Decidability via the tilting correspondence

Konstantinos Kartas

Decidability via the tilting correspondence

Konstantinos Kartas

We prove a relative decidability result for perfectoid fields. This applies to show that the fields $\mathbb{Q}_p(p^{1/p^\infty})$ and $\mathbb{Q}_p(\zeta_{p^\infty})$ are (existentially) decidable relative to the perfect hull of $\mathbb{F}_p((t))$ and $\mathbb{Q}_p^{ab}$ is (existentially) decidable relative to the perfect hull of $\overline{\mathbb{F}}_p((t))$. We also prove some unconditional decidability results in mixed characteristic via reduction to characteristic p.

Introduction

After the decidability of the p-adic numbers $\mathbb{Q}_p$ was established by Ax and Kochen [1965] and independently by Eršov [1965], several decidability questions about local fields and their extensions have been raised and answered:

- In mixed characteristic, Kochen [1975] showed that $\mathbb{Q}_p^{ur}$, the maximal unramified extension of $\mathbb{Q}_p$, is decidable in the language $L_{\mathrm{val}} = \{0, 1, +, \cdot, \mathcal{O}\}$ (see notation). More generally, by work of Eršov [1965], Ziegler [1972], Basarab [1978] and Bélair [1999] and more recently by Lee [2020], Lee and Lee [2021] and Anscombe and Jahnke [2022], we have a good understanding of the model theory of unramified and finitely ramified mixed characteristic henselian fields.

- In positive characteristic, our understanding is much more limited. Nevertheless, by work of Denef and Schoutens [2003], we know that $\mathbb{F}_p((t))$ is *existentially* decidable in $L_{\mathrm{val}}(t) = \{0, 1, t, +, \cdot, \mathcal{O}\}$ (see notation), modulo resolution of singularities.[1] In fact, [Denef and Schoutens 2003, Theorem 4.3] applies to show that (assuming resolution) any finitely ramified extension of $\mathbb{F}_p((t))$ is existentially decidable relative to its residue field. Anscombe and Fehm [2016] showed *unconditionally* that $\mathbb{F}_p((t))$ is existentially decidable but in the language L_{val}, which does not include a constant symbol for t.

Our understanding is less clear for *infinitely ramified* fields and there are many extensions of $\mathbb{Q}_p$ (resp. $\mathbb{F}_p((t))$) of great arithmetic interest, whose decidability problem is still open:

- In mixed characteristic, these include $\mathbb{Q}_p^{ab}$, the maximal abelian extension of $\mathbb{Q}_p$ and the totally ramified extension $\mathbb{Q}_p(\zeta_{p^\infty})$, obtained by adjoining all p^n-th roots of unity. These extensions had already been discussed in Macintyre's survey [1986, page 140] and a conjectural axiomatization of $\mathbb{Q}_p^{ab}$ was given by Koenigsmann [2018, page 55]. Another interesting extension is $\mathbb{Q}_p(p^{1/p^\infty})$, a totally ramified extension of $\mathbb{Q}_p$ obtained by adjoining a compatible system of p-power roots of p.

- In positive characteristic, two very natural infinitely ramified fields are the perfect hulls of $\mathbb{F}_p((t))$ and $\overline{\mathbb{F}}_p((t))$. Both of these fields have been conjectured to be decidable; see [Kuhlmann and Rzepka 2023, page 4] for the former. The recent work of Kuhlmann and Rzepka [2023] ultimately aims at extending Kuhlmann's earlier results for tame fields to cover fields like the ones mentioned above.

The above fields will be the main objects of interest throughout the paper. Their p-adic (resp. t-adic) completions are typical examples of *perfectoid fields* in the sense of Scholze [2012] (see Section 3). A perfectoid field (K, v) is a valued field which is complete with respect to a nondiscrete valuation of rank 1, with residue characteristic equal to $p > 0$ and such that the Frobenius map $\Phi : \mathcal{O}_K/(p) \to \mathcal{O}_K/(p) : x \mapsto x^p$ is surjective. Loosely speaking, the last condition says that one can extract approximate p-power roots of any element in the field. For any perfectoid field K, one can define its tilt $K^\flat$ (see Section 3.2), which intuitively is its *local* function field analogue and serves as a good characteristic p approximation of K. In practice, this means that one can often reduce arithmetic problems about K to arithmetic problems about $K^\flat$. This kind of transfer principle, which works for a fixed p, should be contrasted with the Ax–Kochen/Ershov principle which achieves such a reduction only *asymptotically*, i.e., with the residue characteristic $p \to \infty$. This is explained in detail in Scholze's ICM report; see [Scholze 2014, page 2].

Our main goal is to set the stage for incorporating ideas from perfectoid geometry in the model theory of henselian fields. In the present paper we focus on decidability, although it is conceivable that our methods can be used in different model-theoretic contexts. We will prove the following relative decidability result for the fields discussed above:

---

[1] The formalism of [Denef and Schoutens 2003] does not include a unary predicate $\mathcal{O}$ for the valuation ring. This does not make a difference because of the equivalences $x \in \mathcal{O} \leftrightarrow \exists y(y^2 = 1 + t \cdot x^2)$ (for $p > 2$; replace squares with cubes for $p = 2$) and $x \notin \mathcal{O} \leftrightarrow x^{-1} \in t \cdot \mathcal{O}$.

Corollary A. (a) *Assume $\mathbb{F}_p((t))^{1/p^\infty}$ is decidable (resp. $\exists$-decidable) in $L_{\mathrm{val}}(t)$. Then $\mathbb{Q}_p(p^{1/p^\infty})$ and $\mathbb{Q}_p(\zeta_{p^\infty})$ are decidable (resp. $\exists$-decidable) in L_{val}.*

(b) *Assume $\overline{\mathbb{F}}_p((t))^{1/p^\infty}$ is decidable (resp. $\exists$-decidable) in $L_{\mathrm{val}}(t)$. Then $\mathbb{Q}_p^{ab}$ is decidable (resp. $\exists$-decidable) in L_{val}.*

As usual, we write $\mathbb{F}_p((t))^{1/p^\infty}$ (resp. $\overline{\mathbb{F}}_p((t))^{1/p^\infty}$) for the perfect hull of $\mathbb{F}_p((t))$ (resp. $\overline{\mathbb{F}}_p((t))$), L_{val} for the language of valued fields and $L_{\mathrm{val}}(t)$ for the language L_{val} enriched with a constant symbol for t (see notation). Corollary A is essentially a special case of a general relative decidability result for perfectoid fields, which is discussed next. (Strictly speaking, the fields in Corollary A are not perfectoid but one can still derive Corollary A from Theorem A directly.)

Let F be a perfectoid field of characteristic p (e.g., $F = \widehat{\mathbb{F}_p((t))^{1/p^\infty}}$). An *untilt* of F is a mixed characteristic perfectoid field K together with an isomorphism $\iota : K^\flat \xrightarrow{\cong} F$. In general, there will be many nonisomorphic untilts of F. In fact, there will be too many untilts even up to elementary equivalence (see Proposition 3.6.9(b)), thus shattering the naive guess of K being decidable relative to its tilt $K^\flat$. On a more elementary level, one needs to assume decidability with *parameters* on the positive characteristic side (see Example 4.3.7). Nevertheless, a relative decidability result for perfectoids can be salvaged by asking that K be an R_0-*computable* untilt of F, a notion which is briefly explained below (see Section 4.2 for details).

Write $W(\mathcal{O}_F)$ for the ring of Witt vectors over $\mathcal{O}_F$ (see Section 3.3). An element $\xi \in W(\mathcal{O}_F)$ is identified with its associated Witt vector, which is an infinite sequence $(\xi_0, \xi_1, \dots)$ with $\xi_n \in \mathcal{O}_F$. Whenever there is a *computable* subring $R_0 \subseteq \mathcal{O}_F$ such that $\xi_n \in R_0$ and the function $\mathbb{N} \to R_0 : n \mapsto \xi_n$ is *recursive*, we say that ξ is R_0-computable. As usual, a computable ring R_0 is one whose underlying set is (or may be identified with) a recursive subset of $\mathbb{N}$, so that the ring operations are (or are identified with) recursive functions (e.g., $\mathbb{F}_p[t]$ is a computable ring). An important result by Fargues and Fontaine gives a one-to-one correspondence between untilts of F and certain principal ideals of $W(\mathcal{O}_F)$ (Theorem 3.5.11). An untilt K of F is then said to be an R_0-*computable untilt* whenever there exists an R_0-computable generator $\xi \in W(\mathcal{O}_F)$ of the ideal corresponding to K. We then have the following relative decidability result (the name is due to Scanlon):

Theorem A (perfectoid transfer). *Suppose K is an R_0-computable untilt of F. If F is decidable in $L_{\mathrm{val}}(R_0)$, then K is decidable in L_{val}.*

Here $L_{\mathrm{val}}(R_0)$ is the language L_{val} enriched with constant symbols for the elements of R_0. We also obtain an *existential* version of Theorem A in Section 4.3. The condition on K being an R_0-computable untilt of F is true in virtually all cases of interest (although generically false), even for a natural choice of R_0. Corollary A is an immediate consequence of Theorem A, by considering the t-adic completion of $\mathbb{F}_p((t))^{1/p^\infty}$ (resp. $\overline{\mathbb{F}}_p((t))^{1/p^\infty}$ for part (b)) and $R_0 = \mathbb{F}_p[t]$. The two key ingredients in the proof of Theorem A are:

(1) An (unpublished) Ax–Kochen/Ershov style result by van den Dries for general mixed characteristic henselian fields, presented in Section 2.

(2) The Fargues–Fontaine correspondence described above, presented in Section 3.5.

Let us now outline how these two ingredients will be combined towards the proof of Theorem A. The result by van den Dries will enable us to reduce the decidability of K to the *uniform* decidability of the sequence of residue rings $(\mathcal{O}_K/(p^n))_{n\in\omega}$ and also the decidability of (Γ_v, vp). The Fargues–Fontaine correspondence then gives us a way to interpret each individual residue ring $\mathcal{O}_K/(p^n)$ inside the tilt F (with parameters from R_0), via an interpretation E_n. The assumption on K being R_0-computable will then imply that the sequence of interpretations $(E_n)_{n\in\omega}$ is *uniformly recursive* (see Section 1.2.12). Having assumed the decidability of F in $L_{\mathrm{val}}(R_0)$, this will then give us that the sequence $(\mathcal{O}_K/(p^n))_{n\in\omega}$ is uniformly decidable. The pointed value group (Γ_v, vp) is easily seen to be interpretable in F (with parameters from R_0) and the decidability of K follows.

It should be mentioned that — prior to this work — Rideau, Scanlon and Simon had made a formative observation, namely that the Fargues–Fontaine correspondence already provides us with an interpretation of the valuation ring $\mathcal{O}_K$ in $\mathcal{O}_F$ (with parameters) in the sense of continuous logic. This however does not yield an interpretation in the sense of ordinary first-order logic (see Section 4.1.1). On the other hand, there do exist honest first-order interpretations of the *truncated* residue rings $\mathcal{O}_K/(p^n)$ in $\mathcal{O}_F$, which is why it is convenient to work with them instead.

Theorem A allows us to reduce decidability problems from the mixed characteristic to the positive characteristic world. Although our model-theoretic understanding of the latter is notoriously limited, this method still yields several applications. In Sections 5 and 6 we obtain new *unconditional* decidability results in mixed characteristic via reduction to characteristic p. In Section 5, we prove the following:

Corollary B. *The valued field* $(\mathbb{Q}_p(p^{1/p^\infty}), v_p)$ *(resp.* $(\mathbb{Q}_p(\zeta_{p^\infty}), v_p)$*) admits a maximal immediate extension which is decidable in* L_{val}.

Note that the fields $\mathbb{Q}_p(p^{1/p^\infty})$ and $\mathbb{Q}_p(\zeta_{p^\infty})$ are *not* Kaplansky and have many nonisomorphic maximal immediate extensions, all of which are *tame* in the sense of Kuhlmann [2016] (see Example 5.1.6). Although the work of Kuhlmann yields several decidability results for equal characteristic tame fields, Corollary B is, to my knowledge, the first decidability result for tame fields of mixed characteristic (see Section 5). The proof uses a recent decidability result of Lisinski [2021] for the Hahn field $\mathbb{F}_p((t^\Gamma))$ with $\Gamma = \frac{1}{p^\infty}\mathbb{Z}$ in the language $L_{\mathrm{val}}(t)$, strengthening Kuhlmann's earlier result for $\mathbb{F}_p((t^\Gamma))$ in the language L_{val}; see [Kuhlmann 2016, Theorem 1.6]. Corollary B then follows from Theorem A and basic properties of the tilting equivalence (see Section 3.5.13). It is worth remarking that most maximal immediate extensions of $\mathbb{Q}_p(p^{1/p^\infty})$ (resp. $\mathbb{Q}_p(\zeta_{p^\infty})$) are *undecidable* (see Remark 5.2.3).

Without making essential use of the perfectoid machinery but only the philosophy thereof, we show in Section 6 the following:

Theorem B. *Let K be any of the valued fields $\mathbb{Q}_p(p^{1/p^\infty})$, $\mathbb{Q}_p(\zeta_{p^\infty})$ or $\mathbb{Q}_p^{ab}$. The existential theory of $\mathcal{O}_K/p\mathcal{O}_K$ in the language of rings L_{rings} is decidable.*

The proof is again via reduction to characteristic p, using a recent existential decidability result for equal characteristic henselian valued fields in the language L_{val}, due to Anscombe and Fehm [2016]. However, as Corollary A suggests, if we aim to understand the theories of $\mathbb{Q}_p(p^{1/p^\infty})$, $\mathbb{Q}_p(\zeta_{p^\infty})$ and $\mathbb{Q}_p^{ab}$ via reduction to positive characteristic, we will need stronger results in the language $L_{\mathrm{val}}(t)$ on the characteristic p side. This is also supported by Proposition 7.2.3 which shows that the Diophantine problem for $\mathbb{F}_p((t))^{1/p^\infty}$ in $L_{\mathrm{val}}(t)$ is Turing reducible to the $\forall^1\exists$-theory of $\mathbb{Q}_p(p^{1/p^\infty})$.

An application of a different flavor, which yields an *undecidability* result in mixed characteristic via reduction to characteristic p, was recently found in [Kartas 2023]. Kartas [2023, Theorem A] shows that the asymptotic theory of $\{K : [K : \mathbb{Q}_p] < \infty\}$ in the language L_{val} with a cross-section is undecidable.

Notation

- If (K, v) is a valued field, we denote by $\mathcal{O}_v$ the valuation ring. If the valuation is clear from the context, we shall also write $\mathcal{O}_K$. We write Γ_v for the value group and k_v for the residue field. If the valuation v is clear from the context, we also denote them by Γ and k respectively.

- When (K, v) is of mixed characteristic, we write p for the number $\mathrm{char}(k)$. The notation $p^n\mathcal{O}_v$ stands for the ideal of $\mathcal{O}_v$ generated by the element p^n. If both the field in question and the valuation v are clear from the context, we shall write (p^n) for $p^n\mathcal{O}_v$.

- $\mathbb{Z}_p^{ur}$: The valuation ring of $(\mathbb{Q}_p^{ur}, v_p)$, the maximal unramified extension of $\mathbb{Q}_p$ equipped with the unique extension of the p-adic valuation.

- $\mathbb{Z}_p^{ab}$: The valuation ring of $(\mathbb{Q}_p^{ab}, v_p)$, the maximal Galois extension of $\mathbb{Q}_p$ whose Galois group over $\mathbb{Q}_p$ is abelian.

- We write $\mathbb{F}_p((t))^{1/p^\infty}$ (resp. $\overline{\mathbb{F}}_p((t))^{1/p^\infty}$) for the perfect hull of $\mathbb{F}_p((t))$ (resp. $\overline{\mathbb{F}}_p((t))$).

- For a given language L, we denote by Sent_L the set of L-sentences and by Form_L the set of L-formulas. If M is an L-structure and $A \subseteq M$ is an arbitrary subset, we write $L(A)$ for the language L enriched with a constant symbol c_a for each element $a \in A$. The L-structure M can be updated into an $L(A)$-structure in the obvious way.

- We write $L_{\mathrm{rings}} = \{0, 1, +, \cdot\}$ for the language of rings, $L_{\mathrm{oag}} = \{0, +, <\}$ for the language of ordered abelian groups, $L_{\mathrm{val}} = L_{\mathrm{rings}} \cup \{\mathcal{O}\}$ for the language of valued fields (where $\mathcal{O}$ is a unary predicate for the valuation ring), $L_{\mathrm{val}}(t) = L_{\mathrm{val}} \cup \{t\}$, where t is a constant symbol whose intended interpretation will always be clear from the context. A local ring $(R, \mathfrak{m}_R)$ may be viewed as an L_{lcr}-structure, where L_{lcr} is the *language of local rings* consisting of the language of rings L_{rings} together with a unary predicate $\mathfrak{m}$, whose intended interpretation is the maximal ideal $\mathfrak{m}_R \subseteq R$.

1. Preliminaries

1.1. *Decidability.*

1.1.1. *Introduction.* Fix a *countable* language L and let Sent_L be the set of well-formed L-sentences, identified with $\mathbb{N}$ via some Gödel numbering. Let M be an L-structure. Recall that M is decidable if we have an algorithm to decide whether $M \models \phi$, for any given $\phi \in \text{Sent}_L$. More formally, let $\chi_M : \text{Sent}_L \to \{0, 1\}$ be the characteristic function of $\text{Th}(M) \subseteq \text{Sent}_L$. We say that M is *decidable* if χ_M is recursive.

1.1.2. *Uniform decidability.*

Definition 1.1.3. For each $n \in \mathbb{N}$, let $f_n : \mathbb{N} \to \mathbb{N}$ be a function. The sequence $(f_n)_{n \in \omega}$ is uniformly recursive if the function $\mathbb{N} \times \mathbb{N} \to \mathbb{N} : (n, m) \mapsto f_n(m)$ is recursive.

This concept is best illustrated with a *nonexample*:

Example 1.1.4. Let $A \subseteq \mathbb{N}$ be nonrecursive. For each $n \in \mathbb{N}$, let $\delta_n : \mathbb{N} \to \mathbb{N} : m \mapsto \delta_{nm}$, where δ_{nm} is the Kronecker symbol. For each $n \in \mathbb{N}$, define $f_n : \mathbb{N} \to \mathbb{N}$ to be (1) δ_n if $n \in A$ and (2) identically 0 if $n \notin A$. One readily verifies that each individual f_n is recursive. On the other hand, the sequence $(f_n)_{n \in \omega}$ is not uniformly recursive. Indeed, otherwise we could solve the membership problem for A, using that $n \in A \iff f_n(n) = 1$.

Using some Gödel numbering, we can also state a version of Definition 1.1.3 for sequences of functions $(f_n)_{n \in \omega}$, where $\text{dom}(f_n) = \text{Sent}_L$ (or $\text{cdm}(f_n) = \text{Sent}_L$). One can then define a notion of *uniform decidability* for sequences of L-structures:

Definition 1.1.5. A sequence $(M_n)_{n \in \omega}$ of L-structures is *uniformly decidable* if the sequence of functions $(\chi_{M_n})_{n \in \omega}$ is uniformly recursive, i.e., if the function $\chi : \mathbb{N} \times \text{Sent}_L \to \{0, 1\} : (n, \phi) \mapsto \chi_{M_n}(\phi)$ is recursive.

Remark 1.1.6. If the sequence $\chi : \mathbb{N} \times \text{Sent}_L \to \{0, 1\} : (n, \phi) \mapsto \chi_{M_n}(\phi)$ is recursive when restricted to *existential* sentences, we naturally say that the sequence $(M_n)_{n \in \omega}$ is *uniformly existentially decidable*. Other syntactic variants may be defined analogously.

1.2. *Interpretability.* Our formalism follows closely [Hodges 1993, Section 5.3], where details and proofs may be found.

1.2.1. *Interpretations.* Given a language L, an *unnested* atomic L-formula is one of the form $x = y$ or $x = c$ or $F(\bar{x}) = y$ or $R(\bar{x})$, where x, y are variables, c is a constant symbol, $\bar{x}$ is a tuple of variables, F is a function symbol and R is a relation symbol of the language L.

Definition 1.2.2. An n-dimensional interpretation of an L-structure M in the L'-structure N is a triple $\Gamma = (\partial_\Gamma, \phi \mapsto \phi_\Gamma, f_\Gamma)$ consisting of

(1) an L'-formula $\partial_\Gamma(x_1, \ldots, x_n)$,

(2) a map $\phi \mapsto \phi_\Gamma$, that takes an unnested atomic L-formula $\phi(x_1, \ldots, x_m)$ and sends it to an L'-formula $\phi_\Gamma(\bar{y}_1, \ldots, \bar{y}_m)$, where each $\bar{y}_i$ is an n-tuple of variables,

(3) a surjective map $f_\Gamma : \partial_\Gamma(N^n) \twoheadrightarrow M$,

such that for all unnested atomic L-formulas $\phi(x_1, \ldots, x_m)$ and all $\bar{a}_i \in \partial_\Gamma(N^n)$, we have

$$M \models \phi(f_\Gamma(\bar{a}_1), \ldots, f_\Gamma(\bar{a}_m)) \Longleftrightarrow N \models \phi_\Gamma(\bar{a}_1, \ldots, \bar{a}_m).$$

An *interpretation* of an L-structure M in the L'-structure N is an n-dimensional interpretation Γ, for some $n \in \mathbb{N}$. In that case, we also say that M is *interpretable* in N. The formulas ∂_Γ and ϕ_Γ (for all unnested atomic ϕ) are the *defining formulas* of Γ.

Interpretability is a *transitive* relation on the class of structures, i.e., if the L-structure M is interpretable in the L'-structure N and N is interpretable in the L''-structure P, then there exists a *composite* interpretation of M in P [Hodges 1993, Exercise 2, page 218].

If N is an L'-structure and $\bar{a} = (a_1, \ldots, a_m) \in N^m$, we write $(N, \bar{a})$ for the expansion of N in the language $L(\bar{c})$, which is L together with an m-tuple of constant symbols $(c_1, \ldots, c_m)$ with $c_i^{(N,\bar{a})} = a_i$. If M is interpretable in $(N, \bar{a})$, for some $\bar{a} \in N^m$, we say that M is interpretable in N *with parameters*.

Proposition 1.2.3 [Hodges 1993, reduction Theorem 5.3.2]. *Let Γ be an n-dimensional interpretation of an L-structure M in the L'-structure N. There exists a map $\phi \mapsto \phi_\Gamma$, extending the map of Definition 1.2.2(2), such that for every L-formula $\phi(x_1, \ldots, x_m)$ and all $\bar{a}_i \in \partial_\Gamma(N^n)$, we have that*

$$M \models \phi(f_\Gamma(\bar{a}_1), \ldots, f_\Gamma(\bar{a}_m)) \Longleftrightarrow N \models \phi_\Gamma(\bar{a}_1, \ldots, \bar{a}_m).$$

Proof. We describe how $\phi \mapsto \phi_\Gamma$ is built, for completeness (omitting details). By [Hodges 1993, Corollary 2.6.2], every L-formula is equivalent to one in which all atomic subformulas are unnested. One can then construct $\phi \mapsto \phi_\Gamma$ by induction on the complexity of formulas. The base case is handled by Definition 1.2.2(2). This definition extends inductively according to the following rules:

(1) $(\neg\phi)_\Gamma = \neg(\phi)_\Gamma$.

(2) $\left(\bigwedge_{i=1}^n \phi_i\right)_\Gamma = \bigwedge(\phi_i)_\Gamma$.

(3) $(\forall\phi)_\Gamma = \forall x_1, \ldots, x_n(\partial_\Gamma(x_1, \ldots, x_n) \to \phi_\Gamma)$.

(4) $(\exists\phi)_\Gamma = \exists x_1, \ldots, x_n(\partial_\Gamma(x_1, \ldots, x_n) \wedge \phi_\Gamma)$.

The resulting map satisfies the desired conditions of Proposition 1.2.3. $\square$

Definition 1.2.4. The map $\mathrm{Form}_L \to \mathrm{Form}_{L'} : \phi \mapsto \phi_\Gamma$ constructed in the proof of Proposition 1.2.3 is called the *reduction* map of the interpretation Γ.

1.2.5. *Complexity of interpretations.* The complexity of the defining formulas of an interpretation defines a measure of complexity of the interpretation itself:

Definition 1.2.6 [Hodges 1993, Section 5.4(a)]. An interpretation Γ of an L-structure M in an L'-structure N is quantifier-free if the defining formulas of Γ are quantifier-free. Other syntactic variants are defined analogously (e.g., existential interpretation).

Remark 1.2.7. (a) The reduction map of a positive existential interpretation sends positive existential formulas to positive existential formulas.

(b) The reduction map of an existential interpretation sends *positive existential* formulas to existential formulas but does *not* necessarily send existential formulas to existential formulas.

Lemma 1.2.8. *If the L-structure M is $\exists^+$-interpretable in the L'-structure N and N is $\exists^+$-interpretable in the L''-structure P, then the composite interpretation of M in P is also an $\exists^+$-interpretation.*

Proof. Clear.   □

1.2.9. *Recursive interpretations.*

Definition 1.2.10 [Hodges 1993, Remark 4, page 215]. Suppose L is a recursive language. Let Γ be an interpretation of an L-structure M in the L'-structure N. We say that the interpretation Γ is *recursive* if the map $\phi \mapsto \phi_\Gamma$ on unnested atomic formulas is recursive.

Remark 1.2.11 [Hodges 1993, Remark 4, page 215]. If Γ is a recursive interpretation of an L-structure M in the L'-structure N, then the reduction map of Γ is also recursive.

1.2.12. *Uniformly recursive interpretations.*

Definition 1.2.13. Suppose L and L' are languages. Let $(M_n)_{n\in\omega}$ be a sequence of L-structures and $(N_n)_{n\in\omega}$ be a sequence of L'-structure. For each $n \in \mathbb{N}$, let Γ_n be an interpretation of M_n in N_n. We say that the sequence of interpretations $(\Gamma_n)_{n\in\omega}$ is *uniformly recursive* if the sequence of reduction maps $(\phi \mapsto \phi_{\Gamma_n})_{n\in\omega}$ on unnested atomic formulas is uniformly recursive, i.e., if the map $(n, \phi) \mapsto \phi_{\Gamma_n}$ is recursive.

If an L-structure M is interpretable in the L'-structure N and the latter is decidable, then so is the former. It is not hard to prove the following uniform version:

Proposition 1.2.14. *Suppose L is a recursive language, $(M_n)_{n\in\omega}$ a sequence of L-structures and N is an L'-structure. Suppose N is decidable, Γ_n is an interpretation of M_n in N and the sequence of interpretations $(\Gamma_n)_{n\in\omega}$ is uniformly recursive. Then the sequence $(M_n)_{n\in\omega}$ is uniformly decidable.*

Proof. Rephrasing Proposition 1.2.3 for sentences, yields $\chi_{M_n}(\phi) = \chi_N(\phi_{\Gamma_n})$ for every $\phi \in \mathrm{Sent}_L$. It follows that the map $(n, \phi) \mapsto \chi_{M_n}(\phi)$ is equal to the map $(n, \phi) \mapsto \phi_{\Gamma_n} \mapsto \chi_N(\phi_{\Gamma_n})$ and the latter is recursive as a composition of recursive functions.   □

If the interpretation of M in N is recursive and so is the interpretation of N in P, then the composite interpretation of M in P is recursive as well. Indeed, recursive functions are closed under composition. One also has a uniform version:

Lemma 1.2.15. *Let $(M_n)_{n\in\omega}$ be a sequence of L-structures, $(N_n)_{n\in\omega}$ be a sequence of L'-structures and $(P_n)_{n\in\omega}$ be a sequence of L''-structures. For each $n \in \mathbb{N}$, let Γ_n be an interpretation of M_n in N_n and Δ_n be an interpretation of N_n in P_n and suppose that the sequences of interpretations $(\Gamma_n)_{n\in\omega}$ and $(\Delta_n)_{n\in\omega}$ are uniformly recursive. Let E_n be the composite interpretation of M_n in P_n. Then the sequence of interpretations $(P_n)_{n\in\omega}$ is uniformly recursive.*

Proof. Clear.   □

2. Ax–Kochen/Ershov in mixed characteristic

2.1. *A result by van den Dries.*

2.1.1. *Introduction.* We start with an Ax and Kochen/Ershov style result due to van den Dries (unpublished), which is briefly discussed on page 144 in [van den Dries 2014]. We shall sketch the proof (due to van den Dries), which does not seem to appear anywhere in the published literature. Some references, which use a similar *coarsening* argument, include [van den Dries 1999, page 2], [Anscombe and Jahnke 2022, proof of Corollary 12.3] and [Scanlon 2003, proof of Proposition 9.6]. For background material on coarsenings of valuations, see [van den Dries 2014, Section 7.4].

2.1.2. *Inverse systems.* The formalism of multisorted structures, used in this section, is spelled out in [Scanlon 2003, Section 3]. The proof of van den Dries' Theorem 2.1.5 requires a technical lemma for inverse systems, which we now discuss.

Let $\mathcal{R} = (R_n)_{n \in \omega}$ be a sequence of rings, viewed as a multisorted structure with sorts $(\boldsymbol{R}_n)_{n \in \omega}$, each equipped with the language of rings L_{rings}, and for each $n \in \mathbb{N}$ we have a map $f_n : \boldsymbol{R}_{n+1} \to \boldsymbol{R}_n$. Let T be the theory that requires of $\mathcal{R} = (R_n)_{n \in \omega}$ that the map $f_n : R_{n+1} \to R_n$ be a surjective ring homomorphism with $\mathrm{Ker}(f_n) = p^n R_{n+1}$, i.e., $R_n \cong R_{n+1}/p^n R_{n+1}$. If $\mathcal{R} = (R_n)_{n \in \omega}$ and $\mathcal{S} = (S_n)_{n \in \omega}$ are two models of T, they are isomorphic precisely when there is a *compatible* system $(\phi_n)_{n \in \omega}$ of isomorphisms $\phi_n : R_n \xrightarrow{\cong} S_n$, i.e., such that the diagram commutes

$$
\begin{array}{ccc}
R_{n+1} & \xrightarrow{\phi_{n+1}} & S_{n+1} \\
\downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle g_n} \\
R_n & \xrightarrow{\phi_n} & S_n
\end{array}
$$

for each $n \in \mathbb{N}$. Compatibility of the ϕ_n is essential as there are examples where $R_n \cong S_n$ for each $n \in \mathbb{N}$ but $\mathcal{R} \not\cong \mathcal{S}$ (see e.g., Remark 3.6.10). Somewhat surprisingly, compatibility comes for free in a saturated setting:

Lemma 2.1.3. *Assume* CH. *Let* $\mathcal{R} = (R_n)_{n \in \omega}$ *and* $\mathcal{S} = (S_n)_{n \in \omega}$ *be two models of* T. *Suppose that for each* $n \in \mathbb{N}$, *we have that* $R_n \cong S_n$ *and the rings* R_n, S_n *are saturated with* $|R_n| = |S_n| \leq \aleph_1$. *Then* $\mathcal{R} \cong \mathcal{S}$.

Proof. Let U be a nonprincipal ultrafilter on $\mathbb{N}$ and consider the ultraproducts $R_U = \prod_{n \in \omega} R_n/U$ and $S_U = \prod_{n \in \omega} S_n/U$.

Claim 1. $R_U \cong S_U$.

Proof. Since $R_n \equiv S_n$ for each $n \in \mathbb{N}$, we get that $R_U \equiv S_U$ by Łoś' s Theorem. For each $n \in \mathbb{N}$, let $\mathcal{F}_m = \{n \in \mathbb{N} : |R_n| \geq m\}$. If there exists $m \in \mathbb{N}$ with $\mathcal{F}_m \notin U$, then R_U and S_U are both finite and thus $R_U \cong S_U$. If on the other hand $\mathcal{F}_m \in U$ for all $m \in \mathbb{N}$, then R_U and S_U are both infinite. Since $|R_n| \leq \aleph_1$, we get that $|R_U| \leq \aleph_1^{\aleph_0} = 2^{\aleph_0^2} = 2^{\aleph_0} = \aleph_1$, using the continuum hypothesis (similarly $|S_U| \leq \aleph_1$). Moreover, the ultraproducts R_U and S_U are $\aleph_1$-saturated [Marker 2002, Exercise 4.5.37] and thus saturated of size

$\aleph_1$. Since R_U and S_U are elementary equivalent and both saturated of size $\aleph_1$, we conclude that $R_U \cong S_U$ [loc. cit., Theorem 4.3.20]. □

Next we prove:

Claim 2. For each $n \in \mathbb{N}$, we have $R_U / p^n R_U \cong R_n$.

Proof. Fix $n \in \mathbb{N}$. Since $R_m / p^n R_m \cong R_n$ for $m > n$, we get that $R_U / p^n R_U \equiv R_n$ in L_{rings} by Łoś Theorem. If R_n is finite, then $R_U / p^n R_U \cong R_n$ and we are done. Otherwise, we will have that R_n is saturated of size $\aleph_1$. The same is true for $R_U / p^n R_U$, being interpretable in the structure R_U, which is saturated of size $\aleph_1$ by CH (see the proof of Claim 1). We conclude that $R_U / p^n R_U \cong R_n$ [loc. cit., Theorem 4.3.20]. □

Similarly, for each $n \in \mathbb{N}$, we have $S_U / p^n S_U \cong S_n$. By Claim 1, we obtain $\phi : R_U \xrightarrow{\cong} S_U$. Note that $\phi(p^n R_U) = p^n S_U$, for each $n \in \mathbb{N}$. By Claim 2, this gives rise to a *compatible* system $(\phi_n)_{n \in \omega}$ of isomorphisms $\phi_n : R_n \xrightarrow{\cong} S_n$, which yields $\mathcal{R} \xrightarrow{\cong} \mathcal{S}$. □

2.1.4. *Statement and proof.* The following result will be of fundamental importance for the rest of the paper:

Theorem 2.1.5 (van den Dries). *Let (K, v), (K', v') be two henselian valued fields of mixed characteristic. Then $(K, v) \equiv (K', v')$ in L_{val} if and only if $\mathcal{O}_v / p^n \mathcal{O}_v \equiv \mathcal{O}_{v'} / p^n \mathcal{O}_{v'}$ in L_{rings} for all $n \in \mathbb{N}$ and $(\Gamma_v, vp) \equiv (\Gamma_{v'}, v'p)$ in L_{oag} together with a constant for vp.*

Proof. ⟹: Clear.

⟸: As the statement at hand is *absolute*, we may assume the continuum hypothesis; see [Scanlon 2003, Section 8] or [van den Dries 2014, page 122]. We may therefore assume that both (K, v) and (K', v') are saturated of size $\aleph_1$ [Marker 2002, Corolllary 4.3.13]. By our assumption, we have an isomorphism of ordered abelian groups $(\Gamma_v, vp) \cong (\Gamma_{v'}, v'p)$ and a ring isomorphism $\mathcal{O}_{v'} / p^n \mathcal{O}_{v'} \cong \mathcal{O}_v / p^n \mathcal{O}_v$ for each $n \in \mathbb{N}$. We shall argue that $(K, v) \cong (K', v')$.

Consider the finest coarsening w of v for which the associated residue field k_w has characteristic 0. The corresponding valuation ring is $\mathcal{O}_w = \mathcal{O}_v[\frac{1}{p}]$ and the corresponding value group is $\Gamma_w = \Gamma_v / \text{Conv}(\mathbb{Z}vp)$, where $\text{Conv}(\mathbb{Z}vp)$ is the convex hull of $\mathbb{Z}vp$ in Γ_v. Let $\bar{v}$ be the induced valuation from v on the residue field k_w. We then have that $\mathcal{O}_{\bar{v}} = \mathcal{O}_v / \bigcap_{n \in \omega} p^n \mathcal{O}_v$. We also consider the analogous objects for K'.

Claim 1. We have a ring isomorphism $\mathcal{O}_{\bar{v}} \cong \varprojlim \mathcal{O}_v / p^n \mathcal{O}_v$.

Proof. Consider the ring homomorphism $f : \mathcal{O}_{\bar{v}} \to \varprojlim \mathcal{O}_v / p^n \mathcal{O}_v : x + \bigcap_{n \in \omega} p^n \mathcal{O}_v \mapsto (x + p^n \mathcal{O}_v)_{n \in \omega}$, which is clearly injective. We shall argue that it is also surjective. For a given $(x_n + p^n \mathcal{O}_v)_{n \in \omega} \in \varprojlim \mathcal{O}_v / p^n \mathcal{O}_v$, we may find $x \in \mathcal{O}_v$ with $x \equiv x_n \mod p^n \mathcal{O}_v$, using that $\mathcal{O}_v$ is $\aleph_1$-saturated. It follows that $f(x) = (x_n + p^n \mathcal{O}_v)_{n \in \omega}$. □

Similarly, one obtains an isomorphism $\mathcal{O}_{\bar{v'}} \cong \varprojlim \mathcal{O}_{v'} / p^n \mathcal{O}_{v'}$.
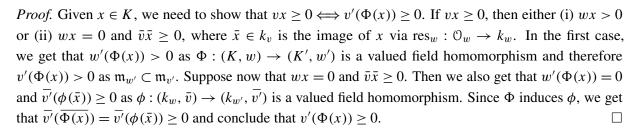
Claim 2. We have an isomorphism of valued fields $\phi : (k_w, \bar{v}) \cong (k_{w'}, \bar{v'})$.

Proof. Let $\mathcal{R} = (\mathcal{O}_v / p^n \mathcal{O}_v)_{n \in \omega}$ and $\mathcal{S} = (\mathcal{O}_{v'} / p^n \mathcal{O}_{v'})_{n \in \omega}$. Then $\mathcal{R} \cong \mathcal{S}$ by Lemma 2.1.3. This yields $\varprojlim \mathcal{O}_v / p^n \mathcal{O}_v \cong \varprojlim \mathcal{O}_{v'} / p^n \mathcal{O}_{v'}$ and thus $\mathcal{O}_{\bar{v}} \cong \mathcal{O}_{\bar{v'}}$ by Claim 1. □

We also have that $\Gamma_w \cong \Gamma_{w'}$, since the isomorphism $(\Gamma_v, vp) \cong (\Gamma_{v'}, v'p)$ descends to the quotients $\Gamma_v/\operatorname{Conv}(\mathbb{Z}vp) \cong \Gamma_{v'}/\operatorname{Conv}(\mathbb{Z}v'p)$. By [van den Dries 2014, Lemma 7.13], the coarsened valued fields (K, w) and (K', w') are henselian too. By the Ax–Kochen/Ershov principle in pure characteristic 0 (see, e.g., [van den Dries 2014, Corollary 5.22]), we get that $(K, w) \equiv (K', w')$ in L_{val}.

Passing once again to elementary extensions, in a suitable language that includes unary predicates for both $\mathcal{O}_v$ and $\mathcal{O}_w$, we may even assume that $(K, w) \cong (K', w')$ to begin with. By stable embeddedness of residue fields for henselian valued fields of pure characteristic 0 (see [van den Dries 2014, Corollary 5.25]), there is even an isomorphism $\Phi : (K, w) \xrightarrow{\cong} (K', w')$ inducing $\phi : (k_w, \bar{v}) \xrightarrow{\cong} (k_{w'}, \bar{v'})$.

Claim 3. The map Φ is an isomorphism of the valued fields (K, v) and (K', v').

Proof. Given $x \in K$, we need to show that $vx \geq 0 \Longleftrightarrow v'(\Phi(x)) \geq 0$. If $vx \geq 0$, then either (i) $wx > 0$ or (ii) $wx = 0$ and $\bar{v}\bar{x} \geq 0$, where $\bar{x} \in k_v$ is the image of x via $\operatorname{res}_w : \mathcal{O}_w \to k_w$. In the first case, we get that $w'(\Phi(x)) > 0$ as $\Phi : (K, w) \to (K', w')$ is a valued field homomorphism and therefore $v'(\Phi(x)) > 0$ as $\mathfrak{m}_{w'} \subset \mathfrak{m}_{v'}$. Suppose now that $wx = 0$ and $\bar{v}\bar{x} \geq 0$. Then we also get that $w'(\Phi(x)) = 0$ and $\bar{v'}(\phi(\bar{x})) \geq 0$ as $\phi : (k_w, \bar{v}) \to (k_{w'}, \bar{v'})$ is a valued field homomorphism. Since Φ induces ϕ, we get that $\bar{v'}(\overline{\Phi(x)}) = \bar{v'}(\phi(\bar{x})) \geq 0$ and conclude that $v'(\Phi(x)) \geq 0$. $\square$

Claim 3 finishes the proof. $\square$

2.2. *Existential AKE in mixed characteristic.* In this section we prove an existential version of Theorem 2.1.5. We first review some known AKE results in the equal characteristic setting.

2.2.1. *Comparison with the equal characteristic case.* For equal characteristic henselian valued fields one has the following simple Ax–Kochen/Ershov principles due to Anscombe and Fehm:

Theorem 2.2.2 [Anscombe and Fehm 2016, Corollary 1.2]. *Let (K, v), (K', v') be two equal characteristic nontrivially valued henselian fields. Then $(K, v) \equiv_\exists (K', v')$ in L_{val} if and only if $k \equiv_\exists k'$ in L_{rings}.*

Theorem 2.2.3 [Anscombe and Fehm 2016, Corollary 7.5]. *Let (K, v) be an equal characteristic henselian valued field. Then $Th_\exists(K, v)$ is decidable in L_{val} if and only if $Th_\exists(k)$ is decidable in L_{rings}.*

Remark 2.2.4. In residue characteristic 0, Theorems 2.2.2 and 2.2.3 were essentially known by work of Ax and Kochen/Ershov prior to the work of Anscombe and Fehm [2016, Remark 7.3].

2.2.5. *Existential AKE in mixed characteristic.* In mixed characteristic, one can easily construct counterexamples of Theorems 2.2.2 and 2.2.3; see [Anscombe and Fehm 2016, Remark 7.6]. It is then natural to ask what an existential AKE principle in mixed characteristic would look like. This will be Theorem 2.2.6 below, whose proof follows closely the proof of Theorem 2.1.5.

We write $\mathfrak{m}_n$ for the maximal ideal of $\mathcal{O}_v/p^n\mathcal{O}_v$ and $(\mathcal{O}_v/p^n\mathcal{O}_v, \mathfrak{m}_n)$ for the local ring, viewed as an L_{lcr}-structure (see notation).

Theorem 2.2.6. *Let (K, v), (K', v') be two henselian valued fields of mixed characteristic. Then the following are equivalent*:

(1) $(K, v) \equiv_\exists (K', v')$ *in* L_{val}.

(2) $\mathcal{O}_v/p^n\mathcal{O}_v \equiv_\exists \mathcal{O}_{v'}/p^n\mathcal{O}_{v'}$ *in* L_{rings} *for all* $n \in \mathbb{N}$.

(3) $(\mathcal{O}_v/p^n\mathcal{O}_v, \mathfrak{m}_n) \equiv_{\exists^+} (\mathcal{O}_{v'}/p^n\mathcal{O}_{v'}, \mathfrak{m}'_n)$ *in* L_{lcr} *for all* $n \in \mathbb{N}$.

Proof. $(1) \Rightarrow (2)$, (3): Clear.

$(2) \Rightarrow (1)$: By symmetry, it will suffice to show that $(K, v) \models \text{Th}_\exists(K', v')$. We may further assume (K', v') is countable by downward Löwenheim and Skolem and that (K, v) is $\aleph_1$-saturated.

We again consider the valuations $w, \bar{v}$ (resp. $w', \bar{v}'$) that were introduced in the proof of Theorem 2.1.5. By our assumption, we have for each $n \in \mathbb{N}$ an embedding of rings $\mathcal{O}_{v'}/p^n\mathcal{O}_{v'} \hookrightarrow \mathcal{O}_v/p^n\mathcal{O}_v$.

Claim. There is an injective ring embedding $\phi : \mathcal{O}_{\bar{v}'} \hookrightarrow \mathcal{O}_{\bar{v}}$.

Proof. Let us fix an enumeration of $\mathcal{O}_{v'}$, say $\mathcal{O}_{v'} = (a_i)_{i \in \mathbb{N}}$. Consider the set of formulas in countably many variables $x = (x_n)_{n \in \omega}$ of the form

$$\Sigma(x) = \{x_i \diamond x_j = x_k(p^n), x_m \square x_\rho(p^n) : a_i \in \mathcal{O}_{v'}, a_i \diamond a_j = a_k(p^n), a_m \square a_\rho(p^n)\}$$

where $\diamond$ is either $+$ or $\cdot$ and $\square$ is either $=$ or $\neq$. Since for each $n \in \mathbb{N}$ we have an embedding of rings $\mathcal{O}_{v'}/p^n\mathcal{O}_{v'} \hookrightarrow \mathcal{O}_v/p^n\mathcal{O}_v$, we get that $\Sigma(x)$ is finitely satisfiable. The ring $\mathcal{O}_v$ is $\aleph_1$-saturated and we thus have $b = (b_n)_{n \in \omega}$ with $b \models \Sigma(x)$. Using that $\mathcal{O}_{\bar{v}} = \mathcal{O}_v/\bigcap_{n \in \omega} p^n\mathcal{O}_v$ (resp. $\mathcal{O}_{\bar{v}'} = \mathcal{O}_{v'}/\bigcap_{n \in \omega} p^n\mathcal{O}_{v'}$), one readily checks that the map $\mathcal{O}_{v'} \to \mathcal{O}_v : a_i \mapsto b_i$ descends to a ring embedding $\mathcal{O}_{\bar{v}'} \hookrightarrow \mathcal{O}_{\bar{v}}$. $\square$

The claim provides us with a valued field embedding $\phi : (k_{w'}, \bar{v}') \hookrightarrow (k_w, \bar{v})$. By the existential Ax–Kochen/Ershov principle in pure characteristic 0 (see Theorem 2.2.2 and Remark 2.2.4), we get that $(K, w) \models \text{Th}_\exists(K', w')$. Replacing K with an $\aleph_1$-saturated extension in a suitable language that includes unary predicates for both $\mathcal{O}_v$ and $\mathcal{O}_w$, we will also have an embedding $(K', w') \hookrightarrow (K, w)$.

By the relative embedding property for equal characteristic 0 henselian valued fields (see in [Kuhlmann 2016, Theorem 7.1] for a more general statement), we can even find $\Phi : (K', w') \hookrightarrow (K, w)$ that induces $\phi : (k_{w'}, \bar{v}') \hookrightarrow (k_w, \bar{v})$. Finally, we get that the map $\Phi : (K', v') \hookrightarrow (K, v)$ is an embedding of valued fields, as in the proof of Claim 3, Theorem 2.1.5.

$(3) \Rightarrow (2)$: For $f(x_1, \ldots, x_m) \in \mathbb{Z}[x_1, \ldots, x_m]$ and $(a_1, \ldots, a_m) \in \mathcal{O}_v^m$, note that $f(a_1, \ldots, a_m) \neq 0 \mod p^n\mathcal{O}_v$ if and only if there exists $y \in \mathfrak{m}_v$ such that $f(a_1, \ldots, a_m) \cdot y = p^n \mod p^{n+1}\mathcal{O}_v$ (similarly for $\mathcal{O}_{v'}$). Consequently, for each $n \in \mathbb{N}$, we see that if $(\mathcal{O}_v/p^{n+1}\mathcal{O}_v, \mathfrak{m}_{n+1}) \equiv_{\exists^+} (\mathcal{O}_{v'}/p^{n+1}\mathcal{O}_{v'}, \mathfrak{m}'_{n+1})$ in L_{lcr}, then $\mathcal{O}_v/p^n\mathcal{O}_v \equiv_\exists \mathcal{O}_{v'}/p^n\mathcal{O}_{v'}$ in L_{rings}. $\square$

2.3. *Decidability.* We now harvest the consequences of Theorems 2.1.5 and 2.2.6 in relation to decidability. Since the countable union of recursive sets is not guaranteed to be recursive, we need to ask not only that each individual $\mathcal{O}_K/(p^n)$ be decidable in L_{rings} but also that the sequence $(\mathcal{O}_K/(p^n))_{n \in \omega}$ be *uniformly decidable* in L_{rings}:

Corollary 2.3.1. *Let (K, v) be a henselian valued field of mixed characteristic. Then the following are equivalent*:

(1) *The valued field (K, v) is decidable in L_{val}.*

(2) *The sequence $(\mathcal{O}_K/(p^n))_{n\in\omega}$ is uniformly decidable in L_{rings} and (Γ_v, vp) is decidable in L_{oag} with a constant for vp.*

Proof. (1) $\Rightarrow$ (2): Clear.

(2) $\Rightarrow$ (1): The identification $\Gamma_v = K^\times/\mathcal{O}^\times$ furnish us with a recursive interpretation E of (Γ_v, vp) in the valued field (K, v). Let also E_n be the natural interpretation of $\mathcal{O}_K/(p^n)$ in the L_{val}-structure (K, v), for each $n \in \mathbb{N}$. If $g_n : \mathrm{Sent}_{L_{\mathrm{rings}}} \to \mathrm{Sent}_{L_{\mathrm{val}}}$ denotes the reduction map of E_n, then one can see that the sequence $(g_n)_{n\in\omega}$ is uniformly recursive (using that E_n is uniform in $n \in \mathbb{N}$). Let

$$\Sigma := \mathrm{Hen}_{(0,p)} \cup \left(\bigcup_{n\in\omega} \{\phi_{E_n} : \mathcal{O}_K/(p^n) \models \phi\} \right) \cup \{\phi_E : (\Gamma_v, vp) \models \phi\}$$

where $\mathrm{Hen}_{(0,p)}$ is a first-order axiom schema capturing Hensel's lemma (see, e.g., [Kuhlmann 2016, page 21]), together with a set of sentences capturing that the valued field has mixed characteristic $(0, p)$.

Claim. The axiomatization Σ is r.e.

Proof. The set $\mathrm{Hen}_{(0,p)}$ is clearly r.e. The set $\{\phi_E : (\Gamma_v, vp) \models \phi\} \subseteq L_{\mathrm{val}}$ is r.e., being the image of a recursive set via the recursive reduction map of E. Since recursively enumerable sets are closed under finite unions, it remains to show that $\bigcup_{n\in\omega}\{\phi_{E_n} : \mathcal{O}_K/(p^n) \models \phi\}$ is r.e.

Let $\chi : \mathbb{N} \times \mathrm{Sent}_{L_{\mathrm{rings}}} \to \mathbb{N}$ be the recursive function associated to the uniformly decidable sequence $(\mathcal{O}_K/(p^n))_{n\in\omega}$. We construct the partial recursive function $\chi' : \mathbb{N} \times \mathrm{Sent}_{L_{\mathrm{rings}}} \to \mathbb{N} \times \mathrm{Sent}_{L_{\mathrm{rings}}}$ which maps $(n, \phi) \mapsto (n, \phi)$ if $\chi(n, \phi) = 1$ and is undefined if $\chi(n, \phi) = 0$. Consider also the recursive function $g : \mathbb{N} \times \mathrm{Sent}_{L_{\mathrm{rings}}} \to \mathrm{Sent}_{L_{\mathrm{val}}} : (n, m) \mapsto g_n(m)$ associated to the uniformly recursive sequence $(g_n)_{n\in\omega}$. Observe that $\bigcup_{n\in\omega}\{\phi_{E_n} : \mathcal{O}_K/(p^n) \models \phi\} = \mathrm{Im}(F)$ where F is the (partial) recursive function $F = g \circ \chi'$. It follows that $\bigcup_{n\in\omega}\{\phi_{E_n} : \mathcal{O}_K/(p^n) \models \phi\}$ is r.e. $\square$

If $(K', v') \models \Sigma$, then $(K', v') \equiv (K, v)$ in L_{val} by Theorem 2.1.5. We therefore get that Σ is a complete axiomatization of (K, v). We conclude that the L_{val}-theory of (K, v) admits a r.e. and complete axiomatization, whence (K, v) is decidable. $\square$

Corollary 2.3.2. *Let (K, v) be a henselian valued field of mixed characteristic. Then the following are equivalent*:

(1) *The valued field (K, v) is $\exists$-decidable in L_{val}.*

(2) *The sequence $(\mathcal{O}_K/(p^n))_{n\in\omega}$ is uniformly $\exists$-decidable in L_{rings}.*

(3) *The sequence $((\mathcal{O}_K/(p^n), \mathfrak{m}_n)_{n\in\omega}$ is uniformly $\exists^+$-decidable in L_{lcr}.*

Proof. Similar to Corollary 2.3.1, ultimately using Theorem 2.2.6. $\square$

3. Perfectoid fields

3.1. *Introduction.*

3.1.1. *Motivation.* The theory of perfectoid fields (and spaces), introduced by Scholze [2012], was initially designed as a means of transferring results available in positive characteristic to mixed characteristic; see [Scholze 2014, Section 1]. It formalizes the earlier Krasner–Kazhdan–Deligne philosophy (due to Krasner [1957], Kazhdan [1986] and Deligne [1984]), of approximating a *highly ramified* mixed characteristic field with a positive characteristic field. Within the framework of perfectoid fields, this kind of approximation becomes precise and robust with the use of the tilting functor (see Section 3.2).

All this is substantially different from the Ax–Kochen method (see, e.g., [van den Dries 2014, 2.20]), which achieves a model-theoretic transfer principle *asymptotically*, i.e., with the residue characteristic $p \to \infty$. The theory of perfectoid fields will allow us to transport decidability information for a fixed residue characteristic (but with high ramification), setting the stage for a different type of model-theoretic transfer principle.

3.1.2. *Definition.*

Definition 3.1.3. A perfectoid field is a complete valued field (K, v) of residue characteristic $p > 0$ such that Γ_v is a dense subgroup of $\mathbb{R}$ and the Frobenius map $\Phi : \mathcal{O}_K/(p) \to \mathcal{O}_K/(p) : x \mapsto x^p$ is surjective.

Example 3.1.4. (a) The p-adic completions of $\mathbb{Q}_p(p^{1/p^\infty})$, $\mathbb{Q}_p(\zeta_{p^\infty})$ and $\mathbb{Q}_p^{ab}$ are mixed characteristic perfectoid fields.

(b) The t-adic completions of $\mathbb{F}_p((t))^{1/p^\infty}$ and $\bar{\mathbb{F}}_p((t))^{1/p^\infty}$ are perfectoid fields of characteristic p.

Remark 3.1.5. In characteristic p, a perfectoid field is simply a perfect, complete nonarchimedean valued field of rank 1.

3.2. *Tilting.*

3.2.1. *Introduction.* A construction, originally due to Fontaine, provides us with a tilting functor that takes any perfectoid field K and transforms it into a perfectoid field $K^\flat$ of characteristic p. We shall now describe this tilting functor. For more details, see [Scholze 2012, Section 3].

3.2.2. *Definition.* Given a perfectoid field (K, v), we shall now define its tilt $(K^\flat, v^\flat)$. Let $\varprojlim_{x \mapsto x^p} K$ be the limit of the inverse system

$$\cdots \xrightarrow{x \mapsto x^p} K \xrightarrow{x \mapsto x^p} K \xrightarrow{x \mapsto x^p} K$$

which is identified as $\varprojlim_{x \mapsto x^p} K = \{(x_n)_{n \in \omega} : x_{n+1}^p = x_n\}$, viewed as a multiplicative monoid via $(x_n)_{n \in \omega} \cdot (y_n)_{n \in \omega} = (x_n \cdot y_n)_{n \in \omega}$. Similarly, one can define the multiplicative monoid $\varprojlim_{x \mapsto x^p} \mathcal{O}_K$.

Let $\varpi \in \mathcal{O}_K$ be such that $0 < v\varpi \leq vp$ (e.g., $\varpi = p$ when char$(K) = 0$ and $\varpi = 0$ when char$(K) = p$) and consider the ring $\varprojlim_\Phi \mathcal{O}_K/(\varpi)$ which is the limit of the inverse system of rings

$$\cdots \xrightarrow{\Phi} \mathcal{O}_K/(\varpi) \xrightarrow{\Phi} \mathcal{O}_K/(\varpi) \xrightarrow{\Phi} \mathcal{O}_K/(\varpi)$$

where $\Phi : \mathcal{O}_K/(\varpi) \to \mathcal{O}_K/(\varpi) : x \mapsto x^p$ is the Frobenius homomorphism.

Lemma 3.2.3 [Scholze 2012, Lemma 3.4(i) and (ii)]. (a) *The ring* $\varprojlim_{\Phi} \mathcal{O}_K/(\varpi)$ *is independent of the choice of* ϖ *and there is a multiplicative isomorphism* $\varprojlim_{x \mapsto x^p} \mathcal{O}_K \xrightarrow{\cong} \varprojlim_{\Phi} \mathcal{O}_K/(\varpi)$. *Moreover, we get a multiplicative morphism* $\sharp : \varprojlim_{\Phi} \mathcal{O}_K/(\varpi) \to \mathcal{O}_K : x \mapsto x^{\sharp}$ *such that if* $x = (x_n + (\varpi))_{n \in \omega}$, *then* $x^{\sharp} \equiv x_0 \mod (\varpi)$.

(b) *There is an element* $\varpi^{\flat} \in \varprojlim_{\Phi} \mathcal{O}_K/(\varpi)$ *with* $v(\varpi^{\flat})^{\sharp} = v\varpi$.

The definition of $\sharp$ goes as follows: Let $x = (x_n)_{n \in \omega} \in \varprojlim_{\Phi} \mathcal{O}_K/(\varpi)$, i.e., $x_n \in \mathcal{O}_K/(\varpi)$ and $x_{n+1}^p = x_n$. Let $\tilde{x}_n \in \mathcal{O}_K$ be an arbitrary lift of $x_n \in \mathcal{O}_K/(\varpi)$. Then the limit $\lim_{n \to \infty} \tilde{x}_n^{p^n}$ exists and is independent of the choice of the $\tilde{x}_n$; see [Scholze 2012, Lemma 3.4(i)]. We define $x^{\sharp} := \lim_{n \to \infty} \tilde{x}_n^{p^n}$.

We now introduce $K^{\flat} := \varprojlim_{\Phi} \mathcal{O}_K/(\varpi)[(\varpi^{\flat})^{-1}]$. A priori this is merely a ring. It is in fact a valued field according to the following:

Lemma 3.2.4 [Scholze 2012, Lemma 3.4(iii)]. (a) *There is a morphism of multiplicative monoids* $K^{\flat} \to K : x \mapsto x^{\sharp}$ (*extending the one of* Lemma 3.2.3(a)), *which induces a morphism of multiplicative monoids* $K^{\flat} \xrightarrow{\cong} \varprojlim_{x \mapsto x^p} K : x \mapsto (x^{\sharp}, (x^{\sharp})^{1/p}, \dots)$. *The map* $v^{\flat} : K^{\flat} \to \Gamma_v \cup \{\infty\} : x \mapsto v x^{\sharp}$ *is a valuation on* $K^{\flat}$, *which makes* $(K^{\flat}, v^{\flat})$ *into a perfectoid field of characteristic* p. *If* $\mathcal{O}_{K^{\flat}}$ *is the valuation ring of* $K^{\flat}$, *then we have a ring isomorphism* $\mathcal{O}_{K^{\flat}} \cong \varprojlim_{\Phi} \mathcal{O}_K/(\varpi)$.

(b) *We have an isomorphism of ordered abelian groups* $(\Gamma_v, v\varpi) \cong (v^{\flat} K^{\flat}, v^{\flat} \varpi^{\flat})$ *and a field isomorphism* $k_v \cong k_{v^{\flat}}$. *Moreover, we have a ring isomorphism* $\mathcal{O}_K/(\varpi) \cong \mathcal{O}_{K^{\flat}}/(\varpi^{\flat})$.

Remark 3.2.5. Lemma 3.2.4(a) allows us to identify the multiplicative underlying monoid of $K^{\flat}$ with $\varprojlim_{x \mapsto x^p} K$. It is not hard to see that, via this identification, addition is described by $(x_n)_{n \in \omega} + (y_n)_{n \in \omega} = (z_n)_{n \in \omega}$, where $z_n = \lim_{m \to \infty} (x_{n+m} + y_{n+m})^{p^m}$.

Definition 3.2.6. We say that the valued field $(K^{\flat}, v^{\flat})$ constructed in Lemma 3.2.4(a) is the tilt of the perfectoid field (K, v).

Remark 3.2.7 [Scholze 2012, Lemma 3.4(iv)]. If (K, v) is a perfectoid field of characteristic p, then $(K^{\flat}, v^{\flat}) \cong (K, v)$.

Example 3.2.8 (see Corollary 4.4.3). In the examples below, $\widehat{K}$ stands for the p-adic (resp. t-adic) completion of the field K depending on whether its characteristic is 0 or p:

(a) $\widehat{\mathbb{Q}_p(p^{1/p^{\infty}})}^{\flat} \cong \widehat{\mathbb{F}_p((t))^{1/p^{\infty}}}$ and $t^{\sharp} = p$.

(b) $\widehat{\mathbb{Q}_p(\zeta_{p^{\infty}})}^{\flat} \cong \widehat{\mathbb{F}_p((t))^{1/p^{\infty}}}$ and $(t+1)^{\sharp} = \zeta_p$.

(c) $\widehat{\mathbb{Q}_p^{ab}}^{\flat} \cong \widehat{\overline{\mathbb{F}}_p((t))^{1/p^{\infty}}}$ and $(t+1)^{\sharp} = \zeta_p$.

Remark 3.2.9. The tilting construction makes sense for nonperfectoid fields as well. However, in the absence of infinite wild ramification, it is too lossy for it to be useful (e.g., $\mathbb{Q}_p^{\flat} = \mathbb{F}_p$).

3.3. *Witt vectors.* We review the basics of Witt vectors. Details and proofs can be found in [Serre 1979, Sections 5 and 6; Kedlaya and Liu 2015, Section 3; van den Dries 2014, Section 6].

3.3.1. *p-rings.*

Definition 3.3.2. A ring R of characteristic p is called perfect if the Frobenius homomorphism $\Phi : R \to R$, $x \mapsto x^p$ is bijective.

Definition 3.3.3. (a) A p-ring is a commutative ring A provided with a filtration $\mathfrak{a}_1 \supset \mathfrak{a}_2 \supset \cdots$ such that $\mathfrak{a}_n \mathfrak{a}_m \subset \mathfrak{a}_{n+m}$ and so that A is Hausdorff and complete with respect to the topology induced by the filtration and $A/\mathfrak{a}_1$ is a perfect ring of characteristic p.

 (b) If in addition $\mathfrak{a}_n = p^n A$ and p is not a zero-divisor, then we say that A is a strict p-ring.

If A is a p-ring, we call the quotient ring A/pA the *residue ring* of the p-ring A and write res : $A \to A/pA$ for the quotient map. A system of *multiplicative representatives* (or simply a system of representatives) is a multiplicative homomorphism $f_A : A/pA \to A$ such that $f_A(\mathrm{res}(x)) = x$. A p-ring A always has a system of representatives $f_A : A/pA \to A$ and when A is strict every element $a \in A$ can be written uniquely in the form $a = \sum_{i=0}^{\infty} f_A(\alpha_i) \cdot p^i$ [Serre 1979, page 37].

Theorem 3.3.4 [Serre 1979, Corollary page 39]. *For every perfect ring R, there exists a unique strict p-ring, denoted by $W(R)$, with residue ring $W(R)/pW(R) \cong R$.*

The ring $W(R)$ is said to be the *ring of Witt vectors* over the ring R. The uniqueness part of Theorem 3.3.4 follows from the next result, which we record here for later use.

Fact 3.3.5 [Kedlaya and Liu 2015, Lemma 3.3.2]. Let A be a strict p-ring and $f_A : A/pA \to A$ be a system of representatives. Let A' be a p-adically complete ring and $\phi : A/pA \to A'/pA'$ be a ring homomorphism. Then there exists a unique ring homomorphism $g : A \to A'$ making the diagram below commute:

$$
\begin{array}{ccc}
A & \xrightarrow{\ g\ } & A' \\
\downarrow & & \downarrow \\
A/pA & \xrightarrow{\ \phi\ } & A'/pA'
\end{array}
$$

where the vertical arrows are the projections modulo p. More precisely, there exists a unique lift $\phi : A/pA \to A'/pA'$ to a multiplicative map $\tilde{\phi} : A/pA \to A'$ and we have $g\left(\sum_{i=0}^{\infty} f_A(\alpha_i) \cdot p^i\right) = \sum_{i=0}^{\infty} \tilde{\phi}(\alpha_i) \cdot p^i$.

3.3.6. *Teichmüller representatives.* There is a system of *Teichmüller representatives* of R in $W(R)$. This is the (unique) multiplicative homomorphism $[\ ] : R \to W(R)$ with the property that $\mathrm{res}([x]) = x$ for all $x \in R$. Explicitly, for $x \in R$ and $n \in \mathbb{N}$, let $x_n \in R$ be such that $x_n^{p^n} = x$ and $\tilde{x}_n \in W(R)$ be an arbitrary lift of x_n. The sequence $(\tilde{x}_n^{p^n})_{n \in \omega}$ is a Cauchy sequence, whose limit is independent of the chosen lifts. We let $[x] := \lim_{n \to \infty} \tilde{x}_n^{p^n}$; see [Serre 1979, Proposition 8 page 35].

It is easy to see that any element $x \in W(R)$ can be written *uniquely* in the form $x = \sum_{n=0}^{\infty} [x_n] \cdot p^n$, for some $x_i \in R$. The vector $(x_0, x_1, \dots) \in R^\omega$ is called the *Teichmüller vector* of x.

3.3.7. *Witt vectors.* The advantage of Witt vectors over Teichmüller vectors, comes from the fact that the ring operations in $W(R)$ have nicer coordinatewise descriptions when using the former (see Section 3.3.8). Write $x \in W(R)$ in the form $x = \sum_{n=0}^{\infty} [x_n^{p^{-n}}] \cdot p^n$, for some $x_i \in R$. The vector $(x_0, x_1, \dots) \in R^{\omega}$ is called the *Witt vector* of x.

3.3.8. *Ring operations.* By the discussion above, the ring $W(R)$ can be thought of as the p-adic analogue of formal power series with coefficients in R. By identifying x with its Witt vector, we see that $W(R)$ has R^{ω} as its underlying set. By [van den Dries 2014, Lemma 6.5], the ring operations are given by

$$(a_0, a_1, \dots) + (b_0, b_1, \dots) = (S_0(a_0, b_0), S_1(a_0, a_1, b_0, b_1), \dots)$$

and

$$(a_0, a_1, \dots) \cdot (b_0, b_1, \dots) = (P_0(a_0, b_0), P_1(a_0, a_1, b_0, b_1), \dots)$$

for suitable polynomials $S_i, P_i \in \mathbb{Z}[x_0, \dots, x_i, y_0, \dots, y_i]$ which are universal, in the sense that they do not depend on R.

Observation 3.3.9. The polynomials S_i (resp. P_i) are *computable*, i.e., the function $\mathbb{N} \to \mathbb{Z}[x_0, y_0, \dots]$, $n \mapsto S_n$ (resp. $\mathbb{N} \to \mathbb{Z}[x_0, y_0, \dots]$, $n \mapsto P_n$) is recursive.

Proof. For $n \in \mathbb{N}$, we introduce the n-th Witt polynomial $W_n(x_0, \dots, x_n) = x_0^{p^n} + px_1^{p^{n-1}} + \dots + p^n \cdot x_n \in \mathbb{Z}[x_0, \dots, x_n]$ [van den Dries 2014, page 135]. The proof of [van den Dries 2014, Lemma 6.5] shows that

$$S_0(x_0, y_0) = x_0 + y_0$$

and

$$W_{n-1}(S_0^p, \dots, S_{n-1}^p) + p^n \cdot S_n = W_{n-1}(S_0(x_0^p, y_0^p), \dots, S_{n-1}(x_0^p, \dots, x_{n-1}^p, y_0^p, \dots, y_{n-1}^p)) + p^n(x_n + y_n),$$

whence the polynomial S_n may be computed recursively from $S_0, \dots, S_{n-1}$. The proof is similar for the P_n. $\qquad \square$

3.4. *Truncated Witt vectors.*

3.4.1. *Definition.* In this paper, we will mostly be working with truncated Witt vectors. These can be thought of as p-adic analogues of truncated power series, i.e., elements of the ring $R[[t]]/(t^n) \cong R[t]/(t^n)$ (over some base ring R). More formally:

Definition 3.4.2. Let R be a perfect ring. Given $n \in \mathbb{N}$, the ring of n-truncated Witt vectors over R is defined as $W_n(R) := W(R)/p^n W(R)$.

3.4.3. *Language.* For a perfect ring R, the pair $(W(R), R)$ (resp. $(W_n(R), R)$) is viewed as a two-sorted structure with sorts $\boldsymbol{W}$ for the Witt ring $W(R)$ (resp. $W_n(R)$) and $\boldsymbol{R}$ for the residue ring R. The sort $\boldsymbol{W}$ is equipped with the language of rings L_{rings}, while the sort $\boldsymbol{R}$ may be equipped with any $L \supseteq L_{\text{rings}}$. We also have a function symbol for the Teichmüller map $[\ ] : \boldsymbol{R} \to \boldsymbol{W}$. For each choice of a language L for the $\boldsymbol{R}$-sort, the resulting language will be denoted by $\langle L_{\text{rings}}, L \rangle$.

3.4.4. *Interpretability in R.*

Lemma 3.4.5. *Let R be a perfect ring, viewed as an L-structure with $L \supseteq L_{\text{rings}}$. For each $n \in \mathbb{N}$, there exists a quantifier-free interpretation Γ_n of the $\langle L_{\text{rings}}, L \rangle$-structure $(W_n(R), R)$ in the L-structure R such that the sequence of interpretations $(\Gamma_n)_{n \in \omega}$ is uniformly recursive.*

Proof. By Section 3.3.8, for $n \in \mathbb{N}$ the underlying set of $W_n(R)$ can be identified with R^n, so we take $\partial_{\Gamma_n}(x_1, \ldots, x_n)$ to be $\bigwedge_{i=1}^{n} x_i = x_i$ and the coordinate map $f_{\Gamma_n} : R^n \to W_n(R)$ as the identity map. The ring operations of $W_n(R)$ are given by

$$(a_0, \ldots, a_{n-1}) + (b_0, \ldots, b_{n-1}) = (S_0(a_0, b_0), \ldots, S_{n-1}(a_0, b_0, \ldots, a_{n-1}, b_{n-1}))$$

and

$$(a_0, \ldots, a_{n-1}) \cdot (b_0, \ldots, b_{n-1}) = (P_0(a_0, b_0), \ldots, P_{n-1}(a_0, b_0, \ldots, a_{n-1}, b_{n-1})),$$

for certain polynomials $S_i, P_i \in \mathbb{Z}[x_0, \ldots, x_i, y_0, \ldots, y_i]$, for $i = 0, \ldots, n-1$. We now need to describe the map $\phi \mapsto \phi_{\Gamma_n}$ on *unnested* atomic L_{rings}-formulas:

(1) If $\phi(x, y, z)$ is the formula $x + y = z$ (here $x, y, z \in \boldsymbol{W}$), we may take $\phi_{\Gamma_n}(\bar{x}, \bar{y}, \bar{z})$ to be the L_{rings}-formula $\bigwedge_{i=0}^{n-1} z_i = S_i(x_0, y_0, \ldots, x_{n-1}, y_{n-1})$.

(2) If $\phi(x, y, z)$ is the formula $x \cdot y = z$ (here $x, y, z \in \boldsymbol{W}$), we may take $\phi_{\Gamma_n}(\bar{x}, \bar{y}, \bar{z})$ to be the L_{rings}-formula $\bigwedge_{i=0}^{n-1} z_i = P_i(x_0, y_0, \ldots, x_{n-1}, y_{n-1})$.

(3) If $\phi(x, y)$ is the formula $[x] = y$ (here $x \in \boldsymbol{R}$ and $y \in \boldsymbol{W}$), we may take $\phi_{\Gamma_n}(x, \bar{y})$ to be the formula $\bigwedge_{i=1}^{n-1} y_i = 0 \wedge y_0 = x$.

(4) If $\phi(\bar{x})$ is an unnested atomic L-formula with variables from the sort $\boldsymbol{R}$, then we may take $\phi_{\Gamma_n}(\bar{x}) := \phi(\bar{x})$ (here $x_i \in \boldsymbol{R}$ for the latter formula).

The above data define a quantifier-free interpretation Γ_n of the $\langle L_{\text{rings}}, L \rangle$-structure $(W_n(R), R)$ in the L-structure R. Moreover, since the polynomials S_i, P_i are computable (Observation 3.3.9), the sequence of interpretations $(\Gamma_n)_{n \in \omega}$ is uniformly recursive. $\square$

3.5. *Untilting.* Expository notes on the material of this section may be found either in Section 5 of the Bourbaki seminar given by Morrow [2019] or in the lecture series notes by Lurie [2018, Lectures 2 and 3].

3.5.1. *Overview.* The functor $K \mapsto K^\flat$ is far from being faithful, i.e., there will be several nonisomorphic mixed characteristic perfectoid fields K that tilt to the same perfectoid field of characteristic p. For example, the p-adic completions of $\mathbb{Q}_p(p^{1/p^\infty})$ and $\mathbb{Q}_p(\zeta_{p^\infty})$ both tilt to the t-adic completion of $\mathbb{F}_p((t))^{1/p^\infty}$. For a perfectoid field F of characteristic p, an *untilt* of F is a pair (K, ι), where (K, v) is a perfectoid field and $\iota : (F, w) \overset{\cong}{\Longrightarrow} (K^\flat, v^\flat)$ is a valued field isomorphism. Fargues and Fontaine give a description of all possible untilts of F in an *intrinsic* fashion, i.e., in a way that uses only arithmetic from F itself (see Theorem 3.5.11). This result will be of vital importance for Theorem A.

3.5.2. *The ring $A_{\inf}$.* Fix any perfectoid field (F, w) of characteristic $p > 0$. We introduce $A_{\inf} := W(\mathcal{O}_F)$, called the *infinitesimal* period ring, which is the ring of Witt vectors over $\mathcal{O}_F$.

Definition 3.5.3. An element $\xi \in A_{\inf}$ is said to be *distinguished* if it is of the form $\xi = \sum_{n=0}^{\infty}[c_n]p^n$, with $c_0 \in \mathfrak{m}_F$ and $c_1 \in \mathcal{O}_F^{\times}$.

In other words, distinguished elements are those of the form $\xi = [\pi] - up$ where $w\pi > 0$ and $u \in A_{\inf}$ is a unit.

Remark 3.5.4. Let $\mathrm{res} : \mathcal{O}_F \to \mathcal{O}_F/\mathfrak{m}_F$ be the residue map and $W(\mathrm{res}) : A_{\inf} \to W(\mathcal{O}_F/\mathfrak{m}_F)$ be the (unique) induced ring homomorphism provided by Fact 3.3.5 that maps $\sum_{n=0}^{\infty}[c_n]p^n \mapsto \sum_{n=0}^{\infty}[\mathrm{res}(c_n)] \cdot p^n$. The element $\xi \in A_{\inf}$ is distinguished precisely when $W(\mathrm{res})(\xi)$ is a unit multiple of p in $W(\mathcal{O}_F/\mathfrak{m}_F)$.

3.5.5. *Distinguished elements and untilts.* We outline how one can go from an untilt of F to an ideal of $A_{\inf}$ generated by a distinguished element and vice versa. Let (K, ι) be an untilt of F, i.e., we have $\iota : (F, w) \xrightarrow{\cong} (K^{\flat}, v^{\flat})$. By Lemma 3.2.3(a), we have a morphism of multiplicative monoids $\sharp : \mathcal{O}_{K^{\flat}} \to \mathcal{O}_K$. We also write $\sharp : \mathcal{O}_F \to \mathcal{O}_K$ for the composite map $\mathcal{O}_F \xrightarrow{\iota} \mathcal{O}_{K^{\flat}} \xrightarrow{\sharp} \mathcal{O}_K$. While $\sharp : \mathcal{O}_F \to \mathcal{O}_K$ is not a ring homomorphism (unless K has characteristic p), it does induce a ring homomorphism $\phi : \mathcal{O}_F \to \mathcal{O}_K/(p) : x \mapsto x^{\sharp} \bmod (p)$. Moreover, ϕ is *surjective* since it descends to an isomorphism $\mathcal{O}_F/(\pi) \xrightarrow{\cong} \mathcal{O}_K/(p)$ for any $\pi \in \mathcal{O}_F$ with $w\pi = vp$. The map θ in the lemma below is important. We shall sketch the proof of the lemma for the convenience of the reader.

Lemma 3.5.6 [Lurie 2018, Lecture 3, Remarks 11–13]. *There exists a ring homomorphism $\theta : A_{\inf} \to \mathcal{O}_K$ inducing ϕ above. Moreover, θ is surjective and $\theta^{-1}(\mathcal{O}_K^{\times}) = A_{\inf}^{\times}$.*

Proof sketch. Apply Fact 3.3.5 with $A = A_{\inf}$, $A' = \mathcal{O}_K$ to get that ϕ lifts uniquely to the ring homomorphism

$$\theta : A_{\inf} \to \mathcal{O}_K : \sum_{n=0}^{\infty}[c_n] \cdot p^n \mapsto \sum_{n=0}^{\infty} c_n^{\sharp} \cdot p^n.$$

We claim that θ is surjective. Recall that ϕ is surjective. Given $x \in \mathcal{O}_K$, we may thus find $c_0 \in \mathcal{O}_F$ such that $x = c_0^{\sharp} + x_1 \cdot p$, for some $x_1 \in \mathcal{O}_K$. Similarly, we may find $c_1 \in \mathcal{O}_F$ such that $x_1 = c_1^{\sharp} + x_2 \cdot p$. We then get that $x = c_0^{\sharp} + c_1^{\sharp} \cdot p + x_2 \cdot p^2$. Continuing this way and since $\mathcal{O}_K$ is p-adically complete, we may write $x = \sum_{n=0}^{\infty} c_n^{\sharp} \cdot p^n$. To show that $\theta^{-1}(\mathcal{O}_K^{\times}) = A_{\inf}^{\times}$, observe that

$$\sum_{n=0}^{\infty}[c_n] \cdot p^n \in A_{\inf}^{\times} \iff c_0 \in \mathcal{O}_F^{\times} \iff c_0^{\sharp} \in \mathcal{O}_K^{\times} \iff \sum_{n=0}^{\infty} c_n^{\sharp} \cdot p^n \in \mathcal{O}_K^{\times} \qquad \square$$

Let us examine the kernel of θ. Let $\pi \in \mathcal{O}_F$ be as above (i.e., such that $w\pi = vp$) and write $\pi^{\sharp} = \bar{u} \cdot p$ for some $\bar{u} \in \mathcal{O}_K^{\times}$. By Lemma 3.5.6, we may find $u \in A_{\inf}^{\times}$ such that $\theta(u) = \bar{u}$. Note that $\xi = [\pi] - u \cdot p \in A_{\inf}$ is a distinguished element and that $\xi \in \mathrm{Ker}(\theta)$. In fact, the following is true:

Proposition 3.5.7 [Lurie 2018, Corollary 17]. *Let (F, w) be a perfectoid field of characteristic p and (K, ι) be an untilt. Let $\theta : A_{\inf} \to \mathcal{O}_K$ be as above. Then $\mathrm{Ker}(\theta)$ is a principal ideal generated by **any** distinguished element $\xi \in \mathrm{Ker}(\theta)$.*

Starting with an untilt (K, ι), we have thus produced an ideal $(\xi_K) \subseteq A_{\inf}$, where ξ_K is a distinguished element in $A_{\inf}$.

Conversely, starting with $(\xi) \subseteq A_{\inf}$, with ξ a distinguished element, we may produce an untilt (K, ι) of F as follows. Write $\theta : A_{\inf} \to A_{\inf}/(\xi)$ for the quotient map. We then have:

Lemma 3.5.8 [Lurie 2018, Lecture 3, page 4, Claim (a)]. *For every $y \in A_{\inf}/(\xi)$, there exists $x \in \mathcal{O}_F$ such that $(y) = (\theta([x]))$.*

Proposition 3.5.9 [Lurie 2018, Proposition 16]. *Let (F, w) be a perfectoid field of characteristic p and $\xi \in A_{\inf}$ a distinguished element. Then the quotient ring $A_{\inf}/(\xi)$ is the valuation ring $\mathcal{O}_K$ of a perfectoid field (K, v) such that $(K^\flat, v^\flat) \cong (F, w)$. The valuation v is such that if $y \in A_{\inf}/(\xi)$, then $vy := wx$ with $x \in \mathcal{O}_F$ so that $(y) = (\theta([x]))$ in $A_{\inf}/(\xi)$.*

The isomorphism $\iota : (F, w) \to (K^\flat, v^\flat)$ of Proposition 3.5.9 is described as follows. If $\xi = [\pi] - up$, then $\mathcal{O}_K/(p) \cong W(\mathcal{O}_F)/(p, [\pi] - up) \cong \mathcal{O}_F/(\pi)$. Passing to inverse limits, this induces a ring isomorphism $\mathcal{O}_{K^\flat} \cong \mathcal{O}_F$, which in turn yields $\iota : F \xrightarrow{\cong} K^\flat$ by passing to fraction fields. Starting with $(\xi) \subseteq A_{\inf}$, we have thus produced an untilt (K, ι).

Definition 3.5.10. Two untilts (K, ι) and (K', ι') are isomorphic when there exists a valued field isomorphism $\phi : K \xrightarrow{\cong} K'$ inducing a commutative diagram

$$
\begin{array}{ccc}
F & \xrightarrow{\iota} & K^\flat \\
\Big\| {\scriptstyle =} & & \Big\downarrow {\scriptstyle \phi^\flat} \\
F & \xrightarrow{\iota'} & K'^\flat
\end{array}
$$

where $\phi^\flat : (x, x^{1/p}, \dots) \mapsto (\phi(x), \phi(x^{1/p}), \dots)$. Let Y_F denote the set of characteristic 0 untilts of (F, w), up to isomorphism.

We write 0 for the isomorphism class of the unique characteristic p untilt of (F, ι), represented by F itself together with the natural isomorphism $\iota : F \xrightarrow{\cong} F^\flat : x \to (x, x^{1/p}, \dots)$, and set $\overline{Y}_F = Y_F \cup \{0\}$ for the set of all untilts of (F, w), up to isomorphism.

Theorem 3.5.11 (Fargues and Fontaine). *Let (F, w) be a perfectoid field of characteristic p. The map $(\xi) \mapsto \operatorname{Frac}(\mathcal{O}_F/(\xi))$ defines a bijective correspondence between the set of ideals $(\xi) \subseteq A_{\inf}$ generated by a distinguished element and the set $\overline{Y}_F$.*

Proof. See [Morrow 2019, Proposition 5.1] or [Lurie 2018, Lecture 2, Corollary 18].   □

Remark 3.5.12. Let (K, ι) be an untilt of (F, w) and $(\xi) = ([\pi] - up)$ be its associated ideal. Note that $(p) = (\theta([\pi]))$ in $A_{\inf}/(\xi)$ and therefore $vp = w\pi$.

3.5.13. *Tilting equivalence.* We emphasized in Section 3.5.1 that untilting is ambiguous, in the sense that there are many ways to untilt a perfectoid field of positive characteristic. However, the ambiguity is eliminated by *fixing a base* perfectoid field K and its associated tilt $K^\flat$. This leads to an equivalence of categories of perfectoid extensions, known as the *tilting equivalence*:

Theorem 3.5.14 (tilting equivalence). *The categories of perfectoid field extensions of K and perfectoid field extensions of $K^\flat$ are equivalent.*

Proof. See [Scholze 2013, Theorem 2.8]. Theorem 5.2 in [loc. cit.] shows a more general result about perfectoid algebras; the case of perfectoid fields follows as a special case by [loc. cit., Lemma 5.21]. □

In the discussion after [loc. cit., Theorem 2.8], Scholze explains that there are two proofs of Theorem 3.5.14:

(1) His original proof in [loc. cit.], using Faltings' almost mathematics; see [loc. cit., Section 4].

(2) An alternative proof which describes the functor $\sharp$ inverse to $\flat$ as $L \mapsto W(\mathcal{O}_L) \otimes_{W(\mathcal{O}_{K^\flat})} K$; see [loc. cit., Remark 5.19].

Let us elaborate more on the second approach, which is in the spirit of Section 3.5 and will be more suitable for us. If $(\xi) \subset W(\mathcal{O}_{K^\flat})$ is the ideal associated to K, then one computes that $W(\mathcal{O}_L) \otimes_{W(\mathcal{O}_{K^\flat})} \mathcal{O}_K = W(\mathcal{O}_L) \otimes_{W(\mathcal{O}_{K^\flat})} W(\mathcal{O}_{K^\flat})/(\xi) = W(\mathcal{O}_L)/(\xi)$. In other words, if L is a perfectoid field extending $K^\flat$, then $L^\sharp$ is simply the untilt of K whose associated ideal in $W(\mathcal{O}_L)$ is (ξ).

3.6. *Space of untilts.* In Section 3.6.3 we exhibit an appealing model-theoretic property of the space Y_F of untilts of (F, w). This will not be used in the rest of the paper. We then study in Section 3.6.5 the size of the space of untilts up to elementary equivalence. We will see that the cardinality is often too big, so that one cannot possibly expect K to be decidable simply relative to $K^\flat$.

3.6.1. *Metric structure on Y_F.* Fargues and Fontaine equip $\overline{Y}_F$ with a metric topology, which allows us to view the space of untilts geometrically. Suppose $x = (K_x, v_x)$ and $y = (K_y, v_y)$ are two "points" of $\overline{Y}_F$, corresponding to the ideals (ξ_x) and (ξ_y) respectively, provided by Theorem 3.5.11. We choose an embedding $\Gamma_w \hookrightarrow \mathbb{R}$, which determines embeddings $\Gamma_{v_y} \hookrightarrow \mathbb{R}$ for all $y \in Y_F$ via the canonical identification $\Gamma_{v_y} \cong \Gamma_w$. One then defines $d(x, y) := |\theta_y(\xi_x)|_y$, where $\theta_y : A_{\inf} \twoheadrightarrow A_{\inf}/(\xi_y) = \mathcal{O}_{K_y}$ is the quotient map and as usual $|a|_y = p^{-v_y(a)}$.

Proposition 3.6.2 [Fargues and Fontaine 2018, Proposition 2.3.2(1)]. *Let $d : \overline{Y}_F \times \overline{Y}_F \to \mathbb{R}$ be as above. The pair $(\overline{Y}_F, d)$ is a complete ultrametric space.*

Proof. See [Fargues and Fontaine 2018, Propositions 2.3.2 and 2.3.4] or [Lurie 2018, Lecture 14, Propositions 6 and 7]. □

Recall that $0 \in \overline{Y}_F$ is the isomorphism class of the untilt corresponding to (F, w) itself together with the natural isomorphism $\iota : F \xrightarrow{\cong} F^\flat : x \to (x, x^{1/p}, \dots)$. We can define a radius function $r(y) := d(0, y)$ for $y \in \overline{Y}_F$, which allows us to think of $\overline{Y}_F$ intuitively as the unit disc with center 0.

3.6.3. *A model-theoretic property of Y_F.* We now show that limits in the punctured unit disc Y_F, with respect to the Fargues–Fontaine metric, agree with limits in the model-theoretic sense:

Proposition 3.6.4. *Let $(x_n)_{n\in\omega}$ be a sequence in Y_F such that $x_n \xrightarrow{d} x$ and $x \neq 0$. Let (K_n, v_n) be the untilt associated to x_n and (K, v) the untilt associated to x. We set*

$$(K^*, v^*) := \prod_{n\in\omega}(K_n, v_n)/U$$

for a nonprincipal ultrafilter U on $\mathbb{N}$. Then $(K^*, v^*) \equiv (K, v)$ in L_{val}.

Proof. Let $(\xi_n) = ([\pi_n] - u_n p)$ be the ideal in $A_{\inf}$ corresponding to (K_n, v_n) and $(\xi) = ([\pi] - up)$ be the ideal corresponding to (K, v). Note that the set $\{v_n p : n \in \mathbb{N}\} \subseteq \Gamma_w$ is bounded from above; otherwise, there would be a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ with $x_{n_k} \to 0$.

Fix $m \in \mathbb{N}$ and let $(\bar{\xi}_n)$ and $(\bar{\xi})$ be the images of the ideals (ξ_n) and (ξ) in $W_m(\mathcal{O}_F)$ via $A_{\inf} \twoheadrightarrow A_{\inf}/(p^m) = W_m(\mathcal{O}_F)$.

Claim. We have that $(\bar{\xi}_n) = (\bar{\xi})$, for sufficiently large n.

Proof. We let $\theta : A_{\inf} \to A_{\inf}/(\xi) = \mathcal{O}_K$ be the quotient map. Since $d(x_n, x) \to 0$, we get that $v(\theta(\xi_n)) \to \infty$. We will thus have that $\xi_n \equiv p^m \cdot \alpha_n \mod (\xi)$, for some $\alpha_n \in A_{\inf}$ and for all $n \gg 0$. Consequently, one gets that $(\bar{\xi}_n) \subseteq (\bar{\xi})$ for $n \gg 0$. Similarly, since $v_n(\theta_n(\xi)) \to \infty$ and $\{v_n p : n \in \mathbb{N}\}$ is bounded, we get that $v_n(\theta_n(\xi)) \geq mv_n p$ for $n \gg 0$. It follows that there exists $\beta_n \in A_{\inf}$ such that $\xi \equiv p^m \cdot \beta_n \mod (\xi_n)$, for $n \gg 0$. We conclude that $(\bar{\xi}_n) = (\bar{\xi})$, for $n \gg 0$. $\square$

It follows that $\mathcal{O}_{K_n}/(p^m) \cong W_m(\mathcal{O}_F)/(\bar{\xi}_n) = W_m(\mathcal{O}_F)/(\bar{\xi}) \cong \mathcal{O}_K/(p^m)$, for sufficiently large n. We also get that (ξ_n) and (ξ) have the same image in $A_{\inf}/(p) \cong \mathcal{O}_F$ for $n \gg 0$ and therefore $(\pi_n) = (\pi)$ for $n \gg 0$. By Lemma 3.2.4(b), we get that $(\Gamma_{v_n}, v_n p) \cong (\Gamma_w, w\pi_n) = (\Gamma_w, w\pi) \cong (\Gamma_v, vp)$, for all sufficiently large n. The conclusion follows from Theorem 2.1.5 and Łoś's Theorem. $\square$

3.6.5. *The space Z_F.* It is natural to consider the set Y_F up to elementary equivalence. More precisely:

Definition 3.6.6. Let (F, w) be a perfectoid field of characteristic p. For $x = (K_x, \iota_x)$, $y = (K_y, \iota_y) \in Y_F$ define the equivalence relation $x \sim y \iff (K_x, v_x) \equiv (K_y, v_y)$ in L_{val}. We define $Z_F := Y_F/\sim$.

Note that the definition of Z_F only takes the underlying valued fields (K, v) into account and not the map ι. We now determine the size of Z_F in a few cases in Proposition 3.6.9. For Proposition 3.6.9(a), we will need the following:

Fact 3.6.7 [Scholze 2012, Proposition 4.3]**.** Let (K, v) be a perfectoid field with $K^\flat$ algebraically closed. Then K is also algebraically closed.

For Proposition 3.6.9(b) we need an algebraic fact. A finite extension L/K has the *unique subfield property* if for every $d \mid [L : K]$, there is a unique subextension F/K such that $[F : K] = d$.

Fact 3.6.8 [Acosta de Orozco and Vélez 1982, Theorem 2.1]**.** Let K be a field, $n \in \mathbb{N}$ be such that $\mathrm{char}(K) \nmid n$, $a \in K$ such that $X^n - a \in K[X]$ is irreducible and set $L = K(a^{1/n})$. Suppose that for every odd prime $p \mid n$, we have that $\zeta_p \notin L \backslash K$ and in case $4 \mid n$ we have $\zeta_4 \notin L \backslash K$. Then L/K has the unique subfield property.

The construction in Proposition 3.6.9(b) is an elaborated version of Scholze's answer [2021] to a closely related mathoverflow question asked by the author.

Proposition 3.6.9. (a) *Let* $(F, w) = (\widehat{\overline{\mathbb{F}_p((t))}}, v_t)$, *the t-adic completion of an algebraic closure of* $\mathbb{F}_p((t))$. *Then* $|Z_F| = 1$.

(b) *Set* $(F, w) = (\widehat{\overline{\mathbb{F}_p((t))}^{1/p^\infty}}, v_t)$. *Then* $|Z_F| = \mathfrak{c}$.

Proof. (a) **First proof.** Let (K, v) be a perfectoid field of characteristic 0 such that $(K^\flat, v^\flat) \cong (\widehat{\overline{\mathbb{F}_p((t))}}, v_t)$. By Fact 3.6.7, we get that K is algebraically closed. By Robinson's completeness of the theory $\mathrm{ACVF}_{(0,p)}$ of algebraically closed valued fields of mixed characteristic $(0, p)$ (see [van den Dries 2014, Corollary 3.34]), we get that $\mathrm{Th}(K, v) = \mathrm{ACVF}_{(0,p)}$ and hence $|Z_F| = 1$.

Second proof. Let $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}_p}}$ be the completed algebraic closure of $\mathbb{Q}_p$. By [Fargues and Fontaine 2014, Remark 2.24], we have a ring isomorphism $\mathcal{O}_K/(p^n) \cong \mathcal{O}_{\mathbb{C}_p}/(p^n)$ for each $n \in \mathbb{N}$. One also has that $(\Gamma_v, vp) \equiv (\Gamma_{\mathbb{C}_p}, v_p p)$ in L_{oag} with a constant for vp, by an easy application of quantifier elimination for the theory ODAG of ordered divisible abelian groups [Marker 2002, Corollary 3.1.17]. We conclude that $(K, v) \equiv (\mathbb{C}_p, v_p)$ from Theorem 2.1.5.

(b) We assume that $p > 2$; we indicate the necessary changes for the case $p = 2$ in the end of the proof. For each $\alpha \in 2^\omega$, we define an algebraic extension K_α of $\mathbb{Q}_p$ as follows. We write $\alpha \restriction n$ for the restriction of α to $n = \{0, 1, \ldots, n-1\}$ (set-theoretically $\alpha \restriction 0 = 0$). We now define inductively:

(1) $K_0 = \mathbb{Q}_p$ and $\pi_0 = p$.

(2) $K_{\alpha \restriction n} = K_{\alpha \restriction (n-1)}(((1+p)^{\alpha(n-1)} \cdot \pi_{\alpha \restriction (n-1)})^{1/p})$ and $\pi_{\alpha \restriction n} = ((1+p)^{\alpha(n-1)} \cdot \pi_{\alpha \restriction (n-1)})^{1/p}$.

Set $\bar{\alpha}_n = \sum_{k=0}^{n-1} \alpha(k) \cdot p^k$ for $n \in \mathbb{N}^{>0}$ and $\bar{\alpha}_0 = 1$ by convention. Note that $X^{p^n} - (1+p)^{\bar{\alpha}_n} p \in \mathbb{Z}_p[X]$ is an Eisenstein (hence irreducible) polynomial and that $K_{\alpha \restriction n} = \mathbb{Q}_p(((1+p)^{\bar{\alpha}_n} p)^{1/p^n})$. We let $K_\alpha = \bigcup_{n \in \omega} K_{\alpha \restriction n}$. Visually, the field K_α is obtained by taking the union of all fields along a certain branch (corresponding to α) in the binary tree below:

Claim 1. For each $\alpha \in 2^\omega$, we have $\zeta_p \notin K_\alpha$ and $(1+p)^{1/p} \notin K_\alpha$.

Proof. Note that $\zeta_p \notin K_\alpha$ since $e(K_{\alpha \restriction n}/\mathbb{Q}_p) = p^n$ while $e(\mathbb{Q}_p(\zeta_p)/\mathbb{Q}_p) = p-1$. Suppose that $(1+p)^{1/p} \in K_{\alpha \restriction n}$, for some $n \in \mathbb{N}$. By Fact 3.6.8, we would have that $K_{\alpha \restriction 1} = \mathbb{Q}_p((1+p)^{1/p})$. If $\alpha(0) = 0$, this

would imply that $\mathbb{Q}_p(p^{1/p}) = \mathbb{Q}_p((1+p)^{1/p})$. If $\alpha(0) = 1$, this would imply that $\mathbb{Q}_p(((1+p) \cdot p)^{1/p}) = \mathbb{Q}_p((1+p)^{1/p})$. In either case, we would get that $\mathbb{Q}_p(p^{1/p}) = \mathbb{Q}_p((1+p)^{1/p})$. However, one sees that $(1+p)^{1/p} \notin \mathbb{Q}_p(p^{1/p})$. Indeed, suppose $a \in \mathbb{Q}_p(p^{1/p})$ is such that $a^p = 1 + p$. Write $a \equiv c_0 + c_1 \cdot p^{1/p} \mod p^{2/p}\mathbb{Z}_p[p^{1/p}]$, for some $c_0, c_1 \in \{0, \ldots, p-1\}$. We then compute

$$a^p \equiv c_0^p + c_1^p \cdot p + c_0^{p-1} \cdot c_1 \cdot p^{1+1/p} \not\equiv 1 + p \mod p^{1+2/p},$$

for any choice of c_0 and c_1. We conclude that $(1+p)^{1/p} \notin K_\alpha$. $\square$

Using Claim 1, we show:

Claim 2. If $\alpha \neq \beta$, then $\widehat{K_\alpha} \not\equiv_{\exists^1} \widehat{K_\beta}$ in L_{rings}.

Proof. Note that $K_\alpha \equiv_{\exists^1} \widehat{K_\alpha}$ (resp. $K_\beta \equiv_{\exists^1} \widehat{K_\beta}$) by Theorem 2.2.6. It suffices to show that $K_\alpha \not\equiv_{\exists^1} K_\beta$ for $\alpha \neq \beta$. Let $n \in \omega$ be least such that $\alpha(n) \neq \beta(n)$ and say $\alpha(n) = 1$. Now if $K_\alpha \equiv_{\exists^1} K_\beta$, there would be $a, b \in K_\alpha$ such that $a^{p^n} = (1+p)^{\bar{\alpha}_n} p$ and $b^{p^n} = (1+p)^{\bar{\beta}_n} p$. Set $c := a/b$ and note that $c^{p^n} = (1+p)^{\bar{\alpha}_n} p / ((1+p)^{\bar{\beta}_n} p) = (1+p)^{p^{n-1}}$. This implies that $(c^p/(1+p))^{p^{n-1}} = 1$. We conclude that either $\zeta_p \in K_\alpha$ or $c^p = 1 + p$, both of which are impossible by Claim 1. $\square$

Next we prove:

Claim 3. For every $\alpha \in 2^\omega$, we have that $\widehat{K_\alpha}^\flat = \widehat{\mathbb{F}_p((t))^{1/p^\infty}}$.

Proof. For every $n \in \mathbb{N}$, we have that $K_{\alpha\restriction(n+1)}/K_{\alpha\restriction n}$ is totally ramified and $\mathcal{O}_{K_{\alpha\restriction(n+1)}} = \mathcal{O}_{K_{\alpha\restriction n}}[\pi_{\alpha\restriction(n+1)}]$ [Serre 1979, Corollary page 19]. It follows that $\mathcal{O}_{K_\alpha} = \mathbb{Z}_p[\{\pi_{\alpha\restriction n} : n \in \mathbb{N}\}]$. We compute

$$\begin{aligned}
\mathcal{O}_{K_\alpha}/(p) &= \mathbb{Z}_p[x_1, x_2, \ldots]/(p, x_1^p - (1+p)^{\alpha(0)}p, x_2^p - (1+p)^{\alpha(1)}x_1, \ldots) \\
&\cong \mathbb{F}_p[x_1, x_2, \ldots]/(x_1^p, x_2^p - x_1, \ldots) \\
&\cong \mathbb{F}_p[x_1^{1/p^\infty}]/(x_1^p) \\
&\overset{t=x_1^p}{=} \mathbb{F}_p[t^{1/p^\infty}]/(t).
\end{aligned}$$

We thus get that $\varprojlim_\Phi \mathcal{O}_{K_\alpha}/(p) \cong \widehat{\mathbb{F}_p[\![t]\!]^{1/p^\infty}}$ and the conclusion follows. $\square$

Claims 2, 3 show that $|Z_F| \geq \mathfrak{c}$. On the other hand, the bound $|Z_F| \leq \mathfrak{c}$ is automatic, as any first-order L_{val}-theory can be identified with a subset of $\text{Sent}_{L_{\text{val}}} \simeq \mathbb{N}$. This finishes the proof in case $p > 2$. For $p = 2$, one first proves a variant of Claim 1, namely that $\zeta_4 \notin K_\alpha$ and $(\pm 3)^{1/2} \notin K_\alpha$. For the former, one can check inductively that $\zeta_4 \notin K_{\alpha\restriction n}$ using Kummer theory for quadratic extensions. The proof of Claim 2 then goes through. Finally, the proof of Claim 3 works verbatim for $p = 2$. $\square$

Remark 3.6.10. Set $(F, w) = (\widehat{\mathbb{F}_p((t))}, v_t)$. Fargues and Fontaine [2014, Remark 2.24] ask whether every untilt (K, v) of (F, w) is isomorphic to $(\mathbb{C}_p, v_p)$. They note in Remark 2.24 that for any untilt (K, v) of (F, w) one has $\mathcal{O}_K/(p^n) \cong \mathcal{O}_{\mathbb{C}_p}/(p^n)$ for each $n \in \mathbb{N}$. However, these isomorphisms are obtained in a noncanonical way and do not necessarily yield a valued field isomorphism between K and $\mathbb{C}_p$. In fact, an example of such a K with $K \not\cong \mathbb{C}_p$ was provided by Kedlaya and Temkin [2018, Theorem 1.3]. This should be contrasted with Proposition 3.6.9(a), saying that all untilts of (F, w) are elementary equivalent.

Remark 3.6.11. In [Anscombe and Fehm 2016, Remark 7.6], the authors write:

> *At present, we do not know of an example of a mixed characteristic henselian valued field (K, v)*
> *for which $\mathrm{Th}_\exists(k_v)$ and $\mathrm{Th}_\exists(\Gamma_v, vp)$ are decidable but $\mathrm{Th}_\exists(K, v)$ is undecidable.*

We note that such an example indeed exists. By Proposition 3.6.9(b) and the fact that there are countably many Turing machines, there must exist an undecidable perfectoid (thus henselian) field (K, v) with $K^\flat \cong \widehat{\mathbb{F}_p((t))^{1/p^\infty}}$. Moreover, the proof even provides us with a field K whose algebraic part is undecidable. On the other hand, by Lemma 3.2.4(b), we have that $k_v \cong \mathbb{F}_p$ and $(\Gamma_v, vp) \cong \left(\frac{1}{p^\infty}\mathbb{Z}, 1\right)$, both of which are decidable — the latter being an easy application of [Robinson and Zakon 1960]. Finally, we note that Dittmann [2022] has recently provided an example which is discretely valued and whose algebraic part is decidable.

Remark 3.6.12. For future use in Proposition 7.1.1, we record here that Claim 3 gives us that $t = (\pi_{\alpha\restriction 0} + (p), \pi_{\alpha\restriction 1} + (p), \dots)$, via the identifications $\widehat{\mathbb{F}_p[\![t]\!]^{1/p^\infty}} \cong \varprojlim_\Phi \mathcal{O}_{K_\alpha}/(p)$.

Question 3.6.13. Is there a perfectoid field (F, w) of characteristic p with $1 < |Z_F| < \mathfrak{c}$?

4. Relative decidability for perfectoid fields

4.1. *Introduction.* We shall now use the results of Section 3 to prove Theorem A and Corollary A.

4.1.1. *Work of Rideau, Scanlon and Simon.* It is my understanding that there is ongoing work by Rideau, Scanlon and Simon, which aims at giving a model-theoretic account of many of the concepts and facts that were discussed in Section 3, in the context of *continuous* logic. In particular, they obtain a biinterpretability result between $\mathcal{O}_K$ and $\mathcal{O}_{K^\flat}$ in the sense of continuous logic. It should be noted that their biinterpretability result was conceived prior to the present paper and in fact influenced the material that is presented here.

4.1.2. *Plan of action.* Without any adjustments, the interpretation of Rideau, Scanlon and Simon does not quite yield an interpretation in the sense of ordinary first-order logic. The problem is that when one converts statements about $\mathcal{O}_K$ to statements about $\mathcal{O}_{K^\flat}$ via $\mathcal{O}_K \cong W(\mathcal{O}_{K^\flat})/(\xi)$, one ends up with infinitely many variables, coming from the Witt vector coordinates. This problem disappears by interpreting one residue ring $\mathcal{O}_K/(p^n)$ at a time, via $\mathcal{O}_K/(p^n) \cong W_n(\mathcal{O}_{K^\flat})/(\xi \mod (p^n))$. This approach is facilitated by Corollary 2.3.1. In Section 2.3, we emphasized that *uniform* decidability of the residue rings is key. As we will see, this eventually comes down to the computability of the distinguished element ξ itself.

4.2. *Computable untilts.*

4.2.1. *Computable Witt vectors.* In analogy with a computable real number, a computable p-adic integer is one for which there is an algorithm which outputs the sequence of its p-adic digits. More precisely, a p-adic integer $a \in \mathbb{Z}_p$ of the form $a = \sum_{n=0}^\infty [\alpha_n] \cdot p^n$ is said to be *computable* precisely when the function $\mathbb{N} \to \mathbb{Z}/p\mathbb{Z} : n \mapsto \alpha_n$ is recursive. The notion of a computable p-adic integer extends naturally to the more general concept of a *computable Witt vector*.

First recall that a *computable* ring R_0 is one whose underlying set is (or is identified with) a *recursive* subset of $\mathbb{N}$, via which the operations $+ : R_0 \times R_0 \to R_0$ and $\cdot : R_0 \times R_0 \to R_0$ are identified with *recursive* functions.

Definition 4.2.2. Fix a perfect ring R and let $R_0 \subseteq R$ a computable subring. Consider the ring $W(R)$, the ring of Witt vectors over R. An element $\xi \in W(R)$ with Witt vector coordinates $(\xi_0, \xi_1, \dots) \in R^\omega$ is said to be R_0-computable if:

(1) For each $n \in \mathbb{N}$, we have that $\xi_n \in R_0$.

(2) The function $\mathbb{N} \to R_0 : n \mapsto \xi_n$ is recursive.

Example 4.2.3. (a) Let $R_0 = R = \mathbb{F}_p$. Then the computable elements of $\mathbb{Z}_p = W(\mathbb{F}_p)$ are the usual computable p-adic integers.

(b) A *nonexample*: Let R be any perfect ring, $R_0 \subseteq R$ any computable subring and $S \subseteq \mathbb{N}$ be a nonrecursive set. The element $\xi = \sum_{n \in S} p^n \in W(R)$ is not computable.

4.2.4. *Computable untilts.* Let (F, w) be a perfectoid field of characteristic p and $A_{\inf} = W(\mathcal{O}_F)$. Let also $R_0 \subseteq \mathcal{O}_F$ be a computable subring.

Remark 4.2.5. The reader is welcome to assume that $F = \widehat{\mathbb{F}_p((t))^{1/p^\infty}}$ or $\widehat{\bar{\mathbb{F}}_p((t))^{1/p^\infty}}$ and $R_0 = \mathbb{F}_p[t]$. This case is enough for the applications presented here, i.e., Corollaries A and B.

We shall now define what it means for an untilt K of F to be R_0-computable:

Definition 4.2.6. Let (K, ι) be an untilt of (F, w) and $R_0 \subseteq \mathcal{O}_F$ a computable subring. We say that (K, v) is an R_0-*computable* untilt of (F, w) if there is an R_0-computable distinguished element $\xi_K \in A_{\inf}$ with $\mathcal{O}_K \cong A_{\inf}/(\xi_K)$.

Example 4.2.7. In Corollary 4.4.6 we will see that $\widehat{\mathbb{Q}_p(p^{1/p^\infty})}$ and $\widehat{\mathbb{Q}_p(\zeta_{p^\infty})}$ are $\mathbb{F}_p[t]$-computable untilts of $\widehat{\mathbb{F}_p((t))^{1/p^\infty}}$. Also, that $\widehat{\mathbb{Q}_p^{ab}}$ is an $\mathbb{F}_p[t]$-computable untilt of $\widehat{\bar{\mathbb{F}}_p((t))^{1/p^\infty}}$.

4.3. *Relative decidability.* Let (K, v) be an untilt of (F, w) and $\xi_K \in A_{\inf}$ be such that $\mathcal{O}_K = A_{\inf}/(\xi_K)$ (Theorem 3.5.11). We write $\bar{\xi}_{K,n}$ for the image of ξ_K in $W_n(\mathcal{O}_F)$ via $A_{\inf} \twoheadrightarrow A_{\inf}/(p^n) \cong W_n(\mathcal{O}_F)$.

4.3.1. *Interpretability.* All the necessary background material related to interpretability is presented in Section 1.2.

Lemma 4.3.2. *For each $n \in \mathbb{N}$, there is a $\exists^+$-interpretation A_n of the local ring $(\mathcal{O}_K/(p^n), \mathfrak{m}_n)$ in the $\langle L_{\mathrm{rings}}, L_{\mathrm{lcr}} \rangle$-structure $((W_n(\mathcal{O}_F), \mathcal{O}_F), \bar{\xi}_{K,n})$ such that the sequence of interpretations $(A_n)_{n \in \omega}$ is uniformly recursive.*

Proof. Fix $n \in \mathbb{N}$. We have that $\mathcal{O}_K/(p^n) \cong A_{\inf}/(p^n, \xi_K) \cong W_n(\mathcal{O}_F)/(\bar{\xi}_{K,n})$. By Proposition 3.5.9, we get that $\mathfrak{m}_n = (\{\theta_n([x]) : x \in \mathfrak{m}_F\})$, where θ_n is the composite map

$$A_{\inf} \xrightarrow{\theta} A_{\inf}/(\xi_K) \xrightarrow{\bmod p^n} W_n(\mathcal{O}_F)/(\bar{\xi}_{K,n}) \xrightarrow{\cong} \mathcal{O}_K/(p^n).$$

We take $\partial_{A_n}(x)$ to be $x \in W$. Let c be a constant symbol with $c^{(W_n(\mathcal{O}_F), \mathcal{O}_F)} = \bar{\xi}_{K,n}$. The reduction map $\mathrm{Form}_{L_{\mathrm{lcr}}} \to \langle L_{\mathrm{rings}}, L_{\mathrm{lcr}} \rangle \cup \{c\} : \phi \mapsto \phi_{A_n}$ on unnested atomic formulas is described as follows:

(1) If $\phi(x, y)$ is $x = y$, then we take ϕ_{A_n} to be the formula $x, y \in W \wedge \exists z \in W (x = y + z \cdot c)$. As usual, the conjunct $x, y \in W$ is an informal way of saying that the variables x, y are taken from the sort W. The cases of $x = 0$ and $x = 1$ are similar.

(2) If $\phi(x, y, z)$ is $x \diamond y = z$, then we take ϕ_{A_n} to be the formula $x, y, z \in W \wedge \exists w \in W (x \diamond y = z + w \cdot c)$, where $\diamond$ is either $\cdot$ or $+$.

(3) If $\phi(x)$ is $x \in \mathfrak{m}$, then we take $\phi_{A_n}(x)$ to be the formula $x \in W \wedge \exists z, w \in W, y \in R$ ($y \in \mathfrak{m} \wedge x = [y] \cdot z + w \cdot c$).

The coordinate map $f_{A_n} : W_n(\mathcal{O}_F) \twoheadrightarrow \mathcal{O}_K/(p^n)$ is the one induced by the map θ_n above. The above data define a recursive $\exists^+$-interpretation A_n of $(\mathcal{O}_K/(p^n), \mathfrak{m}_n)$ in $((W_n(\mathcal{O}_F), \mathcal{O}_F), \bar{\xi}_{K,n})$. The sequence of interpretations $(A_n)_{n \in \omega}$ is uniform in n and thus also (trivially) uniformly recursive. $\square$

Proposition 4.3.3. *Suppose (K, v) is an untilt of (F, w) and that $\xi_K \in A_{\inf}$ is such that $\mathcal{O}_K = A_{\inf}/(\xi_K)$. Let $(\xi_0, \xi_1, \ldots) \in \mathcal{O}_F^{\omega}$ be the Witt vector coordinates of ξ_K. Then:*

(a) *The value group (Γ_v, vp) is recursively interpretable in the valued field $((F, w), \xi_0)$.*

(b) *For each $n \in \mathbb{N}$, there exists a $\exists^+$-interpretation B_n of the local ring $(\mathcal{O}_K/(p^n), \mathfrak{m}_n)$ in the local ring $((\mathcal{O}_F, \mathfrak{m}_F), \xi_0, \ldots, \xi_{n-1})$ such that the sequence of interpretations $(B_n)_{n \in \omega}$ is uniformly recursive.*

Proof. (a) By Lemma 3.2.4(b), we will have that $(\Gamma_v, vp) \cong (\Gamma_w, w\xi_0)$ and the value group $(\Gamma_w, w\xi_0)$ is clearly recursively interpretable in the $L_{\mathrm{val}} \cup \{c\}$-structure (F, w) with $c^{(F,w)} = \xi_0$.

(b) Lemma 4.3.2 provides us with a $\exists^+$-interpretation A_n of the local ring $(\mathcal{O}_K/(p^n), \mathfrak{m}_n)$ in the $\langle L_{\mathrm{rings}}, L_{\mathrm{lcr}} \rangle \cup \{c\}$-structure $((W_n(\mathcal{O}_F), \mathcal{O}_F), \bar{\xi}_{K,n})$ such that the sequence of interpretations $(A_n)_{n \in \omega}$ is uniformly recursive. Lemma 3.4.5 provides us with a quantifier-free interpretation Γ_n of $((W_n(\mathcal{O}_F), \mathcal{O}_F), \bar{\xi}_{K,n})$ in the local ring $((\mathcal{O}_F, \mathfrak{m}_F), \xi_0, \ldots, \xi_{n-1})$, such that the sequence of interpretations $(\Gamma_n)_{n \in \omega}$ is uniformly recursive. Let B_n be the composite interpretation of $(\mathcal{O}_K/(p^n), \mathfrak{m}_n)$ in $((\mathcal{O}_F, \mathfrak{m}_F), \xi_0, \ldots, \xi_{n-1})$. This is also a $\exists^+$-interpretation by Lemma 1.2.8 and the sequence of interpretations $(B_n)_{n \in \omega}$ is uniformly recursive by Lemma 1.2.15. $\square$

4.3.4. *Proof of Theorem A.* Given a computable subring $R_0 \subseteq \mathcal{O}_F$, we write $L_{\mathrm{val}}(R_0)$ for the language L_{val} enriched with a constant c_a for each $a \in R_0$ (see notation). The valued field (F, w) can then be updated to an $L_{\mathrm{val}}(R_0)$-structure with $c_a^{(F,w)} = a$.

Theorem A. *Let (F, w) be a perfectoid field of characteristic p. Suppose $R_0 \subseteq \mathcal{O}_F$ is a computable subring and (K, v) is an R_0-computable untilt of (F, w):*

(a) *If (F, w) is decidable in $L_{\mathrm{val}}(R_0)$, then (K, v) is decidable in L_{val}.*

(b) *If (F, w) is $\exists$-decidable in $L_{\mathrm{val}}(R_0)$, then (K, v) is $\exists$-decidable in L_{val}.*

Proof. (a) By Proposition 4.3.3(a), we get that (Γ_v, vp) is decidable in L_{oag} with a constant for vp. By Proposition 4.3.3(b), for each $n \in \mathbb{N}$, there exists an interpretation B_n of the L_{lcr}-structure $(\mathcal{O}_K/(p^n), \mathfrak{m}_n)$ in the $L_{lcr} \cup \{c_0, \ldots, c_{n-1}\}$-structure $((\mathcal{O}_F, \mathfrak{m}_F), \xi_0, \ldots, \xi_{n-1})$ with $c_m^{(\mathcal{O}_F, \mathfrak{m}_F)} = \xi_m$ for $m = 0, \ldots, n-1$. Moreover, the sequence of interpretations $(B_n)_{n \in \omega}$ is uniformly recursive.

Claim. There exists an interpretation Δ_n of the $L_{lcr} \cup \{c_0, \ldots, c_{n-1}\}$-structure $((\mathcal{O}_F, \mathfrak{m}_F), \xi_0, \ldots, \xi_{n-1})$ in the $L_{val}(R_0)$-structure (F, w) such that the sequence of interpretations $(\Delta_n)_{n \in \omega}$ is uniformly recursive.

Proof. Fix $n \in \mathbb{N}$. We take $\partial_{\Delta_n}(x)$ to be the formula $x \in \mathcal{O}$. The reduction map on unnested atomic formulas is described as follows:

(1) If $\phi(x)$ is the formula $x = c_m$, for some $m = 0, \ldots, n-1$, we take $\phi_{\Delta_n}(x)$ to be the formula $x = c_{\xi_m}$. The formulas $x = y$, $x = 0$ and $x = 1$ are interpreted in the obvious way.

(2) If $\phi(x)$ is the formula $x \in \mathfrak{m}$, we take $\phi_{\Delta_n}(x)$ to be the formula $\exists y(xy = 1 \wedge y \notin \mathcal{O})$.

(3) If $\phi(x, y, z)$ is $x \diamond y = z$, then we take ϕ_{Δ_n} to be the formula $x, y, z \in \mathcal{O} \wedge \phi(x, y, z)$, where $\diamond$ is either $\cdot$ or $+$.

The coordinate map $f_{\Delta_n} : \partial_{\Delta_n}(F) \to \mathcal{O}_F$ is the identity. Since $\mathbb{N} \to R_0 : m \mapsto \xi_m$ is recursive, the reduction map $\mathbb{N} \times \text{Form}_{L_{lcr} \cup \{c_0, \ldots, c_{n-1}\}} \to \text{Form}_{L_{val}(R_0)} : (n, \phi) \mapsto \phi_{\Delta_n}$ restricted to unnested atomic formulas is recursive. By definition, this means that the sequence of interpretations $(\Delta_n)_{n \in \omega}$ is uniformly recursive. $\square$

For each $n \in \mathbb{N}$, let E_n be the composite interpretation of B_n and Δ_n. The sequence $(E_n)_{n \in \omega}$ is uniformly recursive by Lemma 1.2.15. By Proposition 1.2.14, we get that the sequence of rings $(\mathcal{O}_K/(p^n))_{n \in \omega}$ is *uniformly* decidable in L_{rings}. The conclusion follows from Corollary 2.3.1.

(b) For the existential version, one needs to keep track of the complexity of formulas. Since B_n is a $\exists^+$-interpretation and Δ_n is an $\exists$-interpretation, we see that the reduction map $L_{lcr} \to L_{val}(R_0) : \phi \mapsto \phi_{E_n}$ sends $\exists^+$-formulas to $\exists$-formulas (see Remark 1.2.7). It follows that the sequence of local rings $((\mathcal{O}_K/(p^n), \mathfrak{m}_n)_{n \in \omega}$ is uniformly $\exists^+$-decidable in L_{lcr}. We conclude by Corollary 2.3.2. $\square$

The assumption of completeness in Theorem A can be easily relaxed to henselianity:

Corollary 4.3.5. *Let (F, w) be a perfect nontrivially valued field with a rank 1 value group and $R_0 \subseteq \mathcal{O}_F$ be a computable subring. Suppose (K, v) is a mixed characteristic henselian valued field such that $(\widehat{K}, \widehat{v})$ is an R_0-computable untilt of $(\widehat{F}, \widehat{w})$. Then*:

(a) *If (F, w) is decidable in $L_{val}(R_0)$, then (K, v) is decidable in L_{val}.*

(b) *If (F, w) is $\exists$-decidable in $L_{val}(R_0)$, then (K, v) is $\exists$-decidable in L_{val}.*

Proof. Suppose $\widehat{\mathcal{O}}_K = W(\widehat{\mathcal{O}}_F)/(\xi)$, where $\xi = [\pi] - up$, with $\pi \in \mathfrak{m}_F \cap R_0$ and $u \in W(\mathcal{O}_F)^\times$. Note that $W(\widehat{\mathcal{O}}_F)/([\pi] - up) \cong W(\mathcal{O}_F)/([\pi] - up)$; indeed, $W(\mathcal{O}_F)$ is p-adically complete and $[\pi]$ and p are associates in the quotient $W(\mathcal{O}_F)/([\pi] - up)$. For each $n \in \mathbb{N}$, one has that

$$\mathcal{O}_K/(p^n) \cong \widehat{\mathcal{O}}_K/(p^n) \cong W_n(\widehat{\mathcal{O}}_F)/([\pi] - up) \cong W_n(\mathcal{O}_F)/([\pi] - up)$$

We now proceed as in the proof of Theorem A. $\square$

4.3.6. *Remarks on Theorem A.* Note that decidability of K in L_{val} relative to its tilt $K^\flat$ in L_{val}, i.e., without taking parameters into account, is false. Essentially, the point is that p is named in L_{val}, while t is not.

Example 4.3.7. Let $(\Gamma, +, <) \subseteq (\mathbb{R}, +, <)$ be any p-divisible ordered abelian group, which is decidable in L_{oag} but undecidable in $L_{\text{oag}} \cup \{1\}$, where 1 a distinguished element of Γ. For example, let $S \subsetneq \mathbb{P}$ be a nonrecursive set of primes and $S' := \mathbb{P} - S$ be its complement. Denote by $\frac{1}{S}\mathbb{Z}$ the group generated by $\{\frac{1}{s} : s \in S\}$ and consider the ordered abelian group

$$\Gamma = \frac{1}{p^\infty}\left(\frac{1}{S}\mathbb{Z} \oplus \frac{1}{S'}\mathbb{Z}\sqrt{2}\right)$$

equipped with the order induced from the natural embedding $\Gamma \hookrightarrow \mathbb{R}$. This is a regularly dense group, in the sense of Robinson and Zakon [1960], with prime invariants $[\Gamma : p\Gamma] = 1$ and $[\Gamma : q\Gamma] = q$ for $q \neq p$. By [loc. cit.], this group is decidable in L_{oag} but is clearly undecidable in $L_{\text{oag}} \cup \{1\}$. Since Γ is p-divisible, we may form the tame valued field $(F, w) = (\mathbb{F}_p((t^\Gamma)), v_t)$, which is decidable in L_{val} by [Kuhlmann 2016, Theorem 1.4]. However, the untilt (K, v), whose associated Witt vector is $[t] - p$, is undecidable in L_{val}. Indeed, even (Γ_v, vp) is undecidable in L_{oag} with a constant for vp.

Remark 4.3.8. Theorem A holds also for untilts K of F, which have an associated distinguished element ξ_K satisfying a more relaxed condition than the one of Definition 4.2.2. Namely, suppose that $\xi_K = (\xi_0, \xi_1, \dots) \in \mathcal{O}_F^\omega$ and each ξ_n is definable in the valued field (F, w) by a formula $\phi_n(x) \in L_{\text{val}}(R_0)$ with parameters from R_0. Suppose furthermore that the function $\mathbb{N} \to \text{Form}_{L_{\text{val}}(R_0)} : n \mapsto \phi_n(x)$ is recursive. Then the conclusion of Theorem A is still valid, i.e., (K, v) is decidable in L_{val} relative to (F, w) in $L_{\text{val}}(R_0)$.

It is clear that when each $\phi_n(x)$ is a quantifier-free formula, the configuration described in Remark 4.3.8 specializes to the notion of a computable untilt. We have no particular application in mind that requires the level of generality described in Remark 4.3.8, which is one reason we have restricted ourselves to the quantifier-free case (the other being clarity of exposition).

4.4. *Corollary A.* In order to prove Corollary A, we need to calculate the tilts of our fields of interest and compute the associated distinguished elements.

4.4.1. *Computing the distinguished elements.* All computations below are well-known to experts; see, e.g., [Fargues and Fontaine 2014, Example 2.22]. For lack of a detailed reference, we shall spell out the details.

Lemma 4.4.2. *There exist ring isomorphisms*:

(a) $\mathbb{Z}_p[p^{1/p^\infty}]/(p) \cong \mathbb{F}_p[t^{1/p^\infty}]/(t)$ *mapping* $p^{1/p^n} + (p) \mapsto t^{1/p^n} + (t)$.

(b) $\mathbb{Z}_p[\zeta_{p^\infty}]/(p) \cong \mathbb{F}_p[t^{1/p^\infty}]/(t^{p-1})$ *mapping* $\zeta_p + (p) \mapsto t + 1 + (t^{p-1})$.

(c) $\mathbb{Z}_p^{ab}/(p) \cong \bar{\mathbb{F}}_p[t^{1/p^\infty}]/(t^{p-1})$ *mapping* $\zeta_p + (p) \mapsto t + 1 + (t^{p-1})$.

Proof. (a) For each $n \in \mathbb{N}$, the irreducible polynomial of $p^{1/p^{n+1}}$ over $\mathbb{Q}_p(p^{1/p^n})$ is the Eisenstein polynomial $x^p - p^{1/p^n} \in \mathbb{Z}_p[p^{1/p^n}][x]$. We therefore compute

$$\mathbb{Z}_p[p^{1/p^\infty}]/(p) \cong \mathbb{Z}_p[x_1, x_2, \dots]/(p, x_1^p - p, x_2^p - x_1, \dots)$$
$$\cong \mathbb{F}_p[x_1, x_2, \dots]/(x_1^p, x_2^p - x_1, \dots)$$
$$\overset{x_n \mapsto t^{1/p^n}}{\cong} \mathbb{F}_p[t^{1/p^\infty}]/(t)$$

(b) The irreducible polynomial of ζ_p over $\mathbb{Q}_p$ is the cyclotomic polynomial $\Phi_p(x) = x^{p-1} + \cdots + 1$. Moreover, given $n > 1$, the irreducible polynomial of ζ_{p^n} over $\mathbb{Q}_p(\zeta_{p^{n-1}})$ is $x^p - \zeta_{p^{n-1}} \in \mathbb{Z}[\zeta_{p^n}][x]$. We now proceed as in (a) and compute

$$\mathbb{Z}_p[\zeta_{p^\infty}]/(p) \cong \mathbb{Z}_p[x_1, x_2, \dots]/(p, \Phi_p(x_1), x_2^p - x_1, \dots) \overset{x_{n+1} \mapsto x^{1/p^n}}{\cong} \mathbb{F}_p[x^{1/p^\infty}]/(\overline{\Phi}_p(x)).$$

Note that $\overline{\Phi}_p(x) = \frac{x^p - 1}{x - 1} = \frac{(x-1)^p}{x-1} = (x-1)^{p-1}$ and thus

$$\mathbb{Z}_p[\zeta_{p^\infty}]/(p) \cong \mathbb{F}_p[x^{1/p^\infty}]/(x-1)^{p-1} \overset{t = x - 1}{=} \mathbb{F}_p[t^{1/p^\infty}]/(t^{p-1}).$$

(c) By local Kronecker–Weber (see Theorem 14.2 in [Washington 1997] and Proposition 17 in [Serre 1979]), we get that $\mathbb{Z}_p^{ab} = \mathbb{Z}_p^{ur}[\zeta_{p^\infty}]$. We now proceed as in (b). □

Corollary 4.4.3. (a) $\widehat{\mathbb{Q}_p(p^{1/p^\infty})}^\flat \cong \mathbb{F}_p\widehat{((t))^{1/p^\infty}}$ *and* $t^\sharp = p$.

(b) $\widehat{\mathbb{Q}_p(\zeta_{p^\infty})}^\flat \cong \mathbb{F}_p\widehat{((t))^{1/p^\infty}}$ *and* $(t+1)^\sharp = \zeta_p$.

(c) $\widehat{\mathbb{Q}_p^{ab}}^\flat \cong \overline{\mathbb{F}}_p\widehat{((t))^{1/p^\infty}}$ *and* $(t+1)^\sharp = \zeta_p$.

Proof. (a) By Lemma 4.4.2, we have that $\mathbb{Z}_p[p^{1/p^\infty}]/(p) \cong \mathbb{F}_p[t^{1/p^\infty}]/(t)$ via an isomorphism mapping $p^{1/p^n} + (p) \mapsto t^{1/p^n} + (t)$. One can verify directly that $\mathbb{F}_p\widehat{[\![t]\!]^{1/p^\infty}} \to \varprojlim_\Phi \mathbb{F}_p[t^{1/p^\infty}]/(t)$, where $x \mapsto (x \bmod (t), x^{1/p} \bmod (t), \dots)$, is a ring isomorphism. It follows that $\widehat{\mathbb{Q}_p(p^{1/p^\infty})}^\flat \cong \mathbb{F}_p\widehat{((t))^{1/p^\infty}}$ and by definition $t^\sharp = \lim_{n \to \infty}(p^{1/p^n})^{p^n} = p$.

   The proofs of (b) and (c) are similar. □

Proposition 4.4.4 [Fargues and Fontaine 2014, Example 2.22]. *We have the following isomorphisms*:

(a) $\widehat{\mathbb{Z}_p[p^{1/p^\infty}]} \cong W(\mathbb{F}_p\widehat{[\![t]\!]^{1/p^\infty}})/([t] - p)$.

(b) $\widehat{\mathbb{Z}_p[\zeta_{p^\infty}]} \cong W(\mathbb{F}_p\widehat{[\![t]\!]^{1/p^\infty}})/([t+1]^{p-1} + [t+1]^{p-2} + \cdots + 1)$.

(c) $\widehat{\mathbb{Z}_p^{ab}} \cong W(\overline{\mathbb{F}}_p\widehat{[\![t]\!]^{1/p^\infty}})/([t+1]^{p-1} + [t+1]^{p-2} + \cdots + 1)$.

Proof. (a) By Corollary 4.4.3(a), we have that $\widehat{\mathbb{Q}_p(p^{1/p^\infty})}^\flat \cong \mathbb{F}_p\widehat{((t))^{1/p^\infty}}$. Consider the ring homomorphism $\theta : A_{\inf} \twoheadrightarrow \widehat{\mathbb{Z}_p[p^{1/p^\infty}]}$ inducing $\sharp : \mathbb{F}_p\widehat{[\![t]\!]^{1/p^\infty}} \to \widehat{\mathbb{Z}_p[p^{1/p^\infty}]}$. Since $t^\sharp = p$, we get that $\theta([t]) = p$ and therefore $[t] - p \in \mathrm{Ker}(\theta)$. By Proposition 3.5.7, it follows that $\mathrm{Ker}(\theta) = ([t] - p)$.

(b) By Corollary 4.4.3(b), we have that $\widehat{\mathbb{Q}_p(\zeta_{p^\infty})}^{\flat} \cong \widehat{\mathbb{F}_p((t))^{1/p^\infty}}$. mapping $1 + t \mapsto \zeta_p$. Consider the ring homomorphism $\theta : A_{\inf} \twoheadrightarrow \widehat{\mathbb{Z}_p[\zeta_{p^\infty}]}$ inducing $\sharp : \widehat{\mathbb{F}_p[\![t]\!]}^{1/p^\infty} \to \widehat{\mathbb{Z}_p[\zeta_{p^\infty}]}$. Since $(t+1)^\sharp = \zeta_p$, we get that $\theta([t+1]) = \zeta_p$ and therefore $[t+1]^{p-1} + \cdots + [t+1] + 1 \in \mathrm{Ker}(\theta)$. By Proposition 3.5.7, it suffices to show that $\xi = [t+1]^{p-1} + \cdots + [t+1] + 1$ is a distinguished element of $A_{\inf}$. Set $\mathrm{res} : \mathcal{O}_F \twoheadrightarrow \mathcal{O}_F/\mathfrak{m}_F$ for the residue map and $W(\mathrm{res})$ for the unique induced homomorphism $W(\mathrm{res}) : W(R) \to W(R/\mathfrak{m}_R)$ of Fact 3.3.5. We compute that $W(\mathrm{res})(\xi) = 1 + 1 + \cdots + 1 = p$, whence ξ is a distinguished element of $A_{\inf}$ (Remark 3.5.4).

(c) Similar to (b). $\qquad\square$

4.4.5. *Proof of Corollary A.*

Corollary 4.4.6. *We have the following*:

(a) $\widehat{\mathbb{Q}_p(p^{1/p^\infty})}$ and $\widehat{\mathbb{Q}_p(\zeta_{p^\infty})}$ *are* $\mathbb{F}_p[t]$*-computable untilts of* $\widehat{\mathbb{F}_p((t))^{1/p^\infty}}$.

(b) $\widehat{\mathbb{Q}_p^{ab}}$ *is an* $\mathbb{F}_p[t]$*-computable untilt of* $\widehat{\mathbb{F}_p((t))^{1/p^\infty}}$.

Proof. The case of $\widehat{\mathbb{Q}_p(p^{1/p^\infty})}$ is clear. For the other two cases, note that $\xi := [t+1]^{p-1} + [t+1]^{p-2} + \cdots + 1$ is $\mathbb{F}_p[t]$-computable, using the *computable* polynomials $S_0, \ldots, S_n$ for Witt vector addition (Observation 3.3.9). $\qquad\square$

Corollary A. (a) *Assume* $\mathbb{F}_p((t))^{1/p^\infty}$ *is (existentially) decidable in* $L_{\mathrm{val}}(t)$. *Then* $\mathbb{Q}_p(p^{1/p^\infty})$ *and* $\mathbb{Q}_p(\zeta_{p^\infty})$ *are (existentially) decidable in* L_{val}.

(b) *Assume* $\overline{\mathbb{F}}_p((t))^{1/p^\infty}$ *is (existentially) decidable in* $L_{\mathrm{val}}(t)$. *Then* $\mathbb{Q}_p^{ab}$ *is (existentially) decidable in* L_{val}.

Proof. (a) By Corollary 4.4.6 and Corollary 4.3.5.

(b) Similar to (a). $\qquad\square$

5. Applications: Tame fields of mixed characteristic

5.1. *Introduction.*

5.1.1. *Motivation.* We shall prove Corollary B, which to my knowledge is the first decidability result for tame fields of mixed characteristic.

5.1.2. *Preliminaries.* The algebra and model theory of tame fields was introduced and studied by Kuhlmann [2016]. Recall the definition:

Definition 5.1.3. Let (K, v) be a henselian valued field. A finite valued field extension $(K', v')/(K, v)$ is said to be tame if

(1) $([\Gamma' : \Gamma], p) = 1$, where p is the *characteristic exponent* of k, i.e., $p = \mathrm{char}(k)$ if this is positive and $p = 1$ if $\mathrm{char}(k) = 0$.

(2) The residue field extension k'/k is separable.

(3) The extension $(K', v')/(K, v)$ is defectless, meaning that the fundamental equality $[K' : K] = [\Gamma' : \Gamma] \cdot [k' : k]$ holds.

An algebraic valued field extension is said to be tame if every finite subextension is tame.

Definition 5.1.4. A henselian valued field (K, v) is said to be tame if every finite valued field extension $(K', v')/(K, v)$ is tame.

In practice, one often needs a more intrinsic description of tame fields. This is provided by the following:

Proposition 5.1.5 [Kuhlmann 2016, Theorem 3.2]. *Let (K, v) be henselian valued field. Then the following are equivalent*:

(1) (K, v) *is a tame field.*

(2) (K, v) *is algebraically maximal, Γ is p-divisible and k is perfect.*

Example 5.1.6. Using Proposition 5.1.5, one can verify the following:

(a) Any (algebraically) maximal immediate extension of $\mathbb{Q}_p(p^{1/p^\infty})$ or $\mathbb{Q}_p(\zeta_{p^\infty})$ is tame.

(b) Any (algebraically) maximal immediate extension of $\mathbb{F}_p((t))^{1/p^\infty}$ is tame. In particular, the Hahn field $(\mathbb{F}_p((t^\Gamma)), v_t)$ with value group $\Gamma = \frac{1}{p^\infty}\mathbb{Z}$ and residue field $\mathbb{F}_p$ is tame.

5.1.7. *Equal characteristic.* Kuhlmann obtained the following Ax–Kochen/Ershov principle for tame fields of equal characteristic:

Theorem 5.1.8 [Kuhlmann 2016, Theorem 1.4]. *Let (K, v) and (K', v') be two equal characteristic tame fields. Then $(K, v) \equiv (K', v')$ in L_{val} if and only if $k \equiv k'$ in L_{rings} and $\Gamma \equiv \Gamma'$ in L_{oag}.*

Kuhlmann then deduces the following decidability result (among others):

Corollary 5.1.9 [Kuhlmann 2016, Theorem 1.6]. *Set $\Gamma = \frac{1}{p^\infty}\mathbb{Z}$, i.e., for the p-divisible hull of $\mathbb{Z}$. The Hahn field $(\mathbb{F}_p((t^\Gamma)), v_t)$ is decidable in L_{val}.*

In recent work, Lisinski combined results of Kuhlmann [2016] with work of Kedlaya [2006] and obtained the following strengthening of Corollary 5.1.9:

Theorem 5.1.10 [Lisinski 2021, Theorem 1]. *Set $\Gamma = \frac{1}{p^\infty}\mathbb{Z}$. The Hahn field $(\mathbb{F}_p((t^\Gamma)), v_t)$ is decidable in L_t.*

5.1.11. *Mixed characteristic.* Kuhlmann's Theorem 5.1.8 fails as such for mixed characteristic tame fields. A counterexample is given in [Anscombe and Kuhlmann 2016, Theorem 1.5(c)]. The lack of such a principle has been a fundamental obstacle in obtaining decidability results for such fields.

5.2. *Mixed characteristic tame fields.* While we do not know whether $\mathbb{Q}_p(p^{1/p^\infty})$ and $\mathbb{Q}_p(\zeta_{p^\infty})$ are (existentially) decidable, we will show that they admit decidable maximal immediate extensions. These are tame fields by Example 5.1.6(a).

Lemma 5.2.1 [Fargues and Fontaine 2014, Remark 2.23]. *Let (K, v) be a perfectoid field. Then (K, v) is maximal if and only if $(K^\flat, v^\flat)$ is maximal.*

Proof. Immediate from the tilting equivalence Theorem 3.5.14 and the fact that tilting preserves value groups and residue fields; see Lemma 3.2.4(b). □

Corollary B. *The valued field $(\mathbb{Q}_p(p^{1/p^\infty}), v_p)$ (resp. $(\mathbb{Q}_p(\zeta_{p^\infty}), v_p)$) admits a maximal immediate extension which is decidable in L_{val}.*

Proof. Recall from Corollaries 4.4.3(a) and 4.4.4(a) that

$$\widehat{\mathbb{Q}_p(p^{1/p^\infty})}^{\,\flat} \cong \widehat{\mathbb{F}_p((t))^{1/p^\infty}}$$

and that $\xi = [t] - p$. Let $\mathbb{F}_p((t^\Gamma))$ be the Hahn field with value group $\Gamma = \frac{1}{p^\infty}\mathbb{Z}$ and residue field $\mathbb{F}_p$. It is a (nonunique) maximal immediate extension of $\mathbb{F}_p((t))^{1/p^\infty}$. By Theorem 3.5.14, we may form $K = \mathbb{F}_p((t^\Gamma))^\sharp$ as in the diagram below:

$$
\begin{array}{ccc}
K & \xleftarrow{\quad\sharp\quad} & \mathbb{F}_p((t^\Gamma)) \\
\big| & & \big| \\
\widehat{\mathbb{Q}_p(p^{1/p^\infty})} & \xrightarrow{\quad\flat\quad} & \widehat{\mathbb{F}_p((t))^{1/p^\infty}}
\end{array}
$$

such that $\xi_K = [t] - p$ (see Section 3.5.13). By Lemma 5.2.1, we see that (K, v) is a maximal immediate extension of $(\mathbb{Q}_p(p^{1/p^\infty}), v_p)$. By Theorem 5.1.10 and Theorem A, we see that (K, v) is decidable in L_{val}. The proof for $\mathbb{Q}_p(\zeta_{p^\infty})$ is similar. □

Remark 5.2.2. It is also true that $\mathbb{Q}_p^{ab}$ admits a decidable maximal immediate extension (in this case unique) but this already follows from well-known results in the model theory of algebraically maximal Kaplansky fields; see [Kuhlmann 2016, page 4, Part (f)].

In spite of Corollary B, the following is worth noting:

Remark 5.2.3. A tree-like construction similar to Proposition 3.6.9(b) shows that there exist uncountably many pairwise elementary inequivalent maximal immediate extensions of $\mathbb{Q}_p(p^{1/p^\infty})$ (resp. $\mathbb{Q}_p(\zeta_{p^\infty})$). In particular, the valued field $\mathbb{Q}_p(p^{1/p^\infty})$ (resp. $\mathbb{Q}_p(\zeta_{p^\infty})$) has uncountably many *undecidable* maximal immediate extensions. Note that the tilts of all such fields will be maximal immediate extensions of $\mathbb{F}_p((t))^{1/p^\infty}$ and thus will be tame fields. As a consequence of Kuhlmann's Theorem 5.1.8, all of them will be decidable in L_{val}. Nevertheless, they will all be undecidable in $L_{\mathrm{val}}(t)$.

6. Applications: Congruences modulo *p*

6.1. *Introduction.*

6.1.1. *Goal.* We shall now prove Theorem B, which shows the existence of an algorithm that decides whether a system of polynomial equations and inequalities, defined over $\mathbb{Z}$, has a solution modulo p over the valuation rings of our fields of interest.

6.1.2. *Strategy.* The proof is via a local field approximation argument, using the computations of Lemma 4.4.2. One eventually encodes the above problem in the existential theory of the tilt in L_{val}, where the Anscombe–Fehm Theorem (Theorem 2.2.3) applies. This should be contrasted with Corollary A, which requires decidability in the language $L_{\mathrm{val}}(t)$ on the characteristic p side.

6.2. *From residue rings to valuation rings.* The crux of the argument lies in the following:

Proposition 6.2.1. *Let* $f_i(x), g_j(x) \in \mathbb{F}_p[x]$ *be multivariable polynomials in* $x = (x_1, \ldots, x_m)$ *for* $i, j = 1, \ldots, n$. *Then*

$$\mathbb{F}_p[t^{1/p^\infty}]/(t) \models \exists x \bigwedge_{1 \le i, j \le n} (f_i(x) = 0 \wedge g_j(x) \ne 0) \iff \mathbb{F}_p[\![t]\!]^{1/p^\infty} \models \exists x \bigwedge_{1 \le i, j \le n} (v(f_i(x)) > v(g_j(x))).$$

Proof. First observe that

$$\mathbb{F}_p[\![t]\!]^{1/p^\infty}/(t) \cong \varinjlim \mathbb{F}_p[\![t^{1/p^n}]\!]/(t) \cong \varinjlim \mathbb{F}_p[t^{1/p^n}]/(t) \cong \mathbb{F}_p[t^{1/p^\infty}]/(t). \tag{†}$$

$\Rightarrow$: Let $a \in (\mathbb{F}_p[t^{1/p^\infty}]/(t))^m$ be such that $f_i(a) = 0 \wedge g_j(a) \ne 0$, for $1 \le i, j \le n$ and let $\tilde{a}$ be any lift of a in $\mathbb{F}_p[\![t]\!]^{1/p^\infty}$ via the isomorphism (†). We see that $v(f_i(\tilde{a})) \ge 1 > v(g_j(\tilde{a}))$, for all $1 \le i, j \le n$.

$\Leftarrow$: Let $b \in (\mathbb{F}_p[\![t]\!]^{1/p^\infty})^m$ be such that $v(g_j(b)) < v(f_i(b))$ for all $1 \le i, j \le n$. Set $\gamma_1 = \max\{v(g_j(b)) : j = 1, \ldots, n\}$ and $\gamma_2 = \min\{v(f_i(b)) : i = 1, \ldots, n\}$ and consider the open interval $I = (\gamma_1, \gamma_2) \subseteq \frac{1}{p^\infty}\mathbb{Z}^{\ge 0}$. Since $\frac{1}{p^\infty}\mathbb{Z}$ is dense in $\mathbb{R}$, we can find $q \in \frac{1}{p^\infty}\mathbb{Z}$ such that $1 \in qI$.

We now make use of the fact that for each $q \in \frac{1}{p^\infty}\mathbb{Z}^{>0}$, there is an embedding

$$\rho : \mathbb{F}_p((t))^{1/p^\infty} \to \mathbb{F}_p((t))^{1/p^\infty}$$

which maps $t \mapsto t^q$. Indeed, if $q \in \frac{1}{p^N}\mathbb{Z}^{>0}$ for some $N \in \mathbb{N}$, then there exists an embedding $\rho : \mathbb{F}_p((t))^{1/p^N} \to \mathbb{F}_p((t))^{1/p^N}$ mapping $t \mapsto t^q$, exactly as in [Anscombe and Fehm 2016, Remark 7.9]. Such a map can also be extended uniquely to the perfect hull $\mathbb{F}_p((t))^{1/p^\infty}$.

Now let $\rho : \mathbb{F}_p((t))^{1/p^\infty} \to \mathbb{F}_p((t))^{1/p^\infty}$ be as above. Then, since $f_i(x), g_j(x) \in \mathbb{F}_p[x]$, we get

$$v(g_j(\rho(b))) = v(\rho(g_j(b))) = qv(g_j(b)) < qv(f_i(b)) = v(\rho(f_i(b))) = v(f_i(\rho(b)))$$

for all $1 \le i, j \le n$. We may thus replace our witness b with $a = \rho(b)$.

Since $1 \in qI$, we get $f_i(a) = 0 \bmod (t) \wedge g_j(a) \ne 0 \bmod (t)$, for all $i, j = 1, \ldots, n$. The reduction of a modulo (t), seen as a tuple in $\mathbb{F}_p[t^{1/p^\infty}]/t$ via (†), is the desired witness. $\square$

Remark 6.2.2. The same argument used in Proposition 6.2.1 shows that

$$k[t^{1/p^\infty}]/(t^{p-1}) \models \exists x \bigwedge_{1 \le i,j \le n} (f_i(x) = 0 \wedge g_j(x) \neq 0) \iff k[[t]]^{1/p^\infty} \models \exists x \bigwedge_{1 \le i,j \le n} (v(f_i(x)) > v(g_j(x)))$$

where $k = \mathbb{F}_p$ or $\bar{\mathbb{F}}_p$. The argument of Proposition 6.2.1 needs only a slight modification for the converse direction; one needs to take $q \in \frac{1}{p^\infty}\mathbb{Z}$ such that $(p-1) \in qI$ instead of $1 \in qI$ and the same proof goes through.

6.3. _Proof of Theorem B._ We may now prove the following:

Theorem B. _Let K be any of the fields $\mathbb{Q}_p(p^{1/p^\infty})$, $\mathbb{Q}_p(\zeta_{p^\infty})$ or $\mathbb{Q}_p^{ab}$. Then the existential theory of $\mathcal{O}_K/(p)$ is decidable in L_{rings}._

Proof. By the Anscombe–Fehm Theorem, Theorem 2.2.3, the valued fields $(\mathbb{F}_p((t))^{1/p^\infty}, v_t)$ and $(\bar{\mathbb{F}}_p((t))^{1/p^\infty}, v_t)$ are $\exists$-decidable in L_{val}. For $\mathbb{Q}_p(p^{1/p^\infty})$ the conclusion now follows from Lemma 4.4.2(a) and Proposition 6.2.1. For the other two fields, one has to use Lemma 4.4.2(b), (c) and Remark 6.2.2. $\square$

Remark 6.3.1. Note that Corollary A requires decidability in the language $L_{\text{val}}(t)$ on the characteristic p side. However, for the purposes of Theorem B, the Anscombe–Fehm results in L_{val} turned out to be sufficient. This became possible because of Proposition 6.2.1, which "eliminates" any reference to t.

7. Final remarks

7.1. _An almost decidable field._ In Section 2.3, we emphasized that _uniform_ decidability of $(\mathcal{O}_K/(p^n))_{n \in \omega}$ is key for the decidability of (K, v). Indeed, at least by assuming the decidability of $\widehat{\mathbb{F}_p((t))}^{1/p^\infty}$ in $L_{\text{val}}(t)$, Proposition 3.6.9(b) allows us to produce undecidable valued fields (K, v) with each individual residue ring $\mathcal{O}_K/(p^n)$ being decidable.

Proposition 7.1.1. _Assume $\widehat{\mathbb{F}_p((t))}^{1/p^\infty}$ is decidable in $L_{\text{val}}(t)$. Then there exists a mixed characteristic henselian valued field (K, v) such that_:

(1) _The valued field (K, v) is undecidable in L_{val}._

(2) _For each $n \in \mathbb{N}$, the ring $\mathcal{O}_K/(p^n)$ is decidable in L_{rings}._

(3) _The value group (Γ_v, vp) decidable in L_{oag}._

Proof. For the convenience of the reader, we first review the construction from proof of Proposition 3.6.9(b). Given $\alpha \in 2^\omega$, define inductively:

(1) $K_0 = \mathbb{Q}_p$ and $\pi_0 = p$.

(2) $K_{\alpha \restriction n} = K_{\alpha \restriction (n-1)}(((1+p)^{\alpha(n-1)} \cdot \pi_{\alpha \restriction (n-1)})^{1/p})$ and $\pi_{\alpha \restriction n} = ((1+p)^{\alpha(n-1)} \cdot \pi_{\alpha \restriction (n-1)})^{1/p}$.

We let $K_\alpha = \bigcup_{n \in \omega} K_{\alpha \restriction n}$. Set $\bar{\alpha}_n = \sum_{k=0}^{n-1} \alpha(k) \cdot p^k$. Claim 3 of Proposition 3.6.9 shows that $\widehat{K_\alpha}^\flat = \widehat{\mathbb{F}_p((t))}^{1/p^\infty}$. We shall argue below that any $\widehat{K_\alpha}$ satisfies conditions (2) and (3). On the other hand, since $\widehat{K_\alpha} \not\equiv \widehat{K_\beta}$ for $\alpha \neq \beta$ (see Claim 2, Proposition 3.6.9(b)), we will have that some $\widehat{K_\alpha}$ is undecidable in L_{val}.

Fix $\alpha \in 2^\omega$, set $\bar{a} = \sum_{k \geq 0} \alpha(k) \cdot p^k \in \mathbb{Z}_p$ and $\bar{\alpha}_n = \sum_{k=0}^{n-1} \alpha(k) \cdot p^k \in \mathbb{Z}$ for all $n \in \mathbb{N}^{>0}$ ($\bar{\alpha}_0 = 0$ by convention). One sees that $(1+p)^{p^n} \equiv 1 \bmod p^{n+1} \mathbb{Z}_p$, for all $n \in \mathbb{N}$. It follows that the limit $\lim_{n \to \infty} (1+p)^{\bar{\alpha}_n}$ exists in $\mathbb{Z}_p$ and we denote it by $(1+p)^{\bar{a}}$. Fix $n \in \mathbb{N}^{>0}$ for the rest of the proof.

Claim. We have $t^\sharp = (1+p)^{\bar{a}} \cdot p$ and $t^\sharp \equiv (1+p)^{\bar{\alpha}_{n-1}} \cdot p \bmod p^n \mathcal{O}_{K_\alpha}$.

Proof. We have that $t = (\pi_{\alpha\restriction 0} + (p), \pi_{\alpha\restriction 1} + (p), \dots)$, via the identification $\widehat{\mathbb{F}_p[\![t]\!]^{1/p^\infty}} \cong \varprojlim_\Phi \mathcal{O}_{K_\alpha}/(p)$ (see Remark 3.6.12). Since $\pi_{\alpha\restriction n}^{p^n} = (1+p)^{\bar{\alpha}_n} \cdot p$, we compute $t^\sharp = \lim_{n \to \infty} \pi_{\alpha(n)}^{p^n} = \lim_{n \to \infty} (1+p)^{\bar{\alpha}_n} \cdot p = (1+p)^{\bar{a}} \cdot p$. Since $(1+p)^{p^{n-1}} \equiv 1 \bmod p^n \mathcal{O}_{K_\alpha}$, we also get $t^\sharp \equiv (1+p)^{\bar{\alpha}_{n-1}} \cdot p \bmod p^n \mathcal{O}_{K_\alpha}$. $\qquad\square$

As a consequence, we may take $\xi_\alpha = [t] - (1+p)^{\bar{a}} \cdot p$ as a generator of the ideal of A_{inf} associated to K_α. Consider also the distinguished element $\xi' := [t] - (1+p)^{\bar{\alpha}_{n-1}} \cdot p \in A_{\text{inf}}$ and the associated untilt (K', v'). We have an equality of ideals $(p^n, \xi_\alpha) = (p^n, \xi')$ in A_{inf} and thus an isomorphism of rings $\mathcal{O}_{K_\alpha}/(p^n) \cong \mathcal{O}_{K'}/(p^n)$.

Having assumed that the valued field $(\widehat{\mathbb{F}_p(\!(t)\!)^{1/p^\infty}}, v_t)$ is decidable in $L_{\text{val}}(t)$ and since ξ' is $\mathbb{F}_p[t]$-computable, we get that the valued field (K', v') is decidable in L_{val} by Theorem A. In particular, the ring $\mathcal{O}_{K_\alpha}/(p^n) \cong \mathcal{O}_{K'}/(p^n)$ is decidable in L_{rings}. The value group $(\Gamma_\alpha, v_\alpha p) \cong \left(\frac{1}{p^\infty}\mathbb{Z}, 1\right)$ is also decidable in L_{oag} with a constant symbol for 1. Since n was arbitrary, this concludes the proof. $\qquad\square$

Remark 7.1.2. It seems plausible that a similar construction can be carried out with $\mathbb{F}_p(\!(t^\Gamma)\!)$ instead of $\widehat{\mathbb{F}_p(\!(t)\!)^{1/p^\infty}}$, where $\Gamma = \frac{1}{p^\infty}\mathbb{Z}$, thereby recovering the above result unconditionally. Nevertheless, we do not have a working example at present.

7.2. *Reversing the direction.*

7.2.1. Given that our understanding of decidability problems in characteristic p is limited, our philosophy of reducing decidability questions from mixed characteristic to positive characteristic may seem impractical. Nevertheless, we have already seen two applications in Sections 5 and 6. We also mention another application in [Kartas 2023], which proves an *undecidability* result for the asymptotic theory of $\{K : [K : \mathbb{Q}_p] < \infty\}$ in the language L_{val} with a cross-section, again by transposing a result in positive characteristic.

7.2.2. We shall now demonstrate that the characteristic p difficulties in the language $L_{\text{val}}(t)$ are already encoded in the mixed characteristic setting, by showing a relative decidability result in the opposite direction:

Proposition 7.2.3. *If $\mathbb{Q}_p(p^{1/p^\infty})$ is $\forall^1\exists$-decidable in L_{val}, then $\mathbb{F}_p[\![t]\!]^{1/p^\infty}$ is $\exists^+$-decidable in $L_{\text{val}}(t)$.*

Proof. We may focus on $L_{\text{rings}}(t)$ sentences since $x \in \mathcal{O}$ is $\exists^+$-definable in $L_{\text{rings}}(t)$ (see note 1). Let $f_i(X_1, \dots, X_m, T) \in \mathbb{F}_p[X_1, \dots, X_m, T]$ for $i = 1, \dots, n$. We claim that

$$\mathbb{F}_p[\![t]\!]^{1/p^\infty} \models \exists x_1, \dots, x_m \left(\bigwedge_{1 \leq i \leq n} f_i(x_1, \dots, x_m, t) = 0 \right)$$

$$\iff \mathbb{F}_p[t^{1/p^\infty}]/(t) \models \forall y \in \mathfrak{m} \exists x_1, \dots, x_m \left(\bigwedge_{1 \leq i \leq n} f_i(x_1, \dots, x_m, y) = 0 \right).$$

It is enough to prove the claim since $\mathbb{Z}_p[p^{1/p^\infty}]/(p) \cong \mathbb{F}_p[t^{1/p^\infty}]/(t)$ by Lemma 4.4.2(a). We leave it to the reader to write down the $\forall^1\exists$-statement about $\mathbb{Q}_p(p^{1/p^\infty})$ which is equivalent to the one about $\mathbb{F}_p[t^{1/p^\infty}]/(t)$ written above;

$\Rightarrow$: Let $(\alpha_1, \ldots, \alpha_m) \in (\mathbb{F}_p[\![t]\!]^{1/p^\infty})^m$ be such that

$$f_1(\alpha_1, \ldots, \alpha_m, t) = f_2(\alpha_1, \ldots, \alpha_m, t) = \cdots = f_n(\alpha_1, \ldots, \alpha_m, t) = 0.$$

Let $y \in \mathfrak{m} \subset \mathbb{F}_p[\![t]\!]^{1/p^\infty}$ and $\kappa \in \mathbb{N}$ be such that $y^{p^\kappa} \in \mathbb{F}_p[\![t]\!]$. Then there exists a ring endomorphism ρ on $\mathbb{F}_p[\![t]\!]$ mapping $t \mapsto y^{p^\kappa}$, which extends uniquely to $\mathbb{F}_p[\![t]\!]^{1/p^\infty}$. Let $\beta_i := \rho(\alpha_i)$ for $i = 1, \ldots, m$. Since $f_i(X_1, \ldots, X_m, T) \in \mathbb{F}_p[X, T]$, we get

$$f_1(\beta_1, \ldots, \beta_m, y^{p^\kappa}) = f_2(\beta_1, \ldots, \beta_m, y^{p^\kappa}) = \cdots = f_n(\beta_1, \ldots, \beta_m, y^{p^\kappa}) = 0$$

and thus

$$f_1(\beta_1^{1/p^\kappa}, \ldots, \beta_m^{1/p^\kappa}, y) = f_2(\beta_1^{1/p^\kappa}, \ldots, \beta_m^{1/p^\kappa}, y) = \cdots = f_n(\beta_1^{1/p^\kappa}, \ldots, \beta_m^{1/p^\kappa}, y) = 0.$$

In particular, the tuple $(x_1, \ldots, x_m) := (\beta_1^{1/p^\kappa}, \ldots, \beta_m^{1/p^\kappa})$ is a solution modulo (t) to the above system of equations.

$\Leftarrow$: A generalized version of Greenberg's Theorem due to Moret-Bailly [2012, Corollary 1.2.2] shows that

$$f_1(x_1, \ldots, x_m, t) = f_2(x_1, \ldots, x_m, t) = \cdots = f_n(x_1, \ldots, x_m, t) = 0$$

has a solution in $\mathbb{F}_p[\![t]\!]^{1/p^\infty}$ if and only if it has a solution modulo (t^N), for all $N \in \mathbb{N}$. Equivalently, if and only if it has a solution modulo (t^{p^N}) for all $N \in \mathbb{N}$. Using the N-th iterated Frobenius, we see that for any $N \in \mathbb{N}$ we have

$$\mathbb{F}_p[\![t]\!]^{1/p^\infty} \models \exists x_1, \ldots, x_m \left(\bigwedge_{1 \leq i \leq n} f_i(x_1, \ldots, x_m, t) = 0 \bmod t^{p^N} \right)$$

$$\Longleftrightarrow \mathbb{F}_p[\![t]\!]^{1/p^\infty} \models \exists x_1, \ldots, x_m \left(\bigwedge_{1 \leq i \leq n} f_i(x_1, \ldots, x_m, t^{1/p^N}) = 0 \bmod t \right)$$

and the latter is true by assumption, for any $N \in \mathbb{N}$. $\qquad \square$

Remark 7.2.4. By an identical argument, one can prove a similar result for $\mathbb{Q}_p(\zeta_{p^\infty})$ or $\mathbb{Q}_p^{ab}$. In the case of the latter, one gets that $\bar{\mathbb{F}}_p[\![t]\!]^{1/p^\infty}$ is $\exists^+$-decidable in $L_{\text{val}}(t)$, provided that $\mathbb{Q}_p^{ab}$ is $\forall^1\exists$-decidable in L_{val}.

It would be nice to have a version of Proposition 7.2.3 for the full theories. Together with Corollary A, this would yield a Turing equivalence between the theories of $\mathbb{Q}_p(p^{1/p^\infty})$ and $\mathbb{F}_p(\!(t)\!)^{1/p^\infty}$. Nevertheless, Proposition 7.2.3 still suggests that if we eventually want to understand the theories of $\mathbb{Q}_p(p^{1/p^\infty})$, $\mathbb{Q}_p(\zeta_{p^\infty})$ and $\mathbb{Q}_p^{ab}$ (even modest parts of their theories), we would have to face certain characteristic p difficulties, posed in Question 7.2.5 below:

Question 7.2.5. (a) Is $\mathrm{Th}_{\exists^+}(\mathbb{F}_p[\![t]\!]^{1/p^\infty})$ decidable in $L_{\mathrm{val}}(t)$?

(b) Is $\mathrm{Th}_{\exists^+}(\overline{\mathbb{F}}_p[\![t]\!]^{1/p^\infty})$ decidable in $L_{\mathrm{val}}(t)$?

Acknowledgements

References

[Anscombe and Fehm 2016] S. Anscombe and A. Fehm, "The existential theory of equicharacteristic henselian valued fields", *Algebra Number Theory* **10**:3 (2016), 665–683. MR Zbl

[Anscombe and Jahnke 2022] S. Anscombe and F. Jahnke, "The model theory of Cohen rings", *Confluentes Math.* **14**:2 (2022), 1–28. MR Zbl

[Anscombe and Kuhlmann 2016] S. Anscombe and F.-V. Kuhlmann, "Notes on extremal and tame valued fields", *J. Symb. Log.* **81**:2 (2016), 400–416. MR Zbl

[Ax and Kochen 1965] J. Ax and S. Kochen, "Diophantine problems over local fields, II: A complete set of axioms for p-adic number theory", *Amer. J. Math.* **87**:3 (1965), 631–648. MR Zbl

[Basarab 1978] S. A. Basarab, "Some model theory for Henselian valued fields", *J. Algebra* **55**:2 (1978), 191–212. MR Zbl

[Bélair 1999] L. Bélair, "Types dans les corps valués munis d'applications coefficients", *Illinois J. Math.* **43**:2 (1999), 410–425. MR Zbl

[Deligne 1984] P. Deligne, "Les corps locaux de caractéristique p, limites de corps locaux de caractéristique 0", pp. 119–157 in *Représentations des groupes réductifs sur un corps local*, edited by J.-N. Bernstein et al., Travaux en Cours **8**, Hermann, Paris, 1984. MR Zbl

[Denef and Schoutens 2003] J. Denef and H. Schoutens, "On the decidability of the existential theory of $\mathbb{F}_p[\![t]\!]$", pp. 43–60 in *Valuation theory and its applications* (Saskatoon, SK, 1999), vol. 2, edited by F.-V. Kuhlmann et al., Fields Inst. Commun. **33**, Amer. Math. Soc., Providence, RI, 2003. MR Zbl

[Dittmann 2022] P. Dittmann, "Two examples concerning existential undecidability in fields", preprint, 2022. arXiv 2211.01775

[van den Dries 1999] L. van den Dries, "On the elementary theory of rings of Witt vectors with a multiplicative set of representatives for the residue field", *Manuscripta Math.* **98**:2 (1999), 133–137. MR Zbl

[van den Dries 2014] L. van den Dries, "Lectures on the model theory of valued fields", pp. 55–157 in *Model theory in algebra, analysis and arithmetic* (Cetraro, 2012), edited by H. D. Macpherson and C. Toffalori, Lecture Notes in Math. **2111**, Springer, 2014. MR Zbl

[Eršov 1965] J. L. Eršov, "On elementary theories of local fields", *Algebra i Logika Sem.* **4**:2 (1965), 5–30. In Russian. MR

[Fargues and Fontaine 2014] L. Fargues and J.-M. Fontaine, "Vector bundles on curves and p-adic Hodge theory", pp. 17–104 in *Automorphic forms and Galois representations* (Durham, 2011), vol. 2, edited by F. Diamond et al., London Math. Soc. Lecture Note Ser. **415**, Cambridge Univ. Press, 2014. MR Zbl

[Fargues and Fontaine 2018] L. Fargues and J.-M. Fontaine, *Courbes et fibrés vectoriels en théorie de Hodge p-adique*, Astérisque **406**, Soc. Math. de France, Paris, 2018. MR Zbl

[Hodges 1993] W. Hodges, *Model theory*, Encyclopedia of Mathematics and its Applications **42**, Cambridge University Press, 1993. MR Zbl

[Kartas 2023] K. Kartas, "An undecidability result for the asymptotic theory of p-adic fields", *Ann. Pure Appl. Logic* **174**:2 (2023), art. id. 103203. MR Zbl

[Kazhdan 1986] D. Kazhdan, "Representations of groups over close local fields", *J. Analyse Math.* **47** (1986), 175–179. MR Zbl

[Kedlaya 2006] K. S. Kedlaya, "Finite automata and algebraic extensions of function fields", *J. Théor. Nombres Bordeaux* **18**:2 (2006), 379–420. MR Zbl

[Kedlaya and Liu 2015] K. S. Kedlaya and R. Liu, *Relative p-adic Hodge theory: foundations*, Astérisque **371**, Soc. Math. de France, Paris, 2015. MR Zbl

[Kedlaya and Temkin 2018] K. S. Kedlaya and M. Temkin, "Endomorphisms of power series fields and residue fields of Fargues–Fontaine curves", *Proc. Amer. Math. Soc.* **146**:2 (2018), 489–495. MR Zbl

[Kochen 1975] S. Kochen, "The model theory of local fields", pp. 384–425 in *ISILC Logic Conference* (Kiel, 1974), edited by G. H. Müller et al., Lecture Notes in Math. **499**, Springer, 1975. MR Zbl

[Koenigsmann 2018] J. Koenigsmann, "Decidability in local and global fields", pp. 45–59 in *Proceedings of the International Congress of Mathematicians* (Rio de Janeiro, 2018), vol. 2: Invited lectures, edited by B. Sirakov et al., World Sci. Publ., 2018. MR Zbl

[Krasner 1957] M. Krasner, "Approximation des corps valués complets de caractéristique $p \neq 0$ par ceux de caractéristique 0", pp. 129–206 in *Colloque d'algèbre supérieure* (Brussels, 1956), Établissements Ceuterick, Louvain, 1957. MR Zbl

[Kuhlmann 2016] F.-V. Kuhlmann, "The algebra and model theory of tame valued fields", *J. Reine Angew. Math.* **719** (2016), 1–43. MR Zbl

[Kuhlmann and Rzepka 2023] F.-V. Kuhlmann and A. Rzepka, "The valuation theory of deeply ramified fields and its connection with defect extensions", *Trans. Amer. Math. Soc.* **376**:4 (2023), 2693–2738. MR Zbl

[Lee 2020] J. Lee, "Hyperfields, truncated DVRs, and valued fields", *J. Number Theory* **212** (2020), 40–71. MR Zbl

[Lee and Lee 2021] J. Lee and W. Lee, "On the structure of certain valued fields", *Ann. Pure Appl. Logic* **172**:4 (2021), art. id. 102927. MR Zbl

[Lisinski 2021] V. Lisinski, "Decidability of positive characteristic tame Hahn fields in $\mathcal{L}_t$", preprint, 2021. arXiv 2108.04132

[Lurie 2018] J. Lurie, "The Fargues–Fontaine curve", online notes, 2018, available at https://www.math.ias.edu/~lurie/205.html.

[Macintyre 1986] A. Macintyre, "Twenty years of p-adic model theory", pp. 121–153 in *Logic colloquium '84* (Manchester, 1984), edited by J. B. Paris et al., Stud. Logic Found. Math. **120**, North-Holland, Amsterdam, 1986. MR Zbl

[Marker 2002] D. Marker, *Model theory: an introduction*, Graduate Texts in Mathematics **217**, Springer, 2002. MR Zbl

[Moret-Bailly 2012] L. Moret-Bailly, "An extension of Greenberg's theorem to general valuation rings", *Manuscripta Math.* **139**:1-2 (2012), 153–166. MR Zbl

[Morrow 2019] M. Morrow, "The Fargues–Fontaine curve and diamonds", exposé 1150, pp. 533–571 in *Séminaire Bourbaki* 1978/79, Astérisque **414**, Soc. Math. de France, Paris, 2019. MR Zbl

[Acosta de Orozco and Vélez 1982] M. Acosta de Orozco and W. Y. Vélez, "The lattice of subfields of a radical extension", *J. Number Theory* **15**:3 (1982), 388–405. MR Zbl

[Robinson and Zakon 1960] A. Robinson and E. Zakon, "Elementary properties of ordered abelian groups", *Trans. Amer. Math. Soc.* **96** (1960), 222–236. MR Zbl

[Scanlon 2003] T. Scanlon, "Quantifier elimination for the relative Frobenius", pp. 323–352 in *Valuation theory and its applications* (Saskatoon, SK, 1999), vol. 2, edited by F.-V. Kuhlmann et al., Fields Inst. Commun. **33**, Amer. Math. Soc., Providence, RI, 2003. MR Zbl

[Scholze 2012] P. Scholze, "Perfectoid spaces", *Publ. Math. Inst. Hautes Études Sci.* **116** (2012), 245–313. MR Zbl

[Scholze 2013] P. Scholze, "Perfectoid spaces: a survey", pp. 193–227 in *Current developments in mathematics* (Cambridge, MA, 2012), edited by D. Jerison et al., Int. Press, Somerville, MA, 2013. MR Zbl

[Scholze 2014] P. Scholze, "Perfectoid spaces and their applications", pp. 461–486 in *Proceedings of the International Congress of Mathematicians* (Seoul, 2014), vol. 2, edited by S. Y. Jang et al., Kyung Moon Sa, Seoul, 2014. MR Zbl

[Scholze 2021] P. Scholze, "How many untilts?", MathOverflow post, 2021, available at https://mathoverflow.net/q/390522.

[Serre 1979] J.-P. Serre, *Local fields*, Graduate Texts in Mathematics **67**, Springer, 1979. MR Zbl

[Washington 1997] L. C. Washington, *Introduction to cyclotomic fields*, 2nd ed., Graduate Texts in Mathematics **83**, Springer, 1997. MR Zbl

[Ziegler 1972] M. Ziegler, *Die elementare Theorie der henselschen Körper*, Ph.D. thesis, Universität zu Köln, 1972, available at https://tinyurl.com/3ez4sb2a.

kartas@imj-prg.fr   *Institut Mathématique de Jussieu, CNRS/IMJ-PRG, Paris, France*

Algebra & Number Theory

msp.org/ant

Algebra & Number Theory

Volume 18   No. 2   2024