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A number theoretic characterization of E-smooth and (FRS) morphisms: estimates on the number of $\mathbb{Z}/p^k\mathbb{Z}$-points

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We provide uniform estimates on the number of $\mathbb{Z}/p^k\mathbb{Z}$-points lying on fibers of flat morphisms between smooth varieties whose fibers have rational singularities, termed (FRS) morphisms. For each individual fiber, the estimates were known by work of Avni and Aizenbud, but we render them uniform over all fibers. The proof technique for individual fibers is based on Hironaka's resolution of singularities and Denef's formula, but breaks down in the uniform case. Instead, we use recent results from the theory of motivic integration. Our estimates are moreover equivalent to the (FRS) property, just like in the absolute case by Avni and Aizenbud. In addition, we define new classes of morphisms, called E-smooth morphisms ($E \in \mathbb{N}$), which refine the (FRS) property, and use the methods we developed to provide uniform number-theoretic estimates as above for their fibers. Similar estimates are given for fibers of ε-jet flat morphisms, improving previous results by the last two authors.

1. Introduction

1A. *Overview.* Let $\varphi : X \to Y$ be an algebraic morphism between smooth K-varieties, where K is a number field. In this paper we give uniform arithmetic and analytic equivalent characterizations to the (FRS) property of φ, namely to the property of being flat with reduced fibers of rational singularities (see Theorem A). These results can be viewed as a common uniform improvement of the following two theorems:

(1) Theorem A of [2], where bounds were given on the number of $\mathbb{Z}/p^k\mathbb{Z}$-points of reduced local complete intersection schemes which have rational singularities (see also Theorem 1.3).

(2) Theorem 3.4 of [1], where pushforward of smooth measures with respect to φ over non-Archimedean local fields were shown to have bounded density if and only if φ is an (FRS) morphism (see also Theorem 4.3).

In order to prove our uniform characterizations of the (FRS) property, it seems natural to try and adapt the algebro-geometric proof of [2, Theorem A] to the relative case. This fails to work because of unsatisfactory behavior of resolution of singularities in families, with respect to taking points over $\mathbb{Z}$, $\mathbb{Z}/p^k\mathbb{Z}$ and $\mathbb{Z}_p$

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(see Section 1D1). Instead, we prove a model theoretic result of independent interest about approximating suprema of a certain subclass of motivic functions, which we call formally nonnegative functions (see Theorem B). Using Theorem B and by analyzing the jets of φ, we prove Theorem A. Theorem B further strengthens [11, Theorem 2.1.3] in the case of formally nonnegative functions. Finally, we provide uniform estimates on the number of $\mathbb{Z}/p^k\mathbb{Z}$-points lying on fibers of E-smooth morphisms, a new notion we introduce which refines the (FRS) property ($E \in \mathbb{N}$). Uniform estimates are also provided for fibers of ε-jet flat morphisms, achieving optimal bounds; see [20, Theorem 8.18]. See Section 2A1 and Theorems 4.11 and 4.12 for these notions and results.

1B. ***Counting points over $\mathbb{Z}/p^k\mathbb{Z}$: the absolute case.*** Let X be a finite type $\mathbb{Z}$-scheme. The study of the quantity $\#X(\mathbb{Z}/n\mathbb{Z})$, and its asymptotic behavior in $n \in \mathbb{N}$, is a long standing problem in number theory. When $n = p$ is prime, the asymptotic behavior is understood by the Lang–Weil estimates [30], and in particular, the family

$$\left\{\frac{\#X(\mathbb{Z}/p\mathbb{Z})}{p^{\dim X_{\mathbb{Q}}}}\right\}_p$$

is uniformly bounded.

Moving to the case where $n = p^k$ is a prime power (which suffices, by the Chinese remainder theorem), one can observe the following; if X is smooth as a $\mathbb{Z}$-scheme, then an application of Hensel's lemma shows that

$$\left\{\frac{\#X(\mathbb{Z}/p^k\mathbb{Z})}{p^{k\dim X_{\mathbb{Q}}}}\right\}_{p,k}$$

is uniformly bounded in both p and k. On the other hand, taking the nonreduced scheme $X = \operatorname{Spec}\mathbb{Z}[x]/(x^2)$, we see that

$$\frac{\#X(\mathbb{Z}/p^{2k}\mathbb{Z})}{p^{2k\dim X_{\mathbb{Q}}}} = \#X(\mathbb{Z}/p^{2k}\mathbb{Z}) = p^k,$$

which is not uniformly bounded. The following natural question arises.

Question 1.1. Is there a necessary and sufficient condition on X such that $\{\#X(\mathbb{Z}/p^k\mathbb{Z})/p^{k\dim X_{\mathbb{Q}}}\}_{p,k}$ is uniformly bounded?

Aizenbud and Avni [2], relying on results of Mustaţă [33] and Denef [13], gave such a necessary and sufficient condition in the case where $X_{\mathbb{Q}}$ is a local complete intersection.

Definition 1.2. Let K be a field of characteristic 0. A K-scheme of finite type X has *rational singularities* if it is normal and for every resolution of singularities $\pi : \widetilde{X} \to X$, one has

$$R^i\pi_*(O_{\widetilde{X}}) = 0$$

for $i \geq 1$.

Theorem 1.3 (see [2, Theorem A] and [18]). *Let X be a finite type $\mathbb{Z}$-scheme such that $X_{\mathbb{Q}}$ is equidimensional and a local complete intersection. Then the following are equivalent*:

(1) *$X_{\mathbb{Q}}$ has rational singularities* (*and*, *in particular*, *$X_{\mathbb{Q}}$ is reduced*).

(2) *There exists $C > 0$ such that for every prime p and every $k \in \mathbb{N}$ one has*

$$\frac{\# X(\mathbb{Z}/p^k\mathbb{Z})}{p^{k \dim X_{\mathbb{Q}}}} < C.$$

(3) *There exists $C > 0$ such that for every prime p and every $k \in \mathbb{N}$ one has*

$$\left| \frac{\# X(\mathbb{Z}/p^k\mathbb{Z})}{p^{k \dim X_{\mathbb{Q}}}} - \frac{\# X(\mathbb{Z}/p\mathbb{Z})}{p^{\dim X_{\mathbb{Q}}}} \right| < C p^{-1}.$$

1C. ***Counting points over $\mathbb{Z}/p^k\mathbb{Z}$: the relative case.*** Let X and Y be smooth finite type $\mathbb{Z}$-schemes and let $\varphi : X \to Y$ be a dominant morphism. Our goal in this paper is to treat the relative analogue of Question 1.1:

Question 1.4. Is there a necessary and sufficient condition on φ such that the size of each fiber of $\varphi : X(\mathbb{Z}/p^k\mathbb{Z}) \to Y(\mathbb{Z}/p^k\mathbb{Z})$, normalized by $p^{k(\dim X_{\mathbb{Q}} - \dim Y_{\mathbb{Q}})}$, is uniformly bounded when varying p, k and $y \in Y(\mathbb{Z}/p^k\mathbb{Z})$?

Since the Lang–Weil estimates are effective uniformly over all schemes of bounded complexity, Question 1.4 is easily answered in the case where $k = 1$; the condition that $\varphi_{\mathbb{Q}}$ *is flat* is necessary and sufficient; see [20, Theorem 8.4]. For the general case, we use the following notion from [1, Definition II]. By a K-variety with K a field we mean a reduced K-scheme of finite type.

Definition 1.5. Let X and Y be smooth K-varieties, where K is a field with $\mathrm{char}(K) = 0$. We say that a morphism $\varphi : X \to Y$ is (*FRS*) if it is flat and if every fiber of φ has rational singularities.

1D. ***Main results.*** We are now ready to state the main result of this paper.

Theorem A (see Theorem 4.7 for a more general version). *Let $\varphi : X \to Y$ be a dominant morphism between finite type $\mathbb{Z}$-schemes X and Y, with $X_{\mathbb{Q}}, Y_{\mathbb{Q}}$ smooth and geometrically irreducible. Then the following are equivalent*:

(1) *$\varphi_{\mathbb{Q}} : X_{\mathbb{Q}} \to Y_{\mathbb{Q}}$ is* (*FRS*).

(2) *There exists $C_1 > 0$ such that for every prime p, $k \in \mathbb{N}$ and $y \in Y(\mathbb{Z}/p^k\mathbb{Z})$ one has*

$$\frac{\# \varphi^{-1}(y)}{p^{k(\dim X_{\mathbb{Q}} - \dim Y_{\mathbb{Q}})}} < C_1.$$

(3) *There exists $C_2 > 0$ such that for every prime p, $k \in \mathbb{N}$ and $y \in Y(\mathbb{Z}/p^k\mathbb{Z})$ one has*

$$\left| \frac{\# \varphi^{-1}(y)}{p^{k(\dim X_{\mathbb{Q}} - \dim Y_{\mathbb{Q}})}} - \frac{\# \varphi^{-1}(\bar{y})}{p^{(\dim X_{\mathbb{Q}} - \dim Y_{\mathbb{Q}})}} \right| < C_2 p^{-1},$$

where $\bar{y}$ is the image of y under the reduction $Y(\mathbb{Z}/p^k\mathbb{Z}) \to Y(\mathbb{F}_p)$.

(4) *There exists $C_3 > 0$ such that the following hold for every prime p. Let $\mu_{X(\mathbb{Z}_p)}$ and $\mu_{Y(\mathbb{Z}_p)}$ be the canonical measures on $X(\mathbb{Z}_p)$ and $Y(\mathbb{Z}_p)$; see Lemma 4.2. Then the pushforward measure $\varphi_*\mu_{X(\mathbb{Z}_p)}$ has continuous density f_p with respect to $\mu_{Y(\mathbb{Z}_p)}$, and $\|f_p\|_\infty < C_3$.*

Using a jet-scheme characterization of rational singularities by Mustaţă [33; 34], it can be shown that a morphism $\varphi : X \to Y$ between smooth schemes is (FRS) if and only if for each $k \in \mathbb{N}$, every nonempty fiber of the corresponding k-th jet map $J_k(\varphi) : J_k(X) \to J_k(Y)$ is of dimension $\dim J_k(X) - \dim J_k(Y)$ (i.e., $J_k(\varphi)$ is flat) and has a singular locus of codimension at least 1 (see Section 2A1 and Lemma 2.9). Based on this characterization, it is natural to define two variations of the (FRS) property:

- A morphism φ is *ε-jet flat*, for $\varepsilon \in \mathbb{R}_{>0}$, if the fibers of $J_k(\varphi)$ are of dimension at most $\dim J_k(X) - \varepsilon \dim J_k(Y)$, for all $k \in \mathbb{N}$; see [20, Definition 3.22].
- A morphism φ is called *E-smooth* if it is 1-jet flat, and each of the fibers of $J_k(\varphi)$ has singular locus of codimension at least E.

In Section 4C, using methods similar to the proof of Theorem A, we provide uniform estimates on the fibers of E-smooth and ε-jet flat morphisms (see Theorems 4.11 and 4.12). In particular, uniform estimates are given on fibers of flat morphisms whose fibers have terminal or log-canonical singularities.

1D1. *Main difficulties in the proof of Theorem A.* The proof of Theorem 1.3 in [2] proceeds by (locally) embedding X as a complete intersection in $\mathbb{A}^N$ and choosing an embedded resolution of singularities for the pair $(X_\mathbb{Q}, \mathbb{A}^N_\mathbb{Q})$, also called a log-resolution, whose existence follows from [25]. For large p, one can then use Denef's formula [13, Theorem 3.1], to relate $\#X(\mathbb{Z}/p^k\mathbb{Z})$ to $\{\#E_I(\mathbb{F}_p)\}_I$ and numerical data associated to the choice of resolution, where $\{E_I\}_I$ is a collection of constructible subsets built out of the prime divisors $\{E_i\}_{i=1}^M$ appearing in such a resolution. Combined with the Lang–Weil estimates for the E_I, this yields estimates for $\#X(\mathbb{Z}/p^k\mathbb{Z})$. To finally achieve the bounds of Theorem 1.3, one needs the reductions modulo p of the E_I's to be of the expected dimensions over $\mathbb{F}_p$. This can always be done if the prime p is large enough; small primes are treated separately in [18].

If $\varphi_\mathbb{Q} : X_\mathbb{Q} \to Y_\mathbb{Q}$ is (FRS), its fibers are local complete intersections with rational singularities, and one may try to mimic the strategy for Theorem 1.3. The weak point is that this only seems to work for each fiber separately, but does not give the desired uniformity in the choice of fiber. One can try to make this naive fiber-wise strategy more uniform by choosing some simultaneous resolutions of singularities. This can be done by breaking Y into constructible subsets, with resolutions over generic points of the pieces. However, such finite partition of Y into constructible sets does not behave well at all with respect to taking points over the rings $\mathbb{Z}$, $\mathbb{Z}/p^k\mathbb{Z}$, or $\mathbb{Z}_p$. In fact, as far as we can see, the approach with resolutions of singularities in families is hard to adapt to the family situation of Theorem A.

To avoid these difficulties, we use the motivic nature of $\mathbb{Z}/p^k\mathbb{Z}$-point count of the fibers of φ, that is, we use insights from motivic integration and uniform p-adic integration. Let $r_k : Y(\mathbb{Z}_p) \to Y(\mathbb{Z}/p^k\mathbb{Z})$ be the reduction map. Write $d := \dim X_\mathbb{Q} - \dim Y_\mathbb{Q}$. For each prime p, each $y \in Y(\mathbb{Z}_p)$ and each integer

$k \geq 1$ we set

$$g_p(y,k) = \frac{\#\varphi^{-1}(r_k(y))}{p^{kd}} \quad \text{and} \quad \tilde{h}_p(y,k) := g_p(y,k) - g_p(y,1), \tag{1-1}$$

as in the left-hand side of items (2) and (3) of Theorem A. The collections of functions $\{g_p\}_p$, $\{\tilde{h}_p\}_p$ are examples of *motivic functions*, namely in a uniform p-adic sense as in [11], but closely related to genuine motivic constructible functions from [7]. We use motivic integration to extract information on $\{g_p\}_p$ and $\{\tilde{h}_p\}_p$, which in turn allows us to prove Theorem A.

1D2. *Relation of the main number theoretic results to previous results.* Aizenbud and Avni [1; 2] have shown that the (FRS) property of $\varphi_{\mathbb{Q}}$ is equivalent to uniform boundedness of $g_p(y,k)$, when either

(1) one varies over k and p for each fixed y [2, Theorem A];[1] or

(2) one varies over k and y for each fixed p [1, Theorem 3.4].

Using the Lang–Weil estimates, one can further show that the (FRS) property implies uniform boundedness of $g_p(y,k)$ when

(3) one varies over p and y for each fixed k.

The implication (1) ⇒ (2) of Theorem A asserts that if $\varphi_{\mathbb{Q}}$ is (FRS) then $g_p(y,k)$ as above is uniformly bounded when varying over p, k and y simultaneously.

It is worth noting that unlike items (1) and (2), item (3) as above is weaker than the (FRS) property, and is equivalent to 1-jet flatness of $\varphi_{\mathbb{Q}}$ using Lemma 4.13 and Theorem 4.12. In a recent work by Glazer and Hendel, this condition is furthermore shown to be equivalent to $\varphi_{\mathbb{Q}}$ being flat with fibers of semi-log-canonical singularities; see [22, Lemma 6.5, Theorem 6.6] and the discussion therein.

The proofs of the number-theoretic estimates for ε-jet flat and E-smooth morphisms (Theorems 4.11 and 4.12) share similar difficulties with the proof of Theorem A. Theorem 4.12 improves previous bounds for ε-jet flat morphisms: the bounds given in [39, Corollary 2.9] on $g_p(y,k)$ are uniform in k, but not in p and y (see Remark 2.8 for the relation of ε-jet flatness to the log canonical threshold), and the bounds given in [20, Theorem 8.18] are uniform in p, y, k, but are not optimal.

1D3. *Model-theoretic results.* We denote by Loc the collection of all non-Archimedean local fields, by Loc_0 the collection of all $F \in \mathrm{Loc}$ of characteristic zero, and by $\mathrm{Loc}_{\gg}$ the collection of all $F \in \mathrm{Loc}$ with large enough residual characteristic, where "large enough" changes according to our needs.

Let $\mathcal{L}_{\mathrm{DP}}$ denote the Denef–Pas language. This is a first order language with three sorts to account for a valued field F, a residue field k_F and a value group which we identify with $\mathbb{Z}$. An $\mathcal{L}_{\mathrm{DP}}$-definable set $X = \{X_F\}_{F \in \mathrm{Loc}_{\gg}}$ is a collection of subsets $X_F \subseteq F^{n_1} \times k_F^{n_2} \times \mathbb{Z}^{n_3}$ which is uniformly defined using an $\mathcal{L}_{\mathrm{DP}}$-formula.[2] Given $\mathcal{L}_{\mathrm{DP}}$-definable sets X and Y, a collection of functions $\{f : X_F \to Y_F\}_{F \in \mathrm{Loc}_{\gg}}$ is called an ($\mathcal{L}_{\mathrm{DP}}$-)definable function if its graph is definable.

[1] Here "for each fixed y" means for each fixed $y \in Y(\mathbb{Q})$, where p is large enough to allow us to reduce modulo p.

[2] For the notation "$F \in \mathrm{Loc}_{\gg}$" see Section 2B.

Given a definable set $X = \{X_F\}_{F \in \mathrm{Loc}_{\gg}}$, the ring of motivic functions $\mathcal{C}(X)$ is a certain natural class of functions whose building blocks are the definable functions, and is closed under integration. Built on a natural notion of positivity, we define the semiring of formally nonnegative functions $\mathcal{C}_+(X) \subset \mathcal{C}(X)$ (see Definition 2.11). As an example, the collection $\{\varphi_* \mu_F\}_{F \in \mathrm{Loc}_{\gg}}$ of pushforwards of Haar measures μ_F on $\mathcal{O}_F^n$ under any polynomial map φ, as well as $\{g_p\}_p$ above are formally nonnegative motivic functions. The classes $\mathcal{C}_+(X)$ and $\mathcal{C}(X)$ above are uniform p-adic specializations of more genuinely motivic functions defined in [7; 8], but they go by similar methods and theories. See Section 2B for further details on motivic functions.

As a key step towards proving Theorem A, we show the following strengthening of [11, Theorem 2.1.3] for the class of formally nonnegative motivic functions:

Theorem B (Theorem 3.1). *Let f be in $\mathcal{C}_+(X \times W)$, where X and W are $\mathcal{L}_{\mathrm{DP}}$-definable sets. Then there exists a constant $C > 0$, and a function $G \in \mathcal{C}_+(X)$ such that for any $F \in \mathrm{Loc}_{\gg}$ and any $x \in X_F$ such that $w \mapsto f_F(x, w)$ is bounded on W_F, we have*

$$\sup_{w \in W_F} f_F(x, w) \leq G_F(x) \leq C \cdot \sup_{w \in W_F} f_F(x, w).$$

The approximation of suprema given in Theorem B is best possible for the class of formally nonnegative motivic functions $\mathcal{C}_+(X \times W)$, in the sense that one cannot choose C to be a universal constant (see Proposition 3.6). In [11, Theorem 2.1.3], a similar approximation result is shown (for motivic functions in $\mathcal{C}(X \times W)$ and in $\mathcal{C}^{\exp}(X \times W)$), but where the constant C is replaced by q_F^C, with q_F the number of elements in the residue field k_F of F, and where instead of $\sup f_F$ one approximates $\sup |f_F|^2$. For more details on the optimality of these approximation results, see the discussion in Section 3A.

1D4. *Sketch of proof of Theorem A.* To prove Theorem A, we show (1) ⇒ (3) ⇒ (2) ⇒ (4) ⇒ (1). The implications (3) ⇒ (2) ⇒ (4) are rather easy and the implication (4) ⇒ (1) essentially follows from an equivalent analytic characterization of the (FRS) property due to Aizenbud and Avni (see Theorem 4.3). The challenging part of the proof is to show (1) ⇒ (3). Small primes are dealt using Theorem 4.3 and using basic properties of the canonical measure (Lemma 4.2). Thus we may consider only large enough primes p. Let us sketch the main strategy of the proof of (1) ⇒ (2), for large p, which has similar difficulties to (1) ⇒ (3):

(a) We use Theorem 4.3 to show that

$$\sup_{y,k} g_p(y, k) < C(p)$$

for some constant $C(p)$ depending on p.

(b) Since g is a formally nonnegative motivic function (see Definition 2.10), and $g_p(y, k)$ is bounded for each fixed p and k, we may utilize Theorem B to approximate

$$\sup_{y \in Y(\mathbb{Z}_p)} g_p(y, k)$$

for each p and k by $G_p(k)$, for a single motivic function $G = \{G_p : \mathbb{Z}_{\geq 1} \to \mathbb{R}\}_p$.

(c) We use results from [11] on approximate suprema of constructible Presburger functions together with item (a) to deduce that

$$\sup_{k\in\mathbb{Z}_{\geq 1}} G_p(k)$$

can be approximated by $\sum_{l\in L} G_p(l)$ for some finite subset $L \subseteq \mathbb{Z}_{\geq 1}$, with L independent of p.

(d) To deal with $G_p(l)$ for $l \in L$, we use a transfer principle for boundedness of motivic functions from [10] (see Theorem 2.14 below) to reduce to a question about the $\mathbb{F}_p$-fibers of the $(l-1)$-th jet of φ. We then combine Lang–Weil type arguments on the jets of φ, together with a jet-scheme interpretation of the (FRS) property (Proposition 2.3), to deduce that $G_p(l) < C$ for $p \gg 1$, $l \in L$ and some constant $C > 0$ independent of p.

This shows (1) $\Rightarrow$ (2). To prove (1) $\Rightarrow$ (3), we approximate $\tilde{h}_p$ with a motivic function h_p, which unlike $\tilde{h}_p$, is formally nonnegative. We then apply similar steps as above (with a few extra complications) to h_p.

1E. ***Further discussion.*** We now give more context and motivation for Theorem A (for additional details see the references below). The (FRS) property was first introduced and studied in [1; 2], where a very useful analytic interpretation was given as follows. Given a morphism $\varphi : X \to Y$ between smooth $\mathbb{Q}$-varieties, the (FRS) property of φ is characterized by the property that for every $F \in \mathrm{Loc}_0$ and every smooth, compactly supported measure $\mu_{X(F)}$ on $X(F)$, the pushforward measure $\varphi_*(\mu_X)$ on $Y(F)$ has continuous density; see Theorem 4.3 or [1, Theorem 3.4]. Our number theoretic characterization (Theorem A) can be seen as a refinement of this analytic characterization.

These characterizations allow one to use algebro-geometric tools to solve various problems in analysis, probability and group theory. For a motivating example, let $\underline{G}$ be a semisimple algebraic $\mathbb{Q}$-group and let $\varphi_{\mathrm{comm}}^{*t} : \underline{G}^{2t} \to \underline{G}$ be the map $(g_1, \dots, g_{2t}) \mapsto [g_1, g_2] \cdot \dots \cdot [g_{2t-1}, g_{2t}]$, corresponding to the product of t commutator maps. Using the above characterizations and a theorem of Frobenius, one has, see [1, Theorem IV],

$$\varphi_{\mathrm{comm}}^{*t} \text{ is (FRS)} \Rightarrow \#\{N\text{-dimensional irreducible } \mathbb{C}\text{-representations of } \underline{G}(\mathbb{Z}_p)\} = O(N^{2t-2}). \qquad (\star)$$

Aizenbud and Avni showed in [1; 2], that $\varphi_{\mathrm{comm}}^{*21}$ is (FRS) for every $\underline{G}$ as above, which via $(\star)$, confirmed a conjecture of Larsen and Lubotzky [31] about representation growth of compact p-adic and arithmetic groups. These bounds were improved in [3; 20; 24; 29].

The above situation can be generalized as follows. Let $\varphi : X \to \underline{G}$ be a dominant morphism from a smooth $\mathbb{Q}$-variety X to a connected algebraic group $(\underline{G}, \cdot_{\underline{G}})$. We define the *self-convolution* $\varphi * \varphi : X \times X \to \underline{G}$ of φ by $\varphi * \varphi(x_1, x_2) = \varphi(x_1) \cdot_{\underline{G}} \varphi(x_2)$, and write $\varphi^{*t} : X^t \to G$ for the t-th convolution power of φ. Similarly to the usual convolution operation in analysis, this algebraic convolution operation has a smoothing effect on morphisms; in [19; 21], it was shown that $\varphi^{*t} : X^t \to \underline{G}$ has increasingly better singularity properties as t grows, and eventually, φ^{*t} becomes (FRS) for every $t \geq t_0$, for some $t_0 \in \mathbb{N}$.

Moving to the probabilistic picture, let $\mu_{X(\mathbb{Z}_p)}$ and $\mu_{\underline{G}(\mathbb{Z}_p)}$ be the canonical measures on $X(\mathbb{Z}_p)$ and $\underline{G}(\mathbb{Z}_p)$, normalized to have total mass 1. One can then study the collection of random walks on $\underline{G}(\mathbb{Z}_p)$, induced by the pushforward measures $\{\varphi_*\mu_{X(\mathbb{Z}_p)}\}_{p\in\text{primes}}$, by analyzing the convergence rate of their self-convolutions $(\varphi_*\mu_{X(\mathbb{Z}_p)})^{*t}$ to $\mu_{\underline{G}(\mathbb{Z}_p)}$, in the L^r-norm ($r \geq 1$). This rate of convergence can be measured by the notion of *L^q-mixing time*; see, e.g., [32, Chapter 4]. Note that the analytic convolution operation commutes with the algebraic convolution defined above, so that $(\varphi_*\mu_{X(\mathbb{Z}_p)})^{*t} = (\varphi^{*t})_*\mu_{X^t(\mathbb{Z}_p)}$. This makes Theorem A the connecting link between the algebraic and the probabilistic pictures above.

Explicitly, let us denote by t_{alg} the minimal $t \in \mathbb{N}$ such that φ^{*t} is (FRS) and has geometrically irreducible fibers, and call it the *algebraic mixing time of φ*. Then Theorem A, and its general form Theorem 4.7, imply that the algebraic mixing time of φ is equal to the uniform (in $p \gg 1$) L^∞-mixing time of the random walks on $\{\underline{G}(\mathbb{Z}_p)_p\}$ induced by $\{\varphi_*\mu_{X(\mathbb{Z}_p)}\}_p$; see [20, Definition 9.2]. This philosophy was implemented in [20], which motivated this work. There, the authors analyzed the singularity properties of word maps on semisimple algebraic groups, using purely algebraic techniques, and obtained probabilistic results on word measures. In particular, Theorem A completes the proof of [20, Theorems G and 9.3(2)].

1F. *Conventions.*

- Throughout the paper, we use K, K', K'' to denote number fields and $\mathcal{O}_K, \mathcal{O}_{K'}, \mathcal{O}_{K''}$ for their rings of integers. Similarly, local fields and their rings of integers are denoted by F, F', F'' and $\mathcal{O}_F, \mathcal{O}_{F'}, \mathcal{O}_{F''}$, respectively.

- Given a local ring A, a morphism $\varphi : X \to Y$ of schemes X and Y, and given $y \in Y(A)$ (i.e., a morphism $\operatorname{Spec}(A) \to Y$), we denote by $X_{y,\varphi} := \operatorname{Spec}(A) \times_Y X$ the scheme theoretic fiber over y, and simply by $\varphi^{-1}(y) \subseteq X(A)$ the set theoretic fiber of the induced map $\varphi : X(A) \to Y(A)$. Note that if $y \in Y$ is a schematic point, then it can be viewed as $y \in Y(\kappa(y))$, where $\kappa(y)$ is the residue field of y, so that $X_{y,\varphi} := \operatorname{Spec}(\kappa(y)) \times_Y X$.

- Given a K-morphism $\varphi : X \to Y$ between K-varieties X and Y, we denote by X^{sm} (resp. X^{sing}) the smooth (resp. nonsmooth) locus of X. We denote by $X^{\text{sm},\varphi}$ (resp. $X^{\text{sing},\varphi}$) the smooth (resp. nonsmooth) locus of φ in X.

- We denote the base change of an S-scheme X with respect to $S' \to S$ by $X_{S'}$.

2. Preliminaries

2A. *Jet schemes and singularities.* For a thorough discussion of jet schemes see [4, Chapter 3] and [15].

Definition 2.1 [4, Section 3.2]. Let S be a scheme and let X be a scheme over S:

(1) For each $k \in \mathbb{N}$, we define the *k-th jet scheme* of X, denoted $J_k(X/S)$ as the S-scheme representing the functor

$$\mathcal{J}_k(X/S) : W \mapsto \operatorname{Hom}_{S\text{-schemes}}(W \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}(\mathbb{Z}[t]/(t^{k+1})), X),$$

where W is an S-scheme. We write $J_k(X)$ if the scheme S is understood.

(2) Given an S-morphism $\varphi : X \to Y$ and an S-scheme W, the composition with φ induces a map $\mathcal{J}_k(X/S)(W) \to \mathcal{J}_k(Y/S)(W)$, which yields a morphism

$$J_k(\varphi) : J_k(X/S) \to J_k(Y/S),$$

called the *k-th jet* of φ.

(3) For any $k_1 \geq k_2 \in \mathbb{N}$ the reduction map $\mathbb{Z}[t]/(t^{k_1+1}) \to \mathbb{Z}[t]/(t^{k_2+1})$ induces a natural collection of morphisms $\pi^{k_1}_{k_2,X} : J_{k_1}(X/S) \to J_{k_2}(X/S)$ which are called *truncation maps*. Note that the collection $\{J_k(\varphi) : J_k(X/S) \to J_k(Y/S)\}_{k\in\mathbb{N}}$ commutes with $\{\pi^m_{n,X}\}_{m\geq n}$.

(4) The natural map $\mathbb{Z} \to \mathbb{Z}[t]/(t^{m+1})$ induces a *zero section* $s_{m,X} : X \hookrightarrow J_m(X)$. We sometimes write π^m_n and s_m instead of $\pi^m_{n,X}$ and $s_{m,X}$, when X is clear.

In the rest of this subsection, we assume $S = \operatorname{Spec} K$. Mustaţă gave the following interpretation of rational singularities in terms of jet schemes:

Theorem 2.2 [33]. *Let X be a geometrically irreducible, local complete intersection K-variety, with $\operatorname{char}(K) = 0$. Then $J_k(X)$ is geometrically irreducible for all $k \geq 1$ if and only if X has rational singularities.*

Using Theorem 2.2, one can obtain a similar characterization of (FRS) morphisms:

Proposition 2.3 [20, Corollary 3.12; 27]. *Let X and Y be smooth, geometrically irreducible K-varieties, and let $\varphi : X \to Y$ be a K-morphism:*

(1) *Assume $\operatorname{char}(K) = 0$. Then the morphism φ is (FRS) if and only if $J_k(\varphi)$ is flat, with locally integral fibers for each $k \in \mathbb{N}$.*

(2) *The morphism φ is smooth if and only if $J_k(\varphi)$ is smooth for each $k \in \mathbb{N}$.*

Remark 2.4. Let k be a natural number, and K be a field with $\operatorname{char}(K) = 0$ or $\operatorname{char}(K) \gg 1$ (in terms of k). Then the jet scheme $J_k(X)$ of an affine K-scheme $X \subseteq \mathbb{A}^n$ has a simple description; write $X = \operatorname{Spec} K[x_1, \ldots, x_n]/(f_1, \ldots, f_l)$. Then

$$J_k(X) = \operatorname{Spec} K[x_1, \ldots, x_n, x_1^{(1)}, \ldots, x_n^{(1)}, \ldots, x_1^{(k)}, \ldots, x_n^{(k)}]/(\{f_j^{(u)}\}_{j=1,u=0}^{l,k}),$$

where $f_i^{(u)}$ is the u-th formal derivative of f_i. For example, if $f = x_1x_2^2$ then $f^{(1)} = x_1^{(1)}x_2^2 + 2x_1x_2x_2^{(1)}$. Similarly, $J_k(\varphi) = (\varphi, \varphi^{(1)}, \ldots, \varphi^{(k)})$ for a morphism $\varphi : X \to Y$ of affine K-schemes.

The next proposition will be useful in Section 4.

Proposition 2.5. *Let $k \in \mathbb{N}$ and let $\varphi : X \to Y$ be K-morphism as in Proposition 2.3, with $\operatorname{char}(K) = 0$ or $\operatorname{char}(K) \gg 1$ (in terms of k). Then $J_k(X)^{\mathrm{sm},J_k(\varphi)} = J_k(X^{\mathrm{sm},\varphi})$.*

Proof. It follows from Proposition 2.3(2) that $J_k(X^{\mathrm{sm},\varphi}) \subseteq J_k(X)^{\mathrm{sm},J_k(\varphi)}$, so it is left to show the other inclusion. We may assume that X and Y are affine, and that Y admits an étale map $\psi : Y \to \mathbb{A}^m_K$. We may further assume that $Y = \mathbb{A}^m_K$. Indeed, we have

$$J_k(X)^{\mathrm{sm},J_k(\varphi)} = J_k(X)^{\mathrm{sm},J_k(\psi\circ\varphi)} \quad \text{and} \quad J_k(X^{\mathrm{sm},\psi\circ\varphi}) = J_k(X^{\mathrm{sm},\varphi}).$$

By Remark 2.4, we can write $X = \mathrm{Spec}\, K[x_1, \dots, x_{n+l}]/(f_1, \dots, f_l)$, and

$$J_k(X) = \mathrm{Spec}\, K[x_1, \dots, x_{n+l}, \dots, x_1^{(k)}, \dots, x_{n+l}^{(k)}]/(\{f_j^{(u)}\}_{j=1,u=0}^{l,k}).$$

Moreover $J_k(\varphi) = (\varphi, \varphi^{(1)}, \dots, \varphi^{(k)})$ where $\varphi = (f_{l+1}, \dots, f_{l+m}) : X \to \mathbb{A}_K^m$. Write $F_{u(l+m)+j} := f_j^{(u)}$ and $X_{u(n+l)+i} := x_i^{(u)}$, and let $\bar{a} := (a, a^{(1)}, \dots, a^{(n+l)}) \in J_k(X)$. Then $J_k(\varphi)$ is smooth at $\bar{a}$ if and only if the matrix $M = \left(\frac{\partial F_j}{\partial X_i}|_{\bar{a}}\right)_{i=1,j=1,}^{(n+l)(k+1),(l+m)(k+1)}$ is of full rank $(l+m)(k+1)$. Note that M has the shape

$$M = \begin{pmatrix} M_{00} & M_{01} & \cdots & M_{0k} \\ 0 & M_{11} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & M_{kk} \end{pmatrix},$$

where $M_{u_1u_2} = \left(\frac{\partial f_j^{(u_2)}}{\partial x_i^{(u_1)}}|_{\bar{a}}\right)_{i=1,j=1}^{(n+l),(l+m)}$ for $0 \le u_1 \le u_2 \le k$. If M is of full rank, then also $M_{00} = \left(\frac{\partial f_j}{\partial x_i}|_a\right)_{i=1,j=1}^{(n+l),(l+m)}$ must be of full rank, which in turn implies that φ is smooth at a, and the proposition follows. □

Remark 2.6. The case $Y = \mathbb{A}^1$ of Proposition 2.5 has essentially been proven in [17, proof of Theorem 3.3] and [33, Proposition 4.12]; see also [28, page 222]. Proposition 2.5 also relates to [33, Questions 4.10 and 4.11], as follows. Given a local complete intersection variety X, it can be written, locally, as a fiber $\widetilde{X}_{0,\varphi}$ of a flat morphism $\varphi : \widetilde{X} \to \mathbb{A}^m$, with $\widetilde{X}$ smooth. If we assume that $J_k(\varphi)$ is flat for all k, then Proposition 2.5 combined with [23, III, Theorem 10.2] implies that $(\pi^k_{0,\widetilde{X}_{0,\varphi}})^{-1}((\widetilde{X}_{0,\varphi})^{\mathrm{sm}}) = J_k(\widetilde{X}_{0,\varphi})^{\mathrm{sm}}$ for all k, which gives a positive answer to [33, Question 4.11] in this case. If $J_k(\varphi)$ is not flat, one can still effectively describe its smooth locus, but it is harder to describe the smooth locus of its fibers.

2A1. *E-smooth and ε-jet flat morphisms.* We next introduce several properties of morphisms between smooth varieties: ε-flatness, ε-jet flatness, and E-smoothness. The first two notions were first introduced in [20], whereas the E-smoothness notion is new.

Definition 2.7. Let X and Y be smooth, geometrically irreducible K-varieties, and let $\varphi : X \to Y$ be a K-morphism, let $E \ge 1$ be an integer and let $\varepsilon \in \mathbb{R}_{>0}$. Then:

(1) φ is called *ε-flat* if for every $x \in X$ we have $\dim X_{\varphi(x),\varphi} \le \dim X - \varepsilon \dim Y$.

(2) φ is called *ε-jet flat* (resp. jet-flat) if $J_k(\varphi)$ is ε-flat (resp. flat) for every $k \in \mathbb{N}$.

(3) A jet-flat morphism φ is *E-smooth* if for all $k \in \mathbb{Z}_{\ge 0}$ and all $\tilde{x} \in J_k(X)$, the set $(J_k(X)_{J_k(\varphi)(\tilde{x}),J_k(\varphi)})^{\mathrm{sing}}$ is of codimension at least E in $J_k(X)_{J_k(\varphi)(\tilde{x}),J_k(\varphi)}$.

Remark 2.8. (1) By [34], a morphism φ as in Definition 2.7 is ε-jet flat if and only if $\mathrm{lct}(X, X_{\varphi(x),\varphi}) \ge \varepsilon \dim Y$ for all $x \in X$, where $\mathrm{lct}(X, X_{\varphi(x),\varphi})$ is the log-canonical threshold of the pair $(X, X_{\varphi(x),\varphi})$.

(2) In addition, it follows from [33; 16] (see [20, Corollary 3.12]) that if φ is a normal morphism, then it is jet-flat if and only if it is flat and has fibers with log-canonical singularities.

ε-flatness is a quantitative way to measure how close a morphism between smooth varieties is to being flat. Similarly, ε-jet flatness measures how close a morphism is to being jet-flat, which is very close to being an (FRS)-morphism. On the other hand, the starting point of E-smoothness is when φ is jet-flat, and the larger E is, the better the singularities of φ are. This is illustrated in the next lemma.

Lemma 2.9. *Let $\varphi : X \to Y$ be K-morphism between smooth, geometrically irreducible K-varieties*:

(1) *φ is 1-smooth if and only if φ is (FRS).*

(2) *φ is 2-smooth if and only if φ is flat with fibers of terminal singularities.*

Proof. By Proposition 2.3, φ is (FRS) if and only if $J_k(\varphi)$ is flat, with locally integral fibers for each $k \in \mathbb{N}$. By [20, Corollary 3.12(3)], φ is flat with fibers of terminal singularities if and only if $J_k(\varphi)$ is flat, with normal fibers for each $k \in \mathbb{N}$. In particular, in the situation of (1) and (2), for each k, the map $J_k(\varphi)$ is a flat map between smooth varieties, and thus the fibers of $J_k(\varphi)$ are local complete intersections, and hence Cohen–Macaulay. Serre's criterion for normality and reducedness [14, Proposition 5.8.5, Theorem 5.8.6] and [33, Proposition 1.4] now imply items (1) and (2). □

2B. *Motivic functions.* In this subsection we recall the definition and some properties of motivic functions. In order to prove Theorem A, we encode the collection $\{\#\varphi^{-1}(y)\}_{p,k,y\in Y(\mathbb{Z}/p^k\mathbb{Z})}$ using a single motivic function, and utilize this to obtain the desired uniform bounds. We use the notion of motivic functions as was defined and studied in [7; 8; 9; 10]. In order to fully exploit the advantages of the motivic realm, we introduce the class of *formally nonnegative motivic functions*, which is the specialization to local fields of [7, Section 5.3].

Throughout this subsection, we fix a number field K. We use the (three-sorted) *Denef–Pas language*, denoted

$$\mathcal{L}_{\text{DP}} = (\mathcal{L}_{\text{Val}}, \mathcal{L}_{\text{Res}}, \mathcal{L}_{\text{Pres}}, \text{val}, \text{ac}),$$

where:

(1) The valued field sort VF is endowed with the language of rings $\mathcal{L}_{\text{Val}}$, with coefficients in $\mathcal{O}_K$.

(2) The residue field sort RF is endowed with the language of rings $\mathcal{L}_{\text{Res}}$.

(3) The value group sort VG (which we just call $\mathbb{Z}$), is endowed with the Presburger language $\mathcal{L}_{\text{Pres}} = (+, -, \leq, \{\equiv_{\text{mod } n}\}_{n>0}, 0, 1)$ of ordered abelian groups along with constants 0, 1 and a family of relations $\{\equiv_{\text{mod } n}\}_{n>0}$ of congruences modulo n.

(4) $\text{val} : \text{VF} \setminus \{0\} \to \mathbb{Z}$ and $\text{ac} : \text{VF} \to \text{RF}$ are two function symbols.

Let Loc be the collection of all non-Archimedean local fields F with a ring homomorphism $\mathcal{O}_K \to F$. We denote by Loc_0 (resp. Loc_+) the collection of all $F \in \text{Loc}$ of characteristic zero (resp. positive characteristic). For $F \in \text{Loc}$, we denote by $\mathcal{O}_F$ its ring of integer, by k_F its residue field, and by q_F the number of elements in k_F. We use the notation $\mathrm{Loc}_{\gg}$ (resp. $\mathrm{Loc}_{0,\gg}$, $\mathrm{Loc}_{+,\gg}$),[3] for the collection of $F \in \mathrm{Loc}$ (resp. Loc_0, Loc_+) with large enough residual characteristic (depending on some given data).

Given $F \in \mathrm{Loc}$ (and a chosen uniformizer ϖ_F of $\mathcal{O}_F$), we can interpret val and ac as the valuation map $\mathrm{val} : F^\times \to \mathbb{Z}$ and the angular component map $\mathrm{ac} : F \to k_F$, where $\mathrm{ac}(0) = 0$ and $\mathrm{ac}(x) = x \cdot \varpi_F^{-\mathrm{val}(x)}$ mod $\varpi_F \mathcal{O}_F$ for $x \neq 0$. Hence, any formula ϕ in $\mathcal{L}_{\mathrm{DP}}$ with n_1 free VF-variables, n_2 free RF-variables and n_3 free $\mathbb{Z}$-variables, yields a subset $\phi(F) \subseteq F^{n_1} \times k_F^{n_2} \times \mathbb{Z}^{n_3}$. A collection $X = (X_F)_{F \in \mathrm{Loc}_{\gg}}$ with $X_F = \phi(F)$ is called an $\mathcal{L}_{\mathrm{DP}}$-*definable set*. Given $\mathcal{L}_{\mathrm{DP}}$-definable sets X and Y, an $\mathcal{L}_{\mathrm{DP}}$-*definable function* is a collection $f = (f_F : X_F \to Y_F)_{F \in \mathrm{Loc}_{\gg}}$ of functions whose collection of graphs is a definable set. We will often say "definable" instead of "$\mathcal{L}_{\mathrm{DP}}$-definable".

Definition 2.10 [9, Subsections 4.2.4–4.2.5]. Let X be an $\mathcal{L}_{\mathrm{DP}}$-definable set. A collection $f = (f_F)_{F \in \mathrm{Loc}_{\gg}}$ of functions $f_F : X_F \to \mathbb{R}$ is called a *Presburger constructible function*, if it can be written as

$$f_F(x) = \sum_{i=1}^{N_1} q_F^{\alpha_{i,F}(x)} \prod_{j=1}^{N_2} \beta_{ij,F}(x) \prod_{j=1}^{N_3} \frac{1}{1 - q_F^{a_{ij}}},$$

where $N_1, N_2, N_3 \in \mathbb{N}$ and $a_{ij} \in \mathbb{Z}_{<0}$, and $\alpha_i, \beta_{ij} : X \to \mathbb{Z}$ are definable functions. Given f as above, set $\tilde{f}_F : X_F \times \mathbb{R}_{>1} \to \mathbb{R}$ by

$$\tilde{f}_F(x, s) := \sum_{i=1}^{N_1} s^{\alpha_{i,F}(x)} \prod_{j=1}^{N_2} \beta_{ij,F}(x) \prod_{j=1}^{N_3} \frac{1}{1 - s^{a_{ij}}}.$$

We say that f is *formally nonnegative* if $\tilde{f}_F$ takes nonnegative values for every $F \in \mathrm{Loc}_{\gg}$. We denote by $\mathcal{P}(X)$ the ring of Presburger constructible functions on X, and by $\mathcal{P}_+(X)$ the subsemiring of formally nonnegative functions.

Definition 2.11. Let X be an $\mathcal{L}_{\mathrm{DP}}$-definable set. A collection $h = (h_F)_{F \in \mathrm{Loc}_{\gg}}$ of functions $h_F : X_F \to \mathbb{R}$ is called a *motivic function*, if it can be written as

$$h_F(x) = \sum_{i=1}^{N} \#Y_{i,F,x} \cdot f_{i\,F}(x),$$

where:

- $Y_{i,F,x} = \{\xi \in k_F^{r_i} : (x, \xi) \in Y_{i,F}\}$ is the fiber over $x \in X_F$ of a definable set $Y_i \subseteq X \times \mathrm{RF}^{r_i}$ with $r_i \in \mathbb{N}$.
- Each f_i is a Presburger constructible function.

If furthermore every f_i is formally nonnegative, then we call h a *formally nonnegative motivic function*. We denote by $\mathcal{C}(X)$ the ring of motivic functions on X, and by $\mathcal{C}_+(X)$ the subsemiring of formally nonnegative motivic functions.

[3]Our notation for $\mathrm{Loc}_{\gg}$ is slightly more restrictive than the one used in [11]. Here $\mathrm{Loc}_{\gg}$ consists of $\mathrm{Loc}_{0,\gg} \cup \mathrm{Loc}_{+,\gg}$ while in [11], it consisted of $\mathrm{Loc}_0 \cup \mathrm{Loc}_{+,\gg}$.

The classes $\mathcal{C}(X)$ and $\mathcal{C}_+(X)$ defined above are the specialization to local fields of more abstract classes of motivic functions defined in [7, Section 5]; e.g., see the discussion in [9, Section 4.2]). In [7, Theorem 10.1.1], it is shown that these more general classes are preserved under a formal integration operation, and in [8, Section 9] it is shown that this formal integration operation commutes with usual p-adic integration under specialization. This implies the following theorem:

Theorem 2.12. *Let X be an $\mathcal{L}_{\mathrm{DP}}$-definable set, and let f be in $\mathcal{C}_+(X \times \mathrm{VF}^m)$. Assume that for every $F \in \mathrm{Loc}_{\gg}$ and every $x \in X_F$, the function $y \mapsto f_F(x, y)$ belongs to $L^1(F^m)$. Then there exists g in $\mathcal{C}_+(X)$ such that*

$$g_F(x) = \int_{y \in F^m} f_F(x, y)|dy|. \tag{2-1}$$

Remark 2.13. In [9, Theorem 4.3.1] it was shown that the class of motivic functions is preserved under integration in the following stronger sense, namely, that given f in $\mathcal{C}(X \times \mathrm{VF}^m)$, one can find $g \in \mathcal{C}(X)$ such that for every $F \in \mathrm{Loc}_{\gg}$ and $x \in X_F$, if $y \mapsto f_F(x, y)$ belongs to $L^1(F^m)$ then (2-1) holds. This stronger statement relies on an interpolation theorem [9, Theorem 4.3.3] for functions in $\mathcal{C}(X)$. It would be interesting to prove a similar interpolation result for the class of formally nonnegative motivic functions. This will imply the stronger formulation of Theorem 2.12 as in [9, Theorem 4.3.1].

Finally, we need the following transfer result between $\mathrm{Loc}_{0,\gg}$ and $\mathrm{Loc}_{+,\gg}$.

Theorem 2.14 (transfer principle for bounds, [10, Theorem 3.1]). *Let X be an $\mathcal{L}_{\mathrm{DP}}$-definable set, and let $H, G \in \mathcal{C}(X)$ be motivic functions. Then the following holds for $F \in \mathrm{Loc}_{\gg}$; if*

$$|H_F(x)| \le |G_F(x)|,$$

for each $x \in X_F$, then also

$$|H_{F'}(x)| \le |G_{F'}(x)|,$$

for every $F' \in \mathrm{Loc}$ with the same residue field as F, and each $x \in X_{F'}$.

3. An improvement of the approximation of suprema

The main goal of this section is to show the following improvement of [11, Theorem 2.1.3] on approximate suprema. This improvement is made possible by placing ourselves in the special case of formally nonnegative motivic functions and is not possible in the more general situation of [11].

Theorem 3.1 (improved approximation of suprema). *Let f be in $\mathcal{C}_+(X \times W)$, where X and W are definable sets. Then there exist a constant $C > 0$, and a function $G \in \mathcal{C}_+(X)$ such that for any $F \in \mathrm{Loc}_{\gg}$ and any $x \in X_F$ such that $w \mapsto f_F(x, w)$ is bounded on W_F, we have*

$$\sup_{w \in W_F} f_F(x, w) \le G_F(x) \le C \cdot \sup_{w \in W_F} f_F(x, w).$$

The following lemma is immediate:

Lemma 3.2. *Let $\{f_i\}_{i=1}^N$ be in $\mathcal{C}_+(X \times W)$ and set $f = \sum_{i=1}^2 f_i$. Then for $F \in \mathrm{Loc}_{\gg}$, one has*:

$$\frac{1}{N}\sum_{i=1}^{N} \sup_{w\in W_F} f_{i\,F}(x,w) \le \sup_{w\in W_F} f_F(x,w) \le \sum_{i=1}^{N} \sup_{w\in W_F} f_{i\,F}(x,w).$$

Let f be in $\mathcal{C}_+(X \times W)$. By Definition 2.11, we can write $f(x,w) = \sum_{i=1}^2 \#Y_{i,x,w}\cdot g_i(x,w)$, where $g_i \in \mathcal{P}_+(X\times W)$ and $Y_i \subseteq X \times W \times \mathrm{RF}^{r_i}$. Lemma 3.2 thus implies the following:

Corollary 3.3. *Let f be in $\mathcal{C}_+(X \times W)$, where X and W are definable sets*:

(1) *Let $X \times W = \bigsqcup_{i=1}^M C_i$ be a definable partition and set $f_i(x,w) = f(x,w)\cdot 1_{C_i}$. Then it is enough to prove Theorem 3.1 for each f_i.*

(2) *It is enough to prove Theorem 3.1 for f of the form $f = \#Y_{x,w}\cdot g(x,w)$ where $g \in \mathcal{P}_+(X\times W)$.*

Remark 3.4. The key case of Theorem 3.1 is when neither X nor W involve valued field variables. The reduction to this case needs to be done with care. Naively, one can use quantifier elimination to eliminate the valued field variables, but this is problematic since it mixes the valued field variables of X and W, making it hard to take supremum over the variables of W. In order to elude this problem, we will apply cell decomposition iteratively, first taking care of the W variables and then taking care of the X variables.

Proof of Theorem 3.1. Let $f(x,w) = \#Y_{x,w}\cdot g(x,w)$ for some $g \in \mathcal{P}_+(X\times W)$ and $Y \subseteq X\times W\times \mathrm{RF}^r$. Without loss of generality, we may assume that $X = \mathrm{VF}^{n_1}\times\mathrm{RF}^{n_2}\times\mathrm{VG}^{n_3}$ and $W = \mathrm{VF}^{m_1}\times\mathrm{RF}^{m_2}\times\mathrm{VG}^{m_3}$ for some $n_i \ge 0$ and $m_i \ge 0$. We will first reduce to the case where there are no valued field variables, using the following claim.

Claim 1. *We may assume that $X = \mathrm{RF}^{n_2}\times\mathrm{VG}^{n_3}$ and $W = \mathrm{RF}^{m_2}\times\mathrm{VG}^{m_3}$.*

Proof of Claim 1. We first get rid of the valued field variables VF^{m_1} of W. Without loss of generality we may assume that $W = \mathrm{VF}^{m_1}$. By induction, we may further assume that $m_1 = 1$. By [7, Theorem 7.2.1] there exists a definable surjection $\lambda : X \times W \to C \subseteq X \times \mathrm{RF}^s \times \mathbb{Z}^r$ over X as well as $\psi \in \mathcal{C}_+(C)$ such that $f = \psi\circ\lambda$. Note that

$$\sup_{w\in W_F} f_F(x,w) = \sup_{w\in W_F} \psi_F\circ\lambda_F(x,w) = \sup_{(\xi,k)\in k_F^s\times\mathbb{Z}^r} \psi_F(x,\xi,k),$$

up to extending ψ by zero outside C. We may therefore assume that $W = \mathrm{RF}^{m_2}\times\mathrm{VG}^{m_3}$. We next get rid of the valued field variables VF^{n_1} of X, denoted $y := y_1,\ldots,y_{n_1}$. Write $x = (y,\eta,t)\in X$ and $w = (\xi,s)\in W$, with RF-variables η,ξ and VG-variables t,s. By Definition 2.11, f is determined by a finite collection $\alpha_i,\beta_{ij} : X\times W \to \mathbb{Z}$ of definable functions, and by a definable set $Y \subseteq X\times W\times\mathrm{RF}^r$. By quantifier elimination in the valued field variables [36, Theorem 4.1], there exist finitely many polynomials $g_1,\ldots,g_l \in \mathbb{Z}[y_1,\ldots,y_{n_1}]$ such that the graphs of the functions in $\{\alpha_i,\beta_{ij}\}$ can be defined by formulas of the form

$$\bigvee_{i=1}^{L} \chi_i(\xi,\eta,\mathrm{ac}(g_1(y)),\ldots,\mathrm{ac}(g_l(y)))\wedge\theta_i(t,s,t',\mathrm{val}(g_1(y)),\ldots,\mathrm{val}(g_l(y))),$$

and the subset Y can be defined by a formula of the form

$$\bigvee_{i=1}^{L'} \tilde{\chi}_i(\xi, \eta, \xi', \mathrm{ac}(g_1(y)), \dots, \mathrm{ac}(g_l(y))) \wedge \tilde{\theta}_i(t, s, \mathrm{val}(g_1(y)), \dots, \mathrm{val}(g_l(y))),$$

where χ_i and $\tilde{\chi}_i$ are $\mathcal{L}_{\mathrm{Res}}$-formulas, θ_i and $\tilde{\theta}_i$ are $\mathcal{L}_{\mathrm{Pres}}$-formulas, t' is in $\mathbb{Z}$ and ξ' is in RF^r. We now define $\lambda' : X \times W \to \mathrm{RF}^{s'} \times \mathbb{Z}^{r'} \times W$ by $\lambda'(x, w) = (\rho(x), w)$, where $s' := n_2 + l$, $r' := n_3 + l$ and where

$$\rho(x) = \rho(y, \eta, t) := (\eta, \mathrm{ac}(g_1(y)), \dots, \mathrm{ac}(g_l(y)), t, \mathrm{val}(g_1(y)), \dots, \mathrm{val}(g_l(y))).$$

Let C' be the image of λ'. Note we may find definable functions $\tilde{\alpha}_i, \tilde{\beta}_{ij} : C' \to \mathbb{Z}$ and a definable subset $\widetilde{Y} \subseteq C' \times \mathrm{RF}^r$ such that $\alpha_i = \tilde{\alpha}_i \circ \lambda'$, $\beta_i = \tilde{\beta}_{ij} \circ \lambda'$ and $Y = (\lambda' \times \mathrm{Id})^{-1}(\widetilde{Y})$. Using this new definable data, we construct $\psi' \in \mathcal{C}_+(C')$ such that $f = \psi' \circ \lambda'$ and again we have

$$\sup_{w \in W_F} f_F(x, w) = \sup_{w \in W_F} \psi'_F \circ \lambda'_F(x, w) = \sup_{w \in W_F} \psi'_F(\rho(x), w).$$

Hence we have reduced to the case where $X = \mathrm{RF}^{n_2} \times \mathrm{VG}^{n_3}$. This finishes the proof of Claim 1. □

Claim 2. *We may assume that* $X = \mathrm{RF}^{n_2} \times \mathrm{VG}^{n_3}$ *and* $W = \mathrm{RF}^{m_2}$.

Proof of Claim 2. Write $x = (\eta, t)$ and $w = (\xi, s)$ for the variables of $X = \mathrm{RF}^{n_2} \times \mathrm{VG}^{n_3}$ and $W = \mathrm{RF}^{m_2} \times \mathrm{VG}^{m_3}$. We would like to get rid of the value group variables VG^{m_3} of W. Using the (model theoretic) orthogonality of the sorts VG and RF, there is a definable partition of $X \times W$, such that each definable part A is a box $A_1 \times A_2$ with $A_1 \subseteq \mathrm{RF}^{n_2} \times \mathrm{RF}^{m_2}$ and $A_2 \subseteq \mathrm{VG}^{n_3} \times \mathrm{VG}^{m_3}$, and such that on each A, f has the form

$$f_F|_{A_F}(\eta, t, \xi, s) = \#Y_{\eta,\xi} \cdot H_F(t, s),$$

for some $H \in \mathcal{P}_+(\mathrm{VG}^{n_3} \times \mathrm{VG}^{m_3})$ and $Y \subseteq \mathrm{RF}^{n_2} \times \mathrm{RF}^{m_2} \times \mathrm{RF}^r$. By Corollary 3.3, and by our assumption on A, we may assume $f_F = \#Y_{\xi,\eta} \cdot H_F(t, s)$. Note that for each $F \in \mathrm{Loc}_{\gg}$ and each $(\eta, t, \xi) \in X_F \times k_F^{m_2}$ one has

$$\sup_{s \in \mathbb{Z}^{m_3}} f_F(\eta, t, \xi, s) = \#Y_{\eta,\xi} \cdot \sup_{s \in \mathbb{Z}^{m_3}} H_F(t, s).$$

In order to approximate $\sup_{s \in \mathbb{Z}^{m_3}} H_F(t, s)$, it is enough to consider the case where $m_3 = 1$ and proceed by induction on m_3. Using Presburger cell decomposition and rectilinearization (see [6, Theorems 1 and 3]) we may assume that H is in $\mathcal{P}_+(B)$ for $B \subseteq \mathrm{VG}^{n_3} \times \mathbb{N}$ with $B_t := \{s \in \mathbb{N} : (t, s) \in B\}$, such that exactly one of the following holds:

(1) B_t is a finite set for each $t \in \mathbb{Z}^{n_3}$.

(2) $B_t = \mathbb{N}$ or $B_t = \varnothing$ for each $t \in \mathbb{Z}^{n_3}$.

Moreover, H can be taken to be of the form

$$H_F(t, s) = \sum_{i=1}^{N} c_{i,F}(t) s^{a_i} q_F^{b_i s},$$

with $a_i \in \mathbb{N}$ and $b_i \in \mathbb{Z}$ and c_i in $\mathcal{P}(\mathrm{VG}^{n_3})$. Denote by T the image of projection of B to VG^{n_3}. We repeat a part of the argument of the proof of [11, Theorem 2.1.3]. Namely, by [11, Lemmas 2.2.3 and 2.2.4], there exist $m, l \in \mathbb{N}_{\geq 1}$ and finitely many definable functions $h_1, \dots, h_l : T \to \mathbb{N}$ with $h_j(t) \in B_t$ such that for each $t \in T$ for which $s \mapsto H_F(t, s)$ is bounded on B_t, one has

$$\sup_{s \in B_t} H_F(t, s) \leq m \cdot \max_{1 \leq j \leq l} H_F(t, h_j(t)).$$

In particular, setting

$$\widetilde{H}(t) := m \cdot \sum_{j=1}^{l} H(t, h_j(t)) \in \mathcal{P}_+(T)$$

we get

$$\sup_{s \in B_t} H_F(t, s) < \widetilde{H}_F(t) < m \cdot l \cdot \sup_{s \in B_t} H_F(t, s).$$

This finishes the proof of Claim 2. □

Claim 3. *We may assume that* $X = \mathrm{RF}^{n_2}$ *and* $W = \mathrm{RF}^{m_2}$.

Proof. This follows directly by Claim 2, Corollary 3.3, and using the orthogonality of the sorts VG and RF. □

To continue the proof of Theorem 3.1, we may thus assume that $X = \mathrm{RF}^{n_2}$ and $W = \mathrm{RF}^{m_2}$. We may assume, again using Corollary 3.3, that f is of the form $f(x, w) = f(\eta, \xi) = u \cdot \#Y_{\eta,\xi}$, with ξ the coordinate on W, and η on X and $u = \{u_F\}_{F \in \mathrm{Loc}_{\gg}}$ is a motivic number. In particular, for each $\eta \in X_F$,

$$\sup_{w \in W_F} f_F(x, w) = \sup_{\xi \in k_F^{m_2}} f_F(\eta, \xi) = u_F \cdot \sup_{\xi \in k_F^{m_2}} \#Y_{\eta,\xi}.$$

By a definable variant of the Lang–Weil estimates (see [5, main theorem]), there exists a definable partition $X \times W = \bigsqcup_{i=0}^{2} A_i$ and constants $C' > 0$, $d_i \in \mathbb{N}$ and $l_{i1}, l_{i2} \in \mathbb{Z}_{\geq 1}$, such that for each $1 \leq i \leq M$ and each $F \in \mathrm{Loc}_{\gg}$

$$A_{i,F} := \left\{ (\eta, \xi) \in X_F \times W_F : \left| \#Y_{\eta,\xi} - \frac{l_{i1}}{l_{i2}} q_F^{d_i} \right| \leq C' \cdot q_F^{d_i - 1/2} \right\},$$
$$A_{0,F} := \{(\eta, \xi) \in X_F \times W_F : Y_{\eta,\xi} \text{ is empty}\}.$$

Denote by Z_i the projection of A_i to X. For each subset $I \subseteq \{1, \dots, M\}$, let

$$Z_I := \bigcap_{i \in I} Z_i \backslash \bigcup_{j \in I^c} Z_j, \quad \text{with } Z_\varnothing := X \backslash \bigcup_{j=1}^{2} Z_j.$$

Then $X = \bigsqcup_I Z_I$ is a definable partition, and thus we may assume that $X = Z_I$. In this case we have, for $F \in \mathrm{Loc}_{\gg}$,

$$\begin{aligned}\sup_{w\in W_F} f_F(x,w) &= u_F \cdot \sup_{\xi\in k_F^{m_2}} \#Y_{\eta,\xi}\\ &\le u_F \cdot \sum_{i\in I} \sup_{\xi\in k_F^{m_2}} (1_{A_{i,F}} \cdot \#Y_{\eta,\xi})\\ &\le u_F \cdot \sum_{i\in I} 2l_{i1}\cdot q_F^{d_i}\\ &\le u_F \cdot \sum_{i\in I} 4l_{i2}\cdot \sup_{\xi\in k_F^{m_2}} (1_{A_{i,F}} \cdot \#Y_{\eta,\xi})\\ &\le \Big(\sum_{i\in I} 4l_{i2}\Big) \sup_{w\in W_F} f_F(x,w),\end{aligned}$$

where we take zero for the supremum of the empty set. Since $\{u_F \cdot \sum_{i\in I} 2l_{i1}\cdot q_F^{d_i}\}_{F\in\mathrm{Loc}_{\gg}}$ clearly lies in $\mathcal{C}_+(X)$, this finishes the proof of Theorem 3.1. □

3A. *Optimality of the bounds and further remarks.* Let X and W be $\mathcal{L}_{\mathrm{DP}}$-definable sets. Given a subclass $\mathcal{F} \subseteq \mathcal{C}(X\times W)$ of motivic functions, one can ask whether for any $f \in \mathcal{F}$, the function $\{\sup_{w\in W_F} f_F(x,w)\}_{F\in\mathrm{Loc}_{\gg}}$ can be approximated by a motivic function in $\mathcal{C}(X)$ up to a constant C in up to four increasing levels of approximation:

(1) With C depending on F and f.

(2) With C depending on f and independent of F.

(3) With C a universal constant, that is, uniform over all $f \in \mathcal{F}$ and $F \in \mathrm{Loc}_{\gg}$.

(4) With $C = 1 + C' q_F^{-1/2}$ for some C' depending on f and independent of F.

If the class $\mathcal{F}$ satisfies one of the items (i) above, we say that $\mathcal{F}$ admits *an approximation of suprema of type* (i), or $\mathcal{F}$ *is of type* (i). Note that if $\mathcal{F}$ is of type (4) then it is also of type (3), as $C' q_F^{-1/2} < 2$ for $F \in \mathrm{Loc}_{\gg}$. Similarly, type (i) is stronger than type (j) for $j < i$.

Remark 3.5. • The class $\mathcal{C}(X\times W)$ is not of type (1) (and thus of any type). Indeed, take $X = \mathbb{Z}^2$, $W = \{1,2\} \subseteq \mathbb{Z}$ and define $f \in \mathcal{C}(X\times W)$ by $f(x,y,1) = 0$ and $f(x,y,2) = x^2 - y$. Then $\sup_{\,w} f(x,y,w) = \max(0, x^2-y)$ cannot be approximated by a motivic function $g \in \mathcal{C}(X)$ up to a constant depending on F and f. Indeed, such g satisfies $g_F(x,y) = 0$ if and only if $x^2 \le y$, for $F \in \mathrm{Loc}_{\gg}$. For each fixed $F \in \mathrm{Loc}_{\gg}$, the function g_F agrees with a Presburger function on $\mathbb{Z}^2$. By Presburger cell decomposition [6, Theorem 1], we can decompose $\mathbb{Z}^2$ into cells $\mathbb{Z}^2 = \bigsqcup_{i=1}^N C_i$, such that $g_F(x,y)|_{C_i} = \sum_{j=1}^{N_i} c_{ij}(F) q_F^{a_{ij}x+b_{ij}y} x^{k_{ij}} y^{l_{ij}}$, with $a_{ij}, b_{ij} \in \mathbb{Q}$, $k_{ij}, l_{ij} \in \mathbb{N}$ and $c_j(F) \in \mathbb{R}$. Since $Z := \{(x,y) \in \mathbb{Z}^2 : x^2 \le y\} \subseteq \mathbb{Z}^2$ is not Presburger definable, and by the definition of a cell [6, Definiton 2], there is $1 \le i_0 \le N$ such that for some $x_0 \in \mathbb{Z}$, $|C_{i_0} \cap \{(x,y) \in Z : x = x_0\}| = \infty$ and $(x_0, y_0) \in C_{i_0} \cap Z^c \neq \varnothing$ for some $y_0 \in \mathbb{Z}$. Applying [9, Lemma 2.1.7], we get $g_F(x,y)|_{C_{i_0}\cap\{x=x_0\}} \equiv 0$, and thus $g_F(x_0,y_0) = 0$ where $x_0^2 > y_0$, yielding a contradiction.

• The class $\mathcal{C}_+^{\text{weak}}(X\times W) := \{f \in \mathcal{C}(X\times W) : f_F \geq 0\, \forall F \in \text{Loc}_{\gg}\}$ is of type (1), with $C = q_F^{C_0}$ for some $C_0 > 0$ depending only on f. This is a special case treated in the proof of [11, Theorem 2.1.3]. One may wonder whether the class $\mathcal{C}_+^{\text{weak}}(X\times W)$ is of type (2).

Theorem 3.1 shows that the family $\mathcal{C}_+(X\times W)$, which is strictly contained in $\mathcal{C}_+^{\text{weak}}(X\times W)$, is of type (2). The following proposition, which we prove in the Appendix, shows this is the best possible approximation, as already detected by the subclass $\mathcal{P}_+(X\times W)\subseteq \mathcal{C}_+(X\times W)$.

Proposition 3.6. *The families $\mathcal{P}_+(X\times W)$ and $\mathcal{C}_+(X\times W)$ are not of type* (3).

In [11, Theorem 2.1.3], an approximation of suprema result is proven for a more general class $\mathcal{C}^{\exp}(X\times W)$ of motivic exponential functions, which involves additive characters, and which is furthermore built out of functions which are definable in the generalized Denef–Pas language. Due to this larger generality, the approximation shown in [11, Theorem 2.1.3] is a bit weaker than type (1) above; in [11], one approximates $\sup|f|^2$ instead of $\sup f$. This is unavoidable, as already seen in Remark 3.5.

Remark 3.7. One can weaken the definition of approximation as follows. For a function $f \in \mathcal{C}_+(X\times W)$, assume there exist motivic functions $\{g_i\}_{i=1}^m \in \mathcal{C}_+(X)$, with $m\in\mathbb{N}$ such that

$$\max_{1\leq i\leq m}\{g_{i F}(x)\} \leq \sup_{w\in W_F} f_F(x,w) \leq C\cdot \max_{1\leq i\leq m}\{g_{i F}(x)\},$$

where C is as in types (1)-(4) above. Using this weaker form of approximation, we expect $\mathcal{C}_+(X\times W)$ to be of weakened type (4).

One may also weaken (3) by letting the constant C depend on the number of variables running over $X\times W$, and wonder whether $\mathcal{C}_+(X\times W)$ is of type (3) when weakened in this sense.

4. Number theoretic characterization of the (FRS) property

Throughout this section we use the notation of Section 2B. In particular, K is a fixed number field with ring of integers $\mathcal{O}_K$, and Loc denotes the collection of all non-Archimedean local fields F with a ring homomorphism $\mathcal{O}_K\to F$.

We next apply Theorem 3.1 to prove a more general form of Theorem A for $\text{Loc}_{\gg}$, providing a full number theoretic characterization of (FRS) morphisms (Theorem 4.7).

4A. ***An analytic characterization of the (FRS) property.*** Given an $\mathcal{O}_F$-morphism $\varphi: X\to Y$, we denote the natural maps $X(\mathcal{O}_F/\mathfrak{m}_F^k)\to Y(\mathcal{O}_F/\mathfrak{m}_F^k)$ by φ, therefore $\varphi^{-1}(\bar{y})$ is a finite set in $X(\mathcal{O}_F/\mathfrak{m}_F^k)$, for any $\bar{y}\in Y(\mathcal{O}_F/\mathfrak{m}_F^k)$. We denote by $r_{k,Y}: Y(\mathcal{O}_F)\to Y(\mathcal{O}_F/\mathfrak{m}_F^k)$ and by $r_{l,Y}^k: Y(\mathcal{O}_F/\mathfrak{m}_F^k)\to Y(\mathcal{O}_F/\mathfrak{m}_F^l)$ the natural reduction maps for $k\geq l$. When the scheme Y is clear from the context, we omit it from our notation.

Definition 4.1. Let Y be a smooth F-variety, with $F\in\text{Loc}$. A measure μ on $Y(F)$ is called:

(1) *Smooth* if for any $y\in Y(F)$ there exists an analytic neighborhood $U\subseteq Y(F)$ and an analytic diffeomorphism $\psi: U\to\mathcal{O}_F^{\dim Y}$ such that $\psi_*\mu$ is a Haar measure on $\mathcal{O}_F^{\dim Y}$.

(2) *Schwartz* if it is compactly supported and smooth.

Lemma 4.2 [37; 40; 35]. *Let F be in* Loc, *and let Y be a finite type $\mathcal{O}_F$-scheme such that $Y \times_{\operatorname{Spec} \mathcal{O}_F} \operatorname{Spec} F$ is smooth, of pure dimension d. Then there is a unique Schwartz measure $\mu_{Y(\mathcal{O}_F)}$ on $Y(\mathcal{O}_F)$, and there exists $k_0 \in \mathbb{N}$, such that for every $k \geq k_0$ and every $\bar{y} \in Y(\mathcal{O}_F/\mathfrak{m}_F^k)$, one has*

$$\mu_{Y(\mathcal{O}_F)}(r_k^{-1}(\bar{y})) = q_F^{-kd}. \tag{4-1}$$

*The measure $\mu_{Y(\mathcal{O}_F)}$ is referred to as the **canonical measure** on $Y(\mathcal{O}_F)$. In the special case when Y is smooth over $\mathcal{O}_F$, then (4-1) holds for every $k \geq 1$.*

Proof. If Y is affine, then the existence and uniqueness of $\mu_{Y(\mathcal{O}_F)}$ follows from [35, Lemma 3], building on [37, Theorem 9]. In general, let $Y = \bigcup_{i=1}^N U_i$ be an open affine cover by $\mathcal{O}_F$-subschemes U_i. Then $Y(\mathcal{O}_F) = \bigcup_{i=1}^N U_i(\mathcal{O}_F)$. Note that

$$\mu_{U_i(\mathcal{O}_F)}|_{U_i(\mathcal{O}_F)\cap U_j(\mathcal{O}_F)} = \mu_{U_j(\mathcal{O}_F)}|_{U_i(\mathcal{O}_F)\cap U_j(\mathcal{O}_F)}$$

by uniqueness, so we can glue them together to form $\mu_{Y(\mathcal{O}_F)}$. If furthermore Y is smooth over $\mathcal{O}_F$, then by applying Hensel's lemma to (4-1) we can choose $k_0 = 1$; see also [40, Theorem 2.25]. □

Aizenbud and Avni [1] gave an analytic characterization of the (FRS) property:

Theorem 4.3 [1, Theorem 3.4]. *Let $\varphi : X \to Y$ be a map between smooth K-varieties. Then the following are equivalent*:

(1) *φ is (FRS).*

(2) *For any $F \in \mathrm{Loc}_0$ and any Schwartz measure μ on $X(F)$, the measure $\varphi_*(\mu)$ has continuous density with respect to any smooth, nonvanishing measure on $Y(F)$.*

(3) *For any $x \in X(\bar{K})$ and any finite extension K'/K with $x \in X(K')$, there exists $F \in \mathrm{Loc}_0$ containing K', and a nonnegative Schwartz measure μ on $X(F)$ that does not vanish at x such that $\varphi_*(\mu)$ has continuous density with respect to any smooth, nonvanishing measure on $Y(F)$.*

The next result shows a variant of the above characterization holds for local fields of large positive characteristic.

Corollary 4.4. *Let $\varphi : X \to Y$ be a map between smooth K-varieties. Then φ is (FRS) if and only if for every $F \in \mathrm{Loc}_{\gg}$, the measure $\varphi_*(\mu_{X(\mathcal{O}_F)})$ has bounded density with respect to $\mu_{Y(\mathcal{O}_F)}$.*

Proof. Without loss of generality, we may assume that Y is affine. By choosing an $\mathcal{O}_K$-model of Y, we may identify it as an $\mathcal{L}_{\mathrm{DP}}$-definable set. Assume φ is (FRS). For each $F \in \mathrm{Loc}_{\gg}$, write $\tau_F := \varphi_*(\mu_{X(\mathcal{O}_F)})$. By [1, Theorem 3.4(2)], we can write $\tau_F = f_F \cdot \mu_{Y(\mathcal{O}_F)}$ and f_F is continuous, for each $F \in \mathrm{Loc}_{0,\gg}$. Moreover, locally, f can be written as an integral of a motivic function G in $\mathcal{C}_+(Y \times \mathrm{VF}^{\dim X - \dim Y})$, over $\mathrm{VF}^{\dim X - \dim Y}$. By [9, Theorem 4.4.1], it follows that $G_F(y, \cdot)$ is integrable, for each $F \in \mathrm{Loc}_{\gg}$ and $y \in Y(\mathcal{O}_F)$. By Theorem 2.12, f is in $\mathcal{C}_+(Y)$.

Since $Y(\mathcal{O}_F)$ is compact and f_F is continuous, f_F is bounded on $Y(\mathcal{O}_F)$ for each $F \in \mathrm{Loc}_{0,\gg}$. By [38, Appendix B, Theorem 14.6] (or more generally, by [11, Theorem 2.1.2]) there exists $a \in \mathbb{Z}$, such that for each $F \in \mathrm{Loc}_{0,\gg}$ and each $y \in Y(\mathcal{O}_F)$, one has $f_F(y) < q_F^a$. By Theorem 2.14 we thus have $f_F(y) < q_F^a$ for each $F \in \mathrm{Loc}_{\gg}$ and each $y \in Y(\mathcal{O}_F)$, as required. The other direction follows from Theorem 4.3 combined with [19, Lemma 3.15], as in the proof of [19, Proposition 3.16]. □

4B. ***A number-theoretic characterization of the (FRS) property.*** We now recall the Lang–Weil estimates, and set the required notation to state the main theorem.

Definition 4.5. (1) For a finite type $\mathbb{F}_q$-scheme Z, we denote by C_Z the number of its top-dimensional geometrically irreducible components which are defined over $\mathbb{F}_q$.

(2) Let $\varphi : X \to Y$ be a morphism between finite type $\mathbb{Z}$-schemes X and Y, and let $y \in Y(\mathbb{F}_q)$. Then we write $C_{X,q}:=C_{X_{\mathbb{F}_q}}$ and $C_{\varphi,q,y} := C_{(X_{\mathbb{F}_q})_{y,\varphi}}$.

Theorem 4.6 (the Lang–Weil estimates [30]). *For every $M \in \mathbb{N}$, there exists $C(M) > 0$, such that for every prime power q, and any finite type $\mathbb{F}_q$-scheme X of complexity at most M (see, e.g.,* [19, Definition 7.7]*), one has*

$$\left| \frac{\#X(\mathbb{F}_q)}{q^{\dim X}} - C_X \right| < C(M)q^{-1/2}.$$

Let X, Y be finite type $\mathcal{O}_K$-schemes, with X_K, Y_K smooth and geometrically irreducible, and let $\varphi : X \to Y$ be a dominant morphism. Let $\mu_{X(\mathcal{O}_F)}$ and $\mu_{Y(\mathcal{O}_F)}$ be the canonical measures on $X(\mathcal{O}_F)$ and $Y(\mathcal{O}_F)$ for $F \in \mathrm{Loc}$. Since φ is dominant, it follows that $\tau_F := \varphi_*(\mu_{X(\mathcal{O}_F)})$ is absolutely continuous with respect to $\mu_{Y(\mathcal{O}_F)}$, and thus has an L^1-density (see, e.g., [1, Corollary 3.6]), so that $\tau_F = f_F(y) \cdot \mu_{Y(\mathcal{O}_F)}$. When Y is affine, the collection $f = \{f_F : Y(\mathcal{O}_F) \to \mathbb{C}\}_{F \in \mathrm{Loc}_{\gg}}$ can be chosen to be a formally nonnegative motivic function. Indeed, as in the proof of Corollary 4.4, locally, f_F can be written as an integral of a motivic function G in $\mathcal{C}_+(Y \times \mathrm{VF}^{\dim X - \dim Y})$, over $\mathrm{VF}^{\dim X - \dim Y}$. Note there is an open affine subscheme U of Y, such that φ_K is smooth over U_K. Then $G_F(y, \cdot)$ is integrable for every $y \in U(F)$ and $F \in \mathrm{Loc}_{\gg}$. By Theorem 2.12 it follows that $f|_U$ is formally nonnegative. Since $U(F)$ is dense in $Y(F)$ for $F \in \mathrm{Loc}_{\gg}$, by extending $f|_U$ by 0 we get a collection of densities on $\{Y(\mathcal{O}_F)\}_{F \in \mathrm{Loc}_{\gg}}$ which is formally nonnegative.

For $F \in \mathrm{Loc}_{\gg}$, define a function g_F for $y \in Y(\mathcal{O}_F)$ and $k \in \mathbb{Z}_{\geq 1}$ by

$$g_F(y,k) = \frac{1}{\mu_{Y(\mathcal{O}_F)}(B(y,k))} \int_{\tilde{y} \in B(y,k)} f_F(\tilde{y}) \mu_{Y(\mathcal{O}_F)},$$

where $B(y,k) = r_k^{-1}(r_k(y))$. By Theorem 2.12, it follows that $\{g_F : Y(\mathcal{O}_F) \times \mathbb{Z}_{\geq 1} \to \mathbb{C}\}_{F \in \mathrm{Loc}_{\gg}}$ is a formally nonnegative motivic function.

For every $F \in \mathrm{Loc}_{\gg}$, every $y \in Y(\mathcal{O}_F)$ and every $k \in \mathbb{Z}_{\geq 1}$, we have

$$g_F(y,k) = \frac{\varphi_*(\mu_{X(\mathcal{O}_F)})(B(y,k))}{\mu_{Y(\mathcal{O}_F)}(B(y,k))} = \frac{\#\varphi^{-1}(r_k(y))}{q_F^{k(\dim X_K - \dim Y_K)}}, \tag{4-2}$$

where the last equality follows from Lemma 4.2, and the fact that Y is smooth over $\mathcal{O}_F$ for $F \in \mathrm{Loc}_{\gg}$. Set

$$h_F(y,k) = \frac{\#(\varphi^{-1}(r_k(y)) \cap (r_{1,X}^k)^{-1}(X^{\mathrm{sing},\varphi}(k_F)))}{q_F^{k(\dim X_K - \dim Y_K)}}.$$

The asymptotics of the functions h and g, in q_F and k, measure how wild the singularities of φ are. For example, if φ_K is smooth, then $h_F(y,k) \equiv 0$ and $g_F(y,k) < C$ for $F \in \mathrm{Loc}_{\gg}$ and some constant C. On the other hand, if $\varphi : \mathbb{A}^1 \to \mathbb{A}^1$ is the map $x \mapsto x^m$, then $g(0,k) = h(0,k) = q_F^{k-\lceil k/m \rceil}$.

Furthermore, $\{h_F\}_{F \in \mathrm{Loc}_{\gg}}$ is a formally nonnegative motivic function (Proposition 4.9). This is used to prove our main theorem, which we state now.

Theorem 4.7. *Let $\varphi : X \to Y$ be a dominant morphism between finite type $\mathcal{O}_K$-schemes X and Y, with X_K, Y_K smooth and geometrically irreducible. Then the following are equivalent*:

(1) *$\varphi_K : X_K \to Y_K$ is (FRS).*

(2) *There exists $C_1 > 0$, such that for each $F \in \mathrm{Loc}_{\gg}$, $k \in \mathbb{Z}_{\geq 1}$ and $y' \in Y(\mathcal{O}_F)$*
$$h_F(y',k) < C_1 q_F^{-1}.$$

(3) *There exists $C_2 > 0$ such that for each $F \in \mathrm{Loc}_{\gg}$, $k \in \mathbb{Z}_{\geq 1}$ and $y \in Y(\mathcal{O}_F/\mathfrak{m}_F^k)$*
$$\left| \frac{\#\varphi^{-1}(y)}{q_F^{k(\dim X_K - \dim Y_K)}} - \frac{\#\varphi^{-1}(r_1^k(y))}{q_F^{\dim X_K - \dim Y_K}} \right| < C_2 q_F^{-1}.$$

(4) *There exists $C_3 > 0$ such that for each $F \in \mathrm{Loc}_{\gg}$, $k \in \mathbb{Z}_{\geq 1}$ and $y \in Y(\mathcal{O}_F/\mathfrak{m}_F^k)$*
$$\left| \frac{\#\varphi^{-1}(y)}{q_F^{k(\dim X_K - \dim Y_K)}} - C_{\varphi, q_F, r_1^k(y)} \right| < C_3 q_F^{-1/2}.$$

(5) *There exists $C_4 > 0$ such that for each $F \in \mathrm{Loc}_{\gg}$, $\varphi_*(\mu_{X(\mathcal{O}_F)})$ has continuous density f_F with respect to $\mu_{Y(\mathcal{O}_F)}$, and for each $y' \in Y(\mathcal{O}_F)$, one has*
$$|f_F(y') - C_{\varphi, q_F, r_1(y')}| < C_4 q_F^{-1/2}.$$

Before we prove Theorem 4.7, we first show it implies Theorem A.

Proof of Theorem A. We prove (1) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (4) $\Rightarrow$ (1). To prove (3) $\Rightarrow$ (2), we first treat large primes using implication (3) $\Rightarrow$ (4) of Theorem 4.7, and then treat small primes using the Lang–Weil estimates. Implication (4) $\Rightarrow$ (1) follows from Theorem 4.3.

Let us assume that condition (2) holds. By Lemma 4.2, and by condition (2), there exists $C_1 > 0$, such that for every prime p, every $y \in Y(\mathbb{Z}/p^k\mathbb{Z})$ and every $k \geq k_0$, one has
$$\frac{\varphi_*\mu_{X(\mathbb{Z}_p)}(r_k^{-1}(y))}{\mu_{Y(\mathbb{Z}_p)}(r_k^{-1}(y))} = \frac{\mu_{X(\mathbb{Z}_p)}(\varphi^{-1}(r_k^{-1}(y)))}{p^{-k \dim Y_{\mathbb{Q}}}} = \frac{\mu_{X(\mathbb{Z}_p)}(r_k^{-1}(\varphi^{-1}(y)))}{p^{-k \dim Y_{\mathbb{Q}}}} = \frac{\#\varphi^{-1}(y)}{p^{k(\dim X_{\mathbb{Q}} - \dim Y_{\mathbb{Q}})}} < C_1, \quad (4\text{-}3)$$
where $\mu_{X(\mathbb{Z}_p)}$ and $\mu_{Y(\mathbb{Z}_p)}$ are the canonical measures on $X(\mathbb{Z}_p)$ and $Y(\mathbb{Z}_p)$. Let f_p be the density of $\varphi_*\mu_{X(\mathbb{Z}_p)}$ with respect to $\mu_{Y(\mathbb{Z}_p)}$. Combining (4-3) with Lebesgue's differentiation theorem, we get, for almost all $y' \in Y(\mathbb{Z}_p)$,
$$f_p(y') = \lim_{k \to \infty} \frac{\varphi_*\mu_{X(\mathbb{Z}_p)}(r_k^{-1}(r_k(y')))}{\mu_{Y(\mathbb{Z}_p)}(r_k^{-1}(r_k(y')))} < C_1,$$

which implies condition (4).

It is left to prove that (1) ⇒ (3). The case of large primes follows from the implication (1) ⇒ (3) of Theorem 4.7. It is left to prove (3) for a fixed prime p. By Theorem 4.3, we have $f_p < C(p)$ for some $C(p) > 0$. Using (4-3), we deduce that

$$\frac{\#\varphi^{-1}(y)}{p^{k(\dim X_{\mathbb{Q}}-\dim Y_{\mathbb{Q}})}} < C(p), \tag{4-4}$$

for every $k \geq k_0$ and $y \in Y(\mathbb{Z}/p^k\mathbb{Z})$. For $y \in Y(\mathbb{Z}/p^k\mathbb{Z})$ with $k < k_0$ we can take the trivial bound $\#\varphi^{-1}(y) \leq \sum_{l=1}^{k_0} \#X(\mathbb{Z}/p^l\mathbb{Z})$ to deduce (4-4) for every $k \in \mathbb{N}$. Using the triangle inequality, and by applying the trivial upper bound $\#\varphi^{-1}(\bar{y}) < \#X(\mathbb{F}_p)$ for $\bar{y} \in Y(\mathbb{F}_p)$, we deduce (3). □

Remark 4.8. One can easily adapt the proof of Theorem A above to prove a more general statement where the collection $\{\mathbb{Q}_p\}_p$ is replaced with all completions $K_{\mathfrak{p}}$ of a fixed number field K. On the other hand, Theorem 4.7 is definitely not true for all $F \in \mathrm{Loc}$; e.g., take $\varphi(x) = 3x$, and consider unramified extensions of $\mathbb{Q}_3$.

We now move to the proof of Theorem 4.7. We start with the easier implications, and deal with the more challenging implication (1) ⇒ (2) in Section 4B1.

Proof of (2) ⇒ (3) ⇒ (4) ⇒ (5) ⇒ (1) *of Theorem 4.7.* Implication (2) ⇒ (3) Assume that $h_F(y', k) < C_1 q_F^{-1}$ for each $F \in \mathrm{Loc}_{\gg}$, each $k \in \mathbb{N}$ and each $y' \in Y(\mathcal{O}_F)$. Set $y := r_k(y')$ and note that

$$\left|\frac{\#\varphi^{-1}(y)}{q_F^{k(\dim X_K-\dim Y_K)}} - \frac{\#\varphi^{-1}(r_1^k(y))}{q_F^{\dim X_K-\dim Y_K}}\right| = \left|\frac{\#\varphi^{-1}_{|X^{\mathrm{sm},\varphi}}(y))}{q_F^{k(\dim X_K-\dim Y_K)}} + h_F(y',k) - \frac{\#\varphi^{-1}_{|X^{\mathrm{sm},\varphi}}(r_1^k(y))}{q_F^{\dim X_K-\dim Y_K}} - h_F(y',1)\right|$$
$$= |h_F(y',k) - h_F(y',1)| \leq 2C_1 q_F^{-1}.$$

where the second equality follows from Hensel's lemma and the inequality follows from our assumption on h. Since r_k is surjective for $F \in \mathrm{Loc}_{\gg}$, this finishes the proof.

Implication (3) ⇒ (4) Let us first prove that φ_K is flat, assuming condition (3). It is enough to show that $\varphi_{\mathbb{F}_p}$ is flat for infinitely many prime numbers p. Let p be a prime large enough such that $\dim X_K = \dim X_{\mathbb{F}_p}$, $\dim Y_K = \dim Y_{\mathbb{F}_p}$ and such that condition (3) holds for $F = \mathbb{F}_q((t))$ for any q which is a power of p. Note there are infinitely many primes p such that $\mathbb{F}_p$ is a residue field of $\mathcal{O}_K$ for some prime of $\mathcal{O}_K$. Let $x \in X(\mathbb{F}_q)$ for such q and let $\tilde{x} \in J_k(X)(\mathbb{F}_q) \simeq X(\mathbb{F}_q[t]/(t^{k+1}))$ be the image of x under the zero section embedding $X(\mathbb{F}_q) \hookrightarrow J_k(X)(\mathbb{F}_q)$, so that $r_1^k(\varphi(\tilde{x})) = \varphi(x)$. Then by condition (3), we have for any $k \in \mathbb{N}$:

$$\left|\frac{\#\varphi^{-1}(\varphi(\tilde{x}))}{q^{(k+1)(\dim X_K-\dim Y_K)}} - \frac{\#\varphi^{-1}(\varphi(x))}{q^{\dim X_K-\dim Y_K}}\right| < C_2 \cdot q^{-1}. \tag{4-5}$$

Consider $k = 1$. By choosing q to be a suitable power of p we may assume $C_{\varphi,q,\varphi(x)}, C_{J_1(\varphi),q,J_1(\varphi)(\tilde{x})} \geq 1$. Notice that $\#\varphi^{-1}(\varphi(\tilde{x})) = \#J_1(X_{\varphi(x),\varphi_{\mathbb{F}_q}})(\mathbb{F}_q)$. Since $\dim J_1(X_{\varphi(x),\varphi_{\mathbb{F}_q}}) \geq 2\dim X_{\varphi(x),\varphi_{\mathbb{F}_q}}$ we have by (4-5) and by the Lang–Weil estimates that

$$\dim X_{\varphi(x),\varphi_{\mathbb{F}_q}} = \dim X_K - \dim Y_K = \dim X_{\mathbb{F}_q} - \dim Y_{\mathbb{F}_q}.$$

By miracle flatness, we are done.

To prove condition (4), by the triangle inequality, it is enough to find C_3' such that for each $F \in \mathrm{Loc}_{\gg}$, $k \in \mathbb{Z}_{\geq 1}$ and $y \in Y(\mathcal{O}_F/\mathfrak{m}_F^k)$:

$$\left| \frac{\#\varphi^{-1}(r_1^k(y))}{q_F^{\dim X_K - \dim Y_K}} - C_{\varphi, q_F, r_1^k(y)} \right| < C_3' q_F^{-1/2}.$$

This follows from the fact that φ_K is flat, via a relative variant of the Lang–Weil estimates; see, e.g., [20, Theorem 8.4].

Implications (4) $\Rightarrow$ (5) and (5) $\Rightarrow$ (1) Let f_F be the density of $\varphi_*(\mu_{X(\mathcal{O}_F)})$ with respect to $\mu_{Y(\mathcal{O}_F)}$. By Lebesgue's differentiation theorem and condition (4), for almost every $y' \in Y(\mathcal{O}_F)$, we have

$$|f_F(y') - C_{\varphi, q_F, r_1(y')}| = \left| \lim_{k \to \infty} \frac{\mu_{X(\mathcal{O}_F)}(\varphi^{-1}(B(y', k)))}{\mu_{Y(\mathcal{O}_F)}(B(y', k))} - C_{\varphi, q_F, r_1(y')} \right| \tag{4-6}$$

$$= \left| \lim_{k \to \infty} \frac{\#\varphi^{-1}(r_k(y'))}{q_F^{k(\dim X_K - \dim Y_K)}} - C_{\varphi, q_F, r_1(y')} \right| < C_3 q_F^{-1/2}. \tag{4-7}$$

This also shows that f_F is essentially bounded for $F \in \mathrm{Loc}_{\gg}$. By Corollary 4.4 and by Theorem 4.3, it follows that f_F can be chosen to be continuous, so that (4-7) holds for all $y' \in Y(\mathcal{O}_F)$. This implies condition (5), which implies condition (1) using the same Corollary 4.4. □

4B1. *Proof of the implication* (1) $\Rightarrow$ (2). In this section we will prove the remaining implication of Theorem 4.7, namely (1) $\Rightarrow$ (2). We first observe the following:

Proposition 4.9. *Assume that Y is affine. Then h is a formally nonnegative motivic function.*

Proof. We first prove the special case with X affine. Assume that $X \subseteq \mathbb{A}^m$ is the zero locus of $g_1, \ldots, g_l \in \mathcal{O}_K[x_1, \ldots, x_m]$. Since X and Y are affine, the map $\varphi = (f_1, \ldots, f_n) : X \to Y \subseteq \mathbb{A}^n$ is a polynomial map, thus with $f_i \in \mathcal{O}_K[x_1, \ldots, x_m]$. Given $y \in Y(\mathcal{O}_F)$, set

$$S_{y,k,X} := \{x \in \mathcal{O}_F^m : \min_{i,j}\{\mathrm{val}(g_i(x)), \mathrm{val}(f_j(x) - y_j)\} \geq k\}.$$

Now, for any $y \in Y(\mathcal{O}_F)$, we have

$$\#\varphi^{-1}(r_k(y)) = q_F^{km} \int_{\mathcal{O}_F^m} 1_{S_{y,k,X}} |dx_1 \wedge \cdots \wedge dx_m|.$$

Moreover,

$$\#(\varphi^{-1}(r_k(y)) \cap (r_{1,X}^k)^{-1}(X^{\mathrm{sing},\varphi}(k_F))) = q_F^{km} \int_{\mathcal{O}_F^m} 1_{W_{y,k,X}} |dx_1 \wedge \cdots \wedge dx_m|,$$

where

$$W_{y,k,X} := \{x \in S_{y,k,X} : r_1(x) \in X^{\mathrm{sing},\varphi}(k_F)\}.$$

Since $1_{W_{y,X}}$ is formally nonnegative, we get by Theorem 2.12 that h is formally nonnegative as well. Now let $X = \bigcup_{i=1}^N U_i$ be a cover by smooth open affine subschemes U_i. For each i and $F \in \mathrm{Loc}_{\gg}$ write

$V_i := U_i(\mathcal{O}_F)\setminus\bigcup_{j=1}^{i-1} U_j(\mathcal{O}_F)$ and note that

$$\begin{aligned}\#(\varphi^{-1}(r_k(y))\cap(r_{1,X}^k)^{-1}(X^{\mathrm{sing},\varphi}(k_F))) &= \sum_{i=1}^{N}\#((\varphi|_{U_i})^{-1}(r_k(y))\cap(r_{1,X}^k)^{-1}(U_i^{\mathrm{sing},\varphi}(k_F)\cap r_1(V_i)))\\ &= \sum_{i=1}^{N} q_F^{km}\int_{\mathcal{O}_F^m} 1_{W_{y,k,i}}|dx_1\wedge\cdots\wedge dx_m|,\end{aligned}$$

where

$$W_{y,k,i} := \{x\in S_{y,k,U_i} : r_1(x)\in U_i^{\mathrm{sing},\varphi}(k_F)\cap r_1(V_i)\}.$$

This finishes the proof of Proposition 4.9. □

We need one more lemma which we state in the generality of E-smooth morphisms, and which will further be used in the next section.

Lemma 4.10. *Let $E\geq 1$ be an integer, let φ be as in Theorem 4.7 and assume that $\varphi_K : X_K\to Y_K$ is E-smooth. Then, for each $k\in\mathbb{N}$ there exists a constant $C(k)>0$ such that for each $F\in\mathrm{Loc}_{\gg}$, one has*

$$\sup_{y\in Y(\mathcal{O}_F)} h_F(y,k) < C(k)\cdot q_F^{-E}.$$

Proof. Using Theorems 2.14 and 3.1 it is enough to prove the lemma for F lying in $\mathrm{Loc}_{+,\gg}$. By Proposition 2.5 we have

$$J_k(X_{k_F}^{\mathrm{sm},\varphi_{k_F}}) = J_k(X_{k_F})^{\mathrm{sm},J_k(\varphi_{k_F})}, \tag{4-8}$$

for $F\in\mathrm{Loc}_{+,\gg}$. Let $Z_{\tilde{y}} := J_k(X_{k_F})_{\tilde{y},J_k(\varphi_{k_F})}$ be a nonempty fiber of $J_k(\varphi_{k_F})$ over $\tilde{y}\in J_k(Y)(k_F)$. Since $J_k(\varphi_{k_F})$ is flat and by (4-8), we have

$$Z_{\tilde{y}}^{\mathrm{sing}} = Z_{\tilde{y}}\cap J_k(X_{k_F})^{\mathrm{sing},J_k(\varphi_{k_F})} = Z_{\tilde{y}}\cap(\pi_{0,X_{k_F}}^k)^{-1}(X_{k_F}^{\mathrm{sing},\varphi_{k_F}}). \tag{4-9}$$

The E-smoothness of φ_K implies that the right hand side is of codimension at least E in $Z_{\tilde{y}}$. By the definition of h, by the fact that all fibers of $J_k(\varphi_{k_F})$ are of bounded complexity (for a fixed k) and using a relative variant of the Lang–Weil estimates, the lemma follows. □

Proof of the implication (1) ⇒ (2). We may assume that Y is affine. Theorem 3.1 and Proposition 4.9 imply that there exist a constant $C_0>0$ and a motivic function H in $\mathcal{C}_+(\mathbb{Z}_{\geq 1})$ such that

$$\sup_{y\in Y(\mathcal{O}_F)} h_F(y,k) < H_F(k) < C_0\cdot \sup_{y\in Y(\mathcal{O}_F)} h_F(y,k). \tag{4-10}$$

It is thus enough to show that $\sup_k H_F(k) < C_1\cdot q_F^{-1}$ for some constant C_1 which is independent of F.

By Corollary 4.4, by (4-2) and since $h_F\leq g_F$, we deduce that the function $(y,k)\mapsto h_F(y,k)$ is bounded for each $F\in\mathrm{Loc}_{\gg}$. By (4-10) also $k\mapsto H_F(k)$ is bounded for each $F\in\mathrm{Loc}_{\gg}$. As in the proof of Claim 2 of Theorem 3.1, it follows that there exist a finite set L of $\mathbb{Z}_{\geq 1}$ and a constant $C_0'>0$ such that

$$\sup_k H_F(k) \leq C_0'\cdot\sum_{k\in L} H_F(k). \tag{4-11}$$

Using (4-10), (4-11), Lemmas 2.9(1) and 4.10 and by setting $C_1 := C_0 C_0' \cdot \sum_{k\in L} C(k)$, we obtain

$$\sup_k H_F(k) \le C_0' \sum_{k\in L} H_F(k) \le C_0' C_0 \sum_{k\in L} \sup_{y\in Y(\mathcal{O}_F)} h_F(y,k) < C_1 q_F^{-1},$$

for each $F \in \mathrm{Loc}_{\gg}$. This finishes the proof of (1) ⇒ (2). □

4C. *Number-theoretic estimates for E-smooth and ε-jet flat morphisms.* In this subsection we use the improved approximation of suprema (Theorem 3.1), similarly as in Section 4B1, to provide uniform estimates for E-smooth morphisms and ε-jet flat morphisms, improving [20, Theorem 8.18]. We start by giving a characterization of E-smooth morphisms.

Theorem 4.11. *Let $E \ge 1$ be an integer, and let $\varphi : X \to Y$ be a dominant morphism between finite type $\mathcal{O}_K$-schemes X and Y, with X_K, Y_K smooth and geometrically irreducible. Then the following are equivalent*:

(1) *$\varphi_K : X_K \to Y_K$ is E-smooth.*

(2) *There exists $C_1 > 0$, such that, for each $F \in \mathrm{Loc}_{\gg}$, $k \in \mathbb{Z}_{\ge 1}$ and $y' \in Y(\mathcal{O}_F)$,*

$$h_F(y',k) < C_1 q_F^{-E}.$$

(3) *There exists $C_2 > 0$ such that, for each $F \in \mathrm{Loc}_{\gg}$, $k \in \mathbb{Z}_{\ge 1}$ and $y \in Y(\mathcal{O}_F/\mathfrak{m}_F^k)$,*

$$\left| \frac{\#\varphi^{-1}(y)}{q_F^{k(\dim X_K - \dim Y_K)}} - \frac{\#\varphi^{-1}(r_1^k(y))}{q_F^{\dim X_K - \dim Y_K}} \right| < C_2 q_F^{-E}.$$

In particular, when $E = 2$, the conditions above are further equivalent to $\varphi_K : X_K \to Y_K$ being flat with fibers of terminal singularities (*see Lemma 2.9*).

Proof. The proof of (1) ⇒ (2) is identical to the proof of (1) ⇒ (2) in Theorem 4.7, where the only exception is the inequality $\sup_{y\in Y(\mathcal{O}_F)} h_F(y,k) < C_1 q_F^{-E}$ for $F \in \mathrm{Loc}_{\gg}$ which is similarly obtained using Lemma 4.10. (2) ⇒ (3) is similar as in Theorem 4.7.

<u>(3) ⇒ (1)</u> Recall that by Theorem 4.7, condition (3) implies that φ_K is (FRS) and therefore jet-flat, and that

$$|h_F(y',k) - h_F(y',1)| \le C_2 q_F^{-E},$$

for all $F \in \mathrm{Loc}_{\gg}$, $k \in \mathbb{Z}_{\ge 1}$ and $y' \in Y(\mathcal{O}_F)$ (see (2) ⇒ (3) in the proof of Theorem 4.7). Write $W_{y'} := (X_{k_F})_{r_1(y'),\varphi_{k_F}}$. We claim that $(W_{y'})^{\mathrm{sing}}$ is of codimension at least $E+1$ in $W_{y'}$ for all $F \in \mathrm{Loc}_{\gg}$ and $y' \in Y(\mathcal{O}_F)$. Indeed, assume $(W_{y'})^{\mathrm{sing}}$ is of codimension r in $W_{y'}$ with $r \le E$. Identifying $r_1(y') \in Y(k_F)$ with $\tilde{y} := s_1(r_1(y')) = (r_1(y'), 0) \in J_1(Y)(k_F)$ under the zero section embedding $s_1 : Y \hookrightarrow J_1(Y)$, and using (4-9) one has

$$(J_1(X_{k_F})_{\tilde{y},J_1(\varphi_{k_F})})^{\mathrm{sing}} = J_1(W_{y'}) \cap (\pi^1_{0,X_{k_F}})^{-1}(X_{k_F}^{\mathrm{sing},\varphi_{k_F}}) = (\pi^1_{0,W_{y'}})^{-1}(W_{y'}^{\mathrm{sing}}).$$

Since the dimension of the Zariski tangent space of a variety Z at a singular point is larger than $\dim Z$, we have

$$\begin{aligned}\dim(J_1(X_{k_F})_{\tilde{y},J_1(\varphi_{k_F})})^{\mathrm{sing}} &\geq \dim(W_{y'})^{\mathrm{sing}} + \dim X_K - \dim Y_K + 1\\ &\geq \dim W_{y'} - r + \dim X_K - \dim Y_K + 1\\ &\geq \dim J_1(X_{k_F})_{\tilde{y},J_1(\varphi_{k_F})} - r + 1.\end{aligned}$$

Hence $(J_1(X_{k_F})_{\tilde{y},J_1(\varphi_{k_F})})^{\mathrm{sing}}$ is of codimension at most $r-1$. By replacing F with a finite extension, and using the Lang–Weil estimates, one can find $C_3 > 0$ such that

$$h_F(y',1) < C_3 q_F^{-r} \quad \text{and} \quad h_F(y',2) > \tfrac{1}{2} q_F^{-r+1}.$$

But this contradicts $|h_F(y',k) - h_F(y',1)| \leq C_2 q_F^{-E}$. Therefore $h_F(y',1) < C_3 q_F^{-(E+1)}$ for all $F \in \mathrm{Loc}_{\gg}$ and $y' \in Y(\mathcal{O}_F)$. But then by condition (3), we deduce that $h_F(y',k) < C_3 q_F^{-E}$ which implies that φ_K is E-smooth. □

Finally, we provide an estimate on the number of $\mathcal{O}_F/\mathfrak{m}_F^k$-points lying on fibers of ε-jet flat morphisms.

Theorem 4.12. *Let $\varphi : X \to Y$ be a dominant morphism between finite type $\mathcal{O}_K$-schemes X and Y, with X_K, Y_K smooth and geometrically irreducible and let $0 < \varepsilon \leq 1$. Then the following are equivalent:*

(1) *$\varphi_K : X_K \to Y_K$ is ε-jet flat.*

(2) *There exist $C, M > 0$ such that for each $F \in \mathrm{Loc}_{\gg}$, $k \in \mathbb{Z}_{\geq 1}$ and $y \in Y(\mathcal{O}_F/\mathfrak{m}_F^k)$, one has*

$$\frac{\#\varphi^{-1}(y)}{q_F^{k(\dim X_K - \dim Y_K)}} < C \cdot k^M q_F^{k(1-\varepsilon)\dim Y_K}.$$

In particular, when $\varepsilon = 1$ and assuming φ_K has normal fibers, the conditions above are further equivalent to φ_K being flat with fibers of log-canonical singularities (Remark 2.8).

The difficult direction of Theorem 4.12 is (1) ⇒ (2), and it sharpens the bounds given in [20, Theorem 8.18]; the factor $Ck^M q_F^{-k\varepsilon \dim Y_K}$ as in (2) improves a factor of the form $q_F^{-k\varepsilon' \dim Y_K}$ present in [20], where ε' can be taken to be any number such that $\varepsilon' > \varepsilon$. In order to prove these sharper estimates, we use Theorem B, along with the following auxiliary lemma.

Lemma 4.13. *Let $g \in \mathcal{C}_+(\mathbb{Z}_{\geq 1})$ be a formally nonnegative motivic function such that for every $\delta > 0$ and $k \in \mathbb{Z}_{\geq 1}$ we have (varying over $F \in \mathrm{Loc}_{\gg}$)*

$$\lim_{q_F \to \infty} q_F^{-\delta} g_F(k) = 0.$$

Then there exist $M \in \mathbb{N}$ and $C > 0$ such that $g_F(k) < Ck^M$ for every $k \in \mathbb{Z}_{\geq 1}$ and field $F \in \mathrm{Loc}_{\gg}$.

Proof. Since g is formally nonnegative, we may write $g_F = \sum \#Y_{F,i} f_{F,i}$ for $f_{F,i} \in \mathcal{P}_+(\mathbb{Z}_{\geq 1})$ formally nonnegative and $Y_{F,i} \subseteq \mathbb{Z}_{\geq 1} \times \mathrm{RF}^{r_i}$. It is enough to show the claim for a single summand $g_F = \#Y_F f_F$. Using Presburger cell decomposition and the orthogonality of RF and VG, we have a finite partition

$\mathbb{Z}_{\geq 1} = \bigcup A_i$ and we may write $g_F(k)|_A = \sum \#Y_F c_i(q_F) q_F^{a_i k} k^{b_i}$ on each cell A, where $a_i \in \mathbb{Q}$, $b_i \in \mathbb{N}$, $\{(a_i, b_i)\}_{i=1}^N$ are mutually different, and $c_i(q)$ are rational functions in q.

First assume our cell A is finite, in which case it is enough to prove the claim for a fixed $k = k_0$. Using [5, main theorem], we have nonnegative constants d, C_1 and C_2 such that

$$\#Y_F < C_2 q_F^d \text{ for all } F \in \mathrm{Loc}_{\gg} \quad \text{and} \quad C_1 q_F^d < \#Y_F < C_2 q_F^d \tag{†}$$

for infinitely many fields $F \in \mathrm{Loc}_{\gg}$ (with infinitely many residual characteristics).

Therefore, for every $\delta > 0$ and infinitely many fields $F \in \mathrm{Loc}_{\gg}$ we have

$$\lim_{q_F \to \infty} q_F^{-\delta} (C_1 q_F^d) \sum q_F^{a_i k_0} c_i(q_F) k_0^{b_i} \leq \lim_{q_F \to \infty} q_F^{-\delta} g_F(k_0) = 0, \tag{$\triangle$}$$

and thus $\deg_q \left(C_2 q^d \sum q^{a_i k_0} c_i(q) k_0^{b_i} \right) \leq 0$ as a rational function in q. The claim now follows since there exists $C_3 > 0$ such that for every $F \in \mathrm{Loc}_{\gg}$ with q_F large enough,

$$g_F(k_0) = \#Y_F \sum c_i(q_F) q_F^{a_i k_0} k_0^{b_i} < C_2 q_F^d \sum q_F^{a_i k_0} c_i(q_F) k_0^{b_i} < C_3.$$

Now, assume our cell A is infinite and set $a = \max\{a_i\}$. Using $\triangle$ with a general k instead of a fixed k_0, we must have $a \leq 0$, as otherwise for every k large enough $R(q) = C_1 q^d \sum q^{a_i k} c_i(q) k^{b_i}$ is a nonzero rational function in q whose degree is positive, and therefore $\lim_{q_F \to \infty} q_F^{-\delta} R(q_F) \neq 0$ for some $\delta > 0$.

Set $H_F(k) = \sum_{i : a_i = 0} \#Y_F c_i(q_F) k^{b_i}$ and $E_F(k) = \sum_{i : a_i < 0} \#Y_F c_i(q_F) q_F^{a_i k} k^{b_i}$, then we have

$$g_F(k) = H_F(k) + E_F(k) \leq |H_F(k)| + |E_F(k)|.$$

Using (†), we may find a constant C' such that $|E_F(k)| < C'$ for every k large enough and $F \in \mathrm{Loc}_{\gg}$. It is therefore left to take care of $H_F(k)$. We may assume $A = \mathbb{Z}_{\geq 1}$.

We prove by induction on the number of summands N that if $H_F = \sum_{i=1}^N \#Y_F c_i(q_F) k^{b_i}$ is a function satisfying $\lim_{q_F \to \infty} q_F^{-\delta} H_F(k) = 0$ for every k large enough and $\delta > 0$, then there exists a constant $C'' > 0$ such that for every $F \in \mathrm{Loc}_{\gg}$ we have $|\#Y_F c_i(q_F)| < C''$ for all $1 \leq i \leq N$.

For $N = 1$ the claim follows by (†) as before by showing $|\#Y_F c(q_F)|$ is bounded by a rational function of nonpositive q-degree. To prove the claim for $N > 1$, consider the functions

$$\widetilde{H}_{j,F}(k) = H_F(2k) - 2^{b_j} H_F(k) = \sum_{i=1}^N \#Y_F (2^{b_i} - 2^{b_j}) c_i(q_F) k^{b_i}.$$

For each $1 \leq j \leq N$, the function $\widetilde{H}_{j,F}(k)$ has $N-1$ summands and satisfies the induction hypothesis since $H_F(k)$ and $H_F(2k)$ do, and therefore the proof by induction is concluded. Using the triangle inequality, we can now find a bound for $H_F(k)$ as required, proving the lemma. □

Remark 4.14. Note that one may formulate and prove Lemma 4.13 with $g'_F(k) = q_F^{\varepsilon k} g_F(k)$ instead of g, where $\varepsilon \in \mathbb{R}$. The collection $\{q_F^{\varepsilon k} g_F\}_{F \in \mathrm{Loc}_{\gg}}$ may no longer be motivic, but the proof remains the same.

Proof of Theorem 4.12. (2) $\Rightarrow$ (1): Assume towards contradiction that φ_K is not ε-jet flat. Therefore there exist $k \in \mathbb{Z}_{\geq 1}$ and $y \in J_{k-1}(Y_K)(\overline{K})$, such that if $Z_y := J_{k-1}(X_K)_{y, J_{k-1}(\varphi_K)}$ then

$$\dim Z_y > k \dim X_K - \varepsilon k \dim Y_K.$$

The fiber Z_y is defined over a finitely generated algebra D_y over $\mathcal{O}_K$. Let $\mathbb{F}_q$ be a residue field of D_y where $q = p^r$ for $r \in \mathbb{N}$ and prime p large enough. By taking r large enough, we may assume that all of the top-dimensional geometrically irreducible components of $(Z_y)_{\bar{\mathbb{F}}_q}$ are defined over $\mathbb{F}_q$.

Let $\bar{y} \in J_{k-1}(Y_{\mathbb{F}_q})$ be the reduction modulo q of y under the map $D_y \to \mathbb{F}_q$ and let $y' \in Y(\mathbb{F}_q[t]/(t^k))$ be the image of $\bar{y}$ under the natural identification $J_{k-1}(Y)(\mathbb{F}_q) \simeq Y(\mathbb{F}_q[t]/(t^k))$. In particular, we have $\#(Z_{y_{\mathbb{F}_q}})(\mathbb{F}_q) = \#\varphi^{-1}(y')$. The claim now follows using the Lang–Weil estimates for $\#(Z_{y_{\mathbb{F}_q}})(\mathbb{F}_q)$.

It is left to prove (1) $\Rightarrow$ (2). By Theorem 3.1, there exist $G \in \mathcal{C}_+(\mathbb{Z}_{\geq 1})$ and $C' > 1$ such that

$$\sup_{y \in Y(\mathcal{O}_F)} g_F(y,k) < G_F(k) < C' \sup_{y \in Y(\mathcal{O}_F)} g_F(y,k).$$

Set $G'_F(k) := q_F^{-k \dim Y_K (1-\varepsilon)} G_F(k)$. Using the Lang–Weil estimates and Theorem 2.14, we may invoke Lemma 4.13 with $G'_F(k)$ (see Remark 4.14). We therefore get constants $C, M > 0$ such that for each $k \in \mathbb{Z}_{\geq 1}$ and $F \in \mathrm{Loc}_{\gg}$

$$q_F^{-k \dim Y_K (1-\varepsilon)} \sup_{y \in Y(\mathcal{O}_F)} g_F(y,k) < q_F^{-k \dim Y_K (1-\varepsilon)} G_F(k) = G'_F(k) < C k^M.$$

The claim is thus proven. □

Remark 4.15. To conclude the paper, we note that a possible deeper understanding of the estimates in Theorems A, 4.7, 4.11 and 4.12 may come from the results on exponential sums in [12] and may be related to the motivic oscillation index $\mathrm{moi}(\varphi)$ of φ.[4] The motivic oscillation index controls the decay rate of the Fourier transform of $\varphi_*(\mu_{\mathcal{O}_F^n})$; see [12, Proposition 3.11]. In the non-(FRS) case, optimal bounds on the decay rate were given in [12, Theorem 1.5], proving a conjecture of Igusa on exponential sums [26]. Here it can also be shown that $\mathrm{moi}(\varphi)$ controls the explosion rate of the density of the pushforward measure $\varphi_*(\mu_{\mathcal{O}_F^n})$ near a critical point; see, e.g., [20, Theorem 8.18]. The (FRS) case of Igusa's conjecture is open (see the discussion in [12, Section 3.4]), and a potential connection between Theorems 4.7, 4.11 and the $\mathrm{moi}(\varphi)$ could be interesting in that regard.

Appendix: Proof of Proposition 3.6

In this appendix we prove the following:

Proposition A.1 Proposition 3.6. *The families $\mathcal{P}_+(X \times W)$ and $\mathcal{C}_+(X \times W)$ are not of type* (3).

Definition A.2 [6, Definition 1]. Let $X \subseteq \mathbb{Z}^m$ be an $\mathcal{L}_{\mathrm{Pres}}$-definable set. We call a definable function $f : X \to \mathbb{Z}$ $\mathcal{L}_{\mathrm{Pres}}$*-linear* if there exist $\gamma \in \mathbb{Z}$ and integers a_i and $0 \leq c_i < n_i$ for $1 \leq i \leq m$ such that $x_i - c_i \equiv 0 \bmod n_i$ and $f(x_1, \ldots, x_m) = \sum_{i=1}^m a_i((x_i - c_i)/n_i) + \gamma$.

[4] For the definition in the case that $\varphi : \mathbb{A}^n \to \mathbb{A}^1$ is a polynomial; see [12, Section 3.4].

Proof of Proposition A.1. Let $X=\mathbb{Z}_{\geq 1}^m$, $W=\{1,\dots,m\}\subseteq\mathbb{Z}$. Let $p_1=2<p_2=3<\cdots<p_m$ be the first m prime numbers, and take $f(x_1,\dots,x_m,w)=x_w^{p_w}$. We want to show that for every $\epsilon>0$, there is no $g\in\mathcal{C}(X)$ such that

$$\max_{1\leq w\leq m} f_F(x,w)\leq g_F(x)\leq(m-\epsilon)\cdot\max_{1\leq w\leq m} f_F(x,w), \tag{$\star$}$$

for each $F\in\mathrm{Loc}_{\gg}$ and $x\in X_F$. In fact, $\sum_{j=1}^m x_j^{p_j}$ is an optimal approximation (with constant m).

Indeed, assume towards contradiction the existence of $g\in\mathcal{C}(X)$ satisfying ($\star$), for some $\epsilon>0$ and for all $F\in\mathrm{Loc}_{\gg}$. Fix F such that ($\star$) holds, and such that g can be written as in Definition 2.11. By the model theoretic orthogonality of the sorts VG and RF, we may assume all of the definable functions $\alpha_i,\beta_{ij}:X\to\mathbb{Z}$ appearing in the data of g, are $\mathcal{L}_{\mathrm{Pres}}$-definable. Using Presburger cell decomposition [6, Theorem 1], we can decompose X into cells $X=\bigsqcup_{i=1}^N C_i$, such that on each C_i, the definable Presburger functions appearing in g are $\mathcal{L}_{\mathrm{Pres}}$-linear. Note that one of the cells C must have infinite intersection with the set $\{(t^m,t^{m-1},\dots,t):t\in\mathbb{Z}_{\geq 1}\}$. By the definition of a cell [6, Definiton 2], and by possibly restricting into a smaller subcell, we may assume C has the form

$$C=\left\{(x_1,\dots,x_m)\in\mathbb{Z}_{\geq 1}^m:\forall j\left(x_j\geq b+a\sum_{i=j+1}^m x_i\right)\wedge(x_j=c_j \bmod r_j)\right\}, \tag{A-1}$$

for $a,b\in\mathbb{Z}_{\geq 1}$, and integers $0\leq c_j\leq r_j$. Taking a and b divisible enough, the cell C is isomorphic to $\mathbb{Z}_{\geq 1}^m$ by an affine change of coordinates $\varphi:\mathbb{Z}_{\geq 1}^m\to C$, after which $g_F\circ\varphi$ has the form

$$g_F\circ\varphi(e_1,\dots,e_m)=\sum_{i=1}^M \tilde{c}_{i,F}\cdot q_F^{a_{i1}e_1+\cdots+a_{im}e_m}\cdot\prod_{j=1}^m e_j^{b_{ij}}, \tag{$\star\star$}$$

for $\{(a_{i1},\dots,a_{im},b_{i1},\dots,b_{im})\}_i$ mutually different tuples of integers, where $b_{ij}\geq 0$ and $0\neq\tilde{c}_i\in\mathbb{R}$. Since $1/m\leq g_F(x_1,\dots,x_m)/(x_1^{p_1}+\cdots+x_m^{p_m})\leq m$, it follows that $a_{ij}\leq 0$ for all i,j. We can therefore write g_F as

$$g_F(x_1,\dots,x_m)=P_F(x_1,\dots,x_m)+E_F(x_1,\dots,x_m),$$

where $P_F\circ\varphi$ consists of the terms in ($\star\star$) with $a_{i1}=\cdots=a_{im}=0$, i.e., P_F is a polynomial, and $E_F\circ\varphi$ consists of all the terms of ($\star\star$) with $a_{ij}<0$ for some $j\in\{1,\dots,m\}$. Write

$$P_F(x_1,\dots,x_m)=\sum_{j=1}^m d_j x_j^{p_j}+Q_F,$$

where Q_F is the sum of all monomials in P_F which do not belong to the collection $\{x_j^{p_j}\}_{j=1}^m$.

For simplicity, in the following arguments we ignore the congruence relations in (A-1). These arguments can easily be adapted to the general form of (A-1).

Write $\tilde{p} := \prod_{j=1}^{m} p_j$ and let $\tilde{p}_j := \tilde{p}/p_j$. Note that, ignoring potential congruences in (A-1), we have $(t^{\tilde{p}_1}, \dots, t^{\tilde{p}_m}) \in C$ for $t \gg 1$. We now claim that:

(1) $d_j \geq 1$ for all $1 \leq j \leq m$.

(2) $\lim_{t\to\infty}(Q_F(t^{\tilde{p}_1}, \dots, t^{\tilde{p}_m})/t^{\tilde{p}}) = \lim_{t\to\infty}(E_F(t^{\tilde{p}_1}, \dots, t^{\tilde{p}_m})/t^{\tilde{p}}) = 0$, and hence

$$\lim_{t\to\infty} \frac{g_F(t^{\tilde{p}_1}, \dots, t^{\tilde{p}_m})}{t^{\tilde{p}}} = \sum_{j=1}^{m} d_j.$$

Since $\max_{1\leq w\leq m} f_F(t^{\tilde{p}_1}, \dots, t^{\tilde{p}_m}, w) = t^{\tilde{p}}$, items (1) and (2) contradict ($\star$). To prove items (1) and (2), we observe the following:

(a) The cell C contains many asymptotic directions in $\mathbb{Z}^m_{\geq 1}$; indeed, $(t^{l_1}, \dots, t^{l_m})$ is in C for all integers $l_1 > \cdots > l_m \geq 1$ and all $t \gg 1$. Moreover, for each $l_1 > \cdots > l_m \geq 1$, we have

$$\lim_{t\to\infty} E_F(t^{l_1}, \dots, t^{l_m}) = 0. \tag{A-2}$$

(b) Since $(t^{\tilde{p}_1}, \dots, t^{\tilde{p}_m}) \in C$ for $t \gg 1$, and by ($\star$)

$$1 \leq \frac{g_F(t^{\tilde{p}_1}, \dots, t^{\tilde{p}_m})}{\max\{t^{\tilde{p}_1 p_1}, \dots, t^{\tilde{p}_m p_m}\}} = \frac{g_F(t^{\tilde{p}_1}, \dots, t^{\tilde{p}_m})}{t^{\tilde{p}}} \leq m - \epsilon. \tag{A-3}$$

For $\omega \in \mathbb{Q}^m$, we define the *ω-weight* of a monomial $x_1^{n_1} \cdot \cdots \cdot x_m^{n_m}$ to be $\sum_{j=1}^{m} n_j \omega_j$. Formulas (A-2) and (A-3) imply that all monomials $x_1^{n_1} \cdot \cdots \cdot x_m^{n_m}$ appearing in P_F have $(\tilde{p}_1, \dots, \tilde{p}_m)$-weight $\leq \tilde{p}$. Indeed, suppose P_F contains a monomial $\widetilde{Q}_F$ with maximal $(\tilde{p}_1, \dots, \tilde{p}_m)$-weight $\tilde{q} > \tilde{p}$. If $\widetilde{Q}_F$ is the unique monomial of this $(\tilde{p}_1, \dots, \tilde{p}_m)$-weight $\tilde{q}$, then item (a) and (A-3) lead to a contradiction, as $\widetilde{Q}_F(t^{\tilde{p}_1}, \dots, t^{\tilde{p}_m})$ will be the dominant term in $g_F(t^{\tilde{p}_1}, \dots, t^{\tilde{p}_m})$ when $t \gg 1$. If $\widetilde{Q}_F$ is not unique, we take a small perturbation $\tilde{\omega}(d) := (\tilde{p}_1 + 1/d, \dots, \tilde{p}_m + 1/d^m)$ of $(\tilde{p}_1, \dots, \tilde{p}_m)$ with $d > \deg P_F$. Now, each monomial in P_F has a unique $\tilde{\omega}(d)$-weight, and therefore without loss of generality we may assume $\widetilde{Q}_F$ is the monomial of maximal $\tilde{\omega}(d)$-weight in P_F. Taking $t \gg 1$ and applying a variant of (A-3), with $\tilde{\omega}(d)$ instead of $(\tilde{p}_1, \dots, \tilde{p}_m)$, yields a contradiction as before.

(c) The only moniomials with $(\tilde{p}_1, \dots, \tilde{p}_m)$-weight $\tilde{p}$ are $x_1^{p_1}, \dots, x_m^{p_m}$. Indeed, the condition $\sum_{j=1}^{m} n_j \tilde{p}_j = \tilde{p}$ guarantees that each n_j is divisible by p_j.

Item (2) now follows from (A-2) and by (b) and (c) above. We find $\lambda_2, \dots, \lambda_m \in \mathbb{Z}_{\geq 1}$ such that $(t^{\tilde{p}_1}, \lambda_2, \dots, \lambda_m) \in C$ for $t \gg 1$. This implies

$$\lim_{t\to\infty} \frac{g_F(t^{\tilde{p}_1}, \lambda_2, \dots, \lambda_m)}{t^{\tilde{p}}} = \lim_{t\to\infty} \frac{P_F(t^{\tilde{p}_1}, \lambda_2, \dots, \lambda_m)}{t^{\tilde{p}}} = d_1,$$

and hence $d_1 \geq 1$ by ($\star$). More generally, to show that $d_j \geq 1$, we consider

$$(t^{2\tilde{p}_1 - 1}, \dots, t^{2\tilde{p}_{j-1} - 1}, t^{2\tilde{p}_j}, \lambda_{j+1}, \dots, \lambda_m) \in C$$

for $t \gg 1$ (note that $2\tilde{p}_1 - 1 > 2\tilde{p}_2 - 1 > \cdots > 2\tilde{p}_{j-1} - 1 > 2\tilde{p}_j$ for every j). This finishes the proof. $\square$

Remark A.3. Note that without the assumption on the p_i's, one can get tighter approximations than $\sum_{j=1}^{m} x_j^{p_j}$. For example, $\frac{4}{3}(x_1^2 - x_1x_2^2 + x_2^4)$ gives a tighter upper bound for $\max(x_1^2, x_2^4)$, than $x_1^2 + x_2^4$, since $\frac{4}{3}(x_1^2 - x_1x_2^2 + x_2^4) \leq \frac{4}{3}\max(x_1^2, x_2^4)$.

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