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A p-adic Simpson correspondence for rigid analytic varieties

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We establish a p-adic Simpson correspondence in the spirit of Liu and Zhu for rigid analytic varieties X over $\mathbb{C}_p$ with a liftable good reduction by constructing a new period sheaf on $X_{\text{proét}}$. To do so, we use the theory of cotangent complexes described by Beilinson and Bhatt. Then we give an integral decompletion theorem and complete the proof by local calculations. Our construction is compatible with the previous works of Faltings and Liu and Zhu.

1. Introduction

In the theory of complex geometry, for a compact Kähler manifold X, Simpson [1992] established a tensor equivalence between the category of semisimple flat vector bundles on X and the category of polystable Higgs bundles with vanishing Chern classes. Nowadays, such a correspondence is known as nonabelian Hogde theory or the Simpson correspondence. There is a well-established theory of the Simpson correspondence for smooth varieties in characteristic $p > 0$ admitting a lifting modulo p^2 (see [Ogus and Vologodsky 2007]). This leads us to ask for a p-adic analogue of Simpson's correspondence.

The first step is due to Deninger and Werner [2005]. They gave a partial analogue of classical Narasimhan–Seshadri theory by studying parallel transport for vector bundles of curves. At the same time, Faltings [2005] constructed an equivalence between the category of small generalised representations and the category of small Higgs bundles for schemes $\mathfrak{X}_0$ with toroidal singularities over $\mathcal{O}_k$, the ring of integers of some p-adic local field k, under a certain deformation assumption. His method was elaborated and generalised by Abbes, Gros and Tsuji [Abbes et al. 2016] and related to integral p-adic Hodge theory by Morrow and Tsuji [2020]. When X is a rigid analytic space over k, Liu and Zhu [2017] related a

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Higgs bundle on $X_{\hat{\bar{k}},\mathrm{\acute{e}t}}$ to each $\mathbb{Q}_p$-local system on $X_{\mathrm{\acute{e}t}}$ and proved that the resulting Higgs field must be nilpotent (see [Liu and Zhu 2017, Theorem 2.1]). Their work was generalised to the logarithmic case in [Diao et al. 2023b]. However, their Higgs functor is not an equivalence, so it is still open to classify Higgs bundles coming from representations. For smooth rigid spaces X over $\hat{\bar{k}}$, Heuer [2022] established an equivalence between the category of one-dimensional $\hat{\bar{k}}$-representations of the fundamental group $\pi_1(X)$ and the category of pro-finite-étale Higgs bundles. Using his method, Heuer, Mann and Werner [Heuer et al. 2023] constructed a Simpson correspondence for abeloids over $\hat{\bar{k}}$.

In this paper, we establish an equivalence between the category of small generalised representations (Definition 5.1) and the category of small Higgs bundles (Definition 5.2) for rigid analytic varieties X with liftable (see the notation section) good reductions $\mathfrak{X}$ over $\mathcal{O}_{\mathbb{C}_p}$ in the spirit of the work of Liu and Zhu. Our construction is global and the main ingredient is a new overconvergent period sheaf $\mathcal{O}\mathbb{C}^\dagger$ endowed with a canonical Higgs field Θ on $X_{\mathrm{pro\acute{e}t}}$, which can be viewed as a kind of p-adic complete version of the period sheaf $\mathcal{O}\mathbb{C}$ due to Hyodo [1989]. The main theorem is stated as follows:

Theorem 1.1 (Theorem 5.3). *Assume $a \geq 1/(p-1)$. Let $\mathfrak{X}$ be a liftable smooth formal scheme over $\mathcal{O}_{\mathbb{C}_p}$ of relative dimension d with the rigid generic fibre X and $\nu : X_{\mathrm{pro\acute{e}t}} \to \mathfrak{X}_{\mathrm{\acute{e}t}}$ be the natural projection of sites. Then there is an overconvergent period sheaf $\mathcal{O}\mathbb{C}^\dagger$ endowed with a canonical Higgs field Θ such that the following assertions are true*:

(1) *For any a-small generalised representation $\mathcal{L}$ of rank l on $X_{\mathrm{pro\acute{e}t}}$, let $\Theta_{\mathcal{L}} := \mathrm{id}_{\mathcal{L}} \otimes \Theta$ be the induced Higgs field on $\mathcal{L} \otimes_{\widehat{\mathcal{O}}_X} \mathcal{O}\mathbb{C}^\dagger$; then $\mathrm{R}\nu_*(\mathcal{L} \otimes_{\widehat{\mathcal{O}}_X} \mathcal{O}\mathbb{C}^\dagger)$ is discrete. Define $\mathcal{H}(\mathcal{L}) := \nu_*(\mathcal{L} \otimes_{\widehat{\mathcal{O}}_X} \mathcal{O}\mathbb{C}^\dagger)$ and $\theta_{\mathcal{H}(\mathcal{L})} = \nu_*\Theta_{\mathcal{L}}$. Then $(\mathcal{H}(\mathcal{L}), \theta_{\mathcal{H}(\mathcal{L})})$ is an a-small Higgs bundle of rank l.*

(2) *For any a-small Higgs bundle $(\mathcal{H}, \theta_{\mathcal{H}})$ of rank l on $\mathfrak{X}_{\mathrm{\acute{e}t}}$, let $\Theta_{\mathcal{H}} := \mathrm{id}_{\mathcal{H}} \otimes \Theta + \theta_{\mathcal{H}} \otimes \mathrm{id}_{\mathcal{O}\mathbb{C}^\dagger}$ be the induced Higgs field on $\mathcal{H} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}\mathbb{C}^\dagger$ and define*

$$\mathcal{L}(\mathcal{H}, \theta_{\mathcal{H}}) = (\mathcal{H} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}\mathbb{C}^\dagger)^{\Theta_{\mathcal{H}}=0}.$$

Then $\mathcal{L}(\mathcal{H}, \theta_{\mathcal{H}})$ is an a-small generalised representation of rank l.

(3) *The functor $\mathcal{L} \mapsto (\mathcal{H}(\mathcal{L}), \theta_{\mathcal{H}(\mathcal{L})})$ induces an equivalence from the category of a-small generalised representations to the category of a-small Higgs bundles, whose quasi-inverse is given by $(\mathcal{H}, \theta_{\mathcal{H}}) \mapsto \mathcal{L}(\mathcal{H}, \theta_{\mathcal{H}})$. The equivalence preserves tensor products and dualities and identifies the Higgs complexes*

$$\mathrm{HIG}(\mathcal{L} \otimes_{\widehat{\mathcal{O}}_X} \mathcal{O}\mathbb{C}^\dagger, \Theta_{\mathcal{L}}) \simeq \mathrm{HIG}(\mathcal{H}(\mathcal{L}) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}\mathbb{C}^\dagger, \Theta_{\mathcal{H}(\mathcal{L})}).$$

(4) *Let $\mathcal{L}$ be an a-small generalised representation with associated Higgs bundle $(\mathcal{H}, \theta_{\mathcal{H}})$. Then there is a canonical quasi-isomorphism*

$$\mathrm{R}\nu_*(\mathcal{L}) \simeq \mathrm{HIG}(\mathcal{H}, \theta_{\mathcal{H}}),$$

where $\mathrm{HIG}(\mathcal{H}, \theta_{\mathcal{H}})$ is the Higgs complex induced by $(\mathcal{H}, \theta_{\mathcal{H}})$. In particular, $\mathrm{R}\nu_*(\mathcal{L})$ is a perfect complex of $\mathcal{O}_{\mathfrak{X}}[\frac{1}{p}]$-modules concentrated in degree $[0, d]$.

(5) *Assume $f : \mathfrak{X} \to \mathfrak{Y}$ is a smooth morphism between liftable smooth formal schemes over $\mathcal{O}_{\mathbb{C}_p}$. Let $\widetilde{\mathfrak{X}}$ and $\widetilde{\mathfrak{Y}}$ be the fixed A_2-liftings of $\mathfrak{X}$ and $\mathfrak{Y}$, respectively. Assume f lifts to an A_2-morphism $\tilde{f} : \widetilde{\mathfrak{X}} \to \widetilde{\mathfrak{Y}}$. Then the equivalence in* (3) *is compatible with the pull-back along f.*

Note that when $\mathcal{L} = \widehat{\mathcal{O}}_X$, we get $(\mathcal{H}(\widehat{\mathcal{O}}_X), \theta_{\mathcal{H}(\widehat{\mathcal{O}}_X)}) = \big(\mathcal{O}_{\mathfrak{X}}\big[\frac{1}{p}\big], 0\big)$. So our result can be viewed as a generalisation of [Scholze 2013b, Proposition 3.23]. Theorem 1.1(3) also provides a way to compute the pro-étale cohomology for a small generalised representation $\mathcal{L}$. More precisely, we get a quasi-isomorphism

$$\mathrm{R}\Gamma(X_{\text{proét}}, \mathcal{L}) \simeq \mathrm{R}\Gamma(\mathfrak{X}_{\text{ét}}, \mathrm{HIG}(\mathcal{H}(\mathcal{L}), \theta_{\mathcal{H}(\mathcal{L})})).$$

If, in addition, $\mathfrak{X}$ is proper, then we get a finiteness result on pro-étale cohomology of small generalised representations.

Corollary 1.2. *Keep the notation as in Theorem 1.1 and assume $\mathfrak{X}$ is proper. Then for any a-small generalised representation $\mathcal{L}$,* $\mathrm{R}\Gamma(X_{\text{proét}}, \mathcal{L})$ *is concentrated in degree* $[0, 2d]$ *and has cohomologies as finite dimensional $\mathbb{C}_p$-spaces.*

The overconvergent period sheaf $\mathcal{O}\mathbb{C}^{\dagger}$ (with respect to a certain lifting of $\mathfrak{X}$) has $\mathcal{O}\mathbb{C}$ as a subsheaf. Indeed, it is a direct limit of certain p-adic completions of $\mathcal{O}\mathbb{C}$. In particular, when $\mathfrak{X}$ comes from a scheme $\mathfrak{X}_0$ over $\mathcal{O}_k$ and the generalised representation $\mathcal{L}$ comes from a $\mathbb{Z}_p$-local system on the rigid generic fibre X_0 of $\mathfrak{X}_0$, our construction coincides with the work of Liu and Zhu (Remark 5.6). On the other hand, $\mathcal{O}\mathbb{C}^{\dagger}$ is related to an obstruction class $\mathrm{cl}(\mathcal{E}^+)$ solving a certain deformation problem (Remark 2.10 and Proposition 2.14). Since the class $\mathrm{cl}(\mathcal{E}^+)$ is exactly the one used to establish the Simpson correspondence in [Faltings 2005], our construction is compatible with the works of Faltings and Abbes, Gros and Tsuji (Remark 5.5). These answer a question appearing in [Liu and Zhu 2017, Remark 2.5]. Another answer, using a different method, was announced in [Yang and Zuo 2020].

Since we need to take p-adic completions of $\mathcal{O}\mathbb{C}$, we have to find its integral models. Note that $\mathcal{O}\mathbb{C}$ is a direct limit of symmetric products of Faltings' extension, which was constructed for varieties by Faltings [1988] at first and revisited by Scholze [2013a] in the rigid analytic case. So we are reduced to finding an integral version of Faltings' extension. To do so, we use the method of cotangent complexes which was established and developed in [Quillen 1970; Illusie 1971; 1972; Gabber and Ramero 2003], and was systematically used in the p-adic theory by [Scholze 2012; Beilinson 2012; Bhatt 2012]. The proof of Theorem 1.1 is based on some explicit local calculations, especially an integral decompletion theorem (Theorem 3.4) for small representations, which can be regarded as a generalisation of [Diao et al. 2023b, Appendix A].

Notation. Let k be a complete discrete valuation field of mixed characteristics $(0, p)$ with ring of integers $\mathcal{O}_k$ and perfect residue field κ. We normalise the valuation on k by setting $\nu_p(p) = 1$ and the associated norm is given by $\|\cdot\| = p^{-\nu_p(\cdot)}$. We denote by $k_0 = \mathrm{Frac}(\mathrm{W}(\kappa))$ the maximal absolutely unramified subfield of k. Denote by $\mathcal{D}_k = \mathcal{D}_{k/k_0}$ the relative differential ideal of $\mathcal{O}_k$ over $\mathrm{W}(\kappa)$.

Let $\bar{k}$ be a fixed algebraic closure of k and $\mathbb{C}_p = \hat{\bar{k}}$ be its p-adic completion. We denote by $\mathcal{O}_{\mathbb{C}_p}$ (resp. $\mathfrak{m}_{\mathbb{C}_p}$) the ring of integers of $\mathbb{C}_p$ (resp. the maximal ideal of $\mathcal{O}_{\mathbb{C}_p}$). In this paper, when we write $p^a A$ for some $\mathcal{O}_{\mathbb{C}_p}$-module A, we always assume $a \in \mathbb{Q}$. An $\mathcal{O}_{\mathbb{C}_p}$-module M is called *almost vanishing* if it is $\mathfrak{m}_{\mathbb{C}_p}$-torsion, and in this case we write $M^{\mathrm{al}} = 0$. A morphism $f : M \to N$ of $\mathcal{O}_{\mathbb{C}_p}$-modules is *almost injective* (resp. *almost surjective*) if $\mathrm{Ker}(f)^{\mathrm{al}} = 0$ (resp. $\mathrm{Coker}(f)^{\mathrm{al}} = 0$). A morphism is an *almost isomorphism* if it is both almost injective and almost surjective.

We choose a sequence $\{1, \zeta_p, \dots, \zeta_{p^n}, \dots\}$ such that ζ_{p^n} is a primitive p^n-th root of unity in $\bar{k}$ satisfying $\zeta_{p^{n+1}}^p = \zeta_{p^n}$ for every $n \geq 0$. For every $\alpha \in \mathbb{Z}\big[\frac{1}{p}\big] \cap (0, 1)$, one can (uniquely) write $\alpha = (t(\alpha))/p^{n(\alpha)}$ with $\gcd(t(\alpha), p) = 1$ and $n(\alpha) \geq 1$. Then we define $\zeta^\alpha := \zeta_{p^{n(\alpha)}}^{t(\alpha)}$ when $\alpha \neq 0$ and $\zeta^\alpha := 1$ when $\alpha = 0$.

We always fix an element $\rho_k \in \mathbb{C}_p$ with $\nu_p(\rho_k) = \nu_p(\mathcal{D}_k) + 1/(p-1)$. Let $\mathrm{A}_{\mathrm{inf},k} = \mathrm{W}(\mathcal{O}_{\mathbb{C}_p^\flat}) \otimes_{\mathrm{W}(\kappa)} \mathcal{O}_k$ be the period ring of Fontaine. Then there is a surjective homomorphism $\theta_k : \mathrm{A}_{\mathrm{inf},k} \to \mathcal{O}_{\mathbb{C}_p}$ whose kernel is a principal ideal by [Fargues and Fontaine 2018, Proposition 3.1.9]. We fix a generator ξ_k of $\mathrm{Ker}(\theta_k)$. For instance, if $k = k_0$ is absolutely unramified, then we choose $\rho_k = \zeta_p - 1$ and $\xi_k = ([\epsilon]-1)/([\epsilon]^{1/p}-1)$ for $\epsilon = (1, \zeta_p, \zeta_{p^2}, \dots) \in \mathcal{O}_{\mathbb{C}_p}^\flat$. Put $\mathrm{A}_2 = \mathrm{A}_{\mathrm{inf},k}/\xi_k^2$ and denote Fontaine's p-adic analogue of $2\pi i$ by $t = \log[\epsilon]$.

For a p-adic formal scheme $\mathfrak{X}$ over $\mathcal{O}_{\mathbb{C}_p}$, we say it is *smooth* if it is formally smooth and locally of topologically finite type. We say $\mathfrak{X}$ is *liftable* if it admits a lifting $\widetilde{\mathfrak{X}}$ to $\mathrm{Spf}(\mathrm{A}_2)$. In this paper, we always assume $\mathfrak{X}$ is liftable. Let X be the rigid analytic generic fibre of $\mathfrak{X}$ and denote by $\nu : X_{\text{proét}} \to \mathfrak{X}_{\text{ét}}$ the natural projection of sites. Let $\widehat{\mathcal{O}}_X^+$ and $\widehat{\mathcal{O}}_X$ be the completed structure sheaves on $X_{\text{proét}}$ in the sense of [Scholze 2013a, Definition 4.1]. Both of them can be viewed as $\mathcal{O}_{\mathfrak{X}}$-algebras via the projection ν.

Let K be an object in the derived category of complexes of $\mathbb{Z}_p$-modules. We denote by $\hat{K}$ the derived p-adic completion $\mathrm{R}\varprojlim_n K \otimes_{\mathbb{Z}_p} \mathbb{Z}_p/p^n$. In particular, for a morphism $A \to B$ of $\mathbb{Z}_p$-algebras, we denote the derived p-adic completion of cotangent complex $\mathrm{L}_{B/A}$ by $\widehat{\mathrm{L}}_{B/A}$. In this paper, for two complexes K_1 and K_2 of (sheaves of) modules, we write $K_1 \simeq K_2$ if they are quasi-isomorphic. For two modules or sheaves M_1 and M_2, we write $M_1 \cong M_2$ if they are isomorphic.

Organisation. In Section 2, we construct the integral Faltings' extension by using p-complete cotangent complexes and explaining how it is related to deformation theory. At the end of this section we construct the desired overconvergent sheaf. In Section 3, we prove an integral decompletion theorem for small representations. In Section 4, we establish a local Simpson correspondence. We first consider the trivial representation and then reduce the general case to this special case. Finally, in Section 5, we state and prove our main theorem. The Appendix specifies some notation and includes some elementary facts that were used in previous sections.

2. Integral Faltings' extension and period sheaves

We construct the overconvergent period sheaf $\mathcal{O}\mathbb{C}^\dagger$ in this section. To do so, we have to construct an integral version of Faltings' extension.

Integral Faltings' extension. We first discuss the properties of the cotangent complex. The following lemmas are well known, but for the convenience of readers, we include their proofs here.

Lemma 2.1. *Let A be a ring. Suppose that $(f_1, \dots, f_n)$ is a regular sequence in A and generates the ideal $I = (f_1, \dots, f_n)$. Then* $\mathrm{L}_{(A/I)/A} \simeq (I/I^2)[1]$.

Proof. Regard A as a $\mathbb{Z}[X_1, \dots, X_n]$-algebra by mapping X_i to f_i for every i. Since $f_1, \dots, f_n$ is a regular sequence in A, for any $i \geq 1$, we have

$$\mathrm{Tor}_i^{\mathbb{Z}[X_1,\dots,X_n]}(\mathbb{Z}, A) = 0.$$

It follows from [Weibel 1994, 8.8.4] that

$$\mathrm{L}_{(A/I)/A} \simeq \mathrm{L}_{\mathbb{Z}/\mathbb{Z}[X_1,\dots,X_n]} \otimes^L_{\mathbb{Z}[X_1,\dots,X_n]} A.$$

So we may assume $A = \mathbb{Z}[X_1, \dots, X_n]$ and $I = (X_1, \dots, X_n)$. From homomorphisms $\mathbb{Z} \to A \to A/I$ of rings, we get an exact triangle

$$\mathrm{L}_{A/\mathbb{Z}} \otimes^L A/I \longrightarrow \mathrm{L}_{(A/I)/\mathbb{Z}} \longrightarrow \mathrm{L}_{(A/I)/A} \longrightarrow .$$

The middle term is trivial since $A/I = \mathbb{Z}$ and hence we deduce that

$$\mathrm{L}_{(A/I)/A} \simeq (\mathrm{L}_{A/\mathbb{Z}} \otimes^L_A \mathbb{Z})[1] \simeq (I/I^2)[1]. \qquad \square$$

Lemma 2.2. (1) *The map* $\mathrm{dlog} : \mu_{p^\infty} \to \Omega^1_{\mathcal{O}_{\bar{k}}/\mathcal{O}_k}$, $\zeta_{p^n} \mapsto d\zeta_{p^n}/\zeta_{p^n}$ *induces an isomorphism*

$$\mathrm{dlog} : \bar{k}/\rho_k^{-1}\mathcal{O}_{\bar{k}} \otimes \mathbb{Z}_p(1) \to \Omega^1_{\mathcal{O}_{\bar{k}}/\mathcal{O}_k},$$

where $\mathbb{Z}_p(1)$ denotes the Tate twist.

(2) $\mathrm{L}_{\mathcal{O}_{\bar{k}}/\mathcal{O}_k} \simeq \Omega^1_{\mathcal{O}_{\bar{k}}/\mathcal{O}_k}[0]$.

(3) $\widehat{\mathrm{L}}_{\mathcal{O}_{\mathbb{C}_p}/\mathcal{O}_k} \simeq (1/\rho_k)\mathcal{O}_{\mathbb{C}_p}(1)[1]$.

Proof. (1) This is [Fontaine 1982, Théorème 1’].

(2) This is [Beilinson 2012, Theorem 1.3].

(3) This follows from (1) and (2) after taking derived p-completions on both sides. $\square$

Corollary 2.3. (1) $\widehat{\mathrm{L}}_{\mathcal{O}_{\mathbb{C}_p}/A_{\mathrm{inf},k}}[-1] \simeq (1/\rho_k)\mathcal{O}_{\mathbb{C}_p}(1)[0] \simeq \xi_k A_{\mathrm{inf},k}/\xi_k^2 A_{\mathrm{inf},k}[0]$.

(2) $\widehat{\mathrm{L}}_{\mathcal{O}_{\mathbb{C}_p}/A_2} \simeq (1/\rho_k)\mathcal{O}_{\mathbb{C}_p}(1)[1] \oplus (1/\rho_k^2)\mathcal{O}_{\mathbb{C}_p}(2)[2]$.

Proof. (1) Considering the morphisms $\mathcal{O}_k \to A_{\mathrm{inf},k} \to \mathcal{O}_{\mathbb{C}_p}$ of rings, we have an exact triangle

$$\mathrm{L}_{A_{\mathrm{inf},k}/\mathcal{O}_k} \widehat{\otimes}^L_{A_{\mathrm{inf},k}} \mathcal{O}_{\mathbb{C}_p} \to \widehat{\mathrm{L}}_{\mathcal{O}_{\mathbb{C}_p}/\mathcal{O}_k} \to \widehat{\mathrm{L}}_{\mathcal{O}_{\mathbb{C}_p}/A_{\mathrm{inf},k}} \to .$$

Since

$$\widehat{\mathrm{L}}_{A_{\mathrm{inf},k}/\mathcal{O}_k} \simeq \mathrm{L}_{A_{\mathrm{inf}}/\mathrm{W}(\kappa)} \widehat{\otimes}^L_{\mathrm{W}(\kappa)} \mathcal{O}_k = 0,$$

the first quasi-isomorphism follows from Lemma 2.2(3). Now, the second quasi-isomorphism is straightforward from Lemma 2.1.

(2) Considering the morphisms $A_{\mathrm{inf},k} \to A_2 \to \mathcal{O}_{\mathbb{C}_p}$ of rings, we have the exact triangle

$$\mathrm{L}_{A_2/A_{\mathrm{inf},k}} \widehat{\otimes}^L_{A_2} \mathcal{O}_{\mathbb{C}_p} \to \widehat{\mathrm{L}}_{\mathcal{O}_{\mathbb{C}_p}/A_{\mathrm{inf},k}} \to \widehat{\mathrm{L}}_{\mathcal{O}_{\mathbb{C}_p}/A_2} \to .$$

Combining Lemma 2.1 with (1), the above exact triangle reduces to

$$\xi_k^2 A_{\mathrm{inf},k}/\xi_k^4 A_{\mathrm{inf},k} \otimes_{A_2} \mathcal{O}_{\mathbb{C}_p}[1] \to \xi_k A_{\mathrm{inf},k}/\xi_k^2 A_{\mathrm{inf},k}[1] \to \widehat{\mathrm{L}}_{\mathcal{O}_{\mathbb{C}_p}/A_2} \to .$$

Now we complete the proof by noting that the first arrow is trivial. □

We identify $\mathcal{O}_{\mathbb{C}_p}(1)$ with $\mathcal{O}_{\mathbb{C}_p}t$, where t is Fontaine's p-adic analogue of $2\pi i$. It follows from Lemma 2.2(1) that the sequence $\{\mathrm{dlog}(\zeta_{p^n})\}_{n\geq 0}$ can be identified with the element $t \in (1/\rho_k)\mathcal{O}_{\mathbb{C}_p}(1)$. If we regard $A_{\mathrm{inf},k}$ as a subring of $\mathrm{B}_{\mathrm{dR}}^+$ and identify $t\mathrm{B}_{\mathrm{dR}}^+/t^2\mathrm{B}_{\mathrm{dR}}^+$ with $\mathbb{C}_p(1)$, then Corollary 2.3 says that t and $\rho_k\xi_k$ in $\mathbb{C}_p(1)$ differ by a p-adic unit in $\mathcal{O}_{\mathbb{C}_p}^\times$.

Remark 2.4. The corollary is still true if one replaces $\mathbb{C}_p$ by any closed subfield $K \subset \mathbb{C}_p$ containing μ_{p^∞}. All results in this paper hold for K instead of $\mathbb{C}_p$.

Now we construct the integral Faltings' extension in the local case. We fix some notation as follows. Let $\mathfrak{X} = \mathrm{Spf}(R^+)$ be a smooth formal scheme over $\mathrm{Spf}(\mathcal{O}_{\mathbb{C}_p})$ endowed with an étale morphism

$$\square : \mathfrak{X} \to \hat{\mathbb{G}}_m^d = \mathrm{Spf}(\mathcal{O}_{\mathbb{C}_p}\langle \underline{T}^{\pm 1}\rangle),$$

where $\mathcal{O}_{\mathbb{C}_p}\langle \underline{T}^{\pm 1}\rangle = \mathcal{O}_{\mathbb{C}_p}\langle T_1^{\pm 1}, \dots, T_d^{\pm 1}\rangle$. We say $\mathfrak{X}$ is *small* in this case. Let $X = \mathrm{Spa}(R, R^+)$ be the rigid analytic generic fibre of $\mathfrak{X}$ and $X_\infty = \mathrm{Spa}(\widehat{R}_\infty, \widehat{R}_\infty^+)$ be the affinoid perfectoid space associated to the base-change of X along the Galois cover

$$\mathbb{G}_{m,\infty}^d = \mathrm{Spa}(\mathbb{C}_p\langle \underline{T}^{\pm \frac{1}{p^\infty}}\rangle, \mathcal{O}_{\mathbb{C}_p}\langle \underline{T}^{\pm \frac{1}{p^\infty}}\rangle) \to \mathbb{G}_m^d = \mathrm{Spa}(\mathbb{C}_p\langle \underline{T}^{\pm 1}\rangle, \mathcal{O}_{\mathbb{C}_p}\langle \underline{T}^{\pm 1}\rangle).$$

Denote by Γ the Galois group of the cover $X_\infty \to X$ and let γ_i be in Γ satisfying

$$\gamma_i(T_j^{\frac{1}{p^n}}) = \zeta_{p^n}^{\delta_{ij}} T_j^{\frac{1}{p^n}} \tag{2-1}$$

for any $1 \leq i, j \leq d$ and $n \geq 0$. Here, δ_{ij} denotes the Kronecker delta. Then $\Gamma \cong \mathbb{Z}_p\gamma_1 \oplus \cdots \oplus \mathbb{Z}_p\gamma_d$. Let $\widetilde{R}^+$ be a lifting of R^+ along $A_2 \to \mathcal{O}_{\mathbb{C}_p}$. Then the morphisms $\widetilde{R}^+ \to R^+ \to \widehat{R}_\infty^+$ of rings give an exact triangle of p-complete cotangent complexes

$$\mathrm{L}_{R^+/\widetilde{R}^+} \widehat{\otimes}^L_{R^+} \widehat{R}_\infty^+ \to \widehat{\mathrm{L}}_{\widehat{R}_\infty^+/\widetilde{R}^+} \to \widehat{\mathrm{L}}_{\widehat{R}_\infty^+/R^+} \to . \tag{2-2}$$

The first term is easy to handle. Indeed, combining [Weibel 1994, 8.8.4] with Corollary 2.3(2), we deduce that

$$\mathrm{L}_{R^+/\widetilde{R}^+} \widehat{\otimes}^L_{R^+} \widehat{R}_\infty^+ \simeq \frac{1}{\rho_k}\widehat{R}_\infty^+(1)[1] \oplus \frac{1}{\rho_k^2}\widehat{R}_\infty^+(2)[2].$$

Now we compute the third term of (2-2).

Lemma 2.5. *We have* $\widehat{\mathrm{L}}_{\widehat{R}_\infty^+/R^+} \simeq \widehat{\Omega}^1_{R^+} \otimes_{R^+} \widehat{R}_\infty^+[1]$, *where* $\widehat{\Omega}^1_{R^+}$ *denotes the module of formal differentials of* R^+ *over* $\mathcal{O}_{\mathbb{C}_p}$.

Proof. Since R^+ is étale over $\mathcal{O}_{\mathbb{C}_p}\langle \underline{T}^{\pm 1}\rangle$, thanks to [Bhatt et al. 2018, Lemma 3.14], we are reduced to the case $R^+ = \mathcal{O}_{\mathbb{C}_p}\langle \underline{T}^{\pm 1}\rangle$. For any $n \geq 0$, put $A_n^+ = \mathcal{O}_{\mathbb{C}_p}[\underline{T}^{\pm \frac{1}{p^n}}]$ and define $A_\infty^+ = \varinjlim_n A_n^+$. Since all rings involved are p-torsion free, we get

$$\widehat{\mathrm{L}}_{\widehat{R}_\infty^+/R^+} \simeq \widehat{\mathrm{L}}_{A_\infty^+/A_0^+}.$$

By [Illusie 1971, Chapitre II(1.2.3.4)], we see that

$$\mathrm{L}_{A_\infty^+/A_0^+} = \varinjlim_n \mathrm{L}_{A_n^+/A_0^+}.$$

Since all A_n^+'s are smooth over $\mathcal{O}_{\mathbb{C}_p}$, from the exact triangle

$$\mathrm{L}_{A_0^+/\mathcal{O}_{\mathbb{C}_p}} \otimes^L_{A_0^+} A_n^+ \to \mathrm{L}_{A_n^+/\mathcal{O}_{\mathbb{C}_p}} \to \mathrm{L}_{A_n^+/A_0^+} \to,$$

we deduce that

$$\mathrm{L}_{A_n^+/A_0^+} \simeq A_n^+ \otimes_{A_0^+} \frac{1}{p^n}\Omega^1_{A_0^+}/\Omega^1_{A_0^+}[0],$$

where we identify $\Omega^1_{A_n^+}$ with $A_n^+ \otimes_{A_0^+} (1/p^n)\Omega^1_{A_0^+}$. Therefore, we get

$$\mathrm{L}_{A_\infty^+/A_0^+} \simeq A_\infty^+ \otimes_{A_0^+} \Omega^1_{A_0^+} \otimes_{\mathbb{Z}_p} (\mathbb{Q}_p/\mathbb{Z}_p)[0].$$

Now the result follows by taking p-completions. □

Since R^+ admits a lifting $\widetilde{R}^+$ to A_2, the composition

$$\widehat{\mathrm{L}}_{\widehat{R}_\infty^+/R^+} \simeq \widehat{\mathrm{L}}_{\mathrm{A}_2(\widehat{R}_\infty^+)/\widetilde{R}^+} \widehat{\otimes}^L_{\mathrm{A}_2(\widehat{R}_\infty^+)} \widehat{R}_\infty^+ \to \widehat{\mathrm{L}}_{\widehat{R}_\infty^+/\widetilde{R}^+}$$

defines a section of $\widehat{\mathrm{L}}_{\widehat{R}_\infty^+/\widetilde{R}^+} \to \widehat{\mathrm{L}}_{\widehat{R}_\infty^+/R^+}$. Since the exact triangle (2-2) is Γ-equivariant, by taking cohomologies along (2-2), we get the following proposition.

Proposition 2.6. *There exists a Γ-equivariant short exact sequence of $\widehat{R}_\infty^+$-modules*

$$0 \to \frac{1}{\rho_k}\widehat{R}_\infty^+(1) \to E^+ \to \widehat{R}_\infty^+ \otimes_{R^+} \widehat{\Omega}^1_{R^+} \to 0, \tag{2-3}$$

where $E^+ = \mathrm{H}^{-1}(\widehat{\mathrm{L}}_{\widehat{R}_\infty^+/\widetilde{R}^+})$. The above exact sequence admits a (non-Γ-equivariant) section such that $E^+ \cong (1/\rho_k)\widehat{R}_\infty^+(1) \oplus \widehat{R}_\infty^+ \otimes_{R^+} \widehat{\Omega}^1_{R^+}$ as $\widehat{R}_\infty^+$-modules.

Remark 2.7. When R^+ is the base-change of some formal smooth $\mathcal{O}_k$-algebra R_0^+ of topologically finite type along $\mathcal{O}_k \to \mathcal{O}_{\mathbb{C}_p}$, it admits a canonical lifting $\widetilde{R}^+ = R_0^+ \widehat{\otimes}_{\mathcal{O}_k} A_2$. After inverting p, the resulting E^+ becomes the usual Faltings' extension and the corresponding sequence (2-3) is even $\mathrm{Gal}(\bar{k}/k)$-equivariant.

We describe the Γ-action on E^+. For any $1 \leq i \leq d$, by the proof of Lemma 2.5, the compatible sequence $\{\mathrm{dlog}(T_i^{1/p^n})\}_{n\geq 0}$ defines an element $x_i \in E^+$, which goes to $\mathrm{dlog}\, T_i$ via the projection $E^+ \to \widehat{R}_\infty^+ \otimes_{R^+} \widehat{\Omega}^1_{R^+}$. Since Γ acts on T_i's via (2-1), we deduce that, for any $1 \leq i, j \leq d$,

$$\gamma_i(x_j) = x_j + \delta_{ij}.$$

In summary, we have the following proposition.

Proposition 2.8. *The $\widehat{R}^+_\infty$-module E^+ is free of rank $d+1$ and has a basis $t/\rho_k, x_1, \dots, x_d$ such that*

(1) *for any $1 \le i \le d$, x_i is a lifting of $\mathrm{dlog}(T_i) \in \widehat{R}^+_\infty \otimes_{R^+} \widehat{\Omega}^1_{R^+}$ and that*

(2) *for any $1 \le i, j \le d$, $\gamma_i(x_j) = x_j + \delta_{ij} t$.*

Also, let $c : \Gamma \to \mathrm{Hom}_{R^+}\big(\widehat{\Omega}^1_{R^+}, (1/\rho_k)\widehat{R}^+_\infty(1)\big)$ be the map carrying γ_i to $c(\gamma_i)$, which sends $\mathrm{dlog}(T_j)$ to $\delta_{ij}t$. Then the cocycle determined by c in $\mathrm{H}^1\big(\Gamma, \mathrm{Hom}_{R^+}(\widehat{\Omega}^1_{R^+}, (1/\rho_k)\widehat{R}^+_\infty(1))\big)$ coincides with the extension class represented by E^+ in $\mathrm{Ext}^1_\Gamma(\widehat{R}^+_\infty \otimes_{R^+} \widehat{\Omega}^1_{R^+}, (1/\rho_k)\widehat{R}^+_\infty(1))$ via the canonical isomorphism

$$\mathrm{H}^1\bigg(\Gamma, \mathrm{Hom}_{R^+}\bigg(\widehat{\Omega}^1_{R^+}, \frac{1}{\rho_k}\widehat{R}^+_\infty(1)\bigg)\bigg) \cong \mathrm{Ext}^1_\Gamma\bigg(\widehat{R}^+_\infty \otimes_{R^+} \widehat{\Omega}^1_{R^+}, \frac{1}{\rho_k}\widehat{R}^+_\infty(1)\bigg).$$

Proof. It remains to prove the "also" part. By (1), the extension class of E^+ is represented by the cocycle

$$f : \Gamma \to \mathrm{Hom}_{\widehat{R}^+_\infty}\bigg(\widehat{R}^+_\infty \otimes_{R^+} \widehat{\Omega}^1_{R^+}, \frac{1}{\rho_k}\widehat{R}^+_\infty(1)\bigg) \cong \mathrm{Hom}_{R^+}\bigg(\widehat{\Omega}^1_{R^+}, \frac{1}{\rho_k}\widehat{R}^+_\infty(1)\bigg)$$

such that $f(\gamma)(\mathrm{dlog}(T_i)) = \gamma(x_i) - x_i$ for any $\gamma \in \Gamma$ and any i. However, by (2), f is exactly c. We are done. □

Now we extend the above construction to the global case. Let $\mathfrak{X}$ be a smooth formal scheme over $\mathcal{O}_{\mathbb{C}_p}$ with a fixed lifting $\widetilde{\mathfrak{X}}$ to A_2. Denote by X its rigid analytic generic fibre over $\mathbb{C}_p$. We regard both $\mathcal{O}_{\mathfrak{X}}$ and $\mathcal{O}_{\widetilde{\mathfrak{X}}}$ as sheaves on $X_{\text{proét}}$ via the projection $\nu : X_{\text{proét}} \to \mathfrak{X}_{\text{ét}}$ (note that $\mathfrak{X}$ and $\widetilde{\mathfrak{X}}$ have the same étale site). Considering morphisms of sheaves of rings $\mathcal{O}_{\widetilde{\mathfrak{X}}} \to \mathcal{O}_{\mathfrak{X}} \to \widehat{\mathcal{O}}^+_X$, we get an exact triangle

$$\mathrm{L}_{\mathcal{O}_{\mathfrak{X}}/\mathcal{O}_{\widetilde{\mathfrak{X}}}} \widehat{\otimes}^L_{\mathcal{O}_{\mathfrak{X}}} \widehat{\mathcal{O}}^+_X \to \widehat{\mathrm{L}}_{\widehat{\mathcal{O}}^+_X/\mathcal{O}_{\widetilde{\mathfrak{X}}}} \to \mathrm{L}_{\widehat{\mathcal{O}}^+_X/\mathcal{O}_{\mathfrak{X}}} \to . \tag{2-4}$$

Similar to the local case, the first term becomes

$$\mathrm{L}_{\mathcal{O}_{\mathfrak{X}}/\mathcal{O}_{\widetilde{\mathfrak{X}}}} \widehat{\otimes}^L_{\mathcal{O}_{\mathfrak{X}}} \widehat{\mathcal{O}}^+_X \simeq \mathrm{L}_{\mathcal{O}_{\mathbb{C}_p}/A_2} \otimes^L_{\mathcal{O}_{\mathbb{C}_p}} \widehat{\mathcal{O}}^+_X$$

and the composition

$$\widehat{\mathrm{L}}_{\widehat{\mathcal{O}}^+_X/\mathcal{O}_{\widetilde{\mathfrak{X}}}} \simeq \widehat{\mathrm{L}}_{\mathrm{A}_2(\widehat{\mathcal{O}}^+_X)/\mathcal{O}_{\widetilde{\mathfrak{X}}}} \widehat{\otimes}^L_{\mathrm{A}_2(\widehat{\mathcal{O}}^+_X)} \widehat{\mathcal{O}}^+_X \to \widehat{\mathrm{L}}_{\widehat{\mathcal{O}}^+_X/\mathcal{O}_{\widetilde{\mathfrak{X}}}}$$

defines a section of $\widehat{\mathrm{L}}_{\widehat{\mathcal{O}}^+_X/\mathcal{O}_{\widetilde{\mathfrak{X}}}} \to \mathrm{L}_{\widehat{\mathcal{O}}^+_X/\mathcal{O}_{\mathfrak{X}}}$.

We claim that

$$\widehat{\mathrm{L}}_{\widehat{\mathcal{O}}^+_X/\mathcal{O}_{\mathfrak{X}}} \simeq \widehat{\mathcal{O}}^+_X \otimes_{\mathcal{O}_{\mathfrak{X}}} \widehat{\Omega}^1_{\mathfrak{X}}[1]. \tag{2-5}$$

Granting this and taking cohomologies along (2-4), we get the following theorem.

Theorem 2.9. *There is an exact sequence of sheaves of $\widehat{\mathcal{O}}^+_X$-modules*

$$0 \to \frac{1}{\rho_k}\widehat{\mathcal{O}}^+_X(1) \to \mathcal{E}^+ \to \widehat{\mathcal{O}}^+_X \otimes_{\mathcal{O}_{\mathfrak{X}}} \widehat{\Omega}^1_{\mathfrak{X}} \to 0, \tag{2-6}$$

where $\mathcal{E}^+ = \mathrm{H}^{-1}(\widehat{\mathrm{L}}_{\widehat{\mathcal{O}}^+_X/\mathcal{O}_{\widetilde{\mathfrak{X}}}})$.

Remark 2.10. Apply $\mathrm{RHom}(-, (1/\rho_k)\widehat{\mathcal{O}}_X^+(1))$ to the exact triangle (2-4) and consider the induced long exact sequence

$$\cdots \to \mathrm{Ext}^1\Big(\widehat{\mathrm{L}}_{\mathcal{O}_{\mathfrak{X}}/\mathcal{O}_{\widetilde{\mathfrak{X}}}}\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\widehat{\mathcal{O}}_X^+, \frac{1}{\rho_k}\widehat{\mathcal{O}}_X^+(1)\Big) \xrightarrow{\partial} \mathrm{Ext}^2\Big(\widehat{\mathrm{L}}_{\widehat{\mathcal{O}}_X^+/\mathcal{O}_{\mathfrak{X}}}, \frac{1}{\rho_k}\widehat{\mathcal{O}}_X^+(1)\Big) \to \cdots$$

and the commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}^1\big(\widehat{\mathrm{L}}_{\mathcal{O}_{\mathfrak{X}}/\mathcal{O}_{\widetilde{\mathfrak{X}}}}\widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}}\widehat{\mathcal{O}}_X^+, \frac{1}{\rho_k}\widehat{\mathcal{O}}_X^+(1)\big) & \xrightarrow{\ \partial\ } & \mathrm{Ext}^2\big(\widehat{\mathrm{L}}_{\widehat{\mathcal{O}}_X^+/\mathcal{O}_{\mathfrak{X}}}, \frac{1}{\rho_k}\widehat{\mathcal{O}}_X^+(1)\big) \\ \big\downarrow\cong & & \big\downarrow\cong \\ \mathrm{Hom}\big(\frac{1}{\rho_k}\mathcal{O}_{\mathfrak{X}}(1), \frac{1}{\rho_k}\widehat{\mathcal{O}}_X^+(1)\big) & \xrightarrow{\ \partial\ } & \mathrm{Ext}^1\big(\widehat{\mathcal{O}}_X^+\otimes_{\mathcal{O}_{\mathfrak{X}}}\widehat{\Omega}^1_{\mathcal{O}_{\mathfrak{X}}}, \frac{1}{\rho_k}\widehat{\mathcal{O}}_X^+(1)\big) \end{array}$$

Then the extension class $[\mathcal{E}^+]$ associated to $\mathcal{E}^+$ is the image of the natural inclusion $(1/\rho_k)\mathcal{O}_{\mathfrak{X}}(1) \to (1/\rho_k)\widehat{\mathcal{O}}_X^+(1)$ via the connecting map ∂. By construction, it is the obstruction class to lift $\widehat{\mathcal{O}}_X^+$ (as a sheaf of $\mathcal{O}_{\mathfrak{X}}$-algebras) to a sheaf of $\mathcal{O}_{\widetilde{\mathfrak{X}}}$-algebras in the sense of [Illusie 1971, III Proposition 2.1.2.3]. In particular, $\mathcal{E}^+$ depends on the choice of $\widetilde{\mathfrak{X}}$. When $\mathfrak{X}$ comes from a smooth formal scheme $\mathfrak{X}_0$ over $\mathcal{O}_k$ and $\widetilde{\mathfrak{X}}$ is the base-change of $\mathfrak{X}_0$ along $\mathcal{O}_k \to A_2$, the $\mathcal{E}^+$ coincides with the usual Faltings' extension after inverting p. So we call $\mathcal{E}^+$ the *integral Faltings's extension* (with respect to the lifting $\widetilde{\mathfrak{X}}$).

It remains to prove the claim (2-5).

Lemma 2.11. *With the notation as above, we have*

$$\widehat{\mathrm{L}}_{\widehat{\mathcal{O}}_X^+/\mathcal{O}_{\mathfrak{X}}} \simeq \widehat{\mathcal{O}}_X^+ \otimes_{\mathcal{O}_{\mathfrak{X}}} \widehat{\Omega}^1_{\mathfrak{X}}.$$

Proof. Since the problem is local on $X_{\text{proét}}$, by the proof of [Scholze 2013a, Corollary 4.7], we may assume $\mathfrak{X} = \mathrm{Spf}(R)$ is small and are reduced to showing, for any perfectoid affinoid space $U = \mathrm{Spa}(S, S^+) \in X_{\text{proét}}/X_\infty$,

$$\widehat{\mathrm{L}}_{S^+/R^+} \simeq S^+ \otimes_{R^+} \widehat{\Omega}^1_{R^+}. \tag{2-7}$$

Since both S^+ and $\widehat{R}^+_\infty$ are perfectoid rings, by [Bhatt et al. 2018, Lemma 3.14], we have a quasi-isomorphism

$$\widehat{\mathrm{L}}_{\widehat{R}^+_\infty/R^+}\widehat{\otimes}_{\widehat{R}^+_\infty} S^+ \to \widehat{\mathrm{L}}_{S^+/R^+}.$$

Combining this with Lemma 2.5, we get (2-7) as desired. □

Faltings' extension as obstruction class. In this subsection, we shall give another description of the integral Faltings' extension from the perspective of deformation theory. To make the notation clear, in this subsection, for a sheaf S of A_2-algebras, we always identify $\xi_k A_2$ with $(1/\rho_k)S(1)$. Before moving on, we recall some basic results due to Illusie. Although their statements are given in terms of rings, all results still hold for ring topoi.

Let A be a ring with an ideal $I \lhd A$ satisfying $I^2 = 0$. Put $\overline{A} = A/I$ and fix a flat $\overline{A}$-algebra $\overline{B}$. A natural question is whether there exists a flat A-algebra B whose reduction modulo I is $\overline{B}$.

Theorem 2.12 [Illusie 1971, III Proposition 2.1.2.3]. *There is an obstruction class* $\mathrm{cl}\in\mathrm{Ext}^2(\mathrm{L}_{\overline{B}/\overline{A}},\overline{B}\otimes_{\overline{A}}I)$ *such that* $\overline{B}$ *lifts to some flat* A*-algebra* B *if and only if* $\mathrm{cl}=0$. *In this case, the set of isomorphism classes of such deformations forms a torsor under* $\mathrm{Ext}^1(\mathrm{L}_{\overline{B}/\overline{A}},\overline{B}\otimes_{\overline{A}}I)$ *and the group of automorphisms of a fixed deformation is* $\mathrm{Hom}(\mathrm{L}_{\overline{B}/\overline{A}},\overline{B}\otimes_{\overline{A}}I)$.

If B and C are flat A-algebras with reductions $\overline{B}$ and $\overline{C}$, respectively, and if $\bar{f}:\overline{B}\to\overline{C}$ is a morphism of $\overline{A}$-algebras, then one can ask whether there exists a deformation $f:B\to C$ of $\bar{f}$ along $A\to\overline{A}$.

Theorem 2.13 [Illusie 1971, III Proposition 2.2.2]. *There is an obstruction class* $\mathrm{cl}\in\mathrm{Ext}^1(\mathrm{L}_{\overline{B}/\overline{A}},\overline{C}\otimes_{\overline{A}}I)$ *such that* $\bar{f}$ *lifts to a morphism* $f:B\to C$ *if and only if* $\mathrm{cl}=0$. *In this case, the set of all lifts forms a torsor under* $\mathrm{Hom}(\mathrm{L}_{\overline{B}/\overline{A}},\overline{C}\otimes_{\overline{A}}I)$.

We only focus on the case where $(A,I)=(A_2,(\xi))$. Let $\mathfrak{X}$ be a smooth formal scheme over $\mathcal{O}_{\mathbb{C}_p}$ and denote by

$$\mathrm{ob}(\mathfrak{X})\in\mathrm{Ext}^2\Big(\widehat{\mathrm{L}}_{\mathcal{O}_{\mathfrak{X}}/\mathcal{O}_{\mathbb{C}_p}},\frac{1}{\rho_k}\mathcal{O}_{\mathfrak{X}}(1)\Big)$$

the obstruction class to lift $\mathfrak{X}$ to a flat A_2-scheme (see, for example, [Illusie 1971, III Théorème 2.1.7]). Consider the exact triangle

$$\mathrm{L}_{\mathcal{O}_{\mathbb{C}_p}/A_2}\widehat{\otimes}^L_{\mathcal{O}_{\mathbb{C}_p}}\mathcal{O}_{\mathfrak{X}}\to\widehat{\mathrm{L}}_{\mathcal{O}_{\mathfrak{X}}/A_2}\to\widehat{\mathrm{L}}_{\mathcal{O}_{\mathfrak{X}}/\mathcal{O}_{\mathbb{C}_p}}$$

and the induced long exact sequence

$$\cdots\to\mathrm{Ext}^1\Big(\widehat{\mathrm{L}}_{\mathcal{O}_{\mathfrak{X}}/A_2},\frac{1}{\rho_k}\mathcal{O}_{\mathfrak{X}}(1)\Big)\to\mathrm{Ext}^1\Big(\mathrm{L}_{\mathcal{O}_{\mathbb{C}_p}/A_2}\widehat{\otimes}^L_{\mathcal{O}_{\mathbb{C}_p}}\mathcal{O}_{\mathfrak{X}},\frac{1}{\rho_k}\mathcal{O}_{\mathfrak{X}}(1)\Big)\xrightarrow{\partial}\mathrm{Ext}^2\Big(\widehat{\mathrm{L}}_{\mathcal{O}_{\mathfrak{X}}/\mathcal{O}_{\mathbb{C}_p}},\frac{1}{\rho_k}\mathcal{O}_{\mathfrak{X}}(1)\Big)\to\cdots.$$

The obstruction class $\mathrm{ob}(\mathfrak{X})$ is the image of the identity morphism of $(1/\rho_k)\mathcal{O}_{\mathfrak{X}}(1)$ under ∂ via the canonical isomorphism

$$\mathrm{Ext}^1\Big(\mathrm{L}_{\mathcal{O}_{\mathbb{C}_p}/A_2}\widehat{\otimes}^L_{\mathcal{O}_{\mathbb{C}_p}}\mathcal{O}_{\mathfrak{X}},\frac{1}{\rho_k}\mathcal{O}_{\mathfrak{X}}(1)\Big)\cong\mathrm{Hom}\Big(\frac{1}{\rho_k}\mathcal{O}_{\mathfrak{X}}(1),\frac{1}{\rho_k}\mathcal{O}_{\mathfrak{X}}(1)\Big).$$

If $\mathfrak{X}$ is also liftable and $\widetilde{\mathfrak{X}}$ is such a lifting, then $\mathrm{ob}(\mathfrak{X})=0$ and $\widetilde{\mathfrak{X}}$ defines a class

$$[\widetilde{\mathfrak{X}}]\in\mathrm{Ext}^1\Big(\widehat{\mathrm{L}}_{\mathcal{O}_{\mathfrak{X}}/A_2},\frac{1}{\rho_k}\mathcal{O}_{\mathfrak{X}}(1)\Big)$$

which goes to the identity map of $(1/\rho_k)\mathcal{O}_{\mathfrak{X}}(1)$. Indeed, $[\widetilde{\mathfrak{X}}]$ is the image of the identity map of $(1/\rho_k)\mathcal{O}_{\mathfrak{X}}(1)$ via the morphism

$$\mathrm{Ext}^1\Big(\widehat{\mathrm{L}}_{\mathcal{O}_{\mathfrak{X}}/\mathcal{O}_{\widetilde{\mathfrak{X}}}},\frac{1}{\rho_k}\mathcal{O}_{\mathfrak{X}}(1)\Big)\to\mathrm{Ext}^1\Big(\widehat{\mathrm{L}}_{\mathcal{O}_{\mathfrak{X}}/A_2},\frac{1}{\rho_k}\mathcal{O}_{\mathfrak{X}}(1)\Big).$$

We also consider the similar deformation problem for $\widehat{\mathcal{O}}_X^+$. Since $\widehat{\mathcal{O}}_X^+$ is locally perfectoid, thanks to [Bhatt et al. 2018, Lemma 3.14], $\widehat{\mathrm{L}}_{\widehat{\mathcal{O}}_X^+/\mathcal{O}_{\mathbb{C}_p}}=0$ and hence we get a quasi-isomorphism

$$\mathrm{L}_{\mathcal{O}_{\mathbb{C}_p}/A_2}\widehat{\otimes}^L_{\mathcal{O}_{\mathbb{C}_p}}\widehat{\mathcal{O}}_X^+\simeq\widehat{\mathrm{L}}_{\widehat{\mathcal{O}}_X^+/A_2}.$$

In particular, we have an isomorphism

$$\mathrm{Ext}^1\Big(\widehat{\mathrm{L}}_{\widehat{\mathcal{O}}^+_X/A_2}, \frac{1}{\rho_k}\widehat{\mathcal{O}}^+_X(1)\Big) \cong \mathrm{Hom}\Big(\frac{1}{\rho_k}\widehat{\mathcal{O}}^+_X(1), \frac{1}{\rho_k}\widehat{\mathcal{O}}^+_X(1)\Big).$$

Therefore, $\widehat{\mathcal{O}}^+_X$ admits a canonical lifting, which turns out to be $\mathrm{A}_2(\widehat{\mathcal{O}}^+_X)$ and there is a unique class

$$[X] \in \mathrm{Ext}^1\Big(\widehat{\mathrm{L}}_{\widehat{\mathcal{O}}^+_X/A_2}, \frac{1}{\rho_k}\widehat{\mathcal{O}}^+_X(1)\Big)$$

corresponding to the identity map of $(1/\rho_k)\widehat{\mathcal{O}}^+_X(1)$.

Regard $[\widetilde{\mathfrak{X}}]$ and $[X]$ as classes in $\mathrm{Ext}^1(\widehat{\mathrm{L}}_{\mathcal{O}_{\mathfrak{X}}/A_2}, (1/\rho_k)\widehat{\mathcal{O}}^+_X(1))$ via the canonical morphisms induced by $(1/\rho_k)\mathcal{O}_{\mathfrak{X}}(1) \to (1/\rho_k)\widehat{\mathcal{O}}^+_X(1)$ and $\widehat{\mathrm{L}}_{\mathcal{O}_{\mathfrak{X}}/A_2} \to \widehat{\mathrm{L}}_{\widehat{\mathcal{O}}^+_X/A_2}$, respectively. Then as shown in [Illusie 1971, III Proposition 2.2.4], the difference

$$\mathrm{cl}(\mathcal{E}^+) := [\widetilde{\mathfrak{X}}] - [X]$$

belongs to

$$\mathrm{Ext}^1\Big(\widehat{\mathrm{L}}_{\mathcal{O}_{\mathfrak{X}}/\mathcal{O}_{\mathbb{C}_p}}, \frac{1}{\rho_k}\widehat{\mathcal{O}}^+_X(1)\Big) \cong \mathrm{Ext}^1\Big(\widehat{\Omega}^1_{\mathcal{O}_{\mathfrak{X}}/\mathcal{O}_{\mathbb{C}_p}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \widehat{\mathcal{O}}^+_X, \frac{1}{\rho_k}\widehat{\mathcal{O}}^+_X(1)\Big)$$

via the injection

$$\mathrm{Ext}^1\Big(\widehat{\mathrm{L}}_{\mathcal{O}_{\mathfrak{X}}/\mathcal{O}_{\mathbb{C}_p}}, \frac{1}{\rho_k}\widehat{\mathcal{O}}^+_X(1)\Big) \to \mathrm{Ext}^1\Big(\widehat{\mathrm{L}}_{\mathcal{O}_{\mathfrak{X}}/A_2}, \frac{1}{\rho_k}\widehat{\mathcal{O}}^+_X(1)\Big),$$

and $\mathrm{cl}(\mathcal{E}^+)$ is the obstruction answering whether there is an A_2-morphism from $\mathcal{O}_{\widetilde{\mathfrak{X}}}$ to $\mathrm{A}_2(\widehat{\mathcal{O}}^+_X)$ which lifts the $\mathcal{O}_{\mathbb{C}_p}$-morphism $\mathcal{O}_{\mathfrak{X}} \to \widehat{\mathcal{O}}^+_X$ as described in Theorem 2.13.

Recall we have another obstruction class $[\mathcal{E}^+]$ described in Remark 2.10. We claim that it coincides with the class $\mathrm{cl}(\mathcal{E}^+)$ constructed above.

Proposition 2.14. $\mathrm{cl}(\mathcal{E}^+) = [\mathcal{E}^+]$.

Proof. Note that we have a commutative diagram of morphisms of cotangent complexes

$$\begin{array}{ccccccc}
\mathrm{L}_{\mathcal{O}_{\widetilde{\mathfrak{X}}}/\mathrm{A}_2}\widehat{\otimes}^L_{\mathcal{O}_{\widetilde{\mathfrak{X}}}}\widehat{\mathcal{O}}^+_X & \longrightarrow & \mathrm{L}_{\mathcal{O}_{\mathfrak{X}}/\mathrm{A}_2}\widehat{\otimes}^L_{\mathcal{O}_{\mathfrak{X}}}\widehat{\mathcal{O}}^+_X & \xrightarrow{\alpha} & \mathrm{L}_{\mathcal{O}_{\mathfrak{X}}/\mathcal{O}_{\widetilde{\mathfrak{X}}}}\widehat{\otimes}^L_{\mathcal{O}_{\mathfrak{X}}}\widehat{\mathcal{O}}^+_X & \xrightarrow{+1} & \\
\| & & \downarrow{\scriptstyle\beta} & & \downarrow & & \\
\mathrm{L}_{\mathcal{O}_{\widetilde{\mathfrak{X}}}/\mathrm{A}_2}\widehat{\otimes}^L_{\mathcal{O}_{\widetilde{\mathfrak{X}}}}\widehat{\mathcal{O}}^+_X & \longrightarrow & \widehat{\mathrm{L}}_{\widehat{\mathcal{O}}^+_X/\mathrm{A}_2} & \longrightarrow & \widehat{\mathrm{L}}_{\widehat{\mathcal{O}}^+_X/\mathcal{O}_{\widetilde{\mathfrak{X}}}} & \xrightarrow{+1} & \\
& \underset{-1}{\overset{\simeq}{\searrow}} & \downarrow & & \downarrow & & \\
& & \widehat{\mathrm{L}}_{\widehat{\mathcal{O}}^+_X/\mathcal{O}_{\mathfrak{X}}} & = & \widehat{\mathrm{L}}_{\widehat{\mathcal{O}}^+_X/\mathcal{O}_{\mathfrak{X}}} & & \\
& & \downarrow{\scriptstyle +1} & & \downarrow{\scriptstyle +1} & &
\end{array} \qquad (2\text{-}8)$$

where the notation "+1" and "−1" denote the shifts of dimensions.

Consider the resulting diagram from applying $\mathrm{RHom}(-,(1/\rho_k)\widehat{\mathcal{O}}_X^+(1))$ to (2-8). Denote the identity map of $(1/\rho_k)\widehat{\mathcal{O}}_X^+(1)$ by id. By construction, $[\mathcal{E}^+]$ is the image of id via the connecting map induced by the triangle

$$\mathrm{L}_{\mathcal{O}_{\mathfrak{X}}/\mathcal{O}_{\widetilde{\mathfrak{X}}}}\widehat{\otimes}^L_{\mathcal{O}_{\mathfrak{X}}}\widehat{\mathcal{O}}_X^+\to\widehat{\mathrm{L}}_{\widehat{\mathcal{O}}_X^+/\mathcal{O}_{\widetilde{\mathfrak{X}}}}\to\widehat{\mathrm{L}}_{\widehat{\mathcal{O}}_X^+/\mathcal{O}_{\mathfrak{X}}}.$$

By the commutativity of diagram (2-8), $[\mathcal{E}^+]$ is also the image of $\alpha^*(\mathrm{id})$ via the connecting map ∂ induced by the triangle

$$\mathrm{L}_{\mathcal{O}_{\mathfrak{X}}/A_2}\widehat{\otimes}^L_{\mathcal{O}_{\mathfrak{X}}}\widehat{\mathcal{O}}_X^+\to\widehat{\mathrm{L}}_{\widehat{\mathcal{O}}_X^+/A_2}\to\widehat{\mathrm{L}}_{\widehat{\mathcal{O}}_X^+/\mathcal{O}_{\mathfrak{X}}}.$$

On the other hand, by the constructions of $[\widetilde{\mathfrak{X}}]$ and $[X]$, as elements in

$$\mathrm{Ext}^1\Big(\mathrm{L}_{\mathcal{O}_{\mathfrak{X}}/\mathrm{A}_2}\widehat{\otimes}^L_{\mathcal{O}_{\mathfrak{X}}}\widehat{\mathcal{O}}_X^+,\frac{1}{\rho_k}\widehat{\mathcal{O}}_X^+(1)\Big),$$

we have $[\widetilde{\mathfrak{X}}]=\alpha^*(\mathrm{id})$ and $[X]=\beta^*(\mathrm{id})$; here, for the second equality, we identify

$$\mathrm{Hom}\Big(\frac{1}{\rho_k}\widehat{\mathcal{O}}_X^+(1),\frac{1}{\rho_k}\widehat{\mathcal{O}}_X^+(1)\Big)=\mathrm{Ext}^1\Big(\widehat{\mathrm{L}}_{\mathcal{O}_{\mathbb{C}_p}/A_2}\widehat{\otimes}^L_{\mathcal{O}_{\mathbb{C}_p}}\widehat{\mathcal{O}}_X^+,\frac{1}{\rho_k}\widehat{\mathcal{O}}_X^+(1)\Big)$$

with $\mathrm{Ext}^1(\widehat{\mathrm{L}}_{\widehat{\mathcal{O}}_X^+/A_2}\widehat{\otimes}^L_{\mathcal{O}_{\mathbb{C}_p}}\widehat{\mathcal{O}}_X^+,(1/\rho_k)\widehat{\mathcal{O}}_X^+(1))$. So we have

$$\mathrm{cl}(\mathcal{E}^+)=\alpha^*(\mathrm{id})-\beta^*(\mathrm{id})\in\mathrm{Ext}^1\Big(\mathrm{L}_{\mathcal{O}_{\mathfrak{X}}/A_2}\widehat{\otimes}^L_{\mathcal{O}_{\mathfrak{X}}}\widehat{\mathcal{O}}_X^+,\frac{1}{\rho_k}\widehat{\mathcal{O}}_X^+(1)\Big).$$

However, the diagram

$$\begin{array}{ccc}
\widehat{\mathrm{L}}_{\mathrm{A}_2(\widehat{\mathcal{O}}_X^+)/\mathcal{O}_{\widetilde{\mathfrak{X}}}} & \xrightarrow{\quad +1 \quad} & \mathrm{L}_{\mathcal{O}_{\widetilde{\mathfrak{X}}}/\mathrm{A}_2}\widehat{\otimes}^L_{\mathcal{O}_{\widetilde{\mathfrak{X}}}}\widehat{\mathcal{O}}_X^+ \\
\downarrow & & \downarrow \\
\widehat{\mathrm{L}}_{\widehat{\mathcal{O}}_X^+/\mathcal{O}_{\mathfrak{X}}} \xrightarrow{+1} & \mathrm{L}_{\mathcal{O}_{\mathfrak{X}}/\mathrm{A}_2}\widehat{\otimes}^L_{\mathcal{O}_{\widetilde{\mathfrak{X}}}}\widehat{\mathcal{O}}_X^+ & \longrightarrow \mathrm{L}_{\mathcal{O}_{\mathfrak{X}}/\mathcal{O}_{\mathbb{C}_p}}\widehat{\otimes}^L_{\mathcal{O}_{\mathfrak{X}}}\widehat{\mathcal{O}}_X^+
\end{array}$$

induces a commutative diagram

$$\begin{array}{ccc}
\mathrm{Ext}^1\big(\mathrm{L}_{\mathcal{O}_{\mathfrak{X}}/\mathcal{O}_{\mathbb{C}_p}}\widehat{\otimes}^L_{\mathcal{O}_{\mathfrak{X}}}\widehat{\mathcal{O}}_X^+,\tfrac{1}{\rho_k}\widehat{\mathcal{O}}_X^+(1)\big) & \xrightarrow{\subset} & \mathrm{Ext}^1\big(\mathrm{L}_{\mathcal{O}_{\mathfrak{X}}/A_2}\widehat{\otimes}^L_{\mathcal{O}_{\mathfrak{X}}}\widehat{\mathcal{O}}_X^+,\tfrac{1}{\rho_k}\widehat{\mathcal{O}}_X^+(1)\big) \\
& \searrow^{\cong} & \downarrow \partial \\
& & \mathrm{Ext}^2\big(\widehat{\mathrm{L}}_{\widehat{\mathcal{O}}_X^+/\mathcal{O}_{\mathfrak{X}}},\tfrac{1}{\rho_k}\widehat{\mathcal{O}}_X^+(1)\big)
\end{array}$$

In particular, as elements in $\mathrm{Ext}^1(\mathrm{L}_{\mathcal{O}_{\mathfrak{X}}/\mathcal{O}_{\mathbb{C}_p}}\widehat{\otimes}^L_{\mathcal{O}_{\mathfrak{X}}}\widehat{\mathcal{O}}_X^+,(1/\rho_k)\widehat{\mathcal{O}}_X^+(1))$, we have

$$\mathrm{cl}(\mathcal{E}^+)=\partial(\alpha^*(\mathrm{id})-\beta^*(\mathrm{id}))=\partial(\alpha^*(\mathrm{id}))=[\mathcal{E}^+]$$

□

Remark 2.15. When $\mathfrak{X}$ is small affine and comes from a formal scheme over $\mathcal{O}_k$, the obstruction class $\mathrm{cl}(\mathcal{E}^+)$ was considered as a *Higgs–Tate extension associated to* $\widetilde{\mathfrak{X}}$ in [Abbes et al. 2016, I. 4.3].

Example 2.16. *Let* $R^+ = \mathcal{O}_{\mathbb{C}_p}\langle \underline{T}^{\pm 1}\rangle$ *and* $\widetilde{R}^+ = A_2\langle \underline{T}^{\pm 1}\rangle$ *for simplicity. Consider the* A_2*-morphism* $\widetilde{\psi} : \widetilde{R}^+ \to \mathrm{A}_2(\widehat{R}^+_\infty)$, *which sends* T_i *to* $[T_i^\flat]$ *for all* i, *where* $T_i^\flat \in \widehat{R}^{\flat +}_\infty$ *is determined by the compatible sequence* $(T_i^{1/p^n})_{n\geq 0}$. *Then* $\widetilde{\psi}$ *is a lifting of the inclusion* $R^+ \to \widehat{R}^+_\infty$, *but is not* Γ*-equivariant. For any* $\gamma \in \Gamma$, $\gamma \circ \widetilde{\psi}$ *is another lifting. By Theorem 2.13, their difference* $c(\gamma) := \gamma \circ \widetilde{\psi} - \widetilde{\psi}$ *belongs to* $\mathrm{Hom}_{R^+}(\widehat{\Omega}^1_{R^+}, (1/\rho_k)\widehat{R}^+_\infty(1))$. *One can check that, for any* $1 \leq i, j \leq 1$,

$$c(\gamma_i)(\mathrm{dlog}(T_j)) = \frac{(\gamma_i - 1)([T_j^\flat])}{T_j} = \delta_{ij}([\epsilon] - 1) = \delta_{ij} t,$$

where the last equality follows from the fact that $[\epsilon] - 1 - t \in t^2 \mathrm{B}^+_{\mathrm{dR}}$. *By construction, the cocycle* $c : \Gamma \to \mathrm{Hom}_{R^+}(\widehat{\Omega}^1_{R^+}, (1/\rho_k)\widehat{R}^+_\infty(1))$ *is exactly the class* $\mathrm{cl}(\mathcal{E}^+)$. *Comparing this with Proposition 2.8, we deduce that* $\mathrm{cl}(\mathcal{E}^+) = [\mathcal{E}^+]$ *in this case.*

As an application of Proposition 2.14, we study the behaviour of integral Faltings' extension under the pull-back.

Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a formally smooth morphism of liftable smooth formal schemes. Fix liftings $\widetilde{\mathfrak{X}}$ and $\widetilde{\mathfrak{Y}}$ of $\mathfrak{X}$ and $\mathfrak{Y}$, respectively. Denote by $\mathcal{E}^+_X$ and $\mathcal{E}^+_Y$ the corresponding integral Faltings' extensions. Then the pull-back of $\mathcal{E}^+_X$ along the injection

$$f^*\widehat{\Omega}^1_{\mathfrak{Y}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \widehat{\mathcal{O}}^+_X \to \widehat{\Omega}^1_{\mathfrak{X}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \widehat{\mathcal{O}}^+_X$$

defines an extension $\mathcal{E}^+_1$ of $\widehat{\Omega}^1_{\mathfrak{Y}} \otimes_{\mathcal{O}_{\mathfrak{Y}}} \widehat{\mathcal{O}}^+_X \cong f^*\widehat{\Omega}^1_{\mathfrak{Y}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \widehat{\mathcal{O}}^+_X$ by $(1/\rho_k)\widehat{\mathcal{O}}^+_X(1)$.[1] We denote its extension class by

$$\mathrm{cl}_1 \in \mathrm{Ext}^1\left(\widehat{\Omega}^1_{\mathfrak{Y}} \otimes_{\mathcal{O}_{\mathfrak{Y}}} \widehat{\mathcal{O}}^+_X, \frac{1}{\rho_k}\widehat{\mathcal{O}}^+_X(1)\right).$$

On the other hand, the tensor product $\mathcal{E}^+_2 = \mathcal{E}^+_Y \otimes_{\widehat{\mathcal{O}}^+_Y} \widehat{\mathcal{O}}^+_X$ induced by applying $- \otimes_{\widehat{\mathcal{O}}^+_Y} \widehat{\mathcal{O}}^+_X$ to

$$0 \to \frac{1}{\rho_k}\widehat{\mathcal{O}}^+_Y(1) \to \mathcal{E}^+_Y \to \widehat{\mathcal{O}}^+_Y \otimes_{\mathcal{O}_{\mathfrak{Y}}} \widehat{\Omega}^1_{\mathfrak{Y}} \to 0$$

is also an extension of $\widehat{\Omega}^1_{\mathfrak{Y}} \otimes_{\mathcal{O}_{\mathfrak{Y}}} \widehat{\mathcal{O}}^+_X$ by $(1/\rho_k)\widehat{\mathcal{O}}^+_X(1)$ and we denote the associated extension class by

$$\mathrm{cl}_2 \in \mathrm{Ext}^1\left(\widehat{\Omega}^1_{\mathfrak{Y}} \otimes_{\mathcal{O}_{\mathfrak{Y}}} \widehat{\mathcal{O}}^+_X, \frac{1}{\rho_k}\widehat{\mathcal{O}}^+_X(1)\right).$$

Then it is natural to ask whether $\mathcal{E}^+_1 \cong \mathcal{E}^+_2$ (equivalently, $\mathrm{cl}_1 = \mathrm{cl}_2$).

Proposition 2.17. *Keep the notation as above. If* $f : \mathfrak{X} \to \mathfrak{Y}$ *lifts to an* A_2*-morphism* $\tilde{f} : \widetilde{\mathfrak{X}} \to \widetilde{\mathfrak{Y}}$, *then* $\mathrm{cl}_1 = \mathrm{cl}_2$.

We are going to prove this proposition in the rest of this subsection.

[1] Here, the tensor product $\widehat{\Omega}^1_{\mathfrak{Y}} \otimes_{\mathcal{O}_{\mathfrak{Y}}} \widehat{\mathcal{O}}^+_X$ should be understood as $f^{-1}\widehat{\Omega}^1_{\mathfrak{Y}} \otimes_{f^{-1}\mathcal{O}_{\mathfrak{Y}}} \widehat{\mathcal{O}}^+_X$. The same applies to sheaves like $\mathcal{O}^+_X \otimes_{\widehat{\mathcal{O}}^+_Y} \mathcal{E}^+_Y$, $\widehat{\mathcal{O}}^+_X \otimes_{\widehat{\mathcal{O}}^+_Y} \mathcal{O}\mathbb{C}^+_{Y,\rho}$, $\widehat{\mathcal{O}}^+_X \otimes_{\widehat{\mathcal{O}}^+_Y} \mathcal{O}\widehat{\mathbb{C}}^+_{Y,\rho}$, $\widehat{\mathcal{O}}^+_X \otimes_{\widehat{\mathcal{O}}^+_Y} \mathcal{O}\mathbb{C}^{\dagger,+}_{Y,\rho}$.

By Theorem 2.13, there exists an obstruction class

$$\mathrm{cl}(f) \in \mathrm{Ext}^1\Big(\widehat{\mathrm{L}}_{\mathcal{O}_{\mathfrak{Y}}/\mathcal{O}_{\mathbb{C}_p}}, \frac{1}{\rho_k}\mathcal{O}_{\mathfrak{X}}(1)\Big)$$

to lift f along the surjection $A_2 \to \mathcal{O}_{\mathbb{C}_p}$. Before moving on, let us recall the definition of $\mathrm{cl}(f)$.

Let $[\widetilde{\mathfrak{X}}]$ and $[\widetilde{\mathfrak{Y}}]$ be classes defined as before and regard them as elements in $\mathrm{Ext}^1(\widehat{\mathrm{L}}_{\mathcal{O}_{\mathfrak{Y}}/A_2}, (1/\rho_k)\mathcal{O}_{\mathfrak{X}}(1))$ via the obvious morphisms. Then similar to the construction of $\mathrm{cl}(\mathcal{E}^+)$, one can check that

$$\mathrm{cl}(f) = [\widetilde{\mathfrak{X}}] - [\widetilde{\mathfrak{Y}}]$$

via the injection

$$\mathrm{Ext}^1\Big(\widehat{\mathrm{L}}_{\mathcal{O}_{\mathfrak{Y}}/\mathcal{O}_{\mathbb{C}_p}}, \frac{1}{\rho_k}\mathcal{O}_{\mathfrak{X}}(1)\Big) \to \mathrm{Ext}^1\Big(\widehat{\mathrm{L}}_{\mathcal{O}_{\mathfrak{Y}}/A_2}, \frac{1}{\rho_k}\mathcal{O}_{\mathfrak{X}}(1)\Big).$$

For simplicity, we still denote by $\mathrm{cl}(f)$ its image in

$$\mathrm{Ext}^1\Big(\widehat{\mathrm{L}}_{\mathcal{O}_{\mathfrak{Y}}/\mathcal{O}_{\mathbb{C}_p}}, \frac{1}{\rho_k}\widehat{\mathcal{O}}_X^+(1)\Big) \cong \mathrm{Ext}^1\Big(\widehat{\Omega}^1_{\mathfrak{Y}} \otimes_{\mathcal{O}_{\mathfrak{Y}}} \widehat{\mathcal{O}}_X^+, \frac{1}{\rho_k}\widehat{\mathcal{O}}_X^+(1)\Big)$$

via the natural map $(1/\rho_k)\mathcal{O}_{\mathfrak{X}}(1) \to (1/\rho_k)\widehat{\mathcal{O}}_X^+(1)$. Then the following proposition is true.

Proposition 2.18. $$\mathrm{cl}(f) = \mathrm{cl}_1 - \mathrm{cl}_2.$$

Proof. By the constructions of $\mathcal{E}_1^+$ and $\mathcal{E}_2^+$, we see that cl_1 is the image of $\mathrm{cl}(\mathcal{E}_X^+)$ via the morphism

$$\mathrm{Ext}^1\Big(\widehat{\Omega}^1_{\mathfrak{X}}, \frac{1}{\rho_k}\widehat{\mathcal{O}}_X^+(1)\Big) \to \mathrm{Ext}^1\Big(\widehat{\Omega}^1_{\mathfrak{Y}} \otimes_{\mathcal{O}_{\mathfrak{Y}}} \mathcal{O}_{\mathfrak{X}}, \frac{1}{\rho_k}\widehat{\mathcal{O}}_X^+(1)\Big)$$

induced by

$$\mathrm{L}_{\mathcal{O}_{\mathfrak{Y}}/\mathcal{O}_{\mathbb{C}_p}} \widehat{\otimes}^L_{\mathcal{O}_{\mathfrak{Y}}} \mathcal{O}_{\mathfrak{X}} \to \widehat{\mathrm{L}}_{\mathcal{O}_{\mathfrak{X}}/\mathcal{O}_{\mathbb{C}_p}},$$

and that cl_2 is the image of $\mathrm{cl}(\mathcal{E}_Y^+)$ via the morphism

$$\mathrm{Ext}^1\Big(\widehat{\Omega}^1_{\mathfrak{Y}} \otimes_{\mathcal{O}_{\mathfrak{Y}}} \widehat{\mathcal{O}}_Y^+, \frac{1}{\rho_k}\widehat{\mathcal{O}}_Y^+(1)\Big) \to \mathrm{Ext}^1\Big(\widehat{\Omega}^1_{\mathfrak{Y}} \otimes_{\mathcal{O}_{\mathfrak{Y}}} \widehat{\mathcal{O}}_X^+, \frac{1}{\rho_k}\widehat{\mathcal{O}}_X^+(1)\Big)$$

induced by the inclusion $(1/\rho_k)\widehat{\mathcal{O}}_Y^+(1) \to (1/\rho_k)\widehat{\mathcal{O}}_X^+(1)$.

Now by Proposition 2.14, we have

$$\mathrm{cl}_1 - \mathrm{cl}_2 = \mathrm{cl}(\mathcal{E}_X^+) - \mathrm{cl}(\mathcal{E}_Y^+) = ([\widetilde{\mathfrak{X}}] - [\widetilde{\mathfrak{Y}}]) - ([X] - [Y]).$$

However, the inclusion $\widehat{\mathcal{O}}_Y^+ \to \widehat{\mathcal{O}}_X^+$ admits a canonical A_2-lifting, namely $\mathrm{A}_2(\widehat{\mathcal{O}}_Y^+) \to \mathrm{A}_2(\widehat{\mathcal{O}}_X^+)$. So we deduce that $[X] - [Y] = 0$, which completes the proof. □

Now, Proposition 2.17 is a special case of Proposition 2.18.

Corollary 2.19. *Assume $f : \mathfrak{X} \to \mathfrak{Y}$ admits a lifting along $A_2 \to \mathcal{O}_{\mathbb{C}_p}$. Then there is an exact sequence of sheaves of $\widehat{\mathcal{O}}_X^+$-modules*

$$0 \to \widehat{\mathcal{O}}_X^+ \otimes_{\widehat{\mathcal{O}}_Y^+} \mathcal{E}_Y^+ \to \mathcal{E}_X^+ \to \widehat{\mathcal{O}}_X^+ \otimes_{\mathcal{O}_{\mathfrak{X}}} \widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}} \to 0, \tag{2-9}$$

where $\widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}}$ is the module of relative differentials.

Proof. This follows from the Proposition 2.17 combined with the definitions of $\mathcal{E}_1^+$ and $\mathcal{E}_2^+$. □

Period sheaves. Now, we define the desired period sheaf $\mathcal{O}\mathbb{C}^{\dagger}$ as mentioned in Section 1. The construction generalises the previous work of Hyodo [1989].

Let $\mathfrak{X} = \mathrm{Spf}(R^+)$ be a small smooth formal scheme and $\widetilde{\mathfrak{X}} = \mathrm{Spf}(\widetilde{R}^+)$ be a fixed A_2-lifting. Let E^+ be the integral Faltings' extension introduced in Proposition 2.6. Define $E^+_{\rho_k} = \rho_k E^+(-1)$. Then it fits into the exact sequence

$$0 \to \widehat{R}^+_\infty \to E^+_{\rho_k} \to \rho_k \widehat{R}^+_\infty \otimes_{R^+} \widehat{\Omega}^1_{R^+}(-1) \to 0.$$

For any $\rho \in \rho_k \mathcal{O}_{\mathbb{C}_p}$, denote by E^+_ρ the pull-back of $E^+_{\rho_k}$ along the inclusion

$$\rho \widehat{R}^+_\infty \otimes_{R^+} \widehat{\Omega}^1_{R^+}(-1) \to \rho_k \widehat{R}^+_\infty \otimes_{R^+} \widehat{\Omega}^1_{R^+}(-1).$$

Then it fits into the Γ-equivariant exact sequence

$$0 \to \widehat{R}^+_\infty \to E^+_\rho \to \rho \widehat{R}^+_\infty \otimes_{R^+} \widehat{\Omega}^1_{R^+}(-1) \to 0. \tag{2-10}$$

By Proposition 2.8, E^+_ρ admits an $\widehat{R}^+_\infty$-basis $1, (\rho x_1)/t, \dots, (\rho x_d)/t$. Let $E = E^+_\rho\left[\frac{1}{p}\right]$, which fits into the induced exact sequence

$$0 \to \widehat{R}_\infty \to E \to \widehat{R}_\infty \otimes_{R^+} \widehat{\Omega}^1_{R^+}(-1) \to 0.$$

Then it is independent of the choice of ρ and has E^+_ρ as a sub-$\widehat{R}^+_\infty$-module. Also, it admits an $\widehat{R}_\infty$-basis

$$1, y_1 = \frac{x_1}{t}, \dots, y_d = \frac{x_d}{t}$$

such that $\gamma_i(y_j) = y_j + \delta_{ij}$ for any $1 \le i, j \le d$. Define $S_\infty = \varinjlim_n \mathrm{Sym}^n_{\widehat{R}_\infty} E$. Then by similar arguments used in [Hyodo 1989, Section I], we have the following result.

Proposition 2.20. *There exists a canonical Higgs field*

$$\Theta : S_\infty \to S_\infty \otimes_{\widehat{R}_\infty} \widehat{\Omega}^1_{R^+}(-1)$$

on S_∞ such that the induced Higgs complex is a resolution of $\widehat{R}_\infty$. The Higgs field Θ is induced by taking alternative sum along the projection $E \to \widehat{R}_\infty \otimes_{R^+} \widehat{\Omega}^1_{R^+}(-1)$ and if we denote by Y_i the image of y_i in S_∞, then there is a Γ-equivariant isomorphism

$$\iota : S_\infty \xrightarrow{\cong} \widehat{R}_\infty[Y_1, \dots, Y_d]$$

such that $\Theta = \sum_{i=1}^d (\partial/\partial Y_i) \otimes ((\mathrm{dlog}\, T_i)/t)$ via this isomorphism, where $\widehat{R}_\infty[Y_1, \dots, Y_d]$ is the polynomial ring on free variables Y_i's over $\widehat{R}_\infty$.

Since we have $\widehat{R}^+_\infty$-lattices E^+_ρ's of E, inspired by Proposition 2.20, we make the following definition.

Definition 2.21. For any $\rho \in \rho_k \mathcal{O}_{\mathbb{C}_p}$, define

(1) $S^+_{\infty,\rho} = \varinjlim_n \mathrm{Sym}^n_{\widehat{R}^+_\infty} E^+_\rho$;

(2) $\widehat{S}^+_{\infty,\rho} = \varprojlim_n S^+_{\infty,\rho}/p^n$;

(3) $S^{\dagger,+}_\infty = \varinjlim_{\nu_p(\rho) > \nu_p(\rho_k)} \widehat{S}^+_{\infty,\rho}$ and $S^\dagger_\infty = S^{\dagger,+}_\infty\left[\frac{1}{p}\right]$.

For any $\rho_1, \rho_2 \in \rho_k\mathcal{O}_{\mathbb{C}_p}$ satisfying $\nu_p(\rho_1) \geq \nu_p(\rho_2)$, we have $E^+_{\rho_1} \subset E^+_{\rho_2} \subset E$. So Proposition 2.20 implies that $S^+_{\infty,\rho_1} \subset S^+_{\infty,\rho_2} \subset S_\infty$. Moreover, the restriction of Θ to $S^+_{\infty,\rho}$ (for $\rho \in \rho_k\mathcal{O}_{\mathbb{C}_p}$) induces a Higgs field on $S^+_{\infty,\rho}$, which is identified with $\widehat{R}^+_\infty[\rho Y_1, \ldots, \rho Y_d]$ via the canonical isomorphism ι. In this case, we still have $\Theta = \sum_{i=1}^d (\partial/\partial Y_i) \otimes ((\mathrm{dlog}\, T_i)/t)$. Since Θ is continuous, it extends to $\widehat{S}^+_{\infty,\rho}$ and thus we have the following corollary.

Corollary 2.22. *For any $\rho \in \rho_k\mathcal{O}_{\mathbb{C}_p}$, there exists a canonical Higgs field*

$$\Theta : \widehat{S}^+_{\infty,\rho} \to \widehat{S}^+_{\infty,\rho} \otimes_{\widehat{R}^+_\infty} \widehat{\Omega}^1_{R^+}(-1)$$

on $\widehat{S}^+_{\infty,\rho}$. Additionally, there is a Γ-equivariant isomorphism

$$\iota : \widehat{S}^+_{\infty,\rho} \xrightarrow{\cong} \widehat{R}^+_\infty\langle \rho Y_1, \ldots, \rho Y_d\rangle$$

such that

$$\Theta = \sum_{i=1}^d \frac{\partial}{\partial Y_i} \otimes \frac{\mathrm{dlog}\, T_i}{t}$$

via this isomorphism, where $\widehat{R}^+_\infty\langle \rho Y_1, \ldots, \rho Y_d\rangle$ is the p-adic completion of $\widehat{R}^+_\infty[\rho Y_1, \ldots, \rho Y_d]$.

After taking the inductive limit of $\{\rho \in \rho_k\mathcal{O}_{\mathbb{C}_p} | \nu_p(\rho) > \nu_p(\rho_k)\}$, we get the following corollary.

Corollary 2.23. *There exists a canonical Higgs field*

$$\Theta : S^{\dagger,+}_\infty \to S^{\dagger,+}_\infty \otimes_{\widehat{R}^+_\infty} \widehat{\Omega}^1_{R^+}(-1)$$

on $S^{\dagger,+}_\infty$. Additionally, there is a Γ-equivariant isomorphism

$$\iota : S^{\dagger,+}_\infty \xrightarrow{\cong} \varinjlim_{\nu_p(\rho) > \nu_p(\rho_k)} \widehat{R}^+_\infty\langle \rho Y_1, \ldots, \rho Y_d\rangle$$

such that $\Theta = \sum_{i=1}^d (\partial/\partial Y_i) \otimes ((\mathrm{dlog}\, T_i)/t)$ via this isomorphism. After inverting p, the induced Higgs complex

$$\mathrm{HIG}(S^\dagger_\infty, \Theta) : \quad S^\dagger_\infty \xrightarrow{\Theta} S^\dagger_\infty \otimes_{R^+} \widehat{\Omega}^1_{R^+}(-1) \xrightarrow{\Theta} S^\dagger_\infty \otimes_{R^+} \widehat{\Omega}^2_{R^+}(-2) \to \cdots \tag{2-11}$$

is a resolution of $\widehat{R}_\infty$.

Proof. It remains to prove the Higgs complex $\mathrm{HIG}(S^\dagger_\infty, \Theta)$ is a resolution of $\widehat{R}_\infty$. For any $\rho \in \rho_k\mathcal{O}_{\mathbb{C}_p}$, consider the Higgs complexes

$$\mathrm{HIG}(\widehat{S}^+_{\infty,\rho}, \Theta) : \quad \widehat{S}^+_{\infty,\rho} \xrightarrow{\Theta} \widehat{S}^+_{\infty,\rho} \otimes_{R^+} \widehat{\Omega}^1_{R^+}(-1) \xrightarrow{\Theta} \widehat{S}^+_{\infty,\rho} \otimes_{R^+} \widehat{\Omega}^2_{R^+}(-2) \to \cdots$$

and

$$\mathrm{HIG}(S^{\dagger,+}_\infty, \Theta) : \quad S^{\dagger,+}_\infty \xrightarrow{\Theta} \widehat{S}^{\dagger,+}_\infty \otimes_{R^+} \widehat{\Omega}^1_{R^+}(-1) \xrightarrow{\Theta} S^{\dagger,+}_\infty \otimes_{R^+} \widehat{\Omega}^2_{R^+}(-2) \to \cdots .$$

Then we have

$$\mathrm{HIG}(S^\dagger_\infty, \Theta) = \mathrm{HIG}(S^{\dagger,+}_\infty, \Theta)\Big[\frac{1}{p}\Big] = \varinjlim_{\nu_p(\rho) > \nu_p(\rho_k)} \mathrm{HIG}(\widehat{S}^+_{\infty,\rho}, \Theta)\Big[\frac{1}{p}\Big].$$

By Corollary 2.22, $\mathrm{HIG}(\widehat{S}^+_{\infty,\rho}, \Theta)$ is computed by the Koszul complex

$$\mathrm{K}\Big(\widehat{R}^+_\infty\langle\rho Y_1,\dots,\rho Y_d\rangle; \frac{\partial}{\partial Y_1},\dots,\frac{\partial}{\partial Y_d}\Big) \simeq \mathrm{K}\Big(\widehat{R}^+_\infty\langle\rho Y_1\rangle; \frac{\partial}{\partial Y_1}\Big)\widehat{\otimes}^L_{\widehat{R}^+_\infty}\cdots\widehat{\otimes}^L_{\widehat{R}^+_\infty}\mathrm{K}\Big(\widehat{R}^+_\infty\langle\rho Y_d\rangle; \frac{\partial}{\partial Y_d}\Big),$$

via the canonical isomorphism ι. Note that, for any j,

$$\mathrm{H}^i\Big(\mathrm{K}\Big(\widehat{R}^+_\infty\langle\rho Y_j\rangle; \frac{\partial}{\partial Y_j}\Big)\Big) = \begin{cases} \widehat{R}^+_\infty, & i=0,\\ \widehat{R}^+_\infty\langle\Lambda_{j,\rho}\rangle/\widehat{R}^+_\infty\langle\Lambda_{j,\rho}, I, +\rangle, & i=1,\\ 0, & i\geq 2,\end{cases}$$

is derived p-complete by Proposition A.2, where $\widehat{R}^+_\infty\langle\Lambda_{j,\rho}\rangle$ and $\widehat{R}^+_\infty\langle\Lambda_{j,\rho}, I, +\rangle$ are defined as in Definition A.1 for $\Lambda_{j,\rho}=\{\rho^n Y_j^n\}_{n\geq 0}$ and $I=\{\nu_p(n+1)\}_{n\geq 0}$. We deduce that, for any $i\geq 0$,

$$\mathrm{H}^i\Big(\mathrm{K}\Big(\widehat{R}^+_\infty\langle\rho Y_1,\dots,\rho Y_d\rangle; \frac{\partial}{\partial Y_1},\dots,\frac{\partial}{\partial Y_d}\Big)\Big) = \bigwedge^i_{\widehat{R}^+_\infty}\Big(\bigoplus_{j=1}^d \widehat{R}^+_\infty\langle\Lambda_{j,\rho}\rangle/\widehat{R}^+_\infty\langle\Lambda_{j,\rho}, I, +\rangle\Big).$$

In particular, we get

$$\mathrm{H}^0(\mathrm{HIG}(S^{\dagger,+}_\infty,\Theta)) = \varinjlim_{\nu_p(\rho)>\nu_p(\rho_k)} \mathrm{H}^0(\mathrm{HIG}(\widehat{S}^+_{\infty,\rho},\Theta)) = \widehat{R}^+_\infty.$$

It remains to show that, for any $i\geq 1$,

$$\varinjlim_{\nu_p(\rho)>\nu_p(\rho_k)} \mathrm{H}^i(\mathrm{HIG}(\widehat{S}^+_{\infty,\rho},\Theta)) \cong \varinjlim_{\nu_p(\rho)>\nu_p(\rho_k)} \bigwedge^i_{\widehat{R}^+_\infty}\Big(\bigoplus_{j=1}^d \widehat{R}^+_\infty\langle\Lambda_{j,\rho}\rangle/\widehat{R}^+_\infty\langle\Lambda_{j,\rho}, I, +\rangle\Big)$$

is p^∞-torsion. To do so, it suffices to prove that for any $\nu_p(\rho_1)>\nu_p(\rho_2)>\nu_p(\rho_k)$, there is an $N\geq 0$ such that

$$p^N\widehat{R}^+_\infty\langle\Lambda_{j,\rho_1}\rangle\subset\widehat{R}^+_\infty\langle\Lambda_{j,\rho_2}, I, +\rangle.$$

By Remark A.3, we only need to find an N such that the following conditions hold:

(1) For any $i\geq 0$, $N+i\nu_p(\rho_1)-i\nu_p(\rho_2)-\nu_p(i+1)\geq 0$.

(2) $\lim_{i\to+\infty}(N+i\nu_p(\rho_1)-i\nu_p(\rho_2)-\nu_p(i+1))=+\infty$.

Since $\nu_p(\rho_1)>\nu_p(\rho_2)$, such an N exists. This completes the proof. □

Remark 2.24. (1) In the proof of Corollary 2.23, we have seen that for any $\rho\in\rho_k\mathcal{O}_{\mathbb{C}_p}$, the Higgs complex $\mathrm{HIG}\big(S^+_{\infty,\rho}\big[\frac{1}{p}\big],\Theta\big)$ is not a resolution of $\widehat{R}_\infty$.

(2) For any $1\leq i\leq d$, the p^∞-torsion of $\mathrm{H}^i(\mathrm{HIG}(S^{\dagger,+}_\infty,\Theta))$ is unbounded.

Remark 2.25. Since for any $1\leq i,j\leq d$, $\gamma_i(Y_j)=Y_j+\delta_{ij}$, one can check that $\partial/\partial Y_i=\log\gamma_i$ on $S^\dagger_\infty$. So the Higgs field is $\Theta=\sum_{i=1}^d\log\gamma_i\otimes((\mathrm{dlog}\,T_i)/t)$.

Remark 2.26. A similar local construction of $S^{\dagger,+}_\infty$ also appeared in [Abbes et al. 2016, I.4.7].

There is a global analogue by using Theorem 2.9 instead of Proposition 2.6. Put $\mathcal{E}^+_{\rho_k} = \rho_k\mathcal{E}^+(-1)$ and for any $\rho \in \rho_k\mathcal{O}_{\mathbb{C}_p}$, denote by $\mathcal{E}^+_\rho$ the pull-back of $\mathcal{E}^+_{\rho_k}$ along the inclusion

$$\rho\widehat{\mathcal{O}}^+_X \otimes_{\mathcal{O}_{\mathfrak{X}}} \widehat{\Omega}^1_{\mathfrak{X}}(-1) \to \rho_k\widehat{\mathcal{O}}^+_X \otimes_{\mathcal{O}_{\mathfrak{X}}} \widehat{\Omega}^1_{\mathfrak{X}}(-1).$$

Then it fits into the exact sequence

$$0 \to \widehat{\mathcal{O}}^+_X \to \mathcal{E}^+_\rho \to \rho\widehat{\mathcal{O}}^+_X \otimes_{\mathcal{O}_{\mathfrak{X}}} \widehat{\Omega}^1_{\mathfrak{X}}(-1) \to 0. \tag{2-12}$$

As an analogue of Definition 2.21 in the local case, we define period sheaves as follows:

Definition 2.27. For any $\rho \in \rho_k\mathcal{O}_{\mathbb{C}_p}$, define

(1) $\mathcal{O}\mathbb{C}^+_\rho = \varinjlim_n \mathrm{Sym}^n_{\widehat{\mathcal{O}}^+_X} \mathcal{E}^+_\rho$;

(2) $\mathcal{O}\widehat{\mathbb{C}}^+_\rho = \varprojlim_n \mathcal{O}\mathbb{C}^+_\rho/p^n$;

(3) $\mathcal{O}\mathbb{C}^{\dagger,+} = \varinjlim_{\nu_p(\rho)>\nu_p(\rho_k)} \mathcal{O}\widehat{\mathbb{C}}^+_\rho$ and $\mathcal{O}\mathbb{C}^\dagger = \mathcal{O}\mathbb{C}^{\dagger,+}\big[\frac{1}{p}\big]$.

Theorem 2.28. *There is a canonical Higgs field Θ on $\mathcal{O}\mathbb{C}^{\dagger,+}$ such that the induced Higgs complex*

$$\mathrm{HIG}(\mathcal{O}\mathbb{C}^\dagger, \Theta): \ \mathcal{O}\mathbb{C}^\dagger \xrightarrow{\Theta} \mathcal{O}\mathbb{C}^\dagger \otimes_{\mathcal{O}_{\mathfrak{X}}} \widehat{\Omega}^1_{\mathfrak{X}}(-1) \xrightarrow{\Theta} \mathcal{O}\mathbb{C}^\dagger \otimes_{\mathcal{O}_{\mathfrak{X}}} \widehat{\Omega}^2_{\mathfrak{X}}(-2) \to \cdots \tag{2-13}$$

is a resolution of $\widehat{\mathcal{O}}_X$. Additionally, when $\mathfrak{X} = \mathrm{Spf}(R^+)$ is small affine, there is an isomorphism

$$\iota : \mathcal{O}\mathbb{C}^{\dagger,+}_{|X_\infty} \to \varinjlim_{\nu_p(\rho)>\nu_p(\rho_k)} \widehat{\mathcal{O}}^+_X\langle\rho Y_1, \dots, \rho Y_d\rangle_{|X_\infty}$$

such that the Higgs field Θ equals $\sum_{i=1}^d (\partial/\partial Y_i) \otimes ((\mathrm{dlog}\, T_i)/t)$.

Proof. Since the problem is local, we are reduced to Corollary 2.23. □

Finally, we describe the relative version of the above constructions. We assume that $f : \mathfrak{X} \to \mathfrak{Y}$ is a morphism of liftable smooth formal schemes and lifts to an A_2-morphism $\tilde{f} : \widetilde{\mathfrak{X}} \to \widetilde{\mathfrak{Y}}$. Then by Corollary 2.19, for any $\rho \in \rho_k\mathcal{O}_{\mathbb{C}_p}$, we have the exact sequence

$$0 \to \widehat{\mathcal{O}}^+_X \otimes_{\widehat{\mathcal{O}}^+_Y} \mathcal{E}^+_{\rho,Y} \to \mathcal{E}^+_{\rho,X} \to \widehat{\mathcal{O}}^+_X \otimes_{\mathcal{O}_{\mathfrak{X}}} \widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}}(-1) \to 0.$$

By construction of period sheaves in Definition 2.27, we get morphisms of sheaves $\widehat{\mathcal{O}}^+_X \otimes_{\widehat{\mathcal{O}}^+_Y} \mathcal{F}_Y \to \mathcal{F}_X$ for $\mathcal{F} \in \{\mathcal{O}\mathbb{C}^+_\rho, \mathcal{O}\widehat{\mathbb{C}}^+_\rho, \mathcal{O}\mathbb{C}^{\dagger,+}\}$. Also, the natural projection $\mathcal{E}^+_{\rho,X} \to \widehat{\mathcal{O}}^+_X \otimes_{\mathcal{O}_{\mathfrak{X}}} \widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}}(-1)$ induces relative Higgs fields

$$\Theta_{X/Y} : \mathcal{F}_X \to \mathcal{F}_X \otimes_{\mathcal{O}_{\mathfrak{X}}} \widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}}(-1)$$

for $\mathcal{F} \in \{\mathcal{O}\mathbb{C}^+_\rho, \mathcal{O}\widehat{\mathbb{C}}^+_\rho, \mathcal{O}\mathbb{C}^{\dagger,+}\}$. Using similar arguments as above, we get the following proposition.

Proposition 2.29. *Assume that $f : \mathfrak{X} \to \mathfrak{Y}$ is a morphism of liftable smooth formal schemes and lifts to an A_2-morphism $\tilde{f} : \widetilde{\mathfrak{X}} \to \widetilde{\mathfrak{Y}}$. The induced relative Higgs complex*

$$\mathrm{HIG}(\mathcal{O}\mathbb{C}^\dagger_X, \Theta_{X/Y}): \ \mathcal{O}\mathbb{C}^\dagger_X \xrightarrow{\Theta_{X/Y}} \mathcal{O}\mathbb{C}^\dagger_X \otimes_{\mathcal{O}_{\mathfrak{X}}} \widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}}(-1) \xrightarrow{\Theta_{X/Y}} \mathcal{O}\mathbb{C}^\dagger_X \otimes_{\mathcal{O}_{\mathfrak{X}}} \widehat{\Omega}^2_{\mathfrak{X}/\mathfrak{Y}}(-2) \to \cdots$$

is a resolution of $\varinjlim_{\rho,\nu_p(\rho)>\nu_p(\rho_k)}(\widehat{\mathcal{O}}_X^+\widehat{\otimes}_{\widehat{\mathcal{O}}_Y^+}\mathcal{O}\widehat{\mathbb{C}}^+_{\rho,Y})\big[\frac{1}{p}\big]$ *and makes the diagram*

$$\begin{array}{ccccc}
f^*\mathcal{O}\mathbb{C}_Y^\dagger & \xrightarrow{f^*\Theta_Y} & f^*\mathcal{O}\mathbb{C}_Y^\dagger\otimes_{\mathcal{O}_{\mathfrak{Y}}}\widehat{\Omega}^1_{\mathfrak{Y}}(-1) & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \\
\mathcal{O}\mathbb{C}_X^\dagger & \xrightarrow{\Theta_X} & \mathcal{O}\mathbb{C}_X^\dagger\otimes_{\mathcal{O}_{\mathfrak{X}}}\widehat{\Omega}^1_{\mathfrak{X}}(-1) & \longrightarrow & \cdots \\
\downarrow{\scriptstyle\Theta_{X/Y}} & & \downarrow{\scriptstyle\Theta_{X/Y}} & & \\
\mathcal{O}\mathbb{C}_X^\dagger\otimes_{\mathcal{O}_{\mathfrak{X}}}\widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}}(-1) & \xrightarrow{\Theta_{X/Y}} & \mathcal{O}\mathbb{C}_X^\dagger\otimes_{\mathcal{O}_{\mathfrak{X}}}\widehat{\Omega}^2_{\mathfrak{X}/\mathfrak{Y}}(-2) & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \\
\vdots & & \vdots & &
\end{array} \tag{2-14}$$

commute, where $f^*\mathcal{O}\mathbb{C}_Y^\dagger=\widehat{\mathcal{O}}_X\otimes_{\widehat{\mathcal{O}}_Y}\mathcal{O}\mathbb{C}_Y^\dagger$ *and* $f^*\Theta_Y=\mathrm{id}\otimes\Theta_Y$.

Proof. Put $\mathcal{C}:=\varinjlim_{\rho,\nu_p(\rho)>\nu_p(\rho_k)}(\widehat{\mathcal{O}}_X^+\widehat{\otimes}_{\widehat{\mathcal{O}}_Y^+}\mathcal{O}\widehat{\mathbb{C}}^+_{\rho,Y})\big[\frac{1}{p}\big]$. Since f admits a lifting $\tilde{f}$, for any $\rho\in\rho_k\mathcal{O}_{\mathbb{C}_p}$, we have a morphism $\widehat{\mathcal{O}}_X^+\otimes_{\widehat{\mathcal{O}}_Y^+}\mathcal{O}\mathbb{C}^+_{\rho,Y}\to\mathcal{O}\mathbb{C}^+_{\rho,X}$ and hence morphisms $f^*\mathcal{O}\mathbb{C}_Y^\dagger\to\mathcal{C}\to\mathcal{O}\mathbb{C}_X^\dagger$. It remains to show the relative Higgs complex $\mathrm{HIG}(\mathcal{O}\mathbb{C}_X^\dagger,\Theta_{X/Y})$ is a resolution of $\mathcal{C}$ and that the diagram (2-14) commutes. Since the problem is local, we may assume $\mathfrak{Y}=\mathrm{Spf}(S^+)$ and $\mathfrak{X}=\mathrm{Spf}(R^+)$ are both small affine such that the morphism $f:\mathfrak{X}\to\mathfrak{Y}$ is induced by a morphism $S^+\to R^+$ which makes the diagram

$$\begin{array}{ccc}
\mathcal{O}_{\mathbb{C}_p}\langle T_1^{\pm1},\dots,T_d^{\pm1}\rangle & \xrightarrow{\subset} & \mathcal{O}_{\mathbb{C}_p}\langle T_1^{\pm1},\dots,T_d^{\pm1},T_{d+1}^{\pm1},\dots,T_{d+r}^{\pm1}\rangle \\
\downarrow & & \downarrow \\
S^+ & \longrightarrow & R^+
\end{array}$$

commute, where d is the dimension of $\mathfrak{Y}$ over $\mathcal{O}_{\mathbb{C}_p}$, r is the dimension of $\mathfrak{X}$ over $\mathfrak{Y}$ and both vertical maps are étale. Let $\widehat{S}_\infty^+$ and $\widehat{R}_\infty^+$ be the perfectoid rings corresponding to the base-changes of S^+ and R^+ along morphisms

$$\mathcal{O}_{\mathbb{C}_p}\langle T_1^{\pm1},\dots,T_d^{\pm1}\rangle\to\mathcal{O}_{\mathbb{C}_p}\langle T_1^{\pm\frac{1}{p^\infty}},\dots,T_d^{\pm\frac{1}{p^\infty}}\rangle$$

and

$$\mathcal{O}_{\mathbb{C}_p}\langle T_1^{\pm1},\dots,T_d^{\pm1},T_{d+1}^{\pm1},\dots,T_{d+r}^{\pm1}\rangle\to\mathcal{O}_{\mathbb{C}_p}\langle T_1^{\pm\frac{1}{p^\infty}},\dots,T_d^{\pm\frac{1}{p^\infty}},T_{d+1}^{\pm\frac{1}{p^\infty}},\dots,T_{d+r}^{\pm\frac{1}{p^\infty}}\rangle,$$

respectively. Put $Y_\infty=\mathrm{Spa}(\widehat{S}_\infty,\widehat{S}_\infty^+)$ and $X_\infty=\mathrm{Spa}(\widehat{R}_\infty,\widehat{R}_\infty^+)$ with $\widehat{S}_\infty=\widehat{S}_\infty^+\big[\frac{1}{p}\big]$ and $\widehat{R}_\infty=\widehat{R}_\infty^+\big[\frac{1}{p}\big]$. For any $\rho\in\rho_k\mathcal{O}_{\mathbb{C}_p}$, since $\mathcal{E}^+_{\rho,Y}$ fits into the exact sequence

$$0\to\widehat{\mathcal{O}}_X^+\to\widehat{\mathcal{O}}_X^+\otimes_{\widehat{\mathcal{O}}_Y^+}\mathcal{E}^+_{\rho,Y}\to\rho\widehat{\Omega}^1_{\mathfrak{Y}}\otimes_{\mathcal{O}_{\mathfrak{Y}}}\widehat{\mathcal{O}}_X^+(-1)\to 0,$$

we see that $(\widehat{\mathcal{O}}_X^+\otimes_{\widehat{\mathcal{O}}_Y^+}\mathcal{E}_Y^+)(X_\infty)(\subset\mathcal{E}^+_{\rho,X}(X_\infty))$ coincides with $\widehat{R}_\infty^+\otimes_{\widehat{S}_\infty^+}\mathcal{E}^+_{\rho,Y}(Y_\infty)$. This implies that

$$(\widehat{\mathcal{O}}_X^+\otimes_{\widehat{\mathcal{O}}_Y^+}\mathcal{O}\mathbb{C}^+_{\rho,Y})(X_\infty)\cong\widehat{R}_\infty^+[\rho Y_1,\dots,\rho Y_d]$$

such that the induced Higgs field is given by $\sum_{i=0}^{d}(\partial/\partial Y_i)\otimes((\mathrm{dlog}\,T_i)/t)$. On the other hand, we have

$$\mathcal{O}\mathbb{C}^+_{\rho,X}(X_\infty)\cong\widehat{R}^+_\infty[\rho Y_1,\dots,\rho Y_{d+r}]$$

such that the induced Higgs field is given by $\sum_{i=0}^{d+r}(\partial/\partial Y_i)\otimes((\mathrm{dlog}\,T_i)/t)$. So the morphism

$$\widehat{\mathcal{O}}^+_X\otimes_{\widehat{\mathcal{O}}^+_Y}\mathcal{O}\mathbb{C}^+_{\rho,Y}\to\mathcal{O}\mathbb{C}^+_{\rho,X}$$

is compatible with Higgs fields for any $\rho\in\rho_k\mathcal{O}_{\mathbb{C}_p}$. Therefore, for any $\rho\in\rho_k\mathcal{O}_{\mathbb{C}_p}$, we have morphisms of sheaves

$$\widehat{\mathcal{O}}^+_X\otimes_{\widehat{\mathcal{O}}^+_Y}\mathcal{O}\mathbb{C}^+_{\rho,Y}\to\widehat{\mathcal{O}}^+_X\otimes_{\widehat{\mathcal{O}}^+_Y}\mathcal{O}\widehat{\mathbb{C}}^+_{\rho,Y}\to\widehat{\mathcal{O}}^+_X\widehat{\otimes}_{\widehat{\mathcal{O}}^+_Y}\mathcal{O}\widehat{\mathbb{C}}^+_{\rho,Y}\to\mathcal{O}\widehat{\mathbb{C}}^+_{\rho,X}$$

which are all compatible with Higgs fields. After taking direct limits and inverting p, we get morphisms

$$f^*\mathcal{O}\mathbb{C}^\dagger_Y\to\mathcal{C}\to\mathcal{O}\mathbb{C}^\dagger_X$$

of sheaves which are compatible with Higgs fields. In particular, the top two rows of (2-14) form a commutative diagram.

To complete the proof, we have to show that $\mathrm{HIG}(\mathcal{O}\mathbb{C}^\dagger_X,\Theta_{X/Y})$ is a resolution of $\mathcal{C}$. Since we do have a morphism $\mathcal{C}\to\mathrm{HIG}(\mathcal{O}\mathbb{C}^\dagger_X,\Theta_{X/Y})$, we can conclude by checking the exactness locally.

By the "additionally" part of Corollary 2.23, we obtain that

$$\mathcal{O}\mathbb{C}^\dagger_X(X_\infty)=\Big(\varinjlim_{\rho,\nu_p(\rho)>\nu_p(\rho_k)}\widehat{R}^+_\infty\langle\rho Y_1,\dots,\rho Y_{d+r}\rangle\Big)\Big[\frac{1}{p}\Big]$$

with $\Theta_X=\sum_{i=1}^{d+r}(\partial/\partial Y_i)\otimes((\mathrm{dlog}\,T_i)/t)$. A similar argument also shows that

$$\Theta_{X/Y}=\sum_{i=d+1}^{d+r}\frac{\partial}{\partial Y_i}\otimes\frac{\mathrm{dlog}\,T_i}{t}.$$

So the rest of (2-14) commutes. Note that $\mathcal{C}(X_\infty)=(\varinjlim_{\rho,\nu_p(\rho)>\nu_p(\rho_k)}\widehat{R}^+_\infty\langle\rho Y_1,\dots,\rho Y_d\rangle)\big[\frac{1}{p}\big]$. By a similar argument in the proof of Corollary 2.23, we see that $\mathrm{HIG}(\mathcal{O}\mathbb{C}^\dagger_X,\Theta_{X/Y})$ is a resolution of $\mathcal{C}$. □

3. An integral decompletion theorem

In this section, we generalise results in [Diao et al. 2023b, Appendix A] to an integral case which will be used to simplify local calculations. Let $\mathfrak{X}=\mathrm{Spf}(R^+)$, $\widehat{R}^+_\infty$ and Γ be as in the previous section. Throughout this section, we put $\pi=\zeta_p-1$, $r=\nu_p(\pi)=1/(p-1)$ and $c=p^r$. Recall $\nu_p(\rho_k)\ge r$. We begin with some definitions.

Definition 3.1. (1) By a *Banach $\mathcal{O}_{\mathbb{C}_p}$-algebra*, we mean a flat $\mathcal{O}_{\mathbb{C}_p}$-algebra A such that $A\big[\frac{1}{p}\big]$ is a Banach $\mathbb{C}_p$-algebra, and that $A=\big\{a\in A\big[\frac{1}{p}\big]\mid\|a\|\le 1\big\}$.

(2) Assume A is a Banach $\mathcal{O}_{\mathbb{C}_p}$-algebra. For an A-module M, we say it is a *Banach A-module* if $M\big[\frac{1}{p}\big]$ is a Banach $A\big[\frac{1}{p}\big]$-module, and $M=\big\{m\in M\big[\frac{1}{p}\big]\mid\|m\|\le 1\big\}$.

There are some typical examples.

Example 3.2. (1) *If A is a Banach $\mathcal{O}_{\mathbb{C}_p}$-algebra, then any topologically free A-module endowed with the supreme norm is a Banach A-module.*

(2) *The rings R^+ and $\widehat{R}^+_\infty$ are Banach $\mathcal{O}_{\mathbb{C}_p}$-algebras.*

(3) *The $\widehat{R}^+_\infty/R^+$ is a Banach R^+-module.*

Now, we make the definition of (a-trivial) Γ-representations.

Definition 3.3. Assume $a > r$ and $A \in \{R^+, \widehat{R}^+_\infty\}$.

(1) By an *A-representation of* Γ of rank l, we mean a finite free A-module M of rank l endowed with a continuous semilinear Γ-action.

(2) Let M be a representation of Γ of rank l over A. We say M is *a-trivial*, if $M/p^a \cong (A/p^a)^l$ as representations of Γ over A/p^a.

(3) Let M be a representation of Γ of rank l over R^+. We say M is *essentially $(a+r)$-trivial* if M is a-trivial and $M \otimes_{R^+} \widehat{R}^+_\infty$ is $(a+r)$-trivial.

The goal of this section is to prove the following integral decompletion theorem.

Theorem 3.4. *Assume $a > r$. Then the functor $M \mapsto M \otimes_{R^+} \widehat{R}^+_\infty$ induces an equivalence from the category of $(a+r)$-trivial R^+-representations of Γ to the category of $(a+r)$-trivial $\widehat{R}^+_\infty$-representations of Γ. The equivalence preserves tensor products and dualities.*

The first difficulty is to construct the quasi-inverse, namely the decompletion functor, of the functor in Theorem 3.4. To do so, we need to generalise the method adapted in [Diao et al. 2023b] to the small integral case. However, their method only shows the decompletion functor takes values in the category of essentially $(a+r)$-trivial representations. So, the second difficulty is to show the resulting representation is actually $(a+r)$-trivial. The trivialness condition is crucial to overcome both difficulties.

Construction of decompletion functor. We first construct the decompletion functor. From now on, we use $\mathrm{R}\Gamma(\Gamma, M)$ to denote the continuous group cohomology of a p-adically completed R^+-module endowed with a continuous Γ-action. By virtues of [Bhatt et al. 2018, Lemma 7.3], $\mathrm{R}\Gamma(\Gamma, M) = \mathrm{R}\varprojlim_k \mathrm{R}\Gamma(\Gamma, M/p^k)$ can be calculated by the Koszul complex

$$\mathrm{K}(M; \gamma_1 - 1, \dots, \gamma_d - 1): \ M \xrightarrow{(\gamma_1-1,\dots,\gamma_d-1)} M^d \to \cdots .$$

Proposition 3.5. *Assume $a > r$. Let M_∞ be an $(a+r)$-trivial $\widehat{R}^+_\infty$-representation of Γ. Then there exists a finite free R^+-submodule $M \subset M_\infty$ such that the following assertions are true:*

(1) *The finite free A-module M is an essentially $(a+r)$-trivial R^+-representation of Γ such that the natural inclusion $M \hookrightarrow M_\infty$ induces an isomorphism $M \otimes_{R^+} \widehat{R}^+_\infty \cong M_\infty$ of $\widehat{R}^+_\infty$-representations of Γ.*

(2) *The induced morphism $\mathrm{R}\Gamma(\Gamma, M) \to \mathrm{R}\Gamma(\Gamma, M_\infty)$ identifies the former as a direct summand of the latter, whose complement is concentrated in positive degrees and killed by π.*

Remark 3.6. The finite free A-module M is unique up to isomorphism and the functor $M_\infty \mapsto M$ turns out to be the quasi-inverse of the functor $M \mapsto M \otimes R^+_\infty$ described in Theorem 3.4.

Now we prove Proposition 3.5 by using similar arguments in [Diao et al. 2023b]. Since we work on the integral level, so we need to control (p-adic) norms carefully. We start with the following result.

Lemma 3.7. *For any cocycle $f \in \mathrm{C}^\bullet(\Gamma, \widehat{R}_\infty/R)$, there exists a cochain $g \in \mathrm{C}^{\bullet-1}(\Gamma, \widehat{R}_\infty/R)$ such that $dg = f$ and $\|g\| \le c\,\|f\|$.*

Proof. The result follows from the same argument used in the proof of [Diao et al. 2023b, Proposition A.2.2.1], especially the part for checking the condition (3) of [Diao et al. 2023b, Definition A.1.6], by using [Scholze 2013a, Lemma 5.5] instead of [Diao et al. 2023a, Lemma 6.1.7]. □

Since the norm on R (resp. $\widehat{R}_\infty$) is induced by that on R^+ (resp. $\widehat{R}^+_\infty$), there exists a norm-preserving embedding of complexes

$$\mathrm{C}^\bullet(\Gamma, \widehat{R}^+_\infty/R^+) \to \mathrm{C}^\bullet(\Gamma, \widehat{R}_\infty/R).$$

We shall apply Lemma 3.7 via this embedding.

Lemma 3.8. *For any cocycle $f \in \mathrm{C}^\bullet(\Gamma, \widehat{R}^+_\infty/R^+)$, there is a cochain $g \in \mathrm{C}^{\bullet-1}(\Gamma, \widehat{R}^+_\infty/R^+)$ such that $\|g\| \le \|f\|$ and $dg = \pi f$.*

Proof. Regard $\mathrm{C}^\bullet(\Gamma, \widehat{R}^+_\infty/R^+)$ as a subcomplex of $\mathrm{C}^\bullet(\Gamma, \widehat{R}_\infty/R)$ as above. Applying Lemma 3.7 to πf, we get a cochain $g \in \mathrm{C}^{\bullet-1}(\Gamma, \widehat{R}_\infty/R)$ such that $\|g\| \le c\,\|\pi f\|$ and $dg = \pi f$. But $c\,\|\pi f\| = \|f\| \le 1$, so we see $g \in \mathrm{C}^{\bullet-1}(\Gamma, \widehat{R}^+_\infty/R^+)$. □

Lemma 3.9. *Let $(\mathrm{C}^\bullet, d)$ be a complex of Banach modules over a Banach $\mathcal{O}_{\mathbb{C}_p}$-algebra A. Suppose that for every degree s and every cocycle $f \in \mathrm{C}^s$, there exists a $g \in \mathrm{C}^{s-1}$ such that $\|g\| \le \|f\|$ and $dg = \pi f$. Then, for any cochain $f \in \mathrm{C}^s$, there exists an $h \in \mathrm{C}^{s-1}$ such that $\|h\| \le \max(\|f\|/c, \|df\|)$ and $\|\pi^2 f - dh\| \le \|df\|/c$.*

Proof. By assumption, one can choose a $g \in \mathrm{C}^s$ such that $dg = \pi df$ and that $\|g\| \le \|df\|$. Then $(g - \pi f) \in \mathrm{C}^s$ is a cocycle. Using this assumption again, there is an $h \in \mathrm{C}^{s-1}$ satisfying $\|h\| \le \|g - \pi f\|$ and $dh = \pi(g - \pi f)$. Then $\|\pi^2 f - dh\| \le \|g\|/c \le \|df\|/c$ and $\|h\| \le \max(\|df\|, \|f\|/c)$. □

The following lemma is a consequence of Lemmas 3.8 and 3.9.

Lemma 3.10. *For any cochain $f \in \mathrm{C}^\bullet(\Gamma, \widehat{R}^+_\infty/R^+)$, there is a cochain $h \in \mathrm{C}^{\bullet-1}(\Gamma, \widehat{R}^+_\infty/R^+)$ such that $\|h\| \le \max(\|f\|/c, \|df\|)$ and $\|\pi^2 f - dh\| \le \|df\|/c$.*

The following lemma can be viewed as an integral version of [Diao et al. 2023b, Lemma A.1.12].

Lemma 3.11. *We denote $(R^+, \widehat{R}^+_\infty/R^+)$ by (A, M) for simplicity.*

Let $L = \bigoplus_{i=1}^n Ae_i$ be a Banach A-module (with the supreme norm) endowed with a continuous Γ-action. Assume there exists an $R > 1$ such that, for each $\gamma \in \Gamma$ and each i, $\|(\gamma - 1)(e_i)\| \le 1/(Rc)$. Then the following assertions are true:

(1) *For any cocycle* $f \in \mathrm{C}^\bullet(\Gamma, L \otimes_A M)$, *there is a cochain* $g \in \mathrm{C}^{\bullet-1}(\Gamma, L \otimes_A M)$ *such that* $\|g\| \leq \|f\|$ *and* $dg = \pi f$.

(2) *For any cochain* $f \in \mathrm{C}^\bullet(\Gamma, L \otimes_A M)$, *there exists an* $h \in \mathrm{C}^\bullet(\Gamma, L \otimes_A M)$ *such that* $\|h\| \leq \max(\|f\|/c, \|df\|)$ *and* $\|\pi^2 f - dh\| \leq \|df\|/c$.

Proof. We only prove (1) and then (2) follows from Lemma 3.9 directly.

Now, let $f = \sum_{i=1}^n e_i \otimes f_i$ be a cocycle with $f_j \in \mathrm{C}^s(\Gamma, M)$ for all $1 \leq j \leq n$. Then $\|f\| \leq 1$. For any $\gamma_1, \gamma_2, \dots, \gamma_{s+1} \in \Gamma$, we have

$$\Big(\sum_{i=1}^n e_i \otimes df_i\Big)(\gamma_1, \dots, \gamma_{s+1}) = \Big(\sum_{i=1}^n e_i \otimes df_i\Big)(\gamma_1, \dots, \gamma_{s+1}) - df(\gamma_1, \dots, \gamma_{s+1})$$
$$= \sum_{i=1}^n (1-\gamma_1)(e_i) \otimes f_i(\gamma_2, \dots, \gamma_{s+1}).$$

It follows that $\big\|\sum_{i=1}^n e_i \otimes df_i\big\| \leq \|f\|/(Rc)$. In other words, for each $1 \leq j \leq n$, $\|df_j\| \leq \|f\|/(Rc)$. By Lemma 3.10, for every j, there is a $g_j \in \mathrm{C}^{s-1}(\Gamma, M)$ such that $\|g_j\| \leq \max(\|f_j\|/c, \|df_j\|) \leq \|f_j\|/c$ and $\|\pi^2 f_j - dg_j\| \leq \|df_j\|/c \leq \|f\|/(Rc^2)$.

Now, put $g = \sum_{i=1}^n e_i \otimes g_i$. Then $\|g\| \leq \|f\|/c$. On the other hand, we have

$$\pi^2 f - dg = \sum_{i=1}^n e_i \otimes (\pi^2 f_i - dg_i) + \Big(\sum_{i=1}^n e_i \otimes (dg_i - dg)\Big).$$

The first term on the right-hand side is bounded by $\|f\|/(Rc^2)$ and the second term is bounded by $\|g\|/(Rc) \leq \|f\|/(Rc^2)$. Thus $\|\pi^2 f - dg\|$ is bounded by $\|f\|/(Rc^2)$. Then $h_1 := g/\pi$ belongs to $\mathrm{C}^{s-1}(\Gamma, (L \otimes_A M))$ such that $\|h_1\| \leq \|f\|$ and that $\|\pi f - dh_1\| \leq \|f\|/(Rc)$.

Assume we have already $h_1, h_2, \dots, h_t \in \mathrm{C}^{s-1}(\Gamma, L \otimes_A M)$ satisfying

$$\|h_j\| \leq \frac{\|f\|}{R^{j-1}} \quad \text{and} \quad \Big\|\pi f - \sum_{i=1}^j dh_i\Big\| \leq \frac{\|f\|}{R^j c}, \qquad \text{for all } 1 \leq j \leq t.$$

Then $f - \pi^{-1}\sum_{i=1}^t dh_i \in \mathrm{C}^s(\Gamma, L \otimes_A M)$ with norm $\|f - \pi^{-1}\sum_{i=1}^t dh_i\| \leq \|f\|/R^t$. Replacing f by $f - \pi^{-1}\sum_{i=1}^t dh_i$ and proceeding as above, we get an $h_{t+1} \in \mathrm{C}^{s-1}(\Gamma, L \otimes_A M)$ with norm $\|h_{t+1}\| \leq \big\|f - \pi^{-1}\sum_{i=1}^t dh_i\big\| \leq \|f\|/R^t$ such that

$$\Big\|\pi f - \sum_{i=1}^t dh_i - dh_{t+1}\Big\| \leq \frac{\big\|f - \pi^{-1}\sum_{i=1}^t dh_i\big\|}{Rc} \leq \frac{\|f\|}{R^{t+1}c}.$$

Then $\sum_{i=1}^{+\infty} h_i$ converges to an element $h \in \mathrm{C}^{s-1}(\Gamma, L \otimes_A M)$ such that $\pi f = dh$ and that $\|h\| \leq \sup_{j\geq 1}(\|h_j\|) \leq \|f\|$. This implies (1). □

The following lemma is a generalisation of [Diao et al. 2023b, Lemma A.1.14] whose proof is similar.

Lemma 3.12. *Let $A \to B$ be an isometry of Banach $\mathcal{O}_{\mathbb{C}_p}$-algebras. Suppose the natural projection $\mathrm{pr} : B \to B/A$ admits an isometric section $s : B/A \to B$ as Banach modules over A. Then, for all $b_1, b_2 \in B$, we have*

$$\|\mathrm{pr}(b_1 b_2)\| \le \max(\|b_1\|\,\|\mathrm{pr}(b_2)\|, \|b_2\|\,\|\mathrm{pr}(b_1)\|)$$

We shall apply this lemma to the inclusion $R^+ \to \widehat{R}^+_\infty$.

Lemma 3.13. *Denote the triple $(R^+, \widehat{R}^+_\infty)$ by (A, B) for simplicity. Let f be a cocycle in $\mathrm{C}^1(\Gamma, \mathrm{GL}_n(B))$. Suppose there exists an $R > 1$ such that $\|f(\gamma) - 1\| \le 1/(Rc)$ for all $\gamma \in \Gamma$. Let $\bar{f}$ be the image of f in $\mathrm{C}^1(\Gamma, \mathrm{M}_n(B/A))$ (which is not necessary a cocycle). If $\|\bar{f}\| \le 1/(Rc^2)$, then there exists a cocycle $f' \in \mathrm{C}^1(\Gamma, \mathrm{GL}_n(A))$ which is equivalent to f such that $\|f'(\gamma) - 1\| \le 1/(Rc)$ for all $\gamma \in \Gamma$.*

Proof. We proceed as in the proof of [Diao et al. 2023b, Lemma A.1.15]. It is enough to show that there exists an $h \in \mathrm{M}_n(B)$ with $\|h\| \le c\,\|\bar{f}\|$ such that the cocycle

$$g : \gamma \mapsto \gamma(1+h) f(\gamma)(1+h)^{-1}$$

satisfies $\|g(\gamma) - 1\| \le 1/(Rc)$ for all $\gamma \in \Gamma$ and $\|\bar{g}\| \le \|\bar{f}\|/R$ in $\mathrm{C}^1(\Gamma, \mathrm{M}_n(B/A))$.

Granting the claim, by iterating this process, we can find a sequence $h_1, h_2, \dots$ in $\mathrm{M}_n(B)$ with $\|h_n\| \le (c\,\|\bar{f}\|)/R^{n-1} \le 1/(cR^n)$ such that

$$\overline{\gamma\Big(\prod_{i=1}^{n}(1+h_i)\Big) f(\gamma)\Big(\prod_{i=1}^{n}(1+h_i)\Big)^{-1}} \le \frac{\|\bar{f}\|}{R^n}.$$

Set $h = \prod_{i=1}^{+\infty}(1+h_i) \in \mathrm{GL}_n(B)$. Then we have a cocycle

$$f' : \gamma \mapsto \gamma(h) f(\gamma) h^{-1}$$

taking values in $\mathrm{M}_n(A) \cap \mathrm{GL}_n(B)$ such that $\|f'(\gamma) - 1\| \le 1/(Rc)$ for every $\gamma \in \Gamma$. Thus $f' \in \mathrm{GL}_n(A)$ and we prove the lemma.

Now, we prove the claim. Since $f \in \mathrm{C}^1(\Gamma, \mathrm{GL}_n(B))$ is a cocycle, for all $\gamma_1, \gamma_2 \in \Gamma$, we have $f(\gamma_1\gamma_2) = \gamma_1(f(\gamma_2)) f(\gamma_1)$. Using Lemma 3.12, we get

$$\begin{aligned} \|d\bar{f}(\gamma_1, \gamma_2)\| &= \|\overline{\gamma_1 f(\gamma_2) + f(\gamma_1) - f(\gamma_1\gamma_2)}\| \\ &= \|\overline{(\gamma_1 f(\gamma_2) - 1)(f(\gamma_1) - 1) - 1}\| \\ &= \|\overline{(\gamma_1 f(\gamma_2) - 1)(f(\gamma_1) - 1)}\| \le \frac{\|\bar{f}\|}{Rc}. \end{aligned} \tag{3-1}$$

Since $\|\bar{f}\| \le 1/(Rc^2)$, we can apply Lemma 3.10 to $\pi^{-2}\bar{f}$ and get an $\bar{h} \in \mathrm{M}_n(B/A)$ such that

$$\|\bar{h}\| \le \max\Big(\frac{\|\pi^{-2}\bar{f}\|}{c}, \|\pi^{-2} d\bar{f}\|\Big) \le \max(c\|\bar{f}\|, c^2\|d\bar{f}\|) \le c\|\bar{f}\| \le \frac{1}{Rc}.$$

and that

$$\|\bar{f} - d\bar{h}\| \le \frac{\|\pi^{-2} d\bar{f}\|}{c} \le c\,\|d\bar{f}\| \le \frac{\|\bar{f}\|}{R}. \tag{3-2}$$

By assumption, we can lift $\bar{h}$ to an $h \in \mathrm{M}_n(B)$ such that $\|h\| = \|\bar{h}\| \le c\,\|\bar{f}\|$. It follows that for all $\gamma \in \Gamma$, we have

$$\|\gamma(1+h)f(\gamma)(1+h)^{-1} - f(\gamma)\| \le \|h\| \le \frac{1}{Rc}$$

and, therefore,

$$\|\gamma(1+h)f(\gamma)(1+h)^{-1} - 1\| \le \frac{1}{Rc}.$$

Moreover, we have

$$\|\overline{\gamma(1+h)f(\gamma)(1+h)^{-1} - \gamma(1+h)f(\gamma)(1-h)}\| \le \|\bar{h}^2\| \le \frac{c\|\bar{f}\|}{Rc} = \frac{\|\bar{f}\|}{R}. \tag{3-3}$$

By Lemma 3.12, we have

$$\begin{aligned}\|\overline{\gamma(1+h)f(\gamma)(1-h)} &- \bar{f}(\gamma) - \gamma(\bar{h}) + \bar{h}\| \\ &= \|\overline{\gamma(h)(f(\gamma)-1)} - \overline{(f(\gamma)-1)h} - \overline{\gamma(h)f(\gamma)h}\| \le \frac{\|\bar{f}\|}{R}. \end{aligned} \tag{3-4}$$

Combining (3-2), (3-3) and (3-4), we conclude that

$$\|\overline{\gamma(1+h)f(\gamma)(1+h)^{-1}}\| \le \frac{\|\bar{f}\|}{R}$$

which proves the claim as desired. □

Now we are able to prove Proposition 3.5.

Proof of Proposition 3.5. (1) Since $a > r$, we may choose $s > 1$ such that $\|p^{a+r}\| = 1/(sc^2)$. By our assumptions, a basis $\{e_1, e_2, \dots, e_n\}$ of M_∞ determines a cocycle $f \in \mathrm{C}^1(\Gamma, \mathrm{GL}_n(\widehat{R}^+_\infty))$ satisfying $\|f(\gamma)-1\| \le 1/(sc^2)$. In particular, f satisfies the hypothesis of Lemma 3.13. Thus there exists a cocycle $f' \in \mathrm{C}^1(\Gamma, R^+)$ which is equivalent to f such that

$$\|f'(\gamma) - 1\| \le \frac{1}{sc}, \quad \text{for all } \gamma \in \Gamma.$$

Then the cocycle f' defines a finite free sub-R^+-module M of rank n such that

$$M \otimes_{R^+} \widehat{R}^+_\infty \cong M_\infty.$$

(2) By (1), we have $M_\infty \cong M \oplus M \otimes_{R^+} (\widehat{R}^+_\infty/R^+)$. Applying Lemma 3.11(1) to $L = M$, we deduce that $\mathrm{H}^i(\Gamma, M \otimes_{R^+} \widehat{R}^+_\infty/R^+)$ is killed by π for every $i \ge 0$. But $\mathrm{H}^0(\Gamma, M_\infty) = M_\infty^\Gamma$ is π-torsion free, so we get

$$\mathrm{H}^0(\Gamma, M_\infty) = \mathrm{H}^0(\Gamma, M)$$

and complete the proof. □

Up to now, we have constructed a decompletion functor from the category of $(a{+}r)$-trivial $\widehat{R}^+_\infty$-representations of Γ to the category of essentially $(a{+}r)$-trivial R^+-representations of Γ. Now Theorem 3.4 follows from the next proposition directly.

Proposition 3.14. *Every essentially $(a{+}r)$-trivial R^+-representation of Γ is $(a{+}r)$-trivial.*

We give the proof of this proposition in the next subsection.

Essentially $(a+r)$-trivial representation is $(a+r)$-trivial. Throughout this subsection, we always assume $a > r$. For any R^+-module N with a continuous Γ-action, we denote $\mathrm{H}^i(\Gamma, N)$ by $\mathrm{H}^i(N)$ for simplicity.

Now for a fixed essentially $(a+r)$-trivial R^+-representation M of Γ of rank n, we define

$$M_\infty = M \otimes_{R^+} \widehat{R}^+_\infty.$$

Then it is $(a+r)$-trivial and of the form $M_\infty = M \oplus M_{\mathrm{cp}}$ for $M_{\mathrm{cp}} = M \otimes_{R^+} \widehat{R}^+_\infty / R^+$. Since M is a-trivial, by Lemma 3.11, we see that $\mathrm{R}\Gamma(\Gamma, M_{\mathrm{cp}})$ is concentrated in positive degrees and is killed by π. As a consequence, for any $h \geq r$, we have

$$\mathrm{R}\Gamma(\Gamma, M_{\mathrm{cp}}/p^h) \simeq \mathrm{R}\Gamma(\Gamma, M_{\mathrm{cp}})[1].$$

In particular, $\mathrm{R}\Gamma(\Gamma, M_{\mathrm{cp}}/p^h)$ is killed by π. So we deduce that

$$\pi \mathrm{H}^0(M_\infty/p^h) \cong \pi \mathrm{H}^0(M/p^h).$$

Replacing M by $(\widehat{R}^+_\infty)^l$, we get

$$\pi \mathrm{H}^0(\widehat{R}^+_\infty/p^h)^n \cong \pi \mathrm{H}^0(R^+/p^h)^n = (\pi R^+/p^h)^n.$$

Since M_∞ is $(a+r)$-trivial, choose $h = a+r$ and we get

$$\pi \mathrm{H}^0(M/p^{a+r}) \cong \pi \mathrm{H}^0(M_\infty/p^{a+r}) \cong \pi \mathrm{H}^0(\widehat{R}^+_\infty/p^{a+r})^n \cong (\pi R^+/p^{a+r})^n \cong (R^+/p^a)^n.$$

Thus, $\pi \mathrm{H}^0(M/p^{a+r})$ is a free R^+/p^a-module of rank n.

Choose $g_1, \ldots g_n \in \mathrm{H}^0(M/p^{a+r})$ such that $\pi g_1, \ldots, \pi g_n$ is an R^+/p^a-basis of $\pi \mathrm{H}^0(M/p^{a+r})$. We claim that the sub-$R^+/p^{a+r}$-module

$$\sum_{i=1}^n R^+/p^{a+r} \cdot g_i \subset \mathrm{H}^0(M/p^{a+r})$$

is free. For any i, let $\tilde{g}_i \in M$ be a lifting of g_i. Assume $x_1, \ldots, x_n \in R^+$ such that

$$\sum_{i=1}^n x_i \tilde{g}_i \equiv 0 \mod p^{a+r}.$$

Then

$$\sum_{i=1}^n x_i \pi \tilde{g}_i \equiv 0 \mod p^{a+r}.$$

By the choice of g_i's, we deduce that $x_i \in p^a R^+$ for any i. Write $x_i = \pi y_i$ for some $y_i \in R^+$. Then

$$\sum_{i=1}^n y_i \pi \tilde{g}_i \equiv 0 \mod p^{a+r}.$$

So $y_i \in p^a R^+$ and hence $x_i \in p^{a+r} R^+$ for all i. This proves the claim.

It remains to prove $\tilde{g}_1, \dots, \tilde{g}_n$ is an R^+-basis of M. Let $e_1, \dots, e_n$ be an R^+-basis of M. Since M is a-trivial, we get

$$M/p^a = \mathrm{H}^0(M/p^a) = \sum_{i=1}^{n} R^+/p^a e_i.$$

So $\pi e_1, \dots, \pi e_n$ is an R^+/p^{a-r}-basis of $\pi M/p^a$. However, by the choice of $\tilde{g}_i$'s, $\pi\tilde{g}_1, \dots, \pi\tilde{g}_n$ is also an R^+/p^{a-r}-basis of $\pi M/p^a$. Since $a > r$, we deduce that $\tilde{g}_i$'s generate M as an R^+-module. This completes the proof.

4. Local Simpson correspondence

In this section, we establish an equivalence between the category of a-small representations of Γ over $\widehat{R}^+_\infty$ and the category of a-small Higgs modules over R^+. This is a local p-adic Simpson correspondence. Throughout this section, put $r = 1/(p-1)$.

Definition 4.1. Assume $a > r$ and $A \in \{R^+, \widehat{R}^+_\infty\}$. We say a representation M of Γ over A is *a-small* if it is $(a+\nu_p(\rho_k))$-trivial in the sense of Definition 3.3.

Definition 4.2. By a *Higgs module* over R^+, we mean a finite free R^+-module H together with an R^+-linear morphism $\theta : H \to H \otimes_{R^+} \widehat{\Omega}^1_{R^+}(-1)$ such that $\theta \wedge \theta = 0$. A Higgs module (H, θ) is called *a-small*, if θ is divided by $p^{a+\nu_p(\rho_k)}$; that is,

$$\mathrm{Im}(\theta) \subset p^{a+\nu_p(\rho_k)} H \otimes_{R^+} \widehat{\Omega}^1_{R^+}(-1).$$

Let $S^{\dagger,+}_\infty$ with the canonical Higgs field Θ be as in Corollary 2.23. For an a-small representation M over $\widehat{R}^+_\infty$, define

$$\Theta_M = \mathrm{id}_M \otimes \Theta : M \otimes_{\widehat{R}^+_\infty} S^{\dagger,+}_\infty \to M \otimes_{\widehat{R}^+_\infty} S^{\dagger,+}_\infty \otimes_{R^+} \widehat{\Omega}^1_{R^+}(-1). \tag{4-1}$$

Then it is a Higgs field on $M \otimes_{\widehat{R}^+_\infty} S^{\dagger,+}_\infty$. We denote the induced Higgs complex by $\mathrm{HIG}(H \otimes_{R^+} S^{\dagger,+}_\infty, \Theta_H)$. For an a-small Higgs module (H, θ_H), define

$$\Theta_H = \theta_H \otimes \mathrm{id} + \mathrm{id}_H \otimes \Theta : H \otimes_{R^+} S^{\dagger,+}_\infty \to H \otimes_{R^+} S^{\dagger,+}_\infty \otimes_{R^+} \widehat{\Omega}^1_{R^+}(-1). \tag{4-2}$$

Then Θ_H is a Higgs field on $H \otimes_{R^+} S^{\dagger,+}_\infty$. We denote the induced Higgs complex by $\mathrm{HIG}(H \otimes_{R^+} S^{\dagger,+}_\infty, \Theta_H)$. The main theorem in this section is the following local Simpson correspondence.

Theorem 4.3 (local Simpson correspondence). *Assume $a > r$.*

(1) *Let M be an a-small $\widehat{R}^+_\infty$-representation of Γ of rank l. Let $H(M) := (M \otimes_{\widehat{R}^+_\infty} S^{\dagger,+}_\infty)^\Gamma$ and $\theta_{H(M)}$ be the restriction of Θ_M to $H(M)$. Then $(H(M), \theta_{H(M)})$ is an a-small Higgs module of rank l.*

(2) *Let (H, θ_H) be an a-small Higgs module of rank l over R^+. Put $M(H, \theta_H) = (H \otimes_{R^+} S^{\dagger,+}_\infty)^{\Theta_H = 0}$. Then $M(H, \theta_H)$ is an a-small $\widehat{R}^+_\infty$-representation of Γ of rank l.*

(3) *The functor* $M \mapsto (H(M), \theta_{H(M)})$ *induces an equivalence from the category of a-small* $\widehat{R}^+_\infty$*-representations of* Γ *to the category of a-small Higgs modules over* R^+*, whose quasi-inverse is given by* $(H, \theta_H) \mapsto M(H, \theta_H)$*. The equivalence preserves tensor products and dualities.*

(4) *Let* M *be an a-small* $\widehat{R}^+_\infty$*-representation of* Γ *and* (H, θ_H) *be the corresponding Higgs module. Then there is a canonical* Γ*-equivariant isomorphism of Higgs complexes*

$$\mathrm{HIG}(H \otimes_{R^+} S^{\dagger,+}_\infty, \Theta_H) \to \mathrm{HIG}(M \otimes_{\widehat{R}^+_\infty} S^{\dagger,+}_\infty, \Theta_M).$$

Also, there is a canonical quasi-isomorphism

$$\mathrm{R}\Gamma\Big(\Gamma, M\Big[\frac{1}{p}\Big]\Big) \simeq \mathrm{HIG}\Big(H\Big[\frac{1}{p}\Big], \theta_H\Big),$$

where $\mathrm{HIG}\big(H\big[\frac{1}{p}\big], \theta_H\big)$ *is the Higgs complex induced by* (H, θ_H)*.*

The following corollary follows from Theorems 3.4 and 4.3 directly.

Corollary 4.4. *Assume* $a > r$*. The following categories are equivalent*:

(1) *The category of a-small representations of* Γ *over* R^+*.*

(2) *The category of a-small representations of* Γ *over* $\widehat{R}^+_\infty$*.*

(3) *The category of a-small Higgs modules over* R^+*.*

In order to prove the theorem, we need to compute $\mathrm{R}\Gamma(\Gamma, M \otimes_{\widehat{R}^+_\infty} S^{\dagger,+}_\infty)$. By Corollary 2.23, we are reduced to computing $\mathrm{R}\Gamma(\Gamma, M \otimes_{\widehat{R}^+_\infty} \widehat{R}^+_\infty\langle \rho Y_1, \dots, \rho Y_d\rangle)$ for any $\rho \in \rho_k \mathcal{O}_{\mathbb{C}_p}$. So before we move on, let us fix some notation to simplify the calculation.

For any $n \geq 0$, define

$$F_n(Y) = n!\binom{Y}{n} = Y(Y-1)\cdots(Y-n+1) \in \mathbb{Z}[Y].$$

For any $\alpha \in \mathbb{N}\big[\frac{1}{p}\big] \cap (0,1)$, define $\epsilon_\alpha = 1 - \zeta^{-\alpha}$. Then $\nu_p(\rho_k) \geq r \geq \nu_p(\epsilon_\alpha)$.

Calculation in trivial representation case. We are going to compute $\mathrm{R}\Gamma(\Gamma, \widehat{R}^+_\infty\langle \rho Y_1, \dots, \rho Y_d\rangle)$ in this subsection. We assume $d = 1$ first. In this case, $\Gamma = \mathbb{Z}_p\gamma$ and acts on $\widehat{R}^+_\infty\langle \rho Y\rangle$ via $\gamma(Y) = Y + 1$. Note that $\{\rho^n F_n\}_{n\geq 0}$ is a set of topological $\widehat{R}^+_\infty$-basis of $\widehat{R}^+_\infty\langle \rho Y\rangle$ and, for any $n \geq 0$,

$$\gamma(\rho^n F_n) = \rho^n F_n + n\rho \cdot \rho^{n-1} F_{n-1}.$$

So we get a γ-equivariant decomposition

$$\widehat{R}^+_\infty\langle \rho Y\rangle = \widehat{\bigoplus}_{\alpha \in \mathbb{N}[\frac{1}{p}] \cap [0,1)} R^+\langle \rho Y\rangle \cdot T^\alpha.$$

So it suffices to compute $\mathrm{R}\Gamma(\Gamma, R^+\langle \rho Y\rangle \cdot T^\alpha)$ for any α. We only need to consider the Koszul complex

$$\mathrm{K}(R^+\langle \rho Y\rangle \cdot T^\alpha; \gamma - 1) : \ R^+\langle \rho Y\rangle \cdot T^\alpha \xrightarrow{\gamma-1} R^+\langle \rho Y\rangle \cdot T^\alpha.$$

Note that for any α, $\{\rho^n F_n T^\alpha\}_{n\geq 0}$ is a set of topological R^+-basis of $R^+\langle\rho Y\rangle T^\alpha$. So we have

$$(\gamma-1)(\rho^n F_n T^\alpha)=\begin{cases} n\rho\cdot\rho^{n-1}F_{n-1}, & \alpha=0,\\ \zeta^\alpha\epsilon_\alpha T^\alpha\big(\rho^n F_n+n\frac{\rho}{\epsilon_\alpha}\rho^{n-1}F_{n-1}\big), & \alpha\neq 0.\end{cases} \tag{4-3}$$

Put $\Lambda_\rho=\{\rho^n F_n\}_{n\geq 0}$ and $I_\rho=\{\nu_p(\rho(n+1))\}_{n\geq 0}$. Let $R^+\langle\Lambda_\rho\rangle$ and $R^+\langle\Lambda_\rho, I_\rho, +\rangle$ be as in Definition A.1. Then by (4-3), we see that

$$(\gamma-1)(R^+\langle\rho Y\rangle)=R^+\langle\Lambda_\rho, I_\rho, +\rangle$$

and that

$$(\gamma-1)(R^+\langle\rho Y\rangle T^\alpha)\sim\left\{\zeta^\alpha\epsilon_\alpha\left(\rho^n F_n+n\frac{\rho}{\epsilon_\alpha}\rho^{n-1}F_{n-1}\right)\right\}_{n\geq 0}$$

in the sense of Definition A.4. By Proposition A.5, we get

$$(\gamma-1)(R^+\langle\rho Y\rangle T^\alpha)=\epsilon_\alpha(R^+\langle\rho Y\rangle T^\alpha).$$

In summary, we see that for $\alpha\neq 0$, $\mathrm{H}^1(\mathbb{Z}_p\gamma, R^+\langle\rho Y\rangle T^\alpha)$ is killed by ϵ_α and that for $\alpha=0$,

$$\mathrm{H}^1(\mathbb{Z}_p\gamma, R^+\langle\rho Y\rangle)=R^+\langle\rho Y\rangle/R^+\langle\Lambda_\rho, I_\rho, +\rangle.$$

So, keeping the notation as above, we have the following lemma.

Lemma 4.5. (1) *The inclusion $R^+\langle\rho Y\rangle\hookrightarrow\widehat{R}^+_\infty\langle\rho Y\rangle$ identifies $\mathrm{R}\Gamma(\Gamma, R^+\langle\rho Y\rangle)$ with a direct summand of $\mathrm{R}\Gamma(\mathbb{Z}_p\gamma, \widehat{R}^+_\infty\langle\rho Y\rangle)$ whose complement is concentrated in degree 1 and is killed by ζ_p-1.*

(2) $\mathrm{H}^0(\Gamma, R^+\langle\rho Y\rangle)=R^+$ *is independent of ρ.*

(3) $\mathrm{H}^1(\Gamma, R^+\langle\rho Y\rangle)=R^+\langle\rho Y\rangle/R^+\langle\Lambda_\rho, I_\rho, +\rangle$ *is the derived p-adic completion of*

$$\bigoplus_{i\geq 0}R^+/(i+1)\rho R^+.$$

Proof. It remains to compute $\mathrm{H}^0(\Gamma, R^+\langle\rho Y\rangle T^\alpha)$.

When $\alpha\neq 0$, assume $\sum_{n\geq 0}a_n\rho^n F_n T^\alpha$ is γ-invariant. Then we have

$$\sum_{n\geq 0}\zeta^\alpha\epsilon_\alpha\left(a_n+\frac{\rho}{\epsilon_\alpha}(n+1)a_{n+1}\right)\rho^n F_n T^\alpha=0.$$

This implies that, for any $n\geq 0$ and any $m\geq 0$,

$$a_n=(-1)^m\prod_{j=1}^m\left(\frac{\rho}{\epsilon_\alpha}(n+j)\right)a_{n+m}.$$

In particular, $\nu_p(a_n)\geq\sum_{j=1}^m\nu_p(n+j)$ for any $m\geq 0$. This forces $a_n=0$ for any $n\geq 0$.

When $\alpha=0$, assume $\sum_{n\geq 0}a_n\rho^n F_n$ is γ-invariant. Then we have

$$\sum_{n\geq 0}(n+1)\rho a_{n+1}\rho^n F_n=0,$$

which implies $a_n=0$ for any $n\geq 1$. So we have $R^+\langle\rho Y\rangle^\Gamma=R^+$. □

Now we are able to handle the higher dimensional case.

Lemma 4.6. *Identify $\widehat{S}^+_{\infty,\rho}$ with $\widehat{R}^+_\infty\langle\rho Y_1,\dots,\rho Y_d\rangle$.*

(1) *The inclusion $R^+\langle\rho\underline{Y}\rangle\hookrightarrow\widehat{S}^+_{\infty,\rho}$ identifies $\mathrm{R}\Gamma(\Gamma,R^+\langle\rho\underline{Y}\rangle)$ with a direct summand of $\mathrm{R}\Gamma(\Gamma,\widehat{S}^+_{\infty,\rho})$ whose complement is concentrated in degree ≥ 1 and is killed by ζ_p-1.*

(2) *For any $i\geq 0$, we have*

$$\mathrm{H}^i(\Gamma,R^+\langle\rho\underline{Y}\rangle)=\bigwedge^i_{R^+}\Bigl(\bigoplus_{j=1}^d R^+\langle\rho Y_j\rangle/R^+\langle\Lambda_{\rho,j},I_\rho,+\rangle\Bigr)$$

for $\Lambda_{\rho,j}=\{\rho^nF_n(Y_j)\}$ and $I_\rho=\{\nu_p((n+1)\rho)\}_{n\geq 0}$.

Proof. Note that $\mathrm{R}\Gamma(\Gamma,\widehat{R}^+_\infty\langle\rho Y_1,\dots,\rho Y_d\rangle)$ is presented by the Koszul complex

$$\mathrm{K}(\widehat{R}^+_\infty\langle\rho Y_1,\dots,\rho Y_d\rangle;\gamma_1-1,\dots,\gamma_d-1)\simeq\mathrm{K}(\widehat{R}^+_\infty\langle\rho Y_1\rangle;\gamma_1-1)\widehat{\otimes}^L_{\widehat{R}^+_\infty}\cdots\widehat{\otimes}^L_{\widehat{R}^+_\infty}\mathrm{K}(\widehat{R}^+_\infty\langle\rho Y_d\rangle;\gamma_d-1).$$

Since $R^+\langle\rho Y_j\rangle/R^+\langle\Lambda_{\rho,j},I_\rho,+\rangle$ is already derived p-complete, the lemma follows from Lemma 4.5 directly. □

Proposition 4.7. (1) $(S^{\dagger,+}_\infty)^\Gamma=R^+$.

(2) *For any $i\geq 1$, $\mathrm{H}^i(\Gamma,S^{\dagger,+}_\infty)$ is p^∞-torsion.*

Proof. We only need to show, for any $i\geq 1$,

$$\varinjlim_{\nu_p(\rho)>\nu_p(\rho_k)}\mathrm{H}^i(\Gamma,\widehat{S}^+_{\infty,\rho})$$

is p^∞-torsion. However, by Lemma 4.6, this follows from a similar argument as in the proof of Corollary 2.23. □

Calculation in general case. Now, by virtues of Theorem 3.4, we may assume that M is an a-small representation of Γ over R^+. Let $e_1,\dots,e_l$ be an R^+-basis of M and A_j be the matrix of γ_j with respect to the chosen basis for all $1\leq j\leq d$; that is,

$$\gamma_j(e_1,\dots,e_l)=(e_1,\dots,e_l)A_j.$$

Put $B_j=A_j-I$. It is the matrix of γ_j-1 and has p-adic valuation $\nu_p(B_j)\geq a+\nu_p(\rho_k)$ by a-smallness of M. Similar to the trivial representation case, we are reduced to computing $\mathrm{R}\Gamma(\Gamma,M\otimes_{R^+}\widehat{R}^+_\infty\langle\rho Y_1,\dots,\rho Y_d\rangle)$. Note that we still have a Γ-equivariant decomposition

$$M\otimes_{R^+}\widehat{R}^+_\infty\langle\rho Y_1,\dots,\rho Y_d\rangle=\widehat{\bigoplus}_{\underline{\alpha}\in(\mathbb{N}[\frac{1}{p}]\cap[0,1))^d}M\otimes_{R^+}R^+\langle\rho Y_1,\dots,\rho Y_d\rangle\underline{T}^{\underline{\alpha}},$$

where $\underline{T}^{\underline{\alpha}}$ denotes $T_1^{\alpha_1}\cdots T_d^{\alpha_d}$ for any $\underline{\alpha}=(\alpha_1,\dots,\alpha_d)$.

Assume $\underline{\alpha}\neq 0$ at first. Without loss of generality, we assume $\alpha_d\neq 0$. Note that

$$\{e_{i,n}:=e_i\rho^nF_n(Y_d)\underline{T}^{\underline{\alpha}}\}_{1\leq i\leq l,n\geq 0}$$

is a set of topological basis of $M\otimes_{R^+}R^+\langle\rho Y_1,\dots,\rho Y_d\rangle\underline{T}^{\underline{\alpha}}$ over $R^+\langle\rho Y_1,\dots,\rho Y_{d-1}\rangle$. We have

$$(\gamma_d-1)(e_{1,n},\dots,e_{l,n})=\zeta^{\alpha_d}\epsilon_{\alpha_d}\Big((e_{1,n},\dots,e_{l,n})\cdot(\epsilon_{\alpha_d}^{-1}B_d+I)+(e_{1,n-1},\dots,e_{l,n-1})\cdot n\frac{\rho}{\epsilon_{\alpha_d}}A_d\Big).$$

Similar to the trivial representation case, using Proposition A.6, we deduce that

$$\mathrm{R}\Gamma(\mathbb{Z}_p\gamma_d,M\otimes_{R^+}R^+\langle\rho Y_1,\dots,\rho Y_d\rangle\underline{T}^{\underline{\alpha}})\simeq M\otimes_{R^+}R^+\langle\rho Y_1,\dots,\rho Y_d\rangle\underline{T}^{\underline{\alpha}}/\epsilon_{\alpha_d}[-1].$$

Using the Hochschild–Serre spectral sequence, we have the following lemma.

Lemma 4.8. *Assume $\underline{\alpha}\neq 0$. Then the complex $\mathrm{R}\Gamma(\Gamma,M\otimes_{R^+}R^+\langle\rho Y_1,\dots,\rho Y_d\rangle\underline{T}^{\underline{\alpha}})$ is concentrated in positive degrees and is killed by ζ_p-1.*

Now, we focus on the $\underline{\alpha}=0$ case and prove the following proposition.

Proposition 4.9. *Keep the notation as above. Assume $\nu_p(\rho)<a+\nu_p(\rho_k)-r$. Define*

$$H(M):=(M\otimes_{R^+}R^+\langle\rho Y_1,\dots,\rho Y_d\rangle)^{\Gamma}.$$

Then the following assertions are true:

(1) *$H(M)$ is a finite free R^+-module of rank l and is independent of the choice of ρ. More precisely*, *if we define*

$$(h_1,\dots,h_l)=(e_1,\dots,e_l)\sum_{n_1,\dots,n_d\geq 0}\prod_{i=1}^{d}\frac{(-A_i^{-1}B_i)^{n_i}}{n_i!}F_{n_i}(Y_i),$$

then $h_1,\dots,h_l$ is an R^+-basis of $H(M)$.

(2) *The inclusion $H(M)\hookrightarrow M\otimes_{R^+}R^+\langle\rho Y_1,\dots,\rho Y_d\rangle$ induces a Γ-equivariant isomorphism*

$$H(M)\otimes_{R^+}R^+\langle\rho Y_1,\dots,\rho Y_d\rangle\cong M\otimes_{R^+}R^+\langle\rho Y_1,\dots,\rho Y_d\rangle.$$

Proof. We first consider the $d=1$ case. In this case, $\Gamma=\mathbb{Z}_p\gamma$ acts on $R^+\langle\rho Y\rangle$ via $\gamma(Y)=Y+1$. Let $e_1,\dots,e_l$ be a basis of M and A be the matrix of γ associated to the chosen basis. Put $B=A-I$ and then $\nu_p(B)\geq a+\nu_p(\rho_k)>\nu_p(\rho)+r$. Note that $\{\rho^nF_n(Y)\}_{n\geq 0}$ is a set of topological basis of $R^+\langle\rho Y\rangle$.

(1) Assume $x=\sum_{n\geq 0}\underline{e}X_n\rho^nF_n(Y)\in H(M)$, where $X_n\in(R^+)^l$ for any $n\geq 0$ and $\underline{e}$ denotes $(e_1,\dots,e_l)$. Since $\gamma(x)=x$, we deduce that, for any $n\geq 0$,

$$BX_n=-(n+1)\rho AX_{n+1}.$$

In other words, we have

$$X_n=\frac{-A^{-1}B}{n\rho}X_{n-1}=\frac{(-A^{-1}B)^n}{\rho^nn!}X_0.$$

Note that $\nu_p((A^{-1}B)^n/(\rho^nn!))\geq(a+\nu_p(\rho_k)-r-\nu_p(\rho))n$. So we get $(A^{-1}B)^n/(\rho^nn!)\in\mathrm{M}_l(R^+)$ and hence X_n is uniquely determined by X_0. In particular, we have

$$x=\underline{e}\sum_{n\geq 0}\frac{(-A^{-1}B)^n}{\rho^nn!}\rho^nF_n(Y)X_0=\underline{e}\sum_{n\geq 0}\frac{(-A^{-1}B)^n}{n!}F_n(Y)X_0. \tag{4-4}$$

Conversely, any $x \in M \otimes_{R^+} R^+\langle \rho Y_1, \dots, \rho Y_d\rangle$ which is of the form (4-4) for some $X_0 \in (R^+)^l$ is γ-invariant. So we are done.

(2) From the proof of (1), we see that $\sum_{n\geq 0}((-A^{-1}B)^n/(\rho^n n!))\rho^n F_n(Y) \in \mathrm{GL}_l(R^+\langle \rho Y\rangle)$. Thus the h_i's form an $R^+\langle \rho Y\rangle$-basis of $M \otimes_{R^+} R^+\langle \rho Y\rangle$ as desired.

Now, we handle the case for any $d \geq 1$. By what we have proved and by iterating, we get

$$\begin{aligned}\underline{e}(R^+\langle \rho Y_1, \dots, \rho Y_d\rangle)^l &= \underline{e}\sum_{n_d\geq 0}\frac{(-A_d^{-1}B_d)^{n_d}}{n_d!}F_{n_d}(Y_d)(R^+\langle \rho Y_1, \dots, \rho Y_d\rangle)^l\\ &= \underline{e}\sum_{n_{d-1},n_d\geq 0}\frac{(-A_{d-1}^{-1}B_{d-1})^{n_{d-1}}}{n_{d-1}!}F_{n_{d-1}}(Y_{d-1})\frac{(-A_d^{-1}B_d)^{n_d}}{n_d!}F_{n_d}(Y_d)(R^+\langle \rho Y_1, \dots, \rho Y_d\rangle)^l\\ &= \cdots\\ &= \underline{e}\sum_{n_1,\dots,n_d\geq 0}\prod_{i=1}^d\frac{(-A_i^{-1}B_i)^{n_i}}{n_i!}F_{n_i}(Y_i)(R^+\langle \rho Y_1, \dots, \rho Y_d\rangle)^l.\end{aligned}$$

Since $\underline{e}\sum_{n_1,\dots,n_d\geq 0}\prod_{i=1}^d((-A_i^{-1}B_i)^{n_i}/n_i!)F_{n_i}(Y_i)$ forms a Γ-invariant basis, the result follows from $(R^+\langle \rho Y_1, \dots, \rho Y_d\rangle)^\Gamma = R^+$. □

Remark 4.10. Note that if $\nu_p(z) > r$, then

$$(1+z)^Y = \sum_{n\geq 0}\frac{z^n}{n!}F_n(Y).$$

Therefore, for M and ρ as above, as $\nu_p(A_i^{-1}B_j) \geq a > r$, the operator $\prod_{i=1}^d \gamma_i^{-Y_i}$, whose matrix is given by $\sum_{n_1,\dots,n_d\geq 0}\prod_{i=1}^d((-A_i^{-1}B_i)^{n_i}/n_i!)F_{n_i}(Y_i)$, is well defined on $M\otimes_{R^+} R^+\langle \rho Y_1, \dots, \rho Y_d\rangle$. Then the above proposition says that we have $H(M) = \prod_{i=1}^d \gamma_i^{-Y_i}M$. Since $\log(1+z)(1+z)^Y = \sum_{n\geq 0}(z^n/n!)F_n'(Y)$ when $\nu_p(z) > r$, for any $\underline{e}\vec{m} \in M$ with $\vec{m} \in (R^+)^l$ and $1 \leq j \leq d$, we get

$$\begin{aligned}\frac{\partial}{\partial Y_j}\Big(\prod_{i=1}^d \gamma_i^{-Y_i}\underline{e}\vec{m}\Big) &= \underline{e}\frac{\partial}{\partial Y_j}\Big(\sum_{n_1,\dots,n_d\geq 0}\prod_{i=1}^d\frac{(-A_i^{-1}B_i)^{n_i}}{n_i!}F_{n_i}(Y_i)\vec{m}\Big)\\ &= \underline{e}\sum_{n_1,\dots,n_d\geq 0}\frac{(-A_j^{-1}B_j)^{n_j}}{n_j!}F_{n_j}'(Y_j)\prod_{\substack{1\leq i\leq d\\ i\neq j}}\frac{(-A_i^{-1}B_i)^{n_i}}{n_i!}F_{n_i}(Y_i)\vec{m}\\ &= \underline{e}(-\log(A_j)\sum_{n_1,\dots,n_d\geq 0}\prod_{i=1}^d\frac{(-A_i^{-1}B_i)^{n_i}}{n_i!}F_{n_i}(Y_i)\vec{m})\\ &= -\log\gamma_j\prod_{i=1}^d\gamma_i^{-Y_i}\underline{e}\vec{m}.\end{aligned}$$

Corollary 4.11. *Keep the notation as above.*

(1) *Denote by* $\theta_{H(M)}$ *the restriction of* Θ *to* $H(M)$. *Then* $(H(M), \theta_{H(M)})$ *is an* a-*small Higgs module. Also,* $\theta_{H(M)} = \sum_{i=1}^d -\log \gamma_i \otimes ((\mathrm{dlog}\, T_i)/t)$.

(2) *The inclusion* $H(M) \to M \otimes_{R^+} S_\infty^{\dagger,+}$ *induces a* Γ-*equivariant isomorphism*

$$H(M) \otimes_{R^+} S_\infty^{\dagger,+} \cong M \otimes_{R^+} S_\infty^{\dagger,+}$$

and identifies the corresponding Higgs complexes

$$\mathrm{HIG}(H(M) \otimes_{R^+} S_\infty^{\dagger,+}, \Theta_{H(M)}) \cong \mathrm{HIG}(M \otimes_{R^+} S_\infty^{\dagger,+}, \Theta_M).$$

Proof. (1) Since $\Theta = \sum_{i=1}^d (\partial/\partial Y_i) \otimes ((\mathrm{dlog}\, T_i)/t)$, the "Also" part follows from Remark 4.10. Since $\nu_p(B_i) \geq a + \nu_p(\rho_k)$ for all j and $\log \gamma_j = -\sum_{n \geq 1} (-B_j)^n/n$, we see the a-smallness of $(H(M), \theta_{H(M)})$ as $\nu_p(B_i^n/n) \geq a + \nu_p(\rho_k)$ for all n.

(2) This follows from Proposition 4.9(2) and the definition of $\theta_{H(M)}$. □

We have seen how to achieve an a-small Higgs module from an a-small representation. It remains to construct an a-small representation of Γ from an a-small Higgs module.

Proposition 4.12. *Assume* $a > r$. *Let* (H, θ_H) *be an* a-*small Higgs module of rank* l *over* R^+. *Put* $M = (H \otimes_{R^+} S_\infty^{\dagger,+})^{\Theta_H = 0}$.

(1) *The restricted* Γ-*action on* M *makes it an* a-*small* $\widehat{R}_\infty^+$-*representation of rank* l. *Also, if* $\theta_H = \sum_{i=1}^d \theta_i \otimes ((\mathrm{dlog}\, T_i)/t)$, *then* γ_i *acts on* M *via* $\exp(-\theta_i)$.

(2) *The inclusion* $M \hookrightarrow H \otimes_{R^+} S_\infty^{\dagger,+}$ *induces a* Γ-*equivariant isomorphism*

$$M \otimes_{\widehat{R}_\infty^+} S_\infty^{\dagger,+} \cong H \otimes_{R^+} S_\infty^{\dagger,+}$$

and identifies the corresponding Higgs complexes

$$\mathrm{HIG}(M \otimes_{\widehat{R}_\infty^+} S_\infty^{\dagger,+}, \Theta_M) \cong \mathrm{HIG}(H \otimes_{R^+} S_\infty^{\dagger,+}, \Theta_H).$$

Proof. (1) The argument is similar to the proof of Proposition 4.9.

Assume $\rho \in \rho_k \mathfrak{m}_{\mathbb{C}_p}$ such that $a + \nu_p(\rho_k) > \nu_p(\rho) + r$. Let $e_1, \ldots, e_l$ be an R^+-basis of H. We claim that $M = (H \otimes_{R^+} \widehat{R}_\infty^+ \langle \rho Y_1, \ldots, \rho Y_d \rangle)^{\Theta_H = 0}$.

In fact, if $\vec{G} = (G_1, \ldots, G_l)^t \in (\widehat{R}_\infty^+ \langle \rho Y_1, \ldots, \rho Y_d \rangle)^l$ such that $m = \sum_{i=1}^l e_i G_i \in M$, then we see that, for any $1 \leq i \leq d$,

$$\theta_i \vec{G} + \frac{\partial \vec{G}}{\partial Y_i} = 0.$$

This forces $\vec{G} = \prod_{i=1}^d \exp(-\theta_i Y_i) \vec{a}$ for some $\vec{a} \in (\widehat{R}_\infty^+)^l$. Since $\nu_p(\theta_j) \geq a + \nu_p(\rho_k)$, the matrix $\prod_{i=1}^d \exp(-\theta_i Y_i)$ is well defined in $\mathrm{GL}_l(\widehat{R}_\infty^+ \langle \rho Y_1, \ldots, \rho Y_d \rangle)$. This shows that M is finite free of rank l and is independent of the choice of ρ.

Note that $\gamma_i(Y_j) = Y_j + \delta_{ij}$. We see γ_i acts on M via $\exp(-\theta_i)$. Since $\nu_p(\theta_i) \geq a + \nu_p(\rho_k)$, using $\exp(-\theta_i Y_i) = \sum_{n \geq 0} ((-\theta_i)^n/n!) Y_i^n$, we deduce that M is a-small.

(2) This follows from the fact that $\prod_{i=1}^{d}\exp(-\theta_i Y_i)\in \mathrm{GL}_l(\widehat{R}_\infty^+\langle \rho Y_1,\dots,\rho Y_d\rangle)$ and the definition of Γ-action on M. □

Finally, we complete the proof of Theorem 4.3.

Proof of Theorem 4.3. Part (1) was given in Corollary 4.11. Part (2) was proved in Proposition 4.12. The equivalence part of (3) follows from Corollary 4.11(2) (as the θ_i's act via the $-\log\gamma_i$'s) together with Proposition 4.12(2) (as the γ_i's act via the $\exp(-\theta_i)$'s). Elementary linear algebra shows that the equivalence preserves tensor products and dualities. So we only need to prove the "Also" part of (4).

Let M be an a-small representation of Γ over $\widehat{R}_\infty^+$ and (H,θ_H) be the corresponding Higgs module over R^+. By Corollary 2.23, we have quasi-isomorphisms of complexes over $\widehat{R}_\infty$

$$M\Big[\frac{1}{p}\Big]\xrightarrow{\simeq}\mathrm{HIG}(M\otimes_{\widehat{R}_\infty^+}S_\infty^\dagger,\Theta_M)\simeq \mathrm{HIG}(H\otimes_{R^+}S_\infty^\dagger,\Theta_H).$$

Applying $\mathrm{R}\Gamma(\Gamma,\cdot)$, we get a quasi-isomorphism

$$\mathrm{R}\Gamma\Big(\Gamma,M\Big[\frac{1}{p}\Big]\Big)\to \mathrm{R}\Gamma(\Gamma,\mathrm{HIG}(H\otimes_{R^+}S_\infty^\dagger,\Theta_H)).$$

However, it follows from Proposition 4.7 that

$$\mathrm{R}\Gamma(\Gamma,S_\infty^\dagger)\simeq R[0].$$

So we get

$$\mathrm{R}\Gamma(\Gamma,\mathrm{HIG}(H\otimes_{R^+}S_\infty^\dagger,\Theta_H))\simeq \mathrm{HIG}\Big(H\Big[\frac{1}{p}\Big],\theta_H\Big).$$

Therefore, we conclude the desired quasi-isomorphism

$$\mathrm{R}\Gamma\Big(\Gamma,M\Big[\frac{1}{p}\Big]\Big)\simeq \mathrm{HIG}\Big(H\Big[\frac{1}{p}\Big],\theta_H\Big).$$ □

Finally, it is worth pointing out that all results in Theorem 4.3 still hold for $\widehat{S}_{\infty,\rho_k}^+$ instead of $S_\infty^{\dagger,+}$ except the "Also" part of (4) because $\mathrm{HIG}\big(\widehat{S}_{\infty,\rho_k}^+\big[\frac{1}{p}\big],\Theta\big)\neq \widehat{R}_\infty[0]$ and $\mathrm{R}\Gamma\big(\Gamma,\widehat{S}_{\infty,\rho_k}^+\big[\frac{1}{p}\big]\big)\neq R[0]$. For the future use, we give the following proposition.

Proposition 4.13. *Keep the notation as in Theorem 4.3.*

(1) *Let M be an a-small $\widehat{R}_\infty^+$-representation of Γ of rank l. Then $H(M)=(M\otimes_{\widehat{R}_\infty^+}\widehat{S}_{\infty,\rho_k}^+)^\Gamma$ and $\theta_{H(M)}$ is the restriction of Θ_M to $H(M)$.*

(2) *Let (H,θ_H) be an a-small Higgs module of rank l over R^+. Then $M(H,\theta_H)=(H\otimes_{R^+}\widehat{S}_{\infty,\rho_k}^+)^{\Theta_H=0}$.*

(3) *Let M be an a-small $\widehat{R}_\infty^+$-representation of Γ and (H,θ_H) be the corresponding Higgs module. Then there is a canonical Γ-equivariant isomorphism of Higgs complexes*

$$\mathrm{HIG}(H\otimes_{R^+}\widehat{S}_{\infty,\rho_k}^+,\Theta_H)\to \mathrm{HIG}(M\otimes_{\widehat{R}_\infty^+}\widehat{S}_{\infty,\rho_k}^+,\Theta_M).$$

Proof. By Corollary 2.22, we have a Γ-equivariant decomposition

$$\widehat{S}_{\infty,\rho_k}^+=\widehat{\bigoplus}_{\underline{\alpha}\in(\mathbb{N}\cap[0,1))^d}R^+\langle\rho_k Y_1,\dots,\rho_k Y_d\rangle\underline{T}^{\underline{\alpha}}.$$

Let N be the a-small R^+-representation of Γ corresponding to M in the sense of Theorem 3.4. Then $M = N \otimes_{R^+} \widehat{R}^+_\infty$.

(1) Thanks to Lemma 4.8, we have

$$(M \otimes_{\widehat{R}^+_\infty} \widehat{S}^+_{\infty,\rho_k})^\Gamma = (N \otimes_{R^+} R^+\langle \rho_k Y_1, \dots, \rho_k Y_d\rangle)^\Gamma.$$

Since $a > r$, it is automatic that $\nu_p(\rho_k) < a + \nu_p(\rho_k) - r$. So (1) is a consequence of Proposition 4.9.

(2) This follows from the proof of Proposition 4.12(1) directly (because $\nu_p(\rho_k) < a + \nu_p(\rho_k) - r$).

(3) This follows from (1), (2) and Theorem 4.3(4) via the base-change along $S^{\dagger,+}_\infty \to \widehat{S}^+_{\infty,\rho_k}$. □

5. A p-adic Simpson correspondence

Statement and preliminaries. Now, we want to globalise the local Simpson correspondence established in the last section for a liftable smooth formal scheme $\mathfrak{X}$. We fix such an $\mathfrak{X}$ together with an A_2-lifting $\widetilde{\mathfrak{X}}$. Then we have the corresponding integral Faltings' extension $\mathcal{E}^+$ and overconvergent period sheaf $\mathcal{O}\mathbb{C}^{\dagger,+}$. Let X be the rigid analytic generic fibre of $\mathfrak{X}$ and $\nu : X_{\text{proét}} \to \mathfrak{X}_{\text{ét}}$ be the projection of sites. Throughout this section, we assume $r = 1/(p-1)$.

Definition 5.1. Assume $a \ge r$. By an *a-small generalised representation* of rank l on $X_{\text{proét}}$, we mean a sheaf $\mathcal{L}$ of locally finite free $\widehat{\mathcal{O}}_X$-modules of rank l which admits a p-complete sub-$\widehat{\mathcal{O}}^+_X$-module $\mathcal{L}^+$ such that there is an étale covering $\{\mathfrak{X}_i \to \mathfrak{X}\}_{i\in I}$ and rationals $b_i > b > a$ such that, for any i,

$$(\mathcal{L}^+/p^{b_i+\nu_p(\rho_k)})^{\text{al}}_{|X_i} \cong ((\widehat{\mathcal{O}}^+_X/p^{b_i+\nu_p(\rho_k)})^l)^{\text{al}}_{|X_i}$$

is an isomorphism of $(\widehat{\mathcal{O}}^{+\text{al}}_X/p^{b_i+\nu_p(\rho_k)})_{|X_i}$-modules, where $\widehat{\mathcal{O}}^{+\text{al}}_X$ is the almost integral structure sheaf[2] and X_i denotes the rigid analytic generic fibre of $\mathfrak{X}_i$.

Definition 5.2. Assume $a \ge r$. By an *a-small Higgs bundle* of rank l on $\mathfrak{X}_{\text{ét}}$, we mean a sheaf $\mathcal{H}$ of locally finite free $\mathcal{O}_{\mathfrak{X}}[\frac{1}{p}]$-modules of rank l together with an $\mathcal{O}_{\mathfrak{X}}[\frac{1}{p}]$-linear operator $\theta_{\mathcal{H}} : \mathcal{H} \to \mathcal{H} \otimes_{\mathcal{O}_{\mathfrak{X}}} \widehat{\Omega}^1_{\mathfrak{X}}(-1)$ satisfying $\theta_{\mathcal{H}} \wedge \theta_{\mathcal{H}} = 0$ such that it admits a $\theta_{\mathcal{H}}$-preserving $\mathcal{O}_{\mathfrak{X}}$-lattice $\mathcal{H}^+$ — i.e., $\mathcal{H}^+ \subset \mathcal{H}$ is a subsheaf of locally free $\mathcal{O}_{\mathfrak{X}}$-modules with $\mathcal{H}^+[\frac{1}{p}] = \mathcal{H}$ — satisfying the condition

$$\theta_{\mathcal{H}}(\mathcal{H}^+) \subset p^{b+\nu_p(\rho_k)}\mathcal{H}^+ \otimes_{\mathcal{O}_{\mathfrak{X}}} \widehat{\Omega}^1_{\mathfrak{X}}(-1)$$

for some $b > a$.

For any a-small generalised representation, define

$$\Theta_{\mathcal{L}} = \text{id}_{\mathcal{L}} \otimes \Theta : \mathcal{L} \otimes_{\widehat{\mathcal{O}}_X} \mathcal{O}\mathbb{C}^\dagger \to \mathcal{L} \otimes_{\widehat{\mathcal{O}}_X} \mathcal{O}\mathbb{C}^\dagger \otimes_{\mathcal{O}_{\mathfrak{X}}} \widehat{\mathcal{O}}^1_{\mathfrak{X}}(-1).$$

[2]This is the presheaf on $X_{\text{proét}}$ sending each affinoid perfectoid space $U = \text{Spa}(R, R^+)$ to the almost $\mathcal{O}_{\mathbb{C}_p}$-module $R^{+\text{al}}$ in the sense of [Scholze 2012, Section 4]. Since $X_{\text{proét}}$ admits a basis of affinoid perfectoid spaces, the proof of [Scholze 2012, Proposition 7.13] shows that $\widehat{\mathcal{O}}^{+\text{al}}_X$ is a sheaf.

Then $\Theta_{\mathcal{L}}$ is a Higgs field on $\mathcal{L}\otimes_{\widehat{\mathcal{O}}_X}\mathcal{O}\mathbb{C}^\dagger$. Denote the induced Higgs complex by $\mathrm{HIG}(\mathcal{L}\otimes_{\widehat{\mathcal{O}}_X}\mathcal{O}\mathbb{C}^\dagger,\Theta_{\mathcal{L}})$. For any a-small Higgs field $(\mathcal{H},\theta_{\mathcal{H}})$, put

$$\Theta_{\mathcal{H}}=\theta_{\mathcal{H}}\otimes\mathrm{id}+\mathrm{id}_{\mathcal{H}}\otimes\Theta:\mathcal{H}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{O}\mathbb{C}^\dagger\to\mathcal{H}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{O}\mathbb{C}^\dagger\otimes_{\mathcal{O}_{\mathfrak{X}}}\widehat{\Omega}^1_{\mathfrak{X}}(-1).$$

Then $\Theta_{\mathcal{H}}$ is a Higgs field on $\mathcal{H}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{O}\mathbb{C}^\dagger$. Denote the induced Higgs complex by $\mathrm{HIG}(\mathcal{H}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{O}\mathbb{C}^\dagger,\Theta_{\mathcal{H}})$. Then our main theorem is the following p-adic Simpson correspondence.

Theorem 5.3 (p-adic Simpson correspondence). *Keep the notation as above.*

(1) *For any a-small generalised representation $\mathcal{L}$ of rank l on $X_{\text{proét}}$, $\mathrm{R}\nu_*(\mathcal{L}\otimes_{\widehat{\mathcal{O}}_X}\mathcal{O}\mathbb{C}^\dagger)$ is discrete. Define $\mathcal{H}(\mathcal{L}):=\nu_*(\mathcal{L}\otimes_{\widehat{\mathcal{O}}_X}\mathcal{O}\mathbb{C}^\dagger)$ and $\theta_{\mathcal{H}(\mathcal{L})}=\nu_*\Theta_{\mathcal{L}}$. Then $(\mathcal{H}(\mathcal{L}),\theta_{\mathcal{H}(\mathcal{L})})$ is an a-small Higgs bundle of rank l.*

(2) *For any a-small Higgs bundle $(\mathcal{H},\theta_{\mathcal{H}})$ of rank l on $\mathfrak{X}_{\text{ét}}$, put*

$$\mathcal{L}(\mathcal{H},\theta_{\mathcal{H}})=(\mathcal{H}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{O}\mathbb{C}^\dagger)^{\Theta_{\mathcal{H}}=0}.$$

Then $\mathcal{L}(\mathcal{H})$ is an a-small generalised representation of rank l.

(3) *The functor $\mathcal{L}\mapsto(\mathcal{H}(\mathcal{L}),\theta_{\mathcal{H}(\mathcal{L})})$ induces an equivalence from the category of a-small generalised representations to the category of a-small Higgs bundles, whose quasi-inverse is given by $(\mathcal{H},\theta_{\mathcal{H}})\mapsto\mathcal{L}(\mathcal{H},\theta_{\mathcal{H}})$. The equivalence preserves tensor products and dualities and identifies the Higgs complexes*

$$\mathrm{HIG}(\mathcal{L}\otimes_{\widehat{\mathcal{O}}_X}\mathcal{O}\mathbb{C}^\dagger,\Theta_{\mathcal{L}})\simeq\mathrm{HIG}(\mathcal{H}(\mathcal{L})\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{O}\mathbb{C}^\dagger,\Theta_{\mathcal{H}(\mathcal{L})}).$$

(4) *Let $\mathcal{L}$ be an a-small generalised representation with associated Higgs bundle $(\mathcal{H},\theta_{\mathcal{H}})$. Then there is a canonical quasi-isomorphism*

$$\mathrm{R}\nu_*(\mathcal{L})\simeq\mathrm{HIG}(\mathcal{H},\theta_{\mathcal{H}}),$$

where $\mathrm{HIG}(\mathcal{H},\theta_{\mathcal{H}})$ is the Higgs complex induced by $(\mathcal{H},\theta_{\mathcal{H}})$. In particular, $\mathrm{R}\nu_*(\mathcal{L})$ is a perfect complex of $\mathcal{O}_{\mathfrak{X}}[\frac{1}{p}]$-modules concentrated in degree $[0,d]$, where d denotes the dimension of $\mathfrak{X}$ relative to $\mathcal{O}_{\mathbb{C}_p}$.

(5) *Assume $f:\mathfrak{X}\to\mathfrak{Y}$ is a smooth morphism between liftable smooth formal schemes over $\mathcal{O}_{\mathbb{C}_p}$. Let $\widetilde{\mathfrak{X}}$ and $\widetilde{\mathfrak{Y}}$ be the fixed A_2-liftings of $\mathfrak{X}$ and $\mathfrak{Y}$, respectively. Assume f lifts to an A_2-morphism $\tilde{f}:\widetilde{\mathfrak{X}}\to\widetilde{\mathfrak{Y}}$. Then the equivalence in* (3) *is compatible with the pull-back along f.*

Remark 5.4. Assume $\mathcal{L}$ is a sheaf of locally free $\widehat{\mathcal{O}}_X$-modules which becomes a-small after a finite étale base-change $f:\mathfrak{Y}\to\mathfrak{X}$. By étale descent, $\mathrm{R}\nu_*(\mathcal{L}\otimes_{\widehat{\mathcal{O}}_X}\mathcal{O}\mathbb{C}^\dagger)$ is well defined and discrete. Also, $\nu_*(\mathcal{L}\otimes_{\widehat{\mathcal{O}}_X}\mathcal{O}\mathbb{C}^\dagger)$ is a Higgs bundle which becomes an a-small Higgs bundle via pull-back along f. Conversely, if $(\mathcal{H},\theta_{\mathcal{H}})$ is a Higgs bundle on $\mathfrak{X}$ which becomes a-small after taking pull-back along a finite étale morphism f, by pro-étale descent for $\widehat{\mathcal{O}}_X$-bundles, $(\mathcal{H}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{O}\mathbb{C}^\dagger)^{\Theta_{\mathcal{H}}=0}$ is a well defined $\widehat{\mathcal{O}}_X$-bundle. Also, it becomes a-small via the pull-back along f. Therefore, one can establish a p-adic Simpson correspondence in this case.

Remark 5.5. Assume $\mathfrak{X}$ comes from a smooth formal scheme $\mathfrak{X}_0$ over $\mathbb{Z}_p$ and admits an A_2-lifting $\widetilde{\mathfrak{X}}$. Note that Faltings [2005, Definition 2] used Breuil–Kisin twists to define Higgs fields while we use Tate twists, so our smallness conditions on Higgs fields differ from his by a multiplication of (ζ_p-1). By

Proposition 2.14, after choosing a covering $\{\mathfrak{X}_i \to \mathfrak{X}\}_{i\in I}$, the cocycle $\{\theta_{ij}\}_{i,j\in I}$ corresponding to the integral Faltings' extension is exactly the one used in [Faltings 2005, Section 4]. Note that locally we define Higgs fields by $\theta = -\log\gamma$ (Corollary 4.11) while Faltings [2005, Remark(ii)] defined $\theta = \log\gamma$. So our construction is compatible with [Faltings 2005] up to a sign on Higgs fields.

Remark 5.6. Suppose $\mathfrak{X}$ comes from a smooth formal scheme $\mathfrak{X}_0$ over $\mathcal{O}_k$ and $\widetilde{\mathfrak{X}}$ is the base-change of $\mathfrak{X}_0$ along $\mathcal{O}_k \to A_2$. Let $\mathcal{O}\mathbb{C}^\dagger$ be the associated overconvergent period sheaf. By its construction, there is a natural inclusion $\mathcal{O}\mathbb{C} \hookrightarrow \mathcal{O}\mathbb{C}^\dagger$. Now assume $\mathbb{L}$ is a $\mathbb{Z}_p$-local system on $\mathfrak{X}_{\text{ét}}$ and $\mathcal{L} = \mathbb{L}\otimes_{\mathbb{Z}_p}\widehat{\mathcal{O}}_X$ is the corresponding $\widehat{\mathcal{O}}_X$-bundle on $X_{\text{proét}}$. Since the resulting Higgs field is nilpotent by [Liu and Zhu 2017, Theorem 2.1], it can be seen from the proof of Theorem 5.3 that the morphism

$$\nu_*(\mathcal{L}\otimes_{\widehat{\mathcal{O}}_X}\mathcal{O}\mathbb{C}) \to \nu_*(\mathcal{L}\otimes_{\widehat{\mathcal{O}}_X}\mathcal{O}\mathbb{C}^\dagger)$$

is an isomorphism. So our construction is compatible with the work of [Liu and Zhu 2017] in this case.

We do some preparations before proving Theorem 5.3.

Lemma 5.7. *Let $U \in X_{\text{proét}}$ be an affinoid perfectoid and $\mathcal{M}^+$ be a sheaf of p-torsion free $\widehat{\mathcal{O}}_X^+$-modules satisfying one of the following conditions*:

(*a*) *$\mathcal{M}^+_{|U}$ is a sheaf of free $\widehat{\mathcal{O}}^+_{X|U}$-modules.*

(*b*) *$\mathcal{M}^+$ is p-complete and there is an almost isomorphism*

$$(\mathcal{M}^+_{|U}/p^c)^{\mathrm{al}} \cong ((\widehat{\mathcal{O}}^+_{X|U}/p^c)^r)^{\mathrm{al}}$$

for some $c > 0$.

Then the following assertions are true:

(1) *For any $i \geq 1$ and $a > 0$, $H^i(U, \mathcal{M}^+)^{\mathrm{al}} \cong H^i(U, \mathcal{M}^+/p^a)^{\mathrm{al}} = 0$.*

(2) *For any $b > a > 0$, the image of $(\mathcal{M}^+/p^b)(U)$ in $(\mathcal{M}^+/p^a)$ is $\mathcal{M}^+(U)/p^a$.*

(3) *Put $\widehat{\mathcal{M}}^+ = \varprojlim_n \mathcal{M}^+/p^n$. Then $\widehat{\mathcal{M}}^+(U) = \varprojlim_n \mathcal{M}^+(U)/p^n$ and for any $i \geq 1$, $H^i(U, \hat{\mathcal{M}}^+)^{\mathrm{al}} = 0$.*

Proof. By [Scholze 2013a, Lemma 4.10], both (1) and (2) hold for free $\widehat{\mathcal{O}}_X^+$-modules. So we only focus on $\mathcal{M}^+$'s satisfying the second condition.

(1) It is enough to show that for any $i \geq 1$, $H^i(U, \mathcal{M}^+)^{\mathrm{al}} = 0$. Granting this, the rest can be deduced from the long exact sequence induced by

$$0 \to \mathcal{M}^+ \xrightarrow{\times p^a} \mathcal{M}^+ \to \mathcal{M}^+/p^a \to 0.$$

Since $(\mathcal{M}^+_{|U}/p^c)^{\mathrm{al}} \cong ((\widehat{\mathcal{O}}^+_{X|U}/p^c)^r)^{\mathrm{al}}$, by [Scholze 2013a, Lemma 4.10(v)], we deduce that

$$H^i(U, \mathcal{M}^+/p^c)^{\mathrm{al}} = 0$$

for any $i \geq 1$. Consider the exact sequence

$$0 \to \mathcal{M}^+/p^c \xrightarrow{p^{(n-1)c}} \mathcal{M}^+/p^{nc} \to \mathcal{M}^+/p^{(n-1)c} \to 0.$$

By induction on n, we see that for any $i \geq 1$, $H^i(U, \mathcal{M}^+/p^{nc})^{\mathrm{al}} = 0$. Now, the desired result follows from [Scholze 2013a, Lemma 3.18].

(2) Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{M}^+ & \xrightarrow{p^b} & \mathcal{M}^+ & \longrightarrow & \mathcal{M}^+/p^b & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \times p^{b-a}} & & \Big\downarrow & & \Big\downarrow & & \\ 0 & \longrightarrow & \mathcal{M}^+ & \xrightarrow{p^a} & \mathcal{M}^+ & \longrightarrow & \mathcal{M}^+/p^a & \longrightarrow & 0 \end{array}$$

Then by (1), we get the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{M}^+(U)/p^b & \longrightarrow & (\mathcal{M}^+/p^b)(U) & \xrightarrow{\delta_b} & H^1(U, \mathcal{M}^+) & \longrightarrow & 0 \\ & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow{\scriptstyle \times p^{b-a}} & & \\ 0 & \longrightarrow & \mathcal{M}^+(U)/p^a & \longrightarrow & (\mathcal{M}^+/p^a)(U) & \xrightarrow{\delta_a} & H^1(U, \mathcal{M}^+) & \longrightarrow & 0 \end{array}$$

Since the multiplication by p^{b-a} is zero on $H^1(U, \mathcal{M}^+)$, the image of $(\mathcal{M}^+/p^b)(U)$ in $(\mathcal{M}^+/p^a)(U)$ is contained in the kernel of δ_a. In other words, $(\mathcal{M}^+/p^b)(U)$ takes values in $\mathcal{M}^+(U)/p^a$. Now, the result follows.

(3) When $\mathcal{M}^+$ is p-complete, there is nothing to prove. Now, assume $\mathcal{M}^+$ is a free $\widehat{\mathcal{O}}_X^+$-module. The first part follows from (2) and the second part follows from the same argument used in (1). □

Remark 5.8. In this paper, we say a module (or a sheaf of $\widehat{\mathcal{O}}_X^+$-modules) M is p-complete, if $M \cong \mathrm{Rlim}_n\, M \otimes^L_{\mathbb{Z}_p} \mathbb{Z}_p/p^n$. This is different from that $M = \lim_n M/p^n$ in general. However, as mentioned in the paragraph below [Bhatt et al. 2019, Lemma 4.6], if M has bounded p^∞-torsion; that is, $M[p^\infty] = M[p^N]$ for some $N \geq 0$, then saying M is p-complete amounts to saying $M = \lim_n M/p^n$. Indeed, in this case, the pro-systems $\{M \otimes^L_{\mathbb{Z}_p} \mathbb{Z}_p/p^n\}_{n\geq 0}$ and $\{M/p^n\}_{n\geq 0}$ are pro-isomorphic. So we obtain that

$$\mathrm{Rlim}_n\, M \otimes^L_{\mathbb{Z}_p} \mathbb{Z}_p/p^n \simeq \mathrm{Rlim}_n\, M/p^n.$$

Lemma 5.9. *Assume $\mathfrak{X} = \mathrm{Spf}(R^+)$ is small. Define X_∞, $\widehat{R}_\infty^+$ as before. Let $\mathcal{L}^+$ be a sheaf of p-complete and p-torsion free $\widehat{\mathcal{O}}_X^+$-modules such that*

$$(\mathcal{L}^+/p^a)^{\mathrm{al}} \cong ((\widehat{\mathcal{O}}_X^+/p^a)^l)^{\mathrm{al}}$$

for some $a > 0$. Put $M = \mathcal{L}^+(X_\infty)$. Then:

(1) *M is a finite free $\widehat{R}_\infty^+$-module of rank l.*

(2) *For any $0 < b < a$, there is a Γ-equivariant isomorphism $M/p^b \cong (\widehat{R}_\infty^+/p^b)^l$.*

Proof. By Lemma 5.7, we have Γ-equivariant almost isomorphisms

$$M/p^a \xrightarrow{\approx} (\mathcal{L}^+/p^a)(X_\infty) \approx (\widehat{\mathcal{O}}_X^+/p^a)^l(X_\infty) \xleftarrow{\approx} (\widehat{R}_\infty^+/p^a)^l. \tag{5-1}$$

In particular, we get an almost isomorphism $M/p^a \approx (\widehat{R}^+_\infty/p^a)^l$. Denote by $e_1, \dots, e_l$ the standard basis of $(\widehat{R}^+_\infty)^l$.

(1) As mentioned in the paragraph after [Scholze 2013a, Definition 2.2], for any $\epsilon \in \mathbb{Q}_{>0}$, one can find $\mathcal{O}_{\mathbb{C}_p}$-morphisms

$$f : M/p^a \to (\widehat{R}^+_\infty/p^a)^l \quad \text{and} \quad g : (\widehat{R}^+_\infty/p^a)^l \to M/p^a$$

such that $f \circ g = p^\epsilon$ and $g \circ f = p^\epsilon$. In particular, the image of g is $p^\epsilon M/p^a$ and the kernel of g is killed by p^ϵ.

For any i, choose $x_i \in M$ such that

$$x_i \equiv g(e_i) \mod p^a M.$$

Then the x_i's generate

$$p^\epsilon M/p^a \cong M/p^{a-\epsilon}.$$

We claim the x_i's are linear independent over $\widehat{R}^+_\infty/p^{a-\epsilon}$. Granting this, we see $M/p^{a-\epsilon}$ is a finite free $\widehat{R}^+_\infty/p^{a-\epsilon}$-module. Since M is p-torsion free and p-complete by Lemma 5.7(3), by choosing $\epsilon < a$, we deduce that M is finite free of rank l as desired.

So we are reduced to proving the claim. Assume $\lambda_i \in \widehat{R}^+_\infty$ such that $\sum_{i=1}^l \lambda_i x_i \in p^a M$, that is, $g\big(\sum_{i=1}^l \lambda_i e_i\big) \in p^a M$. So $\sum_{i=1}^l \lambda_i e_i \in \operatorname{Ker}(g)$ and thus is killed by p^ϵ. In other words, $p^\epsilon \sum_{i=1}^l \lambda_i e_i \in p^a(\widehat{R}^+_\infty)^l$. This forces $\lambda_i \in p^{a-\epsilon}\widehat{R}^+_\infty$ for any i. So we are done.

(2) By [Scholze 2012, Proposition 4.4], the almost isomorphism $M/p^a \approx (\widehat{R}^+_\infty/p^a)^l$ induces an isomorphism

$$\iota : \mathfrak{m}_{\mathbb{C}_p} \otimes_{\mathcal{O}_{\mathbb{C}_p}} (\widehat{R}^+_\infty/p^a)^l \to \mathfrak{m}_{\mathbb{C}_p} \otimes_{\mathcal{O}_{\mathbb{C}_p}} M/p^a.$$

Since (5-1) is Γ-equivariant, so is ι. Since $\mathfrak{m}_{\mathbb{C}_p}$ is flat over $\mathcal{O}_{\mathbb{C}_p}$, this amounts to a Γ-equivariant isomorphism

$$h : (\mathfrak{m}_{\mathbb{C}_p}\widehat{R}^+_\infty/p^a\mathfrak{m}_{\mathbb{C}_p}\widehat{R}^+_\infty)^l \to \mathfrak{m}_{\mathbb{C}_p}M/p^a\mathfrak{m}_{\mathbb{C}_p}M.$$

Now, for any $\epsilon > 0$, choose $x_{i,\epsilon} \in \mathfrak{m}_{\mathbb{C}_p}M$ such that, for any i,

$$x_{i,\epsilon} \equiv h(p^\epsilon e_i) \mod p^a M.$$

Note that $x_{i,\epsilon}$ is unique modulo $p^a M$. So for $0 < \epsilon' < \epsilon$, we have

$$p^{\epsilon-\epsilon'} x_{i,\epsilon'} \equiv x_{i,\epsilon} \mod p^a M.$$

Assume $\epsilon < a$, we see that $p^{\epsilon-\epsilon'}$ divides $x_{i,\epsilon}$ for any ϵ'. By [Bhatt et al. 2018, Lemma 8.10], R^+ is a topologically free $\mathcal{O}_{\mathbb{C}_p}$-module; therefore, so is $\widehat{R}^+_\infty$. As we have seen that M is a finite free $\widehat{R}^+_\infty$-module, it is also topologically free over $\mathcal{O}_{\mathbb{C}_p}$. This forces that $x_{i,\epsilon}$ is divided by p^ϵ. So we may assume $x_{i,\epsilon} = p^\epsilon y_{i,\epsilon}$ for some $y_{i,\epsilon} \in M$. By construction, $y_{i,\epsilon}$ is unique modulo $p^{a-\epsilon}M$.

Now define $H_\epsilon : (\widehat{R}^+_\infty/p^{a-\epsilon})^l \to M/p^{a-\epsilon}$ by sending e_i to $y_{i,\epsilon}$. By construction of H_ϵ, we see that it is the unique $\widehat{R}^+_\infty$-morphism from $(\widehat{R}^+_\infty/p^{a-\epsilon})^l$ to $M/p^{a-\epsilon}$ whose restriction to $(\mathfrak{m}_{\mathbb{C}_p}\widehat{R}^+_\infty/p^{a-\epsilon})^l$ coincides with h.

We need to show H_ϵ is an isomorphism. However, since M is also finite free, after interchanging M and $(\widehat{R}^+_\infty)^l$ and proceeding as above, we get a unique $G_\epsilon : M/p^{a-\epsilon} \to (\widehat{R}^+_\infty/p^{a-\epsilon})^l$, whose restriction to $\mathfrak{m}_{\mathbb{C}_p}M/p^{a-\epsilon}$ coincides with h^{-1}. Now, a similar argument shows that $H_\epsilon \circ G_\epsilon = \mathrm{id}$ and $G_\epsilon \circ H_\epsilon = \mathrm{id}$. So H_ϵ is an isomorphism.

Finally, since h is Γ-equivariant, by the uniqueness of H_ϵ, we deduce that H_ϵ is also Γ-equivariant. Since ϵ is arbitrary, we are done. □

The following corollary is a special case of Lemma 5.9.

Corollary 5.10. *Assume $\mathfrak{X} = \mathrm{Spf}(R^+)$ is small affine. Let $\mathcal{L}$ be an a-small generalised representation with a sub-$\widehat{\mathcal{O}}^+_X$-sheaf $\mathcal{L}^+$ satisfying $(\mathcal{L}^+/p^{b+\nu_p(\rho_k)})^{\mathrm{al}} \cong ((\widehat{\mathcal{O}}^+_X/p^{b+\nu_p(\rho_k)})^l)^{\mathrm{al}}$ for some $b > a$. Then $\mathcal{L}^+(X_\infty)$ is a b'-small $\widehat{R}^+_\infty$-representation of Γ for any $a < b' < b$.*

Lemma 5.11. *Assume $\mathfrak{X} = \mathrm{Spf}(R^+)$ is affine small. Let $\mathcal{L}^+$ be a sheaf of p-complete and p-torsion free $\widehat{\mathcal{O}}^+_X$-modules such that*

$$(\mathcal{L}^+/p^c)^{\mathrm{al}} \cong ((\widehat{\mathcal{O}}^+_X/p^c)^l)^{\mathrm{al}}$$

for some $c > 0$. Then for any $\mathcal{P}^+ \in \{\mathcal{O}\mathbb{C}^+_\rho, \mathcal{O}\widehat{\mathbb{C}}^+_\rho, \mathcal{O}\mathbb{C}^{\dagger,+}\}$ and for each $i \geq 0$, the natural map

$$H^i(\Gamma, (\mathcal{L}^+ \otimes_{\widehat{\mathcal{O}}^+_X} \mathcal{P}^+)(X_\infty)) \to H^i(X_{\mathrm{pro\acute{e}t}}/\mathfrak{X}, \mathcal{L}^+ \otimes_{\widehat{\mathcal{O}}^+_X} \mathcal{P}^+)$$

is an almost isomorphism. When $i = 0$, it is an isomorphism.

Proof. The proof is similar to [Scholze 2013a, Lemma 5.6; Liu and Zhu 2017, Lemma 2.7]. Denote by $X^{m/X}_\infty$ the m-fold fibre product of X_∞ over X. As X_∞ is a Galois cover of X with Galois group Γ, we have $X^{m/X}_\infty \simeq X_\infty \times \Gamma^{m-1}$. Note that $\widehat{\mathcal{O}}^+_X/p^c$ comes from the étale sheaf $\mathcal{O}^+_X/p^c$ on $X_{\mathrm{\acute{e}t}}$ and that $(\mathcal{L}^+/p^c)^{\mathrm{al}} \cong ((\widehat{\mathcal{O}}^+_X/p^c)^l)^{\mathrm{al}}$. By [Scholze 2013a, Lemma 3.16], for any $i \geq 0$ and $m \geq 1$, we have almost isomorphisms

$$\mathrm{Hom}_{\mathrm{cts}}(\Gamma^{m-1}, H^i(X_\infty, \mathcal{L}^+ \otimes_{\widehat{\mathcal{O}}^+_X} \mathcal{P}^+/p^c)) \to H^i(X^{m/X}_\infty, \mathcal{L}^+ \otimes_{\widehat{\mathcal{O}}^+_X} \mathcal{P}^+/p^c).$$

By induction on n, we have almost isomorphisms

$$\mathrm{Hom}_{\mathrm{cts}}(\Gamma^{m-1}, H^i(X_\infty, \mathcal{L}^+ \otimes_{\widehat{\mathcal{O}}^+_X} \mathcal{P}^+/p^{nc})) \to H^i(X^{m/X}_\infty, \mathcal{L}^+ \otimes_{\widehat{\mathcal{O}}^+_X} \mathcal{P}^+/p^{nc}),$$

for any $n \geq 1$. By letting n go to $+\infty$, we get almost isomorphisms

$$\mathrm{Hom}_{\mathrm{cts}}(\Gamma^{m-1}, H^i(X_\infty, \mathcal{L}^+ \otimes_{\widehat{\mathcal{O}}^+_X} \mathcal{P}^+)) \to H^i(X^{m/X}_\infty, \mathcal{L}^+ \otimes_{\widehat{\mathcal{O}}^+_X} \mathcal{P}^+)$$

for $\mathcal{P}^+ \in \{\mathcal{O}\mathbb{C}^{+,\leq r}_\rho, \mathcal{O}\widehat{\mathbb{C}}^+_\rho\}$, where $\mathcal{O}\mathbb{C}^{+,\leq r}_\rho$ denotes the subsheaf of

$$\mathcal{O}\mathbb{C}^+_\rho \cong \widehat{\mathcal{O}}^+_X[\rho Y_1, \dots, \rho Y_d]$$

consisting of polynomials of degrees $\leq r$. By the coherence of restricted pro-étale topos, $H^i(X_\infty^{m/X}, -)$ commutes with direct limits for all i. Since $\mathcal{O}\mathbb{C}_\rho^+ = \bigcup_{r\geq 0}\mathcal{O}\mathbb{C}_\rho^{+,\leq r}$, we also get desired almost isomorphisms for $\mathcal{P}^+ = \mathcal{O}\mathbb{C}_\rho^+$. A similar argument also works for $\mathcal{P}^+ = \mathcal{O}\mathbb{C}^{\dagger,+} = \bigcup_{\rho,\nu_p(\rho)>\nu_p(\rho_k)}\mathcal{O}\widehat{\mathbb{C}}_\rho^+$. When $i=0$, since both sides are $\mathfrak{m}_{\mathbb{C}_p}$-torsion free, so we get injections.

Now applying the Cartan–Leray spectral sequence to the Galois cover $X_\infty \to X$ and using Lemma 5.7, we conclude that the map

$$H^i(\Gamma_\infty, (\mathcal{L}^+\otimes_{\widehat{\mathcal{O}}_X^+}\mathcal{P}^+)(X_\infty)) \to H^i(X_{\text{proét}}/\mathfrak{X}, \mathcal{L}^+\otimes_{\widehat{\mathcal{O}}_X^+}\mathcal{P}^+)$$

is an almost isomorphism for every $i \geq 0$.

For $i=0$, we know $H^0(X_{\text{proét}}/\mathfrak{X}, \mathcal{L}^+\otimes_{\widehat{\mathcal{O}}_X^+}\mathcal{P}^+)$ is the (0, 0)-term of the Cartan–Leray spectral sequence at the E_2-page, which is the kernel of the map

$$(\mathcal{L}^+\otimes_{\widehat{\mathcal{O}}_X^+}\mathcal{P}^+)(X_\infty) \to (\mathcal{L}^+\otimes_{\widehat{\mathcal{O}}_X^+}\mathcal{P}^+)(X_\infty^{2/X}).$$

On the other hand, $H^0(\Gamma, (\mathcal{L}^+\otimes_{\widehat{\mathcal{O}}_X^+}\mathcal{P}^+)(X_\infty))$ is the kernel of the map

$$(\mathcal{L}^+\otimes_{\widehat{\mathcal{O}}_X^+}\mathcal{P}^+)(X_\infty) \to \mathrm{Hom}_{\mathrm{cts}}(\Gamma, (\mathcal{L}^+\otimes_{\widehat{\mathcal{O}}_X^+}\mathcal{P}^+)(X_\infty)).$$

So the result follows from the injectivity of the map

$$\mathrm{Hom}_{\mathrm{cts}}(\Gamma, (\mathcal{L}^+\otimes_{\widehat{\mathcal{O}}_X^+}\mathcal{P}^+)(X_\infty)) \to (\mathcal{L}^+\otimes_{\widehat{\mathcal{O}}_X^+}\mathcal{P}^+)(X_\infty^{2/X}).$$ □

Proof of Theorem 5.3. Now we are prepared to prove Theorem 5.3.

(1) Let $\mathcal{L}$ be an a-small generalised representation of rank l and $\mathcal{L}^+$ be the sub-$\widehat{\mathcal{O}}_X^+$-sheaf as described in Definition 5.1. Define $\mathcal{H}^+ := \nu_*(\mathcal{L}^+\otimes_{\widehat{\mathcal{O}}_X^+}\mathcal{O}\mathbb{C}^{\dagger,+})$. It suffices to show that $\mathrm{R}^i\nu_*(\mathcal{L}^+\otimes_{\widehat{\mathcal{O}}_X^+}\mathcal{O}\mathbb{C}^{\dagger,+})$ is p^∞-torsion for any $i\geq 1$ and that $\mathcal{H}^+$ satisfies conditions in Definition 5.2. Let $b>a$ and $\{\mathfrak{X}_i\to\mathfrak{X}\}_{i\in I}$ be as in Definition 5.1. Since the problem is local on $\mathfrak{X}_{\text{ét}}$, we are reduced to showing that for any $i\in I$, if we write $\mathfrak{X}_i = \mathrm{Spf}(R_i^+)$, then $H^n(X_{\text{proét}}/\mathfrak{X}_i, \mathcal{L}^+\otimes_{\widehat{\mathcal{O}}_X^+}\mathcal{O}\mathbb{C}^{\dagger,+})$ is p^∞-torsion for any $n\geq 1$ and is a b_i-small Higgs module over R_i^+ for $n=0$ in the sense of Definition 4.2 for some $b_i > b$. So we only need to deal with the case for $\mathfrak{X}$ small affine.

Now we may assume $\mathfrak{X} = \mathrm{Spf}(R^+)$ is affine small itself and that

$$(\mathcal{L}^+/p^{b'})^{\mathrm{al}} \cong ((\widehat{\mathcal{O}}_X^+)^l/p^{b'})^{\mathrm{al}}$$

for some $b'>b$. Let X_∞, $\widehat{R}_\infty^+$ and Γ be as before. By Lemma 5.11, the natural morphism

$$H^i(\Gamma, \mathcal{L}^+(X_\infty)\otimes_{\widehat{R}_\infty^+}S_\infty^{\dagger,+}) \to H^i(X_{\text{proét}}/\mathfrak{X}, \mathcal{L}^+\otimes_{\widehat{\mathcal{O}}_X^+}\mathcal{O}\mathbb{C}^{\dagger,+})$$

is an almost isomorphism for $i\geq 1$ and is an isomorphism for $i=0$. So we are reduced to showing $\mathrm{R}\Gamma(\Gamma, \mathcal{L}^+(X_\infty)\otimes_{\widehat{R}_\infty^+}S_\infty^{\dagger})$ is discrete after inverting p and $H^0(\Gamma, \mathcal{L}^+(X_\infty)\otimes_{\widehat{R}_\infty^+}S_\infty^{\dagger})$ is a b''-small Higgs module for some $b''>b$.

However, by Corollary 5.10, $\mathcal{L}^+(X_\infty)$ is a b''-small $\widehat{R}_\infty^+$-representation of Γ for some fixed $b''>b$. So the result follows from Theorem 4.3(1).

(2) Let $(\mathcal{H},\theta_{\mathcal{H}})$ be an a-small Higgs bundle of rank l and $\mathcal{H}^+$ be the $\mathcal{O}_{\mathfrak{X}}$-lattice as described in Definition 5.2. Fix an a' satisfying $a<a'<b$. Define $\mathcal{L}^+=(\mathcal{H}^+\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{O}\mathbb{C}^{\dagger,+})^{\Theta_{\mathcal{H}}=0}$. Then it is a subsheaf of $\mathcal{L}=(\mathcal{H}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{O}\mathbb{C}^{\dagger})^{\Theta_{\mathcal{H}}=0}$ and hence p-torsion free. We claim that the inclusion $\mathcal{O}\mathbb{C}^{\dagger,+}\to\mathcal{O}\widehat{\mathbb{C}}^+_{\rho_k}$ induces a natural isomorphism

$$(\mathcal{H}^+\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{O}\mathbb{C}^{\dagger,+})^{\Theta_{\mathcal{H}}=0}\to(\mathcal{H}^+\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{O}\widehat{\mathbb{C}}^+_{\rho_k})^{\Theta_{\mathcal{H}}=0}.$$

Indeed, this is a local problem and therefore follows from Proposition 4.13. As $\mathcal{H}^+\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{O}\widehat{\mathbb{C}}^+_{\rho_k}$ is p-complete, by continuity of $\Theta_{\mathcal{H}}$, so is $\mathcal{L}^+$. It remains to prove that $\mathcal{L}^+$ is locally almost trivial modulo $p^{a'+\nu_p(\rho_k)}$.

Assume $\mathfrak{X}=\mathrm{Spf}(R^+)$ is small affine and let X_∞, $\widehat{R}^+_\infty$ and Γ be as before. Shrinking $\mathfrak{X}$ if necessary, we may assume $(\mathcal{H}^+,\theta_{\mathcal{H}})$ is induced by a b'-small Higgs module over R^+ for some $b'>a'$. Then by Theorem 4.3, $\mathcal{L}^+(X_\infty)$ is a b'-small $\widehat{R}^+_\infty$-representation of Γ.

Let us go back to the global case. Choose an étale covering $\{\mathfrak{X}_i\to\mathfrak{X}\}$ of $\mathfrak{X}$ by small affine $\mathfrak{X}_i=\mathrm{Spf}(R_i^+)$ such that on each $\mathfrak{X}_i$, $(\mathcal{H}^+,\theta_{\mathcal{H}^+})$ is induced by a b_i-small Higgs module over R_i^+ for some $b_i>a'$. Denote by $X_{i,\infty}$ the corresponding "X_∞" for $\mathfrak{X}_i$ instead of $\mathfrak{X}$. As above, we have

$$\mathcal{L}^+(X_{i,\infty})/p^{b_i}\cong(\widehat{\mathcal{O}}^+_X(X_{i,\infty})/p^{b_i})^l.$$

Therefore, by the proof of [Scholze 2013a, Lemma 4.10(i)], we get an almost isomorphism

$$(\mathcal{L}^+/p^{b_i})^{\mathrm{al}}_{|X_i}\cong((\widehat{\mathcal{O}}^+_X/p^{b_i})^l)^{\mathrm{al}}_{|X_i}$$

with $b_i>a'>a$ as desired.

(3) Let $\mathcal{L}$ be an a-small generalised representation. There exists a natural morphism of Higgs complexes

$$\iota:\mathrm{HIG}(\mathcal{H}(\mathcal{L})\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{O}\mathbb{C}^\dagger,\Theta_{\mathcal{H}(\mathcal{L})})\to\mathrm{HIG}(\mathcal{L}\otimes_{\widehat{\mathcal{O}}_X}\mathcal{O}\mathbb{C}^\dagger,\Theta_{\mathcal{L}}).$$

By construction of $(\mathcal{H}(\mathcal{L}),\theta_{\mathcal{H}(\mathcal{L})})$, it follows from Theorem 4.3(4) that ι is an isomorphism. Since $\mathcal{O}\mathbb{C}^\dagger$ is a resolution of $\widehat{\mathcal{O}}_X$ by Theorem 2.28, we see that $\mathcal{L}(\mathcal{H}(\mathcal{L}),\theta_{\mathcal{H}(\mathcal{L})})=\mathcal{L}$. The isomorphism

$$(\mathcal{H},\theta_{\mathcal{H}})\to(\mathcal{H}(\mathcal{L}(\mathcal{H})),\theta_{\mathcal{H}(\mathcal{L}(\mathcal{H}))})$$

can be deduced in a similar way. So we get the equivalence as desired.

It remains to show the equivalence preserves products and dualities. But this is a local problem, so we are reduced to Theorem 4.3(3).

(4) This follows from the same arguments in the proof of Theorem 4.3(4). Indeed, combining Theorem 2.28 and the item (3), we have a quasi-isomorphism

$$\mathcal{L}\to\mathrm{HIG}(\mathcal{L}\otimes_{\widehat{\mathcal{O}}_X}\mathcal{O}\mathbb{C}^\dagger,\Theta_{\mathcal{L}})\simeq\mathrm{HIG}(\mathcal{H}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{O}\mathbb{C}^\dagger,\Theta_{\mathcal{H}}).$$

On the other hand, it follows from (1) that there exists a quasi-isomorphism

$$\mathrm{R}\nu_*(\mathrm{HIG}(\mathcal{H}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{O}\mathbb{C}^\dagger,\Theta_{\mathcal{H}}))\simeq\mathrm{HIG}(\mathcal{H},\theta_{\mathcal{H}}).$$

So we get a quasi-isomorphism

$$\mathrm{R}\nu_*(\mathcal{L}) \simeq \mathrm{HIG}(\mathcal{H}, \theta_{\mathcal{H}})$$

as desired.

(5) Since $f : \mathfrak{X} \to \mathfrak{Y}$ admits an A_2-lifting $\tilde{f}$, by Proposition 2.29, we get a morphism $f^*\mathcal{O}\mathbb{C}_Y^\dagger \to \mathcal{O}\mathbb{C}_X^\dagger$ which is compatible with Higgs fields.

Assume $(\mathcal{H}, \theta_{\mathcal{H}})$ is an a-small Higgs field on $\mathfrak{Y}_{\text{ét}}$. Denote by $(f^*\mathcal{H}, f^*\theta_{\mathcal{H}})$ its pull-back along f. By (3), we get the following isomorphisms, which are compatible with Higgs fields:

$$\begin{aligned}\mathcal{L}(f^*\mathcal{H}, f^*\theta_{\mathcal{H}}) \otimes_{\widehat{\mathcal{O}}_X} \mathcal{O}\mathbb{C}_X^\dagger &\cong f^*\mathcal{H} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}\mathbb{C}_X^\dagger \\ &\cong f^*(\mathcal{H} \otimes_{\mathcal{O}_{\mathfrak{Y}}} \mathcal{O}\mathbb{C}_Y^\dagger) \otimes_{f^*\mathcal{O}\mathbb{C}_Y^\dagger} \mathcal{O}\mathbb{C}_X^\dagger \\ &\cong f^*(\mathcal{L}(\mathcal{H}, \theta_{\mathcal{H}}) \otimes_{\widehat{\mathcal{O}}_Y} \mathcal{O}\mathbb{C}_Y^\dagger) \otimes_{f^*\mathcal{O}\mathbb{C}_Y^\dagger} \mathcal{O}\mathbb{C}_X^\dagger \\ &\cong f^*\mathcal{L}(\mathcal{H}, \theta_{\mathcal{H}}) \otimes_{\widehat{\mathcal{O}}_X} \mathcal{O}\mathbb{C}_X^\dagger.\end{aligned}$$

After taking kernels of Higgs fields, we obtain that

$$\mathcal{L}(f^*\mathcal{H}, f^*\theta_{\mathcal{H}}) \cong f^*\mathcal{L}(\mathcal{H}, \theta_{\mathcal{H}}).$$

So the functor $(\mathcal{H}, \theta_{\mathcal{H}}) \to \mathcal{L}(\mathcal{H}, \theta_{\mathcal{H}})$ in (2) is compatible with the pull-back along f. But we have shown it is an equivalence, so its quasi-inverse must commute with the pull-back along f. This completes the proof.

Corollary 5.12. *Assume $\mathfrak{X}$ is a liftable proper smooth formal scheme of relative dimension d over $\mathcal{O}_{\mathbb{C}_p}$. For any small generalised representation $\mathcal{L}$, $\mathrm{R}\Gamma(X_{\text{proét}}, \mathcal{L})$ is concentrated in degree $[0, 2d]$, whose cohomologies are finite dimensional $\mathbb{C}_p$-spaces.*

Proof. Since we have assumed $\mathfrak{X}$ is proper smooth, this follows from Theorem 5.3(4) directly. □

Remark 5.13. Except the item (4), all results in Theorem 5.3 are still true by using $\mathcal{O}\widehat{\mathbb{C}}_{\rho_k}^+$ instead of $\mathcal{O}\mathbb{C}^{\dagger,+}$.

Remark 5.14. In Corollary 5.12, one can also deduce that $\mathrm{R}\Gamma(X_{\text{proét}}, \mathcal{L})$ is concentrated in degree $[0, 2d]$ when $\mathfrak{X}$ is just quasi-compact of relative dimension d over $\mathcal{O}_{\mathbb{C}_p}$. Indeed, in this case, we have

$$\mathrm{R}\Gamma(X_{\text{proét}}, \mathcal{L}) \simeq \mathrm{R}\Gamma(\mathfrak{X}_{\text{ét}}, \mathrm{HIG}(\mathcal{H}, \theta_{\mathcal{H}})) \simeq \mathrm{R}\Gamma(X_{\text{ét}}, \mathrm{HIG}(\mathcal{H}, \theta_{\mathcal{H}}) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{X_{\text{ét}}}),$$

where $\mathrm{HIG}(\mathcal{H}, \theta_{\mathcal{H}}) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{X_{\text{ét}}}$ denotes the induced Higgs complex on $X_{\text{ét}}$. On the other hand, by étale descent, the category of étale vector bundles on $X_{\text{ét}}$ is equivalent to the category of analytic vector bundles on X_{an}, where X_{an} denotes the analytic site of X. So the Higgs complex $\mathrm{HIG}(\mathcal{H}, \theta_{\mathcal{H}}) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{X_{\text{ét}}}$ upgrades to an analytic Higgs complex $\mathrm{HIG}(\mathcal{H}_{\text{an}}, \theta_{\mathcal{H}})$ such that

$$\mathrm{HIG}(\mathcal{H}_{\text{an}}, \theta_{\mathcal{H}}) \otimes_{\mathcal{O}_{X_{\text{an}}}} \mathcal{O}_{X_{\text{ét}}} = \mathrm{HIG}(\mathcal{H}, \theta_{\mathcal{H}}) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{X_{\text{ét}}}.$$

By analytic-étale comparison (see [Fresnel and van der Put 2004, Proposition 8.2.3]), for any coherent $\mathcal{O}_{X_{\mathrm{an}}}$-module $\mathcal{M}$, there is a canonical quasi-isomorphism

$$\mathrm{R}\Gamma(X_{\mathrm{an}}, \mathcal{M}) \simeq \mathrm{R}\Gamma(X_{\text{ét}}, \mathcal{M} \otimes_{\mathcal{O}_{X_{\mathrm{an}}}} \mathcal{O}_{X_{\text{ét}}}).$$

So by considering corresponding spectral sequences of these complexes, we get a quasi-isomorphism

$$\mathrm{R}\Gamma(X_{\mathrm{an}}, \mathrm{HIG}(\mathcal{H}_{\mathrm{an}}, \theta_{\mathcal{H}})) \simeq \mathrm{R}\Gamma(X_{\text{ét}}, \mathrm{HIG}(\mathcal{H}, \theta_{\mathcal{H}}) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{X_{\text{ét}}}).$$

Now, the quasi-compactness of $\mathfrak{X}$ implies that X is a noetherian space. So the result follows from Grothendieck's vanishing theorem [1957, Théorème 3.6.5] directly. The author thanks the anonymous referees for pointing this out.

6. Appendix

We prove some elementary facts used in this paper. Throughout this section, we always assume A is a p-complete flat $\mathcal{O}_{\mathbb{C}_p}$-algebra.

Definition A.1. Let $\Lambda = \{\alpha\}_{\alpha\in\Lambda}$ be an index set and $I = \{i_\alpha\}_\alpha$ be a set of nonnegative real numbers indexed by Λ. Define

(1) $A[\Lambda] = \bigoplus_{\alpha\in\Lambda} A$;

(2) $A\langle\Lambda\rangle = \varprojlim_m A[\Lambda]/p^m A[\Lambda]$;

(3) $A[\Lambda, I] = \bigoplus_{\alpha\in\Lambda} p^{i_\alpha} A$;

(4) $A\langle\Lambda, I\rangle = \varprojlim_m (A[\Lambda, I] + p^m A[\Lambda])/p^m A[\Lambda]$;

(5) $A\langle\Lambda, I, +\rangle = \varprojlim_m A[\Lambda, I]/p^m A[\Lambda, I]$.

Proposition A.2. (1) *$A\langle\Lambda\rangle/A\langle\Lambda, I\rangle$ is the classical p-completion of* $A[\Lambda]/A[\Lambda, I]$.

(2) *$A\langle\Lambda\rangle/A\langle\Lambda, I, +\rangle$ is the derived p-completion of* $A[\Lambda]/A[\Lambda, I]$.

Proof. Since $A\langle\Lambda, I\rangle$ is the closure of $A\langle\Lambda, I, +\rangle$ in $A\langle\Lambda\rangle$ with respect to the p-adic topology, the item (1) follows from (2) directly. So we are reduced to proving (2).

Consider the short exact sequence

$$0 \longrightarrow A[\Lambda, I] \longrightarrow A[\Lambda] \longrightarrow A[\Lambda]/A[\Lambda, I] \longrightarrow 0.$$

For any $n \geq 0$, we get an exact triangle

$$A[\Lambda, I] \otimes^L_{\mathbb{Z}_p} \mathbb{Z}_p/p^n \to A[\Lambda] \otimes^L_{\mathbb{Z}_p} \mathbb{Z}_p/p^n \to (A[\Lambda]/A[\Lambda, I]) \otimes^L_{\mathbb{Z}_p} \mathbb{Z}_p/p^n \to .$$

Applying Rlim_n to this exact triangle and using p-complete flatness of A, we get the exact triangle

$$A\langle\Lambda, I, +\rangle[0] \to A\langle\Lambda\rangle[0] \to K \to,$$

where K denotes the derived p-completion of $A[\Lambda]/A[\Lambda, I]$. Now, the item (2) follows from the injectivity of the map $A\langle\Lambda, I, +\rangle \to A\langle\Lambda\rangle$. □

Remark A.3. For any $(\lambda_\alpha)_{\alpha\in\Lambda}\in\prod_{\alpha\in\Lambda}A$, we write $\lambda_\alpha\xrightarrow{\nu_p}0$, if for any $M>0$ the set $\{\alpha\in\Lambda\,|\,\nu_p(\lambda_\alpha)\leq M\}$ is finite. Then we have

$$A\langle\Lambda, I\rangle=\left\{(\lambda_\alpha)_{\alpha\in\Lambda}\mid \nu_p\Big(\frac{\lambda_\alpha}{p^{i_\alpha}}\Big)\geq 0\right\}$$

and

$$A\langle\Lambda, I, +\rangle=\left\{(\lambda_\alpha)_{\alpha\in\Lambda}\mid \nu_p\Big(\frac{\lambda_\alpha}{p^{i_\alpha}}\Big)\geq 0,\ \frac{\lambda_\alpha}{p^{i_\alpha}}\xrightarrow{\nu_p}0\right\}.$$

Definition A.4. Assume M is a (topologically) free A-module. Let Σ_1 and Σ_2 be two subsets of M.

(1) We write $\Sigma_1\sim\Sigma_2$, if they (topologically) generate the same sub-A-module of M.

(2) We write $\Sigma_1\approx\Sigma_2$, if both of them are sets of (topological) basis of M. In this case, we also write $M\approx\Sigma_1$ if no ambiguity appears.

Proposition A.5. *Fix $\epsilon,\omega\in\mathcal{O}_{\mathbb{C}_p}$. Let M be a (topologically) free A-module with basis $\{x_i\}_{i\geq 0}$. If $N\subset M$ is a submodule such that*

$$N\sim\{\omega(x_i+i\epsilon x_{i-1})\mid i\geq 0\},$$

where $x_{-1}=0$, then $N=\omega M$.

Proof. Put $y_i=x_i+i\epsilon x_{i-1}$ for all i. Then we see that

$$(y_0,y_1,y_2,y_3,\dots)=(x_0,x_1,x_2,x_3,\dots)\cdot X$$

with

$$X=\begin{pmatrix}1&\epsilon&0&0&\cdots\\0&1&2\epsilon&0&\cdots\\0&0&1&3\epsilon&\cdots\\0&0&0&1&\cdots\\\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix}$$

and that

$$(x_0,x_1,x_2,x_3,\dots)=(y_0,y_1,y_2,y_3,\dots)\cdot Y$$

with

$$Y=\begin{pmatrix}1&-\epsilon&2\epsilon^2&-6\epsilon^3&\cdots\\0&1&-2\epsilon&6\epsilon^2&\cdots\\0&0&1&-3\epsilon&\cdots\\0&0&0&1&\cdots\\\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix}$$

The (i,j)-entry of Y is δ_{ij} if $i\geq j$ and is $(-\epsilon)^{j-i}((j-1)!/(i-1)!)$ if $i<j$. Then the proposition follows from the fact $XY=YX=\mathrm{Id}$. □

The following proposition can be proved in the same way.

Proposition A.6. *Fix $\Theta \in \mathrm{M}_l(A)$. Let M be a (topologically) free A-module with basis $\{x_i\}_{i\geq 0}$. Let N be a finite free R-module of rank l with a basis $\{e_1, \dots, e_l\}$. For every $1 \leq j \leq l$ and $i \geq 0$, put $f_{j,i} \in N \otimes_A M$ satisfying*

$$(f_{1,i}, \dots, f_{l,i}) = (e_1 \otimes x_i, \dots, e_l \otimes x_i) + i(e_1 \otimes x_{i-1}, \dots, e_l \otimes x_{i-1})\Theta,$$

where $x_{-1} = 0$. Then $N \otimes_A M \approx \{f_{j,i} \mid 1 \leq j \leq l, i \geq 0\}$.

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