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The Manin–Mumford conjecture and the Tate–Voloch conjecture for a product of Siegel moduli spaces

Congling Qiu

We use perfectoid spaces associated to abelian varieties and Siegel moduli spaces to study torsion points and ordinary CM points. We reprove the Manin–Mumford conjecture, i.e., Raynaud's theorem. We also prove the Tate–Voloch conjecture for a product of Siegel moduli spaces, namely ordinary CM points outside a closed subvariety can not be p-adically too close to it.

1. Introduction

We use the theory of perfectoid spaces to study torsion points in abelian varieties and ordinary CM points in Siegel moduli spaces. The use of perfectoid spaces is inspired by Xie's recent work [2018].

Tate–Voloch conjecture. Our main new result is about ordinary CM points. Let p be a prime number, L the complete maximal unramified extension of $\mathbb{Q}_p$. Let X be a product of Siegel moduli spaces over L with arbitrary level structures.

Theorem 1.1. *Let Z be a closed subvariety of $X_{\bar{L}}$. There exists a constant $c > 0$ such that for every ordinary CM point $x \in X(\bar{L})$, if the distance $d(x, Z)$ from x to Z satisfies $d(x, Z) \le c$, then $x \in Z$.*

The distance $d(x, Z)$ is defined as follows. Let $\|\cdot\|$ be a p-adic norm on $\bar{L}$. Let $\mathfrak{X}$ be an integral model of X over $\mathcal{O}_{\bar{L}}$. Let $\{U_1, \dots, U_n\}$ be a finite open cover of $\mathfrak{X}$ by affine schemes flat over $\mathcal{O}_{\bar{L}}$. Define $d(x, Z)$ to be the supremum of the $\|f(x)\|$ where U_i contains x and $f \in \mathcal{O}_{\mathfrak{X}}(U_i)$ vanishing on $Z \bigcap U_i$. The definition of $d(x, Z)$ depends on the choices of the integral model and the cover. However, the truth

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of Theorem 1.1 does not depend on these choices; see page 986. Moreover, we show that Theorem 1.1 holds for formal subschemes of $\mathfrak{X}$ (with maximal level at p); see Theorem 6.8. For CM points which are canonical liftings, we prove an “almost effective” version; see Theorem 6.13.

It is clear that the same statement in Theorem 1.1 is true replacing $X_{\bar{L}}$ by a closed subvariety. In particular, Theorem 1.1 is in fact equivalent the same statement for X being a single Siegel moduli space, by embedding a product of Siegel moduli spaces into a larger Siegel moduli space.

Remark 1.2. (1) For a power of the modular curve without level structure, Theorem 1.1 was proved by Habegger [2014] by a different method. However, Habegger's proof relies on a result of Pila [2014] (see also [Habegger 2014, Theorem 8]) concerning Zariski closure of a Hecke orbit. As far as we know, it is not available for Siegel moduli spaces yet. Moreover, Habegger's method seems not applicable to formal schemes.

(2) Habegger [2014] also showed that the ordinary condition is necessary.

(3) The original Tate–Voloch conjecture [Tate and Voloch 1996] states that in a semiabelian variety, torsion points outside a closed subvariety can not be p-adically too close to it. This conjecture was proved by Scanlon [1998; 1999] when the semiabelian variety is defined over $\overline{\mathbb{Q}}_p$. Xie [2018] proved dynamical analogs of Tate–Voloch conjecture for projective spaces.

Idea of the proof of Theorem 1.1. It is not hard to reduce Theorem 1.1 to the case that X has maximal level at p; see Lemma 2.15. We sketch the proof of Theorem 1.1 in this case. Relative to the canonical lifting of an ordinary point x in the reduction of X, ordinary CM points in X with reduction x are like p-primary roots of unity relative to 1 in the open unit disc around 1; see Proposition 5.1. This is the Serre–Tate theorem. If we only consider one such disc, Theorem 1.1 follows from a result of Serban [2018]. In general, we need to study all infinitely many Serre–Tate deformation spaces together. In characteristic p, this can be achieved by Chai's [2003] global Serre–Tate theorem; see Section 5. To prove Theorem 1.1, we at first prove a Tate–Voloch type result in a family characteristic p; see Section 6. Then we use the ordinary perfectoid Siegel space associated to X and the perfectoid universal covers of Serre–Tate deformation spaces to translate this result to the desired Theorem 1.1.

Possible generalizations. For Shimura varieties of Hodge type, the ordinary locus in the usual sense could be empty. In this case, we consider the notion of μ-ordinariness; see [Wedhorn 1999]. Then following our strategy, we need three ingredients. At first, a theory of Serre–Tate coordinates for μ-ordinary CM points. For Shimura varieties of Hodge type; see [Hong 2019; Shankar and Zhou 2016]. Secondly, a global theory of Serre–Tate coordinates in characteristic p. For Shimura varieties of PEL type, such results should be known to experts. Thirdly, μ-ordinary perfectoid Shimura varieties. Following Scholze [2015], certain perfectoid Shimura varieties of abelian type are constructed in [Shen 2017]. For universal abelian varieties over Shimura varieties of PEL type, we expect a Tate–Voloch type result for torsion points in fibers over μ-ordinary CM points. Still, we need analogs of the above three ingredients.

Manin–Mumford conjecture. For torsion points in abelian varieties, we reprove Raynaud's theorem, which is also known as the Manin–Mumford conjecture.

Theorem 1.3 [Raynaud 1983b]. *Let F be a number field. Let A be an abelian variety over F and V a closed subvariety of A. If V contains a dense subset of torsion points of A, then V is the translate of an abelian subvariety of A by a torsion point.*

Idea of the proof of Theorem 1.3. We simply consider the case when V does not contain any translate of a nontrivial abelian subvariety. Suppose that A has good reduction at a place of F unramified over a prime number p. Let $[p] : A \to A$ be the morphism multiplication by p. Let Λ_n be a suitable set of reductions of torsions in $[p^n]^{-1}(V)$, and Λ_n^{Zar} its Zariski closure in the base change to $\overline{\mathbb{F}}_p$ of the reduction of A. Use the p-adic perfectoid universal cover of A to lift Λ_n^{Zar} to A. A variant of Scholze's approximation lemma [2012] shows that as n get larger, the liftings are closer to V; see Proposition 2.23. A result of Scanlon [1998] on the Tate–Voloch conjecture for prime-to-p torsions implies that the prime-to-p torsions of these points are in V for n large enough; see Proposition 4.8. Assume that Λ_n is infinite and we deduce a contradiction as follows. A result of Poonen [2005] (see Theorem 4.1) shows that the size of the set of prime-to-p torsions in Λ_n^{Zar} is not small. Then the liftings give a lower bound on the size of the set of prime-to-p torsions in V; see Proposition 4.9. Now consider the l-adic perfectoid space associated to A. By the same approach, we can repeatedly improve such lower bounds. Finally we get a contradiction as A is of finite dimensional.

Remark 1.4. The proofs of Poonen's result and Scanlon's result are independent of Theorem 1.3.

Organization of the Paper. The preliminaries on adic spaces and perfectoid spaces are given in Section 2. We introduce the perfectoid universal cover of an abelian scheme in Section 3. The reader may skip these materials and only come back for references. We set up notations for the proof of Theorem 1.3 in Section 3, then prove Theorem 1.3 in Section 4. We introduce the ordinary perfectoid Siegel space and set up notations for the proof of Theorem 1.1 in Section 5. Then we prove Theorem 1.1 in Section 6.

2. Adic spaces and perfectoid spaces

We briefly recall the theory of adic spaces due to Huber [1993a; 1993b; 1994; 1996], and the generalization by Scholze and Weinstein [2013]. Then we define tube neighborhoods in adic spaces and distance functions. Finally we recall the theory of perfectoid spaces of Scholze [2012] and an approximation lemma due to Scholze.

Let K be a nonarchimedean field, i.e., a complete nondiscrete topological field whose topology is induced by a nonarchimedean norm $\|\cdot\|_K$ ($\|\cdot\|$ for short). Define

$$K^\circ = \{x \in K : \|x\| \le 1\} \quad \text{and} \quad K^{\circ\circ} = \{x \in K : \|x\| < 1\}.$$

Let $\varpi \in K^{\circ\circ} - \{0\}$.

Adic generic fibers of certain formal schemes.

Adic spaces. Let R be a complete Tate K-algebra, i.e., a complete topological K-algebra with a subring $R_0 \subset R$ such that $\{aR_0 : a \in K^\times\}$ forms a basis of open neighborhoods of 0. A subset of R is called bounded if it is contained in a certain aR_0. Let R° be the subring of power bounded elements, i.e., $x \in R^\circ$ if and only if the set of all powers of x form a bounded subset of R. Let $R^+ \subset R^\circ$ be an open integrally closed subring. Such a pair (R, R^+) is called an affinoid K-algebra. Let $\mathrm{Spa}(R, R^+)$ be the topological space whose underlying set is the set of equivalent classes of continuous valuations $|\cdot(x)|$ on R such that $|f(x)| \le 1$ for every $f \in R^+$ and topology is generated by the subsets of the form

$$U\left(\frac{f_1, \dots, f_n}{g}\right) := \{x \in \mathrm{Spa}(R, R^+) : \forall i\, |f_i(x)| \le |g(x)|\}$$

such that $(f_1, \dots, f_n) = R$. There is a natural presheaf on $\mathrm{Spa}(R, R^+)$; see [Huber 1994, page 519]. If this presheaf is a sheaf, then the affinoid K-algebra (R, R^+) is called sheafy, and $\mathrm{Spa}(R, R^+)$ is called an affinoid adic space over K.

Assumption 2.1. If $K^\circ \subset R^+$, for every $x \in \mathrm{Spa}(R, R^+)$, we always choose a representative $|\cdot(x)|$ in the equivalence class of x such that $|f(x)| = \|f\|_K$ for every $f \in K$.

Define a category (V) as in [Scholze 2012, Definition 2.7]. Objects in (V) are triples $(\mathcal{X}, \mathcal{O}_\mathcal{X}, \{|\cdot(x)| : x \in \mathcal{X}\})$ where $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ is a locally ringed topological space whose structure sheaf is a sheaf of complete topological K-algebras, and $|\cdot(x)|$ is an equivalence class of continuous valuations on the stalk of $\mathcal{O}_\mathcal{X}$ at x. Morphisms in (V) are morphisms of locally ringed topological spaces which are continuous K-algebra morphisms on the structure sheaves, and compatible with the valuations on the stalks in the obvious sense.

Definition 2.2. An adic space $\mathcal{X}$ over K is an object in (V) which is locally on $\mathcal{X}$ an affinoid adic space over K. An adic space over $\mathrm{Spa}(K, K^\circ)$ is an adic space over K with a morphism to $\mathrm{Spa}(K, K^\circ)$. A morphism between two adic spaces over $\mathrm{Spa}(K, K^\circ)$ is a morphism in (V) compatible the morphisms to $\mathrm{Spa}(K, K^\circ)$. The set of morphisms $\mathrm{Spa}(K, K^\circ) \to \mathcal{X}$ is denoted by $\mathcal{X}(K, K^\circ)$.

There is a natural inclusion $\mathcal{X}(K, K^\circ) \hookrightarrow \mathcal{X}$ by mapping a morphism $\mathrm{Spa}(K, K^\circ) \to \mathcal{X}$ to its image. We always identify $\mathcal{X}(K, K^\circ)$ as a subset of $\mathcal{X}$ by this inclusion.

Adic generic fibers of certain formal schemes. A Tate K-algebra R is called of topologically finite type (tft for short) if R is a quotient of $K\langle T_1, T_2, \dots, T_n\rangle$. In particular, it is equipped with the ϖ-adic topology. Similarly define K°-algebras of tft. By [Bosch et al. 1984, 5.2.6, Theorem 1] and [Huber 1994, Theorem 2.5], if R is of tft, then an affinoid K-algebra (R, R^+) is sheafy. Similar to the rigid analytic generic fibers of formal schemes over K° [Bosch 2014, 7.4], we naturally have a functor from the category of formal schemes over K° locally of tft to adic spaces over $\mathrm{Spa}(K, K^\circ)$ such that the image of $\mathrm{Spf}\, A$ is $\mathrm{Spa}\big(A\big[\frac{1}{\varpi}\big], A^c\big)$ where A^c is the integral closure of A in $A\big[\frac{1}{\varpi}\big]$. The image of a formal scheme under this functor is called its adic generic fiber.

We are interested in certain infinite covers of abelian schemes and Siegel moduli spaces. They are not of tft. We need to generalize the adic generic fiber functor. In [Scholze and Weinstein 2013], the category of adic spaces over $\mathrm{Spa}(K, K^\circ)$ is enlarged in a sheaf-theoretical way. Moreover, the adic generic fiber functor extends to the category of formal schemes over K° locally admitting a finitely generated ideal of definition.

For our purpose, we only need the following special case. Let $\mathfrak{X}$ be a formal K°-scheme which is covered by affine open formal subschemes $\{\mathrm{Spf}\, A_i : i \in I\}$, where I is an index set, such that each affinoid K-algebra $\bigl(A_i\bigl[\frac{1}{\varpi}\bigr], A_i^c\bigr)$ is sheafy. Then the adic generic fiber $\mathcal{X}$ of $\mathfrak{X}$ is an adic space over $\mathrm{Spa}(K, K^\circ)$ in the sense of Definition 2.2. Indeed, $\mathcal{X}$ is the obtained by gluing the affinoid adic spaces $\mathrm{Spa}\bigl(A_i\bigl[\frac{1}{\varpi}\bigr], A_i^c\bigr)$ in the obvious way. We have an easy consequence.

Lemma 2.3. *Let $\mathcal{X}$ be the adic generic fiber $\mathfrak{X}$. Then there is a natural bijection $\mathfrak{X}(K^\circ) \simeq \mathcal{X}(K, K^\circ)$.*

Tube neighborhoods and distance functions.

Tube neighborhoods. Let $\mathfrak{X} = \mathrm{Spf}\, B$, where B is a flat K°-algebra of tft. Let $\mathfrak{Z}$ be a closed formal subscheme defined by a closed ideal I. Let $\mathcal{X}$ be the adic generic fiber of $\mathfrak{X}$. Then $\mathcal{X} = \mathrm{Spa}(R, R^+)$ where $R = B\bigl[\frac{1}{\varpi}\bigr]$ and R^+ is the integral closure of B in R.

Definition 2.4. For $\epsilon \in K^\times$, the ϵ-neighborhood of $\mathfrak{Z}$ in $\mathcal{X}$ is defined to be the subset

$$\mathcal{Z}_\epsilon := \{x \in \mathcal{X} : |f(x)| \le |\epsilon(x)| \text{ for every } f \in I\}.$$

Remark 2.5. Note that $\mathcal{Z}_\epsilon$ may not be open in $\mathcal{X}$. If I is generated by $\{f_1, \dots, f_n\}$, then $\mathcal{Z}_\epsilon = U((f_1, \dots, f_n, \epsilon)/\epsilon)$ is naturally an open adic subspace of $\mathcal{X}$. In fact, for our applications, we only use this case.

Definition 2.4 immediately implies the following lemmas.

Lemma 2.6. *Let $\mathfrak{Z} = \bigcap_{i=1}^m \mathfrak{Z}_i$, where each $\mathfrak{Z}_i$ is a closed formal subschemes of $\mathfrak{X}$. For $\epsilon \in K^\times$, let $\mathcal{Z}_{i,\epsilon}$ be the ϵ-neighborhood of $\mathfrak{Z}_i$. Then $\mathcal{Z}_\epsilon = \bigcap_{i=1}^m \mathcal{Z}_{i,\epsilon}$.*

Lemma 2.7. *Let $\mathfrak{Z} = \mathfrak{Z}_1 \bigcup \mathfrak{Z}_2$, where $\mathfrak{Z}_1, \mathfrak{Z}_2$ are closed formal subschemes of $\mathfrak{X}$:*

(1) *Then $\mathcal{Z}_{1,\epsilon} \subset \mathcal{Z}_\epsilon$.*

(2) *Suppose that there exists $\delta \in K^{\circ\circ} - \{0\}$ which vanishes on $\mathfrak{Z}_2$. Then $\mathcal{Z}_\epsilon \subset \mathcal{Z}_{1,\epsilon/\delta}$.*

Let X be a K°-scheme locally of finite type, and $\mathfrak{X}$ the ϖ-adic formal completion of X. Let $\mathcal{X}$ be the adic generic fiber of $\mathfrak{X}$. We also call $\mathcal{X}$ the adic generic fiber of X. Let Z be a closed subscheme of X_K. We define tube neighborhoods of Z in $\mathcal{X}$ as follows; see also [Scholze 2012, Proposition 8.7].

Suppose that X is affine. Let $\mathfrak{Z} \subset \mathfrak{X}$ be the closed formal subscheme associated to the schematic closure of Z.

Definition 2.8. For $\epsilon \in K^\times$, the ϵ-neighborhood of Z in $\mathcal{X}$ is defined to be the ϵ-neighborhood of $\mathfrak{Z}$ in $\mathcal{X}$.

Remark 2.9. If the schematic closure of Z has empty special fiber, then $\mathcal{Z}_\epsilon$ is empty.

To define tube neighborhoods in general, we need to glue affinoid pieces. We consider the following relative situation. Let Y be another affine K°-scheme of finite type, and $\Phi : Y \to X$ a K°-morphism. Let W be the preimage of Z which is a closed subscheme of Y_K, and $\mathcal{W}_\epsilon$ its ϵ-neighborhood. By the functoriality of formal completion and taking adic generic fibers, we have an induced morphism $\Psi : \mathcal{Y} \to \mathcal{X}$. From the fact that schematic image is compatible with flat base change (see [Bosch et al. 1990, 2.5, Proposition 2]), we easily deduce the following lemma.

Lemma 2.10. *If $\Phi : Y \to X$ is flat, then $\Psi^{-1}(\mathcal{Z}_\epsilon) = \mathcal{W}_\epsilon$. In particular, if $Y \subset X$ is an open K°-subscheme, $\mathcal{W}_\epsilon = \mathcal{Z}_\epsilon \bigcap \mathcal{Y}$ under the natural inclusion $\mathcal{Y} \hookrightarrow \mathcal{X}$.*

Now we turn to the general case. Let X be an K°-scheme locally of finite type. For an open subscheme $U \subset X$, let Z_U be the restriction of Z to U. Let $S = \{U_i : i \in I\}$ be an affine open cover of X, where I is an index set and each U_i is of finite type over K°. Let $\mathcal{Z}_{U_i,\epsilon}$ be the ϵ-neighborhood of Z_{U_i} in the adic generic fiber $\mathcal{U}_i$ of U_i. Note that each $\mathcal{U}_i$ is naturally an open adic subspace of $\mathcal{X}$.

Definition 2.11. Define the ϵ-neighborhood of Z in $\mathcal{X}$ by $\mathcal{Z}_\epsilon := \bigcup_{U \in S} \mathcal{Z}_{U,\epsilon}$.

As a corollary of Lemma 2.10, this definition is independent of the choice of the cover $\mathcal{U}$.

Distance functions. Let U be an affine open subset of X which is flat over K°. Let I be an ideal of the coordinate ring of U. For $x \in U(K)$, define $d_U(x, I) := \sup\{\|f(x)\| : f \in I\}$. Let $\mathcal{I}$ be the ideal sheaf of the schematic closure of Z in X.

Assume that X is of finite type over K°. Let $\mathcal{U} := \{U_1, \dots, U_n\}$ be a finite affine open cover of X such that each U_i is flat over K°. For $x \in X(K)$, define $d^{\mathcal{U}}(x, Z)$ to be the maximum of $d_{U_i}(x, I)$ over all i such that $x \in U_i$.

Let $x^\circ \in X(K^\circ)$ and x the generic point of x°. Regard x as a point in $\mathcal{X}(K, K^\circ)$ via Lemma 2.3. Let U be an affine open subset of X flat over K° such that $x^\circ \in U(K^\circ)$. We have a tautological relation between the distance function and tube neighborhoods.

Lemma 2.12. *Let $\epsilon \in K^\times$. Then $x \in \mathcal{Z}_{U,\epsilon}$ if and only if $d_U(x, \mathcal{I}(U)) \le \|\epsilon\|$.*

By Lemma 2.10, the number $d_U(x, \mathcal{I}(U))$ does not depend on the choice of U. Define

$$d(x, Z) := d_U(x, \mathcal{I}(U)).$$

Then $d(x, Z) = d^{\mathcal{U}}(x, Z)$ for every finite affine open cover $\mathcal{U}$ of X. Our distance function coincides with the one in the end of [Scanlon 1998, Section 1], which is defined globally.

A finite extension of K has a natural structure of a nonarchimedean field; see [Bosch et al. 1984]. Let $\overline{K}$ be an algebraic closure of K. The above discussion is naturally generalized to $x \in X(\overline{K})$ and $Z \subset X_{\overline{K}}$.

Tate–Voloch type sets. Let X be of finite type over K°.

Definition 2.13. Fix an arbitrary finite affine open cover $\mathcal{U}$ of X by subschemes flat over K°. A set $T \subset X(\overline{K})$ is of Tate–Voloch type if for every closed subscheme Z of $X_{\overline{K}}$, there exists a constant $c > 0$ such that for every $x \in T$, if $d^{\mathcal{U}}(x, Z) \le c$, then $x \in Z(\overline{K})$.

Remark 2.14. Is there always a set of Tate–Voloch type? Let $C \subset X$ be irreducible and flat over K° of relative dimension 1. Choose one point in each residue disk in C. Easy to check that this set of points of X is of Tate–Voloch type. Moreover, we can choose points in residue disks in C chose degrees are unbounded. The following questions are more meaningful. Is there always a Tate–Voloch type set which is Zariski dense in X? Can the points in this set have unbounded the degrees over K? Indeed, the Tate–Voloch type sets in Theorem 1.1 and in the results of Habegger, Scanlon and Xie give positive answers to these two questions.

Let Y be a K°-scheme of finite type, and $\pi : Y \to X$ a finite schematically dominant morphism.

Lemma 2.15. *Let $T \subset X(\overline{K})$ be of Tate–Voloch type and $T' = \pi^{-1}(T) \subset Y(\overline{K})$. Then T' is of Tate–Voloch type.*

Proof. We may assume that $Y = \operatorname{Spec} B$ and $X = \operatorname{Spec} A$ where A is a subring of B. Let L be a finite extension of K. Let Z' be a closed subscheme of Y_L. We need to show that $d(x', Z')$ has a positive lower bound for $x' \in T' - Z'(\overline{K})$. Define the dimension of Z' to be the maximal dimension of the irreducible components of Z'. We allow Z' to be empty, in which case we define its dimension to be -1. We do induction on the dimension of Z'. Then the dimension -1 case is trivial. Now we consider the general case with the hypothesis that the lemma holds for all lower dimensions.

Suppose such a lower bound does not exists, then there exists a sequence of $x'_n \in T' - Z'(\overline{K})$ such that $d(x'_n, Z') \to 0$ as $n \to \infty$. We will find a contradiction. Let Z be the schematic image of Z' by π, $x_n = \pi(x'_n)$. Let the schematic closure of Z in X_{L°(resp. Z' in Y_{L°) be defined by an ideal $J \subset A \otimes L^\circ$ (resp. $I \subset B \otimes L^\circ$). Then $I \otimes L \supset JB \otimes L$. Since $JB \otimes L^\circ$ is finitely generated, there exists a positive integer r such that $I \supset \varpi^r JB \otimes L^\circ$. Thus $d(x_n, Z) \to 0$ as $n \to \infty$. Since T is of Tate–Voloch type, $x_n \in Z(\overline{K})$ for n large enough. We may assume that every $x_n \in Z(\overline{K})$. Since $x'_n \notin Z'$, $\pi^{-1}(Z) = Z' \bigcap Z_1$ where Z_1 is a closed subscheme of Y_L not containing Z' but containing all x'_n. Claim: $d\big(x'_n, Z' \bigcap Z_1\big) \to 0$ as $n \to \infty$. This contradicts the induction hypothesis. Thus $d(x', Z')$ has a positive lower bound for $x' \in T' - Z'(K)$. Now we prove the claim. Let the schematic closure of Z_1 in Y_{L° be defined by an ideal $I_1 \subset B \otimes L^\circ$. Then the schematic closure of $Z' \bigcap Z_1$ is defined by the following ideal of $B \otimes L^\circ$:

$$I_2 := (I_1 \otimes L + I' \otimes L) \bigcap B \otimes L^\circ = (I_1 + I') \otimes L \bigcap B \otimes L^\circ,$$

which is finitely generated. Thus there exists a positive integer s such that $(I_1 + I') \supset \varpi^s I_2$. Now the claim follows from that $d(x'_n, Z') \to 0$ and $x'_n \in Z_1$. □

Perfectoid spaces.

Two perfectoid fields. Instead of recalling the definition of perfectoid fields (see [Scholze 2012, Definition 3.1]), we consider two examples and use them through out this paper.

Let $k = \bar{\mathbb{F}}_p$, $W = W(k)$ the ring of Witt vectors, and $L = W\left[\frac{1}{p}\right]$. For each integer $n \geq 0$, let μ_{p^n} be a primitive p^n-th root of unity in $\bar{L}$ such that $\mu_{p^{n+1}}^p = \mu_{p^n}$. Let

$$L^{\mathrm{cycl}} := \bigcup_{n=1}^{\infty} L(\mu_{p^n}).$$

Let $\varpi = \mu_p - 1$, and K the ϖ-adic completion of L^{cycl}. Then K is a perfectoid field in the sense that

$$K^\circ/\varpi \to K^\circ/\varpi, \quad x \mapsto x^p$$

is surjective; see [Scholze 2012, Definition 3.1]. Let

$$K^\flat = k((t^{1/p^\infty}/))$$

be the t-adic completion of $\bigcup_{n=1}^{\infty} k((t))(t^{1/p^n})$. Then $K^\flat$ is a perfectoid field. Let $\varpi^\flat = t^{1/p}$. Equip $K^\flat$ with the nonarchimedean norm $\|\cdot\|_{K^\flat}$ such that $\|\varpi^\flat\|_{K^\flat} = \|\varpi\|_K$. Consider the morphism

$$K^\circ/\varpi \to K^{\flat\circ}/\varpi^\flat, \quad \mu_{p^n} - 1 \mapsto t^{1/p^n}. \tag{2-1}$$

This morphism is well-defined since

$$(\mu_{p^n} - 1)^{p^m} \simeq \mu_{p^{n-m}} - 1 \pmod{\varpi}$$

for $m < n$. Easy to check this morphism is an isomorphism. We call $K^\flat$ the tilt of K.

Perfectoid spaces. The most important property of a perfectoid K-algebra R is that

$$R^\circ/\varpi \to R^\circ/\varpi, \quad x \mapsto x^p$$

is surjective; see [Scholze 2012, Definition 5.1]. An affinoid K-algebra (R, R^+) is called perfectoid if R is perfectoid. By [loc. cit., Theorem 6.3], an affinoid K-algebra (R, R^+) is sheafy. Define a perfectoid space over K to be an adic space over K locally isomorphic to $\mathrm{Spa}(R, R^+)$, where (R, R^+) is a perfectoid affinoid K-algebra.

By [loc. cit., Theorem 5.2], there is an equivalence between the categories of perfectoid K-algebras and perfectoid $K^\flat$-algebras. By [loc. cit., Lemma 6.2 and Proposition 6.17], this category equivalence induces an equivalence between the categories of perfectoid affinoid K-algebras and perfectoid affinoid $K^\flat$-algebras, as well as an equivalence between the categories of perfectoid spaces over K and perfectoid spaces over $K^\flat$.

The image of an object or a morphism in the category of perfectoid K-algebras, perfectoid affinoid K-algebras, or perfectoid spaces over K is called its tilt.

Two important maps $\sharp$ and ρ. Let R be perfectoid K-algebra and $R^\flat$ its tilt. By [loc. cit., Proposition 5.17], there is a multiplicative homeomorphism $R^\flat \simeq \varprojlim_{x \mapsto x^p} R$. Denote the projection to the first component by

$$R^\flat \to R, \quad f \mapsto f^\sharp.$$

Let (R, R^+) be perfectoid affinoid K-algebra and $(R^\flat, R^{\flat+})$ its tilt. For $x \in \mathrm{Spa}(R, R^+)$, let $\rho(x) \in \mathrm{Spa}(R^\flat, R^{\flat+})$ be the valuation $|f(\rho(x))| = |f^\sharp(x)|$ for $f \in R^\flat$. This defines a map between sets

$$\rho : \mathrm{Spa}(R, R^+) \mapsto \mathrm{Spa}(R^\flat, R^{\flat+}).$$

Note that $\mathrm{Spa}(R^\flat, R^{\flat+})$ is the tilt of $\mathrm{Spa}(R, R^+)$. The definition of ρ glues and we have a map

$$\rho_{\mathcal{X}} : |\mathcal{X}| \simeq |\mathcal{X}^\flat|$$

between the underlying sets of a perfectoid space $\mathcal{X}$ over K and its tilt $\mathcal{X}^\flat$.

Lemma 2.16. (1) *Let $\phi : R \to S$ be a morphism between perfectoid K-algebras, and $\phi^\flat : R^\flat \to S^\flat$ its tilt. Then for every $f \in R^\flat$, we have $\phi^\flat(f)^\sharp = \phi(f^\sharp)$.*

(2) *Let $\Phi : \mathcal{X} \to \mathcal{Y}$ be a morphism between perfectoid spaces over K and $\Phi^\flat$ its tilt. Then as maps between topological spaces, we have*

$$\rho_{\mathcal{Y}} \circ \Phi = \Phi^\flat \circ \rho_{\mathcal{X}}.$$

Proof. (1) follows from the definition of the $\sharp$-map and [Scholze 2012, Theorem 5.2]. (2) follows from (1). □

By (2), the restriction of $\rho_{\mathcal{X}}$ to $\mathcal{X}(K, K^\circ)$ gives the functorial bijection $\mathcal{X}(K, K^\circ) \simeq \mathcal{X}^\flat(K^\flat, K^{\flat\circ})$, which we also denote by $\rho_{\mathcal{X}}$. In the next two paragraphs, we compute $\rho_{\mathcal{X}}$ in two cases.

Tilting and reduction. Let (R, R^+) be a perfectoid affinoid K-algebra and $(R^\flat, R^{\flat+})$ its tilt. Suppose there exists a flat W-algebra S such that:

(1) R^+ is the ϖ-adic completion of $S \otimes_W K^\circ$,

(2) $R^{\flat+}$ is the $\varpi^\flat$-adic completion of $S_k \otimes_k K^{\flat\circ}$.

Let $\phi : S \to W$ be a W-algebra morphism, $\phi_k : S_k \to k$ be its base change. Then ϕ induces a map $\psi : R^+ \to K^\circ$ which further induces a point x of $\mathrm{Spa}(R, R^+)$. Similarly, ϕ_k induces a map $\psi' : R^{\flat+} \to K^{\flat\circ}$ which further induces a point x' of $\mathrm{Spa}(R^\flat, R^{\flat+})$. Then $\psi/\varpi = \psi'/\varpi^\flat$ under the isomorphism $R^+/\varpi \simeq R^{\flat+}/\varpi^\flat$. By [Scholze 2012, Theorem 5.2], ϕ' is the tilt of ϕ and thus we have the following lemma.

Lemma 2.17. *We have $\rho_{\mathrm{Spa}(R,R^+)}(x) = x'$.*

An example: the perfectoid closed unit disc. Let $R = K\langle T^{1/p^\infty}, T^{-1/p^\infty}\rangle$, the $\varpi^\flat$-adic completion of $\bigcup_{r\in\mathbb{Z}_{\geq 0}} K[T^{1/p^r}, T^{-1/p^r}]$. Then R is perfectoid. The tilt $R^\flat$ of R is $K^\flat\langle T^{1/p^\infty}, T^{-1/p^\infty}\rangle$. Let $\mathcal{G}^{\mathrm{perf}} := \mathrm{Spa}(R, R^\circ)$. Then $\mathcal{G}^{\mathrm{perf}}$ is a perfectoid space over $\mathrm{Spa}(K, K^\circ)$, and $\mathcal{G}^{\mathrm{perf},\flat} := \mathrm{Spa}(R^\flat, R^{\flat\circ})$ is its tilt.

Let $c \in \mathbb{Z}_p$, and $m \in \mathbb{Z}_{\geq 0}$. The K°-morphism $R^\circ \to K^\circ$ defined by

$$T^{1/p^n} \to \mu_{p^{m+n}}^c$$

gives a point $x \in \mathcal{G}^{\mathrm{perf}}(K, K^\circ)$. The $K^{\flat\circ}$-morphism $R^{\mathrm{perf}} \to K^\flat$ defined by

$$T^{1/p^n} \to (1 + t^{1/p^{m+n}})^c$$

gives a point $x' \in \mathcal{G}^{\mathrm{perf},\flat}(K^\flat, K^{\flat\circ})$. The following lemma follows from (2-1) and [Scholze 2012, Theorem 5.2].

Lemma 2.18. *We have* $\rho_{\mathcal{G}^{\mathrm{perf}}}(x) = x'$.

Similar result holds for $\mathcal{G}^{l,\mathrm{perf}} = \mathrm{Spa}(R, R^\circ)$ where

$$R = K\langle T_1^{1/p^\infty}, T_1^{-1/p^\infty}, \dots, T_l^{1/p^\infty}, T_l^{-1/p^\infty}\rangle,$$

and its tilt $\mathcal{G}^{l,\mathrm{perf}\,\flat} = \mathrm{Spa}(R^\flat, R^{\flat\circ})$ where

$$R^\flat = K^\flat\langle T_1^{1/p^\infty}, T_1^{-1/p^\infty}, \dots, T_l^{1/p^\infty}, T_l^{-1/p^\infty}\rangle.$$

A variant of Scholze's approximation lemma. The perfectoid fields K, $K^\flat$ and related notations are as on page 987. Let (R, R^+) be a perfectoid affinoid (K, K°)-algebra with tilt $(R^\flat, R^{\flat+})$. Let $\mathcal{X} = \mathrm{Spa}(R, R^+)$ with tilt $\mathcal{X}^\flat = \mathrm{Spa}(R^\flat, R^{\flat+})$. For $f, g \in R$, define $|f(x) - g(x)|$ to be $|(f-g)(x)|$. The following approximation lemma plays an important role in Scholze's work [2012].

Lemma 2.19 [Scholze 2012, Corollary 6.7(1)]. *Let $f \in R^+$. Then for every $c \ge 0$, there exists $g \in R^{\flat+}$ such that for every $x \in \mathcal{X}$, we have*

$$|f(x) - g^\sharp(x)| \le \|\varpi\|^{1/p} \max\{|f(x)|, \|\varpi\|^c\} = \|\varpi\|^{1/p} \max\{|g^\sharp(x)|, \|\varpi\|^c\}. \tag{2-2}$$

Here the map $\sharp$ is as on page 988 (i.e., $|g(\rho(x))| = |g^\sharp(x)|$), and we use $\|\cdot\|$ to denote $\|\cdot\|_K$.

Recall that $k = \bar{\mathbb{F}}_p$. Assume that there exists a k-algebra S, such that $R^{\flat+}$ is the $\varpi^\flat$-adic completion of $S \otimes K^{\flat\circ}$. Then we have natural maps

$$\mathrm{Hom}_k(S, k) \hookrightarrow \mathrm{Hom}_{K^{\flat\circ}}(S \otimes K^{\flat\circ}, K^{\flat\circ}) \simeq \mathcal{X}^\flat(K^\flat, K^{\flat\circ}).$$

Thus we regard $(\mathrm{Spec}\, S)(k)$ as a subset of $\mathcal{X}^\flat$.

Lemma 2.20. *Continue to use the notations in Lemma 2.19. Assume that $c \in \mathbb{Z}\big[\frac{1}{p}\big]$. There exists a finite sum*

$$g_c = \sum_{\substack{i \in \mathbb{Z}[1/p]_{\ge 0},\\ i < 1/p + c}} g_{c,i} \cdot (\varpi^\flat)^i$$

with $g_{c,i} \in S$ and only finitely many $g_{c,i} \ne 0$, such that

$$g - g_c \in (\varpi^\flat)^{1/p+c} R^{\flat+}. \tag{2-3}$$

Proof. There exists a *finite sum* $g' = \sum s_j a_j \in S \otimes K^{\flat\circ}$, where $s_j \in S$ and $a_j \in K^{\flat\circ}$, such that $g - g' \in (\varpi^\flat)^{1/p+c} R^{\flat+}$.

Claim. *Let $a \in K^{\flat\circ}$, then there exists a positive integer N such that*

$$a - \sum_{\substack{h \in (\mathbb{Z}/p^N)_{\ge 0},\\ h < 1/p + c}} \alpha_h \cdot (\varpi^\flat)^h \in (\varpi^\flat)^{1/p+c} K^{\flat\circ}$$

for certain $\alpha_h \in k$.

Indeed, the claim follows from that $K^{\flat\circ}$ is the $\varpi^\flat$-adic completion of $\bigcup_{n=1}^\infty k[\![t]\!][(\varpi^\flat)^{1/p^n}]$. Note that $\{h \in (\mathbb{Z}/p^N)_{\geq 0}, h < 1/p + c\}$ is finite set. So there exists a finite sum

$$g_c = \sum_{\substack{i \in \mathbb{Z}[1/p]_{\geq 0}, \\ i < 1/p + c}} g_{c,i} \cdot (\varpi^\flat)^i$$

with $g_{c,i} \in S$ such that $g' - g_c \in (\varpi^\flat)^{1/p+c} R^{\flat+}$. Then $g - g_c \in (\varpi^\flat)^{1/p+c} R^{\flat+}$. □

Lemma 2.21. *Let g_c be as in Lemma 2.20 and $x \in (\operatorname{Spec} S)(k)$. Regarding $x \in \mathcal{X}^\flat(K^\flat, K^{\flat\circ})$ via the inclusion above. If $|g_c(x)| \leq \|\varpi\|^{1/p+c}$, then $g_{c,i}(x) = 0$ for all i.*

Proof. Since $x \in (\operatorname{Spec} S)(k)$, if $g_{c,i}(x) \neq 0$, then $|g_{c,i}(x)| = 1$. Let $i_0 < 1/p + c$ be the minimal i such that $|g_{c,i}(x)| = 1$. Then $|g_c(x)| = \|\varpi^\flat\|_{K^\flat}^{i_0} > \|\varpi\|^{1/p+c}$, a contradiction. □

Profinite setting. Impose the following assumption.

Assumption 2.22. There are k-algebras $S_0 \subset S_1 \subset \ldots$ such that $S = \bigcup S_n$.

Let $\mathcal{X}_n$ is the adic generic fiber of $\operatorname{Spec} S_n \otimes K^{\flat\circ}$. Then we have a natural morphism

$$\pi_n : \mathcal{X}^\flat \to \mathcal{X}_n.$$

We also use π_n to denote the morphism $(\operatorname{Spec} S)(k) \to (\operatorname{Spec} S_n)(k)$. We have natural maps

$$(\operatorname{Spec} S_n)(k) \hookrightarrow \operatorname{Hom}_{K^{\flat\circ}}(S_n \otimes K^{\flat\circ}, K^{\flat\circ}) \simeq \mathcal{X}_n(K^\flat, K^{\flat\circ})$$

by which we regard $(\operatorname{Spec} S_n)(k)$ as a subset of $\mathcal{X}_n$. For each n, let $\Lambda_n \subset (\operatorname{Spec} S_n)(k)$ be a set of k-points, and Λ_n^{Zar} the Zariski closure of Λ_n in $\operatorname{Spec} S_n$. We have the following maps and inclusions between sets:

$$|\mathcal{X}| \xrightarrow{\rho} |\mathcal{X}^\flat| \xrightarrow{\pi_n} |\mathcal{X}_n| \supset \Lambda_n^{\mathrm{Zar}}(k) \supset \Lambda_n,$$

where ρ is as on page 988.

Let $f \in R^+$, and $\Xi := \{x \in \mathcal{X} : |f(x)| = 0\}$. We have the following variant of Lemma 2.19.

Proposition 2.23. *Assume that $\Lambda_n \subset \pi_n(\rho(\Xi))$ for each n. Then for each $\epsilon \in K^\times$, there exists a positive integer n such that $|f(x)| \leq \|\epsilon\|_K$ for every $x \in (\pi_n \circ \rho)^{-1}(\Lambda_n^{\mathrm{Zar}}(k))$.*

Proof. Choose $c \in \mathbb{Z}_{\geq 0}$ large enough such that $\|\varpi\|_K^{1/p+c} \leq \|\epsilon\|_K$, choose g as in Lemma 2.19 and choose a finite sum

$$g_c = \sum_{\substack{i \in \mathbb{Z}[1/p]_{\geq 0}, \\ i < 1/p + c}} g_{c,i} \cdot (\varpi^\flat)^i$$

as in Lemma 2.20 where $g_{c,i} \in S$ for all i. There exists a positive integer $n(c)$ such that $g_{c,i} \in S_{n(c)}$ for all i by the finiteness of the sum. By the assumption, every element $x \in \Lambda_{n(c)}$ can be written as $\pi_{n(c)} \circ \rho(y)$ where $y \in \Xi$. By (2-2) and (2-3), $|g_c(\rho(y))| \leq \|\varpi\|^{1/p+c}$. Then by Lemma 2.21 and that $\rho(y) \in (\operatorname{Spec} S)(k)$, $g_{c,i}(\rho(y)) = 0$. Since $g_{c,i} \in S_{n(c)}$, $g_{c,i}(x) = 0$. Thus $g_{c,i}$ lies in the ideal defining $\Lambda_{n(c)}^{\mathrm{Zar}}$.

So $g_{c,i}(x)=0$, and thus $g_c(x)=0$, for every $x\in\Lambda^{\mathrm{Zar}}_{n(c)}(k)$. By (2-2) and (2-3), for every $x\in\Lambda^{\mathrm{Zar}}_{n(c)}(k)$, we have

$$|f(\rho^{-1}(\pi_{n(c)}^{-1}(x)))|\leq\|\varpi\|^{1/p+c}\leq\|\epsilon\|. \qquad \square$$

3. Perfectoid universal cover of an abelian scheme

Let K be the perfectoid field on page 987 and $K^\flat$ its tilt. Let $\mathfrak{A}$ be a formal abelian scheme over K°. We first recall the perfectoid universal cover of $\mathfrak{A}$ and its tilt constructed in [Pilloni and Stroh 2016, Lemme A.16]. Then we study the relation between tilting and reduction.

Perfectoid universal cover of an abelian scheme. Let $\mathfrak{A}'$ be a formal abelian scheme over $\operatorname{Spf} K^{\flat\circ}$. Assume that there is an isomorphism

$$\mathfrak{A}\otimes K^\circ/\varpi\simeq\mathfrak{A}'\otimes K^{\flat\circ}/\varpi^\flat \tag{3-1}$$

of abelian schemes over $K^\circ/\varpi\simeq K^{\flat\circ}/\varpi^\flat$. Let

$$\tilde{\mathfrak{A}}:=\varprojlim_{[p]}\mathfrak{A},\quad \tilde{\mathfrak{A}}':=\varprojlim_{[p]}\mathfrak{A}'.$$

Here the transition maps $[p]$ are the morphism multiplication by p and inverse limits exist in the categories of ϖ-adic and $\varpi^\flat$-adic formal schemes; see [Pilloni and Stroh 2016, Lemme A.15]. Index the inverse systems by $\mathbb{Z}_{\geq 0}$. Let $\operatorname{Spf} R_0^+\subset\mathfrak{A}$ be an affine open formal subscheme. Let R_i^+ be the coordinate ring of $([p]^i)^{-1}\operatorname{Spf} R_0^+$, in other words, $\operatorname{Spf} R_i^+=([p]^i)^{-1}\operatorname{Spf} R_0^+$. Let $R_i=R_i^+\big[\frac{1}{\varpi}\big]$, then R_i^+ is integrally closed in R_i. Let R^+ be the ϖ-adic completion of $\bigcup_{i=0}^\infty R_i^+$, $R=R^+\big[\frac{1}{\varpi}\big]$. Let $\operatorname{Spf} R_0'^+\subset\mathfrak{A}'$ be an affine open formal subscheme such that the restriction of (3-1) to $\operatorname{Spf} R_0^+\otimes K^\circ/\varpi$ is an isomorphism to $\operatorname{Spf} R_0'^+\otimes K^{\flat\circ}/\varpi^\flat$. We similarly define $R_i'^+$, R'^+ and R'.

Lemma 3.1 [Pilloni and Stroh 2016, Lemme A.16]. *The affinoid $K^\flat$-algebra (R',R'^+) is perfectoid. So is (R,R^+). Moreover, (R',R'^+) is the tilt of (R,R^+).*

Thus the adic generic fiber $\mathcal{A}^{\mathrm{perf}}$ (resp. $\mathcal{A}'^{\mathrm{perf}}$) of $\tilde{\mathfrak{A}}$ (resp. $\tilde{\mathfrak{A}}'$) is a perfectoid space. Moreover, $\mathcal{A}'^{\mathrm{perf}}$ is the tilt of $\mathcal{A}^{\mathrm{perf}}$. Thus we use $\mathcal{A}^{\mathrm{perf}\flat}$ to denote $\mathcal{A}'^{\mathrm{perf}}$. We call $\mathcal{A}^{\mathrm{perf}}$ (resp. $\mathcal{A}^{\mathrm{perf}\flat}$) the perfectoid universal cover of $\mathfrak{A}$ (resp. $\mathfrak{A}'$). By Lemma 2.3, there are natural bijections

$$\tilde{\mathfrak{A}}(K^\circ)\simeq\mathcal{A}^{\mathrm{perf}}(K,K^\circ),\quad \tilde{\mathfrak{A}}'(K^{\flat\circ})\simeq\mathcal{A}^{\mathrm{perf}\flat}(K^\flat,K^{\flat\circ}).$$

Let $\mathcal{A}$ (resp. $\mathcal{A}'$) be the adic generic fiber of $\mathfrak{A}$ (resp. $\mathfrak{A}'$). By Lemma 2.3, we have natural bijections

$$\mathfrak{A}(K^\circ)\simeq\mathcal{A}(K,K^\circ),\quad \mathfrak{A}'(K^{\flat\circ})\simeq\mathcal{A}'(K^\flat,K^{\flat\circ}).$$

Definition 3.2. The group structures on $\mathcal{A}(K,K^\circ)$, $\mathcal{A}^{\mathrm{perf}}(K,K^\circ)$, $\mathcal{A}'(K^\flat,K^{\flat\circ})$ and $\mathcal{A}^{\mathrm{perf}\flat}(K^\flat,K^{\flat\circ})$ are defined to be the ones induced from the natural bijections above.

By the functoriality of taking adic generic fibers, we have morphisms

$$\pi_n:\mathcal{A}^{\mathrm{perf}}\to\mathcal{A},\quad \pi_n':\mathcal{A}^{\mathrm{perf}\flat}\to\mathcal{A}'$$

for $n \in \mathbb{Z}_{\geq 0}$, and morphisms

$$[p] : \mathcal{A} \to \mathcal{A}, \quad [p] : \mathcal{A}' \to \mathcal{A}'.$$

Consider the following commutative diagram

$$\begin{array}{ccc} \tilde{\mathfrak{A}}(K^\circ) & \xrightarrow{\simeq} & \varprojlim_{[p]} \mathfrak{A}(K^\circ) \\ \downarrow \simeq & & \downarrow \simeq \\ \mathcal{A}^{\mathrm{perf}}(K, K^\circ) & \longrightarrow & \varprojlim_{[p]} \mathcal{A}(K, K^\circ) \end{array} \tag{3-2}$$

where the bottom map is given by the π_n. We immediately have the following lemma.

Lemma 3.3. *The bottom map in* (3-2) *is a group isomorphism.*

Remark 3.4. Indeed, $\mathcal{A}^{\mathrm{perf}}$ serves as certain "limit" of the inverse system $\varprojlim \mathcal{A}$ in the sense of [Scholze and Weinstein 2013, Definition 2.4.1] by [loc. cit., Proposition 2.4.2]. Then Lemma 3.3 also follows from [loc. cit., Proposition 2.4.5].

Now we study torsion points in the inverse limit. We set up some group theoretical convention once for all. Let G be an abelian group. We denote by $G[n]$ the subgroup of elements of orders dividing n and by G_{tor} the subgroup of torsion elements. For a prime p, we use $G[p^\infty]$ to denote the subgroup of p-primary torsion points, and $G_{p'\text{-}\mathrm{tor}}$ to denote the subgroup of prime-to-p torsion points. If H is a *subset* of G, H_{tor} and $H_{p'-\mathrm{tor}}$ to denote the subset $H \bigcap G_{\mathrm{tor}}$ and $H \bigcap G_{p'-\mathrm{tor}}$ when both the definitions of H and G are clear from the context. The following lemma is elementary.

Lemma 3.5. *Let G be an abelian group, then*

$$(\varprojlim_{[p]} G)_{p'-\mathrm{tor}} \simeq \varprojlim_{[p]} G_{p'-\mathrm{tor}}.$$

Lemma 3.6. *There are group isomorphisms*

$$\mathcal{A}^{\mathrm{perf}}(K, K^\circ)_{p'-\mathrm{tor}} \simeq \varprojlim_{[p]} \mathcal{A}(K, K^\circ)_{p'-\mathrm{tor}} \simeq \mathcal{A}(K, K^\circ)_{p'-\mathrm{tor}}$$

where the second isomorphism is the restriction of π_n. Similar result holds for $\mathcal{A}'$ and $\mathcal{A}^{\mathrm{perf}\flat}$.

Proof. The first isomorphism is from Lemmas 3.3 and 3.5. Since $\mathcal{A}(K, K^\circ)[n] \simeq \mathfrak{A}(K^\circ)[n]$ is a finite group, $[p]$ is an isomorphism on $\mathcal{A}(K, K^\circ)[n]$ for every natural number n coprime to p. The second isomorphism follows. □

Proposition 3.7. *The functorial bijection*

$$\rho = \rho_{\mathcal{A}^{\mathrm{perf}}} : \mathcal{A}^{\mathrm{perf}}(K, K^\circ) \simeq \mathcal{A}^{\mathrm{perf}\flat}(K^\flat, K^{\flat\circ})$$

(*see page 988*) *is a group isomorphism.*

Proof. We only show the compatibility of ρ with the multiplication maps, i.e., we show that the following diagram is commutative:

$$\begin{array}{ccc}\mathcal{A}^{\mathrm{perf}}(K,K^\circ)\times\mathcal{A}^{\mathrm{perf}}(K,K^\circ) & \xrightarrow{\rho\times\rho} & \mathcal{A}^{\mathrm{perf}\flat}(K^\flat,K^{\flat\circ})\times\mathcal{A}^{\mathrm{perf}\flat}(K^\flat,K^{\flat\circ})\\ \downarrow & & \downarrow\\ \mathcal{A}^{\mathrm{perf}}(K,K^\circ) & \xrightarrow{\rho} & \mathcal{A}^{\mathrm{perf}\flat}(K^\flat,K^{\flat\circ})\end{array}$$

Here the vertical maps are the multiplication maps on corresponding groups.

Consider the formal abelian schemes $\mathfrak{B}=\mathfrak{A}\times\mathfrak{A}$ and $\mathfrak{B}'=\mathfrak{A}'\times\mathfrak{A}'$. We do the same construction to get their perfectoid universal covers $\mathcal{B}^{\mathrm{perf}}$ and $\mathcal{B}^{\mathrm{perf}\flat}$. The multiplication morphism $\mathfrak{B}\to\mathfrak{A}$ induces $m:\mathcal{B}^{\mathrm{perf}}\to\mathcal{A}^{\mathrm{perf}}$. The multiplication morphism $\mathfrak{B}'\to\mathfrak{A}'$ induces $m':\mathcal{B}^{\mathrm{perf}\flat}\to\mathcal{A}^{\mathrm{perf}\flat}$. By (3-1) and [Scholze 2012, Theorem 5.2], $m'=m^\flat$. By functoriality, we have a commutative diagram:

$$\begin{array}{ccc}\mathcal{B}^{\mathrm{perf}}(K,K^\circ) & \xrightarrow{\rho_{\mathcal{B}^{\mathrm{perf}}}} & \mathcal{B}^{\mathrm{perf}\flat}(K^\flat,K^{\flat\circ})\\ \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle m^\flat}\\ \mathcal{A}^{\mathrm{perf}}(K,K^\circ) & \xrightarrow{\rho_{\mathcal{A}^{\mathrm{perf}}}} & \mathcal{A}^{\mathrm{perf}\flat}(K^\flat,K^{\flat\circ})\end{array}$$

We only need to show that this diagram can be identified with the diagram we want. For example we show that the top horizontal maps in the two diagrams coincide, i.e., a commutative diagram:

$$\begin{array}{ccc}\mathcal{B}^{\mathrm{perf}}(K,K^\circ) & \xrightarrow{\rho_{\mathcal{B}^{\mathrm{perf}}}} & \mathcal{B}^{\mathrm{perf}\flat}(K^\flat,K^{\flat\circ})\\ \downarrow{\scriptstyle\simeq} & & \downarrow{\scriptstyle\simeq}\\ \mathcal{A}^{\mathrm{perf}}(K,K^\circ)\times\mathcal{A}^{\mathrm{perf}}(K,K^\circ) & \xrightarrow{\rho\times\rho} & \mathcal{A}^{\mathrm{perf}}(K,K^\circ)\times\mathcal{A}^{\mathrm{perf}}(K,K^\circ)\end{array}$$

The projection $\mathfrak{B}=\mathfrak{A}\times\mathfrak{A}\to\mathfrak{A}$ to the i-th component, $i=1,2$, induces $p_i:\mathcal{B}^{\mathrm{perf}}\to\mathcal{A}^{\mathrm{perf}}$. Easy to check that

$$p_1\times p_2:\mathcal{B}^{\mathrm{perf}}(K,K^\circ)\to\mathcal{A}^{\mathrm{perf}}(K,K^\circ)\times\mathcal{A}^{\mathrm{perf}}(K,K^\circ)$$

is a group isomorphism by passing to formal schemes. Similarly we have an isomorphism

$$p_1'\times p_2':\mathcal{B}^{\mathrm{perf}\flat}(K^\flat,K^{\flat\circ})\to\mathcal{A}^{\mathrm{perf}}(K,K^\circ)\times\mathcal{A}^{\mathrm{perf}}(K,K^\circ).$$

The commutativity is implied by that $p_i'=p_i^\flat$, which is from (3-1) and [Scholze 2012, Theorem 5.2]. □

Tilting and reduction. Let $k=\overline{\mathbb{F}}_p$ and let $W=W(k)$ be the ring of Witt vectors. Let A be an abelian scheme over W, A_{K° be its base change to K°, $\mathcal{A}$ be the adic generic fiber of A_{K°. Let A_k be the special fiber of A, and A' be the base change $A_k\otimes K^{\flat\circ}$ with adic generic fiber $\mathcal{A}'$. Since

$$A_{K^\circ}\otimes(K^\circ/\varpi)\simeq A\otimes_W k\otimes_k(K^\circ/\varpi)\simeq A'\otimes_{K^{\flat\circ}}(K^{\flat\circ}/\varpi^\flat),$$

we can apply the construction in Lemma 3.1 to the formal completions of $A_k \otimes_k K^\circ$ and A'. Then we have the perfectoid universal cover $\mathcal{A}^{\mathrm{perf}}$ of the ϖ-adic formal completion of A_{K°, the perfectoid universal cover $\mathcal{A}^{\mathrm{perf}\flat}$ of the $\varpi^\flat$-adic formal completion of $A_{K^{\flat\circ}}$, and the morphisms $\pi_n : \mathcal{A}^{\mathrm{perf}} \to \mathcal{A}$, $\pi_n' : \mathcal{A}^{\mathrm{perf}\flat} \to \mathcal{A}'$ for each $n \in \mathbb{Z}_{\geq 0}$. The following well-known results can be deduced from [Serre and Tate 1968].

Lemma 3.8. (1) *The inclusion $A(W) \hookrightarrow A(K^\circ)$ gives an isomorphism $A(W)_{p'-\mathrm{tor}} \simeq A(K^\circ)_{p'-\mathrm{tor}}$.*

(2) *The reduction map gives an isomorphism*

$$\mathrm{red} : A(W)_{p'-\mathrm{tor}} \simeq A(k)_{p'-\mathrm{tor}}.$$

(3) *The natural inclusion $A(k) \hookrightarrow A_{K^{\flat\circ}}(K^{\flat\circ})$ gives an isomorphism $A(k)_{p'-\mathrm{tor}} \simeq A_{K^{\flat\circ}}(K^{\flat\circ})_{p'-\mathrm{tor}}$.*

Now we relate reduction and tilting.

Lemma 3.9. *Let the unindexed maps in the following diagram be the naturals ones*:

$$\begin{array}{ccccccc}
\mathcal{A}(K, K^\circ)_{p'-\mathrm{tor}} & \xleftarrow{\pi_n} & \mathcal{A}^{\mathrm{perf}}(K, K^\circ)_{p'-\mathrm{tor}} & \xrightarrow{\rho} & \mathcal{A}^{\mathrm{perf}\flat}(K^\flat, K^{\flat\circ})_{p'-\mathrm{tor}} & \xrightarrow{\pi_n'} & \mathcal{A}'(K^\flat, K^{\flat\circ})_{p'-\mathrm{tor}} \\
\uparrow & & & & & & \uparrow \\
A_{K^\circ}(K^\circ)_{p'-\mathrm{tor}} & \longleftarrow & A(W)_{p'-\mathrm{tor}} & \xrightarrow{\mathrm{red}} & A(k)_{p'-\mathrm{tor}} & \longrightarrow & A_{K^{\flat\circ}}(K^{\flat\circ})_{p'-\mathrm{tor}}
\end{array}$$

Then each map is a group isomorphism, and the diagram is commutative (*up to inverting the arrows*).

Proof. We may assume $n = 0$. Definition 3.2, Lemma 3.6, Proposition 3.7 and Lemma 3.8 give the isomorphisms. We only need to check the commutativity. And we only need to check the two maps from $A(W)_{p'-\mathrm{tor}}$ to $\mathcal{A}^{\mathrm{perf}\flat}(K^\flat, K^{\flat\circ})_{p'-\mathrm{tor}}$ are the same. This follows from Lemma 2.17. ☐

Similarly, we have the following commutative diagram:

$$\begin{array}{ccccc}
\mathcal{A}^{\mathrm{perf}}(K, K^\circ) & \xrightarrow{\rho} & \mathcal{A}^{\mathrm{perf}\flat}(K^\flat, K^{\flat\circ}) & \searrow^{\simeq} & \\
\uparrow{\scriptstyle\iota} & & & & \\
\varprojlim_{[p]} A(W) & \xrightarrow{\mathrm{red}} & \varprojlim_{[p]} A(k) & \hookrightarrow & \varprojlim_{[p]} \mathcal{A}'(K^\flat, K^{\flat\circ}) \\
\downarrow{\scriptstyle\pi_0} & & \downarrow{\scriptstyle\pi_n'} & & \downarrow{\scriptstyle\pi_n'} \\
\bigcap_{i=0}^\infty p^i A(W) & & A(k) & \hookrightarrow & \mathcal{A}'(K^\flat, K^{\flat\circ}) \\
\downarrow & & \downarrow{\scriptstyle[p^n]} & & \downarrow{\scriptstyle[p^n]} \\
A(W) & \xrightarrow{\mathrm{red}} & A(k) & \hookrightarrow & \mathcal{A}'(K^\flat, K^{\flat\circ})
\end{array} \tag{3-3}$$

Here ι is induced from the inclusion $A(W) \hookrightarrow \mathcal{A}^{\mathrm{perf}}(K, K^\circ)$ and the isomorphism $\mathcal{A}^{\mathrm{perf}}(K, K^\circ) \simeq \varprojlim_{[p]} \mathcal{A}(K, K^\circ)$ (see Lemma 3.3). Here and from now on we regard $\varprojlim_{[p]} A(W)$ as a subset of $\mathcal{A}^{\mathrm{perf}}(K, K^\circ)$ via ι, $A(k)$ as a subset $\mathcal{A}'(K^\flat, K^{\flat\circ})$, and $\varprojlim_{[p]} A(k)$ as a subset of $\mathcal{A}^{\mathrm{perf}\flat}(K^\flat, K^{\flat\circ})$.

4. Proof of Theorem 1.3

In this section, we at first prove a lower bound on prime-to-p torsion points in a subvariety. Then we prove Theorem 1.3. Let $k=\bar{\mathbb{F}}_p$, $W=W(k)$ the ring of Witt vectors, and $L=W\left[\frac{1}{p}\right]$.

Results of Poonen, Raynaud and Scanlon.

Theorem 4.1 [Poonen 2005]. *Let B be an abelian variety defined over k, and V an irreducible closed subvariety of B. Let S be a finite set of primes. Suppose that V generates B, then the composition of*

$$V(k)\hookrightarrow B(k)\xrightarrow{\bigoplus_{l\in S}\mathrm{pr}_l}\bigoplus_{l\in S}B[l^\infty]$$

is surjective, where pr_l is the projection to the l-primary component.

Let A be an abelian scheme over W. Let $T=\bigcap_{n=0}^\infty p^n(A(L)[p^\infty])$, the maximal divisible subgroup of $A(L)[p^\infty]$. Though not needed, as an illustration, we note that by [Raynaud 1983a, Exemples 5.2.3], $T=0$ if the p-rank of A_k is 0 or if A is a "general ordinary abelian variety", and $T=A(L)[p^\infty]\simeq L/\mathbb{Z}_p^{\dim A_L}$ if A is the canonical lifting in Serre–Tate theory; see Section 5.

Lemma 4.2 [Raynaud 1983a, Lemma 5.2.1]. (1) *Let T_o be the subgroup of $A(\bar{L})[p^\infty]$ coming from the connected component of the p-divisible group of A, then $T_o\bigcap T=0$.*

(2) *As a subgroup of $A(\bar{L})[p^\infty]$, T is a $\mathrm{Gal}(\bar{L}/L)$-direct summand.*

Note that

$$\bigcap_{n=0}^\infty p^n(A(W)_{\mathrm{tor}})=A(W)_{p'-\mathrm{tor}}\bigoplus\bigcap_{n=0}^\infty p^n(A(W)[p^\infty]). \tag{4-1}$$

Corollary 4.3. *The following reduction map is injective*

$$\mathrm{red}:\bigcap_{n=0}^\infty p^n(A(W)_{\mathrm{tor}})\to A(k).$$

Let $Z\subset A_L$ be a closed subvariety.

Lemma 4.4 [Raynaud 1983b, 8.2]. *Let T' be a $\mathrm{Gal}(\bar{L}/L)$-direct summand such that as $\mathrm{Gal}(\bar{L}/L)$-modules*

$$A(\bar{L})_{\mathrm{tor}}=A(\bar{L})_{p'-\mathrm{tor}}\bigoplus T\bigoplus T'.$$

If Z does not contain any translate of a nontrivial abelian subvariety of A_L, there exists a positive integer N such that the order of the T'-component of every element in $Z(\bar{L})_{\mathrm{tor}}$ divides p^N.

Remark 4.5. Lemma 4.4 is used by Raynaud [1983b] to reduce the Manin–Mumford conjecture to a theorem (see [loc. cit., Theorem 3.5.1]) obtained by studying p-adic rigid analytic properties of universal vector extension of an abelian variety.

Let K and $K^\flat$ be the perfectoid fields on page 987. Let $\mathcal{A}$ be the adic generic fiber of A_{K°. Let Z^{Zar} be the Zariski closure of Z in A, and $\mathcal{Z}$ the adic generic fiber of $Z^{\mathrm{Zar}}_{K^\circ}$. For $\epsilon \in K^\times$, let $\mathcal{Z}_\epsilon$ be the ϵ-neighborhood of Z_K in $\mathcal{A}$ as in Definition 2.11. By Lemma 2.12, a result of Scanlon [1998] on the Tate–Voloch conjecture implies the following lemma.

Lemma 4.6 [Scanlon 1998]. *There exists $\epsilon \in K^\times$, such that $\mathcal{A}(K, K^\circ)_{p'-\mathrm{tor}} \bigcap \mathcal{Z}_\epsilon \subset \mathcal{Z}$.*

Remark 4.7. The proofs of Poonen's result and Scanlon's result are independent of Theorem 1.3.

A lower bound. Define

$$\Lambda := Z^{\mathrm{Zar}}(W) \bigcap \bigcap_{n=0}^{\infty} p^n(A(W)_{\mathrm{tor}}), \quad \Lambda_\infty := \iota(\pi_0^{-1}(\Lambda)), \tag{4-2}$$

where π_0 and ι are as in the left column of diagram (3-3). Then $\rho(\Lambda_\infty)$ is contained in (the image of) $\varprojlim_{[p]} A(k)$ by diagram (3-3). Now let $\Lambda_n := \pi'_n(\rho(\Lambda_\infty))$. Then Λ_n is contained in (the image of) $A(k)$. Let Λ_n^{Zar} be the Zariski closure of Λ_n in A_k.

Proposition 4.8. *There exists a positive integer n such that*

$$\pi_0(\rho^{-1}(\pi_n'^{-1}(\Lambda_n^{\mathrm{Zar}}(k)_{p'-\mathrm{tor}}))) \bigcap \mathcal{A}(K, K^\circ)_{p'-\mathrm{tor}} \subset \mathcal{Z}.$$

Proof. Let $\mathcal{U}$ be a finite affine open cover of A by affine open subschemes flat over W. Let $U \in \mathcal{U}$. The restriction of $\mathcal{A}^{\mathrm{perf}}$ over the adic generic fiber of U_{K° is a perfectoid space $\mathcal{X} = \mathrm{Spa}(R, R^+)$ whose tilt satisfies Assumption 2.22 (see Lemma 3.1 and the discussion above it). Let $\mathcal{I}$ be the ideal sheaf of Z^{Zar}. Let $f \in \mathcal{I}(U)$. Regard f as in R. By definition of Λ_n, we can apply Proposition 2.23 to f and Λ_n. Varying U in $\mathcal{U}$ and varying f in a finite set of generators of $\mathcal{I}(U)$, Proposition 2.23 implies that for every $\epsilon \in K^\times$, there exists a positive integer n such that

$$\pi_0(\rho^{-1}(\pi_n'^{-1}(\Lambda_n^{\mathrm{Zar}}(k)))) \subset \mathcal{Z}_\epsilon.$$

Then Proposition 4.8 follows from Lemma 4.6. □

Our lower bound on the size of the set of prime-to-p torsions in Z is as follows.

Proposition 4.9. *Let $p > 2$. Assume that Z contains the unit $0 \in A_L$:*

(1) *Assume Λ is infinite. For every prime number $l \neq p$, the image of the composition of*

$$Z^{\mathrm{Zar}}(W)_{p'-\mathrm{tor}} \hookrightarrow A(W)_{p'-\mathrm{tor}} \xrightarrow{\mathrm{pr}_l} A(W)[l^\infty] \tag{4-3}$$

contains a translate of a free $\mathbb{Q}_l/\mathbb{Z}_l$-submodule of $A(W)[l^\infty]$ of rank at least 2. Here the map pr_l is the projection to the l-primary component.

(2) *Assume that the image of the composition of*

$$\Lambda \hookrightarrow A(W)_{\mathrm{tor}} \xrightarrow{\mathrm{pr}_p} A(W)[p^\infty]$$

contains a translate of a free $L/\mathbb{Z}_p$-submodule of rank r. For every prime number $l \neq p$, the image of the composition of (4-3) contains a translate of a free $\mathbb{Q}_l/\mathbb{Z}_l$-submodule of $A(W)[l^\infty]$ of rank $2r$.

Proof. Fix a large n such that

$$\pi_0(\rho^{-1}(\pi_n'^{-1}(\Lambda_n^{\mathrm{Zar}}(k)_{p'-\mathrm{tor}}))) \bigcap \mathcal{A}(K, K^\circ)_{p'-\mathrm{tor}} \subset \mathcal{Z}(K, K^\circ)_{p'-\mathrm{tor}} \tag{4-4}$$

as in Proposition 4.8. Let X be the image of the left hand side of (4-4) via the composition of

$$\mathcal{A}(K, K^\circ)_{p'-\mathrm{tor}} \simeq A(W)_{p'-\mathrm{tor}} \xrightarrow{\mathrm{pr}_l} A(W)[l^\infty].$$

Then X is contained in the image of the composition of (4-3).

To prove (1), we only need to prove the following claim.

Claim. *X contains a translate of a free $\mathbb{Q}_l/\mathbb{Z}_l$-submodule of $A(W)[l^\infty]$ of rank at least 2 for every l.*

By diagram (3-3), we have $\Lambda_0 = \mathrm{red}(\Lambda)$. Since $p > 2$, by Corollary 4.3, Λ_0 is infinite. Since $\Lambda_0 = [p]^n(\Lambda_n)$, Λ_n is infinite. There exists $a \in A(k)$ such that an irreducible component of $\Lambda_n^{\mathrm{Zar}} + a$ (is contained and) generates a nontrivial abelian subvariety A' of A_k. Since Z contains the unit $0 \in A_L$, Λ_n^{Zar} contains the unit $0 \in A_k$ and A' contains a. Let a_p be the p-primary part of a and $a_{p'} = a - a_p$. By Theorem 4.1 (for $\Lambda_n^{\mathrm{Zar}} + a \subset A'$ and $S = \{p, l\}$), the image of

$$\Lambda_n^{\mathrm{Zar}}(k) + a \xrightarrow{\mathrm{pr}_l \bigoplus \mathrm{pr}_p} A(k)[l^\infty] \bigoplus A(k)[p^\infty] \tag{4-5}$$

contains $M \bigoplus \{a_p\}$, where M is a free $\mathbb{Q}_l/\mathbb{Z}_l$-submodule of $A(k)[l^\infty]$ of rank at least 2. Thus

$$(\mathrm{pr}_l \bigoplus \mathrm{pr}_p)(\Lambda_n^{\mathrm{Zar}}(k) + a_{p'}) \supset M \bigoplus \{0\}.$$

We claim

$$\mathrm{pr}_l((\Lambda_n^{\mathrm{Zar}}(k) + a_{p'})_{p'-\mathrm{tor}}) \supset M.$$

Indeed, write $b \in \Lambda_n^{\mathrm{Zar}}(k) + a_{p'}$ as the sum $b_p + b_{p'}$ of p-primary part and prime-to-p part. Then $(\mathrm{pr}_l \bigoplus \mathrm{pr}_p)b = \mathrm{pr}_l(b_{p'}) + b_p$. If this is $x + 0 \in M \bigoplus \{0\}$, then $b_p = 0$, and $b = b_{p'}$. Thus $\mathrm{pr}_l(b) = x \in M$. The claim is proved. By the claim,

$$M - \mathrm{pr}_l(a_{p'}) \subset Y := \mathrm{pr}_l(\Lambda_n^{\mathrm{Zar}}(k)_{p'-\mathrm{tor}}).$$

By Lemma 3.9, X contains the preimage of $[p]^n(Y)$ under the isomorphism $\mathrm{red}: A(W)_{p'-\mathrm{tor}} \simeq A(k)_{p'-\mathrm{tor}}$. Thus we proved the claim above.

To prove (2), we only need to prove the following claim.

Claim. *X contains a translate of a free $\mathbb{Q}_l/\mathbb{Z}_l$-submodule of $A(W)[l^\infty]$ of rank at least $2r$ for every l.*

By diagram (3-3), we have $\Lambda_0 = \mathrm{red}(\Lambda)$. Since $p > 2$, by Corollary 4.3 and the assumption on Λ, $\mathrm{pr}_p(\Lambda_0)$ contains a translate of a free $L/\mathbb{Z}_p$-submodule of rank r. Let $V_1, \dots, V_m$ be the irreducible components of Λ_0^{Zar}. Let A_i be the minimal abelian subvariety of A_k such that a certain translate of A_i contains V_i. Since the p-rank of A_i is at most its dimension, at least one A_i is of dimension at least r. Since $\Lambda_0 = [p]^n(\Lambda_n)$, there exists $a \in A(k)$ such that an irreducible component of $\Lambda_n^{\mathrm{Zar}} + a$ generates an abelian subvariety of A_k of dimension at least r. Then we prove (2) by copying the proof of (1) above, starting from the sentence containing (4-5). The only modification needed is that the rank of M should be at least $2r$. □

The proof of Theorem 1.3. Now we prove Theorem 1.3. By the argument in [Pila and Zannier 2008], we only need to prove the following weaker theorem. We save the symbol A for the proof.

Theorem 4.10. *Let F be number field. Let B be an abelian variety over F and V a closed subvariety of B. If V does not contain any translate of an abelian subvariety of B of positive dimension, then V contains only finitely many torsion points of B.*

Proof. We only need to prove the theorem up to replacing V by a multiple.

Let v be a place of F unramified over a prime number $p > 2$ such that B has good reduction. Let A be the base change to W of the integral smooth model of B over $\mathcal{O}_{F_v}$. Let $Z = V_L \subset A$. By (4-1) and Lemma 4.4, up to replacing V by $[p^N]V$ for N large enough, we may assume that $Z^{\mathrm{Zar}}(W)_{\mathrm{tor}} \subset \bigcap_{n=0}^{\infty} p^n(A(W)_{\mathrm{tor}})$. Thus $\Lambda = Z^{\mathrm{Zar}}(W)_{\mathrm{tor}}$, where Λ is defined as in (4-2). Suppose that V contains infinitely many torsion points. Then Λ is infinite. Up to replacing V by $[p^N]V$, we may assume that Z contains the unit $0 \in A_L$. Now we want to find a contradiction. By Proposition 4.9(1), for every prime number $l \neq p$, the composition of

$$Z(L)_{p'-\mathrm{tor}} \hookrightarrow A(L)_{p'-\mathrm{tor}} \xrightarrow{\mathrm{pr}_l} A(L)[l^\infty]$$

contains a translate of a free $\mathbb{Q}_l/\mathbb{Z}_l$-submodule of $A(L)[l^\infty]$ of rank 2.

Let u be another place of F, unramified over an odd prime number $l \neq p$, such that B has good reduction at u. Let B_o be the reduction. Let M be the completion of the maximal unramified extension of F_u and $\overline{M}$ its algebraic closure. Then the composition

$$V(\overline{M})_{\mathrm{tor}} \hookrightarrow B(\overline{M})_{\mathrm{tor}} \xrightarrow{\mathrm{pr}_l} B(\overline{M})[l^\infty]$$

contains a translate G of a free $\mathbb{Q}_l/\mathbb{Z}_l$-submodule of $B(\overline{M})[l^\infty]$ of rank 2. Let $T = \bigcap_{n=0}^{\infty} l^n(B(M)[l^\infty])$. By Lemma 4.4 (applied to l, M instead of p, L), up to replacing V by $[l^N]V$ for N large enough, G is contained in T. By (4-1) (applied to l, M° instead of p, W), the image of the composition of

$$V(M) \bigcap \bigcap_{n=0}^{\infty} l^n(B(M)_{\mathrm{tor}}) \hookrightarrow B(M) \xrightarrow{\mathrm{pr}_l} B(M)[l^\infty]$$

contains G. By Proposition 4.9(2) (applied to l, M° instead of p, W), for every prime number $q \neq l$, the composition

$$V(M)_{l'-\mathrm{tor}} \hookrightarrow B(M)_{l'-\mathrm{tor}} \xrightarrow{\mathrm{pr}_q} B(M)[q^\infty]$$

contains a translate of a free $\mathbb{Q}_q/\mathbb{Z}_q$-submodule of rank 4. Repeating this process (use more places or only work at v and u), we get a contradiction as A is of finite dimension. □

5. Ordinary perfectoid Siegel space and Serre–Tate theory

Let $\mathbb{A}_f$ be the ring of finite adeles of $\mathbb{Q}$, $U^p \subset \mathrm{GSp}_{2g}(\mathbb{A}_f^p)$ an open compact subgroup contained in the congruence subgroup of level-N for some $N \geq 3$ prime to p. Let $X = X_{g,U^p}$ over $\mathbb{Z}_p$ be the Siegel moduli space of principally polarized g-dimensional abelian varieties over $\mathbb{Z}_p$-schemes with level U^p structure. Let X_o be special fiber of X. We will use the perfectoid fields defined on page 987. We briefly recall some notations. Let $k = \overline{\mathbb{F}}_p$, $W = W(k)$ the ring of Witt vectors, L the fraction field of W, and L^{cycl} the field extension of L by adjoining all p-power-th roots of unity. Let K be the p-adic completion of L^{cycl} which is a perfectoid field. Then $K^\flat = k((t^{1/p^\infty}))$ is the tilt of K. Fix a primitive p^n-th root of unity μ_{p^n} for every positive integer n such that $\mu_{p^{n+1}}^p = \mu_{p^n}$.

Ordinary perfectoid Siegel space. Let $X_o(0) \subset X_o$ be the ordinary locus. Let $\mathfrak{X}(0)$ over $\mathbb{Z}_p$ be the open formal subscheme of the formal completion of X along X_o defined by the condition that every local lifting of the Hasse invariant is invertible; see [Scholze 2015, Definition 3.2.12, Lemma 3.2.13]. Then $\mathfrak{X}(0)/p = X_o(0)$; see [loc. cit., Lemma 3.2.5]. Let $\widehat{X_o(0)_{K^{\flat\circ}}}$ be the $\varpi^\flat$-adic formal completion of $X_o(0)_{K^{\flat\circ}}$. Let $\mathcal{X}(0)$ and $\mathcal{X}'(0)$ be the adic generic fibers of $\mathfrak{X}(0)_{K^\circ}$ and $\widehat{X_o(0)_{K^{\flat\circ}}}$ respectively.

Let $\mathrm{Fr}: X_o(0) \to X_o(0)$ be the (relative) Frobenius morphism (note that $X_o(0)$ is defined over $\mathbb{F}_p$). Let $\mathrm{Fr}^{\mathrm{can}}: \mathfrak{X}(0) \to \mathfrak{X}(0)$ be given by the functor sending an abelian scheme A to its quotient by the connected subgroup scheme of $A[p]$. Then $\mathrm{Fr}^{\mathrm{can}}/p = \mathrm{Fr}$. We also use $\mathrm{Fr}^{\mathrm{can}}$ and Fr to denote their base changes to K° and $K^{\flat\circ}$ respectively. Let

$$\tilde{\mathfrak{X}}(0) := \varprojlim_{\mathrm{Fr}^{\mathrm{can}}} \mathfrak{X}(0)_{K^\circ}, \quad \tilde{\mathfrak{X}}'(0) := \varprojlim_{\mathrm{Fr}} \widehat{X_o(0)_{K^{\flat\circ}}},$$

where the inverse limits are taken in the categories of ϖ-adic and $\varpi^\flat$-adic formal schemes respectively. Here $\varpi = \mu_p - 1$ and $\varpi^\flat = t^{1/p}$. By [Scholze 2015, Corollary 3.2.19], the corresponding adic generic fibers $\mathcal{X}(0)^{\mathrm{perf}}$ and $\mathcal{X}'(0)^{\mathrm{perf}}$ of $\tilde{\mathfrak{X}}(0)$ and $\tilde{\mathfrak{X}}'(0)$ are perfectoid spaces. Moreover, $\mathcal{X}'(0)^{\mathrm{perf}} = \mathcal{X}(0)^{\mathrm{perf},\flat}$, the tilt of $\mathcal{X}(0)^{\mathrm{perf}}$. Then we have the natural projections

$$\pi : \mathcal{X}(0)^{\mathrm{perf}} \to \mathcal{X}(0), \quad \pi' : \mathcal{X}(0)^{\mathrm{perf},\flat} \to \mathcal{X}'(0). \tag{5-1}$$

We also have a natural map between the underlying sets defined on page 988

$$\rho_{\mathcal{X}(0)^{\mathrm{perf}}} : |\mathcal{X}(0)^{\mathrm{perf}}| \to |\mathcal{X}(0)^{\mathrm{perf},\flat}|. \tag{5-2}$$

(The map $\rho_{\mathcal{X}(0)^{\mathrm{perf}}}$ is in fact a homeomorphism and we do not need this fact.)

Classical Serre–Tate theory. We use the adjective "classical" to indicate the Serre–Tate theory [Katz 1981] discussed in this subsection, compared with Chai's global Serre–Tate theory to be discussed in Section 5.

Let R be an Artinian local ring with maximal ideal $\mathfrak{m}$ and residue field k. Let $A/\operatorname{Spec} R$ be an abelian scheme with ordinary special fiber A_k. Let $A_k^\vee$ be the dual abelian variety of A_k. There is a $\mathbb{Z}_p$-module morphism from the product of Tate-modules $T_pA_k \otimes T_pA_k^\vee$ to $1 + \mathfrak{m}$ constructed by Katz [1981]. We call this morphism the classical Serre–Tate coordinate system for $A/\operatorname{Spec} R$. If $A/\operatorname{Spec} R$ is moreover a principally polarized abelian scheme, the Serre–Tate coordinate system for $A/\operatorname{Spec} R$ is a $\mathbb{Z}_p$-module morphism

$$q_{A/\operatorname{Spec} R} : \operatorname{Sym}^2(T_p A_k) \to 1 + \mathfrak{m}. \tag{5-3}$$

Let $x \in X_o(0)(k)$, and let A_x be the corresponding principally polarized abelian variety. Let $\mathfrak{M}_x$ be the formal completion of X at x, and $\mathfrak{A}/\mathfrak{M}_x$ the formal universal deformation of A_x. Then as part of the construction of $q_{A/\operatorname{Spec} R}$, there is an isomorphism of formal schemes over W

$$\mathfrak{M}_x \simeq \operatorname{Hom}_{\mathbb{Z}_p}(\operatorname{Sym}^2(T_p A_x), \hat{\mathbb{G}}_m), \tag{5-4}$$

where $\hat{\mathbb{G}}_m$ is the formal completion of the multiplicative group scheme over W along the unit section. In particular, $\mathfrak{M}_x$ has a formal torus structure. Moreover, if $A_k \simeq A_x$ in (5-3), then (5-3) is the value of (5-4) at the morphism $\operatorname{Spec} R \to \mathfrak{M}_x$ induced by A. Let $\mathcal{O}(\mathfrak{M}_x)$ be the coordinate ring of $\mathfrak{M}_x$, and let $\mathfrak{m}_x$ be the maximal ideal of $\mathcal{O}(\mathfrak{M}_x)$. From (5-4), we have a morphism of $\mathbb{Z}_p$-modules

$$q = q_{\mathfrak{A}/\mathfrak{M}_x} : \operatorname{Sym}^2(T_p A_x) \to 1 + \mathfrak{m}_x.$$

Fix a basis $\xi_1, \dots, \xi_{g(g+1)/2}$ of $\operatorname{Sym}^2(T_p A_x)$.

Proposition 5.1 [de Jong and Noot 1991, 3.2]. *Let F be a finite extension of L with ring of integers F°. Let $y^\circ \in X(F^\circ)$ with generic fiber y. Suppose that $y^\circ \in \mathfrak{M}_x(F^\circ)$. Then y is a CM point if and only if $q(\xi_i)(y^\circ)$ is a p-primary root of unity for $i = 1, \dots, g(g+1)/2$.*

Thus every ordinary CM point is contained in $X(L^{\text{cycl}})$. For an ordinary CM point $y \in X(L^{\text{cycl}})$. there is a unique $y^\circ \in X(K^\circ)$ whose generic fiber is $y_K \in X(K)$. We regard y° as a point in $\mathfrak{X}(0)(K^\circ)$ and y_K as a point in $\mathcal{X}(K, K^\circ)$ via Lemma 2.3.

Definition 5.2. Let $a = (a^{(1)}, \dots, a^{(g(g+1)/2)}) \in \mathbb{Z}_{\geq 0}^{g(g+1)/2}$:

(1) An ordinary CM point $y \in X(L^{\text{cycl}})$ with reduction x is called of order p^a with respect to the basis $\xi_1, \dots, \xi_{g(g+1)/2}$ if $q(\xi_i)(y^\circ)$ is a primitive $p^{a^{(i)}}$-th root of unity for each $i = 1, \dots, g(g+1)/2$. If moreover $q(\xi_i)(y^\circ) = \mu_{p^{a^{(i)}}}$, y is called a μ-generator with respect to the basis $\xi_1, \dots, \xi_{g(g+1)/2}$.

(2) Assume that a is nonincreasing so that $q(\xi_{i+1})(y^\circ)$ is an $r^{(i)}$-th power of $q(\xi_i)(y^\circ)$ for some (nonunique) $r^{(i)} \in \mathbb{Z}_p$, $i = 1, \dots, g(g+1)/2 - 1$. We call $(r^{(1)}, \dots, r^{(g(g+1)/2-1)}) \in \mathbb{Z}_p^{g(g+1)/2-1}$ a ratio of y with respect to the basis $\xi_1, \dots, \xi_{g(g+1)/2}$.

It is clear that if a is nonincreasing, then the usual p-adic absolute value $|r^{(i)}|_p = p^{a^{(i+1)} - a^{(i)}}$.

Let $T_i = q(\xi_i) - 1 \in \mathfrak{m}_x$. Then we have an isomorphism

$$\mathcal{O}(\mathfrak{M}_x) \simeq W[\![T_1, \dots, T_{g(g+1)/2}]\!]. \tag{5-5}$$

Let $\widehat{X_o(0)}_{/x}$ be the formal completion of $X_o(0)$ at x. Restricted to $\widehat{X_o(0)}_{/x}$, (5-5) gives an isomorphism

$$\mathcal{O}(\widehat{X_o(0)}_{/x}) \simeq k[\![T_1, \dots, T_{g(g+1)/2}]\!]. \tag{5-6}$$

Let $U \to X_o(0)$ be an étale morphism, $z \in U(k)$ with image x. Then (5-6) gives an isomorphism

$$\mathcal{O}(\widehat{U}_{/z}) \simeq k[\![T_1, \dots, T_{g(g+1)/2}]\!]. \tag{5-7}$$

Let A_z be the pullback of A_x at z. Then we naturally have $T_pA_z \simeq T_pA_x$. Thus we also regard $\xi_1, \dots, \xi_{g(g+1)/2}$ as a basis of $\mathrm{Sym}^2(T_pA_z)$.

Definition 5.3. We call (5-7) the realization of the classical Serre–Tate coordinate system of $\widehat{U}_{/z}$ at the basis $\xi_1, \dots, \xi_{g(g+1)/2}$ of $\mathrm{Sym}^2(T_pA_z)$.

We have another description of (5-7). Let I_n be a descending sequence of open ideals of $\mathcal{O}(\widehat{U}_{/z})$ defining the topology of $\mathcal{O}(\widehat{U}_{/z})$. Let $R_n := \mathcal{O}(\widehat{U}_{/z})/I_n$, let A_n be the pullback of the formal universal principally polarized abelian scheme over $\mathfrak{M}_x$ to $\mathrm{Spec}\, R_n$ with special fiber A_z. Let

$$q_{A_n/\,\mathrm{Spec}\, R_n} : \mathrm{Sym}^2(T_pA_x^{\mathrm{univ}}) \to R_n^\times$$

be the classical Serre–Tate coordinate system of A_n/R_n. Then $q_{A_n/\,\mathrm{Spec}\, R_n}(\xi_i) - 1 = T_i \pmod{I_n}$. Thus the sequence $\{q_{A_n/\,\mathrm{Spec}\, R_n}(\xi_i) - 1\}_n$ gives an element in $\mathcal{O}(\widehat{U}_{/z}) \simeq \varprojlim_n R_n$, which equals T_i.

Tilts of ordinary CM points. Let $\widehat{X_o(0)_{K^{\flat\circ}}/x}$ be the formal completion of $X_o(0)_{K^{\flat\circ}}$ at x. By (5-6), we have

$$\mathcal{O}(\widehat{X_o(0)_{K^{\flat\circ}}/x}) \simeq K^{\flat\circ}[\![T_1, \dots, T_{g(g+1)/2}]\!]. \tag{5-8}$$

Let $\mathcal{D}_x$ be the adic generic fiber of $\widehat{X_o(0)_{K^{\flat\circ}}/x}$. Then $\mathcal{D}_x$ is an adic subspace of $\mathcal{X}'(0)$ in the sense of Definition 2.2. Moreover, (5-8) and Lemma 2.3 imply an isomorphism

$$\mathcal{D}_x(K^\flat, K^{\flat\circ}) \simeq K^{\flat\circ\circ, g(g+1)/2}. \tag{5-9}$$

Lemma 5.4. *Let $y \in X(L^{\mathrm{cycl}})$ be an ordinary CM point with reduction x:*

(1) *For every $\tilde{y} \in \pi^{-1}(y_K) \subset \mathcal{X}(0)^{\mathrm{perf}}$, we have*

$$\pi' \circ \rho_{\mathcal{X}(0)^{\mathrm{perf}}}(\tilde{y}) \in \mathcal{D}_x.$$

(2) *Let $a = (a^{(1)}, \dots, a^{(g(g+1)/2)}) \in \mathbb{Z}_{\geq 0}^{g(g+1)/2}$ and $I \subset \{1, 2, \dots, g(g+1)/2\}$ the subset of the i such that $a^{(i)} = 0$. Let y be a μ-generator of order p^a with respect to the basis $\xi_1, \dots, \xi_{g(g+1)/2}$ (see Definition 5.2). There exists $\tilde{y} \in \pi^{-1}(y_K)$ such that via the isomorphism* (5-9), *the i-th coordinate of $\pi' \circ \rho_{\mathcal{X}(0)^{\mathrm{perf}}}(\tilde{y})$ is 0 for $i \in I$ and is $t^{1/p^{a^{(i)}}}$ for $i \notin I$.*

Proof. We recall the effect of $\mathrm{Fr}^{\mathrm{can}}$ on $\mathfrak{M}_x$ (see [Katz 1981, 4.1]). Denote $\mathfrak{M}_x$ by $\mathfrak{M}_{A_x}$. Let $\sigma \in \mathrm{Aut}(k)$ be the Frobenius. Let $A_x^{(\sigma)} := A_x \otimes_{k,\sigma} k$ be the base change by σ. Then $\mathrm{Fr}^{\mathrm{can}}$, restricted to $\mathfrak{M}_{A_x}$ gives a morphism $\mathrm{Fr}^{\mathrm{can}} : \mathfrak{M}_{A_x} \to \mathfrak{M}_{A_x^{(\sigma)}}$ over W [loc. cit., page 171]. Let $\sigma(\xi_1), \dots, \sigma(\xi_{g(g+1)/2})$ be the induced basis of $\mathrm{Sym}^2(T_pA_x^{(\sigma)}[p^\infty])$. Then [loc. cit., Lemma 4.1.2] implies that

$$\mathrm{Fr}^{\mathrm{can},*}(q(\sigma(\xi_i))) = q(\xi_i)^p. \tag{5-10}$$

We associate a perfectoid space to $\mathfrak{M}_x$. Let

$$\tilde{\mathfrak{M}}_x := \varprojlim_{\mathrm{Fr}^{\mathrm{can}}} \mathfrak{M}_{A_x^{(\sigma^{-n})}}.$$

By a similar (and easier) proof as the one for [Scholze 2015, Corollary 3.2.19], the adic generic fiber $\mathcal{M}_x^{\mathrm{perf}}$ of $\tilde{\mathfrak{M}}_{x,K^\circ}$ is a perfectoid space. Moreover, let $\mathcal{M}_x'^{\mathrm{perf}}$ be the adic generic fiber of $\varprojlim_{\mathrm{Fr}} \widehat{X_o(0)_{K^{\flat\circ}}/x}$. Then $\mathcal{M}_x'^{\mathrm{perf}}$ is the tilt of $\mathcal{M}_x^{\mathrm{perf}}$. By Lemma 2.16, the tilting process commutes with restriction to an open subspace. Thus to prove Lemma 5.4, we only need to consider the tilting between $\mathcal{M}_x^{\mathrm{perf}}$ and $\mathcal{M}_x'^{\mathrm{perf}}$. Then Lemma 5.4 follows from the cases $c = 0$ and $c = 1$ of Lemma 2.18 (which deals with closed units discs while here we are dealing with open unit discs so that we apply Lemma 2.16 again). □

Global Serre–Tate theory.

The algebraic and geometric formulations. Now we review Chai's [2003] globalization of Serre–Tate coordinate system in characteristic p. Let U be a $\mathbb{F}_p$-scheme. Let A/U be an abelian scheme whose relative dimensions on connected components of U are the same. Define

$$\nu_U = \varprojlim_n \mathrm{Coker}([p^n] : \mathbb{G}_m \to \mathbb{G}_m),$$

which is a $\mathbb{Z}_p$-sheaf on U_{et}.

Example 5.5. (1) Let $m \geq n$ be positive integers, and $U_0 = \mathrm{Spec}\, k[T]/T^{p^n}$. Then the p^m-th power of an element in $(k[T]/T^{p^n})^\times$ with constant term b is b^{p^m}. Thus

$$\nu_{U_0}(U_0) = (k[T]/T^{p^n})^\times / k^\times \simeq 1 + T(k[T]/T^{p^n}). \tag{5-11}$$

(2) Let B be an $\mathbb{F}_p$-algebra, $U = \mathrm{Spec}\, B$ and $U' = \mathrm{Spec}\, B[T]/T^{p^n}$. For $m \geq n$, consider the map

$$B^\times/(B^\times)^{p^m} \bigoplus (1 + TB[T]/T^{p^n}) \to (B[T]/T^{p^n})^\times / ((B[T]/T^{p^n})^\times)^{p^m}$$

defined by $(a, f) \mapsto af$. Easy to check that this is a group isomorphism. In particular,

$$\nu_{U'}(U') \simeq \nu_U(U) \bigoplus (1 + TB[T]/T^{p^n}). \tag{5-12}$$

(3) For every $z \in U(k)$, $\{z\} \times_U U' \simeq U_0$. Then the restriction of the isomorphism (5-12) at z is the isomorphism (5-11).

Suppose A/U is ordinary. Let $T_pA[p^\infty]^{\mathrm{et}}$ be the Tate module attached to the maximal étale quotient of the p-divisible group $A[p^\infty]$. The global Serre–Tate coordinate system for A/U is a homomorphism of $\mathbb{Z}_p$-sheaves

$$q_{A/U} : T_pA[p^\infty]^{\mathrm{et}} \otimes T_pA^\vee[p^\infty]^{\mathrm{et}} \to \nu_U$$

constructed by Chai [2003, 2.5]. Let $U_0 = \mathrm{Spec}\, k[T]/T^{p^n}$. Let A/U_0 be an ordinary abelian scheme, and A_k the special fiber of A. Then

$$T_pA[p^\infty]^{\mathrm{et}} \otimes T_pA^\vee[p^\infty]^{\mathrm{et}} \simeq T_pA_k[p^\infty] \otimes T_pA_k^\vee[p^\infty], \tag{5-13}$$

where the right-hand side is regarded as a constant sheaf.

Lemma 5.6 [Chai 2003, (2.5.1)]. *The morphism of $\mathbb{Z}_p$-modules*

$$T_pA_k[p^\infty]\otimes T_pA_k^\vee[p^\infty]\to \nu_U(U)\simeq 1+T(k[T]/T^{p^n})$$

induced from q_{A/U_0} via (5-13) *coincides with the classical Serre–Tate coordinate system*; *see* (5-3).

The geometric formulation of global Serre–Tate coordinate system is as follows. Let A^{univ} be the universal principally polarized abelian scheme over $X_o(0)$, and $\hat{A}^{\mathrm{univ}}$ the formal completion of A^{univ} along the zero section which is a formal torus over $X_o(0)$. Then the sheaf of polarization-preserving $\mathbb{Z}_p$-homomorphisms between $T_pA^{\mathrm{univ}}[p^\infty]^{\mathrm{et}}$ and $\hat{A}^{\mathrm{univ}}$ is a formal torus over $X_o(0)$ of dimension $g(g+1)/2$. Let us call it $\mathfrak{T}_1$. Let Δ be the diagonal embedding of $X_o(0)$ into $X_o(0)\times X_o(0)$, and let $\mathfrak{T}_2$ be the formal completion of $X_o(0)\times X_o(0)$ along this embedding.

Proposition 5.7 [Chai 2003, Proposition 5.4]. *There is a canonical isomorphism $\mathfrak{T}_1\simeq\mathfrak{T}_2$. In particular, $\mathfrak{T}_2$ has a formal torus structure over the first $X_o(0)$.*

Igusa tower. In order to have sections of the étale $\mathbb{Z}_p$-sheaf $T_pA[p^\infty]^{\mathrm{et}}$ over U, or equivalently to trivialize the formal torus, we need to pass to the Igusa tower, defined as follow. For $n=0,1,\dots,\infty$, let $\mathfrak{I}_n$ be the functor assigning to every k-algebra R the set of isomorphism classes of pairs

$$\{(A,\varepsilon):A\in X_o(0)(R),\quad \varepsilon:A[p^n]\simeq\hat{\mathbb{G}}_{m,R}^g[p^n]\}.$$

By [Hida 2004, 8.1.1], for $n<\infty$ (resp. $n=\infty$) the functor $\mathfrak{I}_n$ is represented by a k-scheme (which we still denote by $\mathfrak{I}_n$) finite (resp. profinite) Galois over $X_o(0)$ with Galois group $\mathrm{GL}_g(\mathbb{Z}/p^n\mathbb{Z})$ (resp. $\mathrm{GL}_g(\mathbb{Z}_p)$). And $\mathfrak{I}_n$ is known as the Igusa scheme of level n.

Realization of the global Serre–Tate coordinate system at a basis. Let U_0 be an affine open subscheme of $X_o(0)$. Let $U=\operatorname{Spec}B:=\mathfrak{I}_\infty|_{U_0}$. Let Δ be the diagonal of $U\times U$. We have two projection maps $\mathrm{pr}_1,\mathrm{pr}_2$ from $\widehat{U\times U}_{/\Delta}$ to the first and second U. For $z\in U(k)$, the restriction of pr_2 induces

$$\mathrm{pr}_1^{-1}(\{z\})\overset{\mathrm{pr}_2}{\simeq}\widehat{U}_{/z}. \tag{5-14}$$

Let $\mathcal{O}(\widehat{U\times U}_{/\Delta})$ be the coordinate ring of $\widehat{U\times U}_{/\Delta}$. Endow $\mathcal{O}(\widehat{U\times U}_{/\Delta})$ a B-algebra structure via pr_1. By Proposition 5.7, we have a (nonunique) B-algebra isomorphism

$$\mathcal{O}(\widehat{U\times U}_{/\Delta})\simeq B[\![T_1,\dots,T_{g(g+1)/2}]\!]. \tag{5-15}$$

Let $A/\widehat{U\times U}_{/\Delta}$ be the pullback of $A^{\mathrm{univ}}|_{U_0}$. Assume that $\mathrm{Sym}^2(T_pA^{\mathrm{univ}}[p^\infty]^{\mathrm{et}})(U)$ is a free $\mathbb{Z}_p$-modules of rank $g(g+1)/2$. Let $\xi_1,\dots,\xi_{g(g+1)/2}$ be a basis of $\mathrm{Sym}^2(T_pA^{\mathrm{univ}}[p^\infty]^{\mathrm{et}})(U)$ (whose existence follows from the definition of $\mathfrak{I}_\infty$ and the polarization). The realization of the global Serre–Tate coordinate system of $A/\widehat{U\times U}_{/\Delta}$ at the basis $\xi_1,\dots,\xi_{g(g+1)/2}$ is a construction of an isomorphism (5-15) as follows.

For the simplicity of notations, let us assume $g=1$. The general case can be dealt in the same way. Let $\xi=\xi_1$ and $T=T_1$. Let

$$U'=\widehat{U\times U}_{/\Delta}/T^{p^n}\simeq\operatorname{Spec}B[\![T]\!]/T^{p^n}$$

and let A_n be the restriction of A to U'. The global Serre–Tate coordinate system of A_n/U' is a homomorphism of $\mathbb{Z}_p$-sheaves over $U_{2,\mathrm{et}}$

$$q_{A_n/U'} : \mathrm{Sym}^2(T_p A_n[p^\infty]^{\mathrm{et}}) \to \nu_{U'}.$$

Note that ξ gives a basis ξ_n of $\mathrm{Sym}^2(T_p A_n[p^\infty]^{\mathrm{et}})(U')$. Then we have

$$q_{A_n/U'}(\xi_n) \in \nu(U') \simeq \nu_U(U) \bigoplus (1 + TB[T]/T^{p^n})$$

where the second isomorphism is (5-12). Consider the morphism

$$\phi_n : \nu(U') \to (1 + TB[T]/T^{p^n}) \hookrightarrow B[T]/T^{p^n}$$

where the first map is the projection and second map is the natural inclusion. Let $T_n^{\mathrm{ST}} \in B[T]/T^{p^n}$ be $\phi_n(q_{A_n/U'}(\xi_n)) - 1$. As n varies, the T_n^{ST} give an element

$$T^{\mathrm{ST}} \in \mathcal{O}(\widehat{U \times U}_{/\Delta}) \simeq \varprojlim_n B[T]/T^{p^n}.$$

We compare the above construction with the realization of the classical Serre–Tate coordinate system. Let $z \in U(k)$. The restriction of A to $\mathrm{pr}_1^{-1}(\{z\})$ is pullback $A^{\mathrm{univ}}|_{\widehat{U}_{/z}}$ of $A^{\mathrm{univ}}|_{U_0}$ to $\widehat{U}_{/z}$ via (5-14). (Thus we may regard A as the family $\{A^{\mathrm{univ}}|_{\widehat{U}_{/z}} : z \in U(k)\}$.) The realization of the classical Serre–Tate coordinate system of $\widehat{U}_{/z}$ at ξ_z (the restriction of ξ at z) gives an element $T_z^{c\,\mathrm{ST}} \in \widehat{U}_{/z}$ and an isomorphism $\widehat{U}_{/z} \simeq \mathrm{Spf}\, k[\![T_z^{c\,\mathrm{ST}}]\!]$ (see Definition 5.3). Here and below, the superscript c indicates "classical".

Lemma 5.8. *The restriction of* T^{ST} *to* $\mathrm{pr}_1^{-1}(\{z\}) \simeq \widehat{U}_{/z}$ *is* $T_z^{c\,\mathrm{ST}}$. *In particular*,

$$\mathcal{O}(\widehat{U \times U}_{/\Delta}) = B[\![T^{\mathrm{ST}}]\!].$$

Proof. The restriction of (5-15) to $\mathrm{pr}_1^{-1}(\{z\}) \simeq \widehat{U}_{/z}$ via (5-14) gives an isomorphism $\mathcal{O}(\widehat{U}_{/z}) \simeq k[\![T]\!]$. Let

$$q_n^c = q^c_{A^{\mathrm{univ}}|_{\widehat{U}_{/z}}/T^{p^n}} : \mathrm{Sym}^2(T_p A_z^{\mathrm{univ}}[p^\infty]) \to 1 + Tk[\![T]\!]/T^{p^n}$$

be the classical Serre–Tate coordinate system of $A^{\mathrm{univ}}|_{\widehat{U}_{/z}}/T^{p^n}$ (see (5-3)). Then the image of $T_z^{c\,\mathrm{ST}}$ in $k[\![T]\!]/T^{p^n}$ is $q_n^c(\xi_z) - 1$. By Example 5.5(3) and Lemma 5.6, $q_n^c(\xi_z)$ equals the restriction of $\phi_n(q_{A_n/U'}(\xi_n))$ at z. Thus the first statement follows. The second statement follows from the first one. □

6. Proof of Theorem 1.1

In this section, we at first prove a Tate–Voloch type result in a family in characteristic p. Combined with the results in Section 5, we prove Theorem 1.1. We continue to use the notations in Section 5.

Tate–Voloch type result in a family in characteristic p. Recall that $k = \overline{\mathbb{F}}_p$ and $K^\flat = k((t^{1/p^\infty}))$. In the proof of Lemma 2.21, we used the following simple fact: let S a k-algebra, $g \in S$ and $x \in (\mathrm{Spec}\, S)(k)$, then $g(x) = 0$ or $|g(x)|_k = 1$ where the valuation $|\cdot|_k$ on k takes value 0 on $0 \in k$ and 1 on $k^\times$. This fact can be naively regarded as an analog of the Tate–Voloch conjecture over k. We want to consider this analog in a family. We need some notations.

Let l be a positive integer. For $d=(d^{(1)},\dots,d^{(l)})\in(p^{\mathbb{Z}_{<0}})^l$, define $t^d:=(t^{d^{(1)}},\dots,t^{d^{(l)}})\in(K^{\flat\circ})^l$. For $c=(c^{(1)},\dots,c^{(l)})\in(\mathbb{Z}_p{}^{\times})^l$, define

$$(1+t^d)^c-1:=((1+t^{d^{(1)}})^{c^{(1)}}-1,\dots,(1+t^{d^{(l)}})^{c^{(l)}}-1)\in(K^{\flat\circ})^l.$$

Fix a sequence $\{d_n\}_{n=1}^{\infty}$ of elements in $(p^{\mathbb{Z}_{<0}})^l$ and a sequence $\{c_n\}_{n=1}^{\infty}$ of elements in $(\mathbb{Z}_p^{\times})^l$. Let

$$y_n=(1+t^{d_n})^{c_n}-1\in(K^{\flat\circ})^l\subset\operatorname{Spec}K^{\flat\circ}[\![T_1,\dots,T_l]\!]. \tag{6-1}$$

Let $\mathbb{N}=\{1,2,\dots\}$ the sequence of positive integers. For $\delta\in(0,1)$ and the given sequence $\{d_n\}_{n=1}^{\infty}$, let

$$\mathbb{N}(\delta)=\{n\in\mathbb{N}:d_n^{(i)}/d_n^{(i+1)}<\delta\}.$$

If $l=1$, we understand $\mathbb{N}(\delta)$ as $\mathbb{N}$.

Proposition 6.1. *Let A be a reduced k-algebra and $V=\operatorname{Spec}A$. Let $\{z_n\}_{n=1}^{\infty}$ be a sequence of (not necessarily distinct) points in $V(k)$. Let $f\in A[\![T_1,\dots,T_l]\!]$ and let $f_{z_n}\in k[\![T_1,\dots,T_l]\!]$ be the restriction of f at z_n. Assume that*

$$\textit{for every infinite subset } \mathbb{N}'\subset\mathbb{N},\ \textit{the set } \{z_n:n\in\mathbb{N}'\} \textit{ is Zariski dense in } V. \tag{$\star$}$$

If $f\neq 0$, then there exists $D_0\in\mathbb{R}_{>0}$ and $\delta_0\in(0,1)$ such that for every $D\geq D_0$ and $\delta\leq\delta_0$, the following set is finite

$$\{n\in\mathbb{N}(\delta):\|f_{z_n}(y_n)\|<\|T_l(y_n)\|^D\}. \tag{6-2}$$

Here $T_l(y_n)$ is, by definition, the l-th coordinate of y_n.

Proof. We do induction on l.

The case $l=1$ is proved as follows. Let $f=\sum_{m\geq 0}a_mT^m$ where $a_m\in A$. Regard a_m as a function on V so that $a_m(z_n)\in k$. Claim: there exists some m such that $a_m(z_n)\neq 0$ for n large enough. Let m_0 be the smallest such m. Then

$$\|f_{z_n}(y_n)\|=\|t^{d_nm_0}\|$$

for n large enough. Let $D_0=m_0$ and we are done. Now we prove the claim by contradiction. Assume that for every m, $a_m(z_n)=0$ for infinitely many n. By assumption ($\star$) and the reducedness of A, $a_m=0$. Thus $f=0$. This is a contradiction.

Now we do the induction. Let $l>1$. We prepare some notations. Let d_n', y_n' be the first $l-1$ components of d_n, y_n respectively. For $\delta\in(0,1)$, we have a subsequence $\mathbb{N}(\delta)'\subset\mathbb{N}$ defined using the sequence $\{d_n'\}_{i=1}^{\infty}$. Then $\mathbb{N}(\delta)'\supset\mathbb{N}(\delta)$.

Assume that $f\neq 0$. Write $f=T_l^{m_1}(g_1+f_1)$ where $g_1\in A[\![T_1,\dots,T_{l-1}]\!]\backslash\{0\}$ and $f_1\in T_lA[\![T_1,\dots,T_l]\!]$. Below, to lighten notation, we abbreviate the subscript z_n. Then for n in the set (6-2), with D and δ to be determined, we have

$$\|g_1(y_n)+f_1(y_n)\| = \|T_l(y_n)\|^{-m_1}\|f(y_n)\| < \|T_l(y_n)\|^{D-m_1}.$$

If $D\geq m_1+1$, then

$$\|g_1(y_n)\| \leq \|g_1(y_n)+f_1(y_n)\| + \|f_1(y_n)\| \leq \|T_l(y_n)\|.$$

Since $\|T_l(y_n)\| < \|T_{l-1}(y_n')\|^{1/\delta}$ and $\|g_1(y_n')\| = \|g_1(y_n)\|$, we have

$$\|g_1(y_n')\| < \|T_{l-1}(y_n')\|^{1/\delta}. \tag{6-3}$$

By the induction hypothesis, there exists $D'>0$ and $\delta_0'\in(0,1)$ such that if $\delta\leq 1/D'$ and $\delta\leq\delta_0'$, $\{n\in\mathbb{N}(\delta)' : \text{(6-3) holds}\}$ is finite. Then (6-2) is finite by choosing $\delta_0=\min\{1/D',\delta_0'\}$. □

Remark 6.2. (1) D_0 and δ_0 are uniform for all choices of $\{c_n\}_{n=1}^{\infty}$. We do not need this fact later.

(2) The proposition is inspired by [Serban 2018, Lemma 2.10]. In the proof of that result, there is a minor imprecision. The following modification is suggested by Serban. Define T_δ in [loc. cit., Lemma 2.10] to be the first set in the intersection but not the entire intersection, so that the statement (2) in loc. cit. is about $T_\delta\cap S_\phi(q^{-1-c})$. The 3rd displayed formula in the proof of [loc. cit., Lemma 2.10] should be removed. Then, on can still get the 5th displayed formula in that proof with slightly more effort.

Closure and limit. We show that assumption ($\star$) in Proposition 6.1 holds in some situations.

Lemma 6.3. *Let $\{B_i\}_{i=0}^{\infty}$ be a system of rings and $B=\varinjlim_i B_i$. Let $f_i:\operatorname{Spec}B\to\operatorname{Spec}B_i$ be the natural morphism. Let $\Lambda\subset\operatorname{Spec}B$ be a subset and $\Lambda_i=f_i(\Lambda)\subset\operatorname{Spec}B_i$. We have the following relation between Zariski closures*:

$$\Lambda^{\mathrm{Zar}} = \bigcap_{i=0}^{\infty} f_i^{-1}(\Lambda_i^{\mathrm{Zar}}). \tag{6-4}$$

Proof. The ideal $I\subset B$ defining Λ^{Zar}, with reduced induced structure as a closed subscheme, is generated by the union of the images I_i in B, where $I_i\subset B_i$ is the ideal of elements whose image in B vanishes on Λ^{Zar}. By the definition of Λ_i, I_i is the ideal defining Λ_i^{Zar}. Then (6-4) follows. □

Let $f:U\to U_0$ be a surjective morphism of schemes. Let $\Lambda_0\subset U_0$ be a subset with Zariski closure Λ_0^{Zar} in U_0. For $s\in\Lambda_0$, choose $z_s\in f^{-1}(s)$. Let $\Lambda=\{z_s : s\in\Lambda_0\}$ with Zariski closure Λ^{Zar} in U.

Lemma 6.4. *Assume that f is closed*:

(1) *The image of Λ^{Zar} in U_0 is Λ_0^{Zar}.*

(2) *Assume that Λ_0^{Zar} is irreducible and U is noetherian. There exists a choice of Λ such that Λ^{Zar} is irreducible.*

(3) *In* (2), *further assume that f is finite and the Zariski closure of every infinite subset of Λ_0 is Λ_0^{Zar}. Then the Zariski closure of every infinite subset of Λ is Λ^{Zar}.*

Proof. (1) is easy and the proof is omitted.

(2) For every member of the finitely many irreducible (so closed) components of $f^{-1}(\Lambda_0^{\mathrm{Zar}})$, its image in Λ_0^{Zar} is a closed subscheme. By the irreducibility of Λ_0^{Zar}, some irreducible component of $f^{-1}(\Lambda_0^{\mathrm{Zar}})$ is surjective to Λ_0^{Zar}. We choose all the z_s in this component.

(3) Note that a finite surjective morphism preserves dimension, and a proper closed subscheme of a noetherian irreducible scheme has a strictly smaller dimension. Then (3) follows from (1) and counting dimensions. □

The last two lemmas imply the following corollary.

Corollary 6.5. *Let the B, B_i be as in Lemma 6.3. Let $U = \mathrm{Spec}\, B$ (not necessary noetherian), $U_0 = \mathrm{Spec}\, B_0$ and $f = f_0$. Assume that each B_i is noetherian and the transition morphisms $\mathrm{Spec}\, B_j \to \mathrm{Spec}\, B_i$ are finite surjective. Assume that the Zariski closure of every infinite subset of Λ_0 is Λ_0^{Zar}. There exists a choice of Λ such that the Zariski closure of every infinite subset of Λ is Λ^{Zar}.*

Later, to fulfill the second assumption of the corollary, we will use the following lemma.

Lemma 6.6. *Let U_0 be a noetherian scheme. For every infinite subset $Y \subset U_0$, there is an infinite subset $\Lambda_0 \subset Y$ such that the Zariski closure of every infinite subset of Λ_0 is Λ_0^{Zar}.*

Proof. By the noetherianness of U_0, there exists a closed subscheme V of U_0 containing an infinite subset Λ_0 of Y such that every proper closed subscheme of V only contains finitely many elements in Y. □

Proof of Theorem 1.1. Let X be a product of Siegel moduli spaces over $\mathbb{Z}_p$ with certain level structures away from p. By Lemma 2.15, Theorem 1.1 follows from the following theorem.

Theorem 6.7. *Let Z be a closed subvariety of $X_{\bar{L}}$. There exists a constant $c > 0$ such that for every ordinary CM point $x \in X(L^{\mathrm{cyc}})$, if $d(x, Z) \le c$, then $x \in Z$.*

Here the distance function $d(x, Z)$ is defined as on page 986 using the integral model X

Proof. We prove Theorem 6.7 when X is a single Siegel moduli space. The general case is proved in the same way or by embedding a product of Siegel moduli spaces into a bigger one. We continue to use the notations in Section 5. In particular, the fields L, L^{cyc}, K and $K^\flat$ below are as in the beginning of Section 5; the formal scheme $\mathfrak{X}(0)$, the adic locus $\mathcal{X}(0)$, the perfectoid spaces $\mathcal{X}(0)^{\mathrm{perf}}$, $\mathcal{X}(0)^{\mathrm{perf}\flat}$ and Frobenius morphism $\mathrm{Fr}^{\mathrm{can}}$ below are as in Section 5. For an ordinary CM point $x \in X(L^{\mathrm{cyc}})$, we the same notation x to denote its base change in $X(K)$. Let x° be the unique K°-point in X whose generic fiber is x.

Suppose that Z is defined over a finite Galois extension F of L. Let $\mathcal{I}$ be the ideal sheaf of the schematic closure of Z in X_{F°. Let $\mathcal{U}$ be an affine open subscheme of X_W, of finite type over W. (This is the only use of a calligraphic font not representing an adic space in this paper.) We only need to find a constant c such that, if an ordinary CM point $x \in X(L^{\mathrm{cyc}})$ satisfies $x^\circ \in \mathcal{U}(K^\circ)$ and $d_{\mathcal{U}_{K^\circ}}(x_K, I) < c$, then $x \in Z$. Here the distance function is as on page 986.

We at first have the following simplification on F. Let $K' = FK$. Suppose $\mathcal{I}(\mathcal{U}_{F^\circ})$ is generated by f_i, $i = 1, \ldots, n$. For $\sigma \in G := \mathrm{Gal}(K'/K)$, f_i^σ is in the coordinate ring of $\mathcal{U}_{K'^\circ}$ and $\|f_i(x_{K'})\| = \|f_i^\sigma(x_{K'})\|$. Let I be the ideal of the coordinate ring of $\mathcal{U}_{K^\circ}$ generated by $\prod_{\sigma\in G} f_i^\sigma$, $i = 1, \ldots, n$. Then

$$d_{\mathcal{U}_{K^\circ}}(x_K, I) = d_{\mathcal{U}_{K'^\circ}}(x_{K'}, \mathcal{I}_{K'^\circ}(\mathcal{U}_{K'^\circ}))^{|G|}.$$

Thus we may assume that $F \subset K$. Equivalently, $F \subset L^{\mathrm{cycl}}$.

Now we reduce Theorem 6.7 to Theorem 6.8 below, which is formulated with affine formal schemes. For an ordinary CM point $x \in X(K)$, we also use x to denote the corresponding point $\mathcal{X}(0)(K, K^\circ)$. Let $\mathfrak{U}$ be the restriction of the ϖ-adic formal completion of $\mathcal{U}$ to $\mathfrak{X}(0)$. By Lemma 2.12, Theorem 6.7 is deduced from Theorem 6.8. □

Theorem 6.8. *Let $\mathfrak{Z}$ be an irreducible closed formal subscheme of $\mathfrak{U}_{F^\circ}$. For a sequence $\{x_n\}_{n=1}^\infty$ of ordinary CM points such that x_n is in the ϵ_n-neighborhood of $\mathfrak{Z}$ and with $\|\epsilon_n\| \to 0$, we have $x_n \in \mathcal{Z}$ for infinitely many n.*

The proof of Theorem 6.8 consists of two bulks: one involves perfectoid spaces and one does not. The perfectoid one is more technical and proves results to be used in the second one. The nonperfectoid one concludes Theorem 6.8. We will present the non-perfectoid one first, on pages 1009 and 1010.

A canonical lifting is an ordinary CM points of order 1 with respect to a (equivalently every) basis, see Definition 5.2(1). The following lemma will be proved in Theorem 6.13 using perfectoid spaces.

Lemma 6.9. *Theorem 6.8 holds if we replace "ordinary CM points" by "canonical liftings".*

Global Serre–Tate coordinate. Before we proceed to the proof of Theorem 6.8, let us recall the realization of the global Serre–Tate coordinate system on page 1004.

Let U_0 be the special fiber of $\mathfrak{U}$. Let $U = \mathrm{Spec}\, B$ be the profinite Galois cover of U_0 defined on page 1004 (and coming from the infinite level Igusa scheme) such that $\mathrm{Sym}^2(T_pA^{\mathrm{univ}}[p^\infty]^{\mathrm{et}})(U)$ is a free $\mathbb{Z}_p$-modules of rank $g(g+1)/2$. Let Δ be the diagonal of $U \times U$. Then by Lemma 5.8, for a basis

$$\xi_1, \ldots, \xi_{g(g+1)/2} \tag{6-5}$$

of $\mathrm{Sym}^2(T_pA^{\mathrm{univ}}[p^\infty]^{\mathrm{et}})(U)$, we have the realization of the global Serre–Tate coordinate system

$$\mathcal{O}(\widehat{U \times U}_{/\Delta}) = B[[T_1^{\mathrm{ST}}, \ldots, T_{g(g+1)/2}^{\mathrm{ST}}]],$$

which has the following property. For every $z \in U(k)$, we have an isomorphism

$$\mathrm{pr}_1^{-1}(\{z\}) \overset{\mathrm{pr}_2}{\simeq} \widehat{U}_{/z} \tag{6-6}$$

as in (5-14), and the corresponding isomorphism

$$\widehat{U}_{/z} \simeq \mathrm{Spf}\, k[[T_{1,z}^{\mathrm{ST}}, \ldots, T_{g(g+1)/2,z}^{\mathrm{ST}}]]. \tag{6-7}$$

Let $T_{i,z}^{\mathrm{ST}}$ be the restriction of T_i^{ST} to $\mathrm{pr}_1^{-1}(\{z\}) \simeq \widehat{U}_{/z}$. Let

$$\xi_{z,1}, \ldots, \xi_{z,g(g+1)/2} \tag{6-8}$$

be the restriction of $\xi_1, \dots, \xi_{g(g+1)/2}$. Then (6-7) coincides with the realization of the classical Serre–Tate coordinate system of $\widehat{U}_{/z}$ at $\xi_{z,1}, \dots, \xi_{z,g(g+1)/2}$; see Definition 5.3.

Proof of Theorem 6.8. After passing to an infinite subsequence, we may assume that $\{\mathrm{red}(x_n)\}_{n=1}^{\infty}$ is a sequence of the same point or pairwise different points. Let $z_n \in U(k)$ be over $\mathrm{red}(x_n) \in U_0(k)$. By Corollary 6.5 and Lemma 6.6, after passing to an infinite subsequence, we may assume the following.

Assumption 6.10. For every infinite subset $\mathbb{N}' \subset \mathbb{N}$, the Zariski closure of the set $\{z_n : n \in \mathbb{N}'\}$ in U is the Zariski closure of the set $\{z_n : n \in \mathbb{N}\}$.

We regarded the basis (6-8) for $z = z_n$ as a basis of $\mathrm{Sym}^2(T_p A_{x_n})$ naturally. Let x_n be of order p^{a_n} with respect to (6-8) (see Definition 5.2(1)) where $a_n = (a_n^{(1)}, \dots, a_n^{(g(g+1)/2)}) \in \mathbb{Z}_{\geq 0}^{g(g+1)/2}$. After passing to an infinite subsequence and permuting the basis (6-5) of $\mathrm{Sym}^2(T_p A^{\mathrm{univ}}[p^\infty]^{\mathrm{et}})(U)$, we may assume that every a_n is nonincreasing; see Definition 5.2(2). Let $l \leq g(g+1)/2$ be a nonnegative integer such that for every n, if $i > l$, then $a_n^{(i)} = 0$. For example, if $l = g(g+1)/2$, the assumption automatically holds; if $l = 0$, we are in the situation of Lemma 6.11.

We will reduce Theorem 6.8 to the case $l = 0$ by using Lemma 6.11 below. We need the "upper triangular change of variables" argument following [Serban 2018]. By "upper triangular change of variables", we indeed mean changing the first l-element of the basis (6-5) of $\mathrm{Sym}^2(T_p A^{\mathrm{univ}}[p^\infty]^{\mathrm{et}})(U)$ via an upper triangular matrix as follows. For $C \in \mathrm{GL}_l(\mathbb{Z}_p)$, $(\xi_1, \dots, \xi_l)C$ combined with $(\xi_{l+1}, \dots, \xi_{(g(g+1)/2)})$ gives a new basis of $\mathrm{Sym}^2(T_p A^{\mathrm{univ}}[p^\infty]^{\mathrm{et}})(U)$. Thus by restriction as in (6-8), we have a new basis of $\mathrm{Sym}^2(T_p A_{x_n})$ for every n. Let x_n be of order $p^{a_n(C)}$ with respect to this new basis, where $a_n(C) \in \mathbb{Z}_{\geq 0}^{g(g+1)/2}$. Then for C upper triangular, $a_n(C)$ is still nonincreasing.

Lemma 6.11. *Assume Assumption 6.10. Assume that for every upper triangular matrix $C \in \mathrm{GL}_l(\mathbb{Z}_p)$, the l-th component (so the i-th component for $i = 1, \dots, l$ as well) of $a_n(C)$ goes to ∞ as $n \to \infty$. Then $x_n \in \mathcal{Z}$ for all $n \in \mathbb{N}$.*

We postpone the proof of Lemma 6.11.

We finish the proof of Theorem 6.8 by induction on the dimension of $\mathfrak{Z}$. If $\mathfrak{Z}$ is empty, define its dimension to be -1. When $\mathfrak{Z}$ is of dimension -1, the theorem is trivial. The induction hypothesis is that the theorem holds for lower dimensions, and it will only be used in the proof of Lemma 6.12(2) below.

By Lemma 6.11 and passing to an infinite subsequence, we may assume that for an upper triangular matrix $C \in \mathrm{GL}_l(\mathbb{Z}_p)$, the l-th component of $a_n(C)$ is bounded. Replacing the basis (6-5) by the new basis that is $(\xi_1, \dots, \xi_l)C$ combined with $(\xi_{l+1}, \dots, \xi_{(g(g+1)/2)})$, we may assume that there is a nonnegative integer m such that for every n, $a_n^{(l)} \leq p^m$. The fact that $a_n^{(i)} = 0$ for $i > l$ does not change.

Lemma 6.12. *Let m be a nonnegative integer. Then the following hold*:

(1) *The adic generic fiber of $(\mathrm{Fr}^{\mathrm{can}})^m(x_n^\circ)$ is in the ϵ_n-neighborhood of the scheme theoretic image $(\mathrm{Fr}^{\mathrm{can}})^m(\mathfrak{Z})$; see* [Kappen 2013, 2.3].

(2) *Assume that $(\mathrm{Fr}^{\mathrm{can}})^m(x_n^\circ) \in (\mathrm{Fr}^{\mathrm{can}})^m(\mathfrak{Z}(K^\circ))$ for infinitely many n, then $x_n^\circ \in \mathfrak{Z}(K^\circ)$ for infinitely many n.*

Proof. To lighten the notations, assume that $m = 1$.

Consider the closed formal subscheme $(\mathrm{Fr}^{\mathrm{can}})^{-1}(\mathrm{Fr}^{\mathrm{can}}(\mathfrak{Z}))$ of $\mathfrak{U}$ which contains $\mathfrak{Z}$. Then x_n° is contained in the ϵ_n-neighborhood of $(\mathrm{Fr}^{\mathrm{can}})^{-1}(\mathrm{Fr}^{\mathrm{can}}(\mathfrak{Z}))$ by Lemma 2.7(1). Then (1) follows from the analog of Lemma 2.10 for formal schemes (which directly follows from Definition 2.4).

For (2), we prove it by contradiction. Let $\{n_i\} \subset \mathbb{N}$ be an infinite subsequence such that $\mathrm{Fr}^{\mathrm{can}}(x_{n_i}^\circ) \in \mathrm{Fr}^{\mathrm{can}}(\mathfrak{Z}(K^\circ))$ and $x_{n_i}^\circ \notin \mathfrak{Z}(K^\circ)$ for n_i large enough. In particular, $(\mathrm{Fr}^{\mathrm{can}})^{-1}(\mathrm{Fr}^{\mathrm{can}}(\mathfrak{Z})) \neq \mathfrak{Z}$. Thus by [Kappen 2013, Proposition 2.10], it is not hard to show that

$$(\mathrm{Fr}^{\mathrm{can}})^{-1}(\mathrm{Fr}^{\mathrm{can}}(\mathfrak{Z})) = \mathfrak{Z} \bigcup \mathfrak{Z}'$$

such that $\mathfrak{Z}'$ does not contain $\mathfrak{Z}$ and $x_{n_i}^\circ \in \mathfrak{Z}'(K^\circ)$. By Lemma 2.6, every x_{n_i} is contained in the ϵ_{n_i}-neighborhood of $\mathfrak{Z} \bigcap \mathfrak{Z}'$. Let $\mathfrak{Z}_1$ be the union of irreducible components of $\mathfrak{Z} \bigcap \mathfrak{Z}'$ which dominate $\mathrm{Spf}\, F^\circ$. By Lemma 2.7(2), there exists $\delta \in K^\circ - \{0\}$ such that every x_{n_i} is contained in the ϵ_{n_i}/δ-neighborhood of $\mathfrak{Z}_1$. Since every irreducible component of $\mathfrak{Z}_1$ has dimension less than the dimension of $\mathfrak{Z}$, by the induction hypothesis, we have $x_{n_i} \in \mathfrak{Z}_1(K^\circ) \subset \mathfrak{Z}(K^\circ)$. This is a contradiction. □

By (5-10) and Lemma 6.12, after passing to an infinite subsequence, we may assume that for every n, if $i \geq l$, then $a_n^{(i)} = 0$, i.e., we may replace l by $l-1$. Continue this process, we may assume that $l = 0$, i.e., $a_n^{(i)} = 0$ for every n and i. Now Theorem 6.8 follows from Lemma 6.11.

Canonical liftings and perfectoid strategy. Now our remaining tasks are the proofs of Lemmas 6.9 and 6.11. For Lemma 6.9, we prove an "almost effective" version of Theorem 6.8 for canonical liftings. In the proof, we use the ordinary perfectoid Siegel space and Scholze's approximation lemma, following a strategy of Xie [2018]. Our later proof of Lemma 6.11 involves a more complicated version of this proof (which in particular uses the global Serre–Tate coordinate).

Let $\mathcal{X}$ be the restriction of $\mathcal{X}(0)^{\mathrm{perf}}$ to the adic generic fiber of $\mathfrak{U}_{K^\circ}$. Then $\mathcal{X} = \mathrm{Spa}(R, R^+)$ where (R, R^+) is a perfectoid affinoid (K, K°)-algebra (there is no need to specify R though it is easy to do so). The restriction of $\mathcal{X}(0)^{\mathrm{perf},\flat}$ to the adic generic fiber of $U_0 \otimes K^{\flat\circ}$ is $\mathcal{X}^\flat = \mathrm{Spa}(R^\flat, R^{\flat+})$, the tilt of $\mathcal{X}$. More concretely, it is given as follows: let S_m be the coordinate ring of $(\mathrm{Fr}^m)^{-1}(U_0)$ with the natural inclusion $S_{m-1} \hookrightarrow S_m$, and $S = \bigcup_m S_m$, then $R^{\flat+}$ is the $\varpi^\flat$-adic completion of $S \otimes K^{\flat\circ}$. Let $\mathcal{X}_m$ be the adic generic fiber of $\mathrm{Spec}\, S_m \otimes K^{\flat\circ}$ and

$$\pi_m : \mathcal{X}^\flat \to \mathcal{X}_m$$

the natural projection. Recall π and π' as defined in (5-1). Then $\pi_0 = \pi'|_{\mathcal{X}^\flat}$ (which has image in $\mathcal{X}_0$). We abbreviate $\pi|_{\mathcal{X}}$ as π (which has image in the adic generic fiber of $\mathfrak{U}_{K^\circ}$.) Let ρ be the restriction of $\rho_{\mathcal{X}(0)^{\mathrm{perf}}}$ (see (5-2)) to $\mathcal{X}$.

For $f \in \mathcal{O}(\mathfrak{U})$ in the defining ideal of $\mathfrak{Z}$, regard f as an element of R^+ by the inclusion $\mathcal{O}(\mathfrak{U}) \subset R^+$. For $c \in \mathbb{Z}_{>0}$, choose g as in Lemma 2.19 (with respect to f) and choose a finite sum

$$g_c = \sum_{\substack{i \in \mathbb{Z}[1/p]_{\geq 0},\\ i < 1/p + c}} g_{c,i} \cdot (\varpi^\flat)^i$$

as in Lemma 2.20 where $g_{c,i} \in S$ for all i. There exists a positive integer $m(c)$ such that $g_{c,i} \in S_{m(c)}$ for all i by the finiteness of the sum. Let $G_c := g_c^{p^{m(c)}}$. Then we have the finite sum

$$G_c = \sum_{\substack{i \in \mathbb{Z}[1/p]_{\geq 0}, \\ i < 1/p+c}} G_{c,i} \cdot (\varpi^\flat)^{p^{m(c)} i}, \tag{6-9}$$

where $G_{c,i} = g_{c,i}^{p^{m(c)}}$. By the construction of the S_n, we have $G_{c,i} \in S_0$. Let I_c be the ideal of S_0 generated by $\{G_{c,i} : i \in \mathbb{Z}[1/p]_{\geq 0}, i < 1/p + c\}$. By the noetherianness of S_0, there exists a positive integer M such that

$$\sum_{c=1}^{\infty} I_c = \sum_{c=1}^{M} I_c. \tag{6-10}$$

For $y \in \mathfrak{X}(0)$ and $\tilde{y} \in \pi^{-1}(y) \subset \mathcal{X}$, $|f(\tilde{y})| = \|f(y)\|$. If $\|f(y)\| \leq \|\varpi\|^{1/p+M}$, by (2-2) and (2-3), we have $|g_c(\pi_{m(c)}(\rho(\tilde{y})))| \leq \|\varpi\|^{1/p+c}$ for $c = 1, \dots, M$. So for $c = 1, \dots, M$, we have

$$|G_c(\pi_0(\rho(\tilde{y})))| = |G_c(\pi_{m(c)}(\rho(\tilde{y})))| \leq \|\varpi\|^{(1/p+c)p^{m(c)}}. \tag{6-11}$$

Theorem 6.13. *Assume that $\{f_1, \dots, f_r\} \subset \mathcal{O}(\mathfrak{U})$ generates the ideal defining $\mathfrak{Z}$. For each f_j, let M_j be the M as in* (6-10) *with f replaced by f_j. Let $\mathbb{M} = \max\{M_j : j = 1, \dots, t\}$. Let y be a canonical lifting in the $\varpi^{1/p+\mathbb{M}}$-neighborhood of $\mathfrak{Z}$. Then $y \in \mathcal{Z}$.*

Proof. Apply Lemma 5.4(2) to y with $a = 1$. Choose $\tilde{y} \in \pi^{-1}(y)$ to be as in Lemma 5.4(2). Then $\pi_0(\rho(\tilde{y})) = \mathrm{red}(y) \in U_0(k)$, where we understand $U_0(k)$ as a subset of $\mathfrak{X}(K^\flat, K^{\flat\circ})$ naturally. Let $f = f_j$ and $M = M_j$ for some j. Then $|f(\tilde{y})| \leq \|\varpi\|^{1/p+M}$ and thus we have (6-11). Similar to Lemma 2.21, by (6-11) and (6-9), we have $G_{c,i}(\pi_0(\rho(\tilde{y}))) = 0$ for every $c = 1, \dots, M$ and the corresponding i. By (6-10), $I_c(\pi_0(\rho(\tilde{y}))) = \{0\}$ for every $c \in \mathbb{Z}_{>0}$. So

$$G_c(\pi_{m(c)}(\rho(\tilde{y}))) = G_c(\pi_0(\rho(\tilde{y}))) = 0$$

for every $c \in \mathbb{Z}_{>0}$. Thus $g_c(\pi_{m(c)}(\rho(\tilde{y}))) = 0$. By (2-2) and (2-3), $|f(\tilde{y})| \leq \|\varpi\|^{1/p+c}$ for every $c \in \mathbb{Z}_{>0}$. Thus $|f(\tilde{y})| = 0$. □

Remark 6.14. The effectivity of $\mathbb{M}$ is essentially determined by the effectivity of the determination of the approximating function g in Lemma 2.19. However, Scholze's proof of Lemma 2.19 uses "almost ring theory" and is not effective. It is meaningful to ask if Lemma 2.19 can be made effective.

Toward the proof of Lemma 6.11. This paragraph closely mimics the proof of Theorem 6.13. Let notations be as above Theorem 6.13 and let $y = x_n$. For every $c = 1, \dots, M$ and a corresponding i, we want to show that $G_{c,i}(\pi_0(\rho(\tilde{x}_n))) = 0$. Then by (6-10), $I_c(\pi_0(\rho(\tilde{x}_n))) = \{0\}$ for every $c \in \mathbb{Z}_{>0}$. So $G_c(\pi_{m(c)}(\rho(\tilde{x}_n))) = G_c(\pi_0(\rho(\tilde{x}_n))) = 0$ for every $c \in \mathbb{Z}_{>0}$. Thus $g_c(\pi_{m(c)}(\rho(\tilde{x}_n))) = 0$. By (2-2) and (2-3), $|f(\tilde{x}_n)| \leq \|\varpi\|^{1/p+c}$ for every $c \in \mathbb{Z}_{>0}$. Thus o $|f(\tilde{x}_n)| = 0$. Let f run over a finite set of generators of the defining ideal of $\mathfrak{Z}$ and choose infinite subsequences successively, we have $x_n \in \mathcal{Z}$ for infinitely many n.

Spaces. For $x \in U_0(k)$ (resp. $U(k)$), let $\mathcal{D}_x$ be the adic generic fiber of the formal completion of $U_0 \otimes K^{\flat\circ}$ (resp. $U \otimes K^{\flat\circ}$) at x. (This coincides with the definition in Section 5.) Equivalently, $\mathcal{D}_x$ is the adic generic fiber of the formal completion of $\widehat{U_0}_{/x} \otimes K^{\flat\circ}$ (resp. $\widehat{U}_{/x} \otimes K^{\flat\circ}$). The following two diagrams summarize the adic spaces/k-schemes and morphisms between them that we use:

$$\begin{array}{ccccc} \mathcal{X} & \xrightarrow{\rho} & \mathcal{X}^\flat & & \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_0} & & \\ \mathcal{X}(0) & & \mathcal{X}_0 & \xleftarrow{(1)} \coprod_{x \in U_0(k)} \mathcal{D}_x & \xleftarrow{(2)} \coprod_{z \in U(k)} \mathcal{D}_z \end{array} \tag{6-12}$$

$$U_0 \xleftarrow{(1')} \coprod_{x \in U_0(k)} \widehat{U_0}_{/x} \xleftarrow{(2')} \coprod_{z \in U(k)} \widehat{U}_{/z} \xleftarrow{(6\text{-}6)} \coprod_{z \in U(k)} \mathrm{pr}_1^{-1}(\{z\}) \xrightarrow{(3)} \widehat{U \times U}_{/\Delta}$$

Here the morphisms (1), (1') and (3) are the natural inclusions. And the morphism (2), when restricted to $\mathcal{D}_z$, $z \in U(k)$, is the natural isomorphism $\mathcal{D}_z \simeq \mathcal{D}_x$ where $x \in U_0(k)$ is the image of z. We have the parallel statement for (2').

Functions. Let $H_{c,i}$ be the image of $G_{c,i}$ in B under the morphism $S_0 = \mathcal{O}(U_0) \to B = \mathcal{O}(U)$, and $H_{c,i,z_n} \in \mathcal{O}(\widehat{U}_{/z_n})$ the image of $H_{c,i}$ under the morphism $B = \mathcal{O}(U) \to \mathcal{O}(\widehat{U}_{/z_n})$.

For $\tilde{x}_n \in \pi^{-1}(x_n) \subset \mathcal{X}$, by Lemma 5.4(1), $\pi_0(\rho(\tilde{x}_n)) \in \mathcal{D}_{\mathrm{red}(x_n)}$. Let y_n be the preimage of $\pi_0(\rho(\tilde{x}_n))$ in $\mathcal{D}_{z_n}$ via the natural isomorphism $\mathcal{D}_{z_n} \simeq \mathcal{D}_{\mathrm{red}(x_n)}$. Then as elements in $K^{\flat\circ}$, we have

$$H_{c,i,z_n}(y_n) = H_{c,i}(y_n) = G_{c,i}(\pi_0(\rho(\tilde{x}_n))).$$

Lemma 6.15. *There is a constant $h_{c,i} < 1$ such that $\|H_{c,i,z_{n_m}}(y_{n_m})\| < h_{c,i}$.*

Proof. If the lemma is not true, let i_0 be the smallest i appearing in the finite sum (6-9) such that $\|H_{c,i,z_{n_m}}(y_{n_m})\| \to 1$ for a subsequence $\{n_m\}_{m=1}^{\infty} \subset \mathbb{N}$. Then (6-9) implies that $\|G_c(\pi_0(\rho(\tilde{x}_{n_m})))\| \to \|\varpi\|^{i_0 p^{m(c)}}$, which contradicts (6-11). □

Let ϕ be the composition of

$$\phi : B = \mathcal{O}(U) \to B \otimes B \to \mathcal{O}(\widehat{U \times U}_{/\Delta}) = B[\![T_1^{\mathrm{ST}}, \ldots, T_{g(g+1)/2}^{\mathrm{ST}}]\!]$$

where the first morphism is $b \mapsto 1 \otimes b$. i.e., ϕ gives the projection $\mathrm{pr}_2 : \widehat{U \times U}_{/\Delta}$ to the second U. Tracking the second diagram of (6-12), we have the following lemma.

Lemma 6.16. *The restriction of $\phi(H_{c,i})$ to $\mathrm{pr}_1^{-1}(\{z_n\}) \overset{\mathrm{pr}_2}{\simeq} \widehat{U_1}_{/z_n}$ in (6-6) is $H_{c,i,z_n} \in \mathcal{O}(\widehat{U_1}_{/z_n})$.*

Proof of Lemma 6.11. We need some notations. For an open subset $O \subset \mathbb{Z}_p^{l-1}$, let $\mathbb{N}(O) \subset \mathbb{N}$ be the subsequence such that the first $l-1$ components of a ratio of x_n with respect to this basis (see Definition 5.2) is in O. If $l = 1$, we understand $\mathbb{N}(O)$ as the whole $\mathbb{N}$ (and we will not need the case $l = 0$). For $r \in \mathbb{Z}_p^{l-1}$ and $\delta \in (0, 1)$, let $\mathbb{N}(r, \delta) = \mathbb{N}(O(r, \delta))$ where $O(r, \delta)$ is the p-adic closed disc centered at r of radius δ.

Now we start to prove Lemma 6.11. By the discussion on page 1012, we only need to prove that for every $n \in \mathbb{N}$, $G_{c,i}(\pi_0(\rho(\tilde{x}_n))) = 0$. Let Spec $A \subset$ Spec B be the Zariski closure of the set $\{z_n : n \in \mathbb{N}\}$. Let f be the image of $\phi(H_{c,i})$ under $B[[T_1^{\mathrm{ST}}, \dots, T_{g(g+1)/2}^{\mathrm{ST}}]] \to A[[T_1^{\mathrm{ST}}, \dots, T_{g(g+1)/2}^{\mathrm{ST}}]]$. By Lemma 6.16, we have

$$G_{c,i}(\pi_0(\rho(\tilde{x}_n))) = H_{c,i,z_n}(y_n) = f(y_n). \tag{6-13}$$

We prove the stronger result $f = 0$ by contradiction.

Assume that $f \neq 0$. We want to apply Proposition 6.1 to f and the y_n. We check the conditions in Proposition 6.1. First, by the compatibility between the Global and classical Serre–Tate coordinates as in the end of page 1009, we use Lemma 5.4(2) to conclude that y_n are as in (6-1) above Proposition 6.1. Second, the assumption ($\star$) in Proposition 6.1 holds by Assumption 6.10. By the assumption that a_n goes to ∞ as $n \to \infty$ in Lemma 6.11, Lemma 6.15 and the second "=" of (6-13), for n large enough, n satisfies the inequality in (6-2) of Proposition 6.1 (for every D). Then by Proposition 6.1, there exists $\delta_0 \in (0, 1)$ such that $\mathbb{N}(0, \delta_0)$ is finite. For a general $r \in \mathbb{Z}_p^{l-1}$, by [Serban 2018, Lemma 2.7], after an "upper triangular change of variables" (as defined above Lemma 6.11), we may use the same proof for $r = 0$ to conclude that there exists $\delta_r \in (0, 1)$ such that $\mathbb{N}(r, \delta_r)$ is finite. By its compactness, $\mathbb{Z}_p^{l-1}$ is the union of p-adic closed discs centered at r of radius δ_r for finitely many r. Then the infinite set $\mathbb{N}$ is the union of the finite sets $\mathbb{N}(r, \delta_r)$ for these finitely many r. This is a contradiction.

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