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On the ordinary Hecke orbit conjecture

Pol van Hoften

We prove the ordinary Hecke orbit conjecture for Shimura varieties of Hodge type at primes of good reduction. We make use of the global Serre–Tate coordinates of Chai as well as recent results of D'Addezio about the monodromy groups of isocrystals. The new ingredients in this paper are a general monodromy theorem for Hecke-stable subvarieties for Shimura varieties of Hodge type, and a rigidity result for the formal completions of ordinary Hecke orbits. Along the way, we show that classical Serre–Tate coordinates can be described using unipotent formal groups, generalising a result of Howe.

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1. Introduction

Let $\mathcal{A}_{g,n}$ be the moduli space of g-dimensional principally polarised abelian varieties (A, λ) with level $n \geq 3$ structure over $\overline{\mathbb{F}}_p$, for a prime number p coprime to n. Recall that there are finite étale prime-to-p Hecke correspondences from $\mathcal{A}_{g,n}$ to itself, and that two points $x, y \in \mathcal{A}_{g,n}(\overline{\mathbb{F}}_p)$ are said to be *in the same prime-to-p Hecke orbit* if they share a preimage under one of these correspondences. Recall the following result of Chai:

Theorem [Chai 1995]. *Let $x \in \mathcal{A}_{g,n}(\overline{\mathbb{F}}_p)$ be a point corresponding to an ordinary principally polarised abelian variety. Then the prime-to-p Hecke orbit of x is Zariski dense in $\mathcal{A}_{g,n}$.*

Our main result is a generalisation of this theorem to Shimura varieties of Hodge type. To state it, we will first introduce some notation.

1.1. ***Main results.*** Let (G, X) be a Shimura datum of Hodge type with reflex field E and let p be a prime number. Let $K_p \subset G(\mathbb{Q}_p)$ be a hyperspecial subgroup and let $K^p \subset G(\mathbb{A}_f^p)$ be a sufficiently small compact open subgroup. Let Sh_G be the special fibre of the canonical integral model of the Shimura variety of level K^pK_p at a prime v above p of E, constructed in [Kisin 2010; Kim and Madapusi Pera 2016].

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Let E_v be the v-adic completion of E, which is a finite extension of $\mathbb{Q}_p$. There is a closed immersion $\mathrm{Sh}_{G,\overline{\mathbb{F}}_p} \to \mathcal{A}_{g,n}$ for some n (see [Xu 2020]), and the intersection $\mathrm{Sh}_{G,\mathrm{ord}}$ of the ordinary locus of $\mathcal{A}_{g,n}$ with Sh_G is nonempty if and only if $E_v = \mathbb{Q}_p$ (see [Lee 2018, Corollary 1.0.2]). Recall that there are prime-to-p Hecke correspondences over Sh_G, which we use to define prime-to-p Hecke orbits.

Theorem I. *If $E_v = \mathbb{Q}_p$, then the prime-to-p Hecke orbit of a point $x \in \mathrm{Sh}_{G,\mathrm{ord}}(\overline{\mathbb{F}}_p)$ is Zariski dense in Sh_G.*

Theorem I generalises results of Maulik–Shankar–Tang, see [Maulik et al. 2022], who deal with GSpin Shimura varieties associated to a quadratic space over $\mathbb{Q}$ and $GU(1, n-1)$ Shimura varieties associated to imaginary quadratic fields E with p split in E; their methods are completely disjoint from ours. There is also work of Shankar [2016] for Shimura varieties of type C, using a group-theoretic version of Chai's strategy of using hypersymmetric points and reducing to the case of Hilbert modular varieties. Shankar crucially proves that the Hodge map $\mathrm{Sh}_{G,\overline{\mathbb{F}}_p} \to \mathcal{A}_{g,n}$ is a closed immersion over the ordinary locus via canonical liftings, whereas we use work of Xu [2020].

Last we mention work of Zhou [2023], who proves the Hecke orbit conjecture for the μ-ordinary locus of certain quaternionic Shimura varieties. Our results do not imply his, but there is some overlap between the cases that we cover.

A fairly direct consequence of Theorem I is a density result for prime-to-p Hecke orbits of an $\overline{\mathbb{F}}_p$-point in the μ-ordinary locus of a Shimura variety of abelian type, at primes v above p of the reflex field E where $E_v = \mathbb{Q}_p$, see Corollary 6.4.1.

1.2. ***Monodromy theorems.*** An important ingredient in our proof is an ℓ-adic monodromy theorem for prime-to-p Hecke-stable subvarieties of special fibres of Shimura varieties, in the style of [Chai 2005, Corollary 3.5]. To state it, let (G, X) be as above and assume for simplicity that G^{ad} is simple over $\mathbb{Q}$. Let V_ℓ be the rational ℓ-adic Tate module of the abelian variety A over Sh_G coming from the map $\mathrm{Sh}_{G,\overline{\mathbb{F}}_p} \to \mathcal{A}_{g,n}$; it is an ℓ-adic local system of rank $2g$.

Theorem II. *Let $Z \subset \mathrm{Sh}_G$ be a smooth locally closed subvariety that is stable under the prime-to-p Hecke operators. Suppose that Z is not contained in the smallest Newton stratum of Sh_G. Let $z \in Z(\overline{\mathbb{F}}_p)$ and let $Z^\circ \subset Z_{\overline{\mathbb{F}}_p}$ be the connected component of Z containing z. Then the neutral component $\mathbb{M}_{\mathrm{geom}}$ of the Zariski closure of the image of the monodromy representation*

$$\rho_{\ell,\mathrm{geom}} : \pi_1^{\acute{e}t}(Z^\circ, z) \to \mathrm{GL}_{2g}(\mathbb{Q}_\ell)$$

corresponding to V_ℓ, is isomorphic to $G^{\mathrm{der}}_{\mathbb{Q}_\ell}$.

This generalises work of Chai [2005] in the Siegel case and others [Kasprowitz 2012; Xiao Xiao 2020] in the PEL case.

In the body of the paper, we work with the integral models of Shimura varieties of Hodge type of level $K_p \subset G(\mathbb{Q}_p)$ constructed in [KMS 2022]. Here K_p is not required to be hyperspecial, for example it is allowed to be any (connected) parahoric subgroup. Our results, namely Theorem 3.2.5 and Corollary 3.2.6, are proved under the assumption that Hypothesis 2.3.1 holds. This hypothesis holds for example when $G_{\mathbb{Q}_p}$ is quasi-split and has no factors of type D, or when K_p is hyperspecial.

We also prove results about irreducible components of smooth locally closed subvarieties that are stable under the prime-to-p Hecke operators, in the style of [Chai 2005, Proposition 4.4], see Theorem 3.4.10. These results are used to prove irreducibility of Ekedahl–Oort strata in [van Hoften 2020].

1.2.1. *An overview of the proof of Theorem II.* Since $Z^\circ \subset \mathrm{Sh}_G$ is defined over a finite field k we can write it as $Z_k^\circ \otimes_k \overline{\mathbb{F}}_p$. We can then consider the Zariski closure $\mathbb{M}$ of the image of

$$\rho_\ell : \pi_1^{\text{ét}}(Z_k^\circ, z) \to \mathrm{GL}_{2g}(\mathbb{Q}_\ell).$$

An argument from [Chai 2005] proves that $\mathbb{M}$ is (isomorphic to) a normal subgroup of $G_{\mathbb{Q}_\ell}$. If $G_{\mathbb{Q}_\ell}^{\text{ad}}$ was a simple-group, then we would be done if we could show that $\mathbb{M}$ was not central in $G_{\mathbb{Q}_\ell}$. However, in general there are *no* primes ℓ such that $G_{\mathbb{Q}_\ell}^{\text{ad}}$ is simple and so at this point we have to deviate from the strategy of [Chai 2005].

Instead, we control $\mathbb{M}$ by studying the centraliser $I_{x,\ell} \subset G_{\mathbb{Q}_\ell}$ of the image of Frobenius elements $\mathrm{Frob}_x \in \pi_1^{\text{ét}}(Z_k^\circ, z)$ corresponding to points $x \in Z_k(\mathbb{F}_q)$. Since the paper [KMS 2022] makes an in-depth study of these Frobenius elements, we can make use of their results about these centralisers. For example, if x is not contained in the basic locus, then they prove that Frob_x is not central. To get more precise results, we need to know that the element $\mathrm{Frob}_x \in G(\mathbb{Q}_\ell)$ is *defined over* $\mathbb{Q}$, which is what Hypothesis 2.3.1 makes precise.

In this way we can show that $\mathbb{M} \subset G_{\mathbb{Q}_\ell}$ is a normal subgroup that surjects onto $G_{\mathbb{Q}_\ell}^{\text{ad}}$. The result about $\mathbb{M}_{\text{geom}} \subset \mathbb{M}$ will be deduced from this.

1.3. ***A sketch of the proof of Theorem I.*** Let $x \in \mathrm{Sh}_G(\overline{\mathbb{F}}_p)$ be an ordinary point, and let Z be the Zariski closure inside $\mathrm{Sh}_{G,\text{ord}}$ of the prime-to-p Hecke orbit of x. Let $y \in Z(\overline{\mathbb{F}}_p)$ be a smooth point of Z. Recall that it follows from the theory of Serre–Tate coordinates that the formal completion $\mathcal{A}_{g,n}^{/y}$ of $\mathcal{A}_{g,n}$ at y is a formal torus. A special case of the main result of [Shankar and Zhou 2021] tells us that

$$S^{/y} := \mathrm{Sh}_{G,\overline{\mathbb{F}}_p}^{/y} \subset \mathcal{A}_{g,n}^{/y}$$

is a formal subtorus. Work of Chai on the deformation theory of ordinary p-divisible groups [Chai 2003] tells us that the dimension of the smallest formal subtorus of $S^{/y}$ containing $Z^{/y}$, is encoded in the unipotent radical of the p-adic monodromy group of the isocrystal $\mathcal{M}$ associated to the universal abelian variety A over Z.

Using Theorem II and results of D'Addezio [2020; 2023], we compute the monodromy group of $\mathcal{M}$ over Z. It follows from this computation that the smallest formal subtorus of $S^{/y}$ containing $Z^{/y}$ is equal to $S^{/y}$.

We conclude by proving that the formal completion $Z^{/y}$ is a formal subtorus of $S^{/y}$. By the rigidity theorem for p-divisible formal groups of [Chai 2008], it suffices to give a representation-theoretic description of the Dieudonné module of $S^{/y}$. Unfortunately, the description of the subtorus $S^{/y}$ coming out of the work of Shankar and Zhou [2021] does not readily lend itself to understanding its Dieudonné module.

Instead, we give a different proof that $S^{/y}$ is a subtorus of $\mathcal{A}_{g,n}^{/y}$. We do this by giving a new description of Serre–Tate coordinates in terms of actions of formal unipotent groups on Rapoport–Zink spaces, generalising results of Howe [2020] in the case $g = 1$. Once we have this perspective, the results of Kim [2019] give an explicit description of the Dieudonné module of the torus $\mathcal{A}_{g,n}^{/y}$ as well as the Dieudonné module of the subtorus $S^{/y}$.

1.4. ***Outline.*** Sections 2 and 3 form the first part of the paper and work in a more general setting than the rest of the paper. In Section 2 we introduce the integral models of Shimura varieties of Hodge type constructed in [KMS 2022]. We recall results and notation from [loc. cit.], in particular, about the Frobenius elements and their centralisers associated to $\overline{\mathbb{F}}_p$-points of these models. In Section 3 we prove monodromy theorems for Hecke-stable subvarieties of the special fibres of these integral models, combining results of [KMS 2022] with ideas of Chai [2005].

Section 4 is a standalone section on Serre–Tate coordinates. In it, we show that the classical Serre–Tate coordinates, as described in [Katz 1981], can be reinterpreted using actions of unipotent formal groups as in [Howe 2020]. This section should be of independent interest.

In Section 5, we specialise to the smooth canonical integral models of Shimura varieties of Hodge type at hyperspecial level, and we moreover assume that the ordinary locus is nonempty. We reprove a result of [Shankar and Zhou 2021], which states that the formal completion of the ordinary locus gives a subtorus of the Serre–Tate torus, and give a group-theoretic description of its Dieudonné module. At the end of this section we also give a short interlude on strongly nontrivial actions of algebraic groups on isocrystals, which we will need to confirm the hypotheses of the rigidity theorem of [Chai 2008].

In Section 6, we put everything together and prove Theorem I. We end by deducing a result for Shimura varieties of abelian type.

2. Integral models of Shimura varieties of Hodge type

Let (G, X) be a Shimura datum of Hodge type. In this section we follow [KMS 2022, Section 1.3] and construct integral models for the Shimura varieties associated to (G, X) in a very general situation. The main goal is to introduce various Frobenius elements $\gamma_{x,m,\ell} \in G(\mathbb{Q}_\ell)$ associated to $\mathbb{F}_{q^m}$-points of these integral models, and to discuss result of [KMS 2022] about their centralisers $I_{x,m,\ell}$. We end by introducing Hypothesis 2.3.1, which will be assumed throughout Section 3, and prove that it holds under minor assumptions.

2.0.1. *Hodge cocharacters.* If (G, X) is a Shimura datum, then for each $x \in X$ there is a cocharacter $\mu_x : \mathbb{G}_{m,\mathbb{C}} \to G_{\mathbb{C}}$, see [KMS 2022, Section 1.2.3] for the precise definition. The $G(\mathbb{C})$-conjugacy class of μ_x does not depend on the choice of x and we will write $\{\mu_X\}$ for this conjugacy class, and denote it by $\{\mu\}$ if X is clear from context. This conjugacy class of cocharacters is defined over a number field $E \subset \mathbb{C}$, called the reflex field.

2.1. ***The construction of integral models.*** For a symplectic space (V, ψ) over $\mathbb{Q}$ we write $\mathcal{G}_V := \mathrm{GSp}(V, \psi)$ for the group of symplectic similitudes of V over $\mathbb{Q}$. It admits a Shimura datum $\mathcal{H}_V$ consisting of the union of the Siegel upper and lower half spaces. Let (G, X) be a Shimura datum of Hodge type with reflex field E and let $(G, X) \to (\mathcal{G}_V, \mathcal{H}_V)$ be a Hodge embedding.

Fix a prime p and choose a $\mathbb{Z}_{(p)}$-lattice $V_{(p)} \subset V$ on which ψ is $\mathbb{Z}_{(p)}$-valued, and write $V_p = V_{(p)} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p$. Write $\mathcal{K}_p \subset \mathcal{G}_V(\mathbb{Q}_p)$ for the stabiliser of V_p in $\mathcal{G}_V(\mathbb{Q}_p)$, and similarly write K_p for the stabiliser of V_p in $G(\mathbb{Q}_p)$.[1] For every sufficiently small compact open subgroup $K^p \subset G(\mathbb{A}_f^p)$ we can find $\mathcal{K}^p \subset \mathcal{G}_V(\mathbb{A}_f^p)$ such that the Hodge embedding induces a closed immersion (see [Kisin 2010, Lemma 2.1.2])

$$\mathbf{Sh}_K(G, X) \to \mathbf{Sh}_{\mathcal{K}}(\mathcal{G}_V, \mathcal{H}_V)_E$$

of Shimura varieties of level $K = K^p K_p$ and $\mathcal{K} = \mathcal{K}^p \mathcal{K}_p$, respectively. We let $\mathcal{S}_{\mathcal{K}}$ over $\mathbb{Z}_{(p)}$ be the moduli-theoretic integral model of $\mathbf{Sh}_{\mathcal{K}}(\mathcal{G}_V, \mathcal{H}_V)$; it is a moduli space of (weakly) polarised abelian schemes (A, λ) up to prime-to-p isogeny with level $\mathcal{K}^p$-structure.

Fix a prime $v \mid p$ of E and let

$$\mathscr{S}_K := \mathscr{S}_K(G, X) \to \mathcal{S}_{\mathcal{K}} \otimes_{\mathbb{Z}_{(p)}} \mathcal{O}_{E,(v)}$$

be the normalisation of the Zariski closure of $\mathbf{Sh}_K(G, X)$ in $\mathcal{S}_{\mathcal{K}} \otimes_{\mathbb{Z}_{(p)}} \mathcal{O}_{E,(v)}$. This construction is compatible with changing the level away from p and we define

$$\mathcal{S}_{\mathcal{K}_p} := \varprojlim_{\mathcal{K}^p \subset \mathcal{G}_V(\mathbb{A}_f^p)} \mathcal{S}_{\mathcal{K}^p \mathcal{K}_p}, \qquad \mathscr{S}_{K_p} := \varprojlim_{K^p \subset G(\mathbb{A}_f^p)} \mathscr{S}_{K^p K_p}.$$

Then, as discussed in [KMS 2022, Section 2.1], the transition maps in both inverse systems are finite étale and moreover $G(\mathbb{A}_f^p)$ acts on $\mathscr{S}_{K_p}$. Let $k = \mathbb{F}_q$ be the residue field of $\mathcal{O}_{E,(v)}$, and write Sh_{G,K_p} for the special fibre of $\mathscr{S}_{K_p}$ and Sh_{G,K^pK_p} for the special fibre of $\mathscr{S}_{K^pK_p}$; these are both schemes over k and $G(\mathbb{A}_f^p)$ acts on Sh_{G,K_p}. We will write $\mathrm{Sh}_{\mathcal{G}_V,\mathcal{K}^p\mathcal{K}_p}$ for the special fibre of $\mathcal{S}_{\mathcal{K}^p\mathcal{K}_p} \otimes_{\mathbb{Z}_{(p)}} \mathcal{O}_{E,(v)}$ and $\mathrm{Sh}_{\mathcal{G}_V,\mathcal{K}_p}$ for the special fibre of $\mathcal{S}_{\mathcal{K}_p} \otimes_{\mathbb{Z}_{(p)}} \mathcal{O}_{E,(v)}$.

Let $\mathbb{V}^p$ be the prime-to-p adelic Tate module of the universal abelian variety $\mathcal{A}$ over $\mathcal{S}_{\mathcal{K}_p}$; this is a pro-étale local system on $\mathcal{S}_{\mathcal{K}_p}$. For a morphism $x : \operatorname{Spec} R \to \mathcal{S}_{\mathcal{K}_p}$ we will write $\mathbb{V}_x^p$ for the pullback along x of $\mathbb{V}^p$. As explained in [KMS 2022, Section 2.1.1] there is a universal isomorphism

$$\epsilon : V \otimes \underline{\mathbb{A}}_f^p \simeq \mathbb{V}^p,$$

sending the symplectic form ψ to an $\underline{\mathbb{A}}_f^{p,\times}$-multiple of the Weil pairing. Here $\underline{\mathbb{A}}_f^p$ denotes the pro-étale sheaf associated to the topological group $\mathbb{A}_f^p$.

[1] It is explained in [KMS 2022, Section 1.3.2] that the collection of subgroups $K_p \subset G(\mathbb{Q}_p)$ that can arise from this construction by varying the symplectic space and the Hodge embedding contains all stabilisers of vertices in the extended Bruhat–Tits building of $G_{\mathbb{Q}_p}$. It is moreover explained in [loc. cit.] that this collection is stable under finite intersections.

2.1.1. *Tensors.* Write $V^{\otimes}$ for the direct sum of $V^{\otimes n} \otimes (V^*)^{\otimes m}$ for all pairs of integers $m \geq 0, n \geq 0$. We will also use this notation later for modules over commutative rings and modules over sheaves of rings.

As in [KMS 2022, Section 1.3.4], we fix tensors $\{s_\alpha \in V\} \subset V^{\otimes}$ such that G is their pointwise stabiliser in $\mathrm{GL}(V)$. Then as explained in [KMS 2022, Sections 1.3.4 and 2.1.2], there are global sections

$$\{s_{\alpha,\mathbb{A}_f^p}\} \in H^0(\mathscr{S}_{K^pK_p}, (\mathbb{V}^p)^{\otimes})$$

such that if we restrict the isomorphism ϵ via $\mathscr{S}_{K_p} \to \mathcal{S}_{\mathcal{K}_p}$ we get an isomorphism

$$\eta : V \otimes \underline{\mathbb{A}}_f^p \to \mathbb{V}^p$$

taking $s_\alpha \otimes 1$ to $s_{\alpha,\mathbb{A}_f^p}$ for all α. In particular, for each $x \in \mathscr{S}_{K_p}(\overline{\mathbb{F}}_p)$ the stabiliser of the tensors $\{s_{\alpha,\mathbb{A}_f^p,x}\}$ in $\mathrm{GL}(\mathbb{V}_x^p)$ is canonically identified with $G \otimes \mathbb{A}_f^p$.

2.1.2. Let $\overline{\mathbb{F}}_p$ denote an algebraic closure of $\mathbb{F}_p$. We will use $\breve{\mathbb{Z}}_p$ to denote the p-typical Witt vectors $W(\overline{\mathbb{F}}_p)$ of $\overline{\mathbb{F}}_p$ and we set $\breve{\mathbb{Q}}_p = \breve{\mathbb{Z}}_p[1/p]$. We let $\sigma : \breve{\mathbb{Z}}_p \to \breve{\mathbb{Z}}_p$ be the automorphism induced by Frobenius on $\overline{\mathbb{F}}_p$, and also denote by σ the induced automorphism of $\breve{\mathbb{Q}}_p$.

Let $x \in \mathrm{Sh}_{G,K^pK_p}(\overline{\mathbb{F}}_p)$ and let $\mathbb{D}_x$ be the rational contravariant Dieudonné module of the p-divisible group $A_x[p^\infty]$ of the abelian variety A_x, equipped with its Frobenius ϕ. By [KMS 2022, Proposition 1.3.7] there are ϕ-invariant tensors $\{s_{\alpha,\mathrm{cris},x}\} \subset \mathbb{D}_x^{\otimes}$ and in [KMS 2022, Section 1.3.8] it is argued that there is an isomorphism $\breve{\mathbb{Q}}_p \otimes V \to \mathbb{D}_x$ sending $1 \otimes s_\alpha$ to $s_{\alpha,\mathrm{cris},x}$. See the statement of [KMS 2022, Proposition 1.3.7] for a characterisation of the tensors $s_{\alpha,\mathrm{cris}}$.

Under such an isomorphism, the Frobenius ϕ corresponds to an element $b_x \in G(\breve{\mathbb{Q}}_p)$, which is well defined up to σ-conjugacy, where $\sigma : G(\breve{\mathbb{Q}}_p) \to G(\breve{\mathbb{Q}}_p)$ is induced by $\sigma : \breve{\mathbb{Q}}_p \to \breve{\mathbb{Q}}_p$. In other words, we can associate to ϕ a well defined element $[b_x]$ of the Kottwitz set $B(G) = B(G_{\mathbb{Q}_p})$ of [Kottwitz 1985]. By [KMS 2022, Lemma 1.3.9], the element $[b_x]$ is contained in the neutral acceptable set $B(G, \{\mu^{-1}\})$ consisting of the $\{\mu^{-1}\}$-admissible elements defined in [KMS 2022, Section 1.1.5]. Here we use $\{\mu\}$ to denote the $G(\overline{\mathbb{Q}}_p)$ conjugacy class of cocharacters induced by the place v of E, where we recall that $\{\mu\}$ was introduced in Section 2.0.1.

It follows from [KMS 2022, Theorem 1.3.14] that there are locally closed subschemes $\mathrm{Sh}_{G,[b],K^pK_p}$ of Sh_{G,K^pK_p}, called *Newton strata*, indexed by $[b] \in B(G, \{\mu^{-1}\})$, such that

$$\mathrm{Sh}_{G,[b],K^pK_p}(\overline{\mathbb{F}}_p) = \{x \in \mathrm{Sh}_{G,K^pK_p}(\overline{\mathbb{F}}_p) \mid [b_x] = [b]\}$$

and such that

$$\overline{\mathrm{Sh}_{G,[b],K^pK_p}} \subset \bigcup_{[b]' \leq [b]} \mathrm{Sh}_{G,[b'],K^pK_p}.$$

Here we are using the partial order $\leq$ on $B(G, \{\mu^{-1}\})$ defined in [Rapoport and Richartz 1996, Section 2.3].

2.2. ***Centralisers.*** Let $x \in \mathrm{Sh}_{G,K_p}(\overline{\mathbb{F}}_p)$ and choose a sufficiently divisible integer m such that the image of x in $\mathrm{Sh}_{G,K^pK_p}(\overline{\mathbb{F}}_p)$ is defined over $\mathbb{F}_{q^m}$. Then the geometric q^m-Frobenius Frob_{q^m} acts on $\mathbb{V}_x^p$ via tensor-preserving automorphisms and therefore determines an element

$$\gamma^p_{x,m} \in G(\mathbb{A}^p_f),$$

which depends on x and m. For $\ell = p$ there is an element $\delta_{x,m} \in G(\mathbb{Q}_{q^m})$, constructed in [KMS 2022, Section 2.1.7], whose class in $B(G_{\mathbb{Q}_p})$ is equal to $[b_x]$. Moreover there is an element $\gamma_{x,m,p} \in G(\mathbb{Q}_{q^m})$ such that

$$\gamma_{x,m,p} = \delta_{x,m}\sigma(\delta_{x,m}) \cdots \sigma^{rm-1}(\delta_{x,m}),$$

where we write $q = p^r$ and where σ denotes the Frobenius on $G(\mathbb{Q}_{q^m})$.

We define $I_{x,\mathbb{A}^p_f} \subset G_{\mathbb{A}^p_f}$ to be the centraliser of $\gamma^p_{x,m}$, which does not depend on m as long as m is sufficiently divisible. We similarly define $I_{x,\ell} \subset G_{\mathbb{Q}_\ell}$ for $\ell \neq p$ to be the centraliser of the projection $\gamma_{x,m,\ell}$ of $\gamma^p_{x,m}$ to $G_{\mathbb{Q}_\ell}$ for sufficiently divisible m.

We define $I_{x,m,p}$ to be the algebraic group over $\mathbb{Q}_p$ whose functor of points is given by

$$R \mapsto \{g \in G(\mathbb{Q}_{q^m} \otimes_{\mathbb{Q}_p} R) \mid g\delta_{x,m} = \delta_{x,m}\sigma(g)\},$$

where σ is induced by $\sigma : G(\mathbb{Q}_{q^m}) \to G(\mathbb{Q}_{q^m})$. As explained in [KMS 2022, Section 2.1.7], the base change $I_{x,m,p} \otimes \mathbb{Q}_{q^m}$ is naturally identified with the centraliser of the semisimple element $\gamma_{x,m,p}$ in $G(\mathbb{Q}_{q^m})$, and $I_{x,m,p}$ is thus reductive. We similarly define $J_{\delta_{x,m}}$ by its functor of points

$$R \mapsto \{g \in G(\breve{\mathbb{Q}}_p \otimes_{\mathbb{Q}_p} R) \mid g\delta_{x,m} = \delta_{x,m}\sigma(g)\}.$$

2.2.1. Consider the decomposition

$$G^{\mathrm{ad}} = \prod_{i=1}^{n} G_i \tag{2.2.1}$$

of G^{ad} into simple groups over $\mathbb{Q}$. Let $\delta_{x,m,i}$ and $\gamma_{x,m,p,i}$ be the images of $\delta_{x,m}$ and $\gamma_{x,m,p}$ in $G_i(\mathbb{Q}_{q^m})$.

Lemma 2.2.2. *Let Z_G be the centre of G. There is a product decomposition*

$$I_{x,m,p}/Z_{G,\mathbb{Q}_p} = \prod_{i=1}^{n} I_{x,m,p,i},$$

where $I_{x,m,p,i}$ represents the functor on $\mathbb{Q}_p$-algebras sending R to

$$\{g \in G_i(\mathbb{Q}_{q^m} \otimes_{\mathbb{Q}_p} R) \mid g\delta_{x,m,i} = \delta_{x,m,i}\sigma(g)\}.$$

Similarly there is a product decomposition

$$J_{\delta_{x,m}}/Z_G \simeq \prod_{i=1}^{n} J_{\delta_{x,m,i}},$$

where $J_{\delta_{x,m,i}}$ represents the functor on $\mathbb{Q}_p$-algebras sending R to

$$\{g \in G_i(\breve{\mathbb{Q}}_p \otimes_{\mathbb{Q}_p} R) \mid g\delta_{x,m,i} = \delta_{x,m,i}\sigma(g)\}.$$

Proof. Consider the commutative diagram

$$\begin{tikzcd} I_{x,m,p} \arrow[r] \arrow[d, hook] & \prod_{i=1}^n I_{x,m,p,i} \arrow[d, hook] \\ \mathrm{Res}_{\mathbb{Q}_{q^m}/\mathbb{Q}_p} G_{\mathbb{Q}_{q^m}} \arrow[r] & \prod_{i=1}^n \mathrm{Res}_{\mathbb{Q}_{q^m}/\mathbb{Q}_p} G_{i,\mathbb{Q}_{q^m}} \end{tikzcd}$$

Since the kernel of the bottom map is central and the bottom map is surjective, it follows that the natural map $I_{x,p} \to \prod_{i=1}^n I_{x,m,p,i}$ is surjective. The kernel is given by the intersection of $I_{x,m,p}$ with the kernel of the bottom map and thus has the following functor of points:

$$R \mapsto \{g \in Z_G(\mathbb{Q}_{q^m} \otimes_{\mathbb{Q}_p} R) \mid g\delta_{x,m,i} = \delta_{x,m,i}\sigma(g)\}.$$

This forces $g = \sigma(g)$ and so $g \in Z_G(R) \subset Z_G(\mathbb{Q}_{q^m} \otimes_{\mathbb{Q}_p} R)$. The same proof shows that there is a product decomposition $J_{\delta_{x,m}}/Z_G \simeq \prod_{i=1}^n J_{\delta_{x,m,i}}$. □

Note that $I_{x,m,p,i} \otimes \mathbb{Q}_{q^m}$ can be identified with the centraliser of $\gamma_{x,m,p,i}$ in $G_{i,\mathbb{Q}_{q^m}}$ as in the beginning of Section 2.2. The centraliser of $\gamma_{x,m,p,i} \in G(\breve{\mathbb{Q}}_p)$ does not depend on m for m sufficiently divisible, and thus the group $I_{x,m,p}$ does not depend on m for m sufficiently divisible. We will write $I_{x,p}$ for the group $I_{x,m,p}$ for sufficiently divisible m and similarly $I_{x,p,i}$ for the group $I_{x,m,p,i}$. We will identify $I_{x,p} \otimes \breve{\mathbb{Q}}_p$ with the centraliser of $\gamma_{x,m,p}$ in $G(\breve{\mathbb{Q}}_p)$ for sufficiently divisible m and similarly we will identify $I_{x,p,i}$ with the centraliser of $\gamma_{x,m,p,i}$ in $G_i(\breve{\mathbb{Q}}_p)$.

2.2.3. Let $x \in \mathrm{Sh}_{G,K_p}(\overline{\mathbb{F}}_p)$ and let $\mathrm{Aut}(A_x)$ be the algebraic group over $\mathbb{Q}$ with functor of points

$$\mathrm{Aut}(A_x)(R) = (\mathrm{End}(A_x) \otimes_{\mathbb{Z}} R)^{\times}.$$

Following [KMS 2022, Section 2.1.3], we define I_x^p to be the largest closed subgroup of $\mathrm{Aut}(A_x)$ that fixes the tensors $s_{\alpha,\mathbb{A}_f^p,x}$ and $I_x \subset I_x^p$ to be the largest closed subgroup that also fixes the tensors $s_{\alpha,\mathrm{cris},x}$. There are natural maps $I_{x,\mathbb{Q}_\ell} \to I_{x,\ell}$ for all (including $\ell = p$), see [KMS 2022, Section 2.1.8] for the $\ell = p$ case.

The groups $I_{x,\ell}$ are connected reductive subgroups of $G_{\mathbb{Q}_\ell}$ and in fact Levi subgroups over $\overline{\mathbb{Q}}_\ell$. By [KMS 2022, Corollary 2.1.9] for all ℓ (including $\ell = p$) the natural map

$$I_{x,\mathbb{Q}_\ell} \to I_{x,\ell}$$

is an isomorphism. This induces a closed immersion of groups $I_{x,\mathbb{Q}_p} \to J_{\delta_{x,m}}$ for some sufficiently divisible m.

2.3. *An assumption.*

We will need to assume the following hypothesis to prove our main monodromy theorems in Section 3.

Hypothesis 2.3.1. For all points $x \in \mathrm{Sh}_{G,K_p}(\overline{\mathbb{F}}_p)$ and for sufficiently divisible m depending on x, there is an element $\gamma_{x,m} \in G(\overline{\mathbb{Q}})$ that is conjugate to $\gamma_{x,m,\ell}$ in $G(\overline{\mathbb{Q}}_\ell)$ for all ℓ (including $\ell = p$). Moreover the $G(\overline{\mathbb{Q}})$-conjugacy class of $\gamma_{x,m}$ is stable under the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

If the $G(\overline{\mathbb{Q}})$-conjugacy class of $\gamma_{x,m}$ contains an element of $G(\mathbb{Q})$, then it is clearly Galois stable. However the converse does not necessarily hold.

Lemma 2.3.2. *The hypothesis holds when K_p is hyperspecial.*

Proof. If K_p is hyperspecial, then [Kisin 2017, Corollary 2.3.1] tells us that there is an element $\gamma_{x,m} \in G(\mathbb{Q})$ that is conjugate to $\gamma_{x,m,\ell}$ in $G(\overline{\mathbb{Q}}_\ell)$ for all ℓ (including $\ell = p$). □

Remark 2.3.3. By [KMS 2022, Corollary 2.2.14], an element $\gamma_{x,m} \in G(\mathbb{Q})$ satisfying the requirements of Hypothesis 2.3.1 exists when $G_{\mathbb{Q}_p}$ is quasi-split and has no factors of type D.

If K_p is very special, the group $G_{\mathbb{Q}_p}$ is tamely ramified and satisfies $p \nmid 2 \cdot \#\pi_1(G^{\mathrm{der}})$ and $\pi_1(G)_I$ is torsion free, where $I \subset \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ is the inertia group, then the existence of an element $\gamma_{x,m} \in G(\mathbb{Q})$ satisfying the requirements of Hypothesis 2.3.1 follows from Theorem I of [van Hoften 2020].

If $p > 2$, if K_p is a very special parahoric subgroup and if the triple (G, X, K_p) is acceptable in the sense of [Kisin and Zhou 2021, Definitions 5.2.6 and 5.2.9], then [Kisin and Zhou 2021, Theorem 6.1.4] proves the existence of an element $\gamma_{x,m} \in G(\overline{\mathbb{Q}})$ satisfying the requirements of Hypothesis 2.3.1.

Remark 2.3.4. When $G_{\mathbb{Q}_p}$ is not quasi-split, one should probably not expect that the $G(\overline{\mathbb{Q}})$-conjugacy class of $\gamma_{x,m}$ always contains an element of $G(\mathbb{Q})$. This is because CM lifts do not exist in general when $G_{\mathbb{Q}_p}$ is not quasi-split. However, we expect Hypothesis 2.3.1 to hold in full generality.

For example, let $x \in \mathrm{Sh}_{G,K_p}(\overline{\mathbb{F}}_p)$ be a point corresponding to the good reduction of an abelian variety *defined over a number field* and assume that $p > 2$. Then [Kisin and Zhou 2021, Theorem 7.2.4] tells us that there is an element $\gamma_{x,m} \in G(\overline{\mathbb{Q}})$ satisfying the requirements of Hypothesis 2.3.1.

2.3.5. We end by deducing a consequence of Hypothesis 2.3.1 that will be used in Section 3. Let G^* denote the quasi-split inner form of G over $\mathbb{Q}$ and let $\Psi : G \otimes \overline{\mathbb{Q}} \to G^* \otimes \overline{\mathbb{Q}}$ be an inner twisting. This means that every $\tau \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ satisfies

$$\Psi(\tau(g)) = h_\tau \tau(\Psi(g)) h_\tau^{-1}$$

for some element $h_\tau \in G^*(\overline{\mathbb{Q}})$. A direct consequence of this definition is that the image under ψ of a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable $G(\overline{\mathbb{Q}})$-conjugacy class is a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable $G^*(\overline{\mathbb{Q}})$-conjugacy class.

Lemma 2.3.6. *Suppose that Hypothesis 2.3.1 holds and let $\gamma_{x,m} \in G(\overline{\mathbb{Q}})$ be the element that is guaranteed to exist by that Hypothesis. Then for sufficiently divisible m the element $\Psi(\gamma_{x,m})$ is $G^*(\overline{\mathbb{Q}})$-conjugate to an element in $G^*(\mathbb{Q})$.*

Proof. If m is sufficiently divisible, then the centraliser of $\gamma_{x,m}$ is connected because this is true for $\gamma_{x,m,\ell}$ and the formation of centralisers commutes with base change. Since G^* is quasi-split and the element $\Psi(\gamma_{x,m})$ is semisimple with connected centraliser, we may apply [Kottwitz 1982, Theorem 4.7(2)] which tells us that the $G^*(\mathbb{Q})$-conjugacy class of $\Psi(\gamma_{x,m})$ contains an element of $G^*(\mathbb{Q})$. □

3. Monodromy of Hecke-invariant subvarieties

In this section we prove an ℓ-adic monodromy theorem in the style of Chai, see [Chai 2005] and [Xiao Xiao 2020; Kasprowitz 2012], for prime-to-p Hecke stable subvarieties of Shimura varieties of Hodge type in characteristic p. We expect the results in this section to be of independent interest, at least beyond the hyperspecial case that we will use in the rest of this article.

In Section 3.1 we establish formal properties of subvarieties Z of Shimura varieties of Hodge type in characteristic p that are stable under prime-to-p Hecke operators. Using techniques from [Chai 2005], we prove that the ℓ-adic monodromy groups of the universal abelian variety over such Z are normal subgroups of $G_{\mathbb{Q}_\ell}$, this is stated as Corollary 3.1.16.

In Section 3.2 we use the results from [KMS 2022] in combination with Hypothesis 2.3.1 to prove Theorem 3.2.5 and Corollary 3.2.6; the latter is a generalisation of Theorem II. In Section 3.3 we combine this theorem with results of D'Addezio [2020] to deduce results about the p-adic monodromy groups of the universal abelian variety over Hecke stable subvarieties.

Finally, in Section 3.4 we prove results about irreducible components of Hecke stable subvarieties in the style of [Chai 2005, Proposition 4.5.4]. We will not use these results in the rest of this article and so this section can safely be skipped for the reader only interested in the proof of Theorem I.

3.1. ***Arithmetic monodromy groups I.*** Let the notation be as in Section 2. In this section we are going to study arithmetic monodromy groups of Hecke stable subvarieties of Sh_{G,K^pK_p}. For maximal generality, we do not assume that these are defined over $k = \mathbb{F}_q$ and so from now on we will implicitly base change the Shimura variety Sh_{G,K^pK_p} to an unspecified finite extension of k, which we will also denote by k.

The morphism $\pi : \mathrm{Sh}_{G,K_p} \to \mathrm{Sh}_{G,K^pK_p}$ is a pro-étale K^p-torsor over Sh_{G,K^pK_p} such that the action of $K^p \subset G(\mathbb{A}_f^p)$ extends to an action of $G(\mathbb{A}_f^p)$. Let $Z \subset \mathrm{Sh}_{G,K^pK_p}$ be a locally closed subscheme and let $\widetilde{Z}$ be the inverse image of Z under π. We say that Z is *stable under the prime-to-p-Hecke operators*, or that Z is *$G(\mathbb{A}_f^p)$-stable*, if $\widetilde{Z}$ is $G(\mathbb{A}_f^p)$-stable.

For the rest of this section ℓ will be used to denote a prime number not equal to p. For such ℓ we let K_ℓ be the image of K^p in $G(\mathbb{Q}_\ell)$ under the projection $G(\mathbb{A}_f^p) \to G(\mathbb{Q}_\ell)$. We let

$$\pi_\ell : \mathrm{Sh}_{G,K_p} \times^{K^p} K_\ell \to \mathrm{Sh}_{G,K^pK_p} \tag{3.1.1}$$

be the induced pro-étale K_ℓ-torsor. For $Z \subset \mathrm{Sh}_{G,K^pK_p}$ a locally closed subscheme, we will write Z_ℓ for the inverse image of Z under π_ℓ. We say that Z is *stable under the ℓ-adic Hecke operators*, or that Z is *$G(\mathbb{Q}_\ell)$-stable*, if Z_ℓ is $G(\mathbb{Q}_\ell)$-stable. When discussing $G(\mathbb{Q}_\ell)$-stable Z we will always implicitly work with $\ell \neq p$. If Z is $G(\mathbb{A}_f^p)$-stable, then it is automatically $G(\mathbb{Q}_\ell)$-stable for all $\ell \neq p$.

All the results in this section will be stated for smooth Z, and the following lemma will be used to reduce to the smooth case in the proof of Theorem I.

Lemma 3.1.1. *Let $Z \subset \mathrm{Sh}_{G,K^pK_p}$ be a locally closed subscheme that is stable under the action of $G(\mathbb{A}_f^p)$ (respectively $G(\mathbb{Q}_\ell)$), then the smooth locus $U \subset Z$ is also stable under this action.*

Proof. For $g \in G(\mathbb{A}_f^p)$ and $K^p \subset G(\mathbb{A}_f^p)$ there is a finite étale correspondence

$$\begin{tikzcd}[column sep=small] & \mathrm{Sh}_{G,(K^p \cap gK^pg^{-1})K_p} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & & \\ \mathrm{Sh}_{G,K^pK_p} & & \mathrm{Sh}_{G,gK^pg^{-1}K_p} \arrow[r, "g"] & \mathrm{Sh}_{G,K^pK_p} \end{tikzcd}$$

and the assumption that $\widetilde{Z}$ is stable under the action of g is equivalent to the statement that the inverse image of Z under p_1 is the same as the inverse image of Z under $g \circ p_2$ for all choices of K^p. Because all the maps in the diagram are finite étale, the same is true for the smooth locus U of Z. Therefore the inverse image $\tilde{U}$ of U under π is stable under the action of $g \in G(\mathbb{A}_f^p)$. □

Lemma 3.1.2. *Let $Z \subset Y \subset \mathrm{Sh}_{G,K^pK_p}$ be locally closed and $G(\mathbb{A}_f^p)$-stable (resp. $G(\mathbb{Q}_\ell)$-stable) subvarieties. Then the closure of Z in Y is also stable under $G(\mathbb{A}_f^p)$ (resp. $G(\mathbb{Q}_\ell)$).*

Proof. This follows in the same way as in the proof of Lemma 3.1.1 from the fact that the prime-to-p Hecke correspondences are finite étale; indeed finite étale maps are open and closed, and thus take closures to closures. □

3.1.3. *Some general topology.* Let $\{X_i\}_{i\in I}$ be a countably indexed cofiltered inverse system of finite type schemes over a field k with surjective affine transition maps. Let $X = \varprojlim_i X_i$ be the inverse limit, it is a nonempty quasi-compact scheme by [Stacks 2020, Lemma 01Z2]. Recall that for a quasi-compact scheme Y there is a profinite topological space $\pi_0(Y)$ of connected components of Y.

Lemma 3.1.4. *The natural map*

$$\pi_0(X) \to \varprojlim_i \pi_0(X_i) \tag{3.1.2}$$

is a homeomorphism.

Proof. The left hand side of (3.1.2) is a profinite topological space by [Stacks 2020, Lemma 0906] and the right hand side of (3.1.2) is visibly an inverse limit of finite sets. Hence both sides are compact Hausdorff topological spaces and to show that the map is a homeomorphism it suffices to show that it is a bijection.

To show that the natural map is a bijection, we construct an explicit inverse. Any compatible system of connected components $\{V_i\}_{i\in I}$ of $\{X_i\}_{i\in I}$ has nonempty and quasi-compact inverse limit $V \subset X$ by [Stacks 2020, Lemma 0A2W]. To prove that V is connected we suppose that there are nonempty open and closed subsets W and W' of V such that $V = W \coprod W'$. Then W and W' are quasi-compact open because V is quasi-compact.

Now [Stacks 2020, Lemma 0A30.(1)] tells us that we can find i and (nonempty) constructible quasi-compact open subsets Z, Z' of V_i such that W is the inverse image of Z under $V \to V_i$ and similarly W' is the inverse image of Z' under $V \to V_i$. In particular, the subsets Z and Z' are disjoint nonempty open subsets of V_i, which gives us a contradiction since V_i is connected.

We have produced a map $\varprojlim_i \pi_0(X_i) \to \pi_0(X)$ and it is not hard to check that it is an inverse of the natural map from the lemma; this concludes the proof. □

Corollary 3.1.5. *Let $Z \subset \mathrm{Sh}_{G,K^pK_p}$ be a $G(\mathbb{A}_f^p)$-stable (resp. $G(\mathbb{Q}_\ell)$-stable) locally closed subscheme of Sh_{G,K^pK_p} and let $\widetilde{Z}$ be as above. Then $\pi_0(\widetilde{Z})$ is equipped with a continuous action of $G(\mathbb{A}_f^p)$ (respectively $\pi_0(Z_\ell)$ is equipped with a continuous action of $G(\mathbb{Q}_\ell)$).*

Proof. The existence of the action follows from the existence of the action on $\widetilde{Z}$ (resp. Z_ℓ). The continuity follows from the continuity of the action of K^p on $\varprojlim_{K^p} Z_{K^p}$ (resp. the continuity of the action of K_ℓ on Z_ℓ) and Lemma 3.1.4. □

The following lemma is only a slight generalisation of [Chai 2005, Lemma 2.8], but we include a proof for the benefit of the reader.

Lemma 3.1.6. *Let X be a second-countable compact Hausdorff topological space with a transitive and continuous action of a locally profinite topological group $\mathbb{G}$. Let $x \in X$ with stabiliser $\mathbb{G}_x \subset \mathbb{G}$, then the orbit map*

$$O : \mathbb{G}/\mathbb{G}_x \to X$$

is a homeomorphism.

Proof. We can write $\mathbb{G}$ as the increasing union of countably many compact open sets, for example by using finite unions of cosets of a compact open subgroup $\mathbb{K} \subset \mathbb{G}$. Since the quotient map $\mathbb{G} \to \mathbb{G}/\mathbb{G}_x$ is open for any topological group, it follows that $\mathbb{G}/\mathbb{G}_x$ can be written as the increasing union of countably many compact open subsets.

Since the orbit map is surjective, the topological space X can be written as a countable union of the compact subsets $O(U)$ for $U \subset \mathbb{G}/\mathbb{G}_x$ compact open. Because X is second-countable it is metrisable by Urysohn's metrisation theorem and thus the Baire category theorem tells us that there exists a compact open subset U of $\mathbb{G}/\mathbb{G}_x$ such that $O(U)$ contains an open subset W of X.

Choose a compact open subset $V \subset U$ such that $O(V) \subset W$. Then $O : V \to O(V)$ is a continuous bijection between compact Hausdorff topological spaces and hence a homeomorphism. Now note that $\mathbb{G}$ acts transitively on both $\mathbb{G}/\mathbb{G}_x$ and on X. Hence by moving around V we see that any point of $y \in \mathbb{G}/\mathbb{G}_x$ has an open neighbourhood V_y such that the natural map $O : V_y \to O(V_y)$ is a homeomorphism, and we conclude that O is a homeomorphism. □

3.1.7. *Lie groups over ℓ-adic local fields.* Recall that a topological group M is called an ℓ-adic Lie group if it admits the (necessarily unique) structure of an ℓ-adic Lie group; see [Glockner 2016, Definition 2.1, Proposition 2.2]. If M is an ℓ-adic Lie group, then by definition there is a finite-dimensional $\mathbb{Q}_\ell$-Lie algebra $\mathrm{Lie}\, M$, an open neighbourhood $U \subset \mathrm{Lie}\, M$ of the identity and an exponential map $\mathrm{Exp} : U \to M$ that is a homeomorphism onto a compact open subgroup of M. For example for an algebraic group H over $\mathbb{Q}_\ell$ the topological group $H(\mathbb{Q}_\ell)$ is an ℓ-adic Lie group with Lie algebra $\mathrm{Lie}\, H(\mathbb{Q}_\ell) = \mathrm{Lie}\, H$.

Lemma 3.1.8. *Let H be an algebraic group over $\mathbb{Q}_\ell$ and let $M \subset H(\mathbb{Q}_\ell)$ be a subgroup that is compact in the subspace topology. Then M is an ℓ-adic Lie group and the morphism $M \to H(\mathbb{Q}_\ell)$ is a morphism of ℓ-adic Lie groups. Moreover, the induced Lie subalgebra $\mathrm{Lie}\, M \subset \mathrm{Lie}\, H(\mathbb{Q}_\ell) = \mathrm{Lie}\, H$ satisfies*

$$[\mathrm{Lie}\, M, \mathrm{Lie}\, M] = [\mathrm{Lie}\, \mathbb{M}, \mathrm{Lie}\, \mathbb{M}],$$

where the bracket notation means the commutator of two Lie subalgebras and where $\mathbb{M}$ *is the Zariski closure of* M.

Proof. The group M is an ℓ-adic Lie group by [Glockner 2016, Proposition 2.3] and the morphism $M \to H(\mathbb{Q}_\ell)$ is a morphism of ℓ-adic Lie groups by [Glockner 2016, Proposition 2.2]. This implies that there is an induced morphism on Lie algebras $\operatorname{Lie} M \to \operatorname{Lie} H(\mathbb{Q}_\ell) = \operatorname{Lie} H$.

Since $M \subset \mathbb{M}(\mathbb{Q}_\ell)$ is Zariski dense, it follows that the smallest algebraic subgroup of H whose Lie algebra contains $\operatorname{Lie} M$ is equal to $\mathbb{M}$; indeed, if there is a smaller algebraic subgroup $\mathbb{M}' \subset \mathbb{M}$ with $\operatorname{Lie} M \subset \operatorname{Lie} \mathbb{M}'$, then we see using the ℓ-adic exponential map that there is a compact open (hence finite index) subgroup of M contained in $\mathbb{M}'(\mathbb{Q}_\ell)$. This contradicts the fact that M is Zariski dense in $\mathbb{M}$.

The fact that the smallest algebraic subgroup of H whose Lie algebra contains $\operatorname{Lie} M$ is equal to $\mathbb{M}$ is expressed as $\mathfrak{a}(\operatorname{Lie} M) = \operatorname{Lie} \mathbb{M}$ in the notation of [Borel 1991, Section 7.1]. By [Borel 1991, Corollary 7.9] we have the following equality of Lie subalgebras of $\operatorname{Lie} H$

$$[\operatorname{Lie} M, \operatorname{Lie} M] = [\mathfrak{a}(\operatorname{Lie} M), \mathfrak{a}(\operatorname{Lie} M)] = [\operatorname{Lie} \mathbb{M}, \operatorname{Lie} \mathbb{M}]. \qquad \square$$

Lemma 3.1.9. *Let* $\mathbb{M}$ *be a semisimple algebraic group over* $\mathbb{Q}_\ell$ *and let* $M \subset \mathbb{M}(\mathbb{Q}_\ell)$ *be a subgroup closed in the* ℓ*-adic topology. If* M *equipped with the subspace topology is compact and* M *is Zariski dense in* $\mathbb{M}$, *then* M *is a compact open subgroup of* $\mathbb{M}(\mathbb{Q}_\ell)$.

Proof. It follows from Lemma 3.1.8 that M is an ℓ-adic Lie group, that $M \to \mathbb{M}(\mathbb{Q}_\ell)$ is a morphism of ℓ-adic Lie groups and that the Lie algebra of M is equal to the Lie algebra of $\mathbb{M}$, since $\mathbb{M}$ is semisimple. Now we can use the exponential map for ℓ-adic Lie groups to show that M contains a compact open subgroup of $\mathbb{M}(\mathbb{Q}_\ell)$. Since M is itself compact, this implies that M is also a compact open subgroup of $\mathbb{M}(\mathbb{Q}_\ell)$. $\square$

3.1.10. The main theorem of Galois theory for schemes tells us that the category of finite-étale covers of a smooth connected scheme Z over k is equivalent to the category of finite sets equipped with a continuous action of $\pi_1^{\text{ét}}(Z, z)$. Under this equivalence, a finite étale cover $f : Y \to Z$ is sent to the finite set $f^{-1}(z)$ equipped with its action of $\pi_1^{\text{ét}}(Z, z)$. In particular, the set of connected components of Y is in bijection with the set of orbits of $\pi_1^{\text{ét}}(Z, z)$ on $f^{-1}(z)$.

If $f : Y \to Z$ is a countably indexed inverse limit of finite étale covers $f_i : Y_i \to Z$ with surjective transition maps, then we can associate to f the profinite set

$$f^{-1}(z) = \varprojlim_i f_i^{-1}(z),$$

equipped with its natural continuous action of $\pi_1^{\text{ét}}(Z, z)$. By Lemma 3.1.4 it follows that the profinite set of orbits of $\pi_1^{\text{ét}}(Z, z)$ on $f^{-1}(z)$ is homeomorphic to the topological space of connected components of Y.

3.1.11. Now let Z be a smooth $G(\mathbb{Q}_\ell)$-stable locally closed subscheme of $\operatorname{Sh}_{G,K^pK_p}$, let $Z^\circ \subset Z$ be a connected component of Z, and let $z \in Z^\circ(\overline{\mathbb{F}}_p)$. Let π_ℓ be as in (3.1.1) and write Z_ℓ for the inverse image of Z under π_ℓ as above; it is stable under the action of $G(\mathbb{Q}_\ell)$ by assumption. Denote by Z_ℓ° the inverse image of Z° under π_ℓ, then $Z_\ell^\circ \to Z^\circ$ is a profinite étale K_ℓ-torsor.

By the Galois theory for schemes discussed above, the cover $\pi_\ell : Z_\ell^\circ \to Z^\circ$ corresponds to the profinite set $\pi_\ell^{-1}(z)$ equipped with its natural action of $\pi_1^{\text{ét}}(Z^\circ, z)$. In particular, the set of connected components of Z_ℓ° corresponds to the set of orbits of $\pi_1^{\text{ét}}(Z^\circ, z)$ on $\pi_\ell^{-1}(z)$.

Choose an element $\tilde{z} \in \pi_\ell^{-1}(z)$. Then using the simply transitive action of K_ℓ on $\pi_\ell^{-1}(z)$ we can identify $\pi_\ell^{-1}(z)$ with K_ℓ; under this identification the chosen element $\tilde{z}$ is send to $1 \in K_\ell$. This defines a continuous group homomorphism

$$\rho_\ell : \pi_1^{\text{ét}}(Z^\circ, z) \to K_\ell,$$

whose conjugacy class does not depend on the choice of $\tilde{z}$. Let $y \in \pi_0(Z_\ell^\circ)$ be the connected component containing $\tilde{z}$. Then the stabiliser of y in K_ℓ is equal to the image of ρ_ℓ.

Let $P_y \subset G(\mathbb{Q}_\ell)$ be the stabiliser of y in $G(\mathbb{Q}_\ell)$. It is a closed topological subgroup by the continuity of the action and the fact that $\pi_0(Z_\ell)$ is Hausdorff. Its intersection with K_ℓ gives us the stabiliser of y in K_ℓ. The action map gives us a continuous map

$$G(\mathbb{Q}_\ell)/P_y \to \pi_0(Z_\ell), \quad g \mapsto g \cdot y$$

with image the orbit $\mathrm{Orb}(y)$ of y.

Lemma 3.1.12. *The orbit* $\mathrm{Orb}(y)$ *is open and closed inside of* $\pi_0(Z_\ell)$. *Moreover the orbit map induces a homeomorphism* $G(\mathbb{Q}_\ell)/P_y \simeq \mathrm{Orb}(y)$; *in particular*, $G(\mathbb{Q}_\ell)/P_y$ *is compact.*

Proof. The identification

$$\pi_0(Z_\ell)/K_\ell \simeq \pi_0(Z).$$

tells us that there are finitely many K_ℓ-orbits on $\pi_0(Z_\ell)$, and that each of them is open and closed. The $G(\mathbb{Q}_\ell)$-orbit of a point y is then a union of finitely many K_ℓ-orbits, and thus also open and closed. Lemma 3.1.12 shows that $\mathrm{Orb}(y)$ is open and closed inside a second-countable profinite topological space. Therefore $\mathrm{Orb}(y)$ is profinite and second-countable. The result now follows from Lemma 3.1.6. □

Let M be the image of ρ_ℓ and let $\mathbb{M}$ be the neutral component of its Zariski closure inside $G(\mathbb{Q}_\ell)$. Let $\rho_{\ell,\mathrm{geom}}$ be the restriction of ρ_ℓ to

$$\pi_1^{\text{ét}}(Z^\circ_{\overline{\mathbb{F}}_p}, z) \subset \pi_1^{\text{ét}}(Z^\circ, z),$$

let M_{geom} be its image and let $\mathbb{M}_{\mathrm{geom}}$ be the neutral component of its Zariski closure inside $G(\mathbb{Q}_\ell)$.

Lemma 3.1.13. *The groups* $\mathbb{M}$ *and* $\mathbb{M}_{\mathrm{geom}}$ *are connected reductive groups over* $\mathbb{Q}_\ell$.

Proof. There is a short exact sequence (e.g., by [D'Addezio 2020, Proposition 3.2.7])

$$1 \to \mathbb{M}'_{\mathrm{geom}} \to \mathbb{M} \to Q \to 1, \tag{3.1.3}$$

where Q is a commutative algebraic group of multiplicative type and where $\mathbb{M}'_{\mathrm{geom}}$ is a closed subgroup of $\mathbb{M}$ with neutral component given by $\mathbb{M}_{\mathrm{geom}}$. In particular, it follows that $\mathbb{M}$ is reductive if $\mathbb{M}_{\mathrm{geom}}$ is reductive. The representation

$$\pi_1^{\text{ét}}(Z^\circ, z) \to K_\ell \to G(\mathbb{Q}_\ell) \to \mathrm{GL}(V)(\mathbb{Q}_\ell)$$

is the monodromy representation of the (rational) ℓ-adic Tate module of the abelian scheme $\pi : A \to \mathrm{Sh}_{G,K^pK_p}$ coming from the Hodge embedding $\mathrm{Sh}_{G,K^pK_p} \to \mathrm{Sh}_{\mathcal{G}_V,\mathcal{K}^p\mathcal{K}_p}$. This is an ℓ-adic sheaf $\mathcal{F}_0$ on Z° which is pure of weight one. Then [Deligne 1980, Theorem 3.4.1.(iii)] tells us that the basechange $\mathcal{F}$ of $\mathcal{F}_0$ to $Z^\circ_{\overline{\mathbb{F}}_p}$ is semisimple. This base change corresponds to the composition of $\rho_{\ell,\text{geom}}$ with $K_\ell \to G(\mathbb{Q}_\ell) \to \mathrm{GL}(V)(\mathbb{Q}_\ell)$. Now [Deligne 1980, Corollary 1.3.9] tells us that $\mathbb{M}_{\text{geom}}$ is a semisimple algebraic group, and thus that it is reductive. □

3.1.14. Let $\mathbb{N}$ be the normaliser of $\mathbb{M}$ in $G_{\mathbb{Q}_\ell}$ and let $\mathbb{N}^\circ$ be its neutral component. The group $\mathbb{N}^\circ$ is a connected reductive group because we are working with reductive groups in characteristic zero; see, e.g., [Conrad et al. 2015, Proposition A.8.12].

Lemma 3.1.15. *The group P_y is contained in $\mathbb{N}$.*

Proof. Let $\gamma \in P_y$, then we want to show that γ normalises $\mathbb{M}$. If V is a compact open subgroup of $G(\mathbb{Q}_\ell)$ contained in K_ℓ, then $V \cap P_y \subset M$. Moreover for every $\gamma \in P_y$ we can find an open subgroup $U \subset G(\mathbb{Q}_\ell)$ such that $\gamma U \gamma^{-1} \subset V$. For example, we can just take the intersection of V with $\gamma V \gamma^{-1}$. For such U the open subgroup $M \cap U$ of M satisfies $\gamma(M \cap U)\gamma^{-1} \subset M$.

Since conjugation by γ is a homeomorphism in the Zariski topology, we see that the Zariski closure of $M \cap U$ is moved under conjugation by γ into the Zariski closure of M. But since $M \cap U$ is an open subgroup of M it is also a closed subgroup and thus compact and thus of finite index in M. This means that the Zariski closure of $M \cap U$ and the Zariski closure of M have the same identity component, both of which are equal to $\mathbb{M}$. Since conjugation preserves 1, this mean it sends $\mathbb{M}$ to $\mathbb{M}$. □

Corollary 3.1.16. *The Zariski closure $\mathbb{M}$ of M is a normal subgroup of $G_{\mathbb{Q}_\ell}$.*

Proof. The group $G(\mathbb{Q}_\ell)/\mathbb{N}(\mathbb{Q}_\ell)$ is compact, because it is a quotient of $G(\mathbb{Q}_\ell)/P_y$ which is compact. Since $\mathbb{N}^\circ(\mathbb{Q}_\ell)$ is finite index in $\mathbb{N}(\mathbb{Q}_\ell)$, it follows that $G(\mathbb{Q}_\ell)/\mathbb{N}^\circ(\mathbb{Q}_\ell)$ is also compact. Since $\mathbb{N}^\circ$ is connected it follows from [Borel and Tits 1965, Propositions 8.4 and 9.3] that it contains a parabolic subgroup of $G_{\mathbb{Q}_\ell}$ and because it is reductive it follows that $\mathbb{N}^\circ = G_{\mathbb{Q}_\ell}$. Therefore $\mathbb{N}^\circ = \mathbb{N} = G_{\mathbb{Q}_\ell}$ and we find that $\mathbb{M}$ is a normal subgroup of $G_{\mathbb{Q}_\ell}$. □

3.2. ***Arithmetic monodromy groups II.*** So far we have not excluded the possibility that $\mathbb{M}$ is contained in the centre of $G_{\mathbb{Q}_\ell}$. In fact, this happens when Z is the supersingular locus inside the modular curve. Thus we will need additional assumptions on Z to prove that $\mathbb{M}$ is not central.

We will show, using the results of [KMS 2022], that if Z contains a point $x \in Z(\mathbb{F}_{q^m})$ not contained in the smallest Newton stratum, then the image of Frob_x under ρ_ℓ is noncentral. If $G^{\text{ad}}_{\mathbb{Q}_\ell}$ were a simple group over $\mathbb{Q}_\ell$, then this would force $\mathbb{M}$ to contain $G^{\text{der}}_{\mathbb{Q}_\ell}$. However G^{ad} is generally not a simple group over $\mathbb{Q}$, and even if it were simple then there would generally be no primes ℓ where $G^{\text{ad}}_{\mathbb{Q}_\ell}$ is simple. To deal with these issues, we will make use of Hypothesis 2.3.1.

3.2.1. Recall that for a point $x \in Z^\circ(\mathbb{F}_{q^m})$, there is a Frobenius element

$$\mathrm{Frob}_x \in \pi_1^{\text{ét}}(Z^\circ, z),$$

whose image under ρ_ℓ is the element $\gamma_{x,m,\ell} \in G(\mathbb{Q}_\ell)$ from Section 2.2. If m is sufficiently divisible, then its centraliser is equal to the group $I_{x,\ell}$.

3.2.2. The decomposition $G^{\mathrm{ad}} = \prod_{i=1}^n G_i$ of (2.2.1) induces maps

$$B(G_{\mathbb{Q}_p}) \to B(G^{\mathrm{ad}}_{\mathbb{Q}_p}) \to \prod_{i=1}^n B(G_{i,\mathbb{Q}_p}).$$

For an element $[b] \in B(G_{\mathbb{Q}_p})$ we will write $[b_i]$ for its image in $B(G_{i,\mathbb{Q}_p})$ under this map. Recall, see[Kret and Shin 2021, Definition 5.3.2], that an element $[b] \in B(G, \{\mu^{-1}\})$ is called $\mathbb{Q}$-*nonbasic* if $[b_i]$ is nonbasic for all i. A Newton stratum $\mathrm{Sh}_{G,[b],K^pK_p}$ is called $\mathbb{Q}$-*nonbasic* if $[b]$ is $\mathbb{Q}$-nonbasic.

Proposition 3.2.3. *Let $x \in Z^\circ(\mathbb{F}_{q^m})$ for some sufficiently divisible m and let $[b] = [b_x] \in B(G, \{\mu^{-1}\})$. Assume that Hypothesis 2.3.1 holds. If $[b_i]$ is nonbasic, then the image of $\rho_\ell(\mathrm{Frob}_{x,m})$ under*

$$G(\mathbb{Q}_\ell) \to G^{\mathrm{ad}}(\mathbb{Q}_\ell) \to G_i(\mathbb{Q}_\ell)$$

is nontrivial. Moreover, the image of $\rho_\ell(\mathrm{Frob}_{x,m})$ in each simple factor of $G_{i,\mathbb{Q}_\ell}$ over $\mathbb{Q}_\ell$ is nontrivial.

Proof. Let m be sufficiently divisible and let $\gamma_{x,m} \in G(\overline{\mathbb{Q}})$ be the element guaranteed to exist by Hypothesis 2.3.1. Let G^* denote the quasi-split inner form of G over $\mathbb{Q}$ and let $\Psi : G \otimes \overline{\mathbb{Q}} \to G^* \otimes \overline{\mathbb{Q}}$ be an inner twisting. Then by Lemma 2.3.6 there is an element $\gamma'_{x,m} \in G^*(\mathbb{Q})$ that is conjugate to $\Psi(\gamma_{x,m})$ in $G^*(\overline{\mathbb{Q}})$. We will write $I'_x \subset G^*$ for the centraliser of $\gamma'_{x,m}$. Recall that by Hypothesis 2.3.1 for all ℓ (including $\ell = p$) the element $\Psi^{-1}(\gamma_{x,m})$ is conjugate to $\gamma_{x,m,\ell}$ in $G(\overline{\mathbb{Q}}_\ell)$.

By the classification of adjoint algebraic groups we can find number fields[2] $F_1, \ldots, F_n$ and absolutely simple adjoint algebraic groups H_i over F_i for each $i = 1, \ldots, n$ such that

$$G^{\mathrm{ad}} = \prod_{i=1}^n \mathrm{Res}_{F_i/\mathbb{Q}} H_i = \prod_{i=1}^n G_i.$$

We have a similar decomposition for $G^{*,\mathrm{ad}}$ with H_i replaced by its quasi-split inner form H_i^* and we will write G_i^* for the restriction of scalars from F_i to $\mathbb{Q}$ of H_i^*.

Let $\gamma'_{x,m,i} \in G_i^*(\mathbb{Q}) = H_i^*(F_i)$ be the image of $\gamma'_{x,m}$ and let $C_{x,i} \subset H_i^*$ be its centraliser. Then there is a product decomposition

$$I'_x/Z_G \simeq \prod_{i=1}^n \mathrm{Res}_{F_i/\mathbb{Q}} C_{x,i} = \prod_{i=1}^n I'_{x,i}.$$

Let $b = b_x \in G(\breve{\mathbb{Q}}_p)$ for sufficiently divisible m be as in Section 2.1.2. Then Equation (2.2.1) shows that there is a product decomposition $J_b/Z_{G,\mathbb{Q}_p} \simeq \prod_{i=1}^n J_{b_i}$, where each $J_{b,i}$ is the twisted centraliser

[2] By the classification of Shimura varieties of abelian type in [Milne 2005, Appendix B], each F_i is totally real.

of the image b_i of b in $G_i(\breve{\mathbb{Q}}_p)$. Moreover the natural inclusion $I_{x,p} \to J_b$ induces closed immersions $I_{x,p,i} \to J_{b_i}$. As in [KMS 2022, Section 1.1.4], there is an inclusion $J_{b,i,\breve{\mathbb{Q}}_p} \to G_{i,\breve{\mathbb{Q}}_p}$, identifying its image with the centraliser $M_{\nu_{b_i}}$ of the fractional cocharacter ν_{b_i} of $G_i \otimes \breve{\mathbb{Q}}_p$ attached to b_i. If $[b_i]$ is not basic, then ν_{b_i} is not central and so $\operatorname{Dim} J_{b,i} = \operatorname{Dim} M_{\nu_{b_i}} < \operatorname{Dim} G_i$ and so $\operatorname{Dim} I_{x,p,i} < \operatorname{Dim} G_i$.

The subgroup

$$\Psi^{-1}(I'_{x,\bar{\mathbb{Q}}_p}) \subset G_{\bar{\mathbb{Q}}_p}$$

can be identified with the centraliser of $\gamma_{x,m,p}$ for m sufficiently divisible, and it follows that

$$\Psi^{-1}(I'_{x,i,\bar{\mathbb{Q}}_p}) \subset G_{i,\bar{\mathbb{Q}}_p}$$

can be identified with the centraliser of the image of $\gamma_{x,m,p,i}$ in $G_{i,\bar{\mathbb{Q}}_p}$. In particular, the group $I'_{x,i,\bar{\mathbb{Q}}_p}$ is conjugate to $I_{x,i,p,\bar{\mathbb{Q}}_p}$ and therefore of the same dimension.

The upshot of the above discussion is that $\operatorname{Dim} I'_{x,i} < \operatorname{Dim} G_i$ if b_i is not basic. It follows that the inclusion $I_{x,i,\mathbb{Q}_\ell} \subset G_{i,\mathbb{Q}_\ell}$ is not an equality for $\ell \neq p$ and thus that the image of $\rho_\ell(\mathrm{Frob}_{x,m})$ in $G_i(\mathbb{Q}_\ell)$ is nontrivial.

To deduce the last statement of Proposition 3.2.3, we note that it suffices to show that the image of $\gamma_{x,m,i}$ in every simple factor of $G_{i,\mathbb{Q}_\ell}$ is noncentral. For this, we fix i and a prime number ℓ.

Then we can write $G^*_{i,\mathbb{Q}_\ell}$ as a product indexed by primes $\mathfrak{p}$ of F_i dividing ℓ

$$G^*_{i,\mathbb{Q}_\ell} = \prod_{\mathfrak{p}\mid\ell} \operatorname{Res}_{F_{i,\mathfrak{p}}/\mathbb{Q}_\ell} H^*_{i,F_{i,\mathfrak{p}}}.$$

The element $\gamma_{x,m,i}$ is noncentral in $H_i^*(F_i)$ and thus also noncentral in $H_i^*(F_{i,\mathfrak{p}})$ for all primes $\mathfrak{p}$ of F_i dividing ℓ, and thus we are done. □

Remark 3.2.4. When b_i is basic then the inclusion $I_{x,i,\mathbb{Q}_p} \subset J_{b,i}$ should be an equality and the image of $\rho_\ell(\mathrm{Frob}_{x,m})$ in $G_i(\mathbb{Q}_\ell)$ should be trivial for all $\ell \neq p$. This is true when K_p is a very special parahoric subgroup, see the proof of [He et al. 2021, Proposition 5.2.10].

We now state and prove our main arithmetic monodromy theorem.

Theorem 3.2.5. *Let $Z \subset \mathrm{Sh}_{G,K^pK_p}$ be a smooth $G(\mathbb{Q}_\ell)$-stable locally closed subvariety. Let $Z^\circ \subset Z$ be a connected component and choose a point $z \in Z^\circ(\overline{\mathbb{F}}_p)$. Let $\mathbb{M}$ be the neutral component of the Zariski closure of the image of*

$$\rho_\ell : \pi_1^{\acute{e}t}(Z^\circ, z) \to K_\ell \to G(\mathbb{Q}_\ell).$$

Assume that Hypothesis 2.3.1 holds. Then $\mathbb{M}$ is a normal subgroup of $G_{\mathbb{Q}_\ell}$ surjecting onto $G_{i,\mathbb{Q}_\ell}$ for all i such that there is a point $x \in Z^\circ(\overline{\mathbb{F}}_p)$ with $b_{x,i}$ nonbasic.

When $(G, X) = (\mathcal{G}_V, \mathcal{H}_V)$, then this is closely related to [Chai 2005, Corollary 3.5].

Proof. Corollary 3.1.16 proves that $\mathbb{M} \subset G_{\mathbb{Q}_\ell}$ is a normal subgroup. For $x \in Z^\circ(\overline{\mathbb{F}}_p)$ we have the Frobenius element $\rho_\ell(\mathrm{Frob}_{x,m}) = \gamma_{x,m,\ell} \in M$, for all sufficiently divisible m, where we recall that M is the image of ρ_ℓ. Thus $\gamma_{x,m,\ell}$ is contained in the Zariski closure of M and thus after replacing m by a power we may assume that $\gamma_{x,m,\ell} \in \mathbb{M}$, the neutral component of the Zariski closure of M.

If $x \in Z^\circ(\overline{\mathbb{F}}_p)$ is a point with $b_{x,i}$ nonbasic, then Proposition 3.2.3 tells us that the image of $\gamma_{x,m,\ell}$ in $G_{i,\mathbb{Q}_\ell}$ is nonzero and thus the image of $\mathbb{M}$ in $G_{i,\mathbb{Q}_\ell}$ is nonzero. This image is moreover normal, so to show that it is equal to $G_{i,\mathbb{Q}_\ell}$ it suffices to show that it maps nontrivially to every simple factor of $G_{i,\mathbb{Q}_\ell}$ over $\mathbb{Q}_\ell$. But this follows from the last line of the statement of Proposition 3.2.3. □

Corollary 3.2.6. *Let $Z \subset \mathrm{Sh}_{G,K^pK_p,\overline{\mathbb{F}}_p}$ be a smooth $G(\mathbb{Q}_\ell)$-stable locally closed subscheme as before, let $Z^\circ \subset Z$ be a connected component and fix $z \in Z^\circ(\overline{\mathbb{F}}_p)$. Let $\mathbb{M}_{\mathrm{geom}}$ be the neutral component of the Zariski closure of the image of*

$$\pi_1^{\acute{e}t}(Z^\circ_{\overline{\mathbb{F}}_p}, z) \to K_\ell \to G(\mathbb{Q}_\ell).$$

Assume that Hypothesis 2.3.1 holds. Then $\mathbb{M}_{\mathrm{geom}}$ is a normal subgroup of $G_{\mathbb{Q}_\ell}$ surjecting onto $G_{i,\mathbb{Q}_\ell}$ for all i such that there is a point $x \in Z^\circ(\overline{\mathbb{F}}_p)$ with $b_{x,i}$ nonbasic. If we can find such a point for all i, then $\mathbb{M}_{\mathrm{geom}} = G^{\mathrm{der}}_{\mathbb{Q}_\ell}$.

Proof. The subscheme Z is defined over a finite extension of k, and so we can speak of its arithmetic monodromy group $\mathbb{M}$. Theorem 3.2.5 tells us that $\mathbb{M}$ surjects onto $G_{i,\mathbb{Q}_\ell}$ for all i such that there is a point $x \in Z^\circ(\overline{\mathbb{F}}_p)$ with $b_{x,i}$ nonbasic. We now claim that $\mathbb{M}_{\mathrm{geom}}$ and $\mathbb{M}$ have the same image in $G_{i,\mathbb{Q}_\ell}$ for all such i.

It follows from the short exact sequence (3.1.3) that $\mathbb{M}_{\mathrm{geom}} \subset \mathbb{M}$ is a normal subgroup with abelian cokernel. Let $\mathbb{M}_{\mathrm{geom},i}$ be the image of $\mathbb{M}_{\mathrm{geom}}$ in $G_{i,\mathbb{Q}_\ell}$ and let $\mathbb{M}_i$ be the image of $\mathbb{M}$ in $G_{i,\mathbb{Q}_\ell}$. Then $\mathbb{M}_{\mathrm{geom},i} \subset \mathbb{M}_i$ is a normal subgroup with abelian cokernel. Given an integer i with $1 \le i \le n$ such that there is a point $x \in Z^\circ(\overline{\mathbb{F}}_p)$ with $b_{x,i}$ nonbasic, then $\mathbb{M}_i = G_{i,\mathbb{Q}_\ell}$ and therefore $\mathbb{M}_i$ has no nontrivial abelian quotients. Thus it follows that the inclusion $\mathbb{M}_{\mathrm{geom},i} \subset \mathbb{M}_i$ is an equality.

If we can find a point x with $b_{x,i}$ nonbasic for all i, then $\mathbb{M}_{\mathrm{geom}}$ surjects onto $G^{\mathrm{ad}}_{\mathbb{Q}_\ell}$ and is moreover semisimple by [Deligne 1980, Corollary 1.3.7]. It must therefore be equal to $G^{\mathrm{der}}_{\mathbb{Q}_\ell}$. □

3.3. *p-adic monodromy groups.* In this subsection we record a consequence of Theorem 3.2.5 in combination with the main results of [D'Addezio 2020; 2023].

Recall the following notions from [D'Addezio 2020, Section 2.2]. Write F-Isoc(S) for the $\mathbb{Q}_p$-linear Tannakian category of F-isocrystals over a smooth finite type scheme S over $\overline{\mathbb{F}}_p$ and write F-Isoc$^\dagger(S)$ for the $\mathbb{Q}_p$-linear Tannakian category of overconvergent F-isocrystals over S. There is a natural fully faithful embedding F-Isoc$^\dagger(S) \subset F$-Isoc(S) which sends an overconvergent F-isocrystal $\mathcal{M}^\dagger$ to the underlying F-isocrystal $\mathcal{M}$. Similarly we write Isoc$^\dagger(S)$ and Isoc(S) for the $\breve{\mathbb{Q}}_p$-linear category of (overconvergent) isocrystals over S. There are natural faithful forgetful functors from (overconvergent) F-isocrystals to (overconvergent) isocrystals.

3.3.1. The morphism $\mathrm{Sh}_{G,K^pK_p} \to \mathrm{Sh}_{\mathcal{G}_V,\mathcal{K}^p\mathcal{K}_p}$ gives us an abelian scheme $\pi : \mathcal{A} \to \mathrm{Sh}_{G,K^pK_p}$ and we consider the F-isocrystal

$$\mathcal{M} := R^1\pi_*\mathcal{O}_{\mathrm{cris},A}[1/p],$$

which is overconvergent by [Étesse 2002, Theorem 7]. Then [KMS 2022, Corollary 1.3.13] proves that there is an exact $\mathbb{Q}_p$-linear tensor functor (the *p-adic realisation functor*)

$$\mathrm{Rel}_p : \mathrm{Rep}_{\mathbb{Q}_p} G \to F\text{-}\mathrm{Isoc}(\mathrm{Sh}_{G,K^pK_p}) \tag{3.3.1}$$

such that the representation $G_{\mathbb{Q}_p} \to \mathcal{G}_V \to \mathrm{GL}_V$ coming from the choice of Hodge embedding is sent to the F-isocrystal $\mathcal{M}$.

Lemma 3.3.2. *This morphism factors via an exact $\mathbb{Q}_p$-linear tensor functor*

$$\mathrm{Rel}_p : \mathrm{Rep}_{\mathbb{Q}_p} G \to F\text{-}\mathrm{Isoc}^{\dagger}(\mathrm{Sh}_{G,K^pK_p}), \tag{3.3.2}$$

which we will also denote by Rel_p.

Proof. Since $F\text{-}\mathrm{Isoc}^{\dagger}(\mathrm{Sh}_{G,K^pK_p}) \subset F\text{-}\mathrm{Isoc}(\mathrm{Sh}_{G,K^pK_p})$ is a full subcategory, it suffices to show that $\mathrm{Rel}_p(W)$ is overconvergent for each representation W of $G_{\mathbb{Q}_p}$. We follow the proof of [KMS 2022, Corollary 1.3.13]. As explained there, each W can be written as the kernel of a map $e : V_{m,n} \to V_{m',n'}$, where

$$V_{m,n} = V^{\otimes m} \otimes V^{*,\otimes n}.$$

Since $\mathcal{M} = \mathrm{Rel}_p(V)$ is overconvergent and the category of overconvergent isocrystals is stable under tensor products, duals and direct sums by [Berthelot 1996, Remark 2.3.3(iii)], we see that $\mathrm{Rel}_p(V_{m,n})$ is overconvergent. Since Rel_p is exact, we see that $\mathrm{Rel}_p(W)$ can be written as the kernel of a map between overconvergent F-isocrystals, and is thus overconvergent. □

Given a smooth locally closed subscheme $Z \subset \mathrm{Sh}_{G,K^pK_p,\overline{\mathbb{F}}_p}$ and a point $z \in Z(\overline{\mathbb{F}}_p)$, there are monodromy groups

$$\mathrm{Mon}(Z, \mathcal{M}, z) \subset \mathrm{Mon}(Z, \mathcal{M}^{\dagger}, z),$$

which are algebraic groups over $\breve{\mathbb{Q}}_p$, see the introduction of [D'Addezio 2023]. They are defined to be the Tannakian groups corresponding to the smallest Tannakian subcategory of $\mathrm{Isoc}(Z)$ respectively $\mathrm{Isoc}^{\dagger}(Z)$ containing $\mathcal{M}$, via the fibre functor ω_z

$$\omega_z : \mathrm{Isoc}(Z) \to \mathrm{Isoc}(\overline{\mathbb{F}}_p) = \mathrm{Vect}_{\breve{\mathbb{Q}}_p}.$$

We will often omit the chosen point z from the notation since the monodromy group does not depend on z up to isomorphism.

Fix an isomorphism $\mathbb{D}_z := \omega_z(\mathcal{M}^{\dagger}) \to V \otimes \breve{\mathbb{Q}}_p$ sending $\omega_z(s_\alpha) = s_{\alpha,\mathrm{cris},z}$ to $s_\alpha \otimes 1$. This identifies the composite

$$\omega_z \circ \mathrm{Rel}_p : \mathrm{Rep}_{\mathbb{Q}_p} G \to \mathrm{Vect}_{\breve{\mathbb{Q}}_p}$$

with the standard fibre functor, tensored up to $\breve{\mathbb{Q}}_p$. Thus if we apply Tannakian duality to (3.3.1) and (3.3.2), we get inclusions

$$\mathrm{Mon}(Z, \mathcal{M}, z) \subset \mathrm{Mon}(Z, \mathcal{M}^{\dagger}, z) \subset G_{\breve{\mathbb{Q}}_p}.$$

Corollary 3.3.3. *Let $Z \subset \mathrm{Sh}_{G,K^pK_p}$ be a smooth locally closed subscheme and assume that there is a prime $\ell \neq p$ such that Z is $G(\mathbb{Q}_\ell)$-stable. Suppose that Z° contains a point x such that $b_{x,i}$ is nonbasic for all i. If Hypothesis 2.3.1 holds, then there is an equality of subgroups of $G_{\breve{\mathbb{Q}}_p}$*

$$\mathrm{Mon}(Z^\circ, \mathcal{M}^\dagger) = G^{\mathrm{der}}_{\breve{\mathbb{Q}}_p}.$$

Proof. Let ℓ be as in the statement of the Corollary. Then it follows from Corollary 3.2.6 that the ℓ-adic monodromy group $\mathbb{M}_{\mathrm{geom}}$ of the abelian variety over Z is equal to $G^{\mathrm{der}}_{\mathbb{Q}_\ell}$. It follows from [D'Addezio 2020, Theorem 1.2.1] (compare [Pál 2022]) that there is an isomorphism of algebraic groups

$$\mathrm{Mon}(Z^\circ, \mathcal{M}^\dagger) \otimes \overline{\mathbb{Q}}_p \simeq G^{\mathrm{der}} \otimes \overline{\mathbb{Q}}_p.$$

Therefore $\mathrm{Mon}(Z^\circ, \mathcal{M}^\dagger) \subset G \otimes \breve{\mathbb{Q}}_p$ is a subgroup which is isomorphic to G^{der} over $\overline{\mathbb{Q}}_p$. It follows that $\mathrm{Mon}(Z^\circ, \mathcal{M}^\dagger)$ is equal to its own derived subgroup and therefore contained in $G^{\mathrm{der}} \otimes \breve{\mathbb{Q}}_p$. This inclusion has to be an isomorphism for dimension reasons, because both groups are connected. □

From now on we will assume that Z is contained in a single Newton stratum $\mathrm{Sh}_{G,[b],K^pK_p}$ of Sh_{G,K^pK_p}. This means that for every representation W of $G_{\mathbb{Q}_p}$ the Newton polygon of $\mathrm{Rel}_p(W)$ is constant. As explained in [D'Addezio 2023, Section 4.3] (cf. [Katz 1979]), this implies that $\mathrm{Rel}_p(W)$ admits a (unique) slope filtration $\mathrm{Rel}_p(W)_\bullet$. There is an induced slope filtration on $\omega_z(\mathrm{Rel}_p(W))$, which gives a fractional cocharacter λ_W of $\mathrm{GL}(\omega_z(\mathrm{Rel}_p(W)))$. Since this construction is functorial in W, it defines a fractional cocharacter λ of $G_{\breve{\mathbb{Q}}_p}$. On the other hand, there is an element $b = b_z \in G(\breve{\mathbb{Q}}_p)$ correspond to the Frobenius of $\mathbb{D}_z = \omega_z(\mathcal{M}^\dagger) = V \otimes \breve{\mathbb{Q}}_p$; let ν_b be the Newton cocharacter of b.

Lemma 3.3.4. *There is an equality $\lambda = \nu_b$.*

Proof. It suffices to show that $\lambda = \nu_b$ after composing with $G \otimes \breve{\mathbb{Q}}_p \to \mathrm{GL}(V) = \mathrm{GL}(\omega_z(\mathcal{M}^\dagger))$. But both of these fractional cocharacters of $\mathrm{GL}(V)$ are per definition the slope cocharacters of the F-isocrystal $\omega_z(\mathcal{M}^\dagger)$. Indeed, this is true for λ per definition and for ν_b by its construction; see [Kottwitz 1985, Section 4]. □

Under our assumption that Z is contained in a single Newton stratum $\mathrm{Sh}_{G,[b],K^pK_p}$ of Sh_{G,K^pK_p} we note that the monodromy group

$$\mathrm{Mon}(Z^\circ, \mathcal{M}) \subset G_{\breve{\mathbb{Q}}_p}$$

of a connected component Z° of Z is contained in the parabolic subgroup $P(\lambda)$ associated to λ, as explained in [D'Addezio 2023, Section 4.1].

Corollary 3.3.5. *Let $Z \subset \mathrm{Sh}_{G,K^pK_p}$ be a smooth locally closed subscheme and assume that there is a prime $\ell \neq p$ such that Z is $G(\mathbb{Q}_\ell)$-stable. Let Z° be a connected component of Z and suppose that Z° is contained in a single $\mathbb{Q}$-nonbasic Newton stratum $\mathrm{Sh}_{G,[b],K^pK_p}$. If Hypothesis 2.3.1 holds, then the p-adic monodromy group*

$$\mathrm{Mon}(Z^\circ, \mathcal{M}) \subset \mathrm{Mon}(Z^\circ, \mathcal{M}^\dagger) = G^{\mathrm{der}}_{\breve{\mathbb{Q}}_p} \subset G_{\breve{\mathbb{Q}}_p}$$

is equal to the intersection

$$G^{\mathrm{der}}_{\breve{\mathbb{Q}}_p} \cap P(\lambda).$$

In particular, the unipotent radical of $\mathrm{Mon}(Z^\circ, \mathcal{M})$ *is isomorphic to the unipotent radical of the parabolic subgroup* P_{ν_b} *of* $G_{\breve{\mathbb{Q}}_p}$.

Proof. The first assertion is a direct consequence of Corollary 3.3.3 and [D'Addezio 2023, Theorem 1.1.1]. The second assertion follows from Lemma 3.3.4 □

3.4. *Irreducible components of Hecke stable subvarieties.* In this section we will study irreducible components of Hecke stable subvarieties and prove results in the style of [Chai 2005, Proposition 4.5.4].[3]The results proved in this section will not be used in the rest of this article, but they are used to prove irreducibility results for EKOR strata in [van Hoften 2020].

Let $\rho : G^{\mathrm{sc}} \to G^{\mathrm{der}}$ be the simply connected cover of the derived group G^{der} of G and note that ρ induces an action of $G^{\mathrm{sc}}(\mathbb{A}_f^p)$ on Sh_{G,K_p}. From now on we will need another assumption:

Hypothesis 3.4.1. For each finite extension F of the reflex field E and any place w of F extending v, the natural maps

$$\pi_0(\mathbf{Sh}_{K^pK_p}(G, X) \otimes_E F) \to \pi_0(\mathscr{S}_{K^pK_p} \otimes_{\mathcal{O}_{E,(v)}} \mathcal{O}_{F,(w)}) \leftarrow \pi_0(\mathrm{Sh}_{G,K^pK_p} \otimes_k k(w))$$

are isomorphisms.

Lemma 3.4.2. *Hypothesis 3.4.1 holds if* Sh_{G,K^pK_p} *has geometrically integral connected components.*

Proof. This is [Madapusi Pera 2019, Corollary 4.1.11]. □

Remark 3.4.3. The variety Sh_{G,K^pK_p} has geometrically integral connected components if K_p is hyperspecial because then the integral models are smooth by work of Kisin [2010]. More generally the Kisin–Pappas integral models [2018] have geometrically integral connected components if K_p is very special; see [Kisin and Pappas 2018, Corollary 4.6.26].

3.4.4. *Connected components.* The following result is well known.

Lemma 3.4.5. *Let* Y_∞ *be a connected component of the scheme*

$$\varprojlim_{U \subset G(\mathbb{A}_f)} \mathbf{Sh}_{U,\mathbb{C}}(G, X).$$

Then Y_∞ *is stable under the action of* $G^{\mathrm{sc}}(\mathbb{A}_f)$.

Proof. This is a direct consequence of the description of connected components of Shimura varieties and strong approximation for $G^{\mathrm{sc}}(\mathbb{Q})$, see [Kisin et al. 2021, Section 5.5.1, Lemma 5.5.4]. □

[3]Our results do not literally generalise Chai's results because he works with $\mathrm{Sp}_{2g}(\mathbb{A}_f^p)$-stable subvarieties while we work with $\mathrm{GSp}_{2g}(\mathbb{A}_f^p)$-stable subvarieties.

Corollary 3.4.6. *Let Y_∞ be a connected component of*

$$\varprojlim_{U^p \subset G(\mathbb{A}_f^p)} \mathbf{Sh}_{U^p K_p, \mathbb{C}}(G, X).$$

Then Y_∞ is stable under the action of $G^{\mathrm{sc}}(\mathbb{A}_f^p)$.

Proof. We consider

$$\varprojlim_{U \subset G(\mathbb{A}_f^p)} \mathbf{Sh}_U(G, X) \to \varprojlim_{U^p \subset G(\mathbb{A}_f^p)} \mathbf{Sh}_{U^p K_p}(G, X)$$

and we let Y'_∞ be a connected component of the left hand side mapping to Y_∞. Then Y'_∞ is stable under the action of $G^{\mathrm{sc}}(\mathbb{A}_f)$ and thus Y_∞ is stable under the action of $G^{\mathrm{sc}}(\mathbb{A}_f^p)$. □

Lemma 3.4.7. *Suppose that Hypothesis 3.4.1 holds and let $Y_\infty \subset \mathrm{Sh}_{G, K_p, \overline{\mathbb{F}}_p}$ be a connected component. Then Y_∞ is stable under the action of $G^{\mathrm{sc}}(\mathbb{A}_f^p)$.*

Proof. It suffices to prove this for Shimura varieties over $\mathbb{C}$, because the connected components are defined over an algebraic closure $\overline{E}$ of the reflex field E and the result can be transported to the special fibre using Hypothesis 3.4.1. The result over $\mathbb{C}$ is Corollary 3.4.6. □

3.4.8. Let $Z \subset \mathrm{Sh}_{G, K^p K_p}$ be a $G(\mathbb{A}_f^p)$-stable locally closed subscheme with inverse image $\widetilde{Z} \subset \mathrm{Sh}_{G, K_p}$. A finite étale cover $X \to Z$ is called $G(\mathbb{A}_f^p)$-equivariant if $\widetilde{X} := \widetilde{Z} \times_Z X$ has an action of $G(\mathbb{A}_f^p)$ making the natural map $\widetilde{X} \to \widetilde{Z}$ equivariant for the action of $G(\mathbb{A}_f^p)$. If Hypothesis 3.4.1 is satisfied, then by Lemma 3.4.7 the group $G^{\mathrm{sc}}(\mathbb{A}_f^p)$ acts on the fibres of

$$\pi_0(\widetilde{X}) \to \pi_0(\mathrm{Sh}_{G, K_p, \overline{\mathbb{F}}_p}).$$

Lemma 3.4.9. *If Hypothesis 3.4.1 holds, then $G(\mathbb{A}_f^p)$ acts continuously on $\pi_0(\widetilde{X})$.*

Proof. The assumption that $X \to Z$ is finite étale implies that $\pi_0(\widetilde{X}) \to \pi_0(\widetilde{Z})$ is a finite map with discrete fibres, and therefore the action of $G(\mathbb{A}_f^p)$ on $\pi_0(\widetilde{X})$ is continuous because the action on $\pi_0(\widetilde{Z})$ is, see Corollary 3.1.5. □

Let Σ be a finite set of places of $\mathbb{Q}$ containing p and containing all places ℓ where G_ℓ^{ad} has a compact factor. From now on we will work with $G(\mathbb{A}_f^p)$-stable subvarieties Z defined over $\overline{\mathbb{F}}_p$ and with geometric monodromy groups.

Theorem 3.4.10. *Let $X \to Z$ be a $G(\mathbb{A}_f^p)$-equivariant finite étale cover of a smooth $G(\mathbb{A}_f^p)$-stable locally closed subscheme $Z \subset \mathrm{Sh}_{G, K^p K_p, \overline{\mathbb{F}}_p}$, and suppose that each connected component of Z intersects a $\mathbb{Q}$-nonbasic Newton stratum. If Hypotheses 2.3.1 and 3.4.1 hold, then $G(\mathbb{A}_f^\Sigma)$ acts trivially on the fibres of*

$$\pi_0(\widetilde{X}) \to \pi_0(\mathrm{Sh}_{G, K_p, \overline{\mathbb{F}}_p}).$$

For a prime $\ell \notin \Sigma$ we will write K_ℓ for the image of $K^p \to G(\mathbb{Q}_\ell)$ and $\pi_\ell : \mathrm{Sh}_{G, K^p K_p, \ell, \overline{\mathbb{F}}_p} \to \mathrm{Sh}_{G, K^p K_p, \overline{\mathbb{F}}_p}$ for the induced K_ℓ-torsor over $\mathrm{Sh}_{G, K^p K_p, \overline{\mathbb{F}}_p}$.

Lemma 3.4.11. *Suppose that Hypothesis 2.3.1 holds and let* $Y_\infty \subset \mathrm{Sh}_{G,K^pK_p,\ell,\overline{\mathbb{F}}_p}$ *be a connected component with image* $Y \subset \mathrm{Sh}_{G,K^pK_p,\overline{\mathbb{F}}_p}$*. Then* $Y_\infty \to Y$ *is a torsor for a compact open subgroup of* $G^{\mathrm{der}}(\mathbb{Q}_\ell)$*.*

Proof. It follows from profinite Galois theory for schemes, see Section 3.1.11, that the stabiliser K_∞ of Y_∞ in $G(\mathbb{Q}_\ell)$ can be identified with the image of

$$\pi_1^{\text{ét}}(Y, y) \to G(\mathbb{Q}_\ell)$$

for some point $y \in Y(\overline{\mathbb{F}}_p)$. If we apply Corollary 3.2.6 and Lemma 3.1.9 to $Z = \mathrm{Sh}_{G,K^pK_p}$, it follows that this image contains a compact open subgroup of $G^{\mathrm{der}}(\mathbb{Q}_\ell)$ and that it is contained in $G^{\mathrm{der}}(\mathbb{Q}_\ell)$. □

Proof of Theorem 3.4.10. We write $Z_\ell \to Z$ for the induced K_ℓ torsor and $X_\ell \to Z_\ell$ for $Z_\ell \times_Z X$. Then the action of $G(\mathbb{A}_f^p)$ on $\widetilde{X}$ and $\widetilde{Z}$ induces an action of $G(\mathbb{Q}_\ell)$ on X_ℓ, and it suffices to show that $G^{\mathrm{sc}}(\mathbb{Q}_\ell)$ acts trivially on the fibres of

$$a_\ell : \pi_0(X_\ell) \to \pi_0(\mathrm{Sh}_{G,K^pK_p,\ell,\overline{\mathbb{F}}_p})$$

for all $\ell \notin \Sigma$.

Let $x \in \pi_0(X_\ell)$ and let Z° be a connected component of Z containing the image of x. Moreover let $Z_\ell^\circ \subset \mathbb{Z}_\ell$ be the inverse image of Z°. Fix a point $z \in Z^\circ(\overline{\mathbb{F}}_p)$, then Hypothesis 2.3.1 and Corollary 3.2.6 tell us that the image of

$$\rho_\ell : \pi_1^{\text{ét}}(Z^\circ, z) \to K_\ell$$

is a compact subgroup $M_{\mathrm{geom},\ell}$ whose Zariski closure $\mathbb{M}_{\mathrm{geom},\ell}$ has neutral component equal to $G_{\mathbb{Q}_\ell}^{\mathrm{der}}$. It follows from Lemma 3.1.9 that the image of ρ_ℓ contains a compact open subgroup $V_\ell \subset G^{\mathrm{der}}(\mathbb{Q}_\ell)$. The upshot of this discussion is that the stabiliser in $G(\mathbb{Q}_\ell)$ of a connected component of Z_ℓ contains a compact open subgroup of $G^{\mathrm{der}}(\mathbb{Q}_\ell)$ and this implies that the stabiliser in $G(\mathbb{Q}_\ell)$ of x contains a compact open subgroup of $G^{\mathrm{der}}(\mathbb{Q}_\ell)$.

Let Y_∞ be a connected component of $\mathrm{Sh}_{G,K^pK_p,\ell,\overline{\mathbb{F}}_p}$ such that the image Y of Y_∞ in Sh_{G,K^pK_p} contains Z°. Then it follows from Hypothesis 2.3.1 and Lemma 3.4.11 that $Y_\infty \to Y$ is a pro-étale torsor for a compact open subgroup $U_\ell \subset G^{\mathrm{der}}$, and from Hypothesis 3.4.1 and Lemma 3.4.7 that Y_∞ is stable under the action of $G^{\mathrm{sc}}(\mathbb{Q}_\ell)$.

We will write $X_\infty \subset X_\ell$ for the inverse image of Y_∞ in X_ℓ and let $X' \subset X$ be its image. Note that $x \in \pi_0(X_\infty)$ by construction. Then $X_\infty \to X'$ is a pro-étale U_ℓ torsor and X_∞ is stable under the action of $G^{\mathrm{sc}}(\mathbb{Q}_\ell)$. This action is moreover continuous by Lemma 3.4.9 and the inclusion

$$\pi_0(X_\infty) \subset \pi_0(X_\ell)$$

is closed since $\{Y_\infty\} \subset \pi_0(\mathrm{Sh}_{G,K^pK_p,\ell,\overline{\mathbb{F}}_p})$ is closed. In particular, the topological space $\pi_0(X_\infty)$ is compact Hausdorff.

Let U'_ℓ be the inverse image of U_ℓ in $G^{\mathrm{sc}}(\mathbb{Q}_\ell)$. Then the quotient

$$U'_\ell \backslash \pi_0(X_\infty) = U_\ell \backslash \pi_0(X_\infty) = \pi_0(X')$$

is finite. This means that there are finitely many (open and closed) U'_ℓ orbits on $\pi_0(X_\infty)$. Therefore the $G^{\mathrm{sc}}(\mathbb{Q}_\ell)$ orbit of x on $\pi_0(X_\infty)$ is a union of finitely many U'_ℓ-orbits and thus closed; in particular it is compact Hausdorff. It then follows from Lemma 3.1.6 that the $G^{\mathrm{sc}}(\mathbb{Q}_\ell)$ orbit of x is homeomorphic to $G^{\mathrm{sc}}(\mathbb{Q}_\ell)/P_x$, where $P_x \subset G^{\mathrm{sc}}(\mathbb{Q}_\ell)$ is the stabiliser of x. In particular, it follows that $G^{\mathrm{sc}}(\mathbb{Q}_\ell)/P_x$ is compact.

The group P_x contains a compact open subgroup of $G^{\mathrm{sc}}(\mathbb{Q}_\ell)$ because the stabiliser of x in $G(\mathbb{Q}_\ell)$ contains a compact open subgroup of $G^{\mathrm{der}}(\mathbb{Q}_\ell)$ and $G^{\mathrm{sc}}(\mathbb{Q}_\ell) \to G^{\mathrm{der}}(\mathbb{Q}_\ell)$ has finite fibres. This implies that $G^{\mathrm{sc}}(\mathbb{Q}_\ell)/P_x$ has the discrete topology, and we conclude that $G^{\mathrm{sc}}(\mathbb{Q}_\ell)/P_x$ is a finite set or equivalently that P_x is a finite index subgroup. The assumption that $G^{\mathrm{sc}}_{\mathbb{Q}_\ell}$ has no compact factors implies, by [Platonov and Rapinchuk 1994, Theorems 7.1 and 7.5], that the group $G^{\mathrm{sc}}(\mathbb{Q}_\ell)$ has no finite index subgroups. Therefore $G^{\mathrm{sc}}(\mathbb{Q}_\ell)/P_x$ is a singleton which is precisely what we wanted to prove. $\square$

4. Serre–Tate coordinates and unipotent group actions

In this section we show that the classical Serre–Tate coordinates, as described in [Katz 1981], can be reinterpreted using the action of a unipotent formal group, as in [Howe 2020]. Our results are more-or-less a direct generalisation of the results of [Howe 2020], except that we construct the action of unipotent formal groups using Rapoport–Zink spaces, while in [loc. cit.] this action is constructed using Igusa varieties.

In Section 4.1, we recall the classical theory of Serre–Tate coordinates following [Katz 1981], which shows that the formal deformation space $\mathrm{Def}(Y)$ of an ordinary p-divisible group Y over $\overline{\mathbb{F}}_p$ has the structure of a commutative formal group. We then compute the scheme-theoretic p-adic Tate-module of the p-divisible group $\mathcal{H}_{0,1}$ associated to this formal group. In Section 4.2 we use Rapoport–Zink spaces to describe an action of the universal cover $\widetilde{\mathcal{H}}_{0,1}$ of $\mathcal{H}_{0,1}$ on the formal scheme $\widehat{\mathrm{Def}}(Y)$ associated to $\mathrm{Def}(Y)$. In Section 4.3 we identify this action with the projection from the universal cover to $\mathcal{H}_{0,1}$ followed by the left-translation action of $\mathcal{H}_{0,1}$ on $\widehat{\mathrm{Def}}(Y)$.

4.0.1. We consider the category Art of Artin local $\breve{\mathbb{Z}}_p$-algebras R such that the natural map $\overline{\mathbb{F}}_p \to R/\mathfrak{m}_R$ is an isomorphism. Here $\mathfrak{m}_R$ is the unique maximal ideal of R and we write $\alpha : R \to \overline{\mathbb{F}}_p$ for the composition of the natural map $R \to R/\mathfrak{m}$ with the inverse of the natural isomorphism $\overline{\mathbb{F}}_p \to R/\mathfrak{m}$. Note that α is functorial for morphisms in Art. We similarly consider the category Nilp of $\breve{\mathbb{Z}}_p$-algebras in which $p^n = 0$ for some n. The category Art is naturally a full subcategory of Nilp.

For a p-divisible group $\mathscr{G}$ over an algebra $R \in \mathrm{Nilp}$ we define the p-adic Tate module to be the functor $T_p\mathscr{G} := \varprojlim_n \mathscr{G}[p^n]$, which is representable by a flat affine scheme over Spec R by [Scholze and Weinstein 2013, Proposition 3.3.1].

4.1. ***Classical Serre–Tate theory.*** Let Y be an ordinary p-divisible group of dimension g and height $2g$ over $\overline{\mathbb{F}}_p$. In other words, let Y be a p-divisible group isomorphic to $(\mathbb{Q}_p/\mathbb{Z}_p)^{\oplus g} \oplus \mu_{p^\infty}^{\oplus g}$.

Let $\mathrm{Def}(Y)$ be the functor on Art sending (R, α) to the set of isomorphism classes of pairs (X, β) where X is a p-divisible group over Spec R and $\beta : X \otimes_{R,\alpha} \overline{\mathbb{F}}_p \to Y_{\overline{\mathbb{F}}_p}$ is an isomorphism. This functor is (pro)-representable by a formally smooth formal scheme $\mathrm{Def}(Y)$ of relative dimension g^2 over $\mathrm{Spf}\, \breve{\mathbb{Z}}_p$.

By [Katz 1981, Theorem 2.1], this functor lifts to a functor valued in abelian groups such that the formal group $\mathrm{Def}(Y)$ is p-divisible.[4]

There is a canonical direct sum decomposition $Y = Y_0 \oplus Y_1$ where Y_0 is the maximal étale quotient of Y and where Y_1 is equal to the formal completion of Y at the origin. Since Y_0 is étale there is a unique lift to a p-divisible formal group over $\breve{\mathbb{Z}}_p$, which we will denote by Y_0^{can}. Similarly Y_1 has a unique lift to a p-divisible formal group over $\breve{\mathbb{Z}}_p$, for example because the Serre dual of Y_1 is étale. We will denote this lift by Y_1^{can} and we will use $Y^{\mathrm{can}} := Y_0^{\mathrm{can}} \oplus Y_1^{\mathrm{can}}$ to denote the canonical lift of Y to $\breve{\mathbb{Z}}_p$.

Let $Y^\vee$ be the Serre-dual of Y and consider the free $\mathbb{Z}_p$-modules of rank g given by $T_pY(\overline{\mathbb{F}}_p)$ and $T_pY^\vee(\overline{\mathbb{F}}_p)$. By [Katz 1981, Theorem 2.1], the formal group $\mathrm{Def}(Y)$ is isomorphic to the functor on Art sending R to

$$\hom\bigl(T_pY(\overline{\mathbb{F}}_p) \otimes_{\mathbb{Z}_p} T_pY^\vee(\overline{\mathbb{F}}_p), \widehat{\mathbb{G}}_m(R)\bigr).$$

Let S be the complete Noetherian local $\breve{\mathbb{Z}}_p$-algebra representing $\mathrm{Def}(Y)$ on Art. Then the abelian group structure on $\mathrm{Def}(Y)$ induces a (continuous) cocommutative Hopf algebra structure on S. In particular the formal scheme $\widehat{\mathrm{Def}}(Y) := \operatorname{Spf} S$, considered as a functor on Nilp, has the structure of a formal group. We will write $\mathcal{H}_{0,1} := \varinjlim \operatorname{Spf} S[p^n]$ for the corresponding p-divisible group over $\operatorname{Spf} \breve{\mathbb{Z}}_p$. Note that it acts via left translation on $\widehat{\mathrm{Def}}(Y)$; we will denote this action by a_{ST} (for Serre–Tate).

Remark 4.1.1. The natural map $\mathcal{H}_{0,1} \to \widehat{\mathrm{Def}}(Y)$ is an isomorphism of formal schemes, since both of them are formally smooth formal schemes of the same dimension. Nevertheless, it is useful to treat them as different objects, for example because the notation $\widehat{\mathrm{Def}}(Y)$ is somewhat unwieldy, especially when passing to universal covers of p-divisible groups.

Lemma 4.1.2. *The p-adic Tate module of $\mathcal{H}_{0,1}$ is isomorphic to the sheaf $\mathscr{H}\!om(Y_0^{\mathrm{can}}, Y_1^{\mathrm{can}})$ on* Nilp *of homomorphisms from Y_0^{can} to Y_1^{can}.*

Proof. Let us prove the stronger assertion that there are isomorphisms $\mathcal{H}_{0,1}[p^n] \simeq \mathscr{H}\!om(Y_0^{\mathrm{can}}, Y_1^{\mathrm{can}})[p^n]$ for all n, compatible with changing n. Note that $\mathcal{H}_{0,1}[p^n]$ is represented by the spectrum of an Artin local $\breve{\mathbb{Z}}_p$-algebra. The same is true for $\mathscr{H}\!om(Y_0^{\mathrm{can}}, Y_1^{\mathrm{can}})[p^n]$, since $\mathscr{H}\!om(\underline{\mathbb{Z}/p^n\mathbb{Z}}, \mu_{p^n}) \simeq \mu_{p^n}$. Thus it suffices to show that the functors $\mathcal{H}_{0,1}[p^n]$ and $\mathscr{H}\!om(Y_0^{\mathrm{can}}, Y_1^{\mathrm{can}})[p^n]$ are isomorphic as functors on Art.

In [Katz 1981, p. 152] it is explained that $\mathrm{Def}(Y)$ is isomorphic to the functor (on Art) sending R to

$$\hom(T_pY(\overline{\mathbb{F}}_p), Y_1^{\mathrm{can}}(R)).$$

Note that $T_pY(\overline{\mathbb{F}}_p) = T_pY_0(\overline{\mathbb{F}}_p) = T_pY_0^{\mathrm{can}}(\overline{\mathbb{F}}_p)$ and that because $T_pY_0^{\mathrm{can}}$ is an inverse limit of étale group schemes, the natural map $T_pY_0^{\mathrm{can}}(R) \to T_pY_0(\overline{\mathbb{F}}_p)$ is an isomorphism for $R \in \mathrm{Art}$. Thus there is a natural isomorphism

$$\hom(T_pY(\overline{\mathbb{F}}_p), Y_1^{\mathrm{can}}(R)) \simeq \hom(T_pY_0^{\mathrm{can}}(R), Y_1^{\mathrm{can}}(R)).$$

[4] Recall that a commutative formal group X is called *p-divisible* if $[p] : X \to X$ is finite flat.

The p^n-torsion of this group is given by

$$\hom(T_pY_0^{\mathrm{can}}(R), Y_1^{\mathrm{can}}(R))[p^n] = \hom(T_pY_0^{\mathrm{can}}(R), Y_1^{\mathrm{can}}[p^n](R)) = \hom(Y_0^{\mathrm{can}}[p^n](R), Y_1^{\mathrm{can}}[p^n](R)).$$

We see that there is an isomorphism $\mathrm{Def}(Y)[p^n] \simeq \mathscr{H}om(Y_0^{\mathrm{can}}[p^n], Y_1^{\mathrm{can}}[p^n])$ of functors on Art, which induces an isomorphism $\mathcal{H}_{0,1}[p^n] \simeq \mathscr{H}om(Y_0^{\mathrm{can}}[p^n], Y_1^{\mathrm{can}}[p^n])$ of functors on Nilp. It is straightforward to check that these isomorphisms are compatible with increasing n, which concludes the proof. □

4.2. ***Rapoport–Zink spaces and unipotent formal groups.*** Let $\widetilde{Y} \to Y$ be the universal cover of Y, defined as the inverse limit of the projective system

$$\varprojlim_{[p]:G\to G} Y.$$

It is representable by a formal scheme by [Scholze and Weinstein 2013, Proposition 3.1.3(iii)]. By the proof of [Caraiani and Scholze 2017, Proposition 4.2.11], the automorphism group functors of Y and $\widetilde{Y}$ on Nilp can be described as follows:

$$\mathbf{Aut}(Y) = \begin{pmatrix} \mathbf{Aut}(Y_0) & 0 \\ \mathscr{H}om(Y_0, Y_1) & \mathbf{Aut}(Y_1) \end{pmatrix}, \quad \mathbf{Aut}(\widetilde{Y}) = \begin{pmatrix} \mathbf{Aut}(\widetilde{Y}_0) & 0 \\ \mathscr{H}om(Y_0, Y_1)[1/p] & \mathbf{Aut}(\widetilde{Y}_1) \end{pmatrix}.$$

Moreover the functors $\mathbf{Aut}(Y_i)$ are pro-étale group schemes which are noncanonically isomorphic to the group schemes associated to the profinite group $\mathrm{GL}_g(\mathbb{Z}_p)$. Let $\widetilde{\mathcal{H}}_{0,1}$ be the universal cover of $\mathcal{H}_{0,1}$. Then by the discussion after [Caraiani and Scholze 2017, Definition 4.1.1], we can identify the fpqc sheaves

$$\widetilde{\mathcal{H}}_{0,1} = (T_p\mathcal{H}_{0,1})[1/p].$$

Moreover, by the proof of [Caraiani and Scholze 2017, Proposition 4.1.2], there is a short exact sequence of fpqc sheaves

$$0 \to T_p\mathcal{H}_{0,1} \to \widetilde{\mathcal{H}}_{0,1} \to \mathcal{H}_{0,1} \to 0.$$

By Lemma 4.1.2, we can identify this with

$$0 \to \mathscr{H}om(Y_0, Y_1) \to \mathscr{H}om(Y_0, Y_1)[1/p] \to \mathcal{H}_{0,1} \to 0.$$

Note that $\mathscr{H}om(Y_0, Y_1)[1/p]$ is isomorphic to $\widetilde{\mathcal{H}}_{0,1}$, and thus representable by a formal scheme by [Scholze and Weinstein 2013, Proposition 3.1.3(iii)] as above. In particular, this means that $\mathbf{Aut}(\widetilde{Y})$ is representable by a formal scheme.

4.2.1. Let RZ_Y be the Rapoport–Zink space associated to Y. It is defined to be the functor on Nilp sending R to the set of isomorphism classes of pairs (X, f), where X is a p-divisible group over Spec R and $f : X \dashrightarrow Y_R$ is a quasi-isogeny (or equivalently, by [Katz 1981, Lemma 1.1.3.3], a quasi-isogeny $f_0 : X_{R/pR} \dashrightarrow Y_{R/pR}$). The functor RZ_Y is representable by a formally smooth formal scheme over $\mathrm{Spf}\, \breve{\mathbb{Z}}_p$ by [Rapoport and Zink 1996, Theorem 2.16]. The group functor $\mathbf{Aut}(\widetilde{Y})$ acts on RZ_Y via postcomposition, where we note that an automorphism of $\widetilde{Y}$ is the same thing as a self-quasi-isogeny of Y.

Let y be the $\overline{\mathbb{F}}_p$-point of RZ_Y corresponding to the identity map $Y \to Y$ and let

$$\mathrm{RZ}_Y^{/y} \subset \mathrm{RZ}_Y$$

be the formal completion of RZ_Y in $\{y\}$, in the sense of formal algebraic spaces as in [Stacks 2020, Tag0GVR]. By definition this is the subfunctor of RZ_Y corresponding to those morphisms $\operatorname{Spec} R \to \mathrm{RZ}_Y$ that factor through $\{y\}$ on the level of topological spaces. In other words, it consists of those morphisms $\operatorname{Spec} R \to \mathrm{RZ}_Y$ such that the induced morphism $\operatorname{Spec} R^{\mathrm{red}} \subset \operatorname{Spec} R \to \mathrm{RZ}_Y$ factors through $y : \operatorname{Spec} \overline{\mathbb{F}}_p \to \mathrm{RZ}_Y$.

In terms of the moduli description, this means that we are looking at those quasi-isogenies $f : X \dashrightarrow Y_R$ such that: There is a (necessarily unique) isomorphism $\beta : X_{R^{\mathrm{red}}} \to Y_{R^{\mathrm{red}}}$ making the following diagram commute:

$$\begin{tikzcd} X_{R^{\mathrm{red}}} \arrow[r, "\beta"] \arrow[d, dashed, "f"] & Y_{R^{\mathrm{red}}} \arrow[d, equal] \\ Y_{R^{\mathrm{red}}} \arrow[r, equal] & Y_{R^{\mathrm{red}}} \end{tikzcd} \tag{4.2.1}$$

Now restrict this moduli description to the full subcategory $\mathrm{Art} \subset \mathrm{Nilp}$. Then $\mathrm{RZ}_Y^{/y}$ can be described as the functor on Art sending (R, α) to the set of isomorphisms classes of triples (X, β, f), where X is a p-divisible group over R equipped with an isomorphism $\beta : X \otimes_{R,\alpha} \overline{\mathbb{F}}_p \to Y$ and where f is a quasi-isogeny $f : X \dashrightarrow Y_R$ such that (4.2.1) commutes.

Lemma 4.2.2. *The natural forgetful map* $\mathrm{RZ}_Y^{/y} \to \mathrm{Def}(Y)$ *sending* (X, β, f) *to* (X, β) *is an isomorphism. In particular, there is an isomorphism of formal schemes* $\widehat{\mathrm{Def}}(Y) \simeq \mathrm{RZ}_Y^{/y}$.

Proof. The commutativity of (4.2.1) expresses the fact that f is a lift of the quasi-isogeny $Y \to Y$ given by the identity. But since quasi-isogenies lift uniquely by [Katz 1981, Lemma 1.1.3.3], the data of f is superfluous and we see that the forgetful map $\mathrm{RZ}_Y^{/y}(R) \to \mathrm{Def}(Y)(R)$ is a bijection for all $R \in \mathrm{Art}$. $\square$

The subgroup

$$\begin{pmatrix} \mathbf{Aut}(Y_0) & 0 \\ \widetilde{\mathcal{H}}_{0,1} & \mathbf{Aut}(Y_1), \end{pmatrix} \subset \mathbf{Aut}(\widetilde{Y})$$

preserves the point $y \in \mathrm{RZ}_Y(\overline{\mathbb{F}}_p)$ and therefore acts on $\widehat{\mathrm{Def}}(Y)$. In particular, the profinite group

$$\mathbf{Aut}(Y_0)(\overline{\mathbb{F}}_p) \times \mathbf{Aut}(Y_1)(\overline{\mathbb{F}}_p) = \mathbf{Aut}(Y)(\overline{\mathbb{F}}_p)$$

acts on $\widehat{\mathrm{Def}}(Y)$. This induces an action of $\mathbf{Aut}(Y)(\overline{\mathbb{F}}_p)$ on $\mathrm{Def}(Y)$ because $\overline{\mathbb{F}}_p$ is an object of $\mathrm{Art} \subset \mathrm{Nilp}$.

Corollary 4.2.3. *This action sends a pair* $(X, \beta) \in \mathrm{Def}(Y)(R)$, *where* X *is a* p*-divisible group over* $\operatorname{Spec} R$ *and* $\beta : X \otimes_{R,\alpha} \overline{\mathbb{F}}_p \to Y_{\overline{\mathbb{F}}_p}$ *is an isomorphism, to* $(X, g \circ \beta)$ *for* $g \in \mathbf{Aut}(Y)(\overline{\mathbb{F}}_p)$.

Proof. This follows from Lemma 4.2.2 and the uniqueness of the isomorphism $\beta : X_{\overline{\mathbb{F}}_p} \to Y$ given $f : X \dashrightarrow Y_R$. $\square$

4.2.4. Since the action of $\widetilde{\mathcal{H}}_{0,1}$ on RZ_Y preserves the point y, there is an induced action

$$\tilde{a}_{\mathrm{RZ}} : \widetilde{\mathcal{H}}_{0,1} \times \widehat{\mathrm{Def}}(Y) \to \widehat{\mathrm{Def}}(Y).$$

The goal of the rest of this section is to prove the following proposition, our proof of which was heavily inspired by the proof of [Howe 2020, Theorem 6.2.1], which deals with the $g = 1$ case.

Proposition 4.2.5. *The action $\tilde{a}_{\mathrm{RZ}}$ factors through an action of $\mathcal{H}_{0,1}$ via the natural quotient map $\widetilde{\mathcal{H}}_{0,1} \to \mathcal{H}_{0,1}$. Moreover the induced action of $\mathcal{H}_{0,1}$ is given by a_{ST}.*

4.3. ***Proof of Proposition 4.2.5.*** Choose isomorphisms

$$T_p Y_0 \simeq \mathbb{Z}_p^{\oplus g}, \quad Y_1 \simeq (\mu_{p^\infty})^{\oplus g},$$

which induce isomorphisms of functors on Art

$$\mathrm{Def}(Y) \simeq \mathscr{H}om(\mathbb{Z}_p^{\oplus g}, (\mu_{p^\infty})^{\oplus g}).$$

In fact if we let $x_1, \dots, x_g \in \mathbb{Z}_p^{\oplus g}$ be the standard basis vectors, then we can in fact identify

$$\mathrm{Def}(Y) \simeq (\mu_{p^\infty})^{\oplus g^2}$$

with coordinates $q_{i,j}$ for $1 \leq i, j \leq g$ and similarly

$$\mathcal{H}_{0,1} \simeq (\mu_{p^\infty})^{\oplus g^2}$$

with coordinates $\zeta_{i,j}$ for $1 \leq i, j \leq g$. For R in Art a morphism

$$\mathrm{Spec}\, R \to \mathrm{Def}(Y)$$

corresponds to elements $q_{i,j} \in 1 + \mathfrak{m}_R$, and the corresponding deformation of Y is the p-divisible group $X_{\underline{q}}$ corresponding to the pushout of (see [Katz 1981, p. 152])

$$0 \to \mathbb{Z}_p^{\oplus g} \to \mathbb{Q}_p^{\oplus g} \to \frac{\mathbb{Q}_p^{\oplus g}}{\mathbb{Z}_p^{\oplus g}} \to 0$$

via the morphism $\mathbb{Z}_{p,R}^{\oplus g} \to \mu_{p^\infty,R}^{\oplus g}$ given by $x_i \mapsto (q_{i,1}, \dots, q_{i,g})$. In fact, there is a pushout diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}^{\oplus g} \arrow[r] \arrow[d] & \mathbb{Z}[1/p]^{\oplus g} \arrow[r] \arrow[d] & \frac{\mathbb{Q}_p^{\oplus g}}{\mathbb{Z}_p^{\oplus g}} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbb{Z}_p^{\oplus g} \arrow[r] & \mathbb{Q}_p^{\oplus g} \arrow[r] & \frac{\mathbb{Q}_p^{\oplus g}}{\mathbb{Z}_p^{\oplus g}} \arrow[r] & 0
\end{tikzcd}$$

and so we can also think of $X_{\underline{q}}$ as the quotient of $\mu_{p^\infty,R}^{\oplus g} \oplus \mathbb{Z}[1/p]^{\oplus g}$ by the image of the map

$$h_{\underline{q}} : \mathbb{Z}^{\oplus g} \to \mu_{p^\infty,R}^{\oplus g} \oplus \mathbb{Z}[1/p]^{\oplus g}$$

given by $x_i \mapsto ((q_{i,1}, \dots, q_{i,g}), (x_i))$.

Let N be an integer such that $q_{i,j}^{p^N} = 1$ for all i, j, which exists since R is Artinian. Then the isogeny

$$\mu_{p^\infty,R}^{\oplus g} \oplus \mathbb{Z}[1/p]^{\oplus g} \to \mu_{p^\infty,R}^{\oplus g} \oplus \mathbb{Z}[1/p]^{\oplus g}, \quad (A, B) \mapsto (p^N A, p^N B)$$

maps $h_{\underline{q}}(\mathbb{Z}^{\oplus g})$ into $h_{\underline{1}}(\mathbb{Z}^{\oplus g})$. Thus it induces a quasi-isogeny

$$f_{\underline{q},N} : X_{\underline{q}} \dashrightarrow X_{\underline{1}} = Y_R, \tag{4.3.1}$$

and the induced quasi-isogeny $X_{\underline{q},\overline{\mathbb{F}}_p} = X_{\underline{1},\overline{\mathbb{F}}_p} \to X_{\underline{1},\overline{\mathbb{F}}_p}$ is given by p^N. It follows that the quasi-isogeny $p^{-N} f_{\underline{q},N}$ is the unique quasi-isogeny lifting the identity $X_{\underline{q},\overline{\mathbb{F}}_p} = X_{\underline{1},\overline{\mathbb{F}}_p} \to X_{\underline{1},\overline{\mathbb{F}}_p}$. Let us write $\underline{q} \in \mathrm{RZ}_Y(R)$ for $p^{-N} f_{\underline{q},N} : X_{\underline{q}} \dashrightarrow Y_R$.

A morphism $\operatorname{Spec} R \to \mathcal{H}_{0,1}$ corresponds to elements $\zeta_{i,j} \in 1 + \mathfrak{m}_R$. The left translation action of $\mathcal{H}_{0,1}$ via the Serre–Tate action is given by

$$a_{\mathrm{ST}}(\underline{\zeta}, \underline{q}) = \underline{\zeta q},$$

where $(\underline{\zeta q})_{i,j} = (\zeta_{i,j} \cdot q_{i,j})$ and where $(\zeta_{i,j} \cdot q_{i,j})$ denotes the multiplication in $\mu_{p^\infty}(R) = 1 + \mathfrak{m}_R$. In terms of p-divisible groups, this correspond to the p-divisible group $X_{\underline{\zeta q}}$. We will write $\underline{\zeta q} \in \mathrm{RZ}_Y(R)$ for the element corresponding to $X_{\underline{\zeta q}}$.

Proof of Proposition 4.2.5. By definition of the action $\tilde{a}_{\mathrm{RZ}}$, it suffices to show that for every fpqc cover $\operatorname{Spec} \tilde{R} \to \operatorname{Spec} R$ and every lift

$$\underline{\tilde{\zeta}} \in \widetilde{\mathcal{H}}_{0,1}(\tilde{R})$$

of $\underline{\zeta} \in \mathcal{H}_{0,1}(\tilde{R})$, we have $\tilde{a}_{\mathrm{RZ}}(\underline{\tilde{\zeta}}, \underline{q}) = \underline{\zeta q}$. There is a universal such lift over the fpqc cover $\tilde{R}$ given by formally adjoining all the p-power roots of all $\zeta_{i,j}$, and it suffices to prove the result for this choice of $\tilde{R}$. Recall that $X_{\underline{q},\tilde{R}}$ is defined as the quotient of

$$\mu_{p^\infty,\tilde{R}}^{\oplus g} \oplus \mathbb{Z}[1/p]^{\oplus g}$$

by the image of the map $h_{\underline{q}}$ which sends the standard basis element $x_i \in \mathbb{Z}^{\oplus g}$ to

$$((q_{i,1}, \dots, q_{i,g}), (x_i)) \in \mu_{p^\infty,\tilde{R}}^{\oplus g} \oplus \mathbb{Z}[1/p]^{\oplus g}.$$

The p-divisible group $X_{\underline{\zeta q},\tilde{R}}$ is defined similarly but then using the map $h_{\underline{\zeta q}}$. The compatible sequence of p-power roots of $\underline{\zeta}$ defined by $\underline{\tilde{\zeta}}$ defines a map

$$\psi_{\underline{\tilde{\zeta}}} : \mu_{p^\infty,\tilde{R}}^{\oplus g} \oplus \mathbb{Z}[1/p]^{\oplus g} \to \mu_{p^\infty,\tilde{R}}^{\oplus g} \oplus \mathbb{Z}[1/p]^{\oplus g}, \quad (A, B) \mapsto (A \cdot L_{\underline{\tilde{\zeta}}}(B), B),$$

where $L_{\underline{\tilde{\zeta}}} : \mathbb{Z}[1/p]^{\oplus g} \to \mu_{p^\infty,\tilde{R}}^{\oplus g}$ is the morphism sending

$$\frac{x_i}{p^n} \mapsto (\zeta_{i,1}^{1/p^n}, \zeta_{i,2}^{1/p^n}, \dots, \zeta_{i,g}^{1/p^n}).$$

It is straightforward to check that this map satisfies

$$\psi_{\underline{\tilde{\zeta}}}(h_{\underline{q}}(\mathbb{Z}^{\oplus g}) = h_{\underline{\zeta q}}(\mathbb{Z}^{\oplus g})$$

and that it thus induces an isomorphism on quotients

$$\phi_{\underline{\tilde{\zeta}}} : X_{\underline{q},\tilde{R}} \to X_{\underline{\zeta q},\tilde{R}}.$$

Choose N sufficiently large such that $\zeta_{i,j}^{p^N} = 1$ and $q_{i,j}^{p^N} = 1$ for all i, j. Let

$$p^{-2N} f_{\underline{\zeta q},2N} : X_{\underline{\zeta q}} \dashrightarrow Y_R$$

be the unique quasi-isogeny lifting the identity map $X_{\underline{\zeta q},\overline{\mathbb{F}}_p} = Y_{\overline{\mathbb{F}}_p} \to Y_{\overline{\mathbb{F}}_p}$ as described in (4.3.1). To prove the proposition it suffices to show that the following diagram commutes:

$$\begin{tikzcd} X_{\underline{q},\tilde{R}} \arrow[r, "\phi_{\underline{\tilde{\zeta}}}"] \arrow[d, dashed, "p^{-N} f_{\underline{q},N}"] & X_{\underline{\zeta q},\tilde{R}} \arrow[d, dashed, "p^{-2N} f_{\underline{\zeta q},2N}"] \\ \left(\dfrac{\mathbb{Q}_p^{\oplus g}}{\mathbb{Z}_p^{\oplus g}}\right)_{\tilde{R}} \oplus \mu_{p^\infty,\tilde{R}}^{\oplus g} \arrow[r, dashed, "\xi_{\underline{\tilde{\zeta}}}"] & \left(\dfrac{\mathbb{Q}_p^{\oplus g}}{\mathbb{Z}_p^{\oplus g}}\right)_{\tilde{R}} \oplus \mu_{p^\infty,\tilde{R}}^{\oplus g} \end{tikzcd} \tag{4.3.2}$$

Here $\xi_{\underline{\tilde{\zeta}}}$ is given by the matrix $\left(\begin{smallmatrix} 1 & 0 \\ \underline{\tilde{\zeta}} & 1 \end{smallmatrix}\right)$, see the beginning of Section 4.2 for the matrix notation. To show that this diagram commutes we consider the auxiliary commutative diagram

$$\begin{tikzcd} \mu_{p^\infty,\tilde{R}}^{\oplus g} \oplus \mathbb{Z}[1/p]^{\oplus g} \arrow[r, "\psi_{\underline{\tilde{\zeta}}}"] \arrow[d, "p^N"] & \mu_{p^\infty,\tilde{R}}^{\oplus g} \oplus \mathbb{Z}[1/p]^{\oplus g} \arrow[d, "p^{2N}"] \\ \mu_{p^\infty,\tilde{R}}^{\oplus g} \oplus \mathbb{Z}[1/p]^{\oplus g} \arrow[r, "p^N \psi_{\underline{\tilde{\zeta}}}"] & \mu_{p^\infty,\tilde{R}}^{\oplus g} \oplus \mathbb{Z}[1/p]^{\oplus g} \end{tikzcd} \tag{4.3.3}$$

The diagram of quasi-isogenies (4.3.2) is obtained from the diagram (4.3.3) by quotienting by the subgroups

$$\begin{tikzcd} h_{\underline{q}}(\mathbb{Z}^{\oplus g}) \arrow[r, "\psi_{\underline{\tilde{\zeta}}}"] \arrow[d, "p^N"] & h_{\underline{\zeta q}}(\mathbb{Z}^{\oplus g}) \arrow[d, "p^{2N}"] \\ h_{\underline{1}}(\mathbb{Z}^{\oplus g}) \arrow[r, "p^N \psi_{\underline{\tilde{\zeta}}}"] & h_{\underline{1}}(\mathbb{Z}^{\oplus g}) \end{tikzcd}$$

and formally inverting certain powers of p. It follows that (4.3.2) is commutative. □

5. The formal neighbourhood of an ordinary point

The goal of this section is to give Serre–Tate coordinates for the formal completions of points in the ordinary locus of Shimura varieties of Hodge type.

In Section 5.1 we specialise to the smooth canonical integral models of Shimura varieties of Hodge type at hyperspecial level, and we moreover assume that the ordinary locus is nonempty. In Section 5.2 we recall a small amount of covariant Dieudonné theory for semiperfect rings, following [Scholze and Weinstein 2013].

In Section 5.4 we prove that the formal completion of the ordinary locus gives a subtorus of the Serre–Tate torus, reproving a special case of [Shankar and Zhou 2021, Theorem 1.1]. We also give a group-theoretic description of the Dieudonné module of the associated p-divisible group. In Section 5.5 we introduce strongly nontrivial actions of algebraic groups on isocrystals, which we will need to confirm the hypotheses of the rigidity theorem of [Chai 2008].

5.1. ***Integral models at hyperspecial level.*** Let the notation be as in Section 2. In particular, we have a Shimura datum (G, X) of Hodge type with reflex field E, a prime p and a prime v of E above p. Moreover there is a symplectic space V and a Hodge embedding $(G, X) \to (\mathcal{G}_V, \mathcal{H}_V)$ and for every sufficiently small $K^p \subset G(\mathbb{A}_f^p)$ there is a sufficiently small $\mathcal{K}^p \subset \mathcal{G}_V(\mathbb{A}_f^p)$ and a finite morphism

$$\mathscr{S}_K := \mathscr{S}_K(G, X) \to \mathcal{S}_{\mathcal{K}} \otimes_{\mathbb{Z}_{(p)}} \mathcal{O}_{E,(v)}. \tag{5.1.1}$$

Recall that there is a $\mathbb{Z}_{(p)}$-lattice $V_{(p)}$ of V on which the symplectic form is $\mathbb{Z}_{(p)}$-valued, and recall that we have defined $K_p \subset G(\mathbb{Q}_p)$ to be its stabiliser. ***From now on we will assume that K_p is a hyperspecial subgroup***, in which case $\mathscr{S}_K$ is the canonical integral model of $\mathbf{Sh}_{K^pK_p}(G, X)$ over $\mathcal{O}_{E_v}$. Moreover the main theorem of [Xu 2020] tells us that the map (5.1.1) is a closed immersion.

5.1.1. The Zariski closure $\mathcal{G}_{\mathbb{Z}_{(p)}}$ of G inside $\mathcal{G}_{V_{(p)}}$ is a reductive group scheme over $\mathbb{Z}_{(p)}$. By [Kisin 2010, Proposition 1.3.2], we can choose tensors $\{s_\alpha\} \subset V_{(p)}^{\otimes}$ whose stabiliser is $\mathcal{G}_{\mathbb{Z}_{(p)}}$. All the results of Section 2 still go through with this choice of tensors.

For $x \in \mathrm{Sh}_{G,K^pK_p}(\overline{\mathbb{F}}_p)$ we have seen in Section 2.1.1 that there are tensors

$$\{s_{\alpha,\mathrm{cris}}\} \subset \mathbb{D}_x^{\otimes},$$

where $\mathbb{D}_x^{\otimes}$ is the rational contravariant Dieudonné module of $A_x[p^\infty]$. Now let $\mathbb{D}(A_x[p^\infty])$ be the *integral* contravariant Dieudonné module. Then as explained in [Shankar and Zhou 2021, Section 6.3], the tensors $\{s_{\alpha,\mathrm{cris}}\}$ lie in

$$\mathbb{D}(A_x[p^\infty])^{\otimes}.$$

It is moreover explained there that there is an isomorphism

$$\mathbb{D}(A_x[p^\infty]) \simeq V_p \otimes_{\mathbb{Z}_{(p)}} \breve{\mathbb{Z}}_p$$

taking $s_{\alpha,\mathrm{cris}}$ to $s_\alpha \otimes 1$.

5.1.2. Let us now drop the level from the notation and write $\mathscr{S}_G$ and $\mathcal{S}_{\mathrm{GSp}}$ respectively for the base changes of $\mathscr{S}_K$ and $\mathcal{S}_{\mathcal{K}}$ respectively to $\breve{\mathbb{Z}}_p$ for some choice of $\mathcal{O}_{E,v} \to \breve{\mathbb{Z}}_p$. Similarly write Sh_G for the special fibre of $\mathscr{S}_G$ and $\mathrm{Sh}_{\mathrm{GSp}}$ for the special fibre of $\mathcal{S}_{\mathrm{GSp}}$. Let $\mathrm{Sh}_{\mathrm{GSp,ord}} \subset \mathrm{Sh}_{\mathrm{GSp}}$ be the dense open ordinary locus and define the ordinary locus of Sh_G by $\mathrm{Sh}_{G,\mathrm{ord}} := \mathrm{Sh}_G \cap \mathrm{Sh}_{\mathrm{GSp,ord}}$. It is an open subset which is nonempty if and only if $E_v = \mathbb{Q}_p$, by [Lee 2018, Corollary 1.0.2]. ***We will assume from now on that $E_v = \mathbb{Q}_p$.***

Lemma 5.1.3. *The ordinary locus $\mathrm{Sh}_{G,\mathrm{ord}}$ is open and dense and equal to the Newton stratum $\mathrm{Sh}_{G,[b],K^pK_p}$ for $[b] \in B(G, \{\mu^{-1}\})$ the maximal element in the partial order introduced in Section 2.1.2.*

The maximal element $[b]$ is known as the *μ-ordinary element*, and the maximal Newton stratum is known as the *μ-ordinary locus*.

Proof. The μ-ordinary locus and the ordinary locus are equal in this case by the proof of [Lee 2018, Corollary 4.3.2], as explained in [Lee 2018, Remark 4.3.3]. The density of the μ-ordinary locus is [Wortmann 2013, Theorem 1.1]; see [KMS 2022, Theorem 3] for a published reference. □

5.1.4. Let $x \in \mathrm{Sh}_G(\overline{\mathbb{F}}_p)$ be an ordinary point and consider the closed immersions of formal neighbourhoods (considered as functors on the category $\mathrm{Nilp}_{\breve{\mathbb{Z}}_p}$ of $\breve{\mathbb{Z}}_p$-algebras where p is nilpotent)

$$\mathscr{S}_G^{/x} := \mathrm{Spf}\, \widehat{\mathcal{O}}_{\mathscr{S}_G, x} \hookrightarrow \mathrm{Spf}\, \widehat{\mathcal{O}}_{\mathcal{S}_{\mathrm{GSp}}, x} =: \mathcal{S}_{\mathrm{GSp}}^{/x}. \tag{5.1.2}$$

Let A be the universal abelian scheme over $\mathcal{S}_{\mathrm{GSp}}$ and let $X = A[p^\infty]$ be the associated p-divisible group over $\mathcal{S}_{\mathrm{GSp}}$. Let $\widehat{\mathrm{Def}}(A_x)$ be the formal deformation space of the abelian variety A_x, that is, the formal scheme representing the functor $\mathrm{Def}(A_x)$ on the category Art of deformations of the abelian variety A_x. Similarly let $\widehat{\mathrm{Def}}(Y)$ be the deformation space of the p-divisible group $X_x =: Y$. There are natural morphisms

$$\mathcal{S}_{\mathrm{GSp}}^{/x} \to \widehat{\mathrm{Def}}(A_x) \to \widehat{\mathrm{Def}}(Y).$$

The first is a closed immersion by the moduli description of $\mathcal{S}_{\mathrm{GSp}}$, and the second morphism is an isomorphism by [Katz 1981, Theorem 1.2.1]. Now [Shankar and Zhou 2021, Theorem 1.1] (see [Noot 1996] for closely related results) implies that the closed formal subscheme

$$\mathscr{S}_G^{/x} \subset \widehat{\mathrm{Def}}(Y)$$

is a p-divisible formal subgroup. The goal of this section is to compute the Dieudonné module of $\mathrm{Sh}_G^{/x}$. We do this by giving a new proof that

$$\mathrm{Sh}_G^{/x} \subset \widehat{\mathrm{Def}}(Y)$$

is a p-divisible formal subgroup, using the methods of Section 4 and results of [Kim 2019].

5.2. *Some covariant Dieudonné theory.*

5.2.1. *A caveat.* In the rest of this subsection we are going to recall some covariant Dieudonné theory for semiperfect rings following [Scholze and Weinstein 2013]. The reason we do this is that the references [Caraiani and Scholze 2017; Kim 2019] are written in this language. Moreover we feel that results such as Lemma 5.2.5 are most naturally stated using the covariant theory.

To avoid potential confusion, we will always write a subscript cov when using covariant Dieudonné theory. The covariant theory and the contravariant theory will interact only once, in Section 5.3, and we will warn the reader again there.

5.2.2. Recall that an $\mathbb{F}_p$-algebra A is *semiperfect* if it is the quotient of a perfect $\mathbb{F}_p$-algebra B and that it is *f-semiperfect* if it is the quotient of a perfect $\mathbb{F}_p$-algebra by a finitely generated ideal. Let A be a semiperfect $\mathbb{F}_p$-algebra and let $A_{\mathrm{cris}}(A)$ be Fontaine's ring of crystalline periods (see [Scholze and Weinstein 2013, Proposition 4.1.3]) with $\varphi : A_{\mathrm{cris}}(A) \to A_{\mathrm{cris}}(A)$ induced by the absolute Frobenius on A.

Definition 5.2.3. A *covariant Dieudonné module* over a semiperfect $\mathbb{F}_p$-algebra A is a pair (M, φ_M), where M is a finite locally free $A_{\mathrm{cris}}(A)$-module and where

$$\varphi_M : \varphi^* M\big[\tfrac{1}{p}\big] \to M\big[\tfrac{1}{p}\big]$$

is an isomorphism such that

$$M \subseteq \varphi_M(M) \subseteq \tfrac{1}{p} M.$$

Remark 5.2.4. Usually one instead asks that

$$pM \subseteq \varphi_M(M) \subseteq M.$$

The reasons for our conventions is that they agree with the conventions in [Caraiani and Scholze 2017; Kim 2019].

A p-divisible group $\mathscr{G}$ over A has a covariant[5] Dieudonné module $(\mathbb{D}_{\mathrm{cov}}(\mathscr{G}), \varphi_{\mathscr{G}})$. For $\operatorname{Spec} A' \to \operatorname{Spec} A$ there is a canonical isomorphism

$$(\mathbb{D}_{\mathrm{cov}}(\mathscr{G}_{A'}), \varphi_{\mathscr{G}_{A'}}) \simeq (\mathbb{D}_{\mathrm{cov}}(\mathscr{G}), \varphi_{\mathscr{G}}) \otimes_{A_{\mathrm{cris}}(A)} A_{\mathrm{cris}}(A').$$

Our covariant Dieudonné modules are normalised as in [Caraiani and Scholze 2017]. In particular, this means that the covariant Dieudonné module of $\mathbb{Q}_p/\mathbb{Z}_p$ over A is $A_{\mathrm{cris}}(A)$ equipped with the trivial Frobenius, and the covariant Dieudonné module of μ_{p^∞} is $A_{\mathrm{cris}}(A)$ equipped with Frobenius given by $1/p$. This also means that the contravariant Dieudonné module is isomorphic to the dual of the covariant Dieudonné module, see [Caraiani and Scholze 2017, footnote on page 692].

Now let $\mathscr{G}$ be a p-divisible group over $\mathbb{F}_p$ with universal cover $\widetilde{\mathscr{G}}$ in the sense of [Scholze and Weinstein 2013, Section 3.1]. If we consider $\widetilde{\mathscr{G}}$ as a functor on Nilp then it is a filtered colimit of spectra of f-semiperfect $\mathbb{F}_p$-algebras by [Scholze and Weinstein 2013, Proposition 3.1.3(iii)] and is thus determined by its restriction to the category of semiperfect $\mathbb{F}_p$-algebras. We can describe it explicitly on the category of f-semiperfect $\mathbb{F}_p$-algebras as follows:

Lemma 5.2.5. *There is a commutative diagram of natural transformation of functors on the category of f-semiperfect $\mathbb{F}_p$-algebras, which evaluated at an object A gives*

$$\widetilde{\mathscr{G}}(A) \xrightarrow{\ \simeq\ } \big(B^+_{\mathrm{cris}}(A) \otimes_{\breve{\mathbb{Q}}_p} \mathbb{D}_{\mathrm{cov}}(\mathscr{G})\big[\tfrac{1}{p}\big]\big)^{\varphi=1},$$

where φ is given by the diagonal Frobenius and where $B^+_{\mathrm{cris}}(A) := A_{\mathrm{cris}}(A)[1/p]$.

[5] We write $\mathbb{D}_{\mathrm{cov}}$ to distinguish from the contravariant Dieudonné theory that we used in Section 3.

Proof. Let A be f-semiperfect, then [Scholze and Weinstein 2013, Theorem 4.1.4] tells us that the covariant Dieudonné module functor over A is fully faithful after inverting p. There is a natural map

$$T_p\mathscr{G}(A) = \mathrm{Hom}_A((\mathbb{Q}_p/\mathbb{Z}_p)_A, \mathscr{G}_A) \to \mathrm{Hom}_{A_{\mathrm{cris}},F}\big(A_{\mathrm{cris}}(A), A_{\mathrm{cris}}(A) \otimes_{\breve{\mathbb{Z}}_p} \mathbb{D}_{\mathrm{cov}}(\mathscr{G})\big)$$
$$\simeq (A_{\mathrm{cris}}(A) \otimes_{\breve{\mathbb{Z}}_p} \mathbb{D}_{\mathrm{cov}}(\mathscr{G}))^{\varphi=1},$$

where the latter bijection is induced by evaluation at 1. After inverting p we get a natural isomorphism

$$\tilde{\mathscr{G}}(A) = \mathrm{Hom}_A((\mathbb{Q}_p/\mathbb{Z}_p)_A, \mathscr{G}_A)\big[\tfrac{1}{p}\big] \to \big(B^+_{\mathrm{cris}}(A) \otimes_{\breve{\mathbb{Q}}_p} \mathbb{D}_{\mathrm{cov}}(\mathscr{G})\big[\tfrac{1}{p}\big]\big)^{\varphi=1}. \qquad \square$$

5.3. ***The Dieudonné module of the Serre–Tate torus.*** Let $x \in \mathrm{Sh}_{G,\mathrm{ord}}(\overline{\mathbb{F}}_p)$ be as above and let $Y = A_x[p^\infty]$ be the corresponding p-divisible group. Recall from Section 4 that $Y = Y_0 \oplus Y_1$ and that both Y_0 and Y_1 lift uniquely to p-divisible groups Y_0^{can} and Y_1^{can} over $\breve{\mathbb{Z}}_p$. Let $\mathrm{Def}(Y)$ be the formal deformation space of Y, considered as a functor on Art together with its extension $\widehat{\mathrm{Def}}(Y)$ to Nilp. We have seen that $\widehat{\mathrm{Def}}(Y)$ has the structure of a p-divisible formal group, and we use $\mathcal{H}_{0,1}$ to denote the corresponding p-divisible group over $\mathrm{Spf}\,\breve{\mathbb{Z}}_p$.

Consider the special fibre $\mathcal{H}_{0,1,\overline{\mathbb{F}}_p}$. Then by Lemma 4.1.2 its p-adic Tate module is given by $\mathscr{H}om(Y_0, Y_1)$. Therefore by [Caraiani and Scholze 2017, Lemmas 4.1.7 and 4.1.8] , we have an isomorphism

$$\mathbb{D}_{\mathrm{cov}}(\mathcal{H}_{0,1,\overline{\mathbb{F}}_p})[1/p] \simeq \mathscr{H}om(\mathbb{D}_{\mathrm{cov}}(Y_0)[1/p], \mathbb{D}_{\mathrm{cov}}(Y_1)[1/p])^{\leq 0},$$

where $\mathscr{H}om$ denotes the internal hom in F-isocrystals and where $(\cdot)^{\leq 0}$ denotes the slope at most 0 part of an F-isocrystal.

5.3.1. Choose an isomorphism (here we use contravariant Dieudonné theory!) $\mathbb{D}(Y) \to V_p \otimes_{\mathbb{Z}_p} \breve{\mathbb{Z}}_p$ sending $s_\alpha \otimes 1$ to $s_{x,\mathrm{cris}}$ as in Section 5.1.1. This induces an isomorphism from $V_p^* \otimes_{\mathbb{Z}_p} \breve{\mathbb{Z}}_p$ to the covariant Dieudonné module $\mathbb{D}_{\mathrm{cov}}(Y)$ and thus gives us Frobenius invariant tensors $\{s_{\alpha,\mathrm{cris}}\} \subset \mathbb{D}_{\mathrm{cov}}(Y)^{\otimes}$. Let $b \in G(\breve{\mathbb{Q}}_p) \subset \mathrm{GL}(V^*)(\breve{\mathbb{Q}}_p)$ be the element corresponding to the Frobenius in $\mathbb{D}_{\mathrm{cov}}(Y)[1/p]$. Then there is an inclusion of F-isocrystals

$$\mathscr{H}om(\mathbb{D}_{\mathrm{cov}}(Y_0)[1/p], \mathbb{D}_{\mathrm{cov}}(Y_1)[1/p]) \subset \mathscr{H}om(\mathbb{D}_{\mathrm{cov}}(Y)[1/p], \mathbb{D}_{\mathrm{cov}}(Y)[1/p]), \tag{5.3.1}$$

which sends $f : \mathbb{D}_{\mathrm{cov}}(Y_0)[1/p] \to \mathbb{D}_{\mathrm{cov}}(Y_1)[1/p]$ to

$$\mathrm{Id} + f : \mathbb{D}_{\mathrm{cov}}(Y_0)[1/p] \oplus \mathbb{D}_{\mathrm{cov}}(Y_1)[1/p] \to \mathbb{D}_{\mathrm{cov}}(Y_0)[1/p] \oplus \mathbb{D}_{\mathrm{cov}}(Y_1)[1/p].$$

The map in equation (5.3.1) realises the source as the slope -1 part of the target.

5.3.2. Write $\mathfrak{gl}(V^*)$ for the Lie algebra of the algebraic group $\mathrm{GL}(V^*) \otimes \breve{\mathbb{Q}}_p$ and identify it with the vector space of endomorphisms of $V^* \otimes \breve{\mathbb{Q}}_p$ equipped with the commutator bracket. We can equip $\mathfrak{gl}(V^*)$ with the structure of an F-isocrystal by letting Frobenius act by conjugation by $b \in \mathrm{GL}(V^*)(\breve{\mathbb{Q}}_p)$. Let us write $(\mathfrak{gl}(V^*), \mathrm{Ad}\, b\sigma)$ for this isocrystal.

Using the isomorphism $V_p^* \otimes_{\mathbb{Z}_p} \breve{\mathbb{Z}}_p \simeq \mathbb{D}_{\mathrm{cov}}(Y)$ as above, we can identify the F-isocrystal on the right hand side of (5.3.1) with $(\mathfrak{gl}(V^*), \mathrm{Ad}\, b\sigma)$. There is a sub-$F$-isocrystal

$$(\mathfrak{g}, \mathrm{Ad}\, b\sigma) \subset (\mathfrak{gl}(V^*), \mathrm{Ad}\, b\sigma),$$

where $\mathfrak{g} = \mathrm{Lie}\, G \otimes \breve{\mathbb{Q}}_p$. By Lemma 5.3.3 below, the subspace $\mathfrak{g} \subset \mathfrak{gl}(V^*)$ is precisely the subspace of those endomorphisms g of $V^* \otimes \breve{\mathbb{Q}}_p$ that satisfy $g^*(s_\alpha \otimes 1) = 0$ for all tensors s_α.

Lemma 5.3.3. *Let C be a field of characteristic zero and let W be a finite dimensional C vector space. Let $H \subset \mathrm{GL}(W)$ be a connected reductive group that is the stabiliser of a collection of tensors $\{t_\alpha\}_{\alpha \in \mathscr{A}} \subset W^{\otimes}$. Then the Lie algebra $\mathfrak{h} \subset \mathfrak{gl}(W)$ is given by the subspace*

$$\{H \in \mathfrak{gl}(W) : H^*(t_\alpha) = 0 \textit{ for all } \alpha \in \mathscr{A}\}.$$

Proof. The Lie algebra is given by the kernel of the map $G(C[\epsilon]/(\epsilon^2)) \to G(C)$. Thus it consists of matrices of the form $1 + \epsilon M$, where $M \in \mathfrak{gl}(W)$, such that for $\alpha \in \mathscr{A}$ we have

$$(1 + \epsilon M)^*(t_\alpha \otimes 1) = t_\alpha.$$

But this is equivalent to $(\epsilon M)^*(t_\alpha \otimes 1) = 0$ or $M^*(t_\alpha) = 0$. □

Let us write

$$(\mathfrak{g}, \mathrm{Ad}\, b\sigma)^{-1} \subset \mathscr{H}om(\mathbb{D}_{\mathrm{cov}}(Y_0)[1/p], \mathbb{D}_{\mathrm{cov}}(Y_1)[1/p]) = \mathbb{D}_{\mathrm{cov}}(\mathcal{H}_{0,1,\overline{\mathbb{F}}_p})[1/p] \tag{5.3.2}$$

for the slope -1 subspace of the F-isocrystal $(\mathfrak{g}, \mathrm{Ad}\, b\sigma)$. Then by [Kim 2019, Lemma 3.1.3] and its proof, there is an inclusion of p-divisible groups

$$\mathcal{H}^G_{0,1,\overline{\mathbb{F}}_p} \subset \mathcal{H}_{0,1,\overline{\mathbb{F}}_p} \tag{5.3.3}$$

inducing (5.3.2) upon taking rational covariant Dieudonné modules. Since both of these p-divisible groups have étale Serre duals, there is a unique lift $\mathcal{H}^G_{0,1}$ of $\mathcal{H}^G_{0,1,\overline{\mathbb{F}}_p}$ to $\breve{\mathbb{Z}}_p$ and a unique lift

$$\mathcal{H}^G_{0,1} \subset \mathcal{H}_{0,1}$$

of the inclusion (5.3.3).

Lemma 5.3.4. *Let A be an f-semiperfect $\mathbb{F}_p$-algebra. Then the inclusion*

$$\widetilde{\mathcal{H}}^G_{0,1,\overline{\mathbb{F}}_p}(A) \subset \widetilde{\mathcal{H}}_{0,1,\overline{\mathbb{F}}_p}(A) = \mathscr{H}om(Y_{0,A}, Y_{1,A})[1/p]$$

identifies $\widetilde{\mathcal{H}}^G_{0,1,\overline{\mathbb{F}}_p}(A)$ with the subspace of those quasi-endomorphisms $f : Y_{0,A} \dashrightarrow Y_{1,A}$ such that the induced quasi-endomorphism

$$g = \left(\begin{smallmatrix} 0 & 0 \\ f & 0 \end{smallmatrix}\right) : Y_A \dashrightarrow Y_A$$

induces an endomorphism $\mathbb{D}_{\mathrm{cov}}(Y_A)[1/p] \to \mathbb{D}_{\mathrm{cov}}(Y_A)[1/p]$ satisfying $g^*(s_{\alpha,\mathrm{cris}} \otimes 1) = 0$.

Proof. This follows from Lemma 5.2.5 in combination with Lemma 5.3.3. □

Remark 5.3.5. The statement of Lemma 5.3.4 contradicts [Kim 2019, Lemma 3.1.3], which implies that the inclusion

$$\widetilde{\mathcal{H}}^G_{0,1,\overline{\mathbb{F}}_p}(A) \subset \widetilde{\mathcal{H}}_{0,1,\overline{\mathbb{F}}_p}(A) = \mathscr{H}om(Y_{0,A}, Y_{1,A})[1/p]$$

identifies $\widetilde{\mathcal{H}}^G_{0,1,\overline{\mathbb{F}}_p}(A)$ with the subspace of those quasi-endomorphisms $f : Y_{0,A} \dashrightarrow Y_{1,A}$ such that the induced quasi-endomorphism

$$g = \left(\begin{smallmatrix} 0 & 0 \\ f & 0 \end{smallmatrix}\right) : Y_A \dashrightarrow Y_A$$

induces an endomorphism $\mathbb{D}_{\mathrm{cov}}(Y_A)[1/p] \to \mathbb{D}_{\mathrm{cov}}(Y_A)[1/p]$ satisfying $g^*(s_{\alpha,\mathrm{cris}} \otimes 1) = s_{\alpha,\mathrm{cris}} \otimes 1$. This cannot be correct because $\widetilde{\mathcal{H}}^G_{0,1,\overline{\mathbb{F}}_p}(A)$ is stable under addition and if g_1, g_2 both satisfy $g^*(s_{\alpha,\mathrm{cris}} \otimes 1) = s_{\alpha,\mathrm{cris}} \otimes 1$ then their sum $g_1 + g_2$ does not.

The following lemma and its corollary essentially follow from [Kim 2019, Proposition 3.2.4]. However the construction there is incorrect because of the error in [Kim 2019, Lemma 3.1.3] pointed out above. Once the subgroup in the statement of Lemma 5.3.6 has been shown to exist with the properties proved in Corollary 5.3.7, the rest of the arguments in [Kim 2019] go through without further changes.

Lemma 5.3.6. *There is a closed subgroup*

$$\mathbf{Aut}_G(\widetilde{Y}) \subset \mathbf{Aut}(\widetilde{Y})$$

such that on A-points for f-semiperfect $\overline{\mathbb{F}}_p$-algebras A, it is the subgroup of those quasi-isogenies $g : Y_A \dashrightarrow Y_A$ that induce isomorphisms $g : \mathbb{D}_{\mathrm{cov}}(Y_A)[1/p] \to \mathbb{D}_{\mathrm{cov}}(Y_A)[1/p]$ satisfying

$$g^*(s_{\alpha,\mathrm{cris}} \otimes 1) = s_{\alpha,\mathrm{cris}} \otimes 1.$$

We will call such quasi-isogenies *tensor-preserving quasi-isogenies.*

Proof. First of all by [Caraiani and Scholze 2017, Lemma 4.2.10] the functor $\mathbf{Aut}(\widetilde{Y})$ satisfies

$$\mathbf{Aut}(\widetilde{Y})(R) = \mathbf{Aut}(\widetilde{Y})(R/p),$$

for all $R \in \mathrm{Nilp}$. Thus we can define a closed subfunctor of $\mathbf{Aut}(\widetilde{Y})$ by specifying its values on $\overline{\mathbb{F}}_p$-algebras.

The matrix description of $\mathbf{Aut}(\widetilde{Y})$ in Section 4.2 gives us a semidirect product decomposition (see [Caraiani and Scholze 2017, Proposition 4.2.11, Remark 4.2.12])

$$\mathbf{Aut}(\widetilde{Y})_{\overline{\mathbb{F}}_p} := \mathscr{H}om(Y_0, Y_1)[1/p] \rtimes \left(\mathbf{Aut}(\widetilde{Y}_0)_{\overline{\mathbb{F}}_p} \times \mathbf{Aut}(\widetilde{Y}_1)_{\overline{\mathbb{F}}_p}\right).$$

Here we are using the map

$$\mathscr{H}om(Y_0, Y_1)[1/p] \to \mathbf{Aut}(\widetilde{Y})_{\overline{\mathbb{F}}_p}, \quad f \mapsto \left(\begin{smallmatrix} 1 & 0 \\ f & 1 \end{smallmatrix}\right)$$

to realise $\mathscr{H}om(Y_0, Y_1)[1/p]$ as the subgroup of lower triangular automorphisms of $\widetilde{Y}$. The condition that $\left(\begin{smallmatrix} 1 & 0 \\ f & 1 \end{smallmatrix}\right) = 1 + f$ satisfies $(1+f)^*(s_{\alpha,\mathrm{cris}} \otimes 1) = s_{\alpha,\mathrm{cris}} \otimes 1$ is equivalent to the condition that $f^*(s_{\alpha,\mathrm{cris}} \otimes 1) = 0$. Thus we see that the intersection $\mathscr{H}om(Y_0, Y_1)[1/p]$ with $\mathbf{Aut}_G(\widetilde{Y})$ is given by

$$\widetilde{\mathcal{H}}^G_{0,1,\overline{\mathbb{F}}_p} \subset \widetilde{\mathcal{H}}_{0,1,\overline{\mathbb{F}}_p} = \mathscr{H}om(Y_0, Y_1)[1/p].$$

By Lemma 5.3.4, this is representable by a closed subgroup.

We can identify the group $(\mathbf{Aut}(\tilde{Y}_1)_{\overline{\mathbb{F}}_p} \times \mathbf{Aut}(\tilde{Y}_0)_{\overline{\mathbb{F}}_p})$ with the locally profinite group scheme associated to the locally profinite group $\mathbf{Aut}(\widetilde{Y})(\overline{\mathbb{F}}_p)$. Using Dieudonné theory, we can identify this locally profinite group with the σ-centraliser of b in $\mathrm{GL}(V^*)(\breve{\mathbb{Q}}_p)$, where we recall that we have fixed an isomorphism $V_p^* \otimes_{\mathbb{Z}_p} \breve{\mathbb{Z}}_p \simeq \mathbb{D}_{\mathrm{cov}}(Y)$ giving rise to $b \in G(\breve{\mathbb{Q}}_p)$. The subgroup of tensor-preserving automorphisms of $\widetilde{Y}$ over $\overline{\mathbb{F}}_p$ can be identified with $J_b(\mathbb{Q}_p)$, the σ-centraliser of $b \in G(\breve{\mathbb{Q}}_p)$, which is a closed subgroup.

Note that $J_b(\mathbb{Q}_p) \subset G(\breve{\mathbb{Q}}_p)$ stabilises $(\mathfrak{g}, \mathrm{Ad}\, b\sigma)^{-1}$ because it acts on $\mathfrak{g}$ via automorphisms that preserve the slope decomposition. Using Lemma 5.3.4 we see that the closed subgroup

$$\widetilde{\mathcal{H}}^G_{0,1,\overline{\mathbb{F}}_p} \rtimes \underline{J_b(\mathbb{Q}_p)} \subset \mathbf{Aut}(\widetilde{Y})_{\overline{\mathbb{F}}_p},$$

has the required properties over $\overline{\mathbb{F}}_p$, and so we are done. $\square$

Since the R-points of $\widetilde{\mathcal{H}}^G_{0,1}$ and $\underline{J_b(\mathbb{Q}_p)}$ both only depend on R/p, we see that

$$\widetilde{\mathcal{H}}^G_{0,1} \rtimes \underline{J_b(\mathbb{Q}_p)} = \mathbf{Aut}_G(\widetilde{Y})$$

describes the unique lift to $\breve{\mathbb{Z}}_p$. This identifies $\widetilde{\mathcal{H}}^G_{0,1}$ with the neutral component $\mathbf{Aut}_G(\widetilde{Y})^\circ$ of $\mathbf{Aut}_G(\widetilde{Y})$.

Corollary 5.3.7. *The identity component*

$$\widetilde{\mathcal{H}}^G_{0,1,\overline{\mathbb{F}}_p} = \mathbf{Aut}_G(\widetilde{Y})^\circ \subset \mathbf{Aut}_G(\widetilde{Y})$$

is isomorphic to $\operatorname{Spf} S$ *where S is the p-adic completion of* $\breve{\mathbb{Z}}_p[\![x_1^{1/p^\infty}, \dots, x_d^{1/p^\infty}]\!]$ *for some d.*

Proof. This is true for $\widetilde{\mathcal{H}}^G_{0,1,\overline{\mathbb{F}}_p}$ because it is the universal cover of a p-divisible group, see [Scholze and Weinstein 2013, Corollary 3.1.5, Section 6.4]. $\square$

5.4. *Serre–Tate coordinates for Hodge type Shimura varieties.* Recall that $x \in \mathrm{Sh}_G(\overline{\mathbb{F}}_p)$ is an ordinary point with associated element $b = b_x \in G(\breve{\mathbb{Q}}_p)$. Recall also from Section 2.1.1 that we have a $G(\overline{\mathbb{Q}}_p)$ conjugacy class of cocharacters $\{\mu\}$ coming from the Shimura datum X and the fixed place v of E.

Lemma 5.4.1. *The conjugacy class of fractional cocharacters $\{\nu_{[b]}\}$ defined by $[b]$ is equal to $\{\mu^{-1}\}$.*

Proof. The ordinary locus is equal to the μ-ordinary locus by Lemma 5.1.3. Therefore we have that $\{\nu_{[b]}\} = \{\bar{\mu}\}$, where $\{\bar{\mu}\}$ is the Galois-average of $\{\mu^{-1}\}$, see [Shankar and Zhou 2021, Section 2.1]. But since $G_{\mathbb{Q}_p}$ is unramified and the local reflex field E_v is equal to $\mathbb{Q}_p$, there is a cocharacter μ defined over $\mathbb{Q}_p$ inducing the conjugacy class of cocharacters $\{\mu\}$. It follows that $\{\bar{\mu}\} = \{\mu^{-1}\}$. $\square$

Let $\{\lambda\}$ be a conjugacy class of (fractional) cocharacters of a connected reductive group H over an algebraically closed field C. Let T be a maximal torus, let λ be a representative of $\{\lambda\}$ factoring through T and let $B \supset T$ be a Borel. Let $\rho \in X^*(T)$ be the half sum of the positive roots with respect to B. Then the pairing $\langle 2\rho, \lambda\rangle$ does not depend on the choice of T, B or λ, and we denote it by $\langle 2\rho, \{\lambda\}\rangle$.

Corollary 5.4.2. *The p-divisible formal group $\mathcal{H}^G_{0,1,\overline{\mathbb{F}}_p}$ has dimension $\langle 2\rho, \{\mu\}\rangle$*

Proof. The dimension of $\mathcal{H}^G_{0,1,\overline{\mathbb{F}}_p}$ is equal to $\langle 2\rho, \nu_{[b]}\rangle$ by [Kim 2019, Proposition 3.1.4], which is equal to $\langle 2\rho, \{\mu^{-1}\}\rangle = \langle 2\rho, \{\mu\}\rangle$ by Lemma 5.4.1. □

Proposition 5.4.3. *The closed formal subscheme*

$$\mathscr{S}_G^{/x} \hookrightarrow \mathcal{S}_{\mathrm{GSp}}^{/x} \hookrightarrow \widehat{\mathrm{Def}}(Y)$$

introduced in (5.1.2), is a p-divisible formal subgroup. The induced inclusion of p-divisible groups

$$\mathscr{S}_G^{/x}[p^\infty] \subset \widehat{\mathrm{Def}}(Y)[p^\infty] = \mathcal{H}_{0,1}$$

induces the inclusion

$$(\mathfrak{g}, \mathrm{Ad}\, b\sigma)^{-1} \subset \mathscr{H}om(\mathbb{D}_{\mathrm{cov}}(Y_0)[1/p], \mathbb{D}_{\mathrm{cov}}(Y_1)[1/p])$$

from (5.3.2) on rational covariant Dieudonné modules of their special fibres.

Proof. By [Kim 2019, Theorem 4.3.1], the closed formal subscheme $\mathscr{S}_G^{/x} \subset \widehat{\mathrm{Def}}(Y)$ is stable under the action of

$$\mathbf{Aut}_G(\widetilde{Y})^\circ \subset \mathbf{Aut}(\widetilde{Y})^\circ.$$

We can identify these groups with

$$\widetilde{\mathcal{H}}^G_{0,1} \subset \widetilde{\mathcal{H}}_{0,1} \tag{5.4.1}$$

By Proposition 4.2.5, the action of $\widetilde{\mathcal{H}}_{0,1}$ on $\widehat{\mathrm{Def}}(Y)$ factors through the natural action of $\mathcal{H}_{0,1}$ on $\widehat{\mathrm{Def}}(Y)$ by left translation, via the natural quotient map

$$\widetilde{\mathcal{H}}_{0,1} \to \mathcal{H}_{0,1}.$$

The inclusion $\mathcal{H}^G_{0,1} \subset \mathcal{H}_{0,1}$ induces an inclusion $T_p\mathcal{H}^G_{0,1} \subset T_p\mathcal{H}_{0,1}$ which induces (5.4.1) after inverting p. This implies that the action of $\widetilde{\mathcal{H}}^G_{0,1}$ on $\mathscr{S}_G^{/x}$ factors through an action of $\mathcal{H}^G_{0,1}$ via the natural quotient map

$$\widetilde{\mathcal{H}}^G_{0,1} \to \mathcal{H}^G_{0,1}.$$

Now consider the closed point $\{x\} \in \mathrm{Sh}_G^{/x}$. Then the associated orbit map gives a closed immersion

$$\mathcal{H}_{0,1,\overline{\mathbb{F}}_p} \hookrightarrow \mathrm{Def}(Y)_{\overline{\mathbb{F}}_p}.$$

This means that we similarly get a closed immersion

$$\mathcal{H}^G_{0,1,\overline{\mathbb{F}}_p} \subset \mathrm{Sh}_G^{/x}.$$

By [Kim 2019, Proposition 3.1.4], the formal scheme $\mathrm{Def}(Y)_{G,\overline{\mathbb{F}}_p}$ has dimension $\langle 2\rho, \{\nu_{[b]}\}\rangle$, which is equal to $\langle 2\rho, \{\mu\}\rangle$ by Lemma 5.4.1, which in turn is equal to the dimension of $\mathrm{Sh}_G^{/x}$. It follows that the orbit map induces an isomorphism

$$\mathcal{H}^G_{0,1,\overline{\mathbb{F}}_p} \to \mathrm{Sh}_G^{/x}$$

and that $\mathrm{Sh}_G^{/x}$ is a formal subgroup of $\mathrm{Def}(Y)_{\overline{\mathbb{F}}_p}$ satisfying the conclusions of the proposition. It remains to show that $\mathscr{S}_G^{/x} \subset \mathrm{Def}(Y)$ is a formal subgroup, which follows from [Shankar and Zhou 2021, Theorem 1.1]. □

5.4.4. *The action of automorphism groups.* Let the notation be as in Section 5. Recall that we have fixed an isomorphism $\mathbb{D}_{\mathrm{cov}}(Y) \simeq V_p^* \otimes \breve{\mathbb{Z}}_p$ sending $s_\alpha \otimes 1$ to $s_{\alpha,\mathrm{cris}}$. This gives us an element $b \in G(\breve{\mathbb{Q}}_p) \subset \mathrm{GL}(V^*)(\breve{\mathbb{Q}}_p)$ corresponding to the Frobenius in $\mathbb{D}_{\mathrm{cov}}(Y)[1/p]$.

Recall from Section 4 that there is an action of $\mathbf{Aut}(\widetilde{Y})$ on RZ_Y. Recall from the discussion before Corollary 4.2.3, that $\underline{\mathbf{Aut}(Y)(\overline{\mathbb{F}}_p)} \subset \mathbf{Aut}(\widetilde{Y})$ preserves the $\overline{\mathbb{F}}_p$-point $y \in \mathrm{RZ}_Y(\overline{\mathbb{F}}_p)$ corresponding to the identity map of Y, and that this induces an action of the profinite group $\mathbf{Aut}(Y)(\overline{\mathbb{F}}_p)$ on $\mathrm{Def}(Y)$. This action is described in Corollary 4.2.3.

5.4.5. Recall that there are closed immersions of topological groups

$$\begin{tikzcd} J_b(\mathbb{Q}_p) \arrow[r, hook] \arrow[d, hook] & G(\breve{\mathbb{Q}}_p) \arrow[d] \\ \mathrm{GL}(V^*)_b(\mathbb{Q}_p) \arrow[r, hook] & \mathrm{GL}(V^*)(\breve{\mathbb{Q}}_p) \end{tikzcd}$$

where $\mathrm{GL}(V^*)_b(\mathbb{Q}_p) = \mathbf{Aut}(\widetilde{Y})(\overline{\mathbb{F}}_p)$ is the σ-centraliser of b in $\mathrm{GL}(V^*)(\breve{\mathbb{Q}}_p)$. Let us write $U_p \subset J_b(\mathbb{Q}_p)$ for the compact open subgroup given by the intersection

$$\mathbf{Aut}_G(\widetilde{Y})(\overline{\mathbb{F}}_p) \cap \mathbf{Aut}(Y)(\overline{\mathbb{F}}_p).$$

Then U_p acts on $\widetilde{\mathcal{H}}^G_{0,1} \subset \mathscr{H}om(Y_0, Y_1)[1/p]$ and preserves the action of $T_p\mathcal{H}^G_{0,1}$, and thus acts on the quotient $\mathcal{H}_{0,1} \simeq \mathscr{S}_G^{/x}$. By Proposition 5.4.3 and the proof of Lemma 5.3.6, the induced action on rational Dieudonné modules can be identified with the natural action of $U_p \subset J_b(\mathbb{Q}_p)$ on

$$(\mathfrak{g}, \mathrm{Ad}\, b\sigma)^{-1} \subset (\mathfrak{g}, \mathrm{Ad}\, b\sigma).$$

In order to apply the rigidity result of Chai [2008] we need to understand this action. We will do this in more generality in the next section.

5.5. ***Strongly nontrivial actions.*** Let G be a connected reductive group over $\mathbb{Q}_p$. Let $b \in G(\breve{\mathbb{Q}}_p)$ be an element and consider the F-isocrystal $(\mathfrak{g}, \mathrm{Ad}\, b\sigma)$, where $\mathfrak{g} = \mathrm{Lie}\, G \otimes \breve{\mathbb{Q}}_p$ equipped with its action of $J_b(\mathbb{Q}_p)$. If we replace b by a σ-conjugate b', then $J_b(\mathbb{Q}_p)$ and $J_{b'}(\mathbb{Q}_p)$ are conjugate in $G(\breve{\mathbb{Q}}_p)$, and there is an isomorphism of isocrystals $(\mathfrak{g}, \mathrm{Ad}\, b\sigma) \simeq (\mathfrak{g}, \mathrm{Ad}\, b'\sigma)$.

Let $\lambda \in \mathbb{Q}$ and let $N_\lambda \subset (\mathfrak{g}, \mathrm{Ad}\, b\sigma)$ be the largest sub-F-isocrystal of slope λ. Then because $J_b(\mathbb{Q}_p)$ acts on $(\mathfrak{g}, \mathrm{Ad}\, b\sigma)$ via F-isocrystal automorphisms, it preserves the subspace N_λ. Let us also denote by b the image of b in $\mathrm{GL}(\mathfrak{g})$, then there is a homomorphism of algebraic groups

$$J_b \to \mathrm{GL}(\mathfrak{g})_b,$$

where $\mathrm{GL}(\mathfrak{g})_b$ denotes the σ-centraliser of b in $\mathrm{GL}(\mathfrak{g})$. There is a parabolic subgroup

$$P(\lambda) \subset \mathrm{GL}(\mathfrak{g})$$

consisting of automorphisms of the $\breve{\mathbb{Q}}_p$-vector space $\mathfrak{g}$ that preserve the slope filtration on the F-isocrystal $(\mathfrak{g}, \mathrm{Ad}\, b\sigma)$, and after potentially replacing b by a σ-conjugate, the image of b lands in $P(\lambda)$. There is thus a group homomorphism

$$J_b \to P(\lambda)_b,$$

where $P(\lambda)_b$ denotes the σ-centraliser of b in $P(\lambda)$. Since N_λ is a graded quotient of the slope filtration of the F-isocrystal $(\mathfrak{g}, \operatorname{Ad} b\sigma)$, there is an induced quotient map $P(\lambda) \to \mathrm{GL}(N_\lambda)$ and this induces a group homomorphism

$$J_b \to \mathrm{GL}(N_\lambda)_b,$$

where $\mathrm{GL}(N_\lambda)_b$ denotes the σ-centraliser of b in $\mathrm{GL}(N_\lambda)$. Let E be the $\mathbb{Q}_p$-algebra of endomorphisms of the F-isocrystal N_λ and let $E^\times$ be the functor on $\mathbb{Q}_p$-algebras given by $R \mapsto (R \otimes E)^\times$. Then there is a natural isomorphism $E^\times \simeq \mathrm{GL}(N_\lambda)_b$.

Let $\mathrm{GL}(E)$ be the general linear group of E considered as a $\mathbb{Q}_p$-vector space and let $E^\times \to \mathrm{GL}(E)$ be the natural map corresponding to the action of E on itself by left translation. Consider E as a $\mathbb{Q}_p$-linear representation of J_b via $J_b \to E^\times$, then the goal of this section is to prove the following result:

Proposition 5.5.1. *Let $T \subset J_b$ be a maximal torus. If $\lambda \neq 0$, then the trivial representation of T does not occur in the representation of T given by E.*

Proof of Proposition 5.5.1. After replacing b by a σ-conjugate we can arrange for it to satisfy (see [Kottwitz 1985, Section 4])

$$b\sigma(b)\cdots\sigma^{r-1}(b) = (r\nu_b)(p)$$

for some r. Here ν_b is the Newton cocharacter of b, which is defined over $\mathbb{Q}_{p^r}$. Let $M_{\nu_b} \subset G \otimes \breve{\mathbb{Q}}_p$ denote the centraliser of the cocharacter ν_b. By [Kim 2019, Proposition 2.2.6], there is a unique isomorphism

$$J_b \otimes \breve{\mathbb{Q}}_p \to M_{\nu_b}$$

such the composition $J_b(\mathbb{Q}_p) \subset J_b(\breve{\mathbb{Q}}_p) \to M_{\nu_b}(\breve{\mathbb{Q}}_p) \subset G(\breve{\mathbb{Q}}_p)$ is the defining inclusion of $J_b(\mathbb{Q}_p)$ as the σ-centraliser of b in $G(\breve{\mathbb{Q}}_p)$.

After tensoring up to $\breve{\mathbb{Q}}_p$, there is a commutative diagram, where L^{reg} is the left regular representation of $\mathrm{GL}(N_\lambda)$ on $\mathrm{GL}(\mathrm{End}(N_\lambda))$,

$$\begin{tikzcd}
 & \mathrm{GL}(E_{\breve{\mathbb{Q}}_p}) \arrow[r, "\simeq"] & \mathrm{GL}(\mathrm{End}(N_\lambda)) \\
J_{b,\breve{\mathbb{Q}}_p} \arrow[r] \arrow[d, "\simeq"] & E^\times_{\breve{\mathbb{Q}}_p} \arrow[u] \arrow[r, "\simeq"] & \mathrm{GL}(N_\lambda) \arrow[u, "\mathrm{L}_{\mathrm{reg}}"'] \\
M_{\nu_b} \arrow[r] & P(\lambda) \arrow[ur] &
\end{tikzcd}$$

If we show that the trivial representation of $T \otimes \breve{\mathbb{Q}}_p$ does not occur in $E \otimes \breve{\mathbb{Q}}_p$, then it follows that the trivial representation of T does not occur in E. The representation $W = \mathrm{End}(N_\lambda)$ of $J_{b,\breve{\mathbb{Q}}_p}$ defined by composition with the left regular representation is a direct sum of copies of the representation N_λ. Therefore it suffices to show that the representation N_λ of $T \otimes \breve{\mathbb{Q}}_p$ does not contain the trivial representation.

We note that $T \otimes \breve{\mathbb{Q}}_p =: T'$ is a maximal torus of M_{ν_b} acting on the associated graded of the slope filtration of the F-isocrystal $(\mathfrak{g}, \operatorname{Ad} b\sigma)$. Since ν_b is a central cocharacter of M_{ν_b} by definition, we see that $(r\nu_b)(p) \in T'(\breve{\mathbb{Q}}_p)$. To determine the slope decomposition of the F-isocrystal $(\mathfrak{g}, \operatorname{Ad} b\sigma)$, it suffices to determine the slope decomposition of the F^r-isocrystal

$$(\mathfrak{g}, (\operatorname{Ad} b\sigma)^r)$$

for some positive integer r.

Let C be an algebraic closure of $\breve{\mathbb{Q}}_p$ and consider the action of T'_C on $\mathfrak{g}_C$ via the adjoint action of G_C. Then we have a decomposition

$$\mathfrak{g}_C \simeq \mathfrak{t}'_C \oplus \Bigl(\bigoplus_{\alpha \in \Phi} U_\alpha\Bigr),$$

where $\Phi \subset X^*(T'_C)$ consists of the simple roots of T_C. There is a similar decomposition

$$\mathfrak{g} \simeq \mathfrak{t}' \oplus \Bigl(\bigoplus_{\alpha_0 \in \Phi_0} U_{\alpha_0}\Bigr), \tag{5.5.1}$$

where $\Phi_0 \in X^*(T'_C)_I$ is the image of Φ and where $I = \operatorname{Gal}(C/\breve{\mathbb{Q}}_p)$ is the inertia group.

Now we choose an integer r with the following properties: the isomorphism $J_b \otimes \breve{\mathbb{Q}}_p \to M_{\nu_b}$ is defined over $\mathbb{Q}_{p^r}$, the equation

$$b\sigma(b) \cdots \sigma^{r-1}(b) = (r\nu_b)(p)$$

is satisfied, and the decomposition (5.5.1) is defined over $\mathbb{Q}_{p^r}$. Then each U_{α_0} is stable under the action of σ^r and $(\operatorname{Ad} b\sigma)^r$ acts on it by $(r\nu_b)(p)\sigma^r$. The operator $\operatorname{Ad} b\sigma$ moreover acts trivially on $\mathfrak{t}'$, and thus for nonzero λ we have that

$$N_\lambda \subset \bigoplus_{\alpha_0} U_{\alpha_0}.$$

After basechanging to C, we see that

$$N_\lambda \subset \bigoplus_{\alpha} U_\alpha.$$

Thus T'_C acts on N_λ via a subset of the nontrivial characters given by the simple roots $\Phi \subset X^*(T'_C)$, and therefore the trivial representation of T' does not occur in N_λ and thus it does not occur in E. □

6. Proof of the main theorems

There are two final ingredients that are introduced in this section. In Section 6.1, we prove the local stabiliser principle of Chai and Oort [2009, Theorem 9.5], which shows that the formal completion of the Zariski closure of a prime-to-p Hecke orbit is stable under the action of a large p-adic Lie group. In Section 6.2.2 we give a summary of results of [Chai 2003], which relates Serre–Tate coordinates of families of ordinary abelian varieties to the p-adic monodromy groups of these abelian varieties. Then in Section 6.3 we put everything together to prove Theorem I. In Section 6.4 we prove Corollary 6.4.1, which is a generalisation of Theorem I to Shimura varieties of abelian type.

We will use the notation introduced in Section 2 and Section 5.1 and moreover we will keep track of the level again. Moreover, all our schemes will now implicitly live over $\overline{\mathbb{F}}_p$. Let $x \in \mathrm{Sh}_{G,K^pK_p}(\overline{\mathbb{F}}_p)$ and let $\tilde{x}$ be a lift of x to $\mathrm{Sh}_{G,K_p}(\overline{\mathbb{F}}_p)$. Then the *prime-to-p Hecke orbit* of x is defined to be the image $H_{K^p}(x) \subset \mathrm{Sh}_{G,K^pK_p}(\overline{\mathbb{F}}_p)$ of the orbit $G(\mathbb{A}_f^p) \cdot \tilde{x} \subset \mathrm{Sh}_{G,K_p}(\overline{\mathbb{F}}_p)$; it does not depend on the choice of lift $\tilde{x}$. For the rest of this section we fix x as above and we let $Z \subset \mathrm{Sh}_{G,\mathrm{ord},K^pK_p}$ be the closure of $H_{K^p}(x)$; note that Z is again $G(\mathbb{A}_f^p)$-stable by Lemma 3.1.2.

6.1. ***Rigidity of Zariski closures of Hecke orbits.*** Let $z \in Z(\overline{\mathbb{F}}_p)$ be a smooth point of Z and let $I_z(\mathbb{Q})$ be the group of self-quasi-isogenies of z respecting the tensors, which was introduced in Section 2.2. Let $Y = A_z[p^\infty]$ and fix a choice of isomorphism $\mathbb{D}_{\mathrm{cov}}(Y) \simeq V_p^* \otimes \breve{\mathbb{Z}}_p$ sending $s_\alpha \otimes 1$ to $s_{\alpha,\mathrm{cris}}$ as in Section 5.3. This gives rise to an element $b_z = b \in G(\breve{\mathbb{Q}}_p)$ and we let $U_p \subset J_b(\mathbb{Q}_p)$ be the compact open subgroup introduced in Section 5.4.4. Let $I_z(\mathbb{Z}_{(p)})$ be the intersection of $I_z(\mathbb{Q})$ with U_p inside $J_b(\mathbb{Q}_p)$. We consider the closed immersion of formal neighbourhoods (where the notation is as in (5.1.2))

$$Z^{/z} \subset \mathrm{Sh}^{/z}_{G,K^pK_p} \subset \mathrm{Sh}^{/z}_{\mathrm{GSp},\mathcal{K}^p\mathcal{K}_p}.$$

The goal of this section is to prove the following result.

Proposition 6.1.1 (local stabiliser principle). *The closed subscheme $Z^{/z} \subset \mathrm{Sh}^{/z}_{G,K^pK_p}$ is stable under the action of $I_z(\mathbb{Z}_{(p)})$ via $I_z(\mathbb{Z}_{(p)}) \to U_p$.*

6.1.2. There is a $G(\mathbb{A}_f^p)$-equivariant closed immersion (using the fact that we have a closed immersion at finite level by the main theorem of [Xu 2020]) $\mathrm{Sh}_{G,K_p} \to \mathrm{Sh}_{\mathcal{G}_V,\mathcal{K}_p}$, where $G(\mathbb{A}_f^p)$ acts on the right hand side via the inclusion $G(\mathbb{A}_f^p) \to \mathcal{G}_V(\mathbb{A}_f^p)$. The space $\mathrm{Sh}_{\mathcal{G}_V,\mathcal{K}_p}$ is a moduli space of (weakly) polarised abelian varieties (A, λ) up to prime-to-p isogeny, equipped with an isomorphism $\mathbb{V}^p A \to V \otimes \underline{\mathbb{A}}_f^p$ compatible with the polarisation up to a scalar in $\underline{\mathbb{A}}_f^{p,\times}$.

Let $\tilde{z}$ be a lift of z to $\mathrm{Sh}_{G,K_p}(\overline{\mathbb{F}}_p)$ as above, which defines an inclusion

$$I_z(\mathbb{Z}_{(p)}) \subset I_z(\mathbb{Q}) \subset G(\mathbb{A}_f^p).$$

The stabiliser in $\mathcal{G}_V(\mathbb{A}_f^p)$ of $\tilde{z} \in \mathrm{Sh}_{\mathcal{G}_V,\mathcal{K}_p}$ is given by $\mathrm{End}_\lambda(A_z)^\times$, which is the group of automorphisms of the abelian variety up to prime-to-p isogeny A that take λ to a $\mathbb{Z}_{(p)}^\times$ multiple of λ.

Lemma 6.1.3. *The stabiliser inside $G(\mathbb{A}_f^p)$ of the point $\tilde{z}$ is equal to $I_z(\mathbb{Z}_{(p)})$.*

Proof. By [KMS 2022, Lemma 2.1.4], the stabiliser is contained in $I_z(\mathbb{Z}_{(p)})$. The stabiliser in $\mathcal{G}_V(\mathbb{A}_f^p)$ of the image of $\tilde{z}$ in $\mathrm{Sh}_{\mathcal{G}_V,\mathcal{K}_p}$ is equal to $\mathrm{End}_\lambda(A_z)^\times$ and thus contains $I_z(\mathbb{Z}_{(p)})$. The result follows. □

In order to prove Proposition 6.1.1, we first prove it for $\mathrm{Sh}_{\mathrm{GSp},\mathcal{K}^p\mathcal{K}_p}$. See [Chai and Oort 2009, Theorem 9.5] for closely related results and arguments.

Let $\mathrm{Sh}^{/\tilde{z}}_{\mathcal{G}_V,\mathcal{K}_p}$ be the formal completion of $\mathrm{Sh}_{\mathcal{G}_V,\mathcal{K}_p}$, considered as a formal algebraic space as in [Stacks 2020, Section 0AIX], and restrict its functor of points to Artin local $\overline{\mathbb{F}}_p$-algebras R with residue field isomorphic to $\overline{\mathbb{F}}_p$. Then $\mathrm{Sh}^{/\tilde{z}}_{\mathcal{G}_V,\mathcal{K}_p}(R)$ is the set of isomorphism classes of (weakly) polarised abelian varieties (A, λ) over R up to prime-to-p isogeny, equipped with an isomorphism $\epsilon : \mathbb{V}^p A \to V \otimes \underline{\mathbb{A}}_f^p$

compatible with the polarisation up to a scalar in $\underline{\mathbb{A}}_f^{p,\times}$, such that after basechanging to $\overline{\mathbb{F}}_p$ we recover the point given by the image of $\tilde{z}$.

This means that there is a (necessarily unique) isomorphism $\beta : A_{\overline{\mathbb{F}}_p} \to A_z$ making the following diagram commute:

$$\begin{tikzcd} \mathbb{V}^p A_{\overline{\mathbb{F}}_p} \arrow[r, "\beta"] \arrow[d, "\epsilon"] & \mathbb{V}^p A_z \arrow[d, "\epsilon_{\tilde{z}}"] \\ V \otimes \mathbb{A}_f^p \arrow[r, equal] & V \otimes \mathbb{A}_f^p \end{tikzcd}$$

The quadruple $(A, \lambda, \beta, \epsilon)$ is uniquely determined by (A, λ, β) because (pro-)étale sheaves on Artin local rings are determined by their restriction to the residue field. In particular, for all $R \in \mathrm{Art}$ the natural forgetful map

$$\mathrm{Sh}^{/\tilde{z}}_{\mathcal{G}_V, \mathcal{K}_p}(R) \to \mathrm{Sh}^{/z}_{\mathrm{GSp}, \mathcal{K}^p \mathcal{K}_p}(R)$$

is an isomorphism. This induces an action of $\mathrm{End}_\lambda(A_z)^\times$ on $\mathrm{Sh}^{/z}_{\mathrm{GSp}, \mathcal{K}^p \mathcal{K}_p}$ that we will now identify.

6.1.4. Recall that there is an inclusion $\mathrm{End}_\lambda(A_z)^\times \subset \mathcal{G}_V(\mathbb{A}_f^p)$ determined by $\tilde{z}$ or rather $\epsilon_{\tilde{z}}$. This means that an automorphism f of A_z acts on $V \otimes \mathbb{A}_f^p$ in a way that makes the following diagram commute:

$$\begin{tikzcd} \mathbb{V}^p A_z \arrow[r, "f"] \arrow[d, "\epsilon_{\tilde{z}}"] & \mathbb{V}^p A_z \arrow[d, "\epsilon_{\tilde{z}}"] \\ V \otimes \mathbb{A}_f^p \arrow[r, "f"] & V \otimes \mathbb{A}_f^p \end{tikzcd}$$

Since $\mathrm{End}_\lambda(A_z)^\times$ stabilises $\tilde{z}$, it acts on $\mathrm{Sh}^{/\tilde{z}}_{\mathcal{G}_V, \mathcal{K}_p}$. This action can be described as follows: An automorphism f sends a triple (A, λ, ϵ) to $(A, \lambda, f \circ \epsilon)$. It is straightforward to check that the unique upgrade $(A, \lambda, f \circ \epsilon)$ to a quadruple $(A, \lambda, \beta', f \circ \epsilon)$ is realised by taking $\beta' = f \circ \beta$. Therefore the induced action of $\mathrm{End}_\lambda(A_z)^\times$ on $\mathrm{Sh}^{/z}_{\mathrm{GSp}, \mathcal{K}^p \mathcal{K}_p}$ is given by $(A, \lambda, \beta) \mapsto (A, \lambda, f \circ \beta)$.

6.1.5. Because deformations of abelian varieties are uniquely determined by deformations of their p-divisible groups, we can also identify

$$\mathrm{Sh}^{/\tilde{z}}_{\mathcal{G}_V, \mathcal{K}_p}(R)$$

with the space of triples (X, λ, β) where (X, λ) is a polarised p-divisible group and β is an isomorphism $(X, \lambda)_{\overline{\mathbb{F}}_p} \to (A_z[p^\infty], \lambda_z)$. The action of $\mathrm{End}_\lambda(A_z)^\times$ is then given by postcomposing β with f. There is a similar description of at finite level, and it follows that the natural map

$$\mathrm{Sh}^{/z}_{\mathrm{GSp}, \mathcal{K}^p \mathcal{K}_p} \subset \mathrm{Def}(A_z[p^\infty])$$

is $\mathrm{End}_\lambda(A_z)^\times$-equivariant, where $\mathrm{End}_\lambda(A_z)^\times$ acts on the right hand side via the inclusion

$$\mathrm{End}_\lambda(A_z)^\times \subset \mathrm{End}(A_z[p^\infty])^\times,$$

followed by the natural action of $\mathrm{End}(A_z[p^\infty])^\times$ on $\mathrm{Def}(A_z[p^\infty])$.

Proof of Proposition 6.1.1. Let $\tilde{z}$ be a lift of z to $\mathrm{Sh}_{G,K_p}(\overline{\mathbb{F}}_p)$ as above, which defines an inclusion

$$I_z(\mathbb{Z}_{(p)}) \subset I_z(\mathbb{Q}) \subset G(\mathbb{A}_f^p).$$

It follows from Lemma 6.1.3 that $I_z(\mathbb{Z}_{(p)}) \subset G(\mathbb{A}_f^p)$ is the stabiliser of the point $\tilde{z}$ under the action of $G(\mathbb{A}_f^p)$. Let $\widetilde{Z}$ be the inverse image of Z in Sh_{G,K_p}, it is stable under the action of $G(\mathbb{A}_f^p)$ by Lemma 3.1.2. There is a commutative diagram

$$\begin{tikzcd} \widetilde{Z} \arrow[r] \arrow[d] & \mathrm{Sh}_{G,K_p} \arrow[r] \arrow[d] & \mathrm{Sh}_{\mathcal{G}_V,\mathcal{K}_p} \arrow[d] \\ Z \arrow[r] & \mathrm{Sh}_{G,K^pK_p} \arrow[r] & \mathrm{Sh}_{\mathrm{GSp},\mathcal{K}^p\mathcal{K}_p} \end{tikzcd}$$

where the top right horizontal map is $G(\mathbb{A}_f^p)$-equivariant via $G(\mathbb{A}_f^p) \to \mathcal{G}_V(\mathbb{A}_f^p)$.

Let $\widetilde{Z}^{/\tilde{z}}$ be the formal completion of $\widetilde{Z}$ at the closed point corresponding to $\widetilde{Z}$, considered as a formal algebraic space as in [Stacks 2020, Section 0AIX]. This is per definition the subfunctor of $\widetilde{Z}$ consisting of those morphisms $\operatorname{Spec} T \to \widetilde{Z}$ that factor through $\tilde{z}$ on the level of topological spaces. Since $I_z(\mathbb{Z}_{(p)})$ stabilises $\tilde{z}$, it acts on $\widetilde{Z}^{/\tilde{z}}$.

By [Stacks 2020, Lemma 0CUF], there is a homeomorphism $|\widetilde{Z}| \simeq \varprojlim_{U^p} |Z_{U^p}|$ and thus we get an isomorphism

$$\widetilde{Z}^{/\tilde{z}} \simeq \varprojlim_{U^p \subset G(\mathbb{A}_f^p)} Z_{U^p}^{/z},$$

where $z \in Z_{U^p}(\overline{\mathbb{F}}_p)$ is the image of $\tilde{z}$ under $\widetilde{Z} \to Z_{U^p}$. The formal algebraic space $Z_{U^p}^{/z}$ can be identified with $\operatorname{Spf} \widehat{\mathcal{O}}_{Z_{U^p},z}$, compatible with changing U^p. Since the transition morphisms are all finite étale, they induce isomorphisms of complete local rings. Therefore, all the transition maps in the inverse system $\varprojlim_{U^p \subset G(\mathbb{A}_f^p)} Z_{U^p}^{/z}$ are isomorphism. We conclude that

$$\widetilde{Z}^{/\tilde{z}} \simeq Z_{U^p}^{/z},$$

and so there is an action of $I_z(\mathbb{Z}_{(p)})$ on $Z_{U^p}^{/z}$. In the same way we can prove that there is an action of $I_z(\mathbb{Z}_{(p)})$ on $\mathrm{Sh}_{G,K^pK_p}^{/z}$. It remains for us to identify this action with the inclusion $I_z(\mathbb{Z}_{(p)}) \to U_p$ followed by the natural action of U_p on $\mathrm{Sh}_{G,K^pK_p}^{/z}$.

Let $\tilde{z}$ be the image of $\tilde{z}$ in $\mathrm{Sh}_{\mathcal{G}_V,\mathcal{K}_p}(\overline{\mathbb{F}}_p)$ and let $z \in \mathrm{Sh}_{\mathrm{GSp},\mathcal{K}^p\mathcal{K}_p}(\overline{\mathbb{F}}_p)$ be its image. Then the stabiliser of $\tilde{z}$ can be identified with the group

$$\mathrm{End}_\lambda(A_z)^\times \subset \mathcal{G}_V(\mathbb{A}_f^p)$$

as before. The discussion above implies that we have an action of $\mathrm{End}_\lambda(A_z)^\times$ on $\mathrm{Sh}_{\mathrm{GSp},\mathcal{K}^p\mathcal{K}_p}^{/z}$ such that the closed immersion

$$\mathrm{Sh}_{G,K^pK_p}^{/z} \subset \mathrm{Sh}_{\mathrm{GSp},\mathcal{K}^p\mathcal{K}_p}^{/z}$$

is $I_z(\mathbb{Z}_{(p)})$-equivariant for the action of $I_z(\mathbb{Z}_{(p)})$ on the right hand side via the map $I_z(\mathbb{Z}_{(p)}) \to \mathrm{End}_\lambda(A_z)^\times$. But we have seen in Section 6.1.5 that the action of $\mathrm{End}_\lambda(A_z)^\times$ on $\mathrm{Sh}^{/z}_{\mathrm{GSp},\mathcal{K}^p\mathcal{K}_p}$ described above agrees with the action of $\mathrm{End}_\lambda(A_z)^\times$ via the inclusion $\mathrm{End}_\lambda(A_z)^\times \to \mathbf{Aut}_\lambda(A_z[p^\infty])(\overline{\mathbb{F}}_p)$.

Note that the following diagram commutes by construction:

$$\begin{tikzcd} I_z(\mathbb{Z}_{(p)}) \arrow[r] \arrow[d] & \mathrm{End}_\lambda(A_z)^\times \arrow[d] \\ U_p \arrow[r] & \mathbf{Aut}_\lambda(A_z[p^\infty])(\overline{\mathbb{F}}_p) \end{tikzcd}$$

Thus we see that $Z^{/z}_{K^p}$ is stable under the action of $I_z(\mathbb{Z}_{(p)})$ on $\mathrm{Sh}^{/x}_{G,K^pK_p}$ given by the inclusion $I_z(\mathbb{Z}_{(p)}) \to U_p$ followed by the natural action of U_p on $\mathrm{Sh}^{/x}_{G,K^pK_p}$. □

Corollary 6.1.6. *Assume that $z \in \mathrm{Sh}_{G,K^pK_p}(\overline{\mathbb{F}}_p)$ is an ordinary point. Then $Z^{/z}$ is a formal subtorus of $\mathrm{Sh}^{/z}_{G,K^pK_p}$.*

Proof. The compact open subgroup $U_p \subset J_b(\mathbb{Q}_p)$ acts on $\mathrm{Sh}^{/z}_{G,K^pK_p}$ as explained in Section 5.4.4. By Proposition 6.1.1 the closed subspace $Z^{/z} \subset \mathrm{Sh}^{/z}_{G,K^pK_p}$ is stable under the action of $I_z(\mathbb{Z}_{(p)}) \subset U_p$ and hence of its closure in U_p. The algebraic group $I_{\mathbb{Q}_p} \subset J_b$ has the same rank as J_b by [Kisin 2017, Corollary 2.1.7]. Let $T \subset I$ be a maximal torus, then [Platonov and Rapinchuk 1994, Theorem 7.9] tells us that the topological closure of $T(\mathbb{Q})$ in $T(\mathbb{Q}_p)$ has finite index in $T(\mathbb{Q}_p)$. It follows from this that the closure of $I_z(\mathbb{Z}_{(p)})$ in U_p contains a compact open subgroup of a maximal torus T of $J_b(\mathbb{Q}_p)$.

Proposition 5.5.1 then tells us that the assumptions of [Chai 2008, Theorem 4.3] are satisfied. This theorem implies that $Z^{/z}$ is a p-divisible formal subgroup of $\mathrm{Sh}^{/z}_{G,K^pK_p}$, in other words, it is a formal subtorus. □

6.2. ***Monodromy of linear subspaces.*** The goal of this section is to prove the following result, which is a consequence of results of [Chai 2003]. Recall that the universal abelian variety A over $\mathrm{Sh}_{\mathrm{GSp},\mathrm{ord},\mathcal{K}^p\mathcal{K}_p}$ gives rise to an F-isocrystal $\mathcal{M}$, see Section 3.3. Let $W \subset \mathrm{Sh}_{\mathrm{GSp},\mathrm{ord},\mathcal{K}^p\mathcal{K}_p}$ be a connected smooth closed subscheme, then we say that W is *linear* at a smooth point $z \in W(\overline{\mathbb{F}}_p)$ if

$$W^{/z} \subset \mathrm{Sh}^{/z}_{\mathrm{GSp},\mathcal{K}^p\mathcal{K}_p}$$

is a p-divisible formal subgroup. Let U_W be the unipotent radical of the monodromy group $\mathrm{Mon}(W, \mathcal{M}, z)$.

Proposition 6.2.1 (Chai). *Let $z \in W(\overline{\mathbb{F}}_p)$ be a smooth point such that W is linear at z. Then we have the inequality*

$$\dim W_z \geq \mathrm{Dim}\, U_W,$$

where $\mathrm{Dim}\, W_z$ *is the dimension of the local ring $\mathcal{O}_{W,z}$.*

Chai proves the stronger statement that this inequality is actually an equality, but we will not need this stronger statement to prove Theorem I.

Our proof of Proposition 6.2.1 is a straightforward application of the results in [Chai 2003, Sections 2–4]. Since [Chai 2003] is an unpublished preprint from 2003, the referee has suggested we include another reference. Thus we give a second proof of Proposition 6.2.1 based on results of [D'Addezio and van Hoften 2022].

6.2.2. For our first proof of Proposition 6.2.1, we need to give a brief summary of [Chai 2003, Sections 2–4]. Consider the closed immersion.

$$W^{/z} \to \mathrm{Sh}^{/z}_{\mathrm{GSp},\mathcal{K}^p\mathcal{K}_p} \hookrightarrow \widehat{\mathrm{Def}}(Y)_{\overline{\mathbb{F}}_p}.$$

Write $R = \widehat{\mathcal{O}_{W,z}}$ and write M for the finite free $\mathbb{Z}_p$-module $T_pY_0(\overline{\mathbb{F}}_p) \otimes_{\mathbb{Z}_p} T_pY_1^\vee(\overline{\mathbb{F}}_p)$. Then the morphism $W^{/z} \to \widehat{\mathrm{Def}}(Y)$ corresponds to an element of

$$\widehat{\mathrm{Def}}(Y)(R) = \mathrm{Hom}(M, \widehat{\mathbb{G}}_m(R)) = \mathrm{Hom}(M, 1 + \mathfrak{m}_R)$$

where the first equality is [Katz 1981, Theorem 2.1]. Thus we get a homomorphism $f : M \to 1 + \mathfrak{m}_R$ and we let $N_z^\vee$ be its kernel. By [Chai 2003, Proposition 4.2.1, Remark 2.5.1], the $\mathbb{Z}_p$-module $N_z^\vee$ is finite free and the quotient $M/N_z^\vee$ is torsion-free. Thus the map

$$W^{/z} \to \mathscr{H}om(M, \widehat{\mathbb{G}}_m)$$

factors through the subtorus

$$\mathscr{H}om(M/N_z^\vee, \widehat{\mathbb{G}}_m) \subset \mathscr{H}om(M, \widehat{\mathbb{G}}_m),$$

which we can also write as $N_z \otimes_{\mathbb{Z}_p} \widehat{\mathbb{G}}_m \subset M^* \otimes_{\mathbb{Z}_p} \widehat{\mathbb{G}}_m$. Here the $*$ denotes taking $\mathbb{Z}_p$-linear dual and the morphism $N_z \to M^*$ is the $\mathbb{Z}_p$-linear dual of the quotient

$$M \to M/N_z^\vee.$$

The following lemma has the same statement as [Chai 2003, Remark 3.14].

Lemma 6.2.3. *The subgroup $N_z \otimes_{\mathbb{Z}_p} \widehat{\mathbb{G}}_m$ is the smallest formal subtorus of $\widehat{\mathrm{Def}}(Y)_{\overline{\mathbb{F}}_p}$ through which the map from* Spf R *factors.*

Proof. A subtorus corresponds to a free $\mathbb{Z}_p$-submodule $N \subset N_z$ such that the quotient N_z/N is torsion free. Write $N^\vee$ for the kernel of the map

$$M \to M/N_z^\vee = N_z^* \to N^*.$$

If

$$\mathrm{Spf}\, R \to N_z \otimes_{\mathbb{Z}_p} \widehat{\mathbb{G}}_m$$

factors through $N \otimes_{\mathbb{Z}_p} \widehat{\mathbb{G}}_m$, then it factors through

$$\mathscr{H}om(M/N^\vee, \widehat{\mathbb{G}}_m) \subset \mathscr{H}om(M, \widehat{\mathbb{G}}_m).$$

Since the kernel of the map $M \to \widehat{\mathbb{G}}_m(R)$ is given by $N_z^\vee$, it follows that $N_z^\vee \subset N^\vee$ and therefore $M/N^\vee = M/N_z^\vee$ and thus $N_z = N$. ◻

Proof of Proposition 6.2.1. We specialise the discussion of Section 6.2.2 to the situation of Proposition 6.2.1. In particular, since $W^{/z}$ is assumed to be a formal subtorus, we are in the situation that

$$W^{/z} = N_z \otimes_{\mathbb{Z}_p} \widehat{\mathbb{G}}_m \subset \widehat{\mathrm{Def}}(Y)_{\mathbb{F}_p}.$$

Chai [2003, Section 4, Theorem 4.4] proves that the dimension of U_W is equal to the rank of N_z. Thus the rank of N_z is certainly bounded from below by the dimension of U_W. But the rank of N_z is also the dimension of the formal scheme $W^{/z}$ which equals the Krull dimension of $\widehat{\mathcal{O}}_{W,z}$ and also the Krull dimension of $\mathcal{O}_{W,z}$, which proves the theorem. $\square$

Second proof of Proposition 6.2.1. By [D'Addezio and van Hoften 2022, Theorem II], the unipotent radical U_W of $\mathrm{Mon}(W, \mathcal{M}, z)$ is isomorphic to the monodromy group

$$\mathrm{Mon}(W^{/z}, \mathcal{M}, z).$$

This monodromy group is defined as in Section 3.3 using the Tannakian category of isocrystals over the formal scheme $W^{/z}$ or equivalently the Tannakian category of isocrystals over the scheme $\operatorname{Spec} \widehat{\mathcal{O}}_{W,z}$, see [D'Addezio and van Hoften 2022, Notation 2.2.5]). Thus it suffices to show that the dimension of $W^{/z}$ is bounded from below by the dimension of $\mathrm{Mon}(W^{/z}, z)$.

Let $Y = A_z[p^\infty]$ as above and write $\mathfrak{a}^+ =: \mathbb{D}_{\mathrm{cov}}(Y)$ and $\mathfrak{a} = \mathfrak{a}^+[1/p]$. Write $\mathfrak{b}^+ \subset \mathfrak{a}^+$ for the covariant Dieudonné module of the p-divisible group associated to $W^{/z}$ and $\mathfrak{b} = \mathfrak{b}^+[1/p]$. Then in the notation of [D'Addezio and van Hoften 2022, Section 5.5] we have

$$W^{/z} = Z(\mathfrak{b}^+).$$

Now [D'Addezio and van Hoften 2022, Theorem 5.5.3] tells us that there is an inclusion of algebraic groups over $\breve{\mathbb{Q}}_p$,

$$\mathrm{Mon}(W^{/z}, \mathcal{M}, z) \subset U(\mathfrak{b}) := \mathfrak{b} \otimes_{\breve{\mathbb{Q}}_p} \mathbb{G}_a.$$

In particular, the height of the isocrystal $\mathfrak{b}$ is bounded from below by the dimension of $\mathrm{Mon}(W^{/z}, \mathcal{M}, z)$. Since $\mathfrak{b}$ has slope 1, it follows that the dimension of the p-divisible group associated to $\mathfrak{b}^+$ is also bounded from below by the dimension of $\mathrm{Mon}(W^{/z}, \mathcal{M}, z)$. $\square$

6.3. *Monodromy and conclusion.* Recall from Section 3.2.2 the maps

$$B(G_{\mathbb{Q}_p}) \to B(G^{\mathrm{ad}}_{\mathbb{Q}_p}) \to \prod_{i=1}^{n} B(G_{i,\mathbb{Q}_p})$$

induced by the decomposition $G^{\mathrm{ad}} = \prod_{i=1}^n G_i$ of (2.2.1). Let $[b_{\mathrm{ord}}] \in B(G, \{\mu^{-1}\})$ be the σ-conjugacy class corresponding to the ordinary locus, and let $[b_{\mathrm{ord},i}]$ be the image of $[b_{\mathrm{ord}}]$ in $B(G_{i,\mathbb{Q}_p})$.

Lemma 6.3.1. *For all i the element $[b_{\mathrm{ord},i}]$ is nonbasic.*

Proof. By the axioms of a Shimura datum, the $G_i(\overline{\mathbb{Q}}_p)$-conjugacy class of cocharacters $\{\mu_i^{-1}\}$ induced by $\{\mu^{-1}\}$ is nontrivial for all i. By Lemma 5.4.1, we have an equality $\{\mu_i^{-1}\} = \{\nu_{[b_{\mathrm{ord},i}]}\}$ and so the Newton cocharacter of $[b_{\mathrm{ord},i}]$ is noncentral for all i. In other words, the σ-conjugacy class $[b_{\mathrm{ord},i}]$ is nonbasic for all i. □

Proof of Theorem I. Let $x \in \mathrm{Sh}_{G,K^pK_p}(\overline{\mathbb{F}}_p)$ be an ordinary point and let Z be the Zariski closure (inside $\mathrm{Sh}_{G,\mathrm{ord},K^pK_p}$) of its prime-to-$p$ Hecke orbit. Then Z is $G(\mathbb{A}_f^p)$-stable by Lemma 3.1.2 and similarly its smooth locus $Z^{\mathrm{sm}} \subset Z$ is $G(\mathbb{A}_f^p)$-stable by Lemma 3.1.1. Let X be the p-divisible group over Z^{sm} of the universal abelian variety and let $\mathcal{M}^{\dagger}$ be the associated overconvergent F-isocrystal, see Section 3.3.

Let $z \in Z^{\mathrm{sm}}(\overline{\mathbb{F}}_p)$ and let $Z^{\circ} \subset Z^{\mathrm{sm}}$ be the connected component containing z. By Lemma 6.3.1, the element $[b_{\mathrm{ord}}]$ is $\mathbb{Q}$-nonbasic and by Lemma 2.3.2, we know that Hypothesis 2.3.1 is satisfied because K_p is hyperspecial. Therefore Corollary 3.3.3 tells us that the monodromy group of $\mathcal{M}^{\dagger}$ over Z° is isomorphic to $G^{\mathrm{der}} \otimes \breve{\mathbb{Q}}_p$. Corollary 3.3.5 tells us that unipotent radical of the monodromy group of $\mathcal{M}$ over Z° is isomorphic to the unipotent radical of the parabolic subgroup $P_{\nu_{[b]}} \subset G \otimes \breve{\mathbb{Q}}_p$ for any choice of $\nu_{[b]} \in \{\nu_{[b]}\}$.

By Lemma 5.4.1, this unipotent radical is isomorphic to the unipotent radical of the parabolic subgroup $P_{\mu} \subset G$ for any choice of representative μ of $\{\mu\}$. This unipotent radical has dimension equal to $\langle 2\rho, \{\mu\}\rangle$ (this notation was introduced after the statement of Lemma 5.4.1).

Corollary 6.1.6 tells us that $Z^{/z}$ is a formal subtorus. Applying Proposition 6.2.1 we see that the Krull dimension of $\mathcal{O}_{Z,z}$ is bounded from below by $\langle 2\rho, \{\mu\}\rangle$. Since the Shimura variety Sh_{G,K^pK_p} also has dimension $\langle 2\rho, \{\mu\}\rangle$, we conclude that

$$Z^{/z} = \mathrm{Sh}_{G,K^pK_p}^{/z}.$$

Because this is true for a dense set of points, it follows that Z is a union of connected components of $\mathrm{Sh}_{G,\mathrm{ord},K^pK_p}$.

By Lemma 5.1.3, the ordinary locus is dense and thus $\pi_0(\mathrm{Sh}_{G,\mathrm{ord},K^pK_p}) = \pi_0(\mathrm{Sh}_{G,K^pK_p})$. Since $G(\mathbb{A}_f^p)$ acts transitively on $\pi_0(\mathrm{Sh}_{G,K_p})$, by [Kisin 2010, Lemma 2.2.5] in combination with [Madapusi Pera 2019, Corollary 4.1.11], it follows that $Z = \mathrm{Sh}_{G,\mathrm{ord},K^pK_p}$. We conclude that the prime-to-p Hecke orbit of x is dense in Sh_{G,K^pK_p} since $\mathrm{Sh}_{G,\mathrm{ord},K^pK_p}$ is dense in Sh_{G,K^pK_p}. □

6.4. ***Consequences for Shimura varieties of abelian type.*** Let (G, X) be a Shimura datum of abelian type with reflex field E, and let $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ be the induced adjoint Shimura datum with reflex field $E^{\mathrm{ad}} \subset E$. Let p be a prime number, let $K_p \subset G(\mathbb{Q}_p)$ be a hyperspecial subgroup and let $K^p \subset G(\mathbb{A}_f^p)$ be a sufficiently small compact open subgroup. Let Sh_{G,K^pK_p} be the special fibre of the canonical integral model of the Shimura variety of level K^pK_p at a prime v above p of E, constructed by Kisin [2010] (see [Kim and Madapusi Pera 2016] for the case $p = 2$).

By [Shen and Zhang 2022, Theorem A], there is an open and dense $G(\mathbb{A}_f^p)$-stable Newton stratum $\mathrm{Sh}_{G,K^pK_p,\mu-\mathrm{ord}}$ in Sh_{G,K^pK_p}, called the *μ-ordinary locus.* If $(G, X) \subset (\mathcal{G}_V, \mathcal{H}_V)$ for some symplectic space V and $E_v = \mathbb{Q}_p$, then the μ-ordinary locus is equal to the ordinary locus by Lemma 5.1.3.

Corollary 6.4.1. *If $E_v^{\mathrm{ad}} = \mathbb{Q}_p$, then the prime-to-$p$ Hecke orbit of $x \in \mathrm{Sh}_{G,K^pK_p,\mu-\mathrm{ord}}(\overline{\mathbb{F}}_p)$ is dense in Sh_{G,K^pK_p}.*

Remark 6.4.2. This corollary is more general than Theorem I even for Shimura varieties of Hodge type. Indeed, there are (many) examples of Shimura data (G, X) of Hodge type and primes p and v such that $E_v \neq \mathbb{Q}_p$ but $E_v^{\mathrm{ad}} = \mathbb{Q}_p$.

Lemma 6.4.3. *Corollary 6.4.1 holds for (G, X) if and only if it holds for $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$.*

Proof. The image K_p^{ad} in $G^{\mathrm{ad}}(\mathbb{Q}_p)$ is a hyperspecial subgroup. We can choose $K^{p,\mathrm{ad}} \subset G^{\mathrm{ad}}(\mathbb{A}_f^p)$ containing the image of K^p such that there is a morphism

$$\mathbf{Sh}_{G,K^pK_p}(G, X) \to \mathbf{Sh}_{G^{\mathrm{ad}},K^{p,\mathrm{ad}}K_p^{\mathrm{ad}}}(G^{\mathrm{ad}}, X^{\mathrm{ad}}) \otimes_{E^{\mathrm{ad}}} E,$$

inducing a morphism on geometric special fibres of integral canonical models

$$\mathrm{Sh}_{G,K^pK_p} \to \mathrm{Sh}_{G^{\mathrm{ad}},K^{p,\mathrm{ad}}K_p^{\mathrm{ad}}}, \tag{6.4.1}$$

where we are taking the canonical integral model of $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ at the place v^{ad} of E^{ad} induced by v. This morphism induces a $G(\mathbb{A}_f^p)$-equivariant morphism

$$\mathrm{Sh}_{G,K_p} \to \mathrm{Sh}_{G^{\mathrm{ad}},K_p^{\mathrm{ad}}},$$

where $G(\mathbb{A}_f^p)$ acts on the left hand side via the natural map $G(\mathbb{A}_f^p) \to G^{\mathrm{ad}}(\mathbb{A}_f^p)$. Since the Newton stratification on Shimura varieties of abelian type can be constructed using the F-crystals with G-structure of Lovering [2017], which are functorial for morphisms of Shimura data, it follows that there is an induced map

$$\mathrm{Sh}_{G,K^pK_p,\mu-\mathrm{ord}} \to \mathrm{Sh}_{G^{\mathrm{ad}},K^{p,\mathrm{ad}}K_p^{\mathrm{ad}},\mu-\mathrm{ord}}.$$

Moreover, since the natural map $B(G, \{\mu^{-1}\}) \to B(G^{\mathrm{ad}}, \{\mu^{-1}\})$ is a bijection as explained in [Kottwitz 1997, Section 6.5], it is in fact true that $\mathrm{Sh}_{G,K^pK_p,\mu-\mathrm{ord}}$ is the inverse image of $\mathrm{Sh}_{G^{\mathrm{ad}},K^{p,\mathrm{ad}}K_p^{\mathrm{ad}},\mu-\mathrm{ord}}$ under (6.4.1). By construction of the integral canonical models of Shimura varieties of abelian type, see [Kisin 2010, Section 3.4.9; 2017, Appendix E.7], the connected components of $\mathrm{Sh}_{G^{\mathrm{ad}},K^{p,\mathrm{ad}}K_p^{\mathrm{ad}}}$ are quotients of connected components of Sh_{G,K^pK_p} by free actions of finite groups. In particular, the map (6.4.1) is finite étale and thus closed.

Because the map (6.4.1) is closed and takes prime-to-p Hecke orbits to prime-to-p Hecke orbits, it must takes Zariski closures of prime-to-p Hecke orbits to Zariski closure of prime-to-p Hecke orbits. Thus for $x \in \mathrm{Sh}_{G,K^pK_p,\mu-\mathrm{ord}}(\overline{\mathbb{F}}_p)$ the Zariski closure of its Hecke orbit in $\mathrm{Sh}_{G,K^pK_p,\mu-\mathrm{ord}}$ has the same dimension as the Zariski closure of its Hecke orbit in $\mathrm{Sh}_{G^{\mathrm{ad}},K^{p,\mathrm{ad}}K_p^{\mathrm{ad}},\mu-\mathrm{ord}}$. Moreover in both cases the prime-to-p Hecke operators act transitively on $\pi_0(\mathrm{Sh}_{G,K^pK_p})$ by [Kisin 2010, Lemma 2.2.5] in combination with [Madapusi Pera 2019, Corollary 4.1.11].[6] Thus prime-to-p Hecke orbits in $\mathrm{Sh}_{G,K^pK_p,\mu-\mathrm{ord}}$ are dense if

[6]The result [Madapusi Pera 2019, Corollary 4.1.11] states that for Shimura varieties of Hodge type and hyperspecial level, Hypothesis 3.4.1 holds. Since the canonical integral models of Shimura varieties of abelian type are constructed from the canonical integral models of Shimura varieties of Hodge type, the statement therefore also holds for canonical integral models of Shimura varieties of abelian type.

and only if their images under (6.4.1) are dense. In particular, if the corollary holds for $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$, then it holds for (G, X).

To prove the converse, we note that a point in the Shimura variety for $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ can, by [Kisin 2010, Lemma 2.2.5] in combination with [Madapusi Pera 2019, Corollary 4.1.11], be moved to a connected component which is in the image of (6.4.1). Therefore every prime-to-p Hecke orbit can be lifted to the Shimura variety for (G, X), and we are done. □

Proof of Corollary 6.4.1. By Lemma 6.4.3, we may assume that G is adjoint. Then by the proof of [Kisin and Pappas 2018, Lemma 4.6.22] we can choose a Shimura datum of Hodge type (G_2, X_2) and a morphism of Shimura data $(G_2, X_2) \to (G, X)$ such that: the group $G_{2,\mathbb{Q}_p}$ is quasi-split and split over an unramified extension and the prime v of E splits in the reflex field $E_2 \supset E$ of (G_2, X_2). The upshot is that we can choose a prime w of E_2 satisfying $E_{2,w} = \mathbb{Q}_p$ and thus the μ-ordinary locus in the special fibre of the canonical integral model for (G_2, X_2) at this prime is equal to the ordinary locus for a choice of Hodge embedding $(\mathcal{G}_V, \mathcal{H}_V)$.

Then Theorem I implies that Corollary 6.4.1 holds for (G_2, X_2) and Lemma 6.4.3 tells us that it also holds for (G, X) which concludes the proof. □

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Locally analytic vector bundles on the Fargues–Fontaine curve

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We develop a version of Sen theory for equivariant vector bundles on the Fargues–Fontaine curve. We show that every equivariant vector bundle canonically descends to a locally analytic vector bundle. A comparison with the theory of (φ, Γ)-modules in the cyclotomic case then recovers the Cherbonnier–Colmez decompletion theorem. Next, we focus on the subcategory of de Rham locally analytic vector bundles. Using the p-adic monodromy theorem, we show that each locally analytic vector bundle $\mathcal{E}$ has a canonical differential equation for which the space of solutions has full rank. As a consequence, $\mathcal{E}$ and its sheaf of solutions $\mathrm{Sol}(\mathcal{E})$ are in a natural correspondence, which gives a geometric interpretation of a result of Berger on (φ, Γ)-modules. In particular, if V is a de Rham Galois representation, its associated filtered (φ, N, G_K)-module is realized as the space of global solutions to the differential equation. A key to our approach is a vanishing result for the higher locally analytic vectors of representations satisfying the Tate–Sen formalism, which is also of independent interest.

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1. Introduction

The study of p-adic Galois representations has been conditioned to an extent by two dogmas. One is the *analytic* dogma; its main idea is to associate to every such representation a (φ, Γ)-module over the Robba ring and to study these objects using p-adic analysis. The other dogma is *geometric*: to every p-adic Galois representation one associates an equivariant vector bundle over the Fargues–Fontaine curve. The aim of this article is, roughly speaking, to find a framework where both analysis and geometry can be carried out. In recent years, much of the theory of p-adic Galois representations has been understood in terms of the

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geometry of the Fargues–Fontaine curve. A notable exception has been the p-adic Langlands program, where the analytic approach plays a crucial role. Thus we are motivated to reduce this discrepancy by introducing corresponding objects on the Fargues–Fontaine curve which are also amenable to analytic methods. These are the locally analytic vector bundles, the main new objects introduced in this article.

We shall now explain this in more detail. Let K be a finite extension of $\mathbb{Q}_p$ with absolute Galois group G_K. Let K_{cyc} be the cyclotomic extension of K and write $\Gamma = \mathrm{Gal}(K_{\mathrm{cyc}}/K)$. For the sake of simplifying the introduction, we shall focus now on the cyclotomic setting, though as we shall explain later, the content of this paper will apply to a wider class of Galois extensions K_∞/K. We have the category $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ of finite dimensional $\mathbb{Q}_p$-representations of G_K.

On the one hand, $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ can be studied via p-adic analysis. To do this, one introduces the Robba ring $\mathcal{R}$, which is the ring of power series over a certain finite extension of $\mathbb{Q}_p$ in a variable T which converge in some annuli $r \le |T| < 1$. It has an action of a Frobenius operator φ as well as an action of Γ. By work of Cherbonnier–Colmez, Fontaine and Kedlaya, it is known that there is a fully faithful embedding

$$\mathrm{Rep}_{\mathbb{Q}_p}(G_K) \hookrightarrow \{(\varphi, \Gamma)\text{-modules over } \mathcal{R}\},$$

with the essential image consisting of the semistable slope 0 objects. If D is a (φ, Γ)-module over $\mathcal{R}$, a fundamental fact is that the Γ-action on D can be differentiated, namely, there is a well defined action of $\mathrm{Lie}(\Gamma)$ on D. Since $\mathrm{Lie}(\Gamma)$ is 1-dimensional, this data is the same as that of a connection ∇ which acts on functions of T by a multiple of d/dT. It is precisely this structure which allows the introduction of p-adic analysis into the picture. For example, in the construction of the p-adic Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ given in [Colmez 2010], the use of this analytic structure is ubiquitous.

On the other hand, $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ can be studied via geometry. The Fargues–Fontaine curve, studied extensively in [Fargues and Fontaine 2018], is defined as the analytic adic space

$$\mathcal{X} = \mathcal{X}(\widehat{K}_{\mathrm{cyc}}) := (\mathrm{Spa}\,\mathrm{A}_{\mathrm{inf}} - \{p[p^\flat] = 0\})/(\varphi^{\mathbb{Z}}, \mathrm{Gal}(\overline{K}/K_{\mathrm{cyc}}))$$

(see Section 3) and has a natural action of Γ. By the work of Fargues and Fontaine, there is a fully faithful embedding

$$\mathrm{Rep}_{\mathbb{Q}_p}(G_K) \hookrightarrow \{\Gamma\text{-equivariant vector bundles on } \mathcal{X}\},$$

again with the essential image consisting of the semistable slope 0 objects. In fact, Fargues and Fontaine show there is an equivalence

$$\{(\varphi, \Gamma)\text{-modules over } \mathcal{R}\} \cong \{\Gamma\text{-equivariant vector bundles on } \mathcal{X}\},$$

compatible with each of the aforementioned embeddings of $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$.

Unfortunately, the action of Γ on an equivariant vector bundle on $\mathcal{X}$ *cannot* be differentiated. This is already true for the structure sheaf $\mathcal{O}_{\mathcal{X}}$. Here is a simplified model of the situation which illustrates why there is no action of $\mathrm{Lie}(\Gamma)$ on $\mathcal{O}_{\mathcal{X}}$. The functions on an open subset of $\mathcal{X}$ can roughly be thought of as power series in T^{1/p^∞} satisfying certain convergence conditions. When we try to apply the operator d/dT to such a power series, the result will often not converge since the derivative

$$d(T^{1/p^n})/dT = (1/p^n)T^{1/p^n-1}$$

grows exponentially larger p-adically as n goes to infinity. Nevertheless, there is a way to single out the sections for which the action of $\mathrm{Lie}(\Gamma)$ does not explode. This is achieved by considering only those sections on which the action of Γ is regular enough. In this toy model picture, this will amount to considering only the power series where the coefficient of the exponent of T^{k/p^n} will decay proportionally to p^n.

More canonically and more generally, these elements for which differentiation is possible are precisely the locally analytic elements. Given an equivariant vector bundle $\tilde{\mathcal{E}}$ on $\mathcal{X}$, there is a subsheaf of locally analytic sections $\tilde{\mathcal{E}}^{\mathrm{la}} \subset \tilde{\mathcal{E}}$. This sheaf is a module over $\mathcal{O}_{\mathcal{X}}^{\mathrm{la}}$ which is preserved under the Γ-action, and, crucially, $\mathrm{Lie}(\Gamma)$ acts on $\tilde{\mathcal{E}}^{\mathrm{la}}$. We are thus naturally led to the definition of a *locally analytic vector bundle on* $\mathcal{X}$: by this we shall mean a locally free $\mathcal{O}_{\mathcal{X}}^{\mathrm{la}}$-module together with a Γ-action. The point is that locally analytic vector bundles capture both analytic and geometric information, both of which has proven important for the study of $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$.

Our first main result is saying that there is no loss of information in this process: each equivariant vector bundle canonically descends to a locally analytic vector bundle.

Theorem A. *The functor $\tilde{\mathcal{E}} \mapsto \tilde{\mathcal{E}}^{\mathrm{la}}$ gives rise to an equivalence of categories from the category of Γ-equivariant vector bundles on $\mathcal{X}$ to the category of locally analytic vector bundles on $\mathcal{X}$. Its inverse is given by the functor $\mathcal{E} \mapsto \mathcal{O}_{\mathcal{X}} \otimes_{\mathcal{O}_{\mathcal{X}}^{\mathrm{la}}} \mathcal{E}$.*

This theorem fits naturally into the framework of Sen theory, as we shall now explain. Let $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$. Then according to Sen's theory, proven in [Sen 1980], there is a canonical isomorphism

$$(V \otimes_{\mathbb{Q}_p} \mathbb{C}_p)^{\mathrm{Gal}(\overline{K}/K_{\mathrm{cyc}})} \cong \widehat{K}_{\mathrm{cyc}} \otimes_{K_{\mathrm{cyc}}} \boldsymbol{D}_{\mathrm{Sen}}(V),$$

where $\boldsymbol{D}_{\mathrm{Sen}}(V)$ is the K_{cyc}-subspace of elements with finite Γ-orbit in $V \otimes_{\mathbb{Q}_p} \mathbb{C}_p$. Later, Fontaine [2004, §3.4] proved an analogue of this theorem for $\boldsymbol{B}_{\mathrm{dR}}^+$: he showed there is an isomorphism

$$(V \otimes_{\mathbb{Q}_p} \boldsymbol{B}_{\mathrm{dR}}^+)^{\mathrm{Gal}(\overline{K}/K_{\mathrm{cyc}})} \cong (\boldsymbol{B}_{\mathrm{dR}}^+)^{\mathrm{Gal}(\overline{K}/K_{\mathrm{cyc}})} \otimes_{K_{\mathrm{cyc}}[[t]]} \boldsymbol{D}_{\mathrm{dif}}^+(V),$$

where $\boldsymbol{D}_{\mathrm{dif}}^+(V)$ is a canonical $K_{\mathrm{cyc}}[[t]]$-submodule of $V \otimes_{\mathbb{Q}_p} \boldsymbol{B}_{\mathrm{dR}}^+$.

In fact, both of these results are implied by Theorem A by specializing at the "point at infinity" $x_\infty \in \mathcal{X}$. Indeed, when $\tilde{\mathcal{E}}$ is the equivariant vector bundle associated to $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ and $\mathcal{E} = \tilde{\mathcal{E}}^{\mathrm{la}}$, specializing the isomorphism $\tilde{\mathcal{E}} \cong \mathcal{O}_{\mathcal{X}} \otimes_{\mathcal{O}_{\mathcal{X}}^{\mathrm{la}}} \mathcal{E}$ at the fiber of x_∞ gives rise to an isomorphism

$$\tilde{\mathcal{E}}_{k(x_\infty)} \cong \mathcal{O}_{\mathcal{X},k(x_\infty)} \otimes_{\mathcal{O}^{\mathrm{la}}_{\mathcal{X},k(x_\infty)}} \mathcal{E}_{k(x_\infty)},$$

which is none other than Sen's theorem. Similarly, there is an isomorphism of the completed stalks at x_∞,

$$\tilde{\mathcal{E}}^{\wedge,+}_{x_\infty} \cong \mathcal{O}^{\wedge,+}_{\mathcal{X},x_\infty} \otimes_{\mathcal{O}^{\mathrm{la},\wedge,+}_{\mathcal{X},x_\infty}} \mathcal{E}^{\wedge,+}_{x_\infty},$$

which recovers Fontaine's theorem. In this way, Theorem A is a sheaf theoretic version of Sen theory on $\mathcal{X}$ which specializes at x_∞ to classical Sen theory.

In the interest of applications, we give a proof of this equivalence not just for the cyclotomic extension, but more generally for any p-adic Lie group $\Gamma = \mathrm{Gal}(K_\infty/K)$, where K_∞ is an infinitely ramified Galois extension of K which contains an unramified twist of the cyclotomic extension. Notably, this condition holds when K_∞ is the extension generated by the torsion points of a formal group.

As we shall explain in the article, these ideas are closely related to the decompletion of (φ, Γ)-modules, especially in the case $K_\infty = K_{\mathrm{cyc}}$. This is not too surprising, because such (φ, Γ)-modules are also obtained by a Sen theory type of idea through the theorem of Cherbonnier and Colmez [1998], and further, these objects relate to $\boldsymbol{D}_{\mathrm{Sen}}$ and $\boldsymbol{D}_{\mathrm{dif}}^+$ in a similar way. In fact, Theorem A is equivalent to the Cherbonnier–Colmez theorem on decompletion of (φ, Γ)-modules (after inverting p). Our proof is not independent from the ideas of Cherbonnier–Colmez, since we still use their trace maps in our arguments. However, it is logically different — more on this below.

First, let us discuss an application of Theorem A, which was a major source of motivation for this work. We give a geometric reinterpretation of Berger's work [2008b] on p-adic differential equations and filtered (φ, N)-modules. In that article, Berger establishes several results regarding de Rham (φ, Γ)-modules (for example, these (φ, Γ)-modules arising from de Rham p-adic Galois representations). To such a (φ, Γ)-module D, Berger associates another (φ, Γ)-module $\mathrm{N}_{\mathrm{dR}}(D)$ (a so called p-adic differential equation), and a $\overline{K}$-vector space of solutions

$$\mathrm{Sol}(D) := \varinjlim_{[L:K]<\infty} (\mathcal{R}_L[\log T] \otimes_{\mathcal{R}} \mathrm{N}_{\mathrm{dR}}(V))^{G_L},$$

where $\mathcal{R}_L$ is the Robba ring with respect to L. The following results can be derived from the main results of [Berger 2008b], for D a de Rham (φ, Γ)-module:

(i) $\mathrm{Sol}(D)$ is a $\overline{K}$-vector space of rank equal to the rank of D.

(ii) There is a canonical isomorphism

$$\mathcal{R}_{\overline{K}}[\log T] \otimes_{K^{\mathrm{un}}} \mathrm{Sol}(D) \cong \mathcal{R}_{\overline{K}}[\log T] \otimes_{\mathcal{R}} \mathrm{N}_{\mathrm{dR}}(V).$$

(iii) $\overline{K} \otimes_{K^{\mathrm{un}}} \mathrm{Sol}(D)$ is canonically isomorphic to $\overline{K} \otimes_K \boldsymbol{D}_{\mathrm{dR}}(D)$.

(iv) $\mathrm{Sol}(D)$ is naturally a filtered (φ, N, G_K)-module.

Furthermore, the functor $D \mapsto \mathrm{Sol}(D)$ gives rise to an equivalence of categories from the category of de Rham (φ, Γ)-modules over $\mathcal{R}$ to the category of filtered (φ, N, G_K)-modules.

The functor of solutions is ultimately understood in [Berger 2008b] by solving the differential equation $\mathrm{Lie}(\Gamma) = 0$, and as such, uses p-adic analysis in a crucial way. It is therefore natural to apply Theorem A to give a geometric interpretation of these results, something previously inaccessible in the framework of vector bundles on the Fargues–Fontaine curve. In fact, when interpreted in a geometric way, [Berger 2008b, théorème A] turns out to be reminiscent of the Riemann–Hilbert correspondence.

Our second main result is the desired geometric interpretation of Berger's results. To describe it, we need to introduce some notation. We have

$$\mathcal{X}_{\log,\overline{K}} := \varprojlim_{[L:K]<\infty} \mathcal{X}_{\log,L},$$

where each $\mathcal{X}_{\log,L}$ is a the analytic line bundle over $\mathcal{X}_L := \mathcal{X}(\hat{L}_{\mathrm{cyc}})$ corresponding to $\mathcal{O}_{\mathcal{X}_L}(1)$, endowed with the projection $p_{\log,L} : \mathcal{X}_{\log,L} \to \mathcal{X}_L$ (see Section 8C). Essentially, $\mathcal{X}_{\log,\overline{K}}$ is obtained by adjoining all $\overline{K}$-scalars and a logarithm to the functions on $\mathcal{X}$. Now let $\mathcal{E}$ be a de Rham locally analytic vector bundle, i.e., suppose that $\dim_K \hat{\mathcal{E}}_{x_\infty}^{\Gamma=1} = \mathrm{rank}(\mathcal{E})$ (see Section 8B). For example, if V is a de Rham p-adic Galois representation, then its associated locally analytic vector bundle is de Rham. To such $\mathcal{E}$, we associate a sheaf $\mathrm{Sol}(\mathcal{E})$ on $\mathcal{X}$, given by

$$\mathrm{Sol}(\mathcal{E}) := \varinjlim_{[L:K]<\infty} p_{\log,L,*}(p_{\log,L}^{*}\mathcal{N}_{\mathrm{dR}}(\mathcal{E}))^{\mathrm{Lie}(\Gamma)=0},$$

where $\mathcal{N}_{\mathrm{dR}}(\mathcal{E})$ is a modification of $\mathcal{E}$ corresponding to the de Rham lattice of $\mathcal{E}$ at x_∞. Roughly speaking, $\mathrm{Sol}(\mathcal{E})$ is the sheaf of solutions to the differential equation $\nabla = 0$ on the modification $\mathcal{N}_{\mathrm{dR}}(\mathcal{E})$. We shall also consider a variant $\mathrm{Sol}^{\varphi}(\mathcal{E})$, which are the solutions on the pullback of $\mathcal{E}$ along the usual covering $\mathcal{Y}_{(0,\infty)} \to \mathcal{X}$ for $\mathcal{Y}_{(0,\infty)} = \mathrm{Spa}\mathrm{A}_{\mathrm{inf}} - \{p[p^\flat] = 0\}/\mathrm{Gal}(\overline{K}/K_{\mathrm{cyc}})$. We then have the following result, by analogy with the results of [Berger 2008b] (see Section 8 for yet more precise statements).

Theorem B. *Let $\mathcal{E}$ be a de Rham locally analytic vector bundle.*

(i) *The sheaf of solutions* $\mathrm{Sol}(\mathcal{E})$ *is locally free over the subsheaf of potentially log smooth sections* $\mathcal{O}_{\mathcal{X}}^{\mathrm{plsm}} \subset \mathcal{O}_{\mathcal{X}}^{\mathrm{la}}$ *and its rank is equal to the rank of* $\mathcal{E}$.

(ii) *There is a canonical isomorphism*

$$\mathcal{O}^{\mathrm{la}}_{\mathcal{X}_{\log,\overline{K}}} \otimes_{\mathcal{O}_{\mathcal{X}}^{\mathrm{plsm}}} \mathrm{Sol}(\mathcal{E}) \xrightarrow{\sim} \mathcal{O}^{\mathrm{la}}_{\mathcal{X}_{\log,\overline{K}}} \otimes_{\mathcal{O}_{\mathcal{X}}^{\mathrm{la}}} \mathcal{N}_{\mathrm{dR}}(\mathcal{E}).$$

(iii) *The stalk of* $\mathrm{Sol}(\mathcal{E})$ *at x_∞ is canonically isomorphic to* $\overline{K} \otimes_K \boldsymbol{D}_{\mathrm{dR}}(\mathcal{E})$.

(iv) *The space of global solutions* $\mathrm{H}^0(\mathcal{Y}_{(0,\infty)}, \mathrm{Sol}^{\varphi}(\mathcal{E}))$ *is naturally a filtered (φ, N, G_K)-module.*

Furthermore, the functor $\mathcal{E} \mapsto \mathrm{H}^0(\mathcal{Y}_{(0,\infty)}, \mathrm{Sol}^{\varphi}(\mathcal{E}))$ *gives rise to an equivalence of categories from the category of de Rham locally analytic vector bundles to the category of filtered (φ, N, G_K)-modules.*

Remark 1.1. (1) In particular, if V is a de Rham representation of G_K with associated locally analytic vector bundle $\mathcal{E}$, then $\mathrm{H}^0(\mathcal{Y}_{(0,\infty)}, \mathrm{Sol}^{\varphi}(\mathcal{E})) = \boldsymbol{D}_{\mathrm{pst}}(V)$ and the stalk $\mathrm{Sol}(\mathcal{E})_{x_\infty}$ is identified with $\overline{K} \otimes_K \boldsymbol{D}_{\mathrm{dR}}(V)$. The localization map corresponds to the natural map $\boldsymbol{D}_{\mathrm{pst}}(V) \to \overline{K} \otimes_K \boldsymbol{D}_{\mathrm{dR}}(V)$.

(2) If $\mathcal{E}$ becomes crystalline after extending K to a finite extension $L \subset K_\infty$, the sheaf $\mathcal{N}_{\mathrm{dR}}(\mathcal{E})^{\nabla=0} \subset \mathrm{Sol}(\mathcal{E})$ is locally free over the subsheaf of smooth sections $\mathcal{O}_{\mathcal{X}}^{\mathrm{sm}} \subset \mathcal{O}_{\mathcal{X}}^{\mathrm{la}}$ of rank equal to the rank of $\mathcal{E}$, and there is a simpler canonical isomorphism

$$\mathcal{O}_{\mathcal{X}}^{\mathrm{la}} \otimes_{\mathcal{O}_{\mathcal{X}}^{\mathrm{sm}}} \mathcal{N}_{\mathrm{dR}}(\mathcal{E})^{\nabla=0} \xrightarrow{\sim} \mathcal{N}_{\mathrm{dR}}(\mathcal{E}).$$

(3) The sheaf $\mathcal{O}_{\mathcal{X}}^{\mathrm{plsm}}$ is much smaller than $\mathcal{O}_{\mathcal{X}}^{\mathrm{la}}$. Though we have not been quite able to show this, $\mathcal{O}_{\mathcal{X}}^{\mathrm{plsm}}$ seems to be "almost" a locally constant sheaf except that the base field becomes slightly larger when localizing; for that reason, we think of $\mathrm{Sol}(\mathcal{E})$ as morally being close to a local system on $\mathcal{X}$. In this sense the (φ, N, G_K)-structure is related to the monodromy of the p-adic differential equation $\nabla = 0$.

Finally, let us discuss the proof of Theorem A. The essential point is to show that if $\tilde{\mathcal{E}}$ is an equivariant vector bundle on $\mathcal{X}$, the natural map $\mathcal{O}_{\mathcal{X}} \otimes_{\mathcal{O}_{\mathcal{X}}^{\mathrm{la}}} \tilde{\mathcal{E}}^{\mathrm{la}} \to \tilde{\mathcal{E}}$ is an isomorphism. Fargues and Fontaine observe that the only point of $\mathcal{X}$ with finite Γ-orbit is x_∞. The idea is then to use a very simple geometric argument: once one knows that $\mathcal{O}_{\mathcal{X}} \otimes_{\mathcal{O}_{\mathcal{X}}^{\mathrm{la}}} \tilde{\mathcal{E}}^{\mathrm{la}} \to \tilde{\mathcal{E}}$ is injective, everything can be understood by arguing locally at x_∞. Indeed, if this map is an isomorphism after localizing and completing along $\mathcal{O}_{\mathcal{X}} \to \widehat{\mathcal{O}}^{+}_{\mathcal{X},x_\infty}$, then the cokernel has to be supported at finitely many points outside x_∞. But these points also form a finite Γ-orbit, so the cokernel cannot be supported anywhere.

It therefore remains to understand the properties of our spaces of locally analytic vectors under certain localizations and completions. To do this, we are naturally led to consider higher locally analytic vectors and their vanishing, and we prove a representation-theoretic result which is of independent interest. To state the result, let G be a p-adic Lie group and let $\widetilde{\Lambda}$ be a Banach ring with a continuous action of G. Assume the topology on $\widetilde{\Lambda}$ is p-adic.

Theorem C. *Suppose G and $\widetilde{\Lambda}$ satisfy the Tate–Sen axioms (TS1)–(TS3) of [Berger and Colmez 2008] as well as an additional axiom (TS4). Then for any finite free $\widetilde{\Lambda}$-semilinear representation M of G, the higher locally analytic vectors $\mathrm{R}^i_{G\text{-la}}(M)$ are zero for $i \geq 1$.*

Here are two special cases of the theorem where we conclude that $\mathrm{R}^i_{G\text{-la}}(M) = 0$ for $i \geq 1$:

(1) If M is a finite dimensional $\widehat{K}_\infty$-module with a semilinear action of Γ, for K_∞ containing an unramified twist of K_{cyc}. In fact, the vanishing of $\mathrm{R}^i_{G\text{-la}}(M)$ can be established for arbitrary K_∞, see Section 5.

(2) If M a finite free $\widetilde{\boldsymbol{B}}_I(\widehat{K}_\infty)$-module with a semilinear action of Γ, under the same assumptions on K_∞.

Note that the vanishing of higher locally analytic vectors is automatic for admissible representations, but the examples above are not admissible. Theorem C illustrates how the Tate–Sen axioms can serve as a substitute for admissibility.

Theorem C is especially useful for making cohomological computations. Here is an example application, which follows directly from the main results of [Rodrigues Jacinto and Rodríguez Camargo 2022] (see Section 5): if M satisfies assumptions of the theorem, then for $i \geq 0$ we have natural isomorphisms

$$\mathrm{H}^i(G, M) \cong \mathrm{H}^i(G, M^{\mathrm{la}}) \cong \mathrm{H}^i(\mathrm{Lie}\, G, M^{\mathrm{la}})^G.$$

Finally, let us mention that in recent work Juan Esteban Rodríguez Camargo [2022] proves similar results to our Theorem C. He then applies them in the setting of rigid adic spaces with fantastic applications to the Calegari–Emerton conjecture, among others.

1A. ***Structure of the article.*** Section 2 contains reminders on locally analytic vectors and their derived functors. In Section 3 we give reminders on the Fargues–Fontaine curve and equivariant vector bundles. In Section 4 we introduce locally analytic bundles and we discuss their basic properties. Section 5 is the longest and most technical section of the paper, in which we prove Theorem C. Theorem A is proved in Section 6. In Section 7 we compare our results to the theory of (φ, Γ)-modules. Finally, in Section 8 we discuss p-adic differential equations on the Fargues–Fontaine curve and explain Theorem B.

At several points in the article we have taken the liberty to raise speculations and ask questions to which we do not yet know the answer.

1B. ***Notation and conventions.*** The field K denotes a finite extension of $\mathbb{Q}_p$. We write $K_{\mathrm{cyc}} = K(\mu_{p^\infty})$ for the cyclotomic extension. Its Galois group $\Gamma_{\mathrm{cyc}} = \mathrm{Gal}(K_{\mathrm{cyc}}/K)$ is an open subgroup of $\mathbb{Z}_p^\times$. We denote by K_∞ an infinitely ramified Galois extension of K with $\Gamma = \mathrm{Gal}(K_\infty/K)$ a p-adic Lie group. If $\overline{K}$ denotes the algebraic closure of K, we let $G_K = \mathrm{Gal}(\overline{K}/K)$ and $H = \mathrm{Gal}(\overline{K}/K_\infty)$ so that $G_K/H = \Gamma$.

The p-adic completion $\widehat{K}_\infty$ of K_∞ is a perfectoid field. Write ϖ for a pseudouniformizer of $\widehat{K}_\infty$ with valuation $\mathrm{val}(\varpi) = p$ that admits a sequence of p-th power roots ϖ^{1/p^n} (such a choice is always possible, and the constructions in this paper never depend on this choice). Let $\varpi^\flat = (\varpi, \varpi^{1/p}, \dots)$ be the corresponding pseudouniformizer of the tilt $\widehat{K}_\infty^\flat$.

Denote by $\mathrm{Lie}(\Gamma)$ the Lie algebra of Γ. It is a finite dimensional $\mathbb{Q}_p$-vector space, and if $v \in \mathrm{Lie}(\Gamma)$ is sufficiently small, we have a corresponding element $\exp(v) \in \Gamma$.

All representations and group actions appearing in this article are assumed to be continuous. Galois cohomology groups are always taken in the continuous sense.

If W is a Banach space over $\mathbb{Q}_p$ we write W^+ for its unit ball.

All completed tensor products appearing in this article are projective. In other words, if V^+ and W^+ are unit balls of two Banach spaces V and W over $\mathbb{Q}_p$, then

$$V^+ \widehat{\otimes}_{\mathbb{Z}_p} W^+ = \varprojlim_n (V^+ \otimes_{\mathbb{Z}_p} W^+)/p^n \quad \text{and} \quad V \widehat{\otimes}_{\mathbb{Q}_p} W = (V^+ \widehat{\otimes}_{\mathbb{Z}_p} W^+)[1/p].$$

2. Locally analytic and pro-analytic vectors

In this section we give reminders on locally analytic and pro-analytic vectors and quote results that will be used in Sections 4–6. We shall freely use our conventions in Section 1B regarding Banach spaces.

2A. ***Locally analytic and pro-analytic vectors.*** We shall say a compact p-adic Lie group G is *small* if there exists a saturated integral valued p-valuation on G which defines its topology and if for some $N \in \mathbb{Z}_{\geq 1}$ there exists an embedding of G into $1 + p^2 M_N(\mathbb{Z}_p)$, the group of N by N matrices congruent to 1 mod p^2. See Sections 23 and 26 of [Schneider 2011] for the first condition. If G is small, there exists an ordered basis $g_1, \dots, g_d$ such that $(x_1, \dots, x_d) \mapsto g_1^{x_1} \cdot \dots \cdot g_1^{x_d}$ gives a homeomorphism of $\mathbb{Z}_p^d$ with G. We then have coordinates on G

$$c = (c_1, \dots, c_d) : G \xrightarrow{\sim} \mathbb{Z}_p^d$$

defined by the inverse map where $c_i(g_1^{x_1}\cdot\dots\cdot g_1^{x_d})=x_i$.

Now let G is an be any compact p-adic Lie group. By [Schneider 2011, Theorem 27.1] and Ado's theorem (see [Pan 2022a, Proposition 2.1.3]), the collection of small open subgroups of G forms a fundamental system of open neighborhoods of the identity element. Let W be a Banach $\mathbb{Q}_p$-linear representation of G (or G-Banach space for short). If H is a small open subgroup of G, choose coordinates c on H and write $c(h)^{\boldsymbol{k}}=\prod_{i=1}^d c_i(h)^{k_i}$ if $\boldsymbol{k}=(k_1,\dots,k_d)$ for $h\in H$. We have the subspace $W^{H\text{-an}}$ of H-analytic vectors in W; it is the subspace of elements $w\in W$ for which there exists a sequence of vectors $\{w_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{N}^d}$ with $w_{\boldsymbol{k}}\to 0$ and

$$h(w)=\sum_{\boldsymbol{k}\in\mathbb{N}^d} c(h)^{\boldsymbol{k}} w_{\boldsymbol{k}}$$

for all $h\in H$. The norm $\|w\|_{H\text{-an}}=\sup_{\boldsymbol{k}}\|w_{\boldsymbol{k}}\|$ makes $W^{H\text{-an}}$ into a Banach space. Note that $W^{H\text{-an}}$ does not depend on the choice of coordinates. We write $W^{\mathrm{la}}=\bigcup_H W^{H\text{-an}}$ for the subspace of locally analytic vectors of W, and endow it with the inductive limit topology, which makes it into an LB space. If W is a Fréchet space whose topology is defined by a countable sequence of seminorms, let W_i be the Hausdorff completion of W for the i-th seminorm, so that $W=\varprojlim W_i$ is a projective limit of Banach spaces. We write $W^{\mathrm{pa}}=\varprojlim W_i^{\mathrm{la}}$ for the subspace of pro-analytic vectors. Finally, we extend the definitions of locally analytic vectors and pro-analytic vectors to LB and LF spaces in the obvious way.

The Lie algebra $\mathrm{Lie}(G)$ acts on each $W^{H\text{-an}}$ (and hence also on W^{la} and W^{pa}) through derivations. This action is given as follows. If $v\in\mathrm{Lie}(G)$ then $\exp(p^k v)\in H$ for $k\gg 0$, and we define

$$\nabla_v(w)=\lim_{k\to\infty}\frac{\exp(p^k v)(w)-w}{p^k}.$$

The operator $\nabla_v : W^{H\text{-an}}\to W^{H\text{-an}}$ is bounded; see [Berger and Colmez 2016, Lemma 2.6].

Locally analytic and pro-analytic vectors behave well when we have a basis of such vectors [Berger and Colmez 2016, Proposition 2.3; Berger 2016, Proposition 2.4]:

Proposition 2.1. *Let B be a Banach or Fréchet G-ring and let W be a free B-module of finite rank, equipped with a B-semilinear action of G. If the B-module W has a basis $w_1,\dots,w_d$ in which the function $G\to\mathrm{GL}_d(B)\subset\mathrm{M}_d(B)$, $g\mapsto\mathrm{Mat}(g)$ is H-analytic* (*resp. locally analytic*, *pro-analytic*), *then* $W^{H\text{-an}}=\bigoplus_{j=1}^d B^{H\text{-an}}\cdot w_i$ $\big($*resp.* $W^{\mathrm{la}}=\bigoplus_{j=1}^d B^{\mathrm{la}}\cdot w_i,\ W^{\mathrm{pa}}=\bigoplus_{j=1}^d B^{\mathrm{pa}}\cdot w_i\big)$.

It will often be useful for us to choose a specific fundamental system of open neighborhoods of G as follows. Fix a small compact open $G_0\subset G$ which with coordinates c. For $n\geq 0$ we set

$$G_n=G^{p^n}=\{g^{p^n}:g\in G_0\}.$$

These are subgroups ([Schneider 2011, Remark 26.9]) which have induced coordinates

$$c|_{G_n}:G_n\xrightarrow{\sim}(p^n\mathbb{Z}_p)^d.$$

The normalization is such that for $w \in W^{G_n\text{-an}}$ we can write

$$g(w) = \sum_{\boldsymbol{k}\in\mathbb{N}^d} c(g)^{\boldsymbol{k}} w_{\boldsymbol{k}}$$

for $g \in G_n$ and $\{w_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{N}^d}$ with $p^{n|\boldsymbol{k}|}w_{\boldsymbol{k}} \to 0$, and the Banach norm is given by

$$\|w\|_{G_n\text{-an}} = \sup_{\boldsymbol{k}} \|p^{n|\boldsymbol{k}|}w_{\boldsymbol{k}}\|.$$

It is easy to check if $w \in W^{G_n\text{-an}}$ then $\|w\|_{G_m\text{-an}} \le \|w\|_{G_{m+1}\text{-an}}$ for $m \ge n$ and $\|w\|_{G_m\text{-an}} = \|w\|$ for $m \gg n$ (see [Berger and Colmez 2016, Lemme 2.4]).

2B. *Rings of analytic functions.* Suppose first that G is small. Let $\mathcal{C}^{\mathrm{an}}(G, \mathbb{Q}_p)$ be the space of analytic functions on G. These are those functions that after pullback by the coordinates $c : G \xrightarrow{\sim} \mathbb{Z}_p^d$ are of the form

$$\boldsymbol{x} = (x_1, \dots, x_d) \mapsto \sum_{\boldsymbol{k}=(k_1,\dots,k_d)\in\mathbb{N}^d} b_{\boldsymbol{k}}\boldsymbol{x}^{\boldsymbol{k}},$$

where $b_{\boldsymbol{k}} \to 0$ as $|\boldsymbol{k}| \to \infty$. The norm $\|f\|_G = \sup_{\boldsymbol{k}\in\mathbb{N}^d} \|b_{\boldsymbol{k}}\|$ makes $\mathcal{C}^{\mathrm{an}}(G, \mathbb{Q}_p)$ into a Banach space. We shall regard $\mathcal{C}^{\mathrm{an}}(G, \mathbb{Q}_p)$ as a G-representation through the left regular action of G.

If now G is any compact p-adic Lie group with a system of small neighborhoods $\{G_n\}_{n\ge0}$ as in Section 2A, we have for each $n \ge 0$ the space of analytic functions $\mathcal{C}^{\mathrm{an}}(G_n, \mathbb{Q}_p)$ on G_n. Using the coordinates $c : G_n \xrightarrow{\sim} (p^n\mathbb{Z}_p)^d$ as in Section 2A, we shall regard $\mathcal{C}^{\mathrm{an}}(G_n, \mathbb{Q}_p)$ as the ring of functions that under the bijection are identified with functions of the form

$$\boldsymbol{x} = (x_1, \dots, x_d) \mapsto \sum_{\boldsymbol{k}=(k_1,\dots,k_d)\in\mathbb{N}^d} b_{\boldsymbol{k}}\boldsymbol{x}^{\boldsymbol{k}},$$

where $p^{n|\boldsymbol{k}|}b_{\boldsymbol{k}} \to 0$ as $|\boldsymbol{k}| \to \infty$. Under this normalization

$$\|f\|_{G_n} = \sup_{\boldsymbol{k}\in\mathbb{N}^d} \|p^{n|\boldsymbol{k}|}b_{\boldsymbol{k}}\|$$

for $f \in \mathcal{C}^{\mathrm{an}}(G_n, \mathbb{Q}_p)$.

The following lemma will be used in Section 5.

Lemma 2.2. *For $k \ge 1$ the subgroup G_{n+k} acts trivially on $\mathcal{C}^{\mathrm{an}}(G_n, \mathbb{Q}_p)^+/p^k$.*

Proof. This is an easy exercise using the coordinates. See [Pan 2022a, Lemma 2.1.2] for the case $k = 1$. □

The following is shown in [Pan 2022a, Proposition 2.1.3] and in its proof (originally in the proof of [Berger and Colmez 2016, théoréme 6.1]).

Proposition 2.3. *Suppose that G is small. There is a dense subspace $\varinjlim_{\ell\in\mathbb{N}} V_\ell \subset \mathcal{C}^{\mathrm{an}}(G, \mathbb{Q}_p)$, where each V_l is a finite-dimensional G-subrepresentation of $\mathcal{C}^{\mathrm{an}}(G, \mathbb{Q}_p)$ with coefficients in $\mathbb{Q}_p$ such that for any $k, \ell \in \mathbb{N}$, we have $V_k \cdot V_\ell \subset V_{k+\ell}$.*

Furthermore, if we fix G and consider small open subgroups $G' \subset G$, we may choose $V_\ell(G') \subset \mathcal{C}^{\mathrm{an}}(G', \mathbb{Q}_p)$ at once for all G' in such a way that the natural map $\mathcal{C}^{\mathrm{an}}(G, \mathbb{Q}_p) \to \mathcal{C}^{\mathrm{an}}(G', \mathbb{Q}_p)$ restricts to $V_\ell(G) \to V_\ell(G')$.

2C. *Higher locally analytic vectors.* Suppose first that G is small and let W be a G-Banach space. There is a G-equivariant isometry

$$W \mathbin{\widehat{\otimes}}_{\mathbb{Q}_p} \mathcal{C}^{\mathrm{an}}(G, \mathbb{Q}_p) \cong \mathcal{C}^{\mathrm{an}}(G, W),$$

where $\mathcal{C}^{\mathrm{an}}(G, W)$ is the space of W-valued analytic functions on G, with its G-Banach structure given by the sup norm and the action $(gf)(x) = g(f(g^{-1}(x))$ for $f \in \mathcal{C}^{\mathrm{an}}(G, W)$. We then have $(\mathcal{C}^{\mathrm{an}}(G, W))^G = W^{G\text{-an}}$, the identification given by $f \mapsto f(1)$. This gives an alternative description of G-analytic vectors that we shall use in what follows.

The functor $W \mapsto W^{G\text{-an}}$ is left exact. Following [Pan 2022a, §2.2; Rodrigues Jacinto and Rodríguez Camargo 2022], define right derived functors for $i \geq 0$:

$$\mathrm{R}^i_{G\text{-an}}(W) = \mathrm{H}^i\big(G, W \mathbin{\widehat{\otimes}}_{\mathbb{Q}_p} \mathcal{C}^{\mathrm{an}}(G, \mathbb{Q}_p)\big)$$

(taking continuous cohomology on the right hand side).

If G is a compact p-adic Lie group with subgroups $\{G_n\}_{n \geq 1}$ as in Sections 2A–2B, taking the colimit over n, there are right derived functors for $W \mapsto W^{G\text{-la}}$ given by

$$\mathrm{R}^i_{G\text{-la}}(W) = \varinjlim_n \mathrm{R}^i_{G_n\text{-an}}(W) = \varinjlim_n \mathrm{H}^i\big(G_n, W \mathbin{\widehat{\otimes}}_{\mathbb{Q}_p} \mathcal{C}^{\mathrm{an}}(G_n, \mathbb{Q}_p)\big).$$

We shall call these groups the higher locally analytic vectors of W. If G is understood from the context we shall just write $\mathrm{R}^i_{\mathrm{la}}$ instead of $\mathrm{R}^i_{G\text{-la}}$.

If

$$0 \to V \to W \to X \to 0$$

is a short exact sequence of G-Banach spaces, then it is strict by the open mapping theorem, and so we have a long exact sequence

$$0 \to V^{\mathrm{la}} \to W^{\mathrm{la}} \to X^{\mathrm{la}} \to \mathrm{R}^1_{\mathrm{la}}(V) \to \mathrm{R}^1_{\mathrm{la}}(W) \to \mathrm{R}^1_{\mathrm{la}}(X) \to \cdots.$$

Lemma 2.4. *Let H be an open subgroup of G and let $H_n = G_n \cap H$. Then for $n \gg 0$ and each $i \geq 0$ there are natural isomorphisms $\mathrm{R}^i_{H_n\text{-an}} \cong \mathrm{R}^i_{G_n\text{-an}}$. In particular, $\mathrm{R}^i_{H\text{-la}} \cong \mathrm{R}^i_{G\text{-la}}$.*

Proof. We have $H_n = G_n$ for $n \gg 0$. □

Suppose that G be a small compact p-adic Lie group, and let H be a small closed normal subgroup. Let W be a G-Banach space. Using the method of Hochshild–Serre we obtain the following spectral sequences.

Proposition 2.5. (i) *There is a spectral sequence*

$$E_2^{ij} = \mathrm{H}^i\big(G/H, \mathrm{H}^j\big(H, W \mathbin{\widehat{\otimes}}_{\mathbb{Q}_p} \mathcal{C}^{\mathrm{an}}(G, \mathbb{Q}_p)\big)\big) \Rightarrow \mathrm{R}^{i+j}_{G\text{-an}}(W).$$

(ii) *There is a spectral sequence*

$$E_2^{ij} = \mathrm{R}^i_{G/H\text{-an}}(\mathrm{H}^j(H, W)) \Rightarrow \mathrm{H}^{i+j}\big(G, W \widehat{\otimes}_{\mathbb{Q}_p} \mathcal{C}^{\mathrm{an}}(G/H, \mathbb{Q}_p)\big).$$

Proof. Apply the Hochshild–Serre spectral sequence to $W \widehat{\otimes}_{\mathbb{Q}_p} \mathcal{C}^{\mathrm{an}}(G, \mathbb{Q}_p)$ and $W \widehat{\otimes}_{\mathbb{Q}_p} \mathcal{C}^{\mathrm{an}}(G/H, \mathbb{Q}_p)$ (see [Rodrigues Jacinto and Rodríguez Camargo 2022, Proposition 5.16]). □

3. Equivariant vector bundles

In this section we give reminders on the Fargues–Fontaine curve and equivariant vector bundles. For more details, see [Fargues and Fontaine 2018, Chapter 9; Scholze and Weinstein 2020, Lectures 12–13].

3A. ***The spaces $\mathcal{Y}_{(0,\infty)}$ and $\mathcal{X}$.*** Let F be a perfectoid field, with tilt $F^\flat$. We have Fontaine's ring $\mathrm{A}_{\mathrm{inf}} = \mathrm{A}_{\mathrm{inf}}(F)$, defined as the Witt vectors of the ring of integers $\mathcal{O}_F^\flat$ of $F^\flat$. Write $\mathrm{Spa}(\mathrm{A}_{\mathrm{inf}})$ for the adic space associated to the Huber pair $(\mathrm{A}_{\mathrm{inf}}, \mathrm{A}_{\mathrm{inf}})$.

Let ϖ be a pseudouniformizer of F, and let f be the residue field of $\mathcal{O}_F$. Then there is a point $x_f \in \mathrm{Spa}(\mathrm{A}_{\mathrm{inf}})$ with residue field f, which is the intersection of the two closed subspaces $\{p = 0\}$ and $\{[\varpi] = 0\}$. We set

$$\mathcal{Y} = \mathcal{Y}(F) = \mathrm{Spa}\mathrm{A}_{\mathrm{inf}} - \{x_f\} \quad \text{and} \quad \mathcal{Y}_{(0,\infty)} = \mathcal{Y}_{(0,\infty)}(F) = \mathrm{Spa}\mathrm{A}_{\mathrm{inf}} - \{p[\varpi] = 0\}.$$

The spaces $\mathcal{Y}$ and $\mathcal{Y}_{(0,\infty)}$ have a Frobenius automorphism φ induced from the Witt vector structure of $\mathrm{A}_{\mathrm{inf}}$.

The space $\mathcal{Y}_{(0,\infty)}$ is a preperfectoid space. The (adic) Fargues–Fontaine curve associated to F is defined as the quotient

$$\mathcal{X} = \mathcal{X}(F) = \mathcal{Y}_{(0,\infty)}(F)/\varphi^{\mathbb{Z}},$$

which makes sense because the Frobenius action is proper and discontinuous. The natural projection $\pi : \mathcal{Y}_{(0,\infty)} \to \mathcal{X}$ is a local isomorphism, so $\mathcal{X}$ is a preperfectoid space, by virtue of $\mathcal{Y}_{(0,\infty)}$ being so. The space $\mathcal{Y}_{(0,\infty)}$ has a canonical point called x_∞, the "point at infinity". It corresponds to the kernel of Fontaine's map

$$\theta : \mathrm{A}_{\mathrm{inf}} \to \mathcal{O}_F, \quad \sum_{n\geq 0} [a_n] p^n \mapsto \sum_{n \geq 0} a_n^\sharp p^n,$$

where for $a \in \mathcal{O}_F$, $a^\sharp$ is defined to be the first coordinate of $a \in \mathcal{O}_F^\flat = \varprojlim_{x \mapsto x^p} \mathcal{O}_F$. Identify x_∞ with its image $\pi(x_\infty) \in \mathcal{X}$. We shall sometimes use the fact that $\ker\theta$ is a principal ideal, generated by $\xi = \varpi - [\varpi^\flat]$ (for example).

If $F = \widehat{K}_\infty$, there is an induced action of the group $\Gamma = \mathrm{Gal}(K_\infty/K)$ on each of the spaces mentioned above, and the map $\mathcal{Y}_{(0,\infty)} \to \mathcal{X}$ is Γ-equivariant. The point $x_\infty \in \mathcal{X}$ is the unique Γ-fixed point; in fact, it is the unique point with finite Γ-orbit [Fargues and Fontaine 2018, Proposition 10.1.1]. From now on, if F is omitted from the notation of $\mathcal{Y}_{(0,\infty)}$ and $\mathcal{X}$, we always take $F = \widehat{K}_\infty$.

3B. ***The spaces $\mathcal{Y}_I$ and $\mathcal{X}_I$.*** It will be fruitful to consider certain open subsets of $\mathcal{Y}_{(0,\infty)}$ and $\mathcal{X}$. By [Scholze and Weinstein 2020, Lecture 12] there is a surjective continuous map $\kappa : \mathcal{Y} \to [0, \infty]$ given by[1]

$$\kappa(x) = \frac{\log |p(\tilde{x})|}{\log |[\varpi^\flat](\tilde{x})|},$$

where $\tilde{x}$ is the maximal generization of x. For each interval $I \subset (0, \infty)$, let $\mathcal{Y}_I$ be the interior of the preimage of $\mathcal{Y}$ under κ. These spaces are Γ-stable if such a Γ action is present. Furthermore, the map φ induces isomorphisms $\varphi : \mathcal{Y}_{pI} \xrightarrow{\sim} \mathcal{Y}_I$. Write $\log(I) = \{\log x : x \in I\}$. Whenever I is sufficiently small so that the inequality $|\log(I)| < \log(p)$ holds, we have $\bar{I} \cap p\bar{I} = 0$ and π maps $\mathcal{Y}_I$ isomorphically onto its image $\pi(\mathcal{Y}_I) = \mathcal{X}_I \subset \mathcal{X}$. Note that $x_\infty \in \mathcal{X}_I$ if and only if I contains an element of $(p-1)p^{\mathbb{Z}}$, because $\kappa(x_\infty) = (p-1)/p$.

For $I \subset (0, \infty)$, we have the coordinate rings

$$\widetilde{\mathrm{B}}_I = \widetilde{\mathrm{B}}_I(\widehat{K}_\infty) = \mathrm{H}^0(\mathcal{Y}_I, \mathcal{O}_{\mathcal{Y}_{(0,\infty)}}).$$

If I is compact, the geometry of $\mathcal{Y}_I$ is simple.

Proposition 3.1. *Suppose $I \subset (0, \infty)$ is a compact interval.*

(i) $\mathcal{Y}_I = \mathrm{Spa}(\widetilde{\mathrm{B}}_I, \tilde{\mathrm{A}}_I)$, *where $\tilde{\mathrm{A}}_I$ is the ring of power bounded elements of $\widetilde{\mathrm{B}}_I$. In particular, $\mathcal{Y}_I$ is affinoid.*

(ii) *$\widetilde{\mathrm{B}}_I$ is a principal ideal domain.*

(iii) *The global sections functor induces an equivalence of categories between vector bundles on $\mathcal{Y}_I$ and finite free $\widetilde{\mathrm{B}}_I$-modules.*

Proof. Parts (i) and (ii) follow from [Fargues and Fontaine 2018, théorème 3.5.1]. Part (iii) follows from [Scholze and Weinstein 2020, Theorem 5.2.8] (originally [Kedlaya and Liu 2015, Theorem 2.7.7]), since finite projective $\widetilde{\mathrm{B}}_I$-modules are finite free. □

3C. ***Equivariant vector bundles.*** The action of Γ on $\mathcal{X}$ gives an automorphism $\gamma : \mathcal{X} \xrightarrow{\sim} \mathcal{X}$ for each $\gamma \in \Gamma$.

Definition 3.2. A Γ-equivariant vector bundle (or simply Γ-vector bundle) on $\mathcal{X}$ is a vector bundle $\tilde{\mathcal{E}}$ on $\mathcal{X}$ equipped with an isomorphism $c_\gamma : \gamma^*\tilde{\mathcal{E}} \xrightarrow{\sim} \tilde{\mathcal{E}}$ for each $\gamma \in \Gamma$ such that the cocycle condition $c_{\gamma_2} \circ \gamma_2^* c_{\gamma_1} = c_{\gamma_1\gamma_2}$ holds for every $\gamma_1, \gamma_2, \in \Gamma$.

Similarly, we have a notion of a (φ, Γ)-vector bundle on $\mathcal{Y}_{(0,\infty)}$. This consists of a Γ-vector bundle $\widetilde{\mathcal{M}}$ on $\mathcal{Y}_{(0,\infty)}$ together with an additional isomorphism $c_\varphi : \varphi^*\widetilde{\mathcal{M}} \xrightarrow{\sim} \widetilde{\mathcal{M}}$ such that $c_\varphi \circ \varphi^* c_\gamma = c_\gamma \circ \gamma^* c_\varphi$ for every $\gamma \in \Gamma$.

Descent along φ gives the following.

Proposition 3.3. *There is an equivalence of categories*

$$\{\Gamma\text{-vector bundles on } \mathcal{X}\} \cong \{(\varphi, \Gamma)\text{-vector bundles on } \mathcal{Y}_{(0,\infty)}\}.$$

[1] Our normalization of κ is the inverse of [loc. cit.].

The equivalence is given by the following functors: If $\tilde{\mathcal{E}}$ is an equivariant vector bundle, we map it to $\mathcal{O}_{\mathcal{Y}_{(0,\infty)}} \otimes_{\mathcal{O}_{\mathcal{X}}} \tilde{\mathcal{E}}$. Conversely, if $\widetilde{\mathcal{M}}$ is a (φ, Γ)-vector bundle on $\mathcal{Y}_{(0,\infty)}$, we map it to $\pi_*(\widetilde{\mathcal{M}})^{\varphi=1}$.

If $\tilde{\mathcal{E}}$ is a Γ-vector bundle on $\mathcal{X}$ and $U \subset \mathcal{X}$ is an open subset stable under Γ, there is an induced action of Γ on $\mathrm{H}^0(U, \tilde{\mathcal{E}})$. In particular, there is a natural action of Γ on $\mathrm{H}^0(\mathcal{X}_I, \tilde{\mathcal{E}})$ when $|\log(I)| < \log(p)$. For a general open subset U, one only has a map

$$c_\gamma : \mathrm{H}^0(U, \mathcal{O}_{\mathcal{X}}) \otimes_{\mathrm{H}^0(\gamma(U), \mathcal{O}_{\mathcal{X}})} \mathrm{H}^0(\gamma(U), \tilde{\mathcal{E}}) \to \mathrm{H}^0(U, \tilde{\mathcal{E}}).$$

Similar remarks apply for (φ, Γ)-equivariant vector bundles on $\mathcal{Y}_{(0,\infty)}$.

Example 3.4. Let V be a finite dimensional $\mathbb{Q}_p$-representation of G_K. Recall that $H = \mathrm{Gal}(K_\infty/K)$. Then by [Fargues and Fontaine 2018, théorème 10.1.5],

$$\tilde{\mathcal{E}}(V) := (V \otimes_{\mathbb{Q}_p} \mathcal{O}_{\mathcal{X}(\mathbb{C}_p)})^H$$

is a Γ-vector bundle on $\mathcal{X}$. More generally, by [loc. cit.], the category of finite dimensional G_K-representations embeds fully faithfully to the category of (φ, Γ)-modules, with essential image the subcategory of étale (φ, Γ)-modules. We can extend the domain of the functor $V \mapsto \tilde{\mathcal{E}}(V)$ from G_K-representations to (φ, Γ)-modules. Conversely, any Γ-vector bundle on $\mathcal{X}$ gives rise to a (φ, Γ)-module, and this correspondence results in a equivalence of categories (see [Fargues and Fontaine 2018, préface, Remark 5.10]). This will be discussed in detail in Section 7.

4. Locally analytic vector bundles

In this section, we introduce the category of locally analytic vector bundles and discuss their basic properties.

4A. ***Locally analytic functions of $\mathcal{Y}_{(0,\infty)}$ and $\mathcal{X}$.*** Let $U \subset \mathcal{X}$ be an open affinoid. Then U is quasicompact and hence stable under the action of a finite index subgroup $\Gamma' \leq \Gamma$. The space of functions $\mathrm{H}^0(U, \mathcal{O}_{\mathcal{X}})$ is a Banach Γ'-ring, and so it makes sense to speak of its subring of Γ'-locally analytic functions. This does not depend on the choice of Γ', and so we shall write $\mathrm{H}^0(U, \mathcal{O}_{\mathcal{X}})^{\mathrm{la}}$ for the Γ'-locally analytic functions in $\mathrm{H}^0(U, \mathcal{O}_{\mathcal{X}})$ for any Γ'. Since taking locally analytic vectors is left exact, these can be glued and we obtain a sheaf of rings $\mathcal{O}_{\mathcal{X}}^{\mathrm{la}}$ on $\mathcal{X}$ that satisfies

$$\mathrm{H}^0(U, \mathcal{O}_{\mathcal{X}}^{\mathrm{la}}) = \mathrm{H}^0(U, \mathcal{O}_{\mathcal{X}})^{\mathrm{la}}$$

for every open affinoid $U \subset \mathcal{X}$.

More generally, suppose U is an open subset of $\mathcal{X}$ which is not necessarily affinoid, but for which there is an increasing cover $U = \bigcup_i U_i$ with each U_i affinoid and a single finite index subgroup $\Gamma' \leq \Gamma$ stabilizing all of the U_i simultaneously. This condition will be satisfied in any situation we shall consider. Then the sections of $\mathcal{O}_{\mathcal{X}}^{\mathrm{la}}$ on U are the pro-analytic functions

$$\mathrm{H}^0(U, \mathcal{O}_{\mathcal{X}}^{\mathrm{la}}) = \varprojlim_i \mathrm{H}^0(U_i, \mathcal{O}_{\mathcal{X}})^{\mathrm{la}} = \mathrm{H}^0(U, \mathcal{O}_{\mathcal{X}})^{\mathrm{pa}}.$$

Lemma 4.1. *The sheaf $\mathcal{O}_{\mathcal{X}}^{\mathrm{la}}$ is stable for the action of Γ on $\mathcal{O}_{\mathcal{X}}$, in the sense that the inclusion $\mathcal{O}_{\mathcal{X}}^{\mathrm{la}} \subset \mathcal{O}_{\mathcal{X}}$ induces isomorphisms*

$$c_\gamma : \gamma^* \mathcal{O}_{\mathcal{X}}^{\mathrm{la}} \xrightarrow{\sim} \mathcal{O}_{\mathcal{X}}^{\mathrm{la}}.$$

Proof. The action of Γ on $\mathcal{O}_{\mathcal{X}}$ gives rise to an isomorphism $c_\gamma : \gamma^*\mathcal{O}_{\mathcal{X}} \xrightarrow{\sim} \mathcal{O}_{\mathcal{X}}$. Upon taking $U \subset \mathcal{X}$ affinoid, evaluating the morphism c_γ at U and taking locally analytic vectors, we get an induced map $c_\gamma(U) : \mathrm{H}^0(U, \gamma^*\mathcal{O}_{\mathcal{X}})^{\mathrm{la}} \xrightarrow{\sim} \mathrm{H}^0(U, \mathcal{O}_{\mathcal{X}})^{\mathrm{la}}$. But this is the same as $\mathrm{H}^0(U, \gamma^*\mathcal{O}_{\mathcal{X}}^{\mathrm{la}}) \xrightarrow{\sim} \mathrm{H}^0(U, \mathcal{O}_{\mathcal{X}}^{\mathrm{la}})$ because of the equality $\mathrm{H}^0(U, \mathcal{O}_{\mathcal{X}}^{\mathrm{la}}) = \mathrm{H}^0(U, \mathcal{O}_{\mathcal{X}})^{\mathrm{la}}$. By writing an arbitrary open set as a union of affinoids, we get the desired induced isomorphism $c_\gamma : \gamma^*\mathcal{O}_{\mathcal{X}}^{\mathrm{la}} \xrightarrow{\sim} \mathcal{O}_{\mathcal{X}}^{\mathrm{la}}$. □

The preceding discussion then applies equally well to $\mathcal{Y}_{(0,\infty)}$, so we have a sheaf $\mathcal{O}_{\mathcal{Y}_{(0,\infty)}}^{\mathrm{la}}$ of locally analytic functions on $\mathcal{Y}_{(0,\infty)}$ endowed with isomorphisms c_γ. Since the φ-action on $\mathcal{Y}_{(0,\infty)}$ commutes with the Γ-action, it preserves the Γ-locally analytic functions, and this gives an isomorphism

$$c_\varphi : \varphi^* \mathcal{O}_{\mathcal{Y}_{(0,\infty)}}^{\mathrm{la}} \xrightarrow{\sim} \mathcal{O}_{\mathcal{Y}_{(0,\infty)}}^{\mathrm{la}}$$

which commutes with the Γ-action as usual.

4B. *A flatness result.* For our application at Section 6 it would be useful to know the inclusion $\mathcal{O}_{\mathcal{X}}^{\mathrm{la}} \subset \mathcal{O}_{\mathcal{X}}$ is flat. We are only able to establish this in the cyclotomic case where $K_\infty = K_{\mathrm{cyc}}$, and only for certain open subsets. Nevertheless, this will suffice for our needs.

So in this subsection suppose $K_\infty = K_{\mathrm{cyc}}$ and let I be a closed interval of the form $I = [r, s]$ with $r \geq (p-1)/p$. We write $\widetilde{\boldsymbol{B}}_{I,\mathrm{cyc}}$ for $\widetilde{\boldsymbol{B}}_I(\widehat{K}_{\mathrm{cyc}})$ of Section 3B. Let K_0' be the maximal unramified extension of $\mathbb{Q}_p$ contained in K_{cyc}. Then we write $\boldsymbol{B}_{I,\mathrm{cyc},K}$ for the ring of power series $f(T) = \sum_{k\in\mathbb{Z}} a_k T^k$ with $a_k \in K_0'$, such that $f(T)$ converges on the nonempty annulus where $|T| \in I$. By a classical result, $\boldsymbol{B}_{I,\mathrm{cyc},K}$ is a principal ideal domain [Lazard 1962, corollaire à proposition 4]. There is an embedding $\boldsymbol{B}_{I,\mathrm{cyc},K} \hookrightarrow \widetilde{\boldsymbol{B}}_{I,\mathrm{cyc}}$ for which $\boldsymbol{B}_{I,\mathrm{cyc},K}$ is Γ_{cyc}-stable. If K is unramified over $\mathbb{Q}_p$, this embedding can be described as follows: the variable T is mapped to $[\varepsilon]-1$, where $\varepsilon = (1, \zeta_p, \zeta_{p^2}, \ldots) \in \widehat{K}_{\mathrm{cyc}}^\flat$. Further, one calculates that $\gamma(T) = (1+T)^{\chi_{\mathrm{cyc}}(\gamma)} - 1$, so $\boldsymbol{B}_{I,\mathrm{cyc},K}$ is indeed stable under the action of Γ_{cyc}.

Proposition 4.2. *Suppose $I = [r, (p-1)p^{k-1}]$ with $k \geq 1$. Then*

(i) $\widetilde{\boldsymbol{B}}_{I,\mathrm{cyc}}^{\mathrm{la}} = \bigcup_{n\geq 0} \varphi^{-n}(\boldsymbol{B}_{p^n I,\mathrm{cyc},K})$,

(ii) *$\widetilde{\boldsymbol{B}}_{I,\mathrm{cyc}}^{\mathrm{la}}$ is a Prüfer domain,*

(iii) *the natural ring morphism $\widetilde{\boldsymbol{B}}_{I,\mathrm{cyc}}^{\mathrm{la}} \to \widetilde{\boldsymbol{B}}_{I,\mathrm{cyc}}$ is flat.*

Proof. Part (i) is [Berger 2016, Theorem 4.4 (2)]. Note that in [loc. cit.] this is stated only for I of the form $[(p-1)p^{l-1}, (p-1)p^{k-1}]$, but the argument given there (see also Section 13 of [Berger 2021]) is valid for any interval of the form $[r, (p-1)p^{k-1}]$. Part (ii) follows, because each $\boldsymbol{B}_{p^n I,\mathrm{cyc}}$ is a principal ideal domain, and an increasing union of such rings is a Prüfer domain. Finally, the ring $\widetilde{\boldsymbol{B}}_{I,\mathrm{cyc}}$ is a domain and hence torsionfree over the subring $\widetilde{\boldsymbol{B}}_{I,\mathrm{cyc}}^{\mathrm{la}}$. Part (iii) is established by recalling that a torsionfree module over a Prüfer domain is flat [Lam 1999, Proposition 4.20]. □

Question 4.3. *To what extent do* (ii) *and* (iii) *of Proposition 4.2 hold for coordinate rings of general open subsets in $\mathcal{X}$ and general K_∞? We do not expect $\widetilde{\boldsymbol{B}}_I^{\mathrm{la}}$ to be a Prüfer domain when Γ has dimension larger than* 1. *Nevertheless, it might still be the case that $\widetilde{\boldsymbol{B}}_I^{\mathrm{la}} \to \widetilde{\boldsymbol{B}}_I$ is flat.*

4C. *Locally analytic vector bundles.*

Definition 4.4. A locally analytic vector bundle on $\mathcal{X}$ is a locally finite free $\mathcal{O}_{\mathcal{X}}^{\mathrm{la}}$-module $\mathcal{E}$ on $\mathcal{X}$ equipped with an isomorphism $c_\gamma : \gamma^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$ for each $\gamma \in \Gamma$ such that the cocycle condition $c_{\gamma_2} \circ \gamma_2^* c_{\gamma_1} = c_{\gamma_1\gamma_2}$ holds for every $\gamma_1, \gamma_2, \in \Gamma$. We require the action to be continuous with respect to the locally analytic topology.

Example 4.5. (1) Let $\tilde{\mathcal{E}}$ be a Γ-vector bundle on $\mathcal{X}$. Define a sheaf $\tilde{\mathcal{E}}^{\mathrm{la}}$ by generalizing the definition of $\mathcal{O}_{\mathcal{X}}^{\mathrm{la}}$. Namely, for every open affinoid $U \subset \mathcal{X}$ choose $\Gamma' \leq \Gamma$ stabilizing U. Then $\mathrm{H}^0(U, \tilde{\mathcal{E}})$ is a Banach Γ'-ring and it makes sense to speak of $\mathrm{H}^0(U, \tilde{\mathcal{E}})^{\mathrm{la}}$, which does not depend on the choice of Γ'. Glue these together to form a sheaf $\tilde{\mathcal{E}}^{\mathrm{la}}$. The sheaf $\tilde{\mathcal{E}}^{\mathrm{la}}$ is an $\mathcal{O}_{\mathcal{X}}^{\mathrm{la}}$ -module with a Γ-action. We shall show in Section 6 that $\tilde{\mathcal{E}}^{\mathrm{la}}$ is locally free and therefore an example of a locally analytic vector bundle.

(2) Conversely, if $\mathcal{E}$ is a locally analytic vector bundle, we can associate to it a Γ-vector bundle $\tilde{\mathcal{E}} = \mathcal{O}_{\mathcal{X}} \otimes_{\mathcal{O}_{\mathcal{X}}^{\mathrm{la}}} \mathcal{E}$. If $U \subset \mathcal{X}$ is an open affinoid such that $\mathcal{E}|_U$ is free, it follows from Proposition 2.1 that

$$\mathrm{H}^0(U, \mathcal{E}) = \mathrm{H}^0(U, \tilde{\mathcal{E}})^{\mathrm{la}},$$

and so $\mathcal{E} = \tilde{\mathcal{E}}^{\mathrm{la}}$. This shows that the functor from Γ-vector bundles to locally analytic vector bundles mapping $\tilde{\mathcal{E}}$ to $\tilde{\mathcal{E}}^{\mathrm{la}}$ is essentially surjective.

It follows from Example 4.5(2) that if $\mathcal{E}$ is a locally analytic vector bundle, we have an action by derivations

$$\mathrm{Lie}(\Gamma) \times \mathcal{E} \to \mathcal{E},$$

or, what amounts to the same, a connection

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathbb{Q}_p} (\mathrm{Lie}\,\Gamma)^\vee$$

satisfying the identity

$$\nabla(fx) = \nabla(f)x + f\nabla(x)$$

for local sections f of $\mathcal{O}_{\mathcal{X}}^{\mathrm{la}}$ and x of $\mathcal{E}$.

Remark 4.6. We emphasize that if $U \subset \mathcal{X}$ is an arbitrary open subset then we have an induced action of $\mathrm{Lie}(\Gamma)$ on $\mathrm{H}^0(U, \mathcal{E})$. This is unlike the Γ-action, which only maps $\mathrm{H}^0(U, \mathcal{E})$ to itself if U is Γ-stable. This is one pleasant aspect of working with locally analytic vector bundles instead of Γ-vector bundles.

Finally, we have the following propositions computing sections of interest. They will not be used elsewhere in the article. We may define a locally analytic φ-vector bundle on $\mathcal{Y}_{(0,\infty)}$ by imitating Definition 4.4. Then given a (φ, Γ)-vector bundle $\widetilde{\mathcal{M}}$ on $\mathcal{Y}_{(0,\infty)}$, one can define a locally analytic φ-vector bundle $\widetilde{\mathcal{M}}^{\mathrm{la}}$ on $\mathcal{Y}_{(0,\infty)}$ as in Example 4.5.

Proposition 4.7. *Let $\tilde{\mathcal{E}}$ (resp. $\widetilde{\mathcal{M}}$) be a Γ-vector bundle on $\mathcal{X}$ (resp. a (φ,Γ)-vector bundle on $\mathcal{Y}_{(0,\infty)}$) and let $\tilde{\mathcal{E}}^{\mathrm{la}}$ (resp. $\widetilde{\mathcal{M}}^{\mathrm{la}}$) be its associated locally analytic vector bundle (resp. locally analytic φ-vector bundle). There are natural isomorphisms*:

(i) $\mathrm{H}^0(\mathcal{Y}_I,\widetilde{\mathcal{M}}^{\mathrm{la}})\cong \mathrm{H}^0(\mathcal{Y}_I,\widetilde{\mathcal{M}})^{\mathrm{la}}$ *for I a closed interval.*

(ii) $\mathrm{H}^0(\mathcal{Y}_I,\widetilde{\mathcal{M}}^{\mathrm{la}})\cong \mathrm{H}^0(\mathcal{Y}_I,\widetilde{\mathcal{M}})^{\mathrm{pa}}$ *for I an open interval.*

(iii) $\mathrm{H}^0(\mathcal{X}_I,\tilde{\mathcal{E}}^{\mathrm{la}})\cong \mathrm{H}^0(\mathcal{X}_I,\tilde{\mathcal{E}})^{\mathrm{la}}$ *for I a closed interval with $|\log(I)|<\log(p)$.*

(iv) $\mathrm{H}^0(\mathcal{X}_I,\tilde{\mathcal{E}}^{\mathrm{la}})\cong \mathrm{H}^0(\mathcal{X}_I,\tilde{\mathcal{E}})^{\mathrm{pa}}$ *for I an open interval with $|\log(I)|<\log(p)$.*

(v) $\mathrm{H}^0(\mathcal{X},\tilde{\mathcal{E}}^{\mathrm{la}})\cong \mathrm{H}^0(\mathcal{X},\tilde{\mathcal{E}})^{\mathrm{la}}$.

(vi) $\mathrm{H}^0(\mathcal{X}-x_\infty,\tilde{\mathcal{E}}^{\mathrm{la}})\cong \mathrm{H}^0(\mathcal{X}-x_\infty,\tilde{\mathcal{E}})^{\mathrm{pa}}$.

Proof. Parts (i) and (iii) are immediate from the definition. For (ii) and (iv), use the coverings $\mathcal{Y}_I=\bigcup_{J\subset I}\mathcal{Y}_J$ and $\mathcal{X}_I=\bigcup_{J\subset I}\mathcal{X}_J$ ranging over $J\subset I$ closed. For (v), consider the covering

$$\mathcal{X}=\mathcal{X}_{[1,\sqrt{p}]}\cup\mathcal{X}_{[\sqrt{p},p]}$$

with intersection $\mathcal{X}_{[\sqrt{p},\sqrt{p}]}\amalg\mathcal{X}_{[1,1]}$ (identifying 1 with p via φ). This yields exact sequences

$$0\to\mathrm{H}^0(\mathcal{X},\tilde{\mathcal{E}}^{\mathrm{la}})\to\mathrm{H}^0(\mathcal{X}_{[1,\sqrt{p}]},\tilde{\mathcal{E}}^{\mathrm{la}})\oplus\mathrm{H}^0(\mathcal{X}_{[\sqrt{p},p]},\tilde{\mathcal{E}}^{\mathrm{la}})\to\mathrm{H}^0(\mathcal{X}_{[\sqrt{p},\sqrt{p}]},\tilde{\mathcal{E}}^{\mathrm{la}})\oplus\mathrm{H}^0(\mathcal{X}_{[1,1]},\tilde{\mathcal{E}}^{\mathrm{la}})$$

and

$$0\to\mathrm{H}^0(\mathcal{X},\tilde{\mathcal{E}})^{\mathrm{la}}\to\mathrm{H}^0(\mathcal{X}_{[1,\sqrt{p}]},\tilde{\mathcal{E}})^{\mathrm{la}}\oplus\mathrm{H}^0(\mathcal{X}_{[\sqrt{p},p]},\tilde{\mathcal{E}})^{\mathrm{la}}\to\mathrm{H}^0(\mathcal{X}_{[\sqrt{p},\sqrt{p}]},\tilde{\mathcal{E}})^{\mathrm{la}}\oplus\mathrm{H}^0(\mathcal{X}_{[1,1]},\tilde{\mathcal{E}})^{\mathrm{la}}.$$

By virtue of (iii) the kernels of these sequences are identified. This proves part (v).

For (vi), use the covering

$$\mathcal{X}-x_\infty=\mathcal{X}_{[1,\sqrt{p}]}\cup(\mathcal{X}_{[\sqrt{p},p]}-x_\infty)$$

with intersection $\mathcal{X}_{[\sqrt{p},\sqrt{p}]}\amalg\mathcal{X}_{[1,1]}$. We may write $\mathcal{X}_{[\sqrt{p},p]}-x_\infty$ as a union of Γ-stable rational open subsets

$$\mathcal{X}_{[\sqrt{p},p]}-\infty=\cup_{n\geq 1}\mathcal{X}_{[\sqrt{p},p]}\{|\xi|\geq p^{-n}\}.$$

Thus

$$\mathrm{H}^0(\mathcal{X}_{[\sqrt{p},p]}-x_\infty,\tilde{\mathcal{E}}^{\mathrm{la}})\cong\mathrm{H}^0(\mathcal{X}_{[\sqrt{p},p]}-x_\infty,\tilde{\mathcal{E}})^{\mathrm{pa}}.$$

Repeating the argument which proved part (v), we conclude. □

We place ourselves in the cyclotomic setting so that $\Gamma=\Gamma_{\mathrm{cyc}}$ and $H=\mathrm{Gal}(\overline{K}/K_{\mathrm{cyc}})$, and we write $\boldsymbol{B}^+_{\mathrm{cris}}(\widehat{K}_{\mathrm{cyc}})=(\boldsymbol{B}^+_{\mathrm{cris}})^H$. Following Section 10.2 of [Fargues and Fontaine 2018], for $n\in\mathbb{Z}$ take $\tilde{\mathcal{E}}=\mathcal{O}_{\mathcal{X}}(n)$ to be the Γ-line bundle corresponding to the graded module

$$\bigoplus_{m\geq 0}\boldsymbol{B}^+_{\mathrm{cris}}(\widehat{K}_{\mathrm{cyc}})^{\varphi=p^{m+n}}.$$

Proposition 4.8.
$$\mathrm{H}^0(\mathcal{X},\mathcal{O}_{\mathcal{X}}(n)^{\mathrm{la}})=\begin{cases}0, & n<0,\\ \mathbb{Q}_p(n), & n\geq 0.\end{cases}$$

Proof. To show this, notice first that

$$\mathrm{H}^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(n)) = \boldsymbol{B}^+_{\mathrm{cris}}(\widehat{K}_{\mathrm{cyc}})^{\varphi=p^n} = \begin{cases} 0, & n<0, \\ \mathbb{Q}_p, & n=0, \\ \boldsymbol{B}^+_{\mathrm{cris}}(\widehat{K}_{\mathrm{cyc}})^{\varphi=p^n}, & n>0. \end{cases}$$

If $n>0$ then by [Fargues and Fontaine 2018, 6.4.2] there is an exact sequence

$$0 \to \mathbb{Q}_p(n) \to \boldsymbol{B}^{+,\varphi=p^n}_{\mathrm{cris}} \to \boldsymbol{B}^+_{\mathrm{dR}}/t^n\boldsymbol{B}^+_{\mathrm{dR}} \to 0.$$

Take H-invariants and locally analytic vectors. By [Berger and Colmez 2016, théorème 4.11] we know that $(\boldsymbol{B}^+_{\mathrm{dR}}/t^n\boldsymbol{B}^+_{\mathrm{dR}})^{H,\mathrm{la}} = K_{\mathrm{cyc}}[\![t]\!]/t^n$, so we are left with an exact sequence

$$0 \to \mathbb{Q}_p(n) \to \boldsymbol{B}^+_{\mathrm{cris}}(\widehat{K}_{\mathrm{cyc}})^{\varphi=p^n,\mathrm{la}} \to K_{\mathrm{cyc}}[\![t]\!]/t^n.$$

Claim. $$\boldsymbol{B}^+_{\mathrm{cris}}(\widehat{K}_{\mathrm{cyc}})^{\varphi=p^n,\mathrm{la}} = \mathbb{Q}_p(n).$$

Note that a similar statement appears in Section 3.3 of [Berger and Colmez 2016] in the case $n=1$. Given the claim the computation is finished because part (v) of Proposition 4.7 implies that

$$\mathrm{H}^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(n)^{\mathrm{la}}) = \boldsymbol{B}^+_{\mathrm{cris}}(\widehat{K}_{\mathrm{cyc}})^{\varphi=p^n,\mathrm{la}} = \begin{cases} 0, & n<0, \\ \mathbb{Q}_p(n), & n\geq 0. \end{cases}$$

To show the claim, take $x \in \boldsymbol{B}^+_{\mathrm{cris}}(\widehat{K}_{\mathrm{cyc}})^{\varphi=p^n,\mathrm{la}}$. Its image in $K_{\mathrm{cyc}}[\![t]\!]/t^n$ is killed by the polynomial

$$P_n(\gamma) := \prod_{i=0}^{n-1} (\chi_{\mathrm{cyc}}(\gamma)^{-i}\gamma - 1)$$

for γ which generates an open subgroup of Γ. It follows that $P_n(\gamma)(x) \in \mathbb{Q}_p(n)$ for this γ. Since $P_n(\gamma)$ acts on $\mathbb{Q}_p(n)$ by a nonzero element we reduce to showing that $\boldsymbol{B}^+_{\mathrm{cris}}(\widehat{K}_{\mathrm{cyc}})^{\varphi=p^n,P_n(\gamma)=0}$ is 0. In fact, if K' is the subfield of K_{cyc} corresponding to $\gamma^{\mathbb{Z}_p} \subset \Gamma$ with maximal unramified subextension K'_0, we shall compute that

$$\boldsymbol{B}_{\mathrm{cris}}(\widehat{K}_{\mathrm{cyc}})^{P_n(\gamma)=0} = \bigoplus_{i=0}^{n-1} K'_0 t^i,$$

and in particular there are no nonzero elements with $\varphi = p^n$.

To show this latter description of the elements killed by $P_n(\gamma)$, we argue by induction. If $n=1$ then $P_n(\gamma) = \gamma - 1$ and the equality follows from the usual description of the Galois invariants of $\boldsymbol{B}_{\mathrm{cris}}$. For $n \geq 2$, we have $P_n(\gamma)/(\gamma-1) = P_{n-1}(\chi_{\mathrm{cyc}}(\gamma)^{-1}\gamma)$ and

$$\boldsymbol{B}_{\mathrm{cris}}(\widehat{K}_{\mathrm{cyc}})^{P_{n-1}(\chi_{\mathrm{cyc}}(\gamma)^{-1}\gamma)=0} = t\boldsymbol{B}_{\mathrm{cris}}(\widehat{K}_{\mathrm{cyc}})^{P_{n-1}(\gamma)=0} = \bigoplus_{i=1}^{n-1} K'_0 t^i.$$

Thus there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K_0' \arrow[r] \arrow[d, "\cong"] & \bigoplus_{i=0}^{n-1} K_0' t^i \arrow[r] \arrow[d] & \bigoplus_{i=1}^{n-1} K_0' t^i \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \boldsymbol{B}_{\mathrm{cris}}(\widehat{K}_{\mathrm{cyc}})^{\gamma-1=0} \arrow[r] & \boldsymbol{B}_{\mathrm{cris}}(\widehat{K}_{\mathrm{cyc}})^{P_n(\gamma)=0} \arrow[r] & \boldsymbol{B}_{\mathrm{cris}}(\widehat{K}_{\mathrm{cyc}})^{P_{n-1}(\chi_{\mathrm{cyc}}(\gamma)^{-1}\gamma)=0} &
\end{tikzcd}$$

whose rows are exact and whose outer vertical maps are isomorphisms. We conclude by the applying the five lemma. □

Remark 4.9. Set $\boldsymbol{B}_e(\widehat{K}_\infty) = \boldsymbol{B}_e^H$ for the usual ring $\boldsymbol{B}_e = \boldsymbol{B}_{\mathrm{cris}}^{\varphi=1}$, so that $\boldsymbol{B}_e \subset \mathrm{H}^0(\mathcal{X} - x_\infty, \mathcal{O}_{\mathcal{X}})$. This inclusion is not an equality: the ring $\boldsymbol{B}_e$ allows only meromorphic functions at x_∞ while in $\mathrm{H}^0(\mathcal{X} - x_\infty, \mathcal{O}_{\mathcal{X}})$ there will be functions with essential singularities. The subring $\boldsymbol{B}_e(\widehat{K}_\infty)^{\mathrm{pa}} \subset \mathrm{H}^0(\mathcal{X} - x_\infty, \mathcal{O}_{\mathcal{X}})^{\mathrm{la}}$ is more tractable and we can understand its structure to an extent. In particular, let us consider the subring $\boldsymbol{B}_e(\widehat{K}_\infty)^{\mathrm{pa}} = \boldsymbol{B}_e \cap \mathrm{H}^0(\mathcal{X} - x_\infty, \mathcal{O}_{\mathcal{X}}^{\mathrm{la}})$ in the case $\Gamma = \Gamma_{\mathrm{cyc}}$. We claim that in fact $\boldsymbol{B}_e(\widehat{K}_\infty)^{\mathrm{pa}} = \mathbb{Q}_p$. To see this, take $x \in \boldsymbol{B}_e(\widehat{K}_\infty)^{\mathrm{pa}}$, and restrict it to $\mathcal{X}_{[\sqrt{p},p]} - x_\infty$. Since $\mathcal{Y}_{[\sqrt{p},p]}$ maps isomorphically onto $\mathcal{X}_{[\sqrt{p},p]}$, the element t gives an element of $\mathrm{H}^0(\mathcal{X}_{[\sqrt{p},p]} - x_\infty, \mathcal{O}_{\mathcal{X}}^{\mathrm{la}})$. Multiplying by a bounded power of t, the function $t^n x$ extends to an element of

$$\mathrm{H}^0(\mathcal{X}_{[\sqrt{p},p]}, \mathcal{O}_{\mathcal{X}}^{\mathrm{la}}) = \mathrm{H}^0(\mathcal{X}_{[\sqrt{p},p]}, \mathcal{O}_{\mathcal{X}})^{\mathrm{la}},$$

which shows that x itself is actually an element of $\boldsymbol{B}_e(\widehat{K}_\infty)^{\mathrm{la}}$, with a pole of order n at x_∞. Therefore, $t^n x \in \mathrm{H}^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(n)^{\mathrm{la}})$ which is equal to $\mathbb{Q}_p(n)$ as was shown in Proposition 4.8. This means x is in $\mathbb{Q}_p$ and so $\boldsymbol{B}_e(\widehat{K}_\infty)^{\mathrm{pa}} = \mathbb{Q}_p$.

Question 4.10. (1) *Is it true that* $\mathrm{H}^0(\mathcal{X} - x_\infty, \mathcal{O}_{\mathcal{X}}^{\mathrm{la}}) = \mathbb{Q}_p$ *if* $\Gamma \neq \Gamma_{\mathrm{cyc}}$ *and* $\dim \Gamma = 1$?

(2) *If* $\dim \Gamma > 1$ *then one can sometimes produce elements in* $\boldsymbol{B}_e(\widehat{K}_\infty)^{\mathrm{la}}$ *which do not belong to* $\mathbb{Q}_p$. *For example, in the Lubin–Tate setting, the element* $(t_{-\sqrt{p}}/t_{\sqrt{p}})^2$ *lies in* $\boldsymbol{B}_e(\widehat{K}_\infty)^{\mathrm{la}}$, *for* $t_{\pm\sqrt{p}}$ *being the analogue of Fontaine's element attached to the uniformizer* $\pi = \pm\sqrt{p}$ (*see Section* 8.3 *of* [Colmez 2002] *for the notation appearing here*). *Is it true that in some generality* $\boldsymbol{B}_e(\widehat{K}_\infty)^{\mathrm{la}}$ *will be* $d-1$ *dimensional for* $d = \dim \Gamma$? *See* [Berger and Colmez 2016, théoréme 6.1] *for a related statement.*

5. Acyclicity of locally analytic vectors for semilinear representations

In this section, we shall prove vanishing the of $\mathrm{R}^i_{\mathrm{la}}$-groups for certain semilinear representations. These results will be used to prove the descent result in Section 6 but are also of independent interest. We follow the strategy of [Pan 2022a], where the case of a trivial representation and a particular family of algebras $\widetilde{\Lambda}$ is treated.

5A. ***Statement of the results.*** To state the main result of this section, we recall the Tate–Sen axioms of [Berger and Colmez 2008, 3]. Let G be a profinite group and let $\widetilde{\Lambda}$ be a G-Banach ring endowed with a valuation val for which the G action is continuous and unitary. We suppose there is a character

$\chi : G \to \mathbb{Z}_p^\times$ with open image and let $H = \ker \chi$. Given an open normal subgroup $G_0 \subset G$ we let $H_0 = G_0 \cap H$ and $\Gamma_{H_0} = G/H_0$.

The Tate–Sen axioms are the following.

(TS1) There exists $c_1 > 0$ such that for any open subgroup $H_1 \subset H_2$ of H_0 there exists $\alpha \in \widetilde{\Lambda}^{H_1}$ with $\mathrm{val}(\alpha) > -c_1$ and $\sum_{\tau \in H_2/H_1} \tau(\alpha) = 1$.

(TS2) There exists $c_2 > 0$ and for each H_0 open in H an integer $n(H_0)$ depending on H_0 such that for $n \geq n(H_0)$, we have the extra data of

- closed subalgebras $\Lambda_{H_0,n} \subset \widetilde{\Lambda}^{H_0}$, and
- trace maps $\mathrm{R}_{H_0,n} : \widetilde{\Lambda}^{H_0} \to \Lambda_{H_0,n}$

satisfying:

(1) For $H_1 \subset H_2$ we have $\Lambda_{H_2,n} \subset \Lambda_{H_1,n}$ and $\mathrm{R}_{H_1,n}|_{\Lambda_{H_2,n}} = \mathrm{R}_{H_2,n}$.

(2) $\mathrm{R}_{H_0,n}$ is $\Lambda_{H_0,n}$-linear and $\mathrm{R}_{H_0,n}(x) = x$ for $x \in \Lambda_{H_0,n}$.

(3) $g(\Lambda_{H_0,n}) = \Lambda_{gH_0g^{-1},n}$ and $g(\mathrm{R}_{H_0,n}(x)) = \mathrm{R}_{gH_0g^{-1},n}(gx)$ if $g \in G$.

(4) $\lim_{n\to\infty} \mathrm{R}_{H_0,n}(x) = x$ for $x \in \widetilde{\Lambda}^{H_0}$.

(5) If $n \geq n(H_0)$ and $x \in \widetilde{\Lambda}^{H_0}$ then $\mathrm{val}(R_{H_0,n}(x)) \geq \mathrm{val}(x) - c_2$.

(TS3) There exists $c_3 > 0$ and for each open normal subgroup G_0 of G an integer $n(G_0) \geq n(H_0)$ such that if $n \geq n(G_0)$ and $\gamma \in \Gamma_{H_0}$ has $n(\gamma) = \mathrm{val}_p(\chi(\gamma) - 1) \leq n$, then $\gamma - 1$ acts invertibly on $\mathrm{X}_{H_0,n} = (1 - \mathrm{R}_{H_0,n})(\widetilde{\Lambda}^{H_0})$ and $\mathrm{val}((\gamma - 1)^{-1}(x)) \geq \mathrm{val}(x) - c_3$.

We introduce an additional possible axiom which does not appear in [Berger and Colmez 2008].

(TS4) For any sufficiently small open normal $G_0 \subset G$ with $H_0 = G_0 \cap H$ and for any $n \geq n(G_0)$, there exists a positive real number $t = t(H_0, n) > 0$ such that if $\gamma \in G_0/H_0$ and $x \in \Lambda_{H_0,n}$ then

$$\mathrm{val}((\gamma - 1)(x)) \geq \mathrm{val}(x) + t.$$

We then have the following result.

Theorem 5.1. *Let M be a finite free $\widetilde{\Lambda}$-semilinear representation of G. Suppose there exists an open subgroup $G_0 \subset G$, a G-stable $\widetilde{\Lambda}^+$-lattice $M^+ \subset M$ and an integer $k > c_1 + 2c_2 + 2c_3$ such that in some basis of M^+, we have $\mathrm{Mat}(g) \in 1 + p^k \mathrm{Mat}_d(\widetilde{\Lambda}^+)$ for every $g \in G_0$. Then*:

(i) *If (TS1)–(TS3) are satisfied then for $i \geq 2$*

$$\mathrm{R}^i_{G\text{-la}}(M) = 0.$$

In fact, $\mathrm{R}^i_{G_0\text{-an}}(M) = 0$ for any sufficiently small open subgroup $G_0 \subset G$.

(ii) *If in addition (TS4) is satisfied then*

$$\mathrm{R}^1_{G\text{-la}}(M) = 0.$$

In fact, for every sufficiently small open subgroup G_0 *there is an open subgroup* $G_1 \subset G_0$ *such that the map* $\mathrm{R}^1_{G_0\text{-an}}(M) \to \mathrm{R}^1_{G_1\text{-an}}(M)$ *is* 0.

(iii) *In particular, if* (TS1)–(TS4) *are satisfied then* M *has no higher locally analytic vectors.*

Remark 5.2. The following was pointed out by the anonymous referee: if the action of G_0 on $\widetilde{\Lambda}$ was locally analytic, then the hypothesis of the existence of M^+ such that G_0 acts trivially mod p^k on it would imply that the action of G_0 on M is locally analytic as well, as it can be deduced from Proposition 2.1 and Lemma 2.2. So the nonlocally analyticity comes only from the coefficients $\widetilde{\Lambda}$.

The following special case is often useful in applications.

Proposition 5.3. *If* G *and* $\widetilde{\Lambda}$ *satisfy* (TS1)–(TS4) *and if in addition the topology on* $\widetilde{\Lambda}$ *is* p*-adic, and if* M *is a finite free* $\widetilde{\Lambda}$*-semilinear representation of* G*, then the higher locally analytic vectors* $\mathrm{R}^i_{\mathrm{la}}(M)$ *vanish for* $i \geq 1$.

Proof. We shall explain how this follows from Theorem 5.1. Indeed, we claim that any finite free $\widetilde{\Lambda}$-semilinear representation of G satisfies the assumptions of the Theorem 5.1 after possibly replacing G by a smaller open subgroup G'. This suffices because, by Lemma 2.4, higher locally analytic vectors do not change when we replace G by G'.

To see why such a G' exists, suppose M is a finite free $\widetilde{\Lambda}$-semilinear representation of G and choose any $\widetilde{\Lambda}$-basis $e_1, \dots, e_d$ of M. If we take $M^+ = \bigoplus_{i=1}^d \widetilde{\Lambda}^+ e_i$ then M^+ is a lattice of M, and by continuity we may find an open subgroup $G' \subset G$ so that $\mathrm{Mat}(g) \in \mathrm{GL}_d(\widetilde{\Lambda}^+)$ for $g \in G'$. This implies that M^+ is G'-stable. Since the topology on $\widetilde{\Lambda}$ is p-adic, we can find an open subgroup $G'_0 \subset G'$ such that $\mathrm{Mat}(g) \in 1 + p^k \mathrm{Mat}_d(\widetilde{\Lambda}^+)$ for every $g \in G'_0$. Thus, the assumptions of Theorem 5.1 hold for this M^+, G' and G'_0. □

Before giving the proof of Theorem 5.1, we record a few applications.

Corollary 5.4. *Suppose* G *and* $\widetilde{\Lambda}$ *satisfy* (TS1)–(TS4) *and let* M *be as in the statement of the theorem. Then for all* $i \geq 0$,

$$\mathrm{H}^i(G, M) \cong \mathrm{H}^i(G, M^{\mathrm{la}}) \cong \mathrm{H}^i(\mathrm{Lie}\, G, M^{\mathrm{la}})^G.$$

Proof. Apply [Rodrigues Jacinto and Rodríguez Camargo 2022, Corollary 1.6 and Theorem 1.7]. □

Two main cases of interest are the following. To state them, we set up some notation first. Let F be an infinitely ramified algebraic extension of K which contains an unramified twist of the cyclotomic extension, i.e., the field extension of K cut out by $\eta\chi_{\mathrm{cyc}}$ for η an unramified character. Suppose also that $\mathrm{Gal}(F/K)$ is a p-adic Lie group. For why we allow an unramified twist of the cyclotomic extension on what follows, see Section 8 of [Berger 2016].

Example 5.5. (1) Take $G = \mathrm{Gal}(F/K)$ and $\widetilde{\Lambda} = \widehat{F}$. Then G and $\widetilde{\Lambda}$ satisfy the axioms (TS1)–(TS3) for arbitrary $c_1 > 0$, $c_2 > 0$ and $c_3 > 1/(p-1)$. See [Berger and Colmez 2008, Proposition 4.1.1] for the case $F = \overline{K}$, which goes back to Tate. For general F the same proof works.

In addition, we claim that G and $\widetilde{\Lambda}$ satisfy the axiom (TS4). Indeed, if G_0 is an open subgroup of G corresponding to a finite extension L of K, then $\Lambda_{H_0,n} = L(\zeta_{p^n})$ and $G_0/H_0 = \mathrm{Gal}(L_{\mathrm{cyc}}/L)$. We take G_0 sufficiently small so that L contains ζ_p. Let $\pi = \zeta_{p^n} - 1$ be the uniformizer of L. For $\gamma \in \mathrm{Gal}(L_{\mathrm{cyc}}/L)$, we have

$$\mathrm{val}((\gamma - 1)(\pi)) = \mathrm{val}(\zeta_{p^n}^{\gamma-1} - 1) = \frac{1}{(p-1)p^{n-2}}.$$

Using the identity $(\gamma-1)(ab) = (\gamma-1)(a)b + \gamma(a)(\gamma-1)(b)$, one then shows by induction that

$$\mathrm{val}((\gamma-1)(\pi^m)) \geq \mathrm{val}(\pi^m) + \frac{1}{p^{n-2}}.$$

If x is any element of $\Lambda_{H_0,n} = L(\zeta_{p^n})$, we may write $x = p^k \pi^m y$ with $k \in \mathbb{Z}$, $m \geq 1$ and $0 \leq \mathrm{val}(y) < \mathrm{val}(\pi)$. Since $\mathcal{O}_L[\zeta_{p^n}] = \mathcal{O}_L[\pi]$, we see by writing y as a polynomial in π that

$$\mathrm{val}(\gamma-1)(y) \geq \mathrm{val}(\pi) + \frac{1}{p^{n-2}}.$$

Using the identity for $\gamma - 1$, we have

$$\begin{aligned}
\mathrm{val}(\gamma-1)(x) &\geq k + \min\bigl(\mathrm{val}((\gamma-1)(\pi^m)y), \mathrm{val}(\pi^m(\gamma-1)(y))\bigr) \\
&\geq k + \min\Bigl(\mathrm{val}(\pi^m) + \mathrm{val}(y) + \frac{1}{p^{n-2}}, \mathrm{val}(\pi^m) + \mathrm{val}(\pi) + \frac{1}{p^{n-2}}\Bigr) \\
&\geq \mathrm{val}(x) + \frac{1}{p^{n-2}},
\end{aligned}$$

so (TS4) holds with $t = 1/p^{n-2}$.

(2) Take $G = \mathrm{Gal}(F/K)$ and for a closed interval $I \subset (p/p-1, \infty)$ let $\widetilde{\Lambda} = \widetilde{\boldsymbol{B}}_I(\widehat{F})$. Then again G and $\widetilde{\Lambda}$ satisfy the axioms (TS1)–(TS4) for arbitrary $c_1 > 0$, $c_2 > 0$ and $c_3 > 1/(p-1)$. Here if $G_0 \subset G$ is an open subgroup corresponding a finite extension L of K then one takes $\Lambda_{H_0,n} = \varphi^{-n}(\boldsymbol{B}_{p^n I,\mathrm{cyc},L})$ with notation as in Section 4B. For (TS1)–(TS3), see [Berger 2008a, Proposition 1.1.12]. Axiom (TS4) follows from [Colmez 2008, Corollary 9.5].

Corollary 5.6. (i) *If M is a finite free $\widehat{F}$-semilinear representation of* $\mathrm{Gal}(F/K)$ *then* $\mathrm{R}^i_{\mathrm{la}}(M) = 0$ *for* $i \geq 1$.

(ii) *If $I \subset (p/p-1, \infty)$ is a closed interval and M is a finite free $\widetilde{\boldsymbol{B}}_I(\widehat{F})$-semilinear representation of* $\mathrm{Gal}(F/K)$ *then* $\mathrm{R}^i_{\mathrm{la}}(M) = 0$ *for $i \geq 1$.*

Proof. In both of these cases the topology on $\widetilde{\Lambda}$ is p-adic, so the theorem applies by Proposition 5.3. □

Remark 5.7. Suppose F/K is any infinitely ramified p-adic Lie extension of K (not necessarily containing an unramified twist of the cyclotomic extension), and let M be a finite free $\widehat{F}$-semilinear representation of $\mathrm{Gal}(F/K)$. Then $\mathrm{R}^i_{\mathrm{la}}(M) = 0$ for $i \geq 1$. To prove this, one is always allowed to replace K by a finite extension. Then the extension FK_{cyc}/F can be assumed to be either trivial or infinite. In the first case, the group $\mathrm{R}^i_{\mathrm{la}}(M)$ vanishes by the corollary. In the second case, one can argue as in the proof of [Pan 2022a, Theorem 3.6.1]. We omit the details since this result will not be used in the article.

The rest of the chapter is devoted to the proof of Theorem 5.1. The proof is inspired by that of [Pan 2022a, Theorem 3.6.1]. The strategy is the following:

(1) In Sections 5B and 5C, we establish some results using (TS1), (TS2) and (TS3) that allow us to descend certain infinite rank $\widetilde{\Lambda}$-semilinear representations of G to $\Lambda^+_{H_k,n}$-semilinear representations of G_0, which are fixed by H_k.

(2) In Section 5D, we apply these results to $\mathcal{C}^{\mathrm{an}}(G, M)$.

(3) Using this and the Hochshild–Serre theorem, we show in Section 5E that $\mathrm{R}^i_{G\text{-la}}(M)$ vanishes when $i \geq 2$, and we give an explicit description for $\mathrm{R}^1_{G\text{-la}}(M)$. It remains to show this latter cohomology group vanishes.

(4) To do this, we decompose $\mathrm{R}^1_{G\text{-la}}(M)$ as a sum of two groups. For the first one, we use an explicit calculation in Section 5F and (TS4) to show its vanishing. For the second one, we show it is zero in Section 5G by using again (TS4) and a computation inspired by Berger and Colmez [2016]. Both of these computations are of a p-adic functional analysis flavor.

5B. *Vanishing of H-cohomology.* If $t \in \mathbb{R}$ we write

$$p^{-t}\widetilde{\Lambda}^+ := \text{elements in } \widetilde{\Lambda} \text{ with val} \geq -t.$$

The first result we shall need for the proof of Theorem 5.1 is the following.

Proposition 5.8. *Suppose that $(G, H, \widetilde{\Lambda})$ satisfies (TS1) for some $c_1 > 0$. If $H_0 \subset H$ is an open subgroup, and $r \geq 1$, we have*

(i) *The natural map $\mathrm{H}^r(H_0, \widetilde{\Lambda}^+) \to \mathrm{H}^r(H_0, p^{-2c_1}\widetilde{\Lambda}^+)$ is 0.*

(ii) *Let M^+ be a finite free $\widetilde{\Lambda}^+$-semilinear representation of H_0 which has an H_0-fixed basis. Then the map $\mathrm{H}^r(H_0, M^+) \to \mathrm{H}^r(H_0, p^{-2c_1}M^+)$ is 0.*

(iii) *Let $M^+ = \widehat{\bigcup_{k\in\mathbb{N}} M_k^+}$ be the completion of an increasing union of finite free $\widetilde{\Lambda}^+$-semilinear representation of H_0, each having an H_0-fixed basis. Then the map $\mathrm{H}^r(H_0, M^+) \to \mathrm{H}^r(H_0, p^{-2c_1}M^+)$ is 0.*

In particular, in each of the cases (i)–(iii) *the rational cohomology $\mathrm{H}^r(H_0, M)$ is equal to zero.*

Proof. We have (i) $\Rightarrow$ (ii), since continuous cohomology commutes with direct sums.

Next, we prove (ii) $\Rightarrow$ (iii). To do this, observe that if $t \in \mathbb{Z}_{\geq 1}$ then $p^t M_k^+$ also a finite free $\widetilde{\Lambda}^+$-semilinear representation of H_0 which has an H_0-fixed basis. Taking long exact cohomologies of the sequences

$$0 \to p^t\Bigl(\bigcup_{k\in\mathbb{N}} M_k^+\Bigr) \to \Bigl(\bigcup_{k\in\mathbb{N}} M_k^+\Bigr) \to M^+/p^tM^+ \to 0$$

and

$$0 \to p^{t-2c_1}\Bigl(\bigcup_{k\in\mathbb{N}} M_k^+\Bigr) \to p^{-2c_1}\Bigl(\bigcup_{k\in\mathbb{N}} M_k^+\Bigr) \to p^{-2c_1}M^+/p^{t-2c_1}M^+ \to 0,$$

we get from (ii) that the natural map

$$\mathrm{H}^r(H_0, M^+/p^t M^+) \to \mathrm{H}^r(H_0, p^{-2c_1}M^+/p^{t-2c_1}M^+)$$

is 0. Now given a cocycle $\xi \in Z^r(H_0, M^+)$, write ξ_0 for its image in $Z^r(H_0, p^{-2c_1}M^+)$. We wish to show that ξ_0 is a coboundary. Choose some fixed $t_0 \geq 3c_1$. Then by virtue of the observation above, the right vertical map of the commutative diagram

$$\begin{tikzcd}
\mathrm{H}^r(H_0, p^{t_0}M^+) \arrow[r] \arrow[d] & \mathrm{H}^r(H_0, M^+) \arrow[r] \arrow[d] & \mathrm{H}^r(H_0, M^+/p^{t_0}M^+) \arrow[d] \\
\mathrm{H}^r(H_0, p^{t_0-2c_1}M^+) \arrow[r] & \mathrm{H}^r(H_0, p^{-2c_1}M^+) \arrow[r] & \mathrm{H}^r(H_0, p^{-2c_1}M^+/p^{t_0-2c_1}M^+)
\end{tikzcd}$$

is 0, which implies that $\xi_0 = \xi_1 + \delta(m_1)$, where m_1 is an $r-1$ cocycle valued in $p^{-2c_1}M^+$ and ξ_1 is an r-cocycle valued in $p^{t_0-2c_1}M^+ \subset p^{c_1}M^+$. Repeating this argument by induction with M^+ replaced with $p^{ic_1}M^+$, we get that we can write $\xi_i = \xi_{i+1} + \delta(m_{i+1})$, where ξ_i is valued in $p^{ic_1}M^+$ and m_{i+1} is valued in $p^{(i-3)c_1}M^+$. Hence the series $\sum_{i=1}^{\infty} m_i$ converges to an $r-1$ cocycle m valued in $p^{-2c_1}M^+$, and we get $\xi_0 = \delta(m)$, as required.

Finally, we prove (i). This statement is probably well known, but for lack of a suitable reference, we provide a proof here. It is essentially a fiber product of the arguments appearing in [Tate 1967, 3.2, Corollary 1; Colmez 2008, Proposition 10.2].

Let $\xi \in Z^r(H_0, \widetilde{\Lambda}^+)$ be an r-cocycle of H_0 valued in $\widetilde{\Lambda}^+$. By a valuation of a cochain we shall mean the infimum of its valuation on elements. Writing δ for the differential, we shall construct a sequence of $r-1$ cochains $x_n \in C^{r-1}(H_0, p^{-2c_1}\widetilde{\Lambda}^+)$ for $n \geq -1$ such that

(1) $\mathrm{val}(\xi - \delta x_n) \geq nc_1$ for $\sigma \in H_0$, and

(2) $\mathrm{val}(x_n - x_{n-1}) \geq (n-2)c_1$ for $n \geq 0$.

This will suffice, since $x_n \to x$ for some $x \in C^{r-1}(H_0, p^{-2c_1}\widetilde{\Lambda}^+)$ which shows that $\xi = \delta x$ is 0 in $\mathrm{H}^r(H_0, p^{-2c_1}\widetilde{\Lambda}^+)$.

To do this, choose $x_{-1} = 0$, which clearly satisfies the first condition. Suppose x_n has been constructed; we construct x_{n+1}. Let ξ_n be the r-cocycle

$$\xi_n := \xi - \delta x_n,$$

which is valued in $p^{nc_1}\widetilde{\Lambda}^+$. Choose $H_1 \subset H_0$ an open subgroup such that for every $\sigma_1, \dots, \sigma_r \in H_0$ and $\sigma \in H_1$ we have

$$\mathrm{val}(\xi_n(\sigma_1, \dots, \sigma_r) - \xi_n(\sigma_1, \dots, \sigma_r\sigma)) \geq (n+2)c_1.$$

Such a choice is possible by the continuity of ξ_n as well as the compactness of H_0.

Now by the axiom (TS1) there is an element $\alpha \in \widetilde{\Lambda}^{H_1}$ such that $\mathrm{val}(\alpha) > -c_1$ and $\sum_{\tau \in H_0/H_1} \tau(\alpha) = 1$. Let S be a system of representatives for H_0/H_1, and define an $r-1$ cochain

$$x_S(\sigma_1, \dots, \sigma_{r-1}) = (-1)^r \sum_{\tau \in S} (\sigma_1 \sigma_2 \cdot \ldots \cdot \sigma_{r-1} \tau)(\alpha) \xi_n(\sigma_1, \dots, \sigma_{r-1}, \tau).$$

Each term in the sum has val $\geq (n-1)c_1$, so $\mathrm{val}(x_S) \geq (n-1)c_1$. In particular, $x_S \in C^{r-1}(H_0, p^{-2c_1}\widetilde{\Lambda}^+)$.

We now compute $(\xi_n - \delta x_S)(\sigma_1, \dots, \sigma_r)$. We have by definition of δ an equation

$$\begin{aligned} \delta x_S(\sigma_1, \dots, \sigma_r) = (-1)^r \sum_{\tau \in S} &(\sigma_1 \cdot \ldots \cdot \sigma_r \tau)(\alpha) \sigma_1(\xi_n(\sigma_2, \dots, \sigma_r, \tau)) \\ &+ \sum_{j=1}^{r-1} (-1)^{j+r} \sum_{\tau \in S} (\sigma_1 \cdot \ldots \cdot \sigma_r \tau)(\alpha) \xi_n(\sigma_1, \dots, \sigma_j \sigma_{j+1}, \dots, \sigma_r, \tau) \\ &+ \sum_{\tau \in S} (\sigma_1 \cdot \ldots \cdot \sigma_{r-1} \tau)(\alpha) \xi_n(\sigma_1, \dots, \sigma_{r-1}, \tau). \end{aligned} \tag{5-1}$$

On the other hand, ξ_n is an r-cocycle, so that $\delta \xi_n(\sigma_1, \dots, \sigma_r, \tau) = 0$ for every $\sigma_1, \dots, \sigma_r$ and τ. Multiplying by $(-1)^r (\sigma_1 \cdot \ldots \cdot \sigma_r \tau)(\alpha)$ and summing over $\tau \in S$, we get the equation

$$\begin{aligned} 0 = (-1)^r \sum_{\tau \in S} &(\sigma_1 \cdot \ldots \cdot \sigma_r \tau)(\alpha) \sigma_1(\xi_n(\sigma_2, \dots, \sigma_r, \tau)) \\ &+ \sum_{j=1}^{r-1} (-1)^{j+r} \sum_{\tau \in S} (\sigma_1 \cdot \ldots \cdot \sigma_r \tau)(\alpha) \xi_n(\sigma_1, \dots, \sigma_j \sigma_{j+1}, \dots, \sigma_r, \tau) \\ &+ \sum_{\tau \in S} (\sigma_1 \cdot \ldots \cdot \sigma_r \tau)(\alpha) \xi_n(\sigma_1, \dots, \sigma_{r-1}, \sigma_r \tau) - \sum_{\tau \in S} (\sigma_1 \cdot \ldots \cdot \sigma_r \tau)(\alpha) \xi_n(\sigma_1, \dots, \sigma_r). \end{aligned} \tag{5-2}$$

Subtracting (5-2) from (5-1), we get

$$\begin{aligned} \delta x_S(\sigma_1, \dots, \sigma_r) = \sum_{\tau \in S} (\sigma_1 \cdot \ldots \cdot \sigma_{r-1} \tau)(\alpha) \xi_n(\sigma_1, \dots, \sigma_{r-1}, \tau) - \sum_{\tau \in S} (\sigma_1 \cdot \ldots \cdot \sigma_r \tau)(\alpha) \xi_n(\sigma_1, \dots, \sigma_{r-1}, \sigma_r \tau) \\ + \sum_{\tau \in S} (\sigma_1 \cdot \ldots \cdot \sigma_r \tau)(\alpha) \xi_n(\sigma_1, \dots, \sigma_r). \end{aligned}$$

Now by choice of α, the last term is simply $\xi_n(\sigma_1, \dots, \sigma_r)$. Thus after rearranging, we have for every $\sigma_1, \dots, \sigma_r \in H_0$ the equation

$$(\xi_n - \delta x_S)(\sigma_1, \dots, \sigma_r) = \sum_{\tau \in S} (\sigma_1 \cdot \ldots \cdot \sigma_{r-1} \tau)(\alpha) \xi_n(\sigma_1, \dots, \sigma_{r-1}, \tau) - \sum_{\tau \in S} (\sigma_1 \cdot \ldots \cdot \sigma_r \tau)(\alpha) \xi_n(\sigma_1, \dots, \sigma_r \tau).$$

For each τ in S, let $\sigma_{r,\tau} \in H_1$ be such that $\tau \sigma_{r,\tau} \in \sigma_r S$. Then the term on the right hand side of the previous equation becomes

$$\sum_{\tau \in S} (\sigma_1 \cdot \ldots \cdot \sigma_{r-1} \tau)(\alpha) \big[\xi_n(\sigma_1, \dots, \sigma_{r-1}, \tau) - \xi_n(\sigma_1, \dots, \tau \sigma_{r,\tau}) \big],$$

so by the choice of H_1 we have

$$\mathrm{val}(\xi - \delta(x_n + x_S)) = \mathrm{val}(\xi_n - \delta x_S) \geq (n+1)c_1.$$

Finally, set $x_{n+1} := x_n + x_S$ where S is arbitrary. The calculations we have done show that $\mathrm{val}(x_{n+1} - x_n) \geq (n-1)c_1$ and $\mathrm{val}(\xi - \delta x_{n+1}) \geq (n+1)c_1$, as required. This concludes the induction and with it the proof. □

5C. *Descent of semilinear representations.* In this subsection we suppose that G and $\widetilde{\Lambda}$ satisfy the axioms (TS1), (TS2) and (TS3).

Given an integer $k > c_1 + 2c_2 + 2c_3$ and an open subgroup $G_0 \subset G$ we write $\mathrm{Mod}^k_{\widetilde{\Lambda}^+}(G, G_0)$ for the category of finite free $\widetilde{\Lambda}^+$-semilinear representations M^+ of G such that in some basis of M^+, we have $\mathrm{Mat}(g) \in 1 + p^k \mathrm{Mat}_d(\widetilde{\Lambda}^+)$ for every $g \in G_0$.

The following will allow us to descend coefficients from $\widetilde{\Lambda}^+$ to the much smaller ring $\Lambda^+_{H_0,n} = \widetilde{\Lambda}^+ \cap \Lambda_{H_0,n}$. It is a simple modification of [Berger and Colmez 2008, Proposition 3.3.1] and is proved in exactly the same way.

Proposition 5.9. *Let $M^+ \in \mathrm{Mod}^k_{\widetilde{\Lambda}^+}(G, G_0)$. Then for $n \geq n(G_0)$ and $H_0 = H \cap G_0$ there exists a unique finite free $\Lambda^+_{H_0,n}$-submodule $\boldsymbol{D}^+_{H_0,n}(M^+)$ of M^+ such that*:

(1) *$\boldsymbol{D}^+_{H_0,n}(M^+)$ is fixed by H_0 and stable by G.*

(2) *The natural map $\widetilde{\Lambda}^+ \otimes_{\Lambda^+_{H_0,n}} \boldsymbol{D}^+_{H_0,n}(M^+) \to M^+$ is an isomorphism. In particular, $\boldsymbol{D}^+_{H_0,n}(M^+)$ is free of rank* $=$ rank M^+.

(3) *$\boldsymbol{D}^+_{H_0,n}(M^+)$ has a basis which is c_3-fixed by G_0/H_0, meaning that for $\gamma \in G_0/H_0$ we have*

$$\mathrm{val}(\mathrm{Mat}(\gamma) - 1) > c_3.$$

Corollary 5.10. *Let $M^+ \in \mathrm{Mod}^k_{\widetilde{\Lambda}^+}(G, G_0)$, $M = M^+ \otimes_{\widetilde{\Lambda}^+} \widetilde{\Lambda}$ and $r \geq 1$. The map*

$$\mathrm{H}^r(H_0, M^+) \to \mathrm{H}^r(H_0, p^{-2c_1} M^+)$$

is 0 *and $H^r(H_0, M) = 0$.*

Proof. This follows from Proposition 5.8 since M^+ has a basis fixed by H_0. □

Lemma 5.11. *Let H_0 be an open subgroup of H, $n \geq n(H_0)$ an integer, $\gamma \in \Gamma_H$ an element such that $n(\gamma) \leq n$ and $B \in \mathrm{M}_{l \times d}(\widetilde{\Lambda}^{H_0})$ a matrix. Let $d \in \mathbb{N} \cup \{\infty\}$. Suppose there are $V_1 \in \mathrm{GL}_l(\Lambda_{H_0,n})$ and $V_2 \in \mathrm{GL}_d(\Lambda_{H_0,n})$ such that* $\mathrm{val}(V_1 - 1), \mathrm{val}(V_2 - 1) > c_3$ *and $\gamma(B) = V_1 B V_2$. Then $B \in \mathrm{M}_{l \times d}(\Lambda_{H_0,n})$.*

Proof. The proof is exactly the same as that of [Berger and Colmez 2008, Lemma 3.2.5]. The only difference between that lemma and the statement appearing here is that there one further assumes $l = d$ and $B \in \mathrm{GL}_d(\widetilde{\Lambda}^{H_0})$, but these assumptions are not used in the proof. In fact, the very same argument shows the result holds for matrices with $d = \infty$, as long as we understand that an infinite matrix has coefficients which tend to zero as the indexes tend to ∞. Namely, if R is a ring with valuation and $l, d \in \mathbb{N} \cup \{\infty\}$, let $\mathrm{M}_{l \times d}(R)$ be the set of matrices $A = (a_{ij})$ of size $l \times d$ and $a_{ij} \in R$ such that $\mathrm{val}(a_{ij}) \to \infty$ as $i + j \to \infty$. The argument then works in the same way. □

Using Lemma 5.11, we have the following description of $\boldsymbol{D}^+_{H_0,n}(M^+)$. It explains why $\boldsymbol{D}^+_{H_0,n}(M^+)$ is functorial in M^+.

Proposition 5.12. *Given $M^+ \in \mathrm{Mod}^k_{\widetilde{\Lambda}^+}(G, G_0)$, the module $\boldsymbol{D}^+_{H_0,n}(M^+)$ is the union of all finitely generated $\Lambda^+_{H_0,n}$-submodules of M^+ which are G-stable, H_0-fixed and admit a c_3-fixed set of generators.*

Proof. Indeed, if we have a submodule generated by c_3-fixed elements $f_1, \dots, f_l$ and if $e_1, \dots, e_d$ is a c_3-fixed basis, write

$$f_i = Be_i$$

for some matrix $B \in \mathrm{M}_{l\times d}(\widetilde{\Lambda}^{H_0,+})$. Then we have

$$\mathrm{Mat}_{f_i}(\gamma)B = \gamma(B)\mathrm{Mat}_{e_i}(\gamma).$$

Here by $\mathrm{Mat}_{f_i}(\gamma)$ we mean any matrix which represents the action in terms of the f_i. It is not a priori unique as the submodule may not be free. Nevertheless, we have $\mathrm{val}(\mathrm{Mat}_{f_i}(\gamma) - 1) > c_3$ by the assumption, and this implies that $\mathrm{Mat}_{f_i}(\gamma)$ is invertible by [Berger and Colmez 2008, Lemma 3.1.2]. So by Lemma 5.11

$$B \in \mathrm{M}_{l\times d}(\Lambda_{H_0,n}) \cap \mathrm{M}_{l\times d}(\widetilde{\Lambda}^{H_0,+}) = \mathrm{M}_{l\times d}(\Lambda^+_{H_0,n}),$$

hence the submodule generated by the f_i is contained in $\boldsymbol{D}^+_{H_0,n}(M^+)$. $\square$

Corollary 5.13. *Let $M^+, N^+ \in \mathrm{Mod}^k_{\widetilde{\Lambda}^+}(G, G_0)$. Then for $n \geq n(G_0)$,*

(i) *There are natural isomorphisms*

$$\boldsymbol{D}^+_{H_0,n}(M^+) \otimes_{\Lambda^+_{H_0,n}} \boldsymbol{D}^+_{H_0,n}(N^+) \xrightarrow{\sim} \boldsymbol{D}^+_{H_0,n}(M^+ \otimes_{\widetilde{\Lambda}^+} N^+),$$
$$\boldsymbol{D}^+_{H_0,n}(M^+) \oplus \boldsymbol{D}^+_{H_0,n}(N^+) \xrightarrow{\sim} \boldsymbol{D}^+_{H_0,n}(M^+ \oplus N^+).$$

(ii) *If $M^+ \subset N^+$ then $\boldsymbol{D}^+_{H_0,n}(M^+) = \boldsymbol{D}^+_{H_0,n}(N^+) \cap M^+$.*

5D. *Descent of $\mathcal{C}^{\mathrm{an}}(G_0, M)$.* From here on G is a compact p-adic Lie group and $G_0 \subset G$ is a small subgroup, as in Section 2. We continue to assume G and $\widetilde{\Lambda}$ satisfy the axioms (TS1), (TS2) and (TS3). The reader may also want to recall our notation and conventions of Section 1B regarding Banach spaces, completions and tensor products.

By Proposition 2.3, we have for $V_l^+ = V_l(G_0) \cap \mathcal{C}^{\mathrm{an}}(G_0, \mathbb{Q}_p)^+$ an equality

$$\widehat{\varinjlim_{l\in\mathbb{N}} V_l^+} = \mathcal{C}^{\mathrm{an}}(G_0, \mathbb{Q}_p)^+.$$

For $M \in \mathrm{Mod}^k_{\widetilde{\Lambda}^+}(G, G_0)$ we have

$$\Bigl(\varinjlim_{l\in\mathbb{N}} M^+ \otimes_{\mathbb{Z}_p} V_l^+\Bigr)^{\wedge} \cong M^+ \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{C}^{\mathrm{an}}(G_0, \mathbb{Q}_p)^+.$$

Each $M^+ \otimes_{\mathbb{Z}_p} V_l^+$ is a finite free $\widetilde{\Lambda}^+$-semilinear representation of G_0. The action of G_k on each of the V_l^+ is trivial mod p^k by Lemma 2.2, and hence its action on $M^+ \otimes V_l^+$ is trivial mod p^k. So if $n \geq n(G_k)$, we may define using Proposition 5.8 a $\Lambda^+_{H_k,n}$-submodule of $M^+ \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{C}^{\mathrm{an}}(G_0, \mathbb{Q}_p)^+$ given by

$$\boldsymbol{D}^+_{H_k,n,\infty}(M^+) := \Big(\varinjlim_{l \in \mathbb{N}} \boldsymbol{D}^+_{H_k,n}(M^+ \otimes V_l^+)\Big)^{\wedge}.$$

The module $\boldsymbol{D}^+_{H_k,n,\infty}(M^+)$ is then G_0-stable and fixed by H_k. By Proposition 5.8 we have natural isomorphisms

$$\widetilde{\Lambda}^+ \otimes_{\Lambda^+_{H_k,n}} \boldsymbol{D}^+_{H_k,n}(M^+ \otimes V_l^+) \xrightarrow{\sim} M^+ \otimes V_l^+.$$

This shows that $\boldsymbol{D}^+_{H_k,n,\infty}(M^+)$ is generated by c_3-fixed elements which give it the sup norm, and there is an isometry

$$\widetilde{\Lambda}^+ \widehat{\otimes}_{\Lambda^+_{H_k,n}} \boldsymbol{D}^+_{H_k,n,\infty}(M^+) \xrightarrow{\sim} M^+ \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{C}^{\mathrm{an}}(G_0, \mathbb{Q}_p)^+.$$

The next proposition follows from Proposition 5.12

Proposition 5.14. *A finitely generated $\Lambda^+_{H_k,n}$-submodule of $M^+ \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{C}^{\mathrm{an}}(G_0, \mathbb{Q}_p)^+$ which is stable by G_0, fixed by H_k and is generated by a c_3-fixed set of elements is contained in $\boldsymbol{D}^+_{H_k,n,\infty}(M^+)$.*

In particular, we have the function log defined, by abuse of notation as the composition of

$$\chi : G_0 \twoheadrightarrow G_0/H_0 \hookrightarrow \mathbb{Z}_p^\times \quad \text{and} \quad \log : \mathbb{Z}_p^\times \to \mathbb{Q}_p.$$

It lies in $\mathcal{C}^{\mathrm{an}}(G_0, \mathbb{Q}_p)^+$. Note that for $g \in G_0$, we have

$$g(\log) = \log + \log(g^{-1}) = \log - \log(g).$$

Lemma 5.15. *The elements* 1 *and* log *of $\widetilde{\Lambda}^+ \widehat{\otimes}\, \mathcal{C}^{\mathrm{an}}(G_0, \mathbb{Q}_p)^+$ lie in $\boldsymbol{D}^+_{H_k,n,\infty}(\widetilde{\Lambda}^+)$.*

Proof. The $\Lambda^+_{H_k,n}$-submodule generated by 1 and log in $\widetilde{\Lambda}^+ \widehat{\otimes}\, \mathcal{C}^{\mathrm{an}}(G_0, \mathbb{Q}_p)^+$ is stable under the G_0 action and fixed by H_k. Furthermore, we claim the elements 1 and log are c_3-fixed by the action of G_k/H_k. This is clear for 1. To show this for log, notice that if $g^{p^k} \in G_k/H_k$ (recalling that $G_k = G_0^{p^k}$) then

$$\mathrm{val}(g^{p^k} - 1)(\log) \geq k > c_1 + 2c_2 + 2c_3 > c_3.$$

We conclude by Proposition 5.14. ◻

Proposition 5.16. (i) *$\boldsymbol{D}^+_{H_k,n,\infty}(\widetilde{\Lambda}^+)$ is a subring of $\widetilde{\Lambda}^+ \widehat{\otimes}\, \mathcal{C}^{\mathrm{an}}(G_0, \mathbb{Q}_p)^+$.*

(ii) *The module structure of $M^+ \widehat{\otimes} \mathcal{C}^{\mathrm{an}}(G_0, \mathbb{Q}_p)^+$ over $\widetilde{\Lambda}^+ \widehat{\otimes} \mathcal{C}^{\mathrm{an}}(G_0, \mathbb{Q}_p)^+$ restricts to a module structure of $\boldsymbol{D}^+_{H_k,n,\infty}(M^+)$ over $\boldsymbol{D}^+_{H_k,n,\infty}(\widetilde{\Lambda}^+)$.*

Proof. $\boldsymbol{D}^+_{H_k,n,\infty}(\widetilde{\Lambda}^+)$ contains 1 by Proposition 5.14. Next, one has the ring and module structure maps

$$\widetilde{\Lambda}^+ \otimes \widetilde{\Lambda}^+ \to \widetilde{\Lambda}^+, \widetilde{\Lambda}^+ \otimes M^+ \to M^+.$$

Applying Proposition 5.12, taking the inductive limit and then taking completions, we get natural maps

$$\boldsymbol{D}^+_{H_k,n,\infty}(\widetilde{\Lambda}^+) \otimes \boldsymbol{D}^+_{H_k,n,\infty}(\widetilde{\Lambda}^+) \to \boldsymbol{D}^+_{H_k,n,\infty}(\widetilde{\Lambda}^+)$$

and

$$\boldsymbol{D}^+_{H_k,n,\infty}(\widetilde{\Lambda}^+) \otimes \boldsymbol{D}^+_{H_k,n,\infty}(M^+) \to \boldsymbol{D}^+_{H_k,n,\infty}(M^+),$$

giving the desired ring and module structures. ◻

5E. *Computation of higher locally analytic vectors, I.* Let $M^+ \in \mathrm{Mod}^k_{\widetilde{\Lambda}^+}(G, G_0)$ and $M = M^+ \otimes_{\widetilde{\Lambda}^+} \widetilde{\Lambda}$. In this subsection we shall do a first simplification towards the computation of the groups $\mathrm{R}^i_{G\text{-la}}(M)$ for $i \geq 1$.

If G_0 is any open subgroup of G, we have $\mathrm{R}^i_{G\text{-la}}(M) = \mathrm{R}^i_{G_0\text{-la}}(M)$ so that if $G_n = G_0^{p^n}$ we have

$$\mathrm{R}^i_{G\text{-la}}(M) = \varinjlim_n \mathrm{H}^i\big(G_n, M \widehat{\otimes}_{\mathbb{Q}_p} \mathcal{C}^{\mathrm{an}}(G_n, \mathbb{Q}_p)\big).$$

Upon possibly making G_0 smaller, we may assume that G_0 is small and that $\chi : G_0/H_0 \to \mathbb{Z}_p^\times$ has image isomorphic to $\mathbb{Z}_p$. Write $\Gamma_n = G_n/H_n$.

Lemma 5.17. *For $i \geq 1$,*

$$\mathrm{H}^i\big(G_n, M \widehat{\otimes}_{\mathbb{Q}_p} \mathcal{C}^{\mathrm{an}}(G_n, \mathbb{Q}_p)\big) \cong \mathrm{H}^i\big(\Gamma_{n+k}, (M \widehat{\otimes}_{\mathbb{Q}_p} \mathcal{C}^{\mathrm{an}}(G_n, \mathbb{Q}_p))^{H_{n+k}}\big).$$

Proof. By the Hochshild–Serre spectral sequence and the vanishing of H_{n+k} cohomologies in (iii) of Proposition 5.8 (taking the inductive system $M^+_{k+k'} = M^+ \otimes V^+_{k+k'}$ for $k' \geq 0$), we have

$$\mathrm{H}^i\big(G_n, M \widehat{\otimes}_{\mathbb{Q}_p} \mathcal{C}^{\mathrm{an}}(G_n, \mathbb{Q}_p)\big) \cong \mathrm{H}^i\big(G_n/H_{n+k}, (M \widehat{\otimes}_{\mathbb{Q}_p} \mathcal{C}^{\mathrm{an}}(G_n, \mathbb{Q}_p))^{H_{n+k}}\big).$$

Now the inclusion $\Gamma_{n+k} \hookrightarrow G_n/H_{n+k}$ induces an isomorphism

$$\mathrm{H}^i\big(G_n/H_{n+k}, (M \widehat{\otimes}_{\mathbb{Q}_p} \mathcal{C}^{\mathrm{an}}(G_n, \mathbb{Q}_p))^{H_{n+k}}\big) \cong \mathrm{H}^i\big(\Gamma_{n+k}, (M \widehat{\otimes}_{\mathbb{Q}_p} \mathcal{C}^{\mathrm{an}}(G_n, \mathbb{Q}_p))^{H_{n+k}}\big).$$

This again follows from Hochshild–Serre, once we notice all the higher cohomologies of G_n/G_{n+k} appearing vanish. This is because G_n/G_{n+k} is finite and the coefficients are rational. □

Corollary 5.18. $\mathrm{R}^i_{G_n\text{-an}}(M) = 0$ *for $i \geq 2$ and $n \geq 0$.*

Proof. Because $\Gamma_{n+k} \cong \mathbb{Z}_p$. □

This proves the first part of Theorem 5.1. It remains to study the 1st derived group

$$\mathrm{R}^1_{G\text{-la}}(M) = \varinjlim_n \mathrm{H}^1\big(\Gamma_{n+k}, (M \widehat{\otimes}_{\mathbb{Q}_p} \mathcal{C}^{\mathrm{an}}(G_n, \mathbb{Q}_p))^{H_{n+k}}\big).$$

Now for $m \geq n(G_{n+k})$, we have by Proposition 5.9 a natural isomorphism

$$\widetilde{\Lambda}^+ \otimes \varinjlim_{\ell} \boldsymbol{D}^+_{H_k,n}(M^+ \otimes V^+_\ell) \cong M^+ \otimes \varinjlim_{\ell \in \mathbb{N}} V^+_\ell.$$

Taking the p-adic completion, we obtain a natural isomorphism

$$\widetilde{\Lambda}^+ \widehat{\otimes}_{\Lambda^+_{H_{n+k},m}} \boldsymbol{D}^+_{H_{n+k},m,\infty}(M^+) \xrightarrow{\sim} M^+ \widehat{\otimes} \mathcal{C}^{\mathrm{an}}(G_n, \mathbb{Q}_p)^+$$

and thus

$$\widetilde{\Lambda}^{+,H_{n+k}} \widehat{\otimes}_{\Lambda^+_{H_{n+k},m}} \boldsymbol{D}^+_{H_{n+k},m,\infty}(M^+) \xrightarrow{\sim} (M^+ \widehat{\otimes} \mathcal{C}^{\mathrm{an}}(G_n, \mathbb{Q}_p)^+)^{H_{n+k}}.$$

On the other hand, recall we have the trace maps

$$\mathrm{R}_{H_{n+k},m} : \widetilde{\Lambda}^{H_{n+k}} \to \Lambda_{H_{n+k},m}$$

which induce for $\mathrm{X}_{H_{n+k},m} = \ker \mathrm{R}_{H_{n+k},m}$ a decomposition

$$\widetilde{\Lambda}^{H_{n+k}} = \Lambda_{H_{n+k},m} \oplus X_{H_{n+k},m}.$$

Therefore, we can decompose

$$\widetilde{\Lambda}^{H_{n+k}} \widehat{\otimes}_{\Lambda_{H_{n+k},m}} \boldsymbol{D}_{H_{n+k},m,\infty}(M) \cong \boldsymbol{D}_{H_{n+k},m,\infty}(M) \oplus \big(\mathrm{X}_{H_{n+k},m} \widehat{\otimes}_{\Lambda_{H_{n+k},m}} \boldsymbol{D}_{H_{n+k},m,\infty}(M)\big),$$

and so we get the description

$$\mathrm{R}^1_{G\text{-la}}(M) = \varinjlim_n \mathrm{H}^1(\Gamma_{n+k}, \boldsymbol{D}_{H_{n+k},m,\infty}(M)) \oplus \mathrm{H}^1\big(\Gamma_{n+k}, \mathrm{X}_{H_{n+k},m} \widehat{\otimes}_{\Lambda_{H_{n+k},m}} \boldsymbol{D}^+_{H_{n+k},m,\infty}(M)\big),$$

where in each object of the direct limit, we take $m \geq n(G_{n+k})$.

5F. *Computation of higher locally analytic vectors, II.* If $m \geq 0$ is an integer and γ is an element of a group, write γ_m for γ^{p^m}. The following simple lemma will be used to compare the behavior of $(\gamma-1)^m$ and $\gamma_m - 1$.

Lemma 5.19. *Let $\ell \geq 0$. The element $X^{p^\ell} - 1$ of the ring $\mathbb{Z}_p[X]$ is in the ideal generated by the elements $p^i(X-1)^{\ell+1-i}$ for $0 \leq i \leq \ell$.*

Proof. For $\ell \geq 1$ we have

$$X^{p^\ell} - 1 = (X^{p^{\ell-1}} - 1)\left(\sum_{i=1}^{p-1} X^{ip^{\ell-1}}\right) = (X^{p^{\ell-1}} - 1)\left(\sum_{i=1}^{p-1} 1 + (X^{ip^{\ell-1}} - 1)\right)$$
$$= (X^{p^{\ell-1}} - 1)\left(p + \sum_{i=1}^{p-1} (X^{ip^{\ell-1}} - 1)\right),$$

so that $X^{p^\ell} - 1$ lies in the ideal

$$(X^{p^{\ell-1}} - 1)(p, (X^{p^{\ell-1}} - 1)) = (p(X^{p^{\ell-1}} - 1), (X^{p^{\ell-1}} - 1)^2).$$

Let I_ℓ be the ideal generated by the elements $p^i(X-1)^{\ell+1-i}$ for $0 \leq i \leq \ell$. It is easy to check that $(pI_{\ell-1}, I^2_{\ell-1})$ is contained in I_ℓ. Hence, induction on ℓ shows that $X^{p^\ell} - 1$ belong to I_ℓ. $\square$

So far we have only used the axioms (TS1), (TS2) and (TS3). We shall now use the final axiom (TS4), which proves us with a positive number $t > 0$.

Proposition 5.20. *If (TS4) holds, then*

(i) *$\Lambda_{H,n}$ is Γ_t-analytic for an open subgroup of Γ depending on t.*

(ii) *There exists an element $s = s(t, c_3) = s(n, m, G_0, c_3)$ such that for $\gamma \in G_{n+k}/H_{n+k}$ we have*

$$(\gamma - 1)\boldsymbol{D}^+_{H_{n+k},m,\infty}(M^+) \subset p^s \boldsymbol{D}^+_{H_{n+k},m,\infty}(M^+).$$

(iii) *$\boldsymbol{D}_{H_{n+k},m,\infty}(M)$ is Γ-analytic for some open subgroup Γ of Γ_{n+k} which depends on n, m, G and c_3.*

Proof. Once (ii) is established, we claim parts (i) and (iii) follow from [Pan 2022a, Example 2.1.9]. Let us elaborate a little bit. Take ℓ large enough so that

$$(\ell - i) + (i+1)t = \ell + t + (t-1)i \geq 2$$

for each $0 \leq i \leq \ell$. Then for such ℓ (which only depends on t) we have by Lemma 5.19

$$(\gamma_\ell - 1)(\Lambda^+_{H,n}) \subset p^2 \Lambda^+_{H,n},$$

so that if $b \in \Lambda_{H,n}$, the series

$$\gamma_\ell^x(b) = \sum_{n \geq 0} \binom{x}{n} (\gamma_\ell - 1)^n(b)$$

converges. This shows b is analytic for the subgroup generated by γ_ℓ. The argument for (iii) given (ii) is similar.

To show part (ii), recall the identity

$$(\gamma-1)(ab)=(\gamma-1)(a)b+\gamma(a)(\gamma-1)(b).$$

Axiom (TS4) implies that if $a\in\Lambda^+_{H,m}$ and $b\in \boldsymbol{D}^+_{H_{n+k},m,\infty}(M^+)$ is c_3-fixed, then ab is $\min(c_3,t)$-fixed. Since the c_3-fixed elements topologically generate $\boldsymbol{D}^+_{H_{n+k},m,\infty}(M^+)$, it follows that every element of $\boldsymbol{D}^+_{H_{n+k},m,\infty}(M^+)$ is $s=\min(c_3,t)$-fixed. □

Using this we can show

Lemma 5.21. *Given n there is m sufficiently large depending only on n (and not on M) such that*

$$\mathrm{H}^1\big(\Gamma_{n+k},\mathrm{X}_{H_{n+k},m}\widehat{\otimes}_{\Lambda_{H_{n+k},m}}\boldsymbol{D}_{H_{n+k},m,\infty}(M)\big)=0.$$

Proof. (This argument is adapted from [Pan 2022a, Lemma 3.6.6].) Fix $m_0\geq n(G_{n+k})$. From the discussion after Corollary 5.18, for $m\geq m_0$ we have a natural isomorphism

$$\widetilde{\Lambda}^{H_{n+k}}\widehat{\otimes}_{\Lambda_{H_{n+k},m}}\boldsymbol{D}_{H_{n+k},m,\infty}(M)\cong\boldsymbol{D}_{H_{n+k},m,\infty}(M)\oplus\big(\mathrm{X}_{H_{n+k},m}\widehat{\otimes}_{\Lambda_{H_{n+k},m}}\boldsymbol{D}_{H_{n+k},m,\infty}(M)\big).$$

By Proposition 5.12, we have an isomorphism

$$\Lambda_{H_{n+k},m}\widehat{\otimes}\,\boldsymbol{D}_{H_{n+k},m_0,\infty}(M)\cong\boldsymbol{D}_{H_{n+k},m,\infty}(M).$$

Let $\mathrm{X}^+_{H_{n+k},m}=\mathrm{X}_{H_{n+k},m}\cap\widetilde{\Lambda}^+$. We get an induced isomorphism

$$\mathrm{X}^+_{H_{n+k},m}\widehat{\otimes}_{\Lambda^+_{H_{n+k},m}}\boldsymbol{D}^+_{H_{n+k},m,\infty}(M^+)\cong\mathrm{X}^+_{H_{n+k},m}\widehat{\otimes}_{\Lambda^+_{H_{n+k},m_0}}\boldsymbol{D}^+_{H_{n+k},m_0,\infty}(M).$$

Let γ be a generator of Γ_{n+k}. By Proposition 5.20, there is some s such that

$$(\gamma-1)\boldsymbol{D}^+_{H_{n+k},m_0,\infty}(M^+)\subset p^s\boldsymbol{D}^+_{H_{n+k},m_0,\infty}(M^+).$$

If ℓ is sufficiently large Proposition 5.20 implies that

$$(\gamma_\ell-1)\boldsymbol{D}^+_{H_{n+k},m_0,\infty}(M^+)\subset p^{2c_3}\boldsymbol{D}^+_{H_{n+k},m_0,\infty}(M^+)$$

(we take $2c_3$ rather than c_3 to take of convergence later in this argument). Choose such an ℓ, and take m large enough so that $n(\gamma_\ell)\leq m$. Then by (TS3) we have $\mathrm{val}((\gamma_\ell-1)^{-1}(x))\geq\mathrm{val}(x)-c_3$ for $x\in\mathrm{X}^+_{H_{n+k},m}$.

We will now show that any element of $\mathrm{X}_{H_{n+k},m}\widehat{\otimes}_{\Lambda_{H_{n+k},m}}\boldsymbol{D}_{H_{n+k},m,\infty}(M)$ is in the image of $\gamma_\ell-1$. This will also imply any element is in the image of $\gamma-1$, since $\gamma_\ell-1$ is divisible by $\gamma-1$, and hence it will further imply that the cohomology

$$\mathrm{H}^1\big(\Gamma_{n+k},\mathrm{X}_{H_{n+k},m}\widehat{\otimes}_{\Lambda_{H_{n+k},m}}\boldsymbol{D}_{H_{n+k},m,\infty}(M)\big)\cong\mathrm{X}_{H_{n+k},m}\widehat{\otimes}_{\Lambda_{H_{n+k},m}}\boldsymbol{D}_{H_{n+k},m,\infty}(M)/(\gamma-1)$$

is 0.

To do this last step, it suffices to show that each simple tensor

$$a\otimes b\in \mathrm{X}^+_{H_{n+k},m}\,\widehat{\otimes}_{\Lambda^+_{H_{n+k},m_0}}\,\boldsymbol{D}^+_{H_{n+k},m_0,\infty}(M^+)\cong \mathrm{X}^+_{H_{n+k},m}\,\widehat{\otimes}_{\Lambda^+_{H_{n+k},m}}\,\boldsymbol{D}^+_{H_{n+k},m,\infty}(M^+)$$

is in the image of $\gamma_\ell-1$. Choose an integer r so that $p^r a$ is in the image of $(\gamma_l-1)^{-1}$ restricted to $\mathrm{X}^+_{H_{n+k},m}$ (choose any $r\geq c_3$). It suffices to show $p^r a\otimes b$ is in the image of $\gamma_\ell-1$. So write $p^r a=(\gamma_\ell-1)^{-1}(c)$ for $c\in \mathrm{X}^+_{H_{n+k},m}$, and consider the series

$$y=\sum_{i=0}^{+\infty}(\gamma_l^{-1}-1)^{-i}(c)\otimes(\gamma_l-1)^i(b)=\sum_{i=0}^{+\infty}\gamma_l^i(1-\gamma_l)^{-i}(c)\otimes(\gamma_l-1)^i(b).$$

This series converges, because by our choices

$$\operatorname{val}((\gamma_\ell-1)^{-1}(x))\geq \operatorname{val}(x)-c_3\quad\text{on } X^+_{H_{n+k},m}\quad\text{and}\quad(\gamma_\ell-1)(x)\geq\operatorname{val}(x)+2c_3\quad\text{on } \boldsymbol{D}^+_{H_{n+k},m_0,\infty}(M^+)!$$

A direct computation then gives

$$(\gamma_\ell-1)(y)=(\gamma_\ell-1)(c)\otimes b=p^r a\otimes b,$$

so $p^r a\otimes b$ is in the image of $\gamma_\ell-1$, as required. □

Combing Lemma 5.21 with the discussion after Corollary 5.18, we get the following description of $\mathrm{R}^1_{G\text{-la}}(M)$.

Proposition 5.22.
$$\mathrm{R}^1_{G\text{-la}}(M)=\varinjlim_{n,m}\mathrm{H}^1\big(\Gamma_{n+k},\boldsymbol{D}_{H_{n+k},m,\infty}(M)\big),$$
where the direct limit is taken over pairs n, m.

5G. *Computation of higher locally analytic vectors, III.* We are now almost ready to prove our theorem. First we prove a lemma that will be used.

Lemma 5.23. *Let $\Gamma=\gamma^{\mathbb{Z}_p}$ and let B be a Banach representation of Γ. Suppose $B=B^{\Gamma\text{-an}}$, and that*

$$\|\gamma-1\|<p^{-1/(p-1)}.$$

Then $\|b\|=\|b\|_{\Gamma\text{-an}}$ for any $b\in B$.

Proof. We have for $x\in\mathbb{Z}_p$ that

$$\gamma^x(b)=\sum\frac{\nabla_\gamma^k(b)}{k!}x^k$$

where $\nabla_\gamma=\log(\gamma)$. By definition

$$\|b\|_{\Gamma\text{-an}}=\sup_{k\geq 0}\{\|\nabla_\gamma^k(b)/k!\|\}.$$

Now recall we have

$$\nabla_\gamma=(\gamma-1)\sum_{m\geq 0}(-1)^m\frac{(\gamma-1)^m}{m+1},$$

so $\|\nabla_\gamma(b)\|\leq\|\gamma-1\|\|b\|$, and more generally

$$\|\nabla_\gamma^k(b)\|\leq\|\gamma-1\|^k\|b\|.$$

It follows that for $k \geq 1$ we have

$$\|\nabla_\gamma^k(b)/k!\| \leq p^{-k/(p-1)}\|\gamma - 1\|^k\|b\| < \|b\|,$$

so that $\|b\|_{\Gamma\text{-an}} = \|b\|$. $\square$

Proof of Theorem 5.1. By Proposition 5.22, $\mathrm{R}^1_{G\text{-la}}(M) = \varinjlim_{n,m} \mathrm{H}^1(\Gamma_{n+k}, \boldsymbol{D}_{H_{n+k},m,\infty}(M))$. Fix n and m. Given $b \in \boldsymbol{D}_{H_{n+k},m,\infty}(M)$ we shall show it becomes zero in some $\mathrm{H}^1(\Gamma_{l+k}, \boldsymbol{D}_{H_{l+k},m',\infty}(M))$ for some $\ell \geq n$, $m' \geq m$ — this will show the direct limit is zero. By Proposition 5.20 we know there is an open subgroup $\Gamma \subset \Gamma_{n+k}$ such that $\boldsymbol{D}_{H_{n+k},m,\infty}(M)$ is Γ-analytic. Writing γ for a generator of Γ, we may take Γ small enough so that $\|\gamma - 1\| < p^{-1/(p-1)}$, and hence Lemma 5.23 applies. Thus, writing $\|\cdot\|_n$ for the norm on $\boldsymbol{D}_{H_{n+k},m,\infty}(M)$ induced from its inclusion into $M \widehat{\otimes} \mathcal{C}^{\mathrm{an}}(G_n, \mathbb{Q}_p)$, we have $\|b\|_n = \|b\|_{\Gamma\text{-an}}$ for $b \in \boldsymbol{D}_{H_{n+k},m,\infty}(M)$. We know there is a real number $D > 0$ such that if $b \in \boldsymbol{D}_{H_{n+k},m,\infty}(M)$ then

$$\|\nabla_\gamma(b)\|_n = \|\nabla_\gamma(b)\|_{\Gamma\text{-an}} \leq D\|b\|_{\Gamma\text{-an}} = D\|b\|_n.$$

Now choose $\ell \geq n$ such that Γ_l has index p^t in Γ, where t is taken large enough so that

$$2p^{1/(p-1)}D \leq p^t.$$

Let $\gamma_t = \gamma^{p^t}$ be the generator of Γ_ℓ, and let $\log_\ell \in \mathcal{C}^{\mathrm{an}}(G_\ell, \mathbb{Q}_p) : G_\ell \twoheadrightarrow G_\ell/H_\ell \to \mathbb{Z}_p$ be the logarithm so that $\log_\ell(\gamma_t) = 1$. Now let $m' \geq m$ be large enough so that $\boldsymbol{D}_{H_{\ell+k},m',\infty}(M)$ is defined. Recall that by Lemma 5.15, $\log_\ell \in \boldsymbol{D}_{H_{\ell+k},m',\infty}(\widetilde{\Lambda}^+)$. Let $\Gamma' \subset \Gamma_{\ell+k}$ be an open subgroup so that $\boldsymbol{D}_{H_{l+k},m',\infty}(M)$ is Γ'-analytic and write p^q for the index of Γ' in $\Gamma_{\ell+k}$. Finally, write γ' for the generator of Γ'. Again by making Γ' smaller we may assume $\|\gamma' - 1\| < p^{-1/(p-1)}$ on $\boldsymbol{D}_{H_{\ell+k},m',\infty}(M)$. We have

$$\gamma' = (\gamma_t^{p^k})^{p^q} = \gamma^{p^{t+k+q}}.$$

Let $z_\ell = \log_\ell / p^{k+q} \in \boldsymbol{D}_{H_{\ell+k},m',\infty}(\widetilde{\Lambda})$, the one computes that $\gamma'(z_\ell) = z_\ell + 1$. Therefore, $\nabla_{\gamma'}(z_\ell) = 1$. Now consider the series

$$bz_\ell - \nabla_{\gamma'}(b)\frac{z_\ell^2}{2!} + \nabla_{\gamma'}^2(b)\frac{z_\ell^3}{3!} - \cdots$$

in $\boldsymbol{D}_{H_{\ell+k},m',\infty}(M)$. We claim first it converges with respect to the norm $\|\cdot\|_\ell$ of $\boldsymbol{D}_{H_{\ell+k},m',\infty}(M)$. Indeed, we have

$$\|z_\ell\|_\ell = p^{k+q}$$

and (noting that $\nabla_{\gamma'}^i = p^{i(t+k+q)}\nabla_\gamma^i$)

$$\|\nabla_{\gamma'}^i(b)\|_\ell = p^{-i(t+k+q)}\|\nabla_\gamma^i(b)\|_\ell \leq p^{-i(t+k+q)}\|\nabla_\gamma^i(b)\|_n \leq p^{-i(t+k+q)}D^i\|b\|_n,$$

so the general term of series has size

$$\|\nabla_{\gamma'}^i(b)/(i+1)! \cdot z_\ell^{i+1}\|_\ell \ll p^{-i(t+k+q)}D^i p^{i(k+q)} p^{i/(p-1)} = (p^{-t}Dp^{1/(p-1)})^i \leq 2^{-i},$$

so the series converges in the in the $\|\cdot\|_\ell$ norm. But then the series must also converge with respect to $\|\cdot\|_{\Gamma'\text{-an}}$ because of Lemma 5.23. So if we write y for the sum of the series, it makes sense to speak of the derivative $\nabla_{\gamma'}(y)$, and one computes that $\nabla_{\gamma'}(y) = b$. So b is in the image of

$$\nabla_{\gamma'} : \boldsymbol{D}_{H_{\ell+k},m',\infty}(M) \to \boldsymbol{D}_{H_{\ell+k},m',\infty}(M),$$

hence also in the image of $\gamma' - 1$, which divides $\nabla_{\gamma'}$. But $\gamma' = \gamma_{t+k}^{p^q}$ so $\gamma_{t+k} - 1$ divides $\gamma' - 1$. It follows that b is also in the image of $\gamma_{t+k} - 1$. This means that b is 0 in

$$\boldsymbol{D}_{H_{\ell+k},m',\infty}(M)/(\gamma_{t+k} - 1) \cong \mathrm{H}^1(\Gamma_{\ell+k}, \boldsymbol{D}_{H_{\ell+k},m',\infty}(M))$$

and we are done! □

Remark 5.24. (1) Since the choices of ℓ and m' did not depend on b, each $\boldsymbol{D}_{H_{n+k},m,\infty}(M)$ maps in its entirety to 0 in some $\boldsymbol{D}_{H_{l+k},m',\infty}(M)$. This shows that M is strongly $\mathfrak{LA}$-acyclic in the sense of [Pan 2022a, §2.2]. After this work was completed, Pan proved that strong $\mathfrak{LA}$-acyclicity is in fact automatic in this setting, see [Pan 2022b, Proposition 2.3.6].

(2) The proof of Theorem 5.1 shows the vanishing of $\varinjlim_{n,m} \mathrm{H}^1\big(\mathrm{Lie}(\Gamma_{n+k}), \boldsymbol{D}_{H_{n+k},m,\infty}(M)\big)$, which is a priori stronger than the vanishing of $\varinjlim_{n,m} \mathrm{H}^1(\Gamma_{n+k}, \boldsymbol{D}_{H_{n+k},m,\infty}(M))$.

6. Descent to locally analytic vectors

Work again in the setting of Sections 3–4. We shall assume in this section that K_∞ contains an unramified twist of the cyclotomic extension. The purpose of this section is to prove the following theorem.

Theorem 6.1. *The functor $\mathcal{E} \mapsto \mathcal{O}_{\mathcal{X}} \otimes_{\mathcal{O}_{\mathcal{X}}^{\mathrm{la}}} \mathcal{E}$ gives rise to an equivalence of categories*

$$\{\textit{locally analytic vector bundles on } \mathcal{X}\} \cong \{\Gamma\textit{-vector bundles on } \mathcal{X}\}.$$

The inverse functor is given by $\tilde{\mathcal{E}} \mapsto \tilde{\mathcal{E}}^{\mathrm{la}}$.

In the rest of this section, we shall prove that given a Γ-vector bundle $\tilde{\mathcal{E}}$ on $\mathcal{X}$, the natural map

$$\mathcal{O}_{\mathcal{X}} \otimes_{\mathcal{O}_{\mathcal{X}}^{\mathrm{la}}} \tilde{\mathcal{E}}^{\mathrm{la}} \to \tilde{\mathcal{E}}$$

is an isomorphism. This is enough for proving Theorem 6.1. Indeed, if this isomorphism is granted, then in particular it follows from Proposition 2.1 that $\tilde{\mathcal{E}}^{\mathrm{la}}$ is locally free over $\mathcal{O}_{\mathcal{X}}^{\mathrm{la}}$, so that the functor $\tilde{\mathcal{E}} \mapsto \tilde{\mathcal{E}}^{\mathrm{la}}$ is valued in the correct category and is fully faithful. On the other hand, it follows from Example 4.5(2) that it is also essentially surjective.

6A. *Computations at the stalk.* In this section, w let $\tilde{\mathcal{E}}$ be a Γ-vector bundle. We have the fiber $\tilde{\mathcal{E}}_{k(x_\infty)}$ at x_∞, a finite dimensional $\widehat{K}_\infty$-semilinear representation of Γ, and the completed stalk $\tilde{\mathcal{E}}_{x_\infty}^{\wedge,+}$, a finite free $\boldsymbol{B}_{\mathrm{dR}}^+(\widehat{K}_\infty) = \boldsymbol{B}_{\mathrm{dR}}^{+,H}$-module. We define

$$\boldsymbol{D}_{\mathrm{Sen}}(\tilde{\mathcal{E}}) = (\tilde{\mathcal{E}}_{k(x_\infty)})^{\mathrm{la}} \quad \text{and} \quad \boldsymbol{D}_{\mathrm{dif}}^+(\tilde{\mathcal{E}}) = (\tilde{\mathcal{E}}_{x_\infty}^{\wedge,+})^{\mathrm{pa}}.$$

If V is a p-adic representation and $\tilde{\mathcal{E}} = \tilde{\mathcal{E}}(V)$ as in Example 3.4, and if $\Gamma = \Gamma_{\mathrm{cyc}}$, then we recover the classical invariant $\boldsymbol{D}_{\mathrm{Sen}}(V)$ according to [Berger and Colmez 2016, théorème 3.2]. The invariant $\boldsymbol{D}_{\mathrm{dif}}^+(V)$

is also recovered, see [Porat 2022, Proposition 3.3.]. It is therefore natural to extend these definitions to arbitrary $\tilde{\mathcal{E}}$ and Γ as we have done here.

There is the following decompletion result.

Theorem 6.2. (i) *The natural map* $\widehat{K}_\infty \otimes_{\widehat{K}_\infty^{\mathrm{la}}} \boldsymbol{D}_{\mathrm{Sen}}(\tilde{\mathcal{E}}) \to \tilde{\mathcal{E}}_{k(x_\infty)}$ *is an isomorphism.*

(ii) *The natural map* $\boldsymbol{B}_{\mathrm{dR}}^+(\widehat{K}_\infty) \otimes_{\boldsymbol{B}_{\mathrm{dR}}^+(\widehat{K}_\infty)^{\mathrm{pa}}} \boldsymbol{D}_{\mathrm{dif}}^+(\tilde{\mathcal{E}}) \to \tilde{\mathcal{E}}_{x_\infty}^{\wedge,+}$ *is an isomorphism.*

Proof. The fiber $\tilde{\mathcal{E}}_{k(x_\infty)}$ is a finite dimensional $\widehat{K}_\infty$-semilinear representation of Γ. So (i) follows from [Berger and Colmez 2016, théorème 3.4]. For (ii), write I_θ for the maximal ideal of $\boldsymbol{B}_{\mathrm{dR}}^+(\widehat{K}_\infty)$. It suffices to prove that for $n \geq 1$ the natural map

$$\boldsymbol{B}_{\mathrm{dR}}^+(\widehat{K}_\infty)/I_\theta^n \otimes_{(\boldsymbol{B}_{\mathrm{dR}}^+/I_\theta^n)^{\mathrm{la}}} (\tilde{\mathcal{E}}_{x_\infty}/I_\theta^n)^{\mathrm{la}} \to \tilde{\mathcal{E}}_{x_\infty}/I_\theta^n \tag{$*$}$$

is an isomorphism.

By Theorem 5.1 (more precisely, Corollary 5.6(i)), we have $\mathrm{R}^1_{\mathrm{la}}(I_\theta^{n-1}\tilde{\mathcal{E}}_{x_\infty}/I_\theta^n) = 0$, so by devissage the map

$$(\tilde{\mathcal{E}}_{x_\infty}/I_\theta^n)^{\mathrm{la}} \to (\tilde{\mathcal{E}}_{x_\infty}/I_\theta)^{\mathrm{la}} = \boldsymbol{D}_{\mathrm{Sen}}(\tilde{\mathcal{E}})$$

is surjective. It follows from the case $n = 1$ and Nakayama's lemma that $(*)$ is surjective too.

For injectivity, we argue as follows. Let $\bar{e}_1, \dots, \bar{e}_d$ be a basis of $\boldsymbol{D}_{\mathrm{Sen}}(\tilde{\mathcal{E}})$ over the field $\widehat{K}_\infty^{\mathrm{la}}$. By what was just proved, we may choose a lifting $e_1, \dots, e_d$ of this basis to $(\tilde{\mathcal{E}}_{x_\infty}/I_\theta^n)^{\mathrm{la}}$. Then $1 \otimes e_1, \dots, 1 \otimes e_d$ generate

$$\boldsymbol{B}_{\mathrm{dR}}^+(\widehat{K}_\infty)/I_\theta^n \otimes_{(\boldsymbol{B}_{\mathrm{dR}}^+/I_\theta^n)^{\mathrm{la}}} (\tilde{\mathcal{E}}_{x_\infty}/I_\theta^n)^{\mathrm{la}}$$

according to Nakayama's lemma.

Now suppose that

$$\sum x_i \otimes e_i \in \boldsymbol{B}_{\mathrm{dR}}^+(\widehat{K}_\infty)/I_\theta^n \otimes_{(\boldsymbol{B}_{\mathrm{dR}}^+/I_\theta^n)^{\mathrm{la}}} (\tilde{\mathcal{E}}_{x_\infty}/I_\theta^n)^{\mathrm{la}}$$

is in the kernel of $(*)$, so its image is 0 mod I_θ^n. Choose a generator ξ of I_θ. Reducing mod I_θ and using the injectivity of $(*)$ for $n = 1$, we get the relation $\sum \bar{x}_i \otimes \bar{e}_i = 0$. As the $\bar{e}_i$ form a basis, each x_i must be divisible by ξ. Writing $x_i = \xi x_i'$, we have

$$\sum x_i \otimes e_i = \sum \xi x_i' \otimes e_i = \xi \sum x_i' \otimes e_i,$$

so the image of

$$\sum x_i' \otimes y_i \in \boldsymbol{B}_{\mathrm{dR}}^+(\widehat{K}_\infty)/I_\theta^{n-1} \otimes_{(\boldsymbol{B}_{\mathrm{dR}}^+/I_\theta^{n-1})^{\mathrm{la}}} (\tilde{\mathcal{E}}_{x_\infty}/I_\theta^{n-1})^{\mathrm{la}}$$

in $\tilde{\mathcal{E}}_{x_\infty}/I_\theta^{n-1}$ is 0. The injectivity now follows from induction. □

Let I be a closed interval with $|\log(I)| < \log(p)$ and let

$$\widetilde{M}_I = \mathrm{H}^0(\mathcal{X}_I, \tilde{\mathcal{E}}).$$

Theorem 5.1 allows us to prove the following Proposition 6.3; we shall subsequently prove a stronger statement in Theorem 6.5.

Proposition 6.3. *There are natural isomorphisms*

$$\boldsymbol{D}_{\mathrm{Sen}}(\tilde{\mathcal{E}}) \cong \widetilde{M}_I^{\mathrm{la}}/(I_\theta \widetilde{M}_I)^{\mathrm{la}} \quad \textit{and} \quad \boldsymbol{D}^+_{\mathrm{dif}}(\tilde{\mathcal{E}}) \cong \varprojlim_n \widetilde{M}_I^{\mathrm{la}}/(I_\theta^n \widetilde{M}_I)^{\mathrm{la}}.$$

Proof. As I_θ is principal, $I_\theta \widetilde{M}_I$ is finite free over $\widetilde{\boldsymbol{B}}_I$. By Corollary 5.6(ii), the cohomology $\mathrm{R}^1_{\mathrm{la}}(I_\theta \widetilde{M}_I)$ vanishes. Applying la to the short exact sequence

$$0 \to I_\theta \widetilde{M}_I \to \widetilde{M}_I \to \widetilde{M}_I I_\theta / \widetilde{M}_I \to 0$$

we get $\widetilde{M}_I^{\mathrm{la}}/(I_\theta \widetilde{M}_I)^{\mathrm{la}} \xrightarrow{\sim} (\widetilde{M}_I / I_\theta \widetilde{M}_I)^{\mathrm{la}} = \boldsymbol{D}_{\mathrm{Sen}}(\tilde{\mathcal{E}})$, which gives the first isomorphism. By the same argument $\widetilde{M}_I^{\mathrm{la}}/(I_\theta^n \widetilde{M}_I)^{\mathrm{la}} \xrightarrow{\sim} (\widetilde{M}_I / I_\theta^n \widetilde{M}_I)^{\mathrm{la}}$ for $n \geq 1$. To get the second isomorphism, take the limit over n. □

6B. *Descent to locally analytic vectors.* In this subsection we will give a proof of Theorem 6.1. We continue with the notation of Section 6A.

We start with the following key proposition, which builds upon all of the work done in Section 4, Section 5 and the previous subsections of Section 6.

Proposition 6.4. *Let $I = [r, (p-1)p^n]$ be an interval with $n \geq 1$ and $|\log(I)| < \log(p)$. Then the natural map*

$$\widetilde{\boldsymbol{B}}_I \otimes_{\widetilde{\boldsymbol{B}}_I^{\mathrm{la}}} \widetilde{M}_I^{\mathrm{la}} \to \widetilde{M}_I \tag{6-1}$$

is an isomorphism.

Proof. First let us explain how to reduce to the cyclotomic case. After an unramified twist, which causes no obstructions to descent, we may assume $K_{\mathrm{cyc}} \subset K_\infty$. Set

$$\widetilde{M}_{I,\mathrm{cyc}} := \widetilde{M}_I^{\mathrm{Gal}(K_\infty/K_{\mathrm{cyc}})}.$$

We then have

$$\widetilde{M}_I \cong \widetilde{\boldsymbol{B}}_I \otimes_{\widetilde{\boldsymbol{B}}_{I,\mathrm{cyc}}} \widetilde{M}_{I,\mathrm{cyc}}$$

(see for example [Berger and Colmez 2008, corollarie 3.2.2]), and if the conclusion of the proposition holds for the cyclotomic case, we have

$$\widetilde{M}_{I,\mathrm{cyc}} \cong \widetilde{\boldsymbol{B}}_{I,\mathrm{cyc}} \otimes_{\widetilde{\boldsymbol{B}}^{\mathrm{la}}_{I,\mathrm{cyc}}} \widetilde{M}^{\mathrm{la}}_{I,\mathrm{cyc}}$$

and hence

$$\widetilde{M}_I \cong \widetilde{\boldsymbol{B}}_I \otimes_{\widetilde{\boldsymbol{B}}^{\mathrm{la}}_{I,\mathrm{cyc}}} \widetilde{M}^{\mathrm{la}}_{I,\mathrm{cyc}}.$$

This shows that $\widetilde{M}_I$ has a basis of locally analytic vectors and by Proposition 2.1 the map (6-1) is an isomorphism.

It remains to establish the proposition in the cyclotomic case where $\widetilde{\boldsymbol{B}}_I = \widetilde{\boldsymbol{B}}_{I,\mathrm{cyc}}$. By Proposition 4.2, $\widetilde{\boldsymbol{B}}_{I,\mathrm{cyc}}$ is flat as a $\widetilde{\boldsymbol{B}}^{\mathrm{la}}_{I,\mathrm{cyc}}$-module. Since $\widetilde{M}^{\mathrm{la}}_{I,\mathrm{cyc}}$ is torsionfree as a $\widetilde{\boldsymbol{B}}^{\mathrm{la}}_{I,\mathrm{cyc}}$-module, it follows from [Stacks

2005–, 0AXM] that $\widetilde{\boldsymbol{B}}_{I,\mathrm{cyc}} \otimes_{\widetilde{\boldsymbol{B}}^{\mathrm{la}}_{I,\mathrm{cyc}}} \widetilde{M}^{\mathrm{la}}_{I,\mathrm{cyc}}$ is also torsionfree. By Proposition 6.3, the completion at $I_\theta \subset \widetilde{\boldsymbol{B}}_{I,\mathrm{cyc}}$ of (6-1) is nothing but the map

$$\boldsymbol{B}^+_{\mathrm{dR}} \otimes_{\boldsymbol{B}^{+,\mathrm{pa}}_{\mathrm{dR}}} \boldsymbol{D}^+_{\mathrm{dif}}(\tilde{\mathcal{E}}) \to \tilde{\mathcal{E}}^{\wedge,+}_{x_\infty},$$

so by Theorem 6.2, the map (6-1) is an isomorphism at least after taking this completion. As $\widetilde{\boldsymbol{B}}_{I,\mathrm{cyc}}$ is a PID (see Proposition 3.1), it follows that (6-1) is injective with cokernel supported at finitely many maximal ideals. These maximal ideals correspond to a finite set of points on $\mathcal{X}$, and this set must form a finite orbit under the action of Γ. But by [Fargues and Fontaine 2018, Proposition 10.1.1], the only point with finite orbit under the Γ-action is x_∞! Thus the cokernel of (6-1) is supported at I_θ. But then it must be 0, as we have just shown the completion at I_θ is an isomorphism. $\square$

Proof of Theorem 6.1. Let U be an open subaffinoid of $\mathcal{X}_I$ for $I = [r, (p-1)p^n]$. Then we claim that the natural map

$$\mathcal{O}_{\mathcal{X}}(U) \otimes_{\mathcal{O}^{\mathrm{la}}_{\mathcal{X}}(U)} \mathrm{H}^0(U, \tilde{\mathcal{E}}^{\mathrm{la}}) \to \mathrm{H}^0(U, \tilde{\mathcal{E}})$$

is an isomorphism. Indeed, we have

$$\mathrm{H}^0(U, \tilde{\mathcal{E}}) \cong \mathcal{O}_{\mathcal{X}}(U) \otimes_{\widetilde{\boldsymbol{B}}_{I,\mathrm{cyc}}} \widetilde{M}_{I,\mathrm{cyc}} \cong \mathcal{O}_{\mathcal{X}}(U) \otimes_{\widetilde{\boldsymbol{B}}^{\mathrm{la}}_{I,\mathrm{cyc}}} \widetilde{M}^{\mathrm{la}}_{I,\mathrm{cyc}}.$$

Thus $\mathrm{H}^0(U, \tilde{\mathcal{E}})$ has a basis of locally analytic elements. By Proposition 2.1, we have an isomorphism

$$\mathcal{O}_{\mathcal{X}}(U) \otimes_{\mathcal{O}_{\mathcal{X}}(U)^{\mathrm{la}}} \mathrm{H}^0(U, \tilde{\mathcal{E}})^{\mathrm{la}} \to \mathrm{H}^0(U, \tilde{\mathcal{E}}),$$

from which the claim follows.

Now let $(\mathcal{O}_{\mathcal{X}} \otimes_{\mathcal{O}^{\mathrm{la}}_{\mathcal{X}}} \tilde{\mathcal{E}}^{\mathrm{la}})^\circ$ be the presheaf on $\mathcal{X}$ sending

$$U \mapsto \mathcal{O}_{\mathcal{X}}(U) \otimes_{\mathcal{O}^{\mathrm{la}}_{\mathcal{X}}(U)} \tilde{\mathcal{E}}^{\mathrm{la}}(U).$$

The $\mathcal{X}_I$ for various I of the form $I = [r, (p-1)p^n]$ with $|\log(I)| < \log(p)$ give a covering of $\mathcal{X}$, so the claim shows that the natural map

$$(\mathcal{O}_{\mathcal{X}} \otimes_{\mathcal{O}^{\mathrm{la}}_{\mathcal{X}}} \tilde{\mathcal{E}}^{\mathrm{la}})^\circ \to \tilde{\mathcal{E}}$$

is an isomorphism on stalks. Theorem 6.1 follows. $\square$

The proof of Theorem 6.1 essentially shows that $\mathcal{E}$ is quasicoherent. This leads to a simple interpretation of $\boldsymbol{D}_{\mathrm{Sen}}$ and $\boldsymbol{D}^+_{\mathrm{dif}}$ in terms of $\mathcal{E}$ as follows. Given a locally analytic vector bundle define

$$\boldsymbol{D}_{\mathrm{Sen}}(\mathcal{E}) = \mathcal{E}_{k(x_\infty)},$$

the fiber of $\mathcal{E}$ at x_∞, and

$$\boldsymbol{D}^+_{\mathrm{dif}}(\mathcal{E}) = \hat{\mathcal{E}}^+_{x_\infty},$$

the completed stalk of $\mathcal{E}$ at x_∞. These would not *a priori* be the same as $\boldsymbol{D}_{\mathrm{Sen}}(\tilde{\mathcal{E}})$ and $\boldsymbol{D}^+_{\mathrm{dif}}(\tilde{\mathcal{E}})$, because quotients in general do not commute with locally analytic vectors, but they do in this case.

Theorem 6.5. *Let $\tilde{\mathcal{E}} = \mathcal{O}_{\mathcal{X}} \otimes_{\mathcal{O}_{\mathcal{X}}^{\mathrm{la}}} \mathcal{E}$. There are natural isomorphisms*

$$\boldsymbol{D}_{\mathrm{Sen}}(\tilde{\mathcal{E}}) \cong \boldsymbol{D}_{\mathrm{Sen}}(\mathcal{E}) \quad \text{and} \quad \boldsymbol{D}^{+}_{\mathrm{dif}}(\tilde{\mathcal{E}}) \cong \boldsymbol{D}^{+}_{\mathrm{dif}}(\mathcal{E}).$$

Proof. For $I = [r, (p-1)p^n]$ with $|\log(I)| < \log(p)$ write $\widetilde{M}_I = \mathrm{H}^0(\mathcal{X}_I, \tilde{\mathcal{E}})$. For any sufficiently small U containing x_∞, the proof of Theorem 6.1 shows that

$$\mathrm{H}^0(U, \mathcal{E}) \cong \mathcal{O}_{\mathcal{X}}(U)^{\mathrm{la}} \otimes_{\widetilde{\boldsymbol{B}}_I^{\mathrm{la}}} \widetilde{M}_I^{\mathrm{la}}.$$

It follows that the quotient $\mathcal{E}_{x_\infty}/m^n_{x_\infty}\mathcal{E}_{x_\infty}$ of the stalk $\mathcal{E}_{x_\infty}$ by the n-th power of the maximal ideal $m_{x_\infty} \subset \mathcal{O}^{\mathrm{la}}_{\mathcal{X},x_\infty}$ is identified with the quotient $\widetilde{M}_I^{\mathrm{la}}/(I_\theta^n \widetilde{M}_I)^{\mathrm{la}}$. Now use Proposition 6.3. $\square$

7. The comparison with (φ, Γ)-modules

In this section, we give reminders on (φ, Γ)-modules and compare them to locally analytic vector bundles. We keep the notation from Section 6 and the assumption that $K^{\eta}_{\mathrm{cyc}} \subset K_\infty$ for some η.

7A. *Galois representations and (φ, Γ)-modules.* Recall the notation from Section 3 and let

$$\widetilde{\boldsymbol{B}}^{\dagger}_{\mathrm{rig}} = \widetilde{\boldsymbol{B}}^{\dagger}_{\mathrm{rig}}(\widehat{K}_\infty) = \varinjlim_r \mathrm{H}^0(\mathcal{Y}_{[r,\infty)}, \mathcal{O}_{\mathcal{Y}}) = \varinjlim_r \varprojlim_{s \geq r} \mathrm{H}^0(\mathcal{Y}_{[r,s]}, \mathcal{O}_{\mathcal{Y}})$$

be the extended Robba ring. The (φ, Γ)-actions on $\mathcal{Y}$ induce actions on $\widetilde{\boldsymbol{B}}^{\dagger}_{\mathrm{rig}}$.

Definition 7.1. A (φ, Γ)-module over $\widetilde{\boldsymbol{B}}^{\dagger}_{\mathrm{rig}}$ is a finite free $\widetilde{\boldsymbol{B}}^{\dagger}_{\mathrm{rig}}$-module with commuting semilinear (φ, Γ)-actions such that in some basis $\mathrm{Mat}(\varphi) \in \mathrm{GL}_d(\widetilde{\boldsymbol{B}}^{\dagger}_{\mathrm{rig}})$.

We can compare these objects to (φ, Γ)-vector bundles using two functors. On the one hand, if $\widetilde{\mathcal{M}}$ is a (φ, Γ)-vector bundle, then $\widetilde{\mathrm{M}}^{\dagger}_{\mathrm{rig}} = \varinjlim_r \mathrm{H}^0(\mathcal{Y}_{[r,\infty)}, \widetilde{\mathcal{M}})$ is a (φ, Γ)-module. Here, the nontrivial thing one needs to check is that $\widetilde{\mathrm{M}}^{\dagger}_{\mathrm{rig}}$ is free, and this follows from $\widetilde{\boldsymbol{B}}^{\dagger}_{\mathrm{rig}}$ being Bézout [Kedlaya 2004, Theorem 3.20].

One the other hand, given a (φ, Γ)-module $\widetilde{\mathrm{M}}^{\dagger}_{\mathrm{rig}}$ we define a (φ, Γ)-vector bundle $\mathrm{FT}(\widetilde{\mathrm{M}}^{\dagger}_{\mathrm{rig}})$ as follows. If $\widetilde{\mathrm{M}}^{\dagger}_{\mathrm{rig}}$ is a (φ, Γ)-module then for every $r \gg 0$ we have a finite free $\widetilde{\boldsymbol{B}}_{[r,\infty)}$-semilinear Γ-representation $\widetilde{\mathrm{M}}_{[r,\infty)}$ together with isomorphisms

$$\varphi^* \widetilde{\boldsymbol{B}}_{[r,\infty)} \otimes_{\widetilde{\boldsymbol{B}}_{[r/p,\infty)}} \widetilde{\mathrm{M}}_{[r/p,\infty)} \xrightarrow{\sim} \widetilde{\mathrm{M}}_{[r,\infty)}$$

as well as identifications

$$\widetilde{\boldsymbol{B}}^{\dagger}_{\mathrm{rig}} \otimes_{\widetilde{\boldsymbol{B}}_{[r,\infty)}} \widetilde{\mathrm{M}}_{[r,\infty)} \xrightarrow{\sim} \widetilde{\mathrm{M}}^{\dagger}_{\mathrm{rig}}.$$

Using the isomorphisms $\varphi : \widetilde{\boldsymbol{B}}_{[r,\infty)} \xrightarrow{\sim} \widetilde{\boldsymbol{B}}_{[r/p,\infty)}$ we can then uniquely extend this to all $r > 0$ by inductively defining $\widetilde{\mathrm{M}}_{[r/p^n,\infty)}$ through the isomorphisms

$$\varphi^* \widetilde{\boldsymbol{B}}_{[r/p^{n-1},\infty)} \otimes_{\widetilde{\boldsymbol{B}}_{[r/p^n,\infty)}} \widetilde{\mathrm{M}}_{[r/p^n,\infty)} \xrightarrow{\sim} \widetilde{\mathrm{M}}_{[r/p^{n-1},\infty)}.$$

Setting for every $r > 0$

$$\mathrm{H}^0(\mathcal{Y}_{[r,\infty)}, \mathrm{FT}(\widetilde{\mathrm{M}}^{\dagger}_{\mathrm{rig}})) := \widetilde{\mathrm{M}}_{[r,\infty)}$$

and for every $s \geq r$

$$\mathrm{H}^0(\mathcal{Y}_{[r,s]}, \mathrm{FT}(\widetilde{\mathrm{M}}^{\dagger}_{\mathrm{rig}})) := \widetilde{\mathrm{M}}_{[r,\infty)} \otimes_{\widetilde{\boldsymbol{B}}_{[r,\infty)}} \widetilde{\boldsymbol{B}}_{[r,s]}$$

we obtain a (φ, Γ)-vector bundle $\mathrm{FT}(\widetilde{\mathrm{M}}^{\dagger}_{\mathrm{rig}})$.

Proposition 7.2. *The functors $\widetilde{\mathcal{M}} \mapsto \varinjlim_r \mathrm{H}^0(\mathcal{Y}_{[r,\infty)}, \widetilde{\mathcal{M}})$ and* FT *induce an equivalence of categories*

$$\{(\varphi, \Gamma)\text{-vector bundles on } \mathcal{Y}_{(0,\infty)}\} \cong \{(\varphi, \Gamma)\text{-modules over } \widetilde{\boldsymbol{B}}^{\dagger}_{\mathrm{rig}}\}.$$

Proof. This is well known. See for example the discussion appearing directly after [Scholze and Weinstein 2020, Definition 13.4.3]. The treatment there is given in the situation where there is no Γ-action present, but the same proof works in our setting. □

The following theorem due to Fontaine and Kedlaya gives the relation of these objects with Galois representations. To formulate it, we need to introduce some terminology. Let y be the point of $\mathcal{Y}$ corresponding to $p = 0$. A (φ, Γ)-module over $\widetilde{\boldsymbol{B}}^{\dagger}_{\mathrm{rig}}$ is called étale if it has a basis for which $\mathrm{Mat}(\varphi) \in \mathrm{GL}_d(\mathcal{O}_{\mathcal{Y},y})$. We also have the notion of a semistable slope 0 vector bundle on $\mathcal{X}$ — we refer the reader to [Fargues and Fontaine 2018, définition 5.5.1, exemple 5.5.2.1].

Theorem 7.3. *The following categories are equivalent.*

1. *Finite dimensional $\mathbb{Q}_p$-representations of G_K.*
2. *Étale (φ, Γ)-modules over $\widetilde{\boldsymbol{B}}^{\dagger}_{\mathrm{rig}}$.*
3. *Γ-vector bundles on $\mathcal{X}$ which are semistable of slope* 0.

Proof. The equivalence of (2) and (3) follows from Proposition 7.2 and Proposition 3.3. The category in (1) is equivalent to (φ, Γ)-modules over $\widetilde{\boldsymbol{B}} = \widehat{\mathcal{O}}_{\mathcal{Y},y}[1/p]$, where $\widehat{\mathcal{O}}_{\mathcal{Y},y}$ is the p-adic completion of $\mathcal{O}_{\mathcal{Y},y}$, by the theorem of Fontaine [1990, théorème 3.4.3 and remarque 3.44(c)]. Next, by a relatively elementary argument, this category is equivalent to the category of (φ, Γ)-modules over $\widetilde{\boldsymbol{B}}^{\dagger}$, see for example [Kedlaya 2015, Theorem 2.4.5] or [de Shalit and Porat 2019, Theorem 4.3]. Finally, one can replace $\widetilde{\boldsymbol{B}}^{\dagger}$ by $\widetilde{\boldsymbol{B}}^{\dagger}_{\mathrm{rig}}$ by [Kedlaya 2004, Proposition 5.11, Corollary 5.12]. See also [Fargues and Fontaine 2018, proposition 11.2.24]. □

7B. *The comparison with locally analytic vector bundles.* Let $\widetilde{\boldsymbol{B}}^{\dagger,\mathrm{pa}}_{\mathrm{rig}}$ be the subring of pro-analytic vectors in $\widetilde{\boldsymbol{B}}^{\dagger}_{\mathrm{rig}}$ for the action of Γ. We have a corresponding version of (φ, Γ)-modules.

Definition 7.4. A (φ, Γ)-module $\mathrm{M}^{\dagger}_{\mathrm{rig}}$ over $\widetilde{\boldsymbol{B}}^{\dagger,\mathrm{pa}}_{\mathrm{rig}}$ is a finite free $\widetilde{\boldsymbol{B}}^{\dagger,\mathrm{pa}}_{\mathrm{rig}}$-module with commuting semilinear (φ, Γ)-actions such that in some basis $\mathrm{Mat}(\varphi) \in \mathrm{GL}_d(\widetilde{\boldsymbol{B}}^{\dagger,\mathrm{pa}}_{\mathrm{rig}})$, and such that the action of Γ is pro-analytic. It is étale if $\widetilde{\boldsymbol{B}}^{\dagger}_{\mathrm{rig}} \otimes_{\widetilde{\boldsymbol{B}}^{\dagger,\mathrm{pa}}_{\mathrm{rig}}} \mathrm{M}^{\dagger}_{\mathrm{rig}}$ is so.

The following theorem explains the relationship between (φ, Γ)-modules and locally analytic vector bundles.

Theorem 7.5. *The following categories are all equivalent.*

1. *(φ, Γ)-modules over $\widetilde{\boldsymbol{B}}^{\dagger}_{\mathrm{rig}}$.*

(2) *(φ, Γ)-modules over $\widetilde{\boldsymbol{B}}^{\dagger,\mathrm{pa}}_{\mathrm{rig}}$.*

(3) *(φ, Γ)-vector bundles over $\mathcal{Y}_{(0,\infty)}$.*

(4) *Locally analytic φ-vector bundles on $\mathcal{Y}_{(0,\infty)}$.*

(5) *Γ-vector bundles on $\mathcal{X}$.*

(6) *Locally analytic vector bundles on $\mathcal{X}$.*

Proof. The equivalences (1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (5) are Propositions 7.2 and 3.3. (4) $\Leftrightarrow$ (6) is similar to Proposition 3.3. The proof of (5) $\Leftrightarrow$ (6) was given in Theorem 6.1, and (3) $\Leftrightarrow$ (4) can be proved in a similar way. It remains to give an equivalence between (2) and (4). The Frobenius trick functor of Section 7A induces a functor

$$\mathrm{FT} : \{(\varphi, \Gamma)\text{-modules over } \widetilde{\boldsymbol{B}}^{\dagger,\mathrm{pa}}_{\mathrm{rig}}\} \to \{\text{Locally analytic } \varphi\text{-vector bundles on } \mathcal{Y}_{(0,\infty)}\}.$$

In the other direction we map a locally analytic φ-vector bundle $\mathcal{M}$ to $\mathcal{M}^{\dagger}_{\mathrm{rig}} = \varinjlim_r \mathrm{H}^0(\mathcal{Y}_{[r,\infty)}, \mathcal{M})$. It is easy to check from the definitions these two are inverses to each other once we know that $\mathcal{M} \mapsto \mathcal{M}^{\dagger}_{\mathrm{rig}}$ is valued in the correct category. So it remains to prove the following:

Claim. *$\mathcal{M}^{\dagger}_{\mathrm{rig}}$ is a (φ, Γ)-module over $\widetilde{\boldsymbol{B}}^{\dagger,\mathrm{pa}}_{\mathrm{rig}}$.*

Proof of Claim. We only need to explain why $\mathcal{M}^{\dagger}_{\mathrm{rig}}$ is a free $\widetilde{\boldsymbol{B}}^{\dagger,\mathrm{pa}}_{\mathrm{rig}}$-module. Since we can always descend along unramified extensions, we may assume $K_{\mathrm{cyc}} \subset K_\infty$. Then $\mathcal{M}$ and $\mathcal{M}^{\dagger}_{\mathrm{rig}}$ are both base changed from their cyclotomic counterparts $\mathcal{M}^{\mathrm{Gal}(K_\infty/K_{\mathrm{cyc}})}$ and $\mathcal{M}^{\dagger,\mathrm{Gal}(K_\infty/K_{\mathrm{cyc}})}_{\mathrm{rig}}$, so we reduce to the cyclotomic case.

To deal with this case, recall the rings $\boldsymbol{B}_{I,\mathrm{cyc}}$ from Section 4. The (cyclotomic) Robba ring is defined as

$$\boldsymbol{B}^{\dagger}_{\mathrm{rig,cyc}} = \varinjlim_{r} \varprojlim_{s \geq r} \boldsymbol{B}_{[r,s],\mathrm{cyc}}.$$

The maps $\boldsymbol{B}_{[r,s],\mathrm{cyc}} \hookrightarrow \widetilde{\boldsymbol{B}}_{I,\mathrm{cyc}}$ of Section 4 induce an embedding $\boldsymbol{B}^{\dagger}_{\mathrm{rig,cyc}} \hookrightarrow \widetilde{\boldsymbol{B}}^{\dagger}_{\mathrm{rig,cyc}} = \widetilde{\boldsymbol{B}}^{\dagger}_{\mathrm{rig}}(\widehat{K}_{\mathrm{cyc}})$. By [Berger 2016, Theorem B] we have

$$\widetilde{\boldsymbol{B}}^{\dagger,\mathrm{pa}}_{\mathrm{rig}} = \bigcup_{n \geq 0} \varphi^{-n}(\boldsymbol{B}^{\dagger}_{\mathrm{rig,cyc}}),$$

and since each $\varphi^{-n}(\boldsymbol{B}^{\dagger}_{\mathrm{rig,cyc}})$ is a Bézout domain [Lazard 1962], the conclusion follows. □

In particular, we recover a decompletion result entirely phrased in terms of (φ, Γ)-modules:

$$\{(\varphi, \Gamma)\text{-modules over } \widetilde{\boldsymbol{B}}^{\dagger}_{\mathrm{rig}}\} \cong \{(\varphi, \Gamma)\text{-modules over } \widetilde{\boldsymbol{B}}^{\dagger,\mathrm{pa}}_{\mathrm{rig}}\}.$$

This result recovers the decompletion theorem of Cherbonnier and Colmez [1998] and Kedlaya [2004].

Theorem 7.6. *If $K_\infty = K_{\mathrm{cyc}}$, base extension induces an equivalence of categories*

$$\{(\varphi, \Gamma)\text{-modules over } \boldsymbol{B}^{\dagger}_{\mathrm{rig,cyc}}\} \cong \{(\varphi, \Gamma)\text{-modules over } \widetilde{\boldsymbol{B}}^{\dagger}_{\mathrm{rig,cyc}}\}.$$

Proof. If M is a (φ, Γ)-module over $\widetilde{\boldsymbol{B}}^{\dagger,\mathrm{pa}}_{\mathrm{rig,cyc}} = \bigcup_n \varphi^{-n}(\boldsymbol{B}^{\dagger}_{\mathrm{rig,cyc}})$ then there exists $n \gg 0$ such that M is defined over $\varphi^{-n}(\boldsymbol{B}^{\dagger}_{\mathrm{rig,cyc}})$. If $e_1, \dots, e_d$ is a basis of M then $\varphi^n(e_1), \dots, \varphi^n(e_d)$ is a basis defined over $\boldsymbol{B}^{\dagger}_{\mathrm{rig,cyc}}$. Therefore the category of (φ, Γ)-modules over $\boldsymbol{B}^{\dagger}_{\mathrm{rig,cyc}}$ is equivalent to the category of (φ, Γ)-modules over $\widetilde{\boldsymbol{B}}^{\dagger,\mathrm{pa}}_{\mathrm{rig,cyc}}$. But this latter category is equivalent to (φ, Γ)-modules over $\widetilde{\boldsymbol{B}}^{\dagger}_{\mathrm{rig,cyc}}$ by Theorem 7.5. □

8. Locally analytic vector bundles and p-adic differential equations

8A. *Modifications of locally analytic vector bundles.* We first introduce the following category. It is the locally analytic version of Berger's category of $\boldsymbol{B}$-pairs; see [Berger 2008a].

Definition 8.1. A locally analytic $\boldsymbol{B}$-pair is a pair $\mathcal{W} = (\mathcal{W}_e, W^+_{\mathrm{dR}})$, where $\mathcal{W}_e$ is a locally free $\mathcal{O}^{\mathrm{la}}_{\mathcal{X}\text{-}\{\infty\}} = \mathcal{O}^{\mathrm{la}}_{\mathcal{X}}|_{\mathcal{X}\text{-}\{\infty\}}$-module with a semilinear Γ-action and $W^+_{\mathrm{dR}} \subset \boldsymbol{B}^{\mathrm{pa}}_{\mathrm{dR}} \otimes_{\mathcal{O}^{\mathrm{la}}_{\mathcal{X}\text{-}\{\infty\}}} \mathcal{W}_e$ is a Γ-stable $\boldsymbol{B}^{+,\mathrm{pa}}_{\mathrm{dR}}$-lattice.

Proposition 8.2. *The functor from locally analytic vector bundles to locally analytic $\boldsymbol{B}$-pairs mapping $\mathcal{E}$ to $(\mathcal{E}|_{\mathcal{X}\text{-}\{\infty\}}, \boldsymbol{D}^+_{\mathrm{dif}}(\mathcal{E}))$ is an equivalence of categories.*

Proof. There is an obvious functor from the category of locally analytic $\boldsymbol{B}$-pairs to the category of $\boldsymbol{B}$-pairs. This leads to a commutative diagram

$$\begin{array}{ccc} \{\text{locally analytic vector bundles}\} & \longrightarrow & \{\text{locally analytic } \boldsymbol{B}\text{-pairs}\} \\ \downarrow \cong & & \downarrow \\ \{\Gamma\text{-vector bundles}\} & \xrightarrow{\ \cong\ } & \{\boldsymbol{B}\text{-pairs}\} \end{array}$$

The left vertical arrow is an equivalence by Theorem 6.1. The lower horizontal arrow is also an equivalence, as explained in [Fargues and Fontaine 2018, §10.1.2]. It follows that the functor from locally analytic $\boldsymbol{B}$-pairs to $\boldsymbol{B}$-pairs is essentially surjective, so every $\boldsymbol{B}$-pair comes from a locally analytic $\boldsymbol{B}$-pair by extending scalars. It now follows from Proposition 2.1 that such a locally analytic $\boldsymbol{B}$-pair is unique. This allows us to define a functor from $\boldsymbol{B}$-pairs to locally analytic $\boldsymbol{B}$-pairs, which gives a quasi-inverse to right vertical morphism. It therefore has to be an equivalence. By commutativity of the diagram, the upper horizontal arrow is also an equivalence, as required. □

Definition 8.3. Given two locally analytic vector bundles $\mathcal{E}_1$ and $\mathcal{E}_2$ we say that $\mathcal{E}_2$ is a modification of $\mathcal{E}_1$ if $\mathcal{E}_1|_{\mathcal{X}\text{-}\{\infty\}} \cong \mathcal{E}_2|_{\mathcal{X}\text{-}\{\infty\}}$.

Note that in particular any Γ-stable $\boldsymbol{B}^{+,\mathrm{pa}}_{\mathrm{dR}}$-lattice $N \subset \boldsymbol{D}_{\mathrm{dif}}(\mathcal{E})$ defines a modification of $\mathcal{E}$ by taking the pair $(\mathcal{E}|_{\mathcal{X}\text{-}\{\infty\}}, N)$.

Remark 8.4. We could have also defined this notion of modification in terms of usual $\boldsymbol{B}$-pairs. Our choice of presentation is meant to illustrate that one can speak of modifications without leaving the locally analytic realm.

8B. ***de Rham and $\mathbb{C}_p$-admissible locally analytic vector bundles.*** Let $\mathcal{E}$ be a locally analytic vector bundle. We say that:

- $\mathcal{E}$ is $\mathbb{C}_p$-admissible if $\dim_K \mathcal{E}_{x_\infty}^{\Gamma=1} = \mathrm{rank}(\mathcal{E})$.
- $\mathcal{E}$ is de Rham if $\boldsymbol{D}_{\mathrm{dR}}(\mathcal{E}) := \dim_K \hat{\mathcal{E}}_{x_\infty}^{\Gamma=1} = \mathrm{rank}(\mathcal{E})$.

If V is a p-adic representation and $\mathcal{E} = \tilde{\mathcal{E}}(V)^{\mathrm{la}}$ then $\mathcal{E}_{x_\infty}^{\Gamma=1} = (\mathbb{C}_p \otimes V)^{G_K}$ and $\boldsymbol{D}_{\mathrm{dR}}(\mathcal{E}) = \boldsymbol{D}_{\mathrm{dR}}(V)$, so this extends the usual definitions.

In what follows, note that $\boldsymbol{D}_{\mathrm{dR}}(\mathcal{E})$ has a natural filtration induced from the I_θ filtration on $\hat{\mathcal{E}}_{x_\infty}$.

Definition 8.5. Suppose $\mathcal{E}$ is de Rham.

(1) $\mathcal{N}_{\mathrm{dR}}(\mathcal{E})$ is the modification of $\mathcal{E}$ given by the lattice $\boldsymbol{D}_{\mathrm{dR}}(\mathcal{E}) \otimes_K \boldsymbol{B}_{\mathrm{dR}}^{+,\mathrm{pa}} \subset \boldsymbol{D}_{\mathrm{dif}}(\mathcal{E})$. It is $\mathbb{C}_p$-admissible.

(2) $\mathcal{M}_{\mathrm{dR}}(\mathcal{E})$ is the locally analytic φ-vector bundle corresponding to $\mathcal{N}_{\mathrm{dR}}(\mathcal{E})$.

8C. ***The surfaces $\mathcal{Y}_{\log,L}$ and $\mathcal{X}_{\log,L}$.*** Fargues and Fontaine [2018, §10.3.3] define a scheme $X_{\log}$. It is a line bundle over the schematic Fargues–Fontaine curve $X_{\mathrm{FF}} = X_{\mathrm{FF}}(\mathbb{C}_p)$ with a natural projection $\pi : X_{\log} \to X$; further, it has a G_K-action and π is G_K-equivariant.

We let $\mathcal{X}_{\log}$ be the analytification of $X_{\log}$. If L is a finite extension of K, we set

$$\mathcal{X}_{\log,L} := \mathcal{X}_{\log}/\mathrm{Gal}(\overline{K}/L_\infty).$$

(Alternatively, this can be defined as the analytification of the quotient of $X_{\log}$ by $\mathrm{Gal}(\overline{K}/L_\infty)$). Similarly, write $\mathcal{Y}_{\log} = \mathcal{Y}_{(0,\infty)} \times_{\mathcal{X}} \mathcal{X}_{\log}$ and $\mathcal{Y}_{\log,L} = \mathcal{Y}_{\log}/\mathrm{Gal}(\overline{K}/L_\infty)$; then $\mathcal{Y}_{\log,L}/\varphi = \mathcal{X}_{\log,L}$. These spaces have an action of $\mathrm{Gal}(L_\infty/L)$, an open subgroup of Γ.

Write p_L (resp. $p_{\log,L}$) for the projection maps $\mathcal{Y}_L \to \mathcal{Y}$ or $\mathcal{X}_L \to \mathcal{X}$ (resp. $\mathcal{Y}_{\log,L} \to \mathcal{Y}$ or $\mathcal{X}_{\log,L} \to \mathcal{X}$). If $I \subset (0,\infty)$ is closed interval, let $\mathcal{Y}_{\log,L,I} = p_{\log,L}^{-1}(\mathcal{Y}_I)$ and similarly $\mathcal{X}_{\log,L,I} = p_{\log,L}^{-1}(\mathcal{X}_I)$ for $\mathcal{X}$ if I is sufficiently small.

Define

$$\widetilde{\boldsymbol{B}}_{\log,L,I} = \mathrm{H}^0(\mathcal{Y}_{\log,L,I}, \mathcal{O}_{\mathcal{Y}_{\log,L}}).$$

As explained in [loc. cit.], there is a natural G_K-equivariant morphism of sheaves

$$d : \mathcal{O}_{X_{\log}} \to \Omega^1_{X_{\log}/X} \cong p_{\log}^* \mathcal{O}_X(-1)$$

which for every vector bundle $\mathcal{E}$ over $\mathcal{X}$ induces an $\mathcal{O}_{\mathcal{X}}$-linear morphism

$$N : p_{\log}^* \mathcal{E} \to p_{\log}^* \mathcal{E} \otimes \Omega^1_{\mathcal{X}_{\log}/\mathcal{X}}.$$

See [Fargues and Fontaine 2018, Lemma 10.3.9] and the subsequent discussion. Similarly, N can be pulled back to $\mathcal{Y}_{\log}$. This then further induces a $\widetilde{\boldsymbol{B}}_{L,I}$-linear differential operator $N : \widetilde{\boldsymbol{B}}_{\log,L,I} \to \widetilde{\boldsymbol{B}}_{\log,L,I}$. If $T \in \widetilde{\boldsymbol{B}}_{\log,L,I}$ is such that $N(T) = 1$ then $\widetilde{\boldsymbol{B}}_{\log,L,I} = \widetilde{\boldsymbol{B}}_{L,I}[T]$ and $N = d/dT$. Such a T exists: if ϖ is any nonunit $\varpi \in \hat{L}_\infty^\times$ and $\varpi^\flat = (\varpi, \varpi^{1/p}, \ldots)$, take $T = \log[\varpi^\flat]$.

Lemma 8.6. *There exists $T \in \widetilde{\boldsymbol{B}}_{\log,L,I}^{\mathrm{la}}$ with $N(T) = 1$. Consequently, $\widetilde{\boldsymbol{B}}_{\log,L,I}^{\mathrm{la}} = \widetilde{\boldsymbol{B}}_{L,I}^{\mathrm{la}}[T]$.*

Proof. The second claim follows the first claim, Proposition 2.1 and the fact that taking locally analytic vectors commutes with filtered colimits. To find such an element T, consider the exact sequence

$$0 \to \widetilde{\boldsymbol{B}}_{L,I} \to \widetilde{\boldsymbol{B}}^{N^2=0}_{\log,L,I} \xrightarrow{N} \widetilde{\boldsymbol{B}}_{L,I} \to 0.$$

After taking locally analytic vectors the sequence stays exact by Theorem 5.1. Thus the sequence

$$0 \to \widetilde{\boldsymbol{B}}^{\mathrm{la}}_{L,I} \to \widetilde{\boldsymbol{B}}^{\mathrm{la},N^2=0}_{\log,L,I} \xrightarrow{N} \widetilde{\boldsymbol{B}}^{\mathrm{la}}_{L,I} \to 0$$

is exact. This means we can lift 1 to an element T with $N(T)=1$, as required. □

Proposition 8.7. *Suppose $\varphi^{\mathbb{Z}}(x_\infty)\cap\mathcal{Y}_I \neq \varnothing$. Then*

(i) *If M is a finite extension of L contained in L_∞, then $\widetilde{\boldsymbol{B}}^{\mathrm{Gal}(L_\infty/M)}_{\log,L,I} = M_0$, where M_0 is the maximal unramified extension of $\mathbb{Q}_p$ contained in M.*

(ii) $\widetilde{\boldsymbol{B}}^{\mathrm{la},\mathrm{Lie}\,\Gamma=0}_{\log,L,I} = L_0'$, *the maximal unramified extension of $\mathbb{Q}_p$ contained in L_∞.*

Proof. Point (i) follows from [Fargues and Fontaine 2018, proposition 10.3.15] and (ii) follows from (i). □

One way to construct de Rham locally analytic vector bundles is as follows. Write $\mathrm{Mod}^{\mathrm{Fil},\varphi,N}_{\mathbb{Q}_p^{\mathrm{un}}}(G_K)$ for the category of finite dimensional vector spaces D over $\mathbb{Q}_p^{\mathrm{un}}$ together with a semilinear action of φ, a monodromy operator N with $\varphi N = pN\varphi$, a filtration on $D\otimes_{\mathbb{Q}_p^{\mathrm{un}}} K^{\mathrm{un}}$ and a discrete action of G_K on D which respects the filtration. For example, if V is a potentially semistable representation then $\boldsymbol{D}_{\mathrm{pst}}(V)$ is an object of $\mathrm{Mod}^{\mathrm{Fil},\varphi}_{\mathbb{Q}_p^{\mathrm{un}}}(G_K)$.

There is a functor

$$\mathcal{E} : \mathrm{Mod}^{\mathrm{Fil},\varphi}_{\mathbb{Q}_p^{\mathrm{un}}}(G_K) \to \{\text{de Rham locally analytic vector bundles}\}$$

defined as follows: Given $D\in\mathrm{Mod}^{\mathrm{Fil},\varphi}_{\mathbb{Q}_p^{\mathrm{un}}}(G_K)$, choose L such that D is defined over L, i.e., $D=\mathbb{Q}_p^{\mathrm{un}}\otimes_{L_0} D_0$. Such an L exists because the action of G_K is discrete. Then $\mathcal{E}(D)$ is defined to be the locally analytic vector bundle corresponding to the pair

$$\big((\mathcal{O}^{\mathrm{la}}_{\mathcal{Y}_{\log,L}-p^{-1}_{\log,L}(\infty)} \otimes_{L_0} D)^{\varphi=1,N=0,\mathrm{Gal}(L_\infty/K_\infty)}, \mathrm{Fil}^0(\boldsymbol{B}^{H_L,\mathrm{pa}}_{\mathrm{dR}} \otimes_{L_0} D_0)^{\mathrm{Gal}(L_\infty/K_\infty)}\big).$$

It is de Rham because

$$D \subset \boldsymbol{B}^{H_K,\mathrm{pa}}_{\mathrm{dR}} \otimes \mathrm{Fil}^0(\boldsymbol{B}^{H_L,\mathrm{pa}}_{\mathrm{dR}} \otimes_{L_0} D_0)^{\mathrm{Gal}(L_\infty/K_\infty)}$$

is fixed by an open subgroup of Γ. If we choose any larger L we get the same pair, so the construction $D\mapsto\mathcal{E}(D)$ is independent of the choice of L.

8D. *Sheaves of smooth functions.* In this subsection we introduce certain sheaves of functions on $\mathcal{X}$. All of these can be defined equally well for $\mathcal{Y}_{(0,\infty)}$.

Definition 8.8. We define the following sheaves of functions on $\mathcal{X}$.

(i) Smooth functions: $\mathcal{O}^{\mathrm{sm}}_{\mathcal{X}} = \mathcal{O}^{\mathrm{la},\mathrm{Lie}\,\Gamma=0}_{\mathcal{X}}$.

(ii) For $[L:K]<\infty$, L-smooth functions: $\mathcal{O}_{\mathcal{X}}^{L\text{-sm}} = p_{L,*}(p_L^*\mathcal{O}_{\mathcal{X}}^{\mathrm{la}})^{\mathrm{Lie}\,\Gamma=0}$.

(iii) For $[L:K]<\infty$, L log-smooth functions: $\mathcal{O}_{\mathcal{X}}^{L\text{-lsm}} = p_{\log,L,*}(p_{\log,L}^*\mathcal{O}_{\mathcal{X}}^{\mathrm{la}})^{\mathrm{Lie}\,\Gamma=0}$.

(iv) Potentially smooth functions: $\mathcal{O}_{\mathcal{X}}^{\mathrm{psm}} = \varinjlim_{[L:K]<\infty} \mathcal{O}_{\mathcal{X}}^{L\text{-sm}}$.

(v) Potentially log-smooth functions: $\mathcal{O}_{\mathcal{X}}^{\mathrm{plsm}} = \varinjlim_{[L:K]<\infty} \mathcal{O}_{\mathcal{X}}^{L\text{-lsm}}$.

The following proposition has been essentially explained to us by Kedlaya.

Proposition 8.9. *Let U be a connected open affinoid subset of $\mathcal{X}$.*

(i) *The sections of each of $\mathcal{O}_{\mathcal{X}}^{\mathrm{sm}}$, $\mathcal{O}_{\mathcal{X}}^{L\text{-sm}}$ and $\mathcal{O}_{\mathcal{X}}^{\mathrm{psm}}$ at U is a field which injects (noncanonically) into $\mathbb{C}_p$.*

(ii) *If $x_\infty \in U$ then there are canonical injections*

$$\mathrm{H}^0(U,\mathcal{O}_{\mathcal{X}}^{\mathrm{sm}}) \hookrightarrow K_\infty, \quad \mathrm{H}^0(U,\mathcal{O}_{\mathcal{X}}^{L\text{-sm}}) \hookrightarrow L_\infty \quad \text{and} \quad \mathrm{H}^0(U,\mathcal{O}_{\mathcal{X}}^{\mathrm{psm}}) \hookrightarrow \overline{K}.$$

(iii) *If $x_\infty \in U$ and $U = \mathcal{X}_I$, we have*

$$\mathrm{H}^0(\mathcal{X}_I,\mathcal{O}_{\mathcal{X}}^{\mathrm{sm}}) = K_0', \quad \mathrm{H}^0(\mathcal{X}_I,\mathcal{O}_{\mathcal{X}}^{L\text{-sm}}) = L_0' \quad \text{and} \quad \mathrm{H}^0(\mathcal{X}_I,\mathcal{O}_{\mathcal{X}}^{\mathrm{psm}}) = K_0^{\mathrm{un}}.$$

(iv) *We have* $\quad \mathcal{O}_{\mathcal{X},x_\infty}^{\mathrm{sm}} = K_\infty, \quad \mathcal{O}_{\mathcal{X},x_\infty}^{L\text{-sm}} = L_\infty \quad$ *and* $\quad \mathcal{O}_{\mathcal{X},x_\infty}^{\mathrm{psm}} = \overline{K}.$

Proof. Each of the assertions (i)–(iv) for $\mathcal{O}_{\mathcal{X}}^{\mathrm{psm}}$ follows from the corresponding assertion for $\mathcal{O}_{\mathcal{X}}^{L\text{-sm}}$. We shall give below arguments proving (i)–(iv) for $\mathcal{O}_{\mathcal{X}}^{\mathrm{sm}}$; the proofs for $\mathcal{O}_{\mathcal{X}}^{L\text{-sm}}$ are the same once K is replaced by L.

After passing to an open subgroup of Γ, we may assume Γ stabilizes U. By [Kedlaya 2016, Theorem 8.8], the ring $\mathcal{O}_{\mathcal{X}}(U)$ is a Dedekind domain. Each rank 1 point x of U defines a maximal ideal of $\mathcal{O}_{\mathcal{X}}(U)$, so $f \in \mathcal{O}_{\mathcal{X}}(U)$ can belong to only finitely many of these points. If $f \in \mathcal{O}_{\mathcal{X}}(U)$ is killed by $\mathrm{Lie}\,\Gamma$ then f is fixed by a finite subgroup of Γ, so these finitely many maximal ideals must form a finite orbit under the Γ-action. But the only rank 1 point with finite orbit is the point x_∞, again by [Fargues and Fontaine 2018, proposition 10.1.1]. So every $f \in \mathcal{O}_{\mathcal{X}}^{\mathrm{sm}}(U)$ either vanishes only at x_∞ or is invertible.

If $x_\infty \notin U$, this proves that $\mathcal{O}_{\mathcal{X}}^{\mathrm{sm}}(U)$ is a field. In particular, it injects into the residue field of each rank 1 point, and there is a dense subset of $\mathcal{X}$ with residue field a subfield of $\mathbb{C}_p$. This proves (i) in this case. On the other hand, if $x_\infty \in U$ then there is a Γ-equivariant embedding of $\mathcal{O}_{\mathcal{X}}^{\mathrm{la}}(U)$ into $\boldsymbol{B}_{\mathrm{dR}}^+(\widehat{K}_\infty)^{\mathrm{la}}$ which gives an embedding of $\mathcal{O}_{\mathcal{X}}^{\mathrm{sm}}(U)$ into $\boldsymbol{B}_{\mathrm{dR}}^+(\widehat{K}_\infty)^{\mathrm{la},\mathrm{Lie}\,\Gamma=0} = K_\infty$. This simultaneously proves (i) and (ii) for $\mathcal{O}_{\mathcal{X}}^{\mathrm{sm}}$.

Next, (iii) follows immediately from Proposition 8.7. For (iv), we have already shown that $\mathcal{O}_{\mathcal{X}}^{\mathrm{sm}}(U) \subset K_\infty$ for each U which contains x_∞, so $\mathcal{O}_{\mathcal{X},x_\infty}^{\mathrm{sm}} \subset K_\infty$. To show the converse inclusion, use the henselian property of local rings of adic spaces [Morel 2019, III.6.3.7] to show first that $K_\infty \subset \mathcal{O}_{\mathcal{X},x_\infty}$. It then follows that $K_\infty \subset \mathcal{O}_{\mathcal{X},x_\infty}^{\mathrm{sm}}$, which concludes the proof. □

We raise a few questions to which we expect a positive answer but have not answered in this article.

Question 8.10. (1) *We can show that $\overline{K} \subset \mathcal{O}_{\mathcal{X}}^{\mathrm{psm}}$ if x is any rank 1 point. Indeed, any untilt of $\mathbb{C}_p^\flat$ is algebraically closed, and one can use this to show that the completed local rings $\boldsymbol{B}_{\mathrm{dR},x}^+$ contain $\overline{K}$. This implies by the same argument that $\overline{K} \subset \mathcal{O}_{\mathcal{X},x}$. But every element of $\overline{K}$ has finite degree over K_0, which is fixed by G_K. This implies that every $x \in \overline{K}$ is fixed by an open subgroup G_K so $\overline{K} \subset \mathcal{O}_{\mathcal{X},x}^{\mathrm{psm}}$.*

Is it true that $\overline{K} = \mathcal{O}_{\mathcal{X},x}^{\mathrm{psm}}$ for any rank 1 point x?

(2) *Is it true that for every connected open affinoid $U \subset \mathcal{X}$, the field $\mathcal{O}_{\mathcal{X}}^{\mathrm{psm}}(U)$ is a finite extension of K_0^{un}? In particular, this would imply a positive answer to question* (1).

(3) *Is it true that $\mathcal{O}_{\mathcal{X}}^{L\text{-sm}} = \mathcal{O}_{\mathcal{X}}^{L\text{-lsm}}$ (and hence $\mathcal{O}_{\mathcal{X}}^{\mathrm{psm}} = \mathcal{O}_{\mathcal{X}}^{\mathrm{plsm}}$)? If $x_\infty \in U$ then $\mathcal{O}_{\mathcal{X}}^{L\text{-sm}}(U) = \mathcal{O}_{\mathcal{X}}^{L\text{-lsm}}(U)$. This can be seen by using the embedding into $\boldsymbol{B}_{\mathrm{dR}}^{+}$ as in the proof of Proposition 8.7.*

8E. ***The solution functor.*** In this subsection, we assume $\mathcal{E}$ is a de Rham locally analytic vector bundle. Given L finite over K, we define the sheaves of solutions on $\mathcal{X}$,

(1) $\mathrm{Sol}_L(\mathcal{E}) := p_{L,*}(p_L^* \mathcal{N}_{\mathrm{dR}}(\mathcal{E}))^{\mathrm{Lie}\,\Gamma=0}$, a module over $\mathcal{O}_{\mathcal{X}}^{L\text{-sm}}$,

(2) $\mathrm{Sol}_{\log,L}(\mathcal{E}) := p_{\log,L,*}(p_{\log,L}^* \mathcal{N}_{\mathrm{dR}}(\mathcal{E}))^{\mathrm{Lie}\,\Gamma=0}$, a module over $\mathcal{O}_{\mathcal{X}}^{L\text{-lsm}}$,

(3) $\mathrm{Sol}(\mathcal{E}) := \varinjlim_{[L:K]<\infty} \mathrm{Sol}_{\log,L}(\mathcal{E})$, a module over $\mathcal{O}_{\mathcal{X}}^{\mathrm{plsm}}$.

We have similar versions of these sheaves on $\mathcal{Y}_{(0,\infty)}$, denoted by $\mathrm{Sol}_*^{\varphi}(\mathcal{E})$ for $* \in \{L, \{\log, L\}, \varnothing\}$. Since the φ action on $\mathcal{Y}_{\log,L}$ is Γ-equivariant, there are natural identifications $\mathrm{Sol}_*(\mathcal{E}) = (\mathrm{Sol}_*^{\varphi}(\mathcal{E}))^{\varphi=1}$ and $\mathrm{Sol}_*^{\varphi}(\mathcal{E}) \cong \mathcal{O}_{\mathcal{Y}_{(0,\infty)}}^{\bullet} \otimes_{\mathcal{O}_{\mathcal{X}}^{\bullet}} \mathrm{Sol}_*(\mathcal{E})$, where $(*, \bullet) = \{(L, L\text{-sm}), (\{\log, L\}, L\text{-lsm}), (\varnothing, \mathrm{plsm})\}$.

To make the link with $\mathcal{E}$ clear, we shall need the following form of the p-adic monodromy theorem due to André [2002], Kedlaya [2004] and Mebkhout [2002].

Proposition 8.11. *There exists a finite extension L over K such that if U is an open subset of $\mathcal{Y}_{[r,\infty)}$ for some $r \gg 0$ then the natural map*

$$\mathcal{O}_{\mathcal{Y}_{\log,L}}^{\mathrm{la}}(p_{\log,L}^{-1}U) \otimes_{\mathcal{O}_{\mathcal{Y}_{(0,\infty)}}^{L\text{-lsm}}(U)} \mathrm{Sol}_{\log,L}^{\varphi}(\mathcal{E})(U) \to \mathcal{O}_{\mathcal{Y}_{\log,L}}^{\mathrm{la}}(p_{\log,L}^{-1}U) \otimes_{\mathcal{O}_{\mathcal{Y}_{(0,\infty)}}^{\mathrm{la}}(U)} \mathcal{M}_{\mathrm{dR}}(\mathcal{E})(U)$$

is an isomorphism. Consequently, if $U \subset \mathcal{X}_I$ for some I then

$$\mathcal{O}_{\mathcal{X}_{\log,L}}^{\mathrm{la}}(p_{\log,L}^{-1}U) \otimes_{\mathcal{O}_{\mathcal{X}}^{L\text{-lsm}}(U)} \mathrm{Sol}_{\log,L}(\mathcal{E})(U) \xrightarrow{\sim} \mathcal{O}_{\mathcal{X}_{\log,L}}^{\mathrm{la}}(p_{\log,L}^{-1}U) \otimes_{\mathcal{O}_{\mathcal{X}}^{\mathrm{la}}(U)} \mathcal{N}_{\mathrm{dR}}(\mathcal{E})(U).$$

Proof. Let $\widetilde{\boldsymbol{D}}_{\mathrm{rig}}^{\dagger}$ be the (φ, Γ)-module corresponding to $\mathcal{M}_{\mathrm{dR}}(\mathcal{E})$. By the p-adic monodromy theorem, we know there is an isomorphism

$$\widetilde{\boldsymbol{B}}_{\log,L}^{\dagger,\mathrm{pa}} \otimes_{L_0'} (\widetilde{\boldsymbol{B}}_{\log,L}^{\dagger,\mathrm{pa}} \otimes_{\widetilde{\boldsymbol{B}}_{\mathrm{rig},K}^{\dagger,\mathrm{pa}}} \widetilde{\boldsymbol{D}}_{\mathrm{rig}}^{\dagger,\mathrm{pa}})^{\mathrm{Lie}\,\Gamma=0} \xrightarrow{\sim} \widetilde{\boldsymbol{B}}_{\log,L}^{\dagger,\mathrm{pa}} \otimes_{\widetilde{\boldsymbol{B}}_{\mathrm{rig},K}^{\dagger,\mathrm{pa}}} \widetilde{\boldsymbol{D}}_{\mathrm{rig}}^{\dagger,\mathrm{pa}}$$

in the cyclotomic setting (see [Berger 2008b, III.2.1]). More generally, we may descend along unramified extensions to give it in the twisted cyclotomic case, and by base changing we get it in our setting as well by the usual argument.

It follows that for $r \gg 0$ we also have an isomorphism

$$\widetilde{\boldsymbol{B}}_{\log,[r,\infty),L}^{\mathrm{pa}} \otimes_{L_0'} (\widetilde{\boldsymbol{B}}_{\log,[r,\infty),L}^{\mathrm{pa}} \otimes_{\widetilde{\boldsymbol{B}}_{[r,\infty),K}^{\mathrm{pa}}} \widetilde{\boldsymbol{D}}_{[r,\infty)}^{\mathrm{pa}})^{\mathrm{Lie}\,\Gamma=0} \xrightarrow{\sim} \widetilde{\boldsymbol{B}}_{\log,[r,\infty),L}^{\mathrm{pa}} \otimes_{\widetilde{\boldsymbol{B}}_{[r,\infty),K}^{\mathrm{pa}}} \widetilde{\boldsymbol{D}}_{[r,\infty)}^{\mathrm{pa}}.$$

Pulling back along Frobenius, we obtain this isomorphism for any r. Then by finding $r \gg 0$ so that $U \subset \mathcal{Y}_{[r,\infty)}$, we can base change the isomorphism along the map $\widetilde{\boldsymbol{B}}_{\log,[r,\infty),L}^{\mathrm{pa}} \to \mathcal{O}_{\mathcal{Y}_{\log,L}}^{\mathrm{la}}(p_{\log,L}^{-1}U)$ to conclude. ☐

Note that whether we need to adjoin log and/or perform a finite extension L of K depends exactly on whether $\mathcal{E}$ becomes crystalline or semistable after restricting G_K to G_L. Applying this observation and taking $\operatorname{Lie}\Gamma = 0$ of both sides of the proposition, we obtain the following.

Theorem 8.12. *The sheaf* $\mathrm{Sol}(\mathcal{E})$ *is a locally free* $\mathcal{O}_{\mathcal{X}}^{\mathrm{plsm}}$*-module of rank equal to* $\operatorname{rank}(\mathcal{E})$. *More precisely*:

(i) *If $\mathcal{E}$ becomes crystalline after restricting G_K to $G_{L'}$ for some $L \subset L' \subset L_\infty$ then* $\mathrm{Sol}_L(\mathcal{E})$ *is a locally free* $\mathcal{O}_{\mathcal{X}}^{L\text{-sm}}$*-module of rank equal to* $\operatorname{rank}(\mathcal{E})$, *and there is a natural isomorphism*

$$\mathcal{O}_{\mathcal{X}_L}^{\mathrm{la}} \otimes_{\mathcal{O}_{\mathcal{X}}^{L\text{-sm}}} \mathrm{Sol}_L(\mathcal{E}) \xrightarrow{\sim} \mathcal{O}_{\mathcal{X}_L}^{\mathrm{la}} \otimes_{\mathcal{O}_{\mathcal{X}}^{\mathrm{la}}} \mathcal{N}_{\mathrm{dR}}(\mathcal{E}).$$

(ii) *If $\mathcal{E}$ becomes semistable after restricting G_K to $G_{L'}$ for some $L \subset L' \subset L_\infty$ then* $\mathrm{Sol}_{\log,L}(\mathcal{E})$ *is a locally free* $\mathcal{O}_{\mathcal{X}}^{L\text{-lsm}}$*-module of rank equal to* $\operatorname{rank}(\mathcal{E})$, *and there is a natural isomorphism*

$$\mathcal{O}_{\mathcal{X}_{\log,L}}^{\mathrm{la}} \otimes_{\mathcal{O}_{\mathcal{X}}^{L\text{-lsm}}} \mathrm{Sol}_{\log,L}(\mathcal{E}) \xrightarrow{\sim} \mathcal{O}_{\mathcal{X}_{\log,L}}^{\mathrm{la}} \otimes_{\mathcal{O}_{\mathcal{X}}^{\mathrm{la}}} \mathcal{N}_{\mathrm{dR}}(\mathcal{E}).$$

Lemma 8.13. *For each sufficiently small open connected affinoid U of $\mathcal{Y}_{(0,\infty)}$ which contains an element of $\varphi^{\mathbb{Z}}(x_\infty)$, and for L large enough so that G_L stabilizes U, there is a natural G_L-embedding* $\mathrm{H}^0(U, \mathrm{Sol}_{\log,L}^{\varphi}(\mathcal{E})) \hookrightarrow L_\infty \otimes_K \boldsymbol{D}_{\mathrm{dR}}(\mathcal{E})$.

Proof. Taking the completed stalk at a φ-translate of x_∞, we obtain an injection

$$\mathcal{O}_{\mathcal{Y}_{\log,L}}^{\mathrm{la}}(p_{\log,L}^{-1}U) \otimes_{\mathcal{O}_{\mathcal{Y}_{(0,\infty)}}^{\mathrm{la}}(U)} \mathcal{M}_{\mathrm{dR}}(\mathcal{E})(U) \hookrightarrow \hat{L}_\infty^{\mathrm{la}} \otimes_{\widehat{K}_\infty^{\mathrm{la}}} \boldsymbol{D}_{\mathrm{dif}}(\mathcal{E}).$$

On the other hand, Proposition 8.7 gives an isomorphism

$$\mathcal{O}_{\mathcal{Y}_{\log,L}}^{\mathrm{la}}(p_{\log,L}^{-1}U) \otimes_{\mathcal{O}_{\mathcal{Y}_{(0,\infty)}}^{L\text{-lsm}}(U)} \mathrm{Sol}_{\log,L}^{\varphi}(\mathcal{E})(U) \xrightarrow{\sim} \mathcal{O}_{\mathcal{Y}_{\log,L}}^{\mathrm{la}}(p_{\log,L}^{-1}U) \otimes_{\mathcal{O}_{\mathcal{Y}_{(0,\infty)}}^{\mathrm{la}}(U)} \mathcal{M}_{\mathrm{dR}}(\mathcal{E})(U).$$

Applying $\operatorname{Lie}\Gamma = 0$ to the composition of these maps gives the desired embedding. □

We can now give an interpretation of the stalk at x_∞:

Proposition 8.14. *There following are each naturally isomorphic to each other.*

(1) *The stalk* $\mathrm{Sol}(\mathcal{E})_{x_\infty}$.

(2) *The stalk* $\mathrm{Sol}(\mathcal{E})_y^{\varphi}$ *for any* $y \in \varphi^{\mathbb{Z}}(x_\infty)$.

(3) $\overline{K} \otimes_K \boldsymbol{D}_{\mathrm{dR}}(\mathcal{E})$.

In particular, $\mathrm{Sol}(\mathcal{E})_{x_\infty}$ *is naturally a filtered $\overline{K}$-representation of G_K of dimension* $\operatorname{rank}(\mathcal{E})$ *and G_K-fixed points* $\boldsymbol{D}_{\mathrm{dR}}(\mathcal{E})$.

Proof. It is clear (1) and (2) are isomorphic. By Lemma 8.13, we have a natural embedding of $\mathrm{Sol}(\mathcal{E})_y$, and hence of $\mathrm{Sol}(\mathcal{E})_{x_\infty}$ into $\overline{K} \otimes_K \boldsymbol{D}_{\mathrm{dR}}(\mathcal{E})$. By Theorem 8.12, $\mathrm{Sol}(\mathcal{E})_{x_\infty}$ is a finite free module of rank equal to $\dim_K \boldsymbol{D}_{\mathrm{dR}}(\mathcal{E})$ over $\mathcal{O}_{\mathcal{X},x_\infty}^{\mathrm{plsm}}$. But by Proposition 8.7 $\mathcal{O}_{\mathcal{X},x_\infty}^{\mathrm{plsm}} = \overline{K}$, so this embedding must be an isomorphism. □

Finally, we consider the global solutions to the differential equation, namely

$$D(\mathcal{E}) = \mathrm{H}^0(\mathcal{Y}_{(0,\infty)}, \mathrm{Sol}^{\varphi}(\mathcal{E})) = \mathrm{H}^0\big(\mathcal{Y}_{(0,\infty)}, \mathcal{O}_{\mathcal{Y}_{(0,\infty)}}^{\mathrm{plsm}} \otimes_{\mathcal{O}_{\mathcal{X}}^{\mathrm{plsm}}} \mathrm{Sol}(\mathcal{E})\big).$$

Proposition 8.15. *$D(\mathcal{E})$ is naturally an object of $\mathrm{Mod}^{\mathrm{Fil},\varphi,N}_{\mathbb{Q}_p^{\mathrm{un}}}(G_K)$ and $\dim_{\mathbb{Q}_p^{\mathrm{un}}} D(\mathcal{E}) = \mathrm{rank}(\mathcal{E})$.*

Proof. We know each $\mathrm{H}^0(\mathcal{Y}_{(0,\infty)}, \mathrm{Sol}^{\varphi}_{\log,L}(\mathcal{E}))$ is an L'_0 vector space for U sufficiently small (independent of L), so $D(\mathcal{E})$ is a $\mathbb{Q}_p^{\mathrm{un}}$-vector space. The filtration is induced from the embedding

$$\mathrm{H}^0(\mathcal{Y}_{(0,\infty)}, \mathrm{Sol}^{\varphi}(\mathcal{E})) \hookrightarrow \mathrm{Sol}(\mathcal{E})_{x_\infty} \cong \overline{K} \otimes_K \boldsymbol{D}_{\mathrm{dR}}(\mathcal{E}).$$

The φ-action is induced from the map $\varphi : \mathcal{Y}_{(0,\infty)} \to \mathcal{Y}_{(0,\infty)}$. The monodromy operator N is induced from the equivariant connection $p^*_{\log,L}\mathcal{M}_{\mathrm{dR}}(\mathcal{E}) \to p^*_{\log,L}\mathcal{N}_{\mathrm{dR}}(\mathcal{E}) \otimes \Omega^1_{\mathcal{Y}_{\log}/\mathcal{Y}_{(0,\infty)}}$. Finally, G_K acts on the smooth elements in $p^*_{\log,L}\mathcal{M}_{\mathrm{dR}}(\mathcal{E})$, and this action is discrete because every element is killed by $\mathrm{Lie}\,\Gamma$, hence by an open subgroup of $\mathrm{Gal}(L_\infty/L)$. To compute the dimension use Theorem 8.12. □

Using this language, Berger's theorem [2008b, théoréme III.2.4] admits the following interpretation.

Theorem 8.16. *The functors $D \mapsto \mathcal{E}(D)$ and $\mathcal{E} \mapsto D(\mathcal{E})$ are mutual inverses and induce an equivalence of categories*

$$\mathrm{Mod}^{\mathrm{Fil},\varphi,N}_{\mathbb{Q}_p^{\mathrm{un}}}(G_K) \cong \{\textit{de Rham locally analytic vector bundles}\}.$$

Remark 8.17. If $\mathcal{E}$ is the locally analytic vector bundle associated to a p-adic representation V, we see that the global-to-local map

$$\mathrm{H}^0(\mathcal{Y}_{(0,\infty)}, \mathrm{Sol}^{\varphi}(\mathcal{E})) \hookrightarrow \mathrm{Sol}(\mathcal{E})_{x_\infty}$$

is nothing but the more familiar map

$$\boldsymbol{D}_{\mathrm{pst}}(V) \hookrightarrow \overline{K} \otimes_K \boldsymbol{D}_{\mathrm{dR}}(V).$$

Question 8.18. *Theorem 8.16 allows us to consider objects of $\mathrm{Mod}^{\mathrm{Fil},\varphi,N}_{\mathbb{Q}_p^{\mathrm{un}}}(G_K)$ as global solutions to p-adic differential equations. The filtration is coming from the behavior of orders of vanishing at $x_\infty = 0$, while the (φ, N, G_K)-structure comes from some sort of monodromy of the map $\varprojlim_L \mathcal{Y}_{\log,L} \to \mathcal{X}$. In our description the space $\varprojlim_L \mathcal{Y}_{\log,L}$ behaves as a substitute for a universal cover of $\mathcal{X}$. It would be interesting if it can be replaced by a more literal cover of $\mathcal{X}$ for which the (φ, N, G_K)-actions can be interpreted as monodromy actions. One could even speculate that in an appropriate sense, the analytic fundamental group of $\mathcal{X}(\mathbb{C}_p)_{\overline{K}}$ should be a tame Weil group with its two dimensions reflecting the φ and N operators.*

We conclude with an example.

Example 8.19. Take $\alpha \in \mathbb{Z}_p^\times$, and given $g \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ let $\xi_\alpha(g) \in \mathbb{Z}_p$ be the element such that $\zeta_{p^n}^{\xi_\alpha(g)} = g(\alpha^{1/p^n})/\alpha^{1/p^n}$ for each $n \geq 1$. The Kummer extension

$$0 \to \mathbb{Q}_p(\chi_{\mathrm{cyc}}) \to V = V_\alpha \to \mathbb{Q}_p \to 0$$

is given by mapping in a basis e, f the element g to the matrix

$$\begin{pmatrix} \chi_{\mathrm{cyc}}(g) & \xi_\alpha(g) \\ 0 & 1 \end{pmatrix}.$$

The associated locally analytic vector bundle $\mathcal{E}$ sits in an exact sequence

$$0 \to \mathcal{O}_{\mathcal{X}}^{\mathrm{la}}(\chi_{\mathrm{cyc}}) \to \mathcal{E} \to \mathcal{O}_{\mathcal{X}}^{\mathrm{la}} \to 0.$$

We have

$$\mathcal{N}_{\mathrm{dR}}(\mathcal{E}) = \mathcal{O}_{\mathcal{X}}^{\mathrm{la}} x \oplus \mathcal{O}_{\mathcal{X}}^{\mathrm{la}} y \cong \mathcal{O}_{\mathcal{X}}^{\mathrm{la}}(1) \oplus \mathcal{O}_{\mathcal{X}}^{\mathrm{la}},$$

where at a neighborhood of x_∞ we have $x = t^{-1}e$ and $y = -\log[\alpha^\flat]t^{-1}e + f$. Thus

$$\mathrm{H}^0(\mathcal{Y}_{(0,\infty)}, \mathrm{Sol}_{\mathbb{Q}_p}^{\varphi}(\mathcal{E})) = \mathrm{H}^0(\mathcal{O}_{\mathcal{Y}_{(0,\infty)}}^{\mathrm{sm}} x \oplus \mathcal{O}_{\mathcal{Y}_{(0,\infty)}}^{\mathrm{sm}} y) = \mathbb{Q}_p x \oplus \mathbb{Q}_p y.$$

The action of φ is given by $\varphi(x) = p^{-1}x$ and $\varphi(y) = y$. This gives the underlying φ-module of $\boldsymbol{D}_{\mathrm{cris}}(V)$.

To get the filtration, we consider the stalk of $\mathrm{Sol}_{\mathbb{Q}_p}(\mathcal{E})$ at x_∞. Observe that Fil^0 consists exactly of these smooth sections which do not have a pole at x_∞. As $\log[\alpha^\flat] \equiv \log_p \alpha \bmod t$, we have $\mathrm{Fil}^0 \,\mathrm{Sol}_{\mathbb{Q}_p}(\mathcal{E})_{x_\infty} = \mathbb{Q}_{p,\mathrm{cyc}}(x \log_p \alpha + y)$ and so the filtration on $\boldsymbol{D}_{\mathrm{cris}}(V)$ is given by

$$\mathrm{Fil}^{-1} = \boldsymbol{D}_{\mathrm{cris}}(V) \supset \mathrm{Fil}^0 = \mathbb{Q}_p(x \log_p \alpha + y) \supset \mathrm{Fil}^1 = 0.$$

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Multiplicity structure of the arc space of a fat point

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The equation $x^m = 0$ defines a fat point on a line. The algebra of regular functions on the arc space of this scheme is the quotient of $k[x, x', x^{(2)}, \dots]$ by all differential consequences of $x^m = 0$. This infinite-dimensional algebra admits a natural filtration by finite-dimensional algebras corresponding to the truncations of arcs. We show that the generating series for their dimensions equals $m/(1-mt)$. We also determine the lexicographic initial ideal of the defining ideal of the arc space. These results are motivated by the nonreduced version of the geometric motivic Poincaré series, multiplicities in differential algebra, and connections between arc spaces and the Rogers–Ramanujan identities. We also prove a recent conjecture put forth by Afsharijoo in the latter context.

1. Introduction

1.1. ***Statement of the main result.*** Let k be a field of characteristic zero. Consider an ideal $I \subset k[\boldsymbol{x}]$, where $\boldsymbol{x} = (x_1, \dots, x_n)$, defining an affine scheme X. We consider the polynomial ring

$$k[\boldsymbol{x}^{(\infty)}] := k[x_i^{(j)} \mid 1 \leqslant i \leqslant n,\ j \geqslant 0]$$

in infinitely many variables $\{x_i^{(j)} \mid 1 \leqslant i \leqslant n,\ j \geqslant 0\}$. This ring is equipped with a k-linear derivation $a \mapsto a'$ defined on the generators by

$$(x_i^{(j)})' = x_i^{(j+1)} \quad \text{for } 1 \leqslant i \leqslant n,\ j \geqslant 0.$$

Then we define the ideal $I^{(\infty)} \subset k[\boldsymbol{x}^{(\infty)}]$ of the arc space of X by

$$I^{(\infty)} := \langle f^{(j)} \mid f \in I,\ j \geqslant 0 \rangle.$$

In this paper, we will focus on the case of a fat point $\mathcal{I}_m := \langle x^m \rangle \subset k[x]$ of multiplicity $m \geqslant 2$. Although the zero set of $\mathcal{I}_m^{(\infty)}$ over k consists of a single point with all the coordinates being zero, the dimension of the corresponding quotient algebra $k[x^{(\infty)}]/\mathcal{I}_m^{(\infty)}$ (the "multiplicity" of the arc space) is infinite.

One can obtain a finer description of the multiplicity structure of $k[x^{(\infty)}]/\mathcal{I}_m^{(\infty)}$ by considering its filtration by finite-dimensional algebras induced by the truncation of arcs

$$k[x^{(\leqslant \ell)}]/\mathcal{I}_m^{(\infty)} := k[x^{(\leqslant \ell)}]/\big(k[x^{(\leqslant \ell)}] \cap \mathcal{I}_m^{(\infty)}\big),$$

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where $x^{(\leqslant \ell)} := \{x, x', \ldots, x^{(\ell)}\}$, and arranging the dimensions of these algebras into a generating series

$$D_{\mathcal{I}_m}(t) := \sum_{\ell=0}^{\infty} \dim_k \bigl(k[x^{(\leqslant \ell)}]/\mathcal{I}_m^{(\infty)}\bigr) \cdot t^{\ell}. \tag{1}$$

The main result of this paper is that

$$D_{\mathcal{I}_m}(t) = \frac{m}{1-mt}. \tag{2}$$

1.2. ***Motivations and related results.*** Our motivation for studying the series (1) comes from three different areas: algebraic geometry, differential algebra, and combinatorics.

(1) From the point of view of *algebraic geometry*, $I^{(\infty)}$ defines the *arc space* $\mathcal{L}(X)$ of the scheme X [Denef and Loeser 2001]. Geometrically, the points of the arc space correspond to the Taylor coefficients of the $k[\![t]\!]$-points of X. The arc space of a variety can be viewed as an infinite-order generalization of the tangent bundle or the space of formal trajectories on the variety. For properties and applications of arc spaces, we refer to [Denef and Loeser 2001; Bourqui et al. 2020].

The *reduced structure* of an arc space is often described by means of the geometric motivic Poincaré series [Denef and Loeser 2001, §2.2]

$$P_X(t) := \sum_{\ell=0}^{\infty} [\pi_\ell(\mathcal{L}(X))] \cdot t^{\ell}, \tag{3}$$

where π_ℓ denotes the projection of $\mathcal{L}(X)$ to the affine subspace with the coordinates $\boldsymbol{x}^{(\leqslant \ell)}$ (i.e., the truncation at order ℓ) and $[Z]$ denotes the class of variety Z in the Grothendieck ring [Denef and Loeser 2001, §2.3]. A fundamental result about these series is the Denef–Loeser theorem [1999, Theorem 1.1] saying that $P_X(t)$ is a rational power series.

The arc spaces may also have a rich *scheme* (*i.e.*, *nilpotent*) *structure*, see [Linshaw and Song 2021; Feigin and Makedonskyi 2020; Dumanski and Feigin 2023], reflecting the geometry of the original scheme [Sebag 2011; Bourqui and Haiech 2021]. In the case of a fat point $\mathcal{I}_m = \langle x^m \rangle \subset k[x]$, we will have $\pi_\ell(\mathcal{L}(X)) \cong \mathbb{A}^0$, so the geometric motivic Poincaré series is equal to

$$P(t) = \frac{[\mathbb{A}^0]}{1-t},$$

where $[\mathbb{A}^0]$ is the class of a point. Note that the series does not depend on the multiplicity m of the point. One way to capture the scheme structure of $\mathcal{L}(X)$ could be to take the components of the projections in (3) with their multiplicities. For example, for the case $\mathcal{I}_m$, one will get

$$\sum_{\ell=0}^{\infty} \dim_k \bigl(k[x^{(\leqslant \ell)}]/\mathcal{I}_m^{(\infty)}\bigr) \cdot [\mathbb{A}^0] \cdot t^{\ell} = D_{\mathcal{I}_m}(t)[\mathbb{A}^0].$$

Our result (2) implies that the series above is rational, as in the Denef–Loeser theorem. Interestingly, the shape of the denominator is different from the one in [Denef and Loeser 2001, Theorem 2.2.1]. The formula above is not the only way to take the multiplicities into account. A related and more popular approach is via Arc Hilbert–Poincaré series [Mourtada 2023, §9]; see also [Mourtada 2014; Bruschek et al. 2013].

(2) *Differential algebra* studies, in particular, differential ideals in $k[\boldsymbol{x}^{(\infty)}]$, that is, ideals closed under derivation. From this point of view, $I^{(\infty)}$ is the differential ideal generated by I. Understanding the structure of the differential ideals $\mathcal{I}_m^{(\infty)}$ is a key component of the low power theorem [Levi 1942; 1945] which provides a constructive way to detect singular solutions of algebraic differential equations in one variable. Besides that, various combinatorial properties of $\mathcal{I}_m^{(\infty)}$ have been studied in differential algebra, see [O'Keefe 1960; Pogudin 2014; Arakawa et al. 2021; Zobnin 2005; 2008; Ait El Manssour and Sattelberger 2023].

While there is a rich dimension theory for solution sets of systems of algebraic differential equations [Kondratieva et al. 1999; Pong 2006; Kolchin 1964], we are not aware of a notion of multiplicity of a solution of such a system. In particular, the existing differential analogue of the Bézout theorem [Binyamini 2017] provides only a bound, unlike the equality in classical Bézout theorem [Hartshorne 1977, Theorem 7.7, Chapter 1]. Our result (2) suggests that one possibility is to define the multiplicity of a solution as the growth rate of multiplicities of its truncations, and this definition will be consistent with the usual algebraic multiplicity for the case of a fat point on a line.

(3) Connections between the multiplicity structure of the arc space of a fat point and Rogers–Ramanujan partition identities from *combinatorics* were pointed out by Bruschek, Mourtada, and Schepers in [2013] (for a recent survey, see [Mourtada 2023, §9]). In particular, they used Hilbert–Poincaré series of similar nature to (1) (motivated by the singularity theory [Mourtada 2014, §4]) to obtain new proofs of the Rogers–Ramanujan identities and their generalizations. In this direction, new results have been obtained recently in [Afsharijoo 2021; Afsharijoo et al. 2023; Bai et al. 2020]. Afsharijoo [2021] used computational experiments to conjecture the initial ideal of $\mathcal{I}_m^{(\infty)}$ with respect to the weighted lexicographic ordering [Afsharijoo 2021, §5] (a special case was already conjectured in [Afsharijoo and Mourtada 2020, §1]). This conjecture would imply a new set of partition identities [Afsharijoo 2021, Conjecture 5.1]. Using combinatorial techniques, some of them have been proved in [Afsharijoo 2021], and the rest were established in [Afsharijoo et al. 2023]; see also [Afsharijoo et al. 2022]. However, the original algebraic conjecture about $\mathcal{I}_m^{(\infty)}$ remained open. As a byproduct of our proof of (2), we prove this conjecture (see Theorem 3.3), thus giving a new proof of one of the main results of [Afsharijoo et al. 2023].

Understanding the structure of the ideal $\mathcal{I}_m^{(\infty)}$ is known to be challenging: for example, its Gröbner basis with respect to the lexicographic ordering is not just infinite but even differentially infinite [Zobnin 2005; Afsharijoo and Mourtada 2020], and the question about the nilpotency index of the $x_i^{(j)}$ modulo $\mathcal{I}_m^{(\infty)}$ posed by Ritt [1950, Appendix, Q.5] remained open for sixty years until the paper of Pogudin [2014]; see also [O'Keefe 1960; Arakawa et al. 2021].

Statement (2) appeared in the Ph.D. thesis of Pogudin [2016, Theorem 3.4.1], but the proof given there was incorrect. We are grateful to Alexey Zobnin for pointing out the error. The proof presented in this paper uses different ideas than the erroneous proof in [Pogudin 2016]. We would like to thank Ilya Dumanski for pointing out that the main dimension result (2) could also be deduced from a combination of Propositions 2.1 and 2.3 from [Feigin and Feigin 2002].

1.3. ***Overview of the proof.*** The key technical tool used in our proofs is a representation of the quotient algebra $k[x^{(\infty)}]/\mathcal{I}_m^{(\infty)}$ as a subalgebra in a certain differential exterior algebra that is constructed in [Pogudin 2014]; see Section 4.1. The injectivity of this representation builds upon the knowledge of a Gröbner basis for $\mathcal{I}_m^{(\infty)}$ with respect to the degree reverse lexicographic ordering [Bruschek et al. 2013; Zobnin 2008; Levi 1942]. We approach (2) as a collection of inequalities

$$m^{\ell+1} \leqslant \dim_k\bigl(k[x^{(\leqslant \ell)}]/\mathcal{I}_m^{(\infty)}\bigr) \leqslant m^{\ell+1} \quad \text{for every } \ell \geqslant 0,\ m \geqslant 1. \tag{4}$$

The starting point of our proof of *the lower bound* uses the insightful conjecture by Afsharijoo [2021, §5] that suggests how the standard monomials of $\mathcal{I}_m^{(\infty)}$ with respect to the lexicographic ordering look like. Using the exterior algebra representation, we prove that these monomials are indeed linearly independent modulo $\mathcal{I}_m^{(\infty)}$, and deduce the lower bound from this; see Section 4.3 and 4.4.

In order to prove *the upper bound* from (4), we represent the image of $k[x^{(\leqslant \ell)}]/\mathcal{I}_m^{(\infty)}$ in the differential exterior algebra as a deformation of an algebra which splits as a direct product of $\ell+1$ algebras of dimension m, thus yielding the desired upper bound; see Section 4.2.

1.4. ***Structure of the paper.*** The rest of the paper is organized as follows: Section 2 contains definitions and notations used to state the main results. Section 3 contains the main results of the paper. The proofs of the results are given in Section 4. Then Section 5 describes computational experiments in [Macaulay2] that we performed to check whether formulas similar to (2) hold for more general fat points in k^n. We formulate some open questions based on the results of these experiments.

2. Preliminaries

Definitions 2.1–2.4 provide necessary background in differential algebra. For further details, we refer to [Kaplansky 1957, Chapter 1] or [Kolchin 1973, §I.1–I.2].

Definition 2.1 (differential rings and fields). A *differential ring* $(R, {}')$ is a commutative ring with a derivation $': R \to R$, that is, a map such that, for all $a, b \in R$, we have $(a+b)' = a'+b'$ and $(ab)' = a'b+ab'$. A *differential field* is a differential ring that is a field. For $i > 0$, $a^{(i)}$ denotes the i-th order derivative of $a \in R$.

Notation 2.2. Let x be an element of a differential ring and $h \in \mathbb{Z}_{\geqslant 0}$. We introduce

$$x^{(<h)} := (x, x', \dots, x^{(h-1)}) \quad \text{and} \quad x^{(\infty)} := (x, x', x'', \dots).$$

Analogously, we can define $x^{(\leqslant h)}$. If $\boldsymbol{x} = (x_1, \dots, x_n)$ is a tuple of elements of a differential ring, then

$$\boldsymbol{x}^{(<h)} := (x_1^{(<h)}, \dots, x_n^{(<h)}) \quad \text{and} \quad \boldsymbol{x}^{(\infty)} := (x_1^{(\infty)}, \dots, x_n^{(\infty)}).$$

Definition 2.3 (differential polynomials). Let R be a differential ring. Consider a ring of polynomials in infinitely many variables

$$R[x^{(\infty)}] := R[x, x', x'', x^{(3)}, \dots],$$

and extend the derivation from R to this ring by $(x^{(j)})' := x^{(j+1)}$. The resulting differential ring is called *the ring of differential polynomials in x over R*. The ring of differential polynomials in several variables is defined by iterating this construction.

Definition 2.4 (differential ideals). Let $S := R[x_1^{(\infty)}, \dots, x_n^{(\infty)}]$ be a ring of differential polynomials over a differential ring R. An ideal $I \subset S$ is called *a differential ideal* if $a' \in I$ for every $a \in I$.

One can verify that, for every $f_1, \dots, f_s \in S$, the ideal

$$\langle f_1^{(\infty)}, \dots, f_s^{(\infty)} \rangle$$

is a differential ideal. Moreover, this is the minimal differential ideal containing $f_1, \dots, f_s$, and we will denote it by $\langle f_1, \dots, f_s \rangle^{(\infty)}$.

Definition 2.5 (fair monomials). (1) For a monomial $m = x^{(h_0)} x^{(h_1)} \cdots x^{(h_\ell)} \in k[x^{(\infty)}]$, we define *the order* and *lowest order*, respectively, as

$$\operatorname{ord} m := \max_{0 \leqslant i \leqslant \ell} h_i \quad \text{and} \quad \operatorname{lord} m := \min_{0 \leqslant i \leqslant \ell} h_i .$$

(2) A monomial $m \in k[x^{(\infty)}]$ is called *fair* (respectively, *strongly fair*) if

$$\operatorname{lord} m \geqslant \deg m - 1 \quad (\text{respectively, } \operatorname{lord} m \geqslant \deg m).$$

We denote the sets of all fair and strongly fair monomials by $\mathcal{F}$ and $\mathcal{F}_s$, respectively. By convention, $1 \in \mathcal{F}$ and $1 \in \mathcal{F}_s$. Note that $\mathcal{F}_s \subset \mathcal{F}$.

(3) For every integers $a, b \geqslant 0$, we define

$$\mathcal{F}_{a,b} := \mathcal{F}^a \cdot \mathcal{F}_s^b,$$

where the product of sets of monomials is the set of pairwise products. In other words, $\mathcal{F}_{a,b}$ is a set of all monomials representable as a product of a fair monomials and b strongly fair monomials.

Remark 2.6. The notion of fair monomials was inspired from the conjectured construction of the initial ideal of $\langle x^i, (x^m)^{(\infty)} \rangle$ given in [Afsharijoo 2021, Conjecture 5.1]. We use the notion to formulate concisely and prove the conjecture (see Theorem 3.3).

Example 2.7. The monomials of order at most two in $\mathcal{F}$ and $\mathcal{F}_s$ are

$$\begin{aligned} \mathcal{F} \cap k[x^{(\leqslant 2)}] &= \{1, x, x', (x')^2, x'x'', x'', (x'')^2, (x'')^3\}, \\ \mathcal{F}_s \cap k[x^{(\leqslant 2)}] &= \{1, x', x'', (x'')^2\}. \end{aligned}$$

Using this, one can produce the monomials of order at most one in $\mathcal{F}_{1,1}$ and $\mathcal{F}_{2,0}$

$$\begin{aligned} \mathcal{F}_{1,1} \cap k[x^{(\leqslant 1)}] &= \{1, x, xx', x', (x')^2, (x')^3\}, \\ \mathcal{F}_{2,0} \cap k[x^{(\leqslant 1)}] &= \{1, x, x^2, xx', x(x')^2, x', (x')^2, (x')^3, (x')^4\} \end{aligned}$$

For example, $(x')^3 \in \mathcal{F}_{1,1}$ can be written as $(x')^2 \cdot x'$, where $(x')^2 \in \mathcal{F}$ and $x' \in \mathcal{F}_s$. Likewise, for the monomials of order at most two, we can write

$$\mathcal{F}_{1,1} \cap k[x^{(\leqslant 2)}] = \big\{1, x, x', x'', xx', xx'', (x')^2, x'x'', (x'')^2, x(x'')^2, (x')^3, (x')^2x'', x'(x'')^2, (x'')^3, \\ (x')^2(x'')^2, x'(x'')^3, (x'')^4, (x'')^5\big\}.$$

3. Main results

The algebra of regular functions on the arc space of a fat point $x^m = 0$ admits a natural filtration by subalgebras induced by the truncation of arcs. Our first main result, Theorem 3.1, gives a simple formula for the dimension of the subalgebra induced by the truncation at order h. Corollary 3.2 gives the generating series for these dimensions, as in (2).

Theorem 3.1. *Let m and h be positive integers and k be a differential field of zero characteristic. Then*

$$\dim_k\bigl(k[x^{(\leqslant h)}]/(k[x^{(\leqslant h)}]\cap\langle x^m\rangle^{(\infty)})\bigr) = m^{h+1}.$$

Corollary 3.2. *Let m be a positive integer and k be a differential field of zero characteristic. Then*

$$\sum_{\ell=0}^{\infty}\dim_k\bigl(k[x^{(\leqslant \ell)}]/\langle x^m\rangle^{(\infty)}\bigr)\cdot t^{\ell} = \frac{m}{1-mt},$$

where $k[x^{(\leqslant \ell)}]/\langle x^m\rangle^{(\infty)} := k[x^{(\leqslant \ell)}]/(k[x^{(\leqslant \ell)}]\cap\langle x^m\rangle^{(\infty)})$.

Given a polynomial ideal and monomial ordering, the monomials which do not appear as leading terms of the elements of the ideal are called *standard* monomials. Motivated by applications to combinatorics, Afsharijoo [2021, §5] used computations experiment to conjecture a description of the standard monomials of $\langle x^m\rangle^{(\infty)}$ with respect to the degree lexicographic ordering. Our second main result, Theorem 3.3, gives such a description and, combined with Lemma 4.10, establishes the conjecture.

Theorem 3.3. *Let k be a differential field of zero characteristic. Consider a degree lexicographic monomial ordering on $k[x^{(\infty)}]$ with the variables ordered as $x < x' < x'' < \cdots$. Let m and i be positive integers with $1\leqslant i\leqslant m$. Then the set of standard monomials of the ideal $\langle x^i, (x^m)^{(\infty)}\rangle$ is $\mathcal{F}_{i-1,m-i}$; see Definition 2.5. Note that, for $i = m$, we obtain the differential ideal $\langle x^m\rangle^{(\infty)}$.*

Corollary 3.4. *Theorem 3.3 also holds for the following orderings*:

- ***purely lexicographic*** *with the variables ordered as in Theorem 3.3*;
- ***weighted lexicographic***: *monomials are first compared by the sum of the orders and then lexicographically as in Theorem 3.3.*

Remark 3.5. The multiplicity of the scheme of polynomial arcs of degree less than h of $x = 0$, defined by $\langle x^m, x^{(h)}\rangle^{(\infty)}$, has been studied in [Ait El Manssour and Sattelberger 2023]. It was shown that this multiplicity, equal to $\dim_k k[x^{(\infty)}]/\langle x^m, x^{(h)}\rangle^{(\infty)}$, is a polynomial in m of degree h which is the Erhart polynomial of some lattice polytope [Ait El Manssour and Sattelberger 2023, Theorem 2.5]. Theorem 3.1 together with a natural surjective morphism $k[x^{(<h)}]/\langle x^m\rangle^{(\infty)}\to k[x^{(\infty)}]/\langle x^m, x^{(h)}\rangle^{(\infty)}$ implies that this polynomial is bounded by m^h.

4. Proofs

4.1. *Key technical tool: embedding into the exterior algebra.*

Notation 4.1. Let k be a field. Then, for $\boldsymbol{\xi} = (\xi_0, \xi_1, \dots, \xi_n)$, we introduce a countable collection of symbols $\{\xi_i^{(j)} \mid 0 \leqslant i \leqslant n,\ j \geqslant 0\}$, and by $\Lambda_k(\boldsymbol{\xi}^{(\infty)})$, we denote the exterior algebra of a k-vector space spanned by these symbols. $\Lambda_k(\boldsymbol{\xi}^{(\infty)})$ is equipped with a structure of a (noncommutative) differential algebra by

$$\big(\xi_j^{(i)}\big)' := \xi_j^{(i+1)} \quad \text{for every } i \geqslant 0 \text{ and } 0 \leqslant j \leqslant n.$$

The next proposition is a minor modification of [Pogudin 2014, Lemma 1]. The proof we will give is a simplification of the proof in [Pogudin 2014, Lemma 1], which will be extended to a proof of Lemma 4.4.

Proposition 4.2. *Let m be a positive integer. Consider $\boldsymbol{\eta} = (\eta_0, \dots, \eta_{m-2})$ and $\boldsymbol{\xi} = (\xi_0, \dots, \xi_{m-2})$. Let*

$$\Lambda := \Lambda_k(\boldsymbol{\eta}^{(\infty)}) \otimes \Lambda_k(\boldsymbol{\xi}^{(\infty)}),$$

which is equipped with a structure of differential algebra (as a tensor product of differential algebras, using the Leibniz rule, that is $(a \otimes b)' := a' \otimes b + a \otimes b'$). Consider a differential homomorphism $\varphi\colon k[x^{(\infty)}] \to \Lambda$ defined by

$$\varphi(x) := \sum_{i=0}^{m-2} \eta_i \otimes \xi_i.$$

Then the kernel of φ is $\langle x^m \rangle^{(\infty)}$.

Example 4.3. Consider the case $m = 3$. Then we will have

$$\varphi(x) = \eta_0 \otimes \xi_0 + \eta_1 \otimes \xi_1.$$

The image of x' will then be

$$\varphi(x') = (\varphi(x))' = \eta_0' \otimes \xi_0 + \eta_0 \otimes \xi_0' + \eta_1' \otimes \xi_1 + \eta_1 \otimes \xi_1'.$$

One can show, for example, that $(x')^4 \notin \langle x^3 \rangle^{(\infty)}$ by showing that $\varphi((x')^4) \neq 0$:

$$\varphi((x')^4) = 24(\eta_0 \wedge \eta_0' \wedge \eta_1 \wedge \eta_1') \otimes (\xi_0 \wedge \xi_0' \wedge \xi_1 \wedge \xi_1') \neq 0.$$

Furthermore, a direct computation shows that $\varphi((x')^5) = 0$. Combined with Proposition 4.2, this implies that $(x')^5 \in \langle x^3 \rangle^{(\infty)}$.

Proof of Proposition 4.2. Consider $(\varphi(x))^m$. This is a sum of tensor products of exterior polynomials of degree m in $m-1$ variables, so it must be zero. Since $(\varphi(x))^m = 0$ and φ is a differential homomorphism, we conclude that $\operatorname{Ker}\varphi \supset \langle x^m \rangle^{(\infty)}$.

Now we will prove the inverse inclusion. We define the weighted degree inverse lexicographic ordering $\prec$ on $k[x^{(\infty)}]$ (see [Zobnin 2008, p. 524]): $M \prec N$ if and only if

- $\operatorname{tord} M < \operatorname{tord} N$, where tord is defined as the sum of the orders, or
- $\operatorname{tord} M = \operatorname{tord} N$ and $\deg M < \deg N$, or
- $\operatorname{tord} M = \operatorname{tord} N$, $\deg M = \deg N$, and N is lexicographically *lower* than M, where the variables are ordered $x < x' < x'' < \cdots$.

For example, we will have $x \prec x' \prec x'' \prec \cdots$ and $xx'' \prec (x')^2$. Then, for every $h \geqslant 0$, the leading monomial of $(x^m)^{(h)}$ with respect to $\prec$ is $(x^{(q)})^{m-r}(x^{(q+1)})^r$, where q and r are the quotient and the reminder of the integer division of h by m, respectively. Let $\mathcal{M}$ be the set of all monomials not divisible by any monomial of the form $(x^{(q)})^{m-r}(x^{(q+1)})^r$. Then we can characterize $\mathcal{M}$ as

$$\mathcal{M} = \left\{x^{(h_0)} \cdots x^{(h_\ell)} \mid h_0 \leqslant \cdots \leqslant h_\ell,\ \forall\, 0 \leqslant i \leqslant \ell - m + 1\colon h_{i+m-1} > h_i + 1\right\}.$$

We will define a linear map ψ from $\mathcal{M}$ to the set of monomials in Λ with the following properties:

(P1) *For every $P \in \mathcal{M}$, we have that $\psi(P) \neq 0$.*

(P2) *For every $P \in \mathcal{M}$, the monomial $\psi(P)$ appears in the polynomial $\varphi(P)$ and, for any $P_0 \in \mathcal{M}$ such that $P_0 \prec P$, the monomial $\psi(P)$ does not appear in the polynomial $\varphi(P_0)$.*

Informally speaking, $\psi(M)$ is the "leading monomial" in $\varphi(M)$. Once such a map ψ has been defined, we can prove the proposition as follows: Let $Q \in \operatorname{Ker}\varphi \setminus \langle x^m \rangle^{(\infty)}$. By replacing Q with the result of the reduction of Q by x^m, $(x^m)'$, ... with respect to $\prec$, we can further assume that all the monomials in Q belong to $\mathcal{M}$[1]. Let Q_0 be the largest of them. By (P1) and (P2), $\varphi(Q_0)$ will involve $\psi(Q_0)$ and $\varphi(Q - Q_0)$ will not, so $\varphi(Q) \neq 0$. This contradicts the assumption that $Q \in \operatorname{Ker}\varphi$. The proposition is proved.

Therefore, it remains to define ψ satisfying (P1) and (P2). We will start with the case $m = 2$ to show the main idea while keeping the notation simple. We define ψ by

$$\psi\left(x^{(h_0)} \cdots x^{(h_\ell)}\right) := \left(\eta^{(0)} \otimes \xi^{(h_0)}\right) \wedge \left(\eta^{(1)} \otimes \xi^{(h_1-1)}\right) \wedge \cdots \wedge \left(\eta^{(\ell)} \otimes \xi^{(h_\ell-\ell)}\right), \tag{5}$$

where $h_0 \leqslant h_1 \leqslant \cdots \leqslant h_\ell$. For proving (P1), we observe that, if $h_{i+1} > h_i + 1$ for all i, then $h_0 < h_1 - 1 < h_2 - 2 < \cdots < h_\ell - \ell$, so there are no coinciding ξ's in (5). The construction for arbitrary m will consist of splitting the monomial into $m - 1$ interlacing submonomials and applying (5) with (η_i, ξ_i) to i-th submonomial. More formally, if $h_0 \leqslant h_1 \leqslant \cdots \leqslant h_\ell$, we define

$$\psi\left(x^{(h_0)} \cdots x^{(h_\ell)}\right) := \prod_{i=0}^{\ell} \left(\eta_{i\,\%\,(m-1)}^{([i/(m-1)])} \otimes \xi_{i\,\%\,(m-1)}^{(h_i - [i/(m-1)])}\right), \tag{6}$$

where $a \,\%\, b$ denotes the remainder of the division of a by b, and $[\alpha]$ denotes the integer part of α. Property (P1) is proved by applying (P1) for $m = 2$ to each submonomial.

For proving (P2), consider $P_0 \in \mathcal{M}$ with $P_0 \preceq P$ and $\psi(P)$ appearing in $\varphi(P_0)$. Since ψ preserves the total orders and doubles the degrees, we have $\operatorname{tord} P_0 = \operatorname{tord} P$ and $\deg P_0 = \deg P$. Let $H := \operatorname{ord} P_0$. Since $P_0 \preceq P$, we have $H \geqslant h_\ell$. Since the maximal orders of η and ξ in $\psi(P)$ do not exceed $[\ell/(m-1)]$ and $h_\ell - [\ell/(m-1)]$, respectively, we have $H \leqslant h_\ell$. Thus, $H = h_\ell$. Applying the same argument recursively to $P/x^{(h_\ell)}$ and $P_0/x^{(h_\ell)}$, we conclude that $P = P_0$.

We will prove that $\varphi(P)$ involves $\psi(P)$ by induction on $\deg P$. The case $\deg P = 0$ is clear. Consider P, with $\deg P > 0$. Similarly to the preceding argument, one can obtain $\psi(P)$ (from $\psi(P/x^{(\ell)})$) only by

[1] Interestingly, although it is known that x^m, $(x^m)'$, ... form a Gröbner basis, we do not really need to use this fact here since a reduction with respect to any set of polynomials is well defined.

taking $\eta_{\ell\,\%\,(m-1)}^{([\ell/(m-1)])} \otimes \xi_{\ell\,\%\,(m-1)}^{(h_\ell - [\ell/(m-1)])}$ (i.e., the last term in (6)) from one of the occurrences of $x^{(h_\ell)}$ in P. Therefore, the coefficient in front of $\psi(P)$ in $\varphi(P)$ will be, up to sign, equal to $\deg_{x^{(h_\ell)}}$ times the coefficient in front of $\psi(P/x^{(h_\ell)})$ in $\varphi(P/x^{(h_\ell)})$. The latter is nonzero by the induction hypothesis. $\square$

Lemma 4.4. *In the notation of Proposition 4.2, let $1 \leqslant r < m$. Then the preimage of the ideal in Λ generated by $\eta_{r-1} \otimes \xi_{r-1}, \dots, \eta_{m-2} \otimes \xi_{m-2}$ under φ is equal to $\langle (x^m)^{(\infty)}, x^r \rangle$.*

Proof. We first prove that the image of x^r belongs to $\langle \eta_{r-1} \otimes \xi_{r-1}, \dots, \eta_{m-2} \otimes \xi_{m-2} \rangle$. This is because $\varphi(x^r)$ is the sum of monomials which are products of r different $\eta_i \otimes \xi_i$. Since there are $m-1$ of them, every such monomial will involve at least one of the last $m-r$ of the $\eta_i \otimes \xi_i$.

Let us consider a polynomial $g \in k[x^{(\infty)}] \setminus \langle (x^m)^{(\infty)}, x^r \rangle$ and prove that $\varphi(g)$ does not belong to $\langle \eta_{r-1} \otimes \xi_{r-1}, \dots, \eta_{m-2} \otimes \xi_{m-2} \rangle$. We can assume that each monomial P of g belongs to

$$\mathcal{M}_r = \{ M \in \mathcal{M} \mid \deg_x M < r \text{ or } 0 < h_{r-1} \}.$$

We will use the map ψ defined in (6). In fact, $\psi(P)$ does not involve the zero-order derivatives of $\xi_{r-1}, \dots, \xi_{m-2}$, since $h_i - [i/(m-1)]$ can only be zero for a monomial in $\mathcal{M}$ only if $i \leqslant r-2$. Thus,

$$\psi(P) \notin \langle \eta_{r-1} \otimes \xi_{r-1}, \dots, \eta_{m-2} \otimes \xi_{m-2} \rangle.$$

Assume that P_0 is the largest summand that appears in g. Then $\varphi(P_0)$ involves $\psi(P_0)$, but $\varphi(g - P_0)$ does not. Therefore, $\varphi(g)$ does not belong to $\langle \eta_{r-1} \otimes \xi_{r-1}, \dots, \eta_{m-2} \otimes \xi_{m-2} \rangle$. $\square$

4.2. *Upper bounds for the dimension.* Throughout the section, we fix a differential field k of zero characteristic.

Proposition 4.5. *Let m, h be positive integers. We denote by $A_{m,h}$ the subalgebra of $k[x^{(\infty)}]/\langle x^m \rangle^{(\infty)}$ generated by the images of $x, x', \dots, x^{(h)}$. Then*

$$\dim A_{m,h} \leqslant m^{h+1}.$$

First we describe a general construction which will be a special case of the so-called associated graded algebra. Let $A = A_0 \oplus A_1 \oplus A_2 \oplus \cdots$ be a $\mathbb{Z}_{\geqslant 0}$-graded algebra over k equipped with a homogeneous derivation of weight one (that is, $A_i' \subseteq A_{i+1}$ for every $i \geqslant 0$). We introduce a map $\operatorname{gr}\colon A \to A$ defined as follows: Consider a nonzero $a \in A$, and let i be the largest index such that $a \in A_i \oplus A_{i+1} \oplus \cdots$. Then we define $\operatorname{gr}(a)$ to be the image of the projection of a onto A_i along $A_{i+1} \oplus A_{i+2} \oplus \cdots$. In other words, we replace each element with its lowest homogeneous component.

Note that gr is not a homomorphism, it is not even a linear map. However, it has two important properties we state as a lemma.

Lemma 4.6. (1) *Let $a_1, \dots, a_n \in A$, and let $p \in k[\mathbf{x}^{(\infty)}]$ be a differential monomial. Then*

$$p(\operatorname{gr}(a_1), \dots, \operatorname{gr}(a_n)) \neq 0 \quad \Longrightarrow \quad \operatorname{gr}(p(a_1, \dots, a_n)) = p(\operatorname{gr}(a_1), \dots, \operatorname{gr}(a_n)).$$

(2) *If $a_1, \dots, a_n \in A$ are k-linearly dependent, then $\operatorname{gr}(a_1), \dots, \operatorname{gr}(a_n)$ also are k-linearly dependent.*

Proof. To prove the first part, one sees that p does not vanish on the lowest homogeneous parts of $a_1, \dots, a_n$, so the homogeneity of the multiplication and derivation imply that taking the lowest homogeneous part commutes with applying p for $a_1, \dots, a_n$.

To prove the second part, let i be the lowest grading appearing among $a_1, \dots, a_n$. Restricting to the component of this weight, one gets a linear relation for $\operatorname{gr}(a_1), \dots, \operatorname{gr}(a_n)$. $\square$

Lemma 4.7. *Let A be a graded differential algebra as above. Consider elements $a_1, \dots, a_n$ in A, and denote the algebras* (*not differential*) *generated by $a_1, \dots, a_n$ and $\operatorname{gr}(a_1), \dots, \operatorname{gr}(a_n)$ by B and B_{gr}, respectively. Then* $\dim B_{\mathrm{gr}} \leqslant \dim B$.

Proof. The algebra B_{gr} is spanned by all the monomials in $\operatorname{gr}(a_1), \dots, \operatorname{gr}(a_n)$. We choose a basis in this spanning set, that is, we consider monomials $p_1, \dots, p_N \in k[x_1, \dots, x_n]$ such that

$$p_1\big(\operatorname{gr}(a_1), \dots, \operatorname{gr}(a_n)\big), \dots, p_N\big(\operatorname{gr}(a_1), \dots, \operatorname{gr}(a_n)\big)$$

form a basis of B_{gr}. The first part of Lemma 4.6 implies that

$$\operatorname{gr}\big(p_i(a_1, \dots, a_n)\big) = p_i\big(\operatorname{gr}(a_1), \dots, \operatorname{gr}(a_n)\big) \quad \text{for every } 1 \leqslant i \leqslant N.$$

Then the second part of Lemma 4.6 implies that $p_1(a_1, \dots, a_n), \dots, p_N(a_1, \dots, a_n)$ are linearly independent. Since they belong to B, we have $\dim B \geqslant N = \dim B_{\mathrm{gr}}$. $\square$

Proof of Proposition 4.5. Let Λ and φ be the exterior algebra and the homomorphism from Proposition 4.2. Proposition 4.2 implies that $A_{m,h}$ is isomorphic to the subalgebra of Λ generated by

$$\sum_{i=0}^{m-2} \eta_i \otimes \xi_i, \ \sum_{i=0}^{m-2} (\eta_i \otimes \xi_i)', \ \sum_{i=0}^{m-2} (\eta_i \otimes \xi_i)'', \dots, \sum_{i=0}^{m-2} (\eta_i \otimes \xi_i)^{(h)}.$$

We define a grading on Λ by setting the weights of $\eta_j^{(i)}$ and $\xi_j^{(i)}$ to be equal to i for every $i \geqslant 0$ and $0 \leqslant j < m-1$. The exterior algebra Λ becomes a graded algebra, and the derivation is homogeneous of weight one.

We fix $h \geqslant 0$ and consider the following elements of Λ:

$$\tilde{\alpha}_{j,i} := (1+\partial)^i \alpha_j \quad \text{for } i \geqslant 0, \ 0 \leqslant j < m-1, \text{ and } \alpha \in \{\eta, \xi\},$$

where ∂ is the operator of differentiation. We introduce

$$v_i := \sum_{j=0}^{m-2} \tilde{\eta}_{j,i} \otimes \tilde{\xi}_{j,i} \quad \text{for } 0 \leqslant i \leqslant h,$$

and let Y_h be the algebra generated by $v_0, \dots, v_h$. For every $0 \leqslant i \leqslant h$, we have $v_i^m = 0$, so Y_h is spanned by the products of the form

$$v_0^{d_0} v_1^{d_1} \dots v_h^{d_h}, \quad \text{where } 0 \leqslant d_0, \dots, d_h < m.$$

Therefore, $\dim Y_h \leqslant m^{h+1}$.

Claim. *There is an invertible $(h+1)\times(h+1)$ matrix M over $\mathbb{Q}$ such that, for $u_0,\dots,u_h$ defined by*

$$(u_0,\dots,u_h)^T := M(v_0,\dots,v_h)^T, \tag{7}$$

we have

$$\operatorname{gr}(u_i)=\sum_{j=0}^{m-2}(\eta_j\otimes\xi_j)^{(i)} \quad \textit{for every } 0\leqslant i\leqslant h.$$

We will first demonstrate how the proposition follows from the claim, and then we prove the claim. Since M is invertible, $u_0,\dots,u_h$ generate Y_h as well. Since $\operatorname{gr}(u_0),\dots,\operatorname{gr}(u_h)$ generate $A_{m,h}$, Lemma 4.7 implies that $m^{h+1}\geqslant\dim Y_h\geqslant\dim A_{m,h}$.

Therefore, it remains to prove the claim. For every $0\leqslant i\leqslant h$, we can write

$$v_i=(1\otimes 1+1\otimes\partial)^i(1\otimes 1+\partial\otimes 1)^i v_0=(1\otimes 1+1\otimes\partial+\partial\otimes 1+\partial\otimes\partial)^i v_0.$$

We set $u_i:=(1\otimes\partial+\partial\otimes 1+\partial\otimes\partial)^i v_0$ for every $0\leqslant i\leqslant h$. Note that, since $1\otimes\partial+\partial\otimes 1$ is just the original derivation on Λ, we have

$$\operatorname{gr}(u_i)=(1\otimes\partial+\partial\otimes 1)^i v_0=v_0^{(i)}=\sum_{j=0}^{m-2}(\eta_j\otimes\xi_j)^{(i)}. \tag{8}$$

By expanding the binomial $(1\otimes 1+(1\otimes\partial+\partial\otimes 1+\partial\otimes\partial))^i$, we can write $v_i=\sum_{j=0}^{i}\binom{i}{j}u_j$. Then we have

$$(v_0,\dots,v_h)^T=\widetilde{M}(u_0,\dots,u_h)^T, \tag{9}$$

where $\widetilde{M}$ is the $(h+1)\times(h+1)$-matrix with the (i,j)-th entry being $\binom{i}{j}$. Since $\widetilde{M}$ is lower-triangular with ones on the diagonal, it is invertible. We set $M:=\widetilde{M}^{-1}$. So we have $(u_0,\dots,u_h)^T:=M(v_0,\dots,v_h)^T$, which together with (8) finishes the proof of the claim. □

By combining the proof of Proposition 4.5 with Lemma 4.4, we can extend Proposition 4.5 as follows:

Corollary 4.8. *Let m,h,i be positive integers with $1\leqslant i\leqslant m$. By $A_{(m,i),h}$ we denote the subalgebra of $k[x^{(\infty)}]/\langle x^i,(x^m)^{(\infty)}\rangle$ generated by the images of $x,x',\dots,x^{(h)}$. Then*

$$\dim A_{(m,i),h}\leqslant i\cdot m^h.$$

Proof. The proof will be a refinement of the proof of Proposition 4.5, and we will use the notation from there. Let π be the canonical homomorphism $\pi:\Lambda\to\Lambda_i:=\Lambda/\langle\xi_{i-1}\otimes\eta_{i-1},\dots,\xi_{m-2}\otimes\eta_{m-2}\rangle$. Since the ideal $\langle\xi_{i-1}\otimes\eta_{i-1},\dots,\xi_{m-2}\otimes\eta_{m-2}\rangle$ is homogeneous with respect to the grading on Λ, there is a natural grading on Λ_i.

We have $A_{(m,i),h}\cong\pi(A_{m,h})$. Since π is a homogeneous homomorphism, $\pi(A_{m,h})$ is generated by $\pi(\operatorname{gr}(u_0)),\dots,\pi(\operatorname{gr}(u_h))$ from (7), so $\dim A_{(m,i),h}=\dim\pi(A_{m,h})\leqslant\dim\pi(Y_h)$. We observe that $\pi(v_0)^i=0$, so $\pi(Y_h)$ is spanned by products of the form

$$\pi(v_0)^{d_0}\pi(v_1)^{d_1}\cdots\pi(v_h)^{d_h},$$

where $0\leqslant d_0<i$ and $0\leqslant d_1,\dots,d_h<m$. Therefore, $\dim\pi(Y_h)\leqslant i\cdot m^h$. □

4.3. *Combinatorial properties of fair monomials.*

Definition 4.9 (nonoverlapping monomials). We say that two monomials $m_1, m_2 \in k[x^{(\infty)}]$ *do not overlap* if $\operatorname{ord} m_1 \leqslant \operatorname{lord} m_2$ or $\operatorname{ord} m_2 \leqslant \operatorname{lord} m_1$.

Lemma 4.10. *Let m, i be integers with $0 \leqslant i \leqslant m$. Let $P \in \mathcal{F}_{i,m-i}$. Then there exist $P_1, \dots, P_i \in \mathcal{F}$ and $P_{i+1}, \dots, P_m \in \mathcal{F}_s$ such that*

$$P = P_1 \cdots P_m \quad \textit{and, for every } 1 \leqslant i < m, \quad \operatorname{ord} P_i \leqslant \operatorname{lord} P_{i+1}.$$

Remark 4.11. Lemma 4.10 implies that the set $\mathcal{F}_{i-1,m-i}$ from Theorem 3.3 coincides with the set of standard monomials conjectured by Afsharijoo [2021, §5].

Proof. Suppose that P can be written as

$$P = \big(x^{(h_{1,0})} \cdots x^{(h_{1,\ell_1})}\big) \cdots \big(x^{(h_{m,0})} \cdots x^{(h_{m,\ell_m})}\big),$$

where each $(x^{(h_{i,0})} \cdots x^{(h_{i,\ell_i})})$ belongs to $\mathcal{F}$ or $\mathcal{F}_s$ and $h_{1,0} \leqslant h_{2,0} \leqslant \cdots \leqslant h_{m,0}$. We first prove that we can make the product to be a product of nonoverlapping monomials.

Let us sort the orders $h_{1,0}, h_{1,1}, \dots, h_{m,\ell_m}$ in the ascending order

$$\big\{(r_{1,0}, \dots, r_{1,\ell_1}); (r_{2,0}, \dots, r_{2,\ell_2}); \dots ; (r_{m,0}, \dots, r_{m,\ell_m})\big\}.$$

Claim. *For all $0 \leqslant i \leqslant m$, we have $h_{i,0} \leqslant r_{i,0}$.*

In the whole list of the $h_{i,j}$, all the numbers to the right from $h_{i,0}$ are $\geqslant h_{i,0}$. Therefore, after sorting, $h_{i,0}$ will either stay or move to the left. Thus, $h_{i,0} \leqslant r_{i,0}$, so the claim is proved.

Hence if $x^{(h_{i,0})} \cdots x^{(h_{i,\ell_i})}$ was a fair (respectively, strongly fair) monomial then $x^{(r_{i,0})} \cdots x^{(r_{i,\ell_i})}$ is a fair (respectively, strongly fair) monomial.

Now we will move all the strongly fair monomials to the right in the decomposition of P. We first prove that, for every $Q = Q_1 Q_2$ such that $Q_1 \in \mathcal{F}_s$, $Q_2 \in \mathcal{F}$, and $\operatorname{ord} Q_1 \leqslant \operatorname{lord} Q_2$, there exist $\widetilde{Q}_1 \in \mathcal{F}_s$ and $\widetilde{Q}_2 \in \mathcal{F}$ such that $Q = \widetilde{Q}_1 \widetilde{Q}_2$ and $\operatorname{ord} \widetilde{Q}_1 \leqslant \operatorname{lord} \widetilde{Q}_2$. Let

$$Q_1 = x^{(h_{1,0})} \cdots x^{(h_{1,\ell_1})} \quad \text{and} \quad Q_2 = x^{(h_{2,0})} \cdots x^{(h_{2,\ell_2})},$$

where $\ell_1 < h_{1,0}$ and $\ell_2 \leqslant h_{2,0}$. If $\ell_2 < h_{2,0}$, then $Q_2 \in \mathcal{F}_s$; so we are done. Otherwise, $\ell_1 + 1 \leqslant h_{1,0}$ implies that $Q_1 x^{(h_{1,0})}$ is a fair monomial, and $\ell_2 - 1 < h_{2,0}$ implies that $Q_2 / x^{(h_{1,0})} \in \mathcal{F}_s$. Thus, we can take $\widetilde{Q}_1 := Q_1 x^{(h_{1,0})}$ and $\widetilde{Q}_2 := Q_2 / x^{(h_{1,0})}$.

Applying the described transformation while possible to the nonoverlapping decomposition of P, one can arrange that the last $m - i$ components are strongly fair. □

Proposition 4.12. *For every positive integers m, h, i with $0 \leqslant i \leqslant m$, the cardinality of $\mathcal{F}_{i,m-i} \cap k[x^{(\leqslant h)}]$ is equal to $(i + 1) \cdot (m + 1)^h$.*

The proof of the proposition will use the following lemma:

Lemma 4.13. *For every integers h and d, we have*

$$\left|\{P \mid P \in \mathcal{F} \cap k[x^{\leqslant h}] \text{ and } \deg P = d\}\right| = \binom{h+1}{d}.$$

If one replaces $\mathcal{F}$ with $\mathcal{F}_s$, the cardinality will be $\binom{h}{d}$.

Proof. Let $x^{(h_0)} \cdots x^{(h_\ell)} \in \mathcal{F}$ such that $\ell \leqslant h_0 \leqslant \cdots \leqslant h_\ell$. We define a map

$$(h_0, \dots, h_\ell) \mapsto (h_0 - \ell, h_1 - \ell - 1, \dots, h_\ell).$$

The map assigns to the orders of a monomial in $\mathcal{F} \cap k[x^{\leqslant h}]$ a list of strictly increasing nonnegative integers not exceeding h. A direct computation shows that this map is a bijection. Since the number of such sequences of length d is equal to the number of subsets of $[0, 1, \dots, h]$ of cardinality d, the number of monomials is $\binom{h+1}{d}$.

The case of $\mathcal{F}_s$ is analogous with the only difference being that the subset will be in $[1, 2, \dots, h]$, thus yielding $\binom{h}{d}$. $\square$

Proof of Proposition 4.12. We will prove the proposition by induction on m. For the base case, we have $\mathcal{F}_{0,0} = \{1\}$, so the statement is true.

Consider $m > 0$, and assume that for all smaller m the proposition is proved. We fix $0 \leqslant i \leqslant m$. Consider a monomial $P \in \mathcal{F}_{i,m-i} \cap k[x^{(\leqslant h)}]$, let $P_1 \cdots P_m$ be a decomposition from Lemma 4.10 with $\deg P_m$ being as large as possible. We denote $\operatorname{tail} P := P_m$ and $\operatorname{head} P := P_1 \cdots P_{m-1}$.

We will show that the map $P \to (\operatorname{head} P, \operatorname{tail} P)$ defines a bijection between $\mathcal{F}_{i,m-i}$ and

$$\begin{aligned} &\text{for } i < m\colon \{(Q_0, Q_1) \in \mathcal{F}_{i,m-i-1} \times \mathcal{F}_s \mid \operatorname{ord} Q_0 \leqslant \deg Q_1\},\\ &\text{for } i = m\colon \{(Q_0, Q_1) \in \mathcal{F}_{m-1,0} \times \mathcal{F} \mid \operatorname{ord} Q_0 < \deg Q_1\}. \end{aligned} \tag{10}$$

We will prove the case $i < m$, as the proof in the case $i = m$ is analogous. First we will show that, for every $P \in \mathcal{F}_{i,m-i}$, we have $\operatorname{ord} \operatorname{head} P \leqslant \deg \operatorname{tail} P$. Assume the contrary, and let $\ell := \operatorname{ord} \operatorname{head} P > \deg \operatorname{tail} P$. Then we will have

$$\operatorname{lord}(x^{(\ell)} \operatorname{tail} P) \geqslant \min(\ell, \operatorname{lord} \operatorname{tail} P) = \ell \geqslant \deg(x^{(\ell)} \operatorname{tail} P).$$

This implies that $x^{(\ell)} \operatorname{tail} P \in \mathcal{F}_s$. Thus, in the decomposition of Lemma 4.10, we could have taken P_m to be $x^{(\ell)} \operatorname{tail} P$. This contradicts the maximality of $\deg \operatorname{tail} P$. In the other direction, if $Q_0 \in \mathcal{F}_{i,m-i-1}$ and $Q_1 \in \mathcal{F}_s$ such that $\operatorname{ord} Q_0 \leqslant \deg Q_1$, then $Q_0 Q_1 \in \mathcal{F}_{i,m-i}$. Moreover, since $x^{(\operatorname{ord} Q_0)} Q_1 \notin \mathcal{F}$, we have $\operatorname{tail}(Q_0 Q_1) = Q_1$.

We will now use the bijection (10) to count the elements in $\mathcal{F}_{i,m-i} \cap k[x^{(\leqslant h)}]$. For $i < m$,

$$\begin{aligned} \left|\mathcal{F}_{i,m-i} \cap k[x^{(\leqslant h)}]\right| &= \sum_{\ell=0}^{h} \left|\mathcal{F}_{i,m-i-1} \cap k[x^{(\leqslant \ell)}]\right| \cdot \left|\{Q_1 \in \mathcal{F}_s \cap k[x^{(\leqslant h)}] \mid \deg Q_1 = \ell\}\right| \\ &= \sum_{\ell=0}^{h} (i+1) \cdot m^\ell \binom{h}{\ell} = (i+1) \cdot (m+1)^h \quad \text{(by Lemma 4.13)}. \end{aligned}$$

For $i = m$:

$$\begin{aligned}\left|\mathcal{F}_{m,0} \cap k[x^{(\leqslant h)}]\right| &= \sum_{\ell=0}^{h+1} \left|\mathcal{F}_{m-1,0} \cap k[x^{(<\ell)}]\right| \cdot \left|\{Q_1 \in \mathcal{F}_s \cap k[x^{(\leqslant h)}] \mid \deg Q_1 = \ell\}\right| \\ &= \sum_{\ell=0}^{h+1} m^{\ell} \binom{h+1}{\ell} = (m+1)^{h+1} \quad \text{(by Lemma 4.13).}\end{aligned}$$

Thus, the proposition is proved. □

4.4. ***Lower bounds for the dimension.***

Notation 4.14. For a differential polynomial $P \in k[\boldsymbol{x}^{(\infty)}]$ and $1 \leqslant i \leqslant n$, we define

- $\operatorname{tord}_{x_i} P$ to be the *total order* of P in x_i, that is, the largest sum of the orders of the derivatives of x_i among the monomials of P;
- $\deg_{x_i^{(\infty)}} P$ to be the *total degree* of P with respect to the variables $x_i, x_i', x_i'', \ldots$.
- We fix a monomial ordering $\prec$ on $k[\boldsymbol{x}^{(\infty)}]$ defined as follows: To each differential monomial $M = x_{i_0}^{(h_0)} x_{i_1}^{(h_1)} \cdots x_{i_\ell}^{(h_\ell)}$ with $(h_0, i_0) \preceq_{\text{lex}} (h_1, i_1) \preceq_{\text{lex}} \cdots \preceq_{\text{lex}} (h_\ell, i_\ell)$, we assign a tuple

$$(\ell,\ h_\ell,\ h_{\ell-1},\ \ldots,\ h_0,\ i_\ell,\ i_{\ell-1},\ \ldots,\ i_0),$$

  and compare monomials by comparing the corresponding tuples lexicographically.

Definition 4.15 (isobaric ideal). An ideal $I \subset k[x^{(\infty)}]$ is called *isobaric* if it can be generated by isobaric polynomials, that is, polynomials with all the monomials having the same total order.

Proposition 4.16. *For $i = 1, 2$, the elements of $\mathcal{F}_{i-1,2-i}$ are the standard monomials modulo $\langle (x^2)^{(\infty)}, x^i \rangle$.*

Proof. We use Proposition 4.2 to obtain the differential homomorphism $\varphi \colon k[x^{(\infty)}] \to \Lambda$ defined by $\varphi(x) = \eta \otimes \xi$ (we will use η and ξ instead of η_0 and ξ_0 for brevity). Let $\tilde{\varphi}$ be the composition of φ with the projection onto $\Lambda / \langle \eta \otimes \xi \rangle$. We will prove the proposition for the elements in $\mathcal{F}_{1,0}$, and the other case can be done in the same way by replacing φ with $\tilde{\varphi}$.

Let $X = x^{(h_0)} \cdots x^{(h_\ell)}$, where $h_0 \leqslant h_1 \leqslant \cdots \leqslant h_\ell$, be an element of $\mathcal{F}_{1,0}$. We will show that a summand

$$B(X) := \left(\eta^{(h_0-\ell)} \wedge \eta^{(h_1-(\ell-1))} \wedge \cdots \wedge \eta^{(h_\ell)}\right) \otimes \left(\xi^{(\ell)} \wedge \xi^{(\ell-1)} \wedge \cdots \wedge \xi' \wedge \xi\right) \tag{11}$$

appears in $\varphi(X)$ with nonzero coefficient. We will prove this by induction on ℓ. The base case $\ell = 0$ is trivial, so let $\ell > 0$. Since $\eta^{(h_0-\ell)}$ may come only from one of the occurrences of $x^{(h_0)}$ in X, we must take $\eta^{(h_0-\ell)} \otimes \xi^{(\ell)}$ from one of the $x^{(h_0)}$. Therefore, the coefficient at $B(X)$ in $\varphi(X)$ is $\deg_{x^{(h_0)}} X$ times the coefficient at $B(X/x^{(h_0)})$ in $\varphi(X/x^{(h_0)})$, which is nonzero by the induction hypothesis.

Let $Y := x^{(s_0)} \cdots x^{(s_{\ell'})}$ be a monomial such that $Y \prec X$. We will prove by contradiction that $B(X)$ does not appear in $\varphi(Y)$. If it does, then $\deg(X) = \deg(Y) = \ell + 1 = \ell' + 1$. Moreover, there exists a permutation σ of $\{0, 1, \ldots, \ell\}$ such that

$$s_i - \sigma(i) = h_i - (\ell - i) \quad \text{for every } 0 \leqslant i \leqslant \ell.$$

The inequality $s_\ell \leqslant h_\ell$ implies $\sigma(\ell) = 0$, and thus, $s_\ell = h_\ell$. Therefore, $s_{\ell-1} \leqslant h_{\ell-1}$, which implies $\sigma(\ell-1) = 1$, and thus, $s_{\ell-1} = h_{\ell-1}$. Continuing in this way, we show that

$$s_i = h_i \quad \text{for all } 0 \le i \le \ell,$$

which contradicts $Y \prec X$. Thus $B(X)$ cannot appear in the $\varphi(Y)$.

Assume that $X \in \mathrm{In}_\prec \langle x^2 \rangle^{(\infty)}$. Then there exist monomials $P_1, \dots P_N$ such that $P_j \prec X$ for all $1 \le j \le N$ and

$$X - \sum_{j=1}^{N} \lambda P_j \in \langle x^2 \rangle^{(\infty)}.$$

Hence, $\varphi(X) - \sum_{j=1}^{N} \lambda_j \varphi(P_j) = 0$. Since $P_j \prec X$ for all $1 \le j \le N$, $B(X)$ cannot be canceled in $\varphi(X) - \sum_{j=1}^{N} \lambda_j \varphi(P_j)$, which is a contradiction. Therefore, X is a standard monomial. $\square$

Lemma 4.17. *Let $I_1 \subset k[y_1^{(\infty)}], \dots, I_s \subset k[y_s^{(\infty)}]$ be ideals, and we denote by M_i the set of the standard monomials modulo I_i with respect to degree lexicographic ordering for $1 \leqslant i \leqslant s$. Then the standard monomials with respect to the ordering $\prec$ (see Notation 4.14) modulo $\langle I_1, \dots, I_s \rangle \subset k[y_1^{(\infty)}, \dots, y_s^{(\infty)}]$ are*

$$M_1 \cdot M_2 \cdots M_s := \{m_1 m_2 \cdots m_s \mid m_1 \in M_1, \dots, m_s \in M_s\}.$$

Proof. For each I_i, consider the reduced Gröbner basis G_i of I_i with respect to the degree lexicographic ordering. For each pair $f, g \in G := G_1 \cup G_2 \cup \ldots \cup G_s$, their S-polynomial is reduced to zero by G

- if f, g belong to the same G_i, due to the fact that G_i is a Gröbner basis;
- otherwise, by the first Buchberger criterion (since f and g have coprime leading monomials). $\square$

Proposition 4.18. *Let $I_1 \subset k[y_1^{(\infty)}], \dots, I_s \subset k[y_s^{(\infty)}]$ be homogeneous and isobaric ideals (not necessarily differential). By M_i we denote the set of standard monomials modulo I_i with respect to the degree lexicographic ordering for $1 \leqslant i \leqslant s$. We define a homomorphism (not necessarily differential)*

$$\varphi\colon k[x^{(\infty)}] \to k[y_1^{(\infty)}, \dots, y_s^{(\infty)}]/\langle I_1, \dots, I_s \rangle$$

by $\varphi(x^{(k)}) := y_1^{(k)} + \cdots + y_s^{(k)}$ and denote $I := \mathrm{Ker}(\varphi)$. Then the elements of

$$M := \left\{ m_1 \dots m_s \mid \forall 1 \leqslant i \leqslant s\colon m_i \in M_i \text{ and } \forall\, 1 \leqslant j < s\colon \operatorname{ord} m_j \leqslant \operatorname{lord} m_{j+1} \right\} \tag{12}$$

are standard monomials modulo I with respect to the ordering $\prec$ (but maybe not all the standard monomials).

Proof. Consider a monomial $P = x^{(h_0)} \cdots x^{(h_\ell)} \in M$, and fix a representation $P = m_1(x), \dots, m_s(x)$ as in (12). Assume that P is a leading monomial of I. Then there exist monomials $P_1, \dots, P_N$ such that

$$P - \sum_{j=1}^{N} \lambda_j P_j \in \mathrm{Ker}\,\varphi \quad \text{and} \quad \forall\, 1 \leqslant j \leqslant N\colon P_j \prec P.$$

Then $\varphi(P) - \sum \lambda_j \varphi(P_j) \in \langle I_1, \dots I_s \rangle$. We define $m := m_1(y_1) m_2(y_2) \cdots m_s(y_s)$.

Claim. *For every monomial $\widetilde{m} \neq m$ in $\varphi(P)$, there exists $1 \leqslant j \leqslant s$ such that either $\deg_{y_j^{(\infty)}} m \neq \deg_{y_j^{(\infty)}} \widetilde{m}$ or $\operatorname{tord}_{y_j} m \neq \operatorname{tord}_{y_j} \widetilde{m}$.*

Assume the contrary, that there exists $\widetilde{m}$ such that, for every $1 \leqslant j \leqslant s$, we have $d_i := \deg_{y_j^{(\infty)}} m = \deg_{y_j^{(\infty)}} \widetilde{m}$ and $\operatorname{tord}_{y_j} m = \operatorname{tord}_{y_j} \widetilde{m}$. We write $\widetilde{m} = \widetilde{m}_1(y_1) \cdots \widetilde{m}_s(y_s)$. Let $1 \leqslant j \leqslant s$ be the largest index such that $m_j \neq \widetilde{m_j}$. Since m_j contains d_j largest derivatives in $m_1(x) \cdots m_j(x) = \widetilde{m}_1(x) \cdots \widetilde{m}_j(x)$ and has the same total order as $\widetilde{m}_j$, we conclude that $m_j = \widetilde{m}_j$. Thus, the claim is proved.

We write the homogeneous and isobaric component of $\sum_{j=1}^{N} \lambda_j \varphi(P_j)$ of the same degree and total order in y_i as m for every $1 \leqslant i \leqslant s$ as $\sum_{i=1}^{M} \mu_i R_i$, where R_i is a differential monomial and $\mu_i \in k$ for every $1 \leqslant i \leqslant M$. Then such a homogeneous and isobaric component of $\varphi(P) - \sum_{j=1}^{N} \lambda_j \varphi(P_j)$ is $Q := m - \sum_{i=1}^{M} \mu_i R_i$ due to the claim. Since, for every $1 \leqslant i \leqslant s$, I_s is homogeneous and isobaric, $Q \in \langle I_1, \ldots, I_s \rangle$.

Note that for every $1 \leqslant i \leqslant M$, the differential monomial R_i is a summand of $\varphi(P_j)$ for some $1 \leqslant j \leqslant N$. Thus, if $P_j = x^{(s_0)} \cdots x^{(s_\ell)}$, then the derivatives that appear in the monomial R_i are of orders $s_0, \ldots, s_\ell$. Hence, $P_j \prec P$ implies $R_j \prec m$. Therefore, m is the leading monomial of Q contradicting Lemma 4.17. ☐

Corollary 4.19. *The elements of $\mathcal{F}_{i-1,m-i}$ are standard monomials modulo $\langle x^i, (x^m)^{(\infty)} \rangle$.*

Proof. We will use Proposition 4.18. Consider the ideals

$$I_1 = \langle y_1^2 \rangle^{(\infty)}, \ldots, I_{i-1} = \langle y_{i-1}^2 \rangle^{(\infty)},\ I_i = \big\langle y_i, (y_i^2)^{(\infty)} \big\rangle, \ldots, I_{m-1} = \big\langle y_{m-1}, (y_{m-1}^2)^{(\infty)} \big\rangle,$$

and define φ as in Proposition 4.18. Lemma 4.4 implies that $\varphi((x^m)^{(k)}) = ((y_1 + \ldots + y_{m-1})^m)^{(k)} = 0$ for every $k \geq 1$ and $\varphi(x^i) = (y_1 + \ldots + y_{i-1})^i = 0$. Therefore, $\langle (x^m)^{(\infty)}, x^i \rangle \subset \operatorname{Ker}(\varphi)$. Proposition 4.16 implies that the standard monomials modulo I_j are the fair monomials for $j < i$ and strongly fair monomials for $i \leqslant j$. Therefore, Proposition 4.18 implies that $\mathcal{F}_{i-1,m-i}$ are standard monomials modulo $\langle x^i, (x^m)^{(\infty)} \rangle$. ☐

4.5. *Putting everything together: proofs of the main results.*

Proof of Theorem 3.1. Consider the images of $\mathcal{F}_{m-1,0} \cap k[x^{(\leqslant h)}]$ in $k[x^{(\infty)}]/\langle x^m \rangle^{(\infty)}$. By Corollary 4.19, they are linearly independent modulo $\langle x^m \rangle^{(\infty)}$. Then Proposition 4.12 implies that the dimension of $k[x^{(\leqslant h)}]/\langle x^m \rangle^{(\infty)}$ is at least m^{h+1}. Together with Proposition 4.5, this implies

$$\dim\big(k[x^{(\leqslant h)}]/\langle x^m \rangle^{(\infty)}\big) = m^{h+1}.$$ ☐

Proof of Theorem 3.3. Fix $h \geqslant 0$. Consider $\mathcal{F}_{i-1,m-i} \cap k[x^{(\leqslant h)}]$. Combining Corollary 4.19, Corollary 4.8, and Proposition 4.12, we show that the image of this set in $k[x^{(\leqslant h)}]/\langle (x^m)^{(\infty)}, x^i \rangle$ forms a basis. Thus, the image of the whole $\mathcal{F}_{i-1,m-i}$ is a basis of $k[x^{(\infty)}]/\langle (x^m)^{(\infty)}, x^i \rangle$. Therefore, by Corollary 4.19, $\mathcal{F}_{i-1,m-i}$ coincides with the set of standard monomials modulo $\langle (x^m)^{(\infty)}, x^i \rangle$. ☐

Proof of Corollary 3.4. Since the ideal $\langle x^i, (x^m)^{(\infty)} \rangle$ is generated by homogeneous and isobaric (that is, weight-homogeneous) polynomials, its Gröbner bases with respect to the purely lexicographic, degree lexicographic, and weighted lexicographic orderings coincide. ☐

5. Computational experiments for more general fat points

In this section, we consider a more general case of a fat point in a n-dimensional space, not just on a line. We used [Macaulay2], in particular, the package Jets [Galetto and Iammarino 2021; 2022] to explore possible analogues of our Theorem 3.1 for this more general case. A related Sage implementation for computing the arc space of an affine scheme with respect to a fat point can be found in [Stout 2017, §9] and [Stout 2014, §5.4].

Let $\boldsymbol{x} = (x_1, \dots, x_n)$, and consider a zero-dimensional ideal $I \subset k[\boldsymbol{x}]$. We will be interested in describing (in particular, in computing the dimension of the quotient ring) $I^{(\infty)} \cap k[\boldsymbol{x}^{(\leqslant h)}]$ for a positive integer h. Since this ideal is the union of the following chain

$$I^{(\leqslant 1)} \cap k[\boldsymbol{x}^{(\leqslant h)}] \subseteq I^{(\leqslant 2)} \cap k[\boldsymbol{x}^{(\leqslant h)}] \subseteq I^{(\leqslant 3)} \cap k[\boldsymbol{x}^{(\leqslant h)}] \subseteq \cdots$$

and $k[\boldsymbol{x}^{(\leqslant h)}]$ is Noetherian, one can compute $I^{(\infty)} \cap k[\boldsymbol{x}^{(\leqslant h)}]$ by computing $I^{(\leqslant H)} \cap k[\boldsymbol{x}^{(\leqslant h)}]$ for large enough H. But how do we determine what H is "large enough"?

- For $I = \langle x^m \rangle \subset k[x]$, the answer is given by Theorem 3.1: if the dimension $k[x^{(\leqslant h)}]/(I^{(\leqslant H)} \cap k[x^{(\leqslant h)}])$ is equal to m^{h+1}, then $I^{(\leqslant H)} \cap k[x^{(\leqslant h)}] = I^{(\infty)} \cap k[x^{(\leqslant h)}]$.
- For general I, we take H to be $1, 2, \dots$, and we stop when we encounter

$$I^{(\leqslant H)} \cap k[\boldsymbol{x}^{(\leqslant h)}] = I^{(\leqslant H+1)} \cap k[\boldsymbol{x}^{(\leqslant h)}].$$

  We conjecture that in this case $I^{(\leqslant H)} \cap k[\boldsymbol{x}^{(\leqslant h)}] = I^{(\infty)} \cap k[\boldsymbol{x}^{(\leqslant h)}]$ (see Question 5.1) but, strictly speaking, we only know that $I^{(\leqslant H)} \cap k[\boldsymbol{x}^{(\leqslant h)}] \subseteq I^{(\infty)} \cap k[\boldsymbol{x}^{(\leqslant h)}]$.

5.1. *Ideals $I = \langle x^m \rangle$.* For ideals of the form $\langle x^m \rangle$, the approach outlined above yields a complete algorithm to compute $I^{(\infty)} \cap k[x^{(\leqslant h)}]$ for any given h and m. We use it for computing examples of Gröbner bases for these ideals with respect to the lexicographic ordering, as shown in Table 1.

5.2. *General fat points.* In this subsection, we consider a general zero-dimensional $I \subset k[\boldsymbol{x}]$ with the zero set of I being the origin. We use the following algorithm following the approach described in the beginning of the section to obtain an upper bound of the dimensions of $k[\boldsymbol{x}^{(\leqslant h)}]/(I^{(\infty)} \cap k[\boldsymbol{x}^{(\leqslant h)}])$.

Step 1: Set $H = 1$.

Step 2: While the dimension of $I^{(\leqslant H)} \cap k[\boldsymbol{x}^{(\leqslant h)}]$ is not zero or $I^{(\leqslant H)} \cap k[\boldsymbol{x}^{(\leqslant h)}] \neq I^{(\leqslant H+1)} \cap k[\boldsymbol{x}^{(\leqslant h)}]$, set $H = H + 1$.

Ideal	Gröbner basis
$\langle x^2 \rangle^{(\infty)} \cap k[x^{(\leqslant 2)}]$	$(x'')^4;\ x'(x'')^2;\ (x')^2 x'';\ (x')^3;\ 2xx'' + (x')^2;\ xx';\ x^2$
$\langle x^3 \rangle^{(\infty)} \cap k[x^{(\leqslant 2)}]$	$(x'')^7;\ x'(x'')^5;\ (x')^2(x'')^4;\ (x')^3(x'')^2;\ (x')^4 x'';\ (x')^5;\ x(x'')^4 + 2(x')^2(x'')^3;$ $3xx'(x'')^2 + (x')^3 x'';\ 6x(x')^2 x'' + (x')^4;\ x(x')^3;\ x^2 x'' + x(x')^2;\ x^2 x';\ x^3$

Table 1. Gröbner bases for $\langle x^m \rangle^{(\infty)} \cap k[x^{(\leqslant h)}]$, where $m = 2, 3$.

Ideal	$h=0$	$h=1$	$h=2$	$h=3$
$\langle x^2, y^2, xy\rangle$	3	9	27	81
$\langle x^2, y^2, xz, yz, z^2-xy\rangle$	5	25	125	—
$\langle x^3, y^2, x^2y\rangle$	5	25	125	—
$\langle x^3, y^2, xy\rangle$	4	16	64	256
$\langle x^3, y^3, x^2y\rangle$	7	49	—	—
$\langle x^4, y^4, x^2y^3\rangle$	14	196	—	—

Table 2. (Bounds for) the dimensions of the truncations of the arc space.

Step 3: Return $\dim\big(k[\boldsymbol{x}^{(\leqslant h)}]/(I^{(\leqslant H)}\cap k[\boldsymbol{x}^{(\leqslant h)}])\big)$.

We expect the resulting bound to be exact (see also Question 5.1), for example, it is exact for $I=\langle x^m\rangle$.

Our implementation of this algorithm in [Macaulay2] is available for download at the following webpage: https://mathrepo.mis.mpg.de/MultiplicityStructureOfArcSpaces. Table 2 shows some of the results we obtained. One can see that the computed dimensions form geometric series with the exponent being the multiplicity of the original ideal exactly as in Theorem 3.1.

However, we have also found ideals for which the generating series of the dimensions is definitely not equal to $m/(1-mt)$, where m is the multiplicity of the ideal. We show some examples of this type in Table 3.

Note that while Table 2 gives only indication that the generating series of the multiplicities for these ideals may be $m/(1-mt)$, Table 3 gives a proof that this is not the case for all the fat points.

5.3. *Open questions.* Based on the results of the computational experiments, we formulate several open questions.

Question 5.1. Let $I\subset k[\boldsymbol{x}]$ be a zero-dimensional ideal with $V(I)$ being a single point. Is it true that, for every integer h

$$\big(I^{(\leqslant H)}\cap k[\boldsymbol{x}^{(\leqslant h)}]=I^{(\leqslant H+1)}\cap k[\boldsymbol{x}^{(\leqslant h)}]\big)\quad\Longrightarrow\quad\big(I^{(\leqslant H)}\cap k[\boldsymbol{x}^{(\leqslant h)}]=I^{(\leqslant\infty)}\cap k[\boldsymbol{x}^{(\leqslant h)}]\big)?$$

Does this statement remain true if we drop the assumption $|V(I)|=1$?

Ideal	$h=0$	$h=1$	$h=2$
$\langle x^3, y^3, xy\rangle$	5	24	115
$\langle x^4, y^3, xy\rangle$	6	33	—
$\langle x^4, y^3, x^2y\rangle$	8	62	—
$\langle x^4, y^4, xy\rangle$	7	42	—
$\langle x^4, y^4, x^2y\rangle$	10	94	—
$\langle x^4, y^4, x^2y^2\rangle$	12	140	—
$\langle x^4, y^6, x^2y^3\rangle$	18	320	—

Table 3. (Bounds for) the dimensions of the truncations of the arc space.

Question 5.2. Let $I \subset k[\boldsymbol{x}]$ be a zero-dimensional ideal with $V(I)$ being a single point of multiplicity m. Is it true that

$$\lim_{h\to\infty} \frac{\dim k[\boldsymbol{x}^{(\leqslant h)}]/I^{(\infty)}}{m^{h+1}} = 1?$$

Question 5.3. Let $I \subset k[\boldsymbol{x}]$ be a zero-dimensional ideal with $V(I)$ being a single point of multiplicity m. Under which conditions it is true that

$$\sum_{h=0}^{\infty} \bigl(\dim k[\boldsymbol{x}^{(\leqslant h)}]/I^{(\infty)}\bigr) \cdot t^h = \frac{m}{1-mt}?$$

More generally, what information about the corresponding scheme can be read off the above generating series?

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Theta correspondence and simple factors in global Arthur parameters

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By using results on poles of L-functions and theta correspondence, we give a bound on b for (χ, b)-factors of the global Arthur parameter of a cuspidal automorphic representation π of a classical group or a metaplectic group where χ is a conjugate self-dual automorphic character and b is an integer which is the dimension of an irreducible representation of $\mathrm{SL}_2(\mathbb{C})$. We derive a more precise relation when π lies in a generic global A-packet.

Introduction

Let F be a number field and let $\mathbb{A}$ be its ring of adeles. Let π be an irreducible cuspidal automorphic representation of a classical group G defined over F. We also treat the case of metaplectic groups in this work. However to avoid excessive notation, we focus on the case of the symplectic groups $G = \mathrm{Sp}(X)$ in this introduction where X is a nondegenerate symplectic space over F. By Arthur's theory of endoscopy [2013], π belongs to a global A-packet associated to an elliptic global A-parameter, which is of the form

$$\boxplus_{i=1}^{r} (\tau_i, b_i)$$

where τ_i is an irreducible self-dual cuspidal automorphic representation of $\mathrm{GL}_{n_i}(\mathbb{A})$ and b_i is a positive integer which represents the unique b_i-dimensional irreducible representation of Arthur's $\mathrm{SL}_2(\mathbb{C})$; see Section 2, for more details.

Jiang [2014] proposed the (τ, b)-theory; see, in particular, Principle 1.2 there. It is a conjecture that uses period integrals to link together automorphic representations in two global A-packets whose global A-parameters are "different" by a (τ, b)-factor. We explain in more details. Let Π_ϕ denote the global A-packet with elliptic global A-parameter ϕ. Let π be an irreducible automorphic representation of $G(\mathbb{A})$ and let σ be an irreducible automorphic representation of $H(\mathbb{A})$, where H is a factor of an endoscopic group of G. Assume that π (resp. σ) occurs in the discrete spectrum. Then it is expected that there exists some kernel function $\mathcal{K}$ depending on G, H and (τ, b) only such that if π and σ satisfy a Gan–Gross–Prasad type of criterion, namely, that the period integral

$$\int_{H(F)\backslash H(\mathbb{A})} \int_{G(F)\backslash G(\mathbb{A})} \mathcal{K}(h, g) f_\sigma(h) \overline{f_\pi(g)}\, dg\, dh \tag{0-1}$$

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is nonvanishing for some choice of $f_\sigma \in \sigma$ and $f_\pi \in \pi$, then π is in the global A-packet Π_ϕ if and only if σ is in the global A-packet Π_{ϕ_2} with $\phi = (\tau, b) \boxplus \phi_2$. Then Jiang [2014, Section 5] proceeds to construct certain kernel functions and then using them, defines endoscopy transfer (by integrating over $H(F) \setminus H(\mathbb{A})$ only in (0-1)) and endoscopy descent (by integrating over $G(F) \setminus G(\mathbb{A})$ only in (0-1)). It is not yet known if these are the kernel functions making the statements of Principle 1.2 in [Jiang 2014] hold. As the kernel functions come from Bessel coefficients or Fourier–Jacobi coefficients as in [Gan et al. 2012, Section 23], we see the nonvanishing of this period integral is analogous to condition (i) in the global Gan–Gross–Prasad conjecture [Gan et al. 2012, Conjecture 24.1].

Jiang [2014, Section 7] suggested that if τ is an automorphic character χ, then the kernel function can be taken to be the theta kernel and endoscopy transfer and endoscopy descent are theta lifts. In this case, the span of

$$\int_{G(F) \setminus G(\mathbb{A})} \mathcal{K}(h, g)\overline{f_\pi(g)}\, dg$$

as f_π runs over π is the theta lift of π. This is an automorphic representation of $H(\mathbb{A})$. Lifting in the other direction is analogous. Assume that the theta lift of π is nonzero. Write ϕ_π for the global A-parameter of π. Then Jiang [2014, Principle 1.2] says that ϕ_π has a (χ, b)-factor and that the global A-parameter of the theta lift of π from G to H should be ϕ_π with the (χ, b)-factor removed. Here b should be of appropriate size relative to G and H. Our work is one step in this direction.

One goal of this article is to expand on the (χ, b)-theory and to present the results of [Mœglin 1997; Ginzburg et al. 2009; Jiang and Wu 2016; 2018; Wu 2022a; 2022b] for various cases in a uniform way. As different reductive dual pairs that occur in theta correspondence have their own peculiarities, the notation and techniques of these papers are adapted to the treatment of their own specific cases. We attempt to emphasize on the common traits of the results which are buried in lengthy and technical proofs in these papers.

After collecting the results on poles of L-functions, poles of Eisenstein series and theta correspondence, we derive a bound for b when b is maximal among all factors of the global A-parameter of π. In addition, we derive an implication on global A-packets. Of course, the heavy lifting was done by the papers mentioned above.

Theorem 0.1 (Corollary 5.3). *The global A-packet attached to the elliptic global A-parameter ϕ cannot have a cuspidal member if ϕ has a (χ, b)-factor with*

$$b > \tfrac{1}{2}\dim_F X + 1, \quad \textit{if } G = \mathrm{Sp}(X).$$

Another way of phrasing this is that we have a bound on the size of b that can occur in a factor of type $(\chi, *)$ in the global A-parameter of a cuspidal automorphic representation. Thus our results have application in getting a Ramanujan bound, which measures the departure of the local components π_v from being tempered for all places v of F, for classical groups and metaplectic groups. This should follow by generalizing the arguments in [Jiang and Liu 2018, Section 5] which treats the symplectic case.

There they first established a bound for b under some conditions on wave front sets. This enables them to control the contribution of GL_1-factors in the global A-parameter to the Ramanujan bound. Our result can supply this ingredient for classical groups and also metaplectic groups unconditionally. Then Jiang and Liu [2018, Section 5] found a Ramanujan bound for π by using the crucial results on the Ramanujan bound for GL_2 in [Kim 2003; Blomer and Brumley 2011].

We describe the idea of the proof of our result. First we relate the existence of a (τ, b)-factor in the elliptic global A-parameter of π to the existence of poles of partial L-functions $L^S(s, \pi \times \tau^\vee)$; see Proposition 2.8. If the global A-parameter of π has a factor (τ, b) where b is maximal among all factors, we can show that the partial L-function $L^S(s, \pi \times \tau^\vee)$ has a pole at $s = \frac{1}{2}(b+1)$. Thus studying the location of poles of $L^S(s, \pi \times \tau^\vee)$ for τ running through all self-dual cuspidal representations of $\mathrm{GL}_n(\mathbb{A}_F)$ can shed light on the size of the b_i that occur in the global A-parameter of π. Then we specialize to the case where τ is a character χ and consider $L^S(s, \pi \times \chi^\vee)$ in what follows.

Next we relate the poles of $L^S(s, \pi \times \chi^\vee)$ to the poles of Eisenstein series attached to the cuspidal datum $\chi \boxtimes \pi$; see Section 3. In fact, in some cases, we use the nonvanishing of $L^S(s, \pi \times \chi^\vee)$ at $s = \frac{1}{2}$ instead; see Proposition 3.1. Then we recall in Theorem 3.5 that the maximal positive pole of the Eisenstein series has a bound which is supplied by the study of global theta lifts. This is enough for showing Corollary 5.3, though we have a more precise result that the maximal positive pole corresponds to the invariant called the lowest occurrence index of π with respect to χ in Theorem 4.4. The lowest occurrence index is the minimum of the first occurrence indices over some Witt towers. For the precise definition see (4-2). We also have a less precise result (Theorem 4.1) relating the first occurrence index of π with respect to certain quadratic spaces to possibly nonmaximal and possibly negative poles of the Eisenstein series.

More precise results can be derived if we assume that π has a generic global A-parameter. This is because we have a more precise result relating poles or nonvanishing of values of the complete L-functions to poles of the Eisenstein series supplied by [Jiang et al. 2013]. Thus we get

Theorem 0.2 (Theorem 6.7). *Let π be a cuspidal member in a generic global A-packet of $G(\mathbb{A}) = \mathrm{Sp}(X)(\mathbb{A})$. Let χ be a self-dual automorphic character of $\mathrm{GL}_1(\mathbb{A})$. Then the following are equivalent*:

(1) *The global A-parameter ϕ_π of π has a $(\chi, 1)$-factor.*

(2) *The complete L-function $L(s, \pi \times \chi^\vee)$ has a pole at $s = 1$.*

(3) *The Eisenstein series $E(g, f_s)$ has a pole at $s = 1$ for some choice of section $f_s \in \mathcal{A}^{Q_1}(s, \chi \boxtimes \pi)$.*

(4) *The lowest occurrence index $\mathrm{LO}_X^\chi(\pi)$ is $\dim X$.*

Here Q_1 is a parabolic subgroup of $\mathrm{Sp}(X_1)$ with Levi subgroup isomorphic to $\mathrm{GL}_1 \times \mathrm{Sp}(X)$, where X_1 is the symplectic space formed from X by adjoining a hyperbolic plane. Roughly speaking, $\mathcal{A}^{Q_1}(s, \chi \boxtimes \pi)$ is a space of automorphic forms on $\mathrm{Sp}(X_1)$ induced from $\chi|\cdot|^s \boxtimes \pi$ viewed as a representation of the parabolic subgroup Q_1. We refer the reader to Section 3 for the precise definition of $\mathcal{A}^{Q_1}(s, \chi \boxtimes \pi)$. We note that the lowest occurrence index $\mathrm{LO}_X^\chi(\pi)$ is an invariant in the theory of theta correspondence related to the invariant called the first occurrence index; see Section 4 for their definitions. We also include a result (Theorem 6.3) that concerns the nonvanishing of $L(s, \pi \times \chi^\vee)$ at $s = \frac{1}{2}$ and the lowest occurrence index. We plan to improve this result in the future by studying a relation between nonvanishing of Bessel or Fourier–Jacobi periods and the lowest occurrence index.

We note that the L-function $L(s, \pi \times \chi^\vee)$ has been well-studied and is intricately entwined with the study of theta correspondence, most prominently in the Rallis inner product formula which says that the inner product of two theta lifts is equal to the residue or value of $L(s, \pi \times \chi^\vee)$ at an appropriate point up to some ramified factors and some abelian L-functions. We refer the reader to [Yamana 2014] which is a culmination of many previous results. In our approach, the Eisenstein series $E(g, f_s)$, which is not of Siegel type, is the key link between $L(s, \pi \times \chi^\vee)$ and the theta lifts.

Now we describe the structure of this article. In Section 1, we set up some basic notation. In Section 2, we define elliptic global A-parameters for classical groups and metaplectic groups and also the global A-packet associated to an elliptic global A-parameter. We show how poles of partial L-functions detect (τ, b)-factors in an elliptic global A-parameter. In Section 3, we define Eisenstein series attached to the cuspidal datum $\chi \boxtimes \pi$ and recall some results on the possible locations of their maximal positive poles. In Section 4, we introduce two invariants of theta correspondence. They are the first occurrence index $\mathrm{FO}_X^{Y,\chi}(\pi)$ and the lowest occurrence index $\mathrm{LO}_X^{\chi}(\pi)$ of π with respect to some data. We relate them to poles of Eisenstein series. Results in Sections 3 and 4 are not new. Our aim is to present the results in a uniform way for easier access. In Section 5, we show a bound for b in (χ, b)-factors of the global A-parameter of π. Finally in Section 6, we consider the case when π has a generic global A-parameter. We show that when $L(s, \pi \times \chi^\vee)$ has a pole at $s = 1$ (resp. $L(s, \pi \times \chi^\vee)$ is nonvanishing at $s = \frac{1}{2}$), the lowest occurrence index is determined.

1. Notation

Let F be a number field and let E be either F or a quadratic field extension of F. Let $\varrho \in \mathrm{Gal}(E/F)$ be the trivial Galois element when $E = F$ and the nontrivial Galois element when $E \neq F$. When $E \neq F$, write $\varepsilon_{E/F}$ for the quadratic character associated to E/F via Class field theory. Let G be an algebraic group over E. We write $\mathrm{R}_{E/F} G$ for the restriction of scalars of Weil. This is an algebraic group over F.

Let ϵ be either 1 or -1. By an ϵ-skew Hermitian space, we mean an E-vector space X together with an F-bilinear pairing

$$\langle \cdot, \cdot \rangle_X : X \times X \to E$$

such that

$$\langle y, x \rangle_X = -\epsilon \langle x, y \rangle_X^\varrho, \quad \langle ax, by \rangle = a \langle x, y \rangle_X b^\varrho$$

for all $a, b \in E$ and $x, y \in X$. We consider the linear transformations of X to act from the right. We follow the notation from [Yamana 2014] closely and we intend to generalize the results here to the quaternionic unitary group case in our future work.

Let X be an ϵ-skew Hermitian space of finite dimension. Then the isometry group of X is one of the following:

(1) The symplectic group $\mathrm{Sp}(X)$ when $E = F$ and $\epsilon = 1$.

(2) The orthogonal group $\mathrm{O}(X)$ when $E = F$ and $\epsilon = -1$.

(3) The unitary group $U(X)$ when $E \neq F$ and $\epsilon = \pm 1$.

We will also consider the metaplectic group. Let v be a place of F and let F_v denote the completion of F at v. Let $\mathbb{A}_F$ (resp. $\mathbb{A}_E$) denote the ring of adeles of F (resp. E). Set $\mathbb{A} := \mathbb{A}_F$. Write $\mathrm{Mp}(X)(F_v)$ (resp. $\mathrm{Mp}(X)(\mathbb{A}_F)$) for the metaplectic double cover of $\mathrm{Sp}(X)(F_v)$ (resp. $\mathrm{Sp}(X)(\mathbb{A}_F)$) defined by Weil [1964]. We note that the functor $\mathrm{Mp}(X)$ is not representable by an algebraic group. We will also need the $\mathbb{C}^1$-extension $\mathrm{Mp}(X)(F_v) \times_{\mu_2} \mathbb{C}^1$ of $\mathrm{Sp}(X)(F_v)$ and we denote it by $\mathrm{Mp}^{\mathbb{C}^1}(F_v)$. Similarly we define $\mathrm{Mp}^{\mathbb{C}^1}(\mathbb{A}_F)$.

Let ψ be a nontrivial automorphic additive character of $\mathbb{A}_F$ which will figure in the Weil representations as well as the global A-parameters for $\mathrm{Mp}(X)$.

For an automorphic representation or admissible representation π, we write $\pi^\vee$ for its contragredient.

2. Global Arthur parameters

First we recall the definition of elliptic global Arthur parameters (A-parameters) for classical groups as well as metaplectic groups; see [Arthur 2013] for the symplectic and the special orthogonal case and we adopt the formulation in [Atobe and Gan 2017] for the case of the (disconnected) orthogonal groups. For the unitary case, see [Mok 2015; Kaletha et al. 2014]. For the metaplectic case, see [Gan and Ichino 2018]. Then we focus on simple factors of global Arthur parameters and relate their presence to poles of partial L-functions. This is a crude first step for detecting (τ, b)-factors in an elliptic global A-parameter according to the "(τ, b)-theory" proposed in [Jiang 2014].

Let $\boldsymbol{G}$ be $\mathrm{U}(X)$, $\mathrm{O}(X)$, $\mathrm{Sp}(X)$ or $\mathrm{Mp}(X)$. Let d denote the dimension of X. Set $\boldsymbol{G}^\circ = \mathrm{SO}(X)$ when $\boldsymbol{G} = \mathrm{O}(X)$. Set $\boldsymbol{G}^\circ = \boldsymbol{G}$ otherwise. Write $\check{\boldsymbol{G}}$ for the (complex) dual group of $\boldsymbol{G}^\circ$. Then

$$\check{\boldsymbol{G}} = \begin{cases} \mathrm{GL}_d(\mathbb{C}) & \text{if } \boldsymbol{G} = \mathrm{U}(X); \\ \mathrm{Sp}_{d-1}(\mathbb{C}) & \text{if } \boldsymbol{G} = \mathrm{O}(X) \text{ and } d \text{ is odd}; \\ \mathrm{SO}_d(\mathbb{C}) & \text{if } \boldsymbol{G} = \mathrm{O}(X) \text{ and } d \text{ is even}; \\ \mathrm{SO}_{d+1}(\mathbb{C}) & \text{if } \boldsymbol{G} = \mathrm{Sp}(X); \\ \mathrm{Sp}_d(\mathbb{C}) & \text{if } \boldsymbol{G} = \mathrm{Mp}(X). \end{cases}$$

An elliptic global A-parameter for $\boldsymbol{G}$ is a finite formal sum of the form

$$\phi = \boxplus_{i=1}^r (\tau_i, b_i), \quad \text{for some positive integer } r$$

where

(1) τ_i is an irreducible conjugate self-dual cuspidal automorphic representation of $\mathrm{GL}_{n_i}(\mathbb{A}_E)$;

(2) b_i is a positive integer which represents the unique b_i-dimensional irreducible representation of Arthur's $\mathrm{SL}_2(\mathbb{C})$

such that:

- $\sum_i n_i b_i = d_{\check{G}}$.
- τ_i is conjugate self-dual of parity $(-1)^{N_{\check{G}}+b_i}$ (see Remark 2.3).
- The factors (τ_i, b_i) are pairwise distinct.

Here $d_{\check{G}}$ is the degree of the standard representation of $\check{\boldsymbol{G}}$ which, explicitly, is

$$d_{\check{G}} = \begin{cases} \dim X & \text{if } \boldsymbol{G} = \mathrm{U}(X); \\ \dim X - 1 & \text{if } \boldsymbol{G} = \mathrm{O}(X) \text{ with } \dim X \text{ odd}; \\ \dim X & \text{if } \boldsymbol{G} = \mathrm{O}(X) \text{ with } \dim X \text{ even}; \\ \dim X + 1 & \text{if } \boldsymbol{G} = \mathrm{Sp}(X); \\ \dim X & \text{if } \boldsymbol{G} = \mathrm{Mp}(X); \end{cases}$$

and

$$N_{\check{G}} = \begin{cases} \dim X \mod 2 & \text{if } \boldsymbol{G} = \mathrm{U}(X); \\ 0 & \text{if } \boldsymbol{G} = \mathrm{O}(X) \text{ with } \dim X \text{ odd}; \\ 1 & \text{if } \boldsymbol{G} = \mathrm{O}(X) \text{ with } \dim X \text{ even}; \\ 1 & \text{if } \boldsymbol{G} = \mathrm{Sp}(X); \\ 0 & \text{if } \boldsymbol{G} = \mathrm{Mp}(X). \end{cases}$$

Remark 2.1. We adopt the notation in [Jiang 2014] and hence we write (τ_i, b_i) rather than $\tau_i \boxtimes \nu_{b_i}$ as is more customary in the literature, so that the quantity b_i, that we study, is more visible.

Remark 2.2. In the unitary case, we basically spell out what $\Psi_2(\mathrm{U}(N), \xi_{\mathbf{1}})$ in [Mok 2015, Definition 2.4.7] is. We have discarded the second factor $\tilde{\psi}$ as it is determined by ψ^N and $\xi_{\mathbf{1}}$ in Mok's notation.

Remark 2.3. (1) For $\boldsymbol{G} = \mathrm{U}(X)$, we say that τ is conjugate self-dual of parity η if the Asai L-function $L(s, \tau, \mathrm{Asai}^{\eta})$ has a pole at $s = 1$. If $\eta = +1$, we also say that τ is conjugate orthogonal and if $\eta = -1$, we also say that τ is conjugate symplectic. The Asai representations come from the decomposition of the twisted tensor product representation of the L-group; see [Mok 2015, (2.2.9) and (2.5.9)] and [Goldberg 1994].

(2) For other cases, we mean self-dual when we write conjugate self-dual. We say that τ is self-dual of parity $+1$ or orthogonal, if $L(s, \tau, \mathrm{Sym}^2)$ has a pole at $s = 1$; we say that τ is self-dual of parity -1 or symplectic, if $L(s, \tau, \wedge^2)$ has a pole at $s = 1$.

(3) The parity is uniquely determined for each irreducible conjugate self-dual cuspidal representation τ.

Let $\Psi_2(\boldsymbol{G})$ denote the set of elliptic global A-parameters of $\boldsymbol{G}$. Let $\phi \in \Psi_2(\boldsymbol{G})$. Via the local Langlands conjecture (which is proved for the general linear groups), at every place v of F, we localize ϕ to get an elliptic local A-parameter,

$$\phi_v : L_{F_v} \times \mathrm{SL}_2(\mathbb{C}) \to \check{\boldsymbol{G}} \rtimes W_{F_v},$$

where W_{F_v} is the Weil group of F_v and L_{F_v} is W_{F_v} if v is archimedean and the Weil–Deligne group $W_{F_v} \times \mathrm{SL}_2(\mathbb{C})$ if v is nonarchimedean. To ϕ_v we associate the local L-parameter $\varphi_{\phi_v} : L_{F_v} \to \check{\boldsymbol{G}} \rtimes W_{F_v}$ given by

$$\varphi_{\phi_v}(w) = \phi_v\left(w, \begin{pmatrix} |w|^{1/2} & \\ & |w|^{-1/2} \end{pmatrix}\right).$$

Let $L^2_{\mathrm{disc}}(\boldsymbol{G})$ denote the discrete part of $L^2(\boldsymbol{G}(F) \setminus \boldsymbol{G}(\mathbb{A}_F))$ when $\boldsymbol{G} \neq \mathrm{Mp}(X)$ and the genuine discrete part of $L^2(\mathrm{Sp}(F) \setminus \mathrm{Mp}(\mathbb{A}_F))$ for $\boldsymbol{G} = \mathrm{Mp}(X)$. Define the full near equivalence class $L^2_{\phi,\psi}(\boldsymbol{G})$ attached to the elliptic global A-parameter ϕ to be the Hilbert direct sum of all irreducible automorphic representations σ occurring in $L^2_{\mathrm{disc}}(\boldsymbol{G})$ such that for almost all v, the local L-parameter of σ_v is φ_{ϕ_v}. We remark that in the $\mathrm{Mp}(X)$-case, the parametrization of σ_v is relative to ψ_v since the local L-parameter of σ_v is attached via the Shimura–Waldspurger correspondence which depends on ψ_v. This is the only case in this article where $L^2_{\phi,\psi}(\boldsymbol{G})$ depends on ψ.

Let $\mathcal{A}_2(\boldsymbol{G})$ denote the dense subspace consisting of automorphic forms in $L^2_{\mathrm{disc}}(\boldsymbol{G})$. Similarly define $\mathcal{A}_{2,\phi,\psi}(\boldsymbol{G})$ to be the dense subspace of $L^2_{\phi,\psi}(\boldsymbol{G})$ consisting of automorphic forms. Then we have a crude form of Arthur's multiplicity formula which decomposes the L^2-discrete spectrum into near equivalence classes indexed by $\Psi_2(\boldsymbol{G})$.

Theorem 2.4. *We have the orthogonal decompositions*

$$L^2_{\mathrm{disc}}(\boldsymbol{G}) = \widehat{\bigoplus_{\phi \in \Psi_2(\boldsymbol{G})}} L^2_{\phi,\psi}(\boldsymbol{G}) \quad \textit{and} \quad \mathcal{A}_2(\boldsymbol{G}) = \bigoplus_{\phi \in \Psi_2(\boldsymbol{G})} \mathcal{A}_{2,\phi,\psi}(\boldsymbol{G}).$$

Remark 2.5. This crude form of Arthur's multiplicity formula has been proved for $\mathrm{Sp}(X)$ and quasisplit $\mathrm{O}(X)$ by Arthur [2013], for $\mathrm{U}(X)$ by [Mok 2015; Kaletha et al. 2014] and for $\mathrm{Mp}(X)$ by [Gan and Ichino 2018]. This is also proved for nonquasisplit even orthogonal (and also unitary groups) in [Chen and Zou 2021] and for nonquasisplit odd orthogonal groups in [Ishimoto 2023]. Thus for all cases needed in this paper, Theorem 2.4 is known.

We have some further remarks on the orthogonal and unitary cases.

Remark 2.6. Arthur's statements use $\mathrm{SO}(X)$ rather than $\mathrm{O}(X)$ and he needs to account for the outer automorphism of $\mathrm{SO}(X)$ when $\dim X$ is even; see the paragraph below [Arthur 2013, Theorem 1.5.2]. The formulation for quasisplit even $\mathrm{O}(X)$ is due to Atobe and Gan [2017, Theorem 7.1(1)]. For odd $\mathrm{O}(X)$, which is isomorphic to $\mathrm{SO}(X) \times \mu_2$, the reformulation of Arthur's result is easy. Let T be a finite set of places of F. Assume that it has even cardinality. Let sgn_T be the automorphic character of $\mu_2(\mathbb{A}_F)$ which is equal to the sign character at places in T and the trivial character at places outside T. These give all the automorphic characters of $\mu_2(\mathbb{A}_F)$. Then every irreducible automorphic representation π of $\mathrm{O}(X)(\mathbb{A}_F)$ is of the form $\pi_0 \boxtimes \mathrm{sgn}_T$ for some irreducible automorphic representation π_0 of $\mathrm{SO}(X)(\mathbb{A}_F)$ and some finite set T of places of even cardinality. A near equivalence class of $\mathrm{O}(X)(\mathbb{A}_F)$ then consists of all irreducible automorphic representations $\pi_0 \boxtimes \mathrm{sgn}_T$ for π_0 running over a near equivalence class of $\mathrm{SO}(X)(\mathbb{A}_F)$ and sgn_T running over all automorphic characters of $\mu_2(\mathbb{A}_F)$.

Remark 2.7. For the $\mathrm{U}(X)$ case, the global A-parameter depends on the choice of a sign and a conjugate self-dual character which determine an embedding of the L-group of $\mathrm{U}(X)$ to the L-group of $\mathrm{R}_{E/F}\,\mathrm{GL}_d$ where we recall that $d := \dim X$. We refer the reader to [Mok 2015, Section 2.1], in particular (2.1.9) there, for details. In this work, we choose the $+1$ sign and the trivial character, which, in Mok's notation, means $\kappa = 1$ and $\chi_\kappa = \mathbf{1}$. Then this corresponds to the standard base change of $\mathrm{U}(X)$ to $\mathrm{R}_{E/F}\,\mathrm{GL}_d$. We note that the L-functions we use below are such that

$$L_v(s, \pi_v \times \tau_v) = L_v(s, \mathrm{BC}(\pi_v) \otimes \tau_v),$$

for all places v, automorphic representations π of $\boldsymbol{G}(\mathbb{A}_F)$ and τ of $\mathrm{R}_{E/F}\,\mathrm{GL}_a(\mathbb{A}_F)$ where BC denotes the standard base change.

By Theorem 2.4, we get

Proposition 2.8. *Let π be an irreducible automorphic representation of $\boldsymbol{G}(\mathbb{A}_F)$ that occurs in $\mathcal{A}_{2,\phi,\psi}(\boldsymbol{G})$. Then*:

(1) *If ϕ has a (τ, b)-factor with b maximal among all factors, then the partial L-function $L^S(s, \pi \times \tau^\vee)$ has a pole at $s = \frac{1}{2}(b+1)$ and this is its maximal pole.*

(2) *if the partial L-function $L^S(s, \pi \times \tau^\vee)$ has a pole at $s = \frac{1}{2}(b'+1)$, then ϕ has a (τ, b)-factor with $b \geq b'$.*

Remark 2.9. In the $\mathrm{Mp}(X)$ case, the L-function depends on ψ, but we suppress it from notation here.

Proof. First we collect some properties of the Rankin–Selberg L-functions for $\mathrm{GL}_m \times \mathrm{GL}_n$. By the Rankin–Selberg method, for an irreducible unitary cuspidal automorphic representation τ, $L^S(s, \tau \times \tau^\vee)$ has a simple pole at $s = 1$ and is nonzero holomorphic for $\mathrm{Re}(s) \geq 1$ and $s \neq 1$; for irreducible unitary cuspidal automorphic representations τ and τ' such that $\tau \not\cong \tau'$, $L^S(s, \tau \times \tau'^\vee)$ is nonzero holomorphic for $\mathrm{Re}(s) \geq 1$. These results can be found in Cogdell's notes [2000] which collect the results from [Jacquet et al. 1983; Jacquet and Shalika 1976; Shahidi 1978; 1980].

Assume that $\phi = \boxplus_{i=1}^r (\tau_i, b_i)$. Then

$$L^S(s, \pi \times \tau^\vee) = \prod_{i=1}^r \prod_{j=0}^{b_i - 1} L^S\big(s - \tfrac{1}{2}(b_i - 1) + j, \tau_i \times \tau^\vee\big),$$

where S is a finite set of places of F outside of which all data are unramified.

Assume that ϕ has a (τ, b)-factor with b maximal among all factors, then by the properties of the Rankin–Selberg L-functions, we see that $L^S(s, \pi \times \tau^\vee)$ has a pole at $s = \frac{1}{2}(b+1)$ and it is maximal.

Next assume that the partial L-function $L^S(s, \pi \times \tau^\vee)$ has a pole at $s = \frac{1}{2}(b'+1)$. If ϕ has no (τ, c)-factor for any $c \in \mathbb{Z}_{>0}$, then $L^S(s, \pi \times \tau^\vee)$ is holomorphic for all $s \in \mathbb{C}$ and we get a contradiction. Thus ϕ has a (τ, b)-factor. We take b maximal among all factors of the form $(\tau, *)$ in ϕ. As b may not be maximal among all simple factors of ϕ, we can only conclude that $L^S(s, \pi \times \tau^\vee)$ is holomorphic for $\mathrm{Re}(s) > \frac{1}{2}(b+1)$. Thus $b' \leq b$. ∎

Given an irreducible cuspidal automorphic representation π, write ϕ_π for the global A-parameter of π. By studying poles of $L^S(s, \pi \times \tau^\vee)$ for varying τ, we can detect the existence of (τ, b)-factors with maximal b in ϕ_π. We would also like to construct an irreducible cuspidal automorphic representation with global A-parameter $\phi_\pi \boxminus (\tau, b)$ which means removing the (τ, b)-factor from ϕ_π if ϕ_π has a (τ, b)-factor. Doing this recursively, we will be able to compute the global A-parameter of a given irreducible cuspidal automorphic representation. In reverse, the construction should produce concrete examples of cuspidal automorphic representations in a given global A-packet with an elliptic global A-parameter. This will be investigated in our future work.

In this article, we focus our attention on the study of poles of $L^S(s, \pi \times \tau^\vee)$ where τ is a conjugate self-dual irreducible cuspidal automorphic representation of $\mathrm{R}_{E/F}\,\mathrm{GL}_1(\mathbb{A})$. Now we write χ for τ to emphasize that we are considering the case of twisting by characters. This case has been well-studied and it is known that the poles of $L^S(s, \pi \times \chi^\vee)$ are intricately related to invariants of theta correspondence via the Rallis inner product formula which relates the inner product of two theta lifts to a residue or a value of the L-function. We refer the readers to [Kudla and Rallis 1994; Wu 2017; Gan et al. 2014; Yamana 2014] for details. One of the key steps is the regularized Siegel–Weil formula which relates a theta integral to a residue or a value of a Siegel–Eisenstein series. Our work considers an Eisenstein series which is not of Siegel type, but which is closely related to $L(s, \pi \times \chi^\vee)$.

3. Eisenstein series attached to $\chi \boxtimes \pi$

In this section we deviate slightly from the notation in Section 2. We use $G(X)$ to denote one of $\mathrm{Sp}(X)$, $\mathrm{O}(X)$ and $\mathrm{U}(X)$. We let $\boldsymbol{G}(X)$ be a cover group of $G(X)$, which means $\boldsymbol{G}(X) = \mathrm{Sp}(X)$ or $\mathrm{Mp}(X)$ if $G(X) = \mathrm{Sp}(X)$, $\boldsymbol{G}(X) = \mathrm{O}(X)$ if $G(X) = \mathrm{O}(X)$ and $\boldsymbol{G}(X) = \mathrm{U}(X)$ if $G(X) = \mathrm{U}(X)$. We adopt similar notation to that in [Mœglin and Waldspurger 1995]. We define Eisenstein series on a larger group of the same type as $\boldsymbol{G}(X)$ and collect some results on their maximal positive poles.

Let π be an irreducible cuspidal automorphic representation of $\boldsymbol{G}(X)(\mathbb{A})$. We always assume that π is genuine when $\boldsymbol{G}(X) = \mathrm{Mp}(X)$. Let χ be a conjugate self-dual automorphic character of $\mathrm{R}_{E/F}\,\mathrm{GL}_1(\mathbb{A}) = \mathbb{A}_E^\times$. When $E \neq F$, we define

$$\epsilon_\chi = \begin{cases} 0 & \text{if } \chi|_{\mathbb{A}_F^\times} = \mathbb{1}; \\ 1 & \text{if } \chi|_{\mathbb{A}_F^\times} = \varepsilon_{E/F}. \end{cases} \tag{3-1}$$

Let a be a positive integer. Let X_a be the ϵ-skew Hermitian space over E that is formed from X by adjoining a-copies of the hyperbolic plane. More precisely, let ℓ_a^+ (resp. ℓ_a^-) be a totally isotropic a-dimensional E-vector space spanned by $e_1^+, \dots, e_a^+$ (resp. $e_1^-, \dots, e_a^-$) such that $\langle e_i^+, e_j^- \rangle = \delta_{ij}$ where δ_{ij} is the Kronecker symbol. Then

$$X_a = \ell_a^+ \oplus X \oplus \ell_a^-$$

with X orthogonal to $\ell_a^+ \oplus \ell_a^-$.

Let $G(X_a)$ be the isometry group of X_a. Let Q_a be the parabolic subgroup of $G(X_a)$ that stabilizes ℓ_a^-. Write $Q_a = M_a N_a$ in the Levi decomposition with N_a being the unipotent radical and M_a the standard Levi subgroup. We have an isomorphism

$$m : \mathrm{R}_{E/F}\,\mathrm{GL}_a \times G(X) \to M_a.$$

where we identify $\mathrm{R}_{E/F}\,\mathrm{GL}_a$ with $\mathrm{R}_{E/F}\,\mathrm{GL}(\ell_a^+)$. Let ρ_{Q_a} be the half sum of the positive roots in N_a, which can be viewed as an element in $\mathfrak{a}_{M_a}^* := \mathrm{Rat}(M_a) \otimes_{\mathbb{Z}} \mathbb{R}$ where $\mathrm{Rat}(M_a)$ is the group of rational characters of M_a. We note that as Q_a is a maximal parabolic subgroup, $\mathfrak{a}_{M_a}^*$ is one-dimensional. Via the Shahidi normalization [2010], we identify $\mathfrak{a}_{M_a}^*$ with $\mathbb{R}$ and thus may regard ρ_{Q_a} as the real number

$$\begin{aligned} &\tfrac{1}{2}(\dim_E X + a), && \text{if } G(X_a) \text{ is unitary;}\\ &\tfrac{1}{2}(\dim_E X + a - 1), && \text{if } G(X_a) \text{ is orthogonal;}\\ &\tfrac{1}{2}(\dim_E X + a + 1), && \text{if } G(X_a) \text{ is symplectic.} \end{aligned}$$

Let $K_{a,v}$ be a good maximal compact subgroup of $G(X_a)(F_v)$ in the sense that the Iwasawa decomposition holds and set $K_a = \prod_v K_{a,v}$.

Let $\mathcal{A}^{Q_a}(s, \chi \boxtimes \pi)$ denote the space of $\mathbb{C}$-valued smooth functions f on $N_a(\mathbb{A})M_a(F) \backslash G(X_a)(\mathbb{A})$ such that:

(1) f is right K_a-finite.

(2) For any $x \in \mathrm{R}_{E/F}\,\mathrm{GL}_a(\mathbb{A})$ and $g \in G(X_a)(\mathbb{A})$ we have

$$f(m(x, I)g) = \chi(\det(x))|\det(x)|_{\mathbb{A}_E}^{s+\rho_{Q_a}} f(g).$$

(3) For any fixed $k \in K_a$, the function $h \mapsto f(m(I, h)k)$ on $G(X)(\mathbb{A})$ is in the space of π.

Now let $\boldsymbol{G}(X) = \mathrm{Mp}(X)$. This case depends on ψ. Let $\widetilde{\mathrm{GL}}_1(F_v)$ be the double cover of $\mathrm{GL}_1(F_v)$ defined as follows. As a set it is $\mathrm{GL}_1(F_v) \times \mu_2$ and the multiplication is given by

$$(g_1, \zeta_1)(g_2, \zeta_2) = (g_1 g_2, \zeta_1\zeta_2(g_1, g_2)_{F_v})$$

which has a Hilbert symbol twist when multiplying the μ_2-part. Analogously we define the double cover $\widetilde{\mathrm{GL}}_1(\mathbb{A})$ of $\mathrm{GL}_1(\mathbb{A})$. Let $\chi_{\psi,v}$ denote the genuine character of $\widetilde{\mathrm{GL}}_1(F_v)$ defined by

$$\chi_{\psi,v}((g, \zeta)) = \zeta\gamma_v(g, \psi_{1/2,v})^{-1}$$

where $\gamma_v(\,\cdot\,, \psi_{1/2,v})$ is a fourth root of unity defined via the Weil index. It is the same one as in [Gan and Ichino 2014, page 521] except that we have put in the subscripts v. Then

$$\chi_\psi((g, \zeta)) = \zeta \prod_v \gamma_v(g_v, \psi_{1/2,v})^{-1}$$

is a genuine automorphic character of $\widetilde{\mathrm{GL}}_1(\mathbb{A})$. Let $\widetilde{K}_a$ denote the preimage of K_a under the projection $\mathrm{Mp}(X_a)(\mathbb{A}) \to \mathrm{Sp}(X_a)(\mathbb{A})$. We will also use $\widetilde{\ }$ to denote the preimages of other subgroups of $\mathrm{Sp}(X_a)(\mathbb{A})$.

Let $\widetilde{m}$ be the isomorphism

$$\widetilde{\mathrm{GL}}_a(\mathbb{A}) \times_{\mu_2} \boldsymbol{G}(X)(\mathbb{A}) \to \widetilde{M}_a(\mathbb{A})$$

that lifts $m : \mathrm{GL}_a(\mathbb{A}) \times G(X)(\mathbb{A}) \to M_a(\mathbb{A})$. Let $\widetilde{\det}$ be the homomorphism

$$\begin{aligned}\widetilde{\mathrm{GL}}_a(\mathbb{A}) &\to \widetilde{\mathrm{GL}}_1(\mathbb{A})\\ (x, \zeta) &\mapsto (\det(x), \zeta).\end{aligned}$$

We keep writing det for the nongenuine homomorphism

$$\begin{aligned}\widetilde{\mathrm{GL}}_a(\mathbb{A}) &\to \mathrm{GL}_1(\mathbb{A})\\ (x, \zeta) &\mapsto \det(x).\end{aligned}$$

Given a nongenuine representation τ of $\widetilde{\mathrm{GL}}_a(\mathbb{A})$, we can twist it by $\chi_\psi \circ \widetilde{\det}$ to get a genuine representation which we denote by $\tau\chi_\psi$.

We remark that there are canonical embeddings of $N_a(\mathbb{A})$ and $\mathrm{Sp}(X_a)(F)$ to $\mathrm{Mp}(X_a)(\mathbb{A})$, so we may regard them as subgroups of $\boldsymbol{G}(X_a)(\mathbb{A})$. Let $\mathcal{A}_\psi^{Q_a}(s, \chi \boxtimes \pi)$ denote the space of $\mathbb{C}$-valued smooth functions f on $N_a(\mathbb{A})M_a(F) \backslash \boldsymbol{G}(X_a)(\mathbb{A})$ such that:

(1) f is right $\widetilde{K}_a$-finite.

(2) For any $x \in \widetilde{\mathrm{GL}}_a(\mathbb{A})$ and $g \in \boldsymbol{G}(X_a)(\mathbb{A})$ we have

$$f(\widetilde{m}(x, I)g) = \chi\chi_\psi(\widetilde{\det}(x))|\det(x)|_{\mathbb{A}_E}^{s+\rho_{Q_a}} f(g).$$

(3) For any fixed $k \in \widetilde{K}_a$, the function $h \mapsto f(\widetilde{m}(I, h)k)$ on $\boldsymbol{G}(X)(\mathbb{A})$ is in the space of π.

To unify notation, we will also write $\mathcal{A}_\psi^{Q_a}(s, \chi \boxtimes \pi)$ for $\mathcal{A}^{Q_a}(s, \chi \boxtimes \pi)$ in the nonmetaplectic case. It should be clear from the context whether we are treating the $\mathrm{Sp}(X)$ case or the $\mathrm{Mp}(X)$ case.

Now return to the general case, so $\boldsymbol{G}(X)$ is one of $\mathrm{Sp}(X)$, $\mathrm{O}(X)$, $\mathrm{U}(X)$ and $\mathrm{Mp}(X)$. Let f_s be a holomorphic section of $\mathcal{A}_\psi^{Q_a}(s, \chi \boxtimes \pi)$. We associate to it the Eisenstein series

$$E_\psi^{Q_a}(g, f_s) := \sum_{\gamma \in Q_a(F) \backslash G(X_a)(F)} f_s(\gamma g).$$

Note that the series is over $\gamma \in Q_a(F) \backslash \mathrm{Sp}(X_a)(F)$ when $\boldsymbol{G}(X) = \mathrm{Mp}(X)$. By Langlands' theory of Eisenstein series [Mœglin and Waldspurger 1995, IV.1], this series is absolutely convergent for $\mathrm{Re}(s) > \rho_{Q_a}$, has meromorphic continuation to the whole s-plane, its poles lie on root hyperplanes and there are only finitely many poles in the positive Weyl chamber. By our identification of $\mathfrak{a}_{M_a}^*$ with $\mathbb{R}$ and the fact that χ is conjugate self-dual, the statements on poles mean that the poles are all real and that there are finitely many poles in the half-plane $\mathrm{Re}(s) > 0$.

We give the setup for any positive integer a, though we will only need $a = 1$ in the statements of our results. However the proofs require "going up the Witt tower" to $\boldsymbol{G}(X_a)$ for a large enough. Since we plan to prove analogous results for quaternionic unitary groups in the future, we keep the setup for general a.

There is a relation between poles of L-functions and the Eisenstein series.

Proposition 3.1. (1) *Assume that the partial L-function $L_\psi^S(s, \pi \times \chi^\vee)$ has its rightmost positive pole at $s = s_0$. Then $E_\psi^{Q_1}(g, f_s)$ has a pole at $s = s_0$.*

(2) *Assume that the partial L-function $L_\psi^S(s, \pi \times \chi^\vee)$ is nonvanishing at $s = \frac{1}{2}$ and is holomorphic for $\mathrm{Re}(s) > \frac{1}{2}$. Assume that*

$$\boldsymbol{G}(X) = \mathrm{U}(X) \textit{ with } \dim X \equiv \epsilon_\chi \pmod 2;$$

$$\boldsymbol{G}(X) = \mathrm{O}(X) \textit{ with } \dim X \textit{ odd};$$

$$\boldsymbol{G}(X) = \mathrm{Mp}(X).$$

Then $E_\psi^{Q_1}(g, f_s)$ has a pole at $s = \frac{1}{2}$.

Remark 3.2. This is [Jiang and Wu 2018, Proposition 2.2] in the symplectic case, [Wu 2022a, Proposition 3.2] in the metaplectic case, [Jiang and Wu 2016, Proposition 2.2] in the unitary case and [Mœglin 1997, Remarque 2] and [Jiang and Wu 2016, Proposition 2.2] in the orthogonal case.

Remark 3.3. The allowed $\boldsymbol{G}(X)$ in item (2) are those for which we have theta dichotomy and epsilon dichotomy (in the local nonarchimedean setting); see [Gan and Ichino 2014, Corollary 9.2, Theorem 11.1].

Remark 3.4. When π is a cuspidal member in a generic global A-packet of $\boldsymbol{G}(X)(\mathbb{A})$, there is a more precise result; see Theorem 6.3 which was proved in [Jiang et al. 2013] and strengthened in [Jiang and Zhang 2020].

We summarize the results on the maximal positive pole of $E_\psi^{Q_1}(g, f_s)$ from [Ginzburg et al. 2009, Theorem 3.1; Jiang and Wu 2016, Theorem 3.1; 2018, Theorem 2,8; Wu 2022a, Theorem 4.2].

Theorem 3.5. *The maximal positive pole of $E_\psi^{Q_1}(g, f_s)$ is of the form*

$$s = \begin{cases} \frac{1}{2}(\dim X + 1 - (2j + \epsilon_\chi)) & \textit{if } \boldsymbol{G}(X) = \mathrm{U}(X); \\ \frac{1}{2}(\dim X - 2j) & \textit{if } \boldsymbol{G}(X) = \mathrm{O}(X); \\ \frac{1}{2}(\dim X + 2 - 2j) & \textit{if } \boldsymbol{G}(X) = \mathrm{Sp}(X); \\ \frac{1}{2}(\dim X + 2 - (2j + 1)) & \textit{if } \boldsymbol{G}(X) = \mathrm{Mp}(X); \end{cases} \tag{3-2}$$

where $j \in \mathbb{Z}$ such that

$$\begin{cases} r_X \le 2j + \epsilon_\chi < \dim X + 1 & \textit{if } \boldsymbol{G}(X) = \mathrm{U}(X); \\ r_X \le 2j < \dim X & \textit{if } \boldsymbol{G}(X) = \mathrm{O}(X); \\ r_X \le 2j < \dim X + 2 & \textit{if } \boldsymbol{G}(X) = \mathrm{Sp}(X); \\ r_X \le 2j + 1 < \dim X + 2 & \textit{if } \boldsymbol{G}(X) = \mathrm{Mp}(X); \end{cases} \tag{3-3}$$

where r_X denotes the Witt index of X.

Remark 3.6. The middle quantities in the inequalities of (3-3) are, in fact, the lowest occurrence index of π in the global theta lift which depends on χ and ψ; see Theorem 4.4. In some cases, the lowest occurrence index turns out to be independent of ψ.

Remark 3.7. To derive the inequalities $r_X \leq \cdots$, we already need to make use of properties of the global theta correspondence. The other parts of the statements can be derived by relating our Eisenstein series to Siegel–Eisenstein series whose poles are completely known. We note that via the Siegel–Weil formula, Siegel–Eisenstein series are related to global theta correspondence.

4. Theta correspondence

We keep the notation of Section 3. First we define the theta lifts and the two invariants called the first occurrence index and the lowest occurrence index. Then we relate the invariants to poles of our Eisenstein series.

Recall that we have taken an ϵ-skew Hermitian space X over E. Let Y be an ϵ-Hermitian space equipped with the form $\langle\,\cdot\,,\,\cdot\,\rangle_Y$. We note that $\langle\,\cdot\,,\,\cdot\,\rangle_Y$ is an F-bilinear pairing

$$\langle\,\cdot\,,\,\cdot\,\rangle_Y : Y \times Y \to E$$

such that

$$\langle y_2, y_1\rangle_Y = \epsilon\langle y_1, y_2\rangle_Y^{\varrho}, \quad \langle y_1 a, y_2 b\rangle_Y = a^{\varrho}\langle y_1, y_2\rangle_Y b$$

for all $a, b \in E$ and $y_1, y_2 \in Y$. Let $G(Y)$ be its isometry group. We note that $G(X)$ acts on X from the right while $G(Y)$ acts on Y from the left. Let W be the vector space $\mathrm{R}_{E/F}(Y \otimes_E X)$ over F and equip it with the symplectic form

$$\langle\,\cdot\,,\,\cdot\,\rangle_W : W \times W \to F$$

given by

$$\langle y_1 \otimes x_1, y_2 \otimes x_2\rangle_W = \mathrm{tr}_{E/F}(\langle y_1, y_2\rangle_Y \langle x_1, x_2\rangle_X^{\varrho}).$$

With this set-up, $G(X)$ and $G(Y)$ form a reductive dual pair inside $\mathrm{Sp}(W)$. Let $W = W^+ \oplus W^-$ be a polarization of W. Let $\mathrm{Mp}^{\mathbb{C}^1}(W)(F_v)$ be the $\mathbb{C}^1$-metaplectic extension of $\mathrm{Sp}(W)(F_v)$. Let ω_v denote the Weil representation of $\mathrm{Mp}^{\mathbb{C}^1}(W)(F_v)$ realized on the space of Schwartz functions $\mathcal{S}(W^+(F_v))$. The Weil representation depends on the additive character ψ_v, but we suppress it from notation. When v is archimedean, we actually take the Fock model [Howe 1989] rather than the full Schwartz space and it is a $(\mathfrak{sp}(W)(F_v), \widetilde{K}_{\mathrm{Sp}(W),v})$-module but we abuse language and call it a representation of $\mathrm{Mp}^{\mathbb{C}^1}(W)(F_v)$. When neither $G(X)$ or $G(Y)$ is an odd orthogonal group, by [Kudla 1994] there exists a homomorphism

$$G(X)(F_v) \times G(Y)(F_v) \to \mathrm{Mp}^{\mathbb{C}^1}(W)(F_v)$$

that lifts the obvious map $G(X)(F_v) \times G(Y)(F_v) \to \mathrm{Sp}(W)(F_v)$. In this case, set $\boldsymbol{G}(X) = G(X)$ (resp. $\boldsymbol{G}(Y) = G(Y)$). When $G(X)$ is an odd orthogonal group, we take $\boldsymbol{G}(Y)(F_v)$ to be the metaplectic double cover of $G(Y)(F_v)$ and set $\boldsymbol{G}(X) = G(X)$. Then by [Kudla 1994] there exists a homomorphism

$$G(X)(F_v) \times \boldsymbol{G}(Y)(F_v) \to \mathrm{Mp}^{\mathbb{C}^1}(W)(F_v)$$

that lifts $G(X)(F_v) \times G(Y)(F_v) \to \mathrm{Sp}(W)(F_v)$. The case is analogous when $G(Y)$ is an odd orthogonal group. In any case, we get a homomorphism

$$\iota_v : \boldsymbol{G}(X)(F_v) \times \boldsymbol{G}(Y)(F_v) \to \mathrm{Mp}^{\mathbb{C}^1}(F_v).$$

It should be clear from the context when $\boldsymbol{G}(X)$ (resp. $\boldsymbol{G}(Y)$) refers to a cover group and when it is not truly a cover. In the unitary case, there are many choices of ι_v. Once we fix χ and an additional character χ_2, then ι_v is fixed. This is worked out in great details in [Harris et al. 1996, Section 1]. Our (χ, χ_2) matches (χ_1, χ_2) in [Harris et al. 1996, (0.2)]. We note that Y should be compatible with χ and χ determines the embedding of $G(X)(\mathbb{A})$ into $\mathrm{Mp}^{\mathbb{C}^1}(\mathbb{A})$ whereas X should be compatible with χ_2 and χ_2 determines the embedding of $G(Y)(\mathbb{A})$ into $\mathrm{Mp}^{\mathbb{C}^1}(\mathbb{A})$. By "compatible", we mean $\epsilon_\chi \equiv \dim Y \pmod 2$ (resp. $\epsilon_{\chi_2} \equiv \dim X \pmod 2$); see [Kudla 1994] for more details. We pull back ω_v to $\boldsymbol{G}(X)(F_v) \times \boldsymbol{G}(Y)(F_v)$ via ι_v and still denote the representation by ω_v.

Denote by ι the adelic analogue of ι_v. We also have the (global) Weil representation ω of $\mathrm{Mp}^{\mathbb{C}^1}(\mathbb{A})$ on the Schwartz space $\mathcal{S}(W^+(\mathbb{A}))$ and its pullback via ι to $\boldsymbol{G}(X)(\mathbb{A}) \times \boldsymbol{G}(Y)(\mathbb{A})$.

Then we can define the theta function which will be used as a kernel function. Let

$$\theta_{X,Y}(g, h, \Phi) := \sum_{w \in W^+(F)} \omega(\iota(g, h))\Phi(w)$$

for $g \in \boldsymbol{G}(X)(\mathbb{A})$, $h \in \boldsymbol{G}(Y)(\mathbb{A})$ and $\Phi \in \mathcal{S}(W^+(\mathbb{A}))$. It is absolutely convergent and is an automorphic form on $\boldsymbol{G}(X)(\mathbb{A}) \times \boldsymbol{G}(Y)(\mathbb{A})$. For $f \in \pi$, set

$$\theta_X^Y(f, \Phi) := \int_{[\boldsymbol{G}(X)]} \overline{f(g)}\theta_{X,Y}(g, h, \Phi)\, dg.$$

Note that we write $[\boldsymbol{G}(X)]$ for $G(X)(F)\backslash G(X)(\mathbb{A})$ when $\boldsymbol{G}(X)$ is not metaplectic and $G(X)(F)\backslash \boldsymbol{G}(X)(\mathbb{A})$ or more explicitly $\mathrm{Sp}(X)(F) \backslash \mathrm{Mp}(X)(\mathbb{A})$ when $\boldsymbol{G}(X)$ is metaplectic. This is an automorphic form on $\boldsymbol{G}(Y)(\mathbb{A})$. It depends on χ and χ_2 in the unitary case and when we want to emphasize the dependency, we will write $\theta_{X,\chi_2}^{Y,\chi}(f, \Phi)$. Let $\Theta_X^Y(\pi)$ denote the space of functions spanned by the $\theta_X^Y(f, \Phi)$ and let $\Theta_{X,,\chi_2}^{Y,\chi}(\pi)$ denote the space of functions spanned by the $\theta_{X,\chi_2}^{Y,\chi}(f, \Phi)$ in the unitary case.

From now on assume that Y is anisotropic (possibly zero), so that it sits at the bottom of its Witt tower. Define Y_r to be the ϵ-Hermitian space formed by adjoining r-copies of the hyperbolic plane to Y. These Y_r form the Witt tower of Y. By the tower property [Rallis 1984; Wu 2013], if the theta lift to $\boldsymbol{G}(Y_r)$ is nonzero then the theta lift to $\boldsymbol{G}(Y_{r'})$ is also nonzero for all $r' \geq r$.

Define the first occurrence index of π in the Witt tower of Y to be

$$\mathrm{FO}_X^{Y,\chi}(\pi) := \begin{cases} \min\{\dim Y_r \mid \Theta_{X,\chi_2}^{Y_r,\chi}(\pi) \neq 0\} & \text{if } \boldsymbol{G}(X) = \mathrm{U}(X); \\ \min\{\dim Y_r \mid \Theta_X^{Y_r}(\pi \otimes (\chi \circ \upsilon)) \neq 0\} & \text{if } \boldsymbol{G}(X) = \mathrm{O}(X); \\ \min\{\dim Y_r \mid \Theta_X^{Y_r}(\pi) \neq 0\} & \text{if } \boldsymbol{G}(X) = \mathrm{Sp}(X) \text{ or } \mathrm{Mp}(X). \end{cases} \tag{4-1}$$

Note that it depends on χ but not on χ_2 in the unitary case as changing χ_2 to another compatible one produces only a character twist on $\Theta_{X,\chi_2}^{Y_r,\chi}(\pi)$. For more details, see [Wu 2022b, (1-1)]. In the orthogonal case, we twist π by $\chi \circ \upsilon$ where υ denotes the spinor norm. If $\boldsymbol{G}(X) = \mathrm{Sp}(X)$ or $\mathrm{Mp}(X)$, we require that $\chi_Y = \chi$ where χ_Y is the quadratic automorphic character of $\mathrm{GL}_1(\mathbb{A})$ associated to Y given by

$$\chi_Y(g) = (g, (-1)^{\dim Y(\dim Y-1)/2} \det\langle\cdot\,,\cdot\rangle_Y),$$

where $(\cdot\,,\cdot)$ is the Hilbert symbol.

Define the lowest occurrence index to be

$$\mathrm{LO}_X^{\chi}(\pi) := \min\{\mathrm{FO}_X^{Y,\chi}(\pi) \mid Y \text{ is compatible with } \chi\}, \tag{4-2}$$

when $\boldsymbol{G}(X) = \mathrm{U}(X)$, $\mathrm{Sp}(X)$ or $\mathrm{Mp}(X)$. Here compatibility means that

$$\begin{aligned} \dim Y \equiv \epsilon_\chi \pmod 2 \quad & \text{if } \boldsymbol{G}(X) = \mathrm{U}(X); \\ \chi_Y = \chi \quad & \text{if } \boldsymbol{G}(X) = \mathrm{Sp}(X) \text{ or } \mathrm{Mp}(X). \end{aligned} \tag{4-3}$$

Define the lowest occurrence index to be

$$\mathrm{LO}_X^{\chi}(\pi) := \min\{\mathrm{FO}_X^{Y,\chi}(\pi \otimes \mathrm{sgn}_T) \mid T \text{ a set of even number of places of } F\}, \tag{4-4}$$

when $\boldsymbol{G}(X) = \mathrm{O}(X)$.

We have the following relations of the first occurrence (resp. the lowest occurrence) and the poles (resp. the maximal positive pole) of the Eisenstein series; see [Jiang and Wu 2018, Corollary 3.9, Theorem 3.10] for the symplectic case, [Wu 2022a, Corollary 6.3, Theorem 6.4] for the metaplectic case, [Jiang and Wu 2016, Corollaries 3.5 and 3.7] for the unitary case and [Ginzburg et al. 2009, Theorems 5.1 and 1.3] for the orthogonal case.

Theorem 4.1. *Let π be an irreducible cuspidal automorphic representation of $\boldsymbol{G}(X)(\mathbb{A})$. Let χ be a conjugate self-dual automorphic character of $\mathrm{R}_{E/F}\,\mathrm{GL}_1(\mathbb{A})$. Let Y be an anisotropic ϵ-Hermitian space that is compatible with χ in the sense of (4-3). Assume that $\mathrm{FO}_X^{Y,\chi}(\pi) = \dim Y + 2r$. Set*

$$s_0 = \begin{cases} \frac{1}{2}(\dim X + 1 - (\dim Y + 2r)) & \text{if } \boldsymbol{G}(X) = \mathrm{U}(X); \\ \frac{1}{2}(\dim X - (\dim Y + 2r)) & \text{if } \boldsymbol{G}(X) = \mathrm{O}(X); \\ \frac{1}{2}(\dim X + 2 - (\dim Y + 2r)) & \text{if } \boldsymbol{G}(X) = \mathrm{Sp}(X) \text{ or } \mathrm{Mp}(X). \end{cases} \tag{4-5}$$

Assume that $s_0 \neq 0$. If $\boldsymbol{G}(X) = \mathrm{O}(X)$ and $s_0 < 0$, further assume that $\frac{1}{2}\dim X < r < \dim X - 2$. Then $s = s_0$ is a pole of the Eisenstein series $E_\psi^{Q_1}(g, f_s)$ for some choice of $f_s \in \mathcal{A}_\psi^{Q_1}(s, \chi \boxtimes \pi)$.

Remark 4.2. Using the notation from Section 2. The quantity s_0 in (4-5) can be written uniformly as

$$\tfrac{1}{2}(d_{\boldsymbol{G}(X)^\vee} - d_{\boldsymbol{G}(Y_r)^\vee} + 1).$$

Remark 4.3. Note that we always have $r \leq \dim X$. The extra condition when $\boldsymbol{G}(X) = \mathrm{O}(X)$ is to avoid treating period integrals over the orthogonal groups of split binary quadratic forms, as our methods cannot deal with the technicality. Theorem 4.1 allows negative s_0. It is possible to detect nonmaximal poles and negative poles of the Eisenstein series by the first occurrence indices.

Theorem 4.4. *Let π be an irreducible cuspidal automorphic representation of $\boldsymbol{G}(X)(\mathbb{A})$. Let χ be a conjugate self-dual automorphic character of $\mathrm{R}_{E/F}\,\mathrm{GL}_1(\mathbb{A})$. Then the maximal positive pole of $E_{\psi}^{Q_1}(g, f_s)$ for f_s running over $\mathcal{A}_{\psi}^{Q_1}(s, \chi \boxtimes \pi)$ is at $s = s_0 \in \mathbb{R}$ if and only if*

$$\mathrm{LO}_X^{\chi}(\pi) = \begin{cases} \dim X + 1 - 2s_0 & \textit{if } \boldsymbol{G}(X) = \mathrm{U}(X); \\ \dim X - 2s_0 & \textit{if } \boldsymbol{G}(X) = \mathrm{O}(X); \\ \dim X + 2 - 2s_0 & \textit{if } \boldsymbol{G}(X) = \mathrm{Sp}(X) \textit{ or } \mathrm{Mp}(X). \end{cases} \tag{4-6}$$

Remark 4.5. Theorem 4.4 does not allow negative s_0.

In Remark 3.7, we mentioned that the part $r_X \leq \cdots$ in Theorem 3.5 is proved by using theta correspondence. What we used is that we always have $\mathrm{LO}_X^{\chi}(\pi) \geq r_X$ by the stable range condition [Rallis 1984, Theorem I.2.1].

5. Application to global Arthur packets

We have derived relations among (χ, b)-factors of global A-parameters, poles of partial L-functions, poles of Eisenstein series and lowest occurrence indices of global theta lifts. Combining these, we have the following implication on global A-packets.

Theorem 5.1. *Let π be an irreducible cuspidal automorphic representation of $\boldsymbol{G}(X)(\mathbb{A})$. Let ϕ_π be its global A-parameter. Let χ be a conjugate self-dual automorphic character of $\mathrm{R}_{E/F}\,\mathrm{GL}_1(\mathbb{A})$. Assume that ϕ_π has a (χ, b)-factor for some positive integer b. Then*

$$b \leq \begin{cases} \dim X - r_X & \textit{if } \boldsymbol{G}(X) = \mathrm{U}(X); \\ \dim X - r_X - 1 & \textit{if } \boldsymbol{G}(X) = \mathrm{O}(X); \\ \dim X - r_X + 1 = \frac{1}{2}\dim X + 1 & \textit{if } \boldsymbol{G}(X) = \mathrm{Sp}(X) \textit{ or } \mathrm{Mp}(X). \end{cases} \tag{5-1}$$

where r_X denotes the Witt index of X.

Proof. If b is not maximal among all factors (τ, b) appearing in ϕ_π, then $b < \frac{1}{2}d_{\boldsymbol{G}(X)^\vee}$. Then it is clear that b satisfies (5-1). Now we assume that b is maximal among all factors appearing in ϕ_π. By Proposition 2.8, $L^S(s, \pi \times \chi^{-1})$ has its rightmost pole at $s = \frac{1}{2}(b+1)$. Then by Proposition 3.1, $E_{\psi}^{Q_1}(g, f_s)$ has a pole at $s = \frac{1}{2}(b+1)$ for some choice of f_s. Assume that $s = \frac{1}{2}(b_1+1)$ is the rightmost pole of the Eisenstein series with $b_1 \geq b$. By Theorem 3.5,

$$\tfrac{1}{2}(b_1+1) \leq \begin{cases} \frac{1}{2}(\dim X + 1 - r_X) & \text{if } \boldsymbol{G}(X) = \mathrm{U}(X); \\ \frac{1}{2}(\dim X - r_X) & \text{if } \boldsymbol{G}(X) = \mathrm{O}(X); \\ \frac{1}{2}(\dim X + 2 - r_X) & \text{if } \boldsymbol{G}(X) = \mathrm{Sp}(X) \text{ or } \mathrm{Mp}(X); \end{cases}$$

or in other words, b_1 is less than or equal to the quantity on the RHS of (5-1). Using the fact that $b \leq b_1$, we get the desired bound for b. □

Remark 5.2. Our result generalizes [Jiang and Liu 2018, Theorem 3.1] for symplectic groups to classical groups and metaplectic groups. In addition, we do not require the assumption on the wave front set in [loc. cit., Theorem 3.1]. This type of result has been used in [loc. cit., Section 5] to find a Ramanujan bound which measures the departure of the local components of a cuspidal π from being tempered.

The metaplectic case has been treated in [Wu 2022a, Theorem 0.1], though the proof is not written down explicitly. Here we supply the detailed arguments for all classical groups and metaplectic groups uniformly.

The corollary below follows immediately from the theorem.

Corollary 5.3. *The global A-packet Π_ϕ attached to the elliptic global A-parameter ϕ cannot have a cuspidal member if ϕ has a (χ, b)-factor with*

$$b > \begin{cases} \dim_E X - r_X & \text{if } \boldsymbol{G}(X) = \mathrm{U}(X); \\ \dim_F X - r_X - 1 & \text{if } \boldsymbol{G}(X) = \mathrm{O}(X); \\ \dim X - r_X + 1 = \frac{1}{2}\dim X + 1 & \text{if } \boldsymbol{G}(X) = \mathrm{Sp}(X) \text{ or } \mathrm{Mp}(X). \end{cases}$$

6. Generic global A-packets

Following the terminology of [Arthur 2013], we say that an elliptic global A-parameter is generic if it is of the form $\phi = \boxplus_{i=1}^{r}(\tau_i, 1)$ and we say a global A-packet is generic if its global A-parameter is generic. Assume that π is a cuspidal member in a generic global A-packet. Then our results can be made more precise. We note that our results for $\mathrm{Mp}(X)$ are conditional on results on normalized intertwining operators; see Assumption 6.1 and Remark 6.2.

First assume that $G(X)$ is quasisplit and that π is globally generic. We explain what we mean by globally generic. We use the same set-up as in [Shahidi 1988, Section 3]. Let B be a Borel subgroup of $G(X)$. Let N denote its unipotent radical and let T be a fixed choice of Levi subgroup of B. Of course, in this case T is a maximal torus of $G(X)$. Let $\bar{F}$ denote an algebraic closure of F. Let Δ denote the set of simple roots of $T(\bar{F})$ in $N(\bar{F})$. Let $\{X_\alpha\}_{\alpha\in\Delta}$ be a $\mathrm{Gal}(\bar{F}/F)$-invariant set of root vectors. Recall that ψ is a fixed nontrivial automorphic character of $\mathbb{A}_F$ which is used in the definitions of the Weil representation and the global A-packets for $\mathrm{Mp}(X)$. It gives rise to generic characters of $N(\mathbb{A})$. We use the one defined as follows. For each place v of F, we define a character $\psi_{N,v}$ of $N(F_v)$. Write an element of $N(F_v)$ as $\prod_{\alpha\in\Delta}\exp(x_\alpha X_\alpha)$ for $x_\alpha \in \bar{F}_v$ such that $\sigma x_\alpha = x_{\sigma\alpha}$ with $\sigma \in \mathrm{Gal}(\bar{F}/F)$. Set

$$\psi_{N,v}\Big(\prod_{\alpha\in\Delta}\exp(x_\alpha X_\alpha)\Big) = \psi_v\Big(\sum_{\alpha\in\Delta} x_\alpha\Big).$$

Let $\psi_N = \otimes_v \psi_{N,v}$. In the $\mathrm{Mp}(X)$ case, we view $N(\mathbb{A})$ as a subgroup of $\mathrm{Mp}(X)(\mathbb{A})$ via the canonical splitting. We require that π is globally generic with respect to the generic character ψ_N of $N(\mathbb{A})$. Thus the notion of global genericity depends on the choice of the generic automorphic character of $N(\mathbb{A})$. However by [Cogdell et al. 2004, Appendix A], the choice has no effect on the L-factors, the ε-factors and the global A-parameter for π in the case of $\boldsymbol{G}(X) = \mathrm{Sp}(X), \mathrm{O}(X), \mathrm{U}(X)$. The case of $\mathrm{Mp}(X)$ is highly dependent on the choice.

When π is globally generic, $b=1$ for every factor (τ, b) in the global A-parameter ϕ_π. This is because the Langlands functorial lift of π is an isobaric sum of conjugate self-dual cuspidal representations of some $\mathrm{R}_{E/F}\,\mathrm{GL}_n(\mathbb{A})$; see Theorem 11.2 of [Ginzburg et al. 2011].

By [Jiang et al. 2013], there is a more precise relation on the poles of L-functions and the poles of Eisenstein series. The set of possible poles of the normalized Eisenstein series is determined by the complete L-function $L(s, \pi \times \chi^\vee)$. From the assumption that π is globally generic, in the right half-plane, $L(s, \pi \times \chi^\vee)$ has at most a simple pole at $s=1$. In fact we only need [loc. cit., Proposition 4.1] rather than the full strength of [loc. cit., Theorem 1.2] which allows the induction datum to be a Speh representation on the general linear group factor of the Levi. By [Jiang and Zhang 2020, Theorem 5.1], [Jiang et al. 2013, Proposition 4.1] can be strengthened to include the case where π is a cuspidal member in a generic global A-packet of $\boldsymbol{G}(X)(\mathbb{A})$ where $\boldsymbol{G}(X) = \mathrm{Sp}(X), \mathrm{O}(X), \mathrm{U}(X)$ does not have to be quasisplit. We rephrase [Jiang et al. 2013, Proposition 4.1] in our context as Theorem 6.3.

First we set up some notation and outline the method for extending [loc. cit., Proposition 4.1] to the case of $\mathrm{Mp}(X)$. Let

$$\rho^+ := \begin{cases} \mathrm{Asai}^\eta \ \text{ where } \eta = (-1)^{\dim X+1} & \text{if } \boldsymbol{G}(X) = \mathrm{U}(X); \\ \wedge^2 & \text{if } \boldsymbol{G}(X) = \mathrm{O}(X) \text{ with } \dim X \text{ odd or if } \boldsymbol{G}(X) = \mathrm{Mp}(X); \\ \mathrm{Sym}^2 & \text{if } \boldsymbol{G}(X) = \mathrm{O}(X) \text{ with } \dim X \text{ even or if } \boldsymbol{G}(X) = \mathrm{Sp}(X); \end{cases} \tag{6-1}$$

and

$$\rho^- := \begin{cases} \mathrm{Asai}^{-\eta} \ \text{ where } \eta = (-1)^{\dim X+1} & \text{if } \boldsymbol{G}(X) = \mathrm{U}(X); \\ \mathrm{Sym}^2 & \text{if } \boldsymbol{G}(X) = \mathrm{O}(X) \text{ with } \dim X \text{ odd or if } \boldsymbol{G}(X) = \mathrm{Mp}(X); \\ \wedge^2 & \text{if } \boldsymbol{G}(X) = \mathrm{O}(X) \text{ with } \dim X \text{ even or if } \boldsymbol{G}(X) = \mathrm{Sp}(X). \end{cases} \tag{6-2}$$

The results of [loc. cit.] do not cover the metaplectic case, but the method should generalize without difficulty. We explain the strategy. First the poles of the Eisenstein series are related to those of the intertwining operators

$$M(w_0, \tau|\cdot|^s \boxtimes \pi) : \mathrm{Ind}_{Q_a(\mathbb{A})}^{\boldsymbol{G}(X_a)(\mathbb{A})}(\tau|\cdot|^s \boxtimes \pi) \to \mathrm{Ind}_{Q_a(\mathbb{A})}^{\boldsymbol{G}(X_1)(\mathbb{A})}(\tau|\cdot|^{-s} \boxtimes \pi)$$

where τ is a conjugate self-dual cuspidal automorphic representation of $\mathrm{GL}_a(\mathbb{A}_E)$ and w_0 is the longest Weyl element in $Q_a \backslash G(X_a)/Q_a$. Then define the normalized intertwining operator

$$N(w_0, \tau|\cdot|^s \boxtimes \pi) := \frac{L(s, \pi \times \tau^\vee) L(2s, \tau, \rho^-)}{L(s+1, \pi \times \tau^\vee) L(2s+1, \tau, \rho^-) \varepsilon(s, \pi \times \tau^\vee) \varepsilon(2s, \tau, \rho^-)} \cdot M(w_0, \tau|\cdot|^s \boxtimes \pi). \tag{6-3}$$

The proof of [loc. cit., Proposition 4.1] relies on the key result that the normalized intertwining operator is holomorphic and nonzero for $\operatorname{Re} s \geq \frac{1}{2}$. Then it boils down to finding the poles of the normalizing factors or equivalently

$$\frac{L(s, \pi \times \tau^\vee) L(2s, \tau, \rho^-)}{L(s+1, \pi \times \tau^\vee) L(2s+1, \tau, \rho^-)}.$$

Once we have the key result available, we expect to have a version of [loc. cit., Proposition 4.1] for the metaplectic groups. Note that our $\rho^{\pm}$ defined in (6-1) and (6-2) is different from the ρ and ρ^{-} in [loc. cit.].

Then by using an inductive formula, we expect to be able to prove [loc. cit., Theorem 1.2] as well. We hope to supply the details in a future work.

Next we allow $G(X)$ to be non-quasisplit. We assume that π is a cuspidal member in a generic global A-packet of $\boldsymbol{G}(X)$. Then by [Jiang and Zhang 2020, Theorem 5.1], (6-3) is holomorphic and nonzero for $\operatorname{Re} s \geq \frac{1}{2}$ when $\boldsymbol{G}(X) = \mathrm{Sp}(X), \mathrm{O}(X), \mathrm{U}(X)$. Then the proof of [Jiang et al. 2013, Proposition 4.1] goes through verbatim for such π. The proof of [Jiang and Zhang 2020, Theorem 5.1] does not generalize readily to the case of $\mathrm{Mp}(X)$ as the relevant results for $\mathrm{Mp}(X)$ are not available.

Thus we make an assumption on the normalized intertwining operator:

Assumption 6.1. The normalized intertwining operator $N(w_0, \chi|\cdot|^s \boxtimes \pi)$ is holomorphic and nonzero for $\operatorname{Re} s \geq \frac{1}{2}$.

Remark 6.2. This is shown to be true by [Jiang and Zhang 2020, Theorem 5.1] when π is a cuspidal member in a generic global A-packet of $\boldsymbol{G}(X)(\mathbb{A})$ for $\boldsymbol{G}(X) = \mathrm{Sp}(X), \mathrm{O}(X), \mathrm{U}(X)$. Thus this is only a condition when $\boldsymbol{G}(X) = \mathrm{Mp}(X)$.

Theorem 6.3. *Assume Assumption 6.1. Let π be a cuspidal member in a generic global A-packet of $\boldsymbol{G}(X)(\mathbb{A})$. Let χ be a conjugate self-dual automorphic character of $\mathrm{R}_{E/F}\,\mathrm{GL}_1(\mathbb{A})$.*

(1) *Assume $\boldsymbol{G}(X) = \mathrm{U}(X)$ with $\epsilon_\chi \not\equiv \dim X \pmod 2$, $\mathrm{O}(X)$ with $\dim X$ even or $\mathrm{Sp}(X)$. Then $L(s, \pi \times \chi^\vee)$ has a pole at $s = 1$ if and only if $E^{Q_1}(g, f_s)$ has a pole at $s = 1$ and it is its maximal pole.*

(2) *Assume $\boldsymbol{G}(X) = \mathrm{U}(X)$ with $\epsilon_\chi \equiv \dim X \pmod 2$, $\mathrm{O}(X)$ with $\dim X$ odd or $\mathrm{Mp}(X)$. Then $L(s, \pi \times \chi^\vee)$ is nonvanishing at $s = \frac{1}{2}$ if and only if $E^{Q_1}_\psi(g, f_s)$ has a pole at $s = \frac{1}{2}$ and it is its maximal pole.*

Remark 6.4. The result of [Jiang et al. 2013] involves normalized Eisenstein series, but the normalization has no impact on the positive poles. The following remarks use the notation in [loc. cit.]. We only need the case $b = 1$ in [loc. cit.] which is Proposition 4.1 there. Furthermore we only apply it in the case where τ is a character. The condition that $L(s, \tau, \rho)$ has a pole at $s = 1$ is automatically satisfied by the requirement on our χ that it is conjugate self-dual of parity $(-1)^{N_{G(X)^\vee}+1}$; see Section 2, especially Remark 2.3.

The global A-parameter ϕ_π can possibly have a $(\chi, 1)$-factor only when χ satisfies the condition that $L(s, \chi, \rho^+)$ has a pole at $s = 1$. Due to the parity condition on factors of an elliptic global A-parameter, in some cases, ϕ_π cannot have a $(\chi, 1)$-factor.

Combining our result (Theorem 4.4) on poles of Eisenstein series and lowest occurrence indices with Theorem 6.3 which gives a precise relation between poles of the complete L-function and those of the Eisenstein series, we get

Theorem 6.5. *Assume Assumption 6.1. Let π be a cuspidal member in a generic global A-packet of $\boldsymbol{G}(X)(\mathbb{A})$. Let χ be a conjugate self-dual automorphic character of $\mathrm{R}_{E/F}\,\mathrm{GL}_1(\mathbb{A})$. In each of the following statements, we consider only those $\boldsymbol{G}(X)$ that are listed*:

(1) *Assume that $L(s, \pi \times \chi^\vee)$ has a pole at $s = 1$. Then*

$$\mathrm{LO}_X^\chi(\pi) = \begin{cases} \dim X - 1 & \text{if } \boldsymbol{G}(X) = \mathrm{U}(X) \text{ and } \epsilon_\chi \not\equiv \dim X \pmod 2; \\ \dim X - 2 & \text{if } \boldsymbol{G}(X) = \mathrm{O}(X) \text{ with } \dim X \text{ even}; \\ \dim X & \text{if } \boldsymbol{G}(X) = \mathrm{Sp}(X). \end{cases}$$

(2) *Assume that $L(s, \pi \times \chi^\vee)$ does not have a pole at $s = 1$. Then*

$$\mathrm{LO}_X^\chi(\pi) \geq \begin{cases} \dim X + 1 & \text{if } \boldsymbol{G}(X) = \mathrm{U}(X) \text{ and } \epsilon_\chi \not\equiv \dim X \pmod 2; \\ \dim X & \text{if } \boldsymbol{G}(X) = \mathrm{O}(X) \text{ with } \dim X \text{ even}; \\ \dim X + 2 & \text{if } \boldsymbol{G}(X) = \mathrm{Sp}(X). \end{cases}$$

(3) *Assume $L\big(\frac{1}{2}, \pi \times \chi^\vee\big) \neq 0$. Then*

$$\mathrm{LO}_X^\chi(\pi) = \begin{cases} \dim X & \text{if } \boldsymbol{G}(X) = \mathrm{U}(X) \text{ and } \epsilon_\chi \equiv \dim X \pmod 2; \\ \dim X - 1 & \text{if } \boldsymbol{G}(X) = \mathrm{O}(X) \text{ with } \dim X \text{ odd}; \\ \dim X + 1 & \text{if } \boldsymbol{G}(X) = \mathrm{Mp}(X). \end{cases}$$

(4) *Assume $L\big(\frac{1}{2}, \pi \times \chi^\vee\big) = 0$. Then*

$$\mathrm{LO}_X^\chi(\pi) \geq \begin{cases} \dim X + 2 & \text{if } \boldsymbol{G}(X) = \mathrm{U}(X) \text{ and } \epsilon_\chi \equiv \dim X \pmod 2; \\ \dim X + 1 & \text{if } \boldsymbol{G}(X) = \mathrm{O}(X) \text{ with } \dim X \text{ odd}; \\ \dim X + 3 & \text{if } \boldsymbol{G}(X) = \mathrm{Mp}(X). \end{cases}$$

Remark 6.6. By the conservation relation for local theta correspondence [Sun and Zhu 2015], there always exists an ϵ-Hermitian space $Z_{[v]}$ over E_v of dimension given by the RHS of the equalities in items (1), (3) such that the local theta lift of π_v to $\boldsymbol{G}(Z_{[v]})$ is nonvanishing. Thus in the case of items (2), (4) and $\boldsymbol{G}(X) \neq \mathrm{O}(X)$, the collection $\{Z_{[v]}\}_v$ for v running over all places of F is always incoherent, i.e., there does not exist an ϵ-Hermitian space Z over E such that the localization Z_v is isomorphic to $Z_{[v]}$ for all v. In the case of items (2), (4) and $\boldsymbol{G}(X) = \mathrm{O}(X)$, we have a nontrivial theta lift of $\pi_v \otimes (\chi_v \circ \upsilon_v) \otimes (\eta_{[v]} \circ \det)$ to $\boldsymbol{G}(Z_{[v]})$ for $\eta_{[v]}$ being the trivial character or the sign character for each place v of F, but the collection $\{\eta_{[v]}\}_v$ is incoherent, i.e., there does not exist an automorphic character η of $\mathbb{A}_F^\times$ such that the localization η_v is equal to $\eta_{[v]}$ for all v; see the definitions of first occurrence (4-1) and lowest occurrence (4-4) for $\mathrm{O}(X)$ for why we have a $(\chi_v \circ \upsilon_v)$-twist. We also note that when π is an irreducible cuspidal automorphic representation and $L\big(\frac{1}{2}, \pi \times \chi^\vee\big) = 0$, it is conjectured that there is an arithmetic version of the Rallis inner product formula which says that the conjectural Beilinson–Bloch height pairing of arithmetic theta lifts (which are cycles on Shimura varieties constructed from an incoherent collection of ϵ-Hermitian spaces) gives the derivative $L'\big(\frac{1}{2}, \pi \times \chi^\vee\big)$ up to some ramified factors and some abelian L-functions. The low rank cases have been proved in [Kudla et al. 2006; Liu 2011a; 2011b]. More recently, the cases of unitary groups of higher rank have been proved in [Li and Liu 2021; 2022], conditional on hypothesis of the modularity of Kudla's generating functions of special cycles.

In terms of "(χ, b)"-factors, we have

Theorem 6.7. *Let $\boldsymbol{G}(X) = \mathrm{U}(X)$ with $\epsilon_\chi \not\equiv \dim X \pmod 2$, $\boldsymbol{G}(X) = \mathrm{O}(X)$ with $\dim X$ even or $\boldsymbol{G}(X) = \mathrm{Sp}(X)$. Let π be a cuspidal member in a generic global A-packet of $\boldsymbol{G}(X)(\mathbb{A})$. Let χ be a conjugate self-dual automorphic character of $\mathrm{R}_{E/F}\,\mathrm{GL}_1(\mathbb{A})$. Then the following are equivalent*:

(1) *The global A-parameter ϕ of π has a $(\chi, 1)$-factor.*

(2) *The complete L-function $L(s, \pi \times \chi^\vee)$ has a pole at $s = 1$ (and this is its maximal pole).*

(3) *The Eisenstein series $E^{Q_1}(g, f_s)$ has a pole at $s = 1$ for some choice of $f_s \in \mathcal{A}^{Q_1}(s, \chi \boxtimes \pi)$ (and this is its maximal pole).*

(4) *The lowest occurrence index $\mathrm{LO}_X^\chi(\pi)$ is*

$$\begin{cases} \dim X - 1 & \text{if } \boldsymbol{G}(X) = \mathrm{U}(X); \\ \dim X - 2 & \text{if } \boldsymbol{G}(X) = \mathrm{O}(X); \\ \dim X & \text{if } \boldsymbol{G}(X) = \mathrm{Sp}(X). \end{cases}$$

Remark 6.8. The statements that the poles are maximal are automatic since π lies in a generic global A-packet. We note that when $\boldsymbol{G}(X) = \mathrm{U}(X)$ with $\epsilon_\chi \equiv \dim X \pmod 2$, $\boldsymbol{G}(X) = \mathrm{O}(X)$ with $\dim X$ odd or $\boldsymbol{G}(X) = \mathrm{Mp}(X)$, ϕ_π cannot have a $(\chi, 1)$-factor as the parity condition is not satisfied.

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Equidistribution theorems for holomorphic Siegel cusp forms of general degree: the level aspect

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This paper is an extension of Kim et al. (2020a), and we prove equidistribution theorems for families of holomorphic Siegel cusp forms of general degree in the level aspect. Our main contribution is to estimate unipotent contributions for general degree in the geometric side of Arthur's invariant trace formula in terms of Shintani zeta functions in a uniform way. Several applications, including the vertical Sato–Tate theorem and low-lying zeros for standard L-functions of holomorphic Siegel cusp forms, are discussed. We also show that the "nongenuine forms", which come from nontrivial endoscopic contributions by Langlands functoriality classified by Arthur, are negligible.

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1. Introduction

Let G be a connected reductive group over $\mathbb{Q}$ and $\mathbb{A}$ the ring of adeles of $\mathbb{Q}$. An equidistribution theorem for a family of automorphic representations of $G(\mathbb{A})$ is one of recent topics in number theory and automorphic representations. After Sauvageot's important results [1997], Shin [2012] proved a so-called limit multiplicity formula which shows that the limit of an automorphic counting measure is the Plancherel measure. It implies the equidistribution of Hecke eigenvalues or Satake parameters at a

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fixed prime in a family of cohomological automorphic forms on $G(\mathbb{A})$. A quantitative version of Shin's result is given by Shin and Templier [2016]. A different approach is discussed in [Finis et al. 2015] for $G = \mathrm{GL}_n$ or SL_n, treating more general automorphic forms which are not necessarily cohomological. Note that in the works of Shin and Shin and Templier, one needs to consider all cuspidal representations in the L-packets. Shin [2012, second paragraph on p. 88] suggested that one can isolate just holomorphic discrete series at infinity. In [Kim et al. 2020a; 2020b], we carried out his suggestion and established equidistribution theorems for holomorphic Siegel cusp forms of degree 2. We should also mention Dalal's work [2022]; see Remark 3.12. See also the related works [Knightly and Li 2019; Kowalski et al. 2012].

In this paper we generalize several equidistribution theorems to holomorphic Siegel cusp forms of general degree. A main tool is Arthur's invariant trace formula, as used in the previous work, but we need a more careful analysis in the computation of unipotent contributions. Let us prepare some notations to explain our results.

Let $G = \mathrm{Sp}(2n)$ be the symplectic group of rank n defined over $\mathbb{Q}$. For an n-tuple of integers $\underline{k} = (k_1, \dots, k_n)$ with $k_1 \geq \cdots \geq k_n > n+1$, let $D^{\mathrm{hol}}_{\underline{l}} = \sigma_{\underline{k}}$ be the holomorphic discrete series representation of $G(\mathbb{R})$ with the Harish-Chandra parameter $\underline{l} = (k_1 - 1, \dots, k_n - n)$ or the Blattner parameter $\underline{k}$.

Let $\mathbb{A}$ (respectively, $\mathbb{A}_f$) be the ring of (respectively, finite) adeles of $\mathbb{Q}$, and $\hat{\mathbb{Z}}$ be the profinite completion of $\mathbb{Z}$. For S_1 a finite set of rational primes, let $S = \{\infty\} \cup S_1$, $\mathbb{Q}_{S_1} = \prod_{p \in S_1} \mathbb{Q}_p$, $\mathbb{A}^S$ be the ring of adeles outside S and $\hat{\mathbb{Z}}^S = \prod_{p \notin S_1} \mathbb{Z}_p$. We denote by $\widehat{G(\mathbb{Q}_{S_1})}$ the unitary dual of $G(\mathbb{Q}_{S_1}) = \prod_{p \in S_1} G(\mathbb{Q}_p)$ equipped with the Fell topology. Fix a Haar measure μ^S on $G(\mathbb{A}^S)$ so that $\mu^S(G(\hat{\mathbb{Z}}^S)) = 1$, and let U be a compact open subgroup of $G(\mathbb{A}^S)$. Consider the algebraic representation $\xi = \xi_{\underline{k}}$ of the highest weight $\underline{k}$ so that it is isomorphic to the minimal K_∞-type of $D^{\mathrm{hol}}_{\underline{l}}$. Let h_U denote the characteristic function of U. Then we define a measure on $\widehat{G(\mathbb{Q}_{S_1})}$ by

$$\hat{\mu}_{U, S_1, \xi, D^{\mathrm{hol}}_{\underline{l}}} := \frac{1}{\mathrm{vol}(G(\mathbb{Q})\backslash G(\mathbb{A}))\cdot \dim \xi} \sum_{\pi^0_{S_1} \in \widehat{G(\mathbb{Q}_{S_1})}} \mu^S(U)^{-1}\, m_{\mathrm{cusp}}(\pi^0_{S_1}; U, \xi, D^{\mathrm{hol}}_{\underline{l}}) \delta_{\pi^0_{S_1}}, \tag{1-1}$$

where $\delta_{\pi^0_{S_1}}$ is the Dirac delta measure supported at $\pi^0_{S_1}$, a unitary representation of $G(\mathbb{Q}_{S_1})$, and

$$m_{\mathrm{cusp}}(\pi^0_{S_1}; U, \xi, D^{\mathrm{hol}}_{\underline{l}}) = \sum_{\substack{\pi \in \Pi(G(\mathbb{A}))^0 \\ \pi_{S_1} \simeq \pi^0_{S_1},\, \pi_\infty \simeq D^{\mathrm{hol}}_{\underline{l}}}} m_{\mathrm{cusp}}(\pi)\ \mathrm{tr}(\pi^S(h_U)), \tag{1-2}$$

where $\Pi(G(\mathbb{A}))^0$ stands for the isomorphism classes of all irreducible unitary cuspidal representations of $G(\mathbb{A})$ and $\pi^S = \otimes'_{p \notin S} \pi_p$.

To state the equidistribution theorem, we need to introduce the Hecke algebra $C^\infty_c(G(\mathbb{Q}_{S_1}))$ which is dense under the map $h \mapsto \hat{h}$, where $\hat{h}(\pi_{S_1}) = \mathrm{tr}(\pi_{S_1}(h))$ is in $\mathcal{F}(\widehat{G(\mathbb{Q}_{S_1})})$ consisting of suitable $\hat{\mu}^{\mathrm{pl}}_{S_1}$-measurable functions on $\widehat{G(\mathbb{Q}_{S_1})}$. (See [Shin 2012, Section 2.3] for that space.)

Let N be a positive integer. Put $S_N = \{p \text{ prime} : p \mid N\}$. We assume that $S_1 \cap S_N = \varnothing$. We denote by $K_p(N)$ the principal congruence subgroup of level N for $G(\mathbb{Z}_p)$ (see (2-3) for the definition), and set $K^S(N) = \prod_{p \notin S} K_p(N)$. For each rational prime p, let us consider the unramified Hecke algebra $\mathcal{H}^{\mathrm{ur}}(G(\mathbb{Q}_p)) \subset C^\infty_c(\mathbb{Q}_p)$, and for each $\kappa > 0$, $\mathcal{H}^{\mathrm{ur}}(G(\mathbb{Q}_p))^\kappa$, the linear subspace of $\mathcal{H}^{\mathrm{ur}}(G(\mathbb{Q}_p))$ consisting of all Hecke elements whose heights are less than κ. (See (2-2).) Let $\mathcal{H}^{\mathrm{ur}}(G(\mathbb{Q}_p))^{\kappa}_{\le 1}$ be the subset of $\mathcal{H}^{\mathrm{ur}}(G(\mathbb{Q}_p))^{\kappa}$ consisting of all Hecke elements whose complex values have absolute values less than 1. Our first main result is

Theorem 1.1. *Fix $\underline{k} = (k_1, \dots, k_n)$ satisfying $k_1 \ge \cdots \ge k_n > n+1$. Fix a positive integer κ. Then there exist constants a, b and $c_0 > 0$ depending only on G such that for each $h_1 = \otimes_{p \in S_1} h_{1,p}$, where $h_{1,p} \in \mathcal{H}^{\mathrm{ur}}(G(\mathbb{Q}_p))^{\kappa}_{\le 1}$, we have*

$$\hat{\mu}_{K^S(N), S_1, \xi, D^{\mathrm{hol}}_{\underline{l}}}(\widehat{h_1}) = \hat{\mu}^{\mathrm{pl}}_{S_1}(\widehat{h_1}) + O\left(\left(\prod_{p \in S_1} p\right)^{a\kappa+b} N^{-n}\right),$$

if $N \ge c_0 \prod_{p \in S_1} p^{2n\kappa}$. Note that the implicit constant of the Landau O-notation is independent of S_1, N and h_1.

Let us apply this theorem to the vertical Sato–Tate theorem and higher level density theorem for standard L-functions of holomorphic Siegel cusp forms.

The principal congruence subgroup $\Gamma(N)$ of level N for $G(\mathbb{Z})$ is obtained by

$$\Gamma(N) = G(\mathbb{Q}) \cap G(\mathbb{R}) K(N),$$

where $K(N) = \prod_{p<\infty} K_p(N)$. Let $S_{\underline{k}}(\Gamma(N))$ be the space of holomorphic Siegel cusp forms of weight $\underline{k}$ with respect to $\Gamma(N)$ (see the next section for a precise definition), and let $HE_{\underline{k}}(N)$ be a basis consisting of all Hecke eigenforms outside N. We can identify $HE_{\underline{k}}(N)$ with a basis of $K(N)$-fixed vectors in the set of cuspidal representations of $G(\mathbb{A})$ whose infinity component is (isomorphic to) $D^{\mathrm{hol}}_{\underline{l}}$. (See the next section for the details.) Put $d_{\underline{k}}(N) = |HE_{\underline{k}}(N)|$. Then we have [Wakatsuki 2018], for some constant $C_{\underline{k}} > 0$,

$$d_{\underline{k}}(N) = C_{\underline{k}} C_N N^{2n^2+n} + O_{\underline{k}}\left(N^{2n^2}\right), \tag{1-3}$$

where $C_N = \prod_{p \mid N} \prod_{i=1}^{n} (1 - p^{-2i})$. Note that $\prod_{i=1}^{n} \zeta(2i)^{-1} < C_N < 1$.

For each $F \in HE_{\underline{k}}(N)$, we denote by $\pi_F = \pi_\infty \otimes \otimes'_p \pi_{F,p}$ the corresponding automorphic cuspidal representation of $G(\mathbb{A})$. Henceforth, we assume that

$$k_1 > \cdots > k_n > n+1. \tag{1-4}$$

Then the Ramanujan conjecture is true, namely, $\pi_{F,p}$ is tempered for any p; see Theorem 4.3. Unfortunately, this assumption forces us to exclude the scalar-valued Siegel cusp forms.

Let $\widehat{G(\mathbb{Q}_p)}^{\mathrm{ur,\,temp}}$ be the subspace of $\widehat{G(\mathbb{Q}_p)}$ consisting of all unramified tempered classes. We denote by $(\theta_1(\pi_{F,p}), \dots, \theta_n(\pi_{F,p}))$ the element of Ω corresponding to $\pi_{F,p}$ under the isomorphism $\widehat{G(\mathbb{Q}_p)}^{\mathrm{ur,\,temp}} \simeq [0,\pi]^n/\mathfrak{S}_n =: \Omega$. Let μ_p be the measure on Ω defined in Section 7.

Theorem 1.2. *Assume (1-4). Fix a prime p. Then the set*

$$\left\{\left(\theta_1(\pi_{F,p}), \dots, \theta_n(\pi_{F,p})\right) \in \Omega : F \in HE_{\underline{k}}(N)\right\}$$

is μ_p-equidistributed in Ω, namely, for each continuous function f on Ω,

$$\lim_{\substack{N \to \infty \\ (p,N)=1}} \frac{1}{d_{\underline{k}}(N)} \sum_{F \in HE_{\underline{k}}(N)} f\left(\theta_1(\pi_{F,p}), \dots, \theta_n(\pi_{F,p})\right) = \int_\Omega f(\theta_1, \dots, \theta_n) \mu_p.$$

By using Arthur's endoscopic classification, we have a finer version of the above theorem. Under the assumption (1-4), the global A-parameter describing π_F, for $F \in HE_{\underline{k}}(N)$, is always semisimple. (See Definition 4.1.) Let $HE_{\underline{k}}(N)^g$ be the subset of $HE_{\underline{k}}(N)$ consisting of F such that the global A-packet containing π_F is associated to a simple global A-parameter. They are Siegel cusp forms which do not come from smaller groups by Langlands functoriality in Arthur's classification. In this paper, we call them genuine forms. Let $HE_{\underline{k}}(N)^{ng}$ be the subset of $HE_{\underline{k}}(N)$ consisting of F such that the global A-packet containing π_F is associated to a nonsimple global A-parameter, i.e., they are Siegel cusp forms which come from smaller groups by Langlands functoriality in Arthur's classification. We call them nongenuine forms. We show that nongenuine forms are negligible. The following result is interesting in its own right. For this, we need some further assumptions on the level N.

Theorem 1.3. *Assume (1-4). We also assume*

(1) *N is an odd prime or*

(2) *N is odd and all prime divisors $p_1, \dots, p_r$ $(r \geq 2)$ of N are congruent to 1 modulo 4 such that $\left(\frac{p_i}{p_j}\right) = 1$ for $i \neq j$, where $\left(\frac{*}{*}\right)$ denotes the Legendre symbol.*

Then

(1) $|HE_{\underline{k}}(N)^g| = C_{\underline{k}} C_N N^{2n^2+n} + O_{n,\underline{k},\epsilon}\big(N^{2n^2+n-1+\epsilon}\big)$ *for any $\epsilon > 0$;*

(2) $|HE_{\underline{k}}(N)^{ng}| = O_{n,\underline{k},\epsilon}\big(N^{2n^2+n-1+\epsilon}\big)$ *for any $\epsilon > 0$;*

(3) *for a fixed prime p, the set $\{(\theta_1(\pi_{F,p}), \dots, \theta_n(\pi_{F,p})) \in \Omega : F \in HE_{\underline{k}}(N)^g\}$ is μ_p-equidistributed in Ω.*

The above assumptions on the level N are necessary in order to estimate nongenuine forms related to nonsplit but quasisplit orthogonal groups in the Arthur's classification by using the transfer theorems for some Hecke elements in the quadratic base change in the ramified case [Yamauchi 2021]. (See Proposition 4.12 for the details.)

Next, we discuss ℓ-level density (where ℓ is a positive integer) for standard L-functions in the level aspect. Let us denote by $\Pi(\mathrm{GL}_n(\mathbb{A}))^0$ the set of all isomorphism classes of irreducible unitary cuspidal representations of $\mathrm{GL}_n(\mathbb{A})$. Keep the assumption on $\underline{k}$ as in (1-4) and the above assumption on the level N. Then F can be described by a global A-parameter $\boxplus_{i=1}^r \pi_i$ with $\pi_i \in \Pi(\mathrm{GL}_{m_i}(\mathbb{A}))^0$ and $\sum_{i=1}^r m_i = 2n+1$. Then we may define the standard L-function of $F \in HE_{\underline{k}}(N)$ by

$$L(s, \pi_F, \mathrm{St}) := \prod_{i=1}^r L(s, \pi_i),$$

which coincides with the classical definition in terms of Satake parameters of F outside N. Then we show unconditionally that the ℓ-level density of the standard L-functions of the family $HE_{\underline{k}}(N)$ has the symmetry type Sp in the level aspect. (See Section 9 for the precise statement. Shin and Templier [2016] showed it under several hypotheses for a family which includes nonholomorphic forms.) Here, in order to obtain lower bounds for conductors, it is necessary to introduce a concept of newforms. This may be of independent interest. Since any local newform theory for $\mathrm{Sp}(2n)$ is unavailable except for $n = 1, 2$, we define the old space $S_{\underline{k}}^{\mathrm{old}}(\Gamma(N))$ to be the intersection of $S_{\underline{k}}(\Gamma(N))$ with the smallest $G(\mathbb{A}_f)$-invariant space of functions on $G(\mathbb{Q})\backslash G(\mathbb{A})$ containing $S_{\underline{k}}(\Gamma(M))$ for all proper divisors M of N. The new space $S_{\underline{k}}^{\mathrm{new}}(\Gamma(N))$ is the orthogonal complement of $S_{\underline{k}}^{\mathrm{old}}(\Gamma(N))$ in $S_{\underline{k}}(\Gamma(N))$ with respect to the Petersson inner product. Then if $F \in S_{\underline{k}}^{\mathrm{new}}(\Gamma(N))$, $q(F) \geq N^{1/2}$ (Theorem 8.3), and if N is squarefree, we can show that $\dim S_{\underline{k}}^{\mathrm{new}}(\Gamma(N)) \geq \zeta(n^2)^{-1} d_{\underline{k}}(N)$ if $n \geq 2$ (Theorem 5.4).

As a corollary, we obtain a result on the order of vanishing of $L(s, \pi_F, \mathrm{St})$ at $s = \frac{1}{2}$, the center of symmetry of the L-function, by using the method of Iwaniec et al. [2000] for holomorphic cusp forms on $\mathrm{GL}_2(\mathbb{A})$ (see also [Brumer 1995] for another formulation related to the Birch–Swinnerton–Dyer conjecture): Let r_F be the order of vanishing of $L(s, \pi_F, \mathrm{St})$ at $s = \frac{1}{2}$. Then we show that under the GRH (generalized Riemann hypothesis), $\sum_{F \in HE_{\underline{k}}(N)} r_F \leq C d_{\underline{k}}(N)$ for some constant $C > 0$. This would be the first result of this kind in Siegel modular forms. We can also show a similar result for the degree 4 spinor L-functions of $\mathrm{GSp}(4)$.

Let us explain our strategy in comparison with the previous works. We choose a test function

$$f = \mu^S(K(N))^{-1} f_\xi h_1 h_{K^S(N)} \in C_c^\infty(G(\mathbb{R})) \otimes \left(\otimes_{p \in S_1} \mathcal{H}^{\mathrm{ur}}(G(\mathbb{Q}_p))_{\leq 1}^{\kappa}\right) \otimes C_c^\infty(G(\mathbb{A}^S))$$

such that f_ξ is a pseudocoefficient of $D_{\underline{l}}^{\mathrm{hol}}$ normalized as $\mathrm{tr}(\pi_\infty(f_\xi)) = 1$. A starting main equality is

$$I_{\mathrm{spec}}(f) = I(f) = I_{\mathrm{geom}}(f),$$

where $I_{\mathrm{spec}}(f)$ (respectively, $I_{\mathrm{geom}}(f)$) is the spectral (respectively, the geometric) side of Arthur's invariant trace $I(f)$. Under the assumption $k_n > n+1$, the spectral side becomes simple by the results of Arthur [1989] and Hiraga [1996], and it is directly related to $S_{\underline{k}}(\Gamma(N))$ because of the choice of a pseudocoefficient of $D_{\underline{l}}^{\mathrm{hol}}$. Now the geometric side is given by

$$I_{\mathrm{geom}}(f) = \sum_{M \in \mathcal{L}} (-1)^{\dim(A_M/A_G)} \frac{|W_0^M|}{|W_0^G|} \sum_{\gamma \in (M(\mathbb{Q}))_{M,\tilde{S}}} a^M(\tilde{S}, \gamma) I_M^G(\gamma, f_\xi) J_M^M(\gamma, h_P), \tag{1-5}$$

where $\tilde{S} = \{\infty\} \sqcup S_N \sqcup S_1$ and $(M(\mathbb{Q}))_{M,\tilde{S}}$ denotes the set of $(M, \tilde{S})$-equivalence classes in $M(\mathbb{Q})$ (see [Arthur 2005, p. 113]); for each M in a finite set $\mathcal{L}$, we choose a parabolic subgroup P such that M is a Levi subgroup of P. (See loc. cit. for details.) Roughly speaking:

- If the test function f is fixed, the terms on (1-5) vanish except for a finite number of $(M, \tilde{S})$-equivalence classes.
- The factor $a^M(\tilde{S}, \gamma)$ is called a global coefficient and it is almost the volume of the centralizer of γ in M if γ is semisimple. The general properties are unknown.
- The factor $I_M^G(\gamma, f_\xi)$ is called an invariant weighted orbital integral, and as the notation shows, it strongly depends on the weight $\underline{k}$ of $\xi = \xi_{\underline{k}}$. Therefore, it is negligible when we consider the level aspect.
- The factor $J_M^M(\gamma, h_P)$ is an orbital integral of γ for $h = \mu^S(K(N))^{-1} h_1 h_{K^S(N)}$.

According to the types of conjugacy classes and M, the geometric side is divided into the terms

$$I_{\mathrm{geom}}(f) = I_1(f) + I_2(f) + I_3(f) + I_4(f),$$

where

- $I_1(f)$: $M = G$ and $\gamma = 1$;
- $I_2(f)$: $M \neq G$ and $\gamma = 1$;
- $I_3(f)$: γ is unipotent, but $\gamma \neq 1$;
- $I_4(f)$: the other contributions.

The first term $I_1(f)$ is $f(1)$ up to constant factors, and the Plancherel formula $\hat{\mu}^{\mathrm{pl}}_{S_1}(\hat{f}) = f(1)$ yields the first term of the equality in Theorem 1.1. The condition $N \geq c_0 \prod_{p \in S_1} p^{2n\kappa}$ in Theorem 1.1 implies that the nonunipotent contribution $I_4(f)$ vanishes by [Shin and Templier 2016, Lemma 8.4]. Therefore, everything is reduced to studying the unipotent contributions $I_2(f)$ and $I_3(f)$. An explicit bound for $I_2(f)$ was given by [Shin and Templier 2016, proof of Theorem 9.16]. However, as for $I_3(f)$, since the number of $(M, \tilde{S})$-equivalence classes in the geometric unipotent conjugacy class of each γ is increasing when N goes to infinity, it is difficult to estimate $I_3(f)$ directly. In the case of GSp(4), we computed unipotent contributions by using case-by-case analysis as in [Kim et al. 2020a]. Here we give a new uniform way to estimate all the unipotent contributions. It is given by a sum of special values of zeta integrals with real characters for spaces of symmetric matrices; see Lemma 3.3 and Theorem 3.7. This formula is a generalization of the dimension formula (see [Shintani 1975; Wakatsuki 2018]) to the trace formula of Hecke operators. By using their explicit formulas [Saito 1999] and analyzing Shintani double zeta functions [Kim et al. 2022], we express the geometric side as a finite sum of products of local integrals and special values of the Hecke L functions with real characters, and then obtain the estimates of the geometric side; see Theorem 3.10.

This paper is organized as follows. In Section 2, we set up some notations. In Section 3, we give key results (see Theorem 3.7 and Theorem 3.10) in estimating trace formulas of Hecke elements. In Section 4, we study Siegel modular forms in terms of Arthur's classification and show that nongenuine forms are negligible. In Section 5, we give a notion of newforms which is necessary to estimate conductors. Sections 6–10 are devoted to proving the main theorems. Finally, in the Appendix, we give an explicit computation of the convolution product of some Hecke elements, which is needed in the computation of ℓ-level density of standard L-functions.

2. Preliminaries

A split symplectic group $G = \mathrm{Sp}(2n)$ over the rational number field $\mathbb{Q}$ is defined by

$$G = \mathrm{Sp}(2n) = \left\{ g \in \mathrm{GL}_{2n} : g \begin{pmatrix} O_n & I_n \\ -I_n & O_n \end{pmatrix} {}^t g = \begin{pmatrix} O_n & I_n \\ -I_n & O_n \end{pmatrix} \right\}.$$

The compact subgroup

$$K_\infty = \left\{ \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in G(\mathbb{R}) \right\}$$

of $G(\mathbb{R})$ is isomorphic to the unitary group $\mathrm{U}(n)$ via the mapping $\left(\begin{smallmatrix} A & -B \\ B & A \end{smallmatrix}\right) \mapsto A + iB$, where $i = \sqrt{-1}$. For each rational prime p, we also set $K_p = G(\mathbb{Z}_p)$ and put $K = \prod_{p \le \infty} K_p$. The compact groups K_v and K are maximal in $G(\mathbb{Q}_v)$ and $G(\mathbb{A})$, respectively,

Holomorphic discrete series of $G(\mathbb{R})$ are parameterized by n-tuples $\underline{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n$ such that $k_1 \ge \dots \ge k_n > n$, which is called the Blattner parameter. We write $\sigma_{\underline{k}}$ for the holomorphic discrete series corresponding to the Blattner parameter $\underline{k} = (k_1, \dots, k_n)$. We also write $D^{\mathrm{hol}}_{\underline{l}}$ for one corresponding to the Harish-Chandra parameter $\underline{l} = (k_1 - 1, k_2 - 2, \dots, k_n - n)$ so that $D^{\mathrm{hol}}_{\underline{l}} = \sigma_{\underline{k}}$.

Let $\mathcal{H}^{\mathrm{ur}}(G(\mathbb{Q}_p))$ denote the unramified Hecke algebra over $G(\mathbb{Q}_p)$, that is,

$$\mathcal{H}^{\mathrm{ur}}(G(\mathbb{Q}_p)) = \left\{\varphi \in C_c^\infty(G(\mathbb{Q}_p)) : \varphi(k_1 x k_2) = \varphi(x)\ \forall k_1, k_2 \in K_p,\ \forall x \in G(\mathbb{Q}_p)\right\}.$$

Let T denote the maximal split $\mathbb{Q}$-torus of G consisting of diagonal matrices. We denote by $X_*(T)$ the group of cocharacters on T over $\mathbb{Q}$. An element e_j in $X_*(T)$ is defined by

$$e_j(x) = \mathrm{diag}(\overbrace{1, \dots, 1}^{j-1}, x, \overbrace{1 \dots, 1}^{n-j+1}, \overbrace{1, \dots, 1}^{j-1}, x^{-1}, \overbrace{1, \dots, 1}^{n-j+1}) \in T, \quad x \in \mathbb{G}_m. \tag{2-1}$$

Then, one has $X_*(T) = \langle e_1, \dots, e_n \rangle$. By the Cartan decomposition, any function in $\mathcal{H}^{\mathrm{ur}}(G(\mathbb{Q}_p))$ is expressed by a linear combination of characteristic functions of double cosets $K_p \lambda(p) K_p$ ($\lambda \in X_*(T)$). A height function $\|\cdot\|$ on $X_*(T)$ is defined by

$$\left\| \prod_{j=1}^n e_j^{m_j} \right\| = \max\left\{|m_j| : 1 \le j \le n\right\}, \quad m_j \in \mathbb{Z}.$$

For each $\kappa \in \mathbb{N}$, we set

$$\mathcal{H}^{\mathrm{ur}}(G(\mathbb{Q}_p))^\kappa = \left\{\varphi \in \mathcal{H}^{\mathrm{ur}}(G(\mathbb{Q}_p)) : \mathrm{Supp}(\varphi) \subset \bigcup_{\mu \in X_*(T),\, \|\mu\| \le \kappa} K_p \mu(p) K_p\right\}. \tag{2-2}$$

Choose a natural number N. We set

$$K_p(N) = \{x \in K_p : x \equiv I_{2n} \mod N\}, \quad K(N) = \prod_{p < \infty} K_p(N). \tag{2-3}$$

One gets a congruence subgroup $\Gamma(N) = G(\mathbb{Q}) \cap G(\mathbb{R}) K(N)$.

Let $\mathfrak{H}_n := \{Z \in M_n(\mathbb{C}) : Z = {}^tZ,\ \mathrm{Im}(Z) > 0\}$. We write $S_{\underline{k}}(\Gamma(N))$ for the space of Siegel cusp forms of weight $\underline{k}$ for $\Gamma(N)$, i.e., $S_{\underline{k}}(\Gamma(N))$ consists of $V_{\underline{k}}$-valued smooth functions F on $G(\mathbb{A})$ satisfying the following conditions:

(i) $F(\gamma g k_\infty k_f) = \rho_{\underline{k}}(k_\infty)^{-1} F(g), \quad g \in G(\mathbb{A}),\ \gamma \in G(\mathbb{Q}),\ k_\infty \in K_\infty,\ k_f \in K(N)$,

(ii) $\rho_{\underline{k}}(g_\infty, iI_n) F|_{G(\mathbb{R})}(g_\infty)$ is holomorphic for $g_\infty \cdot iI_n \in \mathfrak{H}_n$,

(iii) $\max_{g \in G(\mathbb{A})} |F(g)| \ll 1$,

where $\rho_{\underline{k}}$ denotes the finite dimensional irreducible polynomial representation of $\mathrm{U}(n)$ corresponding to $\underline{k}$ together with the representation space $V_{\underline{k}}$ and we set $\rho_{\underline{k}}(g, iI_n) = \rho_{\underline{k}}(iC + D)$ for $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in G(\mathbb{R})$.

Let $\underline{m} = (m_1, \dots, m_n)$, $m_1 | m_2 | \cdots | m_n$, and $D_{\underline{m}} = \operatorname{diag}(m_1, \dots, m_n)$. Let $T(D_{\underline{m}})$ be the Hecke operator defined by the double coset

$$\Gamma(N) \begin{pmatrix} D_{\underline{m}} & 0 \\ 0 & D_{\underline{m}}^{-1} \end{pmatrix} \Gamma(N).$$

Specifically, for each prime p, let $D_{p,\underline{a}} = \operatorname{diag}(p^{a_1}, \dots, p^{a_n})$, with $\underline{a} = (a_1, \dots, a_n)$ and $0 \le a_1 \le \cdots \le a_n$.

Let F be a Hecke eigenform in $S_{\underline{k}}(\Gamma(N))$ with respect to the Hecke operator $T(D_{p,\underline{a}})$ for all $p \nmid N$. (See [Kim et al. 2020a, Section 2.2] for Hecke eigenforms in the case of $n = 2$. One can generalize the contents there to $n \ge 3$.) Then F gives rise to an adelic automorphic form ϕ_F on $\operatorname{Sp}(2n, \mathbb{Q}) \backslash \operatorname{Sp}(2n, \mathbb{A})$, and ϕ_F gives rise to a cuspidal representation π_F which is a direct sum $\pi_F = \pi_1 \oplus \cdots \oplus \pi_r$, where the π_i are irreducible cuspidal representations of $\operatorname{Sp}(2n)$. Since F is an eigenform, the π_i are all near-equivalent to each other. Since we do not have the strong multiplicity one theorem for $\operatorname{Sp}(2n)$, we cannot conclude that π_F is irreducible. However, the strong multiplicity one theorem for GL_n implies that there exists a global A-parameter $\psi \in \Psi(G)$ such that $\pi_i \in \Pi_\psi$ for all i [Schmidt 2018, p. 3088]. (See Section 4 for the definition of the global A-packet.)

On the other hand, given a cuspidal representation π of $\operatorname{Sp}(2n)$ with a $K(N)$-fixed vector and whose infinity component is holomorphic discrete series of lowest weight $\underline{k}$, there exists a holomorphic Siegel cusp form F of weight $\underline{k}$ with respect to $\Gamma(N)$ such that $\pi_F = \pi$. (See [Schmidt 2017, p. 2409] for $n = 2$. One can generalize the contents there to $n \ge 3$.)

We define $HE_{\underline{k}}(N)$ to be a basis of $K(N)$-fixed vectors in the set of cuspidal representations of $\operatorname{Sp}(2n, \mathbb{A})$ whose infinity component is holomorphic discrete series of lowest weight $\underline{k}$, and identify it with a basis consisting of all Hecke eigenforms outside N. In particular, each $F \in HE_{\underline{k}}(N)$ gives rise to an irreducible cuspidal representation π_F of $\operatorname{Sp}(2n)$. Let $\mathcal{F}_{\underline{k}}(N)$ be the set of all isomorphism classes of cuspidal representations of $\operatorname{Sp}(2n)$ such that $\pi^{K(N)} \neq 0$ and $\pi_\infty \simeq \sigma_{\underline{k}}$. Consider the map $\Lambda : HE_{\underline{k}}(N) \longrightarrow \mathcal{F}_{\underline{k}}(N)$, given by $F \longmapsto \pi_F$. It is clearly surjective. For each $\pi = \pi_\infty \otimes \otimes'_p \pi_p \in \mathcal{F}_{\underline{k}}(N)$, set $\pi_f = \otimes'_p \pi_p$. Then we get

$$|\Lambda^{-1}(\pi)| = \dim \pi_f^{K(N)},$$

where $\pi_f^{K(N)} = \{\phi \in \pi_f : \pi_f(k)\phi = \phi \text{ for all } k \in K(N)\}$.

3. Asymptotics of Hecke eigenvalues

For each function $h \in C_c^\infty(K(N) \backslash G(\mathbb{A}_f) / K(N))$, an adelic Hecke operator T_h on $S_{\underline{k}}(\Gamma(N))$ is defined by

$$(T_h F)(g) = \int_{G(\mathbb{A}_f)} F(gx) h(x) \, dx, \qquad F \in S_{\underline{k}}(\Gamma(N)).$$

See [Kim et al. 2020a, pp. 15–16] for the relationship between the classical Hecke operators and adelic Hecke operators for $n = 2$. One can generalize the contents there to $n \ge 3$ easily. Let $f_{\underline{k}}$ denote a pseudocoefficient of $\sigma_{\underline{k}}$ with $\operatorname{tr} \sigma_{\underline{k}}(f_{\underline{k}}) = 1$; see [Clozel and Delorme 1990].

Lemma 3.1. *Suppose $k_n > n+1$ and $h \in C_c^\infty(K(N)\backslash G(\mathbb{A}_f)/K(N))$. The spectral side $I_{\mathrm{spec}}(f_{\underline{k}}h)$ of the invariant trace formula is given by*

$$I_{\mathrm{spec}}(f_{\underline{k}}h) = \sum_{\pi=\sigma_{\underline{k}}\otimes\pi_f\text{, auto. rep. of } G(\mathbb{A})} m_\pi \operatorname{Tr}(\pi_f(h)) = \operatorname{Tr}\big(T_h|_{S_{\underline{k}}(\Gamma(N))}\big),$$

where m_π means the multiplicity of π in the discrete spectrum of $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$.

Proof. The second equality follows from [Wallach 1984]. One can prove the first equality by using the arguments in [Arthur 1989] and the main result in [Hiraga 1996], since it follows from [Hiraga 1996] and $k_n > n+1$ that we obtain $\operatorname{Tr}(\pi_\infty(f_{\underline{k}})) = 0$ for any unitary representation $\pi_\infty(\not\cong \sigma_{\underline{k}})$ of $G(\mathbb{R})$. □

We choose two natural numbers N_1 and N, which are mutually coprime. Suppose that N_1 is squarefree. Set $S_1 = \{p : p \mid N_1\}$. We write h_N for the characteristic function of $\prod_{p\notin S_1\sqcup\{\infty\}} K_p(N)$. For each automorphic representation $\pi = \pi_\infty \otimes \otimes'_p \pi_p$, we set $\pi_{S_1} = \otimes_{p\in S_1}\pi_p$.

Lemma 3.2. *Take a test function h on $G(\mathbb{A}_f)$ as*

$$h = \operatorname{vol}(K(N))^{-1} \times h_1 \otimes h_N, \quad \text{where } h_1 \in \otimes_{p\in S_1}\mathcal{H}^{\mathrm{ur}}(G(\mathbb{Q}_p)). \tag{3-1}$$

Then

$$I_{\mathrm{spec}}(f_{\underline{k}}h) = \sum_{\pi=\sigma_{\underline{k}}\otimes\pi_f\text{, auto. rep. of } G(\mathbb{A})} m_\pi \dim \pi_f^{K(N)} \operatorname{Tr}(\pi_{S_1}(h_1)) = \operatorname{Tr}\big(T_h|_{S_{\underline{k}}(\Gamma(N))}\big).$$

Proof. This lemma immediately follows from Lemma 3.1. □

Let V_r denote the vector space of symmetric matrices of degree r, and define a rational representation ρ of the group $\mathrm{GL}_1 \times \mathrm{GL}_r$ on V_r by $x\cdot\rho(a,m) = a\,{}^t m x m$, where $x \in V_r$ and $(a,m) \in \mathrm{GL}_1\times\mathrm{GL}_r$. The kernel of ρ is given by $\operatorname{Ker}\rho = \{(a^{-2}, aI_r) : a \in \mathrm{GL}_1\}$, and we set

$$H_r = \operatorname{Ker}\rho\backslash(\mathrm{GL}_1\times\mathrm{GL}_r).$$

Then, the pair (H_r, V_r) is a prehomogeneous vector space over $\mathbb{Q}$. For $1\le r\le n$ and $f \in C_c^\infty(G(\mathbb{A}))$ (respectively, $f \in C_c^\infty(G(\mathbb{A}_f))$), we define a function $\Phi_{f,r} \in C_c^\infty(V_r(\mathbb{A}))$ (respectively, $\Phi_{f,r} \in C_c^\infty(V_r(\mathbb{A}_f))$) as

$$\Phi_{f,r}(x) = \int_K f\left(k^{-1}\begin{pmatrix} I_n & * \\ O_n & I_n\end{pmatrix}k\right)dk \quad \left(\text{respectively, } \int_{K_f}\right), \quad \text{where } * = \begin{pmatrix} x & 0 \\ 0 & 0\end{pmatrix} \in V_n.$$

Let $\tilde{f}_{\underline{k}}$ denote the spherical trace function of $\sigma_{\underline{k}}$ with respect to $\rho_{\underline{k}}$ on $G(\mathbb{R})$; see [Wakatsuki 2018, §5.3]. Notice that $\tilde{f}_{\underline{k}}$ is a matrix coefficient of $\sigma_{\underline{k}}$, and so it is not compactly supported. Take a test function $h \in C_c^\infty(G(\mathbb{A}_f))$ and set $\tilde{f} = \tilde{f}_{\underline{k}}h$. Let χ be a real character on $\mathbb{R}_{>0}\mathbb{Q}^\times\backslash\mathbb{A}^\times$. Define a zeta integral $Z_r(\Phi_{\tilde{f},r}, s, \chi)$ by

$$Z_r(\Phi_{\tilde{f},r}, s, \chi) = \int_{H_r(\mathbb{Q})\backslash H_r(\mathbb{A})} |a^r \det(m)^2|^s \chi(a) \sum_{x\in V_r^0(\mathbb{Q})} \Phi_{\tilde{f},r}(x\cdot g)\,dg, \quad g = \rho(a,m),$$

where $V_r^0 = \{x \in V_r : \det(x) \neq 0\}$ and $\mathrm{d}g$ is a Haar measure on $H_r(\mathbb{A})$. The zeta integral $Z_r(\Phi_{\tilde{f},r}, s, \chi)$ is absolutely convergent for the range

$$k_n > 2n, \quad \mathrm{Re}(s) > \frac{r-1}{2}, \quad \begin{cases} \mathrm{Re}(s) < \dfrac{k_n}{2} & \text{if } r = 2, \\ \mathrm{Re}(s) < k_n - \dfrac{r-1}{2} & \text{otherwise,} \end{cases} \tag{3-2}$$

see [Wakatsuki 2018, Proposition 5.15], and $Z(\Phi_{\tilde{f},r}, s, \chi)$ is meromorphically continued to the whole s-plane; see [Shintani 1975; Wakatsuki 2018; Yukie 1993]. The following lemma associates $Z(\Phi_{\tilde{f},r}, s, \chi)$ with the unipotent contribution $I_{\mathrm{unip}}(f) = I_1(f) + I_2(f) + I_3(f)$ of the invariant trace formula.

Lemma 3.3. *Let S_0 be a finite set of finite places of $\mathbb{Q}$. Take a test function $h_{S_0} \in C_c^\infty(G(\mathbb{Q}_{S_0}))$, and let h^{S_0} denote the characteristic function of $\prod_{p \notin S_0 \sqcup \{\infty\}} K_p$. Define a test function $\tilde{f}$ as $\tilde{f} = \tilde{f}_{\underline{k}} h_{S_0} h^{S_0}$. If k_n is sufficiently large ($k_n \gg 2n$), then we have*

$$I_{\mathrm{unip}}\big(f_{\underline{k}} h_{S_0} h^{S_0}\big) = \mathrm{vol}_G \; h_{S_0}(1)\, d_{\underline{k}} + \frac{1}{2} \sum_{r=1}^{n} \sum_{\chi \in \mathscr{X}(S_0)} Z_r\Big(\Phi_{\tilde{f},r}, n - \frac{r-1}{2}, \chi\Big),$$

where $\mathrm{vol}_G = \mathrm{vol}(G(\mathbb{Q})\backslash G(\mathbb{A}))$, $d_{\underline{k}}$ denotes the formal degree of $\sigma_{\underline{k}}$, and $\mathscr{X}(S_0)$ denotes the set consisting of real characters $\chi = \otimes_v \chi_v$ on $\mathbb{R}_{>0}\mathbb{Q}^\times\backslash\mathbb{A}^\times$ such that χ_v is unramified for any $v \notin S_0 \sqcup \{\infty\}$. Note that S_0 may contain S_1 and all prime factors of N.

Remark 3.4. Note that the point $s = n - (r-1)/2$, where $1 \le r \le n$, is contained in the range (3-2), and we have $Z_r(\Phi_{\tilde{f},r}, s, \chi) \equiv 0$ for any real character $\chi \notin \mathscr{X}(S_0)$.

Proof. To study $I_{\mathrm{unip}}(f_{\underline{k}} h_{S_0} h^{S_0})$, we need an additional zeta integral $\tilde{Z}_r(\Phi_{\tilde{f},r}, s)$ defined by

$$\tilde{Z}_r(\Phi_{\tilde{f},r}, s) = \int_{\mathrm{GL}_r(\mathbb{Q})\backslash \mathrm{GL}_r(\mathbb{A})} |\det(m)|^{2s} \sum_{x \in V_r^0(\mathbb{Q})} \Phi_{\tilde{f},r}({}^t m x m)\, \mathrm{d}m.$$

The zeta integral $\tilde{Z}_r(\Phi_{\tilde{f},r}, s)$ is absolutely convergent for the range (3-2), and $\tilde{Z}(\Phi_{\tilde{f},r}, s)$ is meromorphically continued to the whole s-plane; see [Shintani 1975; Wakatsuki 2018; Yukie 1993]. Applying [Wakatsuki 2018, Propositions 3.8 and 3.11, Lemmas 5.10 and 5.16] to $I_{\mathrm{unip}}(f)$, we obtain

$$I_{\mathrm{unip}}\big(f_{\underline{k}} h_{S_0} h^{S_0}\big) = \mathrm{vol}_G \; h_{S_0}(1)\, d_{\underline{k}} + \sum_{r=1}^{n} \tilde{Z}_r\Big(\Phi_{\tilde{f},r}, n - \frac{r-1}{2}\Big) \tag{3-3}$$

for sufficiently large $k_n \gg 2n$. Notice that $f_{\underline{k}}$ is changed to $\tilde{f}_{\underline{k}}$ in the right-hand side of (3-3), and this change is essentially required for the proof of (3-3).

By the same argument as in [Hoffmann and Wakatsuki 2018, (4.9)], we have

$$\tilde{Z}_r(\Phi_{\tilde{f},r}, s) = \frac{1}{2} \sum_{\chi} Z_r(\Phi_{\tilde{f},r}, s, \chi),$$

where χ runs over all real characters on $\mathbb{R}_{>0}\mathbb{Q}^\times\backslash\mathbb{A}^\times$. Suppose that $\chi = \otimes_v \chi_v \notin \mathscr{X}(S_0)$. Then, we can take a prime $p \notin S_0$ such that χ_p is ramified and

$$\Phi_{\tilde{f},r}(a_p x) = \Phi_{\tilde{f},r}(x), \quad \forall a_p \in \mathbb{Z}_p^\times. \tag{3-4}$$

Hence, we get $Z_r(\Phi_{\tilde{f},r}, s, \chi) \equiv 0$, and the proof is completed. $\square$

Remark 3.5. The rational representation ρ of H_r on V_r is faithful, but the representation $x \mapsto {}^t m x m$ of GL_r on V_r is not. Hence, $Z_r(\Phi_{\tilde{f},r}, s, \chi)$ is suitable for Saito's explicit formula [1999], which we use in the proof of Theorem 3.10, but $\tilde{Z}_r(\Phi_{\tilde{f},r}, s)$ is not. This fact is also important for the study of global coefficients in the geometric side; see [Hoffmann and Wakatsuki 2018].

Let ψ be a nontrivial additive character on $\mathbb{Q}\backslash\mathbb{A}$, and a bilinear form $\langle\ ,\ \rangle$ on $V_r(\mathbb{A})$ is defined by $\langle x, y\rangle := \mathrm{Tr}(xy)$. Let $\mathrm{d}x$ denote the self-dual measure on $V_r(\mathbb{A})$ for $\psi(\langle\ ,\ \rangle)$. Then, a Fourier transform of $\Phi \in C^\infty(V_r(\mathbb{A}))$ is defined by

$$\hat{\Phi}(y) = \int_{V_r(\mathbb{A})} \Phi(x)\psi(\langle x, y\rangle)\,\mathrm{d}x, \quad y \in V_r(\mathbb{A}).$$

For each $\Phi_0 \in C_0^\infty(V_r(\mathbb{A}_f))$, we define its Fourier transform $\widehat{\Phi_0}$ in the same manner. The zeta function $Z_r(\Phi_{\tilde{f},r}, s, \mathbb{1})$ satisfies the functional equation [Shintani 1975; Yukie 1993]

$$Z_r(\Phi_{\tilde{f},r}, s, \mathbb{1}) = Z_r\Big(\widehat{\Phi_{\tilde{f},r}}, \frac{r+1}{2} - s, \mathbb{1}\Big), \tag{3-5}$$

where $\mathbb{1}$ denotes the trivial representation on $\mathbb{R}_{>0}\mathbb{Q}^\times\backslash\mathbb{A}^\times$.

Take a test function $\Phi_0 \in C_0^\infty(V_r(\mathbb{A}_f))$ such that $\Phi_0({}^t k x k) = \Phi_0(x)$ holds for any $k \in \prod_{p<\infty} H_r(\mathbb{Z}_p)$ and $x \in V_r(\mathbb{A}_f)$, where $H_r(\mathbb{Z}_p)$ is identified with the projection of $\mathrm{GL}_1(\mathbb{Z}_p)\times\mathrm{GL}_r(\mathbb{Z}_p)$ into $H_r(\mathbb{A}_f)$. We write L_0 for the subset of $V_r(\mathbb{Q})$ which consists of the positive definite symmetric matrices contained in the support of Φ_0. It follows from the condition of Φ_0 that L_0 is invariant for $\Gamma = H_r(\mathbb{Z}) = H_r(\mathbb{Q}) \cap H_r(\hat{\mathbb{Z}})$. Put $\zeta_r(\Phi_0, s) = 1$ for $r = 0$. For $r > 0$, define a Shintani zeta function $\zeta_r(\Phi_0, s)$ as

$$\zeta_r(\Phi_0, s) = \sum_{x \in L_0/\Gamma} \frac{\Phi_0(x)}{\#(\Gamma_x)\det(x)^s},$$

where $\Gamma_x = \{\gamma \in \Gamma : x\cdot\gamma = x\}$. The zeta function $\zeta_r(\Phi_0, s)$ absolutely converges for $\mathrm{Re}(s) > (r+1)/2$, and is meromorphically continued to the whole s-plane; see [Shintani 1975]. Furthermore, $\zeta_r(\Phi_0, s)$ is holomorphic except for possible simple poles at $s = 1, 3/2, \ldots (r+1)/2$.

Lemma 3.6. *Let $1 \le r \le n$, $k_n > 2n$, $h \in C_c^\infty(G(\mathbb{A}_f))$, and take a test function $\tilde{f}$ as $\tilde{f} = \tilde{f}_{\underline{k}} h$. Then, there exists a rational function $\boldsymbol{C}_{n,r}(x_1, \ldots, x_n)$ over $\mathbb{R}$ such that*

$$Z_r\Big(\Phi_{\tilde{f},r}, n - \frac{r-1}{2}, \mathbb{1}\Big) = \boldsymbol{C}_{n,r}(\underline{k}) \times \zeta_r(\widehat{\Phi_{h,r}}, r-n).$$

Proof. This can be proved by the functional equation (3-5) and the same argument as in [Wakatsuki 2018, proof of Lemma 5.16]. □

Note that $\zeta_r(\widehat{\Phi_{h,r}}, s)$ is holomorphic in $\{s \in \mathbb{C} : \mathrm{Re}(s) \le 0\}$, and $\boldsymbol{C}_{n,r}(x_1, \ldots, x_n)$ is explicitly expressed by the Gamma function and the partitions; see [Wakatsuki 2018, (5.17) and Lemma 5.16]. We will use this lemma for the regularization of the range of $\underline{k}$. The zeta integral $Z_r(\Phi_{\tilde{f},r}, n-(r-1)/2, \mathbb{1})$ was defined only for $k_n > 2n$, but the right-hand side of the equality in Lemma 3.6 is available for any $\underline{k}$. In addition, this lemma is necessary to estimate the growth of $I_{\mathrm{unip}}(f)$ with respect to $S = S_1 \sqcup \{\infty\}$. We later define a Dirichlet series $D^S_{m,u_S}(s)$ just before Proposition 3.9, and the series $D^S_{m,u_S}(s)$ appears in the explicit formula of $Z_r(\Phi, s, \mathbb{1})$ when r is even. For the case that r is even and $3 < r < n$, it seems difficult to estimate the growth of its contribution to $Z_r(\Phi_{\tilde{f},r}, n-(r-1)/2, \mathbb{1})$, but we can avoid such difficulty by this lemma, since the part related to $D^S_{m,u_S}(s)$ in Saito's formula [1999, Theorem 3.3] disappears in the special value $\zeta_r(\widehat{\Phi_{h,r}}, r-n)$.

Theorem 3.7. *Suppose $k_n > n+1$. Let $h_1 \in \mathcal{H}^{\mathrm{ur}}(G(\mathbb{Q}_{S_1}))^\kappa = \otimes_{p\in S_1}\mathcal{H}^{\mathrm{ur}}(G(\mathbb{Q}_p))^\kappa$, and let h be a test function on $G(\mathbb{A}_f)$ given as (3-1). Then there exists a positive constant c_0 such that, if $N \geq c_0 N_1^{2n\kappa}$,*

$$\mathrm{Tr}\big(T_h|_{S_{\underline{k}}(\Gamma(N))}\big) = \mathrm{vol}_G\,\mathrm{vol}(K(N))^{-1}h_1(1)d_{\underline{k}} + \frac{1}{2}\sum_{r=1}^{n} C_{n,r}(\underline{k})\zeta_r(\widehat{\Phi_{h,r}}, r-n). \tag{3-6}$$

Proof. Let $f = f_{\underline{k}}h$ and $\tilde{f} = \tilde{f}_{\underline{k}}h$. By Lemma 3.2, it is sufficient to prove that the geometric side $I_{\mathrm{geom}}(f)$ equals the right-hand side of (3-6). If one uses the results in [Arthur 1989] and applies [Shin and Templier 2016, Lemma 8.4] by putting $\Xi : G \subset \mathrm{GL}_m$, $m = 2n$, $B_\Xi = 1$, $c_\Xi = c_0$ in their notations, then one gets $I_{\mathrm{geom}}(f) = I_{\mathrm{unip}}(f)$. Hence, by Lemma 3.3 and putting $h_{S_0}h^{S_0} = h$, we have

$$\mathrm{Tr}\big(T_h|_{S_{\underline{k}}(\Gamma(N))}\big) = \mathrm{vol}_G\,\mathrm{vol}(K(N))^{-1}h_1(1)d_{\underline{k}} + \frac{1}{2}\sum_{r=1}^{n}\sum_{\chi\in\mathscr{X}(S_0)} Z_r\Big(\Phi_{\tilde{f},r}, n-\frac{r-1}{2}, \chi\Big) \tag{3-7}$$

for sufficiently large k_n. Let $\mathscr{M}(a) := \mathrm{diag}(1,\dots,1,a,\dots,a)$, where there are n entries of both 1 and a, for $a \in \mathbb{A}^\times$. For any $a_p \in \mathbb{Z}_p^\times$, $b_p \in \mathbb{Q}_p^\times$, $\mu \in X_*(T)$, we have

$$\mathscr{M}(a_p)^{-1}K_p(N)\mathscr{M}(a_p) = K_p(N) \quad\text{and}\quad \mathscr{M}(a_p)^{-1}\mu(b_p)\mathscr{M}(a_p) = \mu(b_p).$$

Hence, (3-4) holds for any $p<\infty$, and so $Z_r(\Phi_{\tilde{f},r}, n-(r-1)/2, \chi)$ vanishes for any $\chi \neq \mathbb{1}$. Therefore, by Lemma 3.6 we obtain the assertion (3-6) for sufficiently large k_n. By the same argument as in [Wakatsuki 2018, proof of Theorem 5.17], we can prove that this equality (3-6) holds in the range $k_n > n+1$, because the both sides of (3-6) are rational functions of $\underline{k}$ in that range, see Lemma 3.6 and [Wakatsuki 2018, Proposition 5.3]. Thus, the proof is completed. □

Let S denote a finite subset of places of $\mathbb{Q}$, and suppose $\infty \in S$. For each character $\chi = \otimes_v \chi_v$ on $\mathbb{Q}^\times\mathbb{R}_{>0}\backslash\mathbb{A}^\times$, we set

$$L^S(s,\chi) = \prod_{p\notin S} L_p(s,\chi_p), \quad L(s,\chi) = \prod_{p<\infty} L_p(s,\chi_p),$$
$$\zeta^S(s) = L^S(s,\mathbb{1}) = \prod_{p\notin S}(1-p^{-s})^{-1}, \quad\text{and}\quad \zeta(s) = L(s,\mathbb{1}),$$

where $L_p(s,\chi_p) = (1-\chi_p(p)p^{-s})^{-1}$ if χ_p is unramified, and $L_p(s,\chi_p) = 1$ if χ_p is ramified.

Lemma 3.8. *Let $s \in \mathbb{R}$. For $s > 1$,*

$$\zeta^S(s) \leq \zeta(s) \quad\textit{and}\quad (\zeta^S)'(s) \ll \frac{2s\zeta(s)}{s-1},$$

where $(\zeta^S)'(s) = \frac{\mathrm{d}}{\mathrm{d}s}\zeta^S(s)$. For $s \leq -1$,

$$|\zeta^S(s)| \leq (N_S)^{-s}|\zeta(s)|,$$

where $N_S = \prod_{p\in S\setminus\{\infty\}} p$.

Proof. First of all, $(1-p^{-s})^{-1}\ge 1$ for $p\in S$. Hence $\zeta^S(s)\le\zeta(s)$. Let $\log\zeta^S(s)=\sum_{p\notin S}\log(1-p^{-s})^{-1}$. Then

$$\frac{(\zeta^S)'(s)}{\zeta^S(s)}=\sum_{p\notin S}\frac{-p^{-s}\log p}{1-p^{-s}}.$$

If $s>1$, then $1-p^{-s}\ge\frac{1}{2}$. Hence,

$$\left|\frac{(\zeta^S)'(s)}{\zeta^S(s)}\right|\le 2\sum_{p\notin S}p^{-s}\log p\le 2\sum_{p}p^{-s}\log p.$$

By partial summation,

$$\sum_p p^{-s}\log p\le\int_1^\infty\biggl(\sum_{p\le x}\log p\biggr)sx^{-s-1}\,\mathrm{d}x\le\int_1^\infty sx^{-s}\,\mathrm{d}x=\frac{s}{s-1}.$$

Here we use the prime number theorem: $\sum_{p\le x}\log p\sim x$. Therefore, $(\zeta^S)'(s)\ll 2s\zeta(s)/(s-1)$. $\square$

Set $\mathfrak{D}=\{d(\mathbb{Q}^\times)^2 : d\in\mathbb{Q}^\times\}$. For each $d\in\mathfrak{D}$, we denote by $\chi_d=\prod_v\chi_{d,v}$ the quadratic character on $\mathbb{Q}^\times\mathbb{R}_{>0}\backslash\mathbb{A}^\times$ corresponding to the quadratic field $\mathbb{Q}(\sqrt{d}\,)$ via class field theory. If $d=1$, then χ_d means the trivial character $\mathbb{1}$. For each positive even integer m, we set

$$\varphi^S_{d,m}(s)=\zeta^S(2s-m+1)\zeta^S(2s)\frac{L^S(m/2,\chi_d)}{L^S(2s-m/2+1,\chi_d)}N\bigl(\mathfrak{f}^S_d\bigr)^{(m-1)/2-s},$$

where $\mathfrak{f}^S_d$ denotes the conductor of $\chi^S_d=\prod_{p\notin S}\chi_{d,p}$. For each $u_S\in\mathbb{Q}_S=\prod_{v\in S}\mathbb{Q}_v$, one sets

$$\mathfrak{D}(u_S)=\bigl\{d(\mathbb{Q}^\times)^2 : d\in\mathbb{Q}^\times,\ d\in u_S(\mathbb{Q}_S^\times)^2\bigr\}.$$

We need the Dirichlet series

$$D^S_{m,u_S}(s)=\sum_{d(\mathbb{Q}^\times)^2\in\mathfrak{D}(u_S)}\varphi^S_{d,m}(s).$$

The following proposition is a generalization of [Ibukiyama and Saito 2012, Proposition 3.6]:

Proposition 3.9. *Let $m\ge 2$ be an even integer. Suppose $(-1)^{m/2}u_\infty>0$ for $u_S=(u_v)_{v\in S}$ (namely, the term of $d(\mathbb{Q}^\times)^2=(\mathbb{Q}^\times)^2$ does not appear in $D^S_{m,u_S}(s)$ if $(-1)^{m/2}=-1$). The Dirichlet series $D^S_{m,u_S}(s)$ is meromorphically continued to $\mathbb{C}$, and is holomorphic at any $s\in\mathbb{Z}_{\le 0}$.*

Proof. See [Kim et al. 2022, Corollary 4.23] for the case $m>3$. For $m=2$, this statement can be proved by using [Hoffmann and Wakatsuki 2018; Yukie 1992]. $\square$

Theorem 3.10. *Fix a parameter $\underline{k}$ such that $k_n>n+1$. Let $h_1\in\mathcal{H}^{\mathrm{ur}}(G(\mathbb{Q}_{S_1}))^\kappa$, and let $h\in C_c^\infty(G(\mathbb{A}_f))$ be a test function on $G(\mathbb{A}_f)$ given as (3-1). Suppose $\sup_{x\in G(\mathbb{Q}_{S_1})}|h_1(x)|\le 1$. Then, there exist positive constants a, b, and c_0 such that, if $N\ge c_0N_1^{2n\kappa}$,*

$$\mathrm{Tr}\bigl(T_h|_{S_{\underline{k}}(\Gamma(N))}\bigr)=\mathrm{vol}_G\,\mathrm{vol}(K(N))^{-1}h_1(1)d_{\underline{k}}+\mathrm{vol}(K(N))^{-1}O(N_1^{a\kappa+b}N^{-n}).$$

Here the constants a and b do not depend on κ, N_1, or N. See Lemma 3.3 for vol_G and $d_{\underline{k}}$.

Proof. Set

$$I(\tilde{f},r)=\mathrm{vol}(K(N))\times\zeta_r(\widehat{\Phi_{h,r}},r-n),\quad 1\le r\le n.$$

By Theorem 3.7, it is sufficient to prove $I(\tilde{f},r)=O(N_1^{a\kappa+b}N^{-n})$.

Let R be a finite set of places of $\mathbb{Q}$. Take a Haar measure $\mathrm{d}x_\infty$ on $V_r(\mathbb{R})$, and for each prime p, we write $\mathrm{d}x_p$ for the Haar measure on $V_r(\mathbb{Q}_p)$ normalized by $\int_{V_r(\mathbb{Z}_p)}\mathrm{d}x_p=1$. For a test function $\Phi_R\in C_c^\infty(V_r(\mathbb{Q}_R))$ and an $H_r(\mathbb{Q}_R)$-orbit $\mathscr{O}_R\in V_r^0(\mathbb{Q}_R)/H_r(\mathbb{Q}_R)$, we set

$$Z_{r,R}(\Phi_R,s,\mathscr{O}_R)=c_R\int_{\mathscr{O}_R}\Phi_R(x)|\det(x)|_R^{s-(r+1)/2}\,\mathrm{d}x,$$

where $c_R=\prod_{p\in R,\,p<\infty}(1-p^{-1})^{-1}$, $|\cdot|_R=\prod_{v\in R}|\cdot|_v$, and $\mathrm{d}x=\prod_{v\in R}\mathrm{d}x_v$. It is known that $Z_{r,R}(\Phi_R,s,\mathscr{O}_R)$ absolutely converges for $\mathrm{Re}(s)\ge\frac{r+1}{2}$, and is meromorphically continued to the whole s-plane.

Suppose that R does not contain ∞, that is, R consists of primes. Write $\eta_p(x)$ for the Clifford invariant of $x\in V_r^0(\mathbb{Q}_p)$, see [Ikeda 2017, Definition 2.1], and set $\eta_R((x_p)_{p\in R})=\prod_{p\in R}\eta_p(x_p)$. For $\chi=\mathbb{1}_R$ (trivial) or η_R, we put $(\Phi_R\chi)(x)=\Phi_R(x)\,\chi(x)$. It follows from the local functional equation [Ikeda 2017, Theorems 2.1 and 2.2] over $\mathbb{Q}_p$ $(R=\{p\})$ that $Z_{r,p}(\Phi_p\chi,s,\mathscr{O}_p)$ is holomorphic in the range $\mathrm{Re}(s)<0$, and $Z_{r,p}(\Phi_p\chi,s,\mathscr{O}_p)$ possibly has a simple pole at $s=0$. Hence, for any R, $Z_{r,R}(\Phi_R\chi,s,\mathscr{O}_R)$ does not have any pole in the area $\mathrm{Re}(s)<0$, but it may have a pole at $s=0$. Let $\widehat{\Phi_R}$ denote the Fourier transform of $\Phi_R\in C_c^\infty(V_r(\mathbb{Q}_R))$ over $\mathbb{Q}_R$ for $\prod_{v\in R}\psi_v(\langle\ ,\ \rangle)$, where $\psi_v=\psi|_{\mathbb{Q}_v}$.

Define

$$\Phi_{h_1,r}(x)=h_1\left(\begin{pmatrix}I_n&*\\O_n&I_n\end{pmatrix}\right)\in C_c^\infty(V_r(\mathbb{Q}_{S_1})),$$

where $*=\left(\begin{smallmatrix}x&0\\0&0\end{smallmatrix}\right)\in V_n$. Note that this definition is compatible with $\Phi_{\tilde{f},r}$ since h_1 is spherical for $\prod_{p\in S_1}K_p$. Set

$$\mathscr{Z}_r(S_1,h_1)=\sum_{\mathscr{O}_{S_1}\in V_r^0(\mathbb{Q}_{S_1})/H_r(\mathbb{Q}_{S_1})}\left|Z_{r,S_1}(\widehat{\Phi_{h_1,r}}\chi_r,r-n,\mathscr{O}_{S_1})\right|,$$

where

$$\chi_r=\begin{cases}\mathbb{1}_{S_1}&\text{if }(r\text{ is odd and }r<n)\text{ or }r=2<n,\\ \eta_{S_1}&\text{if }r\text{ is even and }2<r<n,\end{cases}$$

and

$$\mathscr{Z}_n(S_1,h_1)=\sum_{\mathscr{O}_{S_1}\in V_r^0(\mathbb{Q}_{S_1})/H_r(\mathbb{Q}_{S_1})}\left|Z_{n,S_1}\left(\Phi_{h_1,n},\frac{n+1}{2},\mathscr{O}_{S_1}\right)\right|\quad\text{if }r=n.$$

It follows from Saito's formula [1999, Theorem 2.1 and §3] that the zeta function $\zeta_r(\widehat{\Phi_{h,r}},s)$ is expressed by a (finite or infinite) sum of Euler products of $Z_{r,p}(\Phi_p\chi_p,s,\mathscr{O}_p)$, with $\chi_p=\mathbb{1}_p$, η_p, or its finite sums, and he explicitly calculated the local zeta function $Z_{r,p}(\Phi_p\chi_p,s,\mathscr{O}_p)$ in [Saito 1997, §2] if Φ_p is the characteristic function of $V(\mathbb{Z}_p)$. We shall prove $I(\tilde{f},r)=O(N_1^{a\kappa+b}N^{-n})$ by using his results.

Case I. Assume r is odd and $r<n$. In the following, we set $S=S_1\sqcup\{\infty\}$. By Saito's formula, we have

$$I(\tilde{f},r)=(\text{constant})\times N^{r(r-1)/2-rn}\times\sum_{\mathscr{O}_{S_1}\in V_r^0(\mathbb{Q}_{S_1})/H_r(\mathbb{Q}_{S_1})} Z_{r,S_1}(\widehat{\Phi_{h_1,r}},r-n,\mathscr{O}_{S_1})$$
$$\times\zeta^S\Big(\frac{r+1}{2}-n\Big)\times\prod_{l=2}^{n}\zeta^S(l)^{-1}\times\prod_{u=1}^{[r/2]}\zeta^S(2u)\zeta^S(2r-2n-2u+1).$$

Therefore, one has

$$|I(\tilde{f},r)|\ll N^{r(r-1)/2-rn}\times N_1^{2n^3}\times\mathscr{Z}_r(S_1,h_1)$$

by using Lemma 3.8.

Case II. Assume r is even and $3<r<n$. By Saito's formula, Proposition 3.9, and Lemma 3.8, one can prove that $|I(\tilde{f},r)|$ is bounded by

$$N^{r(r-1)/2-rn}\times\mathscr{Z}_r(S_1,h_1)\times\left|\zeta^S\Big(\frac{r}{2}\Big)\times\prod_{l=2}^{n}\zeta^S(l)^{-1}\times\prod_{u=1}^{r/2-1}\zeta^S(2u)\times\prod_{u=1}^{r/2}\zeta^S(2r-2n-2u+1)\right|$$
$$\ll N^{r(r-1)/2-rn}\times N_1^{2n^3}\times\mathscr{Z}_r(S_1,h_1)$$

up to a constant. Note that Proposition 3.9 was used for this estimate, since it is necessary to prove the vanishing of the term including $D^S_{r,u_S}(s)$ in the explicit formula [Saito 1999, Theorem 3.3].

Case III. Assume $r=n$. In this case, we should use a method different from Case I and Case II since $Z_{r,S_1}(\widehat{\Phi_{h_1,r}}\chi,s,\mathscr{O}_{S_1})$ may have a simple pole at $s=r-n=0$. Take an n-tuple $\underline{l}=(l_1,\dots,l_n)$, with $l_1\geq\cdots\geq l_n>2n$, and let $n(x)=\left(\begin{smallmatrix}I_n & x\\ O_n & I_n\end{smallmatrix}\right)\in G$ where $x\in V_n$. Recall that $\tilde{f}_{\underline{l}}$ satisfies the following two properties:

(i) $\tilde{f}_{\underline{l}}(k^{-1}gk)=\tilde{f}_{\underline{l}}(g)$, for all $k\in K_\infty$, $g\in G(\mathbb{R})$; see [Wakatsuki 2018, §5.3].

(ii) $\int_{\mathbb{R}}\tilde{f}_{\underline{l}}(g_1^{-1}n_1(t)g_2)\,\mathrm{d}t=0$ for all $g_1,g_2\in G(\mathbb{R})$, where $n_1(t)=n((b_{ij})_{1\leq i,j\leq n})$, $b_{11}=t$, and $b_{ij}=0$ for all $(i,j)\neq(1,1)$; see [Wakatsuki 2018, Lemma 5.9].

By property (i), we can define $\Phi_{\tilde{f}_{\underline{l}},n}(x)=\tilde{f}_{\underline{l}}(n(x))$, where $x\in V_n(\mathbb{R})$.

Lemma 3.11. *For each orbit $\mathscr{O}_\infty\in V_n^0(\mathbb{R})/H_n(\mathbb{R})$, we have $Z_{n,\infty}(\Phi_{\tilde{f}_{\underline{l}},n},(n+1)/2,\mathscr{O}_\infty)=0$.*

Proof. Let $\mathscr{O}_\infty\neq I_n\cdot H_n(\mathbb{R})$, and take a representative element A of $\mathscr{O}_\infty$ as

$$A=\begin{pmatrix}0&0&1\\0&\mathscr{A}&0\\1&0&0\end{pmatrix},\qquad\text{where }\mathscr{A}\in V_{n-2}^0(\mathbb{R}).$$

The orbit $\mathscr{O}_\infty$ is decomposed into $A\cdot\mathrm{GL}_n(\mathbb{R})\sqcup(-A)\cdot\mathrm{GL}_n(\mathbb{R})$. The centralizer $H_{n(A)}$ of $n(A)$ in $H_n(\mathbb{R})$ is given by

$$H_{n(A)}=\{m(h)n(y):h\in\mathrm{O}_A(n),\ y\in V_n(\mathbb{R})\},$$

where

$$m(h)=\begin{pmatrix}{}^th^{-1}&O_n\\O_n&h\end{pmatrix}\quad\text{and}\quad\mathrm{O}_A(n)=\{h\in\mathrm{GL}_n:{}^thAh=A\}.$$

Hence, by property (ii), we have

$$\begin{aligned} Z_{n,\infty}\Big(\Phi_{\tilde{f}_{\underline{l}},n}, \frac{n+1}{2}, \mathscr{O}_\infty\Big) &= \sum_{A'=\pm A} \int_{\mathrm{O}_{A'}(n)\backslash \mathrm{GL}_n(\mathbb{R})} \tilde{f}_{\underline{l}}(m(h)^{-1}n(A')m(h))|\det(h)|^{n+1}\,\mathrm{d}h \\ &= \sum_{A'=\pm A} \int_{\mathscr{N}\mathrm{O}_{A'}(n)\backslash \mathrm{GL}_n(\mathbb{R})} \int_{\mathbb{R}} \tilde{f}_{\underline{l}}(m(h)^{-1}n(A')n_1(2t)m(h))|\det(h)|^{n+1}\,\mathrm{d}t\,\mathrm{d}h \\ &= 0, \end{aligned}$$

where $\mathscr{N} = \{(b_{ij}) : b_{jj} = 1, \text{with } 1 \le j \le n,\ b_{n1} \in \mathbb{R}, \text{ and } b_{ij} = 0 \text{ otherwise}\}$.

In the case $s = (n+1)/2$, we note that $|\det(x)|$ vanishes in the integral of $Z_{n,\infty}(\Phi_{\tilde{f}_{\underline{l}},n}, (n+1)/2, \mathscr{O}_\infty)$. Hence, it follows from property (ii) that

$$\sum_{\mathscr{O}_\infty \in V_n^0(\mathbb{R})/H_n(\mathbb{R})} Z_{n,\infty}\Big(\Phi_{\tilde{f}_{\underline{l}},n}, \frac{n+1}{2}, \mathscr{O}_\infty\Big) = \int_{V_n(\mathbb{R})} \Phi_{\tilde{f}_{\underline{l}},n}(x)\,\mathrm{d}x = 0,$$

and so we also find $Z_{n,\infty}(\Phi_{\tilde{f}_{\underline{l}},n}, (n+1)/2, I_n \cdot H_n(\mathbb{R})) = 0$. □

By Lemmas 3.6 and 3.11, the residue formula [Yukie 1993, Chapter 4] of $Z_n(\Phi, s, \mathbb{1})$ and the same argument as in [Hoffmann and Wakatsuki 2018, proof of Theorem 4.22], we obtain

$$\begin{aligned} \zeta_r(\widehat{\Phi_{h,r}}, 0) &= C_{n,n}(\underline{l})^{-1}\, Z_n\Big(\Phi_{\tilde{f}_{\underline{l}}h,r}, \frac{n+1}{2}, \mathbb{1}\Big) \\ &= C_{n,n}(\underline{l})^{-1}\ \mathrm{vol}(H_n(\mathbb{Q})\backslash H_n(\mathbb{A})^1) \int_{V(\mathbb{R})} \Phi_{\tilde{f}_{\underline{l}},n}(x_\infty)\log|\det(x_\infty)|_\infty\,\mathrm{d}x_\infty \\ &\qquad\qquad \times \int_{V(\mathbb{Q}_{S_1})} \Phi_{h_1,n}(x_{S_1})\,\mathrm{d}x_{S_1}\, N^{-n(n+1)/2}, \end{aligned}$$

where $H_n(\mathbb{A}) = \{(a, m) \in H_n(\mathbb{A}) : |a^n \det(m)^2| = 1\}$. From this, we have

$$|I(\tilde{f}, r)| \ll N^{-n(n+1)/2} \times \mathscr{Z}_r(S_1, h_1).$$

Case IV. Assume $r = 2 < n$. By Saito's formula [Hoffmann and Wakatsuki 2018, Theorem 4.15], we have

$$|I(\tilde{f}, r)| \ll N^{1-2n} \times \mathscr{Z}_2(S_1, h_1) \times \big|\zeta^S(2)^{-1}\zeta^S(3-2n)\big| \times \max_{u_S \in \mathbb{Q}_S^\times/(\mathbb{Q}_S^\times)^2,\, u_\infty < 0} |D^S_{2,u_S}(2-n)|.$$

Hence, it is enough to give an upper bound of $|D^S_{2,u_S}(2-n)|$ for $u_\infty < 0$. Choose a representative element $u_S = (u_v)_{v\in S}$ satisfying $u_p \in \mathbb{Z}_p$, with $p \in S_1$. Take a test function $\Phi = \otimes_v \Phi_v$ such that the support of Φ_∞ is contained in $\{x \in V_2^0(\mathbb{R}) : \det(x) > 0\}$ and Φ_p is the characteristic function of $\mathrm{diag}(1, -u_p) + p^2 V_2(\mathbb{Z}_p)$ (respectively, $V_2(\mathbb{Z}_p)$) for each $p \in S_1$ (respectively, $p \notin S$). Let

$$\Psi(y, yu) = \int_{K_2} \hat{\Phi}\Big({}^t k \begin{pmatrix} 0 & y \\ y & yu \end{pmatrix} k\Big)\,\mathrm{d}k, \qquad K_2 = \mathrm{O}(2,\mathbb{R}) \times \prod_p \mathrm{GL}_2(\mathbb{Z}_p),$$

and we set

$$T(\Phi, s) = \frac{\mathrm{d}}{\mathrm{d}s_1} T(\Phi, s, s_1)\Big|_{s_1=0} \quad \text{and} \quad T(\Phi, s, s_1) = \int_{\mathbb{A}^\times}\int_{\mathbb{A}} |y^2|^s \|(1,u)\|^{s_1}\,\Psi(y, yu)\,\mathrm{d}u\,\mathrm{d}^\times y.$$

By [Shintani 1975, Lemma 1], one obtains $Z_{2,\infty}(\widehat{\Phi_\infty}, n-\frac{1}{2}, \mathscr{O}_\infty)=0$ for any orbit $\mathscr{O}_\infty$ in $V_2^0(\mathbb{R})$. Therefore, from the functional equation [Yukie 1992, Corollary (4.3)], one deduces

$$\left|N_1^{-6} D^S_{2,u_S}(2-n)\right| \ll \left|Z_{2,S}(\Phi_S, 2-n, \mathscr{O}_S)\, D^S_{2,u_S}(2-n)\right| = \left|2^{-1} T\left(\hat{\Phi}, n-\tfrac{1}{2}\right)\right|.$$

By [Yukie 1992, Proposition (2.12) (2)], one gets

$$\left|T\left(\hat{\Phi}, n-\frac{1}{2}\right)\right| \ll N_1^{4n-2} \times \left\{\zeta^S(2n-2) + \left|(\zeta^S)'(2n-2)\right| + \left|\frac{(\zeta^S)'(2n-1)\zeta^S(2n-2)}{\zeta^S(2n-1)}\right|\right\},$$

where $(\zeta^S)'(s)=\frac{\mathrm{d}}{\mathrm{d}s}\zeta^S(s)$, because $\operatorname{Supp}(\hat{\Phi}_p)\subset p^{-2}V(\mathbb{Z}_p)$ for any $p\in S_1$. Therefore, one gets

$$|D^S_{2,u_S}(2-n)| \ll N_1^{4n+4}$$

by Lemma 3.8.

The final task is to prove $\mathscr{Z}_r(S_1, h_1) \ll N_1^{a\kappa+b}$ for some a and b. Using the local functional equations in [Ikeda 2017, Theorem 2.1] (see also [Sweet 1995]), one gets

$$\mathscr{Z}_r(S_1, h_1) \ll N_1^c \times \sum_{\mathscr{O}_{S_1}\in V_r^0(\mathbb{Q}_{S_1})/H_r(\mathbb{Q}_{S_1})} Z_{r,S_1}\left(|\Phi_{h_1,r}|, n-\frac{r-1}{2}, \mathscr{O}_{S_1}\right)$$

for some $c\in\mathbb{N}$. By [Assem 1993, Lemma 2.1.1] and the assumption $\sup_{x\in G(\mathbb{Q}_{S_1})}|h_1(x)|\le 1$, we have

$$|\Phi_{h_1,r}| \le \Phi_{S_1,r,-\kappa},$$

where $\Phi_{S_1,r,-\kappa}$ denotes the characteristic function of $\otimes_{p\in S_1} p^{-\kappa}V_r(\mathbb{Z}_p)$. Hence, by a change of variables, we get

$$\begin{aligned}
Z_{r,S_1}\left(|\Phi_{h_1,r}|, n-\frac{r-1}{2}, \mathscr{O}_{S_1}\right) &\le Z_{r,S_1}\left(\Phi_{S_1,r,-\kappa}, n-\frac{r-1}{2}, \mathscr{O}_{S_1}\right)\\
&= N_1^{\kappa n r-\kappa r(r-1)/2} Z_{r,S_1}\left(\Phi_{S_1,r,0}, n-\frac{r-1}{2}, \mathscr{O}_{S_1}\right)\\
&\le N_1^{\kappa n r-\kappa r(r-1)/2}.
\end{aligned}$$

It follows from classification theory of quadratic forms that $\#(V_r^0(\mathbb{Q}_{S_1})/H_r(\mathbb{Q}_{S_1}))\ll N_1$. Therefore, we obtain a desired upper bound for $\mathscr{Z}_r(S_1,h_1)$. Thus, we obtain $I(\tilde{f}, r)=O(N_1^{a\kappa+b}N^{-n})$. □

Remark 3.12. We give some remarks on Shin and Templier's work [2016] and Dalal's work [2022]. In the setting of [Shin and Templier 2016], they considered "all" cohomological representations as a family which exhausts an L-packet at infinity since they chose the Euler–Poincaré pseudocoefficient at the infinite place. Then there is no contribution from nontrivial unipotent conjugacy classes. Therefore, our work is different from Shin–Templier's work in that we can consider only holomorphic forms in an L-packet.

Shin suggested to consider a family of automorphic representations whose infinite type is any fixed discrete series representation. Dalal [2022] carried it out in the weight aspect by using the stable trace formula. The stabilization allows us to remove the contribution $I_3(f)$ (see Section 1), but instead of $I_3(f)$, the contributions of endoscopic groups have to enter. Dalal obtained a good bound for them by using the concept of hyperendoscopy introduced by Ferrari [2007]. In studying the level aspect, it seems difficult to directly get a sufficient bound for the growth of the hyperendoscopic groups in question; since $\mathrm{Sp}(2n)$ has infinitely many elliptic endoscopic groups

$$\mathrm{SO}(N_1, N_1) \times \mathrm{Sp}(2N_2) \quad \text{and} \quad \mathrm{SO}(N_1+1, N_1-1, E/\mathbb{Q}) \times \mathrm{Sp}(2N_2), \quad N_1 + N_2 = n,$$

where E runs over quadratic extensions of $\mathbb{Q}$ and $\mathrm{SO}(N_1+1, N_1-1, E/\mathbb{Q})$ is the quasisplit orthogonal group attached to $E/\mathbb{Q}$ (see [Arthur 2013, p. 13–14] and [Assem 1998, §4]), it is quite complicated to count the hyperendoscopic groups. (The referee pointed out to us that the essential difficulty in applying hyperendoscopy techniques is in computing endoscopic transfers of indicators of any level subgroup. In particular, answering the transfer problem is necessary to even know which set of groups we are counting.) We also observe the same complication coming from elliptic endoscopic groups in the unipotent terms of the (unstable) Arthur trace formula; see [Hoffmann and Wakatsuki 2018, p. 8]. Assem's results [1993; 1998] make us expect that, for $1 \le r \le n$, some parts of zeta integrals $Z_r(\Phi_{\tilde{f},r}, s, \chi)$ probably correspond to the central contributions of the endoscopic groups $\mathrm{SO}(n-r+1, n-r+1) \times \mathrm{Sp}(2r-2)$ and $\mathrm{SO}(n-r+2, n-r, E/\mathbb{Q}) \times \mathrm{Sp}(2r-2)$. To avoid such complication, we have simplified the unipotent terms in several steps as follows:

- Our method showed the vanishing of a large part of the unipotent terms; see Lemma 3.3 and [Wakatsuki 2018].
- The contributions of $Z_r(\Phi_{\tilde{f},r}, s, \chi)$ vanish when χ is nontrivial; see Theorem 3.7.
- Our careful analysis estimates upper bounds of the contributions of $Z_r(\Phi_{\tilde{f},r}, s, \mathbb{1})$ by using the functional equations; see the proof of Theorem 3.10.

Analogous simplifications should be required even if we use the stable trace formula.

4. Arthur classification of Siegel modular forms

In this section, we study Siegel modular forms in terms of Arthur's classification [2013]; see §1.4 and §1.5 of loc. cit.. Recall $G = \mathrm{Sp}(2n)/\mathbb{Q}$. We call a Siegel cusp form which comes from smaller groups by Langlands functoriality "a nongenuine form". In this section, we estimate the dimension of the space of nongenuine forms and show that they are negligible. This result is interesting in its own right.

Let $F \in HE_{\underline{k}}(N)$, see Section 2, and $\pi = \pi_F$ be the corresponding automorphic representation of $G(\mathbb{A})$. According to Arthur's classification, π can be described by using the global A-packets. Let us recall some notations. A (discrete) global A-parameter is a symbol

$$\psi = \pi_1[d_1] \boxplus \cdots \boxplus \pi_r[d_r]$$

satisfying the following conditions:

(1) for each i, with $1 \le i \le r$, π_i is an irreducible unitary cuspidal self-dual automorphic representation of $\mathrm{GL}_{m_i}(\mathbb{A})$. In particular, the central character ω_i of π_i is trivial or quadratic;

(2) for each i, $d_i \in \mathbb{Z}_{>0}$ and $\sum_{i=1}^{r} m_i d_i = 2n + 1$;

(3) if d_i is odd, then π_i is orthogonal, i.e., $L(s, \pi_i, \mathrm{Sym}^2)$ has a pole at $s = 1$;

(4) if d_i is even, then π_i is symplectic, i.e., $L(s, \pi_i, \wedge^2)$ has a pole at $s = 1$;

(5) $\omega_1^{d_1} \cdots \omega_r^{d_r} = \mathbb{1}$;

(6) if $i \neq j$ and $\pi_i \simeq \pi_j$, then $d_i \neq d_j$.

We say that two global A-parameters $\boxplus_{i=1}^r \pi_i[d_i]$ and $\boxplus_{i=1}^{r'} \pi_i'[d_i']$ are equivalent if $r = r'$ and there exists $\sigma \in \mathfrak{S}_r$ such that $d_i' = d_{\sigma(i)}$ and $\pi_i' = \pi_{\sigma(i)}$. Let $\Psi(G)$ be the set of equivalent classes of global A-parameters. For each $\psi \in \Psi(G)$, one can associate a set Π_ψ of equivalent classes of simple admissible $G(\mathbb{A}_f) \times (\mathfrak{g}, K_\infty)$-modules; see [Arthur 2013]. The set Π_ψ is called a global A-packet for ψ.

Definition 4.1. Let $\psi = \boxplus_{i=1}^r \pi_i[d_i]$ be a global A-parameter.

- ψ is said to be semisimple if $d_1 = \cdots = d_r = 1$; otherwise, ψ is said to be nonsemisimple;
- ψ is said to be simple if $r = 1$ and $d_1 = 1$.

By [Arthur 2013, Theorem 1.5.2] (though our formulation is slightly different from the original one), we have a following decomposition

$$L^2_{\mathrm{disc}}(G(\mathbb{Q})\backslash G(\mathbb{A})) \simeq \bigoplus_{\psi \in \Psi(G)} \bigoplus_{\pi \in \Pi_\psi} m_{\pi,\psi} \pi, \tag{4-1}$$

where $m_{\pi,\psi} \in \{0, 1\}$; see [Atobe 2018, Theorem 2.2] for $m_{\pi,\psi}$. We have the following immediate consequence of (4-1):

Proposition 4.2. *Let $1_{K(N)}$ be the characteristic function of $K(N) \subset G(\mathbb{A}_f)$. Then*

$$S_{\underline{k}}(\Gamma(N)) = \bigoplus_{\psi \in \Psi(G)} \bigoplus_{\substack{\pi \in \Pi_\psi \\ \pi_\infty \simeq \sigma_{\underline{k}}}} m_{\pi,\psi} \pi_f^{K(N)}$$

and

$$|HE_{\underline{k}}(N)| = \mathrm{vol}(K(N))^{-1} \sum_{\psi \in \Psi(G)} \sum_{\substack{\pi \in \Pi_\psi \\ \pi_\infty \simeq \sigma_{\underline{k}}}} m_{\pi,\psi} \, \mathrm{tr}(\pi_f(1_{K(N)})). \tag{4-2}$$

Theorem 4.3. *Assume (1-4). For a global A-parameter $\psi = \boxplus_{i=1}^r \pi_i[d_i]$, suppose that there exists $\pi \in \Pi_\psi$ with $\pi_\infty \simeq \sigma_{\underline{k}}$. Then ψ is semisimple, i.e., $d_i = 1$ for all i, and each π_i is regular algebraic and satisfies the Ramanujan conjecture, i.e., $\pi_{i,p}$ is tempered for any p.*

Proof. By the proof of [Chenevier and Lannes 2019, Corollary 8.5.4], we see that $d_1 = \cdots = d_r = 1$. Hence, ψ is semisimple. Further, by comparing infinitesimal characters $c(\pi_\infty)$, $c(\psi_\infty)$ of π_∞, ψ_∞ respectively, we see that each π_i is regular algebraic by [Chenevier and Lannes 2019, Corollary 6.3.6 and Proposition 8.2.10]. It follows from [Caraiani 2012;2014] that $\pi_{i,p}$ is tempered for any p. □

Therefore, for each finite prime p, the local Langlands parameter at p of π is described as one of the isobaric sum $\boxplus_{i=1}^r \pi_{i,p}$ which is an admissible representation of $\mathrm{GL}_{2n+1}(\mathbb{Q}_p)$.

Definition 4.4. We denote by $HE_{\underline{k}}(N)^{ng}$ the subset of $HE_{\underline{k}}(N)$ consisting of all forms which belong to

$$\bigoplus_{\substack{\psi\in\Psi(G)\\ \psi:\text{nonsimple}}} \bigoplus_{\substack{\pi\in\Pi_\psi\\ \pi_\infty\simeq\sigma_{\underline{k}}}} m_{\pi,\psi}\pi_f^{K(N)},$$

under the isomorphism (4-1). A form in this space is called a nongenuine form.

Similarly, we denote by $HE_{\underline{k}}(N)^{g}$ the subset of $HE_{\underline{k}}(N)$ consisting of all forms which belong to

$$\bigoplus_{\substack{\psi\in\Psi(G)\\ \psi:\text{simple}}} \bigoplus_{\substack{\pi\in\Pi_\psi\\ \pi_\infty\simeq\sigma_{\underline{k}}}} m_{\pi,\psi}\pi_f^{K(N)},$$

under the isomorphism (4-1). A form in this space is called a genuine form.

Definition 4.5. Denote by $\Pi(\mathrm{GL}_n(\mathbb{R}))^c$ the isomorphism classes of all irreducible cohomological admissible $(\mathfrak{gl}_n, O(n))$-modules. For $\tau_\infty\in\Pi(\mathrm{GL}_n(\mathbb{R}))^c$ and a quasicharacter $\chi:\mathbb{Q}^\times\backslash\mathbb{A}^\times\to\mathbb{C}^\times$, we define

$$L^{\text{cusp,ort}}\big(\mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{A}),\tau_\infty,\chi\big):=\bigoplus_{\substack{\pi:\text{orthogonal}\\ \pi_\infty\simeq\tau_\infty,\,\omega_\pi=\chi}} m(\pi)\pi$$

and

$$L^{\text{cusp,ort}}\big(K^{\mathrm{GL}_n}(N),\tau_\infty,\chi\big):=\bigoplus_{\substack{\pi:\text{orthogonal}\\ \pi_\infty\simeq\tau_\infty,\,\omega_\pi=\chi}} m(\pi)\pi^{K^{\mathrm{GL}_n}(N)},$$

where the direct sums are taken over the isomorphism classes of all orthogonal cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A})$ and ω_π stands for the central character of π. The constant $m(\pi)$ is the multiplicity of π in $L^2(\mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{A}))$ which satisfies $m(\pi)\in\{0,1\}$ by [Shalika 1974]. Here, $K^{\mathrm{GL}_n}(N)$ is the principal congruence subgroup of $\mathrm{GL}_n(\hat{\mathbb{Z}})$ of level N. Put

$$l^{\text{cusp,ort}}(n,N,\tau_\infty,\chi):=\dim_{\mathbb{C}}\big(L^{\text{cusp,ort}}(K^{\mathrm{GL}_n}(N),\tau_\infty,\chi)\big)$$

for simplicity. Clearly, $l^{\text{cusp,ort}}(1,N,\tau_\infty,\chi)=|\hat{\mathbb{Z}}^\times/(1+N\hat{\mathbb{Z}})^\times|=\varphi(N)$, where φ stands for Euler's totient function.

Let $P(2n+1)$ be the set of all partitions of $2n+1$ and $P_{\underline{m}}$ be the standard parabolic subgroup of GL_{2n+1} associated to a partition $2n+1=m_1+\cdots+m_r$, and $\underline{m}=(m_1,\ldots,m_r)$.

In order to apply the formula (4-2), it is necessary to study the transfer of Hecke elements in the local Langlands correspondence established by [Arthur 2013, Theorem 1.5.1]. We regard $G=\mathrm{Sp}(2n)$ as a twisted elliptic endoscopic subgroup of GL_{2n+1}; see [Ganapathy and Varma 2017] or [Oi 2023].

Proposition 4.6. *Let N be an odd positive integer. Put $S_N:=\{p \text{ prime}: p\mid N\}$. For the pair (GL_{2n+1},G), the characteristic function of* $\mathrm{vol}(K(N))^{-1}1_{K(N)}$ *as an element of $C_c^\infty(G(\mathbb{Q}_{S_N}))$ is transferred to*

$$\mathrm{vol}\left(K^{\mathrm{GL}_{2n+1}}(N)\right)^{-1}1_{K^{\mathrm{GL}_{2n+1}}(N)}$$

as an element of $C_c^\infty(\mathrm{GL}_{2n+1}(\mathbb{Q}_{S_N}))$.

Proof. It follows from [Ganapathy and Varma 2017, Lemma 8.2.1 (i)]. □

Remark 4.7. Keep the notation in the previous proposition. If Π is the twisted endoscopic transfer of π, then the claim immediately implies

$$\dim_{\mathbb{C}} \pi^{K(N)} \le \dim_{\mathbb{C}} \Pi^{K^{\mathrm{GL}_{2n+1}}(N)}.$$

In fact, we have $\dim_{\mathbb{C}} \pi^{K(N)} = \mathrm{tr}(I_\theta : \Pi^{K^{\mathrm{GL}_{2n+1}}(N)} \to \Pi^{K^{\mathrm{GL}_{2n+1}}(N)})$, where $I_\theta : \Pi \to \Pi$ is the intertwining operator defining the twisted trace. Since I_θ is of finite order, we have the above inequality; see the argument for [Yamauchi 2021, Theorem 1.6].

Applying Proposition 4.6, we have the following:

Proposition 4.8. *Assume (1-4) and N is odd. Then $|HE_{\underline{k}}(N)^{ng}|$ is bounded by*

$$\frac{A_n(N)}{\varphi(N)} \sum_{\substack{\underline{m}=(m_1,\dots,m_r)\in P(2n+1)\\ r\ge 2}} \sum_{\substack{\tau_i\in\Pi(\mathrm{GL}_{m_i}(\mathbb{R}))^c\\ c(\boxplus_{i=1}^r \tau_i)=c(\sigma_{\underline{k}})}} \sum_{\substack{\chi_i:\mathbb{Q}^\times\backslash\mathbb{A}^\times\to\mathbb{C}^\times\\ \chi_i^2=1,\ c(\chi)\mid N}} d_{P_{\underline{m}}}(N) \prod_{i=1}^r l^{\mathrm{cusp,ort}}(m_i, N, \tau_i, \chi_i),$$

where the second sum is indexed by all r-tuples $(\tau_1, \dots, \tau_r)$ such that $\tau_i \in \Pi(\mathrm{GL}_{m_i}(\mathbb{R}))^c$ and $c(\boxplus_{i=1}^r \tau_i) = c(\sigma_{\underline{k}})$, the equality of the infinitesimal characters. Further $c(\chi)$ stands for the conductor of χ and $\varphi(N) = |(\mathbb{Z}/N\mathbb{Z})^\times|$. Here,

(1) $A_n(N) := 2^{(2n+1)\omega(N)}$ *where* $\omega(N) := |\{p \text{ prime} : p \mid N\}|$;

(2) $d_{P_{\underline{m}}}(N) = |P_{\underline{m}}(\mathbb{Z}/N\mathbb{Z})\backslash \mathrm{GL}_{2n+1}(\mathbb{Z}/N\mathbb{Z})| = \mathrm{vol}(K^{\mathrm{GL}_{2n+1}}(N))^{-1}/|P_{\underline{m}}(\mathbb{Z}/N\mathbb{Z})|$.

Proof. Let $\pi = \pi_\infty \otimes \otimes'_p \pi_p$ be an element of Π_ψ for $\psi = \boxplus_{i=1}^r \pi_i$. Let Π_p be the local Langlands correspondence of π_p to $\mathrm{GL}_{2n+1}(\mathbb{Q}_p)$ established by [Arthur 2013, Theorem 1.5.1], and let $\mathcal{L}(\Pi_p) : L_{\mathbb{Q}_p} \to \mathrm{GL}_{2n+1}(\mathbb{C})$ be the local L-parameter of Π_p, where $L_{\mathbb{Q}_p} = W_{\mathbb{Q}_p}$ for each $p < \infty$ and $L_{\mathbb{R}} = W_{\mathbb{R}} \times \mathrm{SL}_2(\mathbb{C})$. Since the localization ψ_p of the global A-parameter ψ at p is tempered by Theorem 4.3, we see that $\mathcal{L}(\Pi_p)$ is equivalent to ψ_p. Since $\mathcal{L}(\Pi_p)$ is independent of $\pi \in \Pi_\psi$ and multiplicity one for $\mathrm{GL}_{2n+1}(\mathbb{A})$ holds, the isobaric sum $\psi = \boxplus_{i=1}^r \pi_i$ as an automorphic representation of $\mathrm{GL}_{2n+1}(\mathbb{A})$ gives rise to a unique global L-parameter on Π_ψ. On the other hand, it follows from [Arthur 2013, Theorem 1.5.1] that $|\Pi_{\psi_p}| \le 2^{2n+1}$ for the local A-packet Π_{ψ_p} at p if $p \mid N$, and Π_{ψ_p} is a singleton if $p \nmid N$. It yields that $|\Pi_\psi| \le 2^{(2n+1)\omega(N)}$. Since the local Langlands correspondence $\pi_p \mapsto \Pi_p$ satisfies the character relation by [Arthur 2013, Theorem 1.5.1], it follows from Proposition 4.6 with Remark 4.7 that for each $\pi \in \Pi_\psi$,

$$\begin{aligned}\dim\big(\pi_f^{K(N)}\big) &= \mathrm{vol}(K(N))^{-1}\,\mathrm{tr}(\pi(1_{K(N)}))\\ &\le \mathrm{vol}(K^{\mathrm{GL}_{2n+1}}(N))^{-1}\,\mathrm{tr}\big((\boxplus_{i=1}^r \pi_i)(1_{K^{\mathrm{GL}_{2n+1}}(N)})\big)\\ &= \dim\big((\boxplus_{i=1}^r \pi_i)_f^{K^{\mathrm{GL}_{2n+1}}(N)}\big),\end{aligned}$$

where we denote by $\pi_f = \otimes'_{p<\infty}\pi_p$ the finite part of the cuspidal representation π. Plugging this into Proposition 4.2, we have

$$|HE_{\underline{k}}(N)^{ng}| = \mathrm{vol}(K(N))^{-1} \sum_{\substack{\psi=\boxplus_{i=1}^r \pi_i\in\Psi(G),\, r\geq 2\\ c(\psi_\infty)=c(\sigma_{\underline{k}})}} \sum_{\pi\in\Pi_\psi} m_{\pi,\psi}\,\mathrm{tr}(\pi_f(1_{K(N)}))$$
$$\leq \frac{A_n(N)}{\varphi(N)} \sum_{\substack{\psi=\boxplus_{i=1}^r \pi_i\in\Psi(G),\, r\geq 2\\ c(\psi_\infty)=c(\sigma_{\underline{k}})}} \dim\big((\boxplus_{i=1}^r \pi_i)_f^{K^{\mathrm{GL}_{2n+1}}(N)}\big),$$

where $1/\varphi(N)$ is inserted because of the condition on the central characters in global A-parameters. Here, $r\geq 2$ is essential to gain the factor $1/\varphi(N)$; see Remark 4.9.

Next we describe $\dim((\boxplus_{i=1}^r \pi_i)_f^{K^{\mathrm{GL}_{2n+1}}(N)})$ in terms of the data (m_i, N, τ_i, χ_i) with $1\leq i\leq r$. Since

$$P_{\underline{m}}(\mathbb{A}_f)\backslash \mathrm{GL}_{2n+1}(\mathbb{A}_f)/K(N) \simeq P_{\underline{m}}(\hat{\mathbb{Z}})\backslash \mathrm{GL}_{2n+1}(\hat{\mathbb{Z}})/K(N) \simeq P_{\underline{m}}(\mathbb{Z}/N\mathbb{Z})\backslash \mathrm{GL}_{2n+1}(\mathbb{Z}/N\mathbb{Z})$$

and a complete system of the representatives can be taken from elements in $\mathrm{GL}_{2n+1}(\hat{\mathbb{Z}})$, and therefore, they normalize $K(N)$. Then a standard method for fixed vectors of an induced representation shows that

$$\dim\big((\boxplus_{i=1}^r \pi_i)_f^{K^{\mathrm{GL}_{2n+1}}(N)}\big) = d_{P_{\underline{m}}}(N) \prod_{i=1}^r \dim\big(\pi_{i,f}^{K^{\mathrm{GL}_{m_i}}(N)}\big),$$

Here, if χ_i is the central character of π_i and $\pi_{i,\infty}\simeq\tau_i$, then $\dim(\pi_{i,f}^{K^{\mathrm{GL}_{m_i}}(N)}) = l^{\mathrm{cusp,ort}}(m_i, N, \tau_i, \chi_i)$. Notice that the conductor of χ_i is a divisor of N. Summing up, we have the claim. $\square$

Remark 4.9. Let $r\geq 2$. The group homomorphism $((\mathbb{Z}/N\mathbb{Z})^\times)^r \to (\mathbb{Z}/N\mathbb{Z})^\times$, $(x_1,\dots,x_r)\mapsto x_1\cdots x_r$, is obviously surjective, and it yields

$$\big|\{(\chi_1,\dots,\chi_r)\in \widehat{(\mathbb{Z}/N\mathbb{Z})^{\times}}^{r} : \chi_1\cdots\chi_r = 1\}\big| = \frac{|\widehat{(\mathbb{Z}/N\mathbb{Z})^{\times}}^{r}|}{\varphi(N)}.$$

This trivial equality explains the appearance of the factor $1/\varphi(N)$ in Proposition 4.8.

Next we study $l^{\mathrm{cusp,ort}}(n, N, \tau, \chi)$ for $\tau\in\Pi(\mathrm{GL}_n(\mathbb{R}))^c$ and for $n\geq 2$. Now if π is a cuspidal representation of GL_{2m+1} which is orthogonal, i.e., $L(s,\pi,\mathrm{Sym}^2)$ has a pole at $s=1$, then π comes from a cuspidal representation τ on $\mathrm{Sp}(2m)$. In this case, the central character ω_π of π is trivial.

If π is a cuspidal representation of GL_{2m} which is orthogonal, i.e., $L(s,\pi,\mathrm{Sym}^2)$ has a pole at $s=1$, then $\omega_\pi^2=1$; If $\omega_\pi=1$, π comes from a cuspidal representation τ on the split orthogonal group $\mathrm{SO}(m,m)$; If $\omega_\pi\neq 1$, then π comes from a cuspidal representation τ on the quasisplit orthogonal group $\mathrm{SO}(m+1,m-1)$.

First we consider the case when χ is trivial in estimating $l^{\mathrm{cusp,ort}}(2n+\delta, N, \tau, \chi)$, where $\delta=0$ or 1. For a positive integer n, let

$$H = \begin{cases} \mathrm{SO}(n,n) & \text{if } G'=\mathrm{GL}_{2n},\\ \mathrm{Sp}(2n) & \text{if } G'=\mathrm{GL}_{2n+1}.\end{cases}$$

We regard H as a twisted elliptic endoscopic subgroup G'.

Proposition 4.10. *Let N be an odd positive integer. For the pair (G', H), the characteristic function of $\mathrm{vol}(K^H(N))^{-1}1_{K^H(N)}$ as an element of $C_c^\infty(H(\mathbb{Q}_{S_N}))$ is transferred to*

$$\mathrm{vol}(K^{G'}(N))^{-1}1_{K^{G'}(N)}$$

as an element of $C_c^\infty(G'(\mathbb{Q}_{S_N}))$.

Proof. It follows from [Ganapathy and Varma 2017, Lemma 8.2.1 (i)]. □

Each cuspidal representation π of $G'(\mathbb{A})$ contributing to $l^{\mathrm{cusp,ort}}(N, \tau, \mathbb{1})$ can be regarded as a simple A-parameter. Also as a cuspidal representation, it strongly descends to a generic cuspidal representation Π_π of $H(\mathbb{A})$ whose L-parameter $\mathcal{L}(\Pi_\tau)$ at infinity of Π_π is same as one of π_∞. In this setting, by [Arthur 2013, Proposition 8.3.2 (b)], the problem is reduced to estimate

$$L^{\mathrm{cusp,gen}}(H, N, \mathcal{L}(\Pi_\tau), \mathbb{1}) := \bigoplus_{\pi \subset L^{\mathrm{cusp,generic,ort}}(H(\mathbb{Q})\backslash H(\mathbb{A}), \mathcal{L}(\Pi_\tau), \mathbb{1})} m(\pi)\pi^{K^H(N)}, \quad m(\pi) \in \{0, 1, 2\},$$

where π runs over all irreducible unitary, cohomological orthogonal cuspidal automorphic representations of $H(\mathbb{A})$ whose L-parameter at infinity is isomorphic to $\mathcal{L}(\Pi_\tau)$ with the central character $\chi = \mathbb{1}$.

Proposition 4.11. *Keep the notations as above. Then*

- $l^{\mathrm{cusp,ort}}(2n+\delta, N, \tau, \mathbb{1}) \le C_n(N)\dim(L^{\mathrm{cusp,gen}}(H, N, \mathcal{L}(\Pi_\tau), \mathbb{1}))$, *where* $C_n(N) := 2^{(2n+\delta)\omega(N)}$ *and*
$$\delta = \begin{cases} 0 & \text{if } G' = \mathrm{GL}_{2n}, \\ 1 & \text{if } G' = \mathrm{GL}_{2n+1}. \end{cases}$$
- $\dim(L^{\mathrm{cusp,gen}}(H, N, \mathcal{L}(\Pi_\tau), \mathbb{1})) \ll c \cdot \mathrm{vol}(K^H(N))^{-1} \sim c N^{\dim(H)}$ *for some $c > 0$, when the infinitesimal character of $\mathcal{L}(\Pi_\tau)$ is fixed and $N \to \infty$.*

Proof. The first claim follows from [Arthur 2013, Proposition 8.3.2 (b)] with a completely similar argument of Proposition 4.8.

The second claim follows from [Savin 1989]. □

Next we consider the case when χ is a quadratic character. In this case, a cuspidal representation π contributing to $L^{\mathrm{cusp,ort}}(K^{\mathrm{GL}_n}(N), \tau_\infty, \chi)$ comes from a cuspidal representation of the quasisplit orthogonal group $\mathrm{SO}(m+1, m-1)$ defined over the quadratic extension associated to χ. However any transfer theorem for Hecke elements in $(\mathrm{GL}_{2m}, \mathrm{SO}(m+1, m-1))$ remains open. To get around this situation, we make use of the transfer theorems for some Hecke elements in the quadratic base change due to Yamauchi [2021]. For this, we need the following assumptions on the level N:

(1) N is an odd prime or

(2) N is odd and all prime divisors $p_1, \dots, p_r$ $(r \ge 2)$ of N are congruent to 1 modulo 4 and $\left(\frac{p_i}{p_j}\right) = 1$ for $i \ne j$, where $\left(\frac{*}{*}\right)$ denotes the Legendre symbol.

These conditions are needed in order that for any quadratic extension $M/\mathbb{Q}$ with the conductor d_M dividing N, there exists an integral ideal $\mathfrak{N}$ of M such that $\mathfrak{N}\mathfrak{N}^\theta = (d_M)$ where θ is the generator of $\mathrm{Gal}(M/\mathbb{Q})$.

Proposition 4.12. *Keep the assumptions on N as above. Then*

$$l^{\mathrm{cusp,ort}}(2m, N, \tau, \chi) \le 2^{2m\cdot\omega(N)} \operatorname{vol}(K^H(N))^{-1},$$

where $H = \mathrm{SO}(m,m)$.

Proof. Let $M/\mathbb{Q}$ be the quadratic extension associated to χ and $\mathcal{O}_M$ the ring of integers of M. Let θ be the generator of $\mathrm{Gal}(M/\mathbb{Q})$. Let $K_M^{\mathrm{GL}_{2m}}(\mathfrak{N})$ be the principal congruence subgroup of $\mathrm{GL}_{2m}(\hat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{O}_M)$ of the level $\mathfrak{N}$. Clearly, the θ-fixed part of $K_M^{\mathrm{GL}_{2m}}(\mathfrak{N})$ is $K^{\mathrm{GL}_{2m}}(d_M)$ where d_M is the conductor of $M/\mathbb{Q}$ and it contains $K^{\mathrm{GL}_{2m}}(N)$ since $d_M|N$. Applying [Yamauchi 2021, Theorem 1.6], we have for a cuspidal representation π of $\mathrm{GL}_{2m}(\mathbb{A})$ and its base change $\Pi := \mathrm{BC}_{M/\mathbb{Q}}(\pi)$ to $\mathrm{GL}_{2m}(\mathbb{A}_M)$,

$$\operatorname{vol}\bigl(K^{\mathrm{GL}_{2m}}(N)\bigr)^{-1} \operatorname{tr}\bigl(\pi(1_{K^{\mathrm{GL}_{2m}}(N)})\bigr) \le \operatorname{vol}\bigl(K_M^{\mathrm{GL}_{2m}}(\mathfrak{N})\bigr)^{-1} \operatorname{tr}\bigl(\Pi(1_{K_M^{\mathrm{GL}_{2m}}(\mathfrak{N})})\bigr).$$

Recall that our π contributing to $L^{\mathrm{cusp,ort}}(2m, N, \tau, \chi)$ is orthogonal, namely, $L(s, \pi, \mathrm{Sym}^2)$ has a pole at $s = 1$. Note that $L(s, \Pi, \mathrm{Sym}^2) = L(s, \pi, \mathrm{Sym}^2)L(s, \pi, \mathrm{Sym}^2 \otimes \chi)$. Now, $L(s, \pi \times (\pi \otimes \chi)) = L(s, \pi, \wedge^2 \otimes \chi)L(s, \pi, \mathrm{Sym}^2 \otimes \chi)$. Suppose Π is cuspidal. Then $\pi \not\simeq \pi \otimes \chi$. So the left-hand side has no zero at $s = 1$, and $L(s, \pi, \mathrm{Sym}^2 \otimes \chi)$ has no zero at $s = 1$. Therefore, $L(s, \Pi, \mathrm{Sym}^2)$ has a pole at $s = 1$.

If Π is noncuspidal, then by Arthur and Clozel [1989], there exists a cuspidal representation τ of $\mathrm{GL}_m(\mathbb{A}_M)$ such that

$$\Pi = \tau \boxplus \tau^\theta.$$

In such a case, if $m = 2$, then $\pi = \mathrm{AI}_M^{\mathbb{Q}}\, \tau$ for some cuspidal representation τ of $\mathrm{GL}_2(\mathbb{A}_M)$; an automorphic induction from $\mathrm{GL}_2(\mathbb{A}_M)$ to $\mathrm{GL}_4(\mathbb{A}_{\mathbb{Q}})$. Since π is cuspidal and orthogonal, τ has to be dihedral. Such π are counted in [Kim et al. 2020b, Section 2.6] and it amounts to $O(N^{11/2+\varepsilon})$ for any $\varepsilon > 0$. This will be negligible because $\operatorname{vol}(K^H(N)) \sim cN^{m(2m-1)} = cN^6$ for some constant $c > 0$. Assume $m \ge 3$. It is easy to see that the dimension of $\bigoplus_{\Pi:\mathrm{noncuspidal}} \Pi_f^{K_M^{\mathrm{GL}_{2m}}(\mathfrak{N})}$ is bounded by

$$O\bigl(N^{m^2-1+m(m+1)/2}\bigr) = O\bigl(N^{3m^2/2+m/2-1}\bigr),$$

where the -1 of $m^2 - 1$ in the exponent of left-hand side in the above equation is inserted because of the fixed central character. Since $\dim \mathrm{SO}(m,m) = m(2m-1)$ and $m \ge 3$, spaces $\Pi_f^{K_M^{\mathrm{GL}_{2m}}(\mathfrak{N})}$ for which Π is noncuspidal are negligible in the estimation. Further, Π is orthogonal with trivial central character. (The central character of Π is $\chi \circ N_{M/\mathbb{Q}} = 1$.) Therefore, we can bound $l^{\mathrm{cusp,ort}}(2m, N, \tau, \chi)$ by

$$l^{\mathrm{cusp,ort}}(2m, \mathfrak{N}, \mathrm{BC}_{M_\infty/\mathbb{R}}(\tau), 1),$$

which is similarly defined for cuspidal representations of $\mathrm{GL}_{2m}(\mathbb{A}_M)$. Applying the argument of the proof of Proposition 4.11 to $(\mathrm{GL}_{2m}/M, \mathrm{SO}(m,m)/M)$, the quantity $l^{\mathrm{cusp,ort}}(2m, N, \tau, \chi)$ is bounded by $2^{2m\omega(\mathfrak{N})} \operatorname{vol}(K^{H_M}(\mathfrak{N}))^{-1}$, where $H_M := \mathrm{SO}(m,m)/M$ and $\omega(\mathfrak{N})$ denotes the number of prime ideals dividing $\mathfrak{N}$. The claim follows from $\mathcal{O}_M/\mathfrak{N} \simeq \mathbb{Z}/N\mathbb{Z}$ since $\operatorname{vol}(K^{H_M}(\mathfrak{N})) = \operatorname{vol}(K^H(N))$ and clearly $\omega(\mathfrak{N}) = \omega(N)$. ∎

Note that for any split reductive group $\mathcal{G}$ over $\mathbb{Q}$ and the principal congruence subgroup $K^{\mathcal{G}}(N)$ of level N, we have that $\mathrm{vol}(K^{\mathcal{G}}(N)) \sim cN^{-\dim\mathcal{G}}$ for some constant $c>0$ as $N\to\infty$. Furthermore, $\omega(N) \ll \log N/(\log\log N)$. Hence $2^{\omega(N)} \ll N^{\epsilon}$, and $A_n(N)=O(N^{\varepsilon})$ and $C_{m_i}(N)=O(N^{\varepsilon})$ for each $1\le i\le r$.

Theorem 4.13. *Assume (1-4). Keep the assumptions on N as in Proposition 4.12. Then* $|HE_{\underline{k}}(N)^{ng}| = O_n(N^{2n^2+n-1+\varepsilon})$ *for any* $\varepsilon>0$. *In particular,*

$$\lim_{N\to\infty}\frac{|HE_{\underline{k}}(N)^{ng}|}{|HE_{\underline{k}}(N)|}=0.$$

Proof. By Proposition 4.8, for each partition $\underline{m}=(m_1,\dots,m_r)$ of $2n+1$, we must only estimate

$$\frac{A_n(N)}{\varphi(N)}d_{P_{\underline{m}}}(N)\prod_{i=1}^{r} l^{\mathrm{cusp,ort}}(m_i,N,\tau_i,\chi_i).$$

By Proposition 4.11 and Proposition 4.12,

$$l^{\mathrm{cusp,ort}}(m_i,N,\tau_i,\chi_i)\ll N^{m_i(m_i-1)/2+\varepsilon}$$

for any $\varepsilon>0$. Further, $d_{P_{\underline{m}}(N)}=O(N^{\dim P_{\underline{m}}\backslash \mathrm{GL}_{2n+1}})=O(N^{\sum_{1\le i<j\le r}m_im_j})$. Note that $\varphi(N)^{-1}=O(N^{-1+\varepsilon})$ for any $\varepsilon>0$. Since

$$\sum_{1\le i<j\le r}m_im_j+\sum_{i=1}^{r}\frac{m_i(m_i-1)}{2}=\frac12\Bigl(\sum_{1\le i,j\le r}m_im_j\Bigr)-\frac12\sum_{i=1}^{r}m_i=\frac12(2n+1)^2-\frac12(2n+1)=2n^2+n,$$

we have the first claim.

The second claim follows from the dimension formula (1-3). □

5. A notion of newforms in $S_{\underline{k}}(\Gamma(N))$

In this section, we introduce a notion of a newform in $S_{\underline{k}}(\Gamma(N))$ with respect to principal congruence subgroups. Since any local newform theory for $\mathrm{Sp}(2n)$ is unavailable except for $n=1,2$, we need a notion of newforms so that we can control a lower bound of conductors for such newforms. This is needed in application to low lying zeros. (See Theorem 8.3 and Lemma 9.3.)

Recall the description

$$S_{\underline{k}}(\Gamma(N))=\bigoplus_{\psi\in\Psi(G)}\ \bigoplus_{\substack{\pi\in\Pi_\psi\\ \pi_\infty\simeq\sigma_{\underline{k}}}} m_{\pi,\psi}\pi_f^{K(N)}$$

in terms of Arthur's classification.

Definition 5.1. The new part (space) of $S_{\underline{k}}(\Gamma(N))$ is defined by

$$S_{\underline{k}}^{\mathrm{new}}(\Gamma(N))=\bigoplus_{\psi\in\Psi(G)}\ \bigoplus_{\substack{\pi=\pi_f\otimes\sigma_{\underline{k}}\in\Pi_\psi\\ \pi^{K(N)}\neq0 \text{ but } \pi^{K(d)}=0 \text{ for any } d|N,\, d\neq N}} m_{\pi,\psi}\pi_f^{K(N)}.$$

The orthogonal complement $S_{\underline{k}}^{\mathrm{old}}(\Gamma(N))$ of $S_{\underline{k}}^{\mathrm{new}}(\Gamma(N))$ in $S_{\underline{k}}(\Gamma(N))$ with respect to Petersson inner product is said to be the old space. Let $HE_{\underline{k}}^{\mathrm{new}}(N)$ be a subset of $HE_{\underline{k}}(N)$ which is a basis of $S_{\underline{k}}^{\mathrm{new}}(\Gamma(N))$.

Remark 5.2. As the referee pointed out, $S_{\underline{k}}^{\mathrm{old}}(\Gamma(N))$ is the intersection of $S_{\underline{k}}(\Gamma(N))$ with the smallest $G(\mathbb{A}_f)$-invariant space of functions on $G(\mathbb{Q})\backslash G(\mathbb{A})$ containing $S_{\underline{k}}(\Gamma(M))$ for all proper divisors M of N.

Set $d_p=(1-p^{-1})^n$, $d_M=\prod_{p|M} d_p$ and $C_p=\prod_{j=1}^n(1-p^{-2j})$, $C_M=\prod_{p|M} C_p$. We set $d_1=1$ and $C_1=1$.

Recall $d_{\underline{k}}(N)=\dim S_{\underline{k}}(\Gamma(N))=C_{\underline{k}}\,C_N\,N^{2n^2+n}+O_{\underline{k}}(N^{2n^2})$.

Lemma 5.3. *Assume that (1-4) holds and N is squarefree. Then we have*

$$d_{\underline{k}}(N)=\sum_{M|N}\dim S_{\underline{k}}^{\mathrm{new}}(\Gamma(M))\left(\frac{N}{M}\right)^{n^2}C_{N/M}\,d_{N/M}^{-1}.$$

Proof. Let $M\mid N$. Take an automorphic representation $\pi=\pi_f\otimes\sigma_{\underline{k}}$ such that $\dim\pi_f^{K(M)}>0$ and $\dim\pi_f^{K(L)}=0$ for any $L\mid M$, $L<M$. Under this condition, π has an intersection with $S_{\underline{k}}^{\mathrm{new}}(\Gamma(M))$, and also with $S_{\underline{k}}(\Gamma(N))$. Let $\pi_f=\otimes_p\pi_p$. By the assumptions and Theorem 4.3, for any prime $p\nmid M$, π_p is tempered spherical, and so π_p is an irreducible induced representation from a Borel subgroup B of $G(\mathbb{Q}_p)$. So $\dim\pi_p^{K_p}=1$. Now $K_p/K_p(p)\simeq\mathrm{Sp}_{2n}(\mathbb{F}_p)$, $\#\mathrm{Sp}_{2n}(\mathbb{F}_p)=p^{2n^2+n}C_p$, and $\#B(\mathbb{F}_p)=p^{n^2+n}d_p$. Hence, $\dim\pi_p^{K_p(p)}=p^{n^2}C_p\,d_p^{-1}$ for all $p\nmid M$. Since N is squarefree, this leads to

$$\dim\pi_f^{K(N)}=\dim\pi_f^{K(M)}\times\left(\frac{N}{M}\right)^{n^2}C_{N/M}\,d_{N/M}^{-1}.$$

Thus, we obtain the assertion. □

Theorem 5.4. *Assume that (1-4) holds and N is squarefree. Then we have*

$$\dim S_{\underline{k}}^{\mathrm{new}}(\Gamma(N))=C_{\underline{k}}C_N N^{2n^2+n}\prod_{p|N}\bigl(1-d_p^{-1}p^{-n^2-n}\bigr)+O_{\underline{k}}\bigl(N^{2n^2}\bigr).$$

Here, $\zeta(n^2)^{-1}<\prod_{p|N}\bigl(1-d_p^{-1}p^{-n^2-n}\bigr)<1$ if $n>1$. If $n=1$, we have $\prod_{p|N}\bigl(1-d_p^{-1}p^{-2}\bigr)>\prod_p\bigl(1-1/(p(p-1))\bigr)=0.374\ldots$.

Proof. Since $C_{N/M}=C_N/C_M$ and $d_{N/M}=d_N/d_M$, from Lemma 5.3, we have

$$d_{\underline{k}}(N)N^{-n^2}C_N^{-1}d_N=\sum_{M|N}\dim S_{\underline{k}}^{\mathrm{new}}(\Gamma(M))M^{-n^2}C_M^{-1}d_M.$$

The Möbius inversion formula gives

$$\dim S_{\underline{k}}^{\mathrm{new}}(\Gamma(N))N^{-n^2}C_N^{-1}d_N=\sum_{M|N}\mu(M)d_{\underline{k}}\left(\frac{N}{M}\right)\left(\frac{N}{M}\right)^{-n^2}C_{N/M}^{-1}d_{N/M},$$

where μ denotes the Möbius function. Therefore,

$$\dim S_{\underline{k}}^{\mathrm{new}}(\Gamma(N))=\sum_{M|N}\mu(M)d_{\underline{k}}\left(\frac{N}{M}\right)M^{n^2}C_M\,d_M^{-1}. \tag{5-1}$$

By [Wakatsuki 2018, Corollary 1.2], there exist constants $C_{\underline{k},r}$ such that $d_{\underline{k}}(N)=\sum_{r=0}^{n} C_{\underline{k},r} C_N\, N^{f(r)}$ if $N>2$, where $f(r)=2n^2+n+\frac{1}{2}r(r-1)-nr$ and $C_{\underline{k},0}=C_{\underline{k}}$. Further, we take two constants D_1 and D_2 so that $d_{\underline{k}}(N)=\sum_{r=0}^{n} C_{\underline{k},r}\, C_N\, N^{f(r)}+D_N$ for $N=1$ or 2. Therefore, by (5-1), we obtain

$$\dim S_{\underline{k}}^{\mathrm{new}}(\Gamma(N))=\sum_{r=0}^{n} C_{\underline{k},r} C_N N^{f(r)} \sum_{M\mid N}\mu(M) d_M^{-1} M^{n^2-f(r)} + \mu(N) N^{n^2} C_N d_N^{-1} D_1+\begin{cases}\mu\left(\frac{N}{2}\right)\left(\frac{N}{2}\right)^{n^2} C_{N/2} d_{N/2}^{-1} D_2 & \text{if } 2\mid N,\\ 0 & \text{if } 2\nmid N.\end{cases}$$

Since N is squarefree,

$$\sum_{M\mid N}\mu(M) d_M^{-1} M^{n^2-f(r)}=\prod_{p\mid N}\left(1-d_p^{-1} p^{n^2-f(r)}\right).$$

Therefore,

$$\dim S_{\underline{k}}^{\mathrm{new}}(\Gamma(N))=\sum_{r=0}^{n} C_{\underline{k},r} C_N N^{f(r)} \prod_{p\mid N}\left(1-d_p^{-1} p^{-f(r)+n^2}\right) + \mu(N) N^{n^2} C_N d_N^{-1} D_1+\begin{cases}\mu\left(\frac{N}{2}\right)\left(\frac{N}{2}\right)^{n^2} C_{N/2} d_{N/2}^{-1} D_2 & \text{if } 2\mid N,\\ 0 & \text{if } 2\nmid N.\end{cases}$$

From this, we obtain the assertion.

Now, $d_p<1$. Hence $\prod_{p\mid N}\left(1-d_p^{-1}p^{n^2-f(r)}\right)<1$. Also $d_p^{-1}<p^n$ since $1/(1-p^{-1})<p$. Therefore, $\prod_{p\mid N}\left(1-d_p^{-1}p^{-n^2-n}\right)>\prod_{p\mid N}\left(1-p^{-n^2}\right)$. Here if $n>1$,

$$\prod_{p\mid N}\left(1-p^{-n^2}\right)^{-1}<\prod_{p}\left(1-p^{-n^2}\right)^{-1}=\zeta(n^2).$$

If $n=1$,

$$\prod_{p\mid N}\left(1-d_p^{-1}p^{-n^2-n}\right)=\prod_{p\mid N}\left(1-\frac{1}{p(p-1)}\right)>\prod_{p}\left(1-\frac{1}{p(p-1)}\right),$$

which is the Artin constant. □

6. Equidistribution theorem of Siegel cusp forms; proof of Theorem 1.1

By the definition in (1-1), we see that

$$\hat{\mu}_{K^S(N),S_1,\xi_{\underline{k}},D_{\underline{l}}^{\mathrm{hol}}}(\widehat{h_1})=\frac{\mathrm{Tr}(T_{h_1}|_{S_{\underline{k}}(\Gamma(N))})}{\mathrm{vol}(G(\mathbb{Q})\backslash G(\mathbb{A}))\cdot\dim\xi_{\underline{k}}}.$$

Notice that $\dim\xi_{\underline{k}}=d_{\underline{k}}$ (under a suitable normalization of the measure). Applying Theorem 3.10 to S_1, we have the claim by the Plancherel formula of Harish-Chandra: $\hat{\mu}_{S_1}^{\mathrm{pl}}(\widehat{h_1})=h_1(1)$.

7. Vertical Sato–Tate theorem for Siegel modular forms: proofs of Theorems 1.2 and 1.3

Suppose that $\underline{k} = (k_1, \ldots, k_n)$ satisfies the condition (1-4). Put $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. For $F \in HE_{\underline{k}}(N)$, consider the cuspidal automorphic representation $\pi = \pi_F = \pi_\infty \otimes \otimes'_p \pi_{F,p}$ of $G(\mathbb{A})$ associated to F. As discussed in the previous section, under the condition (1-4), the A-parameter ψ whose A-packet contains π is semisimple and $\pi_{F,p}$ is tempered for all p. Then if $p \nmid N$, $\pi_{F,p}$ is spherical, and we can write $\pi_{F,p}$ as $\pi_{F,p} = \mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \chi_p$, where $B = TU$ is the upper Borel subgroup and χ_p is a unitary character on $B(\mathbb{Q}_p)$. For each $1 \le j \le n$, put $\alpha_{jp}(\chi_p) := \chi_p(e_j(p^{-1}))$ (see (2-1) for $e_j(p^{-1})$) and by temperedness, we may write $\alpha_{jp}(\chi_p) = e^{\sqrt{-1}\theta_j}$, $\theta_j \in [0, \pi]$. Let $\hat{G} = \mathrm{SO}(2n+1)(\mathbb{C})$ be the complex split orthogonal group over $\mathbb{C}$ associated to the antidiagonal identity matrix. Let $\mathcal{L}(\pi_p) : W_{\mathbb{Q}_p} \to \mathrm{SO}(2n+1)(\mathbb{C})$ be the local Langlands parameter given by

$$\mathcal{L}(\pi_p)(\mathrm{Frob}_p) = (\alpha_{1p}(\chi_p), \ldots, \alpha_{np}(\chi_p), 1, \alpha_{1p}(\chi_p)^{-1}, \ldots, \alpha_{np}(\chi_p)^{-1}),$$

which is called the p-Satake parameter. Put $a^{(i)}(\chi_p) = a^{(i)}_{F,p}(\chi_p) = \frac{1}{2}(\alpha_{ip}(\chi_p) + \alpha_{ip}(\chi_p)^{-1}) = \cos\theta_i$ for $1 \le i \le n$. Let $\widehat{G(\mathbb{Q}_p)}^{\mathrm{ur,\,temp}}$ be the isomorphism classes of unramified tempered representations of $G(\mathbb{Q}_p)$. By [Shin and Templier 2016, Lemma 3.2], we have a topological isomorphism

$$\widehat{G(\mathbb{Q}_p)}^{\mathrm{ur,\,temp}} \xrightarrow{\sim} [0, \pi]^n / \mathfrak{S}_n =: \Omega$$

given by

$$\pi_p = \mathrm{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{Sp}_{2n}(\mathbb{Q}_p)} \chi_p \mapsto \big(\arg(a^{(1)}(\chi_p)), \ldots, \arg(a^{(n)}(\chi_p))\big) =: (\theta_1, \ldots, \theta_n).$$

We denote by $(\theta_1(\pi_{F,p}), \ldots, \theta_n(\pi_{F,p})) \in \Omega$ the corresponding element to $\pi_{F,p}$ under the above isomorphism. Let $\hat{B} = \hat{T}\hat{U}$ be the upper Borel subgroup of $\hat{G} = \mathrm{SO}(2n+1)(\mathbb{C})$. Let $\Delta^+(\hat{G})$ be the set of all positive roots in $X^*(\hat{T}) = \mathrm{Hom}(\hat{T}, \mathrm{GL}_1)$ with respect to $\hat{B}$. We view $(\theta_1, \ldots, \theta_n)$ as parameters of Ω. Let $\mu_p^{\mathrm{pl,\,temp}}$ be the restriction of the Plancherel measure on $\widehat{G(\mathbb{Q}_p)}$ to $\widehat{G(\mathbb{Q}_p)}^{\mathrm{ur,\,temp}}$, and by abusing the notation, we denote by $\mu_p = \mu_p^{\mathrm{pl,\,temp}}$ its pushforward to Ω. Put

$$t := \big(e^{\sqrt{-1}\theta_1}, \ldots, e^{\sqrt{-1}\theta_n}, 1, e^{-\sqrt{-1}\theta_1}, \ldots, e^{-\sqrt{-1}\theta_n}\big)$$

for simplicity. By [Shin and Templier 2016, Proposition 3.3], we have

$$\mu_p^{\mathrm{pl,\,temp}}(\theta_1, \ldots, \theta_n) = W(\theta_1, \ldots, \theta_n)\, d\theta_1 \cdots d\theta_n,$$

where

$$\begin{aligned} W(\theta_1, \ldots, \theta_n) &= \frac{1}{(2\pi)^n}\Big(1 + \frac{1}{p}\Big)^{n^2} \frac{\prod_{\alpha \in \Delta^+(\hat{G})} |1 - e^{\sqrt{-1}\alpha(t)}|^2}{\prod_{\alpha \in \Delta^+(\hat{G})} |1 - p^{-1} e^{\sqrt{-1}\alpha(t)}|^2} \\ &= \frac{1}{(2\pi)^n}\Big(1 + \frac{1}{p}\Big)^{n^2} \frac{\prod_{i=1}^n |1 - e^{\sqrt{-1}\theta_i}|^2 \prod_{\substack{1 \le i < j \le n \\ \varepsilon = \pm 1}} |1 - e^{\sqrt{-1}(\theta_i + \varepsilon\theta_j)}|^2}{\prod_{i=1}^n |1 - p^{-1} e^{\sqrt{-1}\theta_i}|^2 \prod_{\substack{1 \le i < j \le n \\ \varepsilon = \pm 1}} |1 - p^{-1} e^{\sqrt{-1}(\theta_i + \varepsilon\theta_j)}|^2}. \end{aligned}$$

By letting $p \to \infty$, we recover the Sato–Tate measure

$$\mu_\infty^{\mathrm{ST}} = \lim_{p\to\infty} \mu_p^{\mathrm{pl,\,temp}} = \frac{1}{(2\pi)^n} \prod_{i=1}^n \left|1 - e^{\sqrt{-1}\theta_i}\right|^2 \prod_{\substack{1\le i<j\le n\\ \varepsilon=\pm 1}} \left|1 - e^{\sqrt{-1}(\theta_i+\varepsilon\theta_j)}\right|^2 d\theta_1 \cdots d\theta_n.$$

Then Theorems 1.2 and 1.3 follow from Theorems 1.1 and 4.13.

8. Standard L-functions of $\mathrm{Sp}(2n)$

Let $\underline{k} = (k_1, \dots, k_n)$ and $F \in HE_{\underline{k}}(N)$, and let π_F be a cuspidal representation of $G(\mathbb{A})$ associated to F.

Assume (1-4) for $\underline{k}$. By (4-1) and the observation there, the global A-packet Π_ψ containing π_F is associated to a semisimple global A parameter $\psi = \boxplus_{i=1}^r \pi_i$ where π_i is an irreducible cuspidal representation of $\mathrm{GL}_{m_i}(\mathbb{A})$. Then the isobaric sum $\Pi := \boxplus_{i=1}^r \pi_i$ is an automorphic representation of $\mathrm{GL}_{2n+1}(\mathbb{A})$. Therefore, we may define

$$L(s, \pi_F, \mathrm{St}) := L(s, \Pi) = \prod_{i=1}^r L(s, \pi_i).$$

Let $L_p(s, \pi_F, \mathrm{St}) := L(s, \Pi_p) = \prod_{i=1}^r L(s, \pi_{ip})$ be the local p-factor of $L(s, \pi_F, \mathrm{St})$ for each rational prime p.

Let $\pi_F = \pi_\infty \otimes \otimes'_p \pi_p$. For $p \nmid N$, we have that π_p is the spherical representation of $G(\mathbb{Q}_p)$ with the Satake parameter $(\alpha_{1p}, \dots, \alpha_{np}, 1, \alpha_{1p}^{-1}, \dots, \alpha_{np}^{-1})$. Then

$$L_p(s, \pi_F, \mathrm{St})^{-1} = (1 - p^{-s}) \prod_{i=1}^n (1 - \alpha_{ip} p^{-s})(1 - \alpha_{ip}^{-1} p^{-s}).$$

We define the conductor $q(F)$ of F to be the product of the conductors $q(\pi_i)$ of π_i, for $1 \le i \le r$.

Theorem 8.1. *Let $F \in HE_{\underline{k}}(N)$. Then the standard L-function $L(s, \pi_F, \mathrm{St})$ has a meromorphic continuation to all of $\mathbb{C}$. Let*

$$\Lambda(s, \pi_F, \mathrm{St}) = q(F)^{s/2} L_\infty(s, \pi_F, \mathrm{St}) L(s, \pi_F, \mathrm{St}),$$

where $L_\infty(s, \pi_F, \mathrm{St}) = \Gamma_{\mathbb{R}}(s+\epsilon)\Gamma_{\mathbb{C}}(s+k_1-1)\cdots\Gamma_{\mathbb{C}}(s+k_n-n)$,

$$\epsilon = \begin{cases} 0 & \text{if } n \text{ is even},\\ 1 & \text{if } n \text{ is odd},\end{cases}$$

and $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(\frac{s}{2})$, $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$. Then

$$\Lambda(s, \pi_F, \mathrm{St}) = \epsilon(F)\Lambda(1-s, \pi_F, \mathrm{St}),$$

where $\epsilon(F) \in \{\pm 1\}$.

Proof. It follows from the functional equation of $L(s, \Pi)$ by noting that Π is self-dual, and $L(s, \Pi_\infty) = L_\infty(s, \pi_F, \mathrm{St})$ is the local L-function attached to the holomorphic discrete series of the lowest weight $\underline{k}$; see [Kozima 2002]. ☐

The epsilon factor $\epsilon(F)$ turns out to be always 1.

Proposition 8.2. *Let π_F be associated to a semisimple A-parameter. Then $\epsilon(F)=1$.*

Proof. Recall the global A-parameter $\psi=\boxplus_{i=1}^r \pi_i$. Let ω_i be the central character of π_i. Since π_i is orthogonal, its epsilon factor is $\omega_i(-1)$ by [Lapid 2004, Theorem 1]. Hence,

$$\epsilon(F)=\prod_{i=1}^r \omega_i(-1)=\Bigl(\prod_{i=1}^r \omega_i\Bigr)(-1)=\mathbb{1}(-1)=1$$

by the condition on the central character. □

Theorem 8.3. *For any $F\in HE_{\underline{k}}(N)$, the conductor $q(F)$ satisfies $q(F)\le N^{2n+1}$. If $F\in HE_{\underline{k}}^{\mathrm{new}}(N)$, then $q(F)\ge \max\{N\prod_{p|N}p^{-1},\prod_{p|N}p\}$. So if $F\in HE_{\underline{k}}^{\mathrm{new}}(N)$, $q(F)\ge N^{1/2}$.*

Proof. Let π_F be associated to a semisimple global A parameter $\psi=\boxplus_{i=1}^r\pi_i$ where π_i is an irreducible cuspidal representation of $\mathrm{GL}_{m_i}(\mathbb{A})$, and let $\Pi:=\boxplus_{i=1}^r\pi_i$. Let $\Pi=\Pi_\infty\otimes\otimes'_p\Pi_p$. By Proposition 4.6, Π has a nonzero fixed vector by $K^{GL_{2n+1}}(p^{e_p})$, where $e_p=\mathrm{ord}_p(N)$. As in the proof of [Kim et al. 2020a, Lemma 8.1], it implies $\mathrm{depth}(\Pi_p)\le e_p-1$. Hence $q(\Pi_p)\le p^{(2n+1)e_p}$ by [Lansky and Raghuram 2003, Proposition 2.2]. Therefore, $q(F)\le N^{2n+1}$.

If $F\in HE_{\underline{k}}^{\mathrm{new}}(N)$, by Definition 5.1, it is not fixed by $K^{GL_{2n+1}}(p^{e_p-1})$ for each $p\mid N$. By [Miyauchi and Yamauchi 2022, Theorem 1.2], we have $q(\Pi_p)\ge p^{m_i(e_p-1)}$ for some i. In particular, $q(\Pi_p)\ge p^{e_p-1}$ for each $p\mid N$. Hence, $q(F)\ge N\prod_{p|N}p^{-1}$. It is clear that $q(\Pi_p)\ge p$ if $p\mid N$. Hence,

$$q(F)\ge\max\Bigl\{N\cdot\prod_{p|N}p^{-1},\prod_{p|N}p\Bigr\}.$$

Now, $q(F)^2=q(F)\cdot q(F)\ge N$. Hence our result follows. □

Proposition 8.4. *Keep the assumptions on N as in Proposition 4.12. Let $F\in HE_{\underline{k}}(N)$. Then $L(s,\pi_F,\mathrm{St})$ has a pole at $s=1$ if and only if π_F is associated to a semisimple global A-parameter $\psi=1\boxplus\pi_1\boxplus\cdots\boxplus\pi_r$ where π_i is an orthogonal irreducible cuspidal representation of $\mathrm{GL}_{m_i}(\mathbb{A})$, such that if $m_i=1$, π_i is a nontrivial quadratic character. Let $HE_{\underline{k}}(N)^0$ be the subset of $HE_{\underline{k}}(N)$ such that $L(s,\pi_F,\mathrm{St})$ has a pole at $s=1$. Then $|HE_{\underline{k}}(N)^0|=O(N^{2n^2-n+\epsilon})$. So $|HE_{\underline{k}}(N)^0|/|HE_{\underline{k}}(N)|=O(N^{-2n+\epsilon})$.*

This proves [Shin and Templier 2016, Hypothesis 11.2] in our family.

Proof. This follows from the proof of Theorem 4.13, by noting that partitions $\underline{m}=(m_1,\dots,m_r)$ of $2n$ contribute to $HE_{\underline{k}}(N)^0$. □

Böcherer [1986] gave the relationship between Hecke operators and L-functions for level one and scalar-valued Siegel modular forms and it is extended by Shimura [1994a] to a more general setting.

Let $\underline{a}=(a_1,\dots,a_n)$, $0\le a_1\le\cdots\le a_n$, and $D_{p,\underline{a}}=\mathrm{diag}(p^{a_1},\dots,p^{a_n})$. Let F be an eigenform in $HE_{\underline{k}}(N)$ with respect to the Hecke operator $T(D_{p,\underline{a}})$ for all $p\nmid N$, and let $\lambda(F,D_{p,\underline{a}})$ be the eigenvalue.

Then we have the following identity [Shimura 1994a, Theorem 2.9]:

$$\sum_{\underline{a}} \lambda(F, D_{p,\underline{a}}) X^{\sum_{i=1}^n a_i} = \frac{1-X}{1-p^n X} \prod_{i=1}^{n} \frac{(1-p^{2i}X^2)}{(1-\alpha_{ip}p^n X)(1-\alpha_{ip}^{-1}p^n X)}, \tag{8-1}$$

where $\underline{a} = (a_1, \ldots, a_n)$ runs over $0 \le a_1 \le \cdots \le a_n$.

Let $\underline{m} = (m_1, \ldots, m_n)$, $m_1|m_2|\cdots|m_n$, and $D_{\underline{m}} = \mathrm{diag}(m_1, \ldots, m_n)$, and let $\lambda(F, D_{\underline{m}})$ be the eigenvalue of the Hecke operator $T(D_{\underline{m}})$. Let

$$L^N(s, F) = \sum_{\underline{m},\, (m_n, N)=1} \lambda(F, D_{\underline{m}}) \det(D_{\underline{m}})^{-s}.$$

Then

$$L^N(s, F) = \prod_{p \nmid N} L(s, F)_p,$$
$$L(s, F)_p = \sum_{\underline{a}} \lambda(F, D_{p,\underline{a}}) \det(D_{p,\underline{a}})^{-s}.$$

It converges for $\mathrm{Re}(s) > 2n + (k_1 + \cdots + k_n)/n + 1$.

Hence, we have

$$\zeta^N(s)\left[\prod_{i=1}^{n} \zeta^N(2s-2i)\right] L^N(s, F) = L^N(s-n, \pi_F, \mathrm{St}),$$

where $L^N(s, \pi_F, \mathrm{St}) = \prod_{p\nmid N} L_p(s, \pi_F, \mathrm{St})$, and $\zeta^N(s) = \prod_{p\nmid N}(1-p^{-s})^{-1}$.

The central value of $L^N(s, F)$ is at $s = n+\frac{1}{2}$, and $L^N(s, F)$ has a zero at $s = n+\frac{1}{2}$ since $L^N(s, \pi_F, \mathrm{St})$ is holomorphic at $s = \frac{1}{2}$. Theorem 3.10 implies

Theorem 8.5. *For $\underline{m} = (m_1, \ldots, m_n)$, $m_1|m_2|\cdots|m_n$ with $m_n > 1$ and $(m_n, N) = 1$, $N \gg m_n^{2n}$,*

$$\frac{1}{|HE_{\underline{k}}(N)|} \sum_{F \in HE_{\underline{k}}(N)} \lambda(F, D_{\underline{m}}) = O(m_n^{\alpha} N^{-n}),$$

for some constant α.

Proof. Let S_1 be the set of all prime divisors of m_n. Since $m_n > 1$, S_1 is nonempty. The main term of right-hand side in Theorem 3.10 includes $h_1(1)$. Clearly, $h_1(1) = 0$ because the double coset defining the Hecke operator h_1 does not contain any central elements. Since the automorphic counting measure is supported on cuspidal representations, Theorem 3.10 implies the claim. □

Write

$$L^N(s, F) = \sum_{\substack{m=1\\(m,N)=1}}^{\infty} a_F(m) m^{-s} \quad \text{and} \quad L(s, F)_p = \sum_{k=0}^{\infty} a_F(p^k) p^{-ks}$$

for each prime $p \nmid N$. Here $a_F(p^k) = \sum_{\underline{a}} \lambda(F, D_{p,\underline{a}})$, where the sum is over all $\underline{a} = (a_1, \ldots, a_n)$ such that $0 \le a_1 \le \cdots \le a_n$, $a_1 + \cdots + a_n = k$. Hence, for $k > 0$ and $p \nmid N$,

$$\frac{1}{d_{\underline{k}}(N)} \sum_{F \in HE_{\underline{k}}(N)} a_F(p^k) = O(p^{ka} N^{-n}).$$

More generally:

Corollary 8.6. *For $m > 1$, with $(m, N) = 1$, $N \gg m^{2n}$,*

$$\frac{1}{d_{\underline{k}}(N)} \sum_{F \in HE_{\underline{k}}(N)} a_F(m) = O(m^\alpha N^{-n}).$$

Proof. We have $a_F(m) = \sum_{\underline{m}} \lambda(F, D_{\underline{m}})$, where the sum is over all $\underline{m} = (m_1, \dots, m_n)$, $m_1 | m_2 | \cdots | m_n$, $m_1 \cdots m_n = m$. Our assertion follows from Theorem 8.5. □

Write

$$L^N(s, \pi_F, \mathrm{St}) = \sum_{\substack{m=1 \\ (m,N)=1}}^{\infty} \mu_F(m) m^{-s}.$$

Then from (8-1), we have, for $p \nmid N$,

$$\mu_F(p) = \big(a_F(p) + 1\big) p^{-n} \quad \text{and} \quad \mu_F(p^2) = 1 + p^{-2} + \cdots + p^{-2n} + \big(a_F(p^2) + a_F(p)\big) p^{-2n}.$$

More generally, for $p \nmid N$,

$$\mu_F(p^k) = \begin{cases} 1 + p^{-2} h_k(p^{-2}) + p^{-n} \sum_{i=1}^{k} h_{ik}(p^{-1}) a_F(p^i) & \text{if } k \text{ is even}, \\ p^{-n} h'_k(p^{-2}) + p^{-n} \sum_{i=1}^{k} h'_{ik}(p^{-1}) a_F(p^i) & \text{if } k \text{ is odd}, \end{cases}$$

where $h_k, h'_k, h_{ik}, h'_{ik} \in \mathbb{Z}[x]$. Therefore, for $(m, N) = 1$,

$$\mu_F(m) = \prod_{p|m} \big(\delta_{p,m} + p^{-2} h_m^\delta(p^{-1})\big) + \sum_{\substack{u|m \\ u>1}} A_u a_F(u),$$

where

$$A_u \in \mathbb{Q}, \quad h_m^\delta \in \mathbb{Z}[x], \quad \text{and} \quad \delta = \delta_{p,m} = \begin{cases} 1 & \text{if } v_p(m) \text{ is even}, \\ 0 & \text{otherwise}. \end{cases}$$

Therefore, by Corollary 8.6, we have

Theorem 8.7. *Fix $\underline{k} = (k_1, \dots, k_n)$, and let $m = \prod_{p|m} p^{v_p(m)}$ which is coprime to N. Then*

$$\frac{1}{d_{\underline{k}}(N)} \sum_{F \in HE_{\underline{k}}(N)} \mu_F(m) = \prod_{p|m} \big(\delta_{p,m} + p^{-2} h_m^\delta(p^{-1})\big) + O(N^{-n} m^c).$$

This proves [Kim et al. 2020b, Conjecture 6.1 in level aspect] for the Sp(4) case.

9. ℓ-level density of standard L-functions

In this section, we assume (1-4) and keep the assumptions on N in Proposition 4.12. Then we show unconditionally that the ℓ-level density (ℓ a positive integer) of the standard L-functions of the family $HE_{\underline{k}}(N)$ has the symmetry type Sp in the level aspect. Shin and Templier [2016] showed it under several hypotheses with a family which includes nonholomorphic forms.

Under assumption (1-4), F satisfies the Ramanujan conjecture, namely, $|\alpha_{ip}| = 1$ for each i. Let

$$-\frac{L'}{L}(s, \pi_F, \mathrm{St}) = \sum_{m=1}^{\infty} \Lambda(m) b_F(m) m^{-s},$$

where $b_F(p^m) = 1 + \alpha_{1p}^m + \cdots + \alpha_{np}^m + \alpha_{1p}^{-m} + \cdots + \alpha_{np}^{-m}$ when π_p is spherical.

For $F \in HE_{\underline{k}}(N)$, let Π be the Langlands transfer of π_F to GL_{2n+1}. If $F \in HE_{\underline{k}}(N)^g$, then $L(s, \Pi, \wedge^2)$ has no pole at $s = 1$, and $L(s, \Pi, \mathrm{Sym}^2)$ has a simple pole at $s = 1$. Let

$$\begin{aligned} L(s, \Pi \times \Pi) &= \sum \lambda_{\Pi \times \Pi}(n) n^{-s}, \\ L(s, \Pi, \wedge^2) &= \sum \lambda_{\wedge^2(\Pi)}(n) n^{-s}, \\ L(s, \Pi, \mathrm{Sym}^2) &= \sum \lambda_{\mathrm{Sym}^2(\Pi)}(n) n^{-s}. \end{aligned}$$

Then $\mu_F(p^2) = \lambda_{\mathrm{Sym}^2(\Pi)}(p)$ and $\mu_F(p)^2 = \lambda_{\Pi \times \Pi}(p) = \lambda_{\wedge^2(\Pi)}(p) + \lambda_{\mathrm{Sym}^2(\Pi)}(p)$.

Note that $\mu_F(p) = b_F(p)$, and $b_F(p^2) = 2\mu_F(p^2) - \mu_F(p)^2$. Let

$$T(p, \underline{a}) = \Gamma(N) \begin{pmatrix} D_{p,\underline{a}} & 0 \\ 0 & D_{p,\underline{a}}^{-1} \end{pmatrix} \Gamma(N).$$

By Theorem A.1, $T(p, (0, \ldots, 0, 1))^2$, where there are $n - 1$ entries of 0, is a linear combination of

$$T\big(p, (\overbrace{0, \ldots, 0}^{n-1}, 2)\big),\ T\big(p, (\overbrace{0, \ldots, 0}^{n-2}, 1, 1)\big),\ T\big(p, (\overbrace{0, \ldots, 0}^{n-1}, 1)\big),\ T\big(p, \overbrace{(0, \ldots, 0)}^{n}\big) = \Gamma(N) I_{2n} \Gamma(N).$$

Therefore, by Theorem 8.7, if $p \nmid N$,

$$\frac{1}{d_{\underline{k}}(N)} \sum_{F \in HE_{\underline{k}}(N)} \mu_F(p)^2$$

is of the form

$$1 + p^{-1} g(p^{-1}) + O(p^c N^{-n})$$

for some polynomial $g \in \mathbb{Z}[x]$ and $c > 0$. Here the main term $1 + p^{-1} g(p^{-1})$ comes from the coefficient

$$p \sum_{i=0}^{2n-1} p^i \text{ of } T\big(p, \overbrace{(0, \ldots, 0)}^{n}\big)$$

in the linear combination. Here the explicit determination of the coefficient is necessary in our application. Hence, we have

Proposition 9.1. *For some $\alpha > 0$ and $p \nmid N$,*

$$\frac{1}{d_{\underline{k}}(N)} \sum_{F \in HE_{\underline{k}}(N)} b_F(p) = O(p^{-1}) + O(p^{\alpha} N^{-n}), \quad \textit{for } N \gg p^{2n}$$

$$\frac{1}{d_{\underline{k}}(N)} \sum_{F \in HE_{\underline{k}}(N)} b_F(p^2) = 1 + O(p^{-1}) + O(p^{\alpha} N^{-n}), \quad \textit{for } N \gg p^{4n}.$$

Remark 9.2. By a more careful analysis, we can replace the error term $O(N_1^{a\kappa+b}N^{-n})$ in Theorem 3.10 by

$$O\Big(N_1^{n(n+1)/2\kappa+\epsilon}N^{-n(n+1)/2}+N_1^{(2n-1)\kappa+8n-4+\epsilon}N^{1-2n}+N_1^{n\kappa+2n^3+2n-3+\epsilon}N^{-n} \\ +\sum_{r=3}^{n-1}N_1^{\kappa(nr-r(r-1)/2)+(2n-r-1)[r/2]+2n-2r-1+2n^3+\epsilon}N^{r(r-1)/2-nr}\Big),$$

for any $\epsilon>0$. Hence, the first error term $O(p^\alpha N^{-n})$ in Proposition 9.1 can be replaced (by taking $\kappa=1$) by

$$O\Big(p^{n(n+1)/2+\epsilon}N^{-(n^2+n)/2}+p^{10n-5+\epsilon}N^{1-2n}+p^{2n^3+3n-3+\epsilon}N^{-n} \\ +\sum_{r=3}^{n-1}p^{2n^3+2n-1+2nr-r^2-2r+\epsilon}N^{r(r-1)/2-nr}\Big).$$

The second error term $O(p^\alpha N^{-n})$ in Proposition 9.1 can be replaced (by taking $\kappa=2$) by

$$O\Big(p^{n(n+1)+\epsilon}N^{-(n^2+n)/2}+p^{12n-6+\epsilon}N^{1-2n}+p^{2n^3+4n-3+\epsilon}N^{-n} \\ +\sum_{r=3}^{n-1}p^{2n^3+2n-1+3nr-(3/2)(r^2+r)+\epsilon}N^{r(r-1)/2-nr}\Big).$$

We denote the nontrivial zeros of $L(s,\pi_F,\mathrm{St})$ by $\sigma_{F,j}=\frac{1}{2}+\sqrt{-1}\,\gamma_{F,j}$. Without assuming the GRH for $L(s,\pi_F,\mathrm{St})$, we can order them as

$$\cdots\le Re(\gamma_{F,-2})\le Re(\gamma_{F,-1})\le 0\le Re(\gamma_{F,1})\le Re(\gamma_{F,2})\le\cdots.$$

Let $c(F)=q(F)(k_1\cdots k_n)^2$ be the analytic conductor, and let

$$\log c_{\underline{k},N}=\frac{1}{d_{\underline{k}}(N)}\sum_{F\in HE_{\underline{k}}(N)}\log c(F).$$

From Theorems 5.4 and 8.3, we have

Lemma 9.3. *Let $n>1$. We assume that N is squarefree. Then*

$$(k_1\cdots k_n)^2N^{1/(2\zeta(n^2))}\le c_{k,N}\le(k_1\cdots k_n)^2N^{2n+1}.$$

This proves [Shin and Templier 2016, Hypothesis 11.4] in our family. It is used in the proof of (9-1).

Proof. By Theorem 8.3, we have $q(F)\le N^{2n+1}$. It gives rise to the upper bound. If $F\in HE_{\underline{k}}^{\mathrm{new}}(N)$, $q(F)\ge N^{1/2}$ by Theorem 8.3. By Theorem 5.4, $|HE_{\underline{k}}^{\mathrm{new}}(N)|\ge\zeta(n^2)^{-1}|HE_{\underline{k}}(N)|$. Hence,

$$\log c_{\underline{k},N}\ge\log(k_1\cdots k_n)^2+\frac{1}{d_{\underline{k}}(N)}\sum_{F\in HE_{\underline{k}}^{\mathrm{new}}(N)}\log q(F)\ge\log(k_1\cdots k_n)^2+\frac{1}{2\zeta(n^2)}\log N. \qquad\square$$

Consider, for an even Paley–Wiener function ϕ,

$$D(F,\phi)=\sum_{\gamma_{F,j}}\phi\Big(\frac{\gamma_{F,j}}{2\pi}\log c_{\underline{k},N}\Big).$$

Then as in [Kim et al. 2020a, (9.1)],

$$\frac{1}{d_{\underline{k}}(N)}\sum_{F\in HE_{\underline{k}}(N)} D(F,\phi)=\hat{\phi}(0)-\frac{1}{2}\phi(0)-\frac{2}{(\log c_{\underline{k},N})d_{\underline{k}}(N)}\sum_{F\in HE_{\underline{k}}(N)}\sum_{p}\frac{b_F(p)\log p}{\sqrt{p}}\hat{\phi}\Big(\frac{\log p}{\log c_{\underline{k},N}}\Big)$$
$$-\frac{2}{(\log c_{\underline{k},N})d_{\underline{k}}(N)}\sum_{F\in HE_{\underline{k}}(N)}\sum_{p}\frac{(b_F(p^2)-1)\log p}{p}\hat{\phi}\Big(\frac{2\log p}{\log c_{\underline{k},N}}\Big)$$
$$+O\Big(\frac{|HE_{\underline{k}}(N)^0|}{d_{\underline{k}}(N)}\Big)+O\Big(\frac{1}{\log c_{\underline{k},N}}\Big),$$

where $HE_{\underline{k}}(N)^0$ is in Proposition 8.4. (In [Kim et al. 2020a, (9.4)], the term $O\big(|HE_{\underline{k}}(N)^0|/d_{\underline{k}}(N)\big)$ was omitted.)

By Proposition 9.1, we can show as in [Kim et al. 2020a] that for an even Paley–Wiener function ϕ such that the Fourier transform $\hat{\phi}$ of ϕ is supported in $(-\beta,\beta)$, for some $\beta>0$,

$$\frac{1}{d_{\underline{k}}(N)}\sum_{F\in HE_{\underline{k}}(N)} D(F,\phi)=\hat{\phi}(0)-\frac{1}{2}\phi(0)+O\Big(\frac{1}{\log c_{\underline{k},N}}\Big)=\int_{\mathbb{R}}\phi(x)W(\mathrm{Sp})(x)\,\mathrm{d}x+O\Big(\frac{\omega(N)}{\log N}\Big), \quad (9\text{-}1)$$

where $\omega(N)$ is the number of prime factors of N and $W(\mathrm{Sp})(x)=1-(\sin 2\pi x)/(2\pi x)$. (When we exchange two sums, if $p\nmid N$, we use Proposition 9.1. If $p\mid N$, by the Ramanujan bound, $|b_F(p)|\le n$ and $|b_F(p^2)|\le n$. Hence by the trivial bound, we would obtain $\sum_{p|N} b_F(p)\log p/\sqrt{p}\ll\omega(N)$ and $\sum_{p|N} b_F(p^2)\log p/p\ll\omega(N)$.)

In fact, by Remark 9.2, we can take β to be the minimum of

$$\frac{n^2+n}{(2n+1)(n^2+n+1)}-\epsilon,\quad \frac{2n-1}{(2n+1)(10n-9/2)}-\epsilon,\quad \frac{n}{(2n+1)(2n^3+3n-5/2)}-\epsilon,\quad \frac{1}{2n(2n+1)},$$
$$\min_{3\le r\le n-1}\Big\{\frac{nr-r(r-1)/2}{(2n+1)(2nr-r^2-2r+2n^3+2n-1/2)}-\epsilon\Big\}.$$

Namely,

$$\beta=\frac{n}{(2n+1)(2n^3+3n-5/2)}-\epsilon. \quad (9\text{-}2)$$

For a general ℓ, let

$$W(\mathrm{Sp})(x)=\det(K_{-1}(x_j,x_k))_{1\le j\le\ell,1\le k\le\ell},$$

where $K_{-1}(x,y)=\sin\pi(x-y)/\pi(x-y)-\sin\pi(x+y)/\pi(x+y)$. Let $\phi(x_1,\dots,x_\ell)=\phi_1(x_1)\cdots\phi_\ell(x_\ell)$, where each ϕ_i is an even Paley–Wiener function and $\hat{\phi}(u_1,\dots,u_\ell)=\hat{\phi}_1(u_1)\cdots\hat{\phi}_\ell(u_\ell)$. We assume that the Fourier transform $\hat{\phi}_i$ of ϕ_i is supported in $(-\beta,\beta)$ for $i=1,\dots,\ell$. The ℓ-level density function is

$$D^{(\ell)}(F,\phi)=\sum\nolimits^{*}_{j_1,\cdots,j_\ell}\phi\Big(\gamma_{j_1}\frac{\log c_{\underline{k},N}}{2\pi},\gamma_{j_2}\frac{\log c_{\underline{k},N}}{2\pi},\dots,\gamma_{j_\ell}\frac{\log c_{\underline{k},N}}{2\pi}\Big),$$

where $\sum^{*}_{j_1,\dots,j_\ell}$ is over $j_i=\pm1,\pm2,\dots$ with $j_a\ne\pm j_b$ for $a\ne b$. Then as in [Kim et al. 2020b], using Theorem 8.7, we can show

Theorem 9.4. *We assume that N is squarefree. Let $\phi(x_1,\dots,x_\ell)=\phi_1(x_1)\cdots\phi_\ell(x_\ell)$, where each ϕ_i is an even Paley–Wiener function and $\hat\phi(u_1,\dots,u_\ell)=\hat\phi_1(u_1)\cdots\hat\phi_\ell(u_\ell)$. Assume the Fourier transform $\hat\phi_i$ of ϕ_i is supported in $(-\beta,\beta)$ for $i=1,\cdots,\ell$. (See (9-1) for the value of β.) Then*

$$\frac{1}{d_{\underline{k}}(N)}\sum_{F\in HE_{\underline{k}}(N)} D^{(\ell)}(F,\phi)=\int_{\mathbb{R}^\ell}\phi(x)W(\mathrm{Sp})(x)\,\mathrm{d}x+O\Big(\frac{\omega(N)}{\log N}\Big).$$

Remark 9.5. The above theorem is usually stated for Schwartz functions in the literature. But since Schwartz functions approximate any function in L^2-space, the above theorem holds for Payley–Wiener functions, which are in $L^2(\mathbb{R}^n)$, and whose Fourier transforms have compact supports.

10. The order of vanishing of standard L-functions at $s=\frac{1}{2}$

In this section, we show that the average order of vanishing of standard L-functions at $s=\frac{1}{2}$ is bounded under GRH; see [Iwaniec et al. 2000; Brumer 1995]. Under GRH on $L(s,\pi_F,\mathrm{St})$, its zeros are $\frac{1}{2}+\sqrt{-1}\gamma_F$ with $\gamma_F\in\mathbb{R}$.

Theorem 10.1. *Assume the GRH. Assume (1-4) and N is squarefree. Let $r_F=\mathrm{ord}_{s=\frac{1}{2}}L(s,\pi_F,\mathrm{St})$. Then*

$$\frac{1}{d_{\underline{k}}(N)}\sum_{F\in HE_{\underline{k}}(N)} r_F\le C,$$

where $C=\frac{1}{n}(2n+1)\big(2n^3+3n-\frac{5}{2}\big)-\frac{1}{2}+\epsilon$.

Proof. Choose $\phi(x)=\big(2\sin(x\beta/2)/x\big)^2$ for $x\in\mathbb{R}$, where β is from (9-2). Then

$$\hat\phi(x)=\begin{cases}\beta-|x| & \text{if } |x|<\beta,\\ 0 & \text{otherwise.}\end{cases}$$

Since $\phi(x)\ge 0$ for $x\in\mathbb{R}$, from (9-1), we have

$$\frac{1}{d_{\underline{k}}(N)}\sum_{F\in HE_{\underline{k}}(N)} r_F\phi(0)\le\hat\phi(0)-\frac{1}{2}\phi(0)+O\Big(\frac{1}{\log\log N}\Big).$$

Hence, we have

$$\frac{1}{d_{\underline{k}}(N)}\sum_{F\in HE_{\underline{k}}(N)} r_F\le\frac{1}{\beta}-\frac{1}{2}+O\Big(\frac{1}{\log\log N}\Big).\qquad\square$$

We can show a similar result for the spinor L-function of GSp(4). Recall the following from [Kim et al. 2020a]:

Proposition 10.2. *Assume* $(N,11!)=1$.

(1) *(level aspect) Fix k_1,k_2. Then for ϕ whose Fourier transform $\hat\phi$ has support in $(-u,u)$ for some $0<u<1$, as $N\to\infty$ (See [Kim et al. 2020a, Proposition 9.1] for the value of u),*

$$\frac{1}{d_{\underline{k}}(N)}\sum_{F\in HE_{\underline{k}}(N)} D(\pi_F,\phi,\mathrm{Spin})=\hat\phi(0)+\frac{1}{2}\phi(0)+O\Big(\frac{1}{\log\log N}\Big).$$

(2) (*weight aspect*) *Fix* N. *Then for* ϕ *whose Fourier transform* $\hat{\phi}$ *has support in* $(-u,u)$ *for some* $0<u<1$, *as* $k_1+k_2\to\infty$,

$$\frac{1}{d_{\underline{k}}(N)}\sum_{F\in HE_{\underline{k}}(N)} D(\pi_F,\phi,\mathrm{Spin})=\hat{\phi}(0)+\frac{1}{2}\phi(0)+O\Big(\frac{1}{\log((k_1-k_2+2)k_1k_2)}\Big).$$

By a careful analysis, we can show that $v_1=3$, $w_1=6$ in [Kim et al. 2020a, Proposition 8.2] in the level aspect. Hence $u=\frac{1}{40}$ in the level aspect. As in Theorem 10.1, we have

Theorem 10.3. *Let* $G=\mathrm{GSp}(4)$. *Assume the GRH, and let* $r_F=\mathrm{ord}_{s=\frac{1}{2}}L(s,\pi_F,\mathrm{Spin})$. *Then*

$$\frac{1}{d_{\underline{k}}(N)}\sum_{F\in HE_{\underline{k}}(N)} r_F\le\begin{cases}\dfrac{1}{u}+\dfrac{1}{2}+O\Big(\dfrac{1}{\log\log N}\Big) & \textit{level aspect},\\[2ex] \dfrac{1}{u}+\dfrac{1}{2}+O\Big(\dfrac{1}{\log((k_1-k_2+2)k_1k_2)}\Big) & \textit{weight aspect}.\end{cases}$$

Appendix

In this appendix we compute the product $T(p,(0,\dots,0,1))^2$, with $n-1$ entries of 0, from Section 9.

Theorem A.1. *For the Hecke operators, we have*

$$T(p,(\overbrace{0,\dots,0}^{n-1},1))^2=T(p,(\overbrace{0,\dots,0}^{n-1},2))+(p+1)T(p,(\overbrace{0,\dots,0}^{n-2},1,1))+(p^n-1)T(p,(\overbrace{0,\dots,0}^{n-1},1))\\ +\Big(p\sum_{i=0}^{2n-1}p^i\Big)T(p,(\overbrace{0,\dots,0}^{n})).$$

This agrees with [Kim et al. 2020a, (2.7)] when $n=2$. [Note that the coefficient of R_{p^2} there should be replaced with $p^4+p^3+p^2+p$.]

Since $p\nmid N$, we work on $K=\mathrm{Sp}(2n,\mathbb{Z}_p)$ instead of $\Gamma(N)$. Put

$$T_{p,n-1}:=pT(p,(0,\dots,0,1))=K\,\mathrm{diag}(1,\overbrace{p,\dots,p}^{n-1},p^2,\overbrace{p,\dots,p}^{n-1})K\in\mathrm{GSp}(2n,\mathbb{Q}_p).$$

It suffices to consider $T_{p,n-1}^2$. Let us first compute the coset decomposition. Put $\Lambda=\mathrm{GL}_n(\mathbb{Z}_p)$ where the identity element is denoted by 1_n. For any ring R, let $S_n(R)$ be the set of all symmetric matrices of size n defined over R and $M_{m\times n}(R)$ be the set of matrices of size $m\times n$ defined over R. Put

$$M_n(R)=M_{n\times n}(R)$$

for simplicity. For each $D\in M_n(\mathbb{Z}_p)$, we define

$$B(D):=\{B\in M_n(\mathbb{Z}_p)\mid {}^tBD={}^tDB\}.$$

For each $B_1,B_2\in B(D)$, we write $B_1\sim B_2$ if there exists $M\in M_n(\mathbb{Z}_p)$ such that $B_1-B_2=MD$. We denote by $B(D)/\sim$ the set of all equivalence classes of $B(D)$ by the relation $\sim$. We regard $\mathbb{F}_p$

(respectively, $\mathbb{Z}/p^2\mathbb{Z}$) as the subset $\{0, 1, \dots, p-1\}$ (respectively, $\{0, 1, \dots, p^2-1\}$) of $\mathbb{Z}$. Let D_I be the set of the following matrices in $M_n(\mathbb{Z}_p)$:

$$D^I_{n-1} = \mathrm{diag}(\overbrace{p, \dots, p}^{n-1}, 1),$$

$$D^I_s = D^I_s(x) := \left(\begin{array}{c|c|c} p\cdot 1_s & & \\ \hline & 1 & x \\ \hline & & p\cdot 1_{n-1-s} \end{array}\right), \quad 0 \le s \le n-2,\ x \in M_{1\times(n-1-s)}(\mathbb{F}_p),$$

where we fill out zeros in the blank blocks. The cardinality of D_I is $1+p+\cdots+p^{n-1} = (p^n-1)/(p-1)$ which is equal to that of $\Lambda\backslash\Lambda d_{n-1}\Lambda$, where $d_{n-1} = \mathrm{diag}(1, p, \dots, p)$ containing $n-1$ entries of p. Similarly, let D_{II} be the set of the following matrices:

$$D^{II}_{n-1} = \mathrm{diag}(p, \overbrace{1, \dots, 1}^{n-1}),$$

$$D^{II}_s = D^{II}_s(y) := \left(\begin{array}{c|c|c} 1_s & y & \\ \hline & p & \\ \hline & & 1_{n-1-s} \end{array}\right), \quad 1 \le s \le n-1,\ y \in M_{s\times 1}(\mathbb{F}_p).$$

The cardinality of D_{II} is $1+p+\cdots+p^{n-1} = (p^n-1)/(p-1)$ which is equal to that of $\Lambda\backslash\Lambda d_1\Lambda$, where $d_1 = \mathrm{diag}(1, \dots, 1, p)$ containing $n-1$ entries of 1. Finally for each $M \in M_n(\mathbb{Z}_p)$ we denote by $r_p(M)$ the rank of $M \bmod p\mathbb{Z}_p$.

Lemma A.2. *Assume p is odd. The right coset decomposition $T_{p,n-1} = \coprod_{\alpha\in J} K\alpha$ consists of the following elements*:

(1) (*type I*) *We have*

$$\alpha = \alpha_I(D, B) = \begin{pmatrix} p^2\cdot{}^tD^{-1} & B \\ 0_n & D \end{pmatrix},$$

where D runs over the set D_I and B runs over complete representatives of $B(D)/\sim$ such that $r_p(\alpha) = 1$. Further, for each D^I_s, B can be taken over

- *if $s \ne 0$, then $x \ne 0$ and $B = 0$;*
- *if $s = 0$, then $x = 0$ and $B = 0$.*

(2) (*type II*) *We have*

$$\alpha = \alpha_{II}(D, B) = \begin{pmatrix} p\cdot{}^tD^{-1} & B \\ 0_n & pD \end{pmatrix},$$

where D runs over the set D_{II} and B runs over complete representatives of $B(D)/\sim$ such that $r_p(\alpha) = 1$. Further, for each D^{II}_s, B can be taken over.

- *If $s = 0$, then*

$$\begin{pmatrix} B_{22} & B_{23} \\ p\cdot{}^tB_{23} & 0_{n-1} \end{pmatrix},$$

where B_{22} runs over $\mathbb{Z}/p^2\mathbb{Z}$ and B_{23} runs over $M_{1\times(n-1)}(\mathbb{F}_p)$;

- *If $s \neq 0$, for $D_s^{II}(y)$, $y \in M_{s\times 1}(\mathbb{F}_p)$,*

$$\left(\begin{array}{c|c|c} 0_s & p\cdot{}^tB_{21} & 0_{s\times(n-1-s)} \\ \hline B_{21} & B_{22} & B_{23} \\ \hline 0_{(n-1-s)\times s} & p\cdot{}^tB_{23} & 0_{n-1-s} \end{array}\right),$$

where B_{21}, B_{22} and B_{23} run over $M_{1\times s}(\mathbb{F}_p)$, $\mathbb{Z}/p^2\mathbb{Z}$, and $M_{1\times t}(\mathbb{F}_p)$, respectively.

(3) (*type III*) *We have*

$$\alpha = \alpha_{III}(B) = \begin{pmatrix} p1_n & B \\ 0_n & p1_n \end{pmatrix},$$

where B runs over $S_n(\mathbb{F}_p)$ with $r_p(B) = 1$. The number of such B's is $p^n - 1$.

Proof. We just apply the formula [Andrianov 2009, (3.94)]. First we need to compute a complete system of representatives of $\Lambda\backslash\Lambda t\Lambda \simeq (t^{-1}\Lambda t)\cap\Lambda\backslash\Lambda$ for each $t \in \{d_{n-1}, d_1, p1_n\}$ where $d_{n-1} = \mathrm{diag}(1, p, \ldots, p)$ and $d_1 = \mathrm{diag}(1, \ldots, 1, p)$ containing $n-1$ entries of p and 1, respectively. By direct computation, for $t = d_{n-1}$ (respectively, $t = d_1$), it is given by D^I (respectively, D^{II}). For $t = p\cdot 1_n$, it is obviously a singleton.

As for the computation of $B(D)/\sim$, we give details only for $D \in D^I$, and the case of D^{II} is similarly handled. For each $D = D_s^I(x)$, $0 \le s \le n-2$, put

$$A_s = \left(\begin{array}{c|c|c} 1_s & & \\ \hline & 1 & -px \\ \hline & & 1_{n-1-s} \end{array}\right),$$

so that

$$DA_s = \left(\begin{array}{c|c|c} p\cdot 1_s & & \\ \hline & 1 & \\ \hline & & p\cdot 1_{n-1-s} \end{array}\right).$$

Put $A_{n-1} = 1_{2n}$ for $D = D_{n-1}^I$. Then for each $D = D_s^I$, we have a bijection

$$B(D)/\sim \xrightarrow{\simeq} B(DA_s)/\sim, \quad B \mapsto BA_s.$$

Therefore, we may compute $B(DA_s)/\sim$ and convert them by multiplying A_s^{-1} on the right.

We write $B \in B(DA_s)$ as a block matrix

$$B = \begin{array}{c} \begin{array}{ccc} \overbrace{\phantom{B_{11}}}^{s} & \overbrace{\phantom{B_{12}}}^{1} & \overbrace{\phantom{B_{13}}}^{n-1-s} \end{array} \\ \left(\begin{array}{c|c|c} B_{11} & B_{12} & B_{13} \\ \hline B_{21} & B_{22} & B_{23} \\ \hline B_{31} & B_{32} & B_{33} \end{array}\right) \end{array}$$

with respect to the partition $s + 1 + (n-1-s)$ of n where the column is also decomposed as in the row. The relation yields

$$B = \left(\begin{array}{c|c|c} B_{12} & B_{12} & B_{13} \\ \hline p\cdot{}^tB_{12} & B_{22} & p\cdot{}^tB_{32} \\ \hline {}^tB_{13} & B_{32} & B_{33} \end{array}\right),$$

where $B_{11} \in S_s(\mathbb{Z}_p)$, $B_{22} \in \mathbb{Z}_p$, and $B_{33} \in S_{n-1-s}(\mathbb{Z}_p)$. We write $X \in M_n(\mathbb{Z}_p)$ as

$$\begin{array}{ccc} \overbrace{\phantom{X_{11}}}^{s} & \overbrace{\phantom{X_{12}}}^{1} & \overbrace{\phantom{X_{13}}}^{n-1-s} \end{array}$$
$$\left(\begin{array}{c|c|c} X_{11} & X_{12} & X_{13} \\ \hline X_{21} & X_{22} & X_{23} \\ \hline X_{31} & X_{32} & X_{33} \end{array}\right)$$

with respect to the partition $s+1+(n-1-s)$ of n as we have done for B. Then

$$XDA_s = \left(\begin{array}{c|c|c} pX_{11} & X_{12} & pX_{13} \\ \hline pX_{21} & X_{22} & pX_{23} \\ \hline pX_{31} & X_{32} & pX_{33} \end{array}\right).$$

Our matrix B in $B(DA_s)/\sim$ is considered by taking modulo XDA_s for any $X \in M_n(\mathbb{Z}_p)$. Hence B can be, up to equivalence, of the form

$$B = \left(\begin{array}{c|c|c} B_{11} & 0_{s\times 1} & B_{13} \\ \hline 0_{1\times s} & 0 & 0_{1\times(n-1-s)} \\ \hline {}^tB_{13} & 0_{(n-1-s)\times 1} & B_{33} \end{array}\right), \tag{A-1}$$

where B_{11}, B_{33}, and B_{13} belong to $S_s(\mathbb{F}_p)$, $S_{n-1-s}(\mathbb{F}_p)$, and $M_{s\times(n-1-s)}(\mathbb{F}_p)$, respectively. Further, to multiply A_s^{-1} on the right never change anything. Therefore, (A-1) gives a complete system of representatives of $B(D)/\sim$ for $D = D_s^I$. The condition $r_p(\alpha_I(D,B)) = 1$ and the modulo K on the left yield the desired result. For each $D \in D_s^{II}$, a similar computation shows any element of $S(p\cdot D)/\sim$ is given by

$$\begin{array}{ccc} \overbrace{\phantom{B_{11}}}^{s} & \overbrace{\phantom{p\cdot {}^tB_{21}}}^{1} & \overbrace{\phantom{B_{13}}}^{n-1-s} \end{array}$$
$$\left(\begin{array}{c|c|c} B_{11} & p\cdot {}^tB_{21} & B_{13} \\ \hline B_{21} & B_{22} & B_{23} \\ \hline {}^tB_{13} & p\cdot {}^tB_{23} & B_{33} \end{array}\right)$$

modulo the matrices of forms

$$\left(\begin{array}{c|c|c} pX_{11} & p^2X_{12} & pX_{13} \\ \hline pX_{21} & p^2X_{22} & pX_{23} \\ \hline pX_{31} & p^2X_{32} & pX_{33} \end{array}\right).$$

Therefore, $B_{11}, B_{13}, B_{21}, B_{22}, B_{23}$, and B_{33} run over

$$M_s(\mathbb{F}_p), \quad M_{s\times(n-1-s)(\mathbb{F}_p)}, \quad M_{1\times s(\mathbb{F}_p)}, \quad \mathbb{Z}/p^2\mathbb{Z}, \quad M_{1\times(n-1-s)(\mathbb{F}_p)}, \quad \text{and} \quad M_{n-1-s}(\mathbb{F}_p),$$

respectively. The claim now follows from the rank condition $r_p(\alpha_{II}(D,B)) = 1$ and the modulo K on the left again.

As for $D = p1_n$ in the case of type III, it is easy to see that $S(D)/\sim$ is naturally identified with $S_n(\mathbb{F}_p)$. Recall p is an odd prime by assumption. The number of matrices in $S_n(\mathbb{F}_p)$ of rank 1 is given in [MacWilliams 1969, Theorem 2]. □

Recall the right coset decomposition $T_{p,n-1} := K\,\mathrm{diag}(1, p, \dots, p, p^2, p, \dots, p)K = \coprod_{\alpha\in J} K\alpha$, containing two instances of $n-1$ entries of p. For each $\alpha, \beta \in J$, we observe that any element of $K\alpha\beta K$

is of mod p rank at most two and has the similitude p^4. Hence the double coset $K\alpha\beta K$ satisfies $K\alpha\beta K = K\gamma K$, where γ is one of the following four elements:

$$\gamma_1 := \mathrm{diag}(1, \overbrace{p^2, \dots, p^2}^{n-1}, p^4, \overbrace{p^2, \dots, p^2}^{n-1}), \qquad \gamma_3 := \mathrm{diag}(p, \overbrace{p^2, \dots, p^2}^{n-1}, p^3, \overbrace{p^2, \dots, p^2}^{n-1}),$$

$$\gamma_2 := \mathrm{diag}(p, p, \overbrace{p^2, \dots, p^2}^{n-2}, p^3, p^3, \overbrace{p^2, \dots, p^2}^{n-2}), \qquad \gamma_4 := p^2 \cdot I_{2n}$$

Here we use the Weyl elements in K to renormalize the order of entries. Then

$$T_{p,n-1} \cdot T_{p,n-1} = \sum_{i=1}^{4} m(\gamma_i) K\gamma_i K, \tag{A-2}$$

where $m(\gamma_i)$ is defined by

$$m(\gamma_i) := \left|\{(\alpha, \beta) \in J \times J : K\alpha\beta = K\gamma_i\}\right| \tag{A-3}$$

for each $1 \le i \le 4$; see [Shimura 1994b, p. 52]. Let us compute $m(\gamma_i)$ for each γ_i.

Let J_I be the subset of J consisting of the elements

$$\alpha_I^s(x) = \left(\begin{array}{ccc|ccc} p\cdot 1_s & & & & & \\ & p^2 & & & & \\ & -p\cdot {}^t x & p\cdot 1_{n-1-s} & & & \\ \hline & & & p\cdot 1_s & & \\ & & & & 1 & x \\ & & & & & p\cdot 1_{n-1-s} \end{array}\right), \quad 0 \le s \le n-2,\ x \in M_{1\times(n-1-s)}(\mathbb{F}_p)$$

and $\alpha_I^{n-1} = \mathrm{diag}(p^2, p, \dots, p, 1, p, \dots, p)$ containing $n-1$ entries of p both times.

Similarly, let J_{II} be the subset of J consisting of the elements

$$\alpha_{II}^s(y, B_{21}, B_{22}, B_{33}) = \left(\begin{array}{ccc|ccc} p\cdot 1_s & & & 0_s & p\cdot {}^t B_{21} & 0_{s\times(n-1-s)} \\ -{}^t y & 1 & & B_{21} & B_{22} & B_{23} \\ & & p\cdot 1_{n-1-s} & 0_{(n-1-s)\times s} & p\cdot {}^t B_{23} & 0_{n-1-s} \\ \hline & & & p\cdot 1_s & py & \\ & & & & p^2 & \\ & & & & & p\cdot 1_{n-1-s} \end{array}\right),$$

where $1 \le s \le n-1$, $y \in M_{s\times 1}(\mathbb{F}_p)$, and B_{21}, B_{23}, and B_{22} run over $M_{1\times s}(\mathbb{F}_p)$, $M_{1\times(n-1-s)}(\mathbb{F}_p)$, and $\mathbb{Z}/p^2\mathbb{Z}$, respectively. In addition,

$$\alpha_{II}^0(C_{22}, C_{23}) = \left(\begin{array}{cc|cc} 1 & & C_{22} & C_{23} \\ & p\cdot 1_{n-1} & p\cdot {}^t C_{23} & 0_{n-1} \\ \hline & & p^2 & \\ & & & p\cdot 1_{n-1} \end{array}\right), \quad C_{22} \in \mathbb{Z}/p^2\mathbb{Z},\ C_{23} \in M_{1\times(n-1)}(\mathbb{F}_p).$$

Finally, let J_{III} be the subset of J consisting of the elements

$$\alpha_{III}(B) = \left(\begin{array}{c|c} p \cdot 1_n & B \\ \hline & p \cdot 1_n \end{array}\right), \quad B \in S_n(\mathbb{F}_p) \text{ with } r_p(B) = 1.$$

Lemma A.3. *For each* $\alpha \in J$,

$$K\alpha K = K \operatorname{diag}(1, \overbrace{p, \dots, p}^{n-1}, p^2, \overbrace{p, \dots, p}^{n-1})K,$$

and

$$\operatorname{vol}\big(K \operatorname{diag}(1, \overbrace{p, \dots, p}^{n-1}, p^2, \overbrace{p, \dots, p}^{n-1})K\big) = p \sum_{i=0}^{2n-1} p^i,$$

where the measure is normalized as $\operatorname{vol}(K) = 1$.

Proof. Except for the case of type III, it follows from elementary divisor theory. For type III, it follows from [MacWilliams 1969] that the action of $\mathrm{GL}_n(\mathbb{F}_p)$ on the set of all matrices of rank 1 in $S_n(\mathbb{F}_p)$ given by $B \mapsto {}^tXBX$, $X \in \mathrm{GL}_n(\mathbb{F}_p)$ and such a symmetric matrix B has two orbits $O(\operatorname{diag}(1, 0, \dots, 0))$ and $O(\operatorname{diag}(g, 0, \dots, 0))$, both containing $n-1$ entries of 0, where g is a generator of $\mathbb{F}_p^\times$. The claim follows from this and elementary divisor theorem again.

For the latter claim, it is nothing but $|J|$, and we may compute the number of each type. □

Remark A.4. Since $K = \mathrm{Sp}_{2n}(\mathbb{Z}_p)$ contains Weyl elements,

$$\begin{aligned} K \operatorname{diag}(1, \overbrace{p, \dots, p}^{n-1}, p^2, \overbrace{p, \dots, p}^{n-1})K &= K \operatorname{diag}(\overbrace{p, \dots, p}^{i}, 1, \overbrace{p, \dots, p}^{n-i-1}, \overbrace{p, \dots, p}^{i}, p^2, \overbrace{p, \dots, p}^{n-i-1})K \\ &= K \operatorname{diag}(\overbrace{p, \dots, p}^{i}, p^2, \overbrace{p, \dots, p}^{n-i-1}, \overbrace{p, \dots, p}^{i}, 1, \overbrace{p, \dots, p}^{n-i-1})K \end{aligned}$$

for $0 \le i \le n-1$.

Notice that

$$Kd_{n-1}(p)K = K(p^2 \cdot d_{n-1}(p)^{-1})K,$$

where $d_{n-1}(p) := \operatorname{diag}(1, p, \dots, p, p^2, p, \dots, p)$ with $n-1$ entries of p both times.. By definition and Lemma A.3 with Remark A.4, it is easy to see that

$$\begin{aligned} m(\gamma_i) &= \big|\{\beta \in J : \gamma_i\beta^{-1} \in Kd_{n-1}(p)K\}\big| \\ &= \big|\{\beta \in J : \beta \cdot (p^2 \cdot \gamma_i^{-1}) \in Kd_{n-1}(p)K\}\big| \qquad ; \\ &= \big|\{\beta \in J : \beta \cdot (p^2 \cdot \gamma_i^{-1}) \text{ is } p\text{-integral and } r_p(\beta \cdot (p^2 \cdot \gamma_i^{-1})) = 1\}\big| \end{aligned}$$

see [Shimura 1994b, p. 52] for the first equality.

We are now ready to compute the coefficients. For $m(\gamma_1)$, we observe the p-integrality. We see that only $\alpha^0_{II}(C_{22}, C_{23})$ with $C_{22} = 0$ and $C_{23} = 0_{1\times(n-1)}$ can contribute there. Hence, $m(\gamma_1) = 1$.

For $m(\gamma_2)$, we observe the p-integrality and the rank condition. Then only $\alpha^0_{II}(0, 0_{1\times(n-1)})$ and $\alpha^1_{II}(y, 0, 0, 0_{1\times(n-2)})$, with $y \in \mathbb{F}_p$, can do there. Hence $m(\gamma_2) = 1 + p$. For $m(\gamma_3)$, only $\alpha_{III}(B)$, where $B \in S_n(\mathbb{F}_p)$ with $r_p(B) = 1$ contribute. By Lemma A.2-(3), we have $m(\gamma_3) = p^n - 1$.

Finally, we compute $m(\gamma_4)$. Since $p^{-2}\gamma_4 = I_4$, the condition is checked easily. All members of $J = J_I \cup J_{II} \cup J_{III}$ can contribute there. Therefore, we have only to count the number of each type. Hence, we have

$$m(\gamma_4) = \overbrace{1 + p + \cdots + p^{n-1}}^{\text{type I}} + \overbrace{p^{n+1} + p^{n+2} + \cdots + p^{2n}}^{\text{type II}} + \overbrace{p^n - 1}^{\text{type III}} = p \sum_{i=0}^{2n-1} p^i,$$

as desired. Note that $m(\gamma_4)$ is nothing but the volume of $Kd_{n-1}(p)K$; see Lemma A.3.

Recalling $T_{p,n-1} := pT(p, (0, \ldots, 0, 1))$, we have

$$T(p, (0, \ldots, 0, 1))^2 = \sum_{i=1}^{4} m(\gamma_i) K(p^{-2}\gamma_i) K.$$

Note that

$$K(p^{-2}\gamma_1)K = T(p, (\overbrace{0, \ldots, 0}^{n-1}, 2)), \quad K(p^{-2}\gamma_2)K = T(p, (\overbrace{0, \ldots, 0}^{n-2}, 1, 1)),$$
$$K(p^{-2}\gamma_3)K = T(p, (\overbrace{0, \ldots, 0}^{n-1}, 1)), \quad K(p^{-2}\gamma_4)K = T(p, (\overbrace{0, \ldots, 0}^{n})) = KI_{2n}K.$$

We can take K back to $\Gamma(N)$ without changing anything since $p \nmid N$. This proves Theorem A.1.

Remark A.5. We would like to make corrections to [Kim et al. 2020a].

(1) On page 356, line 1, $dx\,dy$ is missing in μ_∞^{ST}. In [25, page 929, line 3], the same typo is repeated.

(2) On page 362, line 12-13, $T_{2,p}^2$ should be a linear combination of four double cosets KMK, where M runs over $\mathrm{diag}(1, p^2, p^4, p^2)$, $\mathrm{diag}(p, p, p^3, p^3)$, $\mathrm{diag}(p, p^2, p^3, p^2)$, and $\mathrm{diag}(p^2, p^2, p^2, p^2)$.

(3) On page 362, the coefficient of R_{p^2} should be $p^4 + p^3 + p^2 + p = p\sum_{i=0}^{3} p^i$ which is the volume of $\mathrm{Sp}(4, \mathbb{Z}_p)\,\mathrm{diag}(1, p^2, p^4, p^2)\,\mathrm{Sp}(4, \mathbb{Z}_p)$ explained in [Roberts and Schmidt 2007, p. 190].

(4) On page 403, Lemma 8.1, the inequality $q(F) \geq N$ is not valid. Similarly, on page 405, Lemma 8.3, the inequality $q(F) \geq N$ is not valid. We need to consider newforms as in Section 5 of this paper. Then for a newform, we obtain the inequality $q(F) \geq N^{1/2}$ and $\log c_{\underline{k},N} \asymp \log N$ is valid as in Lemma 9.3 of this paper.

(5) On page 404, line -5, $N \gg p^{10}$ should be $N \gg p^{20}$.

(6) On page 407, line 3, $N \gg p^{30}$ should be $N \gg p^{10}$.

(7) On page 407, line 8: $N \gg p^{10}$ should be $N \gg p^{20}$.

(8) On page 409, line 10, we need to add $-2\big(G\big(\frac{3}{2}\big) + G\big(-\frac{1}{2}\big)\big)$, in order to account for the poles of $\Lambda(s, \pi_F, \mathrm{Spin})$, and the contour integral is over $\mathrm{Re}(s) = 2$. So, in (9.3), we need to add $O\big(|HE_{\underline{k}}(N)^0|/|HE_{\underline{k}}(N)|\big)$. However, only CAP forms give rise to a pole, and the number of CAP forms in $HE_{\underline{k}}(N)$ is $O(N^{8+\epsilon})$. So it is negligible.

In the case of standard L-functions, the non-CAP and nongenuine forms which give rise to poles are: $1 \boxplus \pi$, where π is an orthogonal cuspidal representation of $GL(4)$ with trivial central character, or $1 \boxplus \pi_1 \boxplus \pi_2$, where the π_i are dihedral cuspidal representations of $GL(2)$. In those cases, by

Proposition 4.11 and [Kim et al. 2020b, Theorem 2.9], we can count such forms without extra conditions on N in Proposition 4.12. So our result is valid as it is written.

Remark A.6. The referee brought to our attention a possible gap in [Sauvageot 1997, p. 181]; see [Dalal 2022, p. 129] and [Nelson and Venkatesh 2021, p. 159]. S.W. Shin communicated to us that the issue has not been fixed at this writing. However, we do not use the result in [Sauvageot 1997], nor any other later results [Dalal 2022; Shin 2012; Shin and Templier 2016] which depend on [Sauvageot 1997].

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