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Twisted derived equivalences and isogenies between K3 surfaces in positive characteristic

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We study isogenies between K3 surfaces in positive characteristic. Our main result is a characterization of K3 surfaces isogenous to a given K3 surface X in terms of certain integral sublattices of the second rational ℓ-adic and crystalline cohomology groups of X. This is a positive characteristic analog of a result of Huybrechts (*Comment. Math. Helv.* **94**:3 (2019), 445–458), and extends results of Yang (*Int. Math. Res. Not.* **2022**:6 (2022), 4407–4450). We give applications to the reduction types of K3 surfaces and to the surjectivity of the period morphism. To prove these results we describe a theory of B-fields and Mukai lattices in positive characteristic, which may be of independent interest. We also prove some results on lifting twisted Fourier–Mukai equivalences to characteristic 0, generalizing results of Lieblich and Olsson (*Ann. Sci. Éc. Norm. Supér.* (4) **48**:5 (2015), 1001–1033).

1. Introduction

The purpose of this paper is to study twisted Fourier–Mukai partners of K3 surfaces in positive characteristics and to develop an isogeny theory for these surfaces which is analogous to that of abelian varieties.

Let k be an algebraically closed field and p be a prime number. When $\operatorname{char} k = p$, we simply write W for the ring of Witt vectors $W(k)$. Let $\widehat{\boldsymbol{Z}}^p$ denote the prime-to-p part of $\widehat{\boldsymbol{Z}}$. For a variety Y over k, we set $\mathrm{H}^*(Y) := \mathrm{H}^*_{\text{ét}}(Y, \widehat{\boldsymbol{Z}})$ if $\operatorname{char} k = 0$, and $\mathrm{H}^*(Y) := \mathrm{H}^*_{\text{ét}}(Y, \widehat{\boldsymbol{Z}}^p) \times \mathrm{H}^*_{\text{cris}}(Y/W)$ if $\operatorname{char} k = p$, and write $\mathrm{H}^*(Y)_{\boldsymbol{Q}} := \mathrm{H}^*(Y) \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$.

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Definition 1.1 (cf. [Yang 2022, Definition 1.1]). Let X and X' be K3 surfaces over k. An isogeny $f : X \rightsquigarrow X'$ is a correspondence, i.e., a $\boldsymbol{Q}$-linear combination of algebraic cycles on $X \times X'$, such that the induced action $\mathrm{H}^2(X')_{\boldsymbol{Q}} \to \mathrm{H}^2(X)_{\boldsymbol{Q}}$ is an isomorphism which preserves the Poincaré pairing. Two isogenies are deemed equivalent if they induce the same map $\mathrm{H}^2(X')_{\boldsymbol{Q}} \xrightarrow{\sim} \mathrm{H}^2(X)_{\boldsymbol{Q}}$.

Our main results concern the existence and uniqueness of isogenies with prescribed cohomological action. We begin with the former. A natural source for isogenies between K3 surfaces is provided by twisted Fourier–Mukai equivalences: For a K3 surface X and Brauer class $\alpha \in \mathrm{Br}(X)$, we denote by $D^b(X, \alpha)$ the bounded derived category of α-twisted sheaves. Given another K3 surface X' and Brauer class α', an equivalence $D^b(X, \alpha) \xrightarrow{\sim} D^b(X', \alpha')$ induces, up to some choices, an isogeny $f : X \rightsquigarrow X'$. We call isogenies which arise this way *primitive derived isogenies*, and compositions of such isogenies *derived isogenies*. The precise definitions are given in Section 4. There we also give a motivic reformulation of the above definition, which will be used for the rest of the paper.

To state our theorems, we denote the K3 lattice $\boldsymbol{U}^{\oplus 3} \oplus \boldsymbol{E}_8^{\oplus 2}$ by Λ and recall Ogus's notion of K3 crystals [1979, Definition 3.1]. Here $\boldsymbol{U}$ denotes the standard hyperbolic plane and $\boldsymbol{E}_8$ denotes the unique unimodular even negative definite lattice of rank 8. Our first theorem is an existence result on derived isogenies:

Theorem 1.2. *Assume* $\mathrm{char}\, k = p \geq 5$. *Let X be a* K3 *surface over k. Endow $\Lambda \otimes W$ with a* K3 *crystal structure and denote it by H_p and let H^p denote $\Lambda \otimes \widehat{\boldsymbol{Z}}^p$.*

Let $\iota : H^p \times H_p \hookrightarrow \mathrm{H}^2(X)_{\boldsymbol{Q}}$ be an isometric embedding which respects the Frobenius actions on H_p and $\mathrm{H}^2_{\mathrm{cris}}(X/W)[1/p]$. There exists a derived isogeny $f : X \rightsquigarrow X'$ to another K3 *surface X' such that $f^*(\mathrm{H}^2(X')) = \mathrm{im}(\iota)$ if and only if ι sends the slope < 1 part of H_p isomorphically onto that of $\mathrm{H}^2_{\mathrm{cris}}(X/W)$.*

We refer the reader to Remark 6.10 for the reason to restrict to $p \geq 5$. The above result is inspired by a theorem of Huybrechts [2019, Theorem 0.1], which can be stated as follows in our terminology:

Theorem 1.3 (Huybrechts). *Let X and X' be two* K3 *surfaces over $\boldsymbol{C}$. Every isomorphism of Hodge structures $\mathrm{H}^2(X', \boldsymbol{Q}) \xrightarrow{\sim} \mathrm{H}^2(X, \boldsymbol{Q})$ which preserves the Poincaré pairings is induced by a derived isogeny $f : X \rightsquigarrow X'$.*

This refines an earlier theorem of Buskin [2019, Theorem 1.1], which affirms a conjecture of Shafarevich. Using the global Torelli theorem and surjectivity of the period map, one checks that Huybrechts' theorem is equivalent to an existence theorem for isogenies: for every K3 surface X over $\boldsymbol{C}$ and every isometric embedding $\iota : \Lambda \hookrightarrow \mathrm{H}^2(X, \boldsymbol{Q})$, there exists another K3 surface X' over $\boldsymbol{C}$ and a derived isogeny $f : X \rightsquigarrow X'$ such that $f^*(\mathrm{H}^2(X', \boldsymbol{Z})) = \mathrm{im}(\iota)$ (see Section 6D). Note that this statement does not involve Hodge structures. Our Theorem 1.2 is a positive characteristic analog for this version of Huybrechts' theorem.

Huybrechts' refinement shows in particular that every isogeny between K3 surfaces over $\boldsymbol{C}$ is equivalent to a derived isogeny. In contrast, the "only if" part of Theorem 1.2 implies that the cohomological actions of derived isogenies in characteristic p obey a certain nontrivial constraint at p. In particular, not every isogeny is equivalent to a derived isogeny. Given this, it is of interest to characterize also the possible cohomological actions of all (not necessarily derived) isogenies. The following result shows that, under some technical assumptions, the "if" part of Theorem 1.2 can be removed for $k = \bar{\boldsymbol{F}}_p$ if one is willing to consider all isogenies:

Theorem 1.4. *Let X, H_p, H^p and ι be as in Theorem 1.2. If $k = \bar{\boldsymbol{F}}_p$ and*

(a) $\mathrm{Pic}(X)$ *has rank ≥ 12 or contains a standard hyperbolic plane, or*

(b) $\mathrm{Pic}(X)$ *contains an ample line bundle L of degree $L^2 < p-4$,*

then there exists another K3 *surface X' over k and an isogeny $f : X \rightsquigarrow X'$ such that $f^*(\mathrm{H}^2(X')) = \mathrm{im}(\iota)$.*

This is a strengthening of [Yang 2022, Theorem 1.4]. We mention that a byproduct in the course of proving the above is a generalization (Theorem 6.18) of Taelman's characterization [2020, Theorem C] of the canonical liftings of ordinary K3 surfaces. Nygaard and Ogus [1985] constructed, for every nonsupersingular K3 surface X, a "section" to the natural morphism $\mathrm{Def}(X) \to \mathrm{Def}(\widehat{\mathrm{Br}}_X)$ from the deformation space of X to that of its formal Brauer group, such that a lifting of $\widehat{\mathrm{Br}}_X$ induces a lifting of X. We call liftings of X which arise this way "Nygaard–Ogus liftings". When X is ordinary, a Nygaard–Ogus lifting is the same as a canonical lifting. Theorem 6.18 gives an integral p-adic Hodge-theoretic characterization of Nygaard–Ogus liftings. See Section 6E for details.

We now describe our uniqueness results. We recall some terminology from [Yang 2022, §6]: an isogeny $f : X \rightsquigarrow X'$ between K3 surfaces is said to be *polarizable* if the induced map $\mathrm{Pic}(X')_{\boldsymbol{Q}} \xrightarrow{\sim} \mathrm{Pic}(X)_{\boldsymbol{Q}}$ sends an ample class to another ample class, and $\boldsymbol{Z}$*-integral* if the induced isomorphism $\mathrm{H}^2(X')_{\boldsymbol{Q}} \xrightarrow{\sim} \mathrm{H}^2(X)_{\boldsymbol{Q}}$ restricts to an isomorphism $\mathrm{H}^2(X') \xrightarrow{\sim} \mathrm{H}^2(X)$. We prove the following Torelli theorem for derived isogenies:

Theorem 1.5. *Assume* $\mathrm{char}\, k \geq 5$. *Let X and X' be* K3 *surfaces over k. A derived isogeny $f : X \rightsquigarrow X'$ is equivalent to the graph of an isomorphism $X' \xrightarrow{\sim} X$, if and only if f is polarizable and $\boldsymbol{Z}$-integral.*

Finally, we remark that Li and Zou [2021] considered derived isogenies and Torelli type theorems for abelian surfaces.

1A. ***Applications to good reductions of* K3 *surfaces.*** We apply our results to study the good reduction conjecture for K3 surfaces:

Conjecture 1.6. Let k be an algebraically closed field of characteristic $p > 0$ and let F be a finite extension of $W[1/p]$. Let X_F be a K3 surface over F such that $\mathrm{H}^2_{\text{ét}}(X_{\bar{F}}, \boldsymbol{Q}_\ell)$ is unramified for some prime $\ell \neq p$. Then, X_F has potentially good reduction.

This conjecture is a K3 analog of the Néron–Ogg–Shafarevich criterion for abelian varieties. It admits many variants (e.g., ones that concern semistable reductions) and is verified in cases when X_F admits a polarization of low degree (see [Matsumoto 2015] and [Liedtke and Matsumoto 2018]). We prove the following:

Theorem 1.7. *Let X_F be as in Conjecture 1.6. Assume $p > 2$ and X_F admits a line bundle of degree prime to p. Then the Gal_F-representation $\mathrm{H}^2_{\text{ét}}(X_{\bar{F}}, \boldsymbol{Q}_p)$ is potentially crystalline. If $p \geq 5$ (resp. $p > 2$) and $\mathrm{H}^2_{\text{ét}}(X_{\bar{F}}, \boldsymbol{Q}_p)$ has potentially good ordinary or (resp. supersingular) reduction, then so does X_F.*

Roughly speaking, the theorem is saying that if the cohomology of X_F predicts that X_F should have potential ordinary or supersingular reduction, then it does. We derive this as a consequence of a more general result (Theorem 8.10), which essentially reduces Conjecture 1.6 to the Hecke orbit conjecture (see Conjecture 8.2), which is a purely Shimura–theoretic statement. In particular, we prove the following.

Theorem 1.8. *Let X_F be an in Conjecture 1.6. Suppose that $p > 2$ and that X_F admits a line bundle of degree prime to p. Assume the Hecke orbit conjecture* (Conjecture 8.2) *holds for all i. Then, X_F has potentially good reduction.*

Our unconditional Theorem 1.7, in the ordinary case, is then a consequence of recent work of Maulik, Shankar, and Tang [Maulik et al. 2022, Theorem 1.4] proving the Hecke orbit conjecture in certain special cases. The supersingular case will be treated by a slightly different argument. Moreover, it seems very likely that a slight generalization of the conjecture can remove the condition on the existence of a prime-to-p line bundle as well, and hence completely affirms Conjecture 1.6.

We remark that nowhere in the proofs of the above results do we directly analyze a degeneration of K3 surfaces, unlike in [Matsumoto 2015] and [Liedtke and Matsumoto 2018]. In particular, we avoid the use of any techniques from the minimal model program. As far as the authors are aware, our method of proving good reduction results by marrying moduli theory of sheaves with density arguments is new in the literature.

After the paper was accepted for publication, Marco D'Addezio and Pol van Hoften proved the Hecke orbit conjecture for Shimura varieties of Hodge type and in particular proved Conjecture 8.2 under a very minor assumption on p [D'Addezio and van Hoften 2022, Section 7.5].

1B. ***Ideas of proof.*** (1) The "only if" part of Theorem 1.2 follows from the general theory of twisted derived equivalences in positive characteristics. The idea for the "if" part is to construct the desired X' together with the isogeny $f : X \rightsquigarrow X'$ by iteratively taking moduli spaces of twisted sheaves on X. This approach is inspired by that of [Huybrechts 2019, Theorem 1.1]. A key technical tool is the theory of B-fields in ℓ-adic and crystalline cohomology, described in Section 2. This allows us to relate classes in $\mathrm{H}^2(X)_{\mathbb{Q}}$ to the Brauer group, and provides a replacement for the Hodge-theoretic B-fields in Huybrechts' proof, although there are some additional complications at p. There are some further technical difficulties caused by the fact that in positive characteristic the cohomology $\mathrm{H}^2(X)_{\mathbb{Q}}$ can only take on adelic coefficients (i.e., $\mathbb{A}_f^p \times W[1/p]$) instead of $\mathbb{Q}$-coefficients. For instance, the Mukai vector which one must specify in order to form a moduli of sheaves is not an adelic object. That is, unlike Brauer classes, one cannot specify a Mukai vector by prescribing its local factors in $\mathrm{H}^2(X)_{\mathbb{Q}}$. We solve these problems by using local–global type results on quadratic forms (e.g., the strong approximation theorem), and the theory of quadratic forms over local rings.

(2) Theorem 1.4 is obtained by the realizing X' as the reduction of a suitable K3 surface in characteristic zero. This strategy is a simultaneous simplification and strengthening of that of [Yang 2022], with the additional input of Theorem 1.2. The characterization of Nygaard–Ogus liftings (Theorem 6.18) is obtained by applying recent advances on integral p-adic Hodge theory from [Bhatt et al. 2018] and

[Cais and Liu 2019] to study deformations of K3 crystals. These techniques for handling crystalline cohomology were unnecessary in Taelman's case [2020], as the deformation of the formal Brauer group of an ordinary K3 is rigid, which is not true for a general finite-height K3. We remark that here the restriction $p \geq 5$ is mainly due to our usage of the deformation theory of K3 crystals.

(3) Theorem 1.5 is a twisted generalization of the derived Torelli theorem of Lieblich and Olsson [2015, Theorem 6.1]. Just as in loc. cit., we prove this result by using a lifting argument to reduce to the global Torelli theorem over $\boldsymbol{C}$. The main difficulty which arises in our generalization is that instead of considering isogenies which arise directly from a (twisted or untwisted) derived equivalence, we are allowing any finite compositions of such. The derived equivalences involved may not be simultaneously liftable to characteristic zero. To overcome this difficulty, we combine the lifting results on derived equivalences with the Kuga–Satake method. This helps us reduce composing isogenies of K3's to composing isogenies of abelian varieties, which is much better understood. There is a technical problem which arises from the usage of Kuga–Satake. Namely, we need to put the relevant K3 surfaces into the same moduli space. However, the K3 surfaces themselves may not have a quasipolarization of a common degree. To overcome this problem, we pass from K3 surfaces to their Hilbert squares, which are treated in [Yang 2023]. The restriction to $p \geq 5$ is imposed because in loc. cit. the second author only treated $\mathrm{K3}^{[n]}$-type varieties when $p > n+1$ for certain technical reasons.

(4) For Theorem 1.7, we first show that the derived prime-to-p isogeny classes of K3's match up with the notion of prime-to-p Hecke orbit on the period domains of Kuga–Satake morphisms, which are some orthogonal Shimura varieties. It follows from some intermediate steps in the proof of Theorem 1.2 that the property of satisfying Conjecture 1.6 is invariant in a prime-to-p derived isogeny class. On the other hand, any X_F which satisfies the hypothesis of Theorem 1.7 produces a mod p point $x(X_F)$ on the period domain, and the set $\mathscr{L}_{\mathrm{bad}} := \{x(X_F) : X_F \text{ violates Conjecture 1.6}\}$ is closed.

If we combine the above observations with the Hecke orbit (HO) conjecture (see Conjecture 8.2), we see that if $\mathscr{L}_{\mathrm{bad}}$ intersects any of the height stratum of the period domains, then it must contain the entirety of that stratum, which is false by a deformation argument. Hence the HO conjecture forces $\mathscr{L}_{\mathrm{bad}}$ to be empty. The HO conjecture is now known for the ordinary locus by the recent work of Maulik, Shankar, and Tang [Maulik et al. 2022] and we will verify it in the superspecial locus for cases relevant to us (Theorem 8.6). This gives Theorem 1.7.

1C. *Plan of paper.* In Section 2, we develop the formalism of B-fields and twisted Mukai lattices in positive characteristic. Section 3 concerns the construction of twisted Chern characters, the twisted Néron–Severi lattice, and the action of a twisted derived equivalence on cohomology. In Section 4 we discuss rational Chow motives and isogenies. In Section 5 we prove some lifting results for twisted derived isogenies. In Section 6, we first prove Theorem 1.2. We then revisit Nygaard–Ogus theory for the point of view of integral p-adic Hodge theory and prove Theorem 1.4. In Section 7, we review the basics of Hilbert squares and the Kuga–Satake period morphism, and then prove Theorem 1.5. Finally, in Section 8, we explain the relationship between our isogeny theory and Hecke orbits, and prove Theorem 1.7.

1D. ***Notation.***

- Let p denote a prime. The letter k denotes a perfect base field of characteristic either 0 or p and ℓ denotes a prime not equal to $\operatorname{char} k$. When $\operatorname{char} k = p$, we write W for $W(k)$ and K for $W[1/p]$.
- If Z is a scheme, we write $\mathrm{H}^i(Z, \mu_n)$ for the flat (fppf) cohomology of the sheaf of n-th roots of unity on Z. If n is coprime to the characteristics of all residue fields of Z, this is equal to the étale cohomology of μ_n.
- We normalize our Chern characters so that the mod m Chern character of a line bundle L is equal to the image of the class of L under the boundary map $\mathrm{H}^1(Z, \boldsymbol{G}_m) \to \mathrm{H}^2(Z, \mu_m)$ of the Kummer sequence.
- Suppose k is a perfect field of characteristic p and S is a k-scheme. If $f : X \to S$ is a scheme, we denote by $\mathrm{H}^j_{\mathrm{cris}}(X)$ the sheaf on $\boldsymbol{Cris}(S/W)$ given by $R^j f_{\mathrm{cris}*}\mathscr{O}_{X/W}$ when S is understood.
- For any integral domain R, and R-modules M and N, an isomorphism $f : M_{\boldsymbol{Q}} \xrightarrow{\sim} N_{\boldsymbol{Q}}$ is said to be R-integral if $f(M) = N$.
- In this paper we only make use of singular, de Rham, étale, flat, and crystalline cohomology. We may omit the subscripts cris, fl, or dR when the choice of the relevant Grothendieck topology is clear from the coefficients.
- For a smooth proper variety Y over k, we let $\mathrm{H}^j(Y)$ denote either $\mathrm{H}^j_{\text{ét}}(Y, \widehat{\boldsymbol{Z}})$ if $\operatorname{char} k = 0$ or $\mathrm{H}^j_{\text{ét}}(Y, \widehat{\boldsymbol{Z}}^p) \times \mathrm{H}^j_{\mathrm{cris}}(Y/W)$ if $\operatorname{char} k = p$.
- Let R be a commutative ring. A quadratic lattice M over R is a free R-module of finite rank equipped with a bilinear symmetric pairing $M \times M \to R$. The pairing is said to be nondegenerate (resp. unimodular or perfect) if the induced map $M \to M^\vee$ is an injection (resp. an isomorphism).

2. B-fields and the twisted Mukai lattice in positive characteristic

Let X be a K3 surface over the complex numbers. Associated to X is the *Mukai lattice* $\widetilde{\mathrm{H}}(X, \boldsymbol{Z})$, which is the direct sum of the singular cohomology groups of X equipped with a certain pairing and Hodge structure. Consider a class $\alpha \in \mathrm{Br}(X)$. Huybrechts and Stellari [2005, Remark 1.3] generalized Mukai's construction to the twisted K3 surface (X, α) by defining the *twisted Mukai lattice* $\widetilde{\mathrm{H}}(X, B, \boldsymbol{Z})$. This construction modifies the Hodge structure on the Mukai lattice in a certain way using an auxiliary choice of a *B-field lift* of α, which is a class $B \in \mathrm{H}^2(X, \boldsymbol{Q})$ whose image in $\mathrm{Br}(X)$ under the exponential map is equal to α.

Suppose now that X is a K3 surface defined over an algebraically closed field of characteristic $p > 0$. After [Lieblich and Olsson 2015], we may consider the ℓ-adic and crystalline realizations of the Mukai motive of X. These are respectively a $\boldsymbol{Z}_l$-lattice $\widetilde{\mathrm{H}}(X, \boldsymbol{Z}_l)$ and a W-lattice $\widetilde{\mathrm{H}}(X/W)$, both of rank 24. In the crystalline setting, $\widetilde{\mathrm{H}}(X/W)$ is equipped with a Frobenius action, which makes $\widetilde{\mathrm{H}}(X/W)$ into a K3 crystal in the sense of Ogus [1979, Definition 3.1]. That this construction makes sense integrally is first observed in [Bragg and Lieblich 2018].

Consider a Brauer class $\alpha \in \mathrm{Br}(X)$. We wish to have an analog of Huybrechts and Stellari's construction of the twisted Mukai lattice in both the ℓ-adic and crystalline settings. The main task is to find the appropriate analog of a B-field lift of a Brauer class in ℓ-adic or crystalline cohomology. The ℓ-adic case is considered in [Lieblich et al. 2014] (we remark that the authors also deal with some additional complications coming from working over a field that is not algebraically closed, which we ignore here). The crystalline case is considered in [Bragg 2021, §3] and [Bragg and Lieblich 2018, §3.4], with the restriction that the Brauer class α is killed by p (rather than a power of p).

In this section we make two contributions. First, we complete the crystalline realization by defining crystalline B-field lifts of classes killed by an arbitrary power of p. We then treat the mixed case, considering all primes simultaneously, and define mixed B-field lifts of Brauer classes whose order is divisible by more than one prime. To assist the reader in connecting these constructions in the Hodge, ℓ-adic, and crystalline settings, we have included a brief summary of the Hodge and ℓ-adic realizations. We have tried to present a perspective which emphasizes the unifying features present in the different settings.

2A. ***Hodge realization.*** Let X be a K3 surface over the complex numbers. We have the exponential exact sequence

$$0 \to \boldsymbol{Z} \to \mathscr{O}_X \xrightarrow{\exp} \mathscr{O}_X^\times \to 1.$$

Consider the induced map $\mathrm{H}^2(X, \mathscr{O}_X) \xrightarrow{\exp} \mathrm{H}^2(X, \mathscr{O}_X^\times)$, which, because $\mathrm{H}^3(X, \boldsymbol{Z}) = 0$, is a surjection. Given a class $v \in \mathrm{H}^2(X, \mathscr{O}_X)$, we note that $\exp(v)$ is contained in the torsion subgroup $\mathrm{H}^2(X, \mathscr{O}_X^\times)_{\mathrm{tors}} = \mathrm{H}^2(X, \boldsymbol{G}_m) = \mathrm{Br}(X)$ if and only if v is contained in the subgroup $\mathrm{H}^2(X, \boldsymbol{Q}) \subset \mathrm{H}^2(X, \mathscr{O}_X)$. Thus, this map restricts to a surjection

$$\exp : \mathrm{H}^2(X, \boldsymbol{Q}) \to \mathrm{Br}(X), \tag{1}$$

which we denote by $B \mapsto \alpha_B = \exp(B)$. According to [Huybrechts and Stellari 2005], a *B-field lift* of a class $\alpha \in \mathrm{Br}(X)$ is a class $B \in \mathrm{H}^2(X, \boldsymbol{Q})$ such that $\alpha_B = \alpha$.

The relationship between B-fields and the Brauer group is expressed in the diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{H}^2(X, \boldsymbol{Z}) & \longrightarrow & \mathrm{H}^2(X, \boldsymbol{Z}) + \mathrm{Pic}(X) \otimes \boldsymbol{Q} & \longrightarrow & \mathrm{Pic}(X) \otimes (\boldsymbol{Q}/\boldsymbol{Z}) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{H}^2(X, \boldsymbol{Z}) & \longrightarrow & \mathrm{H}^2(X, \boldsymbol{Q}) & \longrightarrow & \mathrm{H}^2(X, \boldsymbol{Z}) \otimes (\boldsymbol{Q}/\boldsymbol{Z}) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \dfrac{\mathrm{H}^2(X, \boldsymbol{Q})}{\mathrm{H}^2(X, \boldsymbol{Z}) + \mathrm{Pic}(X) \otimes \boldsymbol{Q}} & \xrightarrow{\sim} & \mathrm{Br}(X) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array} \tag{2}$$

with exact rows and columns. In particular, we see that there are two sources of ambiguity in choosing a B-field lift of a Brauer class, namely, integral classes in $\mathrm{H}^2(X, \boldsymbol{Z})$ and rational classes in $\mathrm{H}^{1,1}(X, \boldsymbol{Q}) = \mathrm{Pic}(X) \otimes \boldsymbol{Q} \subset \mathrm{H}^2(X, \boldsymbol{Q})$.

2B. *ℓ-adic realization.* Let k be an algebraically closed field of arbitrary characteristic. Fix a prime number ℓ, not equal to the characteristic of k. We review the ℓ-adic B-fields and the ℓ-adic realization of the twisted Mukai motive introduced in [Lieblich et al. 2014].

Let X be a K3 surface over k. By duality in étale cohomology, we have $\mathrm{H}^3(X, \mu_{\ell^n}) = 0$ for all $n \geq 1$. It follows that the natural map

$$\mathrm{H}^2(X, \boldsymbol{Z}_\ell(1)) \to \mathrm{H}^2(X, \mu_{\ell^n}) \tag{3}$$

is surjective, and hence we have an identification

$$\mathrm{H}^2(X, \boldsymbol{Z}_\ell(1)) \otimes \boldsymbol{Z}/\ell^n\boldsymbol{Z} \cong \mathrm{H}^2(X, \mu_{\ell^n}).$$

We consider the composition

$$\mathrm{H}^2(X, \boldsymbol{Z}_\ell(1)) \twoheadrightarrow \mathrm{H}^2(X, \mu_{\ell^n}) \twoheadrightarrow \mathrm{Br}(X)[\ell^n], \tag{4}$$

where the second map is induced by the inclusion $\mu_{\ell^n} \subset \boldsymbol{G}_m$.

Definition 2.1. Let $\alpha \in \mathrm{Br}(X)$ be a Brauer class which is killed by a power of ℓ. An *ℓ-adic B-field lift* of α is an element

$$B \in \mathrm{H}^2(X, \boldsymbol{Q}_\ell(1)) \stackrel{\text{def}}{=} \mathrm{H}^2(X, \boldsymbol{Z}_\ell(1)) \otimes_{\mathbf{Z}_\ell} \boldsymbol{Q}_\ell$$

such that if we write $B = a/\ell^n$ for some $a \in \mathrm{H}^2(X, \boldsymbol{Z}_\ell(1))$, then a maps to α under the composition (4).

We give the following alternative description. Define $\mu_{\ell^\infty} = \bigcup_n \mu_{\ell^n} \subset \boldsymbol{G}_m$. The Picard group of X is torsion-free, which implies the vanishing $\mathrm{H}^1(X, \mu_\ell) = 0$. It follows that the inclusions $\mu_{\ell^n} \subset \mu_{\ell^{n+1}}$ induce injections on H^2, and we have a natural identification $\mathrm{H}^2(X, \mu_{\ell^\infty}) = \bigcup_n \mathrm{H}^2(X, \mu_{\ell^n})$. Moreover, for every n we have a commutative diagram

$$\begin{array}{ccc} \mathrm{H}^2(X, \boldsymbol{Z}_\ell(1)) & \xrightarrow{\cdot \ell^m} & \mathrm{H}^2(X, \boldsymbol{Z}_\ell(1)) \\ \downarrow & & \downarrow \\ \mathrm{H}^2(X, \mu_{\ell^n}) & \hookrightarrow & \mathrm{H}^2(X, \mu_{\ell^{n+m}}) \end{array} \tag{5}$$

Taking the direct limit of the maps (3), we get a map

$$\mathrm{H}^2(X, \boldsymbol{Q}_\ell(1)) \to \mathrm{H}^2(X, \mu_{\ell^\infty}). \tag{6}$$

This map may be explicitly described as follows: given $B \in \mathrm{H}^2(X, \boldsymbol{Q}_\ell(1))$, choose $n \geq 0$ such that $\ell^n B \in \mathrm{H}^2(X, \boldsymbol{Z}_\ell(1))$, and map B to the image of $\ell^n B$ under the left map of (4). Note that by the commutativity of (5), this association is well defined, independent of our choice of n. Composing (6) with the natural map $\mathrm{H}^2(X, \mu_{\ell^\infty}) \to \mathrm{Br}(X)$, we get a map

$$\mathrm{H}^2(X, \boldsymbol{Q}_\ell(1)) \to \mathrm{Br}(X). \tag{7}$$

This is the ℓ-adic analog of the exponential map (1). The image of this map is exactly the subgroup $\mathrm{Br}(X)[\ell^\infty] \subset \mathrm{Br}(X)$ consisting of classes killed by some power of ℓ. Furthermore, an ℓ-adic B-field lift of a class $\alpha \in \mathrm{Br}(X)[\ell^\infty]$ (in the sense of Definition 2.1) is exactly a preimage of α under (7). We denote (7) by $B \mapsto \alpha_B$.

The relationship between ℓ-adic B-fields and the Brauer group is expressed by the diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{H}^2(X, \boldsymbol{Z}_\ell(1)) & \longrightarrow & \mathrm{H}^2(X, \boldsymbol{Z}_\ell(1)) + \mathrm{Pic}(X) \otimes \boldsymbol{Q}_\ell & \longrightarrow & \mathrm{Pic}(X) \otimes (\boldsymbol{Q}_\ell/\boldsymbol{Z}_\ell) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{H}^2(X, \boldsymbol{Z}_\ell(1)) & \longrightarrow & \mathrm{H}^2(X, \boldsymbol{Q}_\ell(1)) & \longrightarrow & \mathrm{H}^2(X, \mu_{\ell^\infty}) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \dfrac{\mathrm{H}^2(X, \boldsymbol{Q}_\ell(1))}{\mathrm{H}^2(X, \boldsymbol{Z}_\ell(1)) + \mathrm{Pic}(X) \otimes \boldsymbol{Q}_\ell} & \xrightarrow{\sim} & \mathrm{Br}(X)[\ell^\infty] & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array} \tag{8}$$

with exact rows and columns, where the right-hand column is given by taking the direct limit of the exact sequence induced by the Kummer sequence.

In particular, we have an isomorphism

$$\mathrm{Br}(X)[\ell^\infty] \cong (\boldsymbol{Q}_\ell/\boldsymbol{Z}_\ell)^{\oplus 22-\rho}, \tag{9}$$

where ρ is the Picard rank of X.

2C. ***The twisted ℓ-adic Mukai lattice.*** The *ℓ-adic Mukai lattice* associated to X [Lieblich et al. 2014, Definition 3.3.1] is

$$\widetilde{\mathrm{H}}(X, \boldsymbol{Z}_\ell) = \mathrm{H}^0(X, \boldsymbol{Z}_\ell)(-1) \oplus \mathrm{H}^2(X, \boldsymbol{Z}_\ell) \oplus \mathrm{H}^4(X, \boldsymbol{Z}_\ell)(1),$$

which we equip with the Mukai pairing. Given a class $B \in \mathrm{H}^2(X, \boldsymbol{Q}_\ell)$, we define the associated *twisted ℓ-adic Mukai lattice* to be the submodule

$$\widetilde{\mathrm{H}}(X, \boldsymbol{Z}_\ell, B) = \exp(B)\, \widetilde{\mathrm{H}}(X, \boldsymbol{Z}_\ell) \subset \widetilde{\mathrm{H}}(X, \boldsymbol{Q}_\ell).$$

Here, $\exp(B)$ denotes the isometry $\widetilde{\mathrm{H}}(X, \boldsymbol{Q}_\ell) \to \widetilde{\mathrm{H}}(X, \boldsymbol{Q}_\ell)$ given by

$$(a, b, c) \mapsto \left(a, b + aB, c + b.B + \tfrac{1}{2}aB^2\right). \tag{10}$$

2D. ***Crystalline realization.*** Let k be an algebraically closed field of characteristic $p > 0$ and let X be a K3 surface over k. We will define crystalline B-fields associated to Brauer classes on X whose order is a power of p. There are some new phenomena which present themselves in the crystalline setting that are not present in the Hodge and ℓ-adic theories. In particular, there is a nontrivial interaction between crystalline B-fields and the Frobenius operator on the crystalline cohomology. A related feature is that not every class in rational crystalline cohomology is a crystalline B-field. We give a characterization of which classes are B-fields using only the F–crystal structure on crystalline cohomology in Proposition 2.7. We then construct the crystalline version of the twisted Mukai lattice, and show that this object has a natural structure of a K3 crystal in the sense of Ogus [1979, Definition 3.1]. We conclude with some calculations with the twisted Mukai crystals. In the special case when the Brauer class is killed by p, the results of this section have appeared in [Bragg 2021; Bragg and Lieblich 2018].

Set $W_n = W/p^n W$, so in particular $W_1 = k$. Let $\sigma : k \to k$ be the Frobenius $\lambda \mapsto \lambda^p$. We denote the induced map $\sigma : W \to W$ (abusively) by the same symbol.

2E. *Crystalline B-fields.* We begin by relating the flat cohomology of μ_{p^n} to certain étale cohomology groups. Consider the Kummer sequence

$$1 \to \mu_{p^n} \to \boldsymbol{G}_m \xrightarrow{x \mapsto x^{p^n}} \boldsymbol{G}_m \to 1,$$

which is exact in the fppf topology. Let $\varepsilon : X_{\text{fl}} \to X_{\text{ét}}$ be the natural map from the big fppf site of X to the small étale site of X. By a theorem of Grothendieck, the cohomology of the complex $R\varepsilon_* \boldsymbol{G}_m$ vanishes in all positive degrees. Applying ε_* to the Kummer sequence, we obtain an exact sequence

$$1 \to \boldsymbol{G}_m \xrightarrow{x \mapsto x^{p^n}} \boldsymbol{G}_m \to R^1 \varepsilon_* \mu_{p^n} \to 1$$

of sheaves on the small étale site of X (because X is reduced, the restriction of μ_{p^n} to the small étale site of X is trivial). It follows that

$$R^1 \varepsilon_* \mu_{p^n} = \boldsymbol{G}_m / \boldsymbol{G}_m^{\times p^n},$$

where the quotient is taken in the étale topology. We therefore obtain isomorphisms

$$\mathrm{H}^i(X_{\text{fl}}, \mu_{p^n}) \xrightarrow{\sim} \mathrm{H}^{i-1}(X_{\text{ét}}, \boldsymbol{G}_m / \boldsymbol{G}_m^{\times p^n}). \tag{11}$$

We next relate the étale cohomology groups on the right to crystalline cohomology. We consider the map of étale sheaves

$$d \log : \boldsymbol{G}_m \to \mathrm{W}_n \, \Omega^1_X$$

given by $x \mapsto d\underline{x}/\underline{x}$, where $\underline{x} = (x, 0, 0, \dots)$ is the multiplicative representative of x in $\mathrm{W}_n \, \mathcal{O}_X$. By [Illusie 1971, Proposition I.3.23.2, p. 580] the kernel of $d \log$ is equal to the subsheaf $\boldsymbol{G}_m^{\times p^n} \subset \boldsymbol{G}_m$, so there is an induced injection

$$d \log : \boldsymbol{G}_m / \boldsymbol{G}_m^{\times p^n} \hookrightarrow \mathrm{W}_n \, \Omega^1_X. \tag{12}$$

As the image of $d \log$ is contained in the kernel of d, we have a commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & \boldsymbol{G}_m / \boldsymbol{G}_m^{\times p^n} & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle d \log} & & \downarrow \\ \mathrm{W}_n \, \mathcal{O}_X & \xrightarrow{d} & \mathrm{W}_n \, \Omega^1_X & \xrightarrow{d} & \mathrm{W}_n \, \Omega^2_X \end{array}$$

which we interpret as a map of complexes

$$d\log : \boldsymbol{G}_m/\boldsymbol{G}_m^{\times p^n}[-1] \hookrightarrow \mathrm{W}_n\,\Omega_X^\bullet. \tag{13}$$

An important fact is that the de Rham–Witt complex computes crystalline cohomology, in the sense that there is a canonical isomorphism

$$\mathrm{H}^*(X, \mathrm{W}_n\,\Omega_X^\bullet) \xrightarrow{\sim} \mathrm{H}^*(X/W_n) \tag{14}$$

in each degree [Illusie 1971, Théoréme II.1.4, p. 606]. Taking cohomology of (13) and using the identifications (11) and (14), we find a map

$$d\log : \mathrm{H}^2(X, \mu_{p^n}) \to \mathrm{H}^2(X/W_n). \tag{15}$$

Lemma 2.2. *For each $n \geq 1$, the map (15) is injective.*

Proof. We induct on n. By [Illusie 1971, Corollaire 0.2.1.18, p. 517], there is a short exact sequence

$$1 \to \boldsymbol{G}_m/\boldsymbol{G}_m^{\times p} \xrightarrow{d\log} Z\Omega_X^1 \xrightarrow{W-C} \Omega_{X'}^1 \to 0$$

of étale sheaves, where X' denotes the Frobenius twist of X over k. In particular, from the vanishing of $\mathrm{H}^0(X, \Omega_X^1)$ and the injectivity of $\mathrm{H}^1(X, Z\Omega_X^1) \to \mathrm{H}^2_{\mathrm{dR}}(X/k) = \mathrm{H}^2(X/W_1)$ (a consequence of the degeneration of the Hodge–de Rham spectral sequence) we obtain injectivity of (15) for $n = 1$.

We recall that the crystalline cohomology groups $\mathrm{H}^*(X/W)$ of a K3 surface are torsion-free. This implies in particular that the maps

$$\mathrm{H}^2(X/W) \otimes_{\mathbf{Z}} \mathbf{Z}/p^n\mathbf{Z} \to \mathrm{H}^2(X/W_n)$$

are isomorphisms. Hence, multiplication by p^n on $\mathrm{H}^2(X/W)$ induces a short exact sequence

$$0 \to \mathrm{H}^2(X/k) \xrightarrow{\cdot p^n} \mathrm{H}^2(X/W_{n+1}) \to \mathrm{H}^2(X/W_n) \to 0.$$

We also have a short exact sequence

$$1 \to \mu_p \to \mu_{p^{n+1}} \xrightarrow{\cdot p} \mu_{p^n} \to 1 \tag{16}$$

of fppf groups. We claim that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{H}^2(X, \mu_p) & \longrightarrow & \mathrm{H}^2(X, \mu_{p^{n+1}}) & \xrightarrow{\cdot p} & \mathrm{H}^2(X, \mu_{p^n}) & & \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & \mathrm{H}^2(X/k) & \xrightarrow{\cdot p^n} & \mathrm{H}^2(X/W_{n+1}) & \longrightarrow & \mathrm{H}^2(X/W_n) & \longrightarrow & 0 \end{array} \tag{17}$$

commutes and has exact rows, where the top horizontal row is given by the second cohomology of (16), and the vertical arrows are (15). The exactness of the top row follows from the vanishing of $\mathrm{H}^1(X, \mu_{p^n})$

(we remark that the top right horizontal arrow is surjective if and only if X has finite height). To see the commutativity, note that applying $R^1\varepsilon_*$ to (16) results in the short exact sequence

$$1 \to \boldsymbol{G}_m/\boldsymbol{G}_m^{\times p} \xrightarrow{\cdot p^n} \boldsymbol{G}_m/\boldsymbol{G}_m^{\times p^{n+1}} \to \boldsymbol{G}_m/\boldsymbol{G}_m^{\times p^n} \to 1$$

of étale sheaves. Using diagram (17), the result follows immediately by induction. □

We arrive at a diagram

$$\begin{array}{ccccc} \mathrm{H}^2(X/W) & \xrightarrow{\pi_n} & \mathrm{H}^2(X/W_n) & & \\ & & \cup & & \\ & & \mathrm{H}^2(X, \mu_{p^n}) & \longrightarrow & \mathrm{Br}(X)[p^n] \end{array} \tag{18}$$

where π_n denotes reduction modulo p^n. This is the crystalline analog of (4).

Definition 2.3. Let $\alpha \in \mathrm{Br}(X)$ be a Brauer class which is killed by a power of p. A *crystalline B-field lift of* α is an element

$$B \in \mathrm{H}^2(X/K) \stackrel{\text{def}}{=} \mathrm{H}^2(X/W) \otimes_W K$$

such that if we write $B = a/p^n$ for some $a \in \mathrm{H}^2(X/W)$, then $\pi_n(a)$ is equal to $d\log(\alpha')$ for some $\alpha' \in \mathrm{H}^2(X, \mu_{p^n})$ whose image in $\mathrm{Br}(X)$ is equal to α.

From the surjectivity of the horizontal maps in (18), we see that any p-power torsion Brauer class admits a crystalline B-field lift. However, in contrast to the Hodge and ℓ-adic cases, not every element of $\mathrm{H}^2(X/K)$ is a crystalline B-field lift of a Brauer class, because $\mathrm{H}^2(X, \mu_{p^n})$ is only a subgroup of $\mathrm{H}^2(X/W) \otimes \boldsymbol{Z}/p^n\boldsymbol{Z}$.

Definition 2.4. A class $B \in \mathrm{H}^2(X/K)$ is a *crystalline B-field* if it is a B-field lift of some Brauer class. Let $\mathscr{B}(X) \subset \mathrm{H}^2(X/K)$ denote the subgroup of crystalline B-fields. Let $\mathscr{B}_n(X) \subset \mathscr{B}(X)$ denote the subgroup of crystalline B-fields B such that $p^n B \in \mathrm{H}^2(X/W)$.

We take the direct limit of the maps $\mathscr{B}_n(X) \to \mathrm{H}^2(X, \mu_{p^n})$ to obtain a map

$$\mathscr{B}(X) \twoheadrightarrow \mathrm{H}^2(X, \mu_{p^\infty}), \tag{19}$$

which may be explicitly described exactly as in the étale case (6): given a class $B \in \mathscr{B}(X)$, we choose $n \geq 0$ such that $p^n B \in \mathrm{H}^2(X/W)$, and then reduce modulo p^n. We compose (19) with the map to the Brauer group to obtain a map

$$\mathscr{B}(X) \to \mathrm{Br}(X), \tag{20}$$

which we denote by $B \mapsto \alpha_B$. This is the crystalline analog of the exponential map (1). As in the ℓ-adic case, the image of this map is $\mathrm{Br}(X)[p^\infty] \subset \mathrm{Br}(X)$, and a crystalline B-field lift of a class $\alpha \in \mathrm{Br}(X)[p^\infty]$

is exactly a preimage of α under (20). We have a diagram

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{H}^2(X/W) & \longrightarrow & \mathrm{H}^2(X/W)+\mathrm{Pic}(X)\otimes \boldsymbol{Q}_p & \longrightarrow & \mathrm{Pic}(X)\otimes(\boldsymbol{Q}_p/\boldsymbol{Z}_p) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{H}^2(X/W) & \longrightarrow & \mathscr{B}(X) & \longrightarrow & \mathrm{H}^2(X,\mu_{p^\infty}) & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & \dfrac{\mathscr{B}(X)}{\mathrm{H}^2(X/W)+\mathrm{Pic}(X)\otimes \boldsymbol{Q}_p} & \xrightarrow{\sim} & \mathrm{Br}(X)[p^\infty] & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array} \tag{21}$$

with exact rows and columns.

2F. ***Description of the group of crystalline B-fields.*** We will now give some results describing the subgroup $\mathscr{B}(X)\subset \mathrm{H}^2(X/K)$ more explicitly.

We recall that the *Tate module* of a K3 crystal H (in the sense of Ogus [1979, Definition 3.1]) is the $\boldsymbol{Z}_p$-module $H^{\phi=1}\subset H$ consisting of those elements $h\in H$ satisfying $\phi(h)=h$, where $\phi := p^{-1}\Phi$ and Φ is the Frobenius endomorphism of H. By a result of Illusie [1971, Théorème 5.14, p. 631], if X is a K3 surface then we have an exact sequence

$$0\to \mathrm{H}^2(X,\boldsymbol{Z}_p(1))\to \mathrm{H}^2(X/W)\xrightarrow{p-\Phi} \mathrm{H}^2(X/W)$$

identifying $\mathrm{H}^2(X,\boldsymbol{Z}_p(1))$ with the Tate module $\mathrm{H}^2(X/W)^{\phi=1}$ of the K3 crystal $\mathrm{H}^2(X/W)$, where the left inclusion is given by the inverse limit of the inclusions (15). We have inclusions

$$\mathrm{Pic}(X)\otimes \boldsymbol{Q}_p\subset \mathrm{H}^2(X,\boldsymbol{Q}_p(1))\subset \mathscr{B}(X),$$

where as usual $\mathrm{H}^2(X,\boldsymbol{Q}_p(1))=\mathrm{H}^2(X,\boldsymbol{Z}_p(1))\otimes \boldsymbol{Q}_p$.

Remark 2.5. By analogy with the Lefschetz (1,1) theorem, one might imagine that the inclusion $\mathrm{Pic}(X)\otimes \boldsymbol{Q}_p\subset \mathrm{H}^2(X,\boldsymbol{Q}_p(1))$ is an equality. However, this is frequently false, e.g., for a very general ordinary K3 surface. It is true if X is supersingular, as a consequence of the Tate conjecture for supersingular K3 surfaces (of course, the Tate conjecture is known for all K3 surfaces, but it is only in the supersingular case that there is such a consequence for K3 surfaces over general algebraically closed fields).

Proposition 2.6. *Let X be a* K3 *surface.*

(1) *If X has finite height, then* $\mathscr{B}(X)=\mathrm{H}^2(X/W)+\mathrm{H}^2(X,\boldsymbol{Q}_p(1))$.

(2) *If X is supersingular, then* $\mathscr{B}(X)=\mathscr{B}_1(X)+\mathrm{H}^2(X,\boldsymbol{Q}_p(1))$.

Proof. In either case, we have $\mathrm{H}^2(X/W) \subset \mathscr{B}_1(X) \subset \mathscr{B}(X)$ and $\mathrm{H}^2(X, \boldsymbol{Q}_p(1)) \subset \mathscr{B}(X)$. It follows that in both cases the right-hand side is contained in $\mathscr{B}(X)$. We prove the reverse containments. Consider the commutative diagram

$$\begin{array}{ccc} \mathrm{H}^2(X, \boldsymbol{Z}_p(1)) & \hookrightarrow & \mathrm{H}^2(X/W) \\ \downarrow & & \downarrow \text{mod } p^n \\ \mathrm{H}^2(X, \mu_{p^n}) & \hookrightarrow & \mathrm{H}^2(X/W_n) \end{array} \tag{22}$$

Suppose that X has finite height. Flat duality implies that $\mathrm{H}^3(X, \mu_{p^n}) = 0$ for all $n \geq 1$. Hence, the maps $\mathrm{H}^2(X, \boldsymbol{Z}_p(1)) \to \mathrm{H}^2(X, \mu_{p^n})$ are surjective. It follows that the restriction $\mathrm{H}^2(X, \boldsymbol{Q}_p(1)) \to \mathrm{Br}(X)[p^\infty]$ of the exponential map (20) is surjective. This proves (1). We next prove (2). Suppose that X is supersingular. For each n and i we consider the short exact sequence

$$0 \to \mathrm{U}^i(X, \mu_{p^n}) \to \mathrm{H}^i(X, \mu_{p^n}) \to \mathrm{D}^i(X, \mu_{p^n}) \to 0.$$

As $\mathrm{H}^1(X, \mu_{p^n}) = 0$, flat duality shows that $\mathrm{D}^3(X, \mu_{p^n}) = 0$. Hence, the maps $\mathrm{D}^2(X, \mu_{p^{n+1}}) \to \mathrm{D}^2(X, \mu_{p^n})$ induced by the multiplication $p : \mu_{p^{n+1}} \to \mu_{p^n}$ are surjective. Furthermore, the formal group associated to $\mathrm{U}^2(X, \mu_{p^n})$ is isomorphic to $\widehat{\mathrm{Br}}(X) \cong \widehat{\boldsymbol{G}}_a$, so $\mathrm{U}^2(X, \mu_{p^n}) \cong \boldsymbol{G}_a(k)$. In particular, the groups $\mathrm{U}^2(X, \mu_{p^n})$ are p-torsion, and the maps $\mathrm{U}^2(X, \mu_{p^n}) \to \mathrm{U}^2(X, \mu_{p^{n+1}})$ induced by the inclusion $\mu_{p^n} \subset \mu_{p^{n+1}}$ are isomorphisms. Write $\mathrm{U}^2(X, \mu_{p^\infty})$ for the union of the $\mathrm{U}^2(X, \mu_{p^n})$ and $\mathrm{D}^2(X, \mu_{p^\infty})$ for the union of the $\mathrm{D}^2(X, \mu_{p^n})$. It follows that the composition

$$\mathrm{H}^2(X, \boldsymbol{Q}_p(1)) \to \mathrm{H}^2(X, \mu_{p^\infty}) \to \mathrm{D}^2(X, \mu_{p^\infty})$$

is surjective, and that $\mathrm{U}^2(X, \mu_p) = \mathrm{U}^2(X, \mu_{p^\infty})$. Hence, the exponential map (20) restricts to a surjection $\mathscr{B}_1(X) + \mathrm{H}^2(X, \boldsymbol{Q}_p(1)) \to \mathrm{Br}(X)[p^\infty]$, which proves (2). □

The following describes the subgroup $\mathscr{B}(X) \subset \mathrm{H}^2(X/K)$ in terms of the F-crystal structure on $\mathrm{H}^2(X/W)$, without explicit mention of flat cohomology or the Brauer group. The special case of classes $B \in p^{-1}\,\mathrm{H}^2(X/W)$ is Lemma 3.4.11 of [Bragg and Lieblich 2018].

Proposition 2.7. *A class $B \in \mathrm{H}^2(X/K)$ is a crystalline B-field if and only if*

$$B - \phi(B) \in \mathrm{H}^2(X/W) + \phi(\mathrm{H}^2(X/W)), \tag{23}$$

where $\phi = p^{-1}\Phi$.

Proof. Write $H = \mathrm{H}^2(X/W)$. Suppose that X has finite height. It is immediate from Proposition 2.6(1) that any B-field satisfies the claimed relation. Conversely, suppose that $B = a/p^n$ is an element satisfying (23). Consider the Newton–Hodge decomposition

$$\mathrm{H}^2(X/W) = H_{<1} \oplus H_1 \oplus H_{>1}$$

of $\mathrm{H}^2(X/W)$ into subcrystals with the indicated slopes (see Section 2I below). Write $a = (a_{<1}, a_1, a_{>1})$. We have $pH_{<1} \subset \Phi(H_{<1})$ or, equivalently, $H_{<1} \subset \phi(H_{<1})$ (see for instance [Katz 1979, §1.2]). Consider the map

$$1-\phi : H_{<1} \to \phi(H_{<1}).$$

All slopes of $H_{<1}$ are less than one, so this map is injective. By [Illusie 1971, Lemme II.5.3], it is surjective, and hence an isomorphism. We have $(1-\phi)(a_{<1}) \in p^n\phi(H_{<1})$, so in fact $a_{<1} \in p^n H_{<1}$. We have $\phi(H_{>1}) \subset H_{>1}$. Thus, we have a map

$$1-\phi : H_{>1} \to H_{>1},$$

which as before is both injective and surjective, and hence an isomorphism. We have $(1-\phi)(a_{>1}) \in p^n H_{>1}$, so in fact $a_{>1} \in p^n H_{>1}$. Finally, note that H_1 is a unit root crystal. It follows quickly that $a_1 = p^n h + t$ for some $h \in H_1$ and some t which is fixed by ϕ. We conclude that $B \in \mathscr{B}(X)$. This completes the proof of Proposition 2.7 in the case when X has finite height.

Suppose that X is supersingular. By Lemma 3.4.11 of [Bragg and Lieblich 2018], we have that $\mathscr{B}_1(X)$ consists exactly of those classes $B = a/p$ with $a \in H$ that satisfy (23). By Proposition 2.6(2), any B-field satisfies the claimed relation. We prove the converse. The inclusion of the Tate module is an isogeny, meaning that the map $T \otimes K \to H \otimes K$ is an isomorphism. Thus, the natural map $H \xrightarrow{\sim} H^\vee \to T^\vee \otimes W$ is injective, and we may regard H as a subgroup of the dual lattice $T^\vee \otimes W$. Note that if $h \in H$ and $t \in T$, then $\phi(h).t = \phi(h).\phi(t) = \sigma(h.t)$. It follows that $H + \phi(H) \subset T^\vee \otimes W$. Now, if $B \in \mathrm{H}^2(X/K)$ satisfies the claimed relation, then B is in the kernel of the map $1-\phi : T^\vee \otimes W \to T^\vee \otimes (K/W)$, which is equal to $T^\vee \otimes W + T \otimes \boldsymbol{Q}_p$. We may therefore write $B = B' + t/p^n$ for some $B' \in T^\vee \otimes W$ and some $t \in T$. As t is killed by $1-\phi$, B' also satisfies the relation (23). But by [Ogus 1979, Lemma 3.10], we have $T^\vee \otimes W \subset p^{-1}H$, so $B' \in p^{-1}H$. By Lemma 3.4.11 of [Bragg and Lieblich 2018] we have $B' \in \mathscr{B}(X)$. We also have $t/p^n \in \mathscr{B}(X)$, and we conclude that $B \in \mathscr{B}(X)$, as desired. □

Remark 2.8. One can alternatively prove Proposition 2.7 by generalizing the method of [Bragg and Lieblich 2018, Lemma 3.4.11], which we sketch. This proof has the advantage of avoiding flat duality and being uniform in the height of X. The first step is to understand the cokernel of the map (12). This is described by the short exact sequence [Colliot-Thélène et al. 1983, Lemma 2, p. 779]

$$0 \to \boldsymbol{G}_m/\boldsymbol{G}_m^{\times p^n} \to \mathrm{W}_n\,\Omega^1_X \xrightarrow{1-\mathrm{F}} \mathrm{W}_n\,\Omega^1_X/d(\mathrm{W}_n\,\mathcal{O}_X) \to 0, \tag{24}$$

where 1 denotes the projection and F is the map defined in [Illusie 1971, Proposition II.3.3]. One then proceeds by analyzing the p-adic filtrations on crystalline and de Rham–Witt cohomology.

2G. ***p-primary torsion in the Brauer group.*** We make some observations connecting the group $\mathscr{B}(X)$ of crystalline B-fields to the p-primary torsion in the Brauer group of X. Suppose that X has finite height h. By Proposition 2.6, we have

$$\mathscr{B}(X) = \mathrm{H}^2(X, \boldsymbol{Q}_p(1)) + \mathrm{H}^2(X/W).$$

In particular, (21) induces an isomorphism

$$\frac{\mathrm{H}^2(X, \boldsymbol{Q}_p(1))}{\mathrm{H}^2(X, \boldsymbol{Z}_p(1)) + \mathrm{Pic}(X) \otimes \boldsymbol{Q}_p} \xrightarrow{\sim} \mathrm{Br}(X)[p^\infty]. \tag{25}$$

The slope 1 part of $\mathrm{H}^2(X/W)$ has rank $22-2h$, so we have $\mathrm{H}^2(X, \boldsymbol{Z}_p(1)) \cong \boldsymbol{Z}_p^{\oplus 22-2h}$. Thus, (25) gives an isomorphism

$$\mathrm{Br}(X)[p^\infty] \cong (\boldsymbol{Q}_p/\boldsymbol{Z}_p)^{\oplus 22-\rho-2h}, \tag{26}$$

where ρ is the Picard rank of X. This could also be seen from the fact that, in the finite-height case, the diagram (8) with ℓ replaced by p (and étale cohomology with flat cohomology) still has exact rows and columns.

Remark 2.9. The exponent appearing in the formula (26) for the p-primary torsion of the Brauer group is smaller than that for the l-primary torsion (9) by a factor of $2h$. These "missing" p-primary torsion Brauer classes are the cause of the restriction at p in Theorem 1.2.

We now suppose X is supersingular. By Proposition 2.6, we have

$$\mathscr{B}(X) = \mathscr{B}_1(X) + \mathrm{H}^2(X, \boldsymbol{Q}_p(1)).$$

By the Tate conjecture for supersingular K3 surfaces, the first crystalline Chern character induces an isomorphism $\mathrm{Pic}(X) \otimes \boldsymbol{Z}_p \xrightarrow{\sim} T \otimes \boldsymbol{Z}_p = \mathrm{H}^2(X, \boldsymbol{Z}_p(1))$, and so $\mathrm{H}^2(X, \boldsymbol{Q}_p(1))$ is in the kernel of the crystalline exponential map (20). Write $N = \mathrm{Pic}(X)$. We have $\rho = 22$, so $\mathrm{Br}(X)$ has no prime-to-p torsion (see (9)). We conclude that (20) restricts to a surjection $\mathscr{B}_1(X) \to \mathrm{Br}(X)$. We have a short exact sequence

$$0 \to p^{-1}N/N \to \mathscr{B}_1(X)/H \to \mathrm{Br}(X) \to 0.$$

In particular, $\mathrm{Br}(X)$ is p-torsion. As shown in the proof of Proposition 2.7, we have that $\mathscr{B}_1(X) \subset N^\vee \otimes W + p^{-1}N \subset p^{-1}N \otimes W$, where the latter inclusion holds because discriminant group of N is p-torsion. Let $\mathscr{B}_1(X)^\circ = \mathscr{B}_1(X) \cap (N^\vee \otimes W)$. We have a short exact sequence

$$0 \to N^\vee/N \to \mathscr{B}_1(X)^\circ/H \to \mathrm{Br}(X) \to 0. \tag{27}$$

The subgroup $\mathscr{B}_1(X)^\circ$ can be understood using Ogus's results [1979] on the classification of supersingular K3 crystals. Write $K = H/N \otimes W$ and $V = N^\vee/N \cong \boldsymbol{F}_p^{2\sigma_0}$ (here, σ_0 is the Artin invariant of X). The subspace $K \subset V \otimes k$ is Ogus's *characteristic subspace*, and has dimension σ_0. Let $\phi : V \otimes k \to V \otimes k$ be the map $\phi(v \otimes \lambda) = v \otimes \lambda^p$. Ogus showed that K is totally isotropic and is in a special position with respect to ϕ. Namely, $K + \phi(K)$ has dimension $\sigma_0 + 1$, and $V \otimes k = \sum_i \phi^i(K)$ has dimension $2\sigma_0$. This implies that there exists a *characteristic vector* for K, which is an element $e_1 \in V \otimes k$ such that, writing $e_i = \phi^{i-1}(e_1)$, we have that $\{e_0, \dots, e_{\sigma_0-1}\}$ is a basis for K and $\{e_0, \dots, e_{2\sigma_0-1}\}$ is a basis for $V \otimes k$. We let f_i denote the functional given by pairing with e_i, so that $\{f_0, \dots, f_{\sigma_0-1}\}$ is a basis for $K^\vee = V \otimes k/K$. By Proposition 2.7, the subgroup $\mathscr{B}_1(X)^\circ/H \subset V \otimes k/K$ is the kernel of the map $1 - \phi : V \otimes k/K \to V \otimes k/(K + \phi(K))$. It follows that we have

$$\mathscr{B}_1(X)^\circ/H = \{\lambda f_1 + \lambda^p f_2 + \dots + \lambda^{p^{\sigma_0-1}} f_{\sigma_0-1} \mid \lambda \in k\}.$$

We conclude that $\mathscr{B}_1(X)^\circ/H$ is isomorphic to the underlying additive group $\boldsymbol{G}_a(k)$ of the group field k. The left term of (27) is discrete, and hence there is an isomorphism

$$\mathrm{Br}(X) \cong \boldsymbol{G}_a(k).$$

Remark 2.10. Multiplying by p and then reducing modulo p, the characteristic subspace K is identified with the kernel of the k-linearized first de Rham Chern character $c_1^{\mathrm{dR}} \otimes k : \mathrm{Pic}(X) \otimes k \to \mathrm{H}^2_{\mathrm{dR}}(X/k)$, and the vector space $V \otimes k/K$ is identified with its image. Furthermore, $\mathscr{B}_1(X)/H$ is identified with $\mathrm{H}^2(X, \mu_p)$ (regarded as a subgroup of $\mathrm{H}^2_{\mathrm{dR}}(X/k)$ via the map $d\log$) and $\mathscr{B}_1(X)^\circ/H$ is identified with $\mathrm{U}^2(X, \mu_p)$.

2H. *The twisted Mukai crystal.* We recall the Mukai crystal introduced in [Lieblich and Olsson 2015]. We set

$$\widetilde{\mathrm{H}}(X/W) = \mathrm{H}^0(X/W)(-1) \oplus \mathrm{H}^2(X/W) \oplus \mathrm{H}^4(X/W)(1).$$

As a result of the Tate twists on the first and third factors on the right-hand side, the Frobenius operator $\widetilde{\Phi}$ on $\widetilde{\mathrm{H}}(X/W)$ is given by the formula

$$\widetilde{\Phi}(a, b, c) = (p\sigma(a), \Phi(b), p\sigma(c)),$$

where we have identified H^0 and H^4 with W, and where Φ is the Frobenius operator on $\mathrm{H}^2(X/W)$. We equip $\widetilde{\mathrm{H}}(X/W)$ with the Mukai pairing. It is immediate from the definitions that $\widetilde{\mathrm{H}}(X/W)$ is a K3 crystal of rank 24.

Definition 2.11. Let B be a crystalline B-field. The *twisted Mukai crystal* associated to (X, B) is

$$\widetilde{\mathrm{H}}(X/W, B) = \exp(B)\,\widetilde{\mathrm{H}}(X/W) \subset \widetilde{\mathrm{H}}(X/K).$$

Here, $\exp(B)$ is the isometry of $\widetilde{\mathrm{H}}(X/K)$ defined by the formula (10).

The twisted Mukai crystal has a natural structure of a K3 crystal by the following result.

Theorem 2.12. *Let $B \in \mathscr{B}(X)$ be a crystalline B-field. The endomorphism $\widetilde{\Phi}$ of $\widetilde{\mathrm{H}}(X/K)$ restricts to an endomorphism of $\widetilde{\mathrm{H}}(X/W, B)$. When equipped with the restriction of the Mukai pairing, the twisted Mukai crystal $\widetilde{\mathrm{H}}(X/W, B)$ is a* K3 *crystal of rank* 24.

Proof. When $B \in \mathscr{B}_1(X)$, this is Proposition 3.4.15 of [Bragg and Lieblich 2018]. Using Proposition 2.7, the proof of loc. cit. applies verbatim to give the result for general B-fields as well. □

Note that if $h \in \mathrm{H}^2(X/W)$ then $\exp(h) = \left(1, h, \frac{1}{2}h^2\right) \in \mathrm{H}^*(X/W)$. Thus, as a submodule of $\widetilde{\mathrm{H}}(X/K)$, $\widetilde{\mathrm{H}}(X/W, B)$ depends only on the image of B in $\mathrm{H}^2(X, \mu_{p^n})$. Furthermore, up to isomorphism (of K3 crystals), $\widetilde{\mathrm{H}}(X/W, B)$ only depends on the Brauer class α_B (see [Bragg 2021, Lemma 3.2.4]).

Remark 2.13. For a K3 surface over the complex numbers, Huybrechts and Stellari [2005] define the twisted Mukai lattice $\widetilde{\mathrm{H}}(X, B, \boldsymbol{Z})$ to be equal to the untwisted lattice $\widetilde{\mathrm{H}}(X, \boldsymbol{Z})$ with a modified Hodge structure. This differs from our definition of the twisted Mukai crystal (as well as the twisted ℓ-adic Mukai lattice), as we have defined $\widetilde{\mathrm{H}}(X/W, B)$ by equipping the rational Mukai lattice $\widetilde{\mathrm{H}}(X/K)$ with a nonstandard integral structure, but the same crystal structure. The convention analogous to that of loc. cit. would be to define $\widetilde{\mathrm{H}}(X/W, B)$ to be equal to $\widetilde{\mathrm{H}}(X/W)$ as a W-module, but equipped with the twisted Frobenius operator $\widetilde{\Phi}_B = \exp(-B) \circ \widetilde{\Phi} \circ \exp(B) = \exp(\phi(B) - B)$.

We record the following observation.

Proposition 2.14. *Let X be a* K3 *surface and B be a crystalline B-field. If X has finite height h, then $\widetilde{\mathrm{H}}(X/W, B)$ is a* K3 *crystal of height h and, in particular, is abstractly isomorphic to $\widetilde{\mathrm{H}}(X/W)$. If X is supersingular of Artin invariant σ_0, then $\widetilde{\mathrm{H}}(X/W, B)$ is a supersingular* K3 *crystal whose Artin invariant is equal to either σ_0 if $\alpha_B = 0$ or $\sigma_0 + 1$ if $\alpha_B \neq 0$.*

Proof. Suppose that X has finite height. The defining inclusion $\widetilde{\mathrm{H}}(X/W, B) \subset \widetilde{\mathrm{H}}(X/K)$ is compatible with the pairing and Frobenius. Thus, $\widetilde{\mathrm{H}}(X/W, B)$ and $\widetilde{\mathrm{H}}(X/K)$ are isogenous, and so $\widetilde{\mathrm{H}}(X/W, B)$ has height h. If h is finite, this implies the crystals are isomorphic integrally. Alternatively, we may reason as follows. Because X has finite height, by Proposition 2.6 we may assume B satisfies $B = \phi(B)$. The map $\exp(-B)$ then defines an isomorphism $\widetilde{\mathrm{H}}(X/W, B) \cong \widetilde{\mathrm{H}}(X/W)$ of K3 crystals. If X is supersingular, then the Brauer group of X is p-torsion. As the twisted Mukai crystal depends up to isomorphism only on the class α_B, we may assume that $B \in \mathscr{B}(X)_1$. The result then follows from [Bragg and Lieblich 2018, Corollary 3.4.23]. □

2I. *The Newton–Hodge decomposition of the twisted Mukai crystal.* Let H be a K3 crystal. The *Newton–Hodge decomposition* of H is a canonical direct sum decomposition

$$H = H_{<1} \oplus H_1 \oplus H_{>1}$$

with the following properties. If H has finite height h and rank r, then $H_{<1}$ has slope $1 - 1/h$ and rank h, H_1 has slope 1 and rank $r - 2h$, and $H_{>1}$ has slope $1 + 1/h$ and rank h. Furthermore, H_1 is orthogonal to $H_{<1} \oplus H_{>1}$, and under the pairing, $H_{<1}$ and $H_{>1}$ are dual. If H is supersingular, then $H_1 = H$ and $H_{<1} = H_{>1} = 0$.

Let X be a K3 surface and let B be a crystalline B-field. We will relate the Newton–Hodge decompositions of $\widetilde{\mathrm{H}}(X/W, B)$ and $\mathrm{H}^2(X/W)$.

Proposition 2.15. *As submodules of $\widetilde{\mathrm{H}}(X/K)$,*

$$\begin{aligned} \widetilde{\mathrm{H}}(X/W, B)_{<1} &= \mathrm{H}^2(X/W)_{<1}, \\ \widetilde{\mathrm{H}}(X/W, B)_1 &= \exp(B)\big(\mathrm{H}^0(X/W) \oplus \mathrm{H}^2(X/W)_1 \oplus \mathrm{H}^4(X/W)\big), \\ \widetilde{\mathrm{H}}(X/W, B)_{>1} &= \mathrm{H}^2(X/W)_{>1}. \end{aligned}$$

Proof. If X is supersingular, then $\widetilde{\mathrm{H}}(X/W, B)$ is also supersingular, and the result is trivial. Suppose X has finite height. Write $\mathrm{H}^i = \mathrm{H}^i(X/W)$. By Proposition 2.6, B is congruent modulo H^2 to a B-field B' satisfying $\phi(B') = B'$. As $\widetilde{\mathrm{H}}(X/W, B) = \widetilde{\mathrm{H}}(X/W, B')$, we may assume without loss of generality that B is fixed by ϕ and, in particular, $B \in (\mathrm{H}^2)_1 \otimes K$. We then have that $\widetilde{\mathrm{H}}(X/W, B)_{<1} = \exp(B)\, \widetilde{\mathrm{H}}(X/W)_{<1}$, and similarly for the slope 1 and > 1 parts. It is immediate that the Newton–Hodge decomposition of $\widetilde{\mathrm{H}}(X/W)$ is given by $\widetilde{\mathrm{H}}(X/W)_{<1} = (\mathrm{H}^2)_{<1}$, $\widetilde{\mathrm{H}}(X/W)_1 = \mathrm{H}^0 \oplus (\mathrm{H}^2)_1 \oplus \mathrm{H}^4$, and $\widetilde{\mathrm{H}}(X/W)_{>1} = (\mathrm{H}^2)_{>1}$. The result follows upon noting that $\mathrm{H}_1 \otimes K$ is orthogonal to $(\mathrm{H}^2)_{<1}$ and $(\mathrm{H}^2)_{>1}$, so $\exp(B)(\mathrm{H}^2)_{<1} = (\mathrm{H}^2)_{<1}$ and $\exp(B)(\mathrm{H}^2)_{>1} = (\mathrm{H}^2)_{>1}$. □

Note that if B is a general B-field, the direct sum decomposition of $\widetilde{\mathrm{H}}(X/W, B)_1$ described in the statement of Proposition 2.15 may *not* be preserved by $\widetilde{\Phi}$.

2J. *Mixed realization.* We define B-fields for Brauer classes whose order is not necessarily a prime power. For simplicity we give the definitions only when $\operatorname{char} k = p > 0$.

Definition 2.16. Let $\alpha \in \mathrm{Br}(X)$ be a class of exact order m. Fix a prime q. Let q^n be the largest power of q dividing m, and set $t = m/q^n$. If $q = \ell \neq p$, then an *ℓ-adic B-field lift* of α is an ℓ-adic B-field lift (in the sense of Definition 2.1) of $t\alpha$. Similarly, if $q = p$, then a *crystalline B-field lift* of α is a crystalline B-field lift (in the sense of Definition 2.3) of $t\alpha$.

Definition 2.17. Let $\alpha \in \mathrm{Br}(X)$ be a Brauer class. A *mixed B-field lift of* α is a set $\boldsymbol{B} = \{B_\ell\}_{\ell \neq p} \cup \{B_p\}$ consisting of a choice of an ℓ-adic B-field lift B_ℓ of α for each prime $\ell \neq p$ and a crystalline B-field lift B_p of α (in both cases in the sense of Definition 2.16).

Given a mixed B-field $\boldsymbol{B}$, we write $\boldsymbol{B}^p$ for the component in $\mathrm{H}^2(X, \boldsymbol{A}_f^p)$, and $\boldsymbol{B}_p = B_p$ for the component in $\mathscr{B}(X) \subset \mathrm{H}^2(X/K)$.

We say a few words to explain this definition. Let $\mu_* = \bigcup_m \mu_m$ be the subsheaf of torsion sections of $\boldsymbol{G}_m$. Let $p_0 = p$ and let $p_1, p_2, \dots$ be an enumeration of the remaining primes. We have a canonical isomorphism

$$\mu_{p_0^\infty} \oplus \mu_{p_1^\infty} \oplus \mu_{p_2^\infty} \cdots \cong \mu_*$$

given by multiplication. As described in the introduction, we have

$$\mathrm{H}^2(X, \boldsymbol{A}_f^p) = \prod_{i \geq 1}{}' \mathrm{H}^2(X, \boldsymbol{Q}_{p_i}(1)),$$

where the restricted product on the right-hand side consists of tuples $\{B_i\}$ such that for all but finitely many i we have $B_i \in \mathrm{H}^2(X, \boldsymbol{Z}_{p_i}(1))$. A mixed B-field lift of a class α is a preimage of α under the composition

$$\mathscr{B}(X) \times \mathrm{H}^2(X, \boldsymbol{A}_f^p) \twoheadrightarrow \bigoplus_i \mathrm{H}^2(X, \mu_{p_i^\infty}) \xrightarrow{\sim} \mathrm{H}^2(X, \mu_*) \twoheadrightarrow \mathrm{Br}(X), \tag{28}$$

which we denote by $\boldsymbol{B} \mapsto \alpha_{\boldsymbol{B}}$. Here, the right horizontal map is induced by the inclusion $\mu_* \subset \boldsymbol{G}_m$.

3. Twisted Chern characters and action on cohomology

Let X be a smooth projective variety over a field k and let $\alpha \in \mathrm{Br}(X)$ be a torsion Brauer class. In this section we will define a certain twisted Chern character for α-twisted sheaves on X. This will be a map from the Grothendieck group of coherent α-twisted sheaves on X to the rational Chow group $A^*(X)_{\boldsymbol{Q}}$ of X. There are multiple inequivalent definitions of twisted Chern characters appearing in the literature, several of which are reviewed and compared in [Huybrechts and Stellari 2006, §3]. These all seem to be essentially equivalent in practice. We will use the notion appearing in [Lieblich et al. 2014; Bragg 2021; Bragg and Lieblich 2018], which is also used in [Huybrechts 2019, §2]. This formulation seems to us to be the most flexible, and has a uniform interaction with B-fields in each of the contexts we have considered. We remark that our definition below is described in terms of cocycles in [Bragg 2021, Appendix A.1] and is compared to the twisted Chern characters of Huybrechts and Stellari [2005] in [Bragg 2021, Appendix A.2].

Suppose that $n\alpha = 0$ for some positive integer n. To define our twisted Chern character we will make an auxiliary choice of a preimage $\alpha' \in \mathrm{H}^2(X, \mu_n)$ of α under the surjection

$$\mathrm{H}^2(X, \mu_n) \twoheadrightarrow \mathrm{Br}(X)[n]$$

induced by the inclusion $\mu_n \subset \boldsymbol{G}_m$.

We choose a $\boldsymbol{G}_m$–gerbe $\pi : \mathscr{X} \to X$ with cohomology class α, and identify the category of α-twisted sheaves on X with the category of coherent sheaves on $\mathscr{X}$ of weight 1. We also choose a μ_n-gerbe $\mathscr{X}' \to X$ with cohomology class α', and an isomorphism $\mathscr{X}' \wedge_{\mu_n} \boldsymbol{G}_m \cong \mathscr{X}$ (see [Olsson 2016, Chapter 12.3]). There is then a canonical n-fold twisted invertible sheaf $\mathscr{L}$ on $\mathscr{X}$. Given a locally free α-twisted sheaf $\mathscr{E}$ of finite rank, we note that $\mathscr{E}^{\otimes n} \otimes \mathscr{L}^\vee$ is a 0-twisted sheaf on $\mathscr{X}$. We define

$$\mathrm{ch}^{\alpha'}(\mathscr{E}) = \sqrt[n]{\mathrm{ch}(\pi_*(\mathscr{E}^{\otimes n} \otimes \mathscr{L}^\vee))},$$

where the n-th root is chosen so that rk is positive. One can check that $\mathrm{ch}^{\alpha'}$ depends only on α', and not on the choice of gerbes or on $\mathscr{L}$. We note that ch_0 and ch_1 are given by

$$\mathrm{ch}^{\alpha'}(\mathscr{E}) = (\mathrm{rk}(\mathscr{E}), \pi_*(\det(\mathscr{E}) \otimes \mathscr{L}^\vee), \ldots). \tag{29}$$

Assume that $\mathscr{X}$ has the resolution property, so that every α-twisted sheaf admits a finite resolution by locally free α-twisted sheaves. We then obtain by additivity a map

$$\mathrm{ch}^{\alpha'} : K(X, \alpha) \to A^*(X)_{\boldsymbol{Q}},$$

where $K(X, \alpha)$ denotes the Grothendieck group of the category of α-twisted sheaves. We note that this definition is purely algebraic, and hence makes sense in any characteristic. Furthermore, we did not need α to be topologically trivial, only torsion.

Suppose that X is a K3 surface. We explain the relationship between the choice of α' and the choice of a B-field lift of α. We first observe that, in any of the contexts we have considered, a choice of B-field lift for α determines in particular a choice of preimage of α in $\mathrm{H}^2(X, \mu_n)$. More precisely, a choice of singular B-field lift (if the ground field is the complex numbers) or of a mixed B-field lift determines a preimage in $\mathrm{H}^2(X, \mu_n)$. If α is killed by ℓ^n, then a choice of ℓ-adic B-field lift determines a preimage in $\mathrm{H}^2(X, \mu_{\ell^n})$, and if α is killed by p^n a choice of crystalline B-field lift determines a preimage in $\mathrm{H}^2(X, \mu_{p^n})$. In any of these situations, we write

$$\mathrm{ch}^B(\mathscr{E}) = \mathrm{ch}^{\alpha'}(\mathscr{E}),$$

where α' is the induced preimage. We also set

$$v^B(\mathscr{E}) = \mathrm{ch}^B(\mathscr{E}).\sqrt{\mathrm{td}(X)}.$$

3A. *Twisted Chern characters on twisted* K3 *surfaces.*

Definition 3.1. We assume now that k is an algebraically closed field of characteristic $p > 0$. If X is a K3 surface over k, we define the *extended Néron–Severi group* of X by

$$\widetilde{N}(X) = \langle(1, 0, 0)\rangle \oplus N(X) \oplus \langle(0, 0, 1)\rangle = A^*(X) \subset A^*(X)_{\boldsymbol{Q}}.$$

As the Chern characters of a coherent sheaf on a K3 surface are integral, the extended Néron–Severi group is a natural recipient for the Chern class map, and in fact the Chern class map

$$\mathrm{ch} : K(X) \to \widetilde{N}(X)$$

is an isomorphism. Let $\alpha \in \mathrm{Br}(X)$ be a Brauer class. For two α-twisted sheaves $\mathscr{E}$, $\mathscr{F}$, we have the Riemann–Roch formula

$$\chi(\mathscr{E}, \mathscr{F}) = -\langle v^B(\mathscr{E}), v^B(\mathscr{F})\rangle.$$

We will identify a subgroup of $\widetilde{N}(X) \otimes \boldsymbol{Q}$ which contains the image of the twisted Chern class map

$$\mathrm{ch}^B : K(X, \alpha) \to \widetilde{N}(X) \otimes \boldsymbol{Q}.$$

Definition 3.2. If B is an ℓ-adic B-field, we define the *ℓ-adic twisted Néron–Severi group* by

$$\widetilde{N}(X, B_\ell) = (\widetilde{N}(X) \otimes \mathbf{Z}[\ell^{-1}]) \cap \widetilde{\mathrm{H}}(X, \mathbf{Z}_\ell, B_\ell).$$

If $\mathrm{char}\, k = p$ and B is a crystalline B-field, we define the *crystalline twisted Néron–Severi group* by

$$\widetilde{N}(X, B_p) = (\widetilde{N}(X) \otimes \mathbf{Z}[p^{-1}]) \cap \widetilde{\mathrm{H}}(X/W, B_p),$$

and if $\boldsymbol{B} = \{B_\ell\}_{\ell \neq p} \cup \{B_p\}$ is a mixed B-field, we define the *mixed twisted Néron–Severi group* by

$$\widetilde{N}(X, \boldsymbol{B}) = \left(\bigcap_{\ell \neq p} \widetilde{N}(X, B_\ell)\right) \cap \widetilde{N}(X, B_p),$$

where the intersection is taken inside of $\widetilde{N}(X) \otimes \boldsymbol{Q}$. Note that for all but finitely many primes q the B-field B_q is integral. Hence, the intersection defining $\widetilde{N}(X, \boldsymbol{B})$ is finite.

The restriction of the Mukai pairing on $\widetilde{N}(X) \otimes \boldsymbol{Q}$ to the ℓ-adic twisted Néron–Severi group takes values in $\mathbf{Z}[\ell^{-1}] \cap \mathbf{Z}_\ell = \mathbf{Z}$. Similarly, the Mukai pairing restricts to an integral pairing on the crystalline twisted Néron–Severi group. The following is the crucial integrality result for twisted Chern characters, generalizing the fact that the Chern characters of usual sheaves on K3 surfaces are integral.

Proposition 3.3. *Let X be a* K3 *surface and $\boldsymbol{B}$ a mixed* (*resp. ℓ-adic*, *resp. crystalline*) *B-field lift of a Brauer class $\alpha \in \mathrm{Br}(X)$. For any twisted sheaf $\mathscr{E} \in \boldsymbol{Coh}^{(1)}(X, \alpha)$, the twisted Chern character $\mathrm{ch}^{\boldsymbol{B}}(\mathscr{E})$ lies in the mixed* (*resp. ℓ-adic*, *resp. crystalline*) *twisted Néron–Severi group $\widetilde{N}(X, \boldsymbol{B})$.*

Proof. This is proved in Appendix A of [Bragg 2021] (the quoted statement is written for a crystalline B-field of the form $B = a/p$, but the proof applies essentially unchanged). □

Remark 3.4. The analog of Proposition 3.3 for the Hodge realization follows immediately from the existence of an invertible twisted sheaf in the differentiable category (in fact, this existence is used to define twisted Chern characters in [Huybrechts and Stellari 2005]). The ℓ-adic case is proved in [Lieblich et al. 2014, Lemma 3.3.7] by lifting to characteristic 0.

Proposition 3.5. *For any mixed B-field lift of* α, *the twisted Chern character*

$$\mathrm{ch}^{\boldsymbol{B}} : K(X, \alpha) \to \widetilde{N}(X, \boldsymbol{B})$$

is surjective.

Proof. The analogous result over the complex numbers is [Huybrechts and Stellari 2005, Proposition 1.4]. The proof in our case is identical, up to our differences in convention. □

3B. ***Action on cohomology.*** Let (X, α) and (Y, β) be twisted K3 surfaces over k. Choose mixed B-field lifts $\boldsymbol{B}$ of α and $\boldsymbol{B}'$ of β. As above, we define the twisted Chern character map

$$\mathrm{ch}^{-\boldsymbol{B}\boxplus\boldsymbol{B}'} : K(X \times Y, -\alpha \boxplus \beta) \to \widetilde{N}(X \times Y) \otimes \boldsymbol{Q},$$

and set

$$v^{-\boldsymbol{B}\boxplus\boldsymbol{B}'}(\_) = \mathrm{ch}^{-\boldsymbol{B}\boxplus\boldsymbol{B}'}(\_).\sqrt{\mathrm{td}(X \times Y)}.$$

Let $\Phi_P : D^b(X, \alpha) \to D^b(Y, \beta)$ be a Fourier–Mukai equivalence. We consider the map

$$\Phi_{v^{-\boldsymbol{B}\boxplus\boldsymbol{B}'}(P)} := \pi_{2*}(\pi_1^*(\_) \cup v^{-\boldsymbol{B}\boxplus\boldsymbol{B}'}(P)) : \mathrm{H}^*(X)_{\boldsymbol{Q}} \to \mathrm{H}^*(Y)_{\boldsymbol{Q}}, \tag{30}$$

where $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are the respective projections. Using the same formula, we define maps $\Phi^{\ell}_{v^{-B_\ell\boxplus B'_\ell}(P)}$ on the rational ℓ-adic cohomologies and $\Phi^{\mathrm{cris}}_{v^{-B_p\boxplus B'_p}(P)}$ on rational crystalline cohomology. By definition, these maps are equal to the maps given by restricting (30) to the ℓ-adic and crystalline components of $\mathrm{H}^*_{\boldsymbol{Q}}$.

Theorem 3.6. *Let* $\Phi_P : D^b(X, \alpha) \to D^b(Y, \beta)$ *be a Fourier–Mukai equivalence. The map* (30) *restricts to an isomorphism*

$$\Phi^{\ell}_{v^{-B_\ell\boxplus B'_\ell}(P)} : \widetilde{\mathrm{H}}(X, \mathbf{Z}_\ell, B_\ell) \to \widetilde{\mathrm{H}}(Y, \mathbf{Z}_\ell, B'_\ell) \tag{31}$$

which is compatible with the Mukai pairings for each $\ell \neq p$, *to an isomorphism*

$$\Phi^{\mathrm{cris}}_{v^{-B_p\boxplus B'_p}(P)} : \widetilde{\mathrm{H}}(X/W, B_p) \to \widetilde{\mathrm{H}}(Y/W, B'_p) \tag{32}$$

of K3 *crystals* (*that is*, *an isomorphism of* W*-modules which is compatible with the pairing and Frobenius operators*), *and to an isometry*

$$\Phi_{v^{-\boldsymbol{B}\boxplus\boldsymbol{B}'}(P)} : \widetilde{N}(X, \boldsymbol{B}) \xrightarrow{\sim} \widetilde{N}(Y, \boldsymbol{B}'). \tag{33}$$

Proof. By definition, the map (31) is equal to the correspondence induced by the cycle $v^{-B_\ell\boxplus B'_\ell}(P)$, and the map (32) is equal to the correspondence induced by the cycle $v^{-B_p\boxplus B'_p}(P)$. The compatibility with the pairing, and in the crystalline case, with the Frobenius, is proved exactly as in [Bragg 2021, §3.4]. It remains to show that the correspondences preserve the integral structures. Under the assumption that $p \geq 5$, this is shown in [Bragg 2021, Appendix A]. The result in general can be shown by lifting to characteristic 0, using the techniques of the following section. We omit further details. This proves the claims regarding (31) and (32). To prove the claimed properties of (33), note that the indicated correspondence preserves the subgroups of algebraic cycles, and so restricts to an isomorphism $\widetilde{N}(X) \otimes \boldsymbol{Q} \xrightarrow{\sim} \widetilde{N}(Y) \otimes \boldsymbol{Q}$. The result then follows from the previous claims. □

4. Rational Chow motives and isogenies

Given a smooth proper variety X over and algebraically closed field k, we let $\mathfrak{h}(X)$ denote its rational Chow motive.

Definition 4.1 (cf. [Yang 2022, Definition 1.1]). Let X and X' be K3 surfaces over k. An *isogeny* from X to X' is an isomorphism of motives $f : \mathfrak{h}^2(X') \xrightarrow{\sim} \mathfrak{h}^2(X)$ whose cohomological realization $\mathrm{H}^2(X')_{\boldsymbol{Q}} \to \mathrm{H}^2(X)_{\boldsymbol{Q}}$ preserves the Poincaré pairing. Two isogenies are said to be *equivalent* if they induce the same map $\mathrm{H}^2(X')_{\boldsymbol{Q}} \xrightarrow{\sim} \mathrm{H}^2(X)_{\boldsymbol{Q}}$ (see Section 1D).

Recall [Kahn et al. 2007, 14.2.2] that if X is a K3 surface over an algebraically closed field k then there is a canonical decomposition

$$\mathfrak{h}^2(X) = \mathfrak{h}^2_{\mathrm{alg}}(X) \oplus \mathfrak{h}^2_{\mathrm{tr}}(X)$$

of the Chow motive in degree two into an algebraic part and a transcendental part. The algebraic part $\mathfrak{h}^2_{\mathrm{alg}}(X)$ is isomorphic to $\boldsymbol{L} \otimes \mathrm{NS}(X)$, where $\boldsymbol{L}$ stands for the Lefschetz motive. Similarly, $\mathfrak{h}(X)$ decomposes as $\mathfrak{h}_{\mathrm{alg}}(X) \oplus \mathfrak{h}^2_{\mathrm{tr}}(X)$, where $\mathfrak{h}_{\mathrm{alg}} = \boldsymbol{L}^0 \oplus \mathfrak{h}^2_{\mathrm{alg}} \oplus \boldsymbol{L}^2$.

Now suppose (X, α) and (Y, β) are twisted K3 surfaces with mixed B-field lifts $\boldsymbol{B}$ of α and $\boldsymbol{B}'$ of β. Let $\Phi_P : D^b(X, \alpha) \to D^b(Y, \beta)$ be a Fourier–Mukai equivalence. Following Huybrechts [2019, Theorem 2.1], we have that the correspondence $v^{-\boldsymbol{B} \boxplus \boldsymbol{B}'}(P)$ induces an isomorphism $\mathfrak{h}(X) \xrightarrow{\sim} \mathfrak{h}(Y)$, which restricts to isomorphisms $\mathfrak{h}^2_{\mathrm{tr}}(X) \xrightarrow{\sim} \mathfrak{h}^2_{\mathrm{tr}}(Y)$ and $\mathfrak{h}_{\mathrm{alg}}(X) \xrightarrow{\sim} \mathfrak{h}_{\mathrm{alg}}(Y)$. By Witt's cancellation theorem, we can always find some isomorphism $\mathfrak{h}^2_{\mathrm{alg}}(X) \xrightarrow{\sim} \mathfrak{h}^2_{\mathrm{alg}}(Y)$ that preserves the Poincaré pairing. Adding this and the isomorphism $\mathfrak{h}^2_{\mathrm{tr}}(X) \xrightarrow{\sim} \mathfrak{h}^2_{\mathrm{tr}}(Y)$ induced by $v^{-\boldsymbol{B} \boxplus \boldsymbol{B}'}(P)$, we obtain an isogeny $\mathfrak{h}^2(X) \xrightarrow{\sim} \mathfrak{h}^2(Y)$.

Definition 4.2. Let X, Y be K3 surfaces. An isogeny $f : \mathfrak{h}^2(X) \xrightarrow{\sim} \mathfrak{h}^2(Y)$ is a *primitive derived isogeny* if its restriction $\mathfrak{h}^2_{\mathrm{tr}}(X) \xrightarrow{\sim} \mathfrak{h}^2_{\mathrm{tr}}(Y)$ agrees with the one induced by $v^{-\boldsymbol{B} \boxplus \boldsymbol{B}'}(P)$ for some choices of α, β, $\boldsymbol{B}$, $\boldsymbol{B}'$ and Φ_P as above. A *derived isogeny* is a composition of finitely many primitive derived isogenies.

In particular, note that if there exists a primitive derived isogeny between X and Y, then X and Y are twisted derived equivalent. Twisted derived equivalent K3 surfaces clearly have the same rational Chow motive. In a recent paper, Fu and Vials proved that their motives are moreover isomorphic as Frobenius algebra objects, and over $\boldsymbol{C}$ they also give a motivic characterization of twisted derived equivalent K3's [Fu and Vial 2021, Theorem 1, Corollary 2].

5. Lifting derived isogenies to characteristic 0

The goal of this section is to give some lifting results for primitive derived isogenies. This requires understanding deformations of twisted K3 surfaces and of twisted Fourier–Mukai equivalences in mixed characteristic. Deformations of a twisted K3 surface (X, α) over the complex numbers can be profitably understood in terms of deformations of a pair (X, B), where $B \in \mathrm{H}^2(X, \mathbb{Q})$ is a Hodge B-field lift of α (see for instance [Reinecke 2019]). Considering deformations of (X, B) serves two purposes: first, the B-field B allows one to algebraize formal deformations of the Brauer class, and second, B gives a notion of twisted Chern characters in the deformation family. Suppose now that (X, α) is a twisted K3 surface in positive characteristic. To similarly understand deformations of (X, α) over a base of mixed characteristic, we would need a notion of mixed characteristic B-field lift. The ℓ-adic theory works essentially unchanged in this setting, but the analog of the crystalline theory seems more complicated. We will avoid this issue by using instead of a B-field a simpler object, namely a preimage $\alpha' \in \mathrm{H}^2(X, \mu_n)$ of α under the map $\mathrm{H}^2(X, \mu_n) \to \mathrm{Br}(X)$. The deformation theory of such pairs (X, α') has been considered in [Bragg 2023]: the flat cohomology groups $\mathrm{H}^2(X, \mu_n)$ can be defined relatively in families, and their tangent spaces can be understood in terms of de Rham cohomology. Moreover, it turns out that formal projective deformations of such pairs (X, α') algebraize, and furthermore the class α' can be used to define twisted Chern characters in families. Our approach to the deformation theory of twisted Fourier–Mukai equivalences is based on the techniques of Lieblich and Olsson [2015], which we, in particular, extend to the twisted setting.

Let (X, α) and (Y, β) be twisted K3 surfaces over an algebraically closed field k of characteristic $p > 0$. Let $\Phi_P : D^b(X, \alpha) \cong D^b(Y, \beta)$ be a Fourier–Mukai equivalence induced by a complex $P \in D^b(X \times Y, -\alpha \boxplus \beta)$.

Definition 5.1. The equivalence $\Phi_P : D^b(X, \alpha) \xrightarrow{\sim} D^b(Y, \beta)$ is *filtered* if there exist preimages $\alpha' \in \mathrm{H}^2(X, \mu_n)$ of α and $\beta' \in \mathrm{H}^2(Y, \mu_m)$ of β such that the cohomological transform

$$\Phi_{v^{-\alpha' \boxplus \beta'}(P)} : \widetilde{N}(X)_{\mathbb{Q}} \xrightarrow{\sim} \widetilde{N}(Y)_{\mathbb{Q}}$$

sends $(0, 0, 1)$ to $(0, 0, 1)$.

Note that the condition for being filtered does not depend on the choices of α' and β', and thus is an intrinsic property of Φ_P. We consider the deformation functor $\mathrm{Def}_{(X,\alpha')}$, whose objects over an Artinian local W-algebra A are isomorphism classes of pairs (X_A, α'_A) where X_A is a flat scheme over Spec A such that $X_A \otimes k \cong X$, and $\alpha'_A \in \mathrm{H}^2(X_A, \mu_n)$ is a cohomology class such that $\alpha'_A|_X = \alpha'$ (see [Bragg 2023, Definition 1.1]).

Proposition 5.2. *Suppose that Φ_P is filtered. Given a preimage $\alpha' \in \mathrm{H}^2(X, \mu_n)$ of α, there is a canonically induced preimage $\beta' \in \mathrm{H}^2(Y, \mu_n)$ of β and a morphism*

$$\delta_P : \mathrm{Def}_{(Y,\beta')} \to \mathrm{Def}_{(X,\alpha')}$$

of deformation functors over W (depending on P and α').

Proof. Let $\mathscr{X} \to X$ and $\mathscr{Y} \to Y$ be $\boldsymbol{G}_m$-gerbes representing α and β. The chosen preimage α' corresponds to an n-twisted invertible sheaf $\mathscr{L}$ on $\mathscr{X}$. Using Proposition 3.5, we find a complex of twisted sheaves $\mathscr{E}$ on $\mathscr{X}$ with rank n and $\det \mathscr{E} \cong \mathscr{L}$. Using the assumption that Φ_P is filtered, we see that $\Phi_P(\mathscr{E})$ is a complex of twisted sheaves on $\mathscr{Y}$ of rank n. Thus, its determinant $\mathscr{N} = \det(\Phi_P(\mathscr{E}))$ is an invertible n-twisted sheaf on $\mathscr{Y}$. Note that this implies $n\beta = 0$. We let $\beta' \in \mathrm{H}^2(Y, \mu_n)$ be the preimage of β corresponding to $\mathscr{N}$. Note that the class β' does not depend on our choice of $\mathscr{E}$.

Let $\mathscr{X}' \to X$ and $\mathscr{Y}' \to Y$ be μ_n-gerbes corresponding to α' and β'. Suppose given an Artinian local W-algebra A and a deformation of (Y, β') over A. Up to isomorphism, this is the same as giving a pair $(\mathscr{Y}'_A, \varphi)$, where $\mathscr{Y}'_A$ is a μ_n-gerbe equipped with a flat proper map to $\operatorname{Spec} A$ and $\varphi : \mathscr{Y}'_A \otimes k \cong \mathscr{Y}'$ is an isomorphism of gerbes. We let $\mathscr{D}_{\mathscr{Y}'_A/A}$ be the stack of relatively perfect universally glueable simple $\mathscr{Y}'_A$-twisted complexes over $\operatorname{Spec} A$ with twisted Mukai vector $(0, 0, 1)$ (see [Lieblich and Olsson 2015, Section 5]). We let P' be the pullback of P along the product of the maps $\mathscr{X}' \subset \mathscr{X}$ and $\mathscr{Y}' \subset \mathscr{Y}$. Because Φ_P is filtered, the complex P' induces a map $\mathscr{X}' \to \mathscr{D}_{\mathscr{Y}'_A/A} \otimes k$. By reasoning identical to [Lieblich and Olsson 2015, Lemma 5.5], this map is an open immersion. The image of $\mathscr{X}'$ is contained in the smooth locus of the morphism $\pi \otimes k$, so there is a unique open substack $\mathscr{X}'_A \subset \mathscr{D}_{\mathscr{Y}'_A/A}$ which is flat and proper over $\operatorname{Spec} A$ whose restriction to the closed fiber is isomorphic to $\mathscr{X}'$. Via this isomorphism, the stack $\mathscr{X}'_A$ has a canonical structure of μ_n-gerbe. Thus, given a deformation of (Y, β') over A, we have produced (using the complex P) a deformation of (X, α') over A. This defines a morphism $\mathrm{Def}_{(Y,\beta')} \to \mathrm{Def}_{(X,\alpha')}$. □

We now assume that Φ_P is a filtered Fourier–Mukai equivalence. We fix a preimage $\alpha' \in \mathrm{H}^2(X, \mu_n)$ of α. Let $\beta' \in \mathrm{H}^2(Y, \mu_n)$, and let

$$\delta_P : \mathrm{Def}_{(Y,\beta')} \to \mathrm{Def}_{(X,\alpha')} \tag{34}$$

be the preimage and morphism produced by Proposition 5.2. We continue the notation introduced above, so that $\pi_{\mathscr{X}} : \mathscr{X} \to X$ and $\pi_{\mathscr{Y}} : \mathscr{Y} \to Y$ are $\boldsymbol{G}_m$-gerbes corresponding to α and β, $\mathscr{X}'$ and $\mathscr{Y}'$ are μ_n-gerbes corresponding to α' and β', $\mathscr{L}$ and $\mathscr{N}$ are the corresponding n-fold twisted invertible sheaves on $\mathscr{X}'$ and $\mathscr{Y}'$, and P' is the restriction of P to $\mathscr{X}' \times \mathscr{Y}'$. Let $\boldsymbol{B}_\alpha$ and $\boldsymbol{B}_\beta$ be mixed B-field lifts of α and β such that $n\boldsymbol{B}_\alpha$ and $n\boldsymbol{B}_\beta$ are integral and such that $n\boldsymbol{B}_\alpha \pmod n$ equals α' and $n\boldsymbol{B}_\beta \pmod n$ equals β'. Write Φ for the cohomological transform

$$\Phi = \Phi_{v^{-\alpha' \boxplus \beta'}(P)} : \widetilde{N}(X)_{\boldsymbol{Q}} \to \widetilde{N}(Y)_{\boldsymbol{Q}}.$$

Lemma 5.3. *The transform Φ satisfies $\Phi(0, 0, 1) = (0, 0, 1)$ and $\Phi(1, 0, 0) = (1, 0, 0)$, and restricts to an isometry $N(X) \xrightarrow{\sim} N(Y)$ of integral Néron–Severi lattices.*

Proof. We are assuming that Φ_P is filtered, so we have $\Phi(0, 0, 1) = (0, 0, 1)$. Consider a complex $\mathscr{E}$ of twisted sheaves on $\mathscr{X}$ with rank n and $\det \mathscr{E} \cong \mathscr{L}$. It follows immediately from the definition of the twisted Chern character that $v^{\alpha'}(\mathscr{E}) = (n, 0, s)$ for some integer s. Moreover, we see that the vector

$$\Phi(n, 0, s) = \Phi(v^{\alpha'}(\mathscr{E})) = v^{\beta'}(\Phi_P(\mathscr{E}))$$

has trivial second component. As Φ is an isometry, we conclude that $\Phi(n, 0, s) = (n, 0, s)$. It follows that $\Phi(1, 0, 0) = (1, 0, 0)$, and that Φ restricts to an isometry on the rational Néron–Severi lattices.

We now prove that Φ in fact restricts to an isometry between the integral Néron–Severi lattices. Consider an invertible sheaf L on X. The complex $\Phi_P(\mathscr{E} \otimes \pi^* L)$ of twisted sheaves on $\mathscr{Y}$ has rank n. Let $\mathscr{M}$ be its determinant. Using the formula (29), we see that the pushforward of the (0-twisted) invertible sheaf $\mathscr{M} \otimes \mathscr{N}^\vee$ to Y has cohomology class $\Phi(L)$. In particular, $\Phi(L)$ is in $N(Y)$. □

The following result is our twisted analog of [Lieblich and Olsson 2015, Proposition 6.3].

Proposition 5.4. *The morphism δ_P (34) is an isomorphism, and furthermore has the following properties*:

(1) *For any class $L \in \mathrm{Pic}(X)$, the map δ restricts to an isomorphism*

$$\mathrm{Def}_{(Y,\beta',\Phi(L))} \cong \mathrm{Def}_{(X,\alpha',L)} .$$

(2) *For any augmented Artinian W-algebra A and any lift (X_A, α'_A) of (X, α') over A, there exists a perfect complex $P_A \in D^b(X_A \times_A Y_A, -\alpha_A \boxplus \beta_A)$ lifting P, where $(Y_A, \beta'_A) = \delta^{-1}(X_A, \alpha'_A)$ and α_A and β_A are the Brauer classes associated to α'_A and β'_A.*

Proof. To see that μ_P is an isomorphism, consider the same construction applied to the kernel $Q = P^\vee$ of the inverse Fourier–Mukai transform and the preimage β' of β, which yields a map

$$\mu_Q : \mathrm{Def}_{(X,\alpha')} \xrightarrow{\sim} \mathrm{Def}_{(Y,\beta')} .$$

We claim that μ_P and μ_Q are inverses. This may be verified exactly as in [Lieblich and Olsson 2015, Proposition 6.3]. To see claim (2), note that the restriction along the open immersion

$$\mathscr{X}'_A \times \mathscr{Y}'_A \subset \mathscr{D}_{\mathscr{Y}'_A/A} \times \mathscr{Y}'_A$$

of the universal complex lifts P'. To see (1), suppose that the deformation (X_A, α'_A) is contained in the subfunctor $\mathrm{Def}_{(X,\alpha',L)}$. There is then an invertible sheaf L_A on X_A deforming L. Let $\mathscr{E}_A$ be a relatively perfect complex of α_A-twisted sheaves on X_A with rank n and trivial determinant. Let $\pi_A : \mathscr{X}_A \to X_A$ be the coarse space map. The determinant of the complex $\Phi_{P_A}(\mathscr{E}_A \otimes \pi_A^* L_A)$ is a 0-fold twisted sheaf on $\mathscr{Y}_A$, so its pushforward to Y_A is an invertible sheaf. Moreover, this sheaf has class lifting $\Phi(L)$. □

Definition 5.5. We say that a filtered Fourier–Mukai equivalence Φ_P is *polarized* if there exists B-field lifts $\boldsymbol{B}$ and $\boldsymbol{B}'$ of α and β such that the isometry $\Phi : \mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ (see Lemma 5.3) sends the ample cone C_X of X to the ample cone C_Y of Y.

One checks that the condition to be polarized is independent of the choice of B-field lifts, in the sense that it is verified for one choice of lifts if and only if it is verified for all choices of lifts.

We now prove our main lifting results. By results in [Bragg 2023], the twisted K3 surface (X, α) can be lifted to characteristic 0. Moreover, we can also compatibly lift the preimage α' of α. As a consequence, Proposition 5.4 shows that given such a lift there is an induced formal lift of (Y, β), together with a lift of β' and of the complex P inducing the equivalence. Under the assumption that Φ_P is polarized, we can even produce a (nonformal) lift. We make this precise in the following result.

Theorem 5.6. *Suppose that Φ_P is a filtered polarized Fourier–Mukai equivalence. Let L be an ample line bundle on X. Suppose we are given a complete DVR V with residue field k and a lift (X_V, α'_V, L_V) of (X, α', L) over V. There exists an ample line bundle M on Y, a lift (Y_V, β'_V, M_V) of (Y, β', M) over V, and a perfect complex $P_V \in D^b(X_V \times_V Y_V, -\alpha_V \boxplus \beta_V)$ (where β_V is the image of β'_V in the Brauer group) which induces a Fourier–Mukai equivalence and whose restriction to $D^b(X \times Y, -\alpha \boxplus \beta)$ is quasi-isomorphic to P.*

Proof. Let M be the line bundle on Y corresponding to $\Phi(L)$. By Proposition 5.4, we find compatible deformations $(Y_{V_n}, \beta'_{V_n}, M_{V_n})$ of (Y, β', M) over $V_n = V/\mathfrak{m}^{n+1}$ for each $n \geq 0$, together with compatible perfect complexes $P_{V_n} \in D^b(X_{V_n} \times_{V_n} Y_{V_n}, -\alpha_{V_n} \boxplus \beta_{V_n})$ deforming P, where β_{V_n} is the image of β'_{V_n} in the Brauer group. As Φ_P is polarized, M is ample, so by the Grothendieck existence theorem, there exists a scheme (Y_V, M_V) over V restricting to the (Y_{V_n}, β'_{V_n}). By [Bragg 2023, Proposition 1.4] there exists a class $\beta'_V \in \mathrm{H}^2(Y_V, \mu_n)$ restricting to the β'_{V_n}. Finally, by the Grothendieck existence theorem for perfect complexes [Lieblich 2006, Proposition 3.6.1], there is a perfect complex $P_V \in D^b(X_V \times_V Y_V, -\alpha_V \boxplus \beta_V)$ whose restriction to V_n is quasi-isomorphic to P_{V_n} for each n. Moreover, arguing as in the proof of Theorem 6.1 of [Lieblich and Olsson 2015], we see that the complex P_V induces a Fourier–Mukai equivalence. □

Definition 5.7. Let $\mathcal{X}$ be a K3 surface over a local ring and X be its special fiber. We say that $\mathcal{X}$ is a *perfect lifting* of X if the restriction map $\mathrm{Pic}(\mathcal{X}) \to \mathrm{Pic}(X)$ is an isomorphism.

We remark that if $\mathcal{X}$ as above is over a DVR, the ample and the big and nef cones of the generic fiber are canonically identified with those of the special fiber.

Theorem 5.8. *Let (X, α) and (Y, β) be twisted* K3 *surfaces over k. Let $\Phi_P : D^b(X, \alpha) \xrightarrow{\sim} D^b(Y, \beta)$ be a Fourier–Mukai equivalence. There exists*

(a) *an autoequivalence Φ' of $D^b(Y, \beta)$ which is a composition of spherical twists about (-2)-curves,*

(b) *a DVR V whose fraction field has characteristic 0 and with residue field k,*

(c) *projective lifts (X_V, α_V) and (Y_V, β_V) of (X, α) and (Y, β) over V, and*

(d) *a perfect complex $R_V \in D^b(X_V \times_V Y_V, -\alpha_V \boxplus \beta_V)$ which induces a Fourier–Mukai equivalence and whose restriction to $X \times Y$ is quasi-isomorphic to the kernel R of the equivalence $\Phi' \circ \Phi_P$.*

Moreover, if X and Y have finite height, we may choose the above data so that Φ' is the identity and X_V and Y_V are perfect liftings.

Proof. Choose a preimage $\alpha' \in \mathrm{H}^2(X, \mu_n)$ of α. Given a choice of preimage of β, we obtain an isometry

$$\Phi : \widetilde{N}(X)_Q \xrightarrow{\sim} \widetilde{N}(Y)_Q.$$

Consider the class $v = (\Phi)^{-1}(0, 0, 1) \in \widetilde{\mathrm{NS}}(X)_Q$. Note that this class does not depend on the choice of preimage of β. By [Bragg 2023, Theorem 7.3], we may find a DVR V of characteristic 0 and residue field k and a polarized lift (X_V, α'_V) of (X, α') over V over which the class v extends. Let α_V be the image of α'_V in the Brauer group of X_V. Let $\mathscr{M}_V = \mathscr{M}_{(X_V, \alpha_V)}(v)$ be the relative moduli space of H-stable

α_V-twisted sheaves on $X_V \to \operatorname{Spec} V$ with twisted Mukai vector $v^{\alpha'_V} = v$, where H is a v-generic polarization. Let M_V be the coarse space of $\mathscr{M}_V$. The morphism $M_V \to \operatorname{Spec} V$ is a projective family of K3 surfaces, and there is a class $\gamma_V \in \operatorname{Br}(M_V)$ such that the universal complex Q_V induces an equivalence

$$\Phi_{Q_V} : D^b(M_V, \gamma_V) \xrightarrow{\sim} D^b(X_V, \alpha_V).$$

Let $\gamma \in \operatorname{Br}(M)$ be the restriction of γ_V to M, and let Q be the restriction of Q_V. The Fourier–Mukai equivalence

$$\Phi_P \circ \Phi_Q : D^b(M, \gamma) \xrightarrow{\sim} D^b(Y, \beta)$$

is filtered. As in [Lieblich and Olsson 2015, Lemma 6.2], we may find an autoequivalence Φ' as in the statement of the theorem so that $\Phi' \circ \Phi_P \circ \Phi_Q$ is both filtered and polarized. Let R denote its kernel. Choose a preimage $\gamma'_V \in \mathrm{H}^2(M_V, \mu_m)$ of γ_V, and write γ' for the restriction of γ'_V to M. Let β' be the corresponding lift of β produced by Proposition 5.2. By Theorem 5.6, there is a lift (Y_V, β'_V) of (Y, β') and R_V of R over V, corresponding to the lift (M_V, γ'_V) of (M, γ'). Consider the Fourier–Mukai equivalence

$$\Phi_{R_V} \circ \Phi_{Q_V}^{-1} : D^b(X_V, \alpha_V) \to D^b(Y_V, \beta_V).$$

This equivalence restricts over k to $\Phi' \circ \Phi_P$. By the uniqueness of the kernel, we conclude that R_V restricts to the kernel of the equivalence $\Phi' \circ \Phi_P$, as claimed.

Suppose that X and Y have finite height. We modify the above as follows. Choose α' so that p does not divide $n/\operatorname{ord}(\alpha)$. By [Bragg 2023, Theorem 7.3], we may choose the lift X_V so that the restriction map $\operatorname{Pic}(X_V) \to \operatorname{Pic}(X)$ is an isomorphism. It follows that $\operatorname{Pic}(Y_V) \to \operatorname{Pic}(Y)$ is also an isomorphism. In particular, every (-2)-class in $\operatorname{Pic}(Y)$ extends to Y_V. We now compose Φ_{R_V} with an autoequivalence of $D^b(Y, \beta)$ lifting the inverse of Φ'. The kernel of the resulting equivalence then restricts to P, as desired. □

6. Existence theorems

The goal of this section is to construct isogenies with prescribed action on cohomology. In particular, we will prove Theorems 1.2 and 1.4.

6A. ***Construction of derived isogenies.*** We begin with Theorem 1.2.

Let R be an integral domain whose field of fractions is of characteristic 0 (we have in mind $R = \mathbf{Z}_\ell$ or $R = W$). Set $R_{\mathbf{Q}} := R \otimes_{\mathbf{Z}} \mathbf{Q}$. Let M be a quadratic lattice such that $2^{-1}m^2 \in R$ for every $m \in M$.

Given an element $b \in M$ such that $\langle b, b \rangle \neq 0$, the reflection in b is the isometry $s_b : M_{\mathbf{Q}} \to M_{\mathbf{Q}}$ defined by

$$s_b(x) = x - \frac{2\langle x, b\rangle}{\langle b, b\rangle} b.$$

Let $\widetilde{H}$ be a lattice of the form $R \oplus M \oplus R$ equipped with the Mukai pairing, i.e.,

$$\langle (r, m, s), (r', m', s') \rangle = \langle m, m' \rangle - rs' - r's,$$

and a multiplicative structure given by

$$(r, m, s) \cdot (r', m', s') = (rr', rm' + r'm, rs' + r's + \langle m, m' \rangle).$$

Lemma 6.1. *Let $b \in M$ be a primitive element such that $\langle b, b\rangle \neq 0$. Set $n := \frac{1}{2}b^2$ and $B := b/n \in M_{\boldsymbol{Q}} := M \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$. Let $B' \in M_{\boldsymbol{Q}}$ be another element. If $\Phi : \widetilde{H}_{\boldsymbol{Q}} \xrightarrow{\sim} \widetilde{H}_{\boldsymbol{Q}}$ satisfies*

(a) $\Phi(1, 0, 0) = (0, 0, 1/n)$ *and* $\Phi(0, 0, 1) = (n, 0, 0)$, *and*

(b) $e^B \Phi e^{-B'}$ *is R-integral* (*i.e., restricts to an isometry* $\widetilde{H} \xrightarrow{\sim} \widetilde{H}$),

then $\varphi(M) = s_b(M)$, where $s_b \in \operatorname{Aut}(M_{\boldsymbol{Q}})$ is the reflection in b and φ is the restriction of Φ to $M_{\boldsymbol{Q}}$.

Proof. We extend $r_b := -s_b$ to an isometry $\Psi : \widetilde{H}_{\boldsymbol{Q}} \xrightarrow{\sim} \widetilde{H}_{\boldsymbol{Q}}$ by requiring that Ψ satisfies (a). It is straightforward to verify that $e^B \Psi e^{-B}$ is R-integral:

$$e^B \Psi e^{-B}(0, 0, 1) = e^B \Psi(0, 0, 1) = e^B(n, 0, 0) = (n, b, 1),$$
$$e^B \Psi e^{-B}(1, 0, 0) = e^B \Psi(1, -B, 1/n) = e^B(1, -B, 1/n) = (1, 0, 0),$$
$$e^B \Psi e^{-B}(0, m, 0) = e^B \Psi(0, m, -\langle B, m\rangle) = e^B(n\langle -B, m\rangle, r_b(m), 0) = (\langle -b, m\rangle, -m, 0).$$

We now consider the composition $(e^B \Psi e^{-B})^{-1} \circ (e^B \Phi e^{-B'}) = e^B(\Psi^{-1} \circ \Phi)e^{-B'}$, which has to be R-integral. Direct computation shows

$$\begin{aligned} e^B(\Psi^{-1} \circ \Phi)e^{-B'}(0, m, 0) &= e^B(\Psi^{-1} \circ \Phi)(0, m, -\langle B', m\rangle) \\ &= e^B\big(0, r_b^{-1}(\varphi(m)), -\langle B', m\rangle\big) \\ &= \big(0, r_b^{-1}(\varphi(m)), \langle B, \varphi(m)\rangle - \langle B', m\rangle\big). \end{aligned}$$

As $e^B(\Psi^{-1} \circ \Phi)e^{-B'}$ is R-integral, we deduce that $r_b^{-1} \circ \varphi$ is R-integral, and so $r_b(M) = \varphi(M)$. □

The next result is the key geometric input for the proof of Theorem 1.2. Let k be an algebraically closed field of characteristic p. Given a K3 surface X over k and a class $b \in \mathrm{H}^2(X)$, we let $s_b : \mathrm{H}^2(X)_{\boldsymbol{Q}} \to \mathrm{H}^2(X)_{\boldsymbol{Q}}$ denote the isometry $s_{b_{p_1}} \times s_{b_{p_2}} \times \cdots$. We say that a class $b \in \mathrm{H}^2(X)$ is *primitive* if $nb' = b$ for an integer n and $b' \in \mathrm{H}^2(X)$ implies $n = \pm 1$.

Proposition 6.2. *Let X be a* K3 *surface over k. Let $b \in \mathrm{H}^2(X)$ be a primitive class such that $n := \frac{1}{2}b^2$ is an integer*[1] *and such that b/n is a mixed B-field. There exists a* K3 *surface X' together with a primitive derived isogeny $f : \mathfrak{h}^2(X') \to \mathfrak{h}^2(X)$ such that $f_*(\mathrm{H}^2(X')) = s_b(\mathrm{H}^2(X))$ in $\mathrm{H}^2(X)_{\boldsymbol{Q}}$.*

Proof. Set $\boldsymbol{B} := b/n$ and let $\alpha = \alpha_{\boldsymbol{B}}$ be the Brauer class defined by $\boldsymbol{B}$. Let X' be the moduli space of stable α-twisted sheaves with Mukai vector $v^{\boldsymbol{B}} = (n, 0, 0)$ (where stability is taken with respect to a sufficiently generic polarization). As b is primitive, the class $(n, 0, 0)$ is primitive in $\widetilde{N}(X, \boldsymbol{B})$. Thus, X' is a K3 surface, and there exists a Brauer class $\alpha' \in \operatorname{Br}(X')$ together with an equivalence $\Phi_{\mathscr{E}} : D^b(X', \alpha') \xrightarrow{\sim} D^b(X, \alpha)$. Choose a mixed B-field lift $\boldsymbol{B}'$ of α'. Then the cohomological action $\Phi : \mathrm{H}(X')_{\boldsymbol{Q}} \xrightarrow{\sim} \mathrm{H}(X)_{\boldsymbol{Q}}$ of the algebraic cycle $v^{-\boldsymbol{B}' \boxplus \boldsymbol{B}}(\mathscr{E})$ sends $(0, 0, 1)$ to $(n, 0, 0)$.

Since Φ is an isometry, the vector $u = (\Phi)^{-1}(0, 0, 1/n)$ satisfies $u^2 = 0$ and $\langle u, (0, 0, 1)\rangle = -1$. Therefore, u is necessarily of the form $e^\delta = \big(1, \delta, \frac{1}{2}\delta^2\big)$ for some $\delta \in \mathrm{H}^2_{\text{ét}}(X')_{\boldsymbol{Q}}$. As $(0, 0, 1)$ is an algebraic class, and Φ is induced by an algebraic cycle, we have $\delta \in \mathrm{NS}(X')_{\boldsymbol{Q}}$. After replacing $\boldsymbol{B}'$ by $\boldsymbol{B}' + \delta$,

[1] That is, n is in the image of the diagonal embedding $\boldsymbol{Z} \hookrightarrow W \times \widehat{\boldsymbol{Z}}^p$.

we may assume that Φ sends $(1,0,0)$ to $(0,0,1/n)$. Now we may apply Lemma 6.1 to the ℓ-adic part for each $\ell \neq p$ and to the crystalline part. We conclude that the degree 0 part of the correspondence $v^{-\boldsymbol{B}'\boxplus\boldsymbol{B}}(\mathscr{E})$ sends $\mathrm{H}^2(X')$ to $s_b(\mathrm{H}^2(X))$. □

6B. ***Cartan–Dieudonné theorems and strong approximation.*** To apply Proposition 6.2 towards the proof of Theorem 1.2, we need to show that the reflections s_b about classes $b \in \mathrm{H}^2(X)$ satisfying the conditions of Proposition 6.2 generate a sufficiently large subgroup of isometries of $\mathrm{H}^2(X)_{\boldsymbol{Q}}$. We need two lattice-theoretic inputs. The first is the following generalized Cartan–Dieudonné theorem [Klingenberg 1961, Theorem 2].

Theorem 6.3. *Let R be a local ring with residue characteristic $\neq 2$ and let L be a unimodular quadratic lattice over R. The group $\mathrm{O}(L)$ is generated by the set of reflections s_b, where b ranges over the elements of L such that $b^2 \in R^\times$.*

We also will use the following consequence of the strong approximation theorem. Recall that $\boldsymbol{U}$ denotes the hyperbolic plane, which is a $\boldsymbol{Z}$–lattice of rank 2.

Lemma 6.4. *Let L be a nondegenerate indefinite quadratic lattice over $\boldsymbol{Z}$ of rank ≥ 3. If q is a prime such that $L \otimes \boldsymbol{Z}_q$ contains a copy of $\boldsymbol{U} \otimes \boldsymbol{Z}_q$ as an orthogonal direct summand, then the double quotient*

$$\mathrm{O}(L \otimes \boldsymbol{Q}) \backslash \mathrm{O}(L \otimes \boldsymbol{Q}_q) / \mathrm{O}(L \otimes \boldsymbol{Z}_q)$$

is a singleton.

Proof. This is a slight variant of [Yang 2023, Lemma 2.1.12], whose proof follows from that of [Ogus 1979, Lemma 7.7]. We briefly summarize the argument: Let $\mathsf{K} \subseteq \mathrm{Spin}(L \otimes \boldsymbol{Q}_q)$ be the preimage of $\mathrm{SO}(L \otimes \boldsymbol{Z}_q)$ under the natural map $\mathrm{ad} : \mathrm{Spin} \to \mathrm{SO}$. Using the fact that $L \otimes \boldsymbol{Z}_q$ contains $\boldsymbol{U} \otimes \boldsymbol{Z}_q$ as an orthogonal direct summand, we show that the maps

$$\begin{aligned}\mathrm{Spin}(L \otimes \boldsymbol{Q}) \backslash \mathrm{Spin}(L \otimes \boldsymbol{Q}_q)/\mathsf{K} &\to \mathrm{SO}(L \otimes \boldsymbol{Q}) \backslash \mathrm{SO}(L \otimes \boldsymbol{Q}_q)/\mathrm{SO}(L \otimes \boldsymbol{Z}_q) \\ &\to \mathrm{O}(L \otimes \boldsymbol{Q}) \backslash \mathrm{O}(L \otimes \boldsymbol{Q}_q) / \mathrm{O}(L \otimes \boldsymbol{Z}_q)\end{aligned}$$

are both surjections. Now we conclude using the fact that the first double quotient is a singleton by the strong approximation theorem. □

We now return to the setting of a K3 surface X over an algebraically closed field k of characteristic p.

Lemma 6.5. *Let X be a* K3 *surface over k, and assume that $p \geq 5$. There exists a $\boldsymbol{Z}$–lattice L of rank* 22 *and a primitive indefinite sublattice $L' \subset L$ such that*

(a) *for each $\ell \neq p$, there exists an isometry $L \otimes \boldsymbol{Z}_\ell \cong \mathrm{H}^2(X, \boldsymbol{Z}_\ell)$,*

(b) *there exists an isometry $L' \otimes \boldsymbol{Z}_p \cong T(X) := \mathrm{H}^2(X/W)^{\varphi=1}$, and*

(c) *the double quotients*

$$\mathrm{O}(L \otimes \boldsymbol{Z}_{(p)}) \backslash \mathrm{O}(L \otimes \boldsymbol{A}_f^p) / \mathrm{O}(L \otimes \widehat{\boldsymbol{Z}}^p) \quad \text{and} \quad \mathrm{O}(L' \otimes \boldsymbol{Q}) \backslash \mathrm{O}(L' \otimes \boldsymbol{Q}_p) / \mathrm{O}(L' \otimes \boldsymbol{Z}_p)$$

are both singletons.

Proof. Suppose that X has finite height h. We take $L = \Lambda$ to be the K3 lattice. As L contains a copy of $\boldsymbol{U}$ as an orthogonal direct summand, we may apply [Yang 2023, Lemma 2.1.12] to conclude that the indicated double quotient is a singleton. We will now produce L'. Suppose that $h \leq 9$. By [Ito 2019, Theorem 6.4] (which requires $p \geq 5$), there exists a K3 surface Y over $\overline{\boldsymbol{F}}_p$ such that $h(Y) = h$ and $\rho(Y) = 22 - 2h$. Set $L' = \operatorname{Pic}(Y)$. The existence of a perfect lifting of Y to characteristic zero shows that L' admits a primitive embedding into $L = \Lambda$. Condition (a) is immediate. The embedding $L' \to \mathrm{H}^2(Y/W)$ induces an isomorphism $L' \otimes \boldsymbol{Z}_p \cong T(Y) = T(X)$, giving (b). It remains to check that the double quotient involving L' is a singleton. The pairing on H_1 is perfect, and the inclusion $T(X) \subset \mathrm{H}_1$ induces an isomorphism $T(X) \otimes_{\boldsymbol{Z}_p} W \cong \mathrm{H}_1$, so the discriminant of the pairing on $T(X) \cong L' \otimes \boldsymbol{Z}_p$ is a p-adic unit. As L' has rank ≥ 4, the classification of p-adic lattices [Ogus 1979, Lemma 7.5] implies that $L' \otimes \boldsymbol{Z}_p$ contains a copy of $\boldsymbol{U} \otimes \boldsymbol{Z}_p$ as an orthogonal direct summand. By the Hodge index theorem, L' is indefinite. We conclude using Lemma 6.4. Suppose $h = 10$. We take $L' = \boldsymbol{U}$. This is certainly a primitive sublattice of $L = \Lambda$, and the double quotient involving L' is a singleton. It remains to check that $\boldsymbol{U} \otimes \boldsymbol{Z}_p \cong T(X)$. As explained by Ogus [1983, Remark 1.5], the discriminant of the pairing on H_1 is -1. The same is then true for $T(X)$, because $T(X) \otimes_{\boldsymbol{Z}_p} W \cong \mathrm{H}_1$. By the classification of quadratic lattices over $\boldsymbol{Z}_p$, we conclude that $\boldsymbol{U} \otimes \boldsymbol{Z}_p \cong T(X)$.

Suppose that X is supersingular. Let $L' = L = \Lambda_{\sigma_0}$ be the supersingular K3 lattice of Artin invariant $\sigma_0 = \sigma_0(X)$. The discriminant of the pairing on Λ_{σ_0} is equal to $-p^{2\sigma_0}$, which is an ℓ-adic unit for all $\ell \neq p$, and so (a) holds. Condition (b) is immediate. Finally, by [Ogus 1979, Lemma 7.7], condition (c) holds. □

The following results could be phrased purely in terms of (semi)linear algebra, but for clarity we will maintain the geometric notation.

We recall that $\mathrm{O}(\mathrm{H}^2(X, \boldsymbol{A}_f^p))$ is the subgroup of $\prod_{\ell \neq p} \mathrm{O}(\mathrm{H}^2(X, \boldsymbol{Z}_\ell))$ consisting of those tuples Θ such that Θ_ℓ is ℓ–integral for all but finitely many ℓ (here, we say that Θ_ℓ is ℓ-integral if $\Theta_\ell(\mathrm{H}^2(X, \boldsymbol{Z}_\ell)) = \mathrm{H}^2(X, \boldsymbol{Z}_\ell)$). We let $\mathrm{O}_\Phi(\mathrm{H}^2(X/K))$ be the group of automorphisms of $\mathrm{H}^2(X/K)$ which are isometries with respect to the pairing and which commute with Φ. We set $\mathrm{O}_\Phi(\mathrm{H}^2(X)) = \mathrm{O}_\Phi(\mathrm{H}^2(X/K)) \times \mathrm{O}(\mathrm{H}^2(X, \boldsymbol{A}_f^p))$.

Remark 6.6. Giving an isometric embedding ι as in the statement of Theorem 1.2 is equivalent to giving an isometry $\iota_p : \Lambda \otimes W \hookrightarrow \mathrm{H}^2(X/K)$ of W-modules and for each prime $\ell \neq p$ an isometry $\iota_\ell : \Lambda \otimes \boldsymbol{Z}_\ell \hookrightarrow \mathrm{H}^2(X, \boldsymbol{Q}_\ell)$ of $\boldsymbol{Q}_\ell$-modules such that for all but finitely many ℓ we have $\operatorname{im}(\iota_\ell) = \mathrm{H}^2(X, \boldsymbol{Z}_\ell)$. A similar description holds for the isometric embedding in the statement of Theorem 6.13.

Lemma 6.7. *Suppose that $p \geq 5$. If $\Theta^p \in \mathrm{O}(\mathrm{H}^2(X, \boldsymbol{A}_f^p))$ is an isometry, then there exists a sequence $b_1, \ldots, b_r$ of primitive elements of $\mathrm{H}^2(X)$ such that*

(1) *for each i, $n_i := \frac{1}{2} b_i^2$ is an integer which is not divisible by p, and*

(2) *the isometry $s := s_{b_1} \circ \cdots \circ s_{b_r}$ satisfies $s(\mathrm{H}^2(X, \widehat{\boldsymbol{Z}}^p)) = \Theta^p(\mathrm{H}^2(X, \widehat{\boldsymbol{Z}}^p))$.*

Proof. Let L be a lattice as in Lemma 6.5, and choose an identification $L \otimes \widehat{\boldsymbol{Z}}^p = \mathrm{H}^2(X, \widehat{\boldsymbol{Z}}^p)$. By Lemma 6.4,

$$\mathrm{O}(L \otimes \boldsymbol{Z}_{(p)}) \backslash \mathrm{O}(L \otimes \boldsymbol{A}_f^p) / \mathrm{O}(L \otimes \widehat{\boldsymbol{Z}}^p)$$

is a singleton. Hence, there exists an isometry $\Psi \in \mathrm{O}(L \otimes \mathbf{Z}_{(p)})$ such that $\Psi(L) \otimes \widehat{\mathbf{Z}}^p = \Theta^p(L \otimes \widehat{\mathbf{Z}}^p)$. We apply Theorem 6.3 with $R = \mathbf{Z}_{(p)}$ to produce a sequence $b_1, \ldots, b_r$ of elements of $L \otimes \mathbf{Z}_{(p)}$ such that $b_i^2 \in \mathbf{Z}_{(p)}^\times$ for each i and $\Psi = s_{b_1} \circ \cdots \circ s_{b_r}$. For each i, we may write $b_i = v/m$ for some primitive $v \in L$ and an integer m which is coprime to p. Note that the integer $\frac{1}{2}v^2 = \frac{1}{2}m^2 b_i^2$ is in $\mathbf{Z}_{(p)}^\times$, and hence is not divisible by p. Moreover, we have $s_{b_i} = s_v$. So, by replacing each b_i with the corresponding v, we may arrange that the b_i satisfy (1). Condition (2) holds by construction. □

Lemma 6.8. *Suppose that $p \geq 5$. Let $\Theta_p \in \mathrm{O}_\Phi(\mathrm{H}^2(X/K))$ be an isometry which restricts to the identity on $\mathrm{H}^2(X/W)_{<1}$. There exists a sequence $b_1, \ldots, b_r$ of primitive elements of $\mathrm{H}^2(X)$ such that*

(1) *for each i, $n_i := \frac{1}{2}b_i^2$ is an integer and $\varphi(b_i) = b_i$, and*

(2) *the isometry $s := s_{b_1} \circ \cdots \circ s_{b_r}$ satisfies $s(\mathrm{H}^2(X/W)) = \Theta_p(\mathrm{H}^2(X/W))$.*

Proof. Write $\mathrm{H} = \mathrm{H}^2(X/W)$, and consider the Newton–Hodge decomposition $\mathrm{H} = \mathrm{H}_{<1} \oplus \mathrm{H}_1 \oplus \mathrm{H}_{>1}$ of H. The first and third factors are dual, and orthogonal to H_1. Because Θ_p restricts to the identity on $\mathrm{H}_{<1}$, it must also restrict to the identity on $\mathrm{H}_{>1}$, and hence Θ_p restricts to an element of $\mathrm{O}_\Phi(\mathrm{H}_1) = \mathrm{O}(T(X))$. We fix lattices L, L' as in Lemma 6.5 and an identification $L' \otimes \mathbf{Z}_p = T(X)$. By Lemma 6.4, we may find $\Psi \in \mathrm{O}(L' \otimes \mathbf{Q})$ such that $\Psi(L') \otimes \mathbf{Z}_p = \Theta_p|_{T(X)}(L' \otimes \mathbf{Z}_p)$. By the classical Cartan–Dieudonné theorem, we may find a sequence $b_1, \ldots, b_r$ of elements of $L' \otimes \mathbf{Q}$ such that $\Psi = s_p = s_{b_1} \circ \cdots \circ s_{b_r}$. By scaling, we may assume that each b_i is in L' and is primitive. Note that, as $\mathrm{H}_{<1}$ and $\mathrm{H}_{>1}$ are orthogonal to H_1, the reflections s_{b_i} are the identity on $\mathrm{H}_{<1} \oplus \mathrm{H}_{>1}$. If follows that s satisfies condition (2). □

Lemma 6.9. *Suppose that $p \geq 5$. Let $\Theta \in \mathrm{O}_\Phi(\mathrm{H}^2(X)_{\mathbf{Q}})$ be an isometry such that Θ_p restricts to the identity on $\mathrm{H}^2(X/W)_{<1}$. There exists a sequence $b_1, \ldots, b_m$ of primitive elements of $\mathrm{H}^2(X)$ such that*

(1) *for each i, $n_i := \frac{1}{2}b_i^2$ is an integer and b_i/n_i is a mixed B-field, and*

(2) *the isometry $s := s_{b_1} \circ \cdots \circ s_{b_m}$ satisfies $s(\mathrm{H}^2(X)) = \Theta(\mathrm{H}^2(X))$.*

Proof. We first choose elements $b_1, \ldots, b_r \in \mathrm{H}^2(X)$ by applying Lemma 6.8 to Θ_p. We set $s = s_{b_1} \circ \cdots \circ s_{b_r}$. We apply Lemma 6.7 to $(s^{-1} \circ \Theta)^p$ to obtain elements $b'_1, \ldots, b'_t \in \mathrm{H}^2(X)$. Set $s' = s_{b'_1} \circ \cdots \circ s_{b'_t}$. We claim that the sequence $b_1, \ldots, b_r, b'_1, \ldots, b'_t \in \mathrm{H}^2(X)$ satisfies the desired conditions. We check (1). We have that $n_i := \frac{1}{2}b_i^2$ and $n'_i := \frac{1}{2}(b'_i)^2$ are integers. We have that $\varphi((b_i)_p) = (b_i)_p$, so by Proposition 2.7 each $(b_i)_p/n_i$ is a crystalline B-field. It follows that b_i/n_i is a mixed B-field. As n'_i is not divisible by p, $(b'_i)_p/n'_i$ is in $\mathrm{H}^2(X/W)$, so $(b'_i)_p/n'_i$ is a crystalline B-field, and b'_i/n'_i is a mixed B-field. We have shown that (1) holds. To check (2), note that by construction, we have

$$(s \circ s')^p(\mathrm{H}^2(X, \widehat{\mathbf{Z}}^p)) = \Theta^p(\mathrm{H}^2(X, \widehat{\mathbf{Z}}^p)).$$

Furthermore, as p does not divide $\frac{1}{2}(b'_i)^2$, we have $s'_p(\mathrm{H}^2(X/W)) = \mathrm{H}^2(X/W)$, and so

$$(s \circ s')_p(\mathrm{H}^2(X/W)) = s_p(\mathrm{H}^2(X/W)) = \Theta_p(\mathrm{H}^2(X/W)).$$ □

Proof of Theorem 1.2. We prove the "only if" direction first. Suppose that $f : \mathfrak{h}^2(X') \xrightarrow{\sim} \mathfrak{h}^2(X)$ is a primitive derived isogeny. We may choose Brauer classes $\alpha \in \mathrm{Br}(X)$ and $\alpha' \in \mathrm{Br}(X')$, a Fourier–Mukai equivalence $\Phi_P : D^b(X', \alpha') \xrightarrow{\sim} D^b(X, \alpha)$, and crystalline B-field lifts B and B' of α and α' such that the cohomological transform $\Phi_{v^{-B'\boxplus B}}(P) : \widetilde{\mathrm{H}}(X'/K) \to \widetilde{\mathrm{H}}(X/K)$ and the cohomological realization $\mathrm{H}^2(X'/K) \xrightarrow{\sim} \mathrm{H}^2(X/K)$ of f restrict to the same map $T^2(X'/K) \xrightarrow{\sim} T^2(X/K)$, where $T^2(X/K)$ denotes the orthogonal complement to $\mathrm{NS}(X) \otimes K$ in $\mathrm{H}^2(X/K)$ (not to be confused with the Tate module of $\mathrm{H}^2(X/W)$). By Theorem 3.6, $\Phi_{v^{-B'\boxplus B}}(P)$ restricts to an isomorphism $\widetilde{\mathrm{H}}(X'/W, B') \xrightarrow{\sim} \widetilde{\mathrm{H}}(X/W, B)$ of crystals. Thus, by Proposition 2.15, it induces an isomorphism

$$\mathrm{H}^2(X'/W)_{<1} = \widetilde{\mathrm{H}}(X'/W, B')_{<1} \xrightarrow{\sim} \widetilde{\mathrm{H}}(X/W, B)_{<1} = \mathrm{H}^2(X/W)_{<1}.$$

The transcendental part $T^2(X/K)$ contains $\mathrm{H}^2(X/W)_{<1}$, so the cohomological realization of f also maps the slope < 1 part to the slope < 1 part. This gives the result.

We now prove the "if" direction. For each $\ell \neq p$ fix an isometry $\mathrm{H}^2(X, \mathbf{Z}_\ell) \cong \Lambda \otimes \mathbf{Z}_\ell$. Assume first that the K3 crystals H_p and $\mathrm{H}^2(X/W)$ are abstractly isomorphic. This is the case, for instance, if X has finite height. We fix an isomorphism $\mathrm{H}^2(X/W) \cong H_p$ of K3 crystals. Composing with the given embedding ι and tensoring with $\boldsymbol{Q}$, we find an isometry $\Theta \in \mathrm{O}_\Phi(\mathrm{H}^2(X/K)) \times \mathrm{O}(\mathrm{H}^2(X, \widehat{\mathbf{Z}}^p))$ which maps $\mathrm{H}^2_{\text{ét}}(X, \mathbf{Z}_\ell)$ to $\iota_l(\Lambda \otimes \mathbf{Z}_\ell)$ and $\mathrm{H}^2(X/W)$ to $\iota_p(\Lambda \otimes W)$. By Lemma 6.9, we may find a sequence $b_1, \dots, b_m \in \mathrm{H}^2(X)$ of primitive elements such that for every i, $n_i := \frac{1}{2} b_i^2$ is an integer and b_i/n_i is a mixed B-field, and furthermore the isometry $s := s_{b_1} \circ \cdots \circ s_{b_m}$ satisfies $s(\mathrm{H}^2(X)) = \Theta(\mathrm{H}^2(X))$. The result follows by repeatedly applying Proposition 6.2.

We now consider the case when X is supersingular and H_p and $\mathrm{H}^2(X/W)$ are not isomorphic. This can certainly occur: any two supersingular K3 crystals over k of the same rank and discriminant are isogenous, but by results of Ogus [1979], supersingular K3 crystals themselves have nontrivial moduli. We argue as follows. By the global crystalline Torelli theorem [Ogus 1983], there exists a supersingular K3 surface X' such that $\mathrm{H}^2(X'/W)$ is isomorphic as a K3 crystal to H_p. By Theorem 6.11 below, there exists a derived isogeny $\mathfrak{h}^2(X') \xrightarrow{\sim} \mathfrak{h}^2(X)$, which induces an isometry $\mathrm{H}^2(X'/K) \cong \mathrm{H}^2(X/K)$. We are now reduced to the previous case, and we conclude the result. □

Remark 6.10. The only place where the assumption $p \geq 5$ is used in the above proof is in applying the result of Ito [2019, Theorem 6.4]. If in Theorem 1.2, $H_p = \mathrm{H}^2_{\mathrm{cris}}(X/W)$, i.e., Θ_p as above can be taken to be the identity, then the assumption $p > 2$ suffices. In this case, in producing X' we only need to iteratively take moduli of sheaves twisted by Brauer classes of prime-to-p order.

6C. *Existence in the supersingular case.* We make a few remarks specific to the supersingular case. Here, very strong cohomological results are available: there is a global Torelli theorem [Ogus 1979; 1983; Bragg and Lieblich 2018], as well as a derived Torelli theorem [Bragg 2021]. Together, these give a picture which closely parallels the case of complex K3 surfaces. We will show that any two supersingular K3 surfaces are derived isogenous. More refined results (along the lines of [Huybrechts 2019, Theorem 0.1]) are possible, but we will omit this discussion here.

Theorem 6.11. *Suppose that $p \geq 3$. Let X and Y be two supersingular* K3 *surfaces over k. There exists a derived isogeny $\mathfrak{h}^2(X) \xrightarrow{\sim} \mathfrak{h}^2(Y)$.*

Proof. We use [Bragg and Lieblich 2018, Proposition 5.2.5]: if X is a supersingular K3 surface, then there exists a sequence $X_0, X_1, \dots, X_n$ of supersingular K3 surfaces together with Brauer classes $\alpha_i \in \mathrm{Br}(X_i)$ such that $X_0 = X$, $D^b(X_i, \alpha_i) \cong D^b(X_{i+1}, \alpha_{i+1})$ for each $0 \le i \le n-1$, and $X_n = Z$ is the unique supersingular K3 surface with Artin invariant 1. Applying this to both X and Y, we find derived isogenies

$$\mathfrak{h}^2(X) \xrightarrow{\sim} \mathfrak{h}^2(Z) \xleftarrow{\sim} \mathfrak{h}^2(Y). \qquad \square$$

Remark 6.12. Shioda [1977, Theorem 1.1] showed that supersingular Kummer surfaces are unirational. By a result of Ogus [1979] and the crystalline Torelli theorem these are exactly the supersingular K3 surfaces with Artin invariant $\sigma_0 \le 2$. The Chow motive of a unirational surface is of Tate type. Combining this with Theorem 6.11 we deduce that for any supersingular K3 surface X we have $\mathfrak{h}(X) = \mathfrak{h}_{\mathrm{alg}}(X) = \boldsymbol{L}^0 \oplus \boldsymbol{L}^{\oplus 22} \oplus \boldsymbol{L}^2$ and $\mathfrak{h}^2_{\mathrm{tr}}(X) = 0$. In particular, we have $\mathrm{CH}^2(X) = \boldsymbol{Z}$. This result was first proved by Fakhruddin [2002], using a related method.

6D. ***Existence in characteristic* 0.** It is possible to formulate a purely algebraic analog of Huybrechts' Theorem 1.3 along the lines of Theorem 1.2, valid over any algebraically closed field of characteristic 0.

Theorem 6.13. *Let X be a* K3 *surface over an algebraically closed field of characteristic* 0. *Let $H = \Lambda \otimes \widehat{\boldsymbol{Z}}$. Let $\iota : H \hookrightarrow \mathrm{H}^2(X)_{\boldsymbol{Q}}$ be an isometric embedding. There exists a* K3 *surface X' and a derived isogeny $f : \mathfrak{h}^2(X') \xrightarrow{\sim} \mathfrak{h}^2(X)$ such that $f_*(\mathrm{H}^2(X')) = \mathrm{im}(\iota)$.*

Proof. This can be proved purely algebraically along the same lines as our proof of Theorem 1.2 (but avoiding the extra complications at p). Alternatively, it can be deduced directly from Theorem 1.3. We omit further details. $\square$

6E. ***Nygaard–Ogus theory revisited.*** In preparation for the proof of Theorem 1.4, we briefly recap the deformation theory of K3 crystals and K3 surfaces established in [Nygaard and Ogus 1985, §5]. For the rest of Section 6, assume that k is a perfect field with $\mathrm{char}\, k = p \ge 5$. We refer the reader to the paragraph below the proof of Lemma 4.6 in loc. cit. for this restriction on p. Let $R := k[\varepsilon]/(\varepsilon^e)$ for some e. Recall that a K3 crystal over R is an F-crystal $\boldsymbol{H}$ on $\boldsymbol{Cris}(R/W)$ equipped with a pairing $\boldsymbol{H} \times \boldsymbol{H} \to \mathcal{O}_{R/W}$ and an isotropic line $\mathrm{Fil} \subset \boldsymbol{H}_R$ which satisfy some properties (see Definition 5.1 in loc. cit. for details).[2]

Definition 6.14. Suppose V is a finite flat extension of W such that $V/(p) = R$. A deformation of $\boldsymbol{H}$ to V is a pair $(\boldsymbol{H}, \widetilde{\mathrm{Fil}})$ where $\widetilde{\mathrm{Fil}} \subset \boldsymbol{H}_V$ is an isotropic direct summand which lifts $\mathrm{Fil} \subset \boldsymbol{H}_R$.

Theorem 6.15 (Nygaard and Ogus). *Let X be a* K3 *surface over k and R be as above.*

(a) *The natural map $X_R \mapsto \mathrm{H}^2_{\mathrm{cris}}(X_R)$ defines a bijection between deformations X_R of X to R to deformations of the* K3 *crystal $\mathrm{H}^2_{\mathrm{cris}}(X/W)$ to R, i.e.,* K3 *crystals $\boldsymbol{H}$ over R with $\boldsymbol{H}|_k = \mathrm{H}^2_{\mathrm{cris}}(X/W)$.*

(b) *If X_R is a deformation of X to R, then the map $X_V \mapsto (\mathrm{H}^2_{\mathrm{cris}}(X_R), \mathrm{Fil}^2\, \mathrm{H}^2_{\mathrm{dR}}(X_V/V))$ defines a bijection between deformations X_V of X_R to V and deformations of the* K3 *crystal $\mathrm{H}^2_{\mathrm{cris}}(X_R)$ to V, in the sense of Definition 6.14.*

[2] In fact, [Nygaard and Ogus 1985, Definition 5.1] defined K3 crystals over a more general base which satisfies a technical assumption [Nygaard and Ogus 1985, (4.4.1)]. For our purposes it suffices to consider bases of the form $k[\varepsilon]/(\varepsilon^e)$.

Proof. This follows from [Nygaard and Ogus 1985, Theorem 5.3] and its proof. □

For the rest of Section 6E, X denotes a K3 surface of finite height over k. Recall that there is a canonical slope decomposition (cf. [Nygaard and Ogus 1985, Proposition 5.4])

$$\delta_{\mathrm{can}} : \mathrm{H}^2_{\mathrm{cris}}(X/W) = \mathbb{D}(\widehat{\mathrm{Br}}^*_X) \oplus \mathbb{D}(D^*) \oplus \mathbb{D}(\widehat{\mathrm{Br}}_X)(-1). \tag{35}$$

We define a map $\mathbb{K}$ which sends a deformation of $\widehat{\mathrm{Br}}_X$ to R to a deformation of the K3 crystal $\mathrm{H}^2_{\mathrm{cris}}(X/W)$ to R by setting $\mathbb{K}(G_R) := \mathbb{D}(G^*_R) \oplus \mathbb{D}(D^*_R) \oplus \mathbb{D}(G_R)(-1)$, where D_R denote the canonical lift of D to R. The K3 crystal structure on $\mathbb{K}(G_R)$ is given as follows: Let $\mathscr{P}_{G_R} : \mathbb{D}(G^*_R) \times \mathbb{D}(G_R) \to \mathcal{O}_{R/W}(-1)$ be the canonical pairing and let $\mathscr{P}_{D_R} : \mathbb{D}(D^*_R) \times \mathbb{D}(D^*_R) \to \mathcal{O}_{R/W}(-2)$ be the pairing inherited from that on $\mathbb{D}(D^*)$. The pairing on $\mathbb{K}(G_R)$ is $\mathscr{P}_{G_R}(-1) \oplus \mathscr{P}_{D_R}$. Finally, the isotropic direct summand Fil in $\mathbb{K}(G_R)_R$ is given by $[\mathrm{Fil}^1 \mathbb{D}(G_R)_R](-1)$. We define a decreasing filtration on $\mathbb{K}(G_R)_R$ by setting

$$0 = \mathrm{Fil}^3 \subset \mathrm{Fil}^2 := \mathrm{Fil} \subset \mathrm{Fil}^1 := (\mathrm{Fil}^2)^\perp \subset \mathrm{Fil}^0 = \mathbb{K}(G_R)_R. \tag{36}$$

If we further lift G_R to a p-divisible group G_V for a finite flat extension V of W with $V/(p) = R$, then we can attach a deformation of $\mathbb{K}(G_R)$ to V by setting $\widetilde{\mathrm{Fil}} = [\mathrm{Fil}^1 \mathbb{D}(G_V)_V](-1)$, which we denote by $\mathbb{K}(G_V)$. We define a filtration on $\mathbb{K}(G_V)_V$ using (36) with $\mathbb{K}(G_R)_R$ replaced by $\mathbb{K}(G_V)_V$.

Definition 6.16. If X_V is a formal scheme over $\mathrm{Spf}\, V$ which deforms X, we say X_V is a Nygaard–Ogus lifting if it comes from $\mathbb{K}(G_V)$ for some p-divisible group G_V lifting $\widehat{\mathrm{Br}}_X$ to V via Theorem 6.15. That is, setting $R := V/(p)$, $G_R := (G_V) \otimes R$ and $X_R := (X_V) \otimes R$, we have an isomorphism

$$(\mathrm{H}^2_{\mathrm{cris}}(X_R), \mathrm{Fil}^2\, \mathrm{H}^2_{\mathrm{dR}}(X_V/V)) \xrightarrow{\sim} \mathbb{K}(G_V)$$

lifting δ_{can} in the obvious sense. If X_V is an algebraic space over $\mathrm{Spec}\, V$ which deforms X, then we say X_V is a Nygaard–Ogus lifting if its formal completion at the special fiber is a Nygaard–Ogus lifting.

Proposition 6.17. *If a formal scheme X_V is a Nygaard–Ogus lifting of X, then the natural map* $\mathrm{Pic}(X_V) \to \mathrm{Pic}(X)$ *is an isomorphism. In particular, X_V is algebraizable.*

Proof. See Proposition 4.5 and Remark 4.6 of [Yang 2022]. □

Using integral p-adic Hodge theory, we can characterize Nygaard–Ogus liftings:

Theorem 6.18. *Let F be a finite extension of K with $V := \mathcal{O}_F$. Let X_V be a formal scheme over* $\mathrm{Spf}\, V$ *which lifts X and let X_F denote its rigid-analytic generic fiber. Then X_V is a Nygaard–Ogus lifting if and only if there are* Gal_F*-stable $\mathbf{Z}_p$-sublattices T^0, T^1, T^2 in* $\mathrm{H}^2_{\mathrm{\acute{e}t}}(X_{\bar{F}}, \mathbf{Z}_p)$ *of ranks h, $22-2h$, h respectively, such that, as crystalline* Gal_F*-representations,*

(a) *$T^1(1)$ is unramified,*

(b) *T^0 has Hodge–Tate weight* 1 *with multiplicity* $h-1$ *and* 0 *with multiplicity* 1,

(c) *$T^2(1)$ has Hodge–Tate weight* 1 *with multiplicity* 1 *and* 0 *with multiplicity* $h-1$.

Proof. We recap in the Appendix the results from the integral p-adic Hodge theory used in this proof.

Let S denote Breuil's S-ring. Using the data $\mathrm{Fil}^\bullet \mathrm{H}^2_{\mathrm{dR}}(X_F/F)$, we equip $\mathrm{H}^2_{\mathrm{cris}}(X/W) \otimes_W S_K$ with the structure of an object in $\mathcal{MF}^{\varphi,N}_{S_K}$. For any G_V which lifts $\widehat{\mathrm{Br}}_X$ to V, we set

$$\mathbb{T}_p(G_V) := T_p G_V \oplus T_p D_V \oplus T_p G_V^*(-1).$$

Suppose first that X_V is Nygaard–Ogus, so that it comes from some G_V lifting $G := \widehat{\mathrm{Br}}_X$. Combining (41) and (45), we obtain isomorphisms

$$\mathrm{H}^2_{\text{ét}}(X_{\bar{F}}, \boldsymbol{Z}_p) \underset{\boldsymbol{Z}_p}{\otimes} B_{\mathrm{cris}} \cong \mathrm{H}^2(X/W) \underset{W}{\otimes} B_{\mathrm{cris}} = \mathbb{K}(G) \underset{W}{\otimes} B_{\mathrm{cris}} \cong \mathbb{T}_p(G_V)(-1) \underset{\boldsymbol{Z}_p}{\otimes} B_{\mathrm{cris}},$$

which give rise to a rational isomorphism $\mathrm{H}^2_{\text{ét}}(X_{\bar{F}}, \boldsymbol{Z}_p) \otimes_{\boldsymbol{Z}_p} \boldsymbol{Q}_p \xrightarrow{\sim} \mathbb{T}_p(G_V)(-1) \otimes_{\boldsymbol{Z}_p} \boldsymbol{Q}_p$. We now show that the this restricts to an integral isomorphism

$$\mathrm{H}^2_{\text{ét}}(X_{\bar{F}}, \boldsymbol{Z}_p) \cong \mathbb{T}_p(G_V)(-1). \tag{37}$$

It is easy to check that the object $(\mathbb{K}(G)_K, \mathrm{Fil}^\bullet \mathbb{K}(G_V)_F)$ in MF^{φ}_F admits a decomposition into

$$(\mathbb{D}(\widehat{\mathrm{Br}}^*_X)_K, \mathrm{Fil}^\bullet \mathbb{D}(G_V^*)_F) \oplus (\mathbb{D}(D^*)_K, \mathrm{Fil}^\bullet \mathbb{D}(D_V^*)_F) \oplus (\mathbb{D}(\widehat{\mathrm{Br}}_X)_K, \mathrm{Fil}^\bullet \mathbb{D}(G_V)_F)(-1).$$

By the construction of Nygaard–Ogus liftings, there is an isomorphism

$$\mathbb{D}(G_R^*)_S \oplus \mathbb{D}(D_R^*)_S \oplus \mathbb{D}(G_R)(-1)_S \cong \mathrm{H}^2_{\mathrm{cris}}(X_R)_S$$

of strongly divisible S-modules which is compatible with the isomorphism

$$\mathrm{H}^2(X/W) \underset{W}{\otimes} S_K \cong \mathbb{K}(G) \underset{W}{\otimes} S_K$$

induced by δ_{can}. By applying the functor T_{cris}, we obtain (37), which readily implies the "only if" part of the theorem.

Now we show the "if" part. The proof is essentially a reincarnation of the proof of [Nygaard and Ogus 1985, Proposition 5.5]. The hypothesis implies that there exists an isomorphism $\mathrm{H}^2_{\text{ét}}(X_{\bar{F}}, \boldsymbol{Z}_p) \cong \mathbb{T}_p(G_V)(-1)$ for some G_V which lifts $\widehat{\mathrm{Br}}_X$ to V. By Theorem A.3, there exists a unique isomorphism

$$\mathbb{D}(G_R^*)_S \oplus \mathbb{D}(D_R^*)_S \oplus \mathbb{D}(G_R)(-1)_S \cong \mathrm{H}^2_{\mathrm{cris}}(X_R)_S \tag{38}$$

which gives this isomorphism, $\mathrm{H}^2_{\text{ét}}(X_{\bar{F}}, \boldsymbol{Z}_p) \cong \mathbb{T}_p(G_V)(-1)$, under T_{cris}.

The only thing we need to check is that this isomorphism of S-modules comes from an isomorphism of F-crystals on $\boldsymbol{Cris}(R/W)$

$$\mathbb{K}(G_R) = \mathbb{D}(G_R^*) \oplus \mathbb{D}(D_R^*) \oplus \mathbb{D}(G_R)(-1) \cong \mathrm{H}^2_{\mathrm{cris}}(X_R) \tag{39}$$

which restricts to δ_{can}.

Let e be the ramification degree of V over W and j be any positive number $\leq e$. Set $R_j := R/(\varepsilon^j)$. We claim that there exists a sequence of isomorphisms $\delta_j : \mathbb{K}(G_{R_j}) \cong \mathrm{H}^2_{\mathrm{cris}}(X_R)$ of F-crystals on $\boldsymbol{Cris}(R_j/W)$ such that δ_j is the restriction of δ_{j+1} for each $j < e$ such that $\delta_1 = \delta_{\mathrm{can}}$, and δ_e gives the desired isomorphism (39). Suppose we have constructed δ_j for some $j < e$. Note that (ε^j) is a square-zero ideal in R_{j+1} and we can view R_{j+1} as an object of $\boldsymbol{Cris}(R_j/W)$ by equipping (ε^j) with the trivial PD structure. By [Nygaard and Ogus 1985, Theorem 5.2], to construct δ_{j+1} it suffices to show that $[\mathrm{Fil}^1\,\mathbb{D}(G_{R_{j+1}})_{R_{j+1}}](-1)$ is sent to $\mathrm{Fil}^2\,\mathrm{H}^2_{\mathrm{dR}}(X_{R_{j+1}}/R_{j+1})$ via the composition

$$\mathbb{K}(G_{R_{j+1}})_{R_{j+1}} \cong \mathbb{K}(G_{R_j})_{R_{j+1}} \xrightarrow[(\delta_j)_{R_{j+1}}]{\sim} \mathrm{H}^2_{\mathrm{cris}}(X_{R_j})_{R_{j+1}} \cong \mathrm{H}^2_{\mathrm{dR}}(X_{R_{j+1}}/R_{j+1}).$$

However, this follows directly from the fact that (38) respects the filtrations. Indeed, viewing R_{j+1} as an S-algebra via $S \to \mathcal{O}_F \to R \to R_{j+1}$, we get the above isomorphism by tensoring (38) with R_{j+1}. □

Remark 6.19. When X is ordinary, X_V is Nygaard–Ogus if and only if it is obtained via base change from the canonical lifting, because in this case deformations of $\widehat{\mathrm{Br}}_X$ are completely rigid. Therefore, the above theorem is a generalization of [Taelman 2020, Theorem C] when $p \geq 5$. It also follows from (37) in the above proof that when X_V is a Nygaard–Ogus lifting, for the enlarged formal Brauer group Ψ_{X_V} of X_V, there is a natural injective map of Gal_F-modules

$$T_p\Psi_{X_V} \to \mathrm{H}^2_{\mathrm{\acute{e}t}}(X_{\bar{F}}, \boldsymbol{Z}_p(1)),$$

which generalizes [Taelman 2020, Theorem 2.1]. Indeed, we have $\Psi_{X_V} = \widehat{\mathrm{Br}}_{X_V} \oplus D_V$ for a Nygaard–Ogus lifting.

6F. *Construction of liftable isogenies.* We now prove Theorem 1.4.

Proof of Theorem 1.4. Again write ι^p and ι_p for the prime-to-p and crystalline component of ι. If a Frobenius-preserving isometric embedding $\iota_p : H_p \hookrightarrow \mathrm{H}^2(X/W)$ as in the hypothesis exists, then the K3 crystal H_p has to be abstractly isomorphic to $\mathrm{H}^2(X/W)$ and hence to $\mathbb{K}(\widehat{\mathrm{Br}}_X)$. We choose an isomorphism $H_p \xrightarrow{\sim} \mathbb{K}(\widehat{\mathrm{Br}}_X)$ and consider $(\iota_p)_K := \iota_p \otimes K$ as an isometric automorphism of the F-isocrystal $\mathbb{K}(\widehat{\mathrm{Br}}_X)_K$. Then $(\iota_p)_K$ determines, and is conversely determined by, a pair (h, g), where $h \in \mathrm{End}(\widehat{\mathrm{Br}}_X)[1/p]$ and $g \in \mathrm{End}(D)[1/p]$. Our goal is to produce an isogeny $f : \mathfrak{h}^2(X') \to \mathfrak{h}^2(X)$ for some other K3 surface X' over $k = \bar{\boldsymbol{F}}_p$ such that $f_*(\mathrm{H}^2_{\mathrm{\acute{e}t}}(X', \widehat{\boldsymbol{Z}}^p)) = \iota^p(\Lambda \otimes \widehat{\boldsymbol{Z}}^p)$ and $f_*(\mathrm{H}^2(X'/W)) = (\iota_p)_K(\mathbb{K}(\widehat{\mathrm{Br}}_X))$. By Theorem 1.2, we first reduce to the case when $\iota^p(\Lambda \otimes \widehat{\boldsymbol{Z}}^p) = \mathrm{H}^2_{\mathrm{\acute{e}t}}(X, \widehat{\boldsymbol{Z}}^p)$ and $(\iota_p)_K$ sends the slope 1 part, i.e., $\mathbb{D}(D^*)$, isomorphically onto itself.

By Lubin–Tate theory, for some finite flat extension V of W, there exists a lift G_V of $\widehat{\mathrm{Br}}_X$ to V such that h lifts to $\mathrm{End}(G_V)[1/p]$ [Yang 2022, Lemma 4.8]. Note that $\mathrm{Fil}^\bullet\,\mathbb{K}(G_V)_F$ equips $\mathbb{K}(\widehat{\mathrm{Br}}_X)_K$ with the structure of an object in MF^{φ}_F and $\mathscr{M} := \mathbb{K}(G_V)_S$ defines a strongly divisible S-lattice in the corresponding object $\mathscr{D} := \mathbb{K}(\widehat{\mathrm{Br}}_X) \otimes S_K$ in $\mathcal{MF}^{\varphi,N}_{S_K}$. It is clear that ι_K preserves $\mathrm{Fil}^\bullet\,\mathbb{K}(G_V)_F$ and extends to an automorphism ι_{S_K} of $\mathscr{D}$.

Let X_V be the Nygaard–Ogus lifting of X which corresponds to G_V. We have $T_{\mathrm{cris}}(\mathscr{M}) = \mathrm{H}^2_{\mathrm{\acute{e}t}}(X_{\bar{F}}, \boldsymbol{Z}_p)$ inside $\mathrm{H}^2_{\mathrm{\acute{e}t}}(X_{\bar{F}}, \boldsymbol{Q}_p)$ by the proof of Theorem 6.18, and $V_{\mathrm{cris}}((\iota_p)_K)$ is an automorphism of the Gal_F-module $\mathrm{H}^2_{\mathrm{\acute{e}t}}(X_{\bar{F}}, \boldsymbol{Q}_p)$ which preserves the Poincaré pairing. The image of $\mathrm{H}^2_{\mathrm{\acute{e}t}}(X_{\bar{F}}, \boldsymbol{Z}_p)$ under $V_{\mathrm{cris}}((\iota_p)_K)$ can also be interpreted as $T_{\mathrm{cris}}(\iota_{S_K}(\mathscr{M}))$. Denote this Gal_F-stable $\boldsymbol{Z}_p$-lattice by Λ'_p. By Theorem 6.13, up to replacing F by a finite extension, we can find another K3 surface X' over F with a derived isogeny $f : \mathfrak{h}^2(X') \xrightarrow{\sim} \mathfrak{h}^2(X_F)$ such that

$$f_*(\mathrm{H}^2_{\text{ét}}(X'_{\bar{F}}, \widehat{\mathbf{Z}})) = \Lambda'_p \times \prod_{\ell \neq p} (\Lambda'_\ell := \iota^p(\Lambda \otimes \mathbf{Z}_\ell) = \mathrm{H}^2_{\text{ét}}(X_{\bar{F}}, \mathbf{Z}_\ell)).$$

We argue that f induces an integral isomorphism $\mathrm{Pic}(X'_F) \xrightarrow{\sim} \mathrm{Pic}(X_F)$. Indeed, f induces an isomorphism

$$\mathrm{Pic}(X'_F) \xrightarrow{\sim} \mathrm{Pic}(X_F)_{\mathbf{Q}} \cap \prod_\ell \Lambda'_\ell(1).$$

However, we know that the image of $\mathrm{Pic}(X'_F)$ lies in the unramified part of $\mathrm{H}^2_{\text{ét}}(X_{\bar{F}}, \mathbf{Z}_p(1))$, and the unramified part of $\Lambda'_p(1)$ coincides with that of $\mathrm{H}^2_{\text{ét}}(X_{\bar{F}}, \mathbf{Z}_p(1))$. This implies that the target of the above isomorphism is just $\mathrm{Pic}(X_F)$.

It follows that $\mathrm{Pic}(X'_F)$ also satisfies hypothesis (a), (b) or (c) if $\mathrm{Pic}(X_F) \cong \mathrm{Pic}(X)$ does. For (a) and (c) this is clear; for (b) this follows from [Lieblich et al. 2014, Lemma 2.3.2]. In any case, by [Matsumoto 2015, Theorem 1.1; Ito 2019, §2] and Theorem 8.10 to be proved below, X'_F admits potentially good reduction. Up to replacing F by a further extension, we can find a smooth proper algebraic space X'_V over V such that X'_F is the generic fiber of X'_V. The map induced on crystalline cohomology of special fibers is $D_{\text{cris}}(f)$, which sends $\mathrm{H}^2(X'/W)$ onto $(\iota_p)_K(\mathbb{K}(\widehat{\mathrm{Br}}_X))$. □

7. Uniqueness theorems

In this section we prove Theorem 1.5 by lifting to characteristic 0 (as outlined in the introduction).

7A. ***Shimura varieties.*** Let $p > 2$ be a prime and L be any self-dual quadratic lattice over $\mathbf{Z}_{(p)}$ of rank $m \geq 5$ and signature $(2+, (m-2)-)$. Set $\widetilde{G} := \mathrm{CSpin}(L_{(p)})$, $G := \mathrm{SO}(L_{(p)})$, $\mathcal{K}_p := \mathrm{CSpin}(L \otimes \mathbf{Z}_p)$, $\mathsf{K}_p := \mathrm{SO}(L \otimes \mathbf{Z}_p)$ and $\Omega := \{\omega \in \mathbf{P}(L \otimes \mathbf{C}) : \langle \omega, \omega \rangle = 0, \langle \omega, \bar{\omega} \rangle > 0\}$. Let $\widetilde{\mathscr{S}}_{\mathcal{K}_p}(L)$ (resp. $\mathscr{S}_{\mathsf{K}_p}(L)$) denote the canonical integral model of $\mathrm{Sh}_{\mathcal{K}_p}(\widetilde{G}, \Omega)$ (resp. $\mathrm{Sh}_{\mathsf{K}_p}(G, \Omega)$) over $\mathbf{Z}_{(p)}$ given by [Kisin 2010] (see also [Madapusi Pera 2016, §4]). We choose a compact open subgroup $\mathcal{K}^p$ of $\widetilde{G}(\mathbf{A}_f^p)$ and set $\mathcal{K} = \mathcal{K}_p \mathcal{K}^p$. Similarly, set K^p to be the image of $\mathcal{K}^p$ and $\mathsf{K} := \mathsf{K}_p \mathsf{K}^p$. Denote by $\widetilde{\mathrm{Sh}}_{\mathcal{K}}(L)$, $\widetilde{\mathscr{S}}_{\mathcal{K}}(L)$, $\mathrm{Sh}_{\mathsf{K}}(L)$, and $\mathscr{S}_{\mathsf{K}}(L)$ the stacky quotients $\widetilde{\mathrm{Sh}}_{\mathcal{K}_p}(L)/\mathcal{K}^p$, $\widetilde{\mathscr{S}}_{\mathcal{K}_p}(L)/\mathcal{K}^p$, $\mathrm{Sh}_{\mathsf{K}_p}(L)/\mathsf{K}^p$, and $\mathscr{S}_{\mathsf{K}_p}(L)/\mathsf{K}^p$ respectively.

The model $\widetilde{\mathscr{S}}_{\mathcal{K}}(L)$ is equipped with a universal abelian scheme $\mathscr{A}$ up to prime-to-p isogeny whose cohomology gives rise to sheaves $\mathbf{H}_*$ ($* = B, \mathrm{cris}, \ell, \mathrm{dR}$) on suitable fibers of $\widetilde{\mathscr{S}}(L)$. The abelian scheme $\mathscr{A}$ is equipped with a $\mathrm{Cl}(L)$-action and $\mathbf{Z}/2\mathbf{Z}$-grading, and the sheaves $\mathbf{H}_*$ are equipped with tensors $\boldsymbol{\pi}_* \in \mathbf{H}_*^{\otimes(2,2)}$. We call the triple of $\mathbf{Z}/2\mathbf{Z}$-grading, $\mathrm{Cl}(L)$-action and various realizations of π the *CSpin structures* on $\mathscr{A}$ or $\mathbf{H}_*$. The dual of the images of $\boldsymbol{\pi}_*$ are denoted by $\mathbf{L}_*$. We refer the reader to [Madapusi Pera 2016, §4] for details of these constructions or [Yang 2022, (3.1.3)] for a quick summary.

Here is another way to view the sheaves $\mathbf{L}_*$: On the double quotient $\mathrm{Sh}_{\mathsf{K}}(L)_{\mathbf{C}} = G(\mathbf{Q}) \backslash \Omega \times G(\mathbf{A}_f)/\mathsf{K}$, the standard representation $\mathrm{SO}(L) \to \mathrm{GL}(L)$ produces a variation of $\mathbf{Z}$-Hodge structures [Madapusi Pera 2016, §3.3], which is nothing but $(\mathbf{L}_B, \mathbf{L}_{\mathrm{dR},\mathbf{C}} := \mathbf{L}_{\mathrm{dR}}|_{\mathrm{Sh}_{\mathsf{K}}(L)_{\mathbf{C}}})$. The filtered vector bundle $\mathbf{L}_{\mathrm{dR},\mathbf{C}}$ is commonly called the automorphic vector bundle associated to this representation, and by the general theory of automorphic vector bundles, we know that it admits a canonical descent to the canonical model $\mathrm{Sh}_{\mathsf{K}}(L)$ over the reflex field $\boldsymbol{Q}$. This canonical descent is nothing but $\boldsymbol{L}_{\mathrm{dR}}$ (when restricted to $\mathrm{Sh}_{\mathsf{K}}(L)$). In fact, the pair $(\boldsymbol{L}_B, \boldsymbol{L}_{\mathrm{dR},\boldsymbol{C}})$ is the variation of $\boldsymbol{Z}$-Hodge structures associated to a family of $\boldsymbol{Z}$-motives $\boldsymbol{L}$ over $\mathrm{Sh}_{\mathsf{K}}(L)_{\boldsymbol{C}}$ in the sense of [Madapusi Pera 2015, §1.4]. This family of motives descend to the canonical model $\mathrm{Sh}_{\mathsf{K}}(L)$, whose ℓ-adic realizations give $\boldsymbol{L}_\ell|_{\mathrm{Sh}_{\mathsf{K}}(L)}$ and whose de Rham realization gives $\boldsymbol{L}_{\mathrm{dR}}|_{\mathrm{Sh}_{\mathsf{K}}(L)}$. Once we extend $\mathrm{Sh}_{\mathsf{K}}(L)$ to $\mathscr{S}_{\mathsf{K}}(L)$ over $\boldsymbol{Z}_{(p)}$, these sheaves arising from cohomological realizations of motives over $\mathrm{Sh}_{\mathsf{K}}(L)$ also extend. This motivic point of view is discussed in more detail in [Madapusi Pera 2015, §4.7].

It is explained in [Yang 2022, (3.1.3)] that the sheaves $\boldsymbol{L}_*$ are equipped with an orientation tensor $\boldsymbol{\delta}_* : \underline{\det(L)} \xrightarrow{\sim} \det(\boldsymbol{L}_*)$ $(* = B,\ \ell \neq p)$. Here $\underline{\det(L)}$ denotes the constant sheaf whose stalks are $\det(L)$ on $\widetilde{\mathscr{S}}(L)$ in the appropriate Grothendieck topology. In short, $\boldsymbol{\delta}_*$'s come up because the adjoint representation of $\widetilde{G}$ on $L_{(p)}$ factors through $\mathrm{SO}(L_{(p)})$, i.e., it preserves a choice of orientation δ on $L_{(p)}$. It is possible to discuss de Rham or crystalline realizations of δ, but for our purposes it suffices to use the 2-adic realization $\boldsymbol{\delta}_2$. The sheaves $\boldsymbol{L}_*$ and the tensors $\boldsymbol{\pi}_*$ and $\boldsymbol{\delta}_*$ descend to $\mathscr{S}(L)$.

We will repeatedly make use of the following key fact about $\boldsymbol{L}_{\mathrm{dR}}$ and $\boldsymbol{H}_{\mathrm{dR}}$:

Proposition 7.1. *Let s be any point on $\widetilde{\mathscr{S}}_{\mathsf{K}}(L)$.* $\mathrm{Fil}^1\, \boldsymbol{L}_{\mathrm{dR},s}$ *is one-dimensional, and* $\mathrm{Fil}^1\, \boldsymbol{H}_{\mathrm{dR},s} = \ker(x)$ *for any nonzero element $x \in \mathrm{Fil}^1\, \boldsymbol{L}_{\mathrm{dR},s}$.*

Proof. If $\mathrm{char}\, k(s) = 0$, we can simply base change to $\boldsymbol{C}$ and apply Hodge theory (see [Yang 2022, p. 8]). If $\mathrm{char}\, k(s) = p$, we can check this by a lifting argument or read it off from [Madapusi Pera 2016, §4.9]. □

We recall the definition of a CSpin-isogeny [Yang 2022, Definition 3.2]:

Definition 7.2. Let κ be a perfect field with algebraic closure $\bar{\kappa}$, and let s, s' be κ-points on $\widetilde{\mathscr{S}}_{K_p}(L)$. We call a quasi-isogeny $\mathscr{A}_s \to \mathscr{A}_{s'}$ a *CSpin-isogeny* if it commutes with the CSpin structures, i.e., it respects the $\boldsymbol{Z}/2\boldsymbol{Z}$-grading, $\mathrm{Cl}(L)$-action and sends $\boldsymbol{\pi}_{\ell,s\otimes\bar{\kappa}}$ to $\boldsymbol{\pi}_{\ell,s'\otimes\bar{\kappa}}$ for every $\ell \neq \mathrm{char}\,\kappa$ and in addition $\boldsymbol{\pi}_{\mathrm{cris},s}$ to $\boldsymbol{\pi}_{\mathrm{cris},s'}$ if $\mathrm{char}\,\kappa = p$.

We remark that CSpin-isogenies are stable under liftings and reductions:

Lemma 7.3. *Let κ be a perfect field of characteristic p, and let s, s' be two k-points on $\widetilde{\mathscr{S}}_{\mathsf{K}}(L)$. Let K denote $W(\kappa)[1/p]$ and $F \subseteq \overline{K}$ be a finite extension of K, and let s_F, s'_F be F-valued points on $\widetilde{\mathscr{S}}_{\mathsf{K}}(L)$ which specialize to s, s'. Suppose $\psi_F : \mathscr{A}_{s_F} \to \mathscr{A}_{s'_F}$ is a quasi-isogeny which specializes to $\psi : \mathscr{A}_s \to \mathscr{A}_{s'}$. Then ψ is a CSpin-isogeny if and only if ψ_F is also a CSpin-isogeny.*

Proof. Clearly, ψ respects the $\boldsymbol{Z}/2\boldsymbol{Z}$-grading and the $\mathrm{Cl}(L)$-actions if and only if ψ_F also respects these structures. Let $s_{\overline{K}}$ and $s'_{\overline{K}}$ denote the $\overline{K}$-valued geometric points over s_F and s'_F. To check whether ψ_F sends $\boldsymbol{\pi}_{\ell,s_{\overline{K}}}$ to $\boldsymbol{\pi}_{\ell,s'_{\overline{K}}}$ for every ℓ, it suffices to check this for one ℓ, as one can always take a base change to $\boldsymbol{C}$ and use Betti realizations. Therefore, the only part of the statement which does not follow directly from the smooth and proper base change theorem is that if ψ_F is a CSpin-isogeny, then ψ sends $\boldsymbol{\pi}_{\mathrm{cris},s}$ to $\boldsymbol{\pi}_{\mathrm{cris},s'}$. This follows from [Yang 2022, Remark 3.1]. □

Lemma 7.4. *Let $s_{\boldsymbol{C}}, s'_{\boldsymbol{C}}$ be two $\boldsymbol{C}$-points on $\widetilde{\mathscr{S}}_{\mathsf{K}}(L)$. For every Hodge isometry*

$$g : \boldsymbol{L}_{B,s'_{\boldsymbol{C}}} \otimes \boldsymbol{Q} \xrightarrow{\sim} \boldsymbol{L}_{B,s_{\boldsymbol{C}}} \otimes \boldsymbol{Q}$$

which sends $\boldsymbol{\delta}_{2,s_{\boldsymbol{C}}}$ to $\boldsymbol{\delta}_{2,s'_{\boldsymbol{C}}}$, there exists a CSpin-isogeny $\mathscr{A}_{s_{\boldsymbol{C}}} \xrightarrow{\sim} \mathscr{A}_{s'_{\boldsymbol{C}}}$ which induces g by conjugation.

Proof. From the construction of the local system $\boldsymbol{H}_B$ (see [Madapusi Pera 2016, §3.3]) it is clear that there exists an isomorphism of free $\boldsymbol{Z}_{(p)}$-modules $H \xrightarrow{\sim} \boldsymbol{H}_{B,s_{\boldsymbol{C}}}$ which respects the CSpin-structures, i.e., it respects the $\boldsymbol{Z}/2\boldsymbol{Z}$-grading, $\mathrm{Cl}(L)$-action and sends π to $\boldsymbol{\pi}_{B,s_{\boldsymbol{C}}}$. The same is true for $s'_{\boldsymbol{C}}$, so there exists an isomorphism of $\boldsymbol{Z}_{(p)}$-modules $\psi' : \boldsymbol{H}_{B,s_{\boldsymbol{C}}} \xrightarrow{\sim} \boldsymbol{H}_{B,s'_{\boldsymbol{C}}}$ which respects the CSpin structures. The map $g' : \boldsymbol{L}_{B,s_{\boldsymbol{C}}} \otimes \boldsymbol{Q} \xrightarrow{\sim} \boldsymbol{L}_{B,s'_{\boldsymbol{C}}} \otimes \boldsymbol{Q}$ induced by ψ' by conjugation sends $\boldsymbol{\delta}_{2,s_{\boldsymbol{C}}}$ to $\boldsymbol{\delta}_{2,s'_{\boldsymbol{C}}}$. Therefore, the composition $g^{-1} \circ g'$ lies in $\mathrm{SO}(\boldsymbol{L}_{B,s_{\boldsymbol{C}}} \otimes \boldsymbol{Q})$. Since the natural morphism $\mathrm{CSpin}(L_{\boldsymbol{Q}}) \to \mathrm{SO}(L_{\boldsymbol{Q}})$ is surjective, we may lift $g^{-1} \circ g'$ to an automorphism of $\boldsymbol{H}_{B,s_{\boldsymbol{C}}}$ which preserves the CSpin structures and use it to adjust ψ' to obtain a morphism ψ which induces g by conjugation. It follows from Proposition 7.1 that f preserves the Hodge structures, so that $\tilde{g}$ comes from a CSpin-isogeny. ☐

7B. ***Hilbert squares and period morphisms.*** We will apply the period morphism construction to Hilbert squares of K3 surfaces, so we recollect some basic facts and set up some notation here. Let k be any algebraically closed field of characteristic 0 or $p > 2$, X be any K3 surface over k and $Y := X^{[2]}$ be the Hilbert scheme of two points on X. The lemma below implies that Y is a $\mathrm{K3}^{[2]}$-type variety in the sense of [Yang 2023, Definition 1].

Lemma 7.5. *When* $\mathrm{char}\, k = p > 2$ *or* 0, *Y has the same Hodge numbers as those of a complex* $\mathrm{K3}^{[2]}$*-type variety, and the Hodge–de Rham spectral sequence of Y degenerates at the E_1-page.*

Proof. Let $Y' := \mathrm{Bl}_\Delta(X \times X)$ be the blowup of $X \times X$ along the diagonal $\Delta \subset X \times X$. Let $E \subset Y'$ be the exceptional divisor, which is isomorphic to the projectivization of the tangent bundle of X. There is an action of $\boldsymbol{Z}/2$ on $X \times X$ given by permuting the factors, which lifts to an action on Y' that is trivial on E, and there is a natural map $q : Y' \to Y$ that identifies Y with the quotient $Y'/(\boldsymbol{Z}/2)$. The map q is a double cover branched over the divisor $D = q(E) \subset Y$, which may be described explicitly as the locus of nonreduced subschemes. Using our assumption that 2 is invertible in k, we obtain a canonical direct sum decomposition

$$q_* \mathscr{O}_{Y'} = \mathscr{O}_Y \oplus \mathscr{L},$$

where $\mathscr{L}$ is the cokernel of the pullback map $\mathscr{O}_Y \to q_* \mathscr{O}_{Y'}$. From this and the projection formula we deduce the equality

$$\mathrm{H}^j(Y', q^* \Omega^i_Y) = \mathrm{H}^j(Y, \Omega^i_Y) \oplus \mathrm{H}^j(Y, \Omega^i_Y \otimes \mathscr{L}).$$

All of these data may be defined in a flat family over a flat finite type $\boldsymbol{Z}$-scheme. By semicontinuity, the dimensions of both summands on the right-hand side must be greater than or equal to their corresponding values over the complex numbers. Thus, it will suffice to verify that the groups $\mathrm{H}^j(Y', q^* \Omega^i_Y)$ have the same dimensions as over the complex numbers. This can be done via a direct computation. In more detail, we compute using the identification $q^*\Omega^1_Y \cong \Omega^1_{Y'}(-E)$, which yields isomorphisms

$$q^*\Omega^i_Y \cong \Omega^i_{Y'}(-iE).$$

The cohomology of these sheaves may be related to the Hodge cohomology of X by pushing forward along the blowup morphism $Y' \to X \times X$. The result then follows (eventually) from the fact that the Hodge numbers of X do not depend on the characteristic of the ground field.

The degeneration of the Hodge–de Rham spectral sequence at the E_1-page follows from the fact that $\mathrm{H}^i(Y, \Omega^j_Y) = 0$ for $i + j$ odd. □

Let $\mathsf{H}^*(-)$ be a Weil cohomology with coefficient field $\mathcal{K}$.[3] We will only make use of Betti, ℓ-adic, crystalline, de Rham when appropriate. When there is a specified polarization, let $\mathsf{P}^*(-)$ denote the corresponding primitive cohomology. We will view $\mathrm{NS}(Y)$ as a $\mathbf{Z}$-lattice inside $\mathsf{H}^2(Y)$ via c_1, and will not write c_1 explicitly. $\mathsf{H}^2(Y)$ is equipped with natural *Beauville–Bogomolov forms* (BBF). When $\operatorname{char} k = 0$, these forms are well known. When $\operatorname{char} k = p > n+1$, the étale and crystalline versions of these forms for $\mathrm{K3}^{[n]}$-type varieties were defined in [Yang 2023, §2.1]. Since Y is a Hilbert square on a K3 surface X, as opposed to a general deformation of such a variety, the Beauville–Bogomolov form on Y is easily described by the Poincaré pairing on X: Let δ be the class of the exceptional divisor. Then $\delta^2 = -2$ under the BBF. The incidence correspondence between X and Y embeds $\mathsf{H}^2(X)$ isometrically into $\mathsf{H}^2(Y)$ such that $\mathsf{H}^2(Y)$ admits a natural orthogonal decomposition $\mathsf{H}^2(X) \oplus \mathcal{K}\delta$. Similarly, $\mathrm{NS}(Y)$ decomposes as $\mathrm{NS}(X) \oplus \mathbf{Z}\delta$.

Lemma 7.6. *Let ξ be a polarization on X and ζ be a polarization on Y of the form $m\xi - \delta$. Denote by* $\mathrm{proj}_{\mathsf{P}^2(Y)}\,\delta$ *the projection of δ to $\mathsf{P}^2(Y)$ and by* $\mathrm{Isom}(-,-)$ *the set of isometries between two quadratic lattices. Now let X' be another* K3 *surface over k, take Y', ξ', δ' similarly, and suppose Y' is polarized by $\zeta' := m\xi - \delta'$. There are natural identifications*

$$\begin{aligned}\mathrm{Isom}(\mathsf{P}^2(X), \mathsf{P}^2(X')) &= \{f \in \mathrm{Isom}(\mathsf{H}^2(X), \mathsf{H}^2(X') : f(\xi) = \xi'\} \\ &= \{f \in \mathrm{Isom}(\mathsf{H}^2(Y), \mathsf{H}^2(Y')) : f(\zeta) = \zeta',\, f(\delta) = \delta'\} \\ &= \{f \in \mathrm{Isom}(\mathsf{P}^2(Y), \mathsf{P}^2(Y') : f(\mathrm{proj}_{\mathsf{P}^2(Y)}\,\delta) = \mathrm{proj}_{\mathsf{P}^2(Y')}\,\delta'\}. \end{aligned} \tag{40}$$

Assume now $p \geq 5$ to apply the results of [Yang 2023]. Let X be a K3 surface and $Y := X^{[2]}$. Let ζ be any primitive polarization on Y such that p is prime to the top intersection number ζ^4. Let $\mathrm{Def}(Y;\zeta)$ denote the deformation functor of the pair (Y, ζ), i.e., the functor which sends an Artin W-algebra A to the set of isomorphism classes of the flat deformations of (Y, ζ) over A. We have that $\mathrm{Def}(Y;\zeta)$ is representable by a formal scheme isomorphic to $\mathrm{Spf}(\mathcal{R})$ for $\mathcal{R} := W[\![x_1, \dots, x_{20}]\!]$. Let $(\mathscr{Y}, \boldsymbol{\zeta})$ denote the universal family over $\mathrm{Def}(Y;\zeta)$. Note that $\boldsymbol{\zeta}$ algebraizes $\mathscr{Y}$ into a scheme over $\mathrm{Spec}(\mathcal{R})$. Again we use the symbol $\mathrm{P}^2(-)$ for the primitive cohomologies of (Y, ζ). There are natural pairings on $\mathrm{P}^2(Y, \widehat{\mathbf{Z}}^p)$ and $\mathrm{P}^2(Y/W)$ given by restricting the Beauville–Bogomolov forms (see [Yang 2023, §2.1]).

[3] Here we are using a different font for $\mathsf{H}^*(-)$ to distinguish from the $\mathrm{H}^*(-)$ in Section 1D.

Let $F \subset \overline{K}$ be any finite extension of K and $\tilde{b}$ be any $\mathcal{O}_F$-point on $\mathrm{Def}(X;\zeta)$. Choose an isomorphism $\iota : \overline{K} \xrightarrow{\sim} \boldsymbol{C}$. Let L be the quadratic lattice $\mathrm{P}^2(\mathscr{Y}_{\tilde{b}}(\boldsymbol{C}), \boldsymbol{Z}_{(p)})$, equipped with the restriction of the negative Beauville–Bogomolov form. We remark that since $\mathrm{H}^2(\mathscr{Y}_{\tilde{b}}(\boldsymbol{C}), \boldsymbol{Z})$ is always isomorphic to the lattice $\Lambda^{[2]} := \Lambda \oplus \boldsymbol{Z}(-2)$ and $p \nmid c_1(\boldsymbol{\zeta}_{\tilde{b}_C})^2$, the isomorphism class of L as a quadratic lattice over $\boldsymbol{Z}_{(p)}$ is completely determined by the number $c_1(\boldsymbol{\zeta}_{\tilde{b}_C})^2$ [Milnor and Husemoller 1973, I, Lemma 4.2].

Let b be the closed point of $\mathrm{Def}(X,\zeta)$. We pack the input we need from the Kuga–Satake period morphism into the following proposition:

Proposition 7.7. *Assume $p \geq 5$. There exists a local period morphism $\rho : \mathrm{Spec}\,\mathcal{R} \to \mathscr{S}_{\mathsf{K}_p}(L)$ which identifies $\mathrm{Spec}\,\mathcal{R}$ with the complete local ring $\widehat{\mathcal{O}}_s$ of $s := \rho(b)$ on $\mathscr{S}(L)_W$ such that*:

(a) *There exist an isometry $\alpha_{\mathrm{dR}} : \boldsymbol{P}^2_{\mathrm{dR}} \xrightarrow{\sim} \rho^*\boldsymbol{L}_{\mathrm{dR}}(-1)$ of filtered vector bundles and an isometry $\alpha_{\mathrm{cris}} : \boldsymbol{P}^2_{\mathrm{cris}} \xrightarrow{\sim} \rho^*\boldsymbol{L}_{\mathrm{cris}}(-1)$ of F-crystals that are compatible via the crystalline–de Rham comparison isomorphisms.*

(b) *There is an isometry $\alpha_{\boldsymbol{A}_f,b} : \mathrm{P}^2_{\text{ét}}(\mathscr{Y}_{b,\boldsymbol{A}_f}) \to \boldsymbol{L}_{\boldsymbol{A}_f,b}$ such that for any geometric $\tilde{b}'$ of characteristic zero on $\mathrm{Spec}\,\mathcal{R}$, the pair of isometries $(\alpha_{\boldsymbol{A}_f,\tilde{b}'}, \alpha_{\mathrm{dR},\tilde{b}'})$, where $\alpha_{\boldsymbol{A}_f,\tilde{b}'} : \mathrm{P}^2_{\text{ét}}(\mathscr{Y}_{\tilde{b}'}, \boldsymbol{A}_f) \to \boldsymbol{L}_{\boldsymbol{A}_f,\tilde{b}'}$ is induced by the smooth and proper base change theorem, is absolute Hodge.*

Moreover, for any choice of trivialization $\epsilon_2 : \det(L \otimes \boldsymbol{Q}_2) \xrightarrow{\sim} \det(\mathrm{P}^2_{\text{ét}}(Y, \boldsymbol{Q}_2))$, s can always be chosen such that $\det(\alpha_{2,b})$ sends ϵ_2 to $\boldsymbol{\delta}_{2,s}$.

Proof. See [Yang 2023, §3.3], which is a direct generalization of the results in [Madapusi Pera 2015, §5]. □

Remark 7.8. We remark that in order to construct the local period morphism ρ, we actually have to choose an appropriate $\boldsymbol{Z}$-integral structure for the $\boldsymbol{Z}_{(p)}$-lattice L. However, once it is constructed, we are allowed to forget about the $\boldsymbol{Z}$-integral structure, as the integral models of the relevant Shimura varieties only depend on the $\boldsymbol{Z}_{(p)}$-lattice L.

7C. *Twisted derived Torelli theorem.*

Definition 7.9. Let X and X' be K3 surfaces over an algebraically closed field k of characteristic $p > 0$. Let $f : \mathfrak{h}^2(X') \xrightarrow{\sim} \mathfrak{h}^2(X)$ be an isogeny. We say that f is *liftable* if for some finite extension F of K with $V := \mathcal{O}_F$ and projective schemes X_V and X'_V over V which deform X and X' to V, f lifts to an isogeny $f_F : \mathfrak{h}^2(X'_F) \xrightarrow{\sim} \mathfrak{h}^2(X_F)$. If X and X' are nonsupersingular, we say that f is *perfectly liftable* if X_V and X'_V can be chosen to be perfect liftings.

For the rest of Section 7C, let k be an algebraically closed field of $p \geq 5$.

Lemma 7.10. *Let $(X_0,\xi_0), \dots, (X_m,\xi_m)$ be finitely many nonsupersingular polarized* K3 *surfaces over k and let $f_i : \mathfrak{h}^2(X_i) \xrightarrow{\sim} \mathfrak{h}^2(X_{i+1})$ be a perfectly liftable isogeny which sends ξ_i to ξ_{i+1} for $i = 0, 1, \dots, m-1$. If $f := f_{m-1} \circ \cdots \circ f_0 : (\mathfrak{h}^2(X_0), \xi_0) \xrightarrow{\sim} (\mathfrak{h}^2(X_m), \xi_m)$ induces an integral isomorphism $\mathrm{H}^2_{\mathrm{cris}}(X_0/W) \xrightarrow{\sim} \mathrm{H}^2_{\mathrm{cris}}(X_m/W)$, then f is perfectly liftable to K up to equivalence.*

Proof. Set $Y_i := X_i^{[2]}$ and let δ_i be the exceptional divisor on Y_i. For some number $N \gg 0$, $\zeta_i := p^N \xi_i - \delta_i$ is a polarization on Y_i for each i. The number $\langle \zeta_i, \zeta_i \rangle$ under the Beauville–Bogomolov form on Y_i

is an integer M which is independent of i. Let L denote a $\boldsymbol{Z}_{(p)}$-lattice which is isomorphic to the orthogonal complement of an element $\lambda \in \Lambda^{[2]} \otimes \boldsymbol{Z}_{(p)}$ with $\langle \lambda, \lambda \rangle = M$. We choose trivializations $\epsilon_i : \det(L \otimes \boldsymbol{Q}_2) \xrightarrow{\sim} \det(\mathrm{P}^2_{\text{ét}}(Y_i, \boldsymbol{Q}_2))$ such that f_i sends ϵ_i to ϵ_{i+1}. Let ρ_i denote a local period morphism obtained by applying Proposition 7.7 to (Y_i, ζ_i) and ϵ_i, and let s_i denote the image of the basepoint under ρ_i. Let $\tilde{s}_i$ be a lift of s_i to $\widetilde{\mathscr{S}}_{\mathcal{K}_p}(L)$.

We claim that there exists a CSpin-isogeny $\psi_i : \mathscr{A}_{\tilde{s}_i} \to \mathscr{A}_{\tilde{s}_{i+1}}$ which induces the same isometries

$$\boldsymbol{L}_{\ell,s_i} \xrightarrow{\sim} \boldsymbol{L}_{\ell,s_{i+1}} \quad \text{and} \quad \boldsymbol{L}_{\text{cris},s_i} \xrightarrow{\sim} \boldsymbol{L}_{\text{cris},s_{i+1}}$$

as f_i for each $i = 0, \ldots, m-1$. Indeed, fix an i and let $X_{i,V}$, $X_{i+1,V}$ be perfect liftings of X_i, X_{i+1} over some finite extension V of W such that f lifts to $f_F : X_{i,F} \xrightarrow{\sim} X_{i+1,F}$, where $F = V[1/p]$. Let $Y_{i,V}$, $Y_{i+1,V}$ be the Hilbert squares of $X_{i,V}$, $X_{i+1,V}$. Note that $Y_{i,V}$ and $Y_{i+1,V}$ carry liftings of ζ_i and ζ_{i+1}, so via the local Torelli morphisms ρ_i and ρ_{i+1}, $X_{i,V}$ and $X_{i+1,V}$ induce V-points $s_{i,V}$, $s_{i+1,V}$ on $\mathscr{S}_{\mathsf{K}}(L)$. Lift these points to V-points $\tilde{s}_{i,V}$, $\tilde{s}_{i+1,V}$ on $\widetilde{\mathscr{S}}_{\mathcal{K}}(L)$, which is étale over $\mathscr{S}_{\mathsf{K}}(L)$. Now choose an isomorphism $\bar{F} \xrightarrow{\sim} \boldsymbol{C}$. The isogeny $f_{i,F}(\boldsymbol{C})$ induces a Hodge isometry $\mathrm{P}^2(X_{i,F}(\boldsymbol{C}), \boldsymbol{Q}) \xrightarrow{\sim} \mathrm{P}^2(X_{i+1,F}(\boldsymbol{C}), \boldsymbol{Q})$, which canonically extends to a Hodge isometry $\mathrm{P}^2(Y_{i,F}(\boldsymbol{C}), \boldsymbol{Q}) \xrightarrow{\sim} \mathrm{P}^2(Y_{i+1,F}(\boldsymbol{C}), \boldsymbol{Q})$ via Lemma 7.6. By Proposition 7.7, the latter can be identified with a Hodge isometry $\boldsymbol{L}_{B,s_{i,F}(\boldsymbol{C})} \otimes \boldsymbol{Q} \xrightarrow{\sim} \boldsymbol{L}_{B,s_{i+1,F}(\boldsymbol{C})} \otimes \boldsymbol{Q}$. Note that we have required that f_i send ϵ_i to ϵ_{i+1}. By Lemma 7.4, we obtain a CSpin-isogeny $\psi_{i,\boldsymbol{C}} : \mathscr{A}_{\tilde{s}_{i,F}(\boldsymbol{C})} \xrightarrow{\sim} \mathscr{A}_{\tilde{s}_{i+1,F}(\boldsymbol{C})}$. By Lemma 7.3, $\psi_{i,\boldsymbol{C}}$ specializes to a CSpin-isogeny ψ_i, which can be easily checked to have the desired properties.

By [Lieblich and Maulik 2018, Corollary 4.2], we can find a lifting $X_{0,W}$ of X_0 which also lifts all line bundles on X_0. We transport the induced Hodge filtration on $\mathrm{H}^2_{\text{cris}}(X_0/W)$ to $\mathrm{H}^2_{\text{cris}}(X_m/W)$ using f, which induces a lift $X_{m,W}$ of X_m over W. It is easy to check that $X_{m,W}$ also carries liftings of all line bundles on X_m using [Ogus 1979, Proposition 1.12]. Just as in the previous paragraph, after taking Hilbert squares of the liftings, we obtain via the local period morphisms K-valued points $s_{0,K}$, $s_{m,K}$, $\tilde{s}_{0,K}$, $\tilde{s}_{m,K}$ which lift s_0, s_m, $\tilde{s}_0$, $\tilde{s}_m$. It follows from Proposition 7.1 that the crystalline realization of $\psi := \psi_{m-1} \circ \cdots \circ \psi_0$ preserves the Hodge filtrations of $\mathscr{A}_{s_{0,K}}$ and $\mathscr{A}_{s_{m,K}}$ via the Berthelot–Ogus comparison isomorphisms. By [Berthelot and Ogus 1983, Theorem 3.15], ψ lifts to a CSpin-isogeny $\psi_K : \mathscr{A}_{s_{0,K}} \xrightarrow{\sim} \mathscr{A}_{s_{m,K}}$. Choose an isomorphism $\bar{K} \cong \boldsymbol{C}$. By running the arguments in the preceding paragraph backwards, we obtain a rational Hodge isometry $\mathrm{H}^2(X_{0,K}(\boldsymbol{C}), \boldsymbol{Q}) \xrightarrow{\sim} \mathrm{H}^2(X_{m,K}(\boldsymbol{C}), \boldsymbol{Q})$, which by Huybrechts' theorem [2019, Theorem 0.2] is induced by an isogeny $f_{\boldsymbol{C}}$. We get the desired isogeny f by specializing $f_{\boldsymbol{C}}$. □

Proof of Theorem 1.5.. The forward direction is immediate (and does not need the restriction on p). For the converse, suppose that $f : \mathfrak{h}^2(X') \xrightarrow{\sim} \mathfrak{h}^2(X)$ is polarizable and $\boldsymbol{Z}$-integral. It is easy to see that if X is supersingular, then so is X'. In this case, the result follows from the crystalline Torelli theorem of Ogus [1983, Theorem II] (cf. [Yang 2022, Theorem 6.5]). Therefore, we reduce to the case when X and X' have finite height. We first remark that f maps $\mathrm{NS}(X')$ isomorphically onto $\mathrm{NS}(X)$, so that by the structure of ample cones of K3 surfaces [Ogus 1983, Proposition 1.10], $f(\xi')$ is ample for any ample ξ. By definition, there exists a sequence of K3 surfaces $X' = X_0, \ldots, X_m = X$ over k and primitive derived isogenies $f_i : \mathfrak{h}^2(X_i) \xrightarrow{\sim} \mathfrak{h}^2(X_{i+1})$ such that $f = f_{m-1} \circ \cdots \circ f_0$.

We now show that there exists a sequence $\delta_i : \mathfrak{h}^2(X_i) \xrightarrow{\sim} \mathfrak{h}^2(X_i)$ given by compositions of reflections in (-2)-curves up to a sign and a sequence of ample class $\xi_i \in \mathrm{NS}(X_i)_{\boldsymbol{Q}}$ such that $(\delta_{i+1} \circ f_i)(\xi_i) = \xi_{i+1}$ for each i. We do this by slightly refining the argument of [Yang 2022, Lemma 6.2]. Set δ_0 to be the identity. Choose any ample class $\zeta_0 \in \mathrm{NS}(X_0)_{\boldsymbol{Q}}$ and $\epsilon_0 > 0$, such that the open ball $B(\zeta_0, \epsilon_0)$ centered at ζ_0 of radius ϵ_0 in $\mathrm{NS}(X_0)_{\boldsymbol{R}}$ lies inside the ample cone. By [Ogus 1979, Lemma 7.9], there exists some δ_1, such that $\zeta_1' := \delta_1 \circ f_0(\zeta_0)$ is big and nef. The image of $B(\zeta_0, \epsilon_0)$ in $\mathrm{NS}(X_1)_{\boldsymbol{R}}$ under $\delta_1 \circ f_0$ is an open neighborhood of ζ_1' which necessarily intersects the ample cone of X_1. Therefore, we may now choose ζ_1 together with $\epsilon_1 > 0$ such that $(\delta_1 \circ f_0)^{-1} B(\zeta_1, \epsilon_1) \subseteq B(\zeta_0, \epsilon_0)$. We iterate this process to obtain a sequence of open balls $B(\zeta_i, \epsilon_i) \subset \mathrm{NS}(X_i)_{\boldsymbol{R}}$ which lie inside the ample cones, and a sequence of δ_i's such that $(\delta_{i+1} \circ f_i)^{-1}(B(\zeta_{i+1}, \epsilon_{i+1})) \subseteq B(\zeta_i, \epsilon_i)$. Now we win by choosing an element $\xi_m \in B(\zeta_m, \epsilon_m)$, and iteratively set $\xi_i := (\delta_{i+1} \circ f_i)^{-1}(\xi_{i+1})$. By clearing denominators we may assume that each ξ_i is integral.

Set $\xi = \xi_m$, $\xi' = \xi_0$, $h_i := \delta_{i+1} \circ f_i$ for each $i < m$, and $f' := h_{m-1} \circ \cdots \circ h_0 = f$. For each i, consider $T(X_i) := \mathrm{NS}(X_i)^{\perp} \subset \mathrm{H}^2(X_i)$. Clearly f_i and h_i induce the same maps on transcendental lattices $\mathrm{T}(X_i)_{\boldsymbol{Q}} \xrightarrow{\sim} \mathrm{T}(X_{i+1})_{\boldsymbol{Q}}$. Therefore, f and f' induce the same maps $\mathrm{T}(X')_{\boldsymbol{Q}} \xrightarrow{\sim} \mathrm{T}(X)_{\boldsymbol{Q}}$ but their induced maps $\mathrm{NS}(X') \xrightarrow{\sim} \mathrm{NS}(X)$ may differ by an automorphism of $\mathrm{NS}(X)$ which preserves the ample cone. By Theorem 5.8, each h_i is liftable, so that by Lemma 7.10, $f' : \mathfrak{h}^2(X') \xrightarrow{\sim} \mathfrak{h}^2(X)$ admits a perfect lifting $f'_K : \mathfrak{h}^2(X'_K) \xrightarrow{\sim} \mathfrak{h}^2(X_K)$. Therefore, f', and hence f, lifts to a Hodge isometry $\mathrm{H}^2(X'_K(\boldsymbol{C}), \boldsymbol{Q}) \xrightarrow{\sim} \mathrm{H}^2(X_K(\boldsymbol{C}), \boldsymbol{Q})$ for a chosen isomorphism $\overline{K} \cong \boldsymbol{C}$. Using the smooth and proper base change theorem for étale cohomology, we see that this rational Hodge isometry is $\boldsymbol{Z}[1/p]$-integral. Now we show that it is $\boldsymbol{Z}$-integral. Indeed, we first note that f induces isomorphism $f_p : \mathrm{H}^2_{\text{ét}}(X'_{\overline{K}}, \boldsymbol{Q}_p) \xrightarrow{\sim} \mathrm{H}^2_{\text{ét}}(X_{\overline{K}}, \boldsymbol{Q}_p)$ and $f_{\text{cris}} : \mathrm{H}^2_{\text{cris}}(X'/W)[1/p] \xrightarrow{\sim} \mathrm{H}^2_{\text{cris}}(X/W)[1/p]$. We have $f_p \otimes_{\boldsymbol{Z}_p} B_{\text{cris}} = f_{\text{cris}} \otimes_W B_{\text{cris}}$ under the p-adic comparison isomorphism (see (41) in the Appendix) as it is compatible with cycle class maps, Poincaré duality and trace maps [Ito et al. 2018, Corollary 11.6]. Let S be Breuil's S-ring. Then we have an identification $\mathrm{H}^2_{\text{cris}}(X/W) \otimes_W S = \mathrm{H}^2_{\text{cris}}(X/S)$ and a similar one for X'. Now, we are given that $f_{\text{cris}} \otimes_W B_{\text{cris}}$ sends the S-module $\mathrm{H}^2_{\text{cris}}(X'/S)$ isomorphically onto $\mathrm{H}^2_{\text{cris}}(X/S)$. By [Cais and Liu 2019, Theorem 5.2] (see also Theorem A.5 and Remark A.4 below), f_p sends the $\boldsymbol{Z}_p$-lattice $\mathrm{H}^2_{\text{ét}}(X'_{\overline{K}}, \boldsymbol{Z}_p)$ isomorphically onto $\mathrm{H}^2_{\text{ét}}(X_{\overline{K}}, \boldsymbol{Z}_p)$. Therefore, we have shown that f in fact induces an integral Hodge isometry $\mathrm{H}^2(X'_K(\boldsymbol{C}), \boldsymbol{Z}) \xrightarrow{\sim} \mathrm{H}^2(X_K(\boldsymbol{C}), \boldsymbol{Z})$ which preserves the ample cones. We may now conclude using the global Torelli theorem and [Matsusaka and Mumford 1964, Theorem 2]. ☐

8. Isogenies and Hecke orbits

We briefly recall the definition of prime-to-p Hecke orbit on the orthogonal Shimura varieties. Let Λ be the K3 lattice $\boldsymbol{U}^{\oplus 3} \oplus \boldsymbol{E}_8^{\oplus 2}$, $\lambda \in \Lambda$ be a primitive element with $d := \lambda^2$ and $p > 2$ be a prime such that $p \nmid d$. We shall use the same notation for orthogonal and spinor Shimura varieties as in Section 7A with $L = L_d$ and fix $\mathsf{K}_p = G(\boldsymbol{Z}_p)$. The only difference is that this time L_d has a $\boldsymbol{Z}$-structure, so that the sheaf $\boldsymbol{L}_{\boldsymbol{A}_f^p}$ also has a $\widehat{\boldsymbol{Z}}^p$-structure. Let K_0^p denote the image of $\mathrm{CSpin}(L_d \otimes \widehat{\boldsymbol{Z}}^p)$ in $G(\boldsymbol{A}_f^p)$. More concretely, K_0^p

can be described as the maximal subgroup of $\mathrm{SO}(L_d \otimes \widehat{\boldsymbol{Z}}^p)$ which acts trivially on the discriminant group $\mathrm{disc}(L_d \otimes \widehat{\boldsymbol{Z}}^p) = \mathrm{disc}(L_d)$. A more helpful alternative description for us is that K_0^p can be viewed as the stabilizer of the element $\lambda \otimes 1$ of $\mathrm{SO}(\Lambda \otimes \widehat{\boldsymbol{Z}}^p)$, which can naturally be viewed as a subgroup of $G(\boldsymbol{A}_f^p)$.

The limit $\mathrm{Sh}_{\mathsf{K}_p}(L_d)$ is equipped with a (right) $G(\boldsymbol{A}_f^p)$-action. By the extension property of the canonical integral models, this action extends to $\mathscr{S}_{\mathsf{K}_p}(L_d)$. Recall the complex uniformization of $\mathrm{Sh}_{\mathsf{K}_p}(L_d)$

$$\mathrm{Sh}_{\mathsf{K}_p}(L_d)(\boldsymbol{C}) = G(\boldsymbol{Z}_{(p)})\backslash\Omega \times G(\boldsymbol{A}_f^p),$$

where Ω is the period domain parametrizing Hodge structures of K3 type on L_d [Madapusi Pera 2016, §3.1, 3.2; Yang 2022, Definition 3.1]. Given a point $(\omega, g) \in \Omega \times G(\boldsymbol{A}_f^p)$ and an element $g' \in G(\boldsymbol{A}_f^p)$, g' sends the class of (ω, g) in $\mathrm{Sh}_{\mathsf{K}_p}(L_d)(\boldsymbol{C})$ to that of (ω, gg'). Let k be an algebraically closed field of characteristic 0 or p. Let $\mathsf{M}_{2d,\mathsf{K}_p}$ be the moduli stack over $\boldsymbol{Z}_{(p)}$ of oriented quasipolarized K3 surfaces of degree $2d$ with hyperspecial level structure at p (see [Yang 2022, 3.3.4], where it is denoted by $\widetilde{\mathsf{M}}_{2d,\mathsf{K}_p^{\mathrm{ad}},\mathbb{Z}_{(p)}}$). By the modular interpretation of $\mathsf{M}_{2d,\mathsf{K}_p}$, $\mathsf{M}_{2d,\mathsf{K}_p}(k)$ is in natural bijection with the set of tuples (X, ξ, ϵ, η), where

- (X, ξ) is a quasipolarized K3 surface of degree $2d$ over k,
- ϵ is an isometry
$$\det(L_d \otimes \boldsymbol{Q}_2) \xrightarrow{\sim} \boldsymbol{P}^2_{\text{ét}}(X, \boldsymbol{Q}_2),$$
which naturally extends to an isometry[4]
$$\epsilon^p : \det(L_d \otimes \boldsymbol{A}_f^p) \xrightarrow{\sim} \boldsymbol{P}^2_{\text{ét}}(X, \boldsymbol{A}_f^p),$$
- η is an isometry
$$\Lambda \otimes \widehat{\boldsymbol{Z}}^p \xrightarrow{\sim} \mathrm{H}^2_{\text{ét}}(X, \widehat{\boldsymbol{Z}}^p)$$
which sends $\lambda \otimes 1$ to $c_1(\xi)$ and is compatible with the isometry ϵ^p.

Using these explicit descriptions, it is easy to write down the map $\mathsf{M}_{2d,\mathsf{K}_p}(\boldsymbol{C}) \to \mathrm{Sh}_{\mathsf{K}_p}(L_d)(\boldsymbol{C})$ explicitly: Let (X, ξ, ϵ, η) be the tuple which corresponds to a point $s \in \mathsf{M}_{2d,\mathsf{K}_p}(\boldsymbol{C})$. Choose an isomorphism $\alpha : (\Lambda \otimes \boldsymbol{Z}_{(p)}) \xrightarrow{\sim} (\mathrm{H}^2(X, \boldsymbol{Z}_{(p)}), c_1(\xi))$ which is compatible with ϵ. Then s is sent to the class of $(\omega, \eta^{-1} \circ (\alpha \otimes \boldsymbol{A}_f^p))$, where ω is the Hodge structure on L_d endowed by α. This map is clearly well defined. The integral extension $\mathsf{M}_{2d} \to \mathscr{S}_{\mathsf{K}_0}(L_d)$ is constructed and studied in [Madapusi Pera 2015]. The reader can also look at [Yang 2022, §3.3] for a quick summary of the properties.

Theorem 8.1. *Assume* $\mathrm{char}\, k = p > 2$. *If any point* $x \in \mathscr{S}_{\mathsf{K}_p}(L_d)(k)$ *lies in the image of* $\mathsf{M}_{2d,\mathsf{K}_p}(k)$, *then so does* $x \cdot g$ *for any* $g \in G(\boldsymbol{A}_f^p)$.

Proof. Let $s \in \mathsf{M}_{2d,\mathsf{K}_p}(k)$ be a point such that $x = \rho_{\mathsf{K}}(s)$. Let (X, ξ, η, ϵ) be the tuple which corresponds to s. We view $G(\boldsymbol{A}_f^p)$ as the subgroup of $\mathrm{SO}(\Lambda \otimes \boldsymbol{A}_f^p)$ which fixes $\lambda \otimes 1$.

By Theorem 1.2 and Remark 6.10, there exists a K3 surface X' together with a derived isogeny $f : \mathfrak{h}^2(X') \to \mathfrak{h}^2(X)$ such that $f_*(\mathrm{H}^2(X')) = \mathrm{H}^2_{\mathrm{cris}}(X/W) \times \mathrm{im}(g) \subset \mathrm{H}^2(X)_{\boldsymbol{Q}}$. Moreover, f is a composition

[4] For details on how to obtain this extension, see [Yang 2022, §3.3.3 or Corollary 3.3.7].

of primitive derived isogenies which come from twisted derived equivalences involving Brauer classes of prime-to-p order. Since $f_*(\mathrm{NS}(X')) = f_*(\mathrm{H}^2(X')) \cap \mathrm{NS}(X)_{\boldsymbol{Q}}$, $\xi \in f_*(\mathrm{NS}(X'))$, so that $\mathrm{NS}(X')$ contains a primitive vector of degree $2d$. By [Ogus 1979, Lemma 7.3], we can find a derived auto-isogeny δ on X which is given by reflections in (-2)-curves up to a sign such that $\delta \circ f$ sends ξ to a quasipolarization ξ'. Now we use $\delta \circ f$ to transport (ϵ, η) to similar structures (ϵ', η') on (X', ξ') so that we obtain a point $s' \in \mathsf{M}_{2d,\mathsf{K}^p}(k)$. We claim that $\rho(s') = x \cdot g$. Although $\mathscr{S}_{\mathsf{K}_p}(L_d)$ lacks a direct modular interpretation, we can do this by a lifting argument.

We claim that there exist liftings (X_W, ξ_W) and (X'_W, ξ'_W) of (X, ξ) and (X', ξ') together with an isogeny $(\mathfrak{h}^2(X'_K), \xi'_K) \to (\mathfrak{h}^2(X_K), \xi_K)$ whose étale realization agrees with $\delta \circ f$ via the smooth and proper base change theorem. If X and X' are of finite height, by Theorem 5.8, $\delta \circ f$ can be lifted to an isogeny on the nose. In the supersingular case, we first choose a lifting (X_W, ξ_W). Then X_W induces a Hodge filtration on $\mathrm{H}^2_{\mathrm{cris}}(X/W)$, which can be transported to a filtration on $\mathrm{H}^2_{\mathrm{cris}}(X'/W)$ lifting the one on $\mathrm{H}^2_{\mathrm{dR}}(X'/k)$. By the local Torelli theorem, this defines a lifting X'_W of X'. One easily checks by [Ogus 1979, Proposition 1.12] that ξ' lifts to X'_W. Now we apply [Yang 2023, Lemma 4.3.5] and Theorem 6.13.

Liftings as above induce W-points s_W and s'_W on $\mathsf{M}_{2d,\mathsf{K}_p}$ which lift s and s'. Let $x_W := \rho(s_W)$ and $x'_W := \rho(s'_W)$. Using the $G(\boldsymbol{A}_f^p)$-action, the lifting x_W of x induces a lifting x''_W of $x'' := x \cdot g$. Using the complex uniformization one quickly checks that $x''_W \otimes \boldsymbol{C} = x'_W \otimes \boldsymbol{C}$ for any embedding $K \subset \boldsymbol{C}$. Since $\mathscr{S}_{\mathsf{K}_p}(L_d)$ is a limit of separated schemes, we conclude that $x' = x''$ as desired. □

Choose a small enough compact open $\mathsf{K}^p \subseteq \mathsf{K}_0^p$ such that for $\mathsf{K} := \mathsf{K}_p\mathsf{K}^p$, $\mathscr{S}_{\mathsf{K}}(L_d)$ is a scheme and denote the period morphism $\mathsf{M}_{2d,\mathsf{K}} \to \mathscr{S}_{\mathsf{K}}(L_d)$ by ρ_{K}. For any k-point $x \in \mathscr{S}_{\mathsf{K}}(L_d)$, the image of the $G(\boldsymbol{A}_f^p)$-orbit of a lift $\tilde{x} \in \mathscr{S}_{\mathsf{K}_p}(L_d)(k)$ under the natural projection $\mathscr{S}_{\mathsf{K}_p}(L_d) \to \mathscr{S}_{\mathsf{K}}(L_d)$ is what we call the prime-to-p Hecke orbit of x.

Let $\mathscr{X}$ denote the universal family over $\mathsf{M}_{2d,\mathsf{K}}$. The mod p fiber $\mathsf{M}_{2d,\mathsf{K},\boldsymbol{F}_p}$ (resp. $\mathscr{S}_{\mathsf{K}}(L_d)_{\boldsymbol{F}_p}$) of moduli space $\mathsf{M}_{2d,\mathsf{K}}$ admits a stratification $\mathsf{M}_{2d,\mathsf{K},\boldsymbol{F}_p} = \mathsf{M}^1 \supseteq \mathsf{M}^2 \supseteq \cdots \supseteq \mathsf{M}^{20}$ (resp. $\mathscr{S}_{\mathsf{K}}(L_d)_{\boldsymbol{F}_p} = \mathscr{S}^1 \supseteq \mathscr{S}^2 \supseteq \cdots \supseteq \mathscr{S}^{20}$) such that for $1 \le i \le 10$, a geometric point s lies in M^i (resp. $\mathscr{S}^i$) if and only if $\mathscr{X}_s$ (resp. $\boldsymbol{L}_{\mathrm{cris},s}(-1)$) has height $\ge i$, and for $11 \le i \le 20$, a geometric point s lies in M^i (resp. $\mathscr{S}^i$) if and only if $\mathscr{X}_s$ (resp. $\boldsymbol{L}_{\mathrm{cris},s}(-1)$) is supersingular and has Artin invariant $\le 21 - i$. Set $\mathring{\mathsf{M}}^i := \mathsf{M}^i - \mathsf{M}^{i-1}$ and $\mathring{\mathscr{S}}^i := \mathscr{S}^i - \mathscr{S}^{i-1}$. Heights and Artin invariants are rather classical invariants. For a more modern interpretation in terms of Newton and Ekedahl–Oort (E–O) strata for $\mathscr{S}_{\mathsf{K}}(L_d)_{\boldsymbol{F}_p}$, see for example [Shen 2020, §8.4]. It follows from [Madapusi Pera 2015, Corollary 5.14] that the period morphism respects these stratifications in the sense that $\mathsf{M}^i = \mathscr{S}^i \times_{\mathscr{S}_{\mathsf{K}}(L_d)} \mathsf{M}_{2d,\mathsf{K}}$. We remark that the Zariski closure of the locally closed subscheme $\mathring{\mathscr{S}}^i$ is $\mathscr{S}^i$. By [Shen and Zhang 2022, Corollaries 7.2.2 and 7.3.4], if $1 \le i \le 10$, then $\mathscr{S}^i$ is a central leaf. The locus $\mathscr{S}^{20}$ is the superspecial locus (the unique closed E–O stratum), and is also a central leaf (see [Shen and Zhang 2022, Remark 3.2.2, Examples 6.2.4]).

In our case, the Hecke orbit conjecture predicts the following:

Conjecture 8.2. For $1 \le i \le 10$ or $i = 20$, the prime-to-p Hecke orbit of every $s \in \mathscr{S}^i(\overline{\boldsymbol{F}}_p)$ is Zariski dense in $\mathscr{S}^i$.

We remark that once the above conjecture is known for $\overline{\boldsymbol{F}}_p$, it is automatically true for any algebraically closed field over $\boldsymbol{F}_p$ by a specialization argument. Conjecture 8.2 has been proved by Maulik, Shankar, and Tang [2022, Theorem 1.4] when $i = 1$ and $p \geq 5$. We prove another special case below (Theorem 8.6).

We use N_σ to denote the supersingular lattice of Artin invariant σ. We restrict to considering the $p > 2$ case, when these lattices are characterized by [Huybrechts 2016, §17, Proposition 2.20]. The original reference [Rudakov and Shafarevich 1978] also treated the $p = 2$ case.

Lemma 8.3. *For each $d > 0$ and $i = 0, 1$, there exist a primitive element $\xi \in N_1$ with $\xi^2 = 2d$ and an $\alpha_i \in \mathrm{O}(N_1)$ such that α_i fixes ξ and interchanges the two isotropic lines in $(N_1^\vee/N_1) \otimes \boldsymbol{F}_{p^2}$ and* $\det(\alpha_i) = (-1)^i$.

Proof. The supersingular K3 surface with Artin invariant 1, which is unique up to isomorphism, is given by the desingularization of $A/A[2]$, where $A = E \times E$ for a supersingular elliptic curve E [Ogus 1979, Corollary 7.14]. Since E admits a model over $\boldsymbol{F}_p$, so does X. Let φ be a topological generator of $\mathrm{Gal}_{\boldsymbol{F}_p}$. We fix an isomorphism between N_1 and $\mathrm{NS}(X_{\overline{\boldsymbol{F}}_p})$, so that N is equipped with a $\mathrm{Gal}_{\boldsymbol{F}_p}$-action such that $\mathrm{NS}(X)$ is identified with the φ-invariants N^φ.

Let $\mathrm{NS}(A)(2)$ denote the lattice $\mathrm{NS}(A)$ but with the quadratic form multiplied by a factor of 2. As a result of the Kummer construction, there exist 16 (-2)-curves $\delta_1, \dots, \delta_{16}$ on X and an isometric embedding

$$\mathrm{NS}(A)(2) \oplus \left(\bigoplus_{i=1}^{16} \boldsymbol{Z}\delta_i \right) \hookrightarrow \mathrm{NS}(X).$$

Let $\mu \in \mathrm{NS}(A)(2)$ be a primitive element such that $\mu^2 > 0$. For some coprime numbers a and b, $(a\mu + b\delta_1)^2 = 2d$. The generator φ fixes $\xi := a\mu + b\delta_1$ and interchanges the isotropic lines in $(N_1^\vee/N_1) \otimes \boldsymbol{F}_{p^2}$ (cf. the paragraph below [Liedtke 2016, Examples 4.20]).

Let s_{δ_2} be the reflection in δ_2. Note that s_{δ_2} fixes μ and δ_1, and hence ξ. Moreover, it is not hard to check that s_{δ_2} acts trivially on $N^\vee/N$. Therefore, we can simply set α_0 and α_1 to be φ and $s_{\delta_2} \circ \varphi$, up to permutation. □

Lemma 8.4. $\mathring{\mathsf{M}}^i \neq \varnothing$ *for all i.*

Proof. Each $\mathsf{M}^{i+1} \subseteq \mathsf{M}^i$ is locally cut out by a single equation. $\mathsf{M}_{2d,\mathsf{K},\boldsymbol{F}_p}$ is smooth of dimension 19, and we know that M^{20} is zero-dimensional (cf. [Artin 1974, §7]). Therefore, it suffices to show that $\mathsf{M}^{20} \neq \varnothing$, i.e., there exists a quasipolarization of degree $2d$ on the superspecial K3 surface, which is unique up to isomorphism. This follows from the preceding lemma and [Ogus 1979, Lemma 7.9]. □

Let $\mathcal{K} \subset \widetilde{G}(\boldsymbol{A}_f^p)$ be the preimage of K. Before proceeding we recall that for any geometric point $t \in \widetilde{\mathscr{S}}_{\mathcal{K}}(L_d)$, there is a distinguished subspace $\mathrm{LEnd}(\mathscr{A}_t)$ of $\mathrm{End}(\mathscr{A}_t)$ which consists of the elements whose cohomological realizations lie in $\boldsymbol{L}_{\boldsymbol{A}_f^p,t}$ and $\boldsymbol{L}_{\mathrm{cris},t}$ ([Yang 2022, Definition 3.10]; cf. [Madapusi Pera 2016, Definition 5.11]). When t is on the supersingular locus, the natural maps $\mathrm{LEnd}(\mathscr{A}_t) \otimes \widehat{\boldsymbol{Z}}^p \to \boldsymbol{L}_{\ell,t}$ and $\mathrm{LEnd}(\mathscr{A}_t) \otimes \boldsymbol{Z}_p \to \boldsymbol{L}_{\mathrm{cris},t}^{F=1}$ are isomorphisms [Yang 2023, Proposition 3.2.3].

Lemma 8.5. *Let k be an algebraically closed field with* $\operatorname{char} k = p$. *Let x be a k-point on $\mathscr{S}^i$ for some $i \geq 11$ and t be a k-point on $\widetilde{\mathscr{S}}_{\mathsf{K}}(L_d)$ which lifts x, and set $P := \operatorname{LEnd}(\mathscr{A}_t)$. Then there exists a primitive element $\nu \in N_\sigma$ with $\sigma := 21 - i$ and $\nu^2 = 2d$ such that $P \cong \nu^\perp$.*

Proof. Let $\boldsymbol{Z}\nu$ be a quadratic lattice of rank 1 generated by ν with $\nu^2 = 2d$. By the theory of gluing lattices (see [McMullen 2011, §2] for a quick summary), primitive extensions of $P \oplus \boldsymbol{Z}\nu$ corresponds to the data (G_1, G_2, ϕ), where G_1, G_2 are subgroups of $\operatorname{disc}(P)$ and $\operatorname{disc}(\boldsymbol{Z}\nu)$ and ϕ is an isometry $G_1 \xrightarrow{\sim} G_2$. Therefore, constructing N amounts to choosing appropriate (G_1, G_2, ϕ).

In our case, we take G_1 to be the prime-to-p part of $\operatorname{disc}(P)$, i.e., $\operatorname{disc}(P \otimes \widehat{\boldsymbol{Z}}^p)$, and $G_2 = \operatorname{disc}(\boldsymbol{Z}\nu)$, which is isomorphic to $\boldsymbol{Z}/(2d)\boldsymbol{Z}$ as an abelian group. Then we construct ϕ by a lifting argument: Let x_W be a W-point on $\mathscr{S}(L_d)$ which lifts x and let $x_{\boldsymbol{C}}$ be x_W for some embedding $W \hookrightarrow \boldsymbol{C}$. The period morphism ρ_{K} is known to be surjective on $\boldsymbol{C}$-points, so there exists a quasipolarized K3 surface $(X_{\boldsymbol{C}}, \xi_{\boldsymbol{C}})$ such that the $\boldsymbol{Z}$-Hodge structure $\boldsymbol{L}_{B,x_{\boldsymbol{C}}}$ is naturally identified with $\boldsymbol{P}^2(X_{\boldsymbol{C}}, \boldsymbol{Z})$. We have that the natural map $P \otimes \widehat{\boldsymbol{Z}}^p \to \boldsymbol{L}_{\widehat{\boldsymbol{Z}}^p,x}$ is an isomorphism [Yang 2023, Proposition 3.2.3] and $\boldsymbol{L}_{\widehat{\boldsymbol{Z}}^p,x} \cong \boldsymbol{L}_{B,x_{\boldsymbol{C}}} \otimes \widehat{\boldsymbol{Z}}^p$ by smooth and proper base change and the Artin comparison isomorphisms. Therefore, there is an isomorphism $\beta_1 : G_1 \xrightarrow{\sim} \operatorname{disc}(\boldsymbol{P}^2(X_{\boldsymbol{C}}, \boldsymbol{Z}) \otimes \widehat{\boldsymbol{Z}}^p) = \operatorname{disc}(\boldsymbol{P}^2(X_{\boldsymbol{C}}, \boldsymbol{Z}))$. On the other hand, let $\beta_2 : G_2 \xrightarrow{\sim} \operatorname{disc}(\boldsymbol{Z}\xi_{\boldsymbol{C}})$ be the isomorphism given by sending ν to $\xi_{\boldsymbol{C}} \in \mathrm{H}^2(X_{\boldsymbol{C}}, \boldsymbol{Z})$.

We may transport the gluing data given by the primitive embedding $\boldsymbol{P}^2(X_{\boldsymbol{C}}, \boldsymbol{Z}) \oplus \boldsymbol{Z}\xi \subset \mathrm{H}^2(X_{\boldsymbol{C}}, \boldsymbol{Z})$ to a gluing data ϕ for G_1, G_2 via β_1, β_2. Let N be the lattice given by (G_1, G_2, ϕ). We check that it is a supersingular K3 lattice. Clearly, by our construction, $N \otimes \widehat{\boldsymbol{Z}}^p \cong \Lambda \otimes \widehat{\boldsymbol{Z}}^p$. As P is negative definite, N has signature $(1+, 21-)$. Finally, $\operatorname{disc}(N \otimes \boldsymbol{Z}_p) = \operatorname{disc}(P \otimes \boldsymbol{Z}_p) \cong (\boldsymbol{Z}/p\boldsymbol{Z})^{2\sigma}$ as an abelian group. Therefore, $N \cong N_\sigma$. □

We now prove another special case of Conjecture 8.2:

Theorem 8.6. *Conjecture 8.2 holds for $i = 20$.*

Proof. Take two $\overline{\boldsymbol{F}}_p$-points $x, x' \in \mathscr{S}^{20}$. Choose lifts t, t' for x, x' in $\widetilde{\mathscr{S}}_K(L_d)$. We only need to show that there exists a CSpin-isogeny $\mathscr{A}_t \to \mathscr{A}_{t'}$ which is prime to p. Indeed, this follows from an explicit description of the isogeny classes in $\widetilde{\mathscr{S}}_{\mathsf{K}_p}(L_d)(\overline{\boldsymbol{F}}_p)$ and their images on $\mathscr{S}_{\mathsf{K}_p}(L_d)(\overline{\boldsymbol{F}}_p)$ [Yang 2022, §3.2.3]. Let P and P' denote $\operatorname{LEnd}(\mathscr{A}_t)$ and $\operatorname{LEnd}(\mathscr{A}_{t'})$ respectively.

We first show that every isometry $P_{\boldsymbol{Q}} \xrightarrow{\sim} P'_{\boldsymbol{Q}}$ whose induced isomorphism $\boldsymbol{L}_{2,t} \otimes \boldsymbol{Q} \xrightarrow{\sim} \boldsymbol{L}_{2,t'} \otimes \boldsymbol{Q}$ sends $\delta_{2,t}$ to $\delta_{2,t'}$ is induced by a CSpin-isogeny $\psi : \mathscr{A}_t \to \mathscr{A}_{t'}$ by conjugation. Indeed, by [Yang 2023, Proposition 3.2.4], there exists some CSpin-isogeny $\psi' : \mathscr{A}_t \to \mathscr{A}_{t'}$, which induces some isomorphism $P_{\boldsymbol{Q}} \xrightarrow{\sim} P'_{\boldsymbol{Q}}$ whose induced isomorphism $\boldsymbol{L}_{2,t} \otimes \boldsymbol{Q} \xrightarrow{\sim} \boldsymbol{L}_{2,t'} \otimes \boldsymbol{Q}$ sends $\delta_{2,t}$ to $\delta_{2,t'}$. The group of CSpin-isogenies from $\mathscr{A}_t$ to itself is identified with $\operatorname{CSpin}(P_{\boldsymbol{Q}})$, which surjects to $\operatorname{SO}(P_{\boldsymbol{Q}})$. By composing ψ' with some CSpin-isogeny $\mathscr{A}_t \to \mathscr{A}_t$, we get the desired ψ.

We only need to show that there exists a CSpin-isogeny $\mathscr{A}_t \to \mathscr{A}_{t'}$ which is prime to p. By a Cartan decomposition trick [Yang 2023, Lemma 3.2.6], we only need to show the following claim:

Claim. *There exists an isometry $P \otimes \boldsymbol{Z}_{(p)} \xrightarrow{\sim} P' \otimes \boldsymbol{Z}_{(p)}$ which sends $\delta_{2,x}$ to $\delta_{2,x'}$ and extends to an isomorphism $\boldsymbol{L}_{\mathrm{cris},x} \xrightarrow{\sim} \boldsymbol{L}_{\mathrm{cris},x'}$.*

By Lemma 8.5, for some primitive vectors ξ, ξ' in N_1 with $\xi^2 = (\xi')^2 = 2d$, we have $P \cong \xi^{\perp}$ and $P' \cong (\xi')^{\perp}$. Since some reflection of $N \otimes \boldsymbol{Z}_{(p)}$ takes ξ to ξ' [Milnor and Husemoller 1973, I, Lemma 4.2], $P \otimes \boldsymbol{Z}_{(p)} \cong P' \otimes \boldsymbol{Z}_{(p)}$ as quadratic lattice over $\boldsymbol{Z}_{(p)}$. Now $P \otimes \boldsymbol{Z}_p$ and $P' \otimes \boldsymbol{Z}_p$ are the Tate modules of the supersingular K3 crystals $\boldsymbol{L}_{\mathrm{cris},t}(-1)$ and $\boldsymbol{L}_{\mathrm{cris},t'}(-1)$ respectively. By Ogus's theory of characteristic subspaces [1979, Theorem 3.20], $\boldsymbol{L}_{\mathrm{cris},t}$ (resp. $\boldsymbol{L}_{\mathrm{cris},t'}$) determines an isotropic line of $(P^{\vee}/P) \otimes \overline{\boldsymbol{F}}_p$ (resp. $((P')^{\vee}/P') \otimes \overline{\boldsymbol{F}}_p$) and the isomorphism $P \otimes \boldsymbol{Z}_p \to P' \otimes \boldsymbol{Z}_p$ extends to an isomorphism $\boldsymbol{L}_{\mathrm{cris},t} \xrightarrow{\sim} \boldsymbol{L}_{\mathrm{cris},t'}$ if and only these isotropic lines are respected. Now the claim follows from Lemma 8.3. □

Remark 8.7. As the reader can readily tell, the heart of the above theorem is the claim. Here we have proved the claim in a rather ad hoc way. We go through Lemma 8.5 because there does not seem to be a good classification theory for quadratic lattices over $\boldsymbol{Z}_{(p)}$. Moreover, P and P' are negative definite, so one cannot apply, say, Nikulin's theory to generate automorphisms, which only handles indefinite lattices. Luckily, in our special case, there is a geometric way of constructing the automorphisms we need.

Lemma 8.8. *Let k be an algebraically closed field with $\operatorname{char} k = p > 2$. Let R be a DVR over k with fraction field κ and let X_κ be a supersingular* K3 *surface over κ such that $X_{\bar\kappa}$ has Artin invariant σ_0. There exists a DVR S over k with fraction field L, a finite separable map $R \to S$, and an N_{σ_0}-marked supersingular* K3 *surface X_S over S such that $(X_S)_L \cong (X_\kappa)_L$.*

Proof. By a result of Rudakov and Shafarevich (see [Rudakov and Shafarevich 1976, Theorem 50], and [Bragg and Lieblich 2018, Theorem 5.2.1] for $p = 3$) there exists a DVR S, a finite separable map $R \to S$, and a supersingular K3 surface X_S over S such that $(X_S)_L \cong (X_\kappa)_L$. The Picard scheme $\underline{\mathrm{Pic}}_{X_L}$ is formally étale over $\operatorname{Spec} L$. As $\mathrm{Pic}(X_{\bar L})$ is finitely generated, after taking a further finite separable extension we may ensure that the restriction map $\mathrm{Pic}(X_L) \cong \mathrm{Pic}(X_{\bar L})$ is an isomorphism. Thus, X_L admits an N_{σ_0}-marking. As S is a DVR, we have $\mathrm{Pic}(X_L) = \mathrm{Pic}(X)$, so the generic marking extends uniquely to an N_{σ_0}-marking of X_S. □

Theorem 8.9. *If Conjecture 8.2 holds for i, or $i \geq 11$, then $\mathscr{S}^i \subset \mathrm{im}(\rho_{\mathsf{K}})$.*

Proof. If Conjecture 8.2 holds for i then the conclusion is a direct consequence of Theorem 8.1 and the fact that $\mathrm{im}(\rho_{\mathsf{K}})$ is open. Now assume $i \geq 11$ and take $k = \overline{\boldsymbol{F}}_p$. Note that by Theorem 8.6, $\mathscr{S}^{20} \subset \mathrm{im}(\rho_{\mathsf{K}})$. Since the Zariski closure of $\mathring{\mathscr{S}}^i$ is $\mathscr{S}^i$, the intersection $\mathrm{im}(\rho_{\mathsf{K}}) \cap \mathscr{S}^i$ is open and dense in $\mathscr{S}^i$. Take a closed point $x \in \mathring{\mathscr{S}}^i_k$. Let $\mathcal{R}$ be the ring $k[\![t]\!]$ and $\mathcal{F}$ be its fraction field. Choose an $\mathcal{R}$-valued point $\tilde{x}$ which extends x such that $\tilde{x}_{\mathcal{F}}$ lies in $\mathrm{im}(\rho_{\mathsf{K}}) \cap \mathscr{S}^i$. Such an $\tilde{x}$ can always be found: we can always choose a smooth curve which passes through x and whose generic point lies in $\mathrm{im}(\rho_{\mathsf{K}}) \cap \mathscr{S}^i$. Then we simply take the completion of this curve at x. Let $X_{\mathcal{F}}$ be a supersingular K3 surface over the generic point of $\tilde{x}_{\mathcal{F}}$. Note that the geometric fiber of $X_{\mathcal{F}}$ has Artin invariant $\sigma := 21 - i$. By the preceding lemma, there exists a DVR $\mathcal{R}'$ over $\mathcal{R}$, whose fraction field $\mathcal{F}'$ is a finite extension of $\mathcal{F}$, such that there is an N_σ-marked supersingular K3 surface $\mathcal{X}$ over $\mathcal{R}'$.

We argue that the special fiber $\mathcal{X}_k$ of $\mathcal{X}$ has Artin invariant σ. There are two families of supersingular K3 crystals over $\mathcal{R}'$ (see [Ogus 1979, §5] for the definition): One is obtained by pulling back $\boldsymbol{L}_{\mathrm{cris},\tilde{x}}(-1)$

along $\mathcal{R} \to \mathcal{R}'$. The other is given by $\mathrm{H}^2_{\mathrm{cris}}(\mathcal{X}')$. By construction, these two families agree on the generic fiber. By Proposition 4.6 and Theorem 5.3 of [Ogus 1979], there exists a universal family of supersingular K3 crystals over a smooth projective space $\mathcal{M}$ such that these two families are both obtained by pulling back the universal family along morphisms $\mathcal{R} \to \mathcal{M}$. Since $\mathcal{M}$ is in particular separated, these two morphisms have to agree. Therefore, $\mathrm{H}^2_{\mathrm{cris}}(\mathcal{X}'/\mathcal{R}')$ is precisely the pullback of $\boldsymbol{L}_{\mathrm{cris},\tilde{x}}(-1)$. Now we conclude by the hypothesis that $x \in \mathscr{S}^i_k$.

Now we know that $\mathcal{X}_{\overline{\mathcal{F}}} := \mathcal{X} \otimes \overline{\mathcal{F}}$ and $\mathcal{X}_k$ have the same Artin invariant. This guarantees that the specialization map $\mathrm{Pic}(\mathcal{X}_{\mathcal{F}'}) \to \mathrm{Pic}(\mathcal{X}_k)$ must be an isomorphism, and hence must send the ample cone isomorphically onto the ample cone. Since the big and nef cone is the closure of the ample cone, the quasipolarization on $\mathcal{X}_{\mathcal{F}'}$ extends to a quasipolarization on $\mathcal{X}_k$. This shows that $x \in \mathrm{im}(\rho_{\mathsf{K}})$. □

Finally we discuss some implications of the surjectivity of the period morphism to the good reduction theory of K3 surfaces. As Conjecture 8.2 is known for $i = 1$ and $p \geq 5$ (by [Maulik et al. 2022, Theorem 1.4]), the following result in particular implies the unconditional Theorem 1.7.

Theorem 8.10. *Let k be a perfect field of characteristic $p > 2$. Let F be a finite extension of $K = W[1/p]$. Let X_F be a* K3 *surface over F equipped with a quasipolarization ξ of degree $2d$ with $p \nmid d$. Suppose that the* Gal_F*-action on $\boldsymbol{H}^2_{\text{ét}}(X_{\bar{F}}, \boldsymbol{Q}_\ell)$ is potentially unramified for some $\ell \neq p$. Then we have*:

(a) $\mathrm{H}^2_{\text{ét}}(X_{\bar{F}}, \boldsymbol{A}^p_f)$ *and* $\mathrm{H}^2_{\text{ét}}(X_{\bar{F}}, \boldsymbol{Q}_p)$ *are potentially unramified and crystalline respectively.*

(b) *If* $\mathrm{H}^2_{\text{ét}}(X_{\bar{F}}, \boldsymbol{Q}_p)$ *is crystalline, then* $\mathbb{D}_{\mathrm{cris}}(\mathrm{H}^2_{\text{ét}}(X_{\bar{F}}, \boldsymbol{Q}_p))$ *is a* K3 *crystal.*

(c) *Suppose that the hypothesis of* (b) *is satisfied and* $\mathbb{D}_{\mathrm{cris}}(\mathrm{H}^2_{\text{ét}}(X_{\bar{F}}, \boldsymbol{Q}_p))$ *is a* K3 *crystal of height i. If Conjecture 8.2 holds for i or if $i = \infty$, then X_F has potential good reduction.*

We recall that X_F as above is said to have potential good reduction if, up to replacing F by a finite extension, there exists a smooth proper algebraic space $\mathcal{X}$ over $\mathcal{O}_F$, whose special fiber is a K3 surface over k and whose generic fiber is X_F (cf. [Liedtke and Matsumoto 2018, Definition 2.1]).

Proof. (a) and (b) Up to replacing F by a finite extension, we may equip (X, ξ) with a K-level structure and an orientation so that it is given by an F-point s on $\mathsf{M}_{2d,\mathsf{K}}$, and find a lift $t \in \widetilde{\mathscr{S}}_{\mathsf{K}}(L_d)(F)$ of $\rho_{\mathsf{K}}(s)$. Consider the abelian variety $\mathscr{A}_t$. One easily adapts the argument of Deligne [1981, §6.6] to see that, up to replacing F by a further extension, $\mathscr{A}_t$ admits good reduction. By the extension property of the integral models, we can extend t to an $\mathcal{O}_F$-valued point τ on $\mathscr{S}_{\mathsf{K}}(L_d)$. This implies both (a) and (b).

(c) We have $\tau \otimes k \in \mathscr{S}^i$. If the hypothesis is satisfied, then $\mathscr{S}^i \subset \mathrm{im}(\rho_{\mathsf{K}})$. Now we conclude by the étaleness of ρ_{K}. Indeed, the global Torelli theorem implies that if two $\boldsymbol{C}$-points of $\mathsf{M}_{2d,\mathsf{K}}$ are mapped to the same points under ρ_{K}, then the K3 surfaces they correspond to are (noncanonically) isomorphic. If there is a quasipolarized K3 surface over k whose moduli point is sent to $\tau \otimes k$, then the étaleness of ρ_{K} tells us that there exists an F-point s' of $\mathsf{M}_{2d,\mathsf{K}}$ such that $\rho_{\mathsf{K}}(s) = \rho_{\mathsf{K}}(s')$. Up to replacing F by a finite extension, the K3 surfaces defined by s and s' are isomorphic. □

Appendix: Some results from integral p-adic Hodge theory

We review some basic results in p-adic Hodge theory. Let k be a perfect field of characteristic $p > 0$. We write W for $W(k)$ and K_0 for $W[1/p]$. Let K be a totally ramified extension of K_0 and let π be a uniformizer of its ring of integers $\mathcal{O}_K$.[5] Let G_K denote the absolute Galois group Gal_K. Set $R := \mathcal{O}_K/(p)$.

Let $f : \mathcal{X} \to \operatorname{Spec} \mathcal{O}_K$ be a smooth and proper scheme (more generally, the following discussion applies also when $\mathcal{X}$ is only a formal scheme and $\mathcal{X}_K$ denotes the rigid analytic generic fiber). The subject of p-adic Hodge theory is concerned with how to recover the following tuples of data from one another under suitable assumptions:

(A) The $\boldsymbol{Z}_p$-module $\mathrm{H}^i_{\text{ét}}(\mathcal{X}_{\overline{K}}, \boldsymbol{Z}_p)$ equipped with a G_K-action.

(B) The F-crystal $\boldsymbol{R}^i f_{R,\mathrm{cris}*}\mathcal{O}_{X_R}$ over $\boldsymbol{Cris}(R/W)$ together with the filtered $\mathcal{O}_K$-module $\mathrm{H}^i_{\mathrm{dR}}(\mathcal{X}/\mathcal{O}_K)$.

(B′) The F-crystal $\mathrm{H}^i_{\mathrm{cris}}(\mathcal{X}_k/W)$ together with the filtered $\mathcal{O}_K$-module $\mathrm{H}^i_{\mathrm{dR}}(\mathcal{X}/\mathcal{O}_K)$.

Remark A.1. Let e be the ramification degree of $\mathcal{O}_K$ over W. When $e \le p-1$, $R \cong k[\varepsilon]/\varepsilon^e$ has a PD structure, so that the category of crystals of quasicoherent sheaves over $\boldsymbol{Cris}(R/W)$ is equivalent to that over $\boldsymbol{Cris}(k/W)$ [Berthelot and Ogus 1978, Corollary 6.7]. Therefore, under mild torsion-freeness assumptions on various cohomology modules of $\mathcal{X}$, (B) and (B′) are equivalent data. Moreover, as $\mathcal{O}_K$ is a PD thickening of W, the crystalline de Rham comparison theorem gives us a canonical isomorphism

$$\mathrm{H}^i_{\mathrm{cris}}(\mathcal{X}_k/W) \underset{W}{\otimes} \mathcal{O}_K \cong \mathrm{H}^i_{\mathrm{dR}}(\mathcal{X}/\mathcal{O}_K).$$

If $e > p-1$, then (B) contains strictly more information than (B′). The above isomorphism no longer holds integrally in general. However, there is still a canonical isomorphism after inverting p:

$$\mathrm{H}^i_{\mathrm{cris}}(\mathcal{X}_k/W) \underset{W}{\otimes} K \cong \mathrm{H}^i_{\mathrm{dR}}(\mathcal{X}/\mathcal{O}_K) \underset{\mathcal{O}_K}{\otimes} K.$$

This isomorphism is often called the Berthelot–Ogus isomorphism because it was first introduced in [Berthelot and Ogus 1983]. Below we will often make use of this isomorphism implicitly. Note that in the above isomorphisms, the left-hand side only depends on the special fiber $\mathcal{X}_k$, whereas the right-hand side is equipped with the additional data of a Hodge filtration, which in general depends on the lifting $\mathcal{X}$ of $\mathcal{X}_k$.

Here is an overview of the relationship between the above tuples: The classical (rational) p-adic comparison isomorphisms tell us how to recover (A) and (B′) from one another after inverting p. Integral p-adic Hodge theory (e.g., the seminal paper of Bhatt, Morrow, and Scholze [Bhatt et al. 2018]) tells us how to recover (B) from (A). For our purposes, we are mainly concerned with how to recover (A) from (B). Roughly speaking, the way to do this is to evaluate the F-crystal $\boldsymbol{R}^i f_{R,\mathrm{cris}*}\mathcal{O}_{X_R}$ on a certain PD-thickening S of R (S is often called Breuil's S-ring), so that we obtain an S-module. This S-module is equipped with a Frobenius action from the F-crystal structure on $\boldsymbol{R}^i f_{R,\mathrm{cris}*}\mathcal{O}_{X_R}$, and is moreover equipped with a filtration which absorbs the data of the Hodge filtration on $\mathrm{H}^i_{\mathrm{dR}}(\mathcal{X}/\mathcal{O}_K)$. The main result of

[5] This notation is chosen to be in line with most references in p-adic Hodge theory. In the main text, the letters K and F take the roles of K_0 and K respectively. We apologize for this inconsistency of notation.

[Cais and Liu 2019] tells us that by applying a certain functor (denoted by T_{cris} below) to this S-module, we recover (A). Of course, [Bhatt et al. 2018] already treats the relationship between (A) and (B), but the conclusions there are packaged in a more abstract way.

After inverting p. Let $\mathrm{MF}_K^{\varphi,N}$ denote the category of filtered (φ, N)-modules. An object of this category is a K_0-vector space D which is equipped with

- a Frobenius semilinear injection $\varphi : D \to D$;
- a K_0-linear map $N : D \to D$ such that $N\varphi = pN\varphi$;
- a descending filtration on D_K such that $\mathrm{Fil}^i D_K = D_K$ for $i \ll 0$ and $\mathrm{Fil}^i D_K = 0$ for $i \gg 0$.

Let MF_K^{φ} denote the subcategory with $N = 0$. The motivation to consider this category is that the data in (A) is naturally an object in MF_K^{φ} after inverting p, because there is a canonical Berthelot–Ogus isomorphism $\mathrm{H}^i_{\mathrm{cris}}(\mathcal{X}_k/W) \otimes_W K \cong \mathrm{H}^i_{\mathrm{dR}}(\mathcal{X}/\mathcal{O}_K) \otimes_{\mathcal{O}_K} K$. We will use this isomorphism repeatedly without explicitly mentioning it. We remark that in most references the operator N is in $\mathrm{MF}_K^{\varphi,N}$ to treat varieties with semistable reductions. Since we are assuming good reduction, we may restrict to considering the category MF_K^{φ}.

Let Rep_{G_K} denote the category of G_K-representations over $\boldsymbol{Q}_p$ and let $\mathrm{Rep}_{G_K}^{\mathrm{cris}}$ denote the subcategory of crystalline representations. Given an object $Q \in \mathrm{Rep}_{G_K}^{\mathrm{cris}}$, one may define an object in MF_K^{φ} using the (covariant) Fontaine's functors D_{cris} and D_{dR}, which are defined by $D_{\mathrm{cris}}(Q) = (Q \otimes_{\boldsymbol{Q}_p} B_{\mathrm{cris}})^{G_K}$ and $D_{\mathrm{dR}}(Q) = (Q \otimes_{\boldsymbol{Q}_p} B_{\mathrm{dR}})^{G_K}$. The pair $(D_{\mathrm{cris}}(Q), D_{\mathrm{dR}}(Q))$ are equipped with a Frobenius action and filtrations respectively, and hence define an object in MF_K^{φ}. We abusively denote the resulting functor $\mathrm{Rep}_{G_K} \to \mathrm{MF}_K^{\varphi}$ also by D_{cris}. We define a functor from the essential image of D_{cris} to $\mathrm{Rep}_{G_K}^{\mathrm{cris}}$ by $V_{\mathrm{cris}} = \mathrm{Fil}^0(D \otimes_{K_0} B_{\mathrm{cris}})^{\varphi=1}$. There is an equality of $\boldsymbol{Q}_p$-submodules

$$Q = V_{\mathrm{cris}}(D_{\mathrm{cris}}(Q))$$

of $(Q \otimes_{\boldsymbol{Q}_p} B_{\mathrm{cris}}) \otimes_{K_0} B_{\mathrm{cris}}$, which specifies a natural transformation $V_{\mathrm{cris}} \circ D_{\mathrm{cris}} \Rightarrow \mathrm{id}$ on $\mathrm{Rep}_{G_K}^{\mathrm{cris}}$.[6] The reader may look at [Brinon and Conrad 2009, Part I, Sections 8 and 9] for more details about these objects.

By [Bhatt et al. 2018, Proposition 5.1, Theorem 14.6], there is a p-adic comparison isomorphism

$$\mathrm{H}^i_{\mathrm{cris}}(\mathcal{X}_k/W) \underset{W}{\otimes} B_{\mathrm{cris}} \xrightarrow{\sim} \mathrm{H}^i_{\mathrm{\acute{e}t}}(\mathcal{X}_{\bar{K}}, \boldsymbol{Z}_p) \underset{\boldsymbol{Z}_p}{\otimes} B_{\mathrm{cris}} \tag{41}$$

which respects the Gal_F-actions and filtrations. Therefore, we obtain an isomorphism of objects in MF_K^{φ}

$$D_{\mathrm{cris}}(\mathrm{H}^i_{\mathrm{\acute{e}t}}(\mathcal{X}_{\bar{F}}, \boldsymbol{Q}_p)) \xrightarrow{\sim} (\mathrm{H}^i_{\mathrm{cris}}(\mathcal{X}_k/W)[1/p], \mathrm{H}^i_{\mathrm{dR}}(\mathcal{X}_K/K)). \tag{42}$$

There are multiple rational p-adic comparison isomorphisms of the form (41) (e.g., those constructed earlier by Faltings [1999], Tsuji [1999], and others). We choose to use the one from [Bhatt et al. 2018] because this is the one used in [Cais and Liu 2019], to be cited below. Once we fix this choice of rational p-adic comparison isomorphism, then the isomorphism (42) is also fixed.

[6] Note that the natural transformations between two functors between 1-categories (or locally small categories in the usual sense) do form a *set* (as opposed to a groupoid), so it makes sense to specify an element in this set.

Recovering integral lattices. We now explain how to recover the natural integral lattices in the objects of (42) from one another. Let $\mathfrak{S} := W[\![u]\!]$, and let $\theta : \mathfrak{S} \to \mathcal{O}_K$ be the map sending u to π. Let $\mathrm{Rep}^{\mathrm{cris}\circ}_{\mathrm{Gal}_K}$ denote the category of G_K-stable $\boldsymbol{Z}_p$-lattices in objects of $\mathrm{Rep}^{\mathrm{cris}}_{G_K}$. Let $\mathfrak{M}(-)$ be the functor as in [Kisin 2010, Theorem 1.2.1] which sends an object in $\mathrm{Rep}^{\mathrm{cris}\circ}_{G_K}$ to a Breuil–Kisin module in the sense of [Bhatt et al. 2018, Theorem 4.4], so that there exist canonical isomorphisms

$$\varphi^*\mathfrak{M}(T) \underset{\mathfrak{S}}{\otimes} K_0 \xrightarrow{\sim} D_{\mathrm{cris}}(T[1/p]) \quad \text{and} \quad \varphi^*\mathfrak{M}(T) \underset{\mathfrak{S},\theta}{\otimes} K \xrightarrow{\sim} D_{\mathrm{dR}}(T[1/p]) \tag{43}$$

which preserve Frobenius actions and filtrations respectively. Then we have the following result [Bhatt et al. 2018, Theorem 14.6].

Theorem A.2. *Assume that $\mathrm{H}^i_{\mathrm{cris}}(\mathcal{X}_k/W)$ and $\mathrm{H}^{i+1}(\mathcal{X}_k/W)$ are torsion-free. Then for $T = \mathrm{H}^i_{\text{ét}}(\mathcal{X}_{\overline{K}}, \boldsymbol{Z}_p)$ the isomorphisms (43) map $\mathfrak{M}(T) \otimes_{\mathfrak{S}} W$ and $\mathfrak{M}(T) \otimes_{\mathfrak{S},\theta} \mathcal{O}_K$ isomorphically onto $\mathrm{H}^i_{\mathrm{cris}}(\mathcal{X}_k/W)$ and $\mathrm{H}^i_{\mathrm{dR}}(\mathcal{X}/\mathcal{O}_K)$ respectively, when composed with the isomorphisms in (41).*

We refer the reader also to [Ito et al. 2018, Theorem 3.2] for an exposition which is closer to ours in notation. The above theorem tells us how to recover (B′) from (A). Under the additional assumption that $i < p - 1$, [Cais and Liu 2019, Theorem 5.4] tells us how to recover (A) from (B). Before doing so we need to introduce the intermediate category of Breuil's S-modules, which packages the data of (B) in a different way.

Breuil's S-modules. Let S denote the p-adic completion of the PD envelope of $(\mathfrak{S}, \ker\theta)$. Let S_π denote the ring $W[\![u-\pi]\!]$. Then there is an embedding $\iota : S \hookrightarrow S_\pi$ which sends u to $u - \pi$. Let $f_\pi : S_\pi \to \mathcal{O}_K$ (resp. $f_0 : S \to W$) be the projection which sends $u - \pi$ to 0 (resp. u to 0). Then there is a commutative diagram of W-algebras

$$\begin{array}{ccc} S & \xrightarrow{\ \iota\ } & S_\pi \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f_\pi} \\ W & \longrightarrow & \mathcal{O}_K \end{array}$$

In [Breuil 1997], the above ring S is denoted by S^0_{min}. The letter S in loc. cit. denotes a certain extension of S^0_{min,K_0}. For our purposes, one may simply take $S = S^0_{\mathrm{min},K_0}$ when reading [Breuil 1997]. The letter S in our notation is in line with [Cais and Liu 2019] and [Liu 2008].

Let $\mathcal{MF}^{\varphi,N}_{S_{K_0}}$ denote the category of filtered (φ, N)-modules over S_{K_0}.[7] There is an equivalence of categories

$$\eta : \mathrm{MF}^{\varphi,N}_K \to \mathcal{MF}^{\varphi,N}_{S_{K_0}} \tag{44}$$

which sends $(D, \mathrm{Fil}^\bullet D_K, \varphi, N)$ to an object $(\mathscr{D}, \mathrm{Fil}^\bullet \mathscr{D}, \varphi_{\mathscr{D}}, N_{\mathscr{D}})$ with $\mathscr{D} = D \otimes_W S$ ([Cais and Liu 2019, p. 1215]; see also [Breuil 1997, Theorem 6.1.1]). The quasi-inverse η^{-1} is defined by $(\mathscr{D} \otimes_{f_0} W, \mathscr{D} \otimes_{f_\pi \circ \iota} \mathcal{O}_K)$, for which the Frobenius action and filtration are inherited from those on $\mathscr{D}$. There is a

[7] This is just the category denoted by $\mathcal{MF}(\varphi, N)$ in [Liu 2008, §2.2], except that we have not restricted to positive objects, so that we replace the condition $\mathrm{Fil}^0 \mathscr{D} = \mathscr{D}$ by $\mathrm{Fil}^j \mathscr{D} = \mathscr{D}$ for $j \ll 0$.

canonical natural transformation $\eta^{-1} \circ \eta \Rightarrow \mathrm{id}$ on MF_K^{φ} which underlies the tautological identification of modules

$$(D, D_K) = (D \underset{W}{\otimes} S \underset{f_0}{\otimes} W, D_K \underset{W}{\otimes} S \underset{f_\pi \circ \iota}{\otimes} \mathcal{O}_K).$$

A strongly divisible S-lattice (of height r) in an object $\mathscr{D} \in \mathcal{MF}_{S_K}^{\varphi,N}$ with $\mathrm{Fil}^0 \mathscr{D} = \mathscr{D}$ is an S-lattice such that $\mathscr{M}[1/p] = \mathscr{D}$, $N_{\mathscr{D}}(\mathscr{M}) \subseteq \mathscr{M}$, and $\varphi_{\mathscr{D}}(\mathrm{Fil}^r \mathscr{M}) \subseteq p^r \mathscr{M}$, where $\mathrm{Fil}^r \mathscr{M} := \mathscr{M} \cap \mathrm{Fil}^r \mathscr{D}$. Let $\mathcal{MF}_S^{\varphi,N}$ denote the category of strongly divisible S-lattices in objects of $\mathcal{MF}_{S_{K_0}}^{\varphi,N}$.

Theorem A.3 (Liu). *Suppose that $Q \in \mathrm{Rep}_{G_K}^{\mathrm{cris}}$ has Hodge–Tate weights in $\{0, 1, \dots, p-2\}$. Let $\mathscr{D}$ denote $\eta(Q)$. The covariant functor $T_{\mathrm{cris}} : \mathscr{M} \mapsto \mathrm{Fil}^0(\mathscr{M} \otimes_S A_{\mathrm{cris}})^{\varphi=1}$ defines a bijection between the set of strongly divisible S-lattices in $\mathscr{D}$ and that of G_K-stable $\mathbf{Z}_p$-lattices in $V_{\mathrm{cris}}(D_{\mathrm{cris}}(Q)) = Q$.*

Proof. Theorem 2.3.5 of [Liu 2008] tells us that the above theorem holds for Breuil's functor T_{st}. The contravariant version of this functor is reviewed in Section 2.2 of loc. cit. If we use the superscript (resp. subscript) $*$ to indicate contravariance (resp. covariance), then $T_{\mathrm{st}}^*(-) = T_{*\mathrm{st}}((-)^\vee)$. Proposition 3.5.1 of loc. cit. tells us that $T_{\mathrm{st}}(\mathscr{M}) = T_{\mathrm{cris}}(\mathscr{M})$ as Q is crystalline. □

Remark A.4. Let $\mathcal{C}$ denote the full subcategory of $\mathcal{MF}_S^{\varphi,N}$ whose image in $\mathcal{MF}_{S_{K_0}}^{\varphi,N}$ lies in the essential image of MF_K^{φ} (as a subcategory of $\mathrm{MF}_K^{\varphi,N}$) under η. To sum up, we now have a commutative diagram of categories

$$\begin{array}{ccccccc}
\mathrm{Rep}_{G_K}^{\mathrm{cris}} & \underset{V_{\mathrm{cris}}}{\overset{D_{\mathrm{cris}}}{\rightleftarrows}} & \mathrm{MF}_K^{\varphi} & \hookrightarrow & \mathrm{MF}_K^{\varphi,N} & \underset{\eta^{-1}}{\overset{\eta}{\rightleftarrows}} & \mathcal{MF}_{S_{K_0}}^{\varphi,N} \\
\uparrow & & & & & & \uparrow \\
\mathrm{Rep}_{G_K}^{\mathrm{cris}\circ} & & \xleftarrow{\quad T_{\mathrm{cris}} \quad} & & \mathcal{C} & \hookrightarrow & \mathcal{MF}_S^{\varphi,N}
\end{array}$$

in which the vertical arrows are given by inverting p. Moreover, the natural transformations $V_{\mathrm{cris}} \circ D_{\mathrm{cris}} \Rightarrow \mathrm{id}$ and $\eta^{-1} \circ \eta \Rightarrow \mathrm{id}$ are tautological. By the above theorem, T_{cris} is an equivalence of categories. We remark that since A_{cris} is a W-subalgebra of B_{cris} and the inclusion $A_{\mathrm{cris}} \subseteq B_{\mathrm{cris}}$ respects the filtration and Frobenius structures, $T_{\mathrm{cris}}(\mathscr{M})$ is a priori a $\mathbf{Z}_p$-submodule of $V_{\mathrm{cris}}(D_{\mathrm{cris}}(Q))$. The reason that we emphasize the natural transformations used is to decategorify the language, so that T_{cris}, which is often stated as an equivalence of categories, is concretely an equality of sets.

Theorem A.5 (Cais and Liu). *Assume that $H_{\mathrm{cris}}^i(\mathcal{X}_k/W)$ and $H_{\mathrm{cris}}^{i+1}(\mathcal{X}_k/W)$ are torsion-free and $i \le p-2$. Set $\mathscr{M} := \mathrm{H}_{\mathrm{cris}}^i(\mathcal{X}_R/S)$. Let $\mathfrak{p} : \mathscr{M} \to \mathrm{H}_{\mathrm{cris}}^i(\mathcal{X}_k/W)$ be the canonical projection induced by f_0. Let $\mathscr{D} \in \mathcal{MF}_{S_{K_0}}^{\varphi,N}$ be given by the object $(\mathrm{H}_{\mathrm{cris}}^i(\mathcal{X}_k)_K, \mathrm{Fil}^\bullet \mathrm{H}_{\mathrm{dR}}^i(\mathcal{X}_K/K))$ in MF_K^{φ} via η. Then we have:*

(a) *There is a canonical section s to $\mathfrak{p}[1/p]$ such that s is φ-equivariant and $s \otimes_W S$ induces an isomorphism $\mathscr{M}[1/p] \xrightarrow{\sim} \mathscr{D}$.*

(b) *Under the isomorphism in* (a), *$\mathscr{M}$ defines a strongly divisible S-lattice in $\mathscr{D}$ and $T_{\mathrm{cris}}(\mathscr{M}) = \mathrm{H}_{\mathrm{\acute{e}t}}^i(\mathcal{X}_{\overline{K}}, \mathbf{Z}_p)$.*

Proof. Part (a) is a variant of the Berthelot–Ogus isomorphism [Cais and Liu 2019, Proposition 5.1]. Part (b) follows from [Cais and Liu 2019, Theorem 5.4(2)] and its proof, which proceeds by reducing to proving the equality of two lattices.

Let T be an object of $\mathrm{Rep}_{G_K}^{\mathrm{cris}\circ}$ and let $\mathfrak{M}(T)$ be the Breuil–Kisin module associated to T. Let $M(-)$ be the functor defined by $\varphi^*(\mathfrak{M}(-))$. Then $\underline{\mathscr{M}}(M(T)) := M(T) \otimes_{\mathfrak{S}} S$ can be equipped with additional structures so that it becomes an object in $\mathcal{MF}_S^{\varphi,N}$. The base-change-to-$S$ functor $\underline{\mathscr{M}}$ used here is defined in (3.6) of loc. cit. There is a natural isomorphism $\underline{\mathscr{M}}(M(T))[1/p] \xrightarrow{\sim} \eta(D_{\mathrm{cris}}(T[1/p]))$ which lifts the isomorphism $M(T) \otimes_{\mathfrak{S}} K_0 \xrightarrow{\sim} D_{\mathrm{cris}}(T[1/p])$ in (43). Moreover, T_{cris} sends the strongly divisible S-lattice $\underline{\mathscr{M}}(M(T))$ to T. The reader may also check out the proof of [Snowden 2014, Lemma A.3] for entirely similar considerations.

Now let T be $\mathrm{H}^i_{\mathrm{\acute{e}t}}(\mathcal{X}_{\overline{K}}, \mathbf{Z}_p)$. Since T_{cris} establishes a bijection between strongly divisible S-lattices in $\mathscr{D}$ and G_K-stable $\mathbf{Z}_p$-lattices in $T[1/p]$, one reduces to showing an equality of S-lattices $\mathscr{M} = \underline{\mathscr{M}}(M(T))$ under the isomorphisms

$$\underline{\mathscr{M}}(M(T))[1/p] \cong \mathscr{D} \cong \mathscr{M}[1/p].$$

This is the main step in the proof of [Cais and Liu 2019, Theorem 5.4(2)] (see the second paragraph on page 1226). □

Remark A.6. In the above setting, let $f : \mathcal{X}_R \to \mathrm{Spf}(R)$ be the structure morphism and let $\boldsymbol{H}^i_{\mathrm{cris}}(\mathcal{X}_R)$ denote the F-crystal $\boldsymbol{R}^i f_{\mathrm{cris}*}\mathcal{O}_{\mathcal{X}_R}$. Then $\mathrm{H}^i_{\mathrm{cris}}(\mathcal{X}_R/S)$ (resp. $\mathrm{H}^i_{\mathrm{cris}}(\mathcal{X}_R/\mathcal{O}_K)$) can be viewed as a the S-module given by evaluating $\boldsymbol{H}^i_{\mathrm{cris}}(\mathcal{X}_R)$ on the object S (resp. $\mathcal{O}_K$) of $\boldsymbol{Cris}(R/W)$. The morphism $\theta : S \to \mathcal{O}_K$ defines a canonical isomorphism $\theta^*\boldsymbol{H}^i_{\mathrm{cris}}(\mathcal{X}_R)_S \xrightarrow{\sim} \boldsymbol{H}^i_{\mathrm{cris}}(\mathcal{X}_R)_{\mathcal{O}_K}$. The lifting $\mathcal{X}$ of $\mathcal{X}_R$ to $\mathcal{O}_K$ endows $\boldsymbol{H}^i_{\mathrm{cris}}(\mathcal{X}_R)_{\mathcal{O}_K}$ with a Hodge filtration via the crystalline de Rham comparison $\boldsymbol{H}^i_{\mathrm{cris}}(\mathcal{X}_R)_{\mathcal{O}_K} \cong \mathrm{H}^i_{\mathrm{dR}}(\mathcal{X}/\mathcal{O}_K)$. The S-module $\boldsymbol{H}^i_{\mathrm{cris}}(\mathcal{X}_R)_S$, being an object of $\mathcal{MF}_S^{\varphi,N}$, is also equipped with a natural filtration, which maps isomorphically onto the Hodge filtration on $\boldsymbol{H}^i_{\mathrm{dR}}(\mathcal{X}/\mathcal{O}_K)$. However, note that the filtration on $\boldsymbol{H}^i_{\mathrm{cris}}(\mathcal{X}_R)_S$ is defined in a more formal way, with the Hodge filtration on $\boldsymbol{H}^i_{\mathrm{dR}}(\mathcal{X}/\mathcal{O}_K)$ being the key input. Namely, one first constructs $\mathscr{D}$ out of $(\mathrm{H}^i_{\mathrm{cris}}(\mathcal{X}_k/W), \mathrm{H}^i_{\mathrm{dR}}(\mathcal{X}/\mathcal{O}_K))$, and then defines a filtration on $\mathscr{M}$ by intersecting with $\mathrm{Fil}^\bullet \mathscr{D}$ under the isomorphism in part (a) of the above theorem. One naturally wonders whether this filtration has a more direct cohomological construction. This question is addressed in [Cais and Liu 2019, §6.1]. However, we won't make use of this cohomological interpretation.

Remark A.7. If $\mathcal{X}$ is a smooth proper scheme over $\mathcal{O}_K$, or more generally a smooth proper algebraic space over $\mathcal{O}_K$ whose special and generic fibers are schemes, then the above results hold for $\mathcal{X}_K$ interpreted as the generic fiber in the usual sense. The point is that the analytification of the generic fiber is functorially isomorphic to the rigid analytic generic fiber of the formal completion of $\mathcal{X}$ at the special fiber. The reader may look at [Ito et al. 2018, §11.2] for details.

Applications to p-divisible groups. Let $\mathscr{G}$ be a p-divisible group over $\mathcal{O}_K$ and assume $p \geq 3$. Let $T_p(-)$ denote the Tate module functor, $\mathbb{D}(-)$ denote the contravariant Dieudonné module functor and $\mathscr{G}^*$ denote the Cartier dual of $\mathscr{G}$. There is a p-adic comparison isomorphism

$$\mathbb{D}(\mathscr{G}_k) \underset{W}{\otimes} B_{\mathrm{cris}} \xrightarrow{\sim} T_p\mathscr{G}^*(-1) \underset{\boldsymbol{Z}_p}{\otimes} B_{\mathrm{cris}} \tag{45}$$

which induces an isomorphism $D_{\mathrm{cris}}(T_p\mathscr{G}^*(-1) \otimes_{\boldsymbol{Z}_p} \boldsymbol{Q}_p) \xrightarrow{\sim} \mathbb{D}(\mathscr{G}_k)[1/p]$. $T_{\mathrm{cris}}(\mathbb{D}(\mathscr{G}_R)_S)$ recovers the $\boldsymbol{Z}_p$-lattice $T_p\mathscr{G}^*(-1)$ inside $T_p\mathscr{G}^*(-1) \otimes_{\boldsymbol{Z}_p} \boldsymbol{Q}_p$ [Kisin 2006, Lemma 2.2.4]. Note that $T_p\mathscr{G}^*(-1)$ is canonically isomorphic to $(T_p\mathscr{G})^\vee$.

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