\psi$ and those of representations π^H of H in a corresponding packet Π_ψ^H. The character identities are witnessed through the trace formula $I_{\mathrm{disc},\psi}(f)$. In the case of our parameters with regular infinitesimal character, this distribution computes the trace of convolution by a smooth, compactly supported function f on the subspace of L^2_{disc} spanned by the representations $\pi \in \Pi_\psi$. More specifically, the character identities appear in a decomposition of $I_{\mathrm{disc},\psi}(f)$ referred to as the stabilization of the trace formula (written here in a simplified version for exposition purposes):

$$I_{\mathrm{disc},\psi}(f) = \sum_H S^H_{\mathrm{disc},\psi}(f^H). \tag{2}$$

Here the sum runs over all endoscopic groups H such that ψ factors through ${}^LH$. The distributions $S^H_{\mathrm{disc},\psi}(f^H)$ are stable, meaning that they satisfy a strengthening of the conjugacy-invariance property of characters of representations.

The summands $S^H_{\mathrm{disc},\psi}(f)$, initially defined inductively, can be expanded explicitly as linear combinations of traces $\operatorname{tr}\pi(f)$ of the representations $\pi \in \Pi_\psi$; this is the so-called stable multiplicity formula. We write here a simplified version of the stable multiplicity formula in which we have omitted constants which can be ignored in the asymptotic questions we are concerned with

$$S^H_{\mathrm{disc},\psi}(f^H) = \sum_{\pi\in\Pi_\psi} \xi(\pi, H)\operatorname{tr}\pi(f). \tag{3}$$

The coefficients $\xi(\pi, H)$ arise from characters of a 2-group $\mathcal{S}_\psi$, the group of connected components of the centralizer of the image of ψ. More precisely, there are two mappings

$$\{\text{representations } \pi \in \Pi_\psi\} \to \{\text{characters of } \mathcal{S}_\psi\},$$
$$\{H \text{ such that } \psi \text{ factors through } {}^LH\} \to \{\text{elements of } \mathcal{S}_\psi\},$$

the second of which is a bijection. In this way, the coefficient $\xi(\pi, H)$ in the decomposition of the stable term $S^H_{\mathrm{disc},\psi}(f^H)$ is the value of the character associated to π on the group element corresponding to H.

In this context, the steps of the proof of Theorem 2 can be outlined as:

(i) (Section 5.2) Determine the parameters ψ associated to the packets containing cohomological representations. This relies on work of Arthur [4] and Adams and Johnson [1].

(ii) (Section 4.2) Write the dimension of cohomology as $\sum_{\psi} I_{\mathrm{disc},\psi}(f(\mathfrak{p}^n))$ for a specific function $f(\mathfrak{p}^n)$, summing over the parameters ψ computed in the first step.

(iii) (Section 2.6.2) Fix a cohomological parameter ψ. Use the stabilization of the trace formula to decompose

$$I_{\mathrm{disc},\psi}(f(\mathfrak{p}^n)) = \sum_{H} S^{H}_{\mathrm{disc},\psi}(f(\mathfrak{p}^n)^H).$$

(iv) (Section 3.2) By interpreting the coefficients $\xi(\pi, H)$ appearing in the stable multiplicity formula (3) as values of characters of $\mathcal{S}_{\psi}$, conclude that there is a specific endoscopic group H_{ψ} whose contribution bounds that of all the others in (2), i.e., such that

$$I_{\mathrm{disc},\psi}(f(\mathfrak{p}^n)) \leq K(\psi) S^{H_{\psi}}_{\mathrm{disc},\psi}(f(\mathfrak{p}^n)^{H_{\psi}})$$

for a constant $K(\psi)$ computed in terms $\psi\,|_{\mathrm{SL}_2(\mathbb{C})}$ and of the number of irreducible summands of ψ, and which can be uniformly bounded in terms of the rank N of the unitary group. The group H_{ψ} depends only on $\psi\,|_{\mathrm{SL}_2(\mathbb{C})}$. As such it is determined by the parameter ψ_{∞} and ultimately by the choice of cohomological representations.

(v) (Sections 3.3 and 4.4) Bound the stable trace $S^{H_{\psi}}_{\mathrm{disc},\psi}(f(\mathfrak{p}^n)^{H_{\psi}})$ in terms of the multiplicity $m(\pi^{H_{\psi}}, \mathfrak{p}^n)$ for a family $\pi^{H_{\psi}}$ of representations of H_{ψ}. This relies on the fundamental lemma, proved by Laumon and Ngô for unitary groups [33], but also on a variant for congruence subgroups due to Ferrari [17]. In order to control the discrepancy between $S^{H_{\psi}}_{\mathrm{disc},\psi}$ and $I^{H_{\psi}}_{\mathrm{disc},\psi}$, we make use of the notion of hyperendoscopy, also introduced by Ferrari.

(vi) (Sections 4.3 and 5.3) The representations $\pi^{H_{\psi}}$ obtained via steps (i)-(v) from parameters such that $\psi\,|_{\mathrm{SL}_2(\mathbb{C})} = \nu(2k) \oplus \nu(1)^{N-2k}$ are the product of a discrete series representation and a character. Their limit multiplicity is thus known by results of Savin [43], which gives the desired bounds.

Remark 6. Some comments on possible extensions of the result: the proof exploits the fact that for a global A-parameter ψ, the restriction $\psi\,|_{\mathrm{SL}_2(\mathbb{C})}$ is determined locally at any place. Here, archimedean restrictions associated to cohomological representations propagate to global and everywhere-local restrictions and induce slow rates of growth. But there is nothing special about infinity: similar methods could provide information about automorphic representations which belong to Arthur packets with large Arthur SL_2.

The endoscopic classification was of course proved by Arthur [5] for quasisplit orthogonal and symplectic groups, and Taïbi [46] extended the key result used here, namely the stable multiplicity formula, to some classes of inner forms. It is likely the case that similar methods could be a good starting point to provide analogous bounds for these groups.

The restrictions on the types of representations we deal with are rooted in restrictions on the Arthur parameters we consider. These have two simple pieces which witness opposite extreme behaviors when restricted to the Arthur SL_2. This allows us to obtain bounds by “applying endoscopy once”. To extend the results to e.g., representations associated to global parameters with an arbitrary number of simple pieces, one could iterate the inductive process of steps (iv) and (v) and bound representations coming from hyperendoscopic groups, i.e., endoscopic groups of endoscopic groups.

Our proof method is in the lineage of a body of recent work applying the framework of endoscopy to growth of cohomology. Most notably, bounds on multiplicity growth of all nontempered cohomological representations were obtained by Marshall [34] for $G = U(2, 1)$, and Marshall and Shin [35] for $G = U(N, 1)$ and a level $\mathfrak{n}$ divisible by primes splitting in the CM extension used to define the unitary group.

1.2. ***Conditionality.*** Our results are conditional on the endoscopic classification of representations for inner forms of unitary groups, a result which remains to be fully proved in several ways. As explained in the introduction of [26] and in [38, Section 2.6] the classification depends on upcoming work of Chaudouard and Laumon on the weighted fundamental lemma. It also depends, through its dependency on [5], on several papers of Arthur not yet made public. Moreover, the proof of the classification in [26] is not itself complete: in particular, the results appearing here as Theorems 19 and 23 are only proved for generic parameters. A full proof is expected in [27].

2. L-groups, parameters, and the trace formula

2.1. ***Notation.*** Let E/F be a CM extension of number fields with Galois group $\Gamma_{E/F}$, algebraic closure $\bar{F}$ and absolute Galois groups Γ_F and Γ_E. We denote places of F and E by v and w respectively, and let $E_v = E \otimes_F F_v$ for v a place of F. Let $F_\infty = F \otimes_{\mathbb{Q}} \mathbb{R}$, the product of all archimedean completions of F. Let $\mathcal{O}_F$ and $\mathcal{O}_E$ be rings of integers, and $\mathbb{A}_F$ and $\mathbb{A}_E$ be adèle rings, with $\mathrm{Nm} : \mathbb{A}_E \to \mathbb{A}_F$ the norm map. Let $\mathbb{A}_F^f$ be the finite adèles, so that we have $\mathbb{A}_F = F_\infty \times \mathbb{A}_F^f$.

Fix χ_κ for $\kappa \in \{\pm 1\}$, a pair of Hecke characters of E. We fix χ_{+1} to be trivial and the character χ_{-1} is chosen so that its restriction to $\mathbb{A}_F/F^\times$ is the quadratic character associated to E by class field theory.

If F is a field and G/F is a reductive group, we will denote the center of G by Z_G or by $Z(G)$. If F is global, we denote $G(F_v)$ by G_v and $G(F_\infty)$ by G_∞. For $H \subset G(\mathbb{A}_F)$, we use the notation $H_f = H \cap G(\mathbb{A}_F^f)$. The complexified Lie algebra of G_∞ will be denoted $\mathfrak{g}_\infty$.

2.2. *Unitary groups and their L-groups.*

2.2.1. *Quasisplit unitary groups.* We now introduce unitary groups and their L-groups, following the exposition of Kaletha, Minguez, Shin and White [26, Section 0]. Let E/F be a quadratic algebra: either the CM extension introduced above or one of its localizations E_v/F_v, in which case we have $E_v \simeq F_v \times F_v$ when v is split. If this is the case, fix an identification $E_v = F_v \times F_v$. Let $\sigma \in \mathrm{Aut}_F(E)$ be the nontrivial element of $\Gamma_{E/F}$ if E is a field, and the involution $\sigma(x, y) = (y, x)$ if $E = F \times F$. If E is a split quadratic algebra, set $\Gamma_E := \Gamma_F$. Let Φ_N be the antidiagonal $N \times N$ matrix

$$\Phi_N = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ (-1)^{N-1} & & \end{pmatrix}. \tag{4}$$

Let $U_{E/F}(N)$ (sometimes denoted $U(N)$) be the reductive group over F with $U_{E/F}(\bar{F}) \simeq \mathrm{GL}_N(\bar{F})$, with Galois action

$$\tau_N(g) = \begin{cases} \tau(g), & \tau \in \Gamma_E, \\ \mathrm{Ad}(\Phi_N)\tau(g)^{-t}, & \tau \in \Gamma_F \setminus \Gamma_E. \end{cases}$$

We have $U_{E/F}(N, E) = \mathrm{GL}_N(E)$, and we can identify

$$U_{E/F}(N, F) = \{g \in \mathrm{GL}_N(E) \mid \mathrm{Ad}(\Phi_N)\sigma(g)^{-t} = g\}, \tag{5}$$

a quasisplit unitary group with maximal (nonsplit) torus given by the group of diagonal matrices, and a Borel subgroup consisting of upper-triangular matrices. If $E = F \times F$, we have $U(N) \simeq \mathrm{GL}_N$ and we fix an isomorphism to identify them.

If F is global, we consider the various localizations of $U(N, F)$. If v splits in E, we have $U(N, F_v) \simeq \mathrm{GL}_N(F_v)$. Otherwise $U(N, F_v)$ is a quasisplit unitary group over F_v, which determines it uniquely up to isomorphism as we shall see below.

2.2.2. *Inner forms.* An inner form of $U(N)$ is a pair (G, ξ) consisting of an algebraic group G/F together with an isomorphism $\xi : G(\bar{F}) \to U(N, \bar{F})$ such that for all $\sigma \in \Gamma_F$, the automorphism $\xi^{-1} \circ \sigma \circ \xi \circ \sigma^{-1}$ is inner. Though the choice of ξ is always present, it will sometimes be implicit in our notation. We will denote $U(N)$ by G^* when we want to highlight that it is the quasisplit form of G.

Remark 7. In this article, we always require that the inner forms be groups defined with respect to a Hermitian space over E.

We now discuss which possible groups G can arise as inner forms of $U_{E/F}(N)$ in the cases where F is local or global.

2.2.3. *Local inner forms and the Kottwitz sign.* If v is archimedean, the classification of inner forms is well-known: a unitary group over $F_v = \mathbb{R}$ is determined by its signature $p+q = N$, with $U(p, q) \simeq U(q, p)$. Since the notation $U(N)$ is reserved for quasisplit groups, we denote the compact inner form of $U(N, \mathbb{R})$ by $U_N(\mathbb{R})$.

For v nonarchimedean, the classification of unitary groups coming from Hermitian forms over quadratic algebras over F_v is due to Landherr [31]: If N is odd, there is one class of Hermitian forms up to isomorphism, so the group $U(N, F_v)$ is the unique unitary group of rank N. If N is even, there are two isomorphism classes of unitary groups, only one of which (the one containing $U(N, F_v)$) is quasisplit.

One associates to an inner form G_v of $U_{E_v/F_v}(N)$ a Kottwitz sign $e(G_v)$. We record the formulas for $e(G_v)$ as computed in [29]:

- For $F_v = \mathbb{R}$, let $q(G_v)$ be half the dimension of the symmetric space associated to the group G_v. Then $e(G_v) = (-1)^{q(G_v)-q(G_v^*)}$.
- For F_v nonarchimedean, let $r(G_v)$ be the rank of G_v. Then $e(G_v) = (-1)^{r(G_v)-r(G_v^*)}$.

Kottwitz proves [29] that for G defined over a global field, the local signs cancel out and $\prod_v e(G_v) = 1$.

2.2.4. *Global inner forms.* We describe the classification of global forms of unitary groups, following the discussion in Section 0.3.3 of [26]. For N odd, any collection of local inner twists, quasisplit at all but finitely many places, can be realized as the localization of a global inner twist.

Fo N even, the behavior of the place v in E determines cohomological invariants attached to G_v. For each v, we have $H^1(\Gamma_{F_v}, G_v^{*,\mathrm{ad}}) \simeq \mathbb{Z}/2\mathbb{Z}$. If v is split in E, the invariant of G_v depends on the division algebra D_v such that $G_v = \mathrm{Res}_{F_v}^{D_v} \mathrm{GL}_{M_v}$. Since we only consider unitary groups coming from Hermitian forms, this invariant is always 0 for us. At finite nonsplit places, the quasisplit group $U(N)_v$ and its unique inner form correspond respectively to 0 and 1 in $\mathbb{Z}/2\mathbb{Z}$. At the infinite places, the signature (p, q) determines the invariant $N/2 + q \in \mathbb{Z}/2\mathbb{Z}$. For a collection of local G_v to come from a global unitary group, the all but finitely many nonzero invariants associated to G_v must sum up to zero. Consequently, we have:

Lemma 8. *Let E/F be a CM extension of number fields. There exists an inner form G of $U_{E/F}(N)$ with any choice of signature at the infinite places. Moreover G can be chosen to be quasisplit outside of a set of places of size at most* 1.

Remark 9. The authors of [26] work with a refinement of the notion of inner form. Recall that isomorphism classes of inner forms of G are in bijection with $H^1(\Gamma_F, G^{\mathrm{ad}})$. In addition to this, they introduce the notion of pure inner form, a triple consisting of G, the map ξ, and a cocycle $z \in Z^1(\Gamma_F, G)$ compatible with the inner twist ξ. The map sending a pure inner form to z induces a bijection between isomorphism classes of pure inner forms and $H^1(\Gamma_F, G)$. Inner forms of $U(N)$ which can be realized as pure inner forms are those coming from a Hermitian space, i.e., precisely the groups we work with. We will point out dependency on z whenever it appears: in the normalization of transfer factors, and the pairings associated to local Arthur packets. Due to our rather rudimentary use of the stable trace formula, the choice of pure inner form does not affect our results.

2.2.5. *L-groups.* Throughout, we will work with the Weil group version of the L-group, primarily because it is well-suited to our description of local parameters. In terms of the actual definition of the L-group, this choice is purely cosmetic as the Galois actions involved will always factor through a quotient of order at most 2.

For G/F with F either local or global, fix a root datum. The L-group of G is a semidirect product

$$^{L}G = \hat{G} \rtimes W_F.$$

Here the group $\hat{G}$ is the complex dual group of G, and the action of W_F on $\hat{G}$ is induced by the Galois action on the root datum of G. As a consequence, if G is split then $^{L}G = \hat{G} \times W_F$, and in particular, $^{L}\mathrm{GL}_N(F) = \mathrm{GL}_N(\mathbb{C}) \times W_F$. If G'/F is an inner form of G then by definition $G'(\bar{F}) \simeq G(\bar{F})$ and the corresponding Galois actions differ by an inner automorphism. These induce isomorphisms of root data and Galois actions, and $^{L}G \simeq {}^{L}G'$.

For F global, we will sometimes abuse notation and denote by ${}^L G_v$ the L-group of the base change of G to a completion F_v. In this situation, the embedding $W_{F_v} \to W_F$ induces a map ${}^L G_v \to {}^L G$ which restricts to the identity on $\hat{G}$.

The L-group of $U(N)$ (and of any inner form) is defined as

$${}^L U(N) = \mathrm{GL}_N(\mathbb{C}) \rtimes W_F$$

where W_F acts through the order two quotient $\Gamma_{E/F}$. The nontrivial element $\sigma \in \Gamma_{E/F}$ acts by $\sigma(g) = \Phi_N^{-1} g^{-t} \Phi_N$ of GL_N, where Φ_N is as in (4).

2.2.6. *Morphisms of L-groups.* A morphism of L-groups, or L-morphism, is a continuous morphism

$$\eta : {}^L H \to {}^L G$$

which commutes with the projections onto W_F. In practice, all morphisms of L-groups considered here will be admissible, i.e., induced by an algebraic map $\hat{H} \to \hat{G}$ and such that the image of elements of W_F are semisimple.

Denote the Weil restriction $\mathrm{Res}_F^E \mathrm{GL}_N$ by $G(N)$. In particular, $G(N)(F) = \mathrm{GL}_N(E)$ and $G(N)(E) \simeq \mathrm{GL}_N(E) \times \mathrm{GL}_N(E)$. As such, the connected component $\widehat{G(N)}$ of ${}^L G(N)$ is the product of two copies of $\mathrm{GL}_N(\mathbb{C})$, and W_F acts through $\Gamma_{E/F}$ via the automorphism that interchanges the two factors. Many objects associated to a unitary group $U(N)$ depend on a choice of embedding of L-groups from ${}^L U(N)$ to ${}^L G(N)$.

To define the L-embedding $\eta_\kappa : {}^L U(N) \to {}^L G(N)$, recall the characters χ_κ from Section 2.1. If F is global, we will use these characters, and if $F = F_v$ is local, we will momentarily also denote by χ_κ the restriction of χ_κ to $E_v^\times$. Let I_N be the identity matrix. For each $\kappa \in \{\pm 1\}$ we define η_κ as

$$\begin{aligned} \eta_\kappa(g \rtimes 1) &= (g, {}^t g^{-1}) \rtimes 1, && g \in \hat{G}, \\ \eta_\kappa(I_N \rtimes \sigma) &= (\chi_\kappa(\sigma) I_N, \chi_\kappa^{-1}(\sigma) I_N) \rtimes \sigma, && \sigma \in W_E, \\ \eta_\kappa(I_N \rtimes w_c) &= (\kappa \Phi_N, \Phi_N^{-1}) \rtimes w_c. \end{aligned} \tag{6}$$

We consider a second class of L-embeddings $\xi_{\underline{\kappa}} : {}^L(U(N_1) \times \cdots \times U(N_r)) \to {}^L U(N)$, for $\sum N_i = N$ into ${}^L U(N)$. Put $\kappa_i = (-1)^{N-N_i}$ for each i, and let $\underline{\kappa} = (\kappa_1, \ldots, \kappa_r)$. Given $\underline{\chi}$ with signature $\underline{\kappa}$, and for a choice of w_c as above, the embedding $\xi_{\underline{\kappa}}$ is defined as

$$\begin{aligned} \xi_{\underline{\kappa}}(g_1, \ldots, g_r \rtimes 1) &= \mathrm{diag}(g_1, \ldots, g_r) \rtimes 1, && g_i \in \mathrm{GL}_{N_i}(\mathbb{C}), \\ \xi_{\underline{\kappa}}(I_{N_1}, \ldots, I_{N_r} \rtimes \sigma) &= \mathrm{diag}(\chi_{\kappa_1}(\sigma) I_{N_1}, \ldots, \chi_{\kappa_r}(\sigma) I_{N_r}) \rtimes \sigma, && \sigma \in W_E, \\ \xi_{\underline{\kappa}}(I_{N_1}, \ldots, I_{N_r} \rtimes w_c) &= \mathrm{diag}(\kappa_1 \Phi_{N_1}, \ldots, \kappa_r \Phi_{N_r}) \cdot \Phi_N^{-1} \rtimes w_c. \end{aligned} \tag{7}$$

Note that the composite embedding $\eta_\kappa \circ \xi_{\underline{\kappa}}$ gives an embedding

$$\eta_{\kappa \cdot \underline{\kappa}} : {}^L(U(N_1) \times \cdots \times U(N_r)) \to {}^L G(N) \tag{8}$$

with signature $\kappa \cdot \underline{\kappa} = (\kappa\kappa_1, \ldots, \kappa\kappa_r)$.

Remark 10. The need to consider several embeddings depending on $\underline{\kappa}$ stems from the possibility that parameters for the pair $(U(N), \eta_+)$ may factor through different embeddings of the products of groups $U(N_i)$ associated to different signs.

2.3. *Parameters.*

We introduce the discrete automorphic spectrum of a unitary group G, and the local and global parameters which will classify the (constituents of) automorphic representations, following [5], [26], and [38].

2.3.1. *Automorphic representations.* Let G/F be a reductive group. Fix a closed subgroup $\mathfrak{X} \subset Z_G(\mathbb{A}_F)$ and a maximal compact subgroup of K of $G(\mathbb{A}_F)$, which in turn determines maximal compact subgroups K_v of $G_v = G(F_v)$ for any v. We consider the right-regular representation of $G(\mathbb{A}_F)$ on

$$L^2_{\text{disc}}(G(F)\backslash G(\mathbb{A}_F), \omega),$$

the discrete part of the space of square-integrable functions which transform by ω under the action of $\mathfrak{X}$. We will omit the ω when we allow for any central character. In our initial cases of interest, G/F will be an anisotropic inner form of $U_{E/F}(N)$, the group $\mathfrak{X}$ will be the full center, the central character ω will be trivial, and the entire automorphic spectrum will be discrete. However for induction purposes we will consider arbitrary central character data $(\mathfrak{X}, \omega)$ and allow for $L^2(G(F)\backslash G(\mathbb{A}_F), \omega)$ to have a continuous part. The discrete spectrum decomposes as

$$L^2_{\text{disc}}(G(F)\backslash G(\mathbb{A}_F)) = \bigoplus m(\pi)\pi$$

where $m(\pi)$ denotes the multiplicity of π, and the irreducible constituents are automorphic representations. Each π is a restricted tensor product $\pi = \otimes'_v \pi_v$ with each π_v an irreducible admissible unitary representation of G_v. All but finitely many π_v are spherical with respect to K_v. The representation π_v is said to be tempered if its K_v-finite matrix coefficients belong to $L^{2+\epsilon}(G_v)$ for all $\epsilon > 0$.

After fixing a maximal compact subgroup K_∞ of G_∞, we replace π_∞ by the dense subspace of K_∞-finite smooth vectors, which we view as an admissible $(\mathfrak{g}_\infty, K_\infty)$-module. This is no loss of information since unitary admissible representations are determined by their underlying $(\mathfrak{g}_\infty, K_\infty)$-modules [28, 9.2].

2.3.2. *Local Langlands parameters.* Let F be a local field with Weil group W_F. The Langlands group L_F of F is defined as

$$L_F := \begin{cases} W_F & F \text{ is archimedean,} \\ W_F \times \mathrm{SU}(2, \mathbb{R}) & F \text{ is nonarchimedean.} \end{cases}$$

A local Langlands parameter for the reductive group G/F is a continuous homomorphism $\varphi : L_F \to {}^L G$ satisfying certain conditions (see [9] for a discussion):

(i) The map φ commutes with the projections $L_F \to W_F$ and ${}^L G \to W_F$.

(ii) In the nonarchimedean case, the restriction $\varphi\,|_{\mathrm{SU}(2,\mathbb{C})}$ is algebraic.

(iii) The image of W_F under φ consists of semisimple elements of ${}^L G$.

(iv) If the image of φ in $\hat{G}$ factors through a parabolic subgroup of $\hat{G}$, then this parabolic subgroup must be the dual $\hat{P}$ of a parabolic subgroup P of G.

Continuous maps that satisfy condition (i) are known as L-homomorphisms. If they additionally satisfy (ii)–(iii) they are called *admissible.* If they satisfy (iv), they are called *relevant*, or *G-relevant*. Finally, we say that φ is *bounded* if W_F has bounded image in $\hat{G}$. We will denote the collection of $\hat{G}$-conjugacy classes of Langlands parameters for G by $\Phi(G)$.

2.3.3. *Local Arthur parameters.* In order to describe the nontempered spectrum of G, we consider enhancements of Langlands parameters known as Arthur parameters. These are admissible L-homomorphisms

$$\psi : L_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G$$

such that $\psi\,|_{L_F}$ is bounded. We denote the set of $\hat{G}$-conjugacy classes of Arthur parameters by $\Psi(G)$. We refer to the $\mathrm{SL}_2(\mathbb{C})$ factor in the above product as the "Arthur SL_2", and say that ψ is bounded if it restricts trivially to the Arthur SL_2.

Each Arthur parameter ψ determines a Langlands parameter φ_ψ as follows. Recall (e.g., [47]) that the Weil group W_F is naturally equipped with a norm homomorphism $|\cdot|$ to $\mathbb{C}^\times$. Then φ_ψ is defined as the composition

$$\varphi_\psi : W_F \to {}^L G, \quad \varphi_\psi(\sigma) = \psi\left(\sigma, \begin{pmatrix} |\sigma|^{1/2} & 0 \\ 0 & |\sigma|^{-1/2} \end{pmatrix}\right).$$

We now give a more detailed description of local Arthur parameters in the case where $G = U(N)$, following Section 2.2 of Mok [38]. Specifically, we use the map η_κ introduced in Section 2.2.6 to realize $\Psi(U(N))$ as a set of N-dimensional representations satisfying an appropriate self-duality condition.

We first describe a natural bijection between $\Psi(G(N))$, and $\Psi(\mathrm{GL}_N(E))$. To produce an element of $\Psi(G(N))$, one starts with $\psi \in \Psi(\mathrm{GL}_N(E))$, i.e., an admissible N-dimensional representation of $L_E \times \mathrm{SL}_2(\mathbb{C})$, and promotes it to a L-morphism $\psi' : L_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G(N)$ by choosing $w_c \in W_F \setminus W_E$ and defining

$$\begin{aligned} \psi'(\sigma, g) &= (\psi(\sigma, g), \psi^c(\sigma, g)) \rtimes \sigma, \quad (\sigma, g) \in L_E \times \mathrm{SL}_2(\mathbb{C}) \\ \psi'(w_c) &= (\psi(w_c^2), I_N) \rtimes w_c, \end{aligned}$$

where $\psi^c(\sigma, g) = \psi(w_c^{-1} \sigma w_c, g)$. The resulting bijection $\Psi(G(N)) \simeq \Psi(\mathrm{GL}_N(E))$ is independent of the choice of w_c. Moreover, if $\psi^c \simeq \psi^\vee$ where $\psi^\vee$ is the contragredient of ψ, then ψ is called *conjugate self-dual.* More precisely, the parameter ψ is conjugate self-dual of parity ± 1, depending on the parity of the resulting bilinear form.

The map η_κ introduced in (6) then induces a mapping

$$\eta_{\kappa *} : \Psi(U(N)) \to \Psi(G(N)) \simeq \Psi(\mathrm{GL}_N(E)) \tag{9}$$

which is shown by Mok, following work of Gan, Gross and Prasad [18], to be an injection whose image consists precisely of the subset of $\Psi(\mathrm{GL}_N(E))$ of conjugate self-dual representations of parity $(-1)^{N+1}\kappa$.

2.3.4. *Global Arthur parameters.* In lieu of global parameters, Arthur [5, Section 1.4] introduces formal objects realized by combining cuspidal automorphic representations of GL_N and representations of the Arthur SL_2. Echoing the local discussion, global Arthur parameters are first defined in terms of $G(N)$, and Arthur parameters for $U(N)$ are the ones factoring through a fixed embedding of L-groups.

A global Arthur parameter for GL_N is an unordered sum

$$\psi^N = \boxplus_i \psi_i^{N_i}, \quad \psi_i^{N_i} = \mu_i \boxtimes \nu(m_i).$$

Here μ_i is a cuspidal automorphic representation of $\mathrm{GL}_{n_i}(\mathbb{A}_E)$ and $\nu(m_i)$ is the irreducible m_i-dimensional representation of $\mathrm{SL}_2(\mathbb{C})$, with $m_i n_i = N_i$ and $\sum_i N_i = N$. Departing from our references, we immediately restrict our attention to the set of Arthur parameters such that the $\psi_i^{N_i}$ are pairwise distinct: we denote this set $\Psi(N)$ instead of $\Psi_{\mathrm{ell}}(N)$. The collection $\Psi(N)$ contains a distinguished subset $\Psi_{\mathrm{sim}}(N)$ of simple parameters with a unique summand ψ^N. Following the theorem of Mœglin and Waldspurger [37], this subset $\Psi_{\mathrm{sim}}(N)$ parametrizes the discrete spectrum of GL_N.

We now give the construction of global Arthur parameters for a quasisplit unitary group $G = U(N)$, following Section 1.3.4 of [26]. We start by restricting our attention to the set $\tilde{\Psi}(N) \subset \Psi(N)$ of parameters for which each of the μ_i is *conjugate self-dual*, i.e., satisfies $\mu_i = \overline{\mu_i}^\vee$ where $\bar{\mu} = \mu \circ \sigma$ and $\sigma \in \Gamma_{E/F}$ is nontrivial.

To record the parameter in relation to the embedding η_κ, we introduce the group $\mathcal{L}_\psi$. If ψ^N decomposes as a sum of $\mu_i \boxtimes \nu(m_i)$, we associate to each index a pair $(U_{E/F}(n_i), \eta_{\kappa i})$ as in Section 2.2.6. Here the choice of sign κ_i is determined by μ_i. Then $\mathcal{L}_\psi$ is the fiber product $\mathcal{L}_\psi = \prod_i ({}^L U_{E/F}(n_i) \to W_F)$. There is a natural map $\tilde{\psi}^N : \mathcal{L}_\psi \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G(N)$ given by the direct sum

$$\tilde{\psi}^N = \oplus(\eta_{\kappa i} \otimes \nu(m_i)).$$

A global Arthur parameter for $(U_{E/F}(N), \eta_\kappa)$ is then defined as a pair $\psi = (\psi^N, \tilde{\psi})$ where $\psi^N \in \tilde{\Psi}(N)$, and

$$\tilde{\psi} : \mathcal{L}_\psi \times \mathrm{SL}_2(\mathbb{C}) \to {}^L U_{E/F}(N)$$

is an L-homomorphism such that $\eta_\kappa \circ \tilde{\psi} = \tilde{\psi}^N$. It is useful to remember that ψ^N encodes the arithmetic information of the automorphic representations of GL_{n_i}, and that $\tilde{\psi}$ is an actual homomorphism. As such, we can (and will) discuss the centralizer of the image of $\tilde{\psi}$. Two Arthur parameters are equivalent if the $\tilde{\psi}$ are $\hat{U}(N)$-conjugate, and we denote the set of equivalence classes of ψ as above by $\Psi(U(N), \eta_\kappa)$. Note that we have again broken off from our references in the choice of notation: our set $\Psi(U(N), \eta_\kappa)$ is the one that the authors of [26] denote $\Psi_2(U_{E/F}(N), \eta_\kappa)$. Finally, note that the map $\eta_{\kappa,*}$ sending ψ to ψ^N is an injection: this allows us to view $\Psi(U(N), \eta_\kappa)$ as a subset of $\Psi(N)$. If $(G, \xi_{\underline{\kappa}})$ is a product as in (8), we can similarly define $\Psi(G, \xi_{\underline{\kappa}})$. Via the block-diagonal embedding $\prod_i \mathrm{GL}_{N_i} \hookrightarrow \mathrm{GL}_N$, we can identify $\Psi(G, \xi_{\underline{\kappa}}) \simeq \prod_i \Psi(U(N_i), \eta_{\kappa_i})$.

Remark 11. We have made two constraints on the set of parameters under consideration here which bear highlighting. We require:

(i) That the irreducible summands ψ_i be pairwise distinct. In Mok's description of the parameters in [38, Section 2.4] this amounts to requiring that all the $l_i = 1$.

(ii) That each irreducible summand be conjugate self-dual. This is stricter than requiring ψ to be conjugate self-dual since we could have had $\mu_i^\vee \simeq \mu_j$.

Parameters satisfying these conditions are called *elliptic*. These restrictions will give us control on the group $\mathcal{S}_\psi$ to be introduced below, whose characters determine which products of local representations occur in the discrete spectrum. It is also the case that only the parameters in the set which we denote by $\Psi(U(N), \eta_\kappa)$ correspond to packets whose members actually appear in the decomposition of L^2_{disc}, although this fact is far from obvious and is one of the main theorems in [38] and [26]. Following this result, global elliptic parameters are also called *square-integrable.*

2.3.5. *Localization.* We now describe how a global Arthur parameter $\psi \in \Psi(U(N), \eta_\kappa)$ gives rise to local Arthur parameters ψ_v at each place v. Each cuspidal representation μ of GL_N factors as a restricted tensor product $\mu = \otimes' \mu_v$ over places v of F. The μ_v are admissible representations of $\mathrm{GL}_N(F_v)$. The local Langlands correspondence for GL_N [23; 24; 44] associates to μ_v a parameter $\varphi_{\mu_v} \in \Phi(\mathrm{GL}_N)$. Following [5], we define the localization of ψ at v as the direct sum

$$\psi_v = \oplus_i \psi_{v,i}, \quad \psi_{v,i} = \varphi_{\mu_{v,i}} \otimes \nu(m_i).$$

These localizations a priori only belong to $\Psi(G(N))$. The fact that they are in the image of the map (9) is one of the central theorems of [38].

2.3.6. *Parameters of inner forms.* Let (G, ξ) be an inner form of $G^* = U(N)$. A local Arthur parameter for G is simply a G-relevant parameter for $U(N)$, see Section 2.3.2. Globally, a parameter $\psi \in \Psi(G^*, \eta_\kappa)$ is G-relevant if it is so everywhere locally [26, Section 1.3.7]. We denote by $\Psi(G, \xi)$ the collection of G-relevant parameters in $\Psi(G^*, \eta_\kappa)$. In summary, we have the following chain of inclusions:

$$\Psi(G, \xi) \subset \Psi(G^*, \eta_\kappa) \subset \tilde{\Psi}(N) \subset \Psi(N),$$

where the parameters in $\tilde{\Psi}(N)$ are conjugate self-dual, those in $\Psi(G^*, \eta_\kappa)$ factor through the embedding η_κ, and those in $\Psi(G, \xi)$ are additionally G-relevant.

2.3.7. *Parameters and conjugacy classes.* We attach families of conjugacy classes to objects introduced above, following [5, Section 1.3]. For F global, G reductive, and any finite set S of places of F containing the archimedean ones, let $\mathcal{C}^S(G)$ denote the set of collections $c = \{c_v\}_{v \notin S}$, where each c_v is a semisimple conjugacy class in $\hat{G}$. For two sets S and S', let $c \sim c'$ if $c_v = c'_v$ for almost all v. Denote the set of such equivalence classes by $\mathcal{C}(G)$. As we did for parameters, let $\mathcal{C}(N) := \mathcal{C}(GL_N)$. We associate elements of $\mathcal{C}(G)$ to automorphic representations π of G. Factoring $\pi = \otimes'_v \pi_v$, let $c(\pi) = \{c(\pi_v)\} \in \mathcal{C}(G)$ be the Satake parameters of all the unramified π_v. Note also that an L-embedding $\eta : {}^L G \to {}^L G(N)$ such as those introduced in Section 2.2.6 gives rise to a map $\eta_* : \mathcal{C}(G) \to \mathcal{C}(N)$.

When $G = \mathrm{GL}_N$ one can associate an element of $\mathcal{C}(N)$ to each $\psi \in \Psi(N)$. Starting with simple parameters $\psi \in \Psi_{\text{sim}}(N)$, use the recipe for the representation π_ψ prescribed by Moeglin and Waldspurger's theorem [37] and let $c(\psi) := c(\pi_\psi)$. If ψ is not simple, apply the process to its simple constituents and associate to ψ the conjugacy class coming from the diagonally embedded product of the GL_{N_i} inside of GL_N. This produces a mapping

$$\Psi(N) \to \mathcal{C}(N), \quad \psi \mapsto c(\psi)$$

which is injective, following Jacquet and Shalika [25]. Denote its image by $\mathcal{C}_{\mathrm{aut}}(N)$.

2.3.8. *Stabilizers and quotients.* For ψ either local or global, we have

$$S_\psi := \mathrm{Cent}(\mathrm{Im}(\psi), \hat{G}), \quad \bar{S}_\psi := S_\psi / Z(\hat{G})^{W_F}, \quad \mathcal{S}_\psi := \pi_0(\bar{S}_\psi).$$

As mentioned previously, when ψ is global then $\mathrm{Im}(\psi)$ really means $\mathrm{Im}(\tilde{\psi})$. Localization of parameters $\psi \mapsto \psi_v$ induces a mapping $\mathcal{S}_\psi \to \mathcal{S}_{\psi_v}$. When G is unitary, the groups $\mathcal{S}_\psi$ can be readily computed, as the four authors do in [26, page 63]. In particular, for F global and $\psi \in \Psi(G^*, \eta_\kappa)$ decomposing as $\psi = \boxplus_{i=1}^r \psi_i$, we have

$$\mathcal{S}_\psi = (\mathbb{Z}/2\mathbb{Z})^{r-1}. \tag{10}$$

The reader who looks at the computations in [26] will notice that this is the point where we use the assumptions from Remark 11. Finally, we introduce the element

$$s_\psi := \psi\left(1, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\right) \in S_\psi. \tag{11}$$

We will sometimes conflate s_ψ and its image in the quotient $\mathcal{S}_\psi$.

Remark 12. The authors of [26] work with the centralizer quotient $S_\psi^\natural$, which agrees with $\mathcal{S}_\psi$ for G local and unitary. If the local group G_v is isomorphic to $\mathrm{GL}_{N,v}$ (the only possibility for us at split places, since our unitary groups arise from Hermitian forms) then $S_\psi^\natural \simeq \mathbb{C}^\times$. However, if $G_v = \mathrm{GL}_{N,v}$, then only the trivial character of $S_\psi^\natural$ arises in the character identities, as will be discussed in Section 2.5.6. Thus there is no loss in working instead with the group $\mathcal{S}_\psi = \{1\}$. In the global situation, the characters of $S_\psi^\natural$ that arise all factor through $\mathcal{S}_\psi$ [26, page 89]. Note that we follow Arthur's convention and use the notation $\mathcal{S}_\psi$ instead of $\bar{\mathcal{S}}_\psi$ as in [26].

2.3.9. *Epsilon factors.* The last invariant attached to a global parameter ψ is the character ϵ_ψ of $\mathcal{S}_\psi$, defined by Arthur in [5, Section 1.5]. The definition involves the symplectic root number $\epsilon(1/2, \mu_\alpha)$ of an automorphic L-function $L(s, \mu_\alpha)$ for a product of general linear groups, obtained by composing ψ with the adjoint representation. As such ϵ_ψ encodes arithmetic data in the decomposition of $L^2_{\mathrm{disc}}(G(F)\backslash G(\mathbb{A}_F))$. Note that ϵ_ψ only depends on ψ and in particular is independent of the inner form of G^* under consideration, as discussed in [26, page 89].

2.4. ***Endoscopic data.*** An endoscopic datum for G/F is a triple (ξ, H, s) where:

- s is a semisimple element of $\hat{G}$.
- H/F is a connected, quasisplit group.
- $\xi : {}^L H \to {}^L G$ is an L-embedding.

The triple must satisfy certain conditions, see [26, Section 1.1.1], including that $\xi(\hat{H})$ is the connected component of the centralizer of s in $\hat{G}$. We will work only with *elliptic* endoscopic data, characterized by the requirement that $\xi(Z(\hat{H})^{W_F})^0 \subset (Z(\hat{G}))^{W_F}$. As such, we denote the set of conjugacy classes of elliptic endoscopic data for G by $\mathcal{E}(G)$, dropping the "ell" subscript appearing in our references. An endoscopic datum of G for which ${}^L H \not\simeq {}^L G$ will be called *proper*. We will frequently abuse notation and refer to H as a stand-in for the full datum, and denote the other two elements of the triple by ξ_H and s_H. Lastly, we will also use the formalism of endoscopic data for our unitary groups and denote by $\tilde{\mathcal{E}}(N)$ the set of pairs consisting of a product G of quasisplit unitary groups together with the L-embedding $\xi = \eta_{\kappa,\underline{\kappa}}$ from Section 2.2.6, and by $\tilde{\mathcal{E}}_{\text{sim}}(N)$ the subset of $\tilde{\mathcal{E}}(N)$ for which $G = U(N)$.

For any inner form G of $U_{E/F}(N)$, the set $\mathcal{E}(G)$ consists of pairs

$$(H, \xi) = (U(N_1) \times U(N_2), \xi_{\underline{\kappa}}), \quad N_1, N_2 \geq 0, N_1 + N_2 = N,$$

where $\xi_{\underline{\kappa}}$ was defined in (7). The signature $\underline{\kappa} = ((-1)^{N-N_1}, (-1)^{N-N_2})$ depends on the respective ranks of the groups. The equivalence class of endoscopic data is then uniquely determined by N_1; see [38, Section 2.4].

2.4.1. *Endoscopic data and parameters.* Let F be global and G/Fbe unitary and $\psi = (\psi^N, \tilde{\psi}) \in \Psi(G^*, \eta_\kappa)$ be an Arthur parameter. Let $(H, \xi_H, s_H) \in \mathcal{E}(G)$, and let $\psi^H = (\psi^{N,H}, \tilde{\psi}^H) \in \Psi(H, \eta_\kappa \circ \xi_H)$ be an Arthur parameter for H satisfying $\psi^N = \psi^{N,H}$ and $\tilde{\psi} = \xi_H \circ \tilde{\psi}^H$. In this situation, we will abuse notation and write that $\psi = \xi_H \circ \psi^H$. Since s_H commutes with H, it also commutes with the image of $\tilde{\psi}$. We thus get a mapping

$$(H, \psi^H) \mapsto (\xi_H \circ \psi^H, s_H) \tag{12}$$

from the set of pairs (H, ψ^H) onto the set of pairs consisting of a parameter ψ for G and an element s of the centralizer S_ψ. The importance of the quotient $\mathcal{S}_\psi$ comes from the fact that for each ψ, the map (12) descends to a bijection between $\mathcal{S}_\psi$ and the set of endoscopic data such that ψ factors through ξ_H. We state this result below, under simplifying assumptions: G global unitary and ψ square-integrable.

Lemma 13. *Let F be global and $G^* = U_{E/F}(N)$. Let $\psi = (\tilde{\psi}, \psi^N) \in \Psi(G^*, \eta_\kappa)$. The map* (12) *induces a bijection*

$$(H, \psi^H) \leftrightarrow (\psi, s)$$

where the left-hand side runs over pairs where H stands in for an endoscopic datum (H, ξ, s) and $\psi^H = (\tilde{\psi}^H, \psi^{N,H}) \in \Psi(H, \eta_\kappa \circ \xi_H)$ with $\psi^N = \psi^{N,H}$ and $\tilde{\psi} = \xi \circ \tilde{\psi}^H$, and the right-hand side runs over elements of $\mathcal{S}_\psi$.

Proof. The proof occupies Section 1.4 of [26], and the above statement is a reformulation of Lemma 1.4.3 therein. The square-integrability assumption on ψ implies that S_ψ and a fortiori $\bar{S}_\psi$ are finite. Thus $\bar{S}_\psi = \mathcal{S}_\psi$ and we use the latter. □

2.5. ***Packets.*** Here, we introduce A-packets of representations associated to Arthur parameters, and the character identities relating their traces to those of corresponding representations for endoscopic groups.

2.5.1. *Local Arthur packets.* Let (G, ξ) be a unitary group over a local field. The main local results of Mok [38, Theorem 2.5.1] and Kaletha, Minguez, Shin and White [26, Theorem 1.6.1] associate to each Arthur parameter $\psi \in \Psi(G, \xi)$ a finite set Π_ψ of irreducible unitary representations of $G(F)$ called a local Arthur packet. This packet Π_ψ is empty if ψ is not relevant, and contains only tempered representations when ψ is bounded. Each nonempty Π_ψ is equipped with a pairing

$$\langle \cdot, \cdot \rangle : \mathcal{S}_\psi \times \Pi_\psi \to \{\pm 1\}. \tag{13}$$

In this way, every $\pi \in \Pi_\psi$ gives rise to a character of $\mathcal{S}_\psi$. Unramified representations correspond to the trivial character. The pairing depends on the triple (G, ξ, z) realizing G as a pure inner twist, as discussed in Remark 9.

For F archimedean, all $\pi \in \Pi_\psi$ have the same infinitesimal character. We recall how to compute it from φ_ψ following [39]. The group $W_\mathbb{R}$ is an extension of $\mathbb{C}^\times$ by the group $\langle\sigma\rangle$ of order 2. For each ψ, there is a torus $\hat{T} \in \hat{G}$ such that

$$\varphi_\psi \mid_{\mathbb{C}^\times} (z) = z^\mu \bar{z}^\nu, \quad \mu, \nu \in X_*(\hat{T}).$$

The infinitesimal character of the representations $\pi \in \Pi_\psi$ is then identified with $\mu \in X_*(\hat{T}) \simeq X^*(T)$ via the Harish-Chandra isomorphism.

Lemma 14. *Let $\psi \in \Psi(G)$ be an archimedean Arthur parameter with regular infinitesimal character, and let $H \in \mathcal{E}(G)$ be such that $\psi = \xi_{\underline{\kappa}} \circ \psi^H$ for $\psi^H \in \Psi(H)$. Then the infinitesimal character of ψ^H is also regular.*

Proof. By assumption, $\psi((z, 1), I) = z^\mu \bar{z}^\nu$ and the weights appearing in μ are distinct. The parameter φ factors through $\xi_{\underline{\kappa}} : {}^L H \to {}^L G$. Referring to (6), the restriction of $\xi_{\underline{\kappa}}$ to $\mathbb{C}^\times \subset W_\mathbb{R} \subset {}^L H$ is trivial, since it factors through χ_κ, which takes values in ± 1. Thus the weights of the z-part of $\varphi^H \mid_{\mathbb{C}^\times}$ are also distinct, and the infinitesimal character of the corresponding packet is regular. ☐

For any local F, we record a result initially proved by Mok about the central character of the representations in the packet Π_ψ for the quasisplit group G^*.

Proposition 15 [26, Proposition 1.5.2, 2]. *The Langlands parameter of the central character ω_π : $Z(G^*)(F) \to \mathbb{C}^\times$ of any $\pi \in \Pi_\psi$ is given by the composition*

$$L_F \xrightarrow{\varphi_\psi} {}^L G^* \xrightarrow{(\det \rtimes \mathrm{id}) \circ \eta_\kappa} \mathbb{C}^\times \rtimes W_F.$$

2.5.2. *Global Arthur packets.* Let $\psi \in \Psi(G, \xi)$ be global with localizations ψ_v. The global Arthur packet Π_ψ is then defined as

$$\Pi_\psi = \{\pi = \otimes_v \pi_v \mid \pi_v \in \Pi_{\psi_v}, \langle \cdot, \pi_v \rangle_{\psi_v} = 1 \text{ for almost all } v\}.$$

It is equipped with a pairing

$$\langle \cdot, \cdot \rangle_\psi : \mathcal{S}_\psi \times \Pi_\psi \to \{\pm 1\}, \quad \langle \cdot, \pi \rangle_\psi = \prod_v \langle \cdot, \pi_v \rangle_{\psi_v} \tag{14}$$

determined by the maps $\mathcal{S}_\psi \to \mathcal{S}_{\psi_v}$ induced by localization. We note once again that this pairing depends on the full inner twist (G, ξ). However, the local dependence on the pure inner twist, i.e., the dependency on the cocycle z appearing in the local definition of the pairing, cancels out globally. This is detailed in [26, Section 1.7].

2.5.3. *Test functions.* Continuing with F global, we fix a maximal compact subgroup K of $G(\mathbb{A}_F)$. The group K determines a maximal compact subgroup $K_v \subset G_v$ at each place v: we choose K so that K_v is hyperspecial at all the unramified v. We also fix for each v a Haar measure μ_v on G_v satisfying $\mu_v(K_v) = 1$, and a corresponding measure $\mu = \prod_v \mu_v$ on $G(\mathbb{A}_F)$. The local Hecke algebra $\mathscr{H}(G_v)$ consists of smooth, compactly supported, left and right K_v-finite functions on G_v. We will call its elements local test functions. The global Hecke algebra is the restricted product $\mathscr{H}(G) = \otimes'_v \mathscr{H}(G_v)$: it consists of smooth, compactly supported, K-finite functions. Each such test function is a finite sum of factorizable test functions of the form $f = \prod_v f_v$, where each $f_v \in \mathscr{H}(G_v)$ and all but finitely many f_v are the characteristic function of K_v.

For π_v a smooth, admissible representation of G_v, each $f_v \in \mathcal{H}(G_v)$ gives rise to an operator $\pi_v(f_v)$ on the underlying vector space of π_v, defined as follows:

$$\pi_v(f_v)(x) = \int_{G_v} f_v(g)\pi_v(g)(x)\, d\mu_v.$$

This operator is of trace class, and we denote its trace by $\operatorname{tr} \pi_v(f_v)$. Likewise globally, the algebra $\mathscr{H}(G)$ acts on $L^2_{\mathrm{disc}}(G(F)\backslash G(\mathbb{A}_F))$ and on its irreducible constituents π. We denote the trace of convolution by f by $\operatorname{tr} R(f)$ (when considering the right-regular representation on $L^2_{\mathrm{disc}}(G(F), G(\mathbb{A}_F)))$ or by $\operatorname{tr} \pi(f)$ (when f acts on π irreducible).

2.5.4. *Stable distributions and transfer.* We introduce stable distributions on the local and global Hecke algebras, following Sections 3.1 and 4.2 of [38] respectively. Let $\gamma \in G(F_v)$ and let $G(F_v)_\gamma$ be its centralizer. For $f \in \mathscr{H}(G(F_v))$, let $f_{G(F_v)}(\gamma) := \int_{G(F_v)/G(F_v)_\gamma} f(g\gamma g^{-1}) d\mu_v$ be the *orbital integral* associated to γ and f. It only depends on the $G(F_v)$ conjugacy class of γ.

We now introduce transfer, which makes use of *stable conjugacy classes*: the union of the finitely many conjugacy classes of $G(F_v)$ that are $G(\overline{F_v})$-conjugate. Let $G(F_v)$ first be a quasisplit unitary group. Each stable conjugacy class δ gives rise to a linear functional

$$f_v^G(\delta) = \sum_\gamma \Delta_v(\delta, \gamma) f_{G(F_v)}(\gamma), \tag{15}$$

where the sum is taken over representatives γ of all the conjugacy classes of $G(F_v)$. The factor $\Delta(\delta, \gamma)$ is equal to 1 if $\gamma \in \delta$ and to 0 otherwise. This construction gives a map from $\mathscr{H}(G_v)$ to functions on stable conjugacy classes. Denote the image of this map by $\mathcal{S}(G_v)$. A linear functional on $\mathscr{H}(G_v)$ is said to be *stable* if it factors through $\mathcal{S}(G_v)$.

Now let G_v be an arbitrary unitary group. For each endoscopic group H_v of G_v, the construction of transfer factors by Langlands and Shelstad [32] and Kottwitz and Shelstad [30] gives rise to maps

$\mathscr{H}(G_v) \to \mathcal{S}(H_v)$. The transfer factors are a significantly more delicate generalization of the $\Delta(\delta, \gamma)$ above; in particular, their normalization in [26, Section 1.1.2] (and thus the notion of transfer) depends on the choice of pure inner form as in Remark 9. This provides a system of maps from the Hecke algebras to their stable counterparts, and two functions $f_v \in \mathscr{H}(G_v)$ and $f_v^{H_v} \in \mathscr{H}(H_v)$ will be said to form a *transfer pair* if their images under their respective maps to $\mathcal{S}(H_v)$ agree. Although $f_v^{H_v}$ is not uniquely determined by f_v, we may abuse terminology and refer to a choice of $f_v^{H_v}$ as the transfer of f_v.

To extend the notion of transfer to global test functions, it is first necessary to know that the transfer of characteristic functions of maximal compact subgroups of G_v are the corresponding functions on H_v. This is the fundamental lemma, now a theorem due to Laumon and Ngô [33] in the case of unitary groups, and to Ngô [40] in general, after reductions by Waldspurger [50; 51].

Theorem 16 (fundamental lemma). *Let G_v and H_v be unramified reductive groups over a nonarchimedean local field F_v. Let $K(G_v)$ and $K(H_v)$ be respective choices of hyperspecial maximal compact subgroups. Then their characteristic functions $f_v = 1_{K(G_v)}$ and $f_v^{H_v} = 1_{K(H_v)}$ form a transfer pair.*

With this in mind, the transfer of a factorizable global test function $f = \prod_v f_v \in \mathcal{H}(G_v)$ is the product $f^H = \prod_v f_v^{H_v}$ of its transfers, a definition extended linearly to all of $\mathscr{H}(G)$. We will likewise define the global stable Hecke algebra $\mathcal{S}(G^*) := \otimes'_v \mathcal{S}(G_v^*)$. A linear functional on $\mathscr{H}(G^*)$ is *stable* if it factors through $\mathcal{S}(G^*)$.

2.5.5. *Local character identities.* The transfer of representations between G and its endoscopic groups H is encoded via identities between linear combinations of characters; the coefficients are determined by the pairings (13). We start with distributions $f^G(\psi)$ on $\mathscr{H}(G)$. Let F be local and G^*/F be a quasisplit unitary group or a product thereof, and ψ be an Arthur parameter of G^*. Then Mok attaches a stable linear form to ψ.

Theorem 17 [38, Theorem 3.2.1(a)]. *Let $\psi \in \Psi(G^*)$. Then there exists a unique stable linear form $f \mapsto f^{G^*}(\psi)$ on $\mathscr{H}(G^*)$, determined by transfer properties to* GL_N. *If $G^* = G_1^* \times G_2^*$ and $\psi = \psi_1 \times \psi_2$, then $f^{G^*}(\psi) = f^{G_1^*}(\psi_1) \times f^{G_2^*}(\psi_2)$.*

We will not discuss in detail the character identities relating $f^{G^*}(\psi)$ to traces on GL_N, save for reminding the reader that this distribution is related to the trace $\operatorname{tr} \pi_{\psi,N}(f)$ where $\pi_{\psi,N}$ corresponds to ψ under the Local Langlands Correspondence. We will focus on the relation between the $f^H(\psi^H)$ for the groups $H \in \mathcal{E}(G)$ and the characters of representations in Π_ψ. If $G = G^*$, these identities were established by Mok, and for inner forms by Kaletha, Minguez, Shin and White. Recall that s_ψ is the distinguished element of $\mathcal{S}_\psi$ defined in (11).

Theorem 18 [38, Theorem 3.2.1(b)]. *Let G^* be a quasisplit unitary group, let $\psi \in \Psi(G^*)$, and let Π_ψ be the associated Arthur packet equipped with the pairing of equation* (13). *Let $s_H \in \mathcal{S}_\psi$ be such that (H, ψ^H) correspond to (ψ, s_H) under the correspondence of Lemma 13. Then for a transfer pair (f, f^H) we have*

$$f^H(\psi^H) = \sum_{\pi \in \Pi_\psi} \langle s_\psi s_H, \pi \rangle \operatorname{tr} \pi(f).$$

Theorem 19 [26, Theorem 1.6.1]. *Let (G, ξ) be an inner form of $U(N)$ and let ψ, Π_ψ, H, s_H, and (f, f^H) be as above. Let $e(G)$ be the Kottwitz sign. Then*

$$f^H(\psi^H) = e(G) \sum_{\pi \in \Pi_\psi} \langle s_\psi s_H, \pi \rangle \operatorname{tr} \pi(f).$$

Remark 20. Let us recall a discussion from the introduction: the proofs in [26] are not given in full generality. For example, Theorem 19 is only proved for bounded parameters. The authors of [26] anticipate that the proof will appear in a pair of papers, the first of which [27] should contain the results we use here.

2.5.6. *Local packets for general linear groups.* As discussed in Section 2.2.1, if F is local and corresponds to a place splitting in our global CM extension, then $G \simeq \mathrm{GL}_N$. In this situation the local Arthur packet and the pairing are especially simple.

Theorem 21 [38, Section 2]. *If $G = \mathrm{GL}_N$ and ψ is an Arthur parameter for G, then the packet Π_ψ contains one element: the irreducible representation associated to φ_ψ by the local Langlands correspondence. The character $\langle \cdot, \pi_\psi \rangle$ is trivial.*

We now consider character of identities between representations of G and those of its endoscopic groups. They are alluded to in [38] and [26], but we give a more explicit description based on [45, Section 3.3]. For $G = \mathrm{GL}_N$, stable and regular conjugacy classes coincide, so $\mathcal{S}(G) = \mathcal{H}(G)$. Since the global extension giving rise to our unitary group is CM, we may assume that F is nonarchimedean. If $H = \mathrm{GL}_{N_1} \times \mathrm{GL}_{N_2}$ with $N_1 + N_2 = N$, then the embedding $\xi_{\underline{\kappa}}$ realizes H as a Levi subgroup of G. Let $P = HN$ be a parabolic subgroup of G containing H. Given $f \in \mathcal{H}(G)$, define the constant term along P as

$$f^P(h) := \delta_P^{1/2}(h) \int_N \int_K f(khnk^{-1})\, dk\, dn, \quad h \in H(F).$$

Here the integrals are taken with respect to suitably normalized Haar measures and δ_P is the modulus character. The function f_v^P is smooth and compactly supported, and by results of van Dijk [15], it satisfies the requisite orbital integrals identities to be a transfer of f, so we let $f^H := f^P$. If f is unramified, then f^H is the image of f under the map $\mathcal{H}(G)^{ur} \to \mathcal{H}(H)^{ur}$ induced by the Satake isomorphism. Thus this notion of transfer satisfies the fundamental lemma.

For a parameter ψ of G, we let $f^G(\psi) = \operatorname{tr} \pi_\psi(f)$ [26, Section 1.5] for the unique $\pi_\psi \in \Pi_\psi$ and extend this definition multiplicatively to products of general linear groups. Let π_ψ^H be the unique representation in the packet associated to ψ^H. It follows from the local Langlands correspondence (see for example [23, page 6] and note that the twist therein is accounted for here in the definition of the embedding $\xi_{\underline{\kappa}}$) that $\pi_\psi = \mathcal{I}_P(\pi_\psi^H)$, where $\mathcal{I}_P$ denotes normalized parabolic induction with respect to P. In view of this and of Theorem 21, the local character identities for GL_N amount to an equality of traces between $\operatorname{tr} \pi(f^H)$ and the trace of f on the corresponding induced representation. Again this is a result of van Dijk, which we record below.

Theorem 22 [15, Section 5]. *Let G, H, P, and f^H be as above. Let π be a unitary irreducible representation of H and let $\mathcal{I}_P(\pi)$ be its normalized parabolic induction with respect to P. Then* $\operatorname{tr}\pi(f^H) = \operatorname{tr}\mathcal{I}_P(\pi)(f)$.

2.6. ***The trace formula and its stabilization.*** We now introduce Arthur's trace formula following [26, Section 3] (see [5, Section 3] for a more detailed exposition), focusing on the statements needed for our applications. The rough picture is as follows: for F a number field, and a connected reductive group G/F, the trace formula I_{disc} (sometimes denoted I^G_{disc}) is a distribution on the Hecke algebra $\mathcal{H}(G)$, defined in terms of the traces of intertwining operators on variants of $L^2_{\mathrm{disc}}(G(F)\backslash G(\mathbb{A}_F))$ indexed by a system of Levi subgroups of G. The contribution of the group G itself is the trace $\operatorname{tr} R_{\mathrm{disc}}(f) := \operatorname{tr} R(f)$ introduced in Section 2.5.3. The trace formula admits two decompositions: a spectral one into a sum over the contributions of $\psi \in \Psi(G, \xi)$, and an endoscopic one (or stabilization) into a sum of stable distributions on endoscopic groups. Our proof will follow from the interplay of these two decompositions.

2.6.1. *Contribution of a parameter.* We start by directly introducing the distributions given by the contribution of each Arthur parameter as in [26, Section 3.3]. For following paragraphs, let $(G, \eta_{\kappa\cdot\underline{\kappa}})$ be a pair consisting an inner form of a (possible product of) unitary groups, and an embedding $\eta_{\kappa\cdot\underline{\kappa}} : {}^LG \to {}^LG(N)$ as in Section 2.2.6. When G is an inner form of $U(N)$, we have $\eta_{\kappa\cdot\underline{\kappa}} = \eta_\kappa$.

Recall $\mathcal{C}(G)$, the set of families of conjugacy classes introduced in Section 2.3.7. To any automorphic representation π of G, we can associate an element $c_\pi \in \mathcal{C}(G)$ by letting $c_{\pi,v}$ be the Satake parameter of π_v at all the unramified places v. Likewise, we associate to π an infinitesimal character μ_π. Then for $c \in \mathcal{C}(G)$ and a positive real number t, the distribution $I_{\mathrm{disc},t,c}$ is described in [26, Section 3.1]. It is the restriction of the traces defining I_{disc} to representations π such that $c = c_\pi$, and such that μ_π satisfies $|\operatorname{Im}\mu_\pi| = t$ under a suitable metric. To go from conjugacy classes to parameters, recall that in Section 2.3.7 we identified $\Psi(N)$ with $\mathcal{C}_{\mathrm{aut}}(N) \subset \mathcal{C}(N)$. To each $\psi^N \in \Psi(N)$ is thus associated an element $c(\psi^N) \in \mathcal{C}_{\mathrm{aut}}(N)$ as well as a positive real number $t(\psi^N)$ coming from the infinitesimal character of ψ^N. For each parameter $\psi^N \in \Psi(N)$, we follow [26, Section 3.3] and define

$$I_{\mathrm{disc},\psi^N,\eta_{\kappa\cdot\underline{\kappa}}} = \sum_{\substack{c\mapsto c(\psi^N)\\ t\mapsto t(\psi^N)}} I_{\mathrm{disc},t,c}.$$

The sum runs over the $c \in \mathcal{C}(G)$ that map to $c(\psi^N)$ under the map $\mathcal{C}(G) \to \mathcal{C}(N)$ induced by $\eta_{\kappa\cdot\underline{\kappa}}$. When $G^* = U(N)$, we follow [26, Section 3.3] and shorten $I_{\mathrm{disc},\psi^N,\eta_\kappa}$ to $I_{\mathrm{disc},\psi}$ when $\psi = (\psi^N, \tilde{\psi}) \in \Psi(G^*, \eta_\kappa)$, using the injection $\eta_{\kappa,*}$ of Section 2.3.4. We similarly obtain distributions $\operatorname{tr} R_{\mathrm{disc},c,t}$, $R_{\mathrm{disc},\psi^N,\eta_{\kappa,\underline{\kappa}}}$ and $\operatorname{tr} R_{\mathrm{disc},\psi^N,\eta_\kappa} := \operatorname{tr} R_{\mathrm{disc},\psi}$. If we have $G^* \in \widetilde{\mathcal{E}}_{\mathrm{sim}}(N)$, as well as $(H, \xi_{\underline{\kappa}}) \in \mathcal{E}(G)$ and $\psi \in \Psi(G^*, \eta_\kappa)$, we will also shorten notation and denote $I^H_{\mathrm{disc},\psi} = I^H_{\mathrm{disc},\psi^N,\xi_{\underline{\kappa}}\circ\eta_\kappa}$.

An essential step in the proof of the endoscopic classification of representations is showing that $\operatorname{tr} R_{\mathrm{disc},\psi}$ computes the traces of the representations in Π_ψ, provided that ψ satisfies the two conditions of Remark 11.

Theorem 23 [38, (5.7.27); 26, proof of Theorem 5.0.5]. *Let $\psi \in \Psi(G, \xi)$ be a square-integrable parameter associated to Π_ψ, and let $f \in \mathcal{H}(G)$. Then*

$$\operatorname{tr} R_{\mathrm{disc},\psi}(f) = \sum_{\pi \in \Pi_\psi} m(\pi) \operatorname{tr} \pi(f).$$

In the notation of Section 2.5, the multiplicity $m(\pi)$ is equal to 1 if $\langle \pi, \cdot \rangle_\psi = \epsilon_\psi$ as characters of $\mathcal{S}_\psi$, and 0 otherwise; see [26, Section 1.7]. Note again that [26, Theorem 5.0.5] is stated, but not fully proved, in the case of nongeneric parameters, as mentioned in the introduction and in Remark 20.

Following a result of Bergeron and Clozel, the distributions $\operatorname{tr} R_{\mathrm{disc},\psi}$ and $I_{\mathrm{disc},\psi}$ agree if the infinitesimal character is regular.

Theorem 24 [7, Theorem 6.2]. *Let G be a connected reductive group. Let $\psi \in \Psi(G)$ be a global Arthur parameter such that ψ_∞ has regular infinitesimal character. Then the contributions of the Levi subgroups $M \neq G$ to the distribution $I_{\mathrm{disc},\psi}$ vanish. In particular for all $f \in \mathcal{H}(G)$ we have $I_{\mathrm{disc},\psi}(f) = \operatorname{tr} R_{\mathrm{disc},\psi}(f)$.*

If $H = U(N_1) \times U(N_2)$, and $\psi^H = \psi_1 \times \psi_2$ we can write

$$R_{\mathrm{disc},\psi^H}(f) = R_{\mathrm{disc},\psi_1}(f_1) \cdot R_{\mathrm{disc},\psi_2}(f_2), \quad f = f_1 \times f_2 \in \mathcal{H}(H).$$

Following the above result, we can also write this as $I_{\mathrm{disc},\psi^H}(f)$ provided that ψ^H has regular infinitesimal character.

2.6.2. *Stabilization.* We now recall the identity that drives our theorems: the stabilization of $I_{\mathrm{disc},\psi}$, i.e., its decomposition into sum of stable traces of the transfers f^H of f for the endoscopic groups $H \in \mathcal{E}(G)$. Our references are to Arthur [5], but the versions for unitary groups are formally identical; see for example [26, (3.3.2)] and [38, (4.2.1)]. Recall that $\tilde{\Psi}(N)$ is the set of conjugate self-dual parameters, and $\Psi(G, \xi) \subset \tilde{\Psi}(N)$.

Theorem 25 [5, Corollary 3.3.2(b)]. *Suppose that $\psi \in \tilde{\Psi}(N)$ and let $f \in \mathcal{H}(G)$. Then for each endoscopic datum $(H, \xi_H) \in \mathcal{E}(G)$ there is a constant $\iota(G, H)$ and stable distributions $S^H_{\mathrm{disc},\psi}$ on $\mathcal{H}(H)$, defined inductively, such that*

$$I_{\mathrm{disc},\psi}(f) = \sum_{H \in \mathcal{E}(G)} \iota(G, H) S^H_{\mathrm{disc},\psi}(f^H). \tag{16}$$

Remark 26. For unitary groups, the global factor $\iota(G, H)$ is introduced in [38, Section 4.2] and [26, Section 3.1]. It is independent of the inner form G. If $G = U(N)$ and $H = U(N_1) \times U(N_2)$, then following [38, 4.2] we have

$$\iota(G, H) = \begin{cases} 1, & N_1 N_2 = 0, \\ \frac{1}{2}, & N_1, N_2 \neq 0, \ N_1 \neq N_2, \\ \frac{1}{4}, & N_1 = N_2 \neq 0. \end{cases} \tag{17}$$

3. Upper bounds from the stabilization

In this section we unpack the summands of the stabilization of $I_{\mathrm{disc},\psi}$ and extract upper bounds on the trace of test functions from the character identities.

3.1. ***The stable multiplicity formula.*** Let $\psi = (\psi^N, \tilde{\psi}) \in \Psi(G^*, \eta_\kappa)$. Recall the decomposition from (16):

$$I_{\mathrm{disc},\psi}(f) = \sum_{H \in \mathcal{E}(G)} \iota(G, H) S^H_{\mathrm{disc},\psi}(f^H). \tag{18}$$

The stable multiplicity formula expresses each $S^H_{\mathrm{disc},\psi}$ as a sum of traces. If f_v is a local test function and ψ_v a local parameter, the formula for $f^{H_v}(\psi_v)$ was given in Section 2.5.5. If $f = \prod_v f_v$ and ψ are global, we write $f^H(\psi) := \prod_v f^{H_v}(\psi_v)$. The group $\mathcal{S}_\psi$ and the element s_ψ were defined in Section 2.3.8, and ϵ_ψ in Section 2.3.9. The stable multiplicity formula, only defined for quasisplit groups, is the following expression:

Theorem 27 [38, Theorem 5.1.2]. *For $\psi \in \Psi(G, \eta_\kappa)$, we have*

$$S^G_{\mathrm{disc},\psi}(f) = |\mathcal{S}_\psi|^{-1} \epsilon^G_\psi(s_\psi)\sigma(\bar{S}^0_\psi) f^G(\psi).$$

For any connected reductive group S, the quantity $\sigma(S)$ was defined by Arthur in [5, Section 4.1]. The centralizers S_ψ of our ψ are always finite, so $\bar{S}^0_\psi$ is trivial and $\sigma(\bar{S}^0_\psi) = 1$; see [38, Remark 5.1.4]. The stable multiplicity formula is stated for G a unitary group (in which case the map $\psi \mapsto \psi^N$ is injective), but can be extended to products $H \in \mathcal{E}(G)$. Let $\Psi(H, \psi^N)$ be the set consisting of parameters $\psi^H = (\psi^{N,H}, \tilde{\psi}^H)$ with $\psi^{N,H} = \psi^N$. The stable multiplicity formula for H, given in [38, (5.6.3)], is

$$S^H_{\mathrm{disc},\psi}(f^H) = \sum_{\psi^H \in \Psi(H,\psi^N)} \frac{1}{|\mathcal{S}_{\psi^H}|} \epsilon^H_\psi(s^H_{\psi^H})\sigma(\bar{S}^0_{\psi^H}) f^H(\psi^H). \tag{19}$$

We combine (18) and (19) and rewrite the resulting expression as a sum over pairs (H, ψ^H) to get

$$I_{\mathrm{disc},\psi}(f) = \sum_{(H,\psi^H)} \iota(G, H) \frac{1}{|\mathcal{S}_{\psi^H}|} \epsilon^H_\psi(s^H_{\psi^H})\sigma(\bar{S}^0_{\psi^H}) f^H(\psi^H). \tag{20}$$

We now collect the terms that can be bounded uniformly, and let

$$C(\psi, H) := \iota(G, H)\sigma(\bar{S}^0_{\psi^H})|\mathcal{S}_{\psi^H}|^{-1}. \tag{21}$$

Lemma 28. *Let $\psi \in \Psi(G^*, \eta_\kappa)$ and let $(H, \xi_H, s_H) \in \mathcal{E}(G)$ be an endoscopic datum such that ψ factors through ξ_H. Then*:

(i) *The contribution of (H, ψ^H) to the sum* (20) *is equal to $C(\psi, H)\epsilon^H_\psi(s^H_\psi) f^H(\psi^H)$.*

(ii) *The constant $C(\psi, H)$ is bounded uniformly in ψ and H: it always satisfies $2^{-(N+1)} \le C(\psi, H) \le 1$, where N is the rank of G.*

Proof. Part (i) follows immediately from (20) and it suffices to exhibit the bound on $C(\psi, H)$. As stated above, we have $\sigma(\bar{S}^0_\psi) = 1$ since ψ is elliptic. We also gave uniform bounds on $\iota(G, H)$ in (17) and on $|\mathcal{S}_\psi|$ in (10). ☐

In Lemma 13, we gave a bijection between $\mathcal{S}_\psi$ and the set of pairs (H, ψ^H). We use it to reindex the sum (20) and obtain the expression

$$I_{\mathrm{disc},\psi}(f) = \sum_{s_H \in \mathcal{S}_\psi} C(\psi, s_H)\epsilon^H_\psi(s^H_{\psi^H}) f^H(\psi^H). \tag{22}$$

This sum depends on parameters and representations of H, which we want to rewrite in terms of G. For ϵ_ψ, we use Mok's so-called *endoscopic sign lemma.*

Lemma 29 [38, Lemma 5.6.1]. *Let $(H, \xi, s_H) \in \mathcal{E}(G)$ and $\psi \in \Psi(G^*, \eta_\kappa)$ be such that (H, ψ^H) corresponds to (ψ, s_H). Let $\epsilon^{G^*}_\psi$ and ϵ^H_ψ be the respective characters of ψ and ψ^H. Let $s^H_{\psi^H}$ be the image of $\psi^H(-I)$ in the quotient $\mathcal{S}^H_\psi$ associated to H. Then we have*

$$\epsilon^H_\psi(s^H_{\psi^H}) = \epsilon^{G^*}(s_\psi s_H).$$

We can now rewrite $I_{\mathrm{disc},\psi}(f)$ in a form conducive to extracting bounds.

Proposition 30. *Let $\psi \in \Psi(G, \xi)$, and let $f \in \mathscr{H}(G)$ be factorizable. Then*

$$\begin{aligned} I_{\mathrm{disc},\psi}(f) &= \sum_{s_H \in \mathcal{S}_\psi} C(\psi, s_H)\epsilon^{G^*}_\psi(s_\psi s_H) \prod_v \biggl(\sum_{\pi_v \in \Pi_{\psi_v}} \langle s_{\psi_v} s_{H_v}, \pi_v \rangle \operatorname{tr} \pi_v(f_v) \biggr) \\ &= \sum_{s_H \in \mathcal{S}_\psi} C(\psi, s_H) \sum_{\pi \in \Pi_\psi} \epsilon^{G^*}_\psi(s_\psi s_H) \langle s_\psi s_H, \pi \rangle \operatorname{tr} \pi(f). \end{aligned} \tag{23}$$

Proof. We start from the equality (22):

$$I_{\mathrm{disc},\psi}(f) = \sum_{s_H \in \mathcal{S}_\psi} C(\psi, s_H)\epsilon^H_\psi(s^H_\psi) f^H(\psi^H).$$

The distribution $f^H(\psi^H)$ was defined as $f^H(\psi^H) = \prod_v f^{H_v}_v(\psi^H_v)$. Each local factor can be written in terms of the trace of representations in Π_{ψ_v} by Theorems 18, 19, and 22. In all cases, the identity is

$$f^{H_v}_v(\psi^{H_v}_v) = e(G_v) \sum_{\pi_v \in \Pi_{\psi_v}} \langle s_{\psi_v} s_{H_v}, \pi_v \rangle \operatorname{tr} \pi_v(f_v).$$

The local Kottwitz signs cancel out globally, and using Lemma 29, we rewrite

$$I_{\mathrm{disc},\psi}(f) = \sum_{s_H \in \mathcal{S}_\psi} C(\psi, s_H)\epsilon^{G^*}_\psi(s_\psi s_H) \prod_v \biggl(\sum_{\pi_v \in \Pi_{\psi_v}} \langle s_{\psi_v} s_{H_v}, \pi_v \rangle \operatorname{tr} \pi_v(f_v) \biggr).$$

At all but finitely many v, we have $f_v = 1_{K_v}$ for a hyperspecial maximal compact subgroup K_v. At these places, $\operatorname{tr} \pi_v(f_v)$ is only nonzero on K_v-unramified representations π_v. Unramified local packets contain exactly one unramified representation following [26, Proposition 1.5.2(5)] so we interchange the sum and product to get

$$\prod_v \Bigg(\sum_{\pi_v \in \Pi_{\psi_v}} \langle s_{\psi_v} s_{H_v}, \pi_v \rangle \operatorname{tr} \pi_v(f_v) \Bigg) = \sum_{\pi \in \Pi_\psi} \Bigg(\prod_v \langle s_{\psi_v} s_{H_v}, \pi_v \rangle \Bigg) \operatorname{tr} \pi(f)$$

for $\pi = \otimes_v \pi_v$. Using the definition $\langle \cdot , \pi \rangle := \prod_v \langle \cdot , \pi_v \rangle$, we rewrite

$$I_{\mathrm{disc},\psi}(f) = \sum_{s_H \in \mathcal{S}_\psi} C(\psi, s_H) \sum_{\pi \in \Pi_\psi} \epsilon_\psi^{G^*}(s_\psi s_H) \langle s_\psi s_H, \pi \rangle \operatorname{tr} \pi(f). \qquad \square$$

3.2. ***Upper bounds and the dominant group.*** Recall once more the bijection $(H, \psi^H) \leftrightarrow (\psi, s_H)$ from Section 2.4.1. We will single out one object on either side, and show that for certain f, its contribution to the distribution $I_{\mathrm{disc},\psi}(f)$ bounds the others. Recall that $s_\psi \in \mathcal{S}_\psi$ was the image of the matrix $-I \in \mathrm{SL}_2$ under ψ.

Definition 31. Let (H_ψ, ψ^{H_ψ}) be the pair corresponding to the pair (ψ, s_ψ) containing the distinguished element s_ψ under the bijection $(H, \psi^H) \leftrightarrow (\psi, s_H)$.

Note that it is possible that $H_\psi = G$, for example when ψ is bounded.

Definition 32. Let $\psi \in \Psi(G, \xi)$ and let (H, ξ_H, s_H) be such that ψ factors through ξ_H. Let f be a global test function. Then define

$$S(\psi, s_H, f) = C(\psi, s_H) \sum_{\pi \in \Pi_\psi} \epsilon_\psi^{G^*}(s_\psi s_H) \langle s_\psi s_H, \pi \rangle \operatorname{tr} \pi(f).$$

Proposition 30 can then be reformulated as stating that

$$I_{\mathrm{disc},\psi}(f) = \sum_{s_H \in \mathcal{S}_\psi} S(\psi, s_H, f). \tag{24}$$

Lemma 33. *If $H = H_\psi$, then*

$$S(\psi, s_\psi, f) = C(\psi, s_\psi) \sum_{\pi \in \Pi_\psi} \operatorname{tr}(\pi)(f). \tag{25}$$

Proof. This follows since $s_{H_\psi} = s_\psi$ by definition. Since $\mathcal{S}_\psi \simeq (\mathbb{Z}/2\mathbb{Z})^{r-1}$, this implies that $\epsilon_\psi^{G^*}(s_\psi^2) = 1$ and $\langle s_\psi^2, \pi \rangle = 1$ for all π. $\square$

This allows us to state our main application of the stable trace formula.

Theorem 34. *Let G be a unitary group, let $\psi \in \Psi(G, \xi)$, and let $f \in \mathcal{H}(G)$ be a factorizable test function with $\operatorname{tr} \pi(f)$ real and nonnegative for all $\pi \in \Pi_\psi$. Then there exist a constant $C(\psi)$ such that*

$$I_{\mathrm{disc},\psi}(f) \le C(\psi) S(\psi, s_\psi, f).$$

The constant $C(\psi)$ satisfies $2^{-(N+1)} \le C(\psi) \le 2^{2N}$; it is thus bounded above and below independently of ψ.

Proof. We compare the various terms appearing in (24):

$$I_{\mathrm{disc},\psi}(f) = \sum_{s_H \in \mathcal{S}_\psi} S(\psi, s_H, f).$$

Ignoring for a moment the constants $C(\psi, s_H)$, the summands only differ from one another via the signs $\epsilon_\psi^{G^*}(s_\psi s_H)\langle s_\psi s_H, \pi\rangle \in \{\pm 1\}$ appearing as coefficients of the traces $\operatorname{tr}\pi(f)$. In the term coming from s_ψ, we get

$$S(\psi, s_\psi, f) = C(\psi, s_\psi) \sum_{\pi \in \Pi_\psi} \operatorname{tr}(\pi)(f)$$

from Lemma 33. For any other $s_H \in \mathcal{S}_\psi$, the coefficients $\epsilon_\psi^{G^*}(s_\psi s_H)\langle s_\psi s_H, \pi\rangle$ have the potential to be equal to -1. Thus if $\operatorname{tr}\pi(f) \geq 0$ for all $\pi \in \Pi_\psi$, we have

$$\begin{aligned} S(\psi, s_H, f) &= C(\psi, s_H) \sum_{\pi \in \Pi_\psi} \epsilon_\psi^{G^*}(s_\psi s_H)\langle s_\psi s_H, \pi\rangle \operatorname{tr}\pi(f) \\ &\leq C(\psi, s_H) \sum_{\pi \in \Pi_\psi} \operatorname{tr}(\pi)(f) = \frac{C(\psi, s_H)}{C(\psi, s_\psi)} \cdot S(\psi, s_\psi, f). \end{aligned}$$

Summing over the s_H we get

$$I_{\mathrm{disc},\psi}(f) \leq \left(\frac{\sum_{s_H \in \mathcal{S}_\psi} C(\psi, s_H)}{C(\psi, s_\psi)} \right) S(\psi, s_\psi, f) := C(\psi) S(\psi, s_\psi, f).$$

For the bounds, we showed in Lemma 28 that $2^{-(N+1)} \leq C(\psi, s_H) \leq 1$. As for the cardinality of $\mathcal{S}_\psi$, it is bounded between 1 and 2^{N-1} as we saw in Section 2.3.8. □

In practice, the group H_ψ is easily computed from $\psi\,|_{\mathrm{SL}_2}$.

Lemma 35. *Let* $\psi = \boxplus_i (\mu_i \boxtimes \nu(m_i)) \in \Psi(N)$ *be a global square-integrable Arthur parameter, and let* $N_1 = \sum_{m_i \equiv 1 \bmod 2} m_i$. *Then the group* H_ψ *is*

$$H_\psi = U(N_1) \times U(N - N_1).$$

Proof. By definition $s_\psi = \psi(1, -I) \in \mathrm{GL}_N$. The image of $-I$ under the m-dimensional representation of SL_2 is $(-1)^{m+1} I_m$. Thus $s_\psi = \mathrm{diag}(-I_{N_1}, I_{N_2})$, where $N_1 = \sum_{m_i \equiv 1 \bmod 2} m_i$ and $N_2 = N - N_1$, with centralizer $\mathrm{GL}_{N_1} \times \mathrm{GL}_{N_2}$. □

The image $\psi(\mathrm{SL}_2)$ and the group H_ψ are determined by any localization $\psi_v(\mathrm{SL}_2)$. In Section 5, we will use this, together with the known (archimedean) parameters of cohomological representations, to bound growth of cohomology.

3.3. *Hyperendoscopy.*

We recall the notion of hyperendoscopic datum first introduced by Ferrari [17]. We will use it to bound the expression $S(\psi, s_\psi, f)$. As pointed out by Dalal [13], the results of [17] do not quite hold in full generality, but they do hold for unitary groups, which have simply connected derived subgroups.

Definition 36. A chain of hyperendoscopic data for the (local or global) group G is a collection

$$\mathcal{H} = (G, H_1, \dots, H_q),$$

where H_1 is a proper endoscopic datum for G and H_{i+1} is a proper endoscopic datum for H_i.

The integer $p(\mathcal{H}) = q$ is the *depth* of the datum. Denote

$$\iota(\mathcal{H}) = (-1)^{p(\mathcal{H})}\iota(G, H_1) \cdot \iota(H_1, H_2) \cdot \dots \cdot \iota(H_{q-1}, H_q),$$

and $I^{\mathcal{H}}_{\mathrm{disc}} := I^{H_q}_{\mathrm{disc}}$. As with endoscopic data, two chains of hyperendoscopic data will be considered equivalent if they are conjugate under $\hat{G}$. Ferrari denotes by $\mathcal{HE}(G)$ the collection of equivalence classes of chains of hyperendoscopic data for G. If $\psi \in \Psi(G)$ is an Arthur parameter, we will denote by $\mathcal{HE}(G, \psi)$ the collection of equivalence classes of chains of hyperendoscopic data $\mathcal{H} \in \mathcal{HE}(G)$ such that ψ factors through the embedding $\xi_{p(\mathcal{H})}$ associated to $H_{p(\mathcal{H})}$. Note that the depth of chains in $\mathcal{HE}(G, \psi)$ is bounded above by the number of simple constituents of ψ.

If $\mathcal{H} \in \mathcal{HE}(G)$ is a chain of hyperendoscopic data, and f^G is a test function, we inductively define $f^{H_{i+1}} = (f^{H_i})^{H_{i+1}}$. The function $f^{H_{p(\mathcal{H})}}$ depends on a choice of transfer f^{H_i} at each step. We allow this, but require that our choice of f^{H_i} be consistent over chains that are truncations of one another. The following is the specialization to a parameter ψ of a trick initially discovered by Ferrari [17, 3.4.2].

Proposition 37. *Let $(G, \eta) \in \tilde{\mathcal{E}}(N)$ be quasisplit and let $\psi^N \in \Psi(N)$. Then*

$$S^G_{\mathrm{disc},\psi^N,\eta}(f) = \sum_{\mathcal{H}\in\mathcal{HE}(G,\psi)} \iota(\mathcal{H}) I^{H_q}_{\mathrm{disc},\psi^N,\eta\circ\xi_{p(\mathcal{H})}}(f^{H_q}).$$

Proof. If $\psi^N \in \Psi_{\mathrm{sim}}(N)$, then $I^G_{\mathrm{disc},\psi^N,\eta}(f) = S^G_{\mathrm{disc},\psi^N,\eta}(f)$ and the result holds trivially. Otherwise, we have that

$$S^G_{\mathrm{disc},\psi^N,\eta}(f) = I^G_{\mathrm{disc},\psi^N,\eta}(f) - \sum_{H\in\mathcal{E}(G,\psi^N)} \iota(G, H) S^H_{\mathrm{disc},\psi^N,\eta\circ\xi}(f^H). \tag{26}$$

By induction, for each H in $\mathcal{E}(G, \psi)$, we have

$$S^H_{\mathrm{disc},\psi^N,\eta\circ\xi}(f^H) = \sum_{\mathcal{H}\in\mathcal{HE}(G,\psi)} \iota(\mathcal{H}) I^{H_q}_{\mathrm{disc},\psi^N,\eta\circ\xi_{p(\mathcal{H})}}(f^{H_q}). \tag{27}$$

By construction, each $\mathcal{H} \in \mathcal{HE}(G, \psi^N)$ is obtained from a hyperendoscopic datum $\mathcal{H}' \in \mathcal{HE}(H, \psi^N)$ for some $H \in \mathcal{E}(G, \psi^N)$, and $p(\mathcal{H}) = p(\mathcal{H}') + 1$. Substituting (27) into (26) yields the result. □

Recall that when $G \in \tilde{\mathcal{E}}_{\mathrm{sim}}(N)$, the map $\psi \mapsto \psi^N$ is injective. On the other hand, if H is an product of unitary groups, there could be several parameters ψ^H for H such that $\psi^H \mapsto \psi^N$ under $\eta_{\kappa\cdot\underline{\kappa}}$. From [38, Section 5.6] we see that if $H = H_1 \times H_2$ with $H_i = U(N_i)$, and $f^H = f^{H_1} \times f^{H_2}$, then

$$S^H_{\mathrm{disc},\psi^N,\eta_{\kappa\cdot\underline{\kappa}}}(f^H) = \sum_{\psi^H=\psi_1\times\psi_2,\psi^H\mapsto\psi^N} S^{H_1}_{\mathrm{disc},\psi_1}(f^{H_1}) \times S^{H_2}_{\mathrm{disc},\psi_2}(f^{H_2}).$$

The expression $S(\psi, s_H, f)$ of Definition 32 picks out one of these summands.

Definition 38. Let $H = H_1 \times H_2$ as above, and let $\mathcal{H} \in \mathcal{HE}(H)$ with $p(\mathcal{H}) = q$. Then $H_q = H_{q_1} \times H_{q_2}$ with $H_{q_i} \in \mathcal{HE}(H_i)$. Let $\psi^H = \psi_1 \times \psi_2 \in \Psi(H, \eta_{\kappa\cdot\underline{\kappa}})$. For a test function $f^{H_q} = f^{H_{q_1}} \times f^{H_{q_2}}$, define

$$I^{H_q}_{\mathrm{disc},\psi^H}(f^H) = I^{H_{q1}}_{\mathrm{disc},\psi_1}(f^{H_{q_1}}) \times I^{H_{q2}}_{\mathrm{disc},\psi_2}(f^{H_{q_2}}).$$

Here we use the notation $I^{H_{q_i}}_{\mathrm{disc},\psi_i}$ as in Section 2.6 since $U(N_i) \in \tilde{\mathcal{E}}_{\mathrm{sim}}(N_i)$.

Corollary 39. *Let $H = H_1 \times H_2$ as above and let $\psi^H = \psi_1 \times \psi_2 \in \Psi(H, \xi_{\underline{\kappa}})$ so that (H, ψ^H) corresponds to (ψ^N, s_H) under the correspondence of Lemma 13. Assume that $f^H = f^{H_1} \times f^{H_2}$. Then*

$$S(\psi, s_H, f) = \iota(G, H) \sum_{\mathcal{H} \in \mathcal{HE}(H,\psi)} \iota(\mathcal{H}) I^{H_q}_{\mathrm{disc},\psi^H}(f^{H_q}).$$

Proof. We see in [38, Section 5.6] that the term $S^{H_1}_{\mathrm{disc},\psi_1}(f^{H_1}) \times S^{H_2}_{\mathrm{disc},\psi_2}(f^{H_2})$ is equal to

$$\frac{1}{|\mathcal{S}_{\psi_1}||\mathcal{S}_{\psi_1}|} \epsilon^{H_1}(\psi_1)\epsilon^{H_2}(\psi_2) f^{H_1}(\psi_1) f^{H_2}(\psi_2) = \frac{1}{|\mathcal{S}_\psi|}\epsilon^H(\psi) f^H(\psi).$$

By the argument of Proposition 30, the last expression is equal to

$$\frac{1}{|\mathcal{S}_\psi|} \sum_{\pi \in \Pi_\psi} \epsilon^{G^*}_\psi(s_\psi s_H)\langle s_\psi s_H, \pi\rangle \operatorname{tr}\pi(f) = \frac{S(\psi, s_H, f)\cdot}{|\mathcal{S}_\psi| C(\psi, s_H)} = \frac{S(\psi, s_H, f)}{\iota(G, H)}.$$

Applying Proposition 37 to each factor of $S^{H_1}_{\mathrm{disc},\psi_1}(f^{H_1}) \times S^{H_2}_{\mathrm{disc},\psi_2}(f^{H_2})$, we get

$$\begin{aligned} \frac{S(\psi, s_H, f)}{\iota(G, H)} &= \Bigg(\sum_{\mathcal{H}_1 \in \mathcal{HE}(H_1,\psi_1)} \iota(\mathcal{H}_1) I^{H_{q1}}_{\mathrm{disc},\psi_1}(f^{H_{q_1}}) \Bigg) \cdot \Bigg(\sum_{\mathcal{H}_2 \in \mathcal{HE}(H_2,\psi_2)} \iota(\mathcal{H}_2) I^{H_{q2}}_{\mathrm{disc},\psi_2}(f^{H_{q2}}) \Bigg) \\ &= \sum_{\mathcal{H} \in \mathcal{HE}(H,\psi)} \iota(\mathcal{H}) I^{H_q}_{\mathrm{disc},\psi^H}(f^{H_q}). \end{aligned}$$ □

4. Limit multiplicity

Here we apply the results of the previous section to the limit multiplicity problem.

4.1. ***Level structures.*** Let $\mathcal{O}_E$ and $\mathcal{O}_F$ be the rings of integers of the global fields E and F. We introduce sets of places of F:

- S_f is a finite set of finite places of F, containing the places which ramify in E as well as the places below those where the character χ_- introduced in Section 2.1 is ramified.
- S_∞ is the set of all infinite places of F.
- $S_0 \subsetneq S_\infty$ is a nonempty subset of the infinite places.
- $S = S_f \cup S_\infty$.

Note that the third requirement implies that $F \neq \mathbb{Q}$. Let $\mathfrak{p}$ be an ideal of F with residue characteristic strictly greater than $N^2[F:\mathbb{Q}]+1$, corresponding to a place $v_{\mathfrak{p}} \notin S$. For each finite place v of F, denote by $\mathcal{O}_{F_v}$ the ring of integers of F_v, and let $\hat{\mathcal{O}}_F = \prod_v \mathcal{O}_{F_v}$, and similarly for $\hat{\mathcal{O}}_E$. We define the subgroups $U(N,\mathfrak{p}^n) \subset U(N,\mathbb{A}_F^f)$ to be

$$U(N,\mathfrak{p}^n) := \{g \in U(N,\hat{\mathcal{O}}_F) \subset \mathrm{GL}_N(\hat{\mathcal{O}}_E) \mid g \equiv I_N(\mathfrak{p}^n\mathcal{O}_E)\}.$$

For any finite place v of F, let $U(N,\mathfrak{p}^n)_v = U(N,\mathfrak{p}^n) \cap U(N)_v$. At the expense of possibly enlarging the set S_f, note that for all $v \notin S \cup \{v_{\mathfrak{p}}\}$, the subgroup $U(N,\mathfrak{p}^n)_v$ is a hyperspecial maximal compact subgroup of $U(N)_v$. This gives level structures on the quasisplit group $U(N)$. If $H = U(N_1) \times \cdots \times U(N_r)$ is a product of quasisplit unitary groups, we define level subgroups $H(\mathfrak{p}^n) = U(N_1,\mathfrak{p}^n) \times \cdots \times U(N_r,\mathfrak{p}^n)$.

Let (G,ξ) be an inner form of $U(N,F)$ defined with respect to a Hermitian inner product and with prescribed signatures $U(a_v,b_v)$ at the archimedean places. We require that G_v be compact at the archimedean places contained in S_0: this ensures that the group G is anisotropic. Following Lemma 8, if N is odd, the group G can be chosen so that G_v is quasisplit at all finite places. If N is even, then G is determined by choosing at most one place $v \in S_f$, up to again enlarging S_f. Once that choice is made, the group G can be chosen to be quasisplit away from $\{v\} \cup S_\infty$. In both cases, this group G is realized as an inner form (G,ξ) as in Section 2.2.2.

For each finite $v \notin S_f$, the inner twist induces isomorphisms $\xi_v : G_v \simeq U(N)_v$. For each natural number n, we fix a compact subgroup $K(\mathfrak{p}^n) = \prod_v K_v(\mathfrak{p}^n)$ of $G(\mathbb{A}_F)$ as follows: at all finite $v \notin S$, we let $K_v(\mathfrak{p}^n) = \xi_v^{-1}(U(N,\mathfrak{p}^n)_v)$; at $v \in S_f$, the subgroup $K_v(\mathfrak{p}^n)$ is an arbitrary open compact subgroup fixed once and for all independently of n; at the archimedean places we let $K_v(\mathfrak{p}^n) \simeq U_{a_v}(\mathbb{R}) \times U_{b_v}(\mathbb{R})$ be a maximal compact subgroup. Let $K_f(\mathfrak{p}^n) = \prod_{v<\infty} K_v(\mathfrak{p}^n)$ and $K_\infty(\mathfrak{p}^n) = \prod_{v|\infty} K_v(\mathfrak{p}^n)$. We may use the notation K_v instead of $K_v(\mathfrak{p}^n)$ for $v \neq v_{\mathfrak{p}}$. We extend these definitions to products of unitary groups.

We now define the (cocompact since G is anisotropic) lattices

$$\Gamma(\mathfrak{p}^n) := G(F) \cap K_f(\mathfrak{p}^n).$$

Recall that $G_\infty = \prod_{v|\infty} G_v$ and let $X_G = G_\infty / K_\infty Z_{G_\infty}$. Assume that G_∞ has at least one noncompact factor. The diagonal embedding $\Gamma(\mathfrak{p}^n) \hookrightarrow \prod_{v|\infty} G_v$ induces an action $\Gamma(\mathfrak{p}^n) \curvearrowright X_G$, and we let $X(\mathfrak{p}^n) := \Gamma(\mathfrak{p}^n) \backslash X_G$. We start by comparing them to their disconnected counterparts realized as adelic double quotients. Let

$$Y(\mathfrak{p}^n) = G(F) \backslash G(\mathbb{A}_F) / K(\mathfrak{p}^n) Z_G(\mathbb{A}_F).$$

The quotient $Y(\mathfrak{p}^n)$ is a disjoint union of finitely many connected locally symmetric spaces, each associated to a conjugate ot $K(\mathfrak{p}^n)$. In particular, the summand corresponding to $K(\mathfrak{p}^n)$ is $X(\mathfrak{p}^n)$.

Proposition 40. *Let G be an inner form of $U(N)$ and $Y(\mathfrak{p}^n)$ be defined as above. The cardinality of the set of components $\pi_0(Y(\mathfrak{p}^n))$ is bounded independently of n.*

Proof. We adapt an argument from [16, Section 2]. Considering G as a subgroup of GL_N / E, let $\det : G \to U(1, E/F)$ be the determinant map and let $G^1 = \ker(\det)$. This map induces a fibering of

$Y(\mathfrak{p}^n)$ over

$$U(1,F)\backslash U(1,\mathbb{A}_F)/\det(Z(\mathbb{A}_F)K(\mathfrak{p}^n)).$$

The fibers are adelic double quotients for the group G^1, which is simply connected and has at least one noncompact factor at infinity. So by [41, 7.12], the group G^1 satisfies strong approximation with respect to the set S_∞ and $G^1(F)$ is dense in $G^1(\mathbb{A}_F^f)$, making the fibers connected. Thus we find that

$$\pi_0(Y(\mathfrak{p}^n)) \simeq U(1,F)\backslash U(1,\mathbb{A}_F)/\det(Z(\mathbb{A}_F)K(\mathfrak{p}^n)) = E^1\backslash\mathbb{A}_E^1/\det(Z(\mathbb{A}_F)K(\mathfrak{p}^n)).$$

Now the image $\det(Z(\mathbb{A}_F))$ is the subgroup $(\mathbb{A}_E^1)^N$ of $\mathbb{A}_E^1$. For each finite place w, the factor corresponding to E_w in the quotient $\mathbb{A}_E^1/(\mathbb{A}_E^1)^N$ is a finite set. It follows that by increasing the level in powers of a single prime $\mathfrak{p}$, one can only produce a bounded number of components. □

We now fix a unitary irreducible admissible representation $\pi_\infty = \otimes_{v|\infty}\pi_v$ of G_∞ with trivial central character and such that π_v is the trivial representation if G_v is compact. Denote

$$m(\pi_\infty,\mathfrak{p}^n) := \dim \operatorname{Hom}_{G_\infty}(\pi_\infty, L^2(\Gamma(\mathfrak{p}^n)\backslash G_\infty)). \tag{28}$$

Since $X(\mathfrak{p}^n)$ is one of the connected components of $Y(\mathfrak{p}^n)$, we have

$$m(\pi_\infty,\mathfrak{p}^n) \le \dim \operatorname{Hom}_{G_\infty}(\pi_\infty, L^2(Y(\mathfrak{p}^n))) = \sum_{\pi=\pi_\infty\otimes\pi_f} m(\pi)\dim \pi_f^{K_f(\mathfrak{p}^n)}. \tag{29}$$

We will be interested in the asymptotics of the multiplicities $m(\pi_\infty,\mathfrak{p}^n)$ as $n\to\infty$.

4.2. *Choice of test functions.*

We define test functions whose traces will compute the multiplicity of archimedean representations at level $\mathfrak{p}^n$. Recall that μ_v denotes the Haar measure on G_v.

Definition 41. At each finite place v, let $f_v(\mathfrak{p}^n) := 1_{K_v(\mathfrak{p}^n)}/\mu_v(K_v(\mathfrak{p}^n))$.

Definition 42. Let $v\in S_0$ be an archimedean place such that G_v is compact. Let $f_v(\mathfrak{p}^n)$ be equal to the constant function $f_v = \mu_v(G_v)^{-1}$.

The traces of these test functions count the dimension of spaces of $K(\mathfrak{p}^n)$-fixed vectors. At $v\in S_0$, they only detect the trivial representation and have vanishing trace on all other representations of G_v. We want functions that play the same role at the noncompact archimedean places: they should detect representations π_v contained in a specific subset $\Pi^0_{\psi_v}\subset\Pi_{\psi_v}$ and vanish on $\Pi_{\psi_v}\setminus\Pi^0_{\psi_v}$. The key is that we will only be working with Arthur packets attached to parameters ψ all having one specific ψ_∞. As such, the test function at an infinite place v only needs to isolate $\pi_v\in\Pi^0_v$ from the other finitely many representations in the same packet.

Lemma 43. *Let ψ_v be a local Arthur parameter with associated Arthur packet Π_{ψ_v}. Fix a subset $\Pi^0_{\psi_v}\subset\Pi_{\psi_v}$. Then there exists a function $f_v^0\in\mathcal{H}(G_v)$ such that*

$$\operatorname{tr}\pi_v(f_v^0) = \begin{cases} 1, & \pi_v = \Pi^0_{\psi_v}, \\ 0, & \textit{otherwise}, \end{cases} \qquad \pi_v\in\Pi_v.$$

Proof. This follows directly from linear independence of characters for admissible representations. If v is archimedean this was proved by Harish-Chandra in [22]. □

Definition 44. Let v be a noncompact archimedean place, let ψ_v be an Arthur parameter and fix a subset $\Pi^0_{\psi_v} \subset \Pi_{\psi_v}$. Let $f_v(\mathfrak{p}^n) = f_v(\mathfrak{p}^n, \Pi^0_{\psi_v})$ be the function f^0_v described above.

Definition 45. Let the function $f(\mathfrak{p}^n)$ be defined as $f(\mathfrak{p}^n) = \prod_v f_v(\mathfrak{p}^n_v)$. We will also denote $f_f(\mathfrak{p}^n) = \prod_{v\nmid\infty} f_v(\mathfrak{p}^n_v)$.

Given a choice of ψ_∞ and $\Pi^0_{\psi_\infty}$, the function $f(\mathfrak{p}^n)$ satisfies the assumption of Theorem 34: it is factorizable and has nonnegative trace on $\pi \in \Pi_\psi$.

Proposition 46. *Let $\psi \in \Psi(G, \xi)$. For each $v \in S_\infty \setminus S_0$, fix a subset $\Pi^0_{\psi_v}$ and a corresponding function $f(\mathfrak{p}^n) = f(\mathfrak{p}^n, \Pi^0_{\psi_v})$. Then we have*

$$\operatorname{tr} R_{\mathrm{disc},\psi}(f(\mathfrak{p}^n)) = \sum_{\pi} m(\pi) \dim \pi_f^{K_f(\mathfrak{p}^n)}$$

where the sum is taken over representations $\pi = (\otimes_{v|\infty}\pi_v) \otimes \pi_f \in \Pi_\psi$ such that for archimedean v, the representation π_v is trivial if $v \in S_0$ and $\pi_v \in \Pi^0_v$ otherwise.

Proof. As stated in Theorem 23, the distribution $\operatorname{tr} R_{\mathrm{disc},\psi}(f)$ computes the sum of $\operatorname{tr}\pi(f) = \prod_v \operatorname{tr}\pi_v(f_v)$ over all representations in the packet Π_ψ. At the finite places, the trace of convolution by the characteristic function of a compact open subgroup K_v is equal to the product $\mu_v(K_v) \cdot \dim \pi_v^{K_v}$. For archimedean places $v \in S_0$, the representations π_v are finite-dimensional so the only representation with a K_v-fixed vector is the trivial representation. At $v \in S_\infty \setminus S_0$, the function $f_v(\mathfrak{p}^n)$ was chosen precisely to detect $\pi_v \in \Pi^0_{\psi_v}$. □

The key input allowing us to compare multiplicity growth on G and $H \in \mathcal{E}(H)$ is a fundamental lemma for congruence subgroups, proved by Ferrari [17].

Theorem 47 [17, Theorem 3.2.3]. *Let $\mathfrak{p}$ be a prime of F with localization $F_{v_\mathfrak{p}}$ and residue field $k_\mathfrak{p}$. Let* $\mathrm{Nm}(\mathfrak{p})$ *be the cardinality of $k_\mathfrak{p}$ and let p be its characteristic. Assume that $p > N^2[F : \mathbb{Q}] + 1$. Let $H \in \mathcal{E}(G)$, and $d(G, H) = (\dim G - \dim H)/2$. Then the functions*

$$f_{v_\mathfrak{p}}(\mathfrak{p}^n) = \frac{1_{K_{v_\mathfrak{p}}(\mathfrak{p}^n)}}{\mu_{v_\mathfrak{p}}(K_{v_\mathfrak{p}}(\mathfrak{p}^n))} \quad \textit{and} \quad f^H_{v_\mathfrak{p}}(\mathfrak{p}^n) = \mathrm{Nm}(\mathfrak{p})^{-n\cdot d(G,H)} \frac{1_{K_{v_\mathfrak{p}}(\mathfrak{p}^n)^H}}{\mu_{v_\mathfrak{p}}(K_{v_\mathfrak{p}}(\mathfrak{p}^n)^H)}$$

form a transfer pair.

4.3. ***Adaptation of previous limit multiplicity results.*** Here, we collect all the results so far and import known upper bounds from the literature to prove our main limit multiplicity results. We start with a discussion of central characters.

4.3.1. *Central character data.* Our initial discussion of the discrete spectrum in Section 2.3.1 included the prescription of a subgroup $\mathfrak{X} \subset Z_G(\mathbb{A}_F)$, and we recalled in Proposition 15 that in the case where $G = U_{E/F}(N)$, the central character of representations in a packet Π_ψ is determined explicitly in terms of ψ. We now need to extend this discussion to central characters of $H \in \mathcal{E}(G)$. For this, we will denote $\mathfrak{X}_G = Z_G(\mathbb{A}_F)$, and $\mathfrak{X}_H = Z_H(\mathbb{A}_F)$. As described in [5, Section 3.2], the group $\mathfrak{X}_G$ can be viewed canonically as a subgroup of $\mathfrak{X}_H$, and we can speak of $(\mathfrak{X}_G, \omega)$ as a central character datum of H, though it is not properly speaking a character $Z_H(\mathbb{A}_F)$. This can be extended inductively to $H \in \mathcal{EH}(G)$. From Proposition 15 and the definition of the embeddings in (7), we get the following.

Lemma 48. *Let G be a unitary group and $(H, \xi_H) \in \mathcal{E}(G)$. Let $\psi \in \Psi(G, \xi)$ be associated to the central character datum $(\mathfrak{X}_G, \omega)$, and be such that $\psi = \xi_H \circ \psi^H$. Then the central character datum $(\mathfrak{X}_G, \omega')$ associated to ψ^H is determined by ω and ξ_H.*

4.3.2. *Sets of parameters.*

Definition 49. Let G be a reductive group, ψ_∞ a parameter of G_∞, and $(\mathfrak{X}, \omega)$ a central character. We denote by $\Psi(G, \psi_\infty, \omega)$ the set of $\psi \in \Psi(G)$ such that $(\psi)_\infty = \psi_\infty$ and such that the associated representations in the packet of ψ have central character ω.

Definition 50. Letting G be as above, and $(H, \xi) \in \mathcal{E}(G)$ or $H = G$, we define for $f \in \mathcal{H}(H)$,

$$I^H_{\text{disc},\psi_\infty,\omega}(f) = \sum_{\psi \in \Psi(G,\psi_\infty,\omega)} I^H_{\text{disc},\psi}(f). \tag{30}$$

We will need to rewrite the right-hand side of (30) as a sum over parameters of H. By definition, for each $\psi \in \Psi(G, \eta)$, we have

$$I^H_{\text{disc},\psi}(f) = \sum_{c^H \overset{\eta\circ\xi}{\to} c(\psi)} I^H_{\text{disc},c^H}(f),$$

for $c^H \in \mathcal{C}(H)$. But following the main theorem of the spectral expansion of the trace formula [38, Proposition 4.3.4] applied to each of the simple factors of H, shows that $I^H_{\text{disc},c^H}(f) = 0$ unless $c^H = c(\psi^{H,N})$ is attached to a parameter $\psi^H = (\tilde{\psi}^H, \psi^{N,H})$. By the assumption $c^H \xrightarrow{\eta\circ\xi} c(\psi)$, we mush have $\psi^{H,N} = \psi^N$, and $\tilde{\psi}$ must factor through ${}^L H$. Thus we can rewrite

$$\sum_{\psi \in \Psi(G,\psi_\infty,\omega)} I^H_{\text{disc},\psi}(f) = \sum_{\psi \in \Psi(G,\psi_\infty,\omega)} \sum_{\psi^H \mapsto \psi} I^H_{\text{disc},\psi^H,\omega}(f).$$

In the arguments of this section, we will work with three families of groups, and three sets of parameters, which we describe now.

- The group $G \in \tilde{\mathcal{E}}_{\text{sim}}(N)$ is the inner form of $U_{E/F}(N)$ for which we ultimately want to produce bounds. These will be obtained in Theorem 56 by taking a sum over $\Psi(G, \psi_\infty, 1)$, for the trivial central character of Z_G.

- The group $H = H_\psi = U(N_1) \times U(N_2)$ with $N_1 + N_2 = N$ belongs to the endoscopic datum $(H, \xi, s) \in \mathcal{E}(G)$ whose stable trace gives the upper bounds in Theorem 34. In the lead-up to Theorem 34, for each $\psi \in \Psi(G, \xi)$, we singled out a parameter ψ^H such that $\xi \circ \psi^H = \psi$. Through this choice, the parameter ψ_∞ of G_∞ determines a unique parameter ψ^H_∞ of H_∞. In Proposition 54, the sum will be taken over the set $\Psi(H, \psi^H_\infty, \omega)$ for a suitable central character ω.
- The difference between the distribution giving the upper bounds in Theorem 34 and $I^H_{\mathrm{disc},\psi^H}$ is expressed in Corollary 39 in terms of hyperendoscopic data (H_q, ξ_q) for H. We will consider parameters such that $\xi_q(\psi^{H_q}) \in \Pi(H, \psi^H_\infty, \omega)$. We have $\psi^{H_q}_\infty = \psi^{H^1_q}_\infty \times \psi^{H^2_q}_\infty$ and we will give upper bounds on the multiplicities of representations associated to each of these factors in Propositions 51 and 52.

4.3.3. *Upper bounds for hyperendoscopic groups.* We start by adapting limit multiplicity results of Savin [43], which will form the basis for our inductive proof. Since Savin's result applies to semisimple groups, we pay attention to the central characters and components of locally symmetric spaces. We first give bounds for bounded parameters. The result is stated in terms of any G and H, but will be specialized to $G = U(N_2)$ and $H = H^2_q$.

Proposition 51. *Let $G \in \tilde{\mathcal{E}}_{\mathrm{sim}}(N)$ and $\mathcal{H} \in \mathcal{HE}(G)$ be a hyperendoscopic group. Let $\psi_\infty \in \Psi(G_\infty)$ be a bounded parameter with regular infinitesimal character. Let $(\mathfrak{X}_G, \omega)$ be a central character datum for G such that $\omega\,|_{(\mathfrak{X}_G \cap G_\infty)}$ is the central character associated to ψ_∞ by Proposition 15. Let $v_\mathfrak{p}$ be an unramified finite place of F, associated to the prime $\mathfrak{p}$, and let $f(\mathfrak{p}^n) = \prod_v f_v(\mathfrak{p}^n) \in \mathscr{H}(H)$ be such that*:

- *$f_v(\mathfrak{p}^n)$ is independent of n if $v \neq v_\mathfrak{p}$.*
- *$f_{v_\mathfrak{p}}(\mathfrak{p}^n) = 1_{K(\mathfrak{p}^n)}/\mu(K(\mathfrak{p}^n))$, for $K(\mathfrak{p}^n)$ as in Section 4.1.*
- *$f(\mathfrak{p}^n)$ satisfy the assumptions of Theorem 34.*

Then

$$|I^H_{\mathrm{disc},\psi_\infty,\omega}(f(\mathfrak{p}^n))| \ll Nm(\mathfrak{p}^n)^{\dim H - 1}.$$

Proof. Since the infinitesimal character of ψ_∞ is regular, we can equate

$$I^H_{\mathrm{disc},\psi_\infty,\omega}(f(\mathfrak{p}^n)) = \operatorname{tr} R^H_{\mathrm{disc},\psi_\infty,\omega}(f(\mathfrak{p}^n)) = \sum_{\psi^H_\infty \mapsto \psi_\infty} \sum_{\psi^H \in \Psi(H, \psi^H_\infty, \omega')} \sum_{\pi \in \Pi_{\psi^H}} m(\pi) \operatorname{tr} \pi(f(\mathfrak{p}^n)),$$

where $(\mathfrak{X}, \omega')$ is the central character datum associated to ψ as in Lemma 48. Since the first sum is finite, we will bound

$$\Bigg| \sum_{\psi^H \in \Psi(H, \psi^H_\infty, \omega')} \sum_{\pi \in \Pi_{\psi^H}} m(\pi) \operatorname{tr} \pi(f(\mathfrak{p}^n)) \Bigg|.$$

For each π, we have $\operatorname{tr} \pi(f(\mathfrak{p}^n)) = \prod_v \operatorname{tr} \pi_v(f_v(\mathfrak{p}^n))$. At $v \mid \infty$, the packet is always the same, so $|\operatorname{tr} \pi_\infty(f_\infty)|$ is uniformly bounded. For each finite v, there is an open compact subgroup $K'_v \subset G_v$, depending on n only if $v = v_\mathfrak{p}$, such that $|\operatorname{tr} \pi_v f_v(\mathfrak{p}^n)| \neq 0 \Longrightarrow \dim \pi_v^{K'_v} \neq 0$. Indeed, since $f_v(\mathfrak{p}^n)$ is

K_v-finite, where K_v is a maximal compact subgroup, there is a subgroup $K'_v \leq K_v$ of finite index such that $f_v(\mathfrak{p}^n)$ is K'_v-invariant, so that convolution by f_v is a projection onto $\pi_v^{K'_v}$. At all but finitely many places the group K_v is hyperspecial, we have $f_v(\mathfrak{p}^n) = 1_{K_v}$ and we can take $K'_v = K_v$. Thus we have $|\operatorname{tr} \pi_v f_v| < C(f_v) \dim \pi^{K'_v}$ and by Bernstein's uniform admissibility [8], the right-hand side is bounded uniformly, with $C(f_v(\mathfrak{p}^n)) = 1$ at $v \notin S$. At $v = v_p$, we have $K'_v = K_v(\mathfrak{p}^n)$. Let $K'(\mathfrak{p}^n) = \prod_{v<\infty} K'_v$. From our restriction on the central character, we thus have

$$\left| \sum_{\Psi(H,\psi^H_\infty,\omega')} \sum_{\pi \in \Pi_{\psi^H}} m(\pi) \operatorname{tr} \pi(f(\mathfrak{p}^n)) \right| \leq C(\psi_\infty, S) \sum_{\substack{\pi : \pi_\infty \in \Pi_{\psi^H_\infty} \\ \omega(\pi)=\omega'}} m(\pi) \dim \pi_f^{K'(\mathfrak{p}^n)},$$

where $C(\psi_\infty, S)$ is a constant depending only on ψ_∞ and S.

Since ψ^H_∞ is bounded, any representation $\pi_\infty \in \Pi_{\psi^H_\infty}$ is tempered, which implies that $\pi \in \Pi^H_\psi$ occur in the cuspidal part of the discrete spectrum [52, Theorem 4.3]. Thus for each $\pi_\infty \in \Pi_{\psi^H_\infty}$ we have

$$\sum_{\substack{\pi=\pi_\infty \cdot \pi_f \\ \omega(\pi)=\omega'}} m(\pi) \dim \pi_f^{K'(\mathfrak{p}^n)} \leq \dim \operatorname{Hom}_{H_\infty}(\pi_\infty, L^2_{\text{cusp}}(H(F)\backslash H(\mathbb{A}_F), \omega')^{K'(\mathfrak{p}^n)}).$$

The right-hand side of the inequality is equal to

$$m(\pi_\infty, \mathfrak{p}^n, \omega') := \dim \operatorname{Hom}_{H_\infty}(\pi_\infty, L^2_{\text{cusp}}(H(F)\backslash H(\mathbb{A}_F)/K'(\mathfrak{p}^n), \omega')). \tag{31}$$

The space $Y^*_H(\mathfrak{p}^n) := H(F)\backslash H(\mathbb{A}_F)/K'(\mathfrak{p}^n)$ carries commuting actions of H_∞ and $Z_H(\mathbb{A}_F)$, inducing representations on $L^2_{\text{cusp}}(Y^*_H(\mathfrak{p}^n))$. For n large enough, the character ω' is trivial on $\mathfrak{X}_G \cap K'(\mathfrak{p}^n)$, and thus appears in the representation of $\mathfrak{X}_G$ on $L^2_{\text{cusp}}(Y^*_H(\mathfrak{p}^n))$. It is this ω'-isotypic subspace that we denote by $L^2_{\text{cusp}}(Y^*_H(\mathfrak{p}^n), \omega')$.

To bound $m(\pi_\infty, \mathfrak{p}^n, \omega')$, consider first the case where the central character datum for $\mathfrak{X}_H$ is trivial: this setup is similar to that of Proposition 40. We have $H = U(N_1) \times \cdots \times U(N_r)$; let $H^1 = \mathrm{SU}(N_1) \times \cdots \times \mathrm{SU}(N_r)$. The representation π_∞ of H_∞ restricts to an irreducible representation ρ_∞ of H^1_∞; see [2, Section 2]. Let

$$X_H(\mathfrak{p}^n) = H^1(F)\backslash H^1(\mathbb{A})/K^1(\mathfrak{p}^n), \quad K^1(\mathfrak{p}^n) = K'(\mathfrak{p}^n) \cap G^1(\mathbb{A}).$$

The group H^1 is simply connected and has no compact factors at infinity, so $X_H(\mathfrak{p}^n)$ is connected [41]. Following a result of Savin [43], we have

$$m(\rho_\infty, \mathfrak{p}^n) := \dim \operatorname{Hom}_{H^1_\infty}(\rho_\infty, L^2_{\text{cusp}}(X_H(\mathfrak{p}^n))) \asymp \operatorname{Vol}(X_H(\mathfrak{p}^n)) \asymp Nm(\mathfrak{p})^{n \cdot \dim H^1}.$$

We now consider general central characters. The space $Y^*_H(\mathfrak{p}^n)$ is a disjoint union of finitely many locally symmetric spaces, associated to conjugates of $K^1(\mathfrak{p}^n)$, and the theorem of Savin applies to each of them. Let $T = H/H^1$, and let ν denote the quotient map, through which all central characters factor, see Proposition 15. Following [14, 2.7.1], the set $\pi_0(Y^*_H(\mathfrak{p}^n))$ is a torsor for the finite group

$$T_{\mathfrak{p}^n} := T(\mathbb{A}_F)/T(F)\nu(K'(\mathfrak{p}^n)).$$

Denote by $\mathfrak{X}_{G,\mathfrak{p}^n}$ the image of $\mathfrak{X}_G$ in this quotient. The action of $\mathfrak{X}_{G,\mathfrak{p}^n}$ on $\pi_0(Y^H(\mathfrak{p}^n))$ is induced by multiplication in $T_{\mathfrak{p}^n}$, thus $\pi_0(Y^*_H(\mathfrak{p}^n))$ is a finite union of $[T_{\mathfrak{p}^n} : \mathfrak{X}_{G,\mathfrak{p}^n}]$ many principal homogeneous spaces for $\mathfrak{X}_{G,\mathfrak{p}^n}$. Thus as a $\mathfrak{X}_{G,\mathfrak{p}^n}$-representation, the space $\mathrm{Hom}(\pi_\infty, L^2_{\mathrm{cusp}}(Y^*_H(\mathfrak{p}^n)))$ caries finitely many copies of the regular representation of $\mathfrak{X}_{G,\mathfrak{p}^n}$, and all characters of $\mathfrak{X}_G$ factoring through $\mathfrak{X}_{G,\mathfrak{p}^n}$ occur with equal multiplicity. The group $\mathfrak{X}_G$ is the adelic points of torus of diagonal matrices isomorphic to $U(1)$, and $T \simeq U(1)^r$. So each character ω' of $\mathfrak{X}_G$ factoring through $\mathfrak{X}_{G,\mathfrak{p}^n}$ does so with multiplicity

$$m(\pi_\infty, \mathfrak{p}^n, \omega') = m(\rho_\infty, \mathfrak{p}^n)[T_{\mathfrak{p}^n} : T^H_{\mathfrak{p}^n}] = m(\rho_\infty, \mathfrak{p}^n)\frac{[T(1) : T(\mathfrak{p}^n)]}{[\mathfrak{X}_{G,1} : \mathfrak{X}_{G,\mathfrak{p}^n}]} \asymp \mathrm{Nm}(\mathfrak{p}^n)^{\dim H-1}.$$

Summing over all π_∞ in Π_{ψ_∞}, we conclude. ☐

We now give bounds for parameters where $\psi(\mathrm{SL}_2)$ is maximally large. In the final proof, the group G will be specialized to $U(N_1)$.

Proposition 52. *Let* $G \in \tilde{\mathcal{E}}_{\mathrm{sim}}(N)$, *and* $\psi_\infty \in \Psi(G_\infty)$. *Let* $f(\mathfrak{p}^n) = \prod_v f_v(\mathfrak{p}^n)$ *satisfy the same assumptions as in Proposition 51. Let* $\psi_\infty \in \Psi(G_\infty)$ *be a parameter with regular infinitesimal character and such that* $\psi_\infty|_{\mathrm{SL}_2} = \nu(N)$. *Let* $(\mathfrak{X}_G, \omega)$ *be a central character datum. Then there is a constant* M *depending only on* G, ψ_∞, *and the set* S *of bad places* (*and in particular neither on* n *nor on* ω) *such that*

$$|I^G_{\mathrm{disc},\psi_\infty,\omega}(f(\mathfrak{p}^n))| < M.$$

Proof. The proof is a simplified version of that of Proposition 51. The restriction of the infinitesimal character and on the possible representations at infinity gives

$$|I^G_{\mathrm{disc},\psi_\infty,\omega}(f(\mathfrak{p}^n))| \le C(\psi_\infty, S) \sum_{\substack{\pi : \pi_\infty \in \Pi_{\psi_\infty} \\ \omega(\pi) = \omega}} m(\pi) \dim \pi_f^{K'(\mathfrak{p}^n)}.$$

The assumption on the Arthur SL_2 implies that the representations $\pi_\infty \in \Pi_{\psi_\infty}$ are one-dimensional; see, e.g., [4, Section 5]. Thus they factor through the determinant map ν, and, as above, through the action of the quotient $T(\mathbb{A}_F)/T(F)$. It follows that in this case the multiplicity $m(\pi_\infty, \mathfrak{p}^n, \chi)$ is bounded above by $|T(\mathbb{A}_F)/T(F)\nu(K(\mathfrak{p}^n))|$. Recall here that $\mathfrak{X}_G = Z(\mathbb{A}_F)$. If ω were trivial, then the representations contributing to $m(\pi_\infty, \mathfrak{p}^n, \omega)$ would be bounded above by the size of the quotient $|T(\mathbb{A}_F)/T(\mathbb{Q})\nu(K(\mathfrak{p}^n) \cdot Z(\mathbb{A}_F))|$, which we showed in Proposition 15 to be bounded independently of n. But by the proof of Proposition 51, the representation of $\mathfrak{X}_G$ on $\mathrm{Hom}(\pi_\infty, L^2_{\mathrm{cusp}}(Y^*H(\mathfrak{p}^n)))$ factors through a sum of copies of the regular representation of a finite quotient $\mathfrak{X}_{G,\mathfrak{p}^n}$. As such, all characters of $\mathfrak{X}_G$ appearing in the quotient do so with equal multiplicity. Thus the bound M also holds for ω. ☐

4.4. *Limit multiplicity for G.*

Before we assemble the results for various endoscopic groups, we bound the number of central character data $(\mathfrak{X}_H, \omega)$ of a given level and restriction to $\mathfrak{X}_G$.

Lemma 53. *Let* $H \in \mathcal{E}(G)$. *Let* $(\mathfrak{X}_G, \omega)$ *be a central character datum for* G. *For each* n, *fix a level structure* $K^H_f(\mathfrak{p}^n)$ *as in Section 4.1. Define*

$$\Xi(\omega, \mathfrak{p}^n) = \{(\mathfrak{X}_H, \omega_H) : \omega_H|_{\mathfrak{X}_G} = \omega, \omega_H(\mathfrak{X}_H \cap K^H_f(\mathfrak{p}^n)) = \omega_H(\mathfrak{X}_H \cap Z_H(F)) = 1\}.$$

Then we have $|\Xi(\omega, \mathfrak{p}^n)| \ll \mathrm{Nm}(\mathfrak{p}^n)$.

Proof. Central characters of $H = U_{E/F}(N_1) \times U_{E/F}(N_2)$ are products $\omega_H = \omega_1 \times \omega_2$ of characters of the respective centers. The condition upon restriction to $Z_H(F)$ implies that these are of the form $\omega_i = \theta_i \circ \det$ for θ_i a Hecke character of $\mathbb{A}_E^1$. Given a choice of θ_1, the condition that $\omega_H\,|_{\mathfrak{X}_G} = \omega$ restricts θ_2 to at most N_2 different characters. The restriction on conductor thus implies that $|\Xi(\chi, \mathfrak{p}^n)| \ll |\Xi(\mathfrak{a}\mathfrak{p}^n)|$, where $\Xi(\mathfrak{a}\mathfrak{p}^n)$ consists of Hecke characters θ_1 of $\mathbb{A}_E^1/E^1$ whose conductor divides $\mathfrak{a}\mathfrak{p}^n$; the presence of a conductor away from $\mathfrak{p}$ comes from the possibility that at places $v \in S_f$, the (fixed) subgroup $K_v^H(\mathfrak{p}^n)$ is not maximal. The number of such characters grows like $\mathrm{Nm}(\mathfrak{p}^n)$. □

We now assemble the results of Section 4.3 to give upper bounds for the contribution of parameters where all but one summand have trivial Arthur SL_2. We start by bounding the contribution of each hyperendoscopic group of H_ψ.

Proposition 54. *Let $G \in \tilde{\mathcal{E}}_{\mathrm{sim}}(N)$, and $H = (U_{E/F}(N_1) \times U_{E/F}(N_2), \xi) \in \mathcal{E}(G)$. Let ψ_∞ and ψ_∞^H be such that*:

1. $\psi_\infty = \xi \circ \psi_\infty^H$.
2. $\psi_\infty = \psi_{\infty,1} \oplus \psi_{\infty,2}$ *with* $\psi_{\infty,1}\,|_{\mathrm{SL}_2} = \nu(N_1)$ *and* $\psi_{\infty,2}\,|_{\mathrm{SL}_2} = \nu(1)^{N_2}$.
3. *Each $\psi_{i,\infty}$ factors through ${}^L U(N_i)$.*

Let $(\mathfrak{X}_G, \omega)$ be a central character datum for H consistent with to ψ_∞^H. Assume that $\mathfrak{p}$ is large enough to apply the results of Theorem 47. Let $H_q \in \mathcal{HE}(H)$, and let $f(\mathfrak{p}^n)$ be the sequence of test functions defined in Definition 45. Then

$$|I^{H_q}_{\mathrm{disc},\psi_\infty^H,\omega}(f^{H_q}(\mathfrak{p}^n))| \ll Nm(\mathfrak{p}^n)^{N(N-N_1)}.$$

Proof. Since H_q is a hyperendoscopic group of H we have $H_q = H_q^1 \times H_q^2$, where ψ_i factors through H_q^i. The growth rate in the theorem is defined up to constants, so we can assume that $f^{H_q}(\mathfrak{p}^n) = f^{H_q^1}(\mathfrak{p}^n) \times f^{H_q^2}(\mathfrak{p}^n)$; this is true locally almost everywhere by the Fundamental Lemma. Indeed, at all $v \notin S \cup \{v_\mathfrak{p}\}$, the function $f_v^{H_q}(\mathfrak{p}^n)$ can be taken to be the characteristic function of a hyperspecial maximal compact subgroup of $H_{q,v}$. At $v = v_\mathfrak{p}$, we iterate the conclusion of Theorem 47 to get

$$f_{v_\mathfrak{p}}^{H_q}(\mathfrak{p}^n) = \mathrm{Nm}(\mathfrak{p})^{-n\cdot d(G,H_q)} \frac{1_{K_{v_\mathfrak{p}}(\mathfrak{p}^n)^{H_q}}}{\mu(K_{v_\mathfrak{p}}(\mathfrak{p}^n))} := \frac{\mathrm{Nm}(\mathfrak{p})^{-n\cdot d(G,H_q)}}{\mu(K_{v_p})(\mathfrak{p}^n)/\mu(K_{v_\mathfrak{p}}(\mathfrak{p}^n)^{H_q})} \varphi_{v_\mathfrak{p}}(\mathfrak{p}^n). \tag{32}$$

Write $\varphi_{v_\mathfrak{p}}(\mathfrak{p}^n) = \varphi_{v_\mathfrak{p}}^1(\mathfrak{p}^n) \times \varphi_{v_\mathfrak{p}}^2(\mathfrak{p}^n)$, and for $i = 1, 2$, let

$$\varphi^{H_q^i}(\mathfrak{p}^n) = \varphi_{v_\mathfrak{p}}^i(\mathfrak{p}^n) \cdot \prod_{v \neq v_\mathfrak{p}} f_v^{H_q^i}(\mathfrak{p}^n), \quad \varphi^{H_q}(\mathfrak{p}^n) = \varphi^{H_q^1}(\mathfrak{p}^n) \times \varphi^{H_q^2}(\mathfrak{p}^n).$$

Each of the two functions $\varphi^{H_q^i}(\mathfrak{p}^n)$ satisfies the identical assumptions of Propositions 51 and 52. We also recall that $H = U(N_1) \times U(N_2)$, and we shorten $U(N_i) = H^i$. We also have $\psi^H = \psi_1 \oplus \psi_2$ with ψ_i landing in ${}^L H^i$. Thus if we fix data $(\mathfrak{X}_{H^1}, \omega_1)$ and $(\mathfrak{X}_{H^2}, \omega_2)$ coming from H^1 and H^2 respectively, we find that

$$\begin{aligned}\left|\sum_{\Psi(H,\psi^H_\infty,\omega_1\times\omega_2)} I^{H_q}_{\mathrm{disc},\psi^H}(\varphi^{H_q}(\mathfrak{p}^n))\right| &\le \sum_{\Psi(H,\psi^H_\infty,\omega_1\times\omega_2)} |I^{H^1_q}_{\mathrm{disc},\psi_1}(\varphi^1(\mathfrak{p}^n))| \cdot |I^{H^2_q}_{\mathrm{disc},\psi_2}(\varphi^2(\mathfrak{p}^n))| \\ &= |I^{H^1_q}_{\mathrm{disc},\psi_{\infty,1},\omega_1}(\varphi^1(\mathfrak{p}^n))| \times |I^{H^2_q}_{\mathrm{disc},\psi_{\infty,2},\omega_2}(\varphi^2(\mathfrak{p}^n))| \\ &\ll M \cdot Nm(\mathfrak{p}^n)^{\dim H^2_q - 1}.\end{aligned}$$

The quantity in the left-hand side above isn't quite what we want to measure. First, we want to replace the choice of a pair of central characters $\omega_1 \times \omega_2$ by a sum over all parameters with central character datum $(\mathfrak{X}_G, \omega)$. In Lemma 53, we saw that the number of products $\omega_1 \times \omega_2$ of level $\mathfrak{p}^n$ which restrict to ω on $\mathfrak{X}_G$ is $\ll \mathrm{Nm}(\mathfrak{p}^n)$. Second, we slightly modify the test functions. From (32), we have

$$f^{H_q}(\mathfrak{p}^n) = C(G, H_q, n)\varphi^{H_q}(\mathfrak{p}^n), \quad C(G, H_q, n) \asymp \mathrm{Nm}(\mathfrak{p})^{n \cdot d(G,H_q)}.$$

Thus combining our upper bounds with these modifications we obtain

$$\begin{aligned}|I^{H_q}_{\mathrm{disc},\psi^H_\infty,\omega}(f^{H_q}(\mathfrak{p}^n))| &\ll \mathrm{Nm}(\mathfrak{p}^n)^{(1+d(G,H_q))}\left|\sum_{\psi^H\in\Psi(H,\psi^H_\infty,\omega_1\times\omega_2)} I^{H_q}_{\mathrm{disc},\psi^H}(\varphi^{H_q}(\mathfrak{p}^n))\right| \\ &\ll \mathrm{Nm}(\mathfrak{p}^n)^{d(G,H_q)+\dim H^2_q} \\ &= \mathrm{Nm}(\mathfrak{p}^n)^{\dim(G)/2-\dim(H^1_q)/2+\dim(H^2_q)/2}.\end{aligned}$$

Recall that $\dim G = N^2$, and that since the dual group of H^1_q receives an N_1-dimensional irreducible representation of SL_2, we have $\dim(H^1_q) = N_1^2$. Finally, it follows that $\dim(H^2_q) \le (N - N_1)^2$, which gives us the desired upper bounds. □

Remark 55. Note that the only situation in which this upper bound has a chance of being sharp is when $\dim H^2_q = (N - N_1)^2$, i.e., when $H_q = H$.

We have now collected all the facts leading up to our limit multiplicity theorem.

Theorem 56. *Let ψ_∞ be an Arthur parameter with regular infinitesimal character, and such that $\psi_\infty |_{\mathrm{SL}_2(\mathbb{C})} = \nu(2k) \oplus \nu(1)^{N-2k}$. Let $(\mathfrak{X}_G, 1)$ be the trivial central character. Fix $\Pi^0_{\psi_\infty} \subset \Pi_{\psi_\infty}$. For each $\psi \in \Psi(G, \psi_\infty, 1)$, let*

$$\Pi^0_\psi = \{\pi = \otimes'_v \pi_v \in \Pi_\psi \mid \pi_\infty \in \Pi^0_{\psi_\infty}.\}$$

Then

$$\sum_{\psi\in\Psi(G,\psi_\infty,1)} \sum_{\pi\in\Pi^0_\psi} m(\pi) \dim \pi_f^{K(\mathfrak{p}^n)} \ll \mathrm{Nm}(\mathfrak{p}^n)^{N(N-2k)}. \tag{33}$$

Proof. From Proposition 46, we take $f(\mathfrak{p}^n)$ as in Section 4.2 and write

$$\begin{aligned}\sum_{\psi\in\Psi(G,\psi_\infty,1)}\sum_{\pi\in\Pi_\psi^0} m(\pi)\dim\pi_f^{K(\mathfrak{p}^n)} &= \sum_{\psi\in\Psi(G,\psi_\infty,1)} \operatorname{tr} R_{\mathrm{disc},\psi}(f(\mathfrak{p}^n))\\ &= \sum_{\psi\in\Psi(G,\psi_\infty,1)} I_{\mathrm{disc},\psi}(f(\mathfrak{p}^n)) \qquad \text{(Theorem 24)}\\ &\le \sum_{\psi\in\Psi(G,\psi_\infty,1)} C(\psi)S(\psi,s_{H_\psi},f(\mathfrak{p}^n)),\end{aligned}$$

where the last inequality follows from the results of Section 3.2, where the notation $C(\psi)$ was defined, since $f(\mathfrak{p}^n)$ takes only positive values. The group H_ψ is determined by the localization ψ_v of ψ at any place v, and in particular by ψ_∞. Thus H_ψ is the same for any $\psi\in\Psi(G,\psi_\infty,1)$ since by definition they all localize to the same ψ_∞. By the assumption on ψ_∞, we have $H_\psi = U_{E/F}(2k)\times U_{E/F}(N-2k)$, and the parameters satisfy the assumptions of Proposition 54. To lighten the notation, we denote H_ψ by H and s_ψ by s_H for the end of the proof.

The parameter ψ^H corresponding to s_H under the bijection of Lemma 13, we have shown in Corollary 39 that

$$S(\psi,s_H,f(\mathfrak{p}^n)) = \iota(G,H)\sum_{\mathcal{H}\in\mathcal{HE}(H,\psi)} \iota(\mathcal{H}) I^{H_q}_{\mathrm{disc},\psi^H}(f^{H_q}(\mathfrak{p}^n)).$$

For each summand on the right-hand side, we sum over $\Psi(H,\psi^H,\omega)$, where $(\mathfrak{X}_G,\omega)$ is determined by ψ^H as in Lemma 48. We then apply Proposition 54 with $N_1 = 2k$. Note that we have ensured in Lemma 14 that the infinitesimal character of the representations of $H_{q,\infty}$ associated to all $\psi_\infty^{H_q}$ are regular. This gives us the following bounds:

$$\left|\sum_{\psi\in\Psi(H,\psi_\infty,1)} I^{H_q}_{\mathrm{disc},\psi^H}(f^{H_q}(\mathfrak{p}^n))\right| \ll Nm(\mathfrak{p}^n)^{N(N-2k)}.$$

We conclude by summing over the finitely many $H_q\in\mathcal{HE}(H,\psi)$. □

5. Applications to growth of cohomology

We now apply the results of Section 4 to cohomology of arithmetic groups. This section is concerned with local questions at infinity, and the notation is different from the rest of the paper. From now until Section 5.3, G will be a Lie group.

5.1. *Cohomological representations.*

Given a Lie group G, let $\tilde{G}$ denote the unitary dual of G.

Theorem 57 (Matsushima's formula [36]). *Let G be a connected semisimple Lie group with maximal compact subgroup K and complexified Lie algebra $\mathfrak{g}$. Let $\Gamma\subset G$ be a cocompact lattice and let $X_\Gamma = \Gamma\backslash G/K$. For $\pi\in\tilde{G}$, denote by $m(\pi,\Gamma)$ the multiplicity of π in the right-regular representation of*

G on $L^2(\Gamma\backslash G)$. Then

$$\dim(H^i(X_\Gamma, \mathbb{C})) = \sum_{\pi\in\tilde{G}} m(\pi, \Gamma)\dim(H^i(\mathfrak{g}, K; \pi)).$$

The $H^i(\mathfrak{g}, K; \pi)$ which appear in the right-hand side are the so-called $(\mathfrak{g}, K)$ cohomology groups of π. We say that π is *cohomological* if $H^*(\mathfrak{g}, K; \pi) \neq 0$; such representations were characterized by Vogan and Zuckerman [49].

Theorem 58 [49]. *Let G, $\mathfrak{g}$ be as above. Let K a maximal compact subgroup of G and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ be the corresponding Cartan decomposition with $\mathfrak{k}$ the Lie algebra of K. The group G has finitely many cohomological representations π, and $H^i(\mathfrak{g}, K; \pi) \neq 0$ if and only if*:

(i) *π has the infinitesimal character of the trivial representation of G.*

(ii) $\mathrm{Hom}_K(\pi, \wedge^i\mathfrak{s}) \neq 0$.

Where the action of K on $\wedge^i\mathfrak{s}$ is induced by the adjoint representation.

The results apply only to semisimple groups: they are extended to $U(a, b)$ in [48], and condition (ii) above implies that cohomological representations have trivial central character. Below, we give a concrete parametrization of cohomological representations of $U(a, b)$ in terms of refinements of partitions of $a+b$ which are compatible with the signature (a, b). More details can be found in [6] and [20].

5.1.1. *Cohomological representations and ordered bipartitions.* In [49], cohomological representations $A_\mathfrak{q}$ are built from so-called θ-stable parabolic subalgebras $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$. In [6, Section 5], Bergeron and Clozel show that for $U(a, b)$, the data of the algebra $\mathfrak{q}$ can be encoded in a choice of centralizing Levi subgroup $L(\mathfrak{q}) = \prod U(a_i, b_i) \subset U(a, b)$ whose Lie algebra is $\mathfrak{l}$. Thus $\mathfrak{q}$'s are parametrized by ordered tuples

$$B = ((a_1, b_1), \dots, (a_r, b_r))$$

of pairs of nonnegative integers with $\sum a_i = a$ and $\sum b_i = b$. We call these tuples B *ordered bipartitions* of (a, b) and denote the associated Levi subgroup L_B, and the corresponding representation by π_B.

The ordered bipartitions of (a, b) *almost* parametrize the cohomological representation of $U(a, b)$, but there is redundancy. Specifically, $\pi_B \simeq \pi_{B'}$ if B' has adjacent pairs of the form $(a_1, 0)$, $(a_2, 0)$ (resp. $(0, b_1)(0, b_2)$) which are collapsed into $(a_1 + a_2, 0)$ (resp. $(0, b_1 + b_2)$) in B. We will say that an ordered bipartition is *reduced* if all pairs in which one entry is zero are maximally broken up.

Example 59. The following ordered bipartition is not reduced:

$$((3, 1), (2, 0), (1, 0), (0, 3)).$$

It is associated to the same cohomological representation as the following reduced ordered bipartition:

$$((3, 1)(1, 0)(1, 0)(1, 0)(0, 1)(0, 1)(0, 1)).$$

The cohomology of π_B can be expressed in terms of B.

Proposition 60 [49, Proposition 3.2]. *Let* $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ *be a* θ*-stable parabolic subalgebra and let* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ *be the Cartan decomposition. Let* $R = \dim \mathfrak{u} \cap \mathfrak{s}$. *Then*

$$H^i(\mathfrak{g}, K, A_{\mathfrak{q}}) \simeq \mathrm{Hom}_{\mathfrak{l} \cap \mathfrak{k}}(\wedge^{i-R}\mathfrak{s}, \mathbb{C}).$$

In particular, the smallest nonvanishing degree of cohomology of $A_{\mathfrak{q}}$ is R, which, writing $A_{\mathfrak{q}} = \pi_B$ and referring once more to [6] and [20], is equal to

$$R = \frac{\dim(\mathfrak{s}) - \dim(\mathfrak{s} \cap \mathfrak{l})}{2} = ab - \sum_{i=1}^{r} a_i b_i. \tag{34}$$

In particular, if $a_i b_i = 0$ for all pairs, i.e., if L is compact, then π_B is a discrete series representation and only has cohomology in degree ab.

5.2. ***Arthur parameters of cohomological representations.*** We turn our attention to archimedean parameters ψ whose associated Arthur packets contain cohomological representations. These are obtained via embedding of L-groups from parameters associated to the trivial representation of Levi subgroups of $G = U(a, b)$. The packets associated to these parameters were constructed by Adams and Johnson [1] in conversation with work of Arthur [4], in a language predating the current formulation of the endoscopic classification of representations. Arancibia, Moeglin and Renard [3] have shown that Adams and Johnson's construction yields the same packets as those appearing in the endoscopic classification in [38] and [26].

To begin, note that there is a natural way to associate to an ordered bipartition B of (a, b) an ordered partition P_B of N, namely by letting

$$P_B = (N_1, \dots, N_r), \qquad N_i = a_i + b_i.$$

Let B be an ordered bipartition, and L_B be the associated Levi subgroup. Then $\hat{L} \simeq \prod_i \mathrm{GL}_{N_i}(\mathbb{C}) \hookrightarrow \hat{G}$, is determined by P_B. The description of ${}^L L$, i.e., of the Galois action on $\hat{L}$, is given in Section 2.2.5. Cohomological Arthur parameters depend on an embedding $\xi_{\hat{L},\hat{G}} : {}^L L \hookrightarrow {}^L G$ extending the map $\hat{L} \hookrightarrow \hat{G}$. To define $\xi_{\hat{L},\hat{G}}$, it suffices to give the image of $W_{\mathbb{R}}$ inside of ${}^L G$. Recall that $W_{\mathbb{R}}$ is an extension of $\mathbb{C}^\times$ by a group of order 2, which we write as $\mathbb{C}^\times \sqcup \sigma\mathbb{C}^\times$ with $\sigma^2 = -1$. We give Arthur's construction from Section 5 of [4]. The construction of $A_{\mathfrak{q}}$ in [49] depends on an element α of the Lie algebra $\mathfrak{t}$ of a compact torus. Let T be the torus with Lie algebra $\mathfrak{t}$ and let $\psi_{\hat{L},\hat{G}} : W_{\mathbb{R}} \to {}^L G$ be the map sending $\mathbb{C}^\times$ into $\hat{T}$ so that for any $\lambda^\vee \in X_*(T)$, we have

$$\lambda^\vee(\psi_{\hat{L},\hat{G}}(z)) = z^{\langle \rho_Q, \lambda^\vee \rangle} \bar{z}^{-\langle \rho_Q, \lambda^\vee \rangle}$$

where $\rho_Q = \rho_{\hat{G}} - \rho_{\hat{L}}$, the difference of half-sums of positive roots. Let the element $(1 \rtimes \sigma)$ map to $n_Q \rtimes \sigma$, where for any group G, n_G is an element in the derived group of $\hat{G}$ such that $\mathrm{ad}\, n_G$ interchanges the positive and negative roots of $(\hat{G}, \hat{T})$, and with $n_Q = n_L^{-1} n_G$. Putting this together and denoting the embedding of $\hat{L}$ into $\hat{G}$ by ι, define $\xi_{\hat{L},\hat{G}}(g, w) = \iota(g)\psi_{\hat{L},\hat{G}}(w)$.

Now let $\psi_{0,\hat{L}} : \mathrm{SL}_2(\mathbb{C}) \times W_{\mathbb{R}} \to {}^L L$ be the Arthur parameter of the packet containing the trivial representation of L. It is trivial on $W_{\mathbb{R}}$ and sends SL_2 to the principal SL_2 of $\hat{L}$. Then the Arthur parameter of G corresponding to the Levi subgroup $\hat{L}$ is the composition

$$\psi_{\hat{L}} := \xi_{\hat{L},\hat{G}} \circ \psi_{0,\hat{L}} : \mathrm{SL}_2 \times W_{\mathbb{R}} \to {}^L G.$$

Adams and Johnson [1] and more recently Nair and Prasad [39] have given a description of the packets attached to the parameters $\psi_{\hat{L}}$.

Proposition 61 [1, Section 3.3]. *Let $\hat{L}$ be a Levi subgroup of $\hat{G}$, dual to a Levi $L(\mathfrak{q})$ attached to a θ-stable parabolic subalgebra $\mathfrak{q}$. The parameter $\psi_{\hat{L}} = \xi_{\hat{G},\hat{L}} \circ \psi_{0,\hat{L}}$ corresponds to a packet Π_ψ consisting of the representations $A_{\mathfrak{q}}$ such that $\hat{L}(\mathfrak{q}) = \hat{L}$.*

We now translate the descriptions of the packets $\Pi_{\psi_{\hat{L}}}$ given in [1] and [4] into our parametrization by ordered bipartitions.

Proposition 62. *Let $P = (N_1, \dots, N_r)$ be an ordered partition of N and $\psi_P := \psi_{\hat{L}_P}$ be the corresponding parameter. Then the packet $\Pi_P := \Pi_{\psi_P}$ consists precisely of the cohomological representations π_B associated to bipartitions B such that $P_B = P$.*

Proof. We explained above how L_B gives rise to $\psi_{\hat{L}_B} = \psi_{\hat{L}_{P_B}}$; the parameters $\psi_{\hat{L}_B}$ and $\psi_{\hat{L}_{B'}}$ are equivalent if they are $\hat{G}$-conjugate. The isomorphism classes of representations π_B correspond to Levi subgroups L_B containing the fixed torus T, so we need only consider conjugation by $N_{\hat{G}}(\hat{T})$. This action induces an action of the Weyl group $W(\hat{T}, \hat{G})$ on $\hat{T}$ and on the root datum $(X_*(T), \Delta(T), X_*(\hat{T}), \Delta(\hat{T}))$. Note that the action of conjugation by $\hat{T}$ on cohomological Arthur parameters will only modify $\psi_{\hat{L}}$ by scaling the entries of n_Q. This has no impact on the parameter since n_Q was only specified up to scalars in the construction of $\psi_{\hat{L}}$.

Thus to determine which Levi subgroups $L_{B'}$ give rise to the conjugacy class of $\hat{L}_P$, we consider the action of $W(\hat{G}, \hat{T})$ (denoted $W(\mathfrak{g}, \mathfrak{t})$ in [1]) on the set of ordered bipartitions. Following the description of L_B given in [6], ordered bipartitions are determined ultimately by an element $\alpha \in \mathfrak{t}$. The entries of conjugate elements $w \cdot \alpha$ will have the same values, but these values will be distributed differently among the two pieces of $\mathfrak{t}$ belonging to $U(a)$ and $U(b)$. We denote the values appearing in the entries of α by z_i. The data being preserved by conjugation is the number of entries $a_i + b_i$ which are associated to the same value z_i, as well as the ordering of the z_i. Transitivity of the Weyl group action then ensures that all the possible B such that $P_B = P$ give rise to ψ_P. ☐

5.3. ***Limit multiplicity for cohomological representations.*** We now give results on growth of cohomology. We return our usual notation, in which F is global, $\mathfrak{p}$ is a prime of F, and the subscript "∞" denotes the collection of all the archimedean places. Fix the set S_0 as in Section 4.1 so that it contains all but one archimedean place v_0. Let G be the inner form of $U_{E/F}(N)$ such that $G_{v_0} \simeq U(a, b)$ and all the other factors at infinity are compact. Define the groups $K(\mathfrak{p}^n)$ and $\Gamma(\mathfrak{p}^n)$ as in Section 4.1. By Matsushima's formula and the inequality (29), we have

$$h^i(\mathfrak{p}^n) := \dim(H^i(\Gamma(\mathfrak{p}^n), \mathbb{C}) \leq \sum_{\pi = \mathbf{1}^{|S_0|} \otimes \pi_{v_0} \otimes \pi_f} m(\pi) h^i(\mathfrak{g}_{v_0}, K_{v_0}; \pi_{v_0}) \dim \pi_f^{K_f(\mathfrak{p}^n)}.$$

We can now give our theorem for growth of cohomology.

Theorem 63. *Let ψ_∞ be the cohomological parameter of G_∞ associated to a reordering of* $(2k, 1, \dots, 1)$. *Let*

$$h^i_{\psi_\infty}(\mathfrak{p}^n) = \sum_{\psi \in \Psi(\psi_\infty)} \sum_{\pi \in \Pi_\psi} m(\pi) h^i(\mathfrak{g}_{v_0}, K_{v_0}; \pi_{v_0}) \dim \pi_f^{K_f(\mathfrak{p}^n)}.$$

Then

$$h^i_{\psi_\infty}(\mathfrak{p}^n) \ll \mathrm{Nm}(\mathfrak{p}^n)^{N(N-2k)}.$$

Proof. The possible contribution to cohomology of a given representation π_{v_0} is bounded, so we need only bound the contribution to $m(\pi_\infty, \mathfrak{p}^n)$ coming from packets attached to parameters specializing to ψ_∞, for each $\pi_\infty \simeq \pi_{v_0} \otimes \mathbf{1}^{[F:\mathbb{Q}]-1} \in \Pi^0_{\psi_\infty}$. The result then follows from Theorem 56, provided that cohomological parameters satisfy its assumptions. From Theorem 58 and the following comment, cohomological representations have the (regular) infinitesimal character of the trivial representation, and trivial central character. From the discussion in Section 5.2, reorderings of $(2k, 1, \dots, 1)$ correspond to parameters for which $\psi(\mathrm{SL}_2) = \nu(2k) \oplus \nu(1)^{N-2k}$. Thus the assumptions are satisfied and the result holds. □

Note that the theorem does not in fact bound $m(\pi_\infty, \mathfrak{p}^n)$ for a general $\pi_\infty \in \Pi_{\psi_\infty}$. Indeed, since Arthur packets are not disjoint, the representation π_∞ could also appear in a different Arthur packet whose growth we do not bound. More specifically, if $\pi_\infty = \pi_B \otimes \mathbf{1}^{[F:\mathbb{Q}]-1} \in \Pi^0_{\psi_\infty}$ where B is an ordered bipartition described in Section 5.1.1, it could be the case that B is the reduction of an ordered bipartition B', for example if we had

$$B = ((1,1), (1,0), (1,0), (0,1)), \quad B' = ((1,1), (2,0), (0,1)).$$

On the other hand, if B is not the reduction of another ordered bipartition, then $\pi_\infty = \pi_B$ does not appear in any other archimedean Arthur packet and the theorem produces upper bounds for $m(\pi_\infty, \mathfrak{p}^n)$. We record this discussion below.

Corollary 64. *Let $B = ((a_1, b_1), \dots, (a_r, b_r))$ be a reduced ordered bipartition such that*:

(i) *The associated partition P_B is a reordering of* $(2k, 1, \dots, 1)$.

(ii) *We have $(a_i, b_i) \neq (a_{i+1}, b_{i+1})$ for all i.*

Then

$$m(\pi_B, \mathfrak{p}^n) \ll \mathrm{Nm}(\mathfrak{p}^n)^{N(N-2k)}.$$

Example 65. Assume that $a < b$ in the signature of $U(a, b)$. A family of partitions satisfying the conditions of Corollary 64 are the suitable reorderings of

$$B_j = \begin{cases} ((0,1), \dots, (1,0), (0,1), (a-j, b-j-2), (0,1), (1,0), \dots, (0,1)) & N \text{ even}, \\ ((a-j, b-j-1), (0,1), (1,0), \dots, (0,1), (1,0), (0,1)) & N \text{ odd}, \end{cases}$$

where $1 \leq j \leq a-1$ if N is even (resp. $0 \leq j \leq a-1$ is N is odd.) The computations of Section 5.1.1 show that their lowest degree of cohomology is

$$i = i(N, a, j) = \begin{cases} j(N-j-2)+2a & N \text{ even}, \\ j(N-j-1)+a & N \text{ odd}. \end{cases}$$

Note that $j = \frac{1}{2}(N-2k-2)$ for N even (resp. $j = \frac{1}{2}(N-2k-1)$ for N odd) which gives the family alluded to in the introduction.

Additionally, the smallest $i > 0$ for which $h^i(\mathfrak{p}^n) \neq 0$ is $i = a$. When N is odd, one can check that representations as above with $j = 0$ are the only source of cohomology in degree a. In this situation, we get bounds on Betti numbers.

Corollary 66. *Keeping the assumptions of Theorem 63, assume additionally that N is odd and that $a < b$. Then*

$$h^a(\mathfrak{p}^n) \ll \mathrm{Nm}(\mathfrak{p}^n)^N.$$

Proof. This follows from Theorem 63 with $2k = N-1$ provided that

$$h^a(\mathfrak{p}^n) = \sum_{\psi_\infty} h^a_{\psi_\infty}(\mathfrak{p}^n),$$

where the sum is taken over finitely many parameters ψ_∞ associated to a reordering of $(1, N-1)$. Since there are finitely many Arthur of $U(a, b)$ with cohomological representations, this amounts to showing that representations with cohomology in degree a belong only to packets Π_{ψ_∞} associated to these partitions. Going back to Proposition 60 and the following discussion, in particular to (34), we find that the only representations with cohomology in degree a are of the form π_B for B a reordering of

$$((0,1), (a, b-1)).$$

These cannot be reduced, nor are they the reduction of other ordered bipartitions so by Proposition 62, they only belong to packets associated to parameters corresponding ordered partitions are reorderings of $(1, a+b-1) = (1, N-1)$, which was exactly our requirement. □

5.4. ***Comparison with the Sarnak–Xue conjecture.*** Finally, we compare our results with the conjecture of Sarnak and Xue [42] relating multiplicity growth to decay of matrix coefficients. For an irreducible unitary representation π_∞ of a Lie group G, Sarnak and Xue define

$$p(\pi_\infty) = \inf\{p \geq 2 \mid K\text{-finite matrix coefficients of } \pi_\infty \text{ are in } L^p(G)\}.$$

They then conjecture the following bounds for unitary π_∞:

$$m(\pi_\infty, \mathfrak{p}^n) \ll_\epsilon Vol(X(\mathfrak{p}^n))^{(2/p(\pi_\infty))+\epsilon}.$$

We will now show that the Sarnak and Xue conjecture holds for the representations for which we have proved upper bounds on multiplicity growth.

Proposition 67. *Let $\pi_\infty = \pi_B$ be as in Corollary 64. Then*

$$\frac{2}{p(\pi_B)} \geq \frac{N-2k}{N-1}.$$

Since $m(\pi_B, \mathfrak{p}^n) \ll \mathrm{Nm}(\mathfrak{p}^n)^{N(N-2k)}$ and the volume of $X(\mathfrak{p}^n)$ grows like $\mathrm{Nm}(\mathfrak{p}^n)^{N^2-1}$ we obtain the following.

Corollary 68. *For π_B as in Corollary 64, we have*

$$m(\pi_B, \mathfrak{p}^n) \ll \mathrm{Vol}(X(\mathfrak{p}^n))^{N(N-2k)/(N^2-1)} \ll \mathrm{Vol}(X(\mathfrak{p}^n))^{2/p(\pi_B)}$$

and the Sarnak–Xue conjecture holds.

The remainder of the section sets up and gives a proof of Proposition 67.

5.4.1. *Computation of the rate of decay.* For cohomological representations, we will give bounds on $p(\pi_B)$ from the descriptions of π_B as Langlands quotients given in [49]. For this section we follow the notation of Knapp [28, Sections 7 and 8]. We start by bounding $p(\pi)$ for π an arbitrary Langlands quotient in terms of the inducing data.

We recall the setup for the definition of Langlands quotients. First, fix an Iwasawa decomposition

$$G = KA_0N_0, \quad g = k_g a_g n_g \tag{35}$$

of G. Here K a maximal compact subgroup, A_0 a maximal split torus, N_0 unipotent. By letting $M_0 = Z_K(A_0)$, this gives rise to a minimal parabolic subgroup $S_0 = M_0A_0N_0$. Let $S = MAN$ be parabolic and regular with respect to S_0, which is to say that $A \subset A_0$ is split, M is the no longer necessarily compact Levi component, and $N \subset N_0$ unipotent. Denote by $\mathfrak{a}$ (resp. $\mathfrak{a}_0$) the Lie algebra of A (resp. A_0), and by $\mathfrak{a}_M$ the Lie algebra of the maximal split torus of M, so that $\mathfrak{a}_0 = \mathfrak{a} \oplus \mathfrak{a}_M$. Let ρ_0 be the half-sum of the positive roots of $\mathfrak{a}_0$ in $\mathfrak{g}$. Let σ be a discrete series representation of M and $\nu \in \mathfrak{a}^*$ be real-valued and in the open positive Weyl chamber. Denote by ν_0 the extension of ν to $\mathfrak{a}_0$ by setting it to be zero on $\mathfrak{a}_M$, and let $\rho_0 \in \mathfrak{a}_0^*$ be the half-sum of the positive roots of $\mathfrak{a}_0$ in $\mathfrak{g}$. Denote by $U(S, \sigma, \nu)$ the corresponding parabolically induced representation, and by $J(S, \sigma, \nu)$ its Langlands quotient.

The proof of the upcoming proposition depends on a collection of results from [28]; before stating it we recall the setup and some nomenclature. When studying the decay of matrix coefficients, one introduces a class of so-called *spherical functions* φ_ν^G associated to $\nu \in \mathfrak{a}_0^*$, and defined by

$$\varphi_\nu^G(g) = \int_K e^{-(\nu+\rho_0)H(g^{-1}k)}\, dk$$

where $H(g) = \log(a_g) \in \mathfrak{a}_0$ is defined using the Iwasawa decomposition (35). Paraphrasing Knapp, these are "useful yardsticks" to measure the decay of matrix coefficients. For example, it is known that the K-finite matrix coefficients of discrete series are dominated by φ_0^G. As their name indicates, the spherical functions are K-invariant. A generalization of this notion is that of a *τ-spherical function* associated to a pair $\tau = (\tau_1, \tau_2)$ of representations of K: a τ-spherical function is valued in the space $U_1 \otimes U_2^\vee$ and is left τ_1- and right τ_2-equivariant.

In [28, Section VII.8], Knapp studies representations π by producing and studying an asymptotic expansion of the τ-spherical functions associated to the K-types of π. The functions $F_{\lambda-\rho_0}$ associated to $\lambda \in \mathfrak{a}_0^*$ appearing in this asymptotic expansion control rates of decay of the τ-spherical function in various directions along A_0. The $\lambda - \rho_0$ such that $F_{\lambda-\rho_0}$ contributes nontrivially to the decomposition of the τ-spherical functions of π are called exponents of π. A *leading exponent* of π is an exponent $\mu - \rho_0$ of π, maximal in the sense that for any comparable exponent $\lambda - \rho_0$, the difference $\mu - \lambda$ is a linear combination of simple roots with nonnegative integer coefficients.

Proposition 69. *Let $\omega_1, \dots, \omega_{\dim \mathfrak{a}_0}$ denote the basis of $\mathfrak{a}_0$ dual to the basis of $\mathfrak{a}_0^*$ consisting of the simple roots. Then*

$$p(J(S, \sigma, \nu)) \le \inf\{p \ge 2 \mid p\langle \nu_0 - \rho_0, \omega_j\rangle < -2\langle \rho_0, \omega_j\rangle \textit{ for all } \omega_j\}. \tag{36}$$

Proof. The lemma follows from the combination of results in [28]. From [28, 8.48], the K-finite matrix coefficients of $J(S, \sigma, \nu)$ belong to $L^p(G)$ if and only if for all ω_i and all leading exponents $\mu - \rho_0$ of $J(S, \sigma, \nu)$, the following inequality is satisfied:

$$p\langle \operatorname{Re}\mu - \rho_0, \omega_j\rangle < -2\langle \rho_0, \omega_j\rangle. \tag{37}$$

In [28, 8.47], we see that all leading exponents μ satisfy $\langle \operatorname{Re}\mu, \omega_j\rangle \le \langle \nu_0, \omega_j\rangle$ provided there exists an integer $q \ge 0$ such that the K-finite matrix coefficients of $J(S, \sigma, \nu)$ are bounded above on $\overline{A_0^+}$ by a multiple of $e^{(\nu_0-\rho)(\log a)}(1 + \operatorname{Nm} a)^q$. This upper bound is established for $U(S, \sigma, \nu)$ and a fortiori for $J(S, \sigma, \nu)$ by propositions 7.14 and 7.15 of [28], together with the fact that as a discrete series, the K-finite matrix coefficients of σ are dominated by a multiple of the spherical function φ_0^M. □

We now give an explicit bounds for $p(\pi_B)$ in terms of B, for a class of representations including those of Proposition 67.

Proposition 70. *Let $G = U(a, b)$ with $a + b = N$, and let $B = ((a_1, b_1), \dots, (a_r, b_r))$ be a reduced ordered bipartition of (a, b). Assume that there is a single index k such that $\min\{a_k, b_k\} \neq 0$, and let $N_k = a_k + b_k$. Then*

$$\frac{2}{p(\pi_B)} \ge \frac{N - N_k}{N - 1}.$$

Proof. In light of Proposition 69, we will realize π_B as a Langlands quotient $J(S, \sigma, \nu)$, and show that for the corresponding ν_0 we have

$$\inf\{p \ge 2 \mid p\langle \nu_0 - \rho_0, \omega_j\rangle < -2\langle \rho_0, \omega_j\rangle \text{ for all } \omega_j\} = \frac{2(N-1)}{N - N_k}.$$

To simplify the computations, note that it is equivalent to show that

$$\max_{\omega_j}\left\{\frac{\langle \nu_0, \omega_j\rangle}{\langle \rho_0, \omega_j\rangle}\right\} = \frac{N_k - 1}{N - 1}. \tag{38}$$

We recall the descriptions of cohomological representations as Langlands quotients is given in [49, Section 6]. Let $L_B = \prod_i U(a_i, b_i)$ be the Levi subgroup attached to the representation π_B and fix an Iwasawa decomposition $L = (K \cap L)AN$. By assumption, A has rank $c_k = \min\{a_k, b_k\}$; denote its Lie algebra by $\mathfrak{a}$, and let ν be the half-sum of the roots of $\mathfrak{a}$ in the Lie algebra $\mathfrak{n}$ of N. Let M be the centralizer of A in G, and fix S a choice of parabolic subgroup with Levi MA, such that ν is in the open positive Weyl chamber. Then by [49, Theorem 6.16], there is a discrete series representation σ of M such that $\pi_B \simeq J(S, \sigma, \nu)$.

To conclude, we put ourselves back in the framework of Proposition 69. Let $S_0 = A_0M_0N_0 \subset S$ be a minimal parabolic subgroup. Then if $\mathfrak{a}_M$ is the Lie algebra of a maximal split torus in M, we have $\mathfrak{a}_0 = \mathfrak{a} \oplus \mathfrak{a}_M$ and $\dim \mathfrak{a}_0 = c := \min\{a, b\}$. Let $\alpha_1, \dots, \alpha_c$ be the simple roots of $\mathfrak{a}_0$ in $\mathfrak{g}$. Recall that ν_0 is obtained by extending ν by 0 to $\mathfrak{a}_M$. Thus we can write $\nu_0 = \sum_{j=1}^{c_k} j(N_k - j)\alpha_j$ and $\rho_0 = \sum_{j=1}^{c} j(N - j)\alpha_j$. Since the ω_i are by construction the dual basis to the α_i, we have

$$\frac{\langle \nu_0, \omega_j\rangle}{\langle \rho_0, \omega_j\rangle} = \begin{cases} (N_k - j)/(N - j), & j \le c_k, \\ 0, & j > c_k. \end{cases}$$

The maximum is achieved when $j = 1$. ☐

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A number theoretic characterization of E-smooth and (FRS) morphisms: estimates on the number of $\mathbb{Z}/p^k\mathbb{Z}$-points

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We provide uniform estimates on the number of $\mathbb{Z}/p^k\mathbb{Z}$-points lying on fibers of flat morphisms between smooth varieties whose fibers have rational singularities, termed (FRS) morphisms. For each individual fiber, the estimates were known by work of Avni and Aizenbud, but we render them uniform over all fibers. The proof technique for individual fibers is based on Hironaka's resolution of singularities and Denef's formula, but breaks down in the uniform case. Instead, we use recent results from the theory of motivic integration. Our estimates are moreover equivalent to the (FRS) property, just like in the absolute case by Avni and Aizenbud. In addition, we define new classes of morphisms, called E-smooth morphisms ($E \in \mathbb{N}$), which refine the (FRS) property, and use the methods we developed to provide uniform number-theoretic estimates as above for their fibers. Similar estimates are given for fibers of ε-jet flat morphisms, improving previous results by the last two authors.

1. Introduction

1A. ***Overview.*** Let $\varphi : X \to Y$ be an algebraic morphism between smooth K-varieties, where K is a number field. In this paper we give uniform arithmetic and analytic equivalent characterizations to the (FRS) property of φ, namely to the property of being flat with reduced fibers of rational singularities (see Theorem A). These results can be viewed as a common uniform improvement of the following two theorems:

(1) Theorem A of [2], where bounds were given on the number of $\mathbb{Z}/p^k\mathbb{Z}$-points of reduced local complete intersection schemes which have rational singularities (see also Theorem 1.3).

(2) Theorem 3.4 of [1], where pushforward of smooth measures with respect to φ over non-Archimedean local fields were shown to have bounded density if and only if φ is an (FRS) morphism (see also Theorem 4.3).

In order to prove our uniform characterizations of the (FRS) property, it seems natural to try and adapt the algebro-geometric proof of [2, Theorem A] to the relative case. This fails to work because of unsatisfactory behavior of resolution of singularities in families, with respect to taking points over $\mathbb{Z}$, $\mathbb{Z}/p^k\mathbb{Z}$ and $\mathbb{Z}_p$

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(see Section 1D1). Instead, we prove a model theoretic result of independent interest about approximating suprema of a certain subclass of motivic functions, which we call formally nonnegative functions (see Theorem B). Using Theorem B and by analyzing the jets of φ, we prove Theorem A. Theorem B further strengthens [11, Theorem 2.1.3] in the case of formally nonnegative functions. Finally, we provide uniform estimates on the number of $\mathbb{Z}/p^k\mathbb{Z}$-points lying on fibers of E-smooth morphisms, a new notion we introduce which refines the (FRS) property ($E \in \mathbb{N}$). Uniform estimates are also provided for fibers of ε-jet flat morphisms, achieving optimal bounds; see [20, Theorem 8.18]. See Section 2A1 and Theorems 4.11 and 4.12 for these notions and results.

1B. ***Counting points over $\mathbb{Z}/p^k\mathbb{Z}$: the absolute case.*** Let X be a finite type $\mathbb{Z}$-scheme. The study of the quantity $\#X(\mathbb{Z}/n\mathbb{Z})$, and its asymptotic behavior in $n \in \mathbb{N}$, is a long standing problem in number theory. When $n = p$ is prime, the asymptotic behavior is understood by the Lang–Weil estimates [30], and in particular, the family

$$\left\{\frac{\#X(\mathbb{Z}/p\mathbb{Z})}{p^{\dim X_{\mathbb{Q}}}}\right\}_p$$

is uniformly bounded.

Moving to the case where $n = p^k$ is a prime power (which suffices, by the Chinese remainder theorem), one can observe the following; if X is smooth as a $\mathbb{Z}$-scheme, then an application of Hensel's lemma shows that

$$\left\{\frac{\#X(\mathbb{Z}/p^k\mathbb{Z})}{p^{k\dim X_{\mathbb{Q}}}}\right\}_{p,k}$$

is uniformly bounded in both p and k. On the other hand, taking the nonreduced scheme $X = \operatorname{Spec}\mathbb{Z}[x]/(x^2)$, we see that

$$\frac{\#X(\mathbb{Z}/p^{2k}\mathbb{Z})}{p^{2k\dim X_{\mathbb{Q}}}} = \#X(\mathbb{Z}/p^{2k}\mathbb{Z}) = p^k,$$

which is not uniformly bounded. The following natural question arises.

Question 1.1. Is there a necessary and sufficient condition on X such that $\{\#X(\mathbb{Z}/p^k\mathbb{Z})/p^{k\dim X_{\mathbb{Q}}}\}_{p,k}$ is uniformly bounded?

Aizenbud and Avni [2], relying on results of Mustaţă [33] and Denef [13], gave such a necessary and sufficient condition in the case where $X_{\mathbb{Q}}$ is a local complete intersection.

Definition 1.2. Let K be a field of characteristic 0. A K-scheme of finite type X has *rational singularities* if it is normal and for every resolution of singularities $\pi : \widetilde{X} \to X$, one has

$$R^i\pi_*(O_{\widetilde{X}}) = 0$$

for $i \geq 1$.

Theorem 1.3 (see [2, Theorem A] and [18]). *Let X be a finite type $\mathbb{Z}$-scheme such that $X_{\mathbb{Q}}$ is equidimensional and a local complete intersection. Then the following are equivalent*:

(1) *$X_{\mathbb{Q}}$ has rational singularities (and, in particular, $X_{\mathbb{Q}}$ is reduced).*

(2) *There exists $C > 0$ such that for every prime p and every $k \in \mathbb{N}$ one has*

$$\frac{\#X(\mathbb{Z}/p^k\mathbb{Z})}{p^{k \dim X_{\mathbb{Q}}}} < C.$$

(3) *There exists $C > 0$ such that for every prime p and every $k \in \mathbb{N}$ one has*

$$\left| \frac{\#X(\mathbb{Z}/p^k\mathbb{Z})}{p^{k \dim X_{\mathbb{Q}}}} - \frac{\#X(\mathbb{Z}/p\mathbb{Z})}{p^{\dim X_{\mathbb{Q}}}} \right| < Cp^{-1}.$$

1C. ***Counting points over $\mathbb{Z}/p^k\mathbb{Z}$: the relative case.*** Let X and Y be smooth finite type $\mathbb{Z}$-schemes and let $\varphi : X \to Y$ be a dominant morphism. Our goal in this paper is to treat the relative analogue of Question 1.1:

Question 1.4. Is there a necessary and sufficient condition on φ such that the size of each fiber of $\varphi : X(\mathbb{Z}/p^k\mathbb{Z}) \to Y(\mathbb{Z}/p^k\mathbb{Z})$, normalized by $p^{k(\dim X_{\mathbb{Q}} - \dim Y_{\mathbb{Q}})}$, is uniformly bounded when varying p, k and $y \in Y(\mathbb{Z}/p^k\mathbb{Z})$?

Since the Lang–Weil estimates are effective uniformly over all schemes of bounded complexity, Question 1.4 is easily answered in the case where $k = 1$; the condition that $\varphi_{\mathbb{Q}}$ *is flat* is necessary and sufficient; see [20, Theorem 8.4]. For the general case, we use the following notion from [1, Definition II]. By a K-variety with K a field we mean a reduced K-scheme of finite type.

Definition 1.5. Let X and Y be smooth K-varieties, where K is a field with $\operatorname{char}(K) = 0$. We say that a morphism $\varphi : X \to Y$ is (*FRS*) if it is flat and if every fiber of φ has rational singularities.

1D. ***Main results.*** We are now ready to state the main result of this paper.

Theorem A (see Theorem 4.7 for a more general version). *Let $\varphi : X \to Y$ be a dominant morphism between finite type $\mathbb{Z}$-schemes X and Y, with $X_{\mathbb{Q}}$, $Y_{\mathbb{Q}}$ smooth and geometrically irreducible. Then the following are equivalent*:

(1) *$\varphi_{\mathbb{Q}} : X_{\mathbb{Q}} \to Y_{\mathbb{Q}}$ is (FRS).*

(2) *There exists $C_1 > 0$ such that for every prime p, $k \in \mathbb{N}$ and $y \in Y(\mathbb{Z}/p^k\mathbb{Z})$ one has*

$$\frac{\#\varphi^{-1}(y)}{p^{k(\dim X_{\mathbb{Q}} - \dim Y_{\mathbb{Q}})}} < C_1.$$

(3) *There exists $C_2 > 0$ such that for every prime p, $k \in \mathbb{N}$ and $y \in Y(\mathbb{Z}/p^k\mathbb{Z})$ one has*

$$\left| \frac{\#\varphi^{-1}(y)}{p^{k(\dim X_{\mathbb{Q}} - \dim Y_{\mathbb{Q}})}} - \frac{\#\varphi^{-1}(\bar{y})}{p^{(\dim X_{\mathbb{Q}} - \dim Y_{\mathbb{Q}})}} \right| < C_2 p^{-1},$$

where $\bar{y}$ is the image of y under the reduction $Y(\mathbb{Z}/p^k\mathbb{Z}) \to Y(\mathbb{F}_p)$.

(4) *There exists* $C_3 > 0$ *such that the following hold for every prime* p. *Let* $\mu_{X(\mathbb{Z}_p)}$ *and* $\mu_{Y(\mathbb{Z}_p)}$ *be the canonical measures on* $X(\mathbb{Z}_p)$ *and* $Y(\mathbb{Z}_p)$; *see Lemma 4.2. Then the pushforward measure* $\varphi_*\mu_{X(\mathbb{Z}_p)}$ *has continuous density* f_p *with respect to* $\mu_{Y(\mathbb{Z}_p)}$, *and* $\|f_p\|_\infty < C_3$.

Using a jet-scheme characterization of rational singularities by Mustaţă [33; 34], it can be shown that a morphism $\varphi : X \to Y$ between smooth schemes is (FRS) if and only if for each $k \in \mathbb{N}$, every nonempty fiber of the corresponding k-th jet map $J_k(\varphi) : J_k(X) \to J_k(Y)$ is of dimension $\dim J_k(X) - \dim J_k(Y)$ (i.e., $J_k(\varphi)$ is flat) and has a singular locus of codimension at least 1 (see Section 2A1 and Lemma 2.9). Based on this characterization, it is natural to define two variations of the (FRS) property:

- A morphism φ is *ε-jet flat*, for $\varepsilon \in \mathbb{R}_{>0}$, if the fibers of $J_k(\varphi)$ are of dimension at most $\dim J_k(X) - \varepsilon \dim J_k(Y)$, for all $k \in \mathbb{N}$; see [20, Definition 3.22].
- A morphism φ is called *E-smooth* if it is 1-jet flat, and each of the fibers of $J_k(\varphi)$ has singular locus of codimension at least E.

In Section 4C, using methods similar to the proof of Theorem A, we provide uniform estimates on the fibers of E-smooth and ε-jet flat morphisms (see Theorems 4.11 and 4.12). In particular, uniform estimates are given on fibers of flat morphisms whose fibers have terminal or log-canonical singularities.

1D1. *Main difficulties in the proof of Theorem A.* The proof of Theorem 1.3 in [2] proceeds by (locally) embedding X as a complete intersection in $\mathbb{A}^N$ and choosing an embedded resolution of singularities for the pair $(X_{\mathbb{Q}}, \mathbb{A}^N_{\mathbb{Q}})$, also called a log-resolution, whose existence follows from [25]. For large p, one can then use Denef's formula [13, Theorem 3.1], to relate $\#X(\mathbb{Z}/p^k\mathbb{Z})$ to $\{\#E_I(\mathbb{F}_p)\}_I$ and numerical data associated to the choice of resolution, where $\{E_I\}_I$ is a collection of constructible subsets built out of the prime divisors $\{E_i\}_{i=1}^M$ appearing in such a resolution. Combined with the Lang–Weil estimates for the E_I, this yields estimates for $\#X(\mathbb{Z}/p^k\mathbb{Z})$. To finally achieve the bounds of Theorem 1.3, one needs the reductions modulo p of the E_I's to be of the expected dimensions over $\mathbb{F}_p$. This can always be done if the prime p is large enough; small primes are treated separately in [18].

If $\varphi_{\mathbb{Q}} : X_{\mathbb{Q}} \to Y_{\mathbb{Q}}$ is (FRS), its fibers are local complete intersections with rational singularities, and one may try to mimic the strategy for Theorem 1.3. The weak point is that this only seems to work for each fiber separately, but does not give the desired uniformity in the choice of fiber. One can try to make this naive fiber-wise strategy more uniform by choosing some simultaneous resolutions of singularities. This can be done by breaking Y into constructible subsets, with resolutions over generic points of the pieces. However, such finite partition of Y into constructible sets does not behave well at all with respect to taking points over the rings $\mathbb{Z}$, $\mathbb{Z}/p^k\mathbb{Z}$, or $\mathbb{Z}_p$. In fact, as far as we can see, the approach with resolutions of singularities in families is hard to adapt to the family situation of Theorem A.

To avoid these difficulties, we use the motivic nature of $\mathbb{Z}/p^k\mathbb{Z}$-point count of the fibers of φ, that is, we use insights from motivic integration and uniform p-adic integration. Let $r_k : Y(\mathbb{Z}_p) \to Y(\mathbb{Z}/p^k\mathbb{Z})$ be the reduction map. Write $d := \dim X_{\mathbb{Q}} - \dim Y_{\mathbb{Q}}$. For each prime p, each $y \in Y(\mathbb{Z}_p)$ and each integer

$k \geq 1$ we set

$$g_p(y,k) = \frac{\#\varphi^{-1}(r_k(y))}{p^{kd}} \quad \text{and} \quad \tilde{h}_p(y,k) := g_p(y,k) - g_p(y,1), \tag{1-1}$$

as in the left-hand side of items (2) and (3) of Theorem A. The collections of functions $\{g_p\}_p$, $\{\tilde{h}_p\}_p$ are examples of *motivic functions*, namely in a uniform p-adic sense as in [11], but closely related to genuine motivic constructible functions from [7]. We use motivic integration to extract information on $\{g_p\}_p$ and $\{\tilde{h}_p\}_p$, which in turn allows us to prove Theorem A.

1D2. *Relation of the main number theoretic results to previous results.* Aizenbud and Avni [1; 2] have shown that the (FRS) property of $\varphi_{\mathbb{Q}}$ is equivalent to uniform boundedness of $g_p(y,k)$, when either

(1) one varies over k and p for each fixed y [2, Theorem A];[1] or

(2) one varies over k and y for each fixed p [1, Theorem 3.4].

Using the Lang–Weil estimates, one can further show that the (FRS) property implies uniform boundedness of $g_p(y,k)$ when

(3) one varies over p and y for each fixed k.

The implication (1) $\Rightarrow$ (2) of Theorem A asserts that if $\varphi_{\mathbb{Q}}$ is (FRS) then $g_p(y,k)$ as above is uniformly bounded when varying over p, k and y simultaneously.

It is worth noting that unlike items (1) and (2), item (3) as above is weaker than the (FRS) property, and is equivalent to 1-jet flatness of $\varphi_{\mathbb{Q}}$ using Lemma 4.13 and Theorem 4.12. In a recent work by Glazer and Hendel, this condition is furthermore shown to be equivalent to $\varphi_{\mathbb{Q}}$ being flat with fibers of semi-log-canonical singularities; see [22, Lemma 6.5, Theorem 6.6] and the discussion therein.

The proofs of the number-theoretic estimates for ε-jet flat and E-smooth morphisms (Theorems 4.11 and 4.12) share similar difficulties with the proof of Theorem A. Theorem 4.12 improves previous bounds for ε-jet flat morphisms: the bounds given in [39, Corollary 2.9] on $g_p(y,k)$ are uniform in k, but not in p and y (see Remark 2.8 for the relation of ε-jet flatness to the log canonical threshold), and the bounds given in [20, Theorem 8.18] are uniform in p, y, k, but are not optimal.

1D3. *Model-theoretic results.* We denote by Loc the collection of all non-Archimedean local fields, by Loc_0 the collection of all $F \in \mathrm{Loc}$ of characteristic zero, and by $\mathrm{Loc}_{\gg}$ the collection of all $F \in \mathrm{Loc}$ with large enough residual characteristic, where "large enough" changes according to our needs.

Let $\mathcal{L}_{\mathrm{DP}}$ denote the Denef–Pas language. This is a first order language with three sorts to account for a valued field F, a residue field k_F and a value group which we identify with $\mathbb{Z}$. An $\mathcal{L}_{\mathrm{DP}}$-definable set $X = \{X_F\}_{F \in \mathrm{Loc}_{\gg}}$ is a collection of subsets $X_F \subseteq F^{n_1} \times k_F^{n_2} \times \mathbb{Z}^{n_3}$ which is uniformly defined using an $\mathcal{L}_{\mathrm{DP}}$-formula.[2] Given $\mathcal{L}_{\mathrm{DP}}$-definable sets X and Y, a collection of functions $\{f : X_F \to Y_F\}_{F \in \mathrm{Loc}_{\gg}}$ is called an ($\mathcal{L}_{\mathrm{DP}}$-)definable function if its graph is definable.

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[1] Here "for each fixed y" means for each fixed $y \in Y(\mathbb{Q})$, where p is large enough to allow us to reduce modulo p.

[2] For the notation "$F \in \mathrm{Loc}_{\gg}$" see Section 2B.

Given a definable set $X = \{X_F\}_{F \in \mathrm{Loc}_{\gg}}$, the ring of motivic functions $\mathcal{C}(X)$ is a certain natural class of functions whose building blocks are the definable functions, and is closed under integration. Built on a natural notion of positivity, we define the semiring of formally nonnegative functions $\mathcal{C}_+(X) \subset \mathcal{C}(X)$ (see Definition 2.11). As an example, the collection $\{\varphi_* \mu_F\}_{F \in \mathrm{Loc}_{\gg}}$ of pushforwards of Haar measures μ_F on $\mathcal{O}_F^n$ under any polynomial map φ, as well as $\{g_p\}_p$ above are formally nonnegative motivic functions. The classes $\mathcal{C}_+(X)$ and $\mathcal{C}(X)$ above are uniform p-adic specializations of more genuinely motivic functions defined in [7; 8], but they go by similar methods and theories. See Section 2B for further details on motivic functions.

As a key step towards proving Theorem A, we show the following strengthening of [11, Theorem 2.1.3] for the class of formally nonnegative motivic functions:

Theorem B (Theorem 3.1). *Let f be in $\mathcal{C}_+(X \times W)$, where X and W are $\mathcal{L}_{\mathrm{DP}}$-definable sets. Then there exists a constant $C > 0$, and a function $G \in \mathcal{C}_+(X)$ such that for any $F \in \mathrm{Loc}_{\gg}$ and any $x \in X_F$ such that $w \mapsto f_F(x, w)$ is bounded on W_F, we have*

$$\sup_{w \in W_F} f_F(x, w) \leq G_F(x) \leq C \cdot \sup_{w \in W_F} f_F(x, w).$$

The approximation of suprema given in Theorem B is best possible for the class of formally nonnegative motivic functions $\mathcal{C}_+(X \times W)$, in the sense that one cannot choose C to be a universal constant (see Proposition 3.6). In [11, Theorem 2.1.3], a similar approximation result is shown (for motivic functions in $\mathcal{C}(X \times W)$ and in $\mathcal{C}^{\exp}(X \times W)$), but where the constant C is replaced by q_F^C, with q_F the number of elements in the residue field k_F of F, and where instead of $\sup f_F$ one approximates $\sup |f_F|^2$. For more details on the optimality of these approximation results, see the discussion in Section 3A.

1D4. *Sketch of proof of Theorem A.* To prove Theorem A, we show (1) ⇒ (3) ⇒ (2) ⇒ (4) ⇒ (1). The implications (3) ⇒ (2) ⇒ (4) are rather easy and the implication (4) ⇒ (1) essentially follows from an equivalent analytic characterization of the (FRS) property due to Aizenbud and Avni (see Theorem 4.3). The challenging part of the proof is to show (1) ⇒ (3). Small primes are dealt using Theorem 4.3 and using basic properties of the canonical measure (Lemma 4.2). Thus we may consider only large enough primes p. Let us sketch the main strategy of the proof of (1) ⇒ (2), for large p, which has similar difficulties to (1) ⇒ (3):

(a) We use Theorem 4.3 to show that

$$\sup_{y,k} g_p(y, k) < C(p)$$

for some constant $C(p)$ depending on p.

(b) Since g is a formally nonnegative motivic function (see Definition 2.10), and $g_p(y, k)$ is bounded for each fixed p and k, we may utilize Theorem B to approximate

$$\sup_{y \in Y(\mathbb{Z}_p)} g_p(y, k)$$

for each p and k by $G_p(k)$, for a single motivic function $G = \{G_p : \mathbb{Z}_{\geq 1} \to \mathbb{R}\}_p$.

(c) We use results from [11] on approximate suprema of constructible Presburger functions together with item (a) to deduce that

$$\sup_{k\in\mathbb{Z}_{\geq 1}} G_p(k)$$

can be approximated by $\sum_{l\in L} G_p(l)$ for some finite subset $L \subseteq \mathbb{Z}_{\geq 1}$, with L independent of p.

(d) To deal with $G_p(l)$ for $l \in L$, we use a transfer principle for boundedness of motivic functions from [10] (see Theorem 2.14 below) to reduce to a question about the $\mathbb{F}_p$-fibers of the $(l-1)$-th jet of φ. We then combine Lang–Weil type arguments on the jets of φ, together with a jet-scheme interpretation of the (FRS) property (Proposition 2.3), to deduce that $G_p(l) < C$ for $p \gg 1$, $l \in L$ and some constant $C > 0$ independent of p.

This shows (1) ⇒ (2). To prove (1) ⇒ (3), we approximate $\tilde{h}_p$ with a motivic function h_p, which unlike $\tilde{h}_p$, is formally nonnegative. We then apply similar steps as above (with a few extra complications) to h_p.

1E. ***Further discussion.*** We now give more context and motivation for Theorem A (for additional details see the references below). The (FRS) property was first introduced and studied in [1; 2], where a very useful analytic interpretation was given as follows. Given a morphism $\varphi : X \to Y$ between smooth $\mathbb{Q}$-varieties, the (FRS) property of φ is characterized by the property that for every $F \in \mathrm{Loc}_0$ and every smooth, compactly supported measure $\mu_{X(F)}$ on $X(F)$, the pushforward measure $\varphi_*(\mu_X)$ on $Y(F)$ has continuous density; see Theorem 4.3 or [1, Theorem 3.4]. Our number theoretic characterization (Theorem A) can be seen as a refinement of this analytic characterization.

These characterizations allow one to use algebro-geometric tools to solve various problems in analysis, probability and group theory. For a motivating example, let $\underline{G}$ be a semisimple algebraic $\mathbb{Q}$-group and let $\varphi_{\mathrm{comm}}^{*t} : \underline{G}^{2t} \to \underline{G}$ be the map $(g_1, \dots, g_{2t}) \mapsto [g_1, g_2] \cdot \dots \cdot [g_{2t-1}, g_{2t}]$, corresponding to the product of t commutator maps. Using the above characterizations and a theorem of Frobenius, one has, see [1, Theorem IV],

$$\varphi_{\mathrm{comm}}^{*t} \text{ is (FRS)} \Rightarrow \#\{N\text{-dimensional irreducible } \mathbb{C}\text{-representations of } \underline{G}(\mathbb{Z}_p)\} = O(N^{2t-2}). \qquad (\star)$$

Aizenbud and Avni showed in [1; 2], that $\varphi_{\mathrm{comm}}^{*21}$ is (FRS) for every $\underline{G}$ as above, which via ($\star$), confirmed a conjecture of Larsen and Lubotzky [31] about representation growth of compact p-adic and arithmetic groups. These bounds were improved in [3; 20; 24; 29].

The above situation can be generalized as follows. Let $\varphi : X \to \underline{G}$ be a dominant morphism from a smooth $\mathbb{Q}$-variety X to a connected algebraic group $(\underline{G}, \cdot_{\underline{G}})$. We define the *self-convolution* $\varphi * \varphi : X \times X \to \underline{G}$ of φ by $\varphi * \varphi(x_1, x_2) = \varphi(x_1) \cdot_{\underline{G}} \varphi(x_2)$, and write $\varphi^{*t} : X^t \to G$ for the t-th convolution power of φ. Similarly to the usual convolution operation in analysis, this algebraic convolution operation has a smoothing effect on morphisms; in [19; 21], it was shown that $\varphi^{*t} : X^t \to \underline{G}$ has increasingly better singularity properties as t grows, and eventually, φ^{*t} becomes (FRS) for every $t \geq t_0$, for some $t_0 \in \mathbb{N}$.

Moving to the probabilistic picture, let $\mu_{X(\mathbb{Z}_p)}$ and $\mu_{\underline{G}(\mathbb{Z}_p)}$ be the canonical measures on $X(\mathbb{Z}_p)$ and $\underline{G}(\mathbb{Z}_p)$, normalized to have total mass 1. One can then study the collection of random walks on $\underline{G}(\mathbb{Z}_p)$, induced by the pushforward measures $\{\varphi_* \mu_{X(\mathbb{Z}_p)}\}_{p \in \text{primes}}$, by analyzing the convergence rate of their self-convolutions $(\varphi_* \mu_{X(\mathbb{Z}_p)})^{*t}$ to $\mu_{\underline{G}(\mathbb{Z}_p)}$, in the L^r-norm ($r \geq 1$). This rate of convergence can be measured by the notion of *L^q-mixing time*; see, e.g., [32, Chapter 4]. Note that the analytic convolution operation commutes with the algebraic convolution defined above, so that $(\varphi_* \mu_{X(\mathbb{Z}_p)})^{*t} = (\varphi^{*t})_* \mu_{X^t(\mathbb{Z}_p)}$. This makes Theorem A the connecting link between the algebraic and the probabilistic pictures above.

Explicitly, let us denote by t_{alg} the minimal $t \in \mathbb{N}$ such that φ^{*t} is (FRS) and has geometrically irreducible fibers, and call it the *algebraic mixing time of φ*. Then Theorem A, and its general form Theorem 4.7, imply that the algebraic mixing time of φ is equal to the uniform (in $p \gg 1$) L^∞-mixing time of the random walks on $\{\underline{G}(\mathbb{Z}_p)_p\}$ induced by $\{\varphi_* \mu_{X(\mathbb{Z}_p)}\}_p$; see [20, Definition 9.2]. This philosophy was implemented in [20], which motivated this work. There, the authors analyzed the singularity properties of word maps on semisimple algebraic groups, using purely algebraic techniques, and obtained probabilistic results on word measures. In particular, Theorem A completes the proof of [20, Theorems G and 9.3(2)].

1F. *Conventions.*

- Throughout the paper, we use K, K', K'' to denote number fields and $\mathcal{O}_K$, $\mathcal{O}_{K'}$, $\mathcal{O}_{K''}$ for their rings of integers. Similarly, local fields and their rings of integers are denoted by F, F', F'' and $\mathcal{O}_F$, $\mathcal{O}_{F'}$, $\mathcal{O}_{F''}$, respectively.

- Given a local ring A, a morphism $\varphi : X \to Y$ of schemes X and Y, and given $y \in Y(A)$ (i.e., a morphism $\operatorname{Spec}(A) \to Y$), we denote by $X_{y,\varphi} := \operatorname{Spec}(A) \times_Y X$ the scheme theoretic fiber over y, and simply by $\varphi^{-1}(y) \subseteq X(A)$ the set theoretic fiber of the induced map $\varphi : X(A) \to Y(A)$. Note that if $y \in Y$ is a schematic point, then it can be viewed as $y \in Y(\kappa(y))$, where $\kappa(y)$ is the residue field of y, so that $X_{y,\varphi} := \operatorname{Spec}(\kappa(y)) \times_Y X$.

- Given a K-morphism $\varphi : X \to Y$ between K-varieties X and Y, we denote by X^{sm} (resp. X^{sing}) the smooth (resp. nonsmooth) locus of X. We denote by $X^{\text{sm},\varphi}$ (resp. $X^{\text{sing},\varphi}$) the smooth (resp. nonsmooth) locus of φ in X.

- We denote the base change of an S-scheme X with respect to $S' \to S$ by $X_{S'}$.

2. Preliminaries

2A. ***Jet schemes and singularities.*** For a thorough discussion of jet schemes see [4, Chapter 3] and [15].

Definition 2.1 [4, Section 3.2]. Let S be a scheme and let X be a scheme over S:

(1) For each $k \in \mathbb{N}$, we define the *k-th jet scheme* of X, denoted $J_k(X/S)$ as the S-scheme representing the functor

$$\mathcal{J}_k(X/S) : W \mapsto \operatorname{Hom}_{S\text{-schemes}}(W \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}(\mathbb{Z}[t]/(t^{k+1})), X),$$

where W is an S-scheme. We write $J_k(X)$ if the scheme S is understood.

(2) Given an S-morphism $\varphi : X \to Y$ and an S-scheme W, the composition with φ induces a map $\mathcal{J}_k(X/S)(W) \to \mathcal{J}_k(Y/S)(W)$, which yields a morphism

$$J_k(\varphi) : J_k(X/S) \to J_k(Y/S),$$

called the *k-th jet* of φ.

(3) For any $k_1 \geq k_2 \in \mathbb{N}$ the reduction map $\mathbb{Z}[t]/(t^{k_1+1}) \to \mathbb{Z}[t]/(t^{k_2+1})$ induces a natural collection of morphisms $\pi^{k_1}_{k_2,X} : J_{k_1}(X/S) \to J_{k_2}(X/S)$ which are called *truncation maps*. Note that the collection $\{J_k(\varphi) : J_k(X/S) \to J_k(Y/S)\}_{k\in\mathbb{N}}$ commutes with $\{\pi^m_{n,X}\}_{m\geq n}$.

(4) The natural map $\mathbb{Z} \to \mathbb{Z}[t]/(t^{m+1})$ induces a *zero section* $s_{m,X} : X \hookrightarrow J_m(X)$. We sometimes write π^m_n and s_m instead of $\pi^m_{n,X}$ and $s_{m,X}$, when X is clear.

In the rest of this subsection, we assume $S = \operatorname{Spec} K$. Mustaţă gave the following interpretation of rational singularities in terms of jet schemes:

Theorem 2.2 [33]. *Let X be a geometrically irreducible, local complete intersection K-variety, with* $\operatorname{char}(K) = 0$. *Then $J_k(X)$ is geometrically irreducible for all $k \geq 1$ if and only if X has rational singularities.*

Using Theorem 2.2, one can obtain a similar characterization of (FRS) morphisms:

Proposition 2.3 [20, Corollary 3.12; 27]. *Let X and Y be smooth, geometrically irreducible K-varieties, and let $\varphi : X \to Y$ be a K-morphism:*

1. *Assume* $\operatorname{char}(K) = 0$. *Then the morphism φ is (FRS) if and only if $J_k(\varphi)$ is flat, with locally integral fibers for each $k \in \mathbb{N}$.*
2. *The morphism φ is smooth if and only if $J_k(\varphi)$ is smooth for each $k \in \mathbb{N}$.*

Remark 2.4. Let k be a natural number, and K be a field with $\operatorname{char}(K) = 0$ or $\operatorname{char}(K) \gg 1$ (in terms of k). Then the jet scheme $J_k(X)$ of an affine K-scheme $X \subseteq \mathbb{A}^n$ has a simple description; write $X = \operatorname{Spec} K[x_1, \dots, x_n]/(f_1, \dots, f_l)$. Then

$$J_k(X) = \operatorname{Spec} K[x_1, \dots, x_n, x_1^{(1)}, \dots, x_n^{(1)}, \dots, x_1^{(k)}, \dots, x_n^{(k)}]/(\{f_j^{(u)}\}_{j=1,u=0}^{l,k}),$$

where $f_i^{(u)}$ is the u-th formal derivative of f_i. For example, if $f = x_1x_2^2$ then $f^{(1)} = x_1^{(1)}x_2^2 + 2x_1x_2x_2^{(1)}$. Similarly, $J_k(\varphi) = (\varphi, \varphi^{(1)}, \dots, \varphi^{(k)})$ for a morphism $\varphi : X \to Y$ of affine K-schemes.

The next proposition will be useful in Section 4.

Proposition 2.5. *Let $k \in \mathbb{N}$ and let $\varphi : X \to Y$ be K-morphism as in Proposition 2.3, with* $\operatorname{char}(K) = 0$ *or* $\operatorname{char}(K) \gg 1$ *(in terms of k). Then $J_k(X)^{\mathrm{sm}, J_k(\varphi)} = J_k(X^{\mathrm{sm},\varphi})$.*

Proof. It follows from Proposition 2.3(2) that $J_k(X^{\mathrm{sm},\varphi}) \subseteq J_k(X)^{\mathrm{sm}, J_k(\varphi)}$, so it is left to show the other inclusion. We may assume that X and Y are affine, and that Y admits an étale map $\psi : Y \to \mathbb{A}^m_K$. We may further assume that $Y = \mathbb{A}^m_K$. Indeed, we have

$$J_k(X)^{\mathrm{sm}, J_k(\varphi)} = J_k(X)^{\mathrm{sm}, J_k(\psi\circ\varphi)} \quad \text{and} \quad J_k(X^{\mathrm{sm},\psi\circ\varphi}) = J_k(X^{\mathrm{sm},\varphi}).$$

By Remark 2.4, we can write $X = \operatorname{Spec} K[x_1, \dots, x_{n+l}]/(f_1, \dots, f_l)$, and

$$J_k(X) = \operatorname{Spec} K[x_1, \dots, x_{n+l}, \dots, x_1^{(k)}, \dots, x_{n+l}^{(k)}]/(\{f_j^{(u)}\}_{j=1,u=0}^{l,k}).$$

Moreover $J_k(\varphi) = (\varphi, \varphi^{(1)}, \dots, \varphi^{(k)})$ where $\varphi = (f_{l+1}, \dots, f_{l+m}) : X \to \mathbb{A}_K^m$. Write $F_{u(l+m)+j} := f_j^{(u)}$ and $X_{u(n+l)+i} := x_i^{(u)}$, and let $\bar{a} := (a, a^{(1)}, \dots, a^{(n+l)}) \in J_k(X)$. Then $J_k(\varphi)$ is smooth at $\bar{a}$ if and only if the matrix $M = \bigl(\frac{\partial F_j}{\partial X_i}|_{\bar{a}}\bigr)_{i=1,j=1,}^{(n+l)(k+1),(l+m)(k+1)}$ is of full rank $(l+m)(k+1)$. Note that M has the shape

$$M = \begin{pmatrix} M_{00} & M_{01} & \cdots & M_{0k} \\ 0 & M_{11} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & M_{kk} \end{pmatrix},$$

where $M_{u_1u_2} = \Bigl(\frac{\partial f_j^{(u_2)}}{\partial x_i^{(u_1)}}|_{\bar{a}}\Bigr)_{i=1,j=1}^{(n+l),(l+m)}$ for $0 \le u_1 \le u_2 \le k$. If M is of full rank, then also $M_{00} = \bigl(\frac{\partial f_j}{\partial x_i}|_a\bigr)_{i=1,j=1}^{(n+l),(l+m)}$ must be of full rank, which in turn implies that φ is smooth at a, and the proposition follows. ☐

Remark 2.6. The case $Y = \mathbb{A}^1$ of Proposition 2.5 has essentially been proven in [17, proof of Theorem 3.3] and [33, Proposition 4.12]; see also [28, page 222]. Proposition 2.5 also relates to [33, Questions 4.10 and 4.11], as follows. Given a local complete intersection variety X, it can be written, locally, as a fiber $\widetilde{X}_{0,\varphi}$ of a flat morphism $\varphi : \widetilde{X} \to \mathbb{A}^m$, with $\widetilde{X}$ smooth. If we assume that $J_k(\varphi)$ is flat for all k, then Proposition 2.5 combined with [23, III, Theorem 10.2] implies that $(\pi^k_{0,\widetilde{X}_{0,\varphi}})^{-1}((\widetilde{X}_{0,\varphi})^{\mathrm{sm}}) = J_k(\widetilde{X}_{0,\varphi})^{\mathrm{sm}}$ for all k, which gives a positive answer to [33, Question 4.11] in this case. If $J_k(\varphi)$ is not flat, one can still effectively describe its smooth locus, but it is harder to describe the smooth locus of its fibers.

2A1. *E-smooth and ε-jet flat morphisms.* We next introduce several properties of morphisms between smooth varieties: ε-flatness, ε-jet flatness, and E-smoothness. The first two notions were first introduced in [20], whereas the E-smoothness notion is new.

Definition 2.7. Let X and Y be smooth, geometrically irreducible K-varieties, and let $\varphi : X \to Y$ be a K-morphism, let $E \ge 1$ be an integer and let $\varepsilon \in \mathbb{R}_{>0}$. Then:

(1) φ is called *ε-flat* if for every $x \in X$ we have $\dim X_{\varphi(x),\varphi} \le \dim X - \varepsilon \dim Y$.

(2) φ is called *ε-jet flat* (resp. jet-flat) if $J_k(\varphi)$ is ε-flat (resp. flat) for every $k \in \mathbb{N}$.

(3) A jet-flat morphism φ is *E-smooth* if for all $k \in \mathbb{Z}_{\ge 0}$ and all $\tilde{x} \in J_k(X)$, the set $(J_k(X)_{J_k(\varphi)(\tilde{x}),J_k(\varphi)})^{\mathrm{sing}}$ is of codimension at least E in $J_k(X)_{J_k(\varphi)(\tilde{x}),J_k(\varphi)}$.

Remark 2.8. (1) By [34], a morphism φ as in Definition 2.7 is ε-jet flat if and only if $\operatorname{lct}(X, X_{\varphi(x),\varphi}) \ge \varepsilon \dim Y$ for all $x \in X$, where $\operatorname{lct}(X, X_{\varphi(x),\varphi})$ is the log-canonical threshold of the pair $(X, X_{\varphi(x),\varphi})$.

(2) In addition, it follows from [33; 16] (see [20, Corollary 3.12]) that if φ is a normal morphism, then it is jet-flat if and only if it is flat and has fibers with log-canonical singularities.

ε-flatness is a quantitative way to measure how close a morphism between smooth varieties is to being flat. Similarly, ε-jet flatness measures how close a morphism is to being jet-flat, which is very close to being an (FRS)-morphism. On the other hand, the starting point of E-smoothness is when φ is jet-flat, and the larger E is, the better the singularities of φ are. This is illustrated in the next lemma.

Lemma 2.9. *Let $\varphi : X \to Y$ be K-morphism between smooth, geometrically irreducible K-varieties:*

(1) *φ is 1-smooth if and only if φ is (FRS).*

(2) *φ is 2-smooth if and only if φ is flat with fibers of terminal singularities.*

Proof. By Proposition 2.3, φ is (FRS) if and only if $J_k(\varphi)$ is flat, with locally integral fibers for each $k \in \mathbb{N}$. By [20, Corollary 3.12(3)], φ is flat with fibers of terminal singularities if and only if $J_k(\varphi)$ is flat, with normal fibers for each $k \in \mathbb{N}$. In particular, in the situation of (1) and (2), for each k, the map $J_k(\varphi)$ is a flat map between smooth varieties, and thus the fibers of $J_k(\varphi)$ are local complete intersections, and hence Cohen–Macaulay. Serre's criterion for normality and reducedness [14, Proposition 5.8.5, Theorem 5.8.6] and [33, Proposition 1.4] now imply items (1) and (2). □

2B. ***Motivic functions.*** In this subsection we recall the definition and some properties of motivic functions. In order to prove Theorem A, we encode the collection $\{\#\varphi^{-1}(y)\}_{p,k,y\in Y(\mathbb{Z}/p^k\mathbb{Z})}$ using a single motivic function, and utilize this to obtain the desired uniform bounds. We use the notion of motivic functions as was defined and studied in [7; 8; 9; 10]. In order to fully exploit the advantages of the motivic realm, we introduce the class of *formally nonnegative motivic functions*, which is the specialization to local fields of [7, Section 5.3].

Throughout this subsection, we fix a number field K. We use the (three-sorted) *Denef–Pas language*, denoted

$$\mathcal{L}_{\mathrm{DP}} = (\mathcal{L}_{\mathrm{Val}}, \mathcal{L}_{\mathrm{Res}}, \mathcal{L}_{\mathrm{Pres}}, \mathrm{val}, \mathrm{ac}),$$

where:

(1) The valued field sort VF is endowed with the language of rings $\mathcal{L}_{\mathrm{Val}}$, with coefficients in $\mathcal{O}_K$.

(2) The residue field sort RF is endowed with the language of rings $\mathcal{L}_{\mathrm{Res}}$.

(3) The value group sort VG (which we just call $\mathbb{Z}$), is endowed with the Presburger language $\mathcal{L}_{\mathrm{Pres}} = (+, -, \leq, \{\equiv_{\mathrm{mod}\, n}\}_{n>0}, 0, 1)$ of ordered abelian groups along with constants 0, 1 and a family of relations $\{\equiv_{\mathrm{mod}\, n}\}_{n>0}$ of congruences modulo n.

(4) $\mathrm{val} : \mathrm{VF} \setminus \{0\} \to \mathbb{Z}$ and $\mathrm{ac} : \mathrm{VF} \to \mathrm{RF}$ are two function symbols.

Let Loc be the collection of all non-Archimedean local fields F with a ring homomorphism $\mathcal{O}_K \to F$. We denote by Loc_0 (resp. Loc_+) the collection of all $F \in \mathrm{Loc}$ of characteristic zero (resp. positive characteristic). For $F \in \mathrm{Loc}$, we denote by $\mathcal{O}_F$ its ring of integer, by k_F its residue field, and by q_F the number of elements in k_F. We use the notation $\mathrm{Loc}_{\gg}$ (resp. $\mathrm{Loc}_{0,\gg}$, $\mathrm{Loc}_{+,\gg}$),[3] for the collection of $F \in \mathrm{Loc}$ (resp. Loc_0, Loc_+) with large enough residual characteristic (depending on some given data).

Given $F \in \mathrm{Loc}$ (and a chosen uniformizer ϖ_F of $\mathcal{O}_F$), we can interpret val and ac as the valuation map $\mathrm{val} : F^\times \to \mathbb{Z}$ and the angular component map $\mathrm{ac} : F \to k_F$, where $\mathrm{ac}(0) = 0$ and $\mathrm{ac}(x) = x \cdot \varpi_F^{-\mathrm{val}(x)}$ mod $\varpi_F \mathcal{O}_F$ for $x \neq 0$. Hence, any formula ϕ in $\mathcal{L}_{\mathrm{DP}}$ with n_1 free VF-variables, n_2 free RF-variables and n_3 free $\mathbb{Z}$-variables, yields a subset $\phi(F) \subseteq F^{n_1} \times k_F^{n_2} \times \mathbb{Z}^{n_3}$. A collection $X = (X_F)_{F \in \mathrm{Loc}_{\gg}}$ with $X_F = \phi(F)$ is called an $\mathcal{L}_{\mathrm{DP}}$-*definable set*. Given $\mathcal{L}_{\mathrm{DP}}$-definable sets X and Y, an $\mathcal{L}_{\mathrm{DP}}$-*definable function* is a collection $f = (f_F : X_F \to Y_F)_{F \in \mathrm{Loc}_{\gg}}$ of functions whose collection of graphs is a definable set. We will often say “definable” instead of “$\mathcal{L}_{\mathrm{DP}}$-definable”.

Definition 2.10 [9, Subsections 4.2.4–4.2.5]. Let X be an $\mathcal{L}_{\mathrm{DP}}$-definable set. A collection $f = (f_F)_{F \in \mathrm{Loc}_{\gg}}$ of functions $f_F : X_F \to \mathbb{R}$ is called a *Presburger constructible function*, if it can be written as

$$f_F(x) = \sum_{i=1}^{N_1} q_F^{\alpha_{i,F}(x)} \prod_{j=1}^{N_2} \beta_{ij,F}(x) \prod_{j=1}^{N_3} \frac{1}{1 - q_F^{a_{ij}}},$$

where $N_1, N_2, N_3 \in \mathbb{N}$ and $a_{ij} \in \mathbb{Z}_{<0}$, and $\alpha_i, \beta_{ij} : X \to \mathbb{Z}$ are definable functions. Given f as above, set $\tilde{f}_F : X_F \times \mathbb{R}_{>1} \to \mathbb{R}$ by

$$\tilde{f}_F(x, s) := \sum_{i=1}^{N_1} s^{\alpha_{i,F}(x)} \prod_{j=1}^{N_2} \beta_{ij,F}(x) \prod_{j=1}^{N_3} \frac{1}{1 - s^{a_{ij}}}.$$

We say that f is *formally nonnegative* if $\tilde{f}_F$ takes nonnegative values for every $F \in \mathrm{Loc}_{\gg}$. We denote by $\mathcal{P}(X)$ the ring of Presburger constructible functions on X, and by $\mathcal{P}_+(X)$ the subsemiring of formally nonnegative functions.

Definition 2.11. Let X be an $\mathcal{L}_{\mathrm{DP}}$-definable set. A collection $h = (h_F)_{F \in \mathrm{Loc}_{\gg}}$ of functions $h_F : X_F \to \mathbb{R}$ is called a *motivic function*, if it can be written as

$$h_F(x) = \sum_{i=1}^{N} \# Y_{i,F,x} \cdot f_{i\,F}(x),$$

where:

- $Y_{i,F,x} = \{\xi \in k_F^{r_i} : (x, \xi) \in Y_{i,F}\}$ is the fiber over $x \in X_F$ of a definable set $Y_i \subseteq X \times \mathrm{RF}^{r_i}$ with $r_i \in \mathbb{N}$.
- Each f_i is a Presburger constructible function.

If furthermore every f_i is formally nonnegative, then we call h a *formally nonnegative motivic function*. We denote by $\mathcal{C}(X)$ the ring of motivic functions on X, and by $\mathcal{C}_+(X)$ the subsemiring of formally nonnegative motivic functions.

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[3] Our notation for $\mathrm{Loc}_{\gg}$ is slightly more restrictive than the one used in [11]. Here $\mathrm{Loc}_{\gg}$ consists of $\mathrm{Loc}_{0,\gg} \cup \mathrm{Loc}_{+,\gg}$ while in [11], it consisted of $\mathrm{Loc}_0 \cup \mathrm{Loc}_{+,\gg}$.

The classes $\mathcal{C}(X)$ and $\mathcal{C}_+(X)$ defined above are the specialization to local fields of more abstract classes of motivic functions defined in [7, Section 5]; e.g., see the discussion in [9, Section 4.2]). In [7, Theorem 10.1.1], it is shown that these more general classes are preserved under a formal integration operation, and in [8, Section 9] it is shown that this formal integration operation commutes with usual p-adic integration under specialization. This implies the following theorem:

Theorem 2.12. *Let X be an $\mathcal{L}_{\mathrm{DP}}$-definable set, and let f be in $\mathcal{C}_+(X \times \mathrm{VF}^m)$. Assume that for every $F \in \mathrm{Loc}_{\gg}$ and every $x \in X_F$, the function $y \mapsto f_F(x, y)$ belongs to $L^1(F^m)$. Then there exists g in $\mathcal{C}_+(X)$ such that*

$$g_F(x) = \int_{y \in F^m} f_F(x, y)|dy|. \tag{2-1}$$

Remark 2.13. In [9, Theorem 4.3.1] it was shown that the class of motivic functions is preserved under integration in the following stronger sense, namely, that given f in $\mathcal{C}(X \times \mathrm{VF}^m)$, one can find $g \in \mathcal{C}(X)$ such that for every $F \in \mathrm{Loc}_{\gg}$ and $x \in X_F$, if $y \mapsto f_F(x, y)$ belongs to $L^1(F^m)$ then (2-1) holds. This stronger statement relies on an interpolation theorem [9, Theorem 4.3.3] for functions in $\mathcal{C}(X)$. It would be interesting to prove a similar interpolation result for the class of formally nonnegative motivic functions. This will imply the stronger formulation of Theorem 2.12 as in [9, Theorem 4.3.1].

Finally, we need the following transfer result between $\mathrm{Loc}_{0,\gg}$ and $\mathrm{Loc}_{+,\gg}$.

Theorem 2.14 (transfer principle for bounds, [10, Theorem 3.1]). *Let X be an $\mathcal{L}_{\mathrm{DP}}$-definable set, and let $H, G \in \mathcal{C}(X)$ be motivic functions. Then the following holds for $F \in \mathrm{Loc}_{\gg}$; if*

$$|H_F(x)| \leq |G_F(x)|,$$

for each $x \in X_F$, then also

$$|H_{F'}(x)| \leq |G_{F'}(x)|,$$

for every $F' \in \mathrm{Loc}$ with the same residue field as F, and each $x \in X_{F'}$.

3. An improvement of the approximation of suprema

The main goal of this section is to show the following improvement of [11, Theorem 2.1.3] on approximate suprema. This improvement is made possible by placing ourselves in the special case of formally nonnegative motivic functions and is not possible in the more general situation of [11].

Theorem 3.1 (improved approximation of suprema). *Let f be in $\mathcal{C}_+(X \times W)$, where X and W are definable sets. Then there exist a constant $C > 0$, and a function $G \in \mathcal{C}_+(X)$ such that for any $F \in \mathrm{Loc}_{\gg}$ and any $x \in X_F$ such that $w \mapsto f_F(x, w)$ is bounded on W_F, we have*

$$\sup_{w \in W_F} f_F(x, w) \leq G_F(x) \leq C \cdot \sup_{w \in W_F} f_F(x, w).$$

The following lemma is immediate:

Lemma 3.2. *Let* $\{f_i\}_{i=1}^N$ *be in* $\mathcal{C}_+(X \times W)$ *and set* $f = \sum_{i=1}^2 f_i$. *Then for* $F \in \mathrm{Loc}_{\gg}$, *one has*:

$$\frac{1}{N}\sum_{i=1}^N \sup_{w\in W_F} f_{i\,F}(x,w) \le \sup_{w\in W_F} f_F(x,w) \le \sum_{i=1}^N \sup_{w\in W_F} f_{i\,F}(x,w).$$

Let f be in $\mathcal{C}_+(X \times W)$. By Definition 2.11, we can write $f(x,w) = \sum_{i=1}^2 \#Y_{i,x,w} \cdot g_i(x,w)$, where $g_i \in \mathcal{P}_+(X \times W)$ and $Y_i \subseteq X \times W \times \mathrm{RF}^{r_i}$. Lemma 3.2 thus implies the following:

Corollary 3.3. *Let* f *be in* $\mathcal{C}_+(X \times W)$, *where* X *and* W *are definable sets*:

(1) *Let* $X \times W = \bigsqcup_{i=1}^M C_i$ *be a definable partition and set* $f_i(x,w) = f(x,w) \cdot 1_{C_i}$. *Then it is enough to prove Theorem 3.1 for each* f_i.

(2) *It is enough to prove Theorem 3.1 for* f *of the form* $f = \#Y_{x,w} \cdot g(x,w)$ *where* $g \in \mathcal{P}_+(X \times W)$.

Remark 3.4. The key case of Theorem 3.1 is when neither X nor W involve valued field variables. The reduction to this case needs to be done with care. Naively, one can use quantifier elimination to eliminate the valued field variables, but this is problematic since it mixes the valued field variables of X and W, making it hard to take supremum over the variables of W. In order to elude this problem, we will apply cell decomposition iteratively, first taking care of the W variables and then taking care of the X variables.

Proof of Theorem 3.1. Let $f(x,w) = \#Y_{x,w} \cdot g(x,w)$ for some $g \in \mathcal{P}_+(X \times W)$ and $Y \subseteq X \times W \times \mathrm{RF}^r$. Without loss of generality, we may assume that $X = \mathrm{VF}^{n_1} \times \mathrm{RF}^{n_2} \times \mathrm{VG}^{n_3}$ and $W = \mathrm{VF}^{m_1} \times \mathrm{RF}^{m_2} \times \mathrm{VG}^{m_3}$ for some $n_i \ge 0$ and $m_i \ge 0$. We will first reduce to the case where there are no valued field variables, using the following claim.

Claim 1. *We may assume that* $X = \mathrm{RF}^{n_2} \times \mathrm{VG}^{n_3}$ *and* $W = \mathrm{RF}^{m_2} \times \mathrm{VG}^{m_3}$.

Proof of Claim 1. We first get rid of the valued field variables VF^{m_1} of W. Without loss of generality we may assume that $W = \mathrm{VF}^{m_1}$. By induction, we may further assume that $m_1 = 1$. By [7, Theorem 7.2.1] there exists a definable surjection $\lambda : X \times W \to C \subseteq X \times \mathrm{RF}^s \times \mathbb{Z}^r$ over X as well as $\psi \in \mathcal{C}_+(C)$ such that $f = \psi \circ \lambda$. Note that

$$\sup_{w\in W_F} f_F(x,w) = \sup_{w\in W_F} \psi_F \circ \lambda_F(x,w) = \sup_{(\xi,k)\in k_F^s \times \mathbb{Z}^r} \psi_F(x,\xi,k),$$

up to extending ψ by zero outside C. We may therefore assume that $W = \mathrm{RF}^{m_2} \times \mathrm{VG}^{m_3}$. We next get rid of the valued field variables VF^{n_1} of X, denoted $y := y_1, \dots, y_{n_1}$. Write $x = (y, \eta, t) \in X$ and $w = (\xi, s) \in W$, with RF-variables η, ξ and VG-variables t, s. By Definition 2.11, f is determined by a finite collection $\alpha_i, \beta_{ij} : X \times W \to \mathbb{Z}$ of definable functions, and by a definable set $Y \subseteq X \times W \times \mathrm{RF}^r$. By quantifier elimination in the valued field variables [36, Theorem 4.1], there exist finitely many polynomials $g_1, \dots, g_l \in \mathbb{Z}[y_1, \dots, y_{n_1}]$ such that the graphs of the functions in $\{\alpha_i, \beta_{ij}\}$ can be defined by formulas of the form

$$\bigvee_{i=1}^L \chi_i(\xi, \eta, \mathrm{ac}(g_1(y)), \dots, \mathrm{ac}(g_l(y))) \wedge \theta_i(t, s, t', \mathrm{val}(g_1(y)), \dots, \mathrm{val}(g_l(y))),$$

and the subset Y can be defined by a formula of the form

$$\bigvee_{i=1}^{L'} \tilde{\chi}_i(\xi, \eta, \xi', \mathrm{ac}(g_1(y)), \dots, \mathrm{ac}(g_l(y))) \wedge \tilde{\theta}_i(t, s, \mathrm{val}(g_1(y)), \dots, \mathrm{val}(g_l(y))),$$

where χ_i and $\tilde{\chi}_i$ are $\mathcal{L}_{\mathrm{Res}}$-formulas, θ_i and $\tilde{\theta}_i$ are $\mathcal{L}_{\mathrm{Pres}}$-formulas, t' is in $\mathbb{Z}$ and ξ' is in RF^r. We now define $\lambda' : X \times W \to \mathrm{RF}^{s'} \times \mathbb{Z}^{r'} \times W$ by $\lambda'(x, w) = (\rho(x), w)$, where $s' := n_2 + l$, $r' := n_3 + l$ and where

$$\rho(x) = \rho(y, \eta, t) := (\eta, \mathrm{ac}(g_1(y)), \dots, \mathrm{ac}(g_l(y)), t, \mathrm{val}(g_1(y)), \dots, \mathrm{val}(g_l(y))).$$

Let C' be the image of λ'. Note we may find definable functions $\tilde{\alpha}_i, \tilde{\beta}_{ij} : C' \to \mathbb{Z}$ and a definable subset $\widetilde{Y} \subseteq C' \times \mathrm{RF}^r$ such that $\alpha_i = \tilde{\alpha}_i \circ \lambda'$, $\beta_i = \tilde{\beta}_{ij} \circ \lambda'$ and $Y = (\lambda' \times \mathrm{Id})^{-1}(\widetilde{Y})$. Using this new definable data, we construct $\psi' \in \mathcal{C}_+(C')$ such that $f = \psi' \circ \lambda'$ and again we have

$$\sup_{w \in W_F} f_F(x, w) = \sup_{w \in W_F} \psi'_F \circ \lambda'_F(x, w) = \sup_{w \in W_F} \psi'_F(\rho(x), w).$$

Hence we have reduced to the case where $X = \mathrm{RF}^{n_2} \times \mathrm{VG}^{n_3}$. This finishes the proof of Claim 1. ☐

Claim 2. *We may assume that* $X = \mathrm{RF}^{n_2} \times \mathrm{VG}^{n_3}$ *and* $W = \mathrm{RF}^{m_2}$.

Proof of Claim 2. Write $x = (\eta, t)$ and $w = (\xi, s)$ for the variables of $X = \mathrm{RF}^{n_2} \times \mathrm{VG}^{n_3}$ and $W = \mathrm{RF}^{m_2} \times \mathrm{VG}^{m_3}$. We would like to get rid of the value group variables VG^{m_3} of W. Using the (model theoretic) orthogonality of the sorts VG and RF, there is a definable partition of $X \times W$, such that each definable part A is a box $A_1 \times A_2$ with $A_1 \subseteq \mathrm{RF}^{n_2} \times \mathrm{RF}^{m_2}$ and $A_2 \subseteq \mathrm{VG}^{n_3} \times \mathrm{VG}^{m_3}$, and such that on each A, f has the form

$$f_F|_{A_F}(\eta, t, \xi, s) = \#Y_{\eta,\xi} \cdot H_F(t, s),$$

for some $H \in \mathcal{P}_+(\mathrm{VG}^{n_3} \times \mathrm{VG}^{m_3})$ and $Y \subseteq \mathrm{RF}^{n_2} \times \mathrm{RF}^{m_2} \times \mathrm{RF}^r$. By Corollary 3.3, and by our assumption on A, we may assume $f_F = \#Y_{\xi,\eta} \cdot H_F(t, s)$. Note that for each $F \in \mathrm{Loc}_{\gg}$ and each $(\eta, t, \xi) \in X_F \times k_F^{m_2}$ one has

$$\sup_{s \in \mathbb{Z}^{m_3}} f_F(\eta, t, \xi, s) = \#Y_{\eta,\xi} \cdot \sup_{s \in \mathbb{Z}^{m_3}} H_F(t, s).$$

In order to approximate $\sup_{s \in \mathbb{Z}^{m_3}} H_F(t, s)$, it is enough to consider the case where $m_3 = 1$ and proceed by induction on m_3. Using Presburger cell decomposition and rectilinearization (see [6, Theorems 1 and 3]) we may assume that H is in $\mathcal{P}_+(B)$ for $B \subseteq \mathrm{VG}^{n_3} \times \mathbb{N}$ with $B_t := \{s \in \mathbb{N} : (t, s) \in B\}$, such that exactly one of the following holds:

(1) B_t is a finite set for each $t \in \mathbb{Z}^{n_3}$.

(2) $B_t = \mathbb{N}$ or $B_t = \varnothing$ for each $t \in \mathbb{Z}^{n_3}$.

Moreover, H can be taken to be of the form

$$H_F(t, s) = \sum_{i=1}^{N} c_{i,F}(t) s^{a_i} q_F^{b_i s},$$

with $a_i \in \mathbb{N}$ and $b_i \in \mathbb{Z}$ and c_i in $\mathcal{P}(\mathrm{VG}^{n_3})$. Denote by T the image of projection of B to VG^{n_3}. We repeat a part of the argument of the proof of [11, Theorem 2.1.3]. Namely, by [11, Lemmas 2.2.3 and 2.2.4], there exist $m, l \in \mathbb{N}_{\geq 1}$ and finitely many definable functions $h_1, \dots, h_l : T \to \mathbb{N}$ with $h_j(t) \in B_t$ such that for each $t \in T$ for which $s \mapsto H_F(t,s)$ is bounded on B_t, one has

$$\sup_{s \in B_t} H_F(t,s) \leq m \cdot \max_{1 \leq j \leq l} H_F(t, h_j(t)).$$

In particular, setting

$$\widetilde{H}(t) := m \cdot \sum_{j=1}^{l} H(t, h_j(t)) \in \mathcal{P}_+(T)$$

we get

$$\sup_{s \in B_t} H_F(t,s) < \widetilde{H}_F(t) < m \cdot l \cdot \sup_{s \in B_t} H_F(t,s).$$

This finishes the proof of Claim 2. □

Claim 3. *We may assume that* $X = \mathrm{RF}^{n_2}$ *and* $W = \mathrm{RF}^{m_2}$.

Proof. This follows directly by Claim 2, Corollary 3.3, and using the orthogonality of the sorts VG and RF. □

To continue the proof of Theorem 3.1, we may thus assume that $X = \mathrm{RF}^{n_2}$ and $W = \mathrm{RF}^{m_2}$. We may assume, again using Corollary 3.3, that f is of the form $f(x,w) = f(\eta,\xi) = u \cdot \#Y_{\eta,\xi}$, with ξ the coordinate on W, and η on X and $u = \{u_F\}_{F \in \mathrm{Loc}_{\gg}}$ is a motivic number. In particular, for each $\eta \in X_F$,

$$\sup_{w \in W_F} f_F(x,w) = \sup_{\xi \in k_F^{m_2}} f_F(\eta,\xi) = u_F \cdot \sup_{\xi \in k_F^{m_2}} \#Y_{\eta,\xi}.$$

By a definable variant of the Lang–Weil estimates (see [5, main theorem]), there exists a definable partition $X \times W = \bigsqcup_{i=0}^{2} A_i$ and constants $C' > 0$, $d_i \in \mathbb{N}$ and $l_{i1}, l_{i2} \in \mathbb{Z}_{\geq 1}$, such that for each $1 \leq i \leq M$ and each $F \in \mathrm{Loc}_{\gg}$

$$A_{i,F} := \left\{ (\eta,\xi) \in X_F \times W_F : \left| \#Y_{\eta,\xi} - \frac{l_{i1}}{l_{i2}} q_F^{d_i} \right| \leq C' \cdot q_F^{d_i - 1/2} \right\},$$
$$A_{0,F} := \{(\eta,\xi) \in X_F \times W_F : Y_{\eta,\xi} \text{ is empty}\}.$$

Denote by Z_i the projection of A_i to X. For each subset $I \subseteq \{1,\dots,M\}$, let

$$Z_I := \bigcap_{i \in I} Z_i \setminus \bigcup_{j \in I^c} Z_j, \quad \text{with } Z_{\varnothing} := X \setminus \bigcup_{j=1}^{2} Z_j.$$

Then $X = \bigsqcup_I Z_I$ is a definable partition, and thus we may assume that $X = Z_I$. In this case we have, for $F \in \mathrm{Loc}_{\gg}$,

$$\begin{aligned}\sup_{w\in W_F} f_F(x,w) &= u_F \cdot \sup_{\xi\in k_F^{m_2}} \#Y_{\eta,\xi} \\ &\le u_F \cdot \sum_{i\in I} \sup_{\xi\in k_F^{m_2}} (1_{A_{i,F}} \cdot \#Y_{\eta,\xi}) \\ &\le u_F \cdot \sum_{i\in I} 2l_{i1} \cdot q_F^{d_i} \\ &\le u_F \cdot \sum_{i\in I} 4l_{i2} \cdot \sup_{\xi\in k_F^{m_2}} (1_{A_{i,F}} \cdot \#Y_{\eta,\xi}) \\ &\le \Bigl(\sum_{i\in I} 4l_{i2}\Bigr) \sup_{w\in W_F} f_F(x,w),\end{aligned}$$

where we take zero for the supremum of the empty set. Since $\bigl\{u_F \cdot \sum_{i\in I} 2l_{i1} \cdot q_F^{d_i}\bigr\}_{F\in \mathrm{Loc}_{\gg}}$ clearly lies in $\mathcal{C}_+(X)$, this finishes the proof of Theorem 3.1. □

3A. ***Optimality of the bounds and further remarks.*** Let X and W be $\mathcal{L}_{\mathrm{DP}}$-definable sets. Given a subclass $\mathcal{F} \subseteq \mathcal{C}(X \times W)$ of motivic functions, one can ask whether for any $f \in \mathcal{F}$, the function $\{\sup_{w\in W_F} f_F(x,w)\}_{F\in\mathrm{Loc}_{\gg}}$ can be approximated by a motivic function in $\mathcal{C}(X)$ up to a constant C in up to four increasing levels of approximation:

(1) With C depending on F and f.

(2) With C depending on f and independent of F.

(3) With C a universal constant, that is, uniform over all $f \in \mathcal{F}$ and $F \in \mathrm{Loc}_{\gg}$.

(4) With $C = 1 + C' q_F^{-1/2}$ for some C' depending on f and independent of F.

If the class $\mathcal{F}$ satisfies one of the items (i) above, we say that $\mathcal{F}$ admits *an approximation of suprema of type* (i), or $\mathcal{F}$ *is of type* (i). Note that if $\mathcal{F}$ is of type (4) then it is also of type (3), as $C' q_F^{-1/2} < 2$ for $F \in \mathrm{Loc}_{\gg}$. Similarly, type (i) is stronger than type (j) for $j < i$.

Remark 3.5. • The class $\mathcal{C}(X \times W)$ is not of type (1) (and thus of any type). Indeed, take $X = \mathbb{Z}^2$, $W = \{1,2\} \subseteq \mathbb{Z}$ and define $f \in \mathcal{C}(X \times W)$ by $f(x,y,1) = 0$ and $f(x,y,2) = x^2 - y$. Then $\sup_w f(x,y,w) = \max(0, x^2 - y)$ cannot be approximated by a motivic function $g \in \mathcal{C}(X)$ up to a constant depending on F and f. Indeed, such g satisfies $g_F(x,y) = 0$ if and only if $x^2 \le y$, for $F \in \mathrm{Loc}_{\gg}$. For each fixed $F \in \mathrm{Loc}_{\gg}$, the function g_F agrees with a Presburger function on $\mathbb{Z}^2$. By Presburger cell decomposition [6, Theorem 1], we can decompose $\mathbb{Z}^2$ into cells $\mathbb{Z}^2 = \bigsqcup_{i=1}^N C_i$, such that $g_F(x,y)|_{C_i} = \sum_{j=1}^{N_i} c_{ij}(F) q_F^{a_{ij}x + b_{ij}y} x^{k_{ij}} y^{l_{ij}}$, with $a_{ij}, b_{ij} \in \mathbb{Q}$, $k_{ij}, l_{ij} \in \mathbb{N}$ and $c_j(F) \in \mathbb{R}$. Since $Z := \{(x,y) \in \mathbb{Z}^2 : x^2 \le y\} \subseteq \mathbb{Z}^2$ is not Presburger definable, and by the definition of a cell [6, Definiton 2], there is $1 \le i_0 \le N$ such that for some $x_0 \in \mathbb{Z}$, $|C_{i_0} \cap \{(x,y) \in Z : x = x_0\}| = \infty$ and $(x_0, y_0) \in C_{i_0} \cap Z^c \neq \varnothing$ for some $y_0 \in \mathbb{Z}$. Applying [9, Lemma 2.1.7], we get $g_F(x,y)|_{C_{i_0} \cap \{x = x_0\}} \equiv 0$, and thus $g_F(x_0, y_0) = 0$ where $x_0^2 > y_0$, yielding a contradiction.

- The class $\mathcal{C}_+^{\text{weak}}(X \times W) := \{f \in \mathcal{C}(X \times W) : f_F \geq 0 \, \forall F \in \mathrm{Loc}_{\gg}\}$ is of type (1), with $C = q_F^{C_0}$ for some $C_0 > 0$ depending only on f. This is a special case treated in the proof of [11, Theorem 2.1.3]. One may wonder whether the class $\mathcal{C}_+^{\text{weak}}(X \times W)$ is of type (2).

Theorem 3.1 shows that the family $\mathcal{C}_+(X \times W)$, which is strictly contained in $\mathcal{C}_+^{\text{weak}}(X \times W)$, is of type (2). The following proposition, which we prove in the Appendix, shows this is the best possible approximation, as already detected by the subclass $\mathcal{P}_+(X \times W) \subseteq \mathcal{C}_+(X \times W)$.

Proposition 3.6. *The families $\mathcal{P}_+(X \times W)$ and $\mathcal{C}_+(X \times W)$ are not of type* (3).

In [11, Theorem 2.1.3], an approximation of suprema result is proven for a more general class $\mathcal{C}^{\exp}(X \times W)$ of motivic exponential functions, which involves additive characters, and which is furthermore built out of functions which are definable in the generalized Denef–Pas language. Due to this larger generality, the approximation shown in [11, Theorem 2.1.3] is a bit weaker than type (1) above; in [11], one approximates $\sup|f|^2$ instead of $\sup f$. This is unavoidable, as already seen in Remark 3.5.

Remark 3.7. One can weaken the definition of approximation as follows. For a function $f \in \mathcal{C}_+(X \times W)$, assume there exist motivic functions $\{g_i\}_{i=1}^m \in \mathcal{C}_+(X)$, with $m \in \mathbb{N}$ such that

$$\max_{1 \leq i \leq m} \{g_{i\,F}(x)\} \leq \sup_{w \in W_F} f_F(x, w) \leq C \cdot \max_{1 \leq i \leq m} \{g_{i\,F}(x)\},$$

where C is as in types (1)-(4) above. Using this weaker form of approximation, we expect $\mathcal{C}_+(X \times W)$ to be of weakened type (4).

One may also weaken (3) by letting the constant C depend on the number of variables running over $X \times W$, and wonder whether $\mathcal{C}_+(X \times W)$ is of type (3) when weakened in this sense.

4. Number theoretic characterization of the (FRS) property

Throughout this section we use the notation of Section 2B. In particular, K is a fixed number field with ring of integers $\mathcal{O}_K$, and Loc denotes the collection of all non-Archimedean local fields F with a ring homomorphism $\mathcal{O}_K \to F$.

We next apply Theorem 3.1 to prove a more general form of Theorem A for $\mathrm{Loc}_{\gg}$, providing a full number theoretic characterization of (FRS) morphisms (Theorem 4.7).

4A. ***An analytic characterization of the (FRS) property.*** Given an $\mathcal{O}_F$-morphism $\varphi : X \to Y$, we denote the natural maps $X(\mathcal{O}_F/\mathfrak{m}_F^k) \to Y(\mathcal{O}_F/\mathfrak{m}_F^k)$ by φ, therefore $\varphi^{-1}(\bar{y})$ is a finite set in $X(\mathcal{O}_F/\mathfrak{m}_F^k)$, for any $\bar{y} \in Y(\mathcal{O}_F/\mathfrak{m}_F^k)$. We denote by $r_{k,Y} : Y(\mathcal{O}_F) \to Y(\mathcal{O}_F/\mathfrak{m}_F^k)$ and by $r_{l,Y}^k : Y(\mathcal{O}_F/\mathfrak{m}_F^k) \to Y(\mathcal{O}_F/\mathfrak{m}_F^l)$ the natural reduction maps for $k \geq l$. When the scheme Y is clear from the context, we omit it from our notation.

Definition 4.1. Let Y be a smooth F-variety, with $F \in \mathrm{Loc}$. A measure μ on $Y(F)$ is called:

(1) *Smooth* if for any $y \in Y(F)$ there exists an analytic neighborhood $U \subseteq Y(F)$ and an analytic diffeomorphism $\psi : U \to \mathcal{O}_F^{\dim Y}$ such that $\psi_*\mu$ is a Haar measure on $\mathcal{O}_F^{\dim Y}$.

(2) *Schwartz* if it is compactly supported and smooth.

Lemma 4.2 [37; 40; 35]. *Let F be in* Loc, *and let Y be a finite type $\mathcal{O}_F$-scheme such that $Y \times_{\operatorname{Spec} \mathcal{O}_F} \operatorname{Spec} F$ is smooth, of pure dimension d. Then there is a unique Schwartz measure $\mu_{Y(\mathcal{O}_F)}$ on $Y(\mathcal{O}_F)$, and there exists $k_0 \in \mathbb{N}$, such that for every $k \geq k_0$ and every $\bar{y} \in Y(\mathcal{O}_F/\mathfrak{m}_F^k)$, one has*

$$\mu_{Y(\mathcal{O}_F)}(r_k^{-1}(\bar{y})) = q_F^{-kd}. \tag{4-1}$$

The measure $\mu_{Y(\mathcal{O}_F)}$ is referred to as the ***canonical measure*** *on $Y(\mathcal{O}_F)$. In the special case when Y is smooth over $\mathcal{O}_F$, then* (4-1) *holds for every $k \geq 1$.*

Proof. If Y is affine, then the existence and uniqueness of $\mu_{Y(\mathcal{O}_F)}$ follows from [35, Lemma 3], building on [37, Theorem 9]. In general, let $Y = \bigcup_{i=1}^{N} U_i$ be an open affine cover by $\mathcal{O}_F$-subschemes U_i. Then $Y(\mathcal{O}_F) = \bigcup_{i=1}^{N} U_i(\mathcal{O}_F)$. Note that

$$\mu_{U_i(\mathcal{O}_F)}|_{U_i(\mathcal{O}_F)\cap U_j(\mathcal{O}_F)} = \mu_{U_j(\mathcal{O}_F)}|_{U_i(\mathcal{O}_F)\cap U_j(\mathcal{O}_F)}$$

by uniqueness, so we can glue them together to form $\mu_{Y(\mathcal{O}_F)}$. If furthermore Y is smooth over $\mathcal{O}_F$, then by applying Hensel's lemma to (4-1) we can choose $k_0 = 1$; see also [40, Theorem 2.25]. ☐

Aizenbud and Avni [1] gave an analytic characterization of the (FRS) property:

Theorem 4.3 [1, Theorem 3.4]. *Let $\varphi : X \to Y$ be a map between smooth K-varieties. Then the following are equivalent*:

(1) *φ is (FRS).*

(2) *For any $F \in \mathrm{Loc}_0$ and any Schwartz measure μ on $X(F)$, the measure $\varphi_*(\mu)$ has continuous density with respect to any smooth, nonvanishing measure on $Y(F)$.*

(3) *For any $x \in X(\overline{K})$ and any finite extension K'/K with $x \in X(K')$, there exists $F \in \mathrm{Loc}_0$ containing K', and a nonnegative Schwartz measure μ on $X(F)$ that does not vanish at x such that $\varphi_*(\mu)$ has continuous density with respect to any smooth, nonvanishing measure on $Y(F)$.*

The next result shows a variant of the above characterization holds for local fields of large positive characteristic.

Corollary 4.4. *Let $\varphi : X \to Y$ be a map between smooth K-varieties. Then φ is (FRS) if and only if for every $F \in \mathrm{Loc}_{\gg}$, the measure $\varphi_*(\mu_{X(\mathcal{O}_F)})$ has bounded density with respect to $\mu_{Y(\mathcal{O}_F)}$.*

Proof. Without loss of generality, we may assume that Y is affine. By choosing an $\mathcal{O}_K$-model of Y, we may identify it as an $\mathcal{L}_{\mathrm{DP}}$-definable set. Assume φ is (FRS). For each $F \in \mathrm{Loc}_{\gg}$, write $\tau_F := \varphi_*(\mu_{X(\mathcal{O}_F)})$. By [1, Theorem 3.4(2)], we can write $\tau_F = f_F \cdot \mu_{Y(\mathcal{O}_F)}$ and f_F is continuous, for each $F \in \mathrm{Loc}_{0,\gg}$. Moreover, locally, f can be written as an integral of a motivic function G in $\mathcal{C}_+(Y \times \mathrm{VF}^{\dim X - \dim Y})$, over $\mathrm{VF}^{\dim X - \dim Y}$. By [9, Theorem 4.4.1], it follows that $G_F(y, \cdot)$ is integrable, for each $F \in \mathrm{Loc}_{\gg}$ and $y \in Y(\mathcal{O}_F)$. By Theorem 2.12, f is in $\mathcal{C}_+(Y)$.

Since $Y(\mathcal{O}_F)$ is compact and f_F is continuous, f_F is bounded on $Y(\mathcal{O}_F)$ for each $F \in \mathrm{Loc}_{0,\gg}$. By [38, Appendix B, Theorem 14.6] (or more generally, by [11, Theorem 2.1.2]) there exists $a \in \mathbb{Z}$, such that for each $F \in \mathrm{Loc}_{0,\gg}$ and each $y \in Y(\mathcal{O}_F)$, one has $f_F(y) < q_F^a$. By Theorem 2.14 we thus have $f_F(y) < q_F^a$ for each $F \in \mathrm{Loc}_{\gg}$ and each $y \in Y(\mathcal{O}_F)$, as required. The other direction follows from Theorem 4.3 combined with [19, Lemma 3.15], as in the proof of [19, Proposition 3.16]. ☐

4B. ***A number-theoretic characterization of the (FRS) property.*** We now recall the Lang–Weil estimates, and set the required notation to state the main theorem.

Definition 4.5. (1) For a finite type $\mathbb{F}_q$-scheme Z, we denote by C_Z the number of its top-dimensional geometrically irreducible components which are defined over $\mathbb{F}_q$.

(2) Let $\varphi : X \to Y$ be a morphism between finite type $\mathbb{Z}$-schemes X and Y, and let $y \in Y(\mathbb{F}_q)$. Then we write $C_{X,q}{:=}C_{X_{\mathbb{F}_q}}$ and $C_{\varphi,q,y} := C_{(X_{\mathbb{F}_q})_{y,\varphi}}$.

Theorem 4.6 (the Lang–Weil estimates [30]). *For every $M \in \mathbb{N}$, there exists $C(M) > 0$, such that for every prime power q, and any finite type $\mathbb{F}_q$-scheme X of complexity at most M (see, e.g.,* [19, Definition 7.7]), *one has*

$$\left|\frac{\#X(\mathbb{F}_q)}{q^{\dim X}} - C_X\right| < C(M)q^{-1/2}.$$

Let X, Y be finite type $\mathcal{O}_K$-schemes, with X_K, Y_K smooth and geometrically irreducible, and let $\varphi : X \to Y$ be a dominant morphism. Let $\mu_{X(\mathcal{O}_F)}$ and $\mu_{Y(\mathcal{O}_F)}$ be the canonical measures on $X(\mathcal{O}_F)$ and $Y(\mathcal{O}_F)$ for $F \in \mathrm{Loc}$. Since φ is dominant, it follows that $\tau_F := \varphi_*(\mu_{X(\mathcal{O}_F)})$ is absolutely continuous with respect to $\mu_{Y(\mathcal{O}_F)}$, and thus has an L^1-density (see, e.g., [1, Corollary 3.6]), so that $\tau_F = f_F(y) \cdot \mu_{Y(\mathcal{O}_F)}$. When Y is affine, the collection $f = \{f_F : Y(\mathcal{O}_F) \to \mathbb{C}\}_{F \in \mathrm{Loc}_{\gg}}$ can be chosen to be a formally nonnegative motivic function. Indeed, as in the proof of Corollary 4.4, locally, f_F can be written as an integral of a motivic function G in $\mathcal{C}_+(Y \times \mathrm{VF}^{\dim X - \dim Y})$, over $\mathrm{VF}^{\dim X - \dim Y}$. Note there is an open affine subscheme U of Y, such that φ_K is smooth over U_K. Then $G_F(y, \cdot)$ is integrable for every $y \in U(F)$ and $F \in \mathrm{Loc}_{\gg}$. By Theorem 2.12 it follows that $f|_U$ is formally nonnegative. Since $U(F)$ is dense in $Y(F)$ for $F \in \mathrm{Loc}_{\gg}$, by extending $f|_U$ by 0 we get a collection of densities on $\{Y(\mathcal{O}_F)\}_{F \in \mathrm{Loc}_{\gg}}$ which is formally nonnegative.

For $F \in \mathrm{Loc}_{\gg}$, define a function g_F for $y \in Y(\mathcal{O}_F)$ and $k \in \mathbb{Z}_{\geq 1}$ by

$$g_F(y, k) = \frac{1}{\mu_{Y(\mathcal{O}_F)}(B(y, k))} \int_{\tilde{y} \in B(y,k)} f_F(\tilde{y}) \mu_{Y(\mathcal{O}_F)},$$

where $B(y, k) = r_k^{-1}(r_k(y))$. By Theorem 2.12, it follows that $\{g_F : Y(\mathcal{O}_F) \times \mathbb{Z}_{\geq 1} \to \mathbb{C}\}_{F \in \mathrm{Loc}_{\gg}}$ is a formally nonnegative motivic function.

For every $F \in \mathrm{Loc}_{\gg}$, every $y \in Y(\mathcal{O}_F)$ and every $k \in \mathbb{Z}_{\geq 1}$, we have

$$g_F(y, k) = \frac{\varphi_*(\mu_{X(\mathcal{O}_F)})(B(y, k))}{\mu_{Y(\mathcal{O}_F)}(B(y, k))} = \frac{\#\varphi^{-1}(r_k(y))}{q_F^{k(\dim X_K - \dim Y_K)}}, \tag{4-2}$$

where the last equality follows from Lemma 4.2, and the fact that Y is smooth over $\mathcal{O}_F$ for $F \in \mathrm{Loc}_{\gg}$. Set

$$h_F(y, k) = \frac{\#(\varphi^{-1}(r_k(y)) \cap (r_{1,X}^k)^{-1}(X^{\mathrm{sing},\varphi}(k_F)))}{q_F^{k(\dim X_K - \dim Y_K)}}.$$

The asymptotics of the functions h and g, in q_F and k, measure how wild the singularities of φ are. For example, if φ_K is smooth, then $h_F(y,k) \equiv 0$ and $g_F(y,k) < C$ for $F \in \mathrm{Loc}_{\gg}$ and some constant C. On the other hand, if $\varphi : \mathbb{A}^1 \to \mathbb{A}^1$ is the map $x \mapsto x^m$, then $g(0,k) = h(0,k) = q_F^{k-\lceil k/m \rceil}$.

Furthermore, $\{h_F\}_{F\in\mathrm{Loc}_{\gg}}$ is a formally nonnegative motivic function (Proposition 4.9). This is used to prove our main theorem, which we state now.

Theorem 4.7. *Let $\varphi : X \to Y$ be a dominant morphism between finite type $\mathcal{O}_K$-schemes X and Y, with X_K, Y_K smooth and geometrically irreducible. Then the following are equivalent*:

1. *$\varphi_K : X_K \to Y_K$ is (FRS).*
2. *There exists $C_1 > 0$, such that for each $F \in \mathrm{Loc}_{\gg}$, $k \in \mathbb{Z}_{\geq 1}$ and $y' \in Y(\mathcal{O}_F)$*
$$h_F(y',k) < C_1 q_F^{-1}.$$
3. *There exists $C_2 > 0$ such that for each $F \in \mathrm{Loc}_{\gg}$, $k \in \mathbb{Z}_{\geq 1}$ and $y \in Y(\mathcal{O}_F/\mathfrak{m}_F^k)$*
$$\left| \frac{\#\varphi^{-1}(y)}{q_F^{k(\dim X_K - \dim Y_K)}} - \frac{\#\varphi^{-1}(r_1^k(y))}{q_F^{\dim X_K - \dim Y_K}} \right| < C_2 q_F^{-1}.$$
4. *There exists $C_3 > 0$ such that for each $F \in \mathrm{Loc}_{\gg}$, $k \in \mathbb{Z}_{\geq 1}$ and $y \in Y(\mathcal{O}_F/\mathfrak{m}_F^k)$*
$$\left| \frac{\#\varphi^{-1}(y)}{q_F^{k(\dim X_K - \dim Y_K)}} - C_{\varphi,q_F,r_1^k(y)} \right| < C_3 q_F^{-1/2}.$$
5. *There exists $C_4 > 0$ such that for each $F \in \mathrm{Loc}_{\gg}$, $\varphi_*(\mu_{X(\mathcal{O}_F)})$ has continuous density f_F with respect to $\mu_{Y(\mathcal{O}_F)}$, and for each $y' \in Y(\mathcal{O}_F)$, one has*
$$|f_F(y') - C_{\varphi,q_F,r_1(y')}| < C_4 q_F^{-1/2}.$$

Before we prove Theorem 4.7, we first show it implies Theorem A.

Proof of Theorem A. We prove (1) ⇒ (3) ⇒ (2) ⇒ (4) ⇒ (1). To prove (3) ⇒ (2), we first treat large primes using implication (3) ⇒ (4) of Theorem 4.7, and then treat small primes using the Lang–Weil estimates. Implication (4) ⇒ (1) follows from Theorem 4.3.

Let us assume that condition (2) holds. By Lemma 4.2, and by condition (2), there exists $C_1 > 0$, such that for every prime p, every $y \in Y(\mathbb{Z}/p^k\mathbb{Z})$ and every $k \geq k_0$, one has
$$\frac{\varphi_*\mu_{X(\mathbb{Z}_p)}(r_k^{-1}(y))}{\mu_{Y(\mathbb{Z}_p)}(r_k^{-1}(y))} = \frac{\mu_{X(\mathbb{Z}_p)}(\varphi^{-1}(r_k^{-1}(y)))}{p^{-k\dim Y_{\mathbb{Q}}}} = \frac{\mu_{X(\mathbb{Z}_p)}(r_k^{-1}(\varphi^{-1}(y)))}{p^{-k\dim Y_{\mathbb{Q}}}} = \frac{\#\varphi^{-1}(y)}{p^{k(\dim X_{\mathbb{Q}} - \dim Y_{\mathbb{Q}})}} < C_1, \quad (4\text{-}3)$$
where $\mu_{X(\mathbb{Z}_p)}$ and $\mu_{Y(\mathbb{Z}_p)}$ are the canonical measures on $X(\mathbb{Z}_p)$ and $Y(\mathbb{Z}_p)$. Let f_p be the density of $\varphi_*\mu_{X(\mathbb{Z}_p)}$ with respect to $\mu_{Y(\mathbb{Z}_p)}$. Combining (4-3) with Lebesgue's differentiation theorem, we get, for almost all $y' \in Y(\mathbb{Z}_p)$,
$$f_p(y') = \lim_{k\to\infty} \frac{\varphi_*\mu_{X(\mathbb{Z}_p)}(r_k^{-1}(r_k(y')))}{\mu_{Y(\mathbb{Z}_p)}(r_k^{-1}(r_k(y')))} < C_1,$$
which implies condition (4).

It is left to prove that (1) $\Rightarrow$ (3). The case of large primes follows from the implication (1) $\Rightarrow$ (3) of Theorem 4.7. It is left to prove (3) for a fixed prime p. By Theorem 4.3, we have $f_p < C(p)$ for some $C(p) > 0$. Using (4-3), we deduce that

$$\frac{\#\varphi^{-1}(y)}{p^{k(\dim X_{\mathbb{Q}} - \dim Y_{\mathbb{Q}})}} < C(p), \tag{4-4}$$

for every $k \geq k_0$ and $y \in Y(\mathbb{Z}/p^k\mathbb{Z})$. For $y \in Y(\mathbb{Z}/p^k\mathbb{Z})$ with $k < k_0$ we can take the trivial bound $\#\varphi^{-1}(y) \leq \sum_{l=1}^{k_0} \#X(\mathbb{Z}/p^l\mathbb{Z})$ to deduce (4-4) for every $k \in \mathbb{N}$. Using the triangle inequality, and by applying the trivial upper bound $\#\varphi^{-1}(\bar{y}) < \#X(\mathbb{F}_p)$ for $\bar{y} \in Y(\mathbb{F}_p)$, we deduce (3). ☐

Remark 4.8. One can easily adapt the proof of Theorem A above to prove a more general statement where the collection $\{\mathbb{Q}_p\}_p$ is replaced with all completions $K_{\mathfrak{p}}$ of a fixed number field K. On the other hand, Theorem 4.7 is definitely not true for all $F \in \mathrm{Loc}$; e.g., take $\varphi(x) = 3x$, and consider unramified extensions of $\mathbb{Q}_3$.

We now move to the proof of Theorem 4.7. We start with the easier implications, and deal with the more challenging implication (1) $\Rightarrow$ (2) in Section 4B1.

Proof of (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (1) *of Theorem 4.7.* Implication (2) $\Rightarrow$ (3) Assume that $h_F(y', k) < C_1 q_F^{-1}$ for each $F \in \mathrm{Loc}_{\gg}$, each $k \in \mathbb{N}$ and each $y' \in Y(\mathcal{O}_F)$. Set $y := r_k(y')$ and note that

$$\begin{aligned}\left|\frac{\#\varphi^{-1}(y)}{q_F^{k(\dim X_K - \dim Y_K)}} - \frac{\#\varphi^{-1}(r_1^k(y))}{q_F^{\dim X_K - \dim Y_K}}\right| &= \left|\frac{\#\varphi_{|X^{\mathrm{sm},\varphi}}^{-1}(y))}{q_F^{k(\dim X_K - \dim Y_K)}} + h_F(y', k) - \frac{\#\varphi_{|X^{\mathrm{sm},\varphi}}^{-1}(r_1^k(y))}{q_F^{\dim X_K - \dim Y_K}} - h_F(y', 1)\right| \\ &= |h_F(y', k) - h_F(y', 1)| \leq 2C_1 q_F^{-1}.\end{aligned}$$

where the second equality follows from Hensel's lemma and the inequality follows from our assumption on h. Since r_k is surjective for $F \in \mathrm{Loc}_{\gg}$, this finishes the proof.

Implication (3) $\Rightarrow$ (4) Let us first prove that φ_K is flat, assuming condition (3). It is enough to show that $\varphi_{\mathbb{F}_p}$ is flat for infinitely many prime numbers p. Let p be a prime large enough such that $\dim X_K = \dim X_{\mathbb{F}_p}$, $\dim Y_K = \dim Y_{\mathbb{F}_p}$ and such that condition (3) holds for $F = \mathbb{F}_q((t))$ for any q which is a power of p. Note there are infinitely many primes p such that $\mathbb{F}_p$ is a residue field of $\mathcal{O}_K$ for some prime of $\mathcal{O}_K$. Let $x \in X(\mathbb{F}_q)$ for such q and let $\tilde{x} \in J_k(X)(\mathbb{F}_q) \simeq X(\mathbb{F}_q[t]/(t^{k+1}))$ be the image of x under the zero section embedding $X(\mathbb{F}_q) \hookrightarrow J_k(X)(\mathbb{F}_q)$, so that $r_1^k(\varphi(\tilde{x})) = \varphi(x)$. Then by condition (3), we have for any $k \in \mathbb{N}$:

$$\left|\frac{\#\varphi^{-1}(\varphi(\tilde{x}))}{q^{(k+1)(\dim X_K - \dim Y_K)}} - \frac{\#\varphi^{-1}(\varphi(x))}{q^{\dim X_K - \dim Y_K}}\right| < C_2 \cdot q^{-1}. \tag{4-5}$$

Consider $k = 1$. By choosing q to be a suitable power of p we may assume $C_{\varphi,q,\varphi(x)}, C_{J_1(\varphi),q,J_1(\varphi)(\tilde{x})} \geq 1$. Notice that $\#\varphi^{-1}(\varphi(\tilde{x})) = \#J_1(X_{\varphi(x),\varphi_{\mathbb{F}_q}})(\mathbb{F}_q)$. Since $\dim J_1(X_{\varphi(x),\varphi_{\mathbb{F}_q}}) \geq 2 \dim X_{\varphi(x),\varphi_{\mathbb{F}_q}}$ we have by (4-5) and by the Lang–Weil estimates that

$$\dim X_{\varphi(x),\varphi_{\mathbb{F}_q}} = \dim X_K - \dim Y_K = \dim X_{\mathbb{F}_q} - \dim Y_{\mathbb{F}_q}.$$

By miracle flatness, we are done.

To prove condition (4), by the triangle inequality, it is enough to find C_3' such that for each $F \in \mathrm{Loc}_{\gg}$, $k \in \mathbb{Z}_{\geq 1}$ and $y \in Y(\mathcal{O}_F/\mathfrak{m}_F^k)$:

$$\left| \frac{\#\varphi^{-1}(r_1^k(y))}{q_F^{\dim X_K - \dim Y_K}} - C_{\varphi, q_F, r_1^k(y)} \right| < C_3' q_F^{-1/2}.$$

This follows from the fact that φ_K is flat, via a relative variant of the Lang–Weil estimates; see, e.g., [20, Theorem 8.4].

Implications (4) ⇒ (5) and (5) ⇒ (1) Let f_F be the density of $\varphi_*(\mu_{X(\mathcal{O}_F)})$ with respect to $\mu_{Y(\mathcal{O}_F)}$. By Lebesgue's differentiation theorem and condition (4), for almost every $y' \in Y(\mathcal{O}_F)$, we have

$$\begin{aligned} |f_F(y') - C_{\varphi, q_F, r_1(y')}| &= \left| \lim_{k \to \infty} \frac{\mu_{X(\mathcal{O}_F)}(\varphi^{-1}(B(y', k)))}{\mu_{Y(\mathcal{O}_F)}(B(y', k))} - C_{\varphi, q_F, r_1(y')} \right| \qquad (4\text{-}6) \\ &= \left| \lim_{k \to \infty} \frac{\#\varphi^{-1}(r_k(y'))}{q_F^{k(\dim X_K - \dim Y_K)}} - C_{\varphi, q_F, r_1(y')} \right| < C_3 q_F^{-1/2}. \qquad (4\text{-}7) \end{aligned}$$

This also shows that f_F is essentially bounded for $F \in \mathrm{Loc}_{\gg}$. By Corollary 4.4 and by Theorem 4.3, it follows that f_F can be chosen to be continuous, so that (4-7) holds for all $y' \in Y(\mathcal{O}_F)$. This implies condition (5), which implies condition (1) using the same Corollary 4.4. □

4B1. *Proof of the implication* (1) ⇒ (2). In this section we will prove the remaining implication of Theorem 4.7, namely (1) ⇒ (2). We first observe the following:

Proposition 4.9. *Assume that Y is affine. Then h is a formally nonnegative motivic function.*

Proof. We first prove the special case with X affine. Assume that $X \subseteq \mathbb{A}^m$ is the zero locus of $g_1, \ldots, g_l \in \mathcal{O}_K[x_1, \ldots, x_m]$. Since X and Y are affine, the map $\varphi = (f_1, \ldots, f_n) : X \to Y \subseteq \mathbb{A}^n$ is a polynomial map, thus with $f_i \in \mathcal{O}_K[x_1, \ldots, x_m]$. Given $y \in Y(\mathcal{O}_F)$, set

$$S_{y,k,X} := \{x \in \mathcal{O}_F^m : \min_{i,j}\{\mathrm{val}(g_i(x)), \mathrm{val}(f_j(x) - y_j)\} \geq k\}.$$

Now, for any $y \in Y(\mathcal{O}_F)$, we have

$$\#\varphi^{-1}(r_k(y)) = q_F^{km} \int_{\mathcal{O}_F^m} 1_{S_{y,k,X}} |dx_1 \wedge \cdots \wedge dx_m|.$$

Moreover,

$$\#(\varphi^{-1}(r_k(y)) \cap (r_{1,X}^k)^{-1}(X^{\mathrm{sing},\varphi}(k_F))) = q_F^{km} \int_{\mathcal{O}_F^m} 1_{W_{y,k,X}} |dx_1 \wedge \cdots \wedge dx_m|,$$

where

$$W_{y,k,X} := \{x \in S_{y,k,X} : r_1(x) \in X^{\mathrm{sing},\varphi}(k_F)\}.$$

Since $1_{W_{y,X}}$ is formally nonnegative, we get by Theorem 2.12 that h is formally nonnegative as well. Now let $X = \bigcup_{i=1}^N U_i$ be a cover by smooth open affine subschemes U_i. For each i and $F \in \mathrm{Loc}_{\gg}$ write

$V_i := U_i(\mathcal{O}_F)\setminus \bigcup_{j=1}^{i-1} U_j(\mathcal{O}_F)$ and note that

$$\begin{aligned}\#(\varphi^{-1}(r_k(y)) \cap (r_{1,X}^k)^{-1}(X^{\mathrm{sing},\varphi}(k_F))) &= \sum_{i=1}^{N} \#((\varphi|_{U_i})^{-1}(r_k(y)) \cap (r_{1,X}^k)^{-1}(U_i^{\mathrm{sing},\varphi}(k_F) \cap r_1(V_i)))\\ &= \sum_{i=1}^{N} q_F^{km} \int_{\mathcal{O}_F^m} 1_{W_{y,k,i}} |dx_1 \wedge \cdots \wedge dx_m|,\end{aligned}$$

where

$$W_{y,k,i} := \{x \in S_{y,k,U_i} : r_1(x) \in U_i^{\mathrm{sing},\varphi}(k_F) \cap r_1(V_i)\}.$$

This finishes the proof of Proposition 4.9. ☐

We need one more lemma which we state in the generality of E-smooth morphisms, and which will further be used in the next section.

Lemma 4.10. *Let $E \geq 1$ be an integer, let φ be as in Theorem 4.7 and assume that $\varphi_K : X_K \to Y_K$ is E-smooth. Then, for each $k \in \mathbb{N}$ there exists a constant $C(k) > 0$ such that for each $F \in \mathrm{Loc}_{\gg}$, one has*

$$\sup_{y \in Y(\mathcal{O}_F)} h_F(y,k) < C(k) \cdot q_F^{-E}.$$

Proof. Using Theorems 2.14 and 3.1 it is enough to prove the lemma for F lying in $\mathrm{Loc}_{+,\gg}$. By Proposition 2.5 we have

$$J_k(X_{k_F}^{\mathrm{sm},\varphi_{k_F}}) = J_k(X_{k_F})^{\mathrm{sm},J_k(\varphi_{k_F})}, \tag{4-8}$$

for $F \in \mathrm{Loc}_{+,\gg}$. Let $Z_{\tilde{y}} := J_k(X_{k_F})_{\tilde{y},J_k(\varphi_{k_F})}$ be a nonempty fiber of $J_k(\varphi_{k_F})$ over $\tilde{y} \in J_k(Y)(k_F)$. Since $J_k(\varphi_{k_F})$ is flat and by (4-8), we have

$$Z_{\tilde{y}}^{\mathrm{sing}} = Z_{\tilde{y}} \cap J_k(X_{k_F})^{\mathrm{sing},J_k(\varphi_{k_F})} = Z_{\tilde{y}} \cap (\pi_{0,X_{k_F}}^k)^{-1}(X_{k_F}^{\mathrm{sing},\varphi_{k_F}}). \tag{4-9}$$

The E-smoothness of φ_K implies that the right hand side is of codimension at least E in $Z_{\tilde{y}}$. By the definition of h, by the fact that all fibers of $J_k(\varphi_{k_F})$ are of bounded complexity (for a fixed k) and using a relative variant of the Lang–Weil estimates, the lemma follows. ☐

Proof of the implication (1) $\Rightarrow$ (2). We may assume that Y is affine. Theorem 3.1 and Proposition 4.9 imply that there exist a constant $C_0 > 0$ and a motivic function H in $\mathcal{C}_+(\mathbb{Z}_{\geq 1})$ such that

$$\sup_{y \in Y(\mathcal{O}_F)} h_F(y,k) < H_F(k) < C_0 \cdot \sup_{y \in Y(\mathcal{O}_F)} h_F(y,k). \tag{4-10}$$

It is thus enough to show that $\sup_k H_F(k) < C_1 \cdot q_F^{-1}$ for some constant C_1 which is independent of F.

By Corollary 4.4, by (4-2) and since $h_F \leq g_F$, we deduce that the function $(y,k) \mapsto h_F(y,k)$ is bounded for each $F \in \mathrm{Loc}_{\gg}$. By (4-10) also $k \mapsto H_F(k)$ is bounded for each $F \in \mathrm{Loc}_{\gg}$. As in the proof of Claim 2 of Theorem 3.1, it follows that there exist a finite set L of $\mathbb{Z}_{\geq 1}$ and a constant $C_0' > 0$ such that

$$\sup_k H_F(k) \leq C_0' \cdot \sum_{k \in L} H_F(k). \tag{4-11}$$

Using (4-10), (4-11), Lemmas 2.9(1) and 4.10 and by setting $C_1 := C_0 C_0' \cdot \sum_{k \in L} C(k)$, we obtain

$$\sup_k H_F(k) \le C_0' \sum_{k \in L} H_F(k) \le C_0' C_0 \sum_{k \in L} \sup_{y \in Y(\mathcal{O}_F)} h_F(y, k) < C_1 q_F^{-1},$$

for each $F \in \mathrm{Loc}_{\gg}$. This finishes the proof of (1) ⇒ (2). □

4C. ***Number-theoretic estimates for E-smooth and ε-jet flat morphisms.*** In this subsection we use the improved approximation of suprema (Theorem 3.1), similarly as in Section 4B1, to provide uniform estimates for E-smooth morphisms and ε-jet flat morphisms, improving [20, Theorem 8.18]. We start by giving a characterization of E-smooth morphisms.

Theorem 4.11. *Let $E \ge 1$ be an integer, and let $\varphi : X \to Y$ be a dominant morphism between finite type $\mathcal{O}_K$-schemes X and Y, with X_K, Y_K smooth and geometrically irreducible. Then the following are equivalent*:

(1) *$\varphi_K : X_K \to Y_K$ is E-smooth.*

(2) *There exists $C_1 > 0$, such that, for each $F \in \mathrm{Loc}_{\gg}$, $k \in \mathbb{Z}_{\ge 1}$ and $y' \in Y(\mathcal{O}_F)$,*

$$h_F(y', k) < C_1 q_F^{-E}.$$

(3) *There exists $C_2 > 0$ such that, for each $F \in \mathrm{Loc}_{\gg}$, $k \in \mathbb{Z}_{\ge 1}$ and $y \in Y(\mathcal{O}_F/\mathfrak{m}_F^k)$,*

$$\left| \frac{\#\varphi^{-1}(y)}{q_F^{k(\dim X_K - \dim Y_K)}} - \frac{\#\varphi^{-1}(r_1^k(y))}{q_F^{\dim X_K - \dim Y_K}} \right| < C_2 q_F^{-E}.$$

In particular, when $E = 2$, the conditions above are further equivalent to $\varphi_K : X_K \to Y_K$ being flat with fibers of terminal singularities (*see Lemma 2.9*).

Proof. The proof of (1) ⇒ (2) is identical to the proof of (1) ⇒ (2) in Theorem 4.7, where the only exception is the inequality $\sup_{y \in Y(\mathcal{O}_F)} h_F(y, k) < C_1 q_F^{-E}$ for $F \in \mathrm{Loc}_{\gg}$ which is similarly obtained using Lemma 4.10. (2) ⇒ (3) is similar as in Theorem 4.7.

(3) ⇒ (1) Recall that by Theorem 4.7, condition (3) implies that φ_K is (FRS) and therefore jet-flat, and that

$$|h_F(y', k) - h_F(y', 1)| \le C_2 q_F^{-E},$$

for all $F \in \mathrm{Loc}_{\gg}$, $k \in \mathbb{Z}_{\ge 1}$ and $y' \in Y(\mathcal{O}_F)$ (see (2) ⇒ (3) in the proof of Theorem 4.7). Write $W_{y'} := (X_{k_F})_{r_1(y'), \varphi_{k_F}}$. We claim that $(W_{y'})^{\mathrm{sing}}$ is of codimension at least $E+1$ in $W_{y'}$ for all $F \in \mathrm{Loc}_{\gg}$ and $y' \in Y(\mathcal{O}_F)$. Indeed, assume $(W_{y'})^{\mathrm{sing}}$ is of codimension r in $W_{y'}$ with $r \le E$. Identifying $r_1(y') \in Y(k_F)$ with $\tilde{y} := s_1(r_1(y')) = (r_1(y'), 0) \in J_1(Y)(k_F)$ under the zero section embedding $s_1 : Y \hookrightarrow J_1(Y)$, and using (4-9) one has

$$(J_1(X_{k_F})_{\tilde{y}, J_1(\varphi_{k_F})})^{\mathrm{sing}} = J_1(W_{y'}) \cap (\pi^1_{0, X_{k_F}})^{-1}(X_{k_F}^{\mathrm{sing}, \varphi_{k_F}}) = (\pi^1_{0, W_{y'}})^{-1}(W_{y'}^{\mathrm{sing}}).$$

Since the dimension of the Zariski tangent space of a variety Z at a singular point is larger than $\dim Z$, we have

$$\begin{aligned}\dim(J_1(X_{k_F})_{\tilde{y},J_1(\varphi_{k_F})})^{\mathrm{sing}} &\geq \dim(W_{y'})^{\mathrm{sing}} + \dim X_K - \dim Y_K + 1\\ &\geq \dim W_{y'} - r + \dim X_K - \dim Y_K + 1\\ &\geq \dim J_1(X_{k_F})_{\tilde{y},J_1(\varphi_{k_F})} - r + 1.\end{aligned}$$

Hence $(J_1(X_{k_F})_{\tilde{y},J_1(\varphi_{k_F})})^{\mathrm{sing}}$ is of codimension at most $r-1$. By replacing F with a finite extension, and using the Lang–Weil estimates, one can find $C_3 > 0$ such that

$$h_F(y',1) < C_3 q_F^{-r} \quad \text{and} \quad h_F(y',2) > \tfrac{1}{2} q_F^{-r+1}.$$

But this contradicts $|h_F(y',k) - h_F(y',1)| \leq C_2 q_F^{-E}$. Therefore $h_F(y',1) < C_3 q_F^{-(E+1)}$ for all $F \in \mathrm{Loc}_{\gg}$ and $y' \in Y(\mathcal{O}_F)$. But then by condition (3), we deduce that $h_F(y',k) < C_3 q_F^{-E}$ which implies that φ_K is E-smooth. □

Finally, we provide an estimate on the number of $\mathcal{O}_F/\mathfrak{m}_F^k$-points lying on fibers of ε-jet flat morphisms.

Theorem 4.12. *Let $\varphi : X \to Y$ be a dominant morphism between finite type $\mathcal{O}_K$-schemes X and Y, with X_K, Y_K smooth and geometrically irreducible and let $0 < \varepsilon \leq 1$. Then the following are equivalent*:

(1) *$\varphi_K : X_K \to Y_K$ is ε-jet flat.*

(2) *There exist $C, M > 0$ such that for each $F \in \mathrm{Loc}_{\gg}$, $k \in \mathbb{Z}_{\geq 1}$ and $y \in Y(\mathcal{O}_F/\mathfrak{m}_F^k)$, one has*

$$\frac{\#\varphi^{-1}(y)}{q_F^{k(\dim X_K - \dim Y_K)}} < C \cdot k^M q_F^{k(1-\varepsilon)\dim Y_K}.$$

In particular, when $\varepsilon = 1$ and assuming φ_K has normal fibers, the conditions above are further equivalent to φ_K being flat with fibers of log-canonical singularities (*Remark 2.8*).

The difficult direction of Theorem 4.12 is (1) ⇒ (2), and it sharpens the bounds given in [20, Theorem 8.18]; the factor $Ck^M q_F^{-k\varepsilon \dim Y_K}$ as in (2) improves a factor of the form $q_F^{-k\varepsilon' \dim Y_K}$ present in [20], where ε' can be taken to be any number such that $\varepsilon' > \varepsilon$. In order to prove these sharper estimates, we use Theorem B, along with the following auxiliary lemma.

Lemma 4.13. *Let $g \in \mathcal{C}_+(\mathbb{Z}_{\geq 1})$ be a formally nonnegative motivic function such that for every $\delta > 0$ and $k \in \mathbb{Z}_{\geq 1}$ we have* (*varying over $F \in \mathrm{Loc}_{\gg}$*)

$$\lim_{q_F \to \infty} q_F^{-\delta} g_F(k) = 0.$$

Then there exist $M \in \mathbb{N}$ and $C > 0$ such that $g_F(k) < Ck^M$ for every $k \in \mathbb{Z}_{\geq 1}$ and field $F \in \mathrm{Loc}_{\gg}$.

Proof. Since g is formally nonnegative, we may write $g_F = \sum \# Y_{F,i} f_{F,i}$ for $f_{F,i} \in \mathcal{P}_+(\mathbb{Z}_{\geq 1})$ formally nonnegative and $Y_{F,i} \subseteq \mathbb{Z}_{\geq 1} \times \mathrm{RF}^{r_i}$. It is enough to show the claim for a single summand $g_F = \# Y_F f_F$. Using Presburger cell decomposition and the orthogonality of RF and VG, we have a finite partition

$\mathbb{Z}_{\geq 1} = \bigcup A_i$ and we may write $g_F(k)|_A = \sum \# Y_F c_i(q_F) q_F^{a_i k} k^{b_i}$ on each cell A, where $a_i \in \mathbb{Q}$, $b_i \in \mathbb{N}$, $\{(a_i, b_i)\}_{i=1}^N$ are mutually different, and $c_i(q)$ are rational functions in q.

First assume our cell A is finite, in which case it is enough to prove the claim for a fixed $k = k_0$. Using [5, main theorem], we have nonnegative constants d, C_1 and C_2 such that

$$\# Y_F < C_2 q_F^d \text{ for all } F \in \mathrm{Loc}_{\gg} \quad \text{and} \quad C_1 q_F^d < \# Y_F < C_2 q_F^d \tag{$\dagger$}$$

for infinitely many fields $F \in \mathrm{Loc}_{\gg}$ (with infinitely many residual characteristics).

Therefore, for every $\delta > 0$ and infinitely many fields $F \in \mathrm{Loc}_{\gg}$ we have

$$\lim_{q_F \to \infty} q_F^{-\delta} (C_1 q_F^d) \sum q_F^{a_i k_0} c_i(q_F) k_0^{b_i} \leq \lim_{q_F \to \infty} q_F^{-\delta} g_F(k_0) = 0, \tag{$\triangle$}$$

and thus $\deg_q \bigl(C_2 q^d \sum q^{a_i k_0} c_i(q) k_0^{b_i}\bigr) \leq 0$ as a rational function in q. The claim now follows since there exists $C_3 > 0$ such that for every $F \in \mathrm{Loc}_{\gg}$ with q_F large enough,

$$g_F(k_0) = \# Y_F \sum c_i(q_F) q_F^{a_i k_0} k_0^{b_i} < C_2 q_F^d \sum q_F^{a_i k_0} c_i(q_F) k_0^{b_i} < C_3.$$

Now, assume our cell A is infinite and set $a = \max\{a_i\}$. Using $\triangle$ with a general k instead of a fixed k_0, we must have $a \leq 0$, as otherwise for every k large enough $R(q) = C_1 q^d \sum q^{a_i k} c_i(q) k^{b_i}$ is a nonzero rational function in q whose degree is positive, and therefore $\lim_{q_F \to \infty} q_F^{-\delta} R(q_F) \neq 0$ for some $\delta > 0$.

Set $H_F(k) = \sum_{i : a_i = 0} \# Y_F c_i(q_F) k^{b_i}$ and $E_F(k) = \sum_{i : a_i < 0} \# Y_F c_i(q_F) q_F^{a_i k} k^{b_i}$, then we have

$$g_F(k) = H_F(k) + E_F(k) \leq |H_F(k)| + |E_F(k)|.$$

Using ($\dagger$), we may find a constant C' such that $|E_F(k)| < C'$ for every k large enough and $F \in \mathrm{Loc}_{\gg}$. It is therefore left to take care of $H_F(k)$. We may assume $A = \mathbb{Z}_{\geq 1}$.

We prove by induction on the number of summands N that if $H_F = \sum_{i=1}^N \# Y_F c_i(q_F) k^{b_i}$ is a function satisfying $\lim_{q_F \to \infty} q_F^{-\delta} H_F(k) = 0$ for every k large enough and $\delta > 0$, then there exists a constant $C'' > 0$ such that for every $F \in \mathrm{Loc}_{\gg}$ we have $|\# Y_F c_i(q_F)| < C''$ for all $1 \leq i \leq N$.

For $N = 1$ the claim follows by ($\dagger$) as before by showing $|\# Y_F c(q_F)|$ is bounded by a rational function of nonpositive q-degree. To prove the claim for $N > 1$, consider the functions

$$\widetilde{H}_{j,F}(k) = H_F(2k) - 2^{b_j} H_F(k) = \sum_{i=1}^N \# Y_F (2^{b_i} - 2^{b_j}) c_i(q_F) k^{b_i}.$$

For each $1 \leq j \leq N$, the function $\widetilde{H}_{j,F}(k)$ has $N-1$ summands and satisfies the induction hypothesis since $H_F(k)$ and $H_F(2k)$ do, and therefore the proof by induction is concluded. Using the triangle inequality, we can now find a bound for $H_F(k)$ as required, proving the lemma. □

Remark 4.14. Note that one may formulate and prove Lemma 4.13 with $g'_F(k) = q_F^{\varepsilon k} g_F(k)$ instead of g, where $\varepsilon \in \mathbb{R}$. The collection $\{q_F^{\varepsilon k} g_F\}_{F \in \mathrm{Loc}_{\gg}}$ may no longer be motivic, but the proof remains the same.

Proof of Theorem 4.12. (2) $\Rightarrow$ (1): Assume towards contradiction that φ_K is not ε-jet flat. Therefore there exist $k \in \mathbb{Z}_{\geq 1}$ and $y \in J_{k-1}(Y_K)(\bar{K})$, such that if $Z_y := J_{k-1}(X_K)_{y, J_{k-1}(\varphi_K)}$ then

$$\dim Z_y > k \dim X_K - \varepsilon k \dim Y_K.$$

The fiber Z_y is defined over a finitely generated algebra D_y over $\mathcal{O}_K$. Let $\mathbb{F}_q$ be a residue field of D_y where $q = p^r$ for $r \in \mathbb{N}$ and prime p large enough. By taking r large enough, we may assume that all of the top-dimensional geometrically irreducible components of $(Z_y)_{\bar{\mathbb{F}}_q}$ are defined over $\mathbb{F}_q$.

Let $\bar{y} \in J_{k-1}(Y_{\mathbb{F}_q})$ be the reduction modulo q of y under the map $D_y \to \mathbb{F}_q$ and let $y' \in Y(\mathbb{F}_q[t]/(t^k))$ be the image of $\bar{y}$ under the natural identification $J_{k-1}(Y)(\mathbb{F}_q) \simeq Y(\mathbb{F}_q[t]/(t^k))$. In particular, we have $\#(Z_{y_{\mathbb{F}_q}})(\mathbb{F}_q) = \#\varphi^{-1}(y')$. The claim now follows using the Lang–Weil estimates for $\#(Z_{y_{\mathbb{F}_q}})(\mathbb{F}_q)$.

It is left to prove (1) $\Rightarrow$ (2). By Theorem 3.1, there exist $G \in \mathcal{C}_+(\mathbb{Z}_{\geq 1})$ and $C' > 1$ such that

$$\sup_{y \in Y(\mathcal{O}_F)} g_F(y,k) < G_F(k) < C' \sup_{y \in Y(\mathcal{O}_F)} g_F(y,k).$$

Set $G'_F(k) := q_F^{-k \dim Y_K (1-\varepsilon)} G_F(k)$. Using the Lang–Weil estimates and Theorem 2.14, we may invoke Lemma 4.13 with $G'_F(k)$ (see Remark 4.14). We therefore get constants $C, M > 0$ such that for each $k \in \mathbb{Z}_{\geq 1}$ and $F \in \mathrm{Loc}_{\gg}$

$$q_F^{-k \dim Y_K (1-\varepsilon)} \sup_{y \in Y(\mathcal{O}_F)} g_F(y,k) < q_F^{-k \dim Y_K (1-\varepsilon)} G_F(k) = G'_F(k) < Ck^M.$$

The claim is thus proven. $\square$

Remark 4.15. To conclude the paper, we note that a possible deeper understanding of the estimates in Theorems A, 4.7, 4.11 and 4.12 may come from the results on exponential sums in [12] and may be related to the motivic oscillation index $\mathrm{moi}(\varphi)$ of φ.[4] The motivic oscillation index controls the decay rate of the Fourier transform of $\varphi_*(\mu_{\mathcal{O}_F^n})$; see [12, Proposition 3.11]. In the non-(FRS) case, optimal bounds on the decay rate were given in [12, Theorem 1.5], proving a conjecture of Igusa on exponential sums [26]. Here it can also be shown that $\mathrm{moi}(\varphi)$ controls the explosion rate of the density of the pushforward measure $\varphi_*(\mu_{\mathcal{O}_F^n})$ near a critical point; see, e.g., [20, Theorem 8.18]. The (FRS) case of Igusa's conjecture is open (see the discussion in [12, Section 3.4]), and a potential connection between Theorems 4.7, 4.11 and the $\mathrm{moi}(\varphi)$ could be interesting in that regard.

Appendix: Proof of Proposition 3.6

In this appendix we prove the following:

Proposition A.1 Proposition 3.6. *The families* $\mathcal{P}_+(X \times W)$ *and* $\mathcal{C}_+(X \times W)$ *are not of type* (3).

Definition A.2 [6, Definition 1]. Let $X \subseteq \mathbb{Z}^m$ be an $\mathcal{L}_{\mathrm{Pres}}$-definable set. We call a definable function $f : X \to \mathbb{Z}$ $\mathcal{L}_{\mathrm{Pres}}$-*linear* if there exist $\gamma \in \mathbb{Z}$ and integers a_i and $0 \leq c_i < n_i$ for $1 \leq i \leq m$ such that $x_i - c_i \equiv 0 \bmod n_i$ and $f(x_1, \ldots, x_m) = \sum_{i=1}^m a_i((x_i - c_i)/n_i) + \gamma$.

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[4] For the definition in the case that $\varphi : \mathbb{A}^n \to \mathbb{A}^1$ is a polynomial; see [12, Section 3.4].

Proof of Proposition A.1. Let $X=\mathbb{Z}_{\geq 1}^{m}$, $W=\{1,\dots,m\}\subseteq\mathbb{Z}$. Let $p_1=2<p_2=3<\cdots<p_m$ be the first m prime numbers, and take $f(x_1,\dots,x_m,w)=x_w^{p_w}$. We want to show that for every $\epsilon>0$, there is no $g\in\mathcal{C}(X)$ such that

$$\max_{1\leq w\leq m} f_F(x,w)\leq g_F(x)\leq(m-\epsilon)\cdot\max_{1\leq w\leq m} f_F(x,w), \tag{$\star$}$$

for each $F\in\mathrm{Loc}_{\gg}$ and $x\in X_F$. In fact, $\sum_{j=1}^{m}x_j^{p_j}$ is an optimal approximation (with constant m).

Indeed, assume towards contradiction the existence of $g\in\mathcal{C}(X)$ satisfying ($\star$), for some $\epsilon>0$ and for all $F\in\mathrm{Loc}_{\gg}$. Fix F such that ($\star$) holds, and such that g can be written as in Definition 2.11. By the model theoretic orthogonality of the sorts VG and RF, we may assume all of the definable functions $\alpha_i,\beta_{ij}:X\to\mathbb{Z}$ appearing in the data of g, are $\mathcal{L}_{\mathrm{Pres}}$-definable. Using Presburger cell decomposition [6, Theorem 1], we can decompose X into cells $X=\bigsqcup_{i=1}^{N}C_i$, such that on each C_i, the definable Presburger functions appearing in g are $\mathcal{L}_{\mathrm{Pres}}$-linear. Note that one of the cells C must have infinite intersection with the set $\{(t^m,t^{m-1},\dots,t):t\in\mathbb{Z}_{\geq 1}\}$. By the definition of a cell [6, Definiton 2], and by possibly restricting into a smaller subcell, we may assume C has the form

$$C=\left\{(x_1,\dots,x_m)\in\mathbb{Z}_{\geq 1}^{m}:\forall j\left(x_j\geq b+a\sum_{i=j+1}^{m}x_i\right)\wedge(x_j=c_j \bmod r_j)\right\}, \tag{A-1}$$

for $a,b\in\mathbb{Z}_{\geq 1}$, and integers $0\leq c_j\leq r_j$. Taking a and b divisible enough, the cell C is isomorphic to $\mathbb{Z}_{\geq 1}^{m}$ by an affine change of coordinates $\varphi:\mathbb{Z}_{\geq 1}^{m}\to C$, after which $g_F\circ\varphi$ has the form

$$g_F\circ\varphi(e_1,\dots,e_m)=\sum_{i=1}^{M}\tilde{c}_{i,F}\cdot q_F^{a_{i1}e_1+\cdots+a_{im}e_m}\cdot\prod_{j=1}^{m}e_j^{b_{ij}}, \tag{$\star\star$}$$

for $\{(a_{i1},\dots,a_{im},b_{i1},\dots,b_{im})\}_i$ mutually different tuples of integers, where $b_{ij}\geq 0$ and $0\neq\tilde{c}_i\in\mathbb{R}$. Since $1/m\leq g_F(x_1,\dots,x_m)/(x_1^{p_1}+\cdots+x_m^{p_m})\leq m$, it follows that $a_{ij}\leq 0$ for all i,j. We can therefore write g_F as

$$g_F(x_1,\dots,x_m)=P_F(x_1,\dots,x_m)+E_F(x_1,\dots,x_m),$$

where $P_F\circ\varphi$ consists of the terms in ($\star\star$) with $a_{i1}=\cdots=a_{im}=0$, i.e., P_F is a polynomial, and $E_F\circ\varphi$ consists of all the terms of ($\star\star$) with $a_{ij}<0$ for some $j\in\{1,\dots,m\}$. Write

$$P_F(x_1,\dots,x_m)=\sum_{j=1}^{m}d_jx_j^{p_j}+Q_F,$$

where Q_F is the sum of all monomials in P_F which do not belong to the collection $\{x_j^{p_j}\}_{j=1}^{m}$.

For simplicity, in the following arguments we ignore the congruence relations in (A-1). These arguments can easily be adapted to the general form of (A-1).

Write $\tilde{p} := \prod_{j=1}^{m} p_j$ and let $\tilde{p}_j := \tilde{p}/p_j$. Note that, ignoring potential congruences in (A-1), we have $(t^{\tilde{p}_1}, \dots, t^{\tilde{p}_m}) \in C$ for $t \gg 1$. We now claim that:

(1) $d_j \geq 1$ for all $1 \leq j \leq m$.

(2) $\lim_{t\to\infty}(Q_F(t^{\tilde{p}_1}, \dots, t^{\tilde{p}_m})/t^{\tilde{p}}) = \lim_{t\to\infty}(E_F(t^{\tilde{p}_1}, \dots, t^{\tilde{p}_m})/t^{\tilde{p}}) = 0$, and hence

$$\lim_{t\to\infty} \frac{g_F(t^{\tilde{p}_1}, \dots, t^{\tilde{p}_m})}{t^{\tilde{p}}} = \sum_{j=1}^{m} d_j.$$

Since $\max_{1\leq w\leq m} f_F(t^{\tilde{p}_1}, \dots, t^{\tilde{p}_m}, w) = t^{\tilde{p}}$, items (1) and (2) contradict ($\star$). To prove items (1) and (2), we observe the following:

(a) The cell C contains many asymptotic directions in $\mathbb{Z}^{m}_{\geq 1}$; indeed, $(t^{l_1}, \dots, t^{l_m})$ is in C for all integers $l_1 > \dots > l_m \geq 1$ and all $t \gg 1$. Moreover, for each $l_1 > \dots > l_m \geq 1$, we have

$$\lim_{t\to\infty} E_F(t^{l_1}, \dots, t^{l_m}) = 0. \tag{A-2}$$

(b) Since $(t^{\tilde{p}_1}, \dots, t^{\tilde{p}_m}) \in C$ for $t \gg 1$, and by ($\star$)

$$1 \leq \frac{g_F(t^{\tilde{p}_1}, \dots, t^{\tilde{p}_m})}{\max\{t^{\tilde{p}_1 p_1}, \dots, t^{\tilde{p}_m p_m}\}} = \frac{g_F(t^{\tilde{p}_1}, \dots, t^{\tilde{p}_m})}{t^{\tilde{p}}} \leq m - \epsilon. \tag{A-3}$$

For $\omega \in \mathbb{Q}^m$, we define the *ω-weight* of a monomial $x_1^{n_1} \cdot \dots \cdot x_m^{n_m}$ to be $\sum_{j=1}^{m} n_j \omega_j$. Formulas (A-2) and (A-3) imply that all monomials $x_1^{n_1} \cdot \dots \cdot x_m^{n_m}$ appearing in P_F have $(\tilde{p}_1, \dots, \tilde{p}_m)$-weight $\leq \tilde{p}$. Indeed, suppose P_F contains a monomial $\widetilde{Q}_F$ with maximal $(\tilde{p}_1, \dots, \tilde{p}_m)$-weight $\tilde{q} > \tilde{p}$. If $\widetilde{Q}_F$ is the unique monomial of this $(\tilde{p}_1, \dots, \tilde{p}_m)$-weight $\tilde{q}$, then item (a) and (A-3) lead to a contradiction, as $\widetilde{Q}_F(t^{\tilde{p}_1}, \dots, t^{\tilde{p}_m})$ will be the dominant term in $g_F(t^{\tilde{p}_1}, \dots, t^{\tilde{p}_m})$ when $t \gg 1$. If $\widetilde{Q}_F$ is not unique, we take a small perturbation $\tilde{\omega}(d) := (\tilde{p}_1 + 1/d, \dots, \tilde{p}_m + 1/d^m)$ of $(\tilde{p}_1, \dots, \tilde{p}_m)$ with $d > \deg P_F$. Now, each monomial in P_F has a unique $\tilde{\omega}(d)$-weight, and therefore without loss of generality we may assume $\widetilde{Q}_F$ is the monomial of maximal $\tilde{\omega}(d)$-weight in P_F. Taking $t \gg 1$ and applying a variant of (A-3), with $\tilde{\omega}(d)$ instead of $(\tilde{p}_1, \dots, \tilde{p}_m)$, yields a contradiction as before.

(c) The only moniomials with $(\tilde{p}_1, \dots, \tilde{p}_m)$-weight $\tilde{p}$ are $x_1^{p_1}, \dots, x_m^{p_m}$. Indeed, the condition $\sum_{j=1}^{m} n_j \tilde{p}_j = \tilde{p}$ guarantees that each n_j is divisible by p_j.

Item (2) now follows from (A-2) and by (b) and (c) above. We find $\lambda_2, \dots, \lambda_m \in \mathbb{Z}_{\geq 1}$ such that $(t^{\tilde{p}_1}, \lambda_2, \dots, \lambda_m) \in C$ for $t \gg 1$. This implies

$$\lim_{t\to\infty} \frac{g_F(t^{\tilde{p}_1}, \lambda_2, \dots, \lambda_m)}{t^{\tilde{p}}} = \lim_{t\to\infty} \frac{P_F(t^{\tilde{p}_1}, \lambda_2, \dots, \lambda_m)}{t^{\tilde{p}}} = d_1,$$

and hence $d_1 \geq 1$ by ($\star$). More generally, to show that $d_j \geq 1$, we consider

$$(t^{2\tilde{p}_1 - 1}, \dots, t^{2\tilde{p}_{j-1} - 1}, t^{2\tilde{p}_j}, \lambda_{j+1}, \dots, \lambda_m) \in C$$

for $t \gg 1$ (note that $2\tilde{p}_1 - 1 > 2\tilde{p}_2 - 1 > \dots > 2\tilde{p}_{j-1} - 1 > 2\tilde{p}_j$ for every j). This finishes the proof. □

Remark A.3. Note that without the assumption on the p_i's, one can get tighter approximations than $\sum_{j=1}^{m} x_j^{p_j}$. For example, $\frac{4}{3}(x_1^2 - x_1x_2^2 + x_2^4)$ gives a tighter upper bound for $\max(x_1^2, x_2^4)$, than $x_1^2 + x_2^4$, since $\frac{4}{3}(x_1^2 - x_1x_2^2 + x_2^4) \leq \frac{4}{3}\max(x_1^2, x_2^4)$.

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