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p-groups, p-rank, and semistable reduction of coverings of curves

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We prove various explicit formulas concerning p-rank of p-coverings of pointed semistable curves over discrete valuation rings. In particular, we obtain a full generalization of Raynaud's formula for p-rank of fibers over *nonmarked smooth* closed points in the case of *arbitrary* closed points. As an application, for abelian p-coverings, we give an affirmative answer to an open problem concerning boundedness of p-rank asked by Saïdi more than twenty years ago.

Introduction

Let R be a complete discrete valuation ring with algebraically closed residue field k of characteristic $p > 0$ and $S \stackrel{\text{def}}{=} \operatorname{Spec} R$. Write K for the quotient field of R, $\eta : \operatorname{Spec} K \to S$ for the generic point of S, and $s : \operatorname{Spec} k \to S$ for the closed point of S. Let $\mathscr{X} = (X, D_X)$ be a pointed semistable curve of genus g_X over S. Here, X denotes the underlying semistable curve of $\mathscr{X}$, and D_X denotes the finite (ordered) set of marked points of $\mathscr{X}$. Write $\mathscr{X}_\eta = (X_\eta, D_{X_\eta})$ and $\mathscr{X}_s = (X_s, D_{X_s})$ for the generic fiber and the special fiber of $\mathscr{X}$, respectively. Moreover, we suppose that $\mathscr{X}_\eta$ is a smooth pointed stable curve over η, i.e., D_X satisfies [Knudsen 1983, Definition 1.1(iv)].

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Keywords: p-rank, semistable reduction, pointed semistable curve, pointed semistable covering.

0A. ***Raynaud's formula for p-rank of nonfinite fibers.***

0A1. Let G be a finite group, and let $\mathscr{Y}_\eta = (Y_\eta, D_{Y_\eta})$ be a smooth pointed stable curve over η and $f_\eta : \mathscr{Y}_\eta \to \mathscr{X}_\eta$ a morphism of pointed stable curves over η. Suppose that f_η is a Galois covering whose Galois group is isomorphic to G, that $f_\eta^{-1}(D_{X_\eta}) = D_{Y_\eta}$, and that the branch locus of f_η is contained in D_{X_η}. By replacing S by a finite extension of S (i.e., the spectrum of the normalization of R in a finite extension of K), f_η extends to a *G-pointed semistable covering*

$$f : \mathscr{Y} = (Y, D_Y) \to \mathscr{X}$$

over S (see Definition 1.5 and Proposition 1.6). We write $\mathscr{Y}_s = (Y_s, D_{Y_s})$ for the special fiber of $\mathscr{Y}$ and $f_s : \mathscr{Y}_s \to \mathscr{X}_s$ for the morphism of pointed semistable curves over s induced by f.

Suppose that the order of G is prime to p. Then f_s is a finite, generically étale morphism [SGA 1 1971; Vidal 2001]. On the other hand, suppose that $p \mid \#G$. Then the situation is quite different from that in the case of prime-to-p coverings. The geometry of $\mathscr{Y}_s$ is very complicated and the morphism f_s is not generically étale and, moreover, is *not finite* in general. This kind of phenomenon is called "resolution of nonsingularities" [Tamagawa 2004b] which has many important applications in the theory of arithmetic fundamental groups and anabelian geometry, e.g., [Mochizuki 1996; Lepage 2013; Pop and Stix 2017; Stix 2002].

0A2. M. Raynaud [1990] investigated the geometry of reduction of étale p-group schemes over $\mathscr{X}_\eta$ (i.e., G is a p-group), and proved an explicit formula for the p-rank (see Section 1B3 for the definition of p-rank) of nonfinite fibers of f_s. More precisely, we have the following famous result which is the main theorem of Raynaud.

Theorem 0.1 [Raynaud 1990, Théorèmes 1 et 2]. *Let G be a finite p-group, and let $f : \mathscr{Y} \to \mathscr{X}$ be a G-pointed semistable covering over S and x a closed point of $\mathscr{X}_s$. Suppose that x is a* ***nonmarked smooth*** *point (i.e., $x \notin X_s^{\mathrm{sing}} \cup D_{X_s}$, where X_s^{sing} denote the singular locus of X_s) of $\mathscr{X}_s$. Then we have the following formula for the p-rank of $f^{-1}(x)$:*

$$\sigma(f^{-1}(x)) = 0.$$

In particular, suppose that $\mathscr{X}$ is a smooth pointed stable curve (i.e., X is stable and $D_X = \varnothing$) over S. As a direct consequence of the above formula, the following statements hold:

(i) *The Jacobian of $\mathscr{Y}_\eta$ has potentially good reduction.*

(ii) *The dual semigraph (Section 1B2) of $\mathscr{Y}_s$ is a tree (Section 1A3).*

(iii) *The slopes of the crystalline cohomology of connected components of vertical fibers of f are in* $(0, 1)$.

Remark 0.1.1. If x is *not* a nonmarked smooth point of $\mathscr{X}_s$, $\sigma(f^{-1}(x))$ is not equal to 0 in general. For instance, if x is a singular point of $\mathscr{X}_s$, the dual semigraph of $f^{-1}(x)$ is no longer to be a tree even the simplest case where $G = \mathbb{Z}/p\mathbb{Z}$.

On the other hand, if G is not a p-group, the p-rank of irreducible components of $\mathscr{Y}_s$ cannot be calculated explicitly in general (see Remark 1.4.1).

0B. *Main result.* We maintain the notation introduced in Section 0A. In the present paper, we give a full generalization of Raynaud's formula. Namely, we will prove various formulas for $\sigma(f^{-1}(x))$ where x is an *arbitrary* closed point of $\mathscr{X}_s$. Note that if $f^{-1}(x)$ is finite, then $\sigma(f^{-1}(x)) = 0$ by the definition of p-rank. Moreover, since f is a Galois covering, to calculate $\sigma(f^{-1}(x)) = 0$, we only need to calculate the p-rank of a connected component of $f^{-1}(x)$. Thus, to calculate $\sigma(f^{-1}(x))$, we may assume that $f^{-1}(x)$ is *nonfinite* and *connected*.

0B1. Our main result is the following formulas for $\sigma(f^{-1}(x))$ in terms of the orders of inertia subgroups of irreducible components of $f^{-1}(x)$ which depend only on the action of G on $f^{-1}(x)$ (in the introduction, we do not give the list of definitions of the notation appeared in the main theorem, see Theorems 3.4 and 3.9 for more precise forms):

Theorem 0.2. *Let G be a finite p-group, and let $f : \mathscr{Y} \to \mathscr{X}$ be a G-pointed semistable covering over S and x an* ***arbitrary*** *closed point of $\mathscr{X}_s$. Suppose that $f^{-1}(x)$ is nonfinite and connected. Then we have (see Section 3B3 for $\Gamma_{\mathscr{E}_X}$, Section 3A5 for $\#I_v$, $\#I_e$, and Section 1A1 for $v(\Gamma_{\mathscr{E}_X})$, $e(v)$, $e^{\mathrm{cl}}(\Gamma_{\mathscr{E}_X})$)*

$$\sigma(f^{-1}(x)) = \sum_{v \in v(\Gamma_{\mathscr{E}_X})} \Bigl(1 - \#G/\#I_v + \sum_{e \in e(v)} (\#G/\#I_e)(\#I_e/\#I_v - 1)\Bigr) + \sum_{e \in e^{\mathrm{cl}}(\Gamma_{\mathscr{E}_X})} (\#G/\#I_e - 1).$$

Moreover, suppose that x is a ***singular*** *point of $\mathscr{X}_s$. Then we have a simpler form as follows:*

$$\sigma(f^{-1}(x)) = \sum_{\#I \in \mathcal{I}(x)} \#G/\#I - \sum_{\#J \in \mathcal{J}(x)} \#G/\#J + 1,$$

where $\mathcal{I}(x)$ and $\mathcal{J}(x)$ are the sets of minimal and maximal orders of inertia subgroups associated to x and f (see Definition 3.5(b)), respectively.

0B2. If x is a nonmarked smooth closed point of $\mathscr{X}_s$, Raynaud's formula (i.e., Theorem 0.1) can be deduced by the first formula of Theorem 0.2 (see Section 3B7). If x is a singular closed point of $\mathscr{X}_s$, the p-rank $\sigma(f^{-1}(x))$ had been studied by M. Saïdi [1998a; 1998b] under the assumption where G is a *cyclic* p-group, and his result can be deduced by the second formula of Theorem 0.2 (see Corollary 3.11). Moreover, as an application, in Section 4 of the present paper, by applying the "moreover" part of Theorem 0.2, we give an affirmative answer to an open problem posed by Saïdi (Section 4A) when G is an abelian p-group (see Theorem 4.3).

On the other hand, our approach to proving the formulas for $\sigma(f^{-1}(x))$ is *completely different* from that of Raynaud and Saïdi (Saïdi's method is close to the method of Raynaud), and we calculate $\sigma(f^{-1}(x))$ by introducing a kind of new object which we call *semigraphs with p-rank* (Section 2). Moreover, our method can be used not only for calculating the p-rank of a fiber $f^{-1}(x)$ of a closed point x, but also for *calculating the p-rank $\sigma(\mathscr{Y}_s)$ of the special fiber $\mathscr{Y}_s$* of $\mathscr{Y}$ (see Theorem 3.2 for a formula for $\sigma(\mathscr{Y}_s)$).

0C. ***Strategy of proof.*** We briefly explain the method of proving Theorem 0.2.

0C1. We maintain the notation introduced in Section 0B. To calculate the p-rank $\sigma(f^{-1}(x))$ of $f^{-1}(x)$, we need to calculate (i) the p-rank of the normalizations of irreducible components of $f^{-1}(x)$, and (ii) the Betti number γ_x 1A3 of the dual semigraph Γ_x 1B2 of $f^{-1}(x)$. By using the general theory of semistable curves, (i) can be obtained by using the Deuring–Shafarevich formula (Proposition 1.4).

The major difficulty is (ii). In the cases treated by Raynaud and Saïdi, the geometry of the fiber $f^{-1}(x)$ is well-managed (in fact, Γ_x is a tree when x is a nonmarked smooth point). On the other hand, in the general case (i.e., x is an arbitrary closed point and G is an arbitrary p-group), the geometry of $f^{-1}(x)$ is very complicated, and its dual semigraph is *far from being tree-like*.

0C2. The author observed that we can "avoid" to compute directly the Betti number γ_x of Γ_x if $f^{-1}(x)$ admits a good "deformation" such that the decomposition groups of irreducible components of the deformation are G, and that $\sigma(f^{-1}(x))$ is equal to the p-rank of the deformation. However, in general, such deformations *do not exist* in the theory of algebraic geometry (i.e., we cannot find such deformations in moduli spaces of curves, see Remark 2.4.1).

To overcome this difficulty, we introduce the so-called *semigraphs with p-rank* (Section 2), and define p-rank, coverings, and G-coverings for semigraphs with p-rank. Moreover, we can deform semigraphs with p-rank in a natural way, and prove that the deformations do not change the p-rank of semigraphs with p-rank (Proposition 2.6). Then we may obtain an explicit formula for the p-rank of G-coverings of semigraphs with p-rank (Theorem 2.7). Furthermore, by using the theory of semistable curves, we construct semigraphs with p-rank (Section 3) from G-pointed semistable coverings (in particular, we construct a semigraph with p-rank from $f^{-1}(x)$). Together with some precise analysis of inertia groups (Section 1) of singular points and irreducible components of G-pointed semistable coverings, we obtain Theorem 0.2.

0D. ***Structure of the present paper.*** The present paper is organized as follows. In Section 1, we introduce some notation concerning semigraphs, pointed semistable curves, and pointed semistable coverings. Moreover, we prove some results concerning inertia subgroups of singular points and irreducible components of pointed semistable coverings. In Section 2, we introduce semigraphs with p-rank, and study the p-rank of G-coverings of semigraphs with p-rank. In Section 3, we construct various G-coverings of semigraphs with p-rank from G-pointed semistable coverings. Moreover, by applying the results obtained in Section 2, we obtain various formulas for p-rank concerning G-pointed semistable coverings. In Section 4, we study bounds of p-rank of vertical fibers of G-pointed semistable coverings by using formulas obtained in Section 3.

1. Pointed semistable coverings

In this section, we introduce pointed semistable coverings of pointed semistable curves over discrete valuation rings.

1A. *Semigraphs.* We begin with some general remarks concerning semigraphs; see also [Mochizuki 2006, Section 1].

1A1. A *semigraph* $\mathbb{G}$ consists of the following data:

(i) A set $v(\mathbb{G})$ whose elements we refer to as vertices.

(ii) A set $e(\mathbb{G})$ whose elements we refer to as edges. Moreover, any element $e \in e(\mathbb{G})$ is a set of cardinality 2 satisfying the following property: for each $e \neq e' \in e(\mathbb{G})$, we have $e \cap e' = \varnothing$.

(iii) A set of maps $\{\zeta_e^{\mathbb{G}}\}_{e \in e(\mathbb{G})}$ such that $\zeta_e^{\mathbb{G}} : e \to v(\mathbb{G}) \cup \{v(\mathbb{G})\}$ is a map from the set e to the set $v(\mathbb{G}) \cup \{v(\mathbb{G})\}$, and that $\#((\zeta_e^{\mathbb{G}})^{-1}(\{v(\mathbb{G})\})) \in \{0, 1\}$, where $\#(-)$ denotes the cardinality of $(-)$.

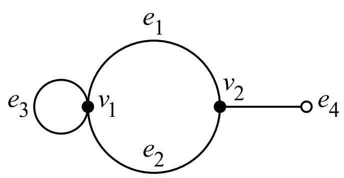
Let $e \in e(\mathbb{G})$ be an edge of $\mathbb{G}$. We shall refer to an element $b \in e$ as a *branch* of the edge e. We shall call that $e \in e(\mathbb{G})$ is *closed* (resp. *open*) if $\#((\zeta_e^{\mathbb{G}})^{-1}(\{v(\mathbb{G})\})) = 0$ (resp. $\#((\zeta_e^{\mathbb{G}})^{-1}(\{v(\mathbb{G})\})) = 1$). Moreover, write $e^{\mathrm{cl}}(\mathbb{G})$ for the set of closed edges of $\mathbb{G}$ and $e^{\mathrm{op}}(\mathbb{G})$ for the set of open edges of $\mathbb{G}$. Note that we have $e(\mathbb{G}) = e^{\mathrm{cl}}(\mathbb{G}) \cup e^{\mathrm{op}}(\mathbb{G})$.

Let $v \in v(\mathbb{G})$ be a vertex of $\mathbb{G}$. Write $b(v)$ for the set of branches $\bigcup_{e \in e(\mathbb{G})} (\zeta_e^{\mathbb{G}})^{-1}(v)$, $e(v)$ for the set of edges which abut to v, and $v(e)$ for the set of vertices which are abutted by e. Note that we have $\#(v(e)) \leq 2$. We shall call a closed edge $e \in e^{\mathrm{cl}}(\mathbb{G})$ *loop* if $\#v(e) = 1$ (i.e., $\#(\zeta_e^{\mathbb{G}}(e)) = 1$). Moreover, we use the notation $e^{\mathrm{lp}}(v)$ to denote the set of loops which abut to v.

Example 1.1. Let us give an example of semigraph to explain the above definitions. We use the notation "$\bullet$" and "$\circ$ with a line segment" to denote a vertex and an open edge, respectively.

Let $\mathbb{G}$ be a semigraph as follows:

$\mathbb{G}$:



Then we have $v(\mathbb{G}) = \{v_1, v_2\}$, $e(\mathbb{G}) = \{e_1, e_2, e_3, e_4\}$, $e^{\mathrm{cl}}(\mathbb{G}) = \{e_1, e_2, e_3\}$, $e^{\mathrm{op}}(\mathbb{G}) = \{e_4\}$, $\zeta_{e_1}^{\mathbb{G}}(e_1) = \zeta_{e_2}^{\mathbb{G}}(e_2) = \{v_1, v_2\}$, $\zeta_{e_3}^{\mathbb{G}}(e_3) = \{v_1\}$, and $\zeta_{e_4}^{\mathbb{G}}(e_4) = \{v_2, \{v(\mathbb{G})\}\}$. Moreover, we have $e^{\mathrm{lp}}(\mathbb{G}) = e^{\mathrm{lp}}(v_1) = \{e_3\}$, $v(e_1) = v(e_2) = \{v_1, v_2\}$, $v(e_3) = \{v_1\}$, $v(e_4) = \{v_2\}$, $e(v_1) = \{e_1, e_2, e_3\}$, and $e(v_2) = \{e_1, e_2, e_4\}$.

1A2. Let $\mathbb{G}$ be a semigraph. We shall call $\mathbb{G}'$ a *subsemigraph* of $\mathbb{G}$ if $\mathbb{G}'$ is a semigraph satisfying the following conditions:

(i) $v(\mathbb{G}')$ (resp. $e(\mathbb{G}')$) is a subset of $v(\mathbb{G})$ (resp. $e(\mathbb{G})$).

(ii) If $e \in e^{\mathrm{cl}}(\mathbb{G}')$, then $\zeta_e^{\mathbb{G}'}(e) \overset{\mathrm{def}}{=} \zeta_e^{\mathbb{G}}(e)$.

(iii) If $e = \{b_1, b_2\} \in e^{\mathrm{op}}(\mathbb{G}')$ such that $\zeta_e^{\mathbb{G}}(b_1) \in v(\mathbb{G}')$ and $\zeta_e^{\mathbb{G}}(b_2) \notin v(\mathbb{G}')$, then $\zeta_e^{\mathbb{G}'}(b_1) \overset{\mathrm{def}}{=} \zeta_e^{\mathbb{G}}(b_1)$ and $\zeta_e^{\mathbb{G}'}(b_2) \overset{\mathrm{def}}{=} \{v(\mathbb{G}')\}$.

Moreover, we define a semigraph $\mathbb{G}\setminus\mathbb{G}'$ as follows:

(i) $v(\mathbb{G}\setminus\mathbb{G}')\overset{\text{def}}{=}v(\mathbb{G})\setminus v(\mathbb{G}')$.

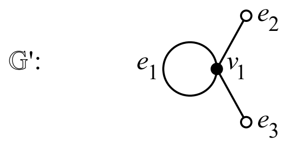
(ii) $e^{\text{cl}}(\mathbb{G}\setminus\mathbb{G}')\overset{\text{def}}{=}\{e\in e^{\text{cl}}(\mathbb{G})\mid v(e)\subseteq v(\mathbb{G}\setminus\mathbb{G}')\text{ in }\mathbb{G}\}$.

(iii) $e^{\text{op}}(\mathbb{G}\setminus\mathbb{G}')\overset{\text{def}}{=}\{e\in e^{\text{cl}}(\mathbb{G})\mid v(e)\cap v(\mathbb{G}')\neq\varnothing\text{ in }\mathbb{G}\text{ and }v(e)\cap v(\mathbb{G}\setminus\mathbb{G}')\neq\varnothing\text{ in }\mathbb{G}\}$
$\cup\,\{e\in e^{\text{op}}(\mathbb{G})\mid v(e)\cap v(\mathbb{G}\setminus\mathbb{G}')\neq\varnothing\text{ in }\mathbb{G}\}$.

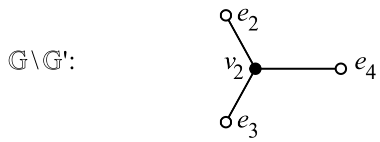
(iv) For each $e=\{b_i\}_{i\in\{1,2\}}\in e^{\text{cl}}(\mathbb{G}\setminus\mathbb{G}')\cup e^{\text{op}}(\mathbb{G}\setminus\mathbb{G}')$, we put

$$\zeta_e^{\mathbb{G}\setminus\mathbb{G}'}(b_i)\overset{\text{def}}{=}\begin{cases}\zeta_e^{\mathbb{G}}(b_i) & \text{if }\zeta_e^{\mathbb{G}}(b_i)\notin v(\mathbb{G}')\text{ and }\zeta_e^{\mathbb{G}}(b_i)\neq\{v(\mathbb{G})\},\\ \{v(\mathbb{G}\setminus\mathbb{G}')\} & \text{otherwise.}\end{cases}$$

Example 1.2. We give some examples to explain the above definition. Let $\mathbb{G}$ be the semigraph of Example 1.1 and $\mathbb{G}'$ be a subsemigraph as follows:



Moreover, the semigraph $\mathbb{G}\setminus\mathbb{G}'$ is the following:



Remark 1.2.1. We explain the motivation of the constructions of $\mathbb{G}'$ and $\mathbb{G}\setminus\mathbb{G}'$. Let $\mathscr{X}=(X,D_X)$ be a pointed semistable curve (Section 1B1) over an algebraically closed field such that the dual semigraph $\Gamma_{\mathscr{X}}$ (Section 1B1) is equal to $\mathbb{G}$ defined in Example 1.1. Write X_{v_1} and X_{v_2} for the irreducible components corresponding to v_1 and v_2, respectively. Then we have the following natural pointed semistable curves:

$$(X_{v_1},D_{X_{v_1}}\overset{\text{def}}{=}X_{v_1}\cap X_{v_2}),\quad (X_{v_2},D_{X_{v_2}}\overset{\text{def}}{=}(X_{v_1}\cap X_{v_2})\cup D_X)$$

whose dual semigraphs are equal to $\mathbb{G}'$ and $\mathbb{G}\setminus\mathbb{G}'$ defined in Example 1.2, respectively.

1A3. A semigraph $\mathbb{G}$ will be called *finite* if $v(\mathbb{G})$ and $e(\mathbb{G})$ are finite. In the present paper, *we only consider finite semigraphs.* Since a semigraph can be regarded as a topological space (i.e., a subspace of $\mathbb{R}^2$), we shall call $\mathbb{G}$ *connected* if $\mathbb{G}$ is connected as a topological space. Moreover, we write

$$\gamma_{\mathbb{G}}\overset{\text{def}}{=}\dim_{\mathbb{C}}(H^1(\mathbb{G},\mathbb{C}))$$

for the Betti number of $\mathbb{G}$, where $\mathbb{C}$ denotes the field of complex numbers. In particular, we shall call $\mathbb{G}$ a *tree* (or $\mathbb{G}$ *tree-like*) if $\gamma_{\mathbb{G}}=0$.

Let $\mathbb{G}$ and $\mathbb{H}$ be two semigraphs. A *morphism* between semigraphs $\mathbb{G}\to\mathbb{H}$ is a collection of maps $v(\mathbb{G})\to v(\mathbb{H})$, $e^{\text{cl}}(\mathbb{G})\to e^{\text{cl}}(\mathbb{H})$, and $e^{\text{op}}(\mathbb{G})\to e^{\text{op}}(\mathbb{H})$ satisfying the following: for each $e_{\mathbb{G}}\in e(\mathbb{G})$, write $e_{\mathbb{H}} \in e(\mathbb{H})$ for the image of $e_{\mathbb{G}}$; then the map $e_{\mathbb{G}} \xrightarrow{\sim} e_{\mathbb{H}}$ is a bijection, and is compatible with the $\{\zeta_e^{\mathbb{G}}\}_{e\in e(\mathbb{G})}$ and $\{\zeta_e^{\mathbb{H}}\}_{e\in e(\mathbb{H})}$.

1B. *Pointed semistable curves.*

1B1. Let $\mathscr{C} \stackrel{\text{def}}{=} (C, D_C)$ be a *pointed semistable curve* over a scheme A, namely, a marked curve over A such that every geometric fiber $C_{\bar{a}}$, $a \in A$, is a semistable curve, and that $D_{C_{\bar{a}}} \subseteq C_{\bar{a}}^{\text{sm}}$, where $C_{\bar{a}}^{\text{sm}}$ denotes the smooth locus of $C_{\bar{a}}$. We shall call C the underlying curve of $\mathscr{C}$ and the finite (ordered) set D_C the set of marked points of $\mathscr{C}$. In particular, we shall call that $\mathscr{C}$ is a *pointed stable curve* if D_C satisfies [Knudsen 1983, Definition 1.1 (iv)].

1B2. Suppose that A is the spectrum of an algebraically closed field. We write $\text{Irr}(C)$ for the set of the irreducible components of C and C^{sing} for the set of singular points (or nodes) of C. We define the *dual semigraph* $\Gamma_{\mathscr{C}}$ of the pointed semistable curve $\mathscr{C}$ to be the following semigraph:

(i) $v(\Gamma_{\mathscr{C}}) \stackrel{\text{def}}{=} \{v_E\}_{E\in \text{Irr}(C)}$.

(ii) $e^{\text{cl}}(\Gamma_{\mathscr{C}}) \stackrel{\text{def}}{=} \{e_s\}_{s\in C^{\text{sing}}}$ and $e^{\text{op}}(\Gamma_{\mathscr{C}}) \stackrel{\text{def}}{=} \{e_m\}_{m\in D_C}$.

(iii) For each $e_s = \{b_s^1, b_s^2\} \in e^{\text{cl}}(\Gamma_{\mathscr{C}})$, $s \in C^{\text{sing}}$, we put

$$\zeta_{e_s}^{\Gamma_{\mathscr{C}}}(e_s) \stackrel{\text{def}}{=} \{v_E \in v(\Gamma_{\mathscr{C}}) \mid s \in E\}.$$

(iv) For each $e_m = \{b_m^1, b_m^2\} \in e^{\text{op}}(\Gamma_{\mathscr{C}})$, $m \in D_C$, we put

$$\zeta_{e_m}^{\Gamma_{\mathscr{C}}}(b_m^1) \stackrel{\text{def}}{=} v_E, \quad \zeta_{e_m}^{\Gamma_{\mathscr{C}}}(b_m^2) \stackrel{\text{def}}{=} \{v(\Gamma_{\mathscr{C}})\},$$

where E is the irreducible component of C satisfying $m \in E$.

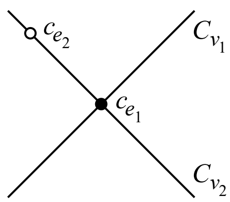
Moreover, we put (see Section 1A3)

$$\gamma_{\mathscr{C}} \stackrel{\text{def}}{=} \gamma_{\Gamma_{\mathscr{C}}} = \dim_{\mathbb{C}}(H^1(\Gamma_{\mathscr{C}}, \mathbb{C})).$$

Let $v \in v(\Gamma_{\mathscr{C}})$ (resp. $e \in e^{\text{cl}}(\Gamma_{\mathscr{C}})$, $e \in e^{\text{op}}(\Gamma_{\mathscr{C}})$). We write C_v (resp. c_e, c_e) for the irreducible component of C corresponding to v (resp. the singular point of C corresponding to e, the marked point of $\mathscr{C}$ corresponding to e) and $\widetilde{C}_v$ for the normalization of C_v.

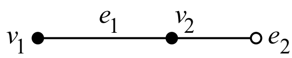
Example 1.3. We give an example to explain dual semigraphs of pointed semistable curves. Let $\mathscr{C} \stackrel{\text{def}}{=} (C, D_C)$ be a pointed semistable curve over k whose irreducible components are C_{v_1} and C_{v_2}, whose node is c_{e_1}, and whose marked point is $c_{e_2} \in C_{v_2}$. We use the notation "•" and "∘" to denote a node and a marked point, respectively. Then $\mathscr{C}$ is as follows:

$\mathscr{C}$:



We write v_1 and v_2 for the vertices of $\Gamma_{\mathscr{C}}$ corresponding to C_{v_1} and C_{v_2}, respectively, e_1 for the closed edge corresponding to c_{e_1}, and e_2 for the open edge corresponding to c_{e_2}. Moreover, we use the notation "$\bullet$" and "$\circ$ with a line segment" to denote a vertex and an open edge, respectively. Then the dual semigraph $\Gamma_{\mathscr{C}}$ of $\mathscr{C}$ is as follows:

$\Gamma_{\mathscr{C}}$:



1B3. Let C be a disjoint union of projective curves over an algebraically closed field of characteristic $p > 0$. We define the *p-rank* (or *Hasse–Witt invariant*) $\sigma(C)$ of C to be

$$\sigma(C) \overset{\text{def}}{=} \dim_{\mathbb{F}_p}(H^1_{\text{ét}}(C, \mathbb{F}_p)).$$

Moreover, let $\mathscr{C} \overset{\text{def}}{=} (C, D_C)$ be a pointed semistable curve over an algebraically closed field of characteristic $p > 0$. Write $\Gamma_{\mathscr{C}}$ for the dual semigraph of $\mathscr{C}$. Then we put

$$\sigma(\mathscr{C}) \overset{\text{def}}{=} \sigma(C) = \gamma_{\mathscr{C}} + \sum_{v \in v(\Gamma_C)} \sigma(\widetilde{C}_v).$$

1B4. Let G be a finite p-group. The p-rank of a Galois covering whose Galois group is isomorphic to G can be calculated by the Deuring–Shafarevich formula (or Crew's formula) as follows:

Proposition 1.4 [Crew 1984, Corollary 1.8]. *Let $h : C' \to C$ be a (possibly ramified) Galois covering of smooth projective curves over an algebraically closed field of characteristic $p > 0$ whose Galois group is a finite p-group G. Then we have*

$$\sigma(C') - 1 = \#G(\sigma(C) - 1) + \sum_{c' \in (C')^{\text{cl}}} (e_{c'} - 1),$$

where $(C')^{\text{cl}}$ denotes the set of closed points of C' and $e_{c'}$ denotes the ramification index at c'.

Remark 1.4.1. We maintain the notation introduced in Proposition 1.4. Suppose that G is *not* a p-group. Then $\sigma(C')$ cannot be calculated explicitly in general. In fact, the p-rank (or more precisely, generalized Hasse–Witt invariants) of prime-to-p étale coverings can almost determine the isomorphism class of C, e.g., [Tamagawa 2004a; Yang 2018].

1C. *Pointed semistable coverings.*

1C1. *Settings.* We fix some notation of the present subsection. Let R be a complete discrete valuation ring with algebraically closed residue field k of characteristic $p > 0$ and K the quotient field. We put $S \overset{\text{def}}{=} \operatorname{Spec} R$. Write η and s for the generic point and the closed point corresponding to the natural morphisms $\operatorname{Spec} K \to S$ and $\operatorname{Spec} k \to S$, respectively. Let $\mathscr{X} \overset{\text{def}}{=} (X, D_X)$ be a pointed semistable curve over S. Write $\mathscr{X}_\eta \overset{\text{def}}{=} (X_\eta, D_{X_\eta})$ for the generic fiber of $\mathscr{X}$, $\mathscr{X}_s \overset{\text{def}}{=} (X_s, D_{X_s})$ for the special fiber of $\mathscr{X}$, and $\Gamma_{\mathscr{X}_s}$ for the dual semigraph of $\mathscr{X}_s$. Moreover, we suppose that $\mathscr{X}_\eta$ is a *smooth pointed stable curve* over η (note that $\mathscr{X}_s$ is not a pointed stable curve in general).

1C2. Let $l : \mathscr{W} \stackrel{\text{def}}{=} (W, D_W) \to \mathscr{X}$ be a morphism of pointed semistable curves over S and G a finite group. We define pointed semistable coverings as follows:

Definition 1.5. The morphism l is called a *pointed semistable covering* (resp. *G-pointed semistable covering*) over S if the morphism

$$l_\eta : \mathscr{W}_\eta \stackrel{\text{def}}{=} (W_\eta, D_{W_\eta}) \to \mathscr{X}_\eta = (X_\eta, D_{X_\eta})$$

over η induced by l on generic fibers is a finite generically étale morphism (resp. a Galois covering whose Galois group is isomorphic to G) such that the following conditions hold:

(i) The branch locus of l_η is contained in D_{X_η}.

(ii) $l_\eta^{-1}(D_{X_\eta}) = D_{W_\eta}$.

(iii) The following universal property holds: if $g : \mathscr{W}' \to \mathscr{X}$ is a morphism of pointed semistable curves over S such that the generic fiber $\mathscr{W}'_\eta$ of $\mathscr{W}'$ and the morphism $g_\eta : \mathscr{W}'_\eta \to \mathscr{X}_\eta$ induced by g on generic fibers are equal to $\mathscr{W}_\eta$ and l_η, respectively, then there exists a unique morphism $h : \mathscr{W}' \to \mathscr{W}$ such that $g = l \circ h$.

We shall call l a *pointed stable covering* (resp. *G-pointed stable covering*) over S if l is a pointed semistable covering (resp. G-pointed semistable covering) over S, and $\mathscr{X}$ is a pointed stable curve over S. We shall call l a *semistable covering* (resp. *stable covering*, *G-semistable covering*, *G-stable covering*) over S if l is a pointed semistable covering (resp. pointed stable covering, G-pointed semistable covering, G-pointed stable covering) over S, and D_X is empty.

1C3. We have the following proposition.

Proposition 1.6. *Let $f_\eta : \mathscr{Y}_\eta \stackrel{\text{def}}{=} (Y_\eta, D_{Y_\eta}) \to \mathscr{X}_\eta$ be a finite morphism of pointed smooth curves over η. Suppose that the branch locus of f_η is contained in D_{X_η} and that $f_\eta^{-1}(D_{X_\eta}) = D_{Y_\eta}$. Then, by replacing S by a finite extension of S, f_η extends to a pointed semistable covering $f : \mathscr{Y} = (Y, D_Y) \to \mathscr{X}$ over S such that the restriction of f to the generic fibers is f_η.*

Proof. The proposition follows from [Liu 2006, Theorem 0.2 and Remark 4.13]. □

Remark 1.6.1. We maintain the notation introduced in Proposition 1.6. In fact, we have that f_η extends *uniquely* to a pointed semistable covering f. Let us explain roughly in this remark.

By adding some marked points, we may obtain a pointed stable curve $\mathscr{X}^{\text{add}} \stackrel{\text{def}}{=} (X^{\text{add}}, D_{X^{\text{add}}})$ whose underlying curve X^{add} is X, and whose set of marked points contains D_X. Write $D_{X_\eta^{\text{add}}}$ for $D_{X^{\text{add}}}|_\eta$, and $D_{Y_\eta^{\text{add}}}$ for $f_\eta^{-1}(D_{X_\eta^{\text{add}}})$. Then $D_{Y_\eta^{\text{add}}}$ contains D_{Y_η}. Moreover, we have a finite morphism of pointed smooth curves

$$f_\eta^{\text{add}} : \mathscr{Y}_\eta^{\text{add}} \to \mathscr{X}_\eta^{\text{add}}$$

over η induced by f_η.

By applying Proposition 1.6 and by replacing S by a finite extension of S, f_η^{add} extends to a pointed semistable covering

$$f^{\text{add}} : \mathscr{Y}^{\text{add}} \overset{\text{def}}{=} (Y^{\text{add}}, D_{Y^{\text{add}}}) \to \mathscr{X}^{\text{add}}$$

over S. Since $\mathscr{X}^{\text{add}}$ is a pointed stable curve over S, we see that $\mathscr{Y}^{\text{add}}$ is a pointed stable model of $\mathscr{Y}_\eta^{\text{add}}$. Then the uniqueness of f^{add} follows from the uniqueness of the pointed stable model $\mathscr{Y}^{\text{add}}$.

We put $D_Y^{\text{ss}} \overset{\text{def}}{=} D_Y^{\text{add}} \setminus D_Y$ and $D_{Y_s}^{\text{ss}} \overset{\text{def}}{=} D_Y^{\text{ss}}|_s$. Let $\text{Con}(Y_s^{\text{add}})$ be the subset of the set of irreducible components of Y_s^{add} consisting of all irreducible components E of Y_s^{add} satisfying the following conditions:

(i) E is isomorphic to $\mathbb{P}_k^1$.

(ii) $E \cap D_{Y_s}^{\text{ss}} \neq \varnothing$ and $E \cap D_Y = \varnothing$.

(iii) $f^{\text{add}}(E)$ is a closed point of $\mathscr{X}^{\text{add}}$.

Note that $\text{Con}(Y_s^{\text{add}})$ may be an empty set. Then by forgetting the marked points D_Y^{ss} and by contracting the irreducible components of $\text{Con}(Y_s^{\text{add}})$ [Bosch et al. 1990, 6.7 Proposition 4], we obtain a pointed semistable curve $\mathscr{Y}$ and a morphism of pointed semistable curves $f : \mathscr{Y} \to \mathscr{X}$ over S induced by f^{add}. We see that f is a pointed semistable covering over S, and that f does not depend on the choices of $D_{X^{\text{add}}}$. Moreover, the uniqueness follows from the uniqueness of f^{add}.

1C4. If a G-pointed semistable covering over S is finite, then it induces a morphism of dual semigraphs of special fibers. More precisely, we have the following result:

Proposition 1.7. *Let G be a finite group, $f : \mathscr{Y} = (Y, D_Y) \to \mathscr{X}$ a **finite** G-pointed semistable covering over S, and $\Gamma_{\mathscr{Y}_s}$ the dual semigraph of $\mathscr{Y}_s$. Then the images of nodes (resp. smooth points) of the special fiber $\mathscr{Y}_s$ of $\mathscr{Y}$ are nodes (resp. smooth points) of $\mathscr{X}_s$. In particular, the map of dual semigraphs $\Gamma_{\mathscr{Y}_s} \to \Gamma_{\mathscr{X}_s}$ induced by the morphism of the special fibers $f_s : \mathscr{Y}_s \to \mathscr{X}_s$ over s induced by f is a morphism of semigraphs 1A3.*

Proof. Let y be a closed point of $\mathscr{Y}$. Write $I_y \subseteq G$ for the inertia subgroup of y. Thus, the natural morphism $\mathscr{Y}/I_y \to \mathscr{X}$ induced by f is étale at the image of y of the quotient morphism $\mathscr{Y} \to \mathscr{Y}/I_y$. Then to verify the proposition, we may assume that $G = I_y$.

If y is a smooth point, then x is a smooth point [Raynaud 1990, Proposition 5]. If y is a node, let Y_1 and Y_2 be the irreducible components (which may be equal) of the underlying curve of the special fiber $\mathscr{Y}_s$ of $\mathscr{Y}$ containing y. Write $D_1 \subseteq G$ and $D_2 \subseteq G$ for the decomposition subgroups of Y_1 and Y_2, respectively. The proof of [Raynaud 1990, Proposition 5] implies the following:

(i) If D_1 and D_2 are not equal to $I_y = G$, then x is a smooth point.

(ii) If $D_1 = D_2 = G$, then x is a node.

Next, we prove that the case (i) will not occur. If D_1 and D_2 are not equal to G, then, for each $\tau \in G \setminus D_1$ (or $\tau \in G \setminus D_2$), we have $\tau(Y_1) = Y_2$ and $\tau(Y_2) = Y_1$. Thus, we obtain $D \overset{\text{def}}{=} D_1 = D_2$. Moreover, D is a normal subgroup of G. By replacing I_y by I_y/D and $\mathscr{Y}$ by $\mathscr{Y}/D$, and by applying the case (ii), we may assume that D is trivial. Then f_s is étale at the generic points of Y_1 and Y_2. Consider the local morphism $f_y : \operatorname{Spec} \mathcal{O}_{\mathscr{Y},y} \to \operatorname{Spec} \mathcal{O}_{\mathscr{X},f(y)}$ induced by f. Since f_y is étale at all the points of $\operatorname{Spec} \mathcal{O}_{\mathscr{Y},y}$ corresponding to the prime ideals of $\mathcal{O}_{\mathscr{Y},y}$ of height 1, the Zariski–Nagata purity theorem implies that f_y is étale. This means that if $f(y)$ is a smooth point, y is a smooth point too. This contradicts our assumption. We complete the proof of the proposition. □

1C5. On the other hand, pointed semistable coverings are not finite morphisms in general.

Definition 1.8. Let $f : \mathscr{Y} \to \mathscr{X}$ be a pointed semistable covering over S. A closed point $x \in \mathscr{X}$ is called a *vertical point associated to* f, or for simplicity, a *vertical point* when there is no fear of confusion, if $f^{-1}(x)$ is not a finite set. The inverse image $f^{-1}(x)$ is called the *vertical fiber associated to* f *and* x.

Remark 1.8.1. We maintain the notation introduced above. Then the specialization homomorphism of admissible fundamental groups of generic fiber and special fiber of $\mathscr{X}$ is not an isomorphism in general. When $\operatorname{char}(K) = 0$, this result follows from $\sigma(\mathscr{X}_s) \le g_X$, where g_X denotes the genus of $\mathscr{X}$. On the other hand, when $\operatorname{char}(K) = p > 0$, this result is highly nontrivial [Tamagawa 2004a, Theorem 0.3; Yang 2020, Theorem 5.2 and Remark 5.2.1]. Then we may ask the following problem:

> By replacing S by a finite extension of S, does there exist a pointed semistable covering $f : \mathscr{Y} \to \mathscr{X}$ over S such that the set of vertical points associated to f is not empty?

Suppose $\operatorname{char}(K) = 0$. The problem was solved by A. Tamagawa [2004b, Theorem 0.2]. In fact, Tamagawa proved a very strong result as following:

> Suppose that $\operatorname{char}(K) = 0$, that k is an algebraic closure of a finite field, and that $\mathscr{X}$ is a pointed *stable* curve over S. Let $x \in \mathscr{X}$ be a closed point of $\mathscr{X}$. Then there exists a pointed stable covering $f : \mathscr{Y} \to \mathscr{X}$ over S such that x is a vertical point associated to f.

Moreover, the author generalized this result to the case where k is an arbitrary algebraically closed field [Yang 2019, Theorem 3.2]. On the other hand, suppose that $\operatorname{char}(K) = p > 0$. The problem was solved by the author when $\mathscr{X}_s$ is irreducible [Yang 2019, Theorem 0.2].

1C6. For the p-rank of vertical fibers of pointed semistable coverings, we have the following famous result proved by Raynaud, which is the main theorem of [Raynaud 1990].

Theorem 1.9 [Raynaud 1990, Théorème 2]. *Let G be a finite p-group, $f : \mathscr{Y} \to \mathscr{X}$ a G-pointed semistable covering over S, and x a vertical point associated to f. If x is a* ***nonmarked smooth*** *point of $\mathscr{X}_s$* (i.e., $x \notin X^{\text{sing}} \cup D_{X_s}$), *then we have $\sigma(f^{-1}(x)) = 0$.*

1C7. In the remainder of the present paper, we will generalize Theorem 1.9 to the case where x is an *arbitrary* (possibly singular) closed point of $\mathscr{X}$. Namely, we will give an explicit formula for p-rank of vertical fibers associated to arbitrary vertical points of G-pointed semistable coverings, where G is a finite p-group.

1D. ***Inertia subgroups and a criterion for vertical fibers.*** In this subsection, we study the relationship between the inertia subgroups of nodes and the inertia subgroups of irreducible components of special fibers of G-pointed semistable coverings. The main result of the present subsection is Proposition 1.12.

1D1. *Settings.* We maintain the settings introduced in Section 1C1.

1D2. Firstly, we have the following lemmas.

Lemma 1.10. *Let G be a finite group, $f : \mathscr{Y} = (Y, D_Y) \to \mathscr{X}$ a **finite** G-pointed semistable covering over S, $\mathscr{Y}_s = (Y_s, D_{Y_s})$ the special fiber of $\mathscr{Y}$, and $y \in \mathscr{Y}_s$ a node. Let Y_1 and Y_2 (which may be equal) be the irreducible components of $\mathscr{Y}_s$ containing y. Write $I_y \subseteq G$ (resp. $I_{Y_1} \subseteq G$, $I_{Y_2} \subseteq G$) for the inertia subgroup of y (resp. Y_1, Y_2). Suppose that G is a p-group. Then the inertia subgroup I_y is generated by I_{Y_1} and I_{Y_2}.*

Proof. Write I for the group generated by I_{Y_1} and I_{Y_2}. Then we have $I \subseteq I_y$. Consider the quotient $\mathscr{Y}/I$. We obtain morphisms of pointed semistable curves $\mu_1 : \mathscr{Y} \to \mathscr{Y}/I$ and $\mu_2 : \mathscr{Y}/I \to \mathscr{X}$ over S such that $\mu_2 \circ \mu_1 = f$. Note that $\mathscr{Y}/I$ is a pointed semistable curve over S [Raynaud 1990, Appendice, Corollaire], and that $\mu_1(y)$ is a node of the special fiber $(\mathscr{Y}/I)_s$ of $\mathscr{Y}/I$ (Proposition 1.7). Moreover, μ_2 is generically étale at the generic points of $\mu_1(Y_1)$ and $\mu_1(Y_2)$. Then by applying the well-known result concerning the structures of étale fundamental groups of nodes of pointed stable curves, e.g., [Tamagawa 2004b, Lemma 2.1(iii)], to the local morphism $\operatorname{Spec} \mathcal{O}_{\mathscr{Y}/I, \mu_1(y)} \to \operatorname{Spec} \mathcal{O}_{\mathscr{X}, f(y)}$ induced by μ_2, we obtain that μ_2 is tamely ramified at $\mu_1(y)$. Moreover, since G is a p-group, μ_2 is étale at $\mu_1(y)$. This means $I_y \subseteq I$. Namely, we have $I_y = I$. We complete the proof of the lemma. □

Lemma 1.11 [Tamagawa 2004b, Propoisiton 4.3(ii)]. *Let G be a finite group, $f : \mathscr{Y} \to \mathscr{X}$ a G-pointed semistable covering over S, and x a node of $\mathscr{X}_s$. Suppose that, for each irreducible component $Z \overset{\text{def}}{=} \overline{\{z\}}$ of $\operatorname{Spec} \widehat{\mathcal{O}}_{\mathscr{X}_s, x}$ and each point w of the fiber $\mathscr{Y} \times_{\mathscr{X}} z$, the natural morphism from the integral closure W^s of Z in $k(w)^s$ to Z is wildly ramified, where $k(w)^s$ denotes the maximal separable subextension of $k(z)$ in $k(w)$. Then x is a vertical point associated to f (i.e., $f^{-1}(x)$ is not finite).*

Remark 1.11.1. Tamagawa [2004b] only treated the case where f is a stable covering. It is easy to see that Tamagawa's proof also holds for pointed semistable coverings.

1D3. Next, we prove a criterion for existence of vertical fibers over nodes as follows:

Proposition 1.12. *Let G be a finite group, $f : \mathscr{Y} = (Y, D_Y) \to \mathscr{X}$ a G-pointed semistable covering over S, $\mathscr{Y}_\eta = (Y_\eta, D_{Y_\eta})$ the generic fiber of $\mathscr{Y}$ over η, $\mathscr{Y}_s = (Y_s, D_{Y_s})$ the special fiber of $\mathscr{Y}$ over s, and x a node of $\mathscr{X}_s$. Write $\psi_2 : \mathscr{Y}' \to \mathscr{X}$ for the normalization morphism of $\mathscr{X}$ in the function field $K(Y)$ induced by the natural injection $K(X) \hookrightarrow K(Y)$ induced by f. We obtain a natural morphism of fiber surfaces $\psi_1 : \mathscr{Y} \to \mathscr{Y}'$ induced by f such that $\psi_2 \circ \psi_1 = f$. Write X_1 and X_2 (which may be equal) for the irreducible components of $\mathscr{X}_s$ containing x. Let $y' \in \psi_2^{-1}(x)_{\text{red}}$, and let Y_1 and Y_2 be the irreducible components of $\mathscr{Y}_s$ such that $y' \in \psi_1(Y_1) \cap \psi_1(Y_2)$. Write $I_{Y_1} \subseteq G$ and $I_{Y_2} \subseteq G$ for the inertia subgroups of Y_1 and Y_2, respectively. Suppose that neither $I_{Y_1} \subseteq I_{Y_2}$ nor $I_{Y_1} \supseteq I_{Y_2}$ holds. Then x is a vertical point associated to f (i.e., $f^{-1}(x)$ is not finite).*

Proof. To verify the proposition, we may assume that x is not a vertical point associated to f. Then $f^{-1}(x)$ is a finite set. Let $a \in \psi_2^{-1}(x)$ and $b \in \psi_1^{-1}(a)$. Thus, ψ_1 induces an isomorphism $\operatorname{Spec} \mathcal{O}_{\mathscr{Y},b} \to \operatorname{Spec} \mathcal{O}_{\mathscr{Y}',a}$. Write y for $\psi_1^{-1}(y')_{\mathrm{red}}$. By replacing $\mathscr{X}$ by the quotient $\mathscr{Y}/D_y$ and G by $D_y \subseteq G$, respectively, where $D_y \subseteq G$ denotes the decomposition group of y, we may assume $f^{-1}(x)_{\mathrm{red}} = \{y\} \subseteq Y_1 \cap Y_2$.

Consider the quotient curve $\mathscr{Y}/I_{Y_1}$ (resp. $\mathscr{Y}/I_{Y_2}$) over S. Note that $\mathscr{Y}/I_{Y_1}$ (resp. $\mathscr{Y}/I_{Y_2}$) is a pointed semistable curve over S. We obtain the following morphisms of pointed semistable curves

$$\lambda_1 : \mathscr{Y} \to \mathscr{Y}/I_{Y_1} \quad (\text{resp. } \lambda_2 : \mathscr{Y} \to \mathscr{Y}/I_{Y_2}),$$
$$\mu_1 : \mathscr{Y}/I_{Y_1} \to \mathscr{X} \quad (\text{resp. } \mu_2 : \mathscr{Y}/I_{Y_2} \to \mathscr{X})$$

over S such that $\mu_1 \circ \lambda_1 = f$ (resp. $\mu_2 \circ \lambda_2 = f$). Note that μ_1 (resp. μ_2) is étale at the generic point of $\lambda_1(Y_1)$ (resp. $\lambda_2(Y_2)$) of degree $\#G/\#I_{Y_1}$ (resp. $\#G/\#I_{Y_2}$).

If μ_1 (resp. μ_2) is also generically étale at the generic point of $\lambda_1(Y_2)$ (resp. $\lambda_2(Y_1)$), then, by applying [Tamagawa 2004b, Lemma 2.1(iii)] to

$$\operatorname{Spec} \widehat{\mathcal{O}}_{\mathscr{Y}/I_{Y_1},\lambda_1(y)} \to \operatorname{Spec} \widehat{\mathcal{O}}_{\mathscr{X},x} \quad (\text{resp. } \operatorname{Spec} \widehat{\mathcal{O}}_{\mathscr{Y}/I_{Y_2},\lambda_2(y)} \to \operatorname{Spec} \widehat{\mathcal{O}}_{\mathscr{X},x}),$$

we obtain that $\operatorname{Spec} \widehat{\mathcal{O}}_{\lambda_1(Y_1),\lambda_1(y)} \to \operatorname{Spec} \widehat{\mathcal{O}}_{X_1,x}$ (resp. $\operatorname{Spec} \widehat{\mathcal{O}}_{\lambda_2(Y_2),\lambda_2(y)} \to \operatorname{Spec} \widehat{\mathcal{O}}_{X_2,x}$) induced by μ_1 (resp. μ_2) is tamely ramified with ramification index t_1 (resp. t_2). Thus, we have $(t_1, p) = 1$ (resp. $(t_2, p) = 1$). On the other hand, since I_{Y_1} (resp. I_{Y_2}) does not contain I_{Y_2} (resp. I_{Y_1}), and I_{Y_2} (resp. I_{Y_1}) is a p-group, we have $p \mid t_1$ (resp. $p \mid t_2$). This is a contradiction. Thus, μ_1 (resp. μ_2) is not generically étale at the generic point of $\lambda_1(Y_2)$ (resp. $\lambda_2(Y_1)$). Thus, the morphism $\operatorname{Spec} \widehat{\mathcal{O}}_{\lambda_1(Y_1),\lambda_1(y)} \to \operatorname{Spec} \widehat{\mathcal{O}}_{X_1,x}$ (resp. $\operatorname{Spec} \widehat{\mathcal{O}}_{\lambda_2(Y_2),\lambda_2(y)} \to \operatorname{Spec} \widehat{\mathcal{O}}_{X_2,x}$) induced by μ_1 (resp. μ_2) is wildly ramified. Lemma 1.11 implies that x is a vertical point associated to f. This contradicts our assumptions. We complete the proof of the proposition. □

The following corollary follows immediately from Lemma 1.10 and Proposition 1.12.

Corollary 1.13. *Let G be a finite group, $f : \mathscr{Y} = (Y, D_Y) \to \mathscr{X}$ a G-pointed semistable covering over S, $\mathscr{Y}_s = (Y_s, D_{Y_s})$ the special fiber of $\mathscr{Y}$, and $y \in \mathscr{Y}_s$ a node. Let Y_1 and Y_2 (which may be equal) be the irreducible components of $\mathscr{Y}_s$ containing y. Write $I_y \subseteq G$ (resp. $I_{Y_1} \subseteq G$, $I_{Y_2} \subseteq G$) for the inertia subgroup of y (resp. Y_1, Y_2). Suppose that f is a* ***finite*** *morphism. Then either $I_{Y_1} \subseteq I_{Y_2}$ or $I_{Y_1} \supseteq I_{Y_2}$ holds. Moreover, if G is a p-group, then the inertia subgroup I_y is equal to either I_{Y_1} or I_{Y_2}.*

2. Semigraphs with *p*-rank

In this section, we develop the theory of semigraphs with p-rank. The main result of the present section is Theorem 2.7.

2A. *Semigraphs with p-rank and their coverings.*

2A1. We define semigraphs with p-rank as follows:

Definition 2.1. Let $\mathbb{G}$ be a semigraph (Section 1A1) and $\sigma_{\mathfrak{G}} : v(\mathbb{G}) \to \mathbb{Z}$ a map. We shall call the pair $\mathfrak{G} \overset{\text{def}}{=} (\mathbb{G}, \sigma_{\mathfrak{G}})$ a *semigraph with p-rank*. Moreover, we call that the semigraph $\mathbb{G}$ is the underlying semigraph of $\mathfrak{G}$, and that the map $\sigma_{\mathfrak{G}}$ is the p-rank map of $\mathfrak{G}$. We define the *p-rank* $\sigma(\mathfrak{G})$ of $\mathfrak{G}$ to be

$$\sigma(\mathfrak{G}) \overset{\text{def}}{=} \sum_{v \in v(\mathbb{G})} \sigma_{\mathfrak{G}}(v) + \gamma_{\mathbb{G}}.$$

A *morphism* of semigraphs with p-rank $\mathfrak{b} : \mathfrak{G}^1 \to \mathfrak{G}^2$ is defined by a morphism of the underlying semigraphs $\beta : \mathbb{G}^1 \to \mathbb{G}^2$. We shall refer to the morphism β as the underlying morphism of $\mathfrak{b}$.

A semigraph with p-rank is called connected if the underlying semigraph $\mathbb{G}$ is a connected semigraph.

Remark 2.1.1. We explain the geometric motivation of the above definitions. Let $\mathscr{X} \overset{\text{def}}{=} (X, D_X)$ be a pointed semistable curve over an algebraically closed field of characteristic $p > 0$. Write $\Gamma_{\mathscr{X}}$ for the dual semigraph (Section 1B2) of $\mathscr{X}$ and we define $\sigma_{\Gamma_{\mathscr{X}}}(v)$, $v \in v(\Gamma_{\mathscr{X}})$, to be the p-rank (Section 1B3) of the normalization of the irreducible component X_v corresponding to v. Then $(\Gamma_{\mathscr{X}}, \sigma_{\Gamma_{\mathscr{X}}})$ is a semigraph with p-rank. On the other hand, a semigraph with p-rank $\mathfrak{G} \overset{\text{def}}{=} (\mathbb{G}, \sigma_{\mathfrak{G}})$ is not arose from a pointed semistable curve in positive characteristic in general since $\sigma_{\mathfrak{G}}$ can attain *negative* integers.

2A2. *Settings.* Let G be a finite p-group of order p^r.

2A3. Let $\mathfrak{b} : \mathfrak{G}^1 \overset{\text{def}}{=} (\mathbb{G}^1, \sigma_{\mathfrak{G}^1}) \to \mathfrak{G}^2 \overset{\text{def}}{=} (\mathbb{G}^2, \sigma_{\mathfrak{G}^2})$ be a morphism of semigraphs with p-rank and $\beta : \mathbb{G}^1 \to \mathbb{G}^2$ the underlying morphism of $\mathfrak{b}$.

Definition 2.2. (a) We shall call that $\mathfrak{b}$ is *p-étale* (resp. *purely inseparable*) at an edge $e \in e(\mathbb{G}^1)$ if $\#\beta^{-1}(\beta(e)) = p$ (resp. $\#\beta^{-1}(\beta(e)) = 1$). We shall call that $\mathfrak{b}$ is *p-generically étale* at $v \in v(\mathbb{G}^1)$ if one of the following conditions holds (see Section 1A1 for $e(v)$):

(Type-I) $\#\beta^{-1}(\beta(v)) = p$ and $\sigma_{\mathfrak{G}^1}(v) = \sigma_{\mathfrak{G}^2}(\beta(v))$.

(Type-II) $\#\beta^{-1}(\beta(v)) = 1$ and

$$\sigma_{\mathfrak{G}^1}(v) - 1 = p(\sigma_{\mathfrak{G}^2}(\beta(v)) - 1) + \sum_{e \in e(v)} \left(\frac{p}{\#\beta^{-1}(\beta(e))} - 1 \right).$$

(b) We shall call that $\mathfrak{b}$ is *purely inseparable* at $v \in v(\mathbb{G}^1)$ if $\#\beta^{-1}(\beta(v)) = 1$, $\mathfrak{b}$ is purely inseparable at each element of $e(v)$, and $\sigma_{\mathfrak{G}^1}(v) = \sigma_{\mathfrak{G}^2}(\beta(v))$.

(c) We shall call that $\mathfrak{b}$ is a *p-covering* if the following conditions hold (see Section 1A1 for $v(e)$):

(i) There exists a $\mathbb{Z}/p\mathbb{Z}$-action (which may be trivial) on $\mathbb{G}^1$ and a trivial $\mathbb{Z}/p\mathbb{Z}$-action on $\mathbb{G}^2$ such that the underlying morphism β of $\mathfrak{b}$ is compatible with the $\mathbb{Z}/p\mathbb{Z}$-actions.

(ii) The natural morphism $\mathbb{G}^1/(\mathbb{Z}/p\mathbb{Z}) \to \mathbb{G}^2$ induced by β is an isomorphism, where $\mathbb{G}^1/(\mathbb{Z}/p\mathbb{Z})$ denotes the quotient semigraph.

(iii) For each $v \in v(\mathbb{G}^1)$, $\mathfrak{b}$ is either p-generically étale or purely inseparable at v.

(iv) Let $e \in e^{\mathrm{cl}}(\mathbb{G}^1)$ and $v(e) = \{v, v'\}$ (note that $v = v'$ if and only if e is a loop (Section 1A1)). Suppose that $\mathfrak{b}$ is p-generically étale at v and v'. Then $\mathfrak{b}$ is p-étale at e.

(v) For each $v \in v(\mathbb{G}^1)$, then $\sigma_{\mathfrak{G}^1}(v) = \sigma_{\mathfrak{G}^1}(\tau(v))$ for each $\tau \in \mathbb{Z}/p\mathbb{Z}$.

Note that the definition of p-coverings implies that the identity morphism of a semigraph with p-rank is a p-covering.

(d) We shall call that $\mathfrak{b}$ is a *covering* if $\mathfrak{b}$ is a composite of p-coverings.

(e) We maintain the notation introduced in Section 2A2. We shall call

$$\Phi : \{1\} = G_r \subset G_{r-1} \subset \cdots \subset G_1 \subset G_0 = G$$

a *maximal normal filtration* of G if G_j is a normal subgroup of G and $G_j/G_{j+1} \cong \mathbb{Z}/p\mathbb{Z}$ for $j \in \{0, \dots, r-1\}$. Note that since G is a p-group, a maximal normal filtration of G exists.

Suppose that $\mathbb{G}^1$ admits a G-action (which may be trivial), that $\mathbb{G}^2$ admits a trivial G-action, and that the underlying morphism β of $\mathfrak{b}$ is compatible with the G-actions. A maximal normal filtration Φ of G induces a sequence of semigraphs:

$$\mathbb{G}^1 = \mathbb{G}_r \xrightarrow{\beta_r} \mathbb{G}_{r-1} \xrightarrow{\beta_{r-1}} \cdots \xrightarrow{\beta_1} \mathbb{G}_0,$$

where $\mathbb{G}_j$, $j \in \{0, \dots, r\}$, denotes the quotient semigraph $\mathbb{G}^1/G_j$. We shall call that $\mathfrak{b}$ is a *G-covering* if there exist a maximal normal filtration Φ of G and a set of p-coverings $\{\mathfrak{b}_j : \mathfrak{G}_j \to \mathfrak{G}_{j-1},\ j = 1, \dots, r\}$ such that the following conditions are satisfied:

(i) The underlying semigraph of $\mathfrak{G}_j$ is equal to $\mathbb{G}_j$ for $j \in \{0, \dots, r\}$ such that $\mathbb{G}_0 = \mathbb{G}^2$.

(ii) The underlying morphism of $\mathfrak{b}_j$ is equal to β_j for $j \in \{1, \dots, r\}$.

(iii) The composite morphism $\mathfrak{b}_1 \circ \cdots \circ \mathfrak{b}_r$ is equal to $\mathfrak{b}$.

(f) Let $\mathfrak{b} : \mathfrak{G}^1 \to \mathfrak{G}^2$ be a G-covering. By the above definition of G-coverings, we obtain a maximal normal filtration Φ of G and a sequence of p-coverings:

$$\Phi_{\mathfrak{G}^1/\mathfrak{G}^2} : \mathfrak{G}^1 = \mathfrak{G}_r \xrightarrow{\mathfrak{b}_r} \mathfrak{G}_{r-1} \xrightarrow{\mathfrak{b}_{r-1}} \cdots \xrightarrow{\mathfrak{b}_1} \mathfrak{G}_0 = \mathfrak{G}^2.$$

We shall call $\Phi_{\mathfrak{G}^1/\mathfrak{G}^2}$ *a sequence of p-coverings induced by* Φ.

Remark 2.2.1. We explain the geometric motivation of the above definitions. Let R be a discrete valuation ring with algebraically closed residue field of characteristic $p > 0$, and let $f : \mathscr{Y} \overset{\text{def}}{=} (Y, D_Y) \to \mathscr{X} \overset{\text{def}}{=} (X, D_X)$ be a *finite* G-pointed semistable covering over R (Definition 1.5). Write $(\Gamma_{\mathscr{Y}_s}, \sigma_{\Gamma_{\mathscr{Y}_s}})$ and $(\Gamma_{\mathscr{X}_s}, \sigma_{\Gamma_{\mathscr{X}_s}})$ for the semigraphs with p-rank associated to the special fibers $\mathscr{Y}_s$ and $\mathscr{X}_s$ of $\mathscr{Y}$ and $\mathscr{X}$ (see Remark 2.1.1), respectively. Then the morphism of special fibers induced by f induces a G-covering $(\Gamma_{\mathscr{Y}_s}, \sigma_{\Gamma_{\mathscr{Y}_s}}) \to (\Gamma_{\mathscr{X}_s}, \sigma_{\Gamma_{\mathscr{X}_s}})$ (see Section 3A).

On the other hand, the definitions of p-étale, purely inseparable, p-generically étale, purely inseparable, and p-coverings of semigraphs with p-rank are motivated by p-étale, purely inseparable, p-generically étale, purely inseparable, and p-coverings of special fibers of finite $\mathbb{Z}/p\mathbb{Z}$-pointed semistable coverings over R. In particular, Definition 2.2(a-Type-II) is motivated by the Deuring-Shafarevich formula (see Proposition 1.4), and Definition 2.2(c-iv) is motivated by the Zariski–Nagata purity theorem of finite $\mathbb{Z}/p\mathbb{Z}$-pointed semistable coverings over R.

2A4. Let $\mathfrak{b} : \mathfrak{G}^1 \to \mathfrak{G}^2$ be a G-covering, $\beta : \mathbb{G}^1 \to \mathbb{G}^2$ the underlying morphism of $\mathfrak{b}$, $v^1 \in v(\mathbb{G}^1)$, and $e^1 \in e(\mathbb{G}^1)$. By the definition of G-coverings, we have a maximal normal filtration Φ of G and a sequence of p-coverings induced by Φ:

$$\Phi_{\mathfrak{G}^1/\mathfrak{G}^2} : \mathfrak{G}^1 = \mathfrak{G}_r \xrightarrow{\mathfrak{b}_r} \mathfrak{G}_{r-1} \xrightarrow{\mathfrak{b}_{r-1}} \cdots \xrightarrow{\mathfrak{b}_1} \mathfrak{G}_0 = \mathfrak{G}^2.$$

Write $\beta_j : \mathbb{G}_j \to \mathbb{G}_{j-1}$, $j \in \{1, \dots, r\}$, for the underlying morphism of $\mathfrak{b}_j$. Write v_j (resp. e_j) for the image $\beta_{j+1} \circ \cdots \circ \beta_r(v^1)$ (resp. $\beta_{j+1} \circ \cdots \circ \beta_r(e^1)$), $j \in \{0, \dots, r-1\}$, and v_r for v^1. We put

$$\#I_{v^1} = p^{\#\{j \in \{1,\dots,r\} \mid \mathfrak{b}_j \text{ is purely inseparable at } v_j\}} \quad \text{and} \quad \#I_{e^1} = p^{\#\{j \in \{1,\dots,r\} \mid \mathfrak{b}_j \text{ is purely inseparable at } e_j\}}.$$

Note that $\#I_{v^1}$ and $\#I_{e^1}$ do not depend on the choice of Φ. Moreover, we put $D_{v^1} \overset{\text{def}}{=} \{\tau \in G \mid \tau(v^1) = v^1\}$, and

$$\#D_{v^1}$$

the cardinality of D_{v^1}.

2A5. We maintain the notation introduced in Section 2A4. If $e^1 \in e(v^1)$, then we have $\#I_{v^1} \mid \#I_{e^1}$. In particular, if e^1 is a loop, then Definition 2.2(c-iv) implies that $\#I_{v^1} = \#I_{e^1}$. Moreover, Definition 2.2 (c-iv) also implies that $\#I_{e^1} \mid \#D_{v^1}$. Write v^2 (resp. e^2) for $\beta(v^1)$ (resp. $\beta(e^1)$). Let $(v^1)'$ (resp. $(e^1)'$) be an arbitrary element of $\beta^{-1}(v^2)$ (resp. $\beta^{-1}(e^2)$). By the action of G on $\mathbb{G}^1$, we have $\#I_{v^1} = \#I_{(v^1)'}$, $\#I_{e^1} = \#I_{(e^1)'}$, and $\#D_{v^1} = \#D_{(v^1)'}$. Thus, we may use the notation $\#I_{v^2}$ (resp. $\#I_{e^2}$, $\#D_{v^2}$) to denote $\#I_{v^1}$ (resp. $\#I_{e^1}$, $\#D_{v^1}$). Namely, $\#I_{v^1}$ (resp. $\#I_{e^1}$, $\#D_{v^1}$) does not depend on the choice of $v^1 \in \beta^{-1}(\beta(v^1))$. Then we have $\#I_{v^2} \mid \#I_{e^2} \mid \#D_{v^2}$.

2A6. We maintain the notation introduced in Sections 2A4 and 2A5. One may compute the p-rank $\sigma_{\mathfrak{G}^1}(v^1)$ by using Definition 2.2(a). Then we have the following Deuring–Shafarevich type formula for the p-rank of G-coverings (see Proposition 1.4 for the Deuring–Shafarevich formula for curves)

$$\begin{aligned}\sigma_{\mathfrak{G}^1}(v^1) - 1 &= (\#D_{v^2}/\#I_{v^2})(\sigma_{\mathfrak{G}^2}(v^2) - 1) + \sum_{e^2 \in e(v^2)} (\#D_{v^2}/\#I_{e^2})(\#I_{e^2}/\#I_{v^2} - 1)\\ &= (\#D_{v^2}/\#I_{v^2})(\sigma_{\mathfrak{G}^2}(v^2) - 1) + \sum_{e^2 \in e(v^2) \setminus e^{\mathrm{lp}}(v^2)} (\#D_{v^2}/\#I_{e^2})(\#I_{e^2}/\#I_{v^2} - 1).\end{aligned}$$

Here, the second equality follows from Definition 2.2(c-iv).

2B. ***An operator concerning coverings.*** In this subsection, we introduce an operator (or a deformation) concerning coverings of semigraphs with p-rank which is a key in our computations of p-rank.

2B1. *Settings.* We fix some notation. Let G be a finite p-group of order p^r, and let $\mathfrak{b} : \mathfrak{G}^1 \overset{\text{def}}{=} (\mathbb{G}^1, \sigma_{\mathfrak{G}^1}) \to \mathfrak{G}^2 \overset{\text{def}}{=} (\mathbb{G}^2, \sigma_{\mathfrak{G}^2})$ be a covering of semigraphs with p-rank (Definition 2.2(d)) and $\beta : \mathbb{G}^1 \to \mathbb{G}^2$ the underlying morphism of $\mathfrak{b}$ (Definition 2.1). We put

$$V^1 \overset{\text{def}}{=} \{v \in v(\mathbb{G}^1) \mid \#\beta^{-1}(\beta(v^1)) = 1\} \subseteq v(\mathbb{G}^1) \quad \text{and} \quad V^2 \overset{\text{def}}{=} \beta(V^1) \subseteq v(\mathbb{G}^2).$$

Moreover, we suppose that $\mathbb{G}^1$, $\mathbb{G}^2$ are *connected*, that $\mathbb{G}^1$ (resp. $\mathbb{G}^2$) admits an action (resp. a trivial action) of G such that β is a G-equivariant, and that $\mathbb{G}^1/G = \mathbb{G}^2$.

2B2. Let $v^2 \in v(\mathbb{G}^2)$ and $v^1 \in \beta^{-1}(v^2)$. Firstly, we define a new semigraph $\mathbb{G}^1_{v^2}$ associated to v^2 as follows (see Example 2.3 below):

(a) Suppose $v^2 \in V^2$. We put $\mathbb{G}^1_{v^2} \overset{\text{def}}{=} \mathbb{G}^1$.

(b) Suppose $v^2 \notin V^2$. We have the following:

(i) $v(\mathbb{G}^1_{v^2}) \overset{\text{def}}{=} (v(\mathbb{G}^1) \setminus \beta^{-1}(v^2)) \sqcup \{v^2_\star\}$, $e^{\text{cl}}(\mathbb{G}^1_{v^2}) \overset{\text{def}}{=} e^{\text{cl}}(\mathbb{G}^1)$, and $e^{\text{op}}(\mathbb{G}^1_{v^2}) \overset{\text{def}}{=} e^{\text{op}}(\mathbb{G}^1)$, where $v^2_\star$ is a new vertex and $\sqcup$ means disjoint union.

(ii) The collection of maps $\{\zeta_e^{\mathbb{G}^1_{v^2}}\}_e$ is as follows:

(1) For each $e \in e^{\text{op}}(\mathbb{G}^1_{v^2}) \overset{\text{def}}{=} e^{\text{op}}(\mathbb{G}^1)$ and $b \in e$ (i.e., a branch of e, see Section 1A1), we put

$$\zeta_e^{\mathbb{G}^1_{v^2}}(b) = \begin{cases} \{v(\mathbb{G}^1_{v^2})\} & \text{if } \zeta_e^{\mathbb{G}^1}(b) = \{v(\mathbb{G}^1)\}, \\ v^2_\star & \text{if } \zeta_e^{\mathbb{G}^1}(b) \in \beta^{-1}(v^2), \\ \zeta_e^{\mathbb{G}^1}(b) & \text{otherwise.} \end{cases}$$

(2) For each $e \in e^{\text{cl}}(\mathbb{G}^1_{v^2}) \overset{\text{def}}{=} e^{\text{cl}}(\mathbb{G}^1)$ and $b \in e$, we put

$$\zeta_e^{\mathbb{G}^1_{v^2}}(b) = \begin{cases} v^2_\star & \text{if } \zeta_e^{\mathbb{G}^1}(b) \in \beta^{-1}(v^2), \\ \zeta_e^{\mathbb{G}^1}(b) & \text{otherwise.} \end{cases}$$

Next, we define a morphism of semigraphs $\beta_{v^2} : \mathbb{G}^1_{v^2} \to \mathbb{G}^2$ as follows (see Example 2.3 below):

(i) For each $v \in v(\mathbb{G}^1_{v^2})$, we put

$$\beta_{v^2}(v) = \begin{cases} v^2 & \text{if } v = v^2_\star, \\ \beta(v) & \text{otherwise.} \end{cases}$$

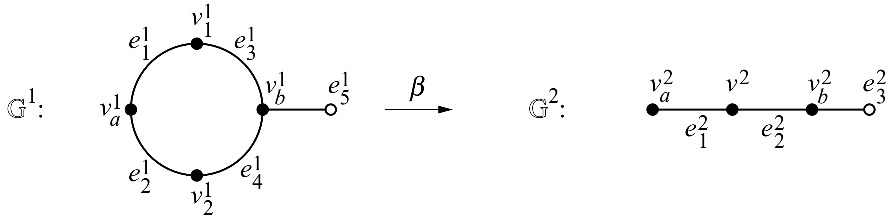
(ii) For each $e \in e(\mathbb{G}^1_{v^2}) = e^{\text{cl}}(\mathbb{G}^1_{v^2}) \cup e^{\text{op}}(\mathbb{G}^1_{v^2})$, we put $\beta_{v^2}(e) \overset{\text{def}}{=} \beta(e)$.

Example 2.3. We give an example to explain the above constructions. We use the notation "•" and "∘ with a line segment" to denote a vertex and an open edge, respectively.

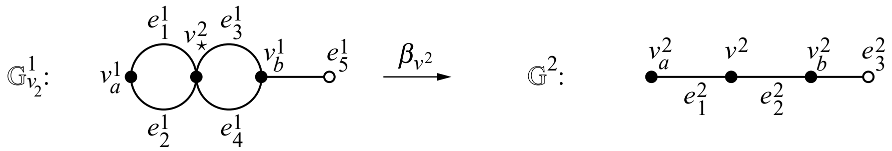
Let $p = 2$, and let $\mathbb{G}^1$, $\mathbb{G}^2$ be the semigraphs below. Moreover, let $\beta : \mathbb{G}^1 \to \mathbb{G}^2$ be a morphism of semigraphs such that

$$\beta(v^1_a) = v^2_a, \quad \beta(v^1_b) = v^2_b, \quad \beta(v^1_1) = \beta(v^1_2) = v^2, \quad \beta(e^1_1) = \beta(e^1_2) = e^2_1, \quad \beta(e^1_3) = \beta(e^1_4) = e^2_2, \quad \beta(e^1_5) = e^2_3.$$

Note that $\mathbb{G}^1$ admits an action of $\mathbb{Z}/2\mathbb{Z}$ such that $\mathbb{G}^1/(\mathbb{Z}/p\mathbb{Z}) = \mathbb{G}^2$. Then we have the following:



By the definitions of $\mathbb{G}^1_{v^2}$ and β_{v^2}, we have the following:



2B3. We maintain the notation introduced in Section 2B2. Next, we define a p-rank map $\sigma_{\mathfrak{G}^1_{v^2}} : v(\mathbb{G}^1_{v^2}) \to \mathbb{Z}$ for $\mathbb{G}^1_{v^2}$ as follows:

(a) Suppose $v^2 \in V^2$. We put $\sigma_{\mathfrak{G}^1_{v^2}} \stackrel{\text{def}}{=} \sigma_{\mathfrak{G}^1}$.

(b) Suppose $v^2 \notin V^2$. Let $v \in v(\mathbb{G}^1_{v^2})$. We have the following:

(i) If $v \neq v^2_\star$, we put $\sigma_{\mathfrak{G}^1_{v^2}}(v) \stackrel{\text{def}}{=} \sigma_{\mathfrak{G}^1}(v)$.

(ii) If $v = v^2_\star$, we put (see Section 1A1 for $e(v^2)$ and Section 2A6 for $\#I_{v^2}$, $\#I_e$)

$$\sigma_{\mathfrak{G}^1_{v^2}}(v^2_\star) \stackrel{\text{def}}{=} (\#G/\#I_{v^2})(\sigma_{\mathfrak{G}^2}(v^2) - 1) + \sum_{e \in e(v^2)} (\#G/\#I_e)(\#I_e/\#I_{v^2} - 1) + 1.$$

2B4. We maintain the notation introduced in Sections 2B2 and 2B3. Let $v^2 \in v(\mathbb{G}^2)$. We define a semigraph with p-rank and a morphism of semigraphs with p-rank associated to $\mathfrak{b} : \mathfrak{G}^1 \to \mathfrak{G}^2$ and v^2, respectively, to be

$$\mathfrak{G}^1_{v^2} \stackrel{\text{def}}{=} (\mathbb{G}^1_{v^2}, \sigma_{\mathfrak{G}^1_{v^2}}), \quad \mathfrak{b}_{v^2} : \mathfrak{G}^1_{v^2} \to \mathfrak{G}^2,$$

where the underlying morphism of $\mathfrak{b}_{v^2}$ is β_{v^2}.

2B5. We maintain the settings introduced in Section 2B1. Let $\mathfrak{G}^i \setminus \{V^i\}$, $i \in \{1, 2\}$, be the (possibly nonconnected) semigraph with p-rank whose underlying semigraph is $\mathbb{G}^i \setminus \{V^i\}$ (in the sense of Section 1A2(b)), and whose p-rank map is $\sigma_{\mathfrak{G}^i}|_{v(\mathbb{G}^i \setminus \{V^i\})}$. We shall call $\mathfrak{b} : \mathfrak{G}^1 \to \mathfrak{G}^2$ a *quasi-G-covering* if the covering $\mathfrak{G}^1 \setminus \{V^1\} \to \mathfrak{G}^2 \setminus \{V^2\}$ induced by $\mathfrak{b}$ is a G-covering.

Definition 2.4. Let $\mathfrak{b} : \mathfrak{G}^1 \to \mathfrak{G}^2$ be a quasi-G-covering of connected semigraphs with p-rank and $v^2 \in v(\mathbb{G}^2)$. We define an operator $\rightleftharpoons^{\text{I}}_{\text{II}} [v^2]$ on $\mathfrak{b} : \mathfrak{G}^1 \to \mathfrak{G}^2$ to be

$$\rightleftharpoons^{\text{I}}_{\text{II}} [v^2](\mathfrak{b} : \mathfrak{G}^1 \to \mathfrak{G}^2) \stackrel{\text{def}}{=} \mathfrak{b}_{v^2} : \mathfrak{G}^1_{v^2} \to \mathfrak{G}^2.$$

Here $\rightleftharpoons^{\text{I}}_{\text{II}}$ means that "from (Type-I) to (Type-II)" in the sense of Definition 2.2(a).

Remark 2.4.1. Suppose that $\mathfrak{b}:\mathfrak{G}^1\to\mathfrak{G}^2$ is a G-covering of semigraphs with p-rank. Then $\sigma_{\mathfrak{G}^1_{v^2}}(v^2_\star)$ is not contained in $\mathbb{Z}_{\geq 0}$ in general. Thus, $\mathfrak{b}_{v^2}:\mathfrak{G}^1_{v^2}\to\mathfrak{G}^2$ cannot be arose from a G-pointed semistable covering in general (see also Remark 2.2.1). On the other hand, in the next subsection, we will see (Proposition 2.6 below) that the operator defined above *does not change* global p-rank (i.e., $\sigma(\mathfrak{G}^1_{v^2})=\sigma(\mathfrak{G}^1)$).

2B6. Let $\mathfrak{b}:\mathfrak{G}^1\to\mathfrak{G}^2$ be a quasi-G-covering and $v^2\in v(\mathbb{G}^2)$. Then the semigraph with p-rank $\mathbb{G}^1_{v^2}$ admits a natural G-action as follows:

(1) The action of G on $v(\mathbb{G}^1_{v^2}\setminus\{v^2_\star\})=v(\mathbb{G}^1)\setminus\beta^{-1}(v^2)$ (resp. $e(\mathbb{G}^1_{v^2})=e(\mathbb{G}^1)$) is the action of G on $v(\mathbb{G}^1)\setminus\beta^{-1}(v^2)$ (resp. $e(\mathbb{G}^1)$) induced by the action of G on $\mathbb{G}^1$.

(2) The action of G on $v^2_\star$ is a trivial action.

We see immediately that $\mathfrak{b}_{v^2}:\mathfrak{G}^1_{v^2}\to\mathfrak{G}^2$ is a quasi-G-covering.

Let $\mathfrak{b}:\mathfrak{G}^1\to\mathfrak{G}^2$ be a G-covering. Suppose that G is an *abelian* p-group. Then together with the G-action defined above, it is easy to check that $\mathfrak{b}_{v^2}:\mathfrak{G}^1_{v^2}\to\mathfrak{G}^2$ is a G-covering.

On the other hand, if G is *not abelian*, then $\mathfrak{b}_{v^2}:\mathfrak{G}^1_{v^2}\to\mathfrak{G}^2$ is *not* a G-covering in general for the following reason. Let $w\stackrel{\text{def}}{=}v^2_\star=\beta^{-1}_{v^2}(v^2)$. With the action of G on $\mathfrak{G}^1_{v^2}$ defined above, if I_{v^1}, $v^1\in\beta^{-1}(v^2)$, is not a normal subgroup of G, then the order $\#I_w$ of the inertia subgroup I_w of w is not equal to $\#I_{v^2}\stackrel{\text{def}}{=}\#I_{v^1}$ (Section 2A5) in general. If $\mathfrak{b}_{v^2}$ is a G-covering, we have (Section 2A6)

$$\sigma_{\mathfrak{G}^1_{v^2}}(w)=(\#G/\#I_w)(\sigma_{\mathfrak{G}^2}(v^2)-1)+\sum_{e\in e(v^2)}(\#G/\#I_e)(\#I_e/\#I_w-1)+1$$

which is not equal to (Section 2B3(b-ii))

$$\#G/\#I_{v^2}(\sigma_{\mathfrak{G}^2}(v^2)-1)+\sum_{e\in e(v^2)}\#G/\#I_e(\#I_e/\#I_{v^2}-1)+1$$

in general if $\#I_w\neq\#I_{v^2}$. This contradicts the definition of $\mathfrak{G}^1_{v^2}$. Thus, $\mathfrak{b}_{v^2}:\mathfrak{G}^1_{v^2}\to\mathfrak{G}^2$ is not a G-covering in general.

2C. *Formula for p-rank of coverings.* In this subsection, we give an explicit formula (i.e., Theorem 2.7) for the p-rank of G-coverings of semigraphs with p-rank.

2C1. *Settings.* We maintain the settings introduced in Section 2B1. Moreover, we assume that $\mathfrak{b}:\mathfrak{G}^1\stackrel{\text{def}}{=}(\mathbb{G}^1,\sigma_{\mathfrak{G}^1})\to\mathfrak{G}^2\stackrel{\text{def}}{=}(\mathbb{G}^2,\sigma_{\mathfrak{G}^2})$ is a quasi-G-covering (Section 2B5).

2C2. Firstly, we have the following lemma.

Lemma 2.5. *Let $i\in\{1,\dots,n\}$, and let $\mathbb{G}$ be a connected semigraph, $\mathbb{G}_i$ a connected subsemigraph of $\mathbb{G}$ 1A2, and $v_i\in v(\mathbb{G}_i)$ a vertex of $\mathbb{G}_i$. Suppose $\mathbb{G}_s\cap\mathbb{G}_t=\varnothing$ for each $s,t\in\{1,\dots,n\}$ if $s\neq t$. Let $\mathbb{G}^{\mathrm{c}}$ be a semigraph defined as follows*:

(i) $v(\mathbb{G}^c) = v(\mathbb{G}) \sqcup \{v^c\}$, $e^{\mathrm{op}}(\mathbb{G}^c) = e^{\mathrm{op}}(\mathbb{G})$, $e^{\mathrm{cl}}(\mathbb{G}^c) = e^{\mathrm{cl}}(\mathbb{G}) \sqcup \{e_i^c\}_{i\in\{1,\dots,n\}}$.

(ii) *Let* $e \in e(\mathbb{G}^c) \setminus \{e_i^c\}_{i\in\{1,\dots,n\}} = e(\mathbb{G})$ *and* $b \in e$ *a branch of* e (*Section 1A1*). *We put*

$$\zeta_e^{\mathbb{G}^c}(b) = \begin{cases} \zeta_e^{\mathbb{G}}(b) & \text{if } \zeta_e^{\mathbb{G}}(b) \neq \{v(\mathbb{G})\}, \\ \{v(\mathbb{G}^c)\} & \text{if } \zeta_e^{\mathbb{G}}(b) = \{v(\mathbb{G})\}. \end{cases}$$

(iii) *Let* $e_i^c = \{b^1_{e_i^c}, b^2_{e_i^c}\}$. *We put* $\zeta^{\mathbb{G}^c}_{e_i^c}(b^1_{e_i^c}) = v_i$, $\zeta^{\mathbb{G}^c}_{e_i^c}(b^2_{e_i^c}) = v^c$.

Then we have (*see Section 1A3 for* $\gamma_{\mathbb{G}}$, $\gamma_{\mathbb{G}^c}$)

$$\gamma_{\mathbb{G}} = \gamma_{\mathbb{G}^c} - n + 1.$$

Proof. The lemma follows from the construction of $\mathbb{G}^c$. □

2C3. We have the following key proposition which says that the operator introduced in Definition 2.4 does not change the p-rank of semigraphs with p-rank.

Proposition 2.6. *We maintain the settings introduced in Section 2C1. Let* $v^2 \in v(\mathbb{G}^2)$ *be an arbitrary vertex of* $\mathbb{G}^2$ *and* $\rightleftharpoons^{\mathrm{I}}_{\mathrm{II}} [v^2](\mathfrak{b} : \mathfrak{G}^1 \to \mathfrak{G}^2) = \mathfrak{b}_{v^2} : \mathfrak{G}^1_{v^2} \to \mathfrak{G}^2$ (*Definition 2.4*). *Then we have*

$$\sigma(\mathfrak{G}^1) = \sigma(\mathfrak{G}^1_{v^2}),$$

where $\sigma(\mathfrak{G}^1)$ *and* $\sigma(\mathfrak{G}^1_{v^2})$ *are the* p-*rank of* $\mathfrak{G}^1$ *and* $\mathfrak{G}^1_{v^2}$, *respectively, defined in Definition 2.1.*

Proof. Suppose $\#\beta^{-1}(v^2) = 1$ (i.e., $v^2 \in V^2$). Then the proposition is trivial since $\mathfrak{G}^1 = \mathfrak{G}^1_{v^2}$. Thus, we may assume $\#\beta^{-1}(v^2) \neq 1$ (i.e., $v^2 \notin V^2$).

Write β_{v^2} for the underlying morphism of $\mathfrak{b}_{v^2}$. Moreover, we put

$$W \overset{\mathrm{def}}{=} \beta^{-1}(v^2),\ \ W^* \overset{\mathrm{def}}{=} \beta^{-1}_{v^2}(v^2) = \{v^2_\star\}.$$

For simplicity, we shall write γ (resp. $\gamma_{\backslash\{v^2\}}$, γ^*, $\gamma^*_{\backslash\{v^2\}}$) for the Betti number (Section 1A3) of $\mathbb{G}^1$ (resp. $\mathbb{G}^1 \setminus W$, $\mathbb{G}^1_{v^2}$, $\mathbb{G}^1_{v^2} \setminus W^*$), where $\mathbb{G}^1 \setminus W$ and $\mathbb{G}^1_{v^2} \setminus W^*$ are semigraphs defined in Section 1A2.

Then we have

$$\sigma(\mathfrak{G}^1) = \gamma_{\backslash\{v^2\}} + \gamma - \gamma_{\backslash\{v^2\}} + \sum_{v\in v(\mathbb{G}^1\setminus W)} \sigma_{\mathfrak{G}^1}(v) + \sum_{v\in W} \sigma_{\mathfrak{G}^1}(v),$$

$$\sigma(\mathfrak{G}^1_{v^2}) = \sigma_{\mathfrak{G}^1_{v^2}}(v^2_\star) + \gamma^*_{\backslash\{v^2\}} + \gamma^* - \gamma^*_{\backslash\{v^2\}} + \sum_{v\in v(\mathbb{G}^1_{v^2}\setminus W^*)} \sigma_{\mathfrak{G}^1_{v^2}}(v).$$

Note that the construction of $\mathfrak{G}^1_{v^2}$ (Sections 2B2, 2B3) implies

$$A \overset{\mathrm{def}}{=} \sum_{v\in v(\mathbb{G}^1\setminus W)} \sigma_{\mathfrak{G}^1}(v) = \sum_{v\in v(\mathbb{G}^1_{v^2}\setminus W^*)} \sigma_{\mathfrak{G}^1_{v^2}}(v),\ \ B \overset{\mathrm{def}}{=} \gamma_{\backslash\{v^2\}} = \gamma^*_{\backslash\{v^2\}}.$$

We calculate $\gamma - \gamma_{\backslash\{v^2\}}$ and $\gamma^* - \gamma^*_{\backslash\{v^2\}}$. By applying Lemma 2.5, it is sufficient to treat the case where $\mathbb{G}^1 \setminus W = \mathbb{G}^1_{v^2} \setminus W^*$ is connected. Then we obtain (see Section 1A1 for $e(v^2)$, $e^{\mathrm{lp}}(v^2)$ and

Sections 2A4 and 2A5 for $\#D_{v^2}, \#I_{v^2}, \#I_e$)

$$\gamma - \gamma_{\setminus\{v^2\}} = (\#G/\#D_{v^2})\Bigg(\Bigg(\sum_{e\in(e(v^2)\cap e^{\mathrm{cl}}(\mathbb{G}^2))\setminus e^{\mathrm{lp}}(v^2)} \#D_{v^2}/\#I_e\Bigg) - 1\Bigg) + \#e^{\mathrm{lp}}(v^2)(\#G/\#I_{v^2}),$$

$$\gamma^* - \gamma^*_{\setminus\{v^2\}} = \Bigg(\sum_{e\in(e(v^2)\cap e^{\mathrm{cl}}(\mathbb{G}^2))\setminus e^{\mathrm{lp}}(v^2)} \#G/\#I_e\Bigg) - 1 + \#e^{\mathrm{lp}}(v^2)(\#G/\#I_{v^2}).$$

On the other hand, for each $v \in W \overset{\mathrm{def}}{=} \beta^{-1}(v^2)$, we have (Section 2A6)

$$\begin{aligned}\sigma_{\mathfrak{G}^1}(v) &= (\#D_{v^2}/\#I_{v^2})(\sigma_{\mathfrak{G}^2}(v^2) - 1) + \sum_{e\in e(v^2)} (\#D_{v^2}/\#I_e)(\#I_e/\#I_{v^2} - 1) + 1 \\ &= (\#D_{v^2}/\#I_{v^2})(\sigma_{\mathfrak{G}^2}(v^2) - 1) + \sum_{e\in e(v^2)\setminus e^{\mathrm{lp}}(v^2)} (\#D_{v^2}/\#I_e)(\#I_e/\#I_{v^2} - 1) + 1.\end{aligned}$$

Moreover, the construction of $\mathfrak{G}^1_{v^2}$ (Section 2B3) implies that

$$\begin{aligned}\sigma_{\mathfrak{G}^1_{v^2}}(v^2_\star) &= (\#G/\#I_{v^2})(\sigma_{\mathfrak{G}^2}(v^2) - 1) + \sum_{e\in e(v^2)} (\#G/\#I_e)(\#I_e/\#I_{v^2} - 1) + 1 \\ &= (\#G/\#I_{v^2})(\sigma_{\mathfrak{G}^2}(v^2) - 1) + \sum_{e\in e(v^2)\setminus e^{\mathrm{lp}}(v^2)} (\#G/\#I_e)(\#I_e/\#I_{v^2} - 1) + 1\end{aligned}$$

We obtain

$$\begin{aligned}\sigma(\mathfrak{G}^1) &= A + B + \sum_{v\in W} \sigma_{\mathfrak{G}^1}(v) + \gamma - \gamma_{\setminus\{v^2\}} \\ &= A + B + \sum_{v\in W}\Bigg((\#D_{v^2}/\#I_{v^2})(\sigma_{\mathfrak{G}^2}(v^2) - 1) + \sum_{e\in e(v^2)\setminus e^{\mathrm{lp}}(v^2)} (\#D_{v^2}/\#I_e)(\#I_e/\#I_{v^2} - 1) + 1\Bigg) \\ &\qquad + (\#G/\#D_{v^2})\Bigg(\Bigg(\sum_{e\in(e(v^2)\cap e^{\mathrm{cl}}(\mathbb{G}^2))\setminus e^{\mathrm{lp}}(v^2)} \#D_{v^2}/\#I_e\Bigg) - 1\Bigg) + \#e^{\mathrm{lp}}(v^2)(\#G/\#I_{v^2}) \\ &= A + B + (\#G/\#D_{v^2})\Bigg((\#D_{v^2}/\#I_{v^2})(\sigma_{\mathfrak{G}^2}(v^2) - 1) + \sum_{e\in e(v^2)\setminus e^{\mathrm{lp}}(v^2)} (\#D_{v^2}/\#I_e)(\#I_e/\#I_{v^2} - 1) + 1\Bigg) \\ &\qquad + (\#G/\#D_{v^2})\Bigg(\Bigg(\sum_{e\in(e(v^2)\cap e^{\mathrm{cl}}(\mathbb{G}^2))\setminus e^{\mathrm{lp}}(v^2)} \#D_{v^2}/\#I_e\Bigg) - 1\Bigg) + \#e^{\mathrm{lp}}(v^2)(\#G/\#I_{v^2}) \\ &= A + B + (\#G/\#I_{v^2})\sigma_{\mathfrak{G}^2}(v^2) - \#G/\#I_{v^2} + \sum_{e\in e(v^2)\setminus e^{\mathrm{lp}}(v^2)} \#G/\#I_{v^2} - \sum_{e\in e(v^2)\setminus e^{\mathrm{lp}}(v^2)} \#G/\#I_e \\ &\qquad + \sum_{e\in(e(v^2)\cap e^{\mathrm{cl}}(\mathbb{G}^2))\setminus e^{\mathrm{lp}}(v^2)} \#G/\#I_e + \#e^{\mathrm{lp}}(v^2)(\#G/\#I_{v^2}) \\ &= A + B + (\#G/\#I_{v^2})\sigma_{\mathfrak{G}^2}(v^2) - \#G/\#I_{v^2} + \sum_{e\in e(v^2)\setminus e^{\mathrm{lp}}(v^2)} \#G/\#I_{v^2} \\ &\qquad - \sum_{e\in(e(v^2)\cap e^{\mathrm{op}}(\mathbb{G}^2))\setminus e^{\mathrm{lp}}(v^2)} \#G/\#I_e + \#e^{\mathrm{lp}}(v^2)(\#G/\#I_{v^2}).\end{aligned}$$

Note that the last equality holds since we have

$$e(v^2)\setminus e^{\mathrm{lp}}(v^2) = ((e(v^2)\cap e^{\mathrm{op}}(\mathbb{G}^2))\setminus e^{\mathrm{lp}}(v^2)) \sqcup ((e(v^2)\cap e^{\mathrm{lp}}(\mathbb{G}^2))\setminus e^{\mathrm{lp}}(v^2)).$$

On the other hand, we obtain

$$\begin{aligned}\sigma(\mathfrak{G}^1_{v^2}) &= A + B + \sigma_{\mathfrak{G}^1}(v^2_\star) + \gamma^* - \gamma^*_{\setminus\{v^2\}}\\ &= A + B + (\#G/\#I_{v^2})(\sigma_{\mathfrak{G}^2}(v^2)-1) + \sum_{e\in e(v^2)\setminus e^{\mathrm{lp}}(v^2)} (\#G/\#I_e)(\#I_e/\#I_{v^2}-1)+1\\ &\qquad + \Bigg(\sum_{e\in(e(v^2)\cap e^{\mathrm{cl}}(\mathbb{G}^2))\setminus e^{\mathrm{lp}}(v^2)} \#G/\#I_e\Bigg) - 1 + \#e^{\mathrm{lp}}(v^2)(\#G/\#I_{v^2})\\ &= A + B + (\#G/\#I_{v^2})\sigma_{\mathfrak{G}^2}(v^2) - \#G/\#I_{v^2} + \sum_{e\in e(v^2)\setminus e^{\mathrm{lp}}(v^2)} \#G/\#I_{v^2}\\ &\qquad - \sum_{e\in(e(v^2)\cap e^{\mathrm{op}}(\mathbb{G}^2))\setminus e^{\mathrm{lp}}(v^2)} \#G/\#I_e + \#e^{\mathrm{lp}}(v^2)(\#G/\#I_{v^2}).\end{aligned}$$

Namely, we have

$$\sigma(\mathfrak{G}^1) = \sigma(\mathfrak{G}^1_{v^2}).$$

We complete the proof of the proposition. □

2C4. The main result of the present section is as follows:

Theorem 2.7. *Let $\mathfrak{b}:\mathfrak{G}^1\to\mathfrak{G}^2$ be a G-covering of connected semigraphs with p-rank (Definition 2.2(e)). Then we have (see Section 1A1 for $e(v)$, $e^{\mathrm{lp}}(v)$)*

$$\begin{aligned}\sigma(\mathfrak{G}^1) = \sum_{v\in v(\mathbb{G}^2)} \Bigg((\#G/\#I_v)(\sigma_{\mathfrak{G}^2}(v)-1) + \sum_{e\in e(v)\setminus e^{\mathrm{lp}}(v)} (\#G/\#I_e)(\#I_e/\#I_v - 1) + 1\Bigg)\\ + \sum_{e\in e^{\mathrm{cl}}(\mathbb{G}^2)\setminus e^{\mathrm{lp}}(\mathbb{G}^2)} (\#G/\#I_e - 1) + \sum_{v\in v(\mathbb{G}^2)} \#e^{\mathrm{lp}}(v)(\#G/\#I_v - 1) + \gamma_{\mathbb{G}^2}.\end{aligned}$$

Proof. By applying Proposition 2.6 and the operator $\rightleftharpoons^{\mathrm{I}}_{\mathrm{II}}$ (Definition 2.4), we may construct a quasi-G-covering $\mathfrak{b}^*:\mathfrak{G}^{1,*}\to\mathfrak{G}^2$ from $\mathfrak{b}$ such that the following conditions are satisfied:

(i) We have $\#(\beta^*)^{-1}(v)=1$ for each $v\in v(\mathbb{G}^2)$, where β^* denotes the underlying morphism of $\mathfrak{b}^*$.

(ii) For each $v\in v(\mathbb{G}^2)$ and $v^*\in(\beta^*)^{-1}(v)$, we have

$$\begin{aligned}\sigma_{\mathfrak{G}^{1,*}}(v^*) &= (\#G/\#I_v)(\sigma_{\mathfrak{G}^2}(v)-1) + \sum_{e\in e(v)} (\#G/\#I_e)(\#I_e/\#I_v-1)+1\\ &= (\#G/\#I_v)(\sigma_{\mathfrak{G}^2}(v)-1) + \sum_{e\in e(v)\setminus e^{\mathrm{lp}}(v)} (\#G/\#I_e)(\#I_e/\#I_v-1)+1.\end{aligned}$$

(iii) $\sigma(\mathfrak{G}^{1,*}) = \sigma(\mathfrak{G}^1)$.

Write $\mathbb{G}^{1,*}$ for the underlying semigraph of $\mathfrak{G}^{1,*}$. We observe that

$$\begin{aligned}\gamma_{\mathbb{G}^{1,*}} &= \gamma_{\mathbb{G}^2} + \sum_{e\in e^{\mathrm{cl}}(\mathbb{G}^2)\setminus e^{\mathrm{lp}}(\mathbb{G}^2)} (\#G/\#I_e - 1) - \sum_{v\in v(\mathbb{G}^2)} \#e^{\mathrm{lp}}(v) + \sum_{v\in v(\mathbb{G}^2)} \#e^{\mathrm{lp}}(v)(\#G/\#I_v)\\ &= \gamma_{\mathbb{G}^2} + \sum_{e\in e^{\mathrm{cl}}(\mathbb{G}^2)\setminus e^{\mathrm{lp}}(\mathbb{G}^2)} (\#G/\#I_e - 1) + \sum_{v\in v(\mathbb{G}^2)} \#e^{\mathrm{lp}}(v)(\#G/\#I_v - 1).\end{aligned}$$

Thus, we obtain

$$\begin{aligned}\sigma(\mathfrak{G}^1) = \sigma(\mathfrak{G}^{1,*}) = \sum_{v\in v(\mathbb{G}^2)} \Bigg((\#G/\#I_v)(\sigma_{\mathfrak{G}^2}(v) - 1) + \sum_{e\in e(v)\setminus e^{\mathrm{lp}}(v)} (\#G/\#I_e)(\#I_e/\#I_v - 1) + 1 \Bigg)\\ + \sum_{e\in e^{\mathrm{cl}}(\mathbb{G}^2)\setminus e^{\mathrm{lp}}(\mathbb{G}^2)} (\#G/\#I_e - 1) + \sum_{v\in v(\mathbb{G}^2)} \#e^{\mathrm{lp}}(v)(\#G/\#I_v - 1) + \gamma_{\mathbb{G}^2}.\end{aligned}$$

This completes the proof of the theorem. □

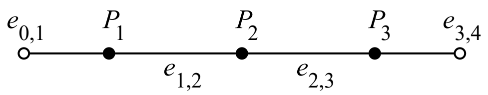
2C5. We introduce a kind of special semigraph. Let n be a positive natural number and $\mathbb{P}_n$ a semigraph (see Example 2.8 below) such that the following conditions are satisfied:

(i) $v(\mathbb{P}_n) = \{P_1, \ldots, P_n\}$, $e^{\mathrm{cl}}(\mathbb{P}_n) = \{e_{1,2}, \ldots, e_{n-1,n}\}$, and $e^{\mathrm{op}}(\mathbb{P}_n) = \{e_{0,1}, e_{n,n+1}\}$.

(ii) $\zeta^{\mathbb{P}_n}_{e_{0,1}}(e_{0,1}) = \{P_1, \{v(\mathbb{P}_n)\}\}$, $\zeta^{\mathbb{P}_n}_{e_{n,n+1}}(e_{n,n+1}) = \{P_n, \{v(\mathbb{P}_n)\}\}$, and $\zeta^{\mathbb{P}_n}_{e_{i,i+1}}(e_{i,i+1}) = \{P_i, P_{i+1}\}$, $i \in \{1, \ldots, n-1\}$.

Example 2.8. We give an example to explain the notion defined above. If $n = 3$, then $\mathbb{P}_3$ is as follows:

$\mathbb{P}_3$:



Definition 2.9. Let $\mathbb{P}_n$ be a semigraph defined above and $\sigma_{\mathfrak{P}_n} : v(\mathbb{P}_n) \to \mathbb{Z}$ a map such that $\sigma_{\mathfrak{P}_n}(P_i) = 0$ for each $i = \{1, \ldots, n\}$. We define a semigraph with p-rank $\mathfrak{P}_n$ to be

$$\mathfrak{P}_n \overset{\mathrm{def}}{=} (\mathbb{P}_n, \sigma_{\mathfrak{P}_n}),$$

and shall call $\mathfrak{P}_n$ an *n-chain*.

Remark 2.9.1. In Section 3C, we will see that n-chains can be naturally arose from quotients of the vertical fibers associated to *singular* vertical points (Definition 1.8) of G-pointed semistable coverings.

2C6. When $\mathfrak{G}^2 = \mathfrak{P}_n$ is a n-chain, Theorem 2.7 has the following important consequence.

Corollary 2.10. *Let $\mathfrak{b} : \mathfrak{G} \to \mathfrak{P}_n$ be a G-covering of connected semigraphs with p-rank. Then we have*

$$\sigma(\mathfrak{G}) = \sum_{i=1}^{n} \#G/\#I_{P_i} - \sum_{i=1}^{n+1} \#G/\#I_{e_{i-1,i}} + 1.$$

Proof. The construction of $\mathbb{P}_n$ implies

$$\sum_{v \in v(\mathbb{P}_n)} \#e^{\mathrm{lp}}(v)(\#G/\#I_v - 1) = \gamma_{\mathbb{P}_n} = 0.$$

Then the corollary follows immediately from Theorem 2.7. □

3. Formulas for p-rank of coverings of curves

In this section, we construct various semigraphs with p-rank from G-pointed semistable coverings. Moreover, we prove various formulas for p-rank concerning G-pointed semistable coverings when G is a finite p-group. More precisely, we prove a formula for p-rank of special fibers (see Theorem 3.2), a formula for p-rank of vertical fibers over vertical points (see Theorem 3.4), and a simpler form of Theorem 3.4 when the vertical points are singular (see Theorem 3.9 which plays a key in Section 4). In particular, Theorems 3.4 and 3.9 generalize Raynaud's result (Theorem 1.9) to the case of *arbitrary closed points*.

3A. *p-rank of special fibers.*

3A1. *Settings.* We maintain the settings introduced in Section 1C1. Let G be a finite p-group of order p^r, and let $f : \mathscr{Y} = (Y, D_Y) \to \mathscr{X} = (X, D_X)$ be a G-pointed semistable covering (Definition 1.5) over S. Moreover, let

$$\Phi : \{1\} = G_r \subset G_{r-1} \subset \cdots \subset G_1 \subset G_0 = G$$

be a maximal normal filtration (Definition 2.2) of G. By applying [Raynaud 1990, Appendice, Corollaire], we have that $\mathscr{X}^{\mathrm{sst}} = (X^{\mathrm{sst}}, D_{X^{\mathrm{sst}}}) \overset{\mathrm{def}}{=} \mathscr{Y}/G$ is a pointed semistable curve over S. Write $h : \mathscr{Y} \to \mathscr{X}^{\mathrm{sst}}$ and $g : \mathscr{X}^{\mathrm{sst}} \to \mathscr{X}$ for the natural morphisms of pointed semistable curves over S induced by f such that $f = g \circ h : \mathscr{Y} \xrightarrow{h} \mathscr{X}^{\mathrm{sst}} \xrightarrow{g} \mathscr{X}$.

3A2. Let $j \in \{0, \ldots, r\}$, [Raynaud 1990, Appendice, Corollaire] implies that $\mathscr{Y}_j \overset{\mathrm{def}}{=} \mathscr{Y}/G_j$ is a pointed semistable curve over S. Then the maximal normal filtration Φ of G induces a sequence of morphism of pointed semistable curves

$$\Phi_{\mathscr{Y}/\mathscr{X}^{\mathrm{sst}}} : \mathscr{Y}_r \overset{\mathrm{def}}{=} \mathscr{Y} \xrightarrow{\phi_r} \mathscr{Y}_{r-1} \xrightarrow{\phi_{r-1}} \cdots \xrightarrow{\phi_1} \mathscr{Y}_0 \overset{\mathrm{def}}{=} \mathscr{X}^{\mathrm{sst}}$$

over S such that $\phi_1 \circ \cdots \circ \phi_r = h$. Note that ϕ_j is a *finite* $\mathbb{Z}/p\mathbb{Z}$-pointed semistable covering over S.

Write $\Gamma_{\mathscr{Y}_j}$ for the dual semigraph (Section 1B2) of the special fiber $(\mathscr{Y}_j)_s$ of $\mathscr{Y}_j$. Then, for each $j \in \{1, \ldots, r\}$, the morphism of the special fibers $(\phi_j)_s : (\mathscr{Y}_j)_s \to (\mathscr{Y}_{j-1})_s$ induces a map of semigraphs $\beta_j : \Gamma_{\mathscr{Y}_j} \to \Gamma_{\mathscr{Y}_{j-1}}$. Moreover, Proposition 1.7 implies that β_j, $j \in \{1, \ldots, r\}$, is a *morphism* of semigraphs.

3A3. *Semigraph with p-rank associated to* $(\mathscr{Y}_j)_s$. Let $v \in v(\Gamma_{\mathscr{Y}_j})$ and $j \in \{0, \ldots, r\}$. We write $\widetilde{Y}_{j,v}$ for the normalization of the irreducible component $Y_{j,v} \subseteq (\mathscr{Y}_j)_s$ corresponding to v. We define a semigraph with p-rank associated to $(\mathscr{Y}_j)_s$ to be

$$\mathfrak{G}_{\mathscr{Y}_j} \overset{\mathrm{def}}{=} (\mathbb{G}_{\mathscr{Y}_j}, \sigma_{\mathfrak{G}_{\mathscr{Y}_j}}), \quad j \in \{0, \ldots, r\},$$

where $\mathbb{G}_{\mathscr{Y}_j} \overset{\mathrm{def}}{=} \Gamma_{\mathscr{Y}_j}$ and $\sigma_{\mathfrak{G}_{\mathscr{Y}_j}}(v) \overset{\mathrm{def}}{=} \sigma(\widetilde{Y}_{j,v})$ for $v \in v(\mathbb{G}_{\mathscr{Y}_j})$.

3A4. *G-covering of semigraphs with p-rank associated to f.* The sequence of pointed semistable coverings $\Phi_{\mathscr{Y}/\mathscr{X}^{\mathrm{sst}}}$ induces a sequence of morphisms of semigraphs with p-rank

$$\Phi_{\mathfrak{G}_{\mathscr{Y}}/\mathfrak{G}_{\mathscr{X}^{\mathrm{sst}}}} : \mathfrak{G}_{\mathscr{Y}} \overset{\mathrm{def}}{=} \mathfrak{G}_{\mathscr{Y}_r} \xrightarrow{\mathfrak{b}_r} \mathfrak{G}_{\mathscr{Y}_{r-1}} \xrightarrow{\mathfrak{b}_{r-1}} \cdots \xrightarrow{\mathfrak{b}_1} \mathfrak{G}_{\mathscr{X}^{\mathrm{sst}}} \overset{\mathrm{def}}{=} \mathfrak{G}_{\mathscr{Y}_0},$$

where $\mathfrak{b}_j : \mathfrak{G}_{\mathscr{Y}_j} \to \mathfrak{G}_{\mathscr{Y}_{j-1}}$, $j \in \{1, \ldots, r\}$, is induced by $\beta_j : \Gamma_{\mathscr{Y}_j} \to \Gamma_{\mathscr{Y}_{j-1}}$. By using the Deuring–Shafarevich formula (Proposition 1.4) and the Zariski–Nagata purity theorem [SGA 1 1971, Exposé X, Théorème de pureté 3.1], we see that $\mathfrak{b}_j$, $j \in \{1, \ldots, r\}$, is a p-covering (Definition 2.2(c)). Moreover, $\mathfrak{b} \overset{\mathrm{def}}{=} \mathfrak{b}_1 \circ \cdots \circ \mathfrak{b}_r$ is a G-covering (Definition 2.2(e)). Then we have

$$\sigma(\mathfrak{G}_{\mathscr{Y}}) = \sigma(\mathscr{Y}_s).$$

Summarizing the discussions above, we obtain the following proposition.

Proposition 3.1. *We maintain the notation introduced above. Let $f : \mathscr{Y} \to \mathscr{X}$ be a G-pointed semistable covering over S and $\mathscr{Y}_s$ the special fiber of $\mathscr{Y}$ over s. Then there exists a G-covering of semigraphs with p-rank $\mathfrak{b} : \mathfrak{G}_{\mathscr{Y}} \to \mathfrak{G}_{\mathscr{X}^{\mathrm{sst}}}$ associated to f (which is constructed above) such that $\sigma(\mathscr{Y}_s) = \sigma_{\mathfrak{G}_{\mathscr{Y}}}(\mathfrak{G}_{\mathscr{Y}})$.*

3A5. We maintain the notation introduced in Section 3A1 and write $\Gamma_{\mathscr{X}_s^{\mathrm{sst}}}$ for the dual semigraph of the special fiber $\mathscr{X}_s^{\mathrm{sst}} = (X_s^{\mathrm{sst}}, D_{X_s^{\mathrm{sst}}})$ of $\mathscr{X}^{\mathrm{sst}}$. Let $v \in v(\Gamma_{\mathscr{X}_s^{\mathrm{sst}}})$ and $e \in e(\Gamma_{\mathscr{X}_s^{\mathrm{sst}}})$ (Section 1A1). We write Y_v and y_e for an irreducible component of $h^{-1}(X_v)_{\mathrm{red}}$ and a closed point of $h^{-1}(x_e)_{\mathrm{red}}$, respectively, where X_v and x_e denote the irreducible component and the closed point of $\mathscr{X}_s^{\mathrm{sst}}$ corresponding to v and e (Section 1B2), respectively. Write $I_{Y_v} \subseteq G$ and $I_{y_e} \subseteq G$ for the inertia subgroup of Y_v and y_e, respectively. Note that since $\#I_{Y_v}$ and $\#I_{y_e}$ do not depend on the choices of Y_v and y_e, respectively, we may denote $\#I_{Y_v}$ and $\#I_{y_e}$ by $\#I_v$ and $\#I_e$, respectively. We put (see Section 1A1 for $v(e)$)

$$\#I_e^{\mathrm{m}} \overset{\mathrm{def}}{=} \max_{v \in v(e)}\{\#I_v\}, \quad e \in e^{\mathrm{cl}}(\Gamma_{\mathscr{X}_s^{\mathrm{sst}}}).$$

Note that Corollary 1.13 implies that $\#I_e = \#I_e^{\mathrm{m}}$.

We have the following formula for p-rank of special fibers of G-pointed stable coverings when G is a finite p-group.

Theorem 3.2. *We maintain the settings introduced above. Let G be a finite p-group, and let $f : \mathscr{Y} \to \mathscr{X}$ be a G-pointed semistable covering over S. Then we have (see Section 1B2 for $\widetilde{X}_v$, Section 1A1 for $e^{\mathrm{cl}}(\Gamma_{\mathscr{X}_s^{\mathrm{sst}}})$, $e^{\mathrm{lp}}(\Gamma_{\mathscr{X}_s^{\mathrm{sst}}})$, $e(v)$, $e^{\mathrm{lp}}(v)$, and Section 1A3 for $\gamma_{\Gamma_{\mathscr{X}_s^{\mathrm{sst}}}}$)*

$$\sigma(\mathscr{Y}_s) = \sum_{v \in v(\Gamma_{\mathscr{X}_s^{\mathrm{sst}}})} \Bigg(1 + (\#G/\#I_v)(\sigma(\widetilde{X}_v) - 1) + \sum_{e \in e(v) \setminus e^{\mathrm{lp}}(v)} (\#G/\#I_e)(\#I_e/\#I_v - 1)\Bigg) + \sum_{e \in e^{\mathrm{cl}}(\Gamma_{\mathscr{X}_s^{\mathrm{sst}}}) \setminus e^{\mathrm{lp}}(\Gamma_{\mathscr{X}_s^{\mathrm{sst}}})} (\#G/\#I_e - 1) + \sum_{v \in v(\Gamma_{\mathscr{X}_s^{\mathrm{sst}}})} \#e^{\mathrm{lp}}(v)(\#G/\#I_v - 1) + \gamma_{\Gamma_{\mathscr{X}_s^{\mathrm{sst}}}}.$$

In particular, if $f : \mathscr{Y} \to \mathscr{X}$ is a G-semistable covering (i.e., $D_X = \varnothing$), then we have

$$\sigma(\mathscr{Y}_s) = \sum_{v \in v(\Gamma_{\mathscr{X}_s^{\mathrm{sst}}})} \left(1 + (\#G/\#I_v)(\sigma(\widetilde{X}_v) - 1) + \sum_{e \in e(v) \setminus e^{\mathrm{lp}}(v)} (\#G/\#I_e^{\mathrm{m}})(\#I_e^{\mathrm{m}}/\#I_v - 1) \right) + \sum_{e \in e^{\mathrm{cl}}(\Gamma_{\mathscr{X}_s^{\mathrm{sst}}}) \setminus e^{\mathrm{lp}}(\Gamma_{\mathscr{X}_s^{\mathrm{sst}}})} (\#G/\#I_e^{\mathrm{m}} - 1) + \sum_{v \in v(\Gamma_{\mathscr{X}_s^{\mathrm{sst}}})} \#e^{\mathrm{lp}}(v)(\#G/\#I_v - 1) + \gamma_{\Gamma_{\mathscr{X}_s^{\mathrm{sst}}}}.$$

Proof. The theorem follows from Theorem 2.7 and Proposition 3.1. □

Remark 3.2.1. Note that it is easy to check that *the formula of Theorem 3.2 depends only on the G-pointed stable coverings.*

3B. *p-rank of vertical fibers.*

3B1. *Settings.* We maintain the settings introduced in Section 3A1. Let x be a *vertical point* (see Definition 1.8) associated to f. Write $\psi : Y' \to X$ for the normalization of X in the function field $K(Y)$ induced by the natural injection $K(X) \hookrightarrow K(Y)$ induced by f. Then Y' admits a natural action of G induced by the action of G on the generic fiber of Y.

Let $y' \in \psi^{-1}(x)$. Write $I_{y'} \subseteq G$ for the inertia subgroup of y'. Proposition 1.6 implies that the morphism of pointed smooth curves $(Y_\eta/I_{y'}, D_{Y_\eta}/I_{y'}) \to \mathscr{X}_\eta$ over η induced by f extends to a pointed semistable covering $\mathscr{Y}_{I_{y'}} \to \mathscr{X}$ over S. In order to calculate the p-rank of $f^{-1}(x)$, since the morphism $\mathscr{Y}_{I_{y'}} \to \mathscr{X}$ is finite étale over x, by replacing $\mathscr{X}$ by $\mathscr{Y}_{I_{y'}}$, we may assume that G is equal to $I_{y'}$. In the remainder of this subsection, we shall assume $G = I_{y'}$ (note that $G = I_{y'}$ if and only if $f^{-1}(x)$ is *connected*).

3B2. Write $\mathscr{X}_s^{\mathrm{sst}} = (X_s^{\mathrm{sst}}, D_{X_s^{\mathrm{sst}}})$ and $\mathscr{Y}_s = (Y_s, D_{Y_s})$ for the special fibers of $\mathscr{X}^{\mathrm{sst}}$ and $\mathscr{Y}$ over s, respectively. By the general theory of semistable curves, $g^{-1}(x)_{\mathrm{red}} \subset X_s^{\mathrm{sst}}$ and $f^{-1}(x)_{\mathrm{red}} = h^{-1}(g^{-1}(x))_{\mathrm{red}} \subset Y_s$ are semistable curves over s, where $(-)_{\mathrm{red}}$ denotes the reduced induced closed subscheme of $(-)$. In particular, the irreducible components of $g^{-1}(x)_{\mathrm{red}}$ are isomorphic to $\mathbb{P}^1_k$.

Write V_X for the set of closed points

$$g^{-1}(x)_{\mathrm{red}} \cap \overline{\{X_s^{\mathrm{sst}} \setminus g^{-1}(x)_{\mathrm{red}}\}},$$

where $\overline{\{X_s^{\mathrm{sst}} \setminus g^{-1}(x)_{\mathrm{red}}\}}$ denotes the topological closure of $X_s^{\mathrm{sst}} \setminus g^{-1}(x)_{\mathrm{red}}$ in X_s^{sst}. Write $V_Y \subset \mathscr{Y}_s$ for the set of closed points $\{h^{-1}(q)_{\mathrm{red}}\}_{q \in V_X}$. We have $\#V_X = 1$ if x is a *smooth point* of $\mathscr{X}_s$, and $\#V_X = 2$ if x is a *node* of $\mathscr{X}_s$.

3B3. We define two pointed semistable curves over s to be

$$\mathscr{E}_X \overset{\mathrm{def}}{=} (g^{-1}(x)_{\mathrm{red}}, (D_{X^{\mathrm{sst}}} \cap g^{-1}(x)_{\mathrm{red}}) \cup V_X) \quad \text{and} \quad \mathscr{E}_Y \overset{\mathrm{def}}{=} (f^{-1}(x)_{\mathrm{red}}, (D_Y \cap f^{-1}(x)_{\mathrm{red}}) \cup V_Y).$$

Then we obtain a *finite* morphism of pointed semistable curves $\rho_{\mathscr{E}_Y/\mathscr{E}_X} : \mathscr{E}_Y \to \mathscr{E}_X$ induced by h. Since $f^{-1}(x)$ is connected, $\mathscr{E}_Y$ admits a natural action of G induced by the action of G on the special fiber $\mathscr{Y}_s$ of $\mathscr{Y}$. Write $\Gamma_{\mathscr{E}_Y}$ and $\Gamma_{\mathscr{E}_X}$ for the dual semigraphs of $\mathscr{E}_Y$ and $\mathscr{E}_X$, respectively. Note that $\Gamma_{\mathscr{E}_X}$ is a tree, and is *not* a n-chain (Definition 2.9) in general if x is not a node. We obtain a map of semigraphs

$$\delta_{\mathscr{E}_Y/\mathscr{E}_X} : \Gamma_{\mathscr{E}_Y} \to \Gamma_{\mathscr{E}_X}$$

induced by $\rho_{\mathscr{E}_Y/\mathscr{E}_X}$. Moreover, Proposition 1.7 implies that the map $\delta_{\mathscr{E}_Y/\mathscr{E}_X} : \Gamma_{\mathscr{E}_Y} \to \Gamma_{\mathscr{E}_X}$ is a morphism of semigraphs.

3B4. *Semigraphs with p-rank associated to $\mathscr{E}_Y$ and $\mathscr{E}_X$.* Let $v \in v(\Gamma_{\mathscr{E}_Y})$. Write $\widetilde{Y}_v$ for the normalization of the irreducible component $Y_v \subseteq \mathscr{E}_Y$ corresponding to v. We define semigraphs with p-rank associated to $\mathscr{E}_Y$ and $\mathscr{E}_X$, respectively, as follows:

$$\mathfrak{E}_Y \overset{\text{def}}{=} (\mathbb{E}_Y, \sigma_{\mathfrak{E}_Y}), \qquad \mathfrak{E}_X \overset{\text{def}}{=} (\mathbb{E}_X, \sigma_{\mathfrak{E}_X}),$$

where $\mathbb{E}_Y \overset{\text{def}}{=} \Gamma_{\mathscr{E}_Y}$, $\mathbb{E}_X \overset{\text{def}}{=} \Gamma_{\mathscr{E}_X}$, $\sigma_{\mathfrak{E}_Y}(v) \overset{\text{def}}{=} \sigma(\widetilde{Y}_v)$ for $v \in v(\mathbb{E}_Y)$, and $\sigma_{\mathfrak{E}_X}(w) \overset{\text{def}}{=} 0$ for $w \in v(\mathbb{E}_X)$.

3B5. *G-coverings of semigraphs with p-rank associated to vertical fibers.* The morphism of dual semigraphs $\delta_{\mathscr{E}_Y/\mathscr{E}_X} : \Gamma_{\mathscr{E}_Y} \to \Gamma_{\mathscr{E}_X}$ induces a morphism of semigraphs with p-rank

$$\mathfrak{d}_{\mathfrak{E}_Y/\mathfrak{E}_X} : \mathfrak{E}_Y \to \mathfrak{E}_X.$$

Moreover, we see that $\mathfrak{d}_{\mathfrak{E}_Y/\mathfrak{E}_X}$ is a G-covering. Then we have

$$\sigma(\mathfrak{E}_Y) = \sigma(f^{-1}(x)_{\text{red}}) = \sigma(f^{-1}(x)).$$

Summarizing the discussions above, we obtain the following proposition.

Proposition 3.3. *We maintain the notation introduced above. Let $f : \mathscr{Y} \to \mathscr{X}$ be a G-pointed semistable covering over S and x a vertical point associated to f. Suppose that $f^{-1}(x)$ is connected. Then there exists a G-covering of semigraphs with p-rank $\mathfrak{d}_{\mathfrak{E}_Y/\mathfrak{E}_X} : \mathfrak{E}_Y \to \mathfrak{E}_X$ associated to f and x (which is constructed above) such that $\sigma(\mathfrak{E}_Y) = \sigma(f^{-1}(x))$.*

3B6. Then we have the following formula for p-rank of vertical fibers.

Theorem 3.4. *We maintain the settings introduced in Section 1C1. Let G be a finite p-group, and let $f : \mathscr{Y} \to \mathscr{X}$ be a G-pointed semistable covering (Definition 1.5) over S and x a vertical point (Definition 1.8) associated to f. We maintain the notation introduced in Sections 3B2 and 3B3. Suppose that $f^{-1}(x)$ is connected. Then we have (see Section 3A5 for $\#I_v$, $\#I_e$, and Section 1A1 for $v(\Gamma_{\mathscr{E}_X})$, $e(v)$, $e^{\text{cl}}(\Gamma_{\mathscr{E}_X})$)*

$$\sigma(f^{-1}(x)) = \sum_{v \in v(\Gamma_{\mathscr{E}_X})} \left(1 - \#G/\#I_v + \sum_{e \in e(v)} (\#G/\#I_e)(\#I_e/\#I_v - 1)\right) + \sum_{e \in e^{\text{cl}}(\Gamma_{\mathscr{E}_X})} (\#G/\#I_e - 1).$$

Proof. The theorem follows from Theorem 2.7 and Proposition 3.3. ☐

3B7. We maintain the notation introduced in Theorem 3.4. We explain that Raynaud's result (i.e., Theorem 1.9) can be directly calculated by using Theorem 3.4 if $x \in X_s \setminus (X_s^{\mathrm{sing}} \cup D_{X_s})$. Note that, since $x \notin D_{X_s}$, we have $g^{-1}(x)_{\mathrm{red}} \cap D_{X_s^{\mathrm{sst}}} = \varnothing$.

Let X_0' be the irreducible component of X_s which contains x. Moreover, we write X_0 for the strict transform of X_0' under the birational morphism $g : \mathscr{X}^{\mathrm{sst}} \to \mathscr{X}$. Then there exists a unique irreducible component $X_1 \subseteq g^{-1}(x)_{\mathrm{red}} \subseteq X_s^{\mathrm{sst}}$ such that $X_0 \cap X_1 \neq \varnothing$. Note that $\#(X_0 \cap X_1) = 1$. Write v_1 for the vertex of $v(\Gamma_{\mathscr{E}_X})$ corresponding to X_1. Since $\Gamma_{\mathscr{E}_X}$ is a connected tree, for each $v \in v(\Gamma_{\mathscr{E}_X})$, there exists a path $l(v_1, v)$ connecting v_1 and v. We define

$$\mathrm{leng}(l(v_1, v)) \overset{\mathrm{def}}{=} \#\{l(v_1, v) \cap v(\Gamma_{\mathscr{E}_X})\}$$

to be the length of the path $l(v_1, v)$. Moreover, for each $v \in v(\Gamma_{\mathscr{E}_X})$, we write

$$l_{v_1,v}$$

for the path such that $\mathrm{leng}(l_{v_1,v}) = \min\{\mathrm{leng}(l(v_1, v))\}_{l(v_1,v)}$.

By applying the general theory of semistable curves, Lemma 1.10, and Corollary 1.13, one may prove the following:

> Let $v, v' \in v(\Gamma_{\mathscr{E}_X})$ and $X_v, X_{v'}$ the irreducible components of $g^{-1}(x)_{\mathrm{red}}$ corresponding to v, v', respectively. Suppose that $\{x_e\} \overset{\mathrm{def}}{=} X_v \cap X_{v'} \neq \varnothing$, and that $\mathrm{leng}(l_{v_1,v}) < \mathrm{leng}(l_{v_1,v'})$. Write $e \in e^{\mathrm{cl}}(\Gamma_{\mathscr{E}_X})$ for the closed edge corresponding to x_e. Then we have $\#I_v = \#I_e$ and $\#I_{v'} \mid \#I_v$.

Note that the inertia subgroup of the unique open edge of $\Gamma_{\mathscr{E}_X}$ (which abuts to v_1) is equal to G. Then Theorem 3.4 implies that $\sigma(f^{-1}(x)) = 0$.

3C. ***p-rank of vertical fibers associated to singular vertical points.*** In this subsection, we will see that Theorem 3.4 has a very simple form if x is a *singular vertical point* which plays a central role in Section 4.

3C1. *Settings.* We maintain the settings introduced in Section 3B1. Moreover, we suppose that the vertical point x is a *node* of $\mathscr{X}_s$. Write X_1' and X_2' (which may be equal) for the irreducible components of $\mathscr{X}_s$ containing x. Write X_1 and X_2 for the strict transforms of X_1' and X_2' under the birational morphism $g : \mathscr{X}^{\mathrm{sst}} \to \mathscr{X}$, respectively.

By the general theory of semistable curves, $g^{-1}(x)_{\mathrm{red}} \subseteq X_s^{\mathrm{sst}}$ is a semistable curve over s and $g^{-1}(x)_{\mathrm{red}} \cap D_{X_s^{\mathrm{sst}}} = \varnothing$. Moreover, the irreducible components of $g^{-1}(x)_{\mathrm{red}}$ are isomorphic to $\mathbb{P}^1_k$. Let C be the semistable subcurve of $g^{-1}(x)_{\mathrm{red}}$ which is a chain of projective lines $\bigcup_{i=1}^n P_i$ such that the following conditions are satisfied:

(i) For any $w, t \in \{1, \dots, n\}$, $P_w \cap P_t = \varnothing$ if $|w-t| \geq 2$, and $P_w \cap P_t$ is reduced to a point if $|w-t| = 1$.

(ii) $P_1 \cap X_1$ (resp. $P_n \cap X_2$) is reduced to a point.

(iii) $C \cap \overline{\{X_s^{\mathrm{sst}} \setminus g^{-1}(x)_{\mathrm{red}}\}} = (P_1 \cap X_1) \cup (P_n \cap X_2)$, where $\overline{\{X_s^{\mathrm{sst}} \setminus g^{-1}(x)_{\mathrm{red}}\}}$ denotes the topological closure of $X_s^{\mathrm{sst}} \setminus g^{-1}(x)_{\mathrm{red}}$ in X_s^{sst}.

Then we have

$$g^{-1}(x)_{\text{red}} = C \cup B,$$

where B denotes the topological closure of $g^{-1}(x)_{\text{red}} \setminus C$ in $g^{-1}(x)_{\text{red}}$. Note that $B \cap C$ are smooth points of C. Then Theorem 1.9 (or Section 3B7) implies that the p-rank of the connected components of $h^{-1}(B)$ are equal to 0. Thus, we have $\sigma(f^{-1}(x)) = \sigma(h^{-1}(C))$.

3C2. We introduce the following notation concerning inertia subgroups of irreducible components of vertical fibers.

Definition 3.5. We maintain the notation introduced above:

(a) Let $\mathcal{V}_x \overset{\text{def}}{=} \{V_0, V_1, \ldots, V_n, V_{n+1}\}$ be a set of irreducible components of the special fiber $\mathscr{Y}_s$ of $\mathscr{Y}$. We shall call $\mathcal{V}_x$ a *collection of vertical fibers* associated to x if the following conditions are satisfied:

(i) $h(V_i) = P_i$ for $i \in \{1, \ldots, n\}$.

(ii) $h(V_0) = X_1$ and $h(V_{n+1}) = X_2$.

(iii) The union $\bigcup_{i=0}^{n+1} V_i \subseteq \mathscr{Y}_s$ is a connected semistable subcurve of $\mathscr{Y}_s$ over s. Note that we have $\left(\bigcup_{i=1}^{n} V_i\right) \cap D_{Y_s} = \varnothing$.

Moreover, we write $I_{V_i} \subseteq G$, $i \in \{0, \ldots, n+1\}$, for the inertia subgroup of V_i, and put

$$\mathcal{I}_{\mathcal{V}_x} \overset{\text{def}}{=} \{I_{V_0}, \ldots, I_{V_{n+1}}\}.$$

Note that Corollary 1.13 implies that either $I_{V_i} \subseteq I_{V_{i+1}}$ or $I_{V_i} \supseteq I_{V_{i+1}}$ holds for $i \in \{0, \ldots, n\}$.

(b) Let $(u, w) \in \{0, \ldots, n+1\} \times \{0, \ldots, n+1\}$ be a pair such that $u \leq w$. We shall call that a group $I^{\min}_{u,w}$ is a *minimal element* of $\mathcal{I}_{\mathcal{V}_x}$ if one of the following conditions are satisfied, where "$\subset$" means that "is a subset which is not equal":

(i) $u = 0$, $w \neq 0$, $w \neq n+1$, and $I^{\min}_{0,w} = I_{V_0} = I_{V_1} = \cdots = I_{V_w} \subset I_{V_{w+1}}$.

(ii) $u \neq 0$, $w = n+1$, and $I_{V_{u-1}} \supset I_{V_u} = I_{V_{u+1}} \cdots = I_{V_{n+1}} = I^{\min}_{u,n+1}$.

(iii) $u \neq 0$, $w \neq n+1$, and $I_{V_{u-1}} \supset I^{\min}_{u,w} = I_{V_u} = I_{V_{u+1}} \cdots = I_{V_w} \subset I_{V_{w+1}}$.

Note that we *do not* define $I^{\min}_{0,0}$. We shall call that a group $J^{\max}_{u,w}$ is a *maximal element* of $\mathcal{I}_{\mathcal{V}_x}$ if one of the following conditions are satisfied:

(i) $(u, w) = (0, n+1)$ and $J^{\max}_{0,n+1} = I_{V_i}$ for all $i \in \{0, \ldots, n+1\}$.

(ii) $u = 0$, $w \neq n+1$, and $J^{\max}_{0,w} = I_{V_0} = I_{V_1} = \cdots = I_{V_w} \supset I_{V_{w+1}}$.

(iii) $u \neq 0$, $w = n+1$, and $I_{V_{u-1}} \subset I_{V_u} = I_{V_{u+1}} \cdots = I_{V_{n+1}} = J^{\max}_{u,n+1}$.

(iv) $u \neq 0$, $w \neq n+1$, and $I_{V_{u-1}} \subset J^{\max}_{u,w} = I_{V_u} = I_{V_{u+1}} \cdots = I_{V_w} \supset I_{V_{w+1}}$.

Moreover, we put

$$\mathcal{I}(x) \overset{\text{def}}{=} \bigsqcup_{I^{\min}_{u,w}\text{: a minimal element of } \mathcal{I}_{\mathcal{V}_x}} \{\#I^{\min}_{u,w}\} \quad \text{and} \quad \mathcal{J}(x) \overset{\text{def}}{=} \bigsqcup_{J^{\max}_{u,w}\text{: a maximal element of } \mathcal{I}_{\mathcal{V}_x}} \{\#J^{\max}_{u,w}\},$$

where $\sqcup$ means disjoint union.

Note that the set $\mathcal{I}(x)$ may be empty (e.g., if $I_{V_0} \subset I_{V_1} \subset \cdots \subset I_{V_{n+1}}$, then $\mathcal{I}(x)$ is empty). On the other hand, since $\#I_{V_i}$, $i \in \{0, \dots, n+1\}$, does not depend on the choice of V_i (i.e., if $h(V_i) = h(V_i')$ for irreducible components V_i, V_i' of $\mathscr{Y}_s$, then $\#I_{V_i} = \#I_{V_i'}$), $\mathcal{I}(x)$ and $\mathcal{J}(x)$ *do not depend on* the choice of $\mathcal{V}_x$.

We shall call $\mathcal{I}(x)$ *the set of minimal orders of inertia subgroups associated to x and f*, and $\mathcal{J}(x)$ *the set of maximal orders of inertia subgroups associated to x and f*, respectively.

3C3. We have the following lemmas.

Lemma 3.6. *We maintain the notation introduced above. Let $y_i \in V_i$ be a closed point and $I_{y_i} \subseteq G$, $i \in \{1, \dots, n\}$ the inertia subgroup of y_i. Write Ray_{V_i}, $i = 1, \dots, n$, for the set of the closed points $h^{-1}(C \cap B)_{\mathrm{red}} \cap V_i$. Then we have $I_{y_i} = I_{V_i}$ for any $y_i \in \mathrm{Ray}_{V_i}$.*

Proof. Since $I_{y_i} \supseteq I_{V_i}$, we only need to prove that $I_{y_i} \subseteq I_{V_i}$. Note that I_{V_i} is a normal subgroup of I_{y_i}. To verify the lemma, by replacing G and $\mathscr{X}^{\mathrm{sst}}$ by I_{y_i} and $\mathscr{Y}/I_{y_i}$, respectively, we may assume $G = I_{y_i}$. Then we have $\#h^{-1}(h(y_i))_{\mathrm{red}} = 1$.

We consider the quotient $\mathscr{Y}/I_{V_i}$. By [Raynaud 1990, Appendice, Corollaire], we have that $\mathscr{Y}/I_{V_i}$ is a pointed semistable curve over S. Write $h_{I_{V_i}}$ for the quotient morphism $\mathscr{Y} \to \mathscr{Y}/I_{V_i}$ and $g_{I_{V_i}}$ for the morphism $\mathscr{Y}/I_{V_i} \to \mathscr{X}^{\mathrm{sst}}$ induced by h such that $h = g_{I_{V_i}} \circ h_{I_{V_i}}$. Write E_{y_i} for the connected component of $h^{-1}(B)_{\mathrm{red}}$ which contains y_i. By contracting $h_{I_{V_i}}(E_{y_i}) \subset \mathscr{Y}/I_{V_i} \times_S s$ (resp. $h(E_{y_i}) \subset \mathscr{X}_s^{\mathrm{sst}}$) [Bosch et al. 1990, 6.7 Proposition 4], we obtain a fiber surface $(\mathscr{Y}/I_{V_i})^{\mathrm{c}}$ and a semistable curve $(\mathscr{X}^{\mathrm{sst}})^c$ over S. Moreover, we have contracting morphisms as follows:

$$c_{h_{I_{V_i}}(E_{y_i})} : \mathscr{Y}/I_{V_i} \to (\mathscr{Y}/I_{V_i})^{\mathrm{c}}, \; c_{h(E_{y_i})} : \mathscr{X}^{\mathrm{sst}} \to (\mathscr{X}^{\mathrm{sst}})^{\mathrm{c}}.$$

Furthermore, we obtain a morphism of fiber surfaces

$$g^{\mathrm{c}}_{I_{V_i}} : (\mathscr{Y}/I_{V_i})^{\mathrm{c}} \to (\mathscr{X}^{\mathrm{sst}})^c$$

induced by $g_{I_{V_i}}$ such that $c_{h(E_{y_i})} \circ g_{I_{V_i}} = g^{\mathrm{c}}_{I_{V_i}} \circ c_{h_{I_{V_i}}(E_{y_i})}$. Note that $(c_{h(E_{y_i})} \circ h)(y_i)$ is a smooth point of the special fiber of $(\mathscr{X}^{\mathrm{sst}})^c$, and $g^{\mathrm{c}}_{I_{V_i}}$ is étale at the generic point of $(c_{h_{I_{V_i}}(E_{y_i})} \circ h_{I_{V_i}})(V_i)$.

We put $y_i^{\mathrm{c}} \stackrel{\mathrm{def}}{=} (c_{h_{I_{V_i}}(E_{y_i})} \circ h_{I_{V_i}})(y_i) \in (\mathscr{Y}/I_{V_i})^{\mathrm{c}}$ and $x_i^c \stackrel{\mathrm{def}}{=} (c_{h(E_{y_i})} \circ h)(y_i) \in (\mathscr{X}^{\mathrm{sst}})^{\mathrm{c}}$. Consider the local morphism

$$g_{y_i^{\mathrm{c}}} : \mathrm{Spec}\, \mathcal{O}_{(\mathscr{Y}/I_{V_i})^{\mathrm{c}}, y_i^{\mathrm{c}}} \to \mathrm{Spec}\, \mathcal{O}_{(\mathscr{X}^{\mathrm{sst}})^{\mathrm{c}}, x_i^{\mathrm{c}}}$$

induced by $g^{\mathrm{c}}_{I_{V_i}}$. Note that [Raynaud 1990, Proposition 1] implies that $\mathrm{Spec}\, \mathcal{O}_{(\mathscr{Y}/I_{V_i})^{\mathrm{c}}, y_i^{\mathrm{c}}} \times_S s$ is irreducible. Then $g_{y_i^{\mathrm{c}}}$ is generically étale at the generic point of $\mathrm{Spec}\, \mathcal{O}_{(\mathscr{Y}/I_{V_i})^{\mathrm{c}}, y_i^{\mathrm{c}}} \times_S s$. Thus, the Zariski–Nagata purity theorem implies that $g_{y_i^{\mathrm{c}}}$ is étale.

If $I_{V_i} \neq I_{y_i}$, then $g_{y_i^{\mathrm{c}}}$ is not an identity. Namely, we have $\#h^{-1}(h(y_i))_{\mathrm{red}} \neq 1$. This contradicts our assumption. Then we obtain $I_{V_i} = I_{y_i}$. We complete the proof of the lemma. □

Lemma 3.7. *We maintain the notation introduced in above. Then we have*

$$G = \langle I_{V_0}, I_{V_{n+1}} \rangle,$$

where $\langle I_{V_0}, I_{V_{n+1}} \rangle$ denotes the subgroup of G generated by I_{V_0} and $I_{V_{n+1}}$.

Proof. Suppose that $G \neq \langle I_{V_0}, I_{V_{n+1}} \rangle$. Since G is a p-group, there exists a normal subgroup $H \subseteq G$ of index p such that $\langle I_{V_0}, I_{V_{n+1}} \rangle \subseteq H$. Write $\mathscr{Y}'$ for the normalization of $\mathscr{X}$ in the function field $K(Y)$ induced by the natural injection $K(X) \hookrightarrow K(Y)$ induced by f. The normalization $\mathscr{Y}'$ admits an action of G induced by the action of G on $\mathscr{Y}$. Consider the quotient $\mathscr{Y}'/H$. Then we obtain a morphism of fiber surfaces $f_H : \mathscr{Y}'/H \to \mathscr{X}$ over S induced by f. Moreover, $\mathscr{Y}'/H$ admits an action of $G/H \cong \mathbb{Z}/p\mathbb{Z}$ induced by the action of G on $\mathscr{Y}'$. Then f_H is generically étale over X'_1 and X'_2. Thus, [Tamagawa 2004b, Lemma 2.1(iii)] implies that f_H is étale above x. Then $f^{-1}(x)$ is not connected. This contradicts our assumptions. We complete the proof of the lemma. □

3C4. We define pointed semistable curves over s as follows:

$$\mathscr{C}_Y \stackrel{\text{def}}{=} (h^{-1}(C)_{\text{red}}, h^{-1}((C \cap X_1) \cup (C \cap X_2))) \quad \text{and} \quad \mathscr{C}_X \stackrel{\text{def}}{=} (C, (C \cap X_1) \cup (C \cap X_2)).$$

Moreover, we have a natural morphism of pointed semistable curves

$$\rho_{\mathscr{C}_Y/\mathscr{C}_X} : \mathscr{C}_Y \to \mathscr{C}_X$$

over s induced by $h : \mathscr{Y} \to \mathscr{X}^{\text{sst}}$. Since $f^{-1}(x)_{\text{red}}$ is connected, $\mathscr{C}_Y$ admits a natural action of G induced by the action of G on $f^{-1}(x)_{\text{red}}$. Write $\Gamma_{\mathscr{C}_Y}$ and $\Gamma_{\mathscr{C}_X}$ for the dual semigraphs of $\mathscr{C}_Y$ and $\mathscr{C}_X$, respectively. Proposition 1.7 implies that the map of semigraphs

$$\delta_{\mathscr{C}_Y/\mathscr{C}_X} : \Gamma_{\mathscr{C}_Y} \to \Gamma_{\mathscr{C}_X}$$

induced by $\rho_{\mathscr{C}_Y/\mathscr{C}_X}$ is a morphism of semigraphs.

3C5. *Semigraphs with p-rank associated to vertical fibers over singular vertical points.* Let $v \in v(\Gamma_{\mathscr{C}_Y})$ and $\widetilde{Y}_v$ the normalization of the irreducible component $Y_v \subseteq \mathscr{C}_Y$ corresponding to v. We define semigraphs with p-rank associated to $\mathscr{C}_Y$ and $\mathscr{C}_X$, respectively, as follows:

$$\mathfrak{C}_Y \stackrel{\text{def}}{=} (\mathbb{C}_Y, \sigma_{\mathfrak{C}_Y}), \qquad \mathfrak{C}_X \stackrel{\text{def}}{=} (\mathbb{C}_X, \sigma_{\mathfrak{C}_X}),$$

where $\mathbb{C}_Y \stackrel{\text{def}}{=} \Gamma_{\mathscr{C}_Y}$, $\mathbb{C}_X \stackrel{\text{def}}{=} \Gamma_{\mathscr{C}_X}$, $\sigma_{\mathfrak{C}_Y}(v) \stackrel{\text{def}}{=} \sigma(\widetilde{Y}_v)$ for $v \in v(\mathbb{C}_Y)$, and $\sigma_{\mathfrak{C}_X}(w) \stackrel{\text{def}}{=} 0$ for $w \in v(\mathbb{C}_X)$.

3C6. *G-coverings of semigraphs with p-rank associated to vertical fibers over singular vertical points.* The morphism of dual semigraphs $\delta_{\mathscr{C}_Y/\mathscr{C}_X} : \Gamma_{\mathscr{C}_Y} \to \Gamma_{\mathscr{C}_X}$ induces a morphism of semigraphs with p-rank

$$\mathfrak{d}_{\mathfrak{C}_Y/\mathfrak{C}_X} : \mathfrak{C}_Y \to \mathfrak{C}_X.$$

Moreover, by Lemma 3.6, we see that $\sigma_{\mathfrak{C}_Y}(v)$ satisfies the Deuring–Shafarevich type formula for $v \in v(\mathbb{C}_Y)$. This implies that $\mathfrak{d}_{\mathfrak{C}_Y/\mathfrak{C}_X}$ is a G-covering of semigraphs with p-rank. Note that by the above construction, $\mathfrak{C}_X$ is an *n-chain* (Definition 2.9). Furthermore, we have

$$\sigma(\mathfrak{C}_Y) = \sigma(h^{-1}(C)) = \sigma(f^{-1}(x)).$$

Summarizing the discussion above, we obtain the following proposition.

Proposition 3.8. *We maintain the notation introduced above. Let $f : \mathscr{Y} \to \mathscr{X}$ be a G-pointed semistable covering over S and $x \in \mathscr{X}_s$ a vertical point associated to f. Suppose that $f^{-1}(x)$ is connected, and that x is a node of $\mathscr{X}_s$. Then there exists a G-covering of semigraphs with p-rank $\mathfrak{d}_{\mathfrak{C}_Y/\mathfrak{C}_X} : \mathfrak{C}_Y \to \mathfrak{C}_X$ associated to f and x (which is constructed above) such that $\mathfrak{C}_X$ is an n-chain and $\sigma(\mathfrak{C}_Y) = \sigma(f^{-1}(x))$.*

3C7. Then we have the following theorem.

Theorem 3.9. *We maintain the settings introduced in Section 1C1. Let G be a finite p-group, and let $f : \mathscr{Y} \to \mathscr{X}$ be a G-pointed semistable covering (Definition 1.5) over S and $x \in \mathscr{X}_s$ a vertical point (Definition 1.8) associated to f. Suppose that $f^{-1}(x)$ is connected, and that x is a node of $\mathscr{X}_s$. Let $\mathcal{I}(x)$ and $\mathcal{J}(x)$ be the sets of minimal and maximal orders of inertia subgroups associated to x and f (Definition 3.5(b)), respectively. Then we have*

$$\sigma(f^{-1}(x)) = \sum_{\#I \in \mathcal{I}(x)} \#G/\#I - \sum_{\#J \in \mathcal{J}(x)} \#G/\#J + 1.$$

Proof. Let $\mathcal{V}_x$ be a collection of vertical fibers associated to x (Definition 3.5(a)). By Proposition 3.8, Corollary 2.10, and Lemma 1.10, we have

$$\sigma(f^{-1}(x)) = \sum_{i=1}^{n} \#G/\#I_{V_i} - \sum_{i=1}^{n+1} \#G/\#\langle I_{V_{i-1}}, I_{V_i}\rangle + 1,$$

where $\langle I_{V_{i-1}}, I_{V_i}\rangle$ denotes the subgroup of G generated by $I_{V_{i-1}}$ and I_{V_i}. The theorem follows from Corollary 1.13 and Lemma 3.7. ☐

3C8. In the remainder of the present subsection, we suppose that G is a cyclic p-group. We show that the formula of Theorem 3.9 coincides with the formula of Saïdi [1998a, Proposition 1]. Since G is an abelian group, I_{V_i}, $i = \{0, \dots, n+1\}$, does not depend on the choice of V_i. Then we may use the notation I_{P_i}, $i \in \{0, \dots, n+1\}$, to denote I_{V_i}.

Lemma 3.10. *We maintain the notation introduced above. If G is a cyclic p-group, then there exists $0 \le u \le n+1$ such that*

$$I_{P_0} \supseteq I_{P_1} \supseteq I_{P_2} \supseteq \cdots \supseteq I_{P_u} \subseteq \cdots \subseteq I_{P_{n-1}} \subseteq I_{P_n} \subseteq I_{P_{n+1}}.$$

Proof. If the lemma is not true, then there exist w, t and v such that $I_{P_v} \neq I_{P_w}$, $I_{P_v} \neq I_{P_t}$ and $I_{P_w} \subset I_{P_{w+1}} = \cdots = I_{P_v} = \cdots = I_{P_{t-1}} \supset I_{P_t}$. Since G is a cyclic group, we may assume $I_{P_w} \supseteq I_{P_t}$. Consider the quotient of $\mathscr{Y}$ by I_{P_w}, we obtain a natural morphism of pointed semistable curves $h_w : \mathscr{Y}/I_{P_w} \to \mathscr{X}^{\mathrm{sst}}$ over S.

We define B_j, $j = \{0, \dots, n+1\}$, to be the union of the connected components of B (Section 3C1) which intersect with P_j nontrivially. By contracting [Bosch et al. 1990, 6.7 Proposition 4]

$$P_{w+1}, \dots, P_{t-1}, B_{w+1}, \dots, B_{t-1},$$
$$(h_w)^{-1}(P_{w+1})_{\mathrm{red}}, \dots, (h_w)^{-1}(P_{t-1})_{\mathrm{red}}, (h_w)^{-1}(B_{w+1})_{\mathrm{red}}, \dots, (h_w)^{-1}(B_{t-1})_{\mathrm{red}},$$

respectively, we obtain a pointed semistable curve $(\mathscr{X}^{\mathrm{sst}})^{\mathrm{c}}$ and a fiber surface $(\mathscr{Y}/I_{P_w})^{\mathrm{c}}$ over S. Write

$$c_{\mathscr{X}^{\mathrm{sst}}}: \mathscr{X}^{\mathrm{sst}} \to (\mathscr{X}^{\mathrm{sst}})^{\mathrm{c}}, \ c_{\mathscr{Y}/I_{P_w}}: \mathscr{Y}/I_{P_w} \to (\mathscr{Y}/I_{P_w})^{\mathrm{c}}$$

for the resulting contracting morphisms, respectively. The morphism h_w induces a morphism of fiber surfaces $h_w^{\mathrm{c}}: (\mathscr{Y}/I_{P_w})^{\mathrm{c}} \to (\mathscr{X}^{\mathrm{sst}})^{\mathrm{c}}$. Then we have the following commutative diagram:

$$\begin{array}{ccc} \mathscr{Y}/I_{P_w} & \xrightarrow{c_{\mathscr{Y}/I_{P_w}}} & (\mathscr{Y}/I_{P_w})^{\mathrm{c}} \\ {\scriptstyle h_w}\downarrow & & \downarrow{\scriptstyle h_w^{\mathrm{c}}} \\ \mathscr{X}^{\mathrm{sst}} & \xrightarrow{c_{\mathscr{X}^{\mathrm{sst}}}} & (\mathscr{X}^{\mathrm{sst}})^{\mathrm{c}} \end{array}$$

Write P_w^{c} and P_t^{c} for the images $c_{\mathscr{X}^{\mathrm{sst}}}(P_w)$ and $c_{\mathscr{X}^{\mathrm{sst}}}(P_t)$, respectively, and x_{wt}^{c} for the closed point $P_w^{\mathrm{c}} \cap P_t^{\mathrm{c}}$. Since h_w^{c} is generically étale above P_w^{c} and P_t^{c}, [Tamagawa 2004b, Lemma 2.1(iii)] implies that $(h_w^{\mathrm{c}})^{-1}(x_{wt}^{\mathrm{c}})_{\mathrm{red}}$ are nodes. Thus, $(\mathscr{Y}/I_{P_w})^{\mathrm{c}}$ is a semistable curve over S, and moreover, h_w^{c} is étale over x_{wt}^{c}. Then the inertia subgroups of the closed points $(h_w^{\mathrm{c}})^{-1}(x_{wt}^{\mathrm{c}})_{\mathrm{red}}$ of the special fiber $(\mathscr{Y}/I_{P_w})_s^{\mathrm{c}}$ of $(\mathscr{Y}/I_{P_w})^{\mathrm{c}}$ are trivial.

On the other hand, since I_{P_w} is a proper subgroup of I_{P_v}, we have that the inertia subgroups of the irreducible components of $h_w^{-1}\big(\bigcup_{j=w+1}^{t-1} P_j\big)_{\mathrm{red}}$ is I_{P_v}/I_{P_w}. Thus, the inertia subgroups of the closed points $c_{\mathscr{Y}/I_{P_w}}(h_w^{-1}\big(\bigcup_{j=w+1}^{t-1} P_j\big)_{\mathrm{red}}) = (h_w^{\mathrm{c}})^{-1}(x_{wt}^{\mathrm{c}})_{\mathrm{red}} \subseteq (\mathscr{Y}/I_{P_w})_s^{\mathrm{c}}$ are not trivial. This is a contradiction. Then we complete the proof of the lemma. □

The above lemma implies the following corollary.

Corollary 3.11. *We maintain the settings introduced in Theorem 3.9. Suppose that G is a cyclic p-group, and that I_{P_0} is equal to G. Then we have*

$$\sigma(f^{-1}(x)) = \#G/\#I_{\min} - \#G/\#I_{P_{n+1}},$$

where $I_{\min}$ denotes the group $\bigcap_{i=0}^{n+1} I_{P_i}$.

Proof. The corollary follows immediately from Theorem 3.9 and Lemma 3.10. □

Remark 3.11.1. The formula in Corollary 3.11 had been obtained by Saïdi [1998a, Proposition 1]. On the other hand, Corollary 3.11 implies immediately that

$$\sigma(f^{-1}(x)) \le \#G - 1$$

when G is a cyclic p-group, which is the main theorem of [Saïdi 1998a, Theorem 1].

4. Bounds of *p*-rank of vertical fibers

In this section, we gives an affirmative answer to an open problem posed by Saïdi concerning bounds of p-rank of vertical fibers posed by Saïdi if G is an arbitrary finite *abelian* p-group. The main result of the present section is Theorem 4.3.

4A. The following was asked by Saïdi [1998a, Question]:

> Let G be a finite p-group, and let $f : \mathscr{Y} \to \mathscr{X}$ be a G-*semistable covering* (i.e., $D_X = \varnothing$, see Definition 1.5) over S and $x \in \mathscr{X}_s$ a vertical point (Definition 1.8) associated to f. Suppose that $f^{-1}(x)$ is connected. Whether or not $\sigma(f^{-1}(x))$ can be bounded by a constant which depends only on $\#G$?

The above problem was solved by Saïdi when G is a cyclic p-group (Remark 3.11.1).

4B. *Settings.* We maintain the settings introduced in Section 1C1 and assume that $\mathscr{X}$ is a *stable curve* over S (i.e., $D_X = \varnothing$). Moreover, when x is a *node* of $\mathscr{X}_s^{\mathrm{sst}}$, let $\mathcal{V}_x$ be a collection of vertical fibers (Definition 3.5) and $\mathcal{I}_{\mathcal{V}_x} \stackrel{\mathrm{def}}{=} \{I_{V_i} \subseteq G\}_{i=\{0,\dots,n+1\}}$ the set of inertia subgroups of V_i (Definition 3.5). Furthermore, in the remainder of the present section, *we assume that G is an finite abelian p-group.*

4C. Since G is abelian, I_{V_i}, $\{i \in \{0, \dots, n+1\}$, does not depend on the choice of V_i. We use the notation I_{P_i} to denote I_{V_i} for $i \in \{0, \dots, n+1\}$. Then we have the following proposition.

Proposition 4.1. *Let I' and I'' be minimal elements of $\mathcal{I}_{\mathcal{V}_x}$ (Definition 3.5(b)) distinct from each other. Then neither $I' \subseteq I''$ nor $I' \supseteq I''$ holds.*

Proof. Without loss of generality, we may assume that $I' = I_{P_a}$ and $I'' = I_{P_b}$ such that $0 \le a < b \le n+1$, $I_{P_a} \neq I_{P_{a+1}}$, and $I_{P_{b-1}} \neq I_{P_b}$. Note that by the definition of minimal elements (Definition 3.5 (b)), $I_{P_{a+1}}$ (resp. $I_{P_{b-1}}$) contains I_{P_a} (resp. I_{P_b}).

If $I' \subseteq I''$, we consider the quotient curve $\mathscr{Y}/I''$. Then we obtain morphisms of semistable curves $\xi_1 : \mathscr{Y} \to \mathscr{Y}/I''$ and $\xi_2 : \mathscr{Y}/I'' \to \mathscr{X}^{\mathrm{sst}}$ such that $\xi_2 \circ \xi_1 = h$. Note that $h(V_a) = P_a$ and $h(V_b) = P_b$, respectively. By contracting $\bigcup_{i=a+1}^{b-1} P_i$ and $\xi_2^{-1}(\bigcup_{i=a+1}^{b-1} P_i)_{\mathrm{red}}$ [Bosch et al. 1990, 6.7 Proposition 4], respectively, we obtain contracting morphisms $c_{\mathscr{X}^{\mathrm{sst}}} : \mathscr{X}^{\mathrm{sst}} \to (\mathscr{X}^{\mathrm{sst}})^{\mathrm{c}}$ and $c_{\mathscr{Y}/I''} : \mathscr{Y}/I'' \to (\mathscr{Y}/I'')^{\mathrm{c}}$, respectively. Moreover, ξ_2 induces a morphism $\xi_2^{\mathrm{c}} : (\mathscr{Y}/I'')^{\mathrm{c}} \to (\mathscr{X}^{\mathrm{sst}})^{\mathrm{c}}$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathscr{Y}/I'' & \xrightarrow{c_{\mathscr{Y}/I''}} & (\mathscr{Y}/I'')^{\mathrm{c}} \\ {\scriptstyle \xi_2}\downarrow & & \downarrow{\scriptstyle \xi_2^{\mathrm{c}}} \\ \mathscr{X}^{\mathrm{sst}} & \xrightarrow{c_{\mathscr{X}^{\mathrm{sst}}}} & (\mathscr{X}^{\mathrm{sst}})^{\mathrm{c}} \end{array}$$

Note that $(\mathscr{X}^{\mathrm{sst}})^{\mathrm{c}}$ is a semistable curve over S.

Since $I' = I_{P_a} \subseteq I'' = I_{P_b}$, ξ_2^{c} is étale at the generic points of $c_{\mathscr{Y}/I''} \circ \xi_1(V_a)$ and $c_{\mathscr{Y}/I''} \circ \xi_1(V_b)$. Thus, by applying the Zariski–Nagata purity theorem and [Tamagawa 2004b, Lemma 2.1(iii)], we obtain that ξ_2^{c} is étale at $c_{\mathscr{Y}/I''} \circ \xi_1(V_a) \cap c_{\mathscr{Y}/I''} \circ \xi_1(V_b)$ (i.e., the inertia group of each point of $c_{\mathscr{Y}/I''} \circ \xi_1(V_a) \cap c_{\mathscr{Y}/I''} \circ \xi_1(V_b)$ is trivial). On the other hand, since $I_{P_{b-1}}$ contains I_{P_b}, the inertia group of each point of $c_{\mathscr{Y}/I''} \circ \xi_1(V_a) \cap c_{\mathscr{Y}/I''} \circ \xi_1(V_b)$ is $I_{P_{b-1}}/I''$. Then we obtain $I_{P_{b-1}} = I''$. This is a contradiction. Then I'' does not contain I'.

Similar arguments to the arguments given in the proof above imply that I' does not contain I''. We complete the proof of the proposition. □

4D. Let N be a finite p-group and H a subgroup of N. Write Sub(−) for the set of the subgroups of (−). We put

$$\#I(H) \overset{\text{def}}{=} \max\big\{\#\mathcal{S} \mid \mathcal{S} \subseteq \mathrm{Sub}(N),\ H \in \mathcal{S},\ \text{ for any } H',\ H'' \in \mathcal{S} \text{ such that } H' \neq H'', \\ \text{neither } H' \subseteq H'' \text{ nor } H' \supseteq H'' \text{ holds}\big\}.$$

Moreover, we put

$$M(N) \overset{\text{def}}{=} \max\{\#I(N')\}_{N' \in \mathrm{Sub}(N)}.$$

For any $1 \leq d \leq \#N$, write $S_d(N)$ for the set of the subgroups of N with order d. Let A be an elementary abelian p-group (i.e., $pA = 0$) such that $\#A = \#N$. We put

$$B(\#N) \overset{\text{def}}{=} \#\,\mathrm{Sub}(A).$$

Note that $B(\#N)$ depends only on $\#N$.

4E. We need a lemma of finite groups.

Lemma 4.2. *Let N be a finite p-group, A an elementary abelian p-group with order $\#N$, and $1 \leq d \leq \#N$ an integer number. Then we have $\#S_d(N) \leq \#S_d(A)$. In particular, we have $M(N) \leq B(\#N)$.*

Proof. Since N is a p-group, N has a nontrivial central subgroup. Fix a central subgroup Z of order p in N. Write $S_d^Z(N)$ (resp. $S_d^{\backslash Z}(N)$) for the set of subgroups of N of order d which contain Z (resp. do not contain Z). If H is a subgroup of N/Z, let $S_d^{(Z,H)}(N)$ be the set of $L \in S_d^{\backslash Z}(N)$ whose projection on N/Z is H. Let $S_d[N/Z]$ be the set of $H \in S_d(N/Z)$ for which $S_d^{(Z,H)}(N) \neq \varnothing$.

Let $H \in S_d[N/Z]$. Then we obtain $\#S_d^{(Z,H)}(N) \leq \#H^1(H, Z) = \#\,\mathrm{Hom}(H^{\mathrm{ab},p}, Z)$, where $(-)^{\mathrm{ab},p}$ denotes $(-)/((-)^p[(-),(-)])$. Moreover, let H' be a subgroup of A of order d and $Z' \cong \mathbb{Z}/p\mathbb{Z}$ a subgroup of A of order p. Then we have

$$\#\,\mathrm{Hom}(H^{\mathrm{ab},p}, Z) \leq \#\,\mathrm{Hom}((H')^{\mathrm{ab},p}, Z').$$

If $d = 1$, the lemma is trivial. Then we may assume that p divides d. We have

$$\begin{aligned}
\#S_d(N) &= \#S_d^Z(N) + \#S_d^{\backslash Z}(N) = \#S_{d/p}(N/Z) + \#S_d^{\backslash Z}(N) \\
&= \#S_{d/p}(N/Z) + \sum_{H \in S_d[N/Z]} \#S_d^{(Z,H)}(N) \\
&\leq \#S_{d/p}(N/Z) + \sum_{H \in S_d[N/Z]} \#(\mathrm{Hom}(H^{\mathrm{ab},p}, Z)) \\
&\leq \#S_{d/p}(N/Z) + \#S_d(N/Z)\#(\mathrm{Hom}((H')^{\mathrm{ab},p}, Z'))
\end{aligned}$$

By induction, we have $\#S_{d/p}(N/Z) \le \#S_{d/p}(A/Z')$ and $\#S_d(N/Z) \le \#S_d(A/Z')$. Moreover, we have

$$\begin{aligned}\#S_d(A) &= \#S_{d/p}(A/Z') + \sum_{H' \in S_d[A/Z']} \#S_d^{(Z',H')}(A) \\ &= \#S_{d/p}(A/Z') + \sum_{H' \in S_d[A/Z']} \#(\mathrm{Hom}((H')^{\mathrm{ab},p}, Z')) \\ &= \#S_{d/p}(A/Z') + \#S_d(A/Z')\#(\mathrm{Hom}((H')^{\mathrm{ab},p}, Z')).\end{aligned}$$

Thus, we obtain

$$\#S_d(N) \le \#S_d(A).$$

This completes the proof of the lemma. □

4F. We have the following result.

Theorem 4.3. *We maintain the settings introduced in Section 1C1. Let G be a finite p-group, and let $f : \mathscr{Y} \to \mathscr{X}$ be a G-semistable covering (i.e., $D_X = \varnothing$, see Definition 1.5) over S and $x \in \mathscr{X}_s$ a vertical point (Definition 1.8) associated to f. Suppose that $f^{-1}(x)$ is connected, and that G is an abelian p-group. Then we have (see Section 4D for $M(G)$, $B(\#G)$)*

$$\sigma(f^{-1}(x)) \le M(G)\#G - 1 \le B(\#G)\#G - 1.$$

In particular, if G is an abelian p-group, then the p-rank $\sigma(f^{-1}(x))$ can be bounded by a constant $B(\#G)$ which depends only on $\#G$.

Proof. If x is a smooth point of the special fiber $\mathscr{X}_s$ of $\mathscr{X}$, then $\sigma(f^{-1}(x)) = 0$ (Theorem 1.9). Thus, we may assume that x is a singular point of $\mathscr{X}_s$.

If $\mathcal{I}(x) = \varnothing$ (Definition 3.5(b)), then Theorem 3.9 implies that $\sigma(f^{-1}(x)) = 0$. If $\mathcal{I}(x) \ne \varnothing$, by applying Theorem 3.9 and Proposition 4.1, we obtain

$$\sigma(f^{-1}(x)) = \sum_{I \in \#I \in \mathcal{I}(x)} \#G/\#I - \sum_{\#J \in \mathcal{J}(x)} \#G/\#J + 1 \le \#\mathcal{I}\#G - 1 \le M(G)\#G - 1 \le B(\#G)\#G - 1. \quad \square$$

Remark 4.3.1. If G is a cyclic p-group, then by the definition of $M(G)$, we have $M(G) = 1$. Thus, if G is a cyclic p-group, we have $\sigma(f^{-1}(x)) \le \#G - 1$. This is the main theorem of [Saïdi 1998a, Theorem 1].

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