Algebra & Number Theory

Volume 18
2024
No. 3

msp

Algebra & Number Theory

msp.org/ant

See inside back cover or msp.org/ant for submission instructions.

The subscription price for 2024 is US $525/year for the electronic version, and $770/year (+$65, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP.

Algebra & Number Theory (ISSN 1944-7833 electronic, 1937-0652 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online.

ANT peer review and production are managed by EditFLOW® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

ALGEBRA AND NUMBER THEORY 18:3 (2024)
https://doi.org/10.2140/ant.2024.18.409

Quotients of admissible formal schemes and adic spaces by finite groups

Bogdan Zavyalov

We give a self-contained treatment of finite group quotients of admissible (formal) schemes and adic spaces that are locally topologically finite type over a locally strongly noetherian adic space.

1. Introduction

1.1. ***Overview.*** This paper studies "geometric quotients" in different geometric setups. Namely, we work in three different situations: flat and locally finite type schemes over a typically nonnoetherian valuation ring, admissible formal schemes over a complete microbial valuation ring (see Definition 3.1.1), and locally topologically finite type adic spaces over a locally strongly noetherian analytic adic space. These 3 different contexts occupy Sections 2, 3, and 4, respectively.

The motivation to study these quotients comes from [Zavyalov 2021a], where we show a refined version of Temkin's local alteration theorem. Our result roughly says that any smooth rigid-space X over an algebraically closed nonarchimedean field C locally admits a formal $\mathcal{O}_C$-model $\mathfrak{X}$ such that $\mathfrak{X}$ is a quotient of a polystable admissible formal $\mathcal{O}_C$-model $\mathfrak{X}'$ by a finite group G (acting freely on the generic fiber). This refined uniformization result is an important technical input in the author's proof of p-adic Poincaré duality in [Zavyalov 2021b].

The actual formulation of this uniformization result is quite technical, and we refer to [Zavyalov 2021a, Theorem 1.4] for the precise formulation. We only mention that, in order to formulate *and* prove this theorem, we had to make sure that a quotient of an admissible formal $\mathcal{O}_C$-scheme by an $\mathcal{O}_C$-action of a

MSC2020: 14A99.
Keywords: adic spaces.

finite group exists as an admissible formal $\mathcal{O}_C$-scheme. This result seems to be missing in the literature, the main difficulty being that the ring $\mathcal{O}_C$ is never noetherian.

Scheme case. Before we deal with quotients of formal schemes and adic spaces, we first discuss quotients of schemes over the base scheme $S = \operatorname{Spec} \mathcal{O}_C$. Even this question is already nontrivial and demonstrates an important source of difficulties in the question of studying quotient spaces in the nonnoetherian situation. The same difficulty will arise in every other setup treated in this paper.

We fix an S-scheme X with an S-action of a finite group G. Then a standard argument constructs the quotient X/G as an S-scheme (under some assumptions); this is carried out in [SGA 1, Exposé V, Section 1] (see also Definition 2.1.1 and Theorem 2.1.15). However, the question of whether, for a finite type S-scheme X, the quotient X/G is of finite type is quite nontrivial.

To explain the main issue, we briefly recall what happens in the classical situation of a finite type R-scheme X with an R-action of a finite group G for some *noetherian* ring R. Under some mild assumptions on X,[1] one can rather easily reduce to the affine situation $X = \operatorname{Spec} A$, where the main work is to show that A^G is of finite type over R. This is done in two steps: one firstly checks that A is a finite A^G-module, and then one uses the Artin–Tate Lemma:

Lemma 1.1.1 [Atiyah and Macdonald 1969, Proposition 7.8]. *Let R be a noetherian ring, and $B \subset C$ an inclusion of R-algebras. Suppose that C is a finite type R-algebra, and C is a finite B-module. Then B is finitely generated over R.*

One may think that probably the Artin–Tate lemma can hold, more generally, over a nonnoetherian base R if C is finitely presented over R. However, this is not the case and the Artin–Tate Lemma fails over any nonnoetherian base:

Example 1.1.2. Let R be a nonnoetherian ring with an ideal I that is not finitely generated. Consider the R-algebra $C := R[\varepsilon]/(\varepsilon^2)$, and the R-subalgebra $B = R \oplus I\varepsilon$. So C is a finitely presented R-algebra, and C is finite as a B-module since it is already finite over R. However, B is not finitely generated R-algebra as that would imply that I is a finitely generated ideal.

Example 1.1.2 shows that the strategy should be appropriately modified in the nonnoetherian situations like schemes over $\mathcal{O}_C$. We deal with this issue by proving a weaker version of the Artin–Tate lemma over any valuation ring k^+ (see Lemma 2.2.3). That proof crucially exploits features of finitely generated algebras over a valuation ring. We emphasize that our argument does use the k^+-flatness assumption in a serious way; we do not know if the quotient of a finitely presented affine k^+-scheme by a finite group action is finitely presented (or finitely generated) over k^+.

Formal schemes and adic spaces. The strategy above can be appropriately modified to work in the world of admissible formal schemes and strongly noetherian adic spaces. In both situations, the main new input is a corresponding version of the Artin–Tate lemma (see Lemmas 3.2.5 and 4.2.4). However, there are

[1] In particular, if X is quasiprojective over R.

issues that are not seen in the scheme case. We explain a few of the main new technical difficulties that arise while proving the result in the world of adic spaces.

Compared to the affine (formal) schemes, the underlying topological space of an affinoid space $\mathrm{Spa}(A, A^+)$ is harder to express in terms of the pair (A, A^+). It is a set of all *valuations* on A with corresponding continuity and integrality conditions. In particular, even if one works with rigid spaces over a nonarchimedean field K, one has to take into account points of higher rank that do not have any immediate geometric meaning. Hence, it takes extra care to identify $\mathrm{Spa}(A^G, A^{+,G})$ with $\mathrm{Spa}(A, A^+)/G$ even on the level of *underlying topological spaces*.

Furthermore, the notion of a topologically finite type (resp. finite) morphism of Tate–Huber pairs is more subtle than its counterpart in the algebraic setup for two different reasons. Firstly, it has a topological aspect that takes some care to work with. Secondly, the notion involves conditions on *both* A and A^+ (see Definitions B.2.1 and B.2.6). Usually, A^+ is nonnoetherian, so it requires some extra work to check the relevant condition on it.

Generality. In the case of adic spaces, we consider spaces that are locally topologically finite type over a strongly noetherian analytic adic space in Section 4. One reason for this level of generality is to include adic spaces that are topologically finite type over $\mathrm{Spa}(k, k^+)$ for a microbial valuation ring k^+ (see Definition 3.1.1). These spaces naturally arise while studying fibers of morphisms of rigid spaces $X \to Y$ over points of Y of *higher rank*.[2] We think that the category of strongly noetherian analytic adic spaces is the natural one to consider.[3] One of its advantages is that it contains both topologically finite type morphisms and morphisms coming from the (not necessary finite) base field extension in rigid geometry.

In the case of formal schemes, the results of Section 3 are written in the generality of admissible formal schemes over a complete, microbial valuation ring k^+ (see Definition 3.1.1). We want to point out that Appendix A contains versions of the main results of Section 3 for a topologically universally adhesive base (see Definition A.3.12). These results are more general and include *both* the cases of formal schemes topologically finite type (and flat) over some k^+ and noetherian formal schemes. However, we prefer to formulate and prove the results in the main body of the paper for admissible formal schemes over k^+ since it simplifies the exposition a lot. We only refer to Appendix A for the necessary changes that have to be made to make the arguments work in the more general adhesive situation.

Likewise, Appendix A has versions of the results of Section 2 over a universally adhesive base (see Definition A.2.1). But we want to point out that a valuation ring k^+ is universally adhesive only if it is microbial (see Lemma A.2.3), so the results of Appendix A do not fully subsume the results of Section 2.2.

[2] Considered as adic spaces.

[3] It may be also reasonable to consider some class of *nonanalytic* adic spaces. However, it is not clear what should be the correct uniform condition on an adic space that would imply the Artin–Tate lemma in both analytic and nonanalytic nonnoetherian setups. Since we never need to use nonanalytic adic spaces in our intended applications, we prefer to work only with analytic adic spaces in this paper.

1.2. ***Comparison with*** **[Hansen 2021].** While writing this paper, we found that similar results for adic spaces were already obtained in [Hansen 2021]. We briefly discuss the main similarities and differences in our approaches.

David Hansen separately discusses two different situations: rigid spaces over a nonarchimedean field K,[4] and general analytic adic spaces. In the former case, he shows that (under some assumptions on X) X/G exists as a rigid space over K for any finite group G. He crucially uses [BGR 1984, Proposition 6.3.3/3] that states that for a K-affinoid affinoid A with a K-action of a finite group G, the ring of invariants A^G is a K-affinoid algebra. The proof of this result uses analytic input: the Weierstrass preparation theorem. In the latter case, he shows that the quotient of X exists as an analytic adic space if the order of G is invertible in $\mathcal{O}_X(X)$. The argument there is based on an averaging trick, so it uses the invertibility assumption in order to be able to divide by $\#G$. We note that if X is a perfectoid space over a perfectoid field, he can drop this invertibility assumption by some other argument. The whole point of the latter case is to be able to work with "big" adic spaces such as perfectoid spaces.

In contrast with Hansen's approach, our methods neither use any nontrivial input from nonarchimedean analysis, nor the averaging trick. What we do is try to imitate the classical algebraic argument based on the Artin–Tate lemma in the setup of strongly noetherian adic spaces. More precisely, we show that if X is a locally topologically finite type adic space (with some other conditions) over a locally strongly noetherian adic space S with an S-action of a finite group G then the quotient X/G exists as a locally topologically finite type adic S-space. Our result does not recover Hansen's result as we do not allow "big" adic spaces such as perfectoid spaces, but it proves a stronger statement in the case of adic spaces locally of finite type over a locally strongly noetherian S as we do not have any assumptions on the order of G. Moreover, even in the case of rigid spaces, it gives a new proof of the existence of X/G as a rigid space that does not use much of analytic theory.

1.3. ***Our results.*** We first study the case of a flat, locally finite type scheme X over a valuation ring k^+ and a k^+-action of a finite group G. We show that X/G exists as a flat, locally finite type k^+-scheme under a mild assumption on X.

Theorem 1.3.1 (Theorems 2.1.15 and 2.2.6). *Let X be a flat, locally finite type k^+-scheme with a k^+-action of a finite group G. Suppose that each point $x \in X$ admits an affine neighborhood V_x containing $G.x$. Then X/G exists as a flat, locally finite type k^+-scheme. Moreover, it satisfies the following properties:*

(1) *$\pi : X \to X/G$ is universal in the category of G-invariant morphisms to locally ringed S-spaces.*

(2) *$\pi : X \to X/G$ is a finite, finitely presented morphism (in particular, it is closed).*

(3) *Fibers of π are exactly the G-orbits.*

(4) *The formation of the quotient X/G commutes with flat base change (see Theorem 2.1.15(4) for the precise statement).*

[4] Defined as locally topologically finite type adic spaces over $\mathrm{Spa}(K, \mathcal{O}_K)$.

We then consider quotients of admissible formal schemes $\mathfrak{X}$ over a complete microbial valuation ring k^+ by a k^+-action of a finite group G. Under similar conditions, we show that $\mathfrak{X}/G$ exists as an admissible formal k^+-scheme and satisfies the expected properties.

Theorem 1.3.2 (Theorem 3.3.4). *Let $\mathfrak{X}$ be an admissible formal k^+-scheme with a k^+-action of a finite group G. Suppose that each point $x \in \mathfrak{X}$ admits an affine neighborhood $\mathfrak{V}_x$ containing $G.x$. Then $\mathfrak{X}/G$ exists as an admissible formal k^+-scheme. Moreover, it satisfies the following properties*:

(1) *$\pi : \mathfrak{X} \to \mathfrak{X}/G$ is universal in the category of G-invariant morphisms to topologically locally ringed spaces over $\mathfrak{S}$.*

(2) *$\pi : \mathfrak{X} \to \mathfrak{X}/G$ is a surjective, finite, topologically finitely presented morphism (in particular, it is closed).*

(3) *Fibers of π are exactly the G-orbits.*

(4) *The formation of the geometric quotient commutes with flat base change (see Theorem 3.3.4(4) for the precise statement).*

Finally, we consider the case of locally topologically finite type adic spaces over a locally strongly noetherian adic space.

Theorem 1.3.3 (Theorem 4.3.4). *Let S be a locally strongly noetherian analytic adic space (see Definition B.2.15), and X a locally topologically finite type adic S-space with an S-action of a finite group G. Suppose that each point $x \in X$ admits an affinoid open neighborhood V_x containing $G.x$. Then X/G exists as a locally topologically finite type adic S-space. Moreover, it satisfies the following properties*:

(1) *$\pi : X \to X/G$ is universal in the category of G-invariant morphisms to topologically locally v-ringed S-spaces (see Definition B.1.1).*

(2) *$\pi : X \to X/G$ is a finite, surjective morphism (in particular, it is closed).*

(3) *Fibers of π are exactly the G-orbits.*

(4) *The formation of the geometric quotient commutes with flat base change (see Theorem 4.3.4(4) for the precise statement).*

The condition that each point $x \in X$ admits an affinoid open neighborhood V_x containing $G.x$ is a much milder in the adic world than in the scheme world. For example, we show that it is automatic if X is a separated rigid-analytic space. This, in particular, implies that a quotient of a separated rigid-analytic space by a finite group action always exists as a rigid-analytic space. In case of a free finite group action, a similar result has been previously obtained in [Conrad and Temkin 2009, Theorem 5.1.1] in the world of Berkovich spaces.

Lemma 1.3.4 (Lemma 4.3.6). *Let K be a nonarchimedean field with the residue field k, X a separated, locally finite type adic* $\mathrm{Spa}(K, \mathcal{O}_K)$*-space, and $\{x_1, \ldots, x_n\}$ is a finite set of points of X. Then there is an open affinoid subset $U \subset X$ containing all x_i.*

Remark 1.3.5. It is reasonable to expect that the assumption of Theorem 1.3.3 is automatic as long as X is S-separated (see Remark 4.3.5). However, the proof of this claim would seem to require a generalization of the main results of [Temkin 2000] to more general adic spaces. This is beyond the scope of this paper.

The natural question is whether these quotients commute with certain functors like formal completion, analytification, and adic generic fiber. We show that this is indeed the case, i.e., the formation of the geometric quotients commutes with the functors mentioned above whenever they are defined. We informally summarize the results below:

Theorem 1.3.6 (Theorems 3.4.1, 4.4.1, and 4.5.3). (1) *Let k^+ be a microbial valuation ring, and X a flat, locally finite type k^+-scheme with a k^+-action of a finite group G. Suppose X satisfies the assumption of Theorem 1.3.1. The natural morphism $\widehat{X}/G \to \widehat{X/G}$ is an isomorphism.*

(2) *Let k^+ be a complete, microbial valuation ring with fraction field k, and $\mathfrak{X}$ an admissible formal k^+-scheme with a k^+-action of a finite group G. Suppose $\mathfrak{X}$ satisfies the assumption of Theorem 1.3.2. The natural morphism $\mathfrak{X}_k/G \to (\mathfrak{X}/G)_k$ is an isomorphism.*

(3) *Let K be a complete rank-*1 *valued field, and X a locally finite type K-scheme with a K-action of a finite group G. Suppose X satisfies the assumption of Theorem 2.1.15. The natural morphism $X^{\mathrm{an}}/G \to (X/G)^{\mathrm{an}}$ is an isomorphism.*

2. Quotients of schemes

2.1. ***Review of classical theory.*** We review the classical theory of quotient of schemes by an action of a finite group. This theory was developed in [SGA 1, Exposé V, Section 1]. We review the main results from there, and present some proofs in a way that will be useful for our later purposes. This section is mostly expository.

For the rest of this section, we fix a base scheme S.

Definition 2.1.1. Let G be a finite group, and X a locally ringed space over S with a right S-action of G. The *geometric quotient* $X/G = (|X/G|, \mathcal{O}_{X/G}, h)$ consists of:

- The topological space $|X/G| := |X|/G$ with the quotient topology. We denote by $\pi : |X| \to |X/G|$ the natural projection.
- The sheaf of rings $\mathcal{O}_{X/G} := (\pi_*\mathcal{O}_X)^G$.
- The morphism $h : X/G \to S$ defined by the pair $(h, h^\#)$, where $h : |X|/G \to S$ is the unique morphism induced by $f : X \to S$ and $h^\#$ is the natural morphism

$$\mathcal{O}_S \to h_*(\mathcal{O}_{X/G}) = h_*((\pi_*\mathcal{O}_X)^G) = (h_*(\pi_*\mathcal{O}_X))^G = (f_*\mathcal{O}_X)^G$$

  that comes from G-invariance of f.

We note that X/G is, a priori, only a ringed space. In the lemma below, we show that it is actually always a locally ringed space:

Lemma 2.1.2. *Let X be a locally ringed space over S with a right S-action of a finite group G. Then X/G is a locally ringed space, and $\pi\colon X \to X/G$ is a map of locally ringed spaces (so $X/G \to S$ is too).*

Remark 2.1.3. This lemma must be well-known, but we do not know any particular reference. We decided to include the proof as it will be a convenient technical tool for us.

Lemma 2.1.2 allows us to construct quotients entirely in the category of locally ringed spaces and not merely in the category of all ringed spaces. The main technical issue with the category of ringed spaces is that locally ringed spaces do not form a full subcategory of it.

Proof of Lemma 2.1.2. We note that the action of G induces a family of ring isomorphisms

$$\mathcal{O}_X(g(U)) \xrightarrow{a_g^U} \mathcal{O}_X(U)$$

for $g \in G$ and open $U \subset X$. Furthermore, for any inclusion of open subsets $V \subset U \subset X$, the diagram

$$\begin{array}{ccc} \mathcal{O}_X(g(U)) & \xrightarrow{a_g^U} & \mathcal{O}_X(U) \\ \downarrow r^{g(U)}_{g(V)} & & \downarrow r^U_V \\ \mathcal{O}_X(g(V)) & \xrightarrow{a_g^V} & \mathcal{O}_X(V) \end{array} \tag{1}$$

is commutative. In particular, G acts on $\mathcal{O}_X(U)$ for any G-stable open $U \subset X$. We describe the stalk $\mathcal{O}_{X/G,\bar{x}}$ for a point $\bar{x} \in X/G$ with a lift $x \in X$ as follows:

$$\mathcal{O}_{X/G,\bar{x}} \simeq \operatorname{colim}_{\{x \in U \subset X \mid g(U)=U \ \forall g \in G\}} \mathcal{O}_X(U)^G. \tag{2}$$

That being said, we wish to show that

$$\pi^\sharp_{\bar{x}}\colon \mathcal{O}_{X/G,\bar{x}} \to \mathcal{O}_{X,x}$$

is a local homomorphism of local rings. This is equivalent to saying that $\mathfrak{m}_x \cap \mathcal{O}_{X/G,\bar{x}}$ is the unique maximal ideal in $\mathcal{O}_{X/G,\bar{x}}$ or, equivalently, that any $f \in \mathcal{O}^\times_{X,x} \cap \mathcal{O}_{X/G,\bar{x}}$ lies in $\mathcal{O}^\times_{X/G,\bar{x}}$.[5]

We use (2) to find a G-stable open $x \in U \subset X$ such that f comes from an element $f_U \in \mathcal{O}_X(U)^G$. The condition that f becomes invertible in $\mathcal{O}_{X,x}$ means that there is an open $x \in V \subset U$ and a function $k_V \in \mathcal{O}_X(V)$ such that

$$k_V \cdot f_U|_V = 1 \in \mathcal{O}_X(V).$$

We set $k_{g(V)} := (a_g^V)^{-1}(k_V) \in \mathcal{O}_X(g(V))$ for $g \in G$. Then G-invariance of f_U and (1) imply that $k_{g(V)} \cdot f_U|_{g(V)} = 1 \in \mathcal{O}_X(g(V))$. Uniqueness of the inverse element and the sheaf axioms imply that $k_{g(V)}$ glue to a section

$$k \in \mathcal{O}_X(W),$$

[5] In what follows, we slightly abuse the notation and write $T \cap \mathcal{O}_{X/G,\bar{x}}$ as an abbreviation for $(\pi^\sharp_{\bar{x}})^{-1}(T) \subset \mathcal{O}_{X/G,\bar{x}}$ for any subset $T \subset \mathcal{O}_{X,x}$.

where $W = \bigcup_{g \in G} g(V)$ is a G-stable open subset of X. Then $k \cdot f_U|_W = 1 \in \mathcal{O}_X(W)$ since this can be checked locally. In particular, k is an inverse in $f_U|_W$, so G-invariance of $f_U|_W$ implies G-invariance of k. In particular, $f_U|_W \in (\mathcal{O}_X(W)^G)^\times$ implying that $f \in \mathcal{O}^\times_{X/G,\bar{x}}$. □

Remark 2.1.4. It is straightforward to see that the pair $(X/G, \pi)$ is a universal object in the category of G-invariant morphisms to locally ringed spaces over S.

Remark 2.1.5. We warn the reader that X/G might not be a scheme even if $S = \operatorname{Spec} \mathbf{C}$ and X is a smooth and proper, connected $\mathbf{C}$-scheme with a $\mathbf{C}$-action of $G = \mathbf{Z}/2\mathbf{Z}$. Namely, Hironaka's example [Olsson 2016, Example 5.3.2] is a smooth and proper, connected 3-fold over $\mathbf{C}$ with a $\mathbf{C}$-action of $\mathbf{Z}/2\mathbf{Z}$ such that there is an orbit $G.x$ that is not contained in any open affine subscheme $U \subset X$. Lemma 2.1.7 below implies that X/G is not a scheme.

Lemma 2.1.6. *Let X be an S-scheme with an S-action of a finite group G. Suppose that each point $x \in X$ admits an open affine subscheme V_x that contains the orbit $G.x$. Then the same holds with X replaced by any G-stable open subscheme $U \subset X$.*

Proof. Let x be a point in U, and V_x an open affine in X that contains $G.x$. Consider $W_x := U \cap V_x$ that is an open (possibly nonaffine) neighborhood of $x \in U$ containing $G.x$. It suffices to show the stronger claim that *any* finite set of points in W_x is contained in an open affine. This follows from [EGA II, Corollaire 4.5.4] as W_x is an open subscheme inside the affine scheme V_x.[6] □

Lemma 2.1.7. *Let R be a noetherian ring, and X a separated, finite type R-scheme with an R-action of a finite group G. Suppose that there is a point $x \in X$ such that the orbit $G.x$ is not contained in any open affine subscheme $U \subset X$. Then X/G is a not a scheme.*

Remark 2.1.8. Lemma 2.1.7 must have been known to experts for a long time. However, we are not aware of any reference for this fact. For example, [SGA 1, Exposé V, Proposition 1.8] discusses only a (rather straightforward) statement that it is impossible for X/G to be a scheme *and* for $\pi : X \to X/G$ to be affine.[7] We strengthen the result and show that X/G is not a scheme without the affineness requirement on π.

Proof. Suppose that X/G is an R-scheme, and consider the image $\bar{x} := \pi(x) \in X/G$. It admits an affine neighborhood $\overline{U} \subset X/G$; this defines an open G-stable subscheme $U := \pi^{-1}(\overline{U}) \subset X$ containing the orbit $G.x$.

Now we note that the morphism $\pi|_U : U \to \overline{U}$ is quasifinite and separated. Indeed, it is separated of finite type since U is separated of finite type over R and $\overline{U}$ is separated; its fibers are finite by construction. Therefore, Zariski's main theorem [EGA IV$_4$, Proposition 18.12.12] implies that $\pi|_U$ is quasiaffine, i.e., the natural morphism

$$U \to \operatorname{Spec} \mathcal{O}_U(U)$$

[6] To use this result, we recall that the structure sheaf $\mathcal{O}_Y$ is ample on any affine scheme Y.

[7] Affineness of $X \to X/G$ is part of the definition of an "admissible" action of G on X introduced in [SGA 1, Exposé V, Definition 1.7].

is a quasicompact open immersion. We note that Spec $\mathcal{O}_U(U)$ naturally admits an action of the group G induced by the action of G on $\mathcal{O}_U$. Trivially, any point $y \in \operatorname{Spec} \mathcal{O}_U(U)$ admits an affine neighborhood containing $G.y$. Thus, Lemma 2.1.6 applied to Spec $\mathcal{O}_U(U)$ and its open subscheme U implies that the same holds for U. As a result, the orbit $G.x$ is contained in some open affine subscheme of X. □

Definition 2.1.1 is useless unless we can verify that X/G is a scheme if X is. The main goal of the rest of the section is to review when this is the case under some (mild) assumptions on X.

We start with the case of an affine scheme $X = \operatorname{Spec} A$ and an affine scheme $S = \operatorname{Spec} R$. Then the natural candidate for the geometric quotient is $Y = \operatorname{Spec} A^G$. There is an evident G-invariant S-map $p\colon X \to Y$ that induces a commutative triangle:

$$\begin{array}{ccc} X & & \\ \downarrow{\scriptstyle \pi} & \searrow^{p} & \\ X/G & \xrightarrow{\ \phi\ } & Y \end{array}$$

We wish to show that ϕ is an isomorphism. Before doing this, we need to recall certain (well-known) properties of G-invariants. We include some proofs for the convenience of the reader.

Lemma 2.1.9. *Let A be an R-algebra with an R-action of a finite group G. Then*:

(1) *The inclusion $A^G \to A$ is integral. In particular, the morphism* Spec $A \to$ Spec A^G *is closed.*

(2) Spec $A \to$ Spec A^G *is surjective, the fibers are exactly G-orbits.*

(3) *If A is of finite type over R. Then $A^G \to A$ is finite.*

Proof. This is [SGA 1, Expose V, Proposition 1.1(i), (ii) and Corollaire 1.5]. We also point out that the results follow from [Atiyah and Macdonald 1969, Exercise 5.12, 5.13], and the observation that an integral, finite type morphism is finite. □

Remark 2.1.10. We warn the reader that Lemma 2.1.9 does not imply that, for a finite type R-algebra A, A^G is of finite type over R (since we allow nonnoetherian R as needed later).

Lemma 2.1.11. *Let R be a ring and A an R-algebra with an R-action of a finite group G. Then the formation of invariants A^G commutes with flat base change, i.e., for any flat R-algebra morphism $A^G \to B$ the natural homomorphism $B \to (B \otimes_{A^G} A)^G$ is an isomorphism.*

Proof. The proof is outlined just after [SGA 1, Exposé V, Proposition 1.9]. □

Proposition 2.1.12. *Let $X =$ Spec A be an affine R-scheme with an R-action of a finite group G. Then the natural map $\phi\colon X/G \to Y =$ Spec A^G is an R-isomorphism of locally ringed spaces. In particular, X/G is an R-scheme.*

Proof. This is shown in [SGA 1, Exposé V, Proposition 1.1(iv)]. We review this argument here as this type of reasoning will be adapted to more sophisticated situations later in the paper.

Step 1. *ϕ is a homeomorphism*: We note that Lemma 2.1.9 ensures that $p\colon X \to \operatorname{Spec} A^G$ is a closed, surjective map with fibers being exactly G-orbits. Thus, $\pi: X \to X/G$ and $p\colon X \to \operatorname{Spec} A^G$ are both topological quotient morphisms with the same fibers (namely, G-orbits). So the induced map f is clearly a homeomorphism.

Step 2. *ϕ is an isomorphism of locally ringed spaces*: We use Lemma 2.1.9 again to check that the morphism of sheaves $\phi^{\#}\colon \mathcal{O}_Y \to \phi_*\mathcal{O}_{X/G}$ is an isomorphism. Using the base of basic affine opens in Y, it suffices to show that the map

$$(A^G)_f \to (A_f)^G \simeq (A \otimes_{A^G} (A^G)_f)^G$$

is an isomorphism for any $f \in A^G$. This follows from Lemma 2.1.11 as $(A^G)_f$ is A^G-flat. □

Now we want to discuss when X/G exists as a scheme in the global set-up without a separatedness assumption. Roughly, we want to cover X by G-stable affines and then deduce the claim from Proposition 2.1.12. In order to do this, we need the following lemma:

Lemma 2.1.13. *Let X be an S-scheme with an S-action of a finite group G. Suppose that for any point $x \in X$ there is an open affine subscheme $V_x \subset X$ that contains the orbit $G.x$. Then each point $x \in X$ has a G-stable open affine neighborhood $U_x \subset X$.*

Proof. The proof is outlined just after [SGA 1, Exposé V, Proposition 1.8], we recall the key steps here. Firstly, Lemma 2.1.6 ensures that one can reduce to the case of an affine base $S = \operatorname{Spec} R$. Then one shows the claim for a separated X, in which case $U_x := \bigcap_{g\in G} g(V_x)$ is affine and does the job. In general, Lemma 2.1.6 guarantees that one can replace X with the *separated* open subscheme $\bigcap_{g\in G} g(V_x)$ and reduce to the separated case. □

We recall one case where the condition of Lemma 2.1.13 is satisfied.

Proposition 2.1.14. *Let $\phi\colon X \to S$ be a locally quasiprojective S-scheme with an S-action of a finite group G.*[8] *Then every point $x \in X$ admits an affine neighborhood containing the orbit $G.x$.*

Proof. The statement is local on S, so we may and do assume that $S = \operatorname{Spec} R$ is affine and there is a quasicompact R-immersion $X \subset \mathbf{P}^N_R$. Then it suffices to show a stronger claim that *any* finite set of points is contained in an open affine. This is shown in [EGA II, Corollaire 4.5.4]. □

Now, we are ready to explain the main existence result [SGA 1, Exposé V, Proposition 1.8]. For later needs, we give a slightly different proof.

[8] i.e., there exists an open covering $S = \cup V_j$ such that each $\phi^{-1}(V_j) \to V_j$ factors through a quasicompact immersion $\phi^{-1}(V_j) \to \mathbf{P}^N_{V_j}$ for some N.

Theorem 2.1.15. *Let X be an S-scheme with an S-action of a finite group G. Suppose that each point $x \in X$ admits an affine neighborhood V_x containing $G.x$. Then X/G is an S-scheme. Moreover, it satisfies the following properties*:

(1) *$\pi : X \to X/G$ is universal in the category of G-invariant morphisms to locally ringed S-spaces.*

(2) *$\pi : X \to X/G$ is an integral, surjective morphism (in particular, it is closed). The morphism π is finite if X is locally of finite type over S.*

(3) *Fibers of π are exactly the G-orbits.*

(4) *The formation of the geometric quotient commutes with flat base change, i.e., for any flat morphism $Z \to X/G$, the geometric quotient $(X \times_{X/G} Z)/G$ is a scheme, and the natural morphism $(X \times_{X/G} Z)/G \to Z$ is an isomorphism.*

Proof. Step 1. *X/G is an S-scheme*: We note that the claim is local on S, so we can use Lemma 2.1.6 to reduce to the case where S is affine. Now Lemma 2.1.13 allows to cover X by G-stable open affine subschemes U_i. Then the construction of the geometric quotient implies that

$$\pi(U_i) \subset X/G$$

is an open subset that is naturally isomorphic to U_i/G, and $\pi^{-1}(U_i/G)$ coincides with U_i. This implies that it suffices to show that U_i/G is a scheme. This was already shown in Proposition 2.1.12.

Step 2. *$\pi : X \to X/G$ is surjective, integral (resp. finite) and fibers are exactly the G-orbits*: Similar to step 1, we can assume that X and S are affine. Then apply Lemma 2.1.9.

Step 3. *$\pi : X \to X/G$ is universal and commutes with flat base change*: The universality is essentially trivial (Remark 2.1.4). To show the latter claim, we can again assume that $X = \operatorname{Spec} A$ and $S = \operatorname{Spec} R$ are affine and it suffices to consider affine Z. Then the claim follows from Lemma 2.1.11 and the identification of X/G with $\operatorname{Spec} A^G$. □

2.2. ***Schemes over a valuation ring k^+.*** The main drawback of Theorem 2.1.15 is that if R is not noetherian we do not know if X/G is finite type over $S = \operatorname{Spec} R$ when X is. This makes this theorem not so useful in practice as we often do not want to leave the realm of finite type morphisms. The main work left is to show that the ring of invariants A^G is finite type over R if A is. If R is noetherian, this problem is resolved using the Artin–Tate lemma (Lemma 1.1.1). The main goal of this section is to generalize it to certain nonnoetherian situations.

For the rest of the section, we fix a valuation ring k^+ with fraction field k and maximal ideal $\mathfrak{m}_k$.

Definition 2.2.1. Let $N \subset M$ be an inclusion of k^+-modules. We say that N is *saturated* in M if the quotient M/N is k^+-torsion free.

Lemma 2.2.2. *Let k^+ be a valuation ring, A a finite type k^+-algebra, and M a finite A-module. Then*:

(1) *A k^+-module N is flat over k^+ if and only if it is torsion free.*

(2) *If M is k^+-flat, it is a finitely presented A-module.*

(3) *If A is k^+-flat, it is a finitely presented k^+-algebra.*

(4) *Let $N \subset M$ be a saturated A-submodule of M. Then N is a finite A-module.*

Proof. By [Matsumura 1986, Theorem 7.7] a k^+-module N is flat if and only if $I \otimes_{k^+} N \to N$ is injective for any *finitely generated* ideal $I \subset k^+$. But such I is principal since k^+ is a valuation ring, so we are done; see also [Stacks, Tag 0539] for a different proof.

The second and third claims are proven in [Stacks, Tag 053E].

Now we show the last claim. We consider the quotient module M/N. The saturatedness assumption says that it is k^+-flat, and it is clearly finite as an A-module. Thus, (2) ensures that M/N is finitely presented over A. So N is a finite A-module as it is the kernel of a homomorphism from a finite module to a finitely presented one; see [Stacks, Tag 0519]). □

Lemma 2.2.3 (nonnoetherian Artin–Tate). *Let $A \to B$ be a finite injective morphism of k^+-algebras. Suppose that B is a finite type k^+-algebra and A is a saturated k^+-submodule of B (in the sense of Definition 2.2.1). Then A is a k^+-algebra of finite type.*

Proof. By assumption, B is of finite type over k^+, so there is a finite set of elements $x_i \in B$ such that the k^+-algebra homomorphism

$$p : k^+[T_1, \dots, T_n] \to B$$

that sends T_i to x_i is surjective. Since B is a finite A-module, we can choose some A-module generators $y_1, \dots, y_m \in B$. The choice of $x_1, \dots, x_n$ and $y_1, \dots, y_m$ implies that there are some $a_{i,j}, a_{i,j,l} \in A$ with the relations

$$x_i = \sum_j a_{i,j} y_i \quad \text{and} \quad y_i y_j = \sum_l a_{i,j,l} y_l.$$

Now consider the k^+-subalgebra A' of A generated by all $a_{i,j}$ and $a_{i,j,l}$. Clearly, A' is of finite type over k^+. Moreover, B is finite over A' as $y_1, \dots, y_m$ are A'-module generators of B.

We use Lemma 2.2.2(4) over A' to ensure that A is finite over A' as it is a saturated A'-submodule of the finite A'-module B. Therefore, A is of finite type over k^+. □

Corollary 2.2.4. *Let A be a flat, finite type k^+-algebra with a k^+-action of a finite group G. Then A^G is a finite type flat k^+-algebra, and the natural morphism $A^G \to A$ is finitely presented.*

Proof. Lemma 2.1.9 gives that A is a finite A^G-module, and A^G is easily seen to be saturated in A using that A is k^+-torsion free (because it is k^+-flat). Therefore, Lemma 2.2.2(1) implies that A^G is k^+-flat and Lemma 2.2.3 ensures that A^G is finite type over k^+. Now Lemma 2.2.2(2) guarantees that A is a finitely presented A^G-module as it is A^G-finite and k^+-flat. Thus, it is finitely presented as an A^G-algebra by [EGA IV$_1$, Proposition 1.4.7]. □

Remark 2.2.5. Lemma 2.2.3 and Corollary 2.2.4 have versions over a universally adhesive base (see Definition A.2.1). We refer to Lemma A.2.5 and Corollary A.2.6 for the precise results.

Theorem 2.2.6. *Let X be a flat, locally finite type k^+-scheme with a k^+-action of a finite group G. Suppose that each point $x \in X$ admits an affine neighborhood V_x containing $G.x$. Then the scheme X/G as in Theorem 2.1.15 is flat and locally finite type over k^+, and the integral surjection $\pi : X \to X/G$ is finite and finitely presented.*

Proof. By construction, X/G is clearly k^+-flat. To show that X/G is locally of finite type and that π is finitely presented, we reduce to the affine case by passing to a G-stable affine open covering of X (see Lemma 2.1.13). Now apply Corollary 2.2.4. □

Remark 2.2.7. Theorem 2.2.6 also has a version for the base scheme S that is universally $\mathcal{I}$-adically adhesive for some quasicoherent ideal of finite type $\mathcal{I}$ (see Definition A.2.7). We refer to Theorem A.2.9.

3. Quotients of admissible formal schemes

We discuss the existence of quotients for some class of formal schemes by an action of a finite group G. The strategy to construct the quotient spaces is close to the one used in Section 2. We first construct the candidate space $\mathfrak{X}/G$ that is, a priori, only a topologically locally ringed space. This construction clearly satisfies the universal property, but it is not clear (and generally false) that $\mathfrak{X}/G$ is a formal scheme. We resolve this issue by first showing that it is a formal scheme if $\mathfrak{X}$ is affine. Then we argue by gluing to prove the claim for a larger class of formal schemes.

There are two main complications compared to Section 2. The first one is that we cannot anymore firstly show that $\mathfrak{X}/G$ is a formal scheme by a very general argument and *then* study its properties under further assumptions, e.g., show that it is flat or (topologically) finite type over the base. The problem can be seen even in the case of an affine formal scheme $\mathfrak{X} = \operatorname{Spf} A$. The proof of Proposition 2.1.12 crucially uses that the localization $(A^G)_f$ is A^G-flat for any $f \in A^G$. The analogue in the world of formal schemes would be that the *completed localization* $(A^G)_{\{f\}} = \widehat{(A^G)_f}$ is A^G-flat. However, this requires some finiteness assumption on A^G in order to hold. Therefore, we need to verify algebraic properties of A^G at *the same time* as constructing the isomorphism $\operatorname{Spf} A/G \simeq \operatorname{Spf} A^G$.

The second, related problem is that one needs to be more careful with certain topological aspects of the theory. For instance, the fiber product of affine formal schemes is given by the *completed* tensor product on the level of corresponding algebras. This is a more delicate functor as it is neither left nor right exact. So we pay extra attention to make sure that these complications do not cause any issues under suitable assumptions.

3.1. *The setup and the candidate space $\mathfrak{X}/G$.*

Definition 3.1.1. A valuation ring k^+ is *microbial* if it has a finitely generated (hence principal) ideal of definition I, i.e., any neighborhood $0 \in U \subset k^+$ open in the valuation topology contains I^n for some n.

Definition 3.1.2. An element $\varpi \in k^+$ is a *pseudouniformizer* if $(\varpi) \subset k^+$ is an ideal of definition in k^+.

Example 3.1.3. Any valuation ring k^+ of finite rank is microbial. This follows from the characterization of microbial valuations in [Huber 1996, Definition 1.1.4(e)] or [Seminar 2015, Proposition 9.1.3(3)].

More generally, a valuation ring $k(x)^+ \subset k(x)$ associated with any point $x \in X$ of an *analytic* adic space (see Definition B.1.3) X is microbial. This can be seen from [Huber 1996, Definition 1.1.4(c)] or [Seminar 2015, Proposition 9.1.3(2)].

For the rest of the section, we fix a complete, microbial valuation ring k^+ with a pseudouniformizer ϖ. We denote by $\mathfrak{S}$ the formal spectrum $\operatorname{Spf} k^+$.

A *formal k^+-scheme* will always mean a ϖ-adic formal k^+-scheme. It is easy to see that this notion does not depend on the choice of an ideal of definition.

Definition 3.1.4. A k^+-algebra A is called *admissible* if A is k^+-flat and topologically of finite type (i.e., there is a surjection $k^+\langle t_1, \dots t_d\rangle \to A$).

A formal k^+-scheme $\mathfrak{X}$ is called *admissible* if it is k^+-flat and locally topologically of finite type.

Remark 3.1.5. (1) We note that there are many (nonequivalent) ways to define flatness in formal geometry. They are all equivalent for a morphism $f : \mathfrak{X} \to \mathfrak{Y}$ of locally topologically finite type formal k^+-schemes.

We prefer to use the following as the definition: f is *flat* if $f^\#_{f(x)} : \mathcal{O}_{\mathfrak{Y}, f(x)} \to \mathcal{O}_{\mathfrak{X},x}$ is flat for all $x \in \mathfrak{X}$ (i.e., f is flat as a morphism of locally ringed spaces). We mention that in the case $f : \operatorname{Spf} B \to \operatorname{Spf} A$ a morphism of affine, topologically finite type formal k^+-schemes, this notion is equivalent to the flatness of $A \to B$. This follows from [Fujiwara and Kato 2018, Proposition I.4.8.1] and Remark A.3.3; see [loc. cit., Section I.2.1(a)] to relate adhesiveness to rigid-noetherianness.

(2) Similarly, a morphism $f : \operatorname{Spf} B \to \operatorname{Spf} A$ of formal k^+-schemes is topologically of finite type if and only if $A \to B$ is topologically of finite type; see [loc. cit., Lemma I.1.7.4].

(3) In particular, if $\mathfrak{X} = \operatorname{Spf} A$ is an admissible formal k^+-scheme, the k^+-algebra A is admissible.

We summarize the main properties of locally topologically finite type formal k^+-schemes in the lemma below:

Lemma 3.1.6. *Let k^+ be a complete, microbial valuation ring, A a topologically finite type k^+-algebra, and M a finite A-module. Then:*

1. *M is ϖ-adically complete. In particular, A is ϖ-adically complete.*
2. *If A is k^+-flat, it is topologically finitely presented.*
3. *If M is k^+-flat, it is finitely presented over A.*
4. *Let $N \subset M$ be a saturated (in the sense of Definition 2.2.1) A-submodule of M. Then N is a finite A-module.*
5. *Let $N \subset M$ be an A-submodule of M. Then the ϖ-adic topology on M restricts to the ϖ-adic topology on N.*
6. *For any element $f \in A$, the completed localization $A_{\{f\}} = \lim_n A_f/\varpi^n A_f$ is A-flat.*

The first five results of this lemma are essentially due to Raynaud and Gruson [1971].

Proof. The first claim is [Bosch 2014, Proposition 7.3/8]. The second is [loc. cit., Corollary 7.3/5]. The third in [loc. cit., Theorem 7.3/4]. The fourth and fifth are covered by [loc. cit., Lemma 7.3/7]. For the last claim, we note that [loc. cit., Proposition 7.3/11] ensures that it suffices to show that

$$A/f^n A \to A_{\{f\}}/f^n A_{\{f\}}$$

is flat for any integer $n \geq 1$. Now [Stacks, Tag 05GG] implies that $A_{\{f\}}/f^n A_{\{f\}} \simeq A_f/f^n A_f$, so the desired statement follows from A-flatness of A_f. □

Definition 3.1.7. Let G be a finite group, and $\mathfrak{X}$ a locally topologically ringed space over $\mathfrak{S}$ with a right $\mathfrak{S}$-action of G. The *geometric quotient* $\mathfrak{X}/G = (|\mathfrak{X}/G|, \mathcal{O}_{\mathfrak{X}/G}, h)$ consists of:

- The topological space $|\mathfrak{X}/G| := |\mathfrak{X}|/G$ with the quotient topology. We denote by $\pi : |\mathfrak{X}| \to |\mathfrak{X}/G|$ the natural projection.
- The sheaf of topological rings $\mathcal{O}_{\mathfrak{X}/G} := (\pi_* \mathcal{O}_{\mathfrak{X}})^G$ with the subspace topology.
- The morphism $h : \mathfrak{X}/G \to \mathfrak{S}$ defined by the pair $(h, h^\#)$, where $h : |\mathfrak{X}|/G \to \mathfrak{S}$ is the unique morphism induced by $f : \mathfrak{X} \to \mathfrak{S}$ and $h^\#$ is the natural morphism

$$\mathcal{O}_{\mathfrak{S}} \to h_*(\mathcal{O}_{\mathfrak{X}/G}) = h_*((\pi_*\mathcal{O}_{\mathfrak{X}})^G) = (h_*(\pi_*\mathcal{O}_{\mathfrak{X}}))^G = (f_*\mathcal{O}_{\mathfrak{X}})^G$$

  that comes from G-invariance of f.

Remark 3.1.8. By construction, $\mathfrak{X}/G$ is a topologically ringed $\mathfrak{S}$-space, and $\pi : \mathfrak{X} \to \mathfrak{X}/G$ is a morphism of topologically ringed $\mathfrak{S}$-spaces. Furthermore, Lemma 2.1.2 ensures that $\mathfrak{X}/G$ is a topologically *locally ringed* $\mathfrak{S}$-space, and π is a morphism of topologically *locally ringed* $\mathfrak{S}$-spaces (so $\mathfrak{X}/G \to \mathfrak{S}$ is too). It is trivial to see that the pair $(\mathfrak{X}/G, \pi)$ is a universal object in the category of G-invariant morphisms to topologically locally ringed $\mathfrak{S}$-spaces.

Our main goal is to show that under some mild assumptions, $\mathfrak{X}/G$ is an admissible formal $\mathfrak{S}$-scheme when $\mathfrak{X}$ is. We start with the case of affine formal schemes and then move to the general case.

3.2. ***Affine case.*** We show that the quotient $\mathfrak{X}/G$ of an admissible affine formal k^+-scheme $\mathfrak{X} = \operatorname{Spf} A$ is canonically isomorphic to $\operatorname{Spf} A^G$ that is, in turn, an admissible formal k^+-scheme. We point out that in contrast to the scheme case, we need firstly to establish that A^G is an admissible k^+-algebra, and only then we can show that $\mathfrak{X}/G$ is isomorphic to $\operatorname{Spf} A^G$. Therefore, we start the section with studying certain properties of the ring of invariants A^G.

Remark 3.2.1. Let (R, I) be a ring with a finitely generated ideal I, M an R-module with the I-adic topology, and $f : M \to M$ an R-linear homomorphism. Then f is automatically continuous in the I-adic topology because $f^{-1}(I^n M) \supset I^n M$.

Lemma 3.2.2. *Let (R, I) be a ring with a finitely generated ideal I, and M an I-adically complete R-module with a closed R-submodule $N \subset M$. Then N is complete in the I-adic topology.*

Proof. Since N is closed in M, it is complete for the subspace topology. All this means by design is that

$$N \to \lim_n N/(I^n M \cap N)$$

is an isomorphism, and we need to justify that this implies that

$$N \to \lim_n N/I^n N$$

is an isomorphism. For this purpose, we will crucially use that I is finitely generated.

We start by considering the diagram:

$$\begin{array}{ccc} N & \xrightarrow{\alpha} & \lim_n N/I^n N \\ & \searrow^{\gamma} & \downarrow \beta \\ & & \lim_n N/(I^n M \cap N) \end{array} \tag{3}$$

Since γ is an isomorphism, we conclude that α is injective. Therefore, it suffices only to justify that α is surjective. By [Stacks, Tag 090S] (which uses that I is finitely generated), it suffices to justify surjectivity of

$$N \to \lim_n N/f^n N$$

for each $f \in I$. Furthermore, [Stacks, Tag 090T] ensures that M is f-adically complete. Therefore, we may assume that $I = (f)$ and show that the natural morphism

$$\alpha\colon N \to \lim_n N/f^n N$$

is surjective. Now we use diagram (3) and the fact that γ is an isomorphism to reduce the question to showing that $\beta\colon \lim_n N/f^n N \to \lim_n N/(f^n M \cap N)$ is injective.

Injectivity of β boils down to showing that, for any sequence of elements $\{a_n \in N\}$ with $a_{n+1} - a_n \in f^n N$ and $a_n \in f^n M \cap N$, we have $a_n \in f^n N$.

The assumption on a_n implies that $a_n = a_{n+1} + f^n x_n$ for some $x_n \in N$. We note that the sum $x_n + f x_{n+1} + f^2 x_{n+2} + \dots$ converges in N because $f^m x_{n+m} \in f^m M \cap N$ for any $m \geq 1$. Let us denote the sum $x_n + f x_{n+1} + f^2 x_{n+2} + \cdots$ by $b_n \in N$. Then we claim that

$$a_n = f^n(x_n + f x_{n+1} + f^2 x_{n+2} + \cdots) = f^n b_n \in f^n N.$$

For this we observe that the partial sums of

$$f^n b_n = f^n(x_n + f x_{n+1} + f^2 x_{n+2} + \cdots)$$

are equal $a_n - a_{n+m}$. Since $a_{n+m} \in f^{n+m} M \cap N$ for any $m \geq 1$ and N is complete in the subspace topology, we conclude

$$a_n = f^n(x_n + f x_{n+1} + f^2 x_{n+2} + \cdots) = f^n b_n$$

finishing the proof. □

Corollary 3.2.3. *Let (R, I) be a ring with a finitely generated ideal I, and A an I-adically complete R-algebra with an R-action of a finite group G.*[9] *Then A^G is complete in the I-adic topology.*

Proof. Note that A^G is closed submodule of A since it is the kernel of the continuous morphism $A \xrightarrow{\alpha-\mathrm{Id}} \prod_{g\in G} A$. Therefore, the result follows directly from Lemma 3.2.2. □

Lemma 3.2.4. *Let A be an admissible k^+-algebra with a k^+-action of a finite group G. Then*:

(1) *A^G is complete in the ϖ-adic topology.*

(2) *A^G is saturated in A.*

(3) *A is finite as an A^G-module.*

Proof. The first claim is Corollary 3.2.3. The second claim is clear by k^+-flatness of A. Thus we only need to show the last claim.

Lemma 2.1.9(1) guarantees that $A^G \to A$ is integral. However, the proof of finiteness in Lemma 2.1.9(3) is not applicable here since A is not necessarily finite type over k^+: it is only topologically finite type.

We now overcome this difficulty. Clearly, the morphism $A^G/\varpi A^G \to A/\varpi A$ is integral. But $A/\varpi A$ is a finite type $k^+/\varpi k^+$-algebra by our assumption, so $A^G/\varpi A^G \to A/\varpi A$ is a finite type morphism. Since an integral map of finite type is finite, we conclude that the morphism $A^G/\varpi A^G \to A/\varpi A$ is finite. Therefore, the successive approximation argument (or [Stacks, Tag 031D]) implies that A is finite as an A^G-module. □

Lemma 3.2.5 (adic Artin–Tate). *Let $A \to B$ be a finite injective morphism of ϖ-adically complete k^+-algebras. Suppose that B is a topologically finite type k^+-algebra and A is a saturated k^+-submodule of B* (*in the sense of Definition 3.1.1*). *Then A is also a topologically finite type k^+-algebra.*

The proof imitates the proof of Lemma 2.2.3; the main new difficulty is that we need to keep track of topological aspects of our algebras in order to work with topologically finite type algebras in a meaningful way.

Proof. Since B is topologically finite type over k^+, we can choose a finite set of elements $x_1, \dots, x_n$ such that the natural k^+-linear continuous homomorphism

$$p\colon k^+\langle T_1, \dots, T_n\rangle \to B$$

that sends T_i to x_i is surjective.

Since B is a finite A-module, we can choose some A-module generators $y_1, \dots, y_m \in B$. The choice of $x_1, \dots, x_n$ and $y_1, \dots, y_m$ implies that there are some $a_{i,j}, a_{i,j,l} \in A$ and relations

$$x_i = \sum_j a_{i,j} y_i \quad \text{and} \quad y_i y_j = \sum_l a_{i,j,l} y_l.$$

We consider the k^+-algebra $A' := k^+\langle T_{i,j}, T_{i,j,l}\rangle$ with a continuous k^+-algebra homomorphism $A' \to A$ that sends $T_{i,j}$ to $a_{i,j}$, and $T_{i,j,l}$ to $a_{i,j,l}$. This map is well-defined as A is ϖ-adically complete.

[9] This action is automatically continuous by Remark 3.2.1.

By definition A' is topologically finite type over k^+, and we claim that B is finite over A' since it is generated by $y_1, \dots, y_m$ as an A'-module. To see this we note that it suffices to show it mod ϖ by successive approximation (or [Stacks, Tag 031D]). However, it is easily seen to be finite mod ϖ due to the relations above.

We use Lemma 3.1.6(4) to conclude that A is finite over A' as a *saturated* submodule of a finite A'-module B. This finishes the proof since a finite algebra over a topologically finite type k^+-algebra is also topologically finite type. □

Corollary 3.2.6. *Let A be an admissible k^+-algebra with a k^+-action of a finite group G. Then A^G is an admissible k^+-algebra, the induced topology on A^G coincides with the ϖ-adic topology, and A is a finitely presented A^G-module.*

Proof. We use Lemma 3.2.4 to see that A^G is ϖ-adically complete, and $A^G \to A$ is saturated. Then Lemma 3.2.5 guarantees that A^G is a topologically finitely generated k^+-algebra. Now A is a finite module over a topologically finitely generated k^+-algebra A^G, so the induced topology on A^G coincides with the ϖ-adic topology by Lemma 3.1.6(5).

Now Lemma 2.2.2(1) implies that A^G is k^+-flat as it is torsion free. Therefore, Lemma 3.1.6(3) guarantees that A is a finitely presented A^G-module. □

Remark 3.2.7. One can show that the ϖ-adic topology on A^G coincides with the induced topology from first principles. But we prefer the proof above as it generalizes better to the topologically universally adhesive situation (see Definition A.3.1).

Namely, Lemma 3.2.5 and Corollary 3.2.6 hold over any I-adically complete base ring R that is topologically universally adhesive (see Definition A.3.1). We refer to Lemma A.3.6 and Corollary A.3.7 for the precise results.

Finally, we are ready to show that $\mathfrak{X}/G$ is an affine admissible formal k^+-scheme if $\mathfrak{X}$ is so.

Proposition 3.2.8. *Let $\mathfrak{X} = \operatorname{Spf} A$ be an affine admissible formal k^+-scheme with a k^+-action of a finite group G. Then the natural map $\phi\colon \mathfrak{X}/G \to \mathfrak{Y} = \operatorname{Spf} A^G$ is a k^+-isomorphism of topologically locally ringed spaces. In particular, $\mathfrak{X}/G$ is an admissible formal k^+-scheme.*

Proof. *Step* 0. $\operatorname{Spf} A^G$ *is an admissible formal k^+-scheme*: The k^+-algebra A is admissible by Remark 3.1.5(3) (and the analogous fact for topologically finitely generated morphisms). Now the claim immediately follows from Corollary 3.2.6.

Step 1. *ϕ is a homeomorphism*: This is completely analogous to step 1 of Proposition 2.1.12. We only need to show that $p\colon \operatorname{Spf} A \to \operatorname{Spf} A^G$ is a surjective, finite morphism with fibers being exactly G-orbits.

Lemma 3.2.4 says that $\operatorname{Spf} A \to \operatorname{Spf} A^G$ is finite. We note that surjectivity of $\operatorname{Spec} A \to \operatorname{Spec} A^G$ obtained in Lemma 2.1.9(2) implies that any prime ideal $\mathfrak{p}$ of A^G lifts to a prime ideal $\mathfrak{P}$ in A. If $\mathfrak{p}$ is open (i.e., it contains ϖ^n for some n), then so is $\mathfrak{P}$. Therefore, the morphism $\operatorname{Spf} A \to \operatorname{Spf} A^G$ is surjective.

Now we note that a prime ideal $\mathfrak{P} \subset A$ is open if and only if so is $g(\mathfrak{P})$ for $g \in G$. So Lemma 2.1.9(2) ensures that the fibers of $\operatorname{Spf} A \to \operatorname{Spf} A^G$ are exactly G-orbits.

Step 2. *ϕ is an isomorphism of topologically locally ringed spaces*: We already know that ϕ is a homeomorphism. So the only thing that we need to show here is that the morphism

$$\mathcal{O}_{\mathfrak{Y}} \to \phi_* \mathcal{O}_{\mathfrak{X}/G}$$

is an isomorphism of topological sheaves. Using the basis of basic affine opens in $\mathfrak{Y}$, it suffices to show that

$$(A^G)_{\{f\}} \to (A_{\{f\}})^G \tag{4}$$

is a topological isomorphism for $f \in A^G$. Corollary 3.2.6 ensures that both sides have the ϖ-adic topology, so we can ignore the topologies.

Now we show that (4) is an (algebraic) isomorphism. We note that

$$A_{\{f\}} \simeq (A^G)_{\{f\}} \widehat{\otimes}_{A^G} A \simeq (A^G)_{\{f\}} \otimes_{A^G} A,$$

where the second isomorphism follows from Lemma 3.1.6(1) and finiteness of A over A^G. Therefore, it suffices to show that the natural morphism

$$(A^G)_{\{f\}} \to ((A^G)_{\{f\}} \otimes_{A^G} A)^G$$

is an isomorphism of k^+-algebras. This follows from Lemmas 2.1.11 and 3.1.6(6). □

Remark 3.2.9. Proposition 3.2.8 can be generalized to the case of an affine, universally adhesive base $\mathfrak{S} = \operatorname{Spf} R$ (see Definition A.3.9). We refer to Proposition A.3.8 for the precise statement.

3.3. ***General case.*** The main goal of this section is to globalize the results of the previous section. This is very close to what we did in the schematic situation in the proof of Theorem 2.1.15.

Lemma 3.3.1. *Let $\mathfrak{X}$ be a formal $\mathfrak{S}$-scheme with an $\mathfrak{S}$-action of a finite group G. Suppose that each point $x \in \mathfrak{X}$ admits an open affine subscheme $\mathfrak{V}_x$ that contains the orbit $G.x$. Then the same holds with $\mathfrak{X}$ replaced by any G-stable open formal subscheme $\mathfrak{U} \subset \mathfrak{X}$.*

Proof. This follows easily from Lemma 2.1.6 as

$$|\mathfrak{X}| \simeq |\mathfrak{X} \times_{\operatorname{Spf} k^+} \operatorname{Spec} k^+/\varpi| \quad \text{and} \quad |\mathfrak{S}| = |\operatorname{Spf} k^+| \simeq |\operatorname{Spec} k^+/\varpi|.$$

Thus, we can reduce the statement to the case of schemes. □

Lemma 3.3.2. *Let $\mathfrak{X}$ be a formal $\mathfrak{S}$-scheme with an $\mathfrak{S}$-action of a finite group G. Suppose that for any point $x \in \mathfrak{X}$ there is an open affine subscheme $\mathfrak{V}_x$ that contains the orbit $G.x$. Then each point $x \in \mathfrak{X}$ has a G-stable open affine neighborhood $\mathfrak{U}_x \subset \mathfrak{X}$.*

Proof. Again, this easily follows from Lemma 2.1.13 as an open subscheme $\mathfrak{U} \subset \mathfrak{X}$ is affine if and only if $\mathfrak{U}_0 := \mathfrak{U} \times_{\operatorname{Spf} k^+} \operatorname{Spec} k^+/\varpi$ is affine [Fujiwara and Kato 2018, Proposition I.4.1.12]. □

Remark 3.3.3. We note that the condition of Lemma 3.3.2 is automatically satisfied if the special fiber $\overline{\mathfrak{X}} := \mathfrak{X} \times_{\operatorname{Spf} k^+} \operatorname{Spec} k^+/\mathfrak{m}_k$ is quasiprojective over $\operatorname{Spec} k^+/\mathfrak{m}_k$. This follows easily from Proposition 2.1.14.

Now we are ready to formulate and prove the main result of this section.

Theorem 3.3.4. *Let $\mathfrak{X}$ be an admissible formal k^+-scheme with a k^+-action of a finite group G. Suppose that each point $x \in \mathfrak{X}$ admits an affine neighborhood $\mathfrak{V}_x$ containing $G.x$. Then $\mathfrak{X}/G$ is an admissible formal k^+-scheme. Moreover, it satisfies the following properties:*

(1) *$\pi : \mathfrak{X} \to \mathfrak{X}/G$ is universal in the category of G-invariant morphisms to topologically locally ringed spaces over $\mathfrak{S}$.*

(2) *$\pi : \mathfrak{X} \to \mathfrak{X}/G$ is a surjective, finite, topologically finitely presented morphism (in particular, it is closed).*

(3) *Fibers of π are exactly the G-orbits.*

(4) *The formation of the geometric quotient commutes with flat base change, i.e., for any flat, topologically finite type k^+-morphism $\mathfrak{Z} \to \mathfrak{X}/G$, the geometric quotient $(\mathfrak{X} \times_{\mathfrak{X}/G} \mathfrak{Z})/G$ is an admissible formal k^+-schemes, and the natural morphism $(\mathfrak{X} \times_{\mathfrak{X}/G} \mathfrak{Z})/G \to \mathfrak{Z}$ is an isomorphism.*

Proof. Step 1. *The geometric quotient $\mathfrak{X}/G$ is an admissible formal k^+-scheme*: The same proof as used in the proof of Theorem 2.1.15 just goes through. We firstly reduce to the case of an affine $\mathfrak{X} = \operatorname{Spf} A$ by choosing a G-stable open affine covering, and then use Proposition 3.2.8 to show the claim in the affine case.

Step 2. *$\pi : \mathfrak{X} \to \mathfrak{X}/G$ is surjective, finite, topologically finitely presented, and fibers are exactly the G-orbits*: The morphism is clearly surjective with fibers being exactly the G-orbits.

To show that it is finite and topologically finitely presented, we can assume that $\mathfrak{X} = \operatorname{Spf} A$ is affine. Lemma 3.2.4 says that $\mathfrak{X} \to \mathfrak{X}/G$ is finite. Corollary 3.2.6 ensures that A is finitely presented as an A^G-module. Therefore, it is topologically finitely presented as an A^G-algebra because [Bosch 2014, Proposition 7.3/10] gives that $A^G \to A$ is topologically finitely presented if and only if $A^G/\varpi^n A^G \to A/\varpi^n A$ is finitely presented for any $n \geq 1$.

Step 3. *$\pi : \mathfrak{X} \to \mathfrak{X}/G$ is universal and commutes with flat base change*: The universality is essentially trivial (see Remark 3.1.8). To show the latter claim, we can again assume that $\mathfrak{X} = \operatorname{Spf} A$ and $\mathfrak{Z} = \operatorname{Spf} B$ are affine. Then the claim boils down to showing that the natural map

$$B \to (A \widehat{\otimes}_{A^G} B)^G$$

is a topological isomorphism. Now we note that Lemma 2.2.2(1) implies that $A \otimes_{A^G} B$ is already ϖ-adically complete as it is a finite module over the topologically finite type k^+-algebra B. Therefore, it suffices to show that the natural map

$$B \to (A \otimes_{A^G} B)^G \tag{5}$$

is a topological isomorphism. Both sides have the ϖ-adic topology by Corollary 3.2.6. So we can ignore the topologies. Now (5) is an isomorphism by Lemma 2.1.11 and flatness of $A^G \to B$ (see Remark 3.1.5). □

Remark 3.3.5. Theorem 3.3.4 can be generalized to the case of a locally universally adhesive base $\mathfrak{S}$ (see Definition A.3.9). We refer to Theorem A.3.15 for the precise statement.

3.4. ***Comparison between the schematic and formal quotients.*** The main goal of this section is to compare the schematic and formal quotients by finite groups actions.

Throughout this section, we fix a microbial valuation ring k^+ and a pseudouniformizer $\varpi \in k^+$. Unlike previous sections, we do not assume that k^+ is complete.

If X is a flat, locally finite type k^+-scheme, we define $\widehat{X}$ to be the formal ϖ-adic completion of X. This is easily seen to be an admissible formal $\hat{k}^+$-scheme with a $\hat{k}^+$-action of G. Using the universal property of geometric quotients, there is a natural morphism $\widehat{X}/G \to \widehat{X/G}$.

Theorem 3.4.1. *Let X be a flat, locally finite type k^+-scheme with a k^+-action of a finite group G. Suppose that any orbit $G.x \subset X$ lies in an affine open subset V_x. The same holds for its ϖ-adic completion $\widehat{X}$ with the induced $\hat{k}^+$-action of G, and the natural morphism:*

$$\widehat{X}/G \to \widehat{X/G}$$

is an isomorphism.

Proof. Step 1. *The condition of Theorem 3.3.4 is satisfied for $\widehat{X}$ with the induced action of G*: Firstly, we observe that $\widehat{X}$ is $\hat{k}^+$-admissible as stated above. Now Lemma 2.1.13 says that our assumption on X implies that there is a covering of $X = \bigcup_{i \in I} U_i$ by affine, open G-stable subschemes. Then $\widehat{X} = \bigcup_{i \in I} \widehat{U}_i$ is an open covering of $\widehat{X}$ by affine, G-stable open formal subschemes. In particular, every orbit lies in an affine open formal subscheme of $\widehat{X}$.

Step 2. *We show that $\widehat{X}/G \to \widehat{X/G}$ is an isomorphism*: We have a commutative diagram

$$\begin{array}{ccc} \widehat{X} & & \\ {\scriptstyle \pi_{\widehat{X}}}\downarrow & \searrow^{\widehat{\pi_X}} & \\ \widehat{X}/G & \xrightarrow{\ \phi\ } & \widehat{X/G} \end{array}$$

of admissible formal $\hat{k}^+$-schemes. We want to show that ϕ is an isomorphism. To prove the claim, we can assume that $X = \operatorname{Spec} A$ is affine by passing to an open covering of X by G-stable affines. Then $X/G \simeq \operatorname{Spec} A^G$, $\widehat{X}/G \simeq \operatorname{Spf} \widehat{A}^G$, and ϕ can be identified with the map

$$\operatorname{Spf}(\widehat{A})^G \to \operatorname{Spf} \widehat{(A^G)}$$

induced by the continuous homomorphism

$$\widehat{A^G} \to (\widehat{A})^G \tag{6}$$

whose source has the ϖ-adic topology by construction and whose target has the ϖ-adic topology by Corollary 3.2.6. So it suffices to show that this map is an isomorphism of abstract rings (ignoring topology) for any flat, finitely generated k^+-flat algebra A.

We note that Corollary 2.2.4 shows that A^G is a finite type k^+-algebra and Lemma 2.1.9(1) shows that A is a finite A^G-module. Therefore, [Bosch 2014, Lemma 7.3/14] implies that the natural homomorphism

$$A \otimes_{A^G} \widehat{A^G} \to \widehat{A}$$

is an (algebraic) isomorphism. Thus we can identify (6) with the natural homomorphism

$$\widehat{A^G} \to (A \otimes_{A^G} \widehat{A^G})^G$$

that is an (algebraic) isomorphism by Lemma 2.1.11 and flatness of the map $A^G \to \widehat{A^G}$; see [Bosch 2014, Lemma 8.2/2]. □

Remark 3.4.2. Theorem 3.4.1 has a version over any topologically universally adhesive base (R, I) (see Definition A.3.1).[10] We refer to Theorem A.3.16 for the precise statement.

4. Quotients of strongly Noetherian adic spaces

We discuss the existence of quotients of some class of analytic adic spaces by an action of a finite group G. We refer the reader to Appendix B for a review of the main definitions and facts from the theory of Huber rings and corresponding adic spaces.

The strategy to construct quotients is close to the one used in Sections 2 and 3. We firstly construct the candidate space X/G that is, a priori, only a topologically locally v-ringed space (see Definition B.1.1). This construction clearly satisfies the universal property, but it is not clear whether X/G is an adic space. We resolve this issue by firstly showing that it is an adic space if X is affinoid. Then we argue by gluing to prove the claim for a larger class of adic spaces.

We point out the two main complications compared to Section 3 (and Section 2). The first new issue that is not seen in the world of formal schemes is that the notion of a finite (resp. topologically finite type) morphism of Huber pairs $(A, A^+) \to (B, B^+)$ is more involved since there is an extra condition on the morphism $A^+ \to B^+$ that makes the theory more subtle (see Definition B.2.1 and B.2.6).

The second issue is that the underlying topological space $\mathrm{Spa}(A, A^+)$ of a Huber pair (A, A^+) is more difficult to express in terms of the pair (A, A^+). It is the set of all valuations on A with corresponding continuity and integrality conditions. So one needs some extra work to identify $\mathrm{Spa}(A^G, A^{+,G})$ with $\mathrm{Spa}(A, A^+)/G$ even in the affinoid case.

4.1. ***The candidate space X/G.*** For the rest of the section we fix a locally strongly noetherian analytic adic space S (see Definition B.2.15).

Example 4.1.1. An example of a strongly noetherian Tate affinoid adic space S is $\mathrm{Spa}(k, k^+)$ for a microbial valuation ring k^+.

Definition 4.1.2. Let G be a finite group and X a valuation locally topologically ringed space over S with a right S-action of G. The *geometric quotient* $X/G = (|X/G|, \mathcal{O}_{X/G}, \{v_{\bar{x}}\}_{\bar{x}\in X/G}, h)$ consists of:

[10] We do not assume that R is I-adically complete.

- The topological space $|X/G| := |X|/G$ with the quotient topology. We denote by $\pi : |X| \to |X/G|$ the natural projection.
- the sheaf of topological rings $\mathcal{O}_{X/G} := (\pi_*\mathcal{O}_X)^G$ with the subspace topology.
- For any $\bar{x} \in X/G$, the valuation $v_{\bar{x}}$ defined as the composition of the natural morphism $k(\bar{x}) \to k(x)$ and the valuation $v_x : k(x) \to \Gamma_{v_x} \cup \{0\}$,[11] where $x \in p^{-1}(\bar{x})$ is any lift of $\bar{x}$.[12]
- The morphism $h : X/G \to S$ defined by the pair $(h, h^\#)$, where $h : |X|/G \to S$ is the unique morphism induced by $f : X \to S$ and $h^\#$ is the natural morphism

$$\mathcal{O}_S \to h_*(\mathcal{O}_{X/G}) = h_*((\pi_*\mathcal{O}_X)^G) = (h_*(\pi_*\mathcal{O}_X))^G = (f_*\mathcal{O}_X)^G$$

  that comes from G-invariance of f.

Remark 4.1.3. We note that Lemma 2.1.2 ensures that X/G is a topologically locally v-ringed $\mathfrak{S}$-space, and $\pi : X \to X/G$ is a morphism of topologically locally v-ringed S-spaces (so $X/G \to S$ is too). It is trivial to see that the pair $(X/G, \pi)$ is a universal object in the category of G-invariant morphisms to topologically locally v-ringed S-spaces.

Our main goal is to show that under some assumptions, X/G is a locally topologically finite type adic S-space when X is. We start with the case of affinoid adic spaces and then move to the general case.

4.2. *Affinoid case.* For the rest of this section, we assume that $S = \mathrm{Spa}(R, R^+)$ is a complete Tate affinoid.

We show that X/G is a topologically finite type adic S-space when $X = \mathrm{Spa}(A, A^+)$ for a topologically finite type complete (R, R^+)-Tate–Huber pair (A, A^+) with an (R, R^+)-action of a finite group G.

We start the section by discussing algebraic properties of the Tate–Huber pair $(A^G, A^{+,G})$. In particular, we show that it is topologically of finite type over (R, R^+) if (R, R^+) is strongly noetherian. The main new input is the "analytic" Artin–Tate Lemma 4.2.4. Then we show that the canonical morphism $X/G \to \mathrm{Spa}(A^G, A^{+,G})$ is an isomorphism. In particular, X/G is an adic space, topologically of finite type over S.

Lemma 4.2.1. *Let (A, A^+) be a complete (R, R^+)-Tate–Huber pair with an (R, R^+)-action of a finite group G. Then*:

(1) *A has a G-stable pair of definition (A_0, ϖ) such that $A_0 \subset A^+$.*

(2) *The subspace topology on (A_0^G, ϖ) coincides with the ϖ-adic topology.*

(3) *(A_0^G, ϖ) is a complete pair of definition of A^G with the subspace topology. In particular, A^G is a Huber ring.*

(4) *$(A^G, A^{+,G})$ with the subspace topology is a Tate–Huber pair.*

[11] Lemma 2.1.2 ensures that $(|X/G|, \mathcal{O}_{X/G})$ is a locally ringed space, so $k(\bar{x})$ is well-defined.

[12] One can show that $v_{\bar{x}}$ is independent of the choice of x similarly to Lemma 2.1.2.

Proof. We note that A is Tate since R is. We choose a pseudouniformizer $\varpi \in R^+$ and a compatible pair of definition (A'_0, ϖ) of A.[13] Then [Huber 1993b, Proposition 1.1] ensures that a subring $A' \subset A$ is a ring of definition if and only if A' is open and bounded. So we can replace A'_0 with $A'_0 \cap A^+$ and ϖ with a power to achieve that $A'_0 \subset A^+$.

Now [loc. cit.] implies that

$$(A_0, \varpi) := \left(\bigcap_{g \in G} g(A'_0), \varpi \right)$$

is a pair of definition in A contained in A^+, and it is G-stable by construction.

To show that the subspace topology in A_0^G coincides with the ϖ-adic topology, it suffices to show that $\varpi^n A_0 \cap A_0^G = \varpi^n A_0^G$. This can be easily seen from the fact that ϖ is a unit in A (and so a nonzero divisor in A_0).

Now we note that A_0^G is complete in the subspace topology since the action of G on A_0 is clearly continuous. Therefore, it is complete in the ϖ-adic topology as these topologies were shown to be equivalent. Also, we note that A_0^G with the subspace topology is clearly open and bounded in A^G, so it is a ring of definition by [Huber 1993b, Proposition 1.1].

Finally, we note that clearly $A^{+,G} \subset A^\circ \cap A^G \subset (A^G)^\circ$ is an open and integrally closed subring of $(A^G)^\circ$. So $(A^G, A^{+,G})$ is a Tate–Huber pair. □

Corollary 4.2.2. *Let (A, A^+) be a complete (R, R^+)-Tate–Huber pair with an (R, R^+)-action of a finite group G. Then the action of G on A is continuous.*

Proof. We choose a G-stable pair of definition (A_0, ϖ) as in Lemma 4.2.1(1). Then it suffices to show that the action of G on A_0 is continuous. This is clear because A_0 carries the ϖ-adic topology. □

Lemma 4.2.3. *Let (A, A^+) be a topologically finite type (see Definition B.2.1) complete (R, R^+)-Tate–Huber pair with an (R, R^+)-action of a finite group G. Then the morphism $(A^G, A^{+,G}) \to (A, A^+)$ is a finite morphism of complete Huber pairs (see Definition B.2.6).*

Proof. Firstly, we note that Lemma 4.2.1 ensures that $(A^G, A^{+,G})$ is a complete Huber–Tate pair, so it makes to ask whether $(A^G, A^{+,G}) \to (A, A^+)$ is a finite morphism of complete Huber pair.

Lemma 2.1.9 gives that the morphisms $A^G \to A$ and $A^{+,G} \to A^+$ are integral. So we only need to show that A is finite as an A^G-module. Lemma B.2.4 (applied to $(R, R^+) \to (A^G, A^{+,G}) \to (A, A^+)$) ensures that $(A^G, A^{+,G}) \to (A, A^+)$ is a topologically finite type morphism of complete Tate–Huber pairs with $A^{+,G} \to A^+$ being integral. Therefore, Lemma B.2.9 implies that $(A^G, A^{+,G}) \to (A, A^+)$ is finite. □

Lemma 4.2.4 (analytic Artin–Tate). *Let (R, R^+) be a strongly noetherian complete Tate–Huber pair, and $i\colon (A, A^+) \to (B, B^+)$ a finite injective morphism of complete Tate–Huber (R, R^+)-pairs. If (B, B^+) is a topologically finite type (R, R^+)-Tate–Huber pair, then so is (A, A^+).*

[13] We abuse the notation and consider ϖ as an element of A via the natural morphism $R \to A$.

The proof of Lemma 4.2.4 imitates the proof of the Adic Artin–Tate Lemma (Lemma 3.2.5), but it is more difficult due to the issue that we need to control the integral aspect of Definition B.2.6. We recommend the reader to look at the proof of Lemma 3.2.5 before reading this proof.

Proof. Step 0. *Preparation for the proof*: We choose a pseudouniformizer $\varpi \in R$ and an open, surjective morphism

$$f\colon R\langle X_1, \dots, X_n\rangle \twoheadrightarrow B$$

such that B^+ is integral over $f(R^+\langle X_1, \dots, X_n\rangle)$. We denote by $x_i \in B^+$ the image $f(X_i)$.

Step 1. *We choose "good" A-module generators $y_1, \dots, y_m$ of B*: Remark B.2.10 implies that there is a compatible choice of rings of definition $A_0 \subset A$, $B'_0 \subset B$ containing all x_i such that B'_0 is a finite A_0-module. Then we choose A_0-module generators $y_1, \dots, y_m$ of B'_0. Since $B \simeq B'_0[\frac{1}{\varpi}]$, $A \simeq A_0[\frac{1}{\varpi}]$, we conclude that $y_1, \dots, y_m$ are also A-module generators of B. The crucial property of this choice of A-module generators is that there exist $a_{i,j}, a_{i,j,k} \in A_0 \subset A^+$ such that

$$x_i = \sum_j a_{i,j} y_j \quad \text{and} \quad y_i y_j = \sum_k a_{i,j,k} y_k.$$

Step 2. *We define another ring of definition B_0*: We consider the unique surjective, continuous R-algebra homomorphism

$$g\colon R\langle X_1, \dots, X_n, Y_1, \dots, Y_m, T_{i,j}, T_{i,j,k}\rangle \to B$$

defined by $g(X_i) = x_i$, $g(Y_j) = y_j$, $g(T_{i,j}) = a_{i,j}$, and $g(T_{i,j,k}) = a_{i,j,k}$. This morphism is automatically open by Remark B.2.3.

We define $B_0 := g(R_0\langle X_1, \dots, X_n, Y_1, \dots, Y_m, T_{i,j}, T_{i,j,k}\rangle)$, where R_0 is a ring of definition in R compatible with A_0; see [Huber 1993b, Corollary 1.3(ii)] for its existence. This is clearly an open and bounded subring of B, so it is a ring of definition.

By construction, B_0 contains $f(R_0\langle X_1, \dots, X_n\rangle)$, and $B_0/\varpi B_0$ is generated as an $R_0/\varpi R_0$-algebra by the classes $\overline{x_i}$, $\overline{y_j}$, $\overline{a_{i,j}}$, and $\overline{a_{i,j,k}}$.

Step 3. *We show that B^+ is integral over R^+B_0*: We note that B^+ is integral over

$$f(R^+\langle X_1, \dots, X_n\rangle) = f(R^+R_0\langle X_1, \dots, X_n\rangle) = R^+ f(R_0\langle X_1, \dots, X_n\rangle).$$

Therefore, it is integral over R^+B_0 since it contains $R^+f(R_0\langle X_1, \dots, X_n\rangle)$ by the previous step.

Step 4. *We show that (B, B^+) is finite over $(R\langle T_{i,j}, T_{i,j,k}\rangle, R^+\langle T_{i,j}, T_{i,j,k}\rangle)$*: We recall that $a_{i,j}, a_{i,j,k} \in A_0 \subset A^+$ for all i, j, k. So, we can use the universal property of restricted power series to define a continuous morphism of complete Tate–Huber pairs:

$$r\colon (R\langle T_{i,j}, T_{i,j,k}\rangle, R^+\langle T_{i,j}, T_{i,j,k}\rangle) \to (A, A^+)$$

as the unique continuous R-algebra morphism such that

$$r(T_{i,j}) = a_{i,j}, r(T_{i,j,k}) = a_{i,j,k}.$$

We also define the morphism

$$t\colon (R\langle T_{i,j}, T_{i,j,k}\rangle, R^+\langle T_{i,j}, T_{i,j,k}\rangle) \to (B, B^+)$$

as the composition of r and i.

We now show that B_0 is finite over $R_0\langle T_{i,j}, T_{i,j,k}\rangle$. Note that this actually makes sense since the natural morphism

$$R_0\langle T_{i,j}, T_{i,j,k}\rangle \to B$$

factors through B_0 by the choice of B_0. We consider the reduction $B_0/\varpi B_0$ and claim that it is finite over

$$R_0\langle T_{i,j}, T_{i,j,k}\rangle/\varpi = (R_0/\varpi)[T_{i,j}, T_{i,j,k}].$$

Indeed, we know that $B_0/\varpi B_0$ is generated as an $R_0/\varpi R_0$-algebra by the elements

$$\overline{x_1}, \dots, \overline{x_n}, \overline{y_1}, \dots, \overline{y_m}, \overline{a_{i,j}}, \overline{a_{i,j,k}}.$$

However, we note that $\overline{a_{i,j}} = \overline{t(T_{i,j})}$ and $\overline{a_{i,j,k}} = \overline{t(T_{i,j,k})}$. Thus, we can conclude that $B_0/\varpi B_0$ is generated as an $(R_0/\varpi R_0)[T_{i,j}, T_{i,j,k}]$-algebra by the elements

$$\overline{x_1}, \dots, \overline{x_n}, \overline{y_1}, \dots, \overline{y_m}.$$

Recall that the choice of x_i and y_j implies that each of $\overline{x_i}$ is a linear combination of $\overline{y_j}$ with coefficients in $\overline{a_{i,j}} = \overline{t(T_{i,j})}$. This implies that $B_0/\varpi B_0$ is generated as an $(R_0/\varpi R_0)[T_{i,j}, T_{i,j,k}]$-algebra by $\overline{y_1}, \dots, \overline{y_m}$. But again, the same argument shows that each product $\overline{y_i}\,\overline{y_j}$ can be expressed as a linear combination of $\overline{y_k}$ with coefficients $\overline{a_{i,j,k}} = \overline{t(T_{i,j,k})}$. This implies that $\overline{y_1}, \dots, \overline{y_m}$ are actually $(R_0/\varpi R_0)[T_{i,j}, T_{i,j,k}]$-module generators for $B_0/\varpi B_0$. Now we use a successive approximation argument (or [Stacks, Tag 031D]) to conclude that B_0 is finite over $R_0\langle T_{i,j}, T_{i,j,k}\rangle$.

We conclude that B is a finite module over $R\langle T_{i,j}, T_{i,j,k}\rangle$ since

$$B = B_0\left[\frac{1}{\varpi}\right] \quad \text{and} \quad R\langle T_{i,j}, T_{i,j,k}\rangle = R_0\langle T_{i,j}, T_{i,j,k}\rangle\left[\frac{1}{\varpi}\right].$$

Thus, we are only left to show that B^+ is integral over $R^+\langle T_{i,j}, T_{i,j,k}\rangle$. Step 3 implies that B^+ is integral over B_0R^+, so it suffices to show that B_0R^+ is integral over $R^+\langle T_{i,j}, T_{i,j,k}\rangle$. But this easily follows from the fact that B_0 is finite over $R_0\langle T_{i,j}, T_{i,j,k}\rangle$.

Step 5. We show that (A, A^+) is finite over $(R\langle T_{i,j}, T_{i,j,k}\rangle, R^+\langle T_{i,j}, T_{i,j,k}\rangle)$: Note that $R\langle T_{i,j}, T_{i,j,k}\rangle$ is noetherian since R is strongly noetherian by assumption. Therefore, we see that A must be a finite $R\langle T_{i,j}, T_{i,j,k}\rangle$-module as a submodule of a finite $R\langle T_{i,j}, T_{i,j,k}\rangle$-module B. Moreover, we see that A^+ is equal to the intersection $B^+ \cap A$ because (B, B^+) is a finite (A, A^+)-Tate–Huber pair. This implies that A^+ is integral over the image $r(R^+\langle T_{i,j}, T_{i,j,k}\rangle)$. We conclude that the complete Huber pair (A, A^+) is finite over $(R\langle T_{i,j}, T_{i,j,k}\rangle, R^+\langle T_{i,j}, T_{i,j,k}\rangle)$. Therefore, it is topologically finite type over (R, R^+) by Lemmas B.2.8 and B.2.4. □

Corollary 4.2.5. *Let (R, R^+) be a strongly noetherian complete Tate–Huber pair and (A, A^+) a topologically finite type complete (R, R^+)-Tate–Huber pair with an (R, R^+)-action of a finite group G. Then the complete Tate–Huber pair $(A^G, A^{+,G})$ is topologically finite type over (R, R^+), and the natural morphism $(A^G, A^{+,G}) \to (A, A^+)$ is a finite morphism of complete Tate–Huber pairs.*

Proof. Lemma 4.2.3 gives that $(A^G, A^{+,G}) \to (A, A^+)$ is a finite morphism of complete Tate–Huber pairs. So Lemma 4.2.4 guarantees that (A^G, A^{+G}) is a topologically finite type complete (R, R^+)-Tate–Huber pair. □

Theorem 4.2.6. *Let (R, R^+) be a strongly noetherian complete Tate–Huber pair and $X = \mathrm{Spa}(A, A^+)$ a topologically finite type affinoid adic $S = \mathrm{Spa}(R, R^+)$-space with an S-action of a finite group G. Then the natural morphism $\phi\colon X/G \to Y = \mathrm{Spa}(A^G, A^{+G})$ is an isomorphism over S. In particular, X/G is a topologically finite type affinoid adic S-space.*

We adapt the proofs of Propositions 2.1.12 and 3.2.8. However, there are certain complications due to the presence of higher rank points. Namely, there are usually many different points $v \in \mathrm{Spa}(A, A^+)$ with the same support $\mathfrak{p}$. Thus in order to study fibers of the map $X \to Y$ we need to work harder than in the algebraic and formal setups.

Proof. Step 0. *Preparation*: The S-action of G on $\mathrm{Spa}(A, A^+)$ induces an (R, R^+)-action of G on (A, A^+). By Corollary 4.2.5, $(A^G, A^{+,G})$ is topologically finite type over (R, R^+). In particular, $Y = \mathrm{Spa}(A^G, A^{+,G})$ is an adic space,[14] and it is topologically finite type over S.

Now we recall that there is a natural map of valuative spaces $p'\colon \mathrm{Spv}\, A \to \mathrm{Spv}\, A^G$, where $\mathrm{Spv}\, A$ (resp. $\mathrm{Spv}\, A^G$) is the set of *all* valuations on the ring A (resp. A^G). We have the commutative diagram

$$\begin{array}{ccc} \mathrm{Spa}(A, A^+) & \xrightarrow{p} & \mathrm{Spa}(A^G, A^{+G}) \\ \downarrow & & \downarrow \\ \mathrm{Spv}\, A & \xrightarrow{p'} & \mathrm{Spv}\, A^G \\ \downarrow & & \downarrow \\ \mathrm{Spec}\, A & \xrightarrow{p''} & \mathrm{Spec}\, A^G \end{array}$$

with the upper vertical maps being the forgetful maps and the lower vertical maps being the maps that send a valuation to its support.

Step 1. *The natural map $p'\colon \mathrm{Spv}\, A \to \mathrm{Spv}\, A^G$ is surjective and G acts transitively on fibers*: Recall that data of a valuation $v \in \mathrm{Spv}\, A$ is the same as data of a prime ideal $\mathfrak{p}_v \subset A$ (its support) and a valuation ring $R_v \subset k(\mathfrak{p})$.

[14]The structure presheaf $\mathcal{O}_Y$ is a sheaf on Y by [Huber 1994, Theorem 2.5].

To show surjectivity of p', pick any valuation $v \in \operatorname{Spv} A^G$; we want to lift it to a valuation of A. We use Lemma 2.1.9 to find a prime ideal $\mathfrak{q} \subset A$ that lifts the support

$$\mathfrak{p}_v := \operatorname{supp}(v) \subset A^G,$$

so $k(\mathfrak{q})$ is finite over $k(\mathfrak{p}_\mathfrak{v})$ since A is A^G-finite.

Now we use [Matsumura 1986, Theorem 10.2] to dominate the valuation ring $R_v \subset k(\mathfrak{p}_v)$ by some valuation ring $R_w \subset k(\mathfrak{q})$. This provides us with a valuation $w\colon A \to \Gamma_w \cup \{0\}$ such that $p'(w) = v$. Therefore, the map p' is surjective.

As for the transitivity of the G-action, we note that Lemma 2.1.9 implies that G acts transitively on the fiber $(p'')^{-1}(\mathfrak{p}_v)$. Furthermore, [Bourbaki 1998, Chapter 5, Section 2, Number 2, Theorem 2] guarantees that, for any prime ideal $\mathfrak{q} \in (p'')^{-1}(\mathfrak{p}_v)$, the stabilizer subgroup

$$G_\mathfrak{q} := \operatorname{Stab}_G(\mathfrak{q})$$

surjects onto the automorphism group $\operatorname{Aut}(k(\mathfrak{q})/k(\mathfrak{p}_v))$. We use [Bourbaki 1998, Chapter 6, Section 8, Number 6, Proposition 6] to see that there is a bijection between the sets

$$\{\text{valuations } w \text{ on } k(\mathfrak{q}) \text{ restricting to } v \text{ on } k(\mathfrak{p}_\mathfrak{v})\} \leftrightarrow \{\text{maximal ideals in } \operatorname{Nr}_{k(\mathfrak{q})}(R_v)\},$$

where $\operatorname{Nr}_{k(\mathfrak{q})}(R_v)$ is the integral closure of R_v in the field $k(\mathfrak{q})$. Now we use [Matsumura 1986, Theorem 9.3(iii)] to conclude that $\operatorname{Aut}(k(\mathfrak{q})/k(\mathfrak{p}_v))$ (and therefore $G_\mathfrak{q}$) acts transitively on the set of maximal ideals of $\operatorname{Nr}_{k(\mathfrak{q})}(R_v)$. As a consequence, $G_\mathfrak{q}$ acts transitively on the set of valuations $w \in p'^{-1}(v)$ with the support $\mathfrak{q}$. Therefore, G acts transitively on $p'^{-1}(v)$ for any $v \in \operatorname{Spv} A^G$.

Step 2. *We show that* $p\colon \operatorname{Spa}(A, A^+) \to \operatorname{Spa}(A^G, A^{+,G})$ *is surjective, and G acts transitively on fibers*: We recall that $\operatorname{Spa}(A, A^+)$ (resp. $\operatorname{Spa}(A^G, A^{+,G})$) is naturally a subset of $\operatorname{Spv}(A)$ (resp. $\operatorname{Spv}(A^G)$). Therefore, it suffices (by step 1) to show that, for any $v \in \operatorname{Spa}(A^G, A^{+,G})$, any $w \in p'^{-1}(v)$ is continuous and $w(A^+) \leq 1$.

It is clear that $w(A^+) \leq 1$ as A^+ is integral over $A^{+,G}$. So we only need to show that the valuation $w \in \operatorname{Spv}(A)$ is continuous.

Lemma 4.2.7. *Let A be a Tate ring with a pair of definition (A_0, ϖ), where ϖ is a pseudouniformizer. Then a valuation $v\colon A \to \Gamma_v \cup \{0\}$ is continuous if and only if*:

- *The value $v(\varpi)$ is cofinal in Γ_v.*
- *$v(a\varpi) < 1$ in Γ_v for any $a \in A_0$.*

Proof. [Seminar 2015, Corollary 9.3.3]. □

We choose a G-stable pair of definition (A_0, ϖ) from Lemma 4.2.1. Then [Bourbaki 1998, Chapter 6, Section 8, Number 1, Proposition 1] gives that Γ_w/Γ_v is a torsion group. Therefore $w(\varpi) = v(\varpi)$ is cofinal in Γ_w if it is cofinal in Γ_v. In particular, $w(\varpi) < 1$.

Now we verify the second condition in Lemma 4.2.7. Since $w(A^+) \le 1$ and $v|_{A^{+,G}} = w|_{A^{+,G}}$,

$$w(a\varpi) = w(a)w(\varpi) < w(a) \le 1$$

for any $a \in A_0 \subset A^+$.

Step 3. *We show that* $\phi\colon X/G \to Y = \mathrm{Spa}(A^G, A^{+,G})$ *is a homeomorphism*: Step 2 implies that ϕ is a bijection. Now note that $p\colon X \to Y$ is a finite, surjective morphism of strongly noetherian adic spaces. Therefore, it is closed by [Huber 1996, Lemma 1.4.5(ii)]. In particular, it is a topological quotient morphism. The map $\pi : X \to X/G$ is a topological quotient morphism by its construction. Hence, ϕ is a homeomorphism.

Step 4. *We show that* ϕ *is an isomorphism of topologically locally* v*-ringed spaces*: Firstly, Remark B.1.2 implies that it suffices to show that the natural morphism

$$\mathcal{O}_Y \to \phi_*\mathcal{O}_{X/G}$$

is an isomorphism of sheaves of topological rings. Using the basis of rational subdomains in Y, it suffices to show that

$$A^G\left\langle \frac{f_1}{s}, \dots \frac{f_n}{s} \right\rangle \to \left(A\left\langle \frac{f_1}{s}, \dots, \frac{f_n}{s} \right\rangle \right)^G \tag{7}$$

is a topological isomorphism for any $f_1, \dots, f_n, s$ generating the unit ideal in A^G. Lemma 4.2.3 gives that (7) is a continuous morphism of complete Tate rings. So the Banach open mapping theorem [Huber 1994, Lemma 2.4(i)] guarantees that it is automatically open (and so a homeomorphism) if it is surjective. Thus, we can ignore the topologies.

Now we show that (7) is an (algebraic) isomorphism. We note that

$$A \otimes_{A^G} A^G\left\langle \frac{f_1}{s}, \dots, \frac{f_n}{s} \right\rangle \simeq A\left\langle \frac{f_1}{s}, \dots, \frac{f_n}{s} \right\rangle,$$

by Corollary B.3.7. Therefore, it suffices to show that

$$A^G\left\langle \frac{f_1}{s}, \dots \frac{f_n}{s} \right\rangle \to \left(A \otimes_{A^G} A^G\left\langle \frac{f_1}{s}, \dots, \frac{f_n}{s} \right\rangle \right)^G$$

is an isomorphism. This follows from Lemma 2.1.11 and flatness of the map $A^G \to A^G\langle \frac{f_1}{s}, \dots, \frac{f_n}{s} \rangle$ obtained in [Huber 1994, Case II.1.(iv) on page 530]. □

4.3. *General case.* The main goal of this section is to globalize the results of the previous section. This is very close to what we did in the formal situation in the proof of Theorem 3.3.4. The main issue is that the adic analogue of Lemma 3.3.2 is more difficult to show.

For the remainder of this section, we fix a locally strongly noetherian analytic adic space S; see Definition B.2.15.

Lemma 4.3.1. *Let $X = \mathrm{Spa}(A, A^+)$ be a preadic Tate affinoid,*[15] *and $V \subset X$ an open preadic subspace. Then any finite set of points $F \subset V$ is contained in an affinoid preadic subspace of V.*

Our proof uses an adic analogue of the theory of "formal" models of rigid spaces in an essential way. It might be possible to justify this claim directly from first principles, but it seems quite difficult due to the fact that the complement $X \setminus V$ does not have a natural structure of a preadic space.

In what follows, for any topological space Z with a map $Z \to \mathrm{Spec}\, A^+$, we denote by $\bar{Z}$ the fiber product $Z \times_{\mathrm{Spec}\, A^+} \mathrm{Spec}\, A^+/\varpi$ in the category of topological spaces.

Proof. First of all, we note that rational subdomains form a basis of the topology of $\mathrm{Spa}(A, A^+)$, and they are quasicompact. Therefore, we can find a quasicompact open subspace $F \subset V' \subset V$, so we may and do assume that V is quasicompact.

We consider the affine open immersion

$$U = \mathrm{Spec}\, A \to S = \mathrm{Spec}\, A^+.$$

And define the category of *U-admissible modifications* $\mathrm{Adm}_{U,S}$ to be the category of projective morphisms $f \colon Y \to S$ that are isomorphisms over U.[16] Then [Bhatt 2017, Theorem 8.1.2 and Remark 8.1.8] (alternatively, one can adapt the proof of [Fujiwara and Kato 2018, Theorem A.4.7] to this situation) shows that

$$\overline{X} := \big(\lim_{f\colon Y \to \mathrm{Spec}\, A^+ \mid f \in \mathrm{Adm}_{U,S}} Y\big) \times_{\mathrm{Spec}\, A^+} \mathrm{Spec}\, A^+/\varpi \simeq \lim_{f \in \mathrm{Adm}_{U,S}} \overline{Y}$$

admits a canonical morphism $\overline{X} \to \mathrm{Spa}(A, A^+)$ that is a homeomorphism. Since $V \subset X$ is quasicompact, [Stacks, Tag 0A2P] implies that there is a U-admissible modification $Y \to \mathrm{Spec}\, A^+$ and a quasicompact open $V' \subset Y$ such that $\pi^{-1}(\overline{V}') = V$ for the projection map $\pi : \overline{X} \to \overline{Y}$.

Now $\overline{V}'$ is a quasiprojective scheme over $\mathrm{Spec}\, A^+/\varpi$. Therefore, [EGA II, Corollaire 4.5.4] implies that there is an open affine subscheme $\overline{W} \subset \overline{V}'$ containing $\pi(F)$. Therefore, $\pi^{-1}(\overline{W}) \subset \mathrm{Spa}(A, A^+)$ contains F, and (the proof of) [Bhatt 2017, Corollary 8.1.7] implies that $\pi^{-1}(\overline{W})$ is affinoid.[17] □

Lemma 4.3.2. *Let X be a preadic space with an action of a finite group G. Suppose that each point $x \in X$ admits an open affinoid preadic subspace V_x that contains the orbit $G.x$. Then the same holds with X replaced by any G-stable open preadic subspace $U \subset X$.*

Proof. The proof is analogous to that of Lemma 2.1.6. One only needs to use Lemma 4.3.1 in place of [EGA II, Corollaire 4.5.4]. □

Lemma 4.3.3. *Let X be a locally topologically finite type adic S-space with an S-action of a finite group G. Suppose that for any point $x \in X$ there is an open affinoid adic subspace $V_x \subset X$ that contains the orbit $G.x$. Then each point $x \in X$ has a G-stable strongly noetherian Tate affinoid neighborhood $U_x \subset X$.*

[15] We do not assume that the structure presheaf $\mathcal{O}_X$ is a sheaf.

[16] We emphasize that a projective morphism is not required to be finitely presented.

[17] The inverse limit giving the preimage in the statement is shown to be affinoid in the proof.

Proof. The proof is similar to that of Lemma 2.1.6. Lemma 4.3.2 allows to reduce to the case S a strongly noetherian Tate affinoid space. Then for a separated X, $U_x := \bigcap_{g\in G} g(V_x)$ is a strongly noetherian Tate affinoid (see Corollary B.7.5) and does the job. In general, Lemma 4.3.2 guarantees that one can replace X with the *separated* open adic subspace $\bigcap_{g\in G} g(V_x)$ and reduce to the separated case. □

Theorem 4.3.4. *Let X be a locally topologically finite type adic S-space with an S-action of a finite group G. Suppose that each point $x \in X$ admits an affinoid open neighborhood V_x containing $G.x$. Then X/G is a locally topologically finite type adic S-space. Moreover, it satisfies the following properties*:

(1) *$\pi : X \to X/G$ is universal in the category of G-invariant morphisms to topologically locally v-ringed S-spaces.*

(2) *$\pi : X \to X/G$ is a finite, surjective morphism (in particular, it is closed).*

(3) *Fibers of π are exactly the G-orbits.*

(4) *The formation of the geometric quotient commutes with flat base change, i.e., for any flat morphism $Z \to X/G$ (see Definition B.4.1) of locally strongly noetherian analytic adic spaces, the geometric quotient $(X \times_{X/G} Z)/G$ is an adic space, and the natural morphism $(X \times_{X/G} Z)/G \to Z$ is an isomorphism.*

Proof. Step 1. *X/G is a topologically locally finite type adic S-space*: As in to step 1 of Theorem 2.1.15, we can use Lemmas 4.3.2 and 4.3.3 to reduce to the case of a strongly noetherian Tate affinoid $S = \mathrm{Spa}(R, R^+)$ and affinoid $X = \mathrm{Spa}(A, A^+)$. Then the claim follows from Theorem 4.2.6.

Step 2. *$\pi : X \to X/G$ is surjective, finite, and fibers are exactly the G-orbits*: As in to step 1, we can assume that X and S are affinoid. Then it follows from Lemma 4.2.3 and Theorem 4.2.6.

Step 3. *$\pi : X \to X/G$ is universal and commutes with flat base change*: The universality is essentially trivial (Remark 2.1.4). To show the latter claim, we can again assume that $X = \mathrm{Spa}(A, A^+)$ and $S = \mathrm{Spa}(R, R^+)$ are strongly noetherian Tate affinoids and it suffices to consider strongly noetherian Tate affinoid $Z = \mathrm{Spa}(B, B^+)$. Then the construction of the quotient implies that it suffices to show that the natural morphism of Tate–Huber pairs

$$(B, B^+) \to ((B, B^+)\widehat{\otimes}_{(A^G, A^{+,G})}(A, A^+))^G =: (C, C^+) \tag{8}$$

is a topological isomorphism.

We can ignore the topologies to show that $B \to B\widehat{\otimes}_{A^G} A$ is a topological isomorphism since its surjectivity would imply openness by the Banach open mapping theorem [Huber 1994, Lemma 2.4(i)]. Now Corollary 4.2.5 and Lemma B.3.6 ensure that $B\widehat{\otimes}_{A^G} A \simeq B \otimes_{A^G} A$. Therefore, it suffices to show that the natural map

$$B \to (B \otimes_{A^G} A)^G$$

is an (algebraic) isomorphism. This follows from Lemma 2.1.11 and flatness of $A \to B$ justified in Lemma B.4.3. □

Remark 4.3.5. As the anonymous referee pointed out, the condition that each point $x \in X$ admits an affinoid open neighborhood V_x containing $G.x$ may be automatic under a very mild assumption on X (it is probably enough to assume that X is separated over S). In the lemma below, we give a proof of this claim for separated rigid-analytic spaces. A more general version of this result would seem to require a generalization of the main results of [Temkin 2000] to more general adic spaces. This is beyond the scope of this paper.

The next lemma uses Berkovich spaces in its proof; the reader unfamiliar with Berkovich spaces can safely ignore this lemma as it is never used in the rest of the paper. We Temkin for suggesting the idea of the following argument.

Lemma 4.3.6. *Let K be a nonarchimedean field with the residue field k, X a separated, locally finite type adic* $\mathrm{Spa}(K, \mathcal{O}_K)$*-space, and $\{x_1, \dots, x_n\}$ is a finite set of points of X. Then there is an open affinoid subset $U \subset X$ containing all x_i.*

Proof. Firstly, we can replace X with a quasicompact open subset containing all x_i to reduce to the case of a quasicompact (and separated) X. Then the underlying topological space of X is spectral, so [Stacks, Tag 0904] implies that it suffices to consider the case when all x_i have the same (unique) rank-1 generalization $x \in X$.

Now since X is separated and quasicompact, [Huber 1996, Lemma 5.1.3] implies that X is taut in the sense of [loc. cit., Definition 5.1.2]. Therefore, [loc. cit, Proposition 8.3.1 and Remark 8.3.2] (or [Henkel 2016, Construction 7.1]) constructs the associated Hausdorff K-strict Berkovich space X^{Ber} with the continuous map of underlying topological spaces

$$\omega\colon X \to X^{\mathrm{Ber}}$$

that sends a point x to its unique rank-1 generalization. In what follows, we will slightly abuse the notation and consider x as both the point of X and X^{Ber} (this is essentially harmless because X^{Ber} is set-theoretically equal to the set of rank-1 points of X).

We consider the germ X^{Ber}_x (see [Berkovich 1993, Section 3.4]) and its reduction $\widetilde{X}^{\mathrm{Ber}}_x = (V_{\tilde{x}}, \widetilde{\mathcal{H}(x)}, \varepsilon)$ (considered as an object of bir_k, see [Temkin 2000, page 4]) that is defined just after [Temkin 2000, Lemma 2.1] (see also [Temkin 2015, Definition 5.7.2.10]). Geometrically, the underlying topological space of the reduction $\widetilde{X}^{\mathrm{Ber}}_x$ is identified with the fiber $\omega^{-1}(\omega(x))$; see [Temkin 2000, Remark 2.6]. In particular, points x_i uniquely correspond to points of $\widetilde{X}^{\mathrm{Ber}}_x$ that we also denote by x_i (by slight abuse of notation).

Now we note that [Temkin 2015, Fact 5.2.2.4] and [Huber 1996, Remarks 1.3.18 and 1.3.19] imply that X^{Ber} is a separated Berkovich space since X is a separated adic space. Therefore, [Temkin 2000, Proposition 2.5] implies that the reduction $\widetilde{X}^{\mathrm{Ber}}_x$ is separated; see [Temkin 2015, Definition/Exercise 5.7.2.8(iv)]. In other words, when $\widetilde{X}^{\mathrm{Ber}}_x$ is considered as an object of Bir_k (see [Temkin 2000, Section 2] for the precise definition and [loc. cit., Proposition 1.4] for the equivalence between Bir_k and bir_k), the underlying birational equivalence class of k-schemes $\widetilde{X}^{\mathrm{Ber}}_x$ is separated (i.e., any representative is separated over k).

Therefore, Chow's lemma [Stacks, Tag 0200] implies that we can find a representative Y of $\widetilde{X}_x^{\mathrm{Ber}}$ that is quasiprojective over k.

Now we recall that the underlying topological space of $\widetilde{X}_x^{\mathrm{Ber}}$ is equal to the cofiltered limit of all representatives of the birational equivalence class $\widetilde{X}_x^{\mathrm{Ber}}$; see [Temkin 2000, Corollary 1.3]. In particular, each point $x_i \in \widetilde{X}_x^{\mathrm{Ber}}$ (uniquely) defines a point $y_i \in Y$. Then [Stacks, Tag 01ZY] implies that there is an open affine subset $V_Y \subset Y$ containing all y_i. Then (by [Temkin 2015, Definition/Exercise 5.7.2.8(ii)]) it gives rise to an affine open subspace $V \subset \widetilde{X}_x^{\mathrm{Ber}}$. Now [Temkin 2000, Theorems 2.4 and 3.1] show that V defines a *good* subdomain $W_x^{\mathrm{Ber}} \subset X_x^{\mathrm{Ber}}$; see [Temkin 2000, page 7] for the definition of a good germ and of a subdomain of a germ. Essentially by definition, this gives rise to a subdomain $x \in W^{\mathrm{Ber}} \subset X^{\mathrm{Ber}}$ such that W is a good Berkovich space. Now the subdomain W^{Ber} defines an open subspace $W \subset X$ (by applying the functor $(-)^{\mathrm{ad}}$ from [Henkel 2016, Construction 7.5]) that contains all the points $x_1, \dots, x_n$ by its construction. Therefore, we may replace X with W to assume that X^{Ber} is good.

Finally, if X^{Ber} is good, there is an open affinoid subspace $x \in U^{\mathrm{Ber}} \subset X^{\mathrm{Ber}}$ by the very definition of goodness.[18] Then $U := \omega^{-1}(U^{\mathrm{Ber}})$ is an open affinoid subspace of X containing $\overline{\{x\}}$. In particular, it contains all point $x_1, \dots, x_n \in \overline{\{x\}}$. □

4.4. *Comparison of adic quotients and formal quotients.* For this section, we fix a complete, microbial valuation ring k^+ (see Definition 3.1.1) with fraction field k, and a choice of a pseudouniformizer ϖ.

We consider the functor of adic generic fiber

$$(-)_k \colon \{\text{admissible formal } k^+\text{-schemes}\} \to \{\text{adic spaces locally of topologically finite type over } \operatorname{Spa}(k, k^+)\}$$

that is defined in [Huber 1996, Section 1.9] (it is denoted by d there). To an affine admissible formal k^+-scheme $\operatorname{Spf}(A)$, this functor assigns the affinoid adic space $\operatorname{Spa}\big(A\big[\frac{1}{\varpi}\big], A^+\big)$ where A^+ is the integral closure of A in $A\big[\frac{1}{\varpi}\big]$.

Let $\mathfrak{X}$ be an admissible formal k^+-scheme with a k^+-action of a finite group G. Then $\mathfrak{X}_k$ is a locally topologically finite type adic $\operatorname{Spa}(k, k^+)$-space with a $\operatorname{Spa}(k, k^+)$-action of G. Using the universal property of geometric quotients, there is a natural morphism $\mathfrak{X}_k/G \to (\mathfrak{X}/G)_k$.

Theorem 4.4.1. *Let $\mathfrak{X}$ be an admissible formal k^+-scheme with a k^+-action of a finite group G. Suppose that any orbit $G.x \subset \mathfrak{X}$ lies in an affine open subset. Then the adic generic fiber $\mathfrak{X}_k$ with the induced* $\operatorname{Spa}(k, k^+)$*-action of G satisfies the assumption of Theorem 4.3.4, and the natural morphism*

$$\mathfrak{X}_k/G \to (\mathfrak{X}/G)_k$$

is an isomorphism.

Proof. As in to step 1 in the proof of Theorem 3.4.1, the condition of Theorem 4.3.4 is satisfied for $\mathfrak{X}_k$ with the induced action of G.

[18] The reader unfamiliar with Berkovich spaces may find it strange, but it is *not* true that a point of a general Berkovich space contains an open affinoid neighborhood.

To show that $\mathfrak{X}_k/G \to (\mathfrak{X}/G)_k$ is an isomorphism, similarly to step 2 in the proof of Theorem 3.4.1 we can assume that $\mathfrak{X} = \operatorname{Spf} A$ is affine. Then we have to show that the natural map

$$\left(A^G\left[\frac{1}{\varpi}\right], (A^G)^+\right) \to \left(A\left[\frac{1}{\varpi}\right]^G, (A^+)^G\right)$$

is an isomorphism of Tate–Huber pairs.

Lemma 2.1.11 implies that the map $A^G\left[\frac{1}{\varpi}\right] \to A\left[\frac{1}{\varpi}\right]^G$ is an algebraic isomorphism. This is a topological isomorphism since both sides have A^G as a ring of definition (see Corollary 3.2.6 and Lemma 4.2.1). Therefore, we are only left to show that the natural map $(A^G)^+ \to (A^+)^G$ is an (algebraic) isomorphism.

Clearly, A^+ is integral over A, and so it is integral over A^G by Lemma 2.1.9(1). Hence, $(A^+)^G$ is integral over $(A^G)^+$. Since $(A^G)^+$ is integrally closed in $A\left[\frac{1}{\varpi}\right]^G = A^G\left[\frac{1}{\varpi}\right]$, we conclude that $(A^G)^+ = (A^+)^G$. □

4.5. ***Comparison of adic quotients and algebraic quotients.*** For this section, we fix a complete, rank-1 valuation ring $\mathcal{O}_K$ with fraction field K, and a choice of a pseudouniformizer ϖ.

A *rigid space* over K always means here an adic space locally topologically finite type over $\operatorname{Spa}(K, \mathcal{O}_K)$. When we need to use classical rigid-analytic spaces, we refer to them as Tate rigid-analytic spaces.

In what follows, for any topologically finite type K-algebra A, we define $\operatorname{Sp} A := \operatorname{Spa}(A, A^\circ)$. We note that [Huber 1994, Lemma 4.4] implies that for any affinoid space $\operatorname{Spa}(A, A^+)$ that is topologically finite type over $\operatorname{Spa}(K, \mathcal{O}_K)$, we have $A^+ = A^\circ$. So this notation does not cause any confusion.

We consider the analytification functor

$$(-)^{\mathrm{an}} \colon \{\text{locally finite type } K\text{-schemes}\} \to \{\text{rigid spaces over } K\}$$

that is defined as a composition

$$(-)^{\mathrm{an}} = r_K \circ (-)^{\mathrm{rig}}$$

of the classical analytification functor $(-)^{\mathrm{rig}}$ (as it is defined in [Bosch 2014, Section 5.4] and the functor r_K that sends a Tate rigid space to the associated adic space; see [Huber 1994, Section 4].

The main issue with the analytification functor is that it does not send affine schemes to affinoid spaces. More precisely, the analytification of an affine scheme $X = \operatorname{Spec} K[T_1, \dots, T_d]/I$ is canonically isomorphic to

$$\bigcup_{n=0}^{\infty} \operatorname{Sp}\left(\frac{K\langle \varpi^n T_1, \dots, \varpi^n T_d\rangle}{I \cdot K\langle \varpi^n T_1, \dots, \varpi^n T_d\rangle}\right) \tag{9}$$

by the discussion after the proof of [Bosch 2014, Lemma 5.4/1]. In particular, $\mathbf{A}_K^{n,\mathrm{an}}$ is not affinoid as it is not quasicompact.

Lemma 4.5.1. *Let X be a rigid space over K with a K-action of a finite group G. Suppose that each point $x \in X$ admits an affinoid open neighborhood V_x containing $G.x$. Then, for any classical point $\bar{x} \in X/G$, the natural map*

$$\mathcal{O}_{X/G,\bar{x}} \to \left(\prod_{x \in \pi^{-1}(\bar{x})} \mathcal{O}_{X,x}\right)^G$$

is an isomorphism.

Proof. Theorem 4.3.4 gives that X/G is a rigid space over K. Lemma 4.3.3 implies that we can assume that $X = \operatorname{Sp} A$ is affinoid, so $X/G \simeq \operatorname{Sp} A^G$ by Theorem 4.2.6; [Huber 1994, Lemma 4.4] guarantees the equality of $+$-rings.

Now, [Bosch 2014, Corollary 4.1/5] implies that $A^G \to \mathcal{O}_{X/G,\bar{x}}$ is flat. Therefore, Lemma 2.1.11 ensures that

$$\mathcal{O}_{X/G,\bar{x}} \simeq (A \otimes_{A^G} \mathcal{O}_{X/G,\bar{x}})^G.$$

Finally, we note that the natural map

$$A \otimes_{A^G} \mathcal{O}_{X/G,\bar{x}} \to \prod_{x \in \pi^{-1}(\bar{x})} \mathcal{O}_{X,x}$$

is an isomorphism by [Conrad 2006, Theorem A.1.3]. □

Corollary 4.5.2. *Let X be a rigid space over K with a K-action of a finite group G. Suppose that each point $x \in X$ admits an affinoid open neighborhood V_x containing $G.x$. Then, for any classical point $\bar{x} \in X/G$, the natural map*

$$\widehat{\mathcal{O}}_{X/G,\bar{x}} \to \Big(\prod_{x \in \pi^{-1}(\bar{x})} \widehat{\mathcal{O}}_{X,x}\Big)^G$$

is an isomorphism.

Proof. By [Conrad 2006, Theorem A.1.3], each $\mathcal{O}_{X,x}$ is $\mathcal{O}_{X/G,\bar{x}}$-finite. Therefore, $\mathcal{O}_{X,x}/\mathfrak{m}_{\bar{x}}\mathcal{O}_{X,x}$ is an artinian $k(\bar{x})$-algebra. Thus, there is n_x such that $\mathfrak{m}_x^{n_x} \subset \mathfrak{m}_{\bar{x}}\mathcal{O}_{X,x}$. This implies that the $\mathfrak{m}_x$-adic topology on $\mathcal{O}_{X,x}$ coincides with the $\mathfrak{m}_{\bar{x}}\mathcal{O}_{X,x}$-adic topology.

Therefore, using that $\mathcal{O}_{X/G,\bar{x}}$ is noetherian [Bosch 2014, Proposition 4.1/6] and $\mathcal{O}_{X,x}$ is finite as an $\mathcal{O}_{X/G,\bar{x}}$-module, we conclude that

$$\prod_{x \in \pi^{-1}(\bar{x})} \widehat{\mathcal{O}}_{X,x} \simeq \Big(\prod_{x \in \pi^{-1}(\bar{x})} \mathcal{O}_{X,x}\Big) \otimes_{\mathcal{O}_{X/G,\bar{x}}} \widehat{\mathcal{O}}_{X/G,\bar{x}}.$$

The claim now follows from Lemmas 2.1.11 and 4.5.1. □

Theorem 4.5.3. *Let X be a locally finite type K-scheme with a K-action of a finite group G. Suppose that any orbit $G.x \subset X$ lies in an affine open subset. Then the analytification X^{an} with the induced K-action of G satisfies the assumption of Theorem 4.3.4, and the natural morphism*

$$\phi\colon X^{\mathrm{an}}/G \to (X/G)^{\mathrm{an}}$$

is an isomorphism.

Proof. Step 1. *The condition of Theorem 4.3.4 is satisfied for X^{an} with the induced action of G*: Firstly, Lemma 2.1.13 says that our assumption on X implies that there is a covering of $X = \bigcup_{i \in I} U_i$ by affine open G-stable subschemes. Then $X^{\mathrm{an}} = \bigcup_{i \in I} U_i^{\mathrm{an}}$ is an open covering by G-stable adic subspaces.

Now we choose i such that $x \in U_i^{\mathrm{an}}$. We note that (9) implies that each U_i^{an} can be written as an ascending union

$$U_i^{\mathrm{an}} = \bigcup_{n=0}^{\infty} U_i^{(n)}$$

of open affinoid subspaces. Since the orbit $G.x$ is finite, it is contained in some $U_i^{(n)}$.

Step 2. *ϕ is a bijection on classical points*: Classical points of $(X/G)^{\mathrm{an}}$ are identified with $(X/G)_0$ the set of closed points of X/G; see [Conrad 1999, Lemma 5.1.2(1)]. Thus the noetherian analogue of Theorem 3.3.4 (or Theorem A.2.9) and the fact that finite morphisms reflect closed points imply that classical points of $(X/G)^{\mathrm{an}}$ are identified with X_0/G.

Similarly, we can use Theorem 4.3.4 and the fact that $\pi_{X^{\mathrm{an}}}$ reflects classical points (as being finite) to conclude that the classical points of X^{an}/G are identified with the set X_0/G.

Step 3. *We reduce to a claim on completed local rings*: We consider the commutative diagram:

$$\begin{array}{ccc} X^{\mathrm{an}} & & \\ \downarrow{\scriptstyle \pi_{X^{\mathrm{an}}}} & \searrow{\scriptstyle \pi_X^{\mathrm{an}}} & \\ X^{\mathrm{an}}/G & \xrightarrow{\ \phi\ } & (X/G)^{\mathrm{an}} \end{array}$$

Since π_X^{an} is a finite, surjective, G-equivariant morphism, we conclude that ϕ is a finite, surjective morphism by Proposition 5.3.1 (that is independent of the present section). Therefore, $\phi_*(\mathcal{O}_{X^{\mathrm{an}}/G})$ is a coherent $\mathcal{O}_{(X/G)^{\mathrm{an}}}$-module by Corollary B.5.3(2) and it suffices to show

$$\mathcal{O}_{(X/G)^{\mathrm{an}}} \to \phi_*(\mathcal{O}_{X^{\mathrm{an}}/G})$$

is an isomorphism. Since *classical points* on a rigid-analytic space reflect isomorphisms of *coherent* sheaves,[19] it suffices to show that

$$\mathcal{O}_{(X/G)^{\mathrm{an}},y} \to (\phi_*(\mathcal{O}_{X^{\mathrm{an}}/G}))_y$$

is an isomorphism for any classical point $y \in (X/G)^{\mathrm{an}}$. Now we use [Conrad 2006, Theorem A.1.3], finiteness and surjectivity of ϕ (that, in particular, implies that ϕ is surjective on classical points), and step 2 to conclude that it suffices to show that the natural map

$$\phi_{\bar{x}}^{\#} \colon \mathcal{O}_{(X/G)^{\mathrm{an}},\phi(\bar{x})} \to \mathcal{O}_{X^{\mathrm{an}}/G,\bar{x}}$$

is an isomorphism at each *classical* point of X^{an}/G. We note that $\phi_{\bar{x}}^{\#}$ is a local homomorphism of noetherian local rings by [Bosch 2014, Proposition 4.1/6]. Thus, [Bourbaki 1998, Chapter III, Section 5.4, Proposition 4] implies that it suffices to show that the morphism

$$\hat{\phi}_{\bar{x}}^{\#} \colon \widehat{\mathcal{O}}_{(X/G)^{\mathrm{an}},\phi(\bar{x})} \to \widehat{\mathcal{O}}_{X^{\mathrm{an}}/G,\bar{x}}$$

[19] This is standard and can be deduced from [BGR 1984, Proposition 9.4.2/6 and Corollary 9.4.2/7].

is an isomorphism, where the completions on both sides are understood with respect to the corresponding maximal ideals.

Step 4. *We show that $\hat{\phi}^\#_{\bar{x}}$ is an isomorphism*: We note Corollary 4.5.2 gives that

$$\widehat{\mathcal{O}}_{X^{\mathrm{an}}/G,\bar{x}} \cong \Bigl(\prod_{x\in\pi^{-1}_{X^{\mathrm{an}}}(\bar{x})} \widehat{\mathcal{O}}_{X^{\mathrm{an}},x}\Bigr)^G.$$

Now we consider the natural morphism of locally ringed spaces $i\colon (X/G)^{\mathrm{an}} \to X/G$. By [Conrad 1999, Lemma 5.1.2(1), (2)], i is a bijection between the sets of classical points of $(X/G)^{\mathrm{an}}$ and closed points of X/G, and the natural morphism

$$\widehat{\mathcal{O}}_{X/G,i(y)} \to \widehat{\mathcal{O}}_{(X/G)^{\mathrm{an}},y}$$

is an isomorphism for any closed point $y \in (X/G)^{\mathrm{an}}$.

We set $\bar{z} := i(\phi(\bar{x}))$. Using finiteness of $X \to X/G$ and Lemma 2.1.11, we see that[20]

$$\widehat{\mathcal{O}}_{(X/G)^{\mathrm{an}},\phi(\bar{x})} \simeq \widehat{\mathcal{O}}_{X/G,\bar{z}} \simeq \Bigl(\prod_{z\in\pi^{-1}_{X}(\bar{z})} \widehat{\mathcal{O}}_{X,z}\Bigr)^G,$$

and $\phi^\#_{\bar{x}}$ identified with the natural map

$$\Bigl(\prod_{z\in\pi^{-1}_{X}(\bar{z})} \widehat{\mathcal{O}}_{X,z}\Bigr)^G \to \Bigl(\prod_{x\in\pi^{-1}_{X^{\mathrm{an}}}(\bar{x})} \widehat{\mathcal{O}}_{X^{\mathrm{an}},x}\Bigr)^G. \tag{10}$$

Finally, we use [Conrad 1999, Lemma 5.1.2(2)] once again to conclude that (10) is an isomorphism. □

5. Properties of the geometric quotients

We discuss some properties of schemes (resp. formal schemes, resp. adic spaces) that are preserved by taking geometric quotients. For instance, one would like to know that X/G is separated (resp. quasiseparated, resp. proper) if X is. This is not entirely obvious as X/G is explicitly constructed only in the affine case, and in general one needs to do some gluing to get X/G. This gluing might, a priori, destroy certain global properties of X such as separatedness. In this section we show that this does not happen for many geometric properties in all schematic, formal and adic setups. We mostly stick to the properties we will need in our paper [Zavyalov 2021a].

5.1. ***Properties of the schematic quotients.*** In this section, we discuss the schematic case. For the rest of the section, we fix a valuation ring k^+. The proofs are written so they will adapt to other settings (formal schemes and adic spaces).

[20]One can repeat the proof of Corollary 4.5.2 using that Lemma 4.5.1 holds on the level of henselian local rings in the scheme world.

Let $f\colon X \to Y$ be a G-invariant morphism of flat, locally finite type k^+-schemes.[21] We assume that any orbit $G.x \subset X$ lies inside some open affine subscheme $V_x \subset X$. In particular, the conditions of Theorem 2.1.15 are satisfied, so it gives that X/G is a k^+-scheme with a finite morphism $\pi\colon X \to X/G$. The universal property of the geometric quotient implies that f factors through π and defines the commutative diagram:

$$\begin{array}{ccc} X & & \\ \big\downarrow{\scriptstyle \pi} & \overset{f}{\searrow} & \\ X/G & \xrightarrow{f'} & Y \end{array}$$

Proposition 5.1.1. *Let $f\colon X \to Y$, a finite group G, and $f'\colon X/G \to Y$ be as above. Then f' is quasicompact* (*resp. quasiseparated*, *resp. separated*, *resp. proper*, *resp. finite*) *if f is so.*

Proof. We note that all these properties are local on Y. Since the formation of X/G commutes with open immersions, we can assume that Y is affine.

Quasicompactness: We suppose that f is quasicompact. Using the fact that Y is affine, we see that quasicompactness of f (resp. f') is equivalent to quasicompactness of the scheme X (resp. X/G). Thus, X is quasicompact. Since π is surjective by Theorem 2.1.15, we conclude that X/G is quasicompact. Therefore, f' is quasicompact as well.

Quasiseparatedness: We suppose that the diagonal morphism $\Delta_X : X \to X \times_Y X$ is quasicompact and consider the following commutative square:

$$\begin{array}{ccc} X & \xrightarrow{\Delta_X} & X \times_Y X \\ \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle \pi \times_Y \pi} \\ X/G & \xrightarrow{\Delta_{X/G}} & X/G \times_Y X/G \end{array} \tag{11}$$

We know that π is finite, so it is quasicompact. Therefore, the morphism $\pi \times_Y \pi$ is quasicompact as well. This implies that the morphism

$$(\pi \times_Y \pi) \circ \Delta_X = \Delta_{X/G} \circ \pi$$

is quasicompact. But we also know that π is surjective, so we see that quasicompactness of $\Delta_{X/G} \circ \pi$ implies quasicompactness of $\Delta_{X/G}$. Thus f' is quasiseparated.

Separatedness: We consider diagram (11) once again. Since π is finite we conclude that $\pi \times_Y \pi$ is finite as well, hence closed. Now we use surjectivity of π to get an equality

$$\Delta_{X/G}(X/G) = (\pi \times_Y \pi)(\Delta_X(X))$$

with $(\pi \times_Y \pi)(\Delta_X(X))$ being closed as image of the closed subset $\Delta_X(X)$ (by separateness of X over affine Y). This shows that $\Delta_{X/G}(X/G)$ is a closed subset of $X/G \times_Y X/G$, so X/G is separated.

[21] Here and later in the paper, a G-invariant morphism means a G-equivariant morphism for some G-action on the source and a trivial G-action on the target.

Properness: We already know that properness of f implies that f' is quasicompact and separated. Also, Theorem 2.2.6 shows that f' is locally of finite type, so it is of finite type. The only thing that we are left to show is that it is universally closed. But this easily follows from universal closedness of f and surjectivity of π.

Finiteness: A finite morphism is proper, so the case of proper morphisms implies that f' is proper. It is also quasifinite as π is surjective and $f = f' \circ \pi$ has finite fibers. Now Zariski's main theorem [EGA IV$_4$, Corollaire 18.12.4] implies that f' is finite. □

We now slightly generalize Proposition 5.1.1 to the case of a G-equivariant morphism f. Namely, we consider a G-equivariant morphism $f\colon X \to Y$ of flat, locally finite type k^+-schemes. We assume that the actions of G on both X and Y satisfy the condition of Theorem 2.2.6. Then the universal property of the geometric quotient implies that f descends to a morphism $f'\colon X/G \to Y/G$ over k^+. We show that various properties of f descend to f'.

Proposition 5.1.2. *Let $f\colon X \to Y$, a finite group G, and $f'\colon X/G \to Y/G$ be as above. Then f' is quasicompact (resp. quasiseparated, resp. separated, resp. proper, resp. a k-modification,[22] resp. finite) if f is so.*

Proof. We start the proof by considering the commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle \pi_X} & & \downarrow{\scriptstyle \pi_Y} \\ X/G & \xrightarrow{f'} & Y/G \end{array}$$

We denote by $h\colon X \to Y/G$ the composition $f' \circ \pi_X = \pi_Y \circ f$. Note that, for all relevant properties $\boldsymbol{P}$ except for k-modification, f satisfies $\boldsymbol{P}$ implies that h satisfies $\boldsymbol{P}$ due to finiteness of π_Y. Thus, all but the k-modification property follow from Proposition 5.1.1 applied to h.

Now suppose that f is a k-modification. We have already proven that f' is a proper map, so we only need to show that its restriction to k-fibers is an isomorphism. This follows from the fact that the formation of the geometric quotient commutes with flat base change such as $\operatorname{Spec} k \to \operatorname{Spec} k^+$. □

Lemma 5.1.3. *Let Y be a flat, locally finite type k^+-scheme, and $f\colon X \to Y$ a G-torsor for a finite group G. The natural morphism $f'\colon X/G \to Y$ is an isomorphism.*

Proof. Since a G-torsor is a finite étale morphism, we see that X is a flat, locally finite type k^+-scheme. Moreover, we note that the conditions of Theorem 2.2.6 are satisfied as f is affine and the action on Y is trivial. Thus, X/G is a flat, locally finite type k^+-scheme. The universal property of the geometric quotient defines the map

$$X/G \to Y$$

[22] A morphism $f\colon X \to Y$ of flat, locally finite type k^+-schemes is called *a k-modification*, if it is proper and the base change $f_k\colon X_k \to Y_k$ is an isomorphism.

that we need to show to be an isomorphism. It suffices to check this étale locally on Y as the formation of X/G commutes with flat base change by Theorem 2.1.15(4). Therefore, it suffices to show that it is an isomorphism after the base change along $X \to Y$. Now, $X \times_Y X$ is a trivial G-torsor over X, so it suffices to show the claim for a trivial G-torsor. This is essentially obvious and follows either from the construction or from the universal property. □

5.2. ***Properties of the formal quotients.*** Similarly to Section 5.1, we discuss that certain properties descend to the geometric quotient in the formal setup. Most proofs are similar to those in Section 5.1.

For the rest of the section, we fix a complete, microbial valuation ring k^+ with a pseudouniformizer ϖ and field of fractions k.

We consider a G-equivariant morphism $f : \mathfrak{X} \to \mathfrak{Y}$ of admissible formal k^+-schemes. We assume that the actions of G on both $\mathfrak{X}$ and $\mathfrak{Y}$ satisfy the condition of Theorem 3.3.4. Then the universal property of the geometric quotient implies that f descends to a morphism $f' : \mathfrak{X}/G \to \mathfrak{Y}/G$ over k^+:

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{f} & \mathfrak{Y} \\ \downarrow{\scriptstyle \pi_{\mathfrak{X}}} & & \downarrow{\scriptstyle \pi_{\mathfrak{Y}}} \\ \mathfrak{X}/G & \xrightarrow{f'} & \mathfrak{Y}/G \end{array}$$

We show that various properties of f descend to f'.

Proposition 5.2.1. *Let $f : \mathfrak{X} \to \mathfrak{Y}$, a finite group G, and $f' : \mathfrak{X}/G \to \mathfrak{Y}/G$ be as above. Then f' is quasicompact (resp. quasiseparated, resp. separated, resp. proper, resp. a rig-isomorphism,*[23] *resp. finite) if f is so.*

Proof. We note that in the case of a quasicompact (resp. quasiseparated, resp. separated, resp. proper) f, the proof of Proposition 5.1.2 works verbatim. We only need to use Theorem 3.3.4 in place of Theorem 2.2.6.

The rig-isomorphism case is easy, akin to the k-modification case in Proposition 5.1.2. We only need to use Theorem 4.4.1 in place of Theorem 2.1.15(4).

Now suppose that f is finite. The proper case implies that f' is proper, and it is clearly quasifinite. Therefore, the mod-ϖ fiber $f'_0 : (\mathfrak{X}/G)_0 \to (\mathfrak{Y}/G)_0$ is finite. Now [Fujiwara and Kato 2018, Proposition I.4.2.3] gives that f' is finite. □

Lemma 5.2.2. *Let $\mathfrak{Y}$ be an admissible formal k^+-scheme, and $f : \mathfrak{X} \to \mathfrak{Y}$ a G-torsor for a finite group G. The natural morphism $f' : \mathfrak{X}/G \to \mathfrak{Y}$ is an isomorphism.*

Proof. The proof of Lemma 5.1.3 adapts to this situation. The only nontrivial fact that we used is that one can check that a morphism is an isomorphism after a finite étale base change (and we use Theorem 3.3.4(4)

[23] A morphism $f : \mathfrak{X} \to \mathfrak{Y}$ of admissible formal k^+-schemes is called *a rig-isomorphism* if the adic generic fiber $f_k : \mathfrak{X}_k \to \mathfrak{Y}_k$ is an isomorphism.

in place of Theorem 2.1.15(4)). This follows from descent for adic, faithfully flat morphisms [Fujiwara and Kato 2018, Proposition I.6.1.5].[24] □

5.3. ***Properties of the adic quotients.*** As in to Sections 5.1 and 5.2, we discuss that certain properties descend to the geometric quotient in the adic setup.

For the rest of the section, we fix a locally strongly noetherian analytic adic space S.

We consider a G-equivariant S-morphism $f : X \to Y$ of locally topologically finite type adic S-spaces. We assume that the actions of G on both X and Y satisfy the condition of Theorem 4.3.4. Then the universal property of the geometric quotient implies that f descends to a morphism $f'\colon X/G \to Y/G$ over S:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle \pi_X} & & \downarrow{\scriptstyle \pi_Y} \\ X/G & \xrightarrow{f'} & Y/G \end{array}$$

We show that various properties of f descend to f'.

Proposition 5.3.1. *Let $f\colon X \to Y$, a finite group G, and $f'\colon X/G \to Y/G$ be as above. Then f' is quasicompact* (*resp. quasiseparated*, *resp. separated*, *resp. proper*, *resp. finite*) *if f is so.*

Proof. The proof is almost identical to that of Proposition 5.1.2. We use Theorem 4.3.4 in place of Theorem 2.2.6 (that is used in the proof of Proposition 5.1.1 that Proposition 5.1.2 relies on). We use [Huber 1996, Proposition 1.5.5] in place of [EGA IV$_4$, Corollaire 18.12.4] to ensure that a quasifinite, proper morphism is finite. □

Lemma 5.3.2. *Let Y be a locally topologically finite type adic S-space*, *and $f : X \to Y$ a G-torsor for a finite group G. The natural morphism $f'\colon X/G \to Y$ is an isomorphism.*

Proof. The proof of Lemma 5.1.3 adapts to this situation. The only nontrivial fact that we used is that one can check that $X/G \to Y$ is an isomorphism after a surjective, flat base change (and we use Theorem 4.3.4(4) in place of Theorem 2.1.15(4)). This follows from Lemma B.4.10 as f' is finite due to Proposition 5.3.1. □

Appendix A: Adhesive rings and boundedness of torsion modules

Let A be a ring with an ideal I. We define the notion of I-torsion part of an A-module and discuss some of its basic properties. Then we define the notion of (universally) adhesive and topologically (universally) adhesive rings. The main original results in Section A are Theorems A.2.9 and A.3.15. The rest is mostly a summary of the results from [Fujiwara and Kato 2018] in a form convenient for the reader.

[24] The case of a finite flat morphism is much easier as the completed tensor product along a finite module coincides with the usual tensor product due to Lemma 3.1.6(1).

A.1. ***I-torsion submodule.***

Definition A.1.1. Let M be an A-module, $a \in A$, and $I \subset A$ an ideal. An element $m \in M$ is *a-torsion* if $a^n m = 0$ for some $n \geq 1$. The set of all a-torsion elements is denoted by $M_{a-\mathrm{tors}}$. An element $m \in M$ is *I-torsion* if m is a-torsion for every $a \in I$. The set of all I-torsion elements is denoted by $M_{I-\mathrm{tors}}$. We say that M is *I-torsion free* if $M_{I-\mathrm{tors}} = 0$. An A-submodule $N \subset M$ is *saturated* if M/N is I-torsion free.

Remark A.1.2. Suppose that $I, J \subset A$ are finitely generated ideals such that $I^n \subset J$ and $J^m \subset I$ for some integers n and m. Then $M_{I-\mathrm{tors}} = M_{J-\mathrm{tors}}$ for any A-module M.

Lemma A.1.3. *Let $A \to B$ be a flat ring homomorphism, and $I \subset A$ a finitely generated ideal, and M an A-module. Then $M_{I-\mathrm{tors}} \otimes_A B \simeq (M \otimes_A B)_{IB-\mathrm{tors}}$.*

Proof. We start by choosing some generators $I = (a_1, \dots, a_n)$. Then

$$M_{I-\mathrm{tors}} = \bigcap M_{a_i-\mathrm{tors}} \quad \text{and} \quad (M \otimes_A B)_{IB-\mathrm{tors}} = \bigcap (M \otimes_A B)_{a_i-\mathrm{tors}}.$$

Therefore, it suffices to show that $M_{a-\mathrm{tors}}$ commutes with flat base change. Now we note that $M_{a-\mathrm{tors}} = \bigcup_n M[a^n]$ where $M[a^n]$ is the submodule of elements annihilated by a^n. It is clear that

$$(M \otimes_A B)[a^n] = M[a^n] \otimes_A B$$

since B is A-flat. This implies that $M_{a-\mathrm{tors}} = (M \otimes_A B)_{a-\mathrm{tors}}$. ☐

Lemma A.1.3 implies that the notion of $M_{I,\mathrm{tors}}$ can be globalized.

Definition A.1.4. Let X be scheme, $\mathcal{I}$ a quasicoherent ideal of finite type, and $\mathcal{M}$ a quasicoherent $\mathcal{O}_X$-module. The *$\mathcal{O}_X$-submodule of $\mathcal{I}$-torsion elements* $\mathcal{M}_{\mathcal{I}-\mathrm{tors}}$ is defined as the sheafification of

$$U \mapsto \mathcal{M}(U)_{\mathcal{I}(U)-\mathrm{tors}}.$$

Remark A.1.5. Lemma A.1.3 implies that $\mathcal{M}_{\mathcal{I}-\mathrm{tors}}$ is a quasicoherent $\mathcal{O}_X$-module. If $X = \operatorname{Spec} A$, $\mathcal{I} = \widetilde{I}$, and $\mathcal{M} = \widetilde{M}$. Then $\mathcal{M}_{\mathcal{I}-\mathrm{tors}} \simeq \widetilde{M_{I-\mathrm{tors}}}$.

Definition A.1.6. Let X be a scheme, and $\mathcal{I}$ a quasicoherent ideal of finite type. We say that X is *$\mathcal{I}$-torsion free* if $\mathcal{O}_{X,\mathcal{I}-\mathrm{tors}} \simeq 0$.

Let $f : X \to Y$ be a morphism of schemes, and $\mathcal{I} \subset \mathcal{O}_Y$ a quasicoherent ideal of finite type. We say that X is *$\mathcal{I}$-torsion free* if X is $\mathcal{I}\mathcal{O}_X$-torsion free.

A.2. ***Universally adhesive schemes.***

Definition A.2.1. A pair (R, I) of a ring and a finitely generated ideal is *adhesive* (or R is *I-adically adhesive*) if $\operatorname{Spec} R \setminus \mathrm{V}(I)$ is noetherian and, for any finite R-module M, the I-torsion submodule $M_{I-\mathrm{tors}}$ (see Definition A.1.1) is R-finite; see [Fujiwara and Kato 2018, Definition 0.8.5.4].

A pair (R, I) is *universally adhesive* if $(R[X_1, \dots, X_d], IR[X_1, \dots, X_d])$ is an adhesive pair for all $d \geq 0$.

Remark A.2.2. A valuation ring k^+ is universally adhesive if it is microbial, in the sense of Definition 3.1.1. More precisely, k^+ is universally ϖ-adically adhesive for any choice of a pseudouniformizer $\varpi \in k^+$. Indeed, [Fujiwara and Kato 2018, Proposition 0.8.5.3] implies that it is sufficient to see that any finite $k^+[X_1, \dots, X_d]$-module M that is ϖ-torsion free (see Definition A.1.1) is finitely presented. This follows from Lemma 2.2.2(2) and the observation that M is torsion free over k^+ if and only if it is ϖ-torsion free.

Lemma A.2.3. *A valuation ring k^+ is I-adically adhesive for some finitely generated ideal I if and only if k^+ is microbial.*

Proof. If k^+ is microbial, we take $I = (\varpi)$ for any pseudouniformizer ϖ. Then k^+ is I-adically adhesive by Remark A.2.2.

Now we suppose that k^+ is adhesive for some finitely generated ideal I. Then $I = (a)$ is principal because k^+ is a valuation ring. Hence, $k^+\left[\frac{1}{a}\right]$ is a noetherian valuation ring by the I-adic adhesiveness of k^+. Therefore, $k^+\left[\frac{1}{a}\right]$ is either a field or discrete valuation ring.

We firstly consider the case $k^+\left[\frac{1}{a}\right]$ is a field. We then observe that $\operatorname{rad}(a)$ is a height-1 prime ideal of k^+ by [Fujiwara and Kato 2018, Propositions 0.6.7.2 and 0.6.7.3]. Therefore, k^+ is microbial by [Huber 1996, Definition 1.1.4] or [Seminar 2015, Proposition 9.1.3].

Now we consider the case $k^+\left[\frac{1}{a}\right]$ a discrete valuation ring. Its maximal ideal $\mathfrak{m}$ is clearly of height-1, so it defines a height-1 prime ideal $\mathfrak{p}$ of k^+. Hence, [loc. cit.] implies that k^+ is microbial. □

Here we summarize the main results about universally adhesive pairs:

Lemma A.2.4. *Let (R, I) be a universally adhesive pair, A a finite type R-algebra, and M a finite A-module:*

(1) *Let $N \subset M$ be a saturated A-submodule of M. Then N is a finite A-module.*

(2) *If M is I-torsion free as an A-module, then it is a finitely presented A-module.*

(3) *If A is I-torsion free as an R-module, then it is a finitely presented R-algebra.*

Proof. We choose some surjective morphism $\varphi\colon R[X_1, \dots, X_d] \to A$. Then the definition of universal adhesiveness says that $R[X_1, \dots, X_d]$ is I-adically adhesive. This easily implies that so is A. Now the first two claims follow from [Fujiwara and Kato 2018, Proposition 0.8.5.3]. To show the last claim, we note that the kernel φ is a saturated submodule of $R[X_1, \dots, X_d]$, so it is a finitely generated ideal by part (1). Therefore, A is finitely presented as an R-algebra. □

Lemma A.2.5. *Let (R, I) be a universally adhesive pair (see Definition A.2.1), and $A \to B$ be a finite injective morphism of R-algebras. Suppose that B is of finite type over R, and that $A \subset B$ is saturated (see Definition A.1.1). Then A is a finite type R-algebra.*

Proof. The proof is analogous to that of Lemma 2.2.3 and again the only difficulty lies in showing that A is finite over A' (as defined in the proof of Lemma 2.2.3). This follows from Lemma A.2.4(1). □

Corollary A.2.6. *Let (R, I) be a universally adhesive pair, and A an I-torsion free, finite type R-algebra with an R-action of a finite group G. The R-flat A^G is a finite type R-algebra, and the natural morphism $A^G \to A$ is finitely presented.*

Proof. The proof of Corollary 2.2.4 works verbatim. One only has to use Lemma A.2.5 in place of Lemma 2.2.3. □

Definition A.2.7. A pair $(X, \mathcal{I})$ of a scheme and a quasicoherent ideal of finite type is *universally adhesive* if there is an open affine covering of $X = \bigcup_{i \in I} \operatorname{Spec} U_i$ such that $(\mathcal{O}(U_i), \mathcal{I}(U_i))$ is universally adhesive for all $i \in I$.

Remark A.2.8. The notion of universal adhesiveness is independent of a choice of affine open covering. This is explained in [Fujiwara and Kato 2018, Propositions 0.8.5.6 and 0.8.6.7]. It essentially follows from Lemma A.1.3 and the fact that noetherianness is local in the fppf topology.

Theorem A.2.9. *Let $(S, \mathcal{I})$ be a universally adhesive pair (in the sense of Definition A.2.7), and X be an $\mathcal{I}$-torsion free, locally finite type S-scheme with an S-action of a finite group G. Suppose that each point $x \in X$ admits an affine neighborhood V_x containing $G.x$. Then the scheme X/G as in Theorem 2.1.15 is $\mathcal{I}$-torsion free and locally finite type over S, and the integral surjection $\pi : X \to X/G$ is finite and finitely presented.*

Proof. The proof of Theorem 2.2.6 goes through if one uses Corollary A.2.6 instead of Corollary 2.2.4. □

A.3. *Universally adhesive formal schemes.*

Definition A.3.1. A pair (R, I) of a ring and a finitely generated ideal is *topologically universally adhesive* (or R is *I-adically topologically universally adhesive*) if (R, I) is universally adhesive, and the pair $(\widehat{R}\langle X_1, \dots, X_d\rangle, I\widehat{R}\langle X_1, \dots, X_d\rangle)$ is adhesive (in the sense of Definition A.2.1) for any $d \geq 0$.

An adically topologized ring R endowed with the adic topology defined by a finitely generated ideal of definition $I \subset R$ is *topologically universally adhesive* if R is I-adically complete, and the pair (R, I) is topologically universally adhesive.

Remark A.3.2. We note that the definition of topologically universally adhesive topological rings is independent of the choice of a finitely generated ideal of definition. For any two ideals of definition I and J, $I^n \subset J$ and $J^m \subset I$ for some integers n and m. Therefore, $M_{I-\text{tors}} = M_{J-\text{tors}}$ by Remark A.1.2.

Remark A.3.3. We note that a microbial valuation ring k^+ is topologically universally adhesive. More precisely, k^+ is topologically universally ϖ-adically adhesive for any choice of a pseudouniformizer $\varpi \in k^+$. This is proven in [Fujiwara and Kato 2018, Theorem 0.9.2.1]. Alternatively, one can easily show the claim from Lemma 3.1.6 and the classical fact that k is strongly noetherian, i.e., $k\langle X_1, \dots, X_d\rangle$ is noetherian for any $d \geq 0$.

Lemma A.3.4. *Let R be a complete, topologically universally I-adically adhesive ring, A be a topologically finite type R-algebra, and M a finite A-module. Then:*

(1) *M is I-adically complete. In particular, A is I-adically complete.*

(2) *Let $N \subset M$ be an A-submodule of M. Then the I-adic topology on M restricts to the I-adic topology on N.*

(3) *Let $N \subset M$ be a saturated A-submodule of M. Then N is a finite A-module.*

(4) *If M is R-flat, it is finitely presented over A.*

(5) *If A is R-flat, it is topologically finitely presented.*

(6) *For any element $f \in A$, the completed localization $A_{\{f\}} = \lim_n A_f/I^n A_f$ is A-flat.*

Proof. We choose a surjection $R\langle T_1, \dots, T_n\rangle \to A$. Then it suffices to show the first claim for $A = R\langle T_1, \dots, T_n\rangle$. We note that A is I-adically adhesive and I-adically complete, and so [Fujiwara and Kato 2018, Proposition 0.8.5.16] (and the discussion after it) implies that any finite A-module is I-adically complete. The second claim follows from [loc. cit., Proposition 0.8.5.16] and the definition of (AP); see [loc. cit., Section 7.4(c), page 161]. The proofs of parts (3)–(5) are similar to the proof of the analogous statements in Lemma A.2.4. The last part is proven in [loc. cit., Proposition I.2.1.2]. □

Definition A.3.5. An algebra A over a complete, topologically universally I-adically adhesive ring R is *admissible* if A is topologically finite type over R and I-torsion free.

Lemma A.3.6. *Let R be an I-adically complete, I-adically topologically universally adhesive ring (see Definition A.3.1), and $A \to B$ be a finite injective morphism of I-adically complete R-algebras. Suppose that B is topologically finite type over R, and $A \subset B$ is saturated in B (See Definition A.1.1). Then A is a topologically finite type R-algebra.*

Proof. The proof is analogous to that of Lemma 3.2.5 and again the only difficulty lies in showing that A is finite over A' (as defined in the proof of Lemma 3.2.5). This follows from Lemma A.3.4(3). □

Corollary A.3.7. *Let R be an I-adically complete, I-adically topologically universally adhesive ring, and A an admissible R-algebra (in the sense of Definition A.3.5) with an R-action of a finite group G. Then A^G is an admissible R-algebra, the induced topology on A^G coincides with the I-adic topology, and A is a finitely presented A^G-module.*

Proof. The proof of Corollary 3.2.6 works verbatim. One only needs to use Lemma A.3.6 in place of Lemma 3.2.5 and Lemma A.3.4 in place of Lemma 3.1.6. □

Proposition A.3.8. *Let R be an I-adically complete, I-adically topologically universally adhesive ring, and $\mathfrak{X} = \operatorname{Spf} A$ an affine admissible formal R-scheme with an R-action of a finite group G. Then the natural map $\phi\colon \mathfrak{X}/G \to \mathfrak{Y} = \operatorname{Spf} A^G$ is an R-isomorphism of topologically locally ringed spaces. In particular, $\mathfrak{X}/G$ is an admissible formal R-scheme.*

Proof. The proof of Proposition 3.2.8 goes through in this more general set-up. The only two differences are that one needs to deduce that A^G is admissible over R (with the induced topology equal to the I-adic) from Corollary A.3.7 instead of Corollary 3.2.6 and one needs to use Lemma A.3.4(6) instead of Lemma 3.1.6(6) to ensure that $(A^G)_{\{f\}}$ is an A^G-flat module. □

Definition A.3.9. A formal scheme $\mathfrak{X}$ is *locally universally adhesive* if there exists an affine open covering $\mathfrak{X} = \bigcup_{i \in I} \mathfrak{U}_i$ such that each $\mathfrak{U}_i$ is isomorphic to $\operatorname{Spf} A$ with A a topologically universally adhesive ring. If $\mathfrak{X}$ is, moreover, quasicompact, we say that $\mathfrak{X}$ is universally adhesive.

Remark A.3.10. Definition A.3.9 is independent of the choice of open covering. More precisely, an affine formal scheme $\mathfrak{X} = \operatorname{Spf} A$ is universally adhesive if and only if A is topologically universally adhesive. This is shown in [Fujiwara and Kato 2018, Propostition I.2.1.9].

Remark A.3.11. Lemma A.3.4(6) can be strengthened to the statement that an adic morphism of affine universally adhesive formal schemes $\operatorname{Spf} B \to \operatorname{Spf} A$ is flat if and only if $A \to B$ is flat.[25] This is proven from [loc. cit., Proposition I.4.8.1].

Definition A.3.12. Let $\mathfrak{S}$ be a universally adhesive formal scheme. An adic $\mathfrak{S}$-scheme $\mathfrak{X}$ is called *admissible* if it is locally of topologically finite type, and there is an affine open covering $\mathfrak{X} = \bigcup_{i \in I} \mathfrak{U}_i$ such that each $\mathfrak{U}_i$ is isomorphic to $\operatorname{Spf} A$ with A an I-torsion free ring for a (and hence any) finitely generated ideal of definition $I \subset A$.

We show that this definition is independent of the choice of a covering.

Lemma A.3.13. *Let $\mathfrak{X} = \operatorname{Spf} A$ be an affine, locally of topologically finite type formal $\mathfrak{S}$-scheme. Then $\mathfrak{X}$ is admissible if and only if A is I-torsion free for a (and hence any) finitely generated ideal of definition I.*

Proof. First of all, we note that Remark A.1.2 implies that Definition A.3.12 is independent of the choice of a finitely generated ideal of definition I. Thus, using that $\mathfrak{X}$ is quasicompact, we can assume that $\mathfrak{X} = \bigcup_{i=1}^n \mathfrak{U}_i = \operatorname{Spf} A_i$ with A_i an I-torsion free A-algebra. Then the morphism $A \to \prod_{i=1}^n A_i$ is faithfully flat by Remark A.3.11 and the fact that all maximal ideals are open in an I-adically complete ring; see [Fujiwara and Kato 2018, Lemma 0.7.2.13]. Now Lemma A.1.3 implies that

$$\left(\prod_{i=1}^n A_i\right)_{I-\mathrm{tors}} \simeq A_{I-\mathrm{tors}} \otimes_A \left(\prod_{i=1}^n A_i\right).$$

Our assumption implies that $\left(\prod_{i=1}^n A_i\right)_{I-\mathrm{tors}} \simeq 0$. Therefore, $A_{I-\mathrm{tors}} \simeq 0$ as $A \to \prod_{i=1}^n A_i$ is faithfully flat. □

Lemma A.3.14. *Let $\mathfrak{S}$ be a universally adhesive formal scheme, and let $\mathfrak{X}$ be an $\mathfrak{S}$-finite, admissible formal $\mathfrak{S}$-scheme. Suppose that $\mathfrak{S}' \to \mathfrak{S}$ is a flat adic morphism of universally adhesive formal schemes. Then $\mathfrak{X}' := \mathfrak{X} \times_{\mathfrak{S}} \mathfrak{S}'$ is an admissible formal $\mathfrak{S}'$-scheme.*

---

[25] We follow [Fujiwara and Kato 2018] and say that an adic morphism of formal schemes $f : \mathfrak{X} \to \mathfrak{Y}$ is flat if and only if $\mathcal{O}_{\mathfrak{Y}, f(x)} \to \mathcal{O}_{\mathfrak{X},x}$ is flat for all $x \in \mathfrak{X}$.

Proof. Lemma A.3.13 ensures that the question is Zariski local on $\mathfrak{X}'$. Thus, we may and do assume that $\mathfrak{S}$, $\mathfrak{S}'$ (and, therefore, $\mathfrak{X}$) are affine. Suppose $\mathfrak{S} = \operatorname{Spf} A$, $\mathfrak{S}' = \operatorname{Spf} A'$, and $\mathfrak{X} = \operatorname{Spf} B$ for an A-algebra B finite as an A-module; see [Fujiwara and Kato 2018, Proposition I.4.2.1]. Choose an ideal of definition $I \subset A$, our assumptions imply that IA' is an ideal of definition in A'. We know that $\mathfrak{X}'$ is given by $\operatorname{Spf} A'\widehat{\otimes}_A B$, so it is easily seen to be a topologically finite type formal $\mathcal{S}'$-scheme. We are only left to check that $A'\widehat{\otimes}_A B$ is I-torsion free.

We note that $A' \otimes_A B$ is finite over A', so it is already IA'-adically complete by Lemma A.3.4(1). Therefore, we conclude that $\mathfrak{X}' \simeq \operatorname{Spf}(A' \otimes_A B)$. Now Remark A.3.11 ensures that A' is A-flat, and so $A' \otimes_A B$ is B-flat. Since B had no I-torsion, the same holds for $A' \otimes_A B$. □

Theorem A.3.15. *Let $\mathfrak{S}$ be a universally adhesive formal scheme (see Definition A.3.9), and $\mathfrak{X}$ an admissible formal $\mathfrak{S}$-scheme (see Definition A.3.12). Suppose that $\mathfrak{X}$ has an $\mathfrak{S}$-action of a finite group G such that each point $x \in \mathfrak{X}$ admits an affine neighborhood $\mathfrak{V}_x$ containing $G.x$. Then $\mathfrak{X}/G$ is an admissible formal $\mathfrak{S}$-scheme. Moreover, it satisfies the following properties:*

(1) *$\pi : \mathfrak{X} \to \mathfrak{X}/G$ is universal in the category of G-invariant morphisms to topologically locally ringed $\mathfrak{S}$-spaces.*

(2) *$\pi : \mathfrak{X} \to \mathfrak{X}/G$ is a finite, surjective, topologically finitely presented morphism (in particular, it is closed).*

(3) *Fibers of π are exactly the G-orbits.*

(4) *The formation of the geometric quotient commutes with flat base change, i.e., for any universally adhesive formal scheme $\mathfrak{Z}$ and a flat adic morphism $\mathfrak{Z} \to \mathfrak{X}/G$, the geometric quotient $(\mathfrak{X} \times_{\mathfrak{X}/G} \mathfrak{Z})/G$ is a formal schemes, and the natural morphism $(\mathfrak{X} \times_{\mathfrak{X}/G} \mathfrak{Z})/G \to \mathfrak{Z}$ is an isomorphism.*

Proof. The proofs of parts (1), (2), and (3) are similar to those of Theorem 3.3.4. The main difference is that one needs to use Proposition A.3.8 in place of Proposition 3.2.8, Lemma A.3.4 in place of Lemma 3.1.6, and [Fujiwara and Kato 2018, Proposition I.2.2.3] in place of [Bosch 2014, Proposition 7.3/10].

We explain part (4) in a bit more detail. We first reduce to the case $\mathfrak{S} = \operatorname{Spf} R$, $\mathfrak{X} = \operatorname{Spf} A$ with A a finite R-module, and $\mathfrak{S}' = \operatorname{Spf} R'$. Then $R \to R'$ is flat by Remark A.3.11. Then Lemma A.3.14 implies that $\mathfrak{X}'$ is $\mathfrak{S}'$-admissible, and then one can repeat the proof of Theorem 3.3.4 using Lemma A.3.4(1) in place of Lemma 3.1.6(1). □

Theorem A.3.16. *Let R be an topologically universally I-adically adhesive ring, and X an I-torsion free, locally finite type R-scheme with a R-action of a finite group G. Suppose that any orbit $G.x \subset X$ lies in an affine open subset V_x. The same holds for its I-adic completion $\widehat{X}$ with the induced $\widehat{R}^{+}$-action of G, and the natural morphism*

$$\widehat{X}/G \to \widehat{X/G}$$

is an isomorphism.

Proof. The proof of Theorem 3.4.1 goes through in this wider generality. The only new nontrivial input is flatness of $A^G \to \widehat{A^G}$. More generally, this flatness holds for any finite type R-algebra B. Namely, any such algebra is I-adically adhesive, so it satisfies the so called BT property; see [Fujiwara and Kato 2018, Definition, Section 0.8.2(a)]) by [loc. cit., Proposition 0.8.5.16]. Therefore, [loc. cit., 8.2.18(i)] implies that $B \to \widehat{B}$ is flat. □

Appendix B: Foundations of adic spaces

The theory of adic spaces still seems to lack a "universal reference" for proofs of all basic questions one might want to use. For example, all of [Huber 1993b; 1994; 1996; Kedlaya and Liu 2019] do not really discuss notions of flat and separated morphisms in detail. The two main goals of this Appendix are to provide the reader with the main definitions we use in the paper, and to give proofs of claims that we need in the paper and that seem difficult to find in the standard literature on the subject.

We stick to the case of analytic adic spaces since this is the only case that we need in this paper.[26]

B.1. ***Basic definitions.*** We start this section by recalling the definition of the category of topologically locally v-ringed spaces $\mathcal{V}$. This category will play the same role as the category of locally ringed spaces plays for the category of schemes. Namely, $\mathcal{V}$ is going to be a sufficiently flexible category with a fully faithful embedding of the category of adic spaces into it.

Definition B.1.1. A *category of topologically locally v-ringed spaces* $\mathcal{V}$ is the category objects of this category are triples $(X, \mathcal{O}_X, \{v_x\}_{x\in X})$ such that:[27]

(1) X is a topological space.

(2) $\mathcal{O}_X$ is a sheaf of topological rings such that the stalk $\mathcal{O}_{X,x}$ is a local ring for all $x \in X$.

(3) v_x is a valuation on the residue field $k(x)$ of $\mathcal{O}_{X,x}$.

Morphisms $f\colon X \to Y$ of objects in $\mathcal{V}$ are defined as maps $(f, f^\#)$ of topologically locally ringed spaces such that the induced maps of residue fields $k(f(x)) \to k(x)$ are compatible with valuations (equivalently, induces a local inclusion between the valuation rings).

Remark B.1.2. The category $\mathcal{V}$ comes with the forgetful functor $F\colon \mathcal{V} \to \mathrm{TLRS}$ to the category of topologically locally ringed spaces. It is clear that this functor is conservative.

Definition B.1.3. We define the *category* AS *of analytic adic spaces* as the full subcategory of $\mathcal{V}$ whose objects are triples $(X, \mathcal{O}_X, \{v_x\}_{x\in X})$ locally isomorphic to $\mathrm{Spa}(A, A^+)$ for a complete Tate–Huber pair (A, A^+). We remind the reader that this requires the pair (A, A^+) to be "sheafy".

[26] We recall that an adic space X is required to be sheafy, i.e., the structure presheaf $\mathcal{O}_X$ must be a sheaf.

[27] Our definition is taken from [Seminar 2015, Definition 13.1.1]. It is different from [Huber 1994, page 521] since the latter requires $\mathcal{O}_X$ to be a sheaf of complete topological rings.

Remark B.1.4. Given any analytic adic space $(X, \mathcal{O}_X, \{v_x\}_{x\in X})$, Huber defined a sheaf $\mathcal{O}_X^+$ as follows

$$\mathcal{O}_X^+(U) = \{f \in \mathcal{O}_X(U) \mid v_x(f) \le 1 \text{ for any } x \in U\}.$$

We note that [Huber 1994, Proposition 1.6] implies that $\mathcal{O}_X^+(X) = A^+$ for any sheafy complete Tate–Huber pair (A, A^+) and $X = \mathrm{Spa}(A, A^+)$.

Remark B.1.5. Note that [Huber 1994, Proposition 2.1(ii)] guarantees that we have a natural identification

$$\mathrm{Hom}_{\mathrm{AS}}(X, \mathrm{Spa}(A, A^+)) = \mathrm{Hom}_{\mathrm{cont}}((A, A^+), (\mathcal{O}_X(X), \mathcal{O}_X^+(X))),$$ [28]

for any analytic adic space X.

Definition B.1.6. A *Tate affinoid adic space* is an object of the category AS that is isomorphic to $\mathrm{Spa}(A, A^+)$ for a complete Tate–Huber pair (A, A^+).

In the following, if we consider a Tate affinoid adic space $X = \mathrm{Spa}(A, A^+)$, we implicitly assume that (A, A^+) is a *complete* Tate–Huber pair.

Remark B.1.7. In general, there are analytic affinoid adic spaces that are not isomorphic to $\mathrm{Spa}(A, A^+)$ for any complete Tate–Huber pair (A, A^+). The analytic condition implies the existence of a pseudouniformizer only *locally* on $\mathrm{Spa}(A, A^+)$, but it does not necessarily exist globally. See [Kedlaya 2019, Example 1.5.7] for an explicit example of an analytic affinoid adic space that is not a Tate affinoid.

B.2. *Finite and topologically finite type morphisms of adic spaces.*

Definition B.2.1. We say that a morphism of complete Tate–Huber pairs $(A, A^+) \to (B, B^+)$ is *topologically of finite type*, if there is a surjective quotient map $f : A\langle T_1, \dots, T_n\rangle \to B$ such that B^+ is integral over $A^+\langle T_1, \dots, T_n\rangle$.

Remark B.2.2. This definition coincides with the definition of topologically finite type morphism of Huber pairs from [Huber 1994, page 533, before Lemma 3.3]. This is stated in [loc. cit., Lemma 3.3(iii)] and it is proven in [Seminar 2015, Proposition 15.3.3].

Remark B.2.3. It turns out that any continuous surjective morphism $f : C \to B$ of complete Tate rings is a quotient mapping. Moreover, it is actually an open map; this is the content of the Banach open mapping theorem [Huber 1994, Lemma 2.4(i)].

There are crucial properties of topologically finite type morphisms that makes them behave similarly to finite type morphisms:

Lemma B.2.4. *Let $f : (A, A^+) \to (B, B^+)$ and $g : (B, B^+) \to (C, C^+)$ be continuous homomorphisms of complete Tate–Huber pairs. If f and g are topologically finite type morphisms then so is $g \circ f$, and if $g \circ f$ is topologically finite type then so is g.*

[28] This is the set of all continuous ring homomorphisms $f : A \to \mathcal{O}_X(X)$ such that $f(A^+) \subset \mathcal{O}_X^+(X)$. We do not claim that $(\mathcal{O}_X(X), \mathcal{O}_X^+(X))$ is a (Tate-)Huber pair.

Proof. This is [Huber 1994, Lemma 3.3(iv)]. □

Definition B.2.5. A morphism of analytic adic spaces $f\colon X \to Y$ is called *locally of topologically finite type*, if there is an open covering of Y by Tate affinoids $\{V_i\}_{i\in I}$ and an open covering of X by Tate affinoids $\{U_i\}_{i\in I}$ such that $f(U_i) \subset V_i$, and $(\mathcal{O}_Y(V_i), \mathcal{O}_Y^+(V_i)) \to (\mathcal{O}_X(U_i), \mathcal{O}_X^+(U_i))$ is topologically of finite type (in the sense of Definition B.2.1). If a morphism f is locally of topologically finite type and quasicompact, it is called *topologically finite type*.

The relation of Definition B.2.5 to Definition B.2.1 for affinoid X and Y is addressed in Theorem B.2.17 under some noetherian condition.

Definition B.2.6. We say that a morphism of complete Tate–Huber pairs $(A, A^+) \to (B, B^+)$ is *finite* if the ring homomorphism $A \to B$ is finite and the ring homomorphism $A^+ \to B^+$ is integral.

Remark B.2.7. Our definition coincides with the definition in Huber's book [1996, (1.4.2)] due to the following (easy) lemma.

Lemma B.2.8. *A finite morphism of complete Tate–Huber pairs $f\colon (A, A^+) \to (B, B^+)$ is of topologically finite type.*

Proof. We choose a set $(y_1, \dots, y_m)$ of A-module generators for B. After multiplying by some power of a pseudouniformizer ϖ we can assume that $y_i \in B^+$ for all i. Then we use the universal property [Huber 1994, Lemma 3.5(i)] to define the continuous surjective morphism

$$g\colon A\langle T_1, \dots, T_m\rangle \to B$$

as the unique continuous A-linear homomorphism such that $f(T_i) = y_i$. It is easily seen to be surjective, and it is open by Remark B.2.3. Moreover, B is integral over $A^+\langle T_1, \dots, T_m\rangle$ since it is even integral over A^+ by the definition of finiteness. □

Lemma B.2.9. *Let $f\colon (A, A^+) \to (B, B^+)$ be a topologically finite type morphism of complete Tate–Huber pairs such that B^+ is integral over A^+. Then there exist rings of definition $A_0 \subset A$ and $B_0 \subset B$ such that $f(A_0) \subset B_0$ and B_0 is finite over A_0. In particular, $(A, A^+) \to (B, B^+)$ is finite.*

Proof. We use Remark B.2.3 to find an open, surjective morphism

$$h\colon A\langle T_1, \dots, T_n\rangle \twoheadrightarrow B.$$

Clearly B^+ is integral over $A^+\langle T_1, \dots, T_n\rangle$. The topological generators $b_i := h(T_i) \in B^+$ are integral over A^+.

Pick monic polynomials $F_i \in A^+[T]$ such that $F_i(b_i) = 0$ for all i. We look at the coefficients $\{a_{i,j}\} \in A^+ \subset A^\circ$ of the polynomials F_i. There are only finitely many of them. We claim that we can find a pair of definition $(A_0, \varpi) \subset A^+$ such that A_0 contains every $a_{i,j}$. Indeed, we pick any ring of definition A_0' in A^+ and consider the subring generated by A_0' and every $a_{i,j}$. It is easy to see that the resulting ring is open and bounded in A, so it is a ring of definition by [Huber 1993b, Proposition 1.1].

Now we define the ring of definition (B_0, ϖ) as the image $h(A_0\langle T_1, \dots, T_n\rangle)$. It is open because h is open, and it is bounded because any morphism of Tate rings preserves boundedness.

We claim that the natural morphism $A_0 \to B_0$ is finite. It suffices to prove that it is finite mod ϖ by successive approximation and completeness. However, it is clearly finite type mod ϖ since it coincides with the composition

$$A_0/\varpi A_0 \to (A_0/\varpi A_0)[T_1, \dots, T_n] \twoheadrightarrow B_0/\varpi B_0,$$

and it is integral since $B_0/\varpi B_0$ is algebraically generated over $A_0/\varpi A_0$ by the residue classes $\overline{b_1}, \dots, \overline{b_n}$ that are integral over $A_0/\varpi A_0$ by construction. Thus this map is integral and finite type, hence finite.

Finally, $(A, A^+) \to (B, B^+)$ is finite since $A \to B$ is equal to the finite map

$$A_0\left[\frac{1}{\varpi}\right] \to B_0\left[\frac{1}{\varpi}\right]. \qquad \square$$

Remark B.2.10. The proof of Lemma B.2.9 actually shows more. We can choose B_0 to contain any finite set of elements $x_1, \dots, x_m \in B^+$. Indeed, the proof just goes through if one replaces $h\colon A\langle T_1, \dots, T_n\rangle \to B$ at the beginning of the proof with the continuous A-algebra morphism

$$h'\colon A\langle T_1, \dots, T_n\rangle\langle X_1, \dots, X_m\rangle \to B$$

satisfying $h'(T_i) = b_i$ and $h'(X_j) = x_j$. Existence of such a morphism follows from the universal property of restricted power series; see [Huber 1994, Lemma 3.5(i)].

Lemma B.2.11. *Let $f\colon (A, A^+) \to (B, B^+)$ be a finite morphism of complete Tate–Huber pairs. If f induces a bijection $A \simeq B$ then f is an isomorphism of Tate–Huber pairs.*

Proof. We note that B^+ is integral over A^+ by the definition of a finite morphism, and f is open by Remark B.2.3. An open continuous bijection is a homeomorphism. So we are only left to show that f induces a bijection $A^+ \simeq B^+$. Now note that A^+ is integrally closed in $A = B$ and B^+ is integral over A^+ because f is assumed to be finite. Thus, $A^+ = B^+$ finishing the proof. $\square$

Definition B.2.12. A morphism of analytic adic spaces $f\colon X \to Y$ is called *finite*, if there is a covering of Y by Tate affinoids $\{V_i\}_{i\in I}$ such that each $U_i := f^{-1}(V_i)$ is an open Tate affinoid subset of X, and the natural morphism $(\mathcal{O}_Y(V_i), \mathcal{O}_Y^+(V_i)) \to (\mathcal{O}_X(U_i), \mathcal{O}_X^+(U_i))$ is finite (in the sense of Definition B.2.6) for all i.

The relation between Definitions B.2.12 and B.2.6 in the case of affinoid X and Y is addressed in Theorem B.2.18 under some noetherian constraints.

Definition B.2.13. A Tate–Huber pair (A, A^+) is called *strongly noetherian* if $A\langle T_1, \dots, T_n\rangle$ is noetherian for all n.

Lemma B.2.14. *Let (A, A^+) be a strongly noetherian complete Tate–Huber pair. A topologically finite type complete (A, A^+)-Tate–Huber pair (B, B^+) is strongly noetherian as well.*

Proof. This is proven in [Huber 1994, Corrolary 3.4]. $\square$

Definition B.2.15. An analytic adic space S is *locally strongly noetherian* if every point $x \in S$ has an affinoid open neighborhood isomorphic to $\mathrm{Spa}(A, A^+)$ for some strongly noetherian complete Tate–Huber pair (A, A^+).

An analytic adic space S is *strongly noetherian* if it is locally strongly noetherian and quasicompact.

Lemma B.2.16. *A Tate affinoid analytic adic space* $\mathrm{Spa}(A, A^+)$ *is strongly noetherian if and only if* (A, A^+) *is a strongly noetherian Tate–Huber pair.*

Proof. The "if" direction is clear. Now suppose that $X := \mathrm{Spa}(A, A^+)$ is strongly noetherian as an analytic adic space. We wish to show that the Tate–Huber pair (A, A^+) is strongly noetherian.

By assumption, each point $x \in X$ has an affinoid open neighborhood $U_x = \mathrm{Spa}(A_x, A_x^+)$ with a strongly complete Tate–Huber pair (A_x, A_x^+). Since rational subdomains form a basis of topology of X and strong noetherianess is preserved by passing to rational subdomains (see Lemma B.2.14), we can assume that each $U_x \subset X$ is a rational subdomain. Then the claim follows from [Kedlaya 2019, Corollary 1.4.19] and sheafiness of A. □

Theorem B.2.17. *Let* $f\colon \mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$ *be a topologically finite type morphism of strongly noetherian Tate affinoids. Then the corresponding map*

$$f^\#\colon (A, A^+) \to (B, B^+)$$

is topologically finite type.

Proof. This is proven in [Huber 1993a, Satz 3.3.23]. □

Theorem B.2.18. *Let* (A, A^+) *be a strongly noetherian Tate–Huber pair, and* $f\colon X \to \mathrm{Spa}(A, A^+)$ *a finite morphism. Then* X *is affinoid and the morphism* $(A, A^+) \to (\mathcal{O}_X(X), \mathcal{O}_X^+(X))$ *is finite.*

Proof. This follows from the combination of [Huber 1993a, Satz 3.6.20 and Korollar 3.12.12]. □

Remark B.2.19. We do not know if Theorems B.2.17 or B.2.18 hold without the extra strong noetherianness assumption.

B.3. ***Completed tensor products.*** The main goal of this section is to prove that under certain assumptions, completed tensor products of Tate rings coincide with usual tensor products. This should be well-known to the experts, but it seems difficult to extract the proof from the existing literature.

For the rest of the section, we fix a complete Tate–Huber pair (A, A^+) with a choice of a pair of definition (A_0, ϖ). We recall the notion of "the natural A-module topology" for a finite A-module M.

Definition B.3.1. A topological A-module structure on M is *natural* if any A-linear map $M \to N$ to a topological A-module N is continuous.

By considering the identity morphism $\mathrm{Id}\colon M \to M$, it is clear that the natural A-module topology is unique, if it exists. It is not, a priori, clear if any module admits a natural topology. However, it turns out that the natural topology actually always exists on a finite A-module.

Lemma B.3.2. *Let M be a finite A-module. There is a topology on M that satisfies the definition of the natural A-module topology.*

Proof. First of all, we claim that the product topology on a finite free module A^n is the natural A-module topology on it. Indeed, it suffices to prove the claim in the case $n = 1$ by the universal property of direct products. But any A-linear map $A \to N$ to a topological A-module is clearly continuous.

Now we deal the case of an arbitrary finitely generated M. We choose a surjective morphism $f\colon A^n \to M$ and provide M with the quotient topology. This is clearly a topological A-module structure. We want to show that any A-linear morphisms $g\colon M \to N$ to a topological A-module N is continuous. We consider the diagram:

$$\begin{array}{ccc} A^n & & \\ {\scriptstyle f}\downarrow & \searrow^{h} & \\ M & \xrightarrow{g} & N \end{array}$$

Then for any open $U \subset N$ we see that $f^{-1}(g^{-1}(U)) = h^{-1}(U)$ is open since h is continuous by the argument above. The definition of quotient topology implies that $g^{-1}(U)$ is open as well. Thus g is indeed continuous. □

Remark B.3.3. We warn the reader that the natural topology on a finite A-module may not be complete as A may have nonclosed ideals.

Lemma B.3.4. *Let M be a finite, complete, first countable topological A-module. Then the topology on M is the natural A-module topology. If (B, B^+) is a finite complete (A, A^+)-Tate–Huber pair, then there is a ring of definition B_0 and a surjective A-linear morphism $p\colon A^n \to B$ with $p(A_0^n) = B_0$.*

Proof. In the case of a finite complete first countable topological A-module M, any surjection $A^n \to M$ must be open by [Huber 1994, Lemma 2.4(i)]. Thus M carries the quotient topology. This topology satisfies the condition of the natural topology by the argument in the last paragraph of Lemma B.3.2.

As for the second claim, we use Lemma B.2.9 to find rings of definition A_0, B_0 such that B_0 is finite over A_0. Choose some generators $b_1, \dots, b_n$ for B_0 over A_0 and consider the morphism $p\colon A^n \to B$ that sends $(a_1, \dots, a_n)$ to $a_1 b_1 + \dots + a_n b_n$. Then clearly $p(A_0^n) = B_0$. □

Finally, we recall that given two morphisms $f\colon A \to B$ and $g\colon A \to C$ of Tate rings there is a canonical way to topologize the tensor product $B \otimes_A C$. Namely, we pick some rings of definition $B_0 \subset B$ and $C_0 \subset C$ such that $f(A_0) \subset B_0$ and $g(A_0) \subset C_0$. Then we topologize $A \otimes_B C$ by requiring the image $(B \otimes_A C)_0 := \mathrm{Im}(B_0 \otimes_{A_0} C_0 \to B \otimes_A C)$ with its ϖ-adic topology to be a ring of definition in $B \otimes_A C$. Then it is straightforward to see that the Tate ring $B \otimes_A C$ ring satisfies the expected universal property in the category of Tate rings.[29] In particular, this shows that this construction does not depend on the choice of rings of definition A_0, B_0, C_0. But we warn the reader that $B \otimes_A C$ need not be (separated and) complete even if A, B and C are; its completion is denoted by $B \widehat{\otimes}_A C$.

[29] See [Huber 1993a, Proposition 2.4.18] or [Seminar 2015, Theorem 5.5.4].

Lemma B.3.5. *Let $f\colon (A, A^+) \to (B, B^+)$ be a finite morphism of complete Tate–Huber pairs, and let $g\colon A \to C$ be any morphisms of Tate rings. Then the topologized tensor product $B \otimes_A C$ has the natural C-module topology.*

We note that this is not automatic from Lemma B.3.4 since $B \otimes_A C$ is not necessarily complete.

Proof. We use Lemma B.3.4 to find a ring of definition $B_0 \subset B$ and a surjection $p\colon A^n \to B$ such that $p(A_0^n) = B_0$. Then after tensoring it with C we get a surjective morphism $C^n \to B \otimes_A C$, and tensoring the surjection $A_0^n \to B_0$ with C_0 we get a surjection $C_0^n \to B_0 \otimes_{A_0} C_0$. Combining these, we get a commutative diagram:

$$\begin{array}{ccccc} & & C^n & \hookleftarrow & C_0^n \\ & \swarrow^{p_C} & & & \downarrow \\ B \otimes_A C & \hookleftarrow & (B \otimes_A C)_0 & \twoheadleftarrow & B_0 \otimes_{A_0} C_0 \end{array}$$

By definition, $(B \otimes_A C)_0$ with its ϖ-adic topology is open in $B \otimes_A C$, so

$$p_C|_{C_0^n}\colon C_0^n \to B \otimes_A C$$

is open onto an open image. Hence, p_C is also open, so $B \otimes_A C$ has the quotient topology via p_C as desired. □

Lemma B.3.6. *Let $f\colon (A, A^+) \to (B, B^+)$ be a finite morphism of complete Tate–Huber pairs with noetherian A. Suppose that $A \to C$ is a continuous morphism of noetherian, complete Tate rings. Then the natural morphism $B \otimes_A C \to B \widehat{\otimes}_A C$ is a topological isomorphism.*

Proof. Lemma B.3.5 implies that $B \otimes_A C$ carries the natural C-module topology. Then we use [Huber 1994, Lemma 2.4(ii)] to conclude that $B \otimes_A C$ is already complete, so the completion map $B \otimes_A C \to B \widehat{\otimes}_A C$ is a topological isomorphism. □

Corollary B.3.7. *Let $f\colon (A, A^+) \to (B, B^+)$ be a finite morphism of complete Tate–Huber pairs with a strongly noetherian Tate ring A. Then the natural morphism*

$$B \otimes_A A\left\langle \frac{f_1}{g}, \dots, \frac{f_n}{g} \right\rangle \to B\left\langle \frac{f_1}{g}, \dots, \frac{f_n}{g} \right\rangle$$

is a topological isomorphism for any choice of elements $f_1, \dots, f_n, g \in A$ generating the unit ideal in A.

Proof. First of all, we note that $A\langle \frac{f_1}{g}, \dots, \frac{f_n}{g} \rangle$ is a complete Tate ring. Moreover, it is noetherian by Lemma B.2.14 so we can apply Lemma B.3.6 with $C = A\langle \frac{f_1}{g}, \dots, \frac{f_n}{g} \rangle$. Thus the question is reduced to showing that the natural morphism

$$B \widehat{\otimes}_A A\left\langle \frac{f_1}{g}, \dots, \frac{f_n}{g} \right\rangle \to B\left\langle \frac{f_1}{g}, \dots, \frac{f_n}{g} \right\rangle$$

is a topological isomorphism. But this easily follows from the universal properties of topologized tensor products (see [Huber 1993a, Proposition 2.4.18] or [Seminar 2015, Theorem 5.5.4]), completions (see

[Seminar 2015, Proposition 7.2.2]), and completed rational localizations (see [Huber 1994, (1.2) on page 517]). □

B.4. ***Flat morphisms of adic spaces.*** We discuss the notion of a flat morphism of adic spaces. This notion is not discussed much in the existing literature, so we provide the reader with some facts that we are using in the paper.

Definition B.4.1. A morphism of analytic adic spaces $f\colon X \to Y$ is called *flat*, if the natural morphism $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is flat for any point $x \in X$.

Similarly to the case of formal schemes, we will soon describe flatness of strongly noetherian Tate affinoids in more concrete terms.

Lemma B.4.2. *Let* $X = \operatorname{Spa}(A, A^+)$ *be a strongly noetherian Tate affinoid adic space, and let* $x \in X$ *be a point corresponding to a valuation* v *with support* $\mathfrak{p}$. *Then the natural morphism* $r_x\colon A_{\mathfrak{p}} \to \mathcal{O}_{X,x}$ *is faithfully flat.*

Proof. We note that rational subdomains form a basis of the topology on an affinoid space, so $\mathcal{O}_{X,x}$ is equal to the filtered colimit of $\mathcal{O}_X(U)$ over all rational subdomains in X containing x. We use [Huber 1994, (II.1), (iv) on page 530] to note that $A \to \mathcal{O}_X(U)$ is flat for each such U. Since flatness is preserved by filtered colimits, we conclude that $A \to \mathcal{O}_{X,x}$ is flat. Note that this implies that $A_{\mathfrak{p}} \to \mathcal{O}_{X,x}$ is flat as well. Indeed, this easily follows from the fact that for any $A_{\mathfrak{p}}$-module M we have isomorphisms

$$M \otimes_{A_{\mathfrak{p}}} \mathcal{O}_{X,x} \cong (M \otimes_A A_{\mathfrak{p}}) \otimes_{A_{\mathfrak{p}}} \mathcal{O}_{X,x} \cong M \otimes_A \mathcal{O}_{X,x}.$$

The discussion above shows that $r_x\colon A_{\mathfrak{p}} \to \mathcal{O}_{X,x}$ is flat, but we also need to show that it is faithfully flat. In order to prove this claim it suffices to show that $A_{\mathfrak{p}} \to \mathcal{O}_{X,x}$ is a local ring homomorphism. We recall that the maximal ideal $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$ is given as

$$\mathfrak{m}_x = \{f \in \mathcal{O}_{X,x} \mid v(f) = 0\}.$$

We need to show $r_x(\mathfrak{p}A_{\mathfrak{p}}) \subset \mathfrak{m}_x$. We pick any element $h \in \mathfrak{p}A_{\mathfrak{p}}$. It can be written as f/s for $f \in \mathfrak{p}$ and $s \in A \setminus \mathfrak{p}$, and we need to check that $v\left(\frac{f}{s}\right) = 0$. The very definition of $\mathfrak{p}$ as the support of v implies that $v(f) = 0$ and $v(s) \neq 0$. Thus

$$v\left(\frac{f}{s}\right) = \frac{v(f)}{v(s)} = 0.$$ □

Lemma B.4.3. *Let* $f\colon \operatorname{Spa}(B, B^+) \to \operatorname{Spa}(A, A^+)$ *be a flat morphism of strongly noetherian Tate affinoid adic spaces. The natural morphism* $f^\sharp\colon A \to B$ *is flat as well.*

Proof. We start the proof by noting that [Huber 1994, Lemma 1.4] implies that for any maximal ideal $\mathfrak{m} \subset B$ there is a valuation $v \in \operatorname{Spa}(B, B^+)$ such that $\operatorname{supp}(v) = \mathfrak{m}$. It is easy to see that

$$\operatorname{supp}(w) = (f^\#)^{-1}(\mathfrak{m}) =: \mathfrak{p},$$

where $w = f(x) \in \mathrm{Spa}(A, A^+)$. We use Lemma B.4.2 to conclude that we have a commutative square

$$\begin{array}{ccc} B_{\mathfrak{m}} & \xrightarrow{r_{\mathfrak{m}}} & \mathcal{O}_{X,v} \\ {\scriptstyle f^{\sharp}_{\mathfrak{p}}}\uparrow & & \uparrow{\scriptstyle f^{\#}_{w}} \\ A_{\mathfrak{p}} & \xrightarrow{r_{\mathfrak{p}}} & \mathcal{O}_{Y,w} \end{array}$$

with $r_{\mathfrak{m}}$ and $r_{\mathfrak{p}}$ being faithfully flat. It is easy to see now that flatness of $f^{\#}_{w}$ implies flatness of $f^{\sharp}_{\mathfrak{p}}$. Finally we note that $\mathfrak{m}$ was an arbitrary maximal ideal in B, so $f^{\sharp}\colon A \to B$ is flat. □

Remark B.4.4. We warn the reader that it is unknown whether Lemma B.4.3 remains true if one drops the strongly noetherian hypothesis. Even the case of rational embeddings is open. However, there are some positive results in this direction in [Kedlaya and Liu 2019, Section 2.4].

Remark B.4.5. Let K be a complete rank-1 valuation field, and

$$f\colon X = \mathrm{Spa}(B, B^{\circ}) \to Y = \mathrm{Spa}(A, A^{\circ})$$

a morphism of topologically finite type affinoid adic $\mathrm{Spa}(K, \mathcal{O}_K)$-spaces. Then f sends classical points to classical points (i.e., $\mathfrak{p}$ defined in the proof of Lemma B.4.3 is maximal). In particular, the proof of Lemma B.4.3 shows that $A \to B$ is flat if

$$f^{\sharp}_{f(x)}\colon \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$$

is flat for any *classical* point $x \in X$.

At this point, we have not given any nontrivial example of a flat morphism. The next lemma gives the main example of interest:

Lemma B.4.6. *Let K be a complete rank-1 valuation ring, and $f^{\sharp}\colon A \to B$ be a flat morphism of topologically finite type K-algebras. Then the corresponding morphism $f\colon X = \mathrm{Spa}(B, B^{\circ}) \to Y = \mathrm{Spa}(A, A^{\circ})$ is flat.*

Proof. Step 1. *The natural morphism $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is flat for any classical point $x \in X$*: Let x correspond to a maximal ideal $\mathfrak{m} \subset B$, and $y = f(x)$ to a maximal ideal $\mathfrak{n} \subset A$. Then [Bosch 2014, Proposition 4.1/2] implies that

$$\widehat{A}_{\mathfrak{n}} \simeq \widehat{\mathcal{O}}_{Y,y}, \quad \widehat{B}_{\mathfrak{m}} \simeq \widehat{\mathcal{O}}_{X,x}$$

where all completions are taken with respect to the corresponding maximal ideal. Furthermore, the rings $A_{\mathfrak{n}}$, $B_{\mathfrak{m}}$, $\mathcal{O}_{Y,y}$, and $\mathcal{O}_{X,x}$ are noetherian by [Bosch 2014, Proposition 3.1/3(i) and 4.1/6]. Therefore, [Stacks, Tag 0523] ensures that flatness of $A_{\mathfrak{n}} \to B_{\mathfrak{m}}$ implies flatness of $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$.

Step 2. *Finish the argument*: We pick a point $x \in X$ with the image $y = f(x)$. Then the morphism

$$f_y^\sharp \colon \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$$

can be rewritten as

$$\operatorname{colim}_{y \in V \subset Y} \mathcal{O}_Y(V) \to \operatorname{colim}_{x \in U \subset X} \mathcal{O}_X(U).$$

Therefore, it suffices to show that the natural morphism

$$\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$$

is flat for any open affinoids $V \subset Y$, $U \subset X$ such that $f(U) \subset V$. Now step 1 ensures that the morphism

$$\mathcal{O}_{V, f(u)} \to \mathcal{O}_{U,u}$$

is flat for any classical $u \in U$. Then Remark B.4.5 ensures that $\mathcal{O}_V(V) \to \mathcal{O}_U(U)$ is flat finishing the proof. □

Remark B.4.7. One can also show that any smooth morphism $f \colon X \to Y$ of locally strongly noetherian adic spaces is flat. The proof is similar to step 2 of the proof of Lemma B.4.6 using [Kedlaya 2019, Lemma 1.1.19(a)] and [Huber 1996, Lemma 1.7.6] for a cofinal system of opens $V \subset Y$, $U \subset X$ such that $f(U) \subset V$.

Remark B.4.8. One can also show that $X_L \to X$ is flat for any extension of nonarchimedean fields $K \subset L$ and a rigid-analytic K-space X. Again, the main input is to show that the morphism

$$A \to A \widehat{\otimes}_K L$$

is flat for any K-affinoid algebra A. This is worked out in [Conrad 1999, Lemma 1.1.5].

Remark B.4.9. We do not know if flatness of $A \to B$ implies flatness of $\operatorname{Spa}(B, B^+) \to \operatorname{Spa}(A, A^+)$ in general (even under the strongly noetherian assumption).

Lemma B.4.10. *Let $f \colon X = \operatorname{Spa}(B, B^+) \to Y = \operatorname{Spa}(A, A^+)$ be a finite morphism of strongly noetherian Tate affinoids, and $g \colon Z = \operatorname{Spa}(C, C^+) \to \operatorname{Spa}(A, A^+)$ be a surjective flat morphism of strongly noetherian Tate affinoids. Then f is an isomorphism if and only if $f' \colon X \times_Y Z \to Z$ is.*

Proof. We note that f' is finite by [Huber 1996, Lemma 1.4.5(i)], so Lemma B.2.11 ensures that it suffices to show that $A \to B$ is a (topological) isomorphism if and only if $C \to C \widehat{\otimes}_A B$ is. We can ignore topologies by Remark B.2.3.

Now we note that Lemma B.3.6 gives that $C \otimes_A B \simeq C \widehat{\otimes}_A B$. Thus, it suffices to show that $A \to B$ is an isomorphism if and only if $C \to C \otimes_A B$ is. This follows from the usual faithfully flat descent as $A \to C$ is flat by Lemma B.4.3, and therefore faithfully flat by [Huber 1994, Lemma 1.4]. □

B.5. ***Coherent sheaves.*** We review the basic theory of coherent sheaves on locally strongly noetherian adic spaces.

We first recall the construction of the $\mathcal{O}_X$-module $\widetilde{M}$ on a strongly noetherian Tate affinoid $X = \mathrm{Spa}(A, A^+)$ associated to a finite A-module M. For each rational subset $U \subset X$, we have

$$\widetilde{M}(U) = \mathcal{O}_X(U) \otimes_A M;$$

[Huber 1994, Theorem 2.5] guarantees that this assignment is indeed a sheaf.

Definition B.5.1. An $\mathcal{O}_X$-module $\mathcal{F}$ on a locally strongly noetherian analytic adic space X is *coherent* if there is an open covering $X = \bigcup_{i \in I} U_i$ by strongly noetherian Tate affinoids such that $\mathcal{F}|_{U_i} \cong \widetilde{M}_i$ for a finite $\mathcal{O}_X(U_i)$-module M_i.

Theorem B.5.2. *Let $X = \mathrm{Spa}(A, A^+)$ be a strongly noetherian Tate affinoid, and $\mathcal{F}$ a coherent $\mathcal{O}_X$-module. Then*:

(1) *there is a unique finite A-module M such that $\mathcal{F} \cong \widetilde{M}$.*

(2) $\mathrm{H}^i(X, \mathcal{F}) = 0$ *for* $i \geq 1$.

Proof. (1) is shown in [Kedlaya 2019, Theorem 1.4.18] (see also [Kedlaya 2019, Definition 1.4.5] for a definition of PCoh_A), and (2) is shown in [Huber 1994, Theorem 2.5]. □

Corollary B.5.3. *Let $f : X \to Y$ be a finite morphism of locally strongly noetherian adic spaces. Then*:

(1) *Coherent $\mathcal{O}_Y$-modules are closed under kernels, cokernels, and extensions in* $\mathrm{Mod}_{\mathcal{O}_Y}$.

(2) *For any coherent $\mathcal{O}_X$-module $\mathcal{F}$, $f_*\mathcal{F}$ is a coherent $\mathcal{O}_Y$-module.*

Proof. It suffices to prove the claim under the additional assumption that Y is a strongly noetherian Tate affinoid. Now both parts easily follow from Theorem B.2.18, Theorem B.5.2 and flatness of $\mathcal{O}_Y(Y) \to \mathcal{O}_Y(U)$ for a rational subdomain $U \subset Y$ [Huber 1994, (II.1), (iv) on page 530]. □

B.6. ***Closed immersions.*** In this section we discuss the notion of closed immersion in the context of locally strongly noetherian adic spaces.

Definition B.6.1. We say that a morphism $f : X \to Y$ of analytic adic spaces is an *open immersion* if f is a homeomorphism of X onto an open subset of Y, and the map $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ is an isomorphism.

Remark B.6.2. Remark B.1.2 ensures that $f : X \to Y$ is an open immersion if and only if f is an isomorphism onto an open adic subspace of Y.

Definition B.6.3. We say that a morphism $f : X \to Y$ of locally strongly noetherian analytic adic spaces is a *closed immersion* if f is a homeomorphism of X onto a closed subset of Y, the map $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is surjective, and the kernel $\mathcal{I} := \ker(\mathcal{O}_Y \to f_*\mathcal{O}_X)$ is coherent.

Remark B.6.4. If $i : X \to Y$ is a closed immersion of (locally strongly noetherian) adic spaces with $\mathcal{I} = \ker(\mathcal{O}_Y \to i_*\mathcal{O}_X)$, then there is a set-theoretic identification

$$|X| = \{y \in Y \mid (i_*\mathcal{O}_X)_y \not\simeq 0\} = \{y \in Y \mid \mathcal{I}_y \not\simeq \mathcal{O}_{Y,y}\}.$$

Lemma B.6.5. *Let $Y = \mathrm{Spa}(A, A^+)$ be a strongly noetherian Tate affinoid,*[30] *and $i\colon X \to Y$ a closed immersion. Then $B := \mathcal{O}_X(X)$ is a complete Tate ring, and the natural morphism $i^*\colon A \to B$ is a topological quotient morphism.*

Proof. The natural morphism $A \to B$ is clearly continuous as $\mathcal{O}_Y \to i_*\mathcal{O}_X$ is a morphism of sheaves of *topological* rings. Therefore, any topologically nilpotent unit $\varpi_A \in A$ defines a topologically nilpotent unit $\varpi := i^*(\varpi_A) \in B$.

We now show that B is a Tate ring. Since X is closed in an affinoid, we conclude that X is quasicompact and quasiseparated. So we choose a finite covering $X = \bigcup_{i=1}^n U_i$ by open affinoid $U_i = \mathrm{Spa}(B_i, B_i^+)$. Then

$$B \subset \prod_{i=1}^n B_i$$

and the topology on B coincides with the subspace topology. Each B_i admits a ring of definition $B_{i,0}$, and we can assume that the topology on every $B_{i,0}$ is the ϖ-adic topology (possibly after replacing ϖ_A with a power). We claim that

$$B_0 := \left(\prod_{i=1}^n B_{i,0}\right) \cap B = \prod_{i=1}^n (B_{i,0} \cap B)$$

is a ring of definition in B. It suffices to show the topology on B_0 induced from $\prod_{i=1}^n B_{i,0}$ coincides with the ϖ-adic topology. This follows from the equalities

$$\varpi^n\left(\left(\prod_{i=1}^n B_{i,0}\right) \cap B\right) = \left(\varpi^n \prod_{i=1}^n B_{i,0}\right) \cap B$$

that, in turn, follow from the fact that ϖ is invertible in B.

Now we address completeness of B. By a similar reason as above, we see that there is a short exact sequence

$$0 \to B \xrightarrow{d} \prod_{i=1}^n B_i \xrightarrow{a} \prod_{i,j} \mathcal{O}_X(U_i \cap U_j)$$

such that d is a topological embedding and a is continuous. Using that X is quasiseparated, we can cover each $U_i \cap U_j$ by a finite number of affinoids $V_{i,j,k}$. Thus, we get a short exact sequence

$$0 \to B \xrightarrow{d} \prod_{i=1}^n B_i \xrightarrow{b} \prod_{i,j,k} \mathcal{O}_X(V_{i,j,k})$$

such that d is a topological embedding and b is continuous. Every $B_i = \mathcal{O}_X(U_i)$ and $\mathcal{O}_X(V_{i,j,k})$ is complete by construction. Therefore, we conclude that B is closed inside the complete Tate ring $\prod_{i=1}^n B_i$. Thus, it is also complete.

[30] Recall that we always implicitly assume that (A, A^+) is complete.

Finally, we show that $A \to B$ is a topological quotient morphism. We consider the short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_Y \to i_*\mathcal{O}_X \to 0.$$

Theorem B.5.2(2) ensures that $\mathrm{H}^1(Y, \mathcal{I}) = 0$, so $A \to B$ is surjective. Now $A \to B$ is a surjective continuous morphism of complete Tate rings, so it is open by Remark B.2.3. In particular, it is a topological quotient morphism. □

Lemma B.6.6. *Let (A, A^+) be a strongly noetherian complete Tate–Huber pair, and let I be an ideal in A. We define $(A^+/I \cap A^+)^c$ to be the integral closure of $A^+/(I \cap A^+)$ in A/I. Then $(A/I, (A^+/I \cap A^+)^c)$ is a complete strongly noetherian Tate–Huber pair, and the morphism*

$$\mathrm{Spa}(A/I, (A^+/I \cap A^+)^c) \to \mathrm{Spa}(A, A^+)$$

is a closed immersion.

Proof. First of all, we note that A/I is complete by [Huber 1994, Proposition 2.4(ii)] and the natural morphism $p\colon A \to A/I$ is open. Now we show that $(A/I, (A^+/I \cap A^+)^c)$ is also a Tate–Huber pair. We choose a pair of definition (A_0, ϖ) with ϖ being a pseudouniformizer in A. Then openness of p implies that $p(A_0)$ is open in A/I. Moreover, its quotient topology coincides with the $p(\varpi)$-adic topology, so it is a ring of definition in A/I. Also, $p(\varpi)$ is a topologically nilpotent unit in A/I, so A/I is a Tate ring. A similar argument shows that $(A^+/I \cap A^+)^c$ is an open subring of A/I that is contained in $(A/I)^\circ$.

We claim that A/I is strongly noetherian. It suffices to show that

$$A\langle T_1, \dots, T_n\rangle \to (A/I)\langle T_1, \dots, T_n\rangle$$

is surjective for each $n \geq 1$. By induction, it suffices to prove the claim for $n = 1$. We pick an element $f \in (A/I)\langle T\rangle$; it can be written as $f = \sum_i \overline{a_i} T^i$ for some $a_i \in A$ such that $\{\overline{a_i}\}$ is a null-system in A/I. This means that for any m there is N_m such that

$$\overline{a_i} \in p(\varpi)^m p(A_0)$$

for any $i \geq N_m$. Thus we can we can find a sequence (b_i) of elements of A such that $\overline{b_i} = \overline{a_i}$ for any $i \geq 0$ and $b_i \in \varpi^m A_0$ for any $i \geq N_m$. This means that $\sum_i b_i T^i$ lies in $A\langle T\rangle$ and its image in $(A/I)\langle T\rangle$ coincides with f.

Now we check that the natural morphism $i\colon X := \mathrm{Spa}(A/I, (A^+/I \cap A^+)^c) \to Y := \mathrm{Spa}(A, A^+)$ is a closed immersion. Firstly, we note that topologically we have an equality

$$i(X) = V(I) := \{x \in \mathrm{Spa}(A, A^+) \mid v_x(I) = 0\}$$

with v_x being the valuation corresponding to a point x. We show that this set is closed. It suffices to show that the set

$$V(f) := \{x \in \mathrm{Spa}(A, A^+) \mid v_x(f) = 0\}$$

is closed for any $f \in I$ since $V(I) = \bigcap_{f \in I} V(f)$. And $V(f)$ is closed as its complement is equal to the union of the rational subdomains

$$Y \setminus V(f) = \bigcup_{n \in \mathbf{N}} Y\left(\frac{\varpi^n}{f}\right).$$

We also need to check that the map $\mathcal{O}_Y \to i_*\mathcal{O}_X$ is surjective with coherent kernel. Clearly,

$$i\colon X = \operatorname{Spa}(A/I, A^+/(I \cap A^+)^c) \to Y = \operatorname{Spa}(A, A^+)$$

is finite, so $i_*\mathcal{O}_X$ is a coherent $\mathcal{O}_Y$-module by Corollary B.5.3(2). Thus, Corollary B.5.3(1) ensures that $\ker(\mathcal{O}_Y \to i_*\mathcal{O}_X)$ is coherent.

Now we show that $\mathcal{O}_Y \to i_*\mathcal{O}_X$ is surjective. It suffices to show that for any rational subdomain $U = Y\left(\frac{f_1}{g}, \dots, \frac{f_n}{g}\right)$ the morphism $\mathcal{O}_Y(U) \to (i_*\mathcal{O}_X)(U)$ is surjective. This boils down to showing that the map

$$A\left\langle \frac{f_1}{g}, \dots, \frac{f_n}{g} \right\rangle \to (A/I)\left\langle \frac{f_1}{g}, \dots, \frac{f_n}{g} \right\rangle$$

is surjective. Consider the commutative diagram

$$\begin{array}{ccc} A\langle T_1, \dots, T_n\rangle & \twoheadrightarrow & (A/I)\langle T_1, \dots, T_n\rangle \\ \downarrow\!\!\!\downarrow & & \downarrow\!\!\!\downarrow \\ A\left\langle \frac{f_1}{g}, \dots, \frac{f_n}{g}\right\rangle & \longrightarrow & (A/I)\left\langle \frac{f_1}{g}, \dots, \frac{f_n}{g}\right\rangle \end{array}$$

where the upper horizontal arrow is surjective by the discussion above. This implies that the lower horizontal arrow is surjective as well. □

Lemma B.6.7. *Let* $f\colon \operatorname{Spa}(B, B^+) \to \operatorname{Spa}(A, A^+)$ *be a morphism of strongly noetherian Tate affinoids, and* $I \subset A$ *an ideal. Then the natural morphism*

$$\operatorname{Spa}(B/IB, (B^+/B^+ \cap IB)^c) \to \operatorname{Spa}(B, B^+) \times_{\operatorname{Spa}(A,A^+)} \operatorname{Spa}(A/I, (A^+/I \cap A^+)^c),$$

is an isomorphism.

Proof. Lemma B.3.6 applied to the finite morphism $A \to A/I$ ensures that[31]

$$\operatorname{Spa}(B, B^+) \times_{\operatorname{Spa}(A,A^+)} \operatorname{Spa}(A/I, (A^+/I \cap A^+)^c) \simeq \operatorname{Spa}(A/I \otimes_A B, (A/I \otimes_A B)^+).$$

Lemma B.6.6 implies that $(A/I, (A^+/I \cap A^+)^c)$ is a complete Tate–Huber pair. Thus Lemma B.3.5 ensures that the tensor product topology on $(A/I) \otimes_A B$ coincides with the natural topology. Lemma B.6.6 also implies that B/IB is a complete Tate ring, in particular, its topology is first countable. Thus, Lemma B.3.4 ensures that its topology coincides with the natural topology. Therefore, the universal property of the natural topology guarantees that the canonical algebraic isomorphism $(A/I) \otimes_A B \simeq B/IB$ preserves topologies on both sides.

[31] In the formula below, $(A/I \otimes_A B)^+$ stands for the integral closure of $\operatorname{Im}((A^+/I \cap A^+)^c \otimes_{A^+} B^+ \to A/I \otimes_A B)$ inside $A/I \otimes_A B$.

Now we recall that

$$((A/I)\otimes_A B)^+ = \mathrm{Im}((A^+/I\cap A^+)^c \otimes_{A^+} B^+ \to (A/I)\otimes_A B)^c = \mathrm{Im}(B^+/(I\cap A^+)B^+ \to B/IB)^c.$$

This admits a natural morphism

$$\mathrm{Im}(B^+/(I\cap A^+)B^+ \to B/IB)^c \to (B^+/(B^+\cap IB))^c$$

that is both injective and surjective. This implies that

$$\begin{aligned}\mathrm{Spa}(B, B^+)\times_{\mathrm{Spa}(A,A^+)} \mathrm{Spa}((A/I), A^+/(I\cap A^+)^c) &\simeq \mathrm{Spa}((A/I)\otimes_A B, (A/I\otimes_A B)^+)\\ &\simeq \mathrm{Spa}(B/IB, (B^+/(B^+\cap IB))^c). \qquad \square\end{aligned}$$

Corollary B.6.8. *Let* $Y = \mathrm{Spa}(A, A^+)$ *be a strongly noetherian Tate affinoid*, $I \subset A$ *an ideal*, *and* $X = \mathrm{Spa}(A/I, (A^+/I\cap A^+)^c)$. *Then the natural map*

$$\widetilde{I} \to \ker(\mathcal{O}_Y \to i_*\mathcal{O}_X)$$

is an isomorphism.

Proof. This follows from the fact that the formation of $\mathrm{Spa}(A/I, (A^+/I\cap A^+)^c)$ commutes with base change by Lemma B.6.7, and the fact that $I\mathcal{O}_Y(U) = I\otimes_A \mathcal{O}_Y(U)$ by A-flatness of $\mathcal{O}_Y(U)$ for a rational subdomain $U\subset Y$. □

Corollary B.6.9. *Let* $Y = \mathrm{Spa}(A, A^+)$ *be a strongly noetherian Tate affinoid*, *and let* $i: X\to Y$ *be a closed immersion. Then it is isomorphic to the closed immersion from* $\mathrm{Spa}(A/I, (A^+/I\cap A^+)^c)$ *for a unique ideal* $I\subset A$.

Proof. Uniqueness of I is easy: Corollary B.6.8 implies that, for a closed immersion

$$X = \mathrm{Spa}(A/I, (A^+/I\cap A^+)^c) \to Y = \mathrm{Spa}(A, A^+),$$

we can recover I as $\Gamma(Y, \mathcal{I})$ for $\mathcal{I} = \ker(\mathcal{O}_Y \to i_*\mathcal{O}_X)$.

Now we show existence of I. We consider the short exact sequence

$$0\to \mathcal{I}\to \mathcal{O}_Y \to i_*\mathcal{O}_X \to 0.$$

Theorem B.5.2(1) implies that $\mathcal{I}\cong \widetilde{I}$ for an ideal $I\subset A$. Moreover, Lemma B.6.5 ensures that $\mathcal{O}_X(X)$ is a complete Tate ring and $\mathcal{O}_X(X)\simeq A/I$ topologically. This isomorphism induces a natural morphism $\phi: X\to \mathrm{Spa}(A/I, (A^+/I\cap A^+)^c)$ by Remark B.1.5.

We first show that ϕ is a homeomorphism. Since both X and $\mathrm{Spa}(A/I, (A^+/I\cap A^+)^c)$ are topologically closed subsets of $\mathrm{Spa}(A, A^+)$, it is sufficient to show that ϕ is a bijection. Now Remark B.6.4 and Corollary B.6.8 imply that both X and $\mathrm{Spa}(A/I, (A^+/I\cap A^+)^c)$ can be topologically identified with the set

$$\{y\in Y \mid \mathcal{J}_y \not\simeq \mathcal{O}_{Y,y}\}.$$

Now we use Remark B.1.2 to ensure that it suffices to show that

$$\phi^{\#}: \mathcal{O}_{\mathrm{Spa}(A/I,(A^+/I\cap A^+)^c)} \to \phi_*\mathcal{O}_X$$

is an isomorphism of sheaves of topological rings. Since $i' \colon \operatorname{Spa}(A/I, (A^+/I \cap A^+)^c) \to \operatorname{Spa}(A, A^+)$ is topologically a closed immersion, it suffices to show that $\phi^\#$ is an isomorphism after applying i'_*, i.e., it suffices to show that the natural morphism

$$i'_* \mathcal{O}_{\operatorname{Spa}(A/I, (A^+/I \cap A^+)^c)} \to i_* \mathcal{O}_X$$

is an isomorphism of sheaves of topological rings. Corollary B.6.8 implies that this is an algebraic isomorphism. We use Remark B.2.3 and Lemma B.6.5 to handle the topological aspect of the isomorphism.[32] □

Corollary B.6.10. *Let $i \colon X \to Y$ be a closed immersion of locally strongly noetherian adic spaces. Then*:

(1) *For any locally topologically finite type morphism $Z \to Y$, the fiber product $Z \times_Y X \to Z$ is a closed immersion.*

(2) *For any closed immersion $i' \colon Z \to X$, the composition $i \circ i' \colon Z \to Y$ is a closed immersion.*

Proof. For the purpose of proving (1), we may assume that X, Y and Z are strongly noetherian Tate affinoids. Then the result follows from Lemma B.6.7 and Corollary B.6.9.

Similarly to prove (2), we may assume that Y is a strongly noetherian Tate affinoid. Then the same holds for X and Z by Corollary B.6.9. It is clear that $Z \to Y$ is a homeomorphism onto its closed image, and that $\mathcal{O}_Y \to (i \circ i')_* \mathcal{O}_Z$ is surjective. Thus, we only need to show that the kernel of that map is coherent. It suffices to show that $(i \circ i')_* \mathcal{O}_Z$ is coherent. Now we note that i and i' are finite by Corollary B.6.9, so $i \circ i'$ is also finite. Therefore, $(i \circ i')_* \mathcal{O}_Z$ is coherent by Corollary B.5.3(2). □

Definition B.6.11. We say that a morphism $f \colon X \to Y$ of analytic adic spaces is a *locally closed immersion* if f can be factored as $j \circ i$ where i is a closed immersion and j is an open immersion.

Lemma B.6.12. *Let $f \colon X \to Y$ and $g \colon Y \to Z$ be locally closed immersions of locally strongly noetherian adic spaces. Then so is $g \circ f$.*

Proof. We first deal with the case f an open immersion and g a closed immersion. In this case the topology on Y is induced from Z, so there is an open adic subspace $U \subset Z$ such that $X = U \cap Z = g^{-1}(U)$. Therefore, we can factor $g \circ f$ as

$$X \xrightarrow{a} U \xrightarrow{b} Z.$$

We note that a is a closed immersion as the restriction of the closed immersion g over $U \subset Z$, and b is an open immersion by construction. Hence, $g \circ f$ is indeed a locally closed immersion.

Now we consider the general case. In this case we can factor f as $j \circ i$ with a closed immersion i and an open immersion j. Similarly, we can factor $g = j' \circ i'$ with a closed immersion j' and an open immersion i'. The argument above implies that the composition $i' \circ j$ can be rewritten as $j'' \circ i''$ for a closed immersion i'' and an open immersion j''. Therefore,

$$g \circ f = j' \circ i' \circ j \circ i = j' \circ j'' \circ i'' \circ i = (j' \circ j'') \circ (i'' \circ i').$$

[32] And an obvious observation that restriction of a closed immersion over an open subspace of the target is again a closed immersion.

Now $i'' \circ i'$ is a closed immersion by Corollary B.6.10(2), and clearly $j' \circ j''$ is an open immersion. Therefore, $g \circ f$ is an immersion. □

Remark B.6.13. The order of the open and closed immersion in Definition B.6.11 is needed to ensure that a composition of immersions is an immersion; the same happens over **C**.

Lemma B.6.14. *Let $f\colon X \to Y$ be a locally closed immersion of analytic adic spaces such that the image $f(X)$ is closed in Y. Then f is a closed immersion.*

Proof. We write f as a composition

$$X \xrightarrow{i} U \xrightarrow{j} Y$$

of a closed immersion i and an open immersion j. Since both i and j are topological embeddings, the same holds for f. Moreover, its image is closed in Y by hypothesis on f. So we are left to show that $\mathcal{I} := \ker(\mathcal{O}_Y \to f_*\mathcal{O}_X)$ is coherent, and $f^\#\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ is surjective.

We use the open covering $Y = U \cup (Y \setminus f(X))$. We know that $\mathcal{I}|_U$ is coherent by assumption, and it is clear that $\mathcal{I}|_{Y\setminus f(X)} \simeq 0$ is coherent. Therefore, we conclude that $\mathcal{I}$ is coherent on Y.

Now we show surjectivity of $f^\sharp$. We note that since f is topologically a closed embedding, we conclude that $(f_*\mathcal{O}_X)_y \cong 0$ for any $y \notin f(X)$. So it suffices to check surjectivity on stalks for $y \in f(X) \subset U$. But then $f^\#_y$ is identified with

$$\mathcal{O}_{Y,y} \cong \mathcal{O}_{U,y} \twoheadrightarrow \mathcal{O}_{X,y}$$

by the assumptions on i and j. □

B.7. *Separated morphisms of adic spaces.*

Definition B.7.1. We say that a locally topologically finite type morphism $f\colon X \to Y$ of locally strongly noetherian analytic adic spaces is *separated*, if the diagonal morphism $\Delta_{X/Y}\colon X \to X \times_Y X$ has closed image

Remark B.7.2. We assume that f is locally topologically finite type to ensure the existence of the fiber product $X \times_Y X$.

Lemma B.7.3. *Let $f\colon X \to Y$ be a locally topologically finite morphism of locally strongly noetherian analytic adic spaces. Then $\Delta_{X/Y}\colon X \to X \times_Y X$ is a locally closed immersion.*

Proof. We cover Y by strongly noetherian Tate affinoids $(U_i)_{i\in I}$, and then we cover the preimages $f^{-1}(U_i)$ by strongly noetherian Tate affinoids $(V_{i,j})_{j\in J_i}$. The construction of fiber products in [Huber 1996, Proposition 1.2.2(a)] implies that $\bigcup_{i,j} V_{i,j} \times_{U_i} V_{i,j}$ is an open subset in $X \times_Y X$ that contains $\Delta_{X/Y}(X)$. Thus in order to show that $\Delta_{X/Y}$ is an immersion, it is suffices to show

$$\alpha\colon X \to \bigcup_{i,j} V_{i,j} \times_{U_i} V_{i,j}$$

is a closed immersion.

Moreover, we note that $\alpha^{-1}(V_{i,j} \times_{U_i} V_{i,j}) = V_{i,j}$ for any $i \in I$, $j \in J_i$. Since the notion of a closed immersion is easily seen to be local on the target, we conclude that it is enough to show that the diagonal morphism is a closed immersion for affinoid spaces $X = \mathrm{Spa}(B, B^+)$ and $Y = \mathrm{Spa}(A, A^+)$. But then the diagonal morphism $X \to X \times_Y X$ coincides with the morphism

$$\Delta_{X/Y} \colon \mathrm{Spa}(B, B^+) \to \mathrm{Spa}(B \widehat{\otimes}_A B, (B \widehat{\otimes}_A B)^+)$$

induced by the natural "multiplication morphism" of Tate–Huber pairs

$$m \colon (B \widehat{\otimes}_A B, (B \widehat{\otimes}_A B)^+) \to (B, B^+)$$

with $(B \widehat{\otimes}_A B)^+$ being the integral closure of $B^+ \widehat{\otimes}_{A^+} B^+$ inside $B \widehat{\otimes}_A B$. Then we see that $\Delta_{X/Y}$ is a closed immersion by Lemma B.6.6 with $I = \ker m$. □

Corollary B.7.4. *Let $f \colon X \to Y$ be a locally topologically finite type, separated morphism of locally strongly noetherian analytic adic spaces. Then the diagonal morphism $\Delta_{X/Y} \colon X \to X \times_Y X$ is a closed immersion.*

Proof. This follows from Lemmas B.6.14 and B.7.3. □

Corollary B.7.5. *Let $f \colon X \to S$ be a locally topologically finite type, separated morphism of analytic adic spaces. Suppose that $S = \mathrm{Spa}(A, A^+)$ is a strongly noetherian Tate affinoid, and that U and V are two open affinoids in X. Then their intersection $U \cap V$ is also an open affinoid in X.*

Proof. Consider the following commutative Cartesian diagram:

$$\begin{array}{ccc} U \cap V & \xrightarrow{i} & U \times_S V \\ \downarrow & & \downarrow \\ X & \xrightarrow{\Delta_{X/S}} & X \times_S X \end{array}$$

Since the map $\Delta_{X/S}$ is a closed immersion by Corollary B.7.4, so is its restriction i. Now we note that U and V are strongly noetherian Tate affinoids by Lemma B.2.14 and Theorem B.2.17. Then $U \times_S V$ is also a strongly noetherian Tate affinoid, so we can apply Corollary B.6.9 to the map i to conclude that $U \cap V$ is affinoid. □

Acknowledgements

We are grateful to B. Bhatt, B. Conrad and S. Petrov for fruitful conversations. We heartily thank M. Temkin for suggesting the argument of Lemma 4.3.6. We express additional gratitude to B. Conrad for reading the first draft of this paper and making lots of suggestions on how to improve the exposition of this paper. We are also very grateful to the anonymous referee who read the paper very carefully and made lots of useful suggestions.

References

[Atiyah and Macdonald 1969] M. F. Atiyah and I. G. Macdonald, *Introduction to commutative algebra*, Addison-Wesley, Reading, MA, 1969. MR Zbl

[Berkovich 1993] V. G. Berkovich, "Étale cohomology for non-archimedean analytic spaces", *Inst. Hautes Études Sci. Publ. Math.* **78** (1993), 5–161. MR Zbl

[BGR 1984] S. Bosch, U. Güntzer, and R. Remmert, *Non-archimedean analysis: a systematic approach to rigid analytic geometry*, Grundl. Math. Wissen. **261**, Springer, 1984. MR Zbl

[Bhatt 2017] B. Bhatt, "Lecture notes for a class on perfectoid spaces", lecture notes, Institute for Advanced Study, 2017, https://www.math.ias.edu/~bhatt/teaching/mat679w17/lectures.pdf.

[Bosch 2014] S. Bosch, *Lectures on formal and rigid geometry*, Lecture Notes in Math. **2105**, Springer, 2014. MR Zbl

[Bourbaki 1998] N. Bourbaki, *Commutative algebra: chapters 1–7*, Springer, 1998. MR Zbl

[Conrad 1999] B. Conrad, "Irreducible components of rigid spaces", *Ann. Inst. Fourier* (*Grenoble*) **49**:2 (1999), 473–541. MR Zbl

[Conrad 2006] B. Conrad, "Relative ampleness in rigid geometry", *Ann. Inst. Fourier* (*Grenoble*) **56**:4 (2006), 1049–1126. MR Zbl

[Conrad and Temkin 2009] B. Conrad and M. Temkin, "Non-archimedean analytification of algebraic spaces", *J. Algebraic Geom.* **18**:4 (2009), 731–788. MR Zbl

[EGA II] A. Grothendieck, "Éléments de géométrie algébrique, II: Étude globale élémentaire de quelques classes de morphismes", *Inst. Hautes Études Sci. Publ. Math.* **8**, 5–222. MR Zbl

[EGA IV_1] A. Grothendieck and J. A. Dieudonné, "Éléments de géométrie algébrique, IV: Étude locale des schémas et des morphismes de schémas, I", *Inst. Hautes Études Sci. Publ. Math.* **20**, 5–259. MR Zbl

[EGA IV_4] A. Grothendieck, "Éléments de géométrie algébrique, IV: Étude locale des schémas et des morphismes de schémas, IV", *Inst. Hautes Études Sci. Publ. Math.* **32**, 5–361. MR Zbl

[Fujiwara and Kato 2018] K. Fujiwara and F. Kato, *Foundations of rigid geometry, I*, European Mathematical Society, Zürich, 2018. MR Zbl

[Hansen 2021] D. Hansen, "Quotients of adic spaces by finite groups", 2021, https://tinyurl.com/4nks368a. To appear in *Math. Res. Letters*.

[Henkel 2016] T. Henkel, "A comparison of adic spaces and Berkovich spaces", preprint, 2016. arXiv 1610.04117

[Huber 1993a] R. Huber, *Bewertungsspektrum und rigide Geometrie*, Regensburger Mathematische Schriften **23**, Universität Regensburg, 1993. MR Zbl

[Huber 1993b] R. Huber, "Continuous valuations", *Math. Z.* **212**:3 (1993), 455–477. MR Zbl

[Huber 1994] R. Huber, "A generalization of formal schemes and rigid analytic varieties", *Math. Z.* **217**:4 (1994), 513–551. MR Zbl

[Huber 1996] R. Huber, *Étale cohomology of rigid analytic varieties and adic spaces*, Aspects of Mathematics **E30**, Vieweg, Braunschweig, 1996. MR Zbl

[Kedlaya 2019] K. S. Kedlaya, "Sheaves, stacks and shtukas", pp. 45–191 in *Perfectoid spaces* (Tuscon, AZ, 2017), edited by B. Cais, Math. Surv. Monogr. **242**, Amer. Math. Soc., Providence, RI, 2019. MR Zbl

[Kedlaya and Liu 2019] K. S. Kedlaya and R. Liu, "Relative p-adic Hodge theory, II: Imperfect period rings", preprint, 2019. arXiv 1602.06899

[Matsumura 1986] H. Matsumura, *Commutative ring theory*, Cambridge Studies in Advanced Mathematics **8**, Cambridge University Press, 1986. MR Zbl

[Olsson 2016] M. Olsson, *Algebraic spaces and stacks*, Amer. Math. Soc. Colloquium Publications **62**, Amer. Math. Soc., Providence, RI, 2016. MR Zbl

[Raynaud and Gruson 1971] M. Raynaud and L. Gruson, "Critères de platitude et de projectivité: techniques de "platification" d'un module", *Invent. Math.* **13** (1971), 1–89. MR Zbl

[Seminar 2015] "Stanford learning seminar", electronic reference, Stanford University, 2015, http://virtualmath1.stanford.edu/~conrad/Perfseminar/.

[SGA 1] A. Grothendieck, *Revêtements étales et groupe fondamental* (Séminaire de Géométrie Algébrique du Bois Marie 1960–1961), Lecture Notes in Math. **224**, Springer. MR Zbl

[Stacks] "The Stacks project", electronic reference, https://stacks.math.columbia.edu.

[Temkin 2000] M. Temkin, "On local properties of non-archimedean analytic spaces", *Math. Ann.* **318**:3 (2000), 585–607. MR Zbl

[Temkin 2015] M. Temkin, "Introduction to Berkovich analytic spaces", pp. 3–66 in *Berkovich spaces and applications* (Paris, 2010), edited by A. Ducros et al., Lecture Notes in Math. **2119**, Springer, 2015. MR Zbl

[Zavyalov 2021a] B. Zavyalov, "Altered local uniformization of rigid-analytic spaces", preprint, 2021. To appear in *Israel Journal of Mathematics*. arXiv 2102.04752

[Zavyalov 2021b] B. Zavyalov, "Mod-p Poincaré duality in p-adic analytic geometry", preprint, 2021. arXiv 2111.01830

Communicated by Antoine Chambert-Loir
Received 2021-10-05 Revised 2023-03-01 Accepted 2023-05-01

bogd.zavyalov@gmail.com School of Mathematics, Institute For Advanced Study, Princeton, NJ, United States

ALGEBRA AND NUMBER THEORY 18:3 (2024)
https://doi.org/10.2140/ant.2024.18.477

Subconvexity bound for GL(3) × GL(2) L-functions: Hybrid level aspect

Sumit Kumar, Ritabrata Munshi and Saurabh Kumar Singh

Let F be a GL(3) Hecke–Maass cusp form of prime level P_1 and let f be a GL(2) Hecke–Maass cuspform of prime level P_2. We will prove a subconvex bound for the GL(3) × GL(2) Rankin–Selberg L-function $L(s, F \times f)$ in the level aspect for certain ranges of the parameters P_1 and P_2.

1. Introduction

In this paper we continue our study of the subconvexity problem for the degree six GL(3) × GL(2) Rankin–Selberg L-functions using the delta symbol approach [Munshi 2018]. In the first paper on this theme Munshi [2022] established subconvex bounds in the t-aspect for these L-functions. Since then the method has been extended by Kumar and Singh together with Sharma and Mallesham (see [Kumar 2023; Kumar et al. 2020; 2022; Sharma and Sawin 2022]), to produce various instances of subconvexity in the spectral aspect and twist aspect. Indeed the delta symbol approach has worked quite well in the t-aspect and the spectral aspect. However its effectiveness and adaptability in the more arithmetic problem of level aspect remains a point of deliberation. In particular, it seems that new inputs are required to tackle the level aspect problem for such L-functions, especially when one of the forms is kept fixed and the level of the other varies. However, as was shown in the lower rank case of Rankin–Selberg convolution of two GL(2) forms [Holowinsky and Munshi 2013], the problem can be more tractable when both the forms vary in certain relative range. The aim of the present paper is to prove such a result for GL(3) × GL(2) Rankin–Selberg convolution.

Theorem 1. *Let P_1 and P_2 be two distinct primes. Let F be a Hecke–Maass cusp form for the congruence subgroup $\Gamma_0(P_1)$ of $\mathrm{SL}(3,\mathbb{Z})$ with trivial nebentypus. Let f be a holomorphic or Maass cusp form for the congruence subgroup $\Gamma_0(P_2)$ of $\mathrm{SL}(2,\mathbb{Z})$ with trivial nebentypus. Let $\mathcal{Q} = P_1^2 P_2^3$ be the arithmetic conductor of the Rankin–Selberg convolution of the above two forms. Then we have*

$$L\left(\tfrac{1}{2}, F \times f\right) \ll \mathcal{Q}^{1/4+\varepsilon}\left(\frac{P_1^{1/4}}{P_2^{3/8}} + \frac{P_2^{1/8}}{P_1^{1/4}}\right).$$

MSC2020: primary 11F66, 11M41; secondary 11F55.
Keywords: subconvexity, Rankin–Selberg L-functions, Hecke–Maass forms.

Note that the convexity bound is given by $\mathcal{Q}^{1/4+\varepsilon}$. Thus the above bound is subconvex in the range

$$P_2^{1/2+\epsilon} < P_1 < P_2^{3/2-\epsilon}.$$

This provides the first instance of a subconvex bound in the level aspect for a degree six L-function which is not a character twist of a fixed L-function. The bound is strongest when P_1 and P_2 are roughly of same size $P_1 \approx P_2$, in which case Theorem 1 gives

$$L\big(\tfrac{1}{2}, F \times f\big) \ll \mathcal{Q}^{1/4-1/40+\varepsilon}.$$

The exponent $\frac{1}{4} - \frac{1}{40}$ appears in other contexts as well and it seems to be the limit of the delta symbol approach. We also note that our proof with some suitable modifications works even in the case of composite levels P_1 and P_2. But to keep the exposition simple and clean we will only give full details for the case of prime levels.

For a detailed introduction to automorphic forms on higher rank groups and for basic analytic properties of Rankin–Selberg convolution L-functions we refer the readers to Goldfeld's book [2006]. Our treatment will be at the level of L-functions, and the Voronoi summation formulae for GL(2) and GL(3) are the only input that we need from the theory of automorphic forms. For a broader introduction to the subconvexity problem and its applications we refer the readers to [Michel 2007; Munshi 2018].

Historically the level aspect subconvexity problem has proved to be more challenging compared to the spectral aspect or the t-aspect, regardless of the method adopted. Indeed Weyl shift is all one needs to prove the t-aspect subconvexity for $\zeta(s)$; see [Weyl 1921]. Whereas Burgess had to nontrivially extend Weyl's ideas and had to invoke Riemann hypothesis for curves over finite fields, to obtain the first level aspect subconvexity result $L\big(\frac{1}{2}, \chi\big) \ll q^{3/16+\varepsilon}$; see [Burgess 1963]. In the 1990s Duke, Friedlander and Iwaniec [Duke et al. 1993; 1994; 2000] used the amplification technique to obtain the level aspect subconvexity for GL(2) L-functions. The amplification method was extended by Kowalski, Michel and Vanderkam [Kowalski et al. 2002] to Rankin–Selberg convolutions $\mathrm{GL}(2) \times \mathrm{GL}(2)$. Venkatesh [2010] used ergodic theory to study orbital integrals, and thus obtained level aspect subconvex bounds for triple products $\mathrm{GL}(2) \times \mathrm{GL}(2) \times \mathrm{GL}(2)$, where two forms are fixed and one varies. A similar technique was also adopted by Michel and Venkatesh [2010] for $\mathrm{GL}(2) \times \mathrm{GL}(2)$ L-functions over any number fields. The level aspect subconvexity problem for any genuine $\mathrm{GL}(d)$ L-function with $d > 2$ remains an important open problem.

Our interest in the subconvexity problem for $\mathrm{GL}(3) \times \mathrm{GL}(2)$ Rankin–Selberg convolution is kindled by two factors. First there is a structural advantage which makes the $\mathrm{GL}(n) \times \mathrm{GL}(n-1)$ L-functions a suitable candidate for analytic number theoretic exploration. Indeed the case of $n = 2$ has been extensively studied in the literature, as we will see below, and we want to extend to the next level $n = 3$. Secondly, $\mathrm{GL}(3) \times \mathrm{GL}(2)$ Rankin–Selberg convolutions appear in important applications, like the quantum unique ergodicity, and so it is important to analyze different aspects of the subconvexity problem for these L-functions with the aim of developing techniques that will eventually work in the required scenarios, e.g., spectral aspect subconvexity for symmetric square L-functions. Finally, let us also stress, that we are motivated to explore the scope of the delta symbol approach to subconvexity and other related problems. After initial success of Munshi [2018], the method has been extended, simplified and generalized by several researchers, e.g., see [Holowinsky and Nelson 2018; Aggarwal 2020; Aggarwal et al. 2020a; 2020b; Kowalski et al. 2020; Kumar 2023; Munshi and Singh 2019; Sharma and Sawin 2022; Lin et al. 2023].

The twists of GL(2) L-functions by Dirichlet characters, or in other words $\mathrm{GL}(2)\times\mathrm{GL}(1)$ L-functions have been studied extensively in the literature, ever since the breakthrough work of Duke, Friedlander and Iwaniec [1993]. Hybrid subconvexity have also been studied for these L-functions. Since this is the lower rank analogue of the L-function we are investigating in this paper, we briefly recall some results in this basic case. Let f be a GL(2) new form of level P_2 and let χ be a primitive Dirichlet character of modulus P_3. Suppose $(P_2, P_3)=1$, then $\mathcal{Q}=P_2P_3^2$ is the arithmetic conductor of $L\big(\frac{1}{2}, f\otimes\chi\big)$. Different methods are now available to prove hybrid subconvexity bound, when the levels of forms vary in a relative range, say $P_2\sim P_3^{\eta}$. Blomer and Harcos [2008] used amplification technique to prove

$$L\big(\tfrac{1}{2}, f\otimes\chi\big)\ll \mathcal{Q}^{1/4+\epsilon}(\mathcal{Q}^{-1/(8(2+\eta))}+\mathcal{Q}^{-1-\eta/(4(2+\eta))})$$

for $0<\eta<1$. Aggarwal, Jo and Nowland [Aggarwal et al. 2018] used classical delta method to prove

$$L\big(\tfrac{1}{2}, f\otimes\chi\big)\ll \mathcal{Q}^{1/4-(2-5\eta)/(20(2+\eta))+\epsilon}$$

for $0<\eta<\frac{2}{5}$. Computing the average of the second moment of $L\big(\frac{1}{2}, f\otimes\chi\big)$ over a family of forms, Hou and Chen [2019] extended the range of η to $0<\eta<\frac{3}{2}-\theta$, where θ is any admissible exponent towards the Petersson–Ramanujan conjecture for the Fourier coefficients. Currently, the result of Hou and Chen yields the widest range $P_2\ll P_3^{3/2-\delta}$, but it falls short of the Burgess bound. In a recent work, Khan [2021] not only extended the range of P_2, but also obtained the Weyl bound in the case of $P_2\sim P_3$. By computing the second moment over a family of $GL(2)$ forms, Khan proved, in the range $P_3\gg P_2^{1/2}$, that

$$\sum_{f\in B_k^\star(P_2)}\big|L\big(\tfrac{1}{2}f\otimes\chi\big)\big|^2\ll_{k,\epsilon}\mathcal{Q}^{\epsilon}(P_2+P_3),$$

where $B_k^\star(P_2)$ denote a basis of holomorphic newforms of level P_2 and weight k, and $\mathcal{Q}=P_2P_3^2$. Recently, during an AIM workshop “Delta symbol and subconvexity”, the first and the third author used the delta symbol approach to prove

$$L\big(\tfrac{1}{2}, f\otimes\chi\big)\ll_{\epsilon}\mathcal{Q}^{\epsilon}\frac{\sqrt{P_2P_3}}{\min\{\sqrt{P_2}, \sqrt{P_3}\}}.$$

This is of same strength as [Khan 2021].

2. The set-up

Let F and f be as in Theorem 1. We will denote the normalized Fourier coefficients of f by $\lambda_f(n)$, and that of F by $\lambda_F(n, r)$. The Rankin–Selberg convolution is given by the absolutely converging Dirichlet series

$$L(s, F \times f) = \sum\sum_{n,r=1}^{\infty} \frac{\lambda_F(n,r)\lambda_f(n)}{(nr^2)^s}$$

in the right half plane $\mathrm{Re}(s) = \sigma > 1$. Here it is also given by a degree six Euler product. This function extends to an entire function and satisfies a functional equation of Riemann type. It is known that this Rankin–Selberg convolution is the standard L-function of a GL(6) automorphic form [Kim and Shahidi 2002].

2A. ***Approximate functional equation.*** The functional equation gives an expression of the central value $L\big(\frac{1}{2}, F\times f\big)$ in terms of rapidly decaying series, the so called approximate functional equation [Iwaniec and Kowalski 2004, Theorem 5.3]. Taking a smooth dyadic subdivision of this expression we get the following.

Lemma 2.1. *Let $\mathcal{Q} = P_1^2 P_2^3$ be the arithmetic conductor attached to the L-function $L\big(\frac{1}{2}, F \times f\big)$. Then, as $\mathcal{Q} \to \infty$, we have*

$$L\big(\tfrac{1}{2}, F \times f\big) \ll_\epsilon \mathcal{Q}^\epsilon \sum_{r \le \mathcal{Q}^{(1+2\epsilon)/4}} \tfrac{1}{r} \sup_{N \le \mathcal{Q}^{1/2+\epsilon}/r^2} \frac{|S_r(N)|}{N^{1/2}} + \mathcal{Q}^{-2021}, \tag{1}$$

where $S_r(N)$ is a sum of the form

$$S_r(N) := \sum_{n=1}^{\infty} \lambda_F(n,r)\lambda_f(n) V\left(\frac{n}{N}\right), \tag{2}$$

for some smooth function V supported in $[1, 2]$ *and satisfying $V^{(j)}(x) \ll_j 1$.*

This is the usual starting point of the delta symbol approach. Thus, to get subconvexity, it is enough to get some cancellation in the sum

$$S_r(N) = \sum_{n=1}^{\infty} \lambda_F(n,r)\lambda_f(n) V(n/N),$$

for N near the generic range $N \asymp \mathcal{Q}^{1/2}$.

2B. ***Delta symbol.*** Next we separate the oscillations involved in $S_r(N)$. For this we will use a Fourier expansion of the Kronecker delta symbol. For any $Q > 1$ one has

$$\delta(n) = \frac{1}{Q} \sum_{1 \le q \le Q} \frac{1}{q} \sideset{}{^\star}\sum_{a \bmod q} e\left(\frac{an}{q}\right) \int_{\mathbb{R}} g(q,x) e\left(\frac{nx}{qQ}\right) \mathrm{d}x,$$

where $g(q,x)$ is a smooth function of x satisfying

$$\begin{aligned} g(q,x) &= 1 + h(q,x), \quad \text{with } h(q,x) = O\big(\tfrac{Q}{q}\big(\tfrac{q}{Q} + |x|\big)^B\big), \\ x^j \frac{\partial^j}{\partial x^j} g(q,x) &\ll \log Q \min\left\{\frac{Q}{q}, \frac{1}{|x|}\right\}, \\ g(q,x) &\ll |x|^{-B}, \end{aligned} \tag{3}$$

for any $B > 1$ and $j \geq 1$. (Here $e(z) = e^{2\pi i z}$.) This expansion of δ is due to Duke, Friedlander and Iwaniec, and one can find details of this in [Iwaniec and Kowalski 2004]. Using the third property of $g(q, x)$, we observe that the effective range of the integration over x is $[-Q^{\varepsilon}, Q^{\varepsilon}]$. Also it follows that if $q \ll Q^{1-\varepsilon}$ and $x \ll Q^{-\varepsilon}$, then $g(q, x)$ can be replaced by 1 at the cost of a negligible error term. In the complimentary range, using second property, we have

$$x^j \frac{\partial^j}{\partial x^j} g(q, x) \ll Q^{\varepsilon}.$$

Finally as in [Munshi 2022], by Parseval and Cauchy, we get

$$\int_{\mathbb{R}} (|g(q, x)| + |g(q, x)|^2)\, \mathrm{d}x \ll Q^{\varepsilon},$$

i.e., $g(q, x)$ has average size "one" in the L^1 and L^2 sense. Applying this expansion and choosing $Q = N^{1/2}$, we get

$$\begin{aligned} S_r(N) &= \sum\sum_{m,n=1}^{\infty} \lambda_F(n, r)\lambda_f(m) V(n/N) W(m/N) \delta(n-m) \\ &= \frac{1}{Q}\int_{\mathbb{R}} \sum_{1\leq q\leq Q} \frac{g(q, x)}{q} \sideset{}{^\star}\sum_{a \bmod q} \sum_{n=1}^{\infty} \lambda_F(n, r) e\left(\frac{na}{q}\right) e\left(\frac{nx}{qQ}\right) V\left(\frac{n}{N}\right) \\ &\qquad\qquad \times \sum_{m=1}^{\infty} \lambda_f(m) e\left(\frac{-ma}{q}\right) e\left(\frac{-mx}{qQ}\right) W\left(\frac{m}{N}\right) \mathrm{d}x. \end{aligned} \tag{4}$$

2C. *Ideas behind the proof.* In this section, we will discuss the method and present a sketch of the proof. For simplicity, let's consider the generic case, i.e., $N = \sqrt{P_1^2 P_2^3}$, $r = 1$ and $q \asymp Q = \sqrt{N}$. Thus $S_r(N)$ in (4) looks like

$$\frac{1}{Q^2} \sum_{q\sim Q} \sideset{}{^\star}\sum_{a \bmod q} \sum_{n\sim N} \lambda_F(n, 1) e\left(\frac{an}{q}\right) \sum_{m\sim N} \lambda_f(m) e\left(\frac{-am}{q}\right).$$

On applying GL(3) Voronoi to the n-sum, the dual length becomes

$$n^{\star} \sim \frac{\text{Conductor}}{\text{Initial Length}} = \frac{Q^3 P_1}{N} = P_1 N^{1/2},$$

and we save

$$\frac{\text{Initial Length}}{\sqrt{\text{Conductor}}} = \frac{N}{Q^{3/2} P_1^{1/2}}.$$

Next we apply GL(2) Voronoi formula to the sum over m. In this case, the dual length (generic) is given by

$$m^{\star} \sim \frac{Q^2 P_2}{N} = P_2,$$

and we save $N/(Q\sqrt{P_2})$ in this step. The resulting character sum is given by

$$\sum_{a \bmod q}^{\star} S(-\bar{a}\overline{P_1}, n^\star; q) e\left(\frac{m^\star \bar{a} \overline{P_2}}{q}\right) = q e\left(\frac{\overline{P_1 m^\star} P_2 n^\star}{q}\right).$$

This reduction of the character sum into an additive character with respect to the GL(3) variable $n^\star$ drives the rest of the argument. We save $\sqrt{Q}$ from the sum over a. Hence, in total, we have saved

$$\frac{N}{Q^{3/2} P_1^{1/2}} \times \frac{N}{Q\sqrt{P_2}} \times \sqrt{Q} = \frac{N}{\sqrt{P_1 P_2}}.$$

In the next step, we apply Cauchy's inequality to the $n^\star$-sum in the following resulting expression:

$$\sum_{q \sim Q} \sum_{n^\star \sim P_1\sqrt{N}} \lambda_F(n, 1) \sum_{m^\star \sim P_2} \lambda_f(m) e\left(\frac{\overline{P_1 m^\star} P_2 n^\star}{q}\right).$$

After Cauchy, we arrive at

$$(P_1\sqrt{N})^{1/2} \left(\sum_{n^\star \sim P_1\sqrt{N}} \left| \sum_{q \sim Q} \sum_{m^\star \sim P_2} \lambda_f(m) e\left(\frac{\overline{P_1 m^\star} P_2 n^\star}{q}\right) \right|^2 \right)^{1/2},$$

in which we seek to save $\sqrt{P_1 P_2}$ and a little more. In the final step, we apply Poisson summation formula to the $n^\star$-sum. In the zero frequency($n^\star = 0$), we save $(QP_2)^{1/2}$ which is sufficient provided

$$(QP_2)^{1/2} > (P_1 P_2)^{1/2} \iff Q > P_1 \iff P_2^{3/2} > P_1.$$

In the nonzero frequency, we save $(P_1\sqrt{N}/\sqrt{Q^2})^{1/2}$. From the additive character inside the modulus, which arises due to a specific feature of GL(3) $\times$ GL(2) L-functions, we also save $\sqrt{Q}$. Thus we save $(P_1\sqrt{N})^{1/2}$, which is sufficient if

$$(P_1\sqrt{N})^{1/2} > (P_1 P_2)^{1/2} \iff P_1 > P_2^{1/2}.$$

Hence, we obtain subconvexity in the range $P_2^{1/2} < P_1 < P_2^{3/2}$. Optimal saving, from Poisson, can be chosen by taking the minimum of the zero and nonzero frequencies savings. Hence

$$S(N) \ll \frac{N\sqrt{P_1 P_2}}{\min\{\sqrt{QP_2}, \sqrt{P_1\sqrt{N}}\}} = \frac{N}{\min\{N^{1/4}/P_1^{1/2}, N^{1/4}/P_2^{1/2}\}},$$

and consequently

$$L\left(\tfrac{1}{2}, F \times f\right) \ll \frac{(P_1^2 P_2^3)^{1/4}}{\min\{P_2^{3/8}/P_1^{1/4}, P_1^{1/4}/P_2^{1/8}\}},$$

which is best possible when $P_1 \asymp P_2 (:= P)$ and $P_1 \neq P_2$. In this case we get

$$L\left(\tfrac{1}{2}, F \times f\right) \ll_\epsilon (P^5)^{1/4 - 1/40 + \epsilon}.$$

3. Voronoi summation formula

Our next step involves applications of summation formulas.

3A. GL(3) *Voronoi.* In this section, we analyze the sum over n using GL(3) Voronoi summation formula. The following Lemma, except for the notations, is taken from [Zhou 2018]. Let F be a Hecke–Maass cusp form of type (ν_1, ν_2) for the congruent subgroup $\Gamma_0(P_1)$ of $SL(3, \mathbb{Z})$ with the trivial character. The Fourier coefficients of F and that of its dual $\tilde{F}$ are related by

$$\lambda_F(r, n) = \lambda_{\tilde{F}}(n, r),$$

for $(nr, P_1) = 1$. Let

$$\alpha_1 = -\nu_1 - 2\nu_2 + 1, \alpha_2 = -\nu_1 + \nu_2, \alpha_3 = 2\nu_1 + \nu_2 - 1$$

be the Langlands parameters for F; see [Goldfeld 2006] for more details. Let g be a compactly supported smooth function on $(0, \infty)$ and $\tilde{g}(s) = \int_0^\infty g(x)x^{s-1}\,dx$ be its Mellin transform. For $\ell = 0$ and 1, we define

$$\gamma_\ell(s) := i^\ell \varepsilon(F) P_1^{1/2+s} \frac{\pi^{-3s-3/2}}{2} \prod_{i=1}^{3} \frac{\Gamma((1+s+\alpha_i+\ell)/2)}{\Gamma((-s-\alpha_i+\ell)/2)},$$

with $|\varepsilon(F)| = 1$. Set $\gamma_\pm(s) = \gamma_0(s) \mp \gamma_1(s)$ and let

$$H_\pm(y) = \frac{1}{2\pi i} \int_{(\sigma)} y^{-s} \frac{\pi^{-3s-3/2}}{2} \gamma_\pm(s) \tilde{g}(-s)\,ds,$$

where $\sigma > -1 + \max\{-\operatorname{Re}(\alpha_1), -\operatorname{Re}(\alpha_2), -\operatorname{Re}(\alpha_3)\}$. Let $G_\pm(y) = P_1^{1/2} H_\pm(y/P_1)$. With the aid of the above terminology, we now state the GL(3) Voronoi summation formula in the following lemma.

Lemma 3.1. *Let $g(x)$ and $\lambda_F(n, r)$ be as above. Let $a, q \in \mathbb{Z}$ with $q > 0$, $(a, q) = 1$, and let $\bar{a}$ be the multiplicative inverse of a modulo q. Suppose $(qr, P_1) = 1$. Then we have*

$$\sum_{n=1}^{\infty} \lambda_F(n, r) e\left(\frac{an}{q}\right) g(n) = q \sum_{\pm} \sum_{n_1 \mid qr} \sum_{n_2=1}^{\infty} \frac{\lambda_F(n_1, n_2)}{n_1 n_2} S(r\bar{a}\overline{P_1}, \pm n_2; qr/n_1) G_\pm\left(\frac{n_1^2 n_2}{q^3 r}\right)$$

where $S(a, b; q)$ is the Kloosterman sum which is defined as

$$S(a, b; q) = \sideset{}{^\star}\sum_{x \bmod q} e\left(\frac{ax + b\bar{x}}{q}\right).$$

Proof. See [Zhou 2018] for the proof. □

To apply Lemma 3.1 in our setup, we need to extract the oscillations of the integral transform. To this end, we state the following lemma.

Lemma 3.2. *Let g be supported in the interval $[X, 2X]$ and let $H_\pm$ be defined as above. Then for any fixed integer $K \geq 1$ and $xX \gg 1$, we have*

$$H_\pm(x) = x\int_0^\infty g(y)\sum_{j=1}^{K}\frac{c_j(\pm)e(3(xy)^{1/3}) + d_j(\pm)e(-3(xy)^{1/3})}{(xy)^{j/3}}\,\mathrm{d}y + O((xX)^{(-K+2)/3}),$$

where $c_j(\pm)$ and $d_j(\pm)$ are some absolute constants depending on α_i, for $i = 1, 2, 3$.

Proof. See Lemma 6.1 of [Li 2009]. □

Plugging the leading term of Lemma 3.2 in Lemma 3.1 and using the resulting expression in (4) we see that the sum over n gets transformed into

$$\frac{N^{2/3}}{P_1^{1/6}qr^{2/3}}\sum_\pm\sum_{n_1\mid qr} n_1^{1/3}\sum_{n_2=1}^{\infty}\frac{\lambda_F(n_1,n_2)}{n_2^{1/3}}S(r\bar{a}\overline{P_1}, \pm n_2; qr/n_1)\mathcal{I}(\,\cdot\,), \tag{5}$$

where

$$\mathcal{I}(\,\cdot\,) = \int_0^\infty V(z)e\left(\frac{Nxz}{qQ} \pm \frac{3(Nn_1^2n_2z)^{1/3}}{P_1^{1/3}qr^{1/3}}\right)\mathrm{d}z.$$

We observe that, using integration by parts repeatedly, the above integral is negligibly small if

$$n_1^2n_2 \gg N^\epsilon\sqrt{N}P_1r = N^\epsilon\frac{P_1Q^3r}{N} =: N_0.$$

In the case when $P_1 \mid qr$, an appropriate Voronoi summation from [Zhou 2018] can still be used. In fact it turns out that our analysis in this paper still goes through with slight modification and the final bound is even better. As such we proceed to present our analysis only in the coprime case.

3B. GL(2) *Voronoi.* In this section, we dualize the sum over m using GL(2) Voronoi summation formula.

Lemma 3.3. *Let $f \in H_k(P_2)$ be a holomorphic Hecke cuspform with Fourier coefficients $\lambda_f(n)$ and trivial nebentypus. Let a and q be integers with $(aP_2, q) = 1$. Let g be a compactly supported smooth bump function on $\mathbb{R}$. Then we have*

$$\sum_{m=1}^{\infty}\lambda_f(m)e\left(\frac{-am}{q}\right)g(n) = \frac{1}{q}\frac{\eta_f(P_2)}{\sqrt{P_2}}\sum_{n=1}^{\infty}\lambda_f(m)e\left(\frac{m\overline{aP_2}}{q}\right)H\left(\frac{m}{P_2q^2}\right), \tag{6}$$

where $a\bar{a} \equiv 1 \pmod q$, $|\eta_f(P_2)| = 1$ and

$$H(y) = 2\pi i^k\int_0^\infty g(x)J_{k-1}(4\pi\sqrt{xy})\,\mathrm{d}x,$$

where J_{k-1} is the J-Bessel function and k is the weight of f.

Proof. See the appendix of [Kowalski et al. 2002]. □

Extracting the oscillations of J_{k-1},

$$J_{k-1}(2\pi x) = e(x)W_{k-1}(x) + e(-x)\overline{W}_{k-1}(x),$$

with

$$x^j \frac{\mathrm{d}^j}{\mathrm{d}x^j} W_{k-1}(x) \ll_{j,k} 1/\sqrt{x},$$

we see that $H(y)$ can be essentially replaced by

$$H(y) = \frac{2\pi i^k}{y^{1/4}} \int_0^\infty g_1(x) e(\pm 2\sqrt{xy})\, \mathrm{d}x,$$

in our analysis, where g_1 is the new weight function which has compact support and $x^j g_1^{(j)}(x) \ll_j 1$, $j \geq 0$. Applying the above lemma, the sum over m in (4) reduces to

$$\frac{N^{3/4}\eta_f(P_2)}{P_2^{1/4}\sqrt{q}} \sum_{m=1}^{\infty} \frac{\lambda_f(m)}{m^{1/4}} e\left(\frac{m a \overline{P_2}}{q}\right) \int_0^\infty W(y) e\left(\frac{-yNx}{qQ}\right) e\left(\frac{\pm 2\sqrt{Nmy}}{qP_2^{1/2}}\right) \mathrm{d}y. \tag{7}$$

Notice the abuse of notation: the weight function W is different from the one in (4). Using stationary phase analysis we observe that the above integral is negligibly small unless

$$m \ll N^\epsilon P_2 = N^\epsilon \frac{Q^2 P_2}{N} =: M_0.$$

Again we will ignore the degenerate case where $P_2 \mid q$ and proceed with the analysis of the generic case. Indeed our analysis works in the degenerate case as well, and the bound that we obtain is even better (as one will expect).

Now plugging (5) and (7) in (4), we arrive at

$$\frac{N^{17/12}\eta_f(P_2)}{P_1^{1/6}P_2^{1/4}Qr^{2/3}} \sum_{1\leq q\leq Q} \frac{1}{q^{5/2}} \sum_{\pm} \sum_{n_1 \mid qr} n_1^{1/3} \sum_{n_2 \ll N_0/n_1^2} \frac{\lambda_F(n_1,n_2)}{n_2^{1/3}} \sum_{m \ll M_0} \frac{\lambda_f(m)}{m^{1/4}} \mathcal{C}(\cdot)\mathfrak{I}(\cdot), \tag{8}$$

where the integral transform is given by

$$\mathfrak{I}(\cdot) = \int_{\mathbb{R}} W(x) g(q,x) \int_0^\infty W(y) \int_0^\infty V(z) e\left(\frac{Nx(z-y)}{qQ} \pm \frac{2\sqrt{Nmy}}{qP_2^{1/2}} \pm \frac{3(Nn_1^2n_2z)^{1/3}}{P_1^{1/3}qr^{1/3}}\right) \mathrm{d}z\, \mathrm{d}y\, \mathrm{d}x,$$

and the character sum is given by

$$\mathcal{C}(\cdot) := \sideset{}{^\star}\sum_{a \bmod q} S(r\bar{a}\overline{P_1}, \pm n_2; qr/n_1) e\left(\frac{m\bar{a}\overline{P_2}}{q}\right) = \sum_{d \mid q} d\mu\left(\frac{q}{d}\right) \sideset{}{^\star}\sum_{\substack{\alpha \bmod qr/n_1 \\ \overline{P_1}n_1\alpha \equiv -m\overline{P_2} \bmod d}} e\left(\pm\frac{\bar{\alpha}n_2}{qr/n_1}\right).$$

4. Cauchy and Poisson

4A. ***Cauchy inequality.*** Now we apply Cauchy's inequality to the n_2-sum in (8). To this end, we split the sum over q into dyadic blocks $q \backsim C$ and further writing $q = q_1 q_2$ with $q_1 \mid (n_1 r)^\infty$, $(n_1 r, q_2) = 1$, we see that $S_r(N)$ is bounded by

$$\sup_{C\ll Q} \frac{N^{17/12}}{P_2^{1/4}P_1^{1/6}Qr^{2/3}C^{5/2}} \sum_{\pm} \sum_{\frac{n_1}{(n_1,r)}\ll C} n_1^{1/3} \sum_{\frac{n_1}{(n_1,r)}|q_1|(n_1r)^\infty} \sum_{n_2\ll N_0/n_1^2} \frac{|\lambda_F(n_1,n_2)|}{n_2^{1/3}} \times \Bigg| \sum_{q_2\backsim C/q_1} \sum_{m\ll M_0} \frac{\lambda_f(m)}{m^{1/4}} \mathcal{C}(\,\cdot\,)\mathfrak{I}(\,\cdot\,)\Bigg|, \tag{9}$$

On applying the Cauchy's inequality to the n_2-sum we arrive at

$$S_r(N) \ll \sup_{C\ll Q} \frac{N^{17/12}}{P_2^{1/4}P_1^{1/6}Qr^{2/3}C^{5/2}} \sum_{\pm} \sum_{\frac{n_1}{(n_1,r)}\ll C} n_1^{1/3}\Theta^{1/2} \sum_{\frac{n_1}{(n_1,r)}|q_1|(n_1r)^\infty} \sqrt{\Omega}, \tag{10}$$

where

$$\Theta = \sum_{n_2\ll N_0/n_1^2} \frac{|\lambda_F(n_1,n_2)|^2}{n_2^{2/3}}, \tag{11}$$

and

$$\Omega = \sum_{n_2\ll N_0/n_1^2} \Bigg| \sum_{q_2\backsim C/q_1} \sum_{m\ll M_0} \frac{\lambda_f(m)}{m^{1/4}} \mathcal{C}(\,\cdot\,)\mathfrak{I}(\,\cdot\,)\Bigg|^2. \tag{12}$$

4B. *Poisson.* We now apply the Poisson summation formula to the n_2-sum in (12). To this end, we smooth out the n_2-sum, i.e., we plug in an appropriate smooth bump function, say, W. Opening the absolute value square, we get

$$\Omega = \sum\sum_{q_2,q_2'\backsim C/q_1} \sum\sum_{m,m'\ll M_0} \frac{\lambda_f(m)\lambda_f(m')}{(mm')^{1/4}} \times \sum_{n_2\in\mathbb{Z}} W\left(\frac{n_2}{N_0/n_1^2}\right)\mathcal{C}(\,\cdot\,)\bar{\mathcal{C}}(\,\cdot\,)\mathfrak{I}(\,\cdot\,)\bar{\mathfrak{I}}(\,\cdot\,).$$

Reducing n_2 modulo $q_1q_2q_2'r/n_1 := \gamma$, and using the change of variable

$$n_2 \mapsto n_2q_1q_2q_2'r/n_1+\beta, \quad \text{with } 0\le\beta<q_1q_2q_2'r/n_1,$$

followed by the Poisson summation formula, we arrive at

$$\Omega = \sum\sum_{q_2,q_2'\backsim C/q_1} \sum\sum_{m,m'\ll M_0} \frac{\lambda_f(m)\lambda_f(m')}{(mm')^{1/4}} \sum_{n_2\in\mathbb{Z}} \sum_{\beta \bmod \gamma} \mathcal{C}(\,\cdot\,)\bar{\mathcal{C}}(\,\cdot\,)\mathcal{J}, \tag{13}$$

where

$$\mathcal{J} = \int_{\mathbb{R}} W\left(\frac{w\gamma+\beta}{N_0/n_1^2}\right)\mathfrak{I}(\,\cdot\,)\bar{\mathfrak{I}}(\,\cdot\,)e(-n_2w)\,\mathrm{d}w.$$

Now changing the variable

$$\frac{w\gamma+\beta}{N_0/n_1^2} \mapsto w,$$

we arrive at

$$\mathcal{J} = \frac{N_0}{n_1^2\gamma}e\left(\frac{n_2\beta}{\gamma}\right)\int_{\mathbb{R}} W(w)\mathfrak{I}(\,\cdot\,)\bar{\mathfrak{I}}(\,\cdot\,)e\left(\frac{-n_2N_0w}{n_1^2\gamma}\right)\mathrm{d}w.$$

Plugging this back in (13), and executing the sum over β, we arrive at

$$\Omega = \frac{N_0}{n_1^2} \sum\sum_{q_2, q_2' \backsim C/q_1} \sum\sum_{m, m' \ll M_0} \frac{\lambda_f(m)\lambda_f(m')}{(mm')^{1/4}} \sum_{n_2 \in \mathbb{Z}} \mathfrak{C}\mathcal{I}, \tag{14}$$

where

$$\mathfrak{C} = \sum_{\substack{d \mid q \\ d' \mid q'}}\sum dd'\mu\left(\frac{q}{d}\right)\mu\left(\frac{q'}{d'}\right) \sideset{}{^\star}\sum_{\substack{\alpha \bmod qr/n_1 \\ \overline{P_1}n_1\alpha \equiv -m\overline{P_2} \bmod d}} \sideset{}{^\star}\sum_{\substack{\alpha' \bmod q'r/n_1 \\ \overline{P_1}n_1\alpha' \equiv -\overline{P_2}m' \bmod d' \\ \pm\bar{\alpha}q_2' \mp \bar{\alpha}' q_2 \equiv -n_2 \bmod q_1q_2q_2'r/n_1}} 1 \tag{15}$$

and

$$\mathcal{I} = \int_{\mathbb{R}} W(w)\mathfrak{J}(\,\cdot\,)\bar{\mathfrak{J}}(\,\cdot\,)e\left(\frac{-n_2N_0w}{n_1q_1q_2q_2'r}\right) \mathrm{d}w. \tag{16}$$

On applying integration by parts, we see that the above integral is negligibly small if

$$n_2 \gg \frac{Q}{q}\frac{n_1q_1q_2q_2'r}{N_0} := N_2. \tag{17}$$

5. Bounding the integral

In this section we will analyze the integral $\mathcal{I}$ given in (16). Recall that the integral $\mathfrak{J}(\,\cdot\,)$ is given by

$$\mathfrak{J}(\cdot) = \int_{\mathbb{R}} W(x)g(q,x)\int_0^\infty W(y)\int_0^\infty V(z)e\left(\frac{Nx(z-y)}{qQ} \pm \frac{2\sqrt{Nmy}}{qP_2^{1/2}} \pm \frac{3(NN_0wz)^{1/3}}{P_1^{1/3}qr^{1/3}}\right) \mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x. \tag{18}$$

Let's first focus on x-integral, i.e.,

$$\int_{\mathbb{R}} W(x)g(q,x)e\left(\frac{Nx(z-y)}{qQ}\right)\mathrm{d}x.$$

In the case, $q \ll Q^{1-\epsilon}$, we split the above integral as follows:

$$\left(\int_{|x| \ll Q^{-\epsilon}} + \int_{|x| \gg Q^{-\epsilon}}\right) W(x)g(q,x)e\left(\frac{Nx(z-y)}{qQ}\right)\mathrm{d}x.$$

For the first part, we can replace $g(q,x)$ by 1 at the cost of a negligible error term (see (3)) so that we essentially have

$$\int_{|x| \ll Q^{-\epsilon}} W(x)e\left(\frac{Nx(z-y)}{qQ}\right)\mathrm{d}x.$$

Using integration by parts, we observe that the above integral is negligibly small unless

$$|z-y| \ll \frac{q}{Q}Q^{\epsilon}.$$

For the second part, using $g^{(j)}(q,x) \ll Q^{\epsilon j}$, we get the restriction $|z-y| \ll \frac{q}{Q}Q^{\epsilon}$. In the other case, i.e., $q \gg Q^{1-\epsilon}$, the condition $|z-y| \ll \frac{q}{Q}Q^{\epsilon}$ is trivially true. Now we write z as $z = y+u$, with $|u| \ll \frac{q}{Q}Q^{\epsilon}$.

Thus the integral $\mathfrak{I}(\cdot)$ up to a negligible error term is given by

$$\int_{\mathbb{R}} W(x)g(q,x)\int_0^\infty \int_{|u|\ll qQ^\epsilon/Q} V(y+u)W(y)e\left(\frac{Nxu}{qQ}\right) \times e\left(\pm\frac{2\sqrt{Nmy}}{qP_2^{1/2}} \pm \frac{3(NN_0w(y+u))^{1/3}}{P_1^{1/3}qr^{1/3}}\right) \mathrm{d}u\,\mathrm{d}y\,\mathrm{d}x. \tag{19}$$

Now we consider the y-integral

$$\int_{\mathbb{R}} V(y+u)W(y)e\left(\pm\frac{2\sqrt{Nmy}}{qP_2^{1/2}} \pm \frac{3(NN_0w(y+u))^{1/3}}{P_1^{1/3}qr^{1/3}}\right) \mathrm{d}y.$$

Expanding $(y+u)^{1/3}$ into the Taylor series

$$(y+u)^{1/3} = y^{1/3} + \frac{u}{3y^{2/3}} - \frac{u^2}{9y^{5/3}} + \cdots,$$

we observe that it is enough to consider only the leading term as

$$\frac{3(NN_0)^{1/3}}{P_1^{1/3}qr^{1/3}}\frac{u}{3y^{2/3}} \ll \frac{Qu}{q} \ll Q^\varepsilon.$$

Thus we are required to analyze the integral

$$I = \int_{\mathbb{R}} W(y)e\left(\pm\frac{2\sqrt{Nmy}}{qP_2^{1/2}} \pm \frac{3(NN_0wy)^{1/3}}{P_1^{1/3}qr^{1/3}}\right) \mathrm{d}y. \tag{20}$$

By stationary phase analysis we see that the integral is negligibly small unless

$$\frac{2\sqrt{Nm}}{qP_2^{1/2}} \asymp \frac{3(NN_0)^{1/3}}{P_1^{1/3}qr^{1/3}} \approx \frac{Q}{q}.$$

Thus the above integral is negligibly small unless $m \sim M_0$ (with M_0 as in Section 3B), in which case the above y-integral is bounded by

$$I \ll \frac{\sqrt{q}}{\sqrt{Q}}.$$

Hence, executing the remaining integrals trivially, and using

$$\int_{\mathbb{R}} |g(q,x)|\,\mathrm{d}x \ll Q^\epsilon,$$

we see that $\mathfrak{I}$ is bounded by

$$\mathfrak{I}(\cdot) \ll q^{3/2}/Q^{3/2}.$$

On substituting this bound in (16), we get

$$\mathcal{I} \ll q^3/Q^3. \tag{21}$$

We record the above discussion in the following lemma.

Lemma 5.1. *Let* I, $\mathfrak{I}(\cdot)$ *and* $\mathcal{I}$ *be as in* (20), (18) *and* (16) *respectively. Then we have*

$$I \ll \frac{\sqrt{q}}{\sqrt{Q}}, \quad \mathfrak{I}(\cdot) \ll q^{3/2}/Q^{3/2}, \quad \text{and} \quad \mathcal{I} \ll q^3/Q^3.$$

6. Character sums

In this section, we will estimate the character sum $\mathfrak{C}$ given in (15),

$$\mathfrak{C} = \sum_{\substack{d \mid q \\ d' \mid q'}}\sum dd'\mu\left(\frac{q}{d}\right)\mu\left(\frac{q'}{d'}\right) \sideset{}{^\star}\sum_{\substack{\alpha \bmod qr/n_1 \\ \overline{P_1}n_1\alpha \equiv -m\overline{P_2} \bmod d}} \sideset{}{^\star}\sum_{\substack{\alpha' \bmod q'r/n_1 \\ \overline{P_1}n_1\alpha' \equiv -\overline{P_2}m' \bmod d' \\ \pm\bar{\alpha}q_2' \mp \bar{\alpha}'q_2 \equiv -n_2 \bmod q_1q_2q_2'r/n_1}} 1. \tag{22}$$

In the case, $n_2 = 0$, the congruence condition

$$\pm\bar{\alpha}q_2' \mp \bar{\alpha}'q_2 \equiv 0 \bmod q_1q_2q_2'r/n_1$$

implies that $q_2 = q_2'$ and $\alpha = \alpha'$. So we can bound the character sum $\mathfrak{C}$ as

$$\mathfrak{C} \ll \sum_{d,d' \mid q}\sum dd' \sideset{}{^\star}\sum_{\substack{\alpha \bmod qr/n_1 \\ \overline{P_1}n_1\alpha \equiv -m\overline{P_2} \bmod d \\ \overline{P_1}n_1\alpha \equiv -\overline{P_2}m' \bmod d'}} 1 \ll \sum_{\substack{d,d' \mid q \\ (d,d') \mid (m-m')}}\sum dd'\frac{qr}{[d,d']}. \tag{23}$$

For $n_2 \neq 0$, we have the following lemma.

Lemma 6.1. *Let* $\mathfrak{C}$ *be as in* (15)*. Then, for* $n_2 \neq 0$, *we have*

$$\mathfrak{C} \ll \frac{q_1^2 r(m,n_1)}{n_1} \sum_{\substack{d_2 \mid (q_2, n_1q_2' \mp mn_2P_1\overline{P_2}) \\ d_2' \mid (q_2', n_1q_2 \pm m'n_2P_1\overline{P_2})}}\sum d_2d_2'.$$

Proof. Let's recall from (15) that

$$\mathfrak{C} = \sum_{\substack{d \mid q \\ d' \mid q'}}\sum dd'\mu\left(\frac{q}{d}\right)\mu\left(\frac{q'}{d'}\right) \sideset{}{^\star}\sum_{\substack{\alpha \bmod qr/n_1 \\ \overline{P_1}n_1\alpha \equiv -m\overline{P_2} \bmod d}} \sideset{}{^\star}\sum_{\substack{\alpha' \bmod q'r/n_1 \\ \overline{P_1}n_1\alpha' \equiv -\overline{P_2}m' \bmod d' \\ \pm\bar{\alpha}q_2' \mp \bar{\alpha}'q_2 \equiv -n_2 \bmod q_1q_2q_2'r/n_1}} 1.$$

Using the Chinese remainder theorem, we observe that $\mathfrak{C}$ can be dominated by a product of two sums $\mathfrak{C} \ll \mathfrak{C}^{(1)}\mathfrak{C}^{(2)}$, where

$$\mathfrak{C}^{(1)} = \sum_{d_1,d_1' \mid q_1}\sum d_1d_1' \sideset{}{^\star}\sum_{\substack{\beta \bmod \frac{q_1r}{n_1} \\ n_1\beta \equiv -mP_1\overline{P_2} \bmod d_1}} \sideset{}{^\star}\sum_{\substack{\beta' \bmod \frac{q_1r}{n_1} \\ n_1\beta' \equiv -m'P_1\overline{P_2} \bmod d_1' \\ \pm\bar{\beta}q_2' \mp \bar{\beta}'q_2 + n_2 \equiv 0 \bmod q_1r/n_1}} 1$$

and

$$\mathfrak{C}^{(2)} = \sum_{\substack{d_2 \mid q_2 \\ d_2' \mid q_2'}}\sum d_2 d_2' \sideset{}{^\star}\sum_{\substack{\beta \bmod q_2 \\ n_1\beta \equiv -mP_1\overline{P_2} \bmod d_2}} \sideset{}{^\star}\sum_{\substack{\beta' \bmod q_2' \\ n_1\beta' \equiv -m'P_1\overline{P_2} \bmod d_2' \\ \pm\overline{\beta}q_2' \mp \overline{\beta'}q_2 + n_2 \equiv 0 \bmod q_2q_2'}} 1.$$

In the second sum $\mathfrak{C}^{(2)}$, since $(n_1, q_2q_2') = 1$, we get $\beta \equiv -m\overline{n_1}P_1\overline{P_2} \bmod d_2$ and $\beta' \equiv -m'\overline{n_1}P_1\overline{P_2} \bmod d_2'$. Now using the congruence modulo q_2q_2', we conclude that

$$\mathfrak{C}^{(2)} \ll \sum_{\substack{d_2 \mid (q_2, n_1q_2' \mp mn_2P_1\overline{P_2}) \\ d_2' \mid (q_2', n_1q_2 \pm m'n_2P_1\overline{P_2})}}\sum d_2d_2'.$$

In the first sum $\mathfrak{C}^{(1)}$, the congruence condition determines β' uniquely in terms of β, and hence

$$\mathfrak{C}^{(1)} \ll \sum_{d_1, d_1' \mid q_1}\sum d_1d_1' \sideset{}{^\star}\sum_{\substack{\beta \bmod q_1r/n_1 \\ n_1\beta \equiv -mP_1\overline{P_2} \bmod d_1}} 1 \ll \frac{rq_1^2(m, n_1)}{n_1}.$$

Hence we have the lemma. ☐

7. Zero frequency

In this section we will estimate the contribution of the zero frequency $n_2 = 0$ to Ω in (14), and thus estimate its total contribution to $S_r(N)$. We have the following lemma.

Lemma 7.1. *Let $S_r(N)$ be as in* (10). *The total contribution of the zero frequency $n_2 = 0$ to $S_r(N)$ is dominated by $O(r^{1/2}N^{3/4}\sqrt{P_1})$.*

Proof. On substituting bounds for $\mathcal{I}$ and $\mathfrak{C}$ from Lemma 5.1 and (23) respectively into (14), we see that the contribution of $n_2 = 0$ to Ω, is bounded by

$$\begin{aligned}
&\ll \frac{N_0C^3}{n_1^2M_0^{1/2}Q^3} \sum_{q_2 \sim C/q_1} qr \sum_{d, d' \mid q}\sum (d, d') \sum_{\substack{m, m' \sim M_0 \\ (d, d') \mid (m - m')}}\sum 1 \\
&\ll \frac{N_0C^3}{n_1^2M_0^{1/2}Q^3} \sum_{q_2 \sim C/q_1} qr \sum_{d, d' \mid q}\sum (M_0(d, d') + M_0^2) \\
&\ll \frac{N_0C^5rM_0^{1/2}}{n_1^2Q^3q_1}(C + M_0).
\end{aligned}$$

Upon substituting this bound for Ω in (10), we get

$$\sup_{C\ll Q} \frac{N^{17/12}}{P_2^{1/4}P_1^{1/6}Qr^{2/3}C^{5/2}} \sum_{\pm} \sum_{\frac{n_1}{(n_1,r)}\ll C} n_1^{1/3}\Theta^{1/2} \sum_{\frac{n_1}{(n_1,r)}|q_1|(n_1r)^\infty} \left(\frac{N_0C^5rM_0^{1/2}}{n_1^2Q^3q_1}(C+M_0)\right)^{1/2}$$
$$\ll \frac{N^{17/12}}{P_2^{1/4}P_1^{1/6}Qr^{2/3}} \frac{N_0^{1/2}r^{1/2}M_0^{1/4}}{Q^{3/2}} \sum_{\frac{n_1}{(n_1,r)}\ll C} \frac{\Theta^{1/2}}{n_1^{2/3}} \sum_{\frac{n_1}{(n_1,r)}|q_1|(n_1r)^\infty} \frac{1}{\sqrt{q_1}}(\sqrt{Q}+\sqrt{M_0})$$
$$\ll \frac{N^{17/12}}{P_2^{1/4}P_1^{1/6}Qr^{2/3}} \frac{N_0^{1/2}r^{1/2}M_0^{1/4}}{Q^{3/2}}\sqrt{Q} \sum_{\frac{n_1}{(n_1,r)}\ll C} \frac{\Theta^{1/2}}{n_1^{7/6}}\sqrt{(n_1,r)}.$$

Note that (as in [Munshi 2022]) we have

$$\sum_{n_1\ll Cr} \frac{(n_1,r)^{1/2}}{n_1^{7/6}}\Theta^{1/2} \ll \left[\sum_{n_1\ll Cr}\frac{(n_1,r)}{n_1}\right]^{1/2}\left[\sum\sum_{n_1^2n_2\le N_0}\frac{|\lambda_F(n_1,n_2)|^2}{(n_1^2n_2)^{2/3}}\right]^{1/2} \ll N_0^{1/6}. \tag{24}$$

Using this bound, we see that the contribution of $n_2=0$ to $S_r(N)$ is bounded by

$$S_r(N) \ll \frac{N^{17/12}}{P_2^{1/4}P_1^{1/6}Qr^{2/3}} \frac{N_0^{1/2}r^{1/2}M_0^{1/4}}{Q^{3/2}}\sqrt{Q}N_0^{1/6} \ll r^{1/2}N^{3/4}\sqrt{P_1}. \qquad \square$$

8. Nonzero frequencies

In this section we will estimate the contribution of the nonzero frequencies $n_2\neq 0$ to Ω in (14). We have the following lemma.

Lemma 8.1. *Let $S_r(N)$ be as in* (10). *The total contribution of $n_2\neq 0$, to $S_r(N)$ is dominated by $O(\sqrt{r}N^{3/4}\sqrt{P_2})$.*

Proof. On plugging in the bounds for the character sums and the integrals from Lemmas 6.1 and 5.1 respectively into (14), we see that the contribution of $n_2\neq 0$ to Ω (which we denote by $\Omega_{\neq 0}$) is bounded by

$$\frac{q_1^2N_0rC^3}{n_1^3M_0^{1/2}Q^3}\sum\sum_{q_2,q_2'\sim\frac{C}{q_1}}\sum\sum_{\substack{d_2\,|\,q_2\\ d_2'\,|\,q_2'}} d_2d_2' \sum\sum\sum_{\substack{m,m'\sim M_0 0\neq n_2\ll N_2\\ n_1q_2'\mp mn_2P_1\overline{P_2}\equiv 0 \bmod d_2\\ n_1q_2\pm m'n_2P_1\overline{P_2}\equiv 0 \bmod d_2'}} (m,n_1).$$

Further writing q_2d_2 in place of q_2 and $q_2'd_2'$ in place of q_2', we arrive at

$$\Omega_{\neq 0} \ll \frac{q_1^2N_0rC^3}{n_1^3M_0^{1/2}Q^3}\sum\sum_{d_2,d_2'\ll C/q_1} d_2d_2' \sum\sum_{\substack{q_2\sim\frac{C}{d_2q_1}\\ q_2'\sim\frac{C}{d_2'q_1}}} \sum\sum\sum_{\substack{m,m'\sim M_0 0\neq n_2\ll N_2\\ n_1q_2'd_2'\mp mn_2P_1\overline{P_2}\equiv 0 \bmod d_2\\ n_1q_2d_2\pm m'n_2P_1\overline{P_2}\equiv 0 \bmod d_2'}} (m,n_1). \tag{25}$$

Let's first assume that $Cn_1/q_1 \ll M_0$. In this case, we count the number of m in the above expression as follows:

$$\sum_{\substack{m \sim M_0 \\ n_1 q_2' d_2' \mp m n_2 P_1 \overline{P_2} \equiv 0 \bmod d_2}} (m, n_1) = \sum_{\ell \mid n_1} \ell \sum_{\substack{m \sim M_0/\ell \\ n_1 q_2' d_2' \bar{\ell} \mp m n_2 P_1 \overline{P_2} \equiv 0 \bmod d_2}} 1 \ll (d_2, n_2)\left(n_1 + \frac{M_0}{d_2}\right).$$

In the above estimate we have used the fact $(d_2, n_1) = 1$. Counting the number of m' in a similar fashion we get that the m-sum and m'-sum in (25) is dominated by

$$(d_2', n_1 q_2 d_2)(d_2, n_2)\left(n_1 + \frac{M_0}{d_2}\right)\left(1 + \frac{M_0}{d_2'}\right).$$

Now substituting the above bound in (25), we arrive at

$$\frac{q_1^2 N_0 r C^3}{n_1^3 M_0^{1/2} Q^3} \sum\sum_{d_2, d_2' \ll \frac{C}{q_1}} d_2 d_2' \sum\sum_{\substack{q_2 \sim \frac{C}{d_2 q_1} \\ q_2' \sim \frac{C}{d_2' q_1}}} \sum_{0 < |n_2| \ll N_2} (d_2', n_1 q_2 d_2)(d_2, n_2)\left(n_1 + \frac{M_0}{d_2}\right)\left(1 + \frac{M_0}{d_2'}\right).$$

Now summing over n_2 and q_2', we get the following expression:

$$\frac{q_1 N_0 r N_2 C^4}{n_1^3 M_0^{1/2} Q^3} \sum\sum_{d_2, d_2' \ll C/q_1} d_2 \sum_{q_2 \sim \frac{C}{d_2 q_1}} (d_2', n_1 q_2 d_2)\left(n_1 + \frac{M_0}{d_2}\right)\left(1 + \frac{M_0}{d_2'}\right).$$

Next we sum over d_2' to arrive at

$$\frac{q_1 N_0 r N_2 C^4}{n_1^3 M_0^{1/2} Q^3} \sum_{d_2 \ll C/q_1} d_2 \sum_{q_2 \sim \frac{C}{d_2 q_1}} \left(n_1 + \frac{M_0}{d_2}\right)\left(\frac{C}{q_1} + M_0\right).$$

Finally executing the remaining sums, we get

$$\begin{aligned}
\Omega_{\neq 0} &\ll \frac{q_1 N_0 r N_2 C^4}{n_1^3 M_0^{1/2} Q^3} \frac{C}{q_1}\left(\frac{Cn_1}{q_1} + M_0\right)\left(\frac{C}{q_1} + M_0\right) \\
&\ll \frac{rC^5}{n_1^3 M_0^{1/2} Q^3} \frac{CQn_1 r}{q_1}\left(\frac{Cn_1}{q_1} + M_0\right)\left(\frac{C}{q_1} + M_0\right) \\
&\ll \frac{r^2 C^6 Q M_0^2}{Q^3 M_0^{1/2}}\left(\frac{1}{n_1^2 q_1}\right) \ll \frac{r^2 C^5 Q^2 M_0^2}{Q^3 M_0^{1/2}}\left(\frac{1}{n_1^2 q_1}\right).
\end{aligned}$$

Upon substituting this bound in place of Ω in (10), we arrive at

$$\sup_C \frac{N^{17/12}}{P_2^{1/4} P_1^{1/6} Q r^{2/3} C^{5/2}} \frac{r C^{5/2} M_0^{3/4}}{\sqrt{Q}} \sum_{\pm} \sum_{\frac{n_1}{(n_1, r)} \ll C} n_1^{1/3} \Theta^{1/2} \sum_{\frac{n_1}{(n_1, r)} |q_1| (n_1 r)^\infty} \frac{1}{\sqrt{n_1^2 q_1}}.$$

Note that (for details see [Munshi 2022])

$$\sum_{\frac{n_1}{(n_1,r)}\ll C} n_1^{1/3}\Theta^{1/2} \sum_{\frac{n_1}{(n_1,r)}|q_1|(n_1 r)^\infty} \frac{1}{\sqrt{n_1^2 q_1}} \ll \sum_{n_1\ll Cr} \frac{(n_1,r)^{1/2}}{n_1^{7/6}}\Theta^{1/2} \ll N_0^{1/6}.$$

On plugging in this estimate, we get

$$\sup_C \frac{N^{17/12}}{P_2^{1/4}P_1^{1/6}Qr^{2/3}C^{5/2}}\frac{rC^{5/2}M_0^{3/4}}{\sqrt{Q}}N_0^{1/6} \ll \sqrt{r}N^{3/4}\sqrt{P_2}.$$

Next we consider the case where $Cn_1/q_1 \gg M_0$. Here our count for m modulo d_2 is not precise and so we need to adopt a different strategy for counting. We consider the first congruence relation in (25)

$$n_1 q_2' d_2' \mp m n_2 P_1 \overline{P_2} \equiv 0 \bmod d_2.$$

Note that

$$n_1 q_2' d_2' P_2 \mp m n_2 P_1 \ll CP_2 n_1/q_1 + M_0 N_2 P_1 \ll CP_2 n_1/q_1 + CP_2 n_1/q_1 \ll CP_2 n_1/q_1.$$

Let

$$n_1 q_2' d_2' P_2 - m n_2 P_1 = h d_2, \quad \text{with } h \ll P_2 n_1. \tag{26}$$

Similarly, we write the second congruence relation as

$$n_1 q_2 d_2 P_2 + m' n_2 P_1 = h' d_2', \quad \text{with } h' \ll P_2 n_1. \tag{27}$$

Using this congruence, we see that the number of d_2' is given by $O((d_2, h'))$. Next we multiply h' and $P_2 q_2' n_1$ into (26) and (27) respectively to arrive at the following equation:

$$m n_2 P_1 h' + h h' d_2 = n_1^2 q_2 q_2' d_2 P_2^2 + P_2 q_2' n_1 m' n_2 P_1. \tag{28}$$

We now rearrange the above equation as follows:

$$P_2 q_2' n_1 m' - m h' = \frac{(hh' - n_1^2 q_2 q_2' P_2^2) d_2}{P_1 n_2} := \frac{\xi}{P_1 n_2}.$$

Reducing this equation modulo h', the number of m' turns out to be

$$O\left((P_2 q_2' n_1, h')\left(1 + \frac{P_2}{h'}\right)\right).$$

Thus we arrive at the following bound for Ω:

$$\frac{q_1^2 N_0 r C^3}{n_1^3 M_0^{1/2} Q^3} \sum_{d_2\sim \frac{C}{q_1}} \frac{C^2}{q_1^2} \sum\sum_{\substack{q_2\sim C^\epsilon \\ q_2'\sim C^\epsilon}} \sum\sum\sum\sum_{\substack{h,h'\ll P_2 n_1 m\sim M_0, n_2\ll N_2 \\ \xi\equiv 0 \bmod P_1 n_2 \\ mh' - \xi/P_1 n_2 \equiv 0 \bmod P_2}} (m, n_1)(d_2, h')(P_2 q_2' n_1, h')\left(1 + \frac{P_2}{h'}\right).$$

Next we count the number of m to get

$$\sum_{\substack{m\sim M_0\\ mh'-\xi/P_1n_2\equiv 0 \bmod P_2}}(m,n_1)=\sum_{\ell\mid n_1}\ell\sum_{\substack{m\sim M_0/\ell\\ mh'-\xi/\ell P_1n_2\equiv 0 \bmod P_2}}1\ll\sum_{\ell\mid n_1}\ell.$$

Also given any ξ (necessarily nonzero) the congruence

$$\xi=(hh'-n_1^2q_2q_2'P_2^2)d_2\equiv 0 \bmod n_2,$$

implies that there are $O(N^{\varepsilon})$ many n_2. We are left with the following expression:

$$\Omega_{\neq 0}\ll\frac{N_0rC^5}{n_1^3M_0^{1/2}Q^3}\sum_{\ell\mid n_1}\ell\sum_{d_2\sim\frac{C}{q_1}}\sum_{\substack{q_2\sim C^{\epsilon}\\ q_2'\sim C^{\epsilon}}}\sum\sum_{\substack{h,h'\ll P_2n_1\\ \xi\equiv 0 \bmod P_1\ell}}(d_2,h')(P_2q_2'n_1,h')\left(1+\frac{P_2}{h'}\right).$$

We now consider the congruence

$$\xi=(hh'-n_1^2q_2q_2'P_2^2)d_2\equiv 0 \bmod P_1\ell.$$

Let's first assume that $d_2\equiv 0 \bmod P_1$. Then first counting the number of d_2 followed by h and h', we see that the number of tuples (h,h',d_2) is given by $O((P_2^2n_1^2C)/(P_1q_1\ell))$. Lastly executing the sum over ℓ, we arrive at

$$\frac{N_0rC^5}{n_1^2M_0^{1/2}Q^3}\frac{P_2^2n_1^2C}{P_1q_1}.$$

Now let $(d_2,\,P_1\ell)=1$. Then we have

$$hh'-n_1^2q_2q_2'P_2^2\equiv 0 \bmod P_1\ell,$$

from which the number of h turns out to be $P_2n_1/P_1\ell$. Next counting the number of d_2 followed by number of h', we see that the number of tuples (h,h',d_2) is given by $O((P_2^2n_1^2C)/(P_1q_1\ell))$. Hence, in this case also, we get the same bound. Thus we conclude that

$$\Omega_{\neq 0}\ll\frac{N_0rC^5}{n_1^2M_0^{1/2}Q^3}\frac{P_2^2n_1^2C}{P_1q_1}.$$

Upon substituting this bound in (10), we arrive at

$$\sup_{C\ll Q}\frac{N^{17/12}}{P_2^{1/4}P_1^{1/6}Qr^{2/3}C^{5/2}}\left(\frac{N_0rC^5}{Q^3M_0^{1/2}}\frac{P_2^2C}{P_1}\right)^{1/2}\sum_{\pm}\sum_{\frac{n_1}{(n_1,r)}\ll C}n_1^{1/3}\Theta^{1/2}\sum_{\frac{n_1}{(n_1,r)}|q_1|(n_1r)^{\infty}}\frac{1}{\sqrt{n_1q_1}}$$
$$\ll\frac{N^{17/12}}{P_2^{1/4}P_1^{1/6}Qr^{2/3}}\left(\frac{N_0r}{Q^3M_0^{1/2}}\frac{P_2^2Q}{P_1}\right)^{1/2}\sum_{\frac{n_1}{(n_1,r)}\ll C}\frac{\Theta^{1/2}}{n_1^{2/3}}(n_1,r)^{1/2}.$$

Now following the argument of [Munshi 2022], we conclude that

$$\sum_{\frac{n_1}{(n_1,r)}\ll C} \frac{\Theta^{1/2}}{n_1^{2/3}}(n_1,r)^{1/2} \ll N_0^{1/6} \sum_{\frac{n_1}{(n_1,r)}\ll C} \frac{(n_1,r)^{1/2}}{n_1} \ll N_0^{1/6}.$$

Hence the contribution of the nonzero frequency to $S_r(N)$ is dominated by

$$\frac{N^{17/12}}{P_2^{1/4}P_1^{1/6}Qr^{2/3}}\left(\frac{N_0 r}{Q^3 M_0^{1/2}}\frac{P_2^2 Q}{P_1}\right)^{1/2} N_0^{1/6} \ll \sqrt{r}N^{3/4}\sqrt{P_2}. \qquad \square$$

9. Conclusion

Finally, plugging bounds from Lemmas 7.1 and 8.1 into Lemma 2.1, we get

$$\begin{aligned} L\left(\tfrac{1}{2}, F\times f\right) &\ll_\epsilon \mathcal{Q}^\epsilon \sum_{r\le \mathcal{Q}^{(1+2\epsilon)/4}} \frac{1}{r} \sup_{N\le \mathcal{Q}^{1/2+\epsilon}/r^2} \sqrt{r}N^{1/4}(\sqrt{P_1}+\sqrt{P_2}) \\ &\ll \sum_{r\le \mathcal{Q}^{(1+2\epsilon)/4}} \frac{1}{r}\mathcal{Q}^{1/8+\epsilon}(\sqrt{P_1}+\sqrt{P_2}) \\ &\ll \mathcal{Q}^{1/8+\epsilon}(\sqrt{P_1}+\sqrt{P_2}) \ll \mathcal{Q}^{1/4+\epsilon}\left(\frac{P_1^{1/4}}{P_2^{3/8}}+\frac{P_2^{1/8}}{P_1^{1/4}}\right). \end{aligned}$$

This establishes Theorem 1.

Acknowledgements

This paper originated from discussions at the AIM online workshop "Delta symbols and subconvexity" held from November 2–6, 2020. The authors wish to thank the American Institute of Mathematics and the organizers of the workshop for their kind invitation. The authors also thank the participants of the workshop, especially Roman Holowinsky and Philippe Michel, for many enlightening conversations. For this work, S. K. Singh was partially supported by D.S.T. inspire faculty fellowship no. DST/INSPIRE/04/2018/000945 and R. Munshi was supported by J.C. Bose fellowship JCB/2021/000018 from SERB DST. Lastly the authors would like to thank the anonymous referee for a careful reading of the paper which helped in improving the exposition of the paper.

References

[Aggarwal 2020] K. Aggarwal, "Weyl bound for GL(2) in t-aspect via a simple delta method", *J. Number Theory* **208** (2020), 72–100. MR Zbl

[Aggarwal et al. 2018] K. Aggarwal, Y. Jo, and K. Nowland, "Hybrid level aspect subconvexity for GL(2) × GL(1) Rankin–Selberg L-functions", *Hardy-Ramanujan J.* **41** (2018), 104–117. MR

[Aggarwal et al. 2020a] K. Aggarwal, R. Holowinsky, Y. Lin, and Z. Qi, "A Bessel delta method and exponential sums for GL(2)", *Q. J. Math.* **71**:3 (2020), 1143–1168. MR Zbl

[Aggarwal et al. 2020b] K. Aggarwal, R. Holowinsky, Y. Lin, and Q. Sun, "The Burgess bound via a trivial delta method", *Ramanujan J.* **53**:1 (2020), 49–74. MR Zbl

[Blomer and Harcos 2008] V. Blomer and G. Harcos, "Hybrid bounds for twisted L-functions", *J. Reine Angew. Math.* **621** (2008), 53–79. MR Zbl

[Burgess 1963] D. A. Burgess, "On character sums and L-series, II", *Proc. London Math. Soc.* (3) **13** (1963), 524–536. MR Zbl

[Duke et al. 1993] W. Duke, J. Friedlander, and H. Iwaniec, "Bounds for automorphic L-functions", *Invent. Math.* **112**:1 (1993), 1–8. MR

[Duke et al. 1994] W. Duke, J. B. Friedlander, and H. Iwaniec, "Bounds for automorphic L-functions, II", *Invent. Math.* **115**:2 (1994), 219–239. Correction in **140**:1 (2000), 227–242. MR Zbl

[Duke et al. 2000] W. Duke, J. Friedlander, and H. Iwaniec, "Erratum: "Bounds for automorphic L-functions, II"", *Invent. Math.* **140**:1 (2000), 227–242. MR

[Goldfeld 2006] D. Goldfeld, *Automorphic forms and L-functions for the group* $\mathrm{GL}(n, \mathbb{R})$, Cambridge Studies in Advanced Mathematics **99**, Cambridge University Press, 2006. MR Zbl

[Holowinsky and Munshi 2013] R. Holowinsky and R. Munshi, "Level aspect subconvexity for Rankin–Selberg L-functions", pp. 311–334 in *Automorphic representations and L-functions* (Mumbai, 2012), edited by D. Prasad et al., Tata Inst. Fundam. Res. Stud. Math. **22**, Tata Inst. Fund. Res., Mumbai, 2013. MR Zbl

[Holowinsky and Nelson 2018] R. Holowinsky and P. D. Nelson, "Subconvex bounds on GL_3 via degeneration to frequency zero", *Math. Ann.* **372**:1-2 (2018), 299–319. MR Zbl

[Hou and Chen 2019] F. Hou and B. Chen, "Level aspect subconvexity for twisted L-functions", *J. Number Theory* **203** (2019), 12–31. MR Zbl

[Iwaniec and Kowalski 2004] H. Iwaniec and E. Kowalski, *Analytic number theory*, Amer. Math. Soc. Colloquium Publications **53**, Amer. Math. Soc., Providence, RI, 2004. MR Zbl

[Khan 2021] R. Khan, "Subconvexity bounds for twisted L-functions", *Q. J. Math.* **72**:3 (2021), 1133–1145. MR Zbl

[Kim and Shahidi 2002] H. H. Kim and F. Shahidi, "Functorial products for $\mathrm{GL}_2 \times \mathrm{GL}_3$ and the symmetric cube for GL_2", *Ann. of Math.* (2) **155**:3 (2002), 837–893. MR Zbl

[Kowalski et al. 2002] E. Kowalski, P. Michel, and J. VanderKam, "Rankin–Selberg L-functions in the level aspect", *Duke Math. J.* **114**:1 (2002), 123–191. MR Zbl

[Kowalski et al. 2020] E. Kowalski, Y. Lin, P. Michel, and W. Sawin, "Periodic twists of GL_3-automorphic forms", *Forum Math. Sigma* **8** (2020), art. id. e15. MR Zbl

[Kumar 2023] S. Kumar, "Subconvexity bound for $\mathrm{GL}(3) \times \mathrm{GL}(2)$ L-functions in $\mathrm{GL}(2)$ spectral aspect", *J. Eur. Math. Soc.* (2023), 1–46. Zbl

[Kumar et al. 2020] S. Kumar, K. Mallesham, and S. K. Singh, "Sub-convexity bound for $\mathrm{GL}(3) \times \mathrm{GL}(2)$ L-functions: $\mathrm{GL}(3)$-spectral aspect", preprint, 2020. arXiv 2006.07819

[Kumar et al. 2022] S. Kumar, K. Mallesham, and S. K. Singh, "Sub-convexity bound for $\mathrm{GL}(3) \times \mathrm{GL}(2)$ L-functions: the depth aspect", *Math. Z.* **301**:3 (2022), 2229–2268. MR Zbl

[Li 2009] X. Li, "The central value of the Rankin–Selberg L-functions", *Geom. Funct. Anal.* **18**:5 (2009), 1660–1695. MR Zbl

[Lin et al. 2023] Y. Lin, P. Michel, and W. Sawin, "Algebraic twists of $\mathrm{GL}_3 \times \mathrm{GL}_2$ L-functions", *Amer. J. Math.* **145**:2 (2023), 585–645. MR Zbl

[Michel 2007] P. Michel, "Analytic number theory and families of automorphic L-functions", pp. 181–295 in *Automorphic forms and applications* (Park City, UT, 2002), edited by P. Sarnak and F. Shahidi, IAS/Park City Math. Ser. **12**, Amer. Math. Soc., Providence, RI, 2007. MR Zbl

[Michel and Venkatesh 2010] P. Michel and A. Venkatesh, "The subconvexity problem for GL_2", *Publ. Math. Inst. Hautes Études Sci.* **111** (2010), 171–271. MR Zbl

[Munshi 2018] R. Munshi, "The subconvexity problem for L-functions", pp. 363–376 in *Proceedings of the International Congress of Mathematicians* (Rio de Janeiro, 2018), vol. 2: Invited lectures, edited by B. Sirakov et al., World Sci. Publ., Hackensack, NJ, 2018. MR Zbl

[Munshi 2022] R. Munshi, "Subconvexity for $\mathrm{GL}(3) \times \mathrm{GL}(2)$ L-functions in t-aspect", *J. Eur. Math. Soc.* (*JEMS*) **24**:5 (2022), 1543–1566. MR Zbl

[Munshi and Singh 2019] R. Munshi and S. K. Singh, "Weyl bound for p-power twist of GL(2) L-functions", *Algebra Number Theory* **13**:6 (2019), 1395–1413. MR Zbl

[Sharma and Sawin 2022] P. Sharma and W. Sawin, "Subconvexity for GL(3) × GL(2) twists", *Adv. Math.* **404**:part B (2022), art. id. 108420. MR Zbl

[Venkatesh 2010] A. Venkatesh, "Sparse equidistribution problems, period bounds and subconvexity", *Ann. of Math.* (2) **172**:2 (2010), 989–1094. MR Zbl

[Weyl 1921] H. Weyl, "Zur Abschätzung von $\zeta(1+it)$", *Math. Z.* **10** (1921), 88–101. Zbl

[Zhou 2018] F. Zhou, "The Voronoi formula on GL(3) with ramification", preprint, 2018. arXiv 1806.10786

Communicated by Philippe Michel
Received 2022-02-05 Revised 2023-01-09 Accepted 2023-05-29

sumit@renyi.hu *Alfred Renyi Institute of Mathematics, Budapest, Hungary*

ritabratamunshi@gmail.com *Stat-Math Unit, Indian Statistical Institute, Kolkata, India*

skumar.bhu12@gmail.com *Department of Mathematics and Statistics, Indian Institute of Technology, Kanpur, India*

ALGEBRA AND NUMBER THEORY 18:3 (2024)
https://doi.org/10.2140/ant.2024.18.499

A categorical Künneth formula for constructible Weil sheaves

Tamir Hemo, Timo Richarz and Jakob Scholbach

We prove a Künneth-type equivalence of derived categories of lisse and constructible Weil sheaves on schemes in characteristic $p > 0$ for various coefficients, including finite discrete rings, algebraic field extensions $E \supset \mathbb{Q}_\ell$, $\ell \neq p$, and their rings of integers $\mathcal{O}_E$. We also consider a variant for ind-constructible sheaves which applies to the cohomology of moduli stacks of shtukas over global function fields.

1. Introduction

The classical Künneth formula expresses the (co-)homology of a product of two spaces X_1 and X_2 in terms of the tensor product of the (co-)homology of the individual factors. For two topological spaces, for example, one has under suitable finiteness hypothesis an isomorphism

$$\bigoplus_{i+j=n} \mathrm{H}^i(X_1, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathrm{H}^j(X_2, \mathbb{Q}) \cong \mathrm{H}^n(X_1 \times X_2, \mathbb{Q}) \tag{1-1}$$

on singular cohomology with rational coefficients. Such cohomology groups are naturally morphism groups in the derived categories of sheaves on these spaces. So one may ask whether the Künneth formula can be extended to a categorical level, that is, whether it is possible to relate the derived categories of sheaves on X_1 and X_2 to those on their product $X_1 \times X_2$. Statements in this direction are referred to as *categorical Künneth formulas* and are known in different contexts: for example, for the respective derived

Richarz is funded by the European Research Council (ERC) under Horizon Europe (grant agreement number 101040935), by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) TRR 326 *Geometry and Arithmetic of Uniformized Structures*, project number 444845124 and the LOEWE professorship in Algebra. Scholbach was supported by Deutsche Forschungsgemeinschaft (DFG), EXC 2044–390685587, Mathematik Münster: Dynamik–Geometrie–Struktur.
MSC2020: 14D24, 14F20, 14F35.
Keywords: Weil sheaves, Künneth formula.

categories of topological sheaves, for D-modules on varieties in characteristic 0 and for quasicoherent sheaves; see [Gaitsgory et al. 2022, Section A.2].

In addition to (1-1) above, categorical Künneth formulas require decomposing a sheaf on $X_1 \times X_2$ into exterior products $M_1 \boxtimes M_2$, with M_1, M_2 being sheaves on X_1, X_2, respectively. For varieties in characteristic $p > 0$, an analogous decomposition for constructible (pro-)étale sheaves fails in general, and so does a categorical Künneth formula in this context; see Example 1.4 below. The main result of the manuscript at hand (see Theorem 1.3) shows how to rectify the failure by adding equivariance data under partial Frobenius morphisms, that is, one arrives at a categorical Künneth formula for constructible Weil sheaves. Our work relies on the analogous result [Drinfeld 1980, Theorem 2.1] for étale fundamental groups known as Drinfeld's lemma; see Section 5C for details and references.

1A. ***Definitions and results.*** Weil sheaves are defined in [Deligne 1980, Definition 1.1.10]. We start by explaining a site-theoretic approach which slightly differs from [Geisser 2004; Lichtenbaum 2005].

Let X be a scheme over a finite field $\mathbb{F}_q$, where q is a p-power. Fix an algebraic closure $\mathbb{F}/\mathbb{F}_q$, and denote by $X_{\mathbb{F}}$ the base change. The partial (q-)Frobenius $\phi_X := \mathrm{Frob}_X \times \mathrm{id}_{\mathbb{F}}$ defines an endomorphism of $X_{\mathbb{F}}$.

Definition 1.1. The *Weil-proétale site* $X^{\mathrm{Weil}}_{\mathrm{proét}}$ is the following site: Objects are pairs (U, φ) consisting of $U \in (X_{\mathbb{F}})_{\mathrm{proét}}$, the proétale site of $X_{\mathbb{F}}$ [Bhatt and Scholze 2015], equipped with an endomorphism $\varphi\colon U \to U$ of $\mathbb{F}$-schemes covering ϕ_X. Morphisms are given by equivariant maps. A family $\{(U_i, \varphi_i) \to (U, \varphi)\}$ of morphisms is a cover if the family $\{U_i \to U\}$ is a cover in $(X_{\mathbb{F}})_{\mathrm{proét}}$.

The Weil-proétale site sits in the sequence of sites

$$(X_{\mathbb{F}})_{\mathrm{proét}} \to X^{\mathrm{Weil}}_{\mathrm{proét}} \to X_{\mathrm{proét}} \tag{1-2}$$

given by the functors $U \leftarrow\!\shortmid (U, \varphi)$ and $(U_{\mathbb{F}}, \phi_U) \leftarrow\!\shortmid U$ in the opposite direction. The maps (1-2) commute over $*_{\mathrm{proét}}$, the proétale site of the point. Thus, for any condensed ring Λ viewed as a sheaf of rings on $*_{\mathrm{proét}}$, we get pullback functors on derived categories of proétale Λ-sheaves

$$\mathrm{D}(X, \Lambda) \to \mathrm{D}(X^{\mathrm{Weil}}, \Lambda) \to \mathrm{D}(X_{\mathbb{F}}, \Lambda).$$

In analogy with the definition of lisse and constructible sheaves (as recalled in Definition 3.1), we introduce the categories of lisse and constructible Weil sheaves $\mathrm{D}_{\mathrm{lis}}(X^{\mathrm{Weil}}, \Lambda) \subset \mathrm{D}_{\mathrm{cons}}(X^{\mathrm{Weil}}, \Lambda)$ as the full subcategories of $\mathrm{D}(X^{\mathrm{Weil}}, \Lambda)$ that are dualizable, resp. that are Zariski locally on X dualizable along a constructible stratification. These categories are equivalent to the corresponding categories of sheaves on the prestack $X_{\mathbb{F}}/\phi_X$, that is, equivalent to the homotopy fixed points of the induced ϕ_X^*-action.

Proposition 1.2 (Propositions 4.4 and 4.11). *The pullback of sheaves along $(X_{\mathbb{F}})_{\mathrm{proét}} \to X^{\mathrm{Weil}}_{\mathrm{proét}}$ induces an equivalence of Λ_*-linear symmetric monoidal stable ∞-categories*

$$\mathrm{D}_\bullet(X^{\mathrm{Weil}}, \Lambda) \xrightarrow{\cong} \mathrm{D}_\bullet(X_{\mathbb{F}}, \Lambda)^{\phi_X^* = \mathrm{id}},$$

for $\bullet \in \{\varnothing, \mathrm{lis}, \mathrm{cons}\}$.

Thus, objects in $D_\bullet(X^{\mathrm{Weil}}, \Lambda)$ are pairs (M, α) with $M \in D_\bullet(X_{\mathbb{F}}, \Lambda)$ and $\alpha \colon M \cong \phi_X^* M$. On the abelian level, we recover the classical approach [Deligne 1980, Definition 1.1.10]. If Λ is a finite discrete ring, then every Weil descent datum on constructible Λ-sheaves is effective so that $D_{\mathrm{cons}}(X^{\mathrm{Weil}}, \Lambda) \cong D_{\mathrm{cons}}(X, \Lambda)$; see Proposition 4.16. However, the categories are not equivalent if $\Lambda = \mathbb{Z}, \mathbb{Z}_\ell, \mathbb{Q}_\ell$, say. This relates to the difference between continuous representations of Galois groups such as $\hat{\mathbb{Z}}$ versus Weil groups such as $\mathbb{Z}$.

For several $\mathbb{F}_q$-schemes $X_1, \dots, X_n$, a similar process is carried out for their product $X := X_1 \times_{\mathbb{F}_q} \cdots \times_{\mathbb{F}_q} X_n$ equipped with the partial Frobenii $\phi_{X_i} \colon X_{\mathbb{F}} \to X_{\mathbb{F}}$, see Section 4B. Generalizing Proposition 1.2, there is an equivalence of Λ_*-linear symmetric monoidal stable ∞-categories

$$D_\bullet(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda) \xrightarrow{\cong} D_\bullet(X_{\mathbb{F}}, \Lambda)^{\phi_{X_1}^* = \mathrm{id}, \dots, \phi_{X_n}^* = \mathrm{id}} \tag{1-3}$$

for $\bullet \in \{\varnothing, \mathrm{lis}, \mathrm{cons}\}$. The category on the left is defined using the Weil-proétale site $(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}})_{\mathrm{pro\acute{e}t}}$ consisting of objects $(U, \varphi_1, \dots, \varphi_n)$ with $U \in (X_{\mathbb{F}})_{\mathrm{pro\acute{e}t}}$ and pairwise commuting endomorphisms $\varphi_i \colon U \to U$ covering the partial Frobenii $\phi_{X_i} \colon X_{\mathbb{F}} \to X_{\mathbb{F}}$ for all $i = 1, \dots, n$. The category on the right is the category of simultaneous homotopy fixed points; see Section 2B. For constructible Weil sheaves, (1-3) relies on decompositions of partial Frobenius invariant cycles in $X_{\mathbb{F}}$; see Proposition 4.8.

The following result is referred to as the categorical Künneth formula for constructible Weil sheaves (or, derived Drinfeld's lemma).

Theorem 1.3 (Theorem 5.2, Remark 5.3). *Let $\mathbb{F}_q$ be a finite field of characteristic $p > 0$. Let $X_1, \dots, X_n$ be finite type $\mathbb{F}_q$-schemes. Let Λ be either a finite discrete ring of prime-to-p torsion, or an algebraic field extension $E \supset \mathbb{Q}_\ell$, $\ell \neq p$, or its ring of integers $\mathcal{O}_E$.*

Then the external tensor product of sheaves $(M_1, \dots, M_n) \mapsto M_1 \boxtimes \cdots \boxtimes M_n$ induces an equivalence

$$D_{\mathrm{cons}}(X_1^{\mathrm{Weil}}, \Lambda) \otimes_{\mathrm{Perf}_{\Lambda_*}} \cdots \otimes_{\mathrm{Perf}_{\Lambda_*}} D_{\mathrm{cons}}(X_n^{\mathrm{Weil}}, \Lambda) \xrightarrow{\cong} D_{\mathrm{cons}}(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda), \tag{1-4}$$

and likewise for the categories of lisse Weil sheaves if, in the case $\Lambda = E$, one assumes the schemes $X_1, \dots, X_n$ to be geometrically unibranch (for example, normal).

This statement can also be recast as the symmetric monoidality of the functor sending a Weil prestack X^{Weil}, which is defined on R-points by $X^{\mathrm{Weil}}(R) := \mathrm{colim}(X(R) \underset{\phi_X}{\overset{\mathrm{id}}{\rightrightarrows}} X(R))$, to its ∞-category of constructible sheaves (Theorem 5.6).

The tensor product of ∞-categories (see Section 2) is formed using the natural Λ_*-linear structures on the categories. We have an analogous equivalence for the categories of lisse Weil sheaves with coefficients Λ in finite discrete p-torsion rings like $\mathbb{Z}/p^m$, $m \geq 1$, see Theorem 5.2. As the following example shows, the use of Weil sheaves is necessary for the essential surjectivity to hold. This behavior is mentioned in the first arXiv version of [Gaitsgory et al. 2022, (0.8)] which is one of the main motivations for our work.

Example 1.4 (compare [SGA 1 2003, Exposé X, Section 1, Remarques 1.10]). Let $X_{1,\mathbb{F}} = X_{2,\mathbb{F}} = \mathbb{A}^1_{\mathbb{F}}$ be the affine line so that $X_{\mathbb{F}} = \mathbb{A}^2_{\mathbb{F}}$ with coordinates denoted by x_1 and x_2. Then

$$U := \{t^p - t = x_1 \cdot x_2\} \longrightarrow \mathbb{A}^2_{\mathbb{F}}$$

defines a finite étale cover with Galois group $\mathbb{Z}/p$. Let $M \in \mathrm{D}_{\mathrm{lis}}(\mathbb{A}^2_{\mathbb{F}}, \Lambda)$ be the sheaf in degree 0 associated with some nontrivial character $\mathbb{Z}/p \to \Lambda_*^\times$. For $\lambda, \mu \in \mathbb{F}$ not differing by a scalar in $\mathbb{F}_p^\times$, the fibers $U|_{\{x_1=\lambda\}}$, $U|_{\{x_1=\mu\}}$ are not isomorphic over $\mathbb{A}^1_{\mathbb{F}}$ by Artin–Schreier theory. Hence, $M \not\cong \phi^*_{X_i} M$ and one can show that $M \not\cong M_1 \boxtimes M_2$ for any $M_i \in \mathrm{D}(\mathbb{A}^1_{\mathbb{F}}, \Lambda)$.

If Λ as above is p-torsion free, then the full faithfulness of (1-4) is a direct consequence of the Künneth formula for $X_{i,\mathbb{F}}$, $i = 1, \dots, n$. For $\Lambda = \mathbb{Z}/p^m$, we use Artin–Schreier theory instead. It would be interesting to see whether the lisse p-torsion case can be extended to constructible sheaves. In both cases, the essential surjectivity relies on a variant of Drinfeld's lemma for Weil group representations, see Theorem 5.9, together with a characterization of partial-Frobenius stable algebraic cycles (Proposition 4.8) as well as a decomposition argument for representations of a product of abstract groups (Proposition 5.12).

With a view towards [Lafforgue 2018], we consider Weil sheaves whose underlying sheaf is ind-constructible, but where the action of the partial Frobenii do not necessarily preserve the constructible pieces. For finite type $\mathbb{F}_q$-schemes $X_1, \dots, X_n$ and Λ as in Theorem 1.3, we consider the category of simultaneous homotopy fixed points

$$\mathrm{D}_\bullet(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda) \overset{\mathrm{def}}{=} \mathrm{D}_\bullet(X_{\mathbb{F}}, \Lambda)^{\phi^*_{X_1}=\mathrm{id}, \dots, \phi^*_{X_n}=\mathrm{id}}$$

for $\bullet \in \{\mathrm{indlis}, \mathrm{indcons}\}$. Then the external tensor product induces a fully faithful functor

$$\mathrm{D}_\bullet(X_1^{\mathrm{Weil}}, \Lambda) \otimes_{\mathrm{Mod}_{\Lambda_*}} \cdots \otimes_{\mathrm{Mod}_{\Lambda_*}} \mathrm{D}_\bullet(X_n^{\mathrm{Weil}}, \Lambda) \longrightarrow \mathrm{D}_\bullet(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda). \tag{1-5}$$

Unlike the case of lisse or constructible sheaves, the functor is not essentially surjective as one can add freely actions by the partial Frobenii, see Remark 6.6. However, we can identify a large class of objects in the essential image of (1-5). When combined with the smoothness results of Xue [2020c, Theorem 4.2.3], we obtain, for example, that the compactly supported cohomology of moduli stacks of shtukas over global function fields lies in the essential image of (1-5); see Section 6B for details.

Remark 1.5. Another motivation for this work is our (Richarz and Scholbach's) ongoing project aiming for a motivic refinement of [Lafforgue 2018]. In this project, we will need a motivic variant of Drinfeld's lemma. Since triangulated categories of motives such as $\mathrm{DM}(X, \mathbb{Q})$ carry t-structures only conditionally, we need a Drinfeld lemma to be a statement about triangulated categories. In conjunction with the conjecture relating Weil-étale motivic cohomology to Weil-étale cohomology [Kahn 2003; Geisser 2004; Lichtenbaum 2005], our results suggest to look for a Drinfeld lemma for constructible Weil motives.

2. Recollections on ∞-categories

Throughout this section, Λ denotes a unital, commutative ring. We briefly collect some notation pertaining to ∞-categories from [Lurie 2017; 2009]. As in [Lurie 2009, Section 5.5.3], Pr^{L} denotes the ∞-category of presentable ∞-categories with colimit-preserving functors. It contains the subcategory $\mathrm{Pr}^{\mathrm{St}} \subset \mathrm{Pr}^{\mathrm{L}}$ consisting of stable ∞-categories.

2A. ***Monoidal aspects.*** The category Pr^{L} carries the Lurie tensor product [Lurie 2017, Section 4.8.1]. This tensor product induces one on the full subcategory $\mathrm{Pr}^{\mathrm{St}} \subset \mathrm{Pr}^{\mathrm{L}}$ consisting of stable ∞-categories [loc. cit., Proposition 4.8.2.18]. For our commutative ring Λ, the ∞-category Mod_Λ of chain complexes of Λ-modules, up to quasiisomorphism, is a commutative monoid in $\mathrm{Pr}^{\mathrm{St}}$ with respect to this tensor product. This structure includes, in particular, the existence of a functor

$$\mathrm{Mod}_\Lambda \times \mathrm{Mod}_\Lambda \to \mathrm{Mod}_\Lambda$$

which, after passing to the homotopy categories is the classical *derived* tensor product on the unbounded derived category of Λ-modules.

We define $\mathrm{Pr}^{\mathrm{St}}_\Lambda$ to be the category of modules, in $\mathrm{Pr}^{\mathrm{St}}$, over Mod_Λ. Noting that modules over Mod_Λ are in particular modules over Sp, the ∞-category of spectra, $\mathrm{Pr}^{\mathrm{St}}_\Lambda$ can be described as the ∞-category consisting of *stable* presentable ∞-categories together with a Λ-linear structure, such that functors are continuous and Λ-linear. Therefore $\mathrm{Pr}^{\mathrm{St}}_\Lambda$ carries a symmetric monoidal structure, whose unit is Mod_Λ. We will also denote by $\mathrm{Pr}^{\mathrm{St}}_\omega$ the category of compactly generated presentable with functors that send compact objects to compact objects (equivalently, those whose right adjoint is continuous).

In order to express monoidal properties of ∞-categories consisting, say, of bounded complexes, recall from [Lurie 2017, Corollary 4.8.1.4 joint with Lemma 5.3.2.11] or [Ben-Zvi et al. 2010, Proposition 4.4] the symmetric monoidal structure on the ∞-category $\mathrm{Cat}^{\mathrm{Ex}}_\infty(\mathrm{Idem})$ of idempotent complete stable ∞-categories and exact functors: it is characterized by

$$D_1 \otimes D_2 \overset{\mathrm{def}}{=} (\mathrm{Ind}(D_1) \otimes \mathrm{Ind}(D_2))^\omega, \tag{2-1}$$

that is, the compact objects in the Lurie tensor product of the Ind-completions. With respect to these monoidal structures, the Ind-completion functor (taking values in compactly generated presentable ∞-categories with the Lurie tensor product) and the functor forgetting the compact generatedness

$$\mathrm{Cat}^{\mathrm{Ex}}_\infty(\mathrm{Idem}) \xrightarrow[\mathrm{Ind}]{\cong} \mathrm{Pr}^{\mathrm{St}}_\omega \longrightarrow \mathrm{Pr}^{\mathrm{St}} \tag{2-2}$$

are both symmetric monoidal [Lurie 2017, Lemmas 5.3.2.9, 5.3.2.11].

The subcategory of compact objects in Mod_Λ is given by perfect complexes of Λ-modules [loc. cit., Proposition 7.2.4.2.]. It is denoted Perf_Λ. Under the equivalence in (2-2), the category $\mathrm{Perf}_\Lambda \in \mathrm{Cat}^{\mathrm{Ex}}_\infty(\mathrm{Idem})$ corresponds to Mod_Λ. Moreover, Perf_Λ is a commutative monoid in $\mathrm{Cat}^{\mathrm{Ex}}_\infty(\mathrm{Idem})$, so that we can consider its category of modules, denoted as $\mathrm{Cat}^{\mathrm{Ex}}_{\infty,\Lambda}(\mathrm{Idem})$. This category inherits a symmetric monoidal structure denoted by $D_1 \otimes_{\mathrm{Perf}_\Lambda} D_2$.

Any stable ∞-category D is canonically enriched over the category of spectra Sp. We write $\mathrm{Hom}_D(\cdot\,,\cdot)$ for the mapping spectrum. Any category in $\mathrm{Pr}^{\mathrm{St}}_\Lambda$ is canonically enriched over Mod_Λ, so that we refer to $\mathrm{Hom}_D(\cdot\,,\cdot) \in \mathrm{Mod}_\Lambda$ as the mapping complex. For example, for $M, N \in \mathrm{Mod}_\Lambda$, then $\mathrm{Hom}_{\mathrm{Mod}_\Lambda}(M, N)$ is commonly also denoted by $\mathrm{RHom}(M, N)$. Its n-th cohomology is the Hom-group $\mathrm{Hom}(M, N[n])$ in the classical derived category.

2B. ***Fixed points of ∞-categories.*** A basic structure in Drinfeld's lemma is the equivariance datum for the partial Frobenii. In this section, we assemble some abstract results where such ∞-categorical constructions are carried out.

Definition 2.1. Let $\phi\colon D \to D$ be an endofunctor in $\mathrm{Cat}_\infty^{\mathrm{Ex}}(\mathrm{Idem})$. The category of *$\phi$-fixed points* is

$$D^{\phi=\mathrm{id}} \overset{\mathrm{def}}{=} \mathrm{Fix}(D,\phi) \overset{\mathrm{def}}{=} \lim(D \underset{\phi}{\overset{\mathrm{id}_D}{\rightrightarrows}} D).$$

Recall that for a symmetric monoidal ∞-category D, a commutative monoid object $\Lambda \in \mathrm{CAlg}(D)$, the forgetful functors $\mathrm{CAlg}(D) \to D$ and $\mathrm{Mod}_\Lambda(D) \to D$ preserve limits [Lurie 2017, Corollaries 3.2.2.5 and 4.2.3.3]. In particular, if D is in addition Λ-linear, that is, an object in $\mathrm{Cat}_{\infty,\Lambda}^{\mathrm{Ex}}(\mathrm{Idem})$, and ϕ is also Λ-linear, then $\mathrm{Fix}(D,\phi)$ admits a natural Λ-linear structure as well.

Because of these facts, we will usually not specify where the limit above is formed. Note that all functors

$$\mathrm{Cat}_\infty^{\mathrm{Ex}}(\mathrm{Idem}) \xrightarrow[\mathrm{Ind}]{\cong} \mathrm{Pr}_\omega^{\mathrm{St}} \xrightarrow{(*)} \mathrm{Pr}^{\mathrm{St}} \longrightarrow \mathrm{Pr}^{\mathrm{L}} \longrightarrow \widehat{\mathrm{Cat}_\infty} \tag{2-3}$$

except for the forgetful functor marked $(*)$ preserve limits; see [Lurie 2017, Corollary 4.2.3.3; 2009, Proposition 5.5.3.13] for the rightmost two functors. To give a concrete example of that failure in our situation, note that $\mathrm{Fix}(D,\mathrm{id}_D) = \mathrm{Fun}(B\mathbb{Z}, D)$, that is, objects are pairs (M,α) consisting of some $M \in D$ and some automorphism $\alpha\colon M \cong M$. Now consider $D = \mathrm{Vect}_\Lambda^{\mathrm{fd}}$, the (abelian) category of finite-dimensional vector spaces over a field Λ. The natural functor

$$\mathrm{Ind}\big(\lim(\mathrm{Vect}_\Lambda^{\mathrm{fd}} \rightrightarrows \mathrm{Vect}_\Lambda^{\mathrm{fd}})\big) \to \lim\big(\mathrm{Ind}(\mathrm{Vect}_\Lambda^{\mathrm{fd}}) \rightrightarrows \mathrm{Ind}(\mathrm{Vect}_\Lambda^{\mathrm{fd}})\big) = \lim\big(\mathrm{Vect}_\Lambda \rightrightarrows \mathrm{Vect}_\Lambda\big)$$

is fully faithful, but *not* essentially surjective: given an automorphism α of an infinite-dimensional vector space M, there need not be a filtration $M = \bigcup M_i$ by finite-dimensional subspaces M_i that is compatible with α.

Fixed point categories inherit t-structures as follows:

Lemma 2.2. *Let $\phi\colon D \to D$ be a functor in* $\mathrm{Cat}_\infty^{\mathrm{Ex}}(\mathrm{Idem})$. *Suppose D carries a t-structure such that ϕ is t-exact. Then* $\mathrm{Fix}(D,\phi)$ *carries a unique t-structure such that the evaluation functor is t-exact. There is a natural equivalence*

$$\mathrm{Fix}(D^\heartsuit,\phi) \xrightarrow{\cong} \mathrm{Fix}(D,\phi)^\heartsuit.$$

Proof. Let us abbreviate $\widetilde{D} := \mathrm{Fix}(D,\phi)$. For $\bullet$ being either "≤ 0" or "≥ 0", we put $\widetilde{D}^\bullet := \mathrm{Fix}(D^\bullet,\phi)$, which is a (nonstable) ∞-category. This is clearly the only choice for a t-structure making ev a t-exact functor. It satisfies the claim about the hearts of the t-structure by definition.

We need to show that it is a t-structure. Being a limit of full subcategories, the categories $\widetilde{D}^\bullet$ are full subcategories of $\widetilde{D}$. Since ϕ, being t-exact, commutes with $\tau_D^{\le 0}$ and $\tau_D^{\ge 0}$, these two functors also yield truncation functors for $\widetilde{D}$. For $M \in \widetilde{D}^{\le 0}$, $N \in \widetilde{D}^{\ge 1}$ (we use cohomological conventions), we have

$$\mathrm{Hom}_{\widetilde{D}}(M,N) = \lim(\mathrm{Hom}_D(M,N) \rightrightarrows \mathrm{Hom}_D(M,N)),$$

where on the right hand side M, N denote the underlying objects in D. Since $M \in D^{\leq 0}$, $N \in D^{\geq 1}$, we have $\mathrm{H}^i \operatorname{Hom}_D(M, N) = 0$ for $i = -1, 0$. Thus, $\mathrm{H}^0 \operatorname{Hom}_{\widetilde{D}}(M, N) = 0$ as well. □

Definition 2.1 can be generalized as follows: Let $\varphi\colon B\mathbb{Z}^n \to \mathrm{Cat}_\infty^{\mathrm{Ex}}(\mathrm{Idem})$ be a diagram. For example, for $n = 1$, this amounts to giving $D = \varphi(*) \in \mathrm{Cat}_\infty^{\mathrm{Ex}}(\mathrm{Idem})$ and an equivalence $\phi = \varphi(1)\colon D \to D$. For $n = 2$, such a datum corresponds to giving D, equivalences $\phi_1, \phi_2\colon D \xrightarrow{\cong} D$ together with an equivalence $\phi_1 \circ \phi_2 \xrightarrow{\cong} \phi_2 \circ \phi_1$. So we define the ∞-category of *simultaneous fixed points* as

$$\mathrm{Fix}(D, \phi_1, \ldots, \phi_n) \overset{\mathrm{def}}{=} \lim \varphi \in \mathrm{Cat}_\infty^{\mathrm{Ex}}(\mathrm{Idem}).$$

Remark 2.3. The statement of Lemma 2.2 carries over verbatim assuming that D has a t-structure and all ϕ_i are t-exact, noting that $B\mathbb{Z}^n = (S^1)^n$ is a finite simplicial set.

Lemma 2.4. *Let $\varphi\colon B\mathbb{Z}^n \to \mathrm{Cat}_\infty^{\mathrm{Ex}}(\mathrm{Idem})$ be a diagram. Denote $D = \varphi(*)$ and $\phi_i = \varphi(e_i)$ for the i-th standard vector $e_i \in \mathbb{Z}^n$. The functor*

$$\mathrm{Fix}(D, \phi_1, \ldots, \phi_n) \to \mathrm{Fix}(\mathrm{Ind}(D), \phi_1, \ldots, \phi_n)$$

induced from the inclusion $D \subset \mathrm{Ind}(D)$ is fully faithful and takes values in compact objects. In particular, it yields a fully faithful functor

$$\mathrm{Ind}(\mathrm{Fix}(D, \phi_1, \ldots, \phi_n)) \to \mathrm{Fix}(\mathrm{Ind}(D), \phi_1, \ldots, \phi_n).$$

Proof. Let $M \in \mathrm{Fix}(D, \phi_1, \ldots, \phi_n)$ and denote its underlying object in D by the same symbol. For every $N \in \mathrm{Fix}(\mathrm{Ind}(D), \phi_1, \ldots,)$, we have a limit diagram of mapping complexes

$$\operatorname{Hom}_{\mathrm{Fix}(\mathrm{Ind}(D))}(M, N) \cong \mathrm{Fix}(\operatorname{Hom}_{\mathrm{Ind}(D)}(M, N), \phi_1, \ldots, \phi_n).$$

Since filtered colimits commute with finite limits in the ∞-category of anima (a.k.a. spaces) [Lurie 2009, Proposition 5.3.3.3.], we see that M is compact in $\mathrm{Fix}(\mathrm{Ind}(D))$ because M is so in $\mathrm{Ind}(D)$. □

Lemma 2.5. *Let $\varphi_i\colon B\mathbb{Z} \to \mathrm{Cat}_{\infty,\Lambda}^{\mathrm{Ex}}(\mathrm{Idem})$, $i = 1, \ldots, n$ be given. Denote $D_i = \varphi_i(*)$, $\phi_i = \varphi_i(1)$ and $\widetilde{D}_i = \mathrm{Ind}(D_i)$. Then there is a canonical equivalence*

$$\mathrm{Fix}(\widetilde{D}_1, \phi_1) \otimes_{\mathrm{Mod}_\Lambda} \cdots \otimes_{\mathrm{Mod}_\Lambda} \mathrm{Fix}(\widetilde{D}_n, \phi_n) \xrightarrow{\cong} \mathrm{Fix}(\widetilde{D}_1 \otimes_{\mathrm{Mod}_\Lambda} \cdots \otimes_{\mathrm{Mod}_\Lambda} \widetilde{D}_n, \phi_1, \ldots, \phi_n).$$

Proof. The categories $\mathrm{Fix}(\widetilde{D}_i, \phi_i)$ are compactly generated: the forgetful functor $U : \mathrm{Fix}(\widetilde{D}_i, \phi_i) \to \widetilde{D}_i = \mathrm{Ind}(D_i)$ preserves colimits, so its left adjoint L preserves compact objects. Moreover, U is conservative, so that the objects $L(d_i)$, for $d_i \in D_i$, form a family of compact generators. Then, we use that any compactly generated category in $\mathrm{Pr}_\Lambda^{\mathrm{St}}$ is dualizable [Lurie 2018, Remark D.7.7.6(1)] so that tensoring with it preserves limits. □

3. Lisse and constructible sheaves

In order to state and prove the categorical Künneth formula for Weil sheaves, we use the framework for lisse and constructible sheaves provided by [Hemo et al. 2023]. For the convenience of the reader, we collect here some basics of the formalism.

Throughout, Λ denotes a condensed ring, for example any T1-topological ring such as discrete rings, algebraic extensions $E/\mathbb{Q}_\ell$ or their ring of integers $\mathcal{O}_E$. In the synopsis below, we refer to the latter choices of Λ as the *standard coefficient rings*. We write Λ_* for the underlying ring. Let $\mathrm{D}(X, \Lambda)$ be the derived category of sheaves of Λ-modules on the proétale site $X_{\text{proét}}$.

Definition 3.1 [Hemo et al. 2023, Definitions 3.3 and 8.1]. For every scheme X and every condensed ring Λ, there are the full subcategories

$$\mathrm{D}_{\mathrm{lis}}(X, \Lambda) \subset \mathrm{D}_{\mathrm{cons}}(X, \Lambda) \subset \mathrm{D}(X, \Lambda). \tag{3-1}$$

By definition, the left hand category of *lisse sheaves* consists of the dualizable objects in the right-most category. An object (henceforth referred to as a sheaf) M in the right hand category is *constructible*, if on any affine $U \subset X$ there is a finite stratification into constructible locally closed subschemes $U_i \subset U$ such that $M|_{U_i}$ is lisse, that is, dualizable. Finally, an *ind-lisse* (respectively, *ind-constructible*) sheaf is a filtered colimit, in the category $\mathrm{D}(X, \Lambda)$, of lisse (respectively, constructible) sheaves. The corresponding full subcategories of $\mathrm{D}(X, \Lambda)$ are denoted by

$$\mathrm{D}_{\mathrm{indlis}}(X, \Lambda) \subset \mathrm{D}_{\mathrm{indcons}}(X, \Lambda) \subset \mathrm{D}(X, \Lambda).$$

For the standard coefficient rings Λ above and quasicompact quasiseparated (qcqs) schemes X, that definition of lisse and constructible sheaves agrees with the classical ones; see [Hemo et al. 2023] for details.

The categories enjoy the following properties:

Synopsis 3.2. (i) Via the natural functor $\mathrm{Mod}_{\Lambda_*} \to \mathrm{D}(X, \Lambda)$, $M \mapsto \underline{M} \otimes_{\underline{\Lambda_*}} \Lambda_X$; see around [Hemo et al. 2023, (3.1)], the category $\mathrm{D}(X, \Lambda)$ is an object in $\mathrm{Pr}^{\mathrm{St}}_{\Lambda_*}$. The functor restricts to a functor $\mathrm{Perf}_{\Lambda_*} \to \mathrm{D}_{\mathrm{lis}}(X, \Lambda)$, and the categories $\mathrm{D}_{\mathrm{lis}}(X, \Lambda) \subset \mathrm{D}_{\mathrm{cons}}(X, \Lambda)$ are objects in $\mathrm{Cat}^{\mathrm{Ex}}_{\infty,\Lambda}(\mathrm{Idem})$. In particular, all categories listed in (3-1) are stable idempotent complete Λ_*-linear ∞-categories.

(ii) The extension-by-zero functor along any constructible locally closed immersion and quasicompact étale morphisms preserves constructibility; see [loc. cit., Lemma 3.4, Corollary 4.6].

(iii) The functors $X \mapsto \mathrm{D}_{\mathrm{cons}}(X, \Lambda)$ and $X \mapsto \mathrm{D}_{\mathrm{lis}}(X, \Lambda)$ satisfy proétale hyperdescent [loc. cit., Corollary 4.7]. (According to [Hansen and Scholze 2023, Theorem 2.2], it also satisfies v-descent, but we will not need this in this paper.) The functor $X \mapsto \mathrm{D}_{\mathrm{indcons}}(X, \Lambda)$, resp. $X \mapsto \mathrm{D}_{\mathrm{indlis}}(X, \Lambda)$ satisfies hyperdescent for quasicompact étale, resp. finite étale covers; see [Hemo et al. 2023, Corollary 8.7].

(iv) If $\Lambda = \operatorname{colim} \Lambda_i$ is a filtered colimit of condensed rings and X is qcqs, then the natural functors

$$\operatorname{colim} \mathrm{D}_{\mathrm{lis}}(X, \Lambda_i) \xrightarrow{\cong} \mathrm{D}_{\mathrm{lis}}(X, \Lambda), \quad \operatorname{colim} \mathrm{D}_{\mathrm{cons}}(X, \Lambda_i) \xrightarrow{\cong} \mathrm{D}_{\mathrm{cons}}(X, \Lambda)$$

are equivalences [loc. cit., Proposition 5.2].

(v) If X is qcqs, then any constructible sheaf is bounded with respect to the t-structure on $\mathrm{D}(X, \Lambda)$ [loc. cit., Corollary 4.11].

(vi) For X locally Noetherian (and much more generally), the t-structure on $\mathrm{D}(X, \Lambda)$ restricts to one on $\mathrm{D}_{\mathrm{lis}}(X, \Lambda)$ and $\mathrm{D}_{\mathrm{cons}}(X, \Lambda)$ provided that Λ is t-admissible in the sense of [loc. cit., Definition 6.1]. Here, t-admissibility is a combination of an algebraic and a topological condition: first, Λ_* needs to be regular coherent (for example, any regular Noetherian ring of finite Krull dimension, but $\mathbb{Z}/\ell^2$ is excluded). The topological condition on the condensed structure of Λ is satisfied for all the standard coefficient rings listed above; see [loc. cit., Theorem 6.2].

(vii) For X locally Noetherian (and again more generally), a sheaf is lisse if and only if it is proétale locally the constant sheaf associated to a perfect complex of Λ_*-modules; see [loc. cit., Theorem 4.13].

(viii) Let X be a qcqs scheme. If the Λ-cohomological dimension is uniformly bounded for all proétale affines $U = \lim_i U_i$ over X, then $\mathrm{Ind}(\mathrm{D}_{\mathrm{cons}}(X, \Lambda)) = \mathrm{D}_{\mathrm{indcons}}(X, \Lambda)$ and likewise for ind-lisse sheaves. If X is of finite type over $\mathbb{F}_q$ or a separably closed field, this condition holds for any of the above standard rings. For discrete p-torsion rings, algebraic extensions $E/\mathbb{Q}_p$ and their ring of integers $\mathcal{O}_E$, this holds for arbitrary qcqs schemes in characteristic p; see [loc. cit., Lemma 8.6, Proposition 8.2].

For schemes $X_1, \dots, X_n$ over a fixed base scheme S (for example, the spectrum of a field) and a condensed ring Λ, we denote the external product in the usual way

$$\boxtimes\colon \mathrm{D}(X_1, \Lambda) \times \cdots \times \mathrm{D}(X_n, \Lambda) \to \mathrm{D}(X_1 \times_S \cdots \times_S X_n, \Lambda),$$
$$(M_1, \dots, M_n) \mapsto M_1 \boxtimes \cdots \boxtimes M_n := p_1^*(M_1) \otimes_{\Lambda_X} \cdots \otimes_{\Lambda_X} p_n^*(M_n).$$

Here $p_i\colon X := X_1 \times_S \cdots \times_S X_n \to X_i$ are the projections. This functor induces the functor

$$\boxtimes\colon \mathrm{D}(X_1, \Lambda) \otimes_{\mathrm{Mod}_{\Lambda_*}} \cdots \otimes_{\mathrm{Mod}_{\Lambda_*}} \mathrm{D}(X_n, \Lambda) \to \mathrm{D}(X_1 \times_S \cdots \times_S X_n, \Lambda), \tag{3-2}$$

in $\mathrm{Pr}^{\mathrm{St}}_{\Lambda_*}$. Here we regard $\mathrm{D}(X_i, \Lambda)$ as objects in $\mathrm{Pr}^{\mathrm{St}}_{\Lambda_*}$, like in (i) in the synopsis above. The external tensor product of constructible sheaves is again constructible, and hence induces a functor

$$\boxtimes\colon \mathrm{D}_{\mathrm{cons}}(X_1, \Lambda) \otimes_{\mathrm{Perf}_{\Lambda_*}} \cdots \otimes_{\mathrm{Perf}_{\Lambda_*}} \mathrm{D}_{\mathrm{cons}}(X_n, \Lambda) \to \mathrm{D}_{\mathrm{cons}}(X_1 \times_S \cdots \times_S X_n, \Lambda), \tag{3-3}$$

in $\mathrm{Cat}^{\mathrm{Ex}}_{\infty, \Lambda_*}(\mathrm{Idem})$ and likewise for the categories of ind-constructible, resp. (ind-)lisse sheaves.

4. Weil sheaves

In this section, we introduce the categories

$$\mathrm{D}_{\mathrm{lis}}(X^{\mathrm{Weil}}, \Lambda) \subset \mathrm{D}_{\mathrm{cons}}(X^{\mathrm{Weil}}, \Lambda) \subset \mathrm{D}(X^{\mathrm{Weil}}, \Lambda)$$

consisting of lisse, resp. constructible, resp. all Weil sheaves. These are the categories featuring in the categorical Künneth formula (Theorem 1.3).

Throughout this section, X is a scheme over a finite field $\mathbb{F}_q$ of characteristic $p > 0$. Unless the contrary is mentioned, we impose no conditions on X. Moreover, Λ is a condensed ring. We fix an algebraic closure $\mathbb{F}$ of $\mathbb{F}_q$, and denote by $X_{\mathbb{F}} := X \times_{\mathbb{F}_q} \operatorname{Spec} \mathbb{F}$ the base change. Denote by ϕ_X (resp. $\phi_{\mathbb{F}}$) the endomorphism of $X_{\mathbb{F}}$ that is the q-Frobenius on X (resp. $\operatorname{Spec} \mathbb{F}$) and the identity on the other factor.

Let

$$\mathrm{D}_{\mathrm{lis}}(X_{\mathbb{F}}, \Lambda) \subset \mathrm{D}_{\mathrm{cons}}(X_{\mathbb{F}}, \Lambda) \subset \mathrm{D}(X_{\mathbb{F}}, \Lambda)$$

be the categories of lisse, resp. constructible, resp. all proétale sheaves of Λ-modules on $X_{\mathbb{F}}$ (Definition 3.1). These categories are objects in $\mathrm{Cat}^{\mathrm{Ex}}_{\infty,\Lambda_*}(\mathrm{Idem})$, that is, Λ_*-linear stable idempotent complete symmetric monoidal ∞-categories where $\Lambda_* = \Gamma(*, \Lambda)$ is the underlying ring.

4A. *The Weil-proétale site.* The Weil-étale topology for schemes over finite field is introduced in [Lichtenbaum 2005]; see also [Geisser 2004]. Our approach for the proétale topology is slightly different:

Definition 4.1. The *Weil-proétale site* of X, denoted by $X^{\mathrm{Weil}}_{\mathrm{proét}}$, is the following site: Objects in $X^{\mathrm{Weil}}_{\mathrm{proét}}$ are pairs (U, φ) consisting of $U \in (X_{\mathbb{F}})_{\mathrm{proét}}$ equipped with an endomorphism $\varphi\colon U \to U$ of $\mathbb{F}$-schemes such that the map $U \to X_{\mathbb{F}}$ intertwines φ and ϕ_X. Morphisms in $X^{\mathrm{Weil}}_{\mathrm{proét}}$ are given by equivariant maps, and a family $\{(U_i, \varphi_i) \to (U, \varphi)\}$ of morphisms is a cover if the family $\{U_i \to U\}$ is a cover in $(X_{\mathbb{F}})_{\mathrm{proét}}$.

Note that $X^{\mathrm{Weil}}_{\mathrm{proét}}$ admits small limits formed componentwise as $\lim(U_i, \varphi_i) = (\lim U_i, \lim \varphi_i)$. In particular, there are limit-preserving maps of sites

$$(X_{\mathbb{F}})_{\mathrm{proét}} \to X^{\mathrm{Weil}}_{\mathrm{proét}} \to X_{\mathrm{proét}} \tag{4-1}$$

given by the functors (in the opposite direction) $U \leftarrow\!\shortmid (U, \varphi)$ and $(U_{\mathbb{F}}, \phi_U) \leftarrow\!\shortmid U$. We denote by $\mathrm{D}(X^{\mathrm{Weil}}, \Lambda)$ the unbounded derived category of sheaves of Λ_X-modules on $X^{\mathrm{Weil}}_{\mathrm{proét}}$. The maps of sites (4-1) induce functors

$$\mathrm{D}(X, \Lambda) \to \mathrm{D}(X^{\mathrm{Weil}}, \Lambda) \to \mathrm{D}(X_{\mathbb{F}}, \Lambda), \tag{4-2}$$

whose composition is the usual pullback functor along $X_{\mathbb{F}} \to X$.

Remark 4.2. The functor $\mathrm{D}(X, \Lambda) \to \mathrm{D}(X^{\mathrm{Weil}}, \Lambda)$ is not an equivalence in general. This relates to the difference between continuous representations Galois versus Weil groups. See, however, Proposition 4.16 for filtered colimits of finite discrete rings Λ.

We have the following basic functoriality: Let $j : U \to X$ be a weakly étale morphism and consider the corresponding object $(U_{\mathbb{F}}, \phi_U)$ of $X^{\mathrm{Weil}}_{\mathrm{proét}}$. Then the slice site $(X^{\mathrm{Weil}}_{\mathrm{proét}})_{/(U_{\mathbb{F}}, \phi_U)}$ is equivalent to $U^{\mathrm{Weil}}_{\mathrm{proét}}$. This gives a functor $(X_{\mathrm{proét}})^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{St}}_{\Lambda}$, $U \mapsto \mathrm{D}(U^{\mathrm{Weil}}, \Lambda)$ which is a hypercomplete sheaf of Λ_*-linear presentable stable categories.

Also, we obtain an adjunction

$$j_! \colon \mathrm{D}(U^{\mathrm{Weil}}, \Lambda) \rightleftarrows \mathrm{D}(X^{\mathrm{Weil}}, \Lambda) \colon j^*$$

that is compatible with the $((j_{\mathbb{F}})_!, (j_{\mathbb{F}})^*)$-adunction under (4-2). The category $\mathrm{D}(X^{\mathrm{Weil}}, \Lambda)$ is equivalent to the category of ϕ_X-equivariant sheaves on $X_{\mathbb{F}}$, as we will now explain.

For each $i \geq 0$, consider the object $(X_i, \Phi_i) \in X^{\mathrm{Weil}}_{\mathrm{proét}}$ with $X_i = \mathbb{Z}^{i+1} \times X_{\mathbb{F}}$ the countably disjoint union of $X_{\mathbb{F}}$, the map $X_i \to X_{\mathbb{F}}$ given by projection and the endomorphism $\Phi_i \colon X_i \to X_i$ given by $(\underline{n}, x) \mapsto (\underline{n} - (1, \dots, 1), \phi_X(x))$ on sections. The inclusion $\mathbb{Z}^i \to \mathbb{Z}^{i+1}$, $\underline{n} \mapsto (0, \underline{n})$ induces a map of schemes $X_{i-1} \to X_i$ where $X_{-1} := X_{\mathbb{F}}$. By pullback, we get a limit-preserving map of sites

$$(X_{i-1})_{\mathrm{proét}} \to (X^{\mathrm{Weil}}_{\mathrm{proét}})_{/(X_i, \Phi_i)}. \tag{4-3}$$

Lemma 4.3. *For each $i \geq 0$, the map* (4-3) *induces an equivalence on the associated* 1*-topoi.*

Proof. As universal homeomorphisms induce equivalences on proétale 1-topoi [Bhatt and Scholze 2015, Lemma 5.4.2], we may assume that X is perfect. In this case, the sites (4-3) are equivalent because ϕ_X is an isomorphism. Explicitly, an inverse is given by sending an object $U \in (X_{i-1})_{\mathrm{proét}}$ to the object $V = \bigsqcup_{\underline{n} \in \mathbb{Z}^{i+1}} V_{\underline{n}}$, $V_{\underline{n}} \to \{\underline{n}\} \times X_{\mathbb{F}}$ defined by

$$V_{\underline{n}} = U_{(n_2 - n_1, \dots, n_{i+1} - n_1)} \times_{X_{\mathbb{F}}, \phi_X^{n_1}} X_{\mathbb{F}},$$

and with endomorphism $\varphi \colon V \to V$ defined by the maps $V_{\underline{n}} = V_{\underline{n} - (1, \dots, 1)} \times_{X_{\mathbb{F}}, \phi_X} X_{\mathbb{F}} \to V_{\underline{n} - (1, \dots, 1)}$. □

Weil sheaves admit the following presentation as the ϕ_X^*-fixed points of $\mathrm{D}(X_{\mathbb{F}}, \Lambda)$, see Definition 2.1.

Proposition 4.4. *The last functor in* (4-2) *induces an equivalence*

$$\mathrm{D}(X^{\mathrm{Weil}}, \Lambda) \cong \lim(\mathrm{D}(X_{\mathbb{F}}, \Lambda) \underset{\phi_X^*}{\overset{\mathrm{id}}{\rightrightarrows}} \mathrm{D}(X_{\mathbb{F}}, \Lambda)). \tag{4-4}$$

Remark 4.5. Objects in (4-4) are pairs (M, α) where $M \in \mathrm{D}(X_{\mathbb{F}}, \Lambda)$ and α is an isomorphism $M \cong \phi_X^* M$. Note that the composition $\phi_X \circ \phi_{\mathbb{F}}$ is the absolute q-Frobenius of $X_{\mathbb{F}}$. In particular, it induces the identity on proétale topoi; see [Bhatt and Scholze 2015, Lemma 5.4.2]. Therefore, replacing ϕ_X^* by $\phi_{\mathbb{F}}^*$ in (4-4) yields an equivalent category.

Proof of Proposition 4.4. The structural morphism $(X_0, \Phi_0) \to (X_{\mathbb{F}}, \phi_X)$ is a cover in $X^{\mathrm{Weil}}_{\mathrm{proét}}$. Its Čech nerve has objects $(X_i, \Phi_i) \in X^{\mathrm{Weil}}_{\mathrm{proét}}$, $i \geq 0$ as above. By descent, there is an equivalence

$$\mathrm{D}(X^{\mathrm{Weil}}, \Lambda) \xrightarrow{\cong} \mathrm{Tot}(\mathrm{D}((X^{\mathrm{Weil}}_{\mathrm{proét}})_{/(X_\bullet, \Phi_\bullet)}, \Lambda)). \tag{4-5}$$

Under Lemma 4.3, the cosimplicial 1-topos associated with $(X^{\mathrm{Weil}}_{\mathrm{proét}})_{/(X_\bullet, \Phi_\bullet)}$ is equivalent to the cosimplicial 1-topos associated with the action of ϕ_X^* on $(X_{\mathbb{F}})_{\mathrm{proét}}$. The equivalence (4-5) then becomes

$$\mathrm{D}(X^{\mathrm{Weil}}, \Lambda) \xrightarrow{\cong} \lim_{B\mathbb{Z}} \mathrm{D}(X_{\mathbb{F}}, \Lambda),$$

for the diagram $B\mathbb{Z} \to \mathrm{Pr}^{\mathrm{St}}_{\Lambda}$ corresponding to the endomorphism ϕ_X^* of $\mathrm{D}(X_{\mathbb{F}}, \Lambda)$. That is, $\mathrm{D}(X^{\mathrm{Weil}}, \Lambda)$ is equivalent to the homotopy fixed points of $\mathrm{D}(X_{\mathbb{F}}, \Lambda)$ with respect to the action of ϕ_X^*, which is our claim. □

4B. ***Weil sheaves on products.*** The discussion of the previous section generalizes to products of schemes as follows. Let $X_1, \dots, X_n$ be schemes over $\mathbb{F}_q$, and denote by $X := X_1 \times_{\mathbb{F}_q} \cdots \times_{\mathbb{F}_q} X_n$ their product. For every $1 \le i \le n$, we have a morphism $\phi_{X_i} \colon X_{i,\mathbb{F}} \to X_{i,\mathbb{F}}$ as in the previous section. We use the notation ϕ_{X_i} to also denote the corresponding map on $X_{\mathbb{F}} = X_{1,\mathbb{F}} \times_{\mathbb{F}} \cdots \times_{\mathbb{F}} X_{n,\mathbb{F}}$ which is ϕ_{X_i} on the i-th factor and the identity on the other factors.

We define the site $(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}})_{\text{proét}}$ whose underlying category consists of tuples $(U, \varphi_1, \dots, \varphi_n)$ with $U \in (X_{\mathbb{F}})_{\text{proét}}$ and pairwise commuting endomorphisms $\varphi_i \colon U \to U$ such that the following diagram commutes

$$\begin{array}{ccc} U & \xrightarrow{\varphi_i} & U \\ \downarrow & & \downarrow \\ X_{\mathbb{F}} & \xrightarrow{\phi_{X_i}} & X_{\mathbb{F}}, \end{array}$$

for all $1 \le i \le n$. As before, we denote by $\mathrm{D}(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda)$ the corresponding derived category of Λ-sheaves.

Using a similar reasoning as in the previous section, we can identify this category of sheaves with the homotopy fixed points

$$\mathrm{D}(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda) \xrightarrow{\cong} \mathrm{Fix}(\mathrm{D}(X_{\mathbb{F}}, \Lambda), \phi_{X_1}^*, \dots, \phi_{X_n}^*) \tag{4-6}$$

of the commuting family of the functors $\phi_{X_i}^*$, see Remark 2.3. Explicitly, for $n = 2$, this is the homotopy limit of the diagram:

$$\begin{array}{ccc} \mathrm{D}(X_{\mathbb{F}}, \Lambda) & \overset{\phi_{X_1}^*}{\underset{\mathrm{id}}{\rightrightarrows}} & \mathrm{D}(X_{\mathbb{F}}, \Lambda) \\ {\scriptstyle \mathrm{id}}\downarrow\;\downarrow{\scriptstyle \phi_{X_2}^*} & & {\scriptstyle \mathrm{id}}\downarrow\;\downarrow{\scriptstyle \phi_{X_2}^*} \\ \mathrm{D}(X_{\mathbb{F}}, \Lambda) & \overset{\phi_{X_1}^*}{\underset{\mathrm{id}}{\rightrightarrows}} & \mathrm{D}(X_{\mathbb{F}}, \Lambda) \end{array}$$

Roughly speaking, objects in the category $\mathrm{D}(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda)$ are given by tuples $(M, \alpha_1, \dots, \alpha_n)$ with $M \in \mathrm{D}(X_{\mathbb{F}}, \Lambda)$ and with pairwise commuting equivalences $\alpha_i \colon M \cong \phi_{X_i}^* M$. That is, equipped with a collection of equivalences $\phi_{X_j}^*(\alpha_i) \circ \alpha_j \simeq \phi_{X_i}^*(\alpha_j) \circ \alpha_i$ for all i, j satisfying higher coherence conditions.

4C. ***Partial-Frobenius stability.*** For schemes $X_1, \dots, X_n$ over $\mathbb{F}_q$, we denote by $X := X_1 \times_{\mathbb{F}_q} \cdots \times_{\mathbb{F}_q} X_n$ their product together with the partial Frobenii $\mathrm{Frob}_{X_i} \colon X \to X$, $1 \le i \le n$. To give a reasonable definition of lisse and constructible Weil sheaves, we need to understand the relation between partial-Frobenius invariant constructible subsets in X and constructible subsets in the single factors X_i.

Definition 4.6. A subset $Z \subset X$ is called *partial-Frobenius invariant* if $\mathrm{Frob}_{X_i}(Z) = Z$ for all $1 \le i \le n$.

The composition $\mathrm{Frob}_{X_1} \circ \cdots \circ \mathrm{Frob}_{X_n}$ is the absolute q-Frobenius on X and thus induces the identity on the topological space underlying X. Therefore, in order to check that $Z \subset X$ is partial-Frobenius invariant, it suffices that, for any fixed i, the subset Z is Frob_{X_j}-invariant for all $j \neq i$. This remark, which also applies to $X_{\mathbb{F}} = X_1 \times_{\mathbb{F}_q} \cdots \times_{\mathbb{F}_q} X_n \times_{\mathbb{F}_q} \operatorname{Spec} \mathbb{F}$, will be used below without further comment.

We first investigate the case of two factors with one being a separably closed field. This eventually rests on Drinfeld's descent result [1987, Proposition 1.1] for coherent sheaves.

Lemma 4.7. *Let X be a qcqs $\mathbb{F}_q$-scheme, and let $k/\mathbb{F}_q$ be a separably closed field. Denote by $p\colon X_k \to X$ the projection. Then $Z \mapsto p^{-1}(Z)$ induces a bijection*

$$\{\textit{constructible subsets in } X\} \leftrightarrow \{\textit{partial-Frobenius invariant, constructible subsets in } X_k\}.$$

Proof. The injectivity is clear because p is surjective. It remains to check the surjectivity. Without loss of generality we may assume that k is algebraically closed, and replace Frob_X by Frob_k which is an automorphism. Given that $Z \mapsto p^{-1}(Z)$ is compatible with passing to complements, unions and localizations on X, we are reduced to proving the bijection for constructible closed subsets Z and for X affine over $\mathbb{F}_q$. By Noetherian approximation (Lemma 4.9), we reduce further to the case where X is of finite type over $\mathbb{F}_q$ and still affine. Now we choose a locally closed embedding $X \to \mathbb{P}^n_{\mathbb{F}_q}$ into projective space. A closed subset $Z' \subset X_k$ is ϕ_k-invariant if and only if its closure inside $\mathbb{P}^n_k$ is so. Hence, it is enough to consider the case where $X = \mathbb{P}^n_{\mathbb{F}_q}$ is the projective space. Let Z' be a closed Frob_k-invariant subset of X_k. When viewed as a reduced subscheme, the isomorphism ϕ_k restricts to an isomorphism of Z'. In particular, $\mathcal{O}_{Z'}$ is a coherent $\mathcal{O}_{X_k}$-module equipped with an isomorphism $\mathcal{O}_{Z'} \cong \phi_k^* \mathcal{O}_{Z'}$. Hence, Drinfeld's descent result [1987, Proposition 1.1] (see also [Kedlaya 2019, Section 4.2] for a recent exposition) yields $Z' = Z_k$ for a unique closed subscheme $Z \subset X$. □

The following proposition generalizes the results [Lau 2004, Lemma 9.2.1] and [Lafforgue 2018, Lemme 8.12] in the case of curves.

Proposition 4.8. *Let $X_1, \dots, X_n$ be qcqs $\mathbb{F}_q$-schemes, and denote $X = X_1 \times_{\mathbb{F}_q} \cdots \times_{\mathbb{F}_q} X_n$. Then any partial-Frobenius invariant constructible closed subset $Z \subset X$ is a finite set-theoretic union of subsets of the form $Z_1 \times_{\mathbb{F}_q} \cdots \times_{\mathbb{F}_q} Z_n$, for appropriate constructible closed subschemes $Z_i \subset X_i$.*

In particular, any partial-Frobenius invariant constructible open subscheme $U \subset X$ is a finite union of constructible open subschemes of the form $U_1 \times_{\mathbb{F}_q} \cdots \times_{\mathbb{F}_q} U_n$, for appropriate constructible open subschemes $U_i \subset X_i$.

Proof. By induction, we may assume $n = 2$. By Noetherian approximation (Lemma 4.9), we reduce to the case where both X_1, X_2 are of finite type over $\mathbb{F}_q$. In the following, all products are formed over $\mathbb{F}_q$, and locally closed subschemes are equipped with their reduced subscheme structure. Let $Z \subset X_1 \times X_2$ be a partial-Frobenius invariant closed subscheme. The complement $U = X_1 \times X_2 \setminus Z$ is also partial-Frobenius invariant.

In the proof, we can replace X_1 (and likewise X_2) by a stratification in the following sense: Suppose $X_1 = A' \sqcup A''$ is a set-theoretic stratification into a closed subset A' with open complement A''. Once we know $Z \cap A' \times X_2 = \bigcup_j Z'_{1j} \times Z'_{2j}$ and $Z \cap A'' \times X_2 = \bigcup_j Z''_{1j} \times Z''_{2j}$ for appropriate closed subschemes

$Z'_{1j} \subset A'$, $Z''_{1j} \subset A''$ and $Z'_{2j}, Z''_{2j} \subset X_2$, we have the set-theoretic equality

$$Z = \bigcup_j Z'_{1i} \times Z'_{2j} \cup \bigcup_j \overline{Z''_{1j}} \times Z''_{2j},$$

where $\overline{Z''_{1j}} \subset X_1$ denotes the scheme-theoretic closure. Here we note that taking scheme-theoretic closures commutes with products because the projections $X_1 \times X_2 \to X_i$ are flat, and that the topological space underlying the scheme-theoretic closure agrees with the topological closure because all schemes involved are of finite type.

The proof is now by Noetherian induction on X_2, the case $X_2 = \varnothing$ being clear (or, if the reader prefers the case where X_2 is zero dimensional reduces to Lemma 4.7). In the induction step, we may assume, using the above stratification argument, that both X_i are irreducible with generic point η_i. We let $\overline{\eta}_i$ be a geometric generic point over η_i, and denote by $p_i \colon X_1 \times X_2 \to X_i$ the two projections. Both p_i are faithfully flat of finite type and in particular open, so that $p_i(U)$ is open in X_i. We have a set-theoretic equality

$$Z = ((X_1 \setminus p_1(U)) \times X_2) \cup (X_1 \times (X_2 \setminus p_2(U))) \cup (Z \cap p_1(U) \times p_2(U)).$$

Once we know $Z \cap p_1(U) \times p_2(U) = \bigcup_j Z_{1j} \times Z_{2j}$ for appropriate closed $Z_{ij} \subset p_i(U)$, we are done. We can therefore replace X_i by $p_i(U)$ and assume that both $p_i \colon U \to X_i$ are surjective.

The base change $U \times_{X_2} \overline{\eta}_2$ is a $\phi_{\overline{\eta}_2}$-invariant subset of $X_1 \times \overline{\eta}_2$. By Lemma 4.7, it is thus of the form $U_1 \times \overline{\eta}_2$ for some open subset $U_1 \subset X_1$. There is an inclusion (of open subschemes of $X_1 \times \eta_2$): $U \times_{X_2} \eta_2 \subset U_1 \times \eta_2$. It becomes a set-theoretic equality, and therefore an isomorphism of schemes, after base change along $\overline{\eta}_2 \to \eta_2$. By faithfully flat descent, this implies that the two mentioned subsets of $X_1 \times \eta_2$ agree. We claim $U_1 = X_1$. Since the projection $U \to X_2$ is surjective, in particular its image contains η_2, so that U_1 is a nonempty subset, and therefore open dense in the irreducible scheme X_1. Let $x_1 \in X_1$ be a point. Since the projection $U \to X_1$ is surjective, $U \cap (\{x_1\} \times X_2)$ is a nonempty open subscheme of $\{x_1\} \times X_2$. So it contains a point lying over (x_1, η_2). We conclude $X_1 \times \eta_2 \subset U$.

We claim that there is a nonempty open subset $A_2 \subset X_2$ such that

$$X_1 \times A_2 \subset U \quad \text{or, equivalently,} \quad X_1 \times (X_2 \setminus A_2) \supset X_1 \times X_2 \setminus U.$$

The underlying topological space of $V = X_1 \times X_2 \setminus U$ is Noetherian and thus has finitely many irreducible components V_j. The closure of the projection $\overline{p_2(V_j)} \subset X_2$ does not contain η_2, since $X_1 \times \eta_2 \subset U$. Thus, $A_2 := \bigcap_j X_1 \setminus \overline{p_2(V_j)}$ satisfies our requirements.

Now we continue by Noetherian induction applied to the stratification $X_2 = A_2 \sqcup (X_2 \setminus A_2)$: We have $Z \cap X_1 \times A_2 = \varnothing$, so that we may replace X_2 by the proper closed subscheme $X_2 \setminus A_2$. Hence, the proposition follows by Noetherian induction. □

The following lemma on Noetherian approximation of partial Frobenius invariant subsets is needed for the reduction to finite type schemes:

Lemma 4.9. *Let $X_1, \dots, X_n$ be qcqs $\mathbb{F}_q$-schemes, and denote $X = X_1 \times_{\mathbb{F}_q} \cdots \times_{\mathbb{F}_q} X_n$. Let $X_i = \lim_j X_{ij}$ be a cofiltered limit of finite type $\mathbb{F}_q$-schemes with affine transition maps, and write $X = \lim_j X_j$, $X_j := X_{1j} \times_{\mathbb{F}_q} \cdots \times_{\mathbb{F}_q} X_{nj}$; see* [Stacks 2017, Tag 01ZA] *for the existence of such presentations. Let $Z \subset X$ be a constructible closed subset. Then the intersection*

$$Z' = \bigcap_{i=1}^{n} \bigcap_{m \in \mathbb{Z}} \mathrm{Frob}_{X_i}^m(Z)$$

is partial Frobenius invariant, constructible closed and there exists an index j and a partial Frobenius invariant closed subset $Z'_j \subset X_j$ such that $Z' = Z'_j \times_{X_j} X$ as sets.

We note that each Frob_{X_i} induces a homeomorphism on the underlying topological space of X so that Z' is well-defined. This lemma applies, in particular, to partial Frobenius invariant constructible closed subsets $Z \subset X$ in which case we have $Z = Z'$.

Proof. As Z is constructible, there exists an index j and a constructible closed subscheme $Z_j \subset X_j$ such that $Z = Z_j \times_{X_j} X$ as sets. We put $Z'_j = \bigcap_{i=1}^n \bigcap_{m \in \mathbb{Z}} \mathrm{Frob}_{X_{ij}}^m(Z_j)$. As X_j is of finite type over $\mathbb{F}_q$, the subset Z'_j is still constructible closed. As partial Frobenii induce bijections on the underlying topological spaces, one checks that $\mathrm{Frob}_{X_{ij}}^m(Z_j) \times_{X_j} X = \mathrm{Frob}_{X_i}^m(Z)$ as sets for all $m \in \mathbb{Z}$. Thus, $Z' = Z'_j \times_{X_j} X$ which, also, is constructible closed because $X \to X_j$ is affine. □

4D. *Lisse and constructible Weil sheaves.* In this subsection, we define the subcategories of lisse and constructible Weil sheaves and establish a presentation similar to (4-4). Let $X_1, \dots, X_n$ be schemes over $\mathbb{F}_q$, and denote $X := X_1 \times_{\mathbb{F}_q} \cdots \times_{\mathbb{F}_q} X_n$. Let Λ be a condensed ring.

Definition 4.10. Let $M \in \mathrm{D}(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda)$:

(1) The Weil sheaf M is called *lisse* if it is dualizable. (Here dualizability refers to the symmetric monoidal structure on $\mathrm{D}(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda)$, given by the derived tensor product of Λ-sheaves on the Weil-proétale topos.)

(2) The Weil sheaf M is called *constructible* if for any open affine $U_i \subset X_i$ there exists a finite subdivision into constructible locally closed subschemes $U_{ij} \subseteq U_i$ such that each restriction $M|_{U_{1j}^{\mathrm{Weil}} \times \cdots \times U_{nj}^{\mathrm{Weil}}} \in \mathrm{D}(U_{1j}^{\mathrm{Weil}} \times \cdots \times U_{nj}^{\mathrm{Weil}}, \Lambda)$ is lisse.

The full subcategories of $\mathrm{D}(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda)$ consisting of lisse, resp. constructible Weil sheaves are denoted by

$$\mathrm{D}_{\mathrm{lis}}(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda) \subset \mathrm{D}_{\mathrm{cons}}(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda).$$

Both categories are idempotent complete stable $\Gamma(X, \Lambda)$-linear symmetric monoidal ∞-categories.

From the presentation (4-6), we get that a Weil sheaf M is lisse if and only if the underlying object $M_{\mathbb{F}} \in \mathrm{D}(X_{\mathbb{F}}, \Lambda)$ is lisse. So (4-6) restricts to an equivalence

$$\mathrm{D}_{\mathrm{lis}}(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda) \cong \mathrm{Fix}(\mathrm{D}_{\mathrm{lis}}(X_{\mathbb{F}}, \Lambda), \phi_{X_1}^*, \dots, \phi_{X_n}^*). \tag{4-7}$$

The same is true for constructible Weil sheaves by the following proposition:

Proposition 4.11. *A Weil sheaf* $M \in \mathrm{D}(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda)$ *is constructible if and only if the underlying sheaf* $M_{\mathbb{F}} \in \mathrm{D}(X_{\mathbb{F}}, \Lambda)$ *is constructible. Consequently*, (4-6) *restricts to an equivalence*

$$\mathrm{D}_{\mathrm{cons}}(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda) \cong \mathrm{Fix}(\mathrm{D}_{\mathrm{cons}}(X_{\mathbb{F}}, \Lambda), \phi_{X_1}^*, \ldots, \phi_{X_n}^*). \tag{4-8}$$

Proof. Clearly, if M is constructible, so is $M_{\mathbb{F}}$ by Definition 4.10. Let $M \in \mathrm{D}(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda)$ such that $M_{\mathbb{F}}$ is constructible. We may assume that all X_i are affine. We claim that there is a finite subdivision $X_{\mathbb{F}} = \bigsqcup X_\alpha$ into constructible locally closed subsets such that $M_{\mathbb{F}}|_{X_\alpha}$ is lisse and such that each X_α is partial Frobenius invariant.

Assuming the claim we finish the argument as follows. By Proposition 4.8, any open stratum $U = X_{j_0} \subset X_{\mathbb{F}}$ is a finite union of subsets of the form $U_{1,\mathbb{F}} \times_{\mathbb{F}} \cdots \times_{\mathbb{F}} U_{n,\mathbb{F}}$ and the restriction of M to each of them is lisse. In particular, the complement $X_{\mathbb{F}} \backslash U$ is defined over $\mathbb{F}_q$ and arises as a finite union of schemes of the form $X' = X'_1 \times_{\mathbb{F}_q} \cdots \times_{\mathbb{F}_q} X'_n$ for suitable qcqs schemes X'_i over $\mathbb{F}_q$. Intersecting each $X'_{\mathbb{F}}$ with the remaining strata $\bigsqcup_{j \neq j_0} X_j$, we conclude by induction on the number of strata.

It remains to prove the claim. We start with any finite subdivision $X_{\mathbb{F}} = \bigsqcup X'_j$ into constructible locally closed subsets such that $M_{\mathbb{F}}|_{X'_j}$ is lisse. Pick an open stratum X'_{j_0}, and set

$$X_{j_0} = \bigcup_{i=1}^{n} \bigcup_{m \in \mathbb{Z}} \phi_{X_i}^m (X'_{j_0}). \tag{4-9}$$

This is a constructible open subset of $X_{\mathbb{F}}$ by Lemma 4.9 applied to its closed complement. Furthermore, $M_{\mathbb{F}}|_{X_{j_0}}$ is lisse by its partial Frobenius equivariance, noting that $\phi_{X_i}^*$ induces equivalences on proétale topoi to treat the negative powers in (4-9). As before, $X_{\mathbb{F}} \backslash X_{j_0}$ is defined over $\mathbb{F}_q$. So replacing X'_j, $j \neq j_0$ by $X'_j \cap (X_{\mathbb{F}} \backslash X_{j_0})$, the claim follows by induction on the number of strata. □

In the case of a single factor $X = X_1$, the preceding discussion implies

$$\mathrm{D}_\bullet(X^{\mathrm{Weil}}, \Lambda) \cong \lim(\mathrm{D}_\bullet(X_{\mathbb{F}}, \Lambda) \underset{\phi_X^*}{\overset{\mathrm{id}}{\rightrightarrows}} \mathrm{D}_\bullet(X_{\mathbb{F}}, \Lambda)), \tag{4-10}$$

for $\bullet \in \{\varnothing, \mathrm{lis}, \mathrm{cons}\}$.

4E. ***Relation with the Weil groupoid.*** In this subsection, we relate lisse Weil sheaves with representations of the Weil groupoid. Throughout, we work with étale fundamental groups as opposed to their proétale variants in order to have Drinfeld's lemma available; see Section 5C. The two concepts differ in general, but agree for geometrically unibranch (for example, normal) Noetherian schemes; see [Bhatt and Scholze 2015, Lemma 7.4.10].

For a Noetherian scheme X, let $\pi_1(X)$ be the *étale fundamental groupoid* of X as defined in [SGA 1 2003, Exposé V, Sections 7 and 9]. Its objects are geometric points of X, and its morphisms are isomorphisms of fiber functors on the finite étale site of X. This is an essentially small category. The automorphism group in $\pi_1(X)$ at a geometric point $x \to X$ is profinite. It is denoted $\pi_1(X, x)$ and called the *étale fundamental group* of (X, x). If X is connected, then the natural map $B\pi_1(X, x) \to \pi_1(X)$ is an equivalence for any $x \to X$. If X is the disjoint sum of schemes X_i, $i \in I$, then $\pi_1(X)$ is the disjoint sum of the $\pi_1(X_i)$, $i \in I$. In this case, if $x \to X$ factors through X_i, then $\pi_1(X, x) = \pi_1(X_i, x)$.

Definition 4.12. Let $X_1, \dots, X_n$ be Noetherian schemes over $\mathbb{F}_q$, and write $X = X_1 \times_{\mathbb{F}_q} \cdots \times_{\mathbb{F}_q} X_n$. The *Frobenius-Weil groupoid* is the stacky quotient

$$\mathrm{FWeil}(X) = \pi_1(X_{\mathbb{F}})/\langle \phi_{X_1}^{\mathbb{Z}}, \dots, \phi_{X_n}^{\mathbb{Z}} \rangle, \tag{4-11}$$

where we use that the partial Frobenii ϕ_{X_i} induce automorphisms on the finite étale site of $X_{\mathbb{F}}$.

For $n = 1$, we denote $\mathrm{FWeil}(X) = \mathrm{Weil}(X)$. Even if X is connected, its base change $X_{\mathbb{F}}$ might be disconnected in which case the action of ϕ_X permutes some connected components. Therefore, fixing a geometric point of $X_{\mathbb{F}}$ is inconvenient, and the reason for us to work with fundamental groupoids as opposed to fundamental groups. The automorphism groups in $\mathrm{Weil}(X)$ carry the structure of locally profinite groups: indeed, if X is connected, then $\mathrm{Weil}(X)$ is, for any choice of a geometric point $x \to X_{\mathbb{F}}$, equivalent to the classifying space of the Weil group $\mathrm{Weil}(X, x)$ from [Deligne 1980, Définition 1.1.10]. Recall that this group sits in an exact sequence of topological groups

$$1 \to \pi_1(X_{\mathbb{F}}, x) \to \mathrm{Weil}(X, x) \to \mathrm{Weil}(\mathbb{F}/\mathbb{F}_q) \simeq \mathbb{Z}, \tag{4-12}$$

where $\pi_1(X_{\mathbb{F}}, x)$ carries its profinite topology and $\mathbb{Z}$ the discrete topology. The topology on the morphism groups in $\mathrm{Weil}(X)$ obtained in this way is independent from the choice of $x \to X_{\mathbb{F}}$. The image of $\mathrm{Weil}(X, x) \to \mathbb{Z}$ is the subgroup $m\mathbb{Z}$ where m is the degree of the largest finite subfield in $\Gamma(X, \mathcal{O}_X)$. In particular, we have $m = 1$ if $X_{\mathbb{F}}$ is connected. Let us add that if $x \to X_{\mathbb{F}}$ is fixed under ϕ_X, then the action of ϕ_X on $\pi_1(X_{\mathbb{F}}, x)$ corresponds by virtue of the formula $\phi_X^* = (\phi_{\mathbb{F}}^*)^{-1}$ to the action of the geometric Frobenius, that is, the inverse of the q-Frobenius in $\mathrm{Weil}(\mathbb{F}/\mathbb{F}_q)$.

Likewise, for every $n \geq 1$, the stabilizers of the Frobenius–Weil groupoid are related to the partial Frobenius–Weil groups introduced in [Drinfeld 1987, Proposition 6.1] and [Lafforgue 2018, Remarque 8.18]. In particular, there is an exact sequence

$$1 \to \pi_1(X_{\mathbb{F}}, x) \to \mathrm{FWeil}(X, x) \to \mathbb{Z}^n,$$

for each geometric point $x \to X_{\mathbb{F}}$. This gives $\mathrm{FWeil}(X)$ the structure of a locally profinite groupoid.

Let Λ be either of the following coherent topological rings: a coherent discrete ring, an algebraic field extension $E \supset \mathbb{Q}_\ell$ for some prime ℓ, or its ring of integers $\mathcal{O}_E \supset \mathbb{Z}_\ell$. For a topological groupoid W, we will denote by $\mathrm{Rep}_\Lambda(W)$ the category of continuous representations of W with values in finitely presented Λ-modules and by $\mathrm{Rep}_\Lambda^{\mathrm{fp}}(W) \subset \mathrm{Rep}_\Lambda(W)$ its full subcategory of representations on finite projective Λ-modules. Here finitely presented Λ-modules M carry the quotient topology induced from the choice of any surjection $\Lambda^n \to M$, $n \geq 0$ and the product topology on Λ^n.

Lemma 4.13. *In the situation above, the category* $\mathrm{Rep}_\Lambda(W)$ *is* Λ_**-linear and abelian. In particular, its full subcategory* $\mathrm{Rep}_\Lambda^{\mathrm{fp}}(W)$ *is* Λ_**-linear and additive.*

Proof. Let W_{disc} be the discrete groupoid underlying W, and denote by $\mathrm{Rep}_\Lambda(W_{\mathrm{disc}})$ the category of W_{disc}-representations on finitely presented Λ-modules. Evidently, this category is Λ_*-linear. It is abelian since Λ is coherent (Synopsis 3.2(vi)); see also [Hemo et al. 2023, Lemma 6.5]. We claim that $\mathrm{Rep}_\Lambda(W) \subset \mathrm{Rep}_\Lambda(W_{\mathrm{disc}})$ is a Λ_*-linear full abelian subcategory. If Λ is discrete (and coherent), then every finitely presented Λ-module carries the discrete topology and the claim is immediate; see also [Stacks 2017, Tag 0A2H]. For $\Lambda = E, \mathcal{O}_E$, one checks that every map of finitely presented Λ-modules is continuous, every surjective map is a topological quotient and every injective map is a closed embedding. For the latter, we use that every finitely presented Λ-module can be written as a countable filtered colimit of compact Hausdorff spaces along injections, and that every injection of compact Hausdorff spaces is a closed embedding. This implies the claim. □

We apply this for W being either of the locally profinite groupoids $\pi_1(X)$, $\pi_1(X_{\mathbb{F}})$ or $\mathrm{FWeil}(X)$. Note that restricting representations along $\pi_1(X_{\mathbb{F}}) \to \mathrm{FWeil}(X)$ induces an equivalence of Λ_*-linear abelian categories

$$\mathrm{Rep}_\Lambda(\mathrm{FWeil}(X)) \cong \mathrm{Fix}(\mathrm{Rep}_\Lambda(\pi_1(X_{\mathbb{F}})), \phi_{X_1}, \dots, \phi_{X_n}), \tag{4-13}$$

and similarly for the Λ_*-linear additive category $\mathrm{Rep}^{\mathrm{fp}}_\Lambda(\mathrm{FWeil}(X))$.

Definition 4.14. For an integer $n \geq 0$, we write $\mathrm{D}^{\{-n,n\}}_{\mathrm{lis}}(X, \Lambda)$ for the full subcategory of $\mathrm{D}_{\mathrm{lis}}(X, \Lambda)$ of objects M such that M and its dual $M^\vee$ lie in degrees $[-n, n]$ with respect to the t-structure on $\mathrm{D}(X, \Lambda)$.

Lemma 4.15. *In the situation above, there is a natural functor*

$$\mathrm{Rep}_\Lambda(\mathrm{FWeil}(X)) \to \mathrm{D}(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda)^\heartsuit, \tag{4-14}$$

that is fully faithful. Moreover, the following properties hold if Λ is either finite discrete or $\Lambda = \mathcal{O}_E$ for $E \supset \mathbb{Q}_\ell$ finite:

(1) *An object M lies in the essential image of* (4-14) *if and only if its underlying sheaf $M_{\mathbb{F}}$ is locally on $(X_{\mathbb{F}})_{\text{proét}}$ isomorphic to $\underline{N} \otimes_{\underline{\Lambda_*}} \Lambda_{X_{\mathbb{F}}}$ for some finitely presented Λ_*-module N.*

(2) *The functor* (4-14) *restricts to an equivalence of Λ_*-linear additive categories*

$$\mathrm{Rep}^{\mathrm{fp}}_\Lambda(\mathrm{FWeil}(X)) \xrightarrow{\cong} \mathrm{D}^{\{0,0\}}_{\mathrm{lis}}(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda).$$

(3) *If Λ_* is regular (so that Λ is t-admissible, see Synopsis 3.2*(vi)), *then* (4-14) *restricts to an equivalence of Λ_*-linear abelian categories*

$$\mathrm{Rep}_\Lambda(\mathrm{FWeil}(X)) \xrightarrow{\cong} \mathrm{D}_{\mathrm{lis}}(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda)^\heartsuit.$$

If all X_i, $i = 1, \dots, n$ are geometrically unibranch, then (1), (2) and (3) hold for general coherent topological rings Λ as above.

Proof. There is a canonical equivalence of topological groupoids $\pi_1(X_{\mathbb{F}}) \cong \widehat{\pi_1^{\text{proét}}(X_{\mathbb{F}})}$ with the profinite completion of the proétale fundamental groupoid; see [Bhatt and Scholze 2015, Lemma 7.4.3]. It follows from [loc. cit., Lemmas 7.4.5, 7.4.7] that restricting representations along $\pi_1^{\text{proét}}(X_{\mathbb{F}}) \to \pi_1(X_{\mathbb{F}})$ induces full embeddings

$$\mathrm{Rep}_\Lambda(\pi_1(X_{\mathbb{F}})) \hookrightarrow \mathrm{Rep}_\Lambda(\pi_1^{\text{proét}}(X_{\mathbb{F}})) \hookrightarrow \mathrm{D}(X_{\mathbb{F}}, \Lambda)^{\heartsuit}, \tag{4-15}$$

that are compatible with the action of ϕ_{X_i} for all $i = 1, \ldots, n$. So we obtain the fully faithful functor (4-14) by passing to fixed points, see (4-13), (4-7) and Lemma 2.2 (see also Remark 2.3).

Part (1) describes the essential image of $\mathrm{Rep}_\Lambda(\pi_1^{\text{proét}}(X_{\mathbb{F}})) \hookrightarrow \mathrm{D}(X_{\mathbb{F}}, \Lambda)^{\heartsuit}$. So if Λ is finite discrete or profinite, then the first functor in (4-15) is an equivalence, and we are done. Part (2) is immediate from (1), noting that an object in the essential image of (4-15) is lisse if and only if its underlying module is finite projective. Likewise, part (3) is immediate from (1), using Synopsis 3.2(vii). Here we need to exclude rings like $\Lambda = \mathbb{Z}/\ell^2$ in order to have a t-structure on lisse sheaves.

Finally, if all X_i are geometrically unibranch, so is $X_{\mathbb{F}}$ which follows from the characterization [Stacks 2017, Tag 0BQ4]. In this case, we get $\pi_1(X_{\mathbb{F}}) \cong \pi_1^{\text{proét}}(X_{\mathbb{F}})$ by [Bhatt and Scholze 2015, Lemma 7.4.10]. This finishes the proof. □

4F. ***Weil-étale versus étale sheaves.*** We end this section with the following description of Weil sheaves with (ind-)finite coefficients. Note that such a simplification in terms of ordinary sheaves is not possible for $\Lambda = \mathbb{Z}, \mathbb{Z}_\ell, \mathbb{Q}_\ell$, say.

Proposition 4.16. *Let X be a qcqs $\mathbb{F}_q$-scheme. Let Λ be a finite discrete ring or a filtered colimit of such rings. Then the natural functors*

$$\mathrm{D}_{\mathrm{lis}}(X, \Lambda) \to \mathrm{D}_{\mathrm{lis}}(X^{\mathrm{Weil}}, \Lambda), \quad \mathrm{D}_{\mathrm{cons}}(X, \Lambda) \to \mathrm{D}_{\mathrm{cons}}(X^{\mathrm{Weil}}, \Lambda),$$

are equivalences.

Proof. Throughout, we repeatedly use that filtered colimits commute with finite limits in Cat_∞. Using compatibility of $\mathrm{D}_{\mathrm{cons}}$ with filtered colimits in Λ (Synopsis 3.2(iv)), we may assume that Λ is finite discrete. By the comparison result with the classical bounded derived category of constructible sheaves [Hemo et al. 2023, Proposition 7.1], we can identify the categories $\mathrm{D}_\bullet(X, \Lambda)$, resp. $\mathrm{D}_\bullet(X_{\mathbb{F}}, \Lambda)$ for $\bullet \in \{\mathrm{lis}, \mathrm{cons}\}$ with full subcategories of the derived category of étale Λ-sheaves $\mathrm{D}(X_{\text{ét}}, \Lambda)$, resp. $\mathrm{D}(X_{\mathbb{F},\text{ét}}, \Lambda)$. Write $X = \lim X_i$ as a cofiltered limit of finite type $\mathbb{F}_q$-schemes X_i with affine transition maps [Stacks 2017, Tag 01ZA]. Using the continuity of étale sites [Stacks 2017, Tag 03Q4], there are natural equivalences

$$\operatorname{colim} \mathrm{D}_\bullet(X_i, \Lambda) \xrightarrow{\cong} \mathrm{D}_\bullet(X, \Lambda), \quad \operatorname{colim} \mathrm{D}_\bullet(X_i^{\mathrm{Weil}}, \Lambda) \xrightarrow{\cong} \mathrm{D}_\bullet(X^{\mathrm{Weil}}, \Lambda) \tag{4-16}$$

for $\bullet \in \{\mathrm{lis}, \mathrm{cons}\}$. Hence, we can assume that X is of finite type over $\mathbb{F}_q$.

To show full faithfulness, we claim more generally that the natural map

$$\mathrm{D}(X_{\text{ét}}, \Lambda) \to \lim(\mathrm{D}(X_{\mathbb{F},\text{ét}}, \Lambda) \underset{\phi_X^*}{\overset{\mathrm{id}}{\rightrightarrows}} \mathrm{D}(X_{\mathbb{F},\text{ét}}, \Lambda)) =: \mathrm{D}(X_{\text{ét}}^{\mathrm{Weil}}, \Lambda)$$

is fully faithful. As Λ is torsion, this is immediate from [Geisser 2004, Corollary 5.2] applied to the inner homomorphisms between sheaves. Let us add that this induces fully faithful functors

$$\mathrm{D}^+(X_{\text{ét}}, \Lambda) \to \mathrm{D}^+(X_{\text{ét}}^{\mathrm{Weil}}, \Lambda) \to \mathrm{D}(X^{\mathrm{Weil}}, \Lambda) \tag{4-17}$$

on bounded below objects; see [Bhatt and Scholze 2015, Proposition 5.2.6(1)].

It remains to prove essential surjectivity. Using a stratification as in Definition 4.10, it is enough to consider the lisse case. Pick $M \in \mathrm{D}_{\mathrm{lis}}(X^{\mathrm{Weil}}, \Lambda)$. It is enough to show that M lies is in the essential image of (4-17), noting that the functor detects dualizability. As M is bounded, this will follow from showing that for every $j \in \mathbb{Z}$, the cohomology sheaf $\mathrm{H}^j(M) \in \mathrm{D}(X^{\mathrm{Weil}}, \Lambda)^{\heartsuit}$ is in the essential image of (4-17).

Fix $j \in \mathbb{Z}$. As M is lisse, the underlying sheaf $\mathrm{H}^j(M)_{\mathbb{F}} \in \mathrm{D}(X_{\mathbb{F}}, \Lambda)^{\heartsuit}$ is proétale-locally constant (Synopsis 3.2(vii)) and valued in finitely presented Λ-modules. By Lemma 4.15(1), it comes from a representation of $\mathrm{Weil}(X)$. Restriction of representations along $\mathrm{Weil}(X) \to \pi_1(X)$ fits into a commutative diagram:

$$\begin{array}{ccc} \mathrm{Rep}_{\Lambda}(\pi_1(X)) & \xrightarrow{\cong} & \mathrm{Rep}_{\Lambda}(\mathrm{Weil}(X)) \\ \downarrow & & \downarrow \\ \mathrm{D}(X_{\text{ét}}, \Lambda)^{\heartsuit} & \longrightarrow & \mathrm{D}(X^{\mathrm{Weil}}, \Lambda)^{\heartsuit} \end{array}$$

where the upper horizontal arrow is an equivalence since Λ is finite. In particular, the object $\mathrm{H}^j(M)$ is in the essential image of the fully faithful functor (4-17). □

5. The categorical Künneth formula

We continue with the notation of Section 4. In particular, $\mathbb{F}_q$ denotes a finite field of characteristic $p > 0$. Recall from Section 2 the tensor product of Λ_*-linear idempotent complete stable ∞-categories. The external tensor product of sheaves $(M_1, \ldots, M_n) \mapsto M_1 \boxtimes \cdots \boxtimes M_n$ as in (3-2) induces a functor

$$\mathrm{D}_{\bullet}(X_1^{\mathrm{Weil}}, \Lambda) \otimes_{\mathrm{Perf}_{\Lambda_*}} \cdots \otimes_{\mathrm{Perf}_{\Lambda_*}} \mathrm{D}_{\bullet}(X_n^{\mathrm{Weil}}, \Lambda) \to \mathrm{D}_{\bullet}(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda), \tag{5-1}$$

for $\bullet \in \{\mathrm{lis}, \mathrm{cons}\}$. Throughout, we consider the following situation. In Remark 5.3 we explain the compatibility of (5-1) with certain (co-)limits in the schemes X_i and coefficients Λ, which allows to relax these assumptions on X and Λ somewhat.

Situation 5.1. The schemes $X_1, \ldots, X_n$ are of finite type over $\mathbb{F}_q$, and Λ is the condensed ring associated with one of the following topological rings:

(a) A finite discrete ring of prime-to-p-torsion.

(b) The ring of integers $\mathcal{O}_E$ of an algebraic field extension $E \supset \mathbb{Q}_\ell$ for $\ell \neq p$ (for example $\overline{\mathbb{Z}}_\ell$).

(c) An algebraic field extension $E \supset \mathbb{Q}_\ell$ for $\ell \neq p$ (for example $\overline{\mathbb{Q}}_\ell$).

(d) A finite discrete p-torsion ring that is flat over $\mathbb{Z}/p^m$ for some $m \geq 1$.

Theorem 5.2. *In Situation 5.1, the functor* (5-1) *is an equivalence in each of the following cases*:

(1) $\bullet = \mathrm{cons}$ *and* Λ *is as in (a), (b) or (c).*

(2) $\bullet = \mathrm{lis}$ *and* Λ *is as in (a), (b), (d) or as in (c) if all* X_i, $i = 1, \dots, n$ *are geometrically unibranch* (*for example*, *normal*).

In the p-torsion free cases (a), (b) and (c), the full faithfulness is a direct consequence of the Künneth formula applied to the $X_{i,\mathbb{F}}$. In the p-torsion case (d), we use Artin–Schreier theory instead. It would be interesting to see whether this part can be extended to constructible sheaves using the mod-p-Riemann–Hilbert correspondence as in, say, [Bhatt and Lurie 2019]. In all cases, the essential surjectivity relies on a variant of Drinfeld's lemma for Weil group representations.

Before turning to the proof of Theorem 5.2, we record the following compatibility of the functor (5-1) with (co-)limits. This can be used to reduce the case of an (infinite) algebraic extension $E \supset \mathbb{Q}_\ell$ in cases (b) and (c) above to the case where $E \supset \mathbb{Q}_\ell$ is finite. In the sequel we will therefore assume E is finite in these cases. Remark 5.3 can further be used to extend Theorem 5.2 to qcqs $\mathbb{F}_q$-schemes X_i and finite discrete rings like $\mathbb{Z}/m$ for any integer $m \geq 1$ in cases (a) and (d).

Remark 5.3 (compatibility of (5-1) with certain (co-)limits). Throughout, we repeatedly use that filtered colimits commute with finite limits in $\mathrm{Cat}^{\mathrm{Ex}}_{\infty,\Lambda_*}(\mathrm{Idem})$: the forgetful functors $\mathrm{Cat}^{\mathrm{Ex}}_{\infty,\Lambda_*}(\mathrm{Idem}) \to \mathrm{Cat}^{\mathrm{Ex}}_{\infty}(\mathrm{Idem}) \to \mathrm{Cat}_\infty$ create these (co)limits [Lurie 2017, Theorem 1.1.4.4; 2009, Corollary 4.4.5.21], and the statement holds in any compactly generated ∞-category, such as Cat_∞ [Bhatt and Mathew 2021, Example 3.6(3)]. We will also throughout use that in all the stable ∞-categories encountered below the tensor product preserves colimits and in particular finite limits:

(1) *Filtered colimits in* Λ. First off, extension of scalars along any map of condensed rings $\Lambda \to \Lambda'$ induces a commutative diagram in $\mathrm{Cat}^{\mathrm{Ex}}_{\infty,\Lambda_*}(\mathrm{Idem})$:

$$\begin{array}{ccc} \mathrm{D}_\bullet(X_1^{\mathrm{Weil}}, \Lambda) \otimes_{\mathrm{Perf}_{\Lambda_*}} \cdots \otimes_{\mathrm{Perf}_{\Lambda_*}} \mathrm{D}_\bullet(X_n^{\mathrm{Weil}}, \Lambda) & \longrightarrow & \mathrm{D}_\bullet(X_1^{\mathrm{Weil}} \cdots \times X_n^{\mathrm{Weil}}, \Lambda) \\ \downarrow & & \downarrow \\ \mathrm{D}_\bullet(X_1^{\mathrm{Weil}}, \Lambda') \otimes_{\mathrm{Perf}_{\Lambda'_*}} \cdots \otimes_{\mathrm{Perf}_{\Lambda'_*}} \mathrm{D}_\bullet(X_n^{\mathrm{Weil}}, \Lambda') & \longrightarrow & \mathrm{D}_\bullet(X_1^{\mathrm{Weil}} \cdots \times X_n^{\mathrm{Weil}}, \Lambda') \end{array}$$

It follows from the compatibility of $\mathrm{D}_{\mathrm{cons}}$ with filtered colimits in Λ (Synopsis 3.2(iv)) that both sides of (5-1) are compatible with filtered colimits in Λ.

(2) *Finite products in* Λ. Let $\Lambda = \prod \Lambda_i$ be a finite product of condensed rings. For any scheme X, the natural map $\mathrm{D}_\bullet(X, \Lambda) \to \prod \mathrm{D}_\bullet(X, \Lambda_i)$ is an equivalence for $\bullet \in \{\varnothing, \mathrm{lis}, \mathrm{cons}\}$, and likewise for Weil sheaves if X is defined over $\mathbb{F}_q$. As $\Lambda_* = \prod \Lambda_{i,*}$, we see that (5-1) is compatible finite products in the coefficients.

(3) *Limits in* X_i *for discrete* Λ. Assume that Λ is finite discrete; see Situation 5.1(a), (d). Let $X_1, \dots, X_n$ be qcqs $\mathbb{F}_q$-schemes. Write each X_i as a cofiltered limit $X_i = \lim X_{ij}$ of finite type $\mathbb{F}_q$-schemes X_{ij} with

affine transition maps [Stacks 2017, Tag 01ZA]. As Λ is finite discrete, we can use the continuity of étale sites as in (4-16) to show that the natural map

$$\mathrm{colim}_j\, \mathrm{D}_\bullet(X_{1j}^{\mathrm{Weil}} \times \cdots \times X_{nj}^{\mathrm{Weil}}, \Lambda) \xrightarrow{\cong} \mathrm{D}_\bullet(X_1^{\mathrm{Weil}} \cdots \times X_n^{\mathrm{Weil}}, \Lambda),$$

is an equivalence for $\bullet \in \{\mathrm{lis}, \mathrm{cons}\}$. Thus, (5-1) is compatible with cofiltered limits of finite type $\mathbb{F}_q$-schemes with affine transition maps.

5A. ***A formulation in terms of prestacks.*** Before turning to the proof, we point out a formulation of the results of the previous subsection in terms of symmetric monoidality of a certain sheaf theory. This formulation makes the connection with constructions in the geometric approaches to the Langlands program [Gaitsgory et al. 2022; Zhu 2021; Lafforgue and Zhu 2019] more manifest. Readers not familiar with prestacks and formulations of sheaf theories on them can safely skip this section. The categories of constructible, resp. lisse Λ-sheaves assemble into a lax symmetric monoidal functor

$$\mathrm{D}_{\bullet,\Lambda} \colon (\mathrm{Sch}_{\mathbb{F}})^{\mathrm{op}} \to \mathrm{Cat}^{\mathrm{Ex}}_{\infty,\Lambda}(\mathrm{Idem}) \quad (\bullet = \mathrm{lis} \text{ or } \mathrm{cons}). \tag{5-2}$$

Namely, as a functor it sends a scheme X to the category of constructible, resp. lisse Λ-sheaves on X, and a morphism $f \colon X \to Y$ to the functor $f^* \colon \mathrm{D}_\bullet(Y, \Lambda) \to \mathrm{D}_\bullet(X, \Lambda)$. These are objects, resp. maps in the ∞-category $\mathrm{Cat}^{\mathrm{Ex}}_{\infty,\Lambda}(\mathrm{Idem}) := \mathrm{Mod}_{\mathrm{Perf}_\Lambda}(\mathrm{Cat}^{\mathrm{Ex}}_{\infty}(\mathrm{Idem}))$, see Section 2A for notation. The lax monoidal structure is given by the external tensor product of sheaves

$$\boxtimes \colon \mathrm{D}_\bullet(X_{\mathrm{proét}}, \Lambda) \otimes_{\mathrm{Perf}_\Lambda} \mathrm{D}_\bullet(Y_{\mathrm{proét}}, \Lambda) \to \mathrm{D}_\bullet((X \times_{\mathbb{F}} Y)_{\mathrm{proét}}, \Lambda).$$

That is, we consider the category of schemes as symmetric monoidal with respect to the fiber product over $\mathbb{F}$, and the external tensor product is natural on X and Y in the appropriate sense; see [Gaitsgory and Lurie 2019, Section 3.1; Gaitsgory and Rozenblyum 2017, Section III.2] for details and precise statements. This functor $\boxtimes$ often fails to be an equivalence, so $\mathrm{D}_{\bullet,\Lambda}$ is not symmetric monoidal. The assertion of Theorem 5.2 is that this issue is resolved by replacing sheaves with Weil sheaves. In order to formulate Theorem 5.2 as the monoidality of a certain functor, we need to replace the category of schemes by a category of objects that model Weil sheaves. We will represent these by taking the appropriate formal quotient by the partial Frobenius automorphism. Such formal quotients can be taken in the category of prestacks.

We denote by $\mathrm{PreStk}_{\mathbb{F}}$ the category of (accessible) functors from the category $\mathrm{CAlg}_{\mathbb{F}}$ of commutative algebras over $\mathbb{F}$ to the ∞-category Ani of Anima. The functor of taking points embeds the category of schemes fully faithfully into $\mathrm{PreStk}_{\mathbb{F}}$. We denote by

$$\mathrm{D}_{\bullet,\Lambda} \colon (\mathrm{PreStk}_{\mathbb{F}})^{\mathrm{op}} \to \mathrm{Cat}^{\mathrm{Ex}}_{\infty,\Lambda}(\mathrm{Idem}) \tag{5-3}$$

the functor obtained by right Kan extension [Lurie 2009, Section 4.3.2] along the inclusion $(\mathrm{Sch}^{\mathrm{fp}}_{\mathbb{F}})^{\mathrm{op}} \subset (\mathrm{PreStk}_{\mathbb{F}})^{\mathrm{op}}$. Concretely, [Lurie 2018, Propositions 6.2.1.9 and 6.2.3.1], given a prestack Y which can be written as a colimit of schemes Y_α over some indexing category A we have a canonical equivalence

$$\mathrm{D}_\bullet(Y, \Lambda) \cong \lim_\alpha \mathrm{D}_\bullet(Y_\alpha, \Lambda). \tag{5-4}$$

This limit is formed in $\mathrm{Cat}^{\mathrm{Ex}}_{\infty,\Lambda}(\mathrm{Idem})$; recall from around (2-3) that the Ind-completion functor to $\mathrm{Cat}^{\mathrm{Ex}}_{\infty,\Lambda}(\mathrm{Idem}) \to \mathrm{Pr}^{\mathrm{St}}_\Lambda$ does *not* preserve (even finite) limits.

With this general sheaf theory in place, we can restrict our attention to the class of prestacks that is relevant to the derived Drinfeld lemma.

Definition 5.4. Let X be a scheme over $\mathbb{F}_q$. The *Weil prestack* is defined as

$$X^{\mathrm{Weil}} := \mathrm{colim}(X \times_{\mathbb{F}_q} \mathbb{F} \underset{\phi_X}{\overset{\mathrm{id}}{\rightrightarrows}} X \times_{\mathbb{F}_q} \mathbb{F}) \in \mathrm{PreStk}_{\mathbb{F}},$$

i.e., it is the prestack sending $R \in \mathrm{CAlg}_{\mathbb{F}}$ to the colimit

$$X^{\mathrm{Weil}}(R) = \mathrm{colim}(X(R) \underset{\phi_X}{\overset{\mathrm{id}}{\rightrightarrows}} X(R)). \tag{5-5}$$

We denote by $\mathrm{Sch}^{\mathrm{fp}}_{\mathrm{Weil}}$ the smallest full monoidal subcategory of $\mathrm{PreStk}_{\mathbb{F}}$ containing the Weil prestacks of finite type schemes $X/\mathbb{F}_q$. Equivalently, this is the full subcategory consisting of finite products of the form $X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}$.

Lemma 5.5. *Let $X_1, \dots, X_n$ be schemes over $\mathbb{F}_q$. There is a canonical equivalence*

$$\mathrm{D}_\bullet(X_1^{\mathrm{Weil}} \times_{\mathbb{F}} \cdots \times_{\mathbb{F}} X_n^{\mathrm{Weil}}) \xrightarrow{\cong} \mathrm{Fix}(\mathrm{D}_\bullet(X_{\mathbb{F}}, \Lambda), \phi^*_{X_1}, \dots, \phi^*_{X_n}). \tag{5-6}$$

Proof. Let $\Phi \colon \mathrm{B}\mathbb{Z}^n \to \mathrm{PreStk}_{\mathbb{F}}$ be the functor corresponding to the commuting automorphisms ϕ_{X_i}. Then the claim follows immediately from the identification of $X_1^{\mathrm{Weil}} \times_{\mathbb{F}} \cdots \times_{\mathbb{F}} X_n^{\mathrm{Weil}}$ with the colimit of Φ (as an object in $\mathrm{PreStk}_{\mathbb{F}}$). □

Theorem 5.6. *Suppose $\bullet$ and Λ are as in Theorem 5.2. Then the restriction of* $\mathrm{D}_{\bullet,\Lambda}$ *to Weil prestacks, i.e., the following composite*

$$\mathrm{D}_{\bullet,\Lambda} \colon (\mathrm{Sch}^{\mathrm{fp}}_{\mathrm{Weil}})^{\mathrm{op}} \subset \mathrm{PreStk}_{\mathbb{F}} \to \mathrm{Cat}^{\mathrm{Ex}}_{\infty,\Lambda}(\mathrm{Idem}), \tag{5-7}$$

is symmetric monoidal.

Proof. As was noted above, the functor in (5-2) is lax symmetric monoidal. By [Torii 2023, Proposition 2.7], the Kan extension in (5-3) is still lax symmetric monoidal. To check its restriction to the (symmetric monoidal) subcategory $\mathrm{Sch}^{\mathrm{fp}}_{\mathrm{Weil}}$ is symmetric monoidal it suffices to show that the lax monoidal maps are in fact isomorphisms. This is precisely the content of Theorem 5.2. □

5B. *Full faithfulness.* In this section, we prove that the functor (5-1) is fully faithful under the conditions of Theorem 5.2. We first consider the p-torsion free cases:

Proposition 5.7. *Let $X_1, \dots, X_n$ and Λ be as in Situation 5.1(a), (b) or (c). Then the functor* (5-1) *is fully faithful for $\bullet \in \{\mathrm{lis}, \mathrm{cons}\}$.*

Proof. For constructible sheaves on $X_{i,\mathbb{F}}$ (as opposed to X_i^{Weil}), this interpretation of the Künneth formula appears already in [Gaitsgory et al. 2022, Section A.2]. Throughout, we drop Λ from the notation. It is enough to verify that for all $M_i, N_i \in \mathrm{D}_{\mathrm{cons}}(X_i^{\mathrm{Weil}})$ the natural map

$$\bigotimes_{i=1}^{n} \mathrm{Hom}_{\mathrm{D}(X_i^{\mathrm{Weil}})}(M_i, N_i) \to \mathrm{Hom}_{\mathrm{D}(X_1^{\mathrm{Weil}}\times\cdots\times X_n^{\mathrm{Weil}})}(M_1 \boxtimes \cdots \boxtimes M_n, N_1 \boxtimes \cdots \boxtimes N_n) \tag{5-8}$$

is an equivalence. As (5-8) is functorial in the objects and compatible with shifts, it suffices, by Definition 4.10, to consider the case where M_i, $i = 1, \dots, n$ is the extension by zero of a lisse Weil Λ-sheaf on some locally closed subscheme $Z_i \subset X_i$. Using the adjunction

$$(\iota_i)_! : \mathrm{D}_{\mathrm{cons}}(Z_i^{\mathrm{Weil}}) \rightleftarrows \mathrm{D}_{\mathrm{cons}}(X_i^{\mathrm{Weil}}) : (\iota_i)^!,$$

and the dualizability of lisse sheaves, we reduce to the case $M_i = \Lambda_{X_i}$, $i = 1, \dots, n$. That is, (5-8) becomes a map of cohomology complexes. By Proposition 4.4, we have

$$\mathrm{R}\Gamma(X_i^{\mathrm{Weil}}, N_i) = \mathrm{Fib}(\mathrm{R}\Gamma(X_{i,\mathbb{F}}, N_i) \xrightarrow{\phi_{X_i}^* - \mathrm{id}} \mathrm{R}\Gamma(X_{i,\mathbb{F}}, N_i)). \tag{5-9}$$

A similar computation holds for the mapping complexes in $\mathrm{D}(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}})$; see (4-6). Such finite limits commute with the tensor product in Mod_Λ. Thus, (5-9) reduces to the Künneth formula

$$\mathrm{R}\Gamma(X_{1,\mathbb{F}}, N_1) \otimes \cdots \otimes \mathrm{R}\Gamma(X_{n,\mathbb{F}}, N_n) \xrightarrow{\cong} \mathrm{R}\Gamma(X_{1,\mathbb{F}} \times_{\mathbb{F}} \cdots \times_{\mathbb{F}} X_{n,\mathbb{F}}, N_1 \boxtimes \cdots \boxtimes N_n),$$

where we use that the X_i are of finite type and the coprimality assumptions on Λ; see [Stacks 2017, Tag 0F1P]. □

Next, we consider the p-torsion case.

Proposition 5.8. *Let $X_1, \dots, X_n$ and Λ be as in Situation 5.1(d). Then the functor* (5-1) *is fully faithful for* $\bullet = \mathrm{lis}$.

Proof. As in the proof of Proposition 5.7, we need to show that the map

$$\bigotimes_{i=1}^{n} \mathrm{R}\Gamma(X_i^{\mathrm{Weil}}, N_i) \to \mathrm{R}\Gamma(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, N_1 \boxtimes \cdots \boxtimes N_n) \tag{5-10}$$

is an equivalence for any $N_i \in \mathrm{D}_{\mathrm{lis}}(X_i^{\mathrm{Weil}})$. Using Zariski descent for both sides, we may assume that each X_i is affine. As Λ is finite discrete (see also the discussion around (4-16)), the invariance of the étale site under perfection reduces us to the case where each X_i is perfect. The proof now proceeds by several reduction steps: (1) Reduce to $N_i = \Lambda_{X_i}$. (2) Reduce to $\Lambda = \mathbb{Z}/p$. (3) Reduce to $q = p$ being a prime. (4) The last step is then an easy computation.

Step 1 (*we may assume $N_i = \Lambda_{X_i}$*). In order to show (5-10) is a quasiisomorphism, it suffices to show this after applying $\tau^{\le r}$ for arbitrary r. The complexes N_i are bounded (Synopsis 3.2(v)). By shifting them appropriately, we may assume $r = 0$. Note that $\mathrm{R}\Gamma(X_i^{\mathrm{Weil}}, N_i) \cong \mathrm{R}\Gamma(X_i, N_i)$; see Proposition 4.16. By right exactness of the tensor product, we have $\tau^{\le 0}\big(\bigotimes_i \mathrm{R}\Gamma(X_i, N_i)\big) = \bigotimes_i \tau^{\le 0}\mathrm{R}\Gamma(X_i, N_i)$. By the comparison with the classical notion of constructible sheaves (for discrete coefficients, see [Hemo et al. 2023, Proposition 7.1] and the discussion preceding it), there is an étale covering $U_i \to X_i$ such that $N_i|_{U_i}$ is perfect-constant. Let $U_{i,\bullet}$ be the Čech nerve of this covering. By étale descent, we have

$$\mathrm{R}\Gamma(X_i, N_i) = \lim_{[j]\in\Delta} \mathrm{R}\Gamma(U_{i,j}, N_i).$$

For each $r \in \mathbb{Z}$, there is some j_r such that

$$\tau^{\leq r} \lim_{[j]\in\Delta} \mathrm{R}\Gamma(U_{i,j}, N_i) = \lim_{[j]\in\Delta,\, j\leq j_r} \tau^{\leq r}\mathrm{R}\Gamma(U_{i,j}, N_i).$$

This can be seen from the spectral sequence (note that it is concentrated in degrees $j \geq 0$ and degrees $j' \geq r$ for some r, since the complexes N_i are bounded from below)

$$\mathrm{H}^{j'}(U_{i,j}, N_i) \Rightarrow \mathrm{H}^{j'+j} \lim_{j\in\Delta} \mathrm{R}\Gamma(U_{i,j}, N_i) = \mathrm{H}^{j'+j}(X_i, N_i).$$

As the tensor product in (5-10) commutes with *finite* limits, we may thus assume that each N_i is perfect-constant. Another dévissage reduces us to the case $N_i = \Lambda_{X_i}$, the constant sheaf itself.

Step 2 (*we may assume* $\Lambda = \mathbb{Z}/p$). By assumption, Λ is flat over $\mathbb{Z}/p^m$ for some $m \geq 1$. We immediately reduce to $\Lambda = \mathbb{Z}/p^m$. For any perfect affine scheme $X = \operatorname{Spec} R$ in characteristic $p > 0$, we claim that $\mathrm{R}\Gamma(X, \mathbb{Z}/p^m) \otimes_{\mathbb{Z}/p^m} \mathbb{Z}/p^r \cong \mathrm{R}\Gamma(X, \mathbb{Z}/p^r)$. Assuming the claim, we finish the reduction step by tensoring (5-10) with the short exact sequence of $\mathbb{Z}/p^m$-modules $0 \to \mathbb{Z}/p^{m-1} \to \mathbb{Z}/p^m \to \mathbb{Z}/p \to 0$, using that finite limits commutes with tensor products. It remains to prove the claim. The Artin–Schreier–Witt exact sequence of sheaves on $X_{\text{ét}}$ yields

$$\mathrm{R}\Gamma(X, \mathbb{Z}/p^m) = [W_m(R) \xrightarrow{F-\mathrm{id}} W_m(R)].$$

Now we use that $W_m(R) \otimes_{\mathbb{Z}/p^m} \mathbb{Z}/p^r \xrightarrow{\cong} W_r(R)$ compatibly with F, which holds since R is perfect. This shows the claim, and we have accomplished Step 2.

Step 3 (*we may assume* q *is prime*). Recall that $q = p^r$ is a prime power. In order to reduce to the case $r = 1$, let $X_i' := X_i$, but now regarded as a scheme over $\mathbb{F}_p$. We have $X'_{i,\mathbb{F}} = \bigsqcup_{i=1}^r X_{i,\mathbb{F}}$. The Galois group $\mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p)$ is generated by the p-Frobenius, which acts by permuting the components in this disjoint union. Thus, we have $\mathrm{D}((X_i')^{\mathrm{Weil}}) = \mathrm{D}(X_i^{\mathrm{Weil}})$. The same reasoning also applies to several factors X_i^{Weil}, so we may assume our ground field to be $\mathbb{F}_p$.

Step 4. Set $R := \bigotimes_{i,\mathbb{F}_p} R_i$, $R_{\mathbb{F}} := R \otimes_{\mathbb{F}_p} \mathbb{F}$. We write ϕ_i for the p-Frobenius on R_i and also for any map on a tensor product involving R_i, by taking the identity on the remaining tensor factors. By Artin–Schreier theory, we have

$$\mathrm{R}\Gamma(X_i^{\mathrm{Weil}}, \mathbb{Z}/p) \overset{*}{=} \mathrm{R}\Gamma(X_i, \mathbb{Z}/p) = [R_i \xrightarrow{\phi_i-\mathrm{id}} R_i],$$

$$\mathrm{R}\Gamma(X_{1,\mathbb{F}} \times_{\mathbb{F}} \cdots \times_{\mathbb{F}} X_{n,\mathbb{F}}, \mathbb{Z}/p) = [R_{\mathbb{F}} \xrightarrow{\phi-\mathrm{id}} R_{\mathbb{F}}],$$

where the equality $*$ follows from Proposition 4.16 and ϕ is the absolute p-Frobenius of $R_{\mathbb{F}}$. Thus, the right hand side in (5-10) is the homotopy orbits of the action of $\mathbb{Z}^{n+1}$ on $R_{\mathbb{F}}$, whose basis vectors act as $\phi_1, \dots, \phi_n$ and ϕ. Note that ϕ is the composite $\phi_{\mathbb{F}} \circ \phi_1 \circ \dots \circ \phi_n$, where $\phi_{\mathbb{F}}$ is the Frobenius on $\mathbb{F}$. Thus, the previously mentioned $\mathbb{Z}^{n+1}$-action on $R_{\mathbb{F}}$ is equivalent to the one where the basis vectors act as $\phi_1, \dots, \phi_n$ and $\phi_{\mathbb{F}}$. We conclude our claim by using that $[R_{\mathbb{F}} \xrightarrow{\mathrm{id} - \phi_{\mathbb{F}}} R_{\mathbb{F}}]$ is quasiisomorphic to $R[0]$. □

5C. *Drinfeld's lemma.* The essential surjectivity in Theorem 5.2 is based on the following variant of Drinfeld's lemma [1980, Theorem 2.1]; see also [Lafforgue 1997, IV.2, Theorem 4; 2018, Lemme 8.11; Lau 2004, Theorem 8.1.4; Kedlaya 2019, Theorem 4.2.12; Heinloth 2018, Lemma 6.3; Scholze and Weinstein 2020, Theorem 16.2.4] for expositions. Its formulation is close to [Lau 2004, Theorem 8.1.4], and in this form is a slight extension of [Lafforgue 2018, Lemme 8.2] for $\mathbb{Z}_\ell$-coefficients and [Xue 2020b, Lemma 3.3.2] for $\mathbb{Q}_\ell$-coefficients. We will drop the coefficient ring Λ from the notation whenever convenient.

Let $X_1, \dots, X_n$ be Noetherian schemes over $\mathbb{F}_q$, and denote $X = X_1 \times_{\mathbb{F}_q} \dots \times_{\mathbb{F}_q} X_n$. Recall the Frobenius–Weil groupoid $\mathrm{FWeil}(X)$, see Definition 4.12. The projections $X_{\mathbb{F}} \to X_{i,\mathbb{F}}$ onto the single factors induce a continuous map of locally profinite groupoids

$$\mu\colon \mathrm{FWeil}(X) \to \mathrm{Weil}(X_1) \times \dots \times \mathrm{Weil}(X_n). \tag{5-11}$$

Theorem 5.9 (version of Drinfelds's lemma). *Let Λ be as in Situation 5.1. Restriction along the map* (5-11) *induces an equivalence*

$$\mathrm{Rep}_\Lambda(\mathrm{Weil}(X_1) \times \dots \times \mathrm{Weil}(X_n)) \xrightarrow{\cong} \mathrm{Rep}_\Lambda(\mathrm{FWeil}(X)), \tag{5-12}$$

between the abelian categories of continuous representations on finitely presented Λ-modules.

Proof. For all objects $x \in \mathrm{FWeil}(X)$, that is, all geometric points $x \to X_{\mathbb{F}}$, passing to the automorphism groups induces a commutative diagram of locally profinite groups:

$$\begin{array}{ccccccc}
1 & \longrightarrow & \pi_1(X_{\mathbb{F}}, x) & \longrightarrow & \mathrm{FWeil}(X, x) & \longrightarrow & \mathbb{Z}^n \\
 & & \downarrow & & \downarrow{\scriptstyle \mu_x} & & \| \\
1 & \longrightarrow & \prod_{i=1}^n \pi_1(X_{i,\mathbb{F}}, x) & \longrightarrow & \prod_{i=1}^n \mathrm{Weil}(X_i, x) & \longrightarrow & \mathbb{Z}^n
\end{array}$$

The left vertical arrow is surjective [Stacks 2017, Tags 0BN6, 0385]. Thus μ_x is surjective as well and hence (5-12) is fully faithful. For essential surjectivity, it remains to show that any continuous representation $\mathrm{FWeil}(X, x) \to \mathrm{GL}(M)$ on a finitely presented Λ-module M factors through μ_x. The key input is Drinfeld's lemma: it implies that μ_x induces an isomorphism on profinite completions. Therefore, it is enough to apply Lemma 5.10 below with $H := \mathrm{FWeil}(X, x) \to \mathrm{Weil}(X_1) \times \dots \times \mathrm{Weil}(X_n) =: G$ and $K := \pi_1(X_{\mathbb{F}}, x)$. This completes the proof of (5-12). □

The following lemma formalizes a few arguments from [Xue 2020b, Section 3.2.3], and we reproduce the proof for the convenience of the reader.

Lemma 5.10 (Drinfeld, Xue). *Let Λ be as in Situation 5.1. Let $\mu\colon H \to G$ be a continuous surjection of locally profinite groups that induces an isomorphism on profinite completions. Assume that there exists a compact open normal subgroup $K \subset H$ containing* $\ker\mu$ *such that H/K is finitely generated and injects into its profinite completion. Then μ induces an equivalence*

$$\mathrm{Rep}_\Lambda(G) \cong \mathrm{Rep}_\Lambda(H)$$

between their categories of continuous representations on finitely presented Λ-modules.

Proof. The case where Λ is finite discrete is obvious, and hence so is the case $\Lambda = \mathcal{O}_E$ for some finite field extension $E \supset \mathbb{Q}_\ell$. The case $\Lambda = E$ is reduced to $\Lambda = \mathbb{Q}_\ell$. As μ is surjective, it remains to show that every continuous representation $\rho\colon H \to \mathrm{GL}(M)$ on a finite-dimensional $\mathbb{Q}_\ell$-vector space factors through G, that is, $\ker\mu \subset \ker\rho$. One shows the following properties:

(1) The group $\ker\mu$ is the intersection over all open subgroups in K which are normal in H.

(2) The group $\ker\rho \cap K$ is a closed normal subgroup in H such that $K/\ker\rho\cap K \cong \rho(K)$ is topologically finitely generated.

These properties imply $\ker\mu \subset \ker\rho \cap K$ as follows: For a finite group L, let $U_L := \cap\ker(K \to L)$ where the intersection is over all continuous morphisms $K \to L$ that are trivial on $\ker\rho \cap K$. Because of the topologically finitely generatedness in (2), this is a finite intersection so that U_L is open in K. Also, it is normal in H, and hence $\ker\mu \subset U_L$ by (1). On the other hand, it is evident that $\ker\rho \cap K = \bigcap_L U_L$ because K is profinite.

For the proof of (1) observe that $\ker\mu$ agrees with the kernel of $H \to H^\wedge \cong G^\wedge$ by our assumption on the profinite completions. Using $\ker\mu \subset K$ and the injection $H/K \to (H/K)^\wedge$ implies (1).

For (2) it is evident that $\ker\rho \cap K$ is a closed normal subgroup in H. Since K is compact, its image $\rho(K)$ is a closed subgroup of the ℓ-adic Lie group $\mathrm{GL}(M)$, hence an ℓ-adic Lie group itself. The final assertion follows from [Serre 1966, Théorème 2]. □

For the overall goal of proving essential surjectivity in Theorem 5.2, we need to investigate how representations of product groups factorize into external tensor products of representations. In view of Lemma 4.13 and its proof, it is enough to consider representations of abstract groups, disregarding the topology. This is done in the next section.

5D. ***Factorizing representations.*** In this subsection, let Λ be a Dedekind domain [Stacks 2017, Tag 034X]. Thus, any submodule N of a finite projective Λ-module M is again finite projective.

Given any group W, we write $\mathrm{Rep}^{\mathrm{fp}}_\Lambda(W)$ for the category of W-representations on finite projective Λ-modules. As in [Curtis and Reiner 1962, Sections 73.8 and 75], we say that such a W-representation M is *fp-simple* if any subrepresentation $0 \neq N \subset M$ has maximal rank. By induction on the rank, every nonzero representation in $\mathrm{Rep}^{\mathrm{fp}}_\Lambda(W)$ admits a nonzero fp-simple subrepresentation. The proof of the following lemma is left to the reader. It parallels [loc. cit., Theorem 75.6].

Lemma 5.11. *A representation $M \in \mathrm{Rep}_\Lambda^{\mathrm{fp}}(W)$ is fp-simple if and only if $M \otimes_\Lambda \mathrm{Frac}(\Lambda)$ is fp-simple* (*hence*, *simple*).

The following proposition will serve in the proof of Theorem 5.2 using Theorem 5.9, where we will need to decompose representations of a product of Weil groups into decompositions of the individual Weil groups.

Proposition 5.12. *Let $W = W_1 \times W_2$ be a product of two groups. Let $M \in \mathrm{Rep}_\Lambda^{\mathrm{fp}}(W)$ be fp-simple. Fix a W_1-subrepresentation $M_1 \subset M$ that is fp-simple. Consider the W_2-representation $M_2 := \mathrm{Hom}_{W_1}(M_1, M)$ and the associated evaluation map*

$$\mathrm{ev}\colon M_1 \boxtimes M_2 \to M.$$

(1) *If Λ is an algebraically closed field*, *then* ev *is an isomorphism and M_2 is simple.*

(2) *If Λ is a perfect field*, *then* ev *is a split surjection and M_2 is semisimple.*

(3) *If Λ is a Dedekind domain of Krull dimension* 1 *with perfect fraction field*, *then there is a short exact sequence*

$$0 \to M \oplus \ker \mathrm{ev} \to M_1 \boxtimes M_2 \to T \to 0, \tag{5-13}$$

where T is Λ-torsion.

Proof. Note that ev is a map in $\mathrm{Rep}_\Lambda^{\mathrm{fp}}(W)$. Its image has maximal rank by the fp-simplicity of M. Thus, if Λ is a field, then it is surjective.

In case (1), we claim that ev is an isomorphism. The following argument was explained to us by Jean-François Dat: For injectivity, observe that $M_1 \boxtimes M_2 = M_1^{\oplus \dim M_2}$ as W_1-representations. Hence, if the kernel of ev is nontrivial, then it contains M_1 as an irreducible constituent. Therefore, it suffices to prove that $\mathrm{Hom}_{W_1}(M_1, \mathrm{ev})$ is injective. Since Λ is algebraically closed, we have $\mathrm{End}_{W_1}(M_1) = \Lambda$ by Schur's lemma. Hence, the composition

$$M_2 = \mathrm{Hom}_{W_1}(\mathrm{End}_{W_1}(M_1), M_2) \cong \mathrm{Hom}_{W_1}(M_1, M_1 \boxtimes M_2) \to \mathrm{Hom}_{W_1}(M_1, M) = M_2$$

is the identity. This shows that $\mathrm{Hom}_{W_1}(M_1, \mathrm{ev})$ is an isomorphism.

In case (2), we claim that $M_1 \boxtimes M_2$ is semisimple, and hence that M appears as a direct summand. Using [Bourbaki 2012, Section 13.4 Corollaire] applied to the group algebras it is enough to show that M_1 and M_2 are absolutely semisimple. Since Λ is perfect, any finite-dimensional representation is semisimple if and only if it is absolutely semisimple; see [loc. cit., Section 13.1]. Hence, it remains to check that $M_{2,\overline{\Lambda}} = M_2 \otimes_\Lambda \overline{\Lambda}$ is semisimple where $\overline{\Lambda}/\Lambda$ is an algebraic closure. The module $M_{2,\overline{\Lambda}} = \mathrm{Hom}_{W_1}(M_{1,\overline{\Lambda}}, M_{\overline{\Lambda}})$ splits as a direct sum according to the simple constituents $\overline{M}_1 \subset M_{1,\overline{\Lambda}}$ and $\overline{M} \subset M_{\overline{\Lambda}}$. Finally, each $\overline{M}_2 = \mathrm{Hom}_{W_1}(\overline{M}_1, \overline{M})$ is either simple or vanishes: If there exists a nonzero W_1-equivariant map $\overline{M}_1 \to \overline{M}$, then it must be injective by the simplicity of $\overline{M}_1$. As $\overline{\Lambda}$ is algebraically closed, the proof of (1) shows that $\overline{M} \cong \overline{M}_1 \boxtimes \overline{M}_2$ so that $\overline{M}_2$ must be simple because $\overline{M}$ is so. This shows that M_2 is absolutely semisimple as well.

In case (3), abbreviate $\Lambda' := \mathrm{Frac}\,\Lambda$, $M' := M \otimes_\Lambda \Lambda'$ and so on. We will repeatedly use that $(-)\otimes_\Lambda \Lambda'$ preserves and detects fp-simplicity of representations, see Lemma 5.11. By (2), the evaluation map $\mathrm{ev}' := \mathrm{ev} \otimes \Lambda'$ admits a Λ'-linear section $\tilde{i}\colon M' \to (M_1 \boxtimes M_2)'$. As M' is finitely presented, there is some $0 \neq \lambda \in \Lambda$ such that $\lambda\tilde{i}$ arises by scalar extension of a map $i\colon M \to M_1 \boxtimes M_2$. By construction, the map $i \oplus \mathrm{incl}\colon M \oplus \ker(\mathrm{ev}) \to M_1 \boxtimes M_2$ is an isomorphism after tensoring with Λ'. So its cokernel is Λ-torsion, and it is injective as both modules at the left are projective (hence Λ-torsion free). This finishes the proof of the proposition. □

5E. *Essential surjectivity.* In this section, we prove the essential surjectivity asserted in Theorem 5.2. Throughout, we freely use the full faithfulness proven in Propositions 5.7 and 5.8.

Recall that $X_1, \dots, X_n$ are finite type $\mathbb{F}_q$-schemes, and write $X := X_1 \times_{\mathbb{F}_q} \cdots \times_{\mathbb{F}_q} X_n$. Let Λ be either a finite discrete ring, a finite field extension $E \supset \mathbb{Q}_\ell$ for $\ell \neq p$ or its ring of integers $\mathcal{O}_E$. Note that this covers all cases from Situation 5.1.

First, we show that it suffices to prove containment in the essential image étale locally.

Lemma 5.13. *Let $U_i \to X_i$ be quasicompact étale surjections for $i = 1, \dots, n$. Then the following properties hold*:

(1) *An object $M \in \mathrm{D}(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda)$ belongs to the full subcategory*

$$\mathrm{D}_{\mathrm{cons}}(X_1^{\mathrm{Weil}}, \Lambda) \otimes_{\mathrm{Perf}_{\Lambda_*}} \cdots \otimes_{\mathrm{Perf}_{\Lambda_*}} \mathrm{D}_{\mathrm{cons}}(X_n^{\mathrm{Weil}}, \Lambda)$$

if and only if its restriction $M|_{U_1^{\mathrm{Weil}} \times \cdots \times U_n^{\mathrm{Weil}}}$ belongs to the full subcategory

$$\mathrm{D}_{\mathrm{cons}}(U_1^{\mathrm{Weil}}, \Lambda) \otimes_{\mathrm{Perf}_{\Lambda_*}} \cdots \otimes_{\mathrm{Perf}_{\Lambda_*}} \mathrm{D}_{\mathrm{cons}}(U_n^{\mathrm{Weil}}, \Lambda) \subset \mathrm{D}(U_1^{\mathrm{Weil}} \times \cdots \times U_n^{\mathrm{Weil}}, \Lambda).$$

(2) *Assume that all $U_i \to X_i$ are finite étale. Then (1) holds for the categories of lisse sheaves.*

Proof. The only if direction in part (1) is clear. Conversely, assume that $M|_{U_1^{\mathrm{Weil}} \times \cdots \times U_n^{\mathrm{Weil}}}$ lies in the essential image of the external tensor product. By étale descent, we have an equivalence

$$\mathrm{D}(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda) \xrightarrow{\cong} \mathrm{Tot}(\mathrm{D}(U_{1,\bullet}^{\mathrm{Weil}} \times \cdots \times U_{n,\bullet}^{\mathrm{Weil}}, \Lambda)).$$

In particular, we get an equivalence $|(j_\bullet)_! \circ j_\bullet^* M| \xrightarrow{\sim} M$ where $j_\bullet := j_{1,\bullet} \times \cdots \times j_{n,\bullet}$ with $j_{i,\bullet}\colon U_{i,\bullet} \to X_i$ for $i = 1, \dots, n$. For each $m \geq 0$, the object $j_m^* M$ lies in

$$\mathrm{D}_{\mathrm{cons}}(U_{1,m}^{\mathrm{Weil}}, \Lambda) \otimes_{\mathrm{Perf}_{\Lambda_*}} \cdots \otimes_{\mathrm{Perf}_{\Lambda_*}} \mathrm{D}_{\mathrm{cons}}(U_{n,m}^{\mathrm{Weil}}, \Lambda).$$

It follows from Synopsis 3.2(ii) that these subcategories are preserved under $(j_m)_!$. So we see

$$(j_m)_! j_m^*(M) \in \mathrm{D}_{\mathrm{cons}}(X_1^{\mathrm{Weil}}, \Lambda) \otimes_{\mathrm{Perf}_{\Lambda_*}} \cdots \otimes_{\mathrm{Perf}_{\Lambda_*}} \mathrm{D}_{\mathrm{cons}}(X_n^{\mathrm{Weil}}, \Lambda)$$

for all $m \geq 0$. For every $m \geq 0$, let M_m denote the realization of the m-th skeleton of the simplicial object $(j_\bullet)_! \circ j_\bullet^* M$ so that we have a natural equivalence $\mathrm{colim}\, M_m \xrightarrow{\cong} M$ in $\mathrm{D}(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda)$. We claim that M is a retract of some M_m, and hence lies in $\mathrm{D}_{\mathrm{cons}}(X_1^{\mathrm{Weil}}, \Lambda) \otimes_{\mathrm{Perf}_{\Lambda_*}} \cdots \otimes_{\mathrm{Perf}_{\Lambda_*}} \mathrm{D}_{\mathrm{cons}}(X_n^{\mathrm{Weil}}, \Lambda)$ by idempotent completeness. To prove the claim, note that the sheaf $M_{\mathbb{F}} \in \mathrm{D}_{\mathrm{cons}}(X_{\mathbb{F}}, \Lambda)$ underlying M is compact in the category of ind-constructible sheaves $\mathrm{D}_{\mathrm{indcons}}(X_{\bar{\mathbb{F}}}, \Lambda)$, see Synopsis 3.2(viii). As taking partial Frobenius fixed points is a finite limit, so commutes with filtered colimits, we see that the natural map of mapping complexes

$$\operatorname{colim} \operatorname{Hom}_{\mathrm{D}(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda)}(M, M_m) \xrightarrow{\cong} \operatorname{Hom}_{\mathrm{D}(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda)}(M, \operatorname{colim} M_m)$$

is an equivalence. In particular, the inverse equivalence $M \xrightarrow{\cong} \operatorname{colim} M_m$ factors through some M_m, presenting M as a retract of M_m. This proves the claim, and hence (1).

For (2), note that if $U_i \to X_i$ are finite étale, then the functors $(j_m)_!$ preserve the lisse categories; see Synopsis 3.2(ii). In particular, for every $m \geq 0$ the object $(j_m)_! j_m^*(M)$ is lisse and so is M_m. We conclude using compactness as before. □

Using Lemma 2.4 and Synopsis 3.2(viii), the fully faithful functor (5-1) uniquely extends to a fully faithful functor

$$\operatorname{Ind}(\mathrm{D}_\bullet(X_1^{\mathrm{Weil}}, \Lambda)) \otimes_{\mathrm{Mod}_{\Lambda_*}} \cdots \otimes_{\mathrm{Mod}_{\Lambda_*}} \operatorname{Ind}(\mathrm{D}_\bullet(X_n^{\mathrm{Weil}}, \Lambda)) \to \mathrm{D}(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda) \tag{5-14}$$

for $\bullet \in \{\mathrm{lis}, \mathrm{cons}\}$. We use this in the following variant of Lemma 5.13.

Lemma 5.14. *The statements (1) and (2) of Lemma 5.13 hold for the functor* (5-14) *with* $\bullet \in \{\mathrm{lis}, \mathrm{cons}\}$. *Namely, to check that an object lies in the essential image of* (5-14), *one can pass to a quasicompact étale cover if* $\bullet = \mathrm{cons}$, *and to a finite étale cover if* $\bullet = \mathrm{lis}$.

Proof. This is immediate from the proof of Lemma 5.13: Arguing as above and using étale descent for ind-constructible, resp. ind-lisse sheaves (Synopsis 3.2(iii)), we see that $M \cong \operatorname{colim} M_m$ with

$$M_m \in \operatorname{Ind}(\mathrm{D}_\bullet(X_1^{\mathrm{Weil}}, \Lambda)) \otimes_{\mathrm{Mod}_{\Lambda_*}} \cdots \otimes_{\mathrm{Mod}_{\Lambda_*}} \operatorname{Ind}(\mathrm{D}_\bullet(X_n^{\mathrm{Weil}}, \Lambda))$$

for all $m \geq 0$ and $\bullet = \mathrm{cons}$, resp. $\bullet = \mathrm{lis}$. As the essential image of (5-14) is closed under colimits, M lies in the corresponding subcategory as well. □

Now we have enough tools to prove the categorical Künneth formula alias derived Drinfeld's lemma:

Proof of Theorem 5.2. In view of Propositions 5.7 and 5.8, it remains to show the essential surjectivity of the external tensor product functor on Weil sheaves (5-1) under the assumptions in Theorem 5.2. Part (1), the case of constructible sheaves, is reduced to part (2), the case of lisse sheaves, by taking a stratification as in Definition 4.10(2) and using the full faithfulness already proven. Here we note that by refining the stratification witnessing the constructibility if necessary, we can even assume all strata to be smooth, so in particular geometrically unibranch. Hence, it remains to prove part (2), that is, the essential surjectivity of the fully faithful functor

$$\boxtimes\colon \mathrm{D}_{\mathrm{lis}}(X_1^{\mathrm{Weil}}, \Lambda) \otimes_{\mathrm{Perf}_{\Lambda_*}} \cdots \otimes_{\mathrm{Perf}_{\Lambda_*}} \mathrm{D}_{\mathrm{lis}}(X_n^{\mathrm{Weil}}, \Lambda) \to \mathrm{D}_{\mathrm{lis}}(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda), \tag{5-15}$$

when either Λ is finite discrete as in cases (a), (d) in Theorem 5.2(2), or $\Lambda = \mathcal{O}_E$ for a finite field extension $E \supset \mathbb{Q}_\ell$, $\ell \neq p$ as in case (b), or $\Lambda = E$ and the X_i are geometrically unibranch as in the remaining case (c). In fact, the latter two cases are easier to handle due to the presence of natural t-structures on the categories of lisse sheaves (Synopsis 3.2(vi)). So we will distinguish two cases below: (1) $\Lambda = \mathcal{O}_E$, or $\Lambda = E$ and all X_i geometrically unibranch. (2) Λ is finite discrete.

Now pick $M \in \mathrm{D}_{\mathrm{lis}}(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda)$. By Synopsis 3.2(v), M is bounded in the standard t-structure on $\mathrm{D}(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda)$. So M is a successive extension of its cohomology sheaves $\mathrm{H}^j(M)$, $j \in \mathbb{Z}$. As M is lisse, Lemma 4.15(1) shows in both cases (1) and (2) that each $\mathrm{H}^j(M)$ comes from a continuous representation on a finitely presented Λ-module in

$$\mathrm{Rep}_\Lambda(\mathrm{FWeil}(X)) \cong \mathrm{Rep}_\Lambda(W), \tag{5-16}$$

where we denote $W := W_1 \times \cdots \times W_n$ with $W_i := \mathrm{Weil}(X_i)$ and the equivalence follows from Theorem 5.9.

Throughout, we repeatedly use that the functor (5-15) is fully faithful, commutes with finite (co-)limits and shifts, and that its essential image is closed under retracts (as the source category is idempotent complete, by definition) and contains all perfect-constant sheaves.

Case 1 (*assume $\Lambda = \mathcal{O}_E$, or $\Lambda = E$ and all X_i geometrically unibranch*). In this case, we have a t-structure on lisse Weil sheaves so that each $\mathrm{H}^j(M)$ belongs to $\mathrm{D}_{\mathrm{lis}}(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda)^\heartsuit$. By induction on the length of M, using the full faithfulness of (5-15), we reduce to the case where M is abelian, that is, a continuous W-representation on a finitely presented Λ-module. The external tensor product induces a commutative diagram:

$$\begin{array}{ccc} \mathrm{Rep}_\Lambda(W_1) \times \cdots \times \mathrm{Rep}_\Lambda(W_n) & \xrightarrow{\boxtimes} & \mathrm{Rep}_\Lambda(W) \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{D}_{\mathrm{lis}}(X_1^{\mathrm{Weil}}, \Lambda)^\heartsuit \times \cdots \times \mathrm{D}_{\mathrm{lis}}(X_n^{\mathrm{Weil}}, \Lambda)^\heartsuit & \xrightarrow{\boxtimes} & \mathrm{D}_{\mathrm{lis}}(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda)^\heartsuit \end{array}$$

where the vertical equivalences are given by Lemma 4.15. Note that M splits into a direct sum $M_{\mathrm{tor}} \oplus M_{\mathrm{fp}}$ where the finitely presented Λ-module underlying M_{tor} is Λ-torsion and M_{fp} is projective. So we can treat either case separately. Using that the essential image of (5-15) is closed under extensions (by full faithfulness) and retracts, the finite projective case is reduced to the fp-simple case and, by Proposition 5.12, to the finite torsion case. Note that the W_i-representations constructed in, say (5-13), are obtained from M_{fp} by taking subquotients and tensor products, so are automatically continuous. Next, as the Λ-module underlying M_{tor} is finite torsion, the Λ-sheaf M_{tor} is perfect-constant along some finite étale cover. So we conclude by Lemma 5.13(2).

Case 2 (*assume Λ is finite discrete as above*). In a nutshell, the argument is similar to the last step in Case 1, but a little more involved due to the absence of natural t-structures on the categories of lisse sheaves in general, see Synopsis 3.2(vi) and [Hemo et al. 2023, Remark 6.9]. More precisely, in the special case, where Λ is a finite field, the argument of case 1) applies, but not so if $\Lambda = \mathbb{Z}/\ell^2$, say. So, instead, we extend (5-15) by passing to Ind-completions to a commutative diagram:

$$\begin{array}{ccc}
\mathrm{D}_{\mathrm{lis}}(X_1^{\mathrm{Weil}}, \Lambda) \otimes_{\mathrm{Perf}_{\Lambda_*}} \cdots \otimes_{\mathrm{Perf}_{\Lambda_*}} \mathrm{D}_{\mathrm{lis}}(X_n^{\mathrm{Weil}}, \Lambda) & \xrightarrow{\boxtimes} & \mathrm{D}_{\mathrm{lis}}(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda) \\
\downarrow & & \downarrow \\
\mathrm{Ind}(\mathrm{D}_{\mathrm{lis}}(X_1^{\mathrm{Weil}}, \Lambda)) \otimes_{\mathrm{Mod}_{\Lambda_*}} \cdots \otimes_{\mathrm{Mod}_{\Lambda_*}} \mathrm{Ind}(\mathrm{D}_{\mathrm{lis}}(X_n^{\mathrm{Weil}}, \Lambda)) & \xrightarrow{\mathrm{Ind}(\boxtimes)} & \mathrm{Ind}(\mathrm{D}_{\mathrm{lis}}(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda))
\end{array}$$

of full subcategories of $\mathrm{D}(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda)$, see the discussion around (5-14). Note that the fully faithful embedding (5-14) factors through $\mathrm{Ind}(\boxtimes)$. Both vertical arrows are the inclusion of the subcategories of compact objects by idempotent completeness of the involved categories and (2-1). Thus, if M lies in the essential image of $\mathrm{Ind}(\boxtimes)$, then it is a retract of a finite colimit of objects in the essential image of $\boxtimes$, so lies itself in this essential image. As M is a successive extension of its cohomology sheaves $\mathrm{H}^j(M)$, it suffices to show

$$\mathrm{H}^j(M) \in \mathrm{Ind}(\mathrm{D}_{\mathrm{lis}}(X_1^{\mathrm{Weil}}, \Lambda)) \otimes_{\mathrm{Mod}_{\Lambda_*}} \cdots \otimes_{\mathrm{Mod}_{\Lambda_*}} \mathrm{Ind}(\mathrm{D}_{\mathrm{lis}}(X_n^{\mathrm{Weil}}, \Lambda)),$$

for all $j \in \mathbb{Z}$. So fix j and denote $N := \mathrm{H}^j(M)$ viewed as a continuous W-representation on a finitely presented Λ-module. As Λ is finite, N comes from a continuous representation of $\pi_1(X_1) \times \cdots \times \pi_1(X_n)$ on which some open subgroup acts trivially. Hence, there exist finite étale surjections $U_i \to X_i$ such that the subgroup $\pi_1(U_1) \times \cdots \times \pi_1(U_n)$ acts trivially on N. In particular, $N|_{U_1^{\mathrm{Weil}} \times \cdots \times U_n^{\mathrm{Weil}}}$ is constant, and hence lies in the essential image of the functor

$$\mathrm{Mod}_R \cong \mathrm{Ind}(\mathrm{Perf}_R) \to \mathrm{Ind}(\mathrm{D}_{\mathrm{lis}}(U_1^{\mathrm{Weil}} \times \cdots \times U_n^{\mathrm{Weil}}, \Lambda)),$$

where $R := \Gamma(\pi_0(U_1) \times \cdots \times \pi_0(U_n), \Lambda)$. As the sets $\pi_0(U_i)$ are finite discrete, each $R_i := \Gamma(\pi_0(U_i), \Lambda)$ is a finite free Λ_*-algebra, and we have $R \cong R_1 \otimes_{\Lambda_*} \cdots \otimes_{\Lambda_*} R_n$. Thus, the external tensor product induces a commutative diagram:

$$\begin{array}{ccc}
\mathrm{Mod}_{R_1} \otimes_{\mathrm{Mod}_{\Lambda_*}} \cdots \otimes_{\mathrm{Mod}_{\Lambda_*}} \mathrm{Mod}_{R_n} & \xrightarrow{\cong} & \mathrm{Mod}_R \\
\downarrow & & \downarrow \\
\mathrm{Ind}(\mathrm{D}_{\mathrm{lis}}(U_1^{\mathrm{Weil}}, \Lambda)) \otimes_{\mathrm{Mod}_{\Lambda_*}} \cdots \otimes_{\mathrm{Mod}_{\Lambda_*}} \mathrm{Ind}(\mathrm{D}_{\mathrm{lis}}(U_n^{\mathrm{Weil}}, \Lambda)) & \xrightarrow{\mathrm{Ind}(\boxtimes)} & \mathrm{Ind}(\mathrm{D}_{\mathrm{lis}}(U_1^{\mathrm{Weil}} \times \cdots \times U_n^{\mathrm{Weil}}, \Lambda))
\end{array}$$

where the upper horizontal arrow is an equivalence. So $N|_{U_1^{\mathrm{Weil}} \times \cdots \times U_n^{\mathrm{Weil}}}$ lies in the essential image of $\mathrm{Ind}(\boxtimes)$, and we conclude by Lemma 5.14 applied to the finite étale covers $U_i \to X_i$ and $\bullet = \mathrm{lis}$. □

6. Ind-constructible Weil sheaves

In this section, we introduce the full subcategories

$$\mathrm{D}_{\mathrm{indlis}}(X^{\mathrm{Weil}}, \Lambda) \subset \mathrm{D}_{\mathrm{indcons}}(X^{\mathrm{Weil}}, \Lambda)$$

of $\mathrm{D}(X^{\mathrm{Weil}}, \Lambda)$ consisting of ind-objects of lisse, resp. constructible sheaves equipped with partial Frobenius action. That is, the partial Frobenius only preserves the ind-system of objects, but not necessarily each member. We will define analogous categories for a product of schemes. Similarly to the lisse, resp. constructible case, there is a fully faithful functor

$$\mathrm{D}_{\mathrm{indcons}}(X_1^{\mathrm{Weil}}, \Lambda) \otimes_{\mathrm{Mod}_{\Lambda_*}} \cdots \otimes_{\mathrm{Mod}_{\Lambda_*}} \mathrm{D}_{\mathrm{indcons}}(X_n^{\mathrm{Weil}}, \Lambda) \to \mathrm{D}_{\mathrm{indcons}}(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda),$$

which, however, will not be an equivalence in general; see Remark 6.6. Nevertheless, we can identify a class of objects that lie in the essential image and that include many cases of interest such as the shtuka cohomology studied in [Lafforgue 2018; Lafforgue and Zhu 2019; Xue 2020b; 2020c].

6A. ***Ind-constructible Weil sheaves.*** Let $\mathbb{F}_q$ be a finite field of characteristic $p > 0$, and fix an algebraic closure $\mathbb{F}$. Let $X_1, \dots, X_n$ be schemes of finite type over $\mathbb{F}_q$. Let Λ be a condensed ring associated with the one of the following topological rings: a discrete coherent torsion ring (for example, a discrete finite ring), an algebraic field extension $E \supset \mathbb{Q}_\ell$, or its ring of integers $\mathcal{O}_E$. We write $X := X_1 \times_{\mathbb{F}_q} \cdots \times_{\mathbb{F}_q} X_n$, and denote by $X_{i,\mathbb{F}} := X_i \times_{\mathbb{F}_q} \operatorname{Spec} \mathbb{F}$ and $X_{\mathbb{F}} := X \times_{\mathbb{F}_q} \operatorname{Spec} \mathbb{F}$ the base change. Recall that under these assumptions, by Synopsis 3.2(viii), we have a fully faithful embedding

$$\mathrm{Ind}(\mathrm{D}_{\mathrm{cons}}(X_{\mathbb{F}}, \Lambda)) \xrightarrow{\cong} \mathrm{D}_{\mathrm{indcons}}(X_{\mathbb{F}}, \Lambda) \subset \mathrm{D}(X_{\mathbb{F}}, \Lambda), \tag{6-1}$$

and likewise for (ind-)lisse sheaves.

Definition 6.1. An object $M \in \mathrm{D}(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda)$ is called *ind-lisse*, resp. *ind-constructible* if the underlying sheaf $M_{\mathbb{F}} \in \mathrm{D}(X_{\mathbb{F}}, \Lambda)$ is ind-lisse, resp. ind-constructible in the sense of Definition 3.1.

We denote by

$$\mathrm{D}_{\mathrm{indlis}}(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda) \subset \mathrm{D}_{\mathrm{indcons}}(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda)$$

the resulting full subcategories of $\mathrm{D}(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda)$ consisting of ind-lisse, resp. ind-constructible objects. Both categories are naturally commutative algebra objects in $\mathrm{Pr}^{\mathrm{St}}_{\Lambda_*}$ (see the notation from Section 2), that is, presentable stable Λ_*-linear symmetric monoidal ∞-categories where $\Lambda_* := \Gamma(*, \Lambda)$ is the ring underlying Λ. It is immediate from Definition 6.1 that the equivalence (4-6) restricts to an equivalence

$$\mathrm{D}_{\bullet}(X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}, \Lambda) \cong \mathrm{Fix}(\mathrm{D}_{\bullet}(X_{\mathbb{F}}, \Lambda), \phi^*_{X_1}, \dots, \phi^*_{X_n})$$

for $\bullet \in \{\mathrm{indlis}, \mathrm{indcons}\}$.

Remark 6.2. Note that we have a fully faithful embedding of $D_{\mathrm{cons}}(X^{\mathrm{Weil}})$ into $\mathrm{D}_{\mathrm{indcons}}(X^{\mathrm{Weil}})$ whose image consists of compact objects. However, the latter category is not generated by this image. Indeed, even in the case of a point, the ind-cons category consists of Λ-modules with an action of an automorphism. This automorphism does not have to fix any finitely generated submodule, which would be the case for any objects generated by the image of the constructible Weil complexes.

Our goal in this section is to obtain a categorical Künneth formula for the categories of ind-lisse, resp. ind-constructible Weil sheaves. In order to state the result, we need the following terminology. Under our assumptions on Λ, each cohomology sheaf $\mathrm{H}^j(M)$, $j \in \mathbb{Z}$ for $M \in \mathrm{D}_{\mathrm{lis}}(X_{\mathbb{F}}, \Lambda)$ is naturally a continuous representation of the proétale fundamental groupoid $\pi_1^{\text{proét}}(X_{\mathbb{F}})$ on a finitely presented Λ-module; see Lemma 4.15. Further, the projections $X_{\mathbb{F}} \to X_{i,\mathbb{F}}$ induce a full surjective map of topological groupoids

$$\pi_1^{\text{proét}}(X_{\mathbb{F}}) \to \pi_1^{\text{proét}}(X_{1,\mathbb{F}}) \times \cdots \times \pi_1^{\text{proét}}(X_{n,\mathbb{F}}). \tag{6-2}$$

Definition 6.3. Let $M \in \mathrm{D}(X_{\mathbb{F}}, \Lambda)$:

(1) The sheaf M is called *split lisse* if it is lisse and the action of $\pi_1^{\text{proét}}(X_{\mathbb{F}})$ on $\mathrm{H}^j(M)$ factors through (6-2) for all $j \in \mathbb{Z}$.

(2) The sheaf M is called *split constructible* if it is constructible and there exists a finite subdivision into locally closed subschemes $X_{i,\alpha} \subseteq X_i$ such that for each $X_\alpha = \prod_i X_{i,\alpha} \subseteq X$, each restriction $M|_{X_\alpha}$ is split lisse.

Definition 6.4. An object $M \in \mathrm{D}(X_1^{\text{Weil}} \times \cdots \times X_n^{\text{Weil}}, \Lambda)$ is called *ind-*(*split lisse*), resp. *ind-*(*split constructible*) if the underlying object $M_{\mathbb{F}} \in \mathrm{D}(X_{\mathbb{F}}, \Lambda)$ is a colimit of split lisse, resp. split constructible objects.

As the category $\mathrm{D}_\bullet(X_{\mathbb{F}}, \Lambda)$, $\bullet \in \{\text{indlis}, \text{indcons}\}$ is cocomplete, every ind-(split lisse) object is ind-lisse, and likewise, every ind-(split constructible) object is ind-constructible.

Theorem 6.5. *Assume that Λ is either a finite discrete ring of prime-to-p torsion, an algebraic field extension $E \supset \mathbb{Q}_\ell$ for $\ell \neq p$, or its ring of integers $\mathcal{O}_E$. Then the functor induced by the external tensor product*

$$\mathrm{D}_\bullet(X_1^{\text{Weil}}, \Lambda) \otimes_{\mathrm{Mod}_{\Lambda_*}} \cdots \otimes_{\mathrm{Mod}_{\Lambda_*}} \mathrm{D}_\bullet(X_n^{\text{Weil}}, \Lambda) \to \mathrm{D}_\bullet(X_1^{\text{Weil}} \times \cdots \times X_n^{\text{Weil}}, \Lambda) \tag{6-3}$$

is fully faithful for $\bullet \in \{\text{indlis}, \text{indcons}\}$. *For* $\bullet = \text{indlis}$, *resp.* $\bullet = \text{indcons}$ *the essential image contains the ind-*(*split lisse*), *resp. ind-*(*split constructible*) *objects.*

Proof. For full faithfulness, it is enough to consider the case $\bullet = \text{indcons}$. Using Lemma 2.5, it remains to show that the functor

$$\bigotimes_i \mathrm{D}_{\text{indcons}}(X_{i,\mathbb{F}},) \cong \mathrm{Ind}\Big(\bigotimes_i \mathrm{D}_{\text{cons}}(X_{i,\mathbb{F}}, \Lambda)\Big) \to \mathrm{D}_\bullet(X_{\mathbb{F}}, \Lambda). \tag{6-4}$$

is fully faithful. In view of (6-1), this is immediate from the Künneth formula for constructible Λ-sheaves as explained in Section 5B.

To identify objects in the essential image, we note that the fully faithful functors (6-3) and (6-4) induce a Cartesian diagram (see Lemma 2.5):

$$\begin{array}{ccc} \bigotimes_i \mathrm{D}_\bullet(X_i^{\text{Weil}},) & \longrightarrow & \mathrm{D}_\bullet(X_1^{\text{Weil}} \times \cdots \times X_n^{\text{Weil}}, \Lambda) \\ \downarrow & & \downarrow \\ \bigotimes_i \mathrm{D}_\bullet(X_{i,\mathbb{F}}, \Lambda) & \longrightarrow & \mathrm{D}_\bullet(X_{\mathbb{F}}, \Lambda) \end{array} \tag{6-5}$$

for $\bullet \in \{\text{indlis}, \text{indcons}\}$. Thus, it is enough to show that the object $M_{\mathbb{F}}$ underlying an ind-split object M lies in the image of the lower horizontal arrow. Since this essential image is closed under colimits, it remains to show it contains the split lisse objects for $\bullet =$ indlis, resp. the split constructible objects for $\bullet =$ indcons.

By the full faithfulness of (6-4), the split constructible case reduces to the split lisse case, see also the proof of Theorem 5.2 in Section 5E. So assume $\bullet =$ indlis and let $M_{\mathbb{F}} \in \mathrm{D}(X_{\mathbb{F}}, \Lambda)$ be split lisse. As each cohomology sheaf $\mathrm{H}^j(M_{\mathbb{F}})$, $j \in \mathbb{Z}$ is at least ind-lisse (see also [Hemo et al. 2023, Remark 8.4]), an induction on the cohomological length of $M_{\mathbb{F}}$ reduces us to show that $\mathrm{H}^j(M_{\mathbb{F}})$ lies in the essential image. By definition, being split lisse implies that the action of $\pi_1^{\text{proét}}(X_{\mathbb{F}})$ on $\mathrm{H}^j(M_{\mathbb{F}})$ factors through $\pi_1^{\text{proét}}(X_{1,\mathbb{F}}) \times \cdots \times \pi_1^{\text{proét}}(X_{n,\mathbb{F}})$. Then the arguments of Section 5E show that $\mathrm{H}^j(M_{\mathbb{F}})$ lies is in the essential image of the lower horizontal arrow in (6-5). We leave the details to the reader. □

Remark 6.6. The functor (6-3) is not essentially surjective in general. To see this, note that the functor $\mathrm{D}_{\text{indcons}}(X^{\text{Weil}}, \Lambda) \to \mathrm{D}_{\text{indcons}}(X_{\mathbb{F}}, \Lambda)$ admits a left adjoint F that adds a free partial Frobenius action. Explicitly, for an object $M \in \mathrm{D}_{\text{indcons}}(X_{\mathbb{F}}, \Lambda)$ the object $F(M)$ has underlying sheaf $F(M)_{\mathbb{F}}$ given by a countable direct sum of copies of M. If M was not originally in the image of the external tensor product (for example, M as in Example 1.4), then $F(M)$ will not be either. This is, however, the only obstacle for essential surjectivity: as noted in the proof of Theorem 6.5, the diagram (6-5) is Cartesian.

6B. ***Cohomology of shtuka spaces.*** Finally, let us mention a key application of Theorem 6.5. Let X be a smooth projective geometrically connected curve over $\mathbb{F}_q$. Let $N \subset X$ be a finite subscheme, and denote its complement by $Y = X \backslash N$. Let $E \supset \mathbb{Q}_\ell$, $\ell \neq p$ be an algebraic field extension containing a fixed square root of q. Let $\mathcal{O}_E$ be its ring of integers and denote by k_E the residue field. Let Λ be any of the topological rings $E, \mathcal{O}_E, k_E$. Let G be a split (for simplicity) reductive group over $\mathbb{F}_q$. We denote by $\widehat{G}$ the Langlands dual group of G considered as a split reductive group over Λ.

In the seminal works [Drinfeld 1980; Lafforgue 2002] ($G = \mathrm{GL}_n$) and [Lafforgue 2018; Lafforgue and Zhu 2019] (general reductive G) on the Langlands correspondence over global function fields, the construction of the $\mathrm{Weil}(Y)$-action on automorphic forms of level N is realized using the cohomology sheaves of moduli stacks of shtukas, defined in [Varshavsky 2004] and [Lafforgue 2018, Section 2]. As explained in [Lafforgue and Zhu 2019; Gaitsgory et al. 2022; Zhu 2021], the output of the geometric construction of Lafforgue can be encoded as a natural transformation

$$\mathrm{H}_{N,I}\colon\ \mathrm{Rep}_\Lambda^{\text{fp}}(\widehat{G}^I) \to \mathrm{Rep}_\Lambda^{\text{cts}}(\mathrm{Weil}(Y)^I), \qquad I \in \mathrm{FinSet} \tag{6-6}$$

of functors $\mathrm{FinSet} \to \mathrm{Cat}$ from the category of finite sets to the category of 1-categories. Here the functor $\mathrm{Rep}_\Lambda^{\text{fp}}(\widehat{G}^\bullet)$ assigns to a finite set I the category of algebraic representations of $\widehat{G}^I$ on finite free Λ-modules, and $\mathrm{Rep}_\Lambda^{\text{cts}}(\mathrm{Weil}(Y)^\bullet)$ the category of continuous representations of $\mathrm{Weil}(Y)^I$ in Λ-modules. In both cases, the transition maps are given by restriction of representations.

Let us recall some elements of its construction. For a finite set I, [Varshavsky 2004] and [Lafforgue 2018, Section 2] define the ind-algebraic stack $\mathrm{Cht}_{N,I}$ classifying I-legged G-shtukas on X with full level-N structure. The morphism sending a G-shtuka to its legs

$$\mathfrak{p}_{N,I}\colon\ \mathrm{Cht}_{N,I} \to Y^I, \tag{6-7}$$

is locally of finite presentation. For every $W \in \mathrm{Rep}^{\mathrm{fp}}_{\Lambda}(\widehat{G}^I)$, there is the normalized Satake sheaf $\mathcal{F}_{N,I,W}$ on $\mathrm{Cht}_{N,I}$; see [Lafforgue 2018, Définition 2.14]. Base changing to $\mathbb{F}$ and taking compactly supported cohomology, we obtain the object

$$\mathcal{H}_{N,I}(W) \overset{\text{def}}{=} (\mathfrak{p}_{N,I,\mathbb{F}})_!(\mathcal{F}_{N,I,W,\mathbb{F}}) \in \mathrm{D}_{\mathrm{indcons}}(Y^I_{\mathbb{F}}, \Lambda);$$

see [Lafforgue 2018, Définition 4.7] and [Xue 2020a, Definition 2.5.1]. Under the normalization of the Satake sheaves, the degree 0 cohomology sheaf

$$\mathrm{H}_{N,I}(W) \overset{\text{def}}{=} \mathrm{H}^0(\mathcal{H}_I(W)) \in \mathrm{D}_{\mathrm{indcons}}(Y^I_{\mathbb{F}}, \Lambda)^{\heartsuit}$$

corresponds to the middle degree compactly supported intersection cohomology of $\mathrm{Cht}_{N,I}$. Using the symmetries of the moduli stacks of shtukas, the sheaf $\mathrm{H}_{N,I}(W)$ is endowed with a partial Frobenius equivariant structure [Lafforgue 2002, Section 6]. So we obtain objects

$$\mathrm{H}_{N,I}(W) \in \mathrm{D}_{\mathrm{indcons}}((Y^{\mathrm{Weil}})^I, \Lambda)^{\heartsuit}. \tag{6-8}$$

Next, using the finiteness [Xue 2020b] and smoothness [Xue 2020c, Theorem 4.2.3] results, the classical Drinfeld's lemma (Theorem 5.9) applies to give objects $\mathrm{H}_{N,I}(W) \in \mathrm{Rep}^{\mathrm{cts}}_{\Lambda}(\mathrm{Weil}(Y)^I)$. The construction of the natural transformation (6-6) encodes the functoriality and fusion satisfied by the objects $\{\mathrm{H}_{N,I}(W)\}$ for varying I and W.

However, in order to analyze construction (6-6) further, it is desirable to upgrade the natural transformation of functors (6-6) to the derived level. Namely, to have construction for the complexes $\{\mathcal{H}_I(W)\}_{I,W}$ and not just for their cohomology sheaves; compare with [Zhu 2021]. Such an upgrade is possible using the derived version of Drinfeld's lemma, as given in the following proposition.

Proposition 6.7. *For $\Lambda \in \{E, \mathcal{O}_E, k_E\}$ and any $W \in \mathrm{Rep}_{\Lambda}(\widehat{G}^I)$, the shtuka cohomology* (6-8) *lies in the essential image of the fully faithful functor*

$$\mathrm{D}_{\mathrm{indlis}}(Y^{\mathrm{Weil}}, \Lambda)^{\otimes I} \to \mathrm{D}_{\mathrm{indcons}}((Y^{\mathrm{Weil}})^I, \Lambda). \tag{6-9}$$

Proof. By [Xue 2020c, Theorem 4.2.3], the ind-constructible sheaf $\mathrm{H}_{N,I}(W)$ is ind-lisse. By [Xue 2020b, Proposition 3.2.15], the action of $\mathrm{FWeil}(Y^I)$ on $\mathrm{H}_{N,I}(W)$ factors through the product $\mathrm{Weil}(Y)^I$. In particular, the action of $\pi_1(X^I_{\mathbb{F}})$ on $\mathrm{H}_{N,I}(W)$ factors through the product $\pi_1(X_{\mathbb{F}})^I$. So it is ind-(split lisse) in the sense of Definition 6.4, and we are done by Theorem 6.5. □

Remark 6.8. One can upgrade the above construction in a homotopy coherent way to show that the whole complex $\mathcal{H}_{N,I}(W)$ lies in $\mathrm{D}_{\mathrm{indcons}}((Y^{\mathrm{Weil}})^I, \Lambda)$. If $N \neq \varnothing$ so that $\mathcal{H}_{N,I}(W)$ is known to be bounded, then Proposition 6.7 implies that $\mathcal{H}_{N,I}(W)$ lies in the essential image of (6-9).

Acknowledgements

We thank Clark Barwick, Jean-François Dat, Christopher Deninger, Rune Haugseng, Claudius Heyer, Peter Schneider, Burt Totaro, Torsten Wedhorn, Alexander Yom Din, and Xinwen Zhu for helpful conversations and email exchanges.

References

[Ben-Zvi et al. 2010] D. Ben-Zvi, J. Francis, and D. Nadler, "Integral transforms and Drinfeld centers in derived algebraic geometry", *J. Amer. Math. Soc.* **23**:4 (2010), 909–966. MR Zbl

[Bhatt and Lurie 2019] B. Bhatt and J. Lurie, "A Riemann–Hilbert correspondence in positive characteristic", *Camb. J. Math.* **7**:1-2 (2019), 71–217. MR Zbl

[Bhatt and Mathew 2021] B. Bhatt and A. Mathew, "The arc-topology", *Duke Math. J.* **170**:9 (2021), 1899–1988. MR Zbl

[Bhatt and Scholze 2015] B. Bhatt and P. Scholze, "The pro-étale topology for schemes", pp. 99–201 in *De la géométrie algébrique aux formes automorphes*, *I*, edited by J.-B. Bost et al., Astérisque **369**, Soc. Math. France, Paris, 2015. MR Zbl

[Bourbaki 2012] N. Bourbaki, *Algèbre, chapitre 8: modules et anneaux semi-simples*, revised 2nd ed., Springer, 2012. MR Zbl

[Curtis and Reiner 1962] C. W. Curtis and I. Reiner, *Representation theory of finite groups and associative algebras*, Pure and Applied Mathematics **11**, Interscience Publishers, New York, 1962. MR Zbl

[Deligne 1980] P. Deligne, "La conjecture de Weil, II", *Inst. Hautes Études Sci. Publ. Math.* **52** (1980), 137–252. MR Zbl

[Drinfeld 1980] V. G. Drinfeld, "Langlands' conjecture for GL(2) over functional fields", pp. 565–574 in *Proceedings of the International Congress of Mathematicians* (Helsinki, 1978), edited by O. Lehto, Acad. Sci. Fennica, Helsinki, 1980. MR Zbl

[Drinfeld 1987] V. G. Drinfeld, "Moduli varieties of F-sheaves", *Funktsional. Anal. i Prilozhen.* **21**:2 (1987), 23–41. In Russian; translated in *Funct. Anal. Appl.* **21**:2 (1987), 107–122. MR

[Gaitsgory and Lurie 2019] D. Gaitsgory and J. Lurie, *Weil's conjecture for function fields*, vol. 1, Annals of Mathematics Studies **199**, Princeton University Press, 2019. MR Zbl

[Gaitsgory and Rozenblyum 2017] D. Gaitsgory and N. Rozenblyum, *A study in derived algebraic geometry, I: Correspondences and duality*, Mathematical Surveys and Monographs **221**, American Mathematical Society, Providence, RI, 2017. MR Zbl

[Gaitsgory et al. 2022] D. Gaitsgory, D. Kazhdan, N. Rozenblyum, and Y. Varshavsky, "A toy model for the Drinfeld–Lafforgue shtuka construction", *Indag. Math.* (*N.S.*) **33**:1 (2022), 39–189. MR Zbl

[Geisser 2004] T. Geisser, "Weil-étale cohomology over finite fields", *Math. Ann.* **330**:4 (2004), 665–692. MR Zbl

[Hansen and Scholze 2023] D. Hansen and P. Scholze, "Relative perversity", *Comm. Amer. Math. Soc.* **3** (2023), 631–668. MR Zbl

[Heinloth 2018] J. Heinloth, "Langlands parameterization over function fields following V. Lafforgue", *Acta Math. Vietnam.* **43**:1 (2018), 45–66. MR Zbl

[Hemo et al. 2023] T. Hemo, T. Richarz, and J. Scholbach, "Constructible sheaves on schemes", *Adv. Math.* **429** (2023), art. id. 109179. MR Zbl

[Kahn 2003] B. Kahn, "Équivalences rationnelle et numérique sur certaines variétés de type abélien sur un corps fini", *Ann. Sci. École Norm. Sup.* (4) **36**:6 (2003), 977–1002. MR Zbl

[Kedlaya 2019] K. S. Kedlaya, "Sheaves, stacks and shtukas", pp. 45–191 in *Perfectoid spaces* (Tuscon, AZ, 2017), edited by B. Cais, Math. Surv. Monogr. **242**, Amer. Math. Soc., Providence, RI, 2019. MR Zbl

[Lafforgue 1997] L. Lafforgue, *Chtoucas de Drinfeld et conjecture de Ramanujan–Petersson*, Astérisque **243**, Soc. Math. France, Paris, 1997. MR Zbl

[Lafforgue 2002] L. Lafforgue, "Chtoucas de Drinfeld et correspondance de Langlands", *Invent. Math.* **147**:1 (2002), 1–241. MR Zbl

[Lafforgue 2018] V. Lafforgue, "Chtoucas pour les groupes réductifs et paramétrisation de Langlands globale", *J. Amer. Math. Soc.* **31**:3 (2018), 719–891. MR Zbl

[Lafforgue and Zhu 2019] V. Lafforgue and X. Zhu, “Décomposition au-dessus des paramétres de Langlands elliptiques”, preprint, 2019. arXiv 1811.07976

[Lau 2004] E. S. Lau, *On generalised $\mathcal{D}$-shtukas*, Ph.D. thesis, Friedrich-Wilhelms-Universität, 2004, available at https://bib.math.uni-bonn.de/downloads/bms/BMS-369.pdf. MR Zbl

[Lichtenbaum 2005] S. Lichtenbaum, “The Weil-étale topology on schemes over finite fields”, *Compos. Math.* **141**:3 (2005), 689–702. MR Zbl

[Lurie 2009] J. Lurie, *Higher topos theory*, Annals of Mathematics Studies **170**, Princeton University Press, 2009. MR Zbl

[Lurie 2017] J. Lurie, “Higher algebra”, preprint, Institute for Advanced Study, 2017, available at https://people.math.harvard.edu/~lurie/papers/HA.pdf.

[Lurie 2018] J. Lurie, “Spectral algebraic geometry”, preprint, Institute for Advanced Study, 2018, available at https://www.math.ias.edu/~lurie/papers/SAG-rootfile.pdf.

[Scholze and Weinstein 2020] P. Scholze and J. Weinstein, *Berkeley lectures on p-adic geometry*, Annals of Mathematics Studies **207**, Princeton University Press, 2020. MR Zbl

[Serre 1966] J.-P. Serre, “Groupes analytiques p-adiques”, exposé 270 in *Séminaire Bourbaki*, 1963/1964, W. A. Benjamin, New York, 1966. Reprinted as pp. 401–410 in *Séminaire Bourbaki* **8**, Soc. Math. France, Paris, 1995. MR Zbl

[SGA 1 2003] A. Grothendieck, *Revêtements étales et groupe fondamental* (Séminaire de Géométrie Algébrique du Bois Marie 1960–1961), Documents Mathématiques (Paris) **3**, Soc. Math. France, Paris, 2003. Updated and annotated reprint of the 1971 original. MR Zbl

[Stacks 2017] “The Stacks project”, electronic reference, 2017, available at http://stacks.math.columbia.edu.

[Torii 2023] T. Torii, “A perfect pairing for monoidal adjunctions”, *Proc. Amer. Math. Soc.* **151**:12 (2023), 5069–5080. MR Zbl

[Varshavsky 2004] Y. Varshavsky, “Moduli spaces of principal F-bundles”, *Selecta Math.* (*N.S.*) **10**:1 (2004), 131–166. MR Zbl

[Xue 2020a] C. Xue, “Cuspidal cohomology of stacks of shtukas”, *Compos. Math.* **156**:6 (2020), 1079–1151. MR Zbl

[Xue 2020b] C. Xue, “Finiteness of cohomology groups of stacks of shtukas as modules over Hecke algebras, and applications”, *Épijournal Géom. Algébrique* **4** (2020), art. id. 6. MR Zbl

[Xue 2020c] C. Xue, “Smoothness of cohomology sheaves of stacks of shtukas”, preprint, 2020. arXiv 2012.12833

[Zhu 2021] X. Zhu, “Coherent sheaves on the stack of Langlands parameters”, preprint, 2021. arXiv 2008.02998

Communicated by Bhargav Bhatt
Received 2022-02-20 Revised 2023-03-23 Accepted 2023-05-29

themo@caltech.edu *Department of Mathematics, California Institute of Technology, Pasadena, CA, United States*

richarz@mathematik.tu-darmstadt.de *Department of Mathematics, Technical University of Darmstadt, Darmstadt, Germany*

jakob.scholbach@unipd.it *Dipartimento di Matematica, University of Padova, Padova, Italy*

ALGEBRA AND NUMBER THEORY 18:3 (2024)
https://doi.org/10.2140/ant.2024.18.537

Generalized Igusa functions and ideal growth in nilpotent Lie rings

Angela Carnevale, Michael M. Schein and Christopher Voll

We introduce a new class of combinatorially defined rational functions and apply them to deduce explicit formulae for local ideal zeta functions associated to the members of a large class of nilpotent Lie rings which contains the free class-2-nilpotent Lie rings and is stable under direct products. Our results unify and generalize a substantial number of previous computations. We show that the new rational functions, and thus also the local zeta functions under consideration, enjoy a self-reciprocity property, expressed in terms of a functional equation upon inversion of variables. We establish a conjecture of Grunewald, Segal, and Smith on the uniformity of normal zeta functions of finitely generated free class-2-nilpotent groups.

1. Introduction

The objective of this paper is twofold. The first aim is to introduce a new class of combinatorially defined multivariate rational functions and to prove that they satisfy a self-reciprocity property, expressed in terms of a functional equation upon inversion of variables. The second is to apply these rational functions to obtain an explicit description of the local ideal zeta functions associated to a class of combinatorially defined Lie rings. We start with a discussion of the latter application before formulating and explaining the new class of rational functions.

1.1. ***Finite uniformity for ideal zeta functions of nilpotent Lie rings.*** Given an additively finitely generated ring $\mathcal{L}$, i.e., a finitely generated $\mathbb{Z}$-module with some biadditive, not necessarily associative multiplication, the ideal zeta function of $\mathcal{L}$ is the Dirichlet generating series

$$\zeta^{\triangleleft}_{\mathcal{L}}(s) = \sum_{I \triangleleft \mathcal{L}} |\mathcal{L} : I|^{-s}, \tag{1-1}$$

where I runs over the (two-sided) ideals of $\mathcal{L}$ of finite additive index in $\mathcal{L}$ and s is a complex variable. Prominent examples of ideal zeta functions include the Dedekind zeta functions, enumerating ideals of rings of integers of algebraic number fields and, in particular, Riemann's zeta function $\zeta(s)$.

MSC2020: 05A15, 11M41, 20E07.
Keywords: subgroup growth, ideal growth, normal zeta functions, ideal zeta functions, Igusa functions, combinatorial reciprocity theorems.

It is not hard to verify that, for a general ring $\mathcal{L}$, the ideal zeta function $\zeta^{\triangleleft}_{\mathcal{L}}(s)$ satisfies an Euler product whose factors are indexed by the rational primes:

$$\zeta^{\triangleleft}_{\mathcal{L}}(s) = \prod_{p \text{ prime}} \zeta^{\triangleleft}_{\mathcal{L}(\mathbb{Z}_p)}(s),$$

where, for a prime p,

$$\zeta^{\triangleleft}_{\mathcal{L}(\mathbb{Z}_p)}(s) = \sum_{I \triangleleft \mathcal{L}(\mathbb{Z}_p)} |\mathcal{L}(\mathbb{Z}_p) : I|^{-s}$$

enumerates the ideals of finite index in the completion $\mathcal{L}(\mathbb{Z}_p) := L \otimes_{\mathbb{Z}} \mathbb{Z}_p$ or, equivalently, the ideals of finite p-power index in $\mathcal{L}$. Here $\mathbb{Z}_p$ denotes the ring of p-adic integers; note that ideals of $\mathcal{L}(\mathbb{Z}_p)$ are, in particular, $\mathbb{Z}_p$-submodules of $\mathcal{L}(\mathbb{Z}_p)$. It is, in contrast, a deep result that the Euler factors $\zeta^{\triangleleft}_{\mathcal{L}(\mathbb{Z}_p)}(s)$ are rational functions in the parameter p^{-s}; see [Grunewald et al. 1988, Theorem 3.5].

Computing these rational functions explicitly for a given ring $\mathcal{L}$ is, in general, a very hard problem. Solving it is usually rewarded by additional insights into combinatorial, arithmetic, or asymptotic aspects of ideal growth. It was shown by du Sautoy and Grunewald [2000] that the problem, in general, involves the determination of the numbers of $\mathbb{F}_p$-rational points of finitely many algebraic varieties defined over $\mathbb{Q}$. Only under additional assumptions on $\mathcal{L}$ may one hope that these numbers are given by finitely many polynomial functions in p. We say that the ideal zeta function of $\mathcal{L}$ is *finitely uniform* if there are finitely many rational functions $W^{\triangleleft}_1(X, Y), \dots, W^{\triangleleft}_N(X, Y) \in \mathbb{Q}(X, Y)$ such that for any prime p there exists $i \in \{1, \dots, N\}$ such that

$$\zeta^{\triangleleft}_{\mathcal{L}(\mathbb{Z}_p)}(s) = W^{\triangleleft}_i(p, p^{-s}).$$

If a single rational function suffices (i.e., $N = 1$), we say that the ideal zeta function of $\mathcal{L}$ is *uniform*. While finite uniformity dominates among low-rank examples, including most of those included in the book [du Sautoy and Woodward 2008] and those computed by Rossmann's [2018] computer algebra package Zeta [2022], it is not ubiquitous: for a nonuniform example in rank 9, see [du Sautoy 2002] and [Voll 2004]. In general, the ideal zeta function of a direct product of rings is not given by a simple function of the ideal zeta functions of the factors. It is not even clear whether (finite) uniformity of the latter implies (finite) uniformity of the former.

1.1.1. *Main results.* We now restrict to the case of Lie rings, namely rings in which the multiplication is antisymmetric and satisfies the Jacobi identity; note that the Jacobi identity holds trivially for all nilpotent rings of class at most two. In this paper we give constructive proofs of (finite) uniformity of ideal zeta functions associated to the members of a large class of nilpotent Lie rings of nilpotency class at most two.

Definition 1.1. Let $\mathfrak{L}$ denote the class of nilpotent Lie rings of nilpotency class at most two which is closed under direct products and contains the following Lie rings:

(1) The free class-2-nilpotent Lie rings $\mathfrak{f}_{2,d}$ on d generators, for $d \geq 2$; see Section 5.2.

(2) The free class-2-nilpotent products $\mathfrak{g}_{d,d'} = \mathbb{Z}^d * \mathbb{Z}^{d'}$, for $d, d' \geq 0$; see Section 5.3.

(3) The higher Heisenberg Lie rings $\mathfrak{h}_d$ for $d \geq 1$; see Section 5.4.

Note that $\mathcal{L}$ contains the free abelian Lie rings $\mathbb{Z}^d = \mathfrak{g}_{d,0} = \mathfrak{g}_{0,d}$.

Our main "global" result produces explicit formulae for almost all Euler factors of the ideal zeta functions associated to Lie rings obtained from the members of $\mathfrak{L}$ by base extension with general rings of integers of number fields. In particular, we show that these zeta functions are finitely uniform and, more precisely, that the variation of the Euler factors is uniform among unramified primes with the same decomposition behavior in the relevant number field.

Theorem 1.2. *Let $\mathcal{L}$ be an element of $\mathfrak{L}$, and let $\boldsymbol{f} = (f_1, \dots, f_g) \in \mathbb{N}^g$ be a g-tuple for some $g \in \mathbb{N}$. There exists an explicitly described rational function $W^{\triangleleft}_{\mathcal{L},\boldsymbol{f}} \in \mathbb{Q}(X, Y)$ such that the following holds*:

Let $\mathcal{O}$ be the ring of integers of a number field of degree n, and set $\mathcal{L}(\mathcal{O}) = \mathcal{L} \otimes \mathcal{O}$. If a rational prime p factorizes in $\mathcal{O}$ as $p\mathcal{O} = \mathfrak{p}_1\mathfrak{p}_2 \cdots \mathfrak{p}_g$, for pairwise distinct prime ideals $\mathfrak{p}_i$ in $\mathcal{O}$ of inertia degrees $(f_1, \dots, f_g)$, then

$$\zeta^{\triangleleft}_{\mathcal{L}(\mathcal{O}),p}(s) = W^{\triangleleft}_{\mathcal{L},\boldsymbol{f}}(p, p^{-s}).$$

In particular, $\zeta^{\triangleleft}_{\mathcal{L}(\mathcal{O})}(s)$ is finitely uniform and $\zeta^{\triangleleft}_{\mathcal{L}}(s) = \zeta^{\triangleleft}_{\mathcal{L}(\mathbb{Z})}(s)$ is uniform. Moreover, the rational function $W^{\triangleleft}_{\mathcal{L},\boldsymbol{f}}$ satisfies the functional equation

$$W^{\triangleleft}_{\mathcal{L},\boldsymbol{f}}(X^{-1}, Y^{-1}) = (-1)^{n \operatorname{rk}_{\mathbb{Z}} \mathcal{L}} X^{\binom{n \operatorname{rk}_{\mathbb{Z}} \mathcal{L}}{2}} Y^{n(\operatorname{rk}_{\mathbb{Z}} \mathcal{L} + \operatorname{rk}_{\mathbb{Z}}(\mathcal{L}/Z(\mathcal{L})))} W^{\triangleleft}_{\mathcal{L},\boldsymbol{f}}(X, Y). \tag{1-2}$$

A special case of Theorem 1.2 establishes part of a conjecture of Grunewald, Segal, and Smith on the normal subgroup growth of free nilpotent groups under extension of scalars. In [Grunewald et al. 1988], they introduced the concept of the *normal zeta function*

$$\zeta^{\triangleleft}_G(s) = \sum_{H \triangleleft G} |G : H|^{-s}$$

of a torsion-free finitely generated nilpotent group G, enumerating the normal subgroups of G of finite index in G. As G is nilpotent, it also satisfies an Euler product decomposition

$$\zeta^{\triangleleft}_G(s) = \prod_{p \text{ prime}} \zeta^{\triangleleft}_{G,p}(s),$$

whose factors enumerate the normal subgroups of G of p-power index. If G has nilpotency class two, then its normal zeta function coincides with the ideal zeta function of the associated Lie ring $\mathcal{L}_G := G/Z(G) \oplus Z(G)$; see the remark on page 206 of [Grunewald et al. 1988] and the more detailed discussion in [Berman et al. 2015, Section 3.1]. Thus, $\zeta^{\triangleleft}_G(s) = \zeta^{\triangleleft}_{\mathcal{L}_G}(s)$. Moreover, every class-2-nilpotent Lie ring $\mathcal{L}$ arises in this way and gives rise to a torsion-free finitely generated nilpotent group $G(\mathcal{L})$; see [Voll 2019, Section 1.2] for details. Theorem 1.2 thus has a direct corollary pertaining to the normal zeta functions of the finitely generated class-2-nilpotent groups corresponding to the Lie rings in $\mathfrak{L}$. Since the groups associated to the free class-2-nilpotent Lie rings $\mathfrak{f}_{2,d}$ are the finitely generated free class-2-nilpotent groups $F_{2,d} = G(\mathfrak{f}_{2,d})$, Theorem 1.2 implies the conjecture on page 188 of [Grunewald et al. 1988] for the case $* = \triangleleft$ and class $c = 2$. The conjecture for normal zeta functions had previously been established only for $d=2$ [Grunewald et al. 1988, Theorem 3]; see also Section 1.1.2. We are not aware of any other case for which the conjecture has been proven or refuted.

For any class-2-nilpotent Lie ring $\mathcal{L}$, it is known [Voll 2010, Theorem C] that the Euler factors of $\zeta^{\triangleleft}_{\mathcal{L}}(s)$ at almost all primes p are realized by rational functions admitting functional equations with the same symmetry factor $(-1)^{\mathrm{rk}_{\mathbb{Z}}\mathcal{L}} X^{\binom{\mathrm{rk}_{\mathbb{Z}}\mathcal{L}}{2}} Y^{\mathrm{rk}_{\mathbb{Z}}\mathcal{L}+\mathrm{rk}_{\mathbb{Z}}(\mathcal{L}/Z(\mathcal{L}))}$. In particular, the functional equation (1-2) of Theorem 1.2 shows that, for the Lie rings $\mathcal{L}(\mathcal{O})$, where $\mathcal{L}$ lies in our class $\mathfrak{L}$ and $\mathcal{O}$ is a number ring, the finitely many primes excluded by [Voll 2010, Theorem C] must ramify in $\mathcal{O}$. We suspect that they are exactly the primes ramifying in $\mathcal{O}$; see Remark 1.5 below.

Theorem 1.2 is a consequence of the following uniform "local" result. Throughout the paper, $\mathfrak{o}$ will denote a compact discrete valuation ring of arbitrary characteristic and residue field of characteristic p and cardinality q. Thus, $\mathfrak{o}$ may, for instance, be a finite extension of the ring $\mathbb{Z}_p$ of p-adic integers (of characteristic zero) or a ring of formal power series of the form $\mathbb{F}_q[\![T]\!]$ (of positive characteristic). The *$\mathfrak{o}$-ideal zeta function*

$$\zeta^{\triangleleft\mathfrak{o}}_{L}(s)=\sum_{I\triangleleft L}|L:I|^{-s}$$

of an $\mathfrak{o}$-algebra L of finite $\mathfrak{o}$-rank is defined as in (1-1), with I ranging over the $\mathfrak{o}$-ideals of L, viz. $(\mathrm{ad}\,L)$-invariant $\mathfrak{o}$-submodules of L. Note that every element $\mathcal{L}$ of $\mathfrak{L}$ may, after tensoring over $\mathbb{Z}$ with $\mathfrak{o}$, be considered a free and finitely generated $\mathfrak{o}$-Lie algebra. Given an $\mathfrak{o}$-module R, we write $L(R)=L\otimes_{\mathfrak{o}}R$.

Theorem 1.3. *Let $\boldsymbol{\mathcal{L}}=(\mathcal{L}_1,\dots,\mathcal{L}_g)$ be a family of elements of $\mathfrak{L}$ and $\boldsymbol{f}=(f_1,\dots,f_g)\in\mathbb{N}^g$. There exists an explicit rational function $W^{\triangleleft}_{\boldsymbol{\mathcal{L}},\boldsymbol{f}}\in\mathbb{Q}(X,Y)$ such that the following holds:*

Let $\mathfrak{o}$ be a compact discrete valuation ring and $(\mathfrak{O}_1,\dots,\mathfrak{O}_g)$ be a family of finite unramified extensions of $\mathfrak{o}$ with inertia degrees $(f_1,\dots,f_g)$. Consider the $\mathfrak{o}$-Lie algebra

$$L=\mathcal{L}_1(\mathfrak{O}_1)\times\cdots\times\mathcal{L}_g(\mathfrak{O}_g).$$

For every finite extension $\mathfrak{O}$ of $\mathfrak{o}$, of inertia degree f over $\mathfrak{o}$, say, the $\mathfrak{O}$-ideal zeta function of $L(\mathfrak{O})$ satisfies

$$\zeta^{\triangleleft\mathfrak{O}}_{L(\mathfrak{O})}(s)=W^{\triangleleft}_{\boldsymbol{\mathcal{L}},\boldsymbol{f}}(q^f,q^{-fs}).$$

The rational function $W^{\triangleleft}_{\boldsymbol{\mathcal{L}},\boldsymbol{f}}$ satisfies the functional equation

$$W^{\triangleleft}_{\boldsymbol{\mathcal{L}},\boldsymbol{f}}(X^{-1},Y^{-1})=(-1)^{N_0}X^{\binom{N_0}{2}}Y^{N_0+N_1}W^{\triangleleft}_{\boldsymbol{\mathcal{L}},\boldsymbol{f}}(X,Y), \tag{1-3}$$

where

$$N_0=\mathrm{rk}_{\mathfrak{o}}L=\sum_{i=1}^{g}f_i\,\mathrm{rk}_{\mathbb{Z}}(\mathcal{L}_i)\quad\textit{and}\quad N_1=\mathrm{rk}_{\mathfrak{o}}(L/Z(L))=\sum_{i=1}^{g}f_i\,\mathrm{rk}_{\mathbb{Z}}(\mathcal{L}_i/Z(\mathcal{L}_i)).$$

Theorem 1.2 is readily deduced from Theorem 1.3. Indeed, let $\mathcal{L}$ be a nilpotent Lie ring as in the statement of Theorem 1.2, and let $\mathcal{O}$ be the ring of integers of a number field. Suppose that the rational prime p is unramified in $\mathcal{O}$ and decomposes as $p\mathcal{O}=\mathfrak{p}_1\mathfrak{p}_2\cdots\mathfrak{p}_g$, where the $\mathfrak{p}_i$ are distinct prime ideals of

$\mathcal{O}$ of inertia degrees f_i. Then $\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}_p \simeq \mathfrak{O}_1 \times \cdots \times \mathfrak{O}_g$, where each $\mathfrak{O}_i/\mathbb{Z}_p$ is an unramified extension of inertia degree f_i. Therefore,

$$\mathcal{L}(\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}_p) \simeq \mathcal{L}(\mathfrak{O}_1) \times \cdots \times \mathcal{L}(\mathfrak{O}_g).$$

Hence, by Theorem 1.3 we have

$$\zeta^{\triangleleft}_{\mathcal{L}(\mathcal{O}),p}(s) = \zeta^{\triangleleft \mathbb{Z}_p}_{\mathcal{L}(\mathcal{O}\otimes_{\mathbb{Z}}\mathbb{Z}_p)}(s) = W^{\triangleleft}_{(\mathcal{L},\ldots,\mathcal{L}),(f_1,\ldots,f_g)}(p, p^{-s})$$

for an explicit rational function $W^{\triangleleft}_{(\mathcal{L},\ldots,\mathcal{L}),(f_1,\ldots,f_g)} \in \mathbb{Q}(X, Y)$. Setting $W^{\triangleleft}_{\mathcal{L},\boldsymbol{f}} = W^{\triangleleft}_{(\mathcal{L},\ldots,\mathcal{L}),(f_1,\ldots,f_g)}$, we obtain Theorem 1.2. The functional equation of Theorem 1.2 follows from that of Theorem 1.3 since $n = \sum_{i=1}^{g} f_i$ as p is unramified in $\mathcal{O}$.

Remark 1.4. Our description of the rational function $W^{\triangleleft}_{\mathcal{L},\boldsymbol{f}}$ is so explicit that one may, in principle, read off the (local) *abscissa of convergence* $\alpha^{\triangleleft \mathfrak{O}}_{L(\mathfrak{O})}$ of $\zeta^{\triangleleft \mathfrak{O}}_{L(\mathfrak{O})}(s)$, viz.

$$\alpha^{\triangleleft \mathfrak{O}}_{L(\mathfrak{O})} := \inf\{\alpha \in \mathbb{R}_{>0} \mid \zeta^{\triangleleft \mathfrak{O}}_{L(\mathfrak{O})}(s) \text{ converges on } \{s \in \mathbb{C} \mid \Re(s) > \alpha\}\} \in \mathbb{Q}_{>0};$$

see Remark 4.23.

Remark 1.5. We emphasize that Theorem 1.3 makes no restriction on the residue characteristic of $\mathfrak{o}$. In this regard it strengthens, for the class of Lie rings under consideration, the result [Voll 2019, Theorem 1.2], which establishes the functional equation (1-3) for all $\mathfrak{o}$ whose residue characteristic avoids finitely many prime numbers; see [Voll 2019, Corollary 1.3] and also [Lee and Voll 2023, Theorem 1.7]. In the global contexts of ideal zeta functions of rings of the form $\mathcal{L}(\mathcal{O})$ for number rings $\mathcal{O}$, Theorem 1.3 shows that the finitely many Euler factors for which the functional equation (1-3) fails must be among those indexed by primes that ramify in $\mathcal{O}$.

In [Schein and Voll 2015, Conjecture 1.4] it was suggested that a functional equation should hold for *all* local factors $\zeta^{\triangleleft}_{\mathfrak{f}_{2,2}(\mathcal{O}),p}(s)$, where $\mathfrak{f}_{2,2}$ is the Heisenberg Lie ring and $\mathcal{O}$ is a number ring; if p ramifies in $\mathcal{O}$, then the symmetry factor must be modified from that of (1-3). Some cases of the conjecture were proved in [Schein and Voll 2016, Corollary 3.13]. There is computational evidence, due to T. Bauer, that other Lie rings in the class $\mathfrak{L}$ also exhibit the remarkable property of the local factors $\zeta^{\triangleleft}_{\mathcal{L}(\mathcal{O}),p}(s)$ at ramified primes p being described by rational functions satisfying functional equations. However, these local factors cannot be computed by the methods of this paper; see Remark 4.8. Bauer's computations, together with the results of this paper, suggest the following natural question: how do the local factors $\zeta^{\triangleleft}_{\mathcal{L}(\mathcal{O}),p}(s)$ behave at ramified primes, and how does the structure of $\mathcal{L}$ govern their behavior?

Another natural problem is to improve upon Definition 1.1 by giving a conceptual characterization of the class of Lie rings to which our method, or a mild generalization thereof, applies. For instance, forthcoming work of T. Bauer extends our argument to explicitly compute the ideal zeta functions of central products of finitely many copies of Lie rings in the class $\mathfrak{L}$. By contrast, nonuniform examples such as those of [du Sautoy 2002; Voll 2004] provide a limit on the applicability of these methods.

1.1.2. *Previous and related work.* Theorems 1.2 and 1.3 generalize and unify several previously known results:

(1) The most classical may be the formula for the $\mathfrak{o}$-ideal zeta function

$$\zeta_{\mathfrak{o}^n}(s) := \zeta^{\triangleleft \mathfrak{o}}_{\mathfrak{o}^n}(s) = \prod_{i=1}^{n} \frac{1}{1-q^{-s+i-1}} \tag{1-4}$$

of the (abelian Lie) ring $\mathfrak{o}^n = \mathfrak{g}_{0,n}(\mathfrak{o}) = \mathfrak{g}_{n,0}(\mathfrak{o})$; see [Grunewald et al. 1988, Proposition 1.1].

(2) The ideal zeta functions of the so-called *Grenham Lie rings* $\mathfrak{g}_{1,d}$ were given in [Voll 2005a, Theorem 5].

(3) Formulae for the ideal zeta functions of the free class-2-nilpotent Lie rings $\mathfrak{f}_{2,d}$ on d generators are the main result of [Voll 2005b].

(4) The paper [Schein and Voll 2015] contains formulae for all local factors of the ideal zeta functions of the Lie rings $\mathfrak{f}_{2,2}(\mathcal{O}) = \mathfrak{g}_{1,1}(\mathcal{O}) = \mathfrak{h}_1(\mathcal{O})$, i.e., the *Heisenberg Lie ring* over an arbitrary number ring $\mathcal{O}$, which are indexed by primes unramified in $\mathcal{O}$. The uniform nature of these functions had already been established in [Grunewald et al. 1988, Theorem 3]. Formulae for factors indexed by nonsplit primes are given in [Schein and Voll 2016].

(5) The ideal zeta functions of the Lie rings $\mathfrak{h}_d \times \mathfrak{o}^r$ were computed in [Grunewald et al. 1988, Proposition 8.4], whereas for the direct products $\mathfrak{h}_d \times \cdots \times \mathfrak{h}_d$ they were computed in [Bauer 2013].

(6) The ideal zeta function of the Lie ring $\mathfrak{g}_{2,2}$ was computed in [Paajanen 2008, Theorem 11.1].

Some of the members of the family of Lie rings $\mathfrak{L}$ have previously been studied in the context of related counting problems, each leading to a different class of zeta functions. We mention specifically four such classes: First, the *subring zeta function* of a (class-2-nilpotent Lie) ring $\mathcal{L}$, enumerating the finite index subrings of $\mathcal{L}$. Second, the *proisomorphic zeta function* of $G(\mathcal{L})$, the finitely generated nilpotent group associated to $\mathcal{L}$ via the Malcev correspondence, enumerating the subgroups of finite index of $G(\mathcal{L})$ whose profinite completions are isomorphic to that of $G(\mathcal{L})$. Third, the *representation zeta function* of $G(\mathcal{L})$, enumerating the twist-isoclasses of complex irreducible representations of $G(\mathcal{L})$. Fourth, the *class number zeta function* of $G(\mathcal{L})$, enumerating the class numbers (i.e., numbers of conjugacy classes) of congruence quotients of this group; see [Lins de Araujo 2019].

The subring zeta functions of the Grenham Lie rings $\mathfrak{g}_{1,d}$ were computed in [Voll 2006]. Those of the free class-2-nilpotent Lie rings $\mathfrak{f}_{2,d}$ are largely unknown, apart from $d = 2$ [Grunewald et al. 1988] and $d = 3$ ([du Sautoy and Woodward 2008, Theorem 2.16], due to G. Taylor). The proisomorphic zeta functions of the members of a combinatorially defined class of groups that includes the Grenham groups $G(\mathfrak{g}_{1,d})$ were computed in [Berman et al. 2018], their normal zeta functions in [Voll 2020]. Moreover, *all* Euler factors of the proisomorphic zeta functions of $G(\mathfrak{f}_{2,d}(\mathcal{O}))$ and $G(\mathfrak{h}_d(\mathcal{O}))$, where $\mathcal{O}$ is an arbitrary number ring, as well as of the base extensions to $\mathcal{O}$ of the groups considered in [Berman et al. 2018] and some other families of nilpotent groups of unbounded class, were computed in [Berman et al. 2022]. The representation zeta functions of the free class-2-nilpotent groups $F_{2,d}(\mathcal{O}) = G(\mathfrak{f}_{2,d}(\mathcal{O}))$ were computed in [Stasinski and Voll 2014, Theorem B], those of the groups $G(\mathfrak{g}_{d,d'}(\mathcal{O}))$ in [Zordan 2022, Theorem A]. In these cases, not only is there a fine Euler decomposition, but the rational function realizing the fine Euler factors is independent of $\mathcal{O}$ and of the prime. The class number zeta functions of the groups $F_{2,d}(\mathcal{O})$ and $G(\mathfrak{g}_{d,d}(\mathcal{O}))$, which may be found in [Lins de Araujo 2020, Corollary 1.5], satisfy the same properties.

Combinatorial structures similar to those employed in the present article were also used in [Rossmann and Voll 2019]. In that paper, they were used to produce explicit formulae for zeta functions enumerating conjugacy classes of the cographical groups defined in [Rossmann and Voll 2019, Section 3.4].

1.1.3. *Methodology.* Our approach to computing the explicit rational functions mentioned in Theorems 1.3 and 1.2 hinges on the following considerations. Fix a prime p and a class-2-nilpotent Lie ring $\mathcal{L}$ and consider, for simplicity, the pro-p completion $L = \mathcal{L}(\mathbb{Z}_p)$ of $\mathcal{L}$. Given a $\mathbb{Z}_p$-sublattice $\Lambda \leq L$, set $\overline{\Lambda} := (\Lambda + L')/L'$ and $\Lambda' := \Lambda \cap L'$. Here we write $L' = [L, L]$ for the commutator subring of L. Clearly, Λ is a $\mathbb{Z}_p$-ideal of L if and only if $[\overline{\Lambda}, L] \subseteq \Lambda'$. This allows us, for fixed $\overline{\Lambda}$, to reduce the problem of enumerating such Λ' to the problem of enumerating subgroups of the finite abelian p-group $L'/[\overline{\Lambda}, L]$. The isomorphism type of the latter is given by the ($\mathbb{Z}_p$)-*elementary divisor type* of $[\overline{\Lambda}, L]$ in L', viz. the partition $\lambda(\Lambda) = (\lambda_1, \dots, \lambda_c)$ with the property that

$$L'/[\overline{\Lambda}, L] \simeq \mathbb{Z}_p/(p^{\lambda_1}) \times \cdots \times \mathbb{Z}_p/(p^{\lambda_c}).$$

For general Lie rings $\mathcal{L}$, controlling this type for varying Λ is a hard problem that may be dealt with by studying suitably defined p-adic integrals with sophisticated tools from algebraic geometry, including Hironaka's resolution of singularities in characteristic zero.

If, however, $\mathcal{L}$ is an element of the class $\mathfrak{L}$, then the elementary divisor type of $[\overline{\Lambda}, L]$ is determined, in a complicated but *combinatorial* manner, by so-called "projection data"; see Definition 4.1. These are the respective elementary divisor types of the projections of $\overline{\Lambda}$ onto various direct summands of L/L'. The technical tool we use to keep track of the resulting infinitude of finite enumerations are the *generalized Igusa functions* introduced in Section 3. An intrinsic advantage of this combinatorial point of view over the general (and typically immensely more powerful) algebro-geometric approach is that, structurally, $\mathbb{Z}_p$ only enters as a compact discrete valuation ring. The effect of passage to various other such local rings, including those of positive characteristic, is therefore easy to control.

For an informal overview of the combinatorial aspects of our approach to counting $\mathfrak{o}$-ideals, see Section 4.1.

1.2. *Counting ideals with generalized Igusa functions.* Our key to Theorem 1.3 is the systematic deployment of a new class of combinatorially defined multivariate rational functions, which we call *generalized Igusa functions*. Expecting that they will be of interest independently of questions pertaining to ideal growth in rings, we explain them here separately.

Generalized Igusa functions interpolate between two well-used classes of rational functions:

(1) A function we refer to as the *Igusa zeta function of degree n* plays a key role in numerous previous computations (for instance [Carnevale et al. 2018; Paajanen 2008; Schein and Voll 2015; 2016; Stasinski and Voll 2014; Voll 2005a; 2005b; 2006; 2020]):

$$I_n(Y; X_1, \dots, X_n) = \sum_{I \subseteq \{1,\dots,n\}} \binom{n}{I}_Y \prod_{i \in I} \frac{X_i}{1 - X_i} \in \mathbb{Q}(Y, X_1, \dots, X_n).$$

Here, $\binom{n}{I}_Y$ denotes the Gaussian multinomial; see (2-2). For instance,

$$\zeta_{\mathfrak{o}^n}(s) = I_n(q^{-1}; ((q^{n-i-s})^i)_{i=1}^n); \tag{1-5}$$

see (1-4) and [Voll 2011, Example 2.20].

(2) In [Schein and Voll 2015], the *weak order zeta function*

$$I_n^{\mathrm{wo}}((X_I)_{I \in \mathcal{P}([n]) \setminus \{\varnothing\}}) = \sum_{I_1 \subsetneq \cdots \subsetneq I_l \subseteq [n]} \prod_{j=1}^{l} \frac{X_{I_j}}{1 - X_{I_j}} \in \mathbb{Q}((X_I)_{I \in \mathcal{P}([n]) \setminus \{\varnothing\}}) \tag{1-6}$$

played a decisive role; see [Schein and Voll 2015, Definition 2.9].

The main protagonist of Section 3 is the *generalized Igusa function* $I_{\underline{n}}^{\mathrm{wo}}(Y_1, \dots, Y_m; \boldsymbol{X})$, a rational function associated to a composition $\underline{n} = (n_1, \dots, n_m)$, with variables $\boldsymbol{X}$ indexed by the subwords of the word $a_1^{n_1} \dots a_m^{n_m}$ in “letters” $a_1, \dots, a_m$; see Definition 3.5 for details. It interpolates between the two classes of rational functions just mentioned: the Igusa function of degree n for the trivial composition (n) and the weak order zeta function for the all-one composition $(1, \dots, 1)$ of n; see Example 3.6.

Remark 1.6. Igusa functions are not to be confused with, but are related to, a class of p-adic integrals known as Igusa’s local zeta function; see [Denef 1991]. For a detailed explanation of the connection between I_n and work of Igusa, as well as further generalizations and applications, see [Klopsch and Voll 2009].

1.3. *Organization and notation.*

1.3.1. In Section 2 we recall a number of preliminary notions and results used to enumerate lattices and finite abelian p-groups. In Section 3 we define the generalized Igusa functions and prove that they satisfy functional equations. In Section 4, these new functions are put to use to compute a general formula (see Theorem 4.21) for local ideal zeta functions of Lie rings satisfying the general combinatorial Hypothesis 4.5. In Section 5 we verify that the members of the class $\mathfrak{L}$ (see Definition 1.1) satisfy Hypothesis 4.5, complete the proof of Theorem 1.3, and attend to a number of special cases.

1.3.2. We write $\mathbb{N} = \{1, 2, \dots\}$ and, for a subset $X \subseteq \mathbb{N}$, set $X_0 = X \cup \{0\}$. For $m, n \in \mathbb{N}_0$ we denote $[n] = \{1, \dots, n\}$, $[n, m] = \{n, n+1, \dots, m\}$, and $(n, m) = \{n+1, \dots, m-1\}$. Given a finite subset $J \subseteq \mathbb{N}_0$, we write $J = \{j_1, \dots, j_r\}_<$ to imply that $j_1 < \cdots < j_r$. We write $J - n$ for the set $\{j - n \mid j \in J\}$. The power set of a set S is denoted $\mathcal{P}(S)$.

A *composition of n with r parts* is a sequence $(\lambda_1, \dots, \lambda_r) \in \mathbb{N}_0^r$ such that $\sum_{i=1}^r \lambda_i = n$. A *partition of n with r parts* is a composition of n with r parts such that $\lambda_1 \geq \cdots \geq \lambda_r$. We occasionally obtain partitions from multisets by arranging their elements in nonascending order. Our notation for the dual partition of a partition λ is λ'. Given partitions $\mu = (\mu_1, \dots, \mu_c)$ and $\lambda = (\lambda_1, \dots, \lambda_c)$ we write $\mu \leq \lambda$ if $\mu_i \leq \lambda_i$ for all $i \in [c]$, i.e., if the Young diagram of μ is included in the Young diagram of λ.

2. Preliminaries

In this preliminary section, we collect some fundamental notions.

2.1. *Gaussian binomials and classical Igusa functions.* For a variable Y and integers $a, b \in \mathbb{N}_0$ with $a \geq b$, the associated *Gaussian binomial* is

$$\binom{a}{b}_Y = \frac{\prod_{i=a-b+1}^{a}(1-Y^i)}{\prod_{i=1}^{b}(1-Y^i)} \in \mathbb{Z}[Y].$$

A simple computation shows that

$$\binom{a}{b}_{Y^{-1}} = Y^{b(b-a)}\binom{a}{b}_Y. \tag{2-1}$$

Given $n \in \mathbb{N}$ and a subset $J = \{j_1, \dots, j_r\}_< \subseteq [n-1]$, the associated *Gaussian multinomial* is defined as

$$\binom{n}{J}_Y = \binom{n}{j_r}_Y \binom{j_r}{j_{r-1}}_Y \cdots \binom{j_2}{j_1}_Y \in \mathbb{Z}[Y]. \tag{2-2}$$

We omit the proof of the following simple lemma, which is similar to [Schein and Voll 2015, Lemma 2.14].

Lemma 2.1. *Let $n \in \mathbb{N}$ and $P = \{p_1, \dots, p_{r-1}\}_< \subseteq J \subseteq [n-1]$. Set $p_0 = 0$ and $p_r = n$. Then*

$$\binom{n}{J}_Y = \binom{n}{P}_Y \prod_{j=1}^{r} \binom{p_j - p_{j-1}}{J \cap (p_{j-1}, p_j) - p_{j-1}}_Y.$$

Definition 2.2 [Schein and Voll 2015, Definition 2.5]. Let $n \in \mathbb{N}$. Given variables Y and $\boldsymbol{X} = (X_1, \dots, X_n)$, we define the *Igusa functions of degree n*

$$I_n(Y; \boldsymbol{X}) = \frac{1}{1-X_n} \sum_{I \subseteq [n-1]} \binom{n}{I}_Y \prod_{i \in I} \frac{X_i}{1-X_i} = \sum_{I \subseteq [n]} \binom{n}{I}_Y \prod_{i \in I} \frac{X_i}{1-X_i} \in \mathbb{Q}(Y, X_1, \dots, X_n),$$

$$I_n^\circ(Y; \boldsymbol{X}) = \frac{X_n}{1-X_n} \sum_{I \subseteq [n-1]} \binom{n}{I}_Y \prod_{i \in I} \frac{X_i}{1-X_i} \in \mathbb{Q}(Y, X_1, \dots, X_n).$$

An important feature of these functions is that they satisfy a functional equation upon inversion of the variables; it is immediate from [Voll 2005a, Theorem 4] that, for all $n \in \mathbb{N}$,

$$I_n(Y^{-1}; \boldsymbol{X}^{-1}) = (-1)^n X_n Y^{-\binom{n}{2}} I_n(Y; \boldsymbol{X}), \tag{2-3}$$

$$I_n^\circ(Y^{-1}; \boldsymbol{X}^{-1}) = (-1)^n X_n^{-1} Y^{-\binom{n}{2}} I_n^\circ(Y; \boldsymbol{X}). \tag{2-4}$$

2.2. ***Subgroups of finite abelian groups, Birkhoff's formula, and Dyck words.*** It is well-known that, given a pair of partitions $\mu \leq \lambda$ and a prime p, the number $a(\lambda, \mu; p)$ of finite abelian p-groups of isomorphism type μ contained in a fixed finite abelian p-group of isomorphism type λ is given by a polynomial in p. More precisely, set

$$\alpha(\lambda, \mu; Y) = \prod_{k \geq 1} Y^{\mu'_k(\lambda'_k - \mu'_k)} \binom{\lambda'_k - \mu'_{k+1}}{\lambda'_k - \mu'_k}_{Y^{-1}} \in \mathbb{Q}[Y], \tag{2-5}$$

where λ' and μ' are the dual partitions of λ and μ, respectively. Then, by a result going back to work of Birkhoff [1935], $a(\lambda, \mu; p) = \alpha(\lambda, \mu; p)$; see [Butler 1994, Lemma 1.4.1], see also [Dyubyuk 1948; Delsarte 1948; Yeh 1948].

In practical applications invoking infinitely many instances of this formula, as in [Schein and Voll 2015; Lee and Voll 2018], it proved advantageous to sort pairs of partitions by their "overlap types" indexed by Dyck words, as we now recall.

Let $c \in \mathbb{N}$. A *Dyck word of length* $2c$ is a word

$$w = \mathbf{0}^{L_1}\mathbf{1}^{M_1}\mathbf{0}^{L_2-L_1}\mathbf{1}^{M_2-M_1}\cdots\mathbf{0}^{L_r-L_{r-1}}\mathbf{1}^{M_r-M_{r-1}}$$

in letters $\mathbf{1}$ and $\mathbf{0}$, both occurring c times each (hence $L_r = M_r = c$), and, crucially, no initial segment of w contains more ones than zeroes (or, equivalently, $M_i \leq L_i$ for all $i \in [r]$). Here, both the L_i and M_i are assumed to be positive. Below, we will use the notational conventions $M_0 = L_0 = 0$ and $L_{r+1} = L_r = c$, $M_{r+1} = M_r = c$. We write $\mathcal{D}_{2c}$ for the set of all Dyck words of length $2c$. See [Schein and Voll 2015, Section 2.4] or [Stanley 1999, Example 6.6.6] for further details on Dyck words.

We say that two partitions λ and μ, both with c parts and satisfying $\mu \leq \lambda$, have *overlap type* $w \in \mathcal{D}_{2c}$, written $w(\lambda, \mu) = w$, if

$$\begin{aligned}\lambda_1 \geq \cdots \geq \lambda_{L_1} \geq \mu_1 \geq \cdots \geq \mu_{M_1} > \lambda_{L_1+1} \geq \cdots \geq \lambda_{L_2} \geq \mu_{M_1+1} \geq \cdots \geq \mu_{M_2} > \\ \cdots > \lambda_{L_{r-1}+1} \geq \cdots \geq \lambda_c \geq \mu_{M_{r-1}+1} \geq \cdots \geq \mu_c.\end{aligned} \tag{2-6}$$

In Definition 4.11 we slightly modify this definition to suit the specific needs of this paper.

2.3. ***Gaussian multinomials and symmetric groups.*** In Section 3, the following Coxeter group theoretic interpretation of the Gaussian multinomials will be useful. Recall that the symmetric group $W = S_n$ of degree n is a Coxeter group, with Coxeter generating system $S = (s_1, \ldots, s_{n-1})$, where $s_i = (i\,i+1)$ denotes the standard transposition. The *Coxeter length* $\ell(w)$ of an element $w \in S_n$ is the length of a shortest word for w with elements from S. We define the *(right) descent set* $\mathrm{Des}(w) = \{i \in [n-1] \mid \ell(ws_i) < \ell(w)\}$. It is well-known [Stanley 2012, Proposition 1.7.1] that the Gaussian multinomials (2-2) satisfy

$$\binom{n}{J}_Y = \sum_{w \in S_n, \mathrm{Des}(w) \subseteq J} Y^{\ell(w)}. \tag{2-7}$$

Let w_0 denote the unique ℓ-longest element in S_n, of length $\ell(w_0) = \binom{n}{2}$. Then, for all $w \in S_n$,

$$\ell(ww_0) = \ell(w_0) - \ell(w), \quad \mathrm{Des}(ww_0) = [n-1] \setminus \mathrm{Des}(w); \tag{2-8}$$

see [Humphreys 1990, Section 1.8].

2.4. ***A note on ramification.*** Let $\mathfrak{o}$ be a compact discrete valuation ring of arbitrary characteristic. Let $\mathfrak{m}$ denote the maximal ideal of $\mathfrak{o}$ and let $\pi \in \mathfrak{o}$ be a uniformizer, i.e., any element such that $\mathfrak{m} = \pi\mathfrak{o}$. Let $\mathfrak{O}$ be a finite extension of $\mathfrak{o}$, with maximal ideal $\mathfrak{M}$ and uniformizer Π. Let $f = [\mathfrak{O}/\mathfrak{M} : \mathfrak{o}/\mathfrak{m}]$ be the inertia degree of the extension $\mathfrak{O}/\mathfrak{o}$, and let e be its ramification index; this means that $\pi\mathfrak{O} = \mathfrak{M}^e$. We will need the following standard fact.

Lemma 2.3. *Let $\mathfrak{O}$ be a finite extension of $\mathfrak{o}$ with ramification index e and inertia degree f. Let $\tau \in \mathbb{N}_0$. Suppose that $\tau = ge + h$, where $g \in \mathbb{N}_0$ and $h \in [e-1]_0$. Then the following isomorphism of $\mathfrak{o}$-modules holds*:

$$\mathfrak{O}/\mathfrak{M}^\tau \simeq (\mathfrak{o}/\mathfrak{m}^{g+1})^{hf} \times (\mathfrak{o}/\mathfrak{m}^g)^{(e-h)f}.$$

In particular, if $\mathfrak{O}/\mathfrak{o}$ is unramified (i.e., $e = 1$), then $\mathfrak{O}/\mathfrak{M}^\tau \simeq (\mathfrak{o}/\mathfrak{m}^\tau)^f$ as $\mathfrak{o}$-modules.

Proof. Let $\beta_1, \dots, \beta_f \in \mathfrak{O}$ be a collection of elements whose reductions modulo $\mathfrak{M}$ constitute an $\mathfrak{o}/\mathfrak{m}$-basis of the residue field $\mathfrak{O}/\mathfrak{M}$. The set $\{\beta_i \Pi^j \mid i \in [f],\ j \in [e-1]_0\}$ provides a basis for $\mathfrak{O}$ as an $\mathfrak{o}$-module; see, for instance, the proof of [Neukirch 1999, Proposition II.6.8]. Now it is clear that $\mathfrak{M}^\tau = \Pi^\tau \mathfrak{O}$ is the $\mathfrak{o}$-linear span of the set

$$\{\pi^{g+1}\beta_i\Pi^j \mid i \in [f],\ j \in [0, h-1]\} \cup \{\pi^g \beta_i \Pi^j \mid i \in [f],\ j \in [h, e-1]\}. \qquad \square$$

Definition 2.4. For $\tau \in \mathbb{N}_0$ and $e, f \in \mathbb{N}$, let $\{\tau\}_{e,f} = \{(g+1)^{(hf)}, g^{((e-h)f)}\}$ be the ef-element multiset consisting of the element $g+1$ with multiplicity hf and the element g with multiplicity $(e-h)f$, where $\tau = ge + h$ and $h \in [e-1]_0$, as in Lemma 2.3.

3. Generalized Igusa functions

In Section 3.1 we introduce generalized Igusa functions and prove that they satisfy functional equations. In Section 3.2 we record an identity involving weak order zeta functions, motivated by our applications of Igusa functions in ideal growth in Section 5.

3.1. ***Generalized Igusa functions and their functional equations.*** Let $\underline{n} = (n_1, \dots, n_m)$ be a composition of $N = \sum_{i=1}^m n_i$ with m parts. Consider the poset $C_{\underline{n}}$ of subwords of the word $v_{\underline{n}} := a_1^{n_1} a_2^{n_2} \dots a_m^{n_m}$ in "letters" $a_1, a_2, \dots, a_m$, each occurring with respective multiplicity n_i. This poset is naturally isomorphic to the lattice

$$C_{n_1} \times \dots \times C_{n_m},$$

the product of the chains of lengths n_i with the product order, which we denote by "$\leq$". We write $\hat{1} = v_{\underline{n}}$ and $\hat{0}$ for the empty word.

We denote by $\mathrm{WO}_{\underline{n}}$ the chain (or order) complex of $C_{\underline{n}}$. An element $V \in \mathrm{WO}_{\underline{n}}$ is a (possibly empty) chain, or flag, of nonempty subwords of $v_{\underline{n}}$, of the form $V = \{v_1 < \dots < v_t\}$. On $\mathrm{WO}_{\underline{n}}$ we consider the

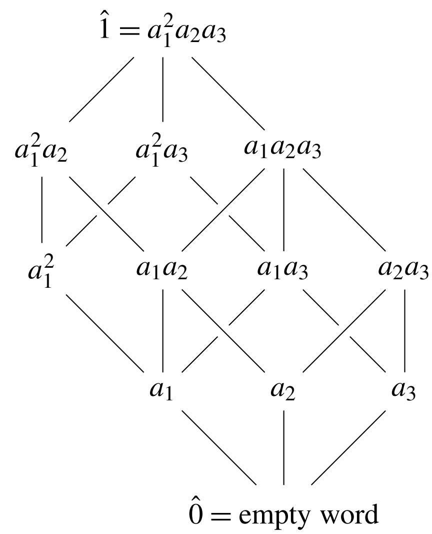

Figure 1. The poset $C_{\underline{n}}$ for $\underline{n} = (2, 1, 1)$.

partial order defined by refinement of flags, also denoted by "$\leq$". Consider the natural map

$$\underline{\pi} : C_{\underline{n}} \to [n_1]_0 \times \cdots \times [n_m]_0,$$
$$v = a_1^{\alpha_1} \ldots a_m^{\alpha_m} \mapsto (\alpha_1, \ldots, \alpha_m) =: (\pi_1(v), \ldots, \pi_m(v)).$$

The degree of the word $v = a_1^{\alpha_1} \ldots a_m^{\alpha_m}$ is $|v| := \sum_{i=1}^m \alpha_i$.

Definition 3.1. We consider the induced morphism of posets

$$\underline{\varphi} : \mathrm{WO}_{\underline{n}} \to \prod_{i=1}^m \mathcal{P}([n_i - 1]),$$
$$V = \{v_1 < \cdots < v_t\} \mapsto (\{\pi_i(v_j) \mid j \in [t]\} \cap [n_i - 1])_{i=1}^m =: (\varphi_i(V))_{i=1}^m.$$

We say that $V \in \mathrm{WO}_{\underline{n}}$ has *full projections* if

$$\underline{\varphi}(V) = ([n_1 - 1], \ldots, [n_m - 1]) =: K.$$

Remark 3.2. We observe that the flag $V = \{v_1 < \cdots < v_t\} \in \mathrm{WO}_{\underline{n}}$ has full projections if, and only if, for all $j \in [t]_0$, the word v_{j+1}/v_j is squarefree, i.e., contains at most one copy of each letter $a_1, \ldots, a_m$. Here we have set $v_0 = \hat{0}$ and $v_{t+1} = \hat{1}$.

Definition 3.3. Let $V = \{v_1 < \cdots < v_t\} \in \mathrm{WO}_{\underline{n}}$. We define

$$W_V(\boldsymbol{X}) = \prod_{j=1}^t \frac{X_{v_j}}{1 - X_{v_j}} \in \mathbb{Q}(X_{v_1}, \ldots, X_{v_t}) \quad \text{and} \quad \binom{\underline{n}}{V}_{\boldsymbol{Y}} = \prod_{i=1}^m \binom{n_i}{\varphi_i(V)}_{Y_i} \in \mathbb{Q}(Y_1, \ldots, Y_m),$$

where $\underline{\varphi}(V) = (\varphi_1(V), \ldots, \varphi_m(V))$.

Example 3.4. Let $\underline{n} = (3, 2, 2)$. The flag $V = \{a_2a_3 < a_1a_2^2a_3\} \in \mathrm{WO}_{(3,2,2)}$ does not have full projections, as $\underline{\varphi}(V) = (\{1\}, \{1\}, \{1\})$. We note that

$$W_V(\boldsymbol{X}) = \frac{X_{a_2a_3}X_{a_1a_2^2a_3}}{(1-X_{a_2a_3})(1-X_{a_1a_2^2a_3})}$$

and

$$\binom{\underline{n}}{V}_{\boldsymbol{Y}} = \binom{3}{1}_{Y_1}\binom{2}{1}_{Y_2}\binom{2}{1}_{Y_3} = (1+Y_1+Y_1^2)(1+Y_2)(1+Y_3).$$

The following is the key combinatorial tool of this paper.

Definition 3.5. The *generalized Igusa function associated with the composition $\underline{n}$* is

$$I_{\underline{n}}^{\mathrm{wo}}(\boldsymbol{Y}; \boldsymbol{X}) := \sum_{V \in \mathrm{WO}_{\underline{n}}} \binom{\underline{n}}{V}_{\boldsymbol{Y}} W_V(\boldsymbol{X}) \in \mathbb{Q}(Y_1, \dots, Y_m, (X_r)_{r \le v_{\underline{n}}}).$$

Example 3.6. (1) For $\underline{n} = (N)$, the trivial composition of N, we recover $I_{(N)}^{\mathrm{wo}}(\boldsymbol{Y}; \boldsymbol{X}) = I_N(Y; \boldsymbol{X})$, the classical Igusa zeta function recalled in Definition 2.2.

(2) For $\underline{n} = (1, \dots, 1)$, the all-one composition of N, we recover $I_{(1,\dots,1)}^{\mathrm{wo}}(\boldsymbol{Y}; \boldsymbol{X}) = I_N^{\mathrm{wo}}(\boldsymbol{X})$, the weak order zeta function recalled in (1-6). We note that the variables $\boldsymbol{Y}$ do not appear in this case, as all the polynomials $\binom{\underline{n}}{V}_{\boldsymbol{Y}}$ are equal to the constant 1.

(3) For $\underline{n} = (2, 1)$ we obtain

$$I_{(2,1)}^{\mathrm{wo}}(\boldsymbol{Y}; \boldsymbol{X}) = \frac{1}{1-X_{a_1^2a_2}}\Bigg(1+\frac{X_{a_2}}{1-X_{a_2}}+\frac{X_{a_1^2}}{1-X_{a_1^2}}+(1+Y_1)\bigg(\frac{X_{a_1}}{1-X_{a_1}}+\frac{X_{a_1a_2}}{1-X_{a_1a_2}}+\frac{X_{a_1}}{1-X_{a_1}}\frac{X_{a_1a_2}}{1-X_{a_1a_2}} + \frac{X_{a_1}}{1-X_{a_1}}\frac{X_{a_1^2}}{1-X_{a_1^2}}+\frac{X_{a_2}}{1-X_{a_2}}\frac{X_{a_1a_2}}{1-X_{a_1a_2}}\bigg)\Bigg).$$

Remark 3.7. Generalized Igusa functions associated with the all-one compositions also coincide with certain instances of generating functions associated with chain partitions in [Beck and Sanyal 2018, Section 4.9].

The following "combinatorial reciprocity theorem" is the main result of this section.

Theorem 3.8. *The generalized Igusa function associated with the composition $\underline{n}$ of $N = \sum_{i=1}^{m} n_i$ satisfies the following functional equation*:

$$I_{\underline{n}}^{\mathrm{wo}}(\boldsymbol{Y}^{-1}; \boldsymbol{X}^{-1}) = (-1)^N X_{v_{\underline{n}}}\left(\prod_{i=1}^{m} Y_i^{-\binom{n_i}{2}}\right) I_{\underline{n}}^{\mathrm{wo}}(\boldsymbol{Y}; \boldsymbol{X}).$$

For the proof of Theorem 3.8 we require a number of preliminary results. The first records simple but crucial "inversion properties" of the rational functions $W_V(\boldsymbol{X})$.

Lemma 3.9. *For all $V \in \mathrm{WO}_{\underline{n}}$,*

$$W_V(\boldsymbol{X}^{-1}) = (-1)^{|V|} \sum_{Q \le V} W_Q(\boldsymbol{X}).$$

Proof. This is a trivial consequence of the observation that

$$\frac{X^{-1}}{1-X^{-1}} = -\left(1+\frac{X}{1-X}\right). \qquad \square$$

We fix some notation used in the rest of this section. We let $\mathrm{WO}^{\times}_{\underline{n}}$ denote the subcomplex of $\mathrm{WO}_{\underline{n}}$ of flags of *proper* subwords of $v_{\underline{n}}$. When dealing with tuples of sets, we will abuse notation and use set theoretical operations for componentwise operations. For instance, for $I = (I_1, \dots, I_m) \in \prod_{i=1}^m \mathcal{P}([n_i - 1])$ we write $I^c := K \setminus I$ for $([n_1 - 1] \setminus I_1, \dots, [n_m - 1] \setminus I_m)$.

The following analogue of [Voll 2006, Lemma 7] is critical for our analysis.

Proposition 3.10. *For all $I \in \prod_{i=1}^m \mathcal{P}([n_i - 1])$,*

$$\sum_{\substack{V \in \mathrm{WO}^{\times}_{\underline{n}} \\ \underline{\varphi}(V) \supseteq I}} W_V(\boldsymbol{X}^{-1}) = (-1)^{N-1} \sum_{\substack{V \in \mathrm{WO}^{\times}_{\underline{n}} \\ \underline{\varphi}(V) \supseteq I^c}} W_V(\boldsymbol{X}). \tag{3-1}$$

Proof. Let $I \in \prod_{i=1}^m \mathcal{P}([n_i - 1])$. The inversion properties established in Lemma 3.9 yield

$$\sum_{\substack{V \in \mathrm{WO}^{\times}_{\underline{n}} \\ \underline{\varphi}(V) \supseteq I}} W_V(\boldsymbol{X}^{-1}) = \sum_{\substack{V \in \mathrm{WO}^{\times}_{\underline{n}} \\ \underline{\varphi}(V) \supseteq I}} (-1)^{|V|} \sum_{Q \le V} W_Q(\boldsymbol{X}) = \sum_{V \in \mathrm{WO}^{\times}_{\underline{n}}} W_V(\boldsymbol{X}) \sum_{\substack{S \ge V \\ \underline{\varphi}(S) \supseteq I}} (-1)^{|S|}.$$

We are left with proving that, for all $V \in \mathrm{WO}^{\times}_{\underline{n}}$,

$$\sum_{\substack{S \ge V \\ \underline{\varphi}(S) \supseteq I}} (-1)^{|S|} = \begin{cases} (-1)^{N-1} & \text{if } \underline{\varphi}(V) \supseteq I^c, \\ 0 & \text{otherwise.} \end{cases} \tag{3-2}$$

Write $V = \{v_1 < \dots < v_t\}$ and set $v_0 := \hat{0}$ and $v_{t+1} := \hat{1}$. Set

$$I_V := I \cup \underline{\varphi}(V) \in \prod_{i=1}^m \mathcal{P}([n_i - 1]).$$

The sum in (3-2) runs over refinements S of the flag V, subject to additional constraints on the projection of S given by I: we say that a refinement S of V is *admissible* if $\underline{\varphi}(S) \supseteq I_V$. As $\underline{\varphi}$ is a poset morphism, the sum in (3-2) runs exactly over the admissible refinements of V.

We will construct such refinements of V "locally". More precisely, let $j \in [t]_0$. We say that S is a *refinement of V between v_j and v_{j+1}* if $S \ge V$ and S and V coincide outside the interval $[v_j, v_{j+1}]$. We further say that $S \ge V$ has *full projections between v_j and v_{j+1}* if $\underline{\varphi}(S \cap [v_j, v_{j+1}])$ is an m-tuple of intervals.

We set

$$I_V^{(j)} := (I_{V,i} \cap [\pi_i(v_j), \pi_i(v_{j+1})])_{i=1}^m \in \prod_{i=1}^m \mathcal{P}([n_i - 1]).$$

Informally, $I_V^{(j)}$ dictates the constraints on a refinement S of V between v_j and v_{j+1}. More precisely, we say that a refinement S of V between v_j and v_{j+1} is *j-admissible* if $\underline{\varphi}(S) \supseteq I_V^{(j)}$. We further define

$$F_j(V, I) := \sum_{\substack{S \geq V \\ j\text{-admissible}}} (-1)^{|S \setminus V|} = \sum_{\substack{S \geq V \\ j\text{-admissible}}} (-1)^{|(S \setminus V) \cup \{v_j, v_{j+1}\}|}.$$

Clearly, given j-admissible refinements V_j of V for all $j \in [t]_0$, the flag $S := \bigcup_{j=0}^{t} V_j$ is an admissible refinement of V and any ("global") admissible refinement of V can be constructed in this way. The sum in (3-2) may thus be rewritten as follows:

$$\sum_{\substack{S \geq V \\ \underline{\varphi}(S) \supseteq I}} (-1)^{|S|} = \sum_{\substack{S \geq V \\ \underline{\varphi}(S) \supseteq I}} (-1)^{|V| + |S \setminus V|} = (-1)^t \sum_{\substack{S \geq V \\ \underline{\varphi}(S) \supseteq I}} (-1)^{|S \setminus V|} = (-1)^t \prod_{j=0}^{t} F_j(V, I). \tag{3-3}$$

We prove (3-2) distinguishing the two cases

(I) $I_V = \underline{\varphi}(V)$ (equivalently, $I \subseteq \underline{\varphi}(V)$) and

(II) $I_V \neq \underline{\varphi}(V)$ (equivalently, $I \setminus \underline{\varphi}(V) \neq \varnothing$).

Case (I). Assume first that $I \subseteq \underline{\varphi}(V)$. In this case, the condition $\underline{\varphi}(S) \supseteq I$ is trivially satisfied for any flag $S \geq V$, as $\underline{\varphi}$ is a poset morphism, and thus any refinement of V is admissible. Moreover, in this case, $\underline{\varphi}(V) \supseteq I^c$ if and only if V has full projections. In other words, (3-2) may be rewritten as follows:

$$\sum_{S \geq V} (-1)^{|S|} = \begin{cases} (-1)^{N-1} & \text{if } V \text{ has full projections,} \\ 0 & \text{otherwise.} \end{cases} \tag{3-4}$$

Let $j \in [t]_0$. As in the case under consideration all local refinements are j-admissible, $F_j(V, I)$ is given in terms of the Möbius function of the interval $[v_j, v_{j+1}]$ in the lattice $C_{\underline{n}}$. Indeed, by Philip Hall's theorem (see, for instance, [Stanley 2012, Proposition 3.8.5]),

$$F_j(V, I) = -\mu(v_j, v_{j+1}) = \begin{cases} (-1)^{|v_{j+1}| - |v_j| + 1} & \text{if } [v_j, v_{j+1}] \text{ is a Boolean algebra,} \\ 0 & \text{otherwise;} \end{cases}$$

see [Stanley 2012, Example 3.8.4]. Using (3-3) we may therefore rewrite the left-hand side of (3-2) as

$$(-1)^t \prod_{j=0}^{t} F_j(V, I) = (-1)^t \prod_{j=0}^{t} (-\mu(v_j, v_{j+1})).$$

It is nonzero if and only if all of its factors are nonzero. The interval $[v_j, v_{j+1}]$ is a Boolean algebra if and only if the word v_{j+1}/v_j is squarefree. By Remark 3.2, this happens for all $j \in [t]_0$ if and only if V

has full projections. In this case we obtain

$$\begin{aligned}\sum_{S\geq V}(-1)^{|S|} &= (-1)^t\sum_{S\geq V}(-1)^{|S\setminus V|}\\ &= (-1)^t\prod_{j=0}^{t}F_j(V,I)\\ &= (-1)^t\prod_{j=0}^{t}(-\mu(v_j,v_{j+1}))\\ &= (-1)^{2t+1}(-1)^{\sum_{j=0}^{t}(|v_{j+1}|-|v_j|)}\\ &= (-1)^{N-1},\end{aligned}$$

proving (3-4) and therefore (3-2) in the case $I\subseteq\underline{\varphi}(V)$.

Case (II). Assume now that $I\setminus\underline{\varphi}(V)\neq\varnothing$. Note that $\underline{\varphi}(V)\supseteq I^c$, the condition invoked in (3-2), holds if and only if $I_V=K$, i.e., if and only if $I_V^{(j)}$ is a tuple of intervals for all $j\in[t]_0$.

We claim that, in the case under consideration, the following holds for all $j\in[t]_0$:

$$F_j(V,I)=\begin{cases}(-1)^{|v_{j+1}|-|v_j|+1} & \text{if } I_V^{(j)} \text{ is a tuple of intervals,}\\ 0 & \text{otherwise.}\end{cases}\tag{3-5}$$

We now prove this claim by induction on the degree of the word v_{j+1}/v_j.

If v_{j+1} covers v_j, then $F_j(V,I)=1$ trivially. So assume that (3-5) holds for $|v_{j+1}/v_j|\leq\ell$, for some $1\leq\ell\in\mathbb{N}$, and suppose that $|v_{j+1}/v_j|=\ell+1$. Let ρ_j denote the number of different letters in v_{j+1}/v_j.

Assume first that $I_V^{(j)}$ is a tuple of intervals, viz.

$$I_V^{(j)}=([\pi_i(v_j),\pi_i(v_{j+1})]\cap[n_i-1])_{i=1}^m.$$

Informally, this means that a j-admissible refinement S of V needs to have full projections between v_j and v_{j+1}. This condition forces the first element of $S\setminus V$ to lie on the ρ_j-dimensional hypercube above v_j: it is obtained by multiplying v_j with at most one copy of each of the ρ_j relevant letters. We may therefore write $F_j(V,I)$ as a sum of $2^{\rho_j}-1$ summands, indexed by the words $v^{(1)},\ldots,v^{(2^{\rho_j}-1)}$ covering v_j in $C_{\underline{n}}$:

$$F_j(V,I)=-\sum_{k=1}^{2^{\rho_j}-1}\sum_{\substack{S\geq V\ j\text{-adm.},\\ \min(S\setminus V)=v^{(k)}}}(-1)^{|S\setminus V|},$$

where, for each $k\in[2^{\rho_j}-1]$, the inner sum is taken over j-admissible refinements S of V having $v^{(k)}$ as smallest element greater than v_j. Each of these sums is known by induction from (3-5). Indeed, since the flags S also have full projections between $v^{(k)}$ and v_{j+1}, we obtain

$$F_j(V,I)=-\sum_{k=1}^{2^{\rho_j}-1}(-1)^{|v_{j+1}|-|v^{(k)}|+1}=(-1)^{|v_{j+1}|-|v_j|+1},$$

establishing (3-5) in the first case.

Suppose now that $I_V^{(j)}$ is not a tuple of intervals. Informally, this means that a j-admissible refinement S of V is not required to have full projections between v_j and v_{j+1}. Without loss of generality we can assume that the first "requirement gap" in $I_V^{(j)}$ is directly above v_j, that is if $\underline{\alpha} = (\alpha_1, \dots, \alpha_m)$ is the m-tuple of (componentwise) minima of $I_V^{(j)} \setminus \underline{\pi}(v_j)$, there is at least one $i \in [m]$ with $\alpha_i > \pi_i(v_j) + 1$. Given a j-admissible refinement S of V, the word $\min(S \setminus V)$, the smallest word in S greater than v_j, clearly belongs to the interval $(v_j, v_{\underline{\alpha}}]$ of subwords of $v_{\underline{\alpha}} := a_1^{\alpha_1} \dots a_m^{\alpha_m}$ which v_j strictly divides. Consider the subset

$$Y := \{v \in (v_j, v_{\underline{\alpha}}] \mid [v, v_{\underline{\alpha}}] \text{ is a Boolean algebra}\}.$$

We rewrite the sum defining $F_j(V, I)$ according to whether or not $\min(S \setminus V) \in Y$:

$$F_j(V, I) = \sum_{\substack{S \geq V \ j\text{-adm.},\\ \min(S \setminus V) \notin Y}} (-1)^{|S \setminus V|} + \sum_{\substack{S \geq V \ j\text{-adm.},\\ \min(S \setminus V) \in Y}} (-1)^{|S \setminus V|}. \tag{3-6}$$

Clearly, the first summand in (3-6) is zero. Indeed, we may further subdivide it by fixing the minimal element $\min(S \setminus V)$. Each of the resulting summands is zero by applying (3-5) inductively to the refined flag $V \cup \{v\}$, replacing v_j by v.

The second summand in (3-6) is zero, too. Indeed, without loss of generality we may assume that

$$I_V^{(j)} = ((\{\pi_i(v_j)\} \cup [\alpha_i, \pi_i(v_{j+1})]) \cap [n_i - 1])_{i=1}^m.$$

(Otherwise, an argument similar to the one for the first summand in (3-6) proves the claim.) Under this assumption, the induction hypothesis yields

$$\sum_{\substack{S \geq V \ j\text{-adm.},\\ \min(S \setminus V) \in Y}} (-1)^{|S \setminus V|} = - \sum_{[v, v_{\underline{\alpha}}] \text{ Boolean}} (-1)^{|v_{j+1}| - |v|} = (-1)^{|v_{j+1}| - |v_{\underline{\alpha}}| + 1} \sum_{Z \subseteq \{0,1\}^{\rho_j}} (-1)^{|Z|} = 0.$$

This proves (3-5) in the second case.

Suppose now $I_V = K$. Since $I_V^{(j)}$ is a tuple of intervals for all $j \in [t]_0$, we get, by (3-5),

$$\sum_{\substack{S \geq V\\ \underline{\varphi}(S) \supseteq I_V}} (-1)^{|S \setminus V|} = (-1)^t \prod_{j=0}^{t} F_j(V, I) = (-1)^{2t+1} (-1)^{\sum_{j=0}^{t} |v_{j+1}| - |v_j|} = (-1)^{N-1}$$

as desired.

Suppose now $I_V \neq K$. This means that there exists $j \in [t]_0$ such that $I_V^{(j)}$ is not a tuple of intervals. By (3-5) we have $F_j(V, I) = 0$, thus the product in (3-3) is also zero, proving (3-2) in the last case. □

Proof of Theorem 3.8. The sum defining the generalized Igusa function can be rewritten as

$$I_{\underline{n}}^{\text{wo}}(\boldsymbol{Y}; \boldsymbol{X}) = \sum_{V \in \text{WO}_{\underline{n}}} \binom{\underline{n}}{V}_{\boldsymbol{Y}} W_V(\boldsymbol{X}) = \frac{1}{1 - X_{v_{\underline{n}}}} \sum_{V \in \text{WO}_{\underline{n}}^{\times}} \binom{\underline{n}}{V}_{\boldsymbol{Y}} W_V(\boldsymbol{X}). \tag{3-7}$$

Inverting the variable in the factor $1/(1-X_{v_{\underline{n}}})$ on the right-hand side of (3-7) simply gives a factor $-X_{v_{\underline{n}}}$. Thus Theorem 3.8 is equivalent to the identity

$$\sum_{V\in\mathrm{WO}_{\underline{n}}^{\times}} \binom{\underline{n}}{V}_{\boldsymbol{Y}^{-1}} W_V(\boldsymbol{X}^{-1}) = (-1)^{N-1}\left(\prod_{i=1}^{m} Y_i^{-\binom{n_i}{2}}\right) \sum_{V\in\mathrm{WO}_{\underline{n}}^{\times}} \binom{\underline{n}}{V}_{\boldsymbol{Y}} W_V(\boldsymbol{X}). \tag{3-8}$$

Writing $\underline{S}_{\underline{n}} = S_{n_1}\times\cdots\times S_{n_m}$, $\underline{w} = (w_1,\dots,w_m)$, $\mathrm{Des}(\underline{w}) = \mathrm{Des}(w_1)\times\cdots\times\mathrm{Des}(w_m)$, and using the identity (2-7), the left-hand side of (3-8) becomes

$$\begin{aligned}\sum_{V\in\mathrm{WO}_{\underline{n}}^{\times}} \binom{\underline{n}}{V}_{\boldsymbol{Y}^{-1}} W_V(\boldsymbol{X}^{-1}) &= \sum_{V\in\mathrm{WO}_{\underline{n}}^{\times}} \Biggl(\sum_{\substack{\underline{w}\in\underline{S}_{\underline{n}}\\ \mathrm{Des}(\underline{w})\subseteq\underline{\varphi}(V)}} \prod_{i=1}^{m} Y_i^{-\ell(w_i)}\Biggr) W_V(\boldsymbol{X}^{-1})\\ &= \sum_{\underline{w}\in\underline{S}_{\underline{n}}}\left(\prod_{i=1}^{m} Y_i^{-\ell(w_i)}\right) \sum_{\substack{V\in\mathrm{WO}_{\underline{n}}^{\times}\\ \underline{\varphi}(V)\supseteq\mathrm{Des}(\underline{w})}} W_V(\boldsymbol{X}^{-1}).\end{aligned}$$

For $i\in[m]$ we denote by $w_0^{(i)}$ the longest element in S_{n_i}, of length $\ell(w_0^{(i)}) = \binom{n_i}{2}$. By Proposition 3.10 and the identities (2-8) we can rewrite

$$\begin{aligned}\sum_{\underline{w}\in\underline{S}_{\underline{n}}}\left(\prod_{i=1}^{m} Y_i^{-\ell(w_i)}\right) \sum_{\substack{V\in\mathrm{WO}_{\underline{n}}^{\times}\\ \underline{\varphi}(V)\supseteq\mathrm{Des}(\underline{w})}} W_V(\boldsymbol{X}^{-1}) &= (-1)^{N-1} \sum_{\underline{w}\in\underline{S}_{\underline{n}}}\left(\prod_{i=1}^{m} Y_i^{-\ell(w_i)}\right) \sum_{\substack{V\in\mathrm{WO}_{\underline{n}}^{\times}\\ \underline{\varphi}(V)\supseteq\mathrm{Des}(\underline{w})^c}} W_V(\boldsymbol{X})\\ &= (-1)^{N-1}\left(\prod_{i=1}^{m} Y_i^{-\binom{n_i}{2}}\right) \sum_{\underline{w}\in\underline{S}_{\underline{n}}}\left(\prod_{i=1}^{m} Y_i^{\ell(w_i w_0^i)}\right) \sum_{\substack{V\in\mathrm{WO}_{\underline{n}}^{\times}\\ \underline{\varphi}(V)\supseteq\mathrm{Des}(\underline{w}\underline{w}_0)}} W_V(\boldsymbol{X})\\ &= (-1)^{N-1}\left(\prod_{i=1}^{m} Y_i^{-\binom{n_i}{2}}\right) \sum_{V\in\mathrm{WO}_{\underline{n}}^{\times}} \binom{\underline{n}}{V}_{\boldsymbol{Y}} W_V(\boldsymbol{X}),\end{aligned}$$

proving (3-8) and thus Theorem 3.8. □

3.2. ***Weak order zeta functions and generalized Igusa functions.*** We record an identity between instances of weak order zeta functions which will be useful in Section 5.3.3 and may be of independent interest. The identity compares instances of weak order zeta functions associated with the all-one-compositions $\underline{g}$ and $\underline{2g}$, with g and $2g$ parts, respectively, and holds when substituting for the variables monomials satisfying certain relations.

In the current section, we call a subword of the word $\hat{1} = v_{\underline{2g}} := a_1\cdots a_{2g}$ *radical* if it is of the form $w = \prod_{i\in\mathcal{J}} a_i a_{i+g}$ for some $\mathcal{J}\subseteq[g]$; see also Definition 4.13. We observe that any subword $r\le v_{\underline{2g}}$ may be written uniquely in the form $r = \sqrt{r}\cdot r'r''$, where $\sqrt{r} = \prod_{i\in\mathcal{I}} a_i a_{i+g}$ is a radical word, whereas $r' = \prod_{i\in\mathcal{I}'} a_i$ and $r'' = \prod_{i\in\mathcal{I}''} a_{i+g}$, and the subsets $\mathcal{I},\mathcal{I}',\mathcal{I}''\subseteq[g]$ are disjoint. Likewise, we define the *radical* $\sqrt{S}$ of a flag $S\in\mathrm{WO}_{\underline{2g}}$ to be the flag of radicals of the words of S.

In the following result, we omit the nonoccurring variable Y from the generalized Igusa functions $I^{\mathrm{wo}}_{\underline{g}}$ and $I^{\mathrm{wo}}_{\underline{2g}}$; see our remark in Example 3.6(2).

Proposition 3.11. *Let $g \in \mathbb{N}$. Suppose that the numerical data $\boldsymbol{y}$ satisfy $y_r = y_{\sqrt{r}} \cdot \prod_{i \in \mathcal{J}' \cup \mathcal{J}''} y_{a_i}$. Then*

$$I^{\mathrm{wo}}_{\underline{2g}}(\boldsymbol{y}) = \left(\prod_{i=1}^{g} \frac{1+y_{a_i}}{1-y_{a_i}}\right) I^{\mathrm{wo}}_{\underline{g}}(\boldsymbol{z}), \tag{3-9}$$

where $z_{\prod_{i\in\mathcal{J}} a_i} = y_{\prod_{i\in\mathcal{J}} a_i a_{i+g}}$ for all $\mathcal{J} \subseteq [g]$.

Proof. By sorting the flags in $\mathrm{WO}_{\underline{2g}}$ by their radicals, we may partition the domain of summation of the left-hand side of (3-9) as follows:

$$\mathrm{WO}_{\underline{2g}} = \bigcup_{R \in \mathrm{WO}_{\underline{g}}} \{S \in \mathrm{WO}_{\underline{2g}} \mid \sqrt{S} = R\}.$$

The claim is equivalent to showing that, for all $R \in \mathrm{WO}_{\underline{g}}$,

$$\sum_{\substack{S \in \mathrm{WO}_{\underline{2g}}:\\ \sqrt{S}=R}} W_S(\boldsymbol{y}) = \left(\prod_{i=1}^{g} \frac{1+y_{a_i}}{1-y_{a_i}}\right) W_R(\boldsymbol{z}) = \prod_{i=1}^{g}\left(1+2\frac{y_{a_i}}{1-y_{a_i}}\right) W_R(\boldsymbol{z}). \tag{3-10}$$

Let $S = \{s_1 < \cdots < s_t\} = \{\sqrt{s_1} \cdot s_1' s_1'' < \cdots < \sqrt{s_t} \cdot s_t' s_t''\} \in \mathrm{WO}_{\underline{2g}}$, where, as above, for $k \in [t]$, $s_k' = \prod_{i \in \mathcal{J}'} a_i$, $s_k'' = \prod_{i \in \mathcal{J}''} a_{i+g}$ and $\sqrt{s_k} = \prod_{i \in \mathcal{J}_k} a_i a_{i+g}$ is radical. Denote $J(S) = \{y_{s_1}, \ldots, y_{s_t}\}$ and, for $j \in [g]$, set $y_{a_j} J(S) := \{y_{a_j} y \mid y \in J(S)\}$. As before we set $s_0 = \hat{0}$ and $s_{t+1} = \hat{1} = v_{\underline{2g}}$.

We claim that, for all $j \in [g]$ and all $S \in \mathrm{WO}_{\underline{2g}}$ with $\sqrt{S} = R$ and the property that, for all $s \in S$ if $a_j \mid s$ or $a_{g+j} \mid s$ then $a_j a_{g+j} \mid s$, the following identity holds:

$$\sum_{\substack{\bar{S} \in \mathrm{WO}_{\underline{2g}}: \sqrt{\bar{S}}=R,\\ J(\bar{S}) \subset J(S) \cup y_{a_j} J(S)}} W_{\bar{S}}(\boldsymbol{y}) = \left(1+2\frac{y_{a_j}}{1-y_{a_j}}\right) W_S(\boldsymbol{y}). \tag{3-11}$$

It is easy to see that (3-10) follows by repeated application of (3-11) for $j \in [g]$.

We prove (3-11) by induction on t, the induction base ($t = 0$) being trivial; we observe that our assumption on the numerical data implies that $y_{a_j} = y_{a_{g+j}}$. The right-hand side may therefore be written as

$$\left(\prod_{l=1}^{i} \frac{y_{s_l}}{1-y_{s_l}}\right)\left(1+\frac{y_{a_j}}{1-y_{a_j}}+\frac{y_{a_{g+j}}}{1-y_{a_{g+j}}}\right)\left(\prod_{l=i+1}^{t} \frac{y_{s_l}}{1-y_{s_l}}\right).$$

The summand 1 in the central factor arises from the flag $\bar{S} = S$, with $W_S(\boldsymbol{y}) = \prod_{i=1}^{t} y_{s_l}/(1-y_{s_l})$. The other two summands account for flags $\bar{S}$ with $J(\bar{S}) = y_{a_j} J(S)$, i.e., for flags whose words differ from those of S by at most an extra factor a_j or a_{g+j} (but not both, as they share with S the radical R), and which do feature at least one such a “augmented” word. We will call such flags *a_j-augmentations* (*of S*).

It remains to show that

$$\sum_{\substack{\overline{S}\in \mathrm{WO}_{\underline{2g}}:\\ a_j\text{-augmentation of } S}} W_{\overline{S}}(\boldsymbol{y}) = \left(\prod_{l=1}^{t} \frac{y_{s_l}}{1-y_{s_l}}\right)\frac{y_{a_j}}{1-y_{a_j}}; \tag{3-12}$$

the argument for a_{g+j} is identical.

We note that there exists a unique $i \in [t]$ such that $a_j \mid s_{i+1}$ but $a_j \nmid s_i$. For all a_j-augmentations $\overline{S}$ of S, the last $t-i$ words coincide with $s_{i+1}, \dots, s_t$. Therefore $\prod_{l=i+1}^{t} y_{s_l}/(1-y_{s_l})$ divides all relevant $W_{\overline{S}}(\boldsymbol{y})$. Without loss of generality we may thus assume that $i=t$, i.e., that *no word of S is divisible by a_j*.

The claimed identity in (3-12) will become clear by interpreting the trivial identity

$$\left(\prod_{l=1}^{t} \frac{y_{s_l}}{1-y_{s_l}}\right)\frac{y_{a_j}}{1-y_{a_j}} = \left(\prod_{l=1}^{t-1} \frac{y_{s_l}}{1-y_{s_l}}\right)\left(\frac{y_{a_j}y_{s_t}}{1-y_{a_j}y_{s_t}} + \frac{y_{s_t}}{1-y_{s_t}}\frac{y_{a_j}y_{s_t}}{1-y_{a_j}y_{s_t}} + \frac{y_{a_j}}{1-y_{a_j}}\frac{y_{a_j}y_{s_t}}{1-y_{a_j}y_{s_t}}\right). \tag{3-13}$$

Informally, the right-hand side of (3-13) reflects the three alternatives for the first occurrence of a_j in an a_j-augmentation of $\overline{S}$:

(1) The first summand arises from the a_j-augmentation $\overline{S} = \{\cdots < s_{t-2} < s_{t-1} < a_j s_t\}$.

(2) The second summand arises from the a_j-augmentation $\overline{S} = \{\cdots < s_{t-1} < s_t < a_j s_t\}$.

(3) The third summand arises from all a_j-augmentations of S whose last *two* words are divisible by a_j, the last one being $a_j s_t$, viz. a_j-augmentations of $S \setminus \{s_t\}$. All the relevant $W_{\overline{S}}(\boldsymbol{y})$ are therefore divisible by $y_{a_j}y_{s_t}/(1-y_{a_j}y_{s_t})$. By induction hypothesis, (3-12) yields

$$\left(\prod_{l=1}^{t-1} \frac{y_{s_l}}{1-y_{s_l}}\right)\frac{y_{a_j}}{1-y_{a_j}} = \sum_{\substack{\overline{S}\in \mathrm{WO}_{\underline{2g}},\\ a_j\text{-augmentation of } S\setminus\{s_t\}}} W_{\overline{S}}(\boldsymbol{y}).$$

This proves the claim, and hence the proposition. □

4. Counting $\mathfrak{o}$-ideals in combinatorially defined $\mathfrak{o}$-Lie algebras

In this section we compute the $\mathfrak{o}$-ideal zeta functions of $\mathfrak{o}$-Lie algebras satisfying a certain combinatorial condition (Hypothesis 4.5) in terms of the generalized Igusa functions introduced in Section 3. This prepares the proof of Theorem 1.3, given in Section 5.

4.1. ***Informal overview.*** We start by summarizing the principal ideas behind our approach, which greatly generalize those of [Schein and Voll 2015]. Let L be an $\mathfrak{o}$-Lie algebra with derived subalgebra $L' = [L, L]$. As noted in Section 1.1.3, if L is class-2-nilpotent, then an $\mathfrak{o}$-sublattice $\Lambda \le L$ is an $\mathfrak{o}$-ideal if $[\overline{\Lambda}, L] \le \Lambda \cap L'$, where $\overline{\Lambda} = (\Lambda + L')/L'$. For simplicity of exposition we will assume, in this overview, that $L' = Z(L)$, i.e., that L has no abelian direct summands. By an argument going back to [Grunewald et al. 1988, Lemma 6.1], the computation of $\zeta_L^{\triangleleft \mathfrak{o}}(s)$ is reduced to a summation over pairs $(\overline{\Lambda}, \mathrm{M})$, where

$\overline{\Lambda} \leq L/L'$ and $\mathrm{M} \leq L'$ are $\mathfrak{o}$-sublattices such that $[\overline{\Lambda}, L] \leq \mathrm{M}$. Recall that the $\mathfrak{O}$-elementary divisor type of a finite-index $\mathfrak{O}$-sublattice $\Lambda \leq \mathfrak{O}^n$, where $\mathfrak{O}$ is a compact discrete valuation ring with maximal ideal $\mathfrak{M}$, is the partition $(\lambda_1, \ldots, \lambda_n)$ such that

$$\mathfrak{O}^n/\Lambda \simeq \mathfrak{O}/\mathfrak{M}^{\lambda_1} \times \cdots \times \mathfrak{O}/\mathfrak{M}^{\lambda_n}.$$

Given the $\mathfrak{o}$-elementary divisor type $\lambda(\overline{\Lambda})$ of $[\overline{\Lambda}, L]$, the lattices M satisfying this condition are enumerated by Birkhoff's formula (2-5).

An essential ingredient of our method, therefore, is an effective description of the $\mathfrak{o}$-elementary divisor type $\lambda(\overline{\Lambda})$ in terms of the structure of $\overline{\Lambda}$. For the $\mathfrak{o}$-Lie algebras considered in this paper, this is accomplished as follows. The quotient L/L' decomposes, as an $\mathfrak{o}$-module, into a direct sum of m components, which are viewed as free modules over finite extensions $\mathfrak{O}_1, \ldots, \mathfrak{O}_m$ of $\mathfrak{o}$. For each component, we consider the $\mathfrak{O}_i$-elementary divisor type $\nu^{(i)}$ of the $\mathfrak{O}_i$-lattice generated by the projection of $\overline{\Lambda}$ onto that component. These are the projection data of Definition 4.1 below. The crucial Hypothesis 4.5 requires that the parts of the partition $\lambda(\overline{\Lambda})$ be given by the minima of term-by-term comparisons among the elementary divisor types appearing in the projection data. Assuming Hypothesis 4.5, we deduce a purely combinatorial expression for $\zeta_L^{\triangleleft \mathfrak{o}}(s)$ in Proposition 4.10.

Analogously to the argument of [Schein and Voll 2015], we break up the sum in Proposition 4.10 into finitely many pieces on which the Gaussian multinomial coefficients — arising via the factors $\beta(\nu^{(i)}; q_i)$ and $\alpha(\lambda(\boldsymbol{\nu}), \mu; q)$, in the notation used there — and the dual partitions occurring in the definition (2-5) of $\alpha(\lambda(\boldsymbol{\nu}), \mu; q)$ are constant. The sum over each piece yields a product of Gaussian multinomials and geometric progressions; these, in turn, are assembled into generalized Igusa functions introduced in Section 3. As in [loc. cit.], Dyck words of fixed length turn out to be suitable indexing objects for the finitely many pieces.

The technical complexity of the current paper, in comparison to [loc. cit.], reflects the fact the translation between projection data and the elementary divisor type $\lambda(\overline{\Lambda})$ is considerably more involved. While the data determining $\lambda(\overline{\Lambda})$ in [loc. cit.] were just a collection of integers, here they are a collection of partitions (the $\nu^{(i)}$ defined above). A more sophisticated combinatorial machinery, viz. the weak orders of Section 3.1, is required to keep track of the relative sizes of the parts of these different partitions; this is necessary in order to specify domains of summation over which the dual partition $\lambda(\overline{\Lambda})'$ is constant.

In Section 4.2 we define the concept of projection data and enumerate lattices $\overline{\Lambda} \leq L/L'$ with fixed projection data. In Section 4.3 we introduce and explain the combinatorial structure behind Hypothesis 4.5 and deduce Proposition 4.10, giving a general formula for $\mathfrak{o}$-ideal zeta functions of $\mathfrak{o}$-Lie algebras satisfying Hypothesis 4.5. In Section 4.4 we state the section's main result, viz. Theorem 4.21, and prove it modulo an auxiliary claim, viz. Proposition 4.20, whose rather technical proof is given in Section 4.5.

Throughout, let $\mathfrak{o}$ be a complete discrete valuation ring with finite residue field of cardinality q, and let $\mathfrak{O}_1, \ldots, \mathfrak{O}_h$ be finite extensions of $\mathfrak{o}$. Let $\pi \in \mathfrak{o}$ be a uniformizer. For each $i \in [h]$, let e_i be the ramification index and f_i be the inertia degree of $\mathfrak{O}_i$ over $\mathfrak{o}$. Let $q_i = q^{f_i}$ be the cardinality of the residue field of $\mathfrak{O}_i$. We write $t = q^{-s}$, where s denotes a complex variable. For each $i \in [h]$, the local ring

$\mathfrak{O}_i$ is a free $\mathfrak{o}$-module of rank $e_i f_i$. Let $(n_1, \dots, n_h) \in \mathbb{N}_0^h$ and set $n = \sum_{i=1}^h e_i f_i n_i$. Consider a family $\tilde{\boldsymbol{\nu}} = (\nu^{(1)}, \dots, \nu^{(h)})$ of partitions $\nu^{(i)}$, each with n_i parts.

4.2. ***Counting lattices with fixed projections.*** Consider the $\mathfrak{o}$-module

$$\Omega = \mathfrak{O}_1^{n_1} \times \cdots \times \mathfrak{O}_h^{n_h}$$

and, for each $i \in [h]$, let $\pi_i : \Omega \to \mathfrak{O}_i^{n_i}$ be the projection onto the i-th component. Choosing an $\mathfrak{O}_i$-basis $(e_1^{(i)}, \dots, e_{n_i}^{(i)})$ of $\mathfrak{O}_i^{n_i}$ and an $\mathfrak{o}$-basis $(\alpha_1^{(i)}, \dots, \alpha_{e_i f_i}^{(i)})$ of each $\mathfrak{O}_i$, it is clear that the collection $\{\alpha_j^{(i)} e_k^{(i)}\}_{ijk}$ constitutes an $\mathfrak{o}$-basis of Ω that allows us to identify Ω with $\mathfrak{o}^n$.

Definition 4.1. For an $\mathfrak{o}$-sublattice $\Lambda \leq \mathfrak{o}^n$, we write $\nu^{(i)} = \nu(\pi_i(\Lambda))$ for the elementary divisor type of the $\mathfrak{O}_i$-sublattice of $\mathfrak{O}_i^{n_i}$ generated by $\pi_i(\Lambda)$. Note that $\nu^{(i)}$ is a partition with n_i parts. The family

$$\boldsymbol{\nu}(\Lambda) = (\nu^{(1)}, \dots, \nu^{(h)})$$

of partitions is called the *projection data* of Λ with respect to Ω.

For any partition $\nu = (\nu_1, \dots, \nu_N)$ with N parts, set $J_\nu = \{d \in [N-1] \mid \nu_d > \nu_{d+1}\}$. For a variable Y, we define

$$\beta(\nu; Y) = \binom{N}{J_\nu}_{Y^{-1}} Y^{\sum_{d=1}^{N-1} d(N-d)(\nu_d - \nu_{d+1})} \in \mathbb{Q}[Y]. \tag{4-1}$$

We observe that $\beta(\nu; Y) = \alpha(\lambda, \nu; Y)$, the "Birkhoff polynomial" (2-5), where λ is any partition whose parts are all at least ν_1. It follows that $\beta(\nu; q)$ enumerates the $\mathfrak{o}$-sublattices of $\mathfrak{o}^N$ of elementary divisor type ν. The following proposition, which is key to our method, generalizes this formula and is analogous to [Schein and Voll 2015, Lemma 2.4]. Recall the formula (1-4) for the zeta function $\zeta_{\mathfrak{o}^n}(s)$ of an abelian (Lie) algebra of finite rank over a compact discrete valuation ring.

Proposition 4.2. *Let $\tilde{\boldsymbol{\nu}} = (\nu^{(1)}, \dots, \nu^{(h)})$ be as above. Then*

$$\sum_{\substack{\Lambda \leq \mathfrak{o}^n \\ \boldsymbol{\nu}(\Lambda) = \tilde{\boldsymbol{\nu}}}} |\mathfrak{o}^n : \Lambda|^{-s} = \frac{\zeta_{\mathfrak{o}^n}(s)}{\prod_{i=1}^h \zeta_{\mathfrak{O}_i^{n_i}}(s)} \left(\prod_{i=1}^h \beta(\nu^{(i)}; q_i) \right) t^{\sum_{i=1}^h \left(\sum_{j=1}^{n_i} \nu_j^{(i)} \right) f_i}.$$

Proof. Recall that for every $i \in [h]$ there is a natural embedding of rings $\iota_i : \mathfrak{O}_i \hookrightarrow \mathrm{Mat}_{e_i f_i}(\mathfrak{o})$ that sends an element $y \in \mathfrak{O}_i$ to the matrix representing the $\mathfrak{o}$-linear operator $x \mapsto xy$ on $\mathfrak{O}_i$ with respect to the chosen $\mathfrak{o}$-basis $\{\alpha_j^{(i)}\}_{j=1}^{e_i f_i}$. Moreover, $\det \iota_i(y) = \mathrm{N}_{\mathfrak{O}_i/\mathfrak{o}}(y)$ for all $y \in \mathfrak{O}_i$. This map extends naturally to an embedding of matrix rings $\mathrm{Mat}_{n_i}(\mathfrak{O}_i) \hookrightarrow \mathrm{Mat}_{e_i f_i n_i}(\mathfrak{o})$ that we continue to denote by ι_i.

Consider the set $\mathcal{H} = \{(H_1, \dots, H_h) \mid \forall i \in [h] : H_i \leq \mathfrak{O}_i^{n_i}\}$. Given $H \in \mathcal{H}$, denote

$$\Sigma_H = \sum_{\substack{\Lambda \leq \mathfrak{o}^n \\ \pi_i(\Lambda) = H_i}} |\mathfrak{o}^n : \Lambda|^{-s}.$$

Thus

$$\sum_{\substack{\Lambda \leq \mathfrak{o}^n \\ \boldsymbol{\nu}(\Lambda) = \tilde{\boldsymbol{\nu}}}} |\mathfrak{o}^n : \Lambda|^{-s} = \sum_{\substack{H \in \mathcal{H} \\ \nu(H_i) = \nu^{(i)}}} \Sigma_H. \tag{4-2}$$

For every $i \in [h]$, let $B_i \in \mathrm{Mat}_{n_i}(\mathfrak{O}_i)$ be a matrix whose rows comprise an $\mathfrak{O}_i$-basis of H_i. Let $B \in \mathrm{Mat}_n(\mathfrak{o})$ be the block-diagonal matrix with blocks $\iota_i(B_i)$. We observe that the map $\mathrm{Mat}_n(\mathfrak{o}) \to \mathrm{Mat}_n(\mathfrak{o})$, $B' \mapsto B'B$ induces a bijection between the set of $\mathfrak{o}$-lattices $\Lambda \leq \mathfrak{o}^n$ such that $\pi_i(\Lambda) = \mathfrak{O}_i^{n_i}$ for all $i \in [h]$ and the set of lattices $\Lambda \leq \mathfrak{o}^n$ such that $\pi_i(\Lambda) = H_i$ for all $i \in [h]$. Furthermore, $\det B = \prod_{i=1}^h \mathrm{N}_{\mathfrak{O}_i\mathfrak{o}}(\det B_i)$; see, for instance, [Kovacs et al. 1999, Theorem 1]. The norms preserve normalized valuation, hence $|\det B|_{\mathfrak{o}} = \prod_{i=1}^h q_i^{-\sum_{j=1}^{n_i} \nu_j^{(i)}}$. We conclude that

$$\Sigma_H = t^{\sum_{i,j} \nu_j^{(i)} f_i} \Sigma_{\mathbf{0}} = \prod_{i=1}^h |\mathfrak{O}_i^{n_i} : H_i|^{-s} \Sigma_{\mathbf{0}}, \tag{4-3}$$

where $\mathbf{0} = (\mathfrak{O}_1^{n_1}, \dots, \mathfrak{O}_h^{n_h}) \in \mathcal{H}$. Thus

$$\zeta_{\mathfrak{o}^n}(s) = \sum_{H \in \mathcal{H}} \Sigma_H = \Sigma_{\mathbf{0}} \sum_{H \in \mathcal{H}} \prod_{i=1}^h |\mathfrak{O}_i^{n_i} : H_i|^{-s} = \Sigma_{\mathbf{0}} \prod_{i=1}^h \zeta_{\mathfrak{O}_i^{n_i}}(s). \tag{4-4}$$

It follows immediately from (4-3) and (4-4) that

$$\Sigma_H = \frac{\zeta_{\mathfrak{o}^n}(s)}{\prod_{i=1}^h \zeta_{\mathfrak{O}_i^{n_i}}(s)} t^{\sum_{i,j} \nu_j^{(i)} f_i},$$

and substitution of this expression into (4-2) implies our claim. □

4.3. ***Rewriting the $\mathfrak{o}$-ideal zeta functions of suitable $\mathfrak{o}$-Lie algebras.*** Now let L be a class-2-nilpotent $\mathfrak{o}$-Lie algebra. We assume that its derived subalgebra L' is isolated, viz. L/L' is torsion-free. Let further $L' \subseteq A \subseteq Z(L)$ be a central, isolated subalgebra. Suppose that

$$L/A \simeq \mathfrak{O}_1^{n_1} \times \cdots \times \mathfrak{O}_h^{n_h}. \tag{4-5}$$

Fixing such an isomorphism, we obtain projections $\pi_i : L/A \to \mathfrak{O}_i^{n_i}$ and are in the setting of Section 4.2. Then c' and c, in the notation of Section 4.2, are the ranks of the free $\mathfrak{o}$-modules L' and A, respectively, whereas $n = \sum_{i=1}^h n_i e_i f_i = \mathrm{rk}_{\mathfrak{o}} L/A$. In particular, $n + c = \mathrm{rk}_{\mathfrak{o}} L$.

Given an $\mathfrak{o}$-sublattice $\Lambda \leq L/A$ of finite index, the commutator $[\Lambda, L]$ is well-defined, as A is central, and of finite index in L'. Let $\lambda(\Lambda)$ be the $\mathfrak{o}$-elementary divisor type of the $\mathfrak{o}$-submodule $[\Lambda, L] \leq L'$.

Definition 4.3. Let $\nu^{(1)} = (\nu_1^{(1)}, \dots, \nu_{n_1}^{(1)})$ and $\nu^{(2)} = (\nu_1^{(2)}, \dots, \nu_{n_2}^{(2)})$ be partitions with n_1 and n_2 parts, respectively. We define $\nu^{(1)} * \nu^{(2)}$ to be the partition whose $n_1 n_2$ parts are obtained from the multiset

$$\{\min\{\nu_k^{(1)}, \nu_\ell^{(2)}\}\}_{k \in [n_1], \ell \in [n_2]}.$$

Given, in addition, $b \in [n_1]$, we define $(\nu^{(1)})^{*b}$ to be the partition whose $\binom{n_1}{b}$ parts are obtained from the multiset

$$\{\min\{\nu_i^{(1)} \mid i \in I\}\}_{I \subseteq [n_1], |I| = b}.$$

We observe that $*$ is an associative binary operation on the set of partitions and that $(\nu^{(1)})^{*2} \neq \nu^{(1)} * \nu^{(1)}$.

Definition 4.4. Let $Z \in \mathbb{N}_0$ and fix, for every $k \in [Z]$, a pair $\widetilde{\mathfrak{S}}_k = (\mathfrak{S}_k, \underline{\sigma}_k)$, where $\mathfrak{S}_k = \{s_{k1}, \dots, s_{k,\tau_k}\} \subseteq [h]$ is a subset of cardinality τ_k and $\underline{\sigma}_k = (\sigma_{k1}, \dots, \sigma_{k,\tau_k}) \in \mathbb{N}^{\tau_k}$.

Given a family $\tilde{\boldsymbol{\nu}} = (\nu^{(1)}, \dots, \nu^{(h)})$ of partitions $\nu^{(i)}$, each with n_i parts, define $\lambda(\tilde{\boldsymbol{\nu}})$ to be the partition obtained from the multiset

$$\bigcup_{k=1}^{Z} \{(\nu^{(s_{k1})})^{*\sigma_{k1}} * \cdots * (\nu^{(s_{k,\tau_k})})^{*\sigma_{k,\tau_k}}\},$$

where $\{\nu^{(i)}\}$ denotes the multiset of parts of the partition $\nu^{(i)}$ and the union is a union of multisets.

We will suppose for the rest of Section 4 that the following assumption on (L, A) holds.

Hypothesis 4.5. *The pairs $\widetilde{\mathfrak{S}}_1, \dots, \widetilde{\mathfrak{S}}_Z$ in Definition 4.4 may be chosen so that for any $\mathfrak{o}$-sublattice $\Lambda \leq L/A$, the equality of partitions $\lambda(\Lambda) = \lambda(\boldsymbol{\nu}(\Lambda))$ holds.*

Comparing the lengths of the partitions $\lambda(\Lambda)$ and $\lambda(\boldsymbol{\nu}(\Lambda))$, we find that Hypothesis 4.5 implies that

$$c' = \sum_{k=1}^{Z} \binom{n_{s_{k1}}}{\sigma_{k1}} \binom{n_{s_{k2}}}{\sigma_{k2}} \cdots \binom{n_{s_{k,\tau_k}}}{\sigma_{k,\tau_k}}.$$

Definition 4.6. Let $\mathfrak{S} = \bigcup_{k=1}^{Z} \mathfrak{S}_k \subseteq [h]$. Let $m = |\mathfrak{S}|$. Renumbering the components in (4-5) if necessary, we may suppose without loss of generality that $\mathfrak{S} = [m]$.

We briefly discuss the motivation for Hypothesis 4.5. It ensures that the elementary divisor type $\lambda(\Lambda)$ depends only on the projection data $\boldsymbol{\nu}(\Lambda)$ and can be described combinatorially in terms of $\boldsymbol{\nu}(\Lambda)$, and that all parts of $\lambda(\Lambda)$ also appear as parts of $\boldsymbol{\nu}(\Lambda)$. This assumption is crucial to our method and enables us to express the $\mathfrak{o}$-ideal zeta function $\zeta_L^{\triangleleft \mathfrak{o}}(s)$ in terms of the generalized Igusa functions of Definition 3.5. A further consequence of Hypothesis 4.5 is a dichotomy among the components of L/A in (4-5). If, on the one hand, $i > m$, then the commutator $[\Lambda, L]$ is independent of the component $\mathfrak{O}_i^{n_i}$; this means that $\mathfrak{O}_i^{n_i}$ lies in the kernel of the projection $\mathrm{pr} : L/A \to L/Z(L)$. If, on the other hand, $i \leq m$, then $\mathrm{pr}(\mathfrak{O}_i^{n_i})$ and $\mathfrak{O}_i^{n_i}$ have the same rank as $\mathfrak{o}$-modules, namely $n_i e_i f_i$. In particular,

$$\sum_{i=1}^{m} n_i e_i f_i = \mathrm{rk}_{\mathfrak{o}}(L/Z(L)). \tag{4-6}$$

This consequence of Hypothesis 4.5 is used in a subtle but crucial way in the proof of Corollary 4.22, which establishes the functional equation satisfied by $\zeta_L^{\triangleleft \mathfrak{o}}(s)$. Indeed, Theorem 4.21 expresses $\zeta_L^{\triangleleft \mathfrak{o}}(s)$ as a sum of finitely many summands. The above observation ensures that each summand satisfies a functional equation with the same symmetry factor.

Remark 4.7. We note that, trivially, Hypothesis 4.5 is stable under direct products.

Remark 4.8. Before proceeding, we observe that Hypothesis 4.5 constrains the extensions $\mathfrak{O}_i$ of $\mathfrak{o}$ to be unramified in natural examples, such as the nonabelian examples considered in Section 5. Suppose that $L = \mathcal{L}_1(\mathfrak{O}_1) \times \cdots \times \mathcal{L}_r(\mathfrak{O}_r)$, where $\mathcal{L}_i$ is a class-2-nilpotent Lie ring and $\mathfrak{O}_i$ is a finite extension of $\mathfrak{o}$ for every $i \in [r]$. Suppose that the subalgebra $L' \leq A \leq Z(L)$ is of the form $A = A_1 \times \cdots \times A_r$, where each A_i is an isolated subalgebra of $\mathcal{L}_i(\mathfrak{O}_i)$; this will be true, for instance, if $A = L'$ or $A = Z(L)$. Then $L/A \simeq \mathcal{L}_1(\mathfrak{O}_1)/A_1 \times \cdots \times \mathcal{L}_r(\mathfrak{O}_r)/A_r$. Suppose, furthermore, that we decompose

$$\begin{aligned}\mathcal{L}_1(\mathfrak{O}_1)/A_1 &\simeq \mathfrak{O}_1^{n_1} \times \cdots \times \mathfrak{O}_1^{n_{N_1}}\\ \mathcal{L}_2(\mathfrak{O}_2)/A_2 &\simeq \mathfrak{O}_2^{n_{N_1+1}} \times \cdots \times \mathfrak{O}_2^{n_{N_2}}\\ &\vdots\\ \mathcal{L}_r(\mathfrak{O}_r)/A_r &\simeq \mathfrak{O}_r^{n_{N_{r-1}+1}} \times \cdots \times \mathfrak{O}_r^{n_{N_r}}\end{aligned}$$

and consider the projection data with respect to the resulting decomposition

$$L/A \simeq \mathfrak{O}_1^{n_1} \times \cdots \times \mathfrak{O}_r^{n_{N_r}}.$$

Here the number of projections is $h = N_r$. Assume that Hypothesis 4.5 is satisfied. We claim that $\mathfrak{O}_i/\mathfrak{o}$ is unramified for all $i \in [r]$ such that $\mathcal{L}_i$ is not abelian.

Indeed, fix uniformizers $\Pi_i \in \mathfrak{O}_i$, let $\tau \in \mathbb{N}$, and consider the lattice

$$\Lambda = \Pi_1^\tau \mathfrak{O}_1^{n_1} \times \cdots \times \Pi_1^\tau \mathfrak{O}_1^{n_{N_1}} \times \Pi_2^\tau \mathfrak{O}_2^{n_{N_1+1}} \times \cdots \times \Pi_r^\tau \mathfrak{O}_r^{n_{N_r}}.$$

The projection data are $\nu_j^{(i)} = \tau$ for all $i \in [N_r]$ and all $j \in [n_i]$. Furthermore, it is clear that

$$[\Lambda, L] = \Pi_1^\tau[\mathcal{L}_1(\mathfrak{O}_1), \mathcal{L}_1(\mathfrak{O}_1)] \times \cdots \times \Pi_r^\tau[\mathcal{L}_r(\mathfrak{O}_r), \mathcal{L}_r(\mathfrak{O}_r)].$$

For every $i \in [r]$, let b_i be the rank of $[\mathcal{L}_i(\mathfrak{O}_i), \mathcal{L}_i(\mathfrak{O}_i)]$ as an $\mathfrak{o}$-module. Then it is immediate from Lemma 2.3 that the partition $\lambda(\Lambda)$ is the disjoint union of the sets $\{\tau\}_{e_i, f_i}$ (see Definition 2.4), with respective multiplicities b_i. Suppose that $\mathcal{L}_i$ is not abelian. Then $b_i > 0$. If, in addition, $e_i \geq 2$, then the elements of $\{\tau\}_{e_i, f_i}$ are not all equal to τ. Hence there are parts of $\lambda(\Lambda)$ that do not appear as parts of the projection data $\tilde{\boldsymbol{\nu}}$, contradicting Hypothesis 4.5.

Definition 4.9. Set $\varepsilon = c - c'$. Given partitions λ and μ with c' and c parts, respectively, we say that $\mu \leq \lambda$ if $\mu \leq \tilde{\lambda}$, where $\tilde{\lambda}$ is any partition with c parts whose parts consist of the c' parts of λ together with any ε integers $\xi_1 \geq \cdots \geq \xi_\varepsilon \geq \mu_1$. By $\alpha(\lambda, \mu; Y)$ we will mean $\alpha(\tilde{\lambda}, \mu; Y)$, the "Birkhoff polynomial" (2-5); note that both definitions are independent of the choice of $\tilde{\lambda}$.

Our objective, which will be attained with Theorem 4.21, is to compute the $\mathfrak{o}$-ideal zeta function of the $\mathfrak{o}$-Lie algebra L. We maintain the notation from above. Recall, in particular, that $n = \sum_{i=1}^h e_i f_i n_i$ is the $\mathfrak{o}$-rank of L/A. Observe that if $\Lambda \leq L/A$ as above, then there exists an $\mathfrak{o}$-sublattice $\mathrm{M} \leq A$ of elementary divisor type μ such that $[\Lambda, L] \leq \mathrm{M}$ if and only if $\mu \leq \lambda(\Lambda)$. Furthermore, as L' is isolated in L, the number of sublattices $\mathrm{M} \leq A$ of elementary divisor type μ that contain $[\Lambda, L]$ is given by $\alpha(\lambda(\Lambda), \mu; q)$.

Recall m from Definition 4.6. Given projection data $\tilde{\boldsymbol{\nu}} = (\nu^{(1)}, \ldots, \nu^{(h)})$, the partition $\lambda(\tilde{\boldsymbol{\nu}})$ depends only on the m-tuple $\boldsymbol{\nu} = (\nu^{(1)}, \ldots, \nu^{(m)})$. Thus we will write $\lambda(\boldsymbol{\nu})$ for $\lambda(\tilde{\boldsymbol{\nu}})$.

Proposition 4.10. *Assuming Hypothesis 4.5, the $\mathfrak{o}$-ideal zeta function of L is given by*

$$\zeta_L^{\triangleleft \mathfrak{o}}(s) = \frac{\zeta_{\mathfrak{o}^n}(s)}{\prod_{i=1}^m \zeta_{\mathfrak{O}_i^{n_i}}(s)} \sum_{\substack{\boldsymbol{\nu},\mu \\ \mu \le \lambda(\boldsymbol{\nu})}} \biggl(\prod_{i=1}^m \beta(\nu^{(i)}; q_i)\biggr) \alpha(\lambda(\boldsymbol{\nu}), \mu; q)(q^n t)^{\sum_{k=1}^c \mu_k} t^{\sum_{i=1}^m (\sum_{j=1}^{n_i} \nu_j^{(i)}) f_i}.$$

Here $\boldsymbol{\nu} = (\nu^{(1)}, \dots, \nu^{(m)})$ runs over all m-tuples of partitions with $n_1, \dots, n_m$ parts, respectively, whereas μ runs over all partitions with c parts. The condition $\mu \le \lambda(\boldsymbol{\nu})$ is to be understood as in Definition 4.9.

Proof. The quotient L/A has $\mathfrak{o}$-rank n, so it follows from [Grunewald et al. 1988, Lemma 6.1] that

$$\zeta_L^{\triangleleft \mathfrak{o}}(s) = \sum_{\Lambda \le L/A} |L/A : \Lambda|^{-s} \sum_{[\Lambda, L] \le \mathrm{M} \le A} |A : \mathrm{M}|^{n-s}.$$

Grouping the lattices $\Lambda \le L/A$ by their projection data $\boldsymbol{\nu}(\Lambda)$, we obtain

$$\zeta_L^{\triangleleft \mathfrak{o}}(s) = \sum_{\tilde{\boldsymbol{\nu}}} \sum_{\substack{\Lambda \le L/A \\ \boldsymbol{\nu}(\Lambda) = \tilde{\boldsymbol{\nu}}}} |L/A : \Lambda|^{-s} \sum_{[\Lambda, L] \le \mathrm{M} \le A} |A : \mathrm{M}|^{n-s}.$$

Setting μ to be the elementary divisor type of M and recalling that $\lambda(\boldsymbol{\nu}(\Lambda))$ is the elementary divisor type of $[\Lambda, L]$ by Hypothesis 4.5, it now follows from Proposition 4.2 that

$$\zeta_L^{\triangleleft \mathfrak{o}}(s) = \frac{\zeta_{\mathfrak{o}^n}(s)}{\prod_{i=1}^h \zeta_{\mathfrak{O}_i^{n_i}}(s)} \sum_{\substack{\tilde{\boldsymbol{\nu}},\mu \\ \mu \le \lambda(\tilde{\boldsymbol{\nu}})}} \biggl(\prod_{i=1}^h \beta(\nu^{(i)}; q_i)\biggr) \alpha(\lambda(\tilde{\boldsymbol{\nu}}), \mu; q)(q^n t)^{\sum_{k=1}^c \mu_k} t^{\sum_{i=1}^h (\sum_{j=1}^{n_i} \nu_j^{(i)}) f_i}.$$

As we observed above, $\alpha(\lambda(\tilde{\boldsymbol{\nu}}), \mu; q)$ depends only on the first m components of the h-tuple $\tilde{\boldsymbol{\nu}}$. Hence the sum in the previous displayed formula may be expressed as

$$\sum_{\substack{\boldsymbol{\nu},\mu \\ \mu \le \lambda(\boldsymbol{\nu})}} \biggl(\prod_{i=1}^m \beta(\nu^{(i)}; q_i)\biggr) \alpha(\lambda(\boldsymbol{\nu}), \mu; q)(q^n t)^{\sum_{k=1}^c \mu_k} t^{\sum_{i=1}^m (\sum_{j=1}^{n_i} \nu_j^{(i)}) f_i} \times \sum_{(\nu^{(m+1)}, \dots, \nu^{(h)})} \biggl(\prod_{i=m+1}^h \beta(\nu^{(i)}; q_i)\biggr) t^{\sum_{i=m+1}^h (\sum_{j=1}^{n_i} \nu_j^{(i)}) f_i}. \tag{4-7}$$

Observing that

$$\sum_{\nu^{(i)}} \beta(\nu^{(i)}; q_i) t^{\sum_{j=1}^{n_i} \nu_j^{(i)} f_i} = \sum_{\mathrm{M} \le \mathfrak{O}_i^{n_i}} [\mathfrak{O}_i^{n_i} : \mathrm{M}]^{-s} = \zeta_{\mathfrak{O}_i^{n_i}}(s),$$

we see that the second sum in (4-7) is equal to $\prod_{i=m+1}^h \zeta_{\mathfrak{O}_i^{n_i}}(s)$. The claim follows. □

Let $w \in \mathcal{D}_{2c}$ be a Dyck word. Recall, from Section 2.2, that w is specified by two r-tuples $(L_1, L_2, \dots, L_r)$ and $(M_1, M_2, \dots, M_r)$ satisfying $L_i - M_i \ge 0$ for all $i \in [r]$ and $L_r = M_r = c$. Recall further that $\varepsilon = c - c'$ and define $\tilde{L}_j = L_j - \varepsilon$ for all $j \in [r]$.

Definition 4.11. Let λ and μ be partitions with c' and c parts, respectively, and let $w \in \mathcal{D}_{2c}$ such that $L_1 \ge \varepsilon$. Fix a partition $\tilde{\lambda}$ with c parts as above; without loss of generality we may take $\xi_\varepsilon \ge \max\{\lambda_1, \mu_1\}$.

We say that λ and μ have *overlap type* w, written $w(\lambda,\mu)=w$, if their parts satisfy the following inequalities:

$$\xi_1\geq\cdots\geq\xi_\varepsilon\geq\lambda_1\geq\cdots\geq\lambda_{\tilde{L}_1}\geq\mu_1\geq\cdots\geq\mu_{M_1}>\lambda_{\tilde{L}_1+1}\geq\cdots\geq\lambda_{\tilde{L}_2}$$
$$\geq\mu_{M_1+1}\geq\cdots\geq\mu_{M_2}>\cdots>\lambda_{\tilde{L}_{r-1}+1}\geq\cdots\geq\lambda_{\tilde{L}_r}\geq\mu_{M_{r-1}+1}\geq\cdots\geq\mu_{M_r}.$$

In other words, $w(\lambda,\mu)=w$ if $w(\tilde{\lambda},\mu)=w$ in the sense of (2-6). Note that $\tilde{L}_1=0$ may occur, if $\varepsilon>0$. Moreover, the set $\mathcal{D}_{2c}$ depends on c and so on the choice of A.

Observe that $\lambda\geq\mu$ if and only if there exists a Dyck word $w\in\mathcal{D}_{2c}$, necessarily unique, such that $w(\lambda,\mu)=w$. Given $w\in\mathcal{D}_{2c}$, we define the function

$$D_w(q,t)=\sum_{\boldsymbol{\nu}}\sum_{\substack{\mu\leq\lambda(\boldsymbol{\nu})\\ w(\lambda(\boldsymbol{\nu}),\mu)=w}}\left(\prod_{i=1}^{m}\beta(\nu^{(i)};q_i)\right)\alpha(\lambda(\boldsymbol{\nu}),\mu;q)(q^nt)^{\sum_{k=1}^{c}\mu_k}t^{\sum_{i=1}^{m}(\sum_{j=1}^{n_i}\nu_j^{(i)})f_i}. \tag{4-8}$$

Remark 4.12. If w is such that $L_1<\varepsilon$, then the above sum is empty and so $D_w(q,t)=0$. In addition, the definition of the partition $\lambda(\boldsymbol{\nu})$ will usually impose some equalities among its parts. Thus, it may happen that the set of projection data $\boldsymbol{\nu}$ whose associated partition $\lambda(\boldsymbol{\nu})$ is compatible with a given Dyck word w is empty even if w satisfies the condition $L_1\geq\varepsilon$ of Definition 4.11. We will see examples of this phenomenon below, e.g., in Section 5.3.2.

Proposition 4.10 now tells us that

$$\zeta_L^{\triangleleft\mathfrak{o}}(s)=\frac{\zeta_{\mathfrak{o}^n}(s)}{\prod_{i=1}^{m}\zeta_{\mathfrak{O}_i^{n_i}}(s)}\sum_{w\in\mathcal{D}_{2c}}D_w(q,t). \tag{4-9}$$

4.4. ***An explicit expression for $\zeta_L^{\triangleleft\mathfrak{o}}(s)$.*** Our aim in this section is to give explicit formulae for the terms $D_w(q,t)$ in (4-9). We will achieve it with Proposition 4.20 — a result whose proof will be given in Section 4.5 — leading to a fully explicit formula for the relevant $\mathfrak{o}$-ideal zeta functions in Theorem 4.21.

We maintain the notation of Section 4.3 and resume some of the notation introduced in Section 3. Consider the composition $\underline{n}=(n_1,\dots,n_m)$ and a family $\boldsymbol{\nu}=(\nu^{(1)},\dots,\nu^{(m)})$ of partitions $\nu^{(i)}$, each with n_i parts. The natural ordering of the elements of the multiset

$$S=\bigcup_{i=1}^{m}\{\nu_j^{(i)}\mid j\in[n_i]\}$$

gives rise to an element $V(\boldsymbol{\nu})\in\mathrm{WO}_{\underline{n}}$. Indeed, the word $v=\prod_{i=1}^{m}a_i^{\alpha_i}\in C_{\underline{n}}$ appears in the flag $V(\boldsymbol{\nu})$ if and only if any element of the multiset

$$S_v=\bigcup_{i=1}^{m}\{\nu_j^{(i)}\mid j\in[\alpha_i]\} \tag{4-10}$$

is larger than any element of the complement $S\setminus S_v=\bigcup_{i=1}^{m}\{\nu_j^{(i)}\mid j\in[\alpha_i+1,n_i]\}$. Given a word $v\in C_{\underline{n}}$, let $m(v)$ denote a minimal element of the multiset S_v. Since, by virtue of Definition 4.4, all parts of $\lambda(\boldsymbol{\nu})$

appear in S, we see that if $k \in \mathbb{N}$ satisfies $\lambda'_k > \lambda'_{k+1}$, then necessarily $k = m(v)$ for some $v \in C_{\underline{n}}$. Here we denote the dual partition of $\lambda(\boldsymbol{\nu})$ by λ' for brevity. Moreover, Hypothesis 4.5 implies that $\lambda'_{m(v)}$ depends only on v and not on the flag $V(\boldsymbol{\nu})$ or on the actual values of the parts $\nu_j^{(i)}$.

Definition 4.13. Let $v \in C_{\underline{n}}$:

(1) Set $\ell(v) = \lambda'_{m(v)}$. In particular, $\ell(v') \leq \ell(v)$ if $v' \leq v$.

(2) We say that v is *radical* if $\ell(v') < \ell(v)$ for all proper subwords $v' < v$.

Note the following explicit formula for $\ell(v)$.

Lemma 4.14. *Let $v = \prod_{i=1}^m a_i^{\alpha_i} \in C_{\underline{n}}$. Then*

$$\ell(v) = \lambda(\boldsymbol{\nu})'_{m(v)} = \sum_{k=1}^{Z} \prod_{j=1}^{\tau_k} \binom{\alpha_{s_{kj}}}{\sigma_{kj}}.$$

Proof. This is a straightforward consequence of Definition 4.4. □

Definition 4.15. Let $w \in \mathcal{D}_{2c}$ be a Dyck word with exactly r letter changes from **0** to **1**; see Section 2.2. A flag $V = \{v_1 < \cdots < v_t\}$ of elements of $C_{\underline{n}}$ is said to be *compatible* with w, or simply w*-compatible*, if:

- $t = r$.
- For all $j \in [r]$, the word v_j is radical and satisfies $\ell(v_j) = \tilde{L}_j$.

Remark 4.16. It follows from Hypothesis 4.5 that all parts of $\boldsymbol{\nu}$ participate in the minima that determine the parts of $\lambda(\boldsymbol{\nu})$. Therefore, the maximal word $\prod_{i=1}^m a_i^{n_i}$ is always radical, and $v_r = \prod_{i=1}^m a_i^{n_i}$ for any w-compatible flag V.

In addition, note that if $\varepsilon > 0$, i.e., if $L' < A$, then some Dyck words $w \in \mathcal{D}_{2c}$ for which there exist w-compatible flags will satisfy $\tilde{L}_1 = 0$. In this case, $v_1 = \varnothing$ for any such flag.

For $w \in \mathcal{D}_{2c}$, let $\mathcal{F}_w$ denote the set of w-compatible flags. It will be convenient to organize the information carried by an element of $\mathcal{F}_w$ in matrix form. Given an element $V = \{v_1 < \cdots < v_r\} \in \mathcal{F}_w$, we let v_0 be the empty word and define ρ_{ij}, for $i \in [m]$ and $j \in [r]$, by

$$\frac{v_j}{v_{j-1}} = \prod_{i=1}^{m} a_i^{\rho_{ij}}.$$

In this way, the flag V gives rise to a matrix $\rho(V) \in \mathrm{Mat}_{m,r}(\mathbb{N}_0)$. Conversely, given a matrix $\rho \in \mathrm{Mat}_{m,r}(\mathbb{N}_0)$, we consider the cumulative sums of its rows: for $i \in [m]$ and $j \in [r]$, define

$$\mathrm{P}_{ij} = \sum_{k=1}^{j} \rho_{ik}. \tag{4-11}$$

Definition 4.17. Let $\mathcal{M}_{\underline{n},w} \subseteq \mathrm{Mat}_{m,r}(\mathbb{N}_0)$ be the set of $(\underline{n}, w)$-*admissible compositions*, namely of matrices ρ satisfying the following two properties:

(1) $\ell\big(\prod_{i=1}^m a_i^{\mathrm{P}_{ij}}\big) = \tilde{L}_j$ for all $j \in [r]$.

(2) The word $\prod_{i=1}^m a_i^{\mathrm{P}_{ij}}$ is radical for all $j \in [r]$.

By Remark 4.16, these properties imply that $\mathrm{P}_{ir} = n_i$ for all $i \in [m]$. Set $w_j = \prod_{i=1}^m a_i^{\mathrm{P}_{ij}}$ for all $j \in [r]$. It is easy to see that the map $\mathcal{F}_w \to \mathcal{M}_{\underline{n},w}$ given by $V \mapsto \rho$ is a bijection, with inverse $\rho \mapsto \{w_1 < \cdots < w_r\}$. Denote

$$\mathrm{P}_i = \{\mathrm{P}_{ij} \mid j \in [r]\}$$

for all $i \in [m]$. For $j \in [r]$, we denote by $\underline{\rho_j}$ the following composition with m parts:

$$\underline{\rho_j} = (\rho_{1j}, \dots, \rho_{mj}). \tag{4-12}$$

Recall from Definition 2.2 the notion of Igusa function and from Definition 3.5 the notion of generalized Igusa function $I^{\mathrm{wo}}_{\underline{\rho_j}}(\boldsymbol{Y}; \boldsymbol{X}) \in \mathbb{Q}(Y_1, \dots, Y_m; (X_v)_{v \le w_j})$.

Definition 4.18. Let $\rho \in \mathcal{M}_{\underline{n},w}$. We define

$$D_{w,\rho}(q,t) = \left(\prod_{i=1}^m \binom{n_i}{\mathrm{P}_i}_{q_i^{-1}}\right) \prod_{j=1}^r \left(\binom{L_j - M_{j-1}}{L_j - M_j}_{q^{-1}} I^{\mathrm{wo}}_{\underline{\rho_j}}(q_1^{-1}, \dots, q_m^{-1}; \boldsymbol{y}^{(j)}) \right)$$
$$\times \prod_{j=1}^{r-1} I^{\circ}_{M_j - M_{j-1}}(q^{-1}; x_{M_{j-1}+1}, \dots, x_{M_j}) I_{M_r - M_{r-1}}(q^{-1}; x_{M_{r-1}+1}, \dots, x_{M_r}),$$

with numerical data defined as follows. For a subword $v = \prod_{i=1}^m a_i^{\alpha_i}$ of $\prod_{i=1}^m a_i^{\rho_{ij}}$ we set $\alpha_i^{(j)} = \mathrm{P}_{i,j-1} + \alpha_i$ and $v^{(j)} = v \cdot w_{j-1} = \prod_{i=1}^m a_i^{\alpha_i^{(j)}}$. Set

$$\delta_v^{(j)} = \begin{cases} 0 & \text{if } \ell(v^{(j)}) = \ell(w_{j-1}), \\ 1 & \text{otherwise,} \end{cases}$$

and define

$$B_v^{(j)} = \begin{cases} \sum_{i=1}^m f_i \alpha_i (n_i - \alpha_i) & \text{if } \delta_v^{(j)} = 0, \\ \sum_{i=1}^m f_i \alpha_i^{(j)} (n_i - \alpha_i^{(j)}) & \text{if } \delta_v^{(j)} = 1. \end{cases}$$

Finally, we set

$$y_v^{(j)} = q^{\delta_v^{(j)} M_{j-1}(n + \ell(v^{(j)}) + \varepsilon - M_{j-1}) + B_v^{(j)}} t^{\sum_{i=1}^m \alpha_i f_i + \delta_v^{(j)} (M_{j-1} + \sum_{i=1}^m \mathrm{P}_{i,j-1} f_i)},$$

where $\ell(v^{(j)})$ is given explicitly by Lemma 4.14. For $k \in [M_{j-1}+1, M_j]$, we set

$$x_k = q^{k(n + L_j - k) + \sum_{i=1}^m f_i \mathrm{P}_{ij}(n_i - \mathrm{P}_{ij})} t^{k + \sum_{i=1}^m f_i \mathrm{P}_{ij}}.$$

Proposition 4.19. *The following functional equation holds*:

$$D_{w,\rho}(q^{-1}, t^{-1}) = (-1)^{c + \sum_{i=1}^m n_i} q^{\binom{n+c}{2} - \binom{n}{2} + \sum_{i=1}^m f_i \binom{n_i}{2}} t^{c + 2\sum_{i=1}^m n_i f_i} D_{w,\rho}(q,t).$$

Proof. The proof is a straightforward computation using the functional equations of

(1) Gaussian binomials (2-1),

(2) classical Igusa functions (2-3), (2-4), and

(3) generalized Igusa functions given in Theorem 3.8,

as well as the definition (4-11) of P_{ij} and the observation that $\mathrm{P}_{ir} = n_i$ for all $i \in [m]$. □

Recall the functions $D_w(q,t)$ introduced in (4-8) and used to describe the $\mathfrak{o}$-ideal zeta function of L in (4-9). The following result, which constitutes the technical heart of the computation of the ideal zeta function $\zeta_L^{\triangleleft\mathfrak{o}}(s)$, relates $D_w(q,t)$ with the explicit functions $D_{w,\rho}(q,t)$ of Definition 4.18. We defer its proof to the next section.

Proposition 4.20. *Let $w \in \mathcal{D}_{2c}$ be a Dyck word. Then*

$$D_w(q,t) = \sum_{\rho \in \mathcal{M}_{\underline{n},w}} D_{w,\rho}(q,t).$$

Theorem 4.21. *The $\mathfrak{o}$-ideal zeta function of L is*

$$\zeta_L^{\triangleleft\mathfrak{o}}(s) = \frac{\zeta_{\mathfrak{o}^n}(s)}{\prod_{i=1}^m \zeta_{\mathfrak{O}_i^{n_i}}(s)} \sum_{w \in \mathcal{D}_{2c}} \sum_{\rho \in \mathcal{M}_{\underline{n},w}} D_{w,\rho}(q,t).$$

Proof. The claim is immediate from (4-9) and Proposition 4.20. □

Corollary 4.22. *Suppose that the extension $\mathfrak{O}_i/\mathfrak{o}$ is unramified for all $i \in [m]$. Then the $\mathfrak{o}$-ideal zeta function of L satisfies the functional equation*

$$\zeta_L^{\triangleleft\mathfrak{o}}(s)\Big|_{q\to q^{-1}} = (-1)^{\mathrm{rk}_\mathfrak{o}(L)} q^{\binom{\mathrm{rk}_\mathfrak{o}(L)}{2}} t^{\mathrm{rk}_\mathfrak{o}(L)+\mathrm{rk}_\mathfrak{o}(L/Z(L))} \zeta_L^{\triangleleft\mathfrak{o}}(s).$$

Proof. Recall that $n + c = \mathrm{rk}_\mathfrak{o}(L/A) + \mathrm{rk}_\mathfrak{o}(A) = \mathrm{rk}_\mathfrak{o}(L)$. Observe that the symmetry factor in Proposition 4.19 is independent of w and ρ. Consequently, the sum $\sum_{w\in\mathcal{D}_{2c}} \sum_{\rho\in\mathcal{M}_{\underline{n},w}} D_{w,\rho}(q,t)$ itself satisfies a functional equation with the same symmetry factor. The remaining factors in Theorem 4.21 satisfy

$$\frac{\zeta_{\mathfrak{o}^n}(s)}{\prod_{i=1}^m \zeta_{\mathfrak{O}_i^{n_i}}(s)}\Big|_{q\to q^{-1}} = \frac{(-1)^n q^{\binom{n}{2}} t^n}{\prod_{i=1}^m (-1)^{n_i} q^{f_i\binom{n_i}{2}} t^{n_i f_i}} \cdot \frac{\zeta_{\mathfrak{o}^n}(s)}{\prod_{i=1}^m \zeta_{\mathfrak{O}_i^{n_i}}(s)}.$$

This yields the functional equation

$$\zeta_L^{\triangleleft\mathfrak{o}}(s)|_{q\to q^{-1}} = (-1)^{\mathrm{rk}_\mathfrak{o}(L)} q^{\binom{\mathrm{rk}_\mathfrak{o}(L)}{2}} t^{\mathrm{rk}_\mathfrak{o}(L)+\sum_{i=1}^m n_i f_i} \zeta_L^{\triangleleft\mathfrak{o}}(s).$$

Since we have assumed that all the extensions $\mathfrak{O}_i/\mathfrak{o}$ are unramified, our claim is now immediate from (4-6). □

Remark 4.23. The explicit formula given in Theorem 4.21 allows one to determine, in principle, the (local) *abscissa of convergence* $\alpha_L^{\triangleleft\mathfrak{o}}$ of the $\mathfrak{o}$-ideal zeta function $\zeta_L^{\triangleleft\mathfrak{o}}(s)$, viz.

$$\alpha_L^{\triangleleft\mathfrak{o}} := \inf\{\alpha \in \mathbb{R}_{>0} \mid \zeta_L^{\triangleleft\mathfrak{o}}(s) \text{ converges on } \{s \in \mathbb{C} \mid \Re(s) > \alpha\}\} \in \mathbb{Q}_{>0}.$$

Indeed, writing the rational function $\zeta_L^{\triangleleft\mathfrak{o}}(s)$ over a common denominator of the form $\prod_{(a,b)\in I}(1-q^a t^b)$, with a, b given by the numerical data given in Definition 4.18, gives

$$\alpha_L^{\triangleleft\mathfrak{o}} = \max\left\{n, \tfrac{a}{b} \mid (a,b) \in I\right\}.$$

This reflects the facts that a/b is the abscissa of convergence of the geometric progression $q^{a-bs}/(1-q^{a-bs})$ and that each of the $D_w(q,t)$ is a nonnegative linear combination of products of such geometric progressions.

Remark 4.24. Observe that if L is replaced by the β-fold direct product L^β, then c is replaced by βc, and the number of summands on the right-hand side of Theorem 4.21 grows superexponentially in β. Cancellations may occur, as in Remark 5.9 below, that cause the complexity of $\zeta^{\triangleleft 0}_{L^\beta}(s)$ to grow less rapidly with respect to β; however, explicit computations in the case of the Heisenberg Lie algebra suggest that the growth can indeed be this rapid.

4.5. ***Proof of Proposition 4.20.*** We start with a lemma involving the notions of Definition 4.13. This observation is simple but crucial to the method of the article.

Lemma 4.25. *Let $v \in C_{\underline{n}}$. There is a unique radical subword $\sqrt{v} \leq v$ such that $\ell(\sqrt{v}) = \ell(v)$.*

Proof. Suppose $v = \prod_{i=1}^m a_i^{\alpha_i}$. If a binomial coefficient $\binom{\alpha}{\sigma}$ is positive, then it will decrease if α is decreased. It follows that if the k-th term in the sum in the statement of Lemma 4.14 is positive, then in any subword $v' \leq v$ satisfying $\ell(v') = \ell(v)$ all the variables $a_{s_{kj}}$ must appear with exponent $\alpha_{s_{kj}}$. Hence we are led to define the set

$$\mathcal{K}_v = \{k \in [Z] \mid \alpha_{s_{kj}} \geq \sigma_{kj} \text{ for all } j \in [\tau_k]\}.$$

Furthermore, we put $\mathfrak{S}_v = \bigcup_{k \in \mathcal{K}_v} \mathfrak{S}_k$ and finally define $\sqrt{v} = \prod_{i \in \mathfrak{S}_v} a_i^{\alpha_i}$. It is clear from the preceding discussion that a subword $v' \leq v$ satisfies $\ell(v') = \ell(v)$ if and only if $\sqrt{v} \leq v' \leq v$. The claimed existence and uniqueness follow. □

Corollary 4.26. *Suppose that $v_1 < v_2$ are two elements of $C_{\underline{n}}$ such that $\ell(v_1) = \ell(v_2)$. Then $\sqrt{v_1} = \sqrt{v_2}$.*

Proof. This is immediate from the construction of $\sqrt{v}$ in the proof of Lemma 4.25. □

Fix a Dyck word $w \in \mathcal{D}_{2c}$. We aim to evaluate the function $D_w(q,t)$ of (4-8). Let $\boldsymbol{\nu} = (\nu^{(1)}, \dots, \nu^{(m)})$ be an m-tuple of partitions, where, for each $i \in [m]$, the partition $\nu^{(i)}$ has n_i parts. Let μ be a partition with c parts such that $\mu \leq \lambda(\boldsymbol{\nu})$ and $w(\lambda(\boldsymbol{\nu}), \mu) = w$, in the sense of Definitions 4.9 and 4.11. To simplify the notation, we write λ for $\lambda(\boldsymbol{\nu})$.

Now let $\{L_j, M_j\}_{j \in [r]}$ be the parameters associated with the Dyck word w. Recall that we have set $L_0 = M_0 = 0$. It follows from the assumption $w(\lambda, \mu) = w$ that $\lambda_{\tilde{L}_j} > \lambda_{\tilde{L}_j + 1}$ for all $j \in [r-1]$, hence that all the positive $\tilde{L}_j$ appear as parts of the dual partition λ'. By the observations before Definition 4.13, there exists a subflag $\kappa_1 < \kappa_2 \cdots < \kappa_r$ of $V(\boldsymbol{\nu})$ such that $\ell(\kappa_j) = \tilde{L}_j$ for every $j \in [r]$; if $\tilde{L}_1 = 0$, then we may take $\kappa_1 = \varnothing$. This subflag need not be unique, and its constituent words need not be radical. However, the flag $\sqrt{\kappa_1} < \cdots < \sqrt{\kappa_r}$ is well-defined by Corollary 4.26. Moreover, it is clear that this flag is an element of $\mathcal{F}_w$ and thus corresponds to an $(\underline{n}, w)$-admissible composition $\rho(\boldsymbol{\nu}) \in \mathcal{M}_{\underline{n},w}$.

For every $\rho \in \mathcal{M}_{\underline{n},w}$ we define the function

$$\Delta_{w,\rho}(q,t) = \sum_{\substack{\boldsymbol{\nu} \\ \rho(\boldsymbol{\nu})=\rho}} \sum_{\substack{\mu \le \lambda(\boldsymbol{\nu}) \\ w(\lambda,\mu)=w}} \left(\prod_{i=1}^{m} \beta(\nu^{(i)}; q_i) \right) \alpha(\lambda(\boldsymbol{\nu}), \mu; q) (q^n t)^{\sum_{k=1}^{c} \mu_k} t^{\sum_{i=1}^{m} \sum_{j=1}^{n_i} \nu_j^{(i)}}. \tag{4-13}$$

Clearly, $D_w(q,t) = \sum_{\rho \in \mathcal{M}_{\underline{n},w}} \Delta_{w,\rho}(q,t)$. Hence, to prove Proposition 4.20 it suffices to show the following:

Lemma 4.27. *The equality $\Delta_{w,\rho}(q,t) = D_{w,\rho}(q,t)$ holds for all $\rho \in \mathcal{M}_{\underline{n},w}$.*

Proof. Fix $\rho \in \mathcal{M}_{\underline{n},w}$. For each $j \in [r]$ we define a multiset

$$\mathcal{S}_j = \bigcup_{i=1}^{m} \{\nu_k^{(i)} \mid k \in [\mathrm{P}_{i,j-1}+1, \mathrm{P}_{ij}]\}.$$

Recall the compositions $\underline{\rho_j}$ defined in (4-12) above, which depend only on ρ. For each $j \in [r]$, the natural ordering of the elements of $\mathcal{S}_j$ provides a weak order $v_j \in \mathrm{WO}_{\underline{\rho_j}}$. Again, these depend only on the projection data $\boldsymbol{\nu}$, so we denote them $v_j(\boldsymbol{\nu})$ and set $\boldsymbol{v}(\boldsymbol{\nu}) = (v_1(\boldsymbol{\nu}), \dots, v_r(\boldsymbol{\nu}))$. As in the previous section, we define $w_j = \prod_{i=1}^{m} a_i^{\mathrm{P}_{ij}} \in C_{\underline{n}}$.

Now fix an r-tuple $(v_1, \dots, v_r) \in \prod_{j=1}^{r} \mathrm{WO}_{\underline{\rho_j}}$. For every $j \in [r]$, suppose that v_j includes the word $\prod_{i=1}^{m} a_i^{\rho_{ij}}$ (except when $\underline{\rho_1}$ is the zero composition, in which case v_1 is empty). Write

$$v_j = \{v_{j1} < v_{j2} < \cdots < v_{j,\ell_j}\}$$

for some $\ell_j \in \mathbb{N}_0$. We define $\tilde{v}_{jk} = w_{j-1} \cdot v_{jk} \in C_{\underline{n}}$. Consider the set $S_{\tilde{v}_{jk}}$ and its minimal element $m(\tilde{v}_{jk})$ as in (4-10). Note that $v_{j,\ell_j} = \prod_{i=1}^{m} a_i^{\rho_{ij}}$ and that consequently $m(\tilde{v}_{j,\ell_j}) = \lambda_{\tilde{L}_j}$. Let $\varepsilon_j \in \mathbb{N}$ be the minimal positive integer such that $\ell(\tilde{v}_{j,\varepsilon_j}) > \tilde{L}_{j-1}$. Then $m(\tilde{v}_{j,\varepsilon_j}) = \lambda_{\tilde{L}_{j-1}+1}$. Observe that $\delta_{v_{jk}}^{(j)} = 0$ in Definition 4.18 if and only if $k < \varepsilon_j$; in this case, $m(\tilde{v}_{jk})$ is equal to a part of $\boldsymbol{\nu}$ that does not appear in the partition $\lambda(\boldsymbol{\nu})$.

For every element $v_{jk} = \prod_{i=1}^{m} a_i^{\gamma_i} \in C_{\underline{\rho_j}}$, define

$$m(v_{jk}) = \min\{\nu_u^{(i)} \mid u \in [\mathrm{P}_{i,j-1}+1, \mathrm{P}_{i,j-1}+\gamma_i]\}.$$

Note that the elements of the set $\{m(v_{jk}) \mid j \in [r], k \in [\ell_j]\}$ are exactly the parts of the projection data $\boldsymbol{\nu}$. Moreover, if $\delta_{v_{jk}}^{(j)} = 1$, then $m(v_{jk}) = m(\tilde{v}_{jk})$. Otherwise, it may happen that $m(v_{jk}) > m(\tilde{v}_{jk})$, as the set defining $m(v_{jk})$ consists entirely of parts of $\boldsymbol{\nu}$ that do not appear in $\lambda(\boldsymbol{\nu})$ and may all be larger than the minimal element of the disjoint set $S_{w_{j-1}}$.

We now define a collection of differences that will provide a convenient parametrization of the pairs $(\boldsymbol{\nu}, \mu)$ that we are considering:

$$s_{jk} = \begin{cases} m(v_{jk}) - m(v_{j,k+1}) & \text{for } k < \ell_j, \\ m(v_{jk}) - \mu_{M_{j-1}+1} & \text{for } k = \ell_j, \end{cases} \quad \text{and} \quad r_k = \begin{cases} \mu_k - m(v_{j+1,\varepsilon_{j+1}}) & \text{for } k \in \{M_1, \dots, M_{r-1}\}, \\ \mu_k & \text{for } k = M_r, \\ \mu_k - \mu_{k+1} & \text{otherwise.} \end{cases}$$

Here the indices of the r_k run over the set $[M_r] = [c]$, whereas the indices of the s_{jk} satisfy $j \in [r]$ and $k \in [\ell_j]$. We emphasize that the r_k have no connection with the parameter r defined earlier. Observe that the s_{jk} and the r_k are all nonnegative integers. Moreover, if we allow all the s_{jk} to run over $\mathbb{N}_0$ and all the r_k to run over $\mathbb{N}$ if $k \in \{M_1, \dots, M_{r-1}\}$ and over $\mathbb{N}_0$ otherwise, then we obtain precisely the pairs $(\boldsymbol{\nu}, \mu)$ satisfying the following three conditions:

(1) $w(\lambda(\boldsymbol{\nu}), \mu) = w$.

(2) $\rho(\boldsymbol{\nu}) = \rho$.

(3) $\boldsymbol{v}(\boldsymbol{\nu}) = (v_1, \dots, v_r)$.

Let $\Delta_{w,\rho,\boldsymbol{v}}(q, t)$ be the function defined by the right-hand side of (4-13), except that the sum runs only over the data $\boldsymbol{\nu}$ satisfying $\boldsymbol{v}(\boldsymbol{\nu}) = (v_1, \dots, v_r)$. Our task is now to rewrite the ingredients of $\Delta_{w,\rho,\boldsymbol{v}}(q, t)$, and hence the function itself, in terms of the parameters s_{jk} and r_k. Consider the following collection of intervals:

$$\begin{aligned} &[\mu_k - r_k + 1, \mu_k], && k \in [c], \\ &[m(v_{jk}) - s_{jk} + 1, m(v_{jk})], && j \in [2, r], k \in [\varepsilon_j, \ell_j]. \end{aligned} \tag{4-14}$$

The reader will easily verify that these intervals are disjoint and that their union is the interval $[1, \mu_1]$. It follows from this observation that

$$\mu_k = \sum_{b=k}^{c} r_b + \sum_{b=j+1}^{r} \sum_{u=\varepsilon_b}^{\ell_b} s_{bu} \tag{4-15}$$

if $k \in [M_{j-1} + 1, M_j]$, whereas if $\nu_d^{(i)} = m(v_{jk})$, then

$$\nu_d^{(i)} = \sum_{u=k}^{\ell_j} s_{ju} + \sum_{b=j+1}^{r} \sum_{u=\varepsilon_b}^{\ell_b} s_{bu} + \sum_{b=M_{j-1}+1}^{c} r_b. \tag{4-16}$$

We now treat the ingredients of $\Delta_{w,\rho,\boldsymbol{v}}(q, t)$, starting with the $\beta(\nu^{(i)}; q_i)$. Since $\rho(\boldsymbol{\nu}) = \rho$, it follows that $\{\mathrm{P}_{ij} \mid j \in [r-1]\} \subseteq J_{\nu^{(i)}}$ for all $i \in [m]$. For every $j \in [r]$ define the set

$$J_{\nu^{(i)}}^{(j)} = \{k - \mathrm{P}_{i,j-1} \mid k \in J_{\nu^{(i)}} \cap (\mathrm{P}_{i,j-1}, \mathrm{P}_{ij})\}.$$

Lemma 2.1 implies that

$$\binom{n_i}{J_{\nu^{(i)}}}_{q_i^{-1}} = \binom{n_i}{\mathrm{P}_i}_{q_i^{-1}} \prod_{j=1}^{r} \binom{\rho_{ij}}{J_{\nu^{(i)}}^{(j)}}_{q_i^{-1}}. \tag{4-17}$$

Using (4-15) and (4-16), the differences $\nu_d^{(i)} - \nu_{d+1}^{(i)}$ appearing in the exponents in $\beta(\nu^{(i)}; q_i)$, as defined in (4-1), can be expressed as sums of distinct parameters s_{jk} and r_k. In particular, we observe that the elements of $J_{\nu^{(i)}}^{(j)}$ are precisely the exponents of the variable a_i that occur in the weak order v_j. It then follows from (4-17) that

$$\prod_{i=1}^{m} \binom{n_i}{J_{\nu^{(i)}}}_{q_i^{-1}} = \prod_{i=1}^{m} \binom{n_i}{\mathrm{P}_i}_{q_i^{-1}} \prod_{j=1}^{r} \binom{\rho_j}{v_j}_{\boldsymbol{Y}},$$

where $\boldsymbol{Y} = (q_1^{-1}, \dots, q_m^{-1})$ and $\binom{\rho_j}{v_j}_{\boldsymbol{Y}}$ is as in Definition 3.3. This completes our analysis of the factors $\beta(\nu^{(i)}; q_i)$.

We now consider the factors $\alpha(\lambda(\boldsymbol{\nu}), \mu; q)$, using the idea behind the proofs of [Schein and Voll 2015, Lemmata 2.16 and 2.17]. The range of parameters k over which the infinite product of (2-5) giving $\alpha(\lambda(\boldsymbol{\nu}), \mu; q) = \alpha(\tilde{\lambda}, \mu; q)$ may have nontrivial factors is precisely $[1, \mu_1]$. Recall that $\tilde{\lambda}'_k = \lambda'_k + \varepsilon$ for all k and observe that the dual partitions $\tilde{\lambda}'$ and μ' are constant on each of the intervals of (4-14). Indeed, if $d \in [\mu_k - r_k + 1, \mu_k]$, where $k \in [M_{j-1} + 1, M_j]$, then $\tilde{\lambda}'_d = L_j$ and $\mu'_d = k$. Similarly, if $d \in [m(v_{jk}) - s_{jk} + 1, m(v_{jk})]$ with $k \in [\varepsilon_j, \ell_j]$, then $\lambda'_d = \ell(\tilde{v}_{jk})$, hence $\tilde{\lambda}'_d = \ell(\tilde{v}_{jk}) + \varepsilon$, and $\mu'_d = M_{j-1}$. By manipulations with Gaussian binomials analogous to those above we find that

$$\prod_{k=1}^{\infty} \binom{\tilde{\lambda}'_k - \mu'_{k+1}}{\tilde{\lambda}'_k - \mu'_k}_{q^{-1}} = \prod_{j=1}^{r} \binom{L_j - M_{j-1}}{L_j - M_j}_{q^{-1}} \binom{M_j - M_{j-1}}{I_\mu^{(j)}}_{q^{-1}},$$

where $I_\mu^{(j)} = \{k - M_{j-1} \mid k \in J_\mu \cap (M_{j-1}, M_j)\} \subset [M_j - M_{j-1} - 1]$. Combining these observations, we obtain

$$\alpha(\lambda(\boldsymbol{\nu}), \mu; q) = \\ \prod_{j=1}^{r} \left(\binom{L_j - M_{j-1}}{L_j - M_j}_{q^{-1}} \binom{M_j - M_{j-1}}{I_\mu^{(j)}}_{q^{-1}} \prod_{k=M_{j-1}+1}^{M_j} q^{k(L_j - k) r_k} \prod_{k=\varepsilon_j}^{\ell_j} q^{M_{j-1}(\ell(\tilde{v}_{jk}) + \varepsilon - M_{j-1}) s_{jk}} \right).$$

The exponents in the remaining factor $(q^n t)^{\sum_{k=1}^{c} \mu_k} t^{\sum_{i=1}^{m} \sum_{j=1}^{n_i} \nu_j^{(i)}}$ of the right-hand side of (4-13) are again readily expressed as sums of parameters r_k and s_{jk} using (4-15) and (4-16). We leave the final assembly as an exercise for the reader. Summing the parameters r_k and s_{jk} over the ranges indicated above, we obtain

$$\Delta_{w,\rho,\boldsymbol{v}}(q,t) = \left(\prod_{i=1}^{m} \binom{n_i}{\mathrm{P}_i}_{q_i^{-1}} \right) \prod_{j=1}^{r} \left(\binom{L_j - M_{j-1}}{L_j - M_j}_{q^{-1}} \binom{\rho_j}{v_j}_{\boldsymbol{Y}} \prod_{k=1}^{\ell_j} \frac{y_{v_{jk}}^{(j)}}{1 - y_{v_{jk}}^{(j)}} \right) \\ \times \prod_{j=1}^{r-1} I^{\circ}_{M_j - M_{j-1}}(q^{-1}; x_{M_{j-1}+1}, \dots, x_{M_j}) \cdot I_{M_r - M_{r-1}}(q^{-1}; x_{M_{r-1}+1}, \dots, x_{M_r}),$$

where the numerical data x_k and $y_{v_{jk}}^{(j)}$ are as given in Definition 4.18. In particular, note that $y_{v_{jk}}^{(j)}$ depends only on the word $\tilde{v}_{jk}$ and not on the weak order v_j. Recall that $I^{\mathrm{wo}}_{\underline{\rho_j}}(q_1^{-1}, \dots, q_m^{-1}; \boldsymbol{y}^{(j)})$ is given in Definition 3.5 as a sum indexed by $\mathrm{WO}_{\underline{\rho_j}}$. Observe that $\Delta_{w,\rho,\boldsymbol{v}}(q, t)$ is equal to the expression for $D_{w,\rho}(q, t)$ in Definition 4.18, except that each factor $I^{\mathrm{wo}}_{\underline{\rho_j}}(q_1^{-1}, \dots, q_m^{-1}; \boldsymbol{y}^{(j)})$ is replaced by the summand corresponding to $v_j \in \mathrm{WO}_{\underline{\rho_j}}$. Summation over all r-tuples $\boldsymbol{v} = (v_1, \dots, v_r) \in \prod_{j=1}^{r} \mathrm{WO}_{\underline{\rho_j}}$ now completes the proof of Lemma 4.27, and hence of Proposition 4.20. □

5. Application to the class $\mathfrak{L}$ — proof of Theorem 1.3

In order to deduce Theorem 1.3 from the results of the previous section, namely Theorem 4.21 and Corollary 4.22, it remains to show that Hypothesis 4.5 is satisfied for $\mathfrak{o}$-Lie algebras L as in the statement of Theorem 1.3. We noted in Remark 4.7 that the hypothesis is stable under direct products. Hence it suffices to verify the hypothesis in the case $L = \mathcal{L}(\mathfrak{O}_1) \times \cdots \times \mathcal{L}(\mathfrak{O}_g)$, where $\mathcal{L}$ is a Lie ring from one of the three defining subclasses in Definition 1.1 and $\mathfrak{O}_i$ is a finite extension of $\mathfrak{o}$, for each $i \in [g]$. It is enough to compute the $\mathfrak{o}$-ideal zeta function of L; indeed, the $\mathfrak{O}$-ideal zeta function of $L(\mathfrak{O})$ is obtained from the $\mathfrak{o}$-ideal zeta function of L by substituting q^f for q, where f is the inertia degree of $\mathfrak{O}/\mathfrak{o}$. This verification (and more) is done in Sections 5.2, 5.3, and 5.4. We recover, *en passant*, the results of previous work by several authors.

5.1. ***Abelian Lie rings.*** It is instructive to consider the output of Theorem 4.21 for the basic example of the abelian $\mathfrak{o}$-Lie algebra $L = \mathfrak{o}^b$. Its zeta function is well-known; see (1-4). Let $A \leq L$ be an $\mathfrak{o}$-sublattice of rank c with a torsion-free quotient $L/A \simeq \mathfrak{o}^n$; here $n = b - c$. Now, let $h \in \mathbb{N}$ and n_i, e_i, f_i, for $i \in [h]$, be natural numbers such that $\sum_{i=1}^{h} n_i e_i f_i = n$, and let $\mathfrak{O}_1, \dots, \mathfrak{O}_h$ be arbitrary finite extensions of $\mathfrak{o}$ with ramification indices e_i and inertia degrees f_i. Then we may express $L/A \simeq \mathfrak{O}_1^{n_1} \times \cdots \times \mathfrak{O}_h^{n_h}$ as in (4-5). Hypothesis 4.5 is satisfied vacuously, as $c' = 0$. Moreover, $m = 0$ in the sense of Definition 4.6. As $\varepsilon = c$, it follows from Remark 4.12 that the only Dyck word $w \in \mathcal{D}_{2c}$ for which $D_w(q,t) \neq 0$ is the "trivial" word $w = \mathbf{0}^c\mathbf{1}^c$. Since the composition $\underline{n}$ is empty, the only $(\underline{n}, w)$-admissible partition is the empty one. We then read off from Theorem 4.21 that

$$\zeta_L^{\triangleleft \mathfrak{o}}(s) = \zeta_{\mathfrak{o}^n}(s) I_c(q^{-1}; x_1, \dots, x_c),$$

where the numerical data are given by $x_k = q^{k(n+c-k)}t^k = q^{k(b-k)}t^k$. Indeed, it is immediate from (1-4) and (1-5) that

$$I_c(q^{-1}; x_1, \dots, x_c) = \zeta_{\mathfrak{o}^c}(s-n) = \prod_{i=n}^{b-1} \frac{1}{1-q^i t} = \frac{\zeta_{\mathfrak{o}^b}(s)}{\zeta_{\mathfrak{o}^n}(s)}.$$

5.2. ***Free class-2-nilpotent Lie rings.*** Let $\mathfrak{f}_{2,d}$ denote the free class-2-nilpotent Lie ring on d generators. If $\mathfrak{O}$ is a finite extension of $\mathfrak{o}$ with ramification index e and inertia degree f, then the derived subalgebra of $\mathfrak{f}_{2,d}(\mathfrak{O})$ is isolated and has $\mathfrak{o}$-rank $\binom{d}{2}ef$ and abelianization of $\mathfrak{o}$-rank def. We will now implement the general framework developed in Section 4 to compute the $\mathfrak{o}$-ideal zeta function of the direct product

$$L = \mathfrak{f}_{2,d_1}(\mathfrak{O}_1) \times \cdots \times \mathfrak{f}_{2,d_m}(\mathfrak{O}_m),$$

where $d_i \in \mathbb{N}$ and $\mathfrak{O}_i$ is a finite extension of $\mathfrak{o}$ for all $i \in [m]$. The abelianization of $\mathfrak{f}_{2,d_i}(\mathfrak{O}_i)$ is isomorphic to $\mathfrak{O}_i^{d_i}$ as an $\mathfrak{o}$-module. Thus L satisfies (4-5), with $A = L' = Z(L)$ and $n_i = d_i$ for every $i \in [m]$. We set $\bar{L} = L/L'$ and let $\pi_i : \bar{L} \to \mathfrak{O}_i^{d_i}$ be the projections as in Section 4.2. Let $\Lambda \leq \bar{L}$ be a finite-index $\mathfrak{o}$-sublattice and $\nu(\pi_i(\Lambda))$ be the elementary divisor type of the $\mathfrak{O}_i$-sublattice of $\mathfrak{O}_i^{d_i}$ generated by $\pi_i(\Lambda)$.

To use the method of the previous section, we must compute the elementary divisor type of the commutator lattice $[\Lambda, L]$.

Lemma 5.1. *Let $L = \mathfrak{f}_{2,d_1}(\mathfrak{O}_1) \times \cdots \times \mathfrak{f}_{2,d_m}(\mathfrak{O}_m)$ and let $\Lambda \leq \bar{L}$ be an $\mathfrak{o}$-sublattice. For every $i \in [m]$, let $\nu^{(i)} = \nu(\pi_i(\Lambda)) = (\nu_1^{(i)}, \ldots, \nu_{d_i}^{(i)})$. Then the $\mathfrak{o}$-elementary divisor type $\lambda(\Lambda)$ of the commutator $[\Lambda, L] \leq L'$ is obtained from the following multiset with $c = \sum_{i=1}^m \binom{d_i}{2} e_i f_i$ elements*:

$$\coprod_{i=1}^{m} \coprod_{1 \leq j < k \leq d_i} \{\min\{\nu_j^{(i)}, \nu_k^{(i)}\}\}_{e_i, f_i}.$$

Proof. Let $(x_1^{(i)}, \ldots, x_{d_i}^{(i)})$ be an $\mathfrak{O}_i$-basis of $\mathfrak{f}_{2,d_i}(\mathfrak{O}_i)$ with respect to which $\pi_i(\Lambda)$ is diagonal:

$$\pi_i(\Lambda) = \langle \Pi_i^{\nu_1^{(i)}} x_1^{(i)}, \ldots, \Pi_i^{\nu_{d_i}^{(i)}} x_{d_i}^{(i)} \rangle_{\mathfrak{O}_i},$$

where $\Pi_i \in \mathfrak{O}_i$ is a uniformizer. Observe that the collection of commutators

$$\{[x_j^{(i)}, x_k^{(i)}]\}_{1 \leq j < k \leq d_i}$$

provides an $\mathfrak{O}_i$-basis of the derived subalgebra of $\mathfrak{f}_{2,d_i}(\mathfrak{O}_i)$. Clearly, the commutator subalgebra $[\pi_i(\Lambda), \pi_i(L)]$ is the $\mathfrak{O}_i$-lattice spanned by the elements $\{\Pi_i^{\nu_j^{(i)}} [x_j^{(i)}, x_k^{(i)}]\}_{j \neq k}$. The $\mathfrak{O}_i$-elementary divisor type of this lattice is the partition with parts $\min\{\nu_j^{(i)}, \nu_k^{(i)}\}$, as observed already just before [Grunewald et al. 1988, Lemma 5.2]. The elementary divisor type of $[\pi_i(\Lambda), \pi_i(L)]$, viewed as a lattice over $\mathfrak{o}$, is given by the multiset

$$\coprod_{1 \leq j < k \leq d_i} \{\min\{\nu_j^{(i)}, \nu_k^{(i)}\}\}_{e_i, f_i}$$

by Lemma 2.3. To complete the proof, we observe that the direct product structure of L implies that $[\Lambda, L] = \bigoplus_{i=1}^m [\pi_i(\Lambda), \pi_i(L)]$. □

Remark 5.2. Observe that $\{\nu\}_{1,f}$ is simply the multiset consisting of the element ν with multiplicity f. Therefore, if the extensions $\mathfrak{O}_i/\mathfrak{o}$ are all unramified (i.e., $e_i = 1$ for all i) then it is immediate from Lemma 5.1 that L satisfies Hypothesis 4.5. Indeed, we may set $Z = \sum_{i=1}^m f_i$ and let the collection $\widetilde{\mathfrak{S}}_1, \ldots, \widetilde{\mathfrak{S}}_Z$ consist of f_i copies of the pair $(\{i\}, 2)$ for every $i \in [m]$. Moreover, our decomposition of L/A satisfies the conditions of Remark 4.8. Therefore, Hypothesis 4.5 necessarily fails if any of the extensions $\mathfrak{O}_i/\mathfrak{o}$ are ramified, and the method of Section 4 is inapplicable. *We therefore assume for the remainder of Section 5.2 that all the $\mathfrak{O}_i$ are unramified over $\mathfrak{o}$.*

As at the beginning of Section 4.4, the possible orderings of the projection data $\boldsymbol{\nu} = (\nu^{(1)}, \ldots, \nu^{(m)})$ are parametrized by the chain complex $\mathrm{WO}_{\underline{n}}$ of $C_{\underline{n}}$. Recall the function $\ell(v)$ of Definition 4.13.

Lemma 5.3. *Let $v = \prod_{i=1}^m a_i^{\alpha_i} \in C_{\underline{n}}$. Then $\ell(v) = \sum_{i=1}^m \binom{\alpha_i}{2} f_i$.*

Proof. Let $i \in [m]$. There are exactly α_i parts of the partition $\nu(\pi_i(\Lambda))$ that are not less than $m(v)$, and hence there are $\binom{\alpha_i}{2}$ pairwise minima that are not less than $m(v)$. Each of these minima appears in $\lambda(\Lambda)$

with multiplicity f_i. Alternatively, apply Lemma 4.14 and the description of the sets $\widetilde{\mathfrak{S}}_1, \dots, \widetilde{\mathfrak{S}}_Z$ given in Remark 5.2 above. □

We now have all the ingredients necessary to apply Definition 4.18 and Theorem 4.21 to obtain an explicit expression for $\zeta_L^{\triangleleft\mathfrak{o}}(s)$.

Example 5.4. We recover an expression for the $\mathbb{Z}_p$-ideal zeta function of $\mathfrak{f}_{2,d}(\mathbb{Z}_p)$, where $d \geq 2$, which was computed by the third author in [Voll 2005b]. The expressions of Theorem 4.21 reduce to a particularly simple form in this case. Here $m = 1$ and $\mathfrak{o} = \mathbb{Z}_p$, and, given a $\mathbb{Z}_p$-sublattice $\Lambda \leq \bar{L}$, there is only one relevant projection datum, namely the elementary divisor type $\nu = (\nu_1, \dots, \nu_d)$ of Λ itself. The derived subalgebra has rank $c = \binom{d}{2}$. In view of Lemma 5.3, the parts of the dual partition $\lambda(\Lambda)' = \lambda(\nu)'$ are all triangular numbers. In particular, if $w \in \mathcal{D}_{2c} = \mathcal{D}_{d(d-1)}$ is a Dyck word, then $D_w(p,t) = 0$ unless all the parameters $L_1, \dots, L_r$ associated to w are triangular numbers.

So suppose that $w \in \mathcal{D}_{d(d-1)}$ is such that $L_j = \binom{\gamma_j}{2}$ for all $j \in [r]$. It is easy to see from Definition 4.17 that there is only one (d, w)-admissible composition, namely $\rho_{1j} = \gamma_j - \gamma_{j-1}$ for all $j \in [r]$ (where we have set $\gamma_0 = 0$). Thus $\mathrm{P}_{1j} = \gamma_j$ for all j. Noting from Example 3.6 that the generalized Igusa function associated to a composition with one part is a classical Igusa function in the sense of Definition 2.2, we read off from Definition 4.18 that

$$D_w(p,t) = \prod_{j=1}^{r} \left(\binom{L_j - M_{j-1}}{L_j - M_j}_{p^{-1}} \binom{d}{\gamma_j}_{p^{-1}} I_{\gamma_j - \gamma_{j-1}}(p^{-1}; y_1^{(j)}, \dots, y_{\gamma_j-\gamma_{j-1}}^{(j)}) \right) \times \prod_{j=1}^{r-1} I^{\circ}_{M_j - M_{j-1}}(p^{-1}; x_{M_{j-1}+1}, \dots, x_{M_j}) \cdot I_{M_r - M_{r-1}}(p^{-1}; x_{M_{r-1}+1}, \dots, x_{M_r}),$$

where

$$y_k^{(j)} = p^{M_{j-1}(d + \binom{\gamma_{j-1}+k}{2}) - M_{j-1}) + (\gamma_{j-1}+k)(d - \gamma_{j-1} - k)} t^{\gamma_{j-1}+k+M_{j-1}},$$
$$x_k = p^{k(d+\binom{\gamma_j}{2}) - k) + \gamma_j(d-\gamma_j)} t^{k+\gamma_j}.$$

Here, as usual, we have $k \in [M_{j-1}+1, M_j]$ in the definition of x_k. Indeed, observe that the only instance of two distinct subwords $v_1, v_2 \leq a_1^d$ satisfying $\ell(v_1) = \ell(v_2)$ is $\ell(\varnothing) = \ell(a_1) = 0$. Thus we always have $\delta_v^{(j)} = 1$ except in the case $\delta_{a_1}^{(1)} = 0$, but it is easy to verify that the uniform expressions given above for the numerical data hold. Finally, by Theorem 4.21,

$$\zeta^{\triangleleft}_{\mathfrak{f}_{2,d}(\mathbb{Z}_p)}(s) = \sum_{w \in \mathcal{D}_{d(d-1)}} D_w(p,t).$$

We leave it as an exercise for the reader to unwind the definitions of [Voll 2005b] and verify that this formula matches [loc. cit., Theorem 4].

5.3. ***Free class-2-nilpotent products of abelian Lie rings.*** Let L_1 and L_2 be abelian Lie rings of ranks d and d', respectively. We denote by $\mathfrak{g}_{d,d'}$ the free class-2-nilpotent product of L_1 and L_2 of nilpotency class at most two. This is the Lie ring version of a group-theoretical construction considered by Levi

[1944] (see also [Golovin 1950]), which is itself a special case of a varietal product as in [Neumann 1967, Section 1.8]. Concretely, a presentation of $\mathfrak{g}_{d,d'}$ is given by

$$\mathfrak{g}_{d,d'} = \langle x_1, \dots, x_d, y_1, \dots, y_{d'}, (z_{ij})_{i\in[d], j\in[d']} \mid [x_i, y_j] = z_{ij} \rangle,$$

where all Lie brackets not following from the relations above vanish.

Example 5.5. (1) $\mathfrak{g}_{1,1}$ is the Heisenberg Lie ring.

(2) $\mathfrak{g}_{d,1}$ is the Grenham Lie ring of degree d.

(3) $\mathfrak{g}_{d,0} = \mathbb{Z}^d$ is the abelian Lie ring of rank d.

(4) $\mathfrak{g}_{d,d} = \mathcal{G}_d$ is the Lie ring featuring in [Stasinski and Voll 2014, Definition 1.2].

We fix $g \in \mathbb{N}$ and g-tuples $\underline{d} = (d_1, \dots, d_g)$ and $\underline{d}' = (d'_1, \dots, d'_g)$ of natural numbers. Let $\mathfrak{O}_1, \dots, \mathfrak{O}_g$ be finite extensions of $\mathfrak{o}$ with ramification indices e_i and inertia degrees f_i, respectively. Consider the $\mathfrak{o}$-Lie algebra

$$L = \mathfrak{g}_{d_1,d'_1}(\mathfrak{O}_1) \times \cdots \times \mathfrak{g}_{d_g,d'_g}(\mathfrak{O}_g).$$

Define $d = \sum_{i=1}^{g} d_i e_i f_i$ and $d' = \sum_{i=1}^{g} d'_i e_i f_i$, and set $c = \sum_{i=1}^{g} d_i d'_i e_i f_i$. Observe that, as an $\mathfrak{o}$-module, L is free of rank $d + d' + c$. Let L' denote the derived subalgebra of L, and let

$$\bar{L} = L/L' \simeq (\mathfrak{O}_1^{d_1} \times \mathfrak{O}_1^{d'_1}) \times (\mathfrak{O}_2^{d_2} \times \mathfrak{O}_2^{d'_2}) \times \cdots \times (\mathfrak{O}_g^{d_g} \times \mathfrak{O}_g^{d'_g})$$

be its abelianization. For each $i \in [g]$, consider the usual basis $\{x_k^{(i)}, y_\ell^{(i)}, z_{k\ell}^{(i)}\}_{k\in[d_i], \ell\in[d'_i]}$ of $\mathfrak{g}_{d_i,d'_i}(\mathfrak{O}_i)$ as an $\mathfrak{O}_i$-module. Consider the natural linear projections

$$\pi_i : \bar{L} \to \langle x_1^{(i)}, \dots, x_{d_i}^{(i)} \rangle_{\mathfrak{O}_i} \simeq \mathfrak{O}_i^{d_i} \quad \text{and} \quad \pi'_i : \bar{L} \to \langle y_1^{(i)}, \dots, y_{d'_i}^{(i)} \rangle_{\mathfrak{O}_i} \simeq \mathfrak{O}_i^{d'_i}.$$

For each $i \in [g]$, fix an $\mathfrak{o}$-basis $(\alpha_1^{(i)}, \dots, \alpha_{e_i f_i}^{(i)})$ of $\mathfrak{O}_i$. Then $\{\alpha_j^{(i)} x_k^{(i)}, \alpha_j^{(i)} y_\ell^{(i)}, \alpha_j^{(i)} z_{k\ell}^{(i)}\}_{k\in[d_i],\ell\in[d'_i],j\in[e_i f_i]}$ is an $\mathfrak{o}$-basis of $\mathfrak{g}_{d_i,d'_i}(\mathfrak{O}_i)$ and the union of these bases is an $\mathfrak{o}$-basis of L.

Let $\Lambda \le \bar{L}$ be an $\mathfrak{o}$-sublattice. For each $i \in [g]$, we let $\nu^{(i)}$, a partition with d_i parts, be the elementary divisor type of the $\mathfrak{O}_i$-sublattice of $\mathfrak{O}_i^{d_i}$ generated by $\pi_i(\Lambda)$. Similarly, we set $\nu^{(i+g)}$ to be the elementary divisor type of the $\mathfrak{O}_i$-sublattice of $\mathfrak{O}_i^{d'_i}$ generated by $\pi'_i(\Lambda)$. In other words,

$$\boldsymbol{\nu} = \boldsymbol{\nu}(\Lambda) = (\nu^{(1)}, \nu^{(1+g)}, \nu^{(2)}, \nu^{(2+g)}, \dots, \nu^{(g)}, \nu^{(2g)}) \tag{5-1}$$

is the projection data of Λ as an $\mathfrak{o}$-sublattice of $\bar{L}$.

Lemma 5.6. *Let $L = \mathfrak{g}_{d_1,d'_1}(\mathfrak{O}_1) \times \cdots \times \mathfrak{g}_{d_g,d'_g}(\mathfrak{O}_g)$ and let $\Lambda \le \bar{L}$ be an $\mathfrak{o}$-sublattice. Let $\boldsymbol{\nu}(\Lambda)$ be as in* (5-1) *above. Then the $\mathfrak{o}$-elementary divisor type $\lambda(\Lambda)$ of the commutator $[\Lambda, L] \le L'$ is obtained from the following multiset with $c = \sum_{i=1}^{g} d_i d'_i e_i f_i$ elements*:

$$\coprod_{i=1}^{g} \coprod_{k=1}^{d_i d'_i} \{(\nu^{(i)} * \nu^{(i+g)})_k\}_{e_i, f_i},$$

where the operation $*$ is explained in Definition 4.3 and the sets $\{a\}_{e_i,f_i}$, for $a \in \mathbb{N}$, are as in Definition 2.4.

Proof. For every $i \in [g]$, let Π_i denote a uniformizer of $\mathfrak{O}_i$. Let $(\xi_1^{(i)}, \dots, \xi_{d_i}^{(i)})$ and $(\upsilon_1^{(i)}, \dots, \upsilon_{d_i'}^{(i)})$ be bases of $\mathfrak{O}_i^{d_i}$ and $\mathfrak{O}_i^{d_i'}$, respectively, such that

$$\langle \pi_i(\Lambda)\rangle_{\mathfrak{O}_i} = \langle \Pi_i^{\nu_1^{(i)}} \xi_1^{(i)}, \dots, \Pi_i^{\nu_{d_i}^{(i)}} \xi_{d_i}^{(i)}\rangle_{\mathfrak{O}_i} \quad \text{and} \quad \langle \pi_i'(\Lambda)\rangle_{\mathfrak{O}_i} = \langle \Pi_i^{\nu_1^{(i+g)}} \upsilon_1^{(i)}, \dots, \Pi_i^{\nu_{d_i'}^{(i+g)}} \upsilon_{d_i'}^{(i)}\rangle_{\mathfrak{O}_i}.$$

Observe that the commutators $[\xi_k^{(i)}, \upsilon_\ell^{(i)}]$ form an $\mathfrak{O}_i$-basis of the subspace $\langle z_{k\ell}^{(i)}\rangle_{\mathfrak{O}_i}$ of L'. Fixing $k \in [d_i]$, we find that

$$[\Pi_i^{\nu_k^{(i)}} \xi_k^{(i)}, \bar{L}] = \bigoplus_{\ell \in [d_i']} \Pi_i^{\nu_k^{(i)}} \mathfrak{O}_i[\xi_k^{(i)}, y_\ell^{(i)}] = \bigoplus_{\ell \in [d_i']} \Pi_i^{\nu_k^{(i)}} \mathfrak{O}_i[\xi_k^{(i)}, \upsilon_\ell^{(i)}].$$

Similarly, for a fixed $\ell \in [d_i']$ we obtain

$$[\Pi_i^{\nu_\ell^{(i+g)}} \upsilon_\ell^{(i)}, \bar{L}] = \bigoplus_{k \in [d_i]} \Pi_i^{\nu_\ell^{(i+g)}} \mathfrak{O}_i[x_k^{(i)}, \upsilon_\ell^{(i)}] = \bigoplus_{k \in [d_i]} \Pi_i^{\nu_\ell^{(i+g)}} \mathfrak{O}_i[\xi_k^{(i)}, \upsilon_\ell^{(i)}].$$

From this we conclude that

$$[\overline{\mathfrak{g}_{d_i,d_i'}(\mathfrak{O}_i)}, \Lambda] = \bigoplus_{k \in [d_i], \ell \in [d_i']} \Pi_i^{\min\{\nu_k^{(i)}, \nu_\ell^{(i+g)}\}} \mathfrak{O}_i[\xi_k^{(i)}, \upsilon_\ell^{(i)}] = \bigoplus_{k \in [d_i], \ell \in [d_i']} \Pi_i^{\min\{\nu_k^{(i)}, \nu_\ell^{(i+g)}\}} \mathfrak{O}_i z_{k\ell}^{(i)}$$

as $\mathfrak{O}_i$-modules, where $\overline{\mathfrak{g}_{d_i,d_i'}(\mathfrak{O}_i)}$ is the abelianization of $\mathfrak{g}_{d_i,d_i'}(\mathfrak{O}_i)$. Therefore,

$$[\bar{L}, \Lambda] = \bigoplus_{i \in [g], k \in [d_i], \ell \in [d_i']} \Pi_i^{\min\{\nu_k^{(i)}, \nu_\ell^{(i+g)}\}} \mathfrak{O}_i z_{k\ell}^{(i)}$$

as $\mathfrak{o}$-modules. The claim follows. ☐

Set $m = 2g$. For $i \in [g]$, set $\mathfrak{O}_{i+g} = \mathfrak{O}_i$ and define $n_i = d_i$ and $n_{i+g} = d_i'$. It is clear from Lemma 5.6 that the Lie ring L fits the general framework of the beginning of Section 4.3. Moreover, we see analogously to Remark 5.2 that if all the $\mathfrak{O}_i$ are unramified over $\mathfrak{o}$, then Hypothesis 4.5 is satisfied. In this case, we take $Z = \sum_{i=1}^{g} f_i$; the collection $\widetilde{\mathfrak{S}}_1, \dots, \widetilde{\mathfrak{S}}_Z$ consists of f_i copies of the pair $((i, i+g), (1, 1))$ for every $i \in [g]$. *Thus we assume for the remainder of this section that all the $\mathfrak{O}_i$ are unramified over $\mathfrak{o}$.*

Consider the composition $\underline{n} = (n_1, \dots, n_{2g})$. Then the natural ordering among all the parts of the projection data $\boldsymbol{\nu} = (\nu^{(1)}, \dots, \nu^{(2g)})$ corresponds to an element of $\mathrm{WO}_{\underline{n}}$.

Lemma 5.7. *Let $v = \prod_{i=1}^{2g} a_i^{\alpha_i} \in C_{\underline{n}}$. Then $\ell(v) = \sum_{i=1}^{g} \alpha_i \alpha_{i+g} f_i$.*

Proof. Let $v \in C_{\underline{n}}$ as above. For any $i \in [g]$, the $d_i d_i'$ parts of $\nu^{(i)} * \nu^{(i+g)}$ are, by definition, the minima $\min\{\nu_k^{(i)}, \nu_\ell^{(i+g)}\}_{k \in [d_i], \ell \in [d_i']}$. Clearly, $\min\{\nu_k^{(i)}, \nu_\ell^{(i+g)}\} \geq m(v)$ if and only if both elements of the pair $(\nu_k^{(i)}, \nu_\ell^{(i+g)})$ are contained in S_v, and it is clear from (4-10) that there are $\alpha_i \alpha_{i+g}$ such pairs. Finally, since we have assumed all $\mathfrak{O}_i/\mathfrak{o}$ to be unramified, every part of $\nu^{(i)} * \nu^{(i+g)}$ appears in $\lambda(\boldsymbol{\nu})$ with multiplicity f_i. Alternatively, use Lemma 4.14. ☐

The $\mathfrak{o}$-ideal zeta function $\zeta_L^{\triangleleft \mathfrak{o}}(s)$ may now be read off from Theorem 4.21.

5.3.1. *Grenham Lie rings over unramified extensions.* As an example, we will treat the case $L = \mathfrak{g}_{d,1}(\mathfrak{O})$, where $\mathfrak{g}_{d,1}$ is the Grenham Lie ring of degree d and $\mathfrak{O}/\mathfrak{o}$ is unramified of degree f. In the case $d = f = 2$, this zeta function was computed previously by Bauer, using methods analogous to those of [Voll 2005a] and quite different from the current paper's approach.

Observe that $L' = Z(L)$, so necessarily we have $A = L'$ and thus $c = c' = df$ and $\varepsilon = 0$ in the notation of Section 4.3. The nonempty radical words $v \in C_{(d,1)}$ are exactly those of the form $v = a_1^{\alpha_1} a_2$ with $\alpha_1 > 0$. If $w \in \mathcal{D}_{2c}$ is a Dyck word with associated parameters $L_1, \dots, L_r$ and $M_1, \dots, M_r$, then clearly there are no $((d, 1), w)$-admissible compositions (recall Definition 4.17) unless all the L_i are divisible by f. Otherwise, there is a unique $((d, 1), w)$-admissible composition $\rho \in \mathrm{Mat}_{2,r}$; it satisfies $\mathrm{P}_{1j} = L_j/f$ and $\mathrm{P}_{2j} = 1$ for all $j \in [r]$. Equivalently, $\rho_{1j} = (L_j - L_{j-1})/f$ for all $j \in [r]$, while $\rho_{21} = 1$ and $\rho_{2j} = 0$ for all $j > 1$.

Let $\mathcal{D}_{2c}(f)$ be the set of Dyck words $w \in \mathcal{D}_{2c}$ such that $f \mid L_i$ for all $i \in [r]$. Given $w \in \mathcal{D}_{2c}(f)$, set $\boldsymbol{L}_w/f = \{L_i/f \mid i \in [r-1]\}$. The following explicit statement is now immediate from Theorem 4.21.

Proposition 5.8. *Let $L = \mathfrak{g}_{d,1}(\mathfrak{O})$, where $\mathfrak{O}/\mathfrak{o}$ is an unramified extension of degree f. Then*

$$\zeta_L^{\triangleleft \mathfrak{o}}(s) = \frac{\zeta_{\mathfrak{o}^{(d+1)f}}(s)}{\zeta_{\mathfrak{O}}(s)\zeta_{\mathfrak{O}^d}(s)} \sum_{w \in \mathcal{D}_{2c}(f)} D_w(q, t),$$

where

$$D_w(q,t) = \binom{d}{\boldsymbol{L}_w/f}_{q^{-f}} \prod_{j=1}^{r} \binom{L_j - M_{j-1}}{L_j - M_j}_{q^{-1}} I^{\mathrm{wo}}_{(L_1/f,1)}(q^{-f}, q^{-f}; \boldsymbol{y}^{(1)})$$
$$\times \prod_{j=2}^{r} I_{(L_j - L_{j-1})/f}(q^{-f}; y_{(L_{j-1}/f)+1}, \dots, y_{L_j/f})$$
$$\times \prod_{j=1}^{r-1} I^{\circ}_{M_j - M_{j-1}}(q^{-1}; x_{M_{j-1}+1}, \dots, x_{M_j}) I_{M_r - M_{r-1}}(q^{-1}; x_{M_{r-1}+1}, \dots, x_{M_r}).$$

Here the numerical data are given by

$$x_k = q^{k((d+1)f + L_j - k) + L_j(d - L_j/f)} t^{k+f+L_j}, \; k \in [M_{j-1}+1, M_j],$$
$$y^{(1)}_{a_1^{\alpha_1} a_2^{\alpha_2}} = q^{f\alpha_1(d-\alpha_1)} t^{f(\alpha_1+\alpha_2)},$$
$$y_k = q^{M_{j-1}((d+k+k(d-k)+1)f - M_{j-1})} t^{f(k+1)+M_{j-1}}, \; k \in [(L_{j-1}/f)+1, L_j/f].$$

Remark 5.9. Using Proposition 5.8 to compute $\zeta^{\triangleleft}_{\mathfrak{g}_{d,1}(\mathbb{Z}_p)}(s)$ produces a sum parametrized by the $\frac{1}{d+1}\binom{2d}{d}$ elements of $\mathcal{D}_{2d}$. Yet [Voll 2005a, Theorem 5], translated to the notation of the present paper, gives the much simpler expression

$$\zeta^{\triangleleft}_{\mathfrak{g}_{d,1}(\mathbb{Z}_p)}(s) = \zeta_{\mathbb{Z}_p^{d+1}}(s) I_d(p^{-1}; z_1, \dots, z_d),$$

where $z_i = p^{i(2d+1-i)} t^{2i+1}$ for $i \in [d]$. We have checked that these expressions coincide for $d \le 3$, but a direct proof of their equality would involve proving an identity of generalized Igusa functions with

Dyck word	flag	$D_{w,\rho}(q,t)$
00001111	a^2b^2	$I^{\mathrm{wo}}_{(2,2)}(q^{-1};\boldsymbol{y})I_4(q^{-1};q^7t^5,q^{12}t^6,q^{15}t^7,q^{16}t^8)$
00100111	$a^2b<a^2b^2$	$\binom{2}{1}^2_{q^{-1}}I^{\mathrm{wo}}_{(2,1)}(q^{-1};\boldsymbol{y})\,\mathrm{gp}(q^6t^4)\,\mathrm{gp}_0(q^7t^5)I_3(q^{-1};q^{12}t^6,q^{15}t^7,q^{16}t^8)$
	$ab^2<a^2b^2$	
00110011	$a^2b<a^2b^2$	$\binom{2}{1}_{q^{-1}}I^{\mathrm{wo}}_{(2,1)}(q^{-1};\boldsymbol{y})I^{\circ}_2(q^{-1};q^6t^4,q^9t^5)\,\mathrm{gp}_0(q^{12}t^6)I_2(q^{-1};q^{15}t^7,q^{16}t^8)$
	$ab^2<a^2b^2$	
01000111	$ab<a^2b^2$	$\binom{2}{1}^2_{q^{-1}}I^{\mathrm{wo}}_{(1,1)}(q^{-1},\boldsymbol{y})\,\mathrm{gp}(q^6t^3)I^{\mathrm{wo}}_{(1,1)}(q^{-1};\boldsymbol{z})I_3(q^{-1};q^{12}t^6,q^{15}t^7,q^{16}t^8)$
01010011	$ab<a^2b<a^2b^2$	$\binom{2}{1}^2_{q^{-1}}I^{\mathrm{wo}}_{(1,1)}(q^{-1};\boldsymbol{y})\,\mathrm{gp}(q^6t^3)\,\mathrm{gp}_0(q^6t^4)\,\mathrm{gp}(q^9t^5)\,\mathrm{gp}_0(q^{12}t^6)I_2(q^{-1};q^{15}t^7,q^{16}t^8)$
	$ab<ab^2<a^2b^2$	

Table 1. Dyck words, together with the associated functions $D_{w,\rho}(q,t)$.

conditions on the numerical data, in the spirit of Proposition 3.11; see also Remark 5.10 below. This example shows that expressions derived from Theorem 4.21 sometimes admit dramatic cancellation.

5.3.2. *The Lie ring* $\mathfrak{g}_{2,2}$. Paajanen [2008, Theorem 11.1] computed the ideal zeta function of the $\mathfrak{o}$-Lie algebra $L=\mathfrak{g}_{2,2}(\mathfrak{o})$. We recover this computation as a special case of our results. By Theorem 4.21 we have

$$\zeta^{\triangleleft\mathfrak{o}}_L(s)=\frac{\zeta_{\mathfrak{o}^4}(s)}{(\zeta_{\mathfrak{o}^2}(s))^2}\sum_{w\in\mathcal{D}_8}\sum_{\rho\in\mathcal{M}_{(2,2),w}}D_{w,\rho}(q,t)=\frac{(1-t)(1-qt)}{(1-q^2t)(1-q^3t)}\sum_{w\in\mathcal{D}_8}\sum_{\rho\in\mathcal{M}_{(2,2),w}}D_{w,\rho}(q,t).$$

There are fourteen Dyck words of length 8, but it is easy to check that there are only five Dyck words $w\in\mathcal{D}_8$ for which there exist w-compatible flags of subwords of the word $a_1^2a_2^2$. For simplicity, for the rest of this example we will write a instead of a_1 and b instead of a_2. We tabulate these Dyck words, together with the associated functions $D_{w,\rho}(q,t)$ in Table 1. Observe that there are three Dyck words with two compatible flags, and that in each of these cases both flags give rise to the same function $D_{w,\rho}(q,t)$. This is a consequence of the symmetries of $L=\mathfrak{g}_{2,2}(\mathfrak{o})$ and is not a general phenomenon.

For brevity, in Table 1, we use the notation $\mathrm{gp}(x)=\frac{x}{1-x}$ and $\mathrm{gp}_0(x)=\frac{1}{1-x}$. Here the numerical data $\boldsymbol{y}$ and $\boldsymbol{z}$ are defined as follows:

$$y_a=y_b=qt,\quad y_{a^2}=y_{b^2}=t^2,\quad y_{ab}=q^2t^2,\quad y_{a^2b}=y_{ab^2}=qt^3,\quad y_{a^2b^2}=t^4,$$
$$z_a=z_b=q^6t^4,\quad z_{ab}=q^7t^5.$$

5.3.3. *The Heisenberg Lie ring.* The relatively free product $\mathfrak{g}_{1,1}$ is the Heisenberg Lie ring $\mathfrak{h}$. This ring is spanned over $\mathbb{Z}$ by three generators x,y,z, with the relations $[x,y]=z$, $[x,z]=[y,z]=0$. It is among the smallest nonabelian nilpotent Lie rings. It was studied by two of the authors in [Schein and Voll 2015], in the case $\mathfrak{o}=\mathbb{Z}_p$; the zeta functions computed there can be recovered as special cases of the analysis in this section. Indeed, consider

$$L = \mathfrak{h}(\mathfrak{O}_1) \times \cdots \times \mathfrak{h}(\mathfrak{O}_g),$$

where the $\mathfrak{O}_i$ are unramified over $\mathfrak{o}$ so that Hypothesis 4.5 holds. Then $c = \sum_{i=1}^{g} f_i$, while $n = 2c$. Note that the quantity denoted n in [loc. cit.] is called c in the current paper. The composition $\underline{n}$ defined just before the statement of Lemma 5.7 is $\underline{n} = (1, 1, \ldots, 1)$, with $2g$ parts. Thus the elements of $C_{\underline{n}}$ correspond to subwords of the word $a_1 \cdots a_{2g}$. The radical subwords are the words of the form $\prod_{i \in J} a_i a_{i+g}$ for some $J \subseteq [g]$. Thus radical subwords are in bijection with subsets of $[g]$. Moreover, if $w \in \mathcal{D}_{2c}$ is a Dyck word, then a w-compatible flag $V = (v_1 < \cdots < v_r) \in \mathcal{F}_w$ corresponds to a sequence of subsets $J_1 \subset \cdots \subset J_r = [g]$ such that $\sum_{i \in J_j \setminus J_{j-1}} f_i = L_j - L_{j-1}$ for all $j \in [r]$. Setting $\mathcal{A}_j = J_j \setminus J_{j-1}$, we obtain precisely the set partitions of $[g]$ that are compatible with w, in the sense of [loc. cit., Definition 3.4]. Recall that the set of set partitions compatible with w was denoted $\mathcal{P}_w$ in [loc. cit.].

We see from Theorem 4.21, applied to $L = \mathfrak{g}_{1,1}(\mathfrak{O}_1) \times \cdots \times \mathfrak{g}_{1,1}(\mathfrak{O}_g)$, that

$$\zeta_L^{\triangleleft \mathfrak{o}}(s) = \frac{\zeta_{\mathfrak{o}^{2c}}(s)}{\prod_{i=1}^{g} \zeta_{\mathfrak{O}_i}(s)^2} \sum_{w \in \mathcal{D}_{2c}} \sum_{\rho \in \mathcal{M}_{\underline{n},w}} D_{w,\rho}(q,t) = \zeta_{\mathfrak{o}^{2c}}(s) \left(\prod_{i=1}^{g} (1 - t^{f_i})^2 \right) \sum_{\substack{w \in \mathcal{D}_{2c} \\ \rho \in \mathcal{M}_{\underline{n},w}}} D_{w,\rho}(q,t).$$

Now set $\mathfrak{o} = \mathbb{Z}_p$; in particular, $q = p$. A comparison with [loc. cit., Equation (2.20)] and the displayed equation immediately before [loc. cit., Theorem 3.6] shows that, to recover the results obtained there, it suffices to prove that if $\rho \in \mathcal{M}_{\underline{n},w}$ is associated to a set partition $\{\mathcal{A}_j\}_{j \in [r]} \in \mathcal{P}_w$, then

$$\left(\prod_{i=1}^{g} (1 - t^{f_i})^2 \right) D_{w,\rho}(p,t) = \left(\prod_{i=1}^{g} (1 - t^{2 f_i}) \right) D_{w,\mathcal{A}}^{\boldsymbol{f}}(p,t), \tag{5-2}$$

where $D_{w,\mathcal{A}}^{\boldsymbol{f}}(p,t)$ is defined by [loc cit., (3.12)].

We read off from Definition 4.18 that, for $\rho \in \mathcal{M}_{\underline{n},w}$,

$$\begin{aligned} D_{w,\rho}(p,t) = \prod_{j=1}^{r} & \left(\binom{L_j - M_{j-1}}{L_j - M_j}_{p^{-1}} I^{\mathrm{wo}}_{\prod_{k \in \mathcal{A}_j} a_k a_{k+g}}(\boldsymbol{y}^{(j)}) \right) \\ & \times \left(\prod_{j=1}^{r-1} I^{\circ}_{M_j - M_{j-1}}(p^{-1}; x_{M_{j-1}+1}, \ldots, x_{M_j}) \right) I_{M_r - M_{r-1}}(p^{-1}; x_{M_{r-1}+1}, \ldots, x_{M_r}), \end{aligned} \tag{5-3}$$

with the numerical data specified there. Since the parameters q_i^{-1} do not actually appear in the relevant generalized Igusa functions, we have omitted them from the notation (just as in Proposition 3.11). Observe that the numerical data x_k in (5-3) match those in the formula for $D_{w,\mathcal{A}}^{\boldsymbol{f}}(p,t)$ given in [loc cit., Theorem 3.6]. Moreover, if $r_{\mathcal{J}} = \prod_{k \in \mathcal{J}} a_k a_{k+g}$ is a radical subword of $\prod_{k \in \mathcal{A}_j} a_k a_{k+g}$, then the numerical datum $y_{r_{\mathcal{J}}}^{(j)}$ matches the numerical datum $y_{\mathcal{J}}^{(j)}$ of [loc. cit., Theorem 3.6]. In addition, we observe that the numerical data of Definition 4.18 satisfy the hypothesis of Proposition 3.11. Recalling from Example 3.6

how to express the weak order Igusa functions of [loc. cit., Definition 2.9] in terms of the generalized Igusa functions of Definition 3.5 above, we find that Proposition 3.11 indeed implies (5-2).

Remark 5.10. Observe that $\mathfrak{h} = \mathfrak{f}_{2,2}$. Thus we can view $L = \mathfrak{f}_{2,2}(\mathfrak{O}_1) \times \cdots \times \mathfrak{f}_{2,2}(\mathfrak{O}_g)$ and obtain an expression for $\zeta_L^{\triangleleft \mathfrak{o}}(s)$ by specializing the analysis of Section 5.2. This expression is not obviously equal to the one obtained above by considering $L = \mathfrak{g}_{1,1}(\mathfrak{O}_1) \times \cdots \times \mathfrak{g}_{1,1}(\mathfrak{O}_g)$ and using the approach of Section 5.3, or to that of [loc. cit., Theorem 3.6]. To verify the equality directly, one has to prove identities between generalized Igusa functions that depend on the numerical data, in the style of Proposition 3.11. We leave this as an exercise for the reader.

5.4. ***The higher Heisenberg Lie rings.*** Let $d \in \mathbb{N}$. The higher Heisenberg Lie ring $\mathfrak{h}_d$ consists of d copies of the Heisenberg Lie ring $\mathfrak{h}$, amalgamated over their centers; in particular $\mathfrak{h}_1 = \mathfrak{h}$. More precisely, $\mathfrak{h}_d$ is spanned over $\mathbb{Z}$ by $2d+1$ elements $x_1, \ldots, x_d, y_1, \ldots, y_d, z$, with the relations $[x_i, y_i] = z$ for all $i \in [d]$; all other pairs of generators commute. Let

$$L = \mathfrak{h}_{d_1}(\mathfrak{O}_1) \times \cdots \times \mathfrak{h}_{d_g}(\mathfrak{O}_g),$$

where $(d_1, \ldots, d_g) \in \mathbb{N}^g$ and each $\mathfrak{O}_i$ is a finite, not necessarily unramified extension of $\mathfrak{o}$. In the case of $d_1 = \cdots = d_g$ and $\mathfrak{o} = \mathfrak{O}_1 = \cdots = \mathfrak{O}_g = \mathbb{Z}_p$, the zeta function $\zeta_L^{\triangleleft \mathfrak{o}}(s)$ was computed by Bauer [2013] in his unpublished M.Sc. thesis by adapting the methods of [Schein and Voll 2015]. Observe that

$$\bar{L} \simeq \mathfrak{O}_1^{d_1} \times \mathfrak{O}_1^{d_1} \times \cdots \times \mathfrak{O}_g^{d_g} \times \mathfrak{O}_g^{d_g} = \underbrace{\mathfrak{O}_1 \times \cdots \times \mathfrak{O}_1}_{2d_1 \text{copies}} \times \cdots \times \underbrace{\mathfrak{O}_g \times \cdots \times \mathfrak{O}_g}_{2d_g \text{copies}}. \tag{5-4}$$

Set $S_i = \sum_{j=1}^{i} 2d_j$. We have naturally expressed $\bar{L}$ as a product of S_g submodules, giving rise to projections $\pi_1, \ldots, \pi_{S_g}$ as in Section 4.3, where $\pi_k : \bar{L} \to \mathfrak{O}_i$ when $S_{i-1} < k \le S_i$. Let $\Lambda \le \bar{L}$ be an $\mathfrak{o}$-sublattice, and let $\boldsymbol{\nu}(\Lambda) = (\nu^{(1)}, \ldots, \nu^{(S_g)})$ be the corresponding projection data with respect to (5-4); each of these S_g partitions has only one part. Note that $L' = Z(L)$ has rank $c = \sum_{i=1}^{g} e_i f_i$ as an $\mathfrak{o}$-module.

Lemma 5.11. *Let $\Lambda \le \bar{L}$ be an $\mathfrak{o}$-sublattice. The $\mathfrak{o}$-elementary divisor type $\lambda(\Lambda)$ of the commutator $[\Lambda, \bar{L}] \le L'$ is obtained from the following multiset with c elements*:

$$\coprod_{i=1}^{g} \{\min\{\nu_1^{(S_{i-1}+1)}, \nu_1^{(S_{i-1}+2)}, \ldots, \nu_1^{(S_i)}\}_{e_i, f_i}\}.$$

Proof. Let $(x_1^{(i)}, \ldots, x_{d_i}^{(i)}, y_1^{(i)}, \ldots, y_{d_i}^{(i)}, z^{(i)})$ be the natural basis of $\mathfrak{h}_{d_i}(\mathfrak{O}_i)$ as an $\mathfrak{O}_i$-module. Let the decomposition (5-4) be such that, for every $k \in [d_i]$, the images of $\pi_{S_{i-1}+k}$ and $\pi_{S_{i-1}+d_i+k}$ are $\mathfrak{O}_i x_k^{(i)}$ and $\mathfrak{O}_i y_k^{(i)}$, respectively. If $\Pi_i \in \mathfrak{O}_i$ is a uniformizer, then it is clear that, for all $i \in [g]$ and all $k \in [d_i]$,

$$[\Lambda, \mathfrak{O}_i x_k^{(i)}] = \Pi_i^{\nu_1^{(S_{i-1}+d_i+k)}} \mathfrak{O}_i z^{(i)} \quad \text{and} \quad [\Lambda, \mathfrak{O}_i y_k^{(i)}] = \Pi_i^{\nu_1^{(S_{i-1}+k)}} \mathfrak{O}_i z^{(i)}.$$

The claim follows. □

It is immediate from the previous lemma that Hypothesis 4.5 is satisfied if all the extensions $\mathfrak{O}_i/\mathfrak{o}$ are unramified. In this case, we set $Z = \sum_{i=1}^{g} f_i$ and take the collection $\widetilde{\mathfrak{S}}_1, \dots, \widetilde{\mathfrak{S}}_Z$ to consist of f_i copies of the pair $([S_{i-1}+1, S_i], (1, 1, \dots, 1))$ for every $i \in [g]$. The following is then given by Lemma 4.14.

Lemma 5.12. *Let* $v = \prod_{k=1}^{S_g} a_k^{\alpha_k} \in C_{\underline{n}}$. *Then* $\ell(v) = \sum_{i=1}^{g} \bigl(\prod_{k=S_{i-1}+1}^{S_i} \alpha_k\bigr) f_i$.

An explicit expression for $\zeta_L^{\triangleleft\mathfrak{o}}(s)$ can now be obtained from Theorem 4.21. In particular,

$$\zeta^{\triangleleft}_{\mathfrak{h}_d(\mathbb{Z}_p)}(s) = \frac{\zeta_{\mathbb{Z}_p^{2d}}(s)}{(\zeta_{\mathbb{Z}_p}(s))^{2d}} I_{2d}^{\mathrm{wo}}((y_I)_{I \in \mathcal{P}([2d]) \setminus \{\varnothing\}}) \frac{1}{1-p^{2d}t^{2d+1}} = \frac{\zeta_{\mathbb{Z}_p^{2d}}(s)}{1-p^{2d}t^{2d+1}},$$

where $y_I = t^{|I|}$. The second equality follows from Lemmata 2.11 and 5.1 of [Schein and Voll 2015]. This recovers [Grunewald et al. 1988, Proposition 8.4]. Note that $\mathfrak{h}_d(\mathbb{Z}_p)$ is a central amalgamation of d copies of $\mathfrak{h}(\mathbb{Z}_p)$. In contrast to the observations of Remark 4.24, the complexity of $\zeta^{\triangleleft}_{\mathfrak{h}_d(\mathbb{Z}_p)}(s)$ grows in a controlled way with d; this is a special case of a general phenomenon [Bauer and Schein 2023, Theorem 1.1].

Acknowledgements

The research of all three authors was supported by a grant from the GIF, the German-Israeli Foundation for Scientific Research and Development (1246/2014). An extended abstract of this work for the FPSAC 2020 conference has appeared as [Carnevale et al. 2020].

Angela Carnevale gratefully acknowledges the support of the Erwin Schrödinger International Institute for Mathematics and Physics (Vienna) and the Irish Research Council through grant no. GOIPD/2018/319. The Emmy Noether Minerva Research Institute at Bar-Ilan University supported a visit by Christopher Voll during the preliminary stages of this project. Angela Carnevale and Christopher Voll are grateful to the University of Auckland for its hospitality during several phases of this project.

We are grateful to Tomer Bauer for sharing with us some computations that provided important initial pointers, and to Tomer Bauer and the anonymous referee for careful readings of the text.

References

[Bauer 2013] T. Bauer, *Computing normal zeta functions of certain groups*, M.Sc. thesis, Bar-Ilan University, 2013.

[Bauer and Schein 2023] T. Bauer and M. M. Schein, "Ideal growth in amalgamated powers of nilpotent rings of class two and zeta functions of quiver representations", *Bull. Lond. Math. Soc.* **55**:3 (2023), 1511–1529. MR Zbl

[Beck and Sanyal 2018] M. Beck and R. Sanyal, *Combinatorial reciprocity theorems: an invitation to enumerative geometric combinatorics*, Graduate Studies in Mathematics **195**, American Mathematical Society, Providence, RI, 2018. MR Zbl

[Berman et al. 2015] M. N. Berman, B. Klopsch, and U. Onn, "On pro-isomorphic zeta functions of D^*-groups of even Hirsch length", preprint, 2015. To appear in *Israel J. Math.* arXiv 1511.06360

[Berman et al. 2018] M. N. Berman, B. Klopsch, and U. Onn, "A family of class-2 nilpotent groups, their automorphisms and pro-isomorphic zeta functions", *Math. Z.* **290**:3-4 (2018), 909–935. MR Zbl

[Berman et al. 2022] M. N. Berman, I. Glazer, and M. M. Schein, "Pro-isomorphic zeta functions of nilpotent groups and Lie rings under base extension", *Trans. Amer. Math. Soc.* **375**:2 (2022), 1051–1100. MR Zbl

[Birkhoff 1935] G. Birkhoff, "Subgroups of abelian groups", *Proc. London Math. Soc.* (2) **38** (1935), 385–401. MR Zbl

[Butler 1994] L. M. Butler, *Subgroup lattices and symmetric functions*, Mem. Amer. Math. Soc. **539**, 1994. MR Zbl

[Carnevale et al. 2018] A. Carnevale, S. Shechter, and C. Voll, "Enumerating traceless matrices over compact discrete valuation rings", *Israel J. Math.* **227**:2 (2018), 957–986. MR Zbl

[Carnevale et al. 2020] A. Carnevale, M. M. Schein, and C. Voll, "Generalized Igusa functions and ideal growth in nilpotent Lie rings", *Sém. Lothar. Combin.* **84B** (2020), art. id. 71. MR Zbl

[Delsarte 1948] S. Delsarte, "Fonctions de Möbius sur les groupes abeliens finis", *Ann. of Math.* (2) **49**:3 (1948), 600–609. MR Zbl

[Denef 1991] J. Denef, "Report on Igusa's local zeta function", exposé 741, pp. 359–386 in *Séminaire Bourbaki*, 1990/91, Astérisque **201-203**, Soc. Math. de France, Paris, 1991. MR Zbl

[du Sautoy 2002] M. du Sautoy, "Counting subgroups in nilpotent groups and points on elliptic curves", *J. Reine Angew. Math.* **549** (2002), 1–21. MR Zbl

[du Sautoy and Grunewald 2000] M. du Sautoy and F. Grunewald, "Analytic properties of zeta functions and subgroup growth", *Ann. of Math.* (2) **152**:3 (2000), 793–833. MR Zbl

[du Sautoy and Woodward 2008] M. du Sautoy and L. Woodward, *Zeta functions of groups and rings*, Lecture Notes in Math. **1925**, Springer, 2008. MR Zbl

[Dyubyuk 1948] P. E. Dyubyuk, "On the number of subgroups of an abelian p-group", *Izv. Akad. Nauk SSSR Ser. Mat.* **12**:4 (1948), 351–378. In Russian. MR Zbl

[Golovin 1950] O. N. Golovin, "Nilpotent products of groups", *Mat. Sb.* (*N.S.*) **27(69)**:3 (1950), 427–454. In Russian; translated in *Amer. Math. Soc. Transl* (*2*) **2** (1956), 89–115. MR Zbl

[Grunewald et al. 1988] F. J. Grunewald, D. Segal, and G. C. Smith, "Subgroups of finite index in nilpotent groups", *Invent. Math.* **93**:1 (1988), 185–223. MR Zbl

[Humphreys 1990] J. E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics **29**, Cambridge University Press, 1990. MR Zbl

[Klopsch and Voll 2009] B. Klopsch and C. Voll, "Igusa-type functions associated to finite formed spaces and their functional equations", *Trans. Amer. Math. Soc.* **361**:8 (2009), 4405–4436. MR Zbl

[Kovacs et al. 1999] I. Kovacs, D. S. Silver, and S. G. Williams, "Determinants of commuting-block matrices", *Amer. Math. Monthly* **106**:10 (1999), 950–952. MR Zbl

[Lee and Voll 2018] S. Lee and C. Voll, "Enumerating graded ideals in graded rings associated to free nilpotent Lie rings", *Math. Z.* **290**:3-4 (2018), 1249–1276. MR Zbl

[Lee and Voll 2023] S. Lee and C. Voll, "Zeta functions of integral nilpotent quiver representations", *Int. Math. Res. Not.* **2023**:4 (2023), 3460–3515. MR Zbl

[Levi 1944] F. W. Levi, "Notes on group-theory, IV-VI", *J. Indian Math. Soc.* (*N.S.*) **8** (1944), 78–91. MR Zbl

[Lins de Araujo 2019] P. M. Lins de Araujo, "Bivariate representation and conjugacy class zeta functions associated to unipotent group schemes, I: Arithmetic properties", *J. Group Theory* **22**:4 (2019), 741–774. MR Zbl

[Lins de Araujo 2020] P. M. Lins de Araujo, "Bivariate representation and conjugacy class zeta functions associated to unipotent group schemes, II: Groups of type F, G, and H", *Internat. J. Algebra Comput.* **30**:5 (2020), 931–975. MR Zbl

[Neukirch 1999] J. Neukirch, *Algebraic number theory*, Grundl. Math. Wissen. **322**, Springer, 1999. MR Zbl

[Neumann 1967] H. Neumann, *Varieties of groups*, Ergebnisse der Math. (2) **37**, Springer, 1967. MR Zbl

[Paajanen 2008] P. M. Paajanen, "Geometric structure of class two nilpotent groups and subgroup growth", preprint, 2008. arXiv 0802.1796

[Rossmann 2018] T. Rossmann, "Computing local zeta functions of groups, algebras, and modules", *Trans. Amer. Math. Soc.* **370**:7 (2018), 4841–4879. MR Zbl

[Rossmann 2022] T. Rossmann, *Zeta, version 0.4.2*, 2022, available at https://torossmann.github.io/Zeta/.

[Rossmann and Voll 2019] T. Rossmann and C. Voll, "Groups, graphs, and hypergraphs: average sizes of kernels of generic matrices with support constraints", 2019. To appear in *Mem. Amer. Math. Soc.* arXiv 1908.09589

[Schein and Voll 2015] M. M. Schein and C. Voll, "Normal zeta functions of the Heisenberg groups over number rings I: The unramified case", *J. Lond. Math. Soc.* (2) **91**:1 (2015), 19–46. MR Zbl

[Schein and Voll 2016] M. M. Schein and C. Voll, "Normal zeta functions of the Heisenberg groups over number rings II: The non-split case", *Israel J. Math.* **211**:1 (2016), 171–195. MR Zbl

[Stanley 1999] R. P. Stanley, *Enumerative combinatorics*, vol. 2, Cambridge Studies in Advanced Mathematics **62**, Cambridge University Press, 1999. MR Zbl

[Stanley 2012] R. P. Stanley, *Enumerative combinatorics*, vol. 1, 2nd ed., Cambridge Studies in Advanced Mathematics **49**, Cambridge University Press, 2012. MR Zbl

[Stasinski and Voll 2014] A. Stasinski and C. Voll, "Representation zeta functions of nilpotent groups and generating functions for Weyl groups of type B", *Amer. J. Math.* **136**:2 (2014), 501–550. MR Zbl

[Voll 2004] C. Voll, "Zeta functions of groups and enumeration in Bruhat–Tits buildings", *Amer. J. Math.* **126**:5 (2004), 1005–1032. MR Zbl

[Voll 2005a] C. Voll, "Functional equations for local normal zeta functions of nilpotent groups", *Geom. Funct. Anal.* **15**:1 (2005), 274–295. MR Zbl

[Voll 2005b] C. Voll, "Normal subgroup growth in free class-2-nilpotent groups", *Math. Ann.* **332**:1 (2005), 67–79. MR Zbl

[Voll 2006] C. Voll, "Counting subgroups in a family of nilpotent semi-direct products", *Bull. London Math. Soc.* **38**:5 (2006), 743–752. MR Zbl

[Voll 2010] C. Voll, "Functional equations for zeta functions of groups and rings", *Ann. of Math.* (2) **172**:2 (2010), 1181–1218. MR Zbl

[Voll 2011] C. Voll, "A newcomer's guide to zeta functions of groups and rings", pp. 99–144 in *Lectures on profinite topics in group theory*, edited by D. Segal, London Math. Soc. Stud. Texts **77**, Cambridge University Press, 2011. MR Zbl

[Voll 2019] C. Voll, "Local functional equations for submodule zeta functions associated to nilpotent algebras of endomorphisms", *Int. Math. Res. Not.* **2019**:7 (2019), 2137–2176. MR Zbl

[Voll 2020] C. Voll, "Ideal zeta functions associated to a family of class-2-nilpotent Lie rings", *Q. J. Math.* **71**:3 (2020), 959–980. MR Zbl

[Yeh 1948] Y. Yeh, "On prime power abelian groups", *Bull. Amer. Math. Soc.* **54** (1948), 323–327. MR Zbl

[Zordan 2022] M. Zordan, "Univariate and bivariate zeta functions of unipotent group schemes of type G", *Internat. J. Algebra Comput.* **32**:4 (2022), 653–682. MR Zbl

Communicated by Victor Reiner
Received 2022-06-13 Revised 2023-02-28 Accepted 2023-04-13

angela.carnevale@universityofgalway.ie *School of Mathematical and Statistical Sciences, University of Galway, Galway, Ireland*

mschein@math.biu.ac.il *Department of Mathematics, Bar Ilan University, Ramat Gan, Israel*

c.voll.98@cantab.net *Faculty of Mathematics, Bielefeld University, Bielefeld, Germany*

ALGEBRA AND NUMBER THEORY 18:3 (2024)
https://doi.org/10.2140/ant.2024.18.583

On Tamagawa numbers of CM tori

Pei-Xin Liang, Yasuhiro Oki, Hsin-Yi Yang and Chia-Fu Yu
Appendix by Jianing Li and Chia-Fu Yu

We investigate the problem of computing Tamagawa numbers of CM tori. This problem arises naturally from the problem of counting polarized abelian varieties with commutative endomorphism algebras over finite fields, and polarized CM abelian varieties and components of unitary Shimura varieties in the works of Achter, Altug, Garcia and Gordon and of Guo, Sheu and Yu, respectively. We make a systematic study on Galois cohomology groups in a more general setting and compute the Tamagawa numbers of CM tori associated to various Galois CM fields. Furthermore, we show that every (positive or negative) power of 2 is the Tamagawa number of a CM tori, proving the analogous conjecture of Ono for CM tori.

1. Introduction

In his two fundamental papers Takashi Ono [1961; 1963b] investigated the arithmetic of algebraic tori. He introduced and explored the class number and Tamagawa number of T, which will be denoted by $h(T)$ and $\tau(T)$ respectively (also see Section 2 for the definitions). One arithmetic significance of these invariants is that the class number $h(\mathbb{G}_{\mathrm{m},k})$ is equal to the class number h_k of the number field k, and the analytic class number formula for k can be reformulated by the simple statement $\tau(\mathbb{G}_{\mathrm{m},k}) = 1$. Thus, the class numbers of algebraic tori can be viewed as generalizations of class numbers of number fields, while Tamagawa numbers play a key role in the extension of analytic class number formulas.

Ono [1963b] showed that $\tau(T) = |H^1(k, X(T))|/|Ш^1(k, T)|$, where $X(T)$ is the group of characters of T and $Ш^1(k, T)$ is the Tate–Shafarevich group of T. Kottwitz [1984] generalized Ono's formula to reductive groups and proved [Kottwitz 1988] the celebrated conjecture of Weil for the Tamagawa number of semisimple simply connected groups. Ono constructed a 15-dimensional algebraic torus with Tamagawa number $\frac{1}{4}$, showing that $\tau(T)$ can be nonintegral and conjectured in [Ono 1963a] that every positive rational number is equal to $\tau(T)$ for some torus T. Ono's conjecture was proved by S. Katayama [1985] for the number field case. For some later studies of class numbers and Tamagawa numbers of algebraic tori we refer to the works of J.-M. Shyr [1977, Theorem 1], S. Katayama [1991], M. Morishita [1991], C. González-Avilés [2008; 2010] and M.-H. Tran [2017].

In this article we are mainly concerned with the problem of computing the Tamagawa numbers of complex multiplication (CM) algebraic tori. CM tori are closely related to the arithmetic of CM abelian varieties and computing their Tamagawa numbers itself is a way of exploring the structure of CM fields.

MSC2020: primary 14K22; secondary 11R29.
Keywords: CM algebraic tori, Tamagawa numbers.

This problem directly contributes to recent works of Achter, Altug, Garcia and Gordon [Achter et al. 2023] and of J. Guo, N. Sheu and the fourth named author [Guo et al. 2022]. In the former one the authors computed the size of an isogeny class of principally polarized abelian varieties over a finite field with commutative endomorphism algebra, and express the number in terms of a discriminant, the Tamagawa number, and the product of Frobenius local densities. In the latter one the authors computed formulas for certain CM abelian varieties and certain polarized abelian varieties over finite fields with commutative endomorphism algebras upon the results of [Xue and Yu 2021]. Using the class number formula for CM tori, they also computed the numbers of connected components of complex unitary Shimura varieties. In the appendix of [Achter et al. 2023], W.-W. Li and T. Rüd have obtained several initial results of the values of $\tau(T)$. Our goals are to prove more cases of CM tori and to determine the range of the values of Tamagawa numbers of all CM tori. With a similar goal but using different methods, T. Rüd [2022] obtains several results along this direction. He provides an algorithm, among others, for giving precise lower bounds and determining possible Tamagawa numbers, and obtains the values $\tau(T)$ for several other CM tori of lower dimension.

We shall describe our results towards computing Tamagawa numbers for a more general class of algebraic tori which include CM tori and then give more detailed results of CM tori. Let k be a global field and $K := \prod_{i=1}^{r} K_i$ be the product of finite separable field extensions K_i of k. Let $E := \prod_{i=1}^{r} E_i$, where each $E_i \subset K_i$ is a subextension of K_i. Denote by $T^K = \prod_i T^{K_i}$ and $T^E = \prod_i T^{E_i}$ the algebraic k-tori associated to the multiplicative groups of K and E, respectively, and let $N_{K/E} = \prod_i N_{K_i/E_i} : T^K \to T^E$ be the norm map. Note that the norm map is surjective; one can check this easily by showing the surjectivity on their $\bar{k}$-rational points (or combining [Morishita 1991, Section 3] with [Conrad et al. 2010, Corollary A.5.4(1), page 507]). Let $\mathbb{G}_{\mathrm{m},k} \to T^E$ be the closed immersion induced by the diagonal embedding $k \hookrightarrow E$; see [Conrad et al. 2010, Proposition A.5.7, page 510]. We regard $\mathbb{G}_{\mathrm{m},k}$ as a k-subtorus of T^E by identifying $\mathbb{G}_{\mathrm{m},k}$ with its image in T^E. We write $T^{K/E,1}$ for the kernel of $N_{K/E}$ and

$$T^{K/E,k} = N_{K/E}^{-1}(\mathbb{G}_{\mathrm{m},k}) := T^K \times_{T^E} \mathbb{G}_{\mathrm{m},k}$$

for the preimage of the k-subtorus $\mathbb{G}_{\mathrm{m},k} \subset T^E$. The tori $T^{K/E,k}$, $T^{K/E,1}$ and $\mathbb{G}_{\mathrm{m},k}$ fit into the following short exact sequence:

$$1 \to T^{K/E,1} \xrightarrow{\iota} T^{K/E,k} \xrightarrow{N_{K/E}} \mathbb{G}_{\mathrm{m},k} \to 1.$$

Let L be the smallest splitting field of $T^{K/E,k}$ and let $G = \mathrm{Gal}(L/k)$. We let $\Lambda := X(T^{K/E,k})$ and $\Lambda^1 := X(T^{K/E,1})$ be the character groups of $T^{K/E,k}$ and $T^{K/E,1}$, respectively. Taking the characters, we have the following exact sequence of G-lattices:

$$0 \to \mathbb{Z} \to \Lambda \to \Lambda^1 \to 0. \tag{1-1}$$

Write $H_i := \mathrm{Gal}(L/K_i)$, $\widetilde{N}_i := \mathrm{Gal}(L/E_i)$, and $N_i^{\mathrm{ab}} := \widetilde{N}_i / D(\widetilde{N}_i) H_i$, where $D(\widetilde{N}_i)$ denotes the commutator group of $\widetilde{N}_i$ (If H_i is normal in $\widetilde{N}_i$ and one puts $N_i := \widetilde{N}_i / H_i$, then N_i^{ab} coincides with the abelianization of the group N_i). When there is no confusion, for brevity we shall write $H^q(A)$ for $H^q(G, A)$ for any G-module A.

Let $\mathrm{Ver}_{G,N_i} : G \to N_i^{\mathrm{ab}}$ denote the transfer from G into N_i^{ab}; see Definition 3.6. For any abelian group H, the Pontryagin dual of H is denoted by $H^\vee$. By Ono's formula (2-3) and the Poitou–Tate duality, we have $\tau(T) = |H^1(G, X(T))|/|Ш^2(G, X(T))|$, where $Ш^i(G, X(T))$ is the i-th Tate–Shafarevich group of $X(T)$.

Let $\mathcal{D}$ be the set of all decomposition groups of G. Denote by

$$r^i_{D,A} : H^i(G, A) \to H^i(D, A), \quad \text{where } D \in \mathcal{D} \tag{1-2}$$

the restriction map for G-module A, and let

$$r^i_{\mathcal{D},A} : H^i(G, A) \to \bigoplus_{D\in\mathcal{D}} H^i(D, A) \tag{1-3}$$

be the restriction map for each G-module A by sending $\xi \mapsto (r^i_{D,A}(\xi))_{D\in\mathcal{D}}$. Put

$$H^2(\mathbb{Z})' := \{x \in H^2(G, \mathbb{Z}) : r^2_{\mathcal{D},\mathbb{Z}}(x) \in \mathrm{Im}(\delta_{\mathcal{D}})\}, \tag{1-4}$$

where $\delta_{\mathcal{D}} : \bigoplus_{D\in\mathcal{D}} H^1(D, \Lambda^1) \to \bigoplus_{D\in\mathcal{D}} H^2(D, \mathbb{Z})$ is the connecting homomorphism induced from (1-1). The group $H^2(\mathbb{Z})'$ plays a similar role of a Selmer group.

Theorem 1.1. *Let the notation be as above*:

(1) *There is a canonical isomorphism* $H^1(G, \Lambda^1) \simeq \bigoplus_i N_i^{\mathrm{ab},\vee}$.

(2) *There is a canonical isomorphism*

$$H^1(G, \Lambda) \simeq \mathrm{Ker}\Big(\sum \mathrm{Ver}^\vee_{G,N_i} : \bigoplus_i N_i^{\mathrm{ab},\vee} \to G^{\mathrm{ab},\vee}\Big),$$

where $\mathrm{Ver}_{G,N_i} : G \to N_i^{\mathrm{ab}}$ *is the transfer map.*

(3) *Assume that* K_i/E_i *is cyclic with Galois group* N_i *for all* i. *Then* $Ш^2(\Lambda) \simeq H^2(\mathbb{Z})'/\mathrm{Im}(\delta)$ *and*

$$\tau(T^{K/E,k}) = \frac{\prod_{i=1}^r |N_i|}{|H^2(\mathbb{Z})'|}. \tag{1-5}$$

(4) *If we further assume that the subgroups* $\widetilde{N}_i$ *and* H_i *are all normal in* G, *and let*

$$\mathrm{Ver}_{G,\boldsymbol{N}} = (\mathrm{Ver}_{G,N_i})_i : G^{\mathrm{ab}} \to \prod_i N_i \quad \textit{and} \quad \mathrm{Ver}_{D,\overline{\boldsymbol{D}}} = (\mathrm{Ver}_{D,\overline{D}_i})_i : D^{\mathrm{ab}} \to \prod_i \overline{D}_i$$

denote the corresponding transfer maps, where $D_i := D \cap \widetilde{N}_i$ *and* $\overline{D}_i := D_i/(D_i \cap H_i) \subset N_i$, *respectively*, *then*

$$H^2(\mathbb{Z})' = \{f \in G^{\mathrm{ab},\vee} : f|_{D^{\mathrm{ab}}} \in \mathrm{Im}(\mathrm{Ver}^\vee_{D,\overline{\boldsymbol{D}}}) \forall D \in \mathcal{D}\},$$

and

$$Ш^2(\Lambda) \simeq \frac{\{f \in G^{\mathrm{ab},\vee} : f|_{D^{\mathrm{ab}}} \in \mathrm{Im}(\mathrm{Ver}^\vee_{D,\overline{\boldsymbol{D}}}) \forall D \in \mathcal{D}\}}{\mathrm{Im}(\mathrm{Ver}^\vee_{G,\boldsymbol{N}})}.$$

A certain case of Theorem 1.1(1) was obtained by Rüd [2022, Proposition 2.5] and he also computed the group $H^1(G, \Lambda)$ explicitly; see Section 3.2 of [loc. cit.].

Using Theorem 1.1(2), we give a different proof of a result of Li and Rüd [Achter et al. 2023, Proposition A.11] which does not rely on Kottwitz's formula; see Corollary 7.2. By (1-5), the ratio $\left(\prod_{i=1}^r |N_i|\right)/\tau(T^{K/E,k})$ is a positive integer; this gives a simple upper bound of $\tau(T^{K/E,k})$.

Observe that $T^{K/E,k}$ is a k-subtorus of $\prod_{i=1}^r T^{K_i/E_i,k}$ and is not equal to $\prod_{i=1}^r T^{K_i/E_i,k}$ when $r > 1$ by the dimension counting. The following theorem gives a sufficient condition for $\tau(T^{K/E,k})$ being equal to $\prod_{i=1}^r \tau(T^{K_i/E_i,k})$.

Theorem 1.2. *Suppose the following conditions hold*: (*a*) *for each* $1 \le i \le r$, *the extension* K_i/k *is Galois with Galois group* G_i, $N_i = \mathrm{Gal}(K_i/E_i)$ *is cyclic and every decomposition group of* G_i *is cyclic*; *and* (*b*) $G \simeq G_1 \times \cdots \times G_r$. *Then we have* $\tau(T^{K/E,k}) = \prod_{i=1}^r \tau(T^{K_i/E_i,k})$.

Now let $K = \prod_{i=1}^r K_i$ be a CM algebra over $\mathbb{Q}$ and $E := K^+$ the $\mathbb{Q}$-subalgebra fixed by the canonical involution ι, and let $T = T^{K/E,\mathbb{Q}}$ be the associated CM torus over $\mathbb{Q}$.

Theorem 1.3. *For any integer* n, *there exists a CM torus* T *over* $\mathbb{Q}$ *such that* $\tau(T) = 2^n$.

Finally, we give a number of results of $\tau(T)$ for Galois CM fields.

Theorem 1.4. *Suppose that* K *is a Galois CM field with Galois group* $G = \mathrm{Gal}(K/\mathbb{Q})$. *Let* $g := [K^+ : \mathbb{Q}]$, $G^+ = \mathrm{Gal}(K^+/\mathbb{Q})$ *and* $T = T^{K/K^+,\mathbb{Q}}$ *be the associated CM torus*:

(1) *If* $K = \mathbb{Q}(\zeta_n) \neq \mathbb{Q}$ *is the* n-*th cyclotomic field with either* $4 \mid n$ *or odd* n, *then*:

- (a) *If* n *is either a power of an odd prime* p *or* $n = 4$, *then* $\tau(T) = 1$.
- (b) *In other cases*, *we have* $\tau(T) = 2$.

(2) *If* G *is abelian*, *then* $\tau(T) \in \{1, 2\}$. *Moreover*, *the following statements hold*:

- (a) *If* g *is odd*, *then* $\tau(T) = 1$.
- (b) *If* g *is even and the exact sequence*

$$1 \to \langle \iota \rangle \to G \to G^+ \to 1 \tag{1-6}$$

splits, *then* $\tau(T) = 2$.

(3) *Let* G *be possibly nonabelian and suppose that the short exact sequence* (1-6) *splits. Then* $\tau(T) \in \{1, 2\}$ *and the following statements hold*:

- (a) *When* g *is odd*, $\tau(T) = 1$.
- (b) *Suppose* g *is even and let* $g^{\mathrm{ab}} := |G^{+\mathrm{ab}}|$, *the cardinality of the abelianization of* G^+:
  - (i) *If* g^{ab} *is even*, *then* $\tau(T) = 2$.
  - (ii) *If* g^{ab} *is odd*, *then there is a unique nonzero element* ξ *in the* 2-*torsion subgroup* $H^2(\Lambda)[2]$ *of* $H^2(\Lambda)$. *Moreover*, $\tau(T) = 1$ *if and only if its restriction* $r_D(\xi) = 0$ *in* $H^2(D, \Lambda)$ *for all* $D \in \mathcal{D}$ *where* r_D *as defined in* (1-2).

(4) *Let P and Q be two odd nonsquare positive integers such that $P-1=a^2$ and $Q-1=Pb^2$ for some integers $a, b \in \mathbb{N}$. Let $K := \mathbb{Q}(\sqrt{\alpha})$ with $\alpha := -(P+\sqrt{P})(Q+\sqrt{Q})$. Then K is a Galois CM field with Galois group Q_8 and*

$$\tau(T) = \begin{cases} \frac{1}{2} & \text{if } \left(\frac{P}{q}\right) = 1 \text{ for all prime } q \mid Q; \\ 2 & \text{otherwise}. \end{cases}$$

(5) *Suppose the Galois extension $K/\mathbb{Q}$ has Galois group D_n of order $2n$. Then n is even and $\tau(T)=2$.*

We mention that some cases of Theorem 1.4 where G is abelian, dihedral or quaternionic were also obtained by Rüd; he also obtained complete results when the degree of (possibly non-Galois) K is less than or equal to 8; see [Rüd 2022, Theorem 1.3, Proposition 1.5 and Examples 5.7 and 5.19]. The overlapping results are obtained by different approaches.

We explain the idea of the proof of Theorem 1.3. First of all, it is rather difficult to construct a CM field such that the Tamagawa number $\tau(T)$ of the associated algebraic torus T is small. Suppose that $K/\mathbb{Q}$ is Galois with Galois group G. T. Rüd [2022] implements a SageMath algorithm for computing $\tau(T)$ and computed the groups $Ш^2_{\mathcal{C}}(G, \Lambda)$ for all 2-groups G of order ≤ 256,[1] where $Ш^2_{\mathcal{C}}(G, \Lambda) := \mathrm{Ker}(H^2(G, \Lambda) \to \prod_C H^2(C, \Lambda))$ and C runs through all cyclic subgroups of G. Based on Rüd's result there is at most one case such that $\tau(T) = \frac{1}{4}$; see [loc. cit., Proposition 5.26]. To get around this, we construct an infinite family of "totally" linearly disjoint Q_8-CM fields $\{K_i\}$ for $i \ge 1$ with $\tau(T_i) = \frac{1}{2}$, that is, the Galois group of the compositum of any finitely many of these Q_8-CM fields K_i is the product of the Galois groups $\mathrm{Gal}(K_i/\mathbb{Q})$. The CM algebra $K := \prod_{i=1}^r K_i$ then satisfies the conditions in Theorem 1.2 and it follows that the CM torus T associated to K satisfies $\tau(T) = \frac{1}{2^r}$.

We point out that the proof of Theorem 1.3 is actually quite tricky. First of all, it follows from [loc. cit.] or Theorem 1.4 that Q_8-CM fields are the simplest ones so that the associated CM tori T can have $\tau(T) = \frac{1}{2}$. On the other hand, Theorem 1.2 requires a condition that every decomposition group of each CM field extension $K_i/\mathbb{Q}$ is cyclic. Fortunately, for any Q_8-CM field K_i with CM torus T_i, one has $\tau(T_i) = \frac{1}{2}$ if and only if this condition for $K_i/\mathbb{Q}$ holds (see Proposition 6.7), so that we can apply Theorem 1.2 to the product of them. On the other hand, one may be wondering whether this condition is superfluous. For this question, we construct linearly disjoint Galois CM fields K_1 and K_2 such that

$$\tau(T_1) = 2, \quad \tau(T_2) = 1, \quad \tau(T) = 1, \tag{1-7}$$

where T_1, T_2 and T are CM tori associated to K_1, K_2 and $K_1 \times K_2$, respectively. This example shows that the cyclicity of decomposition groups of every $K_i/\mathbb{Q}$ is not superfluous.

Theorem 1.3 proves an analogous but more involved conjecture of Ono for CM tori. In the Appendix Jianing Li and the fourth named author show that for any global field k and any positive rational number α, there exists an algebraic torus T over k with $\tau(T) = \alpha$; see Theorem A.8. This extends the result of Katayama [1985] and proves Ono's conjecture for global fields.

[1] See https://toadrush.github.io/tamagawa-cmtori/.

Though our original motivation of investigating Tamagawa numbers of CM tori comes from counting certain abelian varieties and exploring the structure of CM fields, the main part of the problem itself was to compute or investigate their Tate–Shafarevich groups. We explain in Remark 7.10 how the Tate–Shafarevich group of a CM torus also comes into play in the theory of Shimura varieties of PEL-type. Tate–Shafarevich groups measure the failure of the local-global principle for various objects, which is one of main interests in number theory and has been actively studied. For the interested reader's reference, we mention some development on the Tate–Shafarevich group of multinorm one tori, which are different type of tori from the CM tori studied in the present paper. Hürlimann [1984] proved that the multiple norm principle holds for any commutative etale k-algebra of the form $K_1 \times K_2$ in which one component is cyclic and the other one is Galois. Prasad and Rapinchuk [2010] settled the problem of the local-global principle for embeddings of fields with involution into simple algebras with involution, where they also investigated the multiple norm principle. The multiple norm principle has been investigated further by Pollio and Rapinchuk [2013; 2014], Demarche and D. Wei [2014] and D. Wei and F. Xu [2012]. Bayer-Fluckiger, T.-Y. Lee and Parimala [Bayer-Fluckiger et al. 2019] studied the Tate–Shafarevich group of general multinorm one tori in which one of factors is a cyclic extension. They give a simple rule for determining the Tate–Shafarevich group in the case of products of extensions of prime degree p. T.-Y. Lee [2022] computes explicitly the Tate–Shafarevich group for the cases where every factor is a cyclic extension of degree p-power.

This article is organized as follows. Section 2 includes preliminaries and background on Tamagawa numbers of algebraic tori, and some known results of those of CM tori due to Li–Rüd and Guo–Sheu–Yu. Section 3 discusses transfer maps, their extensions and connection with class field theory. In Sections 4 and 5 we compute the Galois cohomology groups of character groups of a class of algebraic tori $T^{K/E,k}$ and $T^{K/E,1}$. Section 6 treats Galois CM tori and in Sections 7 ans 8 we determine the precise ranges of Tamagawa numbers of CM tori. In Section 9 we show there are infinitely many pairs of linearly disjoint Galois CM fields K_1 and K_2 satisfying (1-7). In the Appendix Jianing Li and the fourth author prove Ono's conjecture for global fields.

2. Preliminaries, background and some known results

2A. ***Class numbers and Tamagawa numbers of algebraic tori.*** The cardinality of a set S will be denoted by $|S|$. For any field k, let $\bar{k}$ be a fixed algebraic closure of k, let k^{sep} be the separable closure of k in $\bar{k}$ and denote by $\Gamma_k := \mathrm{Gal}(k^{\mathrm{sep}}/k)$ the Galois group of k. Let $\mathbb{G}_{\mathrm{m},k} := \mathrm{Spec}\, k[X, X^{-1}]$ denote the multiplicative group associated to k with the usual multiplicative group law.

Definition 2.1. (1) A connected linear algebraic group T over a field k is said to be an *algebraic torus* over k if there exists a finite field extension K/k such that there exists an isomorphism $T \otimes_k K \simeq \mathbb{G}_{\mathrm{m},K}^d$ of algebraic groups over K for some positive integer d. Then d is equal to the dimension of T. If T is an algebraic k-torus and K is a field extension of k that satisfies the above property, then K is called a *splitting field* of T. The smallest splitting field (which is unique and Galois, see below) is called *the minimal splitting field of T*.

(2) For any algebraic torus T over k, denote by $X(T) := \mathrm{Hom}_{k^{\mathrm{sep}}}(T \otimes_k k^{\mathrm{sep}}, \mathbb{G}_{\mathrm{m},k^{\mathrm{sep}}})$ the group of characters of T. It is a finite free $\mathbb{Z}$-module of rank d together with a continuous action of the Galois group Γ_k of k.

It is shown in [Ono 1961, Proposition 1.2.1] (also see [Yu 2019] for other proofs) that every algebraic k-torus splits over a finite *separable* field extension K/k. The action of Γ_k on $X(T)$ gives a continuous representation

$$r_T : \Gamma_k \to \mathrm{Aut}(X(T))$$

which factors through a faithful representation of a finite quotient $\mathrm{Gal}(L/k)$ of Γ_k. Here L is the fixed field of the kernel of r_T and is the smallest splitting field of T. In particular, L is a finite Galois extension of k and $X(T)$ can be also regarded as a $\mathrm{Gal}(L/k)$-module.

In the remainder of this article, k denotes a global field. We only discuss finite separable field extensions in this paper. For each place v of k, denote by k_v the completion of k at v, and O_v the ring of integers of k_v if v is finite. For each finite place v, the group $T(k_v)$ of k_v-valued points of T contains a unique maximal open compact subgroup, which is denoted by $T(O_v)$. Let $\mathbb{A}_k$ be the adele ring of k, and let S be a nonempty finite set S of places of k containing all non-Archimedean places if k is a number field. Denote by $U_{T,S} = \prod_{v\in S} T(k_v) \times \prod_{v\notin S} T(O_v)$ the unit group with respect to S and let $\mathrm{Cl}_S(T) := T(\mathbb{A}_k)/T(k)U_{T,S}$ be the *S-class group* of T. By a finiteness theorem of Borel [1963], $\mathrm{Cl}_S(T)$ is a finite group and its cardinality is denoted by $h_S(T)$, called the *S-class number* of T. If k is a number field and $S = \infty$ consists of all non-Archimedean places, we write $U_T := U_{T,\infty}$, $\mathrm{Cl}(T) := T(\mathbb{A}_k)/T(k)U_T$ the *class group* of T and call $h(T) := |\mathrm{Cl}(T)|$ the *class number* of T.

It follows immediately from the definition that if K/k is a finite extension and T_K is an algebraic K-torus, then $h(R_{K/k}T_K) = h(T_K)$, where $R_{K/k}$ denotes the Weil restriction of scalars from K to k. Note that $\dim R_{K/k}T_K = \dim T_K \cdot [K : k]$. It is well known [Ono 1961; 1963b] that

$$X(R_{K/k}T_K) \simeq \mathrm{Ind}_{\Gamma_K}^{\Gamma_k} X(T_K) = \mathrm{Ind}_{\mathrm{Gal}(L/K)}^{\mathrm{Gal}(L/k)} X(T_K),$$

as $\mathrm{Gal}(L/k)$-modules, where L is any finite Galois extension of k over which the algebraic k-torus $R_{K/k}T_K$ splits.

We recall the definition of the Tamagawa number of an algebraic k-torus T. Fix a finite Galois splitting field extension K/k of T with Galois group G. Let χ_T be the character of representation $(X(T)\otimes\mathbb{C}, r_T)$ of G over $\mathbb{C}$, that is, $\chi_T : G \to \mathbb{C}$, $\chi_T(g) := \mathrm{tr}(r_T(g))$ for all $g \in G$. Let

$$L(s, K/k, \chi_T) := \prod_{v\nmid\infty} L_v(s, K/k, \chi_T)$$

be the Artin L-function of the character χ_T, where $L_v(s, K/k, \chi_T)$ is the local Artin L-factor at v. It follows from Brauer's induction theorem (see [Serre 1977, Section 10.5]) that $L(s, K/k, \chi_T)$ has a pole of order a at $s = 1$, where a is the rank of the G-invariant sublattice $X(T)^G$.

Let ω be a nonzero invariant differential form on T of highest degree defined over k. To each place v, one associates a Haar measure ω_v on $T(k_v)$. Then the product of the Haar measures

$$\prod_{v\,|\,\infty} \omega_v \cdot \prod_{v\nmid\infty} (L_v(1, K/k, \chi_T) \cdot \omega_v)$$

converges absolutely and defines a Haar measure on $T(\mathbb{A}_k)$.

Write (N) for the number field case and (F) for the function field case. For (N), let d_k be the discriminant of k. For (F), $k = \mathbb{F}_q(C)$ is the function field of a smooth projective geometrically connected curve over $\mathbb{F}_q$ and let $g(C)$ be the genus of C.

Definition 2.2. Let T be an algebraic torus over a global field k and ω be a nonzero invariant differential form on T defined over k of highest degree. Then

$$\omega_{\mathbb{A},can} := \frac{\prod_{v\,|\,\infty} \omega_v \cdot \prod_{v\nmid\infty} (L_v(1, K/k, \chi_T) \cdot \omega_v)}{\mu_k^{(\dim T)} \cdot \rho_T},$$

defines a Haar measure on $T(\mathbb{A}_k)$, which is called the *Tamagawa measure on* $T(\mathbb{A}_k)$, where

$$\mu_k := \begin{cases} |d_k|^{1/2} & \text{for (N);} \\ q^{g(C)} & \text{for (F),} \end{cases} \quad \text{and} \quad \rho_T := \lim_{s\to 1} (s-1)^a L(s, K/k, \chi_T),$$

and a is the order of the pole of the Artin L-function $L(s, K/k, \chi_T)$ at $s = 1$.

Let $\xi_1, \dots, \xi_a$ be a basis of $X(T)^G$. Define

$$\xi : T(\mathbb{A}_k) \to \mathbb{R}_+^a, \quad x \mapsto (\|\xi_1(x)\|, \dots, \|\xi_a(x)\|),$$

where $\mathbb{R}_+ := \{x \in \mathbb{R}^\times : x > 0\} \subset \mathbb{R}^\times$ is the connected open subgroup. Let $T(\mathbb{A}_k)^1$ denote the kernel of ξ; one has an isomorphism

$$T(\mathbb{A}_k)/T(\mathbb{A}_k)^1 \simeq \operatorname{Im} \xi.$$

For (N), $\operatorname{Im} \xi = R_+^a$ and let $d^\times t := \prod_{i=1}^a dt_i/t_i$ be the canonical measure on $\mathbb{R}_+^a$.

For (F), the image $\operatorname{Im} \xi \subset (q^{\mathbb{Z}})^a$ is a subgroup of finite index and let $d^\times t$ be the counting measure on $\operatorname{Im} \xi$ with measure $(\log q)^a [(q^{\mathbb{Z}})^a : \operatorname{Im} \xi]$ on each point; see [Oesterlé 1984, Definition 5.9, page 24].

Remark 2.3. For (F), incorrectly stating in [Ono 1963b, page 56], the image $\operatorname{Im} \xi$, as pointed out by Tate, is actually not equal to $(q^{\mathbb{Z}})^a$ in general (see [Oesterlé 1984, page 25]), so the modification by the index $[(q^{\mathbb{Z}})^a : \operatorname{Im} \xi]$ is needed. We thank the referee for pointing out this to us.

Let $\omega^1_{\mathbb{A},\mathrm{can}}$ be the unique Haar measure on $T(\mathbb{A}_k)^1$ such that

$$\omega_{\mathbb{A},\mathrm{can}} = \omega^1_{\mathbb{A},\mathrm{can}} \cdot d^\times t, \tag{2-1}$$

that is, for any measurable function F on $T(\mathbb{A}_k)$ one has

$$\int_{T(\mathbb{A}_k)/T(\mathbb{A}_k)^1} \int_{T(\mathbb{A}_k)^1} F(xt)\omega^1_{\mathbb{A},\mathrm{can}} \cdot d^\times t = \int_{T(\mathbb{A}_k)} F(x)\omega_{\mathbb{A},\mathrm{can}}.$$

By a well-known theorem of Borel and Harish-Chandra [Platonov and Rapinchuk 1994, Theorem 5.6], the quotient space $T(\mathbb{A}_k)^1/T(k)$ has finite volume with respect to every Haar measure. In fact $T(\mathbb{A}_k)^1/T(k)$ is the unique maximal compact subgroup of $T(\mathbb{A}_k)/T(k)$, because the group $\mathbb{R}_+^a$ has no nontrivial compact subgroup.

Definition 2.4. Let T be an algebraic torus over a global field k. The *Tamagawa number* $\tau_k(T)$ *of* T is defined by

$$\tau_k(T) := \int_{T(\mathbb{A}_k)^1/T(k)} \omega^1_{\mathbb{A},\mathrm{can}}, \tag{2-2}$$

the volume of $T(\mathbb{A}_k)^1/T(k)$ with respect to $\omega^1_{\mathbb{A},\mathrm{can}}$, where $\omega^1_{\mathbb{A},\mathrm{can}}$ is the Haar measure on $T(\mathbb{A}_k)^1$ defined in (2-1).

One has the following properties ([Ono 1961, Theorem 3.5.1] also see [Ono 1963b, Section 3.2, page 57]):

(i) For any two algebraic k-tori T and T', one has $\tau_k(T \times_k T') = \tau_k(T) \cdot \tau_k(T')$.

(ii) For any finite extension k'/k and any algebraic k'-torus T', one has $\tau_k(R_{k'/k}T') = \tau_{k'}(T')$.

(iii) One has $\tau_k(\mathbb{G}_{\mathrm{m},k}) = 1$.

Note that in the number field case, the last statement (iii) is equivalent to the analytic class number formula [Lang 1994, VIII, Section 2, Theorem 5, page 161].

2B. *Values of Tamagawa numbers.*

Theorem 2.5 (Ono's formula). *Let K/k be a finite Galois extension with Galois group Γ, and T be an algebraic torus over k with splitting field K. Then*

$$\tau_k(T) = \frac{|H^1(\Gamma, X(T))|}{|Ш^1(\Gamma, T)|}, \tag{2-3}$$

where

$$Ш^i(\Gamma, T) := \mathrm{Ker}\Big(H^i(K/k, T) \to \prod_v H^i(K_w/k_v, T)\Big)$$

is the Tate–Shafarevich group associated to $H^i(\Gamma, T)$ for $i \geq 0$ and w is a place of K over v.

Proof. See [Oesterlé 1984, Chapter IV, Corollary 3.3, page 56]. □

Remark 2.6. The cohomology groups $H^1(\Gamma, X(T))$ and $Ш^1(\Gamma, T)$ are independent of the choice of the splitting field K; see [Ono 1963b, Sections 3.3 and 3.4].

According to Ono's formula, the Tamagawa number of any algebraic k-torus is a positive rational number. Ono constructed an infinite family of algebraic $\mathbb{Q}$-tori T with $\tau(T) = \tau_{\mathbb{Q}}(T) = 1/4$, which particularly shows that $\tau(T)$ needs not to be an integer. Ono [1963a] conjectured that every positive rational number can be realized as $\tau_k(T)$ for some algebraic k-torus T. Ono's conjecture was proved by S. Katayama [1985] for the number field case.

For any finite abelian group G, the Pontryagin dual of G is defined to be

$$G^\vee := \mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z}).$$

The Poitou–Tate duality [Platonov and Rapinchuk 1994, Theorem 6.10; Neukirch et al. 2000, Theorem 8.6.8] says that there is a natural isomorphism $Ш^1(\Gamma, T)^\vee \simeq Ш^2(\Gamma, X(T))$. Thus,

$$\tau_k(T) = \frac{|H^1(\Gamma, X(T))|}{|Ш^2(\Gamma, X(T))|}. \tag{2-4}$$

To simplify the notation we shall often suppress the Galois group from Galois cohomology groups and write $H^1(X(T))$ and $Ш^2(X(T))$ for $H^1(\Gamma, X(T))$ and $Ш^2(\Gamma, X(T))$ etc., if there is no risk of confusion.

2C. ***Some known results for CM tori.*** For the convenience of later generalization and investigation, we define here a more general class of k-tori as mentioned in the Introduction.

For every commutative etale k-algebra K, denote by T^K the algebraic k-torus whose group of R-valued points of T^K for any commutative k-algebra R, is

$$T^K(R) = (K \otimes_k R)^\times.$$

Explicitly, if $K = \prod_{i=1}^r K_i$ is a product of finite separable field extensions K_i of k, then

$$T^K = \prod_{i=1}^r T^{K_i} = \prod_{i=1}^r R_{K_i/k}(\mathbb{G}_{m,K_i}),$$

where $R_{K_i/k}$ is the Weil restriction of scalars from K_i to k. Let $E = \prod_{i=1}^r E_i$ be a product of finite subfield extensions $E_i \subset K_i$ over k. Let $N_{K_i/E_i} : T^{K_i} \to T^{E_i}$ be the norm map and put $N = \prod_{i=1}^r N_{K_i/E_i} : T^K \to T^E$. Define $T^{K/E,1} := \mathrm{Ker}\, N$, the kernel of the norm map N, and

$$T^{K/E,k} := \{x \in T^K : N(x) \in \mathbb{G}_{m,k}\},$$

the preimage of $\mathbb{G}_{m,k}$ in T^K under N, where $\mathbb{G}_{m,k} \hookrightarrow T^E$ is viewed as a subtorus of T^E via the diagonal embedding. Then we have the following commutative diagram of algebraic k-tori in which each row is an exact sequence:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & T^{K/E,1} & \xrightarrow{j} & T^K & \xrightarrow{N} & T^E & \longrightarrow & 1 \\ & & \| & & \uparrow{\scriptstyle j_T} & & \uparrow{\scriptstyle \Delta} & & \\ 1 & \longrightarrow & T^{K/E,1} & \xrightarrow{j} & T^{K/E,k} & \xrightarrow{N} & \mathbb{G}_{\mathrm{m},k} & \longrightarrow & 1 \end{array} \tag{2-5}$$

For the rest of this subsection we let $k = \mathbb{Q}$ and $K = \prod_{i=1}^r K_i$ be a CM algebra, where each K_i is a CM field, with the canonical involution ι. The subalgebra $K^+ \subset K$ fixed by ι is the product $K^+ = \prod_{i=1}^r K_i^+$ of the maximal totally real subfields K_i^+ of K_i. Let $T^{K,1} := T^{K/K^+,1} = \mathrm{Ker}\, N_{K/K^+}$ be the associated norm one CM torus and the associated CM torus

$$T^{K,\mathbb{Q}} := T^{K/K^+,\mathbb{Q}}. \tag{2-6}$$

As before, we have the following exact sequence of algebraic tori over $\mathbb{Q}$

$$1 \to T^{K,1} \to T^{K} \xrightarrow{N_{K/K^+}} T^{K^+} \to 1,$$

and a commutative diagram similar to (2-5).

Proposition 2.7. *Let K be a CM algebra and $T = T^{K,\mathbb{Q}}$ the associated CM torus over $\mathbb{Q}$:*

(1) *We have*

$$\tau(T) = \frac{2^r}{n_K},$$

where r is the number of components of K and

$$n_K := [\mathbb{A}^\times : N(T(\mathbb{A})) \cdot \mathbb{Q}^\times]$$

is the global norm index associated to T.

(2) *We have*

$$n_K \mid \prod_{p \in S_{K/K^+}} e_{T,p},$$

where S_{K/K^+} is the finite set of rational primes p with some place $v \mid p$ of K^+ ramified in K, and $e_{T,p} := [\mathbb{Z}_p^\times : N(T(\mathbb{Z}_p))]$. Here $T(\mathbb{Z}_p)$ denotes the unique maximal open compact subgroup of $T(\mathbb{Q}_p)$.

Proof. (1) This is [Guo et al. 2022, Theorem 1.1(1)]. (2) This is [loc. cit., Lemma 4.6(2)]. □

Lemma 2.8. *Let K and T be as in Proposition 2.7:*

(1) *If K contains an imaginary quadratic field, then $n_K \in \{1, 2\}$ and $\tau(T) \in \{2^{r-1}, 2^r\}$.*

(2) *If K contains two distinct imaginary quadratic fields, then $n_K = 1$ and $\tau(T) = 2^r$.*

Proof. This is proved in [Guo et al. 2022, Lemma 4.7] (also see [loc. cit., Section 5.1]) in the case where K is a CM field. The same proof using class field theory also proves the CM algebra case. □

Proposition 2.9. *Let K be a CM field and $T = T^{K,\mathbb{Q}}$ the associated CM torus over $\mathbb{Q}$. Put $g = [K^+ : \mathbb{Q}]$ and let K^{Gal} be the Galois closure of K over $\mathbb{Q}$ with Galois group $\mathrm{Gal}(K^{\mathrm{Gal}}/\mathbb{Q}) = G$:*

(1) *If g is odd, then $H^1(X(T)) = 0$.*

(2) *If $K/\mathbb{Q}$ is Galois and g is odd, then $\tau(T) = 1$.*

(3) *If $K/\mathbb{Q}$ is cyclic, then $\tau(T) = 1$.*

(4) *If $K/\mathbb{Q}$ is Galois of degree 4, then $\tau(T) \in \{1, 2\}$. Moreover, $\tau(T) = 2$ if and only if $G \simeq (\mathbb{Z}/2\mathbb{Z})^2$.*

Proof. See Propositions A.2 and A.12 of [Achter et al. 2023]. □

Note that Proposition 2.9(4) also follows from Proposition 2.9(3) and Lemma 2.8(2).

3. Transfer maps, corestriction maps and extensions

3A. ***Transfer maps.*** Let G be a group and let H be a subgroup of G of finite index. Let $X := G/H$, and let $\varphi : X \to G$ be a section. If $g \in G$ and $x \in X$, the elements $\varphi(gx)$ and $g\varphi(x)$ belong to the same class of mod H; hence there exists a unique element $h^{\varphi}_{g,x} \in H$ such that $g\varphi(x) = \varphi(gx)h^{\varphi}_{g,x}$. Let $\mathrm{Ver}_{G,H}(g) \in H^{\mathrm{ab}}$ be defined by

$$\mathrm{Ver}_{G,H}(g) := \prod_{x\in X} h^{\varphi}_{g,x} \mod D(H),$$

where $D(H) = [H, H]$ is the commutator group of H and the product is computed in $H^{\mathrm{ab}} = H/D(H)$.

By [Serre 2016, Theorem 7.1], the map $\mathrm{Ver}_{G,H} : G \to H^{\mathrm{ab}}$ is a group homomorphism and it does not depend on the choice of the section φ. This homomorphism is called the *transfer* of G into H^{ab} (originally from the term “Verlagerung” in German). One may also view it as a homomorphism $\mathrm{Ver}_{G,H} : G^{\mathrm{ab}} \to H^{\mathrm{ab}}$.

In the literature one also uses the right coset space $X' := H\backslash G$ but this does not effect the result. One can easily show that if $\mathrm{Ver}'_{G,H}$ is the transfer map defined using X', then $\mathrm{Ver}'_{G,H} = \mathrm{Ver}_{G,H}$. One has the following functorial property; see [loc. cit., page 89].

Lemma 3.1. *Let $H \subset G$ and $H' \subset G'$ be subgroups of finite index. If σ be a group homomorphism from the pair (G, H) to (G', H') which induces a bijection $G/H \xrightarrow{\sim} G'/H'$, then the following diagram commutes*:

$$\begin{array}{ccc} G^{\mathrm{ab}} & \xrightarrow{\sigma} & G'^{\,\mathrm{ab}} \\ \downarrow{\scriptstyle \mathrm{Ver}_{G,H}} & & \downarrow{\scriptstyle \mathrm{Ver}_{G',H'}} \\ H^{\mathrm{ab}} & \xrightarrow{\sigma} & H'^{\,\mathrm{ab}} \end{array}$$

Lemma 3.2. *Let G_1 and G_2 be groups, and let $H_i \subset G_i$ be a subgroup of finite index for $i = 1, 2$. Then*

$$\mathrm{Ver}_{G_1\times G_2, H_1\times H_1}(g_1, g_2) = \mathrm{Ver}_{G_1,H_1}(g_1)^{[G_2:H_2]} \cdot \mathrm{Ver}_{G_2,H_2}(g_2)^{[G_1:H_1]}.$$

Proof. Put $G = G_1 \times G_2$, $H = H_1 \times H_2$, $X_i := G_i/H_i$ and $X = G/H = X_1 \times X_2$. Fix a section $\varphi_i : X_i \to G_i$ for each i and let $\varphi = (\varphi_1, \varphi_2) : X \to G$. For $g = (g_1, g_2)$ and $x = (x_1, x_2)$, we have $h^{\varphi}_{g,x} = (h^{\varphi_1}_{g_1,x_1}, h^{\varphi_2}_{g_2,x_2})$. Then in $H^{\mathrm{ab}} = H_1^{\mathrm{ab}} \times H_2^{\mathrm{ab}}$ we have

$$\begin{aligned} \mathrm{Ver}_{G,H}(g) &= \prod_{x\in X} h^{\varphi}_{g,x} = \prod_{x_1\in X_1}\prod_{x_2\in X_2} (h^{\varphi_1}_{g_1,x_1}, h^{\varphi_2}_{g_2,x_2}) \\ &= \prod_{x_1\in X_1} ((h^{\varphi_1}_{g_1,x_1})^{|X_2|}, \mathrm{Ver}_{G_2,H_2}(g_2)) = (\mathrm{Ver}_{G_1,H_1}(g_1)^{|X_2|}, \mathrm{Ver}_{G_2,H_2}(g_2)^{|X_1|}). \end{aligned}$$ □

If G is a finite group and p is a prime, G_p denotes a p-Sylow subgroup of G.

Proposition 3.3. *Let G be a finite group and $N \triangleleft G$ be a cyclic central subgroup of prime order p. Then the transfer $\mathrm{Ver}_{G,N} : G^{\mathrm{ab}} \to N^{\mathrm{ab}}$ is surjective if and only if G_p is cyclic.*

Proof. See Proposition 3.8 of [Rüd 2022]. □

3B. ***Connection of transfer maps with corestriction maps.*** Let $H \subset G$ be a subgroup of finite index, and let A be a G-module. Let

$$f : \mathrm{Ind}_H^G A = \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} A \to A, \quad g \otimes a \mapsto ga$$

be the natural map of G-modules. Applying Galois cohomology $H^i(G, -)$ to the map f and by Shapiro's lemma, we obtain for each $i \geq 0$ a morphism

$$\mathrm{Cor} : H^i(H, A) \to H^i(G, A),$$

called the *corestriction* form H to G.

Applying $H^i(G, -)$ to the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$, one obtains an isomorphism $H^i(G, \mathbb{Z}) \simeq H^{i-1}(G, \mathbb{Q}/\mathbb{Z})$ for all $i \geq 2$. When $i = 2$, this gives the isomorphism

$$H^2(G, \mathbb{Z}) \simeq \mathrm{Hom}(G^{\mathrm{ab}}, \mathbb{Q}/\mathbb{Z}). \tag{3-1}$$

Proposition 3.4. *Let $H \subset G$ be a subgroup of finite index. Through the isomorphism* (3-1) *we have the following commutative diagram*

$$\begin{array}{ccc} H^2(H, \mathbb{Z}) & \xrightarrow{\sim} & \mathrm{Hom}(H^{\mathrm{ab}}, \mathbb{Q}/\mathbb{Z}) \\ \downarrow{\scriptstyle \mathrm{Cor}} & & \downarrow{\scriptstyle \mathrm{Ver}_{G,H}^{\vee}} \\ H^2(G, \mathbb{Z}) & \xrightarrow{\sim} & \mathrm{Hom}(G^{\mathrm{ab}}, \mathbb{Q}/\mathbb{Z}), \end{array}$$

where $\mathrm{Ver}_{G,H}^{\vee}$ *is dual of the transfer* $\mathrm{Ver}_{G,H} : G^{\mathrm{ab}} \to H^{\mathrm{ab}}$. *Moreover, if H is normal in G, the composition $H^{\mathrm{ab}} \to G^{\mathrm{ab}} \to H^{\mathrm{ab}}$ is the norm $N_{G/H}$. Here H^{ab} is viewed as a G-module by conjugation and also as a G/H-module, since H acts trivially on H^{ab}.*

Proof. See [Neukirch et al. 2000, Proposition 1.5.9]. ☐

3C. ***Connection of transfer maps with class field theory.*** Let $k \subset K \subset L$ be three global fields such that the extension L/k is Galois. Put $G = \mathrm{Gal}(L/k)$ and $H = \mathrm{Gal}(L/K) \subset G$. Denote by C_k and C_K the idele class groups of k and K, respectively. The Artin map is a surjective homomorphism $\mathrm{Art}_{L/k} : C_k \to G^{\mathrm{ab}}$; similarly we have $\mathrm{Art}_{L/K} : C_K \to H^{\mathrm{ab}}$. By class field theory we have the following commutative diagrams:

$$\begin{array}{ccc} C_k & \xrightarrow{\mathrm{Art}_{L/k}} & G^{\mathrm{ab}} \\ {\scriptstyle \mathrm{func}}\downarrow & & \downarrow{\scriptstyle \mathrm{Ver}_{G,H}} \\ C_k & \xrightarrow{\mathrm{Art}_{L/k}} & H^{\mathrm{ab}} \end{array} \qquad \begin{array}{ccc} C_k & \xrightarrow{\mathrm{Art}_{L/k}} & G^{\mathrm{ab}} \\ {\scriptstyle N_{K/k}}\uparrow & & \uparrow{\scriptstyle \mathrm{func}} \\ C_k & \xrightarrow{\mathrm{Art}_{L/k}} & H^{\mathrm{ab}} \end{array} \tag{3-2}$$

where func denotes the natural map induced from the inclusion $\mathbb{A}_k^{\times} \hookrightarrow \mathbb{A}_K^{\times}$ or $H \hookrightarrow G$. When K/k is Galois and $L = K$, the second diagram of (3-2) induces a homomorphism

$$\mathrm{Art}_{K/k} : C_k/N_{K/k}(C_K) \to \mathrm{Gal}(K/k)^{\mathrm{ab}},$$

which is an isomorphism by class field theory.

3D. ***Relative transfer maps.*** Let $H \subset N \subset G$ be two subgroups of G of finite index. Let $\widetilde{X} := G/H$ and $X := G/N$ with natural G-equivariant projections $\widetilde{c} : G \to X$ and $c : \widetilde{X} \to X$. Let $\widetilde{\varphi} : X \to G$ be a section, which induces a section $\varphi : X \to \widetilde{X}$. For each $g \in G$ and $x \in X$, let $n^{\widetilde{\varphi}}_{g,x}$ be the unique element in N such that $g\widetilde{\varphi}(x) = \widetilde{\varphi}(gx)n^{\widetilde{\varphi}}_{g,x}$. Since H is a possibly nonnormal subgroup of N, let $(N/H)^{\mathrm{ab}} := \mathrm{Coker}[D(N)H/D(N) \to N/D(N)] \simeq N/D(N)H$. Note that $(N/H)^{\mathrm{ab}}$ is an abelian group which agrees with the abelianization of the group N/H when H is normal in N. Let $\mathrm{Ver}_{G,N/H}(g) \in (N/H)^{\mathrm{ab}}$ be the element defined by

$$\mathrm{Ver}_{G,N/H}(g) := \prod_{x\in X} n^{\widetilde{\varphi}}_{g,x} \mod D(N)H. \tag{3-3}$$

Proposition 3.5. (1) *The map* $\mathrm{Ver}_{G,N/H} : G \to (N/H)^{\mathrm{ab}}$ *does not depend on the choice of the section* $\widetilde{\varphi}$ *and it is a group homomorphism.*

(2) *One has* $\mathrm{Ver}_{G,N/H} = \pi_H \circ \mathrm{Ver}$, *where* $\pi_H : N^{\mathrm{ab}} \to (N/H)^{\mathrm{ab}}$ *is the morphism mod* H.

Proof. Clearly, the statement (1) follows from (2), because Ver does not depend on the choice of $\widetilde{\varphi}$ and is a group homomorphism. (2) By definition $\mathrm{Ver}(g) = \prod_{x\in X} n^{\widetilde{\varphi}}_{g,x} \mod D(N)$, thus $\mathrm{Ver}_{G,N/H} = \pi_H \circ \mathrm{Ver}$. □

Definition 3.6. Let $H \subset N$ be two subgroups of G of finite index. The group homomorphism $\mathrm{Ver}_{G,N/H} : G \to (N/H)^{\mathrm{ab}}$ defined in (3-3) is called the *transfer* of G into $(N/H)^{\mathrm{ab}}$ relative to H. By abuse of notation, we denote the induced map by $\mathrm{Ver}_{G,N/H} : G^{\mathrm{ab}} \to (N/H)^{\mathrm{ab}}$.

One can check directly that the map $\mathrm{Ver}_{G,N/H} : G^{\mathrm{ab}} \to (N/H)^{\mathrm{ab}}$ factors through $\pi_H : G^{\mathrm{ab}} \to (G/H)^{\mathrm{ab}}$, the map modulo H. We denote the induced map by

$$\mathrm{Ver}_{G/H,N/H} : (G/H)^{\mathrm{ab}} \to (N/H)^{\mathrm{ab}}. \tag{3-4}$$

Remark 3.7. If $H \lhd G$ is a normal subgroup, then the induced map $\mathrm{Ver}_{G/H,N/H}$ is the transfer map from G/H to N/H associated to the subgroup $N/H \subset G/H$ of finite index.

Lemma 3.8. *Let* $H \lhd \widetilde{N}$ *be two subgroups of finite index in* G *with* H *normal in* $\widetilde{N}$ *and cyclic quotient* $N = \widetilde{N}/H = \langle\sigma\rangle$ *of order* n. *Fix a lift* $\widetilde{\sigma} \in \widetilde{N}$ *of* σ.

(1) *For each* $g \in G$, *let* $\{x_1, \dots x_r\}$ *be a complete set of double coset representatives for* $\langle g\rangle\backslash G/\widetilde{N}$ *and set* $d_i := d_i(g) = |\langle g\rangle x_i H/H|$, *where* $1 \le i \le r$. *Then*

$$\mathrm{Ver}_{G,N}(g) = \sigma^{\sum_{i=1}^r m(g,x_i)},$$

where $0 \le m(g, x_i) \le n-1$ *is the unique integer such that* $g^{d_i} x_i H = x_i \widetilde{\sigma}^{m(g,x_i)} H$.

(2) *Let* $\sigma^* \in N^\vee := \mathrm{Hom}(N, \mathbb{Q}/\mathbb{Z})$ *be the element defined by* $\sigma^*(\sigma) = 1/n \mod \mathbb{Z}$, $\mathrm{Ver}^\vee_{G,N} : N^\vee \to \mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z})$ *the induced map, and* $f := \mathrm{Ver}^\vee_{G,N}(\sigma^*)$. *Then* $f(g) = \left(\sum_{i=1}^r m(g, x_i)\right)/n \mod \mathbb{Z}$.

Proof. (1) We choose the set of representatives $S = \{g^j x_i : i = 1, \dots, r,\ j = 0, \dots, f_i - 1\}$ of $X = G/\widetilde{N}$, which defines a section $\widetilde{\varphi}$ of the natural projection $G \to G/\widetilde{N}$. Then the image of the element $n^{\widetilde{\varphi}}_{g,x}$ in H is given by

$$n^{\widetilde{\varphi}}_{g,x} \bmod H = \begin{cases} \sigma^{m(g,x_i)} & \text{if } x = g^{d_i-1} x_i H \text{ for some } 1 \le i \le r; \\ 1 & \text{otherwise.} \end{cases}$$

(2) By definition, $f(g) = \sigma^*(\mathrm{Ver}_{G,N}(g)) = \sigma^*(\sigma^{\sum_i m(g,x_i)}) = \left(\sum_i m(g, x_i)\right)/n \bmod \mathbb{Z}$. □

One can show that the integer $m(g, x_i)$ is independent of the choice of double coset representative in $\langle g\rangle x_i H$.

The following lemma will be used in Lemma 5.1.

Lemma 3.9. *Let $G = \prod_{i=1}^r G_i$ be a product of groups G_i and let $N_i \subset G_i$, for each $1 \le i \le r$, be a subgroup of finite index. Put*

$$H_i := G_1 \times \cdots \times G_{i-1} \times \{1\} \times G_{i+1} \times \cdots \times G_r, \quad \widetilde{N}_i := G_1 \times \cdots \times G_{i-1} \times N_i \times G_{i+1} \times \cdots \times G_r.$$

Then the map

$$\prod_{i=1}^r \mathrm{Ver}_{G,\widetilde{N}_i/H_i} : G \to (\widetilde{N}_1/H_1)^{\mathrm{ab}} \times \cdots \times (\widetilde{N}_r/H_r)^{\mathrm{ab}} = N_1^{\mathrm{ab}} \times \cdots \times N_r^{\mathrm{ab}} \tag{3-5}$$

is given by the product of the maps

$$\prod_{i=1}^r \mathrm{Ver}_{G_i,N_i} : \prod_{i=1}^r G_i \to \prod_{i=1}^r N_i^{\mathrm{ab}}.$$

Proof. Let $\mathrm{pr}_i : G \to G_i$ be the i-th projection. Then

$$\mathrm{Ver}_{G,\widetilde{N}_i/H_i} = \mathrm{Ver}_{G/H_i,\widetilde{N}_i/H_i} \circ \mathrm{pr}_i = \mathrm{Ver}_{G_i,N_i} \circ \mathrm{pr}_i,$$

see Remark 3.7. Thus, the map (3-5) is equal to $\prod_i \mathrm{Ver}_{G_i,N_i} \circ (\mathrm{pr}_i)_i$. Since the map $(\mathrm{pr}_i)_i : G \to \prod_i G_i$ is the identity, we show that the map (3-5) is equal to $\prod_i \mathrm{Ver}_{G_i,N_i}$. □

3E. ***Connection of relative transfer maps with class field theory.*** Let $k \subset E \subset K$ be three global fields. Let L/k be a finite Galois extension containing K with Galois group $G = \mathrm{Gal}(L/k)$. Let $H = \mathrm{Gal}(L/K) \subset N = \mathrm{Gal}(L/E)$ be subgroups of G.

The Artin map produces the following isomorphisms (3-2)

$$C_k/N_{L/k}(C_L) \simeq G^{\mathrm{ab}}, \quad C_k/N_{K/k}(C_K) \simeq (G/H)^{\mathrm{ab}},$$
$$C_E/N_{L/E}(C_L) \simeq N^{\mathrm{ab}}, \quad C_E/N_{K/E}(C_K) \simeq (N/H)^{\mathrm{ab}}.$$

We obtain the following commutative diagrams:

$$\begin{array}{ccc} C_k/N_{L/k}(C_L) & \longrightarrow & C_k/N_{K/k}(C_K) \\ \downarrow & & \downarrow \\ C_E/N_{L/E}(C_L) & \longrightarrow & C_E/N_{K/E}(C_K) \end{array} \qquad \begin{array}{ccc} G^{\mathrm{ab}} & \xrightarrow{\pi_H} & (G/H)^{\mathrm{ab}} \\ {\scriptstyle \mathrm{Ver}}\downarrow & & \downarrow{\scriptstyle \mathrm{Ver}_{G/H,N/H}} \\ N^{\mathrm{ab}} & \xrightarrow{\pi_H} & (N/H)^{\mathrm{ab}} \end{array} \tag{3-6}$$

From this we see that $\mathrm{Ver}_{G/H,N/H} : (G/H)^{\mathrm{ab}} \to (N/H)^{\mathrm{ab}}$ does not depend on the choice of the Galois extension L/k.

4. Cohomology groups of algebraic tori

4A. $H^1(\Lambda^1)$ *and* $H^1(\Lambda)$. Let $K = \prod_{i=1}^r K_i$ be a commutative etale k-algebra and $E = \prod_{i=1}^r E_i$ a k-subalgebra. Let $T^{K/E,k}$ and $T^{K/E,1}$ be the k-tori defined in Section 2C. Let L/k be a splitting Galois field extension for T^K, and let $G = \mathrm{Gal}(L/k)$. Put

$$\Lambda := X(T^{K/E,k}) \quad \text{and} \quad \Lambda^1 := X(T^{K/E,1}),$$

which are G-modules. For $1 \le i \le r$, put

$$H_i := \mathrm{Gal}(L/K_i), \tag{4-1}$$

$$\widetilde{N}_i := \mathrm{Gal}(L/E_i), \tag{4-2}$$

$$N_i^{\mathrm{ab}} = (\widetilde{N}_i/H_i)^{\mathrm{ab}} := \widetilde{N}_i/H_i D(\widetilde{N}_i). \tag{4-3}$$

In the case that K_i is Galois over E_i, we let

$$N_i := \widetilde{N}_i/H_i = \mathrm{Gal}(K_i/E_i) \tag{4-4}$$

and then N_i^{ab} coincides with the abelianization of the group N_i, which justifies our notation.

From the diagram (2-5), we obtain the commutative diagram of G-modules:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X(T^E) & \xrightarrow{\widehat{N}} & X(T^K) & \xrightarrow{\widehat{j}} & \Lambda^1 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \widehat{\Delta}} & & \downarrow{\scriptstyle \widehat{j}_T} & & \| & & \\ 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\widehat{N}} & \Lambda & \xrightarrow{\widehat{j}} & \Lambda^1 & \longrightarrow & 0 \end{array} \tag{4-5}$$

We have

$$X(T^K) = \bigoplus_{i=1}^r \mathrm{Ind}_{H_i}^G \mathbb{Z} \quad \text{and} \quad X(T^E) = \bigoplus_{i=1}^r \mathrm{Ind}_{\widetilde{N}_i}^G \mathbb{Z} \tag{4-6}$$

and

$$T^{K/E,1} = \prod_{i=1}^r R_{E_i/k} R^{(1)}_{K_i/E_i} \mathbb{G}_{\mathrm{m},K_i} \quad \text{and} \quad \Lambda^1 = \bigoplus_{i=1}^r \mathrm{Ind}_{\widetilde{N}_i}^G \Lambda^1_{E_i}, \tag{4-7}$$

where

$$\Lambda^1_{E_i} := X(R^{(1)}_{K_i/E_i} \mathbb{G}_{\mathrm{m},K_i}). \tag{4-8}$$

By Shapiro's lemma, one has

$$H^q(G,\Lambda^1)=\bigoplus_{i=1}^{r} H^q(G,\operatorname{Ind}_{\widetilde{N}_i}^{G}\Lambda^1_{E_i})=\bigoplus_{i=1}^{r} H^q(\widetilde{N}_i,\Lambda^1_{E_i}),\quad \forall q\geq 0. \tag{4-9}$$

Note that if K_i/E_i is Galois, then $H^1(\widetilde{N}_i,\Lambda^1_{E_i})=H^1(N_i,\Lambda^1_{E_i})$ (Remark 2.6).

Since the torus $R^{(1)}_{K_i/E_i}\mathbb{G}_{\mathrm{m},K_i}$ is anisotropic, one has $H^0(G,\Lambda^1)=\bigoplus_{i=1}^{r} H^0(\widetilde{N}_i,\Lambda^1_{E_i})=0$.

Taking Galois cohomology to the lower exact sequence of (4-5), we have an exact sequence

$$0\to H^1(G,\Lambda)\to H^1(G,\Lambda^1)\xrightarrow{\delta} H^2(G,\mathbb{Z}). \tag{4-10}$$

Proposition 4.1. *Let the notation be as above*:

(1) *There is a canonical isomorphism* $H^1(G,\Lambda^1)\simeq\bigoplus_i N_i^{\mathrm{ab},\vee}$ (*see* (4-3) *for the definition of* N_i^{ab}).

(2) *Under the canonical isomorphisms* $H^1(G,\Lambda^1)\simeq\bigoplus_i N_i^{\mathrm{ab},\vee}$ *and* $H^2(G,\mathbb{Z})\simeq G^{\mathrm{ab},\vee}$ (3-1), *the map* $\delta: H^1(G,\Lambda^1)\to H^2(G,\mathbb{Z})$ *expresses as*

$$\sum_{i=1}^{r}\mathrm{Ver}^{\vee}_{G,N_i}:\ \bigoplus_{i=1}^{r} N_i^{\mathrm{ab},\vee}\to G^{\mathrm{ab},\vee}, \tag{4-11}$$

where $\mathrm{Ver}_{G,N_i}: G\to N_i^{\mathrm{ab}}$ *is the transfer map. In particular*, $H^1(G,\Lambda)\simeq\operatorname{Ker}\big(\sum\mathrm{Ver}^{\vee}_{G,N_i}\big)$. *Furthermore*, *the map in* (4-11) *factors as*

$$\bigoplus_{i=1}^{r} N_i^{\mathrm{ab},\vee}\xrightarrow{(\mathrm{Ver}^{\vee}_{G/H_i,N_i})_i}\bigoplus_{i=1}^{r}(G/H_i)^{\mathrm{ab},\vee}\xrightarrow{\sum_{i=1}^{r}\pi^{\vee}_{H_i}} G^{\mathrm{ab},\vee}, \tag{4-12}$$

where $\pi_{H_i}: G^{\mathrm{ab}}\to(G/H_i)^{\mathrm{ab}}$ *is the map mod* H_i.

Proof. (1) Taking Galois cohomology of the upper exact sequence of (4-5), we have a long exact sequence of abelian groups

$$H^1(G,X(T^K))\to H^1(G,\Lambda^1)\xrightarrow{\delta} H^2(G,X(T^E))\xrightarrow{\widehat{N}^2} H^2(G,X(T^K)). \tag{4-13}$$

By (4-6), the first term $H^1(G,X(T^K))=\bigoplus_i H^1(H_i,\mathbb{Z})=0$. Using the relations (4-6) and by Shapiro's lemma, we have $H^1(G,\Lambda^1)=\bigoplus_i H^1(\widetilde{N}_i,\Lambda^1_{E_i})$ and the exact sequence (4-13) becomes

$$0\to\bigoplus_{i=1}^{r} H^1(\widetilde{N}_i,\Lambda^1_{E_i})\xrightarrow{\widetilde{\delta}}\bigoplus_{i=1}^{r} H^2(\widetilde{N}_i,\mathbb{Z})\xrightarrow{\mathrm{Res}}\bigoplus_{i=1}^{r} H^2(H_i,\mathbb{Z}). \tag{4-14}$$

It is clear that the following sequence

$$0\to\operatorname{Hom}(N_i^{\mathrm{ab}},\mathbb{Q}/\mathbb{Z})\xrightarrow{\mathrm{Inf}}\operatorname{Hom}(\widetilde{N}_i,\mathbb{Q}/\mathbb{Z})\xrightarrow{\mathrm{Res}}\operatorname{Hom}(H_i,\mathbb{Q}/\mathbb{Z}) \tag{4-15}$$

is exact. Using the canonical isomorphism $H^2(H,\mathbb{Z})\simeq\operatorname{Hom}(H,\mathbb{Q}/\mathbb{Z})$ (3-1) for any group H, we rewrite (4-15) as follows:

$$0\to H^2(N_i^{\mathrm{ab}},\mathbb{Z})\xrightarrow{\mathrm{Inf}} H^2(\widetilde{N}_i,\mathbb{Z})\xrightarrow{\mathrm{Res}} H^2(H_i,\mathbb{Z}). \tag{4-16}$$

Comparing (4-16) and (4-14), there is a unique isomorphism

$$H^1(G,\Lambda^1)=\bigoplus_{i=1}^r H^1(\widetilde{N}_i,\Lambda^1_{E_i})\simeq\bigoplus_{i=1}^r H^2(N_i^{\mathrm{ab}},\mathbb{Z}) \tag{4-17}$$

which fits into the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \bigoplus_{i=1}^r H^2(N_i^{\mathrm{ab}},\mathbb{Z}) & \xrightarrow{\mathrm{Inf}} & \bigoplus_{i=1}^r H^2(\widetilde{N}_i,\mathbb{Z}) & \xrightarrow{\mathrm{Res}} & \bigoplus_{i=1}^r H^2(H_i,\mathbb{Z}) \\ & & \| & & \| & & \| \\ 0 & \longrightarrow & \bigoplus_{i=1}^r H^1(\widetilde{N}_i,\Lambda^1_{E_i}) & \xrightarrow{\widetilde{\delta}} & \bigoplus_{i=1}^r H^2(\widetilde{N}_i,\mathbb{Z}) & \xrightarrow{\mathrm{Res}} & \bigoplus_{i=1}^r H^2(H_i,\mathbb{Z}) \end{array} \tag{4-18}$$

This shows the first statement.

(2) The map $\widehat{\Delta}$ in (4-5) is induced from the restriction of $X(T^E)$ to the subtorus $\mathbb{G}_{\mathrm{m},k}$ and therefore is given by

$$\widehat{\Delta}=\sum_{i=1}^r\widehat{\Delta}_i:\ \bigoplus_{i=1}^r \mathrm{Ind}_{\widetilde{N}_i}^G\,\mathbb{Z}\to\mathbb{Z},\quad \widehat{\Delta}_i(g\otimes n)=n,\quad \forall g\in G, n\in\mathbb{Z}.$$

Thus, by definition, the induced map $\widehat{\Delta}_i^2$ on $H^2(G,-)$ is nothing but the corestriction $\mathrm{Cor}\colon H^2(\widetilde{N}_i,\mathbb{Z})\to H^2(G,\mathbb{Z})$; see Section 3B.

From the diagram (4-5), we obtain the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \bigoplus_{i=1}^r H^1(\widetilde{N}_i,\Lambda^1_{E_i}) & \xrightarrow{\widetilde{\delta}} & \bigoplus_{i=1}^r H^2(\widetilde{N}_i,\mathbb{Z}) & \xrightarrow{\mathrm{Res}} & \bigoplus_{i=1}^r H^2(H_i,\mathbb{Z}) \\ \downarrow & & \| & & \downarrow{\scriptstyle \widehat{\Delta}^2=\mathrm{Cor}} & & \downarrow \\ 0\to H^1(G,\Lambda) & \longrightarrow & H^1(G,\Lambda^1) & \xrightarrow{\delta} & H^2(G,\mathbb{Z}) & \longrightarrow & H^2(G,\Lambda) \end{array} \tag{4-19}$$

With the identification (4-17), we have $\delta=\mathrm{Cor}\circ\widetilde{\delta}=\mathrm{Cor}\circ\mathrm{Inf}$ and the lower long exact sequence of (4-19) becomes

$$0\to H^1(G,\Lambda)\to\bigoplus_{i=1}^r H^2(N_i^{\mathrm{ab}},\mathbb{Z})\xrightarrow{\mathrm{Cor}\circ\mathrm{Inf}} H^2(G,\mathbb{Z})\xrightarrow{\widehat{N}^2} H^2(G,\Lambda). \tag{4-20}$$

By Propositions 3.5 and 3.4, under the isomorphism (3-1) the map $\mathrm{Cor}\circ\mathrm{Inf}\colon H^2(N_i^{\mathrm{ab}},\mathbb{Z})\to H^2(G,\mathbb{Z})$ corresponds to $\mathrm{Ver}_{G,N_i}^\vee : N_i^{\mathrm{ab},\vee}\to G^{\mathrm{ab},\vee}$, where $\mathrm{Ver}_{G,N_i}^\vee$ is the dual of the transfer $\mathrm{Ver}_{G,N_i}: G^{\mathrm{ab}}\to N_i^{\mathrm{ab}}$ relative to H_i. This proves (4-11). Then it follows from (4-20) that $H^1(\Lambda)\simeq\mathrm{Ker}\big(\sum_{i=1}^r\mathrm{Ver}_{G,N_i}^\vee\big)$. As the map $\mathrm{Ver}_{G,N_i}: G^{\mathrm{ab}}\to N_i^{\mathrm{ab}}$ factors as

$$G\xrightarrow{\pi_{H_i}}(G/H_i)^{\mathrm{ab}}\xrightarrow{\mathrm{Ver}_{G/H_i,\widetilde{N}_i/H_i}} N_i^{\mathrm{ab}},$$

the last assertion (4-12) follows. □

Remark 4.2. In terms of class field theory, we have $C_{E_i}/N_{K_i/E_i}(C_{K_i}) \simeq N_i^{\mathrm{ab}}$, $C_k/N_{K_i/k}(C_{K_i}) \simeq (G/H_i)^{\mathrm{ab}}$ and $C_k/N_{L/k}(C_L) \simeq G^{\mathrm{ab}}$. Let L_0 be the compositum of all Galois closures of K_i over k; this is the minimal splitting field of the algebraic torus $T^{K/E,k}$. One has $L_0 \subset L$ and its Galois group $G_0 := \mathrm{Gal}(L_0/k)$ is a quotient of G. Thus, we have the following commutative diagram:

$$\begin{array}{ccccccc} \frac{C_k}{N_{L/k}(C_L)} & \longrightarrow & \frac{C_k}{N_{L_0/k}(C_{L_0})} & \longrightarrow & \prod_i \frac{C_k}{N_{K_i/k}(C_{K_i})} & \longrightarrow & \prod_i \frac{C_{E_i}}{N_{K_i/E_i}(C_{K_i})} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ G^{\mathrm{ab}} & \longrightarrow & G_0^{\mathrm{ab}} & \longrightarrow & \prod_i (G/H_i)^{\mathrm{ab}} & \longrightarrow & \prod_i N_i^{\mathrm{ab}} \end{array} \tag{4-21}$$

Taking the Pontryagin dual, the lower row gives

$$\prod_i N_i^{\mathrm{ab},\vee} \to \prod_i (G/H_i)^{\mathrm{ab},\vee} \to G_0^{\mathrm{ab},\vee} \subset G^{\mathrm{ab},\vee}.$$

From this, we see that the map $\sum \mathrm{Ver}_{G,N_i}^{\vee}$ in (4-11) has image contained in $G_0^{\mathrm{ab},\vee}$. It follows from (4-21) that this map is independent of the choice of the splitting field L; also see Remark 2.6.

4B. $\text{Ш}^2(\Lambda)$. Let $\mathcal{D}$ be the set of all decomposition groups of G. For any G-module A, denote by

$$H^i_{\mathcal{D}}(A) := \prod_{D \in \mathcal{D}} H^i_D(A), \quad H^i_D(A) := H^i(D, A)$$

and

$$r^i_{\mathcal{D},A} = (r^i_{D,A})_{D\in\mathcal{D}} : H^i(G, A) \to H^i_{\mathcal{D}}(A)$$

the restriction map to subgroups D in $\mathcal{D}$ defined in (1-3). By definition, $\text{Ш}^i(A) = \operatorname{Ker} r^i_{\mathcal{D},A}$. We shall write r_D and $r_{\mathcal{D}}$ for $r^i_{D,A}$ (1-2) and $r^i_{\mathcal{D},A}$ (1-3), respectively, if it is clear from the content.

For the remainder of this section, we assume that the extension K_i/E_i *is cyclic* with Galois group N_i for all i. Consider the following commutative diagram

$$\begin{array}{ccccccc} H^1(\Lambda^1) & \xrightarrow{\delta} & H^2(\mathbb{Z}) & \xrightarrow{\widehat{N}} & H^2(\Lambda) & \xrightarrow{\widehat{j}} & H^2(\Lambda^1) \\ \downarrow{\scriptstyle r^1_{\mathcal{D},\Lambda^1}} & & \downarrow{\scriptstyle r^2_{\mathcal{D},\mathbb{Z}}} & & \downarrow{\scriptstyle r^2_{\mathcal{D},\Lambda}} & & \downarrow{\scriptstyle r^2_{\mathcal{D},\Lambda^1}} \\ H^1_{\mathcal{D}}(\Lambda^1) & \xrightarrow{\delta_{\mathcal{D}}} & H^2_{\mathcal{D}}(\mathbb{Z}) & \xrightarrow{\widehat{N}} & H^2_{\mathcal{D}}(\Lambda) & \xrightarrow{\widehat{j}} & H^2_{\mathcal{D}}(\Lambda^1). \end{array} \tag{4-22}$$

Define

$$H^2(\mathbb{Z})' := \{x \in H^2(\mathbb{Z}) : \widehat{N}(x) \in \text{Ш}^2(\Lambda)\} = \{x \in H^2(\mathbb{Z}) : r^2_{\mathcal{D},\mathbb{Z}}(x) \in \mathrm{Im}(\delta_{\mathcal{D}})\}. \tag{4-23}$$

Proposition 4.3. *Assume that K_i/E_i is cyclic with Galois group N_i for all i. Then* $\text{Ш}^2(\Lambda^1) = 0$, $\text{Ш}^2(\Lambda) \simeq H^2(\mathbb{Z})'/\mathrm{Im}(\delta)$ *and*

$$\tau_k(T^{K/E,k}) = \frac{\prod_{i=1}^r |N_i|}{|H^2(\mathbb{Z})'|},$$

where $H^2(\mathbb{Z})'$ is the group defined in (4-23) *and δ is the labeled map in* (4-22).

Proof. It is obvious that $\widehat{j}(Ш^2(\Lambda)) \subset Ш^2(\Lambda^1)$ and then we have a long exact sequence

$$0 \to H^1(\Lambda) \to H^1(\Lambda^1) \xrightarrow{\delta} H^2(\mathbb{Z})' \xrightarrow{\widehat{N}} Ш^2(\Lambda) \xrightarrow{\widehat{j}} Ш^2(\Lambda^1).$$

One has

$$Ш^2(\Lambda^1) = \bigoplus_i Ш^2(G, \operatorname{Ind}_{\widetilde{N}_i}^G \Lambda^1_{E_i}) = \bigoplus_i Ш^2(\widetilde{N}_i, \Lambda^1_{E_i}) = \bigoplus_i Ш^2(N_i, \Lambda^1_{E_i})$$

because $Ш^2(X(T))$ does not depend on the choice of the splitting field. Since K_i/E_i is cyclic, by Chebotarev's density theorem, we have $Ш^2(N_i, \Lambda^1_{E_i}) = 0$ for all i and $Ш^2(\Lambda^1) = 0$. Therefore, one gets the 4-term exact sequence

$$0 \to H^1(\Lambda) \to H^1(\Lambda^1) \xrightarrow{\delta} H^2(\mathbb{Z})' \xrightarrow{\widehat{N}} Ш^2(\Lambda) \to 0. \tag{4-24}$$

From this we obtain $Ш^2(\Lambda) \simeq H^2(\mathbb{Z})'/\operatorname{Im}(\delta)$ and

$$\tau_k(T^{K/E,k}) = \frac{|H^1(\Lambda)|}{|Ш^2(\Lambda)|} = \frac{|H^1(\Lambda^1)|}{|H^2(\mathbb{Z})'|} = \frac{\prod_{i=1}^r |N_i|}{|H^2(\mathbb{Z})'|}. \qquad \square$$

Remark 4.4. Ono [1963b] showed that $\tau(R^{(1)}_{K/k}\mathbb{G}_{\mathrm{m},K}) = [K:k]$ for any cyclic extension K/k. Since $|H^1(K/k, X(R^{(1)}_{K/k}\mathbb{G}_{\mathrm{m},K}))| = [K:k]$, it follows that $Ш^2(K/k, X(R^{(1)}_{K/k}\mathbb{G}_{\mathrm{m},K})) = 0$. This gives an alternative proof of the first statement $Ш^2(\Lambda^1) = 0$ of Proposition 4.3.

In order to compute the groups $H^2(\mathbb{Z})'$ and $Ш^2(\Lambda)$, we describe the maps δ and $\delta_{\mathfrak{D}}$ in the first commutative square of diagram (4-22). Since $\Lambda^1 = \oplus_i \Lambda^1_i$, where $\Lambda^1_i = X(T^{K_i/E_i,1}) = \operatorname{Ind}_{\widetilde{N}_i}^G \Lambda^1_{E_i}$, it suffices to describe the following commutative diagram:

$$\begin{array}{ccc} H^1(\Lambda^1_i) & \xrightarrow{\delta} & H^2(\mathbb{Z}) \\ \downarrow{\scriptstyle r_D} & & \downarrow{\scriptstyle r_D} \\ H^1_D(\Lambda^1_i) & \xrightarrow{\delta_D} & H^2_D(\mathbb{Z}) \end{array}$$

Using the commutative diagram (4-19), the preceding diagram decomposes into two squares:

$$\begin{array}{ccccc} H^1(\operatorname{Ind}_{\widetilde{N}_i}^G \Lambda^1_{E_i}) & \xrightarrow{\widetilde{\delta}} & H^2(\operatorname{Ind}_{\widetilde{N}_i}^G \mathbb{Z}) & \xrightarrow{\widehat{\Delta}^2} & H^2(\mathbb{Z}) \\ \downarrow{\scriptstyle r_D} & & \downarrow{\scriptstyle r_D} & & \downarrow{\scriptstyle r_D} \\ H^1_D(\operatorname{Ind}_{\widetilde{N}_i}^G \Lambda^1_{E_i}) & \xrightarrow{\widetilde{\delta}_D} & H^2_D(\operatorname{Ind}_{\widetilde{N}_i}^G \mathbb{Z}) & \xrightarrow{\widehat{\Delta}^2_D} & H^2_D(\mathbb{Z}) \end{array} \tag{4-25}$$

Proposition 4.5. *Assume that the extension K_i/E_i is cyclic and both the subgroups $\widetilde{N}_i = \mathrm{Gal}(L/E_i)$ (4-2) and $H_i = \mathrm{Gal}(L/K_i)$ (4-1) are normal in G (that is, E_i/k and K_i/k are Galois) for all i. There are natural isomorphisms*

$$H^1(\Lambda^1) \simeq \bigoplus_i H^2(N_i, \mathbb{Z}) \quad \textit{and} \quad H^1_{\mathfrak{D}}(\Lambda^1) = \bigoplus_D \bigoplus_i H^2(\overline{D}_i, \mathbb{Z})^{[G:D\widetilde{N}_i]},$$

where $D_i = D \cap \widetilde{N}_i$ and $\overline{D}_i$ is its image in N_i. Under these identifications the first commutative square of diagram (4-22) *decomposes as the following*:

$$\begin{array}{ccccc} \bigoplus_{i=1}^r H^2(N_i, \mathbb{Z}) & \xrightarrow{\quad\text{Inf}\quad} & \bigoplus_{i=1}^r H^2(\widetilde{N}_i, \mathbb{Z}) & \xrightarrow{\quad\text{Cor}\quad} & H^2(\mathbb{Z}) \\ \downarrow{\scriptstyle r_{\mathcal{D}}} & & \downarrow{\scriptstyle r_{\mathcal{D}}} & & \downarrow{\scriptstyle r_{\mathcal{D}}} \\ \bigoplus_D \bigoplus_{i=1}^r H^2(\overline{D}_i, \mathbb{Z})^{[G:D\widetilde{N}_i]} & \xrightarrow{\text{Inf}} & \bigoplus_D \bigoplus_{i=1}^r H^2(D_i, \mathbb{Z})^{[G:D\widetilde{N}_i]} & \xrightarrow{\text{Cor}} & \bigoplus_D H^2_D(\mathbb{Z}) \end{array} \tag{4-26}$$

Proof. For any element $g \in G$ and any $\widetilde{N}_i$-module X, let X^g be the set X equipped with the $g\widetilde{N}_i g^{-1}$-module structure defined by

$$h' \cdot x := (g^{-1}h'g)x, \quad \text{for } x \in X^g = X \text{ and } h' \in g\widetilde{N}_i g^{-1}.$$

Recall that Mackey's formula [Serre 1977, Section 7.3 Proposition 22, page 58] says that as D-modules one has

$$\operatorname{Ind}_{\widetilde{N}_i}^G X = \bigoplus_{g \in D\backslash G/\widetilde{N}_i} \operatorname{Ind}_{D_i}^D X^g, \tag{4-27}$$

where g runs through double coset representatives for $D\backslash G/\widetilde{N}_i$. Putting $X = \Lambda^1_{E_i}$ or $X = \mathbb{Z}$, we have

$$\operatorname{Ind}_{\widetilde{N}_i}^G \Lambda^1_{E_i} = \bigoplus_{g \in G/D\widetilde{N}_i} \operatorname{Ind}_{D_i}^D (\Lambda^1_{E_i})^g, \quad \operatorname{Ind}_{\widetilde{N}_i}^G \mathbb{Z} = \bigoplus_{g \in G/D\widetilde{N}_i} \operatorname{Ind}_{D_i}^D \mathbb{Z}$$

as $\widetilde{N}_i$ is normal in G.

We first show $(\Lambda^1_{E_i})^g \simeq \Lambda^1_{E_i}$ as $\widetilde{N}_i$-modules. From the exact sequence of algebraic E_i-tori

$$1 \to R^{(1)}_{K_i/E_i}(\mathbb{G}_{\mathrm{m},K_i}) \xrightarrow{j} R_{K_i/E_i}(\mathbb{G}_{\mathrm{m},K_i}) \xrightarrow{N_{K_i/E_i}} \mathbb{G}_{\mathrm{m},E_i} \to 1, \tag{4-28}$$

one has an exact sequence of $\widetilde{N}_i$-modules

$$0 \to \mathbb{Z} \xrightarrow{\widehat{N}} X(R_{K_i/E_i}(\mathbb{G}_{\mathrm{m},K_i})) \xrightarrow{\widehat{j}} \Lambda^1_{E_i} \to 0. \tag{4-29}$$

Since K_i/E_i is Galois, all E_i-tori in (4-28) split over K_i and hence (4-29) becomes an exact sequence of N_i-modules and $X(R_{K_i/E_i}(\mathbb{G}_{\mathrm{m},K_i})) \simeq \operatorname{Ind}_1^{N_i} \mathbb{Z}$ is an induced module. Thus, $\Lambda^1_{E_i} = \operatorname{Coker}(\mathbb{Z} \to \operatorname{Ind}_{H_i}^{\widetilde{N}_i} \mathbb{Z})$ and $\operatorname{Ind}_{H_i}^{\widetilde{N}_i} \mathbb{Z} = \operatorname{Ind}_1^{N_i} \mathbb{Z}$. So it suffices to show that $(\operatorname{Ind}_1^{N_i} \mathbb{Z})^g \simeq \operatorname{Ind}_1^{N_i} \mathbb{Z}$ as N_i-modules. Let $\{n^g = g^{-1}ng\}_{n \in N_i}$ be a $\mathbb{Z}$-basis of $(\operatorname{Ind}_1^{N_i} \mathbb{Z})^g$. The new action of N_i on $(\operatorname{Ind}_1^{N_i} \mathbb{Z})^g$ is given by $h \cdot n^g := h^g n^g = (hn)^g$ for $h, n \in N_i$. So the map $n \mapsto n^g$ gives an isomorphism $\operatorname{Ind}_1^{N_i} \mathbb{Z} \xrightarrow{\sim} (\operatorname{Ind}_1^{N_i} \mathbb{Z})^g$ of N_i-modules, and hence $(\Lambda^1_{E_i})^g \simeq \Lambda^1_{E_i}$.

By Mackay's formula (4-27) and Shapiro's lemma, we obtain the following commutative diagram from (4-25):

$$\begin{array}{ccccc} H^1(\widetilde{N}_i, \Lambda^1_{E_i}) & \xrightarrow{\widetilde{\delta}} & H^2(\widetilde{N}_i, \mathbb{Z}) & \xrightarrow{\mathrm{Cor}} & H^2(\mathbb{Z}) \\ \downarrow r_D & & \downarrow r_D & & \downarrow r_D \\ \bigoplus_g H^1(D_i, (\Lambda^1_{E_i})^g) & \xrightarrow{\widetilde{\delta}_D} & \bigoplus_g H^2(D, \mathbb{Z}) & \xrightarrow{\mathrm{Cor}} & H^2_D(\mathbb{Z}) \end{array} \tag{4-30}$$

Since $(\Lambda^1_{E_i})^g \simeq \Lambda^1_{E_i}$, the bottom row of (4-30) can be expressed as

$$H^1(D_i, \Lambda^1_{E_i})^{[G:D\widetilde{N}_i]} \xrightarrow{\widetilde{\delta}_D} H^2(D_i, \mathbb{Z})^{[G:D\widetilde{N}_i]} \xrightarrow{\mathrm{Cor}} H^2_D(\mathbb{Z}). \tag{4-31}$$

Taking the Galois cohomology $H^*(D_i, -)$ to (4-29), we get an exact sequence

$$0 \to H^1(D_i, \Lambda^1_{E_i}) \xrightarrow{\widetilde{\delta}_{D_i}} H^2(D_i, \mathbb{Z}) \xrightarrow{\widehat{N}} H^2(D_i, \mathrm{Ind}^{\widetilde{N}_i}_{H_i} \mathbb{Z}) \simeq H^2(D_i \cap H_i, \mathbb{Z})^{[\widetilde{N}_i : D_i H_i]}$$

as one has

$$H^2(D_i, \mathrm{Ind}^{\widetilde{N}_i}_{H_i} \mathbb{Z}) = H^2\Big(D_i, \bigoplus_{\widetilde{N}_i / D_i H_i} \mathrm{Ind}^{D_i}_{D_i \cap H_i} \mathbb{Z}\Big) \simeq H^2(D_i \cap H_i, \mathbb{Z})^{[\widetilde{N}_i : D_i H_i]}.$$

Similar to (4-16) and (4-17), using the inflation-restriction exact sequence, we make the following identification $H^1(D_i, \Lambda^1_{E_i}) = H^2(\overline{D}_i, \mathbb{Z})$ and (4-31) becomes

$$H^2(\overline{D}_i, \mathbb{Z})^{[G:D\widetilde{N}_i]} \xrightarrow{\mathrm{Inf}} H^2(D_i, \mathbb{Z})^{[G:D\widetilde{N}_i]} \xrightarrow{\mathrm{Cor}} H^2_D(\mathbb{Z}).$$

This proves the proposition. □

The following proposition gives a group-theoretic description of $Ш^2(\Lambda)$.

Proposition 4.6. *Let the notation and assumptions be as in Proposition 4.5. Let*

$$\mathrm{Ver}_{G,\boldsymbol{N}} = (\mathrm{Ver}_{G,N_i})_i : G^{\mathrm{ab}} \to \prod_i N_i \quad \textit{and} \quad \mathrm{Ver}_{D,\overline{\boldsymbol{D}}} = (\mathrm{Ver}_{D,\overline{D}_i})_i : D^{\mathrm{ab}} \to \prod_i \overline{D}_i$$

denote the corresponding transfer maps, respectively. Then

$$H^2(G, \mathbb{Z})' = \{f \in G^{\mathrm{ab},\vee} : f|_{D^{\mathrm{ab}}} \in \mathrm{Im}(\mathrm{Ver}^\vee_{D,\overline{\boldsymbol{D}}}) \forall D \in \mathcal{D}\}, \tag{4-32}$$

and

$$Ш^2(\Lambda) \simeq \frac{\{f \in G^{\mathrm{ab},\vee} : f|_{D^{\mathrm{ab}}} \in \mathrm{Im}(\mathrm{Ver}^\vee_{D,\overline{\boldsymbol{D}}}) \forall D \in \mathcal{D}\}}{\mathrm{Im}(\mathrm{Ver}^\vee_{G,\boldsymbol{N}})}. \tag{4-33}$$

Proof. We translate the commutative diagram (4-26) in terms of group theory. For each decomposition group $D \in \mathcal{D}$, we have the following corresponding commutative diagram:

$$\begin{array}{ccccc} \prod_{i=1}^r N_i & \xleftarrow{\pi_{\boldsymbol{H}}=(\pi_{H_i})} & \prod_{i=1}^r \widetilde{N}_i^{\mathrm{ab}} & \xleftarrow{\mathrm{Ver}_{G,\boldsymbol{N}}=(\mathrm{Ver}_{G,N_i})} & G^{\mathrm{ab}} \\ \uparrow \text{func} & & \uparrow \text{func} & & \uparrow \text{func} \\ \prod_{i=1}^r \overline{D}_i & \xleftarrow{\pi_{\boldsymbol{D}\cap H}=(\pi_{D_i\cap H_i})} & \prod_{i=1}^r D_i^{\mathrm{ab}} & \xleftarrow{\mathrm{Ver}_{D,\boldsymbol{D}}=(\mathrm{Ver}_{D,D_i})} & D^{\mathrm{ab}}. \end{array}$$

Here we ignore the multiplicity $[G : D\widetilde{N}_i]$ because we are only concerned with the image of the map $\mathrm{Cor} \circ \mathrm{Inf}$ in (4-26) and this does not affect the result. Then $\mathrm{Ver}_{G,\boldsymbol{N}} : G^{\mathrm{ab}} \to \prod_i N_i$ and $\mathrm{Ver}_{D,\overline{\boldsymbol{D}}} : D^{\mathrm{ab}} \to \prod_i \overline{D}_i$ are the respective compositions. By Proposition 4.5, the map δ_D corresponds to $\mathrm{Ver}_{D,\overline{\boldsymbol{D}}}^{\vee}$. From the second description of $H^2(\mathbb{Z})'$ in (4-23), we obtain (4-32).

By Propositions 4.5 and 4.1, the map $\delta : H^1(\Lambda^1) \to H^2(\mathbb{Z})' \subset H^2(\mathbb{Z})$ (4-24) corresponds to $\mathrm{Ver}_{G,\boldsymbol{N}}^{\vee} : \prod_i N_i^{\vee} \to G^{\mathrm{ab},\vee}$. Thus, by Proposition 4.3, we obtain (4-33). This proves the proposition. □

5. Computations of some product cases

We keep the notation in the previous section. In this section we consider the case where the extensions K_i/k are all Galois with Galois group $G_i = \mathrm{Gal}(K_i/k)$. Let $L = K_1 K_2 \cdots K_r$ be the compositum of all K_i over k with Galois group $G = \mathrm{Gal}(L/k)$. Assume that:

$$\begin{aligned} &\text{(i) The canonical map monomorphism } G \to G_1 \times \cdots \times G_r \text{ is an isomorphism.} \\ &\text{(ii) } K_i/E_i \text{ is cyclic with Galois group } N_i := \mathrm{Gal}(K_i/E_i) \text{ for all } i. \end{aligned} \tag{5-1}$$

As before, we put $T = T^{K/E,k}$, $\Lambda := X(T)$, $T^1 = T^{K/E,1}$ and $\Lambda^1 := X(T^1)$. For each $1 \le i \le r$, let

$$T_i := T^{K_i/E_i,k} \quad \text{and} \quad \Lambda_i := X(T_i) \tag{5-2}$$

be the character group of T_i.

5A. $H^1(\Lambda)$ and $H^2(\Lambda^1)$.

Lemma 5.1. *Let $\Lambda_i := X(T_i)$ as in (5-2). We have $H^1(\Lambda) \simeq \oplus_{i=1}^r H^1(\Lambda_i)$.*

Proof. By Proposition 4.1, $H^1(\Lambda)$ is isomorphic to the kernel of the dual of the map $\prod_i \mathrm{Ver}_{G,N_i} : G \to \prod_i N_i^{\mathrm{ab}}$. By Lemma 3.9, $\prod_i \mathrm{Ver}_{G,N_i} = \prod_i \mathrm{Ver}_{G_i,N_i} : \prod_{i=1}^r G_i \to \prod_i N_i^{\mathrm{ab}}$, and therefore the kernel of its dual is equal to the product of abelian groups $H^1(\Lambda_i)$ for $i = 1, \ldots, r$, by Proposition 4.1. This proves the lemma. □

We remark that condition (ii) in (5-1) is not needed in the proof of Lemma 5.1.

Proposition 5.2. *Let $H \subset G$ be a normal subgroup of a finite group G, let A be a G-module, and let $n \ge 1$ be a positive integer. If $H^q(H, A) = 0$ for all $0 < q < n$, then we have a 5-term long exact sequence*

$$0 \to H^n(G/H, A^H) \to H^n(G, A) \to H^n(H, A)^{G/H} \xrightarrow{d} H^{n+1}(G/H, A^H) \to H^{n+1}(G, A).$$

Proof. This follows from the Hochschild–Serre spectral sequence

$$E_2^{p,q} := H^p(G/H, H^q(H, A^H)) \Longrightarrow H^{p+q}(G, A);$$

see [Neukirch et al. 2000, Theorem 2.1.5, page 82 and Proposition 2.1.3, page 81]. □

Proposition 5.3. *Let $K = \prod_{i=1}^r K_i$ and $E = \prod_{i=1}^r E_i$ be as before. Suppose that each K_i/k is Galois with group G_i, and that conditions* (i) *and* (ii) *in* (5-1) *are satisfied. Then*

$$H^2(G, \Lambda^1) \simeq \oplus_{i=1}^r H^2(\widetilde{N}_i, \Lambda^1_{E_i}),$$

where $\Lambda^1_{E_i}$ is defined in (4-8), *and there is a natural homomorphism $v_i : N_i \to (H_i^{\mathrm{ab}})^{|N_i|-1}$ such that $H^2(\widetilde{N}_i, \Lambda^1_{E_i}) \simeq \mathrm{Ker}(v_i^\vee)$. If $r = 1$, that is K/k is a Galois field extension, then $H^2(\Lambda^1) = 0$.*

Proof. By (4-9), we have

$$H^q(G, \Lambda^1) = \bigoplus_{i=1}^r H^q(\widetilde{N}_i, \Lambda^1_{E_i}), \quad \forall q \geq 0.$$

By (4-29), we get the long exact sequence

$$H^q(N_i, \mathrm{Ind}_{H_i}^{\widetilde{N}_i} \mathbb{Z}) \to H^q(N_i, \Lambda^1_{E_i}) \to H^{q+1}(N_i, \mathbb{Z}) \to H^{q+1}(N_i, \mathrm{Ind}_{H_i}^{\widetilde{N}_i} \mathbb{Z}). \tag{5-3}$$

Since $\mathrm{Ind}_{H_i}^{\widetilde{N}_i} \mathbb{Z} = \mathrm{Ind}_1^{N_i} \mathbb{Z}$ is an induced module, it follows from (5-3) that

$$H^2(N_i, \Lambda^1_{E_i}) \xrightarrow{\sim} H^3(N_i, \mathbb{Z}) = H^1(N_i, \mathbb{Z}) = 0. \tag{5-4}$$

Since the algebraic torus T^{K_i} splits over K_i, the H_i-module $\Lambda^1_{E_i} \simeq \mathbb{Z}^{|N_i|-1}$ is trivial and $H^1(H_i, \Lambda^1_{E_i}) = H^1(H_i, \mathbb{Z}^{|N_i|-1}) = 0$. By Proposition 5.2 with $(G, H) = (\widetilde{N}_i, H_i)$, we have the exact sequence

$$0 \to H^2(\widetilde{N}_i, \Lambda^1_{E_i}) \to H^2(H_i, \Lambda^1_{E_i})^{N_i} \xrightarrow{d_i} H^3(N_i, \Lambda^1_{E_i}).$$

Using the same argument as (5-4), we get $H^3(N_i, \Lambda^1_{E_i}) \simeq \mathrm{Hom}(N_i, \mathbb{Q}/\mathbb{Z})$. On the other hand,

$$H^2(H_i, \Lambda^1_{E_i}) \simeq H^2(H_i, \mathbb{Z})^{|N_i|-1} \simeq \mathrm{Hom}(H_i, \mathbb{Q}/\mathbb{Z})^{|N_i|-1}.$$

We now observe that the conjugation action of the group $N_i = \widetilde{N}_i/H$ on $\mathrm{Hom}(H_i, \mathbb{Q}/\mathbb{Z})^{|N_i|-1}$ is trivial. Consequently, we obtain an exact sequence

$$0 \to H^2(\widetilde{N}_i, \Lambda^1_{E_i}) \to \mathrm{Hom}((H_i^{\mathrm{ab}})^{|N_i|-1}, \mathbb{Q}/\mathbb{Z}) \xrightarrow{d_i} \mathrm{Hom}(N_i, \mathbb{Q}/\mathbb{Z}).$$

Let $v_i : N_i \to (H_i^{\mathrm{ab}})^{|N_i|-1}$ be the Pontryagin dual of d_i. Then $H^2(\widetilde{N}_i, \Lambda^1_{E_i}) = \mathrm{Ker}(v_i^\vee)$. □

Corollary 5.4. *Let the notation and assumptions be as in Proposition 5.3. Assume further that the orders of groups G_i are pairwise coprime. Then*

$$H^2(\Lambda^1) \simeq \bigoplus_{i=1}^r \mathrm{Hom}(H_i, \mathbb{Q}/\mathbb{Z})^{[K_i:E_i]-1}.$$

(*In our notation the group $H_1 = \{1\}$ if $r = 1$.*)

Proof. In this case, the maps d_i are all zero. □

Remark 5.5. It would be interesting to describe the map $v_i : N_i \to (H_i^{\mathrm{ab}})^{[K_i/E_i]-1}$ explicitly.

5B. $\tau(T)$. In this subsection we shall further assume that each subgroup N_i, besides being cyclic, is also normal in G_i. This assumption simplifies the description of the group $H^2(\mathbb{Z})'$ through Mackey's formula. Let $\mathfrak{D}_i$ be the decomposition subgroups of G_i for each i.

Lemma 5.6. (1) *The inclusion $\prod_{i=1}^r H^2(G_i, \mathbb{Z})' \subset H^2(\mathbb{Z})'$ holds.*

(2) *Assume that for any $1 \le i \le r$ and $D_i' \in \mathfrak{D}_i$, there exists a member $D \in \mathfrak{D}$ such that $\mathrm{pr}_i(D) = D_i'$ and a section $s_i : D_i' \to D$ of $\mathrm{pr}_i : D \twoheadrightarrow D_i'$ such that the composition $\mathrm{pr}_k \circ s_i : D_i' \to D \to G_k$ is trivial for $k \ne i$. Then $\prod_{i=1}^r H^2(G_i, \mathbb{Z})' = H^2(\mathbb{Z})'$.*

Proof. (1) For each $f = (f_1, \dots, f_r) \in \mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z}) = \prod_{i=1}^r \mathrm{Hom}(G_i, \mathbb{Q}/\mathbb{Z})$, consider the following two conditions:

(a) For each $D \in \mathfrak{D}$, the restriction $f|_D$ of f to D lies in the image of the map $\sum_{i=1}^r \mathrm{Ver}_{D,\bar{D}_i}^\vee$.

(b) For each $1 \le i \le r$ and $D_i' \in \mathfrak{D}_i$, the restriction $f_i|_{D_i'}$ to D_i' lies in the image of the map $\mathrm{Ver}_{D_i', D_i' \cap N_i}^\vee$.

By Proposition 4.6, we have

$$H^2(G, \mathbb{Z})' = \{f \in G^{\mathrm{ab},\vee} : f|_{D^{\mathrm{ab}}} \in \mathrm{Im}(\mathrm{Ver}_{D,\bar{\boldsymbol{D}}}^\vee) \forall D \in \mathfrak{D}\}$$

and it is equivalent to show that the condition (b) implies (a).

Let $D = D_{\mathfrak{P}} \in \mathfrak{D}$ be a decomposition group associated to a prime $\mathfrak{P}$ of L. By definition $D_i = D \cap \widetilde{N}_i$ and $\bar{D}_i = D_i / D_i \cap H_i$. The projection map $\mathrm{pr}_i : G \to G_i$ sends each element σ to its restriction $\sigma|_{K_i}$ to K_i. Let $\mathfrak{P}_i$ denote the prime of K_i below $\mathfrak{P}$ and we have $\mathrm{pr}_i(D_{\mathfrak{P}}) = D_{\mathfrak{P}_i}$. We show that $\mathrm{pr}_i(D_{\mathfrak{P}} \cap \widetilde{N}_i) = D_{\mathfrak{P}_i} \cap N_i$. The inclusion $\subseteq$ is obvious because $\mathrm{pr}_i(D_{\mathfrak{P}} \cap \widetilde{N}_i) \subset \mathrm{pr}_i(D_{\mathfrak{P}_i}) \cap \mathrm{pr}_i(\widetilde{N}_i) = D_{\mathfrak{P}_i} \cap N_i$. For the other inclusion, since $\mathrm{pr}_i : D_{\mathfrak{P}} \to D_{\mathfrak{P}_i}$ is surjective, for each $y \in D_{\mathfrak{P}_i} \cap N_i$ there exists an element $x = (x_i) \in D_{\mathfrak{P}}$ such that $x_i = y$. Since $x_i \in N_i$, x also lies in $\widetilde{N}_i$. This proves the other inclusion. Therefore, $\bar{D}_i = D_{\mathfrak{P}_i} \cap N_i$.

The map $\sum_{i=1}^r \mathrm{Ver}_{D,\bar{D}_i}^\vee$ is dual to the map

$$\prod_{i=1}^r \mathrm{Ver}_{D, D_{\mathfrak{P}_i} \cap N_i} : D \to \prod_{i=1}^r (D_{\mathfrak{P}_i} \cap N_i).$$

Since each $\mathrm{Ver}_{D, D_{\mathfrak{P}_i} \cap N_i} = \mathrm{Ver}_{D_{\mathfrak{P}_i}, D_{\mathfrak{P}_i} \cap N_i} \circ \mathrm{pr}_i$ (3-4), the above map factorizes into the following composition

$$D \xrightarrow{(\mathrm{pr}_i)_i} \prod_{i=1}^r D_{\mathfrak{P}_i} \xrightarrow{\prod_i \mathrm{Ver}_{D_{\mathfrak{P}_i}, D_{\mathfrak{P}_i} \cap N_i}} \prod_{i=1}^r (D_{\mathfrak{P}_i} \cap N_i).$$

Consider the restriction $f|_{\prod_i D_{\mathfrak{P}_i}} = (f_i|_{D_{\mathfrak{P}_i}})$ of f to $\prod_i D_{\mathfrak{P}_i}$, and put $D_i' := D_{\mathfrak{P}_i}$. By condition (b), there exists an element $h_i \in \mathrm{Hom}(D_{\mathfrak{P}_i} \cap N_i, \mathbb{Q}/\mathbb{Z})$ such that $f_i|_{D_{\mathfrak{P}_i}} = \mathrm{Ver}_{D_{\mathfrak{P}_i}, D_{\mathfrak{P}_i} \cap N_i}^\vee(h_i)$. Then

$$f|_{\prod_i D_{\mathfrak{P}_i}} = (\mathrm{Ver}_{D_{\mathfrak{P}_i}, D_{\mathfrak{P}_i} \cap N_i}^\vee(h_i)).$$

Restricting this function to D, we obtain $f|_D = \sum_{i=1}^r \mathrm{Ver}^\vee_{D,\bar{D}_i}(h_i)$. This shows that $f|_D$ lies in the image of $\sum_{i=1}^r \mathrm{Ver}^\vee_{D,\bar{D}_i}$.

(2) By (1), we have $\prod_{i=1}^r H^2(G_i,\mathbb{Z})' \subset H^2(\mathbb{Z})'$. To prove the other inclusion, we must show that each f_i satisfies condition (b) provided f satisfies condition (a), which we assume from now on. For $1 \le i \le r$, let $D \in \mathcal{D}$ be a decomposition group of G over D'_i and $s_i : D'_i \to D$ be a section such that $\mathrm{pr}_k \circ s_i$ is trivial for $k \ne i$. Then (a) implies

$$f|_D = \sum_{k=1}^r \mathrm{Ver}^\vee_{D,\bar{D}_k}(h_k)$$

for some $h_k \in \mathrm{Hom}(\bar{D}_k, \mathbb{Q}/\mathbb{Z})$. Pulling back to D'_i via s_i, we have

$$f_i|_{D'_i} = s_i^* f|_D = \sum_{k=1}^r s_i^* \mathrm{Ver}^\vee_{D,\bar{D}_k}(h_k) = \sum_{k=1}^r s_i^* \mathrm{pr}_k^* \mathrm{Ver}^\vee_{D'_k,\bar{D}_k}(h_k),$$

where $D'_k := \mathrm{pr}_k(D)$ if $k \ne i$. Since $s_i^* \mathrm{pr}_k^* = 1$ if $k = i$ and $s_i^* \mathrm{pr}_k^* = 0$ otherwise, we get $f_i|_{D'_i} = \mathrm{Ver}^\vee_{D'_i,\bar{D}_i}(h_i)$. □

Proposition 5.7. *With the notation and assumptions be as above. Suppose further that for each* $1 \le i \le r$, *every decomposition group of* G_i *is cyclic. Then we have* $\tau(T) = \prod_{i=1}^r \tau(T_i)$, *where* $T_i = T^{K_i/E_i,k}$ (*see* (5-2)).

Proof. We shall show that the assumption in Lemma 5.6(2) holds. Let $D'_i \in \mathcal{D}_i$ be a decomposition group of G_i; by our assumption D'_i is cyclic and let a_i be a generator. Let D be a cyclic subgroup of G generated by $\tilde{a}_i = (1, \dots, 1, a_i, 1, \dots, 1)$ with a_i at the i-th place. Clearly D is the decomposition group of some prime and let $s_i : D'_i \to D$ be the section sending a_i to $\tilde{a}_i$. Clearly for $k \ne i$, $\mathrm{pr}_k \circ s_i : D'_i \to D'_k$ is trivial. By Lemma 5.6, we have $H^2(G,\mathbb{Z})' = \prod_{i=1}^r H^2(G_i,\mathbb{Z})'$.

By Proposition 4.3, we have

$$\tau(T) = \frac{\prod_{i=1}^r |N_i|}{|H^2(\mathbb{Z})'|} \quad \text{and} \quad \tau(T_i) = \frac{|N_i|}{|H^2(G_i,\mathbb{Z})'|}.$$

Therefore, $\tau(T) = \prod_{i=1}^r \tau(T_i)$. □

6. Galois CM fields

We return to the case of CM tori and keep the notation of Section 2C. In this section, we assume that K is a Galois CM field. Let $G = \mathrm{Gal}(K/\mathbb{Q})$, $G^+ := \mathrm{Gal}(K^+/\mathbb{Q})$ and $g := [K^+ : \mathbb{Q}]$. Then one has the short exact sequence

$$1 \to \langle\iota\rangle \to G \to G^+ \to 1. \tag{6-1}$$

Recall the $\mathbb{Q}$-torus $T^{K,\mathbb{Q}}$ (2-6) is the CM torus over $\mathbb{Q}$ associated to the CM field K. We set $T := T^{K,\mathbb{Q}}$.

6A. ***Cyclotomic extensions.***

Proposition 6.1. *Let $K = \mathbb{Q}(\zeta_n) \neq \mathbb{Q}$ be the n-th cyclotomic field with n odd or $4 \mid n$, and T the associated CM torus over $\mathbb{Q}$*:

(1) *If n is either a power of an odd prime p or $n = 4$, then $\tau(T) = 1$.*

(2) *In other cases*, $\tau(T) = 2$.

Proof. (1) In this case the Galois group $G = \mathrm{Gal}(K/\mathbb{Q})$ is cyclic and Proposition 2.9(3) shows that $\tau(T) = 1$.

(2) Suppose first that n is not a power of a prime. By Proposition 2.7(2) the global norm index n_K divides $\prod_{p \in S_{K/K^+}} e_{T,p}$. By [Washington 1997, Proposition 2.15], the quadratic extension K/K^+ is unramified at all finite places of K^+. It follows that $S_{K/K^+} = \varnothing$, and hence that $n_K = 1$. Proposition 2.7(1) then shows $\tau(T) = 2$. Now suppose the other case that $n = 2^v$ with $v \geq 3$. Then K contains $\mathbb{Q}(\zeta_8) = \mathbb{Q}(\sqrt{2}, \sqrt{-1})$, which contains two distinct imaginary quadratic fields. Thus, by Lemma 2.8, we have $n_K = 1$ and hence $\tau(T) = 2$ by Proposition 2.7(1). This completes the proof. □

6B. ***Abelian extensions.***

Proposition 6.2. *Assume that $G = \mathrm{Gal}(K/\mathbb{Q})$ is abelian*:

(1) *We have $\tau(T) \in \{1, 2\}$.*

(2) *If g is odd, then $\tau(T) = 1$.*

(3) *If g is even and* (6-1) *splits, then $\tau(T) = 2$.*

Proof. (1) Write $G = \prod_{i=1}^{\ell} G_i$ as a product of cyclic subgroups G_i. Then there exists i such that the image of ι in the projection $G \to G_i$ is nontrivial, say $i = 1$. Let K_1 be a subfield of K with $\mathrm{Gal}(K_1/\mathbb{Q}) = G_1$, which is a cyclic CM field. Then we have $\tau(T^{K_1,\mathbb{Q}}) = 1$ by Proposition 2.9(3), that is, $n_{K_1} = 2$. Since $K \supset K_1$, we have $n_K \mid n_{K_1}$ and $n_K \in \{1, 2\}$. Therefore, $\tau(T) \in \{1, 2\}$.

(2) This follows from Proposition 2.9.

(3) Since (6-1) splits, $G \simeq \langle \iota \rangle \times G^+$. As g is even, there is an epimorphism

$$\langle \iota \rangle \times G^+ \twoheadrightarrow \langle \iota \rangle \times \mathbb{Z}/2\mathbb{Z}. \tag{6-2}$$

In particular, K contains two distinct imaginary quadratic fields; therefore, $n_K = 1$ and $\tau(T) = 2$ by Lemma 2.8. □

6C. ***Certain Galois extensions.***

Proposition 6.3. *Assume the short exact sequence* (6-1) *splits. Let $g^{\mathrm{ab}} := |G^{+\mathrm{ab}}|$ and $\Lambda := X(T)$. Then $\tau(T) \in \{1, 2\}$. Moreover, the following statements hold*:

(1) *When g is odd*, $\tau(T)=1$.

(2) *When g is even*:

(i) *If g^{ab} is even, then* $\tau(T)=2$.

(ii) *If g^{ab} is odd, then there is a unique nonzero element ξ in the 2-torsion subgroup $H^2(\Lambda)[2]$ of $H^2(\Lambda)$. Moreover, $\tau(T)=1$ if and only if its restriction $r_D(\xi)=0$ in $H^2(D,\Lambda)$ for all $D\in\mathcal{D}$.*

Proof. Since (6-1) splits, $G\simeq\langle\iota\rangle\times G^+$. It follows that K contains an imaginary quadratic field E. Since $n_E=2$, one has $n_K\in\{1,2\}$ and $\tau(T)\in\{1,2\}$. The statement (1) follows from Proposition 2.9(2).

(2) We have $G\simeq\langle\iota\rangle\times G^+$ and $G^{\mathrm{ab}}\simeq\langle\iota\rangle\times G^{+\mathrm{ab}}$. (i) Suppose g^{ab} is even. As (6-2), K contains two distinct imaginary quadratic fields. Thus, $n_K=1$ and $\tau(T)=2$. (ii) Suppose g^{ab} is odd. Put $\Lambda^1=X(T^{K,1})$ and we have the following exact sequence

$$0\to H^1(\Lambda)\to H^1(\Lambda^1)\to H^2(\mathbb{Z})\to H^2(\Lambda)\to H^2(\Lambda^1).$$

We have $H^1(\Lambda^1)\simeq\mathbb{Z}/2\mathbb{Z}$ by Proposition 4.1(1), $H^2(\Lambda^1)=0$ by Proposition 5.3, and $H^1(\Lambda)\simeq\mathbb{Z}/2\mathbb{Z}$ by [Achter et al. 2023, Proposition A.6]. This gives $H^2(\mathbb{Z})\cong H^2(\Lambda)$. The proof of [loc. cit., Lemma A.7] shows that $H^1(G,T)$ is a 2-torsion group. It follows that $Ш^2(\Lambda)$ is also a 2-torsion group. Therefore, $Ш^2(\Lambda)\subset H^2(\Lambda)[2]$. Since g^{ab} is odd,

$$H^2(\Lambda)[2]\simeq H^2(\mathbb{Z})[2]=\mathrm{Hom}(G^{\mathrm{ab}},\mathbb{Q}/\mathbb{Z})[2]=\mathrm{Hom}(\langle\iota\rangle\times G^{+\mathrm{ab}},\mathbb{Q}/\mathbb{Z})[2]\cong\mathbb{Z}/2\mathbb{Z}.$$

Let $\xi\in H^2(\Lambda)[2]$ be the unique nonzero element. Then

$$\tau(T)=1\iff Ш^2(\Lambda)\simeq\mathbb{Z}/2\mathbb{Z}\iff\xi\in Ш^2(\Lambda)\iff r_D(\xi)=0\quad\text{for all } D\in\mathcal{D}.$$

This completes the proof of the proposition. □

Remark 6.4. When G is nonabelian and the short exact sequence (6-1) does not split, we do not know the value of $\tau(T)$ in general. However, if the involution ι is nontrivial on the maximal abelian extension K_{ab} over $\mathbb{Q}$ in K, then K_{ab} is a CM abelian field and it follows from Proposition 6.2 that $\tau(T)\in\{1,2\}$.

It remains to determine the Tamagawa number when G is nonabelian, (6-1) is nonsplit, and K_{ab} is totally real. This includes the cases of $G=Q_8$ and the dihedral groups, for which we treat in the next subsections.

6D. *Q_8-extensions.* The quaternion group $Q_8=\{\pm1,\pm i,\pm j,\pm k\}$ is the group of 8 elements generated by i,j with usual relations $i^2=j^2=-1$ and $ij=-ji=k$. We have the following well-known properties of Q_8.

Lemma 6.5. (1) *The group Q_8 contains 6 elements of order 4, one element of order 2, and the identity. It does not have a subgroup isomorphic to $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$. Thus, every proper subgroup is cyclic.*

(2) *Every nontrivial subgroup contains $\{\pm1\}$, and only subgroup which maps onto $Q_8/\{\pm1\}$ is Q_8.*

(3) *The center $Z(Q_8)=\{\pm1\}=D(Q_8)$, where $D(Q_8)=[Q_8,Q_8]$.*

Proposition 6.6. *Let P and Q be two odd positive integers such that*

$$P-1=a^2, \quad Q-1=Pb^2$$

for some integers $a, b \in \mathbb{N}$. Assume that Q is not a square. Let $K := \mathbb{Q}(\beta)$ be the simple extension of $\mathbb{Q}$ generated by β which satisfies $\beta^2 = \alpha := -(P+\sqrt{P})(Q+\sqrt{Q})$. Then K is a Galois CM field with Galois group Q_8 with maximal totally real field $K^+ = \mathbb{Q}(\sqrt{P}, \sqrt{Q})$. The Galois group $\mathrm{Gal}(K/\mathbb{Q})$ *is generated by τ_1 and τ_2 given by*

$$\tau_1(\beta) = \frac{(\sqrt{P}-1)}{a}\beta, \quad \tau_2(\beta) = \frac{(\sqrt{Q}-1)}{\sqrt{P}a}\beta. \tag{6-3}$$

Moreover, for each prime ℓ, the decomposition group D_ℓ is cyclic except when $\ell \mid Q$. For $\ell \mid Q$, one has

$$D_\ell = \begin{cases} \mathbb{Z}/4\mathbb{Z} & \text{if } \left(\frac{P}{\ell}\right) = 1; \\ Q_8 & \text{if } \left(\frac{P}{\ell}\right) = -1. \end{cases}$$

Proof. Note that $\gcd(P, Q) = 1$ and P is not a square. Then $\mathbb{Q}(\sqrt{P}, \sqrt{Q})$ is a totally real biquadratic field and its Galois group is generated by σ_1 and σ_2, where

$$\sigma_1 : \begin{cases} \sqrt{P} \mapsto -\sqrt{P}, \\ \sqrt{Q} \mapsto \sqrt{Q}, \end{cases} \qquad \sigma_2 : \begin{cases} \sqrt{P} \mapsto \sqrt{P}, \\ \sqrt{Q} \mapsto -\sqrt{Q}. \end{cases}$$

It is clear that α is a totally negative, and hence K is a CM field. The maximal totally real subfield $K^+ = \mathbb{Q}(\alpha) \subset \mathbb{Q}(\sqrt{P}, \sqrt{Q})$. Since the minimal polynomial of α is of degree 4, one has $K^+ = \mathbb{Q}(\sqrt{P}, \sqrt{Q})$. Let $\tau_i \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, $i = 1, 2$, be elements such that $\tau_i|_{K^+} = \sigma_i$. One computes

$$\frac{\tau_1(\beta)^2}{\beta^2} = \frac{\tau_1(\alpha)}{\alpha} = \frac{-\sigma_1(P+\sqrt{P})\sigma_1(Q+\sqrt{Q})}{-(P+\sqrt{P})(Q+\sqrt{Q})} = \frac{(P-\sqrt{P})^2}{P^2-P} = \frac{(\sqrt{P}-1)^2}{P-1} = \frac{(\sqrt{P}-1)^2}{a^2},$$

and obtains

$$\tau_1(\beta) = \pm\frac{(\sqrt{P}-1)}{a}\beta \quad \text{and} \quad \tau_1(K) \subset K.$$

Similarly, one computes $\tau_2(\beta)^2/\beta^2 = (\sqrt{Q}-1)^2/Pb^2$ and obtains

$$\tau_2(\beta) = \pm\frac{(\sqrt{Q}-1)}{\sqrt{P}a}\beta \quad \text{and} \quad \tau_2(K) \subset K.$$

It follows that $K/\mathbb{Q}$ is Galois. Let $\tau_1, \tau_2 \in \mathrm{Gal}(K/\mathbb{Q})$ be defined as in (6-3). One easily computes $\tau_1^2(\beta) = -\beta$, $\tau_2^2(\beta) = -\beta$, $\tau_1\tau_2(\beta) = \iota\tau_2\tau_1(\beta)$ and obtains the relations

$$\tau_1^4 = \tau_2^4 = 1, \quad \tau_1^2 = \tau_2^2, \quad \tau_1\tau_2\tau_1^{-1} = \tau_2^{-1},$$

showing that K is a Q_8-CM field.

Denote by D_ℓ^+ the decomposition group at ℓ in $G^+ = \mathrm{Gal}(K^+/\mathbb{Q})$. Then $D_\ell = Q_8$ if and only if $D_\ell^+ = G^+$. If $\ell \nmid PQ$, then ℓ is unramified in K^+ and D_ℓ^+ is cyclic. Thus, D_ℓ^+ cannot be G^+, $D_\ell \neq Q_8$ and D_ℓ is cyclic. If $\ell \mid P$, then $\left(\frac{Q}{\ell}\right) = \left(\frac{1}{\ell}\right) = 1$. So ℓ is ramified in $\mathbb{Q}(\sqrt{P})$ and splits in $\mathbb{Q}(\sqrt{Q})$. Thus,

$D_\ell^+ \neq G^+$ and hence D_ℓ is cyclic. It remains to treat the case $\ell \mid Q$. If $(\frac{P}{\ell}) = 1$, then $D_\ell^+ \simeq \mathbb{Z}/2\mathbb{Z}$ and hence $D_\ell \simeq \mathbb{Z}/4\mathbb{Z}$. Otherwise, ℓ is ramified in $\mathbb{Q}(\sqrt{Q})$ and inert in $\mathbb{Q}(\sqrt{P})$. One has $D_\ell^+ = G^+$ and $D_\ell = Q_8$. □

Proposition 6.7. *Let K be a CM field which is Galois over $\mathbb{Q}$ with Galois group Q_8. Then*

$$\tau(T^{K,\mathbb{Q}}) = \begin{cases} \frac{1}{2} & \text{if every decomposition group of } K/\mathbb{Q} \text{ is cyclic;} \\ 2 & \text{otherwise.} \end{cases} \tag{6-4}$$

Proof. Put $G = \mathrm{Gal}(K/\mathbb{Q}) \simeq Q_8$ and $N = \langle \iota \rangle$. We shall show that

$$H^1(\Lambda) = H^1(G, \Lambda) = \mathbb{Z}/2\mathbb{Z} \tag{6-5}$$

and

$$Ш^2(\Lambda) = \begin{cases} \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if every decomposition group of } K/\mathbb{Q} \text{ is cyclic;} \\ 0 & \text{otherwise.} \end{cases} \tag{6-6}$$

Then (6-4) follows from (6-5) and (6-6).

By Proposition 4.1 and (4-20), we have an exact sequence

$$0 \to H^1(\Lambda) \to H^1(\Lambda^1) \simeq N^{\mathrm{ab},\vee} \xrightarrow{\mathrm{Ver}_{G,N}^\vee} H^2(\mathbb{Z}) = G^{\mathrm{ab},\vee} \to H^2(\Lambda) \to 0, \tag{6-7}$$

noting that $H^2(G, \Lambda^1) = 0$ when $K/\mathbb{Q}$ is a Galois CM field. Since the 2-Sylow subgroup of Q_8 is not cyclic, the transfer $\mathrm{Ver}_{G,N}$ is not surjective by Proposition 3.3. Therefore, $\mathrm{Ver}_{G,N}^\vee$ is not injective and is zero. This proves $H^1(\Lambda) \simeq \mathbb{Z}/2\mathbb{Z}$ and $H^2(\mathbb{Z})' \simeq Ш^2(\Lambda)$ from (4-24).

If there is a finite place of K whose decomposition group is not cyclic, then we have $Ш^2(\Lambda) = 0$. Therefore, $\tau(T^{K,\mathbb{Q}}) = 2$. On the other hand, suppose that every decomposition group of $K/\mathbb{Q}$ is cyclic. One has $G^{\mathrm{ab}} = G/N \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Recall

$$H^2(\mathbb{Z})' = \{f \in \mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z}) : (*)\ f|_D \in \mathrm{Im}\,\mathrm{Ver}_{D,D\cap N}^\vee, \forall D \in \mathcal{D}\}.$$

Note that $f|_N = 0$, so the restriction to D map $f|_D$ is contained in $\mathrm{Hom}(D/N, \mathbb{Q}/\mathbb{Z})$ whenever $N \subset D$ (also noting that $N \subset D$ if D is not trivial). So if $D = 0$ or $D \simeq \mathbb{Z}/2\mathbb{Z}$, then $f|_D = 0$ and condition $(*)$ above is satisfied automatically. Suppose $D \simeq \mathbb{Z}/4\mathbb{Z}$. Since the 2-Sylow subgroup of D is cyclic, the map $\mathrm{Ver}_{D,N}^\vee : N^\vee \to D^\vee$ is injective and $\mathrm{Im}\,\mathrm{Ver}_{D,N}^\vee$ the unique subgroup of $D^\vee$ of order 2. Then $f|_D \in \mathrm{Hom}(D/N, \mathbb{Q}/\mathbb{Z}) = \mathrm{Im}\,\mathrm{Ver}_{D,N}^\vee$ and condition $(*)$ is also satisfied automatically. Therefore, $Ш^2(\Lambda) \simeq H^2(\mathbb{Z})' = H^2(\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. This proves the proposition. □

Corollary 6.8. *Let P and Q be two odd positive integers, let $\alpha := -(P + \sqrt{P})(Q + \sqrt{Q})$ and let $K = \mathbb{Q}(\sqrt{\alpha})$ be the Galois CM with group Q_8 as in Proposition 6.6. Then*

$$\tau(T^{K,\mathbb{Q}}) = \begin{cases} \frac{1}{2} & \text{if } \left(\frac{P}{q}\right) = 1 \text{ for all primes } q \mid Q; \\ 2 & \text{otherwise.} \end{cases}$$

Proof. By Proposition 6.6, the cyclicity of all decomposition groups of $K/\mathbb{Q}$ is equivalent to the condition $\left(\frac{P}{q}\right) = 1$ for all primes $q \mid Q$. Hence the assertion follows from Proposition 6.7. □

6E. ***The dihedral case.*** Let D_n denote the dihedral group of order $2n$.

Lemma 6.9. *Suppose the CM-field K is Galois over $\mathbb{Q}$ with Galois group G, and let*

$$S := \{\sigma \in G : \iota \notin \langle \sigma \rangle\}.$$

If $|S| \geq |G|/2$, then $n_K \leq 2$. If the inequality is strict, then $n_K = 1$.

Proof. This is due to Jiangwei Xue. Let $L/\mathbb{Q}$ be the class field corresponding to the open subgroup $\mathbb{Q}^\times N(T(\mathbb{A})) \subset \mathbb{A}^\times$ by class field theory. Then $n_K = [\mathbb{A}^\times : \mathbb{Q}^\times N_{L/\mathbb{Q}}(\mathbb{A}_L^\times)] = [L : \mathbb{Q}]$. Suppose p is a rational prime unramified in $K/\mathbb{Q}$ such that the Artin symbol $(p, K/\mathbb{Q})$ lies in S. Since p splits completely in the fixed field $E = K^{D_p}$ of the decomposition group $D_p = \langle (p, K/\mathbb{Q}) \rangle$ of G at p and $\langle \iota \rangle \cap D_p = \{1\}$ (by the definition of S), one has $K = K^+E$ and that every prime v of K^+ lying above p splits in K, and therefore $\mathbb{Q}_p^\times \subset N(T(\mathbb{A})) \subset Q^\times N_{L/\mathbb{Q}}(\mathbb{A}_L^\times)$. It follows from class field theory that p splits completely in L. Thus, the density of S is less than or equal to that of primes splitting completely in L. By the Chebotarev density theorem, we obtain $|S|/|G| \leq 1/[L : \mathbb{Q}]$. Therefore, $n_K \leq |G|/|S|$ and the assertions then follow immediately. □

Proposition 6.10. *Let K be a Galois CM field with group $G = D_n$ and T the associated CM torus over $\mathbb{Q}$. Then n is even and $\tau(T) = 2$.*

Proof. Write $D_n = \langle t, s : t^n = s^2 = 1, sts = t^{-1} \rangle$. One easily sees that the center $Z(D_n) = \{x \in \langle t \rangle : x^2 = 1\}$ contains an element of order 2 if and only if $n = 2m$ is even. Since ι is central of order 2 in D_n, n is even. In this case $|S| = 2m + 1$ and $n_K = 1$ by Lemma 6.9. Therefore, $\tau(T) = 2$. □

Remark 6.11. The criterion in Lemma 6.9 does not help to compute $\tau(T)$ for the case $G = Q_8$ or $G = \mathbb{Z}/2^n\mathbb{Z}$. However, these cases have been treated in Propositions 6.7 and 2.9, respectively.

7. Some nonsimple CM cases

Keep the notation in Section 2C. Write $N_i := \mathrm{Gal}(K_i/K_i^+) = \langle \iota_i \rangle$ with involution ι_i on K_i/K_i^+ and $\iota_i^* \in N_i^\vee = \mathrm{Hom}(N_i, \mathbb{Q}/\mathbb{Z})$ for the unique nontrivial element, that is $\iota_i^*(\iota) = \frac{1}{2} \bmod \mathbb{Z}$. Fix a Galois splitting field L of $T = T^{K,\mathbb{Q}}$ and put $G = \mathrm{Gal}(L/\mathbb{Q})$. For a number field F, denote by $\Sigma_F := \mathrm{Hom}_{\mathbb{Q}}(F, \mathbb{C}) = \mathrm{Hom}_{\mathbb{Q}}(F, \overline{\mathbb{Q}})$ the set of embeddings of F into $\mathbb{C}$. The Galois group $G_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on Σ_F by $\sigma \cdot \phi = \sigma \circ \phi$ for $\sigma \in G_{\mathbb{Q}}$ and $\phi \in \Sigma_F$. Let Φ_i be a CM type of K_i/K_i^+, that is, $\Phi_i \cap (\Phi_i \circ \iota_i) = \varnothing$ and $\Phi_i \cup (\Phi_i \circ \iota_i) = \Sigma_{K_i}$, and for each $g \in G$, set $\Phi_i(g) := \{\phi \in \Phi_i : g\phi \notin \Phi_i\}$.

Lemma 7.1. *Let K be a CM algebra and $T = T^{K,\mathbb{Q}}$ the associated CM torus over $\mathbb{Q}$. Then the map*

$$\bigoplus_{i=1}^{r} \mathrm{Ver}_{G,N_i}^\vee : \bigoplus_{i=1}^{r} N_i^\vee \to G^{\mathrm{ab},\vee} = \mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z})$$

sends each element $\sum_{i=1}^r a_i \iota_i^*$, for $a_i \in \mathbb{Z}$, to the element $f = \sum_{i=1}^r a_i f_i$, where $f_i \in \mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z})$ is the function on G given by $f_i(g) = |\Phi_i(g)|/2 \bmod \mathbb{Z}$.

Proof. We first describe the transfer map $\mathrm{Ver}_{G,N_i}: G \to N_i$. We may regard the CM fields K_i as subfields of $\overline{\mathbb{Q}}$, and put $\widetilde{X}_i := G/H_i = \Sigma_{K_i}$ and $X_i := G/\widetilde{N}_i = \Sigma_{K_i^+}$. Fix a section $\widetilde{\varphi}: X_i \to G$ such that the induced section $\varphi: X_i \to \widetilde{X}_i$ has image Φ_i. Since $N_i = \widetilde{N}_i/H_i$ is abelian, modulo $D(\widetilde{N}_i)H_i$ is the same as modulo H_i. Following the definition of Ver_{G,N_i}, for each $g \in G$,

$$\mathrm{Ver}_{G,N_i}(g) = \prod_{x \in X_i} n^{\widetilde{\varphi}}_{g,x} \mod H_i = \iota_i^{n_i(g)} \in N_i,$$

where the element $n^{\widetilde{\varphi}}_{g,x} \in \widetilde{N}_i$ is defined in Section 3D and $n_i(g) := |\{x \in X_i : n^{\widetilde{\varphi}}_{g,x} \notin H_i\}|$. Let $f_i := \mathrm{Ver}^{\vee}_{G,N_i}(\iota_i^*)$. Then (Lemma 3.8)

$$f_i(g) = \iota_i^*(\mathrm{Ver}_{G,N_i}(g)) = \iota_i^*(\iota_i^{n_i(g)}) = n_i(g)/2 \in \mathbb{Q}/\mathbb{Z}.$$

Thus, it remains to show $n_i(g) = |\Phi_i(g)|$. Let $\widetilde{\Phi}_i := \widetilde{\varphi}(\Sigma_{K_i^+})$. Then we have bijections $\widetilde{\Phi}_i \xrightarrow{\sim} \Phi_i \xrightarrow{\sim} \Sigma_{K_i^+}$. Let $x \in \Sigma_{K_i^+}$, and let $\widetilde{\phi} \in \widetilde{\Phi}_i$ and $\phi \in \Phi_i$ be the corresponding elements. Put $\widetilde{\phi}' := \widetilde{\varphi}(gx)$ and $\phi' = \varphi(gx)$, the image of $\widetilde{\phi}'$. Then $g\widetilde{\phi}$ and $\widetilde{\phi}'$ are elements lying over gx and we have $g\widetilde{\phi} = \widetilde{\phi}' n^{\widetilde{\varphi}}_{g,x}$. So $g\phi = \phi'$ if and only if $n^{\widetilde{\varphi}}_{g,x} \in H_i$. On the other hand, since $\phi' \in \Phi_i$ and two elements $g\phi$ and ϕ' are lying over the same element gx, we have

$$g\phi \notin \Phi_i \iff g\phi \neq \phi' \iff n^{\widetilde{\varphi}}_{g,x} \notin H_i.$$

Therefore, the bijection $\Phi_i \xrightarrow{\sim} \Sigma_{K_i^+}$ gives the bijection of subsets

$$\{\phi \in \Phi_i : g\phi \notin \Phi_i\} \xrightarrow{\sim} \{x \in \Sigma_{K_i^+} : n^{\widetilde{\varphi}}_{g,x} \notin H_i\}.$$

This shows the desired equality $|\Phi_i(g)| = n_i(g)$. □

By Proposition 4.1 and Lemma 7.1, we give an independent proof of the following result of Li and Rüd [Achter et al. 2023, Proposition A.11].

Corollary 7.2. *Let K and T be as in Lemma 7.1 and $\Lambda = X(T)$ be the character group of T. Then*

$$H^1(\Lambda) \simeq \left\{(a_i) \in \{\pm 1\}^r : \sum_{1 \le i \le r, a_i = -1} |\Phi_i(g)| \in 2\mathbb{Z}, \forall g \in G\right\}. \tag{7-1}$$

Proof. Indeed, after making the identity $\oplus_{i=1}^r N_i^{\vee} \simeq (\mathbb{Z}/2\mathbb{Z})^r$ with $\{\pm 1\}^r$, by Lemma 7.1 the map

$$\oplus_{i=1}^r \mathrm{Ver}^{\vee}_{G,N_i}: \oplus_{i=1}^r N_i^{\vee} = \{\pm 1\}^r \to \mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z})$$

sends (a_i) to the function f with

$$f(g) = \sum_{1 \le i \le r, a_i = -1} |\Phi_i(g)|/2 \mod \mathbb{Z}.$$

Thus, $f = 0$ precisely when

$$\sum_{1 \le i \le r, a_i = -1} |\Phi_i(g)| \in 2\mathbb{Z}, \quad \forall g \in G,$$

and (7-1) follows from Proposition 4.1. □

Proposition 7.3. *Let K be a CM algebra and T the associated CM torus over $\mathbb{Q}$. Then there is an isomorphism $\mathbb{A}^\times/\mathbb{Q}^\times N(T(\mathbb{A})) \simeq H^2(\mathbb{Z})'^\vee$.*

Proof. By [Rosengarten 2018, Theorem 1.2.9], we have the following commutative diagram with row exact sequence:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T(\mathbb{Q})_{\text{pro}} & \longrightarrow & T(\mathbb{A})_{\text{pro}} & \longrightarrow & H^2(\mathbb{Q}, \widehat{T})^\vee & \longrightarrow & Ш^1(T) \longrightarrow 0 \\ & & \downarrow N & & \downarrow N & & \downarrow \widehat{N}^\vee & & \downarrow \\ 0 & \longrightarrow & (\mathbb{Q}^\times)_{\text{pro}} & \longrightarrow & (\mathbb{A}^\times)_{\text{pro}} & \longrightarrow & H^2(\mathbb{Q}, \mathbb{Z})^\vee & \longrightarrow & 0 \end{array}$$

where A_{pro} denotes the profinite completion of an abelian group A. It follows from the Poitou–Tate duality and class field theory that:

$$\begin{array}{ccc} (T(\mathbb{A})/T(\mathbb{Q}))_{\text{pro}} & \twoheadrightarrow & (H^2(\mathbb{Q}, \widehat{T})/Ш^2(\mathbb{Q}, \widehat{T}))^\vee \\ \downarrow N & & \downarrow \widehat{N}^\vee \\ (\mathbb{A}^\times/\mathbb{Q}^\times)_{\text{pro}} & \xrightarrow{\sim} & H^2(\mathbb{Q}, \mathbb{Z})^\vee \end{array}$$

By definition $H^2(\mathbb{Q}, \mathbb{Z})' = \operatorname{Ker}(\widehat{N} : H^2(\mathbb{Q}, \mathbb{Z}) \to H^2(\mathbb{Q}, \widehat{T})/Ш^2(\mathbb{Q}, \widehat{T}))$, so we have

$$\operatorname{Coker} N = (T(\mathbb{A})/T(\mathbb{Q})N(T(\mathbb{A})))_{\text{pro}} \xrightarrow{\sim} \operatorname{Coker} \widehat{N}^\vee = H^2(\mathbb{Q}, \mathbb{Z})'^\vee.$$

Since $T(\mathbb{A})/T(\mathbb{Q})N(T(\mathbb{A}))$ is finite, it is equal to its profinite completion. This proves the proposition. □

Remark 7.4. Proposition 7.3 gives a cohomological interpretation of the group $\mathbb{A}^\times/\mathbb{Q}^\times N(T(\mathbb{A}))$. This is reminiscent of the main theorem $\mathbb{A}_k^\times/N_{K/k}(\mathbb{A}_K^\times)k^\times \simeq H^2(G, \mathbb{Z})^\vee$ of class field theory, where K/k is Galois with Galois group G.

Lemma 7.5. *Let K_1 be a CM field and $K := K_1^r$ be the r-copies of K_1. Then $\tau(T^{K,\mathbb{Q}}) = 2^{r-1} \cdot \tau(T^{K_1,\mathbb{Q}})$.*

Proof. This is first proved in [Rüd 2022, Example 8.4, page 2897]; here we give an independent proof. Observe that $N(T^{K,\mathbb{Q}}(\mathbb{A})) = N(\mathbb{A}_K^\times) \cap \mathbb{A}^\times$, where $N = N_{K/K^+}$. To see this, the inclusion $\subseteq$ is clear. If $x \in N(\mathbb{A}_K^\times) \cap \mathbb{A}^\times$, then $x = N(y)$ for some $y \in \mathbb{A}_K^\times$. By definition $y \in T(\mathbb{A})$, and hence $x \in N(T(\mathbb{A}))$. This verifies the other inclusion. It follows that if $K = \prod_{i=1}^r K_i$ is a product of CM fields K_i, then

$$N(T^{K,\mathbb{Q}}(\mathbb{A})) = \mathbb{A}^\times \bigcap_{i=1}^r N_{K_i/K_i^+}(\mathbb{A}_{K_i}^\times)$$

in the sense that the right hand side consists of all elements $x \in \mathbb{A}^\times$ which are contained in $N_{K_i/K_i^+}(\mathbb{A}_{K_i}^\times)$ via the embeddings $\mathbb{A}^\times \hookrightarrow \mathbb{A}_{K_i^+}^\times$ for all i. Thus, if $K_i = K_1$ for all i, then $N(T^{K,\mathbb{Q}}(\mathbb{A})) = \mathbb{A}^\times \cap N_{K_1/K_1^+}(\mathbb{A}_{K_1}^\times) = N(T^{K_1,\mathbb{Q}}(\mathbb{A}))$ and hence $n_K = n_{K_1}$. By Proposition 2.7, $\tau(T^{K,\mathbb{Q}}) = 2^r/n_K = 2^{r-1} \cdot 2/n_{K_1} = 2^{r-1} \cdot \tau(T^{K_1})$. □

Corollary 7.6. *For any integer $n \geq 0$, there exists a CM torus T over $\mathbb{Q}$ such that $\tau(T) = 2^n$.*

Proof. Take $K = E^r$ with $r = n+1 \geq 1$, where E is an imaginary quadratic field. Then $\tau(T^{K,\mathbb{Q}}) = 2^{r-1} \cdot \tau(T^{E,\mathbb{Q}}) = 2^n$. □

Proposition 7.7. *Suppose that CM fields $K_1, \dots, K_r$ satisfy the following:*

(a) *K_i is Galois over $\mathbb{Q}$ with group $G_i \simeq Q_8$ for any i.*

(b) *The Galois group $G = \mathrm{Gal}(L/\mathbb{Q})$ of the compositum $L = K_1 \cdots K_r$ over $\mathbb{Q}$ is isomorphic to $G_1 \times \cdots \times G_r$.*

(c) *Every decomposition group of G_i is cyclic for all $1 \leq i \leq r$.*

Then $\tau(T^{K,\mathbb{Q}}) = \left(\frac{1}{2}\right)^r$.

Proof. By condition (c) and Proposition 6.7, one has

$$\tau(T^{K_i,\mathbb{Q}}) = \tfrac{1}{2} \tag{7-2}$$

for all $1 \leq i \leq r$. Moreover, the equality $[K_i : K_i^+] = 2$ and the conditions (a), (b) and (c) imply that the CM fields $K_1, \dots, K_r$ satisfies the assumptions in Proposition 5.7. See also the conditions (i) and (ii) in Section 5 and the beginning of Section 5B. Hence the assertion follows from Proposition 5.7 and (7-2). □

Theorem 7.8. *For any positive integer r, there exist Q_8-CM fields K_i for $1 \leq i \leq r$ that satisfy the conditions (a), (b) and (c) in Proposition 7.7.*

We will give the proof in the next section. From Proposition 7.7 and Theorem 7.8, we immediately obtain

Corollary 7.9. *For any integer n, there exists a CM torus T over $\mathbb{Q}$ such that $\tau(T) = 2^n$.*

Remark 7.10. Kottwitz [1992] computed the Hasse–Weil zeta function of the moduli spaces S_{K^p} of PEL-type. Using the notation there, it is shown in Section 8 that the algebraic variety S_{K^p} over the reflex field E is a finite disjoint union of the canonical model of Shimura varieties associated to the Shimura datum (G, h^{-1}, K^p) indexed by $\ker^1(\mathbb{Q}, G) := \ker(H^1(\mathbb{Q}, G) \to \prod_{v \leq \infty} H^1(\mathbb{Q}_v, G))$. In Case C and Case A with n even, the set $\ker^1(\mathbb{Q}, G)$ is trivial and there is no difference between the moduli space S_{K^p} and the canonical model of the Shimura variety in question. In Case A with n odd, the set $\ker^1(\mathbb{Q}, G)$ is canonically isomorphic to $\ker^1(\mathbb{Q}, Z)$, where Z is the kernel of the map $F^\times \times \mathbb{Q}^\times \to F_0^\times$ sending (x, t) to $N_{F/F_0}(x)t^{-1}$ and F is the center of the central simple $\mathbb{Q}$-algebra B in the input PEL-datum. The $\mathbb{Q}$-torus Z is exactly the CM torus associated to the CM field F and $\ker^1(\mathbb{Q}, Z)$ is its Tate–Shafarevich group.

Question 7.11. Is Proposition 2.9(2) still true if one drops the condition that $K/\mathbb{Q}$ is Galois?

8. Construction of an effective family of Q_8-CM fields

8A. *Existence of Q_8-extensions of fields.* Let k be a field of characteristic different from 2, and denote by $\mathrm{Br}(k)$ the Brauer group of k. Then we define a pairing

$$(\cdot\,,\cdot)_k\colon k^\times \times k^\times \to \mathrm{Br}(k)$$

by sending $(a, b) \in k^\times \times k^\times$ to the Brauer class of the quaternion algebra

$$\left(\frac{a, b}{k}\right) := k \oplus k\alpha \oplus k\beta \oplus k\alpha\beta,$$

where $\alpha^2 = a$, $\beta^2 = b$ and $\beta\alpha = -\alpha\beta$. By definition, the pairing $(\cdot\,,\cdot)_k$ is symmetric, and the image of $(\cdot\,,\cdot)_k$ is contained in the 2-torsion group of $\mathrm{Br}(k)$.

For $a \in k^\times$, put

$$k_a := k[T]/(T^2 - a).$$

We denote by $N_{k_a/k}\colon k_a \to k$ the norm map of k_a/k.

Proposition 8.1 [Gille and Szamuely 2017, Proposition 1.1.7]. *Let $a, b \in k^\times$. Then the following are equivalent*:

(1) $(a, b)_k = 1$.

(2) $a \in N_{k_b/k}(k_b^\times)$.

(3) $b \in N_{k_a/k}(k_a^\times)$.

The following is one of the most important key to prove Theorem 7.8.

Theorem 8.2 [Kiming 1990, Theorem 4]. *Let $E = k(\sqrt{a}, \sqrt{b})$ be a biquadratic extension of k, where $a, b \in k^\times$. Then the following are equivalent*:

(1) *There is a quadratic extension of K/E such that K/k is a Q_8-extension.*

(2) $(a, a)_k(b, b)_k(a, b)_k = 1$.

8B. *Proof of Theorem 7.8.* First, we recall the properties on the pairings $(\cdot\,,\cdot)_0 := (\cdot\,,\cdot)_{\mathbb{Q}}$ and $(\cdot\,,\cdot)_v := (\cdot\,,\cdot)_{\mathbb{Q}_v}$ for all places v of $\mathbb{Q}$.

Proposition 8.3. *Let v be a place of $\mathbb{Q}$, $a, b \in \mathbb{Q}_v^\times$ and let $\mathbb{Q}_{v,a} := \mathbb{Q}_v[T]/(T^2 - a)$*:

(1) *If v is infinite, then we have $(a, b)_v = 1$ if and only if either a or b is positive.*

(2) *Assume that $v = \ell$ is a nondyadic finite place. If $a, b \in \mathbb{Z}_\ell^\times$, then $(a, b)_\ell = 1$.*

(3) *Under the assumption on v in* (2), *if $a = \ell$ and $b \in \mathbb{Z}_\ell^\times$, then we have $(a, b)_\ell = \left(\frac{b}{\ell}\right)$.*

(4) *If $v = 2$ and $a, b \in 1 + 4\mathbb{Z}_2$, then one has $(a, b)_2 = 1$.*

Proof. (1) This follows from Proposition 8.1 and the equality $N_{\mathbb{C}/\mathbb{R}}(\mathbb{C}^\times) = \mathbb{R}_{>0}$.

(2) Since $\ell \neq 2$ and $a \in \mathbb{Z}_\ell^\times$, we have

$$\mathbb{Q}_{\ell,a} \cong \begin{cases} \mathbb{Q}_\ell \times \mathbb{Q}_\ell & \text{if } a \in (\mathbb{Z}_\ell^\times)^2, \\ \mathbb{Q}_{\ell^2} & \text{if } a \notin (\mathbb{Z}_\ell^\times)^2, \end{cases}$$

where $\mathbb{Q}_{\ell^2}$ denotes the unramified quadratic extension of $\mathbb{Q}_\ell$. Hence $N_{\mathbb{Q}_{\ell,a}/\mathbb{Q}_\ell}(\mathbb{Q}_{\ell,a}^\times)$ contains $\mathbb{Z}_\ell^\times$. Hence the assertion follows from Proposition 8.1 and the assumption $b \in \mathbb{Z}_\ell^\times$.

(3) Since ℓ is not equal to 2, we have

$$N_{\mathbb{Q}_\ell(\sqrt{\ell})/\mathbb{Q}}(\mathbb{Q}_\ell(\sqrt{\ell})^\times) = \langle -\ell \rangle \times (\mathbb{Z}_\ell^\times)^2.$$

Therefore the assertion follows from Proposition 8.1.

(4) Since $1 + 8\mathbb{Z}_2 = (\mathbb{Z}_2^\times)^2$, there is an isomorphism

$$\mathbb{Q}_{2,a} \cong \begin{cases} \mathbb{Q}_2 \times \mathbb{Q}_2 & \text{if } a \in (\mathbb{Z}_2^\times)^2, \\ \mathbb{Q}_4 & \text{if } a \notin (\mathbb{Z}_2^\times)^2. \end{cases}$$

In particular, $N_{\mathbb{Q}_{2,a}/\mathbb{Q}_\ell}(\mathbb{Q}_{2,a}^\times)$ contains $\mathbb{Z}_2^\times$. Hence the assertion follows from $b \in 1+4\mathbb{Z}_2$ and Proposition 8.1. □

The following two lemmas will be used later.

Lemma 8.4. *Let E be a totally real field which is Galois over $\mathbb{Q}$:*

(1) *Let K/E be a quadratic extension such that $K/\mathbb{Q}$ is Galois. Then K is either totally real or CM.*

(2) *If there is a quadratic extension K/E such that $K/\mathbb{Q}$ is Galois, then there is a totally imaginary quadratic extension K'/E such that $K'/\mathbb{Q}$ is Galois and*

$$\mathrm{Gal}(K'/\mathbb{Q}) \cong \mathrm{Gal}(K/\mathbb{Q}).$$

Proof. (1) If K is not totally real, then it is totally complex since $K/\mathbb{Q}$ is Galois. Fix an embedding $\varepsilon \colon K \hookrightarrow \mathbb{C}$, and let ι be the element of $\mathrm{Gal}(K/\mathbb{Q})$ induced by the complex conjugation and ε. Then ι is the unique nontrivial element of $\mathrm{Gal}(K/E) \subset \mathrm{Gal}(K/\mathbb{Q})$ since E is totally real. On the other hand, the assumption that E is Galois implies that $\mathrm{Gal}(K/E)$ is central in $\mathrm{Gal}(K/\mathbb{Q})$. Hence ι is contained in the center of $\mathrm{Gal}(K/\mathbb{Q})$, which implies that K is a CM field.

(2) By (1), we may assume that K is totally real. Write $K = E(\sqrt{\alpha})$, where $\alpha \in E^\times$, and put $K' := E(\sqrt{-\alpha})$. Note that K' corresponds to $\langle (\varepsilon, \iota) \rangle$ under the isomorphism

$$\mathrm{Gal}(K(\sqrt{-1})/\mathbb{Q}) \cong \mathrm{Gal}(K/\mathbb{Q}) \times \mathrm{Gal}(\mathbb{Q}(\sqrt{-1})/\mathbb{Q}).$$

Here ε is the unique nontrivial element of $\mathrm{Gal}(K/E)$, and ι is the complex conjugation on $\mathbb{Q}(\sqrt{-1})$. Note that ε is central in $\mathrm{Gal}(K/\mathbb{Q})$. Then K' is not totally real K' is Galois over $\mathbb{Q}$, and hence it is CM by (1). Moreover, the composite

$$\mathrm{Gal}(K/\mathbb{Q}) \xrightarrow{g\mapsto(g,\mathrm{id}_{\mathbb{Q}(\sqrt{-1})})} \mathrm{Gal}(K/\mathbb{Q}) \times \mathrm{Gal}(\mathbb{Q}(\sqrt{-1})/\mathbb{Q}) \cong \mathrm{Gal}(K(\sqrt{-1})/\mathbb{Q}) \to \mathrm{Gal}(K'/\mathbb{Q})$$

is an isomorphism by the definition of K'. □

For a number field E which is finite Galois over $\mathbb{Q}$, we write for $\mathrm{Ram}(E)$ the set of prime numbers which ramify in E.

Lemma 8.5. *Let r be a positive integer, and $K_1, \dots, K_r$ be number fields which are Galois over $\mathbb{Q}$. For each i, we denote by E_i the maximal abelian subfield of K_i. Assume the following*:

(i) *$[K_i : E_i]$ is a prime number for any $i \geq 1$.*

(ii) *$\mathrm{Ram}(E_i) \cap \mathrm{Ram}(E_j) = \varnothing$ if $i \neq j$.*

Let L be the compositum of $K_1, \dots, K_r$. Then there is an isomorphism

$$\mathrm{Gal}(L/\mathbb{Q}) \cong \prod_{i=1}^{r} \mathrm{Gal}(K_i/\mathbb{Q}).$$

Proof. We give a proof by induction on r. It is trivial if $r = 1$. Next, assume that the assertion holds for $r-1$, that is, there is an isomorphism

$$\mathrm{Gal}(L'/\mathbb{Q}) \cong \prod_{i=1}^{r-1} \mathrm{Gal}(K_i/\mathbb{Q}), \tag{8-1}$$

where $L' := K_1 \cdots K_{r-1}$. We first prove the equality

$$E_r \cap L' = \mathbb{Q}. \tag{8-2}$$

Since E_r is abelian over $\mathbb{Q}$, it is contained in the maximal abelian subfield E' of L'. On the other hand, the induction hypothesis (8-1) implies the equality $E' = E_1 \cdots E_{r-1}$. Hence $E_r \cap L'$ is contained in $E_r \cap (E_1 \cdots E_{r-1})$. However, we have $E_r \cap (E_1 \cdots E_{r-1}) = \mathbb{Q}$ by the assumption (ii) and the global class field theory, and hence (8-2) holds.

Now we prove the equality $K_r \cap L' = \mathbb{Q}$, which gives the desired assertion. If $K_r \cap L' \neq \mathbb{Q}$, then we have $E_r \cap (K_r \cap L') = \mathbb{Q}$ and $E_r \subsetneq E_r \cdot (K_r \cap L') \subset K_r$ by (8-2). Therefore K_r is equal to the compositum of E_r and $K_r \cap L'$ by the assumption (i). Consequently, there is an isomorphism

$$\mathrm{Gal}(K_r/\mathbb{Q}) \cong \mathrm{Gal}(E_r/\mathbb{Q}) \times \mathrm{Gal}(K_r \cap L'/\mathbb{Q}).$$

In particular, $[K_r \cap L' : \mathbb{Q}] = [K_r : E_r]$ is a prime number, and hence $K_r/\mathbb{Q}$ is abelian. This contradicts the assumption (i), which implies the desired equality $K_r \cap L' = \mathbb{Q}$. □

Let $\boldsymbol{L}$ be the set of unordered pairs of prime numbers $\{\ell, \ell'\}$ satisfying the following:

$$\ell \equiv \ell' \equiv 1 \bmod 4, \quad \left(\frac{\ell'}{\ell}\right) = 1.$$

Proposition 8.6. *For any positive integer r, there are r-pairs $\{\ell_1, \ell'_1\}, \dots, \{\ell_r, \ell'_r\}$ in $\boldsymbol{L}$ such that*

$$|\{\ell_1, \dots, \ell_r, \ell'_1, \dots, \ell'_r\}| = 2r.$$

Proof. By the Dirichlet prime number theorem, there are r distinct prime numbers $\ell_1, \dots, \ell_r$ which are congruent to 1 modulo 4. Moreover, the Dirichlet prime number theorem implies the existence of prime numbers $\ell'_1, \dots, \ell'_r$ satisfying the following:

$$\begin{cases} \ell'_i \equiv 1 \mod 4, \left(\frac{\ell'_i}{\ell_i}\right) = 1 & \text{if } i = 1, \\ \ell'_i \equiv 1 \mod 4, \left(\frac{\ell'_i}{\ell_i}\right) = 1, \ell'_i \notin \{\ell'_1, \dots, \ell'_{i-1}\} & \text{if } i \geq 2. \end{cases}$$

Therefore the assertion holds. □

For $\lambda := \{\ell, \ell'\} \in \boldsymbol{L}$, put $K_\lambda^+ := \mathbb{Q}(\sqrt{\ell}, \sqrt{\ell'})$.

Lemma 8.7. ***For any $\lambda \in \boldsymbol{L}$, the decomposition groups of $K_\lambda^+/\mathbb{Q}$ at all finite places are cyclic.***

Proof. Write $\lambda = \{\ell, \ell'\}$. Let v be a finite place of K_λ^+ which lies above a prime number ℓ_0. If $\ell_0 \notin \{\ell, \ell'\}$, then $K_\lambda^+/\mathbb{Q}$ is unramified at v, and hence the assertion holds for v. The assertion for $\ell_0 = \ell$ follows from the assumption $\left(\frac{\ell'}{\ell}\right) = 1$. Finally, in the case $\ell_0 = \ell'$, the statement is a consequence of the equality $\left(\frac{\ell}{\ell'}\right) = \left(\frac{\ell'}{\ell}\right) = 1$. □

Proposition 8.8. *For any $\lambda \in \boldsymbol{L}$, there is a CM field K containing K_λ^+ such that $K/\mathbb{Q}$ is a Q_8-extension.*

Proof. By Lemma 8.4(2), it suffices to prove the existence of Q_8-extension K of $\mathbb{Q}$ containing K_λ^+. Write $\lambda = \{\ell, \ell'\}$. It is equivalent to the equality

$$(\ell, \ell)_0 (\ell', \ell')_0 (\ell, \ell')_0 = 1,$$

which is a consequence of Theorem 8.2. Since the image of the class $(\ell, \ell)_0 (\ell', \ell')_0 (\ell, \ell')_0$ in $\mathrm{Br}(\mathbb{Q}_v)$ is equal to $(\ell, \ell)_v (\ell', \ell')_v (\ell, \ell')_v$, by the Albert–Brauer–Hasse–Noether theorem [Lang 1994, Chapter IX, Section 6, page 195] it is equivalent to prove the following for any place v of $\mathbb{Q}$:

$$(\ell, \ell)_v (\ell', \ell')_v (\ell, \ell')_v = 1. \tag{8-3}$$

Case 1. v **is infinite.** In this case, (8-3) follows from Proposition 8.3(1) since ℓ and ℓ' are positive.

Case 2. $v = \ell_0 \notin \{2, \ell, \ell'\}$**.** The condition $\ell_0 \notin \{\ell, \ell'\}$ implies that ℓ and ℓ' are units in $\mathbb{Z}_{\ell_0}$. Hence, since $\ell_0 \neq 2$, one has

$$(\ell, \ell)_{\ell_0} = (\ell', \ell')_{\ell_0} = (\ell, \ell')_{\ell_0} = 1$$

by Proposition 8.3(2). This implies the equality (8-3).

Case 3. $v = 2$. Since $\ell \equiv \ell' \equiv 1 \mod 4$, Proposition 8.3(4) implies

$$(\ell, \ell)_2 = (\ell', \ell')_2 = (\ell, \ell')_2 = 1.$$

Hence (8-3) holds.

Case 4. $v = \ell$**.** The assumption $\ell \equiv 1 \bmod 4$ is equivalent to the equality $\left(\frac{-1}{\ell}\right) = 1$. Hence

$$(\ell, \ell)_\ell = (\ell, -1)_\ell = 1.$$

On the other hand, the assumption $\left(\frac{\ell'}{\ell}\right) = 1$ implies

$$(\ell, \ell')_\ell = (\ell', \ell')_\ell = 1,$$

which is a consequence of Proposition 8.3(2) and (3). Therefore we obtain the desired equality (8-3).

Case 5. $v = \ell'$**.** Since $\ell' \equiv 1 \bmod 4$ and

$$\left(\frac{\ell}{\ell'}\right) = \left(\frac{\ell'}{\ell}\right) = 1,$$

the assertion (8-3) follows from the same argument as Case 4. □

In the following, we give a proof of Theorem 7.8. By Corollary 7.6, we may assume $n = -r < 0$. Take $\lambda_1 = \{\ell_1, \ell_1'\}, \dots, \lambda_r = \{\ell_r, \ell_r'\} \in \boldsymbol{L}$ satisfying

$$|\{\ell_1, \dots, \ell_r, \ell_1', \dots, \ell_r'\}| = 2r, \tag{8-4}$$

which is possible by Proposition 8.6. Then, Proposition 8.8 implies that there is a Q_8-CM field containing $K_{\lambda_i}^+$ for any $1 \le i \le r$.

The following immediately implies the desired assertion.

Theorem 8.9. *Under the above notations, let K_{λ_i} be a Q_8-CM field containing $K_{\lambda_i}^+$ for each $1 \le i \le r$. Then the CM fields $K_{\lambda_1}, \dots, K_{\lambda_r}$ satisfy the conditions (a), (b) and (c) in Proposition 7.7.*

Proof. From the definition of K_{λ_i} for $1 \le i \le r$, condition (a) in Proposition 7.7 holds.

We shall show that the assumptions (i) and (ii) in Lemma 8.5 hold. By Lemma 6.5(3), for any $1 \le i \le r$, $K_{\lambda_i}^+$ is the maximal abelian subfield of K_{λ_i} and $[K_{\lambda_i} : K_{\lambda_i}^+] = 2$. In particular, assumption (i) holds. On the other hand, since $\mathrm{Ram}(K_{\lambda_i}^+) = \lambda_i$ for any $1 \le i \le r$, the condition (8-4) implies the equality $\mathrm{Ram}(K_{\lambda_i}^+) \cap \mathrm{Ram}(K_{\lambda_j}^+) = \varnothing$ for $i \neq j$. Hence we obtain the assumption (ii), which verifies condition (b) in Proposition 7.7.

Take a finite places w of K_{λ_i} and v of $K_{\lambda_i}^+$ satisfying $w \mid v$. Then Lemma 8.7 implies the cyclicity of the decomposition group of $\mathrm{Gal}(K_{\lambda_i}^+/\mathbb{Q})$ at v. Since $K_{\lambda_i}/\mathbb{Q}$ is a Q_8-extension, the decomposition group at w is cyclic by Lemma 6.5(2). Therefore condition (c) in Proposition 7.7 holds. □

9. Products of two linearly disjoint Galois CM fields

In this section we show the following result.

Theorem 9.1. *There are infinitely many CM algebras $K = K_1 \times K_2$ with linearly disjoint Galois CM fields K_1 and K_2 such that*

$$\tau(T^{K,\mathbb{Q}}) = \tfrac{1}{2} \prod_{i=1}^{2} \tau(T^{K_i,\mathbb{Q}}). \tag{9-1}$$

This theorem shows that the conclusion of Proposition 5.7 is no longer true if one drops the cyclicity of decomposition groups of G_i for all i. We shall use the notations in Section 5B. In particular, $L = K_1K_2$, $G := \mathrm{Gal}(L/\mathbb{Q})$, $G_i := \mathrm{Gal}(K_i/\mathbb{Q})$, $H_i := \mathrm{Gal}(L/K_i)$, $\widetilde{N}_i := \mathrm{Gal}(L/K_i^+)$ and $N_i = \mathrm{Gal}(K_i/K_i^+)$ for $i = 1, 2$.

First, we give a sufficient condition on $K = K_1 \times K_2$ for which Theorem 9.1 holds. Let $\mathcal{C}$ and $\mathcal{C}_i$ be the sets of cyclic subgroups of G and G_i respectively. Then put

$$H^2(\mathbb{Z})'' := \{f \in G^\vee : f|_D \in \mathrm{Im}(\mathrm{Ver}^\vee_{D,\overline{D}}) \text{ for all } D \in \mathcal{C}\},$$
$$H^2(G_i, \mathbb{Z})'' := \{f \in G_i^\vee : f|_{D'} \in \mathrm{Im}(\mathrm{Ver}^\vee_{D',D'\cap N_i}) \text{ for all } D' \in \mathcal{C}_i\}.$$

Lemma 9.2. *If $G \cong G_1 \times G_2$, then $H^2(\mathbb{Z})'' = H^2(G_1, \mathbb{Z})'' \times H^2(G_2, \mathbb{Z})''$.*

Proof. The proof is the same as Lemma 5.6. □

We define a subgroup D_0 of $(\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \times \mathbb{Z}/2\mathbb{Z}$ as follows:

$$D_0 := \langle(\bar{1}, \bar{0}, \bar{1}), (\bar{0}, \bar{1}, \bar{0})\rangle.$$

Here we denote by $(\bar{a}, \bar{b}, \bar{c})$ the element $(a \bmod 4, b \bmod 2, c \bmod 2)$ in $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2 \times \mathbb{Z}/2\mathbb{Z}$ for $a, b, c \in \mathbb{Z}$.

Proposition 9.3. *Assume that Galois CM fields K_1 and K_2 satisfy the following*:

(i) $G_1 \cong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, $N_1 \cong \langle(\bar{2}, \bar{0})\rangle \subset G_1$ *and* $G_2 \cong \mathbb{Z}/2\mathbb{Z}$.

(ii) $G \cong G_1 \times G_2$, *that is*, K_1 *and* K_2 *are linearly disjoint.*

(iii) $\mathcal{D} = \mathcal{C} \cup \{D_0\}$.

Put $K := K_1 \times K_2$. Then we have the following:

$$\tau(T^{K_1,\mathbb{Q}}) = 2, \quad \tau(T^{K_2,\mathbb{Q}}) = 1, \quad \tau(T^{K,\mathbb{Q}}) = 1.$$

In particular, *the equality* (9-1) *holds.*

Proof. By definition, we obtain the following:

$$\widetilde{N}_1 = (2\mathbb{Z}/4\mathbb{Z} \times \{\bar{0}\}) \times \mathbb{Z}/2\mathbb{Z}, \quad H_1 = \{(\bar{0}, \bar{0})\} \times \mathbb{Z}/2\mathbb{Z}, \quad \widetilde{N}_2 = G, \quad H_2 = (\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \times \{\bar{0}\}.$$

Moreover, Proposition 4.3 implies

$$\tau(T^{K_i,\mathbb{Q}}) = \frac{2}{|H^2(G_i, \mathbb{Z})'|}, \quad \tau(T^{K_1\times K_2,\mathbb{Q}}) = \frac{4}{|H^2(\mathbb{Z})'|}.$$

Since G_1 is D_0 modulo G_2, by (iii) G_1 itself is a decomposition group of G_1. By this and that G_1 is not cyclic, one computes that $H^2(G_1, \mathbb{Z})' = 0$, which implies $\tau(T^{K_1,\mathbb{Q}}) = 2/1 = 2$. Moreover, the equality $\tau(T^{K_2,\mathbb{Q}}) = 1$ follows from Proposition 2.9(2) as $[K_2 : \mathbb{Q}] = 2$. In particular, we obtain $|H^2(G_2, \mathbb{Z})'| = 2$, that is,

$$H^2(G_2, \mathbb{Z})' = H^2(G_2, \mathbb{Z})'' = G_2^\vee. \tag{9-2}$$

On the other hand, for the equality $\tau(T^{K,\mathbb{Q}})=1$, it suffices to prove

$$H^2(\mathbb{Z})'=\{f\in G^\vee : f((2\mathbb{Z}/4\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z})\times\{\bar{0}\})=0\}.$$

By direct computation, we have

$$H^2(G_1,\mathbb{Z})''=\{f_1\in G_1^\vee : f(2\mathbb{Z}/4\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z})=0\}.$$

Combining this equality, (9-2) and Lemma 9.2, one has

$$H^2(\mathbb{Z})'\subset H^2(\mathbb{Z})''=H^2(G_1,\mathbb{Z})''\times H^2(G_2,\mathbb{Z})''=\{f\in G^\vee : f((2\mathbb{Z}/4\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z})\times\{\bar{0}\})=0\}.$$

For the reverse inclusion, it suffices to prove

$$\operatorname{Im}(\operatorname{Ver}_{D_0,\bar{\boldsymbol{D}}_0}^\vee)=\{f\in D_0^\vee : f((2\mathbb{Z}/4\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z})\times\{\bar{0}\})=0\}. \tag{9-3}$$

Recall that $\operatorname{Ver}_{D_0,\bar{\boldsymbol{D}}_0}=(\operatorname{Ver}_{D_0,\bar{D}_{0,i}})_{1\le i\le 2}$ and

$$\operatorname{Ver}_{D_0,\bar{D}_{0,i}}=\operatorname{Ver}_{D_0/D_0\cap H_i,\bar{D}_{0,i}}\circ\pi_i,$$

where $\pi_i\colon D_0\to D_0/D_0\cap H_i$ is the canonical surjection. Since one has

$$D_{0,1}=D_0\cap\widetilde{N}_1=2\mathbb{Z}/4\mathbb{Z}\times\{\bar{0}\}\times\mathbb{Z}/2\mathbb{Z},\qquad D_0\cap H_1=\{0\},$$

we obtain that $D_0/D_0\cap H_1$ is not cyclic and $\bar{D}_{0,1}:=D_{0,1}/D_0\cap H_1\cong\mathbb{Z}/2$. Hence Proposition 3.3 implies $\operatorname{Ver}_{D_0,\bar{D}_{0,1}}=0$. On the other hand, we have $\operatorname{Ver}_{D_0,\bar{D}_{0,2}}=\pi_2$ by $D_{0,2}:=D_0\cap\widetilde{N}_2=D_0$. Consequently, the homomorphism $\operatorname{Ver}_{D_0,\bar{\boldsymbol{D}}_0}$ can be written as the composite

$$D_0\xrightarrow{\pi_2}D_0/D_0\cap H_2=\bar{D}_{0,2}\xrightarrow{g_2\mapsto(0,g_2)}\bar{D}_{0,1}\times\bar{D}_{0,2}.$$

Therefore, (9-3) follows from the equality $D_0\cap H_2=(2\mathbb{Z}/4\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z})\times\{\bar{0}\}$. □

Now we construct pairs of CM fields K_1, K_2 satisfying Proposition 9.3.

Lemma 9.4. *Let ℓ be a prime number which is congruent to* 1 *modulo* 4, *and denote by $K_{(\ell)}$ the unique quartic subfield of $\mathbb{Q}(\zeta_\ell)$:*

(1) *The field $K_{(\ell)}$ is CM if and only if $\ell\equiv 5 \bmod 8$.*

(2) *We have $\operatorname{Ram}(K_{(\ell)}/\mathbb{Q})=\{\ell\}$, and ℓ is totally ramified in $K_{(\ell)}$.*

Proof. (1) Recall that $\mathbb{Q}(\zeta_\ell)$ is a CM field which is cyclic of degree $\ell-1$ over $\mathbb{Q}$. Since $K_{(\ell)}$ corresponds to the unique subgroup of $\operatorname{Gal}(\mathbb{Q}(\zeta_\ell)/\mathbb{Q})$ of order $(\ell-1)/4$, it is CM if and only if $(\ell-1)/4$ is an odd number. Finally, the condition $(\ell-1)/4\notin 2\mathbb{Z}$ is equivalent to the desired congruence $\ell\equiv 5 \bmod 8$.

(2) This follows from the equality $\operatorname{Ram}(\mathbb{Q}(\zeta_\ell)/\mathbb{Q})=\{\ell\}$ and that ℓ is totally ramified in $\mathbb{Q}(\zeta_\ell)$. □

Lemma 9.5. *There are infinitely many ordered triples of prime numbers* (ℓ_1, ℓ_2, ℓ_3) *for which the following are satisfied*:

(a) $\ell_1 \equiv 5 \mod 8$.

(b) $\ell_2 \equiv 1 \mod 4$ *and* $\left(\frac{\ell_2}{\ell_1}\right) = -1$.

(c) $\ell_3 \equiv 3 \mod 4$, $\left(\frac{\ell_3}{\ell_2}\right) = 1$ *and* ℓ_3 *splits completely in* $K_{(\ell_1)}$.

Proof. This follows from Dirichlet's prime number theorem. □

For a finite abelian extension $E/\mathbb{Q}$ and a prime number ℓ, we write $D_\ell(E/\mathbb{Q})$ and $I_\ell(E/\mathbb{Q})$ for the decomposition group and the inertia group of $\mathrm{Gal}(E/\mathbb{Q})$ at ℓ respectively. Observe that if E' is a subextension of $E/\mathbb{Q}$ and let $\pi_{E'} : \mathrm{Gal}(E/\mathbb{Q}) \to \mathrm{Gal}(E'/\mathbb{Q})$ denote the natural projection, then $\pi_{E'}(D_\ell(E/\mathbb{Q})) = D_\ell(E'/\mathbb{Q})$ and $\pi_{E'}(I_\ell(E/\mathbb{Q})) = I_\ell(E'/\mathbb{Q})$.

Theorem 9.1 is a consequence of Proposition 9.6 and Lemma 9.7.

Proposition 9.6. *Let* (ℓ_1, ℓ_2, ℓ_3) *be an ordered triple of prime numbers as in Lemma 9.5, and set*

$$K_1 := K_{(\ell_1)}(\sqrt{\ell_2}), \quad K_2 := \mathbb{Q}(\sqrt{-\ell_1\ell_3}).$$

Then the fields K_1, K_2 *are CM and they satisfy the conditions* (i)–(iii) *in Proposition 9.3.*

Proof. By definition, K_2 is an imaginary quadratic field, and hence CM. Moreover, condition (a) in Lemma 9.5 and Lemma 9.4(1) imply that K_1 is also CM.

In the following, we shall prove that the statements (i), (ii) and (iii) in Proposition 9.3 hold. Statement (i) in Proposition 9.3 follows directly from the definitions of K_1 and K_2. To prove statement (ii) in Proposition 9.3, it suffices to prove the equality $K_1 \cap K_2 = \mathbb{Q}$. However, this follows from the fact that ℓ_3 is unramified in K_1 and is totally ramified in K_2. We now show (iii) in Proposition 9.3. It is clear that $L := K_1K_2$ is unramified outside $2, \ell_1, \ell_2$ and ℓ_3. Hence it suffices to compute $D_\ell(L/\mathbb{Q})$ for $\ell \in \{2, \ell_1, \ell_2, \ell_3\}$. First, suppose $\ell = 2$. Then Lemma 9.4(2) implies that 2 is unramified in $K_{(\ell_1)}$. Moreover, by (b) and (c) in Lemma 9.5, we have $\ell_1 \equiv \ell_2 \equiv -\ell_1\ell_3 \equiv 1 \mod 4$. Hence $L/\mathbb{Q}$ is unramified at 2, which implies that $D_2(L/\mathbb{Q})$ is cyclic. Second, assume $\ell = \ell_1$. By assumptions (b) and (c) in Lemma 9.5, we have the following:

$$D_{\ell_1}(K_1/\mathbb{Q}) = G_1, \quad I_{\ell_1}(K_1/\mathbb{Q}) \cong \mathbb{Z}/4\mathbb{Z} \times \{\bar{0}\}, \quad D_{\ell_1}(K_2/\mathbb{Q}) = I_{\ell_1}(K_2/\mathbb{Q}) = G_2.$$

Since $\ell_1 \neq 2$, by local class field theory $I_{\ell_1}(L/\mathbb{Q})$ is a finite quotient of $\mathbb{Z}_{\ell_2}^\times$ and is cyclic. Since $I_{\ell_1}(K_2/\mathbb{Q}) = G_2$, we have

$$I_{\ell_1}(L/\mathbb{Q}) \cong \langle(\bar{1}, \bar{0}, \bar{1})\rangle.$$

The fixed subfield of $I_{\ell_1}(L/\mathbb{Q})$ in L is $L' := \mathbb{Q}(\sqrt{\ell_2}, \sqrt{-\ell_3})$. Indeed, one has

$$L^{\langle(\bar{2},\bar{0},\bar{0})\rangle} = \mathbb{Q}(\sqrt{\ell_1}, \sqrt{\ell_2}, \sqrt{-\ell_1\ell_3}) \quad \text{and} \quad \mathbb{Q}(\sqrt{\ell_1}, \sqrt{\ell_2}, \sqrt{-\ell_1\ell_3})^{\langle(\bar{1},\bar{0},\bar{1})\rangle} = \mathbb{Q}(\sqrt{\ell_2}, \sqrt{-\ell_3}).$$

By (c) in Lemma 9.5, we have $D_{\ell_1}(L'/\mathbb{Q}) = \mathrm{Gal}(L'/\mathbb{Q}(\sqrt{-\ell_3})) \simeq \mathrm{Gal}(\mathbb{Q}(\sqrt{\ell_2})/\mathbb{Q})$, and hence $D_{\ell_1}(L/\mathbb{Q})/I_\ell(L/\mathbb{Q})$ is generated by the image of $(\bar{0}, \bar{1}, \bar{0})$ in $G/I_\ell(L/\mathbb{Q})$. Therefore we obtain $D_{\ell_1}(L/\mathbb{Q}) = D_0$. Third, if $\ell = \ell_2$, then by (b) in Lemma 9.5 we have

$$D_{\ell_2}(K_1/\mathbb{Q}) = G_1, \quad I_{\ell_2}(K_1/\mathbb{Q}) \cong \{0\} \times \mathbb{Z}/2\mathbb{Z}.$$

One computes $\left(\frac{-\ell_1\ell_3}{\ell_2}\right) = -1$ and then

$$D_{\ell_2}(K_2/\mathbb{Q}) = G_2, \quad I_{\ell_2}(K_2/\mathbb{Q}) = \{0\}.$$

Since $I_{\ell_2}(K_2/\mathbb{Q}) = \{0\}$, one has $I_{\ell_2}(K_2/\mathbb{Q}) = \langle(\bar{0}, \bar{1}, \bar{0})\rangle$. By $D_{\ell_2}(K_1/\mathbb{Q}) = G_1$, one sees $D_{\ell_2}(L/\mathbb{Q}) = \langle(\bar{0}, \bar{1}, \bar{0}), (\bar{1}, \bar{0}, \bar{c})\rangle$ for some $\bar{c} \in \mathbb{Z}/2\mathbb{Z}$. Because $D_{\ell_2}(K_2/\mathbb{Q}) = G_2$, we see $\bar{c} = \bar{1}$ and hence $D_{\ell_2}(L/\mathbb{Q}) = D_0$. Finally, suppose $i = 3$. Since ℓ_3 splits completely in K_1, then

$$D_{\ell_3}(L/\mathbb{Q}) = D_{\ell_3}(K_2/\mathbb{Q}) = G_2,$$

and hence it is cyclic. This completes the proof of condition (iii) in Proposition 9.3. □

Lemma 9.7. *Let (ℓ_1, ℓ_2, ℓ_3) and $(\ell'_1, \ell'_2, \ell'_3)$ be ordered triples of prime numbers satisfying (a), (b) and (c) in Lemma 9.5. Put*

$$K := K_{(\ell_1)}(\sqrt{\ell_2}) \times \mathbb{Q}(\sqrt{-\ell_1\ell_3}), \quad K' := K_{(\ell'_1)}(\sqrt{\ell'_2}) \times \mathbb{Q}(\sqrt{-\ell'_1\ell'_3}).$$

Then we have $K \simeq K'$ if and only if $\ell_i = \ell'_i$ for every $1 \le i \le 3$.

Proof. It suffices to prove that $K \simeq K'$ implies $\ell_i = \ell'_i$ for any $1 \le i \le 3$. Assume $K \simeq K'$, which is equivalent to $K_{(\ell_1)}(\sqrt{\ell_2}) \simeq K_{(\ell'_1)}(\sqrt{\ell'_2})$ and $\mathbb{Q}(\sqrt{-\ell_1\ell_3}) \simeq \mathbb{Q}(\sqrt{-\ell'_1\ell'_3})$. Then, by Lemma 9.4(2), one has

$$\{\ell_1, \ell_2\} = \mathrm{Ram}(K_{(\ell_1)}(\sqrt{\ell_2})) = \mathrm{Ram}(K_{(\ell'_1)}(\sqrt{\ell'_2})) = \{\ell'_1, \ell'_2\}.$$

Moreover, the following holds:

$$|I_{\ell_1}(K_{(\ell_1)}(\sqrt{\ell_2})/\mathbb{Q})| = |I_{\ell'_1}(K_{(\ell'_1)}(\sqrt{\ell'_2})/\mathbb{Q})| = 4,$$
$$|I_{\ell_2}(K_{(\ell_1)}(\sqrt{\ell_2})/\mathbb{Q})| = |I_{\ell'_2}(K_{(\ell'_1)}(\sqrt{\ell_2})/\mathbb{Q})| = 2.$$

Hence we must have $\ell_1 = \ell'_1$ and $\ell_2 = \ell'_2$. On the other hand, $\mathbb{Q}(\sqrt{-\ell_1\ell_3}) \simeq \mathbb{Q}(\sqrt{-\ell'_1\ell'_3})$ implies $\ell_1\ell_3 = \ell'_1\ell'_3$ since they are square-free integers. Combining this equality with $\ell_1 = \ell'_1$, we obtain $\ell_3 = \ell'_3$. This completes the proof. □

Appendix: Ono's conjecture on Tamagawa numbers of algebraic tori

by Jianing Li and Chia-Fu Yu

Theorem A.8. *For any global field k and any positive rational number r, there exists a k-torus T such that $\tau(T) = r$.*

Proof. It suffices to construct for each prime ℓ: (a) a k-torus T_1 with $\tau(T_1)=\ell$ and (b) a k-torus T_2 with $\tau(T_2)=\ell^{-2}$. Indeed, the torus $T_3=T_1\times_k T_2$ has Tamagawa number $\tau(T_3)=\ell^{-1}$. Then for an appropriate product T of tori as T_1 or T_3 for different primes ℓ, the Tamagawa number $\tau(T)$ can be equal to any given positive rational number r.

Take a cyclic extension K/k of degree ℓ with Galois group G and let $T_1=R^{(1)}_{K/k}\mathbb{G}_{\mathrm{m},K}$. Then $H^1(G, X(T_1))\simeq H^2(G,\mathbb{Z})\simeq\mathbb{Z}/\ell\mathbb{Z}$ and $\text{Ш}^2(K/k, X(T_1))=0$ by Chebotarev's density theorem. Thus, $\tau(T_1)=\ell$ and (a) is done.

For (b), we follow the idea of Ono; see [Ono 1963b, Section 6.2]. We take an abelian extension K/k with Galois group $G\simeq(\mathbb{Z}/\ell\mathbb{Z})^4$ such that every decomposition group is cyclic. In the number field case, such an extension was constructed by Katayama [1985]. In the function field case, we construct such an extension in Theorem A.9 below. Granting the existence of this extensions K/k in both the number field and function field cases, we consider the k-torus $T_2:=R^{(1)}_{K/k}\mathbb{G}_{\mathrm{m},K}$. Since every decomposition group of G is cyclic, $\text{Ш}^2(G, X(T_2))=H^2(G, X(T_2))\simeq H^3(G,\mathbb{Z})$. By Lyndon's formula [1948, Theorem 6], $H^3(G,\mathbb{Z})\simeq(\mathbb{Z}/\ell\mathbb{Z})^6$. On the other hand, $H^1(G, X(T))\simeq H^2(G,\mathbb{Z})\simeq(\mathbb{Z}/\ell\mathbb{Z})^4$. Thus, $\tau(T_2)=\ell^{-2}$ and (b) is done. □

Theorem A.9. *Let k be a global function field of* char $p>0$. *For any prime ℓ and any positive integer n, there exists an abelian extension of k with Galois group $G\simeq(\mathbb{Z}/\ell\mathbb{Z})^n$ in which every decomposition group is cyclic.*

We shall use the cyclotomic function field and we recall the basic facts following from class field theory. For an explicit construction of these fields by Drinfeld modules, we refer to [Rosen 2002, Chapter 12]. Let $k=\mathbb{F}_p(t)$ be the function field of the projective line over $\mathbb{F}_p$ and let $A=\mathbb{F}_p[t]$. In what follows P, P_i always denote monic irreducible polynomials of A. Let ∞ denote the place of the infinite point and set $V_\infty=\langle t, 1+t^{-1}\mathbb{F}_p[\![t^{-1}]\!]\rangle\subset k_\infty^\times=\mathbb{F}_p((t^{-1}))^\times$. For a monic polynomial $M=\prod_{i=1}^r P_i^{n_i}\in A$, let $K(M)$ be the finite abelian extension of k corresponding to the following open subgroup (with finite index) of $\mathbb{A}_k^\times$

$$U(M)=k^\times\biggl(\prod_{i=1}^r(1+P_i^{n_i}O_{P_i})\times V_\infty\times\prod_{v\nmid M\infty}O_v^\times\biggr).$$

(The field $K(M)$ is called the cyclotomic function field for M.) Thus, we have $\mathbb{A}_k^\times/U(M)\simeq\mathrm{Gal}(K(M)/k)$ via the Artin map. Clearly, P is unramified in $K(M)$ if $P\nmid M$. The decomposition group of ∞ in $K(M)$ is isomorphic to $\mathbb{F}_p^\times$, which is cyclic. There is an isomorphism $(A/M)^\times\cong\mathbb{A}_k^\times/U(M)$ induced by $P \bmod M\mapsto(1,\dots,P,\dots,1) \bmod U(M)$ (P sitting on the place P) for $P\nmid M$, and $a \bmod M\mapsto(1,\dots,1,\dots,a) \bmod U(M)$ for $a\in\mathbb{F}_p^\times\subset(A/M)^\times$ (a sitting on the place ∞). So we have the following isomorphism which maps $P \bmod M$ ($P\nmid M$) to its Frobenius element in $\mathrm{Gal}(K(M)/k)$

$$(A/M)^\times\simeq\mathrm{Gal}(K(M)/k).$$

We will primarily be concerned with the fields $K(P)$ and $K(P^2)$, in which P is totally ramified. If ℓ divides $|(A/P)^\times|$, let $F(P)$ denote the subfield of $K(P)$ fixed by $((A/P)^\times)^\ell$ so that $\mathrm{Gal}(F(P)/k)\simeq\mathbb{Z}/\ell\mathbb{Z}$.

If $a \in A/P$, then $1 + aP \mod P^2 \in (A/P^2)^\times$ is well defined and it generates the subgroup $H(a) = \{1 + aiP \mod P^2 : i = 0, 1, \dots, p-1\}$ of order p when $a \neq 0$. We let $F(P, a)$ denote the subfield of $K(P^2)$ fixed by $\{x \in (A/P^2)^\times : x^{q-1} \in H(a)\}$ where $q = p^{\deg P} = |A/P|$. This group is isomorphic to $H(a) \times (A/P)^\times$; since $(A/P^2)^\times \simeq A/P \times (A/P)^\times$, we have $\mathrm{Gal}(F(P, a)/k) \simeq (\mathbb{Z}/p\mathbb{Z})^{\deg P - 1}$ when $a \neq 0$.

Proof. We first prove the theorem when $k = \mathbb{F}_p(t)$ and $\ell \neq p$. The argument of this case is entirely similar to the case of number fields given in [Katayama 1985]. Choose P_1 such that it splits in $k(\mu_\ell)$ where μ_ℓ is the group of ℓ-th roots of unity. We inductively choose P_r $(1 \le r \le n)$ such that P_r splits completely in the composite field $k(\mu_\ell, \sqrt[\ell]{P_1}, \dots, \sqrt[\ell]{P_{r-1}})F(P_1) \cdots F(P_{r-1})$. Clearly, each P_i has infinitely many ways to choose by density theorems. Now let K be the composite field $F(P_1) \cdots F(P_n)$. The decomposition group of ∞ in K/k is cyclic, since K is a subfield of the cyclotomic function field $K(P_1 \cdots P_r)$. Moreover, if P is not one of the P_i, P is unramified and hence its decomposition group is cyclic. Therefore, to show that K/k is the desired extension, it suffices to show that P_i splits in $F(P_j)$ whenever $i \neq j$. Assume $i < j$. Then P_j splits in $F(P_i)$ by the construction. Conversely, since P_j splits in $k(\sqrt[\ell]{P_i})$, P_i is an ℓ-power in the completion k_{P_j} which implies that $P_i \in ((A/P_j)^\times)^\ell$; hence P_i splits in $F(P_j)$. This proves the theorem when $k = \mathbb{F}_p(t)$ and $\ell \neq p$.

Assume next $k = \mathbb{F}_p(t)$ and $\ell = p$. Take an arbitrary $P_1 \in A$ with $a_1 = 0 \in A/P_1$. Let k_1 be a subfield of $F(P_1, a_1)$ with $[k_1 : k] = p$. We inductively choose (P_r, a_r, k_r) $(1 \le r \le n)$ with $a_r \in A/P_r$ such that $\deg P_r \ge r$ and

$$P_r^{q_i - 1} \equiv 1 + a_i P_i \mod P_i^2 \quad \text{for } i = 1, \dots, r-1, \text{ where } q_i = |A/P_i|.$$

Let $a_{ri} \in A/P_r$ $(i = 1, \dots, r-1)$ such that

$$P_i^{q_r - 1} \equiv 1 + a_{ri} P_r \mod P_r^2 \quad \text{for } i = 1, \dots, r-1.$$

Let $a_r := a_{r1}$, and let k_r be a subfield of $\bigcap_{i=1}^{r-1} F(P_r, a_{ri})$ with $[k_r : k] = p$. Put $K := k_1 \cdots k_r$ and let us show that K/k is the desired extension. We have $\mathrm{Gal}(K/k) \simeq (\mathbb{Z}/p\mathbb{Z})^n$ by considering the ramification. Assume $i < j$. Then P_j splits completely in $F(P_i, a_i)$ by the first congruences and hence splits in k_i; conversely, P_i splits completely in $F(P_j, a_{ji})$ by the second congruences and hence also splits in its subfield k_j. It follows that the decomposition subgroup of P_i $(i = 1, \dots, n)$ in K/k is cyclic. This property also holds for the place ∞, since $K \subset K(P_1^2 \cdots P_n^2)$. This proves the theorem when $k = \mathbb{F}_p(t)$ and $\ell = p$.

Now assume that k is any finite extension of $\mathbb{F}_p(t)$ and ℓ is any prime. Choose $P_1, \dots, P_n$ as above but we additionally require that P_i is unramified in $k/\mathbb{F}_p(t)$, and we define $K/\mathbb{F}_p(t)$ as above. Then $\mathrm{Gal}(Kk/k) \simeq (\mathbb{Z}/\ell\mathbb{Z})^n$. Since each decomposition subgroup for Kk/k is a subgroup of that of $K/\mathbb{F}_p(t)$, the field Kk is the desired extension. This completes the proof of Theorem A.9. □

Acknowledgments

The present project grows up from the 2020 NCTS USRP "Arithmetic of CM tori" where the authors participated. The authors thank Jeff Achter, Ming-Lun Hsieh, Tetsushi Ito, Teruhisa Koshikawa, Wen-Wei Li, Thomas Rüd, Yasuhiro Terakado, Jiangwei Xue, Seidai Yasuda for helpful discussions and their valuable comments and especially to Thomas Rüd who kindly answered the last author's questions and for his inspiring paper [Rüd 2022]. They are grateful to the referees for their careful readings and helpful comments which improve the present paper both in the exposition and mathematics. Liang, Yang and Yu were partially supported by the MoST grant 109-2115-M-001-002-MY3. Oki was supported by JSPS Research Fellowship for Young Scientists and KAKENHI Grant Number 22J00570.

References

[Achter et al. 2023] J. D. Achter, S. A. Altuğ, L. Garcia, and J. Gordon, "Counting abelian varieties over finite fields via Frobenius densities", *Algebra Number Theory* **17**:7 (2023), 1239–1280. MR Zbl

[Bayer-Fluckiger et al. 2019] E. Bayer-Fluckiger, T.-Y. Lee, and R. Parimala, "Hasse principles for multinorm equations", *Adv. Math.* **356** (2019), art. id. 106818. MR Zbl

[Borel 1963] A. Borel, "Some finiteness properties of adele groups over number fields", *Inst. Hautes Études Sci. Publ. Math.* **16** (1963), 5–30. MR Zbl

[Conrad et al. 2010] B. Conrad, O. Gabber, and G. Prasad, *Pseudo-reductive groups*, New Mathematical Monographs **17**, Cambridge University Press, 2010. MR Zbl

[Demarche and Wei 2014] C. Demarche and D. Wei, "Hasse principle and weak approximation for multinorm equations", *Israel J. Math.* **202**:1 (2014), 275–293. MR Zbl

[Gille and Szamuely 2017] P. Gille and T. Szamuely, *Central simple algebras and Galois cohomology*, 2nd ed., Cambridge Studies in Advanced Mathematics **165**, Cambridge University Press, 2017. MR Zbl

[González-Avilés 2008] C. D. González-Avilés, "Chevalley's ambiguous class number formula for an arbitrary torus", *Math. Res. Lett.* **15**:6 (2008), 1149–1165. MR Zbl

[González-Avilés 2010] C. D. González-Avilés, "On Néron–Raynaud class groups of tori and the capitulation problem", *J. Reine Angew. Math.* **648** (2010), 149–182. MR Zbl

[Guo et al. 2022] J.-W. Guo, N.-H. Sheu, and C.-F. Yu, "Class numbers of CM algebraic tori, CM abelian varieties and components of unitary Shimura varieties", *Nagoya Math. J.* **245** (2022), 74–99. MR Zbl

[Hürlimann 1984] W. Hürlimann, "On algebraic tori of norm type", *Comment. Math. Helv.* **59**:4 (1984), 539–549. MR Zbl

[Katayama 1985] S. Katayama, "On the Tamagawa number of algebraic tori", *Sūgaku* **37**:1 (1985), 81–83. In Japanese. MR Zbl

[Katayama 1991] S. Katayama, "Isogenous tori and the class number formulae", *J. Math. Kyoto Univ.* **31**:3 (1991), 679–694. MR Zbl

[Kiming 1990] I. Kiming, "Explicit classifications of some 2-extensions of a field of characteristic different from 2", *Canad. J. Math.* **42**:5 (1990), 825–855. MR Zbl

[Kottwitz 1984] R. E. Kottwitz, "Stable trace formula: cuspidal tempered terms", *Duke Math. J.* **51**:3 (1984), 611–650. MR Zbl

[Kottwitz 1988] R. E. Kottwitz, "Tamagawa numbers", *Ann. of Math.* (2) **127**:3 (1988), 629–646. MR Zbl

[Kottwitz 1992] R. E. Kottwitz, "Points on some Shimura varieties over finite fields", *J. Amer. Math. Soc.* **5**:2 (1992), 373–444. MR Zbl

[Lang 1994] S. Lang, *Algebraic number theory*, 2nd ed., Graduate Texts in Mathematics **110**, Springer, 1994. MR Zbl

[Lee 2022] T.-Y. Lee, "The Tate–Shafarevich groups of multinorm-one tori", *J. Pure Appl. Algebra* **226**:7 (2022), art. id. 106906. MR Zbl

[Lyndon 1948] R. C. Lyndon, "The cohomology theory of group extensions", *Duke Math. J.* **15** (1948), 271–292. MR Zbl

[Morishita 1991] M. Morishita, "On S-class number relations of algebraic tori in Galois extensions of global fields", *Nagoya Math. J.* **124** (1991), 133–144. MR Zbl

[Neukirch et al. 2000] J. Neukirch, A. Schmidt, and K. Wingberg, *Cohomology of number fields*, Grundl. Math. Wissen. **323**, Springer, 2000. MR Zbl

[Oesterlé 1984] J. Oesterlé, "Nombres de Tamagawa et groupes unipotents en caractéristique p", *Invent. Math.* **78**:1 (1984), 13–88. MR Zbl

[Ono 1961] T. Ono, "Arithmetic of algebraic tori", *Ann. of Math.* (2) **74**:1 (1961), 101–139. MR Zbl

[Ono 1963a] T. Ono, "On Tamagawa numbers", *Sūgaku* **15** (1963), 72–81. In Japanese. MR Zbl

[Ono 1963b] T. Ono, "On the Tamagawa number of algebraic tori", *Ann. of Math.* (2) **78**:1 (1963), 47–73. MR Zbl

[Platonov and Rapinchuk 1994] V. Platonov and A. Rapinchuk, *Algebraic groups and number theory*, Pure and Applied Mathematics **139**, Academic Press, Boston, 1994. MR Zbl

[Pollio 2014] T. P. Pollio, "On the multinorm principle for finite abelian extensions", *Pure Appl. Math. Q.* **10**:3 (2014), 547–566. MR Zbl

[Pollio and Rapinchuk 2013] T. P. Pollio and A. S. Rapinchuk, "The multinorm principle for linearly disjoint Galois extensions", *J. Number Theory* **133**:2 (2013), 802–821. MR Zbl

[Prasad and Rapinchuk 2010] G. Prasad and A. S. Rapinchuk, "Local-global principles for embedding of fields with involution into simple algebras with involution", *Comment. Math. Helv.* **85**:3 (2010), 583–645. MR Zbl

[Rosen 2002] M. Rosen, *Number theory in function fields*, Graduate Texts in Mathematics **210**, Springer, 2002. MR Zbl

[Rosengarten 2018] Z. Rosengarten, "Tate duality in positive dimension over function fields", preprint, 2018. arXiv 1805.00522

[Rüd 2022] T. Rüd, "Explicit Tamagawa numbers for certain algebraic tori over number fields", *Math. Comp.* **91**:338 (2022), 2867–2904. MR Zbl

[Serre 1977] J.-P. Serre, *Linear representations of finite groups*, Graduate Texts in Mathematics **42**, Springer, 1977. MR Zbl

[Serre 2016] J.-P. Serre, *Finite groups: an introduction*, Surveys of Modern Mathematics **10**, International Press, Somerville, MA, 2016. MR Zbl

[Shyr 1977] J. M. Shyr, "On some class number relations of algebraic tori", *Michigan Math. J.* **24**:3 (1977), 365–377. MR Zbl

[Tran 2017] M.-H. Tran, "A formula for the S-class number of an algebraic torus", *J. Number Theory* **181** (2017), 218–239. MR Zbl

[Washington 1997] L. C. Washington, *Introduction to cyclotomic fields*, 2nd ed., Graduate Texts in Mathematics **83**, Springer, 1997. MR Zbl

[Wei and Xu 2012] D. Wei and F. Xu, "Integral points for multi-norm tori", *Proc. Lond. Math. Soc.* (3) **104**:5 (2012), 1019–1044. MR Zbl

[Xue and Yu 2021] J. W. Xue and C. F. Yu, "On counting certain abelian varieties over finite fields", *Acta Math. Sin.* (*Engl. Ser.*) **37**:1 (2021), 205–228. MR Zbl

[Yu 2019] C. F. Yu, "Chow's theorem for semi-abelian varieties and bounds for splitting fields of algebraic tori", *Acta Math. Sin.* (*Engl. Ser.*) **35**:9 (2019), 1453–1463. MR Zbl

Communicated by Shou-Wu Zhang
Received 2022-09-30 Revised 2023-03-05 Accepted 2023-05-13

cindy11420@gmail.com *National Tsing Hua University, Taipei, Taiwan*

oki@math.sci.hokudai.ac.jp *Hokkaido University, Sapporo, Japan*

nona01111998@gmail.com *Utrecht University, Utrecht, Netherlands*

chiafu@math.sinica.edu.tw *Institute of Mathematics, Academia Sinica and NCTS, Taipei, Taiwan*

lijn@sdu.edu.cn *Shandong University, Qingdao Campus, Qingdao, China*

chiafu@math.sinica.edu.tw *Institute of Mathematics, Academia Sinica and NCTS, Taipei, Taiwan*

mathematical sciences publishers msp

Guidelines for Authors

Authors may submit manuscripts in PDF format on-line at the Submission page at the ANT website.

Originality. Submission of a manuscript acknowledges that the manuscript is original and and is not, in whole or in part, published or under consideration for publication elsewhere. It is understood also that the manuscript will not be submitted elsewhere while under consideration for publication in this journal.

Language. Articles in *ANT* are usually in English, but articles written in other languages are welcome.

Length There is no a priori limit on the length of an *ANT* article, but *ANT* considers long articles only if the significance-to-length ratio is appropriate. Very long manuscripts might be more suitable elsewhere as a memoir instead of a journal article.

Required items. A brief abstract of about 150 words or less must be included. It should be self-contained and not make any reference to the bibliography. If the article is not in English, two versions of the abstract must be included, one in the language of the article and one in English. Also required are keywords and subject classifications for the article, and, for each author, postal address, affiliation (if appropriate), and email address.

Format. Authors are encouraged to use LaTeX but submissions in other varieties of TeX, and exceptionally in other formats, are acceptable. Initial uploads should be in PDF format; after the refereeing process we will ask you to submit all source material.

References. Bibliographical references should be complete, including article titles and page ranges. All references in the bibliography should be cited in the text. The use of BibTeX is preferred but not required. Tags will be converted to the house format, however, for submission you may use the format of your choice. Links will be provided to all literature with known web locations and authors are encouraged to provide their own links in addition to those supplied in the editorial process.

Figures. Figures must be of publication quality. After acceptance, you will need to submit the original source files in vector graphics format for all diagrams in your manuscript: vector EPS or vector PDF files are the most useful.

Most drawing and graphing packages (Mathematica, Adobe Illustrator, Corel Draw, MATLAB, etc.) allow the user to save files in one of these formats. Make sure that what you are saving is vector graphics and not a bitmap. If you need help, please write to graphics@msp.org with details about how your graphics were generated.

White space. Forced line breaks or page breaks should not be inserted in the document. There is no point in your trying to optimize line and page breaks in the original manuscript. The manuscript will be reformatted to use the journal's preferred fonts and layout.

Proofs. Page proofs will be made available to authors (or to the designated corresponding author) at a Web site in PDF format. Failure to acknowledge the receipt of proofs or to return corrections within the requested deadline may cause publication to be postponed.

Algebra & Number Theory

Volume 18 No. 3 2024