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As a first step toward developing a theory of noncommutative nonlinear elliptic partial differential equations, we analyze noncommutative analogues of Laplace's equation and its variants (some of them nonlinear) over noncommutative tori. Along the way we prove noncommutative analogues of many results in classical analysis, such as Wiener's Theorem on functions with absolutely convergent Fourier series, and standard existence and nonexistence theorems on elliptic functions. We show that many classical methods, including the maximum principle, the direct method of the calculus of variations, and the use of the Leray–Schauder Theorem, have analogues in the noncommutative setting.

1. Introduction

Gelfand's Theorem shows that $X \rightsquigarrow C_0(X)$ sets a contravariant equivalence of categories from the category of locally compact (Hausdorff) spaces and proper maps to the category of commutative C^*-algebras and $*$-homomorphisms. This observation is the key to the whole subject of noncommutative geometry, which is based on the following dictionary:

Classical		Noncommutative
locally compact space	$\leftrightarrow$	C^*-algebra
compact space	$\leftrightarrow$	unital C^*-algebra
vector bundle	$\leftrightarrow$	finitely generated projective module
smooth manifold	$\leftrightarrow$	C^*-algebra with smooth subalgebra
real-valued function	$\leftrightarrow$	self-adjoint element
partial derivative	$\leftrightarrow$	unbounded derivation
integral	$\leftrightarrow$	tracial state

The object of this paper is to begin to use this dictionary to set up a noncommutative theory of elliptic partial differential equations, both linear and nonlinear, along with corresponding aspects of the calculus of variations. Since the theory is still in its infancy, we begin with the very simplest case: Laplace's equation and PDEs closely connected to it, and concentrate on the simplest nontrivial example of a noncommutative manifold, the irrational rotation algebra (or noncommutative 2-torus) A_θ, for $\theta \in \mathbb{R} \smallsetminus \mathbb{Q}$.

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A definition of elliptic partial differential operators, along with the study of one example associated with the irrational rotation algebra, was given in Connes' fundamental paper [1980], but there the emphasis was on pseudodifferential calculus and index theory. Here we focus on other things: variational methods, the maximum principle, an analogue of Wiener's Theorem, tools for treating *nonlinear* equations, the beginnings of a theory of harmonic unitaries, and some aspects of noncommutative complex analysis.

What is the motivation for a noncommutative theory of elliptic PDE? For the most part, it comes from physics. Many of the classical elliptic PDEs arise from variational problems in Riemannian geometry, and are also the field equations of physical theories. But the *uncertainty principle* forces quantum observables to be noncommutative. There is also increasing evidence, as in [Connes and Lott 1990; Chamseddine and Connes 1997; Connes et al. 1998; Seiberg and Witten 1999; Mathai and Rosenberg 2005; 2006], that quantum field theories should allow for the possibility of noncommutative space-times. *Noncommutative sigma-models*, for which the very earliest and simplest investigations are in [Dąbrowski et al. 2000; 2003], will require the noncommutative harmonic map equation, which generalizes the Laplace equation studied in this paper.

We use as our starting point the noncommutative differential geometry of Alain Connes [1980]. This theory only works well with highly symmetric noncommutative spaces, as the smooth elements are taken to be the C^∞ vectors for an action of a Lie group on a C^*-algebra, but this theory is well adapted to the case of the irrational rotation algebra, which carries an ergodic gauge action of the 2-torus $\mathbb{T}^2$.

The outline of this paper is as follows. We begin in Section 2 with the basic properties of the Laplacian on A_θ. Included are analogues of Wiener's theorem (Theorem 2.8) and the maximum principle (Proposition 2.9). In Section 3, we take up the basic properties of Sobolev spaces on A_θ, which are needed for a deeper analysis of some aspects of noncommutative PDEs. We should point out that some of the material of this section has already appeared in [Polishchuk 2006, §3] and in [Luef 2006]. The heart of this paper is contained in Sections 4 and 5, which begin to develop a theory of *nonlinear* elliptic partial differential equations, using methods analogous to those traditional in the theory of nonlinear elliptic PDE. Finally, Section 6 deals with noncommutative complex analysis.

We should mention that another example of noncommutative elliptic PDE and an associated variational problem on noncommutative tori, namely, noncommutative Yang–Mills theory, has already been studied by Connes and Rieffel [Connes and Rieffel 1987; Rieffel 1990]. Furthermore, Theorem 2.8 was previously proved by Gröchenig and Leinert [Gröchenig and Leinert 2004] by another method, and variations on the Gröchenig–Leinert work can be found in [Luef 2006]. In their paper, Gröchenig and Leinert point out some applications to harmonic analysis and wavelet theory, which go off in a somewhat different direction than the applications to mathematical physics which we envisage, though obviously there is some overlap between the two.

2. The linear Laplacian

We will be studying the C^*-algebra A_θ generated by two unitaries U, V satisfying

$$UV = e^{2\pi i\theta} VU.$$

A_θ is simple with unique trace τ if $\theta \in \mathbb{R} \setminus \mathbb{Q}$. (See for example [Rieffel 1981] for a review of the basic facts about A_θ.) The torus $G = \mathbb{T}^2$ acts by

$$\alpha_{(z_1,z_2)}U = z_1 U, \quad \alpha_{(z_1,z_2)}V = z_2 V, \quad |z_1| = |z_2| = 1.$$

The space of C^∞ vectors for the action α is the *smooth irrational rotation algebra*

$$A_\theta^\infty = \left\{ \sum_{m,n} c_{m,n} U^m V^n : c_{m,n} \text{ rapidly decreasing} \right\}.$$

This should be viewed as a noncommutative deformation of the algebra $C^\infty(\mathbb{T}^2)$ of smooth functions on an ordinary 2-torus, and the decomposition of an element of this algebra in terms of multiples of $U^m V^n$ should be viewed as a sort of noncommutative Fourier series decomposition, with $c_{m,n}$ as a sort of Fourier coefficient. For $a \in A_\theta$ but not necessarily in A_θ^∞, the Fourier coefficients $c_{m,n}$ are well defined and satisfy $|c_{m,n}| \leq \|a\|$, since $c_{m,n} = \tau(V^{-n}U^{-m}a)$, but the Fourier series expansion of a is only a formal expansion, and need not converge in the topology of A_θ, just as one has functions in $C(\mathbb{T}^2)$ whose Fourier series do not converge absolutely or even pointwise.

We denote by δ_1 and δ_2 the infinitesimal generators of the actions of the two $\mathbb{T}$ factors in $\mathbb{T}^2$ under α. These are unbounded derivations on A_θ, and map A_θ^∞ to itself. They are given by

$$\delta_1(U) = 2\pi i U, \quad \delta_2(V) = 2\pi i V, \quad \delta_2(U) = \delta_1(V) = 0.$$

These derivations δ_j obviously commute with the adjoint operation $*$, and play the roles of the partial derivatives $\partial/\partial x_j$ in classical analysis on the 2-torus. Since the action α of $\mathbb{T}^2$ preserves the tracial state τ, $\tau \circ \delta_j = 0$, $j = 1, 2$. This fact is the basis for the following Lemma, which we will use many times in the future.

Lemma 2.1 (Integration by parts). *If $a, b \in A_\theta^\infty$, then $\tau(\delta_j(a)b) = -\tau(\delta_j(b)a)$, $j = 1, 2$.*

Proof. We have $0 = \tau(\delta_j(ab)) = \tau(\delta_j(a)b) + \tau(a\delta_j(b))$. □

Definition 2.2. In analogy with the usual notation in analysis, we let

$$\Delta = \delta_1^2 + \delta_2^2.$$

This should be viewed as a noncommutative elliptic partial differential operator. (The notion of ellipticity was defined rigorously in [Connes 1980, p. 602].) Clearly, Δ is a "negative" operator, and its spectrum consists of the numbers $-4\pi^2(m^2+n^2)$, $m, n \in \mathbb{Z}$, with eigenfunctions $U^m V^n$. Via the noncommutative Fourier expansion discussed earlier, the pair $(A_\theta^\infty, \Delta)$ is isomorphic to $C^\infty(\mathbb{T}^2)$ with the usual Laplacian Δ, provided one looks just at the linear structure and forgets the noncommutativity of the multiplication. (This was already observed in [Connes 1980, p. 602].)

Proposition 2.3. *For any $\lambda > 0$ (or not of the form $-4\pi^2 n$ with $n \in \mathbb{N}$), the map $-\Delta + \lambda\colon A_\theta^\infty \to A_\theta^\infty$ is bijective.*

Proof. We have

$$(-\Delta + \lambda)\left(\sum_{m,n} c_{m,n} U^m V^n \right) = \sum_{m,n} \left(4\pi^2(m^2+n^2) + \lambda\right) c_{m,n} U^m V^n.$$

It is immediate that $-\Delta+\lambda$ has no kernel and has an inverse given by the formula

$$\sum_{m,n} c_{m,n} U^m V^n \mapsto \sum_{m,n} \frac{1}{4\pi^2(m^2+n^2)+\lambda} c_{m,n} U^m V^n,$$

since if $c_{m,n}$ is rapidly decreasing, so are the coefficients on the right. □

It is also easy to characterize the image of Δ.

Proposition 2.4. *The image of* $\Delta\colon A_\theta^\infty \to A_\theta^\infty$ *is precisely* $A_\theta^\infty \cap \ker\tau$, *the smooth elements with zero trace.*

Proof. We have $\Delta(\sum_{m,n} c_{m,n}U^mV^n) = -4\pi^2\sum_{m,n}(m^2+n^2)c_{m,n}U^mV^n$, and the factor (m^2+n^2) kills the term with $m=n=0$. Thus the image of Δ is contained in the kernel of τ. Conversely, suppose $a=\sum_{m,n} d_{m,n}U^mV^n$ is an arbitrary element of $A_\theta^\infty \cap \ker\tau$. That means $d_{m,n}$ is rapidly decreasing and $d_{0,0}=0$. Then $d_{m,n}/(m^2+n^2)$ is also rapidly decreasing, and

$$\sum_{m,n}{}' \frac{-d_{m,n}}{4\pi^2\,(m^2+n^2)} U^m V^n,$$

where the $'$ indicates we omit the term with $m=n=0$, converges to an element b of A_θ^∞ with $\Delta b = a$. □

The following consequence is an analogue of a well-known fact about subharmonic functions on compact manifolds.

Corollary 2.5. *If* $a \in A_\theta^\infty$ *is subharmonic* (*i.e., if* $\Delta a \geq 0$), *then* a *is constant.*

Proof. Suppose $a \in A_\theta^\infty$ and $\Delta a \geq 0$. By Proposition 2.4, $\tau(\Delta a)=0$. But τ is a faithful trace, which means that if $b \geq 0$ and $\tau(b)=0$, then $b=0$. Apply this with $b=\Delta a$ and we see that $\Delta a = 0$. This implies a is a scalar multiple of 1. □

For future use, we are also going to want to study other "function spaces" on the noncommutative torus. For example, we have the analogue of the Fourier algebra of functions with absolutely convergent Fourier series.

Definition 2.6. Fix $\theta \in \mathbb{R} \setminus \mathbb{Q}$, and let

$$\mathcal{B}_\theta = \left\{ \sum_{m,n} c_{m,n} U^m V^n : \sum_{m,n} |c_{m,n}| < \infty \right\}.$$

This is obviously a Banach subspace of A_θ with norm $\|\cdot\|_{\ell^1}$ given by the ℓ^1 norm of the coefficients $c_{m,n}$. We also obviously have $\|a\|_{\ell^1} \geq \|a\|$ for $a \in \mathcal{B}_\theta$. ($\|\cdot\|$ will for us always denote the C^*-algebra norm.)

The following lemma, related in spirit to the Sobolev Embedding Theorem [Kazdan 1983, Theorem 1.1], relates the topology of $\mathcal{B}_\theta$ to the subject of Propositions 2.3 and 2.4. More details of noncommutative Sobolev space theory will be taken up in Section 3 below.

Lemma 2.7. *Let* $f \in A_\theta^\infty$. *Then there is a constant* $C>0$ *such that* (*in the notation of Definition* 2.6) $\|f\|_{\ell^1} \leq C\|(-\Delta+1)f\|$. *In particular, the domain of* Δ, *as an unbounded operator on* A_θ, *is contained in* $\mathcal{B}_\theta$.

Proof. Suppose $f = \sum_{m,n} c_{m,n} U^m V^n \in A_\theta^\infty$. Then

$$\|f\|_{\ell^1} = \sum_{m,n} |c_{m,n}| = \sum_{m,n} \bigl(1 + 4\pi^2(m^2+n^2)\bigr)\, c_{m,n} \cdot \frac{a_{m,n}}{1+4\pi^2(m^2+n^2)},$$

where $|a_{m,n}| = 1$. View this as an ℓ^2 inner product and estimate it by Cauchy–Schwarz. We obtain

$$\|f\|_{\ell^1} \le C\|(-1+\Delta)f\|_{\ell^2},$$

where $\|\cdot\|_{\ell^2}$ is the ℓ^2 norm of the sequence of Fourier coefficients (this can also be defined by $\|c\|_{\ell^2} = \tau(c^*c)^{1/2}$) and where

$$C = \Bigl\| \bigl\{ \bigl(1+4\pi^2(m^2+n^2)\bigr)^{-1} \bigr\}_{m,n} \Bigr\|_{\ell^2} = \Biggl(\sum_{m,n} \frac{1}{\bigl(1+4\pi^2(m^2+n^2)\bigr)^2} \Biggr)^{1/2} < \infty.$$

Since the ℓ^2 norm is bounded by the C^*-algebra norm, as $\|c\|_{\ell^2} = \tau(c^*c)^{1/2} \le \|c^*c\|^{1/2} = \|c\|$, the result follows. $\square$

The next result was proved in [Gröchenig and Leinert 2004], using the theory of symmetric L^1-algebras as developed by Leptin, Ludwig, Hulanicki, *et al.* We include a brief proof for the sake of completeness.

Theorem 2.8 (Wiener's Theorem). *The Banach space $\mathcal{B}_\theta$ is a Banach $*$-algebra and is closed under the holomorphic functional calculus of A_θ. Thus if $a \in \mathcal{B}_\theta$ and a is invertible in A_θ, $a^{-1} \in \mathcal{B}_\theta$.*

Proof. Suppose $a = \sum c_{m,n} U^m V^n$ with the sum absolutely convergent. Then

$$a^* = \sum_{m,n} \overline{c_{m,n}} V^{-n} U^{-m} = \sum_{m,n} \overline{c_{m,n}} e^{-2\pi i m n \theta} U^{-m} V^{-n}$$

so $a^* \in \mathcal{B}_\theta$. Similarly, if also $b = \sum d_{m,n} U^m V^n$ (absolutely convergent sum), then ab has Fourier coefficients given by twisted convolution of the Fourier coefficients of a and b, and since the twisting only involves scalars of absolute value 1, the Fourier coefficients of ab are absolutely convergent. More precisely,

$$\begin{aligned} ab &= \Bigl(\sum_{m,n} c_{m,n} U^m V^n\Bigr)\Bigl(\sum_{k,l} d_{k,l} U^k V^l\Bigr) = \sum_{m,n,k,l} c_{m,n} d_{k,l} U^m V^n U^k V^l \\ &= \sum_{m,n,k,l} c_{m,n} d_{k,l} e^{-2\pi i k n \theta} U^{m+k} V^{n+l} = \sum_{p,q} f_{p,q} U^p V^q, \end{aligned}$$

where

$$f_{p,q} = \sum_{m,n} c_{m,n} d_{p-m,q-n} e^{-2\pi i (p-m) n \theta}, \quad \text{so that} \quad |f_{p,q}| \le \sum_{m,n} |c_{m,n}|\,|d_{p-m,q-n}| \le \|c\|_{\ell^1} \|d\|_{\ell^1}.$$

This confirms that $\mathcal{B}_\theta$ is a Banach $*$-algebra and of course a $*$-subalgebra of A_θ.

To prove the analogue of Wiener's Theorem, we unfortunately cannot use the cute proof using the Gelfand transform, since $\mathcal{B}_\theta$ is not commutative. We also cannot use another very elementary proof from [Newman 1975] since this also relies on commutativity. However, Newman's proof is related to the fact — implicit in [Connes 1980, Lemma 1] — that A_θ^∞ is closed under the holomorphic functional calculus of A_θ. To prove this one has to show that if $b \in A_\theta^\infty$ with b invertible in A_θ, then b^{-1} also lies in

A_θ^∞. To prove this fact, iterate the identity $\delta_j(b^{-1}) = -b^{-1}\,\delta_j(b)\,b^{-1}$ to see that b^{-1} lies in the domain of all monomials in δ_1 and δ_2. One might think that since A_θ^∞ is dense in $\mathcal{B}_\theta$, this should be enough to prove Wiener's Theorem for the latter, but this doesn't work, since in general the spectrum and spectral radius functions are only upper semicontinuous, not continuous, on a noncommutative Banach algebra [Newburgh 1951].

To prove the theorem, we rely on an observation of Hulanicki [1972, Proposition 2.5], based on [Raikov 1946, Theorem 5]: if a Banach $*$-algebra B (with isometric involution and a faithful $*$-representation on a Hilbert space) is embedded in its enveloping C^*-algebra A, then the spectra of self-adjoint elements of B are the same whether computed in B or in A if and only if B is symmetric (i.e., for $x \in B$, the spectrum in B of x^*x is contained in $[0, \infty)$). We will apply this with $B = \mathcal{B}_\theta$ and with $A = A_\theta$. Hulanicki also showed [Hulanicki 1970] that the L^1 algebras of discrete nilpotent groups are symmetric. In particular, the L^1 algebra of the discrete Heisenberg group H (with generators a, b, c, where c is central and $aba^{-1}b^{-1} = c$) is symmetric. Thus $\mathcal{B}_\theta$, which is the quotient of $L^1(H)$ by the (self-adjoint) ideal generated by $c - e^{2\pi i\theta}$, is also symmetric. (If B is a symmetric Banach $*$-algebra and J is a closed self-adjoint ideal, then B/J is also symmetric, since if $\dot{x} \in B/J$ is the image of $x \in B$, then the spectrum of $\dot{x}^*\dot{x}$ in B/J is contained in the spectrum of x^*x in B, hence is contained in $[0, \infty)$.) So for $x = x^* \in \mathcal{B}_\theta$, by Hulanicki's observation, if x is invertible in A_θ, $x^{-1} \in \mathcal{B}_\theta$. Suppose $a \in \mathcal{B}_\theta$ and a is invertible in A_θ. Then a^* is also invertible in A_θ, so $x = a^*a \in \mathcal{B}_\theta$ and x is invertible in A_θ. Hence $x^{-1} = a^{-1}a^{*-1} \in \mathcal{B}_\theta$ and $a^{-1} = x^{-1}a^* \in \mathcal{B}_\theta$. □

In the classical theory of the Laplacian, one of the most useful tools is the maximum principle — see, for example, [Kazdan 1983, p. 20]. The following is a noncommutative analogue.

Proposition 2.9 (Maximum principle). *Let $h = h^* \in A_\theta^\infty$, and let $[t_0, t_1]$ be the smallest closed interval containing the spectrum $\sigma(h)$ of h in A_θ, so that $t_1 = \max\{t : t \in \sigma(h)\}$ and $t_0 = \min\{t : t \in \sigma(h)\}$. Then there exists a state φ of A_θ with $\varphi(h) = t_1$, and for such a state, $\varphi(\Delta h) \le 0$. Similarly, there exists a state ψ of A_θ with $\psi(h) = t_0$, and for such a state, $\psi(\Delta h) \ge 0$.*

Proof. The commutative C^*-algebra $C^*(h)$ must have pure states $\widetilde{\varphi}$ and $\widetilde{\psi}$ with $\widetilde{\varphi}(h) = t_1$, $\widetilde{\psi}(h) = t_0$, since $t_0, t_1 \in \sigma(h)$. Extend these to states φ, ψ of the larger C^*-algebra A_θ. Then for $s \in G = \mathbb{T}^2$, the functions $s \mapsto \varphi(\alpha_s(h))$ and $s \mapsto \psi(\alpha_s(h))$ must have a maximum and a minimum, respectively, at the identity element of $\mathbb{T}^2$. (Recall that α is the gauge action by $*$-automorphisms.) Differentiate twice and the result follows by the second derivative test. □

Just as in the classical setting, Laplace's equation arises as the Euler–Lagrange equation of a variational problem.

Definition 2.10. For $a \in A_\theta^\infty$, let

$$E(a) = \tfrac{1}{2}\tau\big(\delta_1(a)^2 + \delta_2(a)^2\big).$$

This is clearly the noncommutative analogue of the classical energy functional

$$f \mapsto \frac{1}{2}\int_M \|\nabla f\|^2 \, d\mathrm{vol}$$

on a compact manifold M.

Proposition 2.11. *The Euler–Lagrange equation for critical points of the energy functional E of Definition* 2.10, *restricted to self-adjoint elements of A_θ^∞, is just Laplace's equation $\Delta a = 0$. Thus the only critical points are the scalar multiples of the identity, which are the points where $E(a) = 0$ and are strict minima for E.*

Proof. This works very much like the classical case. If $a = a^*$ and $h = h^*$, then

$$\left.\frac{d}{dt}\right|_{t=0} E(a+th) = \tfrac{1}{2}\tau\big(\delta_1(a)\delta_1(h) + \delta_1(h)\delta_1(a) + \delta_2(a)\delta_2(h) + \delta_2(h)\delta_2(a)\big).$$

Because of the trace property, we can write this as $\tau(\delta_1(a)\delta_1(h)+\delta_2(a)\delta_2(h))$. For a to be a critical point of E, this must vanish for all choices of h. Integrating by parts using Lemma 2.1, we obtain $\tau(h\Delta(a))=0$ for all h, and since the trace pairing is nondegenerate, we get the Euler–Lagrange equation $\Delta a=0$. Since Δ has pure point spectrum with eigenvalues $-4\pi^2(m^2+n^2)$ and eigenfunctions $U^m V^n$, the equation has the unique solution $a = \lambda 1$, $\lambda \in \mathbb{R}$. These are also the points where E takes its minimum value of 0. □

3. Sobolev spaces

In the treatment of Laplace's equation above, we alluded to the theory of Sobolev spaces. One can develop this theory in the noncommutative setting in complete analogy with the classical case. To simplify the treatment, we deal here only with the L^2 theory, which gives rise to Hilbert spaces. These spaces are convenient for applications to nonlinear elliptic PDE, as we will see in the next section.

Definition 3.1. For $a \in A_\theta$, we define its L^2 norm[1] by

$$\|a\|_{\ell^2} = \tau(a^*a)^{1/2}.$$

We let L^2 or H^0 (this is the Sobolev space of "functions" with 0 derivatives in L^2) be the completion of A_θ in this norm. Obviously this is a Hilbert space, with inner product extending

$$\langle a,\, b\rangle = \tau(b^*a)$$

on A_θ. Also note that the norm of L^2 is simply the ℓ^2 norm for the Fourier coefficients, since if $a \in A_\theta^\infty$ has the Fourier expansion $\sum_{m,n} c_{m,n}U^m V^n$, then

$$\|a\|_{\ell^2}^2 = \tau(a^*a) = \tau\bigg(\sum_{k,l,m,n} \big(c_{m,n}U^m V^n\big)^* c_{k,l}U^k V^l\bigg) = \tau\bigg(\sum_{k,l,m,n} \overline{c_{m,n}}\, c_{k,l} V^{-n}U^{-m}U^k V^l\bigg) = \sum_{m,n} |c_{m,n}|^2.$$

Now let $n \in \mathbb{N}$. We define the Sobolev space[2] H^n of "functions" with n derivatives in L^2 to be the completion of A_θ^∞ in the norm

$$\|a\|_{H^n}^2 = \sum_{0\le|\beta|\le n} \|\delta_\beta(a)\|_{\ell^2}^2.$$

(These spaces are also defined, with slightly different notation, in [Polishchuk 2006, §3].) Here $\beta = \beta_1\beta_2\cdots\beta_{|\beta|}$ runs over sequences with $\beta_j = 1$ or 2 and δ_β means $\delta_{\beta_1}\cdots\delta_{\beta_{|\beta|}}$, a partial derivative of order

[1] This is really the norm for the Hilbert space of the II_1 factor representation of A_θ determined by the trace τ.

[2] Usually this would be called $H^{n,2}$, but we are trying to simplify notation.

$|\beta|$. For example,

$$\|a\|_{H^1}^2 = \|a\|_{\ell^2}^2 + \|\delta_1(a)\|_{\ell^2}^2 + \|\delta_2(a)\|_{\ell^2}^2.$$

The Sobolev space H^n is clearly a Hilbert space, and we obviously have norm-decreasing inclusions $H^n \hookrightarrow H^{n-1}$. Furthermore, it is clear that the Sobolev norms are invariant under taking adjoints and can easily be expressed in terms of the Fourier coefficients; for example, if $a \in A_\theta^\infty$ has the Fourier expansion $\sum_{m,n} c_{m,n} U^m V^n$, then

$$\|a\|_{H^1}^2 = \sum_{m,n} \big(1 + 4\pi^2(m^2+n^2)\big)|c_{m,n}|^2.$$

The next result is the exact analogue of the classical Sobolev embedding theorem [Kazdan 1983, Theorem 1.1] for $\mathbb{T}^2$.

Theorem 3.2 (Sobolev embedding). *The inclusion $H^n \hookrightarrow H^{n-1}$ is compact. The space H^1 is not contained in A_θ, but H^2 has a compact inclusion into $\mathcal{B}_\theta$ (and thus into A_θ).*

Proof. Since the Sobolev norms just depend on the decay of the Fourier coefficients, this follows immediately from the classical Sobolev Embedding Theorem in dimension 2. The inclusion of H^2 into $\mathcal{B}_\theta$ also follows from the estimate

$$\|f\|_{\ell^1} \le C\|(-1+\Delta)f\|_{\ell^2},$$

in the proof of Lemma 2.7, with the compactness coming from the fact that we can approximate by the finite rank operators that truncate the Fourier series after finitely many terms. □

4. Nonlinear problems involving the Laplacian

Somewhat more interesting, and certainly more difficult to treat than the situation of Proposition 2.11, are certain nonlinear problems involving the Laplacian, of the general form $\Delta u = f(u)$. Such problems arise classically from the problem of prescribing the scalar curvature of a metric $e^u g$ obtained by conformally deforming the original metric g on a Riemannian manifold M [Kazdan 1983, Chapters 5, 7]. For example, if g is the usual flat metric on $\mathbb{T}^2$, then the scalar curvature h of the pointwise conformal metric $e^u g$ solves the equation $\Delta u = -he^u$. (This equation is studied in detail in [Kazdan and Warner 1974, §5].) Because of the Gauss–Bonnet theorem on the torus, h must integrate out to 0, so there are no solutions with h a constant unless $h = 0$ and u is a constant. This fact has an exact analogue in our noncommutative setting.

Proposition 4.1. *If $\lambda \in \mathbb{R}$, the equation $\Delta u = -\lambda e^u$ has no solution $u = u^* \in A_\theta^\infty$ unless $\lambda = 0$ and u is a scalar multiple of* 1.

Proof. Suppose $u = u^* \in A_\theta^\infty$. Then $e^u \ge 0$, so if $\lambda \ne 0$, either $\lambda e^u \ge 0$ or $-\lambda e^u \ge 0$. Thus if $\Delta u = -\lambda e^u$, either u or $-u$ is subharmonic. The result now follows from Corollary 2.5. □

Alternative proof. Use the maximum principle, Proposition 2.9. Let $[a, b]$ be the smallest closed interval containing the spectrum of u. Then for any state φ of A_θ, $a \le \varphi(u) \le b$ and $\varphi(e^u) \ge e^a > 0$. If $\Delta u = -\lambda e^u$ and $\lambda > 0$, then by Proposition 2.9, there is a state φ with $\varphi(u) = a$ and $\varphi(\Delta u) \ge 0$, while $\varphi(-\lambda e^u) < 0$, a contradiction. Similarly, if $\lambda < 0$ and $\Delta u = -\lambda e^u$, there is a state φ with $\varphi(u) = b$ and $\varphi(\Delta u) \le 0$, while $\varphi(-\lambda e^u) > 0$, a contradiction. □

Proposition 4.1 suggests that we consider the equation $\Delta u = -\frac{1}{2}\big(he^u + e^u h\big)$ with $h = h^*$ not a scalar. (Note that we have symmetrized the right-hand side to make it self-adjoint, since $u = u^*$ implies Δu is self-adjoint.) Once again, a slight variation on the argument of Proposition 4.1 shows that there is no solution if $h \geq 0$ or if $h \leq 0$; again this is not surprising since one gets the same result in the classical case as a consequence of Gauss–Bonnet.

Proposition 4.2. *If $h \geq 0$ or $h \leq 0$ in A_θ^∞, the equation $\Delta u = -\frac{1}{2}\big(he^u + e^u h\big)$ has no solution $u = u^* \in A_\theta^\infty$ unless $h = 0$ and u is a scalar multiple of* 1.

Proof. This is just like the proof of Proposition 4.1. If $h \geq 0$ and $\Delta u = -\frac{1}{2}\big(he^u + e^u h\big)$, then applying τ to both sides, we get

$$0 = \tau(\Delta u) = -\tau(he^u) = -\tau\left(h^{1/2}e^u h^{1/2}\right). \tag{4-1}$$

Since

$$h^{1/2}e^u h^{1/2} = \left(e^{u/2}h^{1/2}\right)^* \left(e^{u/2}h^{1/2}\right) \geq 0$$

and τ is faithful, that implies $e^{u/2}h^{1/2} = 0$. Since $e^{u/2}$ is invertible, it follows that $h^{1/2} = 0$ and $h = 0$. The case where $h \leq 0$ is almost identical; just replace h by $-h$ and change the sign of the right-hand side of (4-1). $\square$

Unfortunately, the rest of the treatment in [Kazdan and Warner 1974, §5] doesn't extend to our setting, since from the calculation

$$\tau(h) = \tfrac{1}{2}\tau\big(e^{-u}he^u + h\big) = -\tau(e^{-u}\Delta u),$$

it is not clear if $\tau(h) < 0$ follows. (The problem is that we can't commute the various factors that arise from expanding $\delta_j(e^{-u})$ after integration by parts.) But since the main purpose of this section is just to test various techniques and see to what extent they apply to nonlinear noncommutative elliptic PDEs, we will consider instead the following more tractable equation from [Kazdan 1983, Chapter 5]:

$$\Delta u = \mu\, e^u - \lambda, \quad \lambda,\ \mu \in \mathbb{R},\ \lambda,\ \mu > 0. \tag{4-2}$$

Theorem 4.3. *The equation* (4-2) *has the unique solution $t_0 = \ln(\lambda/\mu)$ in $\big(A_\theta^\infty\big)_{s.a.}$.*

Proof. Let

$$\mathcal{L}(u) = E(u) + \tau(\mu\, e^u - \lambda u).$$

Note that for $t \in \mathbb{R}$, $\mu\, e^t - \lambda t$ has an absolute minimum at $t = t_0$, so $\mu\, e^u - \lambda u \geq \lambda(1 - t_0)$ for $u = u^*$ and so $\mathcal{L}(u) \geq \lambda(1 - t_0)$ for $u = u^*$. Furthermore, the Euler–Lagrange equation for a critical point of $\mathcal{L}$ is precisely (4-2), since

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{L}(u + th) = \tau(\delta_1(u)\delta_1(h) + \delta_2(u)\delta_2(h) - \lambda h) + \left.\frac{d}{dt}\right|_{t=0} \tau\big(\mu\, e^{u+th}\big),$$

via the calculation in the proof of Proposition 2.11. Now

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=0} \tau\big(e^{u+th}\big) &= \left.\frac{d}{dt}\right|_{t=0} \sum_{n=0}^{\infty} \frac{1}{n!}\tau\big((u+th)^n\big) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}\tau\big(u^{n-1}h + u^{n-2}hu + \cdots + uhu^{n-2} + hu^{n-1}\big) = \sum_{n=0}^{\infty} \frac{n}{n!}\tau\big(hu^{n-1}\big) = \tau(he^u)\end{aligned}$$

by the invariance of the trace under cyclic permutations of the factors. So applying Lemma 2.1, we see that

$$\left.\frac{d}{dt}\right|_{t=0} \mathscr{L}(u+th) = \tau(-h\Delta(u) - \lambda h + \mu\, he^u) = -\tau\big(h\cdot(\Delta u + \lambda - \mu\, e^u)\big).$$

So nondegeneracy of the trace pairing gives (4-2) as the Euler–Lagrange equation for a critical point of $\mathscr{L}$. It is also clear that t_0 is an absolute minimum for $\mathscr{L}$ and a solution of (4-2). It remains to prove the uniqueness. Suppose u is a solution of (4-2) and write $u = t_0 + v$. Then v satisfies the equation $\Delta v = \lambda(e^v - 1)$, and we need to show $v = 0$. Multiply both sides by v and apply τ. We obtain (using Lemma 2.1)

$$-2E(v) = \tau(v\Delta v) = \lambda\tau(v(e^v - 1)).$$

The left-hand side is ≤ 0, while since $\lambda > 0$ and $t(e^t - 1) \geq 0$ with equality only at $t = 0$, the right-hand side is ≥ 0. Thus $E(v) = 0$, which implies v is a scalar with $v(e^v - 1) = 0$, i.e., $v = 0$. □

With techniques reminiscent of [Kazdan 1983, Chapter 5] we can study a slightly more complicated variant of (4-2).

Theorem 4.4. *Let $a \geq 0$ be invertible in A_θ^∞. Then the equation*

$$\Delta u = \mu\, e^u - a, \quad \mu \in \mathbb{R},\ \mu > 0 \tag{4-3}$$

has a solution $u \in \big(A_\theta^\infty\big)_{s.a.}$.

Without loss of generality (as a result of replacing u by $u - \ln\mu$) we can take $\mu = 1$; that simplifies the calculations and we make this simplification from now on. Some condition on a beyond the fact that $a \geq 0$, for example at least $a \neq 0$, is necessary because of Proposition 4.1, and we see that any solution of (4-3) must satisfy $\tau(e^u) = \tau(a) > 0$.

Proof. Several methods are available for proving existence, but the simplest seems to be to apply the Leray–Schauder Theorem ([Leray and Schauder 1934], [Kazdan 1983, Theorem 5.5]). Consider the family of equations

$$\Delta u = (1-t)\,u + t\,e^u - a, \quad 0 \leq t \leq 1. \tag{4-4}$$

When $t = 0$ this reduces to $\Delta u = u - a$, or $(-\Delta + 1)\,u = a$, which by Proposition 2.3 has the unique solution $u = (-\Delta + 1)^{-1}a$. When $t = 1$, (4-4) reduces to (4-3). We begin by using the maximum principle, Proposition 2.9, which implies an *a priori* bound on solutions of (4-4). (Compare the argument in [Kazdan 1983, pp. 56–57].) Indeed, suppose u satisfies (4-4) for some $0 \leq t \leq 1$, and let $[c, d]$ be the smallest closed interval containing $\sigma(u)$. We may choose a state φ of A_θ with $\varphi(u) = d$, $\varphi(e^u) = e^d$, and by Proposition 2.9, $\varphi(\Delta u) \leq 0$. Since

$$\varphi\big((1-t)\,u + t\,e^u - a\big) = (1-t)\,d + t\,e^d - \varphi(a) \geq (1-t)\,d + t\,e^d - \|a\|,$$

we get a contradiction if $(1-t)\,d + t\,e^d - \|a\| > 0$, which is the case if $d > \|a\|$. So $d \leq \|a\|$. Similarly, we may choose a state ψ of A_θ with $\psi(u) = c$, $\psi(e^u) = e^c$, and by Proposition 2.9, $\psi(\Delta u) \geq 0$. Since

$$\psi\big((1-t)\,u + t\,e^u - a\big) = (1-t)\,c + t\,e^c - \psi(a) \leq (1-t)\,c + t\,e^c - \frac{1}{\|a^{-1}\|},$$

we get a contradiction if $e^c - 1/\|a^{-1}\| < 0$. Thus $e^c - 1/\|a^{-1}\| \ge 0$ and $c \ge -\ln\|a^{-1}\|$. In other words, any solution of (4-4), for any $0 \le t \le 1$, satisfies the *a priori* estimate

$$-\ln\|a^{-1}\| \le u \le \|a\|. \tag{4-5}$$

Now rewrite (4-4) in the form

$$u = (-\Delta + 1)^{-1}\big(a + t\,u - t\,e^u\big).$$

The right-hand side is well-defined and continuous in the C^*-algebra norm topology for $u = (A_\theta)_{\text{s.a.}}$, since $(-\Delta+1)^{-1}$ is bounded by Lemma 2.7. In fact, this Lemma also shows $(-\Delta+1)^{-1}$ is bounded as a map $A_\theta \to \mathcal{B}_\theta$, so as a map $A_\theta \to A_\theta$, it is a limit of operators of finite rank, namely the restrictions of the operator to the span of $\{U^m V^n : m^2 + n^2 \le N\}$, as $N \to \infty$. Thus $(-\Delta+1)^{-1}$ is not only bounded, but also compact. Together with the *a priori* estimate (4-5) and the fact that there is a solution for $t = 0$, this shows that (4-4) satisfies the hypotheses of the Leray–Schauder Theorem. Hence (4-4) has a solution for all $t \in [0,\ 1]$. Thus (4-3) (which is the special case of (4-4) for $t = 1$) has a solution in $\operatorname{dom}\Delta \subseteq A_\theta$, and thus in $\mathcal{B}_\theta$ by Lemma 2.7.

The last step of the proof is *elliptic regularity*. In other words, we need to show that a solution to (4-3), so far only known to be in $\mathcal{B}_\theta$, lies in A_θ^∞. Since $a \in A_\theta^\infty$ and $\mathcal{B}_\theta$ is closed under holomorphic functional calculus (by Theorem 2.8), the right-hand side of (4-3) lies in $\mathcal{B}_\theta$, i.e., has absolutely summable Fourier coefficients. Then (4-3) implies that the Fourier coefficients $c_{m,n}$ of u have even faster decay, namely,

$$\sum_{m,n} (1 + m^2 + n^2)|c_{m,n}| < \infty.$$

Now one can iterate this argument. This is a bit tricky, as at each step one needs a new Banach subalgebra of A_θ to replace $\mathcal{B}$ (we drop the subscript θ for simplicity of notation), so we indicate how this works at the next step, and then sketch how to proceed further. For $u \in \mathcal{B}$ with Fourier coefficients $c_{m,n}$, let

$$\|u\|_1 = \sum_{m,n} (2 + m^2 + n^2)|c_{m,n}|,$$

assuming this converges. We have seen that we know $\|u\|_1 < \infty$. We claim that $\|\cdot\|_1$ is a Banach $*$-algebra norm. This will follow by the argument in the proof of Theorem 2.8 if we can show that

$$\sum_{p,q} (2 + p^2 + q^2) \sum_{m,n} |c_{m,n}|\,|d_{p-m,q-n}| \le \Bigg(\sum_{m,n} (2 + m^2 + n^2)|c_{m,n}|\Bigg)\Bigg(\sum_{l,k} (2 + l^2 + k^2)|d_{l,k}|\Bigg).$$

Comparing the two sides of this inequality, one sees it is equivalent to proving that

$$(2 + p^2 + q^2) \le (2 + m^2 + n^2)(2 + (p-m)^2 + (q-n)^2),$$

or with $\vec{v} = (m, n)$ and $\vec{w} = (p - m, q - n)$ vectors in Euclidean 2-space, that

$$\big(2 + \|\vec{v} + \vec{w}\|^2\big) \le \big(2 + \|\vec{v}\|^2\big)\big(2 + \|\vec{w}\|^2\big).$$

This inequality in turn follows from the standard inequality

$$\|\vec{v} + \vec{w}\|^2 \le \|\vec{v}\|^2 + \|\vec{w}\|^2 + 2\|\vec{v}\| \cdot \|\vec{w}\| \le 2\big(\|\vec{v}\|^2 + \|\vec{w}\|^2\big).$$

This shows the completion of A_θ^∞ in the norm $\|\cdot\|_1$ is a Banach $*$-algebra $\mathcal{B}_1$. Since u and a are in $\mathcal{B}_1$, so is $e^u - a$. By (4-3) again, u has still more rapid decay; its Fourier coefficients satisfy

$$\sum_{m,n}(m^2+n^2)^2|c_{m,n}| < \infty.$$

Now we iterate again using still another Banach $*$-algebra $\mathcal{B}_2$ with the norm

$$\|u\|_2 = \sum_{m,n}\bigl(8+(m^2+n^2)^2\bigr)|c_{m,n}|.$$

Again one has to check that this is a Banach algebra norm, which will follow from the inequalities

$$\begin{aligned} 8+\|\vec{v}+\vec{w}\|^4 &= 8+\bigl(\|\vec{v}+\vec{w}\|^2\bigr)^2 \\ &\le 8+\bigl(2(\|\vec{v}\|^2+\|\vec{w}\|^2)\bigr)^2 \le 8+4\bigl(\|\vec{v}\|^4+\|\vec{w}\|^4+2\|\vec{v}\|^2\cdot\|\vec{v}\|^2\bigr) \\ &\le 8+4\bigl(2(\|\vec{v}\|^4+\|\vec{w}\|^4)\bigr) \le \bigl(8+\|\vec{v}\|^4\bigr)\bigl(8+\|\vec{w}\|^4\bigr). \end{aligned}$$

Thus $\mathcal{B}_2$ is a Banach algebra and $e^u - a \in \mathcal{B}_2$, so that $\Delta u \in \mathcal{B}_2$ and the Fourier coefficients of u decay faster than $(m^2+n^2)^3$, etc. Repeating in this way, we show by induction that $c_{m,n}$ is rapidly decreasing, and thus that $u \in A_\theta^\infty$. $\square$

Sketch of a second proof. One could also approach this problem using "variational methods." By the argument at the beginning of the proof of Theorem 4.3, (4-3) is the Euler–Lagrange equation for critical points of

$$\mathcal{L}(u) = E(u) + \tau(e^u - u\,a) = E(u) + \tau(e^u - a^{1/2}ua^{1/2}).$$

This functional is bounded below since $E(u) \ge 0$ and $\tau(e^u - a^{1/2}ua^{1/2})$ is bounded below (by a constant depending only on a). Indeed, for t and $\lambda > 0$ real, $e^t - \lambda t$ has a global minimum at $t = \ln\lambda$, so $e^t - \lambda t \ge \lambda(1-\ln\lambda)$. If we write $u = u_+ - u_-$ with $u_+u_- = u_-u_+ = 0$ and $u_+,\ u_- \ge 0$, then

$$-\tau(u\,a) = \tau(u_-a) - \tau(u_+a) = -\tau\bigl(u_+^{1/2}au_+^{1/2}\bigr) + \tau\bigl(u_-^{1/2}au_-^{1/2}\bigr) \ge -\tau\bigl(u_+^{1/2}\|a\|u_+^{1/2}\bigr) + 0 = -\|a\|\,\tau(u_+).$$

On the other hand,

$$\tau(e^u) = \tau\bigl(e^{u_+} + e^{-u_-} - 1\bigr) \ge \tau\bigl(e^{u_+}\bigr) - 1,$$

and thus

$$\tau(e^u - u\,a) \ge \tau\bigl(e^{u_+}\bigr) - \|a\|\,\tau(u_+) - 1 = \tau\bigl(e^{u_+} - \|a\|\,u_+\bigr) - 1 \ge \|a\|\,\bigl(1-\ln\|a\|\bigr) - 1.$$

So we will show that $\mathcal{L}$ must have a minimum point, which will be a solution of (4-3).

Choose $u_n = u_n^* \in A_\theta^\infty$ with $\mathcal{L}(u_n)$ decreasing to $\inf\bigl\{\mathcal{L}(u) : u \in \bigl(A_\theta^\infty\bigr)_{\text{s.a.}}\bigr\}$. Since E and $\tau(e^u - a^{1/2}ua^{1/2})$ are separately bounded below, $E(u_n)$ must remain bounded. That means that $\|\delta_j(u_n)\|_{\ell^2}$ remains bounded for $j = 1,\ 2$.

We can also assume that $\|u_n\|_{\ell^2}$ remains bounded. To see this, it is easiest to use a trick (cf. [Kazdan 1983, pp. 56–57]). Because of the *a priori* bound on solutions of (4-3) coming from the maximum principle (see the first proof above), we can modify the function e^u on the right-hand side of the equation and replace it by some C^∞ function that grows linearly for $u \ge \|a\| + 1$ and decays linearly for $u \le -1 - \ln\|a\|$. (This does not affect the maximum principle argument, so the solutions of the modified equation are the same as for the original one.) This has the effect of changing the term $\tau(e^u)$ in the formula for $\mathscr{L}$ to something that outside of a finite interval behaves like a constant times $\tau(u^2)$, which is $\|u\|^2_{\ell^2}$.

Thus we can assume our minimizing sequence u_n is bounded in the Sobolev space H^1. Since the unit ball of a Hilbert space is weakly compact, after passing to a subsequence, we can assume that u_n converges weakly in the Hilbert space H^1, and by Theorem 3.2, strongly in $H^0 = L^2$, to some $u \in H^1$ which is a minimizer for $\mathscr{L}$. (Compare the argument in [Kazdan 1983, Theorem 5.2].) This u is a "weak solution" of our equation and we just need to show it is smooth, i.e., corresponds to a genuine element of A_θ^∞. This requires an elliptic regularity argument similar to the one in the first proof. □

5. Harmonic unitaries

In this section, we discuss the noncommutative analogue of the classical problem of studying harmonic maps $M \to S^1$, where M is a compact Riemannian manifold and S^1 is given its usual metric. This problem was studied and solved in [Eells and Sampson 1964, pp. 128–129]. The homotopy classes of maps $M \to S^1$ are classified by $H^1(M, \mathbb{Z})$. For each homotopy class in $H^1(M, \mathbb{Z})$, we can think of it as an integral class in $H^1(M, \mathbb{R})$, and represent it (by the de Rham and Hodge Theorems) by a unique harmonic 1-form with integral periods. Integrating this 1-form gives a harmonic map $M \to S^1$ in the given homotopy class. This map is not quite unique since we can compose with an isometry (rotation) of the circle, but except for this we have uniqueness. (This follows from [Eells and Sampson 1964, Proposition, p. 123].)

If we dualize a map $M \to S^1$, we obtain a unital $*$-homomorphism $C(S^1) \to C(M)$, which since $C(S^1)$ is the universal C^*-algebra on a single unitary generator, is basically the same as a choice of a unitary element $u \in C(M)$. This analysis suggests that the noncommutative analogue of a harmonic map to S^1 should be a "harmonic" unitary in a noncommutative C^*-algebra A. Each unitary in A defines a class in the topological K-theory group $K_1(A)$, and for A a unital C^*-algebra, every K_1 class is represented by a unitary in $M_n(A)$ for some n, so since we can replace A by $M_n(A)$, the natural problem is to search for a harmonic representative in a given connected component of $U(A)$ (or, passing to the stable limit, in a given K_1 class).

The next level of complexity up from the case where $A = C(M)$ is commutative is the case where $A = C(M, M_n(\mathbb{C}))$ for some n. In this case, a unitary in $U(A)$ is the same thing as a map $M \to U(n)$, and a harmonic unitary should be the same thing as a harmonic map $M \to U(n)$. For example, suppose $M = S^3$ and $n = 2$. Since there are no maps $M \to S^1$ which are not homotopic to a constant, it is natural to look first at smooth maps $f\colon S^3 \to U(2)$ with $\det \circ f\colon S^3 \to \mathbb{T}$ identically equal to 1, i.e., to look at maps $f\colon S^3 \to SU(2) = S^3$, with both copies of S^3 equipped with the standard round metric. This problem is treated in [Eells and Sampson 1964, Proposition, pp. 129–131]. For example, the identity map $S^3 \to S^3 = SU(2) \hookrightarrow U(2)$ is a harmonic map representing the generator of $K_1(A) = K^{-1}(S^3)$. The study of harmonic maps in other homotopy classes, even just in the simple case of $S^3 \to S^3$, is a complicated issue (see, e.g., [Eells and Sampson 1964, Proposition, pp. 129–131] and [Schoen and Uhlenbeck 1984]); however, this is quite tangential to the main theme of this article, so we won't consider it further.

Instead, we consider now the notion of harmonic unitaries in the case of A_θ. Recall first that $K_1(A_\theta) \cong \mathbb{Z}^2$, with U and V as generators [Pimsner and Voiculescu 1980, Corollary 2.5], and that the canonical map $U(A_\theta)/U(A_\theta)_0 \to K_1(A_\theta)$ is an isomorphism [Rieffel 1987].

Definition 5.1. If $u \in A_\theta^\infty$ is unitary, we define the *energy* of u to be

$$E(u) = \tfrac{1}{2}\,\tau\big((\delta_1(u))^*\delta_1(u) + (\delta_2(u))^*\delta_2(u)\big).$$

Obviously this is constructed so as to be ≥ 0. This definition also coincides with the energy defined in Definition 2.10, provided we insert the appropriate $*$'s in the latter (which we can do without changing anything since there we were taking u to be self-adjoint). The unitary u is called *harmonic* if it is a critical point for $E\colon\ U(A_\theta^\infty) \to [0, \infty)$. By the discussion above, a harmonic unitary is the noncommutative analogue of a harmonic circle-valued function on a manifold.

Remark 5.2. Note that in Definition 5.1, $E(u)$ is invariant under multiplication of u by a scalar $\lambda \in \mathbb{T}$. Thus E descends to a functional on the *projective* unitary group $PU(A_\theta^\infty)$ and any sort of uniqueness result for harmonic unitaries can only be up to multiplication of u by a scalar $\lambda \in \mathbb{T}$. This is analogous to what happens in the case of harmonic maps $M \to \mathbb{T}$, where the associated harmonic 1-form is unique but the map itself is only defined up to a constant of integration.

Theorem 5.3. *If $u \in A_\theta^\infty$ is unitary, then u is harmonic if and only if it satisfies the Euler–Lagrange equation*

$$u^*(\Delta u) + (\delta_1(u))^*\,\delta_1(u) + (\delta_2(u))^*\,\delta_2(u) = 0. \tag{5-1}$$

Note that this equation is elliptic (if we drop lower-order terms, it reduces to Laplace's equation $\Delta u = 0$), but highly nonlinear.

Proof. First note that for u unitary, since $u\,u^* = u^*u = 1$, we have

$$\delta_j(u)\,u^* + u\,(\delta_j(u))^* = (\delta_j(u))^*\,u + u^*\,\delta_j(u) = 0,$$

$j = 1,\ 2$. If u is unitary, then any nearby unitary is of the form ue^{ith}, $h = h^*$, and

$$\left.\frac{d}{dt}\right|_{t=0} E(ue^{ith}) = \tfrac{1}{2}\,\tau\Big(-i\delta_1(h)u^*\delta_1(u) + i\delta_1(u)^*u\delta_1(h) \ + \text{ similar expression with } \delta_2\Big).$$

We can use the trace property to move all the $\delta_j(h)$'s to the front. So u is a critical point if and only if for all $h = h^*$,

$$\tau\Big(\delta_1(h)\,\mathrm{Im}\big(u^*\delta_1(u)\big) + \delta_2(h)\,\mathrm{Im}\big(u^*\delta_2(u)\big)\Big) = 0. \tag{5-2}$$

In (5-2), the Im's can be omitted since we have seen that u unitary $\Rightarrow \delta_j(u)^*u$ skew-adjoint. Thus u is harmonic if and only if

$$\tau\Big(\delta_1(h)\,\big(u^*\delta_1(u)\big) + \delta_2(h)\,\big(u^*\delta_2(u)\big)\Big) = 0$$

for all $h = h^*$ in A_θ^∞. Now apply integration by parts (Lemma 2.1). We see that u is harmonic if and only if

$$\tau\Big(h\,\delta_1\big(u^*\delta_1(u)\big) + h\,\delta_2\big(u^*\delta_2(u)\big)\Big) = 0$$

for all $h = h^*$ in A_θ^∞. Since the trace pairing is nondegenerate, the Theorem follows. □

It seems natural to make the following conjecture:

Conjecture 5.4. *In each connected component of $PU(A_\theta^\infty)$, the functional E has a unique minimum, given by scalar multiples of U^nV^m. These are the only harmonic unitaries in this component.*

Unfortunately, because of the complicated nonlinearity in (5-1), plus complications coming from noncommutativity, we have not been able to prove the Conjecture 5.4. However, we have the following partial result. In particular, we see that every connected component in $U(A_\theta^\infty)$ contains a harmonic unitary which is energy-minimizing.

Theorem 5.5. *The scalar multiples of $U^m V^n$ are harmonic and are strict local minima for E. Any harmonic unitary u depending on U alone is a scalar multiple of a power of U. Similarly, any harmonic unitary u depending on V alone is a scalar multiple of a power of V.*

Proof. First suppose u depends on U alone. Then $\delta_2(u) = 0$. So by the proof of Theorem 5.3, if u is harmonic, then $\tau\big(\delta_1(h)\cdot\delta_1(u)^*u\big) = 0$ for all $h = h^*$. This must also hold for general h (not necessarily self-adjoint) since we can split h into its self-adjoint and skew-adjoint parts. Since the range of δ_1 contains U^m unless $m = 0$, $\tau(\delta_1(u)^*u\,U^m)$=0 for $m \neq 0$, which means (since $\delta_1(u)^*u$ depends only on U) that $\delta_1(u)^*u$ is a scalar. Thus u is an eigenfunction for δ_1 and so $u = e^{i\lambda}U^m$ for some m. The case where u depends on V alone is obviously similar.

Next let's examine $u = U^m V^n$. Since $E(U^m V^n) = 2\pi^2(m^2+n^2)$ while

$$(U^m V^n)^*\Delta(U^m V^n) = -4\pi^2(m^2+n^2),$$

u satisfies (5-1) and is therefore harmonic. We show it is a local minimum for E; in fact, the minimum is strict once we pass to $PU(A_\theta^\infty)$. We expand $\delta_j(ue^{ith})$, with $h = h^*$, out to second order in t. Note that with $\delta = \delta_1$ or δ_2,

$$\delta(ue^{ith}) = \delta(u) + it\,\big(\delta(u)h + u\delta(h)\big) - \tfrac{1}{2}t^2\,\big(\delta(u)\,h^2 + u\,\delta(h)\,h + u\,h\,\delta(h)\big) + O(t^3).$$

We substitute this into the formula for $E(ue^{ith})$. The terms linear in t cancel since u is harmonic, and we find that

$$\begin{aligned} E\big(u\,e^{ith}\big) &= 2\pi^2(m^2+n^2) \\ &\quad + t^2\,\tau\Big(\big(\delta_1(u)h + u\delta_1(h)\big)^*\big(\delta_1(u)h + u\delta_1(h)\big) - \tfrac{1}{2}\delta_1(u)^*\big(\delta_1(u)\,h^2 + u\,\delta_1(h)\,h + u\,h\,\delta_1(h)\big) \\ &\qquad - \tfrac{1}{2}(h^2\,\delta_1(u)^* + h\,\delta_1(h)\,u^* + \delta_1(h)\,h\,u^*)\,\delta_1(u) + \text{similar expressions with } \delta_2\Big) + O(t^3). \end{aligned}$$

This actually simplifies considerably since u is an eigenvector for both δ_1 and δ_2, so that $\delta_j(u)^*\delta_j(u)$, $\delta_j(u)^*u$, and $u^*\delta_j(u)$ are all scalars. It turns out that almost everything cancels and one gets

$$\begin{aligned} E(ue^{iht}) &= 2\pi^2(m^2+n^2) + \tfrac{1}{2}t^2\tau\big(\delta_1(h)^2 + \delta_2(h)^2\big) + O(t^3) \\ &= 2\pi^2(m^2+n^2) + t^2\,E(h) + O(t^3). \end{aligned}$$

By Proposition 2.11, the term in t^2 vanishes exactly when h is a constant, and in that case $E(ue^{iht}) = E(u) = 2\pi^2(m^2+n^2)$ (exactly). Otherwise, the coefficient of t^2 is strictly positive and $E(ue^{iht})$ has a strict local minimum at $t = 0$. □

6. The Laplacian and holomorphic geometry

As we have seen, Δ on A_θ behaves very much like the classical Laplacian on $\mathbb{T}^2$. But the Laplacian in (real) dimension 2 is very closely related to holomorphic geometry in complex dimension 1. That suggests that the theory we have developed should be closely related to the Cauchy–Riemann operators ∂ and $\bar{\partial}$ on noncommutative elliptic curves, as developed in references like [Polishchuk and Schwarz 2003; Polishchuk 2004].

In classical analysis (in one complex variable), one usually sets $\bar{\partial} = \frac{1}{2}\big(\frac{\partial}{\partial x_1} + i\,\frac{\partial}{\partial x_2}\big)$, the Cauchy–Riemann operator, with ∂ its complex conjugate. Then $\Delta = 4\,\partial\,\bar{\partial}$. In our situation, the obvious analogue is to set $\bar{\partial} = \frac{1}{2}\big(\delta_1 + i\,\delta_2\big)$.[3] Comparable to Proposition 2.4 is:

Proposition 6.1. *The operator* $\bar{\partial}\colon\ A_\theta^\infty \to A_\theta^\infty$ *has kernel given by scalar multiples of the identity, and restricts to a bijection on* $\ker\tau$.

Proof. Immediate from the fact that if $a = \sum_{m,n} c_{m,n} U^m V^n$, then

$$\bar{\partial}a = \pi\, i \sum_{m,n} (m + in)\, c_{m,n} U^m V^n,$$

together with the characterization of elements of A_θ^∞ in terms of rapidly decreasing Fourier series. □

Thus the noncommutative torus admits no nontrivial global holomorphic functions. This is not surprising since a compact complex manifold admits no nonconstant global holomorphic functions. However, assuming $\tau(f) = 0$, we can solve the *inhomogeneous Cauchy–Riemann equation* $\bar{\partial}u = f$, which in the classical case is related to the proof of the Mittag-Leffler Theorem (see, for example, [Hörmander 1990, Chapter 1]).

In some situations, one is led to the more complicated equation $(\bar{\partial}u)\,u^{-1} = f$, (similar to the one above but with $\bar{\partial}$ replaced by the logarithmic Cauchy–Riemann operator. This equation can be rewritten as $\bar{\partial}u = fu$. Is was already studied (under an alternative convention about whether one should multiply on the left or the right) in a (different) noncommutative context in [Bost 1990], and then by Polishchuk:

Theorem 6.2 [Polishchuk 2006]. *Let* $f \in A_\theta$. *Then the equation* $\bar{\partial}u = fu$ *has a nonzero solution if and only if* $\tau(f) \in \pi i(\mathbb{Z} + i\mathbb{Z})$.

(A slightly different convention is used in the given reference, and in [Polishchuk and Schwarz 2003]: in those works, $\bar{\partial}$ is taken as $(x + iy)\delta_1 + \delta_2$, with $y < 0$. When $x = 0$ and $y = -1$, this is what we have here, up to a constant factor of $-2i$. This constant explains why the result looks different. With our convention, $u = U^m V^n$ solves $\bar{\partial}u = fu$ with $f = \pi i(m + in)$.)

The relevance of Theorem 6.2 concerns the theory of noncommutative *meromorphic* functions. While a compact complex manifold admits no nonconstant global holomorphic functions, it can admit non-constant meromorphic functions, such as (in the case of an elliptic curve) elliptic functions like the Weierstraß $\wp$ function. There are two ways we can view meromorphic functions on a Riemann surface M. On the one hand, they can be considered as ratios of holomorphic sections of holomorphic line bundles $\mathscr{L}$ of M. On the other hand, they can be considered as formal quotients of functions that satisfy the Cauchy–Riemann equation.

These points of view, applied to a noncommutative torus, are equivalent via the following reasoning. A holomorphic vector bundle is defined via its module of (smooth) sections, which is a finitely generated

[3] We could also study different conformal structures on the torus, by changing the i here to another complex number in the upper half-plane, but for the problems we will study here, this makes no essential difference.

projective (right) A_θ^∞-module. This module must be equipped with an operator $\overline{\nabla}$ satisfying the basic axiom

$$\overline{\nabla}(s\cdot a)=\overline{\nabla}(s)\cdot a+s\cdot\bar\partial(a).$$

If we assume the module is A_θ^∞ itself (i.e., the vector bundle is of dimension 1, i.e., a line bundle), then this operator is determined by $f=\overline{\nabla}(1)$, in that for any s,

$$\overline{\nabla}(s)=\overline{\nabla}(1\cdot s)=f\cdot s+1\,\bar\partial(s)=\bar\partial(s)+fs.$$

A holomorphic section of the bundle is then a solution s of $\bar\partial(s)+fs=0$.

On the other hand, the natural definition of meromorphic functions is the following.

Definition 6.3. A *meromorphic function on the noncommutative torus A_θ* is a formal quotient $u^{-1}v$, with $u,\ v\in\operatorname{dom}(\bar\partial)\subset A_\theta$, satisfying the Cauchy–Riemann equation (in the sense to be made precise below). Here we don't want to require that u be invertible in A_θ (otherwise $u^{-1}v$ would be holomorphic, hence constant), so we simply want u to be regular (in the sense of not being either a left or right zero divisor), and the inverse is to be interpreted in a formal sense (or in the maximal ring of quotients [Berberian 1982], the algebra of unbounded operators affiliated to the hyperfinite II_1 factor obtained by completing A_θ in its trace representation). Then the condition that $u^{-1}v$ be meromorphic is that

$$0=\bar\partial(u^{-1}v)=\bar\partial(u^{-1})v+u^{-1}\bar\partial v=-u^{-1}\bar\partial(u)u^{-1}v+u^{-1}\bar\partial v,$$

or (via multiplication by u on the left) that

$$\bar\partial v=fv,\quad \bar\partial u=fu, \tag{6-1}$$

which says precisely that our meromorphic function is a quotient of two holomorphic sections of a holomorphic line bundle with $\overline{\nabla}=\bar\partial+f$. In the other direction, if u and v satisfy (6-1) and u is regular, so that the formal expression $u^{-1}v$ makes sense, then we formally have

$$\begin{aligned}\bar\partial(u^{-1}v)&=\bar\partial(u^{-1})v+u^{-1}\bar\partial v=-u^{-1}\bar\partial(u)u^{-1}v+u^{-1}\bar\partial v\\&=-u^{-1}fuu^{-1}v+u^{-1}fv=-u^{-1}fv+u^{-1}fv=0,\end{aligned}$$

and $u^{-1}v$ is meromorphic.

In accordance with the classical existence theorem of Weierstraß for elliptic functions, we have:

Proposition 6.4. *There exist nonconstant meromorphic functions on the noncommutative torus A_θ, in the sense of Definition* 6.3.

Proof. This follows immediately from the discussion in [Polishchuk 2006, §3], which shows that there are choices for f for which the holomorphic connection $\overline{\nabla}$ is reducible, with a space of holomorphic sections of dimension bigger than 1, and thus there are solutions of (6-1) with u and v not linearly dependent. Note that if this is the case, u cannot be invertible ([Polishchuk 2006, Lemma 3.14]—we also know this independently from Proposition 6.1). But we do require u to be regular, so we need to check that this can be achieved. For example, suppose e is a proper projection in A_θ^∞ ("proper" means $0<\tau(e)<1$) of trace $m+n\theta$ with n relatively prime to both m and $1-m$. The trivial rank-one right A_θ^∞ module splits as $eA_\theta^\infty\oplus(1-e)A_\theta^\infty$, and we can arrange to choose a holomorphic connection on A_θ^∞ that is reducible in a way compatible with this splitting, so that there are 1-dimensional spaces of holomorphic sections on each of eA_θ^∞ and $(1-e)A_\theta^\infty$. By the explicit formulas in [Polishchuk and Schwarz 2003, Proposition 2.5], these come from real-analytic functions in $\mathcal{S}(\mathbb{R})$, and so it's evident that the u that results from putting these together is regular, as by [Berberian 1982], it's enough to observe that its left and right support projections are equal to 1. □

On the other hand, there is also a nonexistence result for meromorphic functions on the (classical) torus: no such nonconstant function exists with only a single simple pole [Ahlfors 1978, Corollary to Theorem 4, p. 271]. We can find an analogue of this in the noncommutative situation also. To explain it, first note that in the sense of distributions on the complex plane, $\bar\partial\left(\frac{1}{z}\right)$ is not zero (if it were, $\frac{1}{z}$ would have a removable singularity, by elliptic regularity), but rather is equal to $\pi\,\delta$, where δ is the Dirac δ-distribution at 0. Suppose there were a meromorphic function f on $\mathbb{T}^2 = \mathbb{C}/(\mathbb{Z}+i\mathbb{Z})$ with at worst one simple pole and no other poles. Then f would be locally integrable and, after translation to move the pole to 0, would define a distribution on $\mathbb{T}^2$ with $\bar\partial(f)$ a multiple of δ. Thus the Fourier series of $\bar\partial(f)$ would be a multiple of the Fourier series of δ, which is $\sum_{m,n} U^m V^n$. And in fact Fourier analysis gives another proof of the nonexistence theorem not using residue calculus. Suppose f were nonconstant. Since a compact complex manifold admits no nonconstant holomorphic functions, f cannot be holomorphic, which means that $\bar\partial f$ must be nonzero in the sense of distributions. Since $\bar\partial(f)$ is a multiple of $\sum_{m,n} U^m V^n$, the proportionality constant, which is also the (0, 0) Fourier coefficient of $\bar\partial f$, must be nonzero. But this is impossible since the Fourier series of any distribution in the image on $\bar\partial$ must have zero constant term. The noncommutative analogue of all this is the following:

Proposition 6.5. *Let f be a distribution in the dual of A_θ^∞. (The distributions consist of formal Fourier series $\sum_{m,n} c_{m,n} U^m V^n$ with $\{c_{m,n}\}$ of tempered growth.) Suppose $\bar\partial f$ is a multiple of $\sum_{m,n} U^m V^n$. Then f is a constant.*

Proof. This follows exactly the lines as the argument above for the classical theorem. If $\bar\partial f$ has formal Fourier expansion $c\sum_{m,n} U^m V^n$, then the (m,n) coefficient, c, must be divisible by $m+in$ for all (m,n). Because of the (0, 0) coefficient, this is only possible if $c=0$. But if $c=0$, then f is in the distributional kernel of $\bar\partial$, which forces all the Fourier coefficients of f to vanish except for the constant term. □

In fact, essentially the same proof proves a slightly more general statement, which in the classical case is equivalent to [Ahlfors 1978, Theorem 4, p. 271]. For the analysis above shows that the sum of the residues of a meromorphic function f on $\mathbb{T}^2$, when the function is considered as a distribution[4], is precisely the constant term in the Fourier series of $\bar\partial f$, up to a factor of π. The analogue of the sum of the residues theorem in the noncommutative world is this:

Proposition 6.6. *Let f be a distribution in the dual of A_θ^∞. Then the constant term in the (formal) Fourier series of $\bar\partial f$ is zero.*

Proof. Essentially the same as before. □

The connection with the main subject of this paper is of course that meromorphic functions w as studied in this section are singular solutions of Laplace's equation $\Delta w = 0$, since $\Delta = 4\,\partial\,\bar\partial$. More precisely, "singular solution" means classically that as a distribution, Δw is not necessarily 0, but has

[4]This requires a comment. A meromorphic function with simple poles is locally integrable, thus defines a distribution in the obvious way. A meromorphic function with higher-order poles is not locally integrable, but can be made into a distribution of principal value integral type. This distribution is not a measure.

countable support. In the noncommutative setting, we do not have a notion of support for a distribution, but the same basic idea applies.

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