Algebra & Number Theory

Volume 18
2024
No. 1

Semisimple algebras and PI-invariants of finite dimensional algebras

Eli Aljadeff and Yakov Karasik

msp

ALGEBRA AND NUMBER THEORY 18:1 (2024)
https://doi.org/10.2140/ant.2024.18.133

Semisimple algebras and PI-invariants of finite dimensional algebras

Eli Aljadeff and Yakov Karasik

Let Γ be the T-ideal of identities of an affine PI-algebra over an algebraically closed field F of characteristic zero. Consider the family $\mathcal{M}_\Gamma$ of finite dimensional algebras Σ with $\mathrm{Id}(\Sigma) = \Gamma$. By Kemer's theory $\mathcal{M}_\Gamma$ is not empty. We show there exists $A \in \mathcal{M}_\Gamma$ with Wedderburn–Malcev decomposition $A \cong A_{ss} \oplus J_A$, where J_A is the Jacobson's radical and A_{ss} is a semisimple supplement with the property that if $B \cong B_{ss} \oplus J_B \in \mathcal{M}_\Gamma$ then A_{ss} is a direct summand of B_{ss}. In particular A_{ss} is unique minimal, thus an invariant of Γ. More generally, let Γ be the T-ideal of identities of a PI algebra and let $\mathcal{M}_{\mathbb{Z}_2,\Gamma}$ be the family of finite dimensional superalgebras Σ with $\mathrm{Id}(E(\Sigma)) = \Gamma$. Here E is the unital infinite dimensional Grassmann algebra and $E(\Sigma)$ is the Grassmann envelope of Σ. Again, by Kemer's theory $\mathcal{M}_{\mathbb{Z}_2,\Gamma}$ is not empty. We prove there exists a superalgebra $A \cong A_{ss} \oplus J_A \in \mathcal{M}_{\mathbb{Z}_2,\Gamma}$ such that if $B \in \mathcal{M}_{\mathbb{Z}_2,\Gamma}$, then A_{ss} is a direct summand of B_{ss} as superalgebras. Finally, we fully extend these results to the G-graded setting where G is a finite group. In particular we show that if A and B are finite dimensional $G_2 := \mathbb{Z}_2 \times G$-graded simple algebras then they are G_2-graded isomorphic if and only if $E(A)$ and $E(B)$ are G-graded PI-equivalent.

1. Introduction

Let F be an algebraically closed field of characteristic zero and $F\langle X\rangle$ the free associative algebra over F on a countable set of variables X. Let Γ be a T-ideal of $F\langle X\rangle$ (i.e., invariant under all algebra endomorphisms of $F\langle X\rangle$). It is easy to see that Γ is in fact the ideal of polynomial identities of a suitable associative algebra (e.g., $\Gamma = \mathrm{Id}(F\langle X\rangle/\Gamma)$). Kemer's representability theorem says that if $\Gamma \neq 0$, then it is the T-ideal of identities of an algebra of the form $E(B)$, the Grassmann envelope of some finite dimensional $\mathbb{Z}_2$-graded algebra $B = B_0 \oplus B_1$ over F. Here $E = E_0 \oplus E_1$ is the infinite dimensional unital Grassmann algebra over F with the natural $\mathbb{Z}_2$-grading and $E(B) = E_0 \otimes B_0 \oplus E_1 \otimes B_1$ viewed as an ungraded algebra. In case Γ is the T-ideal of identities of an affine PI algebra, or equivalently, in case Γ contains a nontrivial Capelli polynomial, Kemer's representability theorem says that $\Gamma = \mathrm{Id}(A)$ where A is a finite dimensional algebra over F. Kemer's representability theorem is the key step towards the positive solution of the Specht problem which claims that every T-ideal is finitely based.

The purpose of this paper is to prove, roughly speaking, that if A is a finite dimensional algebra over an algebraically closed field of characteristic zero F, then the maximal semisimple subalgebra of A, namely a supplement A_{ss} of the Jacobson's radical J_A in A, is "basically uniquely determined" by $\mathrm{Id}(A)$. We

MSC2020: 16R10, 16W50, 16W55.
Keywords: T-ideal, polynomial identities, semisimple, full algebras, graded algebras.

show also that a similar result holds for the algebra $E(B)$, that is, a $\mathbb{Z}_2$-graded semisimple supplement of J_B in a finite dimensional superalgebra B is basically uniquely determined by $\Gamma = \mathrm{Id}(E(B))$. Finally, we extend our results to the G-graded setting where G is a finite group. Before we state the results precisely, we should remark right away that strictly speaking the semisimple part of a finite dimensional algebra cannot be determined by its T-ideal of identities for the simple reason that e.g., $\mathrm{Id}(A) = \mathrm{Id}(A \oplus A)$. So, by "basically uniquely determined" we mean the following.

Theorem 1.1. *Let Γ be a T-ideal of identities and suppose Γ contains a Capelli polynomial c_n for some n. Then there exists a finite dimensional semisimple F-algebra U that satisfies the following conditions*:

(1) *There exists a finite dimensional algebra A over F with* $\mathrm{Id}(A) = \Gamma$ *and such that $A \cong U \oplus J_A$ is its Wedderburn–Malcev decomposition.*

(2) *If B is any finite dimensional algebra over F with* $\mathrm{Id}(B) = \Gamma$ *and B_{ss} is its maximal semisimple subalgebra*, *then U is a direct summand of B_{ss}.*

Clearly, up to an algebra isomorphism, the semisimple algebra U is unique minimal and hence it is an invariant of Γ.

Let A be a finite dimensional algebra over F and let $A \cong A_1 \times \cdots \times A_q \oplus J$ be its Wedderburn–Malcev decomposition where A_i is simple, $i = 1, \ldots, q$, and $J = J_A$ is the Jacobson radical.

Definition 1.2. We say A is *full* if up to a permutation of the simple components $A_1 \cdot J \cdot A_2 \cdots J \cdot A_q \neq 0$.

The following theorem plays a key role in the proof of Theorem 1.1.

Theorem 1.3. *If two full algebras A and B are PI-equivalent then their maximal semisimple subalgebras are isomorphic. In particular this holds in case A and B are fundamental algebras.*

Remark 1.4. Fundamental algebras are special type of full algebras. They are important in Kemer's theory but will not play a role in this paper; see [Aljadeff et al. 2020].

Let us show how Theorem 1.3 follows from Theorem 1.1. Let A_0 be a finite dimensional algebra PI equivalent to A and with minimal semisimple subalgebra U. We show $U \cong A_{ss}$. Recall that for a finite dimensional algebra W, $\exp(W) \leq \dim_F(W_{ss})$ and equality holds if (and only if) W is full. Here $\exp(W)$ is the exponent of the algebra W, an asymptotic PI invariant attached to the T-ideal $\mathrm{Id}(W)$ and so $\exp(A_0) = \exp(A)$; see [Giambruno and Zaicev 1998, Corollary 1]. Furthermore, by Theorem 1.1 we have that U is a direct summand of A_{ss} and the result follows.

For *fundamental algebras* the result of Theorem 1.3 was proved by Procesi [2016, Corollary 3.15]. Procesi's result is based on a geometric construction which corresponds to a T-ideal Γ containing a Capelli polynomial, or equivalently, a T-ideal of identities of a finite dimensional algebra A. Let us comment briefly on Procesi's approach. He considers the coordinate ring $\mathcal{T}_t(Y)$ of the variety of the semisimple representations of the free algebra $F\langle X\rangle$ into the algebra of $t \times t$-matrices over F, where X is a set of cardinality m and t is the *exponent* of Γ; see [Aljadeff et al. 2020, Chapter 21]. The commutative algebra $\mathcal{T}_t(Y)$ acts on the T-ideal K generated by Kemer polynomials of Γ via a quotient algebra $\mathcal{T}_D$, an algebra which is generated by traces. It turns out, and this is a key idea of Kemer [Aljadeff et al. 2016, Section 10], that replacing suitable variables x_i which alternate in a Kemer polynomial f by zx_i (z is an auxiliary variable), it gives rise to the multiplication of f by a trace function. This determines the action of $\mathcal{T}_D$ and hence of $\mathcal{T}_t(Y)$ on K. Finally, it is shown that the support variety W for the $\mathcal{T}_t(Y)$-module K carries the information we need. Indeed, it turns out that if A is any fundamental algebra with $\mathrm{Id}(A) = \Gamma$ and with semisimple part $A_{ss} = A_1 \times \cdots \times A_q$, then the tuple $(t_1, \ldots, t_q)$, where $A_i \cong M_{t_i}(F)$, is an invariant of W.

Our approach instead is mostly combinatorial. It uses a *refined version* of the so called "Kemer's lemma 1" [Aljadeff et al. 2016, Section 6; Kanel-Belov and Rowen 2005, Proposition 4.44; Kemer 1987, Section 2] which deals with full algebras (an important ingredient in Kemer's solution of the Specht problem). We do not use however the more subtle result of Kemer, namely "Kemer's lemma 2" [Aljadeff et al. 2016, Section 7; Kanel-Belov and Rowen 2005, Proposition 4.54; Kemer 1987, Section 2] which concerns with fundamental algebras. The advantage of full algebras comparing to fundamental algebras (beside being a much larger class) is that they are easier to define and in particular they can be characterized without using polynomial identities. This allows us to extend Theorem 1.3 to (1) nonaffine algebras (2) group graded algebras.

Let us turn now to the case where Γ contains no Capelli polynomials. In that case we have the following result.

Theorem 1.5. *Let $\Gamma \leq F\langle X\rangle$ be a nonzero T-ideal and suppose $c_n \notin \Gamma$ for every n. Then there exists a finite dimensional semisimple superalgebra U over F which satisfies the following conditions*:

(1) *There exists a finite dimensional superalgebra A over F with* $\mathrm{Id}(E(A)) = \Gamma$ *and such that $A \cong U \oplus J_A$ is its Wedderburn–Malcev decomposition.*

(2) *If B is any finite dimensional superalgebra over F with* $\mathrm{Id}(E(B)) = \Gamma$ *and B_{ss} is its maximal semisimple subalgebra*, *then U is a direct summand of B_{ss} as superalgebras.*

The proof of Theorem 1.1 is given in the next section (Section 2). In Section 3 we treat the nonaffine case, Theorem 1.5.

In the last two sections of this article we extend the main results to the setting of G-graded T-ideals and G-graded algebras where G is a finite group. The main obstacle here is due to the fact that a G-graded simple algebra A is not determined up to a G-graded isomorphism by the dimensions of the homogeneous components A_g, $g \in G$. The proof uses the extension of Kemer's theory to G-graded algebras where G is a finite group; see [Aljadeff and Kanel-Belov 2010].

Remark 1.6. The extension of the results above to algebras over fields of finite characteristic and in particular over finite fields does not seem to be straightforward. One of the reasons is that *alternation* and *symmetrization*, operations which appear in the proofs, may result as zero multiplication. We refer to the work of Belov, Rowen and Vishne on full quivers of representations of algebras over fields of arbitrary characteristic and more generally over commutative Noetherian domains; see [Belov-Kanel et al. 2010;

2011; 2012]. The notion of full quiver is useful for studying the interactions between the radical and the semisimple component of Zariski closed algebras, a notion that appears in Belov's remarkable solution of the Specht problem for affine algebras over fields of finite characteristic; see [Belov 2010]. We emphasize that such interactions for Zariski closed algebras are considerably more subtle than for finite dimensional algebras over a field of characteristic zero.

2. Preliminaries and proof of the affine case

We start by introducing some combinatorial terminology.

Let $\alpha = (a_1, \dots, a_q)$ be a q-tuple, $q \geq 0$, (or multiset rather, since the order of the a_i will not play a role) of positive integers. For any sub-tuple γ of α we let $\sigma(\gamma) = \sum_{a\in\gamma} a$ be the weight of γ. We set $\sigma(\gamma) = 0$ if γ is the empty tuple.

In what follows the tuple α will correspond to the dimensions of the simple components of a finite dimensional semisimple algebra. More precisely, if A is a finite dimensional algebra over F, we let $A \cong A_1 \times \cdots \times A_q \oplus J_A$ be its Wedderburn–Malcev decomposition. Then $\mathfrak{m}_A = (\dim_F(A_1), \dots, \dim_F(A_q))$ is the tuple corresponding to A. With this notation $\mathfrak{m}_A$ is empty if and only if A is nilpotent.

Definition 2.1. Let $\alpha = (a_1, \dots, a_r)$ and $\beta = (b_1, \dots, b_s)$ be tuples of positive integers. We say β *covers* α if the tuple α may be decomposed into s disjoint, possibly empty, sub-tuples $T_1, \dots, T_s$ such that $\sigma(T_i) \leq b_i$, $i = 1, \dots, s$.

Example 2.2. The tuple (16, 12) covers the tuple (10, 9, 3, 3) but it does not cover the tuple (15, 8, 5).

Note 2.3. (1) The covering relation is antisymmetric.

(2) The covering relation is strictly stronger than majorization.

(3) The covering relation is in fact a partial order relation, if one considers multisets rather than tuples.

Next we recall some definitions and a result from Kemer's theory.

Let A be a finite dimensional algebra over F. Let $A \cong A_{ss} \oplus J_A$ be its Wedderburn–Malcev decomposition where J_A is the Jacobson radical and A_{ss} is a semisimple subalgebra supplementing J_A. The algebra A_{ss} decomposes uniquely (up to permutation) into a direct product of simple algebras $A_1 \times \cdots \times A_q$, where $A_i \cong M_{n_i}(F)$ is the algebra of $n_i \times n_i$-matrices over F. Furthermore, it is well known that all semisimple supplements of J_A in A are isomorphic.

It is clear that in order to test whether a multilinear polynomial p is an identity of A it is sufficient to evaluate the polynomial on a basis of A and so we fix from now on a basis $\mathcal{B} = \{e^i_{k,l}, u_1, \dots, u_d\}$. Here, the elements $\{e^i_{k,l}\}$, $1 \leq k, l \leq n_i$ are the elementary matrices of $M_{n_i}(F)$, $i = 1, \dots, q$, and $\{u_1, \dots, u_d\}$ is a basis of J_A.

Definition 2.4. Let $p = p(x_1, \dots, x_n)$ be a multilinear polynomial. We say an evaluation of p on A is *admissible* if the variables of p assume values only from the basis $\mathcal{B}$. We refer to an evaluation of a variable as *semisimple* (resp. *radical*) if the value is an elementary matrix $e^i_{k,l}$ (resp. an element $u_i \in J_A$).

For the rest of the paper we will consider only admissible evaluations.

Definition 2.5. Let A be a full algebra (Definition 1.2). We say a multilinear polynomial $p(x_1, \dots, x_n)$ is:

(1) A-*weakly full* (or *weakly full* of A or *weakly full* when the algebra in question is clear) if it has a nonzero admissible evaluation on A where elements from all simple components are represented in the evaluation.

(2) A-*full* if every simple component of A_{ss} is represented in *every* admissible nonzero evaluation on A. Also here we may use the terminology *full* of A or just *full*.

(3) A-*strongly full* if every basis element of A_{ss} appears in every admissible nonzero evaluation of p.

Remark 2.6. In this paper we make use of polynomials that are *weakly full* or *strongly full*. We mention *full* polynomials here just for completeness. They appear in Kemer's theory; see [Aljadeff et al. 2016, Definition 5.10].

It is clear that if p is A-strongly full then it is full. Also, every full polynomial is weakly full.
We are interested in the opposite direction. We start with:

Lemma 2.7. *If A is a full algebra then it admits a weakly full polynomial.*

Proof. Let A be as above. Then the multilinear monomial of degree $2q-1$ is weakly full. Indeed, we get a nonzero evaluation where we put q semisimple (resp. $q-1$ radical) values in the odd (resp. even) positions. □

The following theorem is basically *Kemer's lemma* 1; see [Aljadeff et al. 2016].

Theorem 2.8. *The following hold*:

(1) *Every full algebra admits a multilinear strongly full polynomial and therefore admits a full polynomial.*

(2) *Let A be a full algebra and f_0 be a multilinear weakly full polynomial of A. Then there exists a full polynomial f of A in $\langle f_0\rangle_T$, the T-ideal generated by f_0.*

(3) *Let A be a full algebra and f_0 a multilinear weakly full polynomial of A. Then there exists a strongly full polynomial $f \in \langle f_0\rangle_T$ of A.*

Proof. Clearly, the third statement implies the second and together with Lemma 2.7 it implies the first statement. Statement (3) follows from the construction in the proof of Kemer's lemma 1; see [Aljadeff et al. 2016]. □

As we shall need to refer to the precise construction of strongly full polynomials starting from a weakly full polynomial f_0, let us recall their construction here. It is convenient to illustrate first the construction on the weakly full polynomial mentioned above.

Let $A \cong A_1 \times \cdots \times A_q \oplus J_A$, where $A_i \cong M_{n_i}(F)$, $i = 1, \dots, q$ (as above) and suppose that after reordering the simple components we have $A_1 J A_2 \dots J A_q \neq 0$. Let $f_0 = X_1 \cdot w_1 \dots w_{q-1} \cdot X_q$ be a monomial of $2q-1$ variables which is clearly weakly full by the obvious evaluation. Let

$$Z_n = Z_n(x_1, \ldots, x_{n^2}; y_1, \ldots, y_{n^2+1}) = y_1 \cdot x_1 \cdot y_2 \cdots y_{n^2} \cdot x_{n^2} \cdot y_{n^2+1}$$

be a multilinear monomial on $2n^2+1$ variables. For $i = 1, \ldots, q$, we consider k monomials Z_{n_i} in disjoint variables, denoted by $Z_{n_i,l}$, $l = 1, \ldots, k$, where the integer k is sufficiently large and will be determined later. We set $\Delta_i = Z_{n_i,1} \cdots Z_{n_i,k}$, the product of k copies of the monomial Z_{n_i} with disjoint sets of variables. Finally, in view of the inequality $A_1 J A_2 \cdots J A_q \neq 0$ we apply the T-operation and replace the variable X_i by $X_i \cdot \Delta_i$ in the polynomial f_0 (here it is just a monomial) and obtain the monomial

$$\Omega = X_1 \cdot \Delta_1 \cdot w_1 \cdot X_2 \cdot \Delta_2 \cdot w_2 \cdots w_{q-1} \cdot X_q \cdot \Delta_q.$$

We refer to the x's (lower case) in Ω as *designated* variables, the y's as *frame* variables and w's as *bridge* variables. Now, it is not difficult to see that the monomial Ω admits a nonzero evaluation where the x's from $Z_{n_i,l}$ get values consisting of the full basis of the i-th simple component, that is the elementary matrices $\{e^i_{t,s}\}$, the y's from $Z_{n_i,l}$ get values of the form $e^i_{t,t}$ and the w's get radical values which bridge the different simple components. Fixing $r = 1, \ldots, k$, we alternate all x's from the monomials $Z_{n_i,r}$, $i = 1, \ldots, q$, so we obtain k alternating sets of cardinality $\dim_F(A_{ss})$. We denote the polynomial obtained by f_A. We adopt the terminology used in Kemer's theory and refer to each alternating set of designated variables as a *small set*. Moreover, we shall refer to the set of variables x in a small set together with the corresponding frames, that is the y variables that border the x variables, as an *augmented small set*.

Remark 2.9. In Kemer's theory there is also a notion of a *big set*. These are sets which, roughly speaking, involve the alternation of semisimple and bridge variables. We will not make use of big sets here.

Suppose the integer k, namely the number of small sets in f_A, exceeds the nilpotency index of A. Let us show that f_A is a strongly full polynomial of A. We will show that if δ is any admissible *nonzero* evaluation of f_A, then there is at least one small set which assumes precisely a full basis of A_{ss}. Indeed, by the alternation of designated variables we are forced to evaluate each small set on different basis elements and if this is not a full basis of A_{ss}, we have that at least one of the designated variables assumes a radical value. Since k is larger than the nilpotency index of A, we cannot have a radical evaluation in every small set. This shows f_A is strongly full. In fact this proves the last statement of Theorem 2.8 for the weakly full polynomial $f_0 = X_1 \cdot w_1 \cdots w_{q-1} \cdot X_q$.

Let us proceed now to the general case, namely where f_0 is assumed to be an arbitrary multilinear weakly full polynomial of A. Denote by Φ a nonzero evaluation of f_0 which visits every simple component of A. Let us denote the variables of f_0 which assume values from the simple components $A_1, \ldots, A_q$ by $X_1, \ldots, X_q$ respectively. Since the evaluation $\Phi(f_0)$ is nonzero, it is nonzero on one of the monomials of f_0 which we fix from now on and denote it by R_e. We have then that $f_0 = \sum_{\sigma \in S_m} \lambda_\sigma R_\sigma$ where $\lambda_\sigma \in F$ and $\lambda_e = 1$. Here m is the number of variables in f_0. We proceed now as in the previous case, namely replace the variables X_i by $X_i \cdot \Delta_i$ and obtain a polynomial which we denote by Ω. We have that if $f_0 = f_0(X_1, \ldots, X_q; M)$ then $\Omega = f_0(X_1\Delta_1, \ldots, X_q\Delta_q, M) \in \langle f_0 \rangle_T$ where M is a suitable set of variables. By an appropriate evaluation of the monomials Δ_i, $i = 1, \dots, q$, we see that Ω is a nonidentity of A and is clearly weakly full. Finally we alternate the designated variables as above and obtain a polynomial which we denote by f_A. It is not difficult to see that f_A satisfies the third condition of Theorem 2.8 with respect the given weakly full polynomial f_0.

Lemma 2.10 (main lemma-affine). *Notation as above. Suppose A and B are full algebras. Suppose $\mathfrak{m}_B$ does not cover $\mathfrak{m}_A$. Then there exists a strongly full polynomial f_A of A which vanishes on B. In fact, if f_0 is any weakly full polynomial of A then there exists a strongly full polynomial $f_A \in \langle f_0 \rangle_T$ of A which vanishes on B.*

Proof. Let f_A be the strongly full polynomial of A as constructed above in case $f_0 = X_1 \cdot w_1 \cdots w_{q-1} \cdot X_q$. We take a large number of small sets k, exceeding the nilpotency index of B. We claim f_A is an identity of B. We will show that if this is not the case then necessarily B covers A. Let us fix a nonzero evaluation Φ of f_A on B and consider one monomial, which we assume as we may is the monomial Ω of f_A (see the construction above), whose value is nonzero. Note that by the condition on k, there exists an augmented small set, say the j-th set where $j \in \{1, \dots, k\}$, which is free of radical values. It follows that the Φ-values of each segment in $\{Z_{n_1,j}, \dots, Z_{n_q,j}\}$ consist only of semisimple elements in B, and moreover semisimple elements from the same simple component. But because the evaluation of Φ on f_A is nonzero and the variables in the j-th small set alternate, the semisimple values of B must be linearly independent. This implies that B covers A as desired.

In the general case we may argue as follows. Let f_0 be an arbitrary weakly full polynomial of A and let $R_\sigma = R_\sigma(X_1, \dots, X_q; M)$ be any monomial of f_0. Applying the T-operation on R_σ we obtain $\Omega_\sigma = R_\sigma(X_1\Delta_1, \dots, X_q\Delta_q, M) \in \langle R_\sigma \rangle_T$. Next we alternate the designated variables as above and obtain a polynomial which we denote by $(R_\sigma)_A$. As in the first case considered, that is in case where $f_0 = X_1 \cdot w_1 \cdots w_{q-1} \cdot X_q$, we see that if $(R_\sigma)_A$ admits a nonzero evaluation on B, then B covers A. It follows that if f_A admits a nonzero evaluation on B, this is true also for the polynomial $(R_\sigma)_A$, some σ, and so B covers A. □

Corollary 2.11. *Let A and B full algebras. If they are PI-equivalent, then their semisimple parts, A_{ss} and B_{ss} are isomorphic.*

Proof. Indeed, A and B must cover each other. It follows that the tuple of dimensions of the simple components of A and B coincide up to a permutation (see Note 2.3) and hence A_{ss} and B_{ss} are isomorphic. □

In what follows we will need a somewhat stronger statement.

Corollary 2.12. *Let A be a full algebra and $B_1, \dots, B_t$ be a finite family of full algebras, each not covering A. If f_0 is a weakly full polynomial of A then there is a strongly full polynomial $f_A \in \langle f_0 \rangle_T$ of A that vanishes on B_i, $i = 1, \dots, t$. In particular if B is a direct sum of full algebras, each not covering A, then there exists a strongly full polynomial $f_A \in \langle f_0 \rangle_T$ of A which vanishes on B.*

Proof. We only need to pay attention to the number of small sets k in f_A, namely it should exceed the nilpotency index of each J_{B_i}, $i = 1, \dots, t$. $\square$

Recall that any affine PI-algebra A and in particular any finite dimensional algebra is PI-equivalent to a direct sum of full algebras; see for instance [Aljadeff et al. 2016; 2020]. Here we will need a more precise statement.

Definition 2.13. Let A be finite dimensional algebra. We say

$$P(A) = T_1 \oplus \cdots \oplus T_n$$

is a *presentation of A by full algebras* if the following hold:

(1) T_i is full for $i = 1, \dots, n$.

(2) $P(A)$ is PI equivalent to A.

Remark 2.14. Note that an algebra may have two different presentations which are isomorphic as algebras (e.g., a radical direct summand may be attached to different full subalgebras). Thus, when referring to a presentation $P(A)$, we are fixing the set of full algebras $\{T_1, \dots, T_n\}$ up to permutation. Note that if Γ is a T-ideal containing Capelli polynomials we may view $T_1 \oplus \cdots \oplus T_n$ as a presentation of Γ so we may denote it by $P(\Gamma)$.

Proposition 2.15. *Let A be a finite dimensional algebra. Then there exists a presentation $T_1 \oplus \cdots \oplus T_n$ of A. Moreover, there exists such presentation where the semisimple subalgebra $(T_i)_{ss}$ of T_i is a direct summand of A_{ss}, for $i = 1, \dots, n$.*

Proof. In fact the stronger statement follows from the construction in [Aljadeff et al. 2020, Subsection 17.2.4]. Let $A \cong A_1 \times \cdots \times A_q \oplus J_A$ be the Wedderburn–Malcev decomposition. Clearly we may assume A is not full. Consider the subalgebra

$$\mathcal{A}_i = \langle A_1, \dots, A_{i-1}, A_{i+1}, \dots, A_q; J_A \rangle.$$

We claim A and $\mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_q$ are PI-equivalent. Clearly $\mathrm{Id}(A) \subseteq \mathrm{Id}(\mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_q)$. For the converse, if f is a nonidentity of A, it must be a nonidentity of at least one $\mathcal{A}_i$ for otherwise it is a full polynomial of A which implies A is full, contrary to our assumption. The proposition is then proved by induction. $\square$

For any presentation $P(A)$ of A we let $P(A)_{\dim(ss)}$ be the set of *tuples* consisting of the dimensions of the simple components that appear in the different full algebras of $P(A)$ and denote by $P(A)_{\dim(ss),max}$ the set of maximal tuples in $P(A)_{\dim(ss)}$ with respect to covering.

Corollary 2.16. *The set $P(A)_{\dim(ss),\max}$ depends on A but not on the presentation $P(A)$. Hence we can denote the set $P(A)_{\dim(ss),\max}$ by $\mathcal{A}_{\dim(ss),\max}$.*

Proof. Suppose the contrary holds. Let P_1 and P_2 be presentations of A as above. Then without loss of generality there exists a full subalgebra M of P_1 whose tuple is maximal and does not appear as a maximal tuple in P_2. We may assume M is not covered by tuples of P_2 for otherwise M is strictly covered by a tuple of P_2 and in that case we may exchange the roles of P_1 and P_2. Now, by the lemma, there exists a nonidentity polynomial of M which is an identity of every full subalgebra of P_2 and the claim is proved. $\square$

In the next lemma we show we can *fuse* finite dimensional algebras A and B with isomorphic semisimple subalgebras. More generally, suppose the semisimple subalgebra of A is a direct summand of B_{ss}, the semisimple subalgebra of B. We claim $A \times B$ is PI-equivalent to an algebra of the form $B_{ss} \oplus \hat{J}$. Yet more generally, suppose A and B have a common semisimple component U (up to isomorphism), then there exists an algebra C, PI equivalent to $A \times B$, in which the semisimple algebra isomorphic to U appears in C only once. Here is the precise statement.

Lemma 2.17. *Let $A_1 \times \cdots \times A_q \oplus J(A)$ and $B_1 \times \cdots \times B_r \oplus J(B)$ be the Wedderburn–Malcev decompositions of A and B respectively. Suppose $A_1 \times \cdots \times A_k \cong B_1 \times \cdots \times B_k \cong U$. Then $A \times B$ is PI-equivalent to $C = U \times A_{k+1} \times \cdots \times A_q \times B_{k+1} \times \cdots \times B_r \oplus J(A) \oplus J(B)$.*

Proof. We consider the vector space embedding

$$C = U \times A_{k+1} \times \cdots \times A_q \times B_{k+1} \times \cdots \times B_r \oplus J(A) \oplus J(B) \hookrightarrow [U \times A_{k+1} \times \cdots \times A_q \oplus J(A)] \times [U \times B_{k+1} \times \cdots \times B_r \oplus J(B)]$$

where the elements of U are mapped diagonally. It is easy to see that the image is closed under multiplication, yielding an algebra structure on C. As for the polynomial identities the above embedding (now, as algebras) yields $\mathrm{Id}(C) \supseteq \mathrm{Id}(A \times B) = \mathrm{Id}(A) \cap \mathrm{Id}(B)$. On the other hand the algebras A and B are embedded in C and the result follows. $\square$

Definition 2.18. Notation as in the lemma above. We say the algebra C is the fusion of the algebras A and B along U.

Proposition 2.19. *Let $P_1 = P_1(A)$ and $P_2 = P_2(A)$ be presentations of A and let $T_1 \cong (T_1)_{ss} \oplus J_{T_1}$ and $T_2 \cong (T_2)_{ss} \oplus J_{T_2}$ be full subalgebras summands of P_1 and P_2 respectively. Suppose $(T_1)_{ss}$ and $(T_2)_{ss}$, the semisimple parts of T_1 and T_2, are isomorphic and let $U \cong (T_1)_{ss} \cong (T_2)_{ss}$. Let $T_1' = U \oplus J_{T_1} \oplus J_{T_2}$. Then T_1' is full. Furthermore, if we replace $T_1 \cong U \oplus J_{T_1}$ by $T_1' = U \oplus J_{T_1} \oplus J_{T_2}$ in the presentation P_1 we obtain a presentation P_1' of A.*

Proof. From the embedding $U \oplus J_{T_1} \hookrightarrow U \oplus J_{T_1} \oplus J_{T_2}$ we see that every weakly full polynomial of T_1 is weakly full of T_1', so T_1' is full. Furthermore, because $\mathrm{Id}(T_1), \mathrm{Id}(T_2) \supseteq \mathrm{Id}(A)$ we have that $\mathrm{Id}(P_1') \supseteq \mathrm{Id}(A)$. On the other hand $\mathrm{Id}(P_1') \subseteq \mathrm{Id}(P_1)$ $(= \mathrm{Id}(A))$ and the result follows. $\square$

Remark 2.20. Note that fusion of fundamental algebras A and B with isomorphic semisimple subalgebras yields a fundamental algebra; see [Aljadeff et al. 2020] for the definition of fundamental algebras.

Let Γ be the T-ideal of identities of a finite dimensional algebra. Denote by $\mathcal{M}_\Gamma$ the family of presentations $A = T_1 \oplus \cdots \oplus T_n$ of Γ (we simplify the notation slightly and write A rather than $P(A)$ for a presentation of Γ).

In what follows we shall present a procedure in which we iterate 4 steps (numbered 0–3). In each step we replace an algebra $A \in \mathcal{M}_\Gamma$ by an algebra $A' \in \mathcal{M}_\Gamma$ (in particular PI equivalent to A) that is "better" behaved. Then, in one final step (step 4), we construct the algebra A of Theorem 1.1.

Step 0 (deletion): Let $A \in \mathcal{M}_\Gamma$. We delete from A full subalgebras that do not alter $\mathrm{Id}(A)$. Let A_i be a full subalgebra of A. Denote by $\widehat{A_i}$ the summand of A consisting the direct sum of full algebra A_j, $j \neq i$. Then, we delete A_i from the direct sum if $\mathrm{Id}(A_i) \supseteq \mathrm{Id}(\widehat{A_i}) = \cap_{j\neq i} \mathrm{Id}(A_j)$. We abuse notation and simply write the outcome by $F_0(A)$, an operation of type 0 on A, although the operation depends on the choice of the full algebra A_i. Clearly $F_0(A)$ and A are PI equivalent. We write $A = A_{\mathrm{red}_0}$ if *every* operation of type 0 on A is the identity.

Step 1 (fusion): $A \in \mathcal{M}_\Gamma$ and suppose $A = A_{\mathrm{red}_0}$. We fuse full subalgebras with isomorphic semisimple subalgebras. More generally, if A_i and A_j $i \neq j$, are full subalgebras of A and $(A_i)_{ss}$ is a direct summand of $(A_j)_{ss}$, then the operation $F_1 = (F_1)_{A_i,A_j}$ on A is the fusion of A_i and A_j. We abuse notation and simply write the outcome by $F_1(A)$, an operation of type 1 on A, although the operation depends on the choice of the full algebras A_i and A_j. Note that by Proposition 2.19 the algebras $F_1(A)$ and A are PI equivalent. We write $A = A_{\mathrm{red}_{0,1}}$ if every operation of type 0 or 1 on A is the identity.

We come now to a step where we *decompose* full algebras.

Step 2 (decomposition): Let $A \in \mathcal{M}_\Gamma$ and suppose that $A = A_{\mathrm{red}_{0,1}}$. We define an operation of type 2 on A, denoted by F_2, as follows. Choose a full algebra Q appearing in the decomposition of A into full algebras and let $A_{\mathrm{supp}(Q)} = (\hat{Q})_1 \oplus \cdots \oplus (\hat{Q})_n$ be the supplement of Q in A. Note that since $A = A_{\mathrm{red}_{0,1}}$ there is no full algebra component of $A_{\mathrm{supp}(Q)}$ with semisimple part $\cong Q_{ss}$. Suppose there exists a weakly full polynomial p of Q which vanishes on $A_{\mathrm{supp}(Q)}$. In that case we leave the algebra A unchanged, that is $F_2(A) = A$. Otherwise we proceed as follows.

Clearly Q is not nilpotent because A is not nilpotent and $A = A_{\mathrm{red}_{0,1}}$. Let us treat the case where Q_{ss} is simple separately. If Q_{ss} is simple and every weakly full polynomial of Q is a nonidentity of $A_{\mathrm{supp}(Q)}$ we claim $\mathrm{Id}(A) = \mathrm{Id}(A_{\mathrm{supp}(Q)} \oplus J_Q)$ where J_Q is the radical of Q. It is clear that $\mathrm{Id}(A) \subseteq \mathrm{Id}(A_{\mathrm{supp}(Q)} \oplus J_Q)$. Conversely, suppose p is a nonidentity of A. If p is a nonidentity of $A_{\mathrm{supp}(Q)}$ it is also a nonidentity of $\mathrm{Id}(A_{\mathrm{supp}(Q)} \oplus J_Q)$ as needed, so let us assume p is an identity of $A_{\mathrm{supp}(Q)}$. In that case p must be a nonidentity of Q. However, by assumption, p is not weakly full of Q which means here that no indeterminate of p gets a semisimple value in any nonzero evaluation of p. It follows that p is a nonidentity of J_Q and we are done. Suppose now $q > 1$ and let $Q \cong \Delta_1 \times \cdots \times \Delta_q \oplus J_Q$ be the Wedderburn–Malcev decomposition of Q. We are assuming every weakly full polynomial of Q is a nonidentity of $A_{\mathrm{supp}(Q)}$. In that case we claim the following.

Claim 2.21. *We can replace the full subalgebra Q of A by a direct sum of full subalgebras $\mathcal{Q}_1 \oplus \cdots \oplus \mathcal{Q}_q$, where for each $i = 1, \ldots, q$, the semisimple algebra $(\mathcal{Q}_i)_{ss}$ is a proper summand of Q_{ss} (in particular strictly covered by Q) and if $\bar{A}$ denotes the algebra obtained, we have* $\mathrm{Id}(\bar{A}) = \mathrm{Id}(A) = \Gamma$.

Proof. Consider the algebras $\mathcal{Q}_i$, $i = 1, \ldots, q$, obtained from Q by deleting one simple component Δ_i and keeping the radical unchanged. We claim A is PI-equivalent to $A_{\text{supp}(Q)} \oplus \mathcal{Q}_1 \oplus \cdots \oplus \mathcal{Q}_q$. Indeed, it is clear that every identity of the former algebra vanishes on the latter one. Conversely, let p be a nonidentity of the former one. We show it does not vanish on the latter. Clearly, we may assume p vanishes on $A_{\text{supp}(Q)}$ and so, by our assumption above, p is not a weakly full polynomial of Q. This means that p has no nonzero evaluation on Q which visits all simple components of Q and so, being a nonidentity of Q, it must be a nonidentity of $\mathcal{Q}_i$ for some i and hence a nonidentity of the latter. □

We write $A = A_{\text{red}_{0,1,2}}$ if any operation of type 0, 1 or 2 on A is the identity.

Similarly to our notation for the operations F_0 and F_1 above we abuse notation here and simply write $F_2(A) = F_{2,Q}(A)$. It follows from the claim that $F_2(A)$ and A are PI equivalent.

Step 3 (absorption): Fix a presentation $A \in \mathcal{M}_\Gamma$ and suppose $A = A_{\text{red}_{0,1,2}}$. Let $B \in \mathcal{M}_\Gamma$. We denote by F_3^{cond} an operation which replaces, roughly speaking, a full subalgebra Q of A with the fusion of Q with certain full subalgebras of B. More precisely, choose a full subalgebra Q of A and a full subalgebra V of B such that V_{ss} is a direct summand of (possibly isomorphic to) Q_{ss}. Then replace the full subalgebra Q in A by the fusion of Q and V. We denote the outcome by $(F_3^{\text{cond}})_{B,Q,V}(A)$ or simply by $(F_3^{\text{cond}})(A)$. The superscript cond means that this operation is conditional. We define $(F_3)_{B,Q,V}(A)$ as follows. Let $A^{\text{cond}} = (F_3^{\text{cond}})_{B,Q,V}(A)$. If $A^{\text{cond}} = (A^{\text{cond}})_{\text{red}_{0,1,2}}$, we set $(F_3)_{B,Q,V}(A) = A$, otherwise we set $(F_3)_{B,Q,V}(A) = A^{\text{cond}}$. As above we write $F_3(A) = (F_3)_{B,Q,V}(A)$ and have, by Proposition 2.19, that the algebras $F_3(A)$ and A are PI equivalent.

Remark 2.22. The point for introducing the conditional operation is that we want an operation of type 3 to be nontrivial only if an operation of type 0, 1 or 2 has a real effect on A^{cond}. This is to prevent the radical from growing indefinitely.

We write $A = A_{\text{red}_{0,1,2,3}}$ if every operation of type 0, 1, 2 or 3 on A is the identity.

Let us describe now the procedure applied to $A \in \mathcal{M}_\Gamma$:

(1) Apply operations of type 0 on A until any additional operation of type 0 acts as an identity. Denote the outcome by A'.

(2) If there exists an operation of type 1 with $F_1(A') \neq A'$, we apply F_1 on A' and return to step 0 with $A := F_1(A')$. We continue until we get an algebra A'' such that $F_\epsilon(A'') = A''$, $\epsilon = 0, 1$.

(3) If there exists an operation of type 2 with $F_2(A'') \neq A''$, we apply F_2 on A'' and return to step 0. We continue until we get an algebra A''' such that $F_\epsilon(A''') = A'''$, $\epsilon = 0, 1, 2$.

(4) If there exists an operation of type 3 with $F_3(A''') \neq A'''$, we apply F_3 on A''' and return to step 0. We continue until we get an algebra A'''' such that $F_\epsilon(A'''') = A''''$, $\epsilon = 0, 1, 2, 3$.

Theorem 2.23. *For every presentation $A \in \mathcal{M}_\Gamma$ the process above stops. In particular, given a presentation A, applying operations of type $0-3$ we obtain a presentation $\mathcal{A} \in \mathcal{M}_\Gamma$ such that $\mathcal{A} = \mathcal{A}_{\text{red}_{0,1,2,3}}$.*

Before giving the proof let us introduce some notation.

Definition 2.24. (1) We let A_{part} be the multiset (i.e., repetitions are allowed) of unordered tuples whose entries are the dimensions of the simple components of semisimple subalgebras of the full algebras appearing in the decomposition of A. Alternatively, we may think of A_{part} as the multiset of semisimple algebras appearing in the full algebras, summands of A.

(2) Let $A \in \mathcal{M}_\Gamma$. We denote by r_A the number of full subalgebras in the presentation of A.

(3) Let $A \in \mathcal{M}_\Gamma$ with A_{part} as above. If $\sigma = (\sigma_1, \dots, \sigma_m) \in A_{\text{part}}$, i.e., a tuple corresponding to a full algebra, summand of A, we let $n_\sigma = 2^{m^2} \sum_i \sigma_i$ be the weight of σ. Note that the function $f(m) = 2^{m^2}$ satisfies the condition $(m-1)f(m-1) < f(m)$, a condition that will be used later. We let $n_A = n_{A_{\text{part}}} = \sum_{\sigma \in A_{\text{part}}} n_\sigma$ be the *weight* of A.

Proof. We claim:

(1) Let $A \in \mathcal{M}_\Gamma$ and let $\bar{A} = F_\epsilon(A)$, $\epsilon = 0, 1$. If $\bar{A} \neq A$ then $r_{\bar{A}} < r_A$ and $n_{\bar{A}} \leq n_A$.

(2) Let $A \in \mathcal{M}_\Gamma$ and suppose $A = A_{\text{red}_{0,1}}$. Let $\bar{A} = F_2(A)$. If $\bar{A} \neq A$ then $n_{\bar{A}} < n_A$.

The first claim is clear since in these cases we are suppressing a full subalgebra of the presentation of A. Note that if we are suppressing a nilpotent algebra $n_{\bar{A}} = n_A$. For the proof of (2) let $A = A_{\text{red}_{0,1}} \in \mathcal{M}_\Gamma$. This implies no full subalgebras of A are nilpotent unless A is nilpotent, a case we have already addressed (see paragraph above Proposition 2.19). Suppose $F_2(A) \neq A$. This means that one tuple $\sigma = (\sigma_1, \dots, \sigma_m)$, $m \geq 1$ is replaced by m tuples each of which has length $m-1$ and is obtained from σ by deleting σ_i, $i = 1, \dots, m$. It follows that the quantity $2^{(m^2)} \sum_i \sigma_i$, the contribution of σ to n_A, is replaced by $(m-1)2^{((m-1)^2)} \sum_i \sigma_i$. As $(m-1)2^{((m-1)^2)} < 2^{(m^2)}$, the result follows. This proves the second claim.

Consider the pairs $\Theta_A = (n_A, r_A)$, $A \in \mathcal{M}_\Gamma$ with the lexicographic order $\preceq$ (and $\prec$ if the inequality is strict). Let $\bar{A} = F_\epsilon(A)$, $\epsilon = 0, 1, 2$. It follows that if $\bar{A} \neq A$, invoking the claims above, we have $\Theta_{\bar{A}} \prec \Theta_A$. In order to complete the proof of the Theorem we need to treat the operation F_3. We note first that F_3 does not change (and in particular does not increase) Θ_A. Recall that F_3 is effective on A, i.e., $F_3(A) \neq A$, only if $F_\epsilon(F_3(A)) \neq F_3(A)$, $\epsilon = 0, 1, 2$, and also that two operations of type 3 are always separated by an *effective* operation of type $0, 1, 2$. Finally, since the nontrivial operations of type $0, 1, 2$ lower Θ_A the result follows. $\square$

Corollary 2.25. *Given a presentation $A \in \mathcal{M}_\Gamma$, the application of steps $0-3$ to A yields a presentation $\bar{A} \in \mathcal{M}_\Gamma$ with the following properties*:

(1) *If Q is any full subalgebra of $\bar{A}$ then there exists a full subalgebra V of A such that Q_{ss} is a direct summand of V_{ss}.*

(2) *If Q is a full subalgebra of $\bar{A}$, then there is a strongly full polynomial of Q which vanishes on the supplement of Q in $\bar{A}$.*

(3) *If Q is a full subalgebra of $\bar{A}$, and $B \in \mathcal{M}_\Gamma$, then there is a strongly full polynomial of Q which vanishes on every full algebra V of B whose semisimple subalgebra V_{ss} strictly covers Q_{ss} and appears as a summand of the semisimple subalgebra of a full subalgebra of $\bar{A}$.*

Proof. By Theorem 2.23 we may assume $\bar{A} = \bar{A}_{\mathrm{red}_{0,1,2,3}}$ The operations of type 0 and 1 suppress full algebras of A whereas in operation 2 we decompose the semisimple part of a full algebra Q into the direct sum of full algebras whose semisimple part is a direct summand of Q_{ss}. This proves the first statement. Also the second statement follows easily from the construction. Indeed, if this is not the case there is an operation of type 2 which is not the identity on $\bar{A}$ contradicting $\bar{A} = \bar{A}_{\mathrm{red}_{0,1,2,3}}$.

Let us prove the last statement. By the claim we have that if such polynomial does not exist for a suitable full subalgebras V of an algebra $B \in \mathcal{M}_\Gamma$, fusion of V with the corresponding full algebras of $\bar{A}$ generates a decomposition of Q into full algebras whose semisimple algebra is a strict summand of Q_{ss}. This contradicts $\bar{A} = \bar{A}_{\mathrm{red}_{0,1,2,3}}$ and the result follows. □

Remark 2.26. Note that it is possible that a presentation $B \in \mathcal{M}_\Gamma$ contains a full algebra V whose semisimple part V_{ss} does not appear as a direct summand of a full algebra of $\bar{A}$. This does not contradict the last statement of Corollary 2.25.

Let $\bar{A}$ be the algebra obtained from A as in the theorem above and let $\bar{A} = T_1 \oplus \cdots \oplus T_n$ be its decomposition into the direct sum of full algebras. Let $\bar{A}_{\mathrm{part}}$ be the multiset of semsimple algebras appearing in $\bar{A}$, that is $\bar{A}_{\mathrm{part}} = \{(T_i)_{ss}\}_{i=1,\dots,n}$ (see Definition 2.24). Note that here we may replace "multiset" by "set" since at this stage repetitions do not occur.

Our goal is to show $\bar{A}_{\mathrm{part}}$ is uniquely determined by Γ. More precisely

Theorem 2.27. *If $A, B \in \mathcal{M}_\Gamma$ then $\bar{A}_{\mathrm{part}} = \bar{B}_{\mathrm{part}}$.*

Remark 2.28. Note that we know the result for maximal points where $A, B \in \mathcal{M}_\Gamma$ are arbitrary (see Corollary 2.16).

Proof. Suppose the theorem is false and consider the family Ω of all full subalgebras of $\bar{A}$ (resp. $\bar{B}$) whose semisimple part does not appear in $\bar{B}$ (resp. $\bar{A}$). Let $Q \in \Omega$ be maximal with respect to covering and assume without loss of generality that $Q = Q_{\bar{A}}$ is a full subalgebra of $\bar{A}$. Now, by the maximality of $Q_{\bar{A}}$ the semisimple part of every full subalgebra of $\bar{B}$ that strictly covers $Q_{\bar{A}}$ appears in $\bar{A}$. It follows, by Corollary 2.25(3), there exists a full polynomial p which vanishes on every full subalgebra of $\bar{B}$ that strictly covers $Q_{\bar{A}}$. Furthermore, by our construction of strongly full polynomials there exists such p that vanishes on every full subalgebra of $\bar{B}$ that does not cover $Q_{\bar{A}}$ and so p vanishes on $\bar{B}$. This contradicts $\bar{A}$ and $\bar{B}$ are PI equivalent and the theorem is proved. □

Step 4 (merging): In this final step we merge full subalgebras. Let $\bar{A}$ be an algebra as in the theorem. For each isomorphism type of a simple algebra $M_n(F)$ we let d_n be the maximal appearance of $M_n(F)$ in a full subalgebra of $\bar{A}$. Then we let $\mathcal{A}_{\Gamma,ss} = \Lambda_{n_1} \oplus \cdots \oplus \Lambda_{n_t}$ where Λ_{n_i} is the direct sum of d_{n_i} copies $M_{n_i}(F)$. Finally we let $\mathcal{A} \cong \mathcal{A}_{\Gamma,ss} \oplus J_A$, where the direct sum is of vector spaces.

Theorem 2.29. *There is exists an algebra structure on $\mathcal{A}_\Gamma$ so that*:

(1) $\mathrm{Id}(\mathcal{A}_\Gamma) = \Gamma$.

(2) *If B is finite dimensional and* $\mathrm{Id}(B) = \Gamma$ *then $\mathcal{A}_{\Gamma,ss}$ is isomorphic to a direct summand of B_{ss}.*

Proof. For the algebra structure on $\mathcal{A}$ we set the product as follows. The product on $\mathcal{A}_{\Gamma,ss}$ is already determined. Products of radical elements which belong to different full algebras is set to be zero. Let us determine the multiplication of semisimple elements with radicals. Using distributivity we let $z \in J_{\bar{A}_i}$ where $\bar{A}_i$ is a full summand of $\bar{A}$. Choose a summand of $(U_i)_{ss}$ of $\mathcal{A}_{\Gamma,ss}$ isomorphic to $(\bar{A}_i)_{ss}$. Let K be the semisimple supplement of $(U_i)_{ss}$ in $\mathcal{A}_{\Gamma,ss}$, that is

$$(U_i)_{ss} \oplus K \cong \mathcal{A}_{\Gamma,ss}.$$

Then we set the product of z with semsimple elements of $(U_i)_{ss}$ as in $\bar{A}_i$ whereas the multiplication of z with elements of K is set to be zero. Let us show $\mathrm{Id}(\mathcal{A}) = \Gamma$. Each $\bar{A}_i$ is isomorphic to a summand of $\mathcal{A}$ and so $\mathrm{Id}(\bar{A}) \supseteq \mathrm{Id}(\mathcal{A})$. For the opposite inclusion let p be a multilinear nonidentity of $\mathcal{A}$ and fix a nonzero evaluation on $\mathcal{A}$. Since the multiplication of radical elements of different summands $J_{\bar{A}_i}$ and $J_{\bar{A}_j}$ is zero the evaluation may involve at most radicals from $J_{\bar{A}_i}$, for a unique i. For that i, semisimple elements that appear in the evaluation must belong to the summand $(U_i)_{ss}$. We see the polynomial p is a nonidentity of $\bar{A}_i$ and so a nonidentity of $\bar{A}$. For the proof of the second statement, by the construction of $\mathcal{A}_\Gamma$ from $\bar{A}$ we see $\mathcal{A}_{\Gamma,ss}$ is a direct summand of $\bar{A}_{ss}$ and hence, by Theorem 2.27, also of $\bar{B}_{ss}$. Furthermore, we see from step 4 that every Λ_{n_i} is a direct summand of the semisimple part of a full summand of $\bar{A}$ and hence of $\bar{B}$. We complete the proof of the theorem invoking Corollary 2.25(1). $\square$

3. Nonaffine algebras

In this section we prove Theorem 1.5.

We note that the key point in the construction of strongly full polynomials of a finite dimensional full algebra A was the fact that in any nonzero evaluation we were forced to evaluate the designated variables in at least one small set by a *complete* basis of semisimple elements. Then, for such polynomial we showed it is an identity of any full algebra B that does not cover A. Now, if A is a finite dimensional full superalgebra (see [Aljadeff and Kanel-Belov 2010] or Definition 3.3 below), it is not difficult to construct a *super* strongly full polynomial with a similar property, that is, a polynomial p that visits a full basis of the semisimple part of A in every nonzero evaluation. However, this is not what we need. For the proof, we need an *ungraded* polynomial $f_{E(A)}$, nonidentity of $E(A)$, which visits the different supersimple components of A in any nonzero evaluations of the form $\epsilon \otimes u$. Here, $\epsilon = 1 \in E$ or $= \epsilon_i \in E$, where ϵ_i is a generator, and $u \in A$. Furthermore, as in the affine case, we shall need a full basis $\{u\} \subseteq A_{ss}$ to appear in every nonzero evaluation of $f_{E(A)}$. In fact, as in the affine case, we will need to construct such polynomials for $E(A)$ that belong to the T-ideal generated by an arbitrary weakly full polynomial of $E(A)$.

Once we have constructed such polynomials for $E(A)$ where A is a finite dimensional full superalgebra, we will be able to show the analogue of the Main Lemma in the nonaffine setting. The proof of Theorem 1.5 will then follow the same lines of the proof of the affine case.

We start by defining a partial ordering on finite dimensional semisimple $\mathbb{Z}_2$-graded algebras.

Let $A = A_1 \oplus \cdots \oplus A_q$ and $B = B_1 \oplus \cdots \oplus B_s$ be the decompositions of semisimple algebras A and B into direct sum of finite dimensional $\mathbb{Z}_2$-graded simple algebras A_i and B_j respectively. Consider the pair $\mathfrak{m}_A = (\mathfrak{m}_{A,0}, \mathfrak{m}_{A,1})$ where $\mathfrak{m}_{A,0} = (a_{0,1}, \ldots, a_{0,q})$ and $\mathfrak{m}_{A,1} = (a_{1,1}, \ldots, a_{1,q})$ are q-tuples consisting the dimensions of the 0-components and the 1-components of the $\mathbb{Z}_2$-graded simple summands of A. Similarly we have the pair $\mathfrak{m}_B = (\mathfrak{m}_{B,0}, \mathfrak{m}_{B,1})$ and t-tuples $\mathfrak{m}_{B,0} = (b_{0,1}, \ldots, b_{0,t})$ and $\mathfrak{m}_{B,1} = (b_{1,1}, \ldots, b_{1,q})$ for the algebra B.

Definition 3.1. We say B covers A (or $\mathfrak{m}_B$ covers $\mathfrak{m}_A$) if there exists a decomposition of the tuple $(1, \ldots, q)$ into t subsets (possibly empty) such that the sum of the elements of $\mathfrak{m}_{A,0} = (a_{0,1}, \ldots, a_{0,q})$ corresponding to the i-th subset is bounded from above by $b_{0,i}$ and the corresponding sum of odd elements in $\mathfrak{m}_{A,1} = (a_{1,1}, \ldots, a_{1,q})$ is bounded from above by $b_{1,i}$ (same i), $i = 1, \ldots, t$.

Example 3.2. Consider the pair of tuples $\mathfrak{m}_B = (\mathfrak{m}_{B_0}, \mathfrak{m}_{B_1})$ where $\mathfrak{m}_{B_0} = (17, 13)$ and $\mathfrak{m}_{B_1} = (8, 12)$. It covers the pair $\mathfrak{m}_A = (\mathfrak{m}_{A_0}, \mathfrak{m}_{A_1})$ where $\mathfrak{m}_{A_0} = (16, 10, 2)$ and $\mathfrak{m}_{A_1} = (0, 4, 2)$. On the other hand the pair $\mathfrak{m}_B = (\mathfrak{m}_{B_0}, \mathfrak{m}_{B_1})$ where $\mathfrak{m}_{B_0} = (17, 13)$ and $\mathfrak{m}_{B_1} = (8, 12)$ does not cover the pair $\mathfrak{m}_A = (\mathfrak{m}_{A_0}, \mathfrak{m}_{A_1})$ where $\mathfrak{m}_{A_0} = (10, 10, 4)$ and $\mathfrak{m}_{A_1} = (6, 6, 4)$. Note, however, that the tuple $(17, 13)$ (resp. $(8, 12)$) does cover $(10, 10, 4)$ (resp. $(6, 6, 4)$).

Let A be a finite dimensional superalgebra over an algebraically closed field F of characteristic zero. Let $A \cong A_{ss} \oplus J$ be the Wedderburn–Malcev decomposition of A. Let $A_{ss} \cong A_1 \times \cdots \times A_q$ where A_i are supersimple algebras.

Definition 3.3. We say A is full if up to ordering of the supersimple components we have $A_1 \cdot J \cdot A_2 \cdots J \cdot A_q \neq 0$.

Before stating the Main Lemma, let us make precise definitions of admissible evaluations of polynomials as well as weakly full, full and strongly full polynomials of $E(A)$ where A is a finite dimensional full superalgebra.

Let U be a finite dimensional $\mathbb{Z}_2$-simple algebra. It is well known that U is isomorphic to a superalgebra of the form (1) $M_{l,f}(F)$ where the grading is elementary and is determined by an $(l+f)$-tuple with l e's and f σ's, where an elementary matrix $e_{i,j}$ has degree e if $1 \leq i, j \leq l$ or $l+1 \leq i, j \leq l+f$ and degree σ otherwise (2) $FC_2 \otimes M_n(F)$, where FC_2 is the group (super)algebra of $C_2 = \{e, \sigma\}$, and where elements of the form $u_e \otimes e_{i,j}$ have degree e and elements of the form $u_\sigma \otimes e_{i,j}$ have degree σ. Note that the set $\{e_{i,j}\}$ (resp. $\{u_g \otimes e_{i,j}; g \in \{e, \sigma\}\}$) is a basis of $M_{l,f}(F)$ (resp. of $FC_2 \otimes M_n(F)$). We denote by β_{ss} a basis of A_{ss} consisting of all elements of that form. Note that the basis elements in β_{ss} are homogeneous. If U is any simple component of A_{ss}, and z denotes a basis element of U as above, we consider a basis Σ_{ss} of $E(A_{ss})$ consisting of all elements of the form $\epsilon_{i_1} \ldots \epsilon_{i_n} \otimes z$, n is even and $z \in \beta_{ss}$ has degree e (in case $n = 0$, we set $\epsilon_{i_1} \ldots \epsilon_{i_n} = 1$) or $\epsilon_{i_1} \ldots \epsilon_{i_n} \otimes z$, n is odd and $z \in \beta_{ss}$ has degree σ. Here $\epsilon_{i_1}, \ldots, \epsilon_{i_n}$ are different generators of the Grassmann algebra E. Finally, we choose an *homogeneous* basis β_J of the Jacobson radical J of A and consider a basis Σ_J of $E(J)$ consisting of all elements of the form $\epsilon_{i_1} \cdots \epsilon_{i_n} \otimes w$ where (as above) n is even and $w \in \beta_J$ is of degree e or n is odd and $w \in \beta_J$ is of degree σ.

Definition 3.4. Let p be a multilinear polynomial. We say an evaluation of p on $E(A)$ is admissible if all values are taken from Σ_{ss} or Σ_J.

Definition 3.5. Let A be a finite dimensional full superalgebra as above:

(1) We say a multilinear polynomial p is *weakly full* of $E(A)$ if there is an admissible nonzero evaluation of p on $E(A)$ where among the elements $\epsilon_{i_1}\cdots\epsilon_{i_n}\otimes z$, $z\in A_{ss}$ that appear in the evaluation, we have at least one elements z from each $\mathbb{Z}_2$-simple component of A_{ss}.

(2) We say a multilinear polynomial p is *full* of $E(A)$ if all $\mathbb{Z}_2$-simple subalgebras of A_{ss} are represented in every nonzero admissible evaluation of p on $E(A)$. That is, given a nonzero evaluation of p, for every $\mathbb{Z}_2$-simple component A_i, $i=1,\dots,q$, there is a variable of p whose value is of the form $\epsilon_{i_1}\cdots\epsilon_{i_n}\otimes z$ for some $z\in A_i$.

(3) We say a multilinear polynomial p is *strongly full* of $E(A)$ if for every nonzero admissible evaluation of p on $E(A)$ and every $z\in A_{ss}$, there is variable of p whose value is of the form $\epsilon_{i_1}\cdots\epsilon_{i_n}\otimes z$.

The following statement is the main lemma in the nonaffine case.

Lemma 3.6 (main lemma-nonaffine). *Suppose A and B are finite dimensional $\mathbb{Z}_2$-graded full algebras. Suppose $\mathfrak{m}_B$ does not cover $\mathfrak{m}_A$. Then there exists a strongly full polynomial $f_{E(A)}$ of $E(A)$ which is an identity of $E(B)$. Furthermore, if f_0 is an arbitrary weakly full polynomial of $E(A)$, then there exists a strongly full polynomial $f_{E(A)}\in\langle f_0\rangle_T$ of $E(A)$ which is an identity of $E(B)$.*

The proof of the main lemma will be presented in 4 propositions: (1) Construction of a strongly full polynomial $f_{E(A)}$ of $E(A)$ (Propositions 3.8 and 3.9) (2) Construction of a strongly full polynomial $f_{E(A)}\in\langle f_0\rangle_T$ of $E(A)$ where f_0 is an arbitrary weakly full polynomial of $E(A)$ (Proposition 3.10) (3) The polynomial $f_{E(A)}$ is an identity of $E(B)$ (Proposition 3.11).

We start with the construction of a strongly full polynomial of $E(A)$.

Consider the monomial

$$f_0 = X_1\cdot w_1\cdots w_{q-1}\cdot X_q$$

of degree $2q-1$ where the variables are *ungraded*. Note that f_0 is weakly full of $E(A)$ (that is, there is an admissible nonzero evaluation of f_0 which visits every $\mathbb{Z}_2$-simple component of A). We proceed with the construction of a strongly full polynomial $f_{E(A)}$ in $\langle f_0\rangle_T$.

Let d_0 (resp. d_1) be the dimension of the even (resp. odd) homogeneous component of A_{ss}. We consider a diagram composed of two strips of semisimple elements, denoted by $\alpha_{i,j}$ and similarly two strips of variables $x_{i,j}$, horizontal and vertical, where the horizontal strip has d_0 rows and k columns and the vertical strip has d_1 columns and k rows (k to be determined).

$$\boxed{\begin{array}{cccc} \alpha_{1,d_1+1} & \alpha_{1,d_1+2} & \cdots & \alpha_{1,d_1+k} \\ \alpha_{2,d_1+1} & \alpha_{2,d_1+2} & \cdots & \alpha_{2,d_1+k} \\ \vdots & \vdots & & \vdots \\ \alpha_{d_0,d_1+1} & \alpha_{d_0,d_1+2} & \cdots & \alpha_{d_0,d_1+k} \end{array}}$$

$$\boxed{\begin{array}{ccc} \alpha_{d_0+1,1} & \cdots & \alpha_{d_0+1,d_1} \\ \alpha_{d_0+2,1} & \cdots & \alpha_{d_0+2,d_1} \\ \alpha_{d_0+3,1} & \cdots & \alpha_{d_0+3,d_1} \\ \vdots & & \vdots \\ \alpha_{d_0+k,1} & \cdots & \alpha_{d_0+k,d_1} \end{array}}$$

$$\boxed{\begin{array}{cccc} x_{1,d_1+1} & x_{1,d_1+2} & \cdots & x_{1,d_1+k} \\ x_{2,d_1+1} & x_{2,d_1+2} & \cdots & x_{2,d_1+k} \\ \vdots & \vdots & & \vdots \\ x_{d_0,d_1+1} & x_{d_0,d_1+2} & \cdots & x_{d_0,d_1+k} \end{array}}$$

$$\boxed{\begin{array}{ccc} x_{d_0+1,1} & \cdots & x_{d_0+1,d_1} \\ x_{d_0+2,1} & \cdots & x_{d_0+2,d_1} \\ x_{d_0+3,1} & \cdots & x_{d_0+3,d_1} \\ \vdots & & \vdots \\ x_{d_0+k,1} & \cdots & x_{d_0+k,d_1} \end{array}}$$

Remark 3.7. The variables $x_{i,j}$ in the last two strips will appear in the polynomial $f_{E(A)}$ we are about to construct. The role of these strips is to indicate which sets of variables will alternate in $f_{E(A)}$ and which sets of variables will symmetrize. The elements $\alpha_{i,j}$ appearing in the first two strips are the evaluations of the variables $x_{i,j}$.

We construct a *long* monomial consisting of elements of A as follows.

For each $\mathbb{Z}_2$-graded simple component we write a *nonzero* product of the standard basis, namely elements of the form $e_{i,j} \in M_{l,f}(F)$ or $u_g \otimes e_{i,j} \in FC_2 \otimes M_n$ where $g = e, \sigma$. It is known that such a product exists. We refer to these elements as *designated* elements. In order to keep a unified notation we shall replace $e_{i,j} \in M_{l,f}(F)$ by $u_e \otimes e_{i,j}$. Furthermore, we may assume for simplicity that the nonzero product starts (resp. ends) with an element of the form $u_e \otimes e_{1,y}$ (resp. $u_g \otimes e_{x,1}$). Next we border each basis element $u_g \otimes e_{i,j}$ from left (resp. right) with the element $u_e \otimes e_{i,i}$ (resp. $u_e \otimes e_{j,j}$) which we call *frame*, so that the product of the monomial remains nonzero. Let us denote the product above, namely the product corresponding to the $\mathbb{Z}_2$-graded simple algebra A_i by Z_i. We take now the product of k copies of this monomial $Z_{i,1} \cdots Z_{i,k}$. This is clearly nonzero. Next, we bridge the $\mathbb{Z}_2$-graded simple components with appropriate radical values $w_{s,s+1}$ and get a nonzero product as dictated by the expression $A_1 J A_2 \cdots J A_q \neq 0$.

Finally, we tensor the basis elements with Grassmann elements, where even elements of A are tensored with 1 and odd elements are tensored with *different* generators ϵ_i (odd degree). We shall always view these tensors as ungraded elements of $E(A)$ although, abusing language, we will refer to them as even and odd elements respectively.

We obtained a nonzero expression of the form

$$Z_{1,1}\cdots Z_{1,k}\cdot w_{1,2}\cdot Z_{2,1}\cdots Z_{2,k}\cdot w_{2,3}\cdots w_{q-1,q}\cdot Z_{q,1}\cdots Z_{q,k}.$$

Consider the set $U_{\mathrm{even},1}$ of designated even elements in the tuple

$$(Z_{1,1}, Z_{2,1}, \dots, Z_{q,1}).$$

Similarly, we let $U_{\mathrm{even},i}$ be the designated even elements in the tuple $(Z_{1,i}, Z_{2,i}, \dots, Z_{q,i})$, $i = 1, \dots, k$. Observe that the cardinality of $U_{\mathrm{even},i}$ is $d_o = \dim_F A_{ss,0}$. We denote the elements of $U_{\mathrm{even},i}$ by $\alpha_{1,d_1+i}, \dots, \alpha_{d_0,d_1+i}$, that is, as the i-th column of the horizontal strip above. Furthermore, it will be convenient to denote the designated even elements in $(Z_{1,i}, Z_{2,i}, \dots, Z_{q,i})$ in the same order as they appear in the i-th column.

Similar to the even elements above, $U_{\mathrm{odd},j}$ consists of all designated odd elements in the tuple

$$(Z_{1,j}, Z_{2,j}, \dots, Z_{q,j})$$

and we denote them by $\alpha_{d_0+j,1}, \dots, \alpha_{d_0+j,d_1}$, i.e., the elements in the j-th row of the vertical strip.

For each $t = 1, \dots, k$, we alternate the designated (even) elements

$$\alpha_{1,d_1+t}, \dots, \alpha_{d_0,d_1+t}$$

and symmetrize the designated (odd) elements $\alpha_{d_0+t,1}, \dots, \alpha_{d_0+t,d_1}$. We claim the expression obtained is nonzero. Indeed, any *nontrivial* permutation (independently of its sign) of designated even elements will be surrounded by frames where not all match and hence vanishes. Similarly with the odd elements of A. In particular alternating the even elements and symmetrizing the odd elements yields a nonzero value.

We now *symmetrize* the sets of k elements corresponding to the rows of the horizontal strip and *alternate* the sets of k elements corresponding to the columns of the vertical strip. We claim we get a nonzero value. For the proof we may assume each tuple of k even elements are equal and are of the form $u_e \otimes e_{i,j}$ whereas for the odd elements we assume as we may, the elements of each k tuple have the form $\epsilon_{i,j,g} \otimes u_g \otimes e_{i,j}$, $g \in \{e, \sigma\}$, $\epsilon_{i,j,g}$ are generators of the Grassmann algebra and the elements $u_g \otimes e_{i,j}$ of A are equal. It follows that symmetrization of the rows in the horizontal strip and alternation of the columns in the vertical strip yield the multiplication of each monomial by a factor of $(k!)^{d_0}$. In particular, if the corresponding operation is performed on a vanishing product it remains zero whereas, since $\mathrm{char}(F) = 0$, it is nonzero if the operation were performed on a nonvanishing product.

We now replace the elements of $E(A)$ appearing in the monomial

$$Z_{1,1}\cdots Z_{1,k}\cdot w_{1,2}\cdot Z_{2,1}\cdots Z_{2,k}\cdot w_{2,3}\cdots w_{q-1,q}\cdot Z_{q,1}\cdots Z_{q,k}$$

by variables which we call designated variables, frames and bridges. Note that the monomial obtained is in $\langle f_0\rangle_T$ where $f_0 = X_1 \cdot w_1 \cdots w_{q-1} \cdot X_q$. It is convenient to arrange the designated variables $x_{r,s}$ in the two strips in $1-1$ correspondence with the designated elements $\alpha_{r,s} \in E(A)$. Finally, we perform the alternations and symmetrizations on these variables and obtain (by construction) a multilinear nonidentity of $E(A)$ which we denote by $f_{E(A)}$. We summarize the above paragraph in the following proposition.

Proposition 3.8. *Let A be a finite dimensional $\mathbb{Z}_2$-graded algebra over F. Suppose A is full and let $f_{E(A)}$ be as above. Then $f_{E(A)}$ is a nonidentity of $E(A)$. Furthermore, $f_{E(A)} \in \langle f_0 \rangle_T$ where $f_0 = X_1 \cdot w_1 \cdots w_{q-1} \cdot X_q$.*

Proposition 3.9. *For k large enough, the polynomial $f_{E(A)}$ is strongly full of $E(A)$.*

Proof. Suppose this is not the case. We claim that only in a bounded number of columns in the horizontal strip of the diagram we can put either radical elements or odd semisimple elements. Indeed, it is clear that the number of radical values is bounded. If we put arbitrary many odd semisimple values, by the pigeonhole principle, there will be variables in the same row which will get values of the form $\epsilon_{i_1} \cdots \epsilon_{i_n} \otimes a$ and $\epsilon_{j_1} \cdots \epsilon_{j_m} \otimes a$, same a, where n and m are odd. Then the symmetrization of the corresponding variables yields zero. Similarly, in any nonzero evaluation, the number of rows in the vertical strip of the diagram in which we can put radical or even elements is bounded. It follows then that for k large enough there exists a column in the horizontal strip, say the i-th column, which assumes only even elements and there is a j-th row in the vertical strip which assumes only odd elements. But more than that, taking k large enough we may assume $i = j$. It follows that by the alternation of the columns in the horizontal strip (resp. symmetrization of the rows in the vertical strip), in any nonzero evaluation, we are forced to evaluate these on basis elements of the form $\epsilon_{i_1} \cdots \epsilon_{i_n} \otimes a$ where a runs over a full basis of $A_{ss,0}$ (resp. $A_{ss,1}$). This proves the proposition. □

We extend the proposition, namely starting with an arbitrary weakly full polynomial f_0 of $E(A)$.

Proposition 3.10. *Let A be a finite dimensional $\mathbb{Z}_2$-graded algebra over F. Suppose A is full. Let f_0 be a multilinear weakly full polynomial of $E(A)$. Then there exists a polynomial $f_{E(A)} \in \langle f_0 \rangle_T$ which is strongly full of $E(A)$.*

Proof. Let us fix a nonzero admissible evaluation Φ of f_0 in $E(A)$ which visits all $\mathbb{Z}_2$-graded simple components of A_{ss}. Denote by $X_1, \ldots, X_q$ the variables of f_0 which assume values from the q different $\mathbb{Z}_2$-graded simple components of A. Applying the T-operation we replace the variables $X_1, \ldots, X_q$ with $X_1 \Delta_1, \ldots, X_q \Delta_q$ where $\Delta_t = Z_{t,1} \cdots Z_{t,k}$. Finally we alternate and symmetrize the designated variables as above. The polynomial obtained $f_{E(A)} \in \langle f_0 \rangle_T$ is strongly full for the algebra $E(A)$. The proof is similar to the proof above when f_0 is a monomial. Details are omitted. □

Proposition 3.11. *Let A and B be finite dimensional full superalgebras. Suppose B does not cover A. Let $f_{E(A)}$ be the polynomial constructed above. Then for k sufficiently large, the polynomial $f_{E(A)}$ is an identity of $E(B)$. More generally, suppose B is a direct sum of full superalgebras, each not covering A. Then for k sufficiently large, the polynomial $f_{E(A)}$ is an identity of $E(B)$.*

Proof. The proof is similar to the proof in the affine case. In any nonzero evaluation on $E(B)$ we must have an index i which obtains linearly independent semisimple elements of B. If the evaluation is nonzero, we must have a monomial with nonzero value and hence the semisimple elements appearing in each segment must come from the same $\mathbb{Z}_2$-graded simple component of B. We have then that B covers A. Contradiction. □

Corollary 3.12. *Let A and B full superalgebras. If $E(A)$ and $E(B)$ are PI-equivalent then their semisimple parts A_{ss} and B_{ss} are isomorphic.*

Proof. Indeed, B and A cover each other. It follows that the tuple of pairs of *dimensions* of the simple components of A and B coincide (up to a permutation). Finally we note (see below) that the superstructure of a supersimple algebra A is determined by the dimensions of A_0 and A_1 and hence if these coincide, A_{ss} and B_{ss} must be isomorphic as superalgebras. □

For the rest of the proof we follow the proof in the affine case step by step. Along the proof two basic propositions are needed.

Proposition 3.13. *Let A be a finite dimensional superalgebra over F. Then $E(A)$ is PI-equivalent to the direct sum of algebras $E(A_i)$ where A_i is a finite dimensional full superalgebra.*

Proof. Recall that a finite dimensional superalgebra A is PI-equivalent to the direct sum of full superalgebras $\mathfrak{A} = A_1 \oplus \cdots \oplus A_n$. We claim firstly: $E(A)$ and $E(\mathfrak{A})$ are PI-equivalent: Indeed, a superpolynomial f is an identity of A if and only if the superpolynomial f^* is a superidentity of $E(A)$ as a superalgebra where the 0 component is spanned by elements of the form $\epsilon_{i_1} \cdots \epsilon_{i_{2r}} \otimes a_0$ and the 1-component is spanned by elements of the form $\epsilon_{i_1} \cdots \epsilon_{i_{2r+1}} \otimes a_1$; see [Aljadeff et al. 2020, Subsection 19.4.1]. Here ϵ_j is a generator of E, $a_0 \in A^0$, $a_1 \in A^1$, the even and odd elements of A respectively. Then, if $E(A)$ and $E(\mathfrak{A})$ are PI-equivalent as superalgebras, they are PI-equivalent as ungraded algebras. Next we argue that $E(\mathfrak{A}) \cong E(A_1) \oplus \cdots \oplus E(A_n)$ and the proposition is proved. □

The second statement we need is

Proposition 3.14. *Let A and B be finite dimensional supersimple algebras over F. If $\dim_F(A^0) = \dim_F(B^0)$ and $\dim_F(A^1) = \dim_F(B^1)$ then A and B are $\mathbb{Z}_2$-graded isomorphic.*

Proof. Recall that a $\mathbb{Z}_2$-graded simple algebra over an algebraically closed field F of characteristic 0 is isomorphic to $M_{l,f}(F)$, where $l \geq 1$, $f \geq 0$ or $FC_2 \otimes_F M_n(F)$, $n \geq 1$. In the case of $M_{l,f}(F)$ the dimension of the 0-component (resp. 1-component) is $l^2 + f^2$ (resp. $2lf$) and in particular the total dimension is a square number whereas in the case of $FC_2 \otimes_F M_n(F)$ the dimensions of the homogeneous components are each equal to n^2 and hence not a square number. This proves the proposition. □

4. G-graded algebras

In this section we extend the main theorem to the setting of affine G-graded algebras where G is a finite group. The case of nonaffine G-graded algebras is treated in the next section. Here is the precise statement.

Theorem 4.1. *Let G be a finite group and let Γ be a G-graded T-ideal over F. Suppose Γ contains an ungraded Capelli polynomial c_n, some n. Then there exists a finite dimensional semisimple G-graded algebra U over F which satisfies the following conditions*:

(1) *There exists a finite dimensional G-graded algebra A over F with $\mathrm{Id}_G(A) = \Gamma$ and such that $A \cong U \oplus J_A$ is its Wedderburn–Malcev decomposition as G-graded algebras.*

(2) *If B is any finite dimensional G-graded algebra over F with $\mathrm{Id}_G(B) = \Gamma$ and B_{ss} is its maximal semisimple G-graded subalgebra, then U is a direct summand of B_{ss} as G-graded algebras.*

The proof basically follows the main lines of the proof of the ungraded case yet there is a substantial obstacle here due to the fact that G-graded simple algebras are *not determined* up to isomorphism by the dimensions of the corresponding homogeneous components. In the following examples, as usual, F is an algebraically closed field of characteristic zero.

Example 4.2. (1) If G is a finite group, $F^{\alpha}G$ and $F^{\beta}G$, $\alpha, \beta \in H^2(G, F^*)$, are twisted group algebras, then they are G-graded isomorphic if and only if $\alpha = \beta$. Clearly, the dimensions of the homogeneous components equal 1 independently of the cohomology class.

(2) Let $G = \{e, \sigma, \tau, \sigma\tau\}$ be the Klein 4-group. Consider the crossed product grading on $A \cong M_4(F)$, that is the elementary grading determined by the tuple $(e, \sigma, \tau, \sigma\tau)$, and the algebras $B_i \cong F^{\beta_i}G \otimes M_2(F)$, $\beta_1, \beta_2 \in H^2(G, F^*)$. Here β_1 (resp. β_2) is the trivial (resp. nontrivial) cohomology class on G with values on F^*. The dimension of each homogeneous component is 4. The algebras A and B_2 are isomorphic as ungraded algebras ($\cong M_4(F)$) but not isomorphic to $B_1 \cong M_2(F) \oplus M_2(F) \oplus M_2(F) \oplus M_2(F)$. It is easy to see that A and B_2 are nonisomorphic as G-graded algebras; see [Aljadeff and Haile 2014].

Let G be a finite group and let A be a finite dimensional G-graded algebra over F. We decompose A into $A_{ss} \oplus J$ where A_{ss} is a maximal G-graded semisimple algebra which supplements J, the Jacobson radical. The algebra A_{ss} decomposes into a direct product of G-graded simple components $A_1 \times \cdots \times A_q$. As in the ungraded case, the G-graded simple components are uniquely determined up to a G-graded isomorphism.

We start with the definition of the covering relation.

Definition 4.3. Let Q and V be finite dimensional G-graded semisimple algebras over F. We say V covers Q if the G-graded simple components of Q can be decomposed into subsets such that the sum of the dimensions of the corresponding homogeneous components are bounded by the dimensions of the homogeneous components of V. Explicitly, if $Q \cong Q_1 \times \cdots \times Q_q$ and $V \cong V_1 \times \cdots \times V_r$ are the decompositions of Q and V into their G-graded simple components. Let $u_{i,g} = \dim_F(Q_i)_g$ (resp. $v_{j,g} = \dim_F(V_j)_g$) be the dimension of the g-homogeneous component of Q_i (resp. of V_j). Then V covers Q if and only if the indices $1, \dots, q$ can be decomposed into r subsets $\Lambda_1, \dots, \Lambda_r$ such that $\sum_{i\in\Lambda_j} u_{i,g} \leq v_{j,g}$.

Definition 4.4. Let A be a finite dimensional G-graded algebra over F. Let $A \cong A_1 \times \cdots \times A_q \oplus J_A$ be its Wedderburn–Malcev decomposition, where A_i are G-graded simple. We say A is full if up to a permutation of the indices we have $A_1 \cdot J \cdots J \cdot A_q \neq 0$.

Next, we introduce G-graded *weakly full*, *full* and *strongly full* for a given full G-graded algebra A.

Definition 4.5. Let A be a finite dimensional full G-graded algebra. A G-graded polynomial p, nonidentity of A, is A-strongly full if it is homogeneous, multilinear and vanishes when evaluated on A unless every basis element of A_{ss} appears as a value of one of its variables. A G-graded homogeneous polynomial p is A-weakly full if there exists an admissible nonzero evaluation on A that visits each G-graded simple subalgebra of A_{ss}. Finally, p is full if this is so for *every* admissible nonzero evaluation.

Strongly full polynomials were constructed in [Aljadeff and Kanel-Belov 2010]. Nevertheless, we shall need their precise structure so let us recall here their construction.

For each G-graded simple component A_i of A consider a *nonzero* product of all basis elements of A_i. These are elements of the form $u_h \otimes e_{r,s}$, $h \in H$ and $1 \leq r, s \leq m$, whose homogeneous degree is $g_r^{-1} h g_s$. Here, the G-grading on A_i is determined by a triple $(H, \alpha, (g_1, \ldots, g_m))$ where H is a subgroup of G, α is a 2-cocycle representing a class in $H^2(H, F^*)$ and $(p_1, \ldots, p_m) \in G^{(m)}$; see [Bakhturin et al. 2008] and [Aljadeff and Haile 2014, Theorem 1.1]for more on this notation. It is known that such a product exists; see [Aljadeff and Kanel-Belov 2010]. As above we border from right and left each basis element with frames of the form $u_e \otimes e_{i,i}$. We denote such product of basis elements, namely the designated and frame elements, by Z_i. We refer to Z_i as the monomial of basis elements of A_i. We may assume the product starts with an element of the form $u_e \otimes e_{1,1}$ and ends with an element of the form $u_h \otimes e_{r,1}$ and so if $Z_{i,l} = Z_i$, $l = 1, \ldots, k$, we have that the product $Z_{i,1} \cdots Z_{i,k}$ is nonzero. Next we bridge products corresponding to different G-graded simple components by radical (homogeneous) elements w_i. We obtain a nonzero product

$$Z_{1,1} \cdots Z_{1,k} \cdot w_1 Z_{2,1} \cdots Z_{2,k} \cdot w_2 \cdots w_{q-1} \cdot Z_{q,1} \cdots Z_{q,k}.$$

As in the ungraded case we consider the i-th set Λ_i, $i = 1, \ldots, k$ consisting of the designated (semisimple) elements in $Z_{1,i}, \ldots, Z_{q,i}$. We denote by $\Lambda_{i,g}$, $g \in G$, the subset of Λ_i consisting of elements of homogeneous degree g. We claim any nontrivial permutation of designated elements in $\Lambda_{i,g}$ yields a zero product. Clearly, it suffices to consider transpositions T. The claim is clear if T exchanges basis elements which belong to G-graded simple components A_i and A_j with $i \neq j$. Suppose T exchanges basis elements $u_{h_1} \otimes e_{r_1,s_1} \neq u_{h_2} \otimes e_{r_2,s_2}$ of the same G-graded simple component. Since they are of *equal* homogeneous degree, we have that $g_{r_1}^{-1} h_1 g_{s_1} = g_{r_2}^{-1} h_2 g_{s_2}$ and so we must have $(r_1, s_1) \neq (r_2, s_2)$. This implies that frames bordering different designated elements of the same homogeneous degree are different and the claim is proved.

We proceed as in the ungraded case where the monomials consisting of elements of A are replaced by monomials of different graded variables with the corresponding homogeneous degree. The *small* sets here are alternating sets of variables of degree $g \in G$ of cardinality equal the dimension of the g-homogeneous component of A_{ss}. The polynomial obtained is denoted by p. This completes the construction of a G-graded strongly full polynomial of A. As in previous cases we shall need a more general statement.

Proposition 4.6. *Let A be a full G-graded algebra and f_0 a G-graded multilinear polynomial which is weakly full of A. Then there exists a multilinear G-graded strongly full polynomial f_A such that $f_A \in \langle f_0 \rangle_T$.*

Proof. The proof is similar to the proof of Theorem 2.8(3). □

Lemma 4.7. *Let A be a G-graded full algebra and f_A a G-graded strongly full polynomial of A with sufficiently many small sets. If B does not cover A, then f_A is an identity of B.*

Proof. The proof is similar to the proof of Lemma 2.10. □

Note that in the ungraded case this was sufficient in order to deduce that the semisimple subalgebras of A and B are isomorphic.

Theorem 4.8. *Let A and B be finite dimensional G-graded full algebras. Suppose they are G-graded PI-equivalent. Then the maximal semisimple subalgebras A_{ss} and B_{ss} are G-graded isomorphic.*

Proof. By the preceding lemma we know that A and B cover each other and hence the tuples of the dimensions of the homogeneous components of the G-graded simple algebras appearing in the decomposition of A_{ss} and B_{ss} are equal. Our goal is to show the corresponding G-graded simple components are G-graded isomorphic.

For the proof we shall need to insert *suitable* e-central polynomials in the full G-graded polynomial f_A of A constructed above. We recall from [Karasik 2019] that every finite dimensional G-graded simple admits an e-central multilinear polynomial c_A, that is a nonidentity of A, central and G-homogeneous of degree e. Furthermore, it follows from its construction, that the polynomial c_A alternates on certain sets of variables of equal homogeneous degree of cardinality equal $\dim_F(A_g)$, for every $g \in G$. For the proof of Theorem 4.8 we shall need e-central polynomials with some additional properties.

Theorem 4.9. *Let A_i, $i = 1, \dots, q$, be the simple components of A_{ss}. Then there exists a polynomial $m_i(X_G)$ with the following properties*:

(1) *$m_i(X_G)$ is e-central of A_i.*

(2) *$m_i(X_G)$ is an identity of every algebra Σ which satisfies the following conditions*:

- (a) *Σ is finite dimensional G-graded simple.*
- (b) $\dim_F(\Sigma_g) = \dim_F((A_i)_g)$ *for every $g \in G$.*
- (c) $\mathrm{Id}_G(A_i) \not\supseteq \mathrm{Id}_G(\Sigma)$.

Proof. By condition (2c) there is a G-graded homogeneous nonidentity $f_{i,\Sigma}$ of A_i, of homogeneous degree $g \in G$ say, which vanishes on Σ. Then replacing a variable of degree g in an alternating set of c_{A_i} by $f_{i,\Sigma}$ we obtain a nonidentity e-central polynomial $m_{i,\Sigma}(X_G)$ of A_i which vanishes on Σ. Now recall from [Aljadeff and Karasik 2022] that the number of G-graded simple algebras Σ satisfying conditions

(2a) and (2b) above is finite and so, because the nonzero values of $m_{i,\Sigma}(X_G)$ are invertible in F^*, we have that $m_i(X_G) = \Pi_\Sigma m_{i,\Sigma}(X_G)$ is an e-central polynomial of A with the desired properties. $\square$

Finally we insert in f_A polynomials with disjoint sets of variables $m_i(X_G)$ adjacent to each monomial $Z_{i,l}$. This completes the construction of a special strongly full polynomial which we denote by $\mathfrak{f}_A$.

We can complete now the proof of the Theorem 4.8. We are assuming the algebras A and B are PI-equivalent and so by Lemma 4.7, the algebras A and B cover each other. It follows that A_{ss} and B_{ss} have the same number of G-graded simple components. Furthermore, if $A_{ss} \cong A_1 \times \cdots \times A_q$ and $B_{ss} \cong B_1 \times \cdots \times B_q$ then there is a permutation $\sigma \in \mathrm{Sym}(q)$ such that $\dim_F((A_i)_g) = \dim_F((B_{\sigma(i)})_g)$, $i = 1, \ldots, q$ and every $g \in G$.

We claim there is a permutation of the G-graded simple components of B_{ss} such that in addition to the condition above we have that $\mathrm{Id}_G(A_i) \supseteq \mathrm{Id}_G(B_{\sigma(i)})$, $i = 1, \ldots, q$. Suppose not. Then for every permutation σ satisfying the condition above, there is a $j = j(\sigma)$ such that $\mathrm{Id}_G(A_j) \not\supseteq \mathrm{Id}_G(B_{\sigma(j)})$. We will show that the strongly full polynomial $\mathfrak{f}_A$ is an identity of B, in contradiction to the PI-equivalence of A and B. Indeed, evaluating $\mathfrak{f}_A$ on B, the value will be zero unless there is a monomial Z_i, together with the inserted central polynomials, whose value is nonzero. This implies there is a permutation σ of the components of B_{ss} such that the i-th segment of p is evaluated on $B_{\sigma(i)}$. This already implies the condition above on the dimensions. But by assumption there is j such that $\mathrm{Id}_G(A_j) \not\supseteq \mathrm{Id}_G(B_{\sigma(j)})$ and so the central polynomial $m_j(X_G)$ vanishes on $B_{\sigma(j)}$.

We conclude there is a permutation $\sigma \in \mathrm{Sym}(q)$ of the simple components of B_{ss} such that:

(1) $\dim_F((A_i)_g) = \dim_F((B_{\sigma(i)})_g)$, $i = 1, \ldots, q$, and every $g \in G$

(2) $\mathrm{Id}_G(A_i) \supseteq \mathrm{Id}_G(B_{\sigma(i)})$, $i = 1, \ldots, q$.

Our goal is to show that in fact $\mathrm{Id}_G(A_i) = \mathrm{Id}_G(B_{\sigma(i)})$, $i = 1, \ldots, q$. Indeed, this would imply what we need, that is $A_i \cong B_{\sigma(i)}$, $i = 1, \ldots, q$, as G-graded algebras; see [Aljadeff and Haile 2014].

Suppose that G is abelian. In that case let us recall the following general result of O. David [2012].

Theorem 4.10. *Let G be a finite abelian group and let A and B finite dimensional G-graded simple algebras over an algebraically closed field F. Then there is an embedding $A \hookrightarrow B$ as G-graded algebras if and only if* $\mathrm{Id}_G(A) \supseteq \mathrm{Id}_G(B)$.

Clearly, it follows at once from the theorem that G-graded algebras satisfying conditions (1) and (2) above must be G-graded isomorphic. David's result is not known in case G is an arbitrary finite group.

Here, instead, we argue as follows. By symmetry there is a permutation $\tau \in \mathrm{Sym}(q)$ such that:

(1) $\dim_F((B_i)_g) = \dim_F((A_{\tau(i)})_g)$, $i = 1, \ldots, q$ and every $g \in G$.

(2) $\mathrm{Id}_G(B_i) \supseteq \mathrm{Id}_G(A_{\tau(i)})$, $i = 1, \ldots, q$.

Consequently there is a permutation $\rho \in \mathrm{Sym}(q)$ such that A_i and $A_{\rho(i)}$ have equal dimensions of the homogeneous components and $\mathrm{Id}_G(A_i) \supseteq \mathrm{Id}_G(A_{\rho(i)})$. We need to show equality holds. Indeed, we see that $\mathrm{Id}_G(A_i) = \mathrm{Id}_G(A_j)$ for i and j which belong to the same orbit determined by ρ and so, in particular $\mathrm{Id}_G(A_i) = \mathrm{Id}_G(A_{\rho(i)})$, $i = 1, \ldots, q$. $\square$

The remaining steps in the proof of Theorem 4.1 are similar to those in the proof of Theorem 1.1. Details are omitted.

5. PI-equivalence of Grassmann envelopes of finite dimensional G_2-graded algebras

In this section we treat the case where the algebra A is finite dimensional $\mathbb{Z}_2 \times G$-graded and $E(A)$ is the Grassmann envelope of A viewed as a G-graded algebra.

The main result in this case is the following.

Theorem 5.1. *Let G be a finite group. Let Γ be a G-graded T-ideal. Suppose Γ contains a nonzero ungraded polynomial but contains no ungraded Capelli c_n, any n. Then there exists a finite dimensional semisimple $\mathbb{Z}_2 \times G$-graded algebra U over F which satisfies the following conditions*:

(1) *There exists a finite dimensional $\mathbb{Z}_2 \times G$-graded algebra A over F with* $\mathrm{Id}_G(E(A)) = \Gamma$ *and such that $A \cong U \oplus J_A$ is its Wedderburn–Malcev decomposition as $\mathbb{Z}_2 \times G$-graded algebras.*

(2) *If B is any finite dimensional $\mathbb{Z}_2 \times G$-graded algebra over F with* $\mathrm{Id}_G(E(B)) = \Gamma$ *and B_{ss} is its maximal semisimple $\mathbb{Z}_2 \times G$-graded subalgebra*, *then U is a direct summand of B_{ss} as $\mathbb{Z}_2 \times G$-graded algebras.*

The general approach is based on cases that were treated earlier, namely the cases where (1) Γ is a T-ideal of identities of a G-graded affine algebra (2) Γ is a T-ideal of identities of an ungraded nonaffine algebra. It turns out however, that also here there is a substantial difficulty, and this is in the very first step of the general approach (see Theorem 5.2 below). In fact, nearly the entire section will be devoted to the proof of Theorem 5.2.

Before we state the theorem let us set some notation.

Let G be a finite group and denote $G_2 := \mathbb{Z}_2 \times G$. We denote $G_{\text{even}} := 0 \times G$; $G_{\text{odd}} = 1 \times G$ and similarly for a G_2 algebra A we write $A_{\text{even}} = A_{G_{\text{even}}}$; $A_{\text{odd}} = A_{G_{\text{odd}}}$.

Theorem 5.2. *Suppose that A and B are two finite dimensional G_2-graded simple algebras. Then A and B are G_2-graded isomorphic if and only if $E(A)$ and $E(B)$ have the same G-graded identities.*

It is worth noting that the Grassmann $*$ operation allows one to pass from a superidentity of A to a superidentity of $E(A)$ (resp. from a G_2-identity of A to a G_2-identity of $E(A)$). The challenge here lies in transforming a superidentity of A into an ordinary identity of $E(A)$ (resp. from a G_2-identity of A into a G-identity of $E(A)$). The main part of the proof of the above Theorem is to find such a transformation.

We start with the construction of the transformation and in Proposition 5.5 we show the key property that makes it work. We emphasize that the construction and also the Theorem are guaranteed to work only in the case where the algebras in question are finite dimensional G_2-graded simple. In general it is not true that if $E(A)$ and $E(B)$ have the same G-graded identities then A and B have the same G_2-graded identities. An example can be found in [Giambruno and Zaicev 2005, Section 8.2].

The construction we are about to present is a generalization to the G-graded setting of the one in Section 3. Its main property appears in Proposition 5.5. In fact, the previous construction could be applied also here. And if we did, it would enable us to show as above that if $E(A)$ and $E(B)$ have the same G-graded identities then $\dim A_{\bar{g}} = \dim B_{\bar{g}}$ for all $\bar{g}$. However this would not be sufficient here since, as pointed out in the previous section, for general groups G one can easily find examples of nonisomorphic G_2-graded simple algebras having this property.

Let $f = f(X_0; Y_0)$ be a multilinear G_2-graded polynomial, where

$$X_0 = \coprod_{\bar{g} \in G_2} \coprod_{i=1}^{T} X_{\bar{g},i}$$

is a union of T *small sets* of degree $\bar{g}$-variables $X_{\bar{g},i} = \{x_{\bar{g},i}^{(1)}, \ldots, x_{\bar{g},i}^{(\dim A_{\bar{g}})}\}$ (here $\bar{g}$ runs over all of G_2), and $Y_0 = \coprod_{\bar{g} \in G_2} Y_{\bar{g},0}$ are some additional variables. Assume that f has a G_2-graded evaluation $\phi : F\langle X_0; Y_0\rangle \to A$ with the following properties:

(1) For every nontrivial permutation $\sigma \in \prod_{\bar{g} \in G_2} \prod_{i=1}^{T} S_{X_{\bar{g},i}}$ (here S_W is the symmetric group on the set W) the value of $f(\sigma(X_0); Y_0)$ under the evaluation ϕ is 0.

(2) For all $\bar{g} \in G_2$ the value $\phi(x_{\bar{g},i}^{(j)}) =: a_{\bar{g}}^{(j)}$ is independent of $i = 1, \ldots, T$. Furthermore, all $a_{\bar{g}}^{(j)}$, $j = 1, \ldots, \dim A_{\bar{g}}$, are distinct.

We will see later that in the case which is relevant to the proof of Theorem 5.2 it is indeed possible to construct such a polynomial.

Let $k > 0$ be a natural number and consider the polynomial

$$f_k := f(X_1; Y_1) \cdots f(X_k; Y_k),$$

where all X_t and Y_t are disjoint copies of X_0 and Y_0 respectively. Notice that

$$X_t = \coprod_{\bar{g} \in G_2} \coprod_{i=1}^{T} X_{\bar{g},(t-1)T+i}.$$

We extend ϕ to $F\langle X; Y\rangle$, where $X = \coprod_{t=1}^{k} X_t$ and $Y = \coprod_{t=1}^{k} Y_t$, by duplicating the evaluation on X_0 and Y_0 to X_t and Y_t respectively (for all $t = 1, \ldots, k$). As a result, we have in particular for all $\bar{g}$, i and j that $\phi(x_{\bar{g},i}^{(j)}) = a_{\bar{g}}^{(j)}$ (we rely here on property (2)).

For $a \in A$ we set $X_\phi(a) \subset X$ to be all the variables from X which ϕ assigns to them the value a. In other words, $X_\phi(a) = (\phi|_X)^{-1}(a)$. In particular, $X_\phi(a_{\bar{g}}^{(j)}) = \{x_{\bar{g},1}^{(j)}, \ldots, x_{\bar{g},kT}^{(j)}\}$.

Remark 5.3. For every $\bar{g} \in G_2$ we have

$$\coprod_{i=1}^{kT} X_{\bar{g},i} = \coprod_{j=1}^{\dim A_{\bar{g}}} X_\phi(a_{\bar{g}}^{(j)}).$$

One should visualize this equality as "union of columns" (the $X_{\bar{g},i}$'s) = "union of rows" (the $X_\phi(a_{\bar{g}}^{(j)})$) in the matrix

$$\begin{bmatrix} x_{\bar{g},1}^{(1)} & \cdots & x_{\bar{g},kT}^{(1)} \\ \vdots & x_{\bar{g},i}^{(j)} & \vdots \\ x_{\bar{g},1}^{(\dim A_{\bar{g}})} & \cdots & x_{\bar{g},kT}^{(\dim A_{\bar{g}})} \end{bmatrix}.$$

Next, we alternate and symmetrize different subsets of X in the following fashion to obtain a new graded polynomial $s_{k;A}$. For each even (odd) element $\bar{g} \in G_2$ we apply alternation (symmetrization) on all variables $X_{\bar{g},i}$; afterwards we apply symmetrization (alternation) for every set of variables of the form $X_\phi(a)$. All in all, we have

$$s_{k;\phi;A}(f) = \prod_{\bar{g}\in G_{\text{odd}}} \prod_{j=1}^{\dim A_{\bar{g}}} \mathrm{Alt}_{X_\phi(a_{\bar{g}}^{(j)})} \circ \prod_{\bar{g}\in G_{\text{even}}} \prod_{j=1}^{\dim A_{\bar{g}}} \mathrm{Sym}_{X_\phi(a_{\bar{g}}^{(j)})} \circ \prod_{\bar{g}\in G_{\text{odd}}} \prod_{i=1}^{kT} \mathrm{Sym}_{X_{\bar{g},i}} \circ \prod_{\bar{g}\in G_{\text{even}}} \prod_{i=1}^{kT} \mathrm{Alt}_{X_{\bar{g},i}}(f_k).$$

We also consider a "forgetful" operator $F_G^{G_2}$ which transforms G_2-graded polynomials into G-graded polynomials by changing the degree of every variable from $(\epsilon, g) \in G_2$ to $g \in G$. We finally have the G-graded polynomial

$$F_G^{G_2}(s_{k;\phi;A}(f)).$$

We remark that for $g \in G$ the variables $F_G^{G_2}(x_{(0,g),t})$ and $F_G^{G_2}(x_{(1,g),t})$ are two different variables of degree $g \in G$.

Definition 5.4. Let B be a G_2-graded algebra. An evaluation of a G-graded polynomial f on B is called *almost* G_2 if every variable x of f of degree g is evaluated in some $B_{(\epsilon,g)}$.

Furthermore, if B_0 is a subset of B, we say that an evaluation ψ of f on B is a B_0*-evaluation* if every variable of f is evaluated in B_0.

Suppose we have a G_2-graded polynomial f and consider the G-polynomial $F_G^{G_2}(f)$. Note that if ψ is an almost G_2-evaluation of $F_G^{G_2}(f)$, then typically there is no reason that $\deg \psi(F_G^{G_2}(x_{\bar{g}})) = \bar{g}$ (i.e., the parities might not agree). The next Proposition shows that our construction of $F_G^{G_2}(s_{k;\phi;A})(f)$ will ensure that "almost always" the above equality occurs *given that* ψ gives a nonzero value to the polynomial.

Proposition 5.5. *Let B be a finite dimensional G_2-graded algebra. If*

$$\psi : F\langle X_0; Y_0\rangle \to E(B)$$

is a nonzero almost G_2-evaluation, then for every $\bar{g} \in G_2$ we have

$$\deg \psi(F_G^{G_2}(x_{\bar{g},i})) = \bar{g},$$

except possibly for $\dim A \cdot \dim B$ *of the i.*

Furthermore, if there is some $\bar{g}_0 = (\epsilon_0, g_0) \in G_2$ *such that the dimension of* $B_{\bar{g}_0}$ *is strictly smaller than that of* $A_{\bar{g}_0}$, *then for* $k > \dim A \cdot \dim B$, *the polynomial* $F_G^{G_2}(s_{k;\phi;A}(f))$ *is an identity of* $E(B)$.

Proof. We focus on proving the "furthermore" part and along the way we get a proof for the main claim. In order to show that $F_G^{G_2}(s_{k;\phi;A}(f))$ is an identity of $E(B)$, it is enough to show that it is 0 under any almost G_2-evaluation of $F_G^{G_2}(s_{k;\phi;A}(f))$, since this polynomial is multilinear. Let $\psi : F\langle X_0; Y_0\rangle \to E(B)$ be an almost G_2-evaluation of $F_G^{G_2}(s_{k;\phi;A}(f))$.

Suppose that $\psi(F_G^{G_2}(s_{k;\phi;A}(f))) \neq 0$. Then, there is some

$$\sigma \in \prod_{\bar{g}\in G_{\text{odd}}} \prod_{i=1}^{kT} S_{X_{\bar{g},i}} \cdot \prod_{\bar{g}\in G_{\text{even}}} \prod_{i=1}^{kT} S_{X_{\bar{g},i}}$$

such that, under ψ, the polynomial

$$F_G^{G_2}\Bigg(\prod_{\bar{g}\in G_{\text{odd}}} \prod_{j=1}^{\dim A_{\bar{g}}} \mathrm{Alt}_{X_\phi(a_{\bar{g}}^{(j)})} \circ \prod_{\bar{g}\in G_{\text{even}}} \prod_{j=1}^{\dim A_{\bar{g}}} \mathrm{Sym}_{X_\phi(a_{\bar{g}}^{(j)})}(f_k(\sigma(X)); Y)\Bigg) \neq 0.$$

Notice that for all i, the set $X_{\bar{g},i}$ stays the same even after applying σ.

We claim that all *small sets* $F_G^{G_2}(X_{\bar{g}_0,i})$, except possibly $\dim A \cdot \dim B$ of them, have all of their variables assigned to elements of degree $\bar{g}_0$. Indeed, we only need to show that the parity is ϵ_0. Assume that $\epsilon_0 = 0$ (the proof for $\epsilon_0 = 1$ is similar).

If on the contrary there are more than $\dim A \cdot \dim B$ small sets $F_G^{G_2}(X_{\bar{g}_0,i})$ having at least one variable which has an odd evaluation, as $k > \dim A \cdot \dim B \geq \dim A_{\bar{g}_0} \cdot \dim B_{(1,g_0)}$, and in view of Remark 5.3, there is some $l_0 \in \{1, \dots, \dim A_{\bar{g}_0}\}$ such that at least $\dim B_{(1,g_0)}$ distinct variables from $F_G^{G_2}(X_\phi(a_{\bar{g}_0}^{(l_0)}))$ are assigned by ψ values from $B_{(1,g_0)} \otimes E_1$. However as we symmetrize that set, we must get 0 — a contradiction. Notice that we have also proved here the main claim.

Denote by $F_G^{G_2}(X_{\bar{g}_0,i_0})$ a small set with the property from the previous paragraph. Since $\dim B_{\bar{g}_0} < \dim A_{\bar{g}_0}$, the alternation (symmetrization) of size $\dim A_{\bar{g}_0}$ must nullify the polynomial. □

We are now ready to prove Theorem 5.2:

Proof of Theorem 5.2. By [Aljadeff and Haile 2014], A and B are G_2-isomorphic if and only if A and B share the same G_2-identities; see also [Bahturin and Yasumura 2019] for a far reaching generalization of the statement in [Aljadeff and Haile 2014]. As a result, it is enough to show that if A and B are not G_2-PI-equivalent, then $E(A)$ and $E(B)$ are not G-PI-equivalent.

Assume, without loss of generality, that there is a multilinear G_2-polynomial $p(x_{\bar{g}_1,1}, \dots, x_{\bar{g}_n,n})$ which is an identity of B but not of A. We consider the G_2-graded basis $\mathcal{B}_A = \{a_{\bar{g}}^{(j)} : \bar{g} \in G_2,\ j = 1, \dots, \dim_F A_{\bar{g}}\}$ of A as in [Aljadeff and Haile 2014] Theorem 1.1. Let ϕ be a nonzero $\mathcal{B}_A$-evaluation of p. We may also assume that $\phi(p) = \delta$, where δ is a nonzero idempotent of A. In the next few paragraphs we are going to construct a G_2-graded polynomial f from p on which we will perform the construction from the beginning of the section to obtain a polynomial $F_G^{G_2}(s_{k;\phi;A}(f))$ which will be an identity of $E(B)$ and a nonidentity of $E(A)$.

For $i = 1, \dots, n$ let $X(i) = \{x_{\bar{g},i}^{(j)} : \bar{g} \in G_2,\ j = 1, \dots, \dim A_{\bar{g}}\}$ be disjoint variables from the ones of p and set $\phi(x_{\bar{g},i}^{(j)}) = a_{\bar{g}}^{(j)}$. For every i let $j(i)$ be such that $\phi(x_{\bar{g}_i,i}) = a_{\bar{g}_i}^{(j(i))}$. We identify $x_{\bar{g}_i,i}$ with $x_{\bar{g}_i,i}^{(j(i))}$ for every i. We set $X_0 = \coprod_{i=1}^{n} X(i)$.

Similarly to the construction in the proof of Theorem 4.8, one can construct a multilinear G_2-monomial $M = M(X_0; Y)$ with the property that there is an evaluation ϕ_Y of the Y-variables such that the only extension of ϕ_Y to a nonzero $\mathcal{B}_A$-evaluation ϕ_Z of $M(X_0; Y)$ must satisfy $\phi_M|_{X_0} = \phi|_{X_0}$(i.e., ϕ_M also extends ϕ) and if ϕ_M satisfies $\phi_M|_{X_0} = \phi|_{X_0}$ then $\phi_M(M) = \delta$. In what follows we shall denote the unique nonzero evaluation ϕ_M of M by ϕ. Furthermore, one can also arrange that $\phi(M) = \delta$.

Clearly, $\phi(M \cdot p) = \delta$. However, $M \cdot p$ is not multilinear, and so we make some small changes to solve this issue. Consider a new set of variables $z_{\bar{g}_1,1}, \dots, z_{\bar{g}_n,n}$ and replace in Z (only) the variables $z_{\bar{g}_i,i}$ by $x_{\bar{g}_i,i}$ for every i and let M' be the new polynomial. Clearly $M' \cdot p$ is multilinear. We extend ϕ to include all the z-variables by declaring $\phi(z_{\bar{g}_i,i}) = \phi(x_{\bar{g}_i,i})$ so that $\phi(Mp) = \phi(M'p) = \delta$.

Finally let

$$f = M' \cdot p^*,$$

where $*$ is the Grassmann star operation.

We claim that f satisfies properties (1)–(2): By construction property (2) holds. Hence we are left with verifying property (1). Indeed, any nontrivial permutation of any of the variables in some $X(i)$ induces a new evaluation of f, which we call ϕ', that differs from ϕ only on the set $X(i)$. By the construction of M (and M') we get that $\phi'(M') = 0$; showing property (1).

We now consider our final polynomial $F_G^{G_2}(s_{k;\phi;A}(f))$, where $k = n \cdot \dim A \cdot \dim B + 1$. Notice that it is a G-polynomial and that the construction also extends ϕ to an evaluation of all of $s_{k;\phi;A}(f)$ (a G_2-graded evaluation!). We claim that it is an identity of $E(B)$ but not of $E(A)$. It is not an identity of $E(A)$ since we can consider the following G-evaluation ψ in $E(A)$: for every variable v appearing in $s_{k;\phi;A}(f)$ we set

$$\psi(F_G^{G_2}(v)) = \phi(v) \otimes w_v,$$

where $w_v \in E_{\deg v}$ and all the w_v are chosen so that the product of all of them is nonzero. By the definition of $*$, we have that $\psi(F_G^{G_2}(p^*)) = \delta \otimes \prod_{v \in p} w_v$ and so $\psi(F_G^{G_2}(f_k)) = \delta \otimes \prod_{v \in f_k} w_v$. By property (1) of f we conclude that

$$\psi\left(F_G^{G_2}\left(\prod_{\bar{g}\in G_{\text{odd}}} \prod_{i=1}^{kn} \mathrm{Sym}_{X_{g,i}} \circ \prod_{\bar{g}\in G_{\text{even}}} \prod_{i=1}^{kn} \mathrm{Alt}_{X_{\bar{g},i}}(f_k)\right)\right) = \psi(F_G^{G_2}(f_k)) = \delta \otimes \prod_{v\in f_k} w_v.$$

Finally, since ϕ gives the same value $a_{\bar{g}}^{(i)}$ for every variable in $X_\phi(a_{\bar{g}}^{(i)})$, we have that

$$\psi(F_G^{G_2}(s_{k;\phi;A}(f))) = C \cdot \delta \otimes \prod_{v\in f_k} w_v \neq 0,$$

where $C = \prod_{\bar{g}\in G_2} \prod_{i=1}^{\dim A_{\bar{g}}} |X_\phi(a_{\bar{g}}^{(i)})|! = ((kn)!)^{\dim A}$.

We are left with showing that $F_G^{G_2}(s_{k;\phi;A}(f))$ is an identity of $E(B)$. Suppose that $F_G^{G_2}(s_{k;\phi;A}(f))$ is a nonidentity of $E(B)$. Hence there is a nonzero almost G_2-evaluation ψ on $E(B)$. As $\psi(F_G^{G_2}(s_{k;\phi;A}(f))) \neq 0$, there are two permutations

$$\sigma \in \prod_{\bar{g}\in G_2} \prod_{i=1}^{\dim A_{\bar{g}}} S_{X_\phi(a_{\bar{g}}^{(i)})},\ \tau \in \prod_{\bar{g}\in G_2} \prod_{i=1}^{kn} S_{X_{\bar{g},i}}$$

such that

$$\psi(F_G^{G_2}(f_k(\sigma\tau(X), Y, Z))) \neq 0.$$

Clearly, $\sigma\tau(X_i) = \sigma(X_i)$ and σ preserves the G_2-degree. By Proposition 5.5 and the choice of k, there is some $i_0 \in \{1, \dots, k\}$ such that for every $x_{\bar{g}} \in \sigma(X_{i_0})$ we have that $\deg \psi(x_{\bar{g}}) = \bar{g}$. As a result, as p is an identity of B, we can deduce that $\psi(p^*(\sigma(X_{i_0}))) = 0$ and so also $\psi(f(\sigma(X_{i_0}), Y, Z)) = 0$. This clearly forces that $\psi(F_G^{G_2}(f_k(\sigma\tau(X), Y, Z))) = 0$, hence reaching a contradiction. □

We may extend Theorem 5.2 to full $G_2 = \mathbb{Z}_2 \times G$-graded algebras.

Theorem 5.6. *Let A and B be finite dimensional G_2-graded algebras over F. Suppose A and B are full. If $E(A)$ and $E(B)$ are G-graded PI-equivalent then the semisimple parts A_{ss} and B_{ss} are isomorphic as G_2-algebras.*

Proof. For the proof we shall combine the constructions in Section 3 and Section 4, that is for nonaffine ungraded algebras and for affine G-graded algebras, together with the Theorem 5.2. For each G_2-graded simple algebra A_i we let $\mathcal{B}_{A_i}$ be a basis of A_i whose elements are G_2-homogeneous of the form $\{u_h \otimes e_{r,s}\}$. Let K_i denote a nonzero product of the elements in $\mathcal{B}_{A_i}$. We refer to these elements as *designated elements*. Each basis element is bordered by basis elements where for convenience we may assume all but possibly one are of the form $1 \otimes e_{i,j}$. As usual we refer to these as *frame elements*. We may use one of the frame elements so the value of the monomial is an idempotent δ of A. We denote this product by Z_i. We let $Z_{i,j}$, $j = 1, \dots, k$ be a duplicate of the monomial Z_i and let $\bar{Z}_{i,k} = Z_{i,1} \cdot Z_{i,2} \cdots Z_{i,k}$. Here, k is a large integer which needs to be determined. We let $\Theta_l = (a, \dots, a)$ be the k-tuple where a is the l-th element appearing in the monomial K_i. Since the algebra A is full, we have up to ordering of the G_2-graded simple components of A a nonvanishing product $\bar{Z}_{1,k} \cdot w_1 \cdot \bar{Z}_{2,k} \cdots w_{q-1} \cdot \bar{Z}_{q,k} \neq 0$. For every $\bar{g} \in G_2$ we consider k *small sets* , each consisting of $\dim_F(A_{ss})_{\bar{g}}$ designated elements where the j-th small set consists of the designated elements in $K_{1,j}, \dots, K_{q,j}$. We have as in previous cases that any nontrivial permutation on a small set leads to a zero product. Our next step is to tensor even elements with the identity of E (the Grassmann algebra), and odd elements with different generators of E. Note that the product remains nonzero. As in previous cases we will view the elements obtained as G-graded elements but for convenience we will still refer to them using the adjective even or odd. Moreover we shall refer as small sets, a set of the form $(1 \otimes a_1, \dots, 1 \otimes a_m)$ where $(a_1, \dots, a_m)$ is a small set of even homogeneous elements of degree $(0, g)$, $g \in G$ or a set the form $(\epsilon_1 \otimes b_1, \dots, \epsilon_m \otimes b_m)$ where $(b_1, \dots, b_m)$ is a small set of odd homogeneous elements of degree $(1, g)$, $g \in G$. By abuse of notation we keep the notation Θ_l after multiplying the basis elements with Grassmann generators.

Next we alternate and symmetrize small sets of even and odd elements respectively. Then we symmetrize sets $\Theta_l = (a, \dots, a)$ where a is even and alternate sets $\Theta_t = (b, \dots, b)$ where b is odd. One shows the product is nonzero.

Next we replace the designated elements by X variables, the frames by Y's and the bridges by W's where we forget the $\mathbb{Z}_2$-degree, that is X, Y, W are G-graded variables. Clearly by construction we have a nonidentity f of A. Let us denote the nonzero evaluation above by ϕ. As in previous cases with such polynomial one shows that if B_{ss} does not cover A_{ss} as G_2-algebras then f is a nonidentity of $E(A)$ and an identity of $E(B)$ as a G-graded algebras. Thus, since we are assuming $E(A)$ and $E(B)$ are G-graded PI-equivalent we have that A and B cover each other as G_2-graded algebras. We conclude that up to permutation of the simple components of B we have $A \cong A_1 \times \cdots \times A_q \oplus J_A$ and $B \cong B_1 \times \cdots \times B_q \oplus J_B$ where $\dim_F(A_j)_g = \dim_F(B_j)_g$, $g \in G_2$. We want to prove there is a permutation on the G_2-graded simple components of B such that $A_j \cong B_j$ as G_2-graded algebras.

Recall from the Theorem 5.2 above that if $\dim_F(A_j)_g = \dim_F(B_{j'})_g$, all $g \in G_2$, for some j and j', there exists a G-polynomial $p_{j,j'}$ which is a G-graded nonidentity of $E(A_j)$ and an identity of $E(B_{j'})$ unless A_j and $B_{j'}$ are G_2-graded isomorphic. Moreover, we may assume the value of the polynomial $p_{j,j'}$ is the idempotent δ of A we fixed above. Denote by $p_i = \prod_{j'} p_{i,j'}$. We note that p_i is a G-polynomial nonidentity of $E(A_i)$ and an identity of $E(B_{j'})$ for every G_2-graded simple algebra whose dimension of the homogeneous G_2-components are equal to the corresponding dimensions of the homogeneous components of A_j but is not isomorphic to A_j. Finally, we insert to the right of every monomial $Z_{i,l}$ a copy of the polynomial p_i with disjoint variables. The polynomial obtained m_A is a G-graded nonidentity of $E(A)$. By assumption it is a nonidentity of $E(B)$, which forces the existence of a permutation on the G_2-graded simple components of B such that $A_j \cong B_j$ as G_2-graded algebras. This completes the proof. □

We can now complete the proof of Theorem 5.1 as in the proof of Theorem 1.1, that is by performing Steps $0-4$ on the set of finite dimensional G_2-graded algebras A with $\mathrm{Id}_G(E(A)) = \Gamma$ (see Section 2). Details are omitted.

References

[Aljadeff and Haile 2014] E. Aljadeff and D. Haile, "Simple G-graded algebras and their polynomial identities", *Trans. Amer. Math. Soc.* **366**:4 (2014), 1749–1771. MR Zbl

[Aljadeff and Kanel-Belov 2010] E. Aljadeff and A. Kanel-Belov, "Representability and Specht problem for G-graded algebras", *Adv. Math.* **225**:5 (2010), 2391–2428. MR Zbl

[Aljadeff and Karasik 2022] E. Aljadeff and Y. Karasik, "On generic G-graded Azumaya algebras", *Adv. Math.* **399** (2022), Paper No. 108292, 42. MR Zbl

[Aljadeff et al. 2016] E. Aljadeff, A. Kanel-Belov, and Y. Karasik, "Kemer's theorem for affine PI algebras over a field of characteristic zero", *J. Pure Appl. Algebra* **220**:8 (2016), 2771–2808. MR Zbl

[Aljadeff et al. 2020] E. Aljadeff, A. Giambruno, C. Procesi, and A. Regev, *Rings with polynomial identities and finite dimensional representations of algebras*, American Mathematical Society Colloquium Publications **66**, American Mathematical Society, Providence, RI, 2020. MR Zbl

[Bahturin and Yasumura 2019] Y. Bahturin and F. Yasumura, "Graded polynomial identities as identities of universal algebras", *Linear Algebra Appl.* **562** (2019), 1–14. MR Zbl

[Bakhturin et al. 2008] Y. A. Bakhturin, M. V. Zaĭtsev, and S. K. Segal, "Finite-dimensional simple graded algebras", *Mat. Sb.* **199**:7 (2008), 21–40. In Russian; translated in *Sb. Math.* **199**:7–8 (2008), 965–983. MR

[Belov 2010] A. Y. Belov, "Local finite basis property and local representability of varieties of associative rings", *Izv. Ross. Akad. Nauk Ser. Mat.* **74**:1 (2010), 3–134. In Russian; translated in *Izv. Math.* **74**:01 (2010), 1–126. MR Zbl

[Belov-Kanel et al. 2010] A. Belov-Kanel, L. Rowen, and U. Vishne, "Structure of Zariski-closed algebras", *Trans. Amer. Math. Soc.* **362**:9 (2010), 4695–4734. MR Zbl

[Belov-Kanel et al. 2011] A. Belov-Kanel, L. Rowen, and U. Vishne, "Application of full quivers of representations of algebras, to polynomial identities", *Comm. Algebra* **39**:12 (2011), 4536–4551. MR Zbl

[Belov-Kanel et al. 2012] A. Belov-Kanel, L. H. Rowen, and U. Vishne, "Full quivers of representations of algebras", *Trans. Amer. Math. Soc.* **364**:10 (2012), 5525–5569. MR Zbl

[David 2012] O. David, "Graded embeddings of finite dimensional simple graded algebras", *J. Algebra* **367** (2012), 120–141. MR Zbl

[Giambruno and Zaicev 1998] A. Giambruno and M. Zaicev, "On codimension growth of finitely generated associative algebras", *Adv. Math.* **140**:2 (1998), 145–155. MR Zbl

[Giambruno and Zaicev 2005] A. Giambruno and M. Zaicev, *Polynomial identities and asymptotic methods*, Mathematical Surveys and Monographs **122**, American Mathematical Society, Providence, RI, 2005. MR Zbl

[Kanel-Belov and Rowen 2005] A. Kanel-Belov and L. H. Rowen, *Computational aspects of polynomial identities*, Research Notes in Mathematics **9**, A K Peters, Ltd., Wellesley, MA, 2005. MR Zbl

[Karasik 2019] Y. Karasik, "G-graded central polynomials and G-graded Posner's theorem", *Trans. Amer. Math. Soc.* **372**:8 (2019), 5531–5546. MR Zbl

[Kemer 1987] A. R. Kemer, "Finite basability of identities of associative algebras", *Algebra i Logika* **26**:5 (1987), 597–641, 650. MR Zbl

[Prochezi 2016] K. Prochezi, "The geometry of polynomial identities", *Izv. Ross. Akad. Nauk Ser. Mat.* **80**:5 (2016), 103–152. In Russian; translated in *Izv. Math.* **80**:05 (2016), 910–953. MR

Communicated by Jason P. Bell
Received 2022-02-07 Revised 2022-10-30 Accepted 2022-12-07

aljadeff@technion.ac.il *Department of Mathematics, Technion, Haifa, Israel*

theyakov@gmail.com *Department of Mathematics, Technion, Haifa, Israel*

Algebra & Number Theory

msp.org/ant

See inside back cover or msp.org/ant for submission instructions.

The subscription price for 2024 is US $525/year for the electronic version, and $770/year (+$65, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP.

Algebra & Number Theory (ISSN 1944-7833 electronic, 1937-0652 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online.

ANT peer review and production are managed by EditFLOW® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

Algebra & Number Theory

Volume 18 No. 1 2024