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**Free rational curves on low degree hypersurfaces  
and the circle method**

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# Free rational curves on low degree hypersurfaces and the circle method

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We use a function field version of the Hardy–Littlewood circle method to study the locus of free rational curves on an arbitrary smooth projective hypersurface of sufficiently low degree. On the one hand this allows us to bound the dimension of the singular locus of the moduli space of rational curves on such hypersurfaces and, on the other hand, it sheds light on Peyre’s reformulation of the Batyrev–Manin conjecture in terms of slopes with respect to the tangent bundle.

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## 1. Introduction

Let  $X \subset \mathbb{P}^{n-1}$  be a smooth hypersurface of degree  $d \geq 3$ , over a field  $K$  whose characteristic exceeds  $d$  if it is positive. This paper has two aspects. On the one hand, motivated by questions in algebraic geometry, we shall be interested in the locus of points corresponding to free rational curves inside the moduli space  $\mathcal{M}_{0,0}(X, e)$  of degree  $e$  rational curves on  $X$ . On the other hand, by working over a finite field, we shall establish a function field analogue of a recent conjecture due to Peyre [2017] about the distribution of “sufficiently free” rational points of bounded height on Fano varieties.

**1A. Geometry.** The expected dimension of  $\mathcal{M}_{0,0}(X, e)$  is  $(n - d)e + n - 5$ , a fact that is known to hold for generic  $X$  if  $n \geq d + 3$ , thanks to Riedl and Yang [2019]. It follows from work of Browning and Vishe [2017] that  $\mathcal{M}_{0,0}(X, e)$  is irreducible and has the expected dimension for any smooth  $X$ , provided that  $n > (5d - 4)2^{d-1}$ . Our first result strengthens this.

**Theorem 1.1.** *Let  $d \geq 3$ , let  $e \geq 1$  and let  $n > (2d - 1)2^{d-1}$ . Then  $\mathcal{M}_{0,0}(X, e)$  is an irreducible locally complete intersection of the expected dimension.*

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We can also bound the dimension of the singular locus of  $\mathcal{M}_{0,0}(X, e)$ , as follows.

**Theorem 1.2.** *Let  $d \geq 3$ , let  $e \geq 1$  and let  $n > 3(d-1)2^{d-1}$ . Then the space  $\mathcal{M}_{0,0}(X, e)$  is smooth outside a set of codimension at least*

$$\left(\frac{n}{2^{d-2}} - 6(d-1)\right) \left(1 + \left\lfloor \frac{e+1}{d-1} \right\rfloor\right).$$

*In particular, whenever these inequalities are satisfied, it is generically smooth and reduced.*

For  $n \geq 2d+1$  and generic  $X$  of degree  $d \geq 3$ , Harris, Roth and Starr [Harris et al. 2004] have also shown that  $\mathcal{M}_{0,0}(X, e)$  is generically smooth. Note that, provided  $n > 3(d-1)2^{d-1}$ , the codimension goes to  $\infty$  in Theorem 1.2 when either  $e$  or  $n$  does, with  $d$  fixed. Moreover, when both  $e$  and  $n$  are large with respect to  $d$ , the codimension is at least approximately  $1/(2^{d-2}(d-1))$  of the total dimension.

Our work addresses some questions of Eisenbud and Harris [2016, Section 6.8.1] concerning the Fano variety of lines  $F_1(X) = \mathcal{M}_{0,0}(X, 1)$  associated to a smooth hypersurface  $X \subset \mathbb{P}^{n-1}$  of degree  $d$ . Specifically, their question (a) asks whether  $F_1(X)$  is reduced and irreducible if  $n > d+1$  and (b) asks whether the dimension of the singular locus of  $F_1(X)$  can be bounded in terms of  $d$  alone. Theorems 1.1 and 1.2 answer the first question affirmatively for  $n > 3(d-1)2^{d-1}$  and give some weak evidence in support of the second question, by showing that it grows with  $n$  more slowly than the dimension of the whole space. Furthermore, we handle the analogous conjectures with higher degree curves, with no loss in the dependence on  $n$ , meaning that for large enough  $e$  we do better than their predicted bound  $d \leq n/e$ .

By comparison, Starr [2003] has proved that if  $n \geq d+e$  and  $X$  is generic, then  $\mathcal{M}_{0,0}(X, e)$  has canonical singularities, which implies in particular that it is smooth outside a set of codimension at least 2. It does not seem possible that our method will prove that  $\mathcal{M}_{0,0}(X, e)$  has canonical singularities. By [Mustața 2001] and [Lang and Weil 1954] this is equivalent to the conjunction of an infinite sequence of Diophantine estimates (in the spirit of Definition 3.7), but for fixed  $n, d$  and  $e$  it seems unlikely that the circle method is able to handle more than finitely many of them. In unpublished work, Starr and Tian use a bend-and-break approach to produce a less restrictive lower bound for the codimension of the singular locus for a general hypersurface  $X \subset \mathbb{P}^{n-1}$  of degree  $d$ . However, their method never proves a lower bound for the codimension greater than  $n$ , whereas our work achieves this if  $e$  is sufficiently large.

Comparing the various results, we see that Theorem 1.2 holds for a much more restricted range of  $n$  (unless  $e$  is very large relative to  $d$ ) but it is valid for an arbitrary smooth hypersurface, rather than just a general one.

It should be possible to adapt our strategy to prove results about moduli spaces of genus  $g$  curves on  $X$ . However, the codimension we obtain for the whole moduli space will not be any better than the codimension we can prove for the space of maps from a fixed genus  $g$  curve to  $X$ . In particular the codimension will shrink as  $g$  grows, so the bound obtained would only be suitable for  $e$  sufficiently large with respect to  $g$ .

Let  $\mathcal{T}_X$  be the tangent bundle associated to the smooth hypersurface  $X \subset \mathbb{P}^{n-1}$  (as defined in [Hartshorne 1977, page 180], for example). Our remaining result deals specifically with free curves and so we recall the definition here.

**Definition 1.3.** Let  $c : \mathbb{P}^1 \rightarrow X$  be a rational curve and let  $\varrho \in \mathbb{Z}$ . We say that  $c$  is  $\varrho$ -free if  $c^* \mathcal{T}_X \otimes \mathcal{O}_{\mathbb{P}^1}(-\varrho)$  is globally generated.

We shall follow common convention and say that  $c$  is free if it is 0-free, and very free if it is 1-free. One easily checks that this agrees with the standard definition that  $c$  is free if  $c^* \mathcal{T}_X$  is globally generated and very free if  $c^* \mathcal{T}_X$  is ample. The definition of free curves goes back to pioneering work of Kollár, Miyaoka and Mori [Kollár et al. 1992a, Section 1] on rational connectedness for Fano varieties, and they feature heavily in work of Kollár [1996, Section II.3]. We have taken Definition 1.3 from work of Debarre [2001, Definition 4.5], which appears to be the first time that the notion of being  $\varrho$ -free occurs, for varying  $\varrho \in \mathbb{Z}$ .

**Remark 1.4.** If  $c$  is a  $\varrho$ -free rational curve on  $X$  then it follows from Definition 1.3 that  $\deg(c^* \mathcal{T}_X) \geq \text{rank}(c^* \mathcal{T}_X)\varrho$ . In general, the pull-back of the tangent bundle has rank  $n - 2$  and degree  $e(n - d)$ . In this way we see that no degree  $e$  rational curve on  $X$  is ever  $(\lfloor e(n - d)/(n - 2) \rfloor + 1)$ -free. If  $d \geq 2$  then this implies that  $\varrho \leq e$ , for any  $\varrho$ -free rational curve  $\mathbb{P}^1 \rightarrow X$ .

We let  $U_\varrho \subset \mathcal{M}_{0,0}(X, e)$  be the Zariski open set that parametrizes degree  $e$  maps from  $\mathbb{P}^1$  to  $X$  that are  $\varrho$ -free. We write  $Z_\varrho = \mathcal{M}_{0,0}(X, e) \setminus U_\varrho$  for the complement. This is the closed set parametrizing degree  $e$  maps  $\mathbb{P}^1 \rightarrow X$  that are not  $\varrho$ -free. We shall prove the following bound for its dimension.

**Theorem 1.5.** Let  $d \geq 3$  and  $n > 3(d - 1)2^{d-1}$ . Assume that  $\varrho \geq -1$  and

$$e \geq (\varrho + 1) \left( 2 + \frac{1}{d - 2} \right). \quad (1-1)$$

Then

$$\dim Z_\varrho \leq (n - d)e + n - 5 + 2(d - 1) \left\lfloor \frac{\varrho + 1}{2} \right\rfloor - \left( \frac{n}{2^{d-2}} - 6(d - 1) \right) \left( 1 + \left\lfloor \frac{e - \varrho}{d - 1} \right\rfloor - \left\lfloor \frac{\varrho + 1}{2} \right\rfloor \right). \quad (1-2)$$

The notion of free rational curves was originally introduced as a tool to study uniruled and rational connectedness properties of varieties. Taking  $\varrho = 1$  it follows from Theorems 1.1 and 1.5 that  $U_1 \neq \emptyset$  if  $K$  is algebraically closed and  $e$  is sufficiently large. Hence, by appealing to [Debarre 2001, Corollary 4.17], we deduce that any smooth hypersurface  $X \subset \mathbb{P}^{n-1}$  of degree  $d$  is rationally connected if  $d \geq 3$  and  $n > 3(d - 1)2^{d-1}$ . This recovers a weak form of the well-known result, independently due to Campana [1992] and Kollár, Miyaoka and Mori [Kollár et al. 1992b] that Fano varieties are rationally connected. In fact both proofs use reduction to characteristic  $p$ , but they use different properties of characteristic  $p$  varieties, with [Kollár et al. 1992b] relying on Frobenius pull-back and our work using the Lang–Weil estimates.

Theorem 1.2 is derived from Theorem 1.5, which is proved using analytic number theory and builds on an approach employed by Browning and Vishe [2017]. (Theorem 1.1 uses essentially the same approach as [loc. cit.], with one improvement to a key lemma.) One begins by working over a finite field  $K = \mathbb{F}_q$  of characteristic  $> d$ . We bound the dimension of  $Z_\varrho$  by counting the number of points defined over a finite extension of  $\mathbb{F}_q$  that lie in it. In Section 3, we will give an explicit description of this locus in terms

of a system of two Diophantine equations defined over the function field  $\mathbb{F}_q(T)$ . Let  $f \in \mathbb{F}_q[x_1, \dots, x_n]$  be a nonsingular form of degree  $d$  that defines the hypersurface  $X \subset \mathbb{P}^{n-1}$ . Given  $\varrho \in \mathbb{Z}$ , we shall see that the primary counting function of interest to us, denoted  $N_\varrho(q, e, f)$ , is the one that counts vectors  $(\mathbf{g}, \mathbf{h}) \in \mathbb{F}_q[T]^{2n}$ , where  $g_1, \dots, g_n$  have degree at most  $e$  and no common zero, with at least one of degree exactly  $e$ , and where  $h_1, \dots, h_n$  have degree at most  $e - 1 - \varrho$ , such that

$$f(g_1, \dots, g_n) = 0 \quad \text{and} \quad \sum_{i=1}^n h_i \frac{\partial f}{\partial x_i}(g_1, \dots, g_n) = 0. \quad (1-3)$$

Since each partial derivative of  $f$  is a degree  $d - 1$  polynomial, we obtain a linear equation for  $\mathbf{h} \in \mathbb{F}_q[T]^n$  where the coefficients have degree at most  $(d - 1)e$  in  $T$ . Standard heuristics lead us to expect that, for typical  $\mathbf{g}$ , the number of available  $\mathbf{h}$  is  $q^{(e-\varrho)(n-1)-(d-1)e} = q^{e(n-d)-\varrho(n-1)}$ . (In fact, we shall see in Lemmas 3.4 and 3.5 that this is true only if the map  $\mathbb{P}^1 \rightarrow X$  represented by  $\mathbf{g}$  is  $\varrho$ -free.) Thus we expect that  $N_\varrho(q, e, f)$  is approximated by  $q^{e(n-d)-\varrho(n-1)}N(q, e, f)$ , where  $N(q, e, f)$  is the number of vectors  $\mathbf{g} \in \mathbb{F}_q[T]^n$  such that  $f(\mathbf{g}) = 0$ , where  $g_1, \dots, g_n$  have degree at most  $e$  and no common zero, with at least one of degree exactly  $e$ .

In Section 4, we apply the function field version of the Hardy–Littlewood circle method to study the system of degree  $d$  equations (1-3), expressing the number of solutions as an integral of an exponential sum. We shall show that the major arc contribution to this integral cancels almost exactly with the expected approximation  $q^{e(n-d)-\varrho(n-1)}N(q, e, f)$ . In Section 5, we prove an upper bound on all other arcs, taking special care to make all of our implied constants depend explicitly on the size of the finite field  $q$ . The standard way of proceeding involves  $d - 1$  applications of Weyl differencing, a process that would ultimately require  $n > 3(d - 1)2^d$  variables overall. We shall gain a 50% reduction in the number of variables by exploiting the special shape of the Diophantine system (1-3). Finally, we bring everything together and apply the Lang–Weil estimates [Lang and Weil 1954] to turn the bound for  $\#Z_\varrho(\mathbb{F}_q)$  into a bound for the dimension of  $Z_\varrho$ . An application of spreading-out shows that the dimension bound holds over an arbitrary base field  $K$  such that  $\text{char}(K) > d$  if it is positive.

**1B. Arithmetic.** In our geometric investigation of  $Z_\varrho$  we take the point of view that  $e$  and  $\varrho$  are fixed and  $q \rightarrow \infty$ . In this subsection we assume that the finite field is fixed, but we allow the parameters  $e$  and  $\varrho$  to tend to infinity appropriately.

Suppose that  $V$  is a smooth projective geometrically integral Fano variety defined over a number field  $K$ . For suitable Zariski open subsets  $U \subset V$  the Batyrev–Manin conjecture [Franke et al. 1989] makes a precise prediction about the asymptotic behavior of the counting function

$$N_U(B) = \#\{x \in U(K) : H_{\omega_V^{-1}}(x) \leq B\},$$

as  $B \rightarrow \infty$ , where  $H_{\omega_V^{-1}} : V(K) \rightarrow \mathbb{R}$  is an anticanonical height function. These conjectures are flawed, however, since it has been discovered that the presence of Zariski dense thin sets in  $V(K)$  may skew the expected asymptotics. Recently, Peyre [2017] has embarked on an ambitious program to repair the

conjecture by associating a measure of “freeness”  $\ell(x) \in [0, 1]$  to any  $x \in V(K)$  and only counting those rational points for which  $\ell(x) \geq \varepsilon_B$ , where  $\varepsilon_B$  is a function of  $B$  decreasing to zero sufficiently slowly; see [loc. cit., Definition 6.11] for a precise statement. Peyre’s function  $\ell(x)$  is defined using Arakelov geometry and the theory of slopes associated to the tangent bundle  $\mathcal{T}_V$ .

We can lend support to Peyre’s freedom prediction [loc. cit., Section 6] by studying smooth hypersurfaces of low degree in the setting of global fields of positive characteristic. Let  $X \subset \mathbb{P}^{n-1}$  be a smooth hypersurface of degree  $d$  defined over a finite field  $\mathbb{F}_q$  whose characteristic exceeds  $d$ . We put

$$N_X^{\varepsilon\text{-free}}(B) = \#\{x \in X(K) : \ell(x) \geq \varepsilon, H_{-\omega_X}(x) \leq q^B\}, \quad (1-4)$$

where  $K = \mathbb{F}_q(T)$  is the rational function field and  $\ell(x)$  will be defined in Section 6. The expectation is that for a suitable range of  $\varepsilon$ ,  $N_X^{\varepsilon\text{-free}}(B)$  should have the same asymptotic behavior as the usual counting function  $N_X(B)$ , as  $B \rightarrow \infty$ . The following result confirms this and will be proved in Section 6.

**Theorem 1.6.** *Let  $d \geq 3$ , let  $n > 3(d-1)2^{d-1}$  and let*

$$0 \leq \varepsilon < \frac{n-1}{(n-d)(d-1)2^{d-1}}.$$

*Then there exists  $\delta > 0$  such that*

$$N_X^{\varepsilon\text{-free}}(B) = c_X q^B + O(q^{(1-\delta)B}),$$

*as  $B \rightarrow \infty$ , where  $c_X$  is the function field analogue of the constant predicted by Peyre [1995]. Furthermore, the implied constant only depends on  $q$  and  $f$ .*

Note that this result does not require  $\varepsilon_B$  to decrease to zero, but only to stay below some fixed constant. This may be because the hypersurface  $X$  has Picard rank one, since Peyre [2017, Section 7.2] has shown that for the product  $\mathbb{P}^1 \times \mathbb{P}^1$  one requires  $\varepsilon_B \rightarrow 0$  for the asymptotic formula to be true. Finally, one can see from the arguments in Theorem 1.6 that we can take the upper bound for  $\varepsilon$  to be significantly greater than  $(n-1)/((n-d)(d-1)2^{d-1})$  when  $n$  is large. (In fact, the cutoff is allowed to approach  $1/(d+1)$  as  $n \rightarrow \infty$ .)

With appropriate adjustments to the proof of Theorem 1.6, it is also possible to handle the corresponding result for smooth hypersurfaces of low degree defined over  $\mathbb{Q}$ , with Poisson summation taking the place of the Riemann–Roch arguments that feature in Section 3. This is the object of our concurrent work [Browning and Sawin 2020a].

## 2. Examples

As usual,  $X \subset \mathbb{P}^{n-1}$  is assumed to be a smooth hypersurface of degree  $d \geq 3$ , over a field  $K$  whose characteristic is either 0 or  $> d$ . While the latter condition arises very naturally in our argument (as explained in Remark 5.5), the following result shows that the statement of Theorem 1.5 is actually false when it is dropped.

**Lemma 2.1.** *Let  $K = \overline{\mathbb{F}}_p$  for a prime  $p$  and let  $X \subset \mathbb{P}^{n-1}$  be the Fermat hypersurface*

$$x_1^d + \cdots + x_n^d = 0.$$

*Assume that  $p \nmid d$  and  $d \neq ap^r - 1$  for any  $r \in \mathbb{N}$  and  $a \in \{0, \dots, p-1\}$ . Then  $X$  is smooth, none of the curves in  $\mathcal{M}_{0,0}(X, 1)$  are  $(-1)$ -free, and  $\dim \mathcal{M}_{0,0}(X, 1) > 2n - d - 5$ .*

*Proof.* The moduli space of  $n$ -tuples of polynomials of degree  $\leq 1$  satisfying the equation  $x_1^d + \cdots + x_n^d = 0$  is a  $\mathrm{GL}_2$ -bundle over the moduli stack  $\mathcal{M}_{0,0}(X, 1)$  parametrizing lines in  $X$ , because for each line we can choose any basis of the corresponding two-dimensional vector space. Thus its dimension is equal to  $4 + \dim \mathcal{M}_{0,0}(X, 1)$ . This space is cut out by  $d + 1$  equations in  $2n$  variables, where  $\binom{d}{i}$  divides all coefficients of the  $i$ -th equation, for  $0 \leq i \leq d$ . By Lucas' theorem it follows that  $p \mid \binom{d}{i}$  if and only if at least one of the base  $p$  digits of  $i$  is greater than the corresponding base  $p$  digit of  $d$ . In this way we see that  $p \mid \binom{d}{i}$  for some  $0 \leq i \leq d$  if and only if  $d$  does not take the form  $ap^r - 1$  for some  $a \in \{0, \dots, p-1\}$ . But then the space is cut out by fewer than  $d + 1$  equations in  $2n$  variables. This implies that it has dimension greater than  $2n - d - 1$ , whence  $\dim \mathcal{M}_{0,0}(X, 1) > 2n - d - 5$ . Furthermore, since the dimension near each curve is greater than the expected dimension, it follows from Lemma 3.1 that they are not  $(-1)$ -free. Finally, the Fermat hypersurface is smooth over  $K$  if and only if  $p \nmid d$ .  $\square$

This example generalizes a discussion of Debarre [2001, Section 2.15]. It shows that for typical  $p < d$  the statements of Theorems 1.1 and 1.5 are false for fields of characteristic  $p$ .

Returning to the general setting, the following result provides examples of curves that are not  $q$ -free.

**Lemma 2.2.** *Let  $d, m, n \in \mathbb{N}$  with  $d \geq 3$  and  $m \leq n/2$ . Let  $K$  be an infinite field. There exists a nonsingular form  $f(x_1, \dots, x_n)$  over  $K$  of degree  $d$ , such that*

$$f(x_1, \dots, x_m, 0, \dots, 0) = \frac{\partial f}{\partial x_j}(x_1, \dots, x_m, 0, \dots, 0) = 0$$

*for all  $x_1, \dots, x_m$  and all  $j \leq n - m$ . For such a polynomial, every map  $c : \mathbb{P}^1 \rightarrow X$  of degree  $e$  that factors through  $\mathbb{P}^{m-1} \subseteq X \subseteq \mathbb{P}^{n-1}$  fails to be  $(\lfloor e(m-d)/(m-1) \rfloor + 1)$ -free. The moduli space of such rational curves has dimension  $m(e+1) - 4$ .*

Let  $X \subset \mathbb{P}^{n-1}$  be a smooth hypersurface with underlying polynomial  $f$ , as in the lemma. Taking  $m = d$  and  $q = 0$ , we see that when  $n > 2d$  the space  $Z_1$  of non-very-free rational curves  $\mathbb{P}^1 \rightarrow X$  of degree  $e$  has dimension at least  $d(e+1) - 4$ .

*Proof of Lemma 2.2.* Without the nonsingularity condition, the space of such polynomials is linear. The singular polynomials form a closed subset. To prove the existence, it is sufficient to show that this subset has codimension 1. The set of singular polynomials is the projection from the product of this linear space with  $\mathbb{P}^{n-1}$  of the set of pairs of a point and a polynomial singular at that point. For elements in  $\mathbb{P}^{m-1} \subseteq \mathbb{P}^{n-1}$ , the space of polynomials singular at that point has codimension  $m$ , as it is defined by the  $m$  independent conditions  $\frac{\partial f}{\partial x_j}(x_1, \dots, x_m, 0, \dots, 0) = 0$  for  $n - m + 1 \leq j \leq n$ . For all other elements, we claim that the  $n$  conditions  $\frac{\partial f}{\partial x_j}(x_1, \dots, x_n) = 0$  for  $1 \leq j \leq n$  define a codimension  $n$  subspace. To



see this we may take a linear form  $l$  in the last  $n - m$  coordinates that is nonzero at that point. Then the  $n$ -dimensional space of polynomials generated by  $x_j l^{d-1}$  for  $1 \leq j \leq n$  lie in the linear subspace, since  $d - 1 \geq 2$ . But only the zero element in that subspace satisfies all  $n$  conditions. It follows that the singular locus is the union of the projection of a codimension  $m$  bundle on  $\mathbb{P}^{m-1}$  and a codimension  $n$  bundle on its complement in  $\mathbb{P}^{n-1}$ . Thus the singular locus has codimension at least one, as desired.

For the freeness, we use the Euler exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^{n-1}} \rightarrow \mathcal{O}_{\mathbb{P}^{n-1}}(1)^n \rightarrow \mathcal{T}_{\mathbb{P}^{n-1}} \rightarrow 0. \quad (2-1)$$

Consider the map  $\mathcal{O}_{\mathbb{P}^{n-1}}(1)^n \rightarrow \mathcal{O}_{\mathbb{P}^{n-1}}(1)^m$  given by projection onto the last  $m$  factors. Because  $m \leq n/2$  the composition of this projection with the map  $\mathcal{O}_{\mathbb{P}^{n-1}} \rightarrow \mathcal{O}_{\mathbb{P}^{n-1}}(1)^n$  vanishes on  $\mathbb{P}^{m-1}$ . So over  $\mathbb{P}^{m-1}$ , we obtain a map  $\mathcal{T}_{\mathbb{P}^{n-1}} \rightarrow \mathcal{O}_{\mathbb{P}^{n-1}}(1)^m$ .

Next consider the exact sequence  $0 \rightarrow \mathcal{T}_{\mathbb{P}^{n-1}} \rightarrow \mathcal{T}_X \rightarrow \mathcal{O}_X(d) \rightarrow 0$  on  $X$ . The second map of this sequence is the dot product with the derivative of  $f$ . By assumption on  $f$ , restricted to  $\mathbb{P}^{m-1}$ , this map factors through the projection onto the last  $m$  vectors. Hence we obtain an exact sequence

$$0 \rightarrow \mathcal{V} \rightarrow \mathcal{O}_{\mathbb{P}^{m-1}}(1)^m \rightarrow \mathcal{O}_{\mathbb{P}^{m-1}}(d) \rightarrow 0$$

whose kernel  $\mathcal{V}$  is a vector bundle on  $\mathbb{P}^{m-1}$  of degree  $m - d$ , which arises as a quotient of  $\mathcal{T}_X$ .

For  $c: \mathbb{P}^1 \rightarrow X$  a map of degree  $e$  whose image lies in  $\mathbb{P}^{m-1}$ ,  $c^*\mathcal{V}$  is a vector bundle of degree  $e(m - d)$  on  $\mathbb{P}^1$  which arises as a quotient of  $c^*\mathcal{T}_X$ . Because  $c^*\mathcal{V}$  splits as a direct sum of  $m - 1$  line bundles, it must contain some line bundle summand of degree at most  $e(m - d)/(m - 1)$ , and we can round down to the nearest integer. Hence  $c^*\mathcal{T}_X$  has some line bundle summand of degree at most  $\lfloor e(m - d)/(m - 1) \rfloor$  and hence  $c$  is not  $(\lfloor e(m - d)/(m - 1) \rfloor + 1)$ -free.

The dimension estimate is the standard calculation for the moduli space of rational curves in projective space.  $\square$

Even for a general hypersurface there are some non-very-free curves. Indeed, for such a variety, the moduli space of lines has dimension  $2n - d - 5$ , and each line admits a  $(2e + 1)$ -dimensional moduli space of degree  $e$  maps from  $\mathbb{P}^1$  to that line. Because the pull-back of the tangent bundle to a line has rank  $n - 2$  and degree  $n - d$ , it contains some summand of degree at most 0 as soon as  $d \geq 2$ , and so every pull-back of it has a summand of the same degree, and so these degree  $e$  coverings of lines fail to be 1-free. Hence, for a general hypersurface  $X \subset \mathbb{P}^{n-1}$  of degree  $d$ , we have  $\dim Z_1 \geq 2(n + e) - d - 7$ .

These examples show that the dimension of the moduli space of non-very-free curves can grow linearly in  $n$  and it can grow linearly in  $e$ . We do not know if it can grow linearly in  $ne$ , as the dimension of  $\mathcal{M}_{0,0}(X, e)$  does.

### 3. Vector bundles on $\mathbb{P}^1$

Let  $f$  be a homogeneous polynomial of degree  $d$  in  $n$  variables over a field  $K$  and let  $X \subset \mathbb{P}^{n-1}$  be its projective zero locus. Assume that  $X$  is smooth and let  $\mathcal{T}_X$  be its tangent bundle. In this section we

investigate the geometry of  $\varrho$ -free rational curves  $c : \mathbb{P}^1 \rightarrow X$ , in the sense of Definition 1.3. It turns out that there is a natural characterization of the  $(-1)$ -free curves, which we recall here.

**Lemma 3.1.** *A rational curve  $c : \mathbb{P}^1 \rightarrow X$  of degree  $e$  is  $(-1)$ -free if and only if, in a neighborhood of  $c$ , the moduli space of rational curves on  $X$  is smooth of dimension  $(n-d)e + n - 5$ .*

Under the assumptions of Theorem 1.1 or [Riedl and Yang 2019],  $\mathcal{M}_{0,0}(X, e)$  has dimension  $(n-d)e + n - 5$ , so this is simply equivalent to  $\mathcal{M}_{0,0}(X, e)$  being smooth at  $c$ .

*Proof of Lemma 3.1.* The tangent space of the moduli space of rational curves at  $c$  is

$$H^0(\mathbb{P}^1, c^* \mathcal{T}_X) / H^0(\mathbb{P}^1, \mathcal{T}_{\mathbb{P}^1}).$$

Note that  $H^0(\mathbb{P}^1, \mathcal{T}_{\mathbb{P}^1})$  has dimension 3. By Riemann–Roch,

$$\dim H^0(\mathbb{P}^1, c^* \mathcal{T}_X) - \dim H^1(\mathbb{P}^1, c^* \mathcal{T}_X) = \dim(c^* \mathcal{T}_X) + \deg(c^* \mathcal{T}_X) = n - 2 + e(n - d).$$

Hence if  $c$  is a smooth point on a component of dimension  $n - 5 + e(n - d)$  then  $H^0(\mathbb{P}^1, c^* \mathcal{T}_X)$  has dimension  $n - 2 + e(n - d)$  and so  $H^1(\mathbb{P}^1, c^* \mathcal{T}_X)$  vanishes. Thus [Debarre 2001, Remark 4.6] implies that  $c$  is  $(-1)$ -free.

Conversely if  $c$  is  $(-1)$ -free then  $H^1(\mathbb{P}^1, c^* \mathcal{T}_X)$  vanishes by [loc. cit., Remark 4.6], so deformations are unobstructed. Thus the moduli space is smooth at  $c$ , and the dimension of the tangent space to the moduli space is  $n - 5 + e(n - d)$ .  $\square$

Let  $\hat{\mathcal{T}}_X$  be the inverse image of  $\mathcal{T}_X \subseteq \mathcal{T}_{\mathbb{P}^{n-1}}$  under the map  $\mathcal{O}_{\mathbb{P}^{n-1}}(1)^n \rightarrow \mathcal{T}_{\mathbb{P}^{n-1}}$  in the Euler sequence (2-1). This yields

$$0 \rightarrow \mathcal{O}_X \rightarrow \hat{\mathcal{T}}_X \rightarrow \mathcal{T}_X \rightarrow 0,$$

so that in particular  $\hat{\mathcal{T}}_X$  is a vector bundle of rank  $n - 1$  on  $X$ . With this in mind, we refine Definition 1.3 as follows.

**Definition 3.2.** We say that  $c : \mathbb{P}^1 \rightarrow X$  is strongly  $\varrho$ -free if  $c^* \hat{\mathcal{T}}_X \otimes \mathcal{O}_{\mathbb{P}^1}(-\varrho)$  is globally generated.

We thank Paul Nelson for asking a question that suggested the above definition, and which turns out to simplify our argument compared to studying the tangent bundle directly.

**Lemma 3.3.** *If  $c$  is strongly  $\varrho$ -free, then it is  $\varrho$ -free.*

*Proof.* This follows from the fact that  $\mathcal{T}_X$  is a quotient of  $\hat{\mathcal{T}}_X$  and if a vector bundle is globally generated then every quotient is globally generated.  $\square$

**Lemma 3.4.** *We have*

$$\dim H^0(\mathbb{P}^1, c^* \hat{\mathcal{T}}_X \otimes \mathcal{O}_{\mathbb{P}^1}(-1 - \varrho)) \geq e(n - d) - \varrho(n - 1)$$

*with equality if and only if  $c$  is strongly  $\varrho$ -free.*

*Proof.* Because  $\mathcal{I}_X$  is the kernel of the map  $df : \mathcal{I}_{\mathbb{P}^{n-1}} \rightarrow \mathcal{O}_{\mathbb{P}^{n-1}}(d)$ ,  $\hat{\mathcal{I}}_X$  is the kernel of a map  $\mathcal{O}_{\mathbb{P}^{n-1}}(1)^n \rightarrow \mathcal{O}_{\mathbb{P}^{n-1}}(d)$  and hence has degree  $n - d$ . Thus  $c^* \hat{\mathcal{I}}_X$  has degree  $e(n - d)$ . Because it has rank  $n - 1$ , its tensor product with  $\mathcal{O}_{\mathbb{P}^1}(-1 - \varrho)$  has degree  $e(n - d) - \varrho(n - 1) - (n - 1)$ . Hence by Riemann–Roch, the dimension of its space of global sections is

$$\dim H^0(\mathbb{P}^1, c^* \hat{\mathcal{I}}_X \otimes \mathcal{O}_{\mathbb{P}^1}(-1 - \varrho)) = e(n - d) - \varrho(n - 1) + \dim H^1(\mathbb{P}^1, c^* \hat{\mathcal{I}}_X \otimes \mathcal{O}_{\mathbb{P}^1}(-1 - \varrho)).$$

It now suffices to show that  $H^1(\mathbb{P}^1, c^* \hat{\mathcal{I}}_X \otimes \mathcal{O}_{\mathbb{P}^1}(-1 - \varrho))$  vanishes if and only if  $c^* \hat{\mathcal{I}}_X \otimes \mathcal{O}_{\mathbb{P}^1}(-\varrho)$  is globally generated. We can assume that

$$c^* \hat{\mathcal{I}}_X = \bigoplus_{i=1}^{n-1} \mathcal{O}_{\mathbb{P}^1}(k_i).$$

Then  $H^1(\mathbb{P}^1, c^* \hat{\mathcal{I}}_X \otimes \mathcal{O}_{\mathbb{P}^1}(-1 - \varrho)) = 0$  if and only if  $k_i - 1 - \varrho \geq -1$  for all  $i$ , which happens if and only if  $k_i - \varrho \geq 0$  for all  $i$ , which occurs if and only if  $c^* \hat{\mathcal{I}}_X \otimes \mathcal{O}_{\mathbb{P}^1}(-\varrho)$  is globally generated.  $\square$

Vector notation such as  $\mathbf{g}$  or  $\mathbf{h}$  will denote  $n$ -tuples of polynomials in  $T$ . Let  $\mathbf{g}$  be an  $n$ -tuple of polynomials in  $T$  of degree at most  $e$ , at least one of degree  $e$ , with no common zero, and such that  $f(\mathbf{g}) = 0$ . These conditions ensure that  $(g_1 : \dots : g_n)$  defines a degree  $e$  map  $c : \mathbb{P}^1 \rightarrow X$ .

**Lemma 3.5.**  $H^0(\mathbb{P}^1, c^* \hat{\mathcal{I}}_X \otimes \mathcal{O}_{\mathbb{P}^1}(-1 - \varrho))$  is isomorphic to the space of  $n$ -tuples  $\mathbf{h}$  of polynomials in  $T$  of degree  $\leq e - 1 - \varrho$ , such that  $\nabla f(\mathbf{g}) \cdot \mathbf{h} = 0$ .

*Proof.* In this proof it will be convenient to set  $\mathcal{B} = c^* \hat{\mathcal{I}}_X \otimes \mathcal{O}_{\mathbb{P}^1}(-1 - \varrho)$ . We have an exact sequence  $0 \rightarrow \hat{\mathcal{I}}_X \rightarrow \mathcal{O}_{\mathbb{P}^{n-1}}(1)^n \rightarrow \mathcal{O}_{\mathbb{P}^{n-1}}(d) \rightarrow 0$ , with the last map given by multiplication by the gradient of  $f$ . Thus we obtain an exact sequence

$$0 \rightarrow \mathcal{B} \rightarrow c^* \mathcal{O}_{\mathbb{P}^{n-1}}(1)^n \otimes \mathcal{O}_{\mathbb{P}^1}(-1 - \varrho) \rightarrow c^* \mathcal{O}_{\mathbb{P}^{n-1}}(d) \otimes \mathcal{O}_{\mathbb{P}^1}(-1 - \varrho) \rightarrow 0$$

which simplifies to

$$0 \rightarrow \mathcal{B} \rightarrow \mathcal{O}_{\mathbb{P}^1}(e - 1 - \varrho)^n \rightarrow \mathcal{O}_{\mathbb{P}^1}(de - 1 - \varrho) \rightarrow 0,$$

because  $c$  has degree  $e$ . Applying the cohomology long exact sequence, we see that  $H^0(\mathbb{P}^1, \mathcal{B})$  is the kernel of the natural map

$$H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(e - 1 - \varrho)^n) \rightarrow H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(de - 1 - \varrho)),$$

given by multiplication by the gradient of  $f$ . Since  $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(e - 1 - \varrho)^n) = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(e - 1 - \varrho))^n$  is the space of  $n$ -tuples of polynomials of degree at most  $e - 1 - \varrho$ , this is exactly the stated space.  $\square$

We now assume  $K = \mathbb{F}_q$  is a finite field. Thus  $f \in \mathbb{F}_q[x_1, \dots, x_n]$  is a nonsingular form of degree  $d \geq 3$ . We assume throughout that  $\text{char}(\mathbb{F}_q) > d$ .

**Definition 3.6.** Let  $N(q, e, f)$  be the number of tuples of  $n$  polynomials  $g_1, \dots, g_n$  over  $\mathbb{F}_q$ , of degree at most  $e$ , at least one of degree exactly  $e$ , with no common zero, such that  $f(g_1, \dots, g_n) = 0$ .

**Definition 3.7.** For each integer  $\varrho$ , let  $N_\varrho(q, e, f)$  be the number of pairs of a tuple of polynomials  $g_1, \dots, g_n$  over  $\mathbb{F}_q$ , of degree at most  $e$ , at least one of degree exactly  $e$ , with no common zero and a tuple of polynomials  $h_1, \dots, h_n$  over  $\mathbb{F}_q$ , of degree at most  $e - 1 - \varrho$ , such that (1-3) holds.

**Proposition 3.8.** (1) *The number of  $\mathbb{F}_q$ -points on  $\mathcal{M}_{0,0}(X, e)$  is*

$$\frac{N(q, e, f)}{(q - 1)(q^3 - q)}.$$

(2) *The number of  $\mathbb{F}_q$ -points on  $Z_\varrho$  is at most*

$$\frac{N_\varrho(q, e, f)q^{\varrho(n-1)-e(n-d)} - N(q, e, f)}{(q - 1)^2(q^3 - q)}.$$

*Proof.* Each point of  $\mathcal{M}_{0,0}(X, e)$  corresponds to  $|\mathrm{PGL}_2(\mathbb{F}_q)| = q^3 - q$  distinct maps  $\mathbb{P}^1 \rightarrow X$ . Thus in (1) we will count the number of maps  $\mathbb{P}^1 \rightarrow X$ , and in (2) we will count the number of maps  $\mathbb{P}^1 \rightarrow X$  that are not  $\varrho$ -free, and in each case then divide by  $q^3 - q$ .

For (1), it is sufficient to note that for any such tuple  $\mathbf{g}$ ,  $(g_1 : \dots : g_n)$  are the projective coordinates of a degree  $e$  map  $\mathbb{P}^1 \rightarrow X$ . All such maps arise this way, and two tuples define the same map if and only if one is the multiple of the other by a nonzero scalar.

For (2), it follows from Lemma 3.3 that it suffices to consider the space of degree  $e$  maps  $c : \mathbb{P}^1 \rightarrow X$  that are not strongly  $\varrho$ -free. Note that  $N_\varrho(q, e, f)$  is the sum over tuples of polynomials  $(g_1, \dots, g_n)$ , defining maps  $c$ , of  $q$  raised to the dimension of the vector space of possible  $h_1, \dots, h_n$ . By Lemma 3.5 this exponent is

$$\dim H^0(\mathbb{P}^1, c^* \hat{\mathcal{I}}_X \otimes \mathcal{O}_{\mathbb{P}^1}(-1 - \varrho)).$$

By Lemma 3.4,  $q$  to the power of this dimension is equal to  $q^{e(n-d)-\varrho(n-1)}$  if  $c$  is strongly  $\varrho$ -free and is at least  $q^{e(n-d)-\varrho(n-1)+1}$  otherwise. Hence

$$N_\varrho(q, e, f)q^{\varrho(n-1)-e(n-d)} \geq \sum_{\substack{\mathbf{g} \in \mathbb{F}_q[T]^n \\ |\mathbf{g}|=e \\ c \text{ not strongly } \varrho\text{-free}}} q + \sum_{\substack{\mathbf{g} \in \mathbb{F}_q[T]^n \\ |\mathbf{g}|=e \\ c \text{ strongly } \varrho\text{-free}}} 1 = N(q, e, f) + (q - 1) \sum_{\substack{\mathbf{g} \in \mathbb{F}_q[T]^n \\ |\mathbf{g}|=e \\ c \text{ not strongly } \varrho\text{-free}}} 1.$$

The proposition follows on noting that there are  $(q - 1)$  tuples  $\mathbf{g}$  for each map  $c : \mathbb{P}^1 \rightarrow X$ . □

#### 4. The circle method: identification of major arcs

For  $e \geq 1$  we have

$$N(q, e, f) = \#\{\mathbf{g} \in \mathbb{F}_q[T]^n : |\mathbf{g}| = q^e, f(\mathbf{g}) = 0, \gcd(g_1, \dots, g_n) = 1\},$$

where  $\mathbf{g} = (g_1, \dots, g_n)$  and  $|\mathbf{g}| = \max_{1 \leq i \leq n} |g_i|$ . In particular only nonzero vectors  $\mathbf{g}$  occur. Similarly, we may write

$$N_\varrho(q, e, f) = \sum_{\substack{\mathbf{g} \in \mathbb{F}_q[T]^n \\ |\mathbf{g}|=q^e \\ f(\mathbf{g})=0 \\ \gcd(g_1, \dots, g_n)=1}} \sum_{\substack{\mathbf{h} \in \mathbb{F}_q[T]^n \\ |\mathbf{h}| < q^{e-\varrho} \\ \mathbf{h} \cdot \nabla f(\mathbf{g})=0}} 1,$$

where once again we note that only nonzero vectors  $\mathbf{g}$  occur. We may use the function field analogue of the Möbius function  $\mu : \mathbb{F}_q[T] \rightarrow \{0, \pm 1\}$  to detect the coprimality condition  $\gcd(g_1, \dots, g_n) = 1$ . This gives

$$N_\varrho(q, e, f) = \sum_{\substack{k \in \mathbb{F}_q[T] \\ k \text{ monic}}} \mu(k) \sum_{\substack{\mathbf{g} \in \mathbb{F}_q[T]^n \\ 0 < |\mathbf{g}|=q^e/|k| \\ f(\mathbf{g})=0}} \sum_{\substack{\mathbf{h} \in \mathbb{F}_q[T]^n \\ |\mathbf{h}| < q^{e-\varrho} \\ \mathbf{h} \cdot \nabla f(\mathbf{g})=0}} 1 = \sum_{j \geq 0} \sum_{\substack{k \in \mathbb{F}_q[T] \\ |k|=q^j \\ k \text{ monic}}} \mu(k) \sum_{\substack{\mathbf{g} \in \mathbb{F}_q[T]^n \\ 0 < |\mathbf{g}|=q^{e-j} \\ f(\mathbf{g})=0}} \sum_{\substack{\mathbf{h} \in \mathbb{F}_q[T]^n \\ |\mathbf{h}| < q^{e-\varrho} \\ \mathbf{h} \cdot \nabla f(\mathbf{g})=0}} 1.$$

In view of the elementary identity

$$\sum_{\substack{k \in \mathbb{F}_q[T] \\ |k|=q^j \\ k \text{ monic}}} \mu(k) = \begin{cases} 1 & \text{if } j = 0, \\ -q & \text{if } j = 1, \\ 0 & \text{if } j > 1, \end{cases} \tag{4-1}$$

it readily follows that

$$N_\varrho(q, e, f) = \sum_{j \geq 0} c_j N(e - j + 1, e - \varrho),$$

where

$$c_j = \begin{cases} 1 & \text{if } j = 0, \\ -(q + 1) & \text{if } j = 1, \\ q & \text{if } j = 2, \\ 0 & \text{if } j > 2 \end{cases} \tag{4-2}$$

and

$$N(u, v) = \sum_{\substack{\mathbf{g} \in \mathbb{F}_q[T]^n \\ 0 < |\mathbf{g}| < q^u \\ f(\mathbf{g})=0}} \sum_{\substack{\mathbf{h} \in \mathbb{F}_q[T]^n \\ |\mathbf{h}| < q^v \\ \mathbf{h} \cdot \nabla f(\mathbf{g})=0}} 1,$$

for any integers  $u, v \geq 1$ .

We have

$$\sum_{\substack{\mathbf{h} \in \mathbb{F}_q[T]^n \\ |\mathbf{h}| < q^{e-\varrho} \\ \mathbf{h} \cdot \nabla f(\mathbf{g})=0}} 1 = \int_{\mathbb{T}} S(\beta) d\beta,$$

where

$$S(\beta) = \sum_{\substack{\mathbf{h} \in \mathbb{F}_q[T]^n \\ |\mathbf{h}| < q^{e-\varrho}}} \psi(\beta \mathbf{h} \cdot \nabla f(\mathbf{g})).$$

Here the integral is over the space  $\mathbb{T}$  of formal Laurent series in  $T^{-1}$  of degree less than 0, against the Haar measure with total mass 1, and  $\psi$  is the additive character of  $\mathbb{F}_q((T^{-1}))$  that sends a formal Laurent series in  $T^{-1}$  to a fixed nontrivial additive character of  $\mathbb{F}_q$  applied to the coefficient of  $T^{-1}$ . With this notation we now have

$$N_\varrho(q, e, f) = \sum_{j \geq 0} c_j \sum_{\substack{\mathbf{g} \in \mathbb{F}_q[T]^n \\ 0 < |\mathbf{g}| < q^{e-j+1} \\ f(\mathbf{g}) = 0}} \int_{\mathbb{T}} S(\beta) d\beta. \tag{4-3}$$

Our plan will be to define a set of major arcs whose total contribution to  $q^{\varrho(n-1)-e(n-d)}N_\varrho(q, e, f)$  is matched by  $N(q, e, f)$ . We note that the sum over  $\mathbf{g}$  is empty unless  $e \geq j$ , so we will be able to assume this whenever dealing with this sum.

In what follows we shall frequently make use of the basic orthogonality property

$$\sum_{\substack{b \in \mathbb{F}_q[T] \\ |b| < q^B}} \psi(\gamma b) = \begin{cases} q^B & \text{if } \|\gamma\| < q^{-B}, \\ 0 & \text{otherwise,} \end{cases} \tag{4-4}$$

which is valid for any integer  $B \geq 0$  and any  $\gamma \in \mathbb{F}_q((T^{-1}))$ . Here we recall that  $\|\gamma\| = |\sum_{i \leq -1} b_i T^i|$  for any  $\gamma = \sum_{i \leq N} b_i T^i \in \mathbb{F}_q((T^{-1}))$ .

Let  $\mathbf{g} \in \mathbb{F}_q[T]^n$  be a nonzero vector such that  $f(\mathbf{g}) = 0$ . The next result is the first step towards defining the relevant set of major arcs for our problem.

**Lemma 4.1.** *Suppose that  $\beta = a/r + \theta$  for coprime polynomials  $a, r \in \mathbb{F}_q[T]$  such that  $|a| < |r| \leq q^{e-\varrho}$ . Assume that  $|r\theta| < q^{-(d-1)(e-j)}$ . Then*

$$S(\beta) = \begin{cases} q^{n(e-\varrho)} & \text{if } r \mid \gcd(g_1, \dots, g_n)^{d-1} \text{ and } |\theta| < q^{\varrho-e}/|\mathbf{g}|^{d-1}, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* We break the sum into residue classes modulo  $r$ , by writing  $\mathbf{h} = \mathbf{u} + r\mathbf{v}$  for  $|\mathbf{u}| < |r|$  and  $|\mathbf{v}| < q^{e-\varrho}/|r|$ . Then

$$S(\beta) = \sum_{\substack{\mathbf{u} \in \mathbb{F}_q[T]^n \\ |\mathbf{u}| < |r|}} \psi(\beta \mathbf{u} \cdot \nabla f(\mathbf{g})) \sum_{\substack{\mathbf{v} \in \mathbb{F}_q[T]^n \\ |\mathbf{v}| < q^{e-\varrho}/|r|}} \psi(r\theta \mathbf{v} \cdot \nabla f(\mathbf{g}))$$

Since  $|r\theta| < q^{-(d-1)(e-j)}$  we have  $|r\theta \nabla f(\mathbf{g})| \leq |r\theta| q^{(d-1)(e-j)} < 1$ . Thus  $\|r\theta \nabla f(\mathbf{g})\| = |r\theta \nabla f(\mathbf{g})|$  and it follows from (4-4) that

$$\sum_{\substack{\mathbf{v} \in \mathbb{F}_q[T]^n \\ |\mathbf{v}| < q^{e-\varrho}/|r|}} \psi(r\theta \mathbf{v} \cdot \nabla f(\mathbf{g})) = \begin{cases} |r|^{-n} q^{n(e-\varrho)} & \text{if } |\theta \nabla f(\mathbf{g})| < q^{\varrho-e}, \\ 0 & \text{otherwise.} \end{cases}$$

We claim that  $|\nabla f(\mathbf{g})| = |\mathbf{g}|^{d-1}$ . To see this suppose that  $|\mathbf{g}| = q^m$  for a nonnegative integer  $m$  and let  $\mathbf{g}^* \in \mathbb{F}_q^n$  be the (nonzero) leading coefficient of  $\mathbf{g}$ . In particular  $f(\mathbf{g}^*) = 0$  since  $f(\mathbf{g}) = 0$ . Since  $f$  has degree  $d$  it follows that the coefficient of  $T^{m(d-1)}$  in  $\nabla f(\mathbf{g})$  is  $\nabla f(\mathbf{g}^*) \neq \mathbf{0}$ , since  $f$  is nonsingular.

Our argument so far shows that

$$S(\beta) = \begin{cases} |r|^{-n} q^{n(e-\varrho)} T(\beta) & \text{if } |\theta| < q^{\varrho-e}/|\mathbf{g}|^{d-1}, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$T(\beta) = \sum_{\substack{\mathbf{u} \in \mathbb{F}_q[T]^n \\ |\mathbf{u}| < |r|}} \psi(\beta \mathbf{u} \cdot \nabla f(\mathbf{g})).$$

When  $|\theta| < q^{\varrho-e}/|\mathbf{g}|^{d-1}$  it follows that

$$|\theta \mathbf{u} \cdot \nabla f(\mathbf{g})| \leq q^{-1} |\theta r \nabla f(\mathbf{g})| \leq q^{-2+\varrho-e} |r| \leq q^{-2},$$

since  $|r| \leq q^{e-\varrho}$ . Hence, since  $a$  and  $r$  are coprime, we deduce that

$$T(\beta) = \sum_{\substack{\mathbf{u} \in \mathbb{F}_q[T]^n \\ |\mathbf{u}| < |r|}} \psi\left(\frac{a\mathbf{u} \cdot \nabla f(\mathbf{g})}{r}\right) = \begin{cases} |r|^n & \text{if } r \mid \nabla f(\mathbf{g}), \\ 0 & \text{otherwise,} \end{cases}$$

Since  $f$  is a nonsingular form, the statement of the lemma follows on noting that  $r \mid \nabla f(\mathbf{g})$  if and only if  $r \mid \gcd(g_1, \dots, g_n)^{d-1}$ . □

**Lemma 4.2.** *Suppose that  $e \geq \varrho$  and*

$$\frac{a_1}{r_1} + \theta_1 = \frac{a_2}{r_2} + \theta_2,$$

*with  $r_1, r_2 \mid \gcd(g_1, \dots, g_n)^{d-1}$  and  $|\theta_1|, |\theta_2| < q^{\varrho-e}/|\mathbf{g}|^{d-1}$ . Then in fact  $\frac{a_1}{r_1} = \frac{a_2}{r_2}$  (and so  $\theta_1 = \theta_2$ ).*

*Proof.* By clearing denominators, we may assume  $r_1 = r_2 = \gcd(g_1, \dots, g_n)^{d-1}$ . Then  $a_1 - a_2 = \gcd(g_1, \dots, g_n)^{d-1}(\theta_2 - \theta_1)$ , so that

$$|a_1 - a_2| < q^{\varrho-e} \frac{\gcd(g_1, \dots, g_n)^{d-1}}{|\mathbf{g}|^{d-1}} \leq q^{\varrho-e} \leq 1.$$

This implies that  $a_1 = a_2$ , as required. □

We take as major arcs the union

$$\mathfrak{N}_j = \bigcup_{\substack{r \in \mathbb{F}_q[T] \text{ monic} \\ |r| \leq q^{e-\varrho}}} \bigcup_{\substack{|\mathbf{a}| < |r| \\ \gcd(a,r)=1}} \{\beta \in \mathbb{F}_q((T^{-1})) : |r\beta - a| < q^{-(d-1)(e-j)}\}, \tag{4-5}$$

for  $j \geq 0$ . It follows from Lemma 4.1 that  $S(\beta)$  is nonzero for  $\beta \in \mathfrak{N}_j$  if and only if there is some pair  $(a/r, \theta)$  such that  $\beta = a/r + \theta$  and all the conditions

$$|r| \leq q^{e-\varrho}, \quad |\theta| < |r|^{-1} q^{-(d-1)(e-j)}, \quad |a| < |r|, \quad \gcd(a, r) = 1,$$

and

$$r \mid \gcd(g_1, \dots, g_n)^{d-1}, \quad |\theta| < q^{\varrho-e}/|\mathbf{g}|^{d-1}$$

are satisfied. By Lemma 4.2, pairs satisfying these conditions (or even the last three conditions) are unique. Hence we can rewrite the integral over the major arcs as

$$\begin{aligned} \int_{\mathfrak{M}_j} S(\beta) d\beta &= q^{n(e-\varrho)} \sum_{\substack{|r| \leq q^{e-\varrho} \\ r \text{ monic} \\ r \mid \gcd(g_1, \dots, g_n)^{d-1}}} \sum_{\substack{|a| < |r| \\ \gcd(a, r) = 1}} \int_{|\theta| < \min\{q^{e-\varrho}/|\mathbf{g}|^{d-1}, |r|^{-1}q^{-(d-1)(e-j)}\}} d\theta \\ &= q^{n(e-\varrho)} \sum_{\substack{|r| \leq q^{e-\varrho} \\ r \text{ monic} \\ r \mid \gcd(g_1, \dots, g_n)^{d-1}}} \varphi(r) \int_{|\theta| < \min\{q^{e-\varrho}/|\mathbf{g}|^{d-1}, |r|^{-1}q^{-(d-1)(e-j)}\}} d\theta, \end{aligned}$$

for any nonzero vector  $\mathbf{g} \in \mathbb{F}_q[T]^n$  such that  $f(\mathbf{g}) = 0$ , where  $\varphi(r)$  is the function field analogue of the Euler totient function. We want to replace the integral over  $\theta$  by

$$\int_{|\theta| < q^{e-\varrho}/|\mathbf{g}|^{d-1}} d\theta = \frac{q^{e-\varrho}}{|\mathbf{g}|^{d-1}}.$$

The error in doing this is at most this volume multiplied by the indicator function for the inequality

$$|r|^{-1}q^{-(d-1)(e-j)} < q^{e-\varrho}/|\mathbf{g}|^{d-1}.$$

Since  $r \mid \gcd(g_1, \dots, g_n)^{d-1}$  this inequality implies that

$$q^{j+D+1}|\mathbf{g}| \leq |\gcd(g_1, \dots, g_n)|q^e, \tag{4-6}$$

where

$$D = \left\lfloor \frac{e-\varrho}{d-1} \right\rfloor. \tag{4-7}$$

At this point we observe that

$$\sum_{\substack{r \in \mathbb{F}_q[T] \text{ monic} \\ r \mid \gcd(g_1, \dots, g_n)^{d-1}}} \varphi(r) = |\gcd(g_1, \dots, g_n)|^{d-1},$$

since  $\mathbf{g} \neq \mathbf{0}$ . Note that when  $r \mid \gcd(g_1, \dots, g_n)^{d-1}$  and  $|r| > q^{e-\varrho}$  we must have

$$|\gcd(g_1, \dots, g_n)| \geq q^{D+1}, \tag{4-8}$$

with  $D$  as above. Putting everything together it follows that

$$\int_{\mathfrak{M}_j} S(\beta) d\beta = \frac{q^{(n-1)(e-\varrho)} |\gcd(g_1, \dots, g_n)|^{d-1}}{|\mathbf{g}|^{d-1}} (1 + \epsilon_j \mathbf{1}_j(\mathbf{g})) \tag{4-9}$$

for  $\epsilon_j \in [-1, 1]$ , where

$$\mathbf{1}_j(\mathbf{g}) = \begin{cases} 1 & \text{if (4-6) or (4-8) hold,} \\ 0 & \text{otherwise.} \end{cases}$$



Let  $N_\varrho^{\text{major}}(q, e, f)$  denote the contribution to the right-hand side of (4-3) from (4-9) for each  $j$ . We now see that

$$N_\varrho^{\text{major}}(q, e, f) = q^{(n-1)(e-\varrho)} \sum_{j \geq 0} c_j \sum_{\substack{\mathbf{g} \in \mathbb{F}_q[T]^n \\ 0 < |\mathbf{g}| < q^{e-j+1} \\ f(\mathbf{g})=0}} \frac{|\gcd(g_1, \dots, g_n)|^{d-1}}{|\mathbf{g}|^{d-1}} (1 + \epsilon_j \mathbf{1}_j(\mathbf{g})).$$

On noting that  $(n-1)(e-\varrho) - e(d-1) = e(n-d) - \varrho(n-1)$ , the main term is seen to be

$$q^{e(n-d)-\varrho(n-1)} (\tilde{N}(e) - q^d \tilde{N}(e-1)),$$

where for  $u \geq 0$  we set

$$\tilde{N}(u) = \sum_{\substack{\mathbf{g} \in \mathbb{F}_q[T]^n \\ |\mathbf{g}|=q^u \\ f(\mathbf{g})=0}} |\gcd(g_1, \dots, g_n)|^{d-1} = \sum_{\substack{k \in \mathbb{F}_q[T] \\ |k| \leq q^u \\ k \text{ monic}}} |k|^{d-1} N(q, u - \deg(k), f) = \sum_{\ell=0}^u q^{d\ell} N(q, u - \ell, f),$$

in the notation of Definition 3.6. Hence

$$\tilde{N}(e) - q^d \tilde{N}(e-1) = \sum_{\ell=0}^e q^{d\ell} N(q, e - \ell, f) - \sum_{\ell=0}^{e-1} q^{d(\ell+1)} N(q, e - 1 - \ell, f) = N(q, e, f).$$

**Remark 4.3.** The cancellation here is not miraculous. The terms corresponding to  $\mathbf{g}$  with  $|\mathbf{g}| < q^e$  or  $|\gcd(g_1, \dots, g_n)| > 1$  disappear precisely because  $c_j$  were the coefficients defined in (4-2) to sieve out these terms in the first place.

Turning to the error term we can combine (4-6) and (4-8) to deduce that

$$\gcd(g_1, \dots, g_n) \geq q^{D+1} \min(1, q^{j-e} |\mathbf{g}|) = q^{D+1+j-e} |\mathbf{g}|$$

whenever  $\mathbf{1}_j(\mathbf{g}) = 1$ . Hence

$$N_\varrho^{\text{major}}(q, e, f) - q^{e(n-d)-\varrho(n-1)} N(q, e, f) \leq q^{(n-1)(e-\varrho)} \sum_{j \geq 0} |c_j| E_j,$$

where

$$\begin{aligned} E_j &= \sum_{0 \leq u \leq e-j} \sum_{\substack{k \in \mathbb{F}_q[T] \text{ monic} \\ |k| \geq q^{D+1+j-e+u}}} \frac{|k|^{d-1}}{q^{u(d-1)}} \#\{\mathbf{g} \in \mathbb{F}_q[T]^n : |\mathbf{g}| = q^u, f(\mathbf{g}) = 0, k = \gcd(g_1, \dots, g_n)\} \\ &= \sum_{0 \leq u \leq e-j} \sum_{\ell \geq D+1+j-e+u} \frac{q^\ell}{q^{(u-\ell)(d-1)}} N(q, u - \ell, f). \end{aligned}$$

Invoking [Browning and Vishe 2015, Lemma 2.8], we deduce that  $N(q, e, f) = O_f(q^{(e+1)(n-1)})$  for any  $n \geq 3$ , where the implied constant depends at most on  $f$ . Hence, since we may clearly assume that

$n > d + 1$ , it follows that

$$\begin{aligned}
 E_j &\ll_f \sum_{0 \leq u \leq e-j} q^{u(n-d)+n-1} \sum_{\ell \geq D+1+j-e+u} q^{-\ell(n-d-1)} \\
 &\ll_f q^{-(D+1)(n-d-1)} \sum_{0 \leq u \leq e-j} \frac{q^{u(n-d)+n-1}}{q^{(u-e+j)(n-d-1)}} \\
 &\ll_f q^{(e-j)(n-d)+n-1-(D+1)(n-d-1)}.
 \end{aligned} \tag{4-10}$$

The implied constant in this estimate depends only on  $f$  and not on  $q$ . Thus

$$q^{(n-1)(e-\varrho)} \sum_{j \geq 0} |c_j| E_j \ll_f q^{2e(n-d)-\varrho(n-1)+de-e+n-1-(D+1)(n-d-1)}.$$

Putting everything together, we may conclude as follows.

**Lemma 4.4.** *Let  $\varrho \in \mathbb{Z}$  and assume that  $e \geq \varrho$ . Then*

$$N_\varrho^{major}(q, e, f) = q^{e(n-d)-\varrho(n-1)}(N(q, e, f) + O_f(q^{(e+1)(n-1)-(D+1)(n-d-1)})),$$

where  $D$  is given by (4-7).

### 5. The circle method: minor arcs

It remains to study the quantity

$$N_\varrho^{minor}(q, e, f) = \sum_{j \geq 0} c_j \sum_{\substack{\mathbf{g} \in \mathbb{F}_q[T]^n \\ 0 < |\mathbf{g}| < q^{e-j+1} \\ f(\mathbf{g})=0}} \int_{\mathfrak{n}_j} S(\beta) d\beta, \tag{5-1}$$

where  $\mathfrak{n}_j$  is the complement in  $\mathbb{T} = \{\beta \in \mathbb{F}_q((T^{-1})) : |\beta| < 1\}$  of the major arcs  $\mathfrak{M}_j$  that we defined in (4-5). Indeed, in view of Proposition 3.8(2), the following result is now a direct consequence of (4-3) and Lemma 4.4.

**Lemma 5.1.** *Assume that  $d \geq 3$  and  $e \geq \varrho$ . Then*

$$\#Z_\varrho(\mathbb{F}_q) \leq \frac{q^{\varrho(n-1)-e(n-d)}}{(q-1)^2(q^3-q)} N_\varrho^{minor}(q, e, f) + O_f(q^{(e+1)(n-1)-5-(D+1)(n-d-1)}),$$

where  $D$  is given by (4-7).

We have

$$\sum_{\substack{\mathbf{g} \in \mathbb{F}_q[T]^n \\ 0 < |\mathbf{g}| < q^{e-j+1} \\ f(\mathbf{g})=0}} \int_{\mathfrak{n}_j} S(\beta) d\beta = \int_{\mathbb{T}} \int_{\mathfrak{n}_j} (S(\alpha, \beta) - q^{n(e-\varrho)}) d\alpha d\beta, \tag{5-2}$$

where

$$S(\alpha, \beta) = \sum_{\substack{\mathbf{g} \in \mathbb{F}_q[T]^n \\ |\mathbf{g}| < q^{e-j+1}}} \sum_{\substack{\mathbf{h} \in \mathbb{F}_q[T]^n \\ |\mathbf{h}| < q^{e-e}}} \psi(\alpha f(\mathbf{g}) + \beta \mathbf{h} \cdot \nabla f(\mathbf{g})). \quad (5-3)$$

Viewed as polynomials in the  $2n$  variables  $(\mathbf{g}, \mathbf{h})$  the pair of polynomials  $f(\mathbf{g})$  and  $\mathbf{h} \cdot \nabla f(\mathbf{g})$  are homogeneous of degree  $d$ . The obvious thing to do at this point is to apply Weyl differencing  $d - 1$  times in the spirit of Birch. This requires one to work with a simultaneous Diophantine approximation of  $\alpha$  and  $\beta$ , which is somewhat wasteful. It bears fruit provided that

$$2n - \dim V^* > 3(d - 1)2^d,$$

where  $V^*$  is the (affine) ‘‘Birch singular locus’’. In this setting  $V^*$  is the locus of  $(\mathbf{g}, \mathbf{h}) \in \mathbb{A}^{2n}$  such that the pair of vectors  $(\nabla f(\mathbf{g}), \mathbf{0})$  and  $(\mathbf{h} \cdot \nabla^2 f(\mathbf{g}), \nabla f(\mathbf{g}))$  are proportional. Since  $f$  is nonsingular, it follows that  $V^*$  is the set of  $(\mathbf{g}, \mathbf{h}) \in \mathbb{A}^{2n}$  such that  $\mathbf{g} = \mathbf{0}$ , so that  $\dim V^* = n$ . In this way we see that the standard approach would require  $n > 3(d - 1)2^d$  variables overall, although there are additional difficulties associated to having lopsided boxes. In our work we shall exploit the special shape of our polynomials in such a way that our estimates are only sensitive to the Diophantine approximation properties of  $\alpha$  or  $\beta$  independently. This allows us to handle half the number of variables when dealing with the sum  $S(\alpha, \beta)$ .

In what follows it will be convenient to define the monomials

$$P_0(T) = T^{e-j}, \quad P(T) = T^{e-j+1} \quad \text{and} \quad Q(T) = T^{e-e}.$$

Let

$$\mathfrak{M}(J) = \bigcup_{\substack{r \in \mathbb{F}_q[T] \text{ monic} \\ |r| \leq q^J}} \bigcup_{\substack{absa < |r| \\ \gcd(a,r)=1}} \{\alpha \in \mathbb{F}_q((T^{-1})) : |r\alpha - a| < q^J |P_0|^{-d}\}, \quad (5-4)$$

for any integer  $J$ . Note that  $\mathfrak{M}(-1) = \emptyset$ . Let

$$M = \left\lceil \frac{d(e-j)}{2} \right\rceil. \quad (5-5)$$

According to the function field version of Dirichlet’s approximation theorem any element of  $\mathbb{T}$  has a representation  $a/r + \theta$  with  $absa < |r| \leq q^M$  and  $|r\theta| < q^{-M}$ . Hence we can cover  $\mathbb{T}$  by a union of arcs  $\mathfrak{M}(J+1) \setminus \mathfrak{M}(J)$  for integers  $J$  such that  $-1 \leq J \leq M-1$ .

Next, let

$$\mathfrak{N}(K) = \bigcup_{\substack{r \in \mathbb{F}_q[T] \text{ monic} \\ |r| \leq q^K}} \bigcup_{\substack{absa < |r| \\ \gcd(a,r)=1}} \left\{ \beta \in \mathbb{F}_q((T^{-1})) : |r\beta - a| < \frac{q^K}{|P_0|^{d-1}|Q|} \right\}, \quad (5-6)$$

for any integer  $K$ . We note that  $\mathfrak{N}(e - \varrho) = \mathfrak{N}_j$ , in the notation of (4-5). Let

$$N = \left\lceil \frac{(e-j)(d-1) + e - \varrho}{2} \right\rceil. \quad (5-7)$$

It now follows from Dirichlet’s approximation theorem that the minor arcs  $\mathfrak{n}_j$  can be covered by the union of arcs  $\mathfrak{N}(K + 1) \setminus \mathfrak{N}(K)$  for integers  $K$  such that  $e - \varrho \leq K \leq N - 1$ .

Observe in particular that if any minor arcs exist then  $e - \varrho < N$  so

$$(d - 1)(e - j) > e - \varrho. \tag{5-8}$$

We may thus assume (5-8) when dealing with the minor arcs. Keeping the assumptions  $d \geq 3$  and  $e \geq \varrho$ , we see in particular that

$$|P|, |Q| \geq 1.$$

Our plan is to produce two estimates for  $S(\alpha, \beta)$ : one for when  $\alpha$  belongs to  $\mathfrak{M}(J + 1) \setminus \mathfrak{M}(J)$  and one for when  $\beta$  belongs to  $\mathfrak{N}(K + 1) \setminus \mathfrak{N}(K)$ . Before proceeding further we note that

$$\text{meas}(\mathfrak{M}(J)) \leq q^{2J} |P_0|^{-d} \tag{5-9}$$

and

$$\text{meas}(\mathfrak{N}(K)) \leq q^{2K} |P_0|^{-d+1} |Q|^{-1}, \tag{5-10}$$

for any integers  $J, K \geq 0$ .

Suppose that

$$f(\mathbf{x}) = \sum_{i_1, \dots, i_d=1}^n c_{i_1, \dots, i_d} x_{i_1} \dots x_{i_d},$$

with symmetric coefficients  $c_{i_1, \dots, i_d} \in \mathbb{F}_q$ . Associated to  $f$  are the multilinear forms

$$\Psi_i(\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(d-1)}) = d! \sum_{i_1, \dots, i_{d-1}=1}^n c_{i_1, \dots, i_{d-1}, i} x_{i_1}^{(1)} \dots x_{i_{d-1}}^{(d-1)}, \tag{5-11}$$

for  $1 \leq i \leq n$ . Our first estimate for  $S(\alpha, \beta)$  involves summing trivially over  $\mathbf{h}$  and then applying Weyl differencing  $d - 1$  times to the sum over  $\mathbf{g}$ . This eliminates the effect of the lower degree term  $\beta \mathbf{h} \cdot \nabla f(\mathbf{g})$  and leads one to a family of linear exponential sums with phase vectors  $(\alpha \Psi_1(\underline{\mathbf{g}}), \dots, \alpha \Psi_n(\underline{\mathbf{g}}))$ , for  $\underline{\mathbf{g}} = (\mathbf{g}_1, \dots, \mathbf{g}_{d-1}) \in \mathbb{F}_q[T]^{(d-1)n}$ . This approach closely parallels [Browning and Vishe 2017].

An alternative estimate for  $S(\alpha, \beta)$  is obtained by applying Weyl differencing  $d - 2$  times to the sum over  $\mathbf{g}$ . After a further application of Cauchy–Schwarz one then brings the  $\mathbf{h}$ -sum inside, giving a family of linear exponential sums with phase vectors  $(\beta \Psi_1(\underline{\mathbf{g}}), \dots, \beta \Psi_n(\underline{\mathbf{g}}))$ , for  $\underline{\mathbf{g}} \in \mathbb{F}_q[T]^{(d-1)n}$ . This brings the Diophantine properties of  $\beta$  into play but extra difficulties arise from the fact that  $P$  and  $Q$  need not have the same degree.

**5A. Geometry-of-numbers redux.** We shall need to begin by revisiting a function field lattice point counting result that played a key role in [Browning and Vishe 2017]. A lattice in  $\mathbb{F}_q((T^{-1}))^N$  is a set of points of the form  $\mathbf{x} = \Lambda \mathbf{u}$  where  $\Lambda$  is an  $N \times N$  invertible matrix over  $\mathbb{F}_q((T^{-1}))$  and  $\mathbf{u}$  runs over elements of  $\mathbb{F}_q[T]^n$ . Given a lattice  $\Lambda$ , the adjoint lattice is defined as the lattice associated to the inverse transpose matrix  $\Lambda^{-T}$ .

**Remark 5.2.** We can view lattices as vector bundles on  $\mathbb{P}^1$  by viewing the matrix  $\Lambda$  as giving gluing data for gluing the trivial vector bundle on  $\mathbb{A}^1$  and the trivial vector bundle on a formal neighborhood of  $\infty$ , using the Beauville–Laszlo theorem. The adjoint lattice corresponds to the dual vector bundle, and the geometry-of-numbers computations in this section could instead be stated in this language.

Bearing our notation in mind we recall a version of the “shrinking lemma” that is proved in [Browning and Sawin 2020b, Lemma 6.4].

**Lemma 5.3.** *Let  $\gamma$  be a symmetric  $n \times n$  matrix with entries in  $\mathbb{F}_q((T^{-1}))$ . Let  $a, c, s \in \mathbb{Z}$  such that  $c > 0$  and  $s \geq 0$ . Let  $N_{\gamma,a,c}$  be the number of  $\mathbf{x} \in \mathbb{F}_q[T]^n$  such that  $|\mathbf{x}| < q^a$  and  $\|\gamma\mathbf{x}\| < q^{-c}$ . Then*

$$\frac{N_{\gamma,a,c}}{N_{\gamma,a-s,c+s}} \leq q^{ns+n \max(\lfloor (a-c)/2 \rfloor, 0)}.$$

For any  $\alpha \in \mathbb{F}_q((T^{-1}))$  and any  $r > 0$ , we set

$$N(\alpha; r) = \#\{\underline{\mathbf{g}} \in \mathbb{F}_q[T]^{(d-1)n} : |\mathbf{g}_1|, \dots, |\mathbf{g}_{d-1}| < |P|, \|\alpha\Psi_i(\underline{\mathbf{g}})\| < q^{-r} (\forall i \leq n)\}. \quad (5-12)$$

Furthermore, for an integer  $s \geq 0$ , we put

$$N_s(\alpha; r) = \#\{\underline{\mathbf{g}} \in \mathbb{F}_q[T]^{(d-1)n} : |\mathbf{g}_1|, \dots, |\mathbf{g}_{d-1}| < |P|/q^s, \|\alpha\Psi_i(\underline{\mathbf{g}})\| < q^{-r-(d-1)s} (\forall i \leq n)\}.$$

We can use the shrinking lemma to bound the ratio of these two quantities as follows.

**Lemma 5.4.** *For  $r > 0$  and  $s \geq \max(0, e - j + 1 - r)$ , we have*

$$\frac{N(\alpha, r)}{N_s(\alpha, r)} \leq q^{(d-1)ns+n \max(0, \lfloor (e-j+1-r)/2 \rfloor)}.$$

*Proof.* For each  $v \in \{0, \dots, d-1\}$ , let  $N^{(v)}(\alpha, r)$  be the number of vectors  $\underline{\mathbf{g}} \in \mathbb{F}_q[T]^{(d-1)n}$  such that

$$|\mathbf{g}_1|, \dots, |\mathbf{g}_v| < |P|/q^s, \quad |\mathbf{g}_{v+1}|, \dots, |\mathbf{g}_{d-1}| < |P| \quad (5-13)$$

and  $\|\alpha\Psi_i(\underline{\mathbf{g}})\| < q^{-r-v}$ , for  $1 \leq i \leq n$ . Thus we have  $N^{(0)}(\alpha, r) = N(\alpha, r)$  and  $N^{(d-1)}(\alpha, r) = N_s(\alpha, r)$ .

Fix a choice of  $v \in \{1, \dots, d-1\}$  and let  $\mathbf{g}_1, \dots, \mathbf{g}_{v-1}, \mathbf{g}_{v+1}, \dots, \mathbf{g}_{d-1} \in \mathbb{F}_q[T]^n$  such that (5-13) holds. We consider the linear forms

$$L_i(\mathbf{g}) = \alpha\Psi_i(\mathbf{g}_1, \dots, \mathbf{g}_{v-1}, \mathbf{g}, \mathbf{g}_{v+1}, \dots, \mathbf{g}_{d-1}),$$

for  $1 \leq i \leq n$ . These form an  $n \times n$  matrix. Because  $\Psi_i$  is the dualization in one variable of a symmetric  $d$ -linear form, this  $n \times n$  matrix is symmetric. The contribution to  $N^{(v-1)}(\alpha, r)$  from tuples with the chosen  $\mathbf{g}_1, \dots, \mathbf{g}_{v-1}, \mathbf{g}_{v+1}, \dots, \mathbf{g}_{d-1} \in \mathbb{F}_q[T]^n$  is  $N_{\gamma, e-j+1, r+(v-1)s}$  while the contribution to  $N^{(v)}(\alpha, r)$  from tuples of the same form is  $N_{\gamma, e-j+1-s, r+vs}$ . Note that  $r + (v-1)s \geq r > 0$  for  $v \geq 1$  and so Lemma 5.3 is applicable. We deduce that

$$\frac{N^{(v-1)}(\alpha, r)}{N^{(v)}(\alpha, r)} \leq q^{ns+n \max(\lfloor (e-j+1-r-(v-1)s)/2 \rfloor, 0)}$$

for  $1 \leq v \leq d-1$ .

We take the product of this inequality over all  $v$  from 1 to  $d - 1$ . The first term in the exponent contributes  $(d - 1)ns$ . The second contributes  $n \max(\lfloor (e - j + 1 - r)/2 \rfloor, 0)$  for  $v = 1$  and 0 for all other values of  $v$ , on assuming that  $s \geq e - j + 1 - r$ . Thus we get the stated bound.  $\square$

**5B. Weyl differencing.** Our fundamental tool for estimating  $S(\alpha, \beta)$  is Weyl differencing. We recall first that  $|P|, |Q| \geq 1$  in this exponential sum. Appealing to [Browning and Vishe 2017, Equation (5.2)] first, Weyl differencing  $d - 1$  times gives

$$|S(\alpha, \beta)| \leq |P|^n |Q|^n (|P|^{-(d-1)n} N(\alpha, e - j + 1))^{1/2^{d-1}},$$

in the notation of (5-12). Note that as  $N(\alpha, e - j + 1) \geq 1$  and  $2^{d-1} \geq (d - 1)$ , the right side is  $\geq |Q|^n$ . Thus we have

$$|S(\alpha, \beta) - q^{n(e-\varrho)}| \leq 2|P|^n |Q|^n (|P|^{-(d-1)n} N(\alpha, e - j + 1))^{1/2^{d-1}}, \tag{5-14}$$

We can also obtain an upper bound for  $S(\alpha, \beta)$  that only uses information about  $\beta$ . Let us put

$$T(\mathbf{h}) = \sum_{|\mathbf{g}| < |P|} \psi(\alpha f(\mathbf{g}) + \beta \mathbf{h} \cdot \nabla f(\mathbf{g})),$$

so that

$$S(\alpha, \beta) = \sum_{|\mathbf{h}| < |Q|} T(\mathbf{h}),$$

with  $P, Q$  are as before. It follows from Cauchy–Schwarz that

$$|S(\alpha, \beta)|^{2^{d-2}} \leq |Q|^{(2^{d-2}-1)n} \sum_{|\mathbf{h}| < |Q|} |T(\mathbf{h})|^{2^{d-2}}. \tag{5-15}$$

After  $d - 3$  applications of Weyl differencing we obtain

$$|T(\mathbf{h})|^{2^{d-3}} \leq |P|^{(2^{d-3}-d+2)n} \sum_{\mathbf{g}_1, \dots, \mathbf{g}_{d-3}} \left| \sum_{\mathbf{g}} \psi(D(\mathbf{g})) \right|,$$

where  $D(\mathbf{g}) = D_{\mathbf{g}_1, \dots, \mathbf{g}_{d-3}}(\alpha f(\mathbf{g}) + \beta \mathbf{h} \cdot \nabla f(\mathbf{g}))$  and  $D_{\mathbf{g}_1, \dots, \mathbf{g}_{d-3}}$  is the usual differencing operator. Here  $\mathbf{g}_1, \dots, \mathbf{g}_{d-3}, \mathbf{g}$  each run over vectors in  $\mathbb{F}_q[T]^n$  formed from polynomials of degree less than  $e - j + 1$ . A further application of Cauchy–Schwarz now yields

$$|T(\mathbf{h})|^{2^{d-2}} \leq |P|^{(2^{d-2}-d+1)n} \sum_{\mathbf{g}_1, \dots, \mathbf{g}_{d-3}} \left| \sum_{\mathbf{g}} \psi(D(\mathbf{g})) \right|^2.$$

Differencing once more therefore leads to the expression

$$\left| \sum_{\mathbf{g}} \psi(D(\mathbf{g})) \right|^2 = \sum_{\mathbf{g}_{d-2}, \mathbf{g}_{d-1}} \psi(D_{\mathbf{g}_1, \dots, \mathbf{g}_{d-2}}(\alpha f(\mathbf{g}_{d-1}) + \beta \mathbf{h} \cdot \nabla f(\mathbf{g}_{d-1}))),$$

where

$$D_{\mathbf{g}_1, \dots, \mathbf{g}_{d-2}}(\mathbf{h} \cdot \nabla f(\mathbf{g}_{d-1})) = \sum_{i=1}^n h_i \Psi_i(\mathbf{g}_1, \dots, \mathbf{g}_{d-1}),$$

in the notation of (5-11). Returning to (5-15) we ignore the Diophantine approximation properties of  $\alpha$  and instead execute the linear exponential sum over  $\mathbf{h}$ . This leads to the expression

$$|S(\alpha, \beta)| \leq |P|^n |Q|^n (|P|^{-(d-1)n} N(\beta, e - \varrho))^{1/2^{d-2}},$$

in the notation of (5-12). Again,  $N(\beta, e - \varrho) \geq 1$  and  $2^{d-2} \geq (d-1)$  so the right side is  $\geq |Q|^n$ , whence

$$|S(\alpha, \beta) - q^{n(e-\varrho)}| \leq 2|P|^n |Q|^n (|P|^{-(d-1)n} N(\beta, e - \varrho))^{1/2^{d-2}}. \quad (5-16)$$

**Remark 5.5.** When  $\text{char}(\mathbb{F}_q) \leq d$  the polynomials  $\Psi_i$  are identically zero for  $1 \leq i \leq n$ , so that (5-14) and (5-16) give nothing beyond the trivial bound for the exponential sum  $S(\alpha, \beta)$ .

Recall the definitions (5-4) and (5-6) of  $\mathfrak{M}(J)$  and  $\mathfrak{N}(K)$ , respectively. We want to bound the size of  $S(\alpha, \beta)$  when  $\alpha \notin \mathfrak{M}(J)$  and  $\beta \notin \mathfrak{N}(K)$ . To do this it will be convenient to introduce two parameters  $s_1$  and  $s_2$ . Associated to these are the quantities

$$l_1 = e - j + 1 - s_1 \quad \text{and} \quad l_2 = e - j + 1 - s_2.$$

We can use our geometry-of-numbers shrinking result to establishing the following pair of estimates.

**Lemma 5.6.** *Let  $\alpha \notin \mathfrak{M}(J)$  and let  $l_1 \in \mathbb{Z}$  be such that*

$$l_1 \leq 1 + \frac{J}{d-1} \quad \text{and} \quad l_1 \leq e - j + 1.$$

*Then there exists a constant  $c_{d,n} > 0$  such that.*

$$N(\alpha, e - j + 1) \leq c_{d,n} q^{-nl_1} |P|^{(d-1)n}.$$

*Proof.* It follows from Lemma 5.4 that  $N(\alpha, e - j + 1)$  is at most

$$q^{(d-1)ns_1} \#\{\underline{\mathbf{g}} \in \mathbb{F}_q[T]^{(d-1)n} : |\mathbf{g}_1|, \dots, |\mathbf{g}_{d-1}| < |P|/q^{s_1}, \|\alpha\Psi_i(\underline{\mathbf{g}})\| < |P|^{-1}q^{-(d-1)s_1} (\forall i \leq n)\},$$

for any  $s_1 \geq 0$ . Note that  $|P|/q^{s_1} = q^{l_1}$  and  $q^{s_1} = |P_0|/q^{l_1-1}$ . Suppose that

$$|\mathbf{g}_1|, \dots, |\mathbf{g}_{d-1}| < |P|/q^{s_1}$$

and  $\|\alpha\Psi_i(\underline{\mathbf{g}})\| < |P|^{-1}q^{-(d-1)s_1}$  but  $\Psi_i(\underline{\mathbf{g}}) \neq 0$ . Let  $r = \Psi_i(\underline{\mathbf{g}})$  and let  $a$  be the integer part of  $\alpha\Psi_i(\underline{\mathbf{g}})$ , each divided through by any common factors that they might share. Then  $|r| \leq q^{(d-1)(l_1-1)}$  and

$$|r\alpha - a| < |P|^{-1}q^{-(d-1)s_1} = q^{(d-1)(l_1-1)-1} |P|^{-d}.$$

This contradicts the assumption that  $\alpha \notin \mathfrak{M}(J)$ , if  $J \geq (d-1)(l_1-1)$ . Hence, if  $J \geq (d-1)(l_1-1)$  and  $\alpha \notin \mathfrak{M}(J)$ , we have

$$N(\alpha, e - j + 1) \leq q^{(d-1)ns_1} \#\{\underline{\mathbf{g}} \in \mathbb{F}_q[T]^{(d-1)n} : |\mathbf{g}_1|, \dots, |\mathbf{g}_{d-1}| < q^{l_1}, \Psi_i(\underline{\mathbf{g}}) = 0 (\forall i \leq n)\}.$$

The statement of the lemma follows on noting that the remaining cardinality is  $O(q^{(d-2)nl_1})$  for dimensionality reasons, where the implied constant depends only on  $d$  and  $n$ .  $\square$

**Lemma 5.7.** *Let  $\beta \notin \mathfrak{N}(K)$  and let  $l_2 \in \mathbb{Z}$  be such that*

$$l_2 \leq 1 + \frac{K}{d-1} \quad \text{and} \quad l_2 \leq e - j + 1 - \max(0, \varrho - j + 1).$$

*Then there exists a constant  $c_{d,n} > 0$  such that.*

$$N(\beta, e - \varrho) \leq c_{d,n} q^{-nl_2 + n \max(0, \lfloor (\varrho - j + 1)/2 \rfloor)} |P|^{(d-1)n}.$$

*Proof.* This time we take  $r = e - \varrho$  in Lemma 5.4 and deduce that

$$N(\beta, e - \varrho) \leq q^{(d-1)ns_2 + n \max(0, \lfloor (\varrho - j + 1)/2 \rfloor)} \\ \times \#\{\underline{\mathbf{g}} \in \mathbb{F}_q[T]^{(d-1)n} : |\mathbf{g}_1|, \dots, |\mathbf{g}_{d-1}| < q^{l_2}, \|\beta \Psi_i(\underline{\mathbf{g}})\| < |Q|^{-1} q^{-(d-1)s_2} (\forall i \leq n)\},$$

for any  $s_2 \geq \max(0, \varrho - j + 1)$ . Arguing as in the previous result it is simple to check that we must in fact have  $\Psi_i(\underline{\mathbf{g}}) = 0$  for all  $1 \leq i \leq n$  whenever  $\beta \notin \mathfrak{N}(K)$  and  $K \geq (d-1)(l_2 - 1)$ . But then there are  $O(q^{(d-2)nl_2})$  possible vectors  $\underline{\mathbf{g}} \in \mathbb{F}_q[T]^{(d-1)n}$  that contribute. The statement of the lemma follows.  $\square$

In our work we shall take

$$l_1 = 1 + \left\lfloor \frac{J}{d-1} \right\rfloor, \quad l_2 = 1 + \left\lfloor \frac{K}{d-1} \right\rfloor. \quad (5-17)$$

We need to check that the remaining conditions on  $l_1$  and  $l_2$  are satisfied in Lemmas 5.6 and 5.7. To begin with we note that

$$J \leq \left\lceil \frac{d(e-j)}{2} \right\rceil - 1 \leq \frac{d(e-j)}{2} - \frac{1}{2}.$$

Hence for Lemma 5.6 to be applicable it suffices to have

$$d(e-j) - 1 \leq 2(d-1)(e-j).$$

But this is equivalent to  $0 \leq 1 + (d-2)(e-j)$  which follows from (5-8). Next, we note that

$$K \leq \left\lceil \frac{(e-j)(d-1) + e - \varrho}{2} \right\rceil - 1 \leq \frac{(e-j)(d-1) + e - \varrho}{2} - \frac{1}{2},$$

so that Lemma 5.7 is applicable if

$$e - \varrho - 1 \leq (d-1)(e-j - 2 \max(0, \varrho - j + 1)).$$

Thus it suffices to have

$$e - \varrho - 1 \leq (d-1)(e-j) \quad (5-18)$$

and

$$e - \varrho - 1 \leq (d-1)(e+j - 2\varrho - 2). \quad (5-19)$$

However, (5-18) follows from (5-8), so it suffices to assume that (5-19) holds.



Inserting Lemmas 5.6 and 5.7 into our Weyl differencing bounds (5-14) and (5-16), we deduce that there exists a constant  $c_{d,n} > 0$  such that

$$|S(\alpha, \beta) - q^{n(e-\varrho)}| \leq c_{d,n} |P|^n |Q|^n \min(q^{-nl_1/2^{d-1}}, q^{-n(l_2-n \max(0, \lfloor (\varrho-j+1)/2 \rfloor)) / 2^{d-2}}) \\ = c_{d,n} |P|^n |Q|^n / \max(q^{l_1}, q^{2l_2-2 \max(0, \lfloor (\varrho-j+1)/2 \rfloor)}) n / 2^{d-1},$$

whenever  $(\alpha, \beta) \in \mathfrak{M}(J+1) \setminus \mathfrak{M}(J) \times \mathfrak{N}(K+1) \setminus \mathfrak{N}(K)$  and (5-19) holds. We shall proceed under the assumption that the parameter  $l_2$  satisfies

$$l_2 - \max\left(0, \left\lfloor \frac{\varrho - j + 1}{2} \right\rfloor\right) \geq 0. \tag{5-20}$$

This is precisely the circumstance under which our  $\beta$ -treatment is nontrivial. Assume that  $n > (d-1)2^d$ , so that  $2^d(d-1)/n < 1$ . If (5-20) holds we can invoke the inequality  $\max(A, B) \geq A^{2^d(d-1)/n} B^{1-2^d(d-1)/n}$ , which is valid for any  $A, B \geq 1$ . Thus it follows that

$$|S(\alpha, \beta) - q^{n(e-\varrho)}| \leq c_{d,n} \frac{q^{2(d-1)(2l_2-l_1)-4(d-1) \max(0, \lfloor (\varrho-j+1)/2 \rfloor)} |P|^n |Q|^n}{q^{(l_2-\max(0, \lfloor (\varrho-j+1)/2 \rfloor))n/2^{d-2}}}.$$

Returning to (5-2) we see that

$$\sum_{\substack{\mathbf{g} \in \mathbb{F}_q[T]^n \\ 0 < |\mathbf{g}| < q^{e-j+1} \\ f(\mathbf{g})=0}} \int_{n_j} S(\beta) d\beta \leq \sum_{J=-1}^{M-1} \sum_{K=e-\varrho}^{N-1} E(J, K)$$

where we recall from (5-5) and (5-7) that

$$M = \left\lceil \frac{d(e-j)}{2} \right\rceil, \quad N = \left\lceil \frac{(e-j)(d-1) + e - \varrho}{2} \right\rceil,$$

and

$$E(J, K) = \int_{\mathfrak{M}(J+1) \setminus \mathfrak{M}(J)} \int_{\mathfrak{N}(K+1) \setminus \mathfrak{N}(K)} |S(\alpha, \beta) - q^{n(e-\varrho)}| d\alpha d\beta.$$

The measure of all  $(\alpha, \beta)$  in the integral is at most  $q^{4+2J+2K} |P_0|^{-2d+1} |Q|^{-1}$ , by (5-9) and (5-10). Let us consider the total contribution

$$E_{l_1, l_2} = \sum_{J=\max((d-1)(l_1-1), -1)}^{(d-1)l_1-1} \sum_{K=\max((d-1)(l_2-1), e-\varrho)}^{(d-1)l_2-1} E_{J, K},$$

from  $J, K$  associated to integers  $l_1 \geq 0$  and  $l_2 \geq 1$  via (5-17). Then

$$E_{l_1, l_2} \ll \frac{q^{6(d-1)l_2} |P|^n |Q|^{n-1} |P_0|^{-2d+1} q^{-4(d-1) \max(0, \lfloor (\varrho-j+1)/2 \rfloor)}}{q^{(l_2-\max(0, \lfloor (\varrho-j+1)/2 \rfloor))n/2^{d-2}}} \\ = q^{\Delta_j - l_2(n/2^{d-2} - 6(d-1)) + \max(0, \lfloor (\varrho-j+1)/2 \rfloor)(n/2^{d-2} - 4(d-1))},$$

where we have put

$$\Delta_j = (e-j)(n-2d+1) + (e-\varrho)(n-1) + n.$$

Because  $K \geq e - \varrho$ , we have

$$l_2 \geq 1 + \left\lfloor \frac{e - \varrho}{d - 1} \right\rfloor.$$

In particular our condition (5-20) is satisfied when

$$1 + \left\lfloor \frac{e - \varrho}{d - 1} \right\rfloor \geq \max\left(0, \left\lfloor \frac{\varrho - j + 1}{2} \right\rfloor\right). \tag{5-21}$$

Furthermore, assuming  $n > 3(d - 1)2^{d-1}$ , the bound is decreasing in  $l_2$ , so the dominant contribution occurs when

$$l_2 = 1 + \left\lfloor \frac{e - \varrho}{d - 1} \right\rfloor.$$

Since there are  $O(e)$  choices for  $l_1$ , our work has therefore shown that

$$\sum_{\substack{\mathbf{g} \in \mathbb{F}_q[T]^n \\ 0 < |\mathbf{g}| < q^{e-j+1} \\ f(\mathbf{g})=0}} \int_{n_j} S(\beta) d\beta \ll eq^{\Delta_j - \Gamma_j},$$

where

$$\begin{aligned} \Gamma_j &= \left(\frac{n}{2^{d-2}} - 6(d-1)\right) \left(1 + \left\lfloor \frac{e - \varrho}{d - 1} \right\rfloor\right) - \left(\frac{n}{2^{d-2}} - 4(d-1)\right) \max\left(0, \left\lfloor \frac{\varrho - j + 1}{2} \right\rfloor\right) \\ &= \left(\frac{n}{2^{d-2}} - 6(d-1)\right) \left(1 + \left\lfloor \frac{e - \varrho}{d - 1} \right\rfloor - \max\left(0, \left\lfloor \frac{\varrho - j + 1}{2} \right\rfloor\right)\right) - 2(d-1) \max\left(0, \left\lfloor \frac{\varrho - j + 1}{2} \right\rfloor\right). \end{aligned}$$

Thus we certainly require (5-21) to hold in order to expect any saving in our minor arc estimate.

We summarize our argument in the following result.

**Lemma 5.8.** *Let  $d \geq 3$  and  $n > 3(d - 1)2^{d-1}$ . Assume that  $\varrho \geq -1$  and*

$$e \geq \max\left(\varrho + (d - 1) \left\lfloor \frac{\varrho + 1}{2} \right\rfloor, (\varrho + 1) \left(2 + \frac{1}{d - 2}\right)\right). \tag{5-22}$$

Then

$$N_q^{minor}(q, e, f) \ll eq^{\Delta_0 - \Gamma_0}$$

where

$$\Delta_0 = 2e(n - d) - \varrho(n - 1) + n$$

and

$$\Gamma_0 = \left(\frac{n}{2^{d-2}} - 6(d-1)\right) \left(1 + \left\lfloor \frac{e - \varrho}{d - 1} \right\rfloor - \left\lfloor \frac{\varrho + 1}{2} \right\rfloor\right) - 2(d-1) \left\lfloor \frac{\varrho + 1}{2} \right\rfloor.$$

*Proof.* Recall (5-1) and note that  $\Delta_j = \Delta_0 - j(n - 2d + 1)$ . Hence for the range of  $n$  in which we are interested we deduce from (4-2) that

$$|c_j|q^{\Delta_j - \Gamma_j} \ll q^{\Delta_0 - \Gamma_0},$$

for all  $j \geq 0$ . Moreover,  $\Gamma_0$  takes the value recorded in the statement of the lemma when  $\varrho \geq -1$  and the condition (5-22) on  $e$  is enough to ensure that (5-19) and (5-21) both hold for every  $j \in \{0, 1, 2\}$ . (For (5-19) we note that it suffices to have  $e(d-2) \geq (\varrho+1)(2d-3)$ .) This completes the proof.  $\square$

**5C. Deduction of Theorem 1.1.** We assume that  $n > (2d-1)2^{d-1}$ . We revisit the argument deployed in [Browning and Vishe 2015] to establish the irreducibility and dimension of  $\mathcal{M}_{0,0}(X, e)$ . This is based on a counting argument over a finite field  $\mathbb{F}_q$  whose characteristic is greater than the degree  $d$  of the nonsingular form  $f \in \mathbb{F}_q[x_1, \dots, x_n]$  that defines  $X$ . According to [loc. cit., Equation (3.3)], in order to deduce that  $\mathcal{M}_{0,0}(X, e)$  is irreducible and of the expected dimension it suffices to show that

$$\lim_{q \rightarrow \infty} q^{-(n-d)e-n+1} \hat{N}(q, e, f) \leq 1, \quad (5-23)$$

where  $\hat{N}(q, e, f)$  is the number of  $\mathbf{g} \in \mathbb{F}_q[T]^n$  such that  $|\mathbf{g}| < q^{e+1}$  and  $f(\mathbf{g}) = 0$ .

We have

$$\hat{N}(q, e, f) = \int_{\mathbb{T}} S_{\text{BV}}(\alpha) d\alpha,$$

where

$$S_{\text{BV}}(\alpha) = \sum_{\substack{\mathbf{g} \in \mathbb{F}_q[T]^n \\ |\mathbf{g}| < q^{e+1}}} \psi(\alpha f(\mathbf{g})) = q^{-n(e-\varrho)} S(\alpha, 0),$$

in the notation of (5-3), with  $j = 0$ . Take  $j = 0$  in the major arcs  $\mathfrak{M}(J)$  that were defined in (5-4). A straightforward calculation shows that the contribution from the major arc around 0 is

$$\int_{\mathfrak{M}(0)} S_{\text{BV}}(\alpha) d\alpha = \sum_{\substack{\mathbf{g} \in \mathbb{F}_q[T]^n \\ |\mathbf{g}| < q^{e+1}}} \int_{|\theta| < q^{-de}} \psi(\theta f(\mathbf{g})) d\theta = q^{ne-de} (q^{n-1} + O(q^{n/2})).$$

In order to complete the proof of (5-23) it therefore suffices to show that

$$\lim_{q \rightarrow \infty} q^{-(n-d)e-n+1} \sum_{J=0}^{M-1} \int_{\mathfrak{M}(J+1) \setminus \mathfrak{M}(J)} |S_{\text{BV}}(\alpha)| d\alpha < 1$$

where  $M = \lceil de/2 \rceil$  is given by (5-5). To do this we may apply our previous work. Thus it follows from (5-14) and Lemma 5.6 that

$$S_{\text{BV}}(\alpha) \ll |P|^n q^{-nl_1/2^{d-1}},$$

if  $\alpha \notin \mathfrak{M}(J)$  and  $l_1$  is any integer such that  $l_1 \leq 1 + J/(d-1)$  and  $l_1 \leq e+1$ . The choice  $l_1 = 1 + \lfloor J/(d-1) \rfloor$  is acceptable since  $J \leq \lceil de/2 \rceil - 1 \leq (de-1)/2$ , whence

$$l_1 \leq 1 + \frac{de-1}{2(d-1)} = 1 + e,$$

for  $d \geq 3$ . Since  $J \geq 0$  we are clearly only interested in integers  $l_1 \geq 1$ . Appealing to (5-9) to estimate the volume of  $\mathfrak{M}(J + 1)$ , we deduce that for given  $l_1 \geq 1$  the total associated contribution is

$$\begin{aligned} \sum_{J=(d-1)(l_1-1)}^{(d-1)l_1-1} \int_{\mathfrak{M}(J+1) \setminus \mathfrak{M}(J)} |S_{\text{BV}}(\alpha)| d\alpha &\ll \sum_{J=(d-1)(l_1-1)}^{(d-1)l_1-1} q^{2J+2-de} \cdot |P|^n q^{-nl_1/2^{d-1}} \\ &\ll q^{-de+n(e+1)+(2(d-1)-n/2^{d-1})l_1}. \end{aligned}$$

This is decreasing with  $l_1$  if  $n > (d - 1)2^d$  and we may therefore sum over  $l_1 \geq 1$  to finally deduce that

$$q^{-(n-d)e-n+1} \sum_{J=0}^{M-1} \int_{\mathfrak{M}(J+1) \setminus \mathfrak{M}(J)} |S_{\text{BV}}(\alpha)| d\alpha \ll q^{1+2(d-1)-n/2^{d-1}}.$$

The exponent of  $q$  is negative if  $n > (2d - 1)2^{d-1}$ , which thereby concludes the proof of (5-23), whence  $\mathcal{M}_{0,0}(X, e)$  is indeed irreducible and of the expected dimension. It follows from the same method used in [Harris et al. 2004, page 2] that  $\mathcal{M}_{0,0}(X, e)$  is locally a complete intersection. Indeed, since  $\mathcal{M}_{0,0}(X, e)$  is locally the intersection of  $de + 1$  equations in  $\mathcal{M}_{0,0}(\mathbb{P}^{n-1}, e)$ , a smooth stack of dimension  $ne - 4$ , it is a locally complete intersection if and only if its dimension is  $(n - d)e + n - 5$ .

**5D. Deduction of Theorem 1.5.** Assume that  $d \geq 3, n > 3(d-1)2^{d-1}, \varrho \geq -1$ , and  $e \geq (\varrho+1)(2+(1)/d-2)$ . In particular, this implies that  $e \geq \varrho$ , which is needed for Lemma 5.1. In view of Theorem 1.1, the stated bound is trivial unless  $1 + \lfloor (e - \varrho)/(d - 1) \rfloor - \lfloor (\varrho + 1)/2 \rfloor > 0$ , so we may assume that  $\lfloor (e - \varrho)/(d - 1) \rfloor - \lfloor (\varrho + 1)/2 \rfloor \geq 0$  and thus  $e \geq \varrho + (d - 1)\lfloor (\varrho + 1)/2 \rfloor$ . Hence we may assume that (5-22) holds.

Combining Lemmas 5.1 and 5.8 we deduce that

$$\#Z_\varrho(\mathbb{F}_q) \ll e q^{e(n-d)+n-5-\min(\mu_1(n), \mu_2(n))}, \tag{5-24}$$

with

$$\mu_1(n) = \left( \frac{n}{2^{d-2}} - 6(d-1) \right) \left( 1 + D - \left\lfloor \frac{\varrho+1}{2} \right\rfloor \right) - 2(d-1) \left\lfloor \frac{\varrho+1}{2} \right\rfloor$$

and

$$\mu_2(n) = (1 + D)(n - d - 1) - de + e + 1.$$

Here we recall that  $D$  is given by (4-7) as  $\lfloor (e - \varrho)/(d - 1) \rfloor$ .

We claim that  $\mu_1(n) \leq \mu_2(n)$ . They are both increasing affine functions of  $n$ , with  $\mu_1(n)$  of lesser slope than  $\mu_2(n)$ . Hence to check that  $\mu_1(n)$  is the minimum, it suffices to check that  $\mu_2(n) \geq 0$  and  $\mu_1(n) \leq 0$  when  $n = 3(d - 1)2^{d-1}$ . In other words, we must show that

$$3(d - 1)2^{d-1} \geq d + 1 + \frac{e(d - 1) - 1}{1 + D}.$$

To do this, observe that because  $e \geq \varrho + (d - 1)\lfloor (\varrho + 1)/2 \rfloor$  we have  $e \geq ((d + 1)/2)\varrho$ , so that

$$1 + D \geq \frac{e + 1 - \varrho}{d - 1} \geq \frac{e + 1 - (2/(d + 1))e}{d - 1} \geq \frac{e}{d + 1}.$$

Thus

$$d + 1 + \frac{e(d - 1) - 1}{1 + D} \leq d + 1 + \frac{e(d - 1)}{e/(d + 1)} = d(d + 1),$$

so it suffices to check

$$3(d - 1)2^{d-1} \geq d(d + 1).$$

But it is clear that this holds for all  $d \geq 3$ , whence  $\mu_2(n) \geq \mu_1(n)$ .

By [Lang and Weil 1954], it now follows from (5-24) that

$$\dim Z_\varrho \leq e(n - d) + n - 5 - \mu_1(n)$$

for any smooth hypersurface defined over a finite field. For a general hypersurface, we can spread it out to a family defined over a ring finitely generated over  $\mathbb{Z}$ . The dimension of  $Z_\varrho$  in this family is manifestly constant on some open subset of the spectrum of this ring, which must contain a finite-field valued point, so  $\dim Z_\varrho$  is at most  $e(n - d) + n - 5 - \mu_1(n)$  for the generic point and thus for the original hypersurface. This completes the proof of Theorem 1.5.

**5E. Deduction of Theorem 1.2.** We consider the effect of taking  $\varrho = -1$  in Theorem 1.5. Clearly (1-1) is equivalent to  $e \geq 0$  and can be ignored. Note that  $Z_{-1}$  contains the singular locus of  $\mathcal{M}_{0,0}(X, e)$  by [Debarre 2001, Theorem 2.6]. Thus the codimension of the singular locus is at least  $\dim \mathcal{M}_{0,0}(X, e) - \dim Z_{-1}$ . Theorem 1.2 therefore follows from applying Theorem 1.1 to calculate  $\dim \mathcal{M}_{0,0}(X, e)$  and Theorem 1.5 to bound  $\dim Z_{-1}$ .

Because the lower bound for the codimension of the singular locus is strictly positive, the moduli space is generically smooth. Any generically smooth locally complete intersection scheme is reduced, which thereby completes the proof of Theorem 1.1.

### 6. Peyre’s freedom counting function

In this section we prove the asymptotic formula in Theorem 1.6 for the counting function (1-4), by piecing together our work above and the main results in Lee’s thesis [2013]. We have

$$N_X^{\varepsilon\text{-free}}(B) = N_X(B) - E_\varepsilon(B), \tag{6-1}$$

where  $E_\varepsilon(B)$  counts the number of  $x \in X(\mathbb{F}_q(T))$  with  $H_{\omega_V^{-1}}(x) \leq q^B$  such that  $\ell(x) < \varepsilon$ .

Let us begin by studying  $N_X(B)$ . As usual we suppose that  $X$  is defined by a nonsingular form  $f \in \mathbb{F}_q[x_1, \dots, x_n]$  of degree  $d \geq 3$ . It follows from the proof of part (1) of Proposition 3.8 that

$$N_X(B) = \frac{1}{q - 1} \#\{\mathbf{g} \in \mathbb{F}_q[T]^n : \gcd(g_1, \dots, g_n) = 1, |\mathbf{g}|^{n-d} < q^{B+1}, f(\mathbf{g}) = 0\}.$$

Using the Möbius function to detect the coprimality condition we obtain

$$\begin{aligned}
 N_X(B) &= \frac{1}{q-1} \sum_{\substack{k \in \mathbb{F}_q[T] \\ k \text{ monic}}} \mu(k) \#\{\mathbf{g} \in \mathbb{F}_q[T]^n : 0 < |k\mathbf{g}|^{n-d} < q^{B+1}, f(\mathbf{g}) = 0\} \\
 &= \frac{1}{q-1} \sum_{j \geq 0} \sum_{\substack{k \in \mathbb{F}_q[T] \\ |k|=q^j \\ k \text{ monic}}} \mu(k) \#\{\mathbf{g} \in \mathbb{F}_q[T]^n : 0 < |\mathbf{g}|^{n-d} < q^{B+1-j(n-d)}, f(\mathbf{g}) = 0\}.
 \end{aligned}$$

Put  $m = n - (d - 1)2^d$  and assume that  $m > 0$ . Then, on appealing to Lee’s thesis [2013, Theorem 4.1.1], it follows that

$$\#\{\mathbf{g} \in \mathbb{F}_q[T]^n : 0 < |\mathbf{g}|^{n-d} < q^{R+1}, f(\mathbf{g}) = 0\} = q^R(c_f + O(q^{-mR/(2^{d+1}(d-1)(n-d)}))), \tag{6-2}$$

for any  $R > 0$ , where  $c_f$  is the usual product of singular series and singular integral. Using (4-1) to handle the sum over  $j$  and  $k$ , it now follows from (6-2) that there exists  $\delta > 0$  such that

$$N_X(B) = \frac{c_f}{(q-1)\zeta_{\mathbb{F}_q(T)}(n-d)} q^B + O(q^{(1-\delta)B}),$$

where  $\zeta_{\mathbb{F}_q(T)}(s) = (1 - q^{1-s})^{-1}$  is the rational zeta function. Arguing along standard lines (as in Peyre [Peyre 1995, Section 5.4], for example), one readily confirms that this agrees with the Batyrev–Manin–Peyre prediction for the hypersurface  $X$ .

It remains to produce an upper bound for the quantity  $E_\varepsilon(B)$  in (6-1). Let  $x \in X(\mathbb{F}_q(T))$  and suppose that it defines a map  $c : \mathbb{P}^1 \rightarrow X$  of degree  $e$ . Then it follows from [Peyre 2017, Notation 5.7] that

$$\ell(x) = \frac{(n-1)\varrho}{e(n-d)}$$

if and only if  $c$  is  $\varrho$ -free but not  $(\varrho+1)$ -free. (In particular, Remark 1.4 implies that  $\ell(x) \in [0, 1]$ .) We deduce that  $E_\varepsilon(B)$  is at most the number of rational maps from  $\mathbb{P}^1 \rightarrow X$  with degree at most  $B/(n-d)$  which are not  $\varrho$ -free, with

$$\varrho = \left\lfloor \frac{\varepsilon B}{n-1} \right\rfloor + 2. \tag{6-3}$$

We may therefore appeal to the proof of Proposition 3.8(2) to estimate this quantity, finding that

$$E_\varepsilon(B) \leq \frac{N_\varrho(q, B/(n-d), f)q^{\varrho(n-1)-B} - N(q, B/(n-d), f)}{(q-1)^2},$$

with  $\varrho$  given by (6-3). In what follows it will be convenient to set  $e = B/(n-d)$  and to assume that  $e \in \mathbb{N}$ . All of the implied constants that follow are allowed to depend on  $q$  and  $f$ , but not on  $e$  or  $\varrho$ . We seek conditions on  $n$  and  $\varrho$  under which we can deduce that there exists  $\delta > 0$  such that  $E_\varepsilon(B) = O(q^{(1-\delta)e(n-d)})$ .

First we improve our treatment of Lemma 4.4 slightly by revisiting the argument (4-10). Since we no longer care about a dependence on the finite field, rather than invoking a trivial bound we may apply

(6-2) to deduce that  $N(q, u - \ell, f) \ll q^{(u-\ell)(n-d)}$  if  $n > (d-1)2^d$ . But then (4-10) can be replaced by the bound

$$E_j \ll_f \sum_{0 \leq u \leq e-j} q^{u(n-2d+1)} \sum_{\ell \geq D+1+j-e+u} q^{-\ell(n-2d)} \ll_f q^{(e-j)(n-2d+1)-(D+1)(n-2d)},$$

where  $D$  is given by (4-7), whence

$$q^{(n-1)(e-\varrho)} \sum_{j \geq 0} |c_j| E_j \ll_f q^{2e(n-d)-\varrho(n-1)-(D+1)(n-2d)} \ll_f q^{2e(n-d)-\varrho(n-1)-(e-\varrho)(n-2d)/(d-1)}.$$

It now follows from (4-3) and our modified version of Lemma 4.4 that

$$E_\varepsilon(B) \ll q^{e(n-d)-(e-\varrho)(n-2d)/(d-1)} + q^{-e(n-d)+\varrho(n-1)} N_\varrho^{\text{minor}}(q, e, f),$$

provided that  $e \geq \varrho$ . Note that  $\Gamma_0 = \gamma_0 + O_{d,n}(1)$ , with

$$\gamma_0 = \left( \frac{n}{2^{d-2}} - 6(d-1) \right) \left( \frac{e-\varrho}{d-1} - \frac{\varrho}{2} \right) - (d-1)\varrho.$$

Appealing now to Lemma 5.8 we therefore deduce that

$$E_\varepsilon(B) \ll q^{e(n-d)-(e-\varrho)(n-2d)/(d-1)} + e q^{e(n-d)-\gamma_0}$$

if (5-22) holds.

Recall that  $n > 3(d-1)2^{d-1}$ . Then  $n/2^{d-2} - 6(d-1) \geq 2^{-d+2}$  and we can ensure that  $\gamma_0 \geq \delta e$  for a small parameter  $\delta > 0$  (that depends only on  $d$ ) provided that

$$e \geq (d-1)^2 2^{d-1} \varrho. \tag{6-4}$$

This is also enough to ensure that  $(e-\varrho)(n-2d)/(d-1) \geq \delta e$ . This inequality is clearly much stronger than (5-22). The statement of Theorem 1.6 now follows on taking  $e = B/(n-d)$  and noting that the hypothesis on  $\varepsilon$  in the theorem is enough to ensure that (6-4) holds when  $\varrho$  is given by (6-3) and  $B$  is sufficiently large.

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