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One-level density estimates for Dirichlet L-functions with extended support

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We estimate the 1-level density of low-lying zeros of $L(s,\chi)$ with χ ranging over primitive Dirichlet characters of conductor in $\left[\frac{1}{2}Q, Q\right]$ and for test functions whose Fourier transform is supported in $\left(-2-\frac{50}{1093}, 2+\frac{50}{1093}\right)$. Previously, any extension of the support past the range $(-2,2)$ was only known conditionally on deep conjectures about the distribution of primes in arithmetic progressions, beyond the reach of the generalized Riemann hypothesis (e.g., Montgomery's conjecture). Our work provides the first example of a family of L-functions in which the support is unconditionally extended past the "diagonal range" that follows from a straightforward application of the underlying trace formula (in this case orthogonality of characters). We also highlight consequences for nonvanishing of $L(s,\chi)$.

1. Introduction

Motivated by the problem of establishing the nonexistence of Siegel zeros (see [Conrey and Iwaniec 2002] for details), Montgomery [1973] investigated the *vertical* distribution of the zeros of the Riemann zeta function. He showed that under the assumption of the Riemann hypothesis, for any smooth function f with $\operatorname{supp} \hat{f} \subset (-1,1)$,

$$\lim_{T\to\infty} \frac{1}{N(T)} \sum_{T\le\gamma,\gamma'\le 2T} f\left(\frac{\log T}{2\pi}\cdot(\gamma-\gamma')\right) = \int_{\mathbb{R}} f(u)\cdot\left(\delta(u)+1-\left(\frac{\sin 2\pi u}{2\pi u}\right)^2\right)du, \tag{1}$$

where $N(T)$ denotes the number of zeros of the Riemann zeta function up to height T, and γ,γ' are ordinates of the zeros of the Riemann zeta function, and $\delta(u)$ is a Dirac mass at 0. Dyson famously observed that the right-hand side coincides with the pair correlation function of eigenvalues of a random Hermitian matrix.

Dyson's observation leads one to conjecture that the spacings between the zeros of the Riemann zeta function are distributed in the same way as spacings between eigenvalues of a large random Hermitian matrix. Subsequent work of Rudnick and Sarnak [1994] provided strong evidence towards this conjecture by computing (under increasingly restrictive conditions) the n-correlations of the zeros of any given automorphic L-function. Importantly, the work of Rudnick and Sarnak [1996] suggested that the distribution of the zeros of an automorphic L-function is universal and independent of the distribution of its coefficients.

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For number-theoretic applications, the distribution of the so-called "low-lying zeros", that is, zeros close to the central point, is particularly interesting (see, e.g., [Heath-Brown 2004; Young 2006] for various applications; see also [Granville and Soundararajan 2018] and [Watkins 2021], for instance, for results in a different direction). Following the work of Katz and Sarnak [1999] and Iwaniec, Luo and Sarnak [Iwaniec et al. 2000], we believe that the distribution of these low-lying zeros is also universal and predicted by only a few random matrix ensembles (which are either symplectic, orthogonal or unitary).

Specifically, the work of Katz and Sarnak suggests that for any smooth function ϕ and any natural "family" of automorphic objects $\mathcal{F}$,

$$\frac{1}{\#\mathcal{F}}\sum_{\pi\in\mathcal{F}}\sum_{\gamma_\pi}\phi\left(\frac{\log \mathfrak{c}_\pi}{2\pi}\cdot\gamma_\pi\right)\xrightarrow[\#\mathcal{F}\to\infty]{}\int_{\mathbb{R}}\phi(x)K_{\mathcal{F}}(x)\,dx, \tag{2}$$

where γ_π are ordinates of the zeros of the L-function attached to π, $\mathfrak{c}_\pi$ is the analytic conductor of π, and $K_{\mathcal{F}}(x)$ is a function depending only on the "symmetry type" of $\mathcal{F}$. One may wish to consult [Iwaniec et al. 2000] and [Sarnak et al. 2016] for a more detailed discussion.

There is a vast literature providing evidence for (2) (see [Mackall et al. 2016]). Similarly to Montgomery's result (1), all of the results in the literature place a restriction on the support of the Fourier transform of ϕ. This restriction arises from the limitations of the relevant trace formula (in some families it is not always readily apparent what this relevant trace formula is). In practice, an application of the trace formula gives rise to so-called "diagonal" and "off-diagonal" terms. Trivially bounding the off-diagonal terms corresponds to what we call a "straightforward" application of the trace formula.

A central yet extremely difficult problem is to extend the support of $\hat{\phi}$ beyond what a "straightforward" application of the trace formula gives. In fact most works in which the support of $\hat{\phi}$ has been extended further rely on the assumption of various deep hypotheses about primes that sometimes lie beyond the reach of the generalized Riemann hypothesis (GRH).

For example, Iwaniec, Luo and Sarnak show that in the case of holomorphic forms of even weight $\leq K$ one obtains unconditionally a result for $\hat{\phi}$ supported in $(-1, 1)$ and that under the assumption of the generalized Riemann hypothesis this can be enlarged to $(-2, 2)$ (it is observed in [Devin et al. 2022] that assuming GRH only for Dirichlet L-functions is sufficient). Iwaniec, Luo and Sarnak also show that this range can be pushed further to $\operatorname{supp}\hat{\phi}\subset\left(-\frac{22}{9},\frac{22}{9}\right)$ under the additional assumption that, for any $c\geq 1$, $(a, c) = 1$ and $\varepsilon > 0$,

$$\sum_{\substack{p\leq x\\ p\equiv a\ (\mathrm{mod}\, c)}} e(2\sqrt{p}/c)\ll_\varepsilon x^{\frac{1}{2}+\varepsilon}.$$

A similar behaviour is observed on low-lying zeros of dihedral L-functions associated to an imaginary quadratic field [Fouvry and Iwaniec 2003], where an extension of the support is shown to be equivalent to an asymptotic formula on primes with a certain splitting behaviour.

Assuming GRH, Brumer [1992] studied the one-level density of the family of elliptic curves and proved a result for test functions supported in $\left(-\frac{5}{9},\frac{5}{9}\right)$; this corresponds to the "diagonal" range for this family. Heath-Brown [2004] improved this range to $\left(-\frac{2}{3}, \frac{2}{3}\right)$, and Young [2006] pushed the support to $\left(-\frac{7}{9}, \frac{7}{9}\right)$. One-level density estimates for this family have deep implications for average ranks of elliptic curves. In particular, the work of Young was the first to show that, under some reasonable conjectures, a positive proportion of elliptic curves have rank 0 or 1 and thus satisfy the rank part of the Birch and Swinnerton–Dyer conjecture.[1]

As another example, it follows for instance from minor modifications of [Hughes and Rudnick 2003; Chandee et al. 2014] that in the family of primitive Dirichlet characters of modulus $\leq Q$ one can estimate 1-level densities unconditionally for ϕ with $\hat{\phi}$ supported in $(-2, 2)$.[2] As a byproduct of work of Fiorilli and Miller [2015, Theorem 2.8], it follows that for any $\delta \in (0, 2)$, this support can be enlarged to $(-2-\delta, 2+\delta)$ under the following "de-averaging hypothesis":

$$\sum_{\frac{1}{2}Q\leq q\leq Q}\Bigg|\sum_{\substack{p\leq x\\ p\equiv 1\ (\mathrm{mod}\, q)}}\log p-\frac{x}{\varphi(q)}\Bigg|^2 \ll Q^{-\frac{1}{2}\delta}\sum_{\frac{1}{2}Q\leq q\leq Q}\ \sum_{(a,q)=1}\Bigg|\sum_{\substack{p\leq x\\ p\equiv a\ (\mathrm{mod}\, q)}}\log p-\frac{x}{\varphi(q)}\Bigg|^2. \tag{3}$$

In this paper we give a first example of a family of L-functions in which we can *unconditionally* enlarge the support past the "diagonal" range, which would follow from a straightforward application of the trace formula (in this case orthogonality of characters).

Theorem 1. *Let Φ be a smooth function compactly supported in $\left[\frac{1}{2}, 3\right]$, and ϕ be a smooth function such that supp $\hat{\phi} \subset \left(-2-\frac{50}{1093}, 2+\frac{50}{1093}\right)$. Then, as $Q \to \infty$,*

$$\sum_{q}\Phi\Big(\frac{q}{Q}\Big)\sum_{\substack{\chi\ (\mathrm{mod}\, q)\\ \text{primitive}}}\ \sum_{\gamma_\chi}\phi\Big(\frac{\log Q}{2\pi}\gamma_\chi\Big)=\hat{\phi}(0)\sum_{q}\Phi\Big(\frac{q}{Q}\Big)\sum_{\substack{\chi\ (\mathrm{mod}\, q)\\ \text{primitive}}}1+o(Q^2). \tag{4}$$

Here $\frac{1}{2}+i\gamma_\chi$ correspond to nontrivial zeros of $L(s,\chi)$ and since we do not assume the generalized Riemann hypothesis we allow the γ_χ to be complex.

Remark. In stating the theorem we have, for technical simplicity, made a suitable approximation to the conductor $\mathfrak{c}_\pi$ appearing in (2).

Note that ϕ, initially defined on $\mathbb{R}$, is analytically continued to $\mathbb{C}$ by compactness of supp $\hat{\phi}$. Our arguments can be adapted to show that if supp $\hat{\phi} \subset \left(-2-\frac{50}{1093}+\varepsilon, 2+\frac{50}{1093}-\varepsilon\right)$ for some $\varepsilon > 0$, then the error term in (4) is $O(Q^{2-\delta})$ with $\delta = \delta(\varepsilon)$, up to altering slightly the main terms: after applying the explicit formula as in Section 2.2, include the terms of order $\asymp Q^2/\log Q$ into the main term instead of treating them as error terms.

We remark that we make no progress on the "de-averaging hypothesis" (3) of Fiorilli and Miller, which remains a difficult open problem. We estimate the original sum over primes in arithmetic progressions, on average over moduli, by a variant of an argument of Fouvry [1985] and Bombieri, Friedlander and

[1] A stronger conclusion was later reached unconditionally by Bhargava and Shankar [2015] through other methods.

[2] This is in fact the GL(1) analogue of the result of Iwaniec, Luo and Sarnak for holomorphic forms.

Iwaniec [Bombieri et al. 1986] which is based on Linnik's dispersion method. The GRH will be dispensed with by working throughout, as in [Drappeau 2015], with characters of large conductors.

The asymptotic formula (4) is expected to hold true without the extra averaging over q. This extra averaging over q and the cancellation of arguments which comes along play an important role in our arguments.

If the GRH is true for Dirichlet L-functions, then let any $0 < \kappa < \frac{50}{1093}$ be fixed, and let $\lambda > 1$ be small enough that $\kappa' := 2(\lambda - 1) + \lambda\kappa \in \bigl(0, \frac{50}{1093}\bigr)$ as well. Defining

$$\tilde{\phi}(x) = \lambda\left(\frac{\sin \pi(2+\kappa)x}{\pi(2+\kappa)x}\right)^2, \quad \phi = \tilde{\phi} * u,$$

where u is a smooth, positive approximation of unity such that $\phi(0) \geq \lambda^{-1}\tilde{\phi}(0) = 1$, and using the inequality

$$1 - \sum_{\gamma_\chi} \phi\left(\frac{\log Q}{2\pi}\gamma_\chi\right) \leq \mathbf{1}\bigl(L\bigl(\tfrac{1}{2}, \chi\bigr) \neq 0\bigr),$$

we deduce from Theorem 1 that the proportion of nonvanishing $L\bigl(\frac{1}{2}, \chi\bigr)$ with χ ranging over primitive characters of conductor in $\bigl[\frac{1}{2}Q, Q\bigr]$ is at least $1 - \lambda(2+\kappa')^{-1} = 1 - (2+\kappa)^{-1}$ for any $\kappa < \frac{50}{1093}$.

Corollary 2. *Let $\varepsilon \in (0, 10^{-7})$. Assume the generalized Riemann hypothesis for Dirichlet L-functions. Then for all Q large enough, the proportion of primitive characters χ with modulus in $\bigl[\frac{1}{2}Q, Q\bigr]$ for which*

$$L\bigl(\tfrac{1}{2}, \chi\bigr) \neq 0$$

is at least

$$\tfrac{1}{2} + \tfrac{25}{2236} - \varepsilon > 0.51118.$$

Corollary 2 is related to a recent result of Pratt [2019], who showed unconditionally that the proportion of nonvanishing in this family is at least 0.50073. We note that both the arguments of [Pratt 2019] and those presented here eventually rely on bounds of Deshouillers and Iwaniec [1982] on cancellation in sums of Kloosterman sums.

Notation. We call a map $f : \mathbb{R}_+ \to \mathbb{C}$ a *test function* if f is smooth and supported inside $\bigl[\frac{1}{2}, 3\bigr]$.

For $w \in \mathbb{N}$, $n \in \mathbb{Z}$ and $R \geq 1$, we let

$$\mathfrak{u}_R(n, w) := \mathbf{1}_{n \equiv 1 \ (\mathrm{mod}\ w)} - \frac{1}{\varphi(w)} \sum_{\substack{\chi\ (\mathrm{mod}\ w) \\ \mathrm{cond}(\chi) \leq R}} \chi(n).$$

Note the trivial bound

$$|\mathfrak{u}_R(n, w)| \ll \mathbf{1}_{n \equiv 1 \ (\mathrm{mod}\ w)} + \frac{R\tau(w)}{\varphi(w)}. \tag{5}$$

The symbol $n \sim N$ in a summation means $n \in [N, 2N) \cap \mathbb{Z}$. We say that a sequence $(\alpha_n)_n$ is *supported at scale* N if $\alpha_n = 0$ unless $n \sim N$.

The letter ε will denote an arbitrarily small number, whose value may differ at each occurrence. The implied constants will be allowed to depend on ε.

2. Proof of Theorem 1

2.1. *Lemmas on primes in arithmetic progressions.* We will require two results about primes in arithmetic progressions. The first is a standard estimate, obtained from an application of the large sieve.

Lemma 3. *Let $A > 0$, X, Q, $R \geq 2$ satisfy $1 \leq R \leq Q$ and $X \geq Q^2/(\log Q)^A$, and let f be a test function with $\|f^{(j)}\|_\infty \ll_j 1$. Then*

$$\sum_{q \leq Q} \left| \sum_{n \in \mathbb{N}} f\left(\frac{n}{X}\right) \Lambda(n) \mathrm{u}_R(n, q) \right| \ll Q(\log Q)^{O(1)} \sqrt{X} \left(1 + \frac{\sqrt{X}}{RQ} + \frac{X^{\frac{3}{8}}}{Q}\right). \tag{6}$$

The implied constant depends at most on A and the implied constants in the hypothesis.

Proof. By Heath-Brown's combinatorial formula for primes [Iwaniec and Kowalski 2004, Proposition 13.3] (with $K = 2$), we restrict to proving the bound with $\Lambda(n)$ replaced by convolutions of types I and II, of the shape

$$\sum_{\substack{n = m\ell \\ m \sim M}} \sum \alpha_m \quad (M \ll X^{\frac{1}{4}}), \qquad \sum_{\substack{n = m\ell \\ m \sim M}} \sum \alpha_m \beta_\ell \quad (X^{\frac{1}{4}} \ll M \ll X^{\frac{3}{4}}),$$

where $|\alpha_m| \ll (\log X)\tau_4(m)$ and the analogous bound holds for β_ℓ; here we noted that if $m_1 \leq m_2 \leq \sqrt{X}$ and $m_1 m_2 > X^{\frac{1}{4}}$, then either $X^{\frac{1}{4}} < m_1 m_2 \leq X^{\frac{3}{4}}$ or $X^{\frac{1}{4}} \leq m_1 \ll X^{\frac{1}{2}}$. We treat the type I case by the Pólya–Vinogradov inequality [Iwaniec and Kowalski 2004, Theorem 12.5], getting a bound $O(MR^{\frac{3}{2}}(\log Q)^{O(1)})$. We treat the type II case by the large sieve [Iwaniec and Kowalski 2004, Theorem 17.4], getting a contribution $O\big(\sqrt{X}(\log Q)^{O(1)}(Q + \sqrt{M} + \sqrt{X/M} + \sqrt{X}R^{-1})\big)$. □

The second estimate is substantially deeper and we defer its proof to Section 4.

Proposition 4. *Let $\kappa \in \big(0, \frac{50}{1093}\big)$ and $\varepsilon > 0$. Let Ψ and f be test functions, $A > 0$, X, Q, W, $R \geq 1$, and $b \in \mathbb{N}$. Assume that*

$$\frac{Q^2}{(\log Q)^A} \ll X \ll Q^{2+\kappa}, \quad X^{\frac{11}{20}} Q^{-1} \leq R \leq Q^{\frac{2}{3}} X^{-\frac{2}{9}}, \quad b \leq Q^\varepsilon, \quad Q^{1-\varepsilon} \ll W \ll Q,$$

and that $\|f^{(j)}\|_\infty, \|\Psi^{(j)}\|_\infty \ll_j 1$. Then, if $\varepsilon > 0$ is small enough in terms of κ, we have

$$\sum_{w \in \mathbb{N}} \Psi\left(\frac{w}{W}\right) \sum_{n \in \mathbb{N}} \Lambda(n) f\left(\frac{n}{X}\right) \mathrm{u}_R(n, bw) \ll Q^{1-\varepsilon} \sqrt{X}.$$

The implied constant depends at most on κ, A, and the implied constants in the hypotheses.

Proof. See Section 4. □

2.2. *Explicit formula.* We let $\kappa \in \big(0, \frac{50}{1093}\big)$ be such that $\operatorname{supp} \hat{\phi} \subset (-2-\kappa, 2+\kappa)$.

We rewrite the left-hand side of (4) by applying the explicit formula, e.g., [Sica 1998, Theorem 2.2], where the quantity $\Phi(\rho)$ there (not to be confused with our test function) is replaced by $\phi\big(\frac{1}{2\pi i}\big(\rho - \frac{1}{2}\big) \log Q\big)$, so that $F(x) = (1/\log Q)\hat{\phi}(x/\log Q)$. For $q > 1$ and $\chi \pmod q$ primitive, we obtain

$$\sum_{\substack{\rho\in\mathbb{C}\\ \mathrm{Re}(\rho)\in(0,1)\\ L(\rho,\chi)=0}} \phi\Big(\frac{(\rho-\frac{1}{2})\log Q}{2\pi i}\Big) = O\Big(\frac{1}{\log Q}\Big) + \hat{\phi}(0)\frac{\log q}{\log Q} - \frac{1}{\log Q}\sum_{n\geq 1}(\chi(n)+\bar{\chi}(n))\frac{\Lambda(n)}{\sqrt{n}}\hat{\phi}\Big(\frac{\log n}{\log Q}\Big), \tag{7}$$

since the terms I, J appearing in [Sica 1998, Theorem 2.2] satisfy $\big|I\big(\frac{1}{2},b\big)\big| + \big|J\big(\frac{1}{2},b\big)\big| \ll (\log Q)^{-1}$ for $b \in \big\{0, \frac{1}{2}\big\}$ by reasoning similarly as in [Sica 1998, Lemma 3.1]. Let $\Psi(x) = \Phi(x)x$. Summing (7) over χ and q, we see that to conclude it remains to show that

$$S_\phi(Q) := \sum_{q\in\mathbb{N}} \frac{1}{q}\Psi\Big(\frac{q}{Q}\Big) \sum_{\substack{\chi(q)\\ \text{primitive}}} \frac{1}{\log Q}\sum_{n\geq 1}(\chi(n)+\bar{\chi}(n))\frac{\Lambda(n)}{\sqrt{n}}\hat{\phi}\Big(\frac{\log n}{\log Q}\Big) = o(Q). \tag{8}$$

We will in fact obtain the following slightly stronger result.

Proposition 5. *Let $\kappa \in \big(0, \frac{50}{1093}\big)$. For all Q large enough and $\varepsilon > 0$ small enough in terms of κ, we have*

$$S_\phi(Q) = O\Big(\frac{Q}{\log Q}\Big).$$

The implied constant depends on ϕ and ε at most.

We break down the proof of Proposition 5 into the following three sections.

2.3. *Orthogonality and partition of unity.* Applying character orthogonality for primitive characters (see the third display in the proof of Lemma 4.1 of [Bui and Milinovich 2011]), we get

$$S_\phi(Q) = \frac{2}{\log Q}\sum_{v,w}\sum \Psi\Big(\frac{vw}{Q}\Big)\frac{\mu(v)}{v}\frac{\varphi(w)}{w}\sum_{n\equiv 1 \,(\mathrm{mod}\, w)}\frac{\Lambda(n)}{\sqrt{n}}\hat{\phi}\Big(\frac{\log n}{\log Q}\Big). \tag{9}$$

Let V be any test function generating the partition of unity

$$\sum_{j\in\mathbb{Z}} V\Big(\frac{x}{2^j}\Big) = 1$$

for all $x > 0$. Inserting this in (9), we obtain

$$S_\phi(Q) = \frac{2}{\log Q}\sum_{\substack{j\in\mathbb{Z}\\ \frac{1}{2}\leq X:=2^j\leq 2Q^{2+\kappa}}}\sum_{v,w}\sum \Psi\Big(\frac{vw}{Q}\Big)\frac{\mu(v)}{v}\frac{\varphi(w)}{w}\sum_{n\equiv 1 \,(\mathrm{mod}\, w)}\frac{\Lambda(n)}{\sqrt{n}}V\Big(\frac{n}{X}\Big)\hat{\phi}\Big(\frac{\log n}{\log Q}\Big).$$

Set $f_j(x) = x^{-\frac{1}{2}}V(x)\hat{\phi}(\log(2^j x)/\log Q)$ for $\frac{1}{2} \leq 2^j \leq 2Q^{2+\kappa}$. Differentiating the product, we have that for all $k \geq 0$, there exists $C_{\phi,k} \geq 0$ such that $\|f_j^{(k)}\|_\infty \leq C_{\phi,k}$ for all j. We deduce

$$S_\phi(Q) \ll \sup_{1\ll X\ll Q^{2+\kappa}} X^{-\frac{1}{2}}\sup_f |T(Q,X)|,$$

where f varies among test functions subject to $\|f^{(k)}\|_\infty \leq C_{\phi,k}$, and

$$T(Q,X) := \sum_{v,w}\sum \Psi\Big(\frac{vw}{Q}\Big)\frac{\mu(v)}{v}\frac{\varphi(w)}{w}\sum_{n\equiv 1 \,(\mathrm{mod}\, w)}\Lambda(n)f\Big(\frac{n}{X}\Big).$$

We handle the very small values of X by the trivial bound

$$\sum_{n\equiv 1\ (\mathrm{mod}\ w)} \Lambda(n) f\left(\frac{n}{X}\right) \ll \log Q \sum_{\substack{\frac{1}{2}X<n<3X\\ n\neq 1,n\equiv 1\ (\mathrm{mod}\ w)}} 1 \ll \frac{X\log Q}{w},$$

which implies

$$T(Q,X) \ll \frac{X\log Q}{Q} \sum_{vw\asymp Q}\sum 1 \ll X(\log Q)^2.$$

It will therefore suffice to show that for

$$Q^2/(\log Q)^6 \ll X \ll Q^{2+\kappa},$$

we have

$$T(Q,X) \ll \frac{\sqrt{X}Q}{\log Q}.$$

2.4. ***Subtracting the main term.*** We insert the coprimality condition $(n,v)=1$. Since

$$\sum_{v,w}\sum \Psi\left(\frac{vw}{Q}\right)\frac{\mu(v)}{v}\frac{\varphi(w)}{w} \sum_{\substack{n\equiv 1\ (\mathrm{mod}\ w)\\ (n,v)>1}} \Lambda(n) f\left(\frac{n}{X}\right) \ll \sum_{v\ll Q} v^{-1} \sum_{\substack{p\mid v\\ 1\le k\ll \log X}} \left((\log p)\sum_{w\mid p^k-1} 1\right) \ll Q^{1+\varepsilon},$$

we obtain

$$T(Q,X) = \sum_{v,w}\sum \Psi\left(\frac{vw}{Q}\right)\frac{\mu(v)}{v}\frac{\varphi(w)}{w} \sum_{\substack{n\equiv 1\ (\mathrm{mod}\ w)\\ (n,v)=1}} \Lambda(n) f\left(\frac{n}{X}\right) + O(Q^{1+\varepsilon}).$$

Let $1\le R<\frac{1}{2}Q$ so that $R<vw$ for any v,w appearing in the sum. We replace the condition $n\equiv 1\ (\mathrm{mod}\ w)$ by $\mathfrak{u}_R(n,w)$. The difference is

$$\sum_q \frac{1}{q}\Psi\left(\frac{q}{Q}\right) \sum_{\substack{\chi\ (\mathrm{mod}\ q)\\ r=\mathrm{cond}(\chi)\le R\\ r\mid q}} \sum_{(n,q)=1} \Lambda(n) f\left(\frac{n}{X}\right)\chi(n) \sum_{v\mid q/r}\mu(v) = 0$$

since $r<q$ by our choice of R, so that

$$T(Q,X) = \sum_{v,w}\sum \Psi\left(\frac{vw}{Q}\right)\frac{\mu(v)}{v}\frac{\varphi(w)}{w} \sum_{(n,v)=1} \Lambda(n) f\left(\frac{n}{X}\right)\mathfrak{u}_R(n,w) + O(Q^{1+\varepsilon}).$$

We next remove the coprimality condition on n, using the trivial bound (5). For the first term $\mathbf{1}_{n\equiv 1\ (\mathrm{mod}\ w)}$ in $\mathfrak{u}_R(n,w)$, this was already justified above. For the second term, we get

$$\ll RQ^{-1+\varepsilon} \sum_{\substack{v,w\\ vw\asymp Q}}\sum \sum_{p\mid v} \log p \ll RQ^{\varepsilon}.$$

Since $R\ll Q$, both error terms are acceptable. We get

$$T(Q,X) = T(Q,X,R) + O(Q^{1+\varepsilon}),$$

where

$$T(Q,X,R) := \sum_{v,w}\sum \Psi\Big(\frac{vw}{Q}\Big)\frac{\mu(v)}{v}\frac{\varphi(w)}{w}\Delta(w),$$
$$\Delta(w) := \sum_n \Lambda(n) f\Big(\frac{n}{X}\Big)\mathfrak{u}_R(n,w). \tag{10}$$

We are required to show that

$$T(Q,X,R) \ll \frac{\sqrt{X}Q}{\log Q}. \tag{11}$$

2.5. ***Reduction to the critical range.*** We now impose the additional conditions

$$Q^{\frac12\kappa+\varepsilon} \le R \le Q^{\frac12} \quad \text{and} \quad \kappa < \tfrac23. \tag{12}$$

Observe that this κ is the same as that appearing in the statement of Proposition 4. The condition $\kappa < \frac23$ is convenient for applying (6) below, but is rather loose since κ is ultimately required to be much smaller than $\frac23$.

Let $B \in [1, Q^{\frac12}]$ be a parameter. In $T(Q,X,R)$, we write $\varphi(w)/w = \sum_{b|w}\mu(b)/b$ and exchange summation, so that

$$T(Q,X,R) \le \sum_{b,v}\frac{1}{bv}\Big|\sum_w \Psi\Big(\frac{bvw}{Q}\Big)\Delta(bw)\Big| \ll (\log B)^2 \sup_{b,v\le B}\Big|\sum_w \Psi\Big(\frac{bvw}{Q}\Big)\Delta(bw)\Big| + E_1 + E_2,$$

where E_1 (resp. E_2) corresponds to the sum over b, v restricted to $b > B$ (resp. $v > B$). We recall that $\operatorname{supp}\Psi \subset \big[\frac12, 3\big]$ by hypothesis. On the one hand, we have

$$E_1 \ll \sum_{\substack{b,w\\ bw\le 3Q\\ b>B}}\sum \frac1b|\Delta(bw)| \ll Q^{\frac12\varepsilon}B^{-1}\sum_{q\le 3Q}|\Delta(q)| \ll Q^{1+\frac12\varepsilon}\sqrt{X}B^{-1},$$

using (6) along with our hypotheses (12). On the other hand, we have

$$E_2 \ll \sum_{\substack{b,w\\ bw\le 3Q/B}}\sum \frac1b|\Delta(bw)| \ll Q^{\frac12\varepsilon}\sum_{q\le 3Q/B}|\Delta(q)| \ll Q\sqrt{X}(Q^{\frac12\varepsilon}B^{-1} + Q^{-\varepsilon})$$

again by (12) and (6); we have used the bounds $Q^{-1+\varepsilon}\sqrt{X}R^{-1} \ll Q^{-\varepsilon}$ and $Q^{-1+\varepsilon}X^{\frac38} \ll Q^{-\varepsilon}$, which follow from $Q^{\frac12\kappa+\varepsilon} \le R$ and $\kappa < \frac23$ respectively upon reinterpreting ε.

Grouping the above, it will suffice to show that

$$\sum_w \Psi\Big(\frac{bvw}{Q}\Big)\Delta(bw) \ll Q^{1-\varepsilon}\sqrt{X}$$

uniformly for $b, v \le Q^{\varepsilon}$ and test functions Ψ and f. Assume now $\kappa \in \big(0, \frac{50}{1093}\big)$. Then the conditions on R in (12) and in Proposition 4 overlap, so that we may apply Proposition 4 with $W = Q/(bv)$. This gives the above bound, and completes the proof of (11), hence of Proposition 5.

3. Exponential sums estimates

In this section, we work out the modifications to be made to the arguments underlying [Deshouillers and Iwaniec 1982] in order to exploit current knowledge on the spectral gap of the Laplacian on congruence surfaces [Kim and Sarnak 2003]. We will follow the setting in Theorem 2.1 of [Drappeau 2017], since we will need to keep track of the uniformity in q_0. We also take the opportunity to implement the recently described correction to [Bombieri et al. 1986].

Let $\theta \geq 0$ be a bound towards the Petersson–Ramanujan conjecture, in the sense of [Drappeau 2017, (4.6)]. Selberg's $\frac{3}{16}$ theorem corresponds to $\theta \leq \frac{1}{4}$, and the Kim–Sarnak bound [2003] asserts that $\theta \leq \frac{7}{64}$.

Proposition 6. *Let the notation and hypotheses be as in* [Drappeau 2017, Theorem 2.1]. *Then*

$$\sum_{c}\sum_{d}\sum_{n}\sum_{r}\sum_{\substack{s \\ c\equiv c_0 \text{ and } d\equiv d_0 \ (\mathrm{mod}\, q) \\ (qrd,sc)=1}} b_{n,r,s}\, g(c,d,n,r,s)\, \mathrm{e}\Big(n\frac{\overline{rd}}{sc}\Big) \ll_{\varepsilon,\varepsilon_0} (qCDNRS)^{\varepsilon+O(\varepsilon_0)} q^{\frac{3}{2}} K(C,D,N,R,S)\|b_{N,R,S}\|_2,$$

where $\|b_{N,R,S}\|_2^2 = \sum_{n,r,s}|b_{n,r,s}|^2$, *and here*

$$K(C,D,N,R,S)^2 = qCS(RS+N)(C+RD) + C^{1+4\theta}DS((RS+N)R)^{1-2\theta}\Big(1+\frac{qC}{RD}\Big)^{1-4\theta} + D^2NR. \quad (13)$$

Remark 7. The bound of Proposition 6 is monotonically stronger as θ decreases, since the first term is larger than $CS(RS+N)(RD+qC)$. Under the Petersson–Ramanujan conjecture for Maass forms, which predicts that $\theta = 0$ is admissible, the second term in (13) is smaller than the first.

Proof. The proof of the proposition, as with all results of this type, relies on the Kuznetsov formula and large sieve inequalities for coefficients of automorphic forms. The application of the Kuznetsov formula requires one to understand the contribution of holomorphic forms, Eisenstein series, and Maass forms (whether the holomorphic forms appear depends on the sign of the variables inside the Kloosterman sum). We divide these forms into the *exceptional spectrum* and the *regular spectrum*. The exceptional spectrum consists of those (conjecturally nonexistent) Maass forms whose eigenvalues $t_f = \frac{1}{2} + it_f$ have $t_f \in i\mathbb{R}$. By the definition of θ above we have that $|t_f| \leq \theta$ for all f in the exceptional spectrum. The regular spectrum consists of everything that is not exceptional. The contribution of the regular spectrum is handled as in [Drappeau 2017], and does not require any modification here. We improve upon the analysis there in handling the exceptional spectrum by keeping track of the dependence on θ (see the remark made in [Drappeau 2017, p. 703]). The statements which are affected are [Drappeau 2017, Lemma 4.10, Proposition 4.12, Proposition 4.13 and the proof of Theorem 2.1]. The treatment of the exceptional spectrum rests upon a weighted large sieve inequality. These weighted large sieve inequalities are proved, following [Deshouillers and Iwaniec 1982], by an iterative procedure.

With the notation of [Drappeau 2017], the changes to be made are as follows:

- Lemma 4.10 bounds sums of the form

$$\sum_{\substack{q\le Q\\ q_0\mid q}}\sum_{\substack{f\in\mathcal{B}(q,\chi)\\ t_f\in i\mathbb{R}}} Y^{2|t_f|}\Big|\sum_{N<n\le 2N} n^{\frac12}\rho_{f\infty}(n)\Big|^2,$$

and serves to control the first step of the recursion. The bound

$$\sum_{\substack{q\le Q\\ q_0\mid q}}\sum_{\substack{f\in\mathcal{B}(q,\chi)\\ t_f\in i\mathbb{R}}} Y^{2|t_f|}\Big|\sum_{N<n\le 2N} n^{\frac12}\rho_{f\infty}(n)\Big|^2 \ll (QN)^{\varepsilon}(Qq_0^{-1}+N+(NY)^{\frac12})N$$

may be replaced by the bound

$$\ll (QN)^{\varepsilon}\big(Qq_0^{-1}+N+(NY)^{2\theta}(Q^{1-4\theta}+N^{1-4\theta})\big)N.$$

This does not require any change in the recursion argument, but merely the use of the bound $|t_f|\le\theta$ in the very last step [Deshouillers and Iwaniec 1982, page 278], whereby $\sqrt{Y/Y_1}$ is replaced by $(Y/Y_1)^{2\theta}$.

- In Proposition 4.12 one bounds sums of the form

$$\sum_{\substack{m,n,r,s\\ (s,rq)=1}} a_m b_{n,r,s}\sum_{c\in\mathcal{C}(\infty,1/s)}\frac{1}{c}\phi\Big(\frac{4\pi\sqrt{mn}}{c}\Big)S_{\infty,1/s}(m,\pm n;c)$$

in terms of quantities L_{reg} and L_{exc}. In place of

$$L_{\mathrm{exc}}=\Big(1+\sqrt{\frac{N}{RS}}\Big)\sqrt{\frac{1+X^{-1}}{RS}}\Big(\frac{MN}{RS+N}\Big)^{\frac14}\frac{\sqrt{RS}}{1+X}\sqrt{M}\|b_{N,R,S}\|_2,$$

we claim the improved

$$L_{\mathrm{exc}}=q_0^{\frac12-2\theta}\Big(1+\sqrt{\frac{N}{RS}}\Big)\Big(\frac{1+X^{-1}}{RS}\Big)^{2\theta}\Big(\frac{MN}{RS+N}\Big)^{\theta}\Big(1+\frac{M}{RS}\Big)^{\frac12-2\theta}\frac{\sqrt{RS}}{1+X}\sqrt{M}\|b_{N,R,S}\|_2.$$

To obtain this bound one uses the new bound for Lemma 4.10 and follows the arguments of [Deshouillers and Iwaniec 1982, Section 9.1].

- In Proposition 4.13, one bounds

$$\sum_{\substack{c,m,n,r,s\\ (sc,rq)=1}} b_{n,r,s}\bar{\chi}(c)g(c,m,n,r,s)e(mt)S(n\bar{r},\pm m\bar{q};sc)$$

in terms of quantities K_{reg} and K_{exc}. The term

$$K_{\mathrm{exc}}^2=C^3S^2\sqrt{R(N+RS)}$$

can be replaced by

$$K_{\mathrm{exc}}^2=C^{2+4\theta}S^2(R(N+RS))^{1-2\theta}\Big(1+\frac{M}{RS}\Big)^{1-4\theta}.$$

This is seen by using the new definition on L_{exc} in Proposition 4.12, and by keeping track of a factor $q^{-1+2\theta}$ coming from the term $(1+X^{-1})^{2\theta}/(1+X)$.

• Finally, we modify the proof of Theorem 2.1 at two places. First, the bound for $\mathcal{A}_0$ on page 706, as explained in the correction to [Bombieri et al. 1986], is wrong unless further hypotheses on $(b_{n,r,s})$ are imposed. The correct bound in general is

$$\mathcal{A}_0 \ll q^{-2}(\log S)^2 D(NR)^{\frac{1}{2}} \|b_{N,R,S}\|_2,$$

and this yields the term D^2NR instead of D^2NRS^{-1}. Second, our new bound for K_{exc} in Proposition 4.13 gives a contribution $C^{2+4\theta}S^2(R(RS+N))^{1-2\theta}(1+M_1/(RS))^{1-4\theta}$ instead of $C^3S^2\sqrt{R(RS+N)}$ in the definition of L_{exc}^2 and $L^*(M_1)^2$ on page 707 of [Drappeau 2017]. This yields a term $C^{1+4\theta}DS \times ((N+RS)R)^{1-2\theta}(1+qC/(RD))^{1-4\theta}$ instead of $C^2DS\sqrt{(N+RS)R}$ in (4.39) of [Drappeau 2017], and by following the rest of the arguments we deduce our claimed bound. $\square$

4. Primes in arithmetic progressions: proof of Proposition 4

The proof of Theorem 1 relies on Proposition 4, which for the convenience of the reader we recall below.

Proposition 4. *Let $\kappa \in \left(0, \frac{50}{1093}\right)$ and $\varepsilon > 0$. Let Ψ and f be test functions, $A > 0$, $X, Q, W, R \geq 1$, and $b \in \mathbb{N}$. Assume that*

$$\frac{Q^2}{(\log Q)^A} \ll X \ll Q^{2+\kappa}, \quad X^{\frac{11}{20}}Q^{-1} \leq R \leq Q^{\frac{2}{3}}X^{-\frac{2}{9}}, \quad b \leq Q^{\varepsilon}, \quad Q^{1-\varepsilon} \ll W \ll Q,$$

and that $\|f^{(j)}\|_\infty, \|\Psi^{(j)}\|_\infty \ll_j 1$. Then, if $\varepsilon > 0$ is small enough in terms of κ, we have

$$\sum_{w\in\mathbb{N}} \Psi\left(\frac{w}{W}\right) \sum_{n\in\mathbb{N}} \Lambda(n) f\left(\frac{n}{X}\right) \mathfrak{u}_R(n, bw) \ll Q^{1-\varepsilon}\sqrt{X}.$$

The implied constant depends at most on κ, A, and the implied constants in the hypotheses.

Remark 8. What is crucial in our statement is the size of the upper bound, which should be negligible with respect to $Q\sqrt{X}$. On the other hand, we are only interested in values of X larger than Q^2. This is in contrast with most works on primes in arithmetic progressions [Fouvry and Iwaniec 1983; Bombieri et al. 1986; Zhang 2014], where the main challenge is to work with values of X much smaller than Q^2, while only aiming at an error term which is negligible with respect to X. The main point is that in both cases, the large sieve yields an error term which is always too large (see [Iwaniec and Kowalski 2004, Theorem 17.4]), an obstacle which the dispersion method is designed to handle.

In what follows, we will systematically write

$$X = Q^{2+\varpi},$$

so that $-o(1) \leq \varpi \leq \kappa + o(1)$ as $Q \to \infty$.

4.1. ***Combinatorial identity.*** We perform a combinatorial decomposition of the von Mangoldt function into sums of different shapes: type d_1 sums have a long smooth variable, type d_2 sums have two long smooth variables, and type II sums have two rough variables that are neither too small nor too large. We accomplish this decomposition with the Heath-Brown identity and the following combinatorial lemma.

Lemma 9. *Let $\{t_j\}_{1\le j\le J}\in\mathbb{R}$ be nonnegative real numbers such that $\sum_j t_j=1$. Let $\lambda,\sigma,\delta\ge 0$ be real numbers such that*

- $\delta<\frac{1}{12}$,
- $\sigma\le\frac{1}{6}-\frac{1}{2}\delta$,
- $2\lambda+\sigma<\frac{1}{3}$.

Then at least one of the following must occur:

- *Type d_1*: *There exists t_j with $t_j\ge\frac{1}{3}+\lambda$.*
- *Type d_2*: *There exist i, j, k such that $\frac{1}{3}-\delta<t_i,t_j,t_k<\frac{1}{3}+\lambda$, and*
$$\sum_{t_j^*\notin\{t_i,t_j,t_k\}} t_j^*<\sigma.$$
- *Type* II: *There exists $S\subset\{1,\ldots,J\}$ such that*
$$\sigma\le\sum_{j\in S}t_j\le\frac{1}{3}-\delta.$$

Proof. Assume that the type d_1 case and the type II case both fail. Then for every j we have $t_j<\frac{1}{3}+\lambda$, and for every subset S of $\{1,\ldots,J\}$ we have either
$$\sum_{j\in S}t_j<\sigma\quad\text{or}\quad\sum_{j\in S}t_j>\frac{1}{3}-\delta.$$
Let $s_1,\ldots,s_K$ denote those t_j with $\frac{1}{3}-\delta<t_j<\frac{1}{3}+\lambda$. We will show that $K=3$. Let t_j^* be any other t_j, so that $t_j^*\le\frac{1}{3}-\delta$, and therefore $t_j^*<\sigma$. We claim that
$$\sum_j t_j^*<\sigma.$$
If not, then $\sum_j t_j^*>\frac{1}{3}-\delta$. By a greedy algorithm we can find some subcollection S^* of the t_j^* such that
$$\sigma<\sum_{j\in S^*}t_j^*\le 2\sigma.$$
Since $2\sigma\le\frac{1}{3}-\delta$ this subcollection satisfies the type II condition, in contradiction to our assumption.

Now we show that $K=3$. Observe that $K\ge 3$, since if $K\le 2$ we have
$$1=\sum_j t_j=\sum_{i=1}^K s_i+\sum_j t_j^*<2\big(\tfrac{1}{3}+\lambda\big)+\sigma<1.$$
Furthermore, we must have $K\le 3$, since if $K\ge 4$ we have
$$1=\sum_j t_j\ge\sum_{i=1}^K s_i>4\big(\tfrac{1}{3}-\delta\big)>1.$$
This completes the proof. ∎

Using, e.g., the combinatorial identity of Heath-Brown [1982], we deduce the following.

Corollary 10. *Let f be a test function, $u : \mathbb{N} \to \mathbb{C}$ be any map, and $X \geq 1$. Then there exists a sequence $(C_j)_{j\geq 0}$ of positive numbers, depending only on f, such that we have*

$$\left|\sum_{n\in\mathbb{N}} \Lambda(n) f\left(\frac{n}{X}\right) u(n)\right| \ll (\log X)^8 (T_1 + T_2 + T_{\mathrm{II}}), \tag{14}$$

where

$$T_1 = \sup_{\substack{N\gg X^{1/3+\lambda}\\ MN\asymp X}} \sup_{\substack{g\in\mathcal{G}\\ \beta\in\mathcal{S}}} \left|\sum_{\substack{n\in\mathbb{N}\\ m\sim M}}\sum g\left(\frac{n}{N}\right)\beta_m u(mn)\right|, \tag{15}$$

$$T_2 = \sup_{\substack{X^{1/3-\delta}\ll N_2\leq N_1\ll X^{1/3+\lambda}\\ MN_1N_2\asymp X}} \sup_{\substack{g_1,g_2\in\mathcal{G}\\ \beta\in\mathcal{S}}} \left|\sum\sum_{\substack{n_1,n_2\in\mathbb{N}\\ m\sim M}}\sum g_1\left(\frac{n_1}{N_1}\right) g_2\left(\frac{n_2}{N_2}\right)\beta_m u(mn_1n_2)\right|, \tag{16}$$

$$T_{\mathrm{II}} = \sup_{\substack{X^{\sigma}\ll N\ll X^{1/3-\delta}\\ MN\asymp X}} \sup_{\alpha,\beta\in\mathcal{S}} \left|\sum_{\substack{n\sim N\\ m\sim M}}\sum \alpha_m\beta_n u(mn)\right|, \tag{17}$$

where the implied constants are absolute, $\mathcal{G}$ is the set of test functions g satisfying $\|g^{(j)}\|_\infty \leq C_j$, and $\mathcal{S}$ is the set of sequences (β_n) satisfying $|\beta_n| \leq d(n)^8$.

Proof. By the Heath-Brown identity [Iwaniec and Kowalski 2004, Proposition 13.3], there exists bounded coefficients $(c_J)_{1\leq J\leq 4}$ such that

$$\Lambda(n) = \sum_{J=1}^{4} c_J \sum_{\substack{m_1,\ldots,m_J\\ n_1,\ldots,n_J\\ n=m_1\cdots m_J n_1\cdots n_J\\ m_j\leq (3X)^{1/4}}} \log(n_1)\prod_j \mu(m_j)$$

for any n involved in the left-hand side of (14). Let ψ be a test function inducing a partition of unity in the sense that $\sum_{j\in\mathbb{Z}} \psi(x/2^j) = 1$ for all $x > 0$. Then we have

$$\sum_{n\in\mathbb{N}} \Lambda(n) f\left(\frac{n}{X}\right) u(n) = \sum_{J=1}^{4} c_J \sum_{(M_1,\ldots,M_J,N_1,\ldots,N_J)\in U_J} S(M_1,\ldots,M_J,N_1,\ldots,N_J),$$

$$S(M_1,\ldots,N_J) = \sum_{m_1,\ldots,n_J\in\mathbb{N}} \log(n_1)\left(\prod_j \psi\left(\frac{n_j}{N_j}\right)\right)\left(\prod_j \mu^*(m_j)\psi\left(\frac{m_j}{M_j}\right)\right) f\left(\frac{m_1\cdots n_J}{X}\right) u(m_1\cdots n_J),$$

where U_J is the set of $2J$-tuples of powers of 2 such that $\frac{1}{6}X \leq M_1\cdots M_J N_1\cdots N_J \leq 6X$, and $\mu^*(m) = \mu(m)$ if $m \leq (3X)^{\frac{1}{4}}$ and 0 otherwise. We abbreviated $m_1\cdots n_J = m_1\cdots m_J n_1\cdots n_J$. The set U_J has at most $O((\log X)^{2J-1})$ elements. By Lemma 9, for each choice of J and $(M_1,\ldots,N_J) \in U_J$ we have either $N \geq \frac{1}{6}X^{\frac{1}{3}+\lambda}$ for some $N \in \{N_j\}$, or $\frac{1}{6}X^{\frac{1}{3}-\delta} \leq N', N'' \leq 6X^{\frac{1}{3}+\lambda}$ for some $N', N'' \in \{N_j\}$, or $\frac{1}{6}X^{\sigma} \leq N \leq 6X^{\frac{1}{3}-\delta}$ for some subproduct N of N_j and M_j (here we used that for X large enough, we have $(3X)^{\frac14} < \frac16 X^{\frac13-\delta}$). Sorting the sum over J and $(M_1,\dots,N_J)$ according to this trichotomy, and writing $\log(n_1)=\log N_1+\log(n_1/N_1)$, the above is bounded in absolute values by

$$\ll (\log X)^8(T_1^*+T_2^*+T_{\mathrm{II}}^*),$$

where

$$T_1^* = \sup_{\substack{\frac16 X\le MN\le 6X\\ \frac16 X^{1/3+\lambda}\le N\\ |r|\le 8}} \sup_{\substack{g\in\{\psi,\psi\log\}\\ \beta\in\mathcal{S}}} \Bigg|\sum_{\substack{n\in\mathbb{N}\\ m\sim M}} g\Big(\frac{n}{N}\Big)\beta_m f\Big(\frac{mn}{2^r MN}\Big)u(mn)\Bigg|,$$

$$T_2^* = \sup_{\substack{\frac16 X\le N_1N_2M\le 6X\\ \frac16 X^{1/3-\delta}\le N_1,N_2\le 6X^{1/3+\lambda}\\ |r|\le 8}} \sup_{\substack{g_1,g_2\in\{\psi,\psi\log\}\\ \beta\in\mathcal{S}}} \Bigg|\sum_{\substack{n_1,n_2\in\mathbb{N}\\ m\sim M}} g_1\Big(\frac{n_1}{N_1}\Big)g_2\Big(\frac{n_2}{N_2}\Big)\beta_m f\Big(\frac{n_1n_2m}{2^r N_1N_2M}\Big)u(n_1n_2m)\Bigg|,$$

$$T_{\mathrm{II}}^* = \sup_{\substack{\frac16 X\le NM\le 6X\\ \frac16 X^{\sigma}\le N\le 6X^{1/3-\delta}\\ |r|\le 8}} \sup_{\alpha,\beta\in\mathcal{S}} \Bigg|\sum_{\substack{m\sim M\\ n\sim N}} \alpha_m\beta_n f\Big(\frac{mn}{2^r MN}\Big)u(mn)\Bigg|.$$

Here the conditions $m\sim M$ and $n\sim N$ in the sums were added by an additional bounded dichotomy (which is the reason for the presence of the sup over r). Finally, letting $\check f$ be the Mellin transform of f, we have by Mellin inversion $f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\check f(it)x^{-it}\,\mathrm{d}t$, and the map $t\mapsto \check f(it)$ is of Schwartz class on $\mathbb{R}$. In particular, for M, N, r, g, β as in T_1^* we have

$$\Bigg|\sum_{n\in\mathbb{N},m\sim M} g\Big(\frac{n}{N}\Big)\beta_m f\Big(\frac{mn}{2^r MN}\Big)u(mn)\Bigg| \ll \sup_{t\in\mathbb{R}}\Bigg|\sum_{n\in\mathbb{N},m\sim M} g_t\Big(\frac{n}{N}\Big)\beta_{m,t}u(mn)\Bigg|,$$

where $g_t(x)=(1+t^2)\check f(it)x^{-it}g(x)$ (the factor $1+t^2$ being included so that we could write a supremum) and $\beta_{m,t}=m^{-it}\beta_m\in\mathcal{S}$. We note that g_t is a test function satisfying $\|g_t^{(j)}\|_\infty\ll C_j$, where $C_j:=\sup_{0\le k,\ell,m\le j+2}\|t^k\check f(it)\|_\infty\|x^{-\ell}g^{(m)}(x)\|_\infty$ can be bounded in terms of f only. This yields the contribution of T_1 in our claim. The contributions of T_2 and T_{II} are obtained in the same way. □

In what follows, we successively consider T_1, T_2 and T_{II}, which we specialize at

$$u(n):=\sum_{w\in\mathbb{N}}\Psi\Big(\frac{w}{W}\Big)\mathfrak{u}_R(n,bw),$$

and we will write

$$R=X^{\rho}.$$

4.2. *Type d_1 sums.* We suppose M and N are given as in (15). The quantity we wish to bound is

$$T_1(M,N)=\sum_{w}\Psi\Big(\frac{w}{W}\Big)\sum_{\substack{m\sim M\\ (m,bw)=1}}\beta_m\Bigg(\sum_{\substack{n\in\mathbb{N}\\ mn\equiv 1\ (\mathrm{mod}\, bw)}} g\Big(\frac{n}{N}\Big)-\frac{1}{\varphi(bw)}\sum_{\substack{\chi\ (\mathrm{mod}\, bw)\\ \mathrm{cond}(\chi)\le R}}\chi(m)\sum_{(n,bw)=1}\chi(n)g\Big(\frac{n}{N}\Big)\Bigg).$$

By Poisson summation and the classical bound on Gauss sums [Iwaniec and Kowalski 2004, Lemma 3.2],

$$\sum_{n\equiv\bar{m} \pmod{bw}} g\Big(\frac{n}{N}\Big) = \frac{N}{bw}\hat{g}(0) + \frac{N}{bw}\sum_{0<|h|\le W^{1+\varepsilon}/N} \hat{g}\Big(\frac{Nh}{bw}\Big) e\Big(\frac{\bar{m}h}{bw}\Big) + O(Q^{-A}),$$

$$\frac{1}{\varphi(bw)}\sum_{(c,bw)=1} \chi(c) g\Big(\frac{c}{N}\Big) = \frac{N}{bw}\hat{g}(0)\mathbf{1}(\chi=\chi_0) + O\Big(\frac{Q^\varepsilon R^{\frac12}}{W}\Big).$$

Therefore,

$$T_1(M,N) = \frac{N}{b}\sum_w \frac{1}{w}\Psi\Big(\frac{w}{W}\Big)\sum_{\substack{(m,bw)=1\\ m\sim M}} \beta_m \sum_{0<|h|\le W^{1+\varepsilon}/N} \hat{g}\Big(\frac{Nh}{bw}\Big) e\Big(\frac{\bar{m}h}{bw}\Big) + O(MR^{\frac32}Q^\varepsilon).$$

Our goal is to get cancellation in the exponential phases by summing over the smooth variable w. We apply the reciprocity formula

$$\frac{\bar{m}h}{bw} \equiv -\frac{\overline{bw}h}{m} + \frac{h}{mbw} \pmod 1,$$

which implies

$$T_1(M,N) = \frac{N}{b}\sum_w \frac{1}{w}\Psi\Big(\frac{w}{W}\Big)\sum_{\substack{(m,bw)=1\\ m\sim M}} \beta_m \sum_{0<|h|\le W^{1+\varepsilon}/N} \hat{g}\Big(\frac{Nh}{bw}\Big) e\Big(\frac{\overline{bw}h}{m}\Big) + O(MR^{\frac32}Q^\varepsilon + Q^{1+\varepsilon}N^{-1}).$$

We rearrange the sum as

$$\frac{N}{bW}\sum_{\substack{(m,b)=1\\ m\sim M}} \beta_m \sum_{0<|h|\le W^{1+\varepsilon}/N} \sum_{(w,m)=1} \hat{g}\Big(\frac{Nh}{bw}\Big)\frac{W}{w}\Psi\Big(\frac{w}{W}\Big) e\Big(\frac{\overline{bw}h}{m}\Big).$$

By partial summation and a variant of the Weil bound [Drappeau 2015, (2.4)], the sum on w is

$$\ll ((h,m)WM^{-1} + \sqrt{(h,m)}\sqrt{M})Q^\varepsilon.$$

Summing over h and m, we obtain a bound

$$T_1(M,N) \ll Q^{1+\varepsilon} + M^{\frac32}Q^\varepsilon + MR^{\frac32}Q^\varepsilon.$$

This bound is acceptably small provided

$$N \gg \Big(\frac{X}{Q}\Big)^{\frac23+\varepsilon} = X^{\frac13+\frac13\varpi/(2+\varpi)+\varepsilon(1+\varpi)/(2+\varpi)} \quad\text{and}\quad N \gg \frac{X^{\frac12}R^{\frac32}}{Q^{1-2\varepsilon}} = X^{\frac12\varpi/(2+\varpi)+\frac32\rho+2\varepsilon/(2+\varpi)}.$$

These inequalities are satisfied, for all sufficiently small $\varepsilon>0$, under the assumptions

$$\lambda > \frac{\varpi}{3(2+\varpi)} \quad\text{and}\quad \rho < \frac{4+\varpi}{9(2+\varpi)}. \tag{18}$$

We have proved the following.

Lemma 11. *Under the notation and hypotheses of Corollary 10, and assuming (18), we have*

$$T_1 \ll Q^{1-\varepsilon}\sqrt{X}.$$

The implied constant depends on λ, ρ and ϖ.

4.3. ***Type d_2 sums.*** The treatment of the type d_2 sums (16) is nearly identical to [Bombieri et al. 1986, Section 14]. For convenience, we rename (N_1, N_2, M) into (M, N, L) so that we have $MNL \asymp X$. We wish to bound

$$T_2(M,N,L) = \sum_{\ell\sim L}\beta_\ell \sum_{(w,\ell)=1}\Psi\Big(\frac{w}{W}\Big)\Big(\sum_{\substack{m,n\\ \ell mn\equiv 1\ (\mathrm{mod}\, bw)}} g_1\Big(\frac{m}{M}\Big)g_2\Big(\frac{n}{N}\Big) \\ -\frac{1}{\varphi(bw)}\sum_{\substack{\chi\ (\mathrm{mod}\, bw)\\ \mathrm{cond}(\chi)\le R}}\chi(\ell)\sum_{(mn,bw)=1} g_1\Big(\frac{m}{M}\Big)g_2\Big(\frac{n}{N}\Big)\chi(mn)\Big).$$

We perform Poisson summation on the m-sums to get

$$\sum_{m\equiv\overline{\ell n}\ (\mathrm{mod}\, bw)} g_1\Big(\frac{m}{M}\Big) = \frac{M}{bw}\sum_{|h|\le H}\hat g_1\Big(\frac{Mh}{bw}\Big)e\Big(\frac{\overline{\ell n}h}{bw}\Big) + O(Q^{-A}),$$

$$\sum_{(m,bw)=1}\chi(m)g_1\Big(\frac{m}{M}\Big) = \frac{\varphi(bw)}{bw}M\hat g_1(0)\mathbf{1}(\chi=\chi_0) + O(Q^\varepsilon R^{\frac12}),$$

where $H = W^{1+\varepsilon}M^{-1}$. The contribution of the error terms is

$$\ll LNR^{\frac32}Q^\varepsilon.$$

The zero frequency of Poisson summation cancels out. For the nonzero frequencies we employ reciprocity in the form

$$e\Big(\frac{\overline{\ell n}h}{bw}\Big) = e\Big(-\frac{\overline{bw}h}{\ell n}\Big) + O\Big(\frac{H}{LNW}\Big),$$

and the error term contributes a quantity of size $O(Q^{1+\varepsilon})$. We therefore have

$$T_2(M,N,L) = \frac{M}{b}\sum_{\substack{\ell\sim L\\ (\ell,b)=1}}\beta_\ell\sum_{(w,\ell)=1}\frac{1}{w}\Psi\Big(\frac{w}{W}\Big)\sum_{(n,bw)=1} g_2\Big(\frac{n}{N}\Big)\sum_{0<|h|\le H}\hat g_1\Big(\frac{Mh}{bw}\Big)e\Big(-\frac{\overline{bw}h}{\ell n}\Big) \\ + O(Q^{1+\varepsilon} + LNR^{\frac32}Q^\varepsilon). \tag{19}$$

We next separate the variables h and w. We change variables to write

$$\hat g_1\Big(\frac{Mh}{bw}\Big) = \frac{w}{M}\int_{\mathbb{R}} g_1\Big(\frac{wy}{M}\Big)e\Big(-\frac{hy}{b}\Big)\,dy.$$

Since g_1 and Ψ are test functions, the integral is restricted to $y \asymp M/W$. We move the integral to the outside to write the first term of the right-hand side of (19) as

$$\ll \frac{M}{bW}\sup_{y\asymp M/W}\Big|\sum_\ell\beta_\ell\sum_{0<|h|\le H}\mathrm{e}\Big(-\frac{hy}{b}\Big)\sum_w\sum_n\Psi\Big(\frac{w}{W}\Big)g_1\Big(\frac{wy}{M}\Big)g_2\Big(\frac{n}{N}\Big)\mathrm{e}\Big(-\frac{\overline{bw}h}{\ell n}\Big)\Big|. \tag{20}$$

We then use [Deshouillers and Iwaniec 1982, Theorem 12], amended as described in the correction to [Bombieri et al. 1986], more specifically, with the dictionary (the bold symbols denote the variables names from [Deshouillers and Iwaniec 1982])

$$\begin{aligned} \boldsymbol{c}, \boldsymbol{C} &\leftrightarrow n, N, & \boldsymbol{d}, \boldsymbol{D} &\leftrightarrow w, W, \\ \boldsymbol{n}, \boldsymbol{N} &\leftrightarrow h, H, & \boldsymbol{r}, \boldsymbol{R} &\leftrightarrow b', b, \\ \boldsymbol{s}, \boldsymbol{S} &\leftrightarrow \ell, L, & \boldsymbol{b}_{\boldsymbol{n},\boldsymbol{r},\boldsymbol{s}} &\leftrightarrow \mathbf{1}_{b'=b} \mathrm{e}(-hy/b)\beta_\ell. \end{aligned}$$

Since $\lambda < \frac{1}{6}$, we have $H \ll L$ if ε is sufficiently small. Thus, with the same notation, we find the bounds

$$\boldsymbol{K}(\boldsymbol{C}, \boldsymbol{D}, \boldsymbol{N}, \boldsymbol{R}, \boldsymbol{S}) \ll b(NL^2(N+W) + N^2WL^{\frac{3}{2}} + W^2H)^{\frac{1}{2}} \quad \text{and} \quad \|\boldsymbol{b}_{\boldsymbol{N},\boldsymbol{R},\boldsymbol{S}}\|_2 \ll L^\varepsilon (HL)^{\frac{1}{2}}.$$

It will also be easier to sum up the bounds if we assume

$$N \ll W^{1+\varepsilon}. \tag{21}$$

We find

$$T_2(M,N,L) \ll LNR^{\frac{3}{2}}Q^\varepsilon + Q^\varepsilon(\sqrt{X}L + \sqrt{M}NL^{\frac{5}{4}} + L^{\frac{1}{2}}W) \ll LNR^{\frac{3}{2}}Q^\varepsilon + Q^\varepsilon(\sqrt{X}L + \sqrt{M}NL^{\frac{5}{4}}),$$

the second inequality following since $L^{\frac{1}{2}}W \ll X^{\frac{1}{2}}L$. This contribution is acceptable provided

$$M \gg X^{\frac{1}{2}\varpi/(2+\varpi)+\frac{3}{2}\rho+\varepsilon}, \quad MN \gg X^{\frac{1}{2}+\frac{1}{2}\varpi/(2+\varpi)+\varepsilon}, \tag{22}$$

$$M^{\frac{3}{2}}N^{\frac{1}{2}} \gg X^{\frac{1}{2}+\varpi/(2+\varpi)+2\varepsilon}. \tag{23}$$

The bounds (21)–(23) are satisfied if

$$\delta < \frac{1}{12} - \frac{\varpi}{2(2+\varpi)}, \quad \lambda < \frac{1}{6} - \frac{\varpi}{2(2+\varpi)}, \quad \rho < \frac{1}{6}. \tag{24}$$

We therefore conclude the following.

Lemma 12. *Under the notation and hypotheses of Corollary 10, and assuming (24), we have*

$$T_2 \ll Q^{1-\varepsilon}\sqrt{X}.$$

The implied constant depends on λ, δ, ρ and ϖ.

4.4. ***Type II sums.*** In the type II case (17), we wish to prove the bound

$$T_{\mathrm{II}}(M,N) := \sum_w \Psi\Big(\frac{w}{W}\Big) \sum_{m,n}\sum \alpha_m \beta_n \mathfrak{u}_R(mn, bw) \ll \sqrt{X}Q^{1-\varepsilon},$$

where α is supported at scale M, β is supported at scale N, $MN \asymp X$, and $X^\sigma \ll N \ll X^{\frac{1}{3}-\delta}$. We have $|\alpha_m| \le \tau(m)^{O(1)}$, and similarly for β. We use the dispersion method of Linnik [1963], following closely [Fouvry 1985]; see also [Bombieri et al. 1986, Section 10].

We interchange the order of summation and apply the triangle inequality, writing our sum as

$$|T_{\mathrm{II}}(M,N)| \le \sum_m \Big| \sum_w \sum_n \Big|.$$

Applying the Cauchy–Schwarz inequality, we arrive at

$$T_{\mathrm{II}}(M,N)^2 \ll M(\log M)^{O(1)}\mathcal{D}, \tag{25}$$

where

$$\mathcal{D}=\sum_m f\Big(\frac{m}{M}\Big)\Bigg|\sum_{\substack{n,w\\ mn\equiv 1\ (\mathrm{mod}\, bw)}}\sum \Psi\Big(\frac{w}{W}\Big)\beta_n-\frac{1}{\varphi(bw)}\sum_{\substack{\chi\ (\mathrm{mod}\, bw)\\ \mathrm{cond}(\chi)\le R}}\sum_{\substack{n,w\\ (mn,bw)=1}}\sum \Psi\Big(\frac{w}{W}\Big)\beta_n\chi(mn)\Bigg|^2.$$

Here f is some fixed, nonnegative test function majorizing $\mathbf{1}_{[1,2]}$. It suffices to show that

$$\mathcal{D}\ll NQ^{2-\varepsilon}.$$

We open the square and arrive at

$$\mathcal{D}=\mathcal{D}_1-2\,\mathrm{Re}\,\mathcal{D}_2+\mathcal{D}_3, \tag{26}$$

say. We treat each sum $\mathcal{D}_i$ in turn.

4.4.1. *Evaluation of* $\mathcal{D}_3$. By definition we have

$$\mathcal{D}_3:=\sum_m f\Big(\frac{m}{M}\Big)\sum_{\substack{w_1,w_2,n_1,n_2\\ (mn_1,bw_1)=1\\ (mn_2,bw_2)=1}}\sum\sum\sum \sum_{\substack{\chi_1,\chi_2\\ \chi_j\ (\mathrm{mod}\, bw_j)\\ \mathrm{cond}(\chi_j)\le R}}\sum \Psi\Big(\frac{w_1}{W}\Big)\Psi\Big(\frac{w_2}{W}\Big)\beta_{n_1}\overline{\beta_{n_2}}\frac{\chi_1(mn_1)\overline{\chi_2(mn_2)}}{\varphi(bw_1)\varphi(bw_2)}.$$

The computations in [Drappeau 2017, pp. 711–712] can be directly quoted, putting formally

$$\gamma(q)=\mathbf{1}(b\mid q)\Psi(q/(bW)), \tag{27}$$

with the modification that $\mathrm{cond}(\chi_1\overline{\chi_2})\le R^2$ (instead of R, as stated incorrectly in [Drappeau 2017]). Writing $H=Q^\varepsilon b[w_1,w_2]M^{-1}$, we get

$$\begin{aligned}\mathcal{D}_3&=\mathcal{M}_3+O\Bigg(Q^\varepsilon\sum_{\substack{w_1,w_2\asymp W\\ n_1,n_2\asymp N}}\frac{1}{\varphi(bw_1)\varphi(bw_2)}\sum_{\substack{\chi_1,\chi_2\\ \mathrm{cond}(\chi_j)\le R}}\frac{M}{b[w_1,w_2]}\sum_{0<|h|\le H}R\sum_{d\mid(h,b[w_1,w_2])}d\Bigg)\\ &=\mathcal{M}_3+O(Q^\varepsilon N^2R^5),\end{aligned}$$

where the main term is computed as in [Drappeau 2017, p. 712] to be

$$\mathcal{M}_3:=M\hat{f}(0)\sum_{\substack{w_1,w_2,n_1,n_2\\ (n_j,bw_j)=1}}\sum\sum\sum \sum_{\substack{\chi\ \text{primitive}\\ \mathrm{cond}(\chi)\le R\\ \mathrm{cond}(\chi)\mid b(w_1,w_2)}}\Psi\Big(\frac{w_1}{W}\Big)\Psi\Big(\frac{w_2}{W}\Big)\beta_{n_1}\overline{\beta_{n_2}}\chi(n_1\overline{n_2})\frac{\varphi(bw_1w_2)}{bw_1w_2\varphi(bw_1)\varphi(bw_2)}.$$

The error term is acceptable provided

$$NR^5\ll Q^{2-\varepsilon}.$$

Since $N\ll X^{\frac13}$ this is acceptable provided

$$\rho<\frac{4-\varpi}{15(2+\varpi)}. \tag{28}$$

4.4.2. *Evaluation of* $\mathcal{D}_2$. We have

$$\mathcal{D}_2 := \sum_{\substack{w_1,w_2,n_1,n_2\\(n_j,bw_j)=1}}\sum\sum\sum \sum_{\substack{\chi\ (\mathrm{mod}\, bw_2)\\ \mathrm{cond}(\chi)\le R}} \Psi\Big(\frac{w_1}{W}\Big)\Psi\Big(\frac{w_2}{W}\Big)\overline{\beta_{n_1}}\beta_{n_2}\frac{\chi(n_2)}{\varphi(bw_2)} \sum_{\substack{mn_1\equiv 1 (bw_1)\\ (m,w_2)=1}} \chi(m) f\Big(\frac{m}{M}\Big).$$

The computations in [Drappeau 2017, pp. 712–713] can be also quoted directly with the identification (27). We obtain

$$\mathcal{D}_2 = \mathcal{M}_3 + O(R^{\frac{3}{2}}N^2Q^{1+\varepsilon}).$$

This is acceptable if

$$\rho < \frac{2}{3}\lambda + \frac{2(1-\varpi)}{9(2+\varpi)}. \tag{29}$$

4.4.3. *Evaluation of* $\mathcal{D}_1$. We have

$$\mathcal{D}_1 := \sum_{\substack{w_1,w_2,n_1,n_2\\(n_j,bw_j)=1\\ n_1\equiv n_2\ (\mathrm{mod}\, b)}}\sum\sum\sum \Psi\Big(\frac{w_1}{W}\Big)\Psi\Big(\frac{w_2}{W}\Big)\beta_{n_1}\overline{\beta_{n_2}} \sum_{mn_j\equiv 1\ (\mathrm{mod}\, bw_j)} f\Big(\frac{m}{M}\Big).$$

We need to separate the variables w_1, w_2, n_1, n_2 from each other, and this requires a subdivision of the variables. We decompose these variables uniquely, following [Fouvry and Radziwiłł 2018], as follows:

$$\begin{cases} d = (n_1, n_2), & n_1 = dd_1\nu_1 \text{ with } d_1 \mid d^\infty \text{ and } (d,\nu_1) = 1, \quad n_2 = d\nu_2,\\ q_0 = (w_1, w_2), & w_i = q_0 q_i \text{ for } i \in \{1, 2\}.\end{cases}$$

The summation conditions imply

$$(dd_1\nu_1, q_0q_1) = (d\nu_2, q_0q_2) = 1.$$

We therefore have

$$\mathcal{D}_1 = \sum_{(d,b)=1}\sum_{d_1\mid d^\infty}\sum_{(q_0,d)=1} \mathcal{D}_1(d, d_1, q_0),$$

$$\mathcal{D}_1(\cdots) = \sum_{\substack{q_1,q_2,\nu_1,\nu_2\\ (d\nu_1,\nu_2)=(q_1,q_2)=1\\ (q_1q_2,d)=(\nu_1,d)=1\\ (\nu_1,q_1)=(\nu_2,q_2)=(\nu_1\nu_2,bq_0)=1\\ d_1\nu_1\equiv\nu_2\ (\mathrm{mod}\, bq_0)}}\sum\sum\sum \Psi\Big(\frac{q_0q_1}{W}\Big)\Psi\Big(\frac{q_0q_2}{W}\Big)\beta_{dd_1\nu_1}\overline{\beta_{d\nu_2}} \sum_{\substack{mdd_1\nu_1\equiv 1\ (\mathrm{mod}\, bq_0q_1)\\ md\nu_2\equiv 1\ (\mathrm{mod}\, bq_0q_2)}} f\Big(\frac{m}{M}\Big).$$

Using smooth partitions of unity we break the variables into dyadic ranges: $d \asymp D$, $d_1 \asymp D_1$, $q_0 \asymp Q_0$. The contribution from $d \asymp D$ and $d_1 \asymp D_1$ is

$$\ll Q^\varepsilon M \sum_{d\asymp D}\sum_{\substack{d_1\mid d^\infty\\ d_1\asymp D_1}}\sum_{\nu_1\asymp N/dd_1}\sum_{\nu_2\asymp N/d} |\beta_{dd_1\nu_1}||\beta_{d\nu_2}| \ll Q^\varepsilon MN^2 \sum_{d\asymp D}\frac{1}{d^2}\sum_{d_1\mid d^\infty}\frac{\tau(d_1)^{O(1)}}{d_1}\Big(\frac{d_1}{D_1}\Big)^{1-\varepsilon^2}$$

$$\ll Q^\varepsilon MN^2 D_1^{-1+\varepsilon^2} D^{-1},$$

where the sum over q_0 and q_1 was bounded by $O(\tau_3(|mdd_1\nu_1 - 1|)) = O(Q^\varepsilon)$, likewise for the sum over q_2 (note that $md\nu_2 \neq 1$ and $mdd_1\nu_1 \neq 1$). This bound is acceptable provided

$$DD_1 \gg \frac{X}{Q^{2-\varepsilon}}, \tag{30}$$

so we may henceforth assume $DD_1 \ll XQ^{-2+\varepsilon}$.

The contribution from $q_0 \asymp Q_0$ is

$$\begin{aligned}
&\ll Q^\varepsilon \sum_{q_0 \asymp Q_0} \sum_{q_1 \asymp Q/q_0} \sum_{\substack{n_1 \equiv n_2 \, (\mathrm{mod}\, q_0) \\ n_j \asymp N}}\sum \sum_{\substack{m \asymp M \\ m \equiv \overline{n_1} \, (\mathrm{mod}\, q_0 q_1)}} 1 \\
&\ll Q^\varepsilon M \sum_{q_0 \asymp Q_0} \sum_{q_1 \asymp Q/q_0} \frac{1}{q_0 q_1} \sum_{\substack{n_1 \equiv n_2 \, (\mathrm{mod}\, q_0) \\ n_j \asymp N}}\sum 1 \\
&\ll Q^\varepsilon (MN^2 Q_0^{-1} + MN),
\end{aligned}$$

where in the first line the sum over q_2 was again bounded by $\tau(|md\nu_2 - 1|)$. This is acceptable provided

$$N \gg \frac{X}{Q^{2-\varepsilon}} \quad \text{and} \quad Q_0 \gg \frac{X}{Q^{2-\varepsilon}}, \tag{31}$$

so we may henceforth assume $Q_0 \ll XQ^{-2+\varepsilon}$.

We use Poisson summation, following [Drappeau 2017, pp. 714–716]. Let

$$\tilde{q} = bq_0q_1q_2 \quad \text{and} \quad \mu \equiv \begin{cases} \overline{dd_1\nu_1} & (\mathrm{mod}\, bq_0q_1), \\ \overline{d\nu_2} & (\mathrm{mod}\, bq_0q_2). \end{cases}$$

Note that $\tilde{q} \geq \frac{1}{2}W \gg Q^{1-\varepsilon}$. With $H = \tilde{q}^{1+\varepsilon}M^{-1} \ll Q^{2+\varepsilon}/(q_0M)$, we get, for any fixed $A > 0$,

$$\sum_{m \equiv \mu \, (\mathrm{mod}\, \tilde{q})} f\Big(\frac{m}{M}\Big) = \frac{M}{\tilde{q}} \sum_{|h| \leq H} \hat{f}\Big(\frac{hM}{\tilde{q}}\Big) \mathrm{e}\Big(\frac{\mu h}{\tilde{q}}\Big) + O(Q^{-A}). \tag{32}$$

The zero frequency in (32) contributes the main term, which, after summing over d, d_1 and q_0 (and reintegrating the values DD_1 and Q_0 larger than $XQ^{-2+\varepsilon}$ which were discarded earlier), is given by

$$\mathcal{M}_1 := \frac{M}{b} \hat{f}(0) \sum_{\substack{w_1, w_2, n_1, n_2 \\ (n_j, bw_j) = 1 \\ n_1 \equiv n_2 \, (\mathrm{mod}\, b(w_1, w_2))}}\sum\sum\sum \Psi\Big(\frac{w_1}{W}\Big) \Psi\Big(\frac{w_2}{W}\Big) \beta_{n_1} \overline{\beta_{n_2}} \frac{1}{[w_1, w_2]}.$$

The error term in (32) induces in $\mathcal{D}_1(d, d_1, q_0)$ a contribution

$$\ll Q^{-10} N^2,$$

and therefore in $\mathcal{D}_1$ a contribution $O(1)$, which is acceptable.

We solve the congruence conditions on μ by writing

$$d_1\nu_1 - \nu_2 = bq_0t, \quad \mu dd_1\nu_1 = 1 + bq_0q_1\ell, \quad \mu d\nu_2 = 1 + bq_0q_2m,$$

with $t, \ell, m \in \mathbb{Z}$. We deduce

$$\mu d t = q_1 \ell - q_2 m, \quad t = q_1 \nu_2 \ell - q_2 d_1 \nu_1 m.$$

Then we have the equalities, modulo $\mathbb{Z}$,

$$\begin{aligned}
\frac{\mu}{\tilde{q}} = \frac{\mu}{b q_0 q_1 q_2} &= \frac{1}{d d_1 \nu_1 b q_0 q_1 q_2} + \frac{\ell}{d d_1 \nu_1 q_2} \\
&\equiv \frac{1}{d d_1 \nu_1 b q_0 q_1 q_2} + \frac{\ell \overline{d d_1}}{\nu_1 q_2} + \frac{\ell \overline{\nu_1 q_2}}{d d_1} \\
&\equiv \frac{1}{d d_1 \nu_1 b q_0 q_1 q_2} + \frac{t \overline{q_1 \nu_2 d d_1}}{\nu_1 q_2} - \frac{\overline{b q_0 q_1 \nu_1 q_2}}{d d_1} \\
&\equiv \frac{1}{d d_1 \nu_1 b q_0 q_1 q_2} + \frac{d_1 \nu_1 - \nu_2}{b q_0} \frac{\overline{q_1 \nu_2 d d_1}}{\nu_1 q_2} - \frac{\overline{b q_0 q_1 \nu_1 q_2}}{d d_1}.
\end{aligned}$$

By estimating trivially the first term, we have

$$\mathrm{e}\Big(\frac{h\mu}{\tilde{q}}\Big) = \mathrm{e}\Big(h \frac{d_1 \nu_1 - \nu_2}{b q_0} \frac{\overline{q_1 \nu_2 d d_1}}{\nu_1 q_2} - \frac{h \overline{b q_0 q_1 \nu_1 q_2}}{d d_1}\Big) + O\Big(\frac{H q_0}{N W^2}\Big). \tag{33}$$

The error term here is $\ll Q^{\varepsilon} X^{-1}$, which contributes to $\mathcal{D}_1(d, d_1, q_0)$ a quantity

$$\frac{Q^{2+\varepsilon} N}{X q_0^2 d d_1}\Big(1 + \frac{N}{d}\Big),$$

and upon summing over (d, d_1, q_0), this contributes to $\mathcal{D}_1$ a quantity $O(Q^{2+\varepsilon} N^2 X^{-1})$. This error is acceptable if

$$N \ll Q^{2-\varepsilon}. \tag{34}$$

Then we insert the first term of (33) in (32), and insert the Fourier integral. The nonzero frequencies contribute a term

$$\begin{aligned}
\mathcal{R}_1(d, d_1, q_0) := \frac{M q_0}{b W^2} \int \sum_{\substack{q_1, q_2, \nu_1, \nu_2 \\ (d\nu_1, \nu_2) = (q_1, q_2) = 1 \\ (q_1 q_2, d) = (\nu_1, d) = 1 \\ (\nu_1, q_1) = (\nu_2, q_2) = (\nu_1 \nu_2, b q_0) = 1 \\ d_1 \nu_1 \equiv \nu_2 \ (\mathrm{mod}\, b q_0)}} \sum \sum \sum \sum_{0 < |h| \le H} \Psi\Big(\frac{q_0 q_1}{W}\Big) \Psi\Big(\frac{q_0 q_2}{W}\Big) \beta_{d d_1 \nu_1} \overline{\beta_{d \nu_2}} \\
\times f\Big(t \frac{q_0^2 q_1 q_2}{W^2}\Big) \mathrm{e}\Big(h \frac{d_1 \nu_1 - \nu_2}{b q_0} \frac{\overline{q_1 \nu_2 d d_1}}{\nu_1 q_2} - \frac{h \overline{b q_0 q_1 \nu_1 q_2}}{d d_1}\Big) \mathrm{e}\Big(\frac{-h t M q_0}{b W^2}\Big) \,\mathrm{d}t.
\end{aligned}$$

So far, we have obtained under the conditions (31) and (34) the bound

$$\mathcal{D}_1 = \mathcal{M}_1 + \mathcal{R}_1 + O(N Q^{2-\varepsilon}), \quad \text{where } \mathcal{R}_1 := \sum_{\substack{Q_0, D D 1 \ll X Q^{-2+\varepsilon} \\ Q, D, D_1 \text{ dyadic}}} \sum_{\substack{d \asymp D \\ d_1 \asymp D_1 \\ q_0 \asymp Q_0}} \mathcal{R}(d, d_1, q_0).$$

We now restrict the summation over q_1 and q_2 in residue classes modulo dd_1, to account for the oscillatory factors. Let $\lambda_1, \lambda_2 \in (\mathbb{Z}/dd_1\mathbb{Z})^\times$, and

$$b_{\boldsymbol{n},\boldsymbol{r},\boldsymbol{s}} = \sum_{\substack{\nu_1 \\ \nu_1=\boldsymbol{s} \\ \nu_2 dd_1=\boldsymbol{r} \\ (d\nu_1,\nu_2)=(\nu_1\nu_2,bq_0)=1 \\ (\nu_1,d)=1 \\ d_1\nu_1\equiv\nu_2 \pmod{bq_0}}} \sum_{\nu_2} \sum_{\substack{0<|h|\le H \\ h(d_1\nu_1-\nu_2)=bq_0\boldsymbol{n}}} \beta_{dd_1\nu_1}\overline{\beta_{d\nu_2}} \mathrm{e}\Big(-\frac{h\overline{bq_0\lambda_1\nu_1\lambda_2}}{dd_1} - \frac{htMq_0}{bW^2}\Big),$$

$$g(\boldsymbol{c}, \boldsymbol{d}, \boldsymbol{n}, \boldsymbol{r}, \boldsymbol{s}) = \Psi\Big(\frac{q_0\boldsymbol{c}}{W}\Big)\Psi\Big(\frac{q_0\boldsymbol{d}}{W}\Big) f\Big(\frac{tq_0^2\boldsymbol{cd}}{W^2}\Big).$$

Then

$$\mathcal{R}_1(d, d_1, q_0) = \frac{Mq_0}{bW^2}\int_{t\asymp_f 1} \sum_{\lambda_1,\lambda_2 \pmod{dd_1}^*} \tilde{\mathcal{R}}_1(t, (\lambda_j))\,\mathrm{d}t,$$

$$\tilde{\mathcal{R}}_1(t, (\lambda_j)) = \sum_{\substack{\boldsymbol{n},\boldsymbol{r},\boldsymbol{s},\boldsymbol{c},\boldsymbol{d} \\ \boldsymbol{c}\equiv\lambda_1, \boldsymbol{d}\equiv\lambda_2 \pmod{dd_1} \\ (\boldsymbol{sc},\boldsymbol{rd}bdd_1)=1}} b_{\boldsymbol{n},\boldsymbol{r},\boldsymbol{s}}\, g(\boldsymbol{c}, \boldsymbol{d}, \boldsymbol{n}, \boldsymbol{r}, \boldsymbol{s})\mathrm{e}\Big(\frac{\boldsymbol{n}\boldsymbol{r}\overline{\boldsymbol{d}}}{\boldsymbol{sc}}\Big).$$

We apply Proposition 6, with sizes given by

$$\boldsymbol{C} = \boldsymbol{D} = \frac{W}{q_0}, \quad \boldsymbol{S} = \frac{N}{dd_1}, \quad \boldsymbol{R} = Nd_1, \quad \boldsymbol{N} = \frac{HN}{dbq_0}.$$

Let $X = Q^2Y$. Then $Y = Q^\varpi$. Note that

$$\boldsymbol{RS} \asymp N^2D^{-1}, \quad \boldsymbol{N} \ll Q^\varepsilon N^2Y^{-1}D^{-1}Q_0^{-2} \ll Q^\varepsilon \boldsymbol{RS}, \quad \boldsymbol{C} \ll Q^\varepsilon \boldsymbol{RD}.$$

We get

$$\tilde{\mathcal{R}}_1(t, \lambda_j) \ll Q^\varepsilon (DD_1)^{\frac{3}{2}} K \|b_{\boldsymbol{N},\boldsymbol{R},\boldsymbol{S}}\|_2,$$

where

$$Q^{-\varepsilon}K^2 \ll Q^2N^4D^{-1}D_1Q_0^{-2} + Q^{2+4\theta}N^{4-6\theta}D^{-2+2\theta}D_1^{-2\theta}Q_0^{-2-4\theta}\Big(1+\frac{D}{N}\Big)^{1-4\theta} + Q^2N^3Y^{-1}D^{-1}D_1Q_0^{-4}.$$

To bound the term $\|b_{\boldsymbol{N},\boldsymbol{R},\boldsymbol{S}}\|_2$, we assume

$$XQ^{-2+\varepsilon} = o(N), \tag{35}$$

so that $D = o(N)$ by virtue of the line below (30), and the case $d_1\nu_1 = \nu_2$ never occurs in $b_{\boldsymbol{n},\boldsymbol{r},\boldsymbol{s}}$. Then

$$\|b_{\boldsymbol{N},\boldsymbol{R},\boldsymbol{S}}\|_2^2 \le \sum_{\substack{\nu_1,\nu_2,h \\ d_1\nu_1\equiv\nu_2 \pmod{q_0} \\ 0<|h|<H}} |\beta_{dd_1\nu_1}\beta_{d\nu_2}|^2 \ll \frac{Q^{2+\varepsilon}}{Q_0M}\frac{N}{DD_1}\Big(\frac{N}{DQ_0}+1\Big)$$
$$\ll Q^\varepsilon(N^3Y^{-1}D^{-2}D_1^{-1}Q_0^{-2} + N^2Y^{-1}D^{-1}D_1^{-1}Q_0^{-1}).$$

We deduce

$$\tilde{\mathcal{R}}_1(t,(\lambda_j)) \ll Q^{\varepsilon}\sum_{k=1}^{6} Q^{\eta_{k,1}}N^{\eta_{k,2}}Y^{\eta_{k,3}}D^{\eta_{k,4}}D_1^{\eta_{k,5}}Q_0^{\eta_{k,6}},$$

where, for each k, $\eta_k=(\eta_{k,\ell})_{1\le\ell\le 6}$ is given by

$$\{\eta_k\}=\left\{\begin{pmatrix}1\\3\\-\frac{1}{2}\\\frac{1}{2}\\\frac{3}{2}\\-\frac{3}{2}\end{pmatrix},\begin{pmatrix}1\\\frac{7}{2}\\-\frac{1}{2}\\0\\\frac{3}{2}\\-2\end{pmatrix},\begin{pmatrix}2\theta+1\\3-3\theta\\-\frac{1}{2}\\\theta\\1-\theta\\-2\theta-\frac{3}{2}\end{pmatrix},\begin{pmatrix}2\theta+1\\\frac{7}{2}-3\theta\\-\frac{1}{2}\\\theta-\frac{1}{2}\\1-\theta\\-2\theta-2\end{pmatrix},\begin{pmatrix}1\\\frac{5}{2}\\-1\\\frac{1}{2}\\\frac{3}{2}\\-\frac{5}{2}\end{pmatrix},\begin{pmatrix}1\\3\\-1\\0\\\frac{3}{2}\\-3\end{pmatrix}\right\}.$$

Summing over λ_j, integrating over t, and multiplying by $Mq_0/(bW^2)\ll Q^{\varepsilon}N^{-1}YQ_0$, we get

$$\mathcal{R}_1(d,d_1,q_0)\ll Q^{\varepsilon}\sum_{k=1}^{6}Q^{\eta_{k,1}}N^{\eta_{k,2}-1}Y^{\eta_{k,3}+1}D^{\eta_{k,4}+2}D_1^{\eta_{k,5}+2}Q_0^{\eta_{k,6}+1}.$$

We sum over d, d_1 and q_0 in dyadic intervals of lengths D, D_1 and Q_0, obtaining

$$\sum_{\substack{d\asymp D\\ d_1\asymp D_1,\ d_1\mid d^{\infty}\\ q_0\asymp Q_0\\ (d,b)=(q_0,d)=1}}\mathcal{R}_1(d,d_1,q_0)\ll Q^{\varepsilon}\sum_{k=1}^{6}Q^{\eta_{k,1}}N^{\eta_{k,2}-1}Y^{\eta_{k,3}+1}D^{\eta_{k,4}+3}D_1^{\eta_{k,5}+2}Q_0^{\eta_{k,6}+2}.$$

Finally, we sum this dyadically over Q_0, D and D_1 subject to $Q_0+DD_1\ll YQ^{\varepsilon}$. We get

$$\mathcal{R}_1\ll Q^{\varepsilon}\sum_{k=1}^{6}Q^{\eta_{k,1}}N^{\eta_{k,2}-1}Y^{\eta_{k,3}+1+\max(0,\eta_{k,6}+2)+\max(0,\eta_{k,4}+3,\eta_{k,5}+2)}.$$

Here, the terms for $k=5,6$ are majorized by the term $k=1$, so

$$\mathcal{R}_1\ll Q^{\varepsilon}\sum_{k=1}^{4}Q^{\theta_{k,1}}N^{\theta_{k,2}}Y^{\theta_{k,3}},$$

where

$$\{\theta_k\}=\left\{\begin{pmatrix}1\\2\\\frac{9}{2}\end{pmatrix},\begin{pmatrix}1\\\frac{5}{2}\\4\end{pmatrix},\begin{pmatrix}1+2\theta\\2-3\theta\\4-\theta\end{pmatrix},\begin{pmatrix}1+2\theta\\\frac{5}{2}-3\theta\\\frac{7}{2}-\theta\end{pmatrix}\right\}.$$

We conclude that

$$\mathcal{D}_1=\mathcal{M}_1+O(Q^{2-\varepsilon}N)$$

on the condition that $N\ll Q^{-\varepsilon}\min(QY^{-\frac{9}{2}},\ Q^{\frac{2}{3}}Y^{-\frac{8}{3}},\ Q^{(1-2\theta)/(1-3\theta)}Y^{-(4-\theta)/(1-3\theta)},\ Q^{\frac{2}{3}}Y^{-\frac{1}{3}(7-2\theta)/(1-2\theta)})$.

Upon using $\theta \le \frac{7}{64}$, these conditions are implied by

$$N \ll X^{-\varepsilon} \min(X^{\frac{1}{2}(2-9\varpi)/(2+\varpi)}, X^{\frac{1}{43}(50-249\varpi)/(2+\varpi)}, X^{\frac{1}{75}(50-217\varpi)/(2+\varpi)}), \tag{36}$$

and hypotheses (31), (34) and (35).

4.4.4. *Main terms.* The main terms $\mathcal{M}_1$ and $\mathcal{M}_3$, which are real numbers by the symmetry $n_1 \leftrightarrow n_2$, combine to form

$$\mathcal{M}_1 - \mathcal{M}_3$$
$$= M\hat{f}(0) \sum\sum_{w_1, w_2} \Psi\Big(\frac{w_1}{W}\Big)\Psi\Big(\frac{w_2}{W}\Big) \frac{1}{b[w_1, w_2]\varphi(b(w_1, w_2))} \times \sum_{\substack{\chi \text{ prim} \\ \operatorname{cond}(\chi) > R \\ \operatorname{cond}(\chi) \mid b(w_1, w_2)}} \sum\sum_{\substack{n_1, n_2 \\ (n_j, bw_j) = 1}} \beta_{n_1} \overline{\beta_{n_2}} \chi(n_1) \overline{\chi(n_2)}.$$

We may quote the computations in [Drappeau 2017, p. 717], again with the identification (27), to obtain

$$|\mathcal{M}_3 - \mathcal{M}_1| \ll Q^{\varepsilon} M(N + N^2 R^{-2}) \ll Q^{\varepsilon}(X + NXR^{-2}).$$

This is acceptable provided

$$N \gg Q^{\varpi+\varepsilon} \quad \text{and} \quad R \gg Q^{\frac{1}{2}\varpi+\varepsilon}. \tag{37}$$

4.4.5. *Conclusion.* Hypotheses (28), (29), (31), (34), (35), (36) and (37) are all satisfied if

$$\varpi < \frac{1}{8}, \quad \varpi < \sigma < \frac{1}{3} - \delta < \frac{1}{3} - \frac{242\varpi}{75(2+\varpi)}, \quad \frac{\varpi}{2(2+\varpi)} < \rho < \frac{1}{9} - \frac{\varpi}{3(2+\varpi)}. \tag{38}$$

We therefore conclude the following.

Lemma 13. *Under the notation and hypotheses of Corollary 10, assuming (38), we have*

$$T_{\mathrm{II}} \ll \sqrt{X} Q^{1-\varepsilon}.$$

4.5. ***Proof of Proposition 4.*** We combine Lemmas 11, 12, 13 and 9. Setting $\sigma = \varpi + \varepsilon$ and recalling that $\varpi < \frac{1}{8}$, we obtain the conditions

$$\frac{\varpi}{3(2+\varpi)} < \lambda < \frac{1}{6} - \frac{\varpi}{2}, \quad \frac{242\varpi}{75(2+\varpi)} < \delta < \frac{1}{12} - \frac{\varpi}{2(2+\varpi)}, \quad \frac{\varpi}{2(2+\varpi)} < \rho < \frac{1}{9} - \frac{\varpi}{3(2+\varpi)}.$$

The third is implied by our hypothesis on R. The first two can be satisfied whenever $-o(1) \le \varpi < \frac{50}{1093} - o(1)$. This proves Proposition 4.

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References

[Bhargava and Shankar 2015] M. Bhargava and A. Shankar, "Ternary cubic forms having bounded invariants, and the existence of a positive proportion of elliptic curves having rank 0", *Ann. of Math.* **181**:2 (2015), 587–621. MR Zbl

[Bombieri et al. 1986] E. Bombieri, J. B. Friedlander, and H. Iwaniec, "Primes in arithmetic progressions to large moduli", *Acta Math.* **156**:3-4 (1986), 203–251. Correction in arXiv 1903.01371 (2019). MR Zbl

[Brumer 1992] A. Brumer, "The average rank of elliptic curves, I", *Invent. Math.* **109**:3 (1992), 445–472. MR Zbl

[Bui and Milinovich 2011] H. M. Bui and M. B. Milinovich, "Central values of derivatives of Dirichlet L–functions", *Int. J. Number Theory* **7**:2 (2011), 371–388. MR Zbl

[Chandee et al. 2014] V. Chandee, Y. Lee, S.-C. Liu, and M. Radziwiłł, "Simple zeros of primitive Dirichlet L–functions and the asymptotic large sieve", *Q. J. Math.* **65**:1 (2014), 63–87. MR

[Conrey and Iwaniec 2002] B. Conrey and H. Iwaniec, "Spacing of zeros of Hecke L–functions and the class number problem", *Acta Arith.* **103**:3 (2002), 259–312. MR Zbl

[Deshouillers and Iwaniec 1982] J.-M. Deshouillers and H. Iwaniec, "Kloosterman sums and Fourier coefficients of cusp forms", *Invent. Math.* **70**:2 (1982), 219–288. MR Zbl

[Devin et al. 2022] L. Devin, D. Fiorilli, and A. Södergren, "Low-lying zeros in families of holomorphic cusp forms: the weight aspect", *Q. J. Math.* **73**:4 (2022), 1403–1426. MR Zbl

[Drappeau 2015] S. Drappeau, "Théorèmes de type Fouvry–Iwaniec pour les entiers friables", *Compos. Math.* **151**:5 (2015), 828–862. MR Zbl

[Drappeau 2017] S. Drappeau, "Sums of Kloosterman sums in arithmetic progressions, and the error term in the dispersion method", *Proc. Lond. Math. Soc.* **114**:4 (2017), 684–732. MR Zbl

[Fiorilli and Miller 2015] D. Fiorilli and S. J. Miller, "Surpassing the ratios conjecture in the 1–level density of Dirichlet L–functions", *Algebra Number Theory* **9**:1 (2015), 13–52. MR Zbl

[Fouvry 1985] É. Fouvry, "Sur le problème des diviseurs de Titchmarsh", *J. Reine Angew. Math.* **357** (1985), 51–76. MR Zbl

[Fouvry and Iwaniec 1983] É. Fouvry and H. Iwaniec, "Primes in arithmetic progressions", *Acta Arith.* **42**:2 (1983), 197–218. MR Zbl

[Fouvry and Iwaniec 2003] É. Fouvry and H. Iwaniec, "Low-lying zeros of dihedral L–functions", *Duke Math. J.* **116**:2 (2003), 189–217. MR Zbl

[Fouvry and Radziwiłł 2018] É. Fouvry and M. Radziwiłł, "Another application of Linnik dispersion method", *Chebyshevskiĭ Sb.* **19**:3 (2018), 148–163. Zbl

[Granville and Soundararajan 2018] A. Granville and K. Soundararajan, "Large character sums: Burgess's theorem and zeros of L–functions", *J. Eur. Math. Soc.* **20**:1 (2018), 1–14. MR Zbl

[Heath-Brown 1982] D. R. Heath-Brown, "Prime numbers in short intervals and a generalized Vaughan identity", *Canadian J. Math.* **34**:6 (1982), 1365–1377. MR Zbl

[Heath-Brown 2004] D. R. Heath-Brown, "The average analytic rank of elliptic curves", *Duke Math. J.* **122**:3 (2004), 591–623. MR Zbl

[Hughes and Rudnick 2003] C. P. Hughes and Z. Rudnick, "Linear statistics of low-lying zeros of L–functions", *Q. J. Math.* **54**:3 (2003), 309–333. MR Zbl

[Iwaniec and Kowalski 2004] H. Iwaniec and E. Kowalski, *Analytic number theory*, Amer. Math. Soc. Colloq. Publ. **53**, Amer. Math. Soc., Providence, RI, 2004. MR Zbl

[Iwaniec et al. 2000] H. Iwaniec, W. Luo, and P. Sarnak, "Low lying zeros of families of L–functions", *Inst. Hautes Études Sci. Publ. Math.* **91** (2000), 55–131. MR Zbl

[Katz and Sarnak 1999] N. M. Katz and P. Sarnak, *Random matrices, Frobenius eigenvalues, and monodromy*, Amer. Math. Soc. Colloq. Publ. **45**, Amer. Math. Soc., Providence, RI, 1999. MR Zbl

[Kim and Sarnak 2003] H. H. Kim and P. Sarnak, "Refined estimates towards the Ramanujan and Selberg conjectures", (2003). Appendix to "Functoriality for the exterior square of GL_4 and the symmetric fourth of GL_2", *J. Amer. Math. Soc.* **16**:1 (2003), 139–183. MR Zbl

[Linnik 1963] J. V. Linnik, *The dispersion method in binary additive problems*, Amer. Math. Soc., Providence, RI, 1963. MR Zbl

[Mackall et al. 2016] B. Mackall, S. J. Miller, C. Rapti, C. Turnage-Butterbaugh, and K. Winsor, "Some results in the theory of low-lying zeros of families of L–functions", pp. 435–476 in *Families of automorphic forms and the trace formula*, edited by W. Müller et al., Springer, 2016. MR Zbl

[Montgomery 1973] H. L. Montgomery, "The pair correlation of zeros of the zeta function", pp. 181–193 in *Analytic number theory* (St. Louis Univ., 1972), edited by H. G. Diamond, Proc. Sympos. Pure Math. **24**, Amer. Math. Soc., Providence, RI, 1973. MR Zbl

[Pratt 2019] K. Pratt, "Average nonvanishing of Dirichlet L–functions at the central point", *Algebra Number Theory* **13**:1 (2019), 227–249. MR Zbl

[Rudnick and Sarnak 1994] Z. Rudnick and P. Sarnak, "The n–level correlations of zeros of the zeta function", *C. R. Acad. Sci. Paris Sér. I Math.* **319**:10 (1994), 1027–1032. MR Zbl

[Rudnick and Sarnak 1996] Z. Rudnick and P. Sarnak, "Zeros of principal L–functions and random matrix theory", *Duke Math. J.* **81**:2 (1996), 269–322. MR Zbl

[Sarnak et al. 2016] P. Sarnak, S. W. Shin, and N. Templier, "Families of L–functions and their symmetry", pp. 531–578 in *Families of automorphic forms and the trace formula*, edited by W. Müller et al., Springer, 2016. MR Zbl

[Sica 1998] F. Sica, *The order of vanishing of L-functions at the center of the critical strip*, Ph.D. thesis, McGill University, 1998, available at https://www.proquest.com/pagepdf/304479758. MR

[Watkins 2021] M. Watkins, "Comments on Deuring's zero-spacing phenomenon", *J. Number Theory* **218** (2021), 1–43. MR Zbl

[Young 2006] M. P. Young, "Low-lying zeros of families of elliptic curves", *J. Amer. Math. Soc.* **19**:1 (2006), 205–250. MR Zbl

[Zhang 2014] Y. Zhang, "Bounded gaps between primes", *Ann. of Math.* **179**:3 (2014), 1121–1174. MR Zbl

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Multiplicative preprojective algebras are 2-Calabi–Yau

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We prove that multiplicative preprojective algebras, defined by Crawley-Boevey and Shaw, are 2-Calabi–Yau algebras, in the case of quivers containing unoriented cycles. If the quiver is not itself a cycle, we show that the center is trivial, and hence the Calabi–Yau structure is unique. If the quiver is a cycle, we show that the algebra is a noncommutative crepant resolution of its center, the ring of functions on the corresponding multiplicative quiver variety with a type A surface singularity. We also prove that the dg versions of these algebras (arising as certain Fukaya categories) are formal. We conjecture that the same properties hold for all non-Dynkin quivers, with respect to any extended Dynkin subquiver (note that the cycle is the type A case). Finally, we prove that multiplicative quiver varieties — for all quivers — are formally locally isomorphic to ordinary quiver varieties. In particular, they are all symplectic singularities (which implies they are normal and have rational Gorenstein singularities). This includes character varieties of Riemann surfaces with punctures and monodromy conditions. We deduce this from a more general statement about 2-Calabi–Yau algebras (following Bocklandt, Galluzzi, and Vaccarino).

1. Introduction

Multiplicative preprojective algebras have recently gained attention in geometry and topology. These algebras appear in the study of certain wrapped Fukaya categories [29; 30], in the study of microlocal sheaves on rational curves [12], and in the study of generalized affine Hecke algebras [33, Appendix 1]. Their moduli spaces of representations are called multiplicative quiver varieties, and are analogs of Nakajima's quiver varieties. These include character varieties of rank n local systems on closed Riemann surfaces, or on open Riemann surfaces with punctures and monodromy conditions [25; 55; 61]. Multiplicative quiver

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varieties have also been studied from various viewpoints in [9; 13; 20; 22]. A quantization was defined in [39] and further studied in [35].

Historically, Crawley-Boevey and Shaw [25] defined the multiplicative preprojective algebra to view solutions of the Deligne–Simpson problem as irreducible representations of multiplicative preprojective algebras of certain star-shaped quivers. Their paper establishes the foundations for much of this work. For a fixed field k and a quiver Q with vertex set Q_0 and arrow set Q_1 and $q \in (k^\times)^{Q_0}$, Crawley-Boevey and Shaw define

$$\Lambda^q(Q) := \frac{L_Q}{J_Q} := \frac{k\overline{Q}[(1+aa^*)^{-1}]_{a\in\overline{Q}}}{\left(r := \prod_{a\in Q_1}(1+aa^*)(1+a^*a)^{-1} - q\right)},$$

a quotient of the localized path algebra of the double quiver, L_Q, by the two-sided ideal J_Q generated by the single relation, r.

Many of the desirable properties of the (additive) preprojective algebra seem to hold for the multiplicative preprojective algebra. But establishing this rigorously is difficult, as most proof techniques in the additive case (employing the grading on the algebra) are not available in the multiplicative case. In particular, the multiplicative preprojective algebra is not in general a deformation of the ordinary one, nor does it have a useful Hilbert series for a filtration (due to the localization).

In this paper we overcome these difficulties when the quiver contains a cycle, and formulate the general expectations. This is sufficient for applications to multiplicative quiver varieties for *every* quiver.

The main statement is the following:

Conjecture 1.1. *$\Lambda^q(Q)$ is 2-Calabi–Yau for all $q \in (k^\times)^{Q_0}$ and all Q connected and not Dynkin; moreover, it is a prime ring, and the family $\Lambda^q(Q)$ is flat in q.*[1] *If Q is furthermore not extended Dynkin, then $Z(\Lambda^q(Q)) = k$, and the Calabi–Yau structure is unique.*

Here (extended) Dynkin refers to the underlying unoriented graph being of types A, D, or E. We explain how one can reduce the conjecture to the case where Q is extended Dynkin in Section 7D. We carry out this procedure for $Q = \widetilde{A}_n$ and thereby prove the conjecture for all connected quivers containing it.

Theorem 1.2. *$\Lambda^q(Q)$ is 2-Calabi–Yau and prime for any $q \in (k^\times)^{Q_0}$ and any k a field, and Q connected and containing an unoriented cycle. The family of algebras is flat in q. If the quiver properly contains a cycle, then $Z(\Lambda^q(Q)) = k$, and the Calabi–Yau structure is unique.*

This theorem is established in Corollaries 3.20 and 7.12, and the results of Section 8: Propositions 8.5 and 8.6, and Corollary 8.7. Each relies on technical results proven in Section 7. Before outlining the proof techniques, we give four different perspectives on this work:

(I) Symplectic topology: wrapped Fukaya categories. Multiplicative preprojective algebras arise from studying certain wrapped Fukaya categories. Let X_Γ be the Weinstein manifold formed by plumbing cotangent bundles of 2-spheres according to the graph Γ. Ekholm and Lekili [28] and Etgü and Lekili

[1] A prime ring is a noncommutative analog of an integral domain, being a ring R in which $aRb = 0$ implies $a = 0$ or $b = 0$.

[29; 30] produced quasiisomorphisms,

$$\mathcal{W}(X_\Gamma) \xrightarrow[\cong]{[28]} \mathscr{B}_\Gamma \xrightarrow[\cong]{[29;\,30]} \mathscr{L}_\Gamma$$

where $\mathcal{W}(X_\Gamma)$ is the partially wrapped Fukaya category of X_Γ, $\mathscr{B}_\Gamma$ is the Chekanov–Eliashberg dg-algebra and $\mathscr{L}_\Gamma$ is the dg multiplicative preprojective algebra following [36], with $q = 1$; see Definition 4.3.

Since X_Γ is a Liouville manifold, [34, Theorem 1.3] shows that $\mathcal{W}(X_\Gamma)$ is a 2-Calabi–Yau category and hence $\mathscr{L}_\Gamma$ is 2-Calabi–Yau, as a dg-algebra. We establish this result purely algebraically, in the case Γ contains a cycle. In particular, we show in this case that

$$\Lambda^1(\Gamma) = H^0(\mathscr{L}_\Gamma) \xhookrightarrow[=]{\text{Proposition 4.4}} H^*(\mathscr{L}_\Gamma)$$

and hence $\mathscr{L}_\Gamma$ is formal. By Theorem 1.2, the dg multiplicative preprojective algebra $\mathscr{L}_\Gamma$ is formal.[2]

Consequently, deformations of the wrapped Fukaya category, $\mathcal{W}(X_\Gamma)$, as an A_∞-category (respectively Calabi–Yau A_∞-category) over a degree zero base, are given by deformations of $\Lambda^1(\Gamma)$ as an associative algebra (respectively Calabi–Yau algebra). The infinitesimal deformations can be identified with $\mathrm{HH}^2(\Lambda^1(\Gamma))$. Thanks to Theorem 1.2, $\Lambda^1(\Gamma)$ is 2-Calabi–Yau. Hence, Van den Bergh duality identifies $\mathrm{HH}^2(\Lambda^1(\Gamma))$ with $\mathrm{HH}_0(\Lambda^1(\Gamma))$. The techniques in [54] can likely be adapted to compute the latter using the explicit basis for $\Lambda^q(Q)$ computed here. Furthermore, by the 2-Calabi–Yau property, $\mathrm{HH}^3(\Lambda^q(Q)) = 0$, so there are no obstructions to extending to infinite order deformations.

We conjecture that the same holds for every connected, non-Dynkin quiver. More precisely, in addition to Conjecture 1.1, we expect the following:

Conjecture 1.3. *If Q is connected and not Dynkin, then the dg multiplicative preprojective algebra $\Lambda^{dg,q}(Q)$ is quasiisomorphic to $\Lambda^q(Q)$, in degree zero.*

We give the precise definitions and details, as well as proof in the case Q contains a cycle, in Section 4.

(II) Quiver varieties: local structure of multiplicative quiver varieties and moduli spaces attached to 2-Calabi–Yau algebras. Given a dimension vector $\alpha \in \mathbb{N}^{Q_0}$, the affine multiplicative quiver variety is defined as the (coarse) moduli scheme of representations of $\Lambda^q(Q)$. Explicitly, it is the geometric invariant theory quotient of the space of all representations

$$\Lambda^q(Q) \to \bigoplus_i \mathrm{Mat}_{\alpha_i}(k)$$

by the action of $\prod_i \mathrm{GL}(\alpha_i)$ by change of basis. See Section 5 for more details (where we also recall a version incorporating a stability condition).

Properties of multiplicative preprojective algebras determine properties of the corresponding multiplicative quiver varieties. For instance, in Section 7.5 of [55], Tirelli and the second author observe, following [14], that the 2-Calabi–Yau property determines the (formal) local structure of the moduli space of representations. Namely, any formal neighborhood can be identified with the formal neighborhood of

[2] Additionally, since submission of this article, the dg multiplicative preprojective algebra was shown to be 2-Calabi–Yau for all q and all Γ in [15].

the zero representation of the moduli space of representations of some (additive) preprojective algebra. This is proved in more detail here, in Theorem 5.3 (expanding on [14, Section 6], where a similar result is given). Among other applications mentioned in [55], it follows that, when k has characteristic zero, the corresponding multiplicative quiver varieties are normal and are symplectic singularities in the sense of Beauville [3] (in particular, they are normal and have rational Gorenstein singularities); see Corollary 5.5.

This includes (as an open subset) character varieties of Riemann surfaces of positive genus with punctures and prescribed monodromy conditions, as explained in [55, Section 3] (following [25; 61]). (In the case of closed Riemann surfaces, as pointed out in [4], this statement does not require our result, since the group algebra $k[\pi_1(\Sigma)]$ is well-known to be 2-Calabi–Yau.)

One subtle point is that we can describe the local structure of multiplicative quiver varieties for *all* quivers despite the fact that we only prove the 2-Calabi–Yau property for quivers containing a cycle (Theorem 5.4). The key idea is that any quiver can be embedded into a quiver containing a loop and hence any representation of a quiver can be viewed as a representation of a quiver with a cycle. Therefore, its formal neighborhood can be identified with a formal neighborhood of the zero representation of an (additive) preprojective algebra. For detailed definitions, statements, and proofs see Section 5.

(III) Noncommutative algebraic geometry: noncommutative resolutions. Although in the non-Dynkin, nonextended Dynkin case, the center is expected to be trivial (Conjecture 1.1, proved when the quiver contains a cycle), this is far from true in the extended Dynkin case. Indeed, ordinary preprojective algebras of extended Dynkin quivers have a large center, the spectrum of which is a du Val singularity. The algebra itself is a noncommutative crepant resolution of this center. Moreover, this center is the algebra of functions on a natural quiver variety. So it is reasonable to ask if multiplicative preprojective algebras also resolve the corresponding multiplicative quiver variety.

In Shaw's thesis [57], he makes progress towards this question by showing that, for an extended Dynkin quiver Q with extended vertex v, the subalgebra $e_v\Lambda^1(Q)e_v$ is commutative of dimension 2, with a unique singularity at the origin; he expects that (for k of characteristic zero) the singularity there has the corresponding du Val type (see Remark 6.5).

In further analogy to the additive case, it is reasonable to pose the following conjecture:

Conjecture 1.4. *Let Q be extended Dynkin. The algebra $\Lambda^1(Q)$ is a 2-dimensional noncommutative crepant resolution (NCCR) of its center, which is the ring of functions on the associated multiplicative quiver variety $\mathcal{M}_{1,0}(Q,\delta)$. The Satake map $Z(\Lambda^1(Q)) \to e_v\Lambda^1(Q)e_v$ defined by $z \mapsto e_v z$ is an isomorphism.*[3]

See Section 5 for the precise definition of the multiplicative quiver variety. Thanks to our aforementioned results on its local structure, the conjecture implies Shaw's expectation that the singularity of $e_v\Lambda^1(Q)e_v$ is du Val of the corresponding type.

For $Q = \widetilde{A}_n$, we prove the conjecture in Section 6B. In the process, we obtain an explicit description

[3] We use the terminology "Satake" following the analogous one for symplectic reflection algebras at $t = 0$ of Etingof–Ginzburg, itself coming from the map for affine Hecke algebras proposed by Lusztig.

of the center, $Z(\Lambda^1(\widetilde{A}_n))$, which may be of independent interest.

(IV) Representation theory: Kontsevich–Rosenberg principle. A final perspective on this work involves the Kontsevich–Rosenberg Principle which says: a noncommutative geometric structure on an associative algebra A should induce a geometric structure on the representation spaces $\mathrm{Rep}_n(A)$, for all $n \geq 1$. This principle needs adjusting for structures living in the *derived* category of A-modules, as the representation functor is not exact. For a d-Calabi–Yau structure on A, it is shown in [17] and [62] that the derived moduli stack of perfect complexes of A-modules, $\mathbb{R}\,\mathrm{Perf}(A)$, has a canonical $(2-d)$-*shifted* symplectic structure. Since the dg multiplicative preprojective algebra is 2-Calabi–Yau, this implies that its moduli stack of representations has a 0-shifted symplectic structure. By Conjecture 1.3, it is the same as the moduli stack of representations of $\Lambda^q(Q)$ itself. Note that the multiplicative quiver variety can be viewed as a coarse moduli space of semistable representations; so the aforementioned result that this variety locally has the structure of an ordinary quiver variety is a singular analog of the statement on the moduli stack.

We now give a brief overview of the proof of Conjectures 1.1 and 1.3 for quivers containing a cycle. We prove Theorem 1.2 using a complex

$$P_\bullet := \Lambda^q(Q) \otimes_{kQ_0} kQ_0 \otimes_{kQ_0} \Lambda^q(Q) \xrightarrow{\alpha} \Lambda^q(Q) \otimes_{kQ_0} k\overline{Q}_1 \otimes_{kQ_0} \Lambda^q(Q) \xrightarrow{\beta} \Lambda^q(Q) \otimes_{kQ_0} \Lambda^q(Q)$$

defined originally in [25] (following [56, Theorem 10.3] and [26, Corollary 2.11]) and shown to resolve $\Lambda^q(Q)$, *except* for the injectivity of the map α. We show α is injective and then show the dual complex $P_\bullet^\vee$ is a resolution of $\Lambda^q(Q)[-2]$, which implies $\Lambda^q(Q)$ is 2-Calabi–Yau.

First, we establish a chain of implications to reduce the proof to a presentation of the localization L_Q that we call the *strong free product property*, established in Theorem 3.7, see Definition 3.5 or see below for a rough definition. The strong free product property is a version of Anick's weak summand property in the ungraded case; see [1].

To prove the 2-Calabi–Yau property from the strong free product property we show these implications:

$$\begin{gathered}
\text{Strong free product property for } Q: \\
\exists \sigma' : \Lambda^q(Q) *_{kQ_0} kQ_0[t,(q+t)^{-1}] \to L_Q \text{ a linear isomorphism} \\
\Downarrow \text{Section 3A} \\
\text{Weak free product property for } Q:\ \mathrm{gr}(\sigma') : \Lambda^q(Q) *_{kQ_0} kQ_0[t] \to \mathrm{gr}(L_Q) \text{ is an algebra isomorphism} \\
\Downarrow \text{Proposition 3.12} \\
\mathrm{gr}(\sigma')_1 : \Lambda^q(Q) \otimes_{kQ_0} kQ_0[t] \otimes_{kQ_0} \Lambda^q(Q) \to J_Q/J_Q^2 \text{ is an isomorphism of } \Lambda^q(Q)\text{-bimodules} \\
\Downarrow \text{Propositions 3.11 and 3.12} \\
P_\bullet \text{ is a length two projective } \Lambda^q(Q)\text{-bimodule resolution of } \Lambda^q(Q) \\
\Downarrow \text{Theorem 3.17} \\
\Lambda^q(Q) \text{ is 2-Calabi–Yau.}
\end{gathered}$$

Here the isomorphism σ' is determined by a choice of kQ_0-bimodule section $\Lambda^q(Q) \to L_Q$ of the quotient map $L_Q \twoheadrightarrow \Lambda^q(Q)$, but $\mathrm{gr}(\sigma')$ is independent of this choice. The element t maps to the relation, and the filtrations used are the t-adic one on the source and the J_Q-adic filtration on L_Q.

In Section 4, we show that the strong free product property implies that the dg multiplicative preprojective algebra $\Lambda^{dg,q}(Q)$ is formal. Therefore, by the results of this paper, both Conjectures 1.1 and 1.3 would follow from the following more general statement (see Section 7 for precise details).

Conjecture 1.5. *If Q is a connected, non-Dynkin quiver, then σ' as above is a linear isomorphism: $(L_Q, r, \sigma, kQ_0[t, (t+q)^{-1}])$ satisfies the strong free product property.*

Proposition 7.11 proves the conjecture for quivers containing a cycle. This is the technical heart of the paper. Our main technique involves reduction systems over the localized ring $kQ_0[t, (t+q)^{-1}]$. Using the diamond lemma [11], we show these give unique reductions of elements of L_Q to basis elements of the given free product. As a consequence, $\Lambda^q(Q)$ itself obtains the module structure of a free product of the cycle part and the rest of the quiver; see (7-12) for a precise statement.

Remark 1.6. After submission of this article Crawley-Boevey and Yuta Kimura [24, Theorem 1.1] proved that a related, more well-studied algebra, the *deformed preprojective algebra* [23], is 2-Calabi–Yau, in the case that the quiver is connected and non-Dynkin. This algebra is a deformation of the usual (additive) preprojective algebra, given as the quotient of the path algebra of the double quiver by the single relation $\sum_{a\in Q_1} aa^* - a^*a - \sum_{i\in Q_0} \lambda_i e_i$, the case $\lambda_i = 0$ returning the original preprojective algebra. This can also be proved via the techniques of this article, by deducing the strong free product property from the known one for the additive preprojective algebra for non-Dynkin quivers.

Namely, the latter are noncommutative complete intersections [1; 32], shown in [54, Proposition 5.2.1] to be equivalent, in the context of graded algebras, to the (strong or weak) free product property. More generally, let $A = TV/(r)$ be a graded algebra satisfying the free product property. Briefly, this means that we have a section $\sigma : A \to TV$, which we can take to be graded, so that the induced linear map $\sigma' : A *_k k[t] \to TV$ sending t to r is a linear isomorphism. Then for every $\lambda \in k$, σ' also defines a linear isomorphism by the same formula except sending t to $r+\lambda$. This is because, taking a homogeneous basis of A with degree nondecreasing, we obtain a homogeneous basis of TV via the free product, and the substitution $r \mapsto r+\lambda$ is a strictly triangular change of basis. Thus, the algebra $TV/(r+\lambda)$ also satisfies the strong free product property. Note that the same argument given here applies if we replace r by any filtered deformation $r + r'$, with r' in degrees strictly lower than r. They also apply to the quiver context, replacing k by kQ_0, hence imply that the deformed preprojective algebra satisfies the strong free product property. It seems likely this argument can apply to many other interesting algebras.

An outline of the paper is as follows: In Section 2, we give elementary background information on multiplicative preprojective algebras and produce an alternative generating set crucial for our approach to the 2-Calabi–Yau property. In Section 3, we prove the 2-Calabi–Yau property for $\Lambda^q(Q)$ assuming the strong free product property. In Section 4, using the strong free product property, we show that the dg multiplicative preprojective algebra has homology $\Lambda^q(Q)$, concentrated in degree zero. In Section 5, we use the 2-Calabi–Yau property to describe the formal neighborhoods of multiplicative quiver varieties as formal neighborhoods of the zero representation in certain quiver varieties. In Section 6 we use the 2-Calabi–Yau and prime properties in the cycle case, together with work of Shaw, to show the multiplicative preprojective algebra is a noncommutative resolution over its center. In Section 7, we prove the strong free product property first for multiplicative preprojective algebras of cycles, then for partial multiplicative preprojective algebras. Putting the two together, we deduce the strong free product property for connected quivers containing cycles. The key point of the argument, of independent interest, is a construction of bases of these algebras. Finally, in Section 8, we establish the prime property of $\Lambda^q(Q)$ using our explicit bases. We furthermore show that $Z(\Lambda^q(Q)) = k$ for Q connected and properly containing a cycle. This shows that the Calabi–Yau structure in these cases are unique, up to scaling.

2. The multiplicative preprojective algebra

2A. ***Definitions.*** Throughout the paper we fix an arbitrary field k. For each quiver (i.e., directed graph) Q, let Q_0 be the vertex set, Q_1 be the arrow set, and $h, t : Q_1 \to Q_0$ the head and tail maps, respectively. We will assume that Q_0 and Q_1 are finite for convenience, but really only need finitely many arrows incident to each vertex.

Let Q^{op} denote the quiver with the same underlying graph of vertices and edges, but with every arrow in the opposite direction. $\overline{Q}$ denotes the quiver with the same vertex set as Q and Q^{op} and with arrow set $Q_1 \sqcup Q_1^{\mathrm{op}}$. For each arrow $a \in Q_1$, we write a^* for the corresponding arrow in Q_1^{op}, and vice versa. In $\overline{Q}$ we distinguish between arrows in Q and Q^{op} using a function

$$\epsilon : \overline{Q}_1 \to \{\pm 1\}, \quad \epsilon(a) := \begin{cases} 1 & \text{if } a \in Q_1, \\ -1 & \text{if } a \in Q_1^{\mathrm{op}}. \end{cases}$$

For a quiver Q, we denote the path algebra by kQ and follow the convention that paths are concatenated from left to right. We have an inclusion $e_{(-)} : Q_0 \to kQ$ in order to view a vertex $i \in Q_0$ as a length zero path e_i.

For $a \in \overline{Q}_1$, define $g_a := 1 + aa^* \in k\overline{Q}$. Consider the localization $L_Q := k\overline{Q}[g_a^{-1}]_{a\in\overline{Q}_1}$. We write $L := L_Q$, when the quiver is clear from context. Notice, for all $a \in \overline{Q}_1$,

$$g_a a = a + aa^*a = ag_{a^*}. \tag{2-1}$$

This implies

$$g_{a^*}a^* = a^*g_a, \quad g_a^{-1}a = ag_{a^*}^{-1}, \quad g_{a^*}^{-1}a^* = a^*g_a^{-1}.$$

Fixing a total ordering $\leq$ on the set of arrows $\overline{Q}_1$, one can make sense of a product over (subsets of) the arrow set. Using $\leq$ and ϵ we define

$$\rho := \prod_{a\in\overline{Q}_1} g_a^{\epsilon(a)}, \quad l_a := \prod_{b\in\overline{Q}_1, b<a} g_b^{\epsilon(b)}, \quad r_a := \prod_{b\in\overline{Q}_1, b>a} g_b^{\epsilon(b)}.$$

When we need to make the role of the total ordering $\le$ more explicit, we write $\rho_{\le}$ (respectively $l_{a,\le}$ and $r_{a,\le}$) for ρ (respectively l_a and r_a). By definition, l_a and r_a are the subproduct of ρ to the left and right of a, respectively. Therefore,

$$\rho = l_a g_a^{\epsilon(a)} r_a \tag{2-2}$$

for all $a \in \bar{Q}_1$.

Definition 2.1. Fix a quiver Q and $q \in (k^{\times})^{Q_0}$. Consider $\bar{Q}$ and fix an ordering $\le$ on the arrows and a map ϵ as defined above. The *multiplicative preprojective algebra*, $\Lambda^q(Q)$, is defined to be

$$\Lambda^q(Q) := L/J$$

where $L = k\bar{Q}[g_a^{-1}]_{a\in\bar{Q}_1}$ is the localization and J is the two-sided ideal generated by the element $\rho - q$.

Note that q is viewed as an element of $k\bar{Q}$ via $\sum_{i\in Q_0} q_i e_i \in kQ_0 \subset k\bar{Q}$, and as ρ is invertible we need $q_i \neq 0$ so $e_i\Lambda^q(Q) \neq 0$, for all i.

Remark 2.2. The isomorphism class of $\Lambda^q(Q)$ is independent of both the orientation of the quiver and the choice of an ordering on the arrows, by Section 2 in [25].

In the multiplicative preprojective algebra, (2-2) becomes the identity

$$l_a g_a^{\epsilon(a)} r_a = q.$$

Hence

$$r_a l_a = q g_a^{-\epsilon(a)}. \tag{2-3}$$

As mentioned in the introduction, we say that a quiver is (extended) Dynkin, we mean that the underlying unoriented graph is an (extended) type ADE Dynkin diagram. We don't consider nonsimply laced types because, given a quiver, the associated Cartan matrix is $2I - A$ where A is the adjacency matrix of the underlying unoriented graph, which is symmetric.

Example 2.3 (Dynkin case). Let Q be a Dynkin quiver and let R be a commutative ring. Note that the definitions of (multiplicative) preprojective algebra make sense over R. In [42, Section 5], the first named author constructed explicit isomorphisms

$$\Lambda^1(Q) \cong \Pi^0(Q) := R\bar{Q} \Big/ \Big(\sum_{a\in Q_1} [a, a^*]\Big)$$

if 2, 3, and 5 are invertible in R; see also the earlier works [57, Lemma 5.2.1], [22, Corollary 1], [29, Theorem 13], and [47, Section 5]. In particular, for a field k of characteristic zero, we can work over $k[\![\hbar]\!]$ and set $q = e^{\hbar}$. Then Λ^q is a formal deformation of Λ^1. Hence, by [31, Proposition 5.0.2], there exists some $\lambda \in k[\![\hbar]\!]$ such that there is a $k[\![\hbar]\!]$-linear algebra isomorphism

$$\Lambda^q(Q) \cong \Pi^{\lambda}(Q) := k\bar{Q} \Big/ \Big(\sum_{a\in Q_1} [a, a^*] - \sum_{i\in Q_0} \lambda_i e_i\Big).$$

In A types, such an isomorphism holds over any field k and for any actual parameter q. Namely, identifying $(A_n)_0 = \{1, 2, \dots, n\}$ and $(A_n)_1 = \{a_1, a_2, \dots, a_{n-1}\}$ with tail $t(a_i) = i$, the isomorphism $\Lambda^q(A_n) \cong \Pi^\lambda(A_n)$ is given by

$$e_i \mapsto e_i, \quad a_i \mapsto a_i, \quad a_i^* \mapsto \Big(\prod_{j>i} q_j\Big) a_i^*,$$

where $\lambda_i := (q_i - 1)\prod_{j>i} q_j$. Since $\Pi^\lambda(A_n)$ is nonzero if and only if there exists i, j with $i < j$ and $\sum_{\ell=i}^{j} \lambda_\ell = 0$, it follows that $\Lambda^q(A_n)$ is nonzero if and only if $\prod_{\ell=i}^{j} q_\ell = 1$.

2B. *The map θ*. In [25], the important map $\theta : \overline{Q}_1 \to \Lambda^q(Q)$ is defined by $\theta(a) = q^{-1} l_a a r_{a^*}$ and extended to $k\overline{Q}$ by the identity on Q_0 and by requiring θ to be an algebra map. Then Lemma 3.3 in [25] shows that

$$\theta(g_a) = l_a g_a l_a^{-1} \tag{2-4}$$
$$= r_a^{-1} g_a r_a, \tag{2-5}$$

so $\theta(g_a)$ is invertible. Hence θ factors through the localization $L := k\overline{Q}[g_a^{-1}]_{a\in\overline{Q}_1}$. We will show θ descends to the quotient $\Lambda^q(Q)$, with the ordering of the arrows reversed, using the following result.

Lemma 2.4. *Let $\le$ denote a total order on $\overline{Q}_1$ and let $\ge$ denote its opposite ordering, i.e., $a \ge b$ if $b \le a$. Such an order fixes a bijection $\overline{Q}_1 \cong \{a_1, a_2, \dots, a_{|\overline{Q}_1|}\}$. Then*

$$\theta(r_{a_j,\ge}) = l_{a_j,\le} =: l_{a_j} \quad \text{and} \quad \theta(l_{a_j,\ge}) = r_{a_j,\le} =: r_{a_j}$$

for any $a_j \in \overline{Q}_1$.

Proof. We prove $\theta(r_{a_j,\ge}) = l_{a_j}$ by induction on j, where $j = 1$ is the identity $\theta(1) = 1$. Then,

$$\theta(r_{a_{j+1},\ge}) = \theta(g_{a_j}^{\epsilon(a_j)})\theta(r_{a_j,\ge}) \overset{\text{(IH)}}{=} \theta(g_{a_j}^{\epsilon(a_j)}) l_{a_j} \overset{\text{(2-4)}}{=} l_{a_j} g_{a_j}^{\epsilon(a_j)} l_{a_j}^{-1} l_{a_j} = l_{a_j} g_{a_j}^{\epsilon(a_j)} = l_{a_{j+1}}.$$

The second identity is similar and one can formally obtain a proof from the above by exchanging the symbols r and l, the identity (2-4) for (2-5), and the order of the multiplication. □

Corollary 2.5. $\theta(\rho_\ge) = \rho.$

This corollary implies θ descends to a map $\Lambda^q(Q, \ge) \to \Lambda^q(Q, \le)$. Notice that we can similarly define $\theta_\ge : \Lambda^q(Q, \le) \to \Lambda^q(Q, \ge)$.

Proposition 2.6. $\theta_\ge \circ \theta = \mathrm{Id}_{\Lambda^q(Q,\le)}.$

Proof. It suffices to check $\theta_\ge \circ \theta$ is the identity on arrows in Q_1. We have

$$\theta_\ge(\theta(a)) = \theta_\ge(q^{-1} l_a a r_{a^*}) = q^{-1}\theta_\ge(l_a)\theta_\ge(a)\theta_\ge(r_{a^*}),$$

which by Lemma 2.4 equals

$$= q^{-1} r_a \theta_\ge(a) l_{a^*} = q^{-1} r_a (q^{-1} l_a a r_{a^*}) l_{a^*} \overset{\text{(2-3)}}{=\!=} g_a^{-\epsilon(a)} a g_{a^*}^{-\epsilon(a^*)} \overset{\text{(2-1)}}{=} a g_{a^*}^{-\epsilon(a)} g_{a^*}^{-\epsilon(a^*)} = a. \qquad \square$$

3. Calabi–Yau and free product properties

The goal of this section is to prove that $\Lambda^q(Q)$ is 2-Calabi–Yau for Q containing an unoriented cycle. We do so by exhibiting a length two, projective, $\Lambda^q(Q)$-bimodule resolution $P_\bullet$ of $\Lambda^q(Q)$, whose bimodule dual complex $P_\bullet^\vee$ is quasiisomorphic to $\Lambda^q(Q)$. This resolution is due to Crawley-Boevey and Shaw, but they don't state nor prove that it is exact. The main new ingredient we provide is the injectivity of α which relies on the weak free product property. Hence we begin this section with a short digression explaining the strong and weak free product properties.

3A. ***Free product (complete intersection) properties.*** Recall that, if R is a commutative ring and $r \in R$ an element, then there is a dg analog of the quotient $R/(r)$: the Koszul complex $(R[s]/(s^2), d)$ with $d|_R = 0$ and $ds = r$, here $|s| = -1$. Note here that, in spite of the notation, $R[s]/(s^2)$ is the graded-commutative algebra freely generated by R and a single generator s in degree -1. The quotient map $(R[s]/(s^2), d) \to R/(r)$ is a quasiisomorphism if and only if r is a nonzerodivisor.

Thus, in the commutative setting, the nonzerodivisor condition is the correct one for which the Koszul complex (derived imposition of $r = 0$) is equivalent to the quotient algebra.

Now let us pass to the noncommutative setting. If A is an algebra over a ring S and $J = (r)$ an ideal generated by a single relation r, we can form a canonical algebra map,

$$\Phi : A/J *_S S[t] \to \mathrm{gr}_J A, \quad \Phi|_{A/J} = \mathrm{Id}, \Phi(t) = r, \tag{3-1}$$

where gr_J means the associated graded algebra with respect to the J-adic filtration.

Definition 3.1. The pair (A, r) satisfies the *weak free product property* if Φ is an isomorphism.

Remark 3.2. This condition is significantly more subtle than in the commutative case. In particular, it is insufficient for r to be a nonzerodivisor. For example, if $A = k[x]$ and $r = x^2$, then we have $H^1(A * k[s], d) \ni [xs - sx] \neq 0$. (Here A is actually commutative, but we take the noncommutative construction; for a noncommutative example, simply replace A with $k\langle x, y\rangle$.)

The weak free product property is an analog of a *noncommutative complete intersection* (*NCCI*) [32], and closely matches the *weak summand* property from [1] (considered in the graded setting). We have chosen this terminology to make the algebraic property we are using more evident.

When the context is clear, we will sometimes abuse notion and say the quotient A/J, for $J = (r)$, satisfies the weak free product property, even though the choice of A and $r \in A$ is important.

Given an S-bimodule section $\sigma : A/J \to A$ of the quotient map $\pi : A \to A/J$, we can form an associated linear map,

$$\tilde{\sigma} : A/J *_S S[t] \to A, \quad (a_0 t^{m_1} a_1 t^{m_2} \cdots t^{m_n} a_n) \mapsto \sigma(a_0) r^{m_1} \sigma(a_1) r^{m_2} \cdots r^{m_n} \sigma(a_n), \tag{3-2}$$

for $m_i > 0$, for all i. The existence of such a σ (and hence $\tilde{\sigma}$) is automatic if S is separable, as is the case when $S = kQ_0$ below.

By construction, this is $S[t]$-bilinear, where t acts on A by multiplication by r. It also reduces to the identity modulo (t) on the source and J on the target. If (A, r) satisfies the weak free product property, then moreover the completion

$$\widehat{\tilde{\sigma}} : \widehat{A/J *_S S[t]} \to \widehat{A}, \tag{3-3}$$

with respect to the t-adic and J-adic filtrations, is a linear isomorphism.

The goal of the strong free product property is to find a description of A itself as a free product. The first version is the following:

Definition 3.3. The triple (A, r, σ) satisfies the *strong free product property* if $\tilde{\sigma} : A/J *_S S[t] \to A$ is an S-bimodule isomorphism.

Remark 3.4. The choice of σ is important. Let $A = k\langle x, y\rangle$ and $J = (y)$ so $A/J \cong k[x]$. Here $k[t]$ acts on A via $tf := yf$. Consider two different choices

$$\sigma_1, \sigma_2 : k[x] \to k\langle x, y\rangle, \quad \sigma_1(x + (y)) = x, \quad \sigma_2(x + (y)) = x - xy.$$

Then $\tilde{\sigma}_1$ is a linear isomorphism, while $\tilde{\sigma}_2$ is not surjective as

$$x = \sigma_2(x + (y))(1 - y)^{-1} = \sigma_2(x + (y)) \sum_{i \geq 0} y^i \notin \tilde{\sigma}_2(k[x] *_k k[t]).$$

This property is too much to expect in many situations, such as in the presence of rational functions in t. To fix this, let $B = S[t, f^{-1}]$ be a localization of $S[t]$ obtained by inverting some $f \in S^\times + (t)$, such that the map $S[t] \to A$ extends to an algebra map $\tau : B \to A$ (such an extension is necessarily unique). Let $\bar{B} := tB$, so that we have an S-bimodule decomposition $B = S \oplus \bar{B}$. Then $\tilde{\sigma}$ extends to a map $\sigma' : A/J *_S B \to A$, which has the form

$$a_0 b_1 a_1 \cdots b_n a_n \mapsto \sigma(a_0)\tau(b_1) \cdots \tau(b_n)\sigma(a_n), \quad a_i \in A/J, b_i \in \bar{B}. \tag{3-4}$$

Definition 3.5. The quadruple (A, r, σ, B) satisfies the *strong free product property* if σ' is a linear isomorphism.

This definition reduces to the previous definition in the case $B = S[t]$, $\tau(t) = r$.

In this case, it follows by taking associated graded algebras that (A, r) satisfies the weak free product property. Moreover, A is Hausdorff in the J-adic filtration (because the source of σ' is Hausdorff in the t-adic filtration), and σ' is indeed a restriction of $\widehat{\tilde{\sigma}}$.

Remark 3.6. It is important in the definition of σ' to use the natural bimodule complement $\bar{B} = tB$. Here is an example to show why (see also Remark 7.2 for another one, which we naively ran into before realizing our mistake). Let $A = k\langle x, y, z\rangle/(xyz - xz)$, $r = y$, so that $A/J \cong k\langle x, z\rangle/(xz)$. A basis for A/J is given by $\{z^i x^j\}_{i,j \geq 0}$. Let $\sigma : A/J \to A$ be the section preserving this, i.e., $\sigma(z^i x^j + (y, xyz - xz)) = z^i x^j + (xyz - xz)$. Set $B := k[t]$. Then $\sigma' = \tilde{\sigma} : A/J *_k k[t] \to A$ is a linear isomorphism, so (A, r, σ) is a strong free product. However, if we were to instead choose a complement $\bar{B} = (t - 1)B$, then we now have $\sigma'(x(t - 1)z) = 0$, so the map $\sigma' : A/J *_k B \to A$ defined using $\bar{B}$ is not injective. (It is also not surjective, as xz is not in the image.) On the other hand, for general weak free products, using the correct choice $\bar{B} = tB$, σ' is always injective.

Now we return to the setup of Q a connected, non-Dynkin quiver and $q = (k^\times)^{Q_0}$. Let $B := kQ_0[t, (q+t)^{-1}]$ and $\bar{B} := tB = \mathrm{Span}(t^m, (t')^m \mid m \geq 1\}$, for $t' := (q+t)^{-1} - q^{-1}$. We conjecture that, for every such Q, there exists σ such that the quadruple (L_Q, r, σ, B) satisfies the strong free product property. Moreover, in Section 7 we prove this conjecture in the case of quivers containing a cycle.

Theorem 3.7 (Proposition 7.11). *Let Q be a connected quiver containing an unoriented cycle. Let $B = kQ_0[t, (q+t)^{-1}]$ and let r denote the multiplicative preprojective relation. There exists a section σ such that (L_Q, r, σ, B) satisfies the strong free product property.*

The proof of this theorem is technical and uses combinatorial algebraic techniques. Therefore we delay its proof until Section 7, which does not result in circular logic as that section does not depend on results after Section 2.

Remark 3.8. The connectedness assumption can be weakened as follows. If $Q = Q' \sqcup Q''$ then $L_Q = L_{Q'} \oplus L_{Q''}$ and $\Lambda^q(Q) = \Lambda^q(Q') \oplus \Lambda^q(Q'')$ and so by adding sections, the strong free product property for

$$(L_{Q'} \oplus L_{Q''}, r' + r'', \sigma' + \sigma'', B' \oplus B'') = (L_Q, r, \sigma, B)$$

follows from the strong free product properties for $(L_{Q'}, r', \sigma', B')$ and $(L_{Q''}, r'', \sigma'', B'')$. So one only needs the weaker assumption that Q is a quiver with *each* component containing an unoriented cycle. But we state results in the connected setting to simplify the hypotheses.

Corollary 3.9. *Let Q be a connected quiver containing an unoriented cycle. Then $\Lambda^q(Q)$ satisfies the weak free product property. In particular, there exists an isomorphism of graded algebras*

$$\sum_i \varphi_i : \mathrm{gr}(\Lambda^q(Q) *_{kQ_0} kQ_0[t]) \to \mathrm{gr}(L_Q)$$

where the associated graded algebras are taken with respect to the t-adic and J_Q-adic filtrations on $\Lambda^q(Q) *_{kQ_0} kQ_0[t]$ and L_Q respectively.

Remark 3.10. Note that for ordinary preprojective algebras, the free product property was observed in [54, Propositions 5.1.9 and 5.2.1]. In fact, as these algebras are nonnegatively graded with finite-dimensional subspaces in each degree, and one-dimensional in degree zero (connected), the strong and weak free product properties are equivalent (and independent of the choice of graded section σ), as was already observed by Anick [1].[4] Moreover, if A has global dimension at most two, then these conditions are also equivalent to the condition that $A/(r)$ also has global dimension at most two. In the case A is a tensor or path algebra, such algebras were called *noncommutative complete intersections* in [32] due to their close relationship to the condition that representation varieties be complete intersections. For a

[4] Anick works in the graded context over a field rather than kQ_0, but his results generalize to this setting; see [32; 54].

nuanced discussion of this relationship, including sufficient conditions for the representation variety to be a complete intersection, see the introduction and Theorem 24 in [6].

However, in the ungraded case, we only have the implication that we need, that the free product property implies the existence of a length-two projective bimodule resolution. Indeed, the latter property only depends on a piece of the associated graded algebra with respect to the (r)-adic filtration, and in the ungraded case this filtration need not even be Hausdorff. In contrast, the strong free product property implies the Hausdorff condition and gives information about the algebra itself.

Motivated by this, we believe that the strong free product property can be viewed as an ungraded analog of the noncommutative complete intersection property. It is an interesting question to investigate when their representation varieties are complete intersections.

3B. ***A bimodule resolution of Λ.*** In this subsection, we show that for *any* quiver, the weak free product property for $\Lambda^q(Q)$ implies $\Lambda^q(Q)$ has a length two projective bimodule resolution. Consequently, since we establish the weak free product property for connected quivers containing a cycle, we prove $\Lambda^q(Q)$ has Hochschild dimension two for connected quivers containing a cycle. For ease of notation, write $\Lambda := \Lambda^q(Q)$.

Crawley-Boevey and Shaw build a chain complex of Λ-bimodules $P_\bullet = P_2 \xrightarrow{\alpha} P_1 \xrightarrow{\beta} P_0$ where,

$$P_2 = P_0 := \Lambda \otimes_{kQ_0} kQ_0 \otimes_{kQ_0} \Lambda = \langle \eta_v \rangle_{v \in Q_0}, \qquad P_1 := \Lambda \otimes_{kQ_0} k\bar{Q}_1 \otimes_{kQ_0} \Lambda = \langle \eta_a \rangle_{a \in \bar{Q}_1}$$

and

$$\alpha(\eta_v) := \sum_{a \in \bar{Q}_1 : t(a) = v} l_a \Delta_a r_a \quad \text{where} \quad \Delta_a = \begin{cases} \eta_a a^* + a\eta_{a^*} & \text{if } a \in Q_1, \\ -g_a^{-1}(\eta_a a^* + a\eta_{a^*})g_a^{-1} & \text{if } a \in Q_1^{\mathrm{op}}, \end{cases}$$

$$\beta(\eta_a) := a\eta_{t(a)} - \eta_{h(a)}a.$$

We claim that it is a resolution of Λ. To see this, following [25], we first write down an explicit chain map of Λ-bimodule complexes $\psi : P_\bullet \to Q_\bullet$, where $Q_\bullet$ is quasiisomorphic to Λ; we then prove it is an isomorphism. $Q_\bullet$ is the cotangent exact sequence in Corollary 2.11 of [26], but in this context it was defined earlier (and shown quasiisomorphic to Λ) by Schofield [56]. So we have the maps

$$P_\bullet \xrightarrow[{[25]}]{\psi} Q_\bullet \xrightarrow[{[56]}]{\text{quasiiso}} \Lambda.$$

Proposition 3.11 [25, Lemma 3.1]. *For* ***any*** *quiver Q, the following diagram commutes*:

$$\begin{tikzcd} P_0 \arrow[r, "\alpha"] \arrow[d, "\psi_2"] & P_1 \arrow[r, "\beta"] \arrow[d, "\psi_1", "\cong"'] & P_0 \arrow[r, "\gamma"] \arrow[d, "\psi_0", "\cong"'] & \Lambda \arrow[d, "\mathrm{id}", "="'] \\ J/J^2 \arrow[r, "\kappa"] & \Lambda \otimes_L \Omega_{kQ_0}(L) \otimes_L \Lambda \arrow[r, "\lambda"] & \Lambda \otimes_{kQ_0} \Lambda \arrow[r, "\mu"] & \Lambda \end{tikzcd}$$

Where the vertical maps are Λ-bimodule maps defined on generators by,

$$\psi_2(\eta_v) := \rho e_v - q e_v, \quad \psi_1(\eta_a) := 1 \otimes_L [a \otimes_{kQ_0} 1 - 1 \otimes_{kQ_0} a] \otimes_L 1, \quad \psi_0(\eta_v) = e_v \otimes e_v.$$

Here α and β are as defined above and $\gamma(\eta_v) := e_v$. The horizontal maps are defined by

$$\begin{aligned} \kappa(x+J^2) &:= 1\otimes_L\delta(x)\otimes 1 && \textit{where } \delta(x) := x\otimes 1 - 1\otimes x \textit{ for } x\in J,\\ \lambda(1\otimes_L[ab\otimes_{kQ_0}c - a\otimes_{kQ_0}bc]\otimes_L 1) &:= ab\otimes_{kQ_0}c - a\otimes_{kQ_0}bc && \textit{for } a,b,c\in L,\\ \mu(a\otimes b) &:= ab && \textit{for } a,b\in\Lambda. \end{aligned}$$

Since ψ_0 and ψ_1 are Λ-bimodule isomorphisms, it remains to show ψ_2 is a Λ-bimodule isomorphism. We show this using the weak free product property.

Proposition 3.12. *Suppose Λ satisfies the weak free product property. Then $P_\bullet$ is a bimodule resolution of Λ.*

Proof. Taking the $i = 1$ piece of the graded isomorphism

$$\mathrm{gr}(\varphi) = \sum_i \varphi_i : \mathrm{gr}(\Lambda *_{kQ_0} kQ_0[t,(t+q)^{-1}]) \longrightarrow \mathrm{gr}(L_Q)$$

gives an isomorphism of Λ-bimodules

$$\varphi_1 : \Lambda\otimes_{kQ_0} kQ_0\cdot t\otimes_{kQ_0}\Lambda \to J_Q/J_Q^2.$$

Since φ_1 sends $t\mapsto r$, it sends $te_v \mapsto re_v = (\rho - q)e_v$ and hence $\varphi_1 = \psi_2$. We conclude that ψ_2 is an isomorphism of Λ-bimodules and hence $\psi_\bullet : P_\bullet \to Q_\bullet$ is an isomorphism of Λ-bimodule complexes. In particular, $P_\bullet$ is a resolution since $Q_\bullet$ is a resolution. □

For a complex $C_\bullet$ concentrated in nonnegative degrees, define the length by

$$\mathrm{len}(C_\bullet) := \sup\{i\in\mathbb{N} \mid C_i \neq 0\}.$$

For an algebra A, the Hochschild dimension of A is $\mathrm{HH.dim}(A) := \mathrm{len}(\mathrm{HH}_\bullet(A))$ and the global dimension of A, is $\mathrm{gl.dim}(A) := \sup_{M\in A\text{-mod}} \inf_{P_\bullet}\{\mathrm{len}(P_\bullet)\}$ where the infimum is taken over all projective A-module resolutions of M.

Corollary 3.13. *Let Q be a connected quiver containing a cycle. Then*

$$\mathrm{gl.dim}(\Lambda) \le \mathrm{HH.dim}(\Lambda) = 2.$$

Proof. Use $P_\bullet$ to compute $\mathrm{HH}_i(\Lambda)$; $\mathrm{HH}_i(\Lambda) = 0$ for $i > 2$ while $\mathrm{HH}_2(\Lambda) \neq 0$. Therefore $\mathrm{HH.dim}(\Lambda) = 2$. Every left Λ-module, M, has a length two projective left Λ-module resolution $P_\bullet\otimes_\Lambda M$, and hence Λ has global dimension at most two. □

Example 3.14. Note that the inequality in Corollary 3.13 may be strict. If Q is the Jordan quiver (i.e., the quiver with one vertex and one loop) then

$$\Lambda^q(Q) \cong k\langle a, a^*\rangle[(1+a^*a)^{-1}]/(aa^* - qa^*a - (q-1)).$$

The change of variables $x := a$ and $y := a^*/(q-1)$ when $q\neq 1$, identifies $\Lambda^q(Q)$ with a localization of the first quantum Weyl algebra, $k\langle x, y\rangle/(xy - qyx - 1)$, which has global dimension one.

3C. ***The dual complex.*** In this subsection, we show for *any* quiver that if $P_\bullet$ is a resolution of $\Lambda^q(Q)$, then $\Lambda^q(Q)$ is 2-Calabi–Yau. Combining this with the previous subsection, we get that if $\Lambda^q(Q)$ satisfies the weak free product property then $\Lambda^q(Q)$ is 2-Calabi–Yau. In particular, this shows that $\Lambda^q(Q)$ is 2-Calabi–Yau for connected quivers containing a cycle.

First we recall the notion of d-Calabi–Yau algebras [36].

Definition 3.15. A is d-Calabi–Yau if (a) A has finite projective dimension as an A-bimodule; (b) $\mathrm{Ext}^i(A, A\otimes A) = 0$ for $i \neq d$; and (c) there exists an A-bimodule isomorphism

$$\eta : \mathrm{Ext}^d_{A-\mathrm{bimod}}(A, A\otimes A) \to A.$$

The map η is called a d-Calabi–Yau structure.

Remark 3.16. For perfect A-modules, M and N, one has a quasiisomorphism,

$$\mathrm{RHom}_{A-\mathrm{bimod}}(M, N) \xrightarrow{\cong} \mathrm{Hom}_{A-\mathrm{bimod}}(M, A\otimes A) \otimes^{\mathbb{L}}_{A\otimes A^{\mathrm{op}}} N.$$

Taking $M = A^\vee$ and $N = A$ gives $\mathrm{RHom}_{A-\mathrm{bimod}}(A^\vee, A) \cong A \otimes^{\mathbb{L}}_{A\otimes A^{\mathrm{op}}} A$. The isomorphism on the level of d-th homology realizes

$$\eta \in \mathrm{Hom}_{A-\mathrm{bimod}}(A^\vee, A[-d]) =: \mathrm{Ext}^{-d}_{A-\mathrm{bimod}}(A^\vee, A) \cong \mathrm{HH}_d(A)$$

as a class in d-th Hochschild homology.

For dg-algebras, one further equips this structure with a class in negative cyclic homology that lifts the Hochschild homology class of the isomorphism. But, as shown in Proposition 5.7 and explained in Definition 5.9 of [60], for ordinary algebras this additional structure exists uniquely.

We have established $P_\bullet$ as a Λ-bimodule resolution of Λ, if Q is connected and contains a cycle. To show Λ is 2-Calabi–Yau, it suffices to show that its dual complex

$$\mathrm{RHom}_{\Lambda-\mathrm{bimod}}(\Lambda, \Lambda\otimes\Lambda) := \mathrm{Hom}_{\Lambda-\mathrm{bimod}}(P_\bullet, \Lambda\otimes\Lambda) =: P_\bullet^\vee$$

is quasiisomorphic to $\Lambda[-2]$.

Define $\eta_v^\vee \in \mathrm{Hom}_{\Lambda-\mathrm{bimod}}(P_0, \Lambda\otimes\Lambda)$ and $\eta_a^\vee \in \mathrm{Hom}_{\Lambda-\mathrm{bimod}}(P_1, \Lambda\otimes\Lambda)$ by,

$$\eta_v^\vee(\eta_w) := \begin{cases} e_v \otimes e_v & \text{if } v = w, \\ 0 & \text{otherwise }, \end{cases} \qquad \eta_a^\vee(\eta_b) := \begin{cases} e_{t(a)} \otimes e_{h(a)} & \text{if } b = a^*, \\ 0 & \text{otherwise }. \end{cases}$$

These are generators of $P_0^\vee$ and $P_1^\vee$ respectively and give isomorphisms,

$$P_0^\vee \cong \Lambda \otimes_{kQ_0} kQ_0 \otimes_{kQ_0} \Lambda = \langle \eta_v^\vee \rangle, \qquad P_1^\vee \cong \Lambda \otimes_{kQ_0} k\overline{Q}_1 \otimes_{kQ_0} \Lambda = \langle \eta_a^\vee \rangle.$$

Rather than directly study the dual complex $P_\bullet^\vee$, we modify the formulas for $\alpha^\vee$ and $\beta^\vee$ using the map θ, in a way that doesn't affect the homology of the complex. Namely, after choosing generators $\{\xi_v\}$ for $P_0^\vee$ and $\{\xi_a\}$ for $P_1^\vee$, defined below, one can expand

$$\alpha^\vee(\xi_a) = \sum_{v\in Q_0} a'_v \xi_v a''_v, \qquad \beta^\vee(\xi_v) = \sum_{a\in \overline{Q}_1} b'_a \xi_a b''_a,$$

for some $a'_v, a''_v, b'_a, b''_a \in \Lambda$ and then define

$$\alpha_\theta^\vee(\xi_a) := \sum_{v \in Q_0} \theta(a'_v)\xi_v\theta(a''_v), \quad \beta_\theta^\vee(\xi_v) := \sum_{a \in \overline{Q}_1} \theta(b'_a)\xi_a\theta(b''_a).$$

It suffices to show that

$$(P_\bullet^\vee)_\theta := P_0^\vee \xrightarrow{-\beta_\theta^\vee} P_1^\vee \xrightarrow{\alpha_\theta^\vee} P_0^\vee$$

is quasiisomorphic to $\Lambda[-2]$.

We prove this by establishing an isomorphism of Λ-bimodule complexes $\varphi_\bullet : P_\bullet[2] \to (P_\bullet^\vee)_\theta$ following Crawley-Boevey and Shaw, so

$$(P_\bullet^\vee)_\theta \xrightarrow{\varphi_\bullet^{-1}} P_\bullet[2] \xrightarrow{\psi_\bullet[2]} Q_\bullet[2] \xrightarrow{\text{quasiiso}} \Lambda[2].$$

Theorem 3.17. *The following diagram commutes*:

$$\begin{tikzcd}
P_0^\vee \arrow[r, "-\beta_\theta^\vee"] & P_1^\vee \arrow[r, "\alpha_\theta^\vee"] & P_0^\vee \arrow[r, "\gamma\circ\varphi_0^{-1}"] & \Lambda \\
P_0 \arrow[u, "\varphi_0", "\cong"'] \arrow[r, "\alpha"] \arrow[ur, phantom, "\text{(II)}"] & P_1 \arrow[u, "\varphi_1", "\cong"'] \arrow[r, "\beta"] \arrow[ur, phantom, "\text{(I)}"] & P_0 \arrow[u, "\varphi_0", "\cong"'] \arrow[r, "\gamma"] & \Lambda \arrow[u, "\mathrm{id}", "="']
\end{tikzcd}$$

Where the vertical maps are Λ-bimodule isomorphisms defined on generators by,

$$\varphi_0(\eta_v) := \xi_v := q\eta_v^\vee, \quad \varphi_1(\eta_a) := \xi_{a^*} := \begin{cases} l_a\eta_{a^*}^\vee l_{a^*}^{-1} & \textit{if } a \in Q_1^{\mathrm{op}}, \\ -r_{a^*}^{-1}\eta_{a^*}^\vee r_a & \textit{if } a \in Q_1. \end{cases}$$

Note that φ_1 is an invertible map since r_a and l_a are invertible elements of Λ for all $a \in \overline{Q}_1$. The commuting of (I) becomes clear once we compute the maps $\alpha^\vee$, the content of the next lemma.

Lemma 3.18 [25, Lemma 3.2].

$$\alpha^\vee(\eta_a^\vee) = \begin{cases} a^* r_a \eta_{h(a)}^\vee l_a - g_{a^*}^{-1} r_{a^*}\eta_{t(a)}^\vee l_{a^*} g_{a^*}^{-1} a^* & \textit{if } a \in Q_1, \\ r_{a^*}\eta_{t(a)}^\vee l_{a^*} a^* - a^* g_a^{-1} r_a \eta_{h(a)}^\vee l_a g_a^{-1} & \textit{if } a \in Q_1^{\mathrm{op}}. \end{cases}$$

$$\alpha^\vee(\xi_a) = \theta(a^*)\xi_{t(a^*)} - \xi_{h(a^*)}\theta(a^*).$$

$$\alpha_\theta^\vee(\xi_{a^*}) = a\xi_{t(a)} - \xi_{h(a)}a.$$

$$\beta = \varphi_0^{-1} \circ \alpha_\theta^\vee \circ \varphi_1.$$

So square (I) *in Theorem 3.17 commutes.*

Proof. The first two equalities are shown directly in [25] and the last two are clear from the definitions together with Proposition 2.6. □

Proof of Theorem 3.17. By Lemma 3.18, it suffices to show that (II) commutes. While one can similarly compute $\beta^\vee$ directly, such a calculation is unnecessary as the commuting of (II) follows from that of (I).

Indeed, dualizing and applying $(-)_\theta$ to the maps in (I), produces a still commuting diagram:

$$\begin{tikzcd} P_1 \arrow[d, "(\varphi_1)^\vee_\theta"'] & P_0 \arrow[l, "(\alpha^\vee_\theta)^\vee_\theta"'] \arrow[d, "(\varphi_0)^\vee_\theta"] \arrow[dl, phantom, "(\mathrm{I})^\vee_\theta"] \\ P_1^\vee & P_0^\vee \arrow[l, "\beta^\vee_\theta"] \end{tikzcd} = \begin{tikzcd} P_1 \arrow[d, "-\varphi_1"'] & P_0 \arrow[l, "\alpha"'] \arrow[d, "\varphi_0"] \arrow[dl, phantom, "(\mathrm{I})^\vee_\theta"] \\ P_1^\vee & P_0^\vee \arrow[l, "\beta^\vee_\theta"] \end{tikzcd}$$

which shows $\varphi_1 \circ \alpha = -\beta^\vee_\theta \circ \varphi_0$, i.e., (II) commutes.

The equality of maps $(\alpha^\vee_\theta)^\vee_\theta = \alpha$ follows from Proposition 2.6, and $(\varphi_0)^\vee_\theta = \varphi_0^\vee = \varphi_0$ follows from the definitions. For $(\varphi_1)^\vee_\theta = -\varphi_1$, observe that it suffices to show $(\varphi_1)_\theta = -(\varphi_1)^\vee$ and indeed,

$$\begin{aligned}(\varphi_1)_\theta(\eta_{a^*}) = (\xi_a)_\theta &= \begin{cases} \theta(l_{a^*})\eta_a^\vee\theta(l_a^{-1}) & \text{if } \epsilon(a)=1, \\ -\theta(r_{a^*}^{-1})\eta_a^\vee\theta(r_a) & \text{if } \epsilon(a)=1, \end{cases} \\ &= \begin{cases} r_{a^*}\eta_a^\vee r_a^{-1} & \text{if } \epsilon(a)=1, \\ -l_{a^*}^{-1}\eta_a^\vee l_a & \text{if } \epsilon(a)=-1, \end{cases} = -(\varphi_1)^\vee(\eta_{a^*}). \end{aligned} \qquad \square$$

So without conditions on the quiver, we have established:

Corollary 3.19. *If $P_\bullet \to \Lambda$ is exact then $(P_\bullet^\vee)_\theta \to \Lambda[-2]$ is exact and $P_\bullet^\vee \to \Lambda[-2]$ is exact.*

Therefore, the 2-Calabi–Yau property for Λ follows from the a priori weaker Hochschild dimension two property. In the previous subsection, we showed that Λ has Hochschild dimension two for Q connected and containing a cycle.

Corollary 3.20. *If Q is connected and contains a cycle then $\Lambda^q(Q)$ is 2-Calabi–Yau.*

4. Formality of dg multiplicative preprojective algebras

In this section we show that if Q satisfies the strong free product property, then the dg multiplicative preprojective algebra is formal. In particular this proves Conjecture 1.3 in the case Q is connected and contains a cycle. Moreover, it reduces Conjecture 1.3 to the remaining extended Dynkin cases and Conjecture 1.5.

If one views the dg multiplicative preprojective algebra as the central object of study, as in [29; 30], then we are showing one can formally replace it by the non-dg version.

We begin with an elementary lemma. It is not strictly required, but it demonstrates more transparently the construction we will use.

Lemma 4.1. *Let K be a commutative ring. Let A be the dg-algebra defined as a graded algebra to be $K[r]*K[s]$ with $|r|=0$ and $|s|=-1$, product given by concatenation of words, and differential extended as a derivation from the generators $d(s)=r$ and $d(r)=0$. Then A is quasiisomorphic to its cohomology $H^*(A)=K$ concentrated in degree zero. In fact, the identity map is chain homotopic to the augmentation map $A\to K$.*

Proof. Let $h : A \to A[-1]$ be the homotopy with the property $h(rf) = sf$ and $h(sf) = 0$ for all $f \in A$, and $h(K) = 0$. Then $h \circ d + d \circ h - 1_A$ is the projection with kernel K to the augmentation ideal of A. Therefore, it defines a contracting homotopy from A to K. $\square$

In other words, the lemma is observing that A, as the tensor algebra on an acyclic complex $Kr \oplus Ks$, is itself quasiisomorphic to K.

Lemma 4.2. *The dg-algebra A given by*

$$\Lambda^q(Q) *_{kQ_0} kQ_0[r, (r+q)^{-1}] *_{kQ_0} kQ_0[s], \quad \text{with } |r| = 0 \text{ and } |s| = -1$$

and with differential determined by $d(s) = r$ is quasiisomorphic to $\Lambda^q(Q)$ concentrated in degree zero.

Proof. Extending the preceding construction, define a homotopy $h : A \to A[-1]$ by

$$h(frg) = fsg, \quad h(fsg) = 0, \quad h(f(r+q)^{-1}g) = q^{-1}h(fg) - q^{-1}fs(r+q)^{-1}g$$

for $f \in \Lambda^q(Q)$ and $g \in A$. The definition of $h(f(r+q)^{-1}g)$ is chosen to match the formula for $h(frg)$ in the r-adic completion. There is an augmentation $A \twoheadrightarrow \Lambda^q(Q)$ with kernel $(r, s, r' := (r+q)^{-1} - q^{-1})$. Notice that $h \circ d + d \circ h$ is a homotopy from the identity on A to the augmentation $\Lambda^q(Q)$, as it annihilates $\Lambda^q(Q)$ and is the identity on s, r, and r'. $\square$

Definition 4.3. The *dg multiplicative preprojective algebra* is a dg-algebra over kQ_0 defined as a graded algebra by

$$\Lambda^{dg,q}(Q) := L_Q *_{kQ_0} kQ_0[s], \quad |s| = -1, |\alpha| = 0 \text{ for } \alpha \in L_Q.$$

The differential, d, is defined by $d(s) = \rho - q$, $d(L_Q) \equiv 0$, and extended as a kQ_0-linear derivation to $L_Q *_{kQ_0} kQ_0[s]$.

Proposition 4.4. *If $\Lambda^q(Q)$ satisfies the strong free product property,*[5] *then*

$$H_*(\Lambda^{dg,q}(Q)) = H_0(\Lambda^{dg,q}(Q)) \cong \Lambda^q(Q)$$

so in particular $\Lambda^{dg,q}(Q)$ is formal.

By Theorem 3.7, the proposition holds in particular if Q contains a cycle.

Remark 4.5. Note that for the ordinary preprojective algebra, $\Pi(Q)$, the Ginzburg dg-algebra has homology concentrated in degree zero for any non-Dynkin quiver: $\Pi(Q)$ has a length two bimodule resolution (see [50; 18] for the characteristic zero case, and [31] in general) which [1, Theorems 2.6 and 2.9] shows is equivalent for *graded* connected algebras, and [32] observes this extends to the quiver case.

Proof. The strong free product property yields an isomorphism of graded vector spaces,

$$L_Q \cong \Lambda^q(Q) *_{kQ_0} kQ_0[r, (r+q)^{-1}].$$

[5] Meaning $(L_Q, r, \sigma, kQ_0[t, (t+q)^{-1}])$ satisfies the strong free product property for some choice of σ.

Hence, as complexes,

$$\Lambda^{dg,q}(Q) \cong L_Q *_{kQ_0} kQ_0[s] \cong \Lambda^q(Q) *_{kQ_0} kQ_0[r,(r+q)^{-1}] *_{kQ_0} kQ_0[s],$$

which by Lemma 4.2 is quasiisomorphic to $\Lambda^q(Q)$, concentrated in degree zero. It follows that

$$\Lambda^{dg,q}(Q) \cong H_*(\Lambda^{dg,q}(Q)) \cong H_0(\Lambda^{dg,q}(Q)) \cong \Lambda^q(Q)$$

as dg-algebras. □

Remark 4.6. In the presence of Conjecture 1.1, formality of $\Lambda^{dg,q}(Q)$ implies $\Lambda^{dg,q}(Q)$ is 2-Calabi–Yau. Hence by Theorem 1.2, we have shown that $\Lambda^{dg,q}(Q)$ is 2-Calabi–Yau, when Q is connected and contains a cycle. One may be able to adapt the techniques in Section 3 to prove that $\Lambda^{dq,q}(Q)$ is 2-Calabi–Yau, in general. In more detail, writing $\Lambda^{dg} := \Lambda^{dg,q}(Q)$, the role of the $\Lambda^q(Q)$-bimodule resolution, $P_\bullet$, should now be played by the Λ^{dg}-dg-bimodule given by the total complex of

$$\begin{tikzcd} \Lambda^{dg} \otimes_{kQ_0} kQ_0 \cdot s \otimes_{kQ_0} \Lambda^{dg} \arrow[r, "\alpha^{dg}"] \arrow[rr, bend right, "\beta_1^{dg}"'] & \Lambda^{dg} \otimes_{kQ_0} kQ_1 \otimes_{kQ_0} \Lambda^{dg} \arrow[r, "\beta_0^{dg}"] & \Lambda^{dg} \otimes_{kQ_0} \Lambda^{dg}, \end{tikzcd}$$

where $\beta_1^{dg}(a\otimes s\otimes b) = as\otimes b - a\otimes sb$ and α^{dg} (respectively β_0^{dg}) has the same formula as α (respectively β).

Remark 4.7. We are grateful to Georgios Dimitroglou Rizell, who pointed out that our definition differs from that arising in symplectic geometry. Indeed in the derived multiplicative preprojective algebra, $\mathscr{L}_\Gamma$, [30, page 779] define additional variables z_a, ζ_a with z_a invertible and $d(\zeta_a) = z_a - (1 + a^*a)$, and hence $(1 + a^*a)$ is invertible only after taking homology. In contrast, we invert $(1 + a^*a)$ on the chain level in $\Lambda^{dg,q}(Q)$. However, for our main result, Proposition 4.4, this distinction is irrelevant, as we now explain.

We claim that the dg algebra map $\alpha : \mathscr{L}_\Gamma \to \Lambda^{dg,q}(Q)$, given by $\alpha(z_a) = (1 + a^*a)$, $\alpha(\zeta_a) = 0$, and taking arrows to arrows, is a quasiisomorphism. To see this, note that $\mathscr{L}_\Gamma$ can be viewed as a bigraded dg algebra with two differentials: set $|\zeta_a| = (-1, 0)$ and $|s| = (0, -1)$, with horizontal differential $d_H(\zeta_a) = z_a - (1 + a^*a)$, $d_H(s) = 0$, and vertical differential $d_V(\zeta_a) = 0, d_V(s) = \rho - q$. We will show in the next paragraph that the map α induces an isomorphism on horizontal cohomology, that is, $\alpha : (\mathscr{L}_\Gamma, d_H) \to (\Lambda^{dg,q}(Q), 0)$ is a quasiisomorphism. Therefore, α is a morphism of bicomplexes (placing the target in horizontal degree zero), that induces an isomorphism on the first page of the associated spectral sequences. These sequences collapse on the second page. They collapse to the cohomology, since both sequences were third-quadrant (cohomologically) and hence convergent. This proves the claim.

It remains to show that α is an isomorphism on horizontal cohomology. More generally, let A be a graded path algebra on the quiver Q (arrows can be assigned any degrees), and let $S \subseteq A$ be a subset of homogeneous elements; in the case above we have $A := k\overline{Q} *_{kQ_0} kQ_0[s]$ and $S := \{1+a^*a\}_{a\in\overline{Q}_1}$. We wish to compare two localizations. The first is the naive one, $A[f^{-1}]_{f\in S}$. The second is given by replacing A by the quasiisomorphic algebra $\widetilde{A} := A\langle z_f, \zeta_f\rangle_{f\in S}$, with differential $d(\zeta_f) = z_f - f,\ d(z_f) = 0,\ d(A) \equiv 0$. We then consider $\widetilde{A}[z_f^{-1}]_{f\in S}$. To compare these we use the technique of *derived localization*, following [16]. Since A is hereditary with zero differential, by [16, Corollary 4.20, Theorem 5.1], its derived localization by

S is $A[S^{-1}]$ (i.e., it is underived). On the other hand, $\widetilde{A}$ is cofibrant in the category of dg algebras equipped with a morphism from $kQ_0\langle z_f\rangle_{f\in S}$, as it is given by cell attachment (although with nonzero differential). So $\widetilde{A}[z_f^{-1}] = \widetilde{A} *_{kQ_0\langle z_f\rangle} kQ_0\langle z_f, z_f^{-1}\rangle$ is also its derived localization. Now the quasiisomorphism $\widetilde{A} \to A$ is compatible with the morphisms from the path algebra $kQ_0\langle z_f\rangle_{f\in S}$, sending z_f to z_f and to f, respectively. Thus the map $\widetilde{A}[z_f^{-1}]_{f\in S} \to A[f^{-1}]_{f\in S}$ is a quasiisomorphism of derived localizations of A at S.

Note that by combining the two preceding paragraphs, in general, the quasiisomorphism

$$A\langle z_f, z_f^{-1}, \zeta_f\rangle_{f\in S} \to A[f^{-1}]_{f\in S}$$

induces a quasiisomorphism

$$A\langle z_f, z_f^{-1}, \zeta_f, s_i\rangle \to A[f^{-1}]\langle s_i\rangle$$

for any additional arrows s_i and differential $d(s_i)$ compatible with the morphism (only assuming that A is a graded path algebra with S a collection of homogeneous elements). The same is true replacing $A\langle z_f, z_f^{-1}, \zeta_f\rangle$ by any other model of the derived localization of A at S.

Remark 4.8. The dg multiplicative preprojective algebra is called the Legendrian cohomology dg algebra in [29, 3.2] where they establish that it is a multiplicative analog of Ginzburg's dg algebra for a quiver with zero potential defined in [36, 1.4]. It is called a capped Chekanov–Eliashberg algebra in [53, Section 3.2] where they independently prove formality in the case Q is the Jordan quiver and $q = 1$ in [53, Theorem 3.13].

5. Local structure of multiplicative quiver varieties and moduli spaces attached to 2-Calabi–Yau algebras

In this section, we will assume that k is an algebraically closed field of characteristic zero.

We will use our main result to prove, as anticipated in [55, Section 7.5], that multiplicative quiver varieties are étale-locally (or formally locally) isomorphic to ordinary quiver varieties. Our proof uses (a generalization of) a result of Bocklandt, Galluzzi, and Vaccarino in [14] for 2-Calabi–Yau algebras. While our main result is only proved for quivers *with* cycles, we are able to prove this result for *all* quivers. The key idea is to embed any quiver into one containing a new vertex with a cycle, and put the zero vector space at this new vertex. This identifies every multiplicative quiver variety with one for a quiver containing a cycle.

We recall the definition of multiplicative quiver varieties [25; 61; 55], beginning with King's notion of (semi)stability. First, by an algebra over kQ_0, we mean a k-algebra which contains kQ_0 as a subalgebra. Given a module M over such an algebra A, its dimension vector is $\alpha \in \mathbb{N}^{Q_0}$ given by $\alpha_i = \dim e_i M$, $i \in Q_0$. Given a kQ_0-module V, let $\operatorname{Rep}(A, V) := \operatorname{Hom}_{kQ_0\text{-alg}}(A, \operatorname{End}_k(V))$ be the set of A-module structures on V. Let $\operatorname{Rep}_\alpha(A) := \operatorname{Rep}(A, V)$ for $V := \bigoplus_{i\in Q_0} k^{\alpha_i}$, called the representation space of dimension α.

Definition 5.1 [45]. Let Q be a finite quiver. Let A be an algebra over kQ_0, $\theta \in \mathbb{Z}^{Q_0}$ a parameter, $\alpha \in \mathbb{N}^{Q_0}$ a dimension vector. Assume that $\theta \cdot \alpha = 0$. Then an A-module M of dimension vector α is said to be

θ-*semistable* if, for every submodule $N := \{N_i\}_{i \in Q_0}$, with dimension vector $\beta \in \mathbb{N}^{Q_0}$, we have $\beta \cdot \theta \leq 0$. Furthermore M is θ-*stable* if $\beta \cdot \theta < 0$ for all nonzero, proper submodules. Let $Rep^{\theta\text{-ss}}(A, V) \subseteq \mathrm{Rep}(A, V)$ be the subset of θ-semistable module structures, and denote this by $\mathrm{Rep}_\alpha^{\theta\text{-ss}}(A)$ when $V := \bigoplus_{i \in Q_0} k^{\alpha_i}$.

Definition 5.2 [45]. Let Q, α, and θ be as in the definition above, and let A be an algebra over kQ_0. Then the corresponding (semistable) moduli space is

$$\mathcal{M}_\theta(A, \alpha) := \mathrm{Rep}_\alpha^{\theta\text{-ss}}(A) /\!/ \mathrm{GL}(\alpha). \tag{5-1}$$

In the case $A = \Lambda^q$, this is called a *multiplicative quiver variety*, denoted $\mathcal{M}_{q,\theta}(Q, \alpha)$. In the case $A = \Pi^\lambda$ is a (deformed) preprojective algebra, it is called an ordinary quiver variety, denoted $\mathcal{M}_{\lambda,\theta}^{\mathrm{add}}(Q, \alpha)$.

The main results of this section are the following:

Theorem 5.3. *Let A be a 2-Calabi–Yau algebra over kQ_0, and let ρ be a θ-semistable representation of A of dimension α. Then there exists Q', α' such that the formal neighborhood of ρ of the moduli space $\mathcal{M}_\theta(A, \alpha)$ is isomorphic to the formal neighborhood of $\mathcal{M}_{0,0}^{\mathrm{add}}(Q', \alpha')$ at the zero representation.*

Theorem 5.4. *At every point of a multiplicative quiver variety, a formal neighborhood is isomorphic to the formal neighborhood of zero of an ordinary quiver variety.*

In the case where the quiver contains an oriented cycle, Theorem 5.4 follows immediately from Theorem 5.3 and our main result; in general, we need to enlarge the quiver; see Section 5C. Note that the corresponding result for formal neighborhoods of ordinary quiver varieties is known; see [5, Corollary 3.4].

By Artin's approximation theorem [2, Corollary 2.6], we can replace "formal neighborhoods" in the preceding theorems by étale neighborhoods, since we are in the setting of varieties (by which we always mean of finite type) over a field.

Corollary 5.5. *Let A be a 2-Calabi–Yau algebra over kQ_0. Then, all moduli spaces $\mathcal{M}_\theta(A, \alpha)$ are symplectic singularities. In particular, they are normal and have rational Gorenstein singularities. The same holds for all multiplicative quiver varieties.*

The proofs of these results are given in the final subsection.

5A. ***Generalities on completions of 2-Calabi–Yau algebras.*** To prove Theorem 5.3, we will need the following results about the local structure of n-Calabi–Yau algebras at modules M, adapted from [10].

Definition 5.6. Let A and B be A_∞-algebras. B is *minimal* if $m_1^B = 0$. B is further a *minimal model* for A if there exists an A_∞-quasiisomorphism $B \to A$ lifting the identity. We make the same definitions for L_∞ algebras.

In particular, if B is a minimal model for A then $(B, m_2^B) \cong H^*(A)$ as graded algebras. Kadeishvili showed [40] that every A_∞-algebra has a minimal model.

Theorem 5.7 (minimal model theorem). *Let A be an augmented A_∞-algebra over a semisimple k-algebra S. Then, A admits an augmented A_∞-algebra isomorphism $H^*(A)' \oplus C \to A$, where $H^*(A)'$ is an A_∞-algebra which, as a dg algebra, is the cohomology $H^*(A)$ with zero differential, and C is a contractible complex such that all ≥ 2-ary multiplications involving it are zero.*

Similarly, if $\mathfrak{g}$ is an L_∞-algebra over k, then there is an L_∞-isomorphism $H^*(\mathfrak{g})' \oplus \mathfrak{c} \to \mathfrak{g}$, where the underlying ordinary dg Lie algebra of $H^*(\mathfrak{g})'$ is the cohomology of $\mathfrak{g}$ with zero differential, and $\mathfrak{c}$ is a contractible complex such that all ≥ 2-ary multiplications involving it are zero.

Here, an A_∞-algebra A is *augmented* over S if it is of the form $S \oplus \bar{A}$ where S is a subalgebra and $\bar{A}$ is a strict ideal, i.e., all multiplications with $\bar{A}$ as an input land in $\bar{A}$; moreover, we assume that the only nonzero multiplication between S and $\bar{A}$ are the binary operations (i.e., the S-bimodule structure). An *augmented A_∞-morphism* is an A_∞-morphism which is the identity on S, preserves strictly the augmentation ideals, and all higher A_∞-structure maps vanish when one of the inputs is in S.

Remark 5.8. The map $A \to \bar{A}$ gives an equivalence between augmented A_∞-algebras and nonunital A_∞-algebras in the category of S-bimodules. This makes the statements for A_∞ and L_∞- algebras more symmetric. There are also L_∞ analogs of working over a semisimple algebra; for example, we may work with representations of a reductive group. Given an augmented A_∞-algebra over a matrix algebra, the augmentation ideal has an associated L_∞-algebra which is a representation of the general linear group.

Kadeishvili's approach is direct and explicit: he constructs both the A_∞-structure on $H^*(A)'$ and the A_∞-algebra isomorphism $A \to H^*(A)' \oplus C$. For more conceptual treatments, see e.g., Theorem 5.4 of [41] and Remark 4.18 in [21]. For a sketch in the context of L_∞-algebras see Lemma 4.9 of [46].

Remark 5.9. Note that the minimal model theorem is usually stated in the literature for fields, but it is known that the statement and proof generalizes to the case of semisimple algebras over a field.

Definition 5.10. Let A be an A_∞-algebra. We say A is *formal* if there is an augmented A_∞-isomorphism $H^*(A)' \to H^*(A)$, where $H^*(A)$ has zero ℓ-ary multiplication for $\ell \geq 3$.

Definition 5.11. Given a dg associative algebra A with module M, define the *derived Koszul dual algebra with respect to M* to be $E_M(A) := \mathrm{REnd}_A(M)$.

This is only defined up to quasiisomorphism, but it will not matter to us which model is chosen. Note that if A is a Koszul algebra over S, with S the augmentation module, then up to degree conventions, $E_S(A)$ is the completion of the Koszul dual algebra, $A^!$, with respect to the filtration by powers of the augmentation ideal. In this case, A and $A^!$ have an additional weight grading, and $(A^!)^! \cong A$.

Recall that, if A is an n-Calabi–Yau algebra and M a finite-dimensional module, then there is a trace $\lambda : \mathrm{Ext}^n(M, M) \to k$ such that the composition

$$(-,-) : \mathrm{Ext}^i(M, M) \times \mathrm{Ext}^{n-i}(M, M) \xrightarrow{\circ} \mathrm{Ext}^n(M, M) \xrightarrow{\lambda} k$$

is a graded symmetric perfect pairing [44, Lemma 3.4]. Since it is also graded commutative, this says that $\mathrm{Ext}^\bullet(M, M)$ is a *symmetric dg Frobenius* algebra. In the case that $R := \mathrm{End}_A(M)$ is semisimple, this says that $\mathrm{Ext}^n(M, M) \cong R$ as R-modules. Moreover, if we realize R as endomorphisms of a kQ'-representation (i.e., $R \cong \prod_{i \in Q'_0} \mathrm{End}_k(k^{\alpha'_i})$ for some finite set Q'_0 and dimension vector $\alpha' \in \mathbb{N}^{Q'_0}$) then we can write, for $V_i := k^{\alpha'_i}$,

$$\mathrm{Ext}^m(M, M) \cong \bigoplus_{i,j \in Q'_0} \mathrm{Hom}_k(V_i, V_j)^{c^m_{i,j}},$$

for some $c^m_{i,j} \in \mathbb{N}$.

Moreover, in the case of interest, $n = 2$, we only need to consider $m = 1$. Then the pairing on $\mathrm{Ext}^1(M, M)$ is symplectic. By picking an appropriate symplectic basis on $\mathrm{Ext}^1(M, M)$, we can write

$$\mathrm{Ext}^1(M, M) \cong T^*\biggl(\bigoplus_{a \in Q'_1} \mathrm{Hom}_k(V_{t(a)}, V_{h(a)})\biggr),$$

with the standard symplectic structure on the cotangent bundle, for some set Q'_1 of arrows with vertex set Q'_0 (i.e., extending Q'_0 to a quiver $Q' = (Q'_0, Q'_1)$). It turns out that the symplectic pairing on $\mathrm{Ext}^1(M, M)$, and hence the quiver data (Q', α'), completely determines the dg algebra $\mathrm{REnd}(M)$ up to A_∞-isomorphism.

Continue to assume that $R := \mathrm{End}_A(M)$ is semisimple. In this case, the image, call it S, of the action homomorphism $\rho_M : A \to \mathrm{End}_k(M)$ is also semisimple. We could complete A at M, meaning the completion with respect to the filtration by powers of $\ker \rho_M$. This is not necessarily a quasiisomorphism invariant, however. A better way to take the completion is by a double Koszul duality, as $E_M(E_M A)$, where M is viewed as an $E_M A$-module via the augmentation map $\mathrm{REnd}(M) \to \mathrm{End}_A(M)$. The result is certainly complete, and in certain cases it is indeed the completion of A (e.g., for $A = k[x]$ with $M = k$, one obtains $k[\![x]\!]$; see the proof of the next theorem for more cases).

Since S is semisimple, it is Morita equivalent to a direct sum of copies of k, namely kQ'_0 for Q'_0 the set of isomorphism classes of indecomposable summands of M. Then, we can replace the aforementioned "completion" of A by a completed quiver algebra, by replacing M by M', the direct sum of one copy of each nonisomorphic indecomposable summand of M. Then $E_{M'} E_M A$ is augmented over kQ'_0 and is Morita equivalent to the completion of A at M. More precisely, if $V_i = k^{\alpha'_i}$ as before, so that $\mathrm{End}_A(M) = \bigoplus_i \mathrm{End}_k(V_i)$, then $M' = \bigoplus_i V_i$, viewed as an $E_M A$ module via the augmentation $E_M A \to \mathrm{End}_A(M)$.

Theorem 5.12. *Let A be a 2-Calabi–Yau algebra over kQ_0 and M a finite-dimensional module such that $\mathrm{End}_A(M)$ is semisimple. Then $E_M A$ is formal.*

Proof. We deduce this result from [10, Theorem 11.2.1, Corollary 9.3] as follows. The latter gives a formal local characterization of n-Calabi–Yau algebras (more generally for dg exact Calabi–Yau algebras concentrated in nonpositive degrees) for $n \geq 3$. The proof there is valid also in the case $n = 2$, where it yields that the following are equivalent for a *complete augmented* algebra A over kQ_0:

(a) A is a 2-Calabi–Yau algebra.

(b) $E_{kQ_0} A$ is formal and has a nondegenerate trace of degree -2.

In this case, A itself is isomorphic to $E_{kQ_0} \mathrm{REnd}_A(kQ_0)$.

Now, let A be an *ordinary* 2-Calabi–Yau algebra and M a finite-dimensional module with $\mathrm{End}_A(M)$ semisimple. Then $E_M A = \mathrm{REnd}_A(M)$ has a nondegenerate trace of degree -2. We can now apply the aforementioned result to the dg algebra $A' := E_{M'} E_M A$, which is complete and augmented over kQ_0. By construction, $E_{kQ_0} A' \cong E_M A$ (formally, this is because $B := E_M A$ is its own double Koszul dual, as it is augmented, finite-dimensional, and concentrated in positive degrees [10, Proposition A.5.4]). Thus $E_M A$ is formal. □

Remark 5.13. In fact, the proof shows that the following statements are equivalent for an ordinary algebra A and module M with $\mathrm{End}_A(M)$ semisimple:

(a) $E_M A$ is formal and has a nondegenerate trace of degree -2.

(a') $E_M A$ has a nondegenerate trace of degree -2.

(b) The double dual $E_M E_M A$ is 2-Calabi–Yau.

Since the double Koszul dual is Morita equivalent to the completed dg quiver algebra $E_{M'} E_M A$, these statements are also equivalent to this latter algebra being 2-Calabi–Yau.

Remark 5.14. As stated, [10] actually deals with the case of Calabi–Yau dimension $n \geq 3$. In this case, one can also state a version of the theorem: instead of yielding that $E_M A$ is formal, one can only kill the higher A_∞-structures of $\mathrm{Ext}^\bullet(M, M)$ which land in top degree n. The main result of [10] can then be stated as saying that the remaining structure of $E_M A$ is governed by a single cyclically symmetric element called the superpotential.

Remark 5.15. Since submission of this article, Ben Davison [27, Theorem 1.2] has proved a more general formality result for 2-Calabi–Yau categories. As he explains, the reason for the formality is quite simple: the Koszul dual $E_M A$ can be taken to be a cyclic A_∞-algebra which is augmented over $\mathrm{End}_A(M)$. This means that, for $s \in \mathrm{End}_A(M)$, $\langle m_n(a_1, \ldots, a_n), s\rangle = \langle a_1, m_n(a_2, \ldots, a_n, s)\rangle = 0$. This shows that all A_∞-structures landing in top degree (here, degree two) vanish.

Theorem 5.12 implies that the formal moduli problem, based at M, of modules over a 2-Calabi–Yau algebra A is equivalent to that of a dg preprojective algebra. Indeed, using the bar construction, one can realize $E_{M'} \mathrm{Ext}^\bullet(M, M)$ as the completed dg preprojective algebra of the quiver Q', for M' as above. Note that the module M' is a zero representation of this preprojective algebra: all arrows act by zero.

5B. *The representation and moduli schemes.* We are interested rather in the ordinary representation moduli scheme of A, possibly using a nonzero stability condition. In this case, Theorem 5.12 will imply that, when A is 2-Calabi–Yau the formal neighborhood of this scheme at M will be isomorphic to that of the corresponding quiver variety.

To prove this, we use the following generalization of [14, Theorem 6.3], describing the general structure of these schemes whenever A is an algebra with $\mathrm{End}_A(M)$ semisimple. Given (formal) schemes X, Y with actions by a group G, write $X \times^G Y := (X \times Y)/\!/G$ using the diagonal action.

Theorem 5.16. *Let A be an algebra over kQ_0, and $\alpha \in \mathbb{N}^{Q_0}$ a dimension vector. Suppose that $M \in \mathrm{Rep}_\alpha(A)$ is a representation whose GL_α-orbit is closed in some GL_α-stable affine open subset U of $\mathrm{Rep}_\alpha(A)$. Let $A' := H^0 E_{M'} E_M A$ (for M' as above). Then*:

(1) $\mathrm{End}_A(M)$ *is semisimple.*

(2) *There is a GL_α-equivariant isomorphism $\widehat{\mathrm{Rep}_\alpha(A)}_{\mathrm{GL}_\alpha \cdot M} \cong \widehat{\mathrm{Rep}_{\alpha'}(A')}_{M'} \times^{\mathrm{GL}_{\alpha'}} \mathrm{GL}_\alpha$.*

(3) *The formal neighborhood of $[M]$ in $U /\!/ \mathrm{GL}_\alpha$ is isomorphic to the formal neighborhood of $[M']$ in $\mathrm{Rep}_{\alpha'}(A') /\!/ \mathrm{GL}_{\alpha'}$.*

Before we begin the proof of the theorem, as in [14, Section 6], we need to recall some of the formalism of Maurer–Cartan loci. Let $\mathfrak{g}$ be a dg associative or Lie algebra. Then the Maurer–Cartan locus is

$$\mathrm{MC}(\mathfrak{g}) := \left\{ a \in \mathfrak{g}^1 \mid da + \tfrac{1}{2}[a,a] = 0 \right\}.$$

Let $\widehat{\mathrm{MC}}(\mathfrak{g})$ be its formal completion at $0 \in \mathfrak{g}^1$. More generally, given an A_∞ or L_∞-algebra, we can define

$$\widehat{\mathrm{MC}}(\mathfrak{g}) := Z\big(a \mapsto da + \tfrac{1}{2!}[a,a] + \tfrac{1}{3!}[a,a,a] + \cdots\big) \subseteq \widehat{\mathfrak{g}^1},$$

the formal subscheme of $\widehat{\mathfrak{g}^1}$ cut out by the Maurer–Cartan equation (now a power series).

The algebra of functions on this formal scheme is the zeroth Lie algebra cohomology of $\mathfrak{g}^{>0}$, $H^0\,\mathrm{CE}(\mathfrak{g}^{>0}) = H^0(\mathrm{CE}(\mathfrak{g})/((\mathfrak{g}^0)^*))$. Here, the Chevalley–Eilenberg cochain complex is the completed dg symmetric algebra, $\mathrm{CE}(\mathfrak{g}) = (\widehat{\mathrm{Sym}}(\mathfrak{g}^*[-1]), d_{\mathrm{CE}})$, equipped with the Chevalley–Eilenberg differential. For algebras $\mathfrak{g}$ concentrated in positive degrees, this does not depend on A_∞ or L_∞-quasiisomorphisms.

The Maurer–Cartan formal scheme has an infinitesimal action by the Lie algebra $\mathfrak{g}^0$, via gauge equivalence. The gauge action of an element $\xi \in \mathfrak{g}^0$ is recorded by applying the differential and contracting with ξ. The categorical quotient of the Maurer–Cartan formal scheme by this action is defined, on the level of functions, by passing to $\mathfrak{g}^0$-invariant functions. The algebra of functions here is $H^0\,\mathrm{CE}(\mathfrak{g}^{\geq 0})/((H^0\mathfrak{g})^*[-1])$. For algebras $\mathfrak{g}$ concentrated in nonnegative degrees, this quotient does not depend on A_∞ or L_∞-quasiisomorphisms.

Now let A be a kQ_0-algebra and M a module. Consider the nonnegatively graded dg associative algebra of kQ_0-bilinear Hochschild cochains,

$$\mathfrak{g} := \mathrm{HC}_{kQ_0}(A, \mathrm{End}_k(M)) := \bigoplus_{i \geq 0} \mathrm{Hom}_{kQ_0-\mathrm{bimod}}(A^{\otimes_{kQ_0} i}, M),$$

equipped with the usual differential and cup product structure. (We remark that this is well known to be quasiisomorphic to the usual algebra of k-linear Hochschild cochains, since kQ_0 is semisimple.)

Given $a \in \mathfrak{g}^1 = \mathrm{Hom}_{kQ_0-\mathrm{bimod}}(A, \mathrm{End}_k(M))$, we can consider the deformation $(\rho_M + a) : A \to \mathrm{End}_k(M)$, with ρ_M the original module structure. The condition for $\rho_M + a$ to be a module structure is the Maurer–Cartan equation, $da + a^2 = 0$. Hence $\mathrm{MC}(\mathfrak{g}) = \mathrm{Rep}(A, M)$, with zero corresponding to M. Thus $\widehat{\mathrm{MC}}(\mathfrak{g}) = \widehat{\mathrm{Rep}(A, M)}_M$.

Proof of Theorem 5.16. First, to show $\mathrm{End}_A(M)$ is semisimple, we will use Matsushima's criterion [51]:

> If G is a reductive group acting on an affine variety X, then the stabilizer of a point in a closed orbit is reductive.

In the case at hand, $G = \mathrm{GL}_\alpha$ is acting on $X = U$, so the stabilizer $G_M \cong \mathrm{Aut}(M)$ is reductive. So any element $x \in N(\mathrm{End}_A(M))$, the nilradical of $\mathrm{End}_A(M)$, gives rise to an element $1 + x$ in the unipotent radical, which is $\{1\}$ as $\mathrm{Aut}(M)$ is reductive. So $N(\mathrm{End}_A(M)) = 0$, which implies, as $\mathrm{End}_A(M)$ is finite-dimensional, that the Jacobson radical $J(\mathrm{End}_A(M)) = 0$. We conclude that $\mathrm{End}_A(M)$ is semisimple, being Artinian with vanishing Jacobson radical.

To obtain (2), let $\mathfrak{g}$ be the dg algebra of kQ_0-bilinear cochains, $\mathrm{HC}_{kQ_0}(A, \mathrm{End}_k(M))$ as before the proof. As we explained, the completed Maurer–Cartan subscheme $\widehat{\mathrm{MC}}(\mathfrak{g}) = \widehat{\mathrm{MC}}(\mathfrak{g}^{>0})$ is the same as for the minimal model $H^*(\mathfrak{g}^{>0})$ of $\mathfrak{g}^{>0}$ (as these are concentrated in positive degrees). Next, let $\mathfrak{h} := Z^0(\mathfrak{g}) \cong \mathrm{End}_A(M)$, the zero-cycles of $\mathfrak{g}$, which is a reductive Lie subalgebra of $\mathfrak{g}^0$. Its action integrates to the reductive group $H = \mathrm{Aut}_A(M) \cong \mathrm{GL}_{\alpha'}$, so it acts semisimply. Now, we apply Lemma 5.19 below, to obtain a quasiisomorphic L_∞-algebra (in fact A_∞-algebra, see Remark 5.20) $\mathfrak{g}' := \mathfrak{g}^0 \oplus Z^1(\mathfrak{g}) \oplus H^{>1}(\mathfrak{g})$.

Define $\tilde{H}^1(\mathfrak{g})$ to be an H-invariant complement to the one-coboundaries $B^1(\mathfrak{g})$ in $Z^1(\mathfrak{g})$. The L_∞-structure maps $\tilde{H}^1(\mathfrak{g})^m \to H^2(\mathfrak{g})$ in $\mathfrak{g}'$ are the same as the ones on any minimal model $H^*(\mathfrak{g}')$ induced by transfer (as in the proof of Lemma 5.19 below). This gives an embedding of the Maurer–Cartan locus $\widehat{\mathrm{MC}}(H^*\mathfrak{g}) = \widehat{\mathrm{MC}}(H^{>1}\mathfrak{g})$ of the cohomology into the Maurer–Cartan locus of $\mathfrak{g}'$. By Lemma 5.19, this inclusion is compatible with the H-action, which is linear. It is also a formal slice to the infinitesimal $\mathfrak{g}^0$ action on $\widehat{\mathrm{MC}}(\mathfrak{g}')$: the tangent space to this action is $B^1(\mathfrak{g})$, whereas the tangent space to $\widehat{\mathrm{MC}}(\mathfrak{g}')$ is $Z^1(\mathfrak{g})$.

Next let us turn from the formal neighborhood of M in $\mathrm{Rep}_\alpha(A)$ to a formal neighborhood of its GL_α orbit. Luna's slice theorem [48] implies that there is a $(\mathrm{GL}_\alpha)_M = \mathrm{Aut}_A(M) = H$-stable affine subset $V \subseteq U$, such that the action map $\phi : \mathrm{GL}_\alpha \times^H V \to \mathrm{Rep}_\alpha(A)$ induces a GL_α-equivariant isomorphism onto an étale neighborhood of the orbit $\mathrm{GL}_\alpha \cdot M$. Using the fact that $\mathrm{Aut}(M)$ is connected, we have the following identifications. For ease of reading let $\mathrm{FN}(X, Y) := \widehat{Y}_X$ denote the formal neighborhood of X in Y:

$$\mathrm{FN}(\mathrm{GL}_\alpha \cdot M, U) \cong \mathrm{FN}(\mathrm{GL}_\alpha \times^H \{M\}, \mathrm{GL}_\alpha \times^H V) \cong \mathrm{GL}_\alpha \times^H \mathrm{FN}(\{M\}, V).$$

Finally, we showed above that the slice V can be taken to be the Maurer–Cartan locus of $H^{>0}(\mathfrak{g})$. This identifies with $\widehat{\mathrm{Rep}_{\alpha'}(A')_{M'}}$, since the latter is isomorphic to the Maurer–Cartan locus of the minimal model $H^*(\mathfrak{g})$. (Explicitly, since the augmentation ideal of A' acts by zero on M', $\mathrm{End}_{kQ_0}(M') = \mathrm{End}_{A'}(M')$ is the degree zero part of the Hochschild cochain complex of M' with zero differential, so $\mathrm{Ext}_{A'}^{>0}(M', M')$ is quasiisomorphic to $H^{>0}(\mathfrak{g})$.) This completes the proof of (2), as H is identified with $\mathrm{GL}_{\alpha'}$ by definition of α'.

It remains to deduce (3) from (2). First note that, since GL_α is reductive and the orbit $\mathrm{GL}_\alpha \cdot M \subseteq U$ is closed, by Hilbert's theorem, the ideal of $[M]$ in $\mathcal{O}(U /\!/ \mathrm{GL}_\alpha) = \mathcal{O}(U)^{\mathrm{GL}_\alpha}$ is the set of GL_α-invariant functions in the ideal of $\mathrm{GL}_\alpha \cdot M$ in $\mathcal{O}(U)$. Therefore, functions on $\mathrm{FN}([M], U/\!/\, \mathrm{GL}_\alpha)$ are the GL_α-invariant functions in the completion of $\mathcal{O}(U)$ at the fiber $F \subseteq U$ of the projection $U \to U/\!/\, \mathrm{GL}_\alpha$:

$$\mathrm{FN}([M], U/\!/\, \mathrm{GL}_\alpha) \cong \mathrm{FN}(F, U)/\!/\, \mathrm{GL}_\alpha\, .$$

Note that $\mathrm{GL}_\alpha \cdot M \subseteq F$, so we get a further map $\mathrm{FN}(F, U)/\!/\, \mathrm{GL}_\alpha \to \mathrm{FN}(\mathrm{GL}_\alpha \cdot M, U)/\!/\, \mathrm{GL}_\alpha$. We claim that this is an isomorphism. Indeed, let $I_{\mathrm{GL}_\alpha \cdot M} \supseteq I_F$ be the ideals. Then we are considering two different completions of $\mathcal{O}(U/\!/\, \mathrm{GL}_\alpha)$ concentrated at $[M]$, by the systems $\{I_{\mathrm{GL}_\alpha \cdot M}^{\mathrm{GL}_\alpha}\}$ and $\{I_F^{\mathrm{GL}_\alpha}\}$. Since U is irreducible, by Krull's intersection theorem, $\bigcap_{n\geq 0} I^n_{\mathrm{GL}_\alpha \cdot M} = 0$. Hence the systems are both exhaustive. Since $I^n_{[M]}/I^{n+1}_{[M]}$ is finite-dimensional for all n, both systems must yield the $I_{[M]}$-adic completion (equivalently, the completion by all finite-dimensional quotients supported at $[M]$). We deduce

$$\mathrm{FN}([M], U/\!/\, \mathrm{GL}_\alpha) \cong \mathrm{FN}(\mathrm{GL}_\alpha \cdot M, U)/\!/\, \mathrm{GL}_\alpha\, . \tag{5-2}$$

Applying (2), we have

$$\mathrm{FN}(\mathrm{GL}_\alpha \cdot M, U)/\!/\, \mathrm{GL}_\alpha \cong \mathrm{FN}(\mathrm{GL}_{\alpha'} \cdot M', \mathrm{Rep}_{\alpha'}(A'))/\!/\, \mathrm{GL}_{\alpha'}\, .$$

By (5-2) applied to the first and last terms, we obtain finally the desired isomorphism. □

Remark 5.17. Part of the proof is actually showing is that the derived formal moduli stack at $[M]$ of representations of A is identified with the same for the dg algebra $E_{M'} E_M A$ at the zero representation $[M']$. This is true more generally, but under our hypotheses this implies the stated result by taking a truncation and applying Luna's slice theorem.

Remark 5.18. The second statement of the theorem is a strengthened version of the statement in [14] that a formal neighborhood of $[M]$ in $\mathrm{Rep}_\alpha(A)$ identifies with that of $[M']$ in $\mathrm{Rep}_{\alpha'}(A')$ times a formal disc of dimension $\dim \mathrm{GL}_\alpha - \dim \mathrm{GL}_{\alpha'}$. This is because GL_α is smooth, and taking the formal completion at the identity, the product construction here is multiplying by such a formal disc.

The theorem above uses the following lemma:

Lemma 5.19. *Suppose that $\mathfrak{h} \subseteq Z(\mathfrak{g}^0)$ acts on a dg Lie algebra $\mathfrak{g}$ concentrated in nonnegative degrees. Suppose that all $\mathfrak{h}$-subrepresentations have complements (e.g., this is true if the $\mathfrak{h}$ action integrates to an action of a connected reductive group H with Lie algebra $\mathfrak{h}$). Then there is an L_∞-quasiisomorphism*

$$\phi : \mathfrak{g}' := \mathfrak{g}^0 \oplus Z^1(\mathfrak{g}) \oplus H^{>1}(\mathfrak{g}) \to \mathfrak{g},$$

where on the source, all higher brackets

$$\mathfrak{h} \times (\mathfrak{g}')^{\geq 2} \to \mathfrak{g}' \tag{5-3}$$

vanish. The bracket $\mathfrak{g}^0 \times \mathfrak{g}^0 \to \mathfrak{g}^0$ is the original one. Moreover, the linear part $\phi^1 : \mathfrak{g}' \to \mathfrak{g}$ is $\mathfrak{h}$-linear and induces the identity on $\mathfrak{g}^0 \oplus Z^1(\mathfrak{g})$, as well as on cohomology. Finally, $\phi^{\geq 2}$ vanishes on $\mathfrak{h} \times (\mathfrak{g}')^{\geq 1}$.

Proof. We apply the homotopy transfer formulae from [49] (stated for A_∞-algebras but easily adapted to the L_∞ setting). To do this, for each i we pick a decomposition $\mathfrak{g}^i = B^i(\mathfrak{g}) \oplus \tilde{H}^i(\mathfrak{g}) \oplus \mathfrak{q}^i$, with $B^i(\mathfrak{g})$ the i-coboundaries, $\tilde{H}^i(\mathfrak{g})$ an $\mathfrak{h}$-linear complement to $B^i(\mathfrak{g})$ in the i-cocycles $Z^i(\mathfrak{g})$, and $\mathfrak{q}^i$ a $\mathfrak{h}$-linear complement to $Z^i(\mathfrak{g})$ in $\mathfrak{g}^i$. We then define a homotopy $h : \mathfrak{g}^{>1} \to \mathfrak{g}^{>0}$ via the projection $\mathfrak{g}^i \to B^i(\mathfrak{g})$ followed by a $\mathfrak{h}$-linear isomorphism $B^i(\mathfrak{g}) \to \mathfrak{q}^{i-1}$, for $i > 1$, setting $h|_{\mathfrak{g}^{\le 1}} = 0$.

The resulting homotopy is $\mathfrak{h}$-linear and has the property that $t := \mathrm{Id} - (dh + hd)$ is a projection onto the subcomplex $\mathfrak{g}^0 \oplus Z^1(\mathfrak{g}) \oplus \tilde{H}^{>1}(\mathfrak{g})$, which is an $\mathfrak{h}$-subrepresentation. Call this subcomplex $\mathfrak{g}'$. We have an $\mathfrak{h}$-linear decomposition $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{c}$ as complexes, with $\mathfrak{c} = \mathrm{im}(dh + hd)$ a contractible subcomplex (and $\mathfrak{h}$-subrepresentation).

Now use h on all of $\mathfrak{g}$, as in the proof of Theorem 5.7 (see the references above). We obtain a new L_∞-structure on $\mathfrak{g}$, which is L_∞-isomorphic to the original one (with linear part the identity), so that all structures vanish on $\mathfrak{c}$ aside from the differential. The L_∞-structures on $\mathfrak{g}'$ are linear combinations of expressions such as

$$t[a_1, h[a_2, [h[a_3, a_4], h[a_5, a_6]]]],$$

given by iteratively bracketing and applying h, except at the end where t is applied.

By $\mathfrak{h}$-linearity of h, if $x \in \mathfrak{h}$ and $a \in \mathfrak{g}' = \mathrm{im}\, t$, then $h[x, a] = [x, ha] = 0$. Similarly, $t[x, ha] = th[x, a] = 0$. Hence, all contributions to higher brackets $\mathfrak{h} \times \mathfrak{g}^{>1} \to \mathfrak{g}$ vanish. Similarly, $\phi^{>1}$ vanishes on $\mathfrak{h}$ (since $h[x, a] = 0$). By construction ϕ is the identity on $\mathfrak{g}^0 \oplus Z^1(\mathfrak{g})$ and on cohomology. □

Remark 5.20. The lemma has an associative analog with the same proof: let $\mathfrak{g}$ be a dg associative algebra and $\mathfrak{h}$ is a subalgebra for which every $\mathfrak{h}$-subbimodule of $\mathfrak{g}$ admits an $\mathfrak{h}$-complement (e.g., $\mathfrak{g}$ is augmented over a semisimple algebra $\mathfrak{h}$). Then we obtain the same result with an A_∞-quasiisomorphism with higher order parts vanishing on $\mathfrak{h}$, and with higher multiplications on $\mathfrak{g}'$ vanishing on $\mathfrak{h}$. This applies to the situation at hand, so that we could use an A_∞-quasiisomorphism in the proof of Theorem 5.16. However, it makes no difference for the Maurer–Cartan locus. (Actually, this says that the decomposition in Theorem 5.16 enhances to a decomposition of *noncommutative* representation schemes, meaning it describes representations with coefficients in noncommutative Artinian rings.)

5C. *Proof of main results.* In the case where A is 2-Calabi–Yau, we can use Theorem 5.12 (which applies because of part one in Theorem 5.16) and the discussion following it, to refine part three of Theorem 5.16. Namely, we can identify the formal neighborhood of $[M']$ in $\mathrm{Rep}_\alpha(A')/\!/\,\mathrm{GL}_{\alpha'}$ with a formal neighborhood of the zero representation in a quiver variety.

Corollary 5.21. *Let A be a 2-Calabi–Yau algebra over kQ_0 for a quiver Q. Let $\alpha \in \mathbb{N}^{Q_0}$ and let $M \in \mathrm{Rep}_\alpha(A)$, such that $\mathrm{GL}_\alpha \cdot M$ is closed in some GL_α-stable open affine subset, U. Then a formal neighborhood of $[M]$ in $U/\!/\,\mathrm{GL}_\alpha$ is isomorphic to the formal neighborhood $\widehat{\mathcal{M}^{\mathrm{add}}_{0,0}(Q', \alpha')_0}$ of the zero representation in a quiver variety.*

Pick a stability parameter θ. If $M \in \mathrm{Rep}_\alpha(A)$ is θ-semistable, one has the open set $\mathrm{Rep}_\alpha(A)^{\theta\text{-ss}}$, which is a union of GL_α-stable affine open subsets. As M lies in one such affine open subset, M satisfies the hypotheses of Theorem 5.16 and Corollary 5.21. This implies the following corollary:

Corollary 5.22. *Let Q, α, A be as in Theorem 5.16, let $\theta \in \mathbb{Z}^{Q_0}$. Then for every $M \in \mathrm{Rep}_\alpha(A)^{\theta\text{-ss}}$, the conditions of Theorem 5.16 are satisfied. So, the formal neighborhood of $[M]$ in $\mathcal{M}_\theta(A, \alpha)$ is isomorphic to that of zero in $\mathrm{Rep}_{\alpha'}(A')/\!/\,\mathrm{GL}_{\alpha'}$, for A' as in the theorem.*

Proof of Theorem 5.3. Let Q, α, A be as in Corollary 5.21, let $\theta \in \mathbb{Z}^{Q_0}$, and $V := \mathrm{Rep}_\alpha(A)^{\theta\text{-ss}}$. For every $M \in V$ the conditions of Corollary 5.21 are satisfied. So, the formal neighborhood of $[M]$ in $V/\!/\,\mathrm{GL}_\alpha$ is isomorphic to the formal neighborhood $\widehat{\mathcal{M}^{\mathrm{add}}_{0,0}(Q', \alpha')_0}$ of the zero representation in a quiver variety. □

Proof of Theorem 5.4. If the quiver Q contains a cycle, then Theorem 5.4 follows immediately from Theorem 5.3 since $\Lambda^q(Q)$ is 2-Calabi–Yau, by Theorem 1.2.

If Q does not contain a cycle, then build $\widetilde{Q}$ from Q by adding a new vertex i_0, an arrow from i_0 to itself, and an arrow from i_0 to any vertex of Q. If $\alpha \in \mathbb{N}^{Q_0}$ is a dimension vector then define $\tilde{\alpha} \in \mathbb{N}^{\widetilde{Q}_0}$ such that $\tilde{\alpha}|_{Q_0} = \alpha$ and $\tilde{\alpha}_{i_0} = 0$. Note that $\mathrm{Rep}_\alpha(\Lambda^q(Q)) = \mathrm{Rep}_{\tilde{\alpha}}(\Lambda^{\tilde{q}}(\widetilde{Q}))$ where $\tilde{q}$ is similarly such that $\tilde{q}|_{Q_0} = q$ and $\tilde{q}_{i_0} = 1$.

Under this identification, the $\mathrm{GL}_{\tilde{\alpha}} = \mathrm{GL}_\alpha \times \mathrm{GL}_1$ action factors through the projection to GL_α, which identifies the actions on the two varieties. For every $\theta \in \mathbb{Z}^{Q_0}$, extending by zero to $\tilde{\theta}$, one also identifies θ-semistable representations of $\Lambda^q(Q)$ of dimension α with $\tilde{\theta}$-semistable representations of $\Lambda^{\tilde{q}}(\widetilde{Q})$ of dimension $\tilde{\alpha}$. Therefore, $\mathcal{M}_{q,\theta}(Q, \alpha) = \mathcal{M}_{\tilde{q},\tilde{\theta}}(\widetilde{Q}, \tilde{\alpha})$, i.e., the semistable moduli spaces in question are identical. So the result follows in general from the specific case where Q contains a cycle. □

Proof of Corollary 5.5. By [3], a (normal) symplectic singularity is rational Gorenstein. The latter is a formal local property. By [5, Theorem 1.2], ordinary quiver varieties are symplectic singularities. Thus, the moduli spaces in question have rational Gorenstein singularities, and in particular are normal.

Next, thanks to Namikawa [52, Theorem 4], the property of being a (normal) symplectic singularity is a equivalent to having rational Gorenstein singularities and having a symplectic form on the smooth locus. It remains to check the last property. (Note that this property is certainly known for many multiplicative quiver varieties: For instance, Yamakawa [61, Theorem 3.4] showed that the stable locus is smooth symplectic, and this is often the entire smooth locus. For another example, character varieties of Riemann surfaces of genus ≥ 1 (and many of genus zero) have symplectic smooth locus by [55, Section 1.2].)

To see that the smooth locus is symplectic in general, first we can assume that we are in the situation of a 2-Calabi–Yau algebra A (in the case of multiplicative quiver varieties, the proof of Theorem 5.4 in Section 5C identifies the moduli space with one for a 2-Calabi–Yau algebra obtained by enlarging the quiver). At a smooth point of the moduli space, Theorem 5.3 endows the formal neighborhood of the point with a symplectic form, given by the canonical symplectic pairing $\mathrm{Ext}^1(M, M) \times \mathrm{Ext}^1(M, M) \to \mathrm{Ext}^2(M, M) \overset{\mathrm{tr}}{\cong} k$ coming from the Calabi–Yau structure. This is functorial in the point of the moduli space: the Calabi–Yau structure furnishes a fixed A-bimodule isomorphism $A \cong \mathrm{Ext}^2(A, A^e)$. This induces a functorial isomorphism

$$\mathrm{Ext}^2(M, M) \cong H^2(\mathrm{RHom}(A, A^e) \otimes^{\mathbb{L}}_{A^e} \mathrm{End}_k(M)) \to A \otimes_{A^e} \mathrm{End}_k(M) = \frac{\mathrm{End}_k(M)}{[A, \mathrm{End}_k(M)]}.$$

Composing this with the trace map we obtain the functorial trace pairing. □

Remark 5.23. Alternatively, one should be able to construct the symplectic structure on the smooth locus because the latter is an open substack of the symplectic derived moduli stack of representations of $\Lambda^q(Q)$, shown to be symplectic in [17].

6. The multiplicative preprojective algebra of the cycle is an NCCR

The purpose of this section is to prove Conjecture 1.4 in the case where Q is a cycle. We begin with the necessary definitions. Throughout this section, Q denotes an extended Dynkin quiver (not necessarily a cycle).

According to the conjecture, the center of the multiplicative preprojective algebra is the ring of functions on the multiplicative quiver variety $\mathcal{M}_{1,0}(Q, \delta)$. Here δ is the primitive positive imaginary root. In terms of the McKay correspondence, Q is the McKay graph of a finite subgroup $\Gamma < \mathrm{SL}_2(\mathbb{C})$, which means that the vertices are labeled by the irreducible representations of Γ. In these terms, δ_v is the dimension of the irreducible representation of Γ_Q attached to the vertex v. In particular, for the cycle with n vertices, $\Gamma = \mathbb{Z}/n\mathbb{Z}$, and $\delta = (1, \dots, 1)$ is the all ones vector.

We next recall the notion of an NCCR. Van den Bergh [8, Appendix A] originally defined these in to give an alternate proof of Bridgeland's theorem that a flop of three-dimensional smooth varieties induces an equivalence of their bounded derived categories. Van den Bergh later simplified and generalized the definition to the following:

Definition 6.1 [7, Definition 4.1 and Lemma 4.2]. Let R be an Gorenstein commutative integral domain. An algebra A is an *NCCR* over R if:

(1) A is (maximal) Cohen–Macaulay.

(2) A has finite global dimension.

(3) $A \cong \mathrm{End}_R(M)$ for some reflexive module M.[6]

Note that if A is derived equivalent to a *commutative* crepant resolution of $\mathrm{Spec}(R)$, then it will have to satisfy these conditions by [38, Corollary 4.15]. (However, in general, R could admit a commutative crepant resolution but not a noncommutative one, and vice-versa).

In our case, with $\dim R = 2$, it is convenient to observe that we don't have to check the Cohen–Macaulay condition:

Lemma 6.2. *Let R be a normal Noetherian domain of dimension 2 over k. Let M be a finitely generated, reflexive R-module. Then $A := \mathrm{End}_R(M)$ is Cohen–Macaulay.*

[6] Recall an R-module M is *reflexive* if the natural map $M \to \mathrm{Hom}_R(\mathrm{Hom}_R(M, R), R)$ sending $m \in M$ to evaluation on m (i.e., $m \mapsto [\varphi \in \mathrm{Hom}_R(M, R) \mapsto \varphi(m) \in R]$) is an isomorphism.

Proof. Since R is Noetherian and M is finitely generated and reflexive, [58, Lemma 15.23.8] implies that A is reflexive. Since R is 2-dimensional and normal [19, Corollary 3.9] implies that A is Cohen–Macaulay. $\square$

Remark 6.3. Note that, in higher dimensions, while the Cohen–Macaulay property for A is not automatic, it nevertheless can be deduced from the Calabi–Yau property thanks to [37, Theorem 3.2(3)]. This gives an alternative way to handle condition (2) in our situation.

6A. ***Shaw's results on the center.*** While the center of the multiplicative preprojective algebra is in general unknown, in Shaw's thesis [57], he proves the following. Let v be an extending vertex.

Theorem 6.4 [57, Theorem 4.1.1]. *$e_v\Lambda^1(Q)e_v \cong k[X, Y, Z]/(f(X, Y, Z))$ where f has isolated singularity at the origin. Explicitly,*

$$f(X,Y,Z) = \begin{cases} Z^{n+1} + XY + XYZ & \text{if } Q = \widetilde{A}_n, n \geq 1, \\ Z^2 - p_{n-4}(X)XZ + p_{n-5}(X)X^2Y - XY^2 - XYZ & \text{if } Q = \widetilde{D}_n, n \geq 4, \\ Z^2 + X^2Z + Y^3 - XYZ & \text{if } Q = \widetilde{E}_6, \\ Z^2 + Y^3 + X^3Y - XYZ & \text{if } Q = \widetilde{E}_7, \\ Z^2 - Y^3 - X^5 + XYZ & \text{if } Q = \widetilde{E}_8, \end{cases}$$

where $p_{-1}(X) := -1$, $p_0(X) := 0$, and $p_{i+1}(X) := X(p_{i-1}(X) + p_i(X))$ for $i \geq 1$.

Remark 6.5. Shaw expected that the singularities at the origin have the du Val type corresponding to the quiver. Over a field of characteristic zero, Michael Wemyss checked this in E types via Magma. It is also clear that in A types, the singularity is du Val of the same type as the quiver, by the rational substitution $y \mapsto y/(1+z)$. Presumably it can be checked that in type D (over characteristic not equal to two) the singularity also is the corresponding du Val one.

Note that having du Val singularities is equivalent to the statement that the minimal commutative resolution is symplectic, i.e., 2-Calabi–Yau. Thanks to [59], it is also true that if a Gorenstein surface admits an NCCR, then it has du Val singularities. This is another reason to believe Shaw's expectation.

Remark 6.6. Suppose as expected that the singularities are du Val. Then, as in [43], one may construct an NCCR from the minimal resolution. It seems an interesting question to show that this is Morita equivalent to $\Lambda^1(Q)$.

This motivates the final statement in Conjecture 1.4, that the Satake map, $Z(\Lambda^1(Q)) \to e_v\Lambda^1(Q)e_v$, given by $z \mapsto e_v z$, is an isomorphism. With this in place, the above translates into an explicit description of the center.

6B. ***Proof of Conjecture 1.4 for a cycle.*** Fix $n \geq 1$. In the remainder of this section we prove Conjecture 1.4 for $Q = \widetilde{A}_n$. As a consequence, using Shaw's result, we conclude:

Corollary 6.7. *The center of $\Lambda^1(\widetilde{A}_n)$ is isomorphic to $k[X, Y, Z]/(Z^{n+1} + XY + XYZ)$.*

The steps of the proof of Conjecture 1.4 for $\widetilde{A}_n$ are as follows:

(1) First we show that $\Lambda^1(\widetilde{A}_n)$ is isomorphic to an NCCR over $e_0\Lambda^1(\widetilde{A}_n)e_0$.

(2) Then we use the preceding result to establish that the Satake map $Z(\Lambda^1(\widetilde{A}_n)) \to e_0\Lambda^1(\widetilde{A}_n)e_0$, is an isomorphism.

(3) To complete the proof we consider the canonical map $Z(\Lambda^1(\widetilde{A}_n)) \to k[\mathcal{M}_{0,1}(\widetilde{A}_n, \delta)]$, given by associating to a central element and a simple representation the scalar by which the element acts in the representation. We show that this is an isomorphism.

We carry out these steps in the next subsections.

In the first step, we will make use of the prime property for $\Lambda^1(\widetilde{A}_n)$. We state the prime property now, but defer the proof until Section 7, as our proof uses an explicit basis produced in Proposition 7.1.

Remark 6.8. Note that there is no circular logic in the paper, as Section 7 does not rely on any results after Section 2, and hence could instead fit logically between Sections 3A and 3B, whereby every result would be proven in order. We decided that, due to the technical nature of Section 7, whose methods are not used in the preceding material, it would be better to use its results as a black box in Sections 3B–6.

Definition 6.9. Let R be a ring. We say R is *prime* if $rRr' = 0$ implies $r = 0$ or $r' = 0$, for all $r, r' \in R$.

For a commutative ring, this recovers the usual notion of an integral domain, i.e., that the zero ideal is a prime ideal.

Example 6.10. For a nonexample, take $B = \bigoplus_{n\in\mathbb{N}} B_n$ to be a finite-dimensional $\mathbb{N}$-graded algebra not concentrated in degree zero. Then there exists $N \in \mathbb{N}$ such that $B_m = 0$ for all $m > N$ but $B_N \neq 0$. Pick $b \in B_N$ nonzero and notice that $bBb \in \bigoplus_{m\geq 2N} B_m = \{0\}$ since $2N > N$. Hence B is not prime.

In particular for Q Dynkin and $k = \mathbb{C}$, $\Lambda^1(Q) \cong \Pi^0(Q)$ is a finite-dimensional $\mathbb{N}$-graded algebra and therefore not prime. However, for $Q = A_2$ and $q = (1/2, 2) \neq (1, 1)$, then $\Lambda^q(A_2) \cong \Pi^{(-1,1)}(A_2) \cong \mathrm{Mat}_{2\times 2}(k)$ is prime.

Proposition 6.11 (Proposition 7.3). *$\Lambda^q(\widetilde{A}_n)$ is prime for all $n \geq 0$ and all $q \in (k^\times)^{n+1}$.*

6B1. *The NCCR property.* We first show that the multiplicative preprojective algebra is an NCCR (Step 1).

Proposition 6.12. *$\Lambda^1(\widetilde{A}_n)$ is isomorphic to an NCCR over $e_0\Lambda^1(\widetilde{A}_n)e_0$.*

Proof. Define $\Lambda := \Lambda^1(\widetilde{A}_n)$ for ease of notation. Write the vertex set as $\{0, 1, \dots, n\}$ and the arrow set $\{a_0, a_0^*, a_1, a_1^*, \dots, a_n, a_n^*\}$, with $t(a_i) = i = h(a_i^*)$ for $i < n$ but $t(a_n) = 0 = h(a_n^*)$. So the multiplicative preprojective relation at each vertex is

$$e_i(\rho - 1) = \begin{cases} a_0a_0^* + a_na_n^* + a_0a_0^*a_na_n^* & \text{if } i = 0, \\ a_n^*a_n + a_{n-1}^*a_{n-1} + a_n^*a_na_{n-1}^*a_{n-1} & \text{if } i = n, \\ a_ia_i^* - a_{i-1}^*a_{i-1} & \text{otherwise.} \end{cases}$$

Shaw's isomorphism in Theorem 6.4 takes the form

$$a_0a_0^* \mapsto Z \quad a_0a_1 \cdots a_{n-1}a_n^* \mapsto X \quad a_na_{n-1}^*a_{n-2}^* \cdots a_0^* \mapsto Y.$$

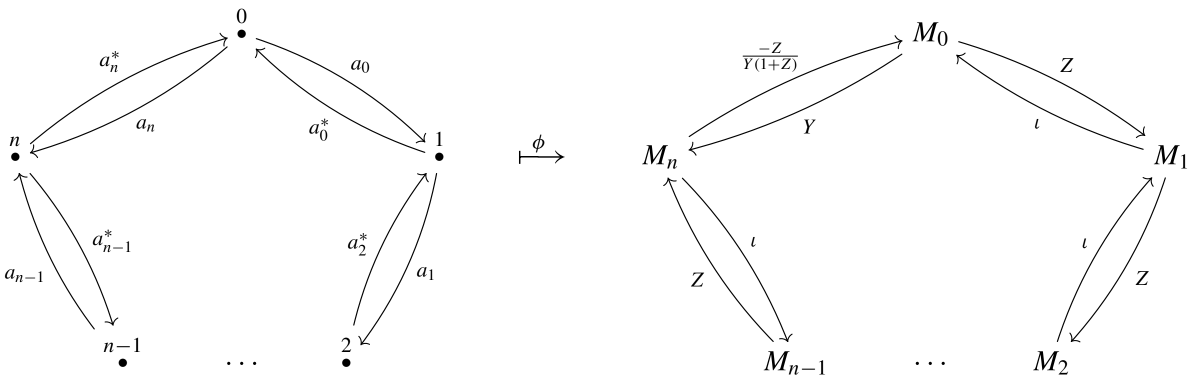
Define $M := e_0\Lambda$ and note that $M = \oplus_{i=0}^{n} M_i$ where $M_i := e_0\Lambda e_i$. Observe that $M_i \cong (Z^i, Y)$, the two-sided ideal generated by Z^i and Y in Λ, as $e_0\Lambda e_0$-modules via a map,

$$\begin{aligned} a_0a_1\cdots a_{i-1} &\mapsto a_0a_1\cdots a_{i-1}a_{i-1}^*a_{i-2}^*\cdots a_0^* = (a_0a_0^*)^i = Z^i \\ a_na_{n-1}^*\cdots a_i^* &\mapsto a_na_{n-1}^*\cdots a_0^* = Y. \end{aligned}$$

Define the map

$$\Lambda \xrightarrow{\phi} \mathrm{End}_{e_0\Lambda e_0}(M),$$

on generators by sending the idempotent e_i at vertex i to the projection map $M \to M_i$, and sending the arrows as follows:



where ι denotes the inclusion map. This map is well-defined at vertex 0 and n since

$$Z + \frac{-YZ}{Y(1+Z)} + \frac{-YZ^2}{Y(1+Z)} = Z + \frac{-Z(1+Z)}{(1+Z)} = Z - Z = 0$$

and at vertex $i \neq 0, n$ since $Z - Z = 0$.

The surjectivity of ϕ follows from the observation that every $e_0\Lambda e_0$-module map of ideals is given by left multiplication by some element of the field of fractions of $e_0\Lambda e_0$. The injectivity follows from the fact that Λ is prime (Proposition 6.11) and injectivity on $e_0\Lambda e_0$, as we now explain.

By definition of primality, for any $a, c \in \Lambda$ both nonzero, there exists $b \in \Lambda$ such that $abc \neq 0$. Fix $\gamma \in \Lambda$ nonzero and take $a = e_0$ and $c = \gamma$ to get a nonzero path $\gamma' \in e_0\Lambda$ containing γ as a subpath. Then take $a = \gamma'$ and $c = e_0$ to get a nonzero path $\gamma'' \in e_0\Lambda e_0$ containing γ as a subpath. Since ϕ is injective on $e_0\Lambda e_0$, $\phi(\gamma'') \neq 0$. Hence $\phi(\gamma) \neq 0$ and ϕ is injective.

To complete the proof that Λ is an NCCR, we need to show that the module $M = e_0\Lambda$ is a reflexive $e_0\Lambda e_0$-module. The computation above shows that

$$\mathrm{Hom}_{e_0\Lambda e_0}(M_i, M_j) \cong e_i\Lambda e_j \cong M_{j-i}$$

as a module over $e_0\Lambda e_0 \cong e_i\Lambda e_i$, so in particular $\mathrm{Hom}_{e_0\Lambda e_0}(M, e_0\Lambda e_0) \cong \oplus_i e_i\Lambda e_0 \cong M$. So M is self-dual and hence reflexive as a $e_0\Lambda e_0$-module. $\square$

6B2. *The center.* Observe, if A is an NCCR over some ring R, then the center $Z(A)$ is an R-algebra. Under suitable hypotheses, they are actually isomorphic. For example, this holds if R an integrally closed Noetherian domain, by Zariski's main theorem (as $\operatorname{Spec} Z(A) \to \operatorname{Spec} R$ is finite and birational).

Instead of using this to establish our isomorphism, we consider an explicit map in the other direction. More generally, suppose A is a ring, $e \in A$ is an idempotent, and $R := eAe$. Then we have a canonical map

$$Z(A) \to R = eAe, \quad z \mapsto ez. \tag{6-1}$$

We call this the "Satake map" following the terminology for Hecke algebras, symplectic reflection algebras, etc.

Under natural conditions, the Satake map is well known to be an isomorphism. Namely, note that eA is an $(eAe) - A$ bimodule, and $\operatorname{End}_{A^{\mathrm{op}}}(eA) = eAe$. Then we have a natural map $A^{\mathrm{op}} \to \operatorname{End}_{eAe}(eA)$.

Lemma 6.13. *Suppose that* (*I*) *the natural map* $A^{\mathrm{op}} \to \operatorname{End}_{eAe}(eA)$ *is an isomorphism, and* (*II*) eAe *is commutative. Then the Satake map* (6-1) *is an isomorphism.*

Proof. We have an identification

$$Z(eAe) \cong \operatorname{End}_{eAe\otimes A^{\mathrm{op}}}(eA) \cong Z(A) \quad z \mapsto ez.$$

Since eAe is commutative, $Z(A) \cong eAe$, via the Satake map. □

Corollary 6.14. *The Satake map* (6-1) *is an isomorphism for* $A = \Lambda^1(\widetilde{A}_n)$ *and* $e = e_v$, *the idempotent at any vertex.*

Proof. This is a direct consequence of Lemma 6.13, once we check hypotheses (I) and (II). Thanks to Proposition 6.12, $A \cong \operatorname{End}_{eAe}(eA)$ so (I) follows from $A \cong A^{\mathrm{op}}$, a consequence of the independence of orientation established in [25, Theorem 1.4]. By Shaw's Theorem 6.4, (II) holds (alternatively, the commutativity of the generators can be checked directly). □

Corollary 6.15. $\Lambda^1(\widetilde{A}_n)$ *is an NCCR over its center.*

Proof. This follows immediately, provided we identify the $Z(\Lambda^1(\widetilde{A}_n))$-module structure on $\Lambda^1(\widetilde{A}_n)$ with left multiplication. Indeed, given $z \in Z(\Lambda^1(\widetilde{A}_n))$ (by tracing through the above maps) its action on $\operatorname{End}_{e\Lambda^1(\widetilde{A}_n)e}(M)$ via the Satake map is multiplication by ez. □

Note that Corollary 6.14 and Theorem 6.4 immediately imply Corollary 6.7.

6B3. *The center as functions on a quiver variety.* It remains to identify the center with the algebra of functions on the multiplicative quiver variety.

In general, given a kQ_0-algebra A and a finite-dimensional kQ_0-module V, we have a canonical algebra homomorphism $\mathrm{ev} : A \to k[\operatorname{Rep}(A, V)] \otimes \operatorname{End}_k(V)$, called "evaluation": $\mathrm{ev}(a)(\rho) = \rho(a)$.

Suppose that $\rho : A \to \operatorname{End}(V)$ is an irreducible representation. Consider $z \in Z(A)$. If k is algebraically closed, then by Schur's Lemma, $\rho(z) = \lambda_{\rho,z} \operatorname{Id}_V$ for some scalar $\lambda_{\rho,z}$. However, we don't assume here that k is algebraically closed. We could fix this by passing to the algebraic closure, but this turns out to be unnecessary as follows.

Lemma 6.16. *Suppose that $v \in Q_0$ is a vertex with $\dim V_v = 1$. Suppose ρ is an irreducible representation. Then $\mathrm{End}(\rho) = k \cdot \mathrm{Id}_V$.*

Proof. If $\phi \in \mathrm{End}(\rho)$, then $\rho(e_v)\phi = \phi\rho(e_v)$. Therefore, ϕ preserves $\rho(e_v)V = V_v$. As this has dimension one, we have $\phi|_{V_v} = \lambda \, \mathrm{Id}_{V_v}$. Now, $\phi - \lambda \, \mathrm{Id}_V$ is not invertible. By Schur's lemma over a general field, this implies that $\phi - \lambda \, \mathrm{Id}_V$ is zero. So $\phi = \lambda \, \mathrm{Id}_V$. □

Corollary 6.17. *Let Q_0, A, V, v and ρ be as in Lemma 6.16. If $z \in Z(A)$, then $\rho(z) \in \mathrm{End}(V)$ is a scalar.*

Proof. Note that $\rho(z) \in \mathrm{End}(\rho)$. Then apply the lemma. □

Corollary 6.18. *Suppose that for some vertex v, we have $V_v = 1$, and moreover that there exists an irreducible representation $A \to \mathrm{End}(V)$. Then the restriction $\mathrm{ev}|_{Z(A)}$ is an algebra map $Z(A) \to k[\mathrm{Rep}(A, V)] \cdot \mathrm{Id}_V$.*

Proof. Let $U \subseteq \mathrm{Rep}(A, V)$ be the locus of representations ρ such that $\mathrm{End}(\rho) = k \cdot \mathrm{Id}_V$. This is a Zariski open subset, since $k \cdot \mathrm{Id}_V$ is always contained in $\mathrm{End}(\rho)$. If $\rho \in \mathrm{Rep}(A, V)$ is irreducible, then by Lemma 6.16, $\rho \in U$. Thus, by our assumptions, U is nonempty. Since $\mathrm{Rep}(A, V)$ is a vector space, it is irreducible. We conclude that U is Zariski dense.

Now, for every $z \in Z(A)$, $\mathrm{ev}(z) : \mathrm{Rep}(A, V) \to \mathrm{End}(V)$ is scalar-valued on U. As U is dense, it is a scalar on all of $\mathrm{Rep}(A, V)$. Hence $\mathrm{ev}(z) \in k[\mathrm{Rep}(A, V)] \otimes \mathrm{Id}$. As z was arbitrary, we obtain the result. □

Back to the situation at hand, for convenience let us orient $\widetilde{A}_n$ clockwise (note that the statement does not depend on orientation). We consider the vector space $V = k^{Q_0}$, which has the property $\dim V_v = 1$ for all $v \in Q_0$. Consider the representation on V where each clockwise arrow is the identity (i.e., the one-by-one matrix [1]) and each counterclockwise arrow is zero. This defines a representation of the localization L_Q that descends to an irreducible representation of $\Lambda^1(Q)$. Therefore, having satisfied the hypotheses of Corollary 6.18, we obtain a canonical map

$$\mathrm{ev}_Z : Z(\Lambda^1(Q)) \to k[\mathcal{M}_{0,1}(Q, \delta)]. \tag{6-2}$$

Proposition 6.19. *The map ev_Z is an isomorphism.*

Proof. To check surjectivity, let $f \in k[\mathcal{M}_{0,1}(Q, \delta)] = k[\mathrm{Rep}_\alpha(\Lambda^1(Q))]^{\mathrm{GL}_\alpha}$. We wish to show that $f \in \mathrm{ev}_Z(Z(\Lambda^1(Q)))$. Note that f is a polynomial in the matrix coefficient functions of the arrows (these are one by one matrices). To be invariant under GL_α, the polynomial must in fact be a polynomial in the functions defined by closed paths in the quiver: each such closed path is canonically a scalar, as it is an endomorphism of a one-dimensional vector space. Thus it suffices to assume that there is a single closed path $a \in e_v \Lambda^1(Q) e_v$ such that $\rho(a) = f(\rho) \cdot \mathrm{Id}_{V_v}$ for all ρ. As the Satake map is an isomorphism (Corollary 6.14), we must have $a = e_v z$ for some $z \in Z(\Lambda^1(Q))$. Then, $\rho(a) = \mathrm{ev}_Z(z) \cdot \mathrm{Id}_{V_v}$. Hence $f(\rho) = \mathrm{ev}_Z(z)$ for all $\rho \in \mathrm{Rep}_\alpha(\Lambda^1(Q))$. This shows that ev_Z is surjective.

By Corollary 6.7 the source is an integral domain. Since we already proved surjectivity, injectivity will follow provided that the target also has dimension at least two. This can be seen by constructing a two-parameter family of representations, e.g., we can take the representations with all clockwise arrows a matrix (a) and all counterclockwise arrows a matrix (b), with $ab \neq -1$. Alternatively, this statement follows from Theorem 5.4. □

7. The strong free product property

In this section, we prove the strong free product property for connected quivers containing a cycle. We first establish the strong free product property for the quivers $\widetilde{A}_n$ for $n \geq 0$ using the diamond lemma to build a section of the quotient map $\pi : L \to \Lambda^q(\widetilde{A}_n)$. Then we establish the more general result using the corresponding result for *partial* multiplicative preprojective algebras; see Section 3A for the prerequisite definitions.

As results in Sections 3B, 3C, 4, 5, 6 rely on results established in this section, the reader should note that we do not use any results beyond Section 3A; see Remark 6.8.

7A. ***The case of cycles.*** Consider the quiver $\widetilde{A}_{n-1}$ with vertex set $(\widetilde{A}_{n-1})_0 := \{0, 1, \dots, n-1\}$ and arrow set $(\widetilde{A}_{n-1})_1 = \{a_0, a_0^*, a_1, a_1^*, \dots, a_{n-1}, a_{n-1}^*\}$ with $t(a_i) = i$ and $h(a_i) = i+1 \pmod{n} i < a_{i+1} < a_j^* < a_{j+1}^*$ for all $i, j \in \{0, 1, \dots, n-2\}$. The multiplicative preprojective algebra for this quiver, with respect to the ordering, is defined to be

$$\Lambda^q(\widetilde{A}_{n-1}) := \frac{k\overline{\widetilde{A}}_{n-1}[(1+a_i a_i^*)^{-1}, (1+a_i^* a_i)^{-1}]_{i=0,\dots,n-1}}{\langle \prod_{i=0}^{n-1}(1+a_i a_i^*) \prod_{i=0}^{n-1}(1+a_i^* a_i)^{-1} - \sum_{i=1}^{n} q_i e_i \rangle} =: \frac{L}{J}.$$

Writing $a := \sum_i a_i$, $a^* := \sum_i a_i^*$, and $q = \sum_i q_i e_i$ since

$$1 + aa^* = 1 + \sum_i a_i a_i^* = \prod_{i=0}^{n-1}(1 + a_i a_i^*), \quad 1 + a^* a = 1 + \sum_i a_i^* a_i = \prod_{i=0}^{n-1}(1 + a_i^* a_i)$$

we have

$$\Lambda^q(\widetilde{A}_{n-1}) := \frac{k\overline{\widetilde{A}}_{n-1}[(1+aa^*)^{-1}, (1+a^*a)^{-1}]}{\langle (1+aa^*)(1+a^*a)^{-1} - q \rangle}.$$

We write $r := (1+aa^*)(1+a^*a)^{-1} - q$ for this relation, S for the degree zero piece $k(\widetilde{A}_{n-1})_0$ of $\Lambda^q(\widetilde{A}_{n-1})$. As in Section 3A, let $B := S[t, (q+t)^{-1}]$ and $\bar{B} = tB$, spanned over S by t^m, $(t')^m$, $m \geq 1$, for $t' := (q+t)^{-1} - q^{-1}$. Let $r' := (q+r)^{-1} - q^{-1}$.

We construct $\sigma : L/(r) *_S B \to L$ so that (L, r, σ', B) satisfies the strong free product property using an explicit basis.

Proposition 7.1. *L is a free left S-module with basis consisting of* 1 *together with all alternating products of elements of the following two sets, for $x := (1 + aa^*)$:*

$$\mathfrak{B} := \{x^m a^\ell, x^m (a^*)^\ell \mid m \in \mathbb{Z}, \ell \in \mathbb{N}\}. \quad \mathfrak{R} := \{r^m, (r')^m \mid m \in \mathbb{N}\}.$$

In particular, $\mathfrak{B}$ *forms a basis for* $\Lambda^q(\widetilde{A}_{n-1}) = L/(r)$, *and* (L, r, σ, B) *satisfies the strong free product property*, *with* σ *induced from the inclusion of* $\mathfrak{B}$ *into* L.

Proof. Note that, for every vertex i, we have $e_i a = a e_j$ for a unique j, and similarly for the elements $a^*, x, y := 1 + a^* a, x^{-1}, y^{-1}$, and by definition, $e_i r = r e_i$. Therefore L is spanned as a *left* S-module by noncommutative monomials in $a, a^*, x, y, x^{-1}, y^{-1}, r$, and r'. Define $\mathcal{M} := \langle a, a^*, x, y, x^{-1}, y^{-1}, r, r' \rangle$ the set of monomials and $\mathcal{P} := S\langle a, a^*, x, y, x^{-1}, y^{-1}, r, r' \rangle$ the set of noncommutative polynomials with coefficients in S.

The set of relations, R, is the two-sided ideal generated by

$$xx^{-1} = 1 = x^{-1}x, \quad yy^{-1} = 1 = y^{-1}y, \quad x = 1 + aa^*, \quad y = 1 + a^*a, \tag{7-1}$$

$$r = xy^{-1} - q, \quad r' = yx^{-1} - q^{-1}. \tag{7-2}$$

So we have the presentation $L \cong \mathcal{P}/R$ and hence $\Lambda^q(\widetilde{A}_{n-1}) \cong \mathcal{P}/(R, r)$.

The idea of the proof is to produce a basis of the *quotient* $L = \mathcal{P}/R$ by realizing it as an S-module *subspace* $\mathcal{P}_{\text{irr}} \subset \mathcal{P}$ spanned by *irreducible* monomials, defined below.

That is, we define an ordering, $\leq$, on the set $\mathcal{M}$. Then we use this ordering to build a system of reductions $\{r_i\}$ from R by reading each relation $R_i \in R$ as an S-module map, r_i, taking the leading term $\operatorname{lt}(R_i)$ to the smaller term $\operatorname{lt}(R_i) - R_i$. We extend r_i to $\mathcal{M}$ via $a\operatorname{lt}(R_i)b \mapsto a(\operatorname{lt}(R_i) - R_i)b$ for $a, b \in \mathcal{M}$. We say $m \in \mathcal{M}$ is *irreducible* (or in normal form) if every reduction is the identity on m or, equivalently, if m doesn't contain the leading term of any relation as a submonomial.

We will show that every $m \in \mathcal{M}$, reduces *uniquely* to normal form, $m' \in \mathcal{P}_{\text{irr}}$, after applying *finitely* many reductions. This implies the S-module map $\mathrm{r} : \mathcal{P} \to \mathcal{P}_{\text{irr}}$ given by S-linear extension of $m \mapsto m'$ is well-defined. Hence r splits the inclusion map $\mathcal{P}_{\text{irr}} \to \mathcal{P}$. As $\ker(\mathrm{r}) = R$, we conclude that r induces an S-module isomorphism $L \cong \mathcal{P}_{\text{irr}}$ and the set of irreducible monomials gives our desired basis.

First we equip $\mathcal{M}$ with an ordering. Fix $w, z, z' \in \mathcal{M}$ and subsets $Z, Z' \subset \mathcal{M}$. Define

$$n_z(w) := \text{ the number of occurrences of } z \text{ in } w, \tag{7-3}$$

$$n_{z,z'}(w) := \text{ the number of occurrences of } z \text{ and } z' \text{ in } w \text{ with } z \text{ appearing before } z', \tag{7-4}$$

$$n_Z(w) := \sum_{z \in Z} n_z(w) \quad \text{and} \quad n_{Z,Z'} := \sum_{z \in Z, z' \in Z'} n_{z,z'}. \tag{7-5}$$

Define a function $N : \mathcal{M} \to \mathbb{N}^5$ taking w to

$$N(w) := (n_a(w), n_{\{a,a^*\},\{x,x^{-1},y,y^{-1}\}}(w), n_{\{ax,ax^{-1}\}}(w), n_{\{y,y^{-1}\}}(w), n_{\{r,r'\}}(w)) \in \mathbb{N}^5. \tag{7-6}$$

Define the ordering $w' \leq w$ in $\mathcal{M}$ if $N(w') \leq N(w)$ in the lexicographical ordering on $\mathbb{N}^5$. This induces an ordering on $\mathcal{P}$, by extending N to $\mathcal{P}$, via $N\big(\sum_i m_i\big) := \max_i\{N(m_i)\}$.

Next, using this ordering, we define a system of reductions from the relations in (7-1), (7-2):

Inverse reductions: $xx^{-1}, x^{-1}x, yy^{-1}, y^{-1}y \mapsto 1$.

Short cycle reductions: $aa^* \mapsto x - 1, a^*a \mapsto y - 1$.

Reordering reductions: $a^*x^{\pm 1} \mapsto y^{\pm 1}a^*$, $ay^{\pm 1} \mapsto x^{\pm 1}a$.

Substitution reductions: $y^{-1} \mapsto x^{-1}(r+q)$, $y \mapsto (r'+q^{-1})x$ (if not preceded by a); $ax \mapsto a(r+q)y$, $ax^{-1} \mapsto ay^{-1}(r'+q^{-1})$.

Reductions in B: $rr', r'r \mapsto -qr' - q^{-1}r$.

By design, if w' is obtained from w by applying a reduction, then $N(w') < N(w)$. This implies that any sequence of reductions terminates in finitely many steps, by the descending chain condition for the lexicographical ordering on $\mathbb{N}^5$.

Next observe that under this reduction system $m \in \mathcal{M}$ is in normal form (or irreducible) if and only if it is alternating in $\mathfrak{B}$ and $\mathfrak{R}$. Therefore, the set of alternating words in $\mathfrak{B}$ and $\mathfrak{R}$ is a spanning set. It remains to show that $m \in \mathcal{M}$ reduces uniquely to normal form, which establishes linear independence.

To prove uniqueness, we need to show whenever w reduces to $r_1(w)$ and $r_2(w)$ that each further reduces to the same irreducible w'. Bergman's diamond lemma says to show uniqueness for all monomials w it suffices to show uniqueness for specific $w = xyz$ where xy and yz are both leading terms for a relation in (7-1), (7-2) [11, Theorem 1.2]. These w are called *overlap ambiguities*. If the two reduced expressions of $w = xyz$ (i.e., $r_1(xy)z$ and $xr_2(yz)$) both further reduce to the same w', we say the overlap ambiguity *resolves*. To complete the proof it suffices to show all overlap ambiguities resolve.

Next, notice that any unresolvable ambiguity involving $y^{\pm 1}$ gives rise to an unresolvable ambiguity not involving $y^{\pm 1}$ by applying the substitution or reordering reductions. So it suffices to check ambiguities in the following smaller system of reductions:

Inverse reductions:

(1) $xx^{-1} \xmapsto{r_1} 1$. (2) $x^{-1}x \xmapsto{r_2} 1$.

Short cycle reductions:

(3) $aa^* \xmapsto{r_3} x-1$. (4) $a^*a \xmapsto{r_4} (r'+q^{-1})x-1$.

Reordering reductions:

(5) $a^*x \xmapsto{r_5} (r'+q^{-1})xa^*$. (6) $ax \xmapsto{r_6} qxa - qar'x$.

(7) $ax^{-1} \xmapsto{r_7} x^{-1}a(r'+q^{-1})$. (8) $a^*x^{-1} \xmapsto{r_8} x^{-1}(r+q)a^*$.

Substitution reductions:

(9) $y^{-1} \mapsto x^{-1}(r+q)$. (10) $y \mapsto (r'+q^{-1})x$.

Reductions in B:

(11) $rr' \mapsto -qr' - q^{-1}r$. (12) $r'r \mapsto -qr' - q^{-1}r$.

The substitution reductions and reductions in B don't overlap with any others, so the only overlap ambiguities are amongst the (1)–(8), involving the generators a, a^*, x, x^{-1} only. The inverse, short cycle, and reordering reductions are quadratic in these generators giving rise to the following 12 cubic overlap ambiguities:

(I) $xx^{-1}x$	(IV) a^*aa^*	(VII) $a^*x^{-1}x$	(X) a^*ax
(II) $x^{-1}xx^{-1}$	(V) a^*xx^{-1}	(VIII) $ax^{-1}x$	(XI) aa^*x^{-1}
(III) aa^*a	(VI) axx^{-1}	(IX) aa^*x	(XII) a^*ax^{-1}

The resolutions of (I) and (II) are immediate (and are completely general, having to do with a basis for $k[x, x^{-1}]$). Here is a summary of the remaining resolutions of ambiguities:

(III)	$(r_3 - r_6 \circ r_4)(aa^*a) = 0$	(VIII)	$(r_8 \circ r_7 - r_2)(ax^{-1}x) = 0$
(IV)	$(r_4 - r_5 \circ r_3)(a^*aa^*) = 0$	(IX)	$(r_3 - r_3 \circ r_6 \circ r_5)(aa^*x) = 0$
(V)	$(r_8 \circ r_5 - r_1)(a^*xx^{-1}) = 0$	(X)	$(r_4 - r_4 \circ r_4 \circ r_5 \circ r_6)(a^*ax) = 0$
(VI)	$(r_7 \circ r_6 - r_1)(axx^{-1}) = 0$	(XI)	$(r_3 - r_3 \circ r_7 \circ r_8)(aa^*x^{-1}) = 0$
(VII)	$(r_5 \circ r_8 - r_2)(a^*x^{-1}x) = 0$	(XII)	$(r_4 - r_4 \circ r_8 \circ r_7)(a^*ax^{-1}) = 0$

We explicitly demonstrate (X), one of the more involved resolutions:

$$\begin{aligned}
a^*ax = (a^*a)x &\xmapsto{r_4} [(r' + q^{-1})x - 1]x = (r' + q^{-1})x^2 - x, \\
a^*ax = a^*(ax) &\xmapsto{r_6} a^*(qxa - qar'x) \\
&\xmapsto{r_4 \circ r_5} q(r' + q^{-1})xa^*a - q((r' + q^{-1})x - 1)r'x \\
&\xmapsto{r_4} q(r' + q^{-1})x((r' + q^{-1})x - 1) - q((r' + q^{-1})x - 1)r'x \\
&\qquad = q(r' + q^{-1})x(q^{-1}x - 1) + qr'x = (r' + q^{-1})x^2 - x. \qquad \square
\end{aligned}$$

Remark 7.2. The choice of $\bar{B}$ was important here. If we instead had defined it so that $(q+t)^{-1} \in \bar{B}$, i.e., if we replace $r' = (q+r)^{-1} - q^{-1} \in \mathfrak{R}$ by $(q+r)^{-1}$, then our desired basis would no longer be linearly independent. Indeed, reducing aa^*a one way, we get $(x-1)a = xa - a$, which is irreducible, whereas the other way we get $a(y-1) = a(q+r)^{-1}x - a$, also irreducible. That is, $xa = a(q+r)^{-1}x$, an equality of two distinct irreducible elements.

Proposition 7.3. *$\Lambda^q(\widetilde{A}_n)$ is prime for all $n \geq 0$ and all $q \in (k^\times)^{n+1}$.*

Proof. We need to show, for every pair $f, g \in \Lambda^q(\widetilde{A}_n)$, both nonzero, there exists some $h \in \Lambda^q(\widetilde{A}_n)$ such that $fhg \neq 0$. It suffices to show that there exists vertices i, j and h such that $e_i f e_j hg \neq 0$, and hence we can take f to be a linear combination of basis elements that all begin at i and end j. By right multiplication by a^{n-j} or $(a^*)^j$, one can take f to be a linear combination of basis elements ending at vertex 0. By left multiplication by a^i or $(a^*)^{n-i}$ and then applying reordering reductions — the q-commutator, $ax - qxa$ is zero, for instance, in $\Lambda^q(\widetilde{A}_n)$ — one can take f to be a linear combination of basis elements starting and ending at vertex 0. In fact, f is of the form $e_0 f_1(x, x^{-1}) f_2(a^{n+1})$, where $f_1 \neq 0$ and f_2 has nonzero constant term. And similarly, we can take $g = e_0 g_1(x, x^{-1}) g_2(a^{n+1})$. Then their product has nonzero term $e_0 f_1(x, x^{-1}) f_2(a^{n+1})(0) g_1(x, x^{-1}) g_2(a^{n+1})(0)$ and hence is nonzero. $\square$

7B. *Partial multiplicative preprojective algebras.* First we define a *partial multiplicative* preprojective algebra following the definition of a partial preprojective algebra by [31, Definition 3.1.1].

Definition 7.4. Fix a quiver Q and $q \in (k^*)^{Q_0}$. Define a partition of the vertex set $Q_0 = \mathcal{B} \sqcup \mathcal{W}$ into a set $\mathcal{B}$ of *black* vertices and a set $\mathcal{W}$ of *white* vertices. The *partial multiplicative preprojective algebra* of $(Q, \mathcal{W})$ is

$$\Lambda^q(Q, \mathcal{W}) := L/(r_{\mathcal{B}}), \quad \text{where} \quad r_{\mathcal{B}} := 1_{\mathcal{B}} r 1_{\mathcal{B}}, \quad \text{for} \quad 1_{\mathcal{B}} := \sum_{j \in \mathcal{B}} e_j.$$

In words, we don't enforce the relations at the white vertices. Hence this algebra interpolates between $\Lambda^q(Q, Q_0) = L$ and $\Lambda^q(Q, \varnothing) = \Lambda^q(Q)$.

Definition 7.5. Let Q be a quiver and let Γ be its underlying graph. Fix $\mathcal{R} \subset Q_0$:

- A subgraph $T \subset \Gamma$ is a *tree* if it is connected and acyclic.
- A tree $T \subset \Gamma$ is *rooted in* $\mathcal{R}$ if it has a single vertex, called the root, in $\mathcal{R}$.
- A *forest rooted in* $\mathcal{R}$ is a disjoint union of trees *rooted in* $\mathcal{R}$.
- A subgraph $S \subset \Gamma$ is *spanning* if the vertex set of S is Q_0.

Notice that every doubled quiver $\overline{Q}$ with $\mathcal{W} \subset Q_0$ nonempty has a spanning forest, F, rooted in $\mathcal{W}$. We view such an F as a subquiver of $\overline{Q}$ by orienting the arrows *towards* the roots, see Figure 1. Since the isomorphism class of $\Lambda^q(Q)$ is independent of the orientation of Q, see Remark 2.2, we can assume that $F_1 \subset Q_1$.

Let $B := \mathcal{B}[t, (t+q)^{-1}]$. Each choice of spanning forest of $\overline{Q}$ rooted at $\mathcal{W}$ gives rise to a linear isomorphism $\sigma' : \Lambda^q(Q, \mathcal{W}) *_{kQ_0} B \to L$ and hence a basis for $\Lambda^q(Q, \mathcal{W}) = L/(r_{\mathcal{B}})$.

Proposition 7.6. *Let Q be a connected quiver and $Q_0 = \mathcal{B} \sqcup \mathcal{W}$ a decomposition into black and white vertices with $\mathcal{W} \neq \varnothing$. Then $(L, r_{\mathcal{B}}, \sigma, B)$ satisfies the strong free product property for some choice of σ.*

In more detail, let $F \subset \overline{Q}$ be a spanning forest rooted in $\mathcal{W}$ with arrows $F_1 \subset Q_1$ directed towards the roots.

A basis for L is given by concatenable words in the set,

$$\{a,\ x_a,\ x_a^{-1} \mid a \in \overline{Q}_1\} \cup \{r_{\mathcal{B}},\ r'_{\mathcal{B}} := (q + r_{\mathcal{B}})^{-1} - q^{-1}\},$$

such that the following subwords do not occur:

$$x_a x_a^{-1},\ x_a^{-1} x_a,\ aa^*,\ a x_{a^*}^{\pm 1} \ \textit{for } a \in \overline{Q}_1, \quad x_a^{\pm 1},\ x_{a^*}^{-1},\ x_{a^*} a^*,\ x_{a^*}^2 \ \textit{for } a \in F_1, \quad r_{\mathcal{B}} r'_{\mathcal{B}},\ r'_{\mathcal{B}} r_{\mathcal{B}}.$$

The words in which $r_{\mathcal{B}}$ and $r'_{\mathcal{B}}$ do not occur form a basis for $\Lambda^q(Q, \mathcal{W}) = L/(r_{\mathcal{B}})$, and the section σ is given by the inclusion of these elements.

Proof. The proof parallels that of Proposition 7.1. Write $r := r_{\mathcal{B}}$ and $r' := r'_{\mathcal{B}}$.

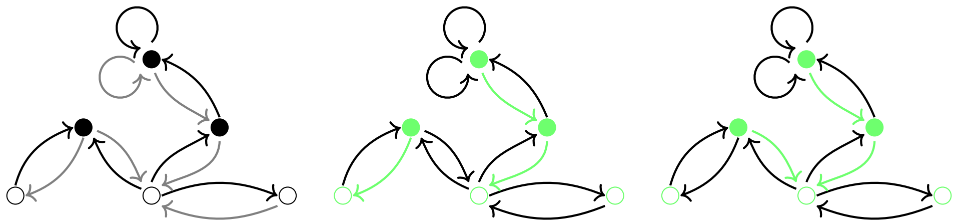


Figure 1. The quiver on the left is a doubled quiver, obtained by adding the gray arrows. It has three white vertices and three black vertices. The middle and right diagrams show two inequivalent spanning forests, in light green, with roots at the white vertices.

Note that L is spanned by the set, $\mathcal{M}$, of concatenable words in $\{a, x_a, x_a^{-1}, r, r' \mid a \in \bar{Q}_1\}$. These words are subject to the following relations, depending on a choice of ordering $\leq$ on the arrows $a \in \bar{Q}_1$:

$$x_a x_a^{-1} = 1 = x_a^{-1} x_a, \quad x_a = 1 + aa^*, \tag{7-7}$$

$$r = \prod_{\substack{a\in\bar{Q}\\ t(a)\in\mathcal{B}}} x_a^{\epsilon(a)} - q, \quad r' = \prod_{\substack{a\in\bar{Q}\\ t(a)\in\mathcal{B}}} x_a^{-\epsilon(a)} - q^{-1}, \tag{7-8}$$

$$rr' = r'r = -qr' - q^{-1}r, \tag{7-9}$$

where recall we write $t(a)$ for the *tail* or source of a, not the target. Define

$$l_a := \prod_{\substack{b\in\bar{Q}\\ b<a,\ t(b)\in\mathcal{B}}} x_b^{\epsilon(b)} \quad \text{and} \quad r_a := \prod_{\substack{b\in\bar{Q}\\ b>a,\ t(b)\in\mathcal{B}}} x_b^{\epsilon(b)}.$$

So for $a \in \bar{Q}_1$ with $t(a) \in \mathcal{B}$ we have the relation

$$l_a(1+aa^*)^{\epsilon(a)} r_a = (r+q)e_{t(a)} \implies x_a^{\epsilon(a)} = l_a^{-1}(r+q)(e_{t(a)})r_a^{-1}.$$

Hence in L, define $\mathrm{red}_a^{\epsilon(a)} := l_a^{-1}(r+q)(e_{t(a)})r_a^{-1}$.

We implement the above relations with the following reductions:

<ins>Inverse reductions</ins>: $x_a x_a^{-1}, x_a^{-1} x_a \mapsto 1$ for $a \in \bar{Q}_1$.

<ins>Short cycle reductions</ins>: $aa^* \mapsto x_a - 1$ for $a \in \bar{Q}_1$.

<ins>Reordering reductions</ins>: $a^* x_a^{\pm} \mapsto x_{a^*}^{\pm} a^*$ for $a \in \bar{Q}_1$.

<ins>Substitution reductions</ins>: $x_a^{\pm} \mapsto \mathrm{red}_a^{\pm}$, $x_{a^*}^{-1} \mapsto 1 - a^* \,\mathrm{red}_a^{-1} a$, $x_{a^*}^2 \mapsto x_{a^*} + a^* \,\mathrm{red}_a\, a$, $x_{a^*} a^* \mapsto a^* \,\mathrm{red}_a$, for $a \in F_1$.

<ins>Reductions in B</ins>: $rr', r'r \mapsto -qr' - q^{-1}r$.

For each word $w \in \mathcal{M}$, use the definition in (7-4) to define a weighted size,

$$\varphi_a(w) := n_{\{a,a^*\}}(w) + \tfrac{3}{2} n_{\{x_a, x_{a^*}\}}(w) + 3n_{\{x_a^{-1}, x_{a^*}^{-1}\}}(w)$$

for each $a \in \overline{Q}_1$. Define a total ordering on the arrows $(\overline{Q}_1, \prec)$ such that,

$$a \prec a' \text{ if } a \in F_1, a' \in \overline{Q}_1 \setminus F_1, \text{ or if } a, a' \in F_1 \text{ with } a' \text{ disconnected from } \mathcal{W} \text{ in } F_1 \setminus \{a\}.$$

Intuitively, we are saying that arrows in the spanning forest come before the rest in the ordering, with arrows closer to the white vertices coming first. Using $\prec$, φ_a, and (7-4), (7-5) define

$$N' : \mathcal{M} \to \mathbb{N}^{(\overline{Q}_1, \prec)} \times \mathbb{N}^2, \quad w \mapsto (2\varphi_a(w), n_{\{a|a\in\overline{Q}_1\},\{x_a|a\in\overline{Q}_1\}}(w), n_{\{r,r'\}}(w)),$$

from which we say $w \le w'$ if $N'(w) \le N'(w')$ in the lexicographical ordering on $\mathbb{N}^{|\overline{Q}_1|+2}$.

Notice, as in Proposition 7.1, that $N'(r_i(w)) < N'(w)$ for any word w and reduction r_i with $r_i(w) \neq w$. First notice that, by design, φ_a decreases under the following reductions:

Inverse reductions: $\varphi_a(x_a x_a^{-1}) = \varphi_a(x_a^{-1} x_a) = 3 + \frac{3}{2} > 0 = \varphi_a(1)$.

Short cycle reductions: $\varphi_a(aa^*) = 2 > \frac{3}{2} = \varphi_a(x_a)$.

Substitution reductions: $\varphi_a(x_a) = \frac{3}{2} > 0 = \varphi_a(\mathrm{red}_a)$, $\varphi_a(x_a^{-1}) = 3 > 0 = \varphi_a(\mathrm{red}_a^{-1})$, $\varphi(x_{a^*}^{-1}) = 3 > 2 = \varphi_a(a^* \,\mathrm{red}_a^{-1}\, a)$, $\varphi(x_{a^*}^2) = 3 > 2 = \varphi_a(a^* \,\mathrm{red}_a\, a)$ and $\varphi(x_{a^*}^2) = 3 > \frac{3}{2} = \varphi_a(x_{a^*})$, $\varphi_a(x_{a^*} a^*) = \frac{5}{2} > 1 = \varphi_a(a^* \,\mathrm{red}_a)$.

For the substitution reductions observe that red_a for $a \in F_1$ has subwords $x_b^{\pm 1}, x_{b^*}^{\pm 1}$ for only $b \in F_1$ which are necessarily *farther* from the root than a, and the remaining arrows are not in the spanning forest. Consequently, φ_a decreasing — despite φ_b increasing for some $b \succ a$ — implies that N' decreases. The reordering reductions preserve all φ_a but decrease $n_{\{a|a\in\overline{Q}_1\},\{x_a|a\in\overline{Q}_1\}}$ by definition, and hence decrease N'. The reductions in B preserve all φ_a and $n_{\{a|a\in\overline{Q}_1\},\{x_a|a\in\overline{Q}_1\}}$ but decrease $n_{\{r,r'\}}$, hence N'.

We conclude that every $w \in \mathcal{M}$ reduces to a kQ_0-linear combination of words without subwords in the leading terms of the reductions:

$$\{x_a x_a^{-1}, x_a^{-1} x_a, aa^*, a x_{a^*}, a x_{a^*}^{-1} \mid a \in \overline{Q}_1\} \cup \{x_{a^*}^{-1}, x_{a^*} a, x_{a^*}^2 \mid a \in F_1\}$$

after applying *finitely* many reductions.

Note that some generators are nonreduced: x_a, x_a^{-1}, and $x_{a^*}^{-1}$ for $a \in F_1$. Therefore, we can put in reductions for each of these and throw out all other reductions involving these generators, provided we check that all the defining relations still reduce to zero. We have the reductions:

(1) $x_a x_a^{-1} \xmapsto{r_1} 1$ for $a \in \overline{Q}_1$.	(5) $a x_{a^*}^{\pm} \xmapsto{r_5} x_a^{\pm} a$ for $a \notin \overline{F}_1$.
(2) $x_a^{-1} x_a \xmapsto{r_2} 1$ for $a \in \overline{Q}_1$.	(6) $a x_{a^*} \xmapsto{r_6} \mathrm{red}_a\, a$ for $a \in F_1$.
(3) $aa^* \xmapsto{r_3} \mathrm{red}_a - 1$ for $a \in F_1$.	(7) $x_{a^*}^2 \xmapsto{r_7} x_{a^*} + a^* \,\mathrm{red}_a\, a$ for $a \in F_1$.
(4) $aa^* \xmapsto{r_4} x_a - 1$ for $a \notin F_1$.	(8) $x_{a^*} a^* \xmapsto{r_8} a^* \,\mathrm{red}_a$ for $a \in F_1$.

Which don't overlap with the remaining reductions:

Substitution reductions: $x_a^{\pm 1} \mapsto \mathrm{red}_a^{\pm 1}$, $x_{a^*}^{-1} \mapsto 1 - a^* x_a^{-1} a$, $a \in F_1$.

Reductions in B: $rr', r'r \mapsto -qr' - q^{-1} r$.

As before, reductions (3) and (4) imply the relations $x_a = 1 + aa^*$, whereas the Substitution Reductions imply the defining relations for r, r'. So this is a valid reduction system.

This reduction system has thirteen ambiguities:

(I) $x_a x_a^{-1} x_a$ for $a \notin \bar{F}_1$.

(II) $x_a^{-1} x_a x_a^{-1}$ for $a \notin \bar{F}_1$.

(III) $a x_{a^*} x_{a^*}^{-1}$ for $a \notin \bar{F}_1$.

(IV) $a x_{a^*}^{-1} x_{a^*}$ for $a \notin \bar{F}_1$.

(V) $a x_{a^*}^2$ for $a \in F_1$.

(VI) $x_{a^*}^2 a^*$ for $a \in F_1$.

(VII) $x_{a^*} a^* a$ for $a \in F_1$.

(VIII) $a x_{a^*} a^*$ for $a \in F_1$.

(IX) $a^* a x_{a^*}$ for $a \in F_1$.

(X) $a^* a x_{a^*}$ for $a \in \bar{Q} \backslash F_1$.

(XI) aa^*a for $a \in \bar{Q}_1 \backslash \bar{F}_1$.

(XII) aa^*a for $a \in F_1$.

(XIII) aa^*a for $a^* \in F_1$.

Which all resolve by the resolutions:

(I) $x_a x_a^{-1} x_a$ for $a \notin \bar{F}_1$.

(II) $(r_2 - r_1)(x_a^{-1} x_a x_a^{-1}) = 0$.

(III) $(r_1 - r_1 \circ r_5 \circ r_5)(a x_{a^*} x_{a^*}^{-1}) = 0$.

(IV) $(r_2 - r_2 \circ r_5 \circ r_5)(a x_{a^*}^{-1} x_{a^*}) = 0$.

(V) $(r_3 \circ r_6 \circ r_7 - r_6 \circ r_6)(a x_{a^*}^2) = 0$.

(VI) $(r_3 \circ r_8 \circ r_7 - r_8 \circ r_8)(x_{a^*}^2 a^*) = 0$.

(VII) $(r_7 \circ r_4 - r_8)(x_{a^*} a^* a) = 0$.

(VIII) $(r_3 \circ r_6 - r_3 \circ r_8)(a x_{a^*} a^*) = 0$.

(IX) $(r_7 \circ r_4 - r_6)(a^* a x_{a^*}) = 0$.

(X) $(r_4 \circ r_5 \circ r_5 - r_4)(a^* a x_{a^*}) = 0$.

(XI) $(r_4 - r_5 \circ r_4)(aa^*a) = 0$.

(XII) $(r_6 \circ r_4 - r_3)(aa^*a) = 0$.

(XIII) $(r_8 \circ r_4 - r_3)(aa^*a) = 0$.

The resolutions of the ambiguities (I)–(IV) and (X)–(XIII) are quick, leaving the computational heart of the calculations with the five resolutions (V)–(IX). Note that the resolutions for (V) and (VI) are identical after swapping the roles of reductions r_6 and r_8, and similarly for (IX) and (VII), leaving three calculations: (V), (VIII), and (IX). These ambiguities express the overlap of r_6 with r_7, r_8, and r_4 respectively and further reduce uniquely to $\mathrm{red}_a^2\, a$, $\mathrm{red}_a(\mathrm{red}_a - 1)$, and $a^* \,\mathrm{red}_a\, a$. $\square$

7C. ***A convenient substitution.*** It will be convenient for us to make the substitutions

$$\overline{x_a^{\pm}} := x_a^{\pm 1} - 1, \tag{7-10}$$

motivated as follows.

Let $A \cong \Lambda^q(Q, \mathcal{W})$ for Q connected, and $\mathcal{W}$ possibly empty. Let I be the ideal generated by all paths beginning and ending at vertices having either $q = 1$ or in $\mathcal{W}$ (if nonempty). Then A/I is nonzero, and we can make use of the I-adic filtration. The modified generators $\overline{x_a^{\pm}}$, for a an arrow in I, have the advantage of lying in the ideal I. As we will show, in the cases Q contains a cycle and $\mathcal{W} \neq \varnothing$, the I-adic filtration is Hausdorff.

Thus, we get an embedding of A into the completion $\widehat{A}_I$, realizing $\overline{x_a^{\pm}}$ as power series with zero constant term. In the special case where $q = 1$ at all black vertices, this embedding sends every modified generator, $\overline{x_a^{\pm}}$, to a noncommutative power series in arrows *with* zero constant term. This completion is closely related to the completion of (partial) additive preprojective algebras with $\lambda = 0$ at all black vertices.

Practically speaking, we only require the above substitution at white vertices to obtain a basis for quivers containing cycles, see Section 7D. But theoretically, we advocate for this substitution at any vertex where we think of q as a deformation parameter based at $q = 1$.

Let us explain how this substitution works in the case of the cycle $\widetilde{A}_n$ (although we do not strictly need it in that case). We formally set $\overline{x^{\pm}} := x^{\pm 1} - 1$ and $\overline{y^{\pm}} := y^{\pm 1} - 1$; then the modified reductions from Section 7A are the following ones:

<u>Inverse reductions</u>: $\overline{x^+}\,\overline{x^-}\,\overline{x^-}\,\overline{x^+} \mapsto -\overline{x^+} - \overline{x^-}$ and $\overline{y^+}\,\overline{y^-}, \overline{y^-}\,\overline{y^+} \mapsto -\overline{y^+} - \overline{y^-}$.

<u>Short cycle reductions</u>: $aa^* \mapsto \overline{x^+}$, $a^*a \mapsto \overline{y^+}$.

<u>Reordering reductions</u>: $a^*\overline{x^{\pm}} \mapsto \overline{x^{\pm}}a^*$, $a\overline{y^{\pm}} \mapsto \overline{y^{\pm}}a$.

<u>Substitution reductions</u>: $\overline{y^-} \mapsto \overline{x^-}(r+q) + r + (q-1)$, (if not preceded by a); $\overline{y^+} \mapsto (r' + q^{-1})\overline{x^+} + r' + (q^{-1} - 1)$ (if not preceded by a); $a\overline{x^+} \mapsto a(r+q)\overline{y^+} + ar + (q-1)a$; $a\overline{x^-} \mapsto a\overline{y^-}(r' + q^{-1}) + ar' + (q^{-1} - 1)a$.

This produces the same ambiguities as before, which resolve in the same way after eliminating the nonreduced generators $\overline{y^{\pm}}$ (another way to say this is that the reductions are the same up to the change of variables, so ambiguities resolve if and only if they did before). The modified ordering function,

$$N^z(w) := (n_a(w), n_{\{a,a^*\},\{\overline{x^+},\overline{x^-},\overline{y^+},\overline{y^-}\}}(w), n_{\{a\overline{x^+},a\overline{x^-}\}}(w), n_{\{\overline{y^+},\overline{y^-}\}}(w)),$$

is strictly decreasing under applications of reductions and hence every term reduces after applying finitely many reductions. So we have proven the following variant of Proposition 7.1:

Proposition 7.7. *Let $Q \cong \widetilde{A}_n$ be a cycle. Then L_Q is a free left kQ_0-module with basis given by alternating words in $\mathfrak{R}$ and $\mathfrak{B}' := \{(\overline{x^{\pm}})^m a^{\ell}, (\overline{x^{\pm}})^m (a^*)^{\ell} \mid m \in \mathbb{N}, \ell \in \mathbb{N}\}$. Hence $\mathfrak{B}'$ is a basis for $\Lambda^q(Q)$.*

In the case of the partial multiplicative preprojective algebra, the modified reductions are as follows:

<u>Inverse reductions</u>: $\overline{x_a^+}\,\overline{x_a^-}, \overline{x_a^-}\,\overline{x_a^+} \mapsto -\overline{x_a^+} - \overline{x_a^-}$ for $a \in \overline{Q}_1$.

<u>Short cycle reductions</u>: $aa^* \mapsto \overline{x_a^+}$ for $a \in \overline{Q}_1$.

<u>Reordering reductions</u>: $a^*\overline{x_a^{\pm}} \mapsto \overline{x_{a^*}^{\pm}}a^*$ for $a \in \overline{Q}_1$.

<u>Substitution reductions</u>: $\overline{x_a^{\pm}} \mapsto \mathrm{red}_a^{\pm} - 1$, $\overline{x_{a^*}^-} \mapsto -a^* \mathrm{red}_a^{-1} a$, $\overline{x_{a^*}^+}^2 \mapsto -\overline{x_{a^*}^+} + a^* \mathrm{red}_a a$, $\overline{x_{a^*}^+}a^* \mapsto a^*(\mathrm{red}_a - 1)$, for $a \in F_1$.

Again, the same ordering function applies here and strictly decreases under these reductions. The ambiguities must resolve since they did before.

Proposition 7.8. *Let $Q, \mathcal{B}, \mathcal{W}$ be as in Proposition 7.6. Then L_Q is a free left kQ_0-module with basis given by concatenable words in the set,*

$$\{a, \overline{x_a^+}, \overline{x_a^-} \mid a \in \overline{Q}_1\} \cup \{r_{\mathcal{B}}, r'_{\mathcal{B}}\},$$

such that the following subwords do not occur:

$$\overline{x_a^+}\,\overline{x_a^-},\ \overline{x_a^-}\,\overline{x_a^+},\ aa^*,\ a\overline{x_{a^*}^\pm}\ \textit{for } a \in \overline{Q}_1, \quad \overline{x_a^\pm},\ \overline{x_{a^*}^-},\ \overline{x_{a^*}^+}a^*,\ \overline{x_{a^*}^+}^2\ \textit{for } a \in F_1, \quad r_{\mathcal{B}}r'_{\mathcal{B}},\ r'_{\mathcal{B}}r_{\mathcal{B}}.$$

A basis for Λ_Q^q *as a free* kQ_0*-module is given by those words above not containing* $r_{\mathcal{B}}, r'_{\mathcal{B}}$.

Remark 7.9. Note that, for the following subsection, we only require the substitutions $\overline{x_a}$ in the case where the arrow a begins at a white vertex (which in particular implies that $a \notin F_1$, although it could be that $a^* \in F_1$). If we only make these substitutions, it is similarly easy to write the above reductions in the case where for certain arrows $\overline{x_a^\pm}$ appears and for others $x_a^{\pm 1}$ appears; we leave this to the reader.

The only thing that we require from the above in the next subsection is the following observation:

$$\textit{Reductions on } 1_{\mathcal{W}} L_Q 1_{\mathcal{W}} \textit{ preserve the augmentation ideal, } \ker(\Lambda^q(Q, \mathcal{W}) \to k\mathcal{W}). \tag{7-11}$$

In other words, any monomial of positive length beginning and ending at white vertices reduces to a linear combination of other such monomials. This was not true with the original generators (e.g., looking at the inverse reductions).

7D. *Quivers containing cycles.* In this section, we prove the strong free product property for a connected quiver containing a cycle, along with providing a natural decomposition and basis for its multiplicative preprojective algebra. In more detail, the multiplicative preprojective algebra decomposes (as a vector space) into a free product of the multiplicative preprojective algebra for the cycle and a *partial* multiplicative preprojective algebra for the complement of the cycle. This technique should extend to the case of general extended Dynkin quivers, hence reducing Conjecture 1.1 to the extended Dynkin case.

Let Q be a connected quiver containing a cycle Q_E, with complement $Q' := Q \setminus Q_E$. Let $\mathcal{W} := (Q_E)_0$, so the vertices of the cycle are white. Fix $q \in (k^*)^{Q_0}$ and a decomposition $q = (q_E, q')$. There is a linear isomorphism

$$\Psi : \Lambda^{q_E}(Q_E) *_{kQ_0} \Lambda^{q'}(Q', \mathcal{W}) \to \Lambda^q(Q). \tag{7-12}$$

We prove this by producing a basis of $\Lambda^q(Q)$ of alternating words in $\Lambda^{q_E}(Q_E)$ and $\Lambda^{q'}(Q', \mathcal{W})$.

Remark 7.10. For the (deformed) *additive* preprojective algebra, the analogous map,

$$\Psi_{\text{add}} : \Pi^{\lambda_E}(Q_E) *_{kQ_0} \Pi^{\lambda'}(Q', \mathcal{W}) \to \Pi^\lambda(Q),$$

is an isomorphism for all connected quivers Q containing an extended Dynkin quiver Q_E. This follows from the proof of [31, Theorem 3.4.2]; see also [54, Section 5], particularly Corollary 5.2.9(ii).

As before, let $B := kQ_0[t, (q+t)^{-1}]$ and $\bar{B} = tB$, which is spanned by elements $\{t^m, (t')^m \mid m \geq 1\}$ where $t' := (q+t)^{-1} - q^{-1}$.

Proposition 7.11. *Let* Q *be a connected quiver containing a cycle* $Q_E \subseteq Q$ ($Q_E \cong \widetilde{A}_{n-1}$). *Then there exists a section* $\sigma : \Lambda^q(Q) \to L$ *such that* (L, r, σ, B) *satisfies the strong free product property.*

In more detail, L_Q is a free left kQ_0-module with basis given by concatenable alternating products in the bases of $\Lambda^{q_E}(Q_E)$ given by Proposition 7.1 or 7.7, of $\Lambda^{q'}(Q', \mathcal{W})$ given by Proposition 7.8, and r^m, $(r')^m$ $(m \geq 1)$.

Corollary 7.12. *Let Q be as in Proposition 7.11. A basis for $\Lambda^q(Q)$ is given by concatenable alternating words in the mentioned bases of $\Lambda^{q_E}(Q_E)$ and $\Lambda^{q'}(Q', \mathcal{W})$. In particular, the family $\Lambda^q(Q)$ defines a free $k[q_i, q_i^{-1}]_{i\in Q_0}$-module, and hence is flat over $(k^\times)^{Q_0}$.*

Remark 7.13. Note in Proposition 7.11 that we only need to replace $x_a^{\pm 1}$ for $\overline{x_a^{\pm}}$ if $a \in \overline{Q'}_1$ begins at a vertex of Q_E. Moreover, making this change to the statement does not affect the proof. On the other hand, we could freely replace $x_a^{\pm 1}$ by $\overline{x_a^{\pm}}$ for *all* arrow in $\overline{Q}_1$, again without changing the proof.

Proof of Proposition 7.11. First we will establish that our proposed basis for L implies the strong free product property. To see this, observe that the set of subwords not containing r_i, r_i' for $i \in Q_0$ form a basis for $\Lambda^q(Q)$. The inclusion of basis elements $\Lambda^q(Q) \to L$ defines a section σ. Using σ define $\sigma' : \Lambda^q(Q) *_{kQ_0} kQ_0[t, (q+t)^{-1}] \to L$ to be the extension of the map taking $t \mapsto r$, $(q+t)^{-1} \mapsto r' + q$, and $p \mapsto \sigma(p)$ for $p \in \Lambda^q(Q)$. Then σ' is clearly a kQ_0-linear isomorphism, and hence (L, r, σ, B) satisfies the strong free product property.

Next we will show that the proposed basis for L implies that there exists a kQ_0-linear isomorphism: $\Psi : \Lambda^{q_E}(Q_E) *_{kQ_0} \Lambda^q(Q', \mathcal{W}) \to \Lambda^q(Q)$. For this, identify:

- $\Lambda^q(Q)$ as the span of words in L without the subwords r_i, r_i'.
- $\Lambda^{q_E}(Q_E)$ as the span of words in $\Lambda^q(Q)$ without the subwords $a, \overline{x_a^{\pm}}$ for $a \in \overline{Q'}_1$.
- $\Lambda^q(Q', \mathcal{W})$ as the span of words in $\Lambda^q(Q)$ without the subwords $b_i, x_{b_i}^{\pm 1}$ for $b_i \in \overline{Q_E}_1$.

Hence there exists kQ_0-linear maps $\iota_1, \iota_2 : \Lambda^{q_E}(Q_E), \Lambda^q(Q', \mathcal{W}) \to \Lambda^q(Q)$ defined by the inclusion of basis elements. These maps determine a unique injective kQ_0-linear map $\Psi := \iota_1 *_{kQ_0} \iota_2 : \Lambda^{q_E}(Q_E) *_{kQ_0} \Lambda^q(Q', \mathcal{W}) \to \Lambda^q(Q)$, which is clearly surjective, hence an isomorphism.

It remains to establish that the given set is indeed a basis for L. By Proposition 7.1 we have a basis $\mathfrak{B}_{Q_E}$ for L_{Q_E} and by Proposition 7.8 we have bases $\mathfrak{B}_{Q'}$ for $L_{Q'}$. Therefore we have a basis of alternating words in $\mathfrak{B}_{Q_E}$ and $\mathfrak{B}_{Q'}$ for $L = L_{Q_E} *_{kQ_0} L_{Q'}$.

However this basis gives rise to a basis for the quotient $L/(\rho_{Q_E} + \rho_{Q'} 1_{\mathcal{B}} - q)$. So we need to show that $L/(\rho_{Q_E} + \rho_{Q'} 1_{\mathcal{B}} - q)$ is isomorphic to $L/(\rho_{Q_E}\rho_{Q'} - q) =: \Lambda^q(Q)$ as kQ_0-modules. Hence we consider the system of reductions combining the systems of reductions from Propositions 7.1 and 7.8. Crucially, we perturb the system of reductions by perturbing the relation $r^{\mathrm{pre}} := \rho_{Q_E} + \rho_{Q'} 1_{\mathcal{B}} - q$ to $r = \rho_{Q_E}\rho_{Q'} - q$.

First observe that this change does nothing to the reductions for $L_{Q'}$, since the transformation is the identity on black vertices. That is, $r^{\mathrm{pre}} 1_{\mathcal{B}} = \rho_{Q'} 1_{\mathcal{B}} - q 1_{\mathcal{B}} = r 1_{\mathcal{B}}$.

For L_{Q_E}, notice $r^{\mathrm{pre}} 1_{\mathcal{W}} = \rho_{Q_E} - q_E$ while $r 1_{\mathcal{W}} = \rho_{Q_E}\rho_{Q'} - q_E$. So we alter each reduction involving $\rho_{Q_E}^{\pm}$ by the transformation

$$\rho_{Q_E} \mapsto \rho_{Q_E}\rho_{Q'} = \rho_{Q_E}(\rho_{Q'} - 1) + \rho_{Q_E}, \quad \rho_{Q_E}^{-1} \mapsto \rho_{Q'}^{-1}\rho_{Q_E}^{-1} = (\rho_{Q'}^{-1} - 1)\rho_{Q_E}^{-1} + \rho_{Q_E}^{-1}.$$

Note that we choose this form for the transformation to emphasize that the new relation splits as a sum of (I) the old relation and (II) a piece in the ideal generated by $\overline{x_a^{\pm}}$ for $a \in \overline{Q'}_1$. This transformation only effects the substitution reductions in the original reduction system for the cycle; see the proof of Proposition 7.1. The substitution reductions become (after applying a reordering reduction) the following:

$$\begin{aligned} y^{-1} &\mapsto x^{-1}(\rho_{Q'}-1)(r+q)+x^{-1}(r+q), && \text{(if not preceded by } a\text{)},\\ y &\mapsto (r'+q^{-1})(\rho_{Q'}^{-1}-1)x+(r'+q^{-1})x && \text{(if not preceded by } a\text{)},\\ ax &\mapsto a(\rho_{Q'}-1)(r+q)y+ary+qya, \\ ax^{-1} &\mapsto ay^{-1}(r'+q^{-1})(\rho_{Q'}^{-1}-1)+ay^{-1}r'+q^{-1}y^{-1}a. \end{aligned}$$

Order monomials in L lexicographically in the orderings N and N' of Propositions 7.1 and 7.6. Then the above reductions strictly decrease the ordering. Here we are using (7-11) from the previous subsection to deduce that the ideal of positive-length monomials beginning and ending at vertices of Q_E is preserved under reductions.

All ambiguities lie either entirely in L_{Q_E} or entirely in $L_{Q'}$. Hence the ambiguities in $L_{Q'}$ resolve as before. The ambiguities in L_{Q_E} still resolve using the same reductions as before perturbing. To see this, note that we have replaced the formal variables $\rho_{Q_E}^{\pm 1}$ (which do not interact with $a, a^*, x^{\pm 1}, y^{\pm 1}$) with the new formal variables $(\rho_{Q_E}\rho_{Q'})^{\pm 1}$.

Since the perturbed system of reductions has all the same leading coefficients as the original, we conclude that L has the desired basis. □

8. The center and primality of multiplicative preprojective algebras

Let Q be a connected quiver strictly containing a cycle. The goal of this section is to complete the proof of Theorem 1.2 by first establishing that $\Lambda^q(Q)$ is prime and then that $Z(\Lambda^q(Q)) = k$ and hence the Calabi–Yau structure is unique up to rescaling.

8A. *Primality of multiplicative preprojective algebras.*

We will show $\Lambda^q(Q)$ is prime by first showing that left multiplication by certain elements is injective on the subspace of concatenable elements.

Lemma 8.1. *Let a denote the sum of all the positively oriented arrows of the cycle in $\overline{Q}_1$. Then left multiplication by a, $L_a : 1_{\mathcal{W}}\Lambda^q(Q) \to 1_{\mathcal{W}}\Lambda^q(Q)$, is injective.*

Proof. Decompose the vertices $Q_0 = \mathcal{B} \sqcup \mathcal{W}$ where the white vertices are in the cycle. Decompose the arrows in $\overline{Q}_1 = \overline{Q_E}_1 \sqcup \overline{Q'}_1$. Define

$$A_+ := \ker(\epsilon_A : \Lambda^{q_E}(Q_E) \to k\mathcal{W}), \quad B_+ := \ker(\epsilon_B : \Lambda^q(Q', \mathcal{W}) \to k\mathcal{W}).$$

Then one can define a descending filtration by $\mathcal{F}_0 = \Lambda^q(Q)$ and $\mathcal{F}_m := \mathrm{Span}(B_+(A_+B_+)^{\geq m}))$ for $m > 0$. Notice $a \in \mathcal{F}_m, b \in \mathcal{F}_\ell$ implies $ab \in \mathcal{F}_{m+\ell}$, so this is an *algebra* filtration.

Consider the exact sequence $B_+ \xhookrightarrow{\iota} \Lambda^q(Q) \xrightarrow{\pi} \Lambda^{q_E}(Q_E)$. The basis of Proposition 7.11 realizes an inclusion $i : \Lambda^{q_E}(Q_E) \to \Lambda^q(Q)$, a kQ_0-module splitting. So for $\alpha, \beta \in \Lambda^{q_E}(Q_E)$, $i(\alpha) \cdot i(\beta) \equiv \alpha \cdot \beta$

modulo the two-sided ideal generated by B_+. Therefore, in the associated graded algebra $gr_{\mathcal{F}}(\Lambda^q(Q)) := \bigoplus_{m=0} \mathcal{F}_m/\mathcal{F}_{m+1}$,

$$i(\alpha)\cdot i(\beta) = \alpha\cdot\beta + \alpha\cdot\beta b'' + \alpha b'\beta + b\alpha\cdot\beta$$

for $b, b', b'' \in B_+$. Therefore, for $b_1 \in B_+$, there exists $b_2 \in B_+$ such that

$$i(\alpha)\cdot i(\beta)b_1 = \alpha\cdot\beta(1+b_2)b_1.$$

Recall A_+ has kQ_0-module basis given by $\{x_a^p,\ x_a^m a^\ell,\ x_a^m(a^*)^\ell \mid m,\ell,p\in\mathbb{Z},\ p\neq 0,\ \ell>0\}$ by Proposition 7.1. In the associated graded algebra $gr_{\mathcal{F}}(\Lambda^q(Q))$, L_a acts on A_+B_+ as follows, for $b \in B_+$:

$$a(x_a^m a^\ell)b = q^m x_a^m a^{\ell+1}b.$$
$$a(x_a^m(a^*)^\ell)b = q^m x_a^m(x_a-1)(a^*)^{\ell-1}b.$$
$$a(x_a^p)b = q^p x_a^p(x_a-1)(\rho_{Q'}^{-1})b.$$

Since L_a is injective on A_+, by Proposition 7.3, we conclude that L_a is injective on the right ideal generated by A_+.

Consider the basis of Proposition 7.11, and write $b \in 1_{\mathcal{W}}B_+$ in this basis. Then ab is again a basis element, and hence L_a takes basis elements injectively to basis elements. We conclude that L_a is injective on the right ideal generated by $1_{\mathcal{W}}B_+$, and therefore on all of $1_{\mathcal{W}}\Lambda^q(Q)$. □

Lemma 8.2. *Right multiplication by a, $R_a : \Lambda^q(Q)1_{\mathcal{W}} \to \Lambda^q(Q)1_{\mathcal{W}}$ is injective.*

The proof is completely analogous, using the same filtration, together with the calculations:

$$b(x_a^m a^\ell)a = bx_a^m a^{\ell+1}.$$
$$b(x_a^m(a^*)^\ell)a = bq^{-\ell+1}x_a^{m+1}(a^*)^{\ell-1} - bx_a^m(a^*)^{\ell-1}.$$
$$b(x_a^p)a = bx_a^p a.$$

Lemma 8.3. *Let $v \in Q_0$. There is unique path $\gamma_{v,w}$ in the spanning forest from v to a white vertex $w \in \mathcal{W}$. Right multiplication by $\gamma_{v,w}$, $R_{\gamma_{v,w}} : \Lambda^q(Q)e_v \to \Lambda^q(Q)e_w$, is injective.*

Proof. We need to show $\alpha\gamma_{h(\alpha),w} \neq 0$ for $\alpha \neq 0$. Consider the basis in Proposition 7.8, consisting of words in a, $\overline{x_a^\pm}$ for a an arrow, without certain disallowed subwords, e.g., aa^* for $a \in \overline{Q}_1$. Note that $\gamma_{h(\alpha),w}$ is a basis element as aa^* cannot appear in a shortest path. Write α as a linear combination of basis elements. Notice $\alpha\gamma_{h(\alpha),w}$ is a linear combination of basis elements *unless* the disallowed subword a^*a is created for some arrow $a \in F_1$. This disallowed subword reduces to $\overline{x_{a^*}^+}$ (which is not itself disallowed since $a^* \notin F_1$, as $a \in F_1$.) Furthermore, the appearance of $\overline{x_{a^*}^+}$ for $a \in F_1$ cannot create the disallowed subwords

$$\text{(I) } \overline{x_{a^*}^+}\,\overline{x_{a^*}^-}, \quad \text{(II) } \overline{x_{a^*}^-}\,\overline{x_{a^*}^+}, \quad \text{(III) } a\overline{x_{a^*}^\pm}, \quad \text{(IV) } \overline{x_{a^*}^+}^2, \quad \text{(V) } \overline{x_{a^*}^+}a^*,$$

for $a \in F_1$, as in each case α or $\gamma_{h(\alpha),w}$ would itself contain a disallowed subword

$$\text{(I) } \overline{x_{a^*}^-}, \quad \text{(II) } \overline{x_{a^*}^-}, \quad \text{(III) } aa^*, \quad \text{(IV) } a\overline{x_{a^*}^+} \text{ or } \overline{x_{a^*}^+}a^*, \quad \text{(V) } aa^*,$$

each a contradiction. We conclude that right multiplication by $\gamma_{h(\alpha),w}$ takes basis elements injectively to basis elements and hence is injective. $\square$

Lemma 8.4. *Let $v \in Q_0$. There is unique path $\gamma_{w,v}$ in the opposite of the spanning forest, F_1^{op}, from $w \in \mathcal{W}$ to v. Left multiplication by $\gamma_{w,v}$, $L_{\gamma_{w,v}} : e_v\Lambda^q(Q) \to e_w\Lambda^q(Q)$, is injective.*

The proof is identical, and follows from the isomorphism $\Lambda^q(Q) \cong \Lambda^q(Q)^{\mathrm{op}}$.

Proposition 8.5. *$\Lambda^q(Q)$ is prime, for Q connected and containing a cycle.*

Proof. Let $\alpha, \beta \in \Lambda^q(Q)$ be nonzero. We will show $\alpha\Lambda^q(Q)\beta \neq 0$ by building an explicit element $\gamma = \gamma_1\gamma_2$ so that $\alpha\gamma\beta \neq 0$. That is, define

$$\gamma_1 := \gamma_{h(\alpha),w}x_a^M a^N, \quad \gamma_2 := a^{N'}x_a^{M'}\gamma_{w,t(\beta)}$$

where $M, M', N, N' \in \mathbb{N}$ are sufficiently large (depending on α and β) and where $\gamma_{h(\alpha),w}$ and $\gamma_{w,t(\beta)}$ are as defined in Lemmas 8.3 and 8.4, respectively.

We will first show that right multiplication by γ_1 is injective on concatenable paths to conclude $\alpha\gamma_1 \neq 0$. Then we will argue that left multiplication by γ_2 is injective on concatenable paths to conclude $\gamma_2\beta \neq 0$. Finally, we will show that $\alpha\gamma_1\gamma_2\beta \neq 0$.

To show $R_{\gamma_1} : \Lambda^q(Q)e_{h(\alpha)} \to \Lambda^q(Q)e_{h(\gamma_1)}$ is injective, it suffices to show that right multiplication by each piece, $\gamma_{h(\alpha),w}$, x_a^M, and a^N, is injective. $R_{\gamma_{h(\alpha),w}}$ is injective by Lemma 8.3, $R_{x_a^M}$ is injective since x_a is invertible, and R_{a^N} is injective by Lemma 8.2.

Similarly, $L_{\gamma_2} : e_{t(\beta)}\Lambda^q(Q) \to e_{t(\gamma_2)}\Lambda^q(Q)$ is injective since L_a, L_{x_a}, and $L_{\gamma_{w,t(\beta)}}$ are injective by Lemma 8.1, invertibility of x_a, and Lemma 8.4, respectively.

Finally notice that $\alpha\gamma_1 \neq 0$ and $\gamma_2\beta \neq 0$ implies $\alpha\gamma_1\gamma_2\beta \neq 0$. To see this, consider the filtration $\mathcal{F}$ defined in the proof of Lemma 8.1. It suffices to show $\alpha\gamma_1\gamma_2\beta \neq 0$ in $\mathrm{gr}_{\mathcal{F}}(\Lambda^q(Q))$. Write $\alpha\gamma_1$ and $\gamma_2\beta$ in the basis of Proposition 7.11 (see the basis in Proposition 7.1). By design $\alpha\gamma_1$ ends with a basis element of the form $x_a^m a^n$ for $m, n > 0$ and $\gamma_2\beta$ begins with a basis element of the form $a^{n'}x_a^{m'}$ for $m', n' > 0$. Their product in $\mathrm{gr}_{\mathcal{F}}(\Lambda^q(Q))$ is the scaled basis element $q^{nm'}x_a^{m+m'}a^{n+n'}$. So $\alpha\gamma_1\gamma_2\beta \neq 0$ in $\mathrm{gr}_{\mathcal{F}}(\Lambda^q(Q))$ and hence in $\Lambda^q(Q)$, completing the proof. $\square$

8B. ***The center of multiplicative preprojective algebras.*** The center of $\Lambda^1(Q)$ depends dramatically on the taxonomy of quiver Q into Dynkin, extended Dynkin, and others:

- For Q Dynkin and k characteristic not 2, 3, or 5, one can compute the center using the isomorphism $\Lambda^1(Q) \cong \Pi(Q)$; see Example 2.3.
- For Q extended Dynkin, Conjecture 1.4 predicts $Z(\Lambda^1(Q)) \cong e_v\Lambda^1(Q)e_v$, which is proven in Section 6B in the case $Q = \widetilde{A}_n$.
- In the remaining cases, Conjecture 1.1 predicts $Z(\Lambda^q(Q)) = k$, for any $q \in (k^*)^{Q_0}$.

The goal of this section is to establish the conjecture in the case Q contains a cycle.

Proposition 8.6. *Let Q be a connected quiver strictly containing a cycle and fix $q \in (k^\times)^{Q_0}$. Then $Z(\Lambda^q(Q)) = k$.*

Proof. Let $z \in Z(\Lambda^q(Q))$. Decompose $z = z_0 + z_+$ into a sum of length zero and positive length paths. First suppose that $z_+ = 0$. Then $z = \sum_{i \in Q_0} c_i e_i$. Note that every individual arrow forms a basis element of Proposition 7.11. Then $za = az$ for every arrow implies that all c_i are equal, as Q is connected.

Now assume $z_+ \neq 0$. Expanding z_+ in the basis of Proposition 7.11, we write $z_+ = \sum_i c_i z_i$, where each z_i is a positive-length alternating word in the cycle and the complement. We claim that each z_i has an arrow not in the cycle. Suppose, by contradiction, there exists j such that z_j consists of only arrows in the cycle. Since Q strictly contains the cycle, there exists an arrow $b \in Q_1$ not in the cycle. And as z_+ commutes with each arrow a_i in the cycle, there exists l such that z_l consists of only arrows in the cycle that ends at $t(b)$. Then $z_+ b = b z_+$. But $z_+ b$ contains a term beginning with $x_a^m a^j$ for some m, j with $(m, j) \neq (0, 0)$. However, $b z_+$ has no term beginning $x_a^m a^j$ unless $(m, j) = (0, 0)$. This contradicts the existence of z_j consisting of only arrows in the cycle.

Since $z_+ \neq 0$, thanks to Lemma 8.3, there exists a vertex i and a path $b = \gamma_{h(z_+),i}$ such that $z_+ b e_i \neq 0$. Therefore also $b z_+ e_i \neq 0$, so $z_+ e_i \neq 0$. By Lemma 8.2, we then have $z_+ a^n \neq 0$ for all n. Hence also $a^n z_+ \neq 0$. Now, for sufficiently large $N \gg 0$, $a^N z_+$ contains basis elements beginning with an arbitrarily high power of the cycle. However, terms of $z_+ a^N$ begin only with powers of the cycle appearing in z_+, since every z_j has a term not in the cycle. These powers are bounded, so this contradicts the assumption that $z_+ \neq 0$. We conclude that z is a scalar multiple of the identity. $\square$

Corollary 8.7. *If Q is connected and properly contains a nonoriented cycle, then $\Lambda^q(Q)$ has a unique, up to scaling, Calabi–Yau structure.*

Proof. Write $\Lambda := \Lambda^q(Q)$. Any two Calabi–Yau structures differ by an invertible map in $\mathrm{Hom}_{\Lambda\text{−bimod}}(\Lambda, \Lambda)$, which is determined by the image of the unit, a central invertible element. So the set of Calabi–Yau structures on Λ, when nonempty, is a $Z(\Lambda)^\times$-torsor. By Proposition 8.6, $Z(\Lambda)^\times = k^\times$, so any two Calabi–Yau structures differ by an invertible scalar. $\square$

This completes the proof of Theorem 1.2.

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References

[1] D. J. Anick, "Noncommutative graded algebras and their Hilbert series", *J. Algebra* **78**:1 (1982), 120–140. MR Zbl

[2] M. Artin, "Algebraic approximation of structures over complete local rings", *Inst. Hautes Études Sci. Publ. Math.* 36 (1969), 23–58. MR Zbl

[3] A. Beauville, "Symplectic singularities", *Invent. Math.* **139**:3 (2000), 541–549. MR Zbl

[4] G. Bellamy and T. Schedler, "Symplectic resolutions of quiver varieties", *Selecta Math.* (*N.S.*) **27**:3 (2021), Paper No. 36, 50. MR Zbl

[5] G. Bellamy and T. Schedler, "Symplectic resolutions of quiver varieties", *Selecta Math.* (*N.S.*) **27**:3 (2021), Paper No. 36, 50. MR Zbl

[6] Y. Berest, G. Felder, and A. Ramadoss, "Derived representation schemes and noncommutative geometry", pp. 113–162 in *Expository lectures on representation theory*, edited by K. Igusa et al., Contemp. Math. **607**, Amer. Math. Soc., Providence, RI, 2014. MR Zbl

[7] M. Van den Bergh, "Non-commutative crepant resolutions", pp. 749–770 in *The legacy of Niels Henrik Abel*, edited by O. A. Laudal and R. Piene, Springer, 2004. MR Zbl

[8] M. Van den Bergh, "Three-dimensional flops and noncommutative rings", *Duke Math. J.* **122**:3 (2004), 423–455. MR Zbl

[9] M. Van den Bergh, "Non-commutative quasi-Hamiltonian spaces", pp. 273–299 in *Poisson geometry in mathematics and physics*, edited by G. Dito et al., Contemp. Math. **450**, Amer. Math. Soc., Providence, RI, 2008. MR Zbl

[10] M. Van den Bergh, "Calabi–Yau algebras and superpotentials", *Selecta Math.* (*N.S.*) **21**:2 (2015), 555–603. MR Zbl

[11] G. M. Bergman, "The diamond lemma for ring theory", *Adv. in Math.* **29**:2 (1978), 178–218. MR

[12] R. Bezrukavnikov and M. Kapranov, "Microlocal sheaves and quiver varieties", *Ann. Fac. Sci. Toulouse Math.* (6) **25**:2-3 (2016), 473–516. MR Zbl

[13] P. Boalch, "Global Weyl groups and a new theory of multiplicative quiver varieties", *Geom. Topol.* **19**:6 (2015), 3467–3536. MR Zbl

[14] R. Bocklandt, F. Galluzzi, and F. Vaccarino, "The Nori–Hilbert scheme is not smooth for 2-Calabi–Yau algebras", *J. Noncommut. Geom.* **10**:2 (2016), 745–774. MR

[15] T. Bozec, D. Calaque, and S. Scherotzke, "Calabi–Yau structures for multiplicative preprojective algebras", preprint, 2021. arXiv 2102.12336

[16] C. Braun, J. Chuang, and A. Lazarev, "Derived localisation of algebras and modules", *Adv. Math.* **328** (2018), 555–622. MR Zbl

[17] C. Brav and T. Dyckerhoff, "Relative Calabi–Yau structures II: shifted Lagrangians in the moduli of objects", *Selecta Math.* (*N.S.*) **27**:4 (2021), Paper No. 63, 45. MR Zbl

[18] S. Brenner, M. C. R. Butler, and A. D. King, "Periodic algebras which are almost Koszul", *Algebr. Represent. Theory* **5**:4 (2002), 331–367. MR Zbl

[19] I. Burban and Y. Drozd, "Maximal Cohen–Macaulay modules over surface singularities", pp. 101–166 in *Trends in representation theory of algebras and related topics*, edited by A. Skowroński, Eur. Math. Soc., Zürich, 2008. MR Zbl

[20] O. Chalykh and M. Fairon, "Multiplicative quiver varieties and generalised Ruijsenaars–Schneider models", *J. Geom. Phys.* **121** (2017), 413–437. MR Zbl

[21] J. Chuang and A. Lazarev, "Feynman diagrams and minimal models for operadic algebras", *J. Lond. Math. Soc.* (2) **81**:2 (2010), 317–337. MR Zbl

[22] W. Crawley-Boevey, "Monodromy for systems of vector bundles and multiplicative preprojective algebras", *Bull. Lond. Math. Soc.* **45**:2 (2013), 309–317. MR Zbl

[23] W. Crawley-Boevey and M. P. Holland, "Noncommutative deformations of Kleinian singularities", *Duke Math. J.* **92**:3 (1998), 605–635. MR

[24] W. Crawley-Boevey and Y. Kimura, "On deformed preprojective algebras", *J. Pure Appl. Algebra* **226**:12 (2022), Paper No. 107130, 22. MR Zbl

[25] W. Crawley-Boevey and P. Shaw, "Multiplicative preprojective algebras, middle convolution and the Deligne–Simpson problem", *Adv. Math.* **201**:1 (2006), 180–208. MR Zbl

[26] J. Cuntz and D. Quillen, "Algebra extensions and nonsingularity", *J. Amer. Math. Soc.* **8**:2 (1995), 251–289. MR Zbl

[27] B. Davison, "Purity and 2-Calabi–Yau categories", preprint, 2021. arXiv 2106.07692

[28] T. Ekholm and Y. Lekili, "Duality between Lagrangian and Legendrian invariants", preprint, 2017. arXiv 1701.01284

[29] T. Etgü and Y. Lekili, "Koszul duality patterns in Floer theory", *Geom. Topol.* **21**:6 (2017), 3313–3389. MR

[30] T. Etgü and Y. Lekili, "Fukaya categories of plumbings and multiplicative preprojective algebras", *Quantum Topol.* **10**:4 (2019), 777–813. MR Zbl

[31] P. Etingof and C.-H. Eu, "Koszulity and the Hilbert series of preprojective algebras", *Math. Res. Lett.* **14**:4 (2007), 589–596. MR Zbl

[32] P. Etingof and V. Ginzburg, "Noncommutative complete intersections and matrix integrals", *Pure Appl. Math. Q.* **3**:1, Special Issue: In honor of Robert D. MacPherson. Part 3 (2007), 107–151. MR Zbl

[33] P. Etingof, A. Oblomkov, and E. Rains, "Generalized double affine Hecke algebras of rank 1 and quantized del Pezzo surfaces", *Adv. Math.* **212**:2 (2007), 749–796. MR Zbl

[34] S. Ganatra, *Symplectic cohomology and duality for the wrapped Fukaya category*, Ph.D. thesis, Massachusetts Institute of Technology, 2012, available at https://www.proquest.com/docview/1238001248. MR

[35] I. Ganev, D. Jordan, and P. Safronov, "The quantum Frobenius for character varieties and multiplicative quiver varieties", preprint, 2019. arXiv 1901.11450

[36] V. Ginzburg, "Calabi–Yau algebras", preprint, 2006. arXiv 0612139

[37] O. Iyama and I. Reiten, "Fomin–Zelevinsky mutation and tilting modules over Calabi–Yau algebras", *Amer. J. Math.* **130**:4 (2008), 1087–1149. MR Zbl

[38] O. Iyama and M. Wemyss, "Singular derived categories of $\mathbb{Q}$-factorial terminalizations and maximal modification algebras", *Adv. Math.* **261** (2014), 85–121. MR Zbl

[39] D. Jordan, "Quantized multiplicative quiver varieties", *Adv. Math.* **250** (2014), 420–466. MR Zbl

[40] T. V. Kadeishvili, "The algebraic structure in the homology of an $A(\infty)$-algebra", *Soobshch. Akad. Nauk Gruzin. SSR* **108**:2 (1982), 249–252. MR Zbl

[41] H. Kajiura, "Noncommutative homotopy algebras associated with open strings", *Rev. Math. Phys.* **19**:1 (2007), 1–99. MR Zbl

[42] D. Kaplan, "Multiplicative preprojective algebras of Dynkin quivers", *J. Pure Appl. Algebra* **227**:1 (2023), Paper No. 107146, 26. MR Zbl

[43] M. Kapranov and E. Vasserot, "Kleinian singularities, derived categories and Hall algebras", *Math. Ann.* **316**:3 (2000), 565–576. MR Zbl

[44] B. Keller, "Deformed Calabi–Yau completions", *J. Reine Angew. Math.* **654** (2011), 125–180. MR Zbl

[45] A. D. King, "Moduli of representations of finite-dimensional algebras", *Quart. J. Math. Oxford Ser.* (2) **45**:180 (1994), 515–530. MR Zbl

[46] M. Kontsevich, "Deformation quantization of Poisson manifolds", *Lett. Math. Phys.* **66**:3 (2003), 157–216. MR Zbl

[47] Y. Lekili and K. Ueda, "Homological mirror symmetry for Milnor fibers of simple singularities", *Algebr. Geom.* **8**:5 (2021), 562–586. MR Zbl

[48] D. Luna, "Slices étales", pp. 81–105 in *Sur les groupes algébriques*, Mém. Soc. Math. France **33**, Soc. Math. France, Paris, 1973. MR Zbl

[49] M. Markl, "Transferring A_∞ (strongly homotopy associative) structures", *Rend. Circ. Mat. Palermo* (2) *Suppl.* 79 (2006), 139–151. MR Zbl

[50] R. Martínez-Villa, "Applications of Koszul algebras: the preprojective algebra", pp. 487–504 in *Representation theory of algebras* (Cocoyoc, 1994), edited by R. Bautista et al., CMS Conf. Proc. **18**, Amer. Math. Soc., Providence, RI, 1996. MR Zbl

[51] Y. Matsushima, "Espaces homogènes de Stein des groupes de Lie complexes", *Nagoya Math. J.* **16** (1960), 205–218. MR Zbl

[52] Y. Namikawa, "Extension of 2-forms and symplectic varieties", *J. Reine Angew. Math.* **539** (2001), 123–147. MR Zbl

[53] G. D. Rizell, T. Ekholm, and D. Tonkonog, "Refined disk potentials for immersed Lagrangian surfaces", *J. Differential Geom.* **121**:3 (2022), 459–539. MR Zbl

[54] T. Schedler, "Zeroth Hochschild homology of preprojective algebras over the integers", *Adv. Math.* **299** (2016), 451–542. MR Zbl

[55] T. Schedler and A. Tirelli, "Symplectic resolutions for multiplicative quiver varieties and character varieties for punctured surfaces", pp. 393–459 in *Representation theory and algebraic geometry—a conference celebrating the birthdays of Sasha Beilinson and Victor Ginzburg*, edited by V. Baranovsky et al., Springer, 2022. MR Zbl

[56] A. H. Schofield, *Representation of rings over skew fields*, London Mathematical Society Lecture Note Series **92**, Cambridge University Press, 1985. MR Zbl

[57] P. Shaw, *Generalisations of preprojective algebras*, Ph.D. thesis, Univeristy of Leeds, 2005.

[58] "The Stacks project", electronic reference, 2005–, available at http://stacks.math.columbia.edu.

[59] J. T. Stafford and M. Van den Bergh, "Noncommutative resolutions and rational singularities", *Michigan Math. J.* **57** (2008), 659–674. MR Zbl

[60] L. de Thanhoffer de Völcsey and M. Van den Bergh, "Calabi–Yau deformations and negative cyclic homology", *J. Noncommut. Geom.* **12**:4 (2018), 1255–1291. MR Zbl

[61] D. Yamakawa, "Geometry of multiplicative preprojective algebra", *Int. Math. Res. Pap. IMRP* (2008), Art. ID rpn008, 77pp. MR Zbl

[62] W.-K. Yeung, "Pre-Calabi–Yau structures and moduli of representations", preprint, 2018. arXiv 1802.05398

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Tautological cycles on tropical Jacobians

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The classical Poincaré formula relates the rational homology classes of tautological cycles on a Jacobian to powers of the class of Riemann theta divisor. We prove a tropical analogue of this formula. Along the way, we prove several foundational results about real tori with integral structures (and, therefore, tropical abelian varieties). For example, we prove a tropical version of the Appell–Humbert theorem. We also study various notions of equivalences between tropical cycles and their relation to one another.

1. Introduction

1A. ***Background.*** Let C be a compact Riemann surface of genus g. Its Jacobian variety J has a number of natural subvarieties $\widetilde{W}_d$ for $d \geq 0$, defined up to translation. The origin is denoted by $\widetilde{W}_0$, the image of the Abel–Jacobi map is denoted by $\widetilde{W}_1$, and $\widetilde{W}_d = \widetilde{W}_{d-1} + \widetilde{W}_1$ is the image of higher symmetric powers of C. One can intersect these subvarieties, add again, pull back or push down under multiplication by integers, and so on. This provides a large supply of algebraic *tautological cycles*, which live naturally in J.

By the Riemann–Roch or Jacobi inversion theorem, one has $\widetilde{W}_g = J$. Riemann's theorem states that $\widetilde{W}_{g-1}$ is a shift of the Riemann theta divisor Θ; see, e.g., [Griffiths and Harris 1978, page 338], [Arbarello et al. 1985, Chapter 1, Section 5], or [Birkenhake and Lange 2004, Theorem 11.2.4]. The classical Poincaré formula gives a refinement of Riemann's theorem; see, e.g., [Griffiths and Harris 1978, page 350], [Arbarello et al. 1985, Chapter 1, Section 5], or [Birkenhake and Lange 2004, Section 11.2]. It states that,

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for $0 \le d \le g$, the classes of $\widetilde{W}_d$ and Θ^{g-d} coincide in rational homology (up to the multiplicative constant $1/(g-d)!$). In other words, the subalgebra of tautological cycles in $H_*(J;\mathbb{Q})$ is generated by the class of Riemann theta divisor. There are also versions of the Poincaré formula over a general field. For example, Lieberman proves a "Weil cohomological equivalence" statement (see [Kleiman 1968, Remark 2A13]), and Mattuck, built on the work of Matsusaka, proves a "numerical equivalence" statement; see [Mattuck 1962, Section 2; Matsusaka 1959].

1B. *Our contribution.* Our main goal in this paper is to prove a tropical analogue of the Poincaré formula. Let Γ be a compact connected metric graph of genus g. Following [Kotani and Sunada 2000; Mikhalkin and Zharkov 2008], one associates to Γ a g-dimensional polarized real torus $\mathrm{Jac}(\Gamma)$, called its tropical Jacobian. There is also a well-behaved theory of divisors, ranks, Abel–Jacobi maps, and Picard groups for metric graphs [Mikhalkin and Zharkov 2008; Gathmann and Kerber 2008; Baker and Norine 2007]. We denote the tropical Abel–Jacobi morphism by $\Phi\colon \Gamma_d \to \mathrm{Jac}(\Gamma)$, which is well-defined up to a translation. Here Γ_d denotes the set of all unordered d-tuples of points of Γ. The image $\widetilde{W}_d = \Phi(\Gamma_d)$ is a polyhedral subset of $\mathrm{Jac}(\Gamma)$ of pure dimension d. Exactly as in the classical situation $\widetilde{W}_d$ may be identified with the *effective locus* $W_d \subseteq \mathrm{Pic}^d(\Gamma)$ via the Abel–Jacobi map. In [Mikhalkin and Zharkov 2008] one also finds the notion of Riemann theta divisor Θ on $\mathrm{Jac}(\Gamma)$, which is closely related to the theory of Voronoi polytopes of lattices. The polyhedral subsets $\widetilde{W}_d$ and Θ of $\mathrm{Jac}(\Gamma)$ support *tropical fundamental cycles* $[\widetilde{W}_d]$ and $[\Theta]$; see Section 8. Recently, the notions of tropical homology, cohomology, and the cycle class map have been developed in [Itenberg et al. 2019] and further studied in [Gross and Shokrieh 2019].

Theorem A (Theorem 9.8 and Corollary 9.10). *For every* $0 \le d \le g$, *we have the equality*

$$[\widetilde{W}_d] = \frac{[\Theta]^{g-d}}{(g-d)!}$$

on $\mathrm{Jac}(\Gamma)$ *modulo tropical homological equivalence. Moreover, the equality also holds modulo numerical equivalence.*

Our proof further provides explicit descriptions of the classes of $\widetilde{W}_d$ and Θ^{g-d} in tropical homology in terms of the combinatorics of the metric graph Γ; see Section 9B and Section 9C.

The Poincaré formula has several interesting, but immediate, consequences.

Corollary B (Corollaries 9.12, 9.13, and 9.15). (a) *There exists a unique* $\mu \in \mathrm{Pic}^{g-1}(\Gamma)$ *such that* $[W_{g-1}] = [\Theta] + \mu$.

(b) *The effective tropical* 0*-cycle obtained from the stable intersection of* $[\widetilde{W}_d]$ *and* $[\widetilde{W}_{g-d}]$ *has degree* $\binom{g}{d}$.

(c) *The tropical* 0*-cycle* $[\Theta]^g$ *has degree* $g!$.

We note that part (a) is a tropical version of Riemann's theorem and has already been proven by Mikhalkin and Zharkov [2008] using other combinatorial techniques. The special case $d = 1$ of part (b) can also be found in [loc. cit.] in the context of the Jacobi inversion theorem, where again the proof is direct and combinatorial. This was essential in the development of break divisors in their paper. Part (c) classically follows from the geometric Riemann–Roch theorem for abelian varieties; see, e.g., [Birkenhake and Lange 2004, Theorem 3.6.3]. In the case where Γ is a chain of loops, (c) has previously been observed in [Cartwright et al. 2015].

Building up to the proof of the Poincaré formula we also prove several foundational results about real tori with integral structures (and, therefore, about tropical abelian varieties) some of which had been used implicitly in previous work on the subject. Most notably, we prove the following tropical version of the Appell–Humbert Theorem:

Theorem C (Theorem 7.2). *Every tropical line bundle on a real torus $N_{\mathbb{R}}/\Lambda$ corresponds to a pair (E, l) of a symmetric form E on $N_{\mathbb{R}}$ with $E(N, \Lambda) \subseteq \mathbb{Z}$ and a morphism $l \in \mathrm{Hom}(N_{\mathbb{R}}, \mathbb{R})$. Two such pairs (E, l) and (E', l') define the same line bundle if and only if $E = E'$ and $(l - l')(N) \subseteq \mathbb{Z}$.*

We also study the relationship between various notions of equivalence of tropical cycles. For example, we prove the following statement.

Theorem D (Propositions 5.8 and 5.11). *Algebraic equivalence implies homological equivalence, and homological equivalence implies numerical equivalence on real tori admitting a "spanning curve".*

1C. *Further directions.* We believe our Poincaré formula is a first step in proving the following ambitious conjecture in tropical Brill–Noether theory. Let $W_d^r \subseteq \mathrm{Pic}^d(\Gamma)$ denote the locus of divisor classes of *degree* d and *rank* at least r; see, e.g., [Cools et al. 2012; Lim et al. 2012].

Conjecture. Assume $\rho = g - (r+1)(g-d+r) \geq 0$. Then there exists a canonical tropical subvariety $Z_d^r \subseteq W_d^r$ of pure dimension ρ such that

$$[Z_d^r] = \left(\prod_{i=0}^{r} \frac{i!}{(g-d+r+i)!}\right)[\Theta]^{g-\rho}.$$

modulo tropical homological equivalence.

Note that our Theorem A precisely establishes this conjecture in the case $r = 0$, in which case $W_d^0 = W_d$ is pure-dimensional by [Gross et al. 2022, Theorem 8.3] (see also Theorem 8.2) and $Z_d^0 = W_d$. Numerical evidence for the conjecture in the case where Γ is a generic chain of loops is given in [Cartwright et al. 2015, Proposition 2.8]. We also remark that a less precise version of this conjecture is posed as a question in [Pflueger 2017, Question 6.2].

As stated above, it follows from the Poincaré formula that the subring of tautological cycles in rational homology is too simple to provide interesting invariants. A celebrated result of Ceresa [1983] implies that for a generic curve C, the class of W_d is *not* proportional to the class of Θ modulo algebraic equivalence. Beauville [2004] (see also [Polishchuk 2005; Marini 2008; Moonen 2009]) has studied results about algebraic equivalence. We believe that the tautological subring of the ring of tropical cycles modulo algebraic equivalence is an interesting object to study. For example, one might hope that this ring is generated by the classes of the W_d for $1 \leq d \leq g-1$. We remark that a tropical version of Ceresa's result has already been established by Zharkov [2015].

As stated in Theorem D, homological equivalence implies numerical equivalence on tropical abelian varieties. We expect this to be true in general on any tropical manifold.

In analogy with Grothendieck's "standard conjecture D" one might also hope that homological equivalence coincides with numerical equivalence, at least in the case of tropical abelian varieties. The analogous classical result has been established by Lieberman [1968]. For *rationally triangulable* smooth projective tropical varieties, this was recently shown by Amini and Piquerez [2020, Theorem 1.3].

1D. *The structure of this paper.* In Sections 2–4 we review the main objects and tools needed to proof the Poincaré formula, including rational polyhedral spaces, tropical cycles, tropical homology, and tropical Jacobians.

In Sections 5–7 we study tropical cycles, tropical homology, and line bundles on real tori. Our results here are of a more foundational nature, and include the Appell–Humbert Theorem. We also study various notions of equivalences of tropical cycles and prove Theorem D.

Finally, in Sections 8–9 we prove the Poincaré formula. In Section 8 we show that the set $\widetilde{W}_i$ has a fundamental cycle. In Section 9 we give explicit expression for both the cycle classes of the $[\widetilde{W}_i]$ and of powers of the theta divisor. Comparing these expressions will finish the proof of Theorem A. The results summarized in Corollary B will be direct consequences of the Poincaré formula.

Notation. We will denote by $\mathbb{N}$ the natural numbers including 0. For an Abelian group A and a topological space X, we will denote by A_X the constant sheaf on X associated to A.

2. Rational polyhedral spaces

The tropical spaces studied in this paper are real tori with integral structures, compact tropical curves, and their Jacobians. They all live inside the category of boundaryless rational polyhedral spaces. We quickly review their definition and refer to [Mikhalkin and Zharkov 2014; Jell et al. 2018; Gross and Shokrieh 2019] for more details.

2A. *Boundaryless rational polyhedral spaces.* A rational polyhedral set in $\mathbb{R}^n$ is a finite union of finite intersections of sets of the form

$$\{x \in \mathbb{R}^n \mid \langle m, x\rangle \leq a\},$$

where $m \in (\mathbb{Z}^n)^*$, $a \in \mathbb{R}$, and $\langle \cdot, \cdot\rangle$ denotes the evaluation pairing. Any such set P comes with a sheaf Aff_P of *integral affine* functions, which are precisely the continuous real-valued functions that are locally (on P) of the form $x \mapsto \langle m, x\rangle + a$ for some $m \in (\mathbb{Z}^n)^*$ and $a \in \mathbb{R}$.

Definition 2.1. A *boundaryless rational polyhedral space* is a pair (X, Aff_X) consisting of a topological space X and a sheaf of continuous real-valued functions Aff_X such that every point $x \in X$ has an open neighborhood U such that there exists a rational polyhedral set P in some $\mathbb{R}^n$, an open subset $V \subseteq P$, and a homeomorphism $f : U \to V$ that induces an isomorphism $f^{-1}(\mathrm{Aff}_P|_V) \cong \mathrm{Aff}_X|_U$ via pulling back functions. Such an isomorphism f is called a *chart* for X. A boundaryless rational polyhedral space that is compact is called a *closed rational polyhedral space*. The sections of Aff_X are called *integral affine functions*.

Remark 2.2. In the literature (for example in [Jell et al. 2018; Gross and Shokrieh 2019]), the notion of rational polyhedral spaces is used for spaces that are locally isomorphic to open subsets of rational polyhedral sets in $\overline{\mathbb{R}}^n$, where $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. This introduces a notion of boundary, which is essential for many applications. For our purposes it is sufficient to consider spaces without boundary. A boundaryless rational polyhedral space is precisely a rational polyhedral space without boundary.

Definition 2.3. (i) A morphism of boundaryless rational polyhedral spaces is a continuous map $f : X \to Y$ such that pullbacks of functions in Aff_Y are in Aff_X.

(ii) A morphism $f : X \to Y$ is called proper if it is a proper map of topological spaces, that is preimages of compact sets are compact.

2B. ***Real tori with integral structures.*** Let N be a lattice, and let $\Lambda \subseteq N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ be a second lattice of full rank, that is such that the induced morphism $\Lambda_{\mathbb{R}} \to N_{\mathbb{R}}$ is an isomorphism. Clearly, $N_{\mathbb{R}}$ gets a well-defined rational polyhedral structure from any isomorphism $N \cong \mathbb{Z}^n$. The *real torus* (*with integral structure*) associated to N and Λ is the quotient $X = N_{\mathbb{R}}/\Lambda$, with the sheaf of affine functions being the one induced by $N_{\mathbb{R}}$. More precisely, if $\pi : N_{\mathbb{R}} \to X$ denotes the quotient map, and $U \subseteq X$ is open, then $\phi : U \to \mathbb{R}$ is in $\mathrm{Aff}_X(U)$ if and only if $\phi \circ \pi \in \mathrm{Aff}_{N_{\mathbb{R}}}(\pi^{-1}U)$. Note that the integral affine structure on X is induced by N and *not* by Λ.

The group law on a real torus X makes it a group object in the category of boundaryless rational polyhedral spaces. In particular, every $x \in X$ defines an automorphism via translation.

Definition 2.4. Let X be a real torus and let $x \in X$. Then the *translation by x* is the morphism

$$t_x : X \to X, \quad y \mapsto x + y.$$

2C. ***Tropical curves.*** A *tropical curve* is a purely 1-dimensional boundaryless rational polyhedral space. With this definition, the underlying space of a tropical curve Γ is a topological graph. In particular, it has a set of vertices (branch points) $V(\Gamma)$ where Γ does not locally look like an open interval in $\mathbb{R}$, and a set of open edges $E(\Gamma)$, which are the connected components of $\Gamma \setminus V(\Gamma)$. The closed edges of Γ are the closures of its open edges and an open edge segment is a connected open subset of an open edge. A tropical curve is *smooth* (see Figure 1) if every point has a neighborhood that is isomorphic to a neighborhood of the origin in a *star-shaped set*, that is a set of the form

$$\bigcup_{0 \le i \le n} \mathbb{R}_{\ge 0} e_i \subseteq \mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}.$$

Here $n > 0$ will denote the valency of the point, we denote by $\mathbf{1}$ the vector whose coordinates are all 1, and e_i denotes the i-th standard basis vector.

Using the integral structure on a compact tropical curve, one can assign lengths to its edges, thus defining a metric graph. Conversely, given a metric graph (a topological graph Γ equipped with an inner

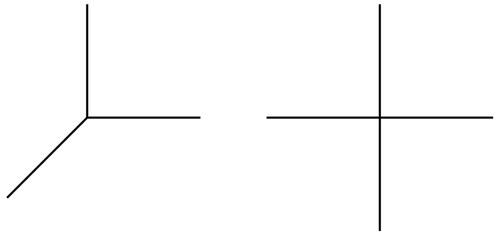

Figure 1. Two tropical curves embedded in $\mathbb{R}^2$. The one to the left is smooth, the one to the right is not.

metric), one can define Aff_Γ as the sheaf of *harmonic* functions on Γ, that is the sheaf of functions whose sum of incoming slopes is 0 at every point. In this way, one obtains a smooth tropical curve $(\Gamma, \mathrm{Aff}_\Gamma)$; see [Mikhalkin and Zharkov 2008, Proposition 3.6].

The *genus* g of a tropical curve Γ is defined as its first Betti number, that is $g = \dim_{\mathbb{R}} H_1(\Gamma; \mathbb{R})$.

Remark 2.5. With our notion of tropical curves, the underlying topological graph is not allowed to have 1-valent vertices. This can be resolved by working in the larger category of polyhedral spaces with boundary mentioned in Remark 2.2 and allowing neighborhoods of ∞ in $\overline{\mathbb{R}}$ as local models for the curves. In this way, tropical curves could have edges of infinite length that end in a 1-valent vertex. But as we will note in Remark 9.11, the results of this paper are easily generalized to apply to compact and connected smooth tropical curves with boundary as well.

Example 2.6. For any positive real number $j \in \mathbb{R}_{>0}$ the sublattice $\mathbb{Z}j$ of $\mathbb{R} = \mathbb{Z}_{\mathbb{R}}$ has full rank. Therefore, the quotient $\Gamma = \mathbb{R}/\mathbb{Z}j$, endowed with the integral affine structure induced by $\mathbb{Z}$, is a 1-dimensional real torus. It is also a smooth tropical curve of genus 1. Its unique edge is both open and closed and it is homeomorphic to the 1-sphere. The length of this edge is given by j, which can be considered as the j-invariant of Γ [Katz et al. 2008].

Example 2.7. Consider the topological space Γ obtained by gluing three intervals $[0, a]$, $[0, b]$, and $[0, c]$ along their lower and upper bounds, respectively. Clearly, Γ is a topological graph with three edges and two vertices. We can view the three intervals as rational polyhedral spaces, so on the interior of the edges of Γ we have a notion of linearity. We can now define Aff_Γ as the sheaf of all continuous functions whose restrictions to the interiors of the intervals are linear, and such that the sum of the outgoing slopes is 0 at the two vertices. With these choices, Γ is the smooth tropical curve associated to the metric graph with three parallel edges of lengths a, b and c. It is depicted in Figure 2.

2D. *Tropical manifolds.* We recall that every loop-free matroid M on a ground set $E(M)$ has an associated tropical linear space L_M, which is a rational polyhedral set in $\mathbb{R}^{E(M)}/\mathbb{R}\mathbf{1}$. We will only consider very special linear spaces and therefore refrain from recalling their precise definition. For our purposes, it suffices to say that $\mathbb{R}^n$ is a tropical linear space for any n, and the 1-dimensional tropical linear spaces are precisely the star-shaped sets appearing in the definition of smooth tropical curves in Section 2C.

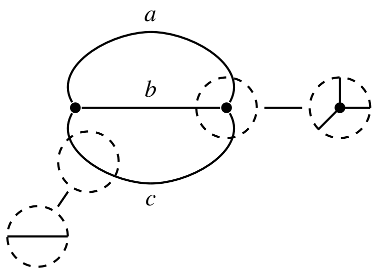

Figure 2. A tropical curve of genus 2 with its local charts and edge lengths.

Definition 2.8. A boundaryless rational polyhedral space X is called a *boundaryless tropical manifold* if it can be covered by charts

$$X \supseteq U \xrightarrow{\cong} V \subseteq L,$$

where U is an open subset of X and V is an open subset of a tropical linear space L.

Since both $\mathbb{R}^n$ and star-shaped sets are tropical linear spaces, it follows that real tori and smooth tropical curves are boundaryless tropical manifolds.

2E. *The cotangent sheaf.*

Definition 2.9. Let X be a boundaryless rational polyhedral space:

(i) The quotient $\mathrm{Aff}_X / \mathbb{R}_X$ is called the *cotangent sheaf* and is denoted by Ω^1_X.

(ii) The *integral tangent space* at a point $x \in X$ is defined as $T^{\mathbb{Z}}_x X = \mathrm{Hom}(\Omega_{X,x}, \mathbb{Z})$.

(iii) The *tangent space* at a point $x \in X$ is defined as $T_x X = (T^{\mathbb{Z}}_x X)_{\mathbb{R}} \cong \mathrm{Hom}(\Omega_{X,x}, \mathbb{R})$.

Example 2.10. Let $X = N_{\mathbb{R}}/\Lambda$ be a real torus. Then Aff_X has no nonconstant global sections because there is no globally defined nonconstant integral affine function on $N_{\mathbb{R}}$ that is Λ-periodic. On the other hand, the quotient $\mathrm{Aff}_X / \mathbb{R}_X = \Omega^1_X$ is isomorphic to the constant sheaf N_X.

By definition, a morphism of boundaryless rational polyhedral spaces $f\colon X \to Y$ induces a morphism $f^{-1}\Omega^1_Y \to \Omega^1_X$. Taking stalks and dualizing induces morphisms on tangent spaces $d_x f\colon T_x X \to T_{f(x)} Y$ for all $x \in X$ that map the integral tangent spaces on X to the integral tangent spaces on Y.

3. Tropical cycles and their tropical cycle classes

We briefly recall the definitions of tropical cycles, tropical (co)homology, and the tropical cycle class map connecting the two. We closely follow [Allermann and Rau 2010; François and Rau 2013; Shaw 2013] regarding tropical cycles and [Itenberg et al. 2019; Mikhalkin and Zharkov 2014; Jell et al. 2018; Gross and Shokrieh 2019] regarding tropical (co)homology and the tropical cycle class map.

3A. ***Tropical cycles.*** For a boundaryless rational polyhedral space X, let us denote by X^{reg} its open subset of points $x \in X$ that have a neighborhood isomorphic (as boundaryless rational polyhedral spaces) to an open subset of $\mathbb{R}^n$ for some $n \in \mathbb{N}$. A tropical k-cycle is a function $A\colon X \to \mathbb{Z}$ such that its support $|A| = \overline{\{x \in X \mid A(x) \neq 0\}}$ is either empty or a purely k-dimensional polyhedral subset of X, A is nonzero precisely on the set $|A|^{\mathrm{reg}}$, on which it is locally constant, and it satisfies the so-called *balancing condition*. The latter is a local condition that is well-known for $X = \mathbb{R}^n$, to which the general case can be reduced. As we will only need it implicitly, we refer to [Allermann and Rau 2010] for details. The sum of two tropical k-cycles on X, considered as a sum of $\mathbb{Z}$-valued functions, is not a tropical k-cycle again in general. However, there exists a unique tropical k-cycle on X that agrees with the sum on the complement of an at most $(k-1)$-dimensional polyhedral subset of X. This makes the set $Z_k(X)$ into an Abelian group. A tropical cycle A is said to be *effective* if it is everywhere nonnegative.

If $f\colon X \to Y$ is a proper morphism of boundaryless rational polyhedral spaces, it induces a *push-forward* $f_*\colon Z_k(X) \to Z_k(Y)$ of tropical cycles. If $A \in Z_k(X)$ is a tropical cycle, then f_*A will be zero outside of the subset $(f|A|)_k \subseteq f|A|$ where the local dimension of $f|A|$ is k. There exists a dense open subset $U \subseteq (f|A|)_k$ such that for each $y \in U$ the fiber $f^{-1}\{y\}$ is finite and contained in $|A|^{\mathrm{reg}}$, and for each such $y \in U$ we have

$$f_*A(y) = \sum_{x \in f^{-1}\{y\}} |\operatorname{coker} d_x f| A(x).$$

Note that the finiteness of $\operatorname{coker} d_x f$ follows from the finiteness of the fiber over y. If X is compact then one can take Y to be a point. Identifying the tropical 0-cycles on a point with $\mathbb{Z}$, the push-forward then defines a morphism $Z_0(X) \to \mathbb{Z}$. The image of a tropical 0-cycle A under this morphism is called the *degree* of A, and it is denoted by $\int_X A$.

If X and Y are boundaryless rational polyhedral spaces, and $A \in Z_k(X)$ and $B \in Z_l(X)$, then the *cross product*

$$A \times B\colon X \times Y \to \mathbb{Z}, \quad (x, y) \mapsto A(x) \cdot B(x)$$

of A and B is a tropical cycle again.

A *rational function* on a boundaryless rational polyhedral space X is a continuous function $\phi\colon X \to \mathbb{R}$ such that ϕ is piecewise affine with integral slopes in every chart. As this is a local condition, rational functions define a sheaf $\mathscr{M}_X$ of Abelian groups. The group of *tropical Cartier divisors* on X is given by $\operatorname{CDiv}(X) = \Gamma(X, \mathscr{M}_X / \operatorname{Aff}_X)$. For every $\phi \in \Gamma(X, \mathscr{M}_X)$ we denote its image in $\operatorname{CDiv}(X)$ by $\operatorname{div}(\phi)$, and refer to it as the associated *principal divisor*. There exists natural bilinear map $\operatorname{CDiv}(X) \times Z_k(X) \to Z_{k-1}(X)$, the intersection pairing of divisors and tropical cycles.

Note that a boundaryless rational polyhedral space X does not automatically have a natural fundamental cycle, that is there is no canonical element in $Z_*(X)$ in general.

Definition 3.1. We will say that a boundaryless rational polyhedral space X *has a fundamental cycle* if X is pure-dimensional and the extension by 0 of the constant function with value 1 on X^{reg} defines a tropical cycle. In that case we will denote this tropical cycle by $[X]$, and refer to it as the *fundamental* *cycle* of X. We will say that a Cartier divisor $D \in \mathrm{CDiv}(X)$ on a tropical space X with fundamental cycle is *effective*, if its *associated Weil divisor* $[D] := D \cdot [X]$ is effective.

If X is a tropical manifold then it has a fundamental cycle $[X]$, which is the unity of the *tropical intersection product* on $Z_*(X)$. The tropical intersection product is compatible with intersections with Cartier divisors in the sense that

$$D \cdot A = [D] \cdot A$$

for every Cartier divisor $D \in \mathrm{CDiv}(X)$ and tropical cycle $A \in Z_*(X)$. Furthermore, the morphism

$$\mathrm{CDiv}(X) \mapsto Z_{\dim(X)-1}(X), \quad D \mapsto [D]$$

is an isomorphism; see [Francois 2013, Corollary 4.9]. If X is locally isomorphic to open subsets of $\mathbb{R}^n$, then a Cartier divisor $D \in \mathrm{CDiv}(X)$ is effective if and only if it is locally given by *concave* rational functions. This follows from the fact that every tropical hypersurface of $\mathbb{R}^n$ is realizable. Here, a rational function is concave if it is the restriction of a concave rational function on $\mathbb{R}^n$ in sufficiently small local charts. Also note that concave functions appear rather than convex ones, because we are using the "min"-convention; see Remark 3.4.

3B. ***Line bundles.*** A *tropical line bundle* on a boundaryless rational polyhedral space X is an Aff_X-torsor. More geometrically, it is a morphism $Y \to X$ of boundaryless rational polyhedral spaces such that locally on X there are trivializations $Y \cong X \times \mathbb{R}$, where two such trivializations are related via the translation by an integral affine function. More precisely, if two trivializations are defined over $U \subseteq X$, then the transition between them is of the form

$$U \times \mathbb{R} \to U \times \mathbb{R}, \quad (u, x) \mapsto (u, x + \phi(u))$$

for some $\phi \in \Gamma(U, \mathrm{Aff}_X)$. The standard argument using Čech cohomology shows that the set of isomorphism classes of tropical line bundles on X is in natural bijection to $H^1(X, \mathrm{Aff}_X)$. In particular, isomorphism classes of tropical line bundles form a group. A rational section of a tropical line bundle $Y \to X$ is a continuous section that is given by a rational function in all trivializations. Exactly as in algebraic geometry, every tropical Cartier divisor D on X defines a line bundle $\mathscr{L}(D)$ on X that comes with a canonical rational section. This defines a bijection between $\mathrm{CDiv}(X)$ and isomorphism classes of pairs $(\mathscr{L}, s)$ of a tropical line bundle $\mathscr{L}$ on X and a rational section s of $\mathscr{L}$.

3C. ***Homology and cohomology.*** Let X be a boundaryless rational polyhedral space. To define the tropical homology and cohomology groups, we need sheaves Ω^p_X of tropical p-forms for $p > 0$. On the open subset X^{reg} it is clear that we would like Ω^p_X to be isomorphic to $\bigwedge^p \Omega^1_X$. However, this is not a suitable definition globally because in general $\bigwedge^p \Omega^1_X$ can be nonzero even for $p > \dim(X)$; see [Gross and Shokrieh 2019, Example 2.9]. One thus defines Ω^p_X as the image of the natural map

$$\bigwedge^p \Omega^1_X \to \iota_*\Big(\bigwedge^p \Omega^1_X|_{X^{\mathrm{reg}}}\Big),$$

where $\iota\colon X^{\mathrm{reg}} \to X$ is the inclusion.

The singular tropical homology groups are defined similar to the integral singular homology groups, but with different coefficients. More precisely, there is a coarsest stratification of X such that the restrictions of the constructible sheaf Ω^1_X is locally constant on all the strata, and only singular simplices are allowed that respect this stratification in the sense that each of their open faces is mapped into a single stratum. The (p,q)-th chain group is then defined as

$$C_{p,q}(X) = \bigoplus_{\sigma\colon \Delta^q \to X \text{ allowable}} \operatorname{Hom}(\Omega^p_X, \mathbb{Z}_{\sigma(\Delta^q)}),$$

where Δ^q denotes the standard q-simplex, the sum runs over all q-simplices respecting the stratification, and $\mathbb{Z}_{\sigma(\Delta^q)}$ denotes the constant sheaf associated to $\mathbb{Z}$ on $\sigma(\Delta^q)$. With the usual boundary operators this defines chain complexes $C_{p,\bullet}$ and the tropical homology groups which are defined as $H_{p,q}(X) = H_q(C_{p,\bullet}(X))$.

Dualizing (over $\mathbb{Z}$) the chain complexes $C_{p,\bullet}(X)$ yields cochain complexes $C^{p,\bullet}(X)$ whose cohomology are the tropical cohomology groups $H^{p,q}(X) = H^q(C^{p,\bullet}(X))$. There is a natural isomorphism

$$H^{p,q}(X) \cong H^q(X, \Omega^p_X).$$

3D. *The first Chern class map.* The quotient map $d\colon \operatorname{Aff}_X \to \Omega^1_X$ of sheaves on a boundaryless rational polyhedral space induces a morphism

$$c_1 := H^1(d)\colon H^1(X, \operatorname{Aff}_X) \to H^1(X, \Omega^1_X) \cong H^{1,1}(X)$$

called the *first Chern class map* from the group of all tropical line bundles on X to the $(1,1)$-tropical cohomology group of X. Using the first Chern class map, any divisor $D \in \operatorname{CDiv}(X)$ has an associated $(1,1)$-cohomology class $c_1(\mathscr{L}(D))$.

3E. *The tropical cycle class map.* Exactly as in algebraic geometry, there is a tropical cycle class map that assigns a class in tropical homology to every tropical cycle. More precisely, on any closed rational polyhedral space X, there exist morphisms

$$\operatorname{cyc}\colon Z_k(X) \to H_{k,k}(X)$$

for every $k \in \mathbb{N}$. We will only need an explicit description of the tropical cycle class map for 1-dimensional tropical cycles, that is when $k = 1$. If $A \in Z_1(X)$, then its support $|A|$ is a compact (not necessarily smooth) tropical curve. For each open edge e of $|A|$ choose a generator $\eta_e \in T_x|A|$ for some $x \in e$. By taking parallel transports of η_e along e we actually obtain a generator for all $T_y|A|$ with $y \in e$. Therefore, η_e defines a morphism $\Omega^1_{|A|} \to \mathbb{Z}_e$ (recall that $\mathbb{Z}_e$ denotes the constant sheaf on e associated to $\mathbb{Z}$), which can be uniquely extended to a morphism $\Omega^1_{|A|} \to \mathbb{Z}_{\bar{e}}$. Precomposing with the morphism $\Omega^1_X \to \Omega^1_{|A|}$ defined by the inclusion $|A| \to X$, one obtains a morphism $\eta_{\bar{e}} \in \operatorname{Hom}(\Omega^1_X, \mathbb{Z}_{\bar{e}})$. To complete the construction, one has to choose a homeomorphism $\gamma_{\bar{e}}\colon \Delta^1 \to \bar{e}$ that parametrizes $\bar{e}$ in the direction specified by η_e. Let us denote the element in $C_{1,1}(X)$ defined by $\gamma_{\bar{e}}$ and $\eta_{\bar{e}}$ by $\gamma_{\bar{e}} \otimes \eta_{\bar{e}}$. Then $\operatorname{cyc}(A)$ is represented by the cycle

$$\sum_e A(e) \cdot \gamma_{\bar{e}} \otimes \eta_{\bar{e}} \in C_{1,1}(X),$$

where the sum runs over all open edges of $|A|$ and $A(e)$ denotes the weight of the tropical 1-cycle A on e.

Example 3.2. Let Γ be the graph from Example 2.7, and denote its edges by e_1, e_2, and e_3. Let v and w be the vertices of Γ and orient all edges from v to w. Let η_i be the primitive tangent direction on e_i in the chosen direction. Then $\operatorname{cyc}[\Gamma]$ is represented by the $(1,1)$-chain

$$\gamma_1 \otimes \eta_1 + \gamma_2 \otimes \eta_2 + \gamma_3 \otimes \eta_3,$$

where γ_i is any path that parametrizes e_i from v to w. This is indeed a cycle. Its boundary is given by

$$w \otimes (\eta_1 + \eta_2 + \eta_3) - v \otimes (\eta_1 + \eta_2 + \eta_3),$$

which vanishes: locally at v (respectively at w), the graph Γ looks like the star-shaped set depicted to the right in Figure 1, and the vectors η_i are the (negatives of the) primitive generators of the rays of the star-shaped set. Since these sum to 0, the boundary is 0.

3F. *Identities in tropical homology.* In [Gross and Shokrieh 2019] we studied various operations on tropical homology and cohomology and showed how to carry over identities known for singular homology to the tropical setting. For example, there are pull-backs of cohomology classes and push-forwards of homology classes along morphisms of boundaryless rational polyhedral spaces, there is a cup product "$\smile$" on tropical cohomology and a cap product "$\frown$" that makes the tropical homology groups a module over the tropical cohomology ring. There also are cross products "$\times$" of both homology and cohomology classes. We will refer the reader to [loc. cit.] for the details regarding these operations. For the reader's convenience, we have summarized the most important identities for the tropical cycle class map in the following theorem:

Theorem 3.3 [Gross and Shokrieh 2019]. *Let X, Y, and Z be closed rational polyhedral spaces, let $f\colon X \to Z$ be a proper morphism, let $A \in Z_*(X)$, $B \in Z_*(Y)$ and $D \in \operatorname{CDiv}(X)$. Then we have*

$$\begin{aligned} \operatorname{cyc}(f_* A) &= f_* \operatorname{cyc}(A), \\ \operatorname{cyc}(A \times B) &= \operatorname{cyc}(A) \times \operatorname{cyc}(B), \qquad \textit{and} \\ \operatorname{cyc}(D \cdot A) &= c_1(\mathscr{L}(D)) \frown \operatorname{cyc}(A). \end{aligned}$$

If X is a closed rational polyhedral space, then the morphism from X to a point defines a morphism $H_{0,0}(X) \to \mathbb{Z}$ by identifying the $(0,0)$-tropical homology group of a point with $\mathbb{Z}$. The image of a tropical cycle $\alpha \in H_{0,0}(X)$ is called the *degree of* α and denoted by $\int_X \alpha$. It is a direct consequence of the first equation in Theorem 3.3 that

$$\int_X A = \int_X \operatorname{cyc}(A)$$

for every $A \in Z_0(X)$.

If X is a closed tropical manifold, then homology and cohomology are dual to each other, in the sense that the morphism

$$H^{*,*}(X) \to H_{*,*}(X), \quad c \mapsto c \frown \mathrm{cyc}[X]$$

is an isomorphism [Jell et al. 2018; Gross and Shokrieh 2019]. In this context one says that c is *Poincaré dual* to $c \frown \mathrm{cyc}[X]$. Poincaré duality allows to define an *intersection product* for tropical homology classes on a closed tropical manifold X. More precisely, if $\alpha, \beta \in H_{*,*}(X)$, and $c \in H^{*,*}(X)$ is Poincaré dual to α, then one defines

$$\alpha \cdot \beta := c \frown \beta.$$

Remark 3.4. Both the intersection pairing between tropical Cartier divisors and tropical cycles, and the tropical cycle class map are not entirely free of choices. The intersection pairing depends on whether one measures incoming or outgoing slopes. When measuring incoming slopes, concave functions define effective principal divisors, whereas when measuring outgoing slopes, convex functions define effective principal divisors. Since minima of linear functions are concave, and maxima of linear functions are convex, one speaks of the “min”- and “max”-conventions, respectively. The cycle class map, on the other hands, depends on a consistent choice of isomorphisms

$$\bigwedge^k N \xrightarrow{\cong} H_k(N_{\mathbb{R}}, N_{\mathbb{R}} \setminus \{0\}; \mathbb{Z})$$

for any lattice N of any rank k; see [Gross and Shokrieh 2019, Section 5].

If one wants Theorem 3.3 to hold, one has to make the choices involved in the definitions of the intersection pairing and the cycle class map consistently. In other words, the choice of either “min”- or “max”-convention will determine the sign of the cycle class map. *In this paper, we will choose the “min”-convention*, because it makes the formulas in Section 9 nicer, but the same formulas hold true in the “max”-convention after appropriately adjusting the sign.

4. Tropical Jacobians

In this section we review the definition of tropical Jacobians, closely following [Mikhalkin and Zharkov 2008]. Let Γ be a compact and connected smooth tropical curve. We write $\Omega_{\mathbb{Z}}(\Gamma) := H^0(\Gamma, \Omega^1_{\Gamma})$ for the group of global integral 1-forms, and $\Omega_{\mathbb{R}}(\Gamma) := \Omega_{\mathbb{Z}}(\Gamma) \otimes_{\mathbb{Z}} \mathbb{R}$ for the group of (real) 1-forms. A 1-form on Γ is completely determined by its restrictions to the edges of Γ, and these restrictions are constant and completely determined by a real number and an orientation of the edge: it will be of the form $r\,dx$, where $r \in \mathbb{R}$, and x is the chart on the edge determined by the orientation. Extracting the data of its restrictions to the edges out of a 1-form gives rise to a natural morphism $\Omega_{\mathbb{R}}(\Gamma) \to C_1(\Gamma; \mathbb{R})$. Since the outgoing primitive direction vectors at any point of Γ (in any chart around that point) sum to 0, the chains in the image of $\Omega_{\mathbb{R}}(\Gamma)$ will in fact be 1-cycles, that is they are mapped to 0 by the boundary morphism. It is not hard to see that the induced map $\Omega_{\mathbb{R}}(\Gamma) \to H_1(\Gamma; \mathbb{R})$ is an isomorphism.

Remark 4.1. Another way to think of the elements of $\Omega_{\mathbb{Z}}(\Gamma)$ is as integral flows. Given $\omega \in \Omega_{\mathbb{Z}}(\Gamma)$, we have already observed that the restriction $\omega|_e$ to an open edge e is determined by a direction and a nonnegative integer. Conversely, a collection of directions and nonnegative integers for every edge in Γ will define a global 1-form if and only if this collection defines a flow.

Global 1-forms on Γ can be integrated on singular 1-chains in Γ. We obtain a pairing

$$\Omega_{\mathbb{R}}(\Gamma) \times C_1(\Gamma; \mathbb{R}) \to \mathbb{R}, \quad (\omega, c) \mapsto \int_c \omega, \tag{4-1}$$

which can be shown to induce a morphism $H_1(\Gamma; \mathbb{R}) \to \Omega_{\mathbb{R}}(\Gamma)^*$. Together with the isomorphism $H_1(\Gamma; \mathbb{R}) \cong \Omega_{\mathbb{R}}(\Gamma)$ from above, we obtain a natural bilinear form E on $H_1(\Gamma; \mathbb{R})$, which can be described explicitly. Namely, for two 1-cycles c_1 and c_2, the pairing $E(c_1, c_2)$ is the weighted length of the intersection of c_1 and c_2, where an oriented line segment occurring in c_1 and c_2 with weights λ and μ, respectively, contributes with weigh $\lambda \cdot \mu$. This bilinear form is clearly symmetric and positive definite. In particular, it is a perfect pairing, and hence the morphism $H_1(\Gamma; \mathbb{R}) \to \Omega_{\mathbb{R}}(\Gamma)^*$ we used to define it is an isomorphism. Via this isomorphism $H_1(\Gamma; \mathbb{Z})$ becomes a sublattice of $\Omega_{\mathbb{R}}(\Gamma)^*$ of full rank, and the positive definite symmetric bilinear form E induces a positive definite symmetric bilinear form Q on $\Omega_{\mathbb{R}}(\Gamma)^*$. The full-rank sublattice of $\Omega_{\mathbb{R}}(\Gamma)^*$ that has integer pairings with the elements of $H_1(\Gamma, \mathbb{Z})$ with respect to Q is precisely $\Omega_{\mathbb{Z}}(\Gamma)^*$.

Definition 4.2. The *tropical Jacobian* associated to the compact and connected smooth tropical curve Γ is the pair consisting of the real torus

$$\mathrm{Jac}(\Gamma) := \Omega_{\mathbb{R}}(\Gamma)^* / H_1(\Gamma; \mathbb{Z})$$

and the bilinear form Q that is defined on the universal cover $\Omega_{\mathbb{R}}(\Gamma)^*$ of $\mathrm{Jac}(\Gamma)$.

Remark 4.3. By the universal coefficient theorem, we also have an isomorphism $H^1(\Gamma; \mathbb{R}) \cong H_1(\Gamma; \mathbb{R})^*$. Together with the isomorphism $\Omega_{\mathbb{R}}(\Gamma) \cong H_1(\Gamma; \mathbb{R})$ from above one obtains an isomorphism $H^1(\Gamma; \mathbb{R}) \cong \Omega_{\mathbb{R}}(\Gamma)^*$. It is therefore also possible to write the Jacobian of Γ as the quotient $H^1(\Gamma; \mathbb{R})/H_1(\Gamma; \mathbb{Z})$.

Now fix a base point $q \in \Gamma$. Given any other point $p \in \Gamma$ there is a path γ_p connecting q to p. As any other path from q to p differs from γ_p by an integral 1-cycle, the class of γ_p in $(C_1(\Gamma; \mathbb{Z})/B_1(\Gamma; \mathbb{Z}))/H_1(\Gamma; \mathbb{Z})$ is independent of the choice of γ_p. Here, $B_1(\Gamma; \mathbb{Z})$ denotes the group of 1-boundaries. Using the pairing (4-1), we obtain an element in $\mathrm{Jac}(\Gamma)$ that only depends on the choice of q. This defines the *Abel–Jacobi map*

$$\Phi_q : \Gamma \to \mathrm{Jac}(\Gamma).$$

Let $p \in \Gamma$, and let U be a sufficiently small connected open neighborhood of p. More precisely, U should be connected and $U \setminus \{p\}$ should be disjoint from $V(\Gamma)$. Then for every $p' \in U \setminus \{p\}$ there exists $r > 0$ and a geodesic path $\gamma : [0, r] \to U$ from p to p'. Let e denote the unique open edge e of Γ containing p', and let η denote the primitive integral tangent vector on e pointing from p towards p'. If

x is any lift of $\Phi_q(p)$ to the universal cover $\Omega_{\mathbb{R}}(\Gamma)^*$, then by definition, p' lifts to $x + r \cdot \delta$, where δ is given by

$$\delta \colon \Omega_{\mathbb{R}}(\Gamma) \to \mathbb{R}, \quad \omega \mapsto \tfrac{1}{r} \int_\gamma \omega = \langle \omega|_e, \eta \rangle.$$

If we identify $\Omega_{\mathbb{R}}(\Gamma)$ with flows on Γ (as in Remark 4.1) then δ is the map assigning to a flow ω on Γ its flow on e in the direction specified by η. In particular, δ is integral, that is $\delta \in \Omega_{\mathbb{Z}}(\Gamma)^*$. This shows that Φ_q is, in fact, a morphism of boundaryless rational polyhedral spaces, and that its action on the tangent space of e is given by

$$\delta = (d\Phi_q)(\eta).$$

Example 4.4. Let Γ be the smooth tropical curve associated to the metric graph that consists of two vertices which are connected by three edges of length 1 (the graph of Example 2.7 with $a = b = c = 1$). It is depicted to the left in Figure 3. We choose one of the vertices as the base point q and orient the edges of Γ such that one edge, call it e_3 is oriented towards q and the other two edges, call them e_1 and e_2, are oriented away from q. The orientations define two simple closed loops c_1 and c_2 in Γ, where c_i first follows e_i and then e_3. These loops define a basis for $H_1(\Gamma; \mathbb{R})$, and hence for $\Omega_{\mathbb{Z}}(\Gamma)$. Let $\delta_1, \delta_2 \in \Omega_{\mathbb{Z}}(\Gamma)^*$ be the dual basis. Since the signed length of $c_i \cap c_j$ is 2 if $i = j$ and 1 if $i \neq j$, the injection $H_1(\Gamma; \mathbb{Z}) \to \Omega_{\mathbb{R}}(\Gamma)^*$ maps c_1 to $(2, 1)$ and c_2 to $(1, 2)$ in the coordinates defined by the basis δ_1, δ_2. If follows that

$$\operatorname{Jac}(\Gamma) = \mathbb{R}^2 \Big/ \mathbb{Z}\binom{1}{2} + \mathbb{Z}\binom{2}{1},$$

where the integral structure is given by $\mathbb{Z}^2 \subseteq \mathbb{R}^2$.

The Abel–Jacobi map sends q to 0 in this quotient. If γ_1 is the geodesic path along e_1 that starts at q, then

$$(d\Phi_q)(\gamma_1(t)) = t \cdot \binom{1}{0} + H_1(\Gamma; \mathbb{Z})$$

for all $t \in [0, 1]$ because the path from q to $\gamma_1(t)$ along e_1 intersects c_1 in an edge segment of length t, and c_2 in a point (an edge segment of length 0). Similarly, if γ_2 is a geodesic path along e_2, and γ_3 is a geodesic path along e_3, both starting at q, then

$$(d\Phi_q)(\gamma_2(t)) = t \cdot \binom{0}{1} + H_1(\Gamma; \mathbb{Z}) \quad \text{and} \quad (d\Phi_q)(\gamma_3(t)) = t \cdot \binom{-1}{-1} + H_1(\Gamma; \mathbb{Z})$$

for all $t \in [0, 1]$.

5. Algebraic, homological, and numerical equivalence

In this section we study different notions of equivalence for tropical cycles on boundaryless rational polyhedral spaces, with a focus on real tori.

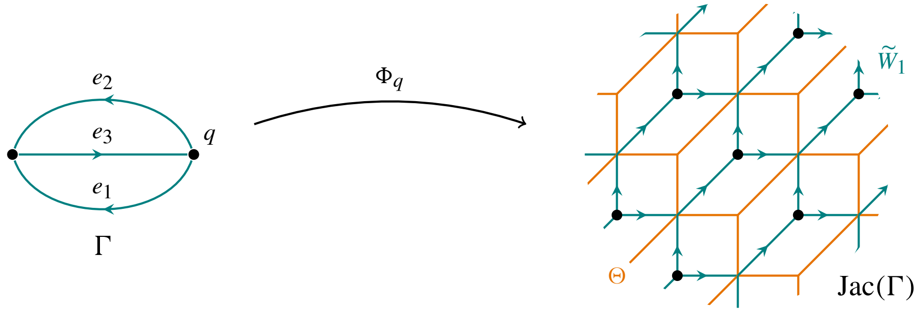

Figure 3. A tropical curve of genus g, the universal cover of its Jacobian, and the sets $\widetilde{W}_1$ (segments with arrows) and Θ (no arrows) lifted to the universal cover. (See Section 8B for the definition of Θ).

5A. ***Algebraic equivalence.*** Following [Zharkov 2015], we make the following definition.

Definition 5.1. Let X be a boundaryless rational polyhedral space. Let R_{alg} be the subgroup of $Z_*(X)$ generated by tropical cycles of the form

$$p_*(q^*(t_0 - t_1) \cdot W),$$

where W is a tropical cycle on $X \times \Gamma$ for some compact and connected smooth tropical curve Γ containing the two points $t_0, t_1 \in \Gamma$, and $p\colon X \times \Gamma \to X$ and $q\colon X \times \Gamma \to \Gamma$ are the natural projections. Note that because Γ is smooth, the difference $t_0 - t_1$ defines a tropical Cartier divisors on Γ (see Section 3A) and tropical Cartier divisors can be pulled-back along any morphism of boundaryless rational polyhedral spaces.

We say that two tropical cycles $A, B \in Z_*(X)$ are *algebraically equivalent*, denoted by $A \sim_{\mathrm{alg}} B$, if their classes in $Z_*(X)/R_{\mathrm{alg}}$ coincide.

Proposition 5.2. *Let X be a boundaryless tropical manifold, let $A, B, C \in Z_*(X)$ be tropical cycles on X, and assume that $A \sim_{\mathrm{alg}} B$. Then*

$$A \cdot C \sim_{\mathrm{alg}} B \cdot C.$$

Proof. By the definition of algebraic equivalence, we may assume that there exists a compact and connected smooth tropical curve Γ, two points $t_0, t_1 \in \Gamma$, and a tropical cycle W on $X \times \Gamma$ such that

$$A - B = p_*(q^*(t_0 - t_1) \cdot W),$$

where p and q denote the projection. Using the projection formula [François and Rau 2013, Theorem 8.3(1)], we see that

$$A \cdot C - B \cdot C = (A - B) \cdot C = p_*(q^*(t_0 - t_1) \cdot (W \cdot (C \times \Gamma))).$$

Applying the definition of algebraic equivalence with W replaced by $W \cdot (C \times \Gamma)$, we obtain that $A \cdot C$ and $B \cdot C$ are algebraically equivalent. $\square$

Definition 5.3. Let $X = N_{\mathbb{R}}/\Lambda$ be a real torus. A *spanning curve* for X is a 1-dimensional polyhedral subset $\Gamma \subseteq X$ such that there exists an effective tropical 1-cycle on X with support Γ, and such that the parallel transports to 0 of the direction vectors of the edges of Γ span $T_0X \cong N_{\mathbb{R}}$. If such a curve exists, we say that X *admits a spanning curve*. See Example 7.4 for an example of a real torus which does *not* admit a spanning curve.

Proposition 5.4. *Let Γ be a compact and connected smooth tropical curve. Then its Jacobian* $\operatorname{Jac}(\Gamma)$ *admits a spanning curve.*

Proof. For any choice of base point $q \in \Gamma$, the image $\Phi_q(\Gamma)$ of Γ under the Abel–Jacobi map is the support of the effective cycle $\Phi_{q*}[\Gamma]$. Using the explicit description given in Section 4 of the tangent directions in $\operatorname{Jac}(\Gamma)$ of the images of the edges of Γ, it follows directly that $\Phi_q(\Gamma)$ is a spanning curve for $\operatorname{Jac}(\Gamma)$. □

Proposition 5.5. *Let $X = N_{\mathbb{R}}/\Lambda$ be a real torus that admits a spanning curve Γ. Let $x \in X$, and recall that we denote by $t_x\colon X \to X$ the translation by x. Then for every tropical cycle $A \in Z_*(X)$ we have*

$$A \sim_{\mathrm{alg}} (t_x)_*A.$$

Proof. By the assumptions on Γ, the point x is in the subgroup of X generated by the differences $y - y'$ for pairs $y, y' \in \Gamma$ contained in the same edge of Γ. Therefore, it suffices to show that $(t_x)_*A \sim_{\mathrm{alg}} (t_{x'})_*A$ for any pair of points x, x' contained in the same edge of Γ. Let Γ_x be the component of Γ containing x. Even though Γ_x is not smooth, it still determines a metric graph G. After a choice of weights that makes Γ into a tropical 1-cycle, the metric graph G is equipped with weights $m\colon E(G) \to \mathbb{Z}_{>0}$ induced by the weights on Γ. Let $\widetilde{G}$ be the metric graph obtained from G replacing each edge e of G by an edge of length $\ell(e)/m(e)$, where $\ell(e)$ denotes the length of e in the metric graph G. If $\widetilde{\Gamma}$ denotes the smooth tropical curve associated to the graph $\widetilde{G}$ (see Section 2C), then there is a natural morphism $f\colon \widetilde{\Gamma} \to |\Gamma_x|$ of rational polyhedral spaces, which is a bijection of the underlying spaces. Let $t, t' \in \widetilde{\Gamma}$ be the unique points with $f(t) = x$ and $f(t') = x'$. Now let

$$g\colon X \times \widetilde{\Gamma} \to X \times \widetilde{\Gamma}, \quad (x, s) \mapsto (x + f(s), s),$$

and denote $W = g_*(A \times [\Gamma]) \in Z_*(X \times \Gamma)$. By construction, if $p\colon X \times \widetilde{\Gamma} \to X$ and $q\colon X \times \widetilde{\Gamma} \to \widetilde{\Gamma}$ denote the projections, we have

$$p_*(q^*(t) \cdot W) = (t_x)_*(A) \quad \text{and} \quad p_*(q^*(t') \cdot W) = (t_{x'})_*(A),$$

finishing the proof. □

5B. *Homological equivalence.*

Definition 5.6. Let X be a closed rational polyhedral space. We say that two tropical cycles A and B are *homologically equivalent*, if $\operatorname{cyc}(A) = \operatorname{cyc}(B)$.

Example 5.7. Let Γ be a compact and connected smooth tropical curve. By definition, we have $H_{0,0}(\Gamma) \cong H_0(\Gamma; \mathbb{Z}) \cong \mathbb{Z}$. It follows that the degree morphism $H_{0,0}(\Gamma) \to \mathbb{Z}$ is an isomorphism. Therefore, the homological equivalence class of a tropical 0-cycle is uniquely determined by its degree. Let $D \in \mathrm{CDiv}(\Gamma)$ be a Cartier divisor on Γ. By Theorem 3.3, we have $\mathrm{cyc}[D] = c_1(\mathscr{L}(D)) \frown [\Gamma]$, and by Poincaré duality this implies that $c_1(\mathscr{L}(D) = 0$ if and only if $\mathrm{cyc}[D] = 0$. By what we just saw, we have $\mathrm{cyc}[D] = 0$ if and only if the degree of D is 0. We see that if $D' \in \mathrm{CDiv}(\Gamma)$ is another Cartier divisor, then $[D]$ and $[D']$ are homologically equivalent if and only if $c_1(\mathscr{L}(D)) = c_1(\mathscr{L}(D'))$, which holds if and only if D and D' have the same degree.

Proposition 5.8. *Algebraic equivalence implies homological equivalence: if A and B are tropical cycles on a closed rational polyhedral space X with $A \sim_{\mathrm{alg}} B$, then $A \sim_{\mathrm{hom}} B$.*

Proof. By the definition of algebraic equivalence, we may assume that there exists a compact and connected smooth tropical curve Γ, two points $t_0, t_1 \in \Gamma$, and a tropical cycle W on $X \times \Gamma$ such that $A - B = p_*(q^*(t_0 - t_1) \cdot W)$. Since $t_0 - t_1$ has degree 0, we have $c_1(\mathscr{L}(t_0 - t_1)) = 0$; see Example 5.7. Therefore, by Theorem 3.3, we have

$$\mathrm{cyc}(A) - \mathrm{cyc}(B) = \mathrm{cyc}(A - B) = p_*(q^* c_1(\mathscr{L}(t_0 - t_1)) \frown \mathrm{cyc}(W)) = 0,$$

finishing the poof. □

Theorem 5.9. *Let X be a real torus admitting a spanning curve, and let $A, B \in Z_*(X)$ be tropical cycles. Then we have*

$$\mathrm{cyc}(A \cdot B) = \mathrm{cyc}(A) \cdot \mathrm{cyc}(B).$$

Proof. As both sides are bilinear in A and B, we may assume that A and B are pure-dimensional, say of dimensions k and l, respectively. By Propositions 5.5 and 5.8, we may replace A by a general translate. Therefore, we can assume that A and B meet transversally, that is that $|A| \cap |B|$ is either empty or of pure dimension $k + l - n$, where $n = \dim(X)$, and $(|A| \cap |B|)^{\mathrm{reg}} = |A|^{\mathrm{reg}} \cap |B|^{\mathrm{reg}}$.

As explained in [Gross and Shokrieh 2019, Remark 5.5], we can view $\mathrm{cyc}(A)$ as an element of the Borel–Moore homology group $H^{BM}_{k,k}(|A|, X)$ with support on $|A|$, and similarly

$$\mathrm{cyc}(B) \in H^{BM}_{l,l}(|A|, X) \quad \text{and} \quad \mathrm{cyc}(A \cdot B) \in H^{BM}_{k+l-n,k+l-n}(|A \cap B|, X).$$

Using Verdier duality [Gross and Shokrieh 2019, Theorem D], the cycle class $\mathrm{cyc}(A)$ is Poincaré dual to a cohomology class with support on $|A|$, that is to an element in $H^{n-k,n-k}_{|A|}(X)$. Therefore, the intersection product $\mathrm{cyc}(A) \cdot \mathrm{cyc}(B)$ is also represented by an element in $H^{BM}_{k+l-n,k+l-n}(|A \cap B|, X)$ and it suffices to prove the equality

$$\mathrm{cyc}(A \cdot B) = \mathrm{cyc}(A) \cdot \mathrm{cyc}(B)$$

in $H^{BM}_{k+l-n,k+l-n}(|A \cap B|, X)$. For dimension reasons, both sides are uniquely determined by their restrictions to $H^{BM}_{k+l-n,k+l-n}(|A \cap B| \cap U, U)$, where U is an open subset of X with $U \cap |A \cap B| = |A \cap B|^{\mathrm{reg}}$ [Gross and Shokrieh 2019, Lemma 4.8(b)]. Combining the facts that $V \mapsto H^{BM}_{k+l-n,k+l-n}(|A \cap B| \cap V, V)$

satisfies the sheaf axioms [loc. cit., Lemma 4.8(b)], X is locally isomorphic to open subsets of $\mathbb{R}^n$, and $(|A|\cap|B|)^{\mathrm{reg}} = |A|^{\mathrm{reg}}\cap|B|^{\mathrm{reg}}$ allows us to further reduce to the case where $U=\mathbb{R}^n$ and A and B are linear subspaces of $\mathbb{R}^n$. In this case, there exist hyperplanes $H_1,\dots,H_{n-k}$ and $H'_1,\dots,H'_{n-l}$, and integers $a, b\in\mathbb{Z}$ such that

$$A = a\cdot H_1\cdots H_{n-k} \quad\text{and}\quad B = b\cdot H'_1\cdots H_{n-l}.$$

Let $\alpha\in H^{n-k,n-k}_{|A|}(X)$ be the Poincaré dual to $\mathrm{cyc}(A)$. Applying [loc. cit., Proposition 5.12] (see also [loc. cit., Remark 5.13]) yields

$$\begin{aligned}\mathrm{cyc}(A\cdot B) &= \mathrm{cyc}((a\cdot H_1\cdots H_{n-k})\cdot(b\cdot H'_1\cdots H_{n-l})\cdot[X])\\ &= \big(a\cdot c_1(\mathscr{L}(H_1))\smile\cdots\smile c_1(\mathscr{L}(H_{n-k}))\big)\frown\big((b\cdot c_1(\mathscr{L}(H'_1))\smile\cdots\smile c_1(\mathscr{L}(H'_{n-l})))\frown\mathrm{cyc}[X]\big)\\ &= \alpha\frown\mathrm{cyc}(B)\\ &= \mathrm{cyc}(A)\cdot\mathrm{cyc}(B),\end{aligned}$$

where the last equality holds by the definition of the intersection product of tropical homology classes. This finishes the proof. □

5C. *Numerical equivalence.*

Definition 5.10. Let X be a closed tropical manifold. Then two tropical cycles $A, B\in Z_*(X)$ on X are *numerically equivalent*, for which we write $A\sim_{\mathrm{num}} B$, if for every tropical cycle $C\in Z_*(X)$ on X we have

$$\int_X A\cdot C = \int_X B\cdot C.$$

Proposition 5.11. *Let X be a real torus admitting a spanning curve, and let $A, B\in Z_*(X)$ with $A\sim_{\mathrm{hom}} B$. Then $A\sim_{\mathrm{num}} B$.*

Proof. Let $C\in Z_*(X)$. By Theorem 5.9, we have

$$\int_X A\cdot C = \int_X \mathrm{cyc}(A\cdot C) = \int_X \mathrm{cyc}(A)\cdot\mathrm{cyc}(C) = \int_X \mathrm{cyc}(B)\cdot\mathrm{cyc}(C) = \int_X \mathrm{cyc}(B\cdot C) = \int_X B\cdot C,$$

from which the assertion follows. □

6. Tropical homology of real tori

Let $X = N_{\mathbb{R}}/\Lambda$ be a real torus. Then the group law and the tropical cross product endow the tropical homology groups with the additional structure of the Pontryagin product.

Definition 6.1. Let X be a real torus with group law $\mu\colon X\times X\to X$. The *tropical Pontryagin product* is defined as the pairing

$$(\alpha,\beta)\mapsto\alpha\star\beta := \mu_*(\alpha\times\beta),$$

where α and β are either elements of $Z_*(X)$ or of $H_{*,*}(X)$. We thus obtain morphisms

$$\star\colon Z_i(X) \otimes_{\mathbb{Z}} Z_k(X) \to Z_{i+k}(X)$$
$$\star\colon H_{i,j}(X) \otimes_{\mathbb{Z}} H_{k,l}(X) \to H_{i+k,j+l}(X)$$

for all choices of natural numbers i, j, k, l. It is not hard to see that $\star$ makes $Z_*(X)$ into a graded abelian group, and $H_{*,*}(X)$ into a bigraded abelian group.

Proposition 6.2. *Let X be a real torus. Then the tropical cycle class map respects Pontryagin products, that is the diagram*

$$\begin{tikzcd} Z_i(X) \otimes_{\mathbb{Z}} Z_j(X) \arrow[r, "\star"] \arrow[d, "\mathrm{cyc} \otimes \mathrm{cyc}"] & Z_{i+j}(X) \arrow[d, "\mathrm{cyc}"] \\ H_{i,i}(X) \otimes_{\mathbb{Z}} H_{j,j}(X) \arrow[r, "\star"] & H_{i+j,i+j}(X) \end{tikzcd}$$

is commutative for all $i, j \in \Sigma$.

Proof. Since the Pontryagin product is defined as the push-forward of a cross product, this follows immediately from the compatibility of the tropical cycle class map with cross products and push-forwards stated in Theorem 3.3. ☐

For the real torus $X = N_{\mathbb{R}}/\Lambda$, we will now describe the group $H_{*,*}(X)$ and the Pontryagin product on it explicitly. First we note that the sheaf Ω^1_X is the constant sheaf M_X associated to the lattice $M = \operatorname{Hom}(N, \mathbb{Z})$, and since $X^{\mathrm{reg}} = X$, we have $\Omega^k_X \cong \left(\bigwedge^k M\right)_X$ for all integers k. By definition of singular tropical homology, we thus have a canonical graded isomorphism

$$H_{*,*}(X) \cong H_*\left(X; \bigwedge^* N\right) \cong H_*(X; \mathbb{Z}) \otimes_{\mathbb{Z}} \bigwedge^* N.$$

The restriction of the Pontryagin product to the first factor $H_*(X; \mathbb{Z}) \cong H_{0,*}(X)$ is precisely the classical Pontryagin product one obtains when one views X as a topological group. But, as a topological group, X is a product of 1-spheres. So using the Künneth theorem one sees that $H_*(X; \mathbb{Z})$ is isomorphic to $\bigwedge H_1(X; \mathbb{Z})$. This is, in fact, an isomorphism of rings, the multiplication of $H_*(X; \mathbb{Z})$ being the Pontryagin product. Finally, because X is the quotient of its universal covering space $N_{\mathbb{R}}$ by the action of Λ, we obtain a natural isomorphism $H_1(X; \mathbb{Z}) \cong \Lambda$. If a tropical 1-cycle in $H_1(X; \mathbb{Z})$ is represented by a loop $\gamma\colon [0, 1] \to X$ then the corresponding element of Λ is given by $\widetilde{\gamma}(1) - \widetilde{\gamma}(0)$ for any lift $\widetilde{\gamma}\colon [0, 1] \to N_{\mathbb{R}}$ of γ to the universal cover. We obtain an isomorphism

$$H_{*,*}(X) \cong \bigwedge^* \Lambda \otimes_{\mathbb{Z}} \bigwedge^* N. \tag{6-1}$$

It is straightforward to check that with this identification, the tropical Pontryagin product on $H_{*,*}(X)$ satisfies

$$(\alpha \otimes \omega) \star (\beta \otimes \xi) = (\alpha \wedge \beta) \otimes (\omega \wedge \xi).$$

By a similar argument, one obtains a description for the tropical cohomology of X that is dual to the description of tropical homology in (6-1). More precisely, one sees that

$$H^{*,*}(X) \cong \bigwedge^* \Lambda^* \otimes_{\mathbb{Z}} \bigwedge^* M, \tag{6-2}$$

and that, with this identification, the tropical cup product on $H^{*,*}(X)$ satisfies

$$(\alpha \otimes \omega) \smile (\beta \otimes \xi) = (\alpha \wedge \beta) \otimes (\omega \wedge \xi).$$

With the descriptions of the tropical homology and the tropical cohomology given in (6-1) and (6-2), the tropical cap product can also be expressed explicitly. More precisely, we have

$$(\alpha \otimes \omega) \frown (\beta \otimes \xi) = (\alpha \lrcorner \beta) \otimes (\omega \lrcorner \xi), \tag{6-3}$$

where "$\lrcorner$" denotes the *interior product* on the exterior algebra.

In bidegree (1, 1) our description of the tropical cohomology of X produces an isomorphism

$$H^{1,1}(X) \cong \Lambda^* \otimes_{\mathbb{Z}} M.$$

We can further identify the right side with $\mathrm{Hom}(\Lambda \otimes_{\mathbb{Z}} N, \mathbb{Z})$, that is with bilinear forms on $N_{\mathbb{R}}$ that have integer values on $\Lambda \times N$.

Convention 6.3. From now on we will always identify, according to the identifications in this section, the cohomology group $H^{1,1}(N_{\mathbb{R}}/\Lambda)$ with the group of bilinear forms on $N_{\mathbb{R}}$ that have integer values on $\Lambda \times N$.

7. Line bundles on real tori

7A. ***Factors of automorphy.*** Let N be a lattice, let $\Lambda \subseteq N_{\mathbb{R}}$ be a lattice of full rank, and let $X = N_{\mathbb{R}}/\Lambda$ be the real torus associated to N and Λ. To describe the tropical line bundles on X we recall from Section 3B that they form a group, canonically identified with $H^1(X, \mathrm{Aff}_X)$. Invoking the results from [Mumford 2008, Appendix to Section 2], together with the fact that the pull-back $\pi^{-1}\mathrm{Aff}_X \cong \mathrm{Aff}_{N_{\mathbb{R}}}$ along the quotient morphism $\pi : N_{\mathbb{R}} \to N_{\mathbb{R}}/\Lambda = X$ has trivial cohomology on $N_{\mathbb{R}}$, we obtain the identification

$$H^1(X, \mathrm{Aff}_X) \cong H^1(\Lambda, \Gamma(N_{\mathbb{R}}, \mathrm{Aff}_{N_{\mathbb{R}}})),$$

where the right side is the first group cohomology group of $\Gamma(N_{\mathbb{R}}, \mathrm{Aff}_{N_{\mathbb{R}}})$, equipped with its natural Λ-action. This is very much akin to the case of complex tori: an element of $H^1(\Lambda, \Gamma(N_{\mathbb{R}}, \mathrm{Aff}_{N_{\mathbb{R}}}))$ can be represented by a *tropical factor of automorphy*, that is a family of integral affine functions indexed by Λ, that, if we represent it as a function $a\colon \Lambda \times N_{\mathbb{R}} \to \mathbb{R}$, satisfies

$$a(\lambda + \mu, x) = a(\lambda, \mu + x) + a(\mu, x) \tag{7-1}$$

for all $\mu, \lambda \in \Lambda$ and $x \in N_{\mathbb{R}}$. Two factors of automorphy represent the same element of $H^1(\Lambda, \Gamma(N_{\mathbb{R}}, \mathrm{Aff}_{N_{\mathbb{R}}}))$ if and only if they differ by a factor of automorphy of the form

$$(\lambda, x) \mapsto l(x+\lambda) - l(x)$$

for some integral affine function $l \in \Gamma(N_{\mathbb{R}}, \mathrm{Aff}_{N_{\mathbb{R}}})$, which happens if and only if they differ by a factor of automorphy of the form

$$(\lambda, x) \mapsto m_{\mathbb{R}}(\lambda),$$

where $m_{\mathbb{R}}$ is the $\mathbb{R}$-linear extension of a linear form $m\colon N \to \mathbb{Z}$.

Any factor of automorphy $a(\,-\,,-\,)$ defines a group action $\lambda.(x, b) = (x+\lambda, b+a(\lambda, x))$ of Λ on the trivial line bundle $N_{\mathbb{R}} \times \mathbb{R}$ on $N_{\mathbb{R}}$. The tropical line bundle on X corresponding to $a(\,-\,,-\,)$ is the quotient $(N_{\mathbb{R}} \times \mathbb{R})/\Lambda$.

7B. *The Appell–Humbert Theorem.* It is easy to check that for every morphism $l \in \mathrm{Hom}(\Lambda, \mathbb{R})$ and every symmetric bilinear form E on $N_{\mathbb{R}}$ with $E(\Lambda \times N) \subseteq \mathbb{Z}$, the family of integral affine functions on $N_{\mathbb{R}}$ defined by

$$a_{E,l}(\lambda, x) = l(\lambda) - E(\lambda, x) - \tfrac{1}{2} E(\lambda, \lambda)$$

is a tropical factor of automorphy. We denote the associated tropical line bundle on X by $\mathscr{L}(E, l)$. The following proposition shows that the first Chern class recovers E from $\mathscr{L}(E, l)$.

Proposition 7.1. *Let E be a symmetric bilinear form on $N_{\mathbb{R}}$ with $E(\Lambda \times N) \subseteq \mathbb{Z}$, and let $l \in \mathrm{Hom}(\Lambda, \mathbb{R})$. Then $c_1(\mathscr{L}(E, l)) = E$, where we identify $H^{1,1}(X)$ with the group of bilinear forms on $N_{\mathbb{R}}$ with integer values on $\Lambda \times N$ according to Convention 6.3.*

Proof. Let $\mathfrak{U} = \{U_\alpha\}_\alpha$ be an open cover of X such that each preimage $\pi^{-1} U_\alpha$ is a union of disjoint open subsets of $N_{\mathbb{R}}$ that map homeomorphically onto U_α. For each α, choose a continuous section $s_\alpha\colon U_\alpha \to \pi^{-1} U_\alpha$ of π. Furthermore, we choose a (necessarily noncontinuous) section $s\colon X \to N_{\mathbb{R}}$ of π. By construction, the line bundle $\mathscr{L}(E, l)$ is represented by the Čech cocycle

$$(U_{\alpha,\beta} \ni x \mapsto a_{E,l}(s_\beta(x) - s_\alpha(x), s_\alpha(x))) \in \check{C}^1(\mathfrak{U}, \mathrm{Aff}_X).$$

Note that $s_\beta - s_\alpha$ has values in Λ and is therefore constant on the connected components of $U_{\alpha,\beta} = U_\alpha \cap U_\beta$ by continuity. In particular, the functions $x \mapsto a_{E,l}(s_\beta(x) - s_\alpha(x), s_\alpha(x))$ are indeed integral affine. By definition, the first Chern class of $\mathscr{L}(E, l)$ is represented by the Čech cocycle obtained by differentiating the transition functions for all α and β. Using the definition of $a_{E,l}$, it follows that $c_1(\mathscr{L}(E, l))$ is represented by the cocycle

$$(U_{\alpha,\beta} \ni x \mapsto -E(s_\beta(x) - s_\alpha(x))) \in \check{C}^1(\mathfrak{U}, \Omega^1_X), \tag{7-2}$$

where we consider E as a function $\Lambda \to N^*$. To compute what this corresponds to under the identification of $H^1(X, \Omega^1_X)$ with $H^1(X; N^*) \cong \Lambda^* \otimes N^*$, we consider the double complex

$$(\check{C}^i(\mathfrak{U}, \mathscr{C}^j(X; N^*)), d_{ij}, \partial_{ij}),$$

where $\mathscr{C}^i(X; N^*)$ denotes the sheafification of the presheaf

$$U \mapsto C^i(U; N^*)$$

and we set $C^{-1}(U; N^*) = N^*$ and $\check{C}^{-1}(\mathfrak{U}, \mathscr{F}) = \Gamma(X, \mathscr{F})$ for any sheaf $\mathscr{F}$. We follow the cocycle of formula (7-2) through the double complex in the zig-zag from the $(1, -1)$ entry to the $(-1, 1)$ entry indicated by the solid arrows in the following diagram:

$$\begin{tikzcd}
 & 0 \arrow[d, dashed] & 0 \arrow[d, dashed] & 0 \arrow[d, dashed] & \\
0 \arrow[r, dashed] & N^* \arrow[r, dashed] \arrow[d, dashed] & C^0(X; N^*) \arrow[r, dashed] \arrow[d, dashed] & C^1(X; N^*) \arrow[r, dashed] \arrow[d] & \cdots \\
0 \arrow[r, dashed] & \check{C}^0(\mathfrak{U}, (N^*)_X) \arrow[r, dashed] \arrow[d, dashed] & \check{C}^0(\mathfrak{U}, \mathscr{C}^0(X; N^*)) \arrow[r] \arrow[d] & \check{C}^0(\mathfrak{U}, \mathscr{C}^1(X; N^*)) \arrow[r, dashed] \arrow[d, dashed] & \cdots \\
0 \arrow[r, dashed] & \check{C}^1(\mathfrak{U}, (N^*)_X) \arrow[r] \arrow[d, dashed] & \check{C}^1(\mathfrak{U}, \mathscr{C}^0(X; N^*)) \arrow[r, dashed] \arrow[d, dashed] & \check{C}^1(\mathfrak{U}, \mathscr{C}^1(X; N^*)) \arrow[r, dashed] \arrow[d, dashed] & \cdots \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

First we apply the differential coming from singular cohomology and obtain

$$((U_{\alpha,\beta} \xleftarrow{x} \{0\}) \mapsto -E(s_\beta(x) - s_\alpha(x))) \in \check{C}^1(\mathfrak{U}, \mathscr{C}^0(X; N^*)).$$

Clearly, this is the image under the differential coming from Čech cohomology of the cochain

$$((U_\alpha \xleftarrow{x} \{0\}) \mapsto -E(s_\alpha(x) - s(x))) \in \check{C}^0(\mathfrak{U}, \mathscr{C}(X; N^*)).$$

Applying the differential of singular cohomology again we obtain

$$((U_\alpha \xleftarrow{\sigma} [0, 1]) \mapsto -E(s_\alpha(\sigma(1)) - s(\sigma(1)) - s_\alpha(\sigma(0)) + s(\sigma(0)))) \in \check{C}^0(\mathfrak{U}, \mathscr{C}^1(X; N^*)).$$

This can be lifted to a singular 1-cochain. Namely, for an arbitrary 1-simplex $\sigma : [0, 1] \to X$ we choose a lift $\sigma' : [0, 1] \to N_{\mathbb{R}}$ and then assign to σ the value

$$-E(\sigma'(1) - s(\sigma(1)) - \sigma'(0) + s(\sigma(0))).$$

This is clearly independent of the choice of σ'. In particular, if the image of σ is contained in U_α, we may choose $\sigma' = s_\alpha \circ \sigma$ and obtain the same cocycle on U_α as before. It is also clear that any loop in X which is the image of a path in $N_{\mathbb{R}}$ from 0 to $\lambda \in \Lambda$ is mapped to $-E(\lambda)$ by this 1-cochain. Therefore, we have $c_1(\mathscr{L}(E, l)) = E$ when identifying $H^{1,1}(X)$ with $\Lambda^* \otimes_{\mathbb{Z}} N^*$ according to Convention 6.3. □

Theorem 7.2 (tropical Appell–Humbert theorem). *Let $\mathscr{L}$ be a tropical line bundle on the real torus $X = N_{\mathbb{R}}/\Lambda$. Then there exists $l \in \mathrm{Hom}(\Lambda, \mathbb{R})$ and a symmetric form E on $N_{\mathbb{R}}$ with $E(\Lambda \times N) \subseteq \mathbb{Z}$ such that $\mathscr{L} \cong \mathscr{L}(E, l)$. Moreover, if we are given another choice of $l' \in \mathrm{Hom}(\Lambda, \mathbb{R})$ and symmetric form E' on $N_{\mathbb{R}}$ with $E'(\Lambda \times N) \subseteq \mathbb{Z}$, then $\mathscr{L} \cong \mathscr{L}(E', l')$ if and only if $E = E'$ and the linear form $(l - l')_{\mathbb{R}} : N_{\mathbb{R}} \to \mathbb{R}$ has integer values on N.*

Proof. We have already seen in Section 7A that there exists a tropical factor of automorphy $a : \Lambda \times N_{\mathbb{R}} \to \mathbb{R}$ such that $\mathscr{L}$ is the line bundle associated to $a(-, -)$. For every $\lambda \in \Lambda$, the function $a(\lambda, -)$ is integral affine, hence its differential $E(\lambda) := -da(\lambda, -)$ defines an element in $\mathrm{Hom}(N, \mathbb{Z})$. Differentiating (7-1), we see that the map $\lambda \mapsto E(\lambda)$ is linear. In other words, E defines a bilinear map on $\Lambda \times N \to \mathbb{Z}$. Therefore, for a suitable function $b\colon \Lambda \to \mathbb{R}$, we have $a(\lambda, x) = -E(\lambda, x) + b(\lambda)$ for all $\lambda \in \Lambda$ and $x \in N_{\mathbb{R}}$. Plugging this into (7-1), we see that $E(\lambda, \mu) = E(\mu, \lambda)$ for all $\lambda, \mu \in \Lambda$, that is that E is, in fact, symmetric. The tropical factor of automorphy $a - a_{E,0}$ is then a family of constant functions, that is we have

$$(a - a_{E,0})(\lambda, x) = l(\lambda)$$

for some function $l\colon \Lambda \to \mathbb{R}$. Applying (7-1) once more we see that l is, in fact, linear. It follows that $a = (a - a_{E,0}) + a_{E,0} = a_{E,l}$. In particular, we have $\mathscr{L} \cong \mathscr{L}(E, l)$.

Now assume we are given a second choice of linear function $l' \in \operatorname{Hom}(\Lambda, \mathbb{R})$ and symmetric form E' on $N_{\mathbb{R}}$ with $E'(\Lambda \times N) \subseteq \mathbb{Z}$ such that $\mathscr{L}(E', l') \cong \mathscr{L}$. We have already seen in Section 7A that this happens if and only if $a_{E,l} - a_{E',l'}$ is of the form $a_{0,m_{\mathbb{R}}|_\Lambda}$ for some linear function $m\colon N \to \mathbb{Z}$. By Proposition 7.1, we have

$$E' = c_1(\mathscr{L}(E', l')) = c_1(\mathscr{L}(E, l)) = E.$$

Therefore, we have $a_{E,l} - a_{E',l'} = a_{0,l-l'}$ and it follows that $(l - l')_{\mathbb{R}}$ has integer values on N. $\square$

Remark 7.3. It follows directly from the tropical Appell–Humbert theorem that there is a bijection between the group of all tropical line bundles with trivial first Chern class and $\Lambda^*_{\mathbb{R}}/N^*$, which is called the dual real torus to X for that reason.

Example 7.4. Let $N = \mathbb{Z}^2$ and let $\Lambda = \mathbb{Z}u_1 + \mathbb{Z}u_2 \subseteq N_{\mathbb{R}} = \mathbb{R}^2$, where

$$u_1 = \begin{pmatrix} 1 \\ \sqrt{6} \end{pmatrix} \quad \text{and} \quad u_2 = \begin{pmatrix} \sqrt{3} \\ \sqrt{2} \end{pmatrix}.$$

We claim that the real torus $X = N_{\mathbb{R}}/\Lambda$ has no spanning curve. Indeed, if there was one, then there existed an effective tropical 1-cycle C in X supported on a spanning curve. Because the tangent directions of C span $N_{\mathbb{R}}$, there is a translate C' of C that intersects C transversally in at least one point. Therefore, we have

$$0 \neq \operatorname{cyc}(C \cdot C') = \operatorname{cyc}(C^2) = \operatorname{cyc}(C)^2,$$

where the first equality follows from Propositions 5.2 and 5.5, and the second one from Theorem 5.9. In particular, we have $\operatorname{cyc}(C) \neq 0$.

As C is a hypersurface in X, it is a tropical Cartier divisor and we have

$$\operatorname{cyc}(C) = c_1(\mathscr{L}(C)) \frown \operatorname{cyc}[X].$$

By Proposition 7.1 and Theorem 7.2, the first Chern class $c_1(\mathscr{L}(C))$ is given by a symmetric from E on $N_{\mathbb{R}}$ with $E(\Lambda \times N) \subseteq \mathbb{Z}$. Let $\mathbb{Z} \ni \alpha_{ij} = E(u_i, e_j)$, where e_i denotes the standard basis of $\mathbb{Z}^2$. Then E is symmetric if and only if

$$\alpha_{11}\sqrt{3} + \alpha_{12}\sqrt{2} = E(u_1, u_2) = E(u_2, u_1) = \alpha_{21} + \alpha_{22}\sqrt{6}.$$

The numbers 1, $\sqrt{2}$, $\sqrt{3}$, and $\sqrt{6}$ being linearly independent over $\mathbb{Q}$ implies that all α_{ij} are zero. Thus, we have $E=0$, which is equivalent to $c_1(\mathscr{L}(C))=0$, which in turn implies that $\operatorname{cyc}(C)=0$, a contradiction.

7C. *Translations of line bundles.*

Proposition 7.5. *Let $X=N_{\mathbb{R}}/\Lambda$ be a real torus, let $l\in\operatorname{Hom}(\Lambda,\mathbb{R})$, and let E be a symmetric bilinear form on $N_{\mathbb{R}}$ with $E(\Lambda\times N)\subseteq\mathbb{Z}$. Furthermore, let $\pi\colon N_{\mathbb{R}}\to X$ be the projection and let $y\in N_{\mathbb{R}}$. Then we have*

$$t_{\pi(y)}^{*}\mathscr{L}(E,l)\cong\mathscr{L}(E,l-E(-,y)).$$

In particular, if the bilinear form E is nondegenerate and $\mathscr{L}'$ is any line bundle on $N_{\mathbb{R}}/\Lambda$ with $c_1(\mathscr{L}')=E$, then there exists $x\in X$ such that $\mathscr{L}'\cong t_x^{*}\mathscr{L}(E,l)$. If, moreover, E restricts to a perfect pairing $\Lambda\times N\to\mathbb{Z}$, then x is unique.

Proof. We recall from above that $\mathscr{L}(E,l)$ can be defined as the quotient of the trivial bundle $N_{\mathbb{R}}\times\mathbb{R}$ by the Λ-action given by $\lambda.(x,b)=(x+\lambda,b+a_{(E,l)}(\lambda,x))$. Since the morphism $\widetilde{t}_y\colon N_{\mathbb{R}}\to N_{\mathbb{R}}$, $x\mapsto x+y$ that induces $t_{\pi(y)}$ on the quotient $N_{\mathbb{R}}/\Lambda$ is Λ-equivariant, the pull-back $t_{\pi(y)}^{*}\mathscr{L}(E,l)$ can be represented as the quotient of

$$\widetilde{t}_y^{*}(N_{\mathbb{R}}\times\mathbb{R})\cong N_{\mathbb{R}}\times\mathbb{R}$$

by the pulled back Λ-action. The action of $\lambda\in\Lambda$ on (x,b) under the pulled back action is obtained by first applying $\widetilde{t}_y$ to the first coordinate, yielding $(x+y,b)$, then applying the Λ-action defined by $a_{(E,l)}$, yielding $(x+y+\lambda,b+a_{(E,l)}(\lambda,x+y))$, and finally applying $\widetilde{t}_y^{-1}$ to the first coordinate, yielding $(x+\lambda,b+a_{(E,l)}(\lambda,x+y))$. So in total, the pulled back action is given by

$$\begin{aligned}\lambda.(x,b)&=(x+\lambda,b+a_{(E,l)}(\lambda,x+y))\\&=\bigl(x+\lambda,b+l(\lambda)-E(\lambda,x+y)-\tfrac{1}{2}E(\lambda,\lambda)\bigr)\\&=\bigl(x+\lambda,b+l(\lambda)-E(\lambda,y)-E(\lambda,x)-\tfrac{1}{2}E(\lambda,\lambda)\bigr)\\&=(x+\lambda,b+a_{(E,l-E(-,y))}),\end{aligned}$$

which is precisely the action on the trivial bundle defined by the factor of automorphy $a_{(E,l-E(-,y))}$. This shows that $t_{\pi(y)}^{*}\mathscr{L}(E,l)=\mathscr{L}(E,l-E(-,y))$.

Now assume that E is nondegenerate and that $\mathscr{L}'$ is any line bundle on X with $c_1(\mathscr{L}')=E$. By Theorem 7.2 and Proposition 7.1, there exists a linear form $l'\colon\Lambda\to\mathbb{R}$ such that $\mathscr{L}'\cong\mathscr{L}(E,l')$. Since E is nondegenerate and $\Lambda_{\mathbb{R}}\cong N_{\mathbb{R}}$, there exists $\tilde{x}\in N_{\mathbb{R}}$ such that $l-l'=E(-,\tilde{x})$. By what we have shown above, we have

$$\mathscr{L}'\cong\mathscr{L}(E,l-E(-,\tilde{x}))\cong t_{\pi(\tilde{x})}^{*}\mathscr{L}(E,l)=t_x^{*}\mathscr{L}(E,l),$$

where $x=\pi(\tilde{x})$. If $x'\in X$ is another point such that $t_{x'}^{*}\mathscr{L}(E,l)\cong\mathscr{L}'$, and $\tilde{x}'\in N_{\mathbb{R}}$ is chosen such that $\pi(\tilde{x}')=x'$, then we have

$$\mathscr{L}(E,l-E(-,\tilde{x}))\cong\mathscr{L}(E,l-E(-,\tilde{x}'))$$

by what we have shown above. This happens if and only if $E(-, \tilde{x} - \tilde{x}')$ has integer values on N by Theorem 7.2. If E restricts to a perfect pairing on $\Lambda \times N$, this happens if and only if $\tilde{x} - \tilde{x}' \in \Lambda$, that is if and only if $x = x'$. □

Remark 7.6. If we call two line bundles on a real torus *tropically equivalent* if they have the same first Chern class, then Proposition 7.5 shows that two tropical line bundles which are translates of each other are tropically equivalent, with the converse being true if their first Chern class is nondegenerate. This is completely analogous to the situation on complex tori, where two line bundles are *analytically equivalent* if they have the same first Chern class [Birkenhake and Lange 2004, Proposition 2.5.3]. If two line bundles on a complex torus are translates of each other, then they are analytically equivalent, with the converse being true if their first Chern class is nondegenerate [loc. cit., Corollary 2.5.4].

7D. *Rational sections of line bundles.* Let $E\colon \Lambda \times N \to \mathbb{Z}$ be bilinear such that $E_{\mathbb{R}}$ is a symmetric bilinear form on $N_{\mathbb{R}}$, and let $l\colon \Lambda \to \mathbb{R}$ be linear. As mentioned above, the tropical line bundle $\mathscr{L}(E, l)$ on X is a quotient of the trivial bundle $N_{\mathbb{R}} \times \mathbb{R}$ by the Λ-action defined by E and l. In particular, the global rational sections of $\mathscr{L}(E, l)$ are precisely those global rational sections of $N_{\mathbb{R}} \times \mathbb{R}$ that are invariant under the Λ-action. More precisely, the global rational sections of $\mathscr{L}(E, l)$ are in bijection with the piecewise linear function $\phi \in \Gamma(N_{\mathbb{R}}, \mathscr{M}_{N_{\mathbb{R}}})$ such that

$$\phi(x + \lambda) = \phi(x) + l(\lambda) - E(\lambda, x) - \tfrac{1}{2} E(\lambda, \lambda). \tag{7-3}$$

The divisor associated to the section of $\mathscr{L}(E, l)$ corresponding to ϕ is precisely the quotient of $\operatorname{div}(\phi)$ by the Λ-action. In particular, this divisor is effective if and only if $\operatorname{div}(\phi)$ is effective, that is if ϕ is concave. Together, concavity and (7-3) put strong constraints on ϕ, or rather its Legendre transform. In fact, it has been shown in [Mikhalkin and Zharkov 2008, Theorem 5.4] that if E is a perfect pairing and $E_{\mathbb{R}}$ is positive definite, these constraints completely determine ϕ up to an additive constant. More precisely, ϕ is given by

$$\phi(x) = \min\bigl\{E(\lambda, x) + \tfrac{1}{2} E(\lambda, \lambda) - l(\lambda) \mid \lambda \in \Lambda\bigr\} + \text{const}$$

in this case (note that this only differs from the formula in [loc. cit.] because we are using the "min"-convention, see Remark 3.4). By the tropical Appell–Humbert theorem it follows that for every line bundle $\mathscr{L}$ on X with $c_1(\mathscr{L}) = E$ there exists a unique effective divisor $D \in \operatorname{CDiv}(X)$ with $\mathscr{L}(D) = \mathscr{L}$.

Proposition 7.7. *Let $X = N_{\mathbb{R}}/\Lambda$ be the real torus associated to a pair of lattices N and $\Lambda \subset N_{\mathbb{R}}$, and let $D, D' \in \operatorname{CDiv}(X)$ be two effective divisors such that $\operatorname{cyc}[D] = \operatorname{cyc}[D']$ is Poincaré dual to $E \in H^{1,1}(X)$ for some perfect pairing $E\colon \Lambda \times N \to \mathbb{Z}$ such that $E_{\mathbb{R}}$ is a positive definite symmetric bilinear form on $N_{\mathbb{R}}$, where we identify $H^{1,1}(X)$ with $\operatorname{Hom}(\Lambda, N^*)$ according to Convention 6.3. Then there exits a unique $x \in X$ such that $t_x^* D = D'$.*

Proof. We have

$$\operatorname{cyc}[D] = \operatorname{cyc}(D \cdot [X]) = c_1(\mathscr{L}(D)) \frown \operatorname{cyc}[X],$$

so $\mathrm{cyc}[D]$ is Poincaré dual to $c_1(\mathscr{L}(D))$, and similarly $\mathrm{cyc}[D']$ is Poincaré dual to $c_1(\mathscr{L}(D'))$. By assumption, it follow that

$$c_1(\mathscr{L}(D)) = c_1(\mathscr{L}(D')) = E.$$

By Proposition 7.5, there exists a unique point $x \in X$ such that $t_x^*(\mathscr{L}(D)) \cong \mathscr{L}(t_x^*(D))$ is isomorphic to $\mathscr{L}(D')$. It follows that the two divisors $t_x^*(D)$ and D' correspond to two concave rational sections of $\mathscr{L}(D')$. But, since $c_1(\mathscr{L}(D')) = E$, these two rational sections differ by a constant. Therefore, $D' = t_x^*(D)$. □

8. Tautological cycles on tropical Jacobians

Classically, the ring of tautological classes on the Jacobian of an algebraic curve is the smallest subring of its Chow group that contains the image of the curve under the Abel–Jacobi map and is invariant under intersection products, Pontryagin products, translations, and the involution map. We will now introduce the most important tropical tautological cycles on a tropical Jacobian.

Throughout this section, Γ will denote a compact connected smooth tropical curve of genus g. We will also fix a base point $q \in \Gamma$ with respect to which we define the Abel–Jacobi map.

8A. ***Effective loci and semibreak divisors.*** Using the group structure on the Jacobian, the Abel–Jacobi map induces morphisms $\Phi_q^d \colon \Gamma^d \to \mathrm{Jac}(\Gamma)$ for all nonnegative integers d.

Definition 8.1. For every integer $0 \le d \le g$ we define

$$\widetilde{W}_d := \Phi_q^d(\Gamma^d).$$

Because Φ_q^d is a proper morphism of boundaryless rational polyhedral spaces, we know that $\widetilde{W}_d$ is an at most d-dimensional boundaryless rational polyhedral subspace of $\mathrm{Jac}(\Gamma)$. By definition, $(\Phi_q^d)_*[\Gamma^d]$ is a tropical d-cycle on $\widetilde{W}_d$. Note that this does not mean that $\widetilde{W}_d$ has dimension d or that it is pure-dimensional as $(\Phi_q^d)_*[\Gamma^d]$ could be 0. All we can say a priori is that the support of $(\Phi_q^d)_*[\Gamma^d]$ is precisely the subset of points of $\widetilde{W}_d$ where the local dimension of $\widetilde{W}_d$ is equal to d.

To show that $\widetilde{W}_d$ in fact is purely d-dimensional we will use the identification of $\mathrm{Jac}(\Gamma)$ with the $\mathrm{Pic}^0(\Gamma)$ given by the tropical Abel–Jacobi theorem [Mikhalkin and Zharkov 2008]. Here, $\mathrm{Pic}(\Gamma)$ denotes the quotient of $\mathrm{CDiv}(\Gamma)$ by the subgroup consisting of all principal divisors, and $\mathrm{Pic}^d(\Gamma)$ denotes the subgroup of $\mathrm{Pic}(\Gamma)$ consisting of the all classes of divisors of degree d. The statement of the tropical Abel–Jacobi theorem is that the Abel–Jacobi map Φ_q induces a bijections $\mathrm{Pic}^d(\Gamma) \to \mathrm{Jac}(\Gamma)$ for $d = 0$, and hence for any d. If W_d denotes the preimage of $\widetilde{W}_d$ in $\mathrm{Pic}^d(\Gamma)$ under the bijection $\mathrm{Pic}^d(\Gamma) \to \mathrm{Jac}(\Gamma)$, then W_d is precisely the set of the classes of effective divisors of degree d. In particular W_d is independent of the base point q. Together with L. Tóthmérész, we have proved the following theorem.

Theorem 8.2 [Gross et al. 2022, Theorem 8.3]. *The subset W_d of $\mathrm{Pic}^d(\Gamma)$ is purely d-dimensional.*

It follows immediately that $\widetilde{W}_d$ is purely d-dimensional as well, and hence that the tropical cycle $(\Phi_q^d)_*[\Gamma^d]$ has support $\widetilde{W}_d$. We will now show that $\widetilde{W}_d$ has a fundamental cycle $[\widetilde{W}_d]$ which we will relate to $(\Phi_q^d)_*[\Gamma^d]$. To do this, we will need the notion of a break and semibreak divisors. A *break divisor* on Γ is an effective divisor B such that there exist g open edge segments $e_1, \dots, e_g \subseteq \Gamma$ and points $q_i \in \bar{e}_i$ such that $\Gamma \setminus \bigcup_i e_i$ is contractible and $B = \sum_i (q_i)$. A *semibreak divisor* is an effective divisor that is dominated by a break divisor, that is an effective divisor D such that there exists an effective divisor E for which $D + E$ is a break divisor; see [Gross et al. 2022].

Proposition 8.3. *Let* $0 \le d \le g$. *Then* $\widetilde{W}_d$ *has a fundamental cycle* $[\widetilde{W}_d]$, *and the equality*

$$(\Phi_q^d)_*[\Gamma^d] = d![\widetilde{W}_d]$$

hold in $Z_*(\mathrm{Jac}(\Gamma))$.

Proof. It suffices to show that $(\Phi_q^d)_*[\Gamma^d]$ has weight $d!$ on all components of $\widetilde{W}_d^{\mathrm{reg}}$. Indeed, if that is the case then $\frac{1}{d!}(\Phi_q)_*[\Gamma^d]$ is a tropical cycle with support $\widetilde{W}_d$ and weight 1 on all components of $\widetilde{W}_d^{\mathrm{reg}}$. But this implies that $\widetilde{W}_d$ has a fundamental cycle and that $(\Phi_q^d)_*[\Gamma^d] = d![\widetilde{W}_d]$.

By the definition of the push-forward, we now have to show that for any $x \in \widetilde{W}_d$ such that $(\Phi_q^d)^{-1}\{x\}$ is finite and contained in $(\Gamma^d)^{\mathrm{reg}}$, the value of $(\Phi_q^d)_*[\Gamma^d]$ at x is $d!$. Let σ be a component of $(\Gamma^d)^{\mathrm{reg}}$. Then there exist open edges $e_1, \dots, e_d$ of Γ such that $\sigma = e_1 \times \cdots \times e_d$. We choose an orientation on each of these d edges. This determines a unique primitive tangent vector η_k on each edge e_k. These d tangent vectors form a basis of the integral tangent space of the product $e_1 \times \cdots \times e_d$. As already noted in Section 4, the image of η_k in the tangent space $\Omega_{\mathbb{Z}}(\Gamma)^*$ of $\mathrm{Jac}(\Gamma)$ is given by

$$(d\Phi_q)(\eta_k) \colon \Omega_{\mathbb{Z}}(\Gamma) \to \mathbb{Z}, \quad \omega \mapsto \langle \omega|_{e_k}, \eta_k \rangle.$$

If we identify $\Omega_{\mathbb{Z}}(\Gamma)$ with integral flows on Γ, as explained in Remark 4.1, then $(d\Phi_q)(\eta_k)$ is the map assigning to an integral flow ω on Γ its flow on e_k in the direction specified by the chosen orientation. Because Φ_q^d is defined as the d-fold sum of Φ_q, we have $(d\Phi_q^d)(\eta_k) = (d\Phi_q)(\eta_k)$. In particular, if $e_k = e_l$ for $k \neq l$, then $(d\Phi_q^d)(\eta_k) = (d\Phi_q^d)(\eta_l)$ which means that Φ_q^d is not injective on σ and $x \notin \Phi_q^d(\sigma)$. We may thus assume that all e_k are distinct. If $\Gamma \setminus \bigcup e_k$ is disconnected, then there exists an $1 \le l \le d$ such that $\Gamma \setminus \cup_{k=1}^l e_k$ has precisely two components C_1 and C_2. For $1 \le k \le l$ let α_k be equal 1 if e_k is oriented such that it leads from C_1 to C_2, and let α_k be equal to -1 if it is oriented the other way. Since the total flow from C_1 to C_2 in any integral flow on Γ is 0, we have

$$\sum_{k=1}^{l} \alpha_k (d\Phi_q^d)(\eta_k) = 0,$$

which means that $d\Phi_q^d$ is not injective on the tangent spaces of σ. Therefore, Φ_q^d is not injective on σ and again $x \notin \Phi_q^d(\sigma)$. If $\Gamma \setminus \bigcup e_k$ is connected, then for each $1 \le k \le d$ there is a simple closed loop in Γ that passes through e_k but not through e_l for $l \neq k$. It follows that for every assignment of values $f \colon \{1, \dots, d\} \to \mathbb{Z}$ there is an integral flow $\omega \in \Omega_{\mathbb{Z}}(\Gamma)$ whose flow on e_k is $f(k)$. This implies that the vectors $(d\Phi_q^d)(\eta_1), \dots, (d\Phi_q^d)(\eta_i)$ span a saturated rank-i sublattice of $\Omega_{\mathbb{Z}}(\Gamma)^*$. Therefore, every point of $(\Phi_q^d)^{-1}\{x\} \cap \sigma$ contributes to the weight of $(\Phi_q^d)_*[\Gamma^d]$ with multiplicity one, and by [Gross et al. 2022, Lemma 8.1] there is at most one of these points. In fact, if $(\Phi_q^d)^{-1}\{x\} \cap \sigma$ is nonempty, then [loc. cit., Lemma 8.1] tells us that all other components σ' of $(\Gamma^d)^{\mathrm{reg}}$ with $(\Phi_q^d)^{-1}\{x\} \cap \sigma' \neq \varnothing$ are obtained from σ via a permutation of coordinates. As there are exactly $d!$ of these permutations, the weight at x is $d!$, finishing the proof. □

As an immediate consequence of Proposition 8.3 we obtain the following corollary.

Corollary 8.4. *The equality of tropical cycles* $\star_{k=1}^d [\widetilde{W}_1] = d![\widetilde{W}_d]$ *holds in* $Z_*(\mathrm{Jac}(\Gamma))$.

Proof. This follows directly from the formulas for $[\widetilde{W}_d]$ and $[\widetilde{W}_1]$ given in Proposition 8.3, and the fact that Φ_q^d is the d-fold sum of Φ_q. □

We have a morphism $H_1(\Gamma; \mathbb{Z}) \to H_1(\mathrm{Jac}(\Gamma); \mathbb{Z})$ induced by the (continuous) Abel–Jacobi map. As noticed in Section 6, there is a natural identification

$$H_1(\mathrm{Jac}(\Gamma); \mathbb{Z}) \cong H_1(\Gamma; \mathbb{Z})$$

coming from the fact that $\mathrm{Jac}(\Gamma) = \Omega_{\mathbb{R}}(\Gamma)^*/H_1(\Gamma; \mathbb{Z})$ is defined by taking a quotient of a real vector space by $H_1(\Gamma; \mathbb{Z})$.

Lemma 8.5. *The morphism*

$$(\Phi_q)_* \colon H_1(\Gamma; \mathbb{Z}) \to H_1(\mathrm{Jac}(\Gamma); \mathbb{Z}) \cong H_1(\Gamma; \mathbb{Z})$$

is the identity.

Proof. Let α be a cycle on Γ representing a class in $H_1(\Gamma; \mathbb{Z})$. We need to show that $(\Phi_q)_*[\alpha] = [\alpha]$. By the Hurewicz theorem, we may assume that it is represented by a loop $\gamma \colon [0, 1] \to \Gamma$ starting and ending at the base point q. By the definition of the Abel–Jacobi map, the path

$$\widetilde{\gamma} \colon [0, 1] \to \Omega_{\mathbb{R}}(\Gamma)^*, \quad t \mapsto \left(\omega \mapsto \int_{\gamma|_{[0,t]}} \omega\right)$$

lifts the composite $\Phi_q \circ \gamma$. Therefore, $(\Phi_q)_*\gamma \in H_1(\mathrm{Jac}(\Gamma); \mathbb{Z})$ is identified with the element $\widetilde{\gamma}(1) - \widetilde{\gamma}(0) = \widetilde{\gamma}(1) \in H_1(\Gamma; \mathbb{Z})$. But this is equal to the image of γ under the embedding $H_1(\Gamma; \mathbb{Z}) \hookrightarrow \Omega_{\mathbb{R}}(\Gamma)^*$. □

8B. ***The tropical Riemann theta divisor.*** Recall from Section 4 that the tropical Jacobian $\mathrm{Jac}(\Gamma) = \Omega_{\mathbb{R}}(\Gamma)^*/H_1(\Gamma; \mathbb{Z})$ of a smooth tropical curve Γ comes equipped with a positive definite symmetric form Q on its universal cover $\Omega_{\mathbb{R}}(\Gamma)^*$ which restricts to a perfect pairing $\Omega_{\mathbb{Z}}(\Gamma)^* \times H_1(\Gamma; \mathbb{Z}) \to \mathbb{Z}$. By Proposition 7.1, the first Chern class of the line bundle $\mathscr{L}(Q, 0)$ is given by Q. As explained in Section 7D, this implies that $\mathscr{L}(Q, 0)$ has, up to an additive constant, a unique concave rational section, the *Riemann theta function*, which defines a unique effective divisor $\Theta \in \mathrm{CDiv}(\mathrm{Jac}(\Gamma))$ with $\mathscr{L}(\Theta) = \mathscr{L}(Q, 0)$. For further details about the Riemann theta function see [Mikhalkin and Zharkov 2008] and see [Foster et al. 2018] for the connection to the nonarchimedean Riemann theta function.

Definition 8.6. The unique effective divisor $\Theta \in \mathrm{CDiv}(\mathrm{Jac}(\Gamma))$ with $\mathscr{L}(\Theta) = \mathscr{L}(Q, 0)$ is called the *tropical Riemann theta divisor* on $\mathrm{Jac}(\Gamma)$.

Note that by construction, we have $c_1(\mathscr{L}(\Theta)) = Q$.

Example 8.7. Figure 3 shows the Θ-divisor for the curve Γ from Example 4.4. It is the image in $\mathrm{Jac}(\Gamma)$ of the boundaries of the Voronoi cells of the lattice points $H_1(\Gamma;\mathbb{Z})$ in $\Omega_{\mathbb{R}}(\Gamma)^*$ with respect to the metric defined by Q.

9. The tropical Poincaré formula

We are finally in a position to prove the Poincaré formula. Our strategy is to give explicit formulas for both sides of the equation. More precisely, we will introduce coordinates on the tropical homology groups of the tropical Jacobian, and will compare the coefficients of both sides of the equation in these coordinates. Throughout this section, Γ will denote a compact and connected smooth tropical curve of genus g, and $e_1, \dots, e_g$ will denote distinct open edges of Γ such that $\Gamma \setminus \left(\bigcup_k e_k\right)$ is contractible. Furthermore, we will assume that we have chosen an orientation on each of the edges $e_1, \dots, e_g$.

9A. ***Bases for the tropical (co)homology of* Jac(Γ).** Recall from Section 6 that there is an isomorphism of rings

$$H_{*,*}(\mathrm{Jac}(\Gamma)) \cong \bigwedge H_1(\Gamma;\mathbb{Z}) \otimes \bigwedge \Omega_{\mathbb{Z}}(\Gamma)^*,$$

where the ring structure on the left side is given by the Pontryagin product. Using this isomorphism, a choice of bases for $H_1(\Gamma;\mathbb{Z})$ and $\Omega_{\mathbb{Z}}(\Gamma)^*$ will induce a basis for $H_{*,*}(\mathrm{Jac}(\Gamma))$. We will use our choice of open edges $e_1, \dots, e_g$ to define bases for these lattices. Let $1 \le k \le g$. The orientation on e_k defines a start and an end point for e_k. Since T is contractible and therefore a tree, there is a path in T from the end to the start point of e_k, and this path is unique up to homotopy. Together with any path in $\bar{e}_k$ from its start to its end point, this defines a *fundamental circuit* $c_k \in H_1(\Gamma;\mathbb{Z})$ that traverses e_k but is disjoint from e_l for $l \neq k$. It is well known, and straightforward to check, that the fundamental circuits $c_1, \dots, c_g$ form a basis of $H_1(\Gamma;\mathbb{Z})$.

To obtain a basis for $\Omega_{\mathbb{Z}}(\Gamma)^*$, let η_k denote the primitive tangent vector on e_k in the direction specified by the orientation, and let $\delta_k = (d\Phi_q)(\eta_k)$. As we observed in Section 4, δ_k can be described as the morphism $\Omega_{\mathbb{Z}}(\Gamma) \to \mathbb{Z}$ assigning to an integral flow on Γ its flow through e_k in the direction specified by the orientation. By definition of the bilinear from Q on $\Omega_{\mathbb{R}}(\Gamma)$, we have $Q(c_k, \delta_l) = 1$ if $k = l$ and $Q(c_k, \delta_l) = 0$ if $k \neq l$, that is $\delta_1, \dots, \delta_g$ is dual to the basis $c_1, \dots, c_g$ with respect to Q. We noticed in Section 4 that $\Omega_{\mathbb{Z}}(\Gamma)^*$ is precisely the set of vectors in $\Omega_{\mathbb{R}}(\Gamma)^*$ that have integral pairing with respect to Q with all elements of $H_1(\Gamma;\mathbb{Z})$. It follows directly that $\delta_1, \dots, \delta_g$ is a basis for $\Omega^*_{\mathbb{Z}}(\Gamma)$.

Similarly, by the isomorphism

$$H^{*,*}(\mathrm{Jac}(\Gamma)) \cong \bigwedge H_1(\Gamma;\mathbb{Z})^* \otimes \bigwedge \Omega_{\mathbb{Z}}(\Gamma)$$

of rings discussed in Section 6, bases for $H_1(\Gamma;\mathbb{Z})^*$ and $\Omega_{\mathbb{Z}}(\Gamma)$ induce a basis for $H^{*,*}(\mathrm{Jac}(\Gamma))$. The bases we will use for these lattices are the dual bases $(c_k^*)_k$ and $(\delta_k^*)_k$ to the bases $(c_k)_k$ and $(\delta_k)_k$.

Note that both $H_{*,*}(\mathrm{Jac}(\Gamma))$ and $H^{*,*}(\mathrm{Jac}(\Gamma))$ are tensor products of skew commutative graded rings. We will use the following notation for elements of special form in groups of this type.

Notation 9.1. Let R_1 and R_2 be two skew-commutative graded rings, let J be a finite set, and let $a\colon J \to R_1$ and $b\colon J \to R_2$ be maps such that for every $j \in J$ the elements $a(j)$ and $b(j)$ are homogeneous of the same degree. Then for any injective map $\sigma\colon \{1, \dots, k\} \to J$, the element

$$\prod_{l=1}^{k} a(\sigma(l)) \otimes \prod_{l=1}^{k} b(\sigma(l))$$

of $R_1 \otimes_{\mathbb{Z}} R_2$ only depends on the image $I := \sigma(\{1, \dots, k\})$. We denote it by

$$\prod_{i\in I} a(i) \otimes \prod_{i \in I} b(i).$$

9B. *Cycle classes of tautological cycles.*

Proposition 9.2. *We have*

$$\mathrm{cyc}[\widetilde{W}_1] = \sum_{k=1}^{g} c_k \otimes \delta_k.$$

Proof. Choose an orientation for every edge e of Γ that coincides with the orientation we have already chosen if $e = e_k$ for some k. Let η_e the primitive tangent vector of e in the direction specified by the orientation, and let $\delta_e = (d\Phi_q)(\eta_e)$. By construction, we have $\delta_{e_k} = \delta_k$ for all $1 \le k \le g$. It follows immediately from the definition of the tropical cycle class map and Theorem 3.3 that $\mathrm{cyc}[\widetilde{W}_1]$ is represented by the $(1, 1)$-cycle

$$\sum_{e \in E(\Gamma)} (\Phi_q)_*(\bar{e}) \otimes \delta_e \in C_{1,1}(\mathrm{Jac}(\Gamma)),$$

where we view the oriented closed edge $\bar{e}$ as a singular 1-simplex by choosing a parametrization compatible with the given orientation. Using that the c_k and the δ_k form dual bases with respect to the bilinear form Q, we see that the above equals

$$\sum_{e \in E(\Gamma)} (\Phi_q)_*(\bar{e}) \otimes \left(\sum_{i=1}^{g} Q(c_i, \delta_e) \cdot \delta_i \right) = \sum_{i=1}^{g} \left(\sum_{e \in E(\Gamma)} Q(c_i, \delta_e) \cdot (\Phi_q)_*(\bar{e}) \right) \otimes \delta_i.$$

Since $Q(c_i, \delta_e)$ is 1 whenever e is on the loop c_i, and 0 otherwise, we have

$$\sum_{e \in E(\Gamma)} Q(c_i, \delta_e)(\Phi_q)_*(\bar{e}) = (\Phi_q)_* c_i,$$

which is equal to c_i by Lemma 8.5. This finishes the proof. $\square$

Remark 9.3. It follows immediately from Proposition 9.2 that the expression

$$\sum_k c_k \otimes \delta_k \in H_1(\Gamma; \mathbb{Z}) \otimes \Omega_{\mathbb{Z}}(\Gamma)^*$$

is independent of the choice of spanning tree used to define the elements c_k and δ_k. On a closer look, it turns out that this independence is more of a feature of linear algebra than a feature of spanning trees. To see this, we observe that the natural isomorphism $H_1(\Gamma;\mathbb{Z}) \cong \Omega_{\mathbb{Z}}(\Gamma)$ identifies the basis $(\delta_k)_k$ with the dual basis of $(c_k)_k$. Therefore, $\sum_k c_k \otimes \delta_k$ is identified with the identity endomorphism on $H_1(\Gamma;\mathbb{Z})$ under the composite

$$H_1(\Gamma;\mathbb{Z}) \otimes \Omega_{\mathbb{Z}}(\Gamma)^* \cong H_1(\Gamma;\mathbb{Z}) \otimes H_1(\Gamma;\mathbb{Z})^* \cong \mathrm{End}(H_1(\Gamma;\mathbb{Z})),$$

which is an invariant of $H_1(\Gamma;\mathbb{Z})$ rather than of Γ.

Lemma 9.4. *We have*

$$\mathrm{cyc}[\widetilde{W}_d] = \sum_{\substack{I\subseteq\{1,\dots,g\}\\ |I|=d}} \bigwedge_{k\in I} c_k \otimes \bigwedge_{k\in I} \delta_k.$$

Proof. Using Propositions 8.3 and 6.2 we obtain

$$d!\,\mathrm{cyc}[\widetilde{W}_d] = d!\,\mathrm{cyc}\Big(\mathop{\bigstar}_{k=1}^{d}[\widetilde{W}_1]\Big) = d!\mathop{\bigstar}_{k=1}^{d}\mathrm{cyc}[\widetilde{W}_1].$$

By Proposition 9.2, this equals

$$\mathop{\bigstar}_{k=1}^{d}\Big(\sum_{l=1}^{g} c_l \otimes \delta_l\Big).$$

Using the description of the Pontryagin product from Section 6, we can rewrite this as

$$\sum_{\sigma} \bigwedge_{k=1}^{d} c_{\sigma(k)} \otimes \bigwedge_{k=1}^{d} \delta_{\sigma(k)},$$

where the sum is over all maps $\sigma : \{1,\dots,d\} \to \{1,\dots,g\}$. Since $\bigwedge \Omega_{\mathbb{Z}}(\Gamma)$ is skew-commutative, only an injective σ would contribute to the sum. If I is the image of an injective σ then, using our Notation 9.1, we have

$$\bigwedge_{k=1}^{d} c_{\sigma(k)} \otimes \bigwedge_{k=1}^{d} \delta_{\sigma(k)} = \bigwedge_{k\in I} c_k \otimes \bigwedge_{k\in I} \delta_k.$$

Since the map $\sigma \mapsto \sigma(I)$, for injective $\sigma : \{1,\dots,d\} \to \{1,\dots,g\}$, is $d!$-to-1, we obtain

$$d!\,\mathrm{cyc}[\widetilde{W}_d] = \sum_{\sigma} \bigwedge_{k=1}^{d} c_{\sigma(k)} \otimes \bigwedge_{k=1}^{d} \delta_{\sigma(k)} = d! \sum_{\substack{I\subseteq\{1,\dots,g\}\\ |I|=d}} \bigwedge_{k\in I} c_k \otimes \bigwedge_{k\in I} \delta_k.$$

The result follows after dividing both sides by $d!$. This division is allowed because the tropical homology groups of $\mathrm{Jac}(\Gamma)$ are torsion-free. $\square$

9C. ***Tropical cycle classes of powers of the theta divisor.***

Lemma 9.5. *We have*

$$c_1(\mathscr{L}(\Theta)) = \sum_{i=1}^{g} c_i^* \otimes \delta_i^*.$$

Proof. As already observed in Section 8B, we have $c_1(\mathscr{L}(\Theta)) = Q$, where we identify

$$H^{1,1}(\mathrm{Jac}(\Gamma)) \cong H^1(\Gamma;\mathbb{Z}) \otimes \Omega_{\mathbb{Z}}(\Gamma)$$

with $\mathrm{Hom}(H_1(\Gamma;\mathbb{Z}) \otimes \Omega_{\mathbb{Z}}(\Gamma), \mathbb{Z})$. Because $(c_k)_k$ and $(\delta_k)_k$ are dual bases with respect to Q, the assertion follows. □

Lemma 9.6. *Let $I \subseteq \{1, \dots, g\}$ with $|I| = d$. Then*

$$\bigwedge_{k\in I} c_k^* \otimes \bigwedge_{k\in I} \delta_k^* \in \bigwedge^d H_1(\Gamma,\mathbb{Z})^* \otimes \bigwedge^d \Omega_{\mathbb{Z}}(\Gamma) \cong H^{d,d}(\mathrm{Jac}(\Gamma))$$

is Poincaré dual to

$$\bigwedge_{k\in\{1,\dots,g\}\setminus I} c_k \otimes \bigwedge_{k\in\{1,\dots,g\}\setminus I} \delta_k \in H_{g-d}(\mathrm{Jac}(\Gamma),\mathbb{Z}) \otimes \bigwedge^{g-d} \Omega_{\mathbb{Z}}(\Gamma)^* \cong H_{g-d,g-d}(\mathrm{Jac}(\Gamma)).$$

Proof. Since $\mathrm{Jac}(\Gamma) = \widetilde{W}_g$, we have

$$\mathrm{cyc}[\mathrm{Jac}(\Gamma)] = \Bigl(\bigwedge_{1\le k\le g} c_k\Bigr) \otimes \Bigl(\bigwedge_{1\le k\le g} \delta_k\Bigr) \tag{9-1}$$

by Lemma 9.4.

Note that for every $\alpha \in H_1(\mathrm{Jac}(\Gamma);\mathbb{Z})^*$, $x \in \bigwedge^i H_1(\mathrm{Jac}(\Gamma);\mathbb{Z})$, and $y \in \bigwedge^j H_1(\mathrm{Jac}(\Gamma);\mathbb{Z})$ we have

$$\alpha \lrcorner (x \wedge y) = (\alpha \lrcorner x) \wedge x + (-1)^i x \wedge (\alpha \lrcorner y)$$

by the properties of the interior product; see [Eisenbud 1995, Proposition A 2.8]. Similarly for every $a \in \Omega_{\mathbb{Z}}(\Gamma)$, $b \in \bigwedge^i \Omega_{\mathbb{Z}}(\Gamma)^*$, and $c \in \bigwedge^j \Omega_{\mathbb{Z}}(\Gamma)^*$ we have

$$a \lrcorner (b \wedge c) = (a \lrcorner b) \wedge c + (-1)^i b \wedge (a \lrcorner c).$$

Using induction, we conclude that

$$\bigwedge_{k\in I} c_k^* \frown \bigwedge_{k\in\{1,\dots,g\}} c_k = \pm \bigwedge_{k\in\{1,\dots,g\}\setminus I} c_k$$

and similarly

$$\bigwedge_{k\in I} \delta_k^* \frown \bigwedge_{k\in\{1,\dots,g\}} \delta_k = \pm \bigwedge_{k\in\{1,\dots,g\}\setminus I} \delta_k,$$

with the sign being the same on the right-hand sides of the two equations as long as we order the sets I and $\{1, \dots, g\}$ consistently in both equations. Combining these identities with the expression (9-1) for the fundamental class of $\mathrm{Jac}(\Gamma)$ and the identity (6-3), it follows that

$$\Big(\bigwedge_{k\in I} c_k^* \otimes \bigwedge_{k\in I} \delta_k^*\Big) \frown [\mathrm{Jac}(\Gamma)] = \bigwedge_{k\in\{1,\dots,g\}\setminus I} c_k \otimes \bigwedge_{k\in\{1,\dots,g\}\setminus I} \delta_k,$$

which is precisely what we needed to show. □

The following result is the tropical analogue of [Birkenhake and Lange 2004, Theorem 4.10.4].

Lemma 9.7. *We have*

$$\mathrm{cyc}([\Theta]^{g-d}) = (g-d)! \sum_{\substack{I\subseteq\{1,\dots,g\}\\ |I|=d}} \bigwedge_{k\in I} c_k \otimes \bigwedge_{k\in I} \delta_k.$$

Proof. Since intersections with divisors is compatible with the tropical cycle class map by Theorem 3.3, we have

$$\mathrm{cyc}([\Theta]^{g-d}) = c_1(\mathscr{L}(\Theta))^{g-d} \frown [\mathrm{Jac}(\Gamma)],$$

that is $\mathrm{cyc}([\Theta]^{g-d})$ is Poincaré dual to $c_1(\mathscr{L}(\Theta))^{g-d}$. By Lemma 9.5 we know that

$$c_1(\mathscr{L}(\Theta)) = \sum_{i=1}^{g} c_i^* \otimes \delta_{e_i}^*.$$

With the description of the cap-product on $H^{*,*}(X)$ given in Section 6, we obtain

$$c_1(\mathscr{L}(\Theta))^{g-d} = (g-d)! \sum_{\substack{I\subseteq\{1,\dots,g\}\\ |I|=g-d}} \bigwedge_{k\in I} c_k^* \otimes \bigwedge_{k\in I} \delta_{e_k}^*$$

similar as in the proof of Lemma 9.4. Applying Lemma 9.6 finishes the proof. □

9D. *The proof of the tropical Poincaré formula.*

Theorem 9.8. *The Poincaré formula holds tropically, that is we have*

$$(g-d)![\widetilde{W}_d] \sim_{\mathrm{hom}} [\Theta]^{g-d}.$$

Remark 9.9. The Poincaré formula is more commonly expressed as

$$[\widetilde{W}_d] \sim_{\mathrm{hom}} \frac{1}{(g-d)!}[\Theta]^{g-d},$$

where we the right side is defined after an extension of scalars to $\mathbb{Q}$. Because the tropical homology groups of Jacobians are torsion-free, this is indeed an equivalent expression of the formula.

Proof. By Lemma 9.4 we have

$$\mathrm{cyc}[\widetilde{W}_d] = \sum_{\substack{I\subseteq\{1,\dots,g\}\\ |I|=d}} \bigwedge_{k\in I} c_k \otimes \bigwedge_{k\in I} \delta_k.$$

On the other hand, by Lemma 9.7 we have

$$\operatorname{cyc}([\Theta]^{g-d}) = (g-d)! \sum_{\substack{I\subseteq\{1,\dots,g\}\\ |I|=d}} \bigwedge_{k\in I} c_k \otimes \bigwedge_{k\in I} \delta_k.$$

It follows immediately that

$$\operatorname{cyc}((g-d)![\widetilde{W}_d]) = \operatorname{cyc}([\Theta]^{g-d}),$$

which is equivalent to saying that $[\widetilde{W}_d]$ and $[\Theta]^{g-d}$ are homologically equivalent. □

Corollary 9.10. *We have*

$$(g-d)![\widetilde{W}_d] \sim_{\text{num}} [\Theta]^{g-d}.$$

Proof. This follows directly from Theorem 9.8 and Proposition 5.11. □

Remark 9.11. We have proved Theorem 9.8 under the assumption that the smooth tropical curve is boundaryless. If Γ is a compact and connected smooth tropical curve with boundary as described in Remark 2.5, then the Poincaré formula holds as well, and the proof in this seemingly more general case can easily be reduced to the boundaryless case. Namely, if Γ' denotes the boundaryless smooth tropical curve obtained from Γ by removing the leaves from Γ, then Γ and Γ' have identical Jacobians, and their theta divisors coincide by definition. Furthermore, the Abel–Jacobi map associated to Γ' contracts all the leaves of Γ', so that the loci $\widetilde{W}_d$ associated to Γ and Γ' coincide as well.

9E. *Consequences of the Poincaré formula.* The tropical Poincaré formula has some interesting immediate consequences. One of them is a tropical version of Riemann's theorem. The statement has appeared before [Mikhalkin and Zharkov 2008], with a different (combinatorial) proof. To state the theorem, recall from Section 8 that the Abel–Jacobi map induces a bijection $\operatorname{Pic}^0(\Gamma) \to \operatorname{Jac}(\Gamma)$. Because all contributions from the chosen base point q cancel in degree 0, this bijection is independent of all choices. In particular, we can view Θ as a divisor on $\operatorname{Pic}^0(\Gamma)$ in a natural way. Also recall from Section 8 that while $\widetilde{W}_d \subseteq \operatorname{Jac}(\Gamma)$ depends on q, the image W_d of Γ^d in $\operatorname{Pic}^d(\Gamma)$ does not.

Corollary 9.12 (tropical Riemann's theorem; see [Mikhalkin and Zharkov 2008, Corollary 8.6]). *There exists a unique* $\mu \in \operatorname{Pic}^{g-1}(\Gamma)$ *such that*

$$[W_{g-1}] = \mu + [\Theta],$$

where we consider $[\Theta]$ *as a tropical cycle in* $\operatorname{Pic}^0(\Gamma)$.

Proof. It suffices to show that there exists a unique $\mu \in \operatorname{Jac}(\Gamma)$ such that $[\widetilde{W}_{g-1}] = (t_\mu)_*[\Theta]$ when considering Θ as a divisor on $\operatorname{Jac}(\Gamma)$. Since $[\widetilde{W}_{g-1}]$ is a codimension-1 tropical cycles on the tropical manifold $\operatorname{Jac}(\Gamma)$, we can view $\widetilde{W}_{g-1}$ as a tropical Cartier divisor as well; see Section 3A. Applying the Poincaré formula (Theorem 9.8) with $d = g-1$ yields

$$\operatorname{cyc}[\widetilde{W}_{g-1}] = \operatorname{cyc}[\Theta].$$

By definition of Θ, the cycle class cyc$[\Theta]$ is Poincaré dual to the element in $H^{1,1}(\mathrm{Jac}(\Gamma))$ corresponding to the linear form Q. As Q restricts to a perfect pairing $H_1(\Gamma;\mathbb{Z})\times\Omega_{\mathbb{Z}}(\Gamma)^*\to\mathbb{Z}$, Proposition 7.7 applies and there a unique $\mu\in\mathrm{Jac}(\Gamma)$ such that $t_\mu^*\widetilde{W}_{g-1}=\Theta$. This is, of course, equivalent to the equality $(t_\mu)_*[\widetilde{W}_{g-1}]=[\Theta]$. □

Corollary 9.13. *For every* $0\le d\le g$, *we have*

$$\int_{\mathrm{Jac}(\Gamma)}[\widetilde{W}_d]\cdot[\widetilde{W}_{g-d}]=\binom{g}{d}.$$

Remark 9.14. In the special case $d=1$ we recover the formula

$$\int_{\mathrm{Jac}(\Gamma)}[\widetilde{W}_1]\cdot[\Theta]=g$$

stated in [Mikhalkin and Zharkov 2008, Theorem 6.5]. Also note that the intersection product $[\widetilde{W}_d]\cdot[\widetilde{W}_{g-d}]$ is effective since one can locally apply the fan displacement rule.

Proof. We apply Poincaré formula (Theorem 9.8) three times, and obtain a chain of equalities

$$[\widetilde{W}_d]\cdot[\widetilde{W}_{g-d}]=\frac{[\Theta]^g}{d!(g-d)!}=\binom{g}{d}[\widetilde{W}_0]$$

that hold modulo homological equivalence. Taking the degree yields the result. □

Corollary 9.15. *We have*

$$\int_{\mathrm{Jac}(\Gamma)}[\Theta]^g=g!.$$

Proof. By the tropical Poincaré formula (Theorem 9.8), we have

$$\int_{\mathrm{Jac}(\Gamma)}[\Theta]^g=g!\int_{\mathrm{Jac}(\Gamma)}[\widetilde{W}_0]=g!.$$

□

Remark 9.16. Classically, the statement of Corollary 9.15 also follows from the geometric Riemann–Roch theorem for Abelian varieties [Birkenhake and Lange 2004, Theorem 3.6.3]. Tropically, it is also possible to prove the statement using the duality of Voronoi and Delaunay decompositions.

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References

[Allermann and Rau 2010] L. Allermann and J. Rau, "First steps in tropical intersection theory", *Math. Z.* **264**:3 (2010), 633–670. MR Zbl

[Amini and Piquerez 2020] O. Amini and M. Piquerez, "Tropical Clemens–Schmid sequence and existence of tropical cycles with a given cohomology class", preprint, 2020. arXiv 2012.13142

[Arbarello et al. 1985] E. Arbarello, M. Cornalba, P. A. Griffiths, and J. Harris, *Geometry of algebraic curves, I*, Grundl. Math. Wissen. **267**, Springer, 1985. MR Zbl

[Baker and Norine 2007] M. Baker and S. Norine, "Riemann–Roch and Abel–Jacobi theory on a finite graph", *Adv. Math.* **215**:2 (2007), 766–788. MR Zbl

[Beauville 2004] A. Beauville, "Algebraic cycles on Jacobian varieties", *Compos. Math.* **140**:3 (2004), 683–688. MR Zbl

[Birkenhake and Lange 2004] C. Birkenhake and H. Lange, *Complex abelian varieties*, 2nd ed., Grundl. Math. Wissen. **302**, Springer, 2004. MR Zbl

[Cartwright et al. 2015] D. Cartwright, D. Jensen, and S. Payne, "Lifting divisors on a generic chain of loops", *Canad. Math. Bull.* **58**:2 (2015), 250–262. MR Zbl

[Ceresa 1983] G. Ceresa, "C is not algebraically equivalent to C^- in its Jacobian", *Ann. of Math.* (2) **117**:2 (1983), 285–291. MR Zbl

[Cools et al. 2012] F. Cools, J. Draisma, S. Payne, and E. Robeva, "A tropical proof of the Brill–Noether theorem", *Adv. Math.* **230**:2 (2012), 759–776. MR Zbl

[Eisenbud 1995] D. Eisenbud, *Commutative algebra*, Graduate Texts in Mathematics **150**, Springer, 1995. MR Zbl

[Foster et al. 2018] T. Foster, J. Rabinoff, F. Shokrieh, and A. Soto, "Non-Archimedean and tropical theta functions", *Math. Ann.* **372**:3-4 (2018), 891–914. MR

[Francois 2013] G. Francois, "Cocycles on tropical varieties via piecewise polynomials", *Proc. Amer. Math. Soc.* **141**:2 (2013), 481–497. MR Zbl

[François and Rau 2013] G. François and J. Rau, "The diagonal of tropical matroid varieties and cycle intersections", *Collect. Math.* **64**:2 (2013), 185–210. MR Zbl

[Gathmann and Kerber 2008] A. Gathmann and M. Kerber, "A Riemann–Roch theorem in tropical geometry", *Math. Z.* **259**:1 (2008), 217–230. MR Zbl

[Griffiths and Harris 1978] P. Griffiths and J. Harris, *Principles of algebraic geometry*, Wiley, New York, 1978. MR Zbl

[Gross and Shokrieh 2019] A. Gross and F. Shokrieh, "A sheaf-theoretic approach to tropical homology", preprint, 2019. To appear in *J. Algebra*. arXiv 1906.09245

[Gross et al. 2022] A. Gross, F. Shokrieh, and L. Tóthmérész, "Effective divisor classes on metric graphs", *Math. Z.* **302**:2 (2022), 663–685. MR Zbl

[Itenberg et al. 2019] I. Itenberg, L. Katzarkov, G. Mikhalkin, and I. Zharkov, "Tropical homology", *Math. Ann.* **374**:1-2 (2019), 963–1006. MR Zbl

[Jell et al. 2018] P. Jell, J. Rau, and K. Shaw, "Lefschetz $(1,1)$-theorem in tropical geometry", *Épijournal Géom. Algébrique* **2** (2018), Art. 11, 27. MR Zbl

[Katz et al. 2008] E. Katz, H. Markwig, and T. Markwig, "The j-invariant of a plane tropical cubic", *J. Algebra* **320**:10 (2008), 3832–3848. MR Zbl

[Kleiman 1968] S. L. Kleiman, "Algebraic cycles and the Weil conjectures", pp. 359–386 in *Dix exposés sur la cohomologie des schémas*, Adv. Stud. Pure Math. **3**, North-Holland, Amsterdam, 1968. MR Zbl

[Kotani and Sunada 2000] M. Kotani and T. Sunada, "Jacobian tori associated with a finite graph and its abelian covering graphs", *Adv. in Appl. Math.* **24**:2 (2000), 89–110. MR Zbl

[Lieberman 1968] D. I. Lieberman, "Numerical and homological equivalence of algebraic cycles on Hodge manifolds", *Amer. J. Math.* **90** (1968), 366–374. MR Zbl

[Lim et al. 2012] C. M. Lim, S. Payne, and N. Potashnik, "A note on Brill–Noether theory and rank-determining sets for metric graphs", *Int. Math. Res. Not.* **2012**:23 (2012), 5484–5504. MR Zbl

[Marini 2008] G. Marini, "Tautological cycles on Jacobian varieties", *Collect. Math.* **59**:2 (2008), 167–190. MR Zbl

[Matsusaka 1959] T. Matsusaka, "On a characterization of a Jacobian variety", *Mem. Coll. Sci. Univ. Kyoto Ser. A. Math.* **32** (1959), 1–19. MR Zbl

[Mattuck 1962] A. Mattuck, "On symmetric products of curves", *Proc. Amer. Math. Soc.* **13** (1962), 82–87. MR Zbl

[Mikhalkin and Zharkov 2008] G. Mikhalkin and I. Zharkov, "Tropical curves, their Jacobians and theta functions", pp. 203–230 in *Curves and abelian varieties* (Athens, GA, 2007), edited by V. Alexeev et al., Contemp. Math. **465**, Amer. Math. Soc., Providence, RI, 2008. MR Zbl

[Mikhalkin and Zharkov 2014] G. Mikhalkin and I. Zharkov, "Tropical eigenwave and intermediate Jacobians", pp. 309–349 in *Homological mirror symmetry and tropical geometry*, edited by R. Castano-Bernard et al., Lect. Notes Unione Mat. Ital. **15**, Springer, 2014. MR Zbl

[Moonen 2009] B. Moonen, "Relations between tautological cycles on Jacobians", *Comment. Math. Helv.* **84**:3 (2009), 471–502. MR Zbl

[Mumford 2008] D. Mumford, *Abelian varieties*, Tata Institute of Fundamental Research Studies in Mathematics **5**, Hindustan Book Agency, New Delhi, 2008. Corrected reprint of the 1974 edition. MR Zbl

[Pflueger 2017] N. Pflueger, "Special divisors on marked chains of cycles", *J. Combin. Theory Ser. A* **150** (2017), 182–207. MR Zbl

[Polishchuk 2005] A. Polishchuk, "Universal algebraic equivalences between tautological cycles on Jacobians of curves", *Math. Z.* **251**:4 (2005), 875–897. MR Zbl

[Shaw 2013] K. M. Shaw, "A tropical intersection product in matroidal fans", *SIAM J. Discrete Math.* **27**:1 (2013), 459–491. MR Zbl

[Zharkov 2015] I. Zharkov, "C is not equivalent to C^- in its Jacobian: a tropical point of view", *Int. Math. Res. Not.* **2015**:3 (2015), 817–829. MR Zbl

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Tautological rings of Shimura varieties and cycle classes of Ekedahl–Oort strata

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We define the tautological ring as the subring of the Chow ring of a Shimura variety generated by all Chern classes of all automorphic bundles. We explain its structure for the special fiber of a good reduction of a Shimura variety of Hodge type and show that it is generated by the cycle classes of the Ekedahl–Oort strata as a vector space. We compute these cycle classes. As applications we get the triviality of ℓ-adic Chern classes of flat automorphic bundles in characteristic 0, an isomorphism of the tautological ring of smooth toroidal compactifications in positive characteristic with the rational cohomology ring of the compact dual of the hermitian domain given by the Shimura datum, and a new proof of Hirzebruch–Mumford proportionality for Shimura varieties of Hodge type.

Introduction

Tautological rings. The Chow ring $A^\bullet(S_K)$ (always with rational coefficients) of a Shimura variety S_K is still a very mysterious object. Here we study the subring generated by all Chern classes of all automorphic bundles on the Shimura variety or on a smooth toroidal compactification of the Shimura variety. In the Siegel case this subring was already studied by van der Geer and Ekedahl [1999; 2009]. Following

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their terminology, we call this subring the *tautological ring*[1] of the Shimura variety or of some toroidal compactification.

More precisely, let $(\boldsymbol{G}, \boldsymbol{X})$ be a Shimura datum and $K \subset \boldsymbol{G}(\mathbb{A}_f)$ a sufficiently small open compact subgroup. The Shimura datum defines a conjugacy class of cocharacters μ of $\boldsymbol{G}$ whose field of definition is the reflex field E of the attached Shimura variety. We denote the canonical model over E of this Shimura variety by $\boldsymbol{S}_K = \mathrm{Sh}_K(\boldsymbol{G}, \boldsymbol{X})$. To simplify the notation here in the introduction let us assume that $\boldsymbol{G}$ does not contain a $\mathbb{Q}$-anisotropic and $\mathbb{R}$-split torus in its center. This condition is automatic if $(\boldsymbol{G}, \boldsymbol{X})$ is of Hodge type. The Borel embedding of the hermitian space $\boldsymbol{X}$ into its compact dual $\boldsymbol{X}^\vee$ induces a morphism

$$\sigma : \boldsymbol{S}_K \to \mathtt{Hdg} := [\boldsymbol{G}\backslash \boldsymbol{X}^\vee]$$

of algebraic stacks over E [Milne 1990, III]. By definition, a vector bundle on $\boldsymbol{S}_K$ is automorphic[2] if it is isomorphic to the pullback of a vector bundle on $\mathtt{Hdg}$. Moreover, it is flat if it is obtained by pullback from a vector bundle on the classifying stack $[\boldsymbol{G}\backslash *]$, i.e., if it is induced by a finite-dimensional representation of $\boldsymbol{G}$ (see Section 5 for details). For a smooth toroidal compactification $\boldsymbol{S}_K^{\mathrm{tor}}$ of $\boldsymbol{S}_K$ given by the choice of a suitable polyhedral cone decomposition, the theory of canonical extensions of automorphic vector bundles shows that there is a canonical extension of σ to $\boldsymbol{S}_K^{\mathrm{tor}}$.

Definition 1 (Definition 5.7). The *tautological ring of* $\boldsymbol{S}_K$ (resp. of $\boldsymbol{S}_K^{\mathrm{tor}}$) is the image of the Chow ring of $\mathtt{Hdg}$ in the Chow ring of $\boldsymbol{S}_K$ (resp. of $\boldsymbol{S}_K^{\mathrm{tor}}$) under pullback via σ.

In the Siegel case, the tautological ring is the subring generated by all Chern classes of the Hodge bundle in the de Rham cohomology of the universal abelian scheme (Example 5.9), which is the definition of van der Geer in this case.

Ekedahl–Oort strata. From now on we assume that $(\boldsymbol{G}, \boldsymbol{X})$ is of Hodge type and that $p > 2$ is a prime of good reduction for the Shimura datum. Then the reductive group $\boldsymbol{G}_{\mathbb{Q}_p}$ has a reductive model $\mathscr{G}$ over $\mathbb{Z}_p$ and hence the algebraic stack $\mathtt{Hdg}$ has a good integral model over the ring of integers of the completion of E at a place above p. Denote by G the special fiber $\mathscr{G}$. Hence G is a reductive group over $\mathbb{F}_p$. Moreover, since the Shimura variety is of Hodge type, there are canonical smooth integral models $\mathscr{S}_K$ and $\mathscr{S}_K^{\mathrm{tor}}$ with special fibers S_K and S_K^{tor} by the work of Vasiu [1999], Kisin [2010], and Kim and Madapusi Pera [2016; 2019] such that the morphism σ extends. Hence we also obtain in characteristic p tautological rings of S_K and S_K^{tor} as images under pullback maps

$$\sigma^* : A^\bullet(\mathtt{Hdg}) \to A^\bullet(S_K^{\mathrm{tor}}), \tag{0.1}$$

where we again denote by $\mathtt{Hdg}$ the special fiber of the above integral model of $\mathtt{Hdg}$. In characteristic p the work of Viehmann and Wedhorn [2013] (for Shimura varieties of PEL type), of Zhang [2018]

[1]One could argue against this terminology: By analogy to the notion of tautological rings of moduli spaces of curves, the tautological ring should be the subring "generated by all interesting classes". But with our definition there are many interesting classes, for instance those of special subvarieties, that are in general not contained in the tautological ring.

[2]More precisely, it is the underlying vector bundle of an automorphic bundle since we ignore the actions by Hecke operators here.

and Wortmann [2013] (for Shimura varieties of Hodge type), and of W. Goldring and Koskivirta [2019a] (for toroidal compactifications of Shimura varieties of Hodge type) shows that the morphism σ factors into

$$\sigma : S_K^{\mathrm{tor}} \xrightarrow{\zeta^{\mathrm{tor}}} G\text{-}\mathtt{Zip}^{\mu} \xrightarrow{\beta} \mathtt{Hdg}, \tag{0.2}$$

where G-$\mathtt{Zip}^{\mu}$ is the stack of G-zips of type μ which was defined and studied in [Pink et al. 2011; 2015]. Here μ is as above, now considered as an element of the set of conjugacy classes of cocharacters of $G_{\overline{\mathbb{F}}_p}$. The stack G-$\mathtt{Zip}^{\mu}$ has a finite stratification by gerbes Z_w, where w runs through a certain subset ${}^{I}W$ of the Weyl group W of G (see Section 3 for a reminder on G-zips). We refer to the Z_w as the Ekedahl–Oort strata in G-$\mathtt{Zip}^{\mu}$. The locally closed subschemes

$$S_w := \zeta^{-1}(Z_w) \subseteq S_K \quad \text{and} \quad S_w^{\mathrm{tor}} := \zeta^{\mathrm{tor},-1}(Z_w) \subseteq S_K^{\mathrm{tor}}$$

are by definition the Ekedahl–Oort strata of S_K and S_K^{tor}. As ζ and ζ^{tor} are smooth by the work of Zhang [2018] and Andreatta [2023], many results proved for $Z_w \subseteq G$-$\mathtt{Zip}^{\mu}$ in [Pink et al. 2011], such as smoothness, a formula for its codimension, or closure relations of the strata, are known to hold also for the Ekedahl–Oort strata S_w and S_w^{tor}. Using a deep result on the existence of Hasse invariants ([Goldring and Koskivirta 2019a]; see also Boxer [2015] in the PEL case of type A and C) we can also prove the following connectedness result on Ekedahl–Oort strata. (From now on, we abbreviate Ekedahl–Oort strata to EO-strata.)

Theorem 2 (Theorem 6.15, Corollary 6.17). (1) *For all $j \geq 1$, the union of all EO-strata of dimension $\leq j$ is geometrically connected in each geometric connected component of the toroidal compactification S_K^{tor}.*

(2) *For Shimura varieties of PEL type, the union of all EO-strata of dimension ≤ 1 is geometrically connected in each geometric connected component of the Shimura variety S_K.*

The first assertion seems to be new even in the Siegel case. Assertion (2) was proved in the Siegel case by Oort [2001, Theorem 1.1].

The tautological ring and the Chow ring of the stack of G-zips. By (0.2), the pullback σ^* is a composition

$$\sigma^* : A^{\bullet}(\mathtt{Hdg}) \xrightarrow{\beta^*} A^{\bullet}(G\text{-}\mathtt{Zip}^{\mu}) \xrightarrow{\zeta^{\mathrm{tor},*}} A^{\bullet}(S_K^{\mathrm{tor}}). \tag{0.3}$$

Brokemper [2018] has given two descriptions for $A^{\bullet}(G\text{-}\mathtt{Zip}^{\mu})$. From his multiplicative description (recalled in Proposition 4.8) we deduce:

Theorem 3 (Theorem 4.16, Lemma 4.2, Corollary 4.12). (1) *The map β^* is surjective and its kernel is generated by all Chern classes in degree > 0 of vector bundles attached to representations of the group G. In particular, the tautological ring of S_K (resp. S_K^{tor}) is equal to the image of ζ^* (resp. $\zeta^{\mathrm{tor},*}$).*

(2) *The graded $\mathbb{Q}$-algebra $A^{\bullet}(G\text{-}\mathtt{Zip}^{\mu})$ is isomorphic to the rational cohomology ring $H^{2\bullet}(\boldsymbol{X}^{\vee}, \mathbb{Q})$ of the complex manifold $\boldsymbol{X}^{\vee}$.*

As a consequence we obtain:

Corollary 4 (Theorem 7.1, Corollary 7.2, Theorem 7.19). *In characteristic $p > 0$, Chern classes of flat automorphic bundles are zero in degree > 0. In characteristic 0, the ℓ-adic Chern classes of flat automorphic bundles are "locally" zero in degree > 0.*

Esnault and Harris [2017] prove in characteristic 0 for compact Shimura varieties (not necessarily of Hodge type) the stronger result that the ℓ-adic Chern classes of flat automorphic bundles are even globally zero, i.e., in the ℓ-adic continuous cohomology with values in the number field over which the automorphic bundle is defined.

One particular important line bundle is the Hodge line bundle $\omega^\flat(\iota) \in \operatorname{Pic}(G\text{-}\mathtt{Zip}^\mu)$ associated to an embedding ι of $(\boldsymbol{G}, \boldsymbol{X})$ in the Siegel Shimura datum. Its pullback to the Shimura variety is the determinant line bundle of the Hodge filtration of the "universal" abelian scheme attached to ι. Combining Corollary 4 with a result of Goldring and Koskivirta [2018] one gets:

Corollary 5 (Proposition 7.5). *Suppose that the adjoint group of $\boldsymbol{G}$ is $\mathbb{Q}$-simple. Then $c_1(\omega^\flat(\iota)) \in A^1(G\text{-}\mathtt{Zip}^\mu)$ does not depend on ι, up to multiplication with positive rational numbers.*

The second description of $A^\bullet(G\text{-}\mathtt{Zip}^\mu)$ by Brokemper (recalled in Proposition 4.14) shows that the classes $[\bar{Z}_w]$ of closures of EO-strata form a $\mathbb{Q}$-basis of $A^\bullet(G\text{-}\mathtt{Zip}^\mu)$. Hence the tautological rings in characteristic p are generated as a $\mathbb{Q}$-vector space by the classes of the closures of EO-strata, which are indexed by the subset ${}^I W$ of the geometric Weyl group W of $\boldsymbol{G}$.

In fact, it is also possible to define classes in $A^\bullet(G\text{-}\mathtt{Zip}^\mu)$ whose pullbacks to the Shimura variety S_K are the classes of the closures of the Newton strata or of central leaves in S_K. In particular, these classes are also contained in the tautological ring of S_K. This will be pursued in another paper.

The technical heart of the paper is to relate both descriptions of Brokemper:

Theorem 6 (Section 4D). *Let G be a reductive group over $\mathbb{F}_p$, where p is any prime ($p = 2$ included), and let μ be a cocharacter of G. There is a concrete algorithm to express, for $w \in {}^I W$, the cycle class $[\bar{Z}_w] \in A^\bullet(G\text{-}\mathtt{Zip}^\mu)$ of the closure of an EO-stratum as a polynomial in Chern classes of vector bundles on* $\mathtt{Hdg}$.

We refer to Section 4D for the meaning of the phrase "there is a concrete algorithm". By pulling back to the Shimura variety or to a toroidal compactification (for $p > 2$) we get the same descriptions of cycle classes of EO-strata in the Chow rings of S_K and S_K^{tor}.

To obtain a description as in Theorem 6, we follow a strategy already used by Ekedahl and van der Geer [2009] in the Siegel case, albeit using a somewhat different language. Following [Goldring and Koskivirta 2019a], we construct a commutative diagram

$$\begin{tikzcd} G\text{-}\mathtt{ZipFlag}^\mu \arrow[r, "\psi"] \arrow[d, "\pi"'] & \mathtt{Brh}_G \arrow[d, "\gamma"] \\ G\text{-}\mathtt{Zip}^\mu \arrow[r, "\beta"] & \mathtt{Hdg} \end{tikzcd} \tag{0.4}$$

where $G\text{-}\mathtt{ZipFlag}^\mu$ is the algebraic stack of flagged G-zips defined by Goldring and Koskivirta [2019a; 2019b] and where $\mathtt{Brh}_G = [B\backslash G/B] = [B\backslash *] \times_{[G\backslash *]} [B\backslash *]$ is the Bruhat stack (which is called the

Schubert stack in the articles of Goldring and Koskivirta). Here $B \subseteq G$ is a Borel subgroup. Then we proceed in three steps.

1. Calculation of cycles of Schubert varieties: In $A^\bullet(\mathtt{Brh}_G)$ there are the classes $[\overline{\mathtt{Brh}}_w]$ of Schubert varieties for $w \in W$. They can be computed as follows. The cycle class of the smallest Schubert variety $[\mathtt{Brh}_e]$ is the class of the diagonal and can be computed by a result of Graham. Then one defines explicit operators δ_w such that $[\overline{\mathtt{Brh}}_w] = \delta_w([\mathtt{Brh}_e])$. This is certainly well known but to our surprise we found this only explained in the literature for classical groups (and sometimes only over the complex numbers). Hence we explain this for arbitrary split reductive groups over an arbitrary field in Section 2.
2. Pullback to G-$\mathtt{ZipFlag}^\mu$: One describes the pullback via ψ explicitly and obtains a description for the cycle classes in $A^\bullet(G\text{-}\mathtt{ZipFlag}^\mu)$ of the closures of $Z_w^\varnothing := \psi^{-1}(\mathtt{Brh}_w)$ (Sections 4A and 4B).
3. Push down to G-$\mathtt{Zip}^\mu$: By a result of Koskivirta [2018], π induces for $w \in {}^IW$ a finite étale map $Z_w^\varnothing \to Z_w$. If $\gamma(w)$ is its degree, we obtain
$$[\bar{Z}_w] = \gamma(w)\pi_*([\bar{Z}_w^\varnothing]).$$
Using a result of Brion [1996] one can describe π_* explicitly (Theorem 4.17). Moreover, we explain how to compute $\gamma(w)$ as the number of $\mathbb{F}_p$-rational points of the flag variety of an explicitly given form of a Levi subgroup of G (Section 3F).

We also introduce the flag space over the Shimura variety (Section 6C) that classifies — roughly speaking — refinements of the Hodge filtration. This generalizes a construction of Ekedahl and van der Geer [2009] and appeared already in Goldring and Koskivirta [2019a; 2019b]. It carries a stratification obtained by pullback from the stratification of the Bruhat stack. From the analogous properties of Schubert varieties, we deduce that the closure of these strata are normal, Cohen–Macaulay, and have only rational singularities. This also generalizes results from these works.

Structure of the tautological ring. By definition the tautological ring is a quotient of $A^\bullet(\mathtt{Hdg})$, and $A^\bullet(\mathtt{Hdg})$ can be described explicitly (Remark 5.8). There is the following conjecture about the tautological ring.

Conjecture 7. The tautological ring of a smooth toroidal compactification $\boldsymbol{S}_K^{\mathrm{tor}}$ (considered as a scheme over some splitting field of $\boldsymbol{G}$) in characteristic zero or S_K^{tor} in characteristic p is isomorphic to the rational cohomology ring of the compact dual $\boldsymbol{X}^\vee$.

By the work of van der Geer [1999] and Esnault and Viehweg [2002], Conjecture 7 is known in characteristic zero in the Siegel case. In Proposition 7.16, we show that this conjecture is equivalent to the property that all Chern classes of positive degree of flat automorphic bundles vanish in the Chow ring of $\boldsymbol{S}_K^{\mathrm{tor}}$ (resp. S_K^{tor}). This equivalence has also been shown in [Esnault and Harris 2017, 1.11] if the Shimura variety is compact. We show that the conjecture always holds in characteristic p:

Theorem 8 (Theorem 7.12). *The map of graded $\mathbb{Q}$-algebras $H^{2\bullet}(X^\vee) \cong A^\bullet(G\text{-}\mathtt{Zip}^\mu) \to A^\bullet(S_K^{\mathrm{tor}})$ is injective.*

Finally, as an immediate application we obtain a very strong form of Hirzebruch–Mumford proportionality in positive characteristic (Theorem 7.20). From this we deduce a new and purely algebraic proof of Hirzebruch–Mumford proportionality for Shimura varieties of Hodge type over $\mathbb{C}$ (Corollary 7.22).

Structure of the paper. The paper starts with a preliminary section in which we recall the notion of Chow groups of quotient stacks and some basic properties of these groups. Then the main body of the paper consists of two parts.

The first part (Sections 2–4) explains how to compute cycle classes of EO-strata in the Chow ring of the stack of G-zips of type μ. This is a purely group-theoretic part and everything is done for arbitrary reductive groups, arbitrary cocharacters, and in arbitrary positive characteristic $p \geq 2$.

In Section 2 we explain how to calculate the cycle classes of Schubert varieties in the Bruhat stack of a split reductive group. All of this is well documented in the literature for classical groups.

Section 3 recalls the stack of G-zips and of G-zips "endowed with a refinement of the Hodge filtration" and defines the commutative diagram (0.4). In Section 4 we explain what maps are induced from this diagram on Chow rings. This allows us to prove Theorem 3 and to give an algorithm for the determination of cycle classes of EO-strata in $A^\bullet(G\text{-}\mathtt{Zip}^\mu)$ (Section 4D). The section closes with stating some easy functoriality properties for maps of reductive groups inducing an isomorphism on adjoint groups.

In the second part of the paper (Sections 5–7) we apply the results from the first part to Shimura varieties of Hodge type. Here we have to make the assumption $p > 2$.

In Section 5 we define the tautological ring for arbitrary Shimura varieties in characteristic 0 and for Shimura varieties of Hodge type in characteristic p, where p is a prime of good reduction.

In Section 6 we recall the definition of EO-strata and prove Theorem 2. Here we also give the definition of the flag space over the Shimura variety and its stratification.

Then we prove in Section 7 the triviality of Chern classes of flat automorphic bundles, the uniqueness of the class of a Hodge line bundle (up to positive scalar), and our results on the structure of the tautological ring and on Hirzebruch–Mumford proportionality.

In the final Section 8 we illustrate our results in the special cases of the Siegel Shimura variety, the Hilbert–Blumenthal variety, and Shimura varieties of Spin type.

Notation. All algebraic spaces and algebraic stacks are assumed to be quasiseparated and of finite type over their respective base.

Notation on reductive groups. Throughout all reductive groups are assumed to be connected (following [SGA 3_{III} 1970]). Let k be a field, and let k^s be a separable closure. Suppose that G is a reductive group over k and that $T \subseteq G$ is a maximal torus, defined over k. Then we denote by $W = (N_G(T)/T)(k^s)$ the Weyl group of (G, T). It carries an action of $\Gamma = \mathrm{Gal}(k^s/k)$ by group automorphisms. We will denote by $X^*(T)$ (resp. $X_*(T)$) the group of characters (resp. cocharacters) of T_{k^s}.

Now suppose that G is quasisplit over k. Then we can choose a Borel pair $T \subseteq B \subseteq G$ defined over k. The choice of B defines a subset $\Sigma \subset W$ of simple reflections and Γ acts by automorphisms of the Coxeter system (W, Σ). We denote by $\ell(\cdot)$ the length function and by $\leq$ the Bruhat order on the

Coxeter system (W, Σ). We choose representatives $\dot{w} \in G(k^s)$ of $w \in W$ such that $(w_1 \dot{w}_2) = \dot{w}_1 \dot{w}_2$ if $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$. We denote by $w_0 \in W$ the element of maximal length and by $e \in W$ the identity.

For any subset $K \subseteq \Sigma$, we denote by W_K the subgroup of W generated by K, and we set

$$^{K}W := \{w \in W \mid \ell(sw) > \ell(w) \text{ for all } s \in K\},$$

which is a system of representatives of $W_K \backslash W$. Let $w_{0,K}$ be the element of maximal length in W_K.

We denote by $\Phi \subset X^*(T)$ (resp. $\Phi^\vee \subset X_*(T)$) the set of roots (resp. coroots) of $(G, T)_{k^s}$ and by $\Phi^+ \subset \Phi$ the set of positive roots given by B (that is, a root α is in Φ^+ if and only if $U_\alpha \subset B$). The based root datum $(X^*(T), \Phi, X_*(T), \Phi^\vee, \Phi^+)$ and the Coxeter system (W, Σ) do not depend on the choice of (T, B), up to unique isomorphism, and are called "the" based root datum of G and "the" Weyl group of G. For a set of simple reflections $K \subset \Sigma$, we denote by $\Phi_K \subset \Phi$ the set of roots that are in the $\mathbb{Z}$-span of the simple roots corresponding to K, and let $\Phi_K^+ := \Phi^+ \cap \Phi_K$.

Let $\mu : \mathbb{G}_{m,k^s} \to G_{k^s}$ be a cocharacter of G_{k^s}. It gives rise to a pair of opposite parabolic subgroups $(P_-(\mu), P_+(\mu))$ and a Levi subgroup $L := L(\mu) = P_-(\mu) \cap P_+(\mu)$ defined by the condition that $\mathrm{Lie}(P_-(\mu))$ $\big($resp. $\mathrm{Lie}(P_+(\mu))\big)$ is the sum of the nonpositive (resp. nonnegative) weight spaces of μ in $\mathrm{Lie}(G)$. On k^s-valued points we have

$$P_+(\mu) = \{g \in G \mid \lim_{t \to 0} \mu(t) g \mu(t)^{-1} \text{ exists}\}, \quad P_-(\mu) = \{g \in G \mid \lim_{t \to \infty} \mu(t) g \mu(t)^{-1} \text{ exists}\},$$

and $L = \mathrm{Cent}_G(\mu)$.

We will also need to consider reductive groups over more general rings than a field. Hence let S be a scheme. To simplify the notation we assume that S is connected. Let G be a reductive group scheme over S, i.e., a smooth affine group scheme over S whose geometric fibers are reductive groups. The map that attaches to $s \in S$ the isomorphism class of the based root datum of the geometric fiber $G_{\bar{s}}$ is locally constant [SGA 3_{III} 1970, Exp. XXII, Proposition 2.8] and hence constant because we assumed S to be connected. Hence we may again speak of "the" based root datum of G. Let (W, Σ) be the Weyl group together with its set of simple reflections of this based root datum. Fix $I \subseteq \Sigma$, and let Par_I be the scheme of parabolic subgroups of G of type I. It is defined étale locally on S because G is split étale locally on S [SGA 3_{III} 1970, Exp. XXII, Corollaire 2.3].

If $\lambda : \mathbb{G}_{m,S'} \to G_{S'}$ is a cocharacter of G defined over some covering $S' \to S$ for the étale topology, then the constructions of the parabolic subgroups $P_+(\lambda)$ and $P_-(\lambda)$ over a field generalize to arbitrary schemes [Conrad 2014, 4.1.7] and yield parabolic subgroups of $G_{S'}$. If I is the type of $P_+(\lambda)$, we also write Par_λ instead of Par_I.

In other words, we say that a parabolic subgroup P of $G_{S'}$ is of type λ if it is locally for the étale topology conjugate to $P_+(\lambda)$. In fact, P is then already locally for the Zariski topology conjugate to $P_+(\lambda)$ by [SGA 3_{III} 1970, Exp. XXVI, Corollaire 5.5].

1. Chow groups of quotient stacks

Let k be a field. All Chow groups in the following will have $\mathbb{Q}$-coefficients.

1A. ***Chow rings of smooth quotient stacks.*** By a quotient stack we will mean a stack of the form $[G\backslash X]$ where X is a quasiseparated algebraic space of finite type over $\mathrm{Spec}(k)$ and G is an affine group scheme of finite type over $\mathrm{Spec}(k)$ which acts on X from the left.

For such X and G, the equivariant Chow groups $A_i^G(X)$ are defined in [Edidin and Graham 1998] as follows: Let $n = \dim X$ and $g = \dim G$. There exists a representation of G on an ℓ-dimensional k-vector space V such that there exists an open subset U of V with complement of codimension strictly bigger than $n - i$ on which G acts freely. For such a U, the quotient $G\backslash(X \times U)$ by the diagonal action exists as an algebraic space and $A_i^G(X)$ is defined to be $A_{i+\ell-g}(G\backslash(X \times U))$. By [Edidin and Graham 1998], this group does not depend on the choice of U, and in fact, by Proposition 16 in that work, the group $A_i([G\backslash X]) := A_{i+g}^G(X)$ depends up to a canonical isomorphism only on the stack $[G\backslash X]$ and not on the chosen presentation of this stack.

A quotient stack is smooth if it admits a presentation as above with X smooth. Suppose that X is in addition separated and equidimensional of dimension n. In this case for $A^i([G\backslash X]) := A_{n-g-i}([G\backslash X])$ on the graded vector space $A^\bullet([G\backslash X]) := \bigoplus_{i\geq 0} A^i([G\backslash X])$ there is a naturally defined cup product turning this group into a graded $\mathbb{Q}$-algebra [Edidin and Graham 1998, Section 2.5]. This construction has been generalized to arbitrary smooth algebraic stacks of finite type over a field by Kresch [1999]. Here we will need only the case of smooth quotient stacks.

By [Edidin and Graham 1998, Proposition 3], the equivariant Chow groups have the same functoriality properties as the usual Chow groups for G-equivariant morphisms $X \to Y$. Every representable morphism $\mathcal{X} \to \mathcal{Y}$ of quotient stacks arises in this way: For a presentation $\mathcal{Y} = [G\backslash Y]$, take $X = \mathcal{X} \times_{\mathcal{Y}} Y$. This is a G-torsor over $\mathcal{X}$ so that $\mathcal{X} = [G\backslash X]$ and by assumption it is representable by an algebraic space. This shows that $A^\bullet(\_)$ is contravariantly functorial for representable morphisms of quotient stacks and covariantly functorial for proper representable morphisms of quotient stacks. In fact, by [Kresch 1999] it is also contravariantly functorial for flat (not necessarily representable) morphisms of smooth quotient stacks.

For an algebraic group H over k, we denote the quotient stack $[H\backslash \mathrm{Spec}(k)]$ by $[H\backslash *]$. This is a smooth algebraic stack over k of dimension $-\dim(H)$. In this paper we will mainly use the following types of morphisms between quotient stacks. For all of them, $A^\bullet(\_)$ is contravariantly functorial.

Example 1.1. Let G and X be as above.

(1) For a quasiseparated algebraic space Y of finite type over k, every morphism $Y \to [G\backslash X]$ is representable.

(2) Let $\mathscr{X}$ be any equidimensional algebraic stack over k. Then every morphism $\mathscr{X} \to [G\backslash *]$ is flat of constant relative dimension. In particular, if $\varphi : H \to G$ is a homomorphism of affine algebraic groups, the canonical morphism $[H\backslash *] \to [G\backslash *]$ is flat of relative dimension $\dim(G) - \dim(H)$.

Let $\varphi : G \to H$ be a map of algebraic groups over k. Let $f : X \to Y$ be a map of quasiseparated algebraic spaces of finite type over k. Suppose that G acts on X and that H acts on Y such that $f(gx) = \varphi(g)f(x)$ for $g \in G(R)$ and $x \in X(R)$ for any k-algebra R. Then f induces a morphism of algebraic stacks $[f] : [G\backslash X] \to [H\backslash Y]$.

Lemma 1.2. (1) *If f is flat, then $[f]$ is flat.*

(2) *If φ is a monomorphism, then $[f]$ is representable.*

Proof. The first assertion is clear, and the second is a very special case of [Stacks 2005–, Tag 04YY]. □

Proposition 1.3. *Let $\mathscr{X} = [G\backslash X]$ be a smooth equidimensional quotient stack over k, and let k' be a Galois extension of k with Galois group Γ. Then the canonical homomorphism*

$$A^\bullet(\mathscr{X}) \to A^\bullet(\mathscr{X}_{k'})^\Gamma$$

is an isomorphism.

Proof. This is well known (e.g., [Brokemper 2018, 1.3.6]) if $\mathscr{X}$ is an algebraic space. In general let $n := \dim(X)$ and $g := \dim(G)$. For $i \geq 0$, choose an ℓ-dimensional representation V of G and an open subset U of V such that G acts freely on U and such that $V \setminus U$ has codimension $> i$. Then

$$A^i(\mathscr{X}) = A_{n-i+\ell-g}(G\backslash(X \times U)) \xrightarrow{\sim} A_{n-i+\ell-g}(G_{k'}\backslash(X_{k'} \times U_{k'}))^\Gamma = A^i(\mathscr{X}_{k'})^\Gamma. \qquad \square$$

We will also use the following result by Brokemper [2018, 1.4.7], which shows that $A^\bullet(\,\cdot\,)$ “ignores unipotent actions”.

Proposition 1.4. *Let $0 \to U \to G \to H \to 0$ be a split exact sequence of linear algebraic groups over k, where U is a smooth connected unipotent group scheme over k. Choose a splitting $H \to G$. Let X be a smooth quasiprojective G-scheme over k and endow X with the H-action via the chosen splitting. Then the pullback map*

$$A^\bullet([G\backslash X]) \to A^\bullet([H\backslash X])$$

is an isomorphism of $\mathbb{Q}$-algebras.

1B. ***A variant of a result of Leray and Hirsch.*** We have the following Leray–Hirsch–type result from [Edidin and Graham 1997]:

Proposition 1.5. *Let $\mathcal{X} \to \mathcal{Y}$ be a representable morphism of smooth quotient stacks over k. Suppose that $\mathcal{Y}$ is connected. Assume that there exists a proper and smooth algebraic space F over k which admits a decomposition into locally closed algebraic subspaces isomorphic to $\mathbb{A}^m_k$ such that $\mathcal{X} \to \mathcal{Y}$ is a Zariski-locally trivial fibration with fiber F, i.e., such that $\mathcal{X} \to \mathcal{Y}$ is Zariski-locally on $\mathcal{Y}$ isomorphic to $\mathcal{Y} \times_k F \to \mathcal{Y}$.*

Then the following hold:

(i) *For any $i \geq 0$, a family of elements of $A^i(\mathcal{X})$ restricts to a basis of $A^i(F')$ for every geometric fiber F' of $\mathcal{X} \to \mathcal{Y}$ if it does so for a single such fiber.*

(ii) *For any $i \geq 0$, there exists a family of elements of $A^i(\mathcal{X})$ which restricts to a basis of $A^i(F')$ for every geometric fiber F' of $\mathcal{X} \to \mathcal{Y}$.*

(iii) *Let $(B_i \subset A^i(\mathcal{X}))_{i\geq 0}$ be a collection of families as in* (ii). *Then $A^\bullet(\mathcal{X})$ is a free module over $A^\bullet(\mathcal{Y})$ and $\bigcup_{i\geq 0} B_i$ is a basis of the $A^\bullet(\mathcal{Y})$-module $A^\bullet(\mathcal{X})$.*

Proof. By taking presentations $\mathcal{X} = [G\backslash X]$ and $\mathcal{Y} = [G\backslash Y]$ as well as suitable $U \subset V$ for G as above the claim reduces to an analogous claim for the morphism $G\backslash(X \times U) \to G\backslash(Y \times U)$ of algebraic spaces. Then the claim is given by [Edidin and Graham 1997, Proposition 6, its proof, and Lemma 1]. □

2. The Bruhat stack and cycle classes of Schubert varieties

From now on we fix a split reductive group scheme G over the field k, a Borel subgroup $B \subset G$ over k and a maximal torus $T \subset B$ over k which is split over k.

2A. ***Chow rings of classifying stacks.*** By [Edidin and Graham 1998, Section 3.2], the Chow rings $A^\bullet([T\backslash *])$, $A^\bullet([B\backslash *])$ and $A^\bullet([G\backslash *])$ are given as follows: Every $\chi \in X^*(T)$ induces a line bundle on $[T\backslash *]$, and we get a morphism $X^*(T) \to A^1([T\backslash *])$ sending χ to the Chern class of this line bundle; see [Edidin and Graham 1998, Section 2.4]. This extends to an isomorphism

$$\mathrm{Sym}(X^*(T)_{\mathbb{Q}}) \xrightarrow{\sim} S := A^\bullet([T\backslash *]).$$

This in fact holds even with $\mathbb{Z}$-coefficients. The canonical homomorphism $A^\bullet([B\backslash *]) \to S = A^\bullet([T\backslash *])$ is an isomorphism (Proposition 1.4). The action of the Weyl group W on T induces an action of W on the abelian group $X^*(T)$ by $(w, \chi) \mapsto \chi \circ \mathrm{int}(w^{-1})$. By functoriality we obtain an action of W on the graded $\mathbb{Q}$-algebra S. Then the natural homomorphism $A^\bullet([G\backslash *]) \to S$ yields an identification

$$A^\bullet([G\backslash *]) \xrightarrow{\sim} S^W \tag{2.1}$$

(recall that we consider rational coefficients).

Example 2.2. Let $G = \mathrm{GL}_n$. Let $T \subseteq G$ be the diagonal torus identified with $\mathbb{G}_m^n$. Then $X^*(T) = \mathbb{Z}^n$ and $A^\bullet([T\backslash *]) = S = \mathbb{Q}[t_1, \dots, t_n]$, where $(t_1, \dots, t_n)$ is the standard basis of $\mathbb{Q}^n = X^*(T)_{\mathbb{Q}}$. Moreover,

$$A^\bullet([G\backslash *]) = \mathbb{Q}[t_1, \dots, t_n]^{S_n} = \mathbb{Q}[\sigma_1, \dots, \sigma_n],$$

where σ_i is the elementary symmetric polynomial of degree i in $t_1, \dots, t_n$.

If $\mathscr{X}$ is a smooth quotient stack and $\mathscr{V}$ is a vector bundle of rank n on $\mathscr{X}$, then $\mathscr{V}$ corresponds to a flat morphism $\alpha_{\mathscr{V}} : \mathscr{X} \to [\mathrm{GL}_n \backslash *]$ of algebraic stacks and the i-th Chern class of $\mathscr{V}$ is given by

$$c_i(\mathscr{V}) = \alpha_{\mathscr{V}}^*(\sigma_i) \in A^\bullet(\mathscr{X}).$$

The determinant $\det : \mathrm{GL}_n \to \mathbb{G}_m$ induces a flat morphism $[\mathrm{GL}_n \backslash *] \to [\mathbb{G}_m\backslash *]$ and hence a pullback morphism of $\mathbb{Q}$-algebras

$$\det^* : A^\bullet([\mathbb{G}_m\backslash *]) = \mathbb{Q}[t_1] \to A^\bullet([\mathrm{GL}_n \backslash *]) = \mathbb{Q}[\sigma_1, \dots, \sigma_n]$$

which sends t_1 to σ_1. In particular,

$$c_1(\mathscr{V}) = c_1(\det \mathscr{V}).$$

Proposition 2.3 [Demazure 1974, 4.6]. *The homomorphism $S \to A^\bullet(G/B)$ sending $\chi \in X^*(T)$ to the Chern class of the induced line bundle on G/B is surjective and its kernel is the ideal J of S generated by the homogeneous elements of S^W of degree > 0.*

2B. *The Chow ring of the Bruhat stack.* We consider the *Bruhat stack*

$$\mathtt{Brh} := \mathtt{Brh}_G := [B\backslash *] \times_{[G\backslash *]} [B\backslash *] \cong [B\backslash G/B],$$

together with its Bruhat decomposition into the locally closed substacks $\mathtt{Brh}_w = [B\backslash BwB/B]$. We are interested in the classes $[\overline{\mathtt{Brh}}_w]$ of the closures $\overline{\mathtt{Brh}}_w$ in $A^\bullet(\mathtt{Brh})$.

Proposition 2.4. (1) *For both natural homomorphisms $S = A^\bullet([B\backslash *]) \to A^\bullet(\mathtt{Brh})$, the module $A^\bullet(\mathtt{Brh})$ is free over S with a basis given by the classes $[\overline{\mathtt{Brh}}_w]$ for $w \in W$.*

(2) *The natural homomorphism $S \otimes_{S^W} S \to A^\bullet(\mathtt{Brh})$ is an isomorphism.*

Proof. Consider $\mathtt{Brh}$ as a G/B-fibration over $[B\backslash *]$ via, say, the first projection. Let $w_0 \in W$ be the longest element. The substack $\mathtt{Brh}_{w_0} \subset \mathtt{Brh}$ is open and given by the open Bruhat cell in G/B. Since the stabilizer of w_0 in $B \times B$ is isomorphic to T, the substack $\mathtt{Brh}_{w_0}$ can be identified with $[T\backslash *]$. Hence it has a natural structure as a U^--torsor over $[B\backslash *]$, where U^- is the unipotent radical of the unique Borel subgroup B^- of G such that $B^- \cap B = T$. The pushout along the open immersion $U^- \hookrightarrow G/B$, $u \mapsto uB$, is isomorphic to $\mathtt{Brh}$. Any U^--torsor is Zariski-locally trivial and hence so is $\mathtt{Brh} \to [B\backslash *]$. Thus $\mathtt{Brh} \to [B\backslash *]$ satisfies the conditions of Proposition 1.5. Hence (1) follows from Proposition 1.5 using the fact that the closures of the Bruhat strata on G/B give a basis of $A^\bullet(G/B)$ (see [Demazure 1974, Corollaire to Proposition 1]).

For (2), we can argue as follows: Consider $S \otimes_{S^W} S$ as an S module via the first factor. The ring S is free over S^W of rank $|W|$ and hence so is $S \otimes_{S^W} S$ over S. The S-module $A^\bullet(\mathtt{Brh})$ is free of rank $|W|$ by (1). Thus $S \otimes_{S^W} S \to A^\bullet(\mathtt{Brh})$ is a homomorphism of free S-modules of the same rank and it suffices to prove that it is surjective. By Proposition 2.3, we can take homogenous elements x_i in S which map to a basis of $A^\bullet(G/B)$. Then by Proposition 1.5 the images of $1 \otimes x_i$ in $A^\bullet(\mathtt{Brh})$ form a basis of $A^\bullet(\mathtt{Brh})$ over S. This proves surjectivity. □

To give a description of the class of $\overline{\mathtt{Brh}}_w$ in $S \otimes_{S^W} S$, we now proceed as follows. We first recall a formula for the class of the diagonal $\mathtt{Brh}_e$ in $\mathtt{Brh}$ by Graham. Then we define explicit operators δ_w on $A^\bullet(\mathtt{Brh})$ such that $[\overline{\mathtt{Brh}}_w] = \delta_w[\mathtt{Brh}_e]$.

2C. *The class of the diagonal.* In [Graham 1997, Theorem 1.1] the following formula for the class of the diagonal $\mathtt{Brh}_e$ in $\mathtt{Brh}$ is proved in the case $k = \mathbb{C}$. The proof given there can be readily adapted to arbitrary fields.

For $w \in W$, let $i_w : S \otimes_{S^W} S \to S$, $r \otimes r' \mapsto rw(r')$. The map $\prod_{w\in W} i_w : S \otimes_{S^W} S \to \prod_{w\in W} S$ is injective because $\operatorname{Spec}(S \otimes_{S^W} S) = \operatorname{Spec}(S) \times_{\operatorname{Spec}(S)/W} \operatorname{Spec}(S)$.

Theorem 2.5 (Graham). *The image of $[\mathtt{Brh}_e]$ under i_e is $\prod_{\alpha\in\Phi^+} \alpha \in S$. The image of $[\mathtt{Brh}_e]$ under i_w for $w \neq 1$ is zero.*

Example 2.6. We recall the results of Fulton on the class of the diagonal for classical groups. For the classical groups GL_n, SO_{2n+1}, Sp_{2n}, and SO_{2n}, we choose the standard maximal torus $T \cong \mathbb{G}_m^n$ and Borel subgroup giving rise to the Weyl group descriptions of page 279 of [Fulton 1996] to obtain $S = \mathrm{Sym}(X^*(T)_{\mathbb{Q}})$ and the roots in $X^*(T)$. We give elements $\widetilde{[\mathtt{Brh}_e]}$ in $S \otimes_{\mathbb{Q}} S = \mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n]$, where x_i and y_i represent the same $\mathbb{G}_m$-factor of T. The images of these elements in $S \otimes_{S^W} S$ are $[\mathtt{Brh}_e]$. As a reference we use [Fulton 1996], where y_j is denoted by y_{n+1-j} and the Schubert variety corresponding to $w = e$ is denoted by Ω_{w_0}.

We fix $n \in \mathbb{N}$ and introduce the following polynomials:

$$\Phi := \Phi_n := \prod_{1 \le i < j \le n} (x_i - y_j) \in S \otimes_{\mathbb{Q}} S,$$
$$\Gamma_k := \det((c_{k+1+j-2i})_{1 \le i,j \le k}) \in \mathbb{Q}[c_{-k+2}, c_{-k+3}, \dots, c_{2k+1}].$$

For instance,

$$\begin{aligned} \Phi_1 &= 1, & \Phi_2 &= x_1 - y_2, \\ \Gamma_1 &= c_1, & \Gamma_2 &= c_1 c_2 - c_0 c_3. \end{aligned} \tag{2.7}$$

We also let $\sigma_1, \dots, \sigma_n$ be the elementary symmetric polynomials in n variables with $\deg(\sigma_i) = i$. We also set $\sigma_0 := 1$.

(A_{n-1}) Let $n \ge 2$. Then

$$\widetilde{[\mathtt{Brh}_e]} = \Phi_n. \tag{2.8}$$

(B_n) Let $n \ge 2$. Then

$$\begin{aligned} \widetilde{[\mathtt{Brh}_e]} &= \Phi_n \Gamma_n, \\ c_i &:= \begin{cases} \frac{1}{2}(\sigma_i(x_1, \dots, x_n) + \sigma_i(y_1, \dots, y_n)) & \text{if } 0 \le i \le n, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{2.9}$$

(C_n) Let $n \ge 2$. Then

$$\begin{aligned} \widetilde{[\mathtt{Brh}_e]} &= \Phi_n \Gamma_n, \\ c_i &:= \begin{cases} \sigma_i(x_1, \dots, x_n) + \sigma_i(y_1, \dots, y_n) & \text{if } 0 \le i \le n, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{2.10}$$

(D_n) Let $n \ge 3$. Then

$$\begin{aligned} \widetilde{[\mathtt{Brh}_e]} &= \Phi_n \Gamma_{n-1}, \\ c_i &:= \begin{cases} \frac{1}{2}(\sigma_i(x_1, \dots, x_n) + \sigma_i(y_1, \dots, y_n)) & \text{if } 0 \le i \le n-1, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{2.11}$$

2D. *The Chevalley formula.* For $(\lambda, \mu) \in X^*(T) \times X^*(T)$, we have a natural line bundle $\mathscr{L}_{\lambda,\mu}$ on $\mathtt{Brh}$ with Chern class $\lambda \otimes \mu$. The following gives a version of a classical formula of Chevalley in this context:

Theorem 2.12 [Goldring and Koskivirta 2019a, Theorem 5.2.2]. (1) *The line bundle* $\mathscr{L}_{\lambda,\mu}$ *has a global section on* $\mathtt{Brh}_w$ *if and only if* $\mu = w^{-1}\lambda$.

(2) *The space* $H^0(\mathtt{Brh}_w, \mathscr{L}_{\lambda,w^{-1}\lambda})$ *has dimension* 1.

(3) *For* $w \in W$, *set* $E_w := \{\alpha \in \Phi^+ \mid ws_\alpha < w, \ell(ws_\alpha) = \ell(w) - 1\}$. *The divisor of any nonzero section of* $\mathscr{L}_{\lambda,w^{-1}\lambda}$ *on* $\mathtt{Brh}_w$ *is equal to*

$$\sum_{\alpha\in E_w} \langle \lambda, \alpha^\vee\rangle [\overline{\mathtt{Brh}}_{ws_\alpha}].$$

Note that the formula in loc. cit. contains an additional minus sign because there the positive roots are defined by the opposite Borel subgroup.

For $w \in W$ and $\lambda \in X^*(T)$, this implies the following relation in $A^\bullet(\mathtt{Brh})$:

$$(\lambda \otimes w^{-1}(\lambda))[\overline{\mathtt{Brh}}_w] = \sum_{\alpha\in E_w} \langle \lambda, \alpha^\vee\rangle [\overline{\mathtt{Brh}}_{ws_\alpha}]. \tag{2.13}$$

2E. *The operators* δ_w. We use certain operators on S and $S \otimes_{S^W} S$: Let n be the semisimple rank of G and $\alpha_1, \dots, \alpha_n$ be the simple roots with respect to T and B. For $1 \le i \le n$, let $s_i := s_{\alpha_i}$ be the simple reflection in W corresponding to α_i and $P_i := B \cup Bs_iB$ the "i-th minimal parabolic" with root system $\{\pm\alpha_i\}$.

Construction 2.14. Let $X_i := [B\backslash *] \times_{[P_i\backslash *]} [B\backslash *]$, and let $p_1, p_2 : X_i \to [B\backslash *]$ be the two projections, which are proper. Then we define $\delta_i : S \to S$ to be the correspondence $p_{1,*} \circ p_2^* : S \to S$.

Construction 2.15. Let $1 \le i \le n$. Consider S as the ring of polynomial functions on $X^*(T)_{\mathbb{Q}}$, that is, the ring of functions $X^*(T)_{\mathbb{Q}} \to \mathbb{Q}$ which, with respect to some (or equivalently any) basis of $X^*(T)$, can be written as a polynomial. For $f \in S$, the element $f - s_{\alpha_i}(f)$ of S vanishes on the hyperplane in $X^*(T)_{\mathbb{Q}}$ given by the vanishing of the coroot $\alpha_i^\vee$. Hence we obtain an element $\tilde{\delta}_i(f) := (f - s_{\alpha_i}(f))/\alpha_i^\vee \in S$. This defines a $\mathbb{Q}$-linear homomorphism $\tilde{\delta}_i : S \to S$.

Theorem 2.16. (1) *For each* $1 \le i \le n$, *we have* $\delta_i = \tilde{\delta}_i$.

(2) *For* $w \in W$, *one gets a well-defined operator* δ_w *on* S *by letting* $\delta_w = \delta_{i_1} \cdots \delta_{i_k}$ *for any decomposition* $w = s_{i_1} \cdots s_{i_k}$ *with* $k = \ell(w)$.

Proof. When $k = \mathbb{C}$ and G is semisimple and simply connected, this is proven in [Bernšteĭn et al. 1973]. See Theorem 5.7 in that work for (1) and Theorem 3.4 there for (2).

The general case can be deduced from this as follows: First, using the functoriality of the various constructions with respect to homomorphisms of reductive groups inducing an isomorphism on adjoint groups one can reduce to the case that G is semisimple and simply connected. Now let $\tilde{k}$ be another algebraically closed base field, $\widetilde{G}$ the reductive group over $\tilde{k}$ with the same root datum as G, with $\widetilde{T}$, $\widetilde{B}$, $\widetilde{\mathtt{Brh}} := \mathtt{Brh}_{\widetilde{G}}$, etc. the corresponding data for $\widetilde{G}$. We have a natural W-equivariant isomorphism $A^\bullet(*/T) \cong A^\bullet(*/\widetilde{T})$ which induces an isomorphism $A^\bullet(\mathtt{Brh}) \cong A^\bullet(\widetilde{\mathtt{Brh}})$. We claim that for $w \in W$, the classes of $\overline{\mathtt{Brh}}_w$ and $\overline{\widetilde{\mathtt{Brh}}}_w$ correspond to each other under this isomorphism. For $w = e$, this follows from Theorem 2.5. From this one deduces the claim by induction on $\ell(w)$ using (2.13).

By taking $\tilde{k} = \mathbb{C}$ this implies the claim. ◻

Remark 2.17. Construction 2.15 shows that one has the following Leibniz-type formula:

$$\begin{aligned}\delta_i(fg) &= \frac{fg - s_{\alpha_i}(f)s_{\alpha_i}(g)}{\alpha_i^\vee} = \frac{(f - s_{\alpha_i}(f))g + s_{\alpha_i}(f)(g - s_{\alpha_i}(g))}{\alpha_i^\vee} \\ &= \delta_i(f)g + s_{\alpha_i}(f)\delta_i(g). \end{aligned} \tag{2.18}$$

Now, for $w \in W$, we define an operator δ_w on $A^\bullet(\mathtt{Brh}) = S \otimes_{S^W} S$ by letting the δ_w just defined on S act on the first factor. For $1 \le i \le n$, the operator $\delta_i = \delta_{s_i}$ on $S \otimes_{S^W} S$ can also be described as follows: Let $\mathtt{Brh}_i$ be the following fiber product:

$$\begin{tikzcd} \mathtt{Brh}_i \arrow[r, "q_2"] \arrow[d, "q_1"] & \mathtt{Brh} \arrow[d] \\ \mathtt{Brh} \arrow[r] & {[P_i\backslash *] \times_{[G\backslash *]} [B\backslash *]} \end{tikzcd} \tag{2.19}$$

Then $\delta_{s_i} = q_{1,*} \circ q_2^* : S \otimes_{S^W} S \to S \otimes_{S^W} S$.

Theorem 2.20. *Let $w \in W$ and $1 \le i \le n$. Then $\delta_{s_i}[\overline{\mathtt{Brh}}_w] = [\overline{\mathtt{Brh}}_{s_i w}]$ if $\ell(s_i w) = \ell(w) + 1$ and $\delta_{s_i}[\overline{\mathtt{Brh}}_w] = 0$ otherwise.*

Proof. We let P_i act on P_i/B and G/B by multiplication from the left and on products of these varieties by the diagonal action. Then the P_i-equivariant diagram

$$\begin{tikzcd} P_i/B \times P_i/B \times G/B \arrow[r, "\pi_{13}"] \arrow[d, "\pi_{23}"] & P_i/B \times G/B \arrow[d, "\pi_2"] \\ P_i/B \times G/B \arrow[r, "\pi_2"] & G/B \end{tikzcd}$$

gives a presentation of (2.19). Here the quotient morphism $P_i/B \times G/B \to \mathtt{Brh} = [B\backslash G/B]$ sends (pB, gB) to $(Bp^{-1}gB)$ and the preimage of $\mathtt{Brh}_w$ is the P_i-orbit

$$O_w := \{(pB, gB) \in P_i/B \times G/B \mid Bp^{-1}gB = BwB\}$$

in $P_i/B \times G/B$. We prove the claim by showing the corresponding claim for the classes of the closed subvarieties $[\overline{O}_w]$ in $A^\bullet(P_i/B \times G/B)$.

The image $\pi_{23}(\pi_{13}^{-1}(O_w))$ is contained in $P_i/B \times P_i wB/B = P_i/B \times (Bs_i BwB/B \cup BwB/B)$. When $s_i w < w$, the latter set is contained in $P_i/B \times \overline{BwB}/B$. Since $\dim(O_w) = \dim(BwB/B) + \dim(P_i/B) > \dim(BwB/B)$, it is then of strictly smaller dimension than $\pi_{23}^{-1}(O_w) = P_i/B \times O_w$. This proves the claim in this case.

Now assume $s_i w > w$. We have $Bs_i BwB = Bs_i wB$ and

$$\pi_{23}(\pi_{13}^{-1}(O_w)) = P_i/B \times P_i wB/B = P_i/B \times (Bs_i wB/B \cup BwB/B).$$

This is a locally closed subset of $\overline{O}_{s_i w}$ of the same dimension, hence it is open in $\overline{O}_{s_i w}$. Thus it suffices to prove that $\pi_{23} : \pi_{13}^{-1}(O_w) \to \pi_{23}(\pi_{13}^{-1}(O_w))$ is an isomorphism. For this it suffices to prove that every fiber of π_{23} above a point in this image consists of a single point. For such a point (pB, gB), the fiber of π_{23}

is isomorphic to $\{qB \in P_i/B \mid Bq^{-1}gB = BwB\}$. When $gB \in BwB/B$, the identity $Bq^{-1}gB = BwB$ implies $q \in B$ and hence the fiber consists of a single point.

Now assume $gB \in Bs_iwB/B$ and let $qB \in P_i/B$ such that $Bq^{-1}gB = BwB$. Then necessarily $q \in Bs_iB$. Then (see [Springer 1998, Lemma 8.3.6]) we can write $q = us_i$ and $g = u's_ibw$ for elements u, u' in the root group U_{α_i} associated to α_i and an element $b \in B$. Then $q^{-1}g = s_iu^{-1}u's_ibw$ with $s_iu^{-1}u's_i \in Bs_iBs_iB = B \cup Bs_iB$. Since $q^{-1}g \in BwB$, we get $s_iu^{-1}u's_i \in B \cap U_{-\alpha_i} = \{e\}$. Thus $u = u'$, which proves that the fiber of π_{23} above (qB, gB) again consists of a single point. This finishes the proof. □

By induction on $\ell(w)$, we get:

Corollary 2.21. *Let $w \in W$. Then $[\overline{\mathtt{Brh}_w}] = \delta_w[\mathtt{Brh}_e]$.*

3. The stacks of G-zips and of flagged G-zips

3A. ***The stack of G-zips.*** We recall the construction of the moduli stack of G-zips as a quotient space; see [Pink et al. 2015, Theorem 1.5]. From now on let k be an algebraic closure of $\mathbb{F}_p$ and G a reductive group scheme over $\mathbb{F}_p$. If X is an object over some $\mathbb{F}_p$-algebra, then we denote by $X^{(p)}$ the pullback of X under the absolute Frobenius. For a scheme X over $\mathbb{F}_p$, we denote by $\varphi : X \to X^{(p)} = X$ its relative Frobenius.

The zip datum. Let $\mu : \mathbb{G}_{m,k} \to G_k$ be a cocharacter of G_k. It gives rise to a pair of opposite parabolic subgroups $(P_-(\mu), P_+(\mu))$ and a Levi subgroup $L := L(\mu) = P_-(\mu) \cap P_+(\mu)$ defined by the condition that $\mathrm{Lie}(P_-(\mu))$ $\big($resp. $\mathrm{Lie}(P_+(\mu))\big)$ is the sum of the nonpositive (resp. nonnegative) weight spaces of μ in $\mathrm{Lie}(G)$. On k-valued points we have

$$P_+(\mu) = \{g \in G \mid \lim_{t\to 0} \mu(t)g\mu(t)^{-1} \text{ exists}\}, \qquad P_-(\mu) = \{g \in G \mid \lim_{t\to \infty} \mu(t)g\mu(t)^{-1} \text{ exists}\},$$

and $L = \mathrm{Cent}_G(\mu)$. We set

$$P := P_-, \qquad Q := (P_+)^{(p)}, \qquad M := L^{(p)} = \mathrm{Cent}_G(\varphi \circ \mu).$$

Hence M is a Levi subgroup of Q.

The stack of G-zips of type μ. Denote the projections to the Levi components $P \to L$ and $Q \to M$ both by $x \mapsto \bar{x}$. The zip group E is defined as

$$E := \{(x, y) \in P \times Q \mid \varphi(\bar{x}) = \bar{y}\}. \tag{3.1}$$

We let $G \times G$ act on G by $(x, y) \cdot g := xgy^{-1}$. By restriction we obtain actions of $P \times Q$ and of E on G.

We denote by

$$G\text{-}\mathtt{Zip}^{\mu} := [E\backslash G]$$

the quotient stack. It is a smooth algebraic stack of dimension 0.

Every morphism $f : G \to G'$ of reductive groups over $\mathbb{F}_p$ yields a morphism of stacks $G\text{-}\mathtt{Zip}^{\mu} \to G'\text{-}\mathtt{Zip}^{f\circ\mu}$. In particular, if $\mu' = \mathrm{int}(h) \circ \mu$ for some $h \in G(k)$, then conjugation with h yields an isomorphism $G\text{-}\mathtt{Zip}^{\mu} \xrightarrow{\sim} G\text{-}\mathtt{Zip}^{\mu'}$. Let κ be the field of definition of the conjugation class of μ. As G is quasisplit, there exists an element in that conjugacy class that is defined over κ. Therefore it is harmless to assume that μ is defined over κ, the field of definition of its conjugacy class. We do assume this from now on. Then the stack $G\text{-}\mathtt{Zip}^{\mu}$ is defined over κ as well.

3B. *Choosing a frame.*

Lemma 3.2. *Let κ be a finite extension of $\mathbb{F}_p$. Let G be a reductive group defined over $\mathbb{F}_p$, let $Q \subseteq G_\kappa$ be a parabolic subgroup, and let $M \subseteq Q$ be a Levi subgroup that is also defined over κ. Then there exists $g \in G(\kappa)$ and a Borel pair $T \subseteq B \subseteq G$ that is already defined over $\mathbb{F}_p$ with $T \subseteq {}^gM$ and $B \subseteq {}^gQ$.*

Proof. As every reductive group over a finite field is quasisplit, we can choose a maximal torus T and a Borel subgroup $B \supseteq T$ defined over $\mathbb{F}_p$. By [SGA 3$_{\text{III}}$ 1970, Exp. XXVI, Lemme 3.8], there exists a parabolic subgroup Q' defined over κ with the same type as Q such that $B \subseteq Q'$. Let M' be the unique Levi subgroup of Q' that contains T. By [SGA 3$_{\text{III}}$ 1970, Exp. XXVI, Corollaire 5.5(iv)] there exists an element $g \in G(\kappa)$ with ${}^gQ = Q'$ and ${}^gM = M'$. □

After replacing μ by some conjugate cocharacter μ', we may (and do) assume by Lemma 3.2 that there exists a Borel pair $T \subseteq B \subseteq G$ defined over $\mathbb{F}_p$ with $B \subseteq Q$ and $T \subseteq M$. If μ is defined over some finite extension κ of $\mathbb{F}_p$, we may assume that its conjugate is also defined over κ. Then T is also a maximal torus of M and hence contains its center. Hence $\varphi \circ \mu$ factors through T. Because T is defined over $\mathbb{F}_p$, also μ itself factors through T. As $B \subseteq Q$, the cocharacter $\varphi \circ \mu$ is B-dominant. Hence μ is also B-dominant because B is defined over $\mathbb{F}_p$.

Recall that we denote by (W, Σ) the Coxeter system associated to (G, B, T). The Frobenius φ on G induces an automorphism of the Coxeter system (W, Σ), which is again denoted by φ (see also Section 3D below). Let $I, J \subseteq \Sigma$ be the set of simple reflections corresponding to the conjugacy classes of P and Q, respectively.

By [Pink et al. 2011, 3.7] (and its proof), we find $z \in G(k)$ with ${}^zT = T$ such that (B, T, z) is a frame for (G, P, L, Q, M, φ) in the sense of [Pink et al. 2011, 3.6], i.e., ${}^zB \subseteq P$ and $\varphi({}^zB \cap L) = B \cap M$. In fact we can and will choose z as follows.

Lemma 3.3. *Let $z \in \mathrm{Norm}_G(T)(k)$ be a lift of $\bar{z} := w_{0,I}w_0 \in W$. Then ${}^zB \subseteq P$ and $\varphi({}^zB \cap L) = B \cap M$.*

Proof. The first claim follows from the fact that ${}^{w_0}B = \varphi({}^{w_0}B) \subset P$. The second claim follows from

$$\varphi({}^zB \cap L) = {}^{\varphi(w_{0,I})}\varphi({}^{w_0}B \cap L) = B \cap M.$$ □

By [Pink et al. 2011, 3.11], the map

$$\varphi \circ \mathrm{int}(z) : (W_I, I) \xrightarrow{\sim} (W_J, J) \tag{3.4}$$

is an isomorphism of Coxeter systems and I, J, and $\varphi \circ \mathrm{int}(z)$ are independent of the choice of the frame.

3C. *Classification of G-zips.* For $w \in W$, let $G_w \subseteq G$ be the E-orbit of $\dot{w}z$. By [Pink et al. 2011, 7.5], there is a bijection

$$ {}^I W \leftrightarrow \{E\text{-orbits in } G\}, \quad w \mapsto G_w, \tag{3.5}$$

and $\dim(G_w) = \ell(w) + \dim(P)$ for $w \in {}^I W$. We call the corresponding locally closed algebraic substack of $G\text{-}\mathtt{Zip}^\mu$,

$$ Z_w := [E \backslash G_w] \subseteq G\text{-}\mathtt{Zip}^\mu, \tag{3.6}$$

the *zip stratum corresponding to* $w \in {}^I W$. One has $\mathrm{codim}_{G\text{-}\mathtt{Zip}^\mu}(Z_w) = \dim(G) - \dim(P) - \ell(w)$.

Let $\overline{G}_w$ be the closure of the E-orbit G_w. We set $\bar{Z}_w := [E \backslash \overline{G}_w]$. This is the unique reduced closed algebraic substack of $G\text{-}\mathtt{Zip}^\mu$ whose underlying topological space is the closure of the one-point topological space underlying Z_w. By [Pink et al. 2011, 6.2], we have

$$ \bar{Z}_w = \bigcup_{w' \preceq w} Z_{w'} \tag{3.7}$$

for a partial order $\preceq$ on ${}^I W$ defined in [loc. cit., 6.2]. Here we will need only the following properties of this partial order (see [He 2007, §3]).

Lemma 3.8. (1) *There exists a unique minimal element in* ${}^I W$, *namely the neutral element* e, *and a unique maximal element in* ${}^I W$, *namely* $w_{0,I} w_0$, *where* w_0 *and* $w_{0,I}$ *are unique elements of maximal length in* W *and in* W_I, *respectively.*

(2) *The partial order* $\preceq$ *is at least as fine as the Bruhat order.*

(3) *Let* $w' \preceq w$. *Then* $\ell(w') \leq \ell(w)$ *and one has* $\ell(w') = \ell(w)$ *if and only if* $w' = w$.

(4) *If* $w' \prec w$ *and there exists no* $u \in {}^I W$ *with* $w' \prec u \prec w$, *then* $\ell(w') = \ell(w) - 1$.

3D. *The action of Frobenius.* Recall that we denote by $\varphi : G \to G$ the relative Frobenius. We also denote by $\sigma : k \to k,\ x \mapsto x^p$ the arithmetic Frobenius. As T and B are defined over $\mathbb{F}_p$, we can identify canonically $T^{(p)}$ with T and $B^{(p)}$ with B. Hence the relative Frobenius induces isogenies $\varphi : T \to T$ and $\varphi : B \to B$.

Set $\mathcal{W} := \mathrm{Norm}_G(T)/T = \pi_0(\mathrm{Norm}_G(T))$, which is a finite étale group scheme over $\mathbb{F}_p$. Then $W = \mathcal{W}(k)$ is the absolute Weyl group. As $\mathrm{Norm}_G(T)$ is also defined over $\mathbb{F}_p$, the relative Frobenius φ induces an automorphism of $\mathcal{W}$ and hence an automorphism φ of the finite group W. As B is defined over $\mathbb{F}_p$, this automorphism preserves the set Σ of simple reflections in W defined by B. By functoriality, σ also defines an automorphism $w \mapsto {}^\sigma w$ of $W = \mathcal{W}(k)$ and we have $\varphi(w) = {}^{\sigma^{-1}} w$ for all $w \in W$. If T is a split torus, then $\varphi = \mathrm{id}$ on W.

We denote by $X^*(T)$ the group of characters of $T \otimes_{\mathbb{F}_p} k$. For $\lambda \in X^*(T)$, we set $\varphi(\lambda) := \lambda \circ \varphi$, which defines an endomorphism φ on the abelian group $X^*(T)$. We denote by $\lambda \mapsto {}^\sigma\lambda$ the canonical action of σ on $X^*(T)$, i.e.,

$$ {}^\sigma\lambda := (\mathrm{id}_{\mathbb{G}_{m,\mathbb{F}_p}} \otimes \sigma) \circ \lambda \circ (\mathrm{id}_T \otimes \sigma^{-1}). $$

Then one has, for $\lambda \in X^*(T)$,

$$\varphi(\lambda) = p\,{}^{\sigma^{-1}}\lambda. \tag{3.9}$$

If T is a split torus, then ${}^{\sigma}\lambda = \lambda$ and $\varphi(\lambda) = p\lambda$ for all $\lambda \in X^*(T)$.

By functoriality, the actions of φ and σ on $X^*(T)$ also induce actions on the graded $\mathbb{Q}$-algebra $S = \mathrm{Sym}(X^*(T))_{\mathbb{Q}}$, and for $f \in S$ of degree d, we have

$$\varphi(f) = p^d\,{}^{\sigma^{-1}}f. \tag{3.10}$$

3E. ***The stack of flagged G-zips of type μ.*** We fix a subset $I_0 \subseteq I$ and let P_0 be the unique parabolic subgroup of G of type I_0 with ${}^zB \subseteq P_0 \subseteq P$. We let E act on $G \times P/P_0$ by

$$(x, y)\cdot(g, aP_0) := (xgy^{-1}, xaP_0)$$

and set

$$G\text{-}\mathtt{ZipFlag}^{\mu,I_0} := [E\backslash(G \times P/P_0)].$$

If $I_0 = \varnothing$, then $P_0 = {}^zB$ and we abbreviate $G\text{-}\mathtt{ZipFlag}^{\mu} := G\text{-}\mathtt{ZipFlag}^{\mu,\varnothing}$. Note that $G\text{-}\mathtt{ZipFlag}^{\mu,I} = G\text{-}\mathtt{Zip}^{\mu}$. For $I_0' \subseteq I_0$, there are canonical projection maps

$$G\text{-}\mathtt{ZipFlag}^{\mu,I_0'} \to G\text{-}\mathtt{ZipFlag}^{\mu,I_0}$$

that are P_0/P_0'-bundles, where P_0' is the unique parabolic subgroup of type I_0' with ${}^zB \subseteq P_0' \subseteq P$. In particular, these maps are proper, smooth, and representable. By taking $I_0' = \varnothing$ and $I_0 = I$, we obtain a projection map

$$\pi : G\text{-}\mathtt{ZipFlag}^{\mu} \to G\text{-}\mathtt{Zip}^{\mu}.$$

Let $L_0 \subset P_0$ be the unique Levi subgroup containing T. We set

$$M_0 := L_0^{(p)} \quad\text{and}\quad Q_0 := M_0B.$$

Then Q_0 is a parabolic subgroup containing B of type

$$J_0 := \varphi({}^zI_0)$$

and M_0 is the unique Levi subgroup of Q_0 containing T. Then (B, T, z) is again a frame for $(G, P_0, L_0, Q_0, M_0, \varphi)$. By [Goldring and Koskivirta 2019b, (3.2.3)], the morphism $G \times P \to G,\ (g, x) \mapsto \bar{x}g\varphi(\bar{x})^{-1}$ induces a smooth representable morphism of algebraic stacks

$$\psi^{I_0} : G\text{-}\mathtt{ZipFlag}^{\mu,I_0} \to \mathtt{Brh}^{I_0} := [P_0\backslash G/Q_0]$$

with irreducible fibers. The maps ψ^{I_0} are compatible with passing to $I_0' \subseteq I_0$.

For $I_0 = \varnothing$, we have $P_0 = {}^zB$ and $Q_0 = B$. Therefore $g \mapsto z^{-1}g$ yields an isomorphism $\mathtt{Brh}^{\varnothing} \xrightarrow{\sim} \mathtt{Brh}_G$ and we denote by ψ the composition

$$\psi : G\text{-}\mathtt{ZipFlag}^{\mu} \xrightarrow{\psi^{\varnothing}} \mathtt{Brh}^{\varnothing} \xrightarrow{\sim} \mathtt{Brh}_G,$$

which is a smooth representable morphism with irreducible fibers. For $w \in W$, we write

$$Z_w^{\varnothing} := \psi^{-1}(\mathtt{Brh}_w) \subseteq G\text{-}\mathtt{ZipFlag}^{\mu}. \tag{3.11}$$

Since ψ is smooth, the $Z_w^{\varnothing}$ form a stratification of $G\text{-}\mathtt{ZipFlag}^{\mu}$ whose closure relation is given by the Bruhat order on W:

$$\overline{Z_w^{\varnothing}} = \bigcup_{w' \leq w} Z_{w'}^{\varnothing}.$$

Proposition 3.12. *The strata $Z_w^{\varnothing}$ are smooth and irreducible. Their closures $\overline{Z_w^{\varnothing}}$ are normal and with only rational singularities. In particular, they are Cohen–Macaulay.*

Proof. As ψ is smooth with irreducible fibers and $\mathtt{Brh}_w$ is smooth and irreducible, the first assertion holds. The smoothness of ψ also implies that $\overline{Z_w^{\varnothing}} = \psi^{-1}(\overline{\mathtt{Brh}_w})$. Hence all remaining assertions follow from the analogous properties for Schubert varieties [Brion and Kumar 2005, 3.2.2, 3.4.3]. □

By [Koskivirta 2018, 2.2.1], we have the following:

Proposition 3.13. *The projection $\pi : G\text{-}\mathtt{ZipFlag}^{\mu} \to G\text{-}\mathtt{Zip}^{\mu}$ induces for $w \in {}^I W$ representable finite étale maps*

$$\pi_w : Z_w^{\varnothing} \to Z_w.$$

Definition 3.14. We set $\gamma(w) := \deg(\pi_w)$.

In the next section we give a description of $\gamma(w)$.

Remark 3.15. Like their name suggests, the spaces $G\text{-}\mathtt{ZipFlag}^{\mu, I_0}$ admit a moduli description as a "flag space" over $G\text{-}\mathtt{Zip}^{\mu}$. Specifically, the stack $G\text{-}\mathtt{ZipFlag}^{\mu}$ is canonically isomorphic to the moduli stack of pairs consisting of a G-zip (I, I_+, I_-, ι) of type μ as in [Pink et al. 2015, Definition 3.1], together with a P_0-subtorsor of the P-torsor I_+. See [Goldring and Koskivirta 2019b, Section 3.1] for details on this construction.

3F. *Calculation of $\gamma(w)$.* Fix $w \in {}^I W$.

The type of $w \in {}^I W$. We recall the following construction from [Pink et al. 2011, §5]. Fix $w \in {}^I W$. Let I_w be the largest subset of I such that

$$\varphi({}^z I_w) = {}^{w^{-1}} I_w$$

and call it the *type of w*. In other words,

$$I_w = \{s \in I \mid (\mathrm{int}(w) \circ \varphi \circ \mathrm{int}(z))^k(s) \in I \text{ for all } k \geq 1\}. \tag{3.16}$$

For instance, as $\varphi({}^z I) = J$, one has $I_e = I$ if and only if $I = J$. Let P_w be the unique parabolic subgroup of type I_w with ${}^z B \subseteq P_w$, and let L_w be the unique Levi subgroup of P_w with $L_w \supseteq T$. As for an arbitrary subset of I, we obtain

$$M_w := {}^{(zw)^{-1}} L_w = L_w^{(p)} \quad \text{and} \quad Q_w := M_w B.$$

Hence Q_w is the unique parabolic subgroup containing B of type J_w, where

$$J_w := {}^{w^{-1}}I_w = \varphi({}^z I_w),$$

and M_w is the unique Levi subgroup of Q_w containing T. Note that M_w (resp. J_w) is denoted by H_w (resp. K_w) in [Pink et al. 2011, §5].

Description of $\gamma(w)$ via flag varieties. Set

$$A_w := \{x \in L_w \mid {}^{zw}\varphi(x) = x\}.$$

Then we have, by [Koskivirta 2018, 2.2.1] and [Pink et al. 2011, 8.1],

$$\gamma(w) = \#(A_w/(A_w \cap {}^z B)). \tag{3.17}$$

Lemma 3.18. (1) $(\mathrm{int}\,(zw) \circ \varphi)(L_w) = L_w$.

(2) $(\mathrm{int}\,(zw) \circ \varphi)(L_w \cap {}^z B) = L_w \cap {}^z B$.

Proof. The first assertion follows from ${}^{zw}\varphi({}^z I_w) = {}^z I_w$. Let us show the second assertion. Both sides are Borel subgroups of L_w which contain T. Hence it suffices to show that they contain the same root subgroups. Let Φ be the set of roots for (G, T), and let Φ^+ be the set of positive roots with respect to B. For a set of simple reflections K, let Φ_K be the set of roots of the standard Levi subgroup L_K of type K. Then $\Phi_K^+ := \Phi_K \cap \Phi^+$ is the system of positive roots given by the Borel subgroup $L_K \cap B$ of L_K.

Because z normalizes T, we can consider its image in W, which we denote again by z. Then the set of roots corresponding to L_w is ${}^z\Phi_{I_w}$ and the set of roots corresponding to $L_w \cap {}^z B$ is ${}^z\Phi^+_{I_w}$. So we must show

$${}^w\varphi({}^z\Phi^+_{I_w}) = \Phi^+_{I_w}.$$

As both sides have the same cardinality, it suffices to show that the left side is contained in the right side. By definition of a frame, we have $\varphi({}^z B \cap L_w) \subseteq B \cap M_w$, and this shows

$${}^w\varphi({}^z\Phi^+_{I_w}) \subseteq {}^w\Phi^+_{J_w} = \Phi^+_{I_w},$$

because ${}^w J_w = I_w$. □

Hence $\mathrm{int}\,zw \circ \varphi$ defines a descent datum from k to $\mathbb{F}_p$ for the reductive group L_w together with its Borel subgroup ${}^z B \cap L_w$. We obtain a reductive group L'_w and a Borel subgroup B'_w defined over $\mathbb{F}_p$ and its full flag variety by $F\ell_w := L'_w/B'_w$. Then we have by (3.17) the following description of $\gamma(w)$.

Proposition 3.19. *For $w \in {}^I W$, one has*

$$\gamma(w) = L'_w(\mathbb{F}_p)/B'_w(\mathbb{F}_p) = F\ell_w(\mathbb{F}_p). \tag{3.20}$$

Here the second identity follows from $H^1(\mathbb{F}_p, B'_w) = 0$.

Remark 3.21. By definition, L'_w is a form defined over $\mathbb{F}_p$ of the standard Levi subgroup of G corresponding to the set of simple reflections I_w. It is split if and only if ${}^w\varphi({}^z s) = s$ for all $s \in I_w$. If the

Dynkin diagram of L_w has no automorphisms (e.g., if it is connected of type B_n, C_n, E_7, E_8, F_4, or G_2), then this is automatic.

If L'_w is split, one obtains, from the decomposition of the flag variety $F\ell_w$ into a disjoint union of Schubert cells, the formula

$$\gamma(w) = \sum_{w \in W_{I_w}} p^{\ell(w)}. \tag{3.22}$$

3G. ***The key diagram.*** The projection $E \to P$, $(x, y) \mapsto x$ is a surjective homomorphism of algebraic groups. We obtain a composition

$$\beta : G\text{-}\mathtt{Zip}^{\mu} = [E \backslash G] \to [E \backslash *] \to [P \backslash *]. \tag{3.23}$$

Finally, we have a morphism $\gamma : \mathtt{Brh}_G \to [P \backslash *]$ defined as the composition

$$\gamma : \mathtt{Brh}_G = [B \backslash *] \times_{[G \backslash *]} [B \backslash *] \xrightarrow{\mathrm{pr}_1} [B \backslash *] \xrightarrow{\sim} [{}^z B \backslash *] \to [P \backslash *], \tag{3.24}$$

where the second map is induced by the isomorphism $b \mapsto zbz^{-1}$ and where the third map is induced by the inclusion ${}^z B \to P$.

The following commutative diagram, where $\alpha := \beta \circ \pi$, will be our key diagram:

$$\begin{tikzcd} G\text{-}\mathtt{ZipFlag}^{\mu} \arrow[r, "\psi"] \arrow[d, "\pi"'] \arrow[dr, "\alpha"] & \mathtt{Brh}_G \arrow[d, "\gamma"] \\ G\text{-}\mathtt{Zip}^{\mu} \arrow[r, "\beta"] & [P \backslash *] \end{tikzcd} \tag{3.25}$$

All morphisms are flat of constant relative dimension. Moreover, π is a $P/{}^z B$-bundle. Note that $P/{}^z B = L/({}^z B \cap L)$ is the full flag variety for L. In particular, π is proper, smooth, and representable.

4. Induced maps of Chow rings

In this section we describe the maps induced by the key diagram (3.25) on Chow rings. If $\mathscr{X}$ is any smooth algebraic quotient stack defined over some subfield k_0 of k, we set $A^\bullet(\mathscr{X}) := A^\bullet(\mathscr{X} \otimes_{k_0} k)$.

4A. ***The Chow ring of G-*** $\mathtt{Zip}^{\mu}$ ***and G-*** $\mathtt{ZipFlag}^{\mu}$**.** We recall the description of Brokemper [2018] of the Chow ring of $A^\bullet(G\text{-}\mathtt{Zip}^{\mu})$:

Recall that $S := \mathrm{Sym}(X^*(T)_{\mathbb{Q}}) = A^\bullet([T \backslash *])$. This is a graded $\mathbb{Q}$-algebra carrying an action by the Weyl group W by graded automorphisms. We also denote by $S_+ := S_{\geq 1}$ the augmentation ideal of S. Let

$$\mathcal{I} := (f - \varphi(f) \mid f \in S_+^W) \subseteq S^W \tag{4.1}$$

be the ideal generated by $f - \varphi(f)$ for $f \in S_+^W$ in S^W. Because we work with rational coefficients, there is also a simpler description of $\mathcal{I}$ (see Remark 4.13 for why the definition in (4.1) is more natural in this context).

Lemma 4.2. $$\mathcal{I} = S_+^W.$$

Proof. We have to show that $S^W_+ \subseteq \mathcal{I}$. Let $f \in S^W$ be of degree $d \geq 1$, and let $s \geq 1$ be an integer such that $T \otimes_{\mathbb{F}_p} \mathbb{F}_{p^s}$ is split. Thus σ^s acts trivially on $X^*(T)$ and hence on S. Then $\varphi^s(f) = p^{ds} f$ by (3.10) and therefore

$$(1 - p^{ds}) f = f - \varphi^s(f) = \sum_{i=1}^{s} (\varphi^{i-1}(f) - \varphi^i(f)) \in \mathcal{I}. \qquad \square$$

For every set $K \subseteq \Sigma$ of simple reflections, S^{W_K} is a finite free S^W-algebra of rank $\#(W/W_K)$, and hence the canonical map

$$S^W/\mathcal{I} \to S^{W_K}/\mathcal{I}S^{W_K}$$

is finite and faithfully flat and, in particular, injective.

We keep the notation from Section 3A. For every type $K \subseteq \Sigma$ of a parabolic subgroup, we denote by K^{o} the opposite type. Then

$$I^{\mathrm{o}} = {}^z I = \varphi^{-1}(J) \tag{4.3}$$

is a set of simple reflections and $LB = {}^z P_I$ is the standard parabolic subgroup of type ${}^z I$.

For a subgroup H of G, we denote by $[H_\varphi \backslash G]$ the quotient stack for the action of H on G by φ-conjugation $(h, g) \mapsto hg\varphi^{-1}(h)$. The following description of the Chow ring of these stacks for $H = T$ and $H = L$ is given by [Brokemper 2018, 2.3.2; 2016, 1.1] and their proofs.

Proposition 4.4. (1) *Consider the homomorphism*

$$S \otimes_{S^W} S \cong A^\bullet([B\backslash G/B]) \cong A^\bullet([T\backslash G/T]) \to A^\bullet([T_\varphi\backslash G]) \tag{4.5}$$

induced by pullback along the quotient morphism $[T_\varphi\backslash G] \to [T\backslash G/T]$ and the homomorphism

$$S \to S \otimes_{S^W} S, \quad f \mapsto f \otimes 1.$$

The composition $S \to A^\bullet([T_\varphi\backslash G])$ of these homomorphisms factors through an isomorphism of graded $\mathbb{Q}$-algebras

$$S/\mathcal{I}S \cong A^\bullet([T_\varphi\backslash G]). \tag{4.6}$$

(2) *The homomorphism $S \otimes_{S^W} S \to S/\mathcal{I}S$ given by (4.5) and (4.6) sends $f \otimes g$ to the class of $f\varphi(g)$.*

(3) *The homomorphism*

$$S^{W_{I^{\mathrm{o}}}}/\mathcal{I}S^{W_{I^{\mathrm{o}}}} \to S/\mathcal{I}S$$

induced by the inclusion $S^{W_{I^{\mathrm{o}}}} \hookrightarrow S$ is injective and free of rank $|W_{I^{\mathrm{o}}}|$.

(4) *The homomorphism $A^\bullet([L_\varphi\backslash G]) \to A^\bullet([T_\varphi\backslash G])$ induced by the quotient morphism $[T_\varphi\backslash G] \to [L_\varphi\backslash G]$ is injective. Under (4.6) it gives an isomorphism of graded $\mathbb{Q}$-algebras*

$$A^\bullet([L_\varphi\backslash G]) \cong S^{W_{I^{\mathrm{o}}}}/\mathcal{I}S^{W_{I^{\mathrm{o}}}}. \tag{4.7}$$

Proposition 4.8 (see [Brokemper 2018, 2.4.4]). (1) *The homomorphism $E \to L$, $x \mapsto \bar{x}$ induces a morphism*

$$G\text{-}\mathtt{Zip}^{\mu} = [E\backslash G] \to [L_{\varphi}\backslash G].$$

Using (4.7), on Chow rings this morphism induces an isomorphism

$$S^{W_{I^{\circ}}}/\mathcal{I}S^{W_{I^{\circ}}} \cong A^{\bullet}([L_{\varphi}\backslash G]) \cong A^{\bullet}(G\text{-}\mathtt{Zip}^{\mu}) \tag{4.9}$$

of graded $\mathbb{Q}$-algebras.

(2) *For the group scheme $E' = E \cap ({}^{z}B \times G)$, we have a natural identification*

$$G\text{-}\mathtt{ZipFlag}^{\mu} = [E\backslash(G \times P/{}^{z}B)] = [E'\backslash G]. \tag{4.10}$$

Under this identification, the homomorphism $E' \to T$, $(x, y) \mapsto \bar{x}$ induces a morphism

$$G\text{-}\mathtt{ZipFlag}^{\mu} \to [T_{\varphi}\backslash G].$$

Using (4.6), on Chow rings this morphism induces an isomorphism

$$S/\mathcal{I}S \cong A^{\bullet}([T_{\varphi}\backslash G]) \cong A^{\bullet}(G\text{-}\mathtt{ZipFlag}^{\mu}) \tag{4.11}$$

of graded $\mathbb{Q}$-algebras.

(3) *Under the isomorphisms (4.9) and (4.11), the homomorphism $\pi^{*} : S^{W_{I^{\circ}}}/\mathcal{I}S^{W_{I^{\circ}}} \to S/\mathcal{I}S$ induced on Chow rings by the projection $\pi : G\text{-}\mathtt{ZipFlag}^{\mu} \to G\text{-}\mathtt{Zip}^{\mu}$ is the one induced by the inclusion $S^{W_{I^{\circ}}} \hookrightarrow S$.*

Proof. The kernels of the surjective homomorphisms $E \to L$ and $E' \to T$ are unipotent. So (1) and (2) follow from Proposition 1.4. Then (3) follows from the compatibility of the various constructions. □

The above results allow us to give a noncanonical identification of $A^{\bullet}(G\text{-}\mathtt{Zip}^{\mu})$ with the rational cohomology ring of a certain flag variety. Let $\boldsymbol{G}_{\mathbb{C}}$ be the reductive group over $\mathbb{C}$ with the same based root datum as G_k, let $\boldsymbol{P}$ be a parabolic subgroup of type I of $\boldsymbol{G}_{\mathbb{C}}$, and set $\boldsymbol{X}^{\vee} := \boldsymbol{G}_{\mathbb{C}}/\boldsymbol{P}_I$. If (G, μ) is induced by a Shimura datum $(\boldsymbol{G}, \boldsymbol{X})$ (see Section 5 below), then $\boldsymbol{X}^{\vee}$ is the compact dual of $\boldsymbol{X}$. This explains the notation. Write

$$H^{2\bullet}(\boldsymbol{X}^{\vee}) := \bigoplus_{i=0}^{d} H^{2i}(\boldsymbol{X}^{\vee}(\mathbb{C}), \mathbb{Q})$$

to denote the cohomology ring of the complex manifold $\boldsymbol{X}^{\vee}(\mathbb{C})$ with rational coefficients. The multiplication is given by cup product. As the cohomology is concentrated in even degree, this is a commutative graded $\mathbb{Q}$-algebra.

Corollary 4.12. *There is an isomorphism of graded $\mathbb{Q}$-algebras $A^{\bullet}(G\text{-}\mathtt{Zip}^{\mu}) \cong H^{2\bullet}(\boldsymbol{X}^{\vee})$.*

Proof. We use the description of $\mathcal{I}$ as in Lemma 4.2. Then we have isomorphisms of graded $\mathbb{Q}$-algebras

$$A^{\bullet}(G\text{-}\mathtt{Zip}^{\mu}) \cong S^{W_{I^{o}}}/\mathcal{I}S^{W_{I^{o}}} \cong S^{W_I}/\mathcal{I}S^{W_I} \cong H^{2\bullet}(\boldsymbol{X}^{\vee}),$$

where the first isomorphism is given by (4.9), the second isomorphism is given by conjugation with the longest element w_0 in the Weyl group, and the third isomorphism holds by [Borel 1953, Theorem 26.1], identifying $X^\vee$ with a quotient of the real compact form of $G_{\mathbb{C}}$. $\square$

Remark 4.13. Recall that we work with $\mathbb{Q}$-coefficients, hence we may use the description of $\mathcal{I}$ in Lemma 4.2. But from the results of Brokemper it follows that the results of Proposition 4.8 even hold with $\mathbb{Z}$-coefficients if one uses the description (4.1) of $\mathcal{I}$ and if the group G is special, i.e., every étale G-torsor is already Zariski-locally trivial. Examples for special groups are GL_n, SL_n, GSp_{2n}, or Sp_{2n}. A nonexample would be PGL_n for $n \geq 2$.

By [Brokemper 2018, 2.4.10, 2.4.11], we also have the following result.

Proposition 4.14. *The ring $A^\bullet(G\text{-}\mathtt{Zip}^\mu)$ is a finite $\mathbb{Q}$-algebra of dimension $\#^I W$. A $\mathbb{Q}$-basis is given by the classes $[\bar{Z}_w]$ of the closures of the E-orbits on G.*

Even for special groups this result cannot be strengthened to integral coefficients, as the more precise description of the integral Chow ring of $G\text{-}\mathtt{Zip}^\mu$ given in [Brokemper 2018, 2.4.12] for $G = \mathrm{GL}_n$ shows. The examples calculated in the Section 8 below suggest that the index of the abelian group generated by the classes $[\bar{Z}_w]$ in the integral Chow ring of $G\text{-}\mathtt{Zip}^\mu$ is of the form $f_{R,\mu}(p)$ for a polynomial $f_{R,\mu} \in \mathbb{Z}[T]$ that depends only on the based root datum R of G with its automorphism given by Frobenius and the cocharacter μ.

4B. ***Pullback maps for the key diagram.*** We now apply $A^\bullet(-)$ as a contravariant functor to the key diagram. Recall that $\mathcal{I} = S_+^W$ is the augmentation ideal of S^W. We have $A^\bullet([P\backslash *]) = A^\bullet([L\backslash *]) = S^{W_{I^\circ}}$ by Proposition 1.4 and (2.1), and

$$A^\bullet(\mathtt{Brh}_G) = S \otimes_{S^W} S = (S \otimes_{\mathbb{Q}} S)/(1 \otimes f - f \otimes 1 \mid f \in \mathcal{I})$$

by Proposition 2.4. Using this, (4.9) and (4.11), we obtain the following commutative diagram of graded $\mathbb{Q}$-algebras by applying $A^\bullet(-)$ to (3.25):

$$\begin{tikzcd} S/\mathcal{I}S & S \otimes_{S^W} S \arrow[l, "\psi^*"'] \\ S^{W_{I^\circ}}/\mathcal{I}S^{W_{I^\circ}} \arrow[u, hook, "\pi^*"] & S^{W_{I^\circ}} \arrow[l, "\beta^*"] \arrow[u, "\gamma^*"'] \arrow[ul, "\alpha^*"'] \end{tikzcd} \tag{4.15}$$

Theorem 4.16. *The morphisms in (4.15) are as follows:*

(1) *The homomorphisms π^* and α^* are induced from the inclusion $S^{W_{I^\circ}} \hookrightarrow S$.*

(2) *The homomorphism β^* is the canonical projection.*

(3) *The homomorphism γ^* is the composition (using (4.3))*

$$\gamma^* : S^{W_{I^\circ}} = z(S^{W_I}) \xrightarrow{z^{-1}} S^{W_I} \xrightarrow{f \mapsto f \otimes 1} S \otimes_{S^W} S.$$

(4) *The homomorphism ψ^* is induced by*

$$f \otimes g \mapsto z(f)\varphi(g).$$

Proof. The description of π^* is given by Proposition 4.8. Since π^* is injective by Proposition 4.4, the descriptions of α^* and β^* will follow from those of ψ^* and γ^* since (4.15) commutes. The description of γ^* follows from the definition of γ and the construction of the isomorphism $A^\bullet(\mathtt{Brh}_G) \cong S \otimes_{S^W} S$.

To verify the description of ψ^*, we consider the following commutative diagram:

$$\begin{tikzcd} G\text{-}\mathtt{ZipFlag}^\mu = [E'\backslash G] \arrow[r, "\psi"] \arrow[d] & \mathtt{Brh}_G = [(B\times B)\backslash G] \arrow[d] \\ {[T_\varphi\backslash G]} \arrow[r] & {[(T\times T)\backslash G]} \end{tikzcd}$$

The morphisms in this diagram are given as follows: The morphism ψ is induced from $G \to G$, $g \mapsto z^{-1}g$ and $E' \to B \times B$, $(x, y) \mapsto ({}^{z^{-1}}x, x)$. Similarly the bottom horizontal morphism is induced from $G \to G$, $g \mapsto z^{-1}g$ and $T \to T \times T$, $t \mapsto ({}^{z^{-1}}t, \varphi(t))$. The left vertical morphism is the one from Proposition 4.8 and the right vertical one is induced from the identity on G and the projection $B \times B \to (B \times B)/\operatorname{rad}^u(B \times B) \cong T \times T$, where rad^u denotes the unipotent radical.

The two vertical morphisms induce isomorphisms on Chow rings. Using Proposition 4.4 one checks that the bottom horizontal morphism induces the morphism $S \otimes_{S^W} S \to S/IS$ which sends $f \otimes g$ to the class of $z(f)\varphi(g)$. This shows what we want. ☐

4C. *Description of π_*.* The morphism $\pi : G\text{-}\mathtt{ZipFlag}^\mu \to G\text{-}\mathtt{Zip}^\mu$, being a $P/{}^zB$-bundle, is proper. Hence under (4.9) and (4.11) it induces a pushforward morphism

$$\pi_* : A^\bullet(G\text{-}\mathtt{ZipFlag}^\mu) \cong S/\mathcal{I}S \to A^\bullet(G\text{-}\mathtt{Zip}^\mu) \cong S^{W_{I^o}}/\mathcal{I}S^{W_{I^o}}.$$

As an application of a general pushforward formula of Brion [1996], we get the following description of π_*:

Theorem 4.17. *The pushforward $\pi_* : S/\mathcal{I}S \to S^{W_{I^o}}/\mathcal{I}S^{W_{I^o}}$ sends the class of $f \in S$ to the class of*

$$\frac{\sum_{w\in W_{I^o}} (-1)^{\ell(w)} w(f)}{\prod_{\alpha\in\Phi^+_{I^o}} \alpha} \in S^{W_{I^o}}.$$

Proof. Consider the following cartesian diagram:

$$\begin{tikzcd} G\text{-}\mathtt{ZipFlag}^\mu \arrow[r, "\psi^\varnothing"] \arrow[d, "\pi"'] & \mathtt{Brh}^\varnothing \cong [{}^zB\backslash G/B] \cong [*/{}^zB] \times_{[*/G]} [*/B] \arrow[r, "\mathrm{pr}_1"] & {[*/{}^zB]} \arrow[d, "\tilde{\pi}"] \\ G\text{-}\mathtt{Zip}^\mu \arrow[rr] & & {[*/P]} \end{tikzcd}$$

On Chow rings this induces the following maps:

$$\begin{tikzcd} S/\mathcal{I}S \arrow[d, bend left, "\pi_*"] & S \arrow[l] \arrow[d, bend left, "\tilde{\pi}_*"] \\ S^{W_{I^o}}/\mathcal{I}S^{W_{I^o}} \arrow[u, bend left, "\pi^*"] & S^{W_{I^o}} \arrow[l] \arrow[u, bend left, "\tilde{\pi}^*"] \end{tikzcd}$$

Hence it suffices to prove the corresponding formula for $\tilde{\pi}_* : A^\bullet([*/{}^zB]) \cong S \to A^\bullet([*/P]) \cong S^{W_{I^\circ}}$. Similarly, using the cartesian diagram

$$\begin{tikzcd} {[*/{}^zB \cap L]} \arrow[r] \arrow[d, "\tilde{\tilde{\pi}}"'] & {[*/{}^zB]} \arrow[d, "\tilde{\pi}"'] \\ {[*/L]} \arrow[r] & {[*/P]} \end{tikzcd}$$

whose horizontal morphisms induce isomorphisms on Chow groups, one reduces to proving the corresponding formula for $\tilde{\tilde{\pi}}_*$. This formula is given by [Brion 1996, Proposition 1.1]. □

The following gives an alternative way of computing the expression in Theorem 4.17:

Lemma 4.18 [Demazure 1973, Lemme 4]. *For $f \in S$, we have*

$$\delta_{w_{0,I^\circ}}(f) = \frac{\sum_{w\in W_{I^\circ}} (-1)^{\ell(w)} w(f)}{\prod_{\alpha\in\Phi^+_{I^\circ}} \alpha},$$

where $\delta_{w_{0,I^\circ}} : S \to S$ is the operator associated to the longest element w_{0,I° of W_{I° by Theorem 2.16.

So the following diagram is commutative:

$$\begin{tikzcd} S \arrow[r, "\delta_{w_{0,I^\circ}}"] \arrow[d] & S \arrow[d] \\ S/\mathcal{I}S \arrow[r, "\pi_*"'] & S^{W_{I^\circ}}/\mathcal{I}S^{W_{I^\circ}} \end{tikzcd}$$

From Proposition 3.13, we now get the following:

Proposition 4.19. *For $w \in {}^IW$, we have $[\bar{Z}_w] = \gamma(w)\pi_*([\bar{Z}^\varnothing_w])$ in $A^\bullet(G\text{-}\mathtt{Zip}^\mu)$.*

4D. ***Computing the cycle classes of the Ekedahl–Oort strata on G-$\mathtt{Zip}^\mu$.*** By putting together the above results we get the following procedure for computing the classes $[\bar{Z}_w]$ in $A^\bullet(G\text{-}\mathtt{Zip}^\mu)$ for $w \in {}^IW$:

For computations, it is convenient to replace the rings appearing in the diagram (4.15) with certain simpler rings mapping surjectively onto them. For this we consider the following diagram of graded algebras, in which all rings are either polynomial rings or subrings of polynomial rings:

$$\begin{tikzcd} S & S \otimes_{\mathbb{Q}} S \arrow[l, "\tilde{\psi}^*"'] \\ S^{W_{I^\circ}} \arrow[u, hook, "\tilde{\pi}^*"'] & S^{W_{I^\circ}} \arrow[l, "\tilde{\beta}^*"] \arrow[u, "\tilde{\gamma}^*"'] \end{tikzcd} \tag{4.20}$$

Here we define the homomorphisms as follows:

(i) The homomorphism $\tilde{\pi}^*$ is the inclusion $S^{W_{I^\circ}} \hookrightarrow S$.

(ii) The homomorphism $\tilde{\beta}^*$ is the identity.

(iii) The homomorphism $\tilde{\gamma}^*$ is the composition

$$\tilde{\gamma}^* : S^{W_{I^o}} = z(S^{W_I}) \xrightarrow{z^{-1}} S^{W_I} \xrightarrow{f \mapsto f \otimes 1} S \otimes_{S^W} S.$$

(iv) The homomorphism $\tilde{\psi}^*$ is given by

$$f \otimes g \mapsto z(f)\varphi(g).$$

Using Theorem 4.16 one readily checks that under the canonical surjections from the objects in the diagram (4.20) to the corresponding objects in the diagram (4.15) these two diagrams are compatible. Similarly, using Theorem 4.17, one checks that the morphism $\pi_* : S/\mathcal{I}S \to S^{W_{I^o}}/\mathcal{I}S^{W_{I^o}}$ lifts to a morphism $\tilde{\pi}_* : S \to S^{W_{I^o}}$ given by the formula from Theorem 4.17.

In the following, for a class c in one of the algebras of (4.15), we will refer to a lift of c to the corresponding algebra in (4.20) as a formula for c. Then, for $w \in {}^I W$, we can compute a formula for $[\bar{Z}_w]$ as follows:

(i) Using the results from Section 2C one finds a formula for the class of the diagonal $\mathtt{Brh}_e$ in $S \otimes S$.

(ii) The operator δ_w on $S \otimes_{S^W} S$ from Section 2E lifts to an operator on $S \otimes S$ by letting the operator δ_w on S from Section 2E act on the first factor of $S \otimes S$. Then, by Corollary 2.21, by applying this operator δ_w to a formula for $[\mathtt{Brh}_e]$ one gets a formula for the class $[\overline{\mathtt{Brh}}_w]$.

(iii) By the definition of the subscheme $Z_w^{\varnothing}$ of G-$\mathtt{ZipFlag}^{\mu}$, the image of a formula for $[\overline{\mathtt{Brh}}_w]$ under the homomorphism $\tilde{\psi}^*$ gives a formula for the class $[\bar{Z}_w^{\varnothing}]$.

(iv) By applying $\tilde{\pi}_*$ to a formula for $[\bar{Z}_w^{\varnothing}]$ one gets a formula for $\pi_*([\bar{Z}_w^{\varnothing}])$.

(v) Using the results from Section 3F one computes the number $\gamma(w)$.

(vi) Using Proposition 3.13 by multiplying the results of the previous two steps we get a formula for $[\bar{Z}_w] = \gamma(w)\pi_*([\bar{Z}_w^{\varnothing}])$.

4E. ***Functoriality in the zip datum.*** To simplify notation it is often convenient for the computations in Section 4D to replace G by some other group $\widetilde{G}$. Here we explain that this is harmless as long as G and $\widetilde{G}$ have the same adjoint group.

Let (G, μ) and $(\widetilde{G}, \tilde{\mu})$ be two pairs consisting of a reductive group over $\mathbb{F}_p$ and a cocharacter defined over the algebraic closure k of $\mathbb{F}_p$. Let

$$f : G \to \widetilde{G}$$

be a map of algebraic groups over $\mathbb{F}_p$ with $f \circ \mu = \tilde{\mu}$. Let κ (resp. $\tilde{\kappa}$) be the field of definition of the conjugacy class of μ (resp. of $\tilde{\mu}$). Then $\tilde{\kappa} \subseteq \kappa$.

Let P and Q be the parabolics and E the zip group attached to (G, μ) as in Section 3A. Let $\widetilde{P}$, $\widetilde{Q}$ and $\widetilde{E}$ be the parabolics and zip group attached similarly to $(\widetilde{G}, \tilde{\mu})$. Then f induces maps $P \to \widetilde{P}$, $Q \to \widetilde{Q}$, and $E \to \widetilde{E}$ and hence a morphism

$$[f] : G\text{-}\mathtt{Zip}^{\mu} \to \widetilde{G}\text{-}\mathtt{Zip}^{\tilde{\mu}} \otimes_{\tilde{\kappa}} \kappa$$

of smooth algebraic quotient stacks over κ. Every map $f : G \to \widetilde{G}$ of algebraic groups can be factorized into the composition of a faithfully flat map $G \to G' = G/\operatorname{Ker}(f)$ and a closed embedding $G' \to \widetilde{G}$. If G is reductive, then G' is reductive. Therefore the following lemma implies, in particular, that the pullback $[f]^*$ on Chow rings exists.

Lemma 4.21. (1) *If f is flat, then $[f]$ is flat.*

(2) *If f is a monomorphism, then $[f]$ is representable.*

Proof. This follows from Lemma 1.2 because if f is a monomorphism, then the induced map $E \to \widetilde{E}$ is also a monomorphism. □

Lemma 4.22. *Suppose that f induces an isomorphism of adjoint groups $G^{\mathrm{ad}} \xrightarrow{\sim} \widetilde{G}^{\mathrm{ad}}$.*

(1) *Let $\widetilde{Z}$ be the radical of $\widetilde{G}$. Let (T, B, z) be a frame as in Section 3B for (G, μ). Set $\widetilde{T} := \widetilde{Z} f(T)$ and $\widetilde{B} := \widetilde{Z} f(B)$. Then $(\widetilde{T}, \widetilde{B}, f(z))$ is a frame for $(\widetilde{G}, \tilde{\mu})$.*

(2) *The map f induces an isomorphism $W \xrightarrow{\sim} \widetilde{W}$ of the Weyl groups with their set of simple reflections attached to (G, B, T) and $(\widetilde{G}, \widetilde{B}, \widetilde{T})$, respectively.*

(3) *The morphism $[f]$ of algebraic stacks induces a homeomorphism of the underlying topological spaces.*

Proof. The hypothesis on f means that $\operatorname{Ker}(f)$ is central and that $\operatorname{Cent}(\widetilde{G}) f(G) = \widetilde{G}$. As $\widetilde{Z} f(G)$ is of finite index in $\operatorname{Cent}(\widetilde{G}) f(G)$ and $\widetilde{G}$ is connected, this implies $\widetilde{Z} f(G) = \widetilde{G}$. As $\widetilde{Z}$ is a torus and clearly commutes with $f(T)$, $\widetilde{T} := \widetilde{Z} f(T)$ is a torus. Its dimension is the reductive rank of $\widetilde{G}$. Hence it is a maximal torus. By hypothesis, f induces a bijection between the roots of (G, T) and of $(\widetilde{G}, \widetilde{T})$. This shows that $\widetilde{Z} f(B)$ is a Borel subgroup and that f induces an isomorphism $W \xrightarrow{\sim} \widetilde{W}$. This implies all remaining assertions. □

We continue to assume that f induces an isomorphism of adjoint groups $G^{\mathrm{ad}} \xrightarrow{\sim} \widetilde{G}^{\mathrm{ad}}$ and use the notation of the lemma. We identify W with $\widetilde{W}$ via the isomorphism induced by f.

We define $\widetilde{G}\text{-}\mathtt{ZipFlag}^{\tilde{\mu}}$ and $\mathtt{Brh}_{\widetilde{G}}$ using $(\widetilde{G}, \widetilde{P}, \widetilde{Q}, \widetilde{B})$. Then using the description of $G\text{-}\mathtt{ZipFlag}^{\mu}$ given in (4.10) one sees that f also induces a map $[\tilde{f}]$ on stacks of flagged G-zips making the diagram

$$\begin{tikzcd} G\text{-}\mathtt{ZipFlag}^{\mu} \arrow[r, "{[\tilde{f}]}"] \arrow[d] & \widetilde{G}\text{-}\mathtt{ZipFlag}^{\tilde{\mu}} \otimes_{\tilde{\kappa}} \kappa \arrow[d] \\ G\text{-}\mathtt{Zip}^{\mu} \arrow[r, "{[f]}"] & \widetilde{G}\text{-}\mathtt{Zip}^{\tilde{\mu}} \otimes_{\tilde{\kappa}} \kappa \end{tikzcd}$$

commute. Moreover, the same arguments as above show that the pullback $[\tilde{f}]^*$ on Chow rings exists.

The key diagram (3.25) and the corresponding diagram of Chow rings (4.15) is functorial for f. The induced map of $\mathbb{Q}$-algebras $\widetilde{S} := \operatorname{Sym}(X^*(\widetilde{T})_{\mathbb{Q}}) \to S$ is equivariant for the action of W. More precisely, if we choose splittings of the exact sequences of tori

$$1 \to \operatorname{Ker}(f)^0 \to T \to T/\operatorname{Ker}(f)^0 \to 1 \quad \text{and} \quad 1 \to f(T) \to \widetilde{T} \to \widetilde{T}/f(T) \to 1,$$

then $\widetilde{S} \to S$ is of the form

$$\widetilde{S} \twoheadrightarrow \mathrm{Sym}(X^*(f(T))_{\mathbb{Q}}) \xrightarrow{\sim} \mathrm{Sym}(X^*(T/\operatorname{Ker}(f)^0)_{\mathbb{Q}}) \hookrightarrow S,$$

where the second map is an isomorphism of $\mathbb{Q}$-algebras with W-action. The map $\widetilde{S} \to S$ is also equivariant for the action of the Frobenius because f is defined over $\mathbb{F}_p$.

Proposition 4.23. *Let $f : G \to \widetilde{G}$ be a map of algebraic groups defined over $\mathbb{F}_p$ that induces an isomorphism on adjoint groups.*

(1) *One has a commutative diagram of $\mathbb{Q}$-linear maps*

$$\begin{tikzcd} A^\bullet(\widetilde{G}\text{-}\mathtt{ZipFlag}^{\tilde{\mu}}) \arrow[r] \arrow[d, "\pi_*"'] & A^\bullet(G\text{-}\mathtt{ZipFlag}^{\mu}) \arrow[d, "\pi_*"] \\ A^\bullet(\widetilde{G}\text{-}\mathtt{Zip}^{\tilde{\mu}}) \arrow[r, "\sim"] & A^\bullet(G\text{-}\mathtt{Zip}^{\mu}) \end{tikzcd} \tag{4.24}$$

where the horizontal maps are the maps of $\mathbb{Q}$-algebras induced by f. The lower horizontal map is an isomorphism.

(2) *For $w \in {}^IW$, the numbers $\gamma(w)$ defined in Definition 3.14 for (G, μ) coincide with those defined for $(\widetilde{G}, \tilde{\mu})$.*

Proof. Under the identifications (4.9) and (4.11) the horizontal maps are both induced by the W-equivariant map $\widetilde{S} \to S$. Hence the commutativity of (4.24) follows from the concrete description of π_* in Theorem 4.17. From Proposition 4.14 and Lemma 4.22(3), we also deduce that the lower horizontal map sends a $\mathbb{Q}$-basis to a $\mathbb{Q}$-basis. In particular, it is an isomorphism.

Let us show (2). The upper horizontal map sends for all $w \in W$ the cycle $[\bar{Z}_w^{\varnothing}]$ defined for $(\widetilde{G}, \tilde{\mu})$ to the cycle $[\bar{Z}_w^{\varnothing}]$ defined for (G, μ) because ψ^* is functorial for f. Hence (2) follows from (1) and Proposition 4.19. ☐

Remark 4.25. One can also show that the classes $[\bar{Z}_w^{\varnothing}]$ for $w \in W$ form a basis of $A^\bullet(G\text{-}\mathtt{ZipFlag}^{\mu})$, and that hence the upper horizontal map in (4.24) is an isomorphism as well.

5. The tautological ring of a Shimura variety

5A. ***Automorphic bundles and the tautological ring in characteristic zero.*** Let $(\boldsymbol{G}, \boldsymbol{X})$ be a Shimura datum. Recall that this means that $\boldsymbol{G}$ is a connected reductive group over $\mathbb{Q}$, that $\boldsymbol{X}$ is a $\boldsymbol{G}(\mathbb{R})$-conjugacy class of homomorphisms $h : \mathbb{S} \to \boldsymbol{G}_{\mathbb{R}}$ of real algebraic groups, where $\mathbb{S} := \operatorname{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{m,\mathbb{C}}$ is $\mathbb{C}^\times$ viewed as a real algebraic group, and that the pair $(\boldsymbol{G}, \boldsymbol{X})$ satisfies a list of axioms [Deligne 1979, 2.1.1].

For $h \in \boldsymbol{X}$, let μ_h be the associated cocharacter of $\boldsymbol{G}_{\mathbb{C}}$, i.e., μ_h is the restriction of

$$h_{\mathbb{C}} : \mathbb{S}_{\mathbb{C}} = \prod_{\mathrm{Gal}(\mathbb{C}/\mathbb{R})} \mathbb{G}_{m,\mathbb{C}} \to G_{\mathbb{C}}$$

to the factor indexed by $\mathrm{id} \in \mathrm{Gal}(\mathbb{C}/\mathbb{R})$. For each faithful finite-dimensional representation $\rho : \boldsymbol{G}_{\mathbb{R}} \to \mathrm{GL}(V)$

of $\boldsymbol{G}$ over $\mathbb{R}$, the Hodge filtration induced by $\rho \circ h$ on V has as stabilizer the parabolic subgroup $P_{-}(\rho \circ \mu_h)$ of $\mathrm{GL}(V_{\mathbb{C}})$ (here we follow the normalizations of [Deligne 1979] using negative μ_h-weights). The $\boldsymbol{G}(\mathbb{C})$-conjugacy class of μ_h has as field of definition a finite extension E of $\mathbb{Q}$, called the reflex field.

Let $\boldsymbol{X}^\vee$ be the compact dual of $\boldsymbol{X}$. Then $\boldsymbol{X}^\vee = \mathrm{Par}_{\boldsymbol{G}_{\mathbb{C}}, \mu_h^{-1}}$ is the scheme of parabolic subgroups of type μ_h^{-1}. It is a projective homogeneous $\boldsymbol{G}$-space and it is defined over E.

For each neat open compact subgroup K of $\boldsymbol{G}(\mathbb{A}_f)$, we denote by $\boldsymbol{S}_K := \mathrm{Sh}_K(\boldsymbol{G}, \boldsymbol{X})$ the canonical model of the attached Shimura variety at level K. This is a smooth quasiprojective scheme over E.

Denote by $\boldsymbol{G}^c$ the quotient of $\boldsymbol{G}$ by the maximal $\mathbb{Q}$-anisotropic $\mathbb{R}$-split torus in the center of $\boldsymbol{G}$. For instance, if $(\boldsymbol{G}, \boldsymbol{X})$ is of Hodge type, then $\boldsymbol{G} = \boldsymbol{G}^c$ and this is the only case that we will use later. But for future reference we explain the following notions and results in full generality. And in general $\boldsymbol{G} \neq \boldsymbol{G}^c$, for instance, if $\boldsymbol{G} = \mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_{2,F}$ for a nontrivial totally real extension F of $\mathbb{Q}$. The action of $\boldsymbol{G}_E$ on the E-scheme $\boldsymbol{X}^\vee$ factors through $\boldsymbol{G}^c_E$.

Milne [1990, III] constructs a diagram of schemes defined over E

$$\begin{tikzcd} & \widetilde{\boldsymbol{S}}_K \arrow[dl, "\pi"'] \arrow[dr, "\tilde{\sigma}"] & \\ \boldsymbol{S}_K & & \boldsymbol{X}^\vee \end{tikzcd} \tag{5.1}$$

where π is a $\boldsymbol{G}^c_E$-torsor and $\tilde{\sigma}$ is $\boldsymbol{G}_E$-equivariant. We set

$$\mathtt{Hdg}_E := [\boldsymbol{G}^c_E \backslash \boldsymbol{X}^\vee], \tag{5.2}$$

which is an algebraic stack over E. The diagram (5.1) corresponds to a morphism of algebraic stacks

$$\sigma : \boldsymbol{S}_K \to \mathtt{Hdg}_E \tag{5.3}$$

making

$$\begin{tikzcd} \widetilde{\boldsymbol{S}}_K \arrow[r, "\tilde{\sigma}"] \arrow[d, "\pi"'] & \boldsymbol{X}^\vee \arrow[d] \\ \boldsymbol{S}_K \arrow[r, "\sigma"] & \mathtt{Hdg}_E \end{tikzcd}$$

cartesian.

Let $\boldsymbol{S}^{\mathrm{tor}}_K$ be a smooth toroidal compactification of $\boldsymbol{S}_K$. Then by [Milne 1990, V, Theorem 6.1] the morphism σ canonically extends to a morphism

$$\sigma^{\mathrm{tor}} : \boldsymbol{S}_K \to \mathtt{Hdg}_E.$$

Note that a vector bundle on the quotient stack $\mathtt{Hdg}_E = [\boldsymbol{G}^c_E \backslash \boldsymbol{X}^\vee]$ is the same as a $\boldsymbol{G}^c$-equivariant vector bundle on $\boldsymbol{X}^\vee$.

Definition 5.4. Let E' be an extension of E. A vector bundle $\mathcal{E}$ on $\boldsymbol{S}_{K,E'}$ (resp. on $\boldsymbol{S}^{\mathrm{tor}}_{K,E'}$) is called an *automorphic bundle* if there exists a vector bundle $\mathscr{E}$ on $[\boldsymbol{G}^c_E \backslash \boldsymbol{X}^\vee]_{E'}$ such that $\mathcal{E} \cong \sigma^*(\mathscr{E})$ (resp. such that $\mathcal{E} \cong (\sigma^{\mathrm{tor}})^*(\mathscr{E})$). Moreover, $(\sigma^{\mathrm{tor}})^*(\mathscr{E})$ is called the *canonical extension* of $\sigma^*(\mathscr{E})$.

Remark 5.5. Suppose that $\boldsymbol{X}^\vee(E') \neq \varnothing$, i.e., there exists a parabolic subgroup $\boldsymbol{P}$ of $\boldsymbol{G}_{E'}$ of type μ^{-1} that is defined over E'. Then the choice of $\boldsymbol{P}$ yields isomorphisms $\boldsymbol{X}^\vee_{E'} \cong \boldsymbol{G}_{E'}/\boldsymbol{P} \cong \boldsymbol{G}^c_{E'}/\boldsymbol{P}^c$, where $\boldsymbol{P}^c$ is the image of $\boldsymbol{P}$ in $\boldsymbol{G}^c_{E'}$. We obtain an isomorphism

$$\mathtt{Hdg}_{E'} \cong [\boldsymbol{P}^c\backslash *].$$

Hence in this case a vector bundle $\mathscr{E}$ on $[\boldsymbol{G}^c_E\backslash \boldsymbol{X}^\vee]_{E'}$ is the same as a finite-dimensional representation (V,η) of $\boldsymbol{P}^c$ over E', and $\sigma^*(\mathscr{E})$ is the automorphic bundle attached to the representation η.

The structure morphism $\boldsymbol{X}^\vee \to \operatorname{Spec} E$ induces a morphism of algebraic stacks

$$\tau : \mathtt{Hdg}_E \to [\boldsymbol{G}^c_E\backslash *].$$

Definition 5.6. Let E' be an extension of E. An automorphic vector bundle on $\boldsymbol{S}_{K,E'}$ (resp. on $\boldsymbol{S}^{\mathrm{tor}}_{K,E'}$) is called *flat* if it is isomorphic to a vector bundle obtained by pullback via $\sigma \circ \tau$ (resp. via $\sigma^{\mathrm{tor}} \circ \tau$) from a vector bundle on $[\boldsymbol{G}^c_{E'}\backslash *]$.

In other words, flat automorphic bundles are those given by representations of G^c. They are endowed with a canonical integrable connection which in the Hodge case is the Gauss–Manin connection.

Definition 5.7. Let E' be a field extension of E. The images of $A^\bullet(\mathtt{Hdg}_{E'})$ in $A^\bullet(\boldsymbol{S}_{K,E'})$ and in $A^\bullet(\boldsymbol{S}^{\mathrm{tor}}_{K,E'})$ are called the *tautological rings* of $\boldsymbol{S}_{K,E'}$ and of $\boldsymbol{S}^{\mathrm{tor}}_{K,E'}$, respectively. They are denoted by $\mathcal{T}_{E'}$ and $\mathcal{T}^{\mathrm{tor}}_{E'}$, respectively.

Remark 5.8. Let E' be a field extension of E, and let E'' be a Galois extension of E' with Galois group Γ. Then Γ acts on $\mathcal{T}_{E''}$ and one has $(\mathcal{T}_{E''})^\Gamma = \mathcal{T}_{E'}$ by Proposition 1.3.

In particular, assume that the reductive group $\boldsymbol{G}$ splits over E''. Then we can choose $\boldsymbol{P} \in \boldsymbol{X}^\vee(E'')$ and

$$A^\bullet(\mathtt{Hdg}_{E''}) \cong A^\bullet([\boldsymbol{P}\backslash *]) = A^\bullet([\boldsymbol{L}\backslash *]) \cong \operatorname{Sym}(X^*(\boldsymbol{T})_{\mathbb{Q}})^{W_{\boldsymbol{L}}},$$

where $\boldsymbol{L}$ is the Levi quotient of $\boldsymbol{P}$, $\boldsymbol{T} \subseteq \boldsymbol{L}$ a maximal torus and $W_{\boldsymbol{L}}$ the Weyl group of $\boldsymbol{L}$. Hence $\mathcal{T}_{E'}$ is a quotient of $\operatorname{Sym}(X^*(\boldsymbol{T})_{\mathbb{Q}})^{\Gamma \ltimes W_{\boldsymbol{L}}}$.

Example 5.9. In the Siegel case, we have $\boldsymbol{G} = \boldsymbol{G}^c = \mathrm{GSp}_{2g}$ and $\boldsymbol{P}$ is a Siegel parabolic subgroup, i.e., the stabilizer of some Lagrangian subspace. We identify $\boldsymbol{S}_K$ with the moduli space of principally polarized abelian varieties of dimension g endowed with some sufficiently fine level structure. Let $f : \mathcal{A} \to \boldsymbol{S}_K$ be the universal abelian scheme over $\boldsymbol{S}_K$.

The Hodge stack $\mathtt{Hdg} = [\boldsymbol{P}\backslash *]$ parametrizes in this case vector bundles together with a symplectic pairing that has values in some line bundle and Lagrangian subbundles. The morphism σ is the classifying map of the de Rham cohomology of $\mathcal{A}$ and its Hodge filtration where the pairing is induced by the principal polarizations.

The projection of $\boldsymbol{P}$ onto its Levi quotient $\boldsymbol{L}$ yields an isomorphism $A^\bullet(\mathtt{Hdg}) \cong A^\bullet([\boldsymbol{L}\backslash *])$. There is an isomorphism $\boldsymbol{L} \cong \mathrm{GL}_g \times \mathbb{G}_m$ for which the projection $\mathrm{GL}_g \times \mathbb{G}_m \to \mathrm{GL}_g$ yields a vector bundle $\Omega^\flat$ on $[\boldsymbol{L}\backslash *]$ whose pullback to $\boldsymbol{S}_K$ is the Hodge filtration bundle $f_*\Omega^1_{\mathcal{A}/\boldsymbol{S}_K}$, and for which the projection $\boldsymbol{L} \to \mathbb{G}_m$ is the restriction of the multiplier character of G and hence the pullback of the corresponding line bundle to S_K is trivial. Therefore in this case the tautological ring is the $\mathbb{Q}$-subalgebra generated by the Chern classes of $f_*\Omega^1_{\mathcal{A}/S_K}$ and our notion agrees with the one introduced in [Ekedahl and van der Geer 2009].

5B. ***Stacks of filtered fiber functors.*** In Remark 5.5 we explained that $\mathtt{Hdg}_{E'}$ is the classifying stack of a certain parabolic subgroup $\boldsymbol{P}^c$ of $\boldsymbol{G}^c$ if such a subgroup can be defined over E'. In this case, $\mathtt{Hdg}_{E'}$ simply classifies $\boldsymbol{P}^c$-torsors. In this subsection we briefly digress to give a moduli-theoretic description of $\mathtt{Hdg}$ in general. This will not be needed in the rest of the article.

Hence, for the moment, let k be any field, let G be a reductive group over k, and let λ be a cocharacter of G defined over some field extension k' of k. Suppose that the conjugacy class of λ is defined over k or, equivalently, that the scheme Par_λ of parabolic subgroups of type λ is defined over k. The reductive group scheme G acts on Par_λ, and we consider the quotient stack

$$\mathtt{Hdg}_{G,\lambda} := [G\backslash \mathrm{Par}_\lambda].$$

Clearly, $\mathtt{Hdg}_{G,\lambda}$ is a smooth algebraic stack over k.

Denote by $\mathtt{Rep}(G)$ the k-linear abelian rigid $\otimes$-category of finite-dimensional representations of G over k. For any k-scheme T, we denote by $\mathtt{FilLF}(T)$ the exact rigid tensor category of filtered finite locally free $\mathscr{O}_T$-modules [Pink et al. 2015, 4C].

Proposition 5.10. *The stack $\mathtt{Hdg}_{G,\lambda}$ is canonically equivalent to the stack $\mathcal{F}_\lambda$ sending a k-scheme T to the groupoid $\mathcal{F}_\lambda(T)$ of exact k-linear $\otimes$-functors* $\mathtt{Rep}(G) \to \mathtt{FilLF}(T)$ *of type* λ (see [Pink et al. 2015, 5.3]).

Proof. First we construct a canonical morphism $\mathcal{F}_\lambda \to \mathtt{Hdg}_{G,\lambda}$ as follows: Let T be a k-scheme and $\varphi : \mathtt{Rep}(G) \to \mathtt{FilLF}(T)$ be an exact k-linear tensor functor of type λ. Similarly, let $\varphi_\lambda : \mathtt{Rep}(G) \to \mathtt{FilLF}(\mathrm{Spec}(k'))$ be the exact k-linear tensor functor induced by the cocharacter λ. Then, by definition, the fact that φ is of type λ means that there exists an fpqc covering T' of $T_{k'}$ over which the functors φ and φ_λ become isomorphic. The group of automorphisms of $\varphi_{\lambda,T'}$ is $P_+(\lambda)_{T'}$. Hence the sheaf $\underline{\mathrm{Isom}}^{\otimes}(\varphi_{\lambda,T'}, \varphi_{T'})$ of tensor isomorphisms $\varphi_{\lambda,T'} \to \varphi_{T'}$ is a right $P_+(\lambda)_{T'}$-torsor over T'. Thus under the canonical isomorphism $[P_+(\lambda)_{T'}\backslash *] \cong (\mathtt{Hdg}_{G,\lambda})_{T'}$ noted above we obtain an object $\mathcal{P}_\varphi$ of $\mathtt{Hdg}_{G,\lambda}(T')$. Since φ is defined over T, there is a canonical descent datum for T'/T on $\varphi_{T'}$. This descent datum induces an analogous descent datum on $\mathcal{P}_\varphi$, so that $\mathcal{P}_\varphi$ descends canonically to an object of $\mathtt{Hdg}_{G,\lambda}(T)$. Finally one checks that the assignment $\varphi \mapsto \mathcal{P}_\varphi$ naturally extends to a morphism of groupoids $\mathcal{F}_\lambda(T) \to \mathtt{Hdg}_{G,\lambda}(T)$ and that for varying T these morphisms are compatible with base change.

To check that the morphism $\mathcal{F}_\lambda \to \mathtt{Hdg}_{G,\lambda}$ is an isomorphism of stacks we may work fpqc-locally on $\mathrm{Spec}(k)$. Hence we may assume that λ is defined over k. Then, under the above isomorphism $\mathtt{Hdg}_{G,\lambda} \cong [P_+(\lambda)\backslash *]$, the claim is given by [Pink et al. 2015, Theorem 5.6]. □

5C. ***The tautological ring in positive characteristic.*** From now on we assume that the Shimura datum $(\boldsymbol{G}, \boldsymbol{X})$ is of Hodge type. Then $\boldsymbol{G} = \boldsymbol{G}^c$. Let p be a prime of good reduction, i.e., there exists a reductive group scheme $\mathscr{G}$ over $\mathbb{Z}_p$ such that $\mathscr{G}_{\mathbb{Q}_p} = \boldsymbol{G}_{\mathbb{Q}_p}$. We fix a neat level structure $K = K^pK_p \subseteq \boldsymbol{G}(\mathbb{A}_f)$ with $K^p \subseteq \boldsymbol{G}(\mathbb{A}_f^p)$ compact open and $K_p = \mathscr{G}(\mathbb{Z}_p) \subseteq \boldsymbol{G}(\mathbb{Q}_p)$ hyperspecial.

Integral models. We fix a place v of the reflex field E over p and denote by E_v the v-adic completion of E. As $\boldsymbol{G}_{\mathbb{Q}_p}$ has a reductive model over $\mathbb{Z}_p$, E_v is an unramified extension $\mathbb{Q}_p$. Let $\mathscr{S}_K$ be the canonical smooth integral model of $\boldsymbol{S}_K$ over the ring of integers O_{E_v} defined by Kisin [2010] and Vasiu [1999] for $p > 2$ and by Kim and Madapusi Pera [2016] for $p = 2$. Let $\breve{\mathbb{Q}}_p$ be the completion of a maximal unramified extension of E_v, and let k be the residue field of the ring of integers of $\breve{\mathbb{Q}}_p$. Let κ be the residue field of O_{E_v}. Then k is an algebraic closure of κ. Let S_K be the special fiber of $\mathscr{S}_K$ over κ, and let G be the special fiber of $\mathscr{G}$. Hence G is a reductive group over $\mathbb{F}_p$.

By definition, E is the field of definition of the conjugacy class of μ_h. As conjugacy classes of cocharacters depend only on the root datum of the reductive group, we can view the $\boldsymbol{G}(\mathbb{C})$-conjugacy class of μ_h also as a $\boldsymbol{G}(\breve{\mathbb{Q}}_p)$-conjugacy $[\mu_h]_{\breve{\mathbb{Q}}_p}$ of cocharacters of $\boldsymbol{G}_{\breve{\mathbb{Q}}_p}$ because $\boldsymbol{G}_{\mathbb{Q}_p}$ splits over an unramified extension. We may also view it as a $G(k)$-conjugacy class $[\mu_h]_k$ of cocharacters of G_k. The field of definition of $[\mu_h]_{\breve{\mathbb{Q}}_p}$ is E_v and the field of definition of $[\mu_h]_k$ is κ.

As $\mathscr{G}$ and G are quasisplit, we may choose an element in $[\mu_h]_{\breve{\mathbb{Q}}_p}$ which extends to a cocharacter μ of $\mathscr{G}$ defined over O_{E_v}. We also denote by μ its reduction modulo p, a cocharacter of G_κ. As μ_h is minuscule, so is μ.

Arithmetic compactifications. We recall some results on integral compactifications by Madapusi Pera [2019]. Let $\mathscr{S}_K^{\mathrm{tor}}$ be some smooth proper toroidal compactification of the integral model $\mathscr{S}_K$. It depends on the choice of a smooth, finite, admissible rational polyhedral cone decomposition. Moreover, let $\mathscr{S}_K^{\mathrm{min}}$ be the minimal compactification of $\mathscr{S}_K$, and let

$$\pi : \mathscr{S}_K^{\mathrm{tor}} \to \mathscr{S}_K^{\mathrm{min}}$$

be the canonical morphism. It is constructed as the Stein factorization of a certain proper morphism [Madapusi Pera 2019, 5.2.1]. In particular, π is proper and has geometrically connected fibers, and for every line bundle $\mathscr{L}$ on $\mathscr{S}_K^{\mathrm{min}}$, one has a canonical isomorphism

$$\mathscr{L} \xrightarrow{\sim} \pi_*\pi^*\mathscr{L}. \tag{5.11}$$

We denote the special fibers over κ of $\mathscr{S}_K^{\mathrm{tor}}$ and of $\mathscr{S}_K^{\mathrm{min}}$ by S_K^{tor} and S_K^{min}, respectively. The restriction of π to the special fibers is again denoted by π.

Recall that a morphism $f : X \to Y$ of finite type between noetherian schemes is called *normal* if it is flat and has geometrically normal fibers. This notion is stable under base change $Y' \to Y$ and if Y is normal and $X \to Y$ is normal, then X is normal.

Lemma 5.12. *The minimal compactification $\mathscr{S}_K^{\mathrm{min}}$ is normal over O_{E_v}.*

Proof. For Shimura varieties of PEL type this is shown in [Lan 2013, 7.2.4.3] using the description of completed local rings of $\mathscr{S}_K^{\mathrm{min}}$ in geometric points [Lan 2013, 7.2.3.17]. But the same description also holds for the minimal compactification for Shimura varieties of Hodge type by [Madapusi Pera 2019, 5.2.8]. □

The tautological ring in characteristic p. The cocharacter $\mu : \mathbb{G}_{m,O_{E_v}} \to \mathscr{G}_{O_{E_v}}$ defines a parabolic subgroup $\mathscr{P} := P_-(\mu)$ of $\mathscr{G}_{O_{E_v}}$, and we set

$$\mathtt{Hdg}_{O_{E_v}} := [\mathscr{P}\backslash *].$$

Then $\mathtt{Hdg}_{O_{E_v}} \otimes_{O_{E_v}} E_v = \mathtt{Hdg}_{E_v}$ by Remark 5.5. We denote by $P := \mathscr{P}_\kappa$ the special fiber of $\mathscr{P}$ which is a parabolic subgroup of G_κ. Then we have

$$\mathtt{Hdg}_\kappa := \mathtt{Hdg}_{O_{E_v}} \otimes_{O_{E_v}} \kappa = [P\backslash *].$$

By [Madapusi Pera 2019, 5.3], the morphisms σ and σ^{tor} extend to morphisms

$$\sigma : \mathscr{S}_K \to \mathtt{Hdg}_{O_{E_v}} \quad \text{and} \quad \sigma^{\mathrm{tor}} : \mathscr{S}_K^{\mathrm{tor}} \to \mathtt{Hdg}_{O_{E_v}} \tag{5.13}$$

of smooth algebraic stacks over O_{E_v}. Let O' be a local finite flat extension of O_{E_v}. As $\mathtt{Hdg}_{O_{E_v}} = [\mathscr{P}\backslash *]$, a vector bundle on $\mathtt{Hdg}_{O'}$ is given by an algebraic representation ρ of the group scheme $\mathscr{P}_{O'}$ on some finite free O'-module. The pullback of such a vector bundle to $\mathscr{S}_{K,O'}$ via σ (resp. to $\mathscr{S}^{\mathrm{tor}}_{K,O'}$ via σ^{tor}) is denoted by

$$\mathscr{V}(\rho) \quad (\text{resp. } \mathscr{V}(\rho)^{\mathrm{tor}}).$$

Again we define vector bundles on $\mathscr{S}_K$ of this form to be *automorphic vector bundles* and $\mathscr{V}(\rho)^{\mathrm{tor}}$ is the canonical extension of $\mathscr{V}(\rho)$ to the toroidal compactification $\mathscr{S}_K^{\mathrm{tor}}$.

The morphisms σ and σ^{tor} induce on special fibers morphisms

$$\sigma : S_K \to \mathtt{Hdg}_\kappa \quad \text{and} \quad \sigma^{\mathrm{tor}} : S_K^{\mathrm{tor}} \to \mathtt{Hdg}_\kappa \tag{5.14}$$

of smooth algebraic stacks over κ. Again we have the notion of an automorphic bundle on S_K and its canonical extension to S_K^{tor}.

We now define the tautological rings in positive characteristic as in characteristic 0.

Definition 5.15. For a field extension κ' of κ, we call the images of $A^\bullet(\mathtt{Hdg}_{\kappa'})$ in $A^\bullet(S_{K,\kappa'})$ and in $A^\bullet(S^{\mathrm{tor}}_{K,\kappa'})$ the *tautological rings* of $S_{K,\kappa'}$ and of $S^{\mathrm{tor}}_{K,\kappa'}$, respectively. We denote them by $\mathcal{T}_{\kappa'}$ and $\mathcal{T}^{\mathrm{tor}}_{\kappa'}$, respectively.

6. Cycle classes of Ekedahl–Oort strata

We continue to use the notation of Section 5C, i.e., $\boldsymbol{S}_K/E$ denotes the Shimura variety attached to a Shimura datum of Hodge type $(\boldsymbol{G}, \boldsymbol{X})$ and a neat open compact subgroup $K \subset \boldsymbol{G}(\mathbb{A}_f)$, $\mathscr{S}_K/O_{E_v}$ denotes its smooth integral model at a prime p of good reduction, S_K/κ its special fiber. We denote by $\mathscr{S}_K^{\mathrm{tor}}$ a fixed smooth proper toroidal compactification of $\mathscr{S}_K$ and by S_K^{tor} its special fiber. Moreover, $\mathscr{G}$ denotes the reductive model of $\boldsymbol{G}_{\mathbb{Q}_p}$ which is endowed with a cocharacter μ defined over O_{E_v}. We denote by (G, μ) the special fiber of $(\mathscr{G}, \mu)$.

From now on we assume that $p > 2$. This hypothesis is only needed for the existence of the smooth morphism ζ and the morphism ζ^{tor} defined below in (6.1) and (6.2). It seems probable that these morphisms also exist with the stated properties for $p = 2$, using ideas from [Kim and Madapusi Pera 2016].

6A. ***Ekedahl–Oort strata.*** From the reductive group G over $\mathbb{F}_p$ and the cocharacter $\mu : \mathbb{G}_{m,\kappa} \to G_\kappa$, we obtain the stack $G\text{-}\mathtt{Zip}^\mu$ recalled in Section 3. We use all notation introduced in Section 3 for this pair (G, μ). In particular, we define $P := P_-(\mu)$, a parabolic subgroup of G of type I which is defined over κ. The choice of P yields an isomorphism $\mathtt{Hdg}_\kappa \cong [P\backslash *]$. The morphism σ (5.14) is a morphism $\sigma : S_K \to [P\backslash *]$.

In a series of papers, Viehmann and Wedhorn [2013], Zhang [2018], and Wortmann [2013] defined (for $p > 2$) a smooth morphism

$$\zeta : S_K \to G\text{-}\mathtt{Zip}^\mu, \tag{6.1}$$

which has also been extended to toroidal compactification

$$\zeta^{\mathrm{tor}} : S_K^{\mathrm{tor}} \to G\text{-}\mathtt{Zip}^\mu \tag{6.2}$$

by Goldring and Koskivirta [2019a, Theorem 6.2.1]. By Andreatta [2023, Theorem 1.2], the morphism ζ^{tor} is smooth as well. Moreover, one has by construction

$$\beta \circ \zeta = \sigma \quad \text{and} \quad \beta \circ \zeta^{\mathrm{tor}} = \sigma^{\mathrm{tor}}, \tag{6.3}$$

where β is the morphism defined in (3.23).

Recall the definition of the zip strata $Z_w \subseteq G\text{-}\mathtt{Zip}^\mu$ (3.6). The *Ekedahl–Oort strata of* S_K (resp. of S_K^{tor}) are defined for $w \in {}^I W$ as

$$S_{K,w} := \zeta^{-1}(Z_w) \quad (\text{resp. } S_{K,w}^{\mathrm{tor}} := (\zeta^{\mathrm{tor}})^{-1}(Z_w)).$$

The smoothness of ζ^{tor} implies the following properties of the EO-strata.

(1) For all $w \in {}^I W$, the $S_{K,w}$ (resp. the $S_{K,w}^{\mathrm{tor}}$) are locally closed smooth subschemes of S_K (resp. S_K^{tor}). They are equidimensional of dimension $\ell(w)$ by [Pink et al. 2011, 5.11].

(2) By (3.7), one has

$$\zeta^{-1}(\bar{Z}_w) = \bar{S}_{K,w} = \bigcup_{w' \preceq w} S_{K,w'} \tag{6.4}$$

and

$$(\zeta^{\mathrm{tor}})^{-1}(\bar{Z}_w) = \overline{S_{K,w}^{\mathrm{tor}}} = \bigcup_{w' \preceq w} S_{K,w'}^{\mathrm{tor}}. \tag{6.5}$$

(3) The map $\zeta^* : A^\bullet(G\text{-}\mathtt{Zip}^\mu) \to A^\bullet(S_K)$ of $\mathbb{Q}$-algebras sends $[\bar{Z}_w]$ to $[\bar{S}_{K,w}]$. Analogously, the map $(\zeta^{\mathrm{tor}})^* : A^\bullet(G\text{-}\mathtt{Zip}^\mu) \to A^\bullet(S_K^{\mathrm{tor}})$ of $\mathbb{Q}$-algebras sends $[\bar{Z}_w]$ to $[\bar{S}_{K,w}^{\mathrm{tor}}]$.

Proposition 6.6. *The tautological rings $\mathcal{T}$ and $\mathcal{T}^{\mathrm{tor}}$ are finite-dimensional $\mathbb{Q}$-algebras that are generated as $\mathbb{Q}$-vector spaces by $[\bar{S}_{K,w}]$ and $[\bar{S}_{K,w}^{\mathrm{tor}}]$ for $w \in {}^I W$, respectively.*

Proof. By (6.3), the tautological rings are the images of $A^\bullet(G\text{-}\mathtt{Zip}^\mu)$. Hence the claim follows from Proposition 4.14. □

We also recall and complement some results from [Goldring and Koskivirta 2019a] about the EO-strata in the minimal compactification. Let $\pi : S_K^{\mathrm{tor}} \to S_K^{\min}$ be the projection, and set

$$S_{K,w}^{\min} := \pi(S_{K,w}^{\mathrm{tor}}), \qquad w \in {}^I W. \tag{6.7}$$

Then the $S_{K,w}^{\min}$ are pairwise disjoint — in other words, $\pi^{-1}(S_{K,w}^{\min}) = S_{K,w}^{\mathrm{tor}}$ — and locally closed in $S_K^{\min}$ by [Goldring and Koskivirta 2019a, 6.3.1]. We endow $S_{K,w}^{\min}$ with the reduced scheme structure.

We will also use the following result of Goldring and Koskivirta.

Theorem 6.8 [Goldring and Koskivirta 2019a]. *The EO-strata $S_{K,w}^{\min}$ in the minimal compactification are affine for all $w \in {}^I W$.*

Theorem 6.8 implies, in particular, that the EO-strata $S_{K,w}$ are quasiaffine for all $w \in {}^I W$, which was known before.

Because π is proper, the closure relation (6.5) implies

$$\overline{S_{K,w}^{\min}} = \bigcup_{w' \preceq w} S_{K,w'}^{\min}. \tag{6.9}$$

Below (Corollary 6.16) we will also show that $S_{K,w}^{\min}$ is equidimensional of dimension $\ell(w)$.

6B. ***Connectedness of unions of Ekedahl–Oort strata.*** In this subsection we show that the smoothness of ζ^{tor} allows us to deduce from [Goldring and Koskivirta 2019a] certain results on the connectedness of EO-strata. These are new even in the Siegel case.

Two lemmas on connectedness. For lack of a reference we collect two probably well-known lemmas.

For a topological space X, we denote by $\pi_0(X)$ the space of connected components of X. This defines a functor π_0 from the category of topological spaces to the category of totally disconnected topological spaces which is left adjoint to the inclusion functor. If X is a noetherian scheme, then $\pi_0(X)$ is a finite discrete space.

Lemma 6.10. *Let $f : X \to Y$ be a continuous map between topological spaces with connected* (and hence nonempty) *fibers. Suppose that the topology on Y is the quotient topology of the topology on X* (as occurs, e.g., if f is closed or open). *Then $\pi_0(f) : \pi_0(Y) \xrightarrow{\sim} \pi_0(X)$ is a homeomorphism.*

Proof. For topological spaces Z and Z', let $\mathcal{C}(Z, Z')$ be the set of continuous maps $Z \to Z'$. Let S be a totally disconnected space. We have functorial bijections

$$\begin{aligned}\mathcal{C}(\pi_0(Y), S) = \mathcal{C}(Y, S) &= \{g \in \mathcal{C}(X, S) \mid g_{|f^{-1}(y)} \text{ is constant for all } y \in Y\} \\ &= \mathcal{C}(X, S) = \mathcal{C}(\pi_0(X), S),\end{aligned}$$

where the first and last equalities hold by adjointness of π_0 and the inclusion functor, the second equality holds because Y carries the quotient topology of X, and the third equality holds because all fibers of f are connected. Therefore $\pi_0(f)$ is a homeomorphism by Yoneda's lemma. □

Let $l \geq 0$ be an integer. Recall that a noetherian scheme X is called *connected in dimension* $\geq l$ if $X \setminus Z$ is connected for every closed subset $Z \subseteq X$ of dimension $< l$. Hence X is connected if and only if X is connected in dimension ≥ 0. A scheme X of finite type over a field k is called *geometrically connected in dimension* $\geq l$ if $X \otimes_k k'$ is connected in dimension $\geq l$ for all field extensions k' of k.

We recall the following variant of a theorem of Grothendieck.

Proposition 6.11. *Let k be a field, let X be a proper k-scheme, and let $D \subseteq X$ be an effective ample divisor. Let $l \geq 1$ be an integer. Suppose that the irreducible components of X have dimension $\geq l+1$ and that X is geometrically connected in dimension $\geq l$. Then the irreducible components of D have dimension $\geq l$, and D is geometrically connected in dimension $\geq l-1$.*

Proof. Let D be the vanishing locus of a section s of an ample line bundle $\mathscr{L}$. Replacing $\mathscr{L}$ and s by some power, we may assume that $\mathscr{L}$ is very ample and hence that X is a closed subscheme of projective space $\mathbb{P}_k^N$ and that $D = X \cap H$ for some hyperplane H. Then the result follows from [SGA 2 2005, Exp. XIII, 2.3]. □

Inheritance of connectedness. For any subset $A \subseteq {}^I W$, we set $Z_A := \bigcup_{w \in A} Z_w$, considered as a subspace of the underlying topological space of $G\text{-}\mathtt{Zip}^\mu$. We also set $S_{K,A} := \bigcup_{w \in A} S_{K,w}$ and define similarly subsets $S_{K,A}^{\mathrm{tor}}$ and $S_{K,A}^{\mathrm{min}}$ of S_K^{tor} and S_K^{min}, respectively. Then $\zeta^{-1}(Z_A) = S_{K,A}$ and $(\zeta^{\mathrm{tor}})^{-1}(Z_A) = S_{K,A}^{\mathrm{tor}}$.

Now let $A \subseteq {}^I W$ be a closed subset, i.e., if $w \in A$ and $w' \in {}^I W$ with $w' \preceq w$, then $w' \in A$. Let A^0 be the set of maximal elements in A with respect to $\preceq$ and set $\partial A := A \setminus A^0$. Then Z_A is closed in $G\text{-}\mathtt{Zip}^\mu$ and Z_{A^0} is open and dense in Z_A. We consider Z_A, Z_{A^0}, and $Z_{\partial A}$ as reduced locally closed algebraic substacks of $G\text{-}\mathtt{Zip}^\mu$.

The subvariety $S_{K,A}$ is closed in S_K, and S_{K,A^0} is open and dense in $S_{K,A}$ by (6.4). Analogous assertions also hold for unions of EO-strata in S_K^{tor} and S_K^{min} by (6.5) and (6.9).

For brevity we say that a scheme X of finite type over a field is *l-gc* if all irreducible components of X have dimension $\geq l+1$, and X is geometrically connected in dimension $\geq l$.

Lemma 6.12. *Let $Y \subseteq S_K^{\mathrm{min}}$ be a closed subscheme, let $A \subseteq {}^I W$ be closed as above, and set $Y_A := Y \cap S_{K,A}^{\mathrm{min}}$. Let $l \geq 1$ be an integer. If Y_A is l-gc, then $Y_{\partial A}$ is $(l-1)$-gc.*

The proof relies heavily on results from [Goldring and Koskivirta 2019a], using Proposition 6.11 as an additional ingredient.

Proof. Let $Y_A^{\mathrm{tor}} := \pi^{-1}(Y_A)$, and let

$$Y_A^{\mathrm{tor}} \xrightarrow{\pi'} Y_A' \xrightarrow{f} Y_A$$

be the Stein factorization of $\pi : Y_A^{\mathrm{tor}} \to Y_A$. As π has geometrically connected fibers, the same holds for the finite morphism f. Hence f is a universal homeomorphism. Therefore Y_A' is l-gc and it suffices to show that $Y_{\partial A}' := f^{-1}(Y_{\partial A})$ is $(l-1)$-gc.

Let ω^{tor} be the Hodge line on S_K^{tor} obtained from some Siegel embedding of the Shimura datum. Let $\omega^{\mathrm{min}} := \pi_* \omega^{\mathrm{tor}}$. By [Madapusi Pera 2019, 5.2.11], ω^{min} extends the Hodge line bundle on S_K, it is ample, and $\pi^*(\omega^{\min}) \cong \omega^{\mathrm{tor}}$. The restrictions of ω^{tor} and $\omega^{\min}$ to Y_A^{tor} and Y_A, respectively, are denoted by $\omega_{Y_A}^{\mathrm{tor}}$ and $\omega_{Y_A}^{\min}$. Set $\omega'_{Y_A} := f^*\omega_{Y_A}^{\min}$. Then for all $N > 1$ one has

$$\pi'_*\omega_{Y_A}^{\mathrm{tor},\otimes N} = \pi'_*\pi'^*(\omega_{Y_A}^{\prime\otimes N}) = \omega_{Y_A}^{\prime\otimes N}, \tag{$*$}$$

where the second equality follows from $\pi'_*(\mathscr{O}_{Y_A^{\mathrm{tor}}}) = \mathscr{O}_{Y'_A}$.

In the special case $Y = S_K^{\min}$ and $A = {}^IW$ one has $\pi' = \pi$ and we also see

$$\pi_*\omega^{\mathrm{tor},\otimes N} \cong \omega^{\min,\otimes N}. \tag{6.13}$$

By [Goldring and Koskivirta 2019a, 6.2.2], there exists an $N \geq 1$ such that for all $w \in {}^IW$ there exist sections $h_w \in \Gamma(\overline{S_{K,w}^{\mathrm{tor}}}, \omega^{\mathrm{tor},\otimes N})$ whose nonvanishing locus is $S_{K,w}^{\mathrm{tor}}$. For $w, w' \in A^0$ with $w \neq w'$,

$$h_w|_{\overline{S_{K,w}^{\mathrm{tor}}} \cap \overline{S_{K,w'}^{\mathrm{tor}}}} = 0 = h_{w'}|_{\overline{S_{K,w}^{\mathrm{tor}}} \cap \overline{S_{K,w'}^{\mathrm{tor}}}},$$

so after passing to some power of N and of h_w, we can glue the sections h_w with $w \in A^0$ to a section $h_A \in \Gamma(S_{K,A}^{\mathrm{tor}}, \omega^{\otimes N})$ whose nonvanishing locus is S_{K,A^0}^{tor} [Goldring and Koskivirta 2019a, 5.2.1]. We denote the restriction of h_A to Y_A^{tor} again by h_A. Using $(*)$ we obtain a section

$$h_A \in \Gamma(Y_A^{\mathrm{tor}}, \omega_{Y_A}^{\mathrm{tor},\otimes N}) = \Gamma(Y'_A, \omega_{Y_A}^{\prime\otimes N})$$

whose vanishing locus in Y'_A is precisely $Y'_{\partial A}$. As $\omega_{Y_A}^{\prime\otimes N}$ is the pullback of $\omega^{\min,\otimes N}$ under a finite morphism $Y'_A \to S_K^{\min}$, it is ample and we conclude by Proposition 6.11. □

Connectedness of the length strata. Let $d := \dim S_K = \dim S_K^{\mathrm{tor}} = \langle 2\rho, \mu\rangle$, where ρ denotes as usual half of the sum of all positive roots on the root datum of G. For $j = 0, \dots, d$, we set

$$S_{K,\leq j}^? := \bigcup_{\ell(w)\leq j} S_{K,w}^? \quad \text{and} \quad S_{K,j}^? := \bigcup_{\ell(w)=j} S_{K,w}^?$$

for $? \in \{\varnothing, \mathrm{tor}, \min\}$. Then $S_{K,\leq j}^?$ is closed in $S_K^?$ and $S_{K,j}^?$ is open and dense in $S_{K,\leq j}^?$ by (6.4), (6.5), and (6.9). We endow them with the reduced subscheme structure. The closed subschemes $S_{K,\leq j}^?$ of $S_K^?$ are called *closed length strata*.

Lemma 6.14. *The schemes $S_{K,j}$ and $S_{K,j}^{\mathrm{tor}}$ are smooth.*

Proof. By Lemma 3.8, no two elements of ${}^IW$ of the same length are comparable with respect to $\preceq$. Hence $S_{K,j}^?$ is the topological sum of the $S_{K,w}^?$ for $w \in {}^IW$ with $\ell(w) = j$. This shows the lemma because $S_{K,w}$ and $S_{K,w}^{\mathrm{tor}}$ are smooth for all $w \in {}^IW$. □

Let X be a scheme of finite type over a field k. By a *geometric connected component of* X we mean a connected component Y of $X_{\bar{k}}$, where $\bar{k}$ is an algebraic closure of k. Then Y is already defined over some finite extension of k.

Theorem 6.15. *Let Y be a geometric connected component of $S_K^{\min}$, and let $Y^{\mathrm{tor}} := \pi^{-1}(Y)$ be the corresponding* (Lemma 6.10) *geometric connected component of S_K^{tor}. Then, for all $j = 1, \dots, d$, the length strata $S_{K,\leq j}^{\mathrm{tor}} \cap Y^{\mathrm{tor}}$ and $S_{K,\leq j}^{\min} \cap Y$ are geometrically connected and equidimensional of dimension j. In particular, they are nonempty.*

Proof. We already know that $S^{\mathrm{tor}}_{K,\le j}$ is equidimensional of dimension j. This shows that $S^{\mathrm{tor}}_{K,\le j}\cap Y^{\mathrm{tor}}$ is either empty or equidimensional of dimension j.

Next we show that $S^{\min}_{K,\le j}\cap Y$ is geometrically connected in dimension $\ge j-1$ and equidimensional of dimension j by descending induction on j. We have $S^{\min}_{K,\le d}=S^{\min}_K$. Because $S^{\min}_K$ is (geometrically) normal (Lemma 5.12), Y is irreducible of dimension d, and in particular, it is geometrically connected in dimension $\ge d-1$. Now let $A_j:=\{w\in {}^IW \mid \ell(w)\le j\}$. Then $A^0_j=\{w\in {}^IW \mid \ell(w)=j\}$ and $\partial A=A_{j-1}$ by Lemma 3.8. Hence by induction we deduce from Lemma 6.12 that $S^{\min}_{K,\le j}\cap Y$ is geometrically connected in dimension $\ge j-1$ and that every irreducible component of $S^{\min}_{K,\le j}\cap Y$ has dimension $\ge j$. On the other hand we have $\dim(Y)\le\dim(S^{\min}_{K,j})=\dim(\pi(S^{\mathrm{tor}}_{K,j}))=j$. Hence $S^{\min}_{K,j}$ is equidimensional of dimension j.

This shows, in particular, that $S^{\min}_{K,\le j}\cap Y$ is nonempty, which implies that

$$S^{\mathrm{tor}}_{K,\le j}\cap Y^{\mathrm{tor}}=\pi^{-1}(S^{\min}_{K,\le j}\cap Y)$$

is nonempty. Moreover, $S^{\mathrm{tor}}_{K,\le j}\cap Y^{\mathrm{tor}}$ is geometrically connected by Lemma 6.10. □

Corollary 6.16. *Each EO-stratum $S^{\min}_{K,w}$ in the minimal compactification is equidimensional of dimension $\ell(w)$.*

Proof. Let $Y\subseteq S^{\min}_{K,w}$ be an irreducible component, and let $\overline{Y}$ be its closure in $S^{\min}_K$. By (6.9), $\overline{Y}$ is an irreducible component of $S^{\min}_{K,\le\ell(w)}$. Hence $\dim(Y)=\dim(\overline{Y})=\ell(w)$ by Theorem 6.15. □

Corollary 6.17. *Let $S^{\mathrm{tor}}_{K,e}$ be the 0-dimensional EO-stratum in S^{tor}_K. Suppose that $S^{\mathrm{tor}}_{K,e}$ is already contained in S_K. Let Y be a geometric connected component of S^{tor}_K. Then the length 1 stratum $S_{K,\le 1}\cap Y$ in $S_K\cap Y$ is geometrically connected.*

The condition that $S^{\mathrm{tor}}_{K,e}$ is contained in S_K is satisfied for all Shimura varieties of PEL type [Goldring and Koskivirta 2019a, 6.4.1], and we expect it to hold in general.

Proof. Let Y and Y' be irreducible components of $S^{\mathrm{tor}}_{K,\le 1}$. As $S^{\mathrm{tor}}_{K,\le 1}\cap Y$ is geometrically connected by Theorem 6.15, it suffices to show that $Y\cap Y'\subseteq S^{\mathrm{tor}}_{K,e}=S_{K,e}$. But this is clear because $S^{\mathrm{tor}}_{K,1}$ is smooth (Lemma 6.14) and hence cannot contain intersection points of irreducible components. □

The following result generalizes [Ekedahl and van der Geer 2009, Proposition 6.1]:

Proposition 6.18. *Let $w\in {}^IW$ and Y an irreducible component of $\overline{S^{\min}_{K,w}}$.*

(i) *The variety Y has dimension $\ell(w)$ and is geometrically connected in dimension $\ge\ell(w)-1$.*

(ii) *The intersection $Y\cap S^{\min}_{K,e}$ is nonempty.*

Proof. (i) As in the proof of Corollary 6.16, the variety Y is an irreducible component of $S^{\min}_{K,\le\ell(w)}$. Hence (i) follows from Theorem 6.15.

(ii) Let $Y^\circ:=Y\cap S^{\min}_{K,w}$. Since Y° is an irreducible component of $S^{\min}_{K,w}$, it is affine by Theorem 6.8. Thus if Y° is closed in $S^{\min}_K$ it has dimension zero, which, using (i), implies $\ell(w)=0$ and hence $w=e$. Otherwise $Y\setminus Y^\circ$ is nonempty. Let Y' be an irreducible component of $Y\setminus Y^\circ=Y\cap\bigcup_{w'\prec w}S^{\min}_{K,w'}$.

Using (i), Lemma 6.12 yields $\dim Y' \geq \ell(w)-1$. On the other hand, the inclusion $Y' \subset \bigcup_{w' \prec w} S^{\min}_{K,w'}$ and Corollary 6.16 yield $\dim Y' \leq \ell(w)-1$. Thus $\dim Y' = \ell(w)-1$, which again by Corollary 6.16 implies that Y' must be an irreducible component of $\overline{S^{\min}_{K,w'}}$ for some $w' \prec w$ with $\ell(w') = \ell(w)-1$. Now we may conclude by induction on $\ell(w)$. ☐

6C. *The flag space over S_K^{tor}.* Let $\pi_K : \mathcal{F}_K^{\mathrm{tor}} \to S_K^{\mathrm{tor}}$ be defined by the following fiber product:

$$\begin{tikzcd} \mathcal{F}_K^{\mathrm{tor}} \arrow[r] \arrow[d, "\pi_K"'] & G\text{-}\mathtt{ZipFlag}^{\mu} \arrow[d, "\pi"] \\ S_K^{\mathrm{tor}} \arrow[r, "\zeta^{\mathrm{tor}}"] & G\text{-}\mathtt{Zip}^{\mu} \end{tikzcd}$$

Similarly, we let $\mathcal{F}_K$ be the restriction of $\mathcal{F}_K^{\mathrm{tor}}$ to S_K. As π is representable by schemes, smooth, and proper, $\mathcal{F}_K^{\mathrm{tor}}$ and $\mathcal{F}_K$ are schemes, and π_K is smooth and proper.

By pulling back the stratification $G\text{-}\mathtt{ZipFlag}^{\mu} = \bigcup_{w \in W} Z_w^{\varnothing}$ (3.11) to $\mathcal{F}_K$ and $\mathcal{F}_K^{\mathrm{tor}}$, we obtain stratifications

$$\mathcal{F}_K = \bigcup_{w \in W} \mathcal{F}_{K,w} \quad \text{and} \quad \mathcal{F}_K^{\mathrm{tor}} = \bigcup_{w \in W} \mathcal{F}_{K,w}^{\mathrm{tor}}.$$

Proposition 6.19. *The strata $\mathcal{F}_{K,w}$ and $\mathcal{F}_{K,w}^{\mathrm{tor}}$ are smooth and equidimensional of dimension $\ell(w)$. Their closures $\overline{\mathcal{F}_{K,w}}$ and $\overline{\mathcal{F}_{K,w}^{\mathrm{tor}}}$ are normal, Cohen–Macaulay, with only rational singularities.*

Proof. Since ζ^{tor} is smooth, this follows from Proposition 3.12. ☐

7. Applications

7A. *Triviality of Chern classes of flat automorphic bundles.* Let E' be an extension of E. By definition, an automorphic bundle over E' is a vector bundle on $\mathrm{Sh}_K(\boldsymbol{G},\boldsymbol{X})_{E'}$ that arises by pullback of a vector bundle on $\mathtt{Hdg}_{E'}$ via the map σ. Recall that such an automorphic bundle is called *flat* if it comes from a vector bundle on $[\boldsymbol{G}_{E'} \backslash *]$ by pullback via the composition

$$\mathrm{Sh}_K(\boldsymbol{G},\boldsymbol{X})_{E'} \xrightarrow{\sigma} \mathtt{Hdg}_{E'} \to [\boldsymbol{G}_{E'} \backslash *],$$

i.e., it is an automorphic bundle associated with a finite-dimensional representation of $\boldsymbol{G}_{E'}$. Similarly, we define what it means for an automorphic bundle (or their canonical extensions to the toroidal compactification) on the integral model or its special fiber to be flat.

Now the theory of Chow rings of G-zips allows us easily to show the following result for flat automorphic bundles on the special fiber.

Theorem 7.1. *Let κ' be an extension of κ, let $\mathscr{V}$ be a flat automorphic bundle on $S_{K,\kappa'}$, and let $\mathscr{V}^{\mathrm{tor}}$ be its canonical extension to $S_{K,\kappa'}^{\mathrm{tor}}$. Then for all $i \geq 1$ the i-th Chern class of $\mathscr{V}$ in $A^\bullet(S_{K,\kappa'})$ and of $\mathscr{V}^{\mathrm{tor}}$ in $A^\bullet(S_{K,\kappa'}^{\mathrm{tor}})$ are zero.*

Proof. As all automorphic bundles are defined over some finite extension of κ, we may assume that κ' is an algebraic extension of κ. By Proposition 1.3, we may assume that $\kappa' = k$ is an algebraic closure of κ. As σ and σ^{tor} both factor through $G\text{-}\mathtt{Zip}^{\mu}$, it suffices to show that under pullback via the composition

$$G\text{-}\mathtt{Zip}^{\mu} \xrightarrow{\beta} \mathtt{Hdg}_k \xrightarrow{\nu} [G_k\backslash *],$$

all elements of degree > 0 in $A^\bullet([G_k\backslash *])$ are sent to 0. Here ν is the canonical projection $\mathtt{Hdg}_K = [P_k\backslash *] \to [G_k\backslash *]$ which induces via pullback on Chow rings the inclusion $A^\bullet([G_k\backslash *]) = S^W \hookrightarrow S^{W_{I^o}} = A^\bullet(\mathtt{Hdg}_k)$. Hence the description of $\mathcal{I}$ in Lemma 4.2 and of β^* in Theorem 4.16 implies the claim. □

Using proper smooth base change we obtain a triviality result for étale Chern classes in characteristic 0 as follows. For a scheme of finite type over a field k, we denote by $H^i(X, \mathbb{Q}_\ell(d))$ the i-th continuous ℓ-adic cohomology with Tate twist defined by Jannsen [1988] or, equivalently, the pro-étale cohomology defined by Bhatt and Scholze [2015]. Here ℓ is a prime different from the characteristic of k. Recall that $\boldsymbol{S}_K$ denotes a Shimura variety of Hodge type in characteristic 0 and that $\boldsymbol{S}_K^{\mathrm{tor}}$ denotes a smooth proper toroidal compactification of $\boldsymbol{S}_K$.

Corollary 7.2. *Let E' be a finite extension of the reflex field E contained in the algebraic closure $\bar{E}$ of E in $\mathbb{C}$. Let $\mathscr{V}$ be a flat automorphic bundle over $\boldsymbol{S}_{K,E'}$, and let $\mathscr{V}^{\mathrm{tor}}$ be its canonical extension to $\boldsymbol{S}^{\mathrm{tor}}_{K,E'}$. Let $p \neq \ell$ be a prime of good reduction for the Shimura datum $(\boldsymbol{G}, \boldsymbol{X})$ and v' a place of E' above p. Then for all $i \geq 1$ the i-th étale Chern classes $c_i(\mathscr{V}) \in H^{2i}(\boldsymbol{S}_{K,E'_{v'}}, \mathbb{Q}_\ell(i))$ and $c_i(\mathscr{V}^{\mathrm{tor}}) \in H^{2i}(\boldsymbol{S}^{\mathrm{tor}}_{K,E'_{v'}}, \mathbb{Q}_\ell(i))$ are zero.*

A stronger version of this statement for continuous cohomology over E' instead of $E'_{v'}$ has been proved by Esnault and Harris [2017] for compact Shimura varieties.

First we note the following fact:

Lemma 7.3. *Let $\mathscr{G}$ be a flat affine group scheme over a Dedekind ring R with quotient field Q. Every representation of $\mathscr{G}_Q$ on a finite-dimensional Q-vector space V extends to a representation of $\mathscr{G}$ on a locally free R-module of finite type.*

Proof. Let $A := \Gamma(\mathscr{G}, \mathscr{O}_{\mathscr{G}})$ be the ring of functions of $\mathscr{G}$. If $\sigma : V \to V \otimes_Q A_Q$ is the comodule map corresponding to the representation in question, then we consider V as a comodule under A via $V \xrightarrow{\sigma} V \otimes_Q \rho_Q = V \otimes_R A$. Then, using the local finiteness of A-comodules, we find an A-sub-comodule $L \subset V$ which is finitely generated over R and which generates V as a Q-vector space. Such an L is torsion-free and hence projective because R is a Dedekind domain. It gives the desired extension. □

Now we prove Corollary 7.2:

Proof. Let v be the restriction of v' to E, and let $\mathscr{S}_K^{\mathrm{tor}}$ be a smooth proper toroidal compactification of $\mathscr{S}_K$ over O_{E_v} with generic fiber $\boldsymbol{S}^{\mathrm{tor}}_{K,E_v}$. Let κ' be the residue field of $O' := O_{E'_{v'}}$. We also use the notation of Section 6. In particular, we denote by $\mathscr{G}$ a reductive model of $\boldsymbol{G}_{\mathbb{Q}_p}$ over $\mathbb{Z}_p$. Let $\mathscr{V}$ be associated to a representation $\rho : \boldsymbol{G}_{E'} \to \mathrm{GL}((E')^n)$. By Lemma 7.3, we can extend the base change $\rho_{E'_{v'}}$ to a dualizable representation $\tilde{\rho}$ of $\mathscr{G}$ over O'.

The special fiber of $\tilde{\rho}$ is then a representation of the split reductive group $G_{\kappa'}$. Let $\mathscr{V}^{\mathrm{tor}}_{\kappa'}$ be the corresponding flat automorphic bundle on $S^{\mathrm{tor}}_{K,\kappa'}$. By construction it lifts to a flat automorphic bundle over $\mathscr{S}^{\mathrm{tor}}_{K,O'}$ whose generic fiber is $\mathscr{V}^{\mathrm{tor}}$. By Theorem 7.1, the i-th Chern class of $\mathscr{V}^{\mathrm{tor}}_{\kappa'}$ in $A^i(S^{\mathrm{tor}}_{K,\kappa'})$ is zero for $i \geq 1$. In particular, its étale cycle class vanishes in

$$H^{2i}(S^{\mathrm{tor}}_{\kappa'}, \mathbb{Q}_\ell(i)) = H^{2i}(\boldsymbol{S}^{\mathrm{tor}}_{K,E'_{v'}}, \mathbb{Q}_\ell(i)),$$

where the equality holds by smooth and proper base change. But this cycle class in the space on the right-hand side is the étale cycle class of $\mathscr{V}^{\mathrm{tor}}$ because the étale cycle class map from Chow groups to étale cohomology is compatible with specialization. By restriction this implies the result for $\boldsymbol{S}_{K,E'_{v'}}$ as well. $\square$

7B. *The Hodge half-line.* As the Shimura datum is of Hodge type there exists a Siegel embedding of G, i.e., an embedding $\iota : G \hookrightarrow \mathrm{GSp}(V)$ of algebraic groups over $\mathbb{F}_p$ such that $\tilde{\mu} := \iota \circ \mu$ is minuscule and the parabolic $P_+(\tilde{\mu})$ is the stabilizer of a Lagrangian subspace $U \subseteq V$. Consider the character

$$\chi(\iota) := \det(V/U)^\vee \tag{7.4}$$

of P, which is defined over κ. It corresponds to a line bundle on the Hodge stack $\mathtt{Hdg}$ over κ. We denote its pullback to $G\text{-}\mathtt{Zip}^\mu$ by $\omega^\flat(\iota)$. We call a class in $A^1(G\text{-}\mathtt{Zip}^\mu)$ a *Hodge line bundle class* if it is the first Chern class of the line bundle $\omega^\flat(\iota)$ given by a symplectic embedding.

Such a Hodge line bundle class is essentially independent of the choice of the embedding by combining Theorem 7.1 with a result of Goldring and Koskivirta.

Proposition 7.5. *Suppose that $\boldsymbol{G}^{\mathrm{ad}}$ is $\mathbb{Q}$-simple. If ι and ι' are two Siegel embeddings, then there exists $\rho \in \mathbb{Q}_{>0}$ such that*

$$c_1(\omega^\flat(\iota)) = \rho c_1(\omega^\flat(\iota')) \in A^1(G\text{-}\mathtt{Zip}^\mu).$$

Proof. By Theorem 7.1, it suffices to show that there exists a character λ of G and $m, n \in \mathbb{Z}_{>0}$ such that $m\chi(\iota) = \lambda + n\chi(\iota')$ as characters of P or, equivalently, of the Levi subgroup L. Let $\widetilde{L}$ be the connected component of the preimage of L in the simply connected cover of the derived group of G, and let $\tilde{\chi}$ and $\tilde{\chi}'$ be the characters obtained from $\chi(\iota)$ and $\chi(\iota')$, respectively, by composition with $\widetilde{L} \to L$. Then it suffices to show there exist $m, n \in \mathbb{Z}_{>0}$ such that $m\tilde{\chi} = n\tilde{\chi}'$. But this is shown in [Goldring and Koskivirta 2018, 1.4.5]. $\square$

It is easy, as was explained to us by Goldring, to give examples where the assertions fail without the assumption that $\boldsymbol{G}^{\mathrm{ad}}$ is $\mathbb{Q}$-simple. Indeed if $\boldsymbol{G} := \{(A, B) \in \mathrm{GL}_{2,\mathbb{Q}} \mid \det(A) = \det(B)\}$ and $\boldsymbol{X}$ is the $\boldsymbol{G}(\mathbb{R})$-conjugacy class of

$$\mathbb{C}^\times \to \boldsymbol{G}(\mathbb{R}), \quad x + iy \mapsto \left(\begin{pmatrix} x & -y \\ y & x \end{pmatrix}, \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \right),$$

then $\mathrm{Sh}_K(\boldsymbol{G}, \boldsymbol{X})$ is the Shimura variety that classifies pairs of elliptic curves (with some level structure). Let GSp_6 be the group of symplectic similitudes over $\mathbb{Q}$ defined by the alternating form

$$\langle x, y \rangle := x_1 y_2 - x_2 y_1 + x_3 y_4 - x_4 y_3 + x_5 y_6 - x_6 y_5 \quad \text{for } x, y \in \mathbb{Q}^6.$$

The embeddings of Shimura data $\boldsymbol{G} \to \mathrm{GSp}_6$ given by

$$(A, B) \mapsto \begin{pmatrix} A & & \\ & A & \\ & & B \end{pmatrix} \quad \text{and} \quad (A, B) \mapsto \begin{pmatrix} A & & \\ & B & \\ & & B \end{pmatrix}$$

then yield the embeddings of $\mathrm{Sh}_K(\boldsymbol{G}, \boldsymbol{X})$ into the moduli space of principally polarized abelian threefolds given by

$$(E_1, E_2) \mapsto E_1^2 \times E_2 \quad \text{and} \quad (E_1, E_2) \mapsto E_1 \times E_2^2.$$

These embeddings then yield Hodge line bundle classes in $A^1(G\text{-}\mathtt{Zip}^\mu)$ that are not multiples of each other.

Let $\mathcal{T}$ and $\mathcal{T}^{\mathrm{tor}}$ be the tautological rings of S_K and S_K^{tor}, respectively.

Definition 7.6. Suppose that $\boldsymbol{G}^{\mathrm{ad}}$ is $\mathbb{Q}$-simple. We call the $\mathbb{Q}_{>0}$ half-line in $A^1(G\text{-}\mathtt{Zip}^\mu)$ generated by $c_1(\omega^\flat(\iota))$ the *Hodge half-line of* $G\text{-}\mathtt{Zip}^\mu$. Its image in the tautological rings $\mathcal{T}_\kappa$ and $\mathcal{T}_\kappa^{\mathrm{tor}}$ is also called the *Hodge half-line*.

By [Madapusi Pera 2019, Theorem 5], we find that the pullback of a Hodge line bundle class to $\mathcal{T}_\kappa$ (resp. to $\mathcal{T}_\kappa^{\mathrm{tor}}$) is generated by the determinant of the sheaf of invariant differentials of the abelian scheme (resp. semiabelian scheme) that is obtained via pullback from the universal abelian (resp. semiabelian) scheme over the Siegel Shimura variety (resp. over a suitable toroidal compactification of the Siegel Shimura variety). In particular, the pullback of a Hodge line bundle class to $\mathcal{T}_\kappa$ is ample.

7C. *Powers of Hodge line bundle classes.* By Propositions 4.8 and 4.14 the Chow ring $A^\bullet(G\text{-}\mathtt{Zip}^\mu)$ is a graded finite-dimensional $\mathbb{Q}$-algebra of dimension $\#^I W$. For $j = 0, \dots, d := \langle 2\rho, \mu\rangle$, the cycle classes $[\bar{Z}_w]$ with $w \in {}^I W$ such that $\ell(w) = d - j$ form a basis of the $\mathbb{Q}$-vector space $A^j(G\text{-}\mathtt{Zip}^\mu)$. In particular, its top-degree part $A^d(G\text{-}\mathtt{Zip}^\mu)$ is 1-dimensional and generated by the unique closed zip stratum $[Z_e]$, which we call the *superspecial stratum*.

Proposition 7.7. *Let $\lambda^\flat \in A^1(G\text{-}\mathtt{Zip}^\mu)$ be a Hodge line bundle class. Then for all $j = 0, \dots, d$ one has*

$$(\lambda^\flat)^{d-j} = \sum_{\substack{w \in {}^I W \\ \ell(w) = j}} \alpha_w [\bar{Z}_w],$$

with $\alpha_w \in \mathbb{Q}_{>0}$. In particular, there exists $\alpha_e \in \mathbb{Q}_{>0}$ such that

$$(\lambda^\flat)^d = \alpha_e [Z_e]. \tag{7.8}$$

Proof. This follows by an easy induction from [Goldring and Koskivirta 2019a, 5.2.2]. □

Remark 7.9. Calculations of examples suggest that the coefficients α_w should be equal for $w \in {}^I W$ with $\ell(w) = j$ if $\boldsymbol{G}^{\mathrm{ad}}$ is $\mathbb{Q}$-simple. We cannot prove this.

By pullback we obtain:

Corollary 7.10. *Let $\lambda^\flat \in A^1(G\text{-}\mathtt{Zip}^\mu)$ be a Hodge line bundle class. Let $\lambda \in \mathcal{T}$ be its pullback. Then for all $j = 0, \ldots, d$ one has*

$$\lambda^{d-j} = \sum_{\substack{w \in {}^I W \\ \ell(w)=j}} \alpha_w [\overline{S_{K,w}}],$$

with $\alpha_w \in \mathbb{Q}_{>0}$. In particular, there exists $\alpha_e \in \mathbb{Q}_{>0}$ such that $\lambda^d = \alpha_e [S_e]$.

7D. ***Description of the tautological ring.*** We now show the pullback map $\zeta^{\text{tor},*} : A^\bullet(G\text{-}\mathtt{Zip}^\mu_k) \to A^\bullet(S^{\text{tor}}_{K,k})$ is always injective. By Proposition 1.3, this also implies the injectivity of $\zeta^{\text{tor},*} : A^\bullet(G\text{-}\mathtt{Zip}^\mu_{\kappa'}) \to A^\bullet(S^{\text{tor}}_{K,\kappa'})$ for every algebraic extension κ' of κ. In particular, we obtain an isomorphism of the tautological ring $\mathcal{T}^{\text{tor}}_{\kappa'}$ with $A^\bullet(G\text{-}\mathtt{Zip}^\mu_{\kappa'})$.

The tool for showing injectivity is the following lemma.

Lemma 7.11. *Let $\alpha : A^\bullet(G\text{-}\mathtt{Zip}^\mu_k) \to T$ be a map of graded $\mathbb{Q}$-algebras. Then α is injective if and only if $\alpha([Z_e]) \neq 0$.*

Proof. It suffices to show that any graded nonzero ideal of $A^\bullet(G\text{-}\mathtt{Zip}^\mu)$ contains $[Z_e]$. By Corollary 4.12, $A^\bullet(G\text{-}\mathtt{Zip}^\mu)$ is isomorphic to the rational cohomology ring of the flag space $\boldsymbol{X}^\vee$ over $\mathbb{C}$. In particular, multiplication yields, for all $j = 0, \ldots, d = \dim \boldsymbol{S}_K$, a perfect pairing

$$A^j(G\text{-}\mathtt{Zip}^\mu) \times A^{d-j}(G\text{-}\mathtt{Zip}^\mu) \to A^d(G\text{-}\mathtt{Zip}^\mu) = \mathbb{Q}[Z_e].$$

This implies our claim. □

Theorem 7.12. *The map $\zeta^{\text{tor},*}$ is injective. One has*

$$\mathcal{T}^{\text{tor}}_k \cong A^\bullet(G\text{-}\mathtt{Zip}^\mu_k) \cong H^{2\bullet}(\boldsymbol{X}^\vee). \tag{7.13}$$

Proof. Let $\lambda^\flat \in A^1(G\text{-}\mathtt{Zip}^\mu)$ be a Hodge line bundle class, say the first Chern class of a line bundle $\omega^\flat$ on $G\text{-}\mathtt{Zip}^\mu$. Let $\omega^{\text{tor}} := \zeta^{\text{tor},*}(\omega^\flat)$. Let $\pi : S^{\text{tor}}_K \to S^{\text{min}}_K$ be the canonical proper birational map to the minimal compactification. By [Madapusi Pera 2019, 5.2.11], there exists an ample line bundle ω^{min} on S^{min}_K such that $\pi^*(\omega^{\text{min}}) \cong \omega^{\text{tor}}$.

By Lemma 7.11 and (7.8), we have to show that

$$\zeta^{\text{tor},*}(c_1(\omega^\flat)^d \cap [G\text{-}\mathtt{Zip}^\mu_k]) = c_1(\omega^{\text{tor}})^d \cap [S^{\text{tor}}_K] \neq 0, \tag{$*$}$$

where the equality holds by [Fulton 1998, 6.6] because ζ^{tor} is a smooth morphism.

The projection formula shows

$$\pi_*(c_1(\omega^{\text{tor}})^d \cap [S^{\text{tor}}_K]) = c_1(\omega^{\text{min}})^d \cap [S^{\text{min}}_K],$$

which is nonzero because ω^{min} is ample and S^{min}_K is proper and of pure dimension d over κ. Hence the left-hand side of $(*)$ is nonzero.

The isomorphisms in (7.13) are then a consequence by using Corollary 4.12. □

It is conjectured that analogously the tautological ring of a smooth toroidal compactification of the Shimura variety in characteristic 0 should be isomorphic to the cohomology ring of the compact dual. Let E' be an algebraic extension of E_v, and let κ' be the residue field of the ring of integers of E'. There is a commutative diagram

$$\begin{tikzcd} A^\bullet(\mathtt{Hdg}_{E'}) \arrow[r, "\sigma^{\mathrm{tor},*}"] \arrow[d, "\cong"'] & A^\bullet(S^{\mathrm{tor}}_{K,E'}) \arrow[d] \\ A^\bullet(\mathtt{Hdg}_{\kappa'}) \arrow[r] & A^\bullet(S^{\mathrm{tor}}_{K,\kappa'}) \end{tikzcd} \tag{7.14}$$

where the vertical arrows are the specialization maps. For the Hodge stacks, one can show that the specialization map is an isomorphism. In particular, the right-hand side specialization map induces a surjective map of $\mathbb{Q}$-algebras

$$\mathrm{sp}^{\mathrm{tor}} : \mathcal{T}^{\mathrm{tor}}_{E'} \to \mathcal{T}^{\mathrm{tor}}_{\kappa'}. \tag{7.15}$$

The analogous diagram to (7.14) with the specialization of $A^\bullet(S_{K,E'}) \to A^\bullet(S_{K,\kappa'})$ as the right vertical arrow yields also a surjective map $\mathrm{sp} : \mathcal{T}_{E'} \to \mathcal{T}_{\kappa'}$.

Proposition 7.16. *Suppose that E' is chosen such that $\kappa' = k$ is algebraically closed. Then the following assertions are equivalent.*

(i) *The map* $\mathrm{sp}^{\mathrm{tor}} : \mathcal{T}^{\mathrm{tor}}_{E'} \to \mathcal{T}^{\mathrm{tor}}_{k}$ *is injective* (and hence yields an isomorphism $\mathcal{T}^{\mathrm{tor}}_{E'} \cong H^{2\bullet}(X^\vee)$ by (7.13)).

(ii) *The composition $A^\bullet([G_{E'}\backslash *]) \to A^\bullet(\mathtt{Hdg}_{E'}) \to A^\bullet(S^{\mathrm{tor}}_K)$ is zero in degree > 0.*

Proof. The commutative diagram (7.14) can be extended to a commutative diagram

$$\begin{tikzcd} A^\bullet([G_{E'}\backslash *]) \arrow[r, hook] \arrow[d, "\cong"'] & A^\bullet(\mathtt{Hdg}_{E'}) \arrow[rr, two heads, "\sigma^{\mathrm{tor},*}"] \arrow[d, "\cong"'] & & \mathcal{T}^{\mathrm{tor}}_{E'} \arrow[d] \\ A^\bullet([G_k\backslash *]) \arrow[r, hook] & A^\bullet(\mathtt{Hdg}_{\kappa'}) \arrow[r, two heads, "\beta^*"'] & A^\bullet(G\text{-}\mathtt{Zip}^\mu) \arrow[r, "\sim", "\zeta^{\mathrm{tor},*}"'] & \mathcal{T}^{\mathrm{tor}}_{k} \end{tikzcd} \tag{7.17}$$

Hence the equivalence follows as the kernel of β^* is generated by the image of $A^{>0}([G_k\backslash *])$ by Theorem 4.16. $\square$

Although we cannot prove this description of the tautological ring in characteristic 0, we can reprove the following analogous statement for cohomology. This was previously known by Chern–Weil theory.

Theorem 7.18. *For the $\mathbb{Q}_\ell$-algebra $H^{2\bullet}(S^{\mathrm{tor}}_{K,\bar{E}}) := \bigoplus_i H^{2i}(S^{\mathrm{tor}}_{K,\bar{E}}, \mathbb{Q}_\ell(i))$, the composition*

$$A^\bullet(\mathtt{Hdg}_{\bar{E}}) \to A^\bullet(S^{\mathrm{tor}}_{K,\bar{E}}) \to H^{2\bullet}(S^{\mathrm{tor}}_{K,\bar{E}})$$

induces an injection $H^{2\bullet}(X^\vee) \hookrightarrow H^{2\bullet}(S^{\mathrm{tor}}_{K,\bar{E}})$.

Proof. The existence of the factorization $A^\bullet(\mathtt{Hdg}_{\bar{E}}) \to H^{2\bullet}(X^\vee) \to H^{2\bullet}(S^{\mathrm{tor}}_{K,\bar{E}})$ is given by Corollary 7.2. To prove injectivity, we may replace $\bar{E}$ by $\bar{\mathbb{Q}}_p$ for some $p \neq \ell$ at which the Shimura variety has good reduction. As in the proof of Corollary 7.2, one then reduces to proving that the morphism $H^{2\bullet}(\boldsymbol{X}^\vee) \cong A^\bullet(G\text{-}\mathtt{Zip}^\mu) \to A^\bullet(S_K^{\mathrm{tor}}) \to H^{2\bullet}(S_K^{\mathrm{tor}})$ is injective in characteristic p. This is given by Theorem 7.12. □

It is conjectured that Theorem 7.18 holds over E instead of $\bar{E}$. This is shown in [Esnault and Harris 2017] for compact Shimura varieties. The strongest statement on the Chern classes of automorphic vector bundles in continuous cohomology which we can obtain with our methods here is the following:

Theorem 7.19. *Let E' be a finite extension of the reflex field E contained in the algebraic closure $\bar{E}$ of E in $\mathbb{C}$. Let $\mathscr{V}$ be a flat automorphic bundle over $\mathbf{S}_{K,E'}$, and let $\mathscr{V}^{\mathrm{tor}}$ be its canonical extension to $\mathbf{S}^{\mathrm{tor}}_{K,E'}$. Let $U \subset \operatorname{Spec}(\mathcal{O}_E)$ be the locus of good reduction of S_K and $\mathscr{S}^{\mathrm{tor}}_{K,U}$ the canonical integral model of $\mathbf{S}^{\mathrm{tor}}_{K,E'}$ over U. Then, for each $i > 0$, the i-th Chern class of $\mathscr{V}^{\mathrm{tor}}$ in $H^{2i}(\mathbf{S}^{\mathrm{tor}}_{K,E'}, \mathbb{Q}_\ell(i))$ lies in the image of the natural map from*

$$\ker\Bigl(H^{2i}(\mathscr{S}^{\mathrm{tor}}_{K,U}, \mathbb{Q}_\ell(i)) \to \bigoplus_{v\in U} H^{2i}(\mathscr{S}^{\mathrm{tor}}_{K,\mathcal{O}_{v'}}, \mathbb{Q}_\ell(i))\Bigr)$$

to $H^{2i}(\mathbf{S}^{\mathrm{tor}}_{K,E'}, \mathbb{Q}_\ell(i))$. (Here $H^{2i}(\mathscr{S}^{\mathrm{tor}}_{K,U}, \mathbb{Q}_\ell(i))$ denotes the continuous or pro-étale cohomology of $\mathscr{S}^{\mathrm{tor}}_{K,U}$.)

Proof. This is proved in the same way as Corollary 7.2: First one uses Lemma 7.3 to extend $\mathscr{V}$ to U, and then proper base change to show that the Chern classes of such an extension lie in the given kernel. □

7E. *Hirzebruch–Mumford proportionality.* The above results immediately imply a very strong form of Hirzebruch–Mumford proportionality in positive characteristic and the usual form of Hirzebruch–Mumford proportionality in characteristic 0.

Recall that an automorphic bundle on $S^{\mathrm{tor}}_{K,k}$ is by definition a vector bundle of the form $\sigma^{\mathrm{tor},*}(\mathscr{E})$ for some vector bundle $\mathscr{E}$ on $\mathtt{Hdg}_k = [G_k \backslash G_k/P_k]$. Let $X^\vee := G_k/P_k$ be the characteristic p version of the compact dual $\boldsymbol{X}^\vee$, and let

$$\rho : X^\vee \to \mathtt{Hdg}_k$$

be the projection. If we consider $\mathscr{E}$ as a G_k-equivariant vector bundle on $X_k^\vee$, then $\rho^*(\mathscr{E})$ is the underlying vector bundle.

Theorem 7.20. *There is an isomorphism $u : A^\bullet(X^\vee) \xrightarrow{\sim} \mathcal{T}_k^{\mathrm{tor}}$ of graded $\mathbb{Q}$-algebras such that for every G_k-equivariant vector bundle $\mathscr{E}$ on $X_k^\vee$ the i-th Chern class of the underlying vector bundle on $X_k^\vee$ is sent by u to the i-th Chern class of the automorphic bundle $\sigma^{\mathrm{tor},*}(\mathscr{E})$.*

Proof. The kernel of the surjective map $\rho^* : A^\bullet(\mathtt{Hdg}_k) \to A^\bullet(X_k^\vee)$ is the same as the kernel of the surjective map $\beta^* : A^\bullet(\mathtt{Hdg}_k) \to A^\bullet(G\text{-}\mathtt{Zip}_k^\mu)$ by Lemma 4.2. Hence we obtain some isomorphism of graded $\mathbb{Q}$-algebras $A^\bullet(X^\vee) \xrightarrow{\sim} A^\bullet(G\text{-}\mathtt{Zip}_k^\mu)$. Composing it with $\zeta^{\mathrm{tor},*} : A^\bullet(G\text{-}\mathtt{Zip}_k^\mu) \to \mathcal{T}_k^{\mathrm{tor}}$, which is an isomorphism by Theorem 7.12, we obtain the desired isomorphism u. □

For a smooth proper equidimensional scheme X over k, we denote by $\int_X : A^{\dim X}(X) \to \mathbb{Q}$ the degree map. Let $\mathbb{Q}[c_1, \dots, c_d]$ be the graded polynomial ring with $\deg(c_i) = i$.

The isomorphism u from Theorem 7.20 induces, in particular, an isomorphism of the 1-dimensional top-degree parts $A^d(X_k^\vee)$ and $\mathcal{T}_k^{\mathrm{tor},d}$, where $d := \dim(X^\vee) = \dim(S_K^{\mathrm{tor}})$. From this we obtain:

Corollary 7.21. *There exists a rational number $R \in \mathbb{Q}^\times$ such that for all classes $\alpha \in A^d(\mathtt{Hdg}_k)$ one has*

$$\int_{X_k^\vee} \rho^*(\alpha) = R \int_{S^{\mathrm{tor}}_{K,k}} \sigma^{\mathrm{tor},*}(\alpha).$$

As specialization of cycles commutes with taking degrees we obtain a new and purely algebraic proof of Hirzebruch–Mumford proportionality in characteristic 0. The original proof of this result is given in [Hirzebruch 1958] and [Mumford 1977].

Corollary 7.22. *There exists a rational number $R \in \mathbb{Q}^\times$ such that for all homogenous $f \in \mathbb{Q}[c_1, \dots, c_d]$ of degree d and all $\boldsymbol{G}_{\mathbb{C}}$-equivariant vector bundles $\mathscr{E}$ on $\boldsymbol{X}^\vee$ one has*

$$\int_{\boldsymbol{X}^\vee} f\big(c_1(\rho^*(\mathscr{E})), \dots, c_d(\rho^*(\mathscr{E}))\big) = R \int_{\boldsymbol{S}^{\mathrm{tor}}_{K,\mathbb{C}}} f\big(c_1(\sigma^{\mathrm{tor},*}(\mathscr{E})), \dots, c_d(\sigma^{\mathrm{tor},*}(\mathscr{E}))\big).$$

Proof. All $\boldsymbol{G}_{\mathbb{C}}$-equivariant vector bundles $\mathscr{E}$ on $\boldsymbol{X}^\vee$ are already defined over some splitting field E' of G that we may assume to be a finite extension of the reflex field. We now choose p and v' as in the proof of Corollary 7.2: let p be a prime number of good reduction for the Shimura datum $(\boldsymbol{G}, \boldsymbol{X})$ such that there exists an unramified place v' of E' over p. Let v be the restriction of v' to E, and let $\mathscr{S}^{\mathrm{tor}}_K$ be a smooth proper toroidal compactification of $\mathscr{S}_K$ over O_{E_v} with generic fiber $\boldsymbol{S}^{\mathrm{tor}}_{K,E_v}$. Consider the commutative diagram

$$\begin{array}{ccccc}
A^\bullet(\boldsymbol{X}^\vee_{E'}) & \xleftarrow{\rho^*} & A^\bullet(\mathtt{Hdg}_{E'}) & \xrightarrow{\sigma^{\mathrm{tor},*}} & A^\bullet(\boldsymbol{S}^{\mathrm{tor}}_{K,E'}) \\
\downarrow{\scriptstyle \mathrm{sp}} & & \downarrow{\scriptstyle \mathrm{sp}} & & \downarrow{\scriptstyle \mathrm{sp}} \\
A^\bullet(X^\vee_{\kappa'}) & \xleftarrow{\rho^*} & A^\bullet(\mathtt{Hdg}_{\kappa'}) & \xrightarrow{\sigma^{\mathrm{tor},*}} & A^\bullet(S^{\mathrm{tor}}_{K,\kappa'})
\end{array} \tag{7.23}$$

where the vertical maps are given by specialization. Then we have

$$\begin{aligned}
\int_{\boldsymbol{X}^\vee} f\big(c_1(\rho^*(\mathscr{E})), \dots, \rho^*(c_d(\mathscr{E}))\big) &= \int_{X^\vee_{\kappa'}} \mathrm{sp}\big(f\big(c_1(\rho^*(\mathscr{E})), \dots, c_d(\rho^*(\mathscr{E}))\big)\big) \\
&= \int_{X^\vee_{\kappa'}} \rho^*\big(\mathrm{sp}\big(f(c_1(\mathscr{E}), \dots, c_d(\mathscr{E}))\big)\big) \\
&= R \int_{S^{\mathrm{tor}}_{K,\kappa'}} \sigma^{\mathrm{tor},*}\big(\mathrm{sp}\big(f(c_1(\mathscr{E}), \dots, c_d(\mathscr{E}))\big)\big) \\
&= R \int_{S^{\mathrm{tor}}_{K,\kappa'}} \mathrm{sp}\big(\sigma^{\mathrm{tor},*}\big(f(c_1(\mathscr{E}), \dots, c_d(\mathscr{E}))\big)\big) \\
&= R \int_{\boldsymbol{S}^{\mathrm{tor}}_{K,\mathbb{C}}} f\big(c_1(\sigma^{\mathrm{tor},*}(\mathscr{E})), \dots, c_d(\sigma^{\mathrm{tor},*}(\mathscr{E}))\big).
\end{aligned}$$

Here the first and the last equality hold because taking the degree commutes with specialization [Fulton 1998, 20.3(a)], and the third equality is a special case of Corollary 7.21. □

The proof shows that the numbers R of Corollaries 7.21 and 7.22 coincide.

8. Examples

For a permutation $\pi \in S_n$, we also write $\pi = [\pi(1), \pi(2), \dots, \pi(n)]$. We will always denote by $\tau_{i,j}$ the transposition of i and j. For any permutation σ, one has $\sigma\tau_{i,j}\sigma^{-1} = \tau_{\sigma(i),\sigma(j)}$.

8A. ***Siegel case.*** Fix $g \geq 1$. We consider the vector space $\mathbb{F}_p^{2g}$ with the symplectic pairing

$$((a_i)_i, (b_i)_i) \mapsto \sum_{1\leq i\leq g} a_i b_{2g+1-i} - \sum_{g+1\leq i\leq 2g} a_i b_{2g+1-i}.$$

We take G to be the resulting group of symplectic similitudes and let μ be the cocharacter of G with weights $(1, \dots, 1, 0, \dots, 0)$ (with each weight having multiplicity g) on the above representation $\mathbb{F}_p^{2g}$ of G.

Let T be the group of diagonal matrices in G. We use the description of the Weyl group W of (G, T) given in [Viehmann and Wedhorn 2013, Section A7], i.e.,

$$W = \{w \in S_{2g} \mid w(i) + w(i^\perp) = 2g + 1\},$$

where $i^\perp := 2g+1-i$. Its simple reflections are $s_i = \tau_{i,i+1}\tau_{2g-i,2g+1-i}$ for $i = 1, \dots, g-1$ and $s_g = \tau_{g,g+1}$. Every element $w \in W$ is uniquely determined by $w(1), \dots, w(g)$. As G is split over $\mathbb{F}_p$, the Frobenius φ acts trivially on W.

We get a frame for the resulting zip datum by taking T to be the above torus, B the group of upper triangular matrices in G, and z the canonical representative of the element $[1+g, \dots, 2g, 1, \dots, g]$ of W. The types I and J of P and Q are both equal to $\{s_1, \dots, s_{g-1}\}$, and

$${}^I W = \{w \in W \mid w^{-1}(1) < \cdots < w^{-1}(g)\} = \{w \in W \mid w^{-1}(g+1) < \cdots < w^{-1}(2g)\}.$$

Elements of T have diagonal entries $(t_1, \dots, t_g, t_g^{-1}, \dots, t_1^{-1})$. Hence if for $1 \leq i \leq g$ we let $x_i \in X^*(T)$ be the character sending such an element to t_i, we obtain a basis $(x_1, \dots, x_g)$ of $X^*(T)$ which induces an isomorphism $S \cong \mathbb{Q}[x_1, \dots, x_g]$. The element $z \in W$ is given by $z(i) = g + i$ for all $i = 1, \dots, g$. It acts on $X^*(T)$ via $x_i \mapsto -x_{g+1-i}$. We have

$$z s_i z^{-1} = s_{g-i} \quad \text{for all } i = 1, \dots, g-1. \tag{8.1}$$

Computation of $\gamma(w)$. For $w \in {}^I W$, set $\sigma_w := \operatorname{int}(wz)$. Then we find

$$I_w = \bigcap_{m\geq 1} I_w^{(m)}, \quad I_w^{(m)} := \{s \in I \mid \sigma_w^k(s) \in I \text{ for all } k = 1, \dots, m\}$$

by (3.16). For instance, $s_i \in I_w^{(1)}$ if and only if $w(g-i)$ and $w(g+1-i)$ are both $\leq g$ or both $\geq g+1$. In this case $w(g+1-i) = w(g-i)+1$ and $\sigma_w(s)$ is $s_{w(g-i)}$ if $w(g-i) \leq g$ and it is $s_{w(g+1-i)^\perp}$ if $w(g-i) \geq g+1$.

We can consider I_w as a subset of vertices of the Dynkin diagram of G and get a subgraph with those edges in the Dynkin diagram of G that have vertices in I_w. As σ_w preserves angles between roots, it is an automorphism of the Dynkin diagram I_w. In particular, it permutes all connected components of I_w. Let $\mathfrak{c}$ be a $\sigma_w^{\mathbb{Z}}$-orbit of such connected components. Choose some connected component C in $\mathfrak{c}$.

Let $m(\mathfrak{c})$ be the number of vertices in C, and let $l(\mathfrak{c})$ be the minimal integer $n \geq 1$ such that $\sigma_w^n(C) = C$. We say that $\mathfrak{c}$ is of linear type if $\sigma^{l(\mathfrak{c})}(w) = w$ for all $w \in C$. Otherwise it is called of unitary type. Then $m(\mathfrak{c})$, $l(\mathfrak{c})$, and the type do not depend on the choice of C.

Now we can calculate $\gamma(w)$ by (3.20) as follows. If $\mathfrak{c}$ is of linear type, then we let $\gamma_{\mathfrak{c}}(w)$ be the number of $\mathbb{F}_p$-valued points in the full flag variety of the scalar restriction of $\mathrm{GL}_{m(\mathfrak{c})+1}$ over $\mathbb{F}_{p^{l(\mathfrak{c})}}$, i.e.,

$$\gamma_{\mathfrak{c}}(w) = \sum_{\pi \in S_{m(\mathfrak{c})+1}} p^{l(\mathfrak{c})\ell(\pi)} = \prod_{1 \leq j \leq m(\mathfrak{c})} \frac{q^{j+1}-1}{q-1},$$

where $q := p^{l(\mathfrak{c})}$. If $\mathfrak{c}$ is of unitary type, then we let $\gamma_{\mathfrak{c}}(w)$ be the number of $\mathbb{F}_p$-valued points in the full flag variety of the scalar restriction of a unitary group in $m+1$ variables over $\mathbb{F}_{p^{l(\mathfrak{c})}}$. To describe this concretely, we let τ be the conjugation with the longest element in the symmetric group $S_{m(\mathfrak{c})+1}$, an automorphism of Coxeter groups of order 2 (except if $m(\mathfrak{c}) = 1$). For $\pi \in S_{m(\mathfrak{c})+1}$, set $\delta(\pi) := p^{2l(\mathfrak{c})\ell(\pi)}$ if $\pi \neq \tau(\pi)$ and $\delta(\pi) := p^{l(\mathfrak{c})\ell(\pi)}$ if $\tau(\pi) = \pi$. Then

$$\gamma_{\mathfrak{c}}(w) = \sum_{\pi \in S_{m(\mathfrak{c})+1}/\tau} \delta(\pi).$$

Altogether we obtain

$$\gamma(w) = \prod_{\mathfrak{c}} \gamma_{\mathfrak{c}}(w),$$

where $\mathfrak{c}$ runs through all orbits of connected components of I_w.

For instance, fix $0 \leq f \leq g$, and let u_f be the permutation

$$u_f := [g+1,\ g+2,\ \ldots,\ g+f,\ 1,\ g+f+1,\ \ldots,\ 2g-1,\ 2,\ \ldots,\ g-f,\ 2g,\ g-f+1,\ \ldots,\ g].$$

Then $u_g = z$ and $\bar{Z}_{u_0}$ is the locus where the p-rank is 0. We have

$$I_{u_f} = I \setminus \{s_1, s_2, \ldots, s_{g-f}\},$$

and in particular, $I_{u_1} = I_{u_0} = \varnothing$. Moreover, I_{u_f} has only one connected component and it is of linear type. Therefore

$$\gamma(u_f) = \prod_{1 \leq j \leq f-1} \frac{p^{j+1}-1}{p-1}.$$

Cycle classes. By Example 2.6, we find that

$$[\mathtt{Brh}_e] = \prod_{1 \leq i < j \leq g} (x_i \otimes 1 - 1 \otimes x_j)\Gamma_g(c_1, \ldots, c_g), \tag{8.2}$$

where $c_i = \sigma_i(x_1, \ldots, x_g) \otimes 1 + 1 \otimes \sigma_i(x_1, \ldots, x_g)$ for $i = 1, \ldots, g$, and we set $c_0 = 2$ and $c_i = 0$ for all $i \notin \{0, \ldots, g\}$.

The operators δ_{s_i} act on S by

$$\delta_{s_i}(f) = \frac{f(x_1, \ldots, x_g) - f(x_1, \ldots, x_{i+1}, x_i, \ldots, x_g)}{x_i - x_{i+1}}$$

for $i = 1, \dots, g-1$, and by

$$\delta_{s_g}(f) = \frac{f(x_1, \dots, x_g) - f(x_1, \dots, x_{g-1}, -x_g)}{2x_g}$$

for $i = g$.

The element z acts on S by $x_i \mapsto -x_{g+1-i}$. Since the torus T is split over $\mathbb{F}_p$, the Frobenius φ acts on S by $x_i \mapsto px_i$. Hence ψ^* sends $x_i \otimes 1$ to $-x_{g+1-i}$ and $1 \otimes x_i$ to px_i. Thus for $w \in W$ we find

$$[\overline{\mathtt{Brh}}_w] = \delta_w\Biggl(\prod_{1\le i<j\le g}(x_i \otimes 1 - 1 \otimes x_j)\Gamma_g(c_1, \dots, c_g)\Biggr)$$

and

$$[\bar{Z}^{\varnothing}_w] = \psi^*([\overline{\mathtt{Brh}}_w]). \tag{8.3}$$

Such a formula is already given in [Ekedahl and van der Geer 2009, Theorem 12.1]. The formula in loc. cit. agrees with (8.3) if one takes the following into account: We believe that in loc. cit. there is a typo and the polynomial should be evaluated at $y_j = p\ell_{g+1-j}$ instead of $y_j = p\ell_j$. Then the formulas agree under the substitution $x_i = \ell_{g+1-i}$.

The case $g = 2$. As an example, let us consider the case $g = 2$. We let

$$\begin{aligned}\Phi &:= x_1 \otimes 1 - 1 \otimes x_2,\\ \Gamma &:= c_1c_2 = ((x_1+x_2)\otimes 1 + 1 \otimes (x_1+x_2))(x_1x_2 \otimes 1 + 1 \otimes x_1x_2),\end{aligned}$$

so that

$$[\mathtt{Brh}_e] = \Phi\Gamma.$$

The set ${}^IW$ consists of the elements $\{e, s_2, s_2s_1, s_2s_1s_2\}$. By applying the operators δ_w, we find

$$\begin{aligned}[\mathtt{Brh}_{s_2}] &= \Phi(x_1 \otimes 1 + 1 \otimes x_1)(x_1 \otimes 1 + 1 \otimes x_2),\\ [\mathtt{Brh}_{s_2s_1}] &= (x_1 \otimes 1 + 1 \otimes x_1)(x_1 \otimes 1 + 1 \otimes x_2),\\ [\mathtt{Brh}_{s_2s_1s_2}] &= x_1 \otimes 1 + 1 \otimes x_1.\end{aligned}$$

Applying ψ^* yields

$$\begin{aligned}[\bar{Z}^{\varnothing}_e] &= -(p^4-1)(x_1+x_2)x_1x_2^2,\\ [\bar{Z}^{\varnothing}_{s_2}] &= -(p^2-1)(px_1-x_2)x_2^2,\\ [\bar{Z}^{\varnothing}_{s_2s_1}] &= (p-1)(px_1-x_2)x_2,\\ [\bar{Z}^{\varnothing}_{s_2s_1s_2}] &= px_1 - x_2.\end{aligned} \tag{8.4}$$

We have $I = I^\circ = \{s_1\}$. Hence, by Theorem 4.17, $\pi_* = \delta_{s_1}$. Since I has only a single element, we see that for $w \in {}^IW$ either $\sigma_w(s_1) = s_1$ and hence $I_w = I$ or $I_w = \varnothing$. In the first case we find $\gamma(w) = p+1$, in the second $\gamma(w) = 1$. Using this we find

$$\gamma(e) = \gamma(s_2s_1s_2) = p+1 \quad \text{and} \quad \gamma(s_2) = \gamma(s_2s_1) = 1. \tag{8.5}$$

Altogether we obtain the following formulas for the classes of the EO-strata:

$$\begin{aligned}
[\bar{Z}_e] &= (p+1)(p^4-1)(x_1+x_2)x_1x_2, \\
[\bar{Z}_{s_2}] &= (p^2-1)((p-1)x_1x_2 - x_1^2 - x_2^2), \\
[\bar{Z}_{s_2s_1}] &= (p-1)(x_1+x_2), \\
[\bar{Z}_{s_2s_1s_2}] &= (p+1)^2.
\end{aligned} \tag{8.6}$$

These formulas agree with the ones given in [Ekedahl and van der Geer 2009, 12.2] with x_i corresponding to ℓ_{g+1-i}, except that it appears that in loc. cit. the rows for s_1s_2 and s_2s_1 should be switched and the entry for $\pi_*([\overline{U}_{s_2}])$ is incorrect.

8B. ***Hilbert–Blumenthal case.*** Fix $d \geq 1$, and let $\widetilde{G} := \mathrm{Res}_{\mathbb{F}_{p^d}/\mathbb{F}_p} \mathrm{GL}_2$. Define G by the cartesian diagram

$$\begin{tikzcd}
G \arrow[r] \arrow[d] & \mathbb{G}_{m,\mathbb{F}_p} \arrow[d] \\
\widetilde{G} \arrow[r, "\det"] & \mathrm{Res}_{\mathbb{F}_{p^d}/\mathbb{F}_p} \mathbb{G}_m
\end{tikzcd}$$

where the right vertical map is the canonical embedding. Let Σ be the set of embeddings $\mathbb{F}_{p^d} \hookrightarrow k$. We fix an embedding ι_0 and identify the set $\mathbb{Z}/d\mathbb{Z}$ with Σ via $i \mapsto \sigma^{-i}\iota_0$, where $\sigma : x \to x^p$ is the arithmetic Frobenius. Let $\tilde{\mu}$ be the cocharacter of $\widetilde{G}_k = \prod_\Sigma \mathrm{GL}_2$ given by $t \mapsto \left(\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right)$ in each component. Then $\tilde{\mu}$ factors through a cocharacter μ of G. Let $\widetilde{T}$ be the standard torus of $\widetilde{G}$, i.e., $\widetilde{T}_k$ is the product of the diagonal tori. For $i \in \mathbb{Z}/d\mathbb{Z}$ and $j = 1, 2$, let $x_j^{(i)}$ be the character

$$\left(\begin{pmatrix} t_1^{(i)} & 0 \\ 0 & t_2^{(i)} \end{pmatrix}\right)_{i\in\mathbb{Z}/d\mathbb{Z}} \mapsto t_j^{(i)}$$

of $\widetilde{T}_k$. Then $\widetilde{S} = \mathrm{Sym}(X^*(\widetilde{T})_{\mathbb{Q}}) = \mathbb{Q}[x_1^{(i)}, x_2^{(i)}; i \in \mathbb{Z}/d\mathbb{Z}]$. The intersection $T = \widetilde{T} \cap G$ is a maximal torus of G and $S = \mathrm{Sym}(X^*(T)_{\mathbb{Q}})$ identifies with the quotient of $\widetilde{S}$ by the ideal generated by $(x_1^{(i)} + x_2^{(i)}) - (x_1^{(i+1)} + x_2^{(i+1)})$ for $i \in \mathbb{Z}/d\mathbb{Z}$. We will compute all cycle classes of EO-strata for $(\widetilde{G}, \tilde{\mu})$. This yields then also the corresponding cycle classes for (G, μ) by Section 4E.

Let $\widetilde{B}$ be the Borel subgroup of $\widetilde{G}$ such that $\widetilde{B}_k$ is the product of groups of upper triangular matrices in GL_2. The Weyl group is $W = \{\pm 1\}^{\mathbb{Z}/d\mathbb{Z}}$ and we have $I = J = \varnothing$. Thus ${}^IW = W$. As a frame for $(\widetilde{G}, \tilde{\mu})$ we choose $(\widetilde{T}, \widetilde{B}, z)$ with z a representative of $(-1, \ldots, -1) \in W$.

By (2.8), we have

$$[\mathtt{Brh}_e] = \prod_{i\in\mathbb{Z}/d\mathbb{Z}} (x_1^{(i)} \otimes 1 - 1 \otimes x_2^{(i)}) \in A^\bullet(\mathtt{Brh}_{\widetilde{G}}).$$

Let $w = (\epsilon_i)_{i\in\mathbb{Z}/d\mathbb{Z}} \in W$. Then $\ell(w) = \#\{i \in \mathbb{Z}/d\mathbb{Z} \mid \epsilon_i = -1\}$. We have

$$[\overline{\mathtt{Brh}}_w] = \prod_{\substack{i\in\mathbb{Z}/d\mathbb{Z} \\ \epsilon_i = 1}} (x_1^{(i)} \otimes 1 - 1 \otimes x_2^{(i)}),$$

and hence

$$[\bar{Z}_w^{\varnothing}] = \psi^*([\overline{\mathtt{Brh}}_w]) = \prod_{\substack{i\in\mathbb{Z}/d\mathbb{Z}\\ \epsilon_i=1}} (x_2^{(i)} - px_2^{(i+1)}).$$

With the notation of Section 3F, we find $I_w = \varnothing$ and $L_w = T$. Hence $\gamma(w) = 1$ for all $w \in W$. Also, π is an isomorphism. Therefore we have isomorphisms

$$A^\bullet(\widetilde{G}\text{-}\mathtt{ZipFlag}^{\tilde{\mu}}) \cong A^\bullet(\widetilde{G}\text{-}\mathtt{Zip}^{\tilde{\mu}}) \cong A^\bullet(G\text{-}\mathtt{Zip}^{\mu}) \cong A^\bullet(G\text{-}\mathtt{ZipFlag}^{\mu})$$

in this case. From the description of $A^\bullet(G\text{-}\mathtt{ZipFlag}^{\mu})$ in Proposition 4.8(2) one deduces easily that $x_2^{(i)} \mapsto z_i$ yields an isomorphism of graded $\mathbb{Q}$-algebras

$$A^\bullet(G\text{-}\mathtt{Zip}^{\mu}) \cong \mathbb{Q}[z_0, \dots, z_{d-1}]/(z_0^2, \dots, z_{d-1}^2).$$

Via this isomorphism we get, for the cycle classes of the $\bar{Z}_w$,

$$[\bar{Z}_w] = \prod_{\substack{i\in\mathbb{Z}/d\mathbb{Z}\\ \epsilon_i=1}} (z_i - pz_{i+1}) \in A^\bullet(G\text{-}\mathtt{Zip}^{\mu}).$$

To describe the Hodge half-line in $A^\bullet(G\text{-}\mathtt{Zip}^{\mu})$ we use the notation from Section 7B. The restriction of the standard embedding ι of G into GSp_{2d} to the maximal torus is given by

$$\left(\begin{pmatrix} t_1^{(i)} & 0 \\ 0 & t_2^{(i)} \end{pmatrix}\right)_{i\in\mathbb{Z}/d\mathbb{Z}} \mapsto \mathrm{diag}(t_1^{(0)}, t_1^{(1)}, \dots, t_1^{(d-1)}, t_2^{(d-1)}, \dots, t_2^{(0)}).$$

Therefore the character $\chi(\iota)$ (see (7.4)) is given by

$$\left(\begin{pmatrix} t_1^{(i)} & 0 \\ 0 & t_2^{(i)} \end{pmatrix}\right)_{i\in\mathbb{Z}/d\mathbb{Z}} \mapsto \prod_{i\in\mathbb{Z}/d\mathbb{Z}} (t_2^{(i)})^{-1},$$

and the Hodge half-line consists of all $\mathbb{Q}_{>0}$-multiples of the class of

$$\lambda := -\sum_{i\in\mathbb{Z}/d\mathbb{Z}} x_2^{(i)} = -(z_0 + \dots + z_{d-1}) \in A^\bullet(G\text{-}\mathtt{Zip}^{\mu}).$$

Hence (as an illustration of Proposition 7.7) we see that

$$[Z_{\le d-1}] = \sum_{i\in\mathbb{Z}/d\mathbb{Z}} (z_i - pz_{i+1}) = (p-1)\lambda \quad \text{and} \quad [Z_e] = (1 + (-1)^d p^d) \prod_{i\in\mathbb{Z}/d\mathbb{Z}} z_i = \frac{p^d + (-1)^d}{d!}\lambda^d.$$

8C. ***The odd spin case.*** We assume that $p > 2$. Let (V, Q) be a quadratic space over $\mathbb{F}_p$ of odd dimension $2n+1 \ge 3$. We denote by $C(V) = C^+(V) \oplus C^-(V)$ its Clifford algebra. It is a $\mathbb{Z}/2\mathbb{Z}$-graded (noncommutative) $\mathbb{F}_p$-algebra of dimension 2^{2n+1} generated as an algebra by the image of the canonical injective $\mathbb{F}_p$-linear map $V \hookrightarrow C(V)$. It is endowed with an involution * uniquely determined by $(v_1 \cdots v_r)^* = v_r \cdots v_1$ for $v_1, \dots, v_r \in V$.

The *spinor similitude group* is the reductive group $G = \mathrm{GSpin}(V)$ over $\mathbb{F}_p$ defined by

$$G(R) = \{g \in C^+(V_R)^\times \mid gV_Rg^{-1} = V_R, g^*g \in R^\times\}.$$

Then $g \mapsto (v \mapsto g \bullet v := gvg^{-1})$ defines a surjective map of reductive groups $G \to \widetilde{G} := \mathrm{SO}(V)$ with kernel $\mathbb{G}_m$. The groups G and $\widetilde{G}$ are both of Dynkin type B_n.

We now assume that we can find an $\mathbb{F}_p$-basis $(v_0, v_1, \ldots, v_{2n})$ such that the matrix of the bilinear form attached to Q with respect to this basis is given by

$$\begin{pmatrix} 1 & & & & & \\ & & & & & 1 \\ & & & & 1 & \\ & & & \iddots & & \\ & & 1 & & & \\ & 1 & & & & \end{pmatrix}.$$

Although there are two isomorphism classes of quadratic spaces over $\mathbb{F}_p$ of dimension $2n + 1$, the associated groups GSpin and SO are isomorphic. Hence our assumption is harmless.

We define the cocharacter $\mu : \mathbb{G}_m \to G$ by

$$\mu(t) = tv_1v_{2n} + v_{2n}v_1.$$

The composition of μ with $G \to \mathrm{SO}(V)$ yields the cocharacter

$$\tilde{\mu} : \mathbb{G}_m \to \mathrm{SO}(V), \quad t \mapsto \mathrm{diag}(1, t, 1, \ldots, 1, t^{-1}).$$

We will compute the cycle classes of EO-strata for $(\widetilde{G}, \tilde{\mu})$. Again by Section 4E this yields then also the corresponding cycle classes for (G, μ).

As a maximal torus $\widetilde{T}$ for $\widetilde{G} = \mathrm{SO}(V)$ we choose

$$\widetilde{T} = \{\mathrm{diag}(1, t_1, \ldots, t_n, t_n^{-1}, \ldots, t_1^{-1}) \mid t_i \in \mathbb{G}_m\}.$$

For $i = 1, \ldots, n$, let x_i be the character $\mathrm{diag}(1, t_1, \ldots, t_n, t_n^{-1}, \ldots, t_1^{-1}) \mapsto t_i$. Then $\widetilde{S} = \mathbb{Q}[x_1, \ldots, x_n]$. The Weyl group W is the group

$$W = \{w \in S_{2n} \mid w(i) + w(2n + 1 - i) = 2n + 1 \text{ for all } i\}$$

acting on T in the standard way via the last $2n$ coordinates. The roots of $(\widetilde{G}, \widetilde{T})$ are given by $\pm x_i \pm x_j$ for $1 \le i \ne j \le n$ and $\pm x_i$ for $i = 1, \ldots, n$. Let $\widetilde{B}$ be the Borel subgroup such that the corresponding simple roots are given by $x_1 - x_2, \ldots, x_{n-1} - x_n, x_n$. Then the set Σ of simple reflections in W corresponding to B is given by $s_1, \ldots, s_{n-1}, s_n$, where s_i is the transposition $\tau_{i,i+1}\tau_{2n-i,2n+1-i}$ for $i = 1, \ldots, n - 1$ and $s_n = \tau_{n,n+1}$.

Let $\tilde{z} \in \widetilde{G}$ be a lift of the element in W with $1 \mapsto 2n$, $2n \mapsto 1$ and $i \mapsto i$ for all $i = 2, \ldots, 2n - 1$. Then $(\widetilde{B}, \widetilde{T}, \tilde{z})$ is a frame by Lemma 3.3.

As I is the set of simple reflections corresponding to simple roots α with $\langle \mu, \alpha \rangle = 0$, we find $I = \{s_2, \ldots, s_n\}$. Hence we find a bijection

$$ {}^I W = \{w^{-1} \in W \mid w(2) < w(3) < \cdots < w(2n-1)\} \xrightarrow{\sim} \{1, \ldots, 2n\}, \quad w \mapsto w^{-1}(1). \tag{8.7} $$

Moreover, $\ell(w) = w^{-1}(1) - 1$ for $w \in {}^I W$. By parts (2) and (4) of Lemma 3.8, this implies that the order $\preceq$ coincides with the Bruhat order on ${}^I W$ and that (8.7) is an isomorphism of ordered sets. There is a concrete reduced expression of $w \in {}^I W$ as a product of simple reflections:

$$ w = \begin{cases} s_1 s_2 \cdots s_{\ell(w)} & \text{if } \ell(w) \leq n; \\ s_1 s_2 \cdots s_n s_{n-1} \cdots s_{2n-\ell(w)} & \text{if } \ell(w) > n. \end{cases} \tag{8.8} $$

For all $s \in I$, one has ${}^z s = s$. As $\widetilde{G}$ is split over $\mathbb{F}_p$, the Frobenius φ acts trivially on W. Therefore, for $w \in {}^I W$, the subset $I_w \subseteq I$ defined in Section 3F is the largest subset such that ${}^w I_w = I_w$. Hence

$$ \begin{gathered} I_e = I, \quad I_{s_1} = I \setminus \{s_2\}, \quad \ldots \quad I_{s_1 \cdots s_{n-1}} = \varnothing, \quad I_{s_1 \cdots s_n} = \varnothing, \\ I_{s_1 \cdots s_n s_{n-1}} = \varnothing, \quad I_{s_1 \cdots s_n s_{n-1} s_{n-2}} = \{s_n\}, \quad \ldots \quad I_{s_1 \cdots s_n \cdots s_1} = I \setminus \{s_2\}. \end{gathered} \tag{8.9} $$

Hence $F\ell_w$ is the flag variety of a split group over $\mathbb{F}_p$ of Dynkin type B_k, where

$$ k = \begin{cases} n - 1 - \ell(w) & \text{if } \ell(w) \leq n-1, \\ 0 & \text{if } \ell(w) = n, \\ \ell(w) - n + 1 & \text{if } \ell(w) \geq n+1. \end{cases} $$

By Example 2.6, we find that

$$ [\mathtt{Brh}_e] = \prod_{1 \leq i < j \leq n} (x_i \otimes 1 - 1 \otimes x_j) \Gamma_n(c_1, \ldots, c_n), \tag{8.10} $$

where $c_i = \frac{1}{2}(\sigma_i(x_1, \ldots, x_n) \otimes 1 + 1 \otimes \sigma_i(x_1, \ldots, x_n))$ for $i = 1, \ldots, n$, and we set $c_0 = 1$ and $c_i = 0$ for all $i \notin \{0, \ldots, n\}$. For instance, if $n = 2, 3$, we find

$$ [\mathtt{Brh}_e] = \begin{cases} (x_1 \otimes 1 - 1 \otimes x_2) c_1 c_2 & \text{if } n = 2, \\ \prod_{1 \leq i < j \leq 3} (x_i \otimes 1 - 1 \otimes x_j) c_3 (c_1 c_2 - c_3) & \text{if } n = 3. \end{cases} \tag{8.11} $$

The operators δ_{s_i} from Section 2E act on $\widetilde{S}$ by

$$ \delta_{s_i}(f) = \frac{f(x_1, \ldots, x_n) - f(x_1, \ldots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \ldots, x_n)}{x_i - x_{i+1}} $$

for $1 \leq i \leq n-1$ and by

$$ \delta_{s_n}(f) = \frac{f(x_1, \ldots, x_n) - f(x_1, \ldots, x_{n-1}, -x_n)}{x_n} $$

for $i = n$.

The homomorphism $\tilde{\psi}^* : \widetilde{S} \otimes \widetilde{S} \to \widetilde{S}$ is given by $x_1 \otimes 1 \mapsto -x_1,\ x_i \otimes 1 \mapsto x_i$ for $i = 2, \ldots, n$ and $1 \otimes x_i \mapsto p x_i$ for $i = 1, \ldots, n$.

Finally, by Lemma 4.18, the operator $\tilde{\pi}_*$ is given by $\delta_{w_{0,I^\circ}}$, where w_{0,I° is the longest element of the Weyl group $W_{I^\circ} = W_{\{s_2, \ldots, s_n\}}$ of type B_{n-1}.

The case $n = 2$. For $i = 1, \dots, 4$, we denote the element of length $i-1$ by $w_i \in {}^I W$. By (8.9), we find

$$\gamma(w_1) = \gamma(w_4) = \#\mathbb{P}^1(\mathbb{F}_p) = p+1 \quad \text{and} \quad \gamma(w_2) = \gamma(w_3) = 1. \tag{8.12}$$

Using the above we find the following formulas for the classes of the closures of the EO-strata:

$$\begin{aligned}
[\bar{Z}_{w_1}] &= (p+1)\tfrac{1}{2}(1-p^2)((p^2+p)x_2^2 + (p-1)x_1^2)x_1, \\
[\bar{Z}_{w_2}] &= \tfrac{1}{2}(p^2-1)(p-1)x_1^2, \\
[\bar{Z}_{w_3}] &= (p-1)x_1, \\
[\bar{Z}_{w_4}] &= (p+1)^2.
\end{aligned} \tag{8.13}$$

Since the Dynkin diagrams of type B_2 and C_2 are isomorphic, by Proposition 4.23, the Chow rings of the associated moduli spaces of G-zips are isomorphic. Indeed one can check that the formulas in (8.13) match those in (8.6) (up to terms in S_+^W) under the isomorphism

$$\mathbb{Q}[x_1, x_2] \to \mathbb{Q}[x_1, x_2], \qquad x_1 \mapsto x_1 + x_2, \quad x_2 \mapsto x_1 - x_2.$$

The case $n = 3$. For $i = 1, \dots, 6$, we denote, as above, the element of length $i-1$ by $w_i \in {}^I W$. By (8.9),

$$\gamma(w_1) = \gamma(w_6) = p^3+2p^2+2p+1, \quad \gamma(w_2) = \gamma(w_5) = \#\mathbb{P}^1(\mathbb{F}_p) = p+1, \quad \gamma(w_3) = \gamma(w_4) = 1. \tag{8.14}$$

Using the above we find the following formulas for the classes of the closures of the EO-strata:

$$\begin{aligned}
[\bar{Z}_{w_1}] &= \tfrac{1}{2}(p^3+2p^2+2p+1)(p^2+p+1)(p+1)^2(p-1) \\
&\qquad \cdot (p^4x_2^2x_3^2 + p^3x_1^2x_2^2 + p^3x_1^2x_3^2 + p^2x_1^4 + p^2x_2^2x_3^2 - 2px_1^4 - px_1^2x_2^2 - px_1^2x_3^2 + x_1^4)x_1, \\
[\bar{Z}_{w_2}] &= -\tfrac{1}{2}(p+1)^3(p-1)^2(p^2x_2^2 + p^2x_3^2 + px_1^2 - x_1^2)x_1^2, \\
[\bar{Z}_{w_3}] &= \tfrac{1}{2}(p^2+p+1)(p+1)(p-1)^3x_1^3, \\
[\bar{Z}_{w_4}] &= (p+1)(p-1)^2x_1^2, \\
[\bar{Z}_{w_5}] &= (p+1)^2(p-1)x_1, \\
[\bar{Z}_{w_6}] &= (p^3+2p^2+2p+1)(p^2+1)(p+1)^2.
\end{aligned}$$

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References

[Andreatta 2023] F. Andreatta, "On two mod p period maps: Ekedahl–Oort and fine Deligne–Lusztig stratifications", *Math. Ann.* **385**:1-2 (2023), 1–40. MR

[Bernšteĭn et al. 1973] I. N. Bernšteĭn, I. M. Gelfand, and S. I. Gelfand, "Schubert cells, and the cohomology of the spaces G/P", *Uspehi Mat. Nauk* **28**:3 (1973), 3–26. In Russian; translated in *Russ. Math. Surv.* **28**:1 (1973), 1–26. MR Zbl

[Bhatt and Scholze 2015] B. Bhatt and P. Scholze, "The pro-étale topology for schemes", pp. 99–201 in *De la géométrie algébrique aux formes automorphes*, *I*: *une collection d'articles en l'honneur du soixantième anniversaire de Gérard Laumon*, edited by J.-B. Bost et al., Astérisque **369**, 2015. MR Zbl

[Borel 1953] A. Borel, "Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts", *Ann. of Math.* (2) **57** (1953), 115–207. MR Zbl

[Boxer 2015] G. A. Boxer, *Torsion in the coherent cohomology of Shimura varieties and Galois representations*, 2015. MR

[Brion 1996] M. Brion, "The push-forward and Todd class of flag bundles", pp. 45–50 in *Parameter spaces* (Warsaw, 1994), edited by P. Pragacz, Banach Center Publ. **36**, Polish Acad. Sci. Inst. Math., Warsaw, 1996. MR Zbl

[Brion and Kumar 2005] M. Brion and S. Kumar, *Frobenius splitting methods in geometry and representation theory*, Progress in Mathematics **231**, Birkhäuser, Boston, 2005. MR Zbl

[Brokemper 2016] D. Brokemper, "On the Chow ring of the classifying space of some Chevalley groups", preprint, 2016. arXiv 1611.07735

[Brokemper 2018] D. Brokemper, "On the Chow ring of the stack of truncated Barsotti–Tate groups", *Pacific J. Math.* **296**:2 (2018), 271–303. MR Zbl

[Conrad 2014] B. Conrad, "Reductive group schemes", pp. 93–444 in *Autour des schémas en groupes*, *I*, Panor. Synthèses **42/43**, Soc. Math. France, Paris, 2014. MR Zbl

[Deligne 1979] P. Deligne, "Variétés de Shimura: interprétation modulaire, et techniques de construction de modèles canoniques", pp. 247–289 in *Automorphic forms*, *representations and L-functions*, *II* (Corvallis, Ore., 1977), edited by A. Borel and W. Casselman, Proc. Sympos. Pure Math. **33**, Amer. Math. Soc., Providence, R.I., 1979. MR Zbl

[Demazure 1973] M. Demazure, "Invariants symétriques entiers des groupes de Weyl et torsion", *Invent. Math.* **21** (1973), 287–301. MR Zbl

[Demazure 1974] M. Demazure, "Désingularisation des variétés de Schubert généralisées", *Ann. Sci. École Norm. Sup.* (4) **7** (1974), 53–88. MR Zbl

[Edidin and Graham 1997] D. Edidin and W. Graham, "Characteristic classes in the Chow ring", *J. Algebraic Geom.* **6**:3 (1997), 431–443. MR Zbl

[Edidin and Graham 1998] D. Edidin and W. Graham, "Equivariant intersection theory", *Invent. Math.* **131**:3 (1998), 595–634. MR Zbl

[Ekedahl and van der Geer 2009] T. Ekedahl and G. van der Geer, "Cycle classes of the E–O stratification on the moduli of abelian varieties", pp. 567–636 in *Algebra*, *arithmetic*, *and geometry*: *in honor of Yu. I. Manin*, *I*, edited by Y. Tschinkel and Y. Zarhin, Progr. Math. **269**, Birkhäuser, Boston, 2009. MR Zbl

[Esnault and Harris 2017] H. Esnault and M. Harris, "Chern classes of automorphic vector bundles", *Pure Appl. Math. Q.* **13**:2 (2017), 193–213. MR Zbl

[Esnault and Viehweg 2002] H. Esnault and E. Viehweg, "Chern classes of Gauss–Manin bundles of weight 1 vanish", *K-Theory* **26**:3 (2002), 287–305. MR Zbl

[Fulton 1996] W. Fulton, "Determinantal formulas for orthogonal and symplectic degeneracy loci", *J. Differential Geom.* **43**:2 (1996), 276–290. MR Zbl

[Fulton 1998] W. Fulton, *Intersection theory*, 2nd ed., Ergebnisse der Math. (3) **2**, Springer, 1998. MR Zbl

[van der Geer 1999] G. van der Geer, "Cycles on the moduli space of abelian varieties", pp. 65–89 in *Moduli of curves and abelian varieties*, edited by C. Faber and E. Looijenga, Aspects Math. **33**, Vieweg Verlag, 1999. MR Zbl

[Goldring and Koskivirta 2018] W. Goldring and J.-S. Koskivirta, "Quasi-constant characters: motivation, classification and applications", *Adv. Math.* **339** (2018), 336–366. MR Zbl

[Goldring and Koskivirta 2019a] W. Goldring and J.-S. Koskivirta, "Strata Hasse invariants, Hecke algebras and Galois representations", *Invent. Math.* **217**:3 (2019), 887–984. MR Zbl

[Goldring and Koskivirta 2019b] W. Goldring and J.-S. Koskivirta, "Stratifications of flag spaces and functoriality", *Int. Math. Res. Not.* **2019**:12 (2019), 3646–3682. MR Zbl

[Graham 1997] W. Graham, "The class of the diagonal in flag bundles", *J. Differential Geom.* **45**:3 (1997), 471–487. MR Zbl

[He 2007] X. He, "The G-stable pieces of the wonderful compactification", *Trans. Amer. Math. Soc.* **359**:7 (2007), 3005–3024. MR Zbl

[Hirzebruch 1958] F. Hirzebruch, "Automorphe Formen und der Satz von Riemann–Roch", pp. 129–144 in *Symposium internacional de topología algebraica*, Universidad Nacional Autónoma de México and UNESCO, Mexico City, 1958. MR Zbl

[Jannsen 1988] U. Jannsen, "Continuous étale cohomology", *Math. Ann.* **280**:2 (1988), 207–245. MR Zbl

[Kim and Madapusi Pera 2016] W. Kim and K. Madapusi Pera, "2-adic integral canonical models", *Forum Math. Sigma* **4** (2016), art. id. e28. MR Zbl

[Kisin 2010] M. Kisin, "Integral models for Shimura varieties of abelian type", *J. Amer. Math. Soc.* **23**:4 (2010), 967–1012. MR Zbl

[Koskivirta 2018] J.-S. Koskivirta, "Normalization of closed Ekedahl–Oort strata", *Canad. Math. Bull.* **61**:3 (2018), 572–587. MR Zbl

[Kresch 1999] A. Kresch, "Cycle groups for Artin stacks", *Invent. Math.* **138**:3 (1999), 495–536. MR Zbl

[Lan 2013] K.-W. Lan, *Arithmetic compactifications of PEL-type Shimura varieties*, London Mathematical Society Monographs Series **36**, Princeton Univ. Press, 2013. MR Zbl

[Madapusi Pera 2019] K. Madapusi Pera, "Toroidal compactifications of integral models of Shimura varieties of Hodge type", *Ann. Sci. Éc. Norm. Supér.* (4) **52**:2 (2019), 393–514. MR Zbl

[Milne 1990] J. S. Milne, "Canonical models of (mixed) Shimura varieties and automorphic vector bundles", pp. 283–414 in *Automorphic forms, Shimura varieties, and L-functions*, *I* (Ann Arbor, MI, 1988), edited by L. Clozel and J. S. Milne, Perspect. Math. **10**, Academic Press, Boston, 1990. MR Zbl

[Mumford 1977] D. Mumford, "Hirzebruch's proportionality theorem in the noncompact case", *Invent. Math.* **42** (1977), 239–272. MR Zbl

[Oort 2001] F. Oort, "A stratification of a moduli space of abelian varieties", pp. 345–416 in *Moduli of abelian varieties* (Texel Island, 1999), edited by C. Faber et al., Progr. Math. **195**, Birkhäuser, Basel, 2001. MR Zbl

[Pink et al. 2011] R. Pink, T. Wedhorn, and P. Ziegler, "Algebraic zip data", *Doc. Math.* **16** (2011), 253–300. MR Zbl

[Pink et al. 2015] R. Pink, T. Wedhorn, and P. Ziegler, "F-zips with additional structure", *Pacific J. Math.* **274**:1 (2015), 183–236. MR Zbl

[SGA 2 2005] A. Grothendieck, *Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux* (Séminaire de Géométrie Algébrique du Bois Marie 1962), Documents Mathématiques **4**, Soc. Math. France, Paris, 2005. Revised reprint of the 1968 French original. MR Zbl

[SGA 3_{III} 1970] M. Demazure and A. Grothendieck, *Schémas en groupes, Tome III: Structure des schémas en groupes réductifs, Exposés XIX–XXVI* (Séminaire de Géométrie Algébrique du Bois Marie 1962–1964), Lecture Notes in Math. **153**, Springer, 1970. MR Zbl

[Springer 1998] T. A. Springer, *Linear algebraic groups*, 2nd ed., Progress in Mathematics **9**, Birkhäuser, Boston, 1998. MR Zbl

[Stacks 2005–] "The Stacks project", electronic reference, 2005–, available at http://stacks.math.columbia.edu.

[Vasiu 1999] A. Vasiu, "Integral canonical models of Shimura varieties of preabelian type", *Asian J. Math.* **3**:2 (1999), 401–518. MR Zbl

[Viehmann and Wedhorn 2013] E. Viehmann and T. Wedhorn, "Ekedahl–Oort and Newton strata for Shimura varieties of PEL type", *Math. Ann.* **356**:4 (2013), 1493–1550. MR Zbl

[Wortmann 2013] D. Wortmann, "The μ-ordinary locus for Shimura varieties of Hodge type", preprint, 2013. arXiv 1310.6444

[Zhang 2018] C. Zhang, "Ekedahl–Oort strata for good reductions of Shimura varieties of Hodge type", *Canad. J. Math.* **70**:2 (2018), 451–480. MR Zbl

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