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A weighted one-level density of families of L-functions

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This paper is devoted to a weighted version of the one-level density of the nontrivial zeros of L-functions, tilted by a power of the L-function evaluated at the central point. Assuming the Riemann hypothesis and the ratio conjecture, for some specific families of L-functions, we prove that the same structure suggested by the density conjecture also holds in this weighted investigation, if the exponent of the weight is small enough. Moreover, we speculate about the general case, conjecturing explicit formulae for the weighted kernels.

1. A weighted version of the one-level density

Let us assume the Riemann hypothesis for all the L-functions that arise. The classical one-level density considers a smooth localization at the central point of the counting function of the nontrivial zeros of an L-function, averaged over a "natural" family of L-functions in the Selberg class[1]. More specifically, given an even and real-valued function f in the Schwartz space[2] and an L-function $L(s)$ in a family $\mathcal{F}$, we consider the quantity

$$\sum_{\gamma_L} f(c(L)\gamma_L), \tag{1-1}$$

where γ_L denotes the imaginary part of a generic nontrivial zero of L and $c(L)$ denotes the log-conductor of $L(s)$ at the central point. We recall that $1/c(L)$ is the mean spacing of the nontrivial zeros of $L(s)$ around $s = \frac{1}{2}$. The one-level density for the family $\mathcal{F}$ is the average of the above quantity over the family, i.e.,

$$\lim_{X\to\infty} \frac{1}{\sum_{L\in\mathcal{F}_X} 1} \sum_{L\in\mathcal{F}_X} \sum_{\gamma_L} f(c(L)\gamma_L), \tag{1-2}$$

with

$$\mathcal{F}_X := \{L \in \mathcal{F} : c(L) \le \log X\}.$$

In the literature, this is also referred to as the "low-lying zeros" density, as the sum (1-1) gives information on the distribution of the zeros of L which are close to the central point. Indeed, if a zero is substantially more than $1/c(L)$ away from the central point, then it does not contribute significantly to the sum (see, e.g., [Iwaniec et al. 2000] for a complete overview).

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[1] We refer, e.g., to [Kaczorowski and Perelli 1999] for the definitions and the basic properties of the Selberg class.

[2] In practice we will see that this condition can be weakened and a decay like $f(x) \ll 1/(1+x^2)$ at infinity will suffice.

Katz and Sarnak [1999] studied a wide variety of families and attached to each of these families of L-functions is a symmetry type (i.e., unitary, symplectic, or orthogonal, hereafter identified by a group G), which should govern the one-level density of the considered family. Namely, the density conjecture predicts that

$$\lim_{X\to\infty}\frac{1}{\sum_{L\in\mathcal{F}_X}1}\sum_{L\in\mathcal{F}_X}\sum_{\gamma_L}f(c(L)\gamma_L)=\int_{-\infty}^{+\infty}f(x)W_{\mathcal{F}}(x)\,dx, \tag{1-3}$$

where $W_{\mathcal{F}}$ equals the one-level density function W_G for the (scaled) limit of

$$G\in\{\mathrm{U}(N),\mathrm{USp}(2N),\mathrm{O}(N),\mathrm{SO}(2N),\mathrm{SO}(2N+1)\}^3,$$

i.e., the kernel appearing in the analogous average in the corresponding random matrix theory setting. In particular, the kernel $W_{\mathcal{F}}$ is predicted to depend on G only. We recall that the function W_G is known for all of the classical compact groups, being

$$\begin{aligned}
W_{\mathrm{U}}(x)&=1,\\
W_{\mathrm{USp}}(x)&=1-\frac{\sin(2\pi x)}{2\pi x},\\
W_{\mathrm{O}}(x)&=1+\frac{1}{2}\delta_0(x),\\
W_{\mathrm{SO}^+}(x)&=1+\frac{\sin(2\pi x)}{2\pi x},\\
W_{\mathrm{SO}^-}(x)&=\delta_0(x)+1-\frac{\sin(2\pi x)}{2\pi x},
\end{aligned}$$

where δ_0 is the Dirac δ-function centered at 0. Examples of one-level density theorems which prove (1-3) in specific cases can be found, e.g., in [Iwaniec et al. 2000; Hughes and Rudnick 2002; 2003; Özlük and Snyder 1999; Conrey 2005; Conrey and Snaith 2007].

In this paper, we investigate a weighted analogue of the one-level density. In particular, we consider a tilted average over the family $\mathcal{F}$ of the quantity (1-1), multiplied by a power of L evaluated at the central point. The philosophy of this tilted average is similar to that of Fazzari [2021a; 2021b]; the weight has the effect of giving more relevance to the L-functions which are large at the central point, near which zeros are expected to be rarer.

More specifically, given $k\in\mathbb{N}$, we are interested in

$$\mathcal{D}_k^{\mathcal{F}}(f)=\mathcal{D}_k^{\mathcal{F}}(f,X):=\frac{1}{\sum\limits_{L\in\mathcal{F}_X}V(L(1/2))^k}\sum_{L\in\mathcal{F}_X}\sum_{\gamma_L}f(c(L)\gamma_L)V\big(L\big(\tfrac{1}{2}\big)\big)^k \tag{1-4}$$

in the limit $X\to\infty$, where V depends on the symmetry type of the family; in particular, $V(z)=|z|^2$ in the unitary case and $V(z)=z$ for the symplectic and orthogonal cases. The quantity $\mathcal{D}_k^{\mathcal{F}}(f)$ links the

[3]Note that the scaled limit of $\mathrm{SO}(2N)$ (respectively, $\mathrm{SO}(2N+1)$) is commonly denoted by SO^+ (respectively, SO^-) in the literature and also in the rest of this paper.

moments to the one-level density, making the connection between nontrivial zeros and the size of $L\left(\frac{1}{2}\right)$ explicit. Indeed, $\mathcal{D}_k^{\mathcal{F}}(f)$ can be seen as a special case of

$$\frac{1}{\sum_{L\in\mathcal{F}_X} V(L(1/2))^k} \sum_{L\in\mathcal{F}_X} g(L)V\left(L\left(\tfrac{1}{2}\right)\right)^k \tag{1-5}$$

where $g(L)$ is a function over the L-functions of a given family $\mathcal{F}$. In the unitary case, for example, we know from Soundararajan's work [2009] that the dominant contribution to the $2k$-th moment comes from those L-functions such that the size of $\left|L\left(\frac{1}{2}\right)\right|$ is about $(\log X)^{k+o(1)}$, which form a thin subset of size about $\#\mathcal{F}_X/(\log T)^{k^2+o(1)}$. Thus, if the function g has size 1, then only these L-functions contribute to the main term of the sum in (1-5). With the choice we made in (1-4), we have $g(L)=\sum_{\gamma_L} f(c(L)\gamma_L)$, which is not bounded but only $\ll c(L)$, by the Riemann–von Mangoldt formula. However, the standard n-th level density [Rudnick and Sarnak 1996] implies that $g(L)\ll c(L)^{\varepsilon}$ for all but $\#\mathcal{F}_X/(\log X)^A$ L-functions in the family, for every $A>0$. Therefore, also in (1-4), we have that only the L-functions such that $\left|L\left(\frac{1}{2}\right)\right| \asymp (\log X)^{k\pm\varepsilon}$ contribute significantly to the main term of the sum. For this reason, for unitary families, $\mathcal{D}_k^{\mathcal{F}}$ can be interpreted as a (weighted) one-level density for the thin subset

$$\left\{L\in\mathcal{F} : (\log X)^{k-\varepsilon} \ll \left|L\left(\tfrac{1}{2}\right)\right| \ll (\log X)^{k+\epsilon}\right\}.$$

Similarly, in the symplectic and orthogonal cases, $\mathcal{D}_k^{\mathcal{F}}$ is a weighted one-level density, focused on the L-functions in the family which are responsible to the k-th moment.

From the computations we perform throughout this paper in some specific cases, we speculate that the structure suggested by the density conjecture (1-3) holds also in the weighted case. Namely, we expect that

$$\mathcal{D}_k^{\mathcal{F}}(f) = \int_{-\infty}^{+\infty} f(x)W_G^k(x)\,dx + O\left(\frac{1}{\log X}\right), \tag{1-6}$$

where the weighted one-level density function W_G^k only depends on k and on the symmetry type of the family $\mathcal{F}$. Note that the superscript k is an index, indicating that we are weighting with the k-th power of $V\left(L\left(\frac{1}{2}\right)\right)$; in particular W_G^k is not the k-th power of W_G.

This kind of weighting naturally appears also in other contexts, such as Kowalski, Saha and Tsimerman's paper [Kowalski et al. 2012]. Given a Siegel modular form F of genus 2, the authors compute the one-level density of the spinor L-functions of F, with a weight ω^F which is essentially the modulus square of the first Fourier coefficient[4] of F. This family is expected to be orthogonal, but with this weight one does not obtain the usual kernel W_{O}. This discrepancy can be explained by Böcherer's conjecture [1986] and [Dickson et al. 2020] (now proved by Furusawa and Morimoto [2021]), which claims that ω^F is proportional to the central value $L\left(\frac{1}{2}, F\right)$. To be more precise, it says that $\omega^F \approx L\left(\frac{1}{2}, F\right)L\left(\frac{1}{2}, F\times\chi_4\right)$. Since $L\left(\frac{1}{2}, F\times\chi_4\right)$ is "uncorrelated" with $L(s, F)$ and with its zeros, then the kernel they obtained is indeed $W_{\mathrm{SO}^+}^1$ (see, e.g., (5-17)[5] and note that weighting with $L\left(\frac{1}{2}, F\right)^k$, the odd part of the family does not contribute, if $k>0$). Moreover, they notice that this kernel is the one that arises from symplectic symmetry types.

[4] I.e., the Fourier coefficient corresponding to the identity matrix.

[5] In [Kowalski et al. 2012], the kernel is written as $1-\delta_0/2$, which is equivalent to $W_{\mathrm{SO}^+}^1$ for test functions whose Fourier transforms are supported in $[-1, 1]$, which is an assumption in [Kowalski et al. 2012].

Thus, the symmetry of the family jumps from O to USp, after weighting with ω^F (see also [Knightly and Reno 2019; Sugiyama 2021] for other examples where this phenomenon of change of symmetry type is observed). This transition can be seen as a particular case of (2-4) below, which conjecturally predicts a relation between the weighted one-level density functions of different symmetry types.

2. Statement of main results

In the following, we focus on three specific families of L-functions, each with a different symmetry type; first we consider the unitary family $\boldsymbol{\zeta} := \{\zeta(\frac{1}{2}+it) : t \in \mathbb{R}\}$, i.e., the continuous family of the Riemann zeta function parametrized by a vertical shift. Then we study the symplectic family $\boldsymbol{L}_\chi$ of quadratic Dirichlet L-functions. Finally, we look at the orthogonal family $\boldsymbol{L}_{\Delta,\chi}$ of the quadratic twists of the L-function associated with the discriminant modular form Δ. For these families, under the assumption of the relevant Riemann hypothesis and ratio conjecture, we perform an asymptotic analysis of $\mathcal{D}_k^{\mathcal{F}}(f)$. Our results confirm our prediction (1-6), for small values of k. We recall that the case $k=0$ is already known in the literature for all of these families, both assuming the ratio conjecture (see [Conrey and Snaith 2007]) and without (for restricted ranges for f, see, e.g., [Hughes and Rudnick 2002; Conrey 2005; Özlük and Snyder 1999]).

We start with the unitary family. Note that, since this is a continuous family, the average over the family in the definition of $\mathcal{D}_k^{\boldsymbol{\zeta}}(f)$ is given by an integration over $t \in [T, 2T]$ instead of the sum in (1-4). In this case, setting

$$\begin{aligned} W_{\mathrm{U}}^0(x) &:= W_{\mathrm{U}}(x) = 1, \\ W_{\mathrm{U}}^1(x) &:= 1 - \frac{\sin^2(\pi x)}{(\pi x)^2} \\ W_{\mathrm{U}}^2(x) &:= 1 - \frac{2+\cos(2\pi x)}{(\pi x)^2} + \frac{3\sin(2\pi x)}{(\pi x)^3} + \frac{3(\cos(2\pi x)-1)}{2(\pi x)^4}, \end{aligned}$$

we prove the following theorem:

Theorem 2.1. *Let us assume the Riemann hypothesis and the ratio conjecture* (*see Conjecture 3.1*). *Let us consider a test function f, which is holomorphic throughout the strip $|\Im(z)| < 2$, real on the real line, even and such that $f(x) \ll 1/(1+x^2)$ as $x \to \infty$. Then, for $k \in \mathbb{N}$ and $k \le 2$, we have*

$$\mathcal{D}_k^{\boldsymbol{\zeta}}(f) = \int_{-\infty}^{+\infty} f(x) W_U^k(x)\, dx + \mathrm{O}\Big(\frac{1}{\log T}\Big).$$

For this unitary family, in [Bettin and Fazzari 2022] we also develop an alternative method built on Hughes and Rudnick's technique [2002], which allows us to show (1-6) unconditionally[6]. Moreover, the analogue of Theorem 2.1 can be proved in the random matrix theory setting without any assumptions, since the formula for the ratios of characteristic polynomials averaged over the unitary group is known

[6]Neither the Riemann Hypothesis nor the ratio conjecture is required. However, this unconditional strategy works only for test functions whose Fourier transform's support is small enough.

unconditionally (see [Conrey et al. 2008, Theorem 4.1] and also [Conrey et al. 2005b; Huckleberry et al. 2016]). Therefore, denoting

$$Z = Z(A,\theta) = \det(I - Ae^{i\theta}) \tag{2-1}$$

for the characteristic polynomial of $N \times N$ matrices A and $\theta_1, \dots, \theta_N$ for the phases of the eigenvalues of A, we prove a result that is the random matrix analogy of Theorem 2.1, essentially with the same proof. When we work on the random matrix theory side, to ensure that the one-level density is well defined, we need the test function to be 2π-periodic. Hence, given $f : \mathbb{R} \to \mathbb{R}$ an even Schwartz function, we define

$$F_N(x) = \sum_{h \in \mathbb{Z}} f\Big(\frac{N}{2\pi}(x - 2\pi h)\Big), \tag{2-2}$$

and we prove the following:

Theorem 2.2. *Let us consider $f : \mathbb{R} \to \mathbb{R}$ an even Schwartz function and F_N as in* (2-2). *Then, for $k \in \mathbb{N}$ and $k \le 2$, we have*

$$\frac{1}{\int_{U(N)} |Z|^{2k}\, d_{\mathrm{Haar}}} \int_{U(N)} \sum_{j=1}^{N} F_N(\theta_j) |Z|^{2k}\, d_{\mathrm{Haar}} \xrightarrow{N \to \infty} \int_{-\infty}^{+\infty} f(x) W_{\mathrm{U}}^k(x)\, dx.$$

In the symplectic case, we compute the weighted one-level density functions for any nonnegative integer $k \le 4$. We set

$$\begin{aligned}
W_{\mathrm{USp}}^0(x) &:= W_{\mathrm{USp}}(x) = 1 - \frac{\sin(2\pi x)}{2\pi x},\\
W_{\mathrm{USp}}^1(x) &:= 1 + \frac{\sin(2\pi x)}{2\pi x} - \frac{2\sin^2(\pi x)}{(\pi x)^2},\\
W_{\mathrm{USp}}^2(x) &:= 1 - \frac{\sin(2\pi x)}{2\pi x} - \frac{24(1-\sin^2(\pi x))}{(2\pi x)^2} + \frac{48\sin(2\pi x)}{(2\pi x)^3} - \frac{96\sin^2(\pi x)}{(2\pi x)^4},\\
W_{\mathrm{USp}}^3(x) &:= 1 + \frac{\sin(2\pi x)}{2\pi x} - \frac{12\sin^2(\pi x)}{(\pi x)^2} - \frac{240\sin(2\pi x)}{(2\pi x)^3}\\
&\qquad - \frac{15(6-10\sin^2(\pi x))}{(\pi x)^4} + \frac{2880\sin(2\pi x)}{(2\pi x)^5} - \frac{90\sin^2(\pi x)}{(\pi x)^6},\\
W_{\mathrm{USp}}^4(x) &:= 1 - \frac{\sin(2\pi x)}{2\pi x} - \frac{10(1+\cos(2\pi x))}{(\pi x)^2} + \frac{90\sin(2\pi x)}{(\pi x)^3} - \frac{15(3-31\cos(2\pi x))}{(\pi x)^4}\\
&\qquad - \frac{1470\sin(2\pi x)}{(\pi x)^5} - \frac{315(1+9\cos(2\pi x))}{(\pi x)^6} + \frac{3150\sin(2\pi x)}{(\pi x)^7} - \frac{1575(1-\cos(2\pi x))}{(\pi x)^8},
\end{aligned}$$

and we prove the following result:

Theorem 2.3. *Let us assume the Riemann hypothesis and the ratio conjecture for the L-functions in the family $\boldsymbol{L}_\chi$ (see Conjecture 4.1). Let us consider a test function f, which is holomorphic throughout the strip $|\Im(z)| < 2$, real on the real line, even and such that $f(x) \ll 1/(1+x^2)$ as $x \to \infty$. Then, for $k \in \mathbb{N}$ and $k \le 4$, we have*

$$\mathcal{D}_k^{\boldsymbol{L}_\chi}(f) = \int_{-\infty}^{+\infty} f(x) W_{\mathrm{USp}}^k(x)\, dx + \mathrm{O}\Big(\frac{1}{\log X}\Big).$$

Also in the symplectic case, with the same proof we also get the corresponding result in the random matrix theory setting unconditionally, as [Conrey et al. 2008, Theorem 4.2] provides the analogue of Conjecture 4.1. Note that, in the symplectic (respectively, orthogonal) case, Z defined as in (2-1) is the characteristic polynomial of $2N \times 2N$ symplectic (respectively, orthogonal) matrices.

Theorem 2.4. *Let us consider $f : \mathbb{R} \to \mathbb{R}$ an even Schwartz function and F_N as in* (2-2)*. Then, for $k \in \mathbb{N}$ and $k \le 4$, we have*

$$\frac{1}{\int_{\mathrm{USp}(2N)} Z^k d_{\mathrm{Haar}}} \int_{\mathrm{USp}(2N)} \sum_{j=1}^{N} F_{2N}(\theta_j)\, Z^k\, d_{\mathrm{Haar}} \xrightarrow{N\to\infty} \int_{-\infty}^{+\infty} f(x) W_{\mathrm{USp}}^k(x)\, dx.$$

Finally, for the (even) orthogonal family $\boldsymbol{L}_{\Delta,\chi}$, we let

$$\begin{aligned}
W_{\mathrm{SO}^+}^0(x) &:= W_{\mathrm{SO}^+}(x) = 1 + \frac{\sin(2\pi x)}{2\pi x},\\
W_{\mathrm{SO}^+}^1(x) &:= 1 - \frac{\sin(2\pi x)}{2\pi x},\\
W_{\mathrm{SO}^+}^2(x) &:= 1 + \frac{\sin(2\pi x)}{2\pi x} - \frac{2\sin^2(\pi x)}{(\pi x)^2},\\
W_{\mathrm{SO}^+}^3(x) &:= 1 - \frac{\sin(2\pi x)}{2\pi x} - \frac{24(1-\sin^2(\pi x))}{(2\pi x)^2} + \frac{48\sin(2\pi x)}{(2\pi x)^3} - \frac{96\sin^2(\pi x)}{(2\pi x)^4},\\
W_{\mathrm{SO}^+}^4(x) &:= 1 + \frac{\sin(2\pi x)}{2\pi x} - \frac{12\sin^2(\pi x)}{(\pi x)^2} - \frac{240\sin(2\pi x)}{(2\pi x)^3}\\
&\qquad - \frac{15(6-10\sin^2(\pi x))}{(\pi x)^4} + \frac{2880\sin(2\pi x)}{(2\pi x)^5} - \frac{90\sin^2(\pi x)}{(\pi x)^6}.
\end{aligned}$$

Notice that there are strong similarities with the symplectic kernels; we will discuss these analogies below. With these notations, we prove the following theorem:

Theorem 2.5. *Let us assume the Riemann hypothesis and the ratio conjecture for the L-functions in the family $\boldsymbol{L}_{\Delta,\chi}$ (see Conjecture 5.1). Let us consider a test function f, which is holomorphic throughout the strip $|\Im(z)| < 2$, real on the real line, even and such that $f(x) \ll 1/(1+x^2)$ as $x \to \infty$. Then, for $k \in \mathbb{N}$ and $k \le 4$, we have*

$$\mathcal{D}_k^{\boldsymbol{L}_{\Delta,\chi}}(f) = \int_{-\infty}^{+\infty} f(x) W_{\mathrm{SO}^+}^k(x)\, dx + \mathrm{O}\Big(\frac{1}{\log X}\Big).$$

Again the analogous result in random matrix theory is instead unconditional (relying on [Conrey et al. 2008, Theorem 4.3] in place of Conjecture 5.1).

Theorem 2.6. *Let us consider $f : \mathbb{R} \to \mathbb{R}$ an even Schwartz function and F_N as in* (2-2)*. Then, for $k \in \mathbb{N}$ and $k \le 4$, we have*

$$\frac{1}{\int_{\mathrm{SO}(2N)} Z^k\, d_{\mathrm{Haar}}} \int_{\mathrm{SO}(2N)} \sum_{j=1}^{N} F_{2N}(\theta_j) Z^k\, d_{\mathrm{Haar}} \xrightarrow{N\to\infty} \int_{-\infty}^{+\infty} f(x) W_{\mathrm{SO}^+}^k(x)\, dx.$$

P_G^k	$G = \mathrm{U}$	$G = \mathrm{USp}$	$G = \mathrm{SO}^+$
$k=0$	0	$-\frac{1}{2}$	$\frac{1}{2}$
$k=1$	$y-1$	$2y-\frac{3}{2}$	$-\frac{1}{2}$
$k=2$	$-2y^3+4y-2$	$-4y^3+6y-\frac{5}{2}$	$2y-\frac{3}{2}$
$k=3$		$12y^5-20y^3+12y-\frac{7}{2}$	$-4y^3+6y-\frac{5}{2}$
$k=4$		$-40y^7+84y^5-60y^3+20y-\frac{9}{2}$	$12y^5-20y^3+12y-\frac{7}{2}$

Table 1. Values of P_G^k obtained for k small.

2A. ***A general conjecture for*** W_G^k**.** Thanks to the explicit expressions we get for the kernels W_G^k in the range $k \le 4$, we can speculate about what happens for any $k \in \mathbb{N}$. First of all, we notice that the Fourier transform of the kernels W_G^k exhibits a structure. From the explicit formulae for W_G^k we get in the range $k \le 4$, $\widehat{W}_G^k$ turns out to be an even function, supported on $[-1, 1]$, uniquely determined by a polynomial on $[0, 1]$. More precisely, we conjecture that

$$\widehat{W}_G^k(y) = \delta_0(y) + P_G^k(|y|)\chi_{[-1,1]}(y), \tag{2-3}$$

where P_G^k is a polynomial depending on k and G only. In particular, in the unitary case and with $k \ge 1$, we expect the degree of P_{U}^k to be $2k-1$ and $P_{\mathrm{U}}^k(0) = -k$, $P_{\mathrm{U}}^k(1) = 0$. For the symplectic family, if $k \ge 1$, we predict P_{USp}^k with degree $2k-1$ and $P_{\mathrm{USp}}^k(0) = -(2k+1)/2$, $P_{\mathrm{USp}}^k(1) = (-1)^{k+1}/2$. Finally for the orthogonal symmetry type, we conjecture the degree of $P_{\mathrm{SO}^+}^k$ to be $2k-3$ and $P_{\mathrm{SO}^+}^k(0) = -(2k-1)/2$, $P_{\mathrm{SO}^+}^k(1) = (-1)^k/2$ for any $k \ge 2$ $\big($the case $k=1$ yields $\widehat{W}_{\mathrm{SO}^+}^1(y) = \delta_0(y) - \frac{1}{2}\big)$. We collect in Table 1 all the values of P_G^k we obtained for k small, which support our speculations. Note that the case $k=0$, corresponding to the first row in the table, was already known in the literature, while all other results are new.

Looking at Table 1, we can detect relations between the weighted one-level density functions with different symmetry types. In particular, from the above discussion, it seems natural to expect that

$$W_{\mathrm{SO}^+}^k(x) = W_{\mathrm{USp}}^{k-1}(x) \tag{2-4}$$

for any $k \in \mathbb{Z}_+$. Moreover, the Fourier transforms of W_G^k suggest that the weighted one-level density function in the unitary case is the average of the symplectic and orthogonal cases; namely, we conjecture that

$$W_{\mathrm{U}}^k(x) = \frac{W_{\mathrm{USp}}^k(x) + W_{\mathrm{SO}^+}^k(x)}{2}. \tag{2-5}$$

We note that also the leading order moment coefficients $f_G(k)$ for the three compact groups U, USp and SO^+ satisfy relations linking them with each other (see [Keating 2005, Equations (6.10) and (6.11)]):

$$f_{\mathrm{SO}^+}(k) = 2^k f_{\mathrm{USp}}(k-1) \quad \text{and} \quad 2^{k^2} f_U(k) = f_{\mathrm{USp}}(k) f_{\mathrm{SO}^+}(k).$$

Equations (2-4) and (2-5) can be seen as the analogue of the above formulae, in the context of the weighted one-level density.

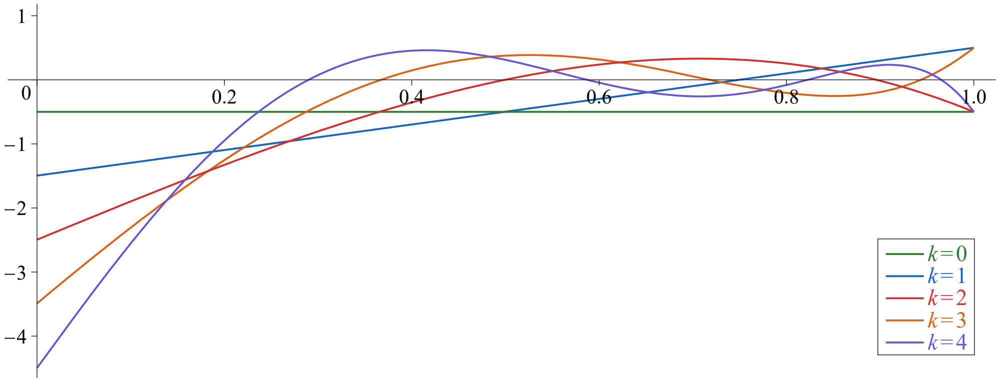

Figure 1. $P^k_{\mathrm{USp}}(y)$, for $y \in [0, 1]$.

Finally, we conjecture an explicit formula for the polynomials P^k_G, which together with (2-3) provides a precise conjecture for the weighted kernels W^k_G. In view of (2-4) and (2-5), it suffices to focus on the symplectic case only. Looking at what happens for $k \le 4$ (see Figure 1), we speculate that for every positive integer k,

$$P^k_{\mathrm{USp}}(y) = -\frac{2k+1}{2} - k(k+1)\sum_{j=1}^{k}(-1)^j c_{j,k}\frac{y^{2j-1}}{2j-1}, \tag{2-6}$$

where the coefficient $c_{j,k}$ is defined by

$$c_{j,k} = \frac{1}{j}\binom{k-1}{j-1}\binom{k+j}{j-1}.$$

We note that the sequence of the $c_{j,k}$ appears in OEIS[7], as the number of diagonal dissections of a convex $(k+2)$-gon into j regions. By Fourier inversion, from (2-3) and (2-6), we get an explicit conjectural formula for W^k_{USp}, namely,

$$W^k_{\mathrm{USp}}(x) = 1 - (2k+1)\frac{\sin(2\pi x)}{2\pi x} + \sum_{j=1}^{k}\frac{k(k+1)}{2^{2j-2}\pi^{2j-1}}\frac{c_{j,k}}{2j-1}\frac{d^{2j-1}}{dx^{2j-1}}\left[\frac{1-\cos(2\pi x)}{2\pi x}\right],$$

see Figure 2.

[7]https://oeis.org/A033282.

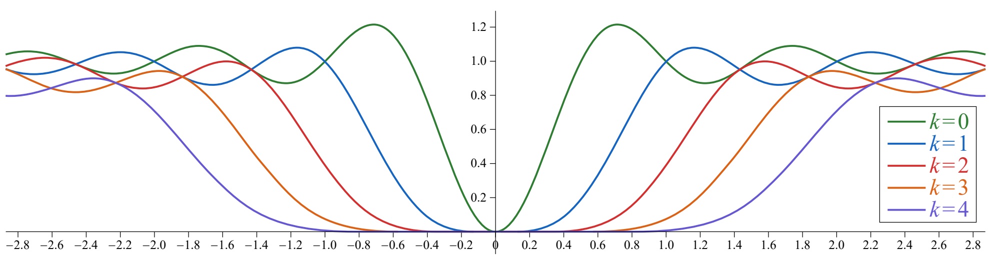

Figure 2. $W^k_{\mathrm{USp}}(x)$ for $k = 0, \ldots, 4$.

From all these discussions, we can formulate the following conjecture:

Conjecture 2.1. *Let us consider a test function f, holomorphic in the strip $|\Im(z)| < 2$, even, real on the real line and such that $f(x) \ll 1/(1+x^2)$ as $x \to \infty$. For any $k \in \mathbb{N}$, given a family $\mathcal{F}$ of L-functions with symmetry type $G \in \{U, \mathrm{USp}, \mathrm{SO}^+\}$, we have*

$$\mathcal{D}_k^{\mathcal{F}}(f) = \int_{-\infty}^{+\infty} f(x) W_G^k(x)\, dx + \mathrm{O}\Big(\frac{1}{\log X}\Big)$$

as $X \to \infty$, where the weighted one-level density function W_G^k depends on k and G only. In addition, the following relations hold:

$$W_{\mathrm{SO}^+}^k(x) = W_{\mathrm{USp}}^{k-1}(x) \quad \textit{and} \quad W_{\mathrm{U}}^k(x) = \frac{W_{\mathrm{USp}}^k(x) + W_{\mathrm{SO}^+}^k(x)}{2}$$

for any $k \in \mathbb{Z}_+$ and $k \in \mathbb{N}$, respectively. Moreover, for every $k \in \mathbb{Z}_+$, in the symplectic case (the others can be recovered by the above relations), we have that

$$\widehat{W}_{\mathrm{USp}}^k(y) = \delta_0(y) + P_{\mathrm{USp}}^k(|y|)\chi_{[-1,1]}(y),$$

where P_{USp}^k is a polynomial of degree $2k-1$, given by

$$P_{\mathrm{USp}}^k(y) = -\frac{2k+1}{2} - k(k+1)\sum_{j=1}^{k}(-1)^j c_{j,k}\frac{y^{2j-1}}{2j-1},$$

with

$$c_{j,k} = \frac{1}{j}\binom{k-1}{j-1}\binom{k+j}{j-1}.$$

2B. ***An expression for $W_G^k(x)$ in terms of hypergeometric functions and its vanishing at $x = 0$.*** We now focus on the behavior of the weighted kernels $W_G^k(x)$ at $x = 0$. For all symmetry types, it seems clear that the order of vanishing of $W_G^k(x)$ for $x \to 0$ increases as k grows. This phenomenon reflects the effect of the weight $V\big(L\big(\tfrac{1}{2}\big)\big)^k$ in the average over the family, which gives more and more relevance to those L-functions that are large at the central point, as k increases. More precisely, for the unitary family we conjecture that

$$W_U^k(x) \sim \frac{\pi^{2k}x^{2k}}{(2k-1)!!(2k+1)!!}, \tag{2-7}$$

as $x \to 0$ and $k \in \mathbb{N}$. In particular, together with (1-6), this suggests that, on weighted average over the considered family, the number of normalized zeros which are less than ε away from the central point is typically $\asymp_k \varepsilon^{2k+1}$. Analogously, the asymptotic behavior of the symplectic and orthogonal kernels can be deduced from (2-7) by (2-4) and (2-5). For small values of k, the behavior of $W_G^k(x)$ at $x = 0$ is outlined in Table 2; the first row was already known in the literature, all the others are new.

In the following conjecture, we condense all the speculations about the behavior of the weighted kernels $W_G^k(x)$ as $x \to 0$:

$W_G^k,\ x\to 0$	$G=U$	$G=\mathrm{USp}$	$G=SO^+$
$k=0$	1	$2\pi^2x^2/3$	2
$k=1$	$\pi^2x^2/3$	$2\pi^4x^4/45$	$2\pi^2x^2/3$
$k=2$	$\pi^4x^4/45$	$2\pi^6x^6/1575$	$2\pi^4x^4/45$
$k=3$		$2\pi^8x^8/99225$	$2\pi^6x^6/1575$
$k=4$		$2\pi^{10}x^{10}/9823275$	$2\pi^8x^8/99225$

Table 2. The behavior of $W_G^k(x)$ at $x=0$ for small values of k.

Conjecture 2.2. *For $G\in\{\mathrm{U},\mathrm{USp},\mathrm{SO}^+\}$ and $k\in\mathbb{N}$, the weighted kernels W_G^k defined in Conjecture 2.1 satisfy the following asymptotic relations as $x\to 0$:*

$$W_U^k(x)\sim\frac{\pi^{2k}x^{2k}}{(2k-1)!!(2k+1)!!},\quad W_{\mathrm{USp}}^k(x)\sim\frac{2\pi^{2(k+1)}x^{2(k+1)}}{(2k+1)!!(2k+3)!!},\quad W_{\mathrm{SO}^+}^k(x)\sim\frac{2\pi^{2k}x^{2k}}{(2k-1)!!(2k+1)!!}.$$

Finally, assuming Conjecture 2.1, we obtain the expansion of $W_G^k(x)$ at $x=0$. In particular, we show that the asymptotic behavior of $W_G^k(x)$ can be deduced from the explicit formulae that we conjectured in Section 2A. In view of (2-4) and (2-5), it suffices to consider the symplectic case only.

Theorem 2.7. *Let us assume Conjecture 2.1. Then for any $k\in\mathbb{N}$, we have*

$$W_{\mathrm{USp}}^k(x)=\sum_{m=1}^{\infty}\beta_{m,k}x^{2m}$$

with

$$\beta_{m,k}=(-1)^{m+1}\frac{(2\pi)^{2m}}{(2m+1)!}\biggl((-1)^k+\frac{k(k+1)}{m+1}\,{}_3F_2\biggl[\begin{matrix}1-k,\ k+2,\ m+1\\ m+2,\ 2\end{matrix};1\biggr]\biggr),$$

where ${}_3F_2$ denotes the generalized hypergeometric function. Moreover, we have

$${}_3F_2\biggl[\begin{matrix}1-k,\ k+2,\ m+1\\ m+2,\ 2\end{matrix};1\biggr]=\begin{cases}\dfrac{(m+1)(-1)^{k+1}}{k(k+1)}, & \text{if } 1\le m\le k,\\[2ex] \dfrac{2(-1)^{k+1}(k-1)!(k+2)!}{(2k+2)!}\left(\binom{2k+1}{k+1}-1\right), & \text{if } m=k+1.\end{cases}$$

In particular, Conjecture 2.2 follows.

3. Proof of Theorems 2.1 and 2.2

We first tilt the Lebesgue measure multiplying by $\left|\zeta\left(\frac{1}{2}+it\right)\right|^2$ and denote it by

$$\langle g\rangle_{|\zeta|^2}:=\frac{1}{T\log T}\int_T^{2T}g(t)\left|\zeta\left(\frac{1}{2}+it\right)\right|^2dt. \tag{3-1}$$

Then we consider f, an even test function, and its Fourier transform

$$\hat{f}(y):=\int_{-\infty}^{+\infty}f(x)e^{-2\pi ixy}\,dx.$$

We recall that Conrey, Farmer and Zirnbauer [Conrey et al. 2008] applied a modification of the recipe for integral moments to the case of ratios getting the following statement, called the ratio conjecture (here, we state this conjecture in a slightly weaker form than in [Conrey et al. 2008], as far as the shifts are concerned):

Conjecture 3.1 [Conrey et al. 2008, Conjecture 5.1]. *Let us denote by $\chi(s)$ the explicit factor in the functional equation $\zeta(s) = \chi(s)\zeta(1-s)$. For any positive integers K, L, Q and R and for any $\alpha_1, \dots, \alpha_{K+L}, \gamma_1, \dots, \gamma_Q, \delta_1, \dots, \delta_R$ complex shifts with real part $\asymp (\log T)^{-1}$ and imaginary part $\ll_\varepsilon T^{1-\varepsilon}$ for every $\varepsilon > 0$, we have*

$$\frac{1}{T}\int_T^{2T} \frac{\prod_{k=1}^{K}\zeta(s+\alpha_k)\prod_{l=K+1}^{K+L}\zeta(1-s-\alpha_l)}{\prod_{q=1}^{Q}\zeta(s+\gamma_q)\prod_{r=1}^{R}\zeta(1-s+\delta_r)}\,dt = \frac{1}{T}\int_T^{2T}\sum_{\sigma\in\Xi_{K,L}}\prod_{k=1}^{K}\frac{\chi(s+\alpha_k)}{\chi(s-\alpha_{\sigma(k)})}Y_U A_\zeta(\cdots)\,dt + O(T^{1/2+\varepsilon}),$$

with $(\cdots) = (\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(K)}; -\alpha_{\sigma(K+1)}, \dots, -\alpha_{\sigma(K+L)}; \gamma; \delta)$, where

$$Y_U(\alpha;\beta;\gamma;\delta) := \frac{\prod_{k=1}^{K}\prod_{l=1}^{L}\zeta(1+\alpha_k+\beta_l)\prod_{q=1}^{Q}\prod_{r=1}^{R}\zeta(1+\gamma_q+\delta_r)}{\prod_{k=1}^{K}\prod_{r=1}^{R}\zeta(1+\alpha_k+\delta_r)\prod_{l=1}^{L}\prod_{q=1}^{Q}\zeta(1+\beta_l+\gamma_q)}$$

and A_ζ is an Euler product, absolutely convergent for all of the variables in small disks around 0, *which is given by*

$$A_\zeta(\alpha;\beta;\gamma;\delta) := \prod_p \frac{\prod_{k=1}^{K}\prod_{l=1}^{L}(1-1/p^{1+\alpha_k+\beta_l})\prod_{q=1}^{Q}\prod_{r=1}^{R}(1-1/p^{1+\gamma_q+\delta_r})}{\prod_{k=1}^{K}\prod_{r=1}^{R}(1-1/p^{1+\alpha_k+\delta_r})\prod_{l=1}^{L}\prod_{q=1}^{Q}(1-1/p^{1+\beta_l+\gamma_q})} \times \sum_{\sum a_k+\sum c_q=\sum b_l+\sum d_r}\frac{\prod\mu(p^{c_q})\prod\mu(p^{d_r})}{p^{\sum(1/2+\alpha_k)a_k+\sum(1/2+\beta_l)b_l+\sum(1/2+\gamma_q)c_q+\sum(1/2+\delta_r)d_r}},$$

while $\Xi_{K,L}$ denotes the subset of permutations $\sigma \in S_{K+L}$ of $\{1, 2, \dots, K+L\}$ for which we have $\sigma(1) < \sigma(2) < \dots < \sigma(K)$ and $\sigma(K+1) < \sigma(K+2) < \dots < \sigma(K+L)$.

By assuming this conjecture about the moments of zeta and denoting

$$N_f(t) = \sum_\gamma f\left(\frac{\log T}{2\pi}(\gamma - t)\right),$$

we can prove the following result:

Proposition 3.1. *Let us assume Conjecture 3.1 and the Riemann hypothesis. We consider a test function $f(z)$ which is holomorphic throughout the strip $|\Im(z)| < 2$, real on the real line, even and such that $f(x) \ll 1/(1+x^2)$ as $x \to \infty$. Then*

$$\mathcal{D}_1^\zeta(f) := \langle N_f\rangle_{|\zeta|^2} = \int_{-\infty}^{+\infty} f(x)W_U^1(x)\,dx + O\left(\frac{1}{\log T}\right),$$

with

$$W_U^1(x) := 1 - \operatorname{sinc}^2(x) = 1 - \frac{\sin^2(\pi x)}{(\pi x)^2}.$$

In addition, with the same strategy as in the proof of Proposition 3.1 (but much longer computations, which can be done by using Sage[8]), we can also study the "fourth moment" case. Namely, we let

$$\langle g\rangle_{|\zeta|^4} := \frac{1}{(T(\log T)^4)/(2\pi^2)}\int_T^{2T} g(t)\Big|\zeta\Big(\frac{1}{2}+it\Big)\Big|^4\,dt, \tag{3-2}$$

and we prove the following result:

Proposition 3.2. *Let us assume Conjecture 3.1 and the Riemann hypothesis. We consider a test function $f(z)$ which is holomorphic throughout the strip $|\Im(z)|<2$, real on the real line, even and such that $f(x)\ll 1/(1+x^2)$ as $x\to\infty$. Then*

$$\mathcal{D}_2^{\zeta}(f) := \langle N_f\rangle_{|\zeta|^4} = \int_{-\infty}^{+\infty} f(x)W_U^2(x)\,dx + O\Big(\frac{1}{\log T}\Big) \tag{3-3}$$

with

$$W_U^2(x) := 1-\frac{2+\cos(2\pi x)}{(\pi x)^2}+\frac{3\sin(2\pi x)}{(\pi x)^3}+\frac{3(\cos(2\pi x)-1)}{2(\pi x)^4}.$$

3A. ***Proof of Proposition 3.1.*** To prove Proposition 3.1, we strongly rely on Conjecture 3.1, which allows us to perform a similar computation as in [Conrey and Snaith 2007, Section 3]. We introduce two parameters $\alpha,\beta\in\mathbb{R}$ of size $\asymp 1/\log T$, we let

$$\boldsymbol{\zeta}_{\alpha,\beta}(t) := \zeta\big(\tfrac{1}{2}+\alpha+it\big)\zeta\big(\tfrac{1}{2}+\beta-it\big) \tag{3-4}$$

and we look at

$$\langle N_f\rangle_{|\zeta|^2}^{\alpha,\beta} := \frac{1}{T\log T}\int_T^{2T}\sum_{\gamma} f\Big(\frac{\log T}{2\pi}(\gamma-t)\Big)\boldsymbol{\zeta}_{\alpha,\beta}(t)\,dt, \tag{3-5}$$

with $\gamma\in\mathbb{R}$ since we are assuming the Riemann hypothesis (we recall that $\rho=\frac{1}{2}+i\gamma$ are the nontrivial zeros of ζ). By the residue theorem, we have that

$$\langle N_f\rangle_{|\zeta|^2}^{\alpha,\beta} = \frac{1}{T\log T}\int_T^{2T}\frac{1}{2\pi i}\Big(\int_{(c)}-\int_{(1-c)}\Big)\frac{\zeta'}{\zeta}(s+it)\cdot f\Big(\frac{-i\log T}{2\pi}\Big(s-\frac{1}{2}\Big)\Big)\,ds\,\boldsymbol{\zeta}_{\alpha,\beta}(t)\,dt, \tag{3-6}$$

where $c\in\big(\frac{1}{2},1\big)$ and $\int_{(c)}$ denotes the integral over the vertical line of those s such that $\Re(s)=c$. We select $c=\frac{1}{2}+\delta$ with $\delta\asymp(\log T)^{-1}$, and we first consider the integral over the c-line

$$\begin{aligned}\mathcal{I} &:= \frac{1}{T\log T}\int_T^{2T}\frac{1}{2\pi i}\int_{(c)}\frac{\zeta'}{\zeta}(s+it)f\Big(\frac{-i\log T}{2\pi}\Big(s-\frac{1}{2}\Big)\Big)\,ds\,\boldsymbol{\zeta}_{\alpha,\beta}(t)\,dt\\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} f\Big(\frac{\log T}{2\pi}(y-i\delta)\Big)\frac{d}{d\gamma}\Big[\frac{I(\alpha;\beta;\delta+iy+\gamma;\delta+iy)}{T\log T}\Big]_{\gamma=0}\,dy,\end{aligned}$$

where

$$I(A;B;C;D) := \int_T^{2T}\frac{\zeta(1/2+A+it)\zeta(1/2+B-it)\zeta(1/2+C+it)}{\zeta(1/2+D+it)}\,dt. \tag{3-7}$$

[8]SageMath, the Sage Mathematics Software System (Version 0.6.3), The Sage Developers, 2021, https://www.sagemath.org.

Moments like (3-7) can be computed thanks to Conjecture 3.1, and it turns out to be

$$I(A;B;C;D)=\int_T^{2T}\left\{\frac{\zeta(1+A+B)\zeta(1+B+C)}{\zeta(1+B+D)}+\left(\frac{t}{2\pi}\right)^{-A-B}\frac{\zeta(1-A-B)\zeta(1-A+C)}{\zeta(1-A+D)}\right.$$
$$\left.+\left(\frac{t}{2\pi}\right)^{-B-C}\frac{\zeta(1+A-C)\zeta(1-B-C)}{\zeta(1-C+D)}\right\}dt+O\big(T^{1/2+\varepsilon}\big) \quad (3\text{-}8)$$

for suitable shifts A, B, C and D, i.e., with real part $\asymp(\log T)^{-1}$ and imaginary part $\ll_\varepsilon T^{1-\varepsilon}$, for every $\varepsilon>0$ (see, e.g., [Conrey and Snaith 2007, Section 2.1]). Notice that the arithmetical factor $A_\zeta(\alpha;\beta;\gamma;\delta)$ from Conjecture 3.1 equals 1 in our case, with $K=2$, $L=1$, $Q=1$ and $R=0$ (this can be easily proven by direct computation or deduced by [Conrey et al. 2005a, Corollary 2.6.2]). We now want to apply (3-8) with $A=\alpha$, $B=\beta$, $C=\delta+iy+\gamma$ and $D=\delta+iy$ and to do so, we need that the imaginary parts of all the shifts are $\ll_\varepsilon T^{1-\varepsilon}$. A standard technique to avoid this issue is splitting the integral over y in two pieces; the contribution to $\mathcal{I}$ coming from $|y|>T^{1-\varepsilon}$ is $\ll T^{-1+\varepsilon}$, thanks to the good decaying of f and to RH, since

$$\frac{1}{T\log T}\int_T^{2T}|\boldsymbol{\zeta}_{\alpha,\beta}(t)|\int_{|y|>T^{1-\varepsilon}}|f(y\log T)|\left|\frac{\zeta'}{\zeta}\Big(\frac12+\delta+iy+it\Big)\right|dy\,dt$$
$$\ll\frac{T^{\varepsilon/100}}{T}\int_T^{2T}\int_{|y|>T^{1-\varepsilon}}\frac{\log(|y|t)}{|y|^2}\,dy\,dt\ll T^{-1+\varepsilon}.$$

Therefore, we can truncate the integral over y at height $T^{1-\varepsilon}$, apply (3-8) and then re-extend the integration over y to infinity with a small error term. Thus, differentiating with respect to γ at $\gamma=0$, moving the path of integration to $\delta=0$ (we are allowed to since now the integral is regular at $\delta=0$), we get

$$\mathcal{I}=\frac{1}{2\pi}\int_{-\infty}^{+\infty}f\Big(\frac{\log T}{2\pi}y\Big)\frac{1}{T\log T}\int_T^{2T}g_{\alpha,\beta}(y;t)\,dt\,dy+O(T^{-1/2+\varepsilon}), \quad (3\text{-}9)$$

with

$$g_{\alpha,\beta}(y;t):=\frac{\zeta(1+\alpha+\beta)\zeta'(1+\beta+iy)}{\zeta(1+\beta+iy)}+\Big(\frac{t}{2\pi}\Big)^{-\alpha-\beta}\frac{\zeta(1-\alpha-\beta)\zeta'(1-\alpha+iy)}{\zeta(1-\alpha+iy)}$$
$$-\Big(\frac{t}{2\pi}\Big)^{-\beta-iy}\zeta(1+\alpha-iy)\zeta(1-\beta-iy). \quad (3\text{-}10)$$

We notice that, when computing this derivative, it is useful to observe that if $f(z)$ is analytic at $z=0$, then (see [Conrey and Snaith 2007, Equation (2.13)])

$$\frac{d}{d\gamma}\left[\frac{f(\gamma)}{\zeta(1-\gamma)}\right]_{\gamma=0}=-f(0).$$

Similarly we deal with the integral over the $(1-c)$-line in (3-6)

$$\mathcal{J}:=\frac{1}{T\log T}\int_T^{2T}\frac{1}{2\pi i}\int_{(1-c)}\frac{\zeta'}{\zeta}(s+it)f\Big(\frac{-i\log T}{2\pi}\Big(s-\frac12\Big)\Big)\,ds\,\boldsymbol{\zeta}_{\alpha,\beta}(t)\,dt$$
$$=\frac{1}{2\pi i}\int_{(c)}f\Big(\frac{-i\log T}{2\pi}\Big(s-\frac12\Big)\Big)\frac{1}{T\log T}\int_T^{2T}\frac{\zeta'}{\zeta}(1-s+it)\boldsymbol{\zeta}_{\alpha,\beta}(t)\,dt\,ds.$$

Using the functional equation

$$\frac{\zeta'}{\zeta}(1-z) = \frac{X'}{X}(z) - \frac{\zeta'}{\zeta}(z),$$

where

$$\frac{X'}{X}(z) := \log \pi - \frac{1}{2}\frac{\Gamma'}{\Gamma}\Big(\frac{z}{2}\Big) - \frac{1}{2}\frac{\Gamma'}{\Gamma}\Big(\frac{1-z}{2}\Big),$$

we express $\mathcal{J}$ as a sum of two terms

$$\mathcal{J} = \mathcal{J}_1 - \mathcal{J}_2, \tag{3-11}$$

with

$$\mathcal{J}_1 := \frac{1}{2\pi i}\int_{(c)} f\bigg(\frac{-i\log T}{2\pi}\Big(s-\frac{1}{2}\Big)\bigg)\frac{1}{T\log T}\int_T^{2T}\frac{X'}{X}(s-it)\boldsymbol{\zeta}_{\alpha,\beta}(t)\,dt\,ds$$

and

$$\mathcal{J}_2 := \frac{1}{2\pi i}\int_{(c)} f\bigg(\frac{-i\log T}{2\pi}\Big(s-\frac{1}{2}\Big)\bigg)\frac{1}{T\log T}\int_T^{2T}\frac{\zeta'}{\zeta}(s-it)\boldsymbol{\zeta}_{\alpha,\beta}(t)\,dt\,ds.$$

With $c = \frac{1}{2}+\delta$, where $\delta \to 0$, it is easy to see that

$$\begin{aligned}\mathcal{J}_1 &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} f\Big(\frac{\log T}{2\pi}y\Big)\frac{1}{T\log T}\int_T^{2T}\frac{X'}{X}\Big(\frac{1}{2}+iy-it\Big)\boldsymbol{\zeta}_{\alpha,\beta}(t)\,dt\,dy\\ &= -\frac{1}{2\pi}\int_{-\infty}^{+\infty} f\Big(\frac{\log T}{2\pi}y\Big)\frac{\log T+O(1)}{T\log T}\int_T^{2T}\boldsymbol{\zeta}_{\alpha,\beta}(t)\,dt\,dy,\end{aligned} \tag{3-12}$$

since, using Stirling's approximation to estimate the gamma-factors, we have (again, we can assume $y \ll T^{1-\varepsilon}$ because of the great decaying of f)

$$\begin{aligned}\frac{X'}{X}\Big(\frac{1}{2}+iy-it\Big) &= -\frac{1}{2}\frac{\Gamma'}{\Gamma}\Big(\frac{1}{4}+\frac{iy}{2}-\frac{it}{2}\Big) - \frac{1}{2}\frac{\Gamma'}{\Gamma}\Big(\frac{1}{4}-\frac{iy}{2}+\frac{it}{2}\Big) + O(1)\\ &= -\frac{1}{2}\log\Big(-\frac{it}{2}\Big) - \frac{1}{2}\log\Big(\frac{it}{2}\Big) + O(1) = -\log T + O(1).\end{aligned}$$

Moreover, with the same choice of c as before, if we set $\alpha = \beta$, we get

$$\mathcal{J}_2 = \mathcal{I}. \tag{3-13}$$

Then (3-6), (3-11) and (3-13) imply that

$$\langle N_f\rangle_{|\zeta|^2}^{\alpha,\alpha} = \mathcal{I}+\mathcal{J} = -\mathcal{J}_1 + 2\mathcal{I} \tag{3-14}$$

and the function $\mathcal{J}_1 = \mathcal{J}_1(\alpha)$ is regular at $\alpha = 0$. We can then take the limit in (3-12), getting

$$\begin{aligned}\lim_{\alpha\to 0}\mathcal{J}_1 &= -\frac{1}{2\pi}\int_{-\infty}^{+\infty} f\Big(\frac{\log T}{2\pi}y\Big)\frac{\log T+O(1)}{T\log T}\int_T^{2T}\Big|\zeta\Big(\frac{1}{2}+it\Big)\Big|^2\,dt\,dy\\ &= -\frac{\log T+O(1)}{2\pi}\int_{-\infty}^{+\infty} f\Big(\frac{\log T}{2\pi}y\Big)\,dy\\ &= -\int_{-\infty}^{+\infty} f(x)\,dx + O\Big(\frac{1}{\log T}\Big)\end{aligned} \tag{3-15}$$

with the change of variable $\log T/(2\pi)y = x$. Lastly, we study the remaining term $\mathcal{I}$, from (3-9) and (3-10). We set $\alpha = \beta = a/\log T$, with $0 < a < 1$. Then we perform the same change of variable $\log T/(2\pi)y = x$ as before, and we get

$$\mathcal{I} = \int_{-\infty}^{+\infty} f(x) \frac{1}{T(\log T)^2} \int_T^{2T} g_{a/\log T, a/\log T}\Big(\frac{2\pi x}{\log T}; t\Big)\, dt\, dx + O(T^{1/2+\varepsilon}).$$

Since $\log(t/(2\pi)) = \log T + O(1)$ as $t \in [T, 2T]$, we have

$$\begin{aligned}\mathcal{I} = \Big(\frac{1}{(\log T)^2} + O\Big(\frac{1}{(\log T)^3}\Big)\Big) \int_{-\infty}^{+\infty} f(x) \Big(\zeta\Big(1+\frac{2a}{\log T}\Big)\frac{\zeta'}{\zeta}\Big(1+\frac{a+2\pi i x}{\log T}\Big) \\ + e^{-2a}\zeta\Big(1-\frac{2a}{\log T}\Big)\frac{\zeta'}{\zeta}\Big(1-\frac{a-2\pi i x}{\log T}\Big) - e^{-a-2\pi i x}\zeta\Big(1+\frac{a-2\pi i x}{\log T}\Big)\zeta\Big(1-\frac{a+2\pi i x}{\log T}\Big)\Big)\, dx, \quad (3\text{-}16)\end{aligned}$$

where the error term is uniform in a. Now, we will prove that the above expression is regular at $a = 0$, showing that

$$\lim_{a\to 0} \mathcal{I} = \int_{-\infty}^{+\infty} f(x)\mathcal{P}(x)\, dx + O\Big(\frac{1}{\log T}\Big), \quad (3\text{-}17)$$

as $T \to \infty$, where

$$\mathcal{P}(x) := \frac{-1 + 2\pi i x + e^{-2\pi i x}}{4\pi^2 x^2}.$$

Intuitively, if we replace each zeta function with its leading term in the expansion at the point 1 given by $\zeta(1+z) \sim 1/z$, we have

$$\begin{aligned}\mathcal{I} &\approx \frac{1}{(\log T)^2} \int_{-\infty}^{+\infty} f(x) \Big(\frac{-(\log T)^2}{2a(a+2\pi i x)} + e^{-2a}\frac{-(\log T)^2}{2a(a-2\pi i x)} + e^{-a-2\pi i x}\frac{(\log T)^2}{(a-2\pi i x)(a+2\pi i x)}\Big)\, dx \\ &= \int_{-\infty}^{+\infty} f(x)\Big(-\frac{1}{2a(a+2\pi i x)} - \frac{e^{-2a}}{2a(a-2\pi i x)} + \frac{e^{-a-2\pi i x}}{(a-2\pi i x)(a+2\pi i x)}\Big)\, dx.\end{aligned}$$

The function inside the parentheses above equals

$$\frac{-a(1+e^{-2a}) + 2\pi i x(1-e^{-2a}) + 2ae^{-a-2\pi i x}}{2a(a^2+4\pi^2 x^2)} = \frac{-1+2\pi i x + e^{-2\pi i x} + O(a)}{4\pi^2 x^2 + O(a^2)},$$

and then tends to $\mathcal{P}(x)$ as $a \to 0$.

To show (3-17) rigorously, we split the integral over x into two parts. We start with the case $x \ll \log T$; from Taylor approximation $f(1+s\pm y) = f(1+s) \pm y f'(1+s) + O_s(y^2)$, we get

$$\begin{aligned}\frac{\zeta'}{\zeta}\Big(1+\frac{2\pi i x}{\log T} \pm \frac{a}{\log T}\Big) &= \frac{\zeta'}{\zeta}\Big(1+\frac{2\pi i x}{\log T}\Big) \pm \frac{a}{\log T}\Big(\frac{\zeta'}{\zeta}\Big)'\Big(1+\frac{2\pi i x}{\log T}\Big) + O_T(a^2) \\ &=: c_1(x) \pm \frac{a}{\log T} c_2(x) + O_T(a^2)\end{aligned}$$

and

$$\zeta\Big(1-\frac{2\pi i x}{\log T} \pm \frac{a}{\log T}\Big) = \zeta\Big(1-\frac{2\pi i x}{\log T}\Big) + O_T(a) =: k(x) + O_T(a)$$

as $a \to 0$, with the notations

$$\begin{aligned} c_1(x) = c_1(x, T) &:= \frac{\zeta'}{\zeta}\Big(1 + \frac{2\pi i x}{\log T}\Big), \\ c_2(x) = c_1(x, T) &:= \Big(\frac{\zeta'}{\zeta}\Big)'\Big(1 + \frac{2\pi i x}{\log T}\Big), \\ k(x) = k(x, T) &:= \zeta\Big(1 - \frac{2\pi i x}{\log T}\Big). \end{aligned}$$

Moreover, we use the asymptotic expansion

$$\zeta(1+z) = \frac{1}{z} + \gamma + O(z), \quad z \to 0, \tag{3-18}$$

and we get

$$\begin{aligned} &\int_{x \ll \log T} f(x)\Big(\zeta\Big(1+\frac{2a}{\log T}\Big)\frac{\zeta'}{\zeta}\Big(1+\frac{a+2\pi i x}{\log T}\Big)+e^{-2a}\zeta\Big(1-\frac{2a}{\log T}\Big)\frac{\zeta'}{\zeta}\Big(1-\frac{a-2\pi i x}{\log T}\Big) \\ &\qquad\qquad -e^{-a-2\pi i x}\zeta\Big(1+\frac{a-2\pi i x}{\log T}\Big)\zeta\Big(1-\frac{a+2\pi i x}{\log T}\Big)\Big)\, dx \\ &= \int_{x \ll \log T} f(x)\Big(\Big[\frac{\log T}{2a}+\gamma+O_T(a)\Big]\Big[c_1(x)+\frac{a}{\log T}c_2(x)+O_T(a^2)\Big] \\ &\quad +e^{-2a}\Big[\frac{-\log T}{2a}+\gamma+O_T(a)\Big]\Big[c_1(x)-\frac{a}{\log T}c_2(x)+O_T(a^2)\Big]-e^{-a-2\pi i x}\big[k(x)+O_T(a)\big]^2\Big)\, dx, \end{aligned}$$

whose limit as $a \to 0$ is

$$\int_{x \ll \log T} f(x)\{c_1(x)\log T + c_2(x) + 2\gamma c_1(x) - e^{-2\pi i x}k(x)^2\}\, dx.$$

By definition of $c_1(x)$, $c_2(x)$ and $k(x)$, the asymptotic expansion (3-18) yields

$$\begin{aligned} c_1(x) &= -\frac{\log T}{2\pi i x} + O(1), \\ c_2(x) &= \frac{(\log T)^2}{(2\pi i x)^2} + O(1), \\ k(x) &= \frac{(\log T)^2}{(2\pi i x)^2} - \frac{2\gamma \log T}{2\pi i x} + O(1), \end{aligned}$$

uniformly for $x \ll \log T$. Then the above is equal to

$$\int_{x \ll \log T} f(x)\Big\{-\frac{(\log T)^2}{2\pi i x} + \frac{(\log T)^2}{(2\pi i x)^2} - e^{-2\pi i x}\frac{(\log T)^2}{(2\pi i x)^2} + O(\log T)\Big\}\, dx$$

(note that the sum $2\gamma c_1(x) - e^{-2\pi i x}k(x)^2$ gives the third term in the parentheses with an error $O(\log T)$, a possible pole at $x = 0$ cancels out), which is

$$= (\log T)^2 \int_{x \ll \log T} f(x)\frac{-1+2\pi i x+e^{-2\pi i x}}{4\pi^2 x^2}\, dx + O(\log T).$$

Finally, we can re-extend the range of integration with a small error term (being $f(x) \ll 1/(1+x^2)$ and $\mathcal{P}(x)$ bounded), getting that the contribution of $x \ll \log T$ in the integral over x in (3-16), in the limit as $a \to 0$, equals

$$(\log T)^2 \int_{-\infty}^{+\infty} f(x)\mathcal{P}(x)\,dx + O(\log T).$$

To prove (3-17), we finally have to bound the contribution of $x \gg \log T$ in the integral on the right-hand side of (3-16), as $a \to 0$; to do so, we use the bounds $\zeta(1+iy) \ll \log y$ (see [Titchmarsh 1986, Theorem 3.5]) and $(\zeta'/\zeta)(1+iy) \ll \log y$ (see [Titchmarsh 1986, Equation (3.11.9)]) for $y \gg 1$, thus the contribution coming from $x \gg \log T$ is equal to

$$\begin{aligned}
\lim_{a\to 0}\int_{x\gg \log T} & f(x)\biggl(\Bigl[\frac{\log T}{2a}+O(1)\Bigr]\Bigl[\frac{\zeta'}{\zeta}\Bigl(1+\frac{2\pi i x}{\log T}\Bigr)+O\Bigl(\frac{a}{\log T}\Bigr)\Bigr] \\
&\quad + e^{-2a}\Bigl[-\frac{\log T}{2a}+O(1)\Bigr]\Bigl[\frac{\zeta'}{\zeta}\Bigl(1+\frac{2\pi i x}{\log T}\Bigr)+O\Bigl(\frac{a}{\log T}\Bigr)\Bigr]+O\bigl((\log x)^2\bigr)\biggr)\,dx \\
&= \int_{x\gg \log T} f(x)\biggl(\log T\,\frac{\zeta'}{\zeta}\Bigl(1+\frac{2\pi i x}{\log T}\Bigr)\lim_{a\to 0}\Bigl[\frac{1-e^{-2a}}{2a}\Bigr]+O\bigl((\log x)^2\bigr)\biggr)\,dx \\
&\ll \int_{x\gg \log T} |f(x)|\bigl(\log T\log x+(\log x)^2\bigr)\,dx \ll \log T\int_{x\gg\log T}\frac{\log x}{x^2}\,dx,
\end{aligned}$$

then (3-17) follows, being $\int_{x\gg \log T}(\log x/x^2)\,dx \ll 1$. Finally, if we decompose $\mathcal{P}(x)$ in even and odd parts

$$\mathcal{P}(x) = -\frac{1}{2}\frac{\sin^2(\pi x)}{(\pi x)^2} - \frac{i\bigl(\sin(2\pi x)-2\pi x\bigr)}{4\pi^2x^2}, \tag{3-19}$$

since f is even and $\mathcal{P}(x)$ bounded, we have

$$\lim_{a\to 0}\mathcal{I} = -\frac{1}{2}\int_{-\infty}^{+\infty} f(x)\frac{\sin^2(\pi x)}{(\pi x)^2}\,dy + O\Bigl(\frac{1}{\log T}\Bigr). \tag{3-20}$$

Putting together (3-14), (3-15) and (3-20), we finally get

$$\langle N_f\rangle_{|\zeta|^2} = \int_{-\infty}^{+\infty} f(x)\biggl(1-\frac{\sin^2(\pi x)}{(\pi x)^2}\biggr)\,dx + O\Bigl(\frac{1}{\log T}\Bigr),$$

as $T \to \infty$, and the theorem has been proved.

3B. ***Proof of Proposition 3.2.*** This proof builds on the same ideas as that of Proposition 3.1, even though we have to handle longer computations; to begin with, we introduce four parameters $\alpha, \beta, \nu, \eta \in \mathbb{R}$ of size $1/\log T$, we let

$$\boldsymbol{\zeta}_{\alpha,\beta,\nu,\eta}(t) := \zeta\bigl(\tfrac{1}{2}+\alpha+it\bigr)\zeta\bigl(\tfrac{1}{2}+\beta+it\bigr)\zeta\bigl(\tfrac{1}{2}+\nu-it\bigr)\zeta\bigl(\tfrac{1}{2}+\eta-it\bigr),$$

and we look at

$$\langle N_f\rangle_{|\zeta|^4}^{\alpha,\beta,\nu,\eta} := \frac{1}{(T(\log T)^4)/(2\pi^2)}\int_T^{2T}\sum_\gamma f\biggl(\frac{\log T}{2\pi}(\gamma-t)\biggr)\boldsymbol{\zeta}_{\alpha,\beta,\nu,\eta}(t)\,dt, \tag{3-21}$$

with $\gamma\in\mathbb{R}$ since we are assuming the Riemann hypothesis. Analogous to (3-14), the residue theorem yields

$$\langle N_f\rangle_{|\zeta|^4}^{\alpha,\beta,\alpha,\beta}=-\mathcal{J}_1+2\mathcal{I}, \tag{3-22}$$

with

$$\mathcal{J}_1=\mathcal{J}_1(\alpha,\beta)=-\int_{-\infty}^{+\infty}f(x)\frac{\log T+O(1)}{(T(\log T)^5)/(2\pi^2)}\int_T^{2T}\boldsymbol{\zeta}_{\alpha,\beta,\alpha,\beta}(t)\,dt\,dx \\ \xrightarrow{\alpha,\beta\to 0}-\int_{-\infty}^{+\infty}f(x)\,dx+O\Big(\frac{1}{\log T}\Big) \tag{3-23}$$

and

$$\begin{aligned}\mathcal{I}&=\frac{1}{2\pi}\int_{-\infty}^{+\infty}f\Big(\frac{\log T}{2\pi}(y-i\delta)\Big)\frac{d}{d\gamma}\bigg[\frac{I(\alpha;\beta;\gamma;\delta+iy;\alpha;\beta)}{T(\log T)^4/(2\pi^2)}\bigg]_{\gamma=\delta+iy}dy\\ &=\int_{-\infty}^{+\infty}f\Big(x-\frac{i\delta\log T}{2\pi}\Big)\\ &\qquad\times\frac{d}{d\gamma}\bigg[\frac{I(a/\log T;b/\log T;\gamma;\delta+2\pi ix/\log T;a/\log T;b/\log T)}{T(\log T)^5/(2\pi^2)}\bigg]_{\gamma=\delta+2\pi ix/\log T}dx,\end{aligned}$$

where $a,b\asymp 1$, $\delta\asymp 1/\log T$ and $I(A;B;C;D;F;G)$ is defined by

$$\int_T^{2T}\frac{\zeta(1/2+A+it)\zeta(1/2+B+it)\zeta(1/2+C+it)\zeta(1/2+F-it)\zeta(1/2+G-it)}{\zeta(1/2+D+it)}\,dt.$$

If the shifts satisfy the conditions prescribed by Conjecture 3.1 then such an integral can be evaluated by using the ratio conjecture. According to the recipe, up to an error $O(T^{1/2+\varepsilon})$, the above moment is a sum of ten pieces, the first being

$$\int_T^{2T}\frac{\zeta(1+A+F)\zeta(1+A+G)\zeta(1+B+F)\zeta(1+B+G)\zeta(1+C+F)\zeta(1+C+G)}{\zeta(1+D+F)\zeta(1+D+G)}\mathcal{A}_{A,B,C,D,F,G}\,dt,$$

where

$$\begin{aligned}\mathcal{A}_{A,B,C,D,F,G}=\prod_p&\Big(1-\frac{1}{p^{1+A+F}}\Big)\Big(1-\frac{1}{p^{1+A+G}}\Big)\Big(1-\frac{1}{p^{1+B+F}}\Big)\Big(1-\frac{1}{p^{1+B+G}}\Big)\\ &\times\Big(1-\frac{1}{p^{1+C+F}}\Big)\Big(1-\frac{1}{p^{1+C+G}}\Big)\Big(1-\frac{1}{p^{1+D+F}}\Big)^{-1}\Big(1-\frac{1}{p^{1+D+G}}\Big)^{-1}\\ &\times\sum_{a+b+c+d=f+g}\frac{\mu(p^d)}{p^{(1/2+A)a+(1/2+B)b+(1/2+C)c+(1/2+D)d+(1/2+F)f+(1/2+G)g}}.\end{aligned}$$

It will be useful to notice that if all the shifts equal zero, then

$$\mathcal{A}:=\mathcal{A}_{0,0,0,0,0,0}=\frac{1}{\zeta(2)};$$

again this can be proven by direct computation or deduced by [Conrey et al. 2005a, Corollary 2.6.2]. All the other nine terms can be recovered from the first one just by swapping the shifts as prescribed by the recipe;

doing so yields a formula for $I(A;\,B;\,C;\,D;\,F;\,G)$ and differentiating with respect to C at $C=D$, we get

$$\begin{aligned}\frac{d}{dC}\big[I(A;\,B;\,C;\,D;\,F;\,G)\big]_{C=D}\\ =\int_T^{2T}\biggl(R_1+\Bigl(\frac{t}{2\pi}\Bigr)^{-A-F}R_2+\Bigl(\frac{t}{2\pi}\Bigr)^{-A-G}R_3+\Bigl(\frac{t}{2\pi}\Bigr)^{-B-F}R_4+\Bigl(\frac{t}{2\pi}\Bigr)^{-B-G}R_5\\ +\Bigl(\frac{t}{2\pi}\Bigr)^{-D-F}R_6+\Bigl(\frac{t}{2\pi}\Bigr)^{-D-G}R_7+\Bigl(\frac{t}{2\pi}\Bigr)^{-A-B-F-G}R_8\\ +\Bigl(\frac{t}{2\pi}\Bigr)^{-A-D-F-G}R_9+\Bigl(\frac{t}{2\pi}\Bigr)^{-B-D-F-G}R_{10}\biggr)\,dt+O\bigl(T^{1/2+\varepsilon}\bigr),\end{aligned} \tag{3-24}$$

with

$$\begin{aligned}R_1&=R_1(A,B,D,F,G)\\ &=\frac{\mathcal{A}_{A,B,D,D,F,G}}{\zeta(1+A+F)\zeta(1+A+G)\zeta(1+B+F)\zeta(1+B+G)\zeta(1+D+F)\zeta(1+D+G)}\\ &\qquad\times\biggl[\frac{\zeta'}{\zeta}(1+D+F)+\frac{\zeta'}{\zeta}(1+D+G)+\frac{\mathcal{A}'_{A,B,D,D,F,G}}{A_{A,B,D,D,F,G}}\biggr],\\ R_2&=R_1(-F,B,D,-A,G),\\ R_3&=R_1(-G,B,D,F,-A),\\ R_4&=R_1(A,-F,D,-B,G),\\ R_5&=R_1(A,-G,D,F,-B),\\ R_6&=R_6(A,B,D,F,G)\\ &=-\frac{\zeta(1+A-D)\zeta(1+A+G)\zeta(1+B-D)\zeta(1+B+G)\zeta(1-F-D)\zeta(1-F+G)}{\zeta(1+D+G)}\mathcal{A}_{A,B,D,D,F,G},\\ R_7&=R_6(A,B,D,G,F),\\ R_8&=R_1(-F,-G,D,-A,-B),\\ R_9&=R_6(-F,B,D,G,-A),\\ R_{10}&=R_6(A,-F,D,G,-B).\end{aligned}$$

If the shifts A, B, D, F, G are $\ll 1/\log T$, the above formula simplifies a lot, since we have

$$R_1=\frac{(-2D-F-G)\mathcal{A}}{(A+F)(A+G)(B+F)(B+G)(D+F)(D+G)}+O\biggl(\frac{(\log T)^5}{\log T}\biggr)$$

and

$$R_6=\frac{-(D+G)\mathcal{A}}{(A-D)(A+G)(B-D)(B+G)(-F-D)(-F+G)}+O\biggl(\frac{(\log T)^5}{\log T}\biggr).$$

As in the proof of Proposition 3.1, by a truncation of the integral over x and Taylor approximations, we can use (3-24) to evaluate $\mathcal{I}$; one can use Sage to carry out this massive computation, getting

$$\lim_{\substack{a\to 0\\ b\to 0}}\mathcal{I}=\int_{-\infty}^{+\infty}f(x)\frac{(\log T)^5\mathcal{A}}{(\log T)^5/(2\pi^2)}h(2\pi ix)\,dx+O\Bigl(\frac{1}{\log T}\Bigr) \tag{3-25}$$

with

$$h(y) := -\frac{y^3 - 2y^2 + 6 - e^{-y}(y^2 + 6y + 6)}{6y^4}.$$

Note that, as in the last section, we moved the path of integration over x to $\delta = 0$, being the integral regular at $\delta = 0$. Therefore, putting together (3-22), (3-23) and (3-25), we get that

$$\begin{aligned}\langle N_f \rangle_{|\zeta|^4} &= \int_{-\infty}^{+\infty} f(x)\Big(1 + 2\frac{2\pi^2}{\zeta(2)} h(2\pi i x)\Big)\,dx + O\Big(\frac{1}{\log T}\Big) \\ &= \int_{-\infty}^{+\infty} f(x)\big(1 + 24h(2\pi i x)\big)\,dx + O\Big(\frac{1}{\log T}\Big) = \int_{-\infty}^{+\infty} f(x) W_U^2(x)\,dx + O\Big(\frac{1}{\log T}\Big),\end{aligned}$$

since f is even.

3C. ***Proof of Theorem 2.2.*** As mentioned in the introduction, the analogue of the ratio conjecture is a theorem in random matrix theory. Therefore, the same machinery described above proves Theorem 2.2, using [Conrey et al. 2008, Theorem 4.1] in place of Conjecture 3.1.

4. Proof of Theorems 2.3 and 2.4

The family $\{L(\frac{1}{2}, \chi_d) : d > 0,\ d \text{ fundamental discriminant}\}$ is a symplectic family, in the sense that it can be modeled by characteristic polynomials of symplectic matrices in the group $\mathrm{USp}(2N)$, if we identify $2N \approx \log(d/\pi)$. Indeed d/π is the analytic conductor of $L(s, \chi_d)$, thus $\log(d/\pi)$ (i.e., the density of zeros) plays the role of $2N$ in the random matrix theory setting[9].

We consider the moments of quadratic Dirichlet L-functions at the critical point $s = \frac{1}{2}$, i.e., the mean value

$$\sum_{d \le X} L\big(\tfrac{1}{2}, \chi_d\big)^k \tag{4-1}$$

in the limit $X \to \infty$, where the summation over d has to be interpreted as the sum over all the positive fundamental discriminants d below X, here and in the following. Also, we will denote by $X^* \sim 1/(2\zeta(2))X$ the number of fundamental discriminants below X. We recall that Jutila [1981] proved asymptotic formulae for the first moment, showing that

$$\sum_{d \le X} L\Big(\frac{1}{2}, \chi_d\Big) \sim \frac{\mathcal{A}}{2} \frac{1}{2\zeta(2)} X \log X, \tag{4-2}$$

where

$$\mathcal{A} = \prod_p \Big(1 - \frac{1}{p(p+1)}\Big) \tag{4-3}$$

and also for the second moment, proving

$$\sum_{d \le X} L\Big(\frac{1}{2}, \chi_d\Big)^2 \sim \frac{\mathcal{B}}{24} \frac{1}{2\zeta(2)} X (\log X)^3, \tag{4-4}$$

[9] See [Conrey et al. 2005a, Conjecture 1.5.3] and the comments below for some clarification concerning the "conductor".

with

$$\mathcal{B} = \prod_p \left(1 - \frac{4p^2 - 3p + 1}{p^3(p+1)}\right). \tag{4-5}$$

It is believed that

$$\sum_{d \le X} L\left(\tfrac{1}{2}, \chi_d\right)^k \sim C_k X (\log X)^{k(k+1)/2}, \tag{4-6}$$

and using analogies with random matrix theory, Keating and Snaith [2000] also conjectured a precise value for the constant C_k. Moreover, the recipe produces a conjectural asymptotic formula with all the main terms for the moments (4-1) with k integer and also for ratios of products of quadratic Dirichlet L-functions (see [Conrey et al. 2008]), which is a symplectic analogue of Conjecture 3.1.

Conjecture 4.1 [Conrey et al. 2008, Conjecture 5.2]. *Let K, Q be two positive integers, $\alpha_1, \dots, \alpha_K$ and $\gamma_1, \dots, \gamma_Q$ be complex shifts with real part $\asymp (\log T)^{-1}$ and imaginary part $\ll_\varepsilon T^{1-\varepsilon}$ for every $\varepsilon > 0$, then*

$$\sum_{d \le X} \frac{\prod_{k=1}^K L(1/2 + \alpha_k, \chi_d)}{\prod_{q=1}^Q L(1/2 + \gamma_q, \chi_d)} = \sum_{d \le X} \sum_{\epsilon \in \{-1,1\}^K} \left(\frac{d}{\pi}\right)^{(1/2)\sum_k (\epsilon_k \alpha_k - \alpha_k)} \prod_{k=1}^K g_S\left(\frac{1}{2} + \frac{\alpha_k - \epsilon_k \alpha_k}{2}\right) Y_S \mathcal{A}_S(\cdots) + O(X^{1/2+\varepsilon}),$$

with $(\cdots) = (\epsilon_1 \alpha_1, \dots, \epsilon_K \alpha_K; \gamma)$, where

$$Y_S(\alpha; \gamma) := \frac{\prod_{j \le k \le K} \zeta(1 + \alpha_j + \alpha_k) \prod_{q < r \le Q} \zeta(1 + \gamma_q + \gamma_r)}{\prod_{k=1}^K \prod_{q=1}^Q \zeta(1 + \alpha_k + \gamma_q)}$$

and $\mathcal{A}_S$ is an Euler product, absolutely convergent for all of the variables in small disks around 0, *which is given by*

$$\mathcal{A}_S(\alpha; \gamma) := \prod_p \frac{\prod_{j \le k \le K} (1 - 1/p^{1+\alpha_j+\alpha_k}) \prod_{q < r \le Q} (1 - 1/p^{1+\gamma_q+\gamma_r})}{\prod_{k=1}^K \prod_{q=1}^Q (1 - 1/p^{1+\alpha_k+\gamma_q})} \times \left(1 + (1 + 1/p)^{-1} \sum_{0 < \sum_k a_k + \sum_q c_q \text{ is even}} \frac{\prod_q \mu(p^{c_q})}{p^{\sum_k a_k (1/2+\alpha_k) + \sum_q c_q (1/2+\gamma_q)}}\right),$$

while

$$g_S(z) := \frac{\Gamma((1-z)/2)}{\Gamma(z/2)}.$$

In particular, for our applications to the weighted one-level density, we are interested in the case where $Q = 1$ and $2 \le K \le 5$.

4A. *Conjecture 4.1 in the case $K = 2$, $Q = 1$.* We start with

$$\sum_{d \le X} \frac{L(1/2 + A, \chi_d) L(1/2 + C, \chi_d)}{L(1/2 + D, \chi_d)}, \tag{4-7}$$

with A, C, D shifts which satisfy the hypotheses prescribed by Conjecture 4.1; by the ratio conjecture, up to a negligible error $O(X^{1/2+\varepsilon})$, this is a sum of four terms and the first is

$$\sum_{d\le X}\frac{\zeta(1+2A)\zeta(1+2C)\zeta(1+A+C)}{\zeta(1+A+D)\zeta(1+C+D)}\mathcal{A}(A,C;D),$$

where

$$\mathcal{A}(A,C;D)=\prod_p\Big(1-\frac{1}{p^{1+2A}}\Big)\Big(1-\frac{1}{p^{1+2C}}\Big)\Big(1-\frac{1}{p^{1+A+C}}\Big)\Big(1-\frac{1}{p^{1+A+D}}\Big)^{-1}\Big(1-\frac{1}{p^{1+C+D}}\Big)^{-1} \times\bigg(1+\frac{p}{p+1}\sum_{0<a+c+d\text{ even}}\frac{\mu(p^d)}{p^{a(1/2+A)+c(1/2+C)+d(1/2+D)}}\bigg).$$

In the following, it will be relevant to notice that for small values of the shifts, then the arithmetical coefficient $\mathcal{A}(A,C;D)$ tends to $\mathcal{A}$, defined in (4-3); this essentially follows from [Conrey et al. 2008, Corollary 6.4]. All the other terms can be easily recovered from the first one, just by changes of sign of the shifts, as the recipe suggests. This yields a formula for (4-7), written as a sum of four pieces; by computing the derivative $\frac{d}{dC}[\cdots]_{C=D}$, we get

$$\sum_{d\le X}\frac{L'}{L}\Big(\frac12+D,\chi_d\Big)L\Big(\frac12+A,\chi_d\Big)=\sum_{d\le X}\bigg(Q_1+\Big(\frac{d}{\pi}\Big)^{-A}g_S\Big(\frac12+A\Big)Q_2+\Big(\frac{d}{\pi}\Big)^{-D}g_S\Big(\frac12+D\Big)Q_3 +\Big(\frac{d}{\pi}\Big)^{-A-D}g_S\Big(\frac12+A+D\Big)Q_4\bigg)+O\big(X^{1/2+\varepsilon}\big), \tag{4-8}$$

with

$$\begin{aligned}Q_1&=\mathcal{A}(A,D;D)\frac{\zeta(1+2A)}{\zeta(1+A+D)}\bigg(\frac{2\zeta'(1+2D)\zeta(1+A+D)}{\zeta(1+2D)}+\frac{\zeta'(1+A+D)\zeta(1+2D)}{\zeta(1+2D)}\\&\qquad-\frac{\zeta'(1+2D)\zeta(1+A+D)}{\zeta(1+2D)}\bigg)+\mathcal{A}'(A,D;D)\zeta(1+2A)\\&=\mathcal{A}(A,D;D)\frac{\zeta(1+2A)}{\zeta(1+A+D)}\bigg(\frac{\zeta'}{\zeta}(1+2D)\zeta(1+A+D)+\zeta'(1+A+D)\bigg)+\mathcal{A}'(A,D;D)\zeta(1+2A),\\Q_2&=\mathcal{A}(-A,D;D)\frac{\zeta(1-2A)}{\zeta(1-A+D)}\bigg(\frac{\zeta'}{\zeta}(1+2D)\zeta(1-A+D)+\zeta'(1-A+D)\bigg)+\mathcal{A}'(-A,D;D)\zeta(1-2A),\\Q_3&=-\mathcal{A}(A,-D;D)\frac{\zeta(1+2A)\zeta(1-2D)\zeta(1+A-D)}{\zeta(1+A+D)},\\Q_4&=-\mathcal{A}(-A,-D;D)\frac{\zeta(1-2A)\zeta(1-2D)\zeta(1-A-D)}{\zeta(1-A+D)}.\end{aligned}$$

Moreover, we notice that if the shifts are $\ll(\log X)^{-1}$, then we can approximate (4-8), getting

$$\begin{aligned}\sum_{d\le X}\frac{L'}{L}\Big(\frac12+D,\chi_d\Big)L\Big(\frac12+A,\chi_d\Big)&\\=\mathcal{A}X^*\bigg(&\frac{-A-3D}{(2A)(2D)(A+D)}+X^{-A}\frac{A-3D}{(-2A)(2D)(-A+D)}\\&+X^{-D}\frac{A+D}{(2A)(2D)(A-D)}+X^{-A-D}\frac{-A+D}{(-2A)(2D)(-A-D)}\bigg)+O(\log X),\end{aligned} \tag{4-9}$$

being that $\mathcal{A}(\pm A,\pm D,D)=\mathcal{A}+O(1/\log X)$ and $\zeta(1+z)=1/z+O(1)$ as $z\to 0$.

4B. ***Conjecture 4.1 in the case $K = 3$, $Q = 1$.*** Now, we study in detail

$$\sum_{d\le X} \frac{L(1/2+A,\chi_d)L(1/2+B,\chi_d)L(1/2+C,\chi_d)}{L(1/2+D,\chi_d)}, \tag{4-10}$$

with A, B, C and D as prescribed by Conjecture 4.1. This time, the asymptotic formula suggested by recipe is a sum of eight terms; the first is

$$\sum_{d\le X} \frac{\zeta(1+2A)\zeta(1+2B)\zeta(1+2C)\zeta(1+A+B)\zeta(1+A+C)\zeta(1+B+C)}{\zeta(1+A+D)\zeta(1+B+D)\zeta(1+C+D)} \mathcal{A}(A,B,C;D),$$

where the (rather horrible) arithmetical coefficient is given by

$$\begin{aligned}
&\mathcal{A}(A,B,C;D)\\
&=\prod_p \Bigl(1-\frac{1}{p^{1+2A}}\Bigr)\Bigl(1-\frac{1}{p^{1+2B}}\Bigr)\Bigl(1-\frac{1}{p^{1+2C}}\Bigr)\Bigl(1-\frac{1}{p^{1+A+B}}\Bigr)\Bigl(1-\frac{1}{p^{1+A+C}}\Bigr)\\
&\quad\times\Bigl(1-\frac{1}{p^{1+B+C}}\Bigr)\Bigl(1-\frac{1}{p^{1+A+D}}\Bigr)^{-1}\Bigl(1-\frac{1}{p^{1+B+D}}\Bigr)^{-1}\Bigl(1-\frac{1}{p^{1+C+D}}\Bigr)^{-1}\\
&\quad\times\Bigl(1+\frac{p}{p+1}\Bigl[\Bigl(1+\frac{1}{p^{1+A+B}}+\frac{1}{p^{1+A+C}}+\frac{1}{p^{1+B+C}}\Bigr)\Bigl(1-\frac{1}{p^{1+2A}}\Bigr)^{-1}\Bigl(1-\frac{1}{p^{1+2B}}\Bigr)^{-1}\Bigl(1-\frac{1}{p^{1+2C}}\Bigr)^{-1}\\
&\qquad-1-\Bigl(\frac{1}{p^{1+A+D}}+\frac{1}{p^{1+B+D}}+\frac{1}{p^{1+C+D}}+\frac{1}{p^{2+A+B+C+D}}\Bigr)\Bigl(1-\frac{1}{p^{1+2A}}\Bigr)^{-1}\Bigl(1-\frac{1}{p^{1+2B}}\Bigr)^{-1}\Bigl(1-\frac{1}{p^{1+2C}}\Bigr)^{-1}\Bigr]\Bigr).
\end{aligned}$$

We notice that, as in the proof of [Conrey et al. 2008, Corollary 6.4], we can prove that the arithmetical coefficient is convergent if all the variables are in small disk around 0, being $\mathcal{A}(\underline{0})=\mathcal{B}$, with $\mathcal{B}$ defined in (4-5). As in the previous example, this gives a formula for (4-10) with all the main terms and error $O(X^{1/2+\varepsilon})$. Differentiating this formula with respect to C at $C=D$, we get

$$\begin{aligned}
&\sum_{d\le X}\frac{L'}{L}\Bigl(\frac12+D,\chi_d\Bigr)L\Bigl(\frac12+A,\chi_d\Bigr)L\Bigl(\frac12+B,\chi_d\Bigr)\\
&\quad=\sum_{d\le X}\Bigl(R_1+\Bigl(\frac{d}{\pi}\Bigr)^{-A}g_S\Bigl(\frac12+A\Bigr)R_2+\Bigl(\frac{d}{\pi}\Bigr)^{-B}g_S\Bigl(\frac12+B\Bigr)R_3+\Bigl(\frac{d}{\pi}\Bigr)^{-D}g_S\Bigl(\frac12+D\Bigr)R_4\\
&\qquad+\Bigl(\frac{d}{\pi}\Bigr)^{-A-B}g_S\Bigl(\frac12+A+B\Bigr)R_5+\Bigl(\frac{d}{\pi}\Bigr)^{-A-D}g_S\Bigl(\frac12+A+D\Bigr)R_6\\
&\qquad+\Bigl(\frac{d}{\pi}\Bigr)^{-B-D}g_S\Bigl(\frac12+B+D\Bigr)R_7+\Bigl(\frac{d}{\pi}\Bigr)^{-A-B-D}g_S\Bigl(\frac12+A+B+D\Bigr)R_8\Bigr)+O(X^{1/2+\varepsilon})
\end{aligned}\tag{4-11}$$

with

$$\begin{aligned}
R_1&=R_1(A,B,D)\\
&=\mathcal{A}(A,B,D;D)\frac{\zeta(1+2A)\zeta(1+2B)\zeta(1+A+B)}{\zeta(1+A+D)\zeta(1+B+D)}\\
&\quad\times\Bigl(\frac{2\zeta'(1+2D)\zeta(1+A+D)\zeta(1+B+D)+\zeta(1+2D)\zeta'(1+A+D)\zeta(1+B+D)}{\zeta(1+2D)}\\
&\qquad+\frac{\zeta(1+2D)\zeta(1+A+D)\zeta'(1+B+D)-\zeta(1+A+D)\zeta(1+B+D)\zeta'(1+2D)}{\zeta(1+2D)}\Bigr)\\
&\qquad+\zeta(1+2A)\zeta(1+2B)\zeta(1+A+B)\mathcal{A}'(A,B,D;D),
\end{aligned}$$

and

$$\begin{aligned}
R_2 &= R_1(-A, B, D),\\
R_3 &= R_1(A, -B, D),\\
R_4 &= R_4(A, B, D)\\
&= -\frac{\zeta(1+2A)\zeta(1+2B)\zeta(1+A+B)\zeta(1-2D)\zeta(1+A-D)\zeta(1+B-D)}{\zeta(1+A+D)\zeta(1+B+D)}\mathcal{A}(A, B, -D; D),\\
R_5 &= R_1(-A, -B, D),\\
R_6 &= R_4(-A, B, D),\\
R_7 &= R_4(A, -B, D),\\
R_8 &= R_4(-A, -B, D).
\end{aligned}$$

If $A, B, D \ll (\log X)^{-1}$ the above formula simplifies a lot, since in this case

$$R_1 = \frac{-AB-3AD-3BD-5D^2}{(2A)(2B)(2D)(A+B)(A+D)(B+D)}\mathcal{B} + O\biggl(\frac{(\log X)^6}{(\log X)^3}\biggr) =: f(A, B, D)\mathcal{B} + O\bigl((\log X)^3\bigr)$$

and

$$R_4 = \frac{-(A+D)(B+D)}{(2A)(2B)(-2D)(A+B)(A-D)(B-D)}\mathcal{B} + O\biggl(\frac{(\log X)^6}{(\log X)^3}\biggr) =: g(A, B, D)\mathcal{B} + O\bigl((\log X)^3\bigr),$$

giving

$$\begin{aligned}
\sum_{d\le X} \frac{L'}{L}\Bigl(\frac{1}{2}+D, \chi_d\Bigr)L\Bigl(\frac{1}{2}+A, \chi_d\Bigr)L\Bigl(\frac{1}{2}+B, \chi_d\Bigr)&\\
= \mathcal{B}X^*\bigl(f(A, B, D)+X^{-A}f(-A, B, D)+X^{-B}f(A, -B, D)+X^{-D}g(A, B, D)&\\
+X^{-A-B}f(-A, -B, D)+X^{-A-D}g(-A, B, D)&\\
+X^{-B-D}g(A, -B, D)+X^{-A-B-D}g(-A, -B, D)\bigr)+O\bigl((\log X)^3\bigr).& \qquad (4\text{-}12)
\end{aligned}$$

Analogous (but longer) formulae can be obtained also in the cases $K = 4$, $Q = 1$ and $K = 5$, $Q = 1$. With exactly the same ideas (but much longer computations) also the case $K > 5$, $Q = 1$ can be dealt.

4C. ***The weighted one-level density for $\{L(\frac{1}{2}, \chi_d)\}_d$.*** We recall that the one-level density for the symplectic family of quadratic Dirichlet L-functions has been studied originally by Özlük and Snyder [1999] and independently by Katz and Sarnak [1999][10], who proved that

$$\lim_{X\to\infty} \frac{1}{X^*}\sum_{d\le X}\sum_{\gamma_d} f\biggl(\frac{\log X}{2\pi}\gamma_d\biggr) = \int_{-\infty}^{+\infty} f(x)\biggl(1-\frac{\sin(2\pi x)}{2\pi x}\biggr)\,dx \qquad (4\text{-}13)$$

under GRH, for any f such that supp $\hat{f} \subset (-2, 2)$. Moreover, Conrey and Snaith [2007] showed (4-13) (also with lower order terms) with no constraint on the support of $\hat{f}$, under the assumption of the

[10] See also *"Zeroes of Zeta Functions, their Spaces and their Spectral Nature"* by Katz and Sarnak, the 1997 preprint version of [Katz and Sarnak 1999].

ratio conjecture; namely, they consider f a test function, holomorphic throughout the strip $|\Im(z)| < 2$, even, real on the real line and such that $f(x) \ll 1/(1+x^2)$ as $x \to \infty$, and they study

$$\mathcal{D}_0^{L_\chi}(f) := \frac{1}{X^*} \sum_{d \le X} \sum_{\gamma_d} f\Big(\frac{\log X}{2\pi} \gamma_d\Big). \tag{4-14}$$

As $X \to \infty$, they show that the above is asymptotic to the right-hand side of (4-14), which matches with the one-level density for the eigenvalues of the matrices from the symplectic group $\mathrm{USp}(2N)$. In particular, we notice that the one-level density function $1 - \sin(2\pi x)/(2\pi x)$ vanishes of order 2 at $x = 0$, being $\sim (2\pi^2/3)x^2$ as $x \to 0$.

Similarly to what we did in Section 3, we now want to compute the weighted one-level density in the symplectic case, tilted by $L\big(\frac{1}{2}, \chi_d\big)$. We note that, differently from what happens in the Riemann zeta function case, here we are allowed to consider the first power as well, as $L\big(\frac{1}{2}, \chi_d\big)$ is real. The analogue of (3-1) in this context is

$$\mathcal{D}_1^{L_\chi}(f) := \frac{1}{\sum_{d \le X} L(1/2, \chi_d)} \sum_{d \le X} \sum_{\gamma_d} f\Big(\frac{\log X}{2\pi} \gamma_d\Big) L\Big(\frac{1}{2}, \chi_d\Big), \tag{4-15}$$

and via ratio conjecture in the form of (4-8) this can be studied asymptotically, as shown in the following result:

Proposition 4.1. *Assume GRH and Conjecture 4.1 for $K = 2$, $Q = 1$. For any test function f, holomorphic in the strip $\Im(z) < 2$, even, real on the real line and such that $f(x) \ll 1/(1+x^2)$ as $x \to \infty$, we have*

$$\mathcal{D}_1^{L_\chi}(f) = \int_{-\infty}^{+\infty} f(x) W_{\mathrm{USp}}^1(x)\, dx + O\Big(\frac{1}{\log X}\Big)$$

as $X \to \infty$, where

$$W_{\mathrm{USp}}^1(x) := 1 + \frac{\sin(2\pi x)}{2\pi x} - \frac{2\sin^2(\pi x)}{(\pi x)^2}.$$

Proof. We start looking at

$$\frac{1}{(\mathcal{A}/2) X^* \log X} \sum_{d \le X} \sum_{\gamma_d} f\Big(\frac{\log X}{2\pi} \gamma_d\Big) L\Big(\frac{1}{2} + \alpha, \chi_d\Big), \tag{4-16}$$

with $\alpha \asymp (\log X)^{-1}$; note that, as $\alpha \to 0$, $\sum_{d \le X} L\big(\frac{1}{2} + \alpha, \chi_d\big)$ tends to $1/(2\zeta(2))(\mathcal{A}/2) X \log X$, which is the normalization $(\mathcal{A}/2) X^* \log X$ we have in (4-16). As usual, we use the Cauchy theorem and the functional equation for $(L'/L)(s, \chi_d)$ to write

$$\frac{1}{(\mathcal{A}/2) X^* \log X} \sum_{d \le X} \sum_{\gamma_d} f\Big(\frac{\log X}{2\pi} \gamma_d\Big) L\Big(\frac{1}{2} + \alpha, \chi_d\Big) = -\mathcal{J}(\alpha) + 2\mathcal{I}(\alpha) + O\Big(\frac{1}{\log X}\Big) \tag{4-17}$$

where

$$\begin{aligned}\mathcal{J}(\alpha) &:= \frac{2}{\mathcal{A} X^* \log X} \frac{1}{2\pi} \int_{-\infty}^{+\infty} (-\log X) f\Big(\frac{y \log X}{2\pi}\Big) \sum_{d \le X} L\Big(\frac{1}{2} + \alpha, \chi_d\Big)\, dy \\ &= \frac{-2}{\mathcal{A} X^* \log X} \int_{-\infty}^{+\infty} f(x) \sum_{d \le X} \Big(L\Big(\frac{1}{2}, \chi_d\Big) + O\Big(\frac{1}{\log X}\Big)\Big)\, dx = -\int_{-\infty}^{+\infty} f(x)\, dx + O\Big(\frac{1}{\log X}\Big)\end{aligned}$$

and

$$\mathcal{I}(\alpha) := \frac{2}{\mathcal{A}X^*(\log X)^2}\int_{-\infty}^{+\infty} f(x)\sum_{d\le X}\frac{L'}{L}\Big(\frac{1}{2}+\delta+\frac{2\pi i x}{\log X},\chi_d\Big)L\Big(\frac{1}{2}+\alpha,\chi_d\Big)\,dx,$$

with $\delta \asymp (\log X)^{-1}$. Now we rely on the assumption of the ratio conjecture (in particular, (4-8) and (4-9)) to compute the sum over d; in particular, in the same way as in the proof of Proposition 3.1, by a truncation of the integral over x and Taylor approximations, we get

$$\mathcal{I}(\alpha) = \frac{2}{(\log X)^2}\int_{-\infty}^{+\infty} f(x) g_X\Big(\alpha,\delta+\frac{2\pi i x}{\log X}\Big)\,dx + O\Big(\frac{1}{\log X}\Big),$$

where

$$g_X(\alpha,w) := \frac{-\alpha-3w}{(2\alpha)(2w)(\alpha+w)} + X^{-\alpha}\frac{\alpha-3w}{(-2\alpha)(2w)(-\alpha+w)} + X^{-w}\frac{\alpha+w}{(2\alpha)(2w)(\alpha-w)} + X^{-\alpha-w}\frac{-\alpha+w}{(-2\alpha)(2w)(-\alpha-w)}.$$

The integral is regular at $\delta = 0$ then, if we denote $\alpha = a/\log X$, we get

$$\mathcal{I}\Big(\frac{a}{\log X}\Big) = \frac{2}{(\log X)^2}\int_{-\infty}^{+\infty} f(x) g_X\Big(\frac{a}{\log X},\frac{2\pi i x}{\log X}\Big)\,dx + O\Big(\frac{1}{\log X}\Big),$$

which is regular at $a = 0$; indeed, if we take the limit as $a \to 0$, we get

$$\mathcal{I}(0) = \lim_{a\to 0}\mathcal{I}\Big(\frac{a}{\log X}\Big) = 2\int_{-\infty}^{+\infty} f(x) g(2\pi i x)\,dx + O\Big(\frac{1}{\log X}\Big)$$

where

$$\begin{aligned} g(w) &:= \lim_{a\to 0}\Big(\frac{-a-3w}{(2a)(2w)(a+w)} + e^{-a}\frac{a-3w}{(-2a)(2w)(-a+w)} + e^{-w}\frac{a+w}{(2a)(2w)(a-w)} + e^{-a-w}\frac{-a+w}{(-2a)(2w)(-a-w)}\Big) \\ &= \frac{-we^{-w}-3w-4e^{-w}+4}{4w^2}. \end{aligned}$$

Then

$$\begin{aligned} \mathcal{D}_1^{L_\chi}(f) &= \lim_{\alpha\to 0}\frac{1}{(\mathcal{A}/2)X^*\log X}\sum_{d\le X}\sum_{\gamma_d} f\Big(\frac{\log X}{2\pi}\gamma_d\Big)L\Big(\frac{1}{2}+\alpha,\chi_d\Big) \\ &= -\mathcal{J}(0) + 2\mathcal{I}(0) + O\Big(\frac{1}{\log X}\Big) = \int_{-\infty}^{+\infty} f(x)\big(1+4g(2\pi i x)\big)\,dx + O\Big(\frac{1}{\log X}\Big), \end{aligned}$$

and since f is even, the main term above equals

$$\int_{-\infty}^{+\infty} f(x)\Big(1+\frac{\sin(2\pi x)}{2\pi x} - \frac{2\sin^2(\pi x)}{(\pi x)^2}\Big)\,dx. \qquad \square$$

Analogously, we can compute the weighted one-level density, tilted by the second power of $L(\frac{1}{2},\chi_d)$, i.e.,

$$\mathcal{D}_2^{L_\chi}(f) := \frac{1}{\sum_{d\le X} L(1/2,\chi_d)^2}\sum_{d\le X}\sum_{\gamma_d} f\Big(\frac{\log X}{2\pi}\gamma_d\Big)L\Big(\frac{1}{2},\chi_d\Big)^2 \tag{4-18}$$

under the assumption of Conjecture 4.1, in the case $K = 3$, $Q = 1$.

Proposition 4.2. *Assume GRH and Conjecture 4.1 for $K = 3$, $Q = 1$. For any function f holomorphic in the strip $\Im(z) < 2$, even, real on the real line and such that $f(x) \ll 1/(1+x^2)$ as $x \to \infty$, we have*

$$\mathcal{D}_2^{\boldsymbol{L}_\chi}(f) = \int_{-\infty}^{+\infty} f(x) W_{\mathrm{USp}}^2(x)\,dx + O\Big(\frac{1}{\log X}\Big)$$

as $X \to \infty$, where

$$W_{\mathrm{USp}}^2(x) := 1 - \frac{\sin(2\pi x)}{2\pi x} - \frac{24(1-\sin^2(\pi x))}{(2\pi x)^2} + \frac{48\sin(2\pi x)}{(2\pi x)^3} - \frac{96\sin^2(\pi x)}{(2\pi x)^4}.$$

Proof. The proof works like that of Proposition 4.1; first, for $\alpha = a/\log X \asymp (\log X)^{-1}$ and for $\beta = b/\log X \asymp (\log X)^{-1}$, we analyze

$$\frac{1}{(\mathcal{B}/24)X^*(\log X)^3} \sum_{d \le X} \sum_{\gamma_d} f\Big(\frac{\log X}{2\pi}\gamma_d\Big) L\Big(\frac{1}{2}+\alpha, \chi_d\Big) L\Big(\frac{1}{2}+\beta, \chi_d\Big), \tag{4-19}$$

which can be written as

$$-\mathcal{J}(\alpha,\beta) + 2\mathcal{I}(\alpha,\beta) + O\Big(\frac{1}{\log X}\Big), \tag{4-20}$$

where

$$\begin{aligned}\mathcal{J}(\alpha,\beta) &:= \frac{-24\log X}{\mathcal{B}X^*(\log X)^3}\frac{1}{2\pi}\int_{-\infty}^{+\infty} f\Big(\frac{y\log X}{2\pi}\Big) \sum_{d\le X} L\Big(\frac{1}{2}+\alpha,\chi_d\Big) L\Big(\frac{1}{2}+\beta,\chi_d\Big)\\ &= -\int_{-\infty}^{+\infty} f(x)\,dx + O\Big(\frac{1}{\log X}\Big)\end{aligned} \tag{4-21}$$

and

$$\mathcal{I}(\alpha,\beta) := \frac{24}{\mathcal{B}X^*(\log X)^4}\int_{-\infty}^{+\infty} f(x) \sum_{d\le X} \frac{L'}{L}\Big(\frac{1}{2}+\delta+\frac{2\pi i x}{\log X}, \chi_d\Big) L\Big(\frac{1}{2}+\alpha,\chi_d\Big) L\Big(\frac{1}{2}+\beta,\chi_d\Big)\,dx,$$

where $\delta \asymp (\log T)^{-1}$, as usual. With the usual machinery, the ratio conjecture (see (4-11) and (4-12)) allows us to evaluate the sum over d; the resulting quantity is regular at $\delta = 0$ and at $\alpha = a/\log X = 0$, $\beta = b/\log X = 0$, thus taking the limit, we get

$$\mathcal{I}(0,0) = 24\int_{-\infty}^{+\infty} f(x) h(2\pi i x)\,dx, \tag{4-22}$$

with

$$h(y) := \frac{y^3 e^{-y} - 5y^3 + 12y^2 e^{-y} + 12y^2 + 48 y e^{-y} + 48 e^{-y} - 48}{48 y^4}.$$

Putting all together, from (4-19), (4-20), (4-21) and (4-22), we finally get

$$\mathcal{D}_2^{\boldsymbol{L}_\chi} = -\mathcal{J}(0,0) + 2\mathcal{I}(0,0) + O\Big(\frac{1}{\log X}\Big) = \int_{-\infty}^{+\infty} f(x)\big(1 + 48h(2\pi i x)\big)\,dx + O\Big(\frac{1}{\log X}\Big).$$

Moreover, since f is even, the main term equals

$$\int_{-\infty}^{+\infty} f(x)\Big(1 - \frac{\sin(2\pi x)}{2\pi x} - \frac{24(1-\sin^2(\pi x))}{(2\pi x)^2} + \frac{48\sin(2\pi x)}{(2\pi x)^3} - \frac{96\sin^2(\pi x)}{(2\pi x)^4}\Big)\,dx. \qquad \square$$

In the same way, we study

$$\mathcal{D}_3^{\boldsymbol{L}_\chi}(f) := \frac{1}{\sum_{d\le X} L(1/2,\chi_d)^3} \sum_{d\le X}\sum_{\gamma_d} f\Big(\frac{\log X}{2\pi}\gamma_d\Big) L\Big(\frac{1}{2},\chi_d\Big)^3, \tag{4-23}$$

assuming Conjecture 4.1 in the case $K=4$, $Q=1$.

Proposition 4.3. *Assume GRH and Conjecture 4.1 for $K=4$, $Q=1$. For any function f holomorphic in the strip $\Im(z)<2$, even, real on the real line and such that $f(x)\ll 1/(1+x^2)$ as $x\to\infty$, we have*

$$\mathcal{D}_3^{\boldsymbol{L}_\chi}(f) = \int_{-\infty}^{+\infty} f(x) W^3_{\mathrm{USp}}(x)\,dx + O\Big(\frac{1}{\log X}\Big)$$

as $X\to\infty$, where

$$W^3_{\mathrm{USp}}(x) := 1 + \frac{\sin(2\pi x)}{2\pi x} - \frac{12\sin^2(\pi x)}{(\pi x)^2} - \frac{240\sin(2\pi x)}{(2\pi x)^3} - \frac{15(6-10\sin^2(\pi x))}{(\pi x)^4} + \frac{2880\sin(2\pi x)}{(2\pi x)^5} - \frac{90\sin^2(\pi x)}{(\pi x)^6}.$$

Proof. We consider $\alpha,\beta,\nu\in\mathbb{R}$ of size $\asymp 1/\log X$. We denote

$$\boldsymbol{L}_{\alpha,\beta,\nu}\big(\tfrac12,\chi_d\big) := L\big(\tfrac12+\alpha,\chi_d\big)L\big(\tfrac12+\beta,\chi_d\big)L\big(\tfrac12+\nu,\chi_d\big),$$

and we look at

$$\frac{1}{\sum_{d\le X} L(1/2,\chi_d)^3} \sum_{d\le X}\sum_{\gamma_d} f\Big(\frac{\log X}{2\pi}\gamma_d\Big) \boldsymbol{L}_{\alpha,\beta,\nu}\Big(\frac{1}{2},\chi_d\Big).$$

With the usual machinery we get that the above equals

$$\int_{-\infty}^{+\infty} f(x)\bigg(1 + 2\frac{1}{\sum_{d\le X} L(1/2,\chi_d)^3} \sum_{d\le X} \frac{L'}{L}\Big(\frac12+\delta+\frac{2\pi i x}{\log X},\chi_d\Big)\boldsymbol{L}_{\alpha,\beta,\nu}\Big(\frac12,\chi_d\Big)\bigg)\,dx$$

up to an error $O(1/\log X)$, with $\delta\asymp 1/\log X$. The remaining sum over d can be evaluated asymptotically by using the ratio conjecture (i.e., Conjecture 4.1 for $K=4$, $Q=1$). This can be done by using Sage to carry out the easy but very long computations. Doing so, letting $\alpha,\beta,\nu\to 0$, we obtain

$$\mathcal{D}_3^{\boldsymbol{L}_\chi}(f) = \int_{-\infty}^{+\infty} f(x)\big(1 + 2\cdot 2880\cdot h(2\pi i x)\big)\,dx + O\Big(\frac{1}{\log X}\Big),$$

with

$$h(y) := \frac{-7y^5+24y^4-240y^2+2880}{5760y^6} + \frac{e^{-y}(-y^5-24y^4-240y^3-1200y^2-2880y-2880)}{5760y^6}.$$

The claim follows, since f is even. □

Finally, we look at

$$\mathcal{D}_4^{\boldsymbol{L}_\chi}(f) := \frac{1}{\sum_{d\le X} L(1/2,\chi_d)^4} \sum_{d\le X}\sum_{\gamma_d} f\Big(\frac{\log X}{2\pi}\gamma_d\Big) L\Big(\frac{1}{2},\chi_d\Big)^4 \tag{4-24}$$

assuming Conjecture 4.1, in the case $K=5$, $Q=1$.

Proposition 4.4. *Assume GRH and Conjecture 4.1 for $K=5$, $Q=1$. For any function f holomorphic in the strip $\Im(z)<2$, even, real on the real line and such that $f(x)\ll 1/(1+x^2)$ as $x\to\infty$, we have*

$$\mathcal{D}_4^{\boldsymbol{L}_\chi}(f)=\int_{-\infty}^{+\infty} f(x)W_{\mathrm{USp}}^4(x)\,dx+O\Big(\frac{1}{\log X}\Big)$$

as $X\to\infty$, where

$$W_{\mathrm{USp}}^4(x):=1-\frac{\sin(2\pi x)}{2\pi x}-\frac{10(1+\cos(2\pi x))}{(\pi x)^2}+\frac{90\sin(2\pi x)}{(\pi x)^3}-\frac{15(3-31\cos(2\pi x))}{(\pi x)^4}$$
$$-\frac{1470\sin(2\pi x)}{(\pi x)^5}-\frac{315(1+9\cos(2\pi x))}{(\pi x)^6}+\frac{3150\sin(2\pi x)}{(\pi x)^7}-\frac{1575(1-\cos(2\pi x))}{(\pi x)^8}.$$

Proof. The proof works in the same way as the previous ones. We consider $\alpha,\beta,\nu,\eta\in\mathbb{R}$ of size $\asymp 1/\log X$, we let

$$\boldsymbol{L}_{\alpha,\beta,\nu,\eta}\big(\tfrac12,\chi_d\big):=L\big(\tfrac12+\alpha,\chi_d\big)L\big(\tfrac12+\beta,\chi_d\big)L\big(\tfrac12+\nu,\chi_d\big)L\big(\tfrac12+\eta,\chi_d\big),$$

and we look at

$$\frac{1}{\sum_{d\le X}L(1/2,\chi_d)^4}\sum_{d\le X}\sum_{\gamma_d} f\Big(\frac{\log X}{2\pi}\gamma_d\Big)\boldsymbol{L}_{\alpha,\beta,\nu,\eta}\Big(\frac12,\chi_d\Big).$$

By the usual manipulations, the above equals

$$\int_{-\infty}^{+\infty} f(x)\Big(1+2\frac{1}{\sum_{d\le X}L(1/2,\chi_d)^4}\sum_{d\le X}\frac{L'}{L}\Big(\frac12+\delta+\frac{2\pi i x}{\log X},\chi_d\Big)\boldsymbol{L}_{\alpha,\beta,\nu,\eta}\Big(\frac12,\chi_d\Big)\Big)\,dx,$$

up to an error $O(1/\log X)$, with $\delta\asymp 1/\log X$. Thanks to Conjecture 4.1 with $K=5$, $Q=1$, the above can be computed asymptotically. As $\alpha,\beta,\nu,\eta\to 0$, with the help of Sage, we then obtain

$$\mathcal{D}_4^{\boldsymbol{L}_\chi}(f)=\int_{-\infty}^{+\infty} f(x)\big(1+2\cdot 4838400\cdot h(2\pi i x)\big)\,dx+O\Big(\frac{1}{\log X}\Big),$$

with

$$h(y):=\frac{-9y^7+40y^6-720y^4+20160y^2-403200}{9676800y^8}+\frac{e^{-y}(y^7+40y^6+720y^5+7440y^4)}{9676800y^8}$$
$$+\frac{e^{-y}(47040y^3+181440y^2+403200y+403200)}{9676800y^8}.$$

Again, being f even, the claim follows. □

4D. ***Proof of Theorem 2.4.*** The same remark as in Section 3C applies here, relying on [Conrey et al. 2008, Theorem 4.2] instead of Conjecture 4.1.

5. Proof of Theorems 2.5 and 2.6

As a last example, we analyze the orthogonal case of the family of quadratic twists of the L-functions associated with the discriminant modular form Δ. which is the unique normalized cusp form of weight 12.

Its Fourier coefficients define the Ramanujan tau function $\tau(n)$, being

$$\Delta(z) = q\prod_{n\geq 1}(1-q^n)^{24} = \sum_{n=1}^{\infty}\tau(n)q^n,$$

with $q = e^{2\pi i z}$. Thus, the L-function associated with Δ is defined by

$$L_\Delta(s) := \sum_{n=1}^{\infty}\frac{\tau^*(n)}{n^s},$$

where $\tau^*(n) = \tau(n)/n^{11/2}$. The family we want to describe is the collection of the quadratic twists of L_Δ, that are

$$L_\Delta(s, \chi_d) = \sum_{n=1}^{\infty}\frac{\chi_d(n)\tau^*(n)}{n^s} = \prod_p\left(1 - \frac{\tau^*(p)\chi_d(p)}{p^s} + \frac{\chi_d(p^2)}{p^{2s}}\right)^{-1},$$

and, for $d > 0$, they satisfy the functional equation

$$\left(\frac{d^2}{4\pi^2}\right)^{s/2}\Gamma\left(s + \frac{11}{2}\right)L_\Delta(s, \chi_d) = \left(\frac{d^2}{4\pi^2}\right)^{(1-s)/2}\Gamma\left(1 - s + \frac{11}{2}\right)L_\Delta(1-s, \chi_d).$$

Finally, we also record that

$$\frac{1}{L_\Delta(s, \chi_d)} = \prod_p\left(1 - \frac{\tau^*(p)\chi_d(p)}{p^s} + \frac{\chi_d(p^2)}{p^{2s}}\right) =: \sum_{n=1}^{\infty}\frac{\chi_d(n)\mu_\Delta(n)}{n^s},$$

where μ_Δ is the multiplicative function defined by $\mu_\Delta(p) = -\tau^*(p)$, $\mu_\Delta(p^2) = 1$ and $\mu_\Delta(p^\alpha) = 0$ if $\alpha \geq 3$.

The family $\{L_\Delta(1/2, \chi_d) : d > 0,\ f.d.\}$ is an even orthogonal family, modeled by the group SO($2N$) with the identification $2N \approx \log(d^2/(4\pi^2))$.

The moments at the central value of L-functions associated with quadratic twists of a modular form have been studied extensively in recent years, but only the first moment [Bump et al. 1990; Iwaniec 1990; Murty and Murty 1991] and partially the second [Soundararajan and Young 2010; Radziwiłł and Soundararajan 2015] have been obtained. It is known that such a family can be either symplectic or orthogonal, depending on the specific L-function we twist; in particular, if we start with the L-function associated with the discriminant modular form Δ, then we are in the latter case. For an orthogonal family $\mathcal{F}$, ordered by the conductor $C(f)$, Conrey and Farmer [2000] and Keating and Snaith [2000] predict that

$$\frac{1}{X^*}\sum_{\substack{f\in\mathcal{F}\\ C(f)\leq X}} L_f\left(\frac{1}{2}\right)^k \sim \frac{f_O(k)}{2}a(k)(\log X^A)^{k(k-1)/2}, \tag{5-1}$$

where the above sum is over the X^* elements of the family $\mathcal{F}$ such that $C(f) \leq X$; $f_O(k)$ is the leading order coefficient of the moments of characteristic polynomials of matrices in $SO(2N)$; $a(k)$ is a constant depending on the particular family involved; A is a constant depending on the functional equation satisfied by the L-functions in the family, in particular on the degree of the relevant parameter in the functional equation for $L_f(s)$ (see [Conrey and Farmer 2000, Equation (1.3)] for further details and examples). Moreover, in this case the recipe [Conrey et al. 2005a] provides a precise formula with all the main terms for any integral moment, extended by [Conrey et al. 2008] to ratios. The ratio conjecture for the orthogonal family of quadratic twists of the discriminant modular form can be stated as follows:

Conjecture 5.1 [Conrey et al. 2008, Conjecture 5.3]. *Let K, Q be two positive integers, $\alpha_1, \dots, \alpha_K$ and $\gamma_1, \dots, \gamma_Q$ be complex shifts with real part $\asymp (\log X)^{-1}$ and imaginary part $\ll_\varepsilon X^{1-\varepsilon}$ for every $\varepsilon > 0$, then*

$$\sum_{d \le X} \frac{\prod_{k=1}^{K} L_\Delta(1/2+\alpha_k, \chi_d)}{\prod_{q=1}^{Q} L_\Delta(1/2+\gamma_q, \chi_d)} = \sum_{d \le X} \sum_{\epsilon \in \{-1,1\}^K} \left(\frac{d^2}{4\pi^2}\right)^{(1/2)\sum_k(\epsilon_k\alpha_k - \alpha_k)} \prod_{k=1}^{K} g_O\left(\frac{1}{2} + \frac{\alpha_k - \epsilon_k\alpha_k}{2}\right) Y_O \mathcal{A}_O(\cdots) + O(X^{1/2+\varepsilon}),$$

with $(\cdots) = (\epsilon_1\alpha_1, \dots, \epsilon_K\alpha_K; \gamma)$, where

$$Y_O(\alpha;\gamma) := \frac{\prod_{j<k\le K}\zeta(1+\alpha_j+\alpha_k)\prod_{q<r\le Q}\zeta(1+\gamma_q+\gamma_r)\prod_{q\le Q}\zeta(1+2\gamma_q)}{\prod_{k=1}^{K}\prod_{q=1}^{Q}\zeta(1+\alpha_k+\gamma_q)}$$

and $\mathcal{A}_O$ is an Euler product, absolutely convergent for all of the variables in small disks around 0, *which is given by*

$$\mathcal{A}_O(\alpha;\gamma) := \prod_p \frac{\prod_{j<k\le K}(1-1/p^{1+\alpha_j+\alpha_k})\prod_{q<r\le Q}(1-1/p^{1+\gamma_q+\gamma_r})\prod_{q\le Q}(1-1/p^{1+2\gamma_q})}{\prod_{k=1}^{K}\prod_{q=1}^{Q}(1-1/p^{1+\alpha_k+\gamma_q})} \times \left(1+\left(1+\frac{1}{p}\right)^{-1} \sum_{0<\sum_k a_k+\sum_q c_q \text{ is even}} \frac{\prod_k \tau^*(p^{a_k})\prod_q \mu_\Delta(p^{c_q})}{p^{\sum_k a_k(1/2+\alpha_k)+\sum_q c_q(1/2+\gamma_q)}}\right)$$

while

$$g_O(s) := \frac{\Gamma(1/2-s+6)}{\Gamma(s-1/2+6)}.$$

In the following, we will analyze the applications of this conjecture to the weighted one-level density, as we did in Section 4C for a symplectic family. To do so, we first look at what Conjecture 5.1 gives in a few specific examples.

5A. ***Conjecture 5.1 in the case $K = 1$, $Q = 0$.*** This is the easiest situation possible, corresponding to the first moment of $L_\Delta\left(\frac{1}{2}, \chi_d\right)$; for A a complex number which satisfies the hypotheses prescribed by Conjecture 5.1, the ratio conjecture yields

$$\frac{1}{X^*}\sum_{d\le X} L_\Delta\left(\frac{1}{2}+A, \chi_d\right) = \mathcal{A}(A) + \left(\frac{d}{2\pi}\right)^{-2A}\frac{\Gamma(6-A)}{\Gamma(6+A)}\mathcal{A}(-A) + O(X^{-1/2+\varepsilon}),$$

with

$$\mathcal{A}(A) := \prod_p\left(1+\frac{p}{p+1}\left[-1+\sum_{m=0}^{\infty}\frac{\tau^*(p^{2m})}{p^{(1/2+A)2m}}\right]\right).$$

We note that $\mathcal{A}(A)$ is regular at $A = 0$; indeed the $m = 0$ and $m = 1$ terms give 1 and $\tau^*(p^2)p^{-1-2A}$ respectively, therefore an approximation for $\mathcal{A}(A)$ would be $\prod_p(1+\tau^*(p^2)/p^{1+2A}+\cdots)$. Differently from the unitary and symplectic cases, where the first term in the corresponding Euler products gives the polar factor $\zeta(1+2A)$, here we would have $L_\Delta(\mathrm{sym}^2, 1+2A)$ the symmetric square of L_Δ, which is well known to be regular and nonzero at 1 (see [Iwaniec 1997, Chapter 13] for a complete overview about the symmetric square and its properties). However, for the sake of brevity, we prefer not to factor out $L_\Delta(\mathrm{sym}^2, 1+2A)$, and we leave the contribution of the symmetric square encoded in the arithmetical factor $\mathcal{A}(A)$, which converges in a small disk around 0. Thus, for $A=0$, we immediately get

$$\frac{1}{X^*}\sum_{d\le X} L_\Delta\Big(\frac{1}{2}, \chi_d\Big) = 2\mathcal{A} + O(X^{-1/2+\varepsilon}), \tag{5-2}$$

where

$$\mathcal{A} := \mathcal{A}(0) = \prod_p \bigg(1 + \frac{p}{p+1}\bigg[-1 + \sum_{m=0}^{\infty} \frac{\tau^*(p^{2m})}{(\sqrt{p})^{2m}}\bigg]\bigg). \tag{5-3}$$

5B. ***Conjecture 5.1 in the case $K=2$, $Q=1$.*** We consider

$$\sum_{d\le X} \frac{L_\Delta(1/2+A, \chi_d) L_\Delta(1/2+C, \chi_D)}{L_\Delta(1/2+D, \chi_d)}, \tag{5-4}$$

with A, C, D shifts satisfying the usual hypotheses prescribed by the ratio conjecture; by Conjecture 5.1, up to a negligible error, this is a sum of four terms and the first is

$$\sum_{d\le X} \frac{\zeta(1+A+C)\zeta(1+2D)}{\zeta(1+A+D)\zeta(1+C+D)} \mathcal{A}(A, C; D),$$

where

$$A\Big(A, C; D\Big) = \prod_p \Big(1 - \frac{1}{p^{1+A+C}}\Big)\Big(1 - \frac{1}{p^{1+2D}}\Big)\Big(1 - \frac{1}{p^{1+A+D}}\Big)^{-1}\Big(1 - \frac{1}{p^{1+C+D}}\Big)^{-1} \\ \times \bigg(1 + \frac{p}{p+1} \sum_{0<a+c+d \text{ even}} \frac{\tau^*(p^a)\tau^*(p^c)\mu(p^d)}{p^{a(1/2+A)+c(1/2+C)+d(1/2+D)}}\bigg).$$

As usual, we note that $\mathcal{A}(A, C; D) \sim \mathcal{A}(0, 0; 0) = \mathcal{A}$ defined in (5-3) as $A, C, D \to 0$; this can be proved by a modification of the proof of [Conrey et al. 2008, Corollary 6.4] or by direct computation. All the other terms can be easily recovered from the first one, then we get a formula for (5-4), written as a sum of four pieces; by computing the derivative $\frac{d}{dC}[\cdots]_{C=D}$, we get

$$\sum_{d\le X} \frac{L'_\Delta}{L_\Delta}\Big(\frac{1}{2}+D, \chi_d\Big) L\Big(\frac{1}{2}+A, \chi_d\Big) = \sum_{d\le X}\bigg(Q_1 + \Big(\frac{d}{2\pi}\Big)^{-2A} g_O\Big(\frac{1}{2}+A\Big) Q_2 + \Big(\frac{d}{2\pi}\Big)^{-2D} g_O\Big(\frac{1}{2}+D\Big) Q_3 \\ + \Big(\frac{d}{2\pi}\Big)^{-2A-2D} g_O\Big(\frac{1}{2}+A+D\Big) Q_4\bigg) + O(X^{1/2+\varepsilon}), \tag{5-5}$$

with

$$Q_1 = \mathcal{A}(A, D; D)\Big(\frac{\zeta'}{\zeta}(1+A+D) - \frac{\zeta'}{\zeta}(1+2D)\Big) + \mathcal{A}'(A, D; D),$$

$$Q_2 = \mathcal{A}(-A, D; D)\Big(\frac{\zeta'}{\zeta}(1-A+D) - \frac{\zeta'}{\zeta}(1+2D)\Big) + \mathcal{A}'(-A, D; D),$$

$$Q_3 = -\mathcal{A}(A, -D; D) \frac{\zeta(1+A-D)\zeta(1+2D)}{\zeta(1+A+D)},$$

$$Q_4 = -\mathcal{A}(-A, -D; D) \frac{\zeta(1-A-D)\zeta(1+2D)}{\zeta(1-A+D)}.$$

Moreover, we notice that if the shifts are of order $\asymp (\log X)^{-1}$, then we can approximate (5-5), getting

$$\sum_{d\le X}\frac{L'_\Delta}{L_\Delta}\Big(\frac{1}{2}+D,\chi_d\Big)L\Big(\frac{1}{2}+A,\chi_d\Big)$$

$$=\mathcal{A}X^*\bigg(\frac{A-D}{(A+D)2D}+X^{-2A}\frac{-A-D}{(-A+D)2D}+X^{-2D}\frac{-A-D}{(A-D)2D}+X^{-2A-2D}\frac{A-D}{(-A-D)2D}\bigg)+O(1),\quad (5\text{-}6)$$

since $\mathcal{A}(\pm A,\pm D,D)=\mathcal{A}+O(1/\log X)$ and $\zeta(1+z)=1/z+O(1)$ as $z\to 0$.

5C. ***Conjecture 5.1 in the case $K=2$, $Q=0$.*** We now analyze closely the second moment of $L_\Delta\big(\frac{1}{2},\chi_d\big)$; we take two complex shifts A, B such that $A, B\asymp(\log X)^{-1}$, and we look at

$$\frac{1}{X^*}\sum_{d\le X}L_\Delta\Big(\frac{1}{2}+A,\chi_d\Big)L_\Delta\Big(\frac{1}{2}+B,\chi_d\Big).$$

By Conjecture 5.1, ignoring the negligible error term $O(X^{1/2+\varepsilon})$, the above is

$$f(A,B)+X^{-2A}f(-A,B)+X^{-2B}f(A,-B)+X^{-2A-2B}f(-A-B),\quad (5\text{-}7)$$

with

$$f(A,B):=\zeta(1+A+B)\prod_p\bigg(1-\frac{1}{p^{1+A+B}}\bigg)\bigg(1+\frac{p}{p+1}\sum_{\substack{m+n>0\\ \text{even}}}\frac{\tau^*(p^m)\tau^*(p^n)}{p^{(1/2+A)m+(1/2+B)n}}\bigg).$$

Since $A, B\asymp(\log X)^{-1}$, we set $a=A\log X\asymp 1$ and $b=B\log X\asymp 1$, so that (5-7) becomes

$$\mathcal{B}\log X\bigg(\frac{1}{a+b}+\frac{e^{-2a}}{-a+b}+\frac{e^{-2b}}{a-b}+\frac{e^{-2a-2b}}{-a-b}\bigg)\bigg(1+O\Big(\frac{1}{\log X}\Big)\bigg),\quad (5\text{-}8)$$

where

$$\mathcal{B}:=\prod_p\bigg(1-\frac{1}{p}\bigg)\bigg(1+\frac{p}{p+1}\sum_{\substack{m+n>0\\ \text{even}}}\frac{\tau^*(p^m)\tau^*(p^n)}{p^{(m+n)/2}}\bigg).\quad (5\text{-}9)$$

The expression in (5-8) is regular at $a=0$ and $b=0$, since the limit of the first parentheses as $a,b\to 0$ equals 4. Therefore, we finally get

$$\frac{1}{X^*}\sum_{d\le X}L_\Delta\Big(\frac{1}{2},\chi_d\Big)^2\sim 4\mathcal{B}\log X.\quad (5\text{-}10)$$

5D. ***Conjecture 5.1 in the case $K=3$, $Q=1$.*** Finally, we look at

$$\sum_{d\le X}\frac{L_\Delta(1/2+A,\chi_d)L_\Delta(1/2+B,\chi_d)L_\Delta(1/2+C,\chi_d)}{L_\Delta(1/2+D,\chi_d)},\quad (5\text{-}11)$$

with A, B, C and D as Conjecture 5.1 prescribes. The first of the eight terms given by the recipe is

$$\sum_{d\le X}\frac{\zeta(1+2D)\zeta(1+A+B)\zeta(1+A+C)\zeta(1+B+C)}{\zeta(1+A+D)\zeta(1+B+D)\zeta(1+C+D)}\mathcal{A}(A,B,C;D),$$

where

$$\mathcal{A}(A,B,C;D)=\prod_p \frac{(1-1/p^{1+2D})(1-1/p^{1+A+B})(1-1/p^{1+A+C})(1-1/p^{1+B+C})}{(1-1/p^{1+A+D})(1-1/p^{1+B+D})(1-1/p^{1+C+D})} \times\biggl(1+\frac{p}{p+1}\sum_{\substack{0<a+b+c+d\\ \text{even}}}\frac{\tau^*(p^a)\tau^*(p^b)\tau^*(p^c)\mu_\Delta(p^d)}{p^{(1/2+A)a+(1/2+B)b+(1/2+C)c+(1/2+D)d}}\biggr)$$

is the arithmetical coefficient, absolutely convergent in small disks around 0, such that $\mathcal{A}(0,0,0;0)=\mathcal{B}$. As in all the previous examples, this gives a formula for (5-11) with all the main terms and error $O(X^{1/2+\varepsilon})$ and differentiating this formula with respect to C at $C=D$, we get

$$\begin{aligned}\sum_{d\le X}\frac{L'_\Delta}{L_\Delta}\Bigl(\frac12+D,\chi_d\Bigr)L_\Delta\Bigl(\frac12+A,\chi_d\Bigr)L_\Delta\Bigl(\frac12+B,\chi_d\Bigr)\\ =\sum_{d\le X}\biggl(R_1+\Bigl(\frac{d}{2\pi}\Bigr)^{-2A}g_O\Bigl(\frac12+A\Bigr)R_2+\Bigl(\frac{d}{2\pi}\Bigr)^{-2B}g_O\Bigl(\frac12+B\Bigr)R_3\\ +\Bigl(\frac{d}{2\pi}\Bigr)^{-2D}g_O\Bigl(\frac12+D\Bigr)R_4+\Bigl(\frac{d}{2\pi}\Bigr)^{-2A-2B}g_O\Bigl(\frac12+A+B\Bigr)R_5\\ +\Bigl(\frac{d}{2\pi}\Bigr)^{-2A-2D}g_O\Bigl(\frac12+A+D\Bigr)R_6+\Bigl(\frac{d}{2\pi}\Bigr)^{-2B-2D}g_O\Bigl(\frac12+B+D\Bigr)R_7\\ +\Bigl(\frac{d}{2\pi}\Bigr)^{-2A-2B-2D}g_O\Bigl(\frac12+A+B+D\Bigr)R_8\biggr)+O(X^{1/2+\varepsilon}),\end{aligned}\tag{5-12}$$

with

$$\begin{aligned}R_1&=R_1(A,B,D)\\ &=\mathcal{A}(A,B,D;D)\zeta(1+A+B)\\ &\quad\times\Bigl(\frac{\zeta'}{\zeta}(1+A+D)+\frac{\zeta'}{\zeta}(1+B+D)-\frac{\zeta'}{\zeta}(1+2D)\Bigr)+\zeta(1+A+B)\mathcal{A}'(A,B,D;D),\\ R_2&=R_1(-A,B,D),\\ R_3&=R_1(A,-B,D),\\ R_4&=R_4(A,B,D)=-\frac{\zeta(1+2D)\zeta(1+A+B)\zeta(1+A-D)\zeta(1+B-D)}{\zeta(1+A+D)\zeta(1+B+D)}\mathcal{A}(A,B,-D;D),\\ R_5&=R_1(-A,-B,D),\\ R_6&=R_4(-A,B,D),\\ R_7&=R_4(A,-B,D),\\ R_8&=R_4(-A,-B,D).\end{aligned}\tag{5-13}$$

If $A,B,D\asymp(\log X)^{-1}$, the above formula simplifies a lot, since

$$R_1=\frac{AB-AD-BD-3D^2}{2D(A+B)(A+D)(B+D)}\mathcal{B}+O\biggl(\frac{(\log X)^4}{(\log X)^3}\biggr)=:f(A,B,D)+O(\log X)$$

and

$$R_4=\frac{(A+D)(B+D)}{(-2D)(A+B)(A-D)(B-D)}\mathcal{B}+\biggl(\frac{(\log X)^4}{(\log X)^3}\biggr)=:g(A,B,D)+O(\log X),$$

giving

$$\sum_{d\le X}\frac{L'_\Delta}{L_\Delta}\Big(\frac{1}{2}+D,\chi_d\Big)L_\Delta\Big(\frac{1}{2}+A,\chi_d\Big)L_\Delta\Big(\frac{1}{2}+B,\chi_d\Big)$$

$$\begin{aligned}&=\mathcal{B}X^*\big(f(A,B,D)+X^{-2A}f(-A,B,D)+X^{-2B}f(A,-B,D)\\&\qquad+X^{-2D}g(A,B,D)+X^{-2A-2B}f(-A,-B,D)+X^{-2A-2D}g(-A,B,D)\\&\qquad+X^{-2B-2D}g(A,-B,D)+X^{-2A-2B-2D}g(-A,-B,D)\big)+O(\log X).\end{aligned}\tag{5-14}$$

Analogous formulae can be obtained in the cases $K=4$, $Q=1$ and $K=5$, $Q=1$. Again, with the same technique, one can get formulae also in the case $K>5$, $Q=1$.

5E. ***The weighted one-level density for $\{L_\Delta(\frac{1}{2},\chi_d)\}_d$.*** In analogy to what we did in Section 4C, we now compute the weighted one-level density for the orthogonal family of quadratic twists of L_Δ. We assume the Riemann hypothesis for the L-functions we are considering, and we denote with $\gamma_{\Delta,d}$ the imaginary part of a generic zero of $L_\Delta(s,\chi_d)$. In the classical case, assuming the ratio conjecture, Conrey and Snaith [2007] proved that

$$\lim_{X\to\infty}\frac{1}{X^*}\sum_{d\le X}\sum_{\gamma_{\Delta,d}}f\Big(\frac{\log X}{\pi}\gamma_{\Delta,d}\Big)=\int_{-\infty}^{+\infty}f(x)\Big(1+\frac{\sin(2\pi x)}{2\pi x}\Big)\,dx\tag{5-15}$$

for any test function f, satisfying the usual properties as in Theorem 2.5. We now use the formulae of the previous section to derive the weighted one-level density; we let

$$\mathcal{D}_1^{L_\Delta,\chi}(f):=\frac{1}{\sum_{d\le X}L_\Delta(1/2,\chi_d)}\sum_{d\le X}\sum_{\gamma_{\Delta,d}}f\Big(\frac{\log X}{\pi}\gamma_{\Delta,d}\Big)L_\Delta\Big(\frac{1}{2},\chi_d\Big),\tag{5-16}$$

and we prove the following result:

Proposition 5.1. *Assume GRH and Conjecture 5.1 for $K=2$, $Q=1$. For any function f holomorphic in the strip $\Im(z)<2$, even, real on the real line and such that $f(x)\ll 1/(1+x^2)$ as $x\to\infty$, we have*

$$\mathcal{D}_1^{L_\Delta,\chi}(f)=\int_{-\infty}^{+\infty}f(x)W^1_{\mathrm{SO}^+}(x)\,dx+O\Big(\frac{1}{\log X}\Big)$$

as $X\to\infty$, where

$$W^1_{\mathrm{SO}^+}(x):=1-\frac{\sin(2\pi x)}{2\pi x}.\tag{5-17}$$

Proof. The strategy of the proof is the same as in the unitary and symplectic cases, thus we will just sketch how the proof works, highlighting the differences with the other cases. For $a\asymp 1/\log X$ a real parameter, we consider the quantity

$$\frac{1}{2\mathcal{A}X^*}\sum_{d\le X}\sum_{\gamma_{\Delta,d}}f\Big(\frac{\log X}{\pi}\gamma_{\Delta,d}\Big)L_\Delta\Big(\frac{1}{2}+\frac{a}{\log X},\chi_d\Big),$$

which can be written as ($\delta\asymp(\log X)^{-1}$)

$$\frac{\log(X^2)}{2\pi}\int_{-\infty}^{+\infty}f\Big(\frac{\log X}{\pi}y\Big)\,dy+2\frac{1}{2\pi}\int_{-\infty}^{+\infty}f\Big(\frac{\log X}{\pi}y\Big)\frac{1}{2\mathcal{A}X^*}\sum_{d\le X}\frac{L'_\Delta}{L_\Delta}\Big(\frac{1}{2}+\delta+iy\Big)L_\Delta\Big(\frac{1}{2}+\frac{a}{\log X},\chi_d\Big)\,dy,$$

with an error $O((\log X)^{-1})$, by using the Cauchy's theorem and the functional equation

$$\frac{L'_\Delta}{L_\Delta}(1-s,\chi_d)=\frac{Y'_\Delta}{Y_\Delta}(s,\chi_d)-\frac{L'_\Delta}{L_\Delta}(s,\chi_d),$$

with $(Y'_\Delta/Y_\Delta)(s,\chi_d)=-\log d^2+O(1)$ (note that the square here is due to the conductor of $L_\Delta(s,\chi_d)$, which is $d^2/(4\pi^2)$). With the change of variable $(\log X/\pi)y=x$ the above equals

$$\int_{-\infty}^{+\infty} f(x)\biggl(1+\frac{1}{2\mathcal{A}X^*\log X}\sum_{d\le X}\frac{L'_\Delta}{L_\Delta}\Bigl(\frac{1}{2}+\delta+\frac{\pi i x}{\log X}\Bigr)L_\Delta(\tfrac{1}{2}+\tfrac{a}{\log X},\chi_d)\biggr)\,dx.$$

Now we use the assumption of the ratio conjecture in the form of (5-5) and (5-6) to evaluate the sum over d, getting

$$\int_{-\infty}^{+\infty} f(x)\biggl(1+\frac{1}{2\log X}h_X\Bigl(\frac{a}{\log X},\frac{\pi i x}{\log X}\Bigr)\biggr)\,dx+O\Bigl(\frac{1}{\log X}\Bigr),$$

with

$$h_X(\alpha,w):=\frac{\alpha-w}{(\alpha+w)2w}+X^{-2\alpha}\frac{-\alpha-w}{(-\alpha+w)2w}+X^{-2w}\frac{-\alpha-w}{(\alpha-w)2w}+X^{-2\alpha-2w}\frac{\alpha-w}{(-\alpha-w)2w}.$$

Letting $a\to 0$, and since $h_X(0,w)=(X^{-2w}-1)/w$, we get

$$\int_{-\infty}^{+\infty} f(x)\biggl(1+\frac{e^{-2\pi i x}-1}{2\pi i x}\biggr)\,dx+O\Bigl(\frac{1}{\log X}\Bigr).$$

Putting all together, since f is even, we finally have

$$\mathcal{D}_1^{\boldsymbol{L}_{\Delta,\chi}}(f)=\int_{-\infty}^{+\infty} f(x)\biggl(1-\frac{\sin(2\pi x)}{2\pi x}\biggr)\,dx+O\Bigl(\frac{1}{\log X}\Bigr).\qquad\square$$

Similarly we compute the analogue of (5-15), tilting by the second power of $L_\Delta\bigl(\frac{1}{2},\chi_d\bigr)$, i.e.,

$$\mathcal{D}_2^{\boldsymbol{L}_{\Delta,\chi}}(f):=\frac{1}{\sum_{d\le X}L_\Delta(1/2,\chi_d)^2}\sum_{d\le X}\sum_{\gamma_{\Delta,d}} f\biggl(\frac{\log X}{\pi}\gamma_{\Delta,d}\biggr)L_\Delta\Bigl(\frac{1}{2},\chi_d\Bigr)^2 \tag{5-18}$$

under the assumption of Conjecture 5.1, in the case $K=3$, $Q=1$. This is achieved in the following proposition:

Proposition 5.2. *Assume GRH and Conjecture 5.1 for $K=3$, $Q=1$. For any function f holomorphic in the strip $\Im(z)<2$, even, real on the real line and such that $f(x)\ll 1/(1+x^2)$ as $x\to\infty$, we have*

$$\mathcal{D}_2^{\boldsymbol{L}_{\Delta,\chi}}(f)=\int_{-\infty}^{+\infty} f(x)W^2_{\mathrm{SO}^+}(x)\,dx+O\Bigl(\frac{1}{\log X}\Bigr)$$

as $X\to\infty$, where

$$W^2_{\mathrm{SO}^+}(x):=1+\frac{\sin(2\pi x)}{\pi x}-\frac{2\sin^2(\pi x)}{(\pi x)^2}.$$

Proof. Again we start with

$$\frac{1}{4\mathcal{B}X^*\log X}\sum_{d\le X}\sum_{\gamma_{\Delta,d}} f\biggl(\frac{\log X}{\pi}\gamma_{\Delta,d}\biggr)L_\Delta\Bigl(\frac{1}{2}+\frac{a}{\log X},\chi_d\Bigr)L_\Delta\Bigl(\frac{1}{2}+\frac{b}{\log X},\chi_d\Bigr),$$

and with the usual machinery, we write it as

$$\int_{-\infty}^{+\infty} f(x)\Big(1+\frac{1}{4\mathcal{B}X^*(\log X)^2}\mathcal{I}\Big(\frac{a}{\log X},\frac{b}{\log X},\delta+\frac{\pi i x}{\log X}\Big)\Big)\,dx + O\Big(\frac{1}{\log X}\Big),$$

with

$$\mathcal{I}(\alpha,\beta,w) := \sum_{d\le X}\frac{L'_\Delta}{L_\Delta}\Big(\frac{1}{2}+w,\chi_d\Big)L_\Delta\Big(\frac{1}{2}+\alpha,\chi_d\Big)L_\Delta\Big(\frac{1}{2}+\beta,\chi_d\Big).$$

Thanks to the assumption of the ratio conjecture (see (5-12) and (5-14)) we are able to evaluate asymptotically the above sum, which is regular at $\alpha,\beta,\delta=0$. More specifically, we have that

$$\lim_{\substack{a\to 0\\ b\to 0\\ \delta\to 0}} \mathcal{I}\Big(\frac{a}{\log X},\frac{b}{\log X},\delta+\frac{y}{\log X}\Big) = \mathcal{B}X^*(\log X)^2 h(y) + O(\log X),$$

with

$$h(y) := \frac{-2ye^{-2y}-6y-4e^{-2y}+4}{y^2}.$$

Then we get

$$\begin{aligned}\mathcal{D}_2^{\boldsymbol{L}_{\Delta,\chi}}(f) &= \int_{-\infty}^{+\infty} f(x)\Big(1+\frac{1}{4}h(\pi i x)\Big)\,dx + O\Big(\frac{1}{\log X}\Big)\\ &= \int_{-\infty}^{+\infty} f(x)\Big(1+\frac{\sin(2\pi x)}{2\pi x}-\frac{\sin^2(\pi x)}{(\pi x)^2}\Big)\,dx + O\Big(\frac{1}{\log X}\Big),\end{aligned}$$

since f is even. □

We go on and define

$$\mathcal{D}_3^{\boldsymbol{L}_{\Delta,\chi}}(f) := \frac{1}{\sum_{d\le X} L_\Delta(1/2,\chi_d)^3}\sum_{d\le X}\sum_{\gamma_{\Delta,d}} f\Big(\frac{\log X}{\pi}\gamma_{\Delta,d}\Big)L_\Delta\Big(\frac{1}{2},\chi_d\Big)^3,$$

analyzing the third-moment case.

Proposition 5.3. *Assume GRH and Conjecture 5.1 for $K=4$, $Q=1$. For any function f holomorphic in the strip $\Im(z)<2$, even, real on the real line and such that $f(x)\ll 1/(1+x^2)$ as $x\to\infty$, we have*

$$\mathcal{D}_3^{\boldsymbol{L}_{\Delta,\chi}}(f) = \int_{-\infty}^{+\infty} f(x)W^3_{\mathrm{SO}^+}(x)\,dx + O\Big(\frac{1}{\log X}\Big)$$

as $X\to\infty$, where

$$W^3_{\mathrm{SO}^+}(x) := 1-\frac{\sin(2\pi x)}{2\pi x}-\frac{24(1-\sin^2(\pi x))}{(2\pi x)^2}+\frac{48\sin(2\pi x)}{(2\pi x)^3}-\frac{96\sin^2(\pi x)}{(2\pi x)^4}.$$

Proof. We introduce the usual real parameters α,β,ν of size $\asymp 1/\log X$, we let

$$\boldsymbol{L}_\Delta^{\alpha,\beta,\nu}\big(\tfrac{1}{2},\chi_d\big) := L_\Delta\big(\tfrac{1}{2}+\alpha,\chi_d\big)L_\Delta\big(\tfrac{1}{2}+\beta,\chi_d\big)L_\Delta\big(\tfrac{1}{2}+\nu,\chi_d\big),$$

and we consider

$$\frac{1}{\sum_{d\le X} L_\Delta(1/2,\chi_d)^3}\sum_{d\le X}\sum_{\gamma_d} f\Big(\frac{\log X}{2\pi}\gamma_d\Big)\boldsymbol{L}_\Delta^{\alpha,\beta,\nu}\Big(\frac{1}{2},\chi_d\Big).$$

With the usual strategy we get that the above equals

$$\int_{-\infty}^{+\infty} f(x)\Big(1+\frac{1}{\sum_{d\le X} L(1/2,\chi_d)^3}\sum_{d\le X}\frac{L'_\Delta}{L_\Delta}\Big(\frac{1}{2}+\delta+\frac{2\pi i x}{\log X},\chi_d\Big)\boldsymbol{L}_\Delta^{\alpha,\beta,\nu}\big(\tfrac{1}{2},\chi_d\big)\Big)\,dx$$

up to an error $O(1/\log X)$, with $\delta \asymp 1/\log X$. We evaluate asymptotically the remaining sum over d thanks to Conjecture 5.1 for $K=4$, $Q=1$), using Sage to carry out the computations. Doing so, letting $\alpha,\beta,\nu\to 0$, we obtain

$$\mathcal{D}_3^{\boldsymbol{L}_{\Delta,\chi}}(f)=\int_{-\infty}^{+\infty} f(x)\Big(1+\frac{1}{2}h(\pi i x)\Big)\,dx+O\Big(\frac{1}{\log X}\Big),$$

with

$$h(y):=\frac{-5y^3+6y^2-6+e^{-2y}(y^3+6y^2+12y+6)}{y^4}.$$

The claim follows, since f is even. □

Finally, in the following result, we study the case $k=4$, given by

$$\mathcal{D}_4^{\boldsymbol{L}_{\Delta,\chi}}(f):=\frac{1}{\sum_{d\le X} L_\Delta(1/2,\chi_d)^4}\sum_{d\le X}\sum_{\gamma_{\Delta,d}} f\Big(\frac{\log X}{\pi}\gamma_{\Delta,d}\Big)L_\Delta\Big(\frac{1}{2},\chi_d\Big)^4:$$

Proposition 5.4. *Assume GRH and Conjecture 5.1 for $K=5$, $Q=1$. For any function f holomorphic in the strip $\Im(z)<2$, even, real on the real line and such that $f(x)\ll 1/(1+x^2)$ as $x\to\infty$, we have*

$$\mathcal{D}_4^{\boldsymbol{L}_{\Delta,\chi}}(f)=\int_{-\infty}^{+\infty} f(x)W^4_{\mathrm{SO}^+}(x)\,dx+O\Big(\frac{1}{\log X}\Big)$$

as $X\to\infty$, where

$$W^4_{\mathrm{SO}^+}(x) := 1+\frac{\sin(2\pi x)}{2\pi x}-\frac{12\sin^2(\pi x)}{(\pi x)^2}-\frac{240\sin(2\pi x)}{(2\pi x)^3}-\frac{15(6-10\sin^2(\pi x))}{(\pi x)^4}+\frac{2880\sin(2\pi x)}{(2\pi x)^5}-\frac{90\sin^2(\pi x)}{(\pi x)^6}.$$

Proof. As usual, if we set

$$\boldsymbol{L}_\Delta^{\alpha,\beta,\nu,\eta}\big(\tfrac{1}{2},\chi_d\big):=L_\Delta\big(\tfrac{1}{2}+\alpha,\chi_d\big)L_\Delta\big(\tfrac{1}{2}+\beta,\chi_d\big)L_\Delta\big(\tfrac{1}{2}+\nu,\chi_d\big)L_\Delta\big(\tfrac{1}{2}+\eta,\chi_d\big),$$

then we express $\mathcal{D}_4^{\boldsymbol{L}_{\Delta,\chi}}(f)$ as the limit for $\alpha,\beta,\nu,\eta\to 0$ of

$$\int_{-\infty}^{+\infty} f(x)\Big(1+\frac{1}{\sum_{d\le X} L(1/2,\chi_d)^4}\sum_{d\le X}\frac{L'_\Delta}{L_\Delta}\Big(\frac{1}{2}+\delta+\frac{2\pi i x}{\log X},\chi_d\Big)\boldsymbol{L}_\Delta^{\alpha,\beta,\nu,\eta}\Big(\frac{1}{2},\chi_d\Big)\Big)\,dx$$

up to an error $O(1/\log X)$, with $\delta\asymp 1/\log X$. The above can be evaluated asymptotically (again Sage is of help in carrying out the computation), and we get

$$\mathcal{D}_4^{\boldsymbol{L}_{\Delta,\chi}}(f)=\int_{-\infty}^{+\infty} f(x)\Big(1+\frac{1}{2}h(\pi i x)\Big)\,dx+O\Big(\frac{1}{\log X}\Big),$$

with

$$h(y):=\frac{-7y^5+12y^4-30y^2+90}{y^6}-\frac{e^{-2y}(y^5+12y^4+60y^3+150y^2+180y+90)}{y^6}.$$

Since f is even, the claim follows. □

Theorem 2.5 follows by Propositions 5.1–5.4.

5F. ***Proof of Theorem 2.6.*** Citing [Conrey et al. 2008, Theorem 4.3] instead of assuming Conjecture 5.1, the same strategy gives an unconditional proof in random matrix theory.

6. Proof of Theorem 2.7

By Fourier inversion, we have that

$$W^k_{\mathrm{USp}}(x) = \int_{-\infty}^{+\infty} \widehat{W}^k_{\mathrm{USp}}(y) e^{2\pi i xy}\,dy = \sum_{n=0}^{\infty}\biggl(\frac{(2\pi i)^n}{n!}\int_{-\infty}^{+\infty}\widehat{W}^k_{\mathrm{USp}}(y)y^n\,dy\biggr)x^n.$$

Moreover, since $\widehat{W}^k_{\mathrm{USp}}$ is even, then $\int_{-\infty}^{+\infty}\widehat{W}^k_{\mathrm{USp}}(y)y^n\,dy = 0$ if n is odd. Hence, by definition of $\widehat{W}^k_{\mathrm{USp}}$,

$$W^k_{\mathrm{USp}}(x) = \sum_{m=0}^{\infty}\beta_{m,k}x^{2m},$$

with

$$\beta_{m,k} = \frac{(2\pi i)^{2m}}{(2m)!}\int_{-\infty}^{+\infty}\biggl(\delta_0(y)+\chi_{[-1,1]}(y)\biggl(-\frac{2k+1}{2}-k(k+1)\sum_{j=1}^{k}(-1)^j c_{j,k}\frac{|y|^{2j-1}}{2j-1}\biggr)\biggr)y^{2m}\,dy,$$

where $\chi_{[-1,1]}$ denotes the indicator function of the interval $[-1,1]$. By computing the integral, being $\int_{-1}^{1}y^{2m}\,dy = 2/(2m+1)$ and $\int_{-1}^{1}y^{2m}|y|^{2j-1}\,dy = 1/(m+j)$, the above yields

$$\beta_{m,k} := \delta_0(m) - \frac{(2\pi i)^{2m}}{(2m)!}\biggl[\frac{2k+1}{2m+1}+k(k+1)\sum_{j=1}^{k}\frac{(-1)^j}{(2j-1)(j+m)}c_{j,k}\biggr].$$

Since

$$c_{j,k} = \frac{1}{j}\binom{k-1}{j-1}\binom{k+j}{j-1} = \frac{j}{k(k+1)}\binom{k+j}{j}\binom{k}{j},$$

we get

$$\beta_{m,k} = \delta_0(m) - \frac{(2\pi i)^{2m}}{(2m)!}\biggl[\frac{2k+1}{2m+1}+S_m(1)\biggr], \tag{6-1}$$

where

$$S_{m,k}(x) := \sum_{j=1}^{\infty}\frac{(-1)^j j}{(2j-1)(j+m)}\binom{k+j}{j}\binom{k}{j}x^j.$$

Now we write the factors in the above sum in terms of the Pochhammer symbol, defined as

$$(a)_0 := 1 \quad\text{and}\quad (a)_n := a(a+1)(a+2)\cdots(a+n-1) \quad\text{for } n \geq 1;$$

namely,

$$\binom{k+j}{j} = \frac{(k+1)_j}{j!},$$
$$(-1)^j\binom{k}{j} = \frac{(-k)_j}{(1)j},$$
$$\frac{j}{(2j-1)(j+m)} = \frac{1}{2m+1}\left(\frac{1}{2j-1}+\frac{m}{j+m}\right) = \frac{1}{2m+1}\left(-\frac{(-1/2)_j}{(1/2)_j}+\frac{(m)_j}{(m+1)_j}\right),$$

so that we have

$$S_{m,k}(x) = \frac{1}{2m+1}\sum_{j=1}^{\infty}\left(-\frac{(-1/2)_j}{(1/2)_j}+\frac{(m)_j}{(m+1)_j}\right)\frac{(-k)_j(k+1)_j}{(1)_j}\frac{x^j}{j!}. \tag{6-2}$$

Reparametrizing the sum and using $(a)_{j+1} = a(a+1)_j$, this gives

$$S_{m,k}(x) = S^1_{m,k}(x) + S^2_{m,k}(x),$$

where

$$S^1_{m,k}(x) := \frac{-k(k+1)x}{2m+1}\sum_{j=0}^{\infty}\frac{(1/2)_j(-k+1)_j(k+2)_j}{(3/2)_j(2)_j}\frac{x^j}{j!}\frac{1}{j+1}$$

and

$$S^2_{m,k}(x) := \frac{-mk(k+1)x}{(2m+1)(m+1)}\sum_{j=0}^{\infty}\frac{(m+1)_j(-k+1)_j(k+2)_j}{(m+2)_j(2)_j}\frac{x^j}{j!}\frac{1}{j+1}.$$

By writing

$$\frac{1}{j+1} = 2\left(1-\frac{2j+1}{2j+2}\right) \quad \text{and} \quad \frac{(1/2)_j}{(3/2)_j} = \frac{1}{2j+1},$$

we get

$$S^1_{m,k}(x) = -\frac{2xk(k+1)}{m+1}\,{}_3F_2\left[\begin{matrix}1-k,\ k+2,\ 1/2\\ 3/2,\ 2\end{matrix};x\right] - \frac{{}_2F_1\left[\begin{smallmatrix}-k,\ k+1\\ 1\end{smallmatrix};x\right]-1}{2m+1},$$

where ${}_pF_q$ denotes the generalized hypergeometric function, defined as

$${}_pF_q\left[\begin{matrix}a_1,\ \ldots,\ a_p\\ b_1,\ \ldots,\ b_q\end{matrix};x\right] := \sum_{n=0}^{\infty}\frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\frac{x^n}{n!}.$$

Similarly, since

$$-\frac{1}{j+1} = \frac{1}{m} - \frac{j+m+1}{(j+1)m},$$

we have

$$S^2_{m,k}(x) = \frac{xk(k+1)}{(2m+1)(m+1)}\,{}_3F_2\left[\begin{matrix}1-k,\ k+2,\ m+1\\ m+2,\ 2\end{matrix};x\right] + \frac{{}_2F_1\left[\begin{smallmatrix}-k,\ k+1\\ 1\end{smallmatrix};x\right]-1}{2m+1}.$$

Therefore, substituting in (6-2) yields

$$S_{m,k}(x) = -\frac{2xk(k+1)}{m+1}\,{}_3F_2\left[\begin{matrix}1-k,\ k+2,\ 1/2\\ 3/2,\ 2\end{matrix};x\right] + \frac{xk(k+1)}{(2m+1)(m+1)}\,{}_3F_2\left[\begin{matrix}1-k,\ k+2,\ m+1\\ m+2,\ 2\end{matrix};x\right]. \tag{6-3}$$

Plugging (6-3) into (6-1), we obtain

$$\beta_{m,k} = \delta_0(m) - \frac{(-1)^m (2\pi)^{2m}}{(2m+1)!}\biggl(2k+1-2k(k+1)\,{}_3F_2\biggl[\begin{matrix}1-k,\ k+2,\ 1/2\\ 3/2,\ 2\end{matrix};1\biggr] + \frac{k(k+1)}{(m+1)}\,{}_3F_2\biggl[\begin{matrix}1-k,\ k+2,\ m+1\\ m+2,\ 2\end{matrix};1\biggr]\biggr). \tag{6-4}$$

Now we need a few lemmas, in order to be able to compute the remaining hypergeometric functions.

Lemma 6.1. *For any $k \in \mathbb{N}$, we have*

$$ {}_3F_2\biggl[\begin{matrix}1-k,\ k+2,\ 1/2\\ 3/2,\ 2\end{matrix};1\biggr] = \begin{cases} \dfrac{1}{k+1}, & \text{if } k \text{ even},\\ \dfrac{1}{k}, & \text{if } k \text{ odd}.\end{cases}$$

Proof. We recall the reduction formula for the generalized hypergeometric function (see, e.g., [Gottschalk and Maslen 1988, Equation (17), when $n = 1$],), being

$$ {}_{A+1}F_{B+1}\biggl[\begin{matrix}a_1,\ \dots,\ a_A,\ c+n\\ b_1,\ \dots,\ b_B,\ c\end{matrix};x\biggr] = \sum_{j=0}^{n}\binom{n}{j}\frac{1}{(c)_j}\frac{\prod_{i=1}^{A}(a_i)_j}{\prod_{i=1}^{B}(b_i)_j}\,{}_AF_B\biggl[\begin{matrix}a_1+j,\ \dots,\ a_A+j\\ b_1+j,\ \dots,\ b_B+j\end{matrix};x\biggr]$$

for any A, B positive integers, $n \in \mathbb{N}$. The left hand side can be then written as

$$\begin{aligned} {}_3F_2\biggl[\begin{matrix}1-k,\ 1/2,\ k+2\\ 3/2,\ 2\end{matrix};1\biggr] &= \sum_{j=0}^{k}\binom{k}{j}\frac{1}{(2)_j}\frac{(1-k)_j(1/2)_j}{(3/2)_j}\,{}_2F_1\biggl[\begin{matrix}1-k+j,\ 1/2+j\\ 3/2+j\end{matrix};1\biggr]\\ &= \sum_{j=0}^{k-1}\binom{k}{j}\frac{1}{(2)_j}\frac{(1-k)_j(1/2)_j}{(3/2)_j}\,{}_2F_1\biggl[\begin{matrix}1-k+j,\ 1/2+j\\ 3/2+j\end{matrix};1\biggr], \end{aligned} \tag{6-5}$$

as $(1-k)_k = 0$. The remaining hypergeometric function can be computed by applying Gauss' summation theorem (see, e.g., [Koepf 2014, Equation (3.1)]), i.e., the formula

$$ {}_2F_1\biggl[\begin{matrix}a,\ b\\ c\end{matrix};1\biggr] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \qquad \Re(c) > \Re(a+b).$$

We recall that if $a = -n$, $n \in \mathbb{N}$, this is the Chu–Vandermonde identity (see, again, [Koepf 2014, immediately below Equation (3.1)])

$$ {}_2F_1\biggl[\begin{matrix}-n,\ b\\ c\end{matrix};1\biggr] = \frac{(c-b)_n}{(c)_n}.$$

This yields

$$ {}_2F_1\biggl[\begin{matrix}1-k+j,\ 1/2+j\\ 3/2+j\end{matrix};1\biggr] = \frac{(k-j-1)!(j+1/2)!}{(k-1/2)!} \tag{6-6}$$

for $k > j$. Plugging (6-6) into (6-5), we get

$$\begin{aligned} {}_3F_2\left[\begin{matrix} 1-k,\ 1/2,\ k+2 \\ 3/2,\ 2 \end{matrix}; 1\right] &= \frac{(k-1)!}{2(k-1/2)!}\sum_{j=0}^{k-1}\binom{k}{j}(-1)^j\frac{(j-1/2)!}{(j+1)!} \\ &= \frac{(k-1)!}{2(k-1/2)!}\sum_{j=0}^{k}\binom{k}{j}(-1)^j\frac{(j-1/2)!}{(j+1)!} - \frac{(-1)^k}{2k(k+1)}. \end{aligned} \tag{6-7}$$

Moreover, since $\left(\frac{1}{2}\right)_j = (1/\sqrt{\pi})(j-\frac{1}{2})!$, $(-1)^j\binom{k}{j} = (-k)_j/j!$ and $(2)_j = (j+1)!$, we have

$$\sum_{j=0}^{k}\binom{k}{j}(-1)^j\frac{(j-1/2)!}{(j+1)!} = \sqrt{\pi}\,{}_2F_1\left[\begin{matrix} -k,\ 1/2 \\ 2 \end{matrix}; 1\right] = 2\frac{(k+1/2)!}{(k+1)!}$$

by applying the Chu–Vandermonde identity. Putting this into (6-7), we finally get

$${}_3F_2\left[\begin{matrix} 1-k,\ 1/2,\ k+2 \\ 3/2,\ 2 \end{matrix}; 1\right] = \frac{k+1/2}{k(k+1)} - \frac{(-1)^k}{2k(k+1)},$$

and the claim follows. □

By using Lemma 6.1, (6-4) becomes

$$\beta_{m,k} = \delta_0(m) - \frac{(-1)^m(2\pi)^{2m}}{(2m+1)!}\left((-1)^k + \frac{k(k+1)}{(m+1)}{}_3F_2\left[\begin{matrix} 1-k,\ k+2,\ m+1 \\ m+2,\ 2 \end{matrix}; 1\right]\right). \tag{6-8}$$

The coefficient $\beta_{0,k}$ can be then computed, thanks to the following lemma:

Lemma 6.2. *For any $k \in \mathbb{N}$ we have*

$${}_3F_2\left[\begin{matrix} 1-k,\ k+2,\ 1 \\ 2,\ 2 \end{matrix}; 1\right] = \begin{cases} 0, & \text{if } k \text{ even}, \\ \dfrac{2}{k(k+1)}, & \text{if } k \text{ odd}. \end{cases}$$

Proof. By definition, we have

$${}_3F_2\left[\begin{matrix} 1-k,\ k+2,\ 1 \\ 2,\ 2 \end{matrix}; 1\right] = \sum_{j=0}^{\infty}\frac{(1-k)_j(k+2)_j(1)_j}{(2)_j(2)_j}\frac{1}{j!} = -\frac{1}{k(k+1)}\sum_{j=0}^{\infty}\frac{(-k)_{j+1}(k+1)_{j+1}}{(1)_{j+1}}\frac{1}{j!},$$

since $(1)_j/(2)_j = 1/(j+1)$, $(1-k)_j = (-k)_{j+1}/(-k)$ and $(2)_j = (1)_{j+1}$. Reparametrizing the series with $l = j+1$, the above yields

$$-\frac{1}{k(k+1)}\left(\sum_{l=0}^{\infty}\frac{(-k)_l(k+1)_l}{(1)_l}\frac{1}{l!} - 1\right) = \frac{1 - {}_2F_1\left[\begin{smallmatrix} -k,\ k+1 \\ 1 \end{smallmatrix}; 1\right]}{k(k+1)}.$$

The claim is then proven, by noticing that ${}_2F_1\left[\begin{smallmatrix} -k,k+1 \\ 1 \end{smallmatrix}; 1\right] = (-1)^k$, thanks to the Chu–Vandermonde identity. □

This implies that $\beta_{0,k}=0$ for any $k\in\mathbb{N}$, proving the first part of Theorem 2.7. To complete our proof, we need to show that

$$\beta_{k+1,k}=\frac{2\pi^{2(k+1)}}{(2k+1)!!(2k+1)!!} \tag{6-9}$$

is the first nonzero coefficient. As a first step, the following lemma shows that $\beta_{i,k}=0$ for all $1\le i\le k$:

Lemma 6.3. *For any $k\in\mathbb{N}$ and for any $1\le m\le k$, we have*

$${}_3F_2\left[\begin{matrix}1-k,\ k+2,\ m+1\\ m+2,\ 2\end{matrix};1\right]=\frac{(m+1)(-1)^{k+1}}{k(k+1)}.$$

Proof. We begin by applying the reduction formula, which yields

$${}_3F_2\left[\begin{matrix}1-k,\ k+2,\ m+1\\ m+2,\ 2\end{matrix};1\right]=\sum_{j=0}^{m-1}\binom{m-1}{j}\frac{1}{(2)_j}\frac{(1-k)_j(k+2)_j}{(m+2)_j}\,{}_2F_1\left[\begin{matrix}1-k+j,\ k+2+j\\ m+2+j\end{matrix};1\right]. \tag{6-10}$$

Moreover, the Chu–Vandermonde identity gives

$${}_2F_1\left[\begin{matrix}1-k+j,\ k+2+j\\ m+2+j\end{matrix};1\right]=\frac{(m-k)_{k-j-1}}{(m+j+2)_{k-j-1}}.$$

Since $(m-k)_{k-j-1}=0$ for all $j<m-1$, only the term $j=m-1$ survives in the sum in (6-10). Hence, we get

$$\begin{aligned}{}_3F_2\left[\begin{matrix}1-k,\ k+2,\ m+1\\ m+2,\ 2\end{matrix};1\right]&=\frac{(1-k)_{m-1}}{(2)_{m-1}}\frac{(k+2)_{m-1}}{(m+2)_{m-1}}\frac{(m-k)_{k-m}}{(2m+1)_{k-m}}\\&=\frac{(-1)^{m-1}(k-1)!}{(k-m)!\,m!}\frac{(k+m)!(m+1)!}{(k+1)!(2m)!}\frac{(-1)^{k-m}(k-m)!(2m)!}{(m+k)!}\\&=\frac{(-1)^{k-1}(k-1)!}{m!}\frac{(m+1)!}{(k+1)!}=\frac{(-1)^{k-1}(m+1)}{k(k+1)},\end{aligned} \tag{6-11}$$

where in the first line we applied the equalities $(m+2)_{m-1}=(2m)!/(m+1)!$, $(k+2)_{m-1}=(k+m)!/(k+1)!$ and $(1-k)_{m-1}=(-1)^{m-1}(k-1)!/(k-m)!$. Similarly also $(m-k)_{k-m}=(-1)^{k-m}(k-m)!$ and $(2m+1)_{k-m}=(k+m)!/(2m)!$. □

Finally, with the following lemma, we can also compute $\beta_{k+1,k}$:

Lemma 6.4. *For any $k\in\mathbb{N}$, we have*

$${}_3F_2\left[\begin{matrix}1-k,\ k+2,\ k+2\\ k+3,\ 2\end{matrix};1\right]=\frac{2(-1)^{k+1}(k-1)!(k+2)!}{(2k+2)!}\left(\binom{2k+1}{k+1}-1\right).$$

Proof. The idea of the proof is similar the one of Lemma 6.3. First we apply the reduction formula in order to write ${}_3F_2\left[\begin{smallmatrix}1-k,k+2,k+2\\ k+3,2\end{smallmatrix};1\right]$ as a finite sum of terms involving ${}_2F_1$, namely,

$${}_3F_2\left[\begin{matrix}1-k,\ k+2,\ k+2\\ k+3,\ 2\end{matrix};1\right]=\sum_{j=0}^{k-1}\binom{k}{j}\frac{1}{(2)_j}\frac{(1-k)_j(k+2)_j}{(k+3)_j}\,{}_2F_1\left[\begin{matrix}1-k+j,\ k+2+j\\ k+3+j\end{matrix};1\right]. \tag{6-12}$$

Note that the term $j=k$ vanishes, as $(1-k)_k=0$. Now we use Gauss' summation theorem and compute the remaining hypergeometric function, i.e.,

$$ {}_2F_1\left[\begin{matrix}1-k+j,\ k+2+j\\ k+3+j\end{matrix};1\right]=\frac{(k+j+2)!}{(2k+1)!}. $$

Plugging this into (6-12), since $(-1)^j\binom{k}{j}=(-k)_j/j!$ and $(k+2)_j=(k+j+1)!/(k+1)!$, we have

$$ \begin{aligned} {}_3F_2\left[\begin{matrix}1-k,\ k+2,\ k+2\\ k+3,\ 2\end{matrix};1\right]&=\frac{(k-1)!(k+2)!}{(2k+1)!}\sum_{j=0}^{k-1}\frac{(-k)_j(k+2)_j}{(2)_j}\frac{1}{j!}\\ &=\frac{(k-1)!(k+2)!}{(2k+1)!}\left({}_2F_1\left[\begin{matrix}-k,\ k+2\\ 2\end{matrix};1\right]-\frac{(-k)_k(k+2)_k}{(2)_k}\frac{1}{k!}\right). \end{aligned} \tag{6-13} $$

Therefore, since ${}_2F_1\left[\begin{smallmatrix}-k,k+2\\ 2\end{smallmatrix};1\right]=(-1)^k/(k+1)$, we get

$$ \begin{aligned} {}_3F_2\left[\begin{matrix}1-k,\ k+2,\ k+2\\ k+3,\ 2\end{matrix};1\right]&=\frac{(k-1)!(k+2)!}{(2k+1)!}\left(\frac{(-1)^k}{k+1}-\frac{(-1)^k(2k+1)!}{((k+1)!)^2}\right)\\ &=\frac{2(k-1)!(k+2)!(-1)^k}{(2k+2)!}\left(1-\frac{(2k+1)!}{(k+1)!\,k!}\right), \end{aligned} $$

and the claim follows. □

To conclude the proof of Theorem 2.7, we just combine (6-8) with Lemma 6.4, getting

$$ \begin{aligned} \beta_{k+1,k}&=\frac{(2\pi)^{2k+2}}{(2k+3)!}\left(1-\frac{k(k+1)}{k+2}\frac{2(k-1)!(k+2)!}{(2k+2)!}\left(\binom{2k+1}{k+1}-1\right)\right)\\ &=\frac{(2\pi)^{2k+2}}{(2k+3)!}\left(1-\frac{2[(k+1)!]^2}{(2k+2)!}\left(\frac{(2k+1)!}{(k+1)!\,k!}-1\right)\right)\\ &=\frac{(2\pi)^{2k+2}}{(2k+3)!}\left(1-\frac{2(k+1)}{2k+2}+\frac{2[(k+1)!]^2}{(2k+2)!}\right)=\frac{(2\pi)^{2k+2}}{(2k+3)!}\frac{(k+1)!\,k!}{(2k+1)!}. \end{aligned} $$

Equation (6-9) follows by the identities $(2k+1)!=2^k k!(2k+1)!!$ and $(2k+3)!=2^{k+1}(k+1)!(2k+3)!!$.

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