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The log product formula

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Let V, W be a pair of smooth varieties. We want to compare curve counts on $V \times W$ with those on V and W. The product formula in Gromov–Witten theory compares the virtual fundamental classes of stable maps to a product $\overline{M}_{g,n}(V \times W)$ to the product of stable maps $\overline{M}_{g,n}(V) \times \overline{M}_{g,n}(W)$. We prove the analogous theorem for *log stable maps* to log smooth varieties V, W.

This extends results of Y.P. Lee and F. Qu, who introduced this formula after K. Behrend. We introduce "log normal cones" and "log virtual fundamental classes," as well as modified versions of standard intersection-theoretic machinery adapted to log geometry.

0. Introduction

The log product formula. The purpose of the present paper is to prove the "product formula" for log Gromov–Witten invariants. We assume the reader is familiar with log geometry at the level of [Ogus 2018].

Let V, W be log smooth, quasiprojective log schemes. The moduli stack of log stable maps [Gross and Siebert 2013; Chen 2014; Abramovich and Chen 2014] $\mathscr{M}^{\ell}_{g,n}(V)$ parametrizes families of fs log smooth curves $C \to S$ with a stable map $C \to V$ of log schemes. These coincide with ordinary stable maps if V has trivial log structure $M_V \simeq \mathcal{O}^*_V$.

Let Q be the fiber product

$$\mathscr{M}^{\ell}_{g,n}(V) \times^{\ell}_{\overline{M}_{g,n}} \mathscr{M}^{\ell}_{g,n}(W)$$

in the category of *fs* (fine and saturated) log algebraic stacks [Ogus 2018, Corollary III.2.1.6], with maps

$$\mathscr{M}^{\ell}_{g,n}(V \times W) \xrightarrow{h} Q \xrightarrow{\widetilde{\Delta}} \mathscr{M}^{\ell}_{g,n}(V) \times \mathscr{M}^{\ell}_{g,n}(W).$$

We define log virtual fundamental class in Definition 3.1. One can endow Q with a log virtual fundamental class in two ways: pushing forward that of $\mathscr{M}^{\ell}_{g,n}(V \times W)$ or pulling back that of $\mathscr{M}^{\ell}_{g,n}(V) \times \mathscr{M}^{\ell}_{g,n}(W)$. The product formula equates these.

Theorem 0.1 (the "log Gromov–Witten product formula"). *The two log virtual fundamental classes are equal in the Chow group A_*Q:*

$$h_*[\mathscr{M}^{\ell}_{g,n}(V \times W)]^{\ell vir} = \Delta^![\mathscr{M}^{\ell}_{g,n}(V) \times \mathscr{M}^{\ell}_{g,n}(W)]^{\ell vir}.$$

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The symbol $\Delta^!$ refers to the log Gysin map of Definition 3.1.

The original, nonlog product formula was established by M. Kontsevich and Yu. Manin in genus zero [Kontsevich and Manin 1996]. It was extended to arbitrary genus by Behrend [1999].

Theorem 0.1 was formulated using ordinary virtual fundamental classes by Lee and Qu [2018] and proved under the assumption that one of V or W has trivial log structure. Like their work and the work of Behrend [1999] before it, our proof centers on this cartesian diagram (Situation 5.5):

$$\begin{array}{ccccc}
\mathscr{M}^{\ell}_{g,n}(V \times W) & \xrightarrow{h} & Q & \xrightarrow[\ulcorner \ell]{} & \mathscr{M}^{\ell}_{g,n}(V) \times \mathscr{M}^{\ell}_{g,n}(W) \\
\downarrow{\scriptstyle c} & \ulcorner \ell & \downarrow & & \downarrow{\scriptstyle a} \\
\mathfrak{D} & \xrightarrow{l} & Q' & \xrightarrow[\ulcorner \ell]{\phi} & \mathfrak{M}_{g,n} \times \mathfrak{M}_{g,n} \\
 & & \downarrow & & \downarrow{\scriptstyle s \times s} \\
 & & \overline{M}_{g,n} & \xrightarrow{\Delta} & \overline{M}_{g,n} \times \overline{M}_{g,n}
\end{array}$$

One applies Costello's formula [2006, Theorem 5.0.1] and commutativity of the Gysin map to this diagram to compare virtual fundamental classes.

In the log setting, one requires this diagram to be cartesian in the 2-category of *fs log algebraic stacks* in order to preserve modular interpretations. The assumption of [Lee and Qu 2018] that V or W have trivial log structure ensures that these squares are *also* cartesian as underlying algebraic stacks.

These fs pullback squares in question likely aren't cartesian on underlying algebraic stacks. Therefore, none of the standard machinery of ordinary Gysin maps and normal cones is valid. This quandary forced us to prove the log analogues of Costello's formula and commutativity for our "log Gysin map." With these modifications, the original proof of K. Behrend essentially still works. We pause to comment on the new technology.

Log normal cones. The *log normal cone* $C^{\ell}_{X/Y} = C_{X/\mathcal{L}Y}$ of a map $f : X \to Y$ of log algebraic stacks is the central object of the present paper. Every log map factors as the composition of a strict and an étale map $X \to \mathcal{L}Y \to Y$, so the cone is determined by two properties:

- It agrees with the ordinary normal cone for strict maps.
- If one can factor f as $X \to Y' \to Y$ with $Y' \to Y$ log étale, the cones are canonically isomorphic:

$$C^{\ell}_{X/Y} \simeq C^{\ell}_{X/Y'}.$$

This object becomes simpler in the presence of charts. Locally, we may assume the map $X \to Y$ has a chart given by a map of Artin cones $\mathscr{A}_P \to \mathscr{A}_Q$. The map $\mathscr{A}_P \to \mathscr{A}_Q$ is log étale, so we can base change across it to get a strict map without altering the log normal cone.

Because this method can lead to radical alterations of the target Y, we recall another strategy that we learned from [Ito et al. 2020, Proposition 2.3.12]. For ordinary schemes, one locally factors a map as a closed immersion composed with a smooth map to get a presentation for the normal cone [Behrend and

Fantechi 1997]. We obtain a similar local factorization (Construction 1.1) into a *strict* closed immersion composed with a log smooth map, and the same presentation exists for the log normal cone.

The above is made more precise in Remark 2.7. The charts and factorizations these techniques require are only locally possible, so we need to know how log normal cones change after étale localization. We encounter a well-known subtlety noticed by W. Bauer [Olsson 2005, Section 7]: The log normal cone isn't invariant under base-changes by log étale maps (Remark 2.13). Our workaround is somewhat different from that of Olsson. These results are at the service of log intersection theory, and we outline a standard package of log virtual fundamental classes and log Gysin maps.

Pushforward and Gysin pullback. The proof of the product formula needs two ingredients: commutativity of Gysin maps and compatibility of pushforward with Gysin maps. The commutativity of Gysin maps readily generalizes to the log setting in Theorem 3.12; on the other hand, compatibility with pushforward simply fails!

Nevertheless, the original proof of the product formula depends on a weak form of this compatibility first introduced by K. Costello [2006, Theorem 5.0.1]. This theorem is false as stated due to a missing properness hypothesis. We fix and generalize the statement in [Herr and Wise 2022] and offer a log generalization in Section 4.

We obtain another partial result towards compatibility of pushforward and Gysin pullback. For a log blowup $p : \widehat{X} \to X$ with a log smoothness assumption, we show $p_*[\widehat{X}]^{\ell vir} = [X]^{\ell vir}$ in Theorem 3.10. The alternative approach of Barrott [2018] may extend our results by modifying the notions of dimension, degree, pushforward, Chow groups, etc. in the log setting. See also [Ranganathan 2022] for an insightful approach to log Chow groups.

D. Ranganathan obtained a version of the log product formula contemporaneously using an explicit blowup instead of our log virtual fundamental class machinery [Ranganathan 2019]. We hope the technology and the strategy of reducing statements about log normal cones to the strict, ordinary case will be of interest.

Context and motivation. A pair (X, D) of a smooth divisor on a smooth variety is an example of a log smooth target V. Jun Li [2001] defined relative stable maps to such a pair (X, D). We instead use the log stable maps $\mathscr{M}^{\ell}_{g,n}(V)$ of Gross and Siebert [2013], and Chen [2014] and Abramovich and Chen [2014].[1] Even if one starts with V and W smooth pairs, their product $V \times W$ is a log smooth log scheme and likely not a smooth pair.

Gross and Siebert define the virtual fundamental class of $\mathscr{M}^{\ell}_{g,n}(V)$ relative to a log variant $\mathcal{L}\mathfrak{M}_{g,n}$ of the stack of prestable curves $\mathfrak{M}_{g,n}$. We take this as a definition of the log virtual fundamental class and related log normal cone. The log virtual fundamental class is then *invariant* under log modifications of the target V. The classes defined in this paper live in ordinary Chow groups $A_*(\cdot)$ but can be refined [Herr et al. 2023] to both large and small log Chow groups [Holmes et al. 2019] or to K theory [Chou et al. 2020].

[1] The forthcoming [Herr et al. 2023] compares relative and log stable maps.

One might try to prove Theorem 0.1 using the ordinary Gysin map $\Delta^!$ instead of the log version. This works if V or W has trivial log structure [Lee and Qu 2018] but is false in general. See [Chou et al. 2020] for counterexamples to the ordinary Gysin map $\Delta^!$ version. Our log Gysin map is necessary to prove Theorem 0.1 partially because it produces classes on the fs fiber product Q instead of the fiber product of underlying schemes.

These log Gysin maps may be of interest wherever the fs fiber product arises in enumerative geometry. For example, fs pullback squares abound in the punctured log stable maps of [Abramovich et al. 2020]. The fs fiber products in [Nabijou and Ranganathan 2022, Section 2.1] are reduced to ordinary fiber products by weak semistable reduction to use ordinary intersection theory. The multiplicativity found in [Holmes et al. 2019] is an example of a log intersection product.

Log Chow groups are still under construction. The log Chow group of X can be defined as a limit or colimit over the Chow groups of log modifications of X [Holmes et al. 2019; Barrott 2018]. The log virtual fundamental class can be refined to lie in log Chow [Chou et al. 2020; Herr et al. 2023]. Rather than making sense of pairing with ψ classes, we simply prove the expected equality of virtual fundamental classes in ordinary Chow here.

See [Molcho and Ranganathan 2021, Section 1.2] for a down-to-earth log intersection product related to our $\Delta^!$.

Our log product formula may compute the log Gromov–Witten invariants of toric varieties, as shown to the author independently by J. Wise and D. Ranganathan. Any pair of toric varieties Y_1, Y_2 of the same dimension are related by a third which is a log blowup of each: $Y_1 \leftarrow \tilde{Y} \to Y_2$. Log virtual fundamental classes and Gromov–Witten invariants are invariant under log blowups [Abramovich and Wise 2018], Theorem 3.10. The log Gromov–Witten invariants of *all* toric varieties of dimension n are essentially the same, and one can compute just one example $(\mathbb{P}^1)^n$.

Conventions. • We *only consider fs log structures*. We therefore use $\mathcal{L}$, $\mathcal{L}Y$ to refer to Olsson's stacks $\mathscr{T}or$, $\mathscr{T}orY$.

• We work over the complex numbers $\mathbb{C}$.

• We adhere to the convention of [Olsson 2003] regarding the use of the term "algebraic stack": we mean a stack in the sense of [Laumon and Moret-Bailly 2000, 3.1] such that

– the diagonal is representable and of finite presentation, and

– there exists a surjective, smooth morphism to it from a scheme.

We do not require the diagonal morphism to be separated.

• By "log algebraic stack," we mean an algebraic stack with a map to $\mathcal{L}$. Maps between them need not lie over $\mathcal{L}$.

• The name "DM stack" means Deligne–Mumford stack and a morphism $f : X \to Y$ of algebraic stacks is (of) "DM-type" or simply "DM" if every Y-scheme $T \to Y$ pulls back to a DM stack $T \times_{f,Y} X$ [Manolache 2012].

- The word "cone" in "log normal cone" refers to a cone stack in the sense of [Behrend and Fantechi 1997].
- Let P be a sharp fs monoid. Write

$$\mathscr{A}_P = [\operatorname{Spec} \mathbb{C}[P] / \operatorname{Spec} \mathbb{C}[P^{gp}]]$$

for the stack quotient in the étale topology endowed with its natural log structure [Abramovich et al. 2017; Cavalieri et al. 2020; Olsson 2003]. Beware that some of these sources first take the dual monoid. This log stack has a notable functor of points for fs log schemes

$$\operatorname{Hom}_{fs}(T, \mathscr{A}_P) = \operatorname{Hom}_{mon}(P, \Gamma(\overline{M}_T)).$$

In particular,

$$\operatorname{Hom}_{fs}(\mathscr{A}_P, \mathscr{A}_Q) = \operatorname{Hom}_{mon}(Q, P).$$

We write $\mathscr{A}$ for $\mathscr{A}_{\mathbb{N}} = [\mathbb{A}^1/\mathbb{G}_m]$. Log algebraic stacks of this form are called "Artin cones." "Artin fans" are log algebraic stacks which admit a strict étale cover by Artin cones. The 2-category of Artin fans is equivalent to a category of "cone stacks" [Cavalieri et al. 2020, Theorem 6.11].

- The present paper concerns analogues of normal cones and pullbacks in the logarithmic category. We use the notation $\ulcorner$, $\times$, C for pullbacks and normal cones of ordinary stacks, and write $\ulcorner\ell$, $\times^\ell$, C^ℓ to distinguish the fs pullbacks and log normal cones. When they happen to coincide, we write $\not\ell$, $\ulcorner\not\ell$, $\times^{\not\ell}$, $C^{\not\ell}$ to emphasize this coincidence.
- Many of our citations could be made to original sources, often written by K. Kato, but we have opted for the book [Ogus 2018]. We have doubled references to Costello's formula [Costello 2006, Theorem 5.0.1; Herr and Wise 2022] where appropriate because the original formulation is incorrect.

1. Preliminaries and the log normal sheaf

The present paper originated with one central construction, which we learned from [Ito et al. 2020, Lemma 2.3.12].

Construction 1.1. The normal cone of a morphism $f : B \to A$ of finite type is constructed by choosing a factorization $B \to B[x_1, \dots, x_r] \twoheadrightarrow A$ inducing a closed immersion into affine r-space

$$\operatorname{Spec} A \hookrightarrow \mathbb{A}^r_B \to \operatorname{Spec} B.$$

The normal cone of f may then be expressed as the quotient of the ordinary normal cone of the closed immersion by the action of the tangent bundle of $\mathbb{A}^r_B \to \operatorname{Spec} B$.

Let $P \to A$ and $Q \to B$ be morphisms from fs monoids to the multiplicative monoids of rings ("prelog rings"). A commutative square

$$\begin{array}{ccc} B & \xrightarrow{f} & A \\ \uparrow & & \uparrow \\ Q & \xrightarrow{\theta} & P \end{array}$$

is a chart of a map between affine log schemes. Assume f is of finite type; θ automatically is by the fs assumption. We will obtain a factorization of the induced log schemes into a *strict closed immersion* followed by a *log smooth map*.

Start with a similar factorization

$$\begin{array}{ccccc} B & \longrightarrow & B[x_1,\ldots,x_r,y_1,\ldots,y_s] & \twoheadrightarrow & A \\ \uparrow & & \uparrow & & \uparrow \\ Q & \longrightarrow & Q_s & \twoheadrightarrow & P \end{array}$$

with $Q_s = Q \oplus \mathbb{N}^s$ mapping to $B[x_1,\ldots x_r, y_1,\ldots y_s]$ by sending the generators of $\mathbb{N}^s$ to the algebra generators $y_1,\ldots y_s$ and $Q_s \to P$ surjective. Define Q_s^θ via the cartesian product:

$$\begin{array}{ccccc} Q_s & \hookrightarrow & Q_s^\theta & \twoheadrightarrow & P \\ \downarrow & & \downarrow & \ulcorner & \downarrow \\ Q_s^{gp} & = & Q_s^{gp} & \longrightarrow & P^{gp} \end{array}$$

By definition, $Q_s^\theta \to P$ is exact, and $Q_s \to Q_s^\theta$ is a "log modification:" an isomorphism on groupifications. Witness also that $Q_s^\theta \to P$ is surjective, so the characteristic monoid map $\overline{Q_s^\theta} \xrightarrow{\sim} \overline{P}$ is an isomorphism [Ogus 2018, Proposition I.4.2.1(5)] and $\mathscr{A}_P \to \mathscr{A}_{Q_s^\theta}$ is strict. Take Spec and Artin cones of monoids to obtain a diagram with strict vertical arrows:

$$\begin{array}{ccccccc} X & \hookrightarrow & X_\theta & \longrightarrow & \mathbb{A}_Y^{r+s} & \longrightarrow & Y \\ \downarrow & & \downarrow & \ulcorner\ell & \downarrow & & \downarrow \\ \mathscr{A}_P & \longrightarrow & \mathscr{A}_{Q_s^\theta} & \longrightarrow & \mathscr{A}_{Q_s} & \longrightarrow & \mathscr{A}_Q \end{array}$$

We've written $Y = \operatorname{Spec} B$, $X = \operatorname{Spec} A$ and introduced the fs pullback X_θ in the diagram. The top row expresses our original map $\operatorname{Spec} f$ as the composition of a strict closed immersion, a log modification, and a smooth and log smooth morphism. The log modification $\mathscr{A}_{Q_s^\theta} \to \mathscr{A}_{Q_s}$ and hence $X_\theta \to \mathbb{A}_Y^{r+s}$ may be expressed as a (strict) open immersion into a log blowup as in [Ogus 2018, Lemma II.1.8.2, Remark II.1.8.5]. Hence $X \subseteq X_\theta$ is a strict closed immersion and $X_\theta \to Y$ is log smooth.

Remark 1.2. Continue in the notation of Construction 1.1. If we began with a morphism of fs log rings with f and θ both surjective, we could omit $Q_s \to B[x_1,\ldots,x_r,y_1,\ldots y_s]$. In that case, we obtain a factorization

$$X \subseteq X_\theta \to Y$$

where $X_\theta \to Y$ is not only log smooth but log étale.

As in [Behrend and Fantechi 1997], we will present the log normal cone locally as $C^\ell_{X/Y} = [C_{X/X_\theta}/T^\ell_{X_\theta/Y}]$ using these factorizations. The difficulty is then piecing together the local descriptions and checking compatibility. In this sense, the heavy lifting has already been done for us by [Manolache 2012]. We spend the rest of this section collecting relevant properties of the log normal sheaf $N^{\ell}_{X/Y}$. When we define the log normal cone $C^{\ell}_{X/Y} \subseteq N^{\ell}_{X/Y}$, its important properties will be locally deduced from such factorizations.

Remark 1.3. An algebraic stack X is DM if and only if the map $X \to \operatorname{Spec} k$ to the base field is of DM-type. If $X \to Y$ is a morphism of DM type and Y admits a stratification by global quotients, then so does X [Manolache 2012, Remark 3.2]. A morphism $f : X \to Y$ of algebraic stacks is of DM type if and only if its diagonal $\Delta_{X/Y} : X \to X \times_Y X$ is unramified [Stacks 2005–, 06N3].

Lemma 1.4. *Let $f : X \to Y$ be a morphism of log algebraic stacks. If the map on underlying stacks is of DM-type, then the induced maps $\mathcal{L}X \to \mathcal{L}Y$ and $X \to \mathcal{L}Y$ are DM-type.*

Proof. The inclusion $X \subseteq \mathcal{L}X$ representing strict maps is open, so it suffices to show that $\mathcal{L}X \to \mathcal{L}Y$ is DM-type.

We will argue that the diagonal of $\mathcal{L}X \to \mathcal{L}Y$ is unramified [Stacks 2005–, 04YW]. The isomorphism $\mathcal{L}X \times_{\mathcal{L}Y} \mathcal{L}X \simeq \mathcal{L}(X \times^{\ell}_{Y} X)$ identifies the diagonal $\Delta_{\mathcal{L}X/\mathcal{L}Y}$ with the result of $\mathcal{L}$ applied to the fs diagonal

$$\Delta^{\ell}_{X/Y} : X \to X \times^{\ell}_{Y} X.$$

Any diagram

$$\begin{array}{ccc} S_0 & \longrightarrow & \mathcal{L}X \\ \big\downarrow & \nearrow\!\!\nearrow & \big\downarrow {\scriptstyle \mathcal{L}\Delta^{\ell}_{X/Y}} \\ S_0' & \longrightarrow & \mathcal{L}(X \times^{\ell}_{Y} X) \end{array}$$

with $S_0 \subseteq S_0'$ a squarezero closed immersion of schemes is equivalent to a diagram

$$\begin{array}{ccc} S & \longrightarrow & X \\ \big\downarrow & \nearrow\!\!\nearrow & \big\downarrow {\scriptstyle \Delta^{\ell}_{X/Y}} \\ S' & \longrightarrow & X \times^{\ell}_{Y} X \end{array}$$

with $S \subseteq S'$ an exact closed immersion of log schemes. Composing with the fsification map $X \times^{\ell}_{Y} X \to X \times_Y X$ sends this square to

$$\begin{array}{ccc} S & \longrightarrow & X \\ \big\downarrow & \nearrow\!\!\nearrow & \big\downarrow {\scriptstyle \Delta^{\ell}_{X/Y}} \\ S' & \longrightarrow & X \times_{Y} X \end{array}$$

in which case the two dashed arrows have the same underlying scheme map because $X \to X \times_Y X$ is unramified by hypothesis. Then the maps on log structure must be the same as well, because

$$(M_X \oplus^{\ell}_{M_Y} M_X)|_{S'} \to (M_X)|_{S'}$$

is an epimorphism. $\square$

Recall the normal sheaf. If $X \subseteq Y$ is a closed embedding with ideal I, the normal sheaf is simply $N_{X/Y} = \operatorname{Spec}\operatorname{Sym}^* I/I^2$. If $X \to Y$ is smooth, then $N_{X/Y} = BT_{X/Y}$ is the classifying space for the tangent bundle. Behrend and Fantechi obtain a normal sheaf more generally by locally factoring $X \to Y$ into a closed immersion into affine space $X \subseteq \mathbb{A}^n_Y \to Y$. If $X \subseteq \mathbb{A}^n_Y$ can be chosen to be a regular closed embedding, the map $X \to Y$ is l.c.i. and the cotangent complex $\mathbb{L}_{X/Y}$ is perfect of amplitude in $[-1, 0]$.

Definition 1.5 (normal sheaf). Let $f : X \to Y$ be a DM type qcqs morphism of algebraic stacks with cotangent complex $\mathbb{L}_{X/Y}$ given by the system of truncations $\{\tau_{\geq -n}\pi^*\mathbb{L}_{X/Y}\}_n$ [Olsson 2007, Theorem 8.1]. The *normal sheaf* is the associated Picard stack [SGA 4_1 1972, XVIII.1.4]

$$h^1/h^0((\mathbb{L}_{X/Y,fl})^\vee)$$

as in [Behrend and Fantechi 1997, Section 2]. An *obstruction theory* for f is a fully faithful functor $N_{X/Y} \subseteq E$ into a vector bundle stack E over X.

Remark 1.6. The Picard stack h^1/h^0 only depends on the truncation $\tau_{\geq -1}$, so we don't need the entire system $\{\tau_{\geq -n}\pi^*\mathbb{L}_{X/Y}\}_n$. Moreover, we can bypass π and pull back directly to the big fppf site

$$X_{fl} \to X_{lis-\text{ét}} \xrightarrow{\pi} X_{\text{ét}}.$$

Factor the map f locally as $X \subseteq M \to Y$ as a closed immersion composed with a smooth map $M \to Y$. Writing $I = I_{X/M}$ for the ideal sheaf, we obtain a map

$$\mathbb{L}_{X/Y} \to [I/I^2 \to \Omega_{M/Y}]$$

which induces isomorphisms on the first two cohomology groups h^{-1}, h^0. This identifies their Picard stacks

$$N_{X/Y} \simeq [N_{X/M}/T_{M/Y}|_X].$$

The functor of points of the normal sheaf on a X-scheme S is given by the category of algebra extensions

$$N_{X/Y}(S) = \underline{\operatorname{Ext}}(\mathbb{L}_{X/Y}|_S, \mathcal{O}_S) = \left\{ \begin{array}{c} \mathcal{O}_Y|_S \searrow \\ \downarrow \\ 0 \to \mathcal{O}_S \longrightarrow \mathcal{A} \longrightarrow \mathcal{O}_X|_S \to 0 \end{array} \quad \begin{array}{c}\text{a squarezero algebra} \\ \text{extension on ét}(T)\end{array} \right\}.$$

Definition 1.7 (log normal sheaf). Let $f : X \to Y$ be a DM type qcqs morphism between log algebraic stacks. Define the log normal sheaf $N^\ell_{X/Y} := N_{X/\mathcal{L}Y} = h^1/h^0(\mathbb{L}^\ell_{X/Y,fl}{}^\vee)$.

We will also have cause to consider $N_{\mathcal{L}X/\mathcal{L}Y}$, and $N_{\mathcal{L}X/\mathcal{L}Y}|_X = N^\ell_{X/Y}$.

Remark 1.8. Locally in X and Y, f factors as $X \subseteq M \to Y$ with $X \subseteq M$ a *strict* closed immersion and $M \to Y$ log smooth by Construction 1.1. We have a similar presentation

$$N^\ell_{X/Y} = [N_{X/M}/T^\ell_{M/Y}|_X].$$

One may alternately define $N^\ell_{X/Y}$ by gluing together these local presentations as in [Behrend and Fantechi 1997, Corollary 3.9]. One checks for $Y \to Z$ étale that the map on normal sheaves $N^\ell_{X/Y} \simeq N^\ell_{X/Z}$ is an isomorphism.

One may alternately use Gabber's notion of log cotangent complex $\mathbb{L}^G_{X/Y}$ because the two agree on truncations $\tau_{\geq -2}\mathbb{L}^\ell_{X/Y} \simeq \tau_{\geq -2}\mathbb{L}^\ell_{X/Y}$ [Olsson 2005, Theorem 8.27]. Gabber's version has the advantage that a distinguished triangle exists for *all* composable pairs of arrows $X \to Y \to Z$.

We claim the functor of points of $N^\ell_{X/Y}$ on an X-scheme S is given by squarezero extensions of algebras

$$\begin{array}{ccccccccc} & & & & \mathcal{O}_Y|_S & & & & \\ & & & & \downarrow & \searrow & & & \\ 0 & \longrightarrow & \mathcal{O}_S & \longrightarrow & \mathcal{A} & \longrightarrow & \mathcal{O}_X|_S & \longrightarrow & 0 \end{array}$$

together with a "log structure" $M_{\mathcal{A}} \to \mathcal{A}$ making

$$\begin{array}{ccc} \mathcal{A}^* & \longrightarrow & \mathcal{O}^*_X|_S \\ \downarrow & & \downarrow \\ M_{\mathcal{A}} & \longrightarrow & M_X|_S \end{array}$$

a pushout. We avoid this perspective because it requires squarezero extensions and log structures on étale-locally ringed topoi (ét(T), $\mathcal{O}_X|_T$), (ét(T), $\mathcal{A}$). The reader may notice a resemblance to "deformations of log structures" in [Illusie 1997] and to the classical notion of squarezero extensions along a map $X \to Y$ reprised in [Olsson 2005, Section 5].

Remark 1.9. The central object of this paper is a subcone stack $C^\ell_{X/Y} \subseteq N^\ell_{X/Y}$ introduced in the next section. This substack has no functor of points, as it is defined by a blowup; see [Khan and Rydh 2018] for a derived workaround. Most of our arguments about $C^\ell_{X/Y}$ go by way of the functor of points of $N^\ell_{X/Y}$, together with a pointwise argument to compare these substacks.

Functoriality of $N^\ell_{X/Y}$. To write down the functoriality of the log normal sheaf, we need to recall some of the machinery of log stacks found in [Olsson 2005].

We denote $\mathcal{L}^i := \mathcal{L}^{[i]}$, the stack of i-simplices of fs log structures. The j-th face map d_j sends

$$(M_0 \to M_1 \to \cdots \to M_{i+1}) \mapsto \begin{cases} (M_1 \to M_2 \to \cdots \to M_{i+1}) & \text{if } j = 0, \\ (M_0 \to \cdots \to M_{j-1} \to M_{j+1} \cdots \to M_{i+1}) & \text{if } j \neq 0, i+1, \\ (M_0 \to \cdots \to M_i) & \text{if } j = i+1. \end{cases}$$

We write $s, t : \mathcal{L}^1 \to \mathcal{L}^0 = \mathcal{L}$ for the "source" d_1 and "target" d_0 maps, respectively. We have an isomorphism $\mathcal{L}^i = \mathcal{L}^1 \times_{t,\mathcal{L},s} \mathcal{L}^1 \times_{t,\mathcal{L},s} \cdots \times_{t,\mathcal{L},s} \mathcal{L}^1$ (i factors).

Endow $\mathcal{L}^i$ with the final tautological log structure, M_{i+1} in the above. All the face maps d_j are strict except $j = i+1$.

We continue [Olsson 2005] to use "$\square$" to denote the category with these objects, arrows, and relations:

$$\begin{array}{ccc} 0 & \longrightarrow & 1 \\ \downarrow & \searrow & \downarrow \\ 2 & \longrightarrow & 3 \end{array}$$

We adopt pictorial mnemonics for fully faithful morphisms of these finite diagrams: $\boxbslash$ means the functor $[2] \subseteq \square$ avoiding 2, etc.

A commutative square

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

should lead to a map $N^{\ell}_{X'/Y'} \to N^{\ell}_{X/Y}$. This square does *not* induce a commutative diagram

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & \times & \downarrow \\ \mathcal{L}Y' & \longrightarrow & \mathcal{L}Y \end{array}$$

so the naive strategy to get a map $N^{\ell}_{X'/Y'} \to N^{\ell}_{X/Y}$ doesn't work. To get around this, Olsson introduces another stack.

Definition 1.10 (compare [Olsson 2005, Lemma 3.12]). Define $\mathcal{V} := \mathcal{L}^1 \times^{\ell}_{t,\mathcal{L},t} \mathcal{L}^1$. Given a scheme T, the points of this stack are cocartesian squares of *fs* log structures:

$$\mathcal{V}(T) := \left\{ \begin{array}{ccc} M_0 & \longrightarrow & M_1 \\ \downarrow & & \downarrow \\ M_2 & \overset{\ell \lrcorner}{\longrightarrow} & M_3 \end{array} \right\}$$

This is the "fsification" of the ordinary pullback $\mathcal{L}^1 \times_{t,\mathcal{L},t} \mathcal{L}^1$, endowed with the non-fs pushout $M_1 \oplus^{mon}_{M_0} M_2$ of the universal log structures.

The natural embedding $\mathcal{V} \to \mathcal{L}^{\square}$ exhibits the squares which are cocartesian as an open substack, as we'll record in Lemma 1.12.

For a morphism $q : Y' \to Y$ of log algebraic stacks, we obtain relative variants

$$\mathcal{V}_q := \mathcal{V} \times_{\urcorner,\mathcal{L}^1} Y', \quad \mathcal{L}^{\square}_q := \mathcal{L}^{\square} \times_{\urcorner,\mathcal{L}^1} Y'.$$

The fs pullback here agrees with the ordinary one because $Y' \to \mathcal{L}^1$ is strict. The points of these stacks over some scheme T are squares

$$\begin{array}{ccc} M_Y|_T & \longrightarrow & M_{Y'}|_T \\ \downarrow & & \downarrow \\ M_0 & \longrightarrow & M_1 \end{array}$$

with those of $\mathcal{V}_q$ required to be cocartesian.

Lemma 1.11. *Let $\mathcal{L}^{arbfine}$ denote the stack of log structures which are fine but not necessarily saturated. The natural monomorphism*

$$\mathcal{L} \hookrightarrow \mathcal{L}^{arbfine}$$

is an open immersion.

Proof. Consider some scheme X and pullback diagram:

$$\begin{array}{ccc} X^{fs} & \longrightarrow & \mathcal{L} \\ \downarrow & \ulcorner & \downarrow \\ X & \longrightarrow & \mathcal{L}^{arbfine} \end{array}$$

Then $X^{fs} \hookrightarrow X$ is a monomorphism, the locus where the stalks of M_X are saturated. After passing to an open cover of X, [Ogus 2018, Theorem II.2.5.4] provides us with a locally finite stratification $X = \bigsqcup_{\sigma \in \Sigma} X_\sigma$ where:

- For each $\sigma \in \Sigma$, $\overline{M}_X|_\sigma$ is constant.
- The cospecialization maps for $x \in \overline{\{\xi\}} \subseteq X$

$$\overline{M}_x \to \overline{M}_\xi$$

are localizations at faces.

The localization of a saturated monoid remains saturated [Ogus 2018, Remark I.1.4.5] and a monoid is saturated if and only if its characteristic monoid is [loc. cit., Proposition I.1.3.5]. We then have that $X^{fs} \subseteq X$ is locally a constructible subset which is closed under generalization, and hence open [Stacks 2005–, Tag 0542]. □

We collect several results of [Olsson 2005] adapted to the fs setting.

Lemma 1.12 [Olsson 2005, Theorem 2.4, Proposition 2.11, Lemma 3.12]. *These statements remain true in the **fs** context*:

(1) *For any finite category Γ, the fibered category $\mathcal{L}^\Gamma$ of diagrams of fs log structures indexed by Γ is an algebraic stack.*

(2) *The simplicial face maps $d_j : \mathcal{L}^{i+1} \to \mathcal{L}^i$ are strict, étale, and DM-type for $j \leq i$.*

(3) *If $[1] \to \square$ avoids the initial object 0 ( or ), it induces a strict étale, DM-type morphism*

$$\mathcal{L}^\square \to \mathcal{L}^1.$$

(4) *If $[2] \to \square$ omits either 1 or 2 ( or ), it induces an étale, DM-type morphism*

$$\mathcal{L}^\square \to \mathcal{L}^2.$$

(5) *The map $\mathcal{V} \subseteq \mathcal{L}^\square$ is an open embedding.*

(6) *Given an fs pullback square*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & \ulcorner \ell & \downarrow \\ Y' & \xrightarrow{q} & Y \end{array}$$

the associated square of stacks

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & \ulcorner & \downarrow \\ \mathcal{V}_q & \longrightarrow & \mathcal{L}Y \end{array}$$

is a pullback.

Proof. Facts (1) through (4) are immediate by Lemma 1.11 and the analogous facts in [Olsson 2005]. The last two follow by the same arguments applied in the fs category. □

Remark 1.13. Apply $\mathcal{L}$ once more to the map $\mathcal{L}Y \to Y$, one gets

$$d_1 : \mathcal{L}^2 Y \to \mathcal{L}Y, \quad (M_Y \to M_0 \to M_1) \mapsto (M_Y \to M_1).$$

The result is étale, so the original $d_1 : \mathcal{L}Y \to Y$ is log étale [Olsson 2003, Theorem 4.6(ii)]. The same reasoning implies $d_{i+1} : \mathcal{L}^{i+1} Y \to \mathcal{L}^i Y$ is log étale in general. In summary, all the face maps are log étale and all but $j = i + 1$ are furthermore strict étale.

Remark 1.14. Given $q : Y' \to Y$ DM, the natural maps

$$\mathcal{V}_q \subseteq \mathcal{L}_q^{\square} \to \mathcal{L}Y'$$

are étale. The second map is the product of the étale map

$$\boxbslash^* : \mathcal{L}^{\square} \to \mathcal{L}^2$$

over $\mathcal{L}^1$ (via $\sqcap$) with Y'.

Definition 1.15. Use Lemma 1.12(6) to turn one commutative square of DM maps into another:

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y' & \xrightarrow{q} & Y. \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathcal{L}_q^{\square} & \longrightarrow & \mathcal{L}Y \end{array}$$

Maps of normal sheaves

$$\varphi : N^{\ell}_{X'/Y'} \simeq N_{X/\mathcal{L}_q^{\square}} \to N^{\ell}_{X/Y}$$

arise from Remark 1.14 and the second square. This coincides with the "natural map"

$$\mathbb{L}^{\ell}_{X/Y}|_{X'} \to \mathbb{L}^{\ell}_{X'/Y'}$$

of [Olsson 2005, (1.1.2)]. We call the composite φ *Olsson's morphism*.

Remark 1.16. In Definition 1.15, if the first square was an fs pullback square, the second factors:

$$\begin{array}{ccccc} X' & & \longrightarrow & & X \\ \downarrow & \ulcorner & & & \downarrow \\ \mathcal{V}_q & \hookrightarrow & \mathcal{L}_q^{\square} & \longrightarrow & \mathcal{L}Y \end{array}$$

Since this square is a pullback, Olsson's morphism

$$\varphi : N^{\ell}_{X'/Y'} \simeq N_{X'/\mathcal{L}_q^{\square}} \simeq N_{X'/\mathcal{V}_q} \hookrightarrow N^{\ell}_{X/Y}|_{X'}$$

is then a closed immersion. This may be checked locally in X, Y, so we assume there is a factorization $X \subseteq X_\theta \to Y$ as in Construction 1.1. Take the fs pullback

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & \ulcorner\ell & \downarrow \\ X'_\theta & \longrightarrow & X_\theta \\ \downarrow & \ulcorner\ell & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

along q to obtain a factorization $X' \subseteq X'_\theta \to Y'$. The strict case gives $N_{X'/X'_\theta} \subseteq N_{X/X_\theta}|_{X'}$ and pullback identifies the tangent spaces $T^{\ell}_{X'_\theta/Y'} \simeq T^{\ell}_{X_\theta/Y}|_{X'_\theta}$.

If $X \to Y$ is also log flat, φ is an isomorphism [Olsson 2005, (1.1(iv))]. This is *not* true if q is log flat, as seen in Remark 2.13. See Lemmas 2.14, 2.15 for the strict case.

Remark 1.17. A commutative square of DM maps may be factored:

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow f \\ Y' & \xrightarrow{q} & Y \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathcal{L}_q^{\square} & \longrightarrow & \mathcal{L}Y \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y \end{array} \tag{1}$$

This induces a commutative square of normal sheaves:

$$\begin{array}{ccc} N^{\ell}_{X'/Y'} \simeq N_{X'/\mathcal{L}_q^{\square}} & \longrightarrow & N^{\ell}_{X/Y} \\ \downarrow & \circ & \downarrow \\ N_{X'/Y'} & \longrightarrow & N_{X/Y} \end{array} \tag{2}$$

The Olsson morphisms are thereby seen to be compatible with the ordinary functoriality of the normal sheaf via the forgetful maps $N^{\ell}_{X/Y} \to N_{X/Y}$.

Now suppose the original square (1) is an fs pullback:

- If q is strict, then (2) is cartesian. This is local in X, Y, so we assume we have a factorization as in Construction 1.1

$$X \subseteq X_\theta \to \mathbb{A}^{r+s}_Y \to Y$$

and take the *ordinary scheme-theoretic* pull back to obtain a similar factorization of $X' \to Y'$ using strictness of q. Since $X \subseteq \mathbb{A}^{r+s}_Y$ is a closed immersion, the statement reduces to the cartesian square of ordinary normal sheaves

$$\begin{array}{ccc} N_{X'/X'_\theta} & \longrightarrow & N_{X/X_\theta} \\ \downarrow & \ulcorner & \downarrow \\ N_{X'/\mathbb{A}^{r+s}_{Y'}} & \longrightarrow & N_{X/\mathbb{A}^{r+s}_Y} \end{array}$$

which may be checked using the functor of points, for example.

- If instead f is strict, then $X' \to \mathcal{V}_q$ factors through Y', and the factorization

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & \ulcorner & \downarrow \\ Y' & \longrightarrow & Y \\ \downarrow & \ulcorner & \hookdownarrow \\ \mathcal{V}_q & \longrightarrow & \mathcal{L}Y \end{array}$$

shows that the vertical arrows of (2) are isomorphisms and the Olsson morphism is the same as the ordinary functoriality of the normal sheaf.

Remark 1.18. Given a commutative square

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y' & \xrightarrow{q} & Y \end{array}$$

of DM maps we can form two other commutative squares out of it:

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathcal{L}^\square_q & \longrightarrow & \mathcal{L}Y \end{array} \qquad \begin{array}{ccc} X' & \longrightarrow & \mathcal{L}X \\ \downarrow & & \downarrow \\ \mathcal{L}Y' & \longrightarrow & \mathcal{L}Y \end{array}$$

They induce morphisms

$$N^\ell_{X'/Y'} \simeq N^\ell_{X'/\mathcal{L}^\square_q} \to N^\ell_{X/Y}|_{X'} \quad \text{and} \quad N^\ell_{X'/Y'} \to N_{\mathcal{L}X/\mathcal{L}Y}|_{X'}.$$

Form the diagram

$$\begin{array}{ccccc} X' & \longrightarrow & \mathcal{L}X & \xrightarrow{s} & X \\ \downarrow & & \downarrow & \ulcorner & \downarrow \\ \mathcal{L}^{\square}_q & \longrightarrow & \mathcal{L}^2 Y & \xrightarrow{d_0} & \mathcal{L}Y \\ \downarrow & & & & \\ \mathcal{L}Y' & & & & \end{array}$$

to see that the two morphisms of normal sheaves are compatible:

$$N^{\ell}_{X'/Y'} \simeq N_{X'/\mathcal{L}^{\square}_q} \to N_{\mathcal{L}X/\mathcal{L}^2 Y}|_{X'} \subseteq N^{\ell}_{X/Y}|_{X'}.$$

L. Barrott pointed out to the author that "Condition (T)" as phrased in [Olsson 2005, (1.5.1)] ensures étale locally that the square

$$\begin{array}{ccc} \mathcal{L}X & \longrightarrow & X \\ \downarrow & \ulcorner & \downarrow \\ \mathcal{L}^2 Y & \longrightarrow & \mathcal{L}Y \end{array}$$

above is Tor-independent.

Lemma 1.19. *Suppose given a pair of commutative squares*

$$\begin{array}{ccccc} X' & \longrightarrow & Y' & \longrightarrow & Z' \\ \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \end{array}$$

of DM-type maps. The diagram

$$\begin{array}{ccccc} & & N^{\ell}_{Y'/Y} & & \\ & \nearrow & & \searrow & \\ N^{\ell}_{X'/X} & & \longrightarrow & & N^{\ell}_{Z'/Z} \end{array}$$

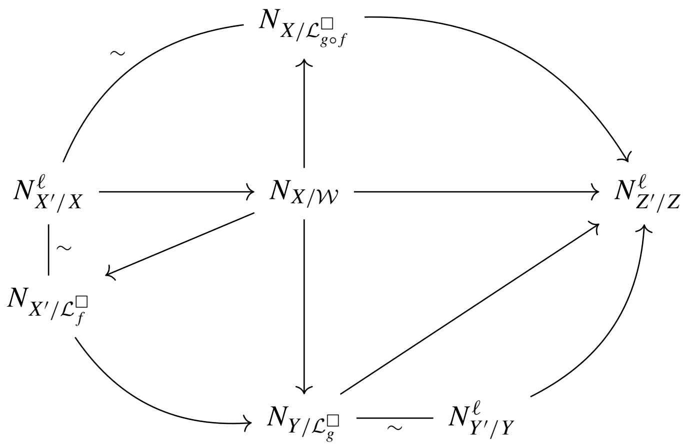
commutes, where all the arrows are Olsson's morphisms.

Proof. Introduce an algebraic X-stack $\mathcal{W}$, with functor of points:

$$\left\{ \begin{array}{ccc} & & \mathcal{W} \\ & \nearrow & \downarrow \\ T & \longrightarrow & X \end{array} \right\} := \left\{ \begin{array}{ccccc} M_2 & \longleftarrow & M_1 & \longleftarrow & M_0 \\ \uparrow & & \uparrow & & \uparrow \\ M_X|_T & \longleftarrow & M_Y|_T & \longleftarrow & M_Z|_T \end{array} \quad \begin{array}{c} \text{commutative diagrams of} \\ \text{fs log structures on } T \end{array} \right\}$$

In other words, $\mathcal{W} := (\mathcal{L}^{\square} \times_{\mathcal{L}^1} \mathcal{L}^{\square}) \times_{\mathcal{L}^2} X$.

All the triangles in this diagram commute by inspection:



Restricting the diagram to $N^{\ell}_{X'/X}$, $N^{\ell}_{Y'/Y}$, and $N^{\ell}_{Z'/Z}$, we get the result. $\square$

Proposition 1.20. *Given $X \xrightarrow{f} Y \xrightarrow{g} Z$ DM-type maps of log algebraic stacks, the Olsson morphisms yield a complex of stacks*

$$N^{\ell}_{X/Y} \to N^{\ell}_{X/Z} \to N^{\ell}_{Y/Z}|_X,$$

in that the composite factors through the vertex.

If h is smooth, $N^{\ell}_{Y/Z} = BT^{\ell}_{Y/Z}$ and rotating the triangle in the derived category yields an exact sequence of cone stacks:

$$T^{\ell}_{Y/Z}|_X \to N^{\ell}_{X/Y} \to N^{\ell}_{X/Z}.$$

Proof. The Olsson morphisms come about from the commutative diagram:

$$\begin{array}{ccccccccc} X & = & X & \longrightarrow & Y & & N^{\ell}_{X/Y} & \longrightarrow & N^{\ell}_{Y/Y}|_X = X \\ \downarrow & & \downarrow & & \downarrow & \rightsquigarrow & \downarrow & & \downarrow{\scriptstyle 0} \\ Y & \longrightarrow & Z & = & Z & & N^{\ell}_{X/Z} & \longrightarrow & N^{\ell}_{Y/Z}|_X \end{array}$$

Use Gabber's log cotangent complex as in Remark 1.8 and rotate to get a distinguished triangle

$$\mathbb{L}^G_{X/Z} \to \mathbb{L}^G_{X/Y} \to \mathbb{L}^G_{Y/Z}|_X[1] \xrightarrow{+1}.$$

Then $\mathbb{L}^{\ell}_{Y/Z} = \Omega^{\ell}_{Y/Z}[0]$ and $h^1/h^0(\mathbb{L}^G_{Y/Z}[1])|_X = h^1/h^0(\mathbb{L}^{\ell}_{Y/Z}[1])|_X = T^{\ell}_{Y/Z}|_X$. $\square$

Remark 1.21. Suppose given a (not necessarily commutative) finite diagram of cones. If the diagram induced by taking abelian hulls is commutative, so was the original.

2. Properties of the log normal cone

We are ready to define the log normal cone. We recall the essential properties of the ordinary normal cone; the rest of the section establishes analogous properties in the log context.

Remark 2.1. Consider a DM-type morphism $f : X \to Y$ of algebraic stacks. K. Behrend and B. Fantechi defined the (intrinsic) normal cone [Behrend and Fantechi 1997, Definition 3.10]

$$C_f = C_{X/Y} \subseteq N_{X/Y};$$

C. Manolache [2012, Definition 2.30] removed their assumptions of smooth Y and DM X. This cone has the following basic properties:

(1) A commutative diagram

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \big\downarrow & & \big\downarrow{\scriptstyle f} \\ Y' & \xrightarrow{q} & Y \end{array}$$

yields a morphism of cones $\varphi : C_{X'/Y'} \to C_{X/Y} \times_X X'$:

- If the square was cartesian, φ is a closed embedding.
- If the square was cartesian and also f or q was flat, φ is an isomorphism.

(2) For a composite

$$X \xrightarrow{f} Y \xrightarrow{g} Z:$$

- If g is l.c.i., $C_{X/Y} = N_{X/Y}$ and the sequence

$$N_{X/Y} \to C_{X/Z} \to C_{Y/Z}|_X$$

of cone stacks is exact.
- If h is smooth, the sequence

$$T_{Y/Z}|_X \to C_{X/Y} \to C_{X/Z}$$

is exact.

(3) Obstruction theories and Gysin pullbacks are obtained by placing the cone in a vector bundle stack $C_{X/Y} \subseteq E$ via an isomorphism $A_*E \simeq A_*X$ called "intersecting with the zero section;" see [Manolache 2012, Section 3; Wise 2011, Proposition 3.6; Kresch 1999, Section 6.2].

Definition 2.2 (log intrinsic normal cone, Olsson morphisms). Let $f : X \to Y$ be a DM-type morphism of log algebraic stacks. We define the *log* (*intrinsic*) *normal cone*

$$C^\ell_{X/Y} := C_{X/\mathcal{L}Y} \subseteq N^\ell_{X/Y}$$

after [Gross and Siebert 2013]. Endow it with the log structure pulled back from X. Given a commutative square of log algebraic stacks and its partner:

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y' & \xrightarrow{q} & Y \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathcal{L}_q^{\square} & \longrightarrow & \mathcal{L}Y \end{array}$$

The latter induces

$$\varphi : C^{\ell}_{X'/Y'} \simeq C_{X'/\mathcal{L}_q^{\square}} \to C^{\ell}_{X/Y}.$$

This is again called the *Olsson morphism*.

Remark 2.3. The map $\mathcal{L}Y \to Y$ has a section $Y \subseteq \mathcal{L}Y$ which is an open immersion. This open immersion represents strict log maps to Y.

As a result, if $X \to Y$ is DM and strict, $C^{\ell}_{X/Y} = C_{X/Y}$ and $N^{\ell}_{X/Y} = N_{X/Y}$. In addition, the Olsson morphisms are the same as the ordinary functoriality of the normal cone (Remarks 1.21 and 1.17).

The Olsson morphism of any fs pullback square is a closed immersion, because it fits into a commutative square of closed immersions from Remark 1.16:

$$\begin{array}{ccc} C^{\ell}_{X'/Y'} & \longrightarrow & C^{\ell}_{X/Y}|_{X'} \\ \big\downarrow & & \big\downarrow \\ N^{\ell}_{X'/Y'} & \hookrightarrow & N^{\ell}_{X/Y}|_{X'} \end{array}$$

Remark 2.4 (short exact sequences of cone stacks). Recall [Behrend and Fantechi 1997, Definition 1.12]. Let E be a vector bundle stack and C, D cone stacks all on some base algebraic stack X. A composable pair of morphisms of cone stacks

$$E \to C \to D$$

is called a *short exact sequence* if:

- $C \to D$ is a smooth epimorphism.
- The square

$$\begin{array}{ccc} E \times C & \xrightarrow{pr_2} & C \\ \downarrow{\scriptstyle \sigma} & & \downarrow \\ C & \longrightarrow & D \end{array}$$

  where pr_2 is the projection and σ the action, is cartesian.

These are equivalent to having $C \simeq E \times_X D$ locally in X.

Note that this definition is *fpqc*-local in the base X [Stacks 2005–, 02VL]. Another reduction we will need applies in case there is a commutative diagram of cone stacks

$$\begin{array}{ccccc} E & \longrightarrow & C & \longrightarrow & D \\ \downarrow & & \downarrow s & & \downarrow t \\ E' & \longrightarrow & C' & \longrightarrow & D' \end{array}$$

with E, E' vector bundles. If the top sequence is exact and the arrows labeled s, t are smooth and surjective, then the bottom is exact. To see this, push out along $E \to E'$ so as to assume $E = E'$ (s, t remain smooth and surjective). The diagram on the left is the pullback along the smooth surjection $D' \to D$ of the one on the right:

$$\begin{array}{ccc} E \times C' & \longrightarrow & C' \\ \downarrow & \ulcorner & \downarrow \\ C' & \longrightarrow & D' \end{array} \qquad \begin{array}{ccc} E \times C & \longrightarrow & C \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

We can verify that $E \times C$ is the pullback after smooth-localizing.

Proposition 2.5. *Suppose* $X \xrightarrow{f} Y \xrightarrow{g} Z$ *are DM maps between log algebraic stacks, and* g *is log smooth. Then*

$$T^{\ell}_{Y/Z}|_X \to C^{\ell}_{X/Y} \to C^{\ell}_{X/Z}$$

is an exact sequence of cone stacks.

Proof. Encode the log structures on the maps via the top row of the diagram:

$$\begin{array}{ccccccc} X & \longrightarrow & \mathcal{L}Y & \longrightarrow & \mathcal{L}^2 Z & \longrightarrow & \mathcal{L}^2 \\ & \searrow & \downarrow & \ulcorner & \downarrow & \ulcorner & \downarrow \\ & & Y & \longrightarrow & \mathcal{L}Z & \longrightarrow & \mathcal{L}^1 \end{array}$$

Since $Y \to \mathcal{L}Z$ is smooth, $\mathcal{L}Y \to \mathcal{L}^2 Z$ is. Moreover, they have the same tangent bundle

$$T^{\ell}_{Y/Z}|_{\mathcal{L}Y} = T_{Y/\mathcal{L}Z}|_{\mathcal{L}Y} = T_{\mathcal{L}Y/\mathcal{L}^2 Z}$$

since the vertical maps are log étale [Ogus 2018, Corollary IV.3.2.4].

Together with the isomorphism $C^{\ell}_{X/Z} \simeq C_{X/\mathcal{L}^2 Z}$, we obtain the exact sequence. ☐

Remark 2.6. In the proof, the composite

$$C^{\ell}_{X/Y} \to C_{X/\mathcal{L}^2 Z} \simeq C^{\ell}_{X/Z}$$

is precisely the Olsson morphism. This is immediate from the diagram:

$$\begin{array}{ccc} X & = & X \\ \downarrow & & \downarrow \\ \mathcal{L}_q^{\square} & \longrightarrow & \mathcal{L}^2 Z \\ \downarrow & \nearrow & \downarrow \\ \mathcal{L}Y & & \mathcal{L}Z \end{array}$$

Remark 2.7. The introduction promised three characterizations of $C^{\ell}_{X/Y}$.

The log intrinsic normal cone is characterized by the strict case of Remark 2.3 and the log étale case of Proposition 2.5. This is because any map $X \to Y$ factors into the strict map $X \to \mathcal{L}Y$ composed with the log étale map $\mathcal{L}Y \to Y$ (Remark 1.13).

We can unpack this definition locally using charts. Suppose a morphism has a global fs chart by Artin cones:

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \mathscr{A}_P & \longrightarrow & \mathscr{A}_Q \end{array}$$

The morphism $\mathscr{A}_P \to \mathscr{A}_Q$ is log étale [Olsson 2003, Corollary 5.23]. Let $W = \mathscr{A}_P \times^{\ell}_{\mathscr{A}_Q} Y$ denote the fs pullback, so that $X \to Y$ factors through a strict map to W and W is log étale over Y. We immediately get

$$C^{\ell}_{X/Y} = C_{X/W}.$$

The reader may be reassured by working locally with this definition. If the reader wants instead to work with charts $\mathrm{Spec}(P \to \mathbb{C}[P])$ in the traditional sense, then log étaleness is no longer immediate and we must check Kato's criteria [Ogus 2018, Corollary IV.3.1.10].

Recall Construction 1.1 — after localizing in the étale topology, we obtain a factorization of any map $X \to Y$ as a *strict* closed immersion followed by a log smooth map

$$X \subseteq X_{\theta} \to Y.$$

Proposition 2.5 therefore locally provides a presentation of the log normal cone:

$$C^{\ell}_{X/Y} = [C_{X/X_{\theta}} / T^{\ell}_{X_{\theta}/Y}].$$

Remark 2.8. We want to work with quasicompact, quasiseparated stacks, but $\mathcal{L}$ is not quasicompact. It *is* quasiseparated in the sense of [Stacks 2005–, 04YW], but [Olsson 2003, Remark 3.17] points out it is not quasiseparated in the sense of [Laumon and Moret-Bailly 2000]; see [Chou et al. 2020, Remark 1.1]. A map $X \to Y$ between algebraic stacks with X quasicompact factors through a quasicompact open substack $U \subseteq Y$.

This ensures that any DM map $X \to Y$ of log stacks with X quasicompact and X, Y quasiseparated factors through $X \to U \to Y$ with $X \to U$ strict, U quasicompact and quasiseparated, and $U \to Y$ log étale.

Example 2.9. We provide an example of Construction 1.1 and Remark 1.2.

Consider the diagonal morphism $\mathbb{A}^1 \xrightarrow{\Delta} \mathbb{A}^2$. The addition map $\mathbb{N}^2 \xrightarrow{+} \mathbb{N}$ gives a chart for Δ.

Denote by B the log blowup of $\mathbb{A}^2$ at the ideal $I \subseteq M_{\mathbb{A}^2}$ generated by $\mathbb{N}^2 \setminus \{0\} \subseteq \mathbb{N}^2$. The pullback $\Delta^* I$ is generated by the image of the composite

$$\mathbb{N}^2 \setminus \{0\} \subseteq \mathbb{N}^2 \xrightarrow{+} \mathbb{N}.$$

The pullback is generated globally by a single element and so Δ factors through the log blowup B.

Name the generators $\mathbb{N}^2 = \mathbb{N}e \oplus \mathbb{N}f$. The log blowup B is covered by two affine opens $D_+(e)$ and $D_+(f)$, on which e and f are invertible.

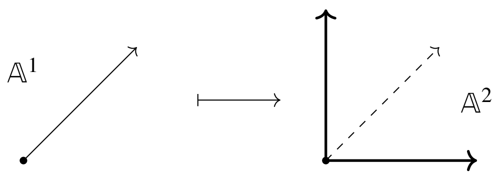
On the chart $D_+(e)$, the morphism $\mathbb{A}^1 \to B$ looks like:

$$\begin{array}{ccc} \mathbb{N} & \longleftarrow & \mathbb{N}e \oplus \mathbb{N}(f-e) \\ \downarrow & & \downarrow \\ \mathbb{C}[t] & \longleftarrow & \mathbb{C}\left[x, \frac{y}{x}\right] \end{array}$$

The horizontal morphisms send $f - e \mapsto 0$ and $\frac{y}{x} \mapsto 1$. Because $(f-e)$ maps to $1 \in \mathbb{C}[t]$, the composite

$$\mathbb{N}e \oplus \mathbb{N}(f-e) \to \mathbb{N} \to \mathbb{C}[t]$$

is another chart for the same log structure on $\mathbb{A}^1$. This means that $\mathbb{A}^1 \to D_+(e)$ is strict. The same discussion applies to $D_+(f)$. In the tropical picture [Cavalieri et al. 2020, Section 2], we subdivided $\mathbb{A}^2$ at the image of the ray corresponding to $\mathbb{A}^1$:



Proposition 2.10. *Consider DM-type morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ between log algebraic stacks. If $C^{\ell}_{X/Y} = N^{\ell}_{X/Y}$, then*

$$N^{\ell}_{X/Y} \to C^{\ell}_{X/Z} \to C_{\mathcal{L}Y/\mathcal{L}Z}|_X$$

is an exact sequence of cone stacks.

Proof. Compare [Behrend and Fantechi 1997, Proposition 3.14].

By Proposition 1.20 and Remark 1.18, this sequence composes to zero. Remark 2.4 allows us to repeatedly *fpqc*-localize in X to check exactness of such a sequence. Localizing along strict smooth covers of Z and strict étale covers of X and Y ensures that the normal cones and sheaf pull back. Reduce to the case where X, Y, and Z are affine log schemes and the map $Y \to Z$ admits a global fs chart. We are therefore in the situation of Construction 1.1.

Reduction to $g : Y \to Z$ strict: Factor $Y \to Z$ into a strict closed immersion composed with a log smooth map

$$Y \subseteq W \twoheadrightarrow Z.$$

We obtain a diagram:

$$\begin{array}{ccccc}
 & & T^{\ell}_{W/Z}|_X & = & T_{\mathcal{L}W/\mathcal{L}Z}|_X \\
 & & \downarrow & & \downarrow \\
N_{X/Y} & \longrightarrow & C^{\ell}_{X/W} & \longrightarrow & C_{\mathcal{L}Y/\mathcal{L}W}|_X \\
\| & & \downarrow & \ulcorner & \downarrow \\
N_{X/Y} & \longrightarrow & C^{\ell}_{X/Z} & \longrightarrow & C_{\mathcal{L}Y/\mathcal{L}Z}|_X
\end{array}$$

Observe that the diagram commutes — the morphism $T^{\ell}_{W/Z}|_X \to C^{\ell}_{X/W}$ in the proof of Proposition 2.5 factors through an identification $T^{\ell}_{W/Z}|_{\mathcal{L}W} \simeq T^{\ell}_{\mathcal{L}W/\mathcal{L}^2 Z}$. Because $\mathcal{L}W \to W$ is log étale, the two tangent spaces are isomorphic [Ogus 2018, IV.3.2.4]. Thus the right square is a pullback. The vertical maps of cones are smooth surjections, so it suffices to show the middle row is exact as in Remark 2.4. We may thereby assume $W = Z$ and $g : Y \to Z$ is a strict closed immersion.

Reduction to $f : X \to Y$ strict: Use Construction 1.1 again to factor $X \to Z$ as a strict closed immersion composed with a log smooth map $X \subseteq W \twoheadrightarrow Z$. The map $X \to W' := W \times_Z Y$ is again a strict closed immersion:

$$\begin{array}{ccccc}
X & \hookrightarrow & W' & \hookrightarrow & W \\
 & \searrow & \downarrow & \ulcorner \ell & \downarrow \\
 & & Y & \hookrightarrow & Z
\end{array} \tag{3}$$

Because the top row is strict, $X \to \mathcal{L}W'$ factors through the open subset $W' \subseteq \mathcal{L}W'$ and

$$C_{\mathcal{L}W'/\mathcal{L}W}|_X = C_{\mathcal{L}W'/\mathcal{L}W}|_{W'}|_X = C^{\ell}_{W'/W}|_X = C_{W'/W}|_X.$$

The fs pullback square in (3) also induces a cartesian square of stacks

$$\begin{array}{ccc}
\mathcal{L}W' & \longrightarrow & \mathcal{L}W \\
\downarrow & \ulcorner & \downarrow \\
\mathcal{L}Y & \longrightarrow & \mathcal{L}Z
\end{array}$$

with $\mathcal{L}W \to \mathcal{L}Z$ smooth. This reveals that

$$C_{\mathcal{L}Y/\mathcal{L}Z}|_{\mathcal{L}W'} = C_{\mathcal{L}W'/\mathcal{L}W}.$$

Putting this together with the above, we have computed

$$C_{\mathcal{L}Y/\mathcal{L}Z}|_X = C_{W'/W}|_X.$$

The factorization (3) gives a diagram:

$$\begin{array}{ccccc} T^{\ell}_{W'/Y}|_X & = & T^{\ell}_{W/Z}|_X & & \\ \downarrow & & \downarrow & & \\ N_{X/W'} & \longrightarrow & C^{\ell}_{X/W} & \longrightarrow & C_{W'/W}|_X \\ \downarrow & & \downarrow & & \| \\ N_{X/Y} & \longrightarrow & C^{\ell}_{X/Z} & \longrightarrow & C_{\mathcal{L}Y/\mathcal{L}Z}|_X \end{array}$$

The composable vertical arrows are the quotients of Proposition 2.5, so the bottom row will be exact if we show the middle row is. The middle row is exact by a relative form of the original [Behrend and Fantechi 1997, Proposition 3.14]. □

Remark 2.11. The exact sequences of cone stacks in Propositions 2.5, 2.10 are natural in morphisms of composable pairs of arrows.

There is a version of Proposition 2.10 for log cotangent complexes that we will use once later on. From any composable pair $X \to Y \to Z$, we get $X \to \mathcal{L}Y \to \mathcal{L}Z$ and $X \to \mathcal{L}Y \to \mathcal{L}^2 Z$. Both result in the same distinguished triangle

$$\mathbb{L}_{\mathcal{L}Y/\mathcal{L}Z}|_X \to \mathbb{L}^{\ell}_{X/Z} \to \mathbb{L}^{\ell}_{X/Y} \to$$

of [Olsson 2007, 8.10].

In the next example, the log normal cone differs from the ordinary scheme-theoretic one.

Example 2.12. In Example 2.9, we considered the log blowup B of $\mathbb{A}^2$ at the origin and the diagonal map. Pull back along the diagonal to get the identity log blowup of $\mathbb{A}^1$:

$$\begin{array}{ccc} \mathbb{A}^1 & \longrightarrow & B \\ \| & \ulcorner\ell & \downarrow \\ \mathbb{A}^1 & \longrightarrow & \mathbb{A}^2 \end{array}$$

Let $\bar{o}_{\mathbb{N}}, \bar{o}_{\mathbb{N}^2}$ both be $\operatorname{Spec} \mathbb{C}$, with log structures coming from $\mathbb{N}$ and $\mathbb{N}^2$, respectively. Then the inclusions of the origins $\bar{o}_{\mathbb{N}} \in \mathbb{A}^1$ and $\bar{o}_{\mathbb{N}^2} \in \mathbb{A}^2$ are strict.

Take the pullback of the above diagram along the inclusion $\bar{o}_{\mathbb{N}^2} \in \mathbb{A}^2$:

$$\begin{array}{ccc} \bar{o}_{\mathbb{N}} & \longrightarrow & D \\ \| & \ulcorner\ell & \downarrow \\ \bar{o}_{\mathbb{N}} & \longrightarrow & \bar{o}_{\mathbb{N}^2} \end{array}$$

The map $D \to \bar{o}_{\mathbb{N}^2}$ is the exceptional divisor of B, which is $\mathbb{P}^1$ with log structure $\overline{M}_x = \mathbb{N}^2$ at the intersections with the axes and $\overline{M}_x = \mathbb{N}$ elsewhere.

To see the log normal cone differ from the ordinary one, compute the normal cones of the arrows in this square: $C^\ell_{\bar{o}_{\mathbb{N}}/\bar{o}_{\mathbb{N}}} = \bar{o}$, $C^\ell_{\bar{o}_{\mathbb{N}}/\bar{o}_{\mathbb{N}^2}} = C^\ell_{\bar{o}_{\mathbb{N}}/D} = \mathbb{A}^1$, and $C^\ell_{D/\bar{o}_{\mathbb{N}^2}} = \mathbb{P}^1$. Although $\bar{o}_{\mathbb{N}}$ and $\bar{o}_{\mathbb{N}^2}$ have the same underlying scheme, the log normal cones of $\bar{o}_{\mathbb{N}}$ over them are different.

Remark 2.13. A handy consequence of Proposition 2.10 is that, if $Y \to Z$ is a DM-type morphism between log algebraic stacks and $Y' \to Y$ is a *strict* étale map, then

$$C^\ell_{Y'/Z} \simeq C^\ell_{Y/Z}|_{Y'}.$$

This is *not* true without the strictness assumption. This is the observation of W. Bauer precluding the existence of a log cotangent complex with all its desiderata; see [Olsson 2005, Section 7].

In general, it need only be a closed immersion. This is because

$$C^\ell_{Y'/Z} \simeq C_{\mathcal{L}Y/\mathcal{L}Z}|_{Y'} \subseteq N_{\mathcal{L}Y/\mathcal{L}Z}|_{Y'} \subseteq N^\ell_{Y/Z}|_{Y'}$$

is a closed immersion which factors through $C^\ell_{Y/Z}|_{Y'}$, as in Remark 1.18.

For a single example, take the log blowup $B \to \mathbb{A}^2$ of the origin $\bar{o} \in \mathbb{A}^2$. The pullback defines a strict pullback square:

$$\begin{array}{ccc} D & \longrightarrow & B \\ \downarrow & \ulcorner\ell & \downarrow \\ \bar{o} & \longrightarrow & \mathbb{A}^2 \end{array}$$

Because the horizontal morphisms are strict, their log normal cones coincide with the ordinary ones. Log blowups are log étale, so we would erroneously be led to conclude that

$$C_{D/B} \overset{?}{=} C_{\bar{o}/\mathbb{A}^2}|_D.$$

The inclusion $D \subseteq B$ is regular, and so is $\bar{o} \in \mathbb{A}^2$, so the normal cones and normal sheaves agree:

$$N_{D/B} = \mathcal{O}_B(D)|_D \quad \text{and} \quad N_{\bar{o}/\mathbb{A}^2}|_D = \mathbb{A}^2_D.$$

The dimensions are different, so they can't be equal.

Lemma 2.14. *Suppose given a strict pullback square*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & \ulcorner\ell & \downarrow \\ Y' & \xrightarrow{q} & Y \end{array}$$

of DM-type morphisms between log algebraic stacks for which q is strict and smooth. Then the Olsson morphism

$$C^\ell_{X'/Y'} \xrightarrow{\sim} C^\ell_{X/Y}|_{X'}$$

is an isomorphism.

Proof. We first note that the Olsson morphism $N^{\ell}_{X'/Y'} \to N^{\ell}_{X/Y}|_{X'}$ on log normal sheaves is an isomorphism. This is clear from the q strict pullback part of Remark 1.17 and the fact that the ordinary normal sheaves are isomorphic.

Now we know that the morphism of cones $C^{\ell}_{X'/Y'} \to C^{\ell}_{X/Y}|_{X'}$ is a closed immersion, and it suffices to show that it is moreover smooth and surjective. We express this map as a composite

$$C^{\ell}_{X'/Y'} \to C^{\ell}_{X'/Y} \to C^{\ell}_{X/Y}|_{X'}.$$

Proposition 2.5 asserts that the first map is smooth and surjective and Proposition 2.10 says the same for the second. $\square$

Lemma 2.15. *Suppose given a pair of fs pullback squares*

$$\begin{array}{ccc} \widetilde{X}' & \longrightarrow & \widetilde{X} \\ \big\downarrow & \ulcorner\ell & \big\downarrow z \\ X' & \longrightarrow & X \\ \big\downarrow & \ulcorner\ell & \big\downarrow \\ Y' & \longrightarrow & Y \end{array}$$

of DM-type morphisms between log algebraic stacks for which z is strict and smooth. Then the diagram of log normal cones

$$\begin{array}{ccc} C^{\ell}_{\widetilde{X}'/Y'} & \longrightarrow & C^{\ell}_{\widetilde{X}/Y} \\ \big\downarrow s' & \ulcorner & \big\downarrow s \\ C^{\ell}_{X'/Y'} & \longrightarrow & C^{\ell}_{X/Y} \end{array}$$

is cartesian and the arrows s, s' are smooth epimorphisms.

Proof. Proposition 2.10 provides a map of short exact sequences of cone stacks:

$$\begin{array}{ccccc} BT^{\ell}_{\widetilde{X}'/X'} & \longrightarrow & C^{\ell}_{\widetilde{X}'/Y'} & \xrightarrow{t'} & C^{\ell}_{X'/Y'}|_{\widetilde{X}'} \\ \| & & \big\downarrow & \ulcorner & \big\downarrow \\ BT^{\ell}_{\widetilde{X}/X}|_{\widetilde{X}'} & \longrightarrow & C^{\ell}_{\widetilde{X}/Y}|_{\widetilde{X}'} & \xrightarrow{\tilde{t}} & C^{\ell}_{X/Y}|_{\widetilde{X}'} \end{array}$$

Witness that the right square is cartesian because [Olsson 2005]

$$T^{\ell}_{\widetilde{X}'/X'} = T^{\ell}_{\widetilde{X}/X}|_{\widetilde{X}'}$$

and that the arrows $t', \tilde{t}$ are clearly smooth epimorphisms. The arrow $\tilde{t}$ is pulled back from the smooth epimorphism $t : C^{\ell}_{\widetilde{X}/Y} \to C^{\ell}_{X/Y}|_{\widetilde{X}}$, so we have the top pullback square:

$$\begin{tikzcd} C^{\ell}_{\widetilde{X}'/Y'} \arrow[r] \arrow[d, "t'"] \arrow[dd, bend right=60, "s'"'] \arrow[dr, phantom, "\ulcorner"] & C^{\ell}_{\widetilde{X}/Y} \arrow[d, "t"] \arrow[dd, bend left=60, "s"] & \\ C^{\ell}_{X'/Y'}|_{\widetilde{X}'} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & C^{\ell}_{X/Y}|_{\widetilde{X}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \widetilde{X} \arrow[d] \\ C^{\ell}_{X'/Y'} \arrow[r] & C^{\ell}_{X/Y} \arrow[r] & X \end{tikzcd}$$

The composite vertical rectangle of cones is the diagram we are after, and so the fact that this square is cartesian is clear. It remains only to note the bent arrows s, s' are smooth epimorphisms because they are the composites of t, t' with pullbacks of the smooth epimorphism $\widetilde{X} \to X$. □

3. Log intersection theory

We develop a log intersection theory package using log cotangent complexes and log normal cones in place of the ordinary ones, closely following [Manolache 2012, Sections 3 and 4].

Definition 3.1 (log perfect obstruction theory). Define a *log perfect obstruction theory* (hereafter "Log POT") for a DM-type morphism $f : X \to Y$ to be a closed immersion of cone stacks

$$C^{\ell}_{X/Y} \subseteq E \quad (\text{equiv. } N^{\ell}_{X/Y} \subseteq E)$$

of the log normal cone into a vector bundle stack E.

Given an fs pullback square

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"'] \arrow[dr, phantom, "\ulcorner \ell"] & X \arrow[d, "f"] \\ Y' \arrow[r] & Y \end{tikzcd}$$

and a Log POT $C^{\ell}_{X/Y} \subseteq E$ for f, the Olsson morphism

$$C^{\ell}_{X'/Y'} \overset{\varphi}{\hookrightarrow} C^{\ell}_{X/Y}|_{X'} \subseteq E|_{X'}$$

defines a *pullback* Log POT.

A related notion of *pullback* Log POT arises when $X' \to X$ is log étale and $f : X \to Y$ any DM-type map. Then Remark 2.13 shows the map

$$C^{\ell}_{X'/Y} \to C^{\ell}_{X/Y}|_{X'}$$

is a closed immersion, and we can compose with an obstruction theory for f to get one for the composite $X' \to X \to Y$.

Given a Log POT $C^\ell_{X/Y} \subseteq E$ for some f, suppose X has a stratification by global quotient stacks and Y is log smooth and equidimensional. Then [Kresch 1999, Proposition 5.3.2] gives us a unique cycle

$$[X, E]^{\ell vir} \in A_* X$$

which pulls back to the class $[C^\ell_{X/Y}] \in A_* E$. This class is called the *log virtual fundamental class* (hereafter "Log VFC").

Remark 3.2. When $\mathcal{L}Y$ is equidimensional, so is $C^\ell_{X/Y}$. The correct definition of the Log VFC requires that the cone be equidimensional. If Y is log smooth, $Y \subseteq \mathcal{L}Y$ is dense. If Y is also equidimensional, we get that $\mathcal{L}Y$ is. This explains our assumptions in Definition 3.1. We don't include these assumptions in the definition of a Log POT only because we may have log Gysin maps more generally.

Nonequidimensional log stacks arise naturally elsewhere in log Gromov–Witten theory. For example, the stacks of punctured log curves $\breve{\mathfrak{M}}$ and punctured maps $\breve{\mathfrak{M}}(\mathscr{X}/B)$ to an Artin fan $\mathscr{X}$ are not equidimensional. They are "idealized log smooth" over the base [Abramovich et al. 2020, Proposition 3.3, Theorem 3.24], which locally entails a composite of a log smooth map and a closed embedding coming from a monoidal ideal.

Definition 3.3 (log Gysin map). Suppose a DM-type $f : X \to Y$ has a Log POT $C^\ell_{X/Y} \subseteq E$. Given a DM-type log map $k : V \to Y$ with V log smooth and equidimensional, form the fs pullback:

$$\begin{array}{ccc} W & \longrightarrow & V \\ \downarrow & \ulcorner \ell & \downarrow k \\ X & \xrightarrow{f} & Y \end{array}$$

The embedding

$$C^\ell_{W/V} \subseteq C^\ell_{X/Y}|_W \subseteq E|_W$$

results in a class

$$[C^\ell_{W/V}, E] \in A_* W.$$

Mimicking [Manolache 2012], we call this "map"

$$f^! = f^!_E$$

the *log Gysin map*.

Remark 3.4. Consider a DM-type morphism $f : X \to Y$ of log algebraic stacks. The cartesian square

$$\begin{array}{ccc} \mathcal{L}X & \xrightarrow{s} & X \\ \downarrow & \ulcorner & \downarrow \\ \mathcal{L}^2 Y & \xrightarrow{d_0} & \mathcal{L}Y \end{array}$$

from Remark 1.18 results in a closed embedding

$$C_{\mathcal{L}X/\mathcal{L}Y} \simeq C_{\mathcal{L}X/\mathcal{L}^2 Y} \subseteq C^\ell_{X/Y}|_{\mathcal{L}X}$$

which we use to canonically extend an obstruction theory $C^{\ell}_{X/Y} \subseteq E$ to a closed embedding

$$C_{\mathcal{L}X/\mathcal{L}Y} \subseteq E|_{\mathcal{L}X}.$$

Now suppose given a composable pair $X \xrightarrow{f} Y \xrightarrow{g} Z$ as above and equip f, g with Log POT's

$$C^{\ell}_{X/Y} \subseteq F, \quad C^{\ell}_{Y/Z} \subseteq G.$$

Define a *compatibility datum* or *compatible triple* for such a pair to be a traditional compatibility datum [Manolache 2012, Definition 4.5] for

$$X \xrightarrow{f} \mathcal{L}Y \xrightarrow{g} \mathcal{L}^2 Z,$$

endowing $\mathcal{L}Y \to \mathcal{L}^2 Z$ with the extended obstruction theory

$$C_{\mathcal{L}Y/\mathcal{L}^2 Z} \subseteq C^{\ell}_{Y/Z}|_{\mathcal{L}Y} \subseteq G|_{\mathcal{L}Y}.$$

This entails a commutative diagram in the derived category of X

$$\begin{array}{ccccccc} \mathcal{G}|_X & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{F} & \xrightarrow{+1} & \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathbb{L}_{\mathcal{L}Y/\mathcal{L}^2 Z}|_X & \longrightarrow & \mathbb{L}^{\ell}_{X/Z} & \longrightarrow & \mathbb{L}^{\ell}_{X/Y} & \xrightarrow{+1} & \end{array}$$

where

- the rows are distinguished triangles,
- the objects $\mathcal{E}, \mathcal{F}, \mathcal{G}$ are of perfect amplitude in $[-1, 0]$, inducing vector bundle stacks E, F, G,
- the vertical arrows are isomorphisms on h^0 and epimorphisms on h^{-1}, inducing obstruction theories $C^{\ell}_{X/Z} \subseteq E$, $C^{\ell}_{X/Y} \subseteq F$, $C_{\mathcal{L}Y/\mathcal{L}^2 Z} \subseteq G|_{\mathcal{L}Y}$.

We offer a couple of basic remarks about our definitions before the examples and theorems.

Remark 3.5. The map $f^!$ just defined takes in log smooth equidimensional stacks DM over Y and produces classes in certain Chow Groups. The operations $f^!$ are refined to log Chow in [Barrott 2018; Herr et al. 2023].

Remark 3.6. Given an fs pullback square

$$\begin{array}{ccc} X' & \longrightarrow & X \\ {\scriptstyle f'}\downarrow & \ulcorner \ell & \downarrow{\scriptstyle f} \\ Y' & \longrightarrow & Y \end{array}$$

of DM maps where f has a Log POT: $C^{\ell}_{X/Y} \subseteq E$, endow f' with the pullback Log POT. Then

$$f^! = f'^!$$

when applied to log smooth, equidimensional log schemes over Y'.

Remark 3.7. If $C^\ell_{X/Y} = N^\ell_{X/Y}$ for a DM morphism $f : X \to Y$, then $N^\ell_{X/Y}$ is a vector bundle stack. One locally factors $X \subseteq X_\theta \to Y$ as in Construction 1.1 and recognizes $C_{X/X_\theta} \simeq N_{X/X_\theta}$, implying $X \subseteq X_\theta$ is regular and N_{X/X_θ} is a vector bundle. We can then take $E = N^\ell_{X/Y}$ as an obstruction theory. If X, Y are equidimensional and Y is log smooth, unwinding definitions shows

$$f^!(Y) = [X],$$

where $[X]$ is the fundamental class of X.

Remark 3.8. Log Gysin maps don't commute with pushforward. Let

$$\begin{array}{ccc} X' & \xrightarrow{p} & X \\ {\scriptstyle f'}\downarrow & \ulcorner\ell & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{q} & Y \end{array}$$

be an fs pullback square. Endow $f : X \to Y$ with a Log POT $C^\ell_{X/Y} \subseteq E$ and give f' the pullback obstruction theory. Then the usual equality [Manolache 2012, Theorem 4.1(i)] can fail:

$$f^! q_* \neq p_* f'^!.$$

Take the square of Example 2.12

$$\begin{array}{ccc} \bar{o}_{\mathbb{N}} & \longrightarrow & D \\ \| & \ulcorner\ell & \downarrow \\ \bar{o}_{\mathbb{N}} & \longrightarrow & \bar{o}_{\mathbb{N}^2} \end{array}$$

and apply both operations to $[\bar{o}_{\mathbb{N}}]$ for a counterexample.

This arises in [Holmes et al. 2019] because pushing forward along various blowups fails to preserves intersection products of DR cycles. This phenomenon was also observed in [Ranganathan 2019].

Remark 3.9. Virtual fundamental classes don't push forward along log blowups: Let $X \to F$ be the morphism from a stack X to its Artin fan (the reader may take a traditional chart instead of F). Choose a finite subdivision $\widehat{F} \to F$, and form the fs pullback:

$$\begin{array}{ccc} \widehat{X} & \longrightarrow & \widehat{F} \\ {\scriptstyle p}\downarrow & \ulcorner\ell & \downarrow \\ X & \longrightarrow & F \end{array}$$

Suppose given a map $f : X \to Y$ with a Log POT $C^\ell_{X/Y} \subseteq E$ and equip $f \circ p : \widehat{X} \to Y$ with the pullback obstruction theory

$$C^\ell_{\widehat{X}/Y} \subseteq C^\ell_{X/Y}|_{\widehat{X}} \subseteq E|_{\widehat{X}}.$$

Then possibly

$$p_*[\widehat{X}, E]^{\ell vir} \neq [X, E]^{\ell vir}.$$

A counterexample is again given by $p : D \to \bar{o}_{\mathbb{N}^2}$, $f : \bar{o}_{\mathbb{N}^2} = \bar{o}_{\mathbb{N}^2}$ as in Example 2.12: $p_*[\mathbb{P}^1] = 0$ for dimension reasons.

This doesn't contradict the main result of [Abramovich and Wise 2018]. We offer a version of their statement in Theorem 3.10.

The rest of this section and the next should reassure the disheartened reader that commonsense formulas of ordinary intersection theory do remain true in the log setting. We regard Remarks 3.8, 3.9 as defects of the usual notion of pushforward p_* in the log setting. The morphisms $\bar{o}_{\mathbb{N}} \to D$, $D \to \bar{o}_{\mathbb{N}^2}$ of Example 2.12 are monomorphisms in the fs category, and $\bar{o}_{\mathbb{N}} \to \bar{o}_{\mathbb{N}^2}$ should be a cycle of *dimension one* in the "two dimensional" log point $\bar{o}_{\mathbb{N}^2}$.

Barrott [2018] introduced log chow groups to correct this defect, in particular via suitable notions of dimension and degree. It also contains compatibility statements between the log notion of p_* and the Gysin map $f^!$ introduced here. See also [Mochizuki 2015].

For now, we content ourselves to use the observation of [Nizioł 2006, Proposition 4.3] that log blowups are birational if the target is log smooth. We will use it to prove that weaker forms of the naïve guesses of Remarks 3.8, 3.9 do hold true, as well as straightforward commutativity of the Gysin maps.

We will need to use Costello's notion [2006, before Theorem 5.0.1] of "pure degree d" to make sense of pushforward on the level of cycles, given by cones embedded in vector bundles. The next theorem allows us to check statements about Log VFC's after a log blowup if the target is log smooth. Its statement and proof are similar to [Abramovich and Wise 2018].

Theorem 3.10. *Suppose given a DM-type map $f : X \to Y$ between locally noetherian algebraic stacks locally of finite type over $\mathbb{C}$ where Y is log smooth and equidimensional. Endow f with a Log POT E and let $X \to F$ be any DM morphism to an Artin Fan. Take the fs pullback along a proper birational map of Artin fans*:

$$\begin{array}{ccc} \widehat{X} & \longrightarrow & \widehat{F} \\ {\scriptstyle p}\downarrow & \ulcorner \ell & \downarrow \\ X & \longrightarrow & F \end{array} \tag{4}$$

For example, $\widehat{F} \to F$ could be a subdivision or a root stack.

Endow $f \circ p$ with the pullback Log POT

$$C^{\ell}_{\widehat{X}/Y} \subseteq C^{\ell}_{X/Y}|_{\widehat{X}} \subseteq E|_{\widehat{X}}.$$

Then

$$p_*[\widehat{X}, E]^{\ell vir} = [X, E]^{\ell vir}$$

Proof. We will actually show that the map

$$t : C^{\ell}_{\widehat{X}/Y} \to C^{\ell}_{X/Y}$$

is of pure degree one. Then the pushforward $A_*E|_{\widehat{X}} \to A_*E$ sends the class of one cone to the other, and "intersecting with the zero section" gives the equality of VFC's.

We will reduce to the case where $X \to F$ is strict. The statement "t is of pure degree one" may be verified étale-locally in X, as we now argue.

Given a strict étale cover $X' \to X$, write $\widehat{X'} := \widehat{X} \times_X X'$. We have a pullback diagram

$$\begin{array}{ccccc} C^{\ell}_{\widehat{X'}/F} & \xrightarrow{t'} & C^{\ell}_{X'/F} & \longrightarrow & X' \\ \downarrow & \ulcorner & \downarrow & \ulcorner & \downarrow \\ C^{\ell}_{\widehat{X}/F} & \xrightarrow{t} & C^{\ell}_{X/F} & \longrightarrow & X \end{array}$$

as in Remark 2.13. Since $X' \to X$ is étale, the other vertical arrows are as well. The property "pure degree one" is smooth-local in the target, so t has it if t' does.

Now étale-localize in X so that $X \to F$ factors through a chart $X \to F_X \to F$ for X. Take the fs pullback along the subdivision $\widehat{F} \to F$:

$$\begin{array}{ccccc} \widehat{X} & \longrightarrow & \widehat{F}_X & \longrightarrow & \widehat{F} \\ \downarrow & \ulcorner \ell & \downarrow & \ulcorner \ell & \downarrow \\ X & \longrightarrow & F_X & \longrightarrow & F \end{array}$$

We can then replace F by F_X in the proof of the theorem and assume $X \to F$ is strict.

Apply the proof of Costello's formula [2006, Theorem 5.0.1] to (4) to conclude

$$t : C^{\ell}_{\widehat{X}/\widehat{F}} \to C^{\ell}_{X/F}$$

is of pure degree one, since $\widehat{F} \to F$ is birational.

Expanding upon (4):

$$\begin{array}{ccccc} \widehat{X} & \longrightarrow & \widehat{F} \times Y & \longrightarrow & \widehat{F} \\ \downarrow & \ulcorner \ell & \downarrow & \ulcorner \ell & \downarrow \\ X & \longrightarrow & F \times Y & \longrightarrow & F \\ & & \downarrow & & \\ & & Y & & \end{array}$$

We get a map of exact sequences of cone stacks:

$$\begin{array}{ccccc} T^{\ell}_{Y}|_{\widehat{X}} & \longrightarrow & C^{\ell}_{\widehat{X}/\widehat{F}\times Y} & \longrightarrow & C^{\ell}_{\widehat{X}/\widehat{F}} \\ \downarrow & & \downarrow \widehat{t} & \ulcorner & \downarrow t \\ T^{\ell}_{Y}|_{X} & \longrightarrow & C^{\ell}_{X/F\times Y} & \longrightarrow & C^{\ell}_{X/F} \end{array}$$

After pulling the bottom row back to $\widehat{X}$, we get the identity on tangent bundles and see that the right square is a pullback. Since the property"of pure degree one" pulls back along smooth maps, the quotient maps in exact sequences of cone stacks are smooth, and t is pure degree one, $\widehat{t}$ is also pure degree one. Because F, $\widehat{F}$ are log étale over a point, $C^{\ell}_{\widehat{X}/\widehat{F}\times Y} = C^{\ell}_{\widehat{X}/\widehat{Y}}$ and $C^{\ell}_{X/F\times Y} = C^{\ell}_{X/Y}$, so the claim is proven. □

Example 3.11. One must be cautious, for Theorem 3.10 is false without the assumption that Y is log smooth. Recall the exceptional divisor $D \to \bar{o}$ of the blowup of $\mathbb{A}^2$ at the origin $\bar{o} = \operatorname{Spec}\mathbb{C}$ from Example 2.12 and its normal cone $C^{\ell}_{D/\bar{o}} = \mathbb{P}^1$.

For the sake of contradiction, let $\widehat{X} = \mathbb{P}^1$ and $X = Y = \bar{o}$ as in the theorem. Endow $C^{\ell}_{\bar{o}/\bar{o}} = \bar{o}$ with the initial Log POT, $E = \bar{o}$. Then

$$[\widehat{X}, E]^{\ell vir} = [D, E]^{\ell vir} = [\mathbb{P}^1] \quad \text{and} \quad [X, E]^{\ell vir} = [\bar{o}, E]^{\ell vir} = [\bar{o}],$$

but again $p_*[\mathbb{P}^1] = 0$ for dimension reasons.

Theorem 3.12 (commutativity of log Gysin map). *Given a composable pair of DM-type maps between log algebraic stacks*

$$X \xrightarrow{f} Y \xrightarrow{g} Z,$$

outfit f, g, and $g \circ f$ with log obstruction theories F, G, E and a compatibility datum (*Remark 3.4*). *Require X to admit stratifications by global quotients.*

If $k : V \to Z$ is a log smooth and equidimensional Z-stack and k is DM-type, take fs pullbacks:

$$\begin{array}{ccccc} T & \longrightarrow & U & \longrightarrow & V \\ \big\downarrow & \ulcorner\ell & \big\downarrow & \ulcorner\ell & \big\downarrow \\ X & \longrightarrow & Y & \longrightarrow & Z \end{array}$$

Then the equality

$$[C^{\ell}_{g\circ f} \subseteq E] = [C^{\ell}_{C^{\ell}_g|_X/C^{\ell}_g} \subseteq F \oplus G|_X] \tag{5}$$

holds on X.

Proof. Pullback via k all obstruction theories and their compatibility datum to reduce to showing the theorem for $k : V = Z$. We essentially apply [Manolache 2012, Theorem 4.8] to $X \to \mathcal{L}Y \to \mathcal{L}^2Z$, endowed with the compatible triple F, G, E by composing with an isomorphism of distinguished triangles:

$$\begin{array}{ccccc} G|_X & \longrightarrow & F & \longrightarrow & E \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ \mathbb{L}_{\mathcal{L}Y/\mathcal{L}Z}|_X & \longrightarrow & \mathbb{L}^{\ell}_{X/Z} & \longrightarrow & \mathbb{L}^{\ell}_{X/Y} \\ \big\downarrow\wr & & \big\| & & \big\downarrow\wr \\ \mathbb{L}_{\mathcal{L}Y/\mathcal{L}^2Z}|_X & \longrightarrow & \mathbb{L}_{X/\mathcal{L}Z} & \longrightarrow & \mathbb{L}_{X/\mathcal{L}^2Z} \end{array}$$

Use Remark 2.8 repeatedly to obtain a strict diagram with U, V quasicompact and étale over the stacks $\mathcal{L}Y$, $\mathcal{L}^2Z$:

$$\begin{array}{ccccc} X & \longrightarrow & U & \longrightarrow & V \\ & \searrow & \downarrow & & \downarrow \\ & & \mathcal{L}Y & \longrightarrow & \mathcal{L}^2Z \end{array}$$

Endow the cone $C_{\mathcal{L}Y/\mathcal{L}Z}$ with the pullback log structure from $\mathcal{L}Y$ and pull it back along the part of the diagram above $\mathcal{L}Y$:

$$\begin{array}{ccc} C_{\mathcal{L}Y/\mathcal{L}Z}|_X = C_{U/V}|_X & \longrightarrow & C_{U/V} \\ & \searrow & \downarrow \\ & & C_{\mathcal{L}Y/\mathcal{L}Z} \end{array}$$

The triangle is strict and the map $C_{U/V} \to C_{\mathcal{L}Y/\mathcal{L}Z}$ is pulled back from the étale $U \to \mathcal{L}Y$, so

$$C^{\ell}_{C_{\mathcal{L}Y/\mathcal{L}Z}|_X/C_{\mathcal{L}Y/\mathcal{L}Z}} = C_{C_{U/V}|_X/C_{U/V}}.$$

Write $i : X \to U$ $j : U \to V$ for the maps. Then the compatibility datum pulls back and [Manolache 2012, Theorem 4.8] gives us

$$(j \circ i)^!_E([V]) = i^!_F \circ j^!_G([V]).$$

Unwinding definitions, this becomes

$$[C_{X/V} \subseteq E] = [C_{C_{U/V}|_X/C_{U/V}} \subseteq F \oplus G|_X]. \tag{6}$$

This may be rewritten as

$$[C^{\ell}_{X/Z} \subseteq E] = [C^{\ell}_{C_{\mathcal{L}Y/\mathcal{L}Z}|_X/C_{\mathcal{L}Y/\mathcal{L}Z}} \subseteq F \oplus G|_X],$$

the claimed equality of classes. □

Remark 3.13. Theorem 3.12 says that

$$(g \circ f)^! = f^! g^!$$

in the sense that any log smooth, equidimensional log stack over Z has rationally equivalent images under these two operations.

Remark 3.14. Consider an fs pullback of DM-type morphisms between log algebraic stacks:

$$\begin{array}{ccc} X' & \xrightarrow{p} & X \\ {\scriptstyle f'}\downarrow & \ulcorner\ell & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{q} & Y \end{array}$$

Write $r : X' \to Y$ for the composite $f \circ p = q \circ f'$. If f, q are endowed with Log POT's $C^{\ell}_{X/Y} \subseteq F$, $C^{\ell}_{Y'/Y} \subseteq E$, how should we give r a Log POT?

The fs pullback square induces a pullback of stacks, which may be reexpressed as a "magic square:"

$$\begin{array}{ccc} \mathcal{L}X' & \longrightarrow & \mathcal{L}X \\ \downarrow & \ulcorner & \downarrow \\ \mathcal{L}Y' & \longrightarrow & \mathcal{L}Y \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} \mathcal{L}X' & \longrightarrow & \mathcal{L}X \times \mathcal{L}Y' \\ \downarrow & \ulcorner & \downarrow \\ \mathcal{L}Y & \longrightarrow & \mathcal{L}Y \times \mathcal{L}Y. \end{array}$$

The magic square induces a closed immersion

$$C_{\mathcal{L}X'/\mathcal{L}Y} \subseteq C_{\mathcal{L}X/\mathcal{L}Y}|_{\mathcal{L}X'} \times_{\mathcal{L}X'} C_{\mathcal{L}Y'/\mathcal{L}Y}|_{\mathcal{L}X'}$$

which pulls back to a closed immersion

$$C^{\ell}_{X'/Y} \subseteq C_{\mathcal{L}X/\mathcal{L}Y}|_{X'} \times_{X'} C_{\mathcal{L}Y'/\mathcal{L}Y}|_{X'}$$

on X'. As in Remark 3.4, we have closed embeddings $C_{\mathcal{L}X/\mathcal{L}Y} \subseteq C^{\ell}_{X/Y}|_{\mathcal{L}X}$, $C_{\mathcal{L}Y'/\mathcal{L}Y} \subseteq C^{\ell}_{Y'/Y}|_{\mathcal{L}Y'}$. We endow r with the Log POT given by the composite

$$C^{\ell}_{X'/Y} \subseteq C_{\mathcal{L}X/\mathcal{L}Y}|_{X'} \times_{X'} C_{\mathcal{L}Y'/\mathcal{L}Y}|_{X'} \subseteq C^{\ell}_{X/Y}|_{X'} \times_{X'} C^{\ell}_{Y'/Y}|_{X'} \subseteq F|_{X'} \times_{X'} E|_{X'}.$$

We now construct a compatibility datum for the triangle $r = q \circ f'$, leaving the reader to apply the same argument to the other triangle $r = f \circ p$. By the definitions of the Log POT's, we have a commutative diagram:

$$\begin{array}{ccccc} C^{\ell}_{X'/Y'} & \longrightarrow & C^{\ell}_{X'/Y} & \longrightarrow & C_{\mathcal{L}Y'/\mathcal{L}Y}|_{X'} \\ \downarrow & & \downarrow & & \downarrow \\ F|_{X'} & \xrightarrow{(0\times id)} & E|_{X'} \times_{X'} F|_{X'} & \longrightarrow & E|_{X'} \end{array}$$

To be clear, the morphism $F|_{X'} \to E|_{X'} \times_{X'} F|_{X'}$ is the vertex map times the identity. It's clear the bottom row comes from a distinguished triangle in the derived category and the top row comes from Remark 2.11.

Corollary 3.15. *Suppose given an fs pullback square*

$$\begin{array}{ccc} X' & \xrightarrow{p} & X \\ {\scriptstyle f'}\downarrow & \ulcorner\ell & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{q} & Y \end{array}$$

of DM-type morphisms between log algebraic stacks which admit stratifications by quotient stacks. Outfit q with a Log POT E and f with a Log POT F; give p, f' the pullback obstruction theories. Then

$$f'^{!} \circ q^{!} = p^{!} \circ f^{!}$$

in the sense that the operations send any log smooth equidimensional input stack to the same class in A_*X'.

Proof. Denote by $r : X' \to Y$ the map $f \circ p = q \circ f'$. Apply Theorem 3.12 to both commutative triangles using the compatibility datum constructed in Remark 3.14 to see that

$$p^{!} \circ f^{!} = r^{!} = f'^{!} \circ q^{!}.$$

□

4. The log Costello formula

This section proves a log analogue of the Costello formula [2006, Theorem 5.0.1]. The original Costello formula is wrong as stated due to a missing properness hypothesis; this is corrected in [Herr and Wise 2022].

Theorem 4.1. *Consider an fs pullback square of DM-type maps between algebraic stacks*:

$$\begin{array}{ccc} X' & \xrightarrow{p} & X \\ \big\downarrow{\scriptstyle f'} & \ulcorner \ell & \big\downarrow{\scriptstyle f} \\ Y' & \xrightarrow{q} & Y \end{array}$$

Assume

- $Y' \to Y$ *is of some pure degree* $d \in \mathbb{Q}$ *as in* [Herr and Wise 2022, Definition 2.3],
- Y', Y *are both log smooth and equidimensional*,
- *all arrows are DM-type and all stacks are locally noetherian and locally finite type over* $\mathbb{C}$,
- X', X *admit stratifications by global quotient stacks* [Kresch 1999] *and*
- q *is proper.*

Endow f *with a log perfect obstruction theory* E *and give* f' *the pullback obstruction theory. Then*

$$p_*[X', E|_{X'}]^{\ell vir} = d \cdot [X, E]^{\ell vir}$$

in the Chow ring of X.

Remark 4.2. Let $Y' \to Y$ be a map between log smooth, equidimensional stacks which is of pure degree d. Let $W \to Y$ be a smooth, log smooth, integral, and saturated morphism and $\widetilde{W} \to W$ a log blowup. Form the fs pullback diagram:

$$\begin{array}{ccc} \widetilde{W}' & \longrightarrow & \widetilde{W} \\ \big\downarrow & \ulcorner \ell & \big\downarrow \\ W' & \longrightarrow & W \\ \big\downarrow & \ulcorner \ell & \big\downarrow \\ Y' & \longrightarrow & Y \end{array}$$

The property "of pure degree d" pulls back along smooth morphisms, so it applies to $W' \to W$. Then [Niziol 2006, Proposition 4.3] shows that $\widetilde{W} \to W$ is birational, so $\widetilde{W}' \to \widetilde{W}$ is also of pure degree d.

Proof of Theorem 4.1. Consider the morphism

$$s : C^{\ell}_{X'/Y'} \to C^{\ell}_{X/Y}.$$

We will prove that s is of pure degree d. Both "of pure degree" and the specific degree d can be checked after pulling back s along a strict, smooth cover of $C^{\ell}_{X/Y}$. Lemmas 2.14, 2.15 show that replacing Y or X by a smooth cover results in such a smooth cover of cones.

We may thereby assume X and Y are log schemes and the map f globally factors as in Construction 1.1

$$X \to X_\theta \to \mathbb{A}^{r+s}_Y \to Y.$$

Note $\mathbb{A}^{r+s}_Y \to Y$ is smooth, log smooth, integral, and saturated, and $X_\theta \to \mathbb{A}^{r+s}_Y$ is a log blowup. We are in the situation of Remark 4.2, so pulling back

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \big\downarrow & \ulcorner \not\ell & \big\downarrow \\ X'_\theta & \longrightarrow & X_\theta \\ \big\downarrow & \ulcorner \ell & \big\downarrow \\ Y' & \longrightarrow & Y \end{array}$$

results in a map $X'_\theta \to X_\theta$ which is pure of degree d along $X \to X_\theta$. The proof of Costello's formula [2006, Theorem 5.0.1] then asserts that

$$C^{\not\ell}_{X'/X'_\theta} \to C^{\not\ell}_{X/X_\theta}$$

is of pure degree d. The short exact sequences of Proposition 2.5

$$\begin{array}{ccccc} T^{\ell}_{X'_\theta/Y'} & \longrightarrow & C^{\not\ell}_{X'/X'_\theta} & \longrightarrow & C^{\ell}_{X'/Y'} \\ \big\downarrow & & \big\downarrow t & & \big\downarrow s \\ T^{\ell}_{X_\theta/Y} & \longrightarrow & C^{\not\ell}_{X/X_\theta} & \longrightarrow & C^{\ell}_{X/Y} \end{array}$$

let us conclude that s is as well. □

Remark 4.3. The original statement of Costello's formula did not require q to be proper. In fact, one can allow q to be simply "pure" in the sense of [Herr and Wise 2022, Definition 2.3]. Without any such assumption, one has counterexamples:

- Let $Y = \mathbb{A}^1$, $Y' = \mathbb{G}_m$, X the origin, and f, q natural inclusions.
- Let $Y = \mathbb{A}^1$ and Y' be the bug-eyed line, $\mathbb{A}^1$ with doubled origin. Let X again be the origin and f, q the natural maps.

Remark 4.4. We prove a K theoretic version of Costello's formula in degree $d = 1$ and the corresponding Hironaka pushforward theorem in [Chou et al. 2020, Theorem 2.7]. That proof is necessarily global, because K theory is sensitive to higher-codimension phenomena. The same proof can be used in Chow.

5. The product formula

Let V, W be log smooth, quasiprojective schemes throughout this section. We denote the stacks of *prestable curves* and *stable curves* which have n-markings and genus g by $\mathfrak{M}_{g,n}$, $\overline{M}_{g,n}$, respectively

[Stacks 2005–, 0DMG]. They are endowed with divisorial log structures coming from the locus of singular curves [Gross and Siebert 2013, 1.5, Appendix A; Kato 2000].

Definition 5.1 (log stable maps). The stack of log stable maps $\mathscr{M}^{\ell}_{g,n}(V)$ has fiber over an fs log scheme T the category of diagrams of fs log schemes

$$\begin{array}{ccc} C & \longrightarrow & V \\ \downarrow & & \\ T & & \end{array}$$

with $C \to T$ a log smooth curve [Kato 2000, Definition 1.2] of genus g and n marked points, such that the underlying diagram of schemes is a stable map of curves.

Remarkably, the log algebraic stack $\mathscr{M}^{\ell}_{g,n}(\operatorname{Spec}\mathbb{C})$ of log curves without a map is isomorphic to the ordinary stack of stable curves $\overline{M}_{g,n}$ with log structure induced by the boundary of degenerate curves [Kato 2000, Theorem 4.5]. The log structures of $\mathscr{M}^{\ell}_{g,n}(V)$ for a general fs target may be more complicated, as they have to do with the "tropical deformation space" of the curve [Gross and Siebert 2013, Example 1.17(1)].

Construction 5.2 [Gross and Siebert 2013, Section 5]. We recall the construction [loc. cit., Section 5] of the natural Log POT for $\mathscr{M}^{\ell}_{g,n}(V) \to \mathfrak{M}_{g,n}$ to clarify differences in notation.

Write $\mathcal{U} \to \mathfrak{M}_{g,n}$ for the universal curve. Define $\mathcal{U}_V$ as the fs pullback, naturally equipped with a tautological map to V:

$$\begin{array}{ccccc} V & \longleftarrow & \mathcal{U}_V & \xrightarrow{\pi_V} & \mathscr{M}^{\ell}_{g,n}(V) \\ & & \downarrow & \ulcorner\ell & \downarrow \\ & & \mathcal{U} & \longrightarrow & \mathfrak{M}_{g,n} \end{array}$$

This diagram induces maps between log cotangent complexes

$$\mathbb{L}^{\ell}_{V}|_{\mathcal{U}_V} \longrightarrow \mathbb{L}^{\ell}_{\mathcal{U}_V/\mathcal{U}} \xleftarrow{t} \mathbb{L}^{\ell}_{\mathscr{M}^{\ell}_{g,n}(V)/\mathfrak{M}_{g,n}}|_{\mathcal{U}_V}.$$

The map $\mathcal{U} \to \mathfrak{M}_{g,n}$ is integral, saturated, and log smooth according to its functor of points, so its underlying map of stacks is flat and the fs pullback square is also an ordinary pullback.

Then t is an isomorphism [Olsson 2005, 1.1(iv)], and the log cotangent complex of V is [loc. cit., 1.1(iii)]

$$\mathbb{L}^{\ell}_{V} = \Omega^{\ell}_{V}[0].$$

We've written [0] to consider a coherent sheaf as a chain complex concentrated in degree 0. Via the isomorphism t and this identification, we have obtained a map

$$\Omega^{\ell}_{V}[0]|_{\mathcal{U}_V} \to \mathbb{L}^{\ell}_{\mathcal{U}/\mathfrak{M}_{g,n}}|_{\mathcal{U}_V}. \tag{7}$$

We need the ordinary relative dualizing sheaf $\omega_{\pi_V^\circ}$ and the identification

$$L\pi_V^!(\,\cdot\,) = \omega_{\pi_V^\circ} \overset{L}{\otimes} L\pi_V^*(\,\cdot\,).$$

Tensor (7) by $\omega_{\pi_V^\circ}$ and use the adjunction

$$\Omega_V^\ell[0]\,|_{\mathcal{U}_V} \overset{L}{\otimes} \omega_{\pi_V^\circ} \longrightarrow L\pi_V^!\mathbb{L}^\ell_{\mathscr{M}^\ell_{g,n}(V)/\mathfrak{M}_{g,n}} \quad \text{and} \quad E(V) := R\pi_{V*}(\Omega_V^\ell[0]\,|_{\mathcal{U}_V} \overset{L}{\otimes} \omega_{\pi_V^\circ}) \longrightarrow \mathbb{L}^\ell_{\mathscr{M}^\ell_{g,n}(V)/\mathfrak{M}_{g,n}}.$$

We won't repeat the verification [Gross and Siebert 2013, Proposition 5.1] that $E(V)$ is a Log POT.

Remark 5.3. The map (7) comes from the map on normal cones

$$C^\ell_{\mathscr{M}^\ell_{g,n}(V)/\mathfrak{M}_{g,n}}|_{\mathcal{U}_V} \xrightarrow{\sim} C^\ell_{\mathcal{U}_V/\mathcal{U}} \longrightarrow BT_V^\ell|_{\mathcal{U}_V}.$$

We needed duality, so we opted for the other perspective.

Remark 5.4 (variants). The reader may choose to work in the relative setting of a log smooth and quasiprojective map $V \to S$. Obstruction Theories are obtained in the same way.

The contact order of a log stable map is locally constant and amounts to another piece of discrete data like the genus or number of marked points. We only fix genus and number of markings to be consistent with [Lee and Qu 2018]. The reader may readily vary the numerical type conditions in our formulas.

We need one more stack, $\mathfrak{D}$: Points of $\mathfrak{D}$ over T are diagrams $(C' \leftarrow C \rightarrow C'')$ of genus g, n-pointed prestable curves over T whose maps are partial stabilizations (they lie over the identities in $\overline{M}_{g,n}$) that don't both contract any component. In other words, $C \to C' \times C''$ itself is a stable map. This stack is only necessary to form an fs pullback square:

Situation 5.5 [Lee and Qu 2018, Section 2]. Recall the fs pullback square:

$$\begin{array}{ccc} \mathscr{M}^\ell_{g,n}(V\times W) & \longrightarrow & \mathscr{M}^\ell_{g,n}(V)\times\mathscr{M}^\ell_{g,n}(W) \\ \downarrow c & \ulcorner\ell & \downarrow a \\ \mathfrak{D} & \xrightarrow{\widetilde{\Delta}} & \mathfrak{M}_{g,n}\times\mathfrak{M}_{g,n} \end{array} \tag{8}$$

Let $C \to V \times W$ be a log stable map over a base T. The maps $(C \to V)$, $(C \to W)$ needn't be stable; denote their stabilizations by $(C' \to V)$, $(C'' \to W)$, respectively.

The top horizontal arrow in (8) sends $(C \to V \times W)$ to the induced log stable maps $(C' \to V, C'' \to W)$. The vertical arrow c sends $(C \to V \times W)$ to the partial stabilizations $(C' \leftarrow C \rightarrow C'')$. The map $\widetilde{\Delta}$ sends a diagram $(C' \leftarrow C \rightarrow C'')$ to the pair of prestable curves C', C''. Finally, a sends a pair of log stable maps $(C' \to V, C'' \to W)$ to the prestable curves (C', C'').

This square has a factorization:

$$\begin{array}{ccccc} \mathscr{M}^{\ell}_{g,n}(V\times W) & \xrightarrow{h} & Q & \xrightarrow{\ulcorner\ell} & \mathscr{M}^{\ell}_{g,n}(V)\times\mathscr{M}^{\ell}_{g,n}(W) \\ \downarrow c & \ulcorner\ell & \downarrow & & \downarrow a \\ \mathfrak{D} & \xrightarrow{l} & Q' & \xrightarrow[\ulcorner\ell]{\phi} & \mathfrak{M}_{g,n}\times\mathfrak{M}_{g,n} \\ & & \downarrow & & \downarrow s\times s \\ & & \overline{M}_{g,n} & \xrightarrow{\Delta} & \overline{M}_{g,n}\times\overline{M}_{g,n} \end{array} \tag{9}$$

Where $s:\mathfrak{M}_{g,n}\to\overline{M}_{g,n}$ stabilizes a prestable curve.

To be clear, $Q=\mathscr{M}^{\ell}_{g,n}(V)\times^{\ell}_{\overline{M}_{g,n}}\mathscr{M}^{\ell}_{g,n}(W)$ and $Q'=\mathfrak{M}_{g,n}\times^{\ell}_{\overline{M}_{g,n}}\mathfrak{M}_{g,n}$ are the analogues of Lee and Qu's [2018] P, $\mathfrak{P}$, etc.

Theorem 5.6 (the "log Gromov–Witten product formula"). *With V, W log smooth, quasiprojective schemes,*

$$h_*[\mathscr{M}^{\ell}_{g,n}(V\times W), E(V\times W)]^{\ell vir}=\Delta^!([\mathscr{M}^{\ell}_{g,n}(V), E(V)]^{\ell vir}\times[\mathscr{M}^{\ell}_{g,n}(W), E(W)]^{\ell vir}).$$

Our proof will be the same as K. Behrend's [1999]: we compute the log normal cone of the map $Q\to Q'$ in two different ways.

Remark 5.7 (on diagram (9)). We equip a with the product $E(V)\boxplus E(W)$ of the natural Log POT's of Construction 5.2, adopting the notation

$$E\boxplus E':=E|_{V\times W}\oplus E'|_{V\times W}.$$

The cotangent complex $\mathbb{L}^{\ell}_{\Delta}$ is of perfect amplitude in [-1, 0] because its source and target are log smooth. Therefore $C^{\ell}_{\Delta}=N^{\ell}_{\Delta}$ serves as a natural Log POT for itself. We equip ϕ with the pullback obstruction theory, resulting in

$$\Delta^!=\phi^!$$

by Remark 3.6. We endow the square bounded by ϕ and a with the natural compatibility datum afforded all such squares as in Remark 3.14.

All of the arrows in diagrams (8) and (9) are of DM-type.

Lemma 5.8. *The stabilization map $s:\mathfrak{M}_{g,n}\to\overline{M}_{g,n}$ is log smooth.*

Proof. The cover $\bigsqcup_m\overline{M}_{g,n+m}\to\mathfrak{M}_{g,n}$ given by forgetting marked points and not stabilizing is strict smooth [Lee and Qu 2018, 1.2.1]. This map is in particular Kummer and surjective, and [Illusie et al. 2013, Theorem 0.2] applies with $\mathbb{P}=$ "log smooth" once we argue that the composite $\bigsqcup_m\overline{M}_{g,n+m}\to\overline{M}_{g,n}$ is log smooth.

The forgetful map $\overline{M}_{g,n+1}\to\overline{M}_{g,n}$ is the universal curve, so it is tautologically log smooth and flat. We see the map $\overline{M}_{g,n+m}\to\overline{M}_{g,n}$ is log smooth by iterating this forgetfulness, and this completes the argument. □

Remark 5.9. The map $\mathfrak{D} \to \mathfrak{M}_{g,n}$ which records the initial curve is log étale since the original map was étale [Behrend 1999, Lemma 4] and ours is the fsification thereof. The stack Q' is log smooth because the map $Q' \to \overline{M}_{g,n}$ is pulled back from $s \times s$.

Given a log étale map $X' \to X$ of log smooth log algebraic stacks with X equidimensional, we claim X' must be as well. The maps $X' \subseteq \mathcal{L}X'$, $X \subseteq \mathcal{L}X$ are dense because of the log smoothness assumption and the map $\mathcal{L}X' \to \mathcal{L}X$ is étale. Thus $\mathcal{L}X$ and $\mathcal{L}X'$ are equidimensional, as well as $X' \subseteq \mathcal{L}X'$. This argument shows that fsification preserves equidimensionality of log smooth stacks, so our fs versions of $\mathfrak{D}$, Q' are equidimensional because the original versions [Behrend 1999] were.

Lemma 5.10. *The obstruction theories $E(V)$, $E(W)$, $E(V \times W)$ are compatible in the sense that*

$$\widetilde{\Delta}^*(E(V) \boxplus E(W)) \simeq E(V \times W).$$

Proof. We completely echo the proof of [Behrend 1999, Proposition 6].

Consider the diagram of universal log curves and tautological maps with the notation:

$$\begin{array}{ccccc}
V & \longleftarrow & & & V \times W \\
\uparrow{\scriptstyle f_V} & & & & \uparrow{\scriptstyle f_{V\times W}} \\
\mathcal{U}_V & \xleftarrow{s_V} & \widetilde{\mathcal{U}}_V & \xleftarrow{q_V} & \mathcal{U}_{V\times W} \\
\downarrow{\scriptstyle \pi_V} & \ell\lrcorner & & \searrow{\scriptstyle \widetilde{\pi}_V} & \downarrow{\scriptstyle \pi_{V\times W}} \\
\mathscr{M}^{\ell}_{g,n}(V) & & \xleftarrow{r_V} & & \mathscr{M}^{\ell}_{g,n}(V \times W)
\end{array}$$

We claim $F \to Rq_{V*}q_V^* F$ is an isomorphism for any vector bundle F on $\mathcal{U}_V$. The map q_V represents partial stabilization. We make the argument for contracting one $\mathbb{P}^1$ at a time.

We first compute that $R^p q_{V*} q_V^* F = 0$ for $p \neq 0$. This claim is local in $\mathcal{U}_V$, so assume F is trivial. The fiber of $R^p q_{V*} q_V^* F$ at a point x is $H^p(q_V^{-1}(x), q_V^* F)$. Hence the fibers $q_V^{-1}(x)$ are either a point or $\mathbb{P}^1$. On each fiber, the cohomology of the trivial vector bundle is concentrated in degree 0 [Stacks 2005–, 01XS]. Not only are F and $q_{V*}q_V^* F$ abstractly isomorphic in that case, but the natural map is an isomorphism [Fantechi et al. 2005, Exercise 9.3.11].

The universal curve π_V is tautologically flat, integral, and saturated. The fs pullback square it belongs to is therefore also an ordinary flat pullback, subject to cohomology and base change [Stacks 2005–, Tag 08IB]. This gives

$$Lr_V^* R\pi_{V*} Lf_V^* \Omega_V = R\widetilde{\pi}_{V*} Ls_V^* Lf_V^* \Omega_V = R\widetilde{\pi}_{V*} Rq_{V*} q_V^* Ls_V^* Lf_V^* \Omega_V = R\pi_{V\times W*} Lf_{V\times W}^* (\Omega_V|_{V\times W}).$$

All the same goes for W. Add the two together to get

$$Lr_V^* R\pi_{V*} Lf_V^* \Omega_V \boxplus Lr_W^* R\pi_{W*} Lf_W^* \Omega_W \;=\; R\pi_{V\times W*} Lf_{V\times W}^* (\Omega_V \boxplus \Omega_W).$$

This is dual to the compatibility we set out to prove, so we are through. □

Proof of Theorem 5.6. Compute the log virtual fundamental class $[Q, E(V) \boxplus E(W)]^{vir}$ in two different ways:

$$\begin{aligned}[Q, E(V) \boxplus E(W)]^{vir} &:= [C^{\ell}_{Q/Q'} \subseteq E(V) \boxplus E(W)] \\ &= a^{!}(Q') \\ &= a^{!}\phi^{!}(\mathfrak{M}_{g,n} \times \mathfrak{M}_{g,n}) \\ &= \phi^{!}a^{!}(\mathfrak{M}_{g,n} \times \mathfrak{M}_{g,n}) \\ &= \Delta^{!}[\mathscr{M}^{\ell}_{g,n}(V) \times \mathscr{M}^{\ell}_{g,n}(W), E(V) \boxplus E(W)]^{vir}.\end{aligned}$$

On the other hand,

$$[Q, E(V) \boxplus E(W)]^{vir} = h_{*}[\mathscr{M}^{\ell}_{g,n}(V \times W), E(V \times W)]^{vir}$$

by the log Costello formula, Theorem 4.1. □

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