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Let F be a $GL(3)$ Hecke–Maass cusp form of prime level P_1 and let f be a $GL(2)$ Hecke–Maass cuspform of prime level P_2 . We will prove a subconvex bound for the $GL(3) \times GL(2)$ Rankin–Selberg L -function $L(s, F \times f)$ in the level aspect for certain ranges of the parameters P_1 and P_2 .

1. Introduction

In this paper we continue our study of the subconvexity problem for the degree six $GL(3) \times GL(2)$ Rankin–Selberg L -functions using the delta symbol approach [Munshi 2018]. In the first paper on this theme Munshi [2022] established subconvex bounds in the t -aspect for these L -functions. Since then the method has been extended by Kumar and Singh together with Sharma and Mallesham (see [Kumar 2023; Kumar et al. 2020; 2022; Sharma and Sawin 2022]), to produce various instances of subconvexity in the spectral aspect and twist aspect. Indeed the delta symbol approach has worked quite well in the t -aspect and the spectral aspect. However its effectiveness and adaptability in the more arithmetic problem of level aspect remains a point of deliberation. In particular, it seems that new inputs are required to tackle the level aspect problem for such L -functions, especially when one of the forms is kept fixed and the level of the other varies. However, as was shown in the lower rank case of Rankin–Selberg convolution of two $GL(2)$ forms [Holowinsky and Munshi 2013], the problem can be more tractable when both the forms vary in certain relative range. The aim of the present paper is to prove such a result for $GL(3) \times GL(2)$ Rankin–Selberg convolution.

Theorem 1. *Let P_1 and P_2 be two distinct primes. Let F be a Hecke–Maass cusp form for the congruence subgroup $\Gamma_0(P_1)$ of $SL(3, \mathbb{Z})$ with trivial nebentypus. Let f be a holomorphic or Maass cusp form for the congruence subgroup $\Gamma_0(P_2)$ of $SL(2, \mathbb{Z})$ with trivial nebentypus. Let $\mathcal{Q} = P_1^2 P_2^3$ be the arithmetic conductor of the Rankin–Selberg convolution of the above two forms. Then we have*

$$L\left(\frac{1}{2}, F \times f\right) \ll \mathcal{Q}^{1/4+\varepsilon} \left(\frac{P_1^{1/4}}{P_2^{3/8}} + \frac{P_2^{1/8}}{P_1^{1/4}} \right).$$

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Keywords: subconvexity, Rankin–Selberg L -functions, Hecke–Maass forms.

Note that the convexity bound is given by $Q^{1/4+\varepsilon}$. Thus the above bound is subconvex in the range

$$P_2^{1/2+\varepsilon} < P_1 < P_2^{3/2-\varepsilon}.$$

This provides the first instance of a subconvex bound in the level aspect for a degree six L -function which is not a character twist of a fixed L -function. The bound is strongest when P_1 and P_2 are roughly of same size $P_1 \approx P_2$, in which case Theorem 1 gives

$$L\left(\frac{1}{2}, F \times f\right) \ll Q^{1/4-1/40+\varepsilon}.$$

The exponent $\frac{1}{4} - \frac{1}{40}$ appears in other contexts as well and it seems to be the limit of the delta symbol approach. We also note that our proof with some suitable modifications works even in the case of composite levels P_1 and P_2 . But to keep the exposition simple and clean we will only give full details for the case of prime levels.

For a detailed introduction to automorphic forms on higher rank groups and for basic analytic properties of Rankin–Selberg convolution L -functions we refer the readers to Goldfeld’s book [2006]. Our treatment will be at the level of L -functions, and the Voronoi summation formulae for $GL(2)$ and $GL(3)$ are the only input that we need from the theory of automorphic forms. For a broader introduction to the subconvexity problem and its applications we refer the readers to [Michel 2007; Munshi 2018].

Historically the level aspect subconvexity problem has proved to be more challenging compared to the spectral aspect or the t -aspect, regardless of the method adopted. Indeed Weyl shift is all one needs to prove the t -aspect subconvexity for $\zeta(s)$; see [Weyl 1921]. Whereas Burgess had to nontrivially extend Weyl’s ideas and had to invoke Riemann hypothesis for curves over finite fields, to obtain the first level aspect subconvexity result $L\left(\frac{1}{2}, \chi\right) \ll q^{3/16+\varepsilon}$; see [Burgess 1963]. In the 1990s Duke, Friedlander and Iwaniec [Duke et al. 1993; 1994; 2000] used the amplification technique to obtain the level aspect subconvexity for $GL(2)$ L -functions. The amplification method was extended by Kowalski, Michel and Vanderkam [Kowalski et al. 2002] to Rankin–Selberg convolutions $GL(2) \times GL(2)$. Venkatesh [2010] used ergodic theory to study orbital integrals, and thus obtained level aspect subconvex bounds for triple products $GL(2) \times GL(2) \times GL(2)$, where two forms are fixed and one varies. A similar technique was also adopted by Michel and Venkatesh [2010] for $GL(2) \times GL(2)$ L -functions over any number fields. The level aspect subconvexity problem for any genuine $GL(d)$ L -function with $d > 2$ remains an important open problem.

Our interest in the subconvexity problem for $GL(3) \times GL(2)$ Rankin–Selberg convolution is kindled by two factors. First there is a structural advantage which makes the $GL(n) \times GL(n-1)$ L -functions a suitable candidate for analytic number theoretic exploration. Indeed the case of $n = 2$ has been extensively studied in the literature, as we will see below, and we want to extend to the next level $n = 3$. Secondly, $GL(3) \times GL(2)$ Rankin–Selberg convolutions appear in important applications, like the quantum unique ergodicity, and so it is important to analyze different aspects of the subconvexity problem for these L -functions with the aim of developing techniques that will eventually work in the required scenarios, e.g., spectral aspect subconvexity for symmetric square L -functions. Finally, let us also stress, that we are

motivated to explore the scope of the delta symbol approach to subconvexity and other related problems. After initial success of Munshi [2018], the method has been extended, simplified and generalized by several researchers, e.g., see [Holowinsky and Nelson 2018; Aggarwal 2020; Aggarwal et al. 2020a; 2020b; Kowalski et al. 2020; Kumar 2023; Munshi and Singh 2019; Sharma and Sawin 2022; Lin et al. 2023].

The twists of $GL(2)$ L -functions by Dirichlet characters, or in other words $GL(2) \times GL(1)$ L -functions have been studied extensively in the literature, ever since the breakthrough work of Duke, Friedlander and Iwaniec [1993]. Hybrid subconvexity have also been studied for these L -functions. Since this is the lower rank analogue of the L -function we are investigating in this paper, we briefly recall some results in this basic case. Let f be a $GL(2)$ new form of level P_2 and let χ be a primitive Dirichlet character of modulus P_3 . Suppose $(P_2, P_3) = 1$, then $Q = P_2 P_3^2$ is the arithmetic conductor of $L(\frac{1}{2}, f \otimes \chi)$. Different methods are now available to prove hybrid subconvexity bound, when the levels of forms vary in a relative range, say $P_2 \sim P_3^\eta$. Blomer and Harcos [2008] used amplification technique to prove

$$L(\frac{1}{2}, f \otimes \chi) \ll Q^{1/4+\epsilon} (Q^{-1/(8(2+\eta))} + Q^{-1-\eta/(4(2+\eta))})$$

for $0 < \eta < 1$. Aggarwal, Jo and Nowland [Aggarwal et al. 2018] used classical delta method to prove

$$L(\frac{1}{2}, f \otimes \chi) \ll Q^{1/4-(2-5\eta)/(20(2+\eta))+\epsilon}$$

for $0 < \eta < \frac{2}{5}$. Computing the average of the second moment of $L(\frac{1}{2}, f \otimes \chi)$ over a family of forms, Hou and Chen [2019] extended the range of η to $0 < \eta < \frac{3}{2} - \theta$, where θ is any admissible exponent towards the Petersson–Ramanujan conjecture for the Fourier coefficients. Currently, the result of Hou and Chen yields the widest range $P_2 \ll P_3^{3/2-\delta}$, but it falls short of the Burgess bound. In a recent work, Khan [2021] not only extended the range of P_2 , but also obtained the Weyl bound in the case of $P_2 \sim P_3$. By computing the second moment over a family of $GL(2)$ forms, Khan proved, in the range $P_3 \gg P_2^{1/2}$, that

$$\sum_{f \in B_k^*(P_2)} |L(\frac{1}{2} f \otimes \chi)|^2 \ll_{k,\epsilon} Q^\epsilon (P_2 + P_3),$$

where $B_k^*(P_2)$ denote a basis of holomorphic newforms of level P_2 and weight k , and $Q = P_2 P_3^2$. Recently, during an AIM workshop “Delta symbol and subconvexity”, the first and the third author used the delta symbol approach to prove

$$L(\frac{1}{2}, f \otimes \chi) \ll_\epsilon Q^\epsilon \frac{\sqrt{P_2 P_3}}{\min\{\sqrt{P_2}, \sqrt{P_3}\}}.$$

This is of same strength as [Khan 2021].

2. The set-up

Let F and f be as in Theorem 1. We will denote the normalized Fourier coefficients of f by $\lambda_f(n)$, and that of F by $\lambda_F(n, r)$. The Rankin–Selberg convolution is given by the absolutely converging Dirichlet

series

$$L(s, F \times f) = \sum_{n,r=1}^{\infty} \frac{\lambda_F(n, r)\lambda_f(n)}{(nr^2)^s}$$

in the right half plane $\text{Re}(s) = \sigma > 1$. Here it is also given by a degree six Euler product. This function extends to an entire function and satisfies a functional equation of Riemann type. It is known that this Rankin–Selberg convolution is the standard L -function of a $\text{GL}(6)$ automorphic form [Kim and Shahidi 2002].

2A. Approximate functional equation. The functional equation gives an expression of the central value $L(\frac{1}{2}, F \times f)$ in terms of rapidly decaying series, the so called approximate functional equation [Iwaniec and Kowalski 2004, Theorem 5.3]. Taking a smooth dyadic subdivision of this expression we get the following.

Lemma 2.1. *Let $Q = P_1^2 P_2^3$ be the arithmetic conductor attached to the L -function $L(\frac{1}{2}, F \times f)$. Then, as $Q \rightarrow \infty$, we have*

$$L(\frac{1}{2}, F \times f) \ll_{\epsilon} Q^{\epsilon} \sum_{r \leq Q^{(1+2\epsilon)/4}} \frac{1}{r} \sup_{N \leq Q^{1/2+\epsilon}/r^2} \frac{|S_r(N)|}{N^{1/2}} + Q^{-2021}, \tag{1}$$

where $S_r(N)$ is a sum of the form

$$S_r(N) := \sum_{n=1}^{\infty} \lambda_F(n, r)\lambda_f(n)V\left(\frac{n}{N}\right), \tag{2}$$

for some smooth function V supported in $[1, 2]$ and satisfying $V^{(j)}(x) \ll_j 1$.

This is the usual starting point of the delta symbol approach. Thus, to get subconvexity, it is enough to get some cancellation in the sum

$$S_r(N) = \sum_{n=1}^{\infty} \lambda_F(n, r)\lambda_f(n)V(n/N),$$

for N near the generic range $N \asymp Q^{1/2}$.

2B. Delta symbol. Next we separate the oscillations involved in $S_r(N)$. For this we will use a Fourier expansion of the Kronecker delta symbol. For any $Q > 1$ one has

$$\delta(n) = \frac{1}{Q} \sum_{1 \leq q \leq Q} \frac{1}{q} \sum_{a \pmod q}^* e\left(\frac{an}{q}\right) \int_{\mathbb{R}} g(q, x)e\left(\frac{nx}{qQ}\right) dx,$$

where $g(q, x)$ is a smooth function of x satisfying

$$\begin{aligned} g(q, x) &= 1 + h(q, x), \quad \text{with } h(q, x) = O\left(\frac{Q}{q}\left(\frac{q}{Q} + |x|\right)^B\right), \\ x^j \frac{\partial^j}{\partial x^j} g(q, x) &\ll \log Q \min\left\{\frac{Q}{q}, \frac{1}{|x|}\right\}, \\ g(q, x) &\ll |x|^{-B}, \end{aligned} \tag{3}$$

for any $B > 1$ and $j \geq 1$. (Here $e(z) = e^{2\pi iz}$.) This expansion of δ is due to Duke, Friedlander and Iwaniec, and one can find details of this in [Iwaniec and Kowalski 2004]. Using the third property of $g(q, x)$, we observe that the effective range of the integration over x is $[-Q^\varepsilon, Q^\varepsilon]$. Also it follows that if $q \ll Q^{1-\varepsilon}$ and $x \ll Q^{-\varepsilon}$, then $g(q, x)$ can be replaced by 1 at the cost of a negligible error term. In the complimentary range, using second property, we have

$$x^j \frac{\partial^j}{\partial x^j} g(q, x) \ll Q^\varepsilon.$$

Finally as in [Munshi 2022], by Parseval and Cauchy, we get

$$\int_{\mathbb{R}} (|g(q, x)| + |g(q, x)|^2) dx \ll Q^\varepsilon,$$

i.e., $g(q, x)$ has average size “one” in the L^1 and L^2 sense. Applying this expansion and choosing $Q = N^{1/2}$, we get

$$\begin{aligned} S_r(N) &= \sum_{m,n=1}^{\infty} \lambda_F(n, r) \lambda_f(m) V(n/N) W(m/N) \delta(n-m) \\ &= \frac{1}{Q} \int_{\mathbb{R}} \sum_{1 \leq q \leq Q} \frac{g(q, x)}{q} \sum_{a \bmod q}^* \sum_{n=1}^{\infty} \lambda_F(n, r) e\left(\frac{na}{q}\right) e\left(\frac{nx}{qQ}\right) V\left(\frac{n}{N}\right) \\ &\quad \times \sum_{m=1}^{\infty} \lambda_f(m) e\left(\frac{-ma}{q}\right) e\left(\frac{-mx}{qQ}\right) W\left(\frac{m}{N}\right) dx. \quad (4) \end{aligned}$$

2C. Ideas behind the proof. In this section, we will discuss the method and present a sketch of the proof. For simplicity, let’s consider the generic case, i.e., $N = \sqrt{P_1^2 P_2^3}$, $r = 1$ and $q \asymp Q = \sqrt{N}$. Thus $S_r(N)$ in (4) looks like

$$\frac{1}{Q^2} \sum_{q \sim Q} \sum_{a \bmod q}^* \sum_{n \sim N} \lambda_F(n, 1) e\left(\frac{an}{q}\right) \sum_{m \sim N} \lambda_f(m) e\left(\frac{-am}{q}\right).$$

On applying $GL(3)$ Voronoi to the n -sum, the dual length becomes

$$n^* \sim \frac{\text{Conductor}}{\text{Initial Length}} = \frac{Q^3 P_1}{N} = P_1 N^{1/2},$$

and we save

$$\frac{\text{Initial Length}}{\sqrt{\text{Conductor}}} = \frac{N}{Q^{3/2} P_1^{1/2}}.$$

Next we apply $GL(2)$ Voronoi formula to the sum over m . In this case, the dual length (generic) is given by

$$m^* \sim \frac{Q^2 P_2}{N} = P_2,$$

and we save $N/(Q\sqrt{P_2})$ in this step. The resulting character sum is given by

$$\sum_{a \bmod q}^* S(-\bar{a} \bar{P}_1, n^*; q) e\left(\frac{m^* \bar{a} \bar{P}_2}{q}\right) = q e\left(\frac{\bar{P}_1 m^* P_2 n^*}{q}\right).$$

This reduction of the character sum into an additive character with respect to the $GL(3)$ variable n^* drives the rest of the argument. We save \sqrt{Q} from the sum over a . Hence, in total, we have saved

$$\frac{N}{Q^{3/2} P_1^{1/2}} \times \frac{N}{Q\sqrt{P_2}} \times \sqrt{Q} = \frac{N}{\sqrt{P_1 P_2}}.$$

In the next step, we apply Cauchy’s inequality to the n^* -sum in the following resulting expression:

$$\sum_{q \sim Q} \sum_{n^* \sim P_1 \sqrt{N}} \lambda_F(n, 1) \sum_{m^* \sim P_2} \lambda_f(m) e\left(\frac{\bar{P}_1 m^* P_2 n^*}{q}\right).$$

After Cauchy, we arrive at

$$(P_1 \sqrt{N})^{1/2} \left(\sum_{n^* \sim P_1 \sqrt{N}} \left| \sum_{q \sim Q} \sum_{m^* \sim P_2} \lambda_f(m) e\left(\frac{\bar{P}_1 m^* P_2 n^*}{q}\right) \right|^2 \right)^{1/2},$$

in which we seek to save $\sqrt{P_1 P_2}$ and a little more. In the final step, we apply Poisson summation formula to the n^* -sum. In the zero frequency ($n^* = 0$), we save $(Q P_2)^{1/2}$ which is sufficient provided

$$(Q P_2)^{1/2} > (P_1 P_2)^{1/2} \iff Q > P_1 \iff P_2^{3/2} > P_1.$$

In the nonzero frequency, we save $(P_1 \sqrt{N} / \sqrt{Q^2})^{1/2}$. From the additive character inside the modulus, which arises due to a specific feature of $GL(3) \times GL(2)$ L -functions, we also save \sqrt{Q} . Thus we save $(P_1 \sqrt{N})^{1/2}$, which is sufficient if

$$(P_1 \sqrt{N})^{1/2} > (P_1 P_2)^{1/2} \iff P_1 > P_2^{1/2}.$$

Hence, we obtain subconvexity in the range $P_2^{1/2} < P_1 < P_2^{3/2}$. Optimal saving, from Poisson, can be chosen by taking the minimum of the zero and nonzero frequencies savings. Hence

$$S(N) \ll \frac{N \sqrt{P_1 P_2}}{\min\{\sqrt{Q P_2}, \sqrt{P_1 \sqrt{N}}\}} = \frac{N}{\min\{N^{1/4}/P_1^{1/2}, N^{1/4}/P_2^{1/2}\}},$$

and consequently

$$L\left(\frac{1}{2}, F \times f\right) \ll \frac{(P_1^2 P_2^3)^{1/4}}{\min\{P_2^{3/8}/P_1^{1/4}, P_1^{1/4}/P_2^{1/8}\}},$$

which is best possible when $P_1 \asymp P_2$ ($:= P$) and $P_1 \neq P_2$. In this case we get

$$L\left(\frac{1}{2}, F \times f\right) \ll_{\epsilon} (P^5)^{1/4-1/40+\epsilon}.$$

3. Voronoi summation formula

Our next step involves applications of summation formulas.

3A. $GL(3)$ Voronoi. In this section, we analyze the sum over n using $GL(3)$ Voronoi summation formula. The following Lemma, except for the notations, is taken from [Zhou 2018]. Let F be a Hecke–Maass cusp form of type (ν_1, ν_2) for the congruent subgroup $\Gamma_0(P_1)$ of $SL(3, \mathbb{Z})$ with the trivial character. The Fourier coefficients of F and that of its dual \tilde{F} are related by

$$\lambda_F(r, n) = \lambda_{\tilde{F}}(n, r),$$

for $(nr, P_1) = 1$. Let

$$\alpha_1 = -\nu_1 - 2\nu_2 + 1, \alpha_2 = -\nu_1 + \nu_2, \alpha_3 = 2\nu_1 + \nu_2 - 1$$

be the Langlands parameters for F ; see [Goldfeld 2006] for more details. Let g be a compactly supported smooth function on $(0, \infty)$ and $\tilde{g}(s) = \int_0^\infty g(x)x^{s-1} dx$ be its Mellin transform. For $\ell = 0$ and 1 , we define

$$\gamma_\ell(s) := i^\ell \varepsilon(F) P_1^{1/2+s} \frac{\pi^{-3s-3/2}}{2} \prod_{i=1}^3 \frac{\Gamma((1+s+\alpha_i+\ell)/2)}{\Gamma((-s-\alpha_i+\ell)/2)},$$

with $|\varepsilon(F)| = 1$. Set $\gamma_\pm(s) = \gamma_0(s) \mp \gamma_1(s)$ and let

$$H_\pm(y) = \frac{1}{2\pi i} \int_{(\sigma)} y^{-s} \frac{\pi^{-3s-3/2}}{2} \gamma_\pm(s) \tilde{g}(-s) ds,$$

where $\sigma > -1 + \max\{-\operatorname{Re}(\alpha_1), -\operatorname{Re}(\alpha_2), -\operatorname{Re}(\alpha_3)\}$. Let $G_\pm(y) = P_1^{1/2} H_\pm(y/P_1)$. With the aid of the above terminology, we now state the $GL(3)$ Voronoi summation formula in the following lemma.

Lemma 3.1. *Let $g(x)$ and $\lambda_F(n, r)$ be as above. Let $a, q \in \mathbb{Z}$ with $q > 0$, $(a, q) = 1$, and let \bar{a} be the multiplicative inverse of a modulo q . Suppose $(qr, P_1) = 1$. Then we have*

$$\sum_{n=1}^\infty \lambda_F(n, r) e\left(\frac{an}{q}\right) g(n) = q \sum_{\pm} \sum_{n_1 | qr} \sum_{n_2=1}^\infty \frac{\lambda_F(n_1, n_2)}{n_1 n_2} S(r\bar{a}\bar{P}_1, \pm n_2; qr/n_1) G_\pm\left(\frac{n_1^2 n_2}{q^3 r}\right)$$

where $S(a, b; q)$ is the Kloosterman sum which is defined as

$$S(a, b; q) = \sum_{x \pmod q}^* e\left(\frac{ax + b\bar{x}}{q}\right).$$

Proof. See [Zhou 2018] for the proof. □

To apply Lemma 3.1 in our setup, we need to extract the oscillations of the integral transform. To this end, we state the following lemma.

Lemma 3.2. *Let g be supported in the interval $[X, 2X]$ and let H_{\pm} be defined as above. Then for any fixed integer $K \geq 1$ and $xX \gg 1$, we have*

$$H_{\pm}(x) = x \int_0^{\infty} g(y) \sum_{j=1}^K \frac{c_j(\pm)e(3(xy)^{1/3}) + d_j(\pm)e(-3(xy)^{1/3})}{(xy)^{j/3}} dy + O((xX)^{-(K+2)/3}),$$

where $c_j(\pm)$ and $d_j(\pm)$ are some absolute constants depending on α_i , for $i = 1, 2, 3$.

Proof. See Lemma 6.1 of [Li 2009]. □

Plugging the leading term of Lemma 3.2 in Lemma 3.1 and using the resulting expression in (4) we see that the sum over n gets transformed into

$$\frac{N^{2/3}}{P_1^{1/6} qr^{2/3}} \sum_{\pm} \sum_{n_1 | qr} n_1^{1/3} \sum_{n_2=1}^{\infty} \frac{\lambda_F(n_1, n_2)}{n_2^{1/3}} S(r\bar{a}P_1, \pm n_2; qr/n_1) \mathcal{I}(\cdot), \tag{5}$$

where

$$\mathcal{I}(\cdot) = \int_0^{\infty} V(z) e\left(\frac{Nxz}{qQ} \pm \frac{3(Nn_1^2 n_2 z)^{1/3}}{P_1^{1/3} qr^{1/3}}\right) dz.$$

We observe that, using integration by parts repeatedly, the above integral is negligibly small if

$$n_1^2 n_2 \gg N^\epsilon \sqrt{N} P_1 r = N^\epsilon \frac{P_1 Q^3 r}{N} =: N_0.$$

In the case when $P_1 | qr$, an appropriate Voronoi summation from [Zhou 2018] can still be used. In fact it turns out that our analysis in this paper still goes through with slight modification and the final bound is even better. As such we proceed to present our analysis only in the coprime case.

3B. GL(2) Voronoi. In this section, we dualize the sum over m using GL(2) Voronoi summation formula.

Lemma 3.3. *Let $f \in H_k(P_2)$ be a holomorphic Hecke cuspform with Fourier coefficients $\lambda_f(n)$ and trivial nebentypus. Let a and q be integers with $(aP_2, q) = 1$. Let g be a compactly supported smooth bump function on \mathbb{R} . Then we have*

$$\sum_{m=1}^{\infty} \lambda_f(m) e\left(\frac{-am}{q}\right) g(n) = \frac{1}{q} \frac{\eta_f(P_2)}{\sqrt{P_2}} \sum_{n=1}^{\infty} \lambda_f(m) e\left(\frac{m\bar{a}P_2}{q}\right) H\left(\frac{m}{P_2 q^2}\right), \tag{6}$$

where $a\bar{a} \equiv 1 \pmod{q}$, $|\eta_f(P_2)| = 1$ and

$$H(y) = 2\pi i^k \int_0^{\infty} g(x) J_{k-1}(4\pi \sqrt{xy}) dx,$$

where J_{k-1} is the J -Bessel function and k is the weight of f .

Proof. See the appendix of [Kowalski et al. 2002]. □

Extracting the oscillations of J_{k-1} ,

$$J_{k-1}(2\pi x) = e(x) W_{k-1}(x) + e(-x) \bar{W}_{k-1}(x),$$

with

$$x^j \frac{d^j}{dx^j} W_{k-1}(x) \ll_{j,k} 1/\sqrt{x},$$

we see that $H(y)$ can be essentially replaced by

$$H(y) = \frac{2\pi i^k}{y^{1/4}} \int_0^\infty g_1(x) e(\pm 2\sqrt{xy}) dx,$$

in our analysis, where g_1 is the new weight function which has compact support and $x^j g_1^{(j)}(x) \ll_j 1$, $j \geq 0$. Applying the above lemma, the sum over m in (4) reduces to

$$\frac{N^{3/4} \eta_f(P_2)}{P_2^{1/4} \sqrt{q}} \sum_{m=1}^\infty \frac{\lambda_f(m)}{m^{1/4}} e\left(\frac{m\bar{a}P_2}{q}\right) \int_0^\infty W(y) e\left(\frac{-yNx}{qQ}\right) e\left(\frac{\pm 2\sqrt{Nmy}}{qP_2^{1/2}}\right) dy. \tag{7}$$

Notice the abuse of notation: the weight function W is different from the one in (4). Using stationary phase analysis we observe that the above integral is negligibly small unless

$$m \ll N^\epsilon P_2 = N^\epsilon \frac{Q^2 P_2}{N} =: M_0.$$

Again we will ignore the degenerate case where $P_2 | q$ and proceed with the analysis of the generic case. Indeed our analysis works in the degenerate case as well, and the bound that we obtain is even better (as one will expect).

Now plugging (5) and (7) in (4), we arrive at

$$\frac{N^{17/12} \eta_f(P_2)}{P_1^{1/6} P_2^{1/4} Q r^{2/3}} \sum_{1 \leq q \leq Q} \frac{1}{q^{5/2}} \sum_{\pm} \sum_{n_1 | qr} n_1^{1/3} \sum_{n_2 \ll N_0/n_1^2} \frac{\lambda_F(n_1, n_2)}{n_2^{1/3}} \sum_{m \ll M_0} \frac{\lambda_f(m)}{m^{1/4}} \mathcal{C}(\cdot) \mathfrak{I}(\cdot), \tag{8}$$

where the integral transform is given by

$$\mathfrak{I}(\cdot) = \int_{\mathbb{R}} W(x) g(q, x) \int_0^\infty W(y) \int_0^\infty V(z) e\left(\frac{Nx(z-y)}{qQ} \pm \frac{2\sqrt{Nmy}}{qP_2^{1/2}} \pm \frac{3(Nn_1^2 n_2 z)^{1/3}}{P_1^{1/3} qr^{1/3}}\right) dz dy dx,$$

and the character sum is given by

$$\mathcal{C}(\cdot) := \sum_{a \bmod q}^* S(r\bar{a}\bar{P}_1, \pm n_2; qr/n_1) e\left(\frac{m\bar{a}\bar{P}_2}{q}\right) = \sum_{d | q} d\mu\left(\frac{q}{d}\right) \sum_{\substack{\alpha \bmod qr/n_1 \\ \bar{P}_1 n_1 \alpha \equiv -m\bar{P}_2 \pmod{d}}}^* e\left(\pm \frac{\bar{\alpha} n_2}{qr/n_1}\right).$$

4. Cauchy and Poisson

4A. Cauchy inequality. Now we apply Cauchy’s inequality to the n_2 -sum in (8). To this end, we split the sum over q into dyadic blocks $q \sim C$ and further writing $q = q_1 q_2$ with $q_1 | (n_1 r)^\infty$, $(n_1 r, q_2) = 1$,

we see that $S_r(N)$ is bounded by

$$\sup_{C \ll Q} \frac{N^{17/12}}{P_2^{1/4} P_1^{1/6} Q r^{2/3} C^{5/2}} \sum_{\pm} \sum_{\substack{n_1 \\ (n_1, r) \ll C}} n_1^{1/3} \sum_{\substack{n_1 \\ (n_1, r) | q_1 | (n_1 r)^\infty}} \sum_{n_2 \ll N_0/n_1^2} \frac{|\lambda_F(n_1, n_2)|}{n_2^{1/3}} \times \left| \sum_{q_2 \sim C/q_1} \sum_{m \ll M_0} \frac{\lambda_f(m)}{m^{1/4}} \mathcal{C}(\cdot) \mathfrak{J}(\cdot) \right|, \quad (9)$$

On applying the Cauchy's inequality to the n_2 -sum we arrive at

$$S_r(N) \ll \sup_{C \ll Q} \frac{N^{17/12}}{P_2^{1/4} P_1^{1/6} Q r^{2/3} C^{5/2}} \sum_{\pm} \sum_{\substack{n_1 \\ (n_1, r) \ll C}} n_1^{1/3} \Theta^{1/2} \sum_{\substack{n_1 \\ (n_1, r) | q_1 | (n_1 r)^\infty}} \sqrt{\Omega}, \quad (10)$$

where

$$\Theta = \sum_{n_2 \ll N_0/n_1^2} \frac{|\lambda_F(n_1, n_2)|^2}{n_2^{2/3}}, \quad (11)$$

and

$$\Omega = \sum_{n_2 \ll N_0/n_1^2} \left| \sum_{q_2 \sim C/q_1} \sum_{m \ll M_0} \frac{\lambda_f(m)}{m^{1/4}} \mathcal{C}(\cdot) \mathfrak{J}(\cdot) \right|^2. \quad (12)$$

4B. Poisson. We now apply the Poisson summation formula to the n_2 -sum in (12). To this end, we smooth out the n_2 -sum, i.e., we plug in an appropriate smooth bump function, say, W . Opening the absolute value square, we get

$$\Omega = \sum_{q_2, q_2' \sim C/q_1} \sum_{m, m' \ll M_0} \frac{\lambda_f(m) \lambda_f(m')}{(mm')^{1/4}} \times \sum_{n_2 \in \mathbb{Z}} W\left(\frac{n_2}{N_0/n_1^2}\right) \mathcal{C}(\cdot) \bar{\mathcal{C}}(\cdot) \mathfrak{J}(\cdot) \bar{\mathfrak{J}}(\cdot).$$

Reducing n_2 modulo $q_1 q_2 q_2' r / n_1 := \gamma$, and using the change of variable

$$n_2 \mapsto n_2 q_1 q_2 q_2' r / n_1 + \beta, \quad \text{with } 0 \leq \beta < q_1 q_2 q_2' r / n_1,$$

followed by the Poisson summation formula, we arrive at

$$\Omega = \sum_{q_2, q_2' \sim C/q_1} \sum_{m, m' \ll M_0} \frac{\lambda_f(m) \lambda_f(m')}{(mm')^{1/4}} \sum_{n_2 \in \mathbb{Z}} \sum_{\beta \pmod{\gamma}} \mathcal{C}(\cdot) \bar{\mathcal{C}}(\cdot) \mathcal{J}, \quad (13)$$

where

$$\mathcal{J} = \int_{\mathbb{R}} W\left(\frac{w\gamma + \beta}{N_0/n_1^2}\right) \mathfrak{J}(\cdot) \bar{\mathfrak{J}}(\cdot) e(-n_2 w) dw.$$

Now changing the variable

$$\frac{w\gamma + \beta}{N_0/n_1^2} \mapsto w,$$

we arrive at

$$\mathcal{J} = \frac{N_0}{n_1^2 \gamma} e\left(\frac{n_2 \beta}{\gamma}\right) \int_{\mathbb{R}} W(w) \mathfrak{J}(\cdot) \bar{\mathfrak{J}}(\cdot) e\left(\frac{-n_2 N_0 w}{n_1^2 \gamma}\right) dw.$$

Plugging this back in (13), and executing the sum over β , we arrive at

$$\Omega = \frac{N_0}{n_1^2} \sum_{q_2, q_2' \sim C/q_1} \sum_{m, m' \ll M_0} \frac{\lambda_f(m)\lambda_f(m')}{(mm')^{1/4}} \sum_{n_2 \in \mathbb{Z}} \mathfrak{E}, \tag{14}$$

where

$$\mathfrak{E} = \sum_{\substack{d|q \\ d'|q'}} dd' \mu\left(\frac{q}{d}\right) \mu\left(\frac{q'}{d'}\right) \sum_{\substack{\alpha \bmod qr/n_1 \\ \bar{P}_1 n_1 \alpha \equiv -m \bar{P}_2 \bmod d \bar{P}_1 n_1 \alpha' \equiv -\bar{P}_2 m' \bmod d' \\ \pm \bar{\alpha} q_2' \mp \bar{\alpha}' q_2 \equiv -n_2 \bmod q_1 q_2 q_2' / n_1}}^* \sum_{\substack{\alpha' \bmod q'r/n_1 \\ \alpha' \bmod q'r/n_1}}^* 1 \tag{15}$$

and

$$\mathcal{I} = \int_{\mathbb{R}} W(w) \mathfrak{J}(\cdot) \bar{\mathfrak{J}}(\cdot) e\left(\frac{-n_2 N_0 w}{n_1 q_1 q_2 q_2' r}\right) dw. \tag{16}$$

On applying integration by parts, we see that the above integral is negligibly small if

$$n_2 \gg \frac{Q}{q} \frac{n_1 q_1 q_2 q_2' r}{N_0} := N_2. \tag{17}$$

5. Bounding the integral

In this section we will analyze the integral \mathcal{I} given in (16). Recall that the integral $\mathfrak{J}(\cdot)$ is given by

$$\mathfrak{J}(\cdot) = \int_{\mathbb{R}} W(x) g(q, x) \int_0^\infty W(y) \int_0^\infty V(z) e\left(\frac{Nx(z-y)}{qQ} \pm \frac{2\sqrt{Nmy}}{qP_2^{1/2}} \pm \frac{3(NN_0wz)^{1/3}}{P_1^{1/3}qr^{1/3}}\right) dz dy dx. \tag{18}$$

Let's first focus on x -integral, i.e.,

$$\int_{\mathbb{R}} W(x) g(q, x) e\left(\frac{Nx(z-y)}{qQ}\right) dx.$$

In the case, $q \ll Q^{1-\epsilon}$, we split the above integral as follows:

$$\left(\int_{|x| \ll Q^{-\epsilon}} + \int_{|x| \gg Q^{-\epsilon}} \right) W(x) g(q, x) e\left(\frac{Nx(z-y)}{qQ}\right) dx.$$

For the first part, we can replace $g(q, x)$ by 1 at the cost of a negligible error term (see (3)) so that we essentially have

$$\int_{|x| \ll Q^{-\epsilon}} W(x) e\left(\frac{Nx(z-y)}{qQ}\right) dx.$$

Using integration by parts, we observe that the above integral is negligibly small unless

$$|z-y| \ll \frac{q}{Q} Q^\epsilon.$$

For the second part, using $g^{(j)}(q, x) \ll Q^{\epsilon j}$, we get the restriction $|z-y| \ll \frac{q}{Q} Q^\epsilon$. In the other case, i.e., $q \gg Q^{1-\epsilon}$, the condition $|z-y| \ll \frac{q}{Q} Q^\epsilon$ is trivially true. Now we write z as $z = y + u$, with $|u| \ll \frac{q}{Q} Q^\epsilon$.

Thus the integral $\mathfrak{I}(\cdot)$ up to a negligible error term is given by

$$\int_{\mathbb{R}} W(x)g(q, x) \int_0^\infty \int_{|u| \ll q Q^\epsilon / Q} V(y+u)W(y)e\left(\frac{Nxu}{qQ}\right) \times e\left(\pm \frac{2\sqrt{Nmy}}{qP_2^{1/2}} \pm \frac{3(NN_0w(y+u))^{1/3}}{P_1^{1/3}qr^{1/3}}\right) du dy dx. \quad (19)$$

Now we consider the y -integral

$$\int_{\mathbb{R}} V(y+u)W(y)e\left(\pm \frac{2\sqrt{Nmy}}{qP_2^{1/2}} \pm \frac{3(NN_0w(y+u))^{1/3}}{P_1^{1/3}qr^{1/3}}\right) dy.$$

Expanding $(y+u)^{1/3}$ into the Taylor series

$$(y+u)^{1/3} = y^{1/3} + \frac{u}{3y^{2/3}} - \frac{u^2}{9y^{5/3}} + \dots,$$

we observe that it is enough to consider only the leading term as

$$\frac{3(NN_0)^{1/3}}{P_1^{1/3}qr^{1/3}} \frac{u}{3y^{2/3}} \ll \frac{Qu}{q} \ll Q^\epsilon.$$

Thus we are required to analyze the integral

$$I = \int_{\mathbb{R}} W(y)e\left(\pm \frac{2\sqrt{Nmy}}{qP_2^{1/2}} \pm \frac{3(NN_0wy)^{1/3}}{P_1^{1/3}qr^{1/3}}\right) dy. \quad (20)$$

By stationary phase analysis we see that the integral is negligibly small unless

$$\frac{2\sqrt{Nm}}{qP_2^{1/2}} \asymp \frac{3(NN_0)^{1/3}}{P_1^{1/3}qr^{1/3}} \approx \frac{Q}{q}.$$

Thus the above integral is negligibly small unless $m \sim M_0$ (with M_0 as in Section 3B), in which case the above y -integral is bounded by

$$I \ll \frac{\sqrt{q}}{\sqrt{Q}}.$$

Hence, executing the remaining integrals trivially, and using

$$\int_{\mathbb{R}} |g(q, x)| dx \ll Q^\epsilon,$$

we see that \mathfrak{I} is bounded by

$$\mathfrak{I}(\cdot) \ll q^{3/2}/Q^{3/2}.$$

On substituting this bound in (16), we get

$$\mathcal{I} \ll q^3/Q^3. \quad (21)$$

We record the above discussion in the following lemma.

Lemma 5.1. *Let $I, \mathfrak{I}(\cdot)$ and \mathcal{I} be as in (20), (18) and (16) respectively. Then we have*

$$I \ll \frac{\sqrt{q}}{\sqrt{Q}}, \quad \mathfrak{I}(\cdot) \ll q^{3/2}/Q^{3/2}, \quad \text{and} \quad \mathcal{I} \ll q^3/Q^3.$$

6. Character sums

In this section, we will estimate the character sum \mathfrak{C} given in (15),

$$\mathfrak{C} = \sum_{\substack{d|q \\ d'|q'}} dd' \mu\left(\frac{q}{d}\right) \mu\left(\frac{q'}{d'}\right) \sum_{\substack{\alpha \pmod{qr/n_1} \\ \bar{P}_1 n_1 \alpha \equiv -m \bar{P}_2 \pmod{d} \\ \pm \bar{\alpha} q'_2 \mp \bar{\alpha}' q_2 \equiv -n_2 \pmod{q_1 q_2 q'_2 r/n_1}}} \sum_{\substack{\alpha' \pmod{q'r/n_1} \\ \bar{P}_1 n_1 \alpha' \equiv -P_2 m' \pmod{d'}}} 1. \tag{22}$$

In the case, $n_2 = 0$, the congruence condition

$$\pm \bar{\alpha} q'_2 \mp \bar{\alpha}' q_2 \equiv 0 \pmod{q_1 q_2 q'_2 r/n_1}$$

implies that $q_2 = q'_2$ and $\alpha = \alpha'$. So we can bound the character sum \mathfrak{C} as

$$\mathfrak{C} \ll \sum_{d, d' | q} dd' \sum_{\substack{\alpha \pmod{qr/n_1} \\ \bar{P}_1 n_1 \alpha \equiv -m \bar{P}_2 \pmod{d} \\ \bar{P}_1 n_1 \alpha \equiv -\bar{P}_2 m' \pmod{d'}}} 1 \ll \sum_{\substack{d, d' | q \\ (d, d') | (m-m')}} dd' \frac{qr}{[d, d']}. \tag{23}$$

For $n_2 \neq 0$, we have the following lemma.

Lemma 6.1. *Let \mathfrak{C} be as in (15). Then, for $n_2 \neq 0$, we have*

$$\mathfrak{C} \ll \frac{q_1^2 r(m, n_1)}{n_1} \sum_{\substack{d_2 | (q_2, n_1 q'_2 \mp m n_2 P_1 \bar{P}_2) \\ d'_2 | (q'_2, n_1 q_2 \pm m' n_2 P_1 \bar{P}_2)}} \sum d_2 d'_2.$$

Proof. Let's recall from (15) that

$$\mathfrak{C} = \sum_{\substack{d|q \\ d'|q'}} dd' \mu\left(\frac{q}{d}\right) \mu\left(\frac{q'}{d'}\right) \sum_{\substack{\alpha \pmod{qr/n_1} \\ \bar{P}_1 n_1 \alpha \equiv -m \bar{P}_2 \pmod{d} \\ \pm \bar{\alpha} q'_2 \mp \bar{\alpha}' q_2 \equiv -n_2 \pmod{q_1 q_2 q'_2 r/n_1}}} \sum_{\substack{\alpha' \pmod{q'r/n_1} \\ \bar{P}_1 n_1 \alpha' \equiv -P_2 m' \pmod{d'}}} 1.$$

Using the Chinese remainder theorem, we observe that \mathfrak{C} can be dominated by a product of two sums $\mathfrak{C} \ll \mathfrak{C}^{(1)} \mathfrak{C}^{(2)}$, where

$$\mathfrak{C}^{(1)} = \sum_{d_1, d'_1 | q_1} d_1 d'_1 \sum_{\substack{\beta \pmod{\frac{q_1 r}{n_1}} \\ n_1 \beta \equiv -m P_1 \bar{P}_2 \pmod{d_1} \\ \pm \bar{\beta} q'_2 \mp \bar{\beta}' q_2 + n_2 \equiv 0 \pmod{q_1 r/n_1}}} \sum_{\substack{\beta' \pmod{\frac{q_1 r}{n_1}} \\ d_1 n_1 \beta' \equiv -m' P_1 \bar{P}_2 \pmod{d'_1}}} 1$$

and

$$\mathfrak{C}^{(2)} = \sum_{\substack{d_2 | q_2 \\ d'_2 | q'_2}} \sum d_2 d'_2 \sum_{\substack{\beta \pmod{q_2} \\ n_1 \beta \equiv -m P_1 \bar{P}_2 \pmod{d_2} \\ \pm \bar{\beta} q'_2 \mp \bar{\beta}' q_2 + n_2 \equiv 0 \pmod{q_2 q'_2}}} \sum_{\substack{\beta' \pmod{q'_2} \\ n_1 \beta' \equiv -m' P_1 \bar{P}_2 \pmod{d'_2}}} 1.$$

In the second sum $\mathfrak{C}^{(2)}$, since $(n_1, q_2 q'_2) = 1$, we get $\beta \equiv -m \bar{n}_1 P_1 \bar{P}_2 \pmod{d_2}$ and $\beta' \equiv -m' \bar{n}_1 P_1 \bar{P}_2 \pmod{d'_2}$. Now using the congruence modulo $q_2 q'_2$, we conclude that

$$\mathfrak{C}^{(2)} \ll \sum_{\substack{d_2 | (q_2, n_1 q'_2 \mp m n_2 P_1 \bar{P}_2) \\ d'_2 | (q'_2, n_1 q_2 \pm m' n_2 P_1 \bar{P}_2)}} \sum d_2 d'_2.$$

In the first sum $\mathfrak{C}^{(1)}$, the congruence condition determines β' uniquely in terms of β , and hence

$$\mathfrak{C}^{(1)} \ll \sum_{d_1, d'_1 | q_1} \sum_{\substack{\beta \pmod{q_1 r / n_1} \\ n_1 \beta \equiv -m P_1 \bar{P}_2 \pmod{d_1}}} d_1 d'_1 \sum 1 \ll \frac{r q_1^2(m, n_1)}{n_1}.$$

Hence we have the lemma. □

7. Zero frequency

In this section we will estimate the contribution of the zero frequency $n_2 = 0$ to Ω in (14), and thus estimate its total contribution to $S_r(N)$. We have the following lemma.

Lemma 7.1. *Let $S_r(N)$ be as in (10). The total contribution of the zero frequency $n_2 = 0$ to $S_r(N)$ is dominated by $O(r^{1/2} N^{3/4} \sqrt{P_1})$.*

Proof. On substituting bounds for \mathcal{I} and \mathfrak{C} from Lemma 5.1 and (23) respectively into (14), we see that the contribution of $n_2 = 0$ to Ω , is bounded by

$$\begin{aligned} &\ll \frac{N_0 C^3}{n_1^2 M_0^{1/2} Q^3} \sum_{q_2 \sim C/q_1} q r \sum_{d, d' | q} \sum_{\substack{m, m' \sim M_0 \\ (d, d') | (m - m')}} 1 \\ &\ll \frac{N_0 C^3}{n_1^2 M_0^{1/2} Q^3} \sum_{q_2 \sim C/q_1} q r \sum_{d, d' | q} (M_0(d, d') + M_0^2) \\ &\ll \frac{N_0 C^5 r M_0^{1/2}}{n_1^2 Q^3 q_1} (C + M_0). \end{aligned}$$

Upon substituting this bound for Ω in (10), we get

$$\begin{aligned} & \sup_{C \ll Q} \frac{N^{17/12}}{P_2^{1/4} P_1^{1/6} Q r^{2/3} C^{5/2}} \sum_{\pm} \sum_{\substack{n_1 \\ (n_1, r) \ll C}} n_1^{1/3} \Theta^{1/2} \sum_{\substack{n_1 \\ (n_1, r) | q_1 | (n_1 r)^\infty}} \left(\frac{N_0 C^5 r M_0^{1/2}}{n_1^2 Q^3 q_1} (C + M_0) \right)^{1/2} \\ & \ll \frac{N^{17/12}}{P_2^{1/4} P_1^{1/6} Q r^{2/3}} \frac{N_0^{1/2} r^{1/2} M_0^{1/4}}{Q^{3/2}} \sum_{\substack{n_1 \\ (n_1, r) \ll C}} \frac{\Theta^{1/2}}{n_1^{2/3}} \sum_{\substack{n_1 \\ (n_1, r) | q_1 | (n_1 r)^\infty}} \frac{1}{\sqrt{q_1}} (\sqrt{Q} + \sqrt{M_0}) \\ & \ll \frac{N^{17/12}}{P_2^{1/4} P_1^{1/6} Q r^{2/3}} \frac{N_0^{1/2} r^{1/2} M_0^{1/4}}{Q^{3/2}} \sqrt{Q} \sum_{\substack{n_1 \\ (n_1, r) \ll C}} \frac{\Theta^{1/2}}{n_1^{7/6}} \sqrt{(n_1, r)}. \end{aligned}$$

Note that (as in [Munshi 2022]) we have

$$\sum_{n_1 \ll Cr} \frac{(n_1, r)^{1/2}}{n_1^{7/6}} \Theta^{1/2} \ll \left[\sum_{n_1 \ll Cr} \frac{(n_1, r)}{n_1} \right]^{1/2} \left[\sum_{n_1^2 n_2 \leq N_0} \frac{|\lambda_F(n_1, n_2)|^2}{(n_1^2 n_2)^{2/3}} \right]^{1/2} \ll N_0^{1/6}. \quad (24)$$

Using this bound, we see that the contribution of $n_2 = 0$ to $S_r(N)$ is bounded by

$$S_r(N) \ll \frac{N^{17/12}}{P_2^{1/4} P_1^{1/6} Q r^{2/3}} \frac{N_0^{1/2} r^{1/2} M_0^{1/4}}{Q^{3/2}} \sqrt{Q} N_0^{1/6} \ll r^{1/2} N^{3/4} \sqrt{P_1}. \quad \square$$

8. Nonzero frequencies

In this section we will estimate the contribution of the nonzero frequencies $n_2 \neq 0$ to Ω in (14). We have the following lemma.

Lemma 8.1. *Let $S_r(N)$ be as in (10). The total contribution of $n_2 \neq 0$, to $S_r(N)$ is dominated by $O(\sqrt{r} N^{3/4} \sqrt{P_2})$.*

Proof. On plugging in the bounds for the character sums and the integrals from Lemmas 6.1 and 5.1 respectively into (14), we see that the contribution of $n_2 \neq 0$ to Ω (which we denote by $\Omega_{\neq 0}$) is bounded by

$$\frac{q_1^2 N_0 r C^3}{n_1^3 M_0^{1/2} Q^3} \sum_{q_2, q_2' \sim \frac{C}{q_1}} \sum_{\substack{d_2 | q_2 \\ d_2' | q_2'}} d_2 d_2' \sum_{\substack{m, m' \sim M_0 0 \neq n_2 \ll N_2 \\ n_1 q_2' \mp m n_2 P_1 \bar{P}_2 \equiv 0 \pmod{d_2} \\ n_1 q_2 \pm m' n_2 P_1 \bar{P}_2 \equiv 0 \pmod{d_2'}}} (m, n_1).$$

Further writing $q_2 d_2$ in place of q_2 and $q_2' d_2'$ in place of q_2' , we arrive at

$$\Omega_{\neq 0} \ll \frac{q_1^2 N_0 r C^3}{n_1^3 M_0^{1/2} Q^3} \sum_{d_2, d_2' \ll C/q_1} d_2 d_2' \sum_{\substack{q_2 \sim \frac{C}{d_2 q_1} \\ q_2' \sim \frac{C}{d_2' q_1}}} \sum_{\substack{m, m' \sim M_0 0 \neq n_2 \ll N_2 \\ n_1 q_2' d_2 \mp m n_2 P_1 \bar{P}_2 \equiv 0 \pmod{d_2} \\ n_1 q_2 d_2 \pm m' n_2 P_1 \bar{P}_2 \equiv 0 \pmod{d_2'}}} (m, n_1). \quad (25)$$

Let's first assume that $Cn_1/q_1 \ll M_0$. In this case, we count the number of m in the above expression as follows:

$$\sum_{\substack{m \sim M_0 \\ n_1 q_2' d_2' \mp mn_2 P_1 \bar{P}_2 \equiv 0 \pmod{d_2}}} (m, n_1) = \sum_{\ell \mid n_1} \sum_{\substack{m \sim M_0/\ell \\ n_1 q_2' d_2' \mp mn_2 P_1 \bar{P}_2 \equiv 0 \pmod{d_2}}} 1 \ll (d_2, n_2) \left(n_1 + \frac{M_0}{d_2} \right).$$

In the above estimate we have used the fact $(d_2, n_1) = 1$. Counting the number of m' in a similar fashion we get that the m -sum and m' -sum in (25) is dominated by

$$(d_2', n_1 q_2 d_2) (d_2, n_2) \left(n_1 + \frac{M_0}{d_2} \right) \left(1 + \frac{M_0}{d_2'} \right).$$

Now substituting the above bound in (25), we arrive at

$$\frac{q_1^2 N_0 r C^3}{n_1^3 M_0^{1/2} Q^3} \sum_{d_2, d_2' \ll \frac{C}{q_1}} d_2 d_2' \sum_{\substack{q_2 \sim \frac{C}{d_2 q_1} \\ q_2' \sim \frac{C}{d_2' q_1}}} \sum_{0 < |n_2| \ll N_2} (d_2', n_1 q_2 d_2) (d_2, n_2) \left(n_1 + \frac{M_0}{d_2} \right) \left(1 + \frac{M_0}{d_2'} \right).$$

Now summing over n_2 and q_2' , we get the following expression:

$$\frac{q_1 N_0 r N_2 C^4}{n_1^3 M_0^{1/2} Q^3} \sum_{d_2, d_2' \ll C/q_1} d_2 \sum_{q_2 \sim \frac{C}{d_2 q_1}} (d_2', n_1 q_2 d_2) \left(n_1 + \frac{M_0}{d_2} \right) \left(1 + \frac{M_0}{d_2'} \right).$$

Next we sum over d_2' to arrive at

$$\frac{q_1 N_0 r N_2 C^4}{n_1^3 M_0^{1/2} Q^3} \sum_{d_2 \ll C/q_1} d_2 \sum_{q_2 \sim \frac{C}{d_2 q_1}} \left(n_1 + \frac{M_0}{d_2} \right) \left(\frac{C}{q_1} + M_0 \right).$$

Finally executing the remaining sums, we get

$$\begin{aligned} \Omega_{\neq 0} &\ll \frac{q_1 N_0 r N_2 C^4}{n_1^3 M_0^{1/2} Q^3} \frac{C}{q_1} \left(\frac{Cn_1}{q_1} + M_0 \right) \left(\frac{C}{q_1} + M_0 \right) \\ &\ll \frac{r C^5}{n_1^3 M_0^{1/2} Q^3} \frac{C Q n_1 r}{q_1} \left(\frac{Cn_1}{q_1} + M_0 \right) \left(\frac{C}{q_1} + M_0 \right) \\ &\ll \frac{r^2 C^6 Q M_0^2}{Q^3 M_0^{1/2}} \left(\frac{1}{n_1^2 q_1} \right) \ll \frac{r^2 C^5 Q^2 M_0^2}{Q^3 M_0^{1/2}} \left(\frac{1}{n_1^2 q_1} \right). \end{aligned}$$

Upon substituting this bound in place of Ω in (10), we arrive at

$$\sup_C \frac{N^{17/12}}{P_2^{1/4} P_1^{1/6} Q r^{2/3} C^{5/2}} \frac{r C^{5/2} M_0^{3/4}}{\sqrt{Q}} \sum_{\pm} \sum_{\substack{n_1 \\ (n_1, r) \ll C}} n_1^{1/3} \Theta^{1/2} \sum_{\substack{n_1 \\ (n_1, r) \mid q_1 | (n_1 r)^\infty}} \frac{1}{\sqrt{n_1^2 q_1}}.$$

Note that (for details see [Munshi 2022])

$$\sum_{\frac{n_1}{(n_1,r)} \ll C} n_1^{1/3} \Theta^{1/2} \sum_{\frac{n_1}{(n_1,r)} |q_1|(n_1r)^\infty} \frac{1}{\sqrt{n_1^2 q_1}} \ll \sum_{n_1 \ll Cr} \frac{(n_1, r)^{1/2}}{n_1^{7/6}} \Theta^{1/2} \ll N_0^{1/6}.$$

On plugging in this estimate, we get

$$\sup_C \frac{N^{17/12}}{P_2^{1/4} P_1^{1/6} Q r^{2/3} C^{5/2}} \frac{r C^{5/2} M_0^{3/4}}{\sqrt{Q}} N_0^{1/6} \ll \sqrt{r} N^{3/4} \sqrt{P_2}.$$

Next we consider the case where $C n_1 / q_1 \gg M_0$. Here our count for m modulo d_2 is not precise and so we need to adopt a different strategy for counting. We consider the first congruence relation in (25)

$$n_1 q_2' d_2' \mp mn_2 P_1 \bar{P}_2 \equiv 0 \pmod{d_2}.$$

Note that

$$n_1 q_2' d_2' P_2 \mp mn_2 P_1 \ll C P_2 n_1 / q_1 + M_0 N_2 P_1 \ll C P_2 n_1 / q_1 + C P_2 n_1 / q_1 \ll C P_2 n_1 / q_1.$$

Let

$$n_1 q_2' d_2' P_2 - mn_2 P_1 = h d_2, \quad \text{with } h \ll P_2 n_1. \tag{26}$$

Similarly, we write the second congruence relation as

$$n_1 q_2 d_2 P_2 + m' n_2 P_1 = h' d_2', \quad \text{with } h' \ll P_2 n_1. \tag{27}$$

Using this congruence, we see that the number of d_2' is given by $O((d_2, h'))$. Next we multiply h' and $P_2 q_2' n_1$ into (26) and (27) respectively to arrive at the following equation:

$$mn_2 P_1 h' + h h' d_2 = n_1^2 q_2 q_2' d_2 P_2^2 + P_2 q_2' n_1 m' n_2 P_1. \tag{28}$$

We now rearrange the above equation as follows:

$$P_2 q_2' n_1 m' - m h' = \frac{(h h' - n_1^2 q_2 q_2' P_2^2) d_2}{P_1 n_2} := \frac{\xi}{P_1 n_2}.$$

Reducing this equation modulo h' , the number of m' turns out to be

$$O\left((P_2 q_2' n_1, h') \left(1 + \frac{P_2}{h'}\right)\right).$$

Thus we arrive at the following bound for Ω :

$$\frac{q_1^2 N_0 r C^3}{n_1^3 M_0^{1/2} Q^3} \sum_{d_2 \sim \frac{C}{q_1}} \frac{C^2}{q_1^2} \sum_{\substack{q_2 \sim C^\epsilon \\ q_2' \sim C^\epsilon}} \sum_{\substack{h, h' \ll P_2 n_1 \\ \xi \equiv 0 \pmod{P_1 n_2} \\ m h' - \xi / P_1 n_2 \equiv 0 \pmod{P_2}}} \sum_{m \sim M_0} \sum_{n_2 \ll N_2} (m, n_1)(d_2, h')(P_2 q_2' n_1, h') \left(1 + \frac{P_2}{h'}\right).$$

Next we count the number of m to get

$$\sum_{\substack{m \sim M_0 \\ mh' - \xi / P_1 n_2 \equiv 0 \pmod{P_2}}} (m, n_1) = \sum_{\ell | n_1} \ell \sum_{\substack{m \sim M_0 / \ell \\ mh' - \xi / \ell P_1 n_2 \equiv 0 \pmod{P_2}}} 1 \ll \sum_{\ell | n_1} \ell.$$

Also given any ξ (necessarily nonzero) the congruence

$$\xi = (hh' - n_1^2 q_2 q_2' P_2^2) d_2 \equiv 0 \pmod{n_2},$$

implies that there are $O(N^\epsilon)$ many n_2 . We are left with the following expression:

$$\Omega_{\neq 0} \ll \frac{N_0 r C^5}{n_1^3 M_0^{1/2} Q^3} \sum_{\ell | n_1} \ell \sum_{d_2 \sim \frac{C}{q_1}} \sum_{\substack{q_2 \sim C^\epsilon \\ q_2' \sim C^\epsilon}} \sum_{\substack{h, h' \ll P_2 n_1 \\ \xi \equiv 0 \pmod{P_1 \ell}}} (d_2, h') (P_2 q_2' n_1, h') \left(1 + \frac{P_2}{h'}\right).$$

We now consider the congruence

$$\xi = (hh' - n_1^2 q_2 q_2' P_2^2) d_2 \equiv 0 \pmod{P_1 \ell}.$$

Let's first assume that $d_2 \equiv 0 \pmod{P_1}$. Then first counting the number of d_2 followed by h and h' , we see that the number of tuples (h, h', d_2) is given by $O((P_2^2 n_1^2 C)/(P_1 q_1 \ell))$. Lastly executing the sum over ℓ , we arrive at

$$\frac{N_0 r C^5}{n_1^2 M_0^{1/2} Q^3} \frac{P_2^2 n_1^2 C}{P_1 q_1}.$$

Now let $(d_2, P_1 \ell) = 1$. Then we have

$$hh' - n_1^2 q_2 q_2' P_2^2 \equiv 0 \pmod{P_1 \ell},$$

from which the number of h turns out to be $P_2 n_1 / P_1 \ell$. Next counting the number of d_2 followed by number of h' , we see that the number of tuples (h, h', d_2) is given by $O((P_2^2 n_1^2 C)/(P_1 q_1 \ell))$. Hence, in this case also, we get the same bound. Thus we conclude that

$$\Omega_{\neq 0} \ll \frac{N_0 r C^5}{n_1^2 M_0^{1/2} Q^3} \frac{P_2^2 n_1^2 C}{P_1 q_1}.$$

Upon substituting this bound in (10), we arrive at

$$\begin{aligned} \sup_{C \ll Q} \frac{N^{17/12}}{P_2^{1/4} P_1^{1/6} Q r^{2/3} C^{5/2}} \left(\frac{N_0 r C^5}{Q^3 M_0^{1/2} P_1} \right)^{1/2} \sum_{\pm} \sum_{\substack{n_1 \\ (n_1, r) \ll C}} n_1^{1/3} \Theta^{1/2} \sum_{\substack{n_1 \\ (n_1, r) | q_1 | (n_1, r)^\infty}} \frac{1}{\sqrt{n_1 q_1}} \\ \ll \frac{N^{17/12}}{P_2^{1/4} P_1^{1/6} Q r^{2/3}} \left(\frac{N_0 r}{Q^3 M_0^{1/2} P_1} \right)^{1/2} \sum_{\substack{n_1 \\ (n_1, r) \ll C}} \frac{\Theta^{1/2}}{n_1^{2/3}} (n_1, r)^{1/2}. \end{aligned}$$

Now following the argument of [Munshi 2022], we conclude that

$$\sum_{\substack{n_1 \\ (n_1, r) \ll C}} \frac{\Theta^{1/2}}{n_1^{2/3}} (n_1, r)^{1/2} \ll N_0^{1/6} \sum_{\substack{n_1 \\ (n_1, r) \ll C}} \frac{(n_1, r)^{1/2}}{n_1} \ll N_0^{1/6}.$$

Hence the contribution of the nonzero frequency to $S_r(N)$ is dominated by

$$\frac{N^{17/12}}{P_2^{1/4} P_1^{1/6} Q r^{2/3}} \left(\frac{N_0 r}{Q^3 M_0^{1/2}} \frac{P_2^2 Q}{P_1} \right)^{1/2} N_0^{1/6} \ll \sqrt{r} N^{3/4} \sqrt{P_2}. \quad \square$$

9. Conclusion

Finally, plugging bounds from Lemmas 7.1 and 8.1 into Lemma 2.1, we get

$$\begin{aligned} L\left(\frac{1}{2}, F \times f\right) &\ll_{\epsilon} Q^{\epsilon} \sum_{r \leq Q^{(1+2\epsilon)/4}} \frac{1}{r} \sup_{N \leq Q^{1/2+\epsilon}/r^2} \sqrt{r} N^{1/4} (\sqrt{P_1} + \sqrt{P_2}) \\ &\ll \sum_{r \leq Q^{(1+2\epsilon)/4}} \frac{1}{r} Q^{1/8+\epsilon} (\sqrt{P_1} + \sqrt{P_2}) \\ &\ll Q^{1/8+\epsilon} (\sqrt{P_1} + \sqrt{P_2}) \ll Q^{1/4+\epsilon} \left(\frac{P_1^{1/4}}{P_2^{3/8}} + \frac{P_2^{1/8}}{P_1^{1/4}} \right). \end{aligned}$$

This establishes Theorem 1.

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