Algebra & Number Theory

Volume 18
2024
No. 3

On Tamagawa numbers of CM tori

Pei-Xin Liang, Yasuhiro Oki, Hsin-Yi Yang and Chia-Fu Yu
Appendix by Jianing Li and Chia-Fu Yu

msp

ALGEBRA AND NUMBER THEORY 18:3 (2024)
https://doi.org/10.2140/ant.2024.18.583

On Tamagawa numbers of CM tori

Pei-Xin Liang, Yasuhiro Oki, Hsin-Yi Yang and Chia-Fu Yu
Appendix by Jianing Li and Chia-Fu Yu

We investigate the problem of computing Tamagawa numbers of CM tori. This problem arises naturally from the problem of counting polarized abelian varieties with commutative endomorphism algebras over finite fields, and polarized CM abelian varieties and components of unitary Shimura varieties in the works of Achter, Altug, Garcia and Gordon and of Guo, Sheu and Yu, respectively. We make a systematic study on Galois cohomology groups in a more general setting and compute the Tamagawa numbers of CM tori associated to various Galois CM fields. Furthermore, we show that every (positive or negative) power of 2 is the Tamagawa number of a CM tori, proving the analogous conjecture of Ono for CM tori.

1. Introduction

In his two fundamental papers Takashi Ono [1961; 1963b] investigated the arithmetic of algebraic tori. He introduced and explored the class number and Tamagawa number of T, which will be denoted by $h(T)$ and $\tau(T)$ respectively (also see Section 2 for the definitions). One arithmetic significance of these invariants is that the class number $h(\mathbb{G}_{\mathrm{m},k})$ is equal to the class number h_k of the number field k, and the analytic class number formula for k can be reformulated by the simple statement $\tau(\mathbb{G}_{\mathrm{m},k}) = 1$. Thus, the class numbers of algebraic tori can be viewed as generalizations of class numbers of number fields, while Tamagawa numbers play a key role in the extension of analytic class number formulas.

Ono [1963b] showed that $\tau(T) = |H^1(k, X(T))|/|\text{Ш}^1(k, T)|$, where $X(T)$ is the group of characters of T and $\text{Ш}^1(k, T)$ is the Tate–Shafarevich group of T. Kottwitz [1984] generalized Ono's formula to reductive groups and proved [Kottwitz 1988] the celebrated conjecture of Weil for the Tamagawa number of semisimple simply connected groups. Ono constructed a 15-dimensional algebraic torus with Tamagawa number $\frac{1}{4}$, showing that $\tau(T)$ can be nonintegral and conjectured in [Ono 1963a] that every positive rational number is equal to $\tau(T)$ for some torus T. Ono's conjecture was proved by S. Katayama [1985] for the number field case. For some later studies of class numbers and Tamagawa numbers of algebraic tori we refer to the works of J.-M. Shyr [1977, Theorem 1], S. Katayama [1991], M. Morishita [1991], C. González-Avilés [2008; 2010] and M.-H. Tran [2017].

In this article we are mainly concerned with the problem of computing the Tamagawa numbers of complex multiplication (CM) algebraic tori. CM tori are closely related to the arithmetic of CM abelian varieties and computing their Tamagawa numbers itself is a way of exploring the structure of CM fields.

MSC2020: primary 14K22; secondary 11R29.
Keywords: CM algebraic tori, Tamagawa numbers.

This problem directly contributes to recent works of Achter, Altug, Garcia and Gordon [Achter et al. 2023] and of J. Guo, N. Sheu and the fourth named author [Guo et al. 2022]. In the former one the authors computed the size of an isogeny class of principally polarized abelian varieties over a finite field with commutative endomorphism algebra, and express the number in terms of a discriminant, the Tamagawa number, and the product of Frobenius local densities. In the latter one the authors computed formulas for certain CM abelian varieties and certain polarized abelian varieties over finite fields with commutative endomorphism algebras upon the results of [Xue and Yu 2021]. Using the class number formula for CM tori, they also computed the numbers of connected components of complex unitary Shimura varieties. In the appendix of [Achter et al. 2023], W.-W. Li and T. Rüd have obtained several initial results of the values of $\tau(T)$. Our goals are to prove more cases of CM tori and to determine the range of the values of Tamagawa numbers of all CM tori. With a similar goal but using different methods, T. Rüd [2022] obtains several results along this direction. He provides an algorithm, among others, for giving precise lower bounds and determining possible Tamagawa numbers, and obtains the values $\tau(T)$ for several other CM tori of lower dimension.

We shall describe our results towards computing Tamagawa numbers for a more general class of algebraic tori which include CM tori and then give more detailed results of CM tori. Let k be a global field and $K := \prod_{i=1}^r K_i$ be the product of finite separable field extensions K_i of k. Let $E := \prod_{i=1}^r E_i$, where each $E_i \subset K_i$ is a subextension of K_i. Denote by $T^K = \prod_i T^{K_i}$ and $T^E = \prod_i T^{E_i}$ the algebraic k-tori associated to the multiplicative groups of K and E, respectively, and let $N_{K/E} = \prod_i N_{K_i/E_i} : T^K \to T^E$ be the norm map. Note that the norm map is surjective; one can check this easily by showing the surjectivity on their $\bar{k}$-rational points (or combining [Morishita 1991, Section 3] with [Conrad et al. 2010, Corollary A.5.4(1), page 507]). Let $\mathbb{G}_{\mathrm{m},k} \to T^E$ be the closed immersion induced by the diagonal embedding $k \hookrightarrow E$; see [Conrad et al. 2010, Proposition A.5.7, page 510]. We regard $\mathbb{G}_{\mathrm{m},k}$ as a k-subtorus of T^E by identifying $\mathbb{G}_{\mathrm{m},k}$ with its image in T^E. We write $T^{K/E,1}$ for the kernel of $N_{K/E}$ and

$$T^{K/E,k} = N_{K/E}^{-1}(\mathbb{G}_{\mathrm{m},k}) := T^K \times_{T^E} \mathbb{G}_{\mathrm{m},k}$$

for the preimage of the k-subtorus $\mathbb{G}_{\mathrm{m},k} \subset T^E$. The tori $T^{K/E,k}$, $T^{K/E,1}$ and $\mathbb{G}_{\mathrm{m},k}$ fit into the following short exact sequence:

$$1 \to T^{K/E,1} \xrightarrow{\iota} T^{K/E,k} \xrightarrow{N_{K/E}} \mathbb{G}_{\mathrm{m},k} \to 1.$$

Let L be the smallest splitting field of $T^{K/E,k}$ and let $G = \mathrm{Gal}(L/k)$. We let $\Lambda := X(T^{K/E,k})$ and $\Lambda^1 := X(T^{K/E,1})$ be the character groups of $T^{K/E,k}$ and $T^{K/E,1}$, respectively. Taking the characters, we have the following exact sequence of G-lattices:

$$0 \to \mathbb{Z} \to \Lambda \to \Lambda^1 \to 0. \tag{1-1}$$

Write $H_i := \mathrm{Gal}(L/K_i)$, $\widetilde{N}_i := \mathrm{Gal}(L/E_i)$, and $N_i^{\mathrm{ab}} := \widetilde{N}_i / D(\widetilde{N}_i) H_i$, where $D(\widetilde{N}_i)$ denotes the commutator group of $\widetilde{N}_i$ (If H_i is normal in $\widetilde{N}_i$ and one puts $N_i := \widetilde{N}_i / H_i$, then N_i^{ab} coincides with the abelianization of the group N_i). When there is no confusion, for brevity we shall write $H^q(A)$ for $H^q(G, A)$ for any G-module A.

Let $\mathrm{Ver}_{G,N_i} : G \to N_i^{\mathrm{ab}}$ denote the transfer from G into N_i^{ab}; see Definition 3.6. For any abelian group H, the Pontryagin dual of H is denoted by $H^\vee$. By Ono's formula (2-3) and the Poitou–Tate duality, we have $\tau(T) = |H^1(G, X(T))|/|\text{Ш}^2(G, X(T))|$, where $\text{Ш}^i(G, X(T))$ is the i-th Tate–Shafarevich group of $X(T)$.

Let $\mathcal{D}$ be the set of all decomposition groups of G. Denote by

$$r^i_{D,A} : H^i(G, A) \to H^i(D, A), \quad \text{where } D \in \mathcal{D} \tag{1-2}$$

the restriction map for G-module A, and let

$$r^i_{\mathcal{D},A} : H^i(G, A) \to \bigoplus_{D\in\mathcal{D}} H^i(D, A) \tag{1-3}$$

be the restriction map for each G-module A by sending $\xi \mapsto (r^i_{D,A}(\xi))_{D\in\mathcal{D}}$. Put

$$H^2(\mathbb{Z})' := \{x \in H^2(G, \mathbb{Z}) : r^2_{\mathcal{D},\mathbb{Z}}(x) \in \mathrm{Im}(\delta_{\mathcal{D}})\}, \tag{1-4}$$

where $\delta_{\mathcal{D}} : \bigoplus_{D\in\mathcal{D}} H^1(D, \Lambda^1) \to \bigoplus_{D\in\mathcal{D}} H^2(D, \mathbb{Z})$ is the connecting homomorphism induced from (1-1). The group $H^2(\mathbb{Z})'$ plays a similar role of a Selmer group.

Theorem 1.1. *Let the notation be as above*:

(1) *There is a canonical isomorphism* $H^1(G, \Lambda^1) \simeq \bigoplus_i N_i^{\mathrm{ab},\vee}$.

(2) *There is a canonical isomorphism*

$$H^1(G, \Lambda) \simeq \mathrm{Ker}\Big(\sum \mathrm{Ver}^\vee_{G,N_i} : \bigoplus_i N_i^{\mathrm{ab},\vee} \to G^{\mathrm{ab},\vee}\Big),$$

where $\mathrm{Ver}_{G,N_i} : G \to N_i^{\mathrm{ab}}$ *is the transfer map.*

(3) *Assume that* K_i/E_i *is cyclic with Galois group* N_i *for all* i. *Then* $\text{Ш}^2(\Lambda) \simeq H^2(\mathbb{Z})'/\mathrm{Im}(\delta)$ *and*

$$\tau(T^{K/E,k}) = \frac{\prod_{i=1}^r |N_i|}{|H^2(\mathbb{Z})'|}. \tag{1-5}$$

(4) *If we further assume that the subgroups* $\widetilde{N}_i$ *and* H_i *are all normal in* G, *and let*

$$\mathrm{Ver}_{G,\boldsymbol{N}} = (\mathrm{Ver}_{G,N_i})_i : G^{\mathrm{ab}} \to \prod_i N_i \quad \textit{and} \quad \mathrm{Ver}_{D,\bar{\boldsymbol{D}}} = (\mathrm{Ver}_{D,\bar{D}_i})_i : D^{\mathrm{ab}} \to \prod_i \bar{D}_i$$

denote the corresponding transfer maps, where $D_i := D \cap \widetilde{N}_i$ *and* $\bar{D}_i := D_i/(D_i \cap H_i) \subset N_i$, *respectively*, *then*

$$H^2(\mathbb{Z})' = \{f \in G^{\mathrm{ab},\vee} : f|_{D^{\mathrm{ab}}} \in \mathrm{Im}(\mathrm{Ver}^\vee_{D,\bar{\boldsymbol{D}}}) \forall D \in \mathcal{D}\},$$

and

$$\text{Ш}^2(\Lambda) \simeq \frac{\{f \in G^{\mathrm{ab},\vee} : f|_{D^{\mathrm{ab}}} \in \mathrm{Im}(\mathrm{Ver}^\vee_{D,\bar{\boldsymbol{D}}}) \forall D \in \mathcal{D}\}}{\mathrm{Im}(\mathrm{Ver}^\vee_{G,\boldsymbol{N}})}.$$

A certain case of Theorem 1.1(1) was obtained by Rüd [2022, Proposition 2.5] and he also computed the group $H^1(G, \Lambda)$ explicitly; see Section 3.2 of [loc. cit.].

Using Theorem 1.1(2), we give a different proof of a result of Li and Rüd [Achter et al. 2023, Proposition A.11] which does not rely on Kottwitz's formula; see Corollary 7.2. By (1-5), the ratio $\left(\prod_{i=1}^r |N_i|\right)/\tau(T^{K/E,k})$ is a positive integer; this gives a simple upper bound of $\tau(T^{K/E,k})$.

Observe that $T^{K/E,k}$ is a k-subtorus of $\prod_{i=1}^r T^{K_i/E_i,k}$ and is not equal to $\prod_{i=1}^r T^{K_i/E_i,k}$ when $r > 1$ by the dimension counting. The following theorem gives a sufficient condition for $\tau(T^{K/E,k})$ being equal to $\prod_{i=1}^r \tau(T^{K_i/E_i,k})$.

Theorem 1.2. *Suppose the following conditions hold*: (*a*) *for each* $1 \le i \le r$, *the extension* K_i/k *is Galois with Galois group* G_i, $N_i = \mathrm{Gal}(K_i/E_i)$ *is cyclic and every decomposition group of* G_i *is cyclic*; *and* (*b*) $G \simeq G_1 \times \cdots \times G_r$. *Then we have* $\tau(T^{K/E,k}) = \prod_{i=1}^r \tau(T^{K_i/E_i,k})$.

Now let $K = \prod_{i=1}^r K_i$ be a CM algebra over $\mathbb{Q}$ and $E := K^+$ the $\mathbb{Q}$-subalgebra fixed by the canonical involution ι, and let $T = T^{K/E,\mathbb{Q}}$ be the associated CM torus over $\mathbb{Q}$.

Theorem 1.3. *For any integer* n, *there exists a CM torus* T *over* $\mathbb{Q}$ *such that* $\tau(T) = 2^n$.

Finally, we give a number of results of $\tau(T)$ for Galois CM fields.

Theorem 1.4. *Suppose that* K *is a Galois CM field with Galois group* $G = \mathrm{Gal}(K/\mathbb{Q})$. *Let* $g := [K^+ : \mathbb{Q}]$, $G^+ = \mathrm{Gal}(K^+/\mathbb{Q})$ *and* $T = T^{K/K^+,\mathbb{Q}}$ *be the associated CM torus*:

(1) *If* $K = \mathbb{Q}(\zeta_n) \neq \mathbb{Q}$ *is the* n-*th cyclotomic field with either* $4 \mid n$ *or odd* n, *then*:
   - (a) *If* n *is either a power of an odd prime* p *or* $n = 4$, *then* $\tau(T) = 1$.
   - (b) *In other cases, we have* $\tau(T) = 2$.

(2) *If* G *is abelian, then* $\tau(T) \in \{1, 2\}$. *Moreover, the following statements hold*:
   - (a) *If* g *is odd, then* $\tau(T) = 1$.
   - (b) *If* g *is even and the exact sequence*
   $$1 \to \langle \iota \rangle \to G \to G^+ \to 1 \tag{1-6}$$
   splits, then $\tau(T) = 2$.

(3) *Let* G *be possibly nonabelian and suppose that the short exact sequence* (1-6) *splits. Then* $\tau(T) \in \{1, 2\}$ *and the following statements hold*:
   - (a) *When* g *is odd*, $\tau(T) = 1$.
   - (b) *Suppose* g *is even and let* $g^{\mathrm{ab}} := |G^{+\mathrm{ab}}|$, *the cardinality of the abelianization of* G^+:
     - (i) *If* g^{ab} *is even, then* $\tau(T) = 2$.
     - (ii) *If* g^{ab} *is odd, then there is a unique nonzero element* ξ *in the* 2-*torsion subgroup* $H^2(\Lambda)[2]$ *of* $H^2(\Lambda)$. *Moreover,* $\tau(T) = 1$ *if and only if its restriction* $r_D(\xi) = 0$ *in* $H^2(D, \Lambda)$ *for all* $D \in \mathfrak{D}$ *where* r_D *as defined in* (1-2).

(4) *Let P and Q be two odd nonsquare positive integers such that $P-1=a^2$ and $Q-1=Pb^2$ for some integers $a, b \in \mathbb{N}$. Let $K := \mathbb{Q}(\sqrt{\alpha})$ with $\alpha := -(P+\sqrt{P})(Q+\sqrt{Q})$. Then K is a Galois CM field with Galois group Q_8 and*

$$\tau(T) = \begin{cases} \frac{1}{2} & \text{if } \left(\frac{P}{q}\right) = 1 \text{ for all prime } q \mid Q; \\ 2 & \text{otherwise.} \end{cases}$$

(5) *Suppose the Galois extension $K/\mathbb{Q}$ has Galois group D_n of order $2n$. Then n is even and $\tau(T)=2$.*

We mention that some cases of Theorem 1.4 where G is abelian, dihedral or quaternionic were also obtained by Rüd; he also obtained complete results when the degree of (possibly non-Galois) K is less than or equal to 8; see [Rüd 2022, Theorem 1.3, Proposition 1.5 and Examples 5.7 and 5.19]. The overlapping results are obtained by different approaches.

We explain the idea of the proof of Theorem 1.3. First of all, it is rather difficult to construct a CM field such that the Tamagawa number $\tau(T)$ of the associated algebraic torus T is small. Suppose that $K/\mathbb{Q}$ is Galois with Galois group G. T. Rüd [2022] implements a SageMath algorithm for computing $\tau(T)$ and computed the groups $Ш^2_{\mathfrak{C}}(G, \Lambda)$ for all 2-groups G of order ≤ 256,[1] where $Ш^2_{\mathfrak{C}}(G, \Lambda) := \operatorname{Ker}(H^2(G, \Lambda) \to \prod_C H^2(C, \Lambda))$ and C runs through all cyclic subgroups of G. Based on Rüd's result there is at most one case such that $\tau(T) = \frac{1}{4}$; see [loc. cit., Proposition 5.26]. To get around this, we construct an infinite family of "totally" linearly disjoint Q_8-CM fields $\{K_i\}$ for $i \ge 1$ with $\tau(T_i) = \frac{1}{2}$, that is, the Galois group of the compositum of any finitely many of these Q_8-CM fields K_i is the product of the Galois groups $\operatorname{Gal}(K_i/\mathbb{Q})$. The CM algebra $K := \prod_{i=1}^r K_i$ then satisfies the conditions in Theorem 1.2 and it follows that the CM torus T associated to K satisfies $\tau(T) = \frac{1}{2^r}$.

We point out that the proof of Theorem 1.3 is actually quite tricky. First of all, it follows from [loc. cit.] or Theorem 1.4 that Q_8-CM fields are the simplest ones so that the associated CM tori T can have $\tau(T) = \frac{1}{2}$. On the other hand, Theorem 1.2 requires a condition that every decomposition group of each CM field extension $K_i/\mathbb{Q}$ is cyclic. Fortunately, for any Q_8-CM field K_i with CM torus T_i, one has $\tau(T_i) = \frac{1}{2}$ if and only if this condition for $K_i/\mathbb{Q}$ holds (see Proposition 6.7), so that we can apply Theorem 1.2 to the product of them. On the other hand, one may be wondering whether this condition is superfluous. For this question, we construct linearly disjoint Galois CM fields K_1 and K_2 such that

$$\tau(T_1) = 2, \quad \tau(T_2) = 1, \quad \tau(T) = 1, \tag{1-7}$$

where T_1, T_2 and T are CM tori associated to K_1, K_2 and $K_1 \times K_2$, respectively. This example shows that the cyclicity of decomposition groups of every $K_i/\mathbb{Q}$ is not superfluous.

Theorem 1.3 proves an analogous but more involved conjecture of Ono for CM tori. In the Appendix Jianing Li and the fourth named author show that for any global field k and any positive rational number α, there exists an algebraic torus T over k with $\tau(T) = \alpha$; see Theorem A.8. This extends the result of Katayama [1985] and proves Ono's conjecture for global fields.

[1] See https://toadrush.github.io/tamagawa-cmtori/.

Though our original motivation of investigating Tamagawa numbers of CM tori comes from counting certain abelian varieties and exploring the structure of CM fields, the main part of the problem itself was to compute or investigate their Tate–Shafarevich groups. We explain in Remark 7.10 how the Tate–Shafarevich group of a CM torus also comes into play in the theory of Shimura varieties of PEL-type. Tate–Shafarevich groups measure the failure of the local-global principle for various objects, which is one of main interests in number theory and has been actively studied. For the interested reader's reference, we mention some development on the Tate–Shafarevich group of multinorm one tori, which are different type of tori from the CM tori studied in the present paper. Hürlimann [1984] proved that the multiple norm principle holds for any commutative etale k-algebra of the form $K_1 \times K_2$ in which one component is cyclic and the other one is Galois. Prasad and Rapinchuk [2010] settled the problem of the local-global principle for embeddings of fields with involution into simple algebras with involution, where they also investigated the multiple norm principle. The multiple norm principle has been investigated further by Pollio and Rapinchuk [2013; 2014], Demarche and D. Wei [2014] and D. Wei and F. Xu [2012]. Bayer-Fluckiger, T.-Y. Lee and Parimala [Bayer-Fluckiger et al. 2019] studied the Tate–Shafarevich group of general multinorm one tori in which one of factors is a cyclic extension. They give a simple rule for determining the Tate–Shafarevich group in the case of products of extensions of prime degree p. T.-Y. Lee [2022] computes explicitly the Tate–Shafarevich group for the cases where every factor is a cyclic extension of degree p-power.

This article is organized as follows. Section 2 includes preliminaries and background on Tamagawa numbers of algebraic tori, and some known results of those of CM tori due to Li–Rüd and Guo–Sheu–Yu. Section 3 discusses transfer maps, their extensions and connection with class field theory. In Sections 4 and 5 we compute the Galois cohomology groups of character groups of a class of algebraic tori $T^{K/E,k}$ and $T^{K/E,1}$. Section 6 treats Galois CM tori and in Sections 7 ans 8 we determine the precise ranges of Tamagawa numbers of CM tori. In Section 9 we show there are infinitely many pairs of linearly disjoint Galois CM fields K_1 and K_2 satisfying (1-7). In the Appendix Jianing Li and the fourth author prove Ono's conjecture for global fields.

2. Preliminaries, background and some known results

2A. ***Class numbers and Tamagawa numbers of algebraic tori.*** The cardinality of a set S will be denoted by $|S|$. For any field k, let $\bar{k}$ be a fixed algebraic closure of k, let k^{sep} be the separable closure of k in $\bar{k}$ and denote by $\Gamma_k := \mathrm{Gal}(k^{\mathrm{sep}}/k)$ the Galois group of k. Let $\mathbb{G}_{\mathrm{m},k} := \operatorname{Spec} k[X, X^{-1}]$ denote the multiplicative group associated to k with the usual multiplicative group law.

Definition 2.1. (1) A connected linear algebraic group T over a field k is said to be an *algebraic torus* over k if there exists a finite field extension K/k such that there exists an isomorphism $T \otimes_k K \simeq \mathbb{G}_{\mathrm{m},K}^d$ of algebraic groups over K for some positive integer d. Then d is equal to the dimension of T. If T is an algebraic k-torus and K is a field extension of k that satisfies the above property, then K is called a *splitting field* of T. The smallest splitting field (which is unique and Galois, see below) is called *the minimal splitting field of T*.

(2) For any algebraic torus T over k, denote by $X(T) := \mathrm{Hom}_{k^{\mathrm{sep}}}(T \otimes_k k^{\mathrm{sep}}, \mathbb{G}_{\mathrm{m},k^{\mathrm{sep}}})$ the group of characters of T. It is a finite free $\mathbb{Z}$-module of rank d together with a continuous action of the Galois group Γ_k of k.

It is shown in [Ono 1961, Proposition 1.2.1] (also see [Yu 2019] for other proofs) that every algebraic k-torus splits over a finite *separable* field extension K/k. The action of Γ_k on $X(T)$ gives a continuous representation

$$r_T : \Gamma_k \to \mathrm{Aut}(X(T))$$

which factors through a faithful representation of a finite quotient $\mathrm{Gal}(L/k)$ of Γ_k. Here L is the fixed field of the kernel of r_T and is the smallest splitting field of T. In particular, L is a finite Galois extension of k and $X(T)$ can be also regarded as a $\mathrm{Gal}(L/k)$-module.

In the remainder of this article, k denotes a global field. We only discuss finite separable field extensions in this paper. For each place v of k, denote by k_v the completion of k at v, and O_v the ring of integers of k_v if v is finite. For each finite place v, the group $T(k_v)$ of k_v-valued points of T contains a unique maximal open compact subgroup, which is denoted by $T(O_v)$. Let $\mathbb{A}_k$ be the adele ring of k, and let S be a nonempty finite set S of places of k containing all non-Archimedean places if k is a number field. Denote by $U_{T,S} = \prod_{v\in S} T(k_v) \times \prod_{v\notin S} T(O_v)$ the unit group with respect to S and let $\mathrm{Cl}_S(T) := T(\mathbb{A}_k)/T(k)U_{T,S}$ be the *S-class group* of T. By a finiteness theorem of Borel [1963], $\mathrm{Cl}_S(T)$ is a finite group and its cardinality is denoted by $h_S(T)$, called the *S-class number* of T. If k is a number field and $S = \infty$ consists of all non-Archimedean places, we write $U_T := U_{T,\infty}$, $\mathrm{Cl}(T) := T(\mathbb{A}_k)/T(k)U_T$ the *class group* of T and call $h(T) := |\mathrm{Cl}(T)|$ the *class number* of T.

It follows immediately from the definition that if K/k is a finite extension and T_K is an algebraic K-torus, then $h(R_{K/k}T_K) = h(T_K)$, where $R_{K/k}$ denotes the Weil restriction of scalars from K to k. Note that $\dim R_{K/k}T_K = \dim T_K \cdot [K : k]$. It is well known [Ono 1961; 1963b] that

$$X(R_{K/k}T_K) \simeq \mathrm{Ind}_{\Gamma_K}^{\Gamma_k} X(T_K) = \mathrm{Ind}_{\mathrm{Gal}(L/K)}^{\mathrm{Gal}(L/k)} X(T_K),$$

as $\mathrm{Gal}(L/k)$-modules, where L is any finite Galois extension of k over which the algebraic k-torus $R_{K/k}T_K$ splits.

We recall the definition of the Tamagawa number of an algebraic k-torus T. Fix a finite Galois splitting field extension K/k of T with Galois group G. Let χ_T be the character of representation $(X(T) \otimes \mathbb{C}, r_T)$ of G over $\mathbb{C}$, that is, $\chi_T : G \to \mathbb{C}$, $\chi_T(g) := \mathrm{tr}(r_T(g))$ for all $g \in G$. Let

$$L(s, K/k, \chi_T) := \prod_{v\nmid\infty} L_v(s, K/k, \chi_T)$$

be the Artin L-function of the character χ_T, where $L_v(s, K/k, \chi_T)$ is the local Artin L-factor at v. It follows from Brauer's induction theorem (see [Serre 1977, Section 10.5]) that $L(s, K/k, \chi_T)$ has a pole of order a at $s = 1$, where a is the rank of the G-invariant sublattice $X(T)^G$.

Let ω be a nonzero invariant differential form on T of highest degree defined over k. To each place v, one associates a Haar measure ω_v on $T(k_v)$. Then the product of the Haar measures

$$\prod_{v\,|\,\infty} \omega_v \cdot \prod_{v\nmid\infty} (L_v(1, K/k, \chi_T)\cdot\omega_v)$$

converges absolutely and defines a Haar measure on $T(\mathbb{A}_k)$.

Write (N) for the number field case and (F) for the function field case. For (N), let d_k be the discriminant of k. For (F), $k = \mathbb{F}_q(C)$ is the function field of a smooth projective geometrically connected curve over $\mathbb{F}_q$ and let $g(C)$ be the genus of C.

Definition 2.2. Let T be an algebraic torus over a global field k and ω be a nonzero invariant differential form on T defined over k of highest degree. Then

$$\omega_{\mathbb{A},can} := \frac{\prod_{v\,|\,\infty} \omega_v \cdot \prod_{v\nmid\infty} (L_v(1, K/k, \chi_T)\cdot\omega_v)}{\mu_k^{(\dim T)}\cdot\rho_T},$$

defines a Haar measure on $T(\mathbb{A}_k)$, which is called the *Tamagawa measure on* $T(\mathbb{A}_k)$, where

$$\mu_k := \begin{cases} |d_k|^{1/2} & \text{for (N);} \\ q^{g(C)} & \text{for (F),} \end{cases} \quad \text{and} \quad \rho_T := \lim_{s\to 1} (s-1)^a L(s, K/k, \chi_T),$$

and a is the order of the pole of the Artin L-function $L(s, K/k, \chi_T)$ at $s = 1$.

Let $\xi_1, \dots, \xi_a$ be a basis of $X(T)^G$. Define

$$\xi : T(\mathbb{A}_k) \to \mathbb{R}_+^a, \quad x \mapsto (\|\xi_1(x)\|, \dots, \|\xi_a(x)\|),$$

where $\mathbb{R}_+ := \{x \in \mathbb{R}^\times : x > 0\} \subset \mathbb{R}^\times$ is the connected open subgroup. Let $T(\mathbb{A}_k)^1$ denote the kernel of ξ; one has an isomorphism

$$T(\mathbb{A}_k)/T(\mathbb{A}_k)^1 \simeq \operatorname{Im}\xi.$$

For (N), $\operatorname{Im}\xi = R_+^a$ and let $d^\times t := \prod_{i=1}^a dt_i/t_i$ be the canonical measure on $\mathbb{R}_+^a$.

For (F), the image $\operatorname{Im}\xi \subset (q^{\mathbb{Z}})^a$ is a subgroup of finite index and let $d^\times t$ be the counting measure on $\operatorname{Im}\xi$ with measure $(\log q)^a[(q^{\mathbb{Z}})^a : \operatorname{Im}\xi]$ on each point; see [Oesterlé 1984, Definition 5.9, page 24].

Remark 2.3. For (F), incorrectly stating in [Ono 1963b, page 56], the image $\operatorname{Im}\xi$, as pointed out by Tate, is actually not equal to $(q^{\mathbb{Z}})^a$ in general (see [Oesterlé 1984, page 25]), so the modification by the index $[(q^{\mathbb{Z}})^a : \operatorname{Im}\xi]$ is needed. We thank the referee for pointing out this to us.

Let $\omega^1_{\mathbb{A},\mathrm{can}}$ be the unique Haar measure on $T(\mathbb{A}_k)^1$ such that

$$\omega_{\mathbb{A},\mathrm{can}} = \omega^1_{\mathbb{A},\mathrm{can}} \cdot d^\times t, \tag{2-1}$$

that is, for any measurable function F on $T(\mathbb{A}_k)$ one has

$$\int_{T(\mathbb{A}_k)/T(\mathbb{A}_k)^1} \int_{T(\mathbb{A}_k)^1} F(xt)\omega^1_{\mathbb{A},\mathrm{can}} \cdot d^\times t = \int_{T(\mathbb{A}_k)} F(x)\omega_{\mathbb{A},\mathrm{can}}.$$

By a well-known theorem of Borel and Harish-Chandra [Platonov and Rapinchuk 1994, Theorem 5.6], the quotient space $T(\mathbb{A}_k)^1/T(k)$ has finite volume with respect to every Haar measure. In fact $T(\mathbb{A}_k)^1/T(k)$ is the unique maximal compact subgroup of $T(\mathbb{A}_k)/T(k)$, because the group $\mathbb{R}_+^a$ has no nontrivial compact subgroup.

Definition 2.4. Let T be an algebraic torus over a global field k. The *Tamagawa number* $\tau_k(T)$ *of* T is defined by

$$\tau_k(T) := \int_{T(\mathbb{A}_k)^1/T(k)} \omega^1_{\mathbb{A},\mathrm{can}}, \tag{2-2}$$

the volume of $T(\mathbb{A}_k)^1/T(k)$ with respect to $\omega^1_{\mathbb{A},\mathrm{can}}$, where $\omega^1_{\mathbb{A},\mathrm{can}}$ is the Haar measure on $T(\mathbb{A}_k)^1$ defined in (2-1).

One has the following properties ([Ono 1961, Theorem 3.5.1] also see [Ono 1963b, Section 3.2, page 57]):

(i) For any two algebraic k-tori T and T', one has $\tau_k(T \times_k T') = \tau_k(T) \cdot \tau_k(T')$.

(ii) For any finite extension k'/k and any algebraic k'-torus T', one has $\tau_k(R_{k'/k}T') = \tau_{k'}(T')$.

(iii) One has $\tau_k(\mathbb{G}_{\mathrm{m},k}) = 1$.

Note that in the number field case, the last statement (iii) is equivalent to the analytic class number formula [Lang 1994, VIII, Section 2, Theorem 5, page 161].

2B. *Values of Tamagawa numbers.*

Theorem 2.5 (Ono's formula). *Let K/k be a finite Galois extension with Galois group Γ, and T be an algebraic torus over k with splitting field K. Then*

$$\tau_k(T) = \frac{|H^1(\Gamma, X(T))|}{|\text{Ш}^1(\Gamma, T)|}, \tag{2-3}$$

where

$$\text{Ш}^i(\Gamma, T) := \mathrm{Ker}\Big(H^i(K/k, T) \to \prod_v H^i(K_w/k_v, T)\Big)$$

is the Tate–Shafarevich group associated to $H^i(\Gamma, T)$ for $i \geq 0$ and w is a place of K over v.

Proof. See [Oesterlé 1984, Chapter IV, Corollary 3.3, page 56]. □

Remark 2.6. The cohomology groups $H^1(\Gamma, X(T))$ and $\text{Ш}^1(\Gamma, T)$ are independent of the choice of the splitting field K; see [Ono 1963b, Sections 3.3 and 3.4].

According to Ono's formula, the Tamagawa number of any algebraic k-torus is a positive rational number. Ono constructed an infinite family of algebraic $\mathbb{Q}$-tori T with $\tau(T) = \tau_{\mathbb{Q}}(T) = 1/4$, which particularly shows that $\tau(T)$ needs not to be an integer. Ono [1963a] conjectured that every positive rational number can be realized as $\tau_k(T)$ for some algebraic k-torus T. Ono's conjecture was proved by S. Katayama [1985] for the number field case.

For any finite abelian group G, the Pontryagin dual of G is defined to be

$$G^\vee := \mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z}).$$

The Poitou–Tate duality [Platonov and Rapinchuk 1994, Theorem 6.10; Neukirch et al. 2000, Theorem 8.6.8] says that there is a natural isomorphism $Ш^1(\Gamma, T)^\vee \simeq Ш^2(\Gamma, X(T))$. Thus,

$$\tau_k(T) = \frac{|H^1(\Gamma, X(T))|}{|Ш^2(\Gamma, X(T))|}. \tag{2-4}$$

To simplify the notation we shall often suppress the Galois group from Galois cohomology groups and write $H^1(X(T))$ and $Ш^2(X(T))$ for $H^1(\Gamma, X(T))$ and $Ш^2(\Gamma, X(T))$ etc., if there is no risk of confusion.

2C. ***Some known results for CM tori.*** For the convenience of later generalization and investigation, we define here a more general class of k-tori as mentioned in the Introduction.

For every commutative etale k-algebra K, denote by T^K the algebraic k-torus whose group of R-valued points of T^K for any commutative k-algebra R, is

$$T^K(R) = (K \otimes_k R)^\times.$$

Explicitly, if $K = \prod_{i=1}^r K_i$ is a product of finite separable field extensions K_i of k, then

$$T^K = \prod_{i=1}^r T^{K_i} = \prod_{i=1}^r R_{K_i/k}(\mathbb{G}_{m,K_i}),$$

where $R_{K_i/k}$ is the Weil restriction of scalars from K_i to k. Let $E = \prod_{i=1}^r E_i$ be a product of finite subfield extensions $E_i \subset K_i$ over k. Let $N_{K_i/E_i} : T^{K_i} \to T^{E_i}$ be the norm map and put $N = \prod_{i=1}^r N_{K_i/E_i} : T^K \to T^E$. Define $T^{K/E,1} := \mathrm{Ker}\, N$, the kernel of the norm map N, and

$$T^{K/E,k} := \{x \in T^K : N(x) \in \mathbb{G}_{m,k}\},$$

the preimage of $\mathbb{G}_{m,k}$ in T^K under N, where $\mathbb{G}_{m,k} \hookrightarrow T^E$ is viewed as a subtorus of T^E via the diagonal embedding. Then we have the following commutative diagram of algebraic k-tori in which each row is an exact sequence:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & T^{K/E,1} & \xrightarrow{j} & T^K & \xrightarrow{N} & T^E & \longrightarrow & 1 \\ & & \| & & \uparrow{\scriptstyle j_T} & & \uparrow{\scriptstyle \Delta} & & \\ 1 & \longrightarrow & T^{K/E,1} & \xrightarrow{j} & T^{K/E,k} & \xrightarrow{N} & \mathbb{G}_{\mathrm{m},k} & \longrightarrow & 1 \end{array} \tag{2-5}$$

For the rest of this subsection we let $k = \mathbb{Q}$ and $K = \prod_{i=1}^r K_i$ be a CM algebra, where each K_i is a CM field, with the canonical involution ι. The subalgebra $K^+ \subset K$ fixed by ι is the product $K^+ = \prod_{i=1}^r K_i^+$ of the maximal totally real subfields K_i^+ of K_i. Let $T^{K,1} := T^{K/K^+,1} = \mathrm{Ker}\, N_{K/K^+}$ be the associated norm one CM torus and the associated CM torus

$$T^{K,\mathbb{Q}} := T^{K/K^+,\mathbb{Q}}. \tag{2-6}$$

As before, we have the following exact sequence of algebraic tori over $\mathbb{Q}$

$$1 \to T^{K,1} \to T^K \xrightarrow{N_{K/K^+}} T^{K^+} \to 1,$$

and a commutative diagram similar to (2-5).

Proposition 2.7. *Let K be a CM algebra and $T = T^{K,\mathbb{Q}}$ the associated CM torus over $\mathbb{Q}$:*

(1) *We have*

$$\tau(T) = \frac{2^r}{n_K},$$

where r is the number of components of K and

$$n_K := [\mathbb{A}^\times : N(T(\mathbb{A})) \cdot \mathbb{Q}^\times]$$

is the global norm index associated to T.

(2) *We have*

$$n_K \mid \prod_{p \in S_{K/K^+}} e_{T,p},$$

where S_{K/K^+} is the finite set of rational primes p with some place $v \mid p$ of K^+ ramified in K, and $e_{T,p} := [\mathbb{Z}_p^\times : N(T(\mathbb{Z}_p))]$. Here $T(\mathbb{Z}_p)$ denotes the unique maximal open compact subgroup of $T(\mathbb{Q}_p)$.

Proof. (1) This is [Guo et al. 2022, Theorem 1.1(1)]. (2) This is [loc. cit., Lemma 4.6(2)]. □

Lemma 2.8. *Let K and T be as in Proposition 2.7:*

(1) *If K contains an imaginary quadratic field, then $n_K \in \{1, 2\}$ and $\tau(T) \in \{2^{r-1}, 2^r\}$.*

(2) *If K contains two distinct imaginary quadratic fields, then $n_K = 1$ and $\tau(T) = 2^r$.*

Proof. This is proved in [Guo et al. 2022, Lemma 4.7] (also see [loc. cit., Section 5.1]) in the case where K is a CM field. The same proof using class field theory also proves the CM algebra case. □

Proposition 2.9. *Let K be a CM field and $T = T^{K,\mathbb{Q}}$ the associated CM torus over $\mathbb{Q}$. Put $g = [K^+ : \mathbb{Q}]$ and let K^{Gal} be the Galois closure of K over $\mathbb{Q}$ with Galois group $\mathrm{Gal}(K^{\mathrm{Gal}}/\mathbb{Q}) = G$:*

(1) *If g is odd, then $H^1(X(T)) = 0$.*

(2) *If $K/\mathbb{Q}$ is Galois and g is odd, then $\tau(T) = 1$.*

(3) *If $K/\mathbb{Q}$ is cyclic, then $\tau(T) = 1$.*

(4) *If $K/\mathbb{Q}$ is Galois of degree 4, then $\tau(T) \in \{1, 2\}$. Moreover, $\tau(T) = 2$ if and only if $G \simeq (\mathbb{Z}/2\mathbb{Z})^2$.*

Proof. See Propositions A.2 and A.12 of [Achter et al. 2023]. □

Note that Proposition 2.9(4) also follows from Proposition 2.9(3) and Lemma 2.8(2).

3. Transfer maps, corestriction maps and extensions

3A. ***Transfer maps.*** Let G be a group and let H be a subgroup of G of finite index. Let $X := G/H$, and let $\varphi : X \to G$ be a section. If $g \in G$ and $x \in X$, the elements $\varphi(gx)$ and $g\varphi(x)$ belong to the same class of mod H; hence there exists a unique element $h^{\varphi}_{g,x} \in H$ such that $g\varphi(x) = \varphi(gx)h^{\varphi}_{g,x}$. Let $\mathrm{Ver}_{G,H}(g) \in H^{\mathrm{ab}}$ be defined by

$$\mathrm{Ver}_{G,H}(g) := \prod_{x \in X} h^{\varphi}_{g,x} \bmod D(H),$$

where $D(H) = [H, H]$ is the commutator group of H and the product is computed in $H^{\mathrm{ab}} = H/D(H)$.

By [Serre 2016, Theorem 7.1], the map $\mathrm{Ver}_{G,H} : G \to H^{\mathrm{ab}}$ is a group homomorphism and it does not depend on the choice of the section φ. This homomorphism is called the *transfer* of G into H^{ab} (originally from the term "Verlagerung" in German). One may also view it as a homomorphism $\mathrm{Ver}_{G,H} : G^{\mathrm{ab}} \to H^{\mathrm{ab}}$.

In the literature one also uses the right coset space $X' := H\backslash G$ but this does not effect the result. One can easily show that if $\mathrm{Ver}'_{G,H}$ is the transfer map defined using X', then $\mathrm{Ver}'_{G,H} = \mathrm{Ver}_{G,H}$. One has the following functorial property; see [loc. cit., page 89].

Lemma 3.1. *Let $H \subset G$ and $H' \subset G'$ be subgroups of finite index. If σ be a group homomorphism from the pair (G, H) to (G', H') which induces a bijection $G/H \xrightarrow{\sim} G'/H'$, then the following diagram commutes*:

$$\begin{array}{ccc} G^{\mathrm{ab}} & \xrightarrow{\sigma} & G'^{\,\mathrm{ab}} \\ \downarrow{\scriptstyle \mathrm{Ver}_{G,H}} & & \downarrow{\scriptstyle \mathrm{Ver}_{G',H'}} \\ H^{\mathrm{ab}} & \xrightarrow{\sigma} & H'^{\,\mathrm{ab}} \end{array}$$

Lemma 3.2. *Let G_1 and G_2 be groups, and let $H_i \subset G_i$ be a subgroup of finite index for $i = 1, 2$. Then*

$$\mathrm{Ver}_{G_1\times G_2, H_1\times H_1}(g_1, g_2) = \mathrm{Ver}_{G_1,H_1}(g_1)^{[G_2:H_2]} \cdot \mathrm{Ver}_{G_2,H_2}(g_2)^{[G_1:H_1]}.$$

Proof. Put $G = G_1 \times G_2$, $H = H_1 \times H_2$, $X_i := G_i/H_i$ and $X = G/H = X_1 \times X_2$. Fix a section $\varphi_i : X_i \to G_i$ for each i and let $\varphi = (\varphi_1, \varphi_2) : X \to G$. For $g = (g_1, g_2)$ and $x = (x_1, x_2)$, we have $h^{\varphi}_{g,x} = (h^{\varphi_1}_{g_1,x_1}, h^{\varphi_2}_{g_2,x_2})$. Then in $H^{\mathrm{ab}} = H_1^{\mathrm{ab}} \times H_2^{\mathrm{ab}}$ we have

$$\begin{aligned} \mathrm{Ver}_{G,H}(g) &= \prod_{x\in X} h^{\varphi}_{g,x} = \prod_{x_1\in X_1}\prod_{x_2\in X_2} (h^{\varphi_1}_{g_1,x_1}, h^{\varphi_2}_{g_2,x_2}) \\ &= \prod_{x_1\in X_1} ((h^{\varphi_1}_{g_1,x_1})^{|X_2|}, \mathrm{Ver}_{G_2,H_2}(g_2)) = (\mathrm{Ver}_{G_1,H_1}(g_1)^{|X_2|}, \mathrm{Ver}_{G_2,H_2}(g_2)^{|X_1|}). \end{aligned}$$ □

If G is a finite group and p is a prime, G_p denotes a p-Sylow subgroup of G.

Proposition 3.3. *Let G be a finite group and $N \lhd G$ be a cyclic central subgroup of prime order p. Then the transfer $\mathrm{Ver}_{G,N} : G^{\mathrm{ab}} \to N^{\mathrm{ab}}$ is surjective if and only if G_p is cyclic.*

Proof. See Proposition 3.8 of [Rüd 2022]. □

3B. *Connection of transfer maps with corestriction maps.* Let $H \subset G$ be a subgroup of finite index, and let A be a G-module. Let

$$f : \mathrm{Ind}_H^G A = \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} A \to A, \quad g \otimes a \mapsto ga$$

be the natural map of G-modules. Applying Galois cohomology $H^i(G, -)$ to the map f and by Shapiro's lemma, we obtain for each $i \geq 0$ a morphism

$$\mathrm{Cor} : H^i(H, A) \to H^i(G, A),$$

called the *corestriction* form H to G.

Applying $H^i(G, -)$ to the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$, one obtains an isomorphism $H^i(G, \mathbb{Z}) \simeq H^{i-1}(G, \mathbb{Q}/\mathbb{Z})$ for all $i \geq 2$. When $i = 2$, this gives the isomorphism

$$H^2(G, \mathbb{Z}) \simeq \mathrm{Hom}(G^{\mathrm{ab}}, \mathbb{Q}/\mathbb{Z}). \tag{3-1}$$

Proposition 3.4. *Let $H \subset G$ be a subgroup of finite index. Through the isomorphism (3-1) we have the following commutative diagram*

$$\begin{array}{ccc} H^2(H, \mathbb{Z}) & \xrightarrow{\sim} & \mathrm{Hom}(H^{\mathrm{ab}}, \mathbb{Q}/\mathbb{Z}) \\ \downarrow{\scriptstyle \mathrm{Cor}} & & \downarrow{\scriptstyle \mathrm{Ver}^\vee_{G,H}} \\ H^2(G, \mathbb{Z}) & \xrightarrow{\sim} & \mathrm{Hom}(G^{\mathrm{ab}}, \mathbb{Q}/\mathbb{Z}), \end{array}$$

where $\mathrm{Ver}^\vee_{G,H}$ *is dual of the transfer* $\mathrm{Ver}_{G,H} : G^{\mathrm{ab}} \to H^{\mathrm{ab}}$. *Moreover, if H is normal in G, the composition $H^{\mathrm{ab}} \to G^{\mathrm{ab}} \to H^{\mathrm{ab}}$ is the norm $N_{G/H}$. Here H^{ab} is viewed as a G-module by conjugation and also as a G/H-module, since H acts trivially on H^{ab}.*

Proof. See [Neukirch et al. 2000, Proposition 1.5.9]. □

3C. *Connection of transfer maps with class field theory.* Let $k \subset K \subset L$ be three global fields such that the extension L/k is Galois. Put $G = \mathrm{Gal}(L/k)$ and $H = \mathrm{Gal}(L/K) \subset G$. Denote by C_k and C_K the idele class groups of k and K, respectively. The Artin map is a surjective homomorphism $\mathrm{Art}_{L/k} : C_k \to G^{\mathrm{ab}}$; similarly we have $\mathrm{Art}_{L/K} : C_K \to H^{\mathrm{ab}}$. By class field theory we have the following commutative diagrams:

$$\begin{array}{ccc} C_k & \xrightarrow{\mathrm{Art}_{L/k}} & G^{\mathrm{ab}} \\ {\scriptstyle \mathrm{func}}\downarrow & & \downarrow{\scriptstyle \mathrm{Ver}_{G,H}} \\ C_k & \xrightarrow{\mathrm{Art}_{L/k}} & H^{\mathrm{ab}} \end{array} \qquad \begin{array}{ccc} C_k & \xrightarrow{\mathrm{Art}_{L/k}} & G^{\mathrm{ab}} \\ {\scriptstyle N_{K/k}}\uparrow & & \uparrow{\scriptstyle \mathrm{func}} \\ C_k & \xrightarrow{\mathrm{Art}_{L/k}} & H^{\mathrm{ab}} \end{array} \tag{3-2}$$

where func denotes the natural map induced from the inclusion $\mathbb{A}_k^\times \hookrightarrow \mathbb{A}_K^\times$ or $H \hookrightarrow G$. When K/k is Galois and $L = K$, the second diagram of (3-2) induces a homomorphism

$$\mathrm{Art}_{K/k} : C_k/N_{K/k}(C_K) \to \mathrm{Gal}(K/k)^{\mathrm{ab}},$$

which is an isomorphism by class field theory.

3D. *Relative transfer maps.* Let $H \subset N \subset G$ be two subgroups of G of finite index. Let $\widetilde{X} := G/H$ and $X := G/N$ with natural G-equivariant projections $\widetilde{c} : G \to X$ and $c : \widetilde{X} \to X$. Let $\widetilde{\varphi} : X \to G$ be a section, which induces a section $\varphi : X \to \widetilde{X}$. For each $g \in G$ and $x \in X$, let $n^{\widetilde{\varphi}}_{g,x}$ be the unique element in N such that $g\widetilde{\varphi}(x) = \widetilde{\varphi}(gx)n^{\widetilde{\varphi}}_{g,x}$. Since H is a possibly nonnormal subgroup of N, let $(N/H)^{\mathrm{ab}} := \mathrm{Coker}[D(N)H/D(N) \to N/D(N)] \simeq N/D(N)H$. Note that $(N/H)^{\mathrm{ab}}$ is an abelian group which agrees with the abelianization of the group N/H when H is normal in N. Let $\mathrm{Ver}_{G,N/H}(g) \in (N/H)^{\mathrm{ab}}$ be the element defined by

$$\mathrm{Ver}_{G,N/H}(g) := \prod_{x\in X} n^{\widetilde{\varphi}}_{g,x} \bmod D(N)H. \tag{3-3}$$

Proposition 3.5. (1) *The map* $\mathrm{Ver}_{G,N/H} : G \to (N/H)^{\mathrm{ab}}$ *does not depend on the choice of the section* $\widetilde{\varphi}$ *and it is a group homomorphism.*

(2) *One has* $\mathrm{Ver}_{G,N/H} = \pi_H \circ \mathrm{Ver}$, *where* $\pi_H : N^{\mathrm{ab}} \to (N/H)^{\mathrm{ab}}$ *is the morphism mod* H.

Proof. Clearly, the statement (1) follows from (2), because Ver does not depend on the choice of $\widetilde{\varphi}$ and is a group homomorphism. (2) By definition $\mathrm{Ver}(g) = \prod_{x\in X} n^{\widetilde{\varphi}}_{g,x} \bmod D(N)$, thus $\mathrm{Ver}_{G,N/H} = \pi_H \circ \mathrm{Ver}$. □

Definition 3.6. Let $H \subset N$ be two subgroups of G of finite index. The group homomorphism $\mathrm{Ver}_{G,N/H} : G \to (N/H)^{\mathrm{ab}}$ defined in (3-3) is called the *transfer* of G into $(N/H)^{\mathrm{ab}}$ relative to H. By abuse of notation, we denote the induced map by $\mathrm{Ver}_{G,N/H} : G^{\mathrm{ab}} \to (N/H)^{\mathrm{ab}}$.

One can check directly that the map $\mathrm{Ver}_{G,N/H} : G^{\mathrm{ab}} \to (N/H)^{\mathrm{ab}}$ factors through $\pi_H : G^{\mathrm{ab}} \to (G/H)^{\mathrm{ab}}$, the map modulo H. We denote the induced map by

$$\mathrm{Ver}_{G/H,N/H} : (G/H)^{\mathrm{ab}} \to (N/H)^{\mathrm{ab}}. \tag{3-4}$$

Remark 3.7. If $H \lhd G$ is a normal subgroup, then the induced map $\mathrm{Ver}_{G/H,N/H}$ is the transfer map from G/H to N/H associated to the subgroup $N/H \subset G/H$ of finite index.

Lemma 3.8. *Let* $H \lhd \widetilde{N}$ *be two subgroups of finite index in* G *with* H *normal in* $\widetilde{N}$ *and cyclic quotient* $N = \widetilde{N}/H = \langle\sigma\rangle$ *of order* n. *Fix a lift* $\widetilde{\sigma} \in \widetilde{N}$ *of* σ.

(1) *For each* $g \in G$, *let* $\{x_1, \dots x_r\}$ *be a complete set of double coset representatives for* $\langle g\rangle\backslash G/\widetilde{N}$ *and set* $d_i := d_i(g) = |\langle g\rangle x_i H/H|$, *where* $1 \le i \le r$. *Then*

$$\mathrm{Ver}_{G,N}(g) = \sigma^{\sum_{i=1}^r m(g,x_i)},$$

where $0 \le m(g,x_i) \le n-1$ *is the unique integer such that* $g^{d_i} x_i H = x_i \widetilde{\sigma}^{m(g,x_i)} H$.

(2) *Let* $\sigma^* \in N^\vee := \mathrm{Hom}(N, \mathbb{Q}/\mathbb{Z})$ *be the element defined by* $\sigma^*(\sigma) = 1/n \bmod \mathbb{Z}$, $\mathrm{Ver}^\vee_{G,N} : N^\vee \to \mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z})$ *the induced map, and* $f := \mathrm{Ver}^\vee_{G,N}(\sigma^*)$. *Then* $f(g) = \big(\sum_{i=1}^r m(g,x_i)\big)/n \bmod \mathbb{Z}$.

Proof. (1) We choose the set of representatives $S = \{g^j x_i : i = 1, \dots, r,\ j = 0, \dots, f_i - 1\}$ of $X = G/\widetilde{N}$, which defines a section $\widetilde{\varphi}$ of the natural projection $G \to G/\widetilde{N}$. Then the image of the element $n^{\widetilde{\varphi}}_{g,x}$ in H is given by

$$n^{\widetilde{\varphi}}_{g,x} \bmod H = \begin{cases} \sigma^{m(g,x_i)} & \text{if } x = g^{d_i - 1} x_i H \text{ for some } 1 \le i \le r; \\ 1 & \text{otherwise.} \end{cases}$$

(2) By definition, $f(g) = \sigma^*(\mathrm{Ver}_{G,N}(g)) = \sigma^*(\sigma^{\sum_i m(g,x_i)}) = \left(\sum_i m(g, x_i)\right)/n \bmod \mathbb{Z}$. □

One can show that the integer $m(g, x_i)$ is independent of the choice of double coset representative in $\langle g \rangle x_i H$.

The following lemma will be used in Lemma 5.1.

Lemma 3.9. *Let $G = \prod_{i=1}^r G_i$ be a product of groups G_i and let $N_i \subset G_i$, for each $1 \le i \le r$, be a subgroup of finite index. Put*

$$H_i := G_1 \times \cdots \times G_{i-1} \times \{1\} \times G_{i+1} \times \cdots \times G_r, \quad \widetilde{N}_i := G_1 \times \cdots \times G_{i-1} \times N_i \times G_{i+1} \times \cdots \times G_r.$$

Then the map

$$\prod_{i=1}^r \mathrm{Ver}_{G,\widetilde{N}_i/H_i} : G \to (\widetilde{N}_1/H_1)^{\mathrm{ab}} \times \cdots \times (\widetilde{N}_r/H_r)^{\mathrm{ab}} = N_1^{\mathrm{ab}} \times \cdots \times N_r^{\mathrm{ab}} \tag{3-5}$$

is given by the product of the maps

$$\prod_{i=1}^r \mathrm{Ver}_{G_i,N_i} : \prod_{i=1}^r G_i \to \prod_{i=1}^r N_i^{\mathrm{ab}}.$$

Proof. Let $\mathrm{pr}_i : G \to G_i$ be the i-th projection. Then

$$\mathrm{Ver}_{G,\widetilde{N}_i/H_i} = \mathrm{Ver}_{G/H_i,\widetilde{N}_i/H_i} \circ \mathrm{pr}_i = \mathrm{Ver}_{G_i,N_i} \circ \mathrm{pr}_i,$$

see Remark 3.7. Thus, the map (3-5) is equal to $\prod_i \mathrm{Ver}_{G_i,N_i} \circ (\mathrm{pr}_i)_i$. Since the map $(\mathrm{pr}_i)_i : G \to \prod_i G_i$ is the identity, we show that the map (3-5) is equal to $\prod_i \mathrm{Ver}_{G_i,N_i}$. □

3E. *Connection of relative transfer maps with class field theory.* Let $k \subset E \subset K$ be three global fields. Let L/k be a finite Galois extension containing K with Galois group $G = \mathrm{Gal}(L/k)$. Let $H = \mathrm{Gal}(L/K) \subset N = \mathrm{Gal}(L/E)$ be subgroups of G.

The Artin map produces the following isomorphisms (3-2)

$$C_k/N_{L/k}(C_L) \simeq G^{\mathrm{ab}}, \quad C_k/N_{K/k}(C_K) \simeq (G/H)^{\mathrm{ab}},$$
$$C_E/N_{L/E}(C_L) \simeq N^{\mathrm{ab}}, \quad C_E/N_{K/E}(C_K) \simeq (N/H)^{\mathrm{ab}}.$$

We obtain the following commutative diagrams:

$$\begin{array}{ccc} C_k/N_{L/k}(C_L) & \longrightarrow & C_k/N_{K/k}(C_K) \\ \downarrow & & \downarrow \\ C_E/N_{L/E}(C_L) & \longrightarrow & C_E/N_{K/E}(C_K) \end{array} \qquad \begin{array}{ccc} G^{\mathrm{ab}} & \xrightarrow{\pi_H} & (G/H)^{\mathrm{ab}} \\ {\scriptstyle \mathrm{Ver}}\downarrow & & \downarrow{\scriptstyle \mathrm{Ver}_{G/H,N/H}} \\ N^{\mathrm{ab}} & \xrightarrow{\pi_H} & (N/H)^{\mathrm{ab}} \end{array} \tag{3-6}$$

From this we see that $\mathrm{Ver}_{G/H,N/H} : (G/H)^{\mathrm{ab}} \to (N/H)^{\mathrm{ab}}$ does not depend on the choice of the Galois extension L/k.

4. Cohomology groups of algebraic tori

4A. ***$H^1(\Lambda^1)$ and $H^1(\Lambda)$.*** Let $K = \prod_{i=1}^r K_i$ be a commutative etale k-algebra and $E = \prod_{i=1}^r E_i$ a k-subalgebra. Let $T^{K/E,k}$ and $T^{K/E,1}$ be the k-tori defined in Section 2C. Let L/k be a splitting Galois field extension for T^K, and let $G = \mathrm{Gal}(L/k)$. Put

$$\Lambda := X(T^{K/E,k}) \quad \text{and} \quad \Lambda^1 := X(T^{K/E,1}),$$

which are G-modules. For $1 \le i \le r$, put

$$H_i := \mathrm{Gal}(L/K_i), \tag{4-1}$$

$$\widetilde{N}_i := \mathrm{Gal}(L/E_i), \tag{4-2}$$

$$N_i^{\mathrm{ab}} = (\widetilde{N}_i/H_i)^{\mathrm{ab}} := \widetilde{N}_i/H_i D(\widetilde{N}_i). \tag{4-3}$$

In the case that K_i is Galois over E_i, we let

$$N_i := \widetilde{N}_i/H_i = \mathrm{Gal}(K_i/E_i) \tag{4-4}$$

and then N_i^{ab} coincides with the abelianization of the group N_i, which justifies our notation.

From the diagram (2-5), we obtain the commutative diagram of G-modules:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X(T^E) & \xrightarrow{\widehat{N}} & X(T^K) & \xrightarrow{\widehat{j}} & \Lambda^1 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \widehat{\Delta}} & & \downarrow{\scriptstyle \widehat{j}_T} & & \| & & \\ 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\widehat{N}} & \Lambda & \xrightarrow{\widehat{j}} & \Lambda^1 & \longrightarrow & 0 \end{array} \tag{4-5}$$

We have

$$X(T^K) = \bigoplus_{i=1}^r \mathrm{Ind}_{H_i}^G \mathbb{Z} \quad \text{and} \quad X(T^E) = \bigoplus_{i=1}^r \mathrm{Ind}_{\widetilde{N}_i}^G \mathbb{Z} \tag{4-6}$$

and

$$T^{K/E,1} = \prod_{i=1}^r R_{E_i/k} R^{(1)}_{K_i/E_i} \mathbb{G}_{\mathrm{m},K_i} \quad \text{and} \quad \Lambda^1 = \bigoplus_{i=1}^r \mathrm{Ind}_{\widetilde{N}_i}^G \Lambda^1_{E_i}, \tag{4-7}$$

where

$$\Lambda^1_{E_i} := X(R^{(1)}_{K_i/E_i} \mathbb{G}_{\mathrm{m},K_i}). \tag{4-8}$$

By Shapiro's lemma, one has

$$H^q(G,\Lambda^1)=\bigoplus_{i=1}^{r} H^q(G,\operatorname{Ind}_{\widetilde{N}_i}^{G}\Lambda^1_{E_i})=\bigoplus_{i=1}^{r} H^q(\widetilde{N}_i,\Lambda^1_{E_i}),\quad \forall q\ge 0. \tag{4-9}$$

Note that if K_i/E_i is Galois, then $H^1(\widetilde{N}_i,\Lambda^1_{E_i})=H^1(N_i,\Lambda^1_{E_i})$ (Remark 2.6).

Since the torus $R^{(1)}_{K_i/E_i}\mathbb{G}_{\mathrm{m},K_i}$ is anisotropic, one has $H^0(G,\Lambda^1)=\bigoplus_{i=1}^{r} H^0(\widetilde{N}_i,\Lambda^1_{E_i})=0$.

Taking Galois cohomology to the lower exact sequence of (4-5), we have an exact sequence

$$0\to H^1(G,\Lambda)\to H^1(G,\Lambda^1)\xrightarrow{\delta} H^2(G,\mathbb{Z}). \tag{4-10}$$

Proposition 4.1. *Let the notation be as above*:

(1) *There is a canonical isomorphism* $H^1(G,\Lambda^1)\simeq\bigoplus_i N_i^{\mathrm{ab},\vee}$ *(see* (4-3) *for the definition of* N_i^{ab}*).*

(2) *Under the canonical isomorphisms* $H^1(G,\Lambda^1)\simeq\bigoplus_i N_i^{\mathrm{ab},\vee}$ *and* $H^2(G,\mathbb{Z})\simeq G^{\mathrm{ab},\vee}$ (3-1), *the map* $\delta: H^1(G,\Lambda^1)\to H^2(G,\mathbb{Z})$ *expresses as*

$$\sum_{i=1}^{r}\mathrm{Ver}^{\vee}_{G,N_i}:\bigoplus_{i=1}^{r} N_i^{\mathrm{ab},\vee}\to G^{\mathrm{ab},\vee}, \tag{4-11}$$

where $\mathrm{Ver}_{G,N_i}: G\to N_i^{\mathrm{ab}}$ *is the transfer map. In particular,* $H^1(G,\Lambda)\simeq\mathrm{Ker}\big(\sum\mathrm{Ver}^{\vee}_{G,N_i}\big)$. *Furthermore, the map in* (4-11) *factors as*

$$\bigoplus_{i=1}^{r} N_i^{\mathrm{ab},\vee}\xrightarrow{(\mathrm{Ver}^{\vee}_{G/H_i,N_i})_i}\bigoplus_{i=1}^{r}(G/H_i)^{\mathrm{ab},\vee}\xrightarrow{\sum_{i=1}^{r}\pi^{\vee}_{H_i}} G^{\mathrm{ab},\vee}, \tag{4-12}$$

where $\pi_{H_i}: G^{\mathrm{ab}}\to(G/H_i)^{\mathrm{ab}}$ *is the map mod* H_i.

Proof. (1) Taking Galois cohomology of the upper exact sequence of (4-5), we have a long exact sequence of abelian groups

$$H^1(G,X(T^K))\to H^1(G,\Lambda^1)\xrightarrow{\delta} H^2(G,X(T^E))\xrightarrow{\widehat{N}^2} H^2(G,X(T^K)). \tag{4-13}$$

By (4-6), the first term $H^1(G,X(T^K))=\bigoplus_i H^1(H_i,\mathbb{Z})=0$. Using the relations (4-6) and by Shapiro's lemma, we have $H^1(G,\Lambda^1)=\bigoplus_i H^1(\widetilde{N}_i,\Lambda^1_{E_i})$ and the exact sequence (4-13) becomes

$$0\to\bigoplus_{i=1}^{r} H^1(\widetilde{N}_i,\Lambda^1_{E_i})\xrightarrow{\widetilde{\delta}}\bigoplus_{i=1}^{r} H^2(\widetilde{N}_i,\mathbb{Z})\xrightarrow{\mathrm{Res}}\bigoplus_{i=1}^{r} H^2(H_i,\mathbb{Z}). \tag{4-14}$$

It is clear that the following sequence

$$0\to\mathrm{Hom}(N_i^{\mathrm{ab}},\mathbb{Q}/\mathbb{Z})\xrightarrow{\mathrm{Inf}}\mathrm{Hom}(\widetilde{N}_i,\mathbb{Q}/\mathbb{Z})\xrightarrow{\mathrm{Res}}\mathrm{Hom}(H_i,\mathbb{Q}/\mathbb{Z}) \tag{4-15}$$

is exact. Using the canonical isomorphism $H^2(H,\mathbb{Z})\simeq\mathrm{Hom}(H,\mathbb{Q}/\mathbb{Z})$ (3-1) for any group H, we rewrite (4-15) as follows:

$$0\to H^2(N_i^{\mathrm{ab}},\mathbb{Z})\xrightarrow{\mathrm{Inf}} H^2(\widetilde{N}_i,\mathbb{Z})\xrightarrow{\mathrm{Res}} H^2(H_i,\mathbb{Z}). \tag{4-16}$$

Comparing (4-16) and (4-14), there is a unique isomorphism

$$H^1(G,\Lambda^1)=\bigoplus_{i=1}^{r}H^1(\widetilde{N}_i,\Lambda^1_{E_i})\simeq\bigoplus_{i=1}^{r}H^2(N_i^{\mathrm{ab}},\mathbb{Z}) \tag{4-17}$$

which fits into the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \bigoplus_{i=1}^r H^2(N_i^{\mathrm{ab}},\mathbb{Z}) & \xrightarrow{\mathrm{Inf}} & \bigoplus_{i=1}^r H^2(\widetilde{N}_i,\mathbb{Z}) & \xrightarrow{\mathrm{Res}} & \bigoplus_{i=1}^r H^2(H_i,\mathbb{Z}) \\ & & \| & & \| & & \| \\ 0 & \longrightarrow & \bigoplus_{i=1}^r H^1(\widetilde{N}_i,\Lambda^1_{E_i}) & \xrightarrow{\widetilde{\delta}} & \bigoplus_{i=1}^r H^2(\widetilde{N}_i,\mathbb{Z}) & \xrightarrow{\mathrm{Res}} & \bigoplus_{i=1}^r H^2(H_i,\mathbb{Z}) \end{array} \tag{4-18}$$

This shows the first statement.

(2) The map $\widehat{\Delta}$ in (4-5) is induced from the restriction of $X(T^E)$ to the subtorus $\mathbb{G}_{\mathrm{m},k}$ and therefore is given by

$$\widehat{\Delta}=\sum_{i=1}^{r}\widehat{\Delta}_i:\ \bigoplus_{i=1}^{r}\mathrm{Ind}_{\widetilde{N}_i}^{G}\mathbb{Z}\to\mathbb{Z},\qquad \widehat{\Delta}_i(g\otimes n)=n,\qquad \forall g\in G,\, n\in\mathbb{Z}.$$

Thus, by definition, the induced map $\widehat{\Delta}_i^2$ on $H^2(G,-)$ is nothing but the corestriction $\mathrm{Cor}\colon H^2(\widetilde{N}_i,\mathbb{Z})\to H^2(G,\mathbb{Z})$; see Section 3B.

From the diagram (4-5), we obtain the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \bigoplus_{i=1}^r H^1(\widetilde{N}_i,\Lambda^1_{E_i}) & \xrightarrow{\widetilde{\delta}} & \bigoplus_{i=1}^r H^2(\widetilde{N}_i,\mathbb{Z}) & \xrightarrow{\mathrm{Res}} & \bigoplus_{i=1}^r H^2(H_i,\mathbb{Z}) \\ \downarrow & & \| & & \downarrow{\scriptstyle \widehat{\Delta}^2=\mathrm{Cor}} & & \downarrow \\ 0\to H^1(G,\Lambda) & \longrightarrow & H^1(G,\Lambda^1) & \xrightarrow{\delta} & H^2(G,\mathbb{Z}) & \longrightarrow & H^2(G,\Lambda) \end{array} \tag{4-19}$$

With the identification (4-17), we have $\delta=\mathrm{Cor}\circ\widetilde{\delta}=\mathrm{Cor}\circ\mathrm{Inf}$ and the lower long exact sequence of (4-19) becomes

$$0\to H^1(G,\Lambda)\to\bigoplus_{i=1}^{r}H^2(N_i^{\mathrm{ab}},\mathbb{Z})\xrightarrow{\mathrm{Cor}\circ\mathrm{Inf}}H^2(G,\mathbb{Z})\xrightarrow{\widehat{N}^2}H^2(G,\Lambda). \tag{4-20}$$

By Propositions 3.5 and 3.4, under the isomorphism (3-1) the map $\mathrm{Cor}\circ\mathrm{Inf}\colon H^2(N_i^{\mathrm{ab}},\mathbb{Z})\to H^2(G,\mathbb{Z})$ corresponds to $\mathrm{Ver}_{G,N_i}^{\vee}:N_i^{\mathrm{ab},\vee}\to G^{\mathrm{ab},\vee}$, where $\mathrm{Ver}_{G,N_i}^{\vee}$ is the dual of the transfer $\mathrm{Ver}_{G,N_i}:G^{\mathrm{ab}}\to N_i^{\mathrm{ab}}$ relative to H_i. This proves (4-11). Then it follows from (4-20) that $H^1(\Lambda)\simeq\mathrm{Ker}\big(\sum_{i=1}^{r}\mathrm{Ver}_{G,N_i}^{\vee}\big)$. As the map $\mathrm{Ver}_{G,N_i}:G^{\mathrm{ab}}\to N_i^{\mathrm{ab}}$ factors as

$$G\xrightarrow{\pi_{H_i}}(G/H_i)^{\mathrm{ab}}\xrightarrow{\mathrm{Ver}_{G/H_i,\widetilde{N}_i/H_i}}N_i^{\mathrm{ab}},$$

the last assertion (4-12) follows. □

Remark 4.2. In terms of class field theory, we have $C_{E_i}/N_{K_i/E_i}(C_{K_i}) \simeq N_i^{\mathrm{ab}}$, $C_k/N_{K_i/k}(C_{K_i}) \simeq (G/H_i)^{\mathrm{ab}}$ and $C_k/N_{L/k}(C_L) \simeq G^{\mathrm{ab}}$. Let L_0 be the compositum of all Galois closures of K_i over k; this is the minimal splitting field of the algebraic torus $T^{K/E,k}$. One has $L_0 \subset L$ and its Galois group $G_0 := \mathrm{Gal}(L_0/k)$ is a quotient of G. Thus, we have the following commutative diagram:

$$\begin{array}{ccccccc} \frac{C_k}{N_{L/k}(C_L)} & \longrightarrow & \frac{C_k}{N_{L_0/k}(C_{L_0})} & \longrightarrow & \prod_i \frac{C_k}{N_{K_i/k}(C_{K_i})} & \longrightarrow & \prod_i \frac{C_{E_i}}{N_{K_i/E_i}(C_{K_i})} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ G^{\mathrm{ab}} & \longrightarrow & G_0^{\mathrm{ab}} & \longrightarrow & \prod_i (G/H_i)^{\mathrm{ab}} & \longrightarrow & \prod_i N_i^{\mathrm{ab}} \end{array} \tag{4-21}$$

Taking the Pontryagin dual, the lower row gives

$$\prod_i N_i^{\mathrm{ab},\vee} \to \prod_i (G/H_i)^{\mathrm{ab},\vee} \to G_0^{\mathrm{ab},\vee} \subset G^{\mathrm{ab},\vee}.$$

From this, we see that the map $\sum \mathrm{Ver}_{G,N_i}^{\vee}$ in (4-11) has image contained in $G_0^{\mathrm{ab},\vee}$. It follows from (4-21) that this map is independent of the choice of the splitting field L; also see Remark 2.6.

4B. $Ш^2(\Lambda)$. Let $\mathfrak{D}$ be the set of all decomposition groups of G. For any G-module A, denote by

$$H^i_{\mathfrak{D}}(A) := \prod_{D \in \mathfrak{D}} H^i_D(A), \quad H^i_D(A) := H^i(D, A)$$

and

$$r^i_{\mathfrak{D},A} = (r^i_{D,A})_{D\in\mathfrak{D}} : H^i(G, A) \to H^i_{\mathfrak{D}}(A)$$

the restriction map to subgroups D in $\mathfrak{D}$ defined in (1-3). By definition, $Ш^i(A) = \mathrm{Ker}\, r^i_{\mathfrak{D},A}$. We shall write r_D and $r_{\mathfrak{D}}$ for $r^i_{D,A}$ (1-2) and $r^i_{\mathfrak{D},A}$ (1-3), respectively, if it is clear from the content.

For the remainder of this section, we assume that the extension K_i/E_i *is cyclic* with Galois group N_i for all i. Consider the following commutative diagram

$$\begin{array}{ccccccc} H^1(\Lambda^1) & \xrightarrow{\delta} & H^2(\mathbb{Z}) & \xrightarrow{\widehat{N}} & H^2(\Lambda) & \xrightarrow{\widehat{j}} & H^2(\Lambda^1) \\ \downarrow r^1_{\mathfrak{D},\Lambda^1} & & \downarrow r^2_{\mathfrak{D},\mathbb{Z}} & & \downarrow r^2_{\mathfrak{D},\Lambda} & & \downarrow r^2_{\mathfrak{D},\Lambda^1} \\ H^1_{\mathfrak{D}}(\Lambda^1) & \xrightarrow{\delta_{\mathfrak{D}}} & H^2_{\mathfrak{D}}(\mathbb{Z}) & \xrightarrow{\widehat{N}} & H^2_{\mathfrak{D}}(\Lambda) & \xrightarrow{\widehat{j}} & H^2_{\mathfrak{D}}(\Lambda^1). \end{array} \tag{4-22}$$

Define

$$H^2(\mathbb{Z})' := \{x \in H^2(\mathbb{Z}) : \widehat{N}(x) \in Ш^2(\Lambda)\} = \{x \in H^2(\mathbb{Z}) : r^2_{\mathfrak{D},\mathbb{Z}}(x) \in \mathrm{Im}(\delta_{\mathfrak{D}})\}. \tag{4-23}$$

Proposition 4.3. *Assume that K_i/E_i is cyclic with Galois group N_i for all i. Then $Ш^2(\Lambda^1) = 0$, $Ш^2(\Lambda) \simeq H^2(\mathbb{Z})'/\mathrm{Im}(\delta)$ and*

$$\tau_k(T^{K/E,k}) = \frac{\prod_{i=1}^r |N_i|}{|H^2(\mathbb{Z})'|},$$

where $H^2(\mathbb{Z})'$ is the group defined in (4-23) and δ is the labeled map in (4-22).

Proof. It is obvious that $\widehat{j}(Ш^2(\Lambda)) \subset Ш^2(\Lambda^1)$ and then we have a long exact sequence

$$0 \to H^1(\Lambda) \to H^1(\Lambda^1) \xrightarrow{\delta} H^2(\mathbb{Z})' \xrightarrow{\widehat{N}} Ш^2(\Lambda) \xrightarrow{\widehat{j}} Ш^2(\Lambda^1).$$

One has

$$Ш^2(\Lambda^1) = \bigoplus_i Ш^2(G, \operatorname{Ind}_{\widetilde{N}_i}^G \Lambda^1_{E_i}) = \bigoplus_i Ш^2(\widetilde{N}_i, \Lambda^1_{E_i}) = \bigoplus_i Ш^2(N_i, \Lambda^1_{E_i})$$

because $Ш^2(X(T))$ does not depend on the choice of the splitting field. Since K_i/E_i is cyclic, by Chebotarev's density theorem, we have $Ш^2(N_i, \Lambda^1_{E_i}) = 0$ for all i and $Ш^2(\Lambda^1) = 0$. Therefore, one gets the 4-term exact sequence

$$0 \to H^1(\Lambda) \to H^1(\Lambda^1) \xrightarrow{\delta} H^2(\mathbb{Z})' \xrightarrow{\widehat{N}} Ш^2(\Lambda) \to 0. \tag{4-24}$$

From this we obtain $Ш^2(\Lambda) \simeq H^2(\mathbb{Z})'/\operatorname{Im}(\delta)$ and

$$\tau_k(T^{K/E,k}) = \frac{|H^1(\Lambda)|}{|Ш^2(\Lambda)|} = \frac{|H^1(\Lambda^1)|}{|H^2(\mathbb{Z})'|} = \frac{\prod_{i=1}^r |N_i|}{|H^2(\mathbb{Z})'|}. \qquad \square$$

Remark 4.4. Ono [1963b] showed that $\tau(R^{(1)}_{K/k}\mathbb{G}_{m,K}) = [K:k]$ for any cyclic extension K/k. Since $|H^1(K/k, X(R^{(1)}_{K/k}\mathbb{G}_{m,K}))| = [K:k]$, it follows that $Ш^2(K/k, X(R^{(1)}_{K/k}\mathbb{G}_{m,K})) = 0$. This gives an alternative proof of the first statement $Ш^2(\Lambda^1) = 0$ of Proposition 4.3.

In order to compute the groups $H^2(\mathbb{Z})'$ and $Ш^2(\Lambda)$, we describe the maps δ and $\delta_{\mathfrak{D}}$ in the first commutative square of diagram (4-22). Since $\Lambda^1 = \oplus_i \Lambda^1_i$, where $\Lambda^1_i = X(T^{K_i/E_i,1}) = \operatorname{Ind}_{\widetilde{N}_i}^G \Lambda^1_{E_i}$, it suffices to describe the following commutative diagram:

$$\begin{array}{ccc} H^1(\Lambda^1_i) & \xrightarrow{\delta} & H^2(\mathbb{Z}) \\ \downarrow r_D & & \downarrow r_D \\ H^1_D(\Lambda^1_i) & \xrightarrow{\delta_D} & H^2_D(\mathbb{Z}) \end{array}$$

Using the commutative diagram (4-19), the preceding diagram decomposes into two squares:

$$\begin{array}{ccccc} H^1(\operatorname{Ind}_{\widetilde{N}_i}^G \Lambda^1_{E_i}) & \xrightarrow{\widetilde{\delta}} & H^2(\operatorname{Ind}_{\widetilde{N}_i}^G \mathbb{Z}) & \xrightarrow{\widehat{\Delta}^2} & H^2(\mathbb{Z}) \\ \downarrow r_D & & \downarrow r_D & & \downarrow r_D \\ H^1_D(\operatorname{Ind}_{\widetilde{N}_i}^G \Lambda^1_{E_i}) & \xrightarrow{\widetilde{\delta}_D} & H^2_D(\operatorname{Ind}_{\widetilde{N}_i}^G \mathbb{Z}) & \xrightarrow{\widehat{\Delta}^2_D} & H^2_D(\mathbb{Z}) \end{array} \tag{4-25}$$

Proposition 4.5. *Assume that the extension K_i/E_i is cyclic and both the subgroups $\widetilde{N}_i = \operatorname{Gal}(L/E_i)$ (4-2) and $H_i = \operatorname{Gal}(L/K_i)$ (4-1) are normal in G (that is, E_i/k and K_i/k are Galois) for all i. There are natural isomorphisms*

$$H^1(\Lambda^1) \simeq \bigoplus_i H^2(N_i, \mathbb{Z}) \quad \textit{and} \quad H^1_{\mathfrak{D}}(\Lambda^1) = \bigoplus_D \bigoplus_i H^2(\overline{D}_i, \mathbb{Z})^{[G:D\widetilde{N}_i]},$$

where $D_i = D \cap \widetilde{N}_i$ and $\overline{D}_i$ is its image in N_i. Under these identifications the first commutative square of diagram (4-22) decomposes as the following:

$$\begin{array}{ccccc} \bigoplus_{i=1}^r H^2(N_i, \mathbb{Z}) & \xrightarrow{\text{Inf}} & \bigoplus_{i=1}^r H^2(\widetilde{N}_i, \mathbb{Z}) & \xrightarrow{\text{Cor}} & H^2(\mathbb{Z}) \\ \downarrow{\scriptstyle r_{\mathfrak{D}}} & & \downarrow{\scriptstyle r_{\mathfrak{D}}} & & \downarrow{\scriptstyle r_{\mathfrak{D}}} \\ \bigoplus_D \bigoplus_{i=1}^r H^2(\overline{D}_i, \mathbb{Z})^{[G:D\widetilde{N}_i]} & \xrightarrow{\text{Inf}} & \bigoplus_D \bigoplus_{i=1}^r H^2(D_i, \mathbb{Z})^{[G:D\widetilde{N}_i]} & \xrightarrow{\text{Cor}} & \bigoplus_D H^2_D(\mathbb{Z}) \end{array} \tag{4-26}$$

Proof. For any element $g \in G$ and any $\widetilde{N}_i$-module X, let X^g be the set X equipped with the $g\widetilde{N}_i g^{-1}$-module structure defined by

$$h' \cdot x := (g^{-1}h'g)x, \quad \text{for } x \in X^g = X \text{ and } h' \in g\widetilde{N}_i g^{-1}.$$

Recall that Mackey's formula [Serre 1977, Section 7.3 Proposition 22, page 58] says that as D-modules one has

$$\operatorname{Ind}_{\widetilde{N}_i}^G X = \bigoplus_{g \in D\backslash G/\widetilde{N}_i} \operatorname{Ind}_{D_i}^D X^g, \tag{4-27}$$

where g runs through double coset representatives for $D\backslash G/\widetilde{N}_i$. Putting $X = \Lambda^1_{E_i}$ or $X = \mathbb{Z}$, we have

$$\operatorname{Ind}_{\widetilde{N}_i}^G \Lambda^1_{E_i} = \bigoplus_{g \in G/D\widetilde{N}_i} \operatorname{Ind}_{D_i}^D (\Lambda^1_{E_i})^g, \quad \operatorname{Ind}_{\widetilde{N}_i}^G \mathbb{Z} = \bigoplus_{g \in G/D\widetilde{N}_i} \operatorname{Ind}_{D_i}^D \mathbb{Z}$$

as $\widetilde{N}_i$ is normal in G.

We first show $(\Lambda^1_{E_i})^g \simeq \Lambda^1_{E_i}$ as $\widetilde{N}_i$-modules. From the exact sequence of algebraic E_i-tori

$$1 \to R^{(1)}_{K_i/E_i}(\mathbb{G}_{\mathrm{m},K_i}) \xrightarrow{j} R_{K_i/E_i}(\mathbb{G}_{\mathrm{m},K_i}) \xrightarrow{N_{K_i/E_i}} \mathbb{G}_{\mathrm{m},E_i} \to 1, \tag{4-28}$$

one has an exact sequence of $\widetilde{N}_i$-modules

$$0 \to \mathbb{Z} \xrightarrow{\widehat{N}} X(R_{K_i/E_i}(\mathbb{G}_{\mathrm{m},K_i})) \xrightarrow{\widehat{j}} \Lambda^1_{E_i} \to 0. \tag{4-29}$$

Since K_i/E_i is Galois, all E_i-tori in (4-28) split over K_i and hence (4-29) becomes an exact sequence of N_i-modules and $X(R_{K_i/E_i}(\mathbb{G}_{\mathrm{m},K_i})) \simeq \operatorname{Ind}_1^{N_i} \mathbb{Z}$ is an induced module. Thus, $\Lambda^1_{E_i} = \operatorname{Coker}(\mathbb{Z} \to \operatorname{Ind}_{H_i}^{\widetilde{N}_i} \mathbb{Z})$ and $\operatorname{Ind}_{H_i}^{\widetilde{N}_i} \mathbb{Z} = \operatorname{Ind}_1^{N_i} \mathbb{Z}$. So it suffices to show that $(\operatorname{Ind}_1^{N_i} \mathbb{Z})^g \simeq \operatorname{Ind}_1^{N_i} \mathbb{Z}$ as N_i-modules. Let $\{n^g = g^{-1}ng\}_{n \in N_i}$ be a $\mathbb{Z}$-basis of $(\operatorname{Ind}_1^{N_i} \mathbb{Z})^g$. The new action of N_i on $(\operatorname{Ind}_1^{N_i} \mathbb{Z})^g$ is given by $h \cdot n^g := h^g n^g = (hn)^g$ for $h, n \in N_i$. So the map $n \mapsto n^g$ gives an isomorphism $\operatorname{Ind}_1^{N_i} \mathbb{Z} \xrightarrow{\sim} (\operatorname{Ind}_1^{N_i} \mathbb{Z})^g$ of N_i-modules, and hence $(\Lambda^1_{E_i})^g \simeq \Lambda^1_{E_i}$.

By Mackay's formula (4-27) and Shapiro's lemma, we obtain the following commutative diagram from (4-25):

$$\begin{array}{ccccc} H^1(\widetilde{N}_i, \Lambda^1_{E_i}) & \xrightarrow{\widetilde{\delta}} & H^2(\widetilde{N}_i, \mathbb{Z}) & \xrightarrow{\mathrm{Cor}} & H^2(\mathbb{Z}) \\ \big\downarrow r_D & & \big\downarrow r_D & & \big\downarrow r_D \\ \bigoplus_g H^1(D_i, (\Lambda^1_{E_i})^g) & \xrightarrow{\widetilde{\delta}_D} & \bigoplus_g H^2(D, \mathbb{Z}) & \xrightarrow{\mathrm{Cor}} & H^2_D(\mathbb{Z}) \end{array} \tag{4-30}$$

Since $(\Lambda^1_{E_i})^g \simeq \Lambda^1_{E_i}$, the bottom row of (4-30) can be expressed as

$$H^1(D_i, \Lambda^1_{E_i})^{[G:D\widetilde{N}_i]} \xrightarrow{\widetilde{\delta}_D} H^2(D_i, \mathbb{Z})^{[G:D\widetilde{N}_i]} \xrightarrow{\mathrm{Cor}} H^2_D(\mathbb{Z}). \tag{4-31}$$

Taking the Galois cohomology $H^*(D_i, -)$ to (4-29), we get an exact sequence

$$0 \to H^1(D_i, \Lambda^1_{E_i}) \xrightarrow{\widetilde{\delta}_{D_i}} H^2(D_i, \mathbb{Z}) \xrightarrow{\widehat{N}} H^2(D_i, \mathrm{Ind}_{H_i}^{\widetilde{N}_i} \mathbb{Z}) \simeq H^2(D_i \cap H_i, \mathbb{Z})^{[\widetilde{N}_i:D_i H_i]}$$

as one has

$$H^2(D_i, \mathrm{Ind}_{H_i}^{\widetilde{N}_i} \mathbb{Z}) = H^2(D_i, \bigoplus_{\widetilde{N}_i/D_i H_i} \mathrm{Ind}_{D_i \cap H_i}^{D_i} \mathbb{Z}) \simeq H^2(D_i \cap H_i, \mathbb{Z})^{[\widetilde{N}_i:D_i H_i]}.$$

Similar to (4-16) and (4-17), using the inflation-restriction exact sequence, we make the following identification $H^1(D_i, \Lambda^1_{E_i}) = H^2(\bar{D}_i, \mathbb{Z})$ and (4-31) becomes

$$H^2(\bar{D}_i, \mathbb{Z})^{[G:D\widetilde{N}_i]} \xrightarrow{\mathrm{Inf}} H^2(D_i, \mathbb{Z})^{[G:D\widetilde{N}_i]} \xrightarrow{\mathrm{Cor}} H^2_D(\mathbb{Z}).$$

This proves the proposition. □

The following proposition gives a group-theoretic description of $Ш^2(\Lambda)$.

Proposition 4.6. *Let the notation and assumptions be as in Proposition 4.5. Let*

$$\mathrm{Ver}_{G,\boldsymbol{N}} = (\mathrm{Ver}_{G,N_i})_i : G^{\mathrm{ab}} \to \prod_i N_i \quad \textit{and} \quad \mathrm{Ver}_{D,\bar{\boldsymbol{D}}} = (\mathrm{Ver}_{D,\bar{D}_i})_i : D^{\mathrm{ab}} \to \prod_i \bar{D}_i$$

denote the corresponding transfer maps, respectively. Then

$$H^2(G, \mathbb{Z})' = \{f \in G^{\mathrm{ab},\vee} : f|_{D^{\mathrm{ab}}} \in \mathrm{Im}(\mathrm{Ver}^\vee_{D,\bar{\boldsymbol{D}}}) \forall D \in \mathcal{D}\}, \tag{4-32}$$

and

$$Ш^2(\Lambda) \simeq \frac{\{f \in G^{\mathrm{ab},\vee} : f|_{D^{\mathrm{ab}}} \in \mathrm{Im}(\mathrm{Ver}^\vee_{D,\bar{\boldsymbol{D}}}) \forall D \in \mathcal{D}\}}{\mathrm{Im}(\mathrm{Ver}^\vee_{G,\boldsymbol{N}})}. \tag{4-33}$$

Proof. We translate the commutative diagram (4-26) in terms of group theory. For each decomposition group $D \in \mathcal{D}$, we have the following corresponding commutative diagram:

$$\begin{array}{ccccc}
\prod_{i=1}^r N_i & \xleftarrow{\pi_{\boldsymbol{H}}=(\pi_{H_i})} & \prod_{i=1}^r \widetilde{N}_i^{\mathrm{ab}} & \xleftarrow{\mathrm{Ver}_{G,\boldsymbol{N}}=(\mathrm{Ver}_{G,N_i})} & G^{\mathrm{ab}} \\
\uparrow \text{func} & & \uparrow \text{func} & & \uparrow \text{func} \\
\prod_{i=1}^r \bar{D}_i & \xleftarrow{\pi_{\boldsymbol{D}\cap \boldsymbol{H}}=(\pi_{D_i\cap H_i})} & \prod_{i=1}^r D_i^{\mathrm{ab}} & \xleftarrow{\mathrm{Ver}_{D,\boldsymbol{D}}=(\mathrm{Ver}_{D,D_i})} & D^{\mathrm{ab}}.
\end{array}$$

Here we ignore the multiplicity $[G : D\widetilde{N}_i]$ because we are only concerned with the image of the map $\mathrm{Cor} \circ \mathrm{Inf}$ in (4-26) and this does not affect the result. Then $\mathrm{Ver}_{G,\boldsymbol{N}} : G^{\mathrm{ab}} \to \prod_i N_i$ and $\mathrm{Ver}_{D,\bar{\boldsymbol{D}}} : D^{\mathrm{ab}} \to \prod_i \bar{D}_i$ are the respective compositions. By Proposition 4.5, the map δ_D corresponds to $\mathrm{Ver}^\vee_{D,\bar{\boldsymbol{D}}}$. From the second description of $H^2(\mathbb{Z})'$ in (4-23), we obtain (4-32).

By Propositions 4.5 and 4.1, the map $\delta : H^1(\Lambda^1) \to H^2(\mathbb{Z})' \subset H^2(\mathbb{Z})$ (4-24) corresponds to $\mathrm{Ver}^\vee_{G,\boldsymbol{N}} : \prod_i N_i^\vee \to G^{\mathrm{ab},\vee}$. Thus, by Proposition 4.3, we obtain (4-33). This proves the proposition. □

5. Computations of some product cases

We keep the notation in the previous section. In this section we consider the case where the extensions K_i/k are all Galois with Galois group $G_i = \mathrm{Gal}(K_i/k)$. Let $L = K_1K_2 \cdots K_r$ be the compositum of all K_i over k with Galois group $G = \mathrm{Gal}(L/k)$. Assume that:

$$\begin{array}{l}\text{(i) The canonical map monomorphism } G \to G_1 \times \cdots \times G_r \text{ is an isomorphism.}\\ \text{(ii) } K_i/E_i \text{ is cyclic with Galois group } N_i := \mathrm{Gal}(K_i/E_i) \text{ for all } i.\end{array} \tag{5-1}$$

As before, we put $T = T^{K/E,k}$, $\Lambda := X(T)$, $T^1 = T^{K/E,1}$ and $\Lambda^1 := X(T^1)$. For each $1 \le i \le r$, let

$$T_i := T^{K_i/E_i,k} \quad \text{and} \quad \Lambda_i := X(T_i) \tag{5-2}$$

be the character group of T_i.

5A. $H^1(\Lambda)$ *and* $H^2(\Lambda^1)$.

Lemma 5.1. *Let $\Lambda_i := X(T_i)$ as in (5-2). We have $H^1(\Lambda) \simeq \oplus_{i=1}^r H^1(\Lambda_i)$.*

Proof. By Proposition 4.1, $H^1(\Lambda)$ is isomorphic to the kernel of the dual of the map $\prod_i \mathrm{Ver}_{G,N_i} : G \to \prod_i N_i^{\mathrm{ab}}$. By Lemma 3.9, $\prod_i \mathrm{Ver}_{G,N_i} = \prod_i \mathrm{Ver}_{G_i,N_i} : \prod_{i=1}^r G_i \to \prod_i N_i^{\mathrm{ab}}$, and therefore the kernel of its dual is equal to the product of abelian groups $H^1(\Lambda_i)$ for $i = 1, \dots, r$, by Proposition 4.1. This proves the lemma. □

We remark that condition (ii) in (5-1) is not needed in the proof of Lemma 5.1.

Proposition 5.2. *Let $H \subset G$ be a normal subgroup of a finite group G, let A be a G-module, and let $n \ge 1$ be a positive integer. If $H^q(H, A) = 0$ for all $0 < q < n$, then we have a 5-term long exact sequence*

$$0 \to H^n(G/H, A^H) \to H^n(G, A) \to H^n(H, A)^{G/H} \xrightarrow{d} H^{n+1}(G/H, A^H) \to H^{n+1}(G, A).$$

Proof. This follows from the Hochschild–Serre spectral sequence

$$E_2^{p,q} := H^p(G/H, H^q(H, A^H)) \Longrightarrow H^{p+q}(G, A);$$

see [Neukirch et al. 2000, Theorem 2.1.5, page 82 and Proposition 2.1.3, page 81]. □

Proposition 5.3. *Let $K = \prod_{i=1}^r K_i$ and $E = \prod_{i=1}^r E_i$ be as before. Suppose that each K_i/k is Galois with group G_i, and that conditions* (i) *and* (ii) *in (5-1) are satisfied. Then*

$$H^2(G, \Lambda^1) \simeq \oplus_{i=1}^r H^2(\widetilde{N}_i, \Lambda^1_{E_i}),$$

where $\Lambda^1_{E_i}$ is defined in (4-8), and there is a natural homomorphism $\upsilon_i : N_i \to (H_i^{\mathrm{ab}})^{|N_i|-1}$ such that $H^2(\widetilde{N}_i, \Lambda^1_{E_i}) \simeq \mathrm{Ker}(\upsilon_i^\vee)$. If $r = 1$, that is K/k is a Galois field extension, then $H^2(\Lambda^1) = 0$.

Proof. By (4-9), we have

$$H^q(G, \Lambda^1) = \bigoplus_{i=1}^r H^q(\widetilde{N}_i, \Lambda^1_{E_i}), \quad \forall q \geq 0.$$

By (4-29), we get the long exact sequence

$$H^q(N_i, \mathrm{Ind}_{H_i}^{\widetilde{N}_i} \mathbb{Z}) \to H^q(N_i, \Lambda^1_{E_i}) \to H^{q+1}(N_i, \mathbb{Z}) \to H^{q+1}(N_i, \mathrm{Ind}_{H_i}^{\widetilde{N}_i} \mathbb{Z}). \tag{5-3}$$

Since $\mathrm{Ind}_{H_i}^{\widetilde{N}_i} \mathbb{Z} = \mathrm{Ind}_1^{N_i} \mathbb{Z}$ is an induced module, it follows from (5-3) that

$$H^2(N_i, \Lambda^1_{E_i}) \xrightarrow{\sim} H^3(N_i, \mathbb{Z}) = H^1(N_i, \mathbb{Z}) = 0. \tag{5-4}$$

Since the algebraic torus T^{K_i} splits over K_i, the H_i-module $\Lambda^1_{E_i} \simeq \mathbb{Z}^{|N_i|-1}$ is trivial and $H^1(H_i, \Lambda^1_{E_i}) = H^1(H_i, \mathbb{Z}^{|N_i|-1}) = 0$. By Proposition 5.2 with $(G, H) = (\widetilde{N}_i, H_i)$, we have the exact sequence

$$0 \to H^2(\widetilde{N}_i, \Lambda^1_{E_i}) \to H^2(H_i, \Lambda^1_{E_i})^{N_i} \xrightarrow{d_i} H^3(N_i, \Lambda^1_{E_i}).$$

Using the same argument as (5-4), we get $H^3(N_i, \Lambda^1_{E_i}) \simeq \mathrm{Hom}(N_i, \mathbb{Q}/\mathbb{Z})$. On the other hand,

$$H^2(H_i, \Lambda^1_{E_i}) \simeq H^2(H_i, \mathbb{Z})^{|N_i|-1} \simeq \mathrm{Hom}(H_i, \mathbb{Q}/\mathbb{Z})^{|N_i|-1}.$$

We now observe that the conjugation action of the group $N_i = \widetilde{N}_i/H$ on $\mathrm{Hom}(H_i, \mathbb{Q}/\mathbb{Z})^{|N_i|-1}$ is trivial. Consequently, we obtain an exact sequence

$$0 \to H^2(\widetilde{N}_i, \Lambda^1_{E_i}) \to \mathrm{Hom}((H_i^{\mathrm{ab}})^{|N_i|-1}, \mathbb{Q}/\mathbb{Z}) \xrightarrow{d_i} \mathrm{Hom}(N_i, \mathbb{Q}/\mathbb{Z}).$$

Let $\upsilon_i : N_i \to (H_i^{\mathrm{ab}})^{|N_i|-1}$ be the Pontryagin dual of d_i. Then $H^2(\widetilde{N}_i, \Lambda^1_{E_i}) = \mathrm{Ker}(\upsilon_i^\vee)$. □

Corollary 5.4. *Let the notation and assumptions be as in Proposition 5.3. Assume further that the orders of groups G_i are pairwise coprime. Then*

$$H^2(\Lambda^1) \simeq \bigoplus_{i=1}^r \mathrm{Hom}(H_i, \mathbb{Q}/\mathbb{Z})^{[K_i:E_i]-1}.$$

(In our notation the group $H_1 = \{1\}$ if $r = 1$.)

Proof. In this case, the maps d_i are all zero. □

Remark 5.5. It would be interesting to describe the map $v_i : N_i \to (H_i^{\mathrm{ab}})^{[K_i/E_i]-1}$ explicitly.

5B. $\tau(T)$. In this subsection we shall further assume that each subgroup N_i, besides being cyclic, is also normal in G_i. This assumption simplifies the description of the group $H^2(\mathbb{Z})'$ through Mackey's formula. Let $\mathcal{D}_i$ be the decomposition subgroups of G_i for each i.

Lemma 5.6. (1) *The inclusion $\prod_{i=1}^r H^2(G_i,\mathbb{Z})' \subset H^2(\mathbb{Z})'$ holds.*

(2) *Assume that for any $1 \le i \le r$ and $D_i' \in \mathcal{D}_i$, there exists a member $D \in \mathcal{D}$ such that $\mathrm{pr}_i(D) = D_i'$ and a section $s_i : D_i' \to D$ of $\mathrm{pr}_i : D \twoheadrightarrow D_i'$ such that the composition $\mathrm{pr}_k \circ s_i : D_i' \to D \to G_k$ is trivial for $k \ne i$. Then $\prod_{i=1}^r H^2(G_i,\mathbb{Z})' = H^2(\mathbb{Z})'$.*

Proof. (1) For each $f = (f_1, \dots, f_r) \in \mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z}) = \prod_{i=1}^r \mathrm{Hom}(G_i, \mathbb{Q}/\mathbb{Z})$, consider the following two conditions:

(a) For each $D \in \mathcal{D}$, the restriction $f|_D$ of f to D lies in the image of the map $\sum_{i=1}^r \mathrm{Ver}^\vee_{D,\bar{D}_i}$.

(b) For each $1 \le i \le r$ and $D_i' \in \mathcal{D}_i$, the restriction $f_i|_{D_i'}$ to D_i' lies in the image of the map $\mathrm{Ver}^\vee_{D_i', D_i'\cap N_i}$.

By Proposition 4.6, we have

$$H^2(G,\mathbb{Z})' = \{f \in G^{\mathrm{ab},\vee} : f|_{D^{\mathrm{ab}}} \in \mathrm{Im}(\mathrm{Ver}^\vee_{D,\bar{\boldsymbol{D}}}) \forall D \in \mathcal{D}\}$$

and it is equivalent to show that the condition (b) implies (a).

Let $D = D_{\mathfrak{P}} \in \mathcal{D}$ be a decomposition group associated to a prime $\mathfrak{P}$ of L. By definition $D_i = D \cap \widetilde{N}_i$ and $\bar{D}_i = D_i/D_i \cap H_i$. The projection map $\mathrm{pr}_i : G \to G_i$ sends each element σ to its restriction $\sigma|_{K_i}$ to K_i. Let $\mathfrak{P}_i$ denote the prime of K_i below $\mathfrak{P}$ and we have $\mathrm{pr}_i(D_{\mathfrak{P}}) = D_{\mathfrak{P}_i}$. We show that $\mathrm{pr}_i(D_{\mathfrak{P}} \cap \widetilde{N}_i) = D_{\mathfrak{P}_i} \cap N_i$. The inclusion $\subseteq$ is obvious because $\mathrm{pr}_i(D_{\mathfrak{P}} \cap \widetilde{N}_i) \subset \mathrm{pr}_i(D_{\mathfrak{P}_i}) \cap \mathrm{pr}_i(\widetilde{N}_i) = D_{\mathfrak{P}_i} \cap N_i$. For the other inclusion, since $\mathrm{pr}_i : D_{\mathfrak{P}} \to D_{\mathfrak{P}_i}$ is surjective, for each $y \in D_{\mathfrak{P}_i} \cap N_i$ there exists an element $x = (x_i) \in D_{\mathfrak{P}}$ such that $x_i = y$. Since $x_i \in N_i$, x also lies in $\widetilde{N}_i$. This proves the other inclusion. Therefore, $\bar{D}_i = D_{\mathfrak{P}_i} \cap N_i$.

The map $\sum_{i=1}^r \mathrm{Ver}^\vee_{D,\bar{D}_i}$ is dual to the map

$$\prod_{i=1}^r \mathrm{Ver}_{D, D_{\mathfrak{P}_i}\cap N_i} : D \to \prod_{i=1}^r (D_{\mathfrak{P}_i} \cap N_i).$$

Since each $\mathrm{Ver}_{D, D_{\mathfrak{P}_i}\cap N_i} = \mathrm{Ver}_{D_{\mathfrak{P}_i}, D_{\mathfrak{P}_i}\cap N_i} \circ \mathrm{pr}_i$ (3-4), the above map factorizes into the following composition

$$D \xrightarrow{(\mathrm{pr}_i)_i} \prod_{i=1}^r D_{\mathfrak{P}_i} \xrightarrow{\prod_i \mathrm{Ver}_{D_{\mathfrak{P}_i}, D_{\mathfrak{P}_i}\cap N_i}} \prod_{i=1}^r (D_{\mathfrak{P}_i} \cap N_i).$$

Consider the restriction $f|_{\prod_i D_{\mathfrak{P}_i}} = (f_i|_{D_{\mathfrak{P}_i}})$ of f to $\prod_i D_{\mathfrak{P}_i}$, and put $D_i' := D_{\mathfrak{P}_i}$. By condition (b), there exists an element $h_i \in \mathrm{Hom}(D_{\mathfrak{P}_i} \cap N_i, \mathbb{Q}/\mathbb{Z})$ such that $f_i|_{D_{\mathfrak{P}_i}} = \mathrm{Ver}^\vee_{D_{\mathfrak{P}_i}, D_{\mathfrak{P}_i}\cap N_i}(h_i)$. Then

$$f|_{\prod_i D_{\mathfrak{P}_i}} = (\mathrm{Ver}^\vee_{D_{\mathfrak{P}_i}, D_{\mathfrak{P}_i}\cap N_i}(h_i)).$$

Restricting this function to D, we obtain $f|_D = \sum_{i=1}^r \mathrm{Ver}^\vee_{D,\bar{D}_i}(h_i)$. This shows that $f|_D$ lies in the image of $\sum_{i=1}^r \mathrm{Ver}^\vee_{D,\bar{D}_i}$.

(2) By (1), we have $\prod_{i=1}^r H^2(G_i,\mathbb{Z})' \subset H^2(\mathbb{Z})'$. To prove the other inclusion, we must show that each f_i satisfies condition (b) provided f satisfies condition (a), which we assume from now on. For $1 \le i \le r$, let $D \in \mathcal{D}$ be a decomposition group of G over D_i' and $s_i : D_i' \to D$ be a section such that $\mathrm{pr}_k \circ s_i$ is trivial for $k \ne i$. Then (a) implies

$$f|_D = \sum_{k=1}^r \mathrm{Ver}^\vee_{D,\bar{D}_k}(h_k)$$

for some $h_k \in \mathrm{Hom}(\bar{D}_k, \mathbb{Q}/\mathbb{Z})$. Pulling back to D_i' via s_i, we have

$$f_i|_{D_i'} = s_i^* f|_D = \sum_{k=1}^r s_i^* \mathrm{Ver}^\vee_{D,\bar{D}_k}(h_k) = \sum_{k=1}^r s_i^* \mathrm{pr}_k^* \mathrm{Ver}^\vee_{D_k',\bar{D}_k}(h_k),$$

where $D_k' := \mathrm{pr}_k(D)$ if $k \ne i$. Since $s_i^* \mathrm{pr}_k^* = 1$ if $k = i$ and $s_i^* \mathrm{pr}_k^* = 0$ otherwise, we get $f_i|_{D_i'} = \mathrm{Ver}^\vee_{D_i',\bar{D}_i}(h_i)$. □

Proposition 5.7. *With the notation and assumptions be as above. Suppose further that for each $1 \le i \le r$, every decomposition group of G_i is cyclic. Then we have $\tau(T) = \prod_{i=1}^r \tau(T_i)$, where $T_i = T^{K_i/E_i,k}$ (see (5-2)).*

Proof. We shall show that the assumption in Lemma 5.6(2) holds. Let $D_i' \in \mathcal{D}_i$ be a decomposition group of G_i; by our assumption D_i' is cyclic and let a_i be a generator. Let D be a cyclic subgroup of G generated by $\tilde{a}_i = (1,\dots,1,a_i,1,\dots,1)$ with a_i at the i-th place. Clearly D is the decomposition group of some prime and let $s_i : D_i' \to D$ be the section sending a_i to $\tilde{a}_i$. Clearly for $k \ne i$, $\mathrm{pr}_k \circ s_i : D_i' \to D_k'$ is trivial. By Lemma 5.6, we have $H^2(G,\mathbb{Z})' = \prod_{i=1}^r H^2(G_i,\mathbb{Z})'$.

By Proposition 4.3, we have

$$\tau(T) = \frac{\prod_{i=1}^r |N_i|}{|H^2(\mathbb{Z})'|} \quad \text{and} \quad \tau(T_i) = \frac{|N_i|}{|H^2(G_i,\mathbb{Z})'|}.$$

Therefore, $\tau(T) = \prod_{i=1}^r \tau(T_i)$. □

6. Galois CM fields

We return to the case of CM tori and keep the notation of Section 2C. In this section, we assume that K is a Galois CM field. Let $G = \mathrm{Gal}(K/\mathbb{Q})$, $G^+ := \mathrm{Gal}(K^+/\mathbb{Q})$ and $g := [K^+ : \mathbb{Q}]$. Then one has the short exact sequence

$$1 \to \langle \iota \rangle \to G \to G^+ \to 1. \tag{6-1}$$

Recall the $\mathbb{Q}$-torus $T^{K,\mathbb{Q}}$ (2-6) is the CM torus over $\mathbb{Q}$ associated to the CM field K. We set $T := T^{K,\mathbb{Q}}$.

6A. *Cyclotomic extensions.*

Proposition 6.1. *Let $K = \mathbb{Q}(\zeta_n) \neq \mathbb{Q}$ be the n-th cyclotomic field with n odd or $4 \mid n$, and T the associated CM torus over $\mathbb{Q}$:*

(1) *If n is either a power of an odd prime p or $n = 4$, then $\tau(T) = 1$.*

(2) *In other cases, $\tau(T) = 2$.*

Proof. (1) In this case the Galois group $G = \mathrm{Gal}(K/\mathbb{Q})$ is cyclic and Proposition 2.9(3) shows that $\tau(T) = 1$.

(2) Suppose first that n is not a power of a prime. By Proposition 2.7(2) the global norm index n_K divides $\prod_{p \in S_{K/K^+}} e_{T,p}$. By [Washington 1997, Proposition 2.15], the quadratic extension K/K^+ is unramified at all finite places of K^+. It follows that $S_{K/K^+} = \varnothing$, and hence that $n_K = 1$. Proposition 2.7(1) then shows $\tau(T) = 2$. Now suppose the other case that $n = 2^v$ with $v \geq 3$. Then K contains $\mathbb{Q}(\zeta_8) = \mathbb{Q}(\sqrt{2}, \sqrt{-1})$, which contains two distinct imaginary quadratic fields. Thus, by Lemma 2.8, we have $n_K = 1$ and hence $\tau(T) = 2$ by Proposition 2.7(1). This completes the proof. □

6B. *Abelian extensions.*

Proposition 6.2. *Assume that $G = \mathrm{Gal}(K/\mathbb{Q})$ is abelian:*

(1) *We have $\tau(T) \in \{1, 2\}$.*

(2) *If g is odd, then $\tau(T) = 1$.*

(3) *If g is even and (6-1) splits, then $\tau(T) = 2$.*

Proof. (1) Write $G = \prod_{i=1}^{\ell} G_i$ as a product of cyclic subgroups G_i. Then there exists i such that the image of ι in the projection $G \to G_i$ is nontrivial, say $i = 1$. Let K_1 be a subfield of K with $\mathrm{Gal}(K_1/\mathbb{Q}) = G_1$, which is a cyclic CM field. Then we have $\tau(T^{K_1,\mathbb{Q}}) = 1$ by Proposition 2.9(3), that is, $n_{K_1} = 2$. Since $K \supset K_1$, we have $n_K \mid n_{K_1}$ and $n_K \in \{1, 2\}$. Therefore, $\tau(T) \in \{1, 2\}$.

(2) This follows from Proposition 2.9.

(3) Since (6-1) splits, $G \simeq \langle\iota\rangle \times G^+$. As g is even, there is an epimorphism

$$\langle\iota\rangle \times G^+ \twoheadrightarrow \langle\iota\rangle \times \mathbb{Z}/2\mathbb{Z}. \tag{6-2}$$

In particular, K contains two distinct imaginary quadratic fields; therefore, $n_K = 1$ and $\tau(T) = 2$ by Lemma 2.8. □

6C. *Certain Galois extensions.*

Proposition 6.3. *Assume the short exact sequence (6-1) splits. Let $g^{\mathrm{ab}} := |G^{+\mathrm{ab}}|$ and $\Lambda := X(T)$. Then $\tau(T) \in \{1, 2\}$. Moreover, the following statements hold:*

(1) *When g is odd*, $\tau(T) = 1$.

(2) *When g is even*:

  (i) *If g^{ab} is even, then* $\tau(T) = 2$.

  (ii) *If g^{ab} is odd, then there is a unique nonzero element ξ in the 2-torsion subgroup $H^2(\Lambda)[2]$ of $H^2(\Lambda)$. Moreover, $\tau(T) = 1$ if and only if its restriction $r_D(\xi) = 0$ in $H^2(D, \Lambda)$ for all $D \in \mathcal{D}$.*

Proof. Since (6-1) splits, $G \simeq \langle\iota\rangle \times G^+$. It follows that K contains an imaginary quadratic field E. Since $n_E = 2$, one has $n_K \in \{1, 2\}$ and $\tau(T) \in \{1, 2\}$. The statement (1) follows from Proposition 2.9(2).

(2) We have $G \simeq \langle\iota\rangle \times G^+$ and $G^{\mathrm{ab}} \simeq \langle\iota\rangle \times G^{+\mathrm{ab}}$. (i) Suppose g^{ab} is even. As (6-2), K contains two distinct imaginary quadratic fields. Thus, $n_K = 1$ and $\tau(T) = 2$. (ii) Suppose g^{ab} is odd. Put $\Lambda^1 = X(T^{K,1})$ and we have the following exact sequence

$$0 \to H^1(\Lambda) \to H^1(\Lambda^1) \to H^2(\mathbb{Z}) \to H^2(\Lambda) \to H^2(\Lambda^1).$$

We have $H^1(\Lambda^1) \simeq \mathbb{Z}/2\mathbb{Z}$ by Proposition 4.1(1), $H^2(\Lambda^1) = 0$ by Proposition 5.3, and $H^1(\Lambda) \simeq \mathbb{Z}/2\mathbb{Z}$ by [Achter et al. 2023, Proposition A.6]. This gives $H^2(\mathbb{Z}) \cong H^2(\Lambda)$. The proof of [loc. cit., Lemma A.7] shows that $H^1(G, T)$ is a 2-torsion group. It follows that $Ш^2(\Lambda)$ is also a 2-torsion group. Therefore, $Ш^2(\Lambda) \subset H^2(\Lambda)[2]$. Since g^{ab} is odd,

$$H^2(\Lambda)[2] \simeq H^2(\mathbb{Z})[2] = \mathrm{Hom}(G^{\mathrm{ab}}, \mathbb{Q}/\mathbb{Z})[2] = \mathrm{Hom}(\langle\iota\rangle \times G^{+\mathrm{ab}}, \mathbb{Q}/\mathbb{Z})[2] \cong \mathbb{Z}/2\mathbb{Z}.$$

Let $\xi \in H^2(\Lambda)[2]$ be the unique nonzero element. Then

$$\tau(T) = 1 \iff Ш^2(\Lambda) \simeq \mathbb{Z}/2\mathbb{Z} \iff \xi \in Ш^2(\Lambda) \iff r_D(\xi) = 0 \quad \text{for all } D \in \mathcal{D}.$$

This completes the proof of the proposition. □

Remark 6.4. When G is nonabelian and the short exact sequence (6-1) does not split, we do not know the value of $\tau(T)$ in general. However, if the involution ι is nontrivial on the maximal abelian extension K_{ab} over $\mathbb{Q}$ in K, then K_{ab} is a CM abelian field and it follows from Proposition 6.2 that $\tau(T) \in \{1, 2\}$.

It remains to determine the Tamagawa number when G is nonabelian, (6-1) is nonsplit, and K_{ab} is totally real. This includes the cases of $G = Q_8$ and the dihedral groups, for which we treat in the next subsections.

6D. *Q_8-extensions.* The quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ is the group of 8 elements generated by i, j with usual relations $i^2 = j^2 = -1$ and $ij = -ji = k$. We have the following well-known properties of Q_8.

Lemma 6.5. (1) *The group Q_8 contains 6 elements of order 4, one element of order 2, and the identity. It does not have a subgroup isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Thus, every proper subgroup is cyclic.*

(2) *Every nontrivial subgroup contains $\{\pm 1\}$, and only subgroup which maps onto $Q_8/\{\pm 1\}$ is Q_8.*

(3) *The center $Z(Q_8) = \{\pm 1\} = D(Q_8)$, where $D(Q_8) = [Q_8, Q_8]$.*

Proposition 6.6. *Let P and Q be two odd positive integers such that*

$$P-1=a^2, \quad Q-1=Pb^2$$

for some integers $a,b\in\mathbb{N}$. Assume that Q is not a square. Let $K:=\mathbb{Q}(\beta)$ be the simple extension of $\mathbb{Q}$ generated by β which satisfies $\beta^2=\alpha:=-(P+\sqrt{P})(Q+\sqrt{Q})$. Then K is a Galois CM field with Galois group Q_8 with maximal totally real field $K^+=\mathbb{Q}(\sqrt{P},\sqrt{Q})$. The Galois group $\mathrm{Gal}(K/\mathbb{Q})$ *is generated by τ_1 and τ_2 given by*

$$\tau_1(\beta)=\frac{(\sqrt{P}-1)}{a}\beta, \quad \tau_2(\beta)=\frac{(\sqrt{Q}-1)}{\sqrt{P}a}\beta. \tag{6-3}$$

Moreover, for each prime ℓ, the decomposition group D_ℓ is cyclic except when $\ell\mid Q$. For $\ell\mid Q$, one has

$$D_\ell=\begin{cases}\mathbb{Z}/4\mathbb{Z} & \textit{if } \left(\frac{P}{\ell}\right)=1;\\ Q_8 & \textit{if } \left(\frac{P}{\ell}\right)=-1.\end{cases}$$

Proof. Note that $\gcd(P,Q)=1$ and P is not a square. Then $\mathbb{Q}(\sqrt{P},\sqrt{Q})$ is a totally real biquadratic field and its Galois group is generated by σ_1 and σ_2, where

$$\sigma_1:\begin{cases}\sqrt{P}\mapsto-\sqrt{P},\\ \sqrt{Q}\mapsto\sqrt{Q},\end{cases} \qquad \sigma_2:\begin{cases}\sqrt{P}\mapsto\sqrt{P},\\ \sqrt{Q}\mapsto-\sqrt{Q}.\end{cases}$$

It is clear that α is a totally negative, and hence K is a CM field. The maximal totally real subfield $K^+=\mathbb{Q}(\alpha)\subset\mathbb{Q}(\sqrt{P},\sqrt{Q})$. Since the minimal polynomial of α is of degree 4, one has $K^+=\mathbb{Q}(\sqrt{P},\sqrt{Q})$. Let $\tau_i\in\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, $i=1,2$, be elements such that $\tau_i|_{K^+}=\sigma_i$. One computes

$$\frac{\tau_1(\beta)^2}{\beta^2}=\frac{\tau_1(\alpha)}{\alpha}=\frac{-\sigma_1(P+\sqrt{P})\sigma_1(Q+\sqrt{Q})}{-(P+\sqrt{P})(Q+\sqrt{Q})}=\frac{(P-\sqrt{P})^2}{P^2-P}=\frac{(\sqrt{P}-1)^2}{P-1}=\frac{(\sqrt{P}-1)^2}{a^2},$$

and obtains

$$\tau_1(\beta)=\pm\frac{(\sqrt{P}-1)}{a}\beta \quad\text{and}\quad \tau_1(K)\subset K.$$

Similarly, one computes $\tau_2(\beta)^2/\beta^2=(\sqrt{Q}-1)^2/Pb^2$ and obtains

$$\tau_2(\beta)=\pm\frac{(\sqrt{Q}-1)}{\sqrt{P}a}\beta \quad\text{and}\quad \tau_2(K)\subset K.$$

It follows that $K/\mathbb{Q}$ is Galois. Let $\tau_1,\tau_2\in\mathrm{Gal}(K/\mathbb{Q})$ be defined as in (6-3). One easily computes $\tau_1^2(\beta)=-\beta$, $\tau_2^2(\beta)=-\beta$, $\tau_1\tau_2(\beta)=\iota\tau_2\tau_1(\beta)$ and obtains the relations

$$\tau_1^4=\tau_2^4=1, \quad \tau_1^2=\tau_2^2, \quad \tau_1\tau_2\tau_1^{-1}=\tau_2^{-1},$$

showing that K is a Q_8-CM field.

Denote by D_ℓ^+ the decomposition group at ℓ in $G^+=\mathrm{Gal}(K^+/\mathbb{Q})$. Then $D_\ell=Q_8$ if and only if $D_\ell^+=G^+$. If $\ell\nmid PQ$, then ℓ is unramified in K^+ and D_ℓ^+ is cyclic. Thus, D_ℓ^+ cannot be G^+, $D_\ell\neq Q_8$ and D_ℓ is cyclic. If $\ell\mid P$, then $\left(\frac{Q}{\ell}\right)=\left(\frac{1}{\ell}\right)=1$. So ℓ is ramified in $\mathbb{Q}(\sqrt{P})$ and splits in $\mathbb{Q}(\sqrt{Q})$. Thus,

$D_\ell^+ \neq G^+$ and hence D_ℓ is cyclic. It remains to treat the case $\ell \mid Q$. If $(\frac{P}{\ell}) = 1$, then $D_\ell^+ \simeq \mathbb{Z}/2\mathbb{Z}$ and hence $D_\ell \simeq \mathbb{Z}/4\mathbb{Z}$. Otherwise, ℓ is ramified in $\mathbb{Q}(\sqrt{Q})$ and inert in $\mathbb{Q}(\sqrt{P})$. One has $D_\ell^+ = G^+$ and $D_\ell = Q_8$. □

Proposition 6.7. *Let K be a CM field which is Galois over $\mathbb{Q}$ with Galois group Q_8. Then*

$$\tau(T^{K,\mathbb{Q}}) = \begin{cases} \frac{1}{2} & \text{if every decomposition group of } K/\mathbb{Q} \text{ is cyclic;} \\ 2 & \text{otherwise.} \end{cases} \tag{6-4}$$

Proof. Put $G = \mathrm{Gal}(K/\mathbb{Q}) \simeq Q_8$ and $N = \langle \iota \rangle$. We shall show that

$$H^1(\Lambda) = H^1(G, \Lambda) = \mathbb{Z}/2\mathbb{Z} \tag{6-5}$$

and

$$Ш^2(\Lambda) = \begin{cases} \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if every decomposition group of } K/\mathbb{Q} \text{ is cyclic;} \\ 0 & \text{otherwise.} \end{cases} \tag{6-6}$$

Then (6-4) follows from (6-5) and (6-6).

By Proposition 4.1 and (4-20), we have an exact sequence

$$0 \to H^1(\Lambda) \to H^1(\Lambda^1) \simeq N^{\mathrm{ab},\vee} \xrightarrow{\mathrm{Ver}_{G,N}^\vee} H^2(\mathbb{Z}) = G^{\mathrm{ab},\vee} \to H^2(\Lambda) \to 0, \tag{6-7}$$

noting that $H^2(G, \Lambda^1) = 0$ when $K/\mathbb{Q}$ is a Galois CM field. Since the 2-Sylow subgroup of Q_8 is not cyclic, the transfer $\mathrm{Ver}_{G,N}$ is not surjective by Proposition 3.3. Therefore, $\mathrm{Ver}_{G,N}^\vee$ is not injective and is zero. This proves $H^1(\Lambda) \simeq \mathbb{Z}/2\mathbb{Z}$ and $H^2(\mathbb{Z})' \simeq Ш^2(\Lambda)$ from (4-24).

If there is a finite place of K whose decomposition group is not cyclic, then we have $Ш^2(\Lambda) = 0$. Therefore, $\tau(T^{K,\mathbb{Q}}) = 2$. On the other hand, suppose that every decomposition group of $K/\mathbb{Q}$ is cyclic. One has $G^{\mathrm{ab}} = G/N \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Recall

$$H^2(\mathbb{Z})' = \{f \in \mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z}) : (*)\ f|_D \in \mathrm{Im}\,\mathrm{Ver}_{D,D\cap N}^\vee, \forall D \in \mathcal{D}\}.$$

Note that $f|_N = 0$, so the restriction to D map $f|_D$ is contained in $\mathrm{Hom}(D/N, \mathbb{Q}/\mathbb{Z})$ whenever $N \subset D$ (also noting that $N \subset D$ if D is not trivial). So if $D = 0$ or $D \simeq \mathbb{Z}/2\mathbb{Z}$, then $f|_D = 0$ and condition $(*)$ above is satisfied automatically. Suppose $D \simeq \mathbb{Z}/4\mathbb{Z}$. Since the 2-Sylow subgroup of D is cyclic, the map $\mathrm{Ver}_{D,N}^\vee : N^\vee \to D^\vee$ is injective and $\mathrm{Im}\,\mathrm{Ver}_{D,N}^\vee$ the unique subgroup of $D^\vee$ of order 2. Then $f|_D \in \mathrm{Hom}(D/N, \mathbb{Q}/\mathbb{Z}) = \mathrm{Im}\,\mathrm{Ver}_{D,N}^\vee$ and condition $(*)$ is also satisfied automatically. Therefore, $Ш^2(\Lambda) \simeq H^2(\mathbb{Z})' = H^2(\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. This proves the proposition. □

Corollary 6.8. *Let P and Q be two odd positive integers, let $\alpha := -(P + \sqrt{P})(Q + \sqrt{Q})$ and let $K = \mathbb{Q}(\sqrt{\alpha})$ be the Galois CM with group Q_8 as in Proposition 6.6. Then*

$$\tau(T^{K,\mathbb{Q}}) = \begin{cases} \frac{1}{2} & \text{if } \left(\frac{P}{q}\right) = 1 \text{ for all primes } q \mid Q; \\ 2 & \text{otherwise.} \end{cases}$$

Proof. By Proposition 6.6, the cyclicity of all decomposition groups of $K/\mathbb{Q}$ is equivalent to the condition $\left(\frac{P}{q}\right) = 1$ for all primes $q \mid Q$. Hence the assertion follows from Proposition 6.7. □

6E. *The dihedral case.* Let D_n denote the dihedral group of order $2n$.

Lemma 6.9. *Suppose the CM-field K is Galois over $\mathbb{Q}$ with Galois group G, and let*

$$S := \{\sigma \in G : \iota \notin \langle \sigma \rangle\}.$$

If $|S| \ge |G|/2$, then $n_K \le 2$. If the inequality is strict, then $n_K = 1$.

Proof. This is due to Jiangwei Xue. Let $L/\mathbb{Q}$ be the class field corresponding to the open subgroup $\mathbb{Q}^\times N(T(\mathbb{A})) \subset \mathbb{A}^\times$ by class field theory. Then $n_K = [\mathbb{A}^\times : \mathbb{Q}^\times N_{L/\mathbb{Q}}(\mathbb{A}_L^\times)] = [L : \mathbb{Q}]$. Suppose p is a rational prime unramified in $K/\mathbb{Q}$ such that the Artin symbol $(p, K/\mathbb{Q})$ lies in S. Since p splits completely in the fixed field $E = K^{D_p}$ of the decomposition group $D_p = \langle (p, K/\mathbb{Q}) \rangle$ of G at p and $\langle \iota \rangle \cap D_p = \{1\}$ (by the definition of S), one has $K = K^+ E$ and that every prime v of K^+ lying above p splits in K, and therefore $\mathbb{Q}_p^\times \subset N(T(\mathbb{A})) \subset Q^\times N_{L/\mathbb{Q}}(\mathbb{A}_L^\times)$. It follows from class field theory that p splits completely in L. Thus, the density of S is less than or equal to that of primes splitting completely in L. By the Chebotarev density theorem, we obtain $|S|/|G| \le 1/[L : \mathbb{Q}]$. Therefore, $n_K \le |G|/|S|$ and the assertions then follow immediately. □

Proposition 6.10. *Let K be a Galois CM field with group $G = D_n$ and T the associated CM torus over $\mathbb{Q}$. Then n is even and $\tau(T) = 2$.*

Proof. Write $D_n = \langle t, s : t^n = s^2 = 1, sts = t^{-1} \rangle$. One easily sees that the center $Z(D_n) = \{x \in \langle t \rangle : x^2 = 1\}$ contains an element of order 2 if and only if $n = 2m$ is even. Since ι is central of order 2 in D_n, n is even. In this case $|S| = 2m + 1$ and $n_K = 1$ by Lemma 6.9. Therefore, $\tau(T) = 2$. □

Remark 6.11. The criterion in Lemma 6.9 does not help to compute $\tau(T)$ for the case $G = Q_8$ or $G = \mathbb{Z}/2^n\mathbb{Z}$. However, these cases have been treated in Propositions 6.7 and 2.9, respectively.

7. Some nonsimple CM cases

Keep the notation in Section 2C. Write $N_i := \mathrm{Gal}(K_i/K_i^+) = \langle \iota_i \rangle$ with involution ι_i on K_i/K_i^+ and $\iota_i^* \in N_i^\vee = \mathrm{Hom}(N_i, \mathbb{Q}/\mathbb{Z})$ for the unique nontrivial element, that is $\iota_i^*(\iota) = \frac{1}{2} \bmod \mathbb{Z}$. Fix a Galois splitting field L of $T = T^{K,\mathbb{Q}}$ and put $G = \mathrm{Gal}(L/\mathbb{Q})$. For a number field F, denote by $\Sigma_F := \mathrm{Hom}_{\mathbb{Q}}(F, \mathbb{C}) = \mathrm{Hom}_{\mathbb{Q}}(F, \overline{\mathbb{Q}})$ the set of embeddings of F into $\mathbb{C}$. The Galois group $G_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on Σ_F by $\sigma \cdot \phi = \sigma \circ \phi$ for $\sigma \in G_{\mathbb{Q}}$ and $\phi \in \Sigma_F$. Let Φ_i be a CM type of K_i/K_i^+, that is, $\Phi_i \cap (\Phi_i \circ \iota_i) = \varnothing$ and $\Phi_i \cup (\Phi_i \circ \iota_i) = \Sigma_{K_i}$, and for each $g \in G$, set $\Phi_i(g) := \{\phi \in \Phi_i : g\phi \notin \Phi_i\}$.

Lemma 7.1. *Let K be a CM algebra and $T = T^{K,\mathbb{Q}}$ the associated CM torus over $\mathbb{Q}$. Then the map*

$$\bigoplus_{i=1}^{r} \mathrm{Ver}_{G,N_i}^\vee : \bigoplus_{i=1}^{r} N_i^\vee \to G^{\mathrm{ab},\vee} = \mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z})$$

sends each element $\sum_{i=1}^r a_i \iota_i^*$, for $a_i \in \mathbb{Z}$, to the element $f = \sum_{i=1}^r a_i f_i$, where $f_i \in \mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z})$ is the function on G given by $f_i(g) = |\Phi_i(g)|/2 \bmod \mathbb{Z}$.

Proof. We first describe the transfer map $\mathrm{Ver}_{G,N_i}: G \to N_i$. We may regard the CM fields K_i as subfields of $\overline{\mathbb{Q}}$, and put $\widetilde{X}_i := G/H_i = \Sigma_{K_i}$ and $X_i := G/\widetilde{N}_i = \Sigma_{K_i^+}$. Fix a section $\widetilde{\varphi}: X_i \to G$ such that the induced section $\varphi: X_i \to \widetilde{X}_i$ has image Φ_i. Since $N_i = \widetilde{N}_i/H_i$ is abelian, modulo $D(\widetilde{N}_i)H_i$ is the same as modulo H_i. Following the definition of Ver_{G,N_i}, for each $g \in G$,

$$\mathrm{Ver}_{G,N_i}(g) = \prod_{x \in X_i} n_{g,x}^{\widetilde{\varphi}} \mod H_i = \iota_i^{n_i(g)} \in N_i,$$

where the element $n_{g,x}^{\widetilde{\varphi}} \in \widetilde{N}_i$ is defined in Section 3D and $n_i(g) := |\{x \in X_i : n_{g,x}^{\widetilde{\varphi}} \notin H_i\}|$. Let $f_i := \mathrm{Ver}_{G,N_i}^{\vee}(\iota_i^*)$. Then (Lemma 3.8)

$$f_i(g) = \iota_i^*(\mathrm{Ver}_{G,N_i}(g)) = \iota_i^*(\iota_i^{n_i(g)}) = n_i(g)/2 \in \mathbb{Q}/\mathbb{Z}.$$

Thus, it remains to show $n_i(g) = |\Phi_i(g)|$. Let $\widetilde{\Phi}_i := \widetilde{\varphi}(\Sigma_{K_i^+})$. Then we have bijections $\widetilde{\Phi}_i \xrightarrow{\sim} \Phi_i \xrightarrow{\sim} \Sigma_{K_i^+}$. Let $x \in \Sigma_{K_i^+}$, and let $\widetilde{\phi} \in \widetilde{\Phi}_i$ and $\phi \in \Phi_i$ be the corresponding elements. Put $\widetilde{\phi}' := \widetilde{\varphi}(gx)$ and $\phi' = \varphi(gx)$, the image of $\widetilde{\phi}'$. Then $g\widetilde{\phi}$ and $\widetilde{\phi}'$ are elements lying over gx and we have $g\widetilde{\phi} = \widetilde{\phi}' n_{g,x}^{\widetilde{\varphi}}$. So $g\phi = \phi'$ if and only if $n_{g,x}^{\widetilde{\varphi}} \in H_i$. On the other hand, since $\phi' \in \Phi_i$ and two elements $g\phi$ and ϕ' are lying over the same element gx, we have

$$g\phi \notin \Phi_i \iff g\phi \neq \phi' \iff n_{g,x}^{\widetilde{\varphi}} \notin H_i.$$

Therefore, the bijection $\Phi_i \xrightarrow{\sim} \Sigma_{K_i^+}$ gives the bijection of subsets

$$\{\phi \in \Phi_i : g\phi \notin \Phi_i\} \xrightarrow{\sim} \{x \in \Sigma_{K_i^+} : n_{g,x}^{\widetilde{\varphi}} \notin H_i\}.$$

This shows the desired equality $|\Phi_i(g)| = n_i(g)$. ☐

By Proposition 4.1 and Lemma 7.1, we give an independent proof of the following result of Li and Rüd [Achter et al. 2023, Proposition A.11].

Corollary 7.2. *Let K and T be as in Lemma 7.1 and $\Lambda = X(T)$ be the character group of T. Then*

$$H^1(\Lambda) \simeq \left\{ (a_i) \in \{\pm 1\}^r : \sum_{1 \le i \le r, a_i = -1} |\Phi_i(g)| \in 2\mathbb{Z}, \forall g \in G \right\}. \tag{7-1}$$

Proof. Indeed, after making the identity $\oplus_{i=1}^r N_i^{\vee} \simeq (\mathbb{Z}/2\mathbb{Z})^r$ with $\{\pm 1\}^r$, by Lemma 7.1 the map

$$\oplus_{i=1}^r \mathrm{Ver}_{G,N_i}^{\vee}: \oplus_{i=1}^r N_i^{\vee} = \{\pm 1\}^r \to \mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z})$$

sends (a_i) to the function f with

$$f(g) = \sum_{1 \le i \le r, a_i = -1} |\Phi_i(g)|/2 \mod \mathbb{Z}.$$

Thus, $f = 0$ precisely when

$$\sum_{1 \le i \le r, a_i = -1} |\Phi_i(g)| \in 2\mathbb{Z}, \quad \forall g \in G,$$

and (7-1) follows from Proposition 4.1. ☐

Proposition 7.3. *Let K be a CM algebra and T the associated CM torus over $\mathbb{Q}$. Then there is an isomorphism $\mathbb{A}^\times/\mathbb{Q}^\times N(T(\mathbb{A})) \simeq H^2(\mathbb{Z})'^\vee$.*

Proof. By [Rosengarten 2018, Theorem 1.2.9], we have the following commutative diagram with row exact sequence:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T(\mathbb{Q})_{\text{pro}} & \longrightarrow & T(\mathbb{A})_{\text{pro}} & \longrightarrow & H^2(\mathbb{Q}, \widehat{T})^\vee & \longrightarrow & Ш^1(T) \longrightarrow 0 \\ & & \downarrow N & & \downarrow N & & \downarrow \widehat{N}^\vee & & \downarrow \\ 0 & \longrightarrow & (\mathbb{Q}^\times)_{\text{pro}} & \longrightarrow & (\mathbb{A}^\times)_{\text{pro}} & \longrightarrow & H^2(\mathbb{Q}, \mathbb{Z})^\vee & \longrightarrow & 0 \end{array}$$

where A_{pro} denotes the profinite completion of an abelian group A. It follows from the Poitou–Tate duality and class field theory that:

$$\begin{array}{ccc} (T(\mathbb{A})/T(\mathbb{Q}))_{\text{pro}} & \twoheadrightarrow & (H^2(\mathbb{Q}, \widehat{T})/Ш^2(\mathbb{Q}, \widehat{T}))^\vee \\ \downarrow N & & \downarrow \widehat{N}^\vee \\ (\mathbb{A}^\times/\mathbb{Q}^\times)_{\text{pro}} & \xrightarrow{\sim} & H^2(\mathbb{Q}, \mathbb{Z})^\vee \end{array}$$

By definition $H^2(\mathbb{Q}, \mathbb{Z})' = \operatorname{Ker}(\widehat{N} : H^2(\mathbb{Q}, \mathbb{Z}) \to H^2(\mathbb{Q}, \widehat{T})/Ш^2(\mathbb{Q}, \widehat{T}))$, so we have

$$\operatorname{Coker} N = (T(\mathbb{A})/T(\mathbb{Q})N(T(\mathbb{A})))_{\text{pro}} \xrightarrow{\sim} \operatorname{Coker} \widehat{N}^\vee = H^2(\mathbb{Q}, \mathbb{Z})'^\vee.$$

Since $T(\mathbb{A})/T(\mathbb{Q})N(T(\mathbb{A}))$ is finite, it is equal to its profinite completion. This proves the proposition. □

Remark 7.4. Proposition 7.3 gives a cohomological interpretation of the group $\mathbb{A}^\times/\mathbb{Q}^\times N(T(\mathbb{A}))$. This is reminiscent of the main theorem $\mathbb{A}_k^\times/N_{K/k}(\mathbb{A}_K^\times)k^\times \simeq H^2(G, \mathbb{Z})^\vee$ of class field theory, where K/k is Galois with Galois group G.

Lemma 7.5. *Let K_1 be a CM field and $K := K_1^r$ be the r-copies of K_1. Then $\tau(T^{K,\mathbb{Q}}) = 2^{r-1} \cdot \tau(T^{K_1,\mathbb{Q}})$.*

Proof. This is first proved in [Rüd 2022, Example 8.4, page 2897]; here we give an independent proof. Observe that $N(T^{K,\mathbb{Q}}(\mathbb{A})) = N(\mathbb{A}_K^\times) \cap \mathbb{A}^\times$, where $N = N_{K/K^+}$. To see this, the inclusion $\subseteq$ is clear. If $x \in N(\mathbb{A}_K^\times) \cap \mathbb{A}^\times$, then $x = N(y)$ for some $y \in \mathbb{A}_K^\times$. By definition $y \in T(\mathbb{A})$, and hence $x \in N(T(\mathbb{A}))$. This verifies the other inclusion. It follows that if $K = \prod_{i=1}^r K_i$ is a product of CM fields K_i, then

$$N(T^{K,\mathbb{Q}}(\mathbb{A})) = \mathbb{A}^\times \bigcap_{i=1}^r N_{K_i/K_i^+}(\mathbb{A}_{K_i}^\times)$$

in the sense that the right hand side consists of all elements $x \in \mathbb{A}^\times$ which are contained in $N_{K_i/K_i^+}(\mathbb{A}_{K_i}^\times)$ via the embeddings $\mathbb{A}^\times \hookrightarrow \mathbb{A}_{K_i^+}^\times$ for all i. Thus, if $K_i = K_1$ for all i, then $N(T^{K,\mathbb{Q}}(\mathbb{A})) = \mathbb{A}^\times \cap N_{K_1/K_1^+}(\mathbb{A}_{K_1}^\times) = N(T^{K_1,\mathbb{Q}}(\mathbb{A}))$ and hence $n_K = n_{K_1}$. By Proposition 2.7, $\tau(T^{K,\mathbb{Q}}) = 2^r/n_K = 2^{r-1} \cdot 2/n_{K_1} = 2^{r-1} \cdot \tau(T^{K_1})$. □

Corollary 7.6. *For any integer $n \geq 0$, there exists a CM torus T over $\mathbb{Q}$ such that $\tau(T) = 2^n$.*

Proof. Take $K = E^r$ with $r = n + 1 \geq 1$, where E is an imaginary quadratic field. Then $\tau(T^{K,\mathbb{Q}}) = 2^{r-1} \cdot \tau(T^{E,\mathbb{Q}}) = 2^n$. □

Proposition 7.7. *Suppose that CM fields $K_1, \dots, K_r$ satisfy the following*:

(a) *K_i is Galois over $\mathbb{Q}$ with group $G_i \simeq Q_8$ for any i.*

(b) *The Galois group $G = \mathrm{Gal}(L/\mathbb{Q})$ of the compositum $L = K_1 \cdots K_r$ over $\mathbb{Q}$ is isomorphic to $G_1 \times \cdots \times G_r$.*

(c) *Every decomposition group of G_i is cyclic for all $1 \leq i \leq r$.*

Then $\tau(T^{K,\mathbb{Q}}) = \left(\frac{1}{2}\right)^r$.

Proof. By condition (c) and Proposition 6.7, one has

$$\tau(T^{K_i,\mathbb{Q}}) = \tfrac{1}{2} \tag{7-2}$$

for all $1 \leq i \leq r$. Moreover, the equality $[K_i : K_i^+] = 2$ and the conditions (a), (b) and (c) imply that the CM fields $K_1, \dots, K_r$ satisfies the assumptions in Proposition 5.7. See also the conditions (i) and (ii) in Section 5 and the beginning of Section 5B. Hence the assertion follows from Proposition 5.7 and (7-2). □

Theorem 7.8. *For any positive integer r, there exist Q_8-CM fields K_i for $1 \leq i \leq r$ that satisfy the conditions (a), (b) and (c) in Proposition 7.7.*

We will give the proof in the next section. From Proposition 7.7 and Theorem 7.8, we immediately obtain

Corollary 7.9. *For any integer n, there exists a CM torus T over $\mathbb{Q}$ such that $\tau(T) = 2^n$.*

Remark 7.10. Kottwitz [1992] computed the Hasse–Weil zeta function of the moduli spaces S_{K^p} of PEL-type. Using the notation there, it is shown in Section 8 that the algebraic variety S_{K^p} over the reflex field E is a finite disjoint union of the canonical model of Shimura varieties associated to the Shimura datum (G, h^{-1}, K^p) indexed by $\ker^1(\mathbb{Q}, G) := \ker(H^1(\mathbb{Q}, G) \to \prod_{v \leq \infty} H^1(\mathbb{Q}_v, G))$. In Case C and Case A with n even, the set $\ker^1(\mathbb{Q}, G)$ is trivial and there is no difference between the moduli space S_{K^p} and the canonical model of the Shimura variety in question. In Case A with n odd, the set $\ker^1(\mathbb{Q}, G)$ is canonically isomorphic to $\ker^1(\mathbb{Q}, Z)$, where Z is the kernel of the map $F^\times \times \mathbb{Q}^\times \to F_0^\times$ sending (x, t) to $N_{F/F_0}(x)t^{-1}$ and F is the center of the central simple $\mathbb{Q}$-algebra B in the input PEL-datum. The $\mathbb{Q}$-torus Z is exactly the CM torus associated to the CM field F and $\ker^1(\mathbb{Q}, Z)$ is its Tate–Shafarevich group.

Question 7.11. Is Proposition 2.9(2) still true if one drops the condition that $K/\mathbb{Q}$ is Galois?

8. Construction of an effective family of Q_8-CM fields

8A. *Existence of Q_8-extensions of fields.* Let k be a field of characteristic different from 2, and denote by $\mathrm{Br}(k)$ the Brauer group of k. Then we define a pairing

$$(\cdot\,,\cdot)_k\colon k^\times \times k^\times \to \mathrm{Br}(k)$$

by sending $(a, b) \in k^\times \times k^\times$ to the Brauer class of the quaternion algebra

$$\left(\frac{a, b}{k}\right) := k \oplus k\alpha \oplus k\beta \oplus k\alpha\beta,$$

where $\alpha^2 = a$, $\beta^2 = b$ and $\beta\alpha = -\alpha\beta$. By definition, the pairing $(\cdot\,,\cdot)_k$ is symmetric, and the image of $(\cdot\,,\cdot)_k$ is contained in the 2-torsion group of $\mathrm{Br}(k)$.

For $a \in k^\times$, put

$$k_a := k[T]/(T^2 - a).$$

We denote by $N_{k_a/k}\colon k_a \to k$ the norm map of k_a/k.

Proposition 8.1 [Gille and Szamuely 2017, Proposition 1.1.7]. *Let $a, b \in k^\times$. Then the following are equivalent*:

(1) $(a, b)_k = 1$.

(2) $a \in N_{k_b/k}(k_b^\times)$.

(3) $b \in N_{k_a/k}(k_a^\times)$.

The following is one of the most important key to prove Theorem 7.8.

Theorem 8.2 [Kiming 1990, Theorem 4]. *Let $E = k(\sqrt{a}, \sqrt{b})$ be a biquadratic extension of k, where $a, b \in k^\times$. Then the following are equivalent*:

(1) *There is a quadratic extension of K/E such that K/k is a Q_8-extension.*

(2) $(a, a)_k(b, b)_k(a, b)_k = 1$.

8B. *Proof of Theorem 7.8.* First, we recall the properties on the pairings $(\cdot\,,\cdot)_0 := (\cdot\,,\cdot)_{\mathbb{Q}}$ and $(\cdot\,,\cdot)_v := (\cdot\,,\cdot)_{\mathbb{Q}_v}$ for all places v of $\mathbb{Q}$.

Proposition 8.3. *Let v be a place of $\mathbb{Q}$, $a, b \in \mathbb{Q}_v^\times$ and let $\mathbb{Q}_{v,a} := \mathbb{Q}_v[T]/(T^2 - a)$*:

(1) *If v is infinite, then we have $(a, b)_v = 1$ if and only if either a or b is positive.*

(2) *Assume that $v = \ell$ is a nondyadic finite place. If $a, b \in \mathbb{Z}_\ell^\times$, then $(a, b)_\ell = 1$.*

(3) *Under the assumption on v in* (2), *if $a = \ell$ and $b \in \mathbb{Z}_\ell^\times$, then we have $(a, b)_\ell = \left(\frac{b}{\ell}\right)$.*

(4) *If $v = 2$ and $a, b \in 1 + 4\mathbb{Z}_2$, then one has $(a, b)_2 = 1$.*

Proof. (1) This follows from Proposition 8.1 and the equality $N_{\mathbb{C}/\mathbb{R}}(\mathbb{C}^\times) = \mathbb{R}_{>0}$.

(2) Since $\ell \neq 2$ and $a \in \mathbb{Z}_\ell^\times$, we have

$$\mathbb{Q}_{\ell,a} \cong \begin{cases} \mathbb{Q}_\ell \times \mathbb{Q}_\ell & \text{if } a \in (\mathbb{Z}_\ell^\times)^2, \\ \mathbb{Q}_{\ell^2} & \text{if } a \notin (\mathbb{Z}_\ell^\times)^2, \end{cases}$$

where $\mathbb{Q}_{\ell^2}$ denotes the unramified quadratic extension of $\mathbb{Q}_\ell$. Hence $N_{\mathbb{Q}_{\ell,a}/\mathbb{Q}_\ell}(\mathbb{Q}_{\ell,a}^\times)$ contains $\mathbb{Z}_\ell^\times$. Hence the assertion follows from Proposition 8.1 and the assumption $b \in \mathbb{Z}_\ell^\times$.

(3) Since ℓ is not equal to 2, we have

$$N_{\mathbb{Q}_\ell(\sqrt{\ell})/\mathbb{Q}}(\mathbb{Q}_\ell(\sqrt{\ell})^\times) = \langle -\ell \rangle \times (\mathbb{Z}_\ell^\times)^2.$$

Therefore the assertion follows from Proposition 8.1.

(4) Since $1 + 8\mathbb{Z}_2 = (\mathbb{Z}_2^\times)^2$, there is an isomorphism

$$\mathbb{Q}_{2,a} \cong \begin{cases} \mathbb{Q}_2 \times \mathbb{Q}_2 & \text{if } a \in (\mathbb{Z}_2^\times)^2, \\ \mathbb{Q}_4 & \text{if } a \notin (\mathbb{Z}_2^\times)^2. \end{cases}$$

In particular, $N_{\mathbb{Q}_{2,a}/\mathbb{Q}_\ell}(\mathbb{Q}_{2,a}^\times)$ contains $\mathbb{Z}_2^\times$. Hence the assertion follows from $b \in 1+4\mathbb{Z}_2$ and Proposition 8.1. □

The following two lemmas will be used later.

Lemma 8.4. *Let E be a totally real field which is Galois over $\mathbb{Q}$:*

(1) *Let K/E be a quadratic extension such that $K/\mathbb{Q}$ is Galois. Then K is either totally real or CM.*

(2) *If there is a quadratic extension K/E such that $K/\mathbb{Q}$ is Galois, then there is a totally imaginary quadratic extension K'/E such that $K'/\mathbb{Q}$ is Galois and*

$$\mathrm{Gal}(K'/\mathbb{Q}) \cong \mathrm{Gal}(K/\mathbb{Q}).$$

Proof. (1) If K is not totally real, then it is totally complex since $K/\mathbb{Q}$ is Galois. Fix an embedding $\varepsilon\colon K \hookrightarrow \mathbb{C}$, and let ι be the element of $\mathrm{Gal}(K/\mathbb{Q})$ induced by the complex conjugation and ε. Then ι is the unique nontrivial element of $\mathrm{Gal}(K/E) \subset \mathrm{Gal}(K/\mathbb{Q})$ since E is totally real. On the other hand, the assumption that E is Galois implies that $\mathrm{Gal}(K/E)$ is central in $\mathrm{Gal}(K/\mathbb{Q})$. Hence ι is contained in the center of $\mathrm{Gal}(K/\mathbb{Q})$, which implies that K is a CM field.

(2) By (1), we may assume that K is totally real. Write $K = E(\sqrt{\alpha})$, where $\alpha \in E^\times$, and put $K' := E(\sqrt{-\alpha})$. Note that K' corresponds to $\langle(\varepsilon, \iota)\rangle$ under the isomorphism

$$\mathrm{Gal}(K(\sqrt{-1})/\mathbb{Q}) \cong \mathrm{Gal}(K/\mathbb{Q}) \times \mathrm{Gal}(\mathbb{Q}(\sqrt{-1})/\mathbb{Q}).$$

Here ε is the unique nontrivial element of $\mathrm{Gal}(K/E)$, and ι is the complex conjugation on $\mathbb{Q}(\sqrt{-1})$. Note that ε is central in $\mathrm{Gal}(K/\mathbb{Q})$. Then K' is not totally real K' is Galois over $\mathbb{Q}$, and hence it is CM by (1). Moreover, the composite

$$\mathrm{Gal}(K/\mathbb{Q}) \xrightarrow{g\mapsto (g,\mathrm{id}_{\mathbb{Q}(\sqrt{-1})})} \mathrm{Gal}(K/\mathbb{Q}) \times \mathrm{Gal}(\mathbb{Q}(\sqrt{-1})/\mathbb{Q}) \cong \mathrm{Gal}(K(\sqrt{-1})/\mathbb{Q}) \to \mathrm{Gal}(K'/\mathbb{Q})$$

is an isomorphism by the definition of K'. □

For a number field E which is finite Galois over $\mathbb{Q}$, we write for $\mathrm{Ram}(E)$ the set of prime numbers which ramify in E.

Lemma 8.5. *Let r be a positive integer, and $K_1, \dots, K_r$ be number fields which are Galois over $\mathbb{Q}$. For each i, we denote by E_i the maximal abelian subfield of K_i. Assume the following*:

(i) $[K_i : E_i]$ *is a prime number for any* $i \geq 1$.

(ii) $\mathrm{Ram}(E_i) \cap \mathrm{Ram}(E_j) = \varnothing$ *if* $i \neq j$.

Let L be the compositum of $K_1, \dots, K_r$. Then there is an isomorphism

$$\mathrm{Gal}(L/\mathbb{Q}) \cong \prod_{i=1}^{r} \mathrm{Gal}(K_i/\mathbb{Q}).$$

Proof. We give a proof by induction on r. It is trivial if $r = 1$. Next, assume that the assertion holds for $r-1$, that is, there is an isomorphism

$$\mathrm{Gal}(L'/\mathbb{Q}) \cong \prod_{i=1}^{r-1} \mathrm{Gal}(K_i/\mathbb{Q}), \tag{8-1}$$

where $L' := K_1 \cdots K_{r-1}$. We first prove the equality

$$E_r \cap L' = \mathbb{Q}. \tag{8-2}$$

Since E_r is abelian over $\mathbb{Q}$, it is contained in the maximal abelian subfield E' of L'. On the other hand, the induction hypothesis (8-1) implies the equality $E' = E_1 \cdots E_{r-1}$. Hence $E_r \cap L'$ is contained in $E_r \cap (E_1 \cdots E_{r-1})$. However, we have $E_r \cap (E_1 \cdots E_{r-1}) = \mathbb{Q}$ by the assumption (ii) and the global class field theory, and hence (8-2) holds.

Now we prove the equality $K_r \cap L' = \mathbb{Q}$, which gives the desired assertion. If $K_r \cap L' \neq \mathbb{Q}$, then we have $E_r \cap (K_r \cap L') = \mathbb{Q}$ and $E_r \subsetneq E_r \cdot (K_r \cap L') \subset K_r$ by (8-2). Therefore K_r is equal to the compositum of E_r and $K_r \cap L'$ by the assumption (i). Consequently, there is an isomorphism

$$\mathrm{Gal}(K_r/\mathbb{Q}) \cong \mathrm{Gal}(E_r/\mathbb{Q}) \times \mathrm{Gal}(K_r \cap L'/\mathbb{Q}).$$

In particular, $[K_r \cap L' : \mathbb{Q}] = [K_r : E_r]$ is a prime number, and hence $K_r/\mathbb{Q}$ is abelian. This contradicts the assumption (i), which implies the desired equality $K_r \cap L' = \mathbb{Q}$. □

Let $\boldsymbol{L}$ be the set of unordered pairs of prime numbers $\{\ell, \ell'\}$ satisfying the following:

$$\ell \equiv \ell' \equiv 1 \bmod 4, \quad \left(\frac{\ell'}{\ell}\right) = 1.$$

Proposition 8.6. *For any positive integer r, there are r-pairs $\{\ell_1, \ell_1'\}, \dots, \{\ell_r, \ell_r'\}$ in $\boldsymbol{L}$ such that*

$$|\{\ell_1, \dots, \ell_r, \ell_1', \dots, \ell_r'\}| = 2r.$$

Proof. By the Dirichlet prime number theorem, there are r distinct prime numbers $\ell_1, \dots, \ell_r$ which are congruent to 1 modulo 4. Moreover, the Dirichlet prime number theorem implies the existence of prime numbers $\ell_1', \dots, \ell_r'$ satisfying the following:

$$\begin{cases} \ell_i' \equiv 1 \bmod 4,\ \left(\frac{\ell_i'}{\ell_i}\right) = 1 & \text{if } i = 1, \\ \ell_i' \equiv 1 \bmod 4,\ \left(\frac{\ell_i'}{\ell_i}\right) = 1,\ \ell_i' \notin \{\ell_1', \dots, \ell_{i-1}'\} & \text{if } i \geq 2. \end{cases}$$

Therefore the assertion holds. □

For $\lambda := \{\ell, \ell'\} \in \boldsymbol{L}$, put $K_\lambda^+ := \mathbb{Q}(\sqrt{\ell}, \sqrt{\ell'})$.

Lemma 8.7. ***For any $\lambda \in \boldsymbol{L}$, the decomposition groups of $K_\lambda^+/\mathbb{Q}$ at all finite places are cyclic.***

Proof. Write $\lambda = \{\ell, \ell'\}$. Let v be a finite place of K_λ^+ which lies above a prime number ℓ_0. If $\ell_0 \notin \{\ell, \ell'\}$, then $K_\lambda^+/\mathbb{Q}$ is unramified at v, and hence the assertion holds for v. The assertion for $\ell_0 = \ell$ follows from the assumption $\left(\frac{\ell'}{\ell}\right) = 1$. Finally, in the case $\ell_0 = \ell'$, the statement is a consequence of the equality $\left(\frac{\ell}{\ell'}\right) = \left(\frac{\ell'}{\ell}\right) = 1$. □

Proposition 8.8. *For any $\lambda \in \boldsymbol{L}$, there is a CM field K containing K_λ^+ such that $K/\mathbb{Q}$ is a Q_8-extension.*

Proof. By Lemma 8.4(2), it suffices to prove the existence of Q_8-extension K of $\mathbb{Q}$ containing K_λ^+. Write $\lambda = \{\ell, \ell'\}$. It is equivalent to the equality

$$(\ell, \ell)_0 (\ell', \ell')_0 (\ell, \ell')_0 = 1,$$

which is a consequence of Theorem 8.2. Since the image of the class $(\ell, \ell)_0 (\ell', \ell')_0 (\ell, \ell')_0$ in $\mathrm{Br}(\mathbb{Q}_v)$ is equal to $(\ell, \ell)_v (\ell', \ell')_v (\ell, \ell')_v$, by the Albert–Brauer–Hasse–Noether theorem [Lang 1994, Chapter IX, Section 6, page 195] it is equivalent to prove the following for any place v of $\mathbb{Q}$:

$$(\ell, \ell)_v (\ell', \ell')_v (\ell, \ell')_v = 1. \tag{8-3}$$

Case 1. v is infinite. In this case, (8-3) follows from Proposition 8.3(1) since ℓ and ℓ' are positive.

Case 2. $v = \ell_0 \notin \{2, \ell, \ell'\}$. The condition $\ell_0 \notin \{\ell, \ell'\}$ implies that ℓ and ℓ' are units in $\mathbb{Z}_{\ell_0}$. Hence, since $\ell_0 \neq 2$, one has

$$(\ell, \ell)_{\ell_0} = (\ell', \ell')_{\ell_0} = (\ell, \ell')_{\ell_0} = 1$$

by Proposition 8.3(2). This implies the equality (8-3).

Case 3. $v = 2$. Since $\ell \equiv \ell' \equiv 1 \bmod 4$, Proposition 8.3(4) implies

$$(\ell, \ell)_2 = (\ell', \ell')_2 = (\ell, \ell')_2 = 1.$$

Hence (8-3) holds.

Case 4. $v = \ell$**.** The assumption $\ell \equiv 1 \mod 4$ is equivalent to the equality $\left(\frac{-1}{\ell}\right) = 1$. Hence

$$(\ell, \ell)_\ell = (\ell, -1)_\ell = 1.$$

On the other hand, the assumption $\left(\frac{\ell'}{\ell}\right) = 1$ implies

$$(\ell, \ell')_\ell = (\ell', \ell')_\ell = 1,$$

which is a consequence of Proposition 8.3(2) and (3). Therefore we obtain the desired equality (8-3).

Case 5. $v = \ell'$**.** Since $\ell' \equiv 1 \mod 4$ and

$$\left(\frac{\ell}{\ell'}\right) = \left(\frac{\ell'}{\ell}\right) = 1,$$

the assertion (8-3) follows from the same argument as Case 4. □

In the following, we give a proof of Theorem 7.8. By Corollary 7.6, we may assume $n = -r < 0$. Take $\lambda_1 = \{\ell_1, \ell_1'\}, \dots, \lambda_r = \{\ell_r, \ell_r'\} \in \boldsymbol{L}$ satisfying

$$|\{\ell_1, \dots, \ell_r, \ell_1', \dots, \ell_r'\}| = 2r, \tag{8-4}$$

which is possible by Proposition 8.6. Then, Proposition 8.8 implies that there is a Q_8-CM field containing $K_{\lambda_i}^+$ for any $1 \le i \le r$.

The following immediately implies the desired assertion.

Theorem 8.9. *Under the above notations, let K_{λ_i} be a Q_8-CM field containing $K_{\lambda_i}^+$ for each $1 \le i \le r$. Then the CM fields $K_{\lambda_1}, \dots, K_{\lambda_r}$ satisfy the conditions* (a), (b) *and* (c) *in Proposition 7.7.*

Proof. From the definition of K_{λ_i} for $1 \le i \le r$, condition (a) in Proposition 7.7 holds.

We shall show that the assumptions (i) and (ii) in Lemma 8.5 hold. By Lemma 6.5(3), for any $1 \le i \le r$, $K_{\lambda_i}^+$ is the maximal abelian subfield of K_{λ_i} and $[K_{\lambda_i} : K_{\lambda_i}^+] = 2$. In particular, assumption (i) holds. On the other hand, since $\mathrm{Ram}(K_{\lambda_i}^+) = \lambda_i$ for any $1 \le i \le r$, the condition (8-4) implies the equality $\mathrm{Ram}(K_{\lambda_i}^+) \cap \mathrm{Ram}(K_{\lambda_j}^+) = \varnothing$ for $i \ne j$. Hence we obtain the assumption (ii), which verifies condition (b) in Proposition 7.7.

Take a finite places w of K_{λ_i} and v of $K_{\lambda_i}^+$ satisfying $w \mid v$. Then Lemma 8.7 implies the cyclicity of the decomposition group of $\mathrm{Gal}(K_{\lambda_i}^+/\mathbb{Q})$ at v. Since $K_{\lambda_i}/\mathbb{Q}$ is a Q_8-extension, the decomposition group at w is cyclic by Lemma 6.5(2). Therefore condition (c) in Proposition 7.7 holds. □

9. Products of two linearly disjoint Galois CM fields

In this section we show the following result.

Theorem 9.1. *There are infinitely many CM algebras $K = K_1 \times K_2$ with linearly disjoint Galois CM fields K_1 and K_2 such that*

$$\tau(T^{K,\mathbb{Q}}) = \tfrac{1}{2} \prod_{i=1}^{2} \tau(T^{K_i,\mathbb{Q}}). \tag{9-1}$$

This theorem shows that the conclusion of Proposition 5.7 is no longer true if one drops the cyclicity of decomposition groups of G_i for all i. We shall use the notations in Section 5B. In particular, $L = K_1K_2$, $G := \mathrm{Gal}(L/\mathbb{Q})$, $G_i := \mathrm{Gal}(K_i/\mathbb{Q})$, $H_i := \mathrm{Gal}(L/K_i)$, $\widetilde{N}_i := \mathrm{Gal}(L/K_i^+)$ and $N_i = \mathrm{Gal}(K_i/K_i^+)$ for $i = 1, 2$.

First, we give a sufficient condition on $K = K_1 \times K_2$ for which Theorem 9.1 holds. Let $\mathcal{C}$ and $\mathcal{C}_i$ be the sets of cyclic subgroups of G and G_i respectively. Then put

$$H^2(\mathbb{Z})'' := \{f \in G^\vee : f|_D \in \mathrm{Im}(\mathrm{Ver}^\vee_{D,\overline{D}}) \text{ for all } D \in \mathcal{C}\},$$
$$H^2(G_i, \mathbb{Z})'' := \{f \in G_i^\vee : f|_{D'} \in \mathrm{Im}(\mathrm{Ver}^\vee_{D', D'\cap N_i}) \text{ for all } D' \in \mathcal{C}_i\}.$$

Lemma 9.2. *If $G \cong G_1 \times G_2$, then $H^2(\mathbb{Z})'' = H^2(G_1, \mathbb{Z})'' \times H^2(G_2, \mathbb{Z})''$.*

Proof. The proof is the same as Lemma 5.6. □

We define a subgroup D_0 of $(\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \times \mathbb{Z}/2\mathbb{Z}$ as follows:

$$D_0 := \langle(\bar{1}, \bar{0}, \bar{1}), (\bar{0}, \bar{1}, \bar{0})\rangle.$$

Here we denote by $(\bar{a}, \bar{b}, \bar{c})$ the element $(a \bmod 4, b \bmod 2, c \bmod 2)$ in $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2 \times \mathbb{Z}/2\mathbb{Z}$ for $a, b, c \in \mathbb{Z}$.

Proposition 9.3. *Assume that Galois CM fields K_1 and K_2 satisfy the following*:

(i) *$G_1 \cong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, $N_1 \cong \langle(\bar{2}, \bar{0})\rangle \subset G_1$ and $G_2 \cong \mathbb{Z}/2\mathbb{Z}$.*

(ii) *$G \cong G_1 \times G_2$, that is, K_1 and K_2 are linearly disjoint.*

(iii) *$\mathcal{D} = \mathcal{C} \cup \{D_0\}$.*

Put $K := K_1 \times K_2$. Then we have the following:

$$\tau(T^{K_1,\mathbb{Q}}) = 2, \quad \tau(T^{K_2,\mathbb{Q}}) = 1, \quad \tau(T^{K,\mathbb{Q}}) = 1.$$

In particular, the equality (9-1) holds.

Proof. By definition, we obtain the following:

$$\widetilde{N}_1 = (2\mathbb{Z}/4\mathbb{Z} \times \{\bar{0}\}) \times \mathbb{Z}/2\mathbb{Z}, \quad H_1 = \{(\bar{0}, \bar{0})\} \times \mathbb{Z}/2\mathbb{Z}, \quad \widetilde{N}_2 = G, \quad H_2 = (\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \times \{\bar{0}\}.$$

Moreover, Proposition 4.3 implies

$$\tau(T^{K_i,\mathbb{Q}}) = \frac{2}{|H^2(G_i, \mathbb{Z})'|}, \quad \tau(T^{K_1\times K_2,\mathbb{Q}}) = \frac{4}{|H^2(\mathbb{Z})'|}.$$

Since G_1 is D_0 modulo G_2, by (iii) G_1 itself is a decomposition group of G_1. By this and that G_1 is not cyclic, one computes that $H^2(G_1, \mathbb{Z})' = 0$, which implies $\tau(T^{K_1,\mathbb{Q}}) = 2/1 = 2$. Moreover, the equality $\tau(T^{K_2,\mathbb{Q}}) = 1$ follows from Proposition 2.9(2) as $[K_2 : \mathbb{Q}] = 2$. In particular, we obtain $|H^2(G_2, \mathbb{Z})'| = 2$, that is,

$$H^2(G_2, \mathbb{Z})' = H^2(G_2, \mathbb{Z})'' = G_2^\vee. \tag{9-2}$$

On the other hand, for the equality $\tau(T^{K,\mathbb{Q}})=1$, it suffices to prove

$$H^2(\mathbb{Z})' = \{f \in G^\vee : f((2\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \times \{\bar{0}\}) = 0\}.$$

By direct computation, we have

$$H^2(G_1, \mathbb{Z})'' = \{f_1 \in G_1^\vee : f(2\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) = 0\}.$$

Combining this equality, (9-2) and Lemma 9.2, one has

$$H^2(\mathbb{Z})' \subset H^2(\mathbb{Z})'' = H^2(G_1,\mathbb{Z})'' \times H^2(G_2,\mathbb{Z})'' = \{f \in G^\vee : f((2\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \times \{\bar{0}\}) = 0\}.$$

For the reverse inclusion, it suffices to prove

$$\mathrm{Im}(\mathrm{Ver}^\vee_{D_0,\bar{\boldsymbol{D}}_0}) = \{f \in D_0^\vee : f((2\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \times \{\bar{0}\}) = 0\}. \tag{9-3}$$

Recall that $\mathrm{Ver}_{D_0,\bar{\boldsymbol{D}}_0} = (\mathrm{Ver}_{D_0,\bar{D}_{0,i}})_{1\le i\le 2}$ and

$$\mathrm{Ver}_{D_0,\bar{D}_{0,i}} = \mathrm{Ver}_{D_0/D_0\cap H_i,\bar{D}_{0,i}} \circ \pi_i,$$

where $\pi_i : D_0 \to D_0/D_0 \cap H_i$ is the canonical surjection. Since one has

$$D_{0,1} = D_0 \cap \widetilde{N}_1 = 2\mathbb{Z}/4\mathbb{Z} \times \{\bar{0}\} \times \mathbb{Z}/2\mathbb{Z}, \quad D_0 \cap H_1 = \{0\},$$

we obtain that $D_0/D_0 \cap H_1$ is not cyclic and $\bar{D}_{0,1} := D_{0,1}/D_0 \cap H_1 \cong \mathbb{Z}/2$. Hence Proposition 3.3 implies $\mathrm{Ver}_{D_0,\bar{D}_{0,1}} = 0$. On the other hand, we have $\mathrm{Ver}_{D_0,\bar{D}_{0,2}} = \pi_2$ by $D_{0,2} := D_0 \cap \widetilde{N}_2 = D_0$. Consequently, the homomorphism $\mathrm{Ver}_{D_0,\bar{\boldsymbol{D}}_0}$ can be written as the composite

$$D_0 \xrightarrow{\pi_2} D_0/D_0 \cap H_2 = \bar{D}_{0,2} \xrightarrow{g_2 \mapsto (0,g_2)} \bar{D}_{0,1} \times \bar{D}_{0,2}.$$

Therefore, (9-3) follows from the equality $D_0 \cap H_2 = (2\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \times \{\bar{0}\}$. □

Now we construct pairs of CM fields K_1, K_2 satisfying Proposition 9.3.

Lemma 9.4. *Let ℓ be a prime number which is congruent to* 1 *modulo* 4, *and denote by $K_{(\ell)}$ the unique quartic subfield of $\mathbb{Q}(\zeta_\ell)$:*

(1) *The field $K_{(\ell)}$ is CM if and only if $\ell \equiv 5 \bmod 8$.*

(2) *We have $\mathrm{Ram}(K_{(\ell)}/\mathbb{Q}) = \{\ell\}$, and ℓ is totally ramified in $K_{(\ell)}$.*

Proof. (1) Recall that $\mathbb{Q}(\zeta_\ell)$ is a CM field which is cyclic of degree $\ell-1$ over $\mathbb{Q}$. Since $K_{(\ell)}$ corresponds to the unique subgroup of $\mathrm{Gal}(\mathbb{Q}(\zeta_\ell)/\mathbb{Q})$ of order $(\ell-1)/4$, it is CM if and only if $(\ell-1)/4$ is an odd number. Finally, the condition $(\ell-1)/4 \notin 2\mathbb{Z}$ is equivalent to the desired congruence $\ell \equiv 5 \bmod 8$.

(2) This follows from the equality $\mathrm{Ram}(\mathbb{Q}(\zeta_\ell)/\mathbb{Q}) = \{\ell\}$ and that ℓ is totally ramified in $\mathbb{Q}(\zeta_\ell)$. □

Lemma 9.5. *There are infinitely many ordered triples of prime numbers* (ℓ_1, ℓ_2, ℓ_3) *for which the following are satisfied*:

(a) $\ell_1 \equiv 5 \bmod 8$.

(b) $\ell_2 \equiv 1 \bmod 4$ *and* $\left(\frac{\ell_2}{\ell_1}\right) = -1$.

(c) $\ell_3 \equiv 3 \bmod 4$, $\left(\frac{\ell_3}{\ell_2}\right) = 1$ *and* ℓ_3 *splits completely in* $K_{(\ell_1)}$.

Proof. This follows from Dirichlet's prime number theorem. □

For a finite abelian extension $E/\mathbb{Q}$ and a prime number ℓ, we write $D_\ell(E/\mathbb{Q})$ and $I_\ell(E/\mathbb{Q})$ for the decomposition group and the inertia group of $\mathrm{Gal}(E/\mathbb{Q})$ at ℓ respectively. Observe that if E' is a subextension of $E/\mathbb{Q}$ and let $\pi_{E'} : \mathrm{Gal}(E/\mathbb{Q}) \to \mathrm{Gal}(E'/\mathbb{Q})$ denote the natural projection, then $\pi_{E'}(D_\ell(E/\mathbb{Q})) = D_\ell(E'/\mathbb{Q})$ and $\pi_{E'}(I_\ell(E/\mathbb{Q})) = I_\ell(E'/\mathbb{Q})$.

Theorem 9.1 is a consequence of Proposition 9.6 and Lemma 9.7.

Proposition 9.6. *Let* (ℓ_1, ℓ_2, ℓ_3) *be an ordered triple of prime numbers as in Lemma 9.5, and set*

$$K_1 := K_{(\ell_1)}(\sqrt{\ell_2}), \quad K_2 := \mathbb{Q}(\sqrt{-\ell_1\ell_3}).$$

Then the fields K_1, K_2 *are CM and they satisfy the conditions* (i)–(iii) *in Proposition 9.3.*

Proof. By definition, K_2 is an imaginary quadratic field, and hence CM. Moreover, condition (a) in Lemma 9.5 and Lemma 9.4(1) imply that K_1 is also CM.

In the following, we shall prove that the statements (i), (ii) and (iii) in Proposition 9.3 hold. Statement (i) in Proposition 9.3 follows directly from the definitions of K_1 and K_2. To prove statement (ii) in Proposition 9.3, it suffices to prove the equality $K_1 \cap K_2 = \mathbb{Q}$. However, this follows from the fact that ℓ_3 is unramified in K_1 and is totally ramified in K_2. We now show (iii) in Proposition 9.3. It is clear that $L := K_1K_2$ is unramified outside $2, \ell_1, \ell_2$ and ℓ_3. Hence it suffices to compute $D_\ell(L/\mathbb{Q})$ for $\ell \in \{2, \ell_1, \ell_2, \ell_3\}$. First, suppose $\ell = 2$. Then Lemma 9.4(2) implies that 2 is unramified in $K_{(\ell_1)}$. Moreover, by (b) and (c) in Lemma 9.5, we have $\ell_1 \equiv \ell_2 \equiv -\ell_1\ell_3 \equiv 1 \bmod 4$. Hence $L/\mathbb{Q}$ is unramified at 2, which implies that $D_2(L/\mathbb{Q})$ is cyclic. Second, assume $\ell = \ell_1$. By assumptions (b) and (c) in Lemma 9.5, we have the following:

$$D_{\ell_1}(K_1/\mathbb{Q}) = G_1, \quad I_{\ell_1}(K_1/\mathbb{Q}) \cong \mathbb{Z}/4\mathbb{Z} \times \{\bar{0}\}, \quad D_{\ell_1}(K_2/\mathbb{Q}) = I_{\ell_1}(K_2/\mathbb{Q}) = G_2.$$

Since $\ell_1 \neq 2$, by local class field theory $I_{\ell_1}(L/\mathbb{Q})$ is a finite quotient of $\mathbb{Z}_{\ell_2}^\times$ and is cyclic. Since $I_{\ell_1}(K_2/\mathbb{Q}) = G_2$, we have

$$I_{\ell_1}(L/\mathbb{Q}) \cong \langle(\bar{1}, \bar{0}, \bar{1})\rangle.$$

The fixed subfield of $I_{\ell_1}(L/\mathbb{Q})$ in L is $L' := \mathbb{Q}(\sqrt{\ell_2}, \sqrt{-\ell_3})$. Indeed, one has

$$L^{\langle(\bar{2},\bar{0},\bar{0})\rangle} = \mathbb{Q}(\sqrt{\ell_1}, \sqrt{\ell_2}, \sqrt{-\ell_1\ell_3}) \quad \text{and} \quad \mathbb{Q}(\sqrt{\ell_1}, \sqrt{\ell_2}, \sqrt{-\ell_1\ell_3})^{\langle(\bar{1},\bar{0},\bar{1})\rangle} = \mathbb{Q}(\sqrt{\ell_2}, \sqrt{-\ell_3}).$$

By (c) in Lemma 9.5, we have $D_{\ell_1}(L'/\mathbb{Q}) = \mathrm{Gal}(L'/\mathbb{Q}(\sqrt{-\ell_3})) \simeq \mathrm{Gal}(\mathbb{Q}(\sqrt{\ell_2})/\mathbb{Q})$, and hence $D_{\ell_1}(L/\mathbb{Q})/I_\ell(L/\mathbb{Q})$ is generated by the image of $(\bar{0}, \bar{1}, \bar{0})$ in $G/I_\ell(L/\mathbb{Q})$. Therefore we obtain $D_{\ell_1}(L/\mathbb{Q}) = D_0$. Third, if $\ell = \ell_2$, then by (b) in Lemma 9.5 we have

$$D_{\ell_2}(K_1/\mathbb{Q}) = G_1, \quad I_{\ell_2}(K_1/\mathbb{Q}) \cong \{0\} \times \mathbb{Z}/2\mathbb{Z}.$$

One computes $\left(\frac{-\ell_1\ell_3}{\ell_2}\right) = -1$ and then

$$D_{\ell_2}(K_2/\mathbb{Q}) = G_2, \quad I_{\ell_2}(K_2/\mathbb{Q}) = \{0\}.$$

Since $I_{\ell_2}(K_2/\mathbb{Q}) = \{0\}$, one has $I_{\ell_2}(K_2/\mathbb{Q}) = \langle(\bar{0}, \bar{1}, \bar{0})\rangle$. By $D_{\ell_2}(K_1/\mathbb{Q}) = G_1$, one sees $D_{\ell_2}(L/\mathbb{Q}) = \langle(\bar{0}, \bar{1}, \bar{0}), (\bar{1}, \bar{0}, \bar{c})\rangle$ for some $\bar{c} \in \mathbb{Z}/2\mathbb{Z}$. Because $D_{\ell_2}(K_2/\mathbb{Q}) = G_2$, we see $\bar{c} = \bar{1}$ and hence $D_{\ell_2}(L/\mathbb{Q}) = D_0$. Finally, suppose $i = 3$. Since ℓ_3 splits completely in K_1, then

$$D_{\ell_3}(L/\mathbb{Q}) = D_{\ell_3}(K_2/\mathbb{Q}) = G_2,$$

and hence it is cyclic. This completes the proof of condition (iii) in Proposition 9.3. □

Lemma 9.7. *Let (ℓ_1, ℓ_2, ℓ_3) and $(\ell'_1, \ell'_2, \ell'_3)$ be ordered triples of prime numbers satisfying (a), (b) and (c) in Lemma 9.5. Put*

$$K := K_{(\ell_1)}\big(\sqrt{\ell_2}\big) \times \mathbb{Q}\big(\sqrt{-\ell_1\ell_3}\big), \quad K' := K_{(\ell'_1)}\Big(\sqrt{\ell'_2}\Big) \times \mathbb{Q}\Big(\sqrt{-\ell'_1\ell'_3}\Big).$$

Then we have $K \simeq K'$ if and only if $\ell_i = \ell'_i$ for every $1 \le i \le 3$.

Proof. It suffices to prove that $K \simeq K'$ implies $\ell_i = \ell'_i$ for any $1 \le i \le 3$. Assume $K \simeq K'$, which is equivalent to $K_{(\ell_1)}(\sqrt{\ell_2}) \simeq K_{(\ell'_1)}(\sqrt{\ell'_2})$ and $\mathbb{Q}(\sqrt{-\ell_1\ell_3}) \simeq \mathbb{Q}(\sqrt{-\ell'_1\ell'_3})$. Then, by Lemma 9.4(2), one has

$$\{\ell_1, \ell_2\} = \mathrm{Ram}(K_{(\ell_1)}(\sqrt{\ell_2})) = \mathrm{Ram}(K_{(\ell'_1)}(\sqrt{\ell'_2})) = \{\ell'_1, \ell'_2\}.$$

Moreover, the following holds:

$$|I_{\ell_1}(K_{(\ell_1)}(\sqrt{\ell_2})/\mathbb{Q})| = |I_{\ell'_1}(K_{(\ell'_1)}(\sqrt{\ell'_2})/\mathbb{Q})| = 4,$$
$$|I_{\ell_2}(K_{(\ell_1)}(\sqrt{\ell_2})/\mathbb{Q})| = |I_{\ell'_2}(K_{(\ell'_1)}(\sqrt{\ell_2})/\mathbb{Q})| = 2.$$

Hence we must have $\ell_1 = \ell'_1$ and $\ell_2 = \ell'_2$. On the other hand, $\mathbb{Q}(\sqrt{-\ell_1\ell_3}) \simeq \mathbb{Q}(\sqrt{-\ell'_1\ell'_3})$ implies $\ell_1\ell_3 = \ell'_1\ell'_3$ since they are square-free integers. Combining this equality with $\ell_1 = \ell'_1$, we obtain $\ell_3 = \ell'_3$. This completes the proof. □

Appendix: Ono's conjecture on Tamagawa numbers of algebraic tori

by Jianing Li and Chia-Fu Yu

Theorem A.8. *For any global field k and any positive rational number r, there exists a k-torus T such that $\tau(T) = r$.*

Proof. It suffices to construct for each prime ℓ: (a) a k-torus T_1 with $\tau(T_1)=\ell$ and (b) a k-torus T_2 with $\tau(T_2)=\ell^{-2}$. Indeed, the torus $T_3=T_1\times_k T_2$ has Tamagawa number $\tau(T_3)=\ell^{-1}$. Then for an appropriate product T of tori as T_1 or T_3 for different primes ℓ, the Tamagawa number $\tau(T)$ can be equal to any given positive rational number r.

Take a cyclic extension K/k of degree ℓ with Galois group G and let $T_1=R^{(1)}_{K/k}\mathbb{G}_{\mathrm{m},K}$. Then $H^1(G,X(T_1))\simeq H^2(G,\mathbb{Z})\simeq\mathbb{Z}/\ell\mathbb{Z}$ and $Ш^2(K/k,X(T_1))=0$ by Chebotarev's density theorem. Thus, $\tau(T_1)=\ell$ and (a) is done.

For (b), we follow the idea of Ono; see [Ono 1963b, Section 6.2]. We take an abelian extension K/k with Galois group $G\simeq(\mathbb{Z}/\ell\mathbb{Z})^4$ such that every decomposition group is cyclic. In the number field case, such an extension was constructed by Katayama [1985]. In the function field case, we construct such an extension in Theorem A.9 below. Granting the existence of this extensions K/k in both the number field and function field cases, we consider the k-torus $T_2:=R^{(1)}_{K/k}\mathbb{G}_{\mathrm{m},K}$. Since every decomposition group of G is cyclic, $Ш^2(G,X(T_2))=H^2(G,X(T_2))\simeq H^3(G,\mathbb{Z})$. By Lyndon's formula [1948, Theorem 6], $H^3(G,\mathbb{Z})\simeq(\mathbb{Z}/\ell\mathbb{Z})^6$. On the other hand, $H^1(G,X(T))\simeq H^2(G,\mathbb{Z})\simeq(\mathbb{Z}/\ell\mathbb{Z})^4$. Thus, $\tau(T_2)=\ell^{-2}$ and (b) is done. □

Theorem A.9. *Let k be a global function field of* char $p>0$. *For any prime ℓ and any positive integer n, there exists an abelian extension of k with Galois group $G\simeq(\mathbb{Z}/\ell\mathbb{Z})^n$ in which every decomposition group is cyclic.*

We shall use the cyclotomic function field and we recall the basic facts following from class field theory. For an explicit construction of these fields by Drinfeld modules, we refer to [Rosen 2002, Chapter 12]. Let $k=\mathbb{F}_p(t)$ be the function field of the projective line over $\mathbb{F}_p$ and let $A=\mathbb{F}_p[t]$. In what follows P, P_i always denote monic irreducible polynomials of A. Let ∞ denote the place of the infinite point and set $V_\infty=\langle t,1+t^{-1}\mathbb{F}_p[\![t^{-1}]\!]\rangle\subset k_\infty^\times=\mathbb{F}_p((t^{-1}))^\times$. For a monic polynomial $M=\prod_{i=1}^r P_i^{n_i}\in A$, let $K(M)$ be the finite abelian extension of k corresponding to the following open subgroup (with finite index) of $\mathbb{A}_k^\times$

$$U(M)=k^\times\Bigl(\prod_{i=1}^{r}(1+P_i^{n_i}O_{P_i})\times V_\infty\times\prod_{v\nmid M\infty}O_v^\times\Bigr).$$

(The field $K(M)$ is called the cyclotomic function field for M.) Thus, we have $\mathbb{A}_k^\times/U(M)\simeq\mathrm{Gal}(K(M)/k)$ via the Artin map. Clearly, P is unramified in $K(M)$ if $P\nmid M$. The decomposition group of ∞ in $K(M)$ is isomorphic to $\mathbb{F}_p^\times$, which is cyclic. There is an isomorphism $(A/M)^\times\cong\mathbb{A}_k^\times/U(M)$ induced by $P \bmod M\mapsto(1,\dots,P,\dots,1) \bmod U(M)$ (P sitting on the place P) for $P\nmid M$, and $a \bmod M\mapsto(1,\dots,1,\dots,a) \bmod U(M)$ for $a\in\mathbb{F}_p^\times\subset(A/M)^\times$ (a sitting on the place ∞). So we have the following isomorphism which maps $P \bmod M$ ($P\nmid M$) to its Frobenius element in $\mathrm{Gal}(K(M)/k)$

$$(A/M)^\times\simeq\mathrm{Gal}(K(M)/k).$$

We will primarily be concerned with the fields $K(P)$ and $K(P^2)$, in which P is totally ramified. If ℓ divides $|(A/P)^\times|$, let $F(P)$ denote the subfield of $K(P)$ fixed by $((A/P)^\times)^\ell$ so that $\mathrm{Gal}(F(P)/k)\simeq\mathbb{Z}/\ell\mathbb{Z}$.

If $a \in A/P$, then $1 + aP \bmod P^2 \in (A/P^2)^\times$ is well defined and it generates the subgroup $H(a) = \{1 + aiP \bmod P^2 : i = 0, 1, \dots, p-1\}$ of order p when $a \neq 0$. We let $F(P, a)$ denote the subfield of $K(P^2)$ fixed by $\{x \in (A/P^2)^\times : x^{q-1} \in H(a)\}$ where $q = p^{\deg P} = |A/P|$. This group is isomorphic to $H(a) \times (A/P)^\times$; since $(A/P^2)^\times \simeq A/P \times (A/P)^\times$, we have $\mathrm{Gal}(F(P, a)/k) \simeq (\mathbb{Z}/p\mathbb{Z})^{\deg P - 1}$ when $a \neq 0$.

Proof. We first prove the theorem when $k = \mathbb{F}_p(t)$ and $\ell \neq p$. The argument of this case is entirely similar to the case of number fields given in [Katayama 1985]. Choose P_1 such that it splits in $k(\mu_\ell)$ where μ_ℓ is the group of ℓ-th roots of unity. We inductively choose P_r $(1 \le r \le n)$ such that P_r splits completely in the composite field $k(\mu_\ell, \sqrt[\ell]{P_1}, \dots, \sqrt[\ell]{P_{r-1}})F(P_1) \cdots F(P_{r-1})$. Clearly, each P_i has infinitely many ways to choose by density theorems. Now let K be the composite field $F(P_1) \cdots F(P_n)$. The decomposition group of ∞ in K/k is cyclic, since K is a subfield of the cyclotomic function field $K(P_1 \cdots P_r)$. Moreover, if P is not one of the P_i, P is unramified and hence its decomposition group is cyclic. Therefore, to show that K/k is the desired extension, it suffices to show that P_i splits in $F(P_j)$ whenever $i \neq j$. Assume $i < j$. Then P_j splits in $F(P_i)$ by the construction. Conversely, since P_j splits in $k(\sqrt[\ell]{P_i})$, P_i is an ℓ-power in the completion k_{P_j} which implies that $P_i \in ((A/P_j)^\times)^\ell$; hence P_i splits in $F(P_j)$. This proves the theorem when $k = \mathbb{F}_p(t)$ and $\ell \neq p$.

Assume next $k = \mathbb{F}_p(t)$ and $\ell = p$. Take an arbitrary $P_1 \in A$ with $a_1 = 0 \in A/P_1$. Let k_1 be a subfield of $F(P_1, a_1)$ with $[k_1 : k] = p$. We inductively choose (P_r, a_r, k_r) $(1 \le r \le n)$ with $a_r \in A/P_r$ such that $\deg P_r \ge r$ and

$$P_r^{q_i - 1} \equiv 1 + a_i P_i \bmod P_i^2 \quad \text{for } i = 1, \dots, r-1, \text{ where } q_i = |A/P_i|.$$

Let $a_{ri} \in A/P_r$ $(i = 1, \dots, r-1)$ such that

$$P_i^{q_r - 1} \equiv 1 + a_{ri} P_r \bmod P_r^2 \quad \text{for } i = 1, \dots, r-1.$$

Let $a_r := a_{r1}$, and let k_r be a subfield of $\bigcap_{i=1}^{r-1} F(P_r, a_{ri})$ with $[k_r : k] = p$. Put $K := k_1 \cdots k_r$ and let us show that K/k is the desired extension. We have $\mathrm{Gal}(K/k) \simeq (\mathbb{Z}/p\mathbb{Z})^n$ by considering the ramification. Assume $i < j$. Then P_j splits completely in $F(P_i, a_i)$ by the first congruences and hence splits in k_i; conversely, P_i splits completely in $F(P_j, a_{ji})$ by the second congruences and hence also splits in its subfield k_j. It follows that the decomposition subgroup of P_i $(i = 1, \dots, n)$ in K/k is cyclic. This property also holds for the place ∞, since $K \subset K(P_1^2 \cdots P_n^2)$. This proves the theorem when $k = \mathbb{F}_p(t)$ and $\ell = p$.

Now assume that k is any finite extension of $\mathbb{F}_p(t)$ and ℓ is any prime. Choose $P_1, \dots, P_n$ as above but we additionally require that P_i is unramified in $k/\mathbb{F}_p(t)$, and we define $K/\mathbb{F}_p(t)$ as above. Then $\mathrm{Gal}(Kk/k) \simeq (\mathbb{Z}/\ell\mathbb{Z})^n$. Since each decomposition subgroup for Kk/k is a subgroup of that of $K/\mathbb{F}_p(t)$, the field Kk is the desired extension. This completes the proof of Theorem A.9. □

Acknowledgments

The present project grows up from the 2020 NCTS USRP "Arithmetic of CM tori" where the authors participated. The authors thank Jeff Achter, Ming-Lun Hsieh, Tetsushi Ito, Teruhisa Koshikawa, Wen-Wei Li, Thomas Rüd, Yasuhiro Terakado, Jiangwei Xue, Seidai Yasuda for helpful discussions and their valuable comments and especially to Thomas Rüd who kindly answered the last author's questions and for his inspiring paper [Rüd 2022]. They are grateful to the referees for their careful readings and helpful comments which improve the present paper both in the exposition and mathematics. Liang, Yang and Yu were partially supported by the MoST grant 109-2115-M-001-002-MY3. Oki was supported by JSPS Research Fellowship for Young Scientists and KAKENHI Grant Number 22J00570.

References

[Achter et al. 2023] J. D. Achter, S. A. Altuğ, L. Garcia, and J. Gordon, "Counting abelian varieties over finite fields via Frobenius densities", *Algebra Number Theory* **17**:7 (2023), 1239–1280. MR Zbl

[Bayer-Fluckiger et al. 2019] E. Bayer-Fluckiger, T.-Y. Lee, and R. Parimala, "Hasse principles for multinorm equations", *Adv. Math.* **356** (2019), art. id. 106818. MR Zbl

[Borel 1963] A. Borel, "Some finiteness properties of adele groups over number fields", *Inst. Hautes Études Sci. Publ. Math.* **16** (1963), 5–30. MR Zbl

[Conrad et al. 2010] B. Conrad, O. Gabber, and G. Prasad, *Pseudo-reductive groups*, New Mathematical Monographs **17**, Cambridge University Press, 2010. MR Zbl

[Demarche and Wei 2014] C. Demarche and D. Wei, "Hasse principle and weak approximation for multinorm equations", *Israel J. Math.* **202**:1 (2014), 275–293. MR Zbl

[Gille and Szamuely 2017] P. Gille and T. Szamuely, *Central simple algebras and Galois cohomology*, 2nd ed., Cambridge Studies in Advanced Mathematics **165**, Cambridge University Press, 2017. MR Zbl

[González-Avilés 2008] C. D. González-Avilés, "Chevalley's ambiguous class number formula for an arbitrary torus", *Math. Res. Lett.* **15**:6 (2008), 1149–1165. MR Zbl

[González-Avilés 2010] C. D. González-Avilés, "On Néron–Raynaud class groups of tori and the capitulation problem", *J. Reine Angew. Math.* **648** (2010), 149–182. MR Zbl

[Guo et al. 2022] J.-W. Guo, N.-H. Sheu, and C.-F. Yu, "Class numbers of CM algebraic tori, CM abelian varieties and components of unitary Shimura varieties", *Nagoya Math. J.* **245** (2022), 74–99. MR Zbl

[Hürlimann 1984] W. Hürlimann, "On algebraic tori of norm type", *Comment. Math. Helv.* **59**:4 (1984), 539–549. MR Zbl

[Katayama 1985] S. Katayama, "On the Tamagawa number of algebraic tori", *Sūgaku* **37**:1 (1985), 81–83. In Japanese. MR Zbl

[Katayama 1991] S. Katayama, "Isogenous tori and the class number formulae", *J. Math. Kyoto Univ.* **31**:3 (1991), 679–694. MR Zbl

[Kiming 1990] I. Kiming, "Explicit classifications of some 2-extensions of a field of characteristic different from 2", *Canad. J. Math.* **42**:5 (1990), 825–855. MR Zbl

[Kottwitz 1984] R. E. Kottwitz, "Stable trace formula: cuspidal tempered terms", *Duke Math. J.* **51**:3 (1984), 611–650. MR Zbl

[Kottwitz 1988] R. E. Kottwitz, "Tamagawa numbers", *Ann. of Math.* (2) **127**:3 (1988), 629–646. MR Zbl

[Kottwitz 1992] R. E. Kottwitz, "Points on some Shimura varieties over finite fields", *J. Amer. Math. Soc.* **5**:2 (1992), 373–444. MR Zbl

[Lang 1994] S. Lang, *Algebraic number theory*, 2nd ed., Graduate Texts in Mathematics **110**, Springer, 1994. MR Zbl

[Lee 2022] T.-Y. Lee, "The Tate–Shafarevich groups of multinorm-one tori", *J. Pure Appl. Algebra* **226**:7 (2022), art. id. 106906. MR Zbl

[Lyndon 1948] R. C. Lyndon, "The cohomology theory of group extensions", *Duke Math. J.* **15** (1948), 271–292. MR Zbl

[Morishita 1991] M. Morishita, "On S-class number relations of algebraic tori in Galois extensions of global fields", *Nagoya Math. J.* **124** (1991), 133–144. MR Zbl

[Neukirch et al. 2000] J. Neukirch, A. Schmidt, and K. Wingberg, *Cohomology of number fields*, Grundl. Math. Wissen. **323**, Springer, 2000. MR Zbl

[Oesterlé 1984] J. Oesterlé, "Nombres de Tamagawa et groupes unipotents en caractéristique p", *Invent. Math.* **78**:1 (1984), 13–88. MR Zbl

[Ono 1961] T. Ono, "Arithmetic of algebraic tori", *Ann. of Math.* (2) **74**:1 (1961), 101–139. MR Zbl

[Ono 1963a] T. Ono, "On Tamagawa numbers", *Sūgaku* **15** (1963), 72–81. In Japanese. MR Zbl

[Ono 1963b] T. Ono, "On the Tamagawa number of algebraic tori", *Ann. of Math.* (2) **78**:1 (1963), 47–73. MR Zbl

[Platonov and Rapinchuk 1994] V. Platonov and A. Rapinchuk, *Algebraic groups and number theory*, Pure and Applied Mathematics **139**, Academic Press, Boston, 1994. MR Zbl

[Pollio 2014] T. P. Pollio, "On the multinorm principle for finite abelian extensions", *Pure Appl. Math. Q.* **10**:3 (2014), 547–566. MR Zbl

[Pollio and Rapinchuk 2013] T. P. Pollio and A. S. Rapinchuk, "The multinorm principle for linearly disjoint Galois extensions", *J. Number Theory* **133**:2 (2013), 802–821. MR Zbl

[Prasad and Rapinchuk 2010] G. Prasad and A. S. Rapinchuk, "Local-global principles for embedding of fields with involution into simple algebras with involution", *Comment. Math. Helv.* **85**:3 (2010), 583–645. MR Zbl

[Rosen 2002] M. Rosen, *Number theory in function fields*, Graduate Texts in Mathematics **210**, Springer, 2002. MR Zbl

[Rosengarten 2018] Z. Rosengarten, "Tate duality in positive dimension over function fields", preprint, 2018. arXiv 1805.00522

[Rüd 2022] T. Rüd, "Explicit Tamagawa numbers for certain algebraic tori over number fields", *Math. Comp.* **91**:338 (2022), 2867–2904. MR Zbl

[Serre 1977] J.-P. Serre, *Linear representations of finite groups*, Graduate Texts in Mathematics **42**, Springer, 1977. MR Zbl

[Serre 2016] J.-P. Serre, *Finite groups: an introduction*, Surveys of Modern Mathematics **10**, International Press, Somerville, MA, 2016. MR Zbl

[Shyr 1977] J. M. Shyr, "On some class number relations of algebraic tori", *Michigan Math. J.* **24**:3 (1977), 365–377. MR Zbl

[Tran 2017] M.-H. Tran, "A formula for the S-class number of an algebraic torus", *J. Number Theory* **181** (2017), 218–239. MR Zbl

[Washington 1997] L. C. Washington, *Introduction to cyclotomic fields*, 2nd ed., Graduate Texts in Mathematics **83**, Springer, 1997. MR Zbl

[Wei and Xu 2012] D. Wei and F. Xu, "Integral points for multi-norm tori", *Proc. Lond. Math. Soc.* (3) **104**:5 (2012), 1019–1044. MR Zbl

[Xue and Yu 2021] J. W. Xue and C. F. Yu, "On counting certain abelian varieties over finite fields", *Acta Math. Sin.* (*Engl. Ser.*) **37**:1 (2021), 205–228. MR Zbl

[Yu 2019] C. F. Yu, "Chow's theorem for semi-abelian varieties and bounds for splitting fields of algebraic tori", *Acta Math. Sin.* (*Engl. Ser.*) **35**:9 (2019), 1453–1463. MR Zbl

Communicated by Shou-Wu Zhang
Received 2022-09-30 Revised 2023-03-05 Accepted 2023-05-13

cindy11420@gmail.com *National Tsing Hua University, Taipei, Taiwan*

oki@math.sci.hokudai.ac.jp *Hokkaido University, Sapporo, Japan*

nona01111998@gmail.com *Utrecht University, Utrecht, Netherlands*

chiafu@math.sinica.edu.tw *Institute of Mathematics, Academia Sinica and NCTS, Taipei, Taiwan*

lijn@sdu.edu.cn *Shandong University, Qingdao Campus, Qingdao, China*

chiafu@math.sinica.edu.tw *Institute of Mathematics, Academia Sinica and NCTS, Taipei, Taiwan*

Algebra & Number Theory

msp.org/ant

See inside back cover or msp.org/ant for submission instructions.

The subscription price for 2024 is US $525/year for the electronic version, and $770/year (+$65, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP.

Algebra & Number Theory (ISSN 1944-7833 electronic, 1937-0652 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online.

ANT peer review and production are managed by EditFLOW® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

Algebra & Number Theory

Volume 18 No. 3 2024