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By using results on poles of L-functions and theta correspondence, we give a bound on b for (χ, b)-factors of the global Arthur parameter of a cuspidal automorphic representation π of a classical group or a metaplectic group where χ is a conjugate self-dual automorphic character and b is an integer which is the dimension of an irreducible representation of $\mathrm{SL}_2(\mathbb{C})$. We derive a more precise relation when π lies in a generic global A-packet.

Introduction

Let F be a number field and let $\mathbb{A}$ be its ring of adeles. Let π be an irreducible cuspidal automorphic representation of a classical group G defined over F. We also treat the case of metaplectic groups in this work. However to avoid excessive notation, we focus on the case of the symplectic groups $G = \mathrm{Sp}(X)$ in this introduction where X is a nondegenerate symplectic space over F. By Arthur's theory of endoscopy [2013], π belongs to a global A-packet associated to an elliptic global A-parameter, which is of the form

$$\boxplus_{i=1}^{r}(\tau_i, b_i)$$

where τ_i is an irreducible self-dual cuspidal automorphic representation of $\mathrm{GL}_{n_i}(\mathbb{A})$ and b_i is a positive integer which represents the unique b_i-dimensional irreducible representation of Arthur's $\mathrm{SL}_2(\mathbb{C})$; see Section 2, for more details.

Jiang [2014] proposed the (τ, b)-theory; see, in particular, Principle 1.2 there. It is a conjecture that uses period integrals to link together automorphic representations in two global A-packets whose global A-parameters are "different" by a (τ, b)-factor. We explain in more details. Let Π_ϕ denote the global A-packet with elliptic global A-parameter ϕ. Let π be an irreducible automorphic representation of $G(\mathbb{A})$ and let σ be an irreducible automorphic representation of $H(\mathbb{A})$, where H is a factor of an endoscopic group of G. Assume that π (resp. σ) occurs in the discrete spectrum. Then it is expected that there exists some kernel function $\mathcal{K}$ depending on G, H and (τ, b) only such that if π and σ satisfy a Gan–Gross–Prasad type of criterion, namely, that the period integral

$$\int_{H(F)\backslash H(\mathbb{A})}\int_{G(F)\backslash G(\mathbb{A})}\mathcal{K}(h,g)\,f_\sigma(h)\,\overline{f_\pi(g)}\,dg\,dh \tag{0-1}$$

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is nonvanishing for some choice of $f_\sigma \in \sigma$ and $f_\pi \in \pi$, then π is in the global A-packet Π_ϕ if and only if σ is in the global A-packet Π_{ϕ_2} with $\phi = (\tau, b) \boxplus \phi_2$. Then Jiang [2014, Section 5] proceeds to construct certain kernel functions and then using them, defines endoscopy transfer (by integrating over $H(F) \setminus H(\mathbb{A})$ only in (0-1)) and endoscopy descent (by integrating over $G(F) \setminus G(\mathbb{A})$ only in (0-1)). It is not yet known if these are the kernel functions making the statements of Principle 1.2 in [Jiang 2014] hold. As the kernel functions come from Bessel coefficients or Fourier–Jacobi coefficients as in [Gan et al. 2012, Section 23], we see the nonvanishing of this period integral is analogous to condition (i) in the global Gan–Gross–Prasad conjecture [Gan et al. 2012, Conjecture 24.1].

Jiang [2014, Section 7] suggested that if τ is an automorphic character χ, then the kernel function can be taken to be the theta kernel and endoscopy transfer and endoscopy descent are theta lifts. In this case, the span of

$$\int_{G(F) \setminus G(\mathbb{A})} \mathcal{K}(h, g) \overline{f_\pi(g)}\, dg$$

as f_π runs over π is the theta lift of π. This is an automorphic representation of $H(\mathbb{A})$. Lifting in the other direction is analogous. Assume that the theta lift of π is nonzero. Write ϕ_π for the global A-parameter of π. Then Jiang [2014, Principle 1.2] says that ϕ_π has a (χ, b)-factor and that the global A-parameter of the theta lift of π from G to H should be ϕ_π with the (χ, b)-factor removed. Here b should be of appropriate size relative to G and H. Our work is one step in this direction.

One goal of this article is to expand on the (χ, b)-theory and to present the results of [Mœglin 1997; Ginzburg et al. 2009; Jiang and Wu 2016; 2018; Wu 2022a; 2022b] for various cases in a uniform way. As different reductive dual pairs that occur in theta correspondence have their own peculiarities, the notation and techniques of these papers are adapted to the treatment of their own specific cases. We attempt to emphasize on the common traits of the results which are buried in lengthy and technical proofs in these papers.

After collecting the results on poles of L-functions, poles of Eisenstein series and theta correspondence, we derive a bound for b when b is maximal among all factors of the global A-parameter of π. In addition, we derive an implication on global A-packets. Of course, the heavy lifting was done by the papers mentioned above.

Theorem 0.1 (Corollary 5.3). *The global A-packet attached to the elliptic global A-parameter ϕ cannot have a cuspidal member if ϕ has a (χ, b)-factor with*

$$b > \tfrac{1}{2} \dim_F X + 1, \quad \text{if } G = \mathrm{Sp}(X).$$

Another way of phrasing this is that we have a bound on the size of b that can occur in a factor of type $(\chi, *)$ in the global A-parameter of a cuspidal automorphic representation. Thus our results have application in getting a Ramanujan bound, which measures the departure of the local components π_v from being tempered for all places v of F, for classical groups and metaplectic groups. This should follow by generalizing the arguments in [Jiang and Liu 2018, Section 5] which treats the symplectic case.

There they first established a bound for b under some conditions on wave front sets. This enables them to control the contribution of GL_1-factors in the global A-parameter to the Ramanujan bound. Our result can supply this ingredient for classical groups and also metaplectic groups unconditionally. Then Jiang and Liu [2018, Section 5] found a Ramanujan bound for π by using the crucial results on the Ramanujan bound for GL_2 in [Kim 2003; Blomer and Brumley 2011].

We describe the idea of the proof of our result. First we relate the existence of a (τ, b)-factor in the elliptic global A-parameter of π to the existence of poles of partial L-functions $L^S(s, \pi \times \tau^\vee)$; see Proposition 2.8. If the global A-parameter of π has a factor (τ, b) where b is maximal among all factors, we can show that the partial L-function $L^S(s, \pi \times \tau^\vee)$ has a pole at $s = \frac{1}{2}(b+1)$. Thus studying the location of poles of $L^S(s, \pi \times \tau^\vee)$ for τ running through all self-dual cuspidal representations of $\mathrm{GL}_n(\mathbb{A}_F)$ can shed light on the size of the b_i that occur in the global A-parameter of π. Then we specialize to the case where τ is a character χ and consider $L^S(s, \pi \times \chi^\vee)$ in what follows.

Next we relate the poles of $L^S(s, \pi \times \chi^\vee)$ to the poles of Eisenstein series attached to the cuspidal datum $\chi \boxtimes \pi$; see Section 3. In fact, in some cases, we use the nonvanishing of $L^S(s, \pi \times \chi^\vee)$ at $s = \frac{1}{2}$ instead; see Proposition 3.1. Then we recall in Theorem 3.5 that the maximal positive pole of the Eisenstein series has a bound which is supplied by the study of global theta lifts. This is enough for showing Corollary 5.3, though we have a more precise result that the maximal positive pole corresponds to the invariant called the lowest occurrence index of π with respect to χ in Theorem 4.4. The lowest occurrence index is the minimum of the first occurrence indices over some Witt towers. For the precise definition see (4-2). We also have a less precise result (Theorem 4.1) relating the first occurrence index of π with respect to certain quadratic spaces to possibly nonmaximal and possibly negative poles of the Eisenstein series.

More precise results can be derived if we assume that π has a generic global A-parameter. This is because we have a more precise result relating poles or nonvanishing of values of the complete L-functions to poles of the Eisenstein series supplied by [Jiang et al. 2013]. Thus we get

Theorem 0.2 (Theorem 6.7). *Let π be a cuspidal member in a generic global A-packet of $G(\mathbb{A}) = \mathrm{Sp}(X)(\mathbb{A})$. Let χ be a self-dual automorphic character of $\mathrm{GL}_1(\mathbb{A})$. Then the following are equivalent*:

(1) *The global A-parameter ϕ_π of π has a $(\chi, 1)$-factor.*

(2) *The complete L-function $L(s, \pi \times \chi^\vee)$ has a pole at $s = 1$.*

(3) *The Eisenstein series $E(g, f_s)$ has a pole at $s = 1$ for some choice of section $f_s \in \mathcal{A}^{Q_1}(s, \chi \boxtimes \pi)$.*

(4) *The lowest occurrence index $\mathrm{LO}_X^\chi(\pi)$ is $\dim X$.*

Here Q_1 is a parabolic subgroup of $\mathrm{Sp}(X_1)$ with Levi subgroup isomorphic to $\mathrm{GL}_1 \times \mathrm{Sp}(X)$, where X_1 is the symplectic space formed from X by adjoining a hyperbolic plane. Roughly speaking, $\mathcal{A}^{Q_1}(s, \chi \boxtimes \pi)$ is a space of automorphic forms on $\mathrm{Sp}(X_1)$ induced from $\chi|\cdot|^s \boxtimes \pi$ viewed as a representation of the parabolic subgroup Q_1. We refer the reader to Section 3 for the precise definition of $\mathcal{A}^{Q_1}(s, \chi \boxtimes \pi)$. We note that the lowest occurrence index $\mathrm{LO}_X^\chi(\pi)$ is an invariant in the theory of theta correspondence related

to the invariant called the first occurrence index; see Section 4 for their definitions. We also include a result (Theorem 6.3) that concerns the nonvanishing of $L(s, \pi \times \chi^\vee)$ at $s = \frac{1}{2}$ and the lowest occurrence index. We plan to improve this result in the future by studying a relation between nonvanishing of Bessel or Fourier–Jacobi periods and the lowest occurrence index.

We note that the L-function $L(s, \pi \times \chi^\vee)$ has been well-studied and is intricately entwined with the study of theta correspondence, most prominently in the Rallis inner product formula which says that the inner product of two theta lifts is equal to the residue or value of $L(s, \pi \times \chi^\vee)$ at an appropriate point up to some ramified factors and some abelian L-functions. We refer the reader to [Yamana 2014] which is a culmination of many previous results. In our approach, the Eisenstein series $E(g, f_s)$, which is not of Siegel type, is the key link between $L(s, \pi \times \chi^\vee)$ and the theta lifts.

Now we describe the structure of this article. In Section 1, we set up some basic notation. In Section 2, we define elliptic global A-parameters for classical groups and metaplectic groups and also the global A-packet associated to an elliptic global A-parameter. We show how poles of partial L-functions detect (τ, b)-factors in an elliptic global A-parameter. In Section 3, we define Eisenstein series attached to the cuspidal datum $\chi \boxtimes \pi$ and recall some results on the possible locations of their maximal positive poles. In Section 4, we introduce two invariants of theta correspondence. They are the first occurrence index $\mathrm{FO}_X^{Y,\chi}(\pi)$ and the lowest occurrence index $\mathrm{LO}_X^{\chi}(\pi)$ of π with respect to some data. We relate them to poles of Eisenstein series. Results in Sections 3 and 4 are not new. Our aim is to present the results in a uniform way for easier access. In Section 5, we show a bound for b in (χ, b)-factors of the global A-parameter of π. Finally in Section 6, we consider the case when π has a generic global A-parameter. We show that when $L(s, \pi \times \chi^\vee)$ has a pole at $s = 1$ (resp. $L(s, \pi \times \chi^\vee)$ is nonvanishing at $s = \frac{1}{2}$), the lowest occurrence index is determined.

1. Notation

Let F be a number field and let E be either F or a quadratic field extension of F. Let $\varrho \in \mathrm{Gal}(E/F)$ be the trivial Galois element when $E = F$ and the nontrivial Galois element when $E \neq F$. When $E \neq F$, write $\varepsilon_{E/F}$ for the quadratic character associated to E/F via Class field theory. Let G be an algebraic group over E. We write $\mathrm{R}_{E/F}G$ for the restriction of scalars of Weil. This is an algebraic group over F.

Let ϵ be either 1 or -1. By an ϵ-skew Hermitian space, we mean an E-vector space X together with an F-bilinear pairing

$$\langle \cdot, \cdot \rangle_X : X \times X \to E$$

such that

$$\langle y, x \rangle_X = -\epsilon \langle x, y \rangle_X^{\varrho}, \quad \langle ax, by \rangle = a \langle x, y \rangle_X b^{\varrho}$$

for all $a, b \in E$ and $x, y \in X$. We consider the linear transformations of X to act from the right. We follow the notation from [Yamana 2014] closely and we intend to generalize the results here to the quaternionic unitary group case in our future work.

Let X be an ϵ-skew Hermitian space of finite dimension. Then the isometry group of X is one of the following:

(1) The symplectic group $\mathrm{Sp}(X)$ when $E = F$ and $\epsilon = 1$.

(2) The orthogonal group $\mathrm{O}(X)$ when $E = F$ and $\epsilon = -1$.

(3) The unitary group $U(X)$ when $E \neq F$ and $\epsilon = \pm 1$.

We will also consider the metaplectic group. Let v be a place of F and let F_v denote the completion of F at v. Let $\mathbb{A}_F$ (resp. $\mathbb{A}_E$) denote the ring of adeles of F (resp. E). Set $\mathbb{A} := \mathbb{A}_F$. Write $\mathrm{Mp}(X)(F_v)$ (resp. $\mathrm{Mp}(X)(\mathbb{A}_F)$) for the metaplectic double cover of $\mathrm{Sp}(X)(F_v)$ (resp. $\mathrm{Sp}(X)(\mathbb{A}_F)$) defined by Weil [1964]. We note that the functor $\mathrm{Mp}(X)$ is not representable by an algebraic group. We will also need the $\mathbb{C}^1$-extension $\mathrm{Mp}(X)(F_v) \times_{\mu_2} \mathbb{C}^1$ of $\mathrm{Sp}(X)(F_v)$ and we denote it by $\mathrm{Mp}^{\mathbb{C}^1}(F_v)$. Similarly we define $\mathrm{Mp}^{\mathbb{C}^1}(\mathbb{A}_F)$.

Let ψ be a nontrivial automorphic additive character of $\mathbb{A}_F$ which will figure in the Weil representations as well as the global A-parameters for $\mathrm{Mp}(X)$.

For an automorphic representation or admissible representation π, we write $\pi^\vee$ for its contragredient.

2. Global Arthur parameters

First we recall the definition of elliptic global Arthur parameters (A-parameters) for classical groups as well as metaplectic groups; see [Arthur 2013] for the symplectic and the special orthogonal case and we adopt the formulation in [Atobe and Gan 2017] for the case of the (disconnected) orthogonal groups. For the unitary case, see [Mok 2015; Kaletha et al. 2014]. For the metaplectic case, see [Gan and Ichino 2018]. Then we focus on simple factors of global Arthur parameters and relate their presence to poles of partial L-functions. This is a crude first step for detecting (τ, b)-factors in an elliptic global A-parameter according to the "(τ, b)-theory" proposed in [Jiang 2014].

Let $\boldsymbol{G}$ be $\mathrm{U}(X)$, $\mathrm{O}(X)$, $\mathrm{Sp}(X)$ or $\mathrm{Mp}(X)$. Let d denote the dimension of X. Set $\boldsymbol{G}^\circ = \mathrm{SO}(X)$ when $\boldsymbol{G} = \mathrm{O}(X)$. Set $\boldsymbol{G}^\circ = \boldsymbol{G}$ otherwise. Write $\check{\boldsymbol{G}}$ for the (complex) dual group of $\boldsymbol{G}^\circ$. Then

$$\check{\boldsymbol{G}} = \begin{cases} \mathrm{GL}_d(\mathbb{C}) & \text{if } \boldsymbol{G} = \mathrm{U}(X); \\ \mathrm{Sp}_{d-1}(\mathbb{C}) & \text{if } \boldsymbol{G} = \mathrm{O}(X) \text{ and } d \text{ is odd}; \\ \mathrm{SO}_d(\mathbb{C}) & \text{if } \boldsymbol{G} = \mathrm{O}(X) \text{ and } d \text{ is even}; \\ \mathrm{SO}_{d+1}(\mathbb{C}) & \text{if } \boldsymbol{G} = \mathrm{Sp}(X); \\ \mathrm{Sp}_d(\mathbb{C}) & \text{if } \boldsymbol{G} = \mathrm{Mp}(X). \end{cases}$$

An elliptic global A-parameter for $\boldsymbol{G}$ is a finite formal sum of the form

$$\phi = \boxplus_{i=1}^r (\tau_i, b_i), \quad \text{for some positive integer } r$$

where

(1) τ_i is an irreducible conjugate self-dual cuspidal automorphic representation of $\mathrm{GL}_{n_i}(\mathbb{A}_E)$;

(2) b_i is a positive integer which represents the unique b_i-dimensional irreducible representation of Arthur's $\mathrm{SL}_2(\mathbb{C})$

such that:

- $\sum_i n_i b_i = d_{\check{\boldsymbol{G}}}$.
- τ_i is conjugate self-dual of parity $(-1)^{N_{\check{G}}+b_i}$ (see Remark 2.3).
- The factors (τ_i, b_i) are pairwise distinct.

Here $d_{\check{\boldsymbol{G}}}$ is the degree of the standard representation of $\check{\boldsymbol{G}}$ which, explicitly, is

$$d_{\check{\boldsymbol{G}}} = \begin{cases} \dim X & \text{if } \boldsymbol{G} = \mathrm{U}(X); \\ \dim X - 1 & \text{if } \boldsymbol{G} = \mathrm{O}(X) \text{ with } \dim X \text{ odd}; \\ \dim X & \text{if } \boldsymbol{G} = \mathrm{O}(X) \text{ with } \dim X \text{ even}; \\ \dim X + 1 & \text{if } \boldsymbol{G} = \mathrm{Sp}(X); \\ \dim X & \text{if } \boldsymbol{G} = \mathrm{Mp}(X); \end{cases}$$

and

$$N_{\check{\boldsymbol{G}}} = \begin{cases} \dim X \bmod 2 & \text{if } \boldsymbol{G} = \mathrm{U}(X); \\ 0 & \text{if } \boldsymbol{G} = \mathrm{O}(X) \text{ with } \dim X \text{ odd}; \\ 1 & \text{if } \boldsymbol{G} = \mathrm{O}(X) \text{ with } \dim X \text{ even}; \\ 1 & \text{if } \boldsymbol{G} = \mathrm{Sp}(X); \\ 0 & \text{if } \boldsymbol{G} = \mathrm{Mp}(X). \end{cases}$$

Remark 2.1. We adopt the notation in [Jiang 2014] and hence we write (τ_i, b_i) rather than $\tau_i \boxtimes \nu_{b_i}$ as is more customary in the literature, so that the quantity b_i, that we study, is more visible.

Remark 2.2. In the unitary case, we basically spell out what $\Psi_2(\mathrm{U}(N), \xi_{\mathbf{1}})$ in [Mok 2015, Definition 2.4.7] is. We have discarded the second factor $\tilde{\psi}$ as it is determined by ψ^N and $\xi_{\mathbf{1}}$ in Mok's notation.

Remark 2.3. (1) For $\boldsymbol{G} = \mathrm{U}(X)$, we say that τ is conjugate self-dual of parity η if the Asai L-function $L(s, \tau, \mathrm{Asai}^{\eta})$ has a pole at $s = 1$. If $\eta = +1$, we also say that τ is conjugate orthogonal and if $\eta = -1$, we also say that τ is conjugate symplectic. The Asai representations come from the decomposition of the twisted tensor product representation of the L-group; see [Mok 2015, (2.2.9) and (2.5.9)] and [Goldberg 1994].

(2) For other cases, we mean self-dual when we write conjugate self-dual. We say that τ is self-dual of parity $+1$ or orthogonal, if $L(s, \tau, \mathrm{Sym}^2)$ has a pole at $s = 1$; we say that τ is self-dual of parity -1 or symplectic, if $L(s, \tau, \wedge^2)$ has a pole at $s = 1$.

(3) The parity is uniquely determined for each irreducible conjugate self-dual cuspidal representation τ.

Let $\Psi_2(\boldsymbol{G})$ denote the set of elliptic global A-parameters of $\boldsymbol{G}$. Let $\phi \in \Psi_2(\boldsymbol{G})$. Via the local Langlands conjecture (which is proved for the general linear groups), at every place v of F, we localize ϕ to get an elliptic local A-parameter,

$$\phi_v : L_{F_v} \times \mathrm{SL}_2(\mathbb{C}) \to \check{\boldsymbol{G}} \rtimes W_{F_v},$$

where W_{F_v} is the Weil group of F_v and L_{F_v} is W_{F_v} if v is archimedean and the Weil–Deligne group $W_{F_v} \times \mathrm{SL}_2(\mathbb{C})$ if v is nonarchimedean. To ϕ_v we associate the local L-parameter $\varphi_{\phi_v} : L_{F_v} \to \check{\boldsymbol{G}} \rtimes W_{F_v}$ given by

$$\varphi_{\phi_v}(w) = \phi_v\left(w, \begin{pmatrix} |w|^{1/2} & \\ & |w|^{-1/2} \end{pmatrix}\right).$$

Let $L^2_{\mathrm{disc}}(\boldsymbol{G})$ denote the discrete part of $L^2(\boldsymbol{G}(F) \setminus \boldsymbol{G}(\mathbb{A}_F))$ when $\boldsymbol{G} \neq \mathrm{Mp}(X)$ and the genuine discrete part of $L^2(\mathrm{Sp}(F) \setminus \mathrm{Mp}(\mathbb{A}_F))$ for $\boldsymbol{G} = \mathrm{Mp}(X)$. Define the full near equivalence class $L^2_{\phi,\psi}(\boldsymbol{G})$ attached to the elliptic global A-parameter ϕ to be the Hilbert direct sum of all irreducible automorphic representations σ occurring in $L^2_{\mathrm{disc}}(\boldsymbol{G})$ such that for almost all v, the local L-parameter of σ_v is φ_{ϕ_v}. We remark that in the $\mathrm{Mp}(X)$-case, the parametrization of σ_v is relative to ψ_v since the local L-parameter of σ_v is attached via the Shimura–Waldspurger correspondence which depends on ψ_v. This is the only case in this article where $L^2_{\phi,\psi}(\boldsymbol{G})$ depends on ψ.

Let $\mathcal{A}_2(\boldsymbol{G})$ denote the dense subspace consisting of automorphic forms in $L^2_{\mathrm{disc}}(\boldsymbol{G})$. Similarly define $\mathcal{A}_{2,\phi,\psi}(\boldsymbol{G})$ to be the dense subspace of $L^2_{\phi,\psi}(\boldsymbol{G})$ consisting of automorphic forms. Then we have a crude form of Arthur's multiplicity formula which decomposes the L^2-discrete spectrum into near equivalence classes indexed by $\Psi_2(\boldsymbol{G})$.

Theorem 2.4. *We have the orthogonal decompositions*

$$L^2_{\mathrm{disc}}(\boldsymbol{G}) = \widehat{\bigoplus_{\phi \in \Psi_2(\boldsymbol{G})}} L^2_{\phi,\psi}(\boldsymbol{G}) \quad \textit{and} \quad \mathcal{A}_2(\boldsymbol{G}) = \bigoplus_{\phi \in \Psi_2(\boldsymbol{G})} \mathcal{A}_{2,\phi,\psi}(\boldsymbol{G}).$$

Remark 2.5. This crude form of Arthur's multiplicity formula has been proved for $\mathrm{Sp}(X)$ and quasisplit $\mathrm{O}(X)$ by Arthur [2013], for $\mathrm{U}(X)$ by [Mok 2015; Kaletha et al. 2014] and for $\mathrm{Mp}(X)$ by [Gan and Ichino 2018]. This is also proved for nonquasisplit even orthogonal (and also unitary groups) in [Chen and Zou 2021] and for nonquasisplit odd orthogonal groups in [Ishimoto 2023]. Thus for all cases needed in this paper, Theorem 2.4 is known.

We have some further remarks on the orthogonal and unitary cases.

Remark 2.6. Arthur's statements use $\mathrm{SO}(X)$ rather than $\mathrm{O}(X)$ and he needs to account for the outer automorphism of $\mathrm{SO}(X)$ when $\dim X$ is even; see the paragraph below [Arthur 2013, Theorem 1.5.2]. The formulation for quasisplit even $\mathrm{O}(X)$ is due to Atobe and Gan [2017, Theorem 7.1(1)]. For odd $\mathrm{O}(X)$, which is isomorphic to $\mathrm{SO}(X) \times \mu_2$, the reformulation of Arthur's result is easy. Let T be a finite set of places of F. Assume that it has even cardinality. Let sgn_T be the automorphic character of $\mu_2(\mathbb{A}_F)$ which is equal to the sign character at places in T and the trivial character at places outside T. These give all the automorphic characters of $\mu_2(\mathbb{A}_F)$. Then every irreducible automorphic representation π of $\mathrm{O}(X)(\mathbb{A}_F)$ is of the form $\pi_0 \boxtimes \mathrm{sgn}_T$ for some irreducible automorphic representation π_0 of $\mathrm{SO}(X)(\mathbb{A}_F)$ and some finite set T of places of even cardinality. A near equivalence class of $\mathrm{O}(X)(\mathbb{A}_F)$ then consists of all irreducible automorphic representations $\pi_0 \boxtimes \mathrm{sgn}_T$ for π_0 running over a near equivalence class of $\mathrm{SO}(X)(\mathbb{A}_F)$ and sgn_T running over all automorphic characters of $\mu_2(\mathbb{A}_F)$.

Remark 2.7. For the $\mathrm{U}(X)$ case, the global A-parameter depends on the choice of a sign and a conjugate self-dual character which determine an embedding of the L-group of $\mathrm{U}(X)$ to the L-group of $\mathrm{R}_{E/F}\,\mathrm{GL}_d$ where we recall that $d := \dim X$. We refer the reader to [Mok 2015, Section 2.1], in particular (2.1.9) there, for details. In this work, we choose the $+1$ sign and the trivial character, which, in Mok's notation, means $\kappa = 1$ and $\chi_\kappa = \mathbf{1}$. Then this corresponds to the standard base change of $\mathrm{U}(X)$ to $\mathrm{R}_{E/F}\,\mathrm{GL}_d$. We note that the L-functions we use below are such that

$$L_v(s, \pi_v \times \tau_v) = L_v(s, \mathrm{BC}(\pi_v) \otimes \tau_v),$$

for all places v, automorphic representations π of $\boldsymbol{G}(\mathbb{A}_F)$ and τ of $\mathrm{R}_{E/F}\,\mathrm{GL}_a(\mathbb{A}_F)$ where BC denotes the standard base change.

By Theorem 2.4, we get

Proposition 2.8. *Let π be an irreducible automorphic representation of $\boldsymbol{G}(\mathbb{A}_F)$ that occurs in $\mathcal{A}_{2,\phi,\psi}(\boldsymbol{G})$. Then*:

(1) *If ϕ has a (τ, b)-factor with b maximal among all factors, then the partial L-function $L^S(s, \pi \times \tau^\vee)$ has a pole at $s = \frac{1}{2}(b+1)$ and this is its maximal pole.*

(2) *if the partial L-function $L^S(s, \pi \times \tau^\vee)$ has a pole at $s = \frac{1}{2}(b'+1)$, then ϕ has a (τ, b)-factor with $b \geq b'$.*

Remark 2.9. In the $\mathrm{Mp}(X)$ case, the L-function depends on ψ, but we suppress it from notation here.

Proof. First we collect some properties of the Rankin–Selberg L-functions for $\mathrm{GL}_m \times \mathrm{GL}_n$. By the Rankin–Selberg method, for an irreducible unitary cuspidal automorphic representation τ, $L^S(s, \tau \times \tau^\vee)$ has a simple pole at $s = 1$ and is nonzero holomorphic for $\mathrm{Re}(s) \geq 1$ and $s \neq 1$; for irreducible unitary cuspidal automorphic representations τ and τ' such that $\tau \ncong \tau'$, $L^S(s, \tau \times \tau'^\vee)$ is nonzero holomorphic for $\mathrm{Re}(s) \geq 1$. These results can be found in Cogdell's notes [2000] which collect the results from [Jacquet et al. 1983; Jacquet and Shalika 1976; Shahidi 1978; 1980].

Assume that $\phi = \boxplus_{i=1}^r (\tau_i, b_i)$. Then

$$L^S(s, \pi \times \tau^\vee) = \prod_{i=1}^{r} \prod_{j=0}^{b_i - 1} L^S\big(s - \tfrac{1}{2}(b_i - 1) + j, \tau_i \times \tau^\vee\big),$$

where S is a finite set of places of F outside of which all data are unramified.

Assume that ϕ has a (τ, b)-factor with b maximal among all factors, then by the properties of the Rankin–Selberg L-functions, we see that $L^S(s, \pi \times \tau^\vee)$ has a pole at $s = \frac{1}{2}(b+1)$ and it is maximal.

Next assume that the partial L-function $L^S(s, \pi \times \tau^\vee)$ has a pole at $s = \frac{1}{2}(b'+1)$. If ϕ has no (τ, c)-factor for any $c \in \mathbb{Z}_{>0}$, then $L^S(s, \pi \times \tau^\vee)$ is holomorphic for all $s \in \mathbb{C}$ and we get a contradiction. Thus ϕ has a (τ, b)-factor. We take b maximal among all factors of the form $(\tau, *)$ in ϕ. As b may not be maximal among all simple factors of ϕ, we can only conclude that $L^S(s, \pi \times \tau^\vee)$ is holomorphic for $\mathrm{Re}(s) > \frac{1}{2}(b+1)$. Thus $b' \leq b$. □

Given an irreducible cuspidal automorphic representation π, write ϕ_π for the global A-parameter of π. By studying poles of $L^S(s, \pi \times \tau^\vee)$ for varying τ, we can detect the existence of (τ, b)-factors with maximal b in ϕ_π. We would also like to construct an irreducible cuspidal automorphic representation with global A-parameter $\phi_\pi \boxminus (\tau, b)$ which means removing the (τ, b)-factor from ϕ_π if ϕ_π has a (τ, b)-factor. Doing this recursively, we will be able to compute the global A-parameter of a given irreducible cuspidal automorphic representation. In reverse, the construction should produce concrete examples of cuspidal automorphic representations in a given global A-packet with an elliptic global A-parameter. This will be investigated in our future work.

In this article, we focus our attention on the study of poles of $L^S(s, \pi \times \tau^\vee)$ where τ is a conjugate self-dual irreducible cuspidal automorphic representation of $\mathrm{R}_{E/F}\,\mathrm{GL}_1(\mathbb{A})$. Now we write χ for τ to emphasize that we are considering the case of twisting by characters. This case has been well-studied and it is known that the poles of $L^S(s, \pi \times \chi^\vee)$ are intricately related to invariants of theta correspondence via the Rallis inner product formula which relates the inner product of two theta lifts to a residue or a value of the L-function. We refer the readers to [Kudla and Rallis 1994; Wu 2017; Gan et al. 2014; Yamana 2014] for details. One of the key steps is the regularized Siegel–Weil formula which relates a theta integral to a residue or a value of a Siegel–Eisenstein series. Our work considers an Eisenstein series which is not of Siegel type, but which is closely related to $L(s, \pi \times \chi^\vee)$.

3. Eisenstein series attached to $\chi \boxtimes \pi$

In this section we deviate slightly from the notation in Section 2. We use $G(X)$ to denote one of $\mathrm{Sp}(X)$, $\mathrm{O}(X)$ and $\mathrm{U}(X)$. We let $\boldsymbol{G}(X)$ be a cover group of $G(X)$, which means $\boldsymbol{G}(X) = \mathrm{Sp}(X)$ or $\mathrm{Mp}(X)$ if $G(X) = \mathrm{Sp}(X)$, $\boldsymbol{G}(X) = \mathrm{O}(X)$ if $G(X) = \mathrm{O}(X)$ and $\boldsymbol{G}(X) = \mathrm{U}(X)$ if $G(X) = \mathrm{U}(X)$. We adopt similar notation to that in [Mœglin and Waldspurger 1995]. We define Eisenstein series on a larger group of the same type as $\boldsymbol{G}(X)$ and collect some results on their maximal positive poles.

Let π be an irreducible cuspidal automorphic representation of $\boldsymbol{G}(X)(\mathbb{A})$. We always assume that π is genuine when $\boldsymbol{G}(X) = \mathrm{Mp}(X)$. Let χ be a conjugate self-dual automorphic character of $\mathrm{R}_{E/F}\,\mathrm{GL}_1(\mathbb{A}) = \mathbb{A}_E^\times$. When $E \neq F$, we define

$$\epsilon_\chi = \begin{cases} 0 & \text{if } \chi|_{\mathbb{A}_F^\times} = \mathbb{1}; \\ 1 & \text{if } \chi|_{\mathbb{A}_F^\times} = \varepsilon_{E/F}. \end{cases} \tag{3-1}$$

Let a be a positive integer. Let X_a be the ϵ-skew Hermitian space over E that is formed from X by adjoining a-copies of the hyperbolic plane. More precisely, let ℓ_a^+ (resp. ℓ_a^-) be a totally isotropic a-dimensional E-vector space spanned by $e_1^+, \dots, e_a^+$ (resp. $e_1^-, \dots, e_a^-$) such that $\langle e_i^+, e_j^- \rangle = \delta_{ij}$ where δ_{ij} is the Kronecker symbol. Then

$$X_a = \ell_a^+ \oplus X \oplus \ell_a^-$$

with X orthogonal to $\ell_a^+ \oplus \ell_a^-$.

Let $G(X_a)$ be the isometry group of X_a. Let Q_a be the parabolic subgroup of $G(X_a)$ that stabilizes ℓ_a^-. Write $Q_a = M_a N_a$ in the Levi decomposition with N_a being the unipotent radical and M_a the standard Levi subgroup. We have an isomorphism

$$m : \mathrm{R}_{E/F}\,\mathrm{GL}_a \times G(X) \to M_a.$$

where we identify $\mathrm{R}_{E/F}\,\mathrm{GL}_a$ with $\mathrm{R}_{E/F}\,\mathrm{GL}(\ell_a^+)$. Let ρ_{Q_a} be the half sum of the positive roots in N_a, which can be viewed as an element in $\mathfrak{a}_{M_a}^* := \mathrm{Rat}(M_a) \otimes_{\mathbb{Z}} \mathbb{R}$ where $\mathrm{Rat}(M_a)$ is the group of rational characters of M_a. We note that as Q_a is a maximal parabolic subgroup, $\mathfrak{a}_{M_a}^*$ is one-dimensional. Via the Shahidi normalization [2010], we identify $\mathfrak{a}_{M_a}^*$ with $\mathbb{R}$ and thus may regard ρ_{Q_a} as the real number

$$\begin{aligned} \tfrac{1}{2}(\dim_E X + a), &\quad \text{if } G(X_a) \text{ is unitary;} \\ \tfrac{1}{2}(\dim_E X + a - 1), &\quad \text{if } G(X_a) \text{ is orthogonal;} \\ \tfrac{1}{2}(\dim_E X + a + 1), &\quad \text{if } G(X_a) \text{ is symplectic.} \end{aligned}$$

Let $K_{a,v}$ be a good maximal compact subgroup of $G(X_a)(F_v)$ in the sense that the Iwasawa decomposition holds and set $K_a = \prod_v K_{a,v}$.

Let $\mathcal{A}^{Q_a}(s, \chi \boxtimes \pi)$ denote the space of $\mathbb{C}$-valued smooth functions f on $N_a(\mathbb{A})M_a(F) \setminus G(X_a)(\mathbb{A})$ such that:

(1) f is right K_a-finite.

(2) For any $x \in \mathrm{R}_{E/F}\,\mathrm{GL}_a(\mathbb{A})$ and $g \in G(X_a)(\mathbb{A})$ we have

$$f(m(x, I)g) = \chi(\det(x))|\det(x)|_{\mathbb{A}_E}^{s+\rho_{Q_a}} f(g).$$

(3) For any fixed $k \in K_a$, the function $h \mapsto f(m(I, h)k)$ on $G(X)(\mathbb{A})$ is in the space of π.

Now let $\boldsymbol{G}(X) = \mathrm{Mp}(X)$. This case depends on ψ. Let $\widetilde{\mathrm{GL}}_1(F_v)$ be the double cover of $\mathrm{GL}_1(F_v)$ defined as follows. As a set it is $\mathrm{GL}_1(F_v) \times \mu_2$ and the multiplication is given by

$$(g_1, \zeta_1)(g_2, \zeta_2) = (g_1 g_2, \zeta_1 \zeta_2 (g_1, g_2)_{F_v})$$

which has a Hilbert symbol twist when multiplying the μ_2-part. Analogously we define the double cover $\widetilde{\mathrm{GL}}_1(\mathbb{A})$ of $\mathrm{GL}_1(\mathbb{A})$. Let $\chi_{\psi,v}$ denote the genuine character of $\widetilde{\mathrm{GL}}_1(F_v)$ defined by

$$\chi_{\psi,v}((g, \zeta)) = \zeta \gamma_v(g, \psi_{1/2,v})^{-1}$$

where $\gamma_v(\cdot, \psi_{1/2,v})$ is a fourth root of unity defined via the Weil index. It is the same one as in [Gan and Ichino 2014, page 521] except that we have put in the subscripts v. Then

$$\chi_\psi((g, \zeta)) = \zeta \prod_v \gamma_v(g_v, \psi_{1/2,v})^{-1}$$

is a genuine automorphic character of $\widetilde{\mathrm{GL}}_1(\mathbb{A})$. Let $\widetilde{K}_a$ denote the preimage of K_a under the projection $\mathrm{Mp}(X_a)(\mathbb{A}) \to \mathrm{Sp}(X_a)(\mathbb{A})$. We will also use $\widetilde{\ }$ to denote the preimages of other subgroups of $\mathrm{Sp}(X_a)(\mathbb{A})$.

Let $\widetilde{m}$ be the isomorphism

$$\widetilde{\mathrm{GL}}_a(\mathbb{A}) \times_{\mu_2} \boldsymbol{G}(X)(\mathbb{A}) \to \widetilde{M}_a(\mathbb{A})$$

that lifts $m : \mathrm{GL}_a(\mathbb{A}) \times G(X)(\mathbb{A}) \to M_a(\mathbb{A})$. Let $\widetilde{\det}$ be the homomorphism

$$\widetilde{\mathrm{GL}}_a(\mathbb{A}) \to \widetilde{\mathrm{GL}}_1(\mathbb{A})$$
$$(x, \zeta) \mapsto (\det(x), \zeta).$$

We keep writing det for the nongenuine homomorphism

$$\widetilde{\mathrm{GL}}_a(\mathbb{A}) \to \mathrm{GL}_1(\mathbb{A})$$
$$(x, \zeta) \mapsto \det(x).$$

Given a nongenuine representation τ of $\widetilde{\mathrm{GL}}_a(\mathbb{A})$, we can twist it by $\chi_\psi \circ \widetilde{\det}$ to get a genuine representation which we denote by $\tau\chi_\psi$.

We remark that there are canonical embeddings of $N_a(\mathbb{A})$ and $\mathrm{Sp}(X_a)(F)$ to $\mathrm{Mp}(X_a)(\mathbb{A})$, so we may regard them as subgroups of $\boldsymbol{G}(X_a)(\mathbb{A})$. Let $\mathcal{A}_\psi^{Q_a}(s, \chi \boxtimes \pi)$ denote the space of $\mathbb{C}$-valued smooth functions f on $N_a(\mathbb{A})M_a(F) \setminus \boldsymbol{G}(X_a)(\mathbb{A})$ such that:

(1) f is right $\widetilde{K}_a$-finite.

(2) For any $x \in \widetilde{\mathrm{GL}}_a(\mathbb{A})$ and $g \in \boldsymbol{G}(X_a)(\mathbb{A})$ we have

$$f(\widetilde{m}(x, I)g) = \chi\chi_\psi(\widetilde{\det}(x))|\det(x)|_{\mathbb{A}_E}^{s+\rho_{Q_a}} f(g).$$

(3) For any fixed $k \in \widetilde{K}_a$, the function $h \mapsto f(\widetilde{m}(I, h)k)$ on $\boldsymbol{G}(X)(\mathbb{A})$ is in the space of π.

To unify notation, we will also write $\mathcal{A}_\psi^{Q_a}(s, \chi \boxtimes \pi)$ for $\mathcal{A}^{Q_a}(s, \chi \boxtimes \pi)$ in the nonmetaplectic case. It should be clear from the context whether we are treating the $\mathrm{Sp}(X)$ case or the $\mathrm{Mp}(X)$ case.

Now return to the general case, so $\boldsymbol{G}(X)$ is one of $\mathrm{Sp}(X)$, $\mathrm{O}(X)$, $\mathrm{U}(X)$ and $\mathrm{Mp}(X)$. Let f_s be a holomorphic section of $\mathcal{A}_\psi^{Q_a}(s, \chi \boxtimes \pi)$. We associate to it the Eisenstein series

$$E_\psi^{Q_a}(g, f_s) := \sum_{\gamma \in Q_a(F) \setminus G(X_a)(F)} f_s(\gamma g).$$

Note that the series is over $\gamma \in Q_a(F) \setminus \mathrm{Sp}(X_a)(F)$ when $\boldsymbol{G}(X) = \mathrm{Mp}(X)$. By Langlands' theory of Eisenstein series [Mœglin and Waldspurger 1995, IV.1], this series is absolutely convergent for $\mathrm{Re}(s) > \rho_{Q_a}$, has meromorphic continuation to the whole s-plane, its poles lie on root hyperplanes and there are only finitely many poles in the positive Weyl chamber. By our identification of $\mathfrak{a}_{M_a}^*$ with $\mathbb{R}$ and the fact that χ is conjugate self-dual, the statements on poles mean that the poles are all real and that there are finitely many poles in the half-plane $\mathrm{Re}(s) > 0$.

We give the setup for any positive integer a, though we will only need $a = 1$ in the statements of our results. However the proofs require "going up the Witt tower" to $\boldsymbol{G}(X_a)$ for a large enough. Since we plan to prove analogous results for quaternionic unitary groups in the future, we keep the setup for general a.

There is a relation between poles of L-functions and the Eisenstein series.

Proposition 3.1. (1) *Assume that the partial L-function $L^S_\psi(s, \pi \times \chi^\vee)$ has its rightmost positive pole at $s = s_0$. Then $E^{Q_1}_\psi(g, f_s)$ has a pole at $s = s_0$.*

(2) *Assume that the partial L-function $L^S_\psi(s, \pi \times \chi^\vee)$ is nonvanishing at $s = \frac{1}{2}$ and is holomorphic for $\mathrm{Re}(s) > \frac{1}{2}$. Assume that*

$$\boldsymbol{G}(X) = \mathrm{U}(X) \text{ with } \dim X \equiv \epsilon_\chi \pmod 2;$$
$$\boldsymbol{G}(X) = \mathrm{O}(X) \text{ with } \dim X \text{ odd};$$
$$\boldsymbol{G}(X) = \mathrm{Mp}(X).$$

Then $E^{Q_1}_\psi(g, f_s)$ has a pole at $s = \frac{1}{2}$.

Remark 3.2. This is [Jiang and Wu 2018, Proposition 2.2] in the symplectic case, [Wu 2022a, Proposition 3.2] in the metaplectic case, [Jiang and Wu 2016, Proposition 2.2] in the unitary case and [Mœglin 1997, Remarque 2] and [Jiang and Wu 2016, Proposition 2.2] in the orthogonal case.

Remark 3.3. The allowed $\boldsymbol{G}(X)$ in item (2) are those for which we have theta dichotomy and epsilon dichotomy (in the local nonarchimedean setting); see [Gan and Ichino 2014, Corollary 9.2, Theorem 11.1].

Remark 3.4. When π is a cuspidal member in a generic global A-packet of $\boldsymbol{G}(X)(\mathbb{A})$, there is a more precise result; see Theorem 6.3 which was proved in [Jiang et al. 2013] and strengthened in [Jiang and Zhang 2020].

We summarize the results on the maximal positive pole of $E^{Q_1}_\psi(g, f_s)$ from [Ginzburg et al. 2009, Theorem 3.1; Jiang and Wu 2016, Theorem 3.1; 2018, Theorem 2,8; Wu 2022a, Theorem 4.2].

Theorem 3.5. *The maximal positive pole of $E^{Q_1}_\psi(g, f_s)$ is of the form*

$$s = \begin{cases} \frac{1}{2}(\dim X + 1 - (2j + \epsilon_\chi)) & \text{if } \boldsymbol{G}(X) = \mathrm{U}(X); \\ \frac{1}{2}(\dim X - 2j) & \text{if } \boldsymbol{G}(X) = \mathrm{O}(X); \\ \frac{1}{2}(\dim X + 2 - 2j) & \text{if } \boldsymbol{G}(X) = \mathrm{Sp}(X); \\ \frac{1}{2}(\dim X + 2 - (2j + 1)) & \text{if } \boldsymbol{G}(X) = \mathrm{Mp}(X); \end{cases} \tag{3-2}$$

where $j \in \mathbb{Z}$ such that

$$\begin{cases} r_X \le 2j + \epsilon_\chi < \dim X + 1 & \text{if } \boldsymbol{G}(X) = \mathrm{U}(X); \\ r_X \le 2j < \dim X & \text{if } \boldsymbol{G}(X) = \mathrm{O}(X); \\ r_X \le 2j < \dim X + 2 & \text{if } \boldsymbol{G}(X) = \mathrm{Sp}(X); \\ r_X \le 2j + 1 < \dim X + 2 & \text{if } \boldsymbol{G}(X) = \mathrm{Mp}(X); \end{cases} \tag{3-3}$$

where r_X denotes the Witt index of X.

Remark 3.6. The middle quantities in the inequalities of (3-3) are, in fact, the lowest occurrence index of π in the global theta lift which depends on χ and ψ; see Theorem 4.4. In some cases, the lowest occurrence index turns out to be independent of ψ.

Remark 3.7. To derive the inequalities $r_X \leq \cdots$, we already need to make use of properties of the global theta correspondence. The other parts of the statements can be derived by relating our Eisenstein series to Siegel–Eisenstein series whose poles are completely known. We note that via the Siegel–Weil formula, Siegel–Eisenstein series are related to global theta correspondence.

4. Theta correspondence

We keep the notation of Section 3. First we define the theta lifts and the two invariants called the first occurrence index and the lowest occurrence index. Then we relate the invariants to poles of our Eisenstein series.

Recall that we have taken an ϵ-skew Hermitian space X over E. Let Y be an ϵ-Hermitian space equipped with the form $\langle \cdot, \cdot \rangle_Y$. We note that $\langle \cdot, \cdot \rangle_Y$ is an F-bilinear pairing

$$\langle \cdot, \cdot \rangle_Y : Y \times Y \to E$$

such that

$$\langle y_2, y_1 \rangle_Y = \epsilon \langle y_1, y_2 \rangle_Y^{\varrho}, \quad \langle y_1 a, y_2 b \rangle_Y = a^{\varrho} \langle y_1, y_2 \rangle_Y b$$

for all $a, b \in E$ and $y_1, y_2 \in Y$. Let $G(Y)$ be its isometry group. We note that $G(X)$ acts on X from the right while $G(Y)$ acts on Y from the left. Let W be the vector space $\mathrm{R}_{E/F}(Y \otimes_E X)$ over F and equip it with the symplectic form

$$\langle \cdot, \cdot \rangle_W : W \times W \to F$$

given by

$$\langle y_1 \otimes x_1, y_2 \otimes x_2 \rangle_W = \mathrm{tr}_{E/F}(\langle y_1, y_2 \rangle_Y \langle x_1, x_2 \rangle_X^{\varrho}).$$

With this set-up, $G(X)$ and $G(Y)$ form a reductive dual pair inside $\mathrm{Sp}(W)$. Let $W = W^+ \oplus W^-$ be a polarization of W. Let $\mathrm{Mp}^{\mathbb{C}^1}(W)(F_v)$ be the $\mathbb{C}^1$-metaplectic extension of $\mathrm{Sp}(W)(F_v)$. Let ω_v denote the Weil representation of $\mathrm{Mp}^{\mathbb{C}^1}(W)(F_v)$ realized on the space of Schwartz functions $\mathcal{S}(W^+(F_v))$. The Weil representation depends on the additive character ψ_v, but we suppress it from notation. When v is archimedean, we actually take the Fock model [Howe 1989] rather than the full Schwartz space and it is a $(\mathfrak{sp}(W)(F_v), \widetilde{K}_{\mathrm{Sp}(W),v})$-module but we abuse language and call it a representation of $\mathrm{Mp}^{\mathbb{C}^1}(W)(F_v)$. When neither $G(X)$ or $G(Y)$ is an odd orthogonal group, by [Kudla 1994] there exists a homomorphism

$$G(X)(F_v) \times G(Y)(F_v) \to \mathrm{Mp}^{\mathbb{C}^1}(W)(F_v)$$

that lifts the obvious map $G(X)(F_v) \times G(Y)(F_v) \to \mathrm{Sp}(W)(F_v)$. In this case, set $\boldsymbol{G}(X) = G(X)$ (resp. $\boldsymbol{G}(Y) = G(Y)$). When $G(X)$ is an odd orthogonal group, we take $\boldsymbol{G}(Y)(F_v)$ to be the metaplectic double cover of $G(Y)(F_v)$ and set $\boldsymbol{G}(X) = G(X)$. Then by [Kudla 1994] there exists a homomorphism

$$G(X)(F_v) \times \boldsymbol{G}(Y)(F_v) \to \mathrm{Mp}^{\mathbb{C}^1}(W)(F_v)$$

that lifts $G(X)(F_v) \times G(Y)(F_v) \to \mathrm{Sp}(W)(F_v)$. The case is analogous when $G(Y)$ is an odd orthogonal group. In any case, we get a homomorphism

$$\iota_v : \boldsymbol{G}(X)(F_v) \times \boldsymbol{G}(Y)(F_v) \to \mathrm{Mp}^{\mathbb{C}^1}(F_v).$$

It should be clear from the context when $\boldsymbol{G}(X)$ (resp. $\boldsymbol{G}(Y)$) refers to a cover group and when it is not truly a cover. In the unitary case, there are many choices of ι_v. Once we fix χ and an additional character χ_2, then ι_v is fixed. This is worked out in great details in [Harris et al. 1996, Section 1]. Our (χ, χ_2) matches (χ_1, χ_2) in [Harris et al. 1996, (0.2)]. We note that Y should be compatible with χ and χ determines the embedding of $G(X)(\mathbb{A})$ into $\mathrm{Mp}^{\mathbb{C}^1}(\mathbb{A})$ whereas X should be compatible with χ_2 and χ_2 determines the embedding of $G(Y)(\mathbb{A})$ into $\mathrm{Mp}^{\mathbb{C}^1}(\mathbb{A})$. By "compatible", we mean $\epsilon_\chi \equiv \dim Y \pmod 2$ (resp. $\epsilon_{\chi_2} \equiv \dim X \pmod 2$); see [Kudla 1994] for more details. We pull back ω_v to $\boldsymbol{G}(X)(F_v) \times \boldsymbol{G}(Y)(F_v)$ via ι_v and still denote the representation by ω_v.

Denote by ι the adelic analogue of ι_v. We also have the (global) Weil representation ω of $\mathrm{Mp}^{\mathbb{C}^1}(\mathbb{A})$ on the Schwartz space $\mathcal{S}(W^+(\mathbb{A}))$ and its pullback via ι to $\boldsymbol{G}(X)(\mathbb{A}) \times \boldsymbol{G}(Y)(\mathbb{A})$.

Then we can define the theta function which will be used as a kernel function. Let

$$\theta_{X,Y}(g, h, \Phi) := \sum_{w \in W^+(F)} \omega(\iota(g, h))\Phi(w)$$

for $g \in \boldsymbol{G}(X)(\mathbb{A})$, $h \in \boldsymbol{G}(Y)(\mathbb{A})$ and $\Phi \in \mathcal{S}(W^+(\mathbb{A}))$. It is absolutely convergent and is an automorphic form on $\boldsymbol{G}(X)(\mathbb{A}) \times \boldsymbol{G}(Y)(\mathbb{A})$. For $f \in \pi$, set

$$\theta_X^Y(f, \Phi) := \int_{[\boldsymbol{G}(X)]} \overline{f(g)} \theta_{X,Y}(g, h, \Phi)\, dg.$$

Note that we write $[\boldsymbol{G}(X)]$ for $G(X)(F)\backslash G(X)(\mathbb{A})$ when $\boldsymbol{G}(X)$ is not metaplectic and $G(X)(F)\backslash \boldsymbol{G}(X)(\mathbb{A})$ or more explicitly $\mathrm{Sp}(X)(F) \setminus \mathrm{Mp}(X)(\mathbb{A})$ when $\boldsymbol{G}(X)$ is metaplectic. This is an automorphic form on $\boldsymbol{G}(Y)(\mathbb{A})$. It depends on χ and χ_2 in the unitary case and when we want to emphasize the dependency, we will write $\theta_{X,\chi_2}^{Y,\chi}(f, \Phi)$. Let $\Theta_X^Y(\pi)$ denote the space of functions spanned by the $\theta_X^Y(f, \Phi)$ and let $\Theta_{X,,\chi_2}^{Y,\chi}(\pi)$ denote the space of functions spanned by the $\theta_{X,\chi_2}^{Y,\chi}(f, \Phi)$ in the unitary case.

From now on assume that Y is anisotropic (possibly zero), so that it sits at the bottom of its Witt tower. Define Y_r to be the ϵ-Hermitian space formed by adjoining r-copies of the hyperbolic plane to Y. These Y_r form the Witt tower of Y. By the tower property [Rallis 1984; Wu 2013], if the theta lift to $\boldsymbol{G}(Y_r)$ is nonzero then the theta lift to $\boldsymbol{G}(Y_{r'})$ is also nonzero for all $r' \geq r$.

Define the first occurrence index of π in the Witt tower of Y to be

$$\mathrm{FO}_X^{Y,\chi}(\pi) := \begin{cases} \min\{\dim Y_r \mid \Theta_{X,\chi_2}^{Y_r,\chi}(\pi) \neq 0\} & \text{if } \boldsymbol{G}(X) = \mathrm{U}(X); \\ \min\{\dim Y_r \mid \Theta_X^{Y_r}(\pi \otimes (\chi \circ \upsilon)) \neq 0\} & \text{if } \boldsymbol{G}(X) = \mathrm{O}(X); \\ \min\{\dim Y_r \mid \Theta_X^{Y_r}(\pi) \neq 0\} & \text{if } \boldsymbol{G}(X) = \mathrm{Sp}(X) \text{ or } \mathrm{Mp}(X). \end{cases} \tag{4-1}$$

Note that it depends on χ but not on χ_2 in the unitary case as changing χ_2 to another compatible one produces only a character twist on $\Theta_{X,\chi_2}^{Y_r,\chi}(\pi)$. For more details, see [Wu 2022b, (1-1)]. In the orthogonal

case, we twist π by $\chi \circ \upsilon$ where υ denotes the spinor norm. If $\boldsymbol{G}(X) = \mathrm{Sp}(X)$ or $\mathrm{Mp}(X)$, we require that $\chi_Y = \chi$ where χ_Y is the quadratic automorphic character of $\mathrm{GL}_1(\mathbb{A})$ associated to Y given by

$$\chi_Y(g) = (g, (-1)^{\dim Y(\dim Y-1)/2} \det\langle \cdot, \cdot \rangle_Y),$$

where $(\cdot, \cdot)$ is the Hilbert symbol.

Define the lowest occurrence index to be

$$\mathrm{LO}_X^{\chi}(\pi) := \min\{\mathrm{FO}_X^{Y,\chi}(\pi) \mid Y \text{ is compatible with } \chi\}, \tag{4-2}$$

when $\boldsymbol{G}(X) = \mathrm{U}(X)$, $\mathrm{Sp}(X)$ or $\mathrm{Mp}(X)$. Here compatibility means that

$$\begin{aligned} \dim Y &\equiv \epsilon_\chi \pmod 2 && \text{if } \boldsymbol{G}(X) = \mathrm{U}(X); \\ \chi_Y &= \chi && \text{if } \boldsymbol{G}(X) = \mathrm{Sp}(X) \text{ or } \mathrm{Mp}(X). \end{aligned} \tag{4-3}$$

Define the lowest occurrence index to be

$$\mathrm{LO}_X^{\chi}(\pi) := \min\{\mathrm{FO}_X^{Y,\chi}(\pi \otimes \mathrm{sgn}_T) \mid T \text{ a set of even number of places of } F\}, \tag{4-4}$$

when $\boldsymbol{G}(X) = \mathrm{O}(X)$.

We have the following relations of the first occurrence (resp. the lowest occurrence) and the poles (resp. the maximal positive pole) of the Eisenstein series; see [Jiang and Wu 2018, Corollary 3.9, Theorem 3.10] for the symplectic case, [Wu 2022a, Corollary 6.3, Theorem 6.4] for the metaplectic case, [Jiang and Wu 2016, Corollaries 3.5 and 3.7] for the unitary case and [Ginzburg et al. 2009, Theorems 5.1 and 1.3] for the orthogonal case.

Theorem 4.1. *Let π be an irreducible cuspidal automorphic representation of $\boldsymbol{G}(X)(\mathbb{A})$. Let χ be a conjugate self-dual automorphic character of $\mathrm{R}_{E/F}\,\mathrm{GL}_1(\mathbb{A})$. Let Y be an anisotropic ϵ-Hermitian space that is compatible with χ in the sense of* (4-3)*. Assume that $\mathrm{FO}_X^{Y,\chi}(\pi) = \dim Y + 2r$. Set*

$$s_0 = \begin{cases} \frac{1}{2}(\dim X + 1 - (\dim Y + 2r)) & \text{if } \boldsymbol{G}(X) = \mathrm{U}(X); \\ \frac{1}{2}(\dim X - (\dim Y + 2r)) & \text{if } \boldsymbol{G}(X) = \mathrm{O}(X); \\ \frac{1}{2}(\dim X + 2 - (\dim Y + 2r)) & \text{if } \boldsymbol{G}(X) = \mathrm{Sp}(X) \text{ or } \mathrm{Mp}(X). \end{cases} \tag{4-5}$$

Assume that $s_0 \neq 0$. If $\boldsymbol{G}(X) = \mathrm{O}(X)$ and $s_0 < 0$, further assume that $\frac{1}{2}\dim X < r < \dim X - 2$. Then $s = s_0$ is a pole of the Eisenstein series $E_\psi^{Q_1}(g, f_s)$ for some choice of $f_s \in \mathcal{A}_\psi^{Q_1}(s, \chi \boxtimes \pi)$.

Remark 4.2. Using the notation from Section 2. The quantity s_0 in (4-5) can be written uniformly as

$$\tfrac{1}{2}(d_{\boldsymbol{G}(X)^\vee} - d_{\boldsymbol{G}(Y_r)^\vee} + 1).$$

Remark 4.3. Note that we always have $r \leq \dim X$. The extra condition when $\boldsymbol{G}(X) = \mathrm{O}(X)$ is to avoid treating period integrals over the orthogonal groups of split binary quadratic forms, as our methods cannot deal with the technicality. Theorem 4.1 allows negative s_0. It is possible to detect nonmaximal poles and negative poles of the Eisenstein series by the first occurrence indices.

Theorem 4.4. *Let π be an irreducible cuspidal automorphic representation of $\boldsymbol{G}(X)(\mathbb{A})$. Let χ be a conjugate self-dual automorphic character of $\mathrm{R}_{E/F}\,\mathrm{GL}_1(\mathbb{A})$. Then the maximal positive pole of $E_\psi^{Q_1}(g, f_s)$ for f_s running over $\mathcal{A}_\psi^{Q_1}(s, \chi \boxtimes \pi)$ is at $s = s_0 \in \mathbb{R}$ if and only if*

$$\mathrm{LO}_X^\chi(\pi) = \begin{cases} \dim X + 1 - 2s_0 & \text{if } \boldsymbol{G}(X) = \mathrm{U}(X); \\ \dim X - 2s_0 & \text{if } \boldsymbol{G}(X) = \mathrm{O}(X); \\ \dim X + 2 - 2s_0 & \text{if } \boldsymbol{G}(X) = \mathrm{Sp}(X) \text{ or } \mathrm{Mp}(X). \end{cases} \tag{4-6}$$

Remark 4.5. Theorem 4.4 does not allow negative s_0.

In Remark 3.7, we mentioned that the part $r_X \le \cdots$ in Theorem 3.5 is proved by using theta correspondence. What we used is that we always have $\mathrm{LO}_X^\chi(\pi) \ge r_X$ by the stable range condition [Rallis 1984, Theorem I.2.1].

5. Application to global Arthur packets

We have derived relations among (χ, b)-factors of global A-parameters, poles of partial L-functions, poles of Eisenstein series and lowest occurrence indices of global theta lifts. Combining these, we have the following implication on global A-packets.

Theorem 5.1. *Let π be an irreducible cuspidal automorphic representation of $\boldsymbol{G}(X)(\mathbb{A})$. Let ϕ_π be its global A-parameter. Let χ be a conjugate self-dual automorphic character of $\mathrm{R}_{E/F}\,\mathrm{GL}_1(\mathbb{A})$. Assume that ϕ_π has a (χ, b)-factor for some positive integer b. Then*

$$b \le \begin{cases} \dim X - r_X & \text{if } \boldsymbol{G}(X) = \mathrm{U}(X); \\ \dim X - r_X - 1 & \text{if } \boldsymbol{G}(X) = \mathrm{O}(X); \\ \dim X - r_X + 1 = \frac{1}{2}\dim X + 1 & \text{if } \boldsymbol{G}(X) = \mathrm{Sp}(X) \text{ or } \mathrm{Mp}(X). \end{cases} \tag{5-1}$$

where r_X denotes the Witt index of X.

Proof. If b is not maximal among all factors (τ, b) appearing in ϕ_π, then $b < \frac{1}{2} d_{\boldsymbol{G}(X)^\vee}$. Then it is clear that b satisfies (5-1). Now we assume that b is maximal among all factors appearing in ϕ_π. By Proposition 2.8, $L^S(s, \pi \times \chi^{-1})$ has its rightmost pole at $s = \frac{1}{2}(b+1)$. Then by Proposition 3.1, $E_\psi^{Q_1}(g, f_s)$ has a pole at $s = \frac{1}{2}(b+1)$ for some choice of f_s. Assume that $s = \frac{1}{2}(b_1+1)$ is the rightmost pole of the Eisenstein series with $b_1 \ge b$. By Theorem 3.5,

$$\tfrac{1}{2}(b_1+1) \le \begin{cases} \frac{1}{2}(\dim X + 1 - r_X) & \text{if } \boldsymbol{G}(X) = \mathrm{U}(X); \\ \frac{1}{2}(\dim X - r_X) & \text{if } \boldsymbol{G}(X) = \mathrm{O}(X); \\ \frac{1}{2}(\dim X + 2 - r_X) & \text{if } \boldsymbol{G}(X) = \mathrm{Sp}(X) \text{ or } \mathrm{Mp}(X); \end{cases}$$

or in other words, b_1 is less than or equal to the quantity on the RHS of (5-1). Using the fact that $b \le b_1$, we get the desired bound for b. □

Remark 5.2. Our result generalizes [Jiang and Liu 2018, Theorem 3.1] for symplectic groups to classical groups and metaplectic groups. In addition, we do not require the assumption on the wave front set in

[loc. cit., Theorem 3.1]. This type of result has been used in [loc. cit., Section 5] to find a Ramanujan bound which measures the departure of the local components of a cuspidal π from being tempered.

The metaplectic case has been treated in [Wu 2022a, Theorem 0.1], though the proof is not written down explicitly. Here we supply the detailed arguments for all classical groups and metaplectic groups uniformly.

The corollary below follows immediately from the theorem.

Corollary 5.3. *The global A-packet Π_ϕ attached to the elliptic global A-parameter ϕ cannot have a cuspidal member if ϕ has a (χ, b)-factor with*

$$b > \begin{cases} \dim_E X - r_X & \text{if } \boldsymbol{G}(X) = \mathrm{U}(X); \\ \dim_F X - r_X - 1 & \text{if } \boldsymbol{G}(X) = \mathrm{O}(X); \\ \dim X - r_X + 1 = \frac{1}{2}\dim X + 1 & \text{if } \boldsymbol{G}(X) = \mathrm{Sp}(X) \text{ or } \mathrm{Mp}(X). \end{cases}$$

6. Generic global *A*-packets

Following the terminology of [Arthur 2013], we say that an elliptic global A-parameter is generic if it is of the form $\phi = \boxplus_{i=1}^{r}(\tau_i, 1)$ and we say a global A-packet is generic if its global A-parameter is generic. Assume that π is a cuspidal member in a generic global A-packet. Then our results can be made more precise. We note that our results for $\mathrm{Mp}(X)$ are conditional on results on normalized intertwining operators; see Assumption 6.1 and Remark 6.2.

First assume that $G(X)$ is quasisplit and that π is globally generic. We explain what we mean by globally generic. We use the same set-up as in [Shahidi 1988, Section 3]. Let B be a Borel subgroup of $G(X)$. Let N denote its unipotent radical and let T be a fixed choice of Levi subgroup of B. Of course, in this case T is a maximal torus of $G(X)$. Let $\bar{F}$ denote an algebraic closure of F. Let Δ denote the set of simple roots of $T(\bar{F})$ in $N(\bar{F})$. Let $\{X_\alpha\}_{\alpha\in\Delta}$ be a $\mathrm{Gal}(\bar{F}/F)$-invariant set of root vectors. Recall that ψ is a fixed nontrivial automorphic character of $\mathbb{A}_F$ which is used in the definitions of the Weil representation and the global A-packets for $\mathrm{Mp}(X)$. It gives rise to generic characters of $N(\mathbb{A})$. We use the one defined as follows. For each place v of F, we define a character $\psi_{N,v}$ of $N(F_v)$. Write an element of $N(F_v)$ as $\prod_{\alpha\in\Delta}\exp(x_\alpha X_\alpha)$ for $x_\alpha \in \bar{F}_v$ such that $\sigma x_\alpha = x_{\sigma\alpha}$ with $\sigma \in \mathrm{Gal}(\bar{F}/F)$. Set

$$\psi_{N,v}\Big(\prod_{\alpha\in\Delta}\exp(x_\alpha X_\alpha)\Big) = \psi_v\Big(\sum_{\alpha\in\Delta} x_\alpha\Big).$$

Let $\psi_N = \otimes_v \psi_{N,v}$. In the $\mathrm{Mp}(X)$ case, we view $N(\mathbb{A})$ as a subgroup of $\mathrm{Mp}(X)(\mathbb{A})$ via the canonical splitting. We require that π is globally generic with respect to the generic character ψ_N of $N(\mathbb{A})$. Thus the notion of global genericity depends on the choice of the generic automorphic character of $N(\mathbb{A})$. However by [Cogdell et al. 2004, Appendix A], the choice has no effect on the L-factors, the ε-factors and the global A-parameter for π in the case of $\boldsymbol{G}(X) = \mathrm{Sp}(X), \mathrm{O}(X), \mathrm{U}(X)$. The case of $\mathrm{Mp}(X)$ is highly dependent on the choice.

When π is globally generic, $b=1$ for every factor (τ, b) in the global A-parameter ϕ_π. This is because the Langlands functorial lift of π is an isobaric sum of conjugate self-dual cuspidal representations of some $\mathrm{R}_{E/F}\,\mathrm{GL}_n(\mathbb{A})$; see Theorem 11.2 of [Ginzburg et al. 2011].

By [Jiang et al. 2013], there is a more precise relation on the poles of L-functions and the poles of Eisenstein series. The set of possible poles of the normalized Eisenstein series is determined by the complete L-function $L(s, \pi\times\chi^\vee)$. From the assumption that π is globally generic, in the right half-plane, $L(s,\pi\times\chi^\vee)$ has at most a simple pole at $s=1$. In fact we only need [loc. cit., Proposition 4.1] rather than the full strength of [loc. cit., Theorem 1.2] which allows the induction datum to be a Speh representation on the general linear group factor of the Levi. By [Jiang and Zhang 2020, Theorem 5.1], [Jiang et al. 2013, Proposition 4.1] can be strengthened to include the case where π is a cuspidal member in a generic global A-packet of $\boldsymbol{G}(X)(\mathbb{A})$ where $\boldsymbol{G}(X)=\mathrm{Sp}(X), \mathrm{O}(X), \mathrm{U}(X)$ does not have to be quasisplit. We rephrase [Jiang et al. 2013, Proposition 4.1] in our context as Theorem 6.3.

First we set up some notation and outline the method for extending [loc. cit., Proposition 4.1] to the case of $\mathrm{Mp}(X)$. Let

$$\rho^+ := \begin{cases} \mathrm{Asai}^\eta \text{ where } \eta=(-1)^{\dim X+1} & \text{if } \boldsymbol{G}(X)=\mathrm{U}(X);\\ \wedge^2 & \text{if } \boldsymbol{G}(X)=\mathrm{O}(X) \text{ with } \dim X \text{ odd or if } \boldsymbol{G}(X)=\mathrm{Mp}(X);\\ \mathrm{Sym}^2 & \text{if } \boldsymbol{G}(X)=\mathrm{O}(X) \text{ with } \dim X \text{ even or if } \boldsymbol{G}(X)=\mathrm{Sp}(X); \end{cases} \tag{6-1}$$

and

$$\rho^- := \begin{cases} \mathrm{Asai}^{-\eta} \text{ where } \eta=(-1)^{\dim X+1} & \text{if } \boldsymbol{G}(X)=\mathrm{U}(X);\\ \mathrm{Sym}^2 & \text{if } \boldsymbol{G}(X)=\mathrm{O}(X) \text{ with } \dim X \text{ odd or if } \boldsymbol{G}(X)=\mathrm{Mp}(X);\\ \wedge^2 & \text{if } \boldsymbol{G}(X)=\mathrm{O}(X) \text{ with } \dim X \text{ even or if } \boldsymbol{G}(X)=\mathrm{Sp}(X). \end{cases} \tag{6-2}$$

The results of [loc. cit.] do not cover the metaplectic case, but the method should generalize without difficulty. We explain the strategy. First the poles of the Eisenstein series are related to those of the intertwining operators

$$M(w_0, \tau|\cdot|^s\boxtimes\pi) : \mathrm{Ind}_{Q_a(\mathbb{A})}^{\boldsymbol{G}(X_a)(\mathbb{A})}(\tau|\cdot|^s\boxtimes\pi) \to \mathrm{Ind}_{Q_a(\mathbb{A})}^{\boldsymbol{G}(X_1)(\mathbb{A})}(\tau|\cdot|^{-s}\boxtimes\pi)$$

where τ is a conjugate self-dual cuspidal automorphic representation of $\mathrm{GL}_a(\mathbb{A}_E)$ and w_0 is the longest Weyl element in $Q_a\backslash G(X_a)/Q_a$. Then define the normalized intertwining operator

$$N(w_0,\tau|\cdot|^s\boxtimes\pi) := \frac{L(s,\pi\times\tau^\vee)L(2s,\tau,\rho^-)}{L(s+1,\pi\times\tau^\vee)L(2s+1,\tau,\rho^-)\varepsilon(s,\pi\times\tau^\vee)\varepsilon(2s,\tau,\rho^-)}\cdot M(w_0,\tau|\cdot|^s\boxtimes\pi). \tag{6-3}$$

The proof of [loc. cit., Proposition 4.1] relies on the key result that the normalized intertwining operator is holomorphic and nonzero for $\mathrm{Re}\, s\ge\frac{1}{2}$. Then it boils down to finding the poles of the normalizing factors or equivalently

$$\frac{L(s,\pi\times\tau^\vee)L(2s,\tau,\rho^-)}{L(s+1,\pi\times\tau^\vee)L(2s+1,\tau,\rho^-)}.$$

Once we have the key result available, we expect to have a version of [loc. cit., Proposition 4.1] for the metaplectic groups. Note that our $\rho^{\pm}$ defined in (6-1) and (6-2) is different from the ρ and ρ^{-} in [loc. cit.].

Then by using an inductive formula, we expect to be able to prove [loc. cit., Theorem 1.2] as well. We hope to supply the details in a future work.

Next we allow $G(X)$ to be non-quasisplit. We assume that π is a cuspidal member in a generic global A-packet of $\boldsymbol{G}(X)$. Then by [Jiang and Zhang 2020, Theorem 5.1], (6-3) is holomorphic and nonzero for $\operatorname{Re} s \geq \frac{1}{2}$ when $\boldsymbol{G}(X) = \mathrm{Sp}(X), \mathrm{O}(X), \mathrm{U}(X)$. Then the proof of [Jiang et al. 2013, Proposition 4.1] goes through verbatim for such π. The proof of [Jiang and Zhang 2020, Theorem 5.1] does not generalize readily to the case of $\mathrm{Mp}(X)$ as the relevant results for $\mathrm{Mp}(X)$ are not available.

Thus we make an assumption on the normalized intertwining operator:

Assumption 6.1. The normalized intertwining operator $N(w_0, \chi|\cdot|^s \boxtimes \pi)$ is holomorphic and nonzero for $\operatorname{Re} s \geq \frac{1}{2}$.

Remark 6.2. This is shown to be true by [Jiang and Zhang 2020, Theorem 5.1] when π is a cuspidal member in a generic global A-packet of $\boldsymbol{G}(X)(\mathbb{A})$ for $\boldsymbol{G}(X) = \mathrm{Sp}(X), \mathrm{O}(X), \mathrm{U}(X)$. Thus this is only a condition when $\boldsymbol{G}(X) = \mathrm{Mp}(X)$.

Theorem 6.3. *Assume Assumption 6.1. Let π be a cuspidal member in a generic global A-packet of $\boldsymbol{G}(X)(\mathbb{A})$. Let χ be a conjugate self-dual automorphic character of $\mathrm{R}_{E/F}\,\mathrm{GL}_1(\mathbb{A})$.*

(1) *Assume $\boldsymbol{G}(X) = \mathrm{U}(X)$ with $\epsilon_\chi \not\equiv \dim X \pmod 2$, $\mathrm{O}(X)$ with $\dim X$ even or $\mathrm{Sp}(X)$. Then $L(s, \pi \times \chi^{\vee})$ has a pole at $s = 1$ if and only if $E^{Q_1}(g, f_s)$ has a pole at $s = 1$ and it is its maximal pole.*

(2) *Assume $\boldsymbol{G}(X) = \mathrm{U}(X)$ with $\epsilon_\chi \equiv \dim X \pmod 2$, $\mathrm{O}(X)$ with $\dim X$ odd or $\mathrm{Mp}(X)$. Then $L(s, \pi \times \chi^{\vee})$ is nonvanishing at $s = \frac{1}{2}$ if and only if $E^{Q_1}_{\psi}(g, f_s)$ has a pole at $s = \frac{1}{2}$ and it is its maximal pole.*

Remark 6.4. The result of [Jiang et al. 2013] involves normalized Eisenstein series, but the normalization has no impact on the positive poles. The following remarks use the notation in [loc. cit.]. We only need the case $b = 1$ in [loc. cit.] which is Proposition 4.1 there. Furthermore we only apply it in the case where τ is a character. The condition that $L(s, \tau, \rho)$ has a pole at $s = 1$ is automatically satisfied by the requirement on our χ that it is conjugate self-dual of parity $(-1)^{N_{\boldsymbol{G}(X)^{\vee}}+1}$; see Section 2, especially Remark 2.3.

The global A-parameter ϕ_π can possibly have a $(\chi, 1)$-factor only when χ satisfies the condition that $L(s, \chi, \rho^{+})$ has a pole at $s = 1$. Due to the parity condition on factors of an elliptic global A-parameter, in some cases, ϕ_π cannot have a $(\chi, 1)$-factor.

Combining our result (Theorem 4.4) on poles of Eisenstein series and lowest occurrence indices with Theorem 6.3 which gives a precise relation between poles of the complete L-function and those of the Eisenstein series, we get

Theorem 6.5. *Assume Assumption 6.1. Let π be a cuspidal member in a generic global A-packet of $\boldsymbol{G}(X)(\mathbb{A})$. Let χ be a conjugate self-dual automorphic character of $\mathrm{R}_{E/F}\,\mathrm{GL}_1(\mathbb{A})$. In each of the following statements, we consider only those $\boldsymbol{G}(X)$ that are listed*:

(1) *Assume that $L(s,\pi\times\chi^\vee)$ has a pole at $s=1$. Then*

$$\mathrm{LO}_X^\chi(\pi)=\begin{cases}\dim X-1 & \text{if } \boldsymbol{G}(X)=\mathrm{U}(X) \text{ and } \epsilon_\chi\not\equiv\dim X \pmod 2;\\ \dim X-2 & \text{if } \boldsymbol{G}(X)=\mathrm{O}(X) \text{ with } \dim X \text{ even};\\ \dim X & \text{if } \boldsymbol{G}(X)=\mathrm{Sp}(X).\end{cases}$$

(2) *Assume that $L(s,\pi\times\chi^\vee)$ does not have a pole at $s=1$. Then*

$$\mathrm{LO}_X^\chi(\pi)\geq\begin{cases}\dim X+1 & \text{if } \boldsymbol{G}(X)=\mathrm{U}(X) \text{ and } \epsilon_\chi\not\equiv\dim X \pmod 2;\\ \dim X & \text{if } \boldsymbol{G}(X)=\mathrm{O}(X) \text{ with } \dim X \text{ even};\\ \dim X+2 & \text{if } \boldsymbol{G}(X)=\mathrm{Sp}(X).\end{cases}$$

(3) *Assume $L\left(\frac{1}{2},\pi\times\chi^\vee\right)\neq 0$. Then*

$$\mathrm{LO}_X^\chi(\pi)=\begin{cases}\dim X & \text{if } \boldsymbol{G}(X)=\mathrm{U}(X) \text{ and } \epsilon_\chi\equiv\dim X \pmod 2;\\ \dim X-1 & \text{if } \boldsymbol{G}(X)=\mathrm{O}(X) \text{ with } \dim X \text{ odd};\\ \dim X+1 & \text{if } \boldsymbol{G}(X)=\mathrm{Mp}(X).\end{cases}$$

(4) *Assume $L\left(\frac{1}{2},\pi\times\chi^\vee\right)=0$. Then*

$$\mathrm{LO}_X^\chi(\pi)\geq\begin{cases}\dim X+2 & \text{if } \boldsymbol{G}(X)=\mathrm{U}(X) \text{ and } \epsilon_\chi\equiv\dim X \pmod 2;\\ \dim X+1 & \text{if } \boldsymbol{G}(X)=\mathrm{O}(X) \text{ with } \dim X \text{ odd};\\ \dim X+3 & \text{if } \boldsymbol{G}(X)=\mathrm{Mp}(X).\end{cases}$$

Remark 6.6. By the conservation relation for local theta correspondence [Sun and Zhu 2015], there always exists an ϵ-Hermitian space $Z_{[v]}$ over E_v of dimension given by the RHS of the equalities in items (1), (3) such that the local theta lift of π_v to $\boldsymbol{G}(Z_{[v]})$ is nonvanishing. Thus in the case of items (2), (4) and $\boldsymbol{G}(X)\neq\mathrm{O}(X)$, the collection $\{Z_{[v]}\}_v$ for v running over all places of F is always incoherent, i.e., there does not exist an ϵ-Hermitian space Z over E such that the localization Z_v is isomorphic to $Z_{[v]}$ for all v. In the case of items (2), (4) and $\boldsymbol{G}(X)=\mathrm{O}(X)$, we have a nontrivial theta lift of $\pi_v\otimes(\chi_v\circ\upsilon_v)\otimes(\eta_{[v]}\circ\det)$ to $\boldsymbol{G}(Z_{[v]})$ for $\eta_{[v]}$ being the trivial character or the sign character for each place v of F, but the collection $\{\eta_{[v]}\}_v$ is incoherent, i.e., there does not exist an automorphic character η of $\mathbb{A}_F^\times$ such that the localization η_v is equal to $\eta_{[v]}$ for all v; see the definitions of first occurrence (4-1) and lowest occurrence (4-4) for $\mathrm{O}(X)$ for why we have a $(\chi_v\circ\upsilon_v)$-twist. We also note that when π is an irreducible cuspidal automorphic representation and $L\left(\frac{1}{2},\pi\times\chi^\vee\right)=0$, it is conjectured that there is an arithmetic version of the Rallis inner product formula which says that the conjectural Beilinson–Bloch height pairing of arithmetic theta lifts (which are cycles on Shimura varieties constructed from an incoherent collection of ϵ-Hermitian spaces) gives the derivative $L'\left(\frac{1}{2},\pi\times\chi^\vee\right)$ up to some ramified factors and some abelian L-functions. The low rank cases have been proved in [Kudla et al. 2006; Liu 2011a; 2011b]. More recently, the cases

of unitary groups of higher rank have been proved in [Li and Liu 2021; 2022], conditional on hypothesis of the modularity of Kudla's generating functions of special cycles.

In terms of "(χ, b)"-factors, we have

Theorem 6.7. *Let $\boldsymbol{G}(X) = \mathrm{U}(X)$ with $\epsilon_\chi \not\equiv \dim X \pmod 2$, $\boldsymbol{G}(X) = \mathrm{O}(X)$ with $\dim X$ even or $\boldsymbol{G}(X) = \mathrm{Sp}(X)$. Let π be a cuspidal member in a generic global A-packet of $\boldsymbol{G}(X)(\mathbb{A})$. Let χ be a conjugate self-dual automorphic character of $\mathrm{R}_{E/F}\,\mathrm{GL}_1(\mathbb{A})$. Then the following are equivalent*:

(1) *The global A-parameter ϕ of π has a $(\chi, 1)$-factor.*

(2) *The complete L-function $L(s, \pi \times \chi^\vee)$ has a pole at $s = 1$ (and this is its maximal pole).*

(3) *The Eisenstein series $E^{Q_1}(g, f_s)$ has a pole at $s = 1$ for some choice of $f_s \in \mathcal{A}^{Q_1}(s, \chi \boxtimes \pi)$ (and this is its maximal pole).*

(4) *The lowest occurrence index $\mathrm{LO}^\chi_X(\pi)$ is*

$$\begin{cases} \dim X - 1 & \text{if } \boldsymbol{G}(X) = \mathrm{U}(X); \\ \dim X - 2 & \text{if } \boldsymbol{G}(X) = \mathrm{O}(X); \\ \dim X & \text{if } \boldsymbol{G}(X) = \mathrm{Sp}(X). \end{cases}$$

Remark 6.8. The statements that the poles are maximal are automatic since π lies in a generic global A-packet. We note that when $\boldsymbol{G}(X) = \mathrm{U}(X)$ with $\epsilon_\chi \equiv \dim X \pmod 2$, $\boldsymbol{G}(X) = \mathrm{O}(X)$ with $\dim X$ odd or $\boldsymbol{G}(X) = \mathrm{Mp}(X)$, ϕ_π cannot have a $(\chi, 1)$-factor as the parity condition is not satisfied.

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References

[Arthur 2013] J. Arthur, *The endoscopic classification of representations: orthogonal and symplectic groups*, Amer. Math. Soc. Colloq. Publ. **61**, Amer. Math. Soc., Providence, RI, 2013. MR Zbl

[Atobe and Gan 2017] H. Atobe and W. T. Gan, "On the local Langlands correspondence and Arthur conjecture for even orthogonal groups", *Represent. Theory* **21** (2017), 354–415. MR Zbl

[Blomer and Brumley 2011] V. Blomer and F. Brumley, "On the Ramanujan conjecture over number fields", *Ann. of Math.* (2) **174**:1 (2011), 581–605. MR Zbl

[Chen and Zou 2021] R. Chen and J. Zou, "Arthur's multiplicity formula for even orthogonal and unitary groups", preprint, 2021. arXiv 2103.07956

[Cogdell 2000] J. W. Cogdell, "Notes on L-functions for GL_n", Lectures given at the School on Automorphic Forms on $\mathrm{GL}(n)$, The Abdus Salam International Centre for Theoretical Physics, 2000, available at https://indico.ictp.it/event/a02086.

[Cogdell et al. 2004] J. W. Cogdell, H. H. Kim, I. I. Piatetski-Shapiro, and F. Shahidi, "Functoriality for the classical groups", *Publ. Math. Inst. Hautes Études Sci.* **99** (2004), 163–233. MR Zbl

[Gan and Ichino 2014] W. T. Gan and A. Ichino, "Formal degrees and local theta correspondence", *Invent. Math.* **195**:3 (2014), 509–672. MR Zbl

[Gan and Ichino 2018] W. T. Gan and A. Ichino, "The Shimura–Waldspurger correspondence for Mp_{2n}", *Ann. of Math.* (2) **188**:3 (2018), 965–1016. MR Zbl

[Gan et al. 2012] W. T. Gan, B. H. Gross, and D. Prasad, "Symplectic local root numbers, central critical L values, and restriction problems in the representation theory of classical groups", pp. 1–109 in *Sur les conjectures de Gross et Prasad*, *I*, Astérisque **346**, Soc. Math. France, Paris, 2012. MR Zbl

[Gan et al. 2014] W. T. Gan, Y. Qiu, and S. Takeda, "The regularized Siegel–Weil formula (the second term identity) and the Rallis inner product formula", *Invent. Math.* **198**:3 (2014), 739–831. MR Zbl

[Ginzburg et al. 2009] D. Ginzburg, D. Jiang, and D. Soudry, "Poles of L-functions and theta liftings for orthogonal groups", *J. Inst. Math. Jussieu* **8**:4 (2009), 693–741. MR Zbl

[Ginzburg et al. 2011] D. Ginzburg, S. Rallis, and D. Soudry, *The descent map from automorphic representations of* $\mathrm{GL}(n)$ *to classical groups*, World Sci., Hackensack, NJ, 2011. MR Zbl

[Goldberg 1994] D. Goldberg, "Some results on reducibility for unitary groups and local Asai L-functions", *J. Reine Angew. Math.* **448** (1994), 65–95. MR Zbl

[Harris et al. 1996] M. Harris, S. S. Kudla, and W. J. Sweet, "Theta dichotomy for unitary groups", *J. Amer. Math. Soc.* **9**:4 (1996), 941–1004. MR Zbl

[Howe 1989] R. Howe, "Transcending classical invariant theory", *J. Amer. Math. Soc.* **2**:3 (1989), 535–552. MR Zbl

[Ishimoto 2023] H. Ishimoto, "The endoscopic classification of representations of non-quasi-split odd special orthogonal groups", preprint, 2023. arXiv 2301.12143

[Jacquet and Shalika 1976] H. Jacquet and J. A. Shalika, "A non-vanishing theorem for zeta functions of GL_n", *Invent. Math.* **38**:1 (1976), 1–16. MR Zbl

[Jacquet et al. 1983] H. Jacquet, I. I. Piatetskii-Shapiro, and J. A. Shalika, "Rankin–Selberg convolutions", *Amer. J. Math.* **105**:2 (1983), 367–464. MR Zbl

[Jiang 2014] D. Jiang, "Automorphic integral transforms for classical groups, I: Endoscopy correspondences", pp. 179–242 in *Automorphic forms and related geometry*: *assessing the legacy of I. I. Piatetski-Shapiro* (New Haven, CT, 2012), edited by J. W. Cogdell et al., Contemp. Math. **614**, Amer. Math. Soc., Providence, RI, 2014. MR Zbl

[Jiang and Liu 2018] D. Jiang and B. Liu, "Fourier coefficients and cuspidal spectrum for symplectic groups", pp. 211–244 in *Geometric aspects of the trace formula* (Elmau, Germany, 2016), edited by W. Müller et al., Springer, 2018. MR Zbl

[Jiang and Wu 2016] D. Jiang and C. Wu, "On (χ, b)-factors of cuspidal automorphic representations of unitary groups, I", *J. Number Theory* **161** (2016), 88–118. MR Zbl

[Jiang and Wu 2018] D. Jiang and C. Wu, "Periods and (χ, b)-factors of cuspidal automorphic forms of symplectic groups", *Israel J. Math.* **225**:1 (2018), 267–320. MR Zbl

[Jiang and Zhang 2020] D. Jiang and L. Zhang, "Arthur parameters and cuspidal automorphic modules of classical groups", *Ann. of Math.* (2) **191**:3 (2020), 739–827. MR Zbl

[Jiang et al. 2013] D. Jiang, B. Liu, and L. Zhang, "Poles of certain residual Eisenstein series of classical groups", *Pacific J. Math.* **264**:1 (2013), 83–123. MR Zbl

[Kaletha et al. 2014] T. Kaletha, A. Minguez, S. W. Shin, and P.-J. White, "Endoscopic classification of representations: inner forms of unitary groups", preprint, 2014. arXiv 1409.3731

[Kim 2003] H. H. Kim, "Functoriality for the exterior square of GL_4 and the symmetric fourth of GL_2", *J. Amer. Math. Soc.* **16**:1 (2003), 139–183. MR Zbl

[Kudla 1994] S. S. Kudla, "Splitting metaplectic covers of dual reductive pairs", *Israel J. Math.* **87**:1-3 (1994), 361–401. MR Zbl

[Kudla and Rallis 1994] S. S. Kudla and S. Rallis, "A regularized Siegel–Weil formula: the first term identity", *Ann. of Math.* (2) **140**:1 (1994), 1–80. MR Zbl

[Kudla et al. 2006] S. S. Kudla, M. Rapoport, and T. Yang, *Modular forms and special cycles on Shimura curves*, Annals of Mathematics Studies **161**, Princeton University Press, 2006. MR Zbl

[Li and Liu 2021] C. Li and Y. Liu, "Chow groups and L-derivatives of automorphic motives for unitary groups", *Ann. of Math.* (2) **194**:3 (2021), 817–901. MR Zbl

[Li and Liu 2022] C. Li and Y. Liu, "Chow groups and L-derivatives of automorphic motives for unitary groups, II", *Forum Math. Pi* **10** (2022), art. id. e5. MR Zbl

[Liu 2011a] Y. Liu, "Arithmetic theta lifting and L-derivatives for unitary groups, I", *Algebra Number Theory* **5**:7 (2011), 849–921. MR Zbl

[Liu 2011b] Y. Liu, "Arithmetic theta lifting and L-derivatives for unitary groups, II", *Algebra Number Theory* **5**:7 (2011), 923–1000. MR Zbl

[Mœglin 1997] C. Mœglin, "Non nullité de certains relêvements par séries théta", *J. Lie Theory* **7**:2 (1997), 201–229. MR Zbl

[Mœglin and Waldspurger 1995] C. Mœglin and J.-L. Waldspurger, *Spectral decomposition and Eisenstein series*, Cambridge Tracts in Mathematics **113**, Cambridge University Press, 1995. MR

[Mok 2015] C. P. Mok, *Endoscopic classification of representations of quasi-split unitary groups*, Mem. Amer. Math. Soc. **1108**, Amer. Math. Soc., Providence, RI, 2015. MR Zbl

[Rallis 1984] S. Rallis, "On the Howe duality conjecture", *Compositio Math.* **51**:3 (1984), 333–399. MR Zbl

[Shahidi 1978] F. Shahidi, "Functional equation satisfied by certain L-functions", *Compositio Math.* **37**:2 (1978), 171–207. MR Zbl

[Shahidi 1980] F. Shahidi, "On nonvanishing of L-functions", *Bull. Amer. Math. Soc.* (*N.S.*) **2**:3 (1980), 462–464. MR Zbl

[Shahidi 1988] F. Shahidi, "On the Ramanujan conjecture and finiteness of poles for certain L-functions", *Ann. of Math.* (2) **127**:3 (1988), 547–584. MR Zbl

[Shahidi 2010] F. Shahidi, *Eisenstein series and automorphic L-functions*, Amer. Math. Soc. Colloq. Publ. **58**, Amer. Math. Soc., Providence, RI, 2010. MR Zbl

[Sun and Zhu 2015] B. Sun and C.-B. Zhu, "Conservation relations for local theta correspondence", *J. Amer. Math. Soc.* **28**:4 (2015), 939–983. MR Zbl

[Weil 1964] A. Weil, "Sur certains groupes d'opérateurs unitaires", *Acta Math.* **111** (1964), 143–211. MR Zbl

[Wu 2013] C. Wu, "Irreducibility of theta lifting for unitary groups", *J. Number Theory* **133**:10 (2013), 3296–3318. MR Zbl

[Wu 2017] C. Wu, "A critical case of Rallis inner product formula", *Sci. China Math.* **60**:2 (2017), 201–222. MR Zbl

[Wu 2022a] C. Wu, "Periods and (χ, b)-factors of cuspidal automorphic forms of metaplectic groups", *J. Number Theory* **241** (2022), 262–296. MR Zbl

[Wu 2022b] C. Wu, "Poles of Eisenstein series and theta lifts for unitary groups", *Pacific J. Math.* **317**:1 (2022), 207–237. MR Zbl

[Yamana 2014] S. Yamana, "L-functions and theta correspondence for classical groups", *Invent. Math.* **196**:3 (2014), 651–732. MR Zbl

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