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Shintani–Barnes cocycles and values of the zeta functions of algebraic number fields

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We construct a new Eisenstein cocycle, called the Shintani–Barnes cocycle, which specializes in a uniform way to the values of the zeta functions of general number fields at positive integers. Our basic strategy is to generalize the construction of the Eisenstein cocycle presented in the work of Vlasenko and Zagier by using some recent techniques developed by Bannai, Hagihara, Yamada, and Yamamoto in their study of the polylogarithm for totally real fields. We also closely follow the work of Charollois, Dasgupta, and Greenberg. In fact, one of the key ingredients which enables us to deal with general number fields is the introduction of a new technique, called the "exponential perturbation", which is a slight modification of the Q-perturbation studied in their work.

1. Introduction

It is classically known that the Hecke integral formula [1917] expresses the zeta function of a number field of degree g as an integral of the Eisenstein series over a certain torus orbit on the locally symmetric space for $\mathrm{SL}_g(\mathbb{Z})$.

In some special cases, typically in the case where the number field is totally real, it is known that such an integral formula has a cohomological interpretation, and this often enables us to access the algebraic properties of the special values of the zeta function. More precisely, one can construct a certain $(g-1)$-cocycle on $\mathrm{SL}_g(\mathbb{Z})$ which can be thought as an algebraic counterpart of the Eisenstein series, and a $(g-1)$-cycle on $\mathrm{SL}_g(\mathbb{Z})$ which can be thought as an algebraic counterpart of the torus orbit, so that their pairing gives the value of the zeta function of a given totally real number field. Such a

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cocycle is often called the Eisenstein cocycle. Actually, many different kinds of Eisenstein cocycles have been constructed and studied by Harder [1987], Sczech [1993], Nori [1995], Solomon [1998], Hill [2007], Vlasenko and Zagier [2013], Charollois, Dasgupta, and Greenberg [Charollois et al. 2015], Beilinson, Kings, and Levin [Beilinson et al. 2018], Bergeron, Charollois, and Garcia [Bergeron et al. 2020], Flórez, Karabulut, and Wong [Flórez et al. 2019], Lim and Park [2019], Bannai, Hagihara, Yamada and Yamamoto [Bannai et al. 2023], and Sharifi and Venkatesh [2020], and various applications have been obtained. However, the number fields previously treated are basically limited to totally real fields or totally imaginary fields. The aim of this paper is to propose a new formulation in which we can treat all number fields in a uniform way.

1.1. ***Shintani cocycles.*** Among these many kinds of construction of the Eisenstein cocycle, a method we use in this paper is called Shintani's method, and the Eisenstein cocycles constructed by Shintani's method are often called the Shintani cocycles;[1] see [Solomon 1998; Hill 2007; Charollois et al. 2015; Lim and Park 2019; Bannai et al. 2023]. Roughly speaking, a Shintani cocycle is constructed as a family of objects (e.g., functions, formal power series, distributions, etc.) indexed by rational cones in $\mathbb{R}^g$. Therefore, what we do in this paper is basically the following:

(1) Define a certain object "ψ_C" for each rational cone $C \subset \mathbb{R}^g$.

(2) Prove that the family $(\psi_C)_C$ satisfies the "cocycle relation".

(3) Prove that the cohomology class defined by $(\psi_C)_C$ specializes to the special values of the zeta function of a given number field.

Let $g, k \geq 1$ be integers. In this paper, we say that a matrix $Q \in \mathrm{GL}_g(\mathbb{Q})$ is *irreducible* if its characteristic polynomial is irreducible over $\mathbb{Q}$. In Section 6, for a rational open cone

$$C_I = \sum_{i=1}^{g} \mathbb{R}_{>0}\alpha_i \subset \mathbb{R}^g$$

generated by $I = (\alpha_1, \dots, \alpha_g) \in (\mathbb{Q}^g - \{0\})^g$, and an irreducible matrix $Q \in \mathrm{GL}_g(\mathbb{Q})$, we consider a holomorphic function

$$\psi^Q_{kg,I}(y) := \mathrm{sgn}(I) \sum_{x \in C^Q_I \cap \mathbb{Z}^g - \{0\}} \frac{1}{\langle x, y\rangle^{g+kg}}$$

on

$$\Bigl\{ y \in \mathbb{C}^g \mid \text{there exists } \lambda \in \mathbb{C}^\times \text{ such that for all } i \in \{1, \dots, g\}, \mathrm{Re}(\langle \alpha_i, \lambda y\rangle) > 0 \Bigr\} \subset \mathbb{C}^g - \{0\},$$

where

- $\mathrm{sgn}(I) = \mathrm{sgn}(\det(\alpha_1, \dots, \alpha_g)) \in \{0, \pm 1\}$,
- the bracket $\langle x, y\rangle = {}^t x y$ denotes the dot product,
- C^Q_I is the "exponential Q-perturbation" of the cone C_I (Section 5.1).

[1]The terminology seems to depend on the authors. We adopt this convention in this paper.

Then we prove that the collection $(\psi^Q_{kg,I})_{I,Q}$ defines a class

$$[\Psi_{kg}] \in H^{g-1}(Y^\circ, \mathrm{SL}_g(\mathbb{Z}), \mathscr{F}^{\Xi}_{kg})$$

of the equivariant cohomology of a certain $\mathrm{SL}_g(\mathbb{Z})$-equivariant sheaf $\mathscr{F}^{\Xi}_{kg}$ on $Y^\circ := \mathbb{C}^g - \boldsymbol{i}\mathbb{R}^g$; see Section 3 and Theorem 6.2.5. We call our Shintani cocycle the Shintani–Barnes cocycle because the function $\psi^Q_{kg,I}(y)$ is essentially the Barnes zeta function.

Then for a number field $F/\mathbb{Q}$ of degree g, a fractional ideal $\mathfrak{a} \subset F$, and a continuous map $\chi : F^\times_{\mathbb{R}} \to \mathbb{Z}$, we construct a specialization map

$$H^{g-1}(Y^\circ, \mathrm{SL}_g(\mathbb{Z}), \mathscr{F}^{\Xi}_{kg}) \to H^{g-1}_{\mathrm{sing}}(F^\times_{\mathbb{R}}/\mathcal{O}^\times_{F,+}, \mathbb{C}) \to \mathbb{C},$$

using a certain integral operator; see (8-11). The image of the Shintani–Barnes cocycle $[\Psi_{kg}]$ under this specialization map can be computed using the classical Hurwitz formula (Proposition 7.1.3, Example 7.2.4) and a version of the Shintani cone decomposition (Proposition 8.2.1). As a result, we prove that the class $[\Psi_{kg}]$ maps to the value of the partial zeta function,

$$\pm \frac{\sqrt{D_{\mathcal{O}_F}}\, N\mathfrak{a}(k!)^g}{(g+gk-1)!} \zeta_{\mathcal{O}_F}(\boldsymbol{\varepsilon}^{k+1}\chi, \mathfrak{a}^{-1}, k+1),$$

under the specialization map, where $\boldsymbol{\varepsilon} : F^\times_{\mathbb{R}} \to \{\pm 1\}$ is the sign character; see Theorem 8.3.2.

The idea of using the Barnes zeta functions is based on the work of Vlasenko and Zagier [2013] dealing with the values of the zeta functions of real quadratic fields at positive integers, and the idea of constructing the Shintani cocycle as a Čech cocycle of an equivariant sheaf is based on the work of Bannai, Hagihara, Yamada, and Yamamoto [Bannai et al. 2023], in which the higher-dimensional polylogarithm associated to a totally real field is studied. Moreover, the concept of the exponential Q-perturbation C^Q_I of a cone C_I is a slight modification of the Q-perturbation studied by Charollois, Dasgupta, Greenberg [Charollois et al. 2015] and Yamamoto [2010]. We use irreducible matrices $Q \in \mathrm{GL}_g(\mathbb{Q})$ instead of the "irrational vectors" used in [Charollois et al. 2015]. These three ideas are the main ingredients in this paper which enable us to deal with general number fields.

1.2. ***Structure of the paper.*** Sections 2–5 are devoted to preparing some tools that are necessary for the definition of the Shintani–Barnes cocycle. More precisely, in Section 2 we review some elementary facts about irreducible matrices of $\mathrm{GL}_g(\mathbb{Q})$ and their relationship to number fields. In Section 3 we introduce the sheaves $\mathscr{F}_d$ and $\mathscr{F}^{\Xi}_d$ on $Y^\circ = \mathbb{C}^g - \boldsymbol{i}\mathbb{R}^g$, and examine the basic properties of these sheaves. Then in Section 4 we compute the equivariant cohomology groups of these sheaves using the equivariant Čech complex. In Section 5 we introduce the notion of the exponential perturbation, and prove the cocycle relation satisfied by rational cones. Based on these preparations, in Section 6 we give the definition of the Shintani–Barnes cocycle.

The remaining sections (Sections 7 and 8) are devoted to showing that we can obtain the special values of the zeta functions as a specialization of the Shintani–Barnes cocycle. In Section 7 we first introduce a certain integral operator, and construct the first half of the specialization map. In Section 8 we finish the construction of the specialization map using a version of the Shintani cone decomposition, and finally prove the main result, Theorem 8.3.2.

2. Preliminaries

Conventions.

- Throughout the paper we fix an integer $g \geq 1$.
- For a ring R, a vector $x \in R^g$ is always regarded as a column vector, and the matrix algebra $M_g(R)$ acts on R^g by the matrix multiplication from the left.
- For $x_1, \ldots, x_g \in R^g$, we often regard $(x_1, \ldots, x_g)$ as a $g \times g$-matrix whose columns are $x_1, \ldots, x_g$.
- For $\gamma \in M_g(R)$, its transpose is denoted by ${}^t\gamma \in M_g(R)$.
- The bracket
$$\langle\ ,\ \rangle : R^g \times R^g \to R, \quad (x, y) \mapsto \langle x, y\rangle = {}^t x y$$
denotes the standard scalar product (the dot product, not a Hermitian product even if $R = \mathbb{C}$).
- If A and B are sets, then $A - B$ denotes the relative complement of B in A.
- Let $\{S_\lambda\}_{\lambda \in \Lambda}$ be a family of sets. For $s \in \prod_{\lambda\in\Lambda} S_\lambda$, the λ-component of s is often denoted by $s_\lambda \in S_\lambda$.

2.1. ***Irreducible matrices.*** In this subsection we review some basic facts about irreducible matrices of $\mathrm{GL}_g(\mathbb{Q})$. We say that a matrix $Q \in \mathrm{GL}_g(\mathbb{Q})$ is *irreducible over* $\mathbb{Q}$ if the characteristic polynomial of Q is an irreducible polynomial over $\mathbb{Q}$. We often drop "over $\mathbb{Q}$" if it is obvious from the context. Let

$$\Xi := \{Q \in \mathrm{GL}_g(\mathbb{Q}) \mid Q \text{ is irreducible over } \mathbb{Q}\}$$

denote the set of irreducible matrices of $\mathrm{GL}_g(\mathbb{Q})$. The group $\mathrm{GL}_g(\mathbb{Q})$ acts on Ξ by the conjugate action. For $Q \in \Xi$ and $\gamma \in \mathrm{GL}_g(\mathbb{Q})$, let

$$[\gamma](Q) := \gamma Q \gamma^{-1} \in \Xi$$

denote this conjugate action.

Now, for $Q \in \Xi$, let

$$\Gamma_Q := \mathrm{Stab}_{\mathrm{SL}_g(\mathbb{Z})}(Q) = \left\{\gamma \in \mathrm{SL}_g(\mathbb{Z}) \mid [\gamma](Q) = \gamma Q \gamma^{-1} = Q\right\}$$

denote the subgroup of $\mathrm{SL}_g(\mathbb{Z})$ stabilizing Q. Moreover, let

$$F_Q := \mathbb{Q}[Q] \subset M_g(\mathbb{Q}) \quad \text{and} \quad \mathcal{O}_Q := F_Q \cap M_g(\mathbb{Z}) \subset F_Q$$

denote the subalgebras of $M_g(\mathbb{Q})$ generated by Q over $\mathbb{Q}$ and its "$M_g(\mathbb{Z})$-part" respectively.

Lemma 2.1.1. *Let $Q \in \Xi$, and let $f_Q(X) \in \mathbb{Q}[X]$ be the characteristic polynomial of Q.*

(1) *Q has g distinct eigenvalues in $\mathbb{C}$, and hence Q is diagonalizable in* $\mathrm{GL}_g(\mathbb{C})$.

(2) *There are no nonzero proper Q-stable $\mathbb{Q}$-subspaces of $\mathbb{Q}^g$.*

(3) *For any nonzero vector* $x \in \mathbb{Q}^g - \{0\}$, *the map*

$$F_Q \xrightarrow{\sim} \mathbb{Q}^g, \quad \gamma \mapsto \gamma x$$

is an isomorphism of $\mathbb{Q}$*-vector spaces.*

(4) *The* $\mathbb{Q}$*-algebra* F_Q *is a field of degree* g *over* $\mathbb{Q}$, *and we have*

$$N_{F_Q/\mathbb{Q}}(\gamma) = \det \gamma$$

for $\gamma \in F_Q$, *where* $N_{F_Q/\mathbb{Q}}$ *is the norm of the field extension* $F_Q/\mathbb{Q}$.

(5) *We have*

$$F_Q = \{\gamma \in M_g(\mathbb{Q}) \mid \gamma Q = Q\gamma\}.$$

(6) *We have*

$$\Gamma_Q = \{\gamma \in \mathcal{O}_Q \mid N_{F_Q/\mathbb{Q}}(\gamma) = 1\} \subset \mathcal{O}_Q^\times,$$

i.e., Γ_Q *is the norm-one unit group of* $\mathcal{O}_Q$.

(7) *The action of* Γ_Q *on* $\mathbb{Q}^g - \{0\}$ *is free, i.e., for any* $x \in \mathbb{Q}^g - \{0\}$ *and* $\gamma \in \Gamma_Q$, *we have* $\gamma x = x$ *if and only if* $\gamma = 1$.

Proof. (1) This follows from the fact that $f_Q(X)$ is an irreducible polynomial over $\mathbb{Q}$.

(2) This also follows from the irreducibility of $f_Q(X)$. Indeed, if $V \subset \mathbb{Q}^g$ is a Q-stable $\mathbb{Q}$-subspace, then the characteristic polynomial of $Q|_V$ divides $f_Q(X)$.

(3) and (4) First, since $x \neq 0$, the image of the map

$$F_Q \to \mathbb{Q}^g, \quad \gamma \mapsto \gamma x$$

is a nonzero Q-stable $\mathbb{Q}$-subspace. Hence, by (2), this map is surjective. Now, again since $f_Q(X)$ is an irreducible polynomial over $\mathbb{Q}$, we see that $F_Q \simeq \mathbb{Q}[X]/(f_Q(X))$ is a field of degree g over $\mathbb{Q}$. Therefore, by comparing the dimension, we find that the above map is an isomorphism. The identity $N_{F_Q/\mathbb{Q}}(\gamma) = \det \gamma$ is nothing but the definition of the norm.

(5) Let F'_Q denote the right-hand side. The inclusion $F_Q \subset F'_Q$ is obvious. We compare the dimension. First we have

$$F'_Q \otimes_{\mathbb{Q}} \mathbb{C} \subset F''_Q := \{\gamma \in M_g(\mathbb{C}) \mid \gamma Q = Q\gamma\}.$$

Then, by (1), the right-hand side F''_Q is simultaneously diagonalizable in $M_g(\mathbb{C})$. Therefore, F''_Q is isomorphic to the space of diagonal matrices. Thus we find

$$\dim_{\mathbb{Q}} F'_Q = \dim_{\mathbb{C}} F'_Q \otimes_{\mathbb{Q}} \mathbb{C} \leq \dim_{\mathbb{C}} F''_Q = g = \dim_{\mathbb{Q}} F_Q,$$

and hence we obtain $F_Q = F'_Q$.

(6) This follows directly from (4) and (5).

(7) By (6), we see that $\Gamma_Q \subset F_Q^\times$, and by (3) and (4), we see that $F_Q^\times$ acts freely on $\mathbb{Q}^g - \{0\}$. □

2.2. ***Review on number fields.*** In this subsection we take a closer look at the relationship between irreducible matrices and number fields.

Let $F/\mathbb{Q}$ be a number field of degree g, and let

$$\tau_1, \dots, \tau_g : F \hookrightarrow \mathbb{C}$$

be the field embeddings of F into $\mathbb{C}$, i.e., $\{\tau_1, \dots, \tau_g\} = \mathrm{Hom}_{\mathrm{field}}(F, \mathbb{C})$.[2] Let $\mathcal{O} \subset F$ be an order in F, i.e., $\mathcal{O} \subset F$ is a subring which is a finitely generated $\mathbb{Z}$-module and generates F over $\mathbb{Q}$. Let $\mathfrak{a} \subset F$ be a proper fractional $\mathcal{O}$-ideal, i.e., $\mathfrak{a} \subset F$ is a finitely generated $\mathcal{O}$-submodule such that

$$\{\alpha \in F \mid \alpha\mathfrak{a} \subset \mathfrak{a}\} = \mathcal{O}. \tag{2-1}$$

Let $w_1, \dots, w_g \in \mathfrak{a}$ be a basis of $\mathfrak{a}$ over $\mathbb{Z}$, and put

$$w := {}^t(w_1, \dots, w_g) \in F^g \quad \text{and} \quad w^{(i)} := \tau_i(w) = {}^t(\tau_i(w_1), \dots, \tau_i(w_g)) \in \mathbb{C}^g$$

for $i = 1, \dots, g$. We define the norm polynomial $N_w(x) = N_w(x_1, \dots, x_g) \in \mathbb{Q}[x_1, \dots, x_g]$ with respect to this basis by

$$N_w(x) := \prod_{i=1}^{g} \langle x, w^{(i)} \rangle \in \mathbb{Q}[x_1, \dots, x_g],$$

where $x = (x_1, \dots, x_g)$. The situation can be summarized in the following diagram:

$$\begin{array}{ccccccc}
x & \in & \mathbb{Z}^g & \subset & \mathbb{Q}^g & \xrightarrow{\;N_w\;} & \mathbb{Q} \\
\downarrow & & \downarrow\wr & & \downarrow\wr & & \| \\
\langle x, w \rangle & \in & \mathfrak{a} & \subset & F & \xrightarrow[N_{F/\mathbb{Q}}]{} & \mathbb{Q}
\end{array}$$

Moreover, let

$$\rho_w : F \to M_g(\mathbb{Q})$$

be the regular representation of F with respect to the basis $w_1, \dots, w_g$, i.e., for $\alpha \in F$ and $x \in \mathbb{Q}^g$, we have

$$\langle \rho_w(\alpha)x, w \rangle = \alpha\langle x, w \rangle = \langle x, \alpha w \rangle \in F. \tag{2-2}$$

Dual objects. Let $w_1^*, \dots, w_g^* \in F$ be the dual basis of $w_1, \dots, w_g$ with respect to the field trace $\mathrm{Tr}_{F/\mathbb{Q}}$, i.e.,

$$\mathrm{Tr}_{F/\mathbb{Q}}(w_i w_j^*) = \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases}$$

Then it is easy to see that $w_1^*, \dots, w_g^*$ form a $\mathbb{Z}$-basis of a proper fractional $\mathcal{O}$-ideal

$$\mathfrak{a}^* := \{\alpha \in F \mid \mathrm{Tr}_{F/\mathbb{Q}}(\alpha\mathfrak{a}) \subset \mathbb{Z}\}.$$

---

[2] At this stage we don't make a distinction between real embeddings and complex embeddings. Later, in Section 8.2, we will make such a distinction for convenience.

We define

$$w^* := {}^t(w_1^*, \dots, w_g^*) \in F^g,$$
$$w^{*(i)} := \tau_i(w^*) = {}^t(\tau_i(w_1^*), \dots, \tau_i(w_g^*)) \in \mathbb{C}^g,$$
$$N_{w^*}(x) := \prod_{i=1}^{g} \langle x, w^{*(i)} \rangle \in \mathbb{Q}[x_1, \dots, x_g],$$

and

$$\rho_{w^*} : F \to M_g(\mathbb{Q})$$

in the same way as above, starting from the dual basis $w_1^*, \dots, w_g^*$.

Lemma 2.2.1. *Let $\theta \in F^\times$ be an element such that $F = \mathbb{Q}(\theta)$. Put $Q = \rho_w(\theta) \in \mathrm{GL}_g(\mathbb{Q})$.*

(1) *We have $Q \in \Xi$. Conversely, any element of Ξ can be obtained in this way.*

(2) *The regular representation $\rho_w : F \to M_g(\mathbb{Q})$ induces isomorphisms*

$$\begin{array}{ccc} F & \xrightarrow[\sim]{\rho_w} & F_Q \\ \cup & & \cup \\ \mathcal{O} & \xrightarrow[\sim]{} & \mathcal{O}_Q \\ \cup & & \cup \\ \mathcal{O}^1 & \xrightarrow[\sim]{} & \Gamma_Q \end{array}$$

where $\mathcal{O}^1 := \{u \in \mathcal{O}^\times \mid N_{F/\mathbb{Q}}(u) = 1\}$ is the norm-one unit group of $\mathcal{O}$.

(3) *$w^{*(1)}, \dots, w^{*(g)} \in \mathbb{C}^g$ are the dual basis of $w^{(1)}, \dots, w^{(g)} \in \mathbb{C}^g$ with respect to the scalar product $\langle\ ,\ \rangle$, i.e., we have*

$$\langle w^{*(i)}, w^{(j)} \rangle = \delta_{ij}.$$

(4) *For $\alpha \in F$, we have*

$$\rho_{w^*}(\alpha) = {}^t\rho_w(\alpha).$$

(5) *Let $\alpha \in F$. Then $w^{(i)}$ is an eigenvector of ${}^t\rho_w(\alpha)$ with eigenvalue $\tau_i(\alpha)$.*

(6) *Let $\alpha \in F$. Then $w^{*(i)}$ is an eigenvector of $\rho_w(\alpha)$ with eigenvalue $\tau_i(\alpha)$.*

(7) *For $\gamma \in \Gamma_Q$, we have*

$$N_w(\gamma x) = N_w(x) \quad \textit{and} \quad N_{w^*}({}^t\gamma x) = N_{w^*}(x).$$

Proof. (1) Since θ generates F, the characteristic polynomial of $Q = \rho_w(\theta)$ is irreducible, and hence $Q \in \Xi$. The latter half of the statement follows from Lemma 2.1.1(3), (4). Indeed, for $Q \in \Xi$, fix a nonzero vector $x \in \mathbb{Q}^g$ and take a basis $w_1, \dots, w_g \in F_Q$ corresponding to the standard basis of $\mathbb{Q}^g$ via the isomorphism

$$F_Q \xrightarrow{\sim} \mathbb{Q}^g, \quad \gamma \mapsto \gamma x.$$

Let $\mathfrak{a} \subset F_Q$ be the subset corresponding to $\mathbb{Z}^g \subset \mathbb{Q}^g$ under this isomorphism. Then we easily see that $\mathfrak{a}$ is a proper $\mathcal{O}_Q$-ideal and that ρ_w is the natural inclusion $F_Q \hookrightarrow M_g(\mathbb{Q})$. Hence we find that $Q = \rho_w(Q)$.

(2) The first isomorphism $F \xrightarrow{\sim} F_Q$ is obvious. The second isomorphism follows from (2-1), and the third follows from Lemma 2.1.1(6).

(3) Put

$$W := (w^{(1)}, \dots, w^{(g)}) = (\tau_j(w_i))_{ij} \in M_g(\mathbb{C}) \quad \text{and} \quad W^* := (w^{*(1)}, \dots, w^{*(g)}) = (\tau_j(w_i^*))_{ij} \in M_g(\mathbb{C}).$$

Then, by definition, we have

$$W{}^tW^* = (\mathrm{Tr}_{F/\mathbb{Q}}(w_i w_j^*))_{ij} = 1 \in M_g(\mathbb{C}), \tag{2-3}$$

and hence

$$(\langle w^{*(i)}, w^{(j)} \rangle)_{ij} = {}^tW^* W = 1.$$

(4)–(6) First, by (2-2), we have

$$\langle x, \alpha w \rangle = \langle \rho_w(\alpha)x, w \rangle = \langle x, {}^t\rho_w(\alpha)w \rangle \in F$$

for all $x \in \mathbb{Q}^g$. Therefore, we find that $\alpha w = {}^t\rho_w(\alpha)w \in F^g$. By applying τ_i, we obtain (5). In particular,

$$W \mathrm{diag}(\tau_1(\alpha), \dots, \tau_g(\alpha)) = {}^t\rho_w(\alpha)W, \tag{2-4}$$

where $\mathrm{diag}(\tau_1(\alpha), \dots, \tau_g(\alpha)) \in M_g(\mathbb{C})$ is the diagonal matrix with diagonal entries $\tau_1(\alpha), \dots, \tau_g(\alpha)$. Similarly, we have

$$W^* \mathrm{diag}(\tau_1(\alpha), \dots, \tau_g(\alpha)) = {}^t\rho_{w^*}(\alpha)W^*. \tag{2-5}$$

On the other hand, by using (2-3) and (2-4) we also find that

$$\mathrm{diag}(\tau_1(\alpha), \dots, \tau_g(\alpha)){}^tW^* = {}^tW^{*t}\rho_w(\alpha),$$

and hence, by taking the transpose, we have

$$W^* \mathrm{diag}(\tau_1(\alpha), \dots, \tau_g(\alpha)) = \rho_w(\alpha)W^*. \tag{2-6}$$

By comparing (2-5) and (2-6), we obtain (4) and (6).

(7) This follows from (2), (5), and (6). Indeed, take $u \in \mathcal{O}^1$ such that $\rho_w(u) = \gamma$. Then we have

$$N_w(\gamma x) = \prod_{i=1}^{g} \langle \gamma x, w^{(i)} \rangle = \prod_{i=1}^{g} \langle x, {}^t\rho_w(u)w^{(i)} \rangle = N_{F/\mathbb{Q}}(u)N_w(x) = N_w(x).$$

The statement for $N_{w^*}(x)$ can be proved similarly. □

3. The space Y° and the sheaves $\mathscr{F}_d$ and $\mathscr{F}_d^\Xi$

3.1. *Definitions.* Let $\mathbb{P}^{g-1}(\mathbb{C}) = (\mathbb{C}^g - \{0\})/\mathbb{C}^\times$ be the complex projective $(g-1)$–space, and let

$$\pi_{\mathbb{C}} : \mathbb{C}^g - \{0\} \to \mathbb{P}^{g-1}(\mathbb{C})$$

be the natural projection. We define an open subset Y° of $\mathbb{C}^g - \{0\}$ by

$$Y^\circ := \mathbb{C}^g - \boldsymbol{i}\mathbb{R}^g \subset \mathbb{C}^g - \{0\},$$

where $\boldsymbol{i} \in \mathbb{C}$ is the imaginary unit. The group $\mathrm{GL}_g(\mathbb{Q})$ acts on $\mathbb{C}^g - \{0\}$, Y°, and $\mathbb{P}^{g-1}(\mathbb{C})$ by the matrix action from the left. For an integer $d \geq 0$, we define a sheaf $\mathscr{F}_d$ on Y° as

$$\mathscr{F}_d := \pi_{\mathbb{C}}^{-1} \Omega^{g-1}_{\mathbb{P}^{g-1}(\mathbb{C})}(-d)|_{Y^\circ},$$

where $\Omega^{g-1}_{\mathbb{P}^{g-1}(\mathbb{C})}(-d)$ is the $(-d)$-th Serre twist of the sheaf $\Omega^{g-1}_{\mathbb{P}^{g-1}(\mathbb{C})}$ of holomorphic $(g-1)$-forms on $\mathbb{P}^{g-1}(\mathbb{C})$, and $\pi_{\mathbb{C}}^{-1}$ is the inverse image functor of sheaves. Furthermore, we define

$$\mathscr{F}_d^\Xi := \underline{\mathrm{Hom}}(\underline{\mathbb{Z}[\Xi]}, \mathscr{F}_d) \simeq \prod_{Q \in \Xi} \mathscr{F}_d,$$

where $\underline{\mathbb{Z}[\Xi]}$ is the constant sheaf associated to the free abelian group $\mathbb{Z}[\Xi]$ generated by the set Ξ of irreducible matrices of $\mathrm{GL}_g(\mathbb{Q})$, and $\underline{\mathrm{Hom}}$ is the sheaf Hom. For $Q \in \Xi$, let

$$\mathrm{ev}_Q : \mathscr{F}_d^\Xi \to \mathscr{F}_d \tag{3-1}$$

denote the evaluation map at Q. See Remark 3.1.1 below.

Remark 3.1.1. (1) More generally, for a sheaf $\mathscr{F}$ (of abelian groups) on Y°, we define

$$\mathscr{F}^\Xi := \underline{\mathrm{Hom}}(\underline{\mathbb{Z}[\Xi]}, \mathscr{F}).$$

Note that for an open subset $U \subset Y^\circ$, we have

$$\Gamma(U, \underline{\mathrm{Hom}}(\underline{\mathbb{Z}[\Xi]}, \mathscr{F})) = \mathrm{Hom}(\underline{\mathbb{Z}[\Xi]}|_U, \mathscr{F}|_U) = \mathrm{Hom}(\mathbb{Z}[\Xi], \Gamma(U, \mathscr{F})) = \mathrm{Map}(\Xi, \Gamma(U, \mathscr{F})).$$

Then the evaluation map $\mathrm{ev}_Q : \mathscr{F}^\Xi \to \mathscr{F}$ is given by

$$\mathrm{ev}_Q : \Gamma(U, \mathscr{F}^\Xi) = \mathrm{Map}(\Xi, \Gamma(U, \mathscr{F})) \to \Gamma(U, \mathscr{F}), \quad \phi \mapsto \phi(Q).$$

(2) By (1) we also see that $\mathscr{F}^\Xi \simeq \prod_{Q \in \Xi} \mathscr{F}$.

(3) The sheaf $\mathscr{F}_d^\Xi$ is an analogue of the group $\mathcal{N}$ considered in [Charollois et al. 2015].

Remark 3.1.2. The sections of the sheaf $\Omega^{g-1}_{\mathbb{P}^{g-1}(\mathbb{C})}(-d)$ on an open subset $U \subset \mathbb{P}^{g-1}(\mathbb{C})$ can be described as follows. First, let ω be a holomorphic $(g-1)$-form on $\mathbb{C}^g - \{0\}$ defined by

$$\omega(y_1, \dots, y_g) := \sum_{i=1}^{g} (-1)^{i-1} y_i \, dy_1 \wedge \cdots \wedge \check{dy_i} \wedge \cdots \wedge dy_g$$

for $y = {}^t(y_1, \dots, y_g) \in \mathbb{C}^g - \{0\}$, where $\check{dy_i}$ means that dy_i is omitted.

Then we have

$$\Gamma(U, \Omega^{g-1}_{\mathbb{P}^{g-1}(\mathbb{C})}(-d)) \simeq \{f\omega \mid f \text{ holomorphic function on } \pi_{\mathbb{C}}^{-1}(U) \text{ such that } f(\lambda y) = \lambda^{-g-d} f(y) \text{ for all } \lambda \in \mathbb{C}^{\times}\}. \tag{3-2}$$

In this paper we use this as a definition of the sheaf $\Omega^{g-1}_{\mathbb{P}^{g-1}(\mathbb{C})}(-d)$.

The sheaf $\Omega^{g-1}_{\mathbb{P}^{g-1}(\mathbb{C})}(-d)$ has a natural $\mathrm{GL}_g(\mathbb{Q})$-equivariant structure via the pullback of differential forms. Since $\pi_{\mathbb{C}}$ is a $\mathrm{GL}_g(\mathbb{Q})$-equivariant map, this induces $\mathrm{GL}_g(\mathbb{Q})$-equivariant structures on $\mathscr{F}_d$ and $\mathscr{F}_d^{\Xi}$. We describe these $\mathrm{GL}_g(\mathbb{Q})$-equivariant structures more explicitly in Section 3.3.

3.2. ***A vanishing result.*** Here our aim is to compute the cohomology group $H^q(U, \pi_{\mathbb{C}}^{-1}\Omega^{g-1}_{\mathbb{P}^{g-1}(\mathbb{C})}(-d))$ for convex open subsets $U \subset \mathbb{C}^g - \{0\}$. Actually, we will show that

$$H^q(U, \pi_{\mathbb{C}}^{-1}\Omega^{g-1}_{\mathbb{P}^{g-1}(\mathbb{C})}(-d)) = 0$$

for $q \geq 1$, and also give an explicit description of $H^0(U, \pi_{\mathbb{C}}^{-1}\Omega^{g-1}_{\mathbb{P}^{g-1}(\mathbb{C})}(-d))$.

Let

$$\mathbb{D} := \{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0\}$$

be the right half-plane. We start with the following elementary lemma.

Lemma 3.2.1. *Let X be a paracompact manifold, and let $\mathrm{pr}_1 : X \times \mathbb{D} \to X$ be the first projection. Let $U \subset X \times \mathbb{D}$ be an open subset such that for any $x \in X$, the set*

$$\{z \in \mathbb{D} \mid (x, z) \in U\}$$

is a nonempty convex subset of $\mathbb{D}$. Then there exists a continuous section $s : X \to U$ of $\mathrm{pr}_1|_U : U \to X$ such that $s \circ \mathrm{pr}_1$ is homotopic to the identity map id_U over X, i.e., there exists a continuous map

$$h : [0, 1] \times U \to U$$

such that $h(0, u) = s \circ \mathrm{pr}_1(u)$, $h(1, u) = u$, and $\mathrm{pr}_1 \circ h(t, u) = \mathrm{pr}_1(u)$ for $t \in [0, 1]$ and $u \in U$.

Proof. In order to construct a section, it suffices to construct a continuous map

$$f : X \to \mathbb{D}$$

such that $(x, f(x)) \in U$ for all $x \in X$. First, by assumption, for each $x \in X$ we can take $z_x \in \mathbb{D}$ such that $(x, z_x) \in U$. Then there exist an open neighborhood $U_x \subset X$ of x and an open neighborhood $V_x \subset \mathbb{D}$ of z_x such that $U_x \times V_x \subset U$. Since $X = \bigcup_{x \in X} U_x$ and X is paracompact, there exists a subset $\Lambda \subset X$ such that $\{U_\lambda\}_{\lambda \in \Lambda}$ is a locally finite open covering of X. Note that for $x \in U_\lambda$, we have

$$(x, z_\lambda) \in U_\lambda \times V_\lambda \subset U.$$

By using the paracompactness once again, there exists a partition of unity with respect to the open covering $\{U_\lambda\}_{\lambda \in \Lambda}$, i.e., a collection $\{\phi_\lambda\}_{\lambda \in \Lambda}$ of continuous maps

$$\phi_\lambda : X \to [0, 1]$$

such that $\operatorname{supp}(\phi_\lambda) \subset U_\lambda$ and $\sum_{\lambda\in\Lambda} \phi_\lambda(x) = 1$ for all $x \in X$. Put

$$f := \sum_{\lambda\in\Lambda} z_\lambda \phi_\lambda : X \to \mathbb{D}.$$

Then, by the convexity assumption, we see that

$$(x, f(x)) = \left(x, \sum_{\lambda\in\Lambda} z_\lambda \phi_\lambda(x)\right) \in U$$

for all $x \in X$. Thus we obtain a section

$$s : X \to U, \quad x \mapsto (x, f(x)).$$

Again by the convexity assumption, we see that $s \circ \mathrm{pr}_1$ is homotopic to the identity map id_U over X. Indeed,

$$h : [0, 1] \times U \to U, \quad (t, (x, z)) \mapsto (x, tz + (1 - t) f(x))$$

gives a homotopy between $s \circ \mathrm{pr}_1$ and id_U over X. □

Lemma 3.2.2. *Let $U \subset \mathbb{C}^g - \{0\}$ be a convex open subset.*

(1) *There exists $x \in \mathbb{C}^g - \{0\}$ such that $U \subset V_x := \{y \in \mathbb{C}^g - \{0\} \mid \mathrm{Re}(\langle x, y\rangle) > 0\}$.*

(2) *The projection $\pi_{\mathbb{C}}|_U : U \to \pi_{\mathbb{C}}(U)$ has a continuous section $s : \pi_{\mathbb{C}}(U) \to U$ such that $s \circ \pi_{\mathbb{C}}|_U$ is homotopic to the identity map id_U over $\pi_{\mathbb{C}}(U)$.*

(3) *The image $\pi_{\mathbb{C}}(U)$ is a Stein manifold.*

Proof. (1) By the so-called hyperplane separation theorem [Rudin 1991, Theorem 3.4(a)] applied to U and $\{0\}$, there exist $x \in \mathbb{C}^g - \{0\}$ and $\mu \in \mathbb{R}$ such that

$$0 = \mathrm{Re}(\langle x, 0\rangle) \le \mu < \mathrm{Re}(\langle x, y\rangle)$$

for all $y \in U$, and hence $U \subset V_x = \{y \in \mathbb{C}^g - \{0\} \mid \mathrm{Re}(\langle x, y\rangle) > 0\}$.

(2) We first construct a section $s_x : \pi_{\mathbb{C}}(V_x) \to V_x$ of $\pi_{\mathbb{C}}|_{V_x}$ as follows. Set

$$V_x^1 := \{y \in \mathbb{C}^g - \{0\} \mid \langle x, y\rangle = 1\} \subset V_x.$$

Then we easily see that $\pi_{\mathbb{C}}|_{V_x^1} : V_x^1 \xrightarrow{\sim} \pi_{\mathbb{C}}(V_x)$ is a biholomorphism. Thus we define

$$s_x := (\pi_{\mathbb{C}}|_{V_x^1})^{-1} : \pi_{\mathbb{C}}(V_x) \xrightarrow{\sim} V_x^1 \subset V_x$$

to be the inverse map of $\pi_{\mathbb{C}}|_{V_x^1}$, which is clearly a section of $\pi_{\mathbb{C}}|_{V_x}$. Then we have a trivialization φ of $\pi_{\mathbb{C}}|_{V_x}$

$$\begin{array}{ccc} \pi_{\mathbb{C}}(V_x) \times \mathbb{D} & \xrightarrow[\sim]{\varphi} & V_x \\ & {}_{\mathrm{pr}_1}\searrow \quad \swarrow_{\pi_{\mathbb{C}}|_{V_x}} & \\ & \pi_{\mathbb{C}}(V_x) & \end{array}$$

defined by $\varphi(z, \lambda) := \lambda s_x(z)$ for $(z, \lambda) \in \pi_{\mathbb{C}}(V_x) \times \mathbb{D}$.

Therefore, it suffices to construct a continuous section s' of

$$p := \mathrm{pr}_1|_{\varphi^{-1}(U)} : \varphi^{-1}(U) \xrightarrow{\mathrm{pr}_1} \pi_{\mathbb{C}}(U)$$

such that $s' \circ p$ is homotopic to $\mathrm{id}_{\varphi^{-1}(U)}$ over $\pi_{\mathbb{C}}(U)$. By Lemma 3.2.1, it suffices to show the following:

Claim. *For any $z \in \pi_{\mathbb{C}}(U)$, the set*

$$\mathbb{D}_z := \{\lambda \in \mathbb{D} \mid (z, \lambda) \in \varphi^{-1}(U)\}$$

is a nonempty convex subset of $\mathbb{D}$.

Proof of claim. Let $z \in \pi_{\mathbb{C}}(U)$. The set $\mathbb{D}_z$ is obviously nonempty. Suppose that $\lambda, \lambda' \in \mathbb{D}_z$, i.e., $\lambda s_x(z), \lambda' s_x(z) \in U$. Then for $t \in [0, 1]$, we have $(t\lambda + (1-t)\lambda')s_x(z) \in U$ because U is convex, and hence $t\lambda + (1-t)\lambda' \in \mathbb{D}_z$. □

(3) From the above argument, we see that $\pi_{\mathbb{C}}(U)$ is an open subset of

$$\pi_{\mathbb{C}}(V_x) \simeq V_x^1 \simeq \mathbb{C}^{g-1}.$$

Since every pseudoconvex open subset of $\mathbb{C}^{g-1}$ is a Stein manifold (see [Hörmander 1973, Theorem 4.2.8, Example after Definition 5.1.3]), it suffices to see that $\pi_{\mathbb{C}}(U)$ is pseudoconvex. This follows, for example, from [Hörmander 1994, Proposition 4.6.3, Theorem 4.6.8]. (Use [Hörmander 1994, Theorem 4.6.8] for $X = U$, $z_0 = 0$, and $L(y) = \langle x, y\rangle$. Note that a convex set U is obviously $\mathbb{C}$ convex.) □

Proposition 3.2.3. *Let $U \subset \mathbb{C}^g - \{0\}$ be a convex open subset.*

(1) *The natural map*

$$H^q(\pi_{\mathbb{C}}(U), \Omega^{g-1}_{\mathbb{P}^{g-1}(\mathbb{C})}(-d)) \xrightarrow{\sim} H^q(U, \pi_{\mathbb{C}}^{-1}\Omega^{g-1}_{\mathbb{P}^{g-1}(\mathbb{C})}(-d))$$

is an isomorphism for all $q \geq 0$.

(2) *Under this identification, we have*

$$\Gamma(U, \pi_{\mathbb{C}}^{-1}\Omega^{g-1}_{\mathbb{P}^{g-1}(\mathbb{C})}(-d))$$
$$= \{f\omega \mid f \text{ holomorphic function on } \pi_{\mathbb{C}}^{-1}(\pi_{\mathbb{C}}(U)) \text{ such that } f(\lambda y) = \lambda^{-g-d} f(y) \text{ for all } \lambda \in \mathbb{C}^{\times}\}.$$

(3) *For all $q \geq 1$, we have*

$$H^q(U, \pi_{\mathbb{C}}^{-1}\Omega^{g-1}_{\mathbb{P}^{g-1}(\mathbb{C})}(-d)) = 0.$$

Proof. (1) This follows from Lemma 3.2.2(2) and [Kashiwara and Schapira 1990, Corollary 2.7.7(ii)].

(2) This follows directly from (1) and the description of $\Omega^{g-1}_{\mathbb{P}^{g-1}(\mathbb{C})}(-d)$; see Remark 3.1.2.

(3) By Lemma 3.2.2(3), we know $\pi_{\mathbb{C}}(U)$ is a Stein manifold. Moreover, $\Omega^{g-1}_{\mathbb{P}^{g-1}(\mathbb{C})}(-d)$ is a coherent sheaf on $\mathbb{P}^{g-1}(\mathbb{C})$. So (3) follows from (1) and Cartan's Theorem B; see [Hörmander 1973, Theorem 7.4.3]. □

3.3. $\mathrm{GL}_g(\mathbb{Q})$*-equivariant structures.* In this subsection we explicitly describe the $\mathrm{GL}_g(\mathbb{Q})$-equivariant structures on $\mathscr{F}_d$ and $\mathscr{F}_d^{\Xi}$.

In this paper, for a subgroup $G \subset \mathrm{GL}_g(\mathbb{Q})$ and a sheaf $\mathscr{F}$ (of abelian groups) on Y°, we define a G-equivariant structure on $\mathscr{F}$ to be a collection $\{[\gamma]\}_{\gamma\in G}$ of isomorphisms

$$[\gamma] : \mathscr{F} \xrightarrow{\sim} ({}^t\gamma)_*\mathscr{F}$$

subject to the conditions

(i) $[1] = \mathrm{id}_{\mathscr{F}}$,

(ii) $[\gamma_1\gamma_2] = ({}^t\gamma_2)_*[\gamma_1] \circ [\gamma_2]$ for all $\gamma_1, \gamma_2 \in G$.

Here, ${}^t\gamma$ is the transpose matrix of γ, and $({}^t\gamma)_*\mathscr{F}$ (resp. $({}^t\gamma_2)_*[\gamma_1]$) is the direct image of $\mathscr{F}$ (resp. $[\gamma_1]$) with respect to the map ${}^t\gamma : Y^\circ \to Y^\circ$ (resp. ${}^t\gamma_2 : Y^\circ \to Y^\circ$).[3]

The $\mathrm{GL}_g(\mathbb{Q})$-equivariant structure on $\mathscr{F}_d$ can be defined as follows. First, by Proposition 3.2.3(2),

$$\Gamma(U, \mathscr{F}_d) = \{f\omega \mid f \text{ holomorphic function on } \pi_{\mathbb{C}}^{-1}(\pi_{\mathbb{C}}(U)) \text{ such that } f(\lambda y) = \lambda^{-g-d} f(y) \text{ for all } \lambda \in \mathbb{C}^\times\}$$

for a convex open subset $U \subset Y^\circ$, where

$$\omega(y_1, \dots, y_g) := \sum_{i=1}^{g} (-1)^{i-1} y_i \, dy_1 \wedge \cdots \wedge \check{dy_i} \wedge \cdots \wedge dy_g.$$

Lemma 3.3.1. *For $\gamma \in \mathrm{GL}_g(\mathbb{Q})$, we have*

$$\omega(\gamma y) = \det(\gamma)\omega(y).$$

Proof. It suffices to prove the identity for elementary matrices γ. This case can be checked easily. □

Definition 3.3.2. For $\gamma \in \mathrm{GL}_g(\mathbb{Q})$ and a convex open subset $U \subset Y^\circ$, let $[\gamma]_U$ denote the pullback map

$$[\gamma]_U : \Gamma(U, \mathscr{F}_d) \xrightarrow{\sim} \Gamma(U, ({}^t\gamma)_*\mathscr{F}_d) = \Gamma(({}^t\gamma)^{-1}U, \mathscr{F}_d),$$
$$f(y)\omega(y) \longmapsto f({}^t\gamma y)\omega({}^t\gamma y) = \det(\gamma) f({}^t\gamma y)\omega(y).$$

Here $f({}^t\gamma y)$ is regarded as a holomorphic function of $y \in ({}^t\gamma)^{-1}\pi_{\mathbb{C}}^{-1}(\pi_{\mathbb{C}}(U)) = \pi_{\mathbb{C}}^{-1}\big(\pi_{\mathbb{C}}(({}^t\gamma)^{-1}U)\big)$. We may drop the subscript U and write as $[\gamma] = [\gamma]_U$ if there is no confusion.

Lemma 3.3.3. (1) *Let $V, U \subset Y^\circ$ be convex open subsets such that $V \subset U$, and let $s \in \Gamma(U, \mathscr{F}_d)$ be a section. Then we have*

$$[\gamma]_U(s)|_V = [\gamma]_V(s|_V)$$

in $\Gamma(V, ({}^t\gamma)_*\mathscr{F}_d)$.

(2) *The collection $\{[\gamma]_U \mid U \subset Y^\circ \text{ convex open}\}$ defines an isomorphism of sheaves*

$$[\gamma] : \mathscr{F}_d \xrightarrow{\sim} ({}^t\gamma)_*\mathscr{F}_d.$$

(3) *The collection $\{[\gamma]\}_{\gamma\in\mathrm{GL}_g(\mathbb{Q})}$ defines a $\mathrm{GL}_g(\mathbb{Q})$-equivariant structure on $\mathscr{F}_d$.*

Proof. (1) is clear, and (2) follows from (1) since convex open subsets form a basis of open subsets of Y°. We prove (3).

[3] We consider the action of ${}^t\gamma$ on Y° instead of γ since it is more convenient later when we use the identity $\langle \gamma x, y\rangle = \langle x, {}^t\gamma y\rangle$.

Condition (i) of the definition is obvious.

Let $U \subset Y^\circ$ be a convex open subset, and let $s(y) = f(y)\omega(y) \in \Gamma(U, \mathscr{F}_d)$ be a section. Then for $\gamma_1, \gamma_2 \in \mathrm{GL}_g(\mathbb{Q})$, we have

$$({}^t\gamma_2)_*[\gamma_1] \circ [\gamma_2](s(y)) = [\gamma_1]_{{}^t\gamma_2^{-1}U} \circ [\gamma_2]_U(s(y)) = [\gamma_1]_{{}^t\gamma_2^{-1}U}(s({}^t\gamma_2 y)) = s({}^t\gamma_2{}^t\gamma_1 y) = [\gamma_1\gamma_2](s(y)).$$

Since convex open subsets form a basis of open subsets of Y°, this shows condition (ii). □

This describes the $\mathrm{GL}_g(\mathbb{Q})$-equivariant structure on $\mathscr{F}_d$. Next we describe the $\mathrm{GL}_g(\mathbb{Q})$-equivariant structure on $\mathscr{F}_d^\Xi$. First, note that the conjugate action

$$[\gamma] : \mathbb{Z}[\Xi] \xrightarrow{\sim} \mathbb{Z}[\Xi], \quad Q \mapsto [\gamma](Q) = \gamma Q \gamma^{-1}$$

of $\mathrm{GL}_g(\mathbb{Q})$ on $\mathbb{Z}[\Xi]$ naturally induces a $\mathrm{GL}_g(\mathbb{Q})$-equivariant structure on the associated constant sheaf $\underline{\mathbb{Z}[\Xi]}$. Therefore, for a $\mathrm{GL}_g(\mathbb{Q})$-equivariant sheaf $\mathscr{F}$, the sheaf

$$\mathscr{F}^\Xi = \underline{\mathrm{Hom}}(\underline{\mathbb{Z}[\Xi]}, \mathscr{F})$$

has a natural $\mathrm{GL}_g(\mathbb{Q})$-equivariant structure induced from those of $\underline{\mathbb{Z}[\Xi]}$ and $\mathscr{F}$. In particular, we obtain a $\mathrm{GL}_g(\mathbb{Q})$-equivariant structure on $\mathscr{F}_d^\Xi$.

More concretely, for an open subset $U \subset Y^\circ$ and a section

$$\phi \in \Gamma(U, \mathscr{F}^\Xi) = \mathrm{Map}(\Xi, \Gamma(U, \mathscr{F}))$$

(see Remark 3.1.1) the $\mathrm{GL}_g(\mathbb{Q})$-equivariant structure on $\mathscr{F}^\Xi$ can be computed as

$$[\gamma](\phi)(Q) = [\gamma]\big(\phi\big([\gamma^{-1}](Q)\big)\big) = [\gamma](\phi(\gamma^{-1}Q\gamma))$$

for $\gamma \in \mathrm{GL}_g(\mathbb{Q})$ and $Q \in \Xi$. In particular, we see that for $Q \in \Xi$, the evaluation map

$$\mathrm{ev}_Q : \mathscr{F}^\Xi \to \mathscr{F}$$

(see Remark 3.1.1) is a Γ_Q-equivariant map, where $\Gamma_Q = \mathrm{Stab}_{\mathrm{SL}_g(\mathbb{Z})}(Q) \subset \mathrm{SL}_g(\mathbb{Z})$ is the stabilizer of $Q \in \Xi$ in $\mathrm{SL}_g(\mathbb{Z})$.

4. Equivariant cohomology

Recall that $\Gamma_Q = \mathrm{Stab}_{\mathrm{SL}_g(\mathbb{Z})}(Q) \subset \mathrm{SL}_g(\mathbb{Z})$ denotes the stabilizer of $Q \in \Xi$ in $\mathrm{SL}_g(\mathbb{Z})$. In this section we compute the equivariant cohomology groups

$$H^q(Y^\circ, \Gamma_Q, \mathscr{F}_d) \quad \text{and} \quad H^q(Y^\circ, \mathrm{SL}_g(\mathbb{Z}), \mathscr{F}_d^\Xi)$$

using the equivariant Čech complexes; see Corollary 4.3.4. We closely follow the argument in [Bannai et al. 2023].

Here, for a subgroup $G \subset \mathrm{GL}_g(\mathbb{Q})$, the equivariant cohomology

$$H^q(Y^\circ, G, -) : \mathbf{Sh}(Y^\circ, G) \to \mathbf{Ab}$$

is defined to be the right derived functor of the G-invariant global section functor

$$\Gamma(Y^\circ, G, -) : \mathbf{Sh}(Y^\circ, G) \to \mathbf{Ab}, \quad \mathscr{F} \mapsto \Gamma(Y^\circ, \mathscr{F})^G,$$

where $\mathbf{Sh}(Y^\circ, G)$ is the category of G-equivariant sheaves on Y°, $\mathbf{Ab}$ is the category of abelian groups, and $\Gamma(Y^\circ, \mathscr{F})^G$ is the G-invariant part of the global section $\Gamma(Y^\circ, \mathscr{F})$.

4.1. *Open covering.* In this subsection we introduce a certain $\mathrm{GL}_g(\mathbb{Q})$-stable open covering of Y°. For $\alpha \in \mathbb{C}^g - \{0\}$, we define an open subset $V_\alpha \subset \mathbb{C}^g - \{0\}$ by

$$V_\alpha := \{ y \in \mathbb{C}^g \mid \mathrm{Re}(\langle \alpha, y \rangle) > 0 \} \subset \mathbb{C}^g - \{0\}.$$

Clearly, $V_\alpha \subset \mathbb{C}^g - \{0\}$ is a convex open subset. Let

$$X_{\mathbb{Q}} := \mathbb{Q}^g - \{0\}$$

denote the set of all nonzero rational vectors on which $\mathrm{GL}_g(\mathbb{Q})$ acts by the matrix multiplication from the left. Then we easily see that

$$Y^\circ = \bigcup_{\alpha \in X_{\mathbb{Q}}} V_\alpha.$$

Let $\mathscr{X}_{\mathbb{Q}} := \{V_\alpha\}_{\alpha \in X_{\mathbb{Q}}}$ denote this open covering of Y°. For $r \geq 0$ and $I = (\alpha_1, \dots, \alpha_r) \in (X_{\mathbb{Q}})^r$, set

$$V_I := \bigcap_{i=1}^{r} V_{\alpha_i} = \left\{ y \in Y^\circ \mid \mathrm{Re}(\langle \alpha_i, y \rangle) > 0 \text{ for all } i \right\} \subset Y^\circ.$$

In the case $r = 0$, we set $(X_{\mathbb{Q}})^0 = \{\varnothing\}$ and $V_\varnothing = Y^\circ$ by convention. Let

$$j_I : V_I \hookrightarrow Y^\circ$$

denote the inclusion map.

First, we show that $\mathscr{X}_{\mathbb{Q}} = \{V_\alpha\}_{\alpha \in X_{\mathbb{Q}}}$ is a $\mathrm{GL}_g(\mathbb{Q})$-stable open covering. Note that the group $\mathrm{GL}_g(\mathbb{Q})$ acts diagonally on $(X_{\mathbb{Q}})^r$. For $I = (\alpha_1, \dots, \alpha_r) \in (X_{\mathbb{Q}})^r$ and $\gamma \in \mathrm{GL}_g(\mathbb{Q})$, let

$$\gamma I = (\gamma\alpha_1, \dots, \gamma\alpha_r) \in (X_{\mathbb{Q}})^r$$

denote this diagonal action of γ on I.

Lemma 4.1.1. *For $r \geq 0$, $I = (\alpha_1, \dots, \alpha_r) \in (X_{\mathbb{Q}})^r$, and $\gamma \in \mathrm{GL}_g(\mathbb{Q})$, we have*

$$V_{\gamma I} = {}^t\gamma^{-1} V_I.$$

In other words, we have the following commutative diagram:

$$\begin{array}{ccc} V_I & \xrightarrow{j_I} & Y^\circ \\ {\scriptstyle {}^t\gamma^{-1}} \downarrow \wr & & \downarrow {\scriptstyle {}^t\gamma^{-1}} \\ V_{\gamma I} & \xrightarrow{j_{\gamma I}} & Y^\circ \end{array}$$

Proof. For $y \in Y^\circ$, we have $y \in V_{\gamma I}$ if and only if

$$0 < \mathrm{Re}(\langle \gamma\alpha_i, y\rangle) = \mathrm{Re}(\langle \alpha_i, {}^t\gamma y\rangle)$$

for all $i \in \{1, \dots, r\}$. This proves the lemma. □

4.2. ***The equivariant Čech complex.*** Let $\mathscr{F}$ be a $\mathrm{GL}_g(\mathbb{Q})$-equivariant sheaf on Y°. We consider the $\mathrm{GL}_g(\mathbb{Q})$-equivariant "sheaf Čech complex"

$$\mathscr{C}^\bullet(\mathcal{X}_\mathbb{Q}, \mathscr{F}) : \mathscr{C}^0(\mathcal{X}_\mathbb{Q}, \mathscr{F}) \xrightarrow{\mathrm{d}^0} \mathscr{C}^1(\mathcal{X}_\mathbb{Q}, \mathscr{F}) \xrightarrow{\mathrm{d}^1} \mathscr{C}^2(\mathcal{X}_\mathbb{Q}, \mathscr{F}) \xrightarrow{\mathrm{d}^2} \cdots$$

defined as follows. For $q \geq 0$, put

$$\mathscr{C}^q(\mathcal{X}_\mathbb{Q}, \mathscr{F}) := \prod_{I \in (X_\mathbb{Q})^{q+1}} j_{I*} j_I^{-1}\mathscr{F},$$

where j_{I*} (resp. j_I^{-1}) is the direct image (resp. inverse image) functor induced by the inclusion map $j_I : V_I \hookrightarrow Y^\circ$. By Lemma 4.1.1, the $\mathrm{GL}_g(\mathbb{Q})$-equivariant structure

$$[\gamma] : \mathscr{F} \xrightarrow{\sim} ({}^t\gamma)_*\mathscr{F}$$

of $\mathscr{F}$ induces isomorphisms

$$[\gamma] : j_{I*} j_I^{-1}\mathscr{F} \xrightarrow{\sim} j_{I*} j_I^{-1}({}^t\gamma)_*\mathscr{F} \simeq ({}^t\gamma)_* j_{\gamma I*} j_{\gamma I}^{-1}\mathscr{F} \quad \text{and} \quad [\gamma] : \mathscr{C}^q(\mathcal{X}_\mathbb{Q}, \mathscr{F}) \xrightarrow{\sim} ({}^t\gamma)_*\mathscr{C}^q(\mathcal{X}_\mathbb{Q}, \mathscr{F}).$$

We easily see that this defines a $\mathrm{GL}_g(\mathbb{Q})$-equivariant structure on $\mathscr{C}^q(\mathcal{X}_\mathbb{Q}, \mathscr{F})$. More concretely, for an open subset $U \subset Y^\circ$ and a section

$$s = (s_I)_{I \in (X_\mathbb{Q})^{q+1}} \in \Gamma(U, \mathscr{C}^q(\mathcal{X}_\mathbb{Q}, \mathscr{F})) = \prod_{I \in (X_\mathbb{Q})^{q+1}} \Gamma(U \cap V_I, \mathscr{F}),$$

we have

$$([\gamma](s))_I = [\gamma](s_{\gamma^{-1}I}), \tag{4-1}$$

where $([\gamma](s))_I$ is the I-th component of $[\gamma](s)$.

The differential map

$$\mathrm{d}^q : \mathscr{C}^q(\mathcal{X}_\mathbb{Q}, \mathscr{F}) \to \mathscr{C}^{q+1}(\mathcal{X}_\mathbb{Q}, \mathscr{F})$$

is given by

$$(\mathrm{d}^q(s))_{(\alpha_0,\dots,\alpha_{q+1})} = \sum_{i=0}^{q+1} (-1)^i s_{(\alpha_0,\dots,\check{\alpha}_i,\dots,\alpha_{q+1})}|_{U \cap V_{(\alpha_0,\dots,\alpha_{q+1})}}$$

for an open subset $U \subset Y^\circ$ and a section $s = (s_I)_{I \in (X_\mathbb{Q})^{q+1}} \in \Gamma(U, \mathscr{C}^q(\mathcal{X}_\mathbb{Q}, \mathscr{F}))$. Here $\check{\alpha}_i$ means that α_i is omitted. Moreover, there is a map

$$\mathrm{d}^{-1} : \mathscr{F} \to \mathscr{C}^0(\mathcal{X}_\mathbb{Q}, \mathscr{F}) = \prod_{\alpha \in X_\mathbb{Q}} j_{\alpha*} j_\alpha^{-1}\mathscr{F}$$

induced by the natural maps $\mathscr{F} \to j_{\alpha*} j_\alpha^{-1}\mathscr{F}$.

Then we have the following.

Lemma 4.2.1. (1) *For $q \geq -1$, the differential map d^q is a $\mathrm{GL}_g(\mathbb{Q})$-equivariant map, i.e., $[\gamma] \circ \mathrm{d}^q = \mathrm{d}^q \circ [\gamma]$ for $\gamma \in \mathrm{GL}_g(\mathbb{Q})$.*

(2) *For any $\alpha_0 \in X_{\mathbb{Q}}$, the sequence*

$$0 \longrightarrow \mathscr{F}|_{V_{\alpha_0}} \xrightarrow{\mathrm{d}^{-1}} \mathscr{C}^0(\mathcal{X}_{\mathbb{Q}}, \mathscr{F})|_{V_{\alpha_0}} \xrightarrow{\mathrm{d}^0} \mathscr{C}^1(\mathcal{X}_{\mathbb{Q}}, \mathscr{F})|_{V_{\alpha_0}} \longrightarrow \cdots$$

is homotopic to zero. In particular, the sequence

$$0 \longrightarrow \mathscr{F} \xrightarrow{\mathrm{d}^{-1}} \mathscr{C}^0(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}) \xrightarrow{\mathrm{d}^0} \mathscr{C}^1(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}) \longrightarrow \cdots$$

is an exact sequence of $\mathrm{GL}_g(\mathbb{Q})$-equivariant sheaves since $Y^\circ = \bigcup_{\alpha_0 \in X_{\mathbb{Q}}} V_{\alpha_0}$.

Proof. (1) Let $U \subset Y^\circ$ be an open subset, and let

$$s = (s_I)_{I \in (X_{\mathbb{Q}})^{q+1}} \in \Gamma(U, \mathscr{C}^q(\mathcal{X}_{\mathbb{Q}}, \mathscr{F})) = \prod_{I \in (X_{\mathbb{Q}})^{q+1}} \Gamma(U \cap V_I, \mathscr{F})$$

be a section. Let $J = (\alpha_0, \dots, \alpha_{q+1}) \in (X_{\mathbb{Q}})^{q+2}$, and put $J^{(i)} := (\alpha_0, \dots, \check{\alpha}_i, \dots, \alpha_{q+1}) \in (X_{\mathbb{Q}})^{q+1}$ for $i = 0, \dots, q+1$. Then we have

$$\begin{aligned}\big(\mathrm{d}^q([\gamma](s))\big)_J &= \sum_{i=0}^{q+1} (-1)^i [\gamma](s_{\gamma^{-1}J^{(i)}})|_{{}^t\gamma^{-1}U \cap V_J} = \sum_{i=0}^{q+1} (-1)^i [\gamma](s_{\gamma^{-1}J^{(i)}}|_{U \cap V_{\gamma^{-1}J}}) \\ &= [\gamma]\Bigg(\sum_{i=0}^{q+1} (-1)^i s_{\gamma^{-1}J^{(i)}}|_{U \cap V_{\gamma^{-1}J}}\Bigg) = \big([\gamma](\mathrm{d}^q(s))\big)_J.\end{aligned}$$

(2) See [Godement 1973, Théorème 5.2.1] or [Stacks 2005–, Lemma 02FU]. Although they prove only the exactness of the sequence, we can prove the statement in this lemma using essentially the same argument. See also [Kashiwara and Schapira 1990, Lemma 2.8.2, Remark 2.8.3]. □

By applying the additive functor

$$\underline{\mathrm{Hom}}(\underline{\mathbb{Z}[\Xi]}, -) : \mathbf{Sh}(Y^\circ, \mathrm{GL}_g(\mathbb{Q})) \to \mathbf{Sh}(Y^\circ, \mathrm{GL}_g(\mathbb{Q})), \quad \mathscr{G} \mapsto \mathscr{G}^\Xi := \underline{\mathrm{Hom}}(\underline{\mathbb{Z}[\Xi]}, \mathscr{G}),$$

we obtain the following.

Corollary 4.2.2. *The sequence*

$$0 \longrightarrow \mathscr{F}^\Xi \xrightarrow{\mathrm{d}^{-1}} \mathscr{C}^0(\mathcal{X}_{\mathbb{Q}}, \mathscr{F})^\Xi \xrightarrow{\mathrm{d}^0} \mathscr{C}^1(\mathcal{X}_{\mathbb{Q}}, \mathscr{F})^\Xi \longrightarrow \cdots$$

is an exact sequence of $\mathrm{GL}_g(\mathbb{Q})$-equivariant sheaves.

Proof. Since the homotopy is preserved by the additive functor, by Lemma 4.2.1(2), we see that for any $\alpha_0 \in X_{\mathbb{Q}}$, the sequence

$$0 \longrightarrow \mathscr{F}^\Xi|_{V_{\alpha_0}} \xrightarrow{\mathrm{d}^{-1}} \mathscr{C}^0(\mathcal{X}_{\mathbb{Q}}, \mathscr{F})^\Xi|_{V_{\alpha_0}} \xrightarrow{\mathrm{d}^0} \mathscr{C}^1(\mathcal{X}_{\mathbb{Q}}, \mathscr{F})^\Xi|_{V_{\alpha_0}} \longrightarrow \cdots$$

is homotopic to zero, and hence exact. □

Now, by taking the global section, set

$$C^q(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}) := \Gamma(Y^\circ, \mathscr{C}^q(\mathcal{X}_{\mathbb{Q}}, \mathscr{F})) = \prod_{I\in (X_{\mathbb{Q}})^{q+1}} \Gamma(V_I, \mathscr{F}).$$

Then we obtain a complex

$$C^\bullet(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}) : C^0(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}) \xrightarrow{\mathrm{d}^0} C^1(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}) \xrightarrow{\mathrm{d}^1} C^2(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}) \xrightarrow{\mathrm{d}^2} \cdots$$

of $\mathrm{GL}_g(\mathbb{Q})$-modules. Note that this is the usual Čech complex associated to the open covering $\mathcal{X}_{\mathbb{Q}}$. Furthermore, set

$$C^q(\mathcal{X}_{\mathbb{Q}}, \mathscr{F})^\Xi := \Gamma(Y^\circ, \mathscr{C}^q(\mathcal{X}_{\mathbb{Q}}, \mathscr{F})^\Xi) = \mathrm{Map}(\Xi, C^q(\mathcal{X}_{\mathbb{Q}}, \mathscr{F})).$$

Then we obtain another complex

$$C^\bullet(\mathcal{X}_{\mathbb{Q}}, \mathscr{F})^\Xi : C^0(\mathcal{X}_{\mathbb{Q}}, \mathscr{F})^\Xi \xrightarrow{\mathrm{d}^0} C^1(\mathcal{X}_{\mathbb{Q}}, \mathscr{F})^\Xi \xrightarrow{\mathrm{d}^1} C^2(\mathcal{X}_{\mathbb{Q}}, \mathscr{F})^\Xi \xrightarrow{\mathrm{d}^2} \cdots$$

of $\mathrm{GL}_g(\mathbb{Q})$-modules. For $Q \in \Xi$, the evaluation map

$$\mathrm{ev}_Q : C^\bullet(\mathcal{X}_{\mathbb{Q}}, \mathscr{F})^\Xi \to C^\bullet(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}) \tag{4-2}$$

is a Γ_Q-equivariant morphism of complexes.

4.3. ***Acyclicity.*** Our aim here is to prove the acyclicity of the sheaves $\mathscr{C}^q(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}_d)$ and $\mathscr{C}^q(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}_d)^\Xi$; see Proposition 4.3.3. Then we can compute the equivariant cohomology groups $H^q(Y^\circ, \Gamma_Q, \mathscr{F}_d)$ and $H^q(Y^\circ, \mathrm{SL}_g(\mathbb{Z}), \mathscr{F}_d^\Xi)$ using the Čech complexes $C^\bullet(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}_d)$ and $C^\bullet(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}_d)^\Xi$; see Corollary 4.3.4.

Lemma 4.3.1. *Let $r \geq 1$ and $I = (\alpha_1, \dots, \alpha_r) \in (X_{\mathbb{Q}})^r$.*

(1) *For all $q \geq 1$, we have*

$$H^q(V_I, \mathscr{F}_d) = 0.$$

(2) *For all $q \geq 1$, we have*

$$R^q j_{I*}(j_I^{-1}\mathscr{F}_d) = 0,$$

where $R^q j_{I*}$ (resp. j_I^{-1}) is the higher direct image (resp. inverse image) functor induced by the inclusion map $j_I : V_I \hookrightarrow Y^\circ$.

(3) *For any open subset $U \subset Y^\circ$ and $q \geq 0$, we have an isomorphism*

$$H^q(U, j_{I*}j_I^{-1}\mathscr{F}_d) \xrightarrow{\sim} H^q(U \cap V_I, \mathscr{F}_d).$$

Proof. (1) This follows directly from Proposition 3.2.3(3) since V_I is convex.

(2) Let $x \in Y^\circ$. Since convex open subsets form a basis of open subsets of Y°, we have

$$(R^q j_{I*}(j_I^{-1}\mathscr{F}_d))_x = \varinjlim_{x\in U \text{ convex}} H^q(U \cap V_I, j_I^{-1}\mathscr{F}_d) = \varinjlim_{x\in U \text{ convex}} H^q(U \cap V_I, \mathscr{F}_d) = 0.$$

Here the last vanishing follows from Proposition 3.2.3(3). This proves (2).

(3) This follows from (2) and the Leray spectral sequence. ☐

Proposition 4.3.2. *For $q \geq 0$, the sheaves $\mathscr{C}^q(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}_d)$ and $\mathscr{C}^q(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}_d)^{\Xi}$ are $\Gamma(Y^\circ, -)$-acyclic, i.e.,*

$$H^p(Y^\circ, \mathscr{C}^q(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}_d)) = 0 \quad \textit{and} \quad H^p(Y^\circ, \mathscr{C}^q(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}_d)^{\Xi}) = 0 \qquad \textit{for } p \geq 1.$$

Proof. We imitate the argument in [Bannai et al. 2023, Proposition 3.4, Lemma 3.5]. For $I \in (X_{\mathbb{Q}})^{q+1}$, put $\mathscr{F}_I := j_{I*} j_I^{-1} \mathscr{F}_d$, and let

$$0 \to \mathscr{F}_I \to \mathscr{I}_I^\bullet$$

be an injective resolution of $\mathscr{F}_I$. First we show that

$$0 \to \mathscr{C}^q(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}_d) = \prod_{I \in (X_{\mathbb{Q}})^{q+1}} \mathscr{F}_I \to \prod_{I \in (X_{\mathbb{Q}})^{q+1}} \mathscr{I}_I^\bullet, \tag{4-3}$$

$$0 \to \mathscr{C}^q(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}_d)^{\Xi} = \Bigl(\prod_{I \in (X_{\mathbb{Q}})^{q+1}} \mathscr{F}_I\Bigr)^{\Xi} \to \Bigl(\prod_{I \in (X_{\mathbb{Q}})^{q+1}} \mathscr{I}_I^\bullet\Bigr)^{\Xi} \tag{4-4}$$

are both injective resolutions of $\mathscr{C}^q(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}_d)$ and $\mathscr{C}^q(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}_d)^{\Xi}$ respectively. It is clear that

$$\prod_{I \in (X_{\mathbb{Q}})^{q+1}} \mathscr{I}_I^p \quad \text{and} \quad \Bigl(\prod_{I \in (X_{\mathbb{Q}})^{q+1}} \mathscr{I}_I^p\Bigr)^{\Xi} \simeq \prod_{Q \in \Xi} \prod_{I \in (X_{\mathbb{Q}})^{q+1}} \mathscr{I}_I^p$$

are injective sheaves because they are products of injective sheaves; see Remark 3.1.1(2). We must show the exactness of (4-3) and (4-4). Let $U \subset Y^\circ$ be any convex open subset. By Lemma 4.3.1(3) and Proposition 3.2.3(3), we have

$$H^p(U, \mathscr{F}_I) \xrightarrow{\sim} H^p(U \cap V_I, \mathscr{F}_d) = 0$$

for $p \geq 1$. Therefore, we find that

$$0 \to \mathscr{F}_I(U) \to \mathscr{I}_I^\bullet(U)$$

is exact because $H^p(U, \mathscr{F}_I)$ is the cohomology of this complex. Hence,

$$0 \to \prod_{I \in (X_{\mathbb{Q}})^{q+1}} \mathscr{F}_I(U) \to \prod_{I \in (X_{\mathbb{Q}})^{q+1}} \mathscr{I}_I^\bullet(U) \quad \text{and} \quad 0 \to \prod_{Q \in \Xi} \prod_{I \in (X_{\mathbb{Q}})^{q+1}} \mathscr{F}_I(U) \to \prod_{Q \in \Xi} \prod_{I \in (X_{\mathbb{Q}})^{q+1}} \mathscr{I}_I^\bullet(U)$$

are also exact. Since convex open subsets of Y° form a basis of open subsets, we obtain the exactness of (4-3) and (4-4).

Then for $p \geq 1$, we have

$$\begin{aligned} H^p(Y^\circ, \mathscr{C}^q(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}_d)) &\simeq H^p\Bigl(\Gamma\Bigl(Y^\circ, \prod_{I \in (X_{\mathbb{Q}})^{q+1}} \mathscr{I}_I^\bullet\Bigr)\Bigr) \\ &\simeq \prod_{I \in (X_{\mathbb{Q}})^{q+1}} H^p(\Gamma(Y^\circ, \mathscr{I}_I^\bullet)) \\ &\simeq \prod_{I \in (X_{\mathbb{Q}})^{q+1}} H^p(Y^\circ, \mathscr{F}_I) \quad \simeq \prod_{I \in (X_{\mathbb{Q}})^{q+1}} H^p(V_I, \mathscr{F}_d) = 0, \end{aligned}$$

and similarly,

$$H^p(Y^\circ, \mathscr{C}^q(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}_d)^{\Xi}) \simeq H^p\Big(\Gamma\Big(Y^\circ, \Big(\prod_{I\in(X_{\mathbb{Q}})^{q+1}} \mathscr{I}_I^\bullet\Big)^{\Xi}\Big)\Big)$$
$$\simeq \prod_{Q\in\Xi}\prod_{I\in(X_{\mathbb{Q}})^{q+1}} H^p(\Gamma(Y^\circ, \mathscr{I}_I^\bullet)) \simeq \prod_{Q\in\Xi}\prod_{I\in(X_{\mathbb{Q}})^{q+1}} H^p(V_I, \mathscr{F}_d) = 0. \qquad \square$$

Proposition 4.3.3. (1) *Let $Q \in \Xi$. For $q \geq 0$, the sheaf $\mathscr{C}^q(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}_d)$ is $\Gamma(Y^\circ, \Gamma_Q, -)$-acyclic, i.e.,*

$$H^p(Y^\circ, \Gamma_Q, \mathscr{C}^q(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}_d)) = 0$$

for $p \geq 1$. In particular, the complex

$$0 \to \mathscr{F}_d \xrightarrow{\mathrm{d}^{-1}} \mathscr{C}^\bullet(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}_d)$$

gives a $\Gamma(Y^\circ, \Gamma_Q, -)$-acyclic resolution of $\mathscr{F}_d$.

(2) *For $q \geq 0$, the sheaf $\mathscr{C}^q(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}_d)^{\Xi}$ is $\Gamma(Y^\circ, \mathrm{SL}_g(\mathbb{Z}), -)$-acyclic, i.e., we have*

$$H^p(Y^\circ, \mathrm{SL}_g(\mathbb{Z}), \mathscr{C}^q(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}_d)^{\Xi}) = 0$$

for $p \geq 1$. In particular, the complex

$$0 \to \mathscr{F}_d^{\Xi} \xrightarrow{\mathrm{d}^{-1}} \mathscr{C}^\bullet(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}_d)^{\Xi}$$

gives a $\Gamma(Y^\circ, \mathrm{SL}_g(\mathbb{Z}), -)$-acyclic resolution of $\mathscr{F}_d^{\Xi}$.

Proof. (1) First note that the functor $\Gamma(Y^\circ, \Gamma_Q, -)$ is a composition of two left exact functors $\Gamma(Y^\circ, -)$ and $(-)^{\Gamma_Q}$. Moreover, $\Gamma(Y^\circ, -)$ sends injective objects to injective objects. Therefore, we have a spectral sequence

$$E_2^{ab} = H^a\big(\Gamma_Q, H^b(Y^\circ, \mathscr{C}^q(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}_d))\big) \Rightarrow H^{a+b}(Y^\circ, \Gamma_Q, \mathscr{C}^q(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}_d)),$$

where $H^a(\Gamma_Q, -)$ is the usual group cohomology of Γ_Q. Now, by Proposition 4.3.2, we already have

$$H^b(Y^\circ, \mathscr{C}^q(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}_d)) = 0 \quad \text{for all } b \geq 1.$$

Therefore, it suffices to show

$$H^a\big(\Gamma_Q, \Gamma(Y^\circ, \mathscr{C}^q(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}_d))\big) = H^a(\Gamma_Q, C^q(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}_d)) = 0 \quad \text{for all } a \geq 1.$$

Actually, we will prove that $C^q(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}_d)$ is a coinduced Γ_Q-module. First, recall that

$$C^q(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}_d) = \prod_{I\in(X_{\mathbb{Q}})^{q+1}} \Gamma(V_I, \mathscr{F}_d),$$

and that Γ_Q acts freely on $(X_{\mathbb{Q}})^{q+1}$ by Lemma 2.1.1(7). Let $A \subset (X_{\mathbb{Q}})^{q+1}$ be a system of representatives of $\Gamma_Q\backslash(X_{\mathbb{Q}})^{q+1}$, and set

$$M := \prod_{I\in A} \Gamma(V_I, \mathscr{F}_d).$$

Then recall that the $\mathrm{GL}_g(\mathbb{Q})$-equivariant structure on $\mathscr{F}_d$ gives an isomorphism

$$[\gamma] : \Gamma(V_I, \mathscr{F}_d) \xrightarrow{\sim} \Gamma(({}^t\gamma)^{-1} V_I, \mathscr{F}_d) = \Gamma(V_{\gamma I}, \mathscr{F}_d) \tag{4-5}$$

for each $I \in (X_{\mathbb{Q}})^{q+1}$ and $\gamma \in \mathrm{GL}_g(\mathbb{Q})$; see Lemma 4.1.1. Therefore, for each $\gamma \in \Gamma_Q$, we have an isomorphism

$$M = \prod_{I \in A} \Gamma(V_I, \mathscr{F}_d) \xrightarrow{\sim} \prod_{I \in A} \Gamma(V_{\gamma I}, \mathscr{F}_d), \quad (s_I)_{I \in A} \mapsto ([\gamma](s_I))_{I \in A},$$

and hence we obtain an isomorphism

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[\Gamma_Q], M) = \prod_{\gamma \in \Gamma_Q} M \xrightarrow{\sim} \prod_{\gamma \in \Gamma_Q} \prod_{I \in A} \Gamma(V_{\gamma I}, \mathscr{F}_d) = C^q(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}_d).$$

Since this is clearly a Γ_Q-equivariant isomorphism, we see $C^q(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}_d)$ is a coinduced Γ_Q-module.

(2) This can be proved similarly. First, by the spectral sequence

$$E_2^{ab} = H^a\big(\mathrm{SL}_g(\mathbb{Z}), H^b(Y^\circ, \mathscr{C}^q(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}_d)^{\Xi})\big) \Rightarrow H^{a+b}\big(Y^\circ, \mathrm{SL}_g(\mathbb{Z}), \mathscr{C}^q(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}_d)^{\Xi}\big)$$

and Proposition 4.3.2, it suffices to show

$$H^a\big(\mathrm{SL}_g(\mathbb{Z}), C^q(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}_d)^{\Xi}\big) = 0 \quad \text{for all } a \geq 1.$$

Again, we will prove that

$$C^q(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}_d)^{\Xi} \simeq \prod_{Q \in \Xi} \prod_{I \in (X_{\mathbb{Q}})^{q+1}} \Gamma(V_I, \mathscr{F}_d)$$

is a coinduced $\mathrm{SL}_g(\mathbb{Z})$-module. Note that the action of $\mathrm{SL}_g(\mathbb{Z})$ on $\Xi \times (X_{\mathbb{Q}})^{q+1}$ is free. Indeed, if

$$\gamma(Q, I) = ([\gamma](Q), \gamma I) = (Q, I),$$

then it follows that $\gamma \in \Gamma_Q$, and hence $\gamma = 1$, since the action of Γ_Q on $(X_{\mathbb{Q}})^{q+1}$ is free. Let $A' \subset \Xi \times (X_{\mathbb{Q}})^{q+1}$ be a system of representatives of $\mathrm{SL}_g(\mathbb{Z}) \backslash (\Xi \times (X_{\mathbb{Q}})^{q+1})$, and set

$$M' := \prod_{(Q,I) \in A'} \Gamma(V_I, \mathscr{F}_d).$$

Then again by using (4-5), we obtain an isomorphism

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[\mathrm{SL}_g(\mathbb{Z})], M') = \prod_{\gamma \in \mathrm{SL}_g(\mathbb{Z})} M' \xrightarrow{\sim} \prod_{\gamma \in \mathrm{SL}_g(\mathbb{Z})} \prod_{(Q,I) \in A'} \Gamma(V_{\gamma I}, \mathscr{F}_d) \simeq C^q(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}_d)^{\Xi}$$

of $\mathrm{SL}_g(\mathbb{Z})$-modules. Thus we find that $C^q(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}_d)^{\Xi}$ is a coinduced $\mathrm{SL}_g(\mathbb{Z})$-module. □

Corollary 4.3.4. (1) *Let $Q \in \Xi$. For $q \geq 0$, we have*

$$H^q(Y^\circ, \Gamma_Q, \mathscr{F}_d) \simeq H^q\big(\Gamma(Y^\circ, \Gamma_Q, \mathscr{C}^\bullet(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}_d))\big) = H^q(C^\bullet(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}_d)^{\Gamma_Q}),$$

where the second and third H^q are the cohomology of complexes.

(2) *For $q \geq 0$, we have*

$$H^q(Y^\circ, \mathrm{SL}_g(\mathbb{Z}), \mathscr{F}_d^{\Xi}) \simeq H^q\big(\Gamma(Y^\circ, \mathrm{SL}_g(\mathbb{Z}), \mathscr{C}^\bullet(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}_d)^{\Xi})\big) = H^q\big(\mathrm{Map}_{\mathrm{SL}_g(\mathbb{Z})}(\Xi, C^\bullet(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}_d))\big),$$

where $\mathrm{Map}_{\mathrm{SL}_g(\mathbb{Z})}(-,-)$ *is the set of* $\mathrm{SL}_g(\mathbb{Z})$*-equivariant maps.*

(3) *For $Q \in \Xi$, we have the commutative diagram*

$$\begin{array}{ccc} H^q(Y^\circ, \mathrm{SL}_g(\mathbb{Z}), \mathscr{F}_d^{\Xi}) & \xrightarrow{\mathrm{ev}_Q} & H^q(Y^\circ, \Gamma_Q, \mathscr{F}_d) \\ \wr\big| & & \big|\wr \\ H^q\big(\mathrm{Map}_{\mathrm{SL}_g(\mathbb{Z})}(\Xi, C^\bullet(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}_d))\big) & \xrightarrow{\mathrm{ev}_Q} & H^q(C^\bullet(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}_d)^{\Gamma_Q}) \end{array}$$

where the two ev_Q *are the evaluation maps induced by* (3-1) *and* (4-2).

We end this section with one more corollary, concerning an operation which shifts the index $d \geq 0$ of $\mathscr{F}_d$.

Corollary 4.3.5. *Let $P(y_1, \dots, y_g) \in \mathbb{C}[y_1, \dots, y_g]$ be a homogeneous polynomial of degree $d' \leq d$ such that*

$$P({}^t\gamma y) = P(y) \quad \textit{for all } \gamma \in \Gamma_Q.$$

Then the multiplication by P,

$$P : C^q(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}_d) \to C^q(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}_{d-d'}), \quad (s_I(y))_{I \in (X_{\mathbb{Q}})^{q+1}} \mapsto (P(y)s_I(y))_{I \in (X_{\mathbb{Q}})^{q+1}},$$

gives a Γ_Q-equivariant map of complexes, and hence induces a map

$$P : H^q(Y^\circ, \Gamma_Q, \mathscr{F}_d) \to H^q(Y^\circ, \Gamma_Q, \mathscr{F}_{d-d'}).$$

Example 4.3.6. A typical example of such a Γ_Q-invariant homogeneous polynomial P is the norm polynomial N_{w^*} defined in Section 2.2; see Lemma 2.2.1. More generally, let $k \geq 1$ be an integer. Under the notation in Lemma 2.2.1, the k-th power $N_{w^*}^k$ of the norm polynomial N_{w^*} is a Γ_Q-invariant homogeneous polynomial of degree kg. In particular, we have a map

$$N_{w^*}^k : H^q(Y^\circ, \Gamma_Q, \mathscr{F}_{kg}) \to H^q(Y^\circ, \Gamma_Q, \mathscr{F}_0).$$

5. Cones and the exponential perturbation

In this section we introduce the notion of exponential perturbation, which is a modification of the so-called upper closure or Q-perturbation (Colmez perturbation) used in [Yamamoto 2010; Bannai et al. 2023; Charollois et al. 2015]. This is one of the key ingredients enabling us to deal with general number fields.

For $r \geq 0$, $I = (\alpha_1, \dots, \alpha_r) \in (\mathbb{R}^g - \{0\})^r$, let

$$C_I := \sum_{i=1}^{r} \mathbb{R}_{>0}\alpha_i \subset \mathbb{R}^g$$

denote the *open cone generated by* $\alpha_1, \dots, \alpha_r$. In the case $r = 0$ and $I = \varnothing$, we set $C_\varnothing := \{0\}$.

Remark 5.0.1. We follow the convention to call C_I an "open" cone although it is not necessarily an open subset of $\mathbb{R}^g$. Note that, however, C_I is open in $\mathrm{Span}_{\mathbb{R}}\{\alpha_1, \dots, \alpha_r\}$, where $\mathrm{Span}_{\mathbb{R}}\{\alpha_1, \dots, \alpha_r\} \subset \mathbb{R}^g$ is the $\mathbb{R}$-subspace spanned by $\alpha_1, \dots, \alpha_r$; see Lemma 5.2.4.

Recall that $X_{\mathbb{Q}} := \mathbb{Q}^g - \{0\}$ denotes the set of nonzero vectors of $\mathbb{Q}^g$. In this paper we fix the terminology concerning cones as follows.

Definition 5.0.2. (1) An open cone C_I is called *rational* if we can take $I \in (X_{\mathbb{Q}})^r$.

(2) An open cone C_I is called *simplicial* if $\alpha_1, \dots, \alpha_r$ are linearly independent over $\mathbb{R}$.

(3) We refer to a subset of $\mathbb{R}^g$ which can be written as a disjoint union of a finite number of rational simplicial open cones as a *rational constructible cone*.

5.1. ***The exponential perturbation.*** Recall that

$$\Xi = \{Q \in \mathrm{GL}_g(\mathbb{Q}) \mid Q \text{ is irreducible over } \mathbb{Q}\}$$

denotes the set of irreducible matrices of $\mathrm{GL}_g(\mathbb{Q})$; see Section 2.1.

Definition 5.1.1. For $Q \in \Xi$ and a subset $A \subset \mathbb{R}^g$, we define the *exponential Q-perturbation* A^Q of A as

$$A^Q := \{x \in \mathbb{R}^g \mid \text{there exists } \delta > 0 \text{ such that for all } \varepsilon \in (0, \delta),\ \exp(\varepsilon Q)x \in A\},$$

where $\exp(\varepsilon Q) \in \mathrm{GL}_g(\mathbb{R})$ is the matrix exponential of $\varepsilon Q \in \mathrm{GL}_g(\mathbb{R})$.

Remark 5.1.2. This exponential Q-perturbation is defined by considering the perturbation of $x \in \mathbb{R}^g$ by the matrix action of $\exp(\varepsilon Q)$, and we call this process the *exponential perturbation*. The original Q-perturbation used in [Charollois et al. 2015] is the perturbation of x by the vectors $Q \in \mathbb{R}^g$ whose components are linearly independent over $\mathbb{Q}$.

Lemma 5.1.3. *Let $Q \in \Xi$.*

(1) *Let $A, B \subset \mathbb{R}^g$ be subsets such that $A \subset B$. Then we have*

$$A^Q \subset B^Q.$$

(2) *Let $A_1, \dots, A_m \subset \mathbb{R}^g$ be subsets. Then we have*

$$(A_1 \cap \cdots \cap A_m)^Q = A_1^Q \cap \cdots \cap A_m^Q.$$

In particular, if $A_1 \cap \cdots \cap A_m = \varnothing$, then $A_1^Q \cap \cdots \cap A_m^Q = \varnothing$.

Proof. (1) is obvious. We prove (2). The inclusion $\subset$ is clear. We prove $\supset$. Let $x \in A_1^Q \cap \cdots \cap A_m^Q$. Then, by definition, there exist $\delta_1, \dots, \delta_m > 0$ such that

$$\exp((0, \delta_i)Q)x \subset A_i$$

for $i = 1, \dots, m$. Put $\delta := \min\{\delta_1, \dots, \delta_m\} > 0$. Then we have

$$\exp((0, \delta)Q)x \subset A_1 \cap \cdots \cap A_m,$$

and hence $x \in (A_1 \cap \cdots \cap A_m)^Q$. □

In the following, we study the exponential Q-perturbation C_I^Q of rational open cones C_I, which play an important role in the construction of our Shintani cocycle.

Lemma 5.1.4. *For* $r \geq 0$, $I = (\alpha_1, \dots, \alpha_r) \in (X_{\mathbb{Q}})^r$, $Q \in \Xi$, *and* $\gamma \in \mathrm{GL}_g(\mathbb{Q})$, *we have*

$$\gamma(C^Q_{\gamma^{-1}I}) = C_I^{[\gamma](Q)},$$

where $[\gamma](Q) = \gamma Q \gamma^{-1} \in \Xi$.

Proof. Indeed, for $x \in \mathbb{R}^g$ and $\varepsilon > 0$, we see that

$$\exp(\varepsilon[\gamma](Q))x \in C_I \iff \exp(\varepsilon\gamma Q\gamma^{-1})x \in C_I \iff \exp(\varepsilon Q)\gamma^{-1}x \in \gamma^{-1}(C_I) = C_{\gamma^{-1}I}.$$

This proves the lemma. □

5.2. *Rationality.* The aim of this subsection is to prove the following proposition:

Proposition 5.2.1. *Let* $0 \leq r \leq g$, $I = (\alpha_1, \dots, \alpha_r) \in (X_{\mathbb{Q}})^r$, *and* $Q \in \Xi$.

(1) *Suppose* $\dim_{\mathbb{Q}} \mathrm{Span}_{\mathbb{Q}}\{\alpha_1, \dots, \alpha_r\} \leq g-1$. *Then*

$$C_I^Q = \begin{cases} \{0\} & \text{if } 0 \in C_I, \\ \varnothing & \text{if } 0 \notin C_I. \end{cases}$$

(2) *The exponential Q-perturbation C_I^Q of the rational open cone C_I generated by I is a rational constructible cone, i.e., a disjoint union of a finite number of rational simplicial open cones.*

To prove this proposition, we first prepare several lemmas. In the following, for $\alpha \in \mathbb{R}^g - \{0\}$, we put

$$U_{\alpha,\pm} := \{x \in \mathbb{R}^g \mid \pm\langle x, \alpha\rangle > 0\} \quad \text{and} \quad H_\alpha := \{x \in \mathbb{R}^g \mid \langle x, \alpha\rangle = 0\}.$$

We start with recalling the following fact.

Lemma 5.2.2 [Shintani 1976, Section 1.2; Hida 1993, pp. 68–69, Lemma 1]. (1) *Let $W \subset \mathbb{Q}^g$ be a $\mathbb{Q}$-subspace, and let $l_1, \dots, l_m \in \mathbb{Q}^g - \{0\}$. Then the subset*

$$X = \{x \in W \otimes_{\mathbb{Q}} \mathbb{R} \subset \mathbb{R}^g \mid \langle x, l_i\rangle > 0 \text{ for } i = 1, \dots, m\} \subset \mathbb{R}^g$$

is a rational constructible cone.

(2) *Let $C, C' \subset \mathbb{R}^g$ be rational constructible cones. Then $C \cup C'$, $C \cap C'$, and $C - C'$ are rational constructible cones.*

Proof. See [Shintani 1976, Lemma 2, Corollary to Lemma 2] and [Hida 1993, pp. 68–69, Lemma 1]. Although, in [Hida 1993], it is assumed that the total space is of the form $F \otimes_{\mathbb{Q}} \mathbb{R}$ for a number field F and that W is a subspace generated by elements in F, the proof there does not use this special assumption. □

The following is the key lemma of this section.

Lemma 5.2.3. *Let $Q \in \Xi$ and $\alpha \in \mathbb{Q}^g - \{0\}$. For $k \geq 0$, put*

$$H_{\pm}^{(k)} := \bigl\{x \in \mathbb{R}^g \mid \pm\langle x, {}^tQ^k\alpha\rangle > 0 \text{ and } \langle x, {}^tQ^j\alpha\rangle = 0 \text{ for } 0 \leq j \leq k-1\bigr\}.$$

Note that $H_{\pm}^{(0)} = U_{\alpha,\pm}$ by definition.

(1) *There exists $k_0 \geq 0$ such that $H_{\pm}^{(k)} = \varnothing$ for all $k \geq k_0 + 1$. Moreover, we have*

$$\mathbb{R}^g - \{0\} = \bigsqcup_{k=0}^{k_0} (H_+^{(k)} \sqcup H_-^{(k)}),$$

where $\bigsqcup$ and $\sqcup$ denote the disjoint union.

(2) *For all $k \geq 0$, the sets $H_+^{(k)}$ and $H_-^{(k)}$ are rational constructible cones.*

(3) *For all $k \geq 0$, we have $H_+^{(k)} \subset (H_+^{(0)})^Q = (U_{\alpha,+})^Q$ and $H_-^{(k)} \subset (H_-^{(0)})^Q = (U_{\alpha,-})^Q$.*

(4) *We have $H_\alpha^Q = \{0\}$ and*

$$\mathbb{R}^g - \{0\} = (U_{\alpha,+})^Q \sqcup (U_{\alpha,-})^Q.$$

In particular, $\mathbb{R}^g = H_\alpha^Q \sqcup (U_{\alpha,+})^Q \sqcup (U_{\alpha,-})^Q$.

(5) *We have*

$$(U_{\alpha,+})^Q = \bigsqcup_{k=0}^{k_0} H_+^{(k)} \quad \text{and} \quad (U_{\alpha,-})^Q = \bigsqcup_{k=0}^{k_0} H_-^{(k)}.$$

In particular, $(U_{\alpha,+})^Q$ and $(U_{\alpha,-})^Q$ are rational constructible cones.

Proof. (1) and (2) For $k \geq 0$, put

$$H^{(k)} := \{x \in \mathbb{R}^g \mid \langle x, {}^tQ^j\alpha\rangle = 0 \text{ for } 0 \leq j \leq k-1\}.$$

Then we have a descending chain

$$\mathbb{R}^g = H^{(0)} \supset H^{(1)} \supset H^{(2)} \supset \cdots$$

of $\mathbb{R}$-vector spaces. Note that the subspaces $H^{(k)}$ are all defined over $\mathbb{Q}$ since we have ${}^tQ^j\alpha \in \mathbb{Q}^g - \{0\}$ for $j \geq 0$. Since $\mathbb{R}^g$ is a finite-dimensional vector space, there exists $k_0 \geq 0$ such that $H^{(k)} = H^{(k_0+1)}$ for all $k \geq k_0 + 1$.

Claim. $$H^{(k_0+1)} = 0.$$

Proof of claim. Indeed, let $x \in H^{(k_0+1)} = H^{(k_0+2)}$. Then we have

$$\langle Qx, {}^tQ^j\alpha\rangle = \langle x, {}^tQ^{j+1}\alpha\rangle = 0 \quad \text{for } 0 \leq j \leq k_0,$$

and hence $Qx \in H^{(k_0+1)}$. Therefore, $H^{(k_0+1)}$ is a Q-stable subspace of $\mathbb{R}^g$ defined over $\mathbb{Q}$. Moreover, since $\alpha \neq 0$, we have

$$H^{(k_0+1)} \subset H^{(1)} \subsetneq \mathbb{R}^g.$$

Therefore, we obtain $H^{(k_0+1)} = 0$ by Lemma 2.1.1(2). □

Now (1) follows from the fact

$$H^{(k)} - H^{(k+1)} = H_+^{(k)} \sqcup H_-^{(k)} \quad \text{for all } k \geq 0,$$

and (2) follows from Lemma 5.2.2(1).

(3) Let $x \in H_+^{(k)}$. Then we have

$$\langle \exp(\varepsilon Q)x, \alpha \rangle = \sum_{m \geq k} \frac{\langle x, {}^t Q^m \alpha \rangle}{m!} \varepsilon^m.$$

Now since $\langle x, {}^t Q^k \alpha \rangle > 0$, there exists $\delta > 0$ such that

$$\langle \exp(\varepsilon Q)x, \alpha \rangle = \sum_{m \geq k} \frac{\langle x, {}^t Q^m \alpha \rangle}{m!} \varepsilon^m > 0$$

for all $\varepsilon \in (0, \delta)$. Hence $x \in (H_+^{(0)})^Q$. The inclusion $H_-^{(k)} \subset (H_-^{(0)})^Q$ can be proved similarly.

(4) First, by Lemma 5.1.3(2), we see $(U_{\alpha,+})^Q \cap (U_{\alpha,-})^Q = \varnothing$, and $H_\alpha^Q \cap (U_{\alpha,\pm})^Q = \varnothing$. On the other hand, we obviously have $0 \in H_\alpha^Q$, and hence $0 \notin (U_{\alpha,\pm})^Q$. Therefore, by (1) and (3), we obtain

$$\mathbb{R}^g - \{0\} \subset (U_{\alpha,+})^Q \sqcup (U_{\alpha,-})^Q \subset \mathbb{R}^g - \{0\}.$$

Thus we find $\mathbb{R}^g - \{0\} = (U_{\alpha,+})^Q \sqcup (U_{\alpha,-})^Q$ and $H_\alpha^Q = \{0\}$.

(5) The first part follows from (1), (3), and (4). Then the latter part follows from (2). □

Lemma 5.2.4. *Let $I = (\alpha_1, \dots, \alpha_r) \in (X_{\mathbb{Q}})^r$ such that $\alpha_1, \dots, \alpha_r \in \mathbb{Q}^g - \{0\}$ are linearly independent. Note that we automatically have $r \leq g$.*

(1) *There exist $\alpha'_1, \dots, \alpha'_r, \beta'_1, \dots, \beta'_{g-r} \in \mathbb{Q}^g - \{0\}$ such that*

$$C_I = \left(\bigcap_{i=1}^{r} U_{\alpha'_i,+} \right) \cap \left(\bigcap_{i=1}^{g-r} H_{\beta'_i} \right).$$

(2) *Let $Q \in \Xi$. Then we have*

$$\mathbb{R}^g = C_I^Q \sqcup (\mathbb{R}^g - C_I)^Q.$$

Proof. (1) Put $W := \mathrm{Span}_{\mathbb{Q}}\{\alpha_1, \dots, \alpha_r\} \subset \mathbb{Q}^g$, and let $W^\perp \subset \mathbb{Q}^g$ be its orthogonal complement with respect to the scalar product $\langle\ ,\ \rangle$. Let $\alpha'_1, \dots, \alpha'_r \in W$ be the dual basis of $\alpha_1, \dots, \alpha_r$ in W with respect to $\langle\ ,\ \rangle$, i.e.,

$$\langle \alpha_i, \alpha'_j \rangle = \begin{cases} 1 & (i = j), \\ 0 & (i \neq j), \end{cases}$$

and let $\beta'_1, \dots, \beta'_{g-r} \in W^\perp$ be a basis of $W^\perp$ over $\mathbb{Q}$. Then $\alpha'_1, \dots, \alpha'_r, \beta'_1, \dots, \beta'_{g-r}$ satisfy the desired property. Indeed, let $\beta_1, \dots, \beta_{g-r} \in W^\perp$ be the dual basis of $\beta'_1, \dots, \beta'_{g-r}$ in $W^\perp$, and let $x \in \mathbb{R}^g$. Since $\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_{g-r}$ form a basis of $\mathbb{R}^g$, we have

$$x = \sum_{i=1}^{r} c_i \alpha_i + \sum_{j=1}^{g-r} d_j \beta_j$$

for some $c_i, d_j \in \mathbb{R}$. Then we have $x \in C_I$ if and only if

$$\langle x, \alpha'_i \rangle = c_i > 0 \quad \text{and} \quad \langle x, \beta'_j \rangle = d_j = 0 \qquad \text{for all } i, j.$$

This proves (1).

(2) Using (1), we take $\alpha'_1, \dots, \alpha'_r, \beta'_1, \dots, \beta'_{g-r} \in \mathbb{Q}^g - \{0\}$ such that

$$C_I = \left(\bigcap_{i=1}^{r} U_{\alpha'_i,+} \right) \cap \left(\bigcap_{i=1}^{g-r} H_{\beta'_i} \right). \tag{5-1}$$

We then have

$$\mathbb{R}^g - C_I = \bigcup_{i=1}^{r} (U_{\alpha'_i,-} \cup H_{\alpha'_i}) \cup \bigcup_{i=1}^{g-r} (U_{\beta'_i,+} \cup U_{\beta'_i,-}).$$

By taking the exponential Q-perturbation and using Lemma 5.1.3(1), we obtain

$$\bigcup_{i=1}^{r} \big((U_{\alpha'_i,-})^Q \cup H^Q_{\alpha'_i} \big) \cup \bigcup_{i=1}^{g-r} \big((U_{\beta'_i,+})^Q \cup (U_{\beta'_i,-})^Q \big) \subset (\mathbb{R}^g - C_I)^Q. \tag{5-2}$$

On the other hand, by (5-1) and Lemmas 5.1.3(2) and 5.2.3(4), we obtain

$$\begin{aligned} \mathbb{R}^g - C_I^Q &= \mathbb{R}^g - \left(\left(\bigcap_{i=1}^{r} (U_{\alpha'_i,+})^Q \right) \cap \left(\bigcap_{i=1}^{g-r} H^Q_{\beta'_i} \right) \right) \\ &= \bigcup_{i=1}^{r} \big((U_{\alpha'_i,-})^Q \cup H^Q_{\alpha'_i} \big) \cup \bigcup_{i=1}^{g-r} \big((U_{\beta'_i,+})^Q \cup (U_{\beta'_i,-})^Q \big). \end{aligned} \tag{5-3}$$

Thus, by combining (5-2) and (5-3), we find that $\mathbb{R}^g - C_I^Q \subset (\mathbb{R}^g - C_I)^Q$, and hence $\mathbb{R}^g = C_I^Q \cup (\mathbb{R}^g - C_I)^Q$. Finally, since we have $C_I^Q \cap (\mathbb{R}^g - C_I)^Q = \varnothing$ by Lemma 5.1.3(2), we obtain $\mathbb{R}^g = C_I^Q \sqcup (\mathbb{R}^g - C_I)^Q$. □

Proof of Proposition 5.2.1. (1) Since $\mathrm{Span}_{\mathbb{Q}}\{\alpha_1, \dots, \alpha_r\} \subsetneq \mathbb{Q}^g$, there exists $\beta \in \mathbb{Q}^g - \{0\}$ such that

$$C_I \subset \mathrm{Span}_{\mathbb{R}}\{\alpha_1, \dots, \alpha_r\} \subset H_\beta.$$

Therefore, by Lemmas 5.1.3(1) and 5.2.3(4), we have either $C_I^Q = \varnothing$ or $C_I^Q = \{0\}$. Then it is clear that $C_I^Q = \{0\}$ if and only if $0 \in C_I$. This proves (1).

(2) Since $\varnothing$ and $\{0\}$ are obviously rational constructible cones, we may assume $\alpha_1, \dots, \alpha_r$ generates $\mathbb{R}^g$. In particular, we have $r = g$ and C_I is a rational simplicial open cone. By Lemma 5.2.4(1), there exist $\alpha'_1, \dots, \alpha'_g \in \mathbb{Q}^g - \{0\}$ such that

$$C_I = \bigcap_{i=1}^{g} U_{\alpha'_i,+}.$$

Then, by Lemma 5.1.3(2), we have

$$C_I^Q = \bigcap_{i=1}^{g} (U_{\alpha'_i,+})^Q.$$

Now, we already know that $(U_{\alpha'_i,+})^Q$ is a rational constructible cone by Lemma 5.2.3(5), and hence C_I^Q is also a rational constructible cone by Lemma 5.2.2(2). □

5.3. *Cocycle relation.*

Definition 5.3.1. (1) For a subset $A \subset \mathbb{R}^g$, let

$$\mathbf{1}_A : \mathbb{R}^g \to \mathbb{R}, \quad x \mapsto \begin{cases} 0 & \text{if } x \notin A, \\ 1 & \text{if } x \in A \end{cases}$$

denote the characteristic function of A.

(2) For $I = (\alpha_1, \dots, \alpha_g) \in (\mathbb{R}^g - \{0\})^g$, we set

$$\operatorname{sgn}(I) := \operatorname{sgn}\det(\alpha_1, \dots, \alpha_g) \in \{-1, 0, 1\},$$

where $(\alpha_1, \dots, \alpha_g)$ is regarded as an element in $M_g(\mathbb{R})$. We assume $\operatorname{sgn} 0 := 0$.

(3) Let $r \geq 1$ and $I = (\alpha_1, \dots, \alpha_r) \in (\mathbb{R}^g - \{0\})^r$. We say that $x \in \mathbb{R}^g$ is *in general position relative to* I if x is not contained in any proper $\mathbb{R}$-subspace of $\mathbb{R}^g$ generated by a subset of $\{\alpha_1, \dots, \alpha_r\}$.

Remark 5.3.2. The condition "in general position relative to I" is slightly more strict than the condition "generic with respect to $\{\alpha_1, \dots, \alpha_r\}$" in the sense of Yamamoto [2010, p. 471]. Actually, this difference is not important at all, but we adopt this definition since it is more useful in this paper.

Lemma 5.3.3. *Let* $r \geq 1$, $I = (\alpha_1, \dots, \alpha_r) \in (X_{\mathbb{Q}})^r$, $x \in \mathbb{R}^g - \{0\}$, *and* $Q \in \Xi$. *Then there exists* $\delta > 0$ *such that* $\exp(\varepsilon Q)x$ *is in general position relative to* I *for all* $\varepsilon \in (0, \delta)$.

Proof. Let $W_1, \dots, W_m \subsetneq \mathbb{R}^g$ be all the proper $\mathbb{R}$-subspaces which can be generated by some subset of $\{\alpha_1, \dots, \alpha_r\}$. In particular, $y \in \mathbb{R}^g$ is in general position relative to I if and only if $y \notin \bigcup_{j=1}^m W_j$.

Take $\beta_1, \dots, \beta_m \in \mathbb{Q}^g - \{0\}$ such that $W_j \subset H_{\beta_j}$ for $j = 1, \dots, m$. (See Section 5.2 for the definition of H_{β_j}.) Then, by Lemma 5.2.3(4), for each j, there exists $\delta_j > 0$ such that

$$\exp((0, \delta_j)Q)x \subset U_{\beta_j,+} \cup U_{\beta_j,-} = \mathbb{R}^g - H_{\beta_j}.$$

Put $\delta := \min\{\delta_1, \dots, \delta_m\} > 0$. Then for all $\varepsilon \in (0, \delta)$, we have

$$\exp(\varepsilon Q)x \notin \bigcup_{j=1}^m H_{\beta_j} \supset \bigcup_{j=1}^m W_j,$$

and hence $\exp(\varepsilon Q)x$ is in general position relative to I. □

The following is the main proposition of this subsection.

Proposition 5.3.4. *Let* $J = (\alpha_0, \dots, \alpha_g) \in (X_{\mathbb{Q}})^{g+1}$ *and* $Q \in \Xi$. *Assume that there exists* $y \in \mathbb{R}^g - \{0\}$ *such that for all* $i = 0, \dots, g$ *we have* $\langle \alpha_i, y \rangle > 0$. *Then we have*

$$\sum_{i=0}^g (-1)^i \operatorname{sgn}(J^{(i)}) \mathbf{1}_{C^Q_{J^{(i)}}}(x) = 0$$

for $x \in \mathbb{R}^g - \{0\}$, *where* $J^{(i)} = (\alpha_0, \dots, \check{\alpha}_i, \dots, \alpha_g) \in (X_{\mathbb{Q}})^g$.

Proof. Take such $y \in \mathbb{R}^g - \{0\}$. We will reduce the problem to the "generic case". First, we claim that for each $i = 0, \dots, g$, there exists $\delta_i > 0$ such that

$$\exp((0, \delta_i)Q)x \subset C_{J^{(i)}} \quad \text{or} \quad \exp((0, \delta_i)Q)x \subset \mathbb{R}^g - C_{J^{(i)}}.$$

Indeed, if $\alpha_0, \dots, \check{\alpha}_i, \dots, \alpha_g$ (α_i is omitted) are linearly independent, then this follows directly from Lemma 5.2.4(2). On the other hand, if $\alpha_0, \dots, \check{\alpha}_i, \dots, \alpha_g$ are linearly dependent, then we have $\mathrm{Span}_{\mathbb{Q}}\{\alpha_0, \dots, \check{\alpha}_i, \dots, \alpha_g\} \subsetneq \mathbb{Q}^g$, and hence there exists $\alpha \in \mathbb{Q}^g - \{0\}$ such that $C_{J^{(i)}} \subset H_\alpha$. Therefore, by Lemma 5.2.3(4) along with Lemma 5.1.3, we find

$$\mathbb{R}^g - \{0\} = (U_{\alpha,+})^Q \sqcup (U_{\alpha,-})^Q \subset (\mathbb{R}^g - C_{J^{(i)}})^Q,$$

and we can take such $\delta_i > 0$.

Consequently, for $i = 0, \dots, g$, we obtain

$$\mathbf{1}_{C^Q_{J^{(i)}}}(x) = \mathbf{1}_{C_{J^{(i)}}}(\exp(\varepsilon Q)x) \quad \text{for all } \varepsilon \in (0, \delta_i).$$

On the other hand, by Lemma 5.3.3, there exists $\delta > 0$ such that for all $\varepsilon \in (0, \delta)$, $\exp(\varepsilon Q)x$ is in general position relative to J. Set $\varepsilon_0 := \frac{1}{2}\min\{\delta_0, \dots, \delta_g, \delta\}$, and put $x' := \exp(\varepsilon_0 Q)x$. Then

- $\mathbf{1}_{C^Q_{J^{(i)}}}(x) = \mathbf{1}_{C_{J^{(i)}}}(x')$ for $i = 0, \dots, g$,
- x' is in general position relative to J.

Therefore, it suffices to prove

$$\sum_{i=0}^{g} (-1)^i \operatorname{sgn}(J^{(i)}) \mathbf{1}_{C_{J^{(i)}}}(x') = 0 \tag{5-4}$$

for any x' which is in general position relative to J. First, if $\langle x', y \rangle \le 0$, then we have

$$\mathbf{1}_{C_{J^{(i)}}}(x') = 0 \quad \text{for all } i \in \{0, \dots, g\}$$

because $\langle \alpha_i, y \rangle > 0$ for all $i = 0, \dots, g$. Therefore, we may assume $\langle x', y \rangle > 0$. In this case, the identity (5-4) follows from [Yamamoto 2010, Proposition 6.2].

Indeed, let $\gamma \in \mathrm{GL}_g(\mathbb{R})$ such that ${}^t\gamma e_g = y$, where $e_g = {}^t(0, \dots, 0, 1) \in \mathbb{R}^g$. Then

- $\gamma x', \gamma\alpha_0, \dots, \gamma\alpha_g \in \mathcal{H} := \{v \in \mathbb{R}^g \mid \langle v, e_g \rangle > 0\}$,
- $\gamma x'$ is in general position relative to γJ,
- $\operatorname{sgn}(\gamma J^{(i)}) = \operatorname{sgn}(\det(\gamma)) \operatorname{sgn}(J^{(i)})$,
- $\mathbf{1}_{C_{J^{(i)}}}(x') = \mathbf{1}_{C_{\gamma J^{(i)}}}(\gamma x')$,

and hence we can use [Yamamoto 2010, Proposition 6.2]. This completes the proof. □

Remark 5.3.5. It is also possible to prove the last part using [Charollois et al. 2015, Theorem 2.1].

6. Construction of the Shintani–Barnes cocycle

Recall that for $d \geq 0$, we have sheaves

$$\mathscr{F}_d = \pi_{\mathbb{C}}^{-1}\Omega^{g-1}_{\mathbb{P}^{g-1}(\mathbb{C})}(-d)|_{Y^\circ} \quad \text{and} \quad \mathscr{F}_d^{\Xi} = \underline{\operatorname{Hom}}(\underline{\mathbb{Z}[\Xi]}, \mathscr{F}_d) \simeq \prod_{Q\in\Xi} \mathscr{F}_d$$

on $Y^\circ = \mathbb{C}^g - \boldsymbol{i}\mathbb{R}^g$. In this section we construct a certain cohomology class in $H^{g-1}(Y^\circ, \mathrm{SL}_g(\mathbb{Z}), \mathscr{F}_d^{\Xi})$ using the Čech complex $\mathscr{C}^\bullet(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}_d)^{\Xi}$.

6.1. ***Barnes zeta function associated to C_I^Q.*** Recall that for $I = (\alpha_1, \dots, \alpha_g) \in (X_{\mathbb{Q}})^g$, the open subset $V_I \subset Y^\circ$ is defined as

$$V_I = \{y \in Y^\circ \mid \mathrm{Re}(\langle \alpha_i, y\rangle) > 0 \text{ for } i = 1, \dots, g\},$$

and we have

$$\Gamma(V_I, \mathscr{F}_d) = \{f\omega \mid f \text{ holomorphic function on } \pi_{\mathbb{C}}^{-1}(\pi_{\mathbb{C}}(V_I)) \text{ such that } f(\lambda y) = \lambda^{-g-d} f(y) \text{ for all } \lambda \in \mathbb{C}^\times\}$$

by Proposition 3.2.3. Note that $\pi_{\mathbb{C}}^{-1}(\pi_{\mathbb{C}}(V_I)) \subset \mathbb{C}^g - \{0\}$ is an open subset of the following form:

$$\pi_{\mathbb{C}}^{-1}(\pi_{\mathbb{C}}(V_I)) = \{y \in \mathbb{C}^g \mid \text{there exists } \lambda \in \mathbb{C}^\times \text{ such that } \lambda y \in V_I\} \subset \mathbb{C}^g - \{0\}.$$

Definition 6.1.1. For $d \geq 1$, $I = (\alpha_1, \dots, \alpha_g) \in (X_{\mathbb{Q}})^g$, $Q \in \Xi$, and $y \in \pi_{\mathbb{C}}^{-1}(\pi_{\mathbb{C}}(V_I))$, set

$$\psi_{d,I}^Q(y) := \mathrm{sgn}(I) \sum_{x \in C_I^Q \cap \mathbb{Z}^g - \{0\}} \frac{1}{\langle x, y\rangle^{g+d}}, \tag{6-1}$$

where $\mathrm{sgn}(I) = \mathrm{sgn}\det(\alpha_1, \dots, \alpha_g) \in \{-1, 0, 1\}$; see Section 5.3.

Proposition 6.1.2. *The infinite series* (6-1) *converges absolutely and locally uniformly for* $y \in \pi_{\mathbb{C}}^{-1}(\pi_{\mathbb{C}}(V_I))$. *In particular*, $\psi_{d,I}^Q$ *is a holomorphic function on* $\pi_{\mathbb{C}}^{-1}(\pi_{\mathbb{C}}(V_I))$. *Moreover, we have*

$$\psi_{d,I}^Q(\lambda y) = \lambda^{-g-d}\psi_{d,I}^Q(y)$$

for all $\lambda \in \mathbb{C}^\times$ *and* $y \in \pi_{\mathbb{C}}^{-1}(\pi_{\mathbb{C}}(V_I))$.

Proof. If $\mathrm{sgn}(I) = 0$, then by Proposition 5.2.1(1), we see that $C_I^Q \cap \mathbb{Z}^g - \{0\} = \varnothing$, and hence the sum is zero. (In particular, the series converges.) Therefore, we may assume that $\alpha_1, \dots, \alpha_g$ form a basis of $\mathbb{Q}^g$. Furthermore, since $\mathrm{sgn}(I)$ and C_I^Q do not change if we replace α_i by its multiple by positive integers, we may assume that $\alpha_1, \dots, \alpha_g \in \mathbb{Z}^g - \{0\}$.

Let $y \in \pi_{\mathbb{C}}^{-1}(\pi_{\mathbb{C}}(V_I))$ and take $\lambda \in \mathbb{C}^\times$ such that $\lambda y \in V_I$. Then take a relatively compact open neighborhood $U \subset V_I$ of λy, i.e., U is an open neighborhood of λy such that its closure $\overline{U}$ is compact and $\overline{U} \subset V_I$. Since $y \in \lambda^{-1}\overline{U} \subset \pi_{\mathbb{C}}^{-1}(\pi_{\mathbb{C}}(V_I))$, it suffices to show that (6-1) converges absolutely and uniformly on $\lambda^{-1}\overline{U}$.

First, note that by the definition of C_I^Q, we have

$$C_I^Q \subset \overline{C}_I = \sum_{i=1}^{g} \mathbb{R}_{\geq 0}\alpha_i,$$

where $\overline{C}_I$ is the closed cone generated by I. Put

$$R_I := \sum_{i=1}^{g} [0, 1)\alpha_i .$$

Then we see

- $C_I^Q \cap \mathbb{Z}^g \subset \overline{C}_I \cap \mathbb{Z}^g = \bigl\{x + \sum_{i=1}^{g} n_i \alpha_i \mid x \in R_I \cap \mathbb{Z}^g,\ n_i \in \mathbb{Z}_{\geq 0}\bigr\}$,
- $R_I \cap \mathbb{Z}^g$ is a finite set,
- $\bigl\{\mathrm{Re}(\langle x, y'\rangle) \mid x \in R_I \cap \mathbb{Z}^g - \{0\},\ y' \in \overline{U}\bigr\}$ is a compact subset of $\mathbb{R}_{>0}$.

Therefore, set

$$b := \min\bigl\{\mathrm{Re}(\langle x, y'\rangle) \mid x \in R_I \cap \mathbb{Z}^g - \{0\},\ y' \in \overline{U}\bigr\} > 0.$$

Moreover, for $i = 1, \dots, g$, set

$$a_i := \min\bigl\{\mathrm{Re}(\langle \alpha_i, y'\rangle) \mid y' \in \overline{U}\bigr\} > 0.$$

Then for $y'' = \lambda^{-1} y' \in \lambda^{-1}\overline{U}$, where $y' \in \overline{U}$, we have

$$\begin{aligned}
&\sum_{x \in C_I^Q \cap \mathbb{Z}^g - \{0\}} \left| \frac{1}{\langle x, y''\rangle^{g+d}} \right| \\
&\leq |\lambda|^{g+d} \sum_{x \in \overline{C}_I \cap \mathbb{Z}^g - \{0\}} \frac{1}{|\langle x, y'\rangle|^{g+d}} \\
&\leq |\lambda|^{g+d} \sum_{x \in \overline{C}_I \cap \mathbb{Z}^g - \{0\}} \frac{1}{\bigl(\mathrm{Re}(\langle x, y'\rangle)\bigr)^{g+d}} \\
&\leq |\lambda|^{g+d} \sum_{\substack{x' \in R_I \cap \mathbb{Z}^g,\, (n_1, \dots, n_g) \in (\mathbb{Z}_{\geq 0})^g, \\ x' + \sum_{i=1}^{g} n_i \alpha_i \neq 0}} \frac{1}{\bigl(\mathrm{Re}(\langle x', y'\rangle) + \sum_{i=1}^{g} n_i \,\mathrm{Re}(\langle \alpha_i, y'\rangle)\bigr)^{g+d}} \\
&\leq |\lambda|^{g+d} \sum_{(n_1, \dots, n_g) \in (\mathbb{Z}_{\geq 0})^g - \{0\}} \frac{1}{\bigl(\sum_{i=1}^{g} n_i a_i\bigr)^{g+d}} + |\lambda|^{g+d} \#(R_I \cap \mathbb{Z}^g - \{0\}) \sum_{(n_1, \dots, n_g) \in (\mathbb{Z}_{\geq 0})^g} \frac{1}{\bigl(b + \sum_{i=1}^{g} n_i a_i\bigr)^{g+d}},
\end{aligned}$$

where $\#(R_I \cap \mathbb{Z}^g - \{0\})$ is the order of the finite set $R_I \cap \mathbb{Z}^g - \{0\}$. It is now clear that the last two series converge for $d \geq 1$. The last statement in the proposition follows directly from the definition. □

Remark 6.1.3. Since C_I^Q is a rational constructible cone (see Proposition 5.2.1), we see that $\psi_{d,I}^Q$ can be written as a sum of a finite number of the Barnes zeta functions; see [Barnes 1904; Yamamoto 2010]. Conceptually, we may also view $\psi_{d,I}^Q$ as a decomposed piece of the "Eisenstein series"

$$\psi_d(y) = \sum_{x \in \mathbb{Z}^g - \{0\}} \frac{1}{\langle x, y\rangle^{g+d}},$$

which coincides with the classical holomorphic Eisenstein series of weight $2 + d$ if $g = 2$, $d \geq 2$ is even, and $y = (1, z)$ with $\mathrm{Im}(z) > 0$, but does not converge if $g \geq 3$. Therefore, the following construction of the Shintani–Barnes cocycle can be seen as a cohomological realization of this (generally) nonconvergent Eisenstein series.

Corollary 6.1.4. *Let $d \geq 1$. For $I = (\alpha_1, \dots, \alpha_g) \in (X_{\mathbb{Q}})^g$ and $Q \in \Xi$, we have*

$$\psi^{Q}_{d,I}\omega \in \Gamma(V_I, \mathscr{F}_d), \quad \text{where } \omega(y) = \sum_{i=1}^{g} (-1)^{i-1} y_i \, dy_1 \wedge \cdots \wedge \check{dy_i} \wedge \cdots \wedge dy_g.$$

Proof. This follows directly from Propositions 3.2.3(2) and 6.1.2. □

6.2. *The Shintani–Barnes cocycle.*

Definition 6.2.1. For $d \geq 1$, we define a map $\Psi_d : \Xi \to C^{g-1}(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}_d)$ by

$$\Psi_d(Q) := (\psi^{Q}_{d,I}\omega)_{I \in (X_{\mathbb{Q}})^g} \in C^{g-1}(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}_d) = \prod_{I \in (X_{\mathbb{Q}})^g} \Gamma(V_I, \mathscr{F}_d) \qquad \text{for } Q \in \Xi.$$

We aim to show that Ψ_d defines a class in $H^{g-1}(Y^\circ, \mathrm{SL}_g(\mathbb{Z}), \mathscr{F}_d^{\Xi})$ via Corollary 4.3.4.

Proposition 6.2.2. *The map Ψ_d is a $\mathrm{SL}_g(\mathbb{Z})$-equivariant map, i.e., we have*

$$\Psi_d\big([\gamma](Q)\big) = [\gamma](\Psi_d(Q))$$

for $Q \in \Xi$ and $\gamma \in \mathrm{SL}_g(\mathbb{Z})$. In other words, we have

$$\Psi_d \in \mathrm{Map}_{\mathrm{SL}_g(\mathbb{Z})}(\Xi, C^{g-1}(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}_d)) = \Gamma\big(Y^\circ, \mathrm{SL}_g(\mathbb{Z}), \mathscr{C}^{g-1}(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}_d)^{\Xi}\big).$$

Proof. Let $I = (\alpha_1, \dots, \alpha_g) \in (X_{\mathbb{Q}})^g$. We need to show

$$\Psi_d([\gamma](Q))_I = \big([\gamma](\Psi_d(Q))\big)_I \in \Gamma(V_I, \mathscr{F}_d),$$

where $\Psi_d([\gamma](Q))_I$ $\big($resp. $\big([\gamma](\Psi_d(Q))\big)_I\big)$ is the I-th component of $\Psi_d\big([\gamma](Q)\big)$ $\big($resp. $[\gamma](\Psi_d(Q))\big)$ as always. Indeed, we have

$$\begin{aligned}
\big([\gamma](\Psi_d(Q))\big)_I(y) &= \big([\gamma](\psi^{Q}_{d,\gamma^{-1}I}\omega)\big)(y) \\
&= \psi^{Q}_{d,\gamma^{-1}I}({}^t\gamma y)\omega({}^t\gamma y) \\
&= \mathrm{sgn}(\gamma^{-1}I) \sum_{x \in C^{Q}_{\gamma^{-1}I} \cap \mathbb{Z}^g - \{0\}} \frac{\omega({}^t\gamma y)}{\langle x, {}^t\gamma y\rangle^{g+d}} \\
&= \mathrm{sgn}(\det(\gamma^{-1}))\,\mathrm{sgn}(I)\det({}^t\gamma) \sum_{x \in C^{Q}_{\gamma^{-1}I} \cap \mathbb{Z}^g - \{0\}} \frac{\omega(y)}{\langle \gamma x, y\rangle^{g+d}} \\
&= \mathrm{sgn}(I) \sum_{x \in \gamma(C^{Q}_{\gamma^{-1}I}) \cap \mathbb{Z}^g - \{0\}} \frac{\omega(y)}{\langle x, y\rangle^{g+d}} \\
&= \mathrm{sgn}(I) \sum_{x \in C^{[\gamma](Q)}_{I} \cap \mathbb{Z}^g - \{0\}} \frac{\omega(y)}{\langle x, y\rangle^{g+d}} = \Psi_d([\gamma](Q))_I(y)
\end{aligned}$$

for $y \in \pi_{\mathbb{C}}^{-1}(\pi_{\mathbb{C}}(V_I))$. Here, the first and second equalities follow from the definition of $[\gamma]$ (see (4-1) and Definition 3.3.2), the fourth equality follows from Lemma 3.3.1, and the sixth equality follows from Lemma 5.1.4. ☐

Corollary 6.2.3. *For $Q \in \Xi$, we have*

$$\Psi_d(Q) \in C^{g-1}(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}_d)^{\Gamma_Q} = \Gamma(Y^\circ, \Gamma_Q, \mathscr{C}^{g-1}(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}_d)).$$

Proof. Because Γ_Q is the stabilizer of Q in $\mathrm{SL}_g(\mathbb{Z})$ and Ψ_d is a $\mathrm{SL}_g(\mathbb{Z})$-equivariant map, it follows that $\Psi_d(Q)$ is a Γ_Q-invariant element. ☐

Proposition 6.2.4. (1) *Let $Q \in \Xi$. We have*

$$\mathrm{d}^{g-1}(\Psi_d(Q)) = 0$$

under the differential map

$$\mathrm{d}^{g-1} : C^{g-1}(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}_d) \to C^{g}(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}_d).$$

(2) *We have*

$$\mathrm{d}^{g-1}(\Psi_d) = 0$$

under the differential map

$$\mathrm{d}^{g-1} : \Gamma\big(Y^\circ, \mathrm{SL}_g(\mathbb{Z}), \mathscr{C}^{g-1}(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}_d)^{\Xi}\big) \to \Gamma\big(Y^\circ, \mathrm{SL}_g(\mathbb{Z}), \mathscr{C}^{g}(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}_d)^{\Xi}\big).$$

In the following, we refer to Ψ_d as the Shintani–Barnes cocycle.

Proof. (1) Let $J = (\alpha_0, \dots, \alpha_g) \in (X_{\mathbb{Q}})^{g+1}$. For $i = 0, \dots, g$, put $J^{(i)} = (\alpha_0, \dots, \check{\alpha}_i, \dots, \alpha_g) \in (X_{\mathbb{Q}})^{g}$. We need to show

$$\big(\mathrm{d}^{g-1}(\Psi_d(Q))\big)_J = \sum_{i=0}^{g} (-1)^i \Psi_d(Q)_{J^{(i)}}|_{V_J} = 0. \tag{6-2}$$

First if $V_J = \varnothing$, then (6-2) is obvious because $\Gamma(\varnothing, \mathscr{F}_d) = 0$. Assume $V_J \neq \varnothing$, and take $y' \in V_J$. Then we have $\langle \alpha_i, \mathrm{Re}(y')\rangle = \mathrm{Re}(\langle \alpha_i, y'\rangle) > 0$ for all $i = 0, \dots, g$, and hence the assumption in Proposition 5.3.4 is satisfied. Therefore, by Proposition 5.3.4, we find

$$\begin{aligned}
\sum_{i=0}^{g} (-1)^i \Psi_d(Q)_{J^{(i)}}|_{V_J}(y) &= \sum_{i=0}^{g} (-1)^i \operatorname{sgn}(J^{(i)}) \sum_{x \in C^Q_{J^{(i)}} \cap \mathbb{Z}^g - \{0\}} \frac{1}{\langle x, y\rangle^{g+d}} \omega(y) \\
&= \sum_{x \in \mathbb{Z}^g - \{0\}} \left(\sum_{i=0}^{g} (-1)^i \operatorname{sgn}(J^{(i)}) \mathbf{1}_{C^Q_{J^{(i)}}}(x) \right) \frac{\omega(y)}{\langle x, y\rangle^{g+d}} \\
&= 0
\end{aligned}$$

for $y \in \pi_{\mathbb{C}}^{-1}(\pi_{\mathbb{C}}(V_J))$. This proves (1).

(2) This follows from (1). ☐

We obtain the following.

Theorem 6.2.5. *For $d \geq 1$, the Shintani–Barnes cocycle Ψ_d defines a class*

$$[\Psi_d] \in H^{g-1}(Y^\circ, \mathrm{SL}_g(\mathbb{Z}), \mathscr{F}_d^{\Xi}).$$

Moreover, for $Q \in \Xi$, the element $\Psi_d(Q) \in C^{g-1}(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}_d)^{\Gamma_Q}$ defines a class

$$[\Psi_d(Q)] \in H^{g-1}(Y^\circ, \Gamma_Q, \mathscr{F}_d),$$

and we have

$$\mathrm{ev}_Q([\Psi_d]) = [\Psi_d(Q)].$$

Proof. This follows from Corollary 4.3.4, Proposition 6.2.2, Corollary 6.2.3, and Proposition 6.2.4. □

7. Integration

The goal of the remaining sections is to construct a specialization map (8-11), and prove that the Shintani–Barnes cocycle class $[\Psi_d]$ specializes to the special value of the zeta functions of number fields; see Theorem 8.3.2.

Let $Q \in \Xi$ be fixed throughout this section. In this section we define an integral map

$$\int_Q : H^q(Y^\circ, \Gamma_Q, \mathscr{F}_0) \to H^q_Q(Y^\circ, \Gamma_Q, \mathbb{C}),$$

where $H^q_Q(Y^\circ, \Gamma_Q, \mathbb{C})$ is a certain auxiliary cohomology group defined later; see Section 7.2. This group $H^q_Q(Y^\circ, \Gamma_Q, \mathbb{C})$ will be studied more closely in Section 8 using a topological method.

7.1. ***Integration and the Hurwitz formula.*** For $q \geq 0$, let

$$\Delta^q := \left\{ (t_1, \dots, t_{q+1}) \in \mathbb{R}^{q+1} \;\middle|\; \sum_{i=1}^{q+1} t_i = 1,\ t_i \geq 0 \right\}$$

denote the standard q-simplex. Note that we can also embed Δ^q into $\mathbb{R}^q$ by

$$\Delta^q \hookrightarrow \mathbb{R}^q, \quad (t_1, \dots, t_{q+1}) \mapsto (t_2, \dots, t_{q+1}),$$

and we equip Δ^q with an orientation induced from the standard orientation of $\mathbb{R}^q$. Moreover, for $\xi_1, \dots, \xi_{q+1} \in \mathbb{C}^g - \{0\}$, let

$$\sigma_{(\xi_1, \dots, \xi_{q+1})} : \Delta^q \to \mathbb{C}^g, \quad (t_1, \dots, t_{q+1}) \mapsto \sum_{i=1}^{q+1} t_i \xi_i$$

denote the affine q-simplex with vertices $\xi_1, \dots, \xi_{q+1}$, and let

$$|\sigma_{(\xi_1, \dots, \xi_{q+1})}| := \sigma_{(\xi_1, \dots, \xi_{q+1})}(\Delta^q) \subset \mathbb{C}^g$$

denote the image of $\sigma_{(\xi_1, \dots, \xi_{q+1})}$.

Now, let $U \subset \mathbb{C}^g - \{0\}$ be a convex open subset and let $\xi_1, \dots, \xi_g \in U$ be a basis of $\mathbb{C}^g$. Then for a homogeneous holomorphic function f on $\pi_{\mathbb{C}}^{-1}(\pi_{\mathbb{C}}(U))$ of degree $-g$, (i.e., $f(\lambda y) = \lambda^{-g} f(y)$ for all $\lambda \in \mathbb{C}^\times$), we consider the integral

$$\int_{\sigma_{(\xi_1,\dots,\xi_g)}} f\omega := \int_{\Delta^{g-1}} (\sigma_{(\xi_1,\dots,\xi_g)})^*(f\omega), \tag{7-1}$$

where

$$\omega(y) = \sum_{i=1}^{g} (-1)^{i-1} y_i \, dy_1 \wedge \cdots \wedge \check{dy_i} \wedge \cdots \wedge dy_g.$$

Here note that $f\omega$ is a holomorphic $(g-1)$-form on $\pi_{\mathbb{C}}^{-1}(\pi_{\mathbb{C}}(U)) \supset U$, and we have $|\sigma_{(\xi_1,\dots,\xi_g)}| \subset U$ since U is convex.

Remark 7.1.1. Via the identification (3-2), the above $f\omega$ corresponds to a holomorphic $(g-1)$-form on $\pi_{\mathbb{C}}(U) \subset \mathbb{P}^{g-1}(\mathbb{C})$. More precisely, there is a holomorphic $(g-1)$-form η on $\pi_{\mathbb{C}}(U) \subset \mathbb{P}^{g-1}(\mathbb{C})$ such that

$$(\pi_{\mathbb{C}})^*\eta = f\omega.$$

Then we see that the integral (7-1) is actually an integral on $\mathbb{P}^{g-1}(\mathbb{C})$:

$$\int_{\sigma_{(\xi_1,\dots,\xi_g)}} f\omega = \int_{\pi_{\mathbb{C}} \circ \sigma_{(\xi_1,\dots,\xi_g)}} \eta.$$

Lemma 7.1.2. *Let $U \subset \mathbb{C}^g - \{0\}$ be a convex open subset, and let $\xi_1, \dots, \xi_g \in \mathbb{C}^g - \{0\}$ be a basis of $\mathbb{C}^g$ such that*

$$\xi_1, \dots, \xi_g \in U.$$

Furthermore, let $\lambda_1, \dots, \lambda_g \in \mathbb{C}^\times$ be any complex numbers such that

$$\lambda_1\xi_1, \dots, \lambda_g\xi_g \in U.$$

Then for a homogeneous holomorphic function f on $\pi_{\mathbb{C}}^{-1}(\pi_{\mathbb{C}}(U))$ of degree $-g$, we have

$$\int_{\sigma_{(\xi_1,\dots,\xi_g)}} f\omega = \int_{\sigma_{(\lambda_1\xi_1,\dots,\lambda_g\xi_g)}} f\omega.$$

Proof. Let

$$h : [0, 1] \times \Delta^{g-1} \to U, \quad (u, t) \mapsto u\sigma_{(\xi_1,\dots,\xi_g)}(t) + (1-u)\sigma_{(\lambda_1\xi_1,\dots,\lambda_g\xi_g)}(t) = \sum_{i=1}^{g} (u + (1-u)\lambda_i) t_i \xi_i \tag{7-2}$$

be a homotopy between $\sigma_{(\xi_1,\dots,\xi_g)}$ and $\sigma_{(\lambda_1\xi_1,\dots,\lambda_g\xi_g)}$. Note that we have $h(u, t) \in U$ because U is convex. We regard h as a singular g-chain in a usual way using the standard decomposition of the prism $[0, 1] \times \Delta^{g-1}$; see [Hatcher 2002, Section 2.1, Proof of 2.10]. Then we have

$$\partial h = \sigma_{(\xi_1,\dots,\xi_g)} - \sigma_{(\lambda_1\xi_1,\dots,\lambda_g\xi_g)} + h',$$

where

$$h' : [0, 1] \times \partial\Delta^{g-1} \to U, \quad (u, t) \mapsto h(u, t),$$

which is also regarded as a singular $(g-1)$-chain. Let $\xi_1^*, \dots, \xi_g^* \in \mathbb{C}^g$ be the dual basis of $\xi_1, \dots, \xi_g$, and let

$$Z := \bigcup_{i=1}^{g} \{y \in \mathbb{C}^g \mid \langle \xi_i^*, y\rangle = 0\}$$

be the union of hyperplanes defined by $\xi_1^*, \dots, \xi_g^*$. Then, by (7-2), we easily see

$$h'([0,1] \times \partial\Delta^{g-1}) \subset Z.$$

Now, by Remark 7.1.1, there exists a holomorphic $(g-1)$-form η on $\pi_{\mathbb{C}}(U)$ such that

$$(\pi_{\mathbb{C}})^*\eta = f\omega.$$

In particular, we have

$$d(f\omega) = (\pi_{\mathbb{C}})^*(d\eta) = 0,$$

where d is the usual derivative of differential forms. Moreover, we also have

$$\int_{h'} f\omega = \int_{\pi_{\mathbb{C}}\circ h'} \eta = 0$$

because $\pi_{\mathbb{C}} \circ h'$ is contained in a divisor $\pi_{\mathbb{C}}(Z-\{0\}) \subset \mathbb{P}^{g-1}(\mathbb{C})$. Therefore, we obtain

$$0 = \int_h d(f\omega) = \int_{\partial h} f\omega = \int_{\sigma_{(\xi_1,\dots,\xi_g)}} f\omega - \int_{\sigma_{(\lambda_1\xi_1,\dots,\lambda_g\xi_g)}} f\omega + \int_{h'} f\omega = \int_{\sigma_{(\xi_1,\dots,\xi_g)}} f\omega - \int_{\sigma_{(\lambda_1\xi_1,\dots,\lambda_g\xi_g)}} f\omega.$$

This completes the proof. □

An important example of such an integral is the following Hurwitz formula (see [Hurwitz 1922; Sczech 1993]), which is also known as the Feynman parametrization.

Proposition 7.1.3 [Hurwitz 1922]. *Let $x \in \mathbb{C}^g - \{0\}$, and let $\xi_1, \dots, \xi_g \in \mathbb{C}^g - \{0\}$ be a basis of $\mathbb{C}^g$ such that*

$$\xi_1, \dots, \xi_g \in V_x = \{y \in \mathbb{C}^g - \{0\} \mid \mathrm{Re}(\langle x, y\rangle) > 0\}.$$

(1) *We have*

$$\int_{\sigma_{(\xi_1,\dots,\xi_g)}} \frac{\omega(y)}{\langle x, y\rangle^g} = \frac{1}{(g-1)!} \frac{\det(\xi_1, \dots, \xi_g)}{\langle x, \xi_1\rangle \cdots \langle x, \xi_g\rangle}. \tag{$*$}$$

(2) *Let $\xi_1^*, \dots, \xi_g^* \in \mathbb{C}^g$ be the dual basis of $\xi_1, \dots, \xi_g$, and let $\underline{k} = (k_1, \dots, k_g) \in (\mathbb{Z}_{\geq 0})^g$. Then*

$$\int_{\sigma_{(\xi_1,\dots,\xi_g)}} \langle \xi_1^*, y\rangle^{k_1} \cdots \langle \xi_g^*, y\rangle^{k_g} \frac{\omega(y)}{\langle x, y\rangle^{g+|\underline{k}|}} = \frac{\underline{k}!}{(g+|\underline{k}|-1)!} \frac{\det(\xi_1, \dots, \xi_g)}{\langle x, \xi_1\rangle^{k_1+1} \cdots \langle x, \xi_g\rangle^{k_g+1}},$$

where $|\underline{k}| := k_1 + \cdots + k_g$ and $\underline{k}! := k_1! \cdots k_g!$.

Proof. (1) Let $W := (\xi_1, \dots, \xi_g) \in \mathrm{GL}_g(\mathbb{C})$ be the matrix whose columns are $\xi_1, \dots, \xi_g$ so that the $(g-1)$-simplex $\sigma_{(\xi_1,\dots,\xi_g)}$ is represented by the linear transformation W, i.e., we have $\sigma_{(\xi_1,\dots,\xi_g)}(t_1, \dots, t_g) = W^t(t_1, \dots, t_g)$ for $(t_1, \dots, t_g) \in \Delta^{g-1} \subset \mathbb{R}^g$. Then

$$\int_{\sigma_{(\xi_1,\dots,\xi_g)}} \frac{\omega(y)}{\langle x, y\rangle^g} = \int_{\Delta^{g-1}} \frac{\omega(Wy)}{\langle x, Wy\rangle^g} = \det W \int_{\Delta^{g-1}} \frac{\omega(y)}{\langle {}^tWx, y\rangle^g}.$$

For $i = 1, \dots, g$, put

$$a_i := \langle x, \xi_i \rangle \neq 0,$$

and let $e_1, \dots, e_g \in \mathbb{C}^g$ be the standard basis, i.e., $e_i = {}^t(0, \dots, 0, \overset{i}{1}, 0, \dots, 0)$. Then we find

$$\begin{aligned}
\det W \int_{\Delta^{g-1}} \frac{\omega(y)}{\langle {}^tWx, y\rangle^g} &= \frac{\det W}{a_1 \cdots a_g} \int_{\Delta^{g-1}} \frac{\omega((a_1 y_1, \dots, a_g y_g))}{(a_1 y_1 + \cdots + a_g y_g)^g} \\
&= \frac{\det W}{a_1 \cdots a_g} \int_{\sigma(a_1 e_1, \dots, a_g e_g)} \frac{\omega(y)}{(y_1 + \cdots + y_g)^g} \\
&= \frac{\det W}{a_1 \cdots a_g} \int_{\sigma(e_1, \dots, e_g)} \frac{\omega(y)}{(y_1 + \cdots + y_g)^g} \\
&= \frac{\det W}{a_1 \cdots a_g} \int_{\sigma(e_1, \dots, e_g)} \omega(y) \\
&= \frac{1}{(g-1)!} \frac{\det W}{a_1 \cdots a_g}.
\end{aligned}$$

Here, the third equality follows from Lemma 7.1.2, and the last equality follows from an elementary computation. This proves (1).

(2) First note that for fixed $\xi_1, \dots, \xi_g$, the formula $(*)$ can be seen as an equality of holomorphic functions in the x-variable. Thus, for $1 \le i \le g$, we consider a linear differential operator

$$D_i := \left\langle \xi_i^*, \frac{\partial}{\partial x} \right\rangle = \xi_{i1}^* \frac{\partial}{\partial x_1} + \cdots + \xi_{ig}^* \frac{\partial}{\partial x_g},$$

where ξ_{ij}^* is the j-th component of ξ_i^*. Then we can compute the action of D_i on the both sides of $(*)$ using the formula

$$D_i \frac{1}{\langle x, y\rangle^n} = -n \langle \xi_i^*, y\rangle \frac{1}{\langle x, y\rangle^{n+1}},$$

where $y \in \mathbb{C}^g$, $\langle x, y\rangle \neq 0$, and $n \ge 1$. Now (2) follows from (1) by applying to $(*)$ the operator

$$D_1^{k_1} \cdots D_g^{k_g}. \qquad \square$$

Remark 7.1.4. The right-hand side of the Hurwitz formula (Proposition 7.1.3) is exactly the building block of Sczech's Eisenstein cocycle [Sczech 1993].

7.2. *The integral map $\int_Q$.* Let $Q \in \Xi$, and let $\theta^{(1)}, \dots, \theta^{(g)} \in \mathbb{C}$ be the distinct eigenvalues of ${}^tQ$. Note that by Lemma 2.1.1(1), ${}^tQ$ has g distinct eigenvalues.

We will introduce an auxiliary cohomology group $H_Q^q(Y^\circ, \Gamma_Q, \mathbb{C})$ and define the integral map $\int_Q$.

Definition 7.2.1. Let $q \ge 0$. We say that $I \in (X_{\mathbb{Q}})^{q+1}$ is *Q-admissible* if we can take a system of eigenvectors $\xi_1, \dots, \xi_g$ of ${}^tQ$ in V_I, i.e., if

$$\text{there exists } \xi_1, \dots, \xi_g \in V_I \text{ such that } {}^tQ\xi_i = \theta^{(i)}\xi_i \text{ for } i = 1, \dots, g.$$

We define $(X_{\mathbb{Q}})_Q^{q+1}$ to be the set of all Q-admissible elements of $(X_{\mathbb{Q}})^{q+1}$.

Recall that

$$\Gamma(V_I, \mathscr{F}_0) = \{ f\omega \mid f \text{ holomorphic function on } \pi_{\mathbb{C}}^{-1}(\pi_{\mathbb{C}}(V_I)) \text{ such that } f(\lambda y) = \lambda^{-g} f(y) \text{ for all } \lambda \in \mathbb{C}^\times \}.$$

Definition 7.2.2. For $q \geq 0$ and a Q-admissible $I \in (X_{\mathbb{Q}})_Q^{q+1}$, we define a map

$$\int_{Q,I} : \Gamma(V_I, \mathscr{F}_0) \to \mathbb{C}, \quad s \mapsto \int_{Q,I} s \tag{7-3}$$

as follows. Take $\xi_1, \dots, \xi_g \in V_I$ such that ${}^t Q \xi_i = \theta^{(i)} \xi_i$ for $i = 1, \dots, g$, and define

$$\int_{Q,I} f\omega := \int_{\sigma(\xi_1, \dots, \xi_g)} f\omega$$

for $f\omega \in \Gamma(V_I, \mathscr{F}_0)$. Note that by Lemma 7.1.2, the map $\int_{Q,I}$ is independent of the choice of the eigenvectors $\xi_1, \dots, \xi_g$.

Remark 7.2.3. Strictly speaking, the map $\int_Q$ is depending on the (fixed) choice of the order of the eigenvalues $\theta^{(1)}, \dots, \theta^{(g)}$ up to sign.

Example 7.2.4. Let the notation be the same as in Section 2.2. Furthermore, let $\theta \in F^\times$ and $Q = \rho_w(\theta) \in \Xi$ be as in Lemma 2.2.1, and let $I \in (X_{\mathbb{Q}})_Q^g$.

(1) For $k \geq 0$ and $x \in C_I^Q - \{0\}$, we have

$$N_{w^*}(y)^k \frac{\omega(y)}{\langle x, y \rangle^{g+kg}} \in \Gamma(V_I, \mathscr{F}_0),$$

and

$$\int_{Q,I} N_{w^*}(y)^k \frac{\omega(y)}{\langle x, y \rangle^{g+kg}} = \frac{(k!)^g}{(g+kg-1)!} \frac{\det(w^{(1)}, \dots, w^{(g)})}{N_w(x)^{k+1}}.$$

(2) For $k \geq 1$, we have

$$N_{w^*}(y)^k \psi_{kg,I}^Q(y) \omega(y) \in \Gamma(V_I, \mathscr{F}_0),$$

and

$$\int_{Q,I} N_{w^*}(y)^k \psi_{kg,I}^Q(y) \omega(y) = \frac{(k!)^g \det(w^{(1)}, \dots, w^{(g)})}{(g+kg-1)!} \operatorname{sgn}(I) \sum_{x \in C_I^Q \cap \mathbb{Z}^g - \{0\}} \frac{1}{N_w(x)^{k+1}}.$$

Proof. (1) First, since $x \in C_I^Q - \{0\}$, we easily see $\operatorname{Re}(\langle x, y \rangle) > 0$ for all $y \in V_I$, i.e., $V_I \subset V_x$. In particular, $\langle x, y \rangle \neq 0$ for all $y \in \pi_{\mathbb{C}}^{-1}(\pi_{\mathbb{C}}(V_I))$, and hence we obtain the first assertion. Now, by Lemma 2.2.1(5), we know $w^{(1)}, \dots, w^{(g)} \in \mathbb{C}^g$ are the eigenvectors of ${}^t Q$ with eigenvalues $\theta^{(1)} := \tau_1(\theta), \dots, \theta^{(g)} := \tau_g(\theta) \in \mathbb{C}$, respectively. Take $\mu_1, \dots, \mu_g \in \mathbb{C}^\times$ so that $\xi_1 := \mu_1 w^{(1)}, \dots, \xi_g := \mu_g w^{(g)} \in V_I$. This is possible since I is Q-admissible. Then, by Lemma 2.2.1(3), we see that $\xi_1^* := \mu_1^{-1} w^{*(1)}, \dots, \xi_g^* := \mu_g^{-1} w^{*(g)}$ form the dual basis of $\xi_1, \dots, \xi_g$. Thus, by Proposition 7.1.3, we find

$$\begin{aligned}\int_{Q,I} N_{w^*}(y)^k \frac{\omega(y)}{\langle x, y\rangle^{g+kg}} &= \int_{\sigma(\xi_1,\dots,\xi_g)} \prod_{i=1}^{g} \langle \mu_i \xi_i^*, y\rangle^k \frac{\omega(y)}{\langle x, y\rangle^{g+kg}} \\ &= (\mu_1 \cdots \mu_g)^k \frac{(k!)^g}{(g+kg-1)!} \frac{\det(\xi_1, \dots, \xi_g)}{\prod_{i=1}^{g} \langle x, \xi_i\rangle^{k+1}} \\ &= (\mu_1 \cdots \mu_g)^k \frac{(k!)^g}{(g+kg-1)!} \frac{\det(\mu_1 w^{(1)}, \dots, \mu_g w^{(g)})}{\prod_{i=1}^{g} \langle x, \mu_i w^{(i)}\rangle^{k+1}} \\ &= \frac{(k!)^g}{(g+kg-1)!} \frac{\det(w^{(1)}, \dots, w^{(g)})}{N_w(x)^{k+1}}.\end{aligned}$$

(2) The first assertion follows from Proposition 6.1.2. The integral formula follows from (1) by taking the sum over $x \in C_I^Q \cap \mathbb{Z}^g - \{0\}$. □

Next, we extend the map (7-3) to the cohomology group.

Lemma 7.2.5. *Let $I = (\alpha_0, \dots, \alpha_q) \in (X_{\mathbb{Q}})^{q+1}$.*

(1) *If $q \geq 1$ and I is Q-admissible, then so is $I^{(i)} = (\alpha_0, \dots, \check{\alpha}_i, \dots, \alpha_q)$ for $i = 0, \dots, q$.*

(2) *Let $\gamma \in \Gamma_Q$. If I is Q-admissible, then so is γI, i.e., $(X_{\mathbb{Q}})_Q^{q+1}$ is a Γ_Q-stable subset of $(X_{\mathbb{Q}})^{q+1}$.*

Proof. (1) This follows from the fact $V_I = V_{I^{(i)}} \cap V_{\alpha_i} \subset V_{I^{(i)}}$.

(2) Take $\xi_1, \dots, \xi_g \in V_I$ such that ${}^tQ\xi_i = \theta^{(i)}\xi_i$ for $i = 1, \dots, g$. Then since ${}^tQ\,{}^t\gamma = {}^t\gamma\,{}^tQ$, we see that ${}^t\gamma^{-1}\xi_1, \dots, {}^t\gamma^{-1}\xi_g$ are again eigenvectors of ${}^tQ$ with eigenvalues $\theta^{(1)}, \dots, \theta^{(g)}$ respectively. On the other hand, by Lemma 4.1.1, we have

$${}^t\gamma^{-1}\xi_i \in {}^t\gamma^{-1} V_I = V_{\gamma I}$$

for $i = 1, \dots, g$. Thus we find that ${}^t\gamma^{-1}\xi_1, \dots, {}^t\gamma^{-1}\xi_g$ are a system of eigenvectors of ${}^tQ$ in $V_{\gamma I}$. □

For a Γ_Q-equivariant sheaf $\mathscr{F}$ on Y°, set

$$C_Q^q(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}) := \prod_{I \in (X_{\mathbb{Q}})_Q^{q+1}} \Gamma(V_I, \mathscr{F}) \quad \text{and} \quad {}_QC^q(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}) := \prod_{\substack{I \in (X_{\mathbb{Q}})^{q+1}, \\ I \notin (X_{\mathbb{Q}})_Q^{q+1}}} \Gamma(V_I, \mathscr{F}).$$

Then we have a natural short exact sequence

$$0 \to {}_QC^q(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}) \to C^q(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}) \xrightarrow{p_Q} C_Q^q(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}) \to 0, \tag{7-4}$$

where p_Q is the natural projection. By Lemma 7.2.5, we easily see that ${}_QC^\bullet(\mathcal{X}_{\mathbb{Q}}, \mathscr{F})$ becomes a Γ_Q-equivariant subcomplex of $C^\bullet(\mathcal{X}_{\mathbb{Q}}, \mathscr{F})$, and hence $C_Q^\bullet(\mathcal{X}_{\mathbb{Q}}, \mathscr{F})$ has a natural structure of Γ_Q-equivariant complex induced from that of $C^\bullet(\mathcal{X}_{\mathbb{Q}}, \mathscr{F})$. For a subgroup $\Gamma \subset \Gamma_Q$, we define

$$H_Q^q(Y^\circ, \Gamma, \mathscr{F}) := H^q(C_Q^\bullet(\mathcal{X}_{\mathbb{Q}}, \mathscr{F})^\Gamma)$$

to be the q-th cohomology group of the complex $C_Q^\bullet(\mathcal{X}_{\mathbb{Q}}, \mathscr{F})^\Gamma$.

Now, by taking the product of (7-3) over $I \in (X_{\mathbb{Q}})_Q^{q+1}$, we define

$$\int_Q : C_Q^q(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}_0) \to C_Q^q(\mathcal{X}_{\mathbb{Q}}, \mathbb{C}), \quad (s_I)_{I \in (X_{\mathbb{Q}})_Q^{q+1}} \mapsto \left(\int_{Q,I} s_I \right)_{I \in (X_{\mathbb{Q}})_Q^{q+1}}.$$

Here $\mathbb{C}$ is regarded as a constant sheaf associated to $\mathbb{C}$ with the trivial Γ_Q-equivariant structure.

Proposition 7.2.6. *The map*

$$\int_Q : C_Q^\bullet(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}_0) \to C_Q^\bullet(\mathcal{X}_{\mathbb{Q}}, \mathbb{C})$$

is a morphism of Γ_Q*-equivariant complexes, and hence induces a map*

$$\int_Q : H_Q^q(Y^\circ, \Gamma_Q, \mathscr{F}_0) \to H_Q^q(Y^\circ, \Gamma_Q, \mathbb{C})$$

for $q \geq 0$*.*

Proof. First we must show $\int_Q \circ \mathrm{d}^q = \mathrm{d}^q \circ \int_Q$ for $q \geq 0$. Let $J = (\alpha_0, \dots, \alpha_{q+1}) \in (X_{\mathbb{Q}})_Q^{q+2}$, and let $\xi_1, \dots, \xi_g \in V_J$ be a system of eigenvectors of ${}^tQ$ with eigenvalues $\theta^{(1)}, \dots, \theta^{(g)}$ respectively. Then for $s = (s_I)_{I \in (X_{\mathbb{Q}})_Q^{q+1}} \in C_Q^q(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}_0)$, we have

$$\begin{aligned}\left(\int_Q d^q(s) \right)_J = \int_{Q,J} (d^q(s))_J &= \int_{\sigma(\xi_1,\dots,\xi_g)} \sum_{i=0}^{q+1} (-1)^i s_{J^{(i)}}|_{V_J} \\ &= \sum_{i=0}^{q+1} (-1)^i \int_{\sigma(\xi_1,\dots,\xi_g)} s_{J^{(i)}} = \sum_{i=0}^{q+1} (-1)^i \left(\int_Q s \right)_{J^{(i)}} = \left(d^q \left(\int_Q s \right) \right)_J,\end{aligned}$$

where $J^{(i)} = (\alpha_0, \dots, \check{\alpha}_i, \dots, \alpha_{q+1})$.

Next we must show $\int_Q \circ [\gamma] = [\gamma] \circ \int_Q$ for $\gamma \in \Gamma_Q$. Let $J = (\alpha_1, \dots, \alpha_{q+1}) \in (X_{\mathbb{Q}})_Q^{q+1}$, and let again $\xi_1, \dots, \xi_g \in V_J$ be a system of eigenvectors of ${}^tQ$ with eigenvalues $\theta^{(1)}, \dots, \theta^{(g)}$ respectively. Then as in the proof of Lemma 7.2.5, we see that ${}^t\gamma\xi_1, \dots, {}^t\gamma\xi_g$ are eigenvectors of ${}^tQ$ in $V_{\gamma^{-1}J}$ with eigenvalues $\theta^{(1)}, \dots, \theta^{(g)}$ respectively. Therefore, for $s = (s_I)_{I \in (X_{\mathbb{Q}})_Q^{q+1}} \in C_Q^q(\mathcal{X}_{\mathbb{Q}}, \mathscr{F}_0)$, we have

$$\begin{aligned}\left(\int_Q [\gamma](s) \right)_J = \int_{Q,J} ([\gamma](s))_J &= \int_{\sigma(\xi_1,\dots,\xi_g)} s_{\gamma^{-1}J}({}^t\gamma y) \\ &= \int_{\sigma({}^t\gamma\xi_1,\dots,{}^t\gamma\xi_g)} s_{\gamma^{-1}J}(y) \\ &= \int_{Q,\gamma^{-1}J} s_{\gamma^{-1}J} = \left(\int_Q s \right)_{\gamma^{-1}J} = \left([\gamma] \left(\int_Q s \right) \right)_J.\end{aligned}$$

This completes the proof. □

Let $\int_Q$ also denote the composition

$$\int_Q : H^q(Y^\circ, \Gamma_Q, \mathscr{F}_0) \xrightarrow{p_Q} H_Q^q(Y^\circ, \Gamma_Q, \mathscr{F}_0) \xrightarrow{\int_Q} H_Q^q(Y^\circ, \Gamma_Q, \mathbb{C}), \tag{7-5}$$

where p_Q is the natural map induced from the projection p_Q in (7-4). See also Corollary 4.3.4.

8. Specialization to the zeta values

In this section we compute the group $H^q_Q(Y^\circ, \Gamma_Q, \mathbb{C})$ explicitly, and show that we can get the values of the zeta function as a specialization of the Shintani–Barnes cocycle $[\Psi_d]$.

First we return to the setting in Section 2.2. Let

- $F/\mathbb{Q}$ be a number field of degree g,
- $\tau_1, \dots, \tau_g : F \hookrightarrow \mathbb{C}$ be the field embeddings of F into $\mathbb{C}$,
- $\mathcal{O} \subset F$ be an order,
- $\mathfrak{a} \subset F$ be a proper fractional $\mathcal{O}$-ideal,
- $w_1, \dots, w_g \in \mathfrak{a}$ be a basis of $\mathfrak{a}$ over $\mathbb{Z}$,
- $w := {}^t(w_1, \dots, w_g) \in F^g$, and $w^{(i)} := \tau_i(w) = {}^t(\tau_i(w_1), \dots, \tau_i(w_g)) \in \mathbb{C}^g$,
- $\rho_w : F \to M_g(\mathbb{Q})$ be the regular representation with respect to

$$w : \mathbb{Q}^g \xrightarrow{\sim} F, \quad x \mapsto \langle x, w\rangle,$$

- $N_w(x_1, \dots, x_g) \in \mathbb{Q}[x_1, \dots, x_g]$ be the norm polynomial with respect to w,
- $w_1^*, \dots, w_g^* \in F$ be the dual basis of $w_1, \dots, w_g$ with respect to the trace $\mathrm{Tr}_{F/\mathbb{Q}}$,
- $w^*, w^{*(i)}, N_{w^*}, \rho_{w^*}$ be the dual objects obtained from $w_1^*, \dots, w_g^*$.

Take $\theta \in F^\times$ such that $F = \mathbb{Q}(\theta)$ and put $Q := \rho_w(\theta) \in \Xi$. Also, set $\theta^{(1)} := \tau_1(\theta), \dots, \theta^{(g)} := \tau_g(\theta) \in \mathbb{C}^\times$ to be the eigenvalues of ${}^tQ$. We fix this notation.

8.1. ***Computation of $H^q_Q(Y^\circ, \Gamma_Q, \mathbb{C})$.*** Define

$$T_w := \{x \in \mathbb{R}^g \mid N_w(x) \neq 0\} \subset \mathbb{R}^g - \{0\}$$

to be the set of real vectors whose norm is nonzero. By Lemma 2.2.1(7), it is clear that T_w is a Γ_Q-stable subset of $\mathbb{R}^g - \{0\}$. Note that under the isomorphism

$$w : \mathbb{R}^g \xrightarrow{\sim} F_\mathbb{R} := F \otimes_\mathbb{Q} \mathbb{R}, \quad x \mapsto \langle x, w\rangle,$$

T_w corresponds to $F_\mathbb{R}^\times = \{\alpha \in F_\mathbb{R} \mid N_{F/\mathbb{Q}}(\alpha) \neq 0\}$, i.e.,

$$w : T_w \xrightarrow{\sim} F_\mathbb{R}^\times. \tag{8-1}$$

The aim of this subsection is to obtain an isomorphism

$$H^q_Q(Y^\circ, \Gamma_Q, \mathbb{C}) \xleftarrow{\sim} H^q(T_w/\Gamma_Q, \mathbb{C}) \simeq H^q(F_\mathbb{R}^\times/\mathcal{O}^1, \mathbb{C}), \tag{8-2}$$

where the last two cohomology groups are the usual singular cohomology groups.

As in Section 7, for $I=(\alpha_1,\dots,\alpha_{q+1})\in(X_{\mathbb{Q}})^{q+1}$, let

$$\sigma_I:\Delta^q\to\mathbb{R}^g,\quad t=(t_1,\dots,t_{q+1})\mapsto\sum_{i=1}^{q+1}\alpha_i t_i$$

denote the affine q-simplex with vertices $\alpha_1,\dots,\alpha_{q+1}$, and let $|\sigma_I|:=\sigma_I(\Delta^q)\subset\mathbb{R}^g$ denote the image of σ_I. The following lemma enables us to compute the group $H^q_Q(Y^\circ,\Gamma_Q,\mathbb{C})$ using these simplices.

Lemma 8.1.1. *Let $q\ge 0$ and $I=(\alpha_1,\dots,\alpha_{q+1})\in(X_{\mathbb{Q}})^{q+1}$. The following conditions are equivalent*:

(i) *I is Q-admissible.*

(ii) $|\sigma_I|\subset T_w$.

To prove this lemma, recall the following fact:

Lemma 8.1.2. *Let $A\subset\mathbb{C}$ be a convex compact subset. The following conditions are equivalent*:

(i) $0\notin A$.

(ii) *There exists $\lambda\in\mathbb{C}^\times$ such that* $\mathrm{Re}(\lambda A)\subset\mathbb{R}_{>0}$.

Proof. This follows from [Rudin 1991, Theorem 3.4(b)]. ☐

Proof of Lemma 8.1.1. First, by Lemma 2.2.1(5), we know that $w^{(1)},\dots,w^{(g)}$ are the eigenvectors of ${}^tQ$ with eigenvalues $\theta^{(1)},\dots,\theta^{(g)}$ respectively. Therefore,

I is Q-admissible

$\iff$ for all $j\in\{1,\dots,g\}$ there exists $\lambda_j\in\mathbb{C}^\times$ such that $\lambda_j w^{(j)}\in V_I$

$\iff$ for all $j\in\{1,\dots,g\}$ there exists $\lambda_j\in\mathbb{C}^\times$ such that for all $i\in\{1,\dots,q+1\}$, $\mathrm{Re}(\langle\alpha_i,\lambda_j w^{(j)}\rangle)>0$

$\iff$ for all $j\in\{1,\dots,g\}$ there exists $\lambda_j\in\mathbb{C}^\times$ such that $\mathrm{Re}(\lambda_j\langle|\sigma_I|,w^{(j)}\rangle)\subset\mathbb{R}_{>0}$

$\overset{*}{\iff}$ $0\notin\langle|\sigma_I|,w^{(j)}\rangle$ for all $j\in\{1,\dots,g\}$

$\iff$ $N_w(x)\ne 0$ for all $x\in|\sigma_I|$

$\iff$ $|\sigma_I|\subset T_w$.

Note that the fourth equivalence $\overset{*}{\iff}$ follows from Lemma 8.1.2 since $\langle|\sigma_I|,w^{(i)}\rangle\subset\mathbb{C}$ is a convex compact subset. This proves the lemma. ☐

For $q\ge 0$, let $\Sigma_q:=\{\sigma:\Delta^q\to T_w \text{ continuous}\}$ denote the set of singular q-simplices in T_w, and let

$$S_q:=\mathbb{Z}[\Sigma_q]$$

denote the group of singular q-chains of T_w. For $j=1,\dots,q+1$, let

$$\delta^q_j:\Delta^{q-1}\to\Delta^q,\quad (t_1,\dots,t_q)\mapsto(t_1,\dots,t_{j-1},0,t_j,\dots,t_q)$$

denote the j-th face map. Then we have a boundary map $\partial : S_q \to S_{q-1}$ which maps $\sigma \in \Sigma_q$ to

$$\partial\sigma = \sum_{j=1}^{q+1} (-1)^{j-1} \sigma \circ \delta_j^q \in S_{q-1}.$$

The action of Γ_Q on T_w naturally induces an action of Γ_Q on S_q, and we have a Γ_Q-equivariant singular chain complex $S_\bullet$. Moreover, let

$$K_q := \mathbb{Z}[(X_{\mathbb{Q}})_Q^{q+1}]$$

denote the free abelian group generated by $(X_{\mathbb{Q}})_Q^{q+1}$. By Lemma 7.2.5(2), we have a natural action of Γ_Q on K_q. Then, by Lemma 8.1.1, we have a natural injective homomorphism

$$K_q \hookrightarrow S_q, \quad I \mapsto \sigma_I,$$

which is clearly a Γ_Q-equivariant map. In the following, we identify K_q with a Γ_Q-submodule

$$\mathbb{Z}\big[\sigma_I \mid I \in (X_{\mathbb{Q}})_Q^{q+1}\big] = \mathbb{Z}\big[\sigma_I \mid I \in (X_{\mathbb{Q}})^{q+1}, |\sigma_I| \subset T_w\big] \subset S_q$$

of S_q via this injective map. Then, by Lemma 7.2.5(1), we see that the boundary map ∂ maps K_q to K_{q-1}, and hence $K_\bullet \subset S_\bullet$ becomes a Γ_Q-equivariant subcomplex of $S_\bullet$.

Note that we have a natural isomorphism

$$K_{\mathbb{C}}^\bullet := \mathrm{Hom}_{\mathbb{Z}}(K_\bullet, \mathbb{C}) \simeq \prod_{I \in (X_{\mathbb{Q}})_Q^{\bullet+1}} \mathbb{C} = C_Q^\bullet(\mathcal{X}_{\mathbb{Q}}, \mathbb{C})$$

of Γ_Q-equivariant complexes, and hence

$$H_Q^q(Y^\circ, \Gamma_Q, \mathbb{C}) \simeq H^q((K_{\mathbb{C}}^\bullet)^{\Gamma_Q}).$$

Therefore, in order to obtain (8-2), we compare $K_\bullet$ and $S_\bullet$.

Proposition 8.1.3. (1) *Let $\Gamma \subset \Gamma_Q$ be a subgroup. For $q \geq 0$, the quotient group S_q/K_q is an induced Γ-module.*

(2) *The inclusion map*

$$K_\bullet \hookrightarrow S_\bullet$$

is a quasi-isomorphism. In other words, the quotient complex $S_\bullet/K_\bullet$ is exact.

Proof. (1) This is clear since we have

$$S_q/K_q \simeq \mathbb{Z}[\sigma \in \Sigma_q \mid \sigma \notin K_q]$$

and $\Gamma \subset \Gamma_Q$ acts freely on the basis $\{\sigma \in \Sigma_q \mid \sigma \notin K_q\}$.

(2) This kind of fact may be well known to experts, but here we give a proof for the sake of completeness of the paper. First take any finite open covering

$$T_w = \bigcup_{k=1}^N U_k$$

of T_w such that U_k is a convex open subset of T_w for all k. The existence of such a covering can be easily seen from the identification $w : T_w \xrightarrow{\sim} F_{\mathbb{R}}^{\times}$.

We will prove that the quotient complex $S_\bullet/K_\bullet$ is exact. Let $q \geq 0$ and let $a \in S_q$ such that $\partial a \in K_{q-1}$. We need to show the following:

Aim. *There exist $\eta \in S_{q+1}$ and $b \in K_q$ such that $a = \partial\eta + b$.*

Suppose $a \in S_q$ is of the form

$$a = \sum_{i=1}^{r} c_i \sigma_i,$$

where σ_i are distinct singular q-simplices in T_w, and $c_i \in \mathbb{Z}$. By using the barycentric subdivision if necessary, without loss of generality we may assume

$$\text{for each } i \in \{1, \dots, r\} \text{ there exists } \kappa_i \in \{1, \dots, N\} \text{ such that } \sigma_i(\Delta^q) \subset U_{\kappa_i}. \tag{8-3}$$

Indeed, let

$$S : S_n \to S_n \quad \text{and} \quad T : S_n \to S_{n+1}$$

be the subdivision operator and the chain homotopy between S and id_{S_n} defined as in [Hatcher 2002, Section 2.1, Proof of Proposition 2.21]. Then taking into account the fact that the barycenter of any $\sigma_I \in K_n$ ($I \in (X_{\mathbb{Q}})_Q^{n+1}$) belongs to $\mathbb{Q}^g \cap |\sigma_I|$, we easily see that S (resp. T) maps K_n to K_n (resp. K_{n+1}). Hence we have

$$\partial S(a) = S(\partial a) \in K_{q-1} \quad \text{and} \quad a - S(a) = \partial T(a) + T(\partial a) \in \partial S_{q+1} + K_q.$$

Therefore, we can replace a with its (iterated) barycentric subdivision $S^m(a)$ (m sufficiently large) until we have (8-3).

We fix such κ_i for each $i = 1, \dots, r$.

Step 1: In order to "approximate" σ_i by the elements in K_q, we first approximate their vertices "simultaneously". For $i = 1, \dots, r$ and $j = 1, \dots, q+1$, let $v_{ij} \in U_{\kappa_i} \subset T_w$ denote the j-th vertex of σ_i, i.e.,

$$v_{ij} = \sigma_i(0, \dots, 0, \overset{j}{1}, 0, \dots, 0) \in T_w.$$

Then for $i = 1, \dots, r$ and $j = 1, \dots, q+1$, take $v'_{ij} \in U_{\kappa_i} \cap \mathbb{Q}^g$ satisfying the following conditions:

(V1) If $v_{ij} \in \mathbb{Q}^g$, then $v'_{ij} = v_{ij}$.

(V2) If $v_{ij} = v_{mn}$ for some $i, m \in \{1, \dots, r\}$ and $j, n \in \{1, \dots, q+1\}$, then $v'_{ij} = v'_{mn}$. (In other words, if the j-th vertex of σ_i and the n-th vertex of σ_m are the same, then v'_{ij} and v'_{mn} are the same as well.)

This is possible because $\mathbb{Q}^g$ is dense in $\mathbb{R}^g$. Then set

$$I_i := (v'_{i1}, \dots, v'_{i,q+1}) \in (X_{\mathbb{Q}})^{q+1} \quad \text{for } i = 1, \dots, r \qquad \text{and} \qquad a' := \sum_{i=1}^{r} c_i \sigma_{I_i}.$$

Since U_{κ_i} is convex, we have $\sigma_{I_i} \subset U_{\kappa_i} \subset T_w$, and hence $\sigma_{I_i} \in K_q$. Therefore, we see that $a' \in K_q$.

Now, recall that for $j = 1, \dots, q+1$,

$$\delta_j^q : \Delta^{q-1} \to \Delta^q, \quad (t_1, \dots, t_q) \mapsto (t_1, \dots, t_{j-1}, 0, t_j, \dots, t_q)$$

denotes the j-th face map. Then, by the conditions (V1) and (V2), we have the following:

(F1) If $\sigma_i \circ \delta_j^q \in K_{q-1}$, then $\sigma_{I_i} \circ \delta_j^q = \sigma_i \circ \delta_j^q$.

(F2) If $\sigma_i \circ \delta_j^q = \sigma_m \circ \delta_n^q$ for $i, m \in \{1, \dots, r\}$ and $j, n \in \{1, \dots, q+1\}$, then $\sigma_{I_i} \circ \delta_j^q = \sigma_{I_m} \circ \delta_n^q$. (In other words, if the j-th face of σ_i and the n-th face of σ_m are the same, then the j-th face of σ_{I_i} and the n-th face of σ_{I_m} are the same as well.)

Step 2: Next we consider the homotopy between a and a'. For $i = 1, \dots, r$, let

$$h_i : [0, 1] \times \Delta^q \to T_w, \quad (u, t) \mapsto u\sigma_i(t) + (1-u)\sigma_{I_i}(t)$$

be a homotopy between σ_i and σ_{I_i}. Note that since U_{κ_i} is convex, we have

$$h_i([0, 1] \times \Delta^q) \subset U_{\kappa_i}.$$

The homotopy h_i defines a $(q+1)$-chain $\eta_i \in S_{q+1}$in a usual way using the standard decomposition of the prism $[0, 1] \times \Delta^q$. More precisely, for $j = 1, \dots, q+1$, put

$$\epsilon_j^q : \Delta^{q+1} \to [0, 1] \times \Delta^q, \quad (t_1, \dots, t_{q+2}) \mapsto \Bigg(\sum_{m \geq j+1} t_m, (t_1, \dots, t_{j-1}, t_j + t_{j+1}, t_{j+2}, \dots, t_{q+2})\Bigg).$$

Using these maps, the $(q+1)$-chain $\eta_i \in S_{q+1}$ is defined as

$$\eta_i := \sum_{j=1}^{q+1} (-1)^{j-1} h_i \circ \epsilon_j^q.$$

Set $\eta := \sum_{i=1}^r c_i \eta_i \in S_{q+1}$.

Step 3: Now we examine the assumption $\partial a \in K_{q-1}$. First, we have

$$\partial a = \sum_{i=1}^r \sum_{j=1}^{q+1} (-1)^{j-1} c_i \sigma_i \circ \delta_j^q.$$

For each singular $(q-1)$-simplex $\sigma \in \Sigma_{q-1}$, set

$$C_\sigma := \sum_{\substack{i=1,\dots,r, \\ j=1,\dots,q+1, \\ \sigma_i \circ \delta_j^q = \sigma}} (-1)^{j-1} c_i \in \mathbb{Z}.$$

In the case where the index set of the sum is empty, we set $C_\sigma = 0$ by convention. Then we can rewrite ∂a as

$$\partial a = \sum_{\sigma \in \Sigma_{q-1}} C_\sigma \sigma.$$

Then, by the assumption $\partial a \in K_{q-1}$, we find that $C_\sigma = 0$ for all $\sigma \notin K_{q-1}$ since the set Σ_{q-1} of singular $(q-1)$-simplices is a basis of S_{q-1}.

Step 4: Next we compute the boundary of the homotopy $\eta \in S_{q+1}$. By an elementary computation we see

$$\partial \eta_i = \sigma_i - \sigma_{I_i} - \sum_{j=1}^{q+1} \sum_{m=1}^{q} (-1)^{j+m} h_{ij} \circ \epsilon_m^{q-1},$$

where

$$h_{ij} : [0, 1] \times \Delta^{q-1} \to T_w, \quad (u, t) \mapsto u\sigma_i \circ \delta_j^q(t) + (1-u)\sigma_{I_i} \circ \delta_j^q(t)$$

is a homotopy between $\sigma_i \circ \delta_j^q$ and $\sigma_{I_i} \circ \delta_j^q$; see [Hatcher 2002, Section 2.1, Proof of 2.10].

Now, by the properties (F1) and (F2), we see the following:

(H1) If $\sigma_i \circ \delta_j^q \in K_{q-1}$, then $h_{ij}(u, t) = \sigma_i \circ \delta_j^q(t)$ for $(u, t) \in [0, 1] \times \Delta^{q-1}$.

(H2) If $\sigma_i \circ \delta_j^q = \sigma_m \circ \delta_n^q$ for $i, m \in \{1, \ldots, r\}$ and $j, n \in \{1, \ldots, q+1\}$, then $h_{ij} = h_{mn}$.

Then for each singular $(q-1)$-simplex $\sigma \in \Sigma_{q-1}$, we define a map

$$h_\sigma : [0, 1] \times \Delta^{q-1} \to T_w$$

as follows: If σ is of the form $\sigma = \sigma_i \circ \delta_j^q$ for some $i \in \{1, \ldots, r\}$ and $j \in \{1, \ldots, q+1\}$, we set $h_\sigma := h_{ij}$. This is well defined by the property (H2). If σ is not of the form $\sigma_i \circ \delta_j^q$, then simply set $h_\sigma(u, t) := \sigma(t)$ for $(u, t) \in [0, 1] \times \Delta^{q-1}$.

Then we find

$$\begin{aligned}
\partial \eta &= a - a' - \sum_{i=1}^{r} \sum_{j=1}^{q+1} \sum_{m=1}^{q} (-1)^{j+m} c_i h_{ij} \circ \epsilon_m^{q-1} \\
&= a - a' - \sum_{i=1}^{r} \sum_{j=1}^{q+1} \sum_{m=1}^{q} (-1)^{j+m} c_i h_{\sigma_i \circ \delta_j^q} \circ \epsilon_m^{q-1} \\
&= a - a' - \sum_{\sigma \in \Sigma_{q-1}} \sum_{m=1}^{q} (-1)^{m-1} h_\sigma \circ \epsilon_m^{q-1} \sum_{\substack{i=1,\ldots,r, \\ j=1,\ldots,q+1, \\ \sigma_i \circ \delta_j^q = \sigma}} (-1)^{j-1} c_i \\
&= a - a' - \sum_{\sigma \in \Sigma_{q-1}} C_\sigma \sum_{m=1}^{q} (-1)^{m-1} h_\sigma \circ \epsilon_m^{q-1} \\
&= a - a' - \sum_{\sigma \in \Sigma_{q-1} \cap K_{q-1}} C_\sigma \sum_{m=1}^{q} (-1)^{m-1} h_\sigma \circ \epsilon_m^{q-1}.
\end{aligned}$$

Note that the last equality holds since we have $C_\sigma = 0$ for $\sigma \notin K_{q-1}$. Moreover, by the property (H1), we easily see that if $\sigma = \sigma_i \circ \delta_j^q \in K_{q-1}$, then $h_\sigma \circ \epsilon_m^{q-1} \in K_q$ for all $m = 1, \ldots, q$. Therefore, by setting

$$b := a' + \sum_{\sigma \in \Sigma_{q-1} \cap K_{q-1}} C_\sigma \sum_{m=1}^{q} (-1)^{m-1} h_\sigma \circ \epsilon_m^{q-1} \in K_q,$$

we obtain the desired identity $a = \partial \eta + b$. □

Let $S_{\mathbb{C}}^{\bullet} := \mathrm{Hom}_{\mathbb{Z}}(S_{\bullet}, \mathbb{C})$ denote the singular cochain complex of T_w with coefficients in $\mathbb{C}$.

Corollary 8.1.4. *Let $\Gamma \subset \Gamma_Q$ be a subgroup.*

(1) *The map $K_{\bullet} \hookrightarrow S_{\bullet}$ induces a quasi-isomorphism*

$$(K_{\bullet})_{\Gamma} \to (S_{\bullet})_{\Gamma},$$

where $(-)_{\Gamma}$ denotes the Γ-coinvariant part. In particular, we obtain an isomorphism

$$H_q((K_{\bullet})_{\Gamma}) \xrightarrow{\sim} H_q(T_w/\Gamma, \mathbb{Z}).$$

(2) *The map $K_{\bullet} \hookrightarrow S_{\bullet}$ induces a quasi-isomorphism*

$$(S_{\mathbb{C}}^{\bullet})^{\Gamma} \to (K_{\mathbb{C}}^{\bullet})^{\Gamma}.$$

In particular, we obtain an isomorphism

$$H^q(T_w/\Gamma, \mathbb{C}) \xrightarrow{\sim} H^q((K_{\mathbb{C}}^{\bullet})^{\Gamma}) \simeq H_Q^q(Y^{\circ}, \Gamma, \mathbb{C}).$$

Proof. First note that since the action of Γ_Q on T_w is free and properly discontinuous, the singular homology $H_q(T_w/\Gamma, \mathbb{Z})$ (resp. singular cohomology $H^q(T_w/\Gamma, \mathbb{C})$) can be computed by the equivariant singular homology (resp. equivariant singular cohomology), i.e., we have

$$H_q(T_w/\Gamma, \mathbb{Z}) \simeq H_q((S_{\bullet})_{\Gamma}) \quad \text{and} \quad H^q(T_w/\Gamma, \mathbb{C}) \simeq H^q((S_{\mathbb{C}}^{\bullet})^{\Gamma}).$$

See [Cartan and Eilenberg 1956, Chapter XVI, Section 9].

(1) We consider the tautological exact sequence

$$0 \to K_q \to S_q \to S_q/K_q \to 0. \tag{8-4}$$

By Proposition 8.1.3(1), we obtain a short exact sequence

$$0 = H_1(\Gamma, S_q/K_q) \to (K_q)_{\Gamma} \to (S_q)_{\Gamma} \to (S_q/K_q)_{\Gamma} \to 0,$$

where $H_1(\Gamma, -)$ is the first group homology of Γ. This induces a long exact sequence

$$\cdots \to H_{q+1}((S_{\bullet}/K_{\bullet})_{\Gamma}) \to H_q((K_{\bullet})_{\Gamma}) \to H_q((S_{\bullet})_{\Gamma}) \to H_q((S_{\bullet}/K_{\bullet})_{\Gamma}) \to \cdots.$$

Therefore, it remains to show

$$H_q((S_{\bullet}/K_{\bullet})_{\Gamma}) = 0$$

for $q \geq 0$. Indeed, by Proposition 8.1.3, we see that

$$\cdots \to S_2/K_2 \to S_1/K_1 \to S_0/K_0 \to 0 \tag{8-5}$$

is an exact sequence of induced Γ-modules. Therefore, (8-5) can be seen as a $(-)_{\Gamma}$-acyclic resolution of 0. Thus we see

$$H_q((S_{\bullet}/K_{\bullet})_{\Gamma}) = H_q(\Gamma, 0) = 0$$

for all $q \geq 0$.

(2) This can be proved similarly. By applying $\mathrm{Hom}_{\mathbb{Z}}(-,\mathbb{C})$ to (8-4), we obtain a short exact sequence

$$0 \to (S_q/K_q)^\vee_{\mathbb{C}} \to S^q_{\mathbb{C}} \to K^q_{\mathbb{C}} \to 0,$$

where $(S_q/K_q)^\vee_{\mathbb{C}} := \mathrm{Hom}_{\mathbb{Z}}(S_q/K_q,\mathbb{C})$. Then, by Proposition 8.1.3(1), we see that $(S_q/K_q)^\vee_{\mathbb{C}}$ is a coinduced Γ-module, and hence we obtain another short exact sequence

$$0 \to ((S_q/K_q)^\vee_{\mathbb{C}})^\Gamma \to (S^q_{\mathbb{C}})^\Gamma \to (K^q_{\mathbb{C}})^\Gamma \to H^1(\Gamma,(S_q/K_q)^\vee_{\mathbb{C}}) = 0.$$

Furthermore, this exact sequence induces a long exact sequence

$$\cdots \to H^q\big(((S_\bullet/K_\bullet)^\vee_{\mathbb{C}})^\Gamma\big) \to H^q((S^\bullet_{\mathbb{C}})^\Gamma) \to H^q((K^\bullet_{\mathbb{C}})^\Gamma) \to H^{q+1}\big(((S_\bullet/K_\bullet)^\vee_{\mathbb{C}})^\Gamma\big) \to \cdots.$$

Therefore, it remains to show that

$$H^q\big(((S_\bullet/K_\bullet)^\vee_{\mathbb{C}})^\Gamma\big) = 0$$

for $q \geq 0$. Indeed, by applying $\mathrm{Hom}_{\mathbb{Z}}(-,\mathbb{C})$ to (8-5), we see that

$$0 \to (S_0/K_0)^\vee_{\mathbb{C}} \to (S_1/K_1)^\vee_{\mathbb{C}} \to (S_2/K_2)^\vee_{\mathbb{C}} \to \cdots$$

is a $(-)^\Gamma$-acyclic resolution of 0, and hence

$$H^q\big(((S_\bullet/K_\bullet)^\vee_{\mathbb{C}})^\Gamma\big) \simeq H^q(\Gamma, 0) = 0$$

for all $q \geq 0$. □

As a result, for a subgroup $\Gamma \subset \Gamma_Q$ and a homology class $\mathfrak{z} \in H_{g-1}(T_w/\Gamma,\mathbb{Z})$, we can define an evaluation map

$$\langle \mathfrak{z},\ \rangle : H^{g-1}_Q(Y^\circ,\Gamma_Q,\mathbb{C}) \simeq H^{g-1}(T_w/\Gamma_Q,\mathbb{C}) \to H^{g-1}(T_w/\Gamma,\mathbb{C}) \xrightarrow{\langle \mathfrak{z},\ \rangle} \mathbb{C} \tag{8-6}$$

by taking the pairing with $\mathfrak{z}$.

8.2. ***Shintani decomposition.*** Using Corollary 8.1.4, here we construct a cone decomposition of a homology class $\mathfrak{z} \in H_{g-1}(T_w/\Gamma,\mathbb{Z})$. See Proposition 8.2.1 and Remark 8.2.2. We need such a cone decomposition in order to compute the specialization of the Shintani–Barnes cocycle.

Recall that $\tau_1,\dots,\tau_g$ are the field embeddings of F into $\mathbb{C}$. Clearly, τ_i extends to

$$\tau_i : F_{\mathbb{R}} = F \otimes \mathbb{R} \to \mathbb{C}.$$

Let F_{τ_i} denote the completion of F with respect to the embedding τ_i. In the following, we assume for simplicity that $\tau_1,\dots,\tau_{r_1}$ are the real embeddings, i.e., $F_{\tau_i} = \mathbb{R}$ for $i = 1,\dots,r_1$, and $\tau_{r_1+1},\dots,\tau_g$ are the nonreal embeddings, i.e., $F_{\tau_i} = \mathbb{C}$ for $i = r_1+1,\dots,g$.

For $\mu = (\mu_1,\dots,\mu_{r_1}) \in \{\pm 1\}^{r_1}$ $(:= \{-1,1\}^{r_1})$, set

$$F^\times_{\mathbb{R},\mu} := \{x \in F^\times_{\mathbb{R}} \mid \mu_i\tau_i(x) > 0 \ \text{ for } \ i = 1,\dots,r_1\}.$$

Clearly, $\{F^\times_{\mathbb{R},\mu} \mid \mu \in \{\pm 1\}^{r_1}\}$ are the connected components of $F^\times_{\mathbb{R}}$, and we have $F^\times_{\mathbb{R}} = \bigsqcup_{\mu\in\{\pm1\}^{r_1}} F^\times_{\mathbb{R},\mu}$. Then let $T_{w,\mu} \subset T_w$ be the connected component of T_w corresponding to $F^\times_{\mathbb{R},\mu}$ via the identification (8-1):

$$w : T_w \xrightarrow{\sim} F^\times_{\mathbb{R}}.$$

If $\mu = (1, 1, \dots, 1)$, then $F^\times_{\mathbb{R},\mu}$ is the totally positive component of $F^\times_{\mathbb{R}}$, and simply denoted by $F^\times_{\mathbb{R},+}$. Furthermore, let

$$F^\times_+ := F^\times \cap F^\times_{\mathbb{R},+} = \{x \in F^\times \mid \tau_i(x) > 0 \ \text{ for } \ i = 1, \dots, r_1\},$$
$$\mathcal{O}^\times_+ := \mathcal{O}^\times \cap F^\times_{\mathbb{R},+} = \{u \in \mathcal{O}^\times \mid \tau_i(u) > 0 \ \text{ for } \ i = 1, \dots, r_1\}$$

denote the totally positive parts of $F^\times$ and $\mathcal{O}^\times$ respectively, and let $\Gamma^+_Q \subset \Gamma_Q$ be the image of $\mathcal{O}^\times_+$ under the isomorphism

$$\rho_w : \mathcal{O}^1 \xrightarrow{\sim} \Gamma_Q$$

(see Section 2.2).

By Dirichlet's unit theorem, we know that

$$T_w/\mathbb{R}_{>0}\Gamma^+_Q \simeq F^\times_{\mathbb{R}}/\mathbb{R}_{>0}\mathcal{O}^\times_+$$

is compact, and its connected components

$$T_{w,\mu}/\mathbb{R}_{>0}\Gamma^+_Q \simeq F^\times_{\mathbb{R},\mu}/\mathbb{R}_{>0}\mathcal{O}^\times_+ \quad \text{for } \ \mu \in \{\pm 1\}^{r_1}$$

are homeomorphic to $(g-1)$-dimensional topological tori. Therefore, we have

$$H_{g-1}(T_w/\,\Gamma^+_Q, \mathbb{Z}) \simeq H_{g-1}(T_w/\mathbb{R}_{>0}\Gamma^+_Q, \mathbb{Z}) \simeq \mathbb{Z}^{\{\pm1\}^{r_1}}. \tag{8-7}$$

Here the first isomorphism is a canonical isomorphism induced from the projection

$$T_w/\,\Gamma^+_Q \to T_w/\mathbb{R}_{>0}\Gamma^+_Q,$$

which is clearly a homotopy equivalence. In order to fix the second isomorphism of (8-7), we equip $T_w/\mathbb{R}_{>0}\Gamma^+_Q$ with an orientation as follows.

Orientation. Set

$$\boldsymbol{T}_\mu := T_{w,\mu}/\mathbb{R}_{>0}\Gamma^+_Q \subset \boldsymbol{T} := T_w/\mathbb{R}_{>0}\Gamma^+_Q$$

for simplicity. Recall that an orientation of a $(g-1)$-dimensional manifold X is defined as a system $(\nu_x)_{x\in X}$ of generators $\nu_x \in H_{g-1}(X, X-\{x\}, \mathbb{Z}) \simeq \mathbb{Z}$ with a certain compatibility; see [Hatcher 2002, Section 3.3]. Note that giving a generator ν_x of $H_{g-1}(X, X-\{x\}, \mathbb{Z}) \simeq \mathbb{Z}$ is equivalent to giving an isomorphism

$$o_x : H_{g-1}(X, X-\{x\}, \mathbb{Z}) \xrightarrow{\sim} \mathbb{Z}, \quad \nu_x \mapsto 1.$$

We first fix an orientation of the $(g-1)$-sphere $\boldsymbol{S}^{g-1} = (\mathbb{R}^g-\{0\})/\mathbb{R}_{>0}$ as follows. Let $x \in \mathbb{R}^g-\{0\}$ and let $\bar{x} \in \boldsymbol{S}^{g-1}$ be its image. Moreover, let $I = (\alpha_1, \dots, \alpha_g) \in (X_{\mathbb{Q}})^g$ such that $0 \notin |\sigma_I|$ and $x \notin \partial C_I$, where ∂C_I is the boundary of the cone C_I. Then we see

$$\bar{\sigma}_I : \Delta^{g-1} \xrightarrow{\sigma_I} \mathbb{R}^g-\{0\} \to \boldsymbol{S}^{g-1}$$

defines a class $[\bar{\sigma}_I] \in H_{g-1}(\boldsymbol{S}^{g-1}, \boldsymbol{S}^{g-1} - \{\bar{x}\}, \mathbb{Z})$. We fix the isomorphism $o_{\bar{x}}$ so that we have

$$o_{\bar{x}}([\bar{\sigma}_I]) = \operatorname{sgn}(I)\mathbf{1}_{C_I}(x)$$

for all such I, where $\operatorname{sgn}(I) = \operatorname{sgn}(\det I) \in \{0, \pm 1\}$. This defines an orientation of $\boldsymbol{S}^{g-1}$. Then this orientation of $\boldsymbol{S}^{g-1}$ induces orientations of $T_w/\mathbb{R}_{>0} \subset \boldsymbol{S}^{g-1}$ and $\boldsymbol{T} = T_w/\mathbb{R}_{>0}\Gamma_Q^+$ because the action of Γ_Q^+ on $T_w/\mathbb{R}_{>0}$ is free, properly discontinuous, and orientation-preserving. More explicitly, for $x \in T_w$ and its image $\boldsymbol{x} \in \boldsymbol{T}$, the local orientation isomorphism

$$o_{\boldsymbol{x}} : H_{g-1}(\boldsymbol{T}, \boldsymbol{T} - \{\boldsymbol{x}\}, \mathbb{Z}) \xrightarrow{\sim} \mathbb{Z}$$

can be computed as follows. let $I = (\alpha_1, \dots, \alpha_g) \in (X_{\mathbb{Q}})_Q^g$ such that $\gamma x \notin \partial C_I$ for all $\gamma \in \Gamma_Q^+$. Then

$$\boldsymbol{\sigma}_I : \Delta^{g-1} \xrightarrow{\sigma_I} T_w \to \boldsymbol{T}$$

defines a class $[\boldsymbol{\sigma}_I] \in H_{g-1}(\boldsymbol{T}, \boldsymbol{T} - \{\boldsymbol{x}\}, \mathbb{Z})$, and we have

$$o_{\boldsymbol{x}}([\boldsymbol{\sigma}_I]) = \operatorname{sgn}(I) \sum_{\gamma \in \Gamma_Q^+} \mathbf{1}_{C_I}(\gamma x). \tag{8-8}$$

Now, since $\{\boldsymbol{T}_\mu \mid \mu \in \{\pm 1\}^{r_1}\}$ are the connected components of $\boldsymbol{T}$, this orientation defines isomorphisms

$$o_\mu : H_{g-1}(\boldsymbol{T}_\mu, \mathbb{Z}) \xrightarrow{\sim} \mathbb{Z}, \quad \mu \in \{\pm 1\}^{r_1},$$
$$o = \bigoplus_\mu o_\mu : H_{g-1}(\boldsymbol{T}, \mathbb{Z}) \simeq \bigoplus_{\mu \in \{\pm 1\}^{r_1}} H_{g-1}(\boldsymbol{T}_\mu, \mathbb{Z}) \xrightarrow{\sim} \bigoplus_{\mu \in \{\pm 1\}^{r_1}} \mathbb{Z}$$

such that for all $\boldsymbol{x} \in \boldsymbol{T}_\mu$, the following diagram is commutative:

$$\begin{array}{ccc} H_{g-1}(\boldsymbol{T}_\mu, \mathbb{Z}) & \xrightarrow[\sim]{o_\mu} & \mathbb{Z} \\ \downarrow{\scriptstyle \mathrm{loc}_{\boldsymbol{x}}} & & \| \\ H_{g-1}(\boldsymbol{T}, \boldsymbol{T} - \{\boldsymbol{x}\}, \mathbb{Z}) & \xrightarrow[\sim]{o_{\boldsymbol{x}}} & \mathbb{Z} \end{array} \tag{8-9}$$

Here the left vertical arrow is the natural localization map; see [Hatcher 2002, Theorem 3.26, Lemma 3.27].

For $\chi = (\chi_\mu)_\mu \in \bigoplus_{\mu \in \{\pm 1\}^{r_1}} \mathbb{Z}$, let

$$\mathfrak{z}_\chi \in H_{g-1}(\boldsymbol{T}, \mathbb{Z}) \simeq H_{g-1}(T_w/\,\Gamma_Q^+, \mathbb{Z})$$

denote the class such that $o(\mathfrak{z}_\chi) = \chi$. Note that if $\mathfrak{z}_\mu$ denotes the fundamental class of $\boldsymbol{T}_\mu$, then $\mathfrak{z}_\chi$ can be written as $\mathfrak{z}_\chi = \sum_\mu \chi_\mu \mathfrak{z}_\mu$.

Proposition 8.2.1. *Let* $\chi = (\chi_\mu)_\mu \in \bigoplus_{\mu \in \{\pm 1\}^{r_1}} \mathbb{Z}$.

(1) *There exists*

$$\Phi = \sum_{i=1}^{r} c_i \sigma_{I_i} \in K_{g-1} = \mathbb{Z}[\sigma_I \mid I \in (X_{\mathbb{Q}})_Q^g] \subset S_{g-1}$$

which represents the homology class $\mathfrak{z}_\chi \in H_{g-1}(T_w/\,\Gamma_Q^+, \mathbb{Z})$, *where* $I_1, \dots, I_r \in (X_{\mathbb{Q}})_Q^g$, *and* $c_i \in \mathbb{Z}$.

(2) *Then for* $x \in \mathbb{R}^g - \{0\}$, *we have*

$$\sum_{\gamma \in \Gamma_Q^+} \sum_{i=1}^{r} c_i \operatorname{sgn}(I_i) \mathbf{1}_{C_{I_i}^Q}(\gamma x) = \chi(x) \mathbf{1}_{T_w}(x),$$

where χ *is regarded as a locally constant function* $\chi : T_w \to \mathbb{Z}$ *which has value* χ_μ *on* $T_{w,\mu}$, *i.e.,* $\chi(x) = \chi_\mu$ *for* $x \in T_{w,\mu}$.

Proof. (1) This is a direct consequence of Corollary 8.1.4(1).

(2) First note that we have

$$\mathbf{1}_{C_{I_i}^Q}(\gamma x) = \mathbf{1}_{\gamma^{-1} C_{I_i}^Q}(x) = \mathbf{1}_{C_{\gamma^{-1} I_i}^Q}(x)$$

for $\gamma \in \Gamma_Q^+$. Now, since the action of Γ_Q^+ on $T_w / \mathbb{R}_{>0}$ is properly discontinuous, the collection $\{\gamma^{-1} C_{I_i}\}_{i,\gamma}$ of subsets of T_w is locally finite. Therefore, as in the proof of Proposition 5.3.4, by using Lemma 5.3.3, we can find $\delta > 0$ such that

$$\exp(\varepsilon Q) x \notin \partial C_{\gamma^{-1} I_i}$$

for all $\varepsilon \in (0, 2\delta)$, $i = 1, \dots, r$, and $\gamma \in \Gamma_Q^+$. Set

$$x' := \exp(\delta Q) x.$$

Then we have

$$\mathbf{1}_{C_{I_i}^Q}(\gamma x) = \mathbf{1}_{C_{\gamma^{-1} I_i}^Q}(x) = \mathbf{1}_{C_{\gamma^{-1} I_i}}(x') = \mathbf{1}_{C_{I_i}}(\gamma x').$$

Moreover, by using Lemma 2.2.1(5), we see that $\exp(\delta Q)$ preserves the connected components $T_{w,\mu}$ of T_w, and hence we have

$$\chi(x) \mathbf{1}_{T_w}(x) = \chi(x') \mathbf{1}_{T_w}(x').$$

Therefore, it suffices to show

$$\sum_{\gamma \in \Gamma_Q^+} \sum_{i=1}^{r} c_i \operatorname{sgn}(I_i) \mathbf{1}_{C_{I_i}}(\gamma x') = \chi(x') \mathbf{1}_{T_w}(x'). \tag{8-10}$$

First, by Lemma 8.1.1, all of the terms in (8-10) are 0 if $x' \notin T_w$. Therefore, we assume $x' \in T_{w,\mu}$ for some $\mu \in \{\pm 1\}^{r_1}$. Set

$$\boldsymbol{T}_\mu := T_{w,\mu} / \mathbb{R}_{>0} \Gamma_Q^+ \quad \text{and} \quad \boldsymbol{x}' := \mathbb{R}_{>0} \Gamma_Q^+ x' \in \boldsymbol{T}_\mu.$$

Then, by (8-8), we see that the image of Φ under the localization map

$$o_{\boldsymbol{x}'} \circ \operatorname{loc}_{\boldsymbol{x}'} : H_{g-1}(T_{w,\mu} / \Gamma_Q^+, \mathbb{Z}) \simeq H_{g-1}(\boldsymbol{T}_\mu, \mathbb{Z}) \xrightarrow{\operatorname{loc}_{\boldsymbol{x}'}} H_{g-1}(\boldsymbol{T}, \boldsymbol{T} - \{\boldsymbol{x}'\}, \mathbb{Z}) \xrightarrow[\sim]{o_{\boldsymbol{x}'}} \mathbb{Z}$$

is equal to

$$\sum_{\gamma \in \Gamma_Q^+} \sum_{i=1}^{r} c_i \operatorname{sgn}(I_i) \mathbf{1}_{C_{I_i}}(\gamma x').$$

On the other hand, by (8-9), $o_{\boldsymbol{x}'} \circ \operatorname{loc}_{\boldsymbol{x}'}(\Phi) = o_\mu(\mathfrak{z}_\chi) = \chi_\mu$ because Φ represents $\mathfrak{z}_\chi$. This completes the proof. □

Remark 8.2.2. In the case where $\mathfrak{z}_\chi = \mathfrak{z}_\mu$ is the fundamental class of a connected component $\boldsymbol{T}_\mu$, Proposition 8.2.1 says that

$$\sum_{i=1}^{r} c_i \operatorname{sgn}(I_i)\mathbf{1}_{C^Q_{I_i}}$$

gives a signed fundamental domain for $T_{w,\mu}/\Gamma_Q^+$ in the sense of Charollois, Dasgupta, and Greenberg [Charollois et al. 2015, Definition 2.4], which is a "weighted version" of the Shintani cone decomposition; see also [Diaz y Diaz and Friedman 2014; Espinoza and Friedman 2020].

Remark 8.2.3. Let the notation χ, $\mathfrak{z}_\chi$, and $\Phi = \sum_{i=1}^r c_i \sigma_{I_i}$ be the same as in Proposition 8.2.1. We can compute the evaluation map

$$\langle \mathfrak{z}_\chi, \ \rangle : H_Q^{g-1}(Y^\circ, \Gamma_Q, \mathbb{C}) \simeq H^{g-1}(T_w/\Gamma_Q, \mathbb{C}) \to H^{g-1}(T_w/\Gamma_Q^+, \mathbb{C}) \xrightarrow{\langle \mathfrak{z}_\chi, \ \rangle} \mathbb{C}$$

(see (8-6)) explicitly as follows. Let

$$s = (s_I)_{I \in (X_\mathbb{Q})^g_Q} \in C_Q^{g-1}(\mathcal{X}_\mathbb{Q}, \mathbb{C}) = \prod_{I \in (X_\mathbb{Q})^g_Q} \mathbb{C}$$

be a Γ_Q-invariant cocycle and let $[s] \in H_Q^{g-1}(Y^\circ, \Gamma_Q, \mathbb{C})$ be the class represented by s. Then we have

$$\langle \mathfrak{z}_\chi, [s] \rangle = \sum_{i=1}^{r} c_i s_{I_i}.$$

8.3. *Values of the zeta functions.* Recall that F is a number field of degree g, $\mathcal{O}$ is an order in F, and $\mathfrak{a} \subset F$ is a proper fractional $\mathcal{O}$-ideal.

Definition 8.3.1. (1) For a continuous map

$$\chi : F_\mathbb{R}^\times = (F \otimes_\mathbb{Q} \mathbb{R})^\times \to \mathbb{Z},$$

let

$$\zeta_\mathcal{O}(\chi, \mathfrak{a}^{-1}, s) := \sum_{x \in (\mathfrak{a} - \{0\})/\mathcal{O}_+^\times} \frac{\chi(x)}{|N_{F/\mathbb{Q}}(x)|^s}, \quad \operatorname{Re}(s) > 1$$

denote the partial zeta function associated to χ and a proper fractional $\mathcal{O}$-ideal $\mathfrak{a}^{-1}$. Here, note that χ is constant on each connected component of $F_\mathbb{R}^\times$, and thus invariant under the action of $\mathcal{O}_+^\times$.

(2) Let

$$\boldsymbol{\varepsilon} : F_\mathbb{R}^\times \to \{\pm 1\}, \quad x \mapsto \frac{N_{F/\mathbb{Q}}(x)}{|N_{F/\mathbb{Q}}(x)|}$$

denote the sign character.

Now, let $k \geq 1$, and let $\chi \in \bigoplus_{\mu \in \{\pm 1\}^{r_1}} \mathbb{Z}$. Note that χ can be regarded as a continuous map $\chi : F_\mathbb{R}^\times \to \mathbb{Z}$ via

$$\chi : F_\mathbb{R}^\times \to F_\mathbb{R}^\times / F_{\mathbb{R},+}^\times \simeq \{\pm 1\}^{r_1} \xrightarrow{\chi} \mathbb{Z}.$$

So far, we have defined the following series of maps between cohomology groups:

$$\begin{array}{ccc}
H^{g-1}(Y^\circ, \mathrm{SL}_g(\mathbb{Z}), \mathscr{F}_{kg}^{\Xi}) & \ni & [\Psi_{kg}] \\
\downarrow \mathrm{ev}_Q & & \Big\downarrow \\
H^{g-1}(Y^\circ, \Gamma_Q, \mathscr{F}_{kg}) & & \Big| \\
\downarrow N_{w^*}^k & & \Big| \\
H^{g-1}(Y^\circ, \Gamma_Q, \mathscr{F}_0) & & \Big| \\
\downarrow \int_Q & & \Big| \\
H_Q^{g-1}(Y^\circ, \Gamma_Q, \mathbb{C}) & & \Big| \\
\downarrow \langle \mathfrak{z}_\chi, \ \rangle & & \Big\downarrow \\
\mathbb{C} & \ni & \left\langle \mathfrak{z}_\chi, \int_Q N_{w^*}^k \mathrm{ev}_Q([\Psi_{kg}]) \right\rangle
\end{array} \tag{8-11}$$

See Corollary 4.3.4, Example 4.3.6, (7-5), and Remark 8.2.3 for the definitions of these maps.

Theorem 8.3.2. *We have*

$$\left\langle \mathfrak{z}_\chi, \int_Q N_{w^*}^k \mathrm{ev}_Q([\Psi_{kg}]) \right\rangle = \frac{(k!)^g \det(w^{(1)}, \dots, w^{(g)})}{(g+gk-1)!} \zeta_{\mathcal{O}}(\boldsymbol{\varepsilon}^{k+1}\chi, \mathfrak{a}^{-1}, k+1),$$

where $\boldsymbol{\varepsilon}^{k+1}\chi(x) = \boldsymbol{\varepsilon}(x)^{k+1}\chi(x)$.

Proof. By Hurwitz' formula (Example 7.2.4), we see that the class

$$\int_Q N_{w^*}^k \mathrm{ev}_Q([\Psi_{kg}]) \in H_Q^{g-1}(Y^\circ, \Gamma_Q, \mathbb{C})$$

is represented by

$$\left(\int_{Q,I} N_{w^*}(y)^k \psi_{kg,I}^Q(y)\omega(y) \right)_{I \in (X_{\mathbb{Q}})_Q^g} = \left(\frac{(k!)^g \det(w^{(1)}, \dots, w^{(g)})}{(g+gk-1)!} \operatorname{sgn}(I) \sum_{x \in C_I^Q \cap \mathbb{Z}^g - \{0\}} \frac{1}{N_w(x)^{k+1}} \right)_{I \in (X_{\mathbb{Q}})_Q^g}.$$

On the other hand, by Proposition 8.2.1(1), we can take a representative

$$\Phi = \sum_{i=1}^r c_i \sigma_{I_i} \in K_{g-1} = \mathbb{Z}[\sigma_I \mid I \in (X_{\mathbb{Q}})_Q^g] \subset S_{g-1}$$

of $\mathfrak{z}_\chi \in H_{g-1}(T_w/\Gamma_Q^+, \mathbb{Z})$. Then, by using Remark 8.2.3 and Proposition 8.2.1(2), we find

$$\begin{aligned}
\left\langle \mathfrak{z}_\chi, \int_Q N_{w*}^k \mathrm{ev}_Q([\Psi_{kg}]) \right\rangle &= \frac{(k!)^g \det(w^{(1)}, \dots, w^{(g)})}{(g+gk-1)!} \sum_{i=1}^r c_i \operatorname{sgn}(I_i) \sum_{x \in C_{I_i}^Q \cap \mathbb{Z}^g - \{0\}} \frac{1}{N_w(x)^{k+1}} \\
&= \frac{(k!)^g \det(w^{(1)}, \dots, w^{(g)})}{(g+gk-1)!} \sum_{x \in \mathbb{Z}^g - \{0\}} \sum_{i=1}^r c_i \operatorname{sgn}(I_i) \mathbf{1}_{C_{I_i}^Q}(x) \frac{1}{N_w(x)^{k+1}} \\
&= \frac{(k!)^g \det(w^{(1)}, \dots, w^{(g)})}{(g+gk-1)!} \sum_{x \in (\mathbb{Z}^g - \{0\})/\Gamma_Q^+} \sum_{\gamma \in \Gamma_Q^+} \sum_{i=1}^r c_i \operatorname{sgn}(I_i) \mathbf{1}_{C_{I_i}^Q}(\gamma x) \frac{1}{N_w(x)^{k+1}} \\
&= \frac{(k!)^g \det(w^{(1)}, \dots, w^{(g)})}{(g+gk-1)!} \sum_{x \in (\mathbb{Z}^g - \{0\})/\Gamma_Q^+} \frac{\chi(x)}{N_w(x)^{k+1}} \\
&= \frac{(k!)^g \det(w^{(1)}, \dots, w^{(g)})}{(g+gk-1)!} \sum_{x \in (\mathfrak{a} - \{0\})/\mathcal{O}_+^\times} \frac{\boldsymbol{\varepsilon}(x)^{k+1} \chi(x)}{|N_{F/\mathbb{Q}}(x)|^{k+1}}. \qquad \square
\end{aligned}$$

Remark 8.3.3. It is easy to see that

$$\det(w^{(1)}, \dots, w^{(g)})^2 = D_{\mathcal{O}} N\mathfrak{a}^2,$$

where $D_{\mathcal{O}}$ is the discriminant of the order $\mathcal{O}$. Moreover, we also know that $\operatorname{sgn}(D_{\mathcal{O}}) = (-1)^{r_2}$, where r_2 is the number of complex places of F. Therefore, by permuting the order of the embeddings $\tau_1, \dots, \tau_g$ if necessary, we have

$$\det(w^{(1)}, \dots, w^{(g)}) = \boldsymbol{i}^{r_2} \sqrt{|D_{\mathcal{O}}|} N\mathfrak{a},$$

where $\boldsymbol{i} \in \mathbb{C}$ is the imaginary unit. Hence (under a suitable ordering of $\tau_1, \dots, \tau_g$), Theorem 8.3.2 can be also written as

$$\left\langle \mathfrak{z}_\chi, \int_Q N_{w*}^k \mathrm{ev}_Q([\Psi_{kg}]) \right\rangle = \boldsymbol{i}^{r_2} \frac{\sqrt{|D_{\mathcal{O}}|} N\mathfrak{a} (k!)^g}{(g+gk-1)!} \zeta_{\mathcal{O}}(\boldsymbol{\varepsilon}^{k+1}\chi, \mathfrak{a}^{-1}, k+1).$$

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