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LOCAL ESTIMATES AND GLOBAL CONTINUITIES IN LEBESGUE SPACES FOR BILINEAR OPERATORS

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In this paper, we first prove some local estimates for bilinear operators (closely related to the bilinear Hilbert transform and similar singular operators) with truncated symbol. Such estimates, in accordance with the Heisenberg uncertainty principle correspond to a description of "off-diagonal decay". In addition they allow us to prove global continuities in Lebesgue spaces for bilinear operators with spatial dependent symbol.

1. Introduction

The simplest bilinear operator is the pointwise product Π, defined by

$$\Pi(f, g)(x) := f(x)g(x),$$

for all $f, g \in \mathcal{S}(\mathbb{R})$. The Hölder inequalities give us the continuities on Lebesgue spaces for this operator. So for all exponents $p, q, r \in (0, \infty]$ such that

$$\frac{1}{p}+\frac{1}{q}=\frac{1}{r}, \tag{1-1}$$

the operator Π is continuous from $L^p(\mathbb{R}) \times L^q(\mathbb{R})$ into $L^r(\mathbb{R})$. Also a natural question appears: How can we modify this bilinear operation and simultaneous keep these continuities?

First let T be a bilinear operator, acting from $\mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R})$ into $\mathcal{S}'(\mathbb{R})$. It is well known that we have a spatial representation of T with a kernel $K \in \mathcal{S}'(\mathbb{R}^3)$ and a frequency representation with a symbol $\sigma \in \mathcal{S}'(\mathbb{R}^3)$ such that (in distributional sense)

$$\begin{aligned} T(f, g)(x) &= \int_{\mathbb{R}^2} K(x, y, z) f(y) g(z)\, dy\, dz \\ &= \int_{\mathbb{R}^2} e^{ix(\alpha+\beta)} \sigma(x, \alpha, \beta) \widehat{f}(\alpha)\widehat{g}(\beta)\, d\alpha\, d\beta, \end{aligned} \tag{1-2}$$

for all $f, g \in \mathcal{S}(\mathbb{R})$. In the rest of this paper, we denote by T_σ the operator associated to the symbol σ. The kernel and the symbol are related by the relation

$$K(x, y, z) = \int_{\mathbb{R}^2} e^{i(\alpha(x-y)+\beta(x-z))} \sigma(x, \alpha, \beta)\, d\alpha\, d\beta.$$

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For example, the product operator Π is given by the symbol

$$\sigma(x,\alpha,\beta)=1.$$

One of the first classes of bilinear symbols to be studied was the class of symbols satisfying the *bilinear Hörmander condition*: For all $a,b,c\geq 0$,

$$\left|\partial_x^a\partial_\alpha^b\partial_\beta^c\sigma(x,\alpha,\beta)\right|\lesssim(1+|\alpha|+|\beta|)^{-b-c}. \tag{1-3}$$

The corresponding operators T_σ were studied by R. Coifman and Y. Meyer [1978; 1975], C. Kenig and E. M. Stein [1999] and recently by L. Grafakos and R. Torres [2002]. We know that under (1-3), the operator T_σ is bounded from $L^p(\mathbb{R})\times L^q(\mathbb{R})$ into $L^r(\mathbb{R})$ for all exponents p,q,r satisfying (1-1) and $1<p,q<\infty$. In fact if the symbol is x-independent, one can just assume an homogeneous decay in (1-3) (that is with $(|\alpha|+|\beta|)^{-b-c}$) and then these operators can be decomposed with paraproducts, which were first exploited by J. M. Bony [1981] and R. Coifman and Y. Meyer [1978]. The paraproducts are studied with the linear tools (the Calderón–Zygmund decomposition, the Littlewood–Paley theory and the concept of Carleson measure). In order to get the continuities for x-dependent symbols, pointwise estimates of the bilinear kernel are used. Mainly for a symbol σ satisfying (1-3), integrations by parts allow us to obtain

$$|K(x,y,z)|\lesssim(1+|x-y|+|x-z|)^{-M} \tag{1-4}$$

for any large enough integer M. This estimate is very useful and precisely describes the "off-diagonal decrease" of the operator. Such an information helps us to reduce the study of x-dependent symbols to the study of x-independent symbols (and so to the study of paraproducts). Through these ideas, this first class of symbols are well understood nowadays. We note that this reduction (using pointwise estimates on the kernel) has already been used in the linear case to study the pseudo-differential operators of the well-known class $\mathrm{op}(S^0_{1,0})$. Thus "off-diagonal estimates" play an important role.

Since the work of A. Calderón [1965; 1977] in the 70's about the L^2 boundedness of commutators and Cauchy integrals, more singular bilinear operators have appeared. Mainly, he showed that the commutators and Cauchy integrals can be decomposed by using the bilinear Hilbert transforms. The bilinear Hilbert transform H_{λ_1,λ_2} is defined by

$$H_{\lambda_1,\lambda_2}(f,g)(x):=p.v.\int_{\mathbb{R}}f(x-\lambda_1y)g(x-\lambda_2y)\frac{dy}{y},$$

for all $f,g\in\mathcal{S}(\mathbb{R})$. The x-independent symbol is

$$\sigma(\alpha,\beta)=i\pi\,\mathrm{sign}(\lambda_1\alpha+\lambda_2\beta)$$

and so is singular on a whole line in the frequency plane. A. Calderón conjectured that these operators are continuous on Lebesgue spaces. This famous conjecture was first partially solved by M. Lacey and C. Thiele [1997a; 1997b; 1998; 1999]. Then some uniform (with respect to the parameters λ_1 and λ_2) continuities were shown in [Grafakos and Li 2004; Li 2006]. These proofs use a technical time frequency analysis, which was proven by C. Muscalu, T. Tao and C. Thiele [2002a; 2002b; 2004] and independently by J. Gilbert and A. Nahmod [2000; 2002]. They also get a very important result in the study of bilinear operators: continuities in Lebesgue spaces for more singular operators than those of the first class. We are interested by these bilinear operators and we will deal with them and some "smooth spatial perturbations". So we replace in (1-3) the quantity

$$|\alpha| + |\beta| = d((\alpha, \beta), 0)$$

by the lower quantity $d((\alpha, \beta), \Delta)$, where Δ is a line in the frequency plane:

$$\Delta := \{(\alpha, \beta) \in \mathbb{R}^2, \lambda_1\alpha + \lambda_2\beta = 0\}.$$

We assume that Δ is nondegenerate, that is, λ_1 and λ_2 are nonvanishing reals and not equal, in order that Δ be a graph over the three variables α, β and $\alpha+\beta$. We assume that the symbol σ satisfies

$$\left|\partial_x^a \partial_\alpha^b \partial_\beta^c \sigma(x, \alpha, \beta)\right| \lesssim (1 + |\lambda_1\alpha + \lambda_2\beta|)^{-b-c}, \tag{1-5}$$

for all $a, b, c \geq 0$. In the previous mentioned papers, the main result is this: If σ is x-independent and satisfies (1-5) (or the homogeneous version) then T_σ is continuous from $L^p(\mathbb{R}) \times L^q(\mathbb{R})$ into $L^r(\mathbb{R})$ for every exponents $p, q, r \in (0, \infty]$ satisfying

$$0 < \frac{1}{r} = \frac{1}{p} + \frac{1}{q} < \frac{3}{2} \quad \text{and} \quad 1 < p, q \leq \infty.$$

So there is a natural question (asked in [Bényi et al. 2006]): If an x-dependent symbol satisfies (1-5), is the operator T_σ continuous from $L^p(\mathbb{R}) \times L^q(\mathbb{R})$ into $L^r(\mathbb{R})$ with the same exponents p, q and r? A. Benyi, C. Demeter, A. Nahmod, R. Torres, C. Thiele and P. Villarroya [2007] proved a general result for singular integral kernels. As an example, they can apply their result to pseudo-differential operators associated to symbols

$$\sigma(x, \alpha, \beta) = \tau(x, \lambda_1\alpha + \lambda_2\beta)$$

with τ in the class $S^0_{1,0}$ because of a modulation invariant condition imposed. Here we are able to treat general symbols satisfying (1-5) and complete the answer to the question in [Bényi et al. 2006]. These operators do not fall under the scope of [Benyi et al. 2007] because they do not have modulation invariance. On the other hand, the general operators in [Benyi et al. 2007] cannot be realized as pseudo-differential bilinear operators with symbols satisfying (1-5) because of the minimal regularity assumptions required in the kernels.

With this aim, we would like to use the same arguments as for the symbols satisfying (1-3), where we have seen the important role of the "off-diagonal decay" of the bilinear kernel, obtained with integrations by parts. For our more singular operators, integration by parts does not work: To obtain a description of "off-diagonal estimates" is the most important difficulty.

We now come to our main result. For notation, we denote the norm in $L^p(E)$ for any measurable set $E \subset \mathbb{R}$ by

$$\|\cdot\|_{p,E,dx}$$

(or $\|\cdot\|_{p,E}$ if there is no confusion for the variable). For an interval I, we set

$$C_k(I) := \left\{x \in \mathbb{R},\ 2^k \leq 1 + \frac{d(x, I)}{|I|} < 2^{k+1}\right\}$$

the scaled corona around I. So we have

$$C_0(I) = 2I \quad \text{and} \quad C_k(I) \subset 2^{k+1} I.$$

We will first prove:

Theorem 1.1. *Let Δ be a nondegenerate line of the frequency plane. Let p, q be exponents such that*

$$1 < p, q \leq \infty \quad \textit{and} \quad 0 < \frac{1}{r} = \frac{1}{q} + \frac{1}{p} < \frac{3}{2}.$$

Then for all $\delta \geq 1$, there is a constant

$$C = C(p, q, r, \Delta, \delta)$$

such that for all interval $I \subset \mathbb{R}$, for all symbol $\sigma \in C^\infty(\mathbb{R}^3)$ satisfying for all $a, b, c \geq 0$,

$$\left|\partial_x^a \partial_\alpha^b \partial_\beta^c \sigma(x, \alpha, \beta)\right| \lesssim \left(|I|^{-1} + d((\alpha, \beta), \Delta)\right)^{-b-c}, \tag{1-6}$$

we have the following local estimate: For all functions $f, g \in \mathcal{S}(\mathbb{R})$,

$$\left(\frac{1}{|I|}\int_I |T_\sigma(f, g)(x)|^r dx\right)^{1/r} \leq C \left(\sum_{k\geq 0} 2^{-k\delta}\left(\frac{1}{|2^{k+1}I|}\int_{C_k(I)} |f(x)|^p dx\right)^{1/p}\right)\left(\sum_{k\geq 0} 2^{-k\delta}\left(\frac{1}{|2^{k+1}I|}\int_{C_k(I)} |g(x)|^q dx\right)^{1/q}\right).$$

In particular, with the Hardy–Littlewood operator M_{HL}, we have

$$\left(\frac{1}{|I|}\int_I |T_\sigma(f, g)(x)|^r dx\right)^{1/r} \lesssim \inf_I M_{HL}(|f|^p)^{1/p} \inf_I M_{HL}(|g|^q)^{1/q} \lesssim \|f\|_\infty \|g\|_\infty.$$

The weight

$$\left(|I|^{-1} + d((\alpha, \beta), \Delta)\right)^{-N}$$

is not integrable over the whole frequency plane (even if N is large enough due to the modulation invariance) and therefore we cannot have a pointwise estimate of the bilinear kernel (such as (1-4) when we assume (1-3)). So such a result is interesting because it precisely describes "off-diagonal estimates" for the bilinear operator:

Corollary 1.2. *With the same notations as Theorem 1.1, for all large enough δ, there exists a constant*

$$C = C(p, q, r, \Delta, \delta)$$

such that for any measurable sets $E, F \subset \mathbb{R}$ we have for all functions $f \in L^p(E)$ and $g \in L^q(F)$:

$$\|T_\sigma(f, g)\|_{r,I} \leq C\left(1 + \frac{d(I, E)}{|I|}\right)^{-\delta}\left(1 + \frac{d(I, F)}{|I|}\right)^{-\delta} \|f\|_{p,E} \|g\|_{q,F}.$$

This corollary is a direct application of Theorem 1.1. So in spite of the fact that the symbol could be much more singular than those satisfying only (1-3), we almost obtain the pointwise estimate (1-4). Here we have a description of the same fast decrease for the bilinear kernel, not with a pointwise estimate, but with local estimates at the scale $|I|$. These local estimates are less precise than the pointwise estimate but we will see that they are sufficient and they can play the same role.

We note that Theorem 1.1 is in accordance with the Heisenberg's Uncertainty Principle, which tells us that if we want to localize at the scale $|I|$ in the spatial domain, we cannot localize in the frequency domain at a lower scale than $|I|^{-1}$. For example, our Theorem 1.1 applies if the symbol is supported in the domain

$$\left\{(\alpha, \beta),\ d((\alpha, \beta), \Delta) \geq |I|^{-1}\right\}$$

and it is this case that we consider first in the proof. In fact in (1-6), we allow instead a nice behavior around the line Δ. With this point of view, we could call Theorem 1.1 an "high frequency estimate". In this expression, the term "frequency" corresponds to the distance between the point (α, β) to the line of singularity Δ. We prefer the expression "local estimates", because we will use the fast spatial decay in order to get the following result.

Theorem 1.3. *Let Δ be a nondegenerate line of the frequency plane. Let p and q be exponents such that*

$$1 < p, q \leq \infty \quad \textit{and} \quad 0 < \frac{1}{r} = \frac{1}{q} + \frac{1}{p} < \frac{3}{2}.$$

For all symbol $\sigma \in C^{\infty}(\mathbb{R}^3)$ satisfying for all $a, b, c \geq 0$,

$$\left|\partial_x^a \partial_\alpha^b \partial_\beta^c \sigma(x, \alpha, \beta)\right| \lesssim \left(1 + d((\alpha, \beta), \Delta)\right)^{-b-c},$$

the associated operator T_σ is bounded from $L^p(\mathbb{R}) \times L^q(\mathbb{R})$ into $L^r(\mathbb{R})$.

This result answers a question of [Bényi et al. 2006]. In addition it will allow us to define a bilinear pseudo-differential calculus, based on these operators: In our next paper [Bernicot 2008], we will define classes for bilinear pseudo-differential operators of order (m_1, m_2) and study their action on Sobolev spaces. In order to carry on the work of [Bényi et al. 2006], we will give rules of symbolic calculus for the duality and the composition and also complete the construction of a bilinear pseudo-differential calculus.

Remark 1.4. The proof of Theorem 1.1 is a shake between a localization argument and the "classical" time-frequency analysis used for these bilinear operators. So it is quite easy to obtain a version of our Theorems 1.1 and 1.3 for $(n-1)$-linear operators T_σ with a nondegenerate space Δ of dimension $k < \frac{n}{2}$, by following the ideas of [Muscalu et al. 2002a]. By using the results of [Terwilleger 2007], we are able to obtain the same results for a multidimensional problem and by using the uniform estimates of [Muscalu et al. 2002b], it seems possible to obtain uniform (with respect to the nondegenerate line Δ) local estimates.

The plan of this article is as follows. We first prove Theorem 1.1 in Section 2 for x-independent symbols. Then in Section 3 we get the same result for maximal bilinear operators and we conclude the proof of Theorem 1.1 in the general case. Then in Section 4, we use these local estimates to obtain global continuities for bilinear operators in weighted Lebesgue and Sobolev spaces and in particular we prove Theorem 1.3.

2. Proof of Theorem 1.1 for x-independent symbol

In this section, we assume that the symbol σ is x-independent and is supported on the domain

$$\left\{(\alpha, \beta),\ d((\alpha, \beta), \Delta) \geq |I|^{-1}\right\}.$$

We divide the proof into two subsections. First, we will quickly recall the decomposition of our bilinear operator T_σ by combinatorial model sums. So we will reduce the problem to a study of the "restricted weak type" for some localized trilinear forms. Then we will study them in the proof of Theorem 2.4 (see page 9).

Reduction to the study of discrete models. First of all, we define and recall the time-frequency tools (see for example [Muscalu et al. 2004]):

Definition 2.1. A *tile* is a rectangle (that is, a product of two intervals) $I \times \omega$ of area one. A *tritile* s is a rectangle $s = I_s \times \omega_s$ of bounded area, which contains three tiles $s_i = I_{s_i} \times \omega_{s_i}$ $(i = 1, 2, 3)$ such that, for all $i, j \in \{1, 2, 3\}$,

$$I_{s_i} = I_s \qquad \text{and} \qquad i \neq j \Rightarrow \omega_{s_i} \cap \omega_{s_j} = \varnothing.$$

A set $\{I\}_{I \in \mathcal{J}}$ of real intervals is called a *grid* if for all $k \in \mathbb{Z}$,

$$\sum_{\substack{I \in \mathcal{J} \\ 2^{k-1} \leq |I| \leq 2^{k+1}}} \mathbf{1}_I \lesssim \mathbf{1}_{\mathbb{R}}, \tag{2-1}$$

where the implicit constant is independent of k and of the grid. So a grid has the same structure as the dyadic grid.

Let $\mathbf{Q}$ be a set of tritiles. It is called a *collection* if

- $\{I_s,\ s \in \mathbf{Q}\}$ is a grid,
- $\mathcal{J} := \{\omega_s,\ s \in \mathbf{Q}\} \cup \bigcup_{i=1}^{3} \{\omega_{s_i},\ s \in \mathbf{Q}\}$ is a grid, and
- $\omega_{s_i} \subsetneq \varpi \in \mathcal{J} \Rightarrow$ for all $j \in \{1, 2, 3\},\ \omega_{s_j} \subset \varpi$.

Now we can define the wave packet for a tile.

Definition 2.2. Let Φ be a smooth function such that

$$\|\Phi\|_2 = 1 \quad \text{and} \quad \operatorname{supp}(\widehat{\Phi}) \subset [-\tfrac{1}{2}, \tfrac{1}{2}].$$

For $P = I \times \omega$ a tile, we set

$$\Phi_P(x) := |I|^{-1/2} \Phi\Big(\frac{x - c(I)}{|I|}\Big) e^{ixc(\omega)},$$

where for U an interval we denote by $c(U)$ its center. So Φ_P is normalized in the $L^2(\mathbb{R})$ space, concentrated in space around I and its spectrum is exactly contained in ω.

Nowadays it is well known (see for example [Bilyk and Grafakos 2006a; 2006b]) that the operator T_σ of Theorem 1.1 admits a decomposition

$$T_\sigma(f, g)(x) := \sum_{u=(u_1,u_2,u_3) \in \mathbb{Z}^3} (1 + |u|^2)^{-N} \sum_{s \in \mathbf{S}_u} |I_s|^{-1/2} \epsilon_s(u) \big\langle (\tau_{u_1}\phi)_{s_1}, f \big\rangle \big\langle (\tau_{u_2}\phi)_{s_2}, g \big\rangle (\tau_{u_3}\phi)_{s_3}(x),$$

where $\mathbf{S}_u$ is a collection of tritiles depending on u, $\epsilon_s(u)$ are bounded reals for $s \in \mathbf{S}_u$, and N is an integer as large as we want. We write τ_v for the translation operator

$$\tau_v(f)(x) = f(x - v).$$

The coefficients $\epsilon_s(u)$ are uniformly bounded with respect to the parameter u and the implicit constant in (2-1) (for the definition of a grid) is bounded by the estimates of the symbol σ.

By using the assumption that σ is supported in

$$\{(\alpha,\beta),\ |\alpha-\beta|\geq |I|^{-1}\},$$

we have the very important property

$$|\omega_s|\gtrsim |I|^{-1}, \tag{2-2}$$

which is equivalent to

$$|I_s|\lesssim |I|,$$

for all $u\in\mathbb{Z}^3$, and for all $s\in\mathbf{S}_u$.

So Theorem 1.1 is a consequence of the following theorem.

Theorem 2.3. *Let $\mathbf{S}$ be a collection of tritiles satisfying the property (2-2), $(\epsilon_s)_{s\in\mathbf{S}}$ bounded reals and $(\phi^i)_{i=1,2,3}$ smooth functions whose spectrum is contained in $[-\frac{1}{2},\frac{1}{2}]$. We denote $T_{\mathbf{S}}$ the bilinear operator defined by*

$$T_{\mathbf{S}}(f,g)(x):=\sum_{s\in\mathbf{S}}|I_s|^{-1/2}\epsilon_s\langle\phi^1_{s_1},f\rangle\langle\phi^2_{s_2},g\rangle\phi^3_{s_3}(x).$$

Then for the exponents (p,q,r) of Theorem 1.1 and for all $\delta\geq 1$, we have the local estimate

$$\left(\int_I |T_{\mathcal{S}}(f,g)|^r\right)^{1/r}\lesssim\left(\sum_{k\geq 0}2^{-k(1/p+\delta)}\|f\|_{p,C_k(I)}\right)\left(\sum_{k\geq 0}2^{-k(1/q+\delta)}\|g\|_{q,C_k(I)}\right).$$

In addition the implicit constant depends on the functions ϕ^i by the parameters

$$c_M(\phi^i):=\sup_{x\in\mathbb{R}}\sum_{0\leq k\leq M}(1+|x|)^M\left|(\phi^i)^k(x)\right|$$

for $M=M(p,q,r,\delta)$ a large enough integer.

In order to show this result, we need to decompose the functions f and g around the interval I. The interval I being fixed, we omit it in the notation for convenience and for $i\in\mathbb{N}$, we set the corona $C_i:=C_i(I)$. With the property (2-2), we have the decomposition

$$T_{\mathbf{S}}(f,g)=\sum_{k_1,k_2\geq 0}T^{k_1,k_2}_{\mathbf{S},0}(f,g)+\sum_{\substack{k_1,k_2\geq 0\\ l\leq 0}}T^{k_1,k_2,l}_{\mathbf{S},1}(f,g) \tag{2-3}$$

with

$$T^{k_1,k_2}_{\mathbf{S},0}(f,g)(x):=\sum_{\substack{s\in\mathbf{S}\\ I_s\subset 2I}}|I_s|^{-1/2}\epsilon_s\langle\phi^1_{s_1},f\mathbf{1}_{C_{k_1}}\rangle\langle\phi^2_{s_2},g\mathbf{1}_{C_{k_2}}\rangle\phi^3_{s_3}(x),$$

$$T^{k_1,k_2,l}_{\mathbf{S},1}(f,g)(x):=\sum_{\substack{s\in\mathbf{S},\ I_s\not\subset 2I\\ 2^l|I|\leq|I_s|<2^{l+1}|I|}}|I_s|^{-1/2}\epsilon_s\langle\phi^1_{s_1},f\mathbf{1}_{C_{k_1}}\rangle\langle\phi^2_{s_2},g\mathbf{1}_{C_{k_2}}\rangle\phi^3_{s_3}(x).$$

Due to the important property (2-2), we only have to consider tiles s with $|I_s|\leq|I|$. The other terms (corresponding to $l>0$) cannot be studied as we are going to do, according to the Heisenberg Uncertainty Principle.

Starting on page 9, we shall prove the following theorem.

Theorem 2.4. *Let (p, q, r) be exponents as in Theorem 1.1. The operators $T^j_{\mathbf{S},i}$ are continuous from $L^p(\mathbb{R}) \times L^q(\mathbb{R})$ into $L^r(I)$. For convenience, we denote*

$$C(T^j_{\mathbf{S},i}) := \|T^j_{\mathbf{S},i}\|_{L^p \times L^q \to L^r}$$

and we omit the exponents. Then these continuity bounds satisfy

$$\begin{aligned} C(T^{k_1,k_2}_{\mathbf{S},0}) &\lesssim c_M(\phi^1) c_M(\phi^2) c_M(\phi^3)\, 2^{-\delta'(k_1+k_2)}, \\ C(T^{k_1,k_2,l}_{\mathbf{S},1}) &\lesssim c_M(\phi^1) c_M(\phi^2) c_M(\phi^3)\, 2^{-\delta'(|l|+k_1+k_2)} \end{aligned}$$

for any large enough real δ', with an integer $M = M(p, q, r, \delta')$.

We claim that Theorem 2.3 is a consequence of Theorem 2.4.

Proof of Theorem 2.3. By using Theorem 2.4 and the decomposition (2-3), we have that for all functions $f, g \in \mathcal{S}(\mathbb{R})$,

(i) if $r \geq 1$, then

$$\|T_{\mathbf{S}}(f, g)\|_{r,I} \lesssim \sum_{k_1,k_2 \geq 0} C(T^{k_1,k_2}_{\mathbf{S},0}) \|f \mathbf{1}_{C_{k_1}}\|_p \|g \mathbf{1}_{C_{k_2}}\|_q + \sum_{\substack{k_1,k_2 \geq 0 \\ l \leq 0}} C(T^{k_1,k_2,l}_{\mathbf{S},1}) \|f \mathbf{1}_{C_{k_1}}\|_q \|g \mathbf{1}_{C_{k_2}}\|_r;$$

(ii) if $r < 1$, then

$$\|T_{\mathbf{S}}(f, g)\|^r_{r,I} \lesssim \sum_{k_1,k_2 \geq 0} C(T^{k_1,k_2}_{\mathbf{S},0})^r \|f \mathbf{1}_{C_{k_1}}\|^r_p \|g \mathbf{1}_{C_{k_2}}\|^r_q + \sum_{\substack{k_1,k_2 \geq 0 \\ l \leq 0}} C(T^{k_1,k_2,l}_{\mathbf{S},1})^r \|f \mathbf{1}_{C_{k_1}}\|^r_p \|g \mathbf{1}_{C_{k_2}}\|^r_q.$$

Case (i) ($r \geq 1$): With the estimate of $C(T^{k_1,k_2}_{\mathbf{S},0})$ and $C(T^{k_1,k_2,l}_{\mathbf{S},1})$ given by Theorem 2.4, we obtain

$$\begin{aligned} \|T_{\mathbf{S}}(f, g)\|_{r,I} &\lesssim \sum_{k_1,k_2 \geq 0} 2^{-\delta'(k_1+k_2)} \|f \mathbf{1}_{C_{k_1}}\|_p \|g \mathbf{1}_{C_{k_2}}\|_q + \sum_{\substack{k_1,k_2 \geq 0 \\ l \leq 0}} 2^{-\delta'(k_1+k_2+|l|)} \|f \mathbf{1}_{C_{k_1}}\|_p \|g \mathbf{1}_{C_{k_2}}\|_q \\ &\lesssim \sum_{k_1,k_2 \geq 0} 2^{-\delta'(k_1+k_2)} \|f \mathbf{1}_{C_{k_1}}\|_p \|g \mathbf{1}_{C_{k_2}}\|_q. \end{aligned}$$

Hence by using that δ' is as large as we want, the conclusion follows for case (i).

Case (ii) ($r \leq 1$): We have

$$\begin{aligned} \|T_{\mathbf{S}}(f, g)\|^r_{r,I} &\lesssim \sum_{k_1,k_2 \geq 0} 2^{-r\delta'(k_1+k_2)} \|f \mathbf{1}_{C_{k_1}}\|^r_p \|g \mathbf{1}_{C_{k_2}}\|^r_q + \sum_{\substack{k_1,k_2 \geq 0 \\ l \leq 0}} 2^{-r\delta'(k_1+k_2+|l|)} \|f \mathbf{1}_{C_{k_1}}\|^r_p \|g \mathbf{1}_{C_{k_2}}\|^r_q \\ &\lesssim \sum_{k_1,k_2 \geq 0} 2^{-r\delta'(k_1+k_2)} \|f \mathbf{1}_{C_{k_1}}\|^r_p \|g \mathbf{1}_{C_{k_2}}\|^r_q. \end{aligned}$$

By using Hölder's inequality and $\rho > 0$ such that

$$\frac{1}{p} + \rho,\ \frac{1}{q} + \rho < 1,$$

we obtain

$$\begin{aligned}\|T_{\mathbf{S}}(f,g)\|_{r,I} &\lesssim \Big(\sum_{k_1\geq 0} 2^{-k_1 p(\delta'-1)(\rho+1/p)}\|f\mathbf{1}_{C_{k_1}}\|_p^p\Big)^{1/p}\Big(\sum_{k_2\geq 0} 2^{-k_2 q(\delta'-1)(\rho+1/q)}\|g\mathbf{1}_{C_{k_2}}\|_q^q\Big)^{1/q} \\ &\lesssim \Big(\sum_{k_1\geq 0} 2^{-k_1(\delta'-1)(\rho+1/p)}\|f\mathbf{1}_{C_{k_1}}\|_p\Big)\Big(\sum_{k_2\geq 0} 2^{-k_2(\delta'-1)(\rho+1/q)}\|g\mathbf{1}_{C_{k_2}}\|_q\Big).\end{aligned}$$

This corresponds to the desired result (the real δ' being as large as we want) for case **(ii)**. □

We have also reduced the proof of Theorem 1.1 (for our particular symbol σ) to that of Theorem 2.4.

Proof of Theorem 2.4. By using "duality", to prove Theorem 2.4, we have to estimate the trilinear form defined on $\mathcal{S}(\mathbb{R})\times\mathcal{S}(\mathbb{R})\times\mathcal{S}(\mathbb{R})$ by

$$\Lambda_i^j(f_1,f_2,f_3) := \langle T^j_{\mathbf{S},i}(f_1,f_2), f_3\mathbf{1}_I\rangle = \sum_{s\in\mathbf{Q}_i^j} |I_s|^{-1/2}\epsilon_s\langle\phi^1_{s_1}, f_1\mathbf{1}_{C_{k_1}}\rangle\langle\phi^2_{s_2}, f_2\mathbf{1}_{C_{k_2}}\rangle\langle\phi^3_{s_3}, f_3\mathbf{1}_I\rangle, \tag{2-4}$$

where $\mathbf{Q}_i^j$ is a collection of tritiles, depending on $T^j_{\mathbf{S},i}$.

We need to define the usual tools of time-frequency analysis.

Definition 2.5. We have already defined the tritiles. For $j\in\{1,2,3\}$ an index and $t\in\mathbf{S}$ a tritile, a collection $\mathbf{T}$ of tritiles is called a *j-tree* with *top t* if for all $s\in\mathbf{T}$,

$$I_s\subset I_t \quad\text{and}\quad \omega_{t_j}\subset\omega_{s_j}.$$

Then we set

$$I_{\mathbf{T}} := I_t,$$

the time-interval of the tree $\mathbf{T}$. A collection $\mathbf{T}$ of tritiles is called a *tree* if there exists an index $j\in\{1,2,3\}$ such that $\mathbf{T}$ is a j-tree. For $\mathbf{T}$ a j-tree, we define the *size* of the function f_j over this tree by

$$\operatorname{size}_j(\mathbf{T}) := \Big(\frac{1}{|I_{\mathbf{T}}|}\sum_{s\in\mathbf{T}}\big|\langle f_j,\phi^j_{s_j}\rangle\big|^2\Big)^{1/2}.$$

For $\mathbf{Q}$ a collection of tritiles, we define the *global size* by

$$\operatorname{size}^*_j(\mathbf{Q}) = \sup\big\{\operatorname{size}_k(\mathbf{T}) :\ \mathbf{T}\subset\mathbf{Q},\ \mathbf{T}\text{ is a k-tree},\ k\neq j\big\}.$$

The quantity $|I_{\mathbf{T}}|^{1/2}\operatorname{size}_j(\mathbf{T})$ corresponds to the norm of the function f_j in the space L^2, after being restricted on the tree $\mathbf{T}$ in the time-frequency space.

We recall the (abstract) [Muscalu et al. 2004, Proposition 6.5], where [Muscalu et al. 2004, Lemma 6.7] is used to estimate the quantities $\widetilde{energy}_j$.

Proposition 2.6. *Let $(\theta_j)_{1\leq j\leq 3}$ be three exponents of $(0,1)$ satisfying*

$$\theta_1+\theta_2+\theta_3=1.$$

Then there exists a constant $C = C(\theta_i)$ such that for all collection $\mathbf{Q}$ of tritiles, we have

$$\left|\sum_{s\in\mathbf{Q}} |I_s|^{-1/2} \prod_{i=1}^{3} \langle \phi^i_{s_i}, f_i\rangle\right| \le C \prod_{i=1}^{3} \operatorname{size}^*_i(\mathbf{Q})^{\theta_i} \|f_i\|_2^{1-\theta_i}.$$

This result is the main idea of this time-frequency analysis. To prove it, we use a stopping-time argument in order to build an "orthogonal" covering of the time-frequency space with trees of $\mathbf{Q}$.

Now we recall the notion of *restricted weak type* for trilinear forms.

Definition 2.7. For E a Borelian set of $\mathbb{R}$, we write

$$F(E) := \{f \in \mathcal{S}(\mathbb{R}) : \text{ for all } x \in \mathbb{R},\ |f(x)| \le \mathbf{1}_E(x)\}.$$

Let Λ be a trilinear form defined on $\mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R})$. Let p_1, p_2, p_3 be exponents of $\mathbb{R}^*$, possibly negative. We say that Λ is of *restricted weak type* (p_1, p_2, p_3) if there exists a constant C such that for all measurable sets E_1, E_2, E_3 of finite measure, we can find a substantial subset $E'_\beta \subset E_\beta$ (that is, $|E'_\beta| \ge \frac{|E_\beta|}{2}$) for $\beta \in \{1, 2, 3\}$ such that for all $f_\beta \in F(E'_\beta)$,

$$|\Lambda(f_1, f_2, f_3)| \le C \prod_{\beta=1}^{3} |E_\beta|^{1/p_\beta} \tag{2-5}$$

and $E'_\beta = E_\beta$ if $p_\beta > 0$. The best constant in (2-5) is called *the bound of restricted type* and will be denoted by $C(\Lambda)$.

By the real interpolation theory for trilinear forms of restricted weak type [Muscalu et al. 2002b, Lemmas 3.6, 3.7, 3.8, 3.9, 3.10 and 3.11], Theorem 2.4 is a consequence of the following result (which is a stronger continuity result).

Theorem 2.8. *Let p_1, p_2, p_3 be nonvanishing reals such that*

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$$

and there exists a unique index $\alpha \in \{1, 2, 3\}$ with $-\frac{1}{2} < \frac{1}{p_\alpha} < 0$, and $\frac{1}{2} < \frac{1}{p_\beta} < 1$ for $\beta \ne \alpha$. Then the trilinear forms Λ^j_i defined by (2-4) are of restricted weak type (p_1, p_2, p_3). In addition the bounds of restricted type $C(\Lambda^j_i)$ satisfy

$$C(\Lambda_0^{k_1,k_2}) \lesssim c_M(\phi^1) c_M(\phi^2) c_M(\phi^3) 2^{-\delta'(k_1+k_2)},$$
$$C(\Lambda_1^{k_1,k_2,l}) \lesssim c_M(\phi^1) c_M(\phi^2) c_M(\phi^3) 2^{-\delta'(|l|+k_1+k_2)}$$

for any real $\delta' \ge 1$ with $M = M(\delta', p_i)$ a large enough integer.

Proof. The exponents $(p_\beta)_\beta$ and the index $\alpha \in \{1, 2, 3\}$ are fixed for the proof. Let E_1, E_2 and E_3 be measurable sets of finite measure. First we construct the substantial subset $E'_\alpha \subset E_\alpha$. Denote

$$U := \bigcup_{i=1}^{3} \left\{x \in \mathbb{R},\ M_{HL}(\mathbf{1}_{E_i})(x) > \eta \frac{|E_i|}{|E_\alpha|}\right\}.$$

By using Hardy–Littlewood Theorem, there exists a numerical constant η such that

$$|U| \leq \frac{|E_\alpha|}{2}.$$

We set also $E'_\alpha = E_\alpha \setminus U$. It is interesting to note that the set E'_α does not depend on the form Λ_i^j. Now we fix the functions $f_\beta \in F(E'_\beta)$ for $\beta \in \{1, 2, 3\}$ and we shall prove the inequality (2-5). The proof is divided in three parts: In the first step we use general estimates for collections of tritiles, in the second step we will use specific estimates adapted to the above collections of tritiles and then we will conclude in the third step.

First step: a general estimate. Let $\mathbf{P}$ be an "abstract" collection of tritiles, then for $k \geq 0$ we set $\mathbf{P}_k$ the subcollection

$$\mathbf{P}_k := \left\{ s \in \mathbf{P},\quad 2^k \leq 1 + \frac{d(I_s, U^c)}{|I_s|} < 2^{k+1} \right\}.$$

These collections form a partition of $\mathbf{P}$:

$$\mathbf{P} = \bigsqcup_{k \geq 0} \mathbf{P}_k.$$

For each $k \geq 0$, we can apply Proposition 2.6 to the collection $\mathbf{Q} = \mathbf{P}_k$. So for any choice of exponents $0 < \theta_1, \theta_2, \theta_3 < 1$ with

$$\sum_{\beta=1}^{3} \theta_\beta = 1,$$

we obtain

$$\Lambda(P_k) := \left| \sum_{s \in \mathbf{P}_k} |I_s|^{-1/2} \epsilon_s \prod_{\beta=1}^{3} \langle f_\beta, \phi_{s_\beta}^\beta \rangle \right| \lesssim \prod_{\beta=1}^{3} (\text{size}_\beta^*(\mathbf{P}_k))^{\theta_\beta} \|f_\beta\|_2^{1-\theta_\beta}.$$

In order to estimate the quantities $\text{size}_\beta^*(\mathbf{P}_k)$, we recall [Muscalu et al. 2002b, Lemma 7.8].

Lemma 2.9. *For all integer N as large as we want, there exists a constant $C = C(N)$ such that for all collection $\mathbf{Q}$ of tritiles, for all $\beta \in \{1, 2, 3\}$, we have*

$$\text{size}_\beta^*(\mathbf{Q}) \leq C \sup_{s \in \mathbf{Q}} \frac{1}{|I_s|} \int_{\mathbb{R}} \left(1 + \frac{d(x, I_s)}{|I_s|}\right)^{-N} |f_\beta(x)|\, dx.$$

Then for $\mathbf{Q} = \mathbf{P}_k$, by using the definition of the sets U and E'_α, we have

$$\text{size}_\beta^*(\mathbf{P}_k) \lesssim 2^k \frac{|E_\beta|}{|E_\alpha|}, \quad \text{and} \quad \text{size}_\alpha^*(\mathbf{P}_k) \lesssim 2^{-Nk}.$$

for all $\beta \neq \alpha$. As f_β belongs to $F(E_\beta)$, we have

$$\|f_\beta\|_2 \leq |E_\beta|^{1/2}.$$

So for $0 < \epsilon < 1$ and N an integer as large as we want, we get

$$\Lambda(P_k) \lesssim \prod_{\beta\neq\alpha} \Big(2^k \frac{|E_\beta|}{|E_\alpha|}\Big)^{\theta_\beta(1-\epsilon)} |E_\beta|^{(1-\theta_\beta)/2} 2^{Nk\theta_\alpha(1-\epsilon)} |E_\alpha|^{(1-\theta_\alpha)/2} \prod_{\beta=1}^{3} (\mathrm{size}^*_\beta(\mathbf{P}_k))^{\theta_\beta\epsilon}$$
$$\lesssim 2^{-k} \Big(\prod_{\beta\neq\alpha} |E_\beta|^{(1+\theta_\beta)/2-\epsilon\theta_\beta} |E_\alpha|^{(\theta_\alpha-1)/2+\epsilon(1-\theta_\alpha)}\Big)\Big(\prod_{\beta=1}^{3} (\mathrm{size}^*_\beta(\mathbf{P}_k))^{\theta_\beta\epsilon}\Big).$$

By definition of size^*_β, $\mathbf{P}_k$ is a subcollection of $\mathbf{P}$ so for all $\beta \in \{1, 2, 3\}$,

$$\mathrm{size}^*_\beta(\mathbf{P}_k) \leq \mathrm{size}^*_\beta(\mathbf{P}).$$

We can also compute the sum over $k \geq 0$ and we obtain

$$\Lambda(P) := \Big|\sum_{s\in\mathbf{P}} |I_s|^{-1/2}\epsilon_s \prod_{\beta=1}^{3} \langle f_\beta, \phi^\beta_{s_\beta}\rangle\Big| \leq \sum_{k\geq 0} P_k$$
$$\lesssim \Big(\prod_{\beta\neq\alpha} |E_\beta|^{(1+\theta_\beta)/2-\epsilon\theta_\beta} |E_\alpha|^{(\theta_\alpha-1)/2+\epsilon(1-\theta_\alpha)}\Big)\Big(\prod_{\beta=1}^{3} (\mathrm{size}^*_\beta(\mathbf{P}))^{\theta_\beta\epsilon}\Big). \tag{2-6}$$

The first term is "good", according to the wished global continuity. In the next step, we will use another estimate of the quantities size^*_β, which will be adapted to our specific trilinear forms Λ_i^j and which allow us to obtain the desired decays.

Second step: use of the specific form of our trilinear forms Λ_i^j.

First case: the forms Λ_1^j.

In this case, we use another decomposition

$$\Lambda_1^{k_1,k_2,l}(f_1, f_2, f_3) \leq \sum_{\substack{I_0 \not\subseteq 2I \\ 2^{l-1}|I|\leq|I_0|\leq 2^{l+1}|I|}} \Lambda_1^{k_1,k_2,l}(I_0)(f_1, f_2, f_3),$$

where I_0 is an interval of $\mathbb{R}$ and

$$\Lambda_1^{k_1,k_2,l}(I_0)(f_1, f_2, f_3) := \sum_{\substack{s\in\mathbf{S} \\ I_s=I_0}} |I_s|^{-1/2}\epsilon_s \langle f_1 \mathbf{1}_{C_{k_1}}, \phi^1_{s_1}\rangle\langle f_2\mathbf{1}_{C_{k_2}}, \phi^2_{s_2}\rangle\langle \mathbf{1}_I f_3, \phi^3_{s_3}\rangle.$$

Let I_0 be fixed and denote

$$2^l = \frac{|I_0|}{|I|}.$$

The collection of tritiles associated to $\Lambda_1^{k_1,k_2,l}(I_0)$ is also

$$\mathbf{P} := \{s \in \mathbf{S},\ I_s = I_0\}.$$

For all $s \in \mathbf{P}$, by using $f_3 \in F(E_3')$, we have

$$\frac{1}{|I_s|}\int_I |f_3(x)|\Big(1+\frac{d(x,I_s)}{|I_s|}\Big)^{-N}dx \le \frac{1}{|I_s|}\int_I\Big(1+\frac{d(x,I_s)}{|I_s|}\Big)^{-N}dx$$
$$\le \frac{|I|}{|I_0|}\Big(1+\frac{d(I,I_0)}{|I_0|}\Big)^{-N}.$$

Then Lemma 2.9 gives us

$$\operatorname{size}_3^*(\mathbf{P}) \lesssim 2^{-l}\Big(1+\frac{d(I,I_0)}{|I_0|}\Big)^{-N}.$$

By the same reasoning, we obtain for $f_1 \in F(E_1')$ and $s \in \mathbf{P}$,

$$\frac{1}{|I_s|}\int_{C_{k_1}} |f_1(x)|\Big(1+\frac{d(x,I_0)}{|I_s|}\Big)^{-N}dx \le \frac{1}{|I_0|}\int_{C_{k_1}}\Big(1+\frac{d(x,I_0)}{|I_0|}\Big)^{-N}dx \le 2^{k_1-l}\Big(1+\frac{d(C_{k_1},I_0)}{|I_0|}\Big)^{-N}.$$

And so we get

$$\operatorname{size}_1^*(\mathbf{P}) \lesssim 2^{k_1-l}\Big(1+\frac{d(C_{k_1},I_0)}{|I_0|}\Big)^{-N}.$$

Likely, we have

$$\operatorname{size}_2^*(\mathbf{P}) \lesssim 2^{k_2-l}\Big(1+\frac{d(C_{k_2},I_0)}{|I_0|}\Big)^{-N}.$$

With $\theta_1+\theta_2+\theta_3 = 1$ and Lemma 2.9, we can estimate

$$\operatorname{size}_1^*(\mathbf{P})^{\theta_1}\ \operatorname{size}_2^*(\mathbf{P})^{\theta_2}\ \operatorname{size}_3^*(\mathbf{P})^{\theta_3} \lesssim 2^{\theta_1 k_1+\theta_2 k_2 - l}A(I_0), \tag{2-7}$$

where $A(I_0)$ is the product of three terms

$$A(I_0) := \Big(1+\frac{d(I,I_0)}{|I_0|}\Big)^{-N\theta_3}\Big(1+\frac{d(C_{k_1},I_0)}{|I_0|}\Big)^{-N\theta_1}\Big(1+\frac{d(C_{k_2},I_0)}{|I_0|}\Big)^{-N\theta_2}.$$

We are going to get four different estimates for $A(I_0)$.

To keep the information about the position of I_0, we first have

$$A(I_0) \le \Big(1+\frac{d(I,I_0)}{|I_0|}\Big)^{-N\theta_3}. \tag{2-8}$$

By using

$$d(I,I_0)+d(C_{k_1},I_0) \gtrsim d(I,C_{k_1}) \gtrsim 2^{k_1}|I| \simeq 2^{k_1-l}|I_0|$$

and the fact that $2^l \le 1$, we obtain

$$A(I_0) \lesssim (1+2^{k_1-l})^{-N\min\{\theta_1,\theta_3\}} \lesssim 2^{-k_1 N\min\{\theta_1,\theta_3\}} \tag{2-9}$$

and likely

$$A(I_0) \lesssim 2^{-k_2 N\min\{\theta_2,\theta_3\}}. \tag{2-10}$$

As $I_0 \nsubseteq 2I$ and $2^l \le 1$, $d(I_0,I) \ge |I|$ and hence

$$\Big(1+\frac{d(I,I_0)}{|I_0|}\Big)^{-N} \lesssim \Big(\frac{|I_0|}{|I|}\Big)^{N}.$$

So we get
$$A(I_0) \lesssim \left(\frac{|I_0|}{|I|}\right)^{N\theta_3} \lesssim 2^{lN\theta_3}. \tag{2-11}$$
Taking the geometric mean of (2-8), (2-9), (2-10) and (2-11) (with another exponent N which is as large as we want), we obtain
$$A(I_0) \lesssim 2^{-(k_1+k_2+|l|)N}\left(1+\frac{d(I, I_0)}{|I_0|}\right)^{-N}. \tag{2-12}$$
With the help of (2-6) and (2-7), we finally estimate
$$\begin{aligned}\left|\Lambda_1^{k_1,k_2,l}(f_1, f_2, f_3)\right| &\lesssim \sum_{I_0}\left|\Lambda_1^{k_1,k_2,l}(I_0)(f_1, f_2, f_3)\right| \\ &\lesssim \sum_{I_0}\left(\prod_{\beta\neq\alpha}|E_\beta|^{(1+\theta_\beta)/2-\epsilon\theta_\beta}|E_\alpha|^{(\theta_\alpha-1)/2+\epsilon(1-\theta_\alpha)}\right)2^{\epsilon(k_1+k_2+|l|)}A(I_0)^\epsilon.\end{aligned}$$
From (2-12), the sum over the interval I_0 with $|I_0| = 2^l|I|$ is bounded. For N a large enough exponent (not exactly the same), we have
$$\left|\Lambda_1^{k_1,k_2,l}(f_1, f_2, f_3)\right| \lesssim \left(\prod_{\beta\neq\alpha}|E_\beta|^{(1+\theta_\beta)/2-\epsilon\theta_\beta}|E_\alpha|^{(\theta_\alpha-1)/2+\epsilon(1-\theta_\alpha)}\right)\tilde{C}(\Lambda_1^{k_1,k_2,l}),$$
where
$$\tilde{C}(\Lambda_1^{k_1,k_2,l}) := 2^{-N\epsilon(k_1+k_2+|l|)}. \tag{2-13}$$
<u>Second case</u>: the forms Λ_0^j.

We use the same principle. We are interested in
$$\Lambda_0^{k_1,k_2}(f_1, f_2, f_3) := \sum_{\substack{s\in\mathbf{S}\\ I_s\subset 2I}} |I_s|^{-1/2}\epsilon_s\langle f_1\mathbf{1}_{C_{k_1}}, \phi_{s_1}^1\rangle\langle f_2\mathbf{1}_{C_{k_2}}, \phi_{s_2}^2\rangle\langle f_3, \phi_{s_3}^3\rangle.$$
So now we choose the collection
$$\mathbf{P} := \{s \in \mathbf{S}, \quad I_s \subset 2I\}.$$
For all $s \in \mathbf{P}$,
$$\frac{1}{|I_s|}\int_I\left(1+\frac{d(x, I_s)}{|I_s|}\right)^{-N}dx \leq 1$$
and so with Lemma 2.9 we have
$$\mathrm{size}_3^*(\mathbf{P}) \lesssim 1.$$
For f_1, we use that
$$\frac{1}{|I_s|}\int_{C_{k_1}}\left(1+\frac{d(x, I_s)}{|I_s|}\right)^{-N}dx \lesssim \left(1+\frac{d(C_{k_1}, I)}{|I|}\right)^{-(N-2)}$$
to conclude
$$\mathrm{size}_1^*(\mathbf{P}) \lesssim 2^{-k_1(N-2)}.$$
By the same argument for f_2, we have
$$\mathrm{size}_2^*(\mathbf{P}) \lesssim 2^{-k_2(N-2)}.$$

In this case, we can also estimate (with N another large enough integer)

$$\mathrm{size}_1^*(\mathbf{P})^{\theta_1}\,\mathrm{size}_2^*(\mathbf{P})^{\theta_2}\,\mathrm{size}_3^*(\mathbf{P})^{\theta_3} \leq 2^{-(k_1+k_2)N}.$$

With (2-6), we finally obtain

$$\Lambda_0^{k_1,k_2}(f_1,f_2,f_3) \lesssim \Big(\prod_{\beta\neq\alpha} |E_\beta|^{(1+\theta_\beta)/2-\epsilon\theta_\beta}|E_\alpha|^{(\theta_\alpha-1)/2+\epsilon(1-\theta_\alpha)}\Big)\tilde{C}(\Lambda_0^{k_1,k_2}),$$

where

$$\tilde{C}(\Lambda_0^{k_1,k_2}) := 2^{-N(k_1+k_2)\epsilon}. \tag{2-14}$$

Third step: end of the proof. For the trilinear form Λ_i^j, we have obtain a bound $C = \tilde{C}(\Lambda_i^j)$ such that for all functions $f_\beta \in F(E'_\beta)$ we have

$$\big|\Lambda_i^j(f_1,f_2,f_3)\big| \lesssim \tilde{C}(\Lambda_i^j)\Big(\prod_{\beta\neq\alpha} |E_\beta|^{(1+\theta_\beta)/2-\epsilon\theta_\beta}|E_\alpha|^{(\theta_\alpha-1)/2+\epsilon(1-\theta_\alpha)}\Big).$$

Let $(p_\beta)_\beta$ be the exponents of Theorem 2.8. Then we shall show that we can find $\theta_1,\theta_2,\theta_3 \in (0,1)$ and $\epsilon > 0$ such that for all $\beta \neq \alpha$,

$$\frac{1+\theta_\beta}{2} - \epsilon\theta_\beta = \frac{1}{p_\beta}, \quad \text{and} \quad \frac{\theta_\alpha - 1}{2} + \epsilon(1-\theta_\alpha) = \frac{1}{p_\alpha}.$$

Let $\gamma > 0$ be a real satisfying

$$\left|\frac{1}{2} - \frac{1}{p_\beta}\right| < \frac{1}{2+\gamma},$$

for all $\beta \neq \alpha$. This is possible because $1 < p_\beta < 2$ for $\beta \neq \alpha$. We begin to choose $\theta_\alpha \in (0,1)$ such that

$$1 > \theta_\alpha > \max\Big\{\theta_\alpha^0 := \frac{p_\alpha + (2+\gamma)}{p_\alpha}, 0\Big\},$$

and

$$\min\Big\{-\frac{1}{2+\gamma} = \frac{1}{p_\alpha(1-\theta_\alpha^0)}, \frac{1}{p_\alpha}\Big\} > \frac{1}{p_\alpha(1-\theta_\alpha)} > -\frac{1}{2}.$$

This is possible because p_α is negative and satisfies

$$\frac{1}{p_\alpha} > -\frac{1}{2}.$$

Then we get ϵ by

$$\epsilon := \frac{1}{2} + \frac{1}{p_\alpha(1-\theta_\alpha)} \in (0, \tfrac{1}{2}) \subset (0,1).$$

We now define θ_β for $\beta \neq \alpha$ by

$$\theta_\beta := \frac{\dfrac{1}{p_\beta} - \dfrac{1}{2}}{\dfrac{1}{2} - \epsilon}.$$

We have $1 < p_\beta < 2$ and $0 < \epsilon < \frac{1}{2}$, so $0 < \theta_\beta$ and

$$0 < \theta_\beta = \frac{\dfrac{1}{p_\beta} - \dfrac{1}{2}}{\dfrac{-1}{p_\alpha(1-\theta_\alpha)}} < \frac{\dfrac{1}{2+\gamma}}{\dfrac{1}{2+\gamma}} = 1.$$

Consequently, we have solved the system of equations for the exponents. With this choice, we obtain for all $f_1 \in F(E_1')$, $f_2 \in F(E_2')$, $f_3 \in F(E_3')$,

$$\Lambda_i^j(f_1, f_2, f_3) \lesssim \tilde{C}(\Lambda_i^j) \prod_{\beta=1}^{3} |E_i|^{1/p_\beta},$$

where $\tilde{C}(\Lambda_i^j)$ are defined in (2-13) and (2-14). So Λ_i^j is of restricted weak type and we have the following estimate for $C(\Lambda_i^j)$:

$$C(\Lambda_i^j) \lesssim \tilde{C}(\Lambda_i^j).$$

In addition the parameter N in (2-13) and (2-14) is as large as we want, and we have also obtained the desired estimates on $C(\Lambda_i^j)$. $\square$

By using the concept of "restricted weak type", we can have a "stronger" result than Theorem 1.1.

Theorem 2.10. *Let T and p, q, r be an operator and exponents of Theorem 1.1. Then for all $\delta \geq 1$, there exists a constant*

$$C = C(p, q, r, \delta)$$

(independent on the interval I) such that for all sets E_3 of finite measure, there exists a substantial subset $E_3' \subset E_3$ satisfying that for all functions $f \in \mathcal{S}(\mathbb{R})$, $g \in \mathcal{S}(\mathbb{R})$ and $h \in F(E_3')$,

$$|\langle T(f, g), h\mathbf{1}_I\rangle| \leq C\left(\sum_{k\geq 0} 2^{-k(1/p+\delta)} \|f\mathbf{1}_{2^k I}\|_p\right)\left(\sum_{k\geq 0} 2^{-k(1/q+\delta)} \|g\mathbf{1}_{2^k I}\|_q\right)|E_3|^{1/r'}.$$

When $r > 1$, this result is stronger than Theorem 1.1 but less practicable. We now prove it because it will be useful in the sequel.

Proof. The proof is almost the same as the previous one, so we shall only explain the modifications. We always study the trilinear form

$$\Lambda(f, g, h) := \langle T(f, g), h\mathbf{1}_I\rangle.$$

In page 6 we saw that the study of Λ can be reduced to the study of the model sum

$$\Lambda(f, g, h) = \sum_{s\in\mathbf{S}} |I_s|^{-1/2} \epsilon_s \langle\phi_{s_1}, f\rangle\langle\phi_{s_2}, g\rangle\langle\phi_{s_3}, h\mathbf{1}_I\rangle,$$

where $\mathbf{S}$ is a general collection of tritiles. Then we have decomposed this sum with (2-3) by

$$\Lambda(f, g, h) = \sum_{k_1,k_2\geq 0} \Lambda_0^{k_1,k_2}(f, g, h) + \sum_{\substack{k_1,k_2\geq 0\\ l\leq 0}} \Lambda_1^{k_1,k_2,l}(f, g, h).$$

By Theorem 2.4, we have shown that the trilinear forms Λ_i^j are of restricted weak type (p, q, r') and we have obtained estimates on their bounds. The construction of the substantial subset $E'_\alpha = E'_3$ does not depend on the trilinear form Λ_i^j, so we can deduce that our trilinear form Λ is always of restricted weak type. Also for measurable sets E_1, E_2, E_3 of finite measure, there exists a substantial subset $E'_3 \subset E_3$ such that for all functions $f \in F(E_1)$, $g \in F(E_2)$ and $h \in F(E'_3)$,

$$|\Lambda(f, g, h)| \lesssim |E_3|^{1/r'} \left(\sum_{k_1, k_2 \geq 0} 2^{-\delta'(k_1+k_2)} |E_1 \cap C_{k_1}|^{1/p} |E_2 \cap C_{k_2}|^{1/q} \right).$$

Here δ' is an exponent as large as we want. Over each corona, by using the real interpolation on the exponents p and q (so r is fixed), we obtain also the desired result. $\square$

Having obtained our main result for the x-independent symbols, we will extend our result for maximal operators and for x-dependent symbols in the next section.

3. More general bilinear operators

Let us name our "off-diagonal estimates" for convenience.

Definition 3.1. Let T be an operator (maybe non-bilinear) acting from $\mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R})$ into $\mathcal{S}'(\mathbb{R})$. For $p, q, r \in (0, \infty]$ exponents such that

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q},$$

we say that T satisfies *"off-diagonal estimates" at the scale L and at the order δ*, in short

$$T \in \mathcal{O}_{L,\delta}(L^p \times L^q, L^r),$$

if there exists a constant $C = C(p, q, r, L, \delta)$ such that for all functions $f, g \in \mathcal{S}(\mathbb{R})$ and all interval I of length $|I| = L$, we have

$$\|T(f, g)\|_{r,I} \leq C \left(\sum_{k \geq 0} 2^{-k(\delta+1/p)} \|f\|_{p,2^{k+1}I} \right) \left(\sum_{k \geq 0} 2^{-k(\delta+1/q)} \|g\|_{q,2^{k+1}I} \right).$$

Remark 3.2. Equivalently, an operator T satisfies "off-diagonal estimates" at the scale L and at the order δ if there exists a constant $C = C(p, q, r, L, \delta)$ such that for all functions $f, g \in \mathcal{S}(\mathbb{R})$ and all interval I of length $|I| = L$, we have

$$\|T(f, g)\|_{r,I} \leq C \left(\sum_{k \geq 0} 2^{-k(\delta+1/p)} \|f\|_{p,C_k(I)} \right) \left(\sum_{k \geq 0} 2^{-k(\delta+1/q)} \|g\|_{q,C_k(I)} \right).$$

This is a better way to describe the "off-diagonal decay" of an operator T and these properties can be described as in Corollary 1.2.

First we generalize the previous result for maximal operators.

"Off-diagonal estimates" for maximal bilinear operators.

Theorem 3.3. *Let Δ be a nondegenerate line in the frequency plane. Let $p, q \in (1, \infty]$ be exponents such that*

$$0 < \frac{1}{r} = \frac{1}{q} + \frac{1}{p} < \frac{3}{2}.$$

For all $\delta \geq 1$, $L > 0$, for all symbol σ supported in

$$\{(\alpha, \beta),\ d((\alpha, \beta), \Delta) \geq L^{-1}\}$$

satisfying for all $b, c \geq 0$,

$$\left|\partial_\alpha^b \partial_\beta^c \sigma(\alpha, \beta)\right| \lesssim |d((\alpha, \beta), \Delta)|^{-b-c}$$

and for all smooth function ϕ, which is equal to 1 around 0, the maximal bilinear operator

$$T_{max}(f, g)(x) := \sup_{r>0} \left| \int e^{ix(\alpha+\beta)} \widehat{f}(\alpha)\widehat{g}(\beta)\, \sigma(\alpha, \beta)\big(1 - \phi(r(\alpha - \beta))\big)\, d\alpha\, d\beta \right|$$

satisfies "off-diagonal estimates" at the scale L and at the order δ:

$$T_{max} \in \mathcal{O}_{L,\delta}(L^p \times L^q, L^r).$$

In addition the implicit constant can be uniformly bounded by $L > 0$.

Theorem 3.4. *For the same exponents, we have the same continuities for the maximal bilinear operator (at the scale L)*

$$M^L(f, g)(x) := \sup_{0<r\leq L} \frac{1}{r} \int_{|t|\leq r} |f(x-t)g(x+t)|\, dt.$$

Theorem 3.5. *Let K be a kernel on $\mathbb{R}$ satisfying Hörmander's conditions, then the maximal bilinear operator*

$$T^L_{max}(f, g)(x) := \sup_{0<\epsilon<r<L} \left| \int_{\epsilon\leq|y|\leq r} f(x-y)g(x+y)K(y)\, dy \right|$$

satisfies the same local estimates

$$T^L_{max} \in \mathcal{O}_{L,\delta}(L^p \times L^q, L^r)$$

for the exponents p, q, r as of Theorem 3.3.

Proof. The proof of these three theorems is a shake between the proof of our Theorem 1.1 and an additional maximality argument. The maximal truncation in the physical space (Theorems 3.4 and 3.5) is a little more complex than the maximal truncation in the frequency space (Theorem 3.3). So we deal with the last two theorems and just explain the modifications to prove them. The maximal version of the different arguments has been shown first by M. Lacey [2000] and then improved by C. Demeter, T. Tao and C. Thiele [2005]. In these articles, the authors study the behavior of the maximal averages (like in Theorem 3.4). [Demeter et al. 2005, Remark 1.6] specifies the similarity between the operators of Theorems 3.4 and 3.5. So in fact the previous three theorems are an illustration of the same ideas, and we will not detail them.

The reduction on page 6 is based on the decomposition of the bilinear operator by discrete models. For our maximal operators, the same reduction is shown in [Demeter et al. 2005, Theorem 4.4] and the important condition (2-2) for the tiles is always satisfied. Then the maximal version of Proposition

2.6 is given in [Demeter et al. 2005] too (there is a new factor in the different inequalities but it is not important). We have exactly the same version of Lemma 2.9 for maximal bilinear operators [Demeter et al. 2005, Proposition 6.2]. Using these technical modifications, we can prove Theorem 1.1 and obtain its maximal versions: Theorems 3.3, 3.4 and 3.5. □

Proof of Theorem 1.1 for x-dependent symbols. In this subsection, we prove the "off-diagonal estimates" of Theorem 1.1 in the case where the symbol σ depends on the spatial variable x and also we complete the proof of our main result.

Theorem 3.6. *Let Δ be a nondegenerate line of the frequency space. Let $\sigma \in C^\infty(\mathbb{R}^3)$ be a symbol satisfying for all $a, b, c \geq 0$,*

$$\left|\partial_x^a \partial_\alpha^b \partial_\beta^c \sigma(x, \alpha, \beta)\right| \lesssim \big(1 + d((\alpha, \beta), \Delta)\big)^{-b-c}.$$

Then the bilinear operator T_σ (defined on $\mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R})$ by (1-2)) verifies

$$T_\sigma \in \mathbb{O}_{1,\delta}(L^p \times L^q, L^r)$$

for any $\delta \geq 0$ and any exponents p, q, r such that

$$0 < \frac{1}{r} = \frac{1}{p} + \frac{1}{q} < \frac{3}{2} \quad \textit{and} \quad 1 < p, q \leq \infty.$$

Our assumptions for the symbol correspond to the class $BS^0_{1,0;\theta}$ of [Bényi et al. 2006], where the angle $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \setminus \{0, -\frac{\pi}{4}\}$ is given by the line

$$\Delta := \{(\alpha, \beta),\ \beta = \alpha \tan\theta\}.$$

For convenience, we will deal in the proof only with the case $\theta = \frac{\pi}{4}$. The important fact is that the singular quantity $(\beta - \alpha \tan\theta)$ does not correspond to the quantity $\alpha + \beta$, which appears in the exponential term of (1-2). The limit and particular case $\theta = -\frac{\pi}{4}$ is studied in [Bényi et al. 2006].

Proof. The proof is quite technical. We will also assume that $r \geq 1$ (which allows us to simplify a few arguments). Then we will explain in Remark 3.7 how to modify the proof to obtain the same result when $r < 1$.

So we fix an interval I of length $|I| = 1$. We use a decomposition of the symbol σ. Let Φ be a smooth function on $\mathbb{R}$ such that if $|x| \leq 1$ then

$$\Phi(x) = 1 \quad \text{and} \quad \operatorname{supp}(\Phi) \subset [-2, 2].$$

We also have

$$\begin{aligned}\sigma(x, \alpha, \beta) &= \sigma(x, \alpha, \beta)(1 - \Phi(\alpha - \beta)) + \sigma(x, \alpha, \beta)\Phi(\alpha, \beta) \\ &:= \sigma^\infty(x, \alpha, \beta) + \sigma^0(x, \alpha, \beta).\end{aligned}$$

(i) The case of the symbol σ^∞.

We have an operator associated to this symbol

$$T^\infty(f, g)(x) := \int_{\mathbb{R}^2} e^{ix(\alpha+\beta)} \widehat{f}(\alpha)\widehat{g}(\beta)\,\sigma(x, \alpha, \beta)\,(1 - \Phi(\alpha - \beta))\,d\alpha\,d\beta,$$

which can been written as

$$T^\infty(f,g)(x) = U_x(f,g)(x),$$

with U defined by

$$U_y(f,g)(x) := \int_{\mathbb{R}^2} e^{ix(\alpha+\beta)} \widehat{f}(\alpha)\widehat{g}(\beta)\sigma(y,\alpha,\beta)(1-\Phi(\alpha-\beta))\, d\alpha\, d\beta.$$

By using the Sobolev embedding

$$W^{1,r}(I) \hookrightarrow L^\infty(I)$$

because $r \geq 1$, we get

$$|T^\infty(f,g)(x)| \leq \|U_y(f,g)(x)\|_{\infty, y\in I} \lesssim \sum_{k=0}^{1} \|\partial_y^k U_y(f,g)(x)\mathbf{1}_I(y)\|_{r,dy}.$$

for all $x \in I$. Then by integrating for $x \in I$ and using Fubini's Theorem, we obtain

$$\|T^\infty(f,g)\|_{r,I} \lesssim \sum_{k=0}^{1} \| \|\partial_y^k U_y(f,g)\|_{r,I}\|_{r,I,dy}.$$

We can fix $k \in \{0,1\}$ and $y \in I$. Then we have

$$\|\partial_y^k U_y(f,g)\|_{r,dx} \lesssim \|V(f,g)\|_{r,I},$$

where V is the bilinear operator defined by

$$V(f,g)(x) := \int_{\mathbb{R}^2} e^{ix(\alpha+\beta)} \widehat{f}(\alpha)\widehat{g}(\beta)\partial_y^k\sigma(y,\alpha,\beta)(1-\Phi(\alpha-\beta))\, d\alpha\, d\beta.$$

So $V = T_\tau$ is the bilinear operator associated to the x-independent symbol

$$\tau(\alpha,\beta) := \partial_y^k \sigma(y,\alpha,\beta)(1-\Phi(\alpha-\beta)).$$

From the assumptions about σ, the symbol τ satisfies $\left|\partial_\alpha^b \partial_\beta^c \tau(\alpha,\beta)\right| \lesssim |\alpha-\beta|^{-n-p}$ for all $b, c \geq 0$. In addition, τ is supported in the domain $\{(\alpha,\beta), |\alpha-\beta| \geq 1\}$. We can also apply Theorem 1.1 proved in Section 2 for x-independent symbol. For all $\delta \geq 1$, we have an "off-diagonal estimate" at the scale 1,

$$\|V(f,g)\|_{r,I} \lesssim \left(\sum_{k_1\geq 0} 2^{-k_1(1/p+\delta)}\|f\|_{p,2^{k_1}I}\right)\left(\sum_{k_2\geq 0} 2^{-k_2(1/q+\delta)}\|g\|_{q,2^{k_2}I}\right).$$

All theses estimates are uniform with respect to $k \in \{0,1\}$ and $y \in I$, so we get

$$\|T^\infty(f,g)\|_{r,I} \lesssim \left(\sum_{k_1\geq 0} 2^{-k_1(1/p+\delta)}\|f\|_{p,2^{k_1}I}\right)\left(\sum_{k_2\geq 0} 2^{-k_2(1/q+\delta)}\|g\|_{q,2^{k_2}I}\right). \tag{3-1}$$

So we have shown the desired estimates for this first term.

(ii) The case of the symbol σ^0. The associated operator is given by

$$T^0(f,g)(x) := \int_{\mathbb{R}^2} e^{ix(\alpha+\beta)} \widehat{f}(\alpha)\widehat{g}(\beta)\sigma(x,\alpha,\beta)\Phi(\alpha,\beta)\, d\alpha\, d\beta.$$

We use the same arguments as for the first point. So we have to study the operator V defined by

$$V(f,g)(x) := \int_{\mathbb{R}^2} e^{ix(\alpha+\beta)} \widehat{f}(\alpha)\widehat{g}(\beta)\partial_y^k \sigma(y,\alpha,\beta)\Phi(\alpha-\beta)\, d\alpha\, d\beta.$$

The parameters $k \in \{0, 1\}$ and $y \in I$ are fixed. The symbol associated to this operator is supported on

$$\{(\alpha,\beta),\ |\alpha-\beta| \leq 2\}.$$

That is why we use modulations to move this support:

$$\begin{aligned} V(f,g)(x) &= \int_{\mathbb{R}^2} e^{ix(\alpha+\beta)} \widehat{f}(\alpha+3)\widehat{g}(\beta-3)\partial_y^k\sigma(y,\alpha+3,\beta-3)\Phi(\alpha-\beta+6)\, d\alpha\, d\beta \\ &= \int_{\mathbb{R}^2} e^{ix(\alpha+\beta)} \widehat{e^{3i\cdot} f}(\alpha)\widehat{e^{-3i\cdot} g}(\beta)\partial_y^k\sigma(y,\alpha+3,\beta-3)\Phi(\alpha-\beta+6)\, d\alpha\, d\beta. \end{aligned}$$

Also V is now the bilinear operator, applied to the modulated functions $e^{3i\cdot}f$ and $e^{-3i\cdot}g$, whose (x-independent) symbol

$$\tau(\alpha,\beta) := \partial_y^k\sigma(y,\alpha+3,\beta-3)\Phi(\alpha-\beta+6)$$

is supported on

$$\{(\alpha,\beta),\ |\alpha-\beta+6| \leq 2\} \subset \{(\alpha,\beta),\ 1 \leq |\alpha-\beta| \leq 8\}$$

and satisfies for all $b, c \geq 0$,

$$\left|\partial_\alpha^b\partial_\beta^c\tau(\alpha,\beta)\right| \lesssim \max_{0\leq j\leq b}\max_{0\leq i\leq c}(1+|\alpha-\beta+6|)^{-i-j}\mathbf{1}_{1\leq|\alpha-\beta|\leq 8} \lesssim \mathbf{1}_{1\leq|\alpha-\beta|\leq 8} \lesssim \mathbf{1}_{1\leq|\alpha-\beta|\leq 8}|\alpha-\beta|^{-b-c}.$$

Also we can use Theorem 1.1 (proved in Section 2 for x-independent symbol) again and we obtain

$$\|V(f,g)\|_{r,I} \lesssim \left(\sum_{k_1\geq 0} 2^{-k_1(1/p+\delta)}\|f\|_{p,I}\right)\left(\sum_{k_2\geq 0} 2^{-k_2(1/q+\delta)}\|g\|_{q,I}\right). \qquad \square$$

Remark 3.7. We want to explain here how to modify the previous proof when $r < 1$. When we study bilinear operators with $r < 1$, we have to use the associated trilinear form and the concept of "restricted weak type" (see Definition 2.7). These two arguments allow us to get around the lack of the triangular inequality in the space L^r. Let

$$\Lambda(f,g,h) := \langle T(f,g), h\rangle.$$

We have

$$\Lambda(f,g,h) = \int_{\mathbb{R}}\int_{\mathbb{R}^2} e^{ix(\alpha+\beta)}\sigma(x,\alpha,\beta)\widehat{f}(\alpha)\widehat{g}(\beta)h(x)\, d\alpha\, d\beta dx.$$

We use the same decomposition of σ, getting the trilinear forms Λ^∞ and Λ^0. Let us study first Λ^∞ and fix an interval I of length $|I| = 1$. We take a function $h \in \mathcal{S}(\mathbb{R})$, which is supported on I. We use again the Sobolev embedding $W^{1,1}(I) \hookrightarrow L^\infty(I)$. By writing

$$|\Lambda^\infty(f,g,h)| \leq \int_{\mathbb{R}} \|U_y(f,g)(x)\mathbf{1}_I(y)\|_{\infty,I,dy}|h(x)|\mathbf{1}_I(x)\, dx,$$

we can also obtain

$$|\Lambda^{\infty}(f,g,h)| \lesssim \int_I \int_I |U_y(f,g)(x)|\,|h(x)|\,dx\,dy + \int_I \int_I |\partial_y U_y(f,g)(x)|\,|h(x)|\,dx\,dy.$$

Then when $y \in I$ and $k \in \{0,1\}$ are fixed, we find again the quantities

$$\int_I \left|\partial_y^k U_y(f,g)(x)\right| |h(x)|\,dx.$$

Now the bilinear operator $\partial_y^k U_y$ is associated to an x-independent symbol, which verifies the good assumptions. We can also use Theorem 2.10 in order to obtain the wished estimates (3-1) in a "restricted weak type sense" for the exponent r. We produce the same modifications to study Λ^0. By noticing that the way to construct the substantial subset (in the definition of restricted weak type) does not depend on the trilinear form, we can deduce that the trilinear form Λ satisfies (3-1) in a "restricted weak type sense" too. Then we use interpolation on the exponent r, to obtain exactly (3-1), which allows us to conclude.

4. Continuities for bilinear operators satisfying "off-diagonal estimates"

Recall that in the linear case, by using the maximal sharp function, we can prove weighted continuities for linear operator with the Muckenhoupt weights. In the bilinear case, we do not have a good substitute to the maximal sharp function. That is why we shall use the previous "off-diagonal estimates" to obtain weighted global continuities on Lebesgue spaces and in particular to prove Theorem 1.3.

First we want to give an application of these "off-diagonal estimates". Recall that in the previous sections, we have proved that our bilinear operators (and maximal bilinear operators) satisfy these "off-diagonal estimates" at any order. The time-frequency analysis does not work for functions in the L^∞ space. So we do not know if our operators T are bounded from $L^\infty \times L^\infty$ in BMO. However these local estimates give a weak result about the behavior of $T(f,g)$ when the two functions f and g belong to L^∞.

Proposition 4.1. *Let f, g be two functions of $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and fix $r \in (1,\infty)$. If there exist $L > 0$, $\delta \geq 1$ and $p, q > 1$ such that an operator*

$$T \in \mathbb{O}_{\delta,L}(L^p \times L^q, L^r),$$

then we have

$$\limsup_{|I|\to\infty} \left(\frac{1}{|I|}\int_I |T(f,g)|^r\right)^{1/r} \lesssim \|f\|_\infty \|g\|_\infty.$$

Here we take the limit when I is an interval with $|I| \to \infty$ and the implicit constant does not depend on the two functions f and g and on the parameter L.

Proof. We set

$$I_i := [iL, (i+1)L[$$

for all $i \in \mathbb{Z}$. Then for I with $|I| \gg L$, we get

$$\int_I |T(f,g)|^r \leq \sum_{\substack{i\in\mathbb{Z} \\ I_i \cap I \neq \varnothing}} \int_{I_i} |T(f,g)|^r.$$

However, the number of indices i which appears in the sum is bounded by $|I|/L$, so by using the local estimate we get

$$\int_I |T(f,g)|^r \lesssim \sum_{\substack{i\in\mathbb{Z}\\ I_i\cap I\neq\varnothing}} \frac{L}{|I_i|}\int_{I_i} |T(f,g)|^r \lesssim \sum_{\substack{i\in\mathbb{Z}\\ I_i\cap I\neq\varnothing}} L\|f\|_\infty^r\|g\|_\infty^r \lesssim |I|\|f\|_\infty^r\|g\|_\infty^r.$$

The second inequality is due to the fact that

$$|I_i|^{1/r}\|T(f,g)\|_{r,I_i} \lesssim \inf_{x\in I_i} M_{HL}(f)(x) \inf_{x\in I_i} M_{HL}(g)(x) \lesssim \|f\|_\infty\|g\|_\infty.$$

So we obtain

$$\Big(\frac{1}{|I|}\int_I |T_{max}(f,g)|^r\Big)^{1/r} \lesssim \|f\|_\infty\|g\|_\infty$$

uniformly with L for $|I|$ large enough. $\square$

Let us now define our weights.

Definition 4.2. Let $\theta > 0$ and $l > 0$ be fixed. We set that a nonnegative function ω belongs to the *class* $\mathbb{P}_\theta(l)$ if there exists a constant C such that for all interval I of length $|I| = l$ and for all integer $k \geq 0$, we have

$$2^{-k\theta}\sup_{x\in I}\omega(x) \leq C\inf_{2^kI}\omega(x). \tag{4-1}$$

We claim that a function $\omega \in \mathbb{P}_\theta(l)$ is likely to be a polynomial function whose degree is less than θ and is almost constant at the scale l. We show in the next example that these classes are not empty.

Example 4.3. For all $\theta > 0$ and $\alpha \in [0,\theta)$, the functions

$$x \mapsto 1, \qquad x \mapsto (1+|x|)^\alpha \qquad \text{and} \qquad x \mapsto (1+|x|)^{-\alpha}$$

belong to the class $\mathbb{P}_\theta(1)$. The proof is easy and is left to the reader.

Remark 4.4. In fact, it is easy to prove that a weight ω belongs to the class $\mathbb{P}_\theta(l)$ if and only if there exists a constant C such that for all $x, y \in \mathbb{R}$,

$$\omega(x) \leq C\Big(1+\frac{|x-y|}{l}\Big)^\theta\omega(y).$$

We cannot compare these weights with the Muckenhoupt weights, because for $\omega \in \mathbb{P}_\theta(l)$ we have information only at the scale l.

Theorem 4.5. *Let T be a bilinear operator and $p, q, r \in (0,\infty)$ be exponents satisfying*

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q} \qquad \textit{and} \qquad 1 \leq p, q.$$

For $\delta > 0$ and $l > 0$, if T satisfies "off-diagonal estimates" at the order δ and at the scale l, then for all $\omega \in \mathbb{P}_\theta(l)$ with $0 \leq \theta < \delta\max\{r, 1\}$, the operator T is continuous from $L^p(\omega) \times L^q(\omega)$ into $L^r(\omega)$.

Proof. To check this, we recall that for all interval I of length $|I| = l$,

$$\left(\int_I |T(f,g)|^r\right)^{1/r} \lesssim \left(\sum_{k\geq 0} 2^{-k(1/p+\delta)} \|f\|_{p,2^kI}\right)\left(\sum_{k\geq 0} 2^{-k(1/q+\delta)} \|g\|_{q,2^kI}\right). \tag{4-2}$$

Then we decompose the whole space $\mathbb{R}$ with the disjoint intervals I_i defined by

$$I_i = [il, (i+1)l]$$

for $i \in \mathbb{Z}$. So we have

$$\|T(f,g)\|_{r,wdx} = \| \|T(f,g)\|_{r,wdx,I_i} \|_{r,i\in\mathbb{Z}}.$$

Let $i \in \mathbb{Z}$ be fixed. We use (4-1) and (4-2) to obtain

$$\begin{aligned}\|T(f,g)\|_{r,wdx,I_i} &\leq \|w\|_{\infty,I_i}^{1/r} \|T(f,g)\|_{r,I_i} \\ &\lesssim \|w\|_{\infty,I_i}^{1/r} \left(\sum_{k\geq 0} 2^{-k(1/p+\delta)} \|f\|_{p,2^kI_i}\right)\left(\sum_{k\geq 0} 2^{-k(1/q+\delta)} \|g\|_{q,2^kI_i}\right).\end{aligned}$$

We estimate the first sum with

$$\begin{aligned}\|w\|_{\infty,I_i}^{1/p} \left(\sum_{k\geq 0} 2^{-k(1/p+\delta)} \|f\|_{p,2^kI_i}\right) &\lesssim \sum_{k\geq 0} 2^{-k(1/p+\delta)} \|w\|_{\infty,I_i}^{1/p} \|f\|_{p,2^kI_i} \\ &\lesssim \sum_{k\geq 0} 2^{-k(1/p+\delta)} 2^{k\theta/p} \inf_{2^kI_i} \omega^{1/p} \|f\|_{p,2^kI_i} \\ &\lesssim \sum_{k\geq 0} 2^{-k(1/p+\delta-\theta/p)} \|f\|_{p,wdx,2^kI_i}.\end{aligned}$$

The second term is studied by the same way. By summing over $i \in \mathbb{Z}$, we get

$$\|T(f,g)\|_{r,wdx} \lesssim \left\| \left(\sum_{k\geq 0} 2^{-k(1/p+\delta-\theta/p)} \|f\|_{p,wdx,2^kI_i}\right)\left(\sum_{k\geq 0} 2^{-k(1/q+\delta-\theta/q)} \|g\|_{q,wdx,2^kI_i}\right) \right\|_{r,i\in\mathbb{Z}}.$$

With the help of Hölder's and Minkowski's inequalities, we obtain

$$\|T(f,g)\|_{r,wdx} \lesssim \left(\sum_{k\geq 0} 2^{-k(1/p+\delta-\theta/p)} \| \|f\|_{p,wdx,2^kI_i} \|_{p,i\in\mathbb{Z}}\right)\left(\sum_{k\geq 0} 2^{-k(1/q+\delta-\theta/q)} \| \|g\|_{q,wdx,2^kI_i} \|_{q,i\in\mathbb{Z}}\right).$$

However the collection of sets $(2^kI_i)_i$ is a 2^k-covering, so

$$\|T(f,g)\|_{r,wdx} \lesssim \left(\sum_{k\geq 0} 2^{-k(\delta-\theta/p)} \|f\|_{p,wdx}\right)\left(\sum_{k\geq 0} 2^{-k(\delta-\theta/q)} \|g\|_{q,wdx}\right).$$

Then we conclude with the fact that $p, q > 1$ and hence

$$\max\left\{\frac{\theta}{p}, \frac{\theta}{q}\right\} \leq \begin{cases} \dfrac{\theta}{r} < \delta & \text{if } r \geq 1, \\ \theta < \delta & \text{if } r \leq 1. \end{cases}$$

□

Remark 4.6. Since it is obvious that the weight $\omega(x) = 1$ belongs to the class $\mathbb{P}_\theta(L)$, we have also proved that the operators of Theorem 1.1 and the maximal operators of Theorems 3.3, 3.4 and 3.5 are bounded in classical Lebesgue spaces.

Definition 4.7. Let ω be a weight on $\mathbb{R}$. For all $m \geq 0$ and $p \in (1,\infty)$, we set $W^{m,p}(\omega)$ for the Sobolev space on $\mathbb{R}$ with the weight ω, defined as the set of distributions $f \in \mathcal{S}'(\mathbb{R})$ such that

$$J_m(f) \in L^p(\omega),$$

where $J_m := (\mathrm{Id} - \Delta)^{m/2}$.

We complete this result with a proposition in Sobolev spaces:

Proposition 4.8. *Let Δ be a nondegenerate line, ω be a weight in $\bigcup_{\theta\geq 0} \mathbb{P}_\theta(1)$ and $\sigma \in C^\infty(\mathbb{R}^3)$ be a symbol satisfying*

$$\left|\partial_x^a \partial_\alpha^b \partial_\beta^c \sigma(x,\alpha,\beta)\right| \lesssim \left(1 + d((\alpha,\beta),\Delta)\right)^{-b-c},$$

for all $a, b, c \geq 0$. Let p, q and r be exponents satisfying

$$0 < \frac{1}{r} = \frac{1}{p} + \frac{1}{q} < \frac{3}{2} \quad \textit{and} \quad 1 < p, q < \infty.$$

Then the bilinear operator T_σ (defined on $\mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R})$ by (1-2)) satisfies

$$\|D^{(n)} T_\sigma(f,g)\|_{L^r(\omega)} \lesssim \sum_{\substack{0\leq i,j\leq n \\ i+j\leq n}} \|D^{(i)} f\|_{L^p(\omega)} \|D^{(j)} g\|_{L^q(\omega)}, \tag{4-3}$$

for all integer $n \geq 0$ and for all functions $f, g \in \mathcal{S}(\mathbb{R})$. Here we write $D^{(i)}$ for the differentiation operator of order i. Also T_σ is continuous from $W^{m,p}(\omega) \times W^{m,q}(\omega)$ into $W^{m,r}(\omega)$ for all real $m \geq 0$.

Proof. Let us begin to prove (4-3). The two functions f and g are smooth so we can differentiate the integral defining $T_\sigma(f,g)$. It is also easy to check that

$$D^{(1)} T_\sigma(f,g) = T_\sigma(D^{(1)} f, g) + T_\sigma(f, D^{(1)} g) + T_{\partial_x \sigma}(f,g).$$

Then for higher orders, we get

$$D^{(n)} T_\sigma(f,g) = \sum_{\substack{0\leq i,j,k\leq n \\ i+j+k=n}} T_{\partial_x^k \sigma}(D^{(i)} f, D^{(j)} g).$$

By using the previous Theorems 1.1 and 4.5, we obtain (4-3). We can also deduce a weaker estimate

$$\|D^{(n)} T_\sigma(f,g)\|_{r,\omega} \lesssim \|f\|_{W^{n,p}(\omega)} \|g\|_{W^{n,q}(\omega)},$$

for all $f, g \in \mathcal{S}(\mathbb{R})$. By density (see Lemma 4.9), the operator T_σ can be continuously extended from $W^{n,p}(\omega) \times W^{n,q}(\omega)$ into $W^{n,r}(\omega)$. Then we will use interpolation to extend this result when n is not an integer. The exponents p, q and r are fixed and we study the bilinear operator T_σ. We have shown that T_σ is continuous from $W^{n,p}(\omega) \times W^{n,q}(\omega)$ into $W^{n,r}(\omega)$, for all integer n. By using bilinear interpolation (with Lemma 4.9) on n, we finish the proof. (The theory of multilinear interpolation is studied in [Lions and Peetre 1964, Chapter 4] for the real case and in [Bergh and Löfström 1976, Theorem 4.4.1] for the complex case.) □

Lemma 4.9. *For all weight*

$$\omega \in \bigcup_{\theta \geq 0} \mathbb{P}_\theta(1),$$

all exponent $p \in (1, \infty)$ *and all real* $s \geq 0$, *the space* $\mathcal{S}(\mathbb{R})$ *is a dense subspace in* $W^{s,p}(\omega)$. *In addition, the collection of Sobolev spaces* $(W^{s,p}(\omega))_{s\geq 0}$ *form an interpolation scale.*

Proof. Let ω be a fixed weight in $\bigcup_{\theta\geq 0} \mathbb{P}_\theta(1)$. We have seen in Remark 4.4 that ω has a polynomial growth. Since $J_s(\mathcal{S}(\mathbb{R})) = \mathcal{S}(\mathbb{R})$, we have the inclusion $\mathcal{S}(\mathbb{R}) \subset W^{s,p}(\omega)$. We recall that

$$J_s := (\mathrm{Id} - \Delta)^{s/2}.$$

In addition, we have that

$$L^p(\omega) \subset \mathcal{S}'(\mathbb{R}),$$

so we can compute the operator J_{-s} on the space $L^p(\omega)$. We finally obtain that J_s is an automorphism from $W^{s,p}(\omega)$ to $L^p(\omega)$ and an isomorphism on $\mathcal{S}(\mathbb{R})$. As $\mathcal{S}(\mathbb{R})$ is dense in $L^p(\omega)$, we get the density of $\mathcal{S}(\mathbb{R})$ into the Sobolev space $W^{s,p}(\omega)$.

For the interpolation claim, we omit the details. The classical proof for complex interpolation with $\omega = 1$ can easily be extended to the general case. □

Remark 4.10. From the fact that the weight $\omega(x) = 1$ belongs to the class $\mathbb{P}_\theta(1)$, we have also proved that the operators of Theorem 1.3 satisfy an Hölder's inequality in Sobolev spaces.

Remark 4.11. Also with the notation of [Bényi et al. 2006], we have proved continuities for all operators associated to symbols $\sigma \in BS^0_{1,0;\theta}$. In addition, we have described the action of these operators on Sobolev spaces. This is an interesting improvement of the last article and it incites us to obtain new results in order to continue the construction of a bilinear pseudo-differential calculus. We will do it in a next paper [Bernicot 2008] by introducing new larger symbolic classes of bilinear symbols of order (m_1, m_2) and studying rules of a bilinear symbolic calculus.

About continuities in Lebesgue spaces, a question is still open: What about the classes $BS^0_{\rho,\delta;\theta}$ (defined in [Bényi et al. 2006])?

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