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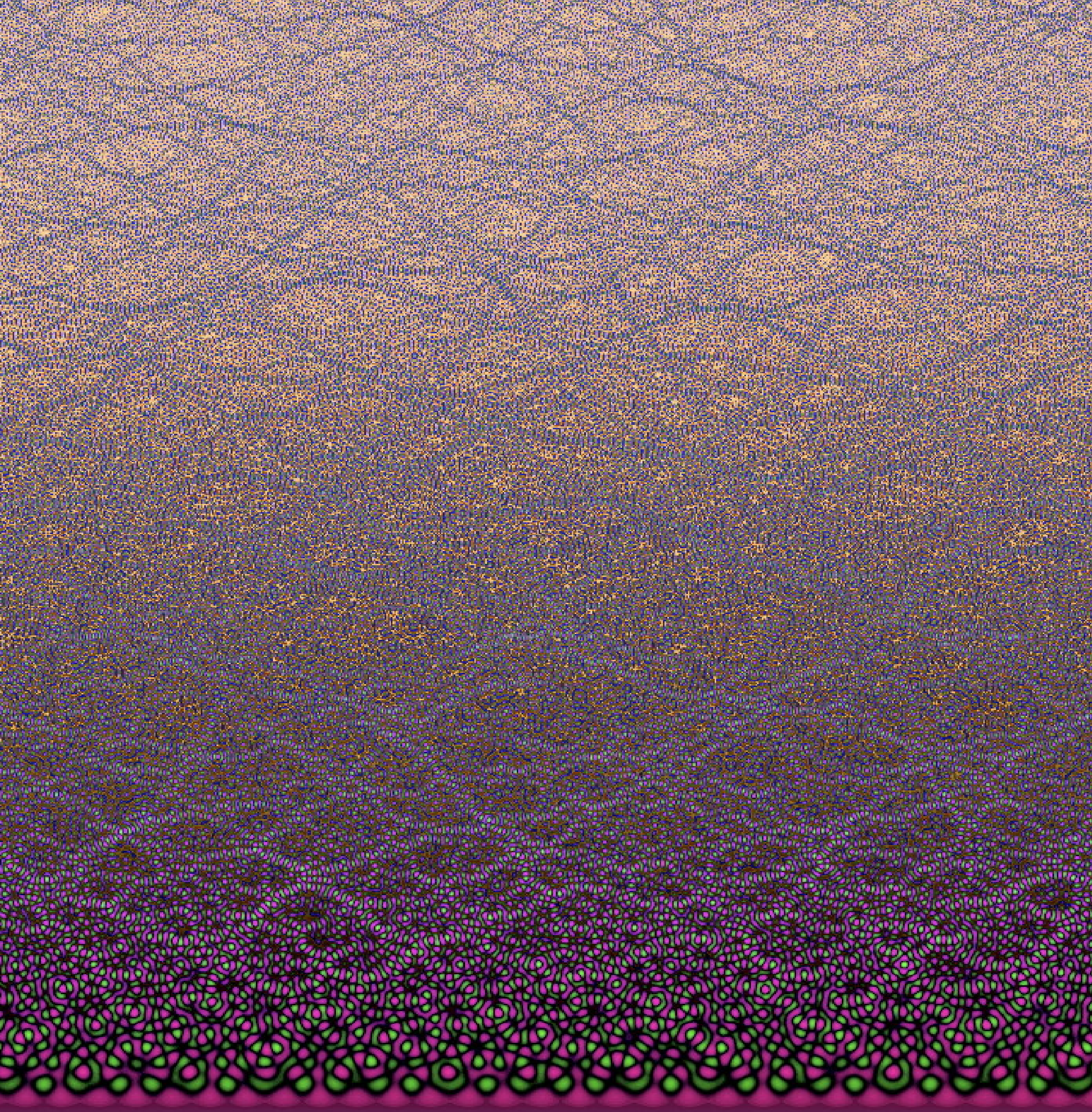

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LOCAL ESTIMATES AND GLOBAL CONTINUITIES IN LEBESGUE SPACES FOR BILINEAR OPERATORS

FRÉDÉRIC BERNICOT

In this paper, we first prove some local estimates for bilinear operators (closely related to the bilinear Hilbert transform and similar singular operators) with truncated symbol. Such estimates, in accordance with the Heisenberg uncertainty principle correspond to a description of "off-diagonal decay". In addition they allow us to prove global continuities in Lebesgue spaces for bilinear operators with spatial dependent symbol.

1. Introduction

The simplest bilinear operator is the pointwise product Π, defined by

$$\Pi(f, g)(x) := f(x)g(x),$$

for all $f, g \in \mathcal{S}(\mathbb{R})$. The Hölder inequalities give us the continuities on Lebesgue spaces for this operator. So for all exponents $p, q, r \in (0, \infty]$ such that

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r}, \tag{1-1}$$

the operator Π is continuous from $L^p(\mathbb{R}) \times L^q(\mathbb{R})$ into $L^r(\mathbb{R})$. Also a natural question appears: How can we modify this bilinear operation and simultaneous keep these continuities?

First let T be a bilinear operator, acting from $\mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R})$ into $\mathcal{S}'(\mathbb{R})$. It is well known that we have a spatial representation of T with a kernel $K \in \mathcal{S}'(\mathbb{R}^3)$ and a frequency representation with a symbol $\sigma \in \mathcal{S}'(\mathbb{R}^3)$ such that (in distributional sense)

$$\begin{aligned} T(f, g)(x) &= \int_{\mathbb{R}^2} K(x, y, z) f(y) g(z)\, dy\, dz \\ &= \int_{\mathbb{R}^2} e^{ix(\alpha+\beta)} \sigma(x, \alpha, \beta) \widehat{f}(\alpha) \widehat{g}(\beta)\, d\alpha\, d\beta, \end{aligned} \tag{1-2}$$

for all $f, g \in \mathcal{S}(\mathbb{R})$. In the rest of this paper, we denote by T_σ the operator associated to the symbol σ. The kernel and the symbol are related by the relation

$$K(x, y, z) = \int_{\mathbb{R}^2} e^{i(\alpha(x-y)+\beta(x-z))} \sigma(x, \alpha, \beta)\, d\alpha\, d\beta.$$

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For example, the product operator Π is given by the symbol

$$\sigma(x,\alpha,\beta)=1.$$

One of the first classes of bilinear symbols to be studied was the class of symbols satisfying the *bilinear Hörmander condition*: For all $a,b,c\geq 0$,

$$\left|\partial_x^a\partial_\alpha^b\partial_\beta^c\sigma(x,\alpha,\beta)\right|\lesssim(1+|\alpha|+|\beta|)^{-b-c}. \tag{1-3}$$

The corresponding operators T_σ were studied by R. Coifman and Y. Meyer [1978; 1975], C. Kenig and E. M. Stein [1999] and recently by L. Grafakos and R. Torres [2002]. We know that under (1-3), the operator T_σ is bounded from $L^p(\mathbb{R})\times L^q(\mathbb{R})$ into $L^r(\mathbb{R})$ for all exponents p,q,r satisfying (1-1) and $1<p,q<\infty$. In fact if the symbol is x-independent, one can just assume an homogeneous decay in (1-3) (that is with $(|\alpha|+|\beta|)^{-b-c}$) and then these operators can be decomposed with paraproducts, which were first exploited by J. M. Bony [1981] and R. Coifman and Y. Meyer [1978]. The paraproducts are studied with the linear tools (the Calderón–Zygmund decomposition, the Littlewood–Paley theory and the concept of Carleson measure). In order to get the continuities for x-dependent symbols, pointwise estimates of the bilinear kernel are used. Mainly for a symbol σ satisfying (1-3), integrations by parts allow us to obtain

$$|K(x,y,z)|\lesssim(1+|x-y|+|x-z|)^{-M} \tag{1-4}$$

for any large enough integer M. This estimate is very useful and precisely describes the "off-diagonal decrease" of the operator. Such an information helps us to reduce the study of x-dependent symbols to the study of x-independent symbols (and so to the study of paraproducts). Through these ideas, this first class of symbols are well understood nowadays. We note that this reduction (using pointwise estimates on the kernel) has already been used in the linear case to study the pseudo-differential operators of the well-known class $\mathrm{op}(S^0_{1,0})$. Thus "off-diagonal estimates" play an important role.

Since the work of A. Calderón [1965; 1977] in the 70's about the L^2 boundedness of commutators and Cauchy integrals, more singular bilinear operators have appeared. Mainly, he showed that the commutators and Cauchy integrals can be decomposed by using the bilinear Hilbert transforms. The bilinear Hilbert transform H_{λ_1,λ_2} is defined by

$$H_{\lambda_1,\lambda_2}(f,g)(x):=p.v.\int_{\mathbb{R}}f(x-\lambda_1 y)g(x-\lambda_2 y)\frac{dy}{y},$$

for all $f,g\in\mathcal{S}(\mathbb{R})$. The x-independent symbol is

$$\sigma(\alpha,\beta)=i\pi\ \mathrm{sign}(\lambda_1\alpha+\lambda_2\beta)$$

and so is singular on a whole line in the frequency plane. A. Calderón conjectured that these operators are continuous on Lebesgue spaces. This famous conjecture was first partially solved by M. Lacey and C. Thiele [1997a; 1997b; 1998; 1999]. Then some uniform (with respect to the parameters λ_1 and λ_2) continuities were shown in [Grafakos and Li 2004; Li 2006]. These proofs use a technical time frequency analysis, which was proven by C. Muscalu, T. Tao and C. Thiele [2002a; 2002b; 2004] and independently by J. Gilbert and A. Nahmod [2000; 2002]. They also get a very important result in the study of bilinear operators: continuities in Lebesgue spaces for more singular operators than those of the first class. We are interested by these bilinear operators and we will deal with them and some "smooth spatial perturbations". So we replace in (1-3) the quantity

$$|\alpha| + |\beta| = d((\alpha, \beta), 0)$$

by the lower quantity $d((\alpha, \beta), \Delta)$, where Δ is a line in the frequency plane:

$$\Delta := \{(\alpha, \beta) \in \mathbb{R}^2, \lambda_1\alpha + \lambda_2\beta = 0\}.$$

We assume that Δ is nondegenerate, that is, λ_1 and λ_2 are nonvanishing reals and not equal, in order that Δ be a graph over the three variables α, β and $\alpha + \beta$. We assume that the symbol σ satisfies

$$\left|\partial_x^a \partial_\alpha^b \partial_\beta^c \sigma(x, \alpha, \beta)\right| \lesssim (1 + |\lambda_1\alpha + \lambda_2\beta|)^{-b-c}, \tag{1-5}$$

for all $a, b, c \geq 0$. In the previous mentioned papers, the main result is this: If σ is x-independent and satisfies (1-5) (or the homogeneous version) then T_σ is continuous from $L^p(\mathbb{R}) \times L^q(\mathbb{R})$ into $L^r(\mathbb{R})$ for every exponents $p, q, r \in (0, \infty]$ satisfying

$$0 < \frac{1}{r} = \frac{1}{p} + \frac{1}{q} < \frac{3}{2} \quad \text{and} \quad 1 < p, q \leq \infty.$$

So there is a natural question (asked in [Bényi et al. 2006]): If an x-dependent symbol satisfies (1-5), is the operator T_σ continuous from $L^p(\mathbb{R}) \times L^q(\mathbb{R})$ into $L^r(\mathbb{R})$ with the same exponents p, q and r? A. Benyi, C. Demeter, A. Nahmod, R. Torres, C. Thiele and P. Villarroya [2007] proved a general result for singular integral kernels. As an example, they can apply their result to pseudo-differential operators associated to symbols

$$\sigma(x, \alpha, \beta) = \tau(x, \lambda_1\alpha + \lambda_2\beta)$$

with τ in the class $S^0_{1,0}$ because of a modulation invariant condition imposed. Here we are able to treat general symbols satisfying (1-5) and complete the answer to the question in [Bényi et al. 2006]. These operators do not fall under the scope of [Benyi et al. 2007] because they do not have modulation invariance. On the other hand, the general operators in [Benyi et al. 2007] cannot be realized as pseudo-differential bilinear operators with symbols satisfying (1-5) because of the minimal regularity assumptions required in the kernels.

With this aim, we would like to use the same arguments as for the symbols satisfying (1-3), where we have seen the important role of the "off-diagonal decay" of the bilinear kernel, obtained with integrations by parts. For our more singular operators, integration by parts does not work: To obtain a description of "off-diagonal estimates" is the most important difficulty.

We now come to our main result. For notation, we denote the norm in $L^p(E)$ for any measurable set $E \subset \mathbb{R}$ by

$$\|\cdot\|_{p,E,dx}$$

(or $\|\cdot\|_{p,E}$ if there is no confusion for the variable). For an interval I, we set

$$C_k(I) := \left\{x \in \mathbb{R},\ 2^k \leq 1 + \frac{d(x, I)}{|I|} < 2^{k+1}\right\}$$

the scaled corona around I. So we have

$$C_0(I)=2I \quad \text{and} \quad C_k(I)\subset 2^{k+1}I.$$

We will first prove:

Theorem 1.1. *Let Δ be a nondegenerate line of the frequency plane. Let p,q be exponents such that*

$$1<p,q\leq\infty \quad \text{and} \quad 0<\frac{1}{r}=\frac{1}{q}+\frac{1}{p}<\frac{3}{2}.$$

Then for all $\delta\geq 1$, there is a constant

$$C=C(p,q,r,\Delta,\delta)$$

such that for all interval $I\subset\mathbb{R}$, for all symbol $\sigma\in C^\infty(\mathbb{R}^3)$ satisfying for all $a,b,c\geq 0$,

$$\left|\partial_x^a\partial_\alpha^b\partial_\beta^c\sigma(x,\alpha,\beta)\right|\lesssim\left(|I|^{-1}+d((\alpha,\beta),\Delta)\right)^{-b-c}, \tag{1-6}$$

we have the following local estimate: For all functions $f,g\in\mathcal{S}(\mathbb{R})$,

$$\left(\frac{1}{|I|}\int_I|T_\sigma(f,g)(x)|^r\,dx\right)^{1/r} \leq C\left(\sum_{k\geq 0}2^{-k\delta}\left(\frac{1}{|2^{k+1}I|}\int_{C_k(I)}|f(x)|^p\,dx\right)^{1/p}\right)\left(\sum_{k\geq 0}2^{-k\delta}\left(\frac{1}{|2^{k+1}I|}\int_{C_k(I)}|g(x)|^q\,dx\right)^{1/q}\right).$$

In particular, with the Hardy–Littlewood operator M_{HL}, we have

$$\left(\frac{1}{|I|}\int_I|T_\sigma(f,g)(x)|^r\,dx\right)^{1/r}\lesssim\inf_I M_{HL}(|f|^p)^{1/p}\inf_I M_{HL}(|g|^q)^{1/q}\lesssim\|f\|_\infty\|g\|_\infty.$$

The weight

$$\left(|I|^{-1}+d((\alpha,\beta),\Delta)\right)^{-N}$$

is not integrable over the whole frequency plane (even if N is large enough due to the modulation invariance) and therefore we cannot have a pointwise estimate of the bilinear kernel (such as (1-4) when we assume (1-3)). So such a result is interesting because it precisely describes "off-diagonal estimates" for the bilinear operator:

Corollary 1.2. *With the same notations as Theorem 1.1, for all large enough δ, there exists a constant*

$$C=C(p,q,r,\Delta,\delta)$$

such that for any measurable sets $E,F\subset\mathbb{R}$ we have for all functions $f\in L^p(E)$ and $g\in L^q(F)$:

$$\|T_\sigma(f,g)\|_{r,I}\leq C\left(1+\frac{d(I,E)}{|I|}\right)^{-\delta}\left(1+\frac{d(I,F)}{|I|}\right)^{-\delta}\|f\|_{p,E}\|g\|_{q,F}.$$

This corollary is a direct application of Theorem 1.1. So in spite of the fact that the symbol could be much more singular than those satisfying only (1-3), we almost obtain the pointwise estimate (1-4). Here we have a description of the same fast decrease for the bilinear kernel, not with a pointwise estimate, but with local estimates at the scale $|I|$. These local estimates are less precise than the pointwise estimate but we will see that they are sufficient and they can play the same role.

We note that Theorem 1.1 is in accordance with the Heisenberg's Uncertainty Principle, which tells us that if we want to localize at the scale $|I|$ in the spatial domain, we cannot localize in the frequency domain at a lower scale than $|I|^{-1}$. For example, our Theorem 1.1 applies if the symbol is supported in the domain

$$\left\{(\alpha,\beta),\ d((\alpha,\beta),\Delta)\geq |I|^{-1}\right\}$$

and it is this case that we consider first in the proof. In fact in (1-6), we allow instead a nice behavior around the line Δ. With this point of view, we could call Theorem 1.1 an "high frequency estimate". In this expression, the term "frequency" corresponds to the distance between the point (α,β) to the line of singularity Δ. We prefer the expression "local estimates", because we will use the fast spatial decay in order to get the following result.

Theorem 1.3. *Let Δ be a nondegenerate line of the frequency plane. Let p and q be exponents such that*

$$1<p,q\leq\infty \quad \text{and} \quad 0<\frac{1}{r}=\frac{1}{q}+\frac{1}{p}<\frac{3}{2}.$$

For all symbol $\sigma\in C^\infty(\mathbb{R}^3)$ satisfying for all $a,b,c\geq 0$,

$$\left|\partial_x^a\partial_\alpha^b\partial_\beta^c\sigma(x,\alpha,\beta)\right|\lesssim\left(1+d((\alpha,\beta),\Delta)\right)^{-b-c},$$

the associated operator T_σ is bounded from $L^p(\mathbb{R})\times L^q(\mathbb{R})$ into $L^r(\mathbb{R})$.

This result answers a question of [Bényi et al. 2006]. In addition it will allow us to define a bilinear pseudo-differential calculus, based on these operators: In our next paper [Bernicot 2008], we will define classes for bilinear pseudo-differential operators of order (m_1,m_2) and study their action on Sobolev spaces. In order to carry on the work of [Bényi et al. 2006], we will give rules of symbolic calculus for the duality and the composition and also complete the construction of a bilinear pseudo-differential calculus.

Remark 1.4. The proof of Theorem 1.1 is a shake between a localization argument and the "classical" time-frequency analysis used for these bilinear operators. So it is quite easy to obtain a version of our Theorems 1.1 and 1.3 for $(n-1)$-linear operators T_σ with a nondegenerate space Δ of dimension $k<\frac{n}{2}$, by following the ideas of [Muscalu et al. 2002a]. By using the results of [Terwilleger 2007], we are able to obtain the same results for a multidimensional problem and by using the uniform estimates of [Muscalu et al. 2002b], it seems possible to obtain uniform (with respect to the nondegenerate line Δ) local estimates.

The plan of this article is as follows. We first prove Theorem 1.1 in Section 2 for x-independent symbols. Then in Section 3 we get the same result for maximal bilinear operators and we conclude the proof of Theorem 1.1 in the general case. Then in Section 4, we use these local estimates to obtain global continuities for bilinear operators in weighted Lebesgue and Sobolev spaces and in particular we prove Theorem 1.3.

2. Proof of Theorem 1.1 for x-independent symbol

In this section, we assume that the symbol σ is x-independent and is supported on the domain

$$\left\{(\alpha,\beta),\ d((\alpha,\beta),\Delta)\geq |I|^{-1}\right\}.$$

We divide the proof into two subsections. First, we will quickly recall the decomposition of our bilinear operator T_σ by combinatorial model sums. So we will reduce the problem to a study of the "restricted weak type" for some localized trilinear forms. Then we will study them in the proof of Theorem 2.4 (see page 9).

Reduction to the study of discrete models. First of all, we define and recall the time-frequency tools (see for example [Muscalu et al. 2004]):

Definition 2.1. A *tile* is a rectangle (that is, a product of two intervals) $I \times \omega$ of area one. A *tritile* s is a rectangle $s = I_s \times \omega_s$ of bounded area, which contains three tiles $s_i = I_{s_i} \times \omega_{s_i}$ $(i = 1, 2, 3)$ such that, for all $i, j \in \{1, 2, 3\}$,

$$I_{s_i} = I_s \qquad \text{and} \qquad i \neq j \Rightarrow \omega_{s_i} \cap \omega_{s_j} = \varnothing.$$

A set $\{I\}_{I\in\mathcal{J}}$ of real intervals is called a *grid* if for all $k \in \mathbb{Z}$,

$$\sum_{\substack{I\in\mathcal{J}\\ 2^{k-1}\leq|I|\leq 2^{k+1}}} \mathbf{1}_I \lesssim \mathbf{1}_{\mathbb{R}}, \tag{2-1}$$

where the implicit constant is independent of k and of the grid. So a grid has the same structure as the dyadic grid.

Let $\mathbf{Q}$ be a set of tritiles. It is called a *collection* if

- $\{I_s,\ s \in \mathbf{Q}\}$ is a grid,
- $\mathcal{J} := \{\omega_s,\ s \in \mathbf{Q}\} \cup \bigcup_{i=1}^{3} \{\omega_{s_i},\ s \in \mathbf{Q}\}$ is a grid, and
- $\omega_{s_i} \subsetneq \varpi \in \mathcal{J} \Rightarrow$ for all $j \in \{1, 2, 3\},\ \omega_{s_j} \subset \varpi$.

Now we can define the wave packet for a tile.

Definition 2.2. Let Φ be a smooth function such that

$$\|\Phi\|_2 = 1 \quad \text{and} \quad \operatorname{supp}(\widehat{\Phi}) \subset [-\tfrac{1}{2}, \tfrac{1}{2}].$$

For $P = I \times \omega$ a tile, we set

$$\Phi_P(x) := |I|^{-1/2}\Phi\Big(\frac{x - c(I)}{|I|}\Big)e^{ixc(\omega)},$$

where for U an interval we denote by $c(U)$ its center. So Φ_P is normalized in the $L^2(\mathbb{R})$ space, concentrated in space around I and its spectrum is exactly contained in ω.

Nowadays it is well known (see for example [Bilyk and Grafakos 2006a; 2006b]) that the operator T_σ of Theorem 1.1 admits a decomposition

$$T_\sigma(f, g)(x) := \sum_{u=(u_1,u_2,u_3)\in\mathbb{Z}^3} (1 + |u|^2)^{-N} \sum_{s\in\mathbf{S}_u} |I_s|^{-1/2}\epsilon_s(u)\big\langle(\tau_{u_1}\phi)_{s_1}, f\big\rangle\big\langle(\tau_{u_2}\phi)_{s_2}, g\big\rangle(\tau_{u_3}\phi)_{s_3}(x),$$

where $\mathbf{S}_u$ is a collection of tritiles depending on u, $\epsilon_s(u)$ are bounded reals for $s \in \mathbf{S}_u$, and N is an integer as large as we want. We write τ_v for the translation operator

$$\tau_v(f)(x) = f(x - v).$$

The coefficients $\epsilon_s(u)$ are uniformly bounded with respect to the parameter u and the implicit constant in (2-1) (for the definition of a grid) is bounded by the estimates of the symbol σ.

By using the assumption that σ is supported in

$$\{(\alpha, \beta),\ |\alpha - \beta| \geq |I|^{-1}\},$$

we have the very important property

$$|\omega_s| \gtrsim |I|^{-1}, \tag{2-2}$$

which is equivalent to

$$|I_s| \lesssim |I|,$$

for all $u \in \mathbb{Z}^3$, and for all $s \in \mathbf{S}_u$.

So Theorem 1.1 is a consequence of the following theorem.

Theorem 2.3. *Let* $\mathbf{S}$ *be a collection of tritiles satisfying the property* (2-2), $(\epsilon_s)_{s\in\mathbf{S}}$ *bounded reals and* $(\phi^i)_{i=1,2,3}$ *smooth functions whose spectrum is contained in* $[-\frac{1}{2}, \frac{1}{2}]$. *We denote* $T_{\mathbf{S}}$ *the bilinear operator defined by*

$$T_{\mathbf{S}}(f, g)(x) := \sum_{s\in\mathbf{S}} |I_s|^{-1/2} \epsilon_s \langle \phi^1_{s_1}, f\rangle \langle \phi^2_{s_2}, g\rangle \phi^3_{s_3}(x).$$

Then for the exponents (p, q, r) *of Theorem 1.1 and for all* $\delta \geq 1$, *we have the local estimate*

$$\left(\int_I |T_{\mathcal{S}}(f, g)|^r\right)^{1/r} \lesssim \left(\sum_{k\geq 0} 2^{-k(1/p+\delta)} \|f\|_{p,C_k(I)}\right)\left(\sum_{k\geq 0} 2^{-k(1/q+\delta)} \|g\|_{q,C_k(I)}\right).$$

In addition the implicit constant depends on the functions ϕ^i *by the parameters*

$$c_M(\phi^i) := \sup_{x\in\mathbb{R}} \sum_{0\leq k\leq M} (1 + |x|)^M \left|(\phi^i)^k(x)\right|$$

for $M = M(p, q, r, \delta)$ *a large enough integer.*

In order to show this result, we need to decompose the functions f and g around the interval I. The interval I being fixed, we omit it in the notation for convenience and for $i \in \mathbb{N}$, we set the corona $C_i := C_i(I)$. With the property (2-2), we have the decomposition

$$T_{\mathbf{S}}(f, g) = \sum_{k_1,k_2\geq 0} T^{k_1,k_2}_{\mathbf{S},0}(f, g) + \sum_{\substack{k_1,k_2\geq 0\\ l\leq 0}} T^{k_1,k_2,l}_{\mathbf{S},1}(f, g) \tag{2-3}$$

with

$$T^{k_1,k_2}_{\mathbf{S},0}(f, g)(x) := \sum_{\substack{s\in\mathbf{S}\\ I_s\subset 2I}} |I_s|^{-1/2} \epsilon_s \langle \phi^1_{s_1}, f\mathbf{1}_{C_{k_1}}\rangle \langle \phi^2_{s_2}, g\mathbf{1}_{C_{k_2}}\rangle \phi^3_{s_3}(x),$$

$$T^{k_1,k_2,l}_{\mathbf{S},1}(f, g)(x) := \sum_{\substack{s\in\mathbf{S},\ I_s\not\subset 2I\\ 2^l|I|\leq|I_s|<2^{l+1}|I|}} |I_s|^{-1/2} \epsilon_s \langle \phi^1_{s_1}, f\mathbf{1}_{C_{k_1}}\rangle \langle \phi^2_{s_2}, g\mathbf{1}_{C_{k_2}}\rangle \phi^3_{s_3}(x).$$

Due to the important property (2-2), we only have to consider tiles s with $|I_s| \leq |I|$. The other terms (corresponding to $l > 0$) cannot be studied as we are going to do, according to the Heisenberg Uncertainty Principle.

Starting on page 9, we shall prove the following theorem.

Theorem 2.4. *Let (p,q,r) be exponents as in Theorem 1.1. The operators $T^j_{\mathbf{S},i}$ are continuous from $L^p(\mathbb{R})\times L^q(\mathbb{R})$ into $L^r(I)$. For convenience, we denote*

$$C(T^j_{\mathbf{S},i}) := \|T^j_{\mathbf{S},i}\|_{L^p\times L^q\to L^r}$$

and we omit the exponents. Then these continuity bounds satisfy

$$C(T^{k_1,k_2}_{\mathbf{S},0}) \lesssim c_M(\phi^1)c_M(\phi^2)c_M(\phi^3)\, 2^{-\delta'(k_1+k_2)},$$
$$C(T^{k_1,k_2,l}_{\mathbf{S},1}) \lesssim c_M(\phi^1)c_M(\phi^2)c_M(\phi^3)\, 2^{-\delta'(|l|+k_1+k_2)}$$

for any large enough real δ', with an integer $M=M(p,q,r,\delta')$.

We claim that Theorem 2.3 is a consequence of Theorem 2.4.

Proof of Theorem 2.3. By using Theorem 2.4 and the decomposition (2-3), we have that for all functions $f,g\in\mathcal{S}(\mathbb{R})$,

(i) if $r\geq 1$, then

$$\|T_{\mathbf{S}}(f,g)\|_{r,I} \lesssim \sum_{k_1,k_2\geq 0} C(T^{k_1,k_2}_{\mathbf{S},0})\|f\mathbf{1}_{C_{k_1}}\|_p\|g\mathbf{1}_{C_{k_2}}\|_q + \sum_{\substack{k_1,k_2\geq 0\\ l\leq 0}} C(T^{k_1,k_2,l}_{\mathbf{S},1})\|f\mathbf{1}_{C_{k_1}}\|_q\|g\mathbf{1}_{C_{k_2}}\|_r;$$

(ii) if $r<1$, then

$$\|T_{\mathbf{S}}(f,g)\|^r_{r,I} \lesssim \sum_{k_1,k_2\geq 0} C(T^{k_1,k_2}_{\mathbf{S},0})^r\|f\mathbf{1}_{C_{k_1}}\|^r_p\|g\mathbf{1}_{C_{k_2}}\|^r_q + \sum_{\substack{k_1,k_2\geq 0\\ l\leq 0}} C(T^{k_1,k_2,l}_{\mathbf{S},1})^r\|f\mathbf{1}_{C_{k_1}}\|^r_p\|g\mathbf{1}_{C_{k_2}}\|^r_q.$$

Case (i) ($r\geq 1$): With the estimate of $C(T^{k_1,k_2}_{\mathbf{S},0})$ and $C(T^{k_1,k_2,l}_{\mathbf{S},1})$ given by Theorem 2.4, we obtain

$$\begin{aligned}\|T_{\mathbf{S}}(f,g)\|_{r,I} &\lesssim \sum_{k_1,k_2\geq 0} 2^{-\delta'(k_1+k_2)}\|f\mathbf{1}_{C_{k_1}}\|_p\|g\mathbf{1}_{C_{k_2}}\|_q + \sum_{\substack{k_1,k_2\geq 0\\ l\leq 0}} 2^{-\delta'(k_1+k_2+|l|)}\|f\mathbf{1}_{C_{k_1}}\|_p\|g\mathbf{1}_{C_{k_2}}\|_q\\ &\lesssim \sum_{k_1,k_2\geq 0} 2^{-\delta'(k_1+k_2)}\|f\mathbf{1}_{C_{k_1}}\|_p\|g\mathbf{1}_{C_{k_2}}\|_q.\end{aligned}$$

Hence by using that δ' is as large as we want, the conclusion follows for case (i).

Case (ii) ($r\leq 1$): We have

$$\begin{aligned}\|T_{\mathbf{S}}(f,g)\|^r_{r,I} &\lesssim \sum_{k_1,k_2\geq 0} 2^{-r\delta'(k_1+k_2)}\|f\mathbf{1}_{C_{k_1}}\|^r_p\|g\mathbf{1}_{C_{k_2}}\|^r_q + \sum_{\substack{k_1,k_2\geq 0\\ l\leq 0}} 2^{-r\delta'(k_1+k_2+|l|)}\|f\mathbf{1}_{C_{k_1}}\|^r_p\|g\mathbf{1}_{C_{k_2}}\|^r_q\\ &\lesssim \sum_{k_1,k_2\geq 0} 2^{-r\delta'(k_1+k_2)}\|f\mathbf{1}_{C_{k_1}}\|^r_p\|g\mathbf{1}_{C_{k_2}}\|^r_q.\end{aligned}$$

By using Hölder's inequality and $\rho>0$ such that

$$\frac{1}{p}+\rho,\ \frac{1}{q}+\rho<1,$$

we obtain

$$\|T_{\mathbf{S}}(f,g)\|_{r,I} \lesssim \Big(\sum_{k_1\geq 0} 2^{-k_1 p(\delta'-1)(\rho+1/p)}\|f\mathbf{1}_{C_{k_1}}\|_p^p\Big)^{1/p}\Big(\sum_{k_2\geq 0} 2^{-k_2 q(\delta'-1)(\rho+1/q)}\|g\mathbf{1}_{C_{k_2}}\|_q^q\Big)^{1/q}$$
$$\lesssim \Big(\sum_{k_1\geq 0} 2^{-k_1(\delta'-1)(\rho+1/p)}\|f\mathbf{1}_{C_{k_1}}\|_p\Big)\Big(\sum_{k_2\geq 0} 2^{-k_2(\delta'-1)(\rho+1/q)}\|g\mathbf{1}_{C_{k_2}}\|_q\Big).$$

This corresponds to the desired result (the real δ' being as large as we want) for case **(ii)**. □

We have also reduced the proof of Theorem 1.1 (for our particular symbol σ) to that of Theorem 2.4.

Proof of Theorem 2.4. By using "duality", to prove Theorem 2.4, we have to estimate the trilinear form defined on $\mathcal{S}(\mathbb{R})\times\mathcal{S}(\mathbb{R})\times\mathcal{S}(\mathbb{R})$ by

$$\Lambda_i^j(f_1,f_2,f_3) := \langle T^j_{\mathbf{S},i}(f_1,f_2), f_3\mathbf{1}_I\rangle = \sum_{s\in\mathbf{Q}_i^j} |I_s|^{-1/2}\epsilon_s\langle\phi^1_{s_1}, f_1\mathbf{1}_{C_{k_1}}\rangle\langle\phi^2_{s_2}, f_2\mathbf{1}_{C_{k_2}}\rangle\langle\phi^3_{s_3}, f_3\mathbf{1}_I\rangle, \tag{2-4}$$

where $\mathbf{Q}_i^j$ is a collection of tritiles, depending on $T^j_{\mathbf{S},i}$.

We need to define the usual tools of time-frequency analysis.

Definition 2.5. We have already defined the tritiles. For $j\in\{1,2,3\}$ an index and $t\in\mathbf{S}$ a tritile, a collection $\mathbf{T}$ of tritiles is called a *j-tree* with *top t* if for all $s\in\mathbf{T}$,

$$I_s\subset I_t \quad\text{and}\quad \omega_{t_j}\subset\omega_{s_j}.$$

Then we set

$$I_{\mathbf{T}} := I_t,$$

the time-interval of the tree $\mathbf{T}$. A collection $\mathbf{T}$ of tritiles is called a *tree* if there exists an index $j\in\{1,2,3\}$ such that $\mathbf{T}$ is a j-tree. For $\mathbf{T}$ a j-tree, we define the *size* of the function f_j over this tree by

$$\operatorname{size}_j(\mathbf{T}) := \Big(\frac{1}{|I_{\mathbf{T}}|}\sum_{s\in\mathbf{T}}\big|\langle f_j,\phi^j_{s_j}\rangle\big|^2\Big)^{1/2}.$$

For $\mathbf{Q}$ a collection of tritiles, we define the *global size* by

$$\operatorname{size}_j^*(\mathbf{Q}) = \sup\big\{\operatorname{size}_k(\mathbf{T}) :\ \mathbf{T}\subset\mathbf{Q},\ \mathbf{T}\text{ is a k-tree},\ k\neq j\big\}.$$

The quantity $|I_{\mathbf{T}}|^{1/2}\operatorname{size}_j(\mathbf{T})$ corresponds to the norm of the function f_j in the space L^2, after being restricted on the tree $\mathbf{T}$ in the time-frequency space.

We recall the (abstract) [Muscalu et al. 2004, Proposition 6.5], where [Muscalu et al. 2004, Lemma 6.7] is used to estimate the quantities $\widetilde{energy}_j$.

Proposition 2.6. *Let $(\theta_j)_{1\leq j\leq 3}$ be three exponents of $(0,1)$ satisfying*

$$\theta_1+\theta_2+\theta_3 = 1.$$

Then there exists a constant $C = C(\theta_i)$ such that for all collection $\mathbf{Q}$ *of tritiles, we have*

$$\left|\sum_{s\in\mathbf{Q}} |I_s|^{-1/2}\prod_{i=1}^{3}\langle\phi_{s_i}^i, f_i\rangle\right| \le C\prod_{i=1}^{3}\mathrm{size}_i^*(\mathbf{Q})^{\theta_i}\,\|f_i\|_2^{1-\theta_i}.$$

This result is the main idea of this time-frequency analysis. To prove it, we use a stopping-time argument in order to build an "orthogonal" covering of the time-frequency space with trees of $\mathbf{Q}$.

Now we recall the notion of *restricted weak type* for trilinear forms.

Definition 2.7. For E a Borelian set of $\mathbb{R}$, we write

$$F(E) := \{f \in \mathcal{S}(\mathbb{R}) : \text{ for all } x \in \mathbb{R},\ |f(x)| \le \mathbf{1}_E(x)\}.$$

Let Λ be a trilinear form defined on $\mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R})$. Let p_1, p_2, p_3 be exponents of $\mathbb{R}^*$, possibly negative. We say that Λ is of *restricted weak type* (p_1, p_2, p_3) if there exists a constant C such that for all measurable sets E_1, E_2, E_3 of finite measure, we can find a substantial subset $E'_\beta \subset E_\beta$ (that is, $|E'_\beta| \ge \frac{|E_\beta|}{2}$) for $\beta \in \{1, 2, 3\}$ such that for all $f_\beta \in F(E'_\beta)$,

$$|\Lambda(f_1, f_2, f_3)| \le C\prod_{\beta=1}^{3}|E_\beta|^{1/p_\beta} \tag{2-5}$$

and $E'_\beta = E_\beta$ if $p_\beta > 0$. The best constant in (2-5) is called *the bound of restricted type* and will be denoted by $C(\Lambda)$.

By the real interpolation theory for trilinear forms of restricted weak type [Muscalu et al. 2002b, Lemmas 3.6, 3.7, 3.8, 3.9, 3.10 and 3.11], Theorem 2.4 is a consequence of the following result (which is a stronger continuity result).

Theorem 2.8. *Let p_1, p_2, p_3 be nonvanishing reals such that*

$$\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}=1$$

and there exists a unique index $\alpha \in \{1, 2, 3\}$ with $-\frac{1}{2} < \frac{1}{p_\alpha} < 0$, and $\frac{1}{2} < \frac{1}{p_\beta} < 1$ for $\beta \ne \alpha$. Then the trilinear forms Λ_i^j defined by (2-4) are of restricted weak type (p_1, p_2, p_3). In addition the bounds of restricted type $C(\Lambda_i^j)$ satisfy

$$C(\Lambda_0^{k_1,k_2}) \lesssim c_M(\phi^1)c_M(\phi^2)c_M(\phi^3)2^{-\delta'(k_1+k_2)},$$
$$C(\Lambda_1^{k_1,k_2,l}) \lesssim c_M(\phi^1)c_M(\phi^2)c_M(\phi^3)2^{-\delta'(|l|+k_1+k_2)}$$

for any real $\delta' \ge 1$ with $M = M(\delta', p_i)$ a large enough integer.

Proof. The exponents $(p_\beta)_\beta$ and the index $\alpha \in \{1, 2, 3\}$ are fixed for the proof. Let E_1, E_2 and E_3 be measurable sets of finite measure. First we construct the substantial subset $E'_\alpha \subset E_\alpha$. Denote

$$U := \bigcup_{i=1}^{3}\left\{x \in \mathbb{R},\ M_{HL}(\mathbf{1}_{E_i})(x) > \eta\frac{|E_i|}{|E_\alpha|}\right\}.$$

By using Hardy–Littlewood Theorem, there exists a numerical constant η such that

$$|U| \leq \frac{|E_\alpha|}{2}.$$

We set also $E'_\alpha = E_\alpha \setminus U$. It is interesting to note that the set E'_α does not depend on the form Λ_i^j. Now we fix the functions $f_\beta \in F(E'_\beta)$ for $\beta \in \{1, 2, 3\}$ and we shall prove the inequality (2-5). The proof is divided in three parts: In the first step we use general estimates for collections of tritiles, in the second step we will use specific estimates adapted to the above collections of tritiles and then we will conclude in the third step.

First step: a general estimate. Let $\mathbf{P}$ be an "abstract" collection of tritiles, then for $k \geq 0$ we set $\mathbf{P}_k$ the subcollection

$$\mathbf{P}_k := \left\{ s \in \mathbf{P}, \quad 2^k \leq 1 + \frac{d(I_s, U^c)}{|I_s|} < 2^{k+1} \right\}.$$

These collections form a partition of $\mathbf{P}$:

$$\mathbf{P} = \bigsqcup_{k \geq 0} \mathbf{P}_k.$$

For each $k \geq 0$, we can apply Proposition 2.6 to the collection $\mathbf{Q} = \mathbf{P}_k$. So for any choice of exponents $0 < \theta_1, \theta_2, \theta_3 < 1$ with

$$\sum_{\beta=1}^{3} \theta_\beta = 1,$$

we obtain

$$\Lambda(P_k) := \left| \sum_{s \in \mathbf{P}_k} |I_s|^{-1/2} \epsilon_s \prod_{\beta=1}^{3} \langle f_\beta, \phi_{s_\beta}^\beta \rangle \right| \lesssim \prod_{\beta=1}^{3} (\mathrm{size}^*_\beta(\mathbf{P}_k))^{\theta_\beta} \| f_\beta \|_2^{1-\theta_\beta}.$$

In order to estimate the quantities $\mathrm{size}^*_\beta(\mathbf{P}_k)$, we recall [Muscalu et al. 2002b, Lemma 7.8].

Lemma 2.9. *For all integer N as large as we want, there exists a constant $C = C(N)$ such that for all collection $\mathbf{Q}$ of tritiles, for all $\beta \in \{1, 2, 3\}$, we have*

$$\mathrm{size}^*_\beta(\mathbf{Q}) \leq C \sup_{s \in \mathbf{Q}} \frac{1}{|I_s|} \int_{\mathbb{R}} \left(1 + \frac{d(x, I_s)}{|I_s|}\right)^{-N} |f_\beta(x)| \, dx.$$

Then for $\mathbf{Q} = \mathbf{P}_k$, by using the definition of the sets U and E'_α, we have

$$\mathrm{size}^*_\beta(\mathbf{P}_k) \lesssim 2^k \frac{|E_\beta|}{|E_\alpha|}, \quad \text{and} \quad \mathrm{size}^*_\alpha(\mathbf{P}_k) \lesssim 2^{-Nk}.$$

for all $\beta \neq \alpha$. As f_β belongs to $F(E_\beta)$, we have

$$\| f_\beta \|_2 \leq |E_\beta|^{1/2}.$$

So for $0 < \epsilon < 1$ and N an integer as large as we want, we get

$$\Lambda(P_k) \lesssim \prod_{\beta\neq\alpha}\Big(2^k\frac{|E_\beta|}{|E_\alpha|}\Big)^{\theta_\beta(1-\epsilon)}|E_\beta|^{(1-\theta_\beta)/2}2^{Nk\theta_\alpha(1-\epsilon)}|E_\alpha|^{(1-\theta_\alpha)/2}\prod_{\beta=1}^{3}(\mathrm{size}^*_\beta(\mathbf{P}_k))^{\theta_\beta\epsilon}$$
$$\lesssim 2^{-k}\Big(\prod_{\beta\neq\alpha}|E_\beta|^{(1+\theta_\beta)/2-\epsilon\theta_\beta}|E_\alpha|^{(\theta_\alpha-1)/2+\epsilon(1-\theta_\alpha)}\Big)\Big(\prod_{\beta=1}^{3}(\mathrm{size}^*_\beta(\mathbf{P}_k))^{\theta_\beta\epsilon}\Big).$$

By definition of size^*_β, $\mathbf{P}_k$ is a subcollection of $\mathbf{P}$ so for all $\beta\in\{1,2,3\}$,

$$\mathrm{size}^*_\beta(\mathbf{P}_k)\leq \mathrm{size}^*_\beta(\mathbf{P}).$$

We can also compute the sum over $k\geq 0$ and we obtain

$$\Lambda(P) := \Big|\sum_{s\in\mathbf{P}}|I_s|^{-1/2}\epsilon_s\prod_{\beta=1}^{3}\langle f_\beta,\phi^\beta_{s_\beta}\rangle\Big| \leq \sum_{k\geq 0}P_k$$
$$\lesssim \Big(\prod_{\beta\neq\alpha}|E_\beta|^{(1+\theta_\beta)/2-\epsilon\theta_\beta}|E_\alpha|^{(\theta_\alpha-1)/2+\epsilon(1-\theta_\alpha)}\Big)\Big(\prod_{\beta=1}^{3}(\mathrm{size}^*_\beta(\mathbf{P}))^{\theta_\beta\epsilon}\Big). \quad (2\text{-}6)$$

The first term is "good", according to the wished global continuity. In the next step, we will use another estimate of the quantities size^*_β, which will be adapted to our specific trilinear forms Λ_i^j and which allow us to obtain the desired decays.

Second step: use of the specific form of our trilinear forms Λ_i^j.

First case: the forms Λ_1^j.

In this case, we use another decomposition

$$\Lambda_1^{k_1,k_2,l}(f_1,f_2,f_3)\leq \sum_{\substack{I_0\not\subseteq 2I\\ 2^{l-1}|I|\leq|I_0|\leq 2^{l+1}|I|}}\Lambda_1^{k_1,k_2,l}(I_0)(f_1,f_2,f_3),$$

where I_0 is an interval of $\mathbb{R}$ and

$$\Lambda_1^{k_1,k_2,l}(I_0)(f_1,f_2,f_3) := \sum_{\substack{s\in\mathbf{S}\\ I_s=I_0}}|I_s|^{-1/2}\epsilon_s\langle f_1\mathbf{1}_{C_{k_1}},\phi^1_{s_1}\rangle\langle f_2\mathbf{1}_{C_{k_2}},\phi^2_{s_2}\rangle\langle \mathbf{1}_I f_3,\phi^3_{s_3}\rangle.$$

Let I_0 be fixed and denote

$$2^l=\frac{|I_0|}{|I|}.$$

The collection of tritiles associated to $\Lambda_1^{k_1,k_2,l}(I_0)$ is also

$$\mathbf{P}:=\{s\in\mathbf{S},\ \ I_s=I_0\}.$$

For all $s \in \mathbf{P}$, by using $f_3 \in F(E_3')$, we have

$$\frac{1}{|I_s|}\int_I |f_3(x)|\Big(1+\frac{d(x,I_s)}{|I_s|}\Big)^{-N} dx \le \frac{1}{|I_s|}\int_I \Big(1+\frac{d(x,I_s)}{|I_s|}\Big)^{-N} dx \le \frac{|I|}{|I_0|}\Big(1+\frac{d(I,I_0)}{|I_0|}\Big)^{-N}.$$

Then Lemma 2.9 gives us

$$\text{size}_3^*(\mathbf{P}) \lesssim 2^{-l}\Big(1+\frac{d(I,I_0)}{|I_0|}\Big)^{-N}.$$

By the same reasoning, we obtain for $f_1 \in F(E_1')$ and $s \in \mathbf{P}$,

$$\frac{1}{|I_s|}\int_{C_{k_1}} |f_1(x)|\Big(1+\frac{d(x,I_0)}{|I_s|}\Big)^{-N} dx \le \frac{1}{|I_0|}\int_{C_{k_1}} \Big(1+\frac{d(x,I_0)}{|I_0|}\Big)^{-N} dx \le 2^{k_1-l}\Big(1+\frac{d(C_{k_1},I_0)}{|I_0|}\Big)^{-N}.$$

And so we get

$$\text{size}_1^*(\mathbf{P}) \lesssim 2^{k_1-l}\Big(1+\frac{d(C_{k_1},I_0)}{|I_0|}\Big)^{-N}.$$

Likely, we have

$$\text{size}_2^*(\mathbf{P}) \lesssim 2^{k_2-l}\Big(1+\frac{d(C_{k_2},I_0)}{|I_0|}\Big)^{-N}.$$

With $\theta_1+\theta_2+\theta_3=1$ and Lemma 2.9, we can estimate

$$\text{size}_1^*(\mathbf{P})^{\theta_1}\ \text{size}_2^*(\mathbf{P})^{\theta_2}\ \text{size}_3^*(\mathbf{P})^{\theta_3} \lesssim 2^{\theta_1 k_1+\theta_2 k_2-l} A(I_0), \tag{2-7}$$

where $A(I_0)$ is the product of three terms

$$A(I_0) := \Big(1+\frac{d(I,I_0)}{|I_0|}\Big)^{-N\theta_3}\Big(1+\frac{d(C_{k_1},I_0)}{|I_0|}\Big)^{-N\theta_1}\Big(1+\frac{d(C_{k_2},I_0)}{|I_0|}\Big)^{-N\theta_2}.$$

We are going to get four different estimates for $A(I_0)$.

To keep the information about the position of I_0, we first have

$$A(I_0) \le \Big(1+\frac{d(I,I_0)}{|I_0|}\Big)^{-N\theta_3}. \tag{2-8}$$

By using

$$d(I,I_0)+d(C_{k_1},I_0) \gtrsim d(I,C_{k_1}) \gtrsim 2^{k_1}|I| \simeq 2^{k_1-l}|I_0|$$

and the fact that $2^l \le 1$, we obtain

$$A(I_0) \lesssim (1+2^{k_1-l})^{-N\min\{\theta_1,\theta_3\}} \lesssim 2^{-k_1 N\min\{\theta_1,\theta_3\}} \tag{2-9}$$

and likely

$$A(I_0) \lesssim 2^{-k_2 N\min\{\theta_2,\theta_3\}}. \tag{2-10}$$

As $I_0 \nsubseteq 2I$ and $2^l \le 1$, $d(I_0,I) \ge |I|$ and hence

$$\Big(1+\frac{d(I,I_0)}{|I_0|}\Big)^{-N} \lesssim \Big(\frac{|I_0|}{|I|}\Big)^{N}.$$

So we get

$$A(I_0) \lesssim \Big(\frac{|I_0|}{|I|}\Big)^{N\theta_3} \lesssim 2^{lN\theta_3}. \tag{2-11}$$

Taking the geometric mean of (2-8), (2-9), (2-10) and (2-11) (with another exponent N which is as large as we want), we obtain

$$A(I_0) \lesssim 2^{-(k_1+k_2+|l|)N}\Big(1+\frac{d(I,I_0)}{|I_0|}\Big)^{-N}. \tag{2-12}$$

With the help of (2-6) and (2-7), we finally estimate

$$\begin{aligned}\big|\Lambda_1^{k_1,k_2,l}(f_1,f_2,f_3)\big| &\lesssim \sum_{I_0}\big|\Lambda_1^{k_1,k_2,l}(I_0)(f_1,f_2,f_3)\big| \\ &\lesssim \sum_{I_0}\Big(\prod_{\beta\neq\alpha}|E_\beta|^{(1+\theta_\beta)/2-\epsilon\theta_\beta}|E_\alpha|^{(\theta_\alpha-1)/2+\epsilon(1-\theta_\alpha)}\Big)2^{\epsilon(k_1+k_2+|l|)}A(I_0)^\epsilon.\end{aligned}$$

From (2-12), the sum over the interval I_0 with $|I_0|=2^l|I|$ is bounded. For N a large enough exponent (not exactly the same), we have

$$\big|\Lambda_1^{k_1,k_2,l}(f_1,f_2,f_3)\big| \lesssim \Big(\prod_{\beta\neq\alpha}|E_\beta|^{(1+\theta_\beta)/2-\epsilon\theta_\beta}|E_\alpha|^{(\theta_\alpha-1)/2+\epsilon(1-\theta_\alpha)}\Big)\tilde{C}(\Lambda_1^{k_1,k_2,l}),$$

where

$$\tilde{C}(\Lambda_1^{k_1,k_2,l}) := 2^{-N\epsilon(k_1+k_2+|l|)}. \tag{2-13}$$

<u>Second case</u>: the forms Λ_0^j.

We use the same principle. We are interested in

$$\Lambda_0^{k_1,k_2}(f_1,f_2,f_3) := \sum_{\substack{s\in\mathbf{S}\\ I_s\subset 2I}}|I_s|^{-1/2}\epsilon_s\langle f_1\mathbf{1}_{C_{k_1}},\phi^1_{s_1}\rangle\langle f_2\mathbf{1}_{C_{k_2}},\phi^2_{s_2}\rangle\langle f_3,\phi^3_{s_3}\rangle.$$

So now we choose the collection

$$\mathbf{P} := \{s\in\mathbf{S},\quad I_s\subset 2I\}.$$

For all $s\in\mathbf{P}$,

$$\frac{1}{|I_s|}\int_I\Big(1+\frac{d(x,I_s)}{|I_s|}\Big)^{-N}dx \le 1$$

and so with Lemma 2.9 we have

$$\mathrm{size}_3^*(\mathbf{P}) \lesssim 1.$$

For f_1, we use that

$$\frac{1}{|I_s|}\int_{C_{k_1}}\Big(1+\frac{d(x,I_s)}{|I_s|}\Big)^{-N}dx \lesssim \Big(1+\frac{d(C_{k_1},I)}{|I|}\Big)^{-(N-2)}$$

to conclude

$$\mathrm{size}_1^*(\mathbf{P}) \lesssim 2^{-k_1(N-2)}.$$

By the same argument for f_2, we have

$$\mathrm{size}_2^*(\mathbf{P}) \lesssim 2^{-k_2(N-2)}.$$

In this case, we can also estimate (with N another large enough integer)

$$\mathrm{size}_1^*(\mathbf{P})^{\theta_1}\,\mathrm{size}_2^*(\mathbf{P})^{\theta_2}\,\mathrm{size}_3^*(\mathbf{P})^{\theta_3} \le 2^{-(k_1+k_2)N}.$$

With (2-6), we finally obtain

$$\Lambda_0^{k_1,k_2}(f_1, f_2, f_3) \lesssim \Big(\prod_{\beta\neq\alpha} |E_\beta|^{(1+\theta_\beta)/2-\epsilon\theta_\beta} |E_\alpha|^{(\theta_\alpha-1)/2+\epsilon(1-\theta_\alpha)}\Big)\tilde{C}(\Lambda_0^{k_1,k_2}),$$

where

$$\tilde{C}(\Lambda_0^{k_1,k_2}) := 2^{-N(k_1+k_2)\epsilon}. \tag{2-14}$$

Third step: end of the proof. For the trilinear form Λ_i^j, we have obtain a bound $C = \tilde{C}(\Lambda_i^j)$ such that for all functions $f_\beta \in F(E'_\beta)$ we have

$$\big|\Lambda_i^j(f_1, f_2, f_3)\big| \lesssim \tilde{C}(\Lambda_i^j)\bigg(\prod_{\beta\neq\alpha} |E_\beta|^{(1+\theta_\beta)/2-\epsilon\theta_\beta} |E_\alpha|^{(\theta_\alpha-1)/2+\epsilon(1-\theta_\alpha)}\bigg).$$

Let $(p_\beta)_\beta$ be the exponents of Theorem 2.8. Then we shall show that we can find $\theta_1, \theta_2, \theta_3 \in (0, 1)$ and $\epsilon > 0$ such that for all $\beta \neq \alpha$,

$$\frac{1+\theta_\beta}{2} - \epsilon\theta_\beta = \frac{1}{p_\beta}, \quad \text{and} \quad \frac{\theta_\alpha - 1}{2} + \epsilon(1-\theta_\alpha) = \frac{1}{p_\alpha}.$$

Let $\gamma > 0$ be a real satisfying

$$\Big|\frac{1}{2} - \frac{1}{p_\beta}\Big| < \frac{1}{2+\gamma},$$

for all $\beta \neq \alpha$. This is possible because $1 < p_\beta < 2$ for $\beta \neq \alpha$. We begin to choose $\theta_\alpha \in (0, 1)$ such that

$$1 > \theta_\alpha > \max\Big\{\theta_\alpha^0 := \frac{p_\alpha + (2+\gamma)}{p_\alpha}, 0\Big\},$$

and

$$\min\Big\{-\frac{1}{2+\gamma} = \frac{1}{p_\alpha(1-\theta_\alpha^0)}, \frac{1}{p_\alpha}\Big\} > \frac{1}{p_\alpha(1-\theta_\alpha)} > -\frac{1}{2}.$$

This is possible because p_α is negative and satisfies

$$\frac{1}{p_\alpha} > -\frac{1}{2}.$$

Then we get ϵ by

$$\epsilon := \frac{1}{2} + \frac{1}{p_\alpha(1-\theta_\alpha)} \in (0, \tfrac{1}{2}) \subset (0, 1).$$

We now define θ_β for $\beta \neq \alpha$ by

$$\theta_\beta := \frac{\dfrac{1}{p_\beta} - \dfrac{1}{2}}{\dfrac{1}{2} - \epsilon}.$$

We have $1 < p_\beta < 2$ and $0 < \epsilon < \frac{1}{2}$, so $0 < \theta_\beta$ and

$$0 < \theta_\beta = \frac{\dfrac{1}{p_\beta} - \dfrac{1}{2}}{\dfrac{-1}{p_\alpha(1-\theta_\alpha)}} < \frac{\dfrac{1}{2+\gamma}}{\dfrac{1}{2+\gamma}} = 1.$$

Consequently, we have solved the system of equations for the exponents. With this choice, we obtain for all $f_1 \in F(E'_1)$, $f_2 \in F(E'_2)$, $f_3 \in F(E'_3)$,

$$\Lambda_i^j(f_1, f_2, f_3) \lesssim \tilde{C}(\Lambda_i^j) \prod_{\beta=1}^{3} |E_i|^{1/p_\beta},$$

where $\tilde{C}(\Lambda_i^j)$ are defined in (2-13) and (2-14). So Λ_i^j is of restricted weak type and we have the following estimate for $C(\Lambda_i^j)$:

$$C(\Lambda_i^j) \lesssim \tilde{C}(\Lambda_i^j).$$

In addition the parameter N in (2-13) and (2-14) is as large as we want, and we have also obtained the desired estimates on $C(\Lambda_i^j)$. □

By using the concept of "restricted weak type", we can have a "stronger" result than Theorem 1.1.

Theorem 2.10. *Let T and p, q, r be an operator and exponents of Theorem 1.1. Then for all $\delta \geq 1$, there exists a constant*

$$C = C(p, q, r, \delta)$$

(*independent on the interval I*) *such that for all sets E_3 of finite measure, there exists a substantial subset $E'_3 \subset E_3$ satisfying that for all functions $f \in \mathcal{S}(\mathbb{R})$, $g \in \mathcal{S}(\mathbb{R})$ and $h \in F(E'_3)$,*

$$|\langle T(f, g), h\mathbf{1}_I\rangle| \leq C\Bigg(\sum_{k\geq 0} 2^{-k(1/p+\delta)} \|f\mathbf{1}_{2^k I}\|_p\Bigg)\Bigg(\sum_{k\geq 0} 2^{-k(1/q+\delta)} \|g\mathbf{1}_{2^k I}\|_q\Bigg)|E_3|^{1/r'}.$$

When $r > 1$, this result is stronger than Theorem 1.1 but less practicable. We now prove it because it will be useful in the sequel.

Proof. The proof is almost the same as the previous one, so we shall only explain the modifications. We always study the trilinear form

$$\Lambda(f, g, h) := \langle T(f, g), h\mathbf{1}_I\rangle.$$

In page 6 we saw that the study of Λ can be reduced to the study of the model sum

$$\Lambda(f, g, h) = \sum_{s\in\mathbf{S}} |I_s|^{-1/2} \epsilon_s \langle\phi_{s_1}, f\rangle\langle\phi_{s_2}, g\rangle\langle\phi_{s_3}, h\mathbf{1}_I\rangle,$$

where $\mathbf{S}$ is a general collection of tritiles. Then we have decomposed this sum with (2-3) by

$$\Lambda(f, g, h) = \sum_{k_1,k_2\geq 0} \Lambda_0^{k_1,k_2}(f, g, h) + \sum_{\substack{k_1,k_2\geq 0\\ l\leq 0}} \Lambda_1^{k_1,k_2,l}(f, g, h).$$

By Theorem 2.4, we have shown that the trilinear forms Λ_i^j are of restricted weak type (p, q, r') and we have obtained estimates on their bounds. The construction of the substantial subset $E'_\alpha = E'_3$ does not depend on the trilinear form Λ_i^j, so we can deduce that our trilinear form Λ is always of restricted weak type. Also for measurable sets E_1, E_2, E_3 of finite measure, there exists a substantial subset $E'_3 \subset E_3$ such that for all functions $f \in F(E_1)$, $g \in F(E_2)$ and $h \in F(E'_3)$,

$$|\Lambda(f, g, h)| \lesssim |E_3|^{1/r'} \left(\sum_{k_1, k_2 \geq 0} 2^{-\delta'(k_1+k_2)} |E_1 \cap C_{k_1}|^{1/p} |E_2 \cap C_{k_2}|^{1/q} \right).$$

Here δ' is an exponent as large as we want. Over each corona, by using the real interpolation on the exponents p and q (so r is fixed), we obtain also the desired result. $\square$

Having obtained our main result for the x-independent symbols, we will extend our result for maximal operators and for x-dependent symbols in the next section.

3. More general bilinear operators

Let us name our "off-diagonal estimates" for convenience.

Definition 3.1. Let T be an operator (maybe non-bilinear) acting from $\mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R})$ into $\mathcal{S}'(\mathbb{R})$. For $p, q, r \in (0, \infty]$ exponents such that

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q},$$

we say that T satisfies *"off-diagonal estimates" at the scale L and at the order δ*, in short

$$T \in \mathcal{O}_{L,\delta}(L^p \times L^q, L^r),$$

if there exists a constant $C = C(p, q, r, L, \delta)$ such that for all functions $f, g \in \mathcal{S}(\mathbb{R})$ and all interval I of length $|I| = L$, we have

$$\|T(f, g)\|_{r,I} \leq C \left(\sum_{k \geq 0} 2^{-k(\delta+1/p)} \|f\|_{p,2^{k+1}I} \right) \left(\sum_{k \geq 0} 2^{-k(\delta+1/q)} \|g\|_{q,2^{k+1}I} \right).$$

Remark 3.2. Equivalently, an operator T satisfies "off-diagonal estimates" at the scale L and at the order δ if there exists a constant $C = C(p, q, r, L, \delta)$ such that for all functions $f, g \in \mathcal{S}(\mathbb{R})$ and all interval I of length $|I| = L$, we have

$$\|T(f, g)\|_{r,I} \leq C \left(\sum_{k \geq 0} 2^{-k(\delta+1/p)} \|f\|_{p,C_k(I)} \right) \left(\sum_{k \geq 0} 2^{-k(\delta+1/q)} \|g\|_{q,C_k(I)} \right).$$

This is a better way to describe the "off-diagonal decay" of an operator T and these properties can be described as in Corollary 1.2.

First we generalize the previous result for maximal operators.

"Off-diagonal estimates" for maximal bilinear operators.

Theorem 3.3. *Let Δ be a nondegenerate line in the frequency plane. Let $p, q \in (1, \infty]$ be exponents such that*

$$0 < \frac{1}{r} = \frac{1}{q} + \frac{1}{p} < \frac{3}{2}.$$

For all $\delta \geq 1$, $L > 0$, for all symbol σ supported in

$$\{(\alpha, \beta),\ d((\alpha, \beta), \Delta) \geq L^{-1}\}$$

satisfying for all $b, c \geq 0$,

$$\left|\partial_\alpha^b \partial_\beta^c \sigma(\alpha, \beta)\right| \lesssim |d((\alpha, \beta), \Delta)|^{-b-c}$$

and for all smooth function ϕ, which is equal to 1 around 0, the maximal bilinear operator

$$T_{max}(f, g)(x) := \sup_{r>0} \left| \int e^{ix(\alpha+\beta)} \widehat{f}(\alpha)\widehat{g}(\beta)\, \sigma(\alpha, \beta)\big(1 - \phi(r(\alpha - \beta))\big)\, d\alpha\, d\beta \right|$$

satisfies "off-diagonal estimates" at the scale L and at the order δ:

$$T_{max} \in \mathcal{O}_{L,\delta}(L^p \times L^q, L^r).$$

In addition the implicit constant can be uniformly bounded by $L > 0$.

Theorem 3.4. *For the same exponents, we have the same continuities for the maximal bilinear operator (at the scale L)*

$$M^L(f, g)(x) := \sup_{0<r\leq L} \frac{1}{r} \int_{|t|\leq r} |f(x-t)g(x+t)|\, dt.$$

Theorem 3.5. *Let K be a kernel on $\mathbb{R}$ satisfying Hörmander's conditions, then the maximal bilinear operator*

$$T^L_{max}(f, g)(x) := \sup_{0<\epsilon<r<L} \left| \int_{\epsilon\leq|y|\leq r} f(x-y)g(x+y)K(y)\, dy \right|$$

satisfies the same local estimates

$$T^L_{max} \in \mathcal{O}_{L,\delta}(L^p \times L^q, L^r)$$

for the exponents p, q, r as of Theorem 3.3.

Proof. The proof of these three theorems is a shake between the proof of our Theorem 1.1 and an additional maximality argument. The maximal truncation in the physical space (Theorems 3.4 and 3.5) is a little more complex than the maximal truncation in the frequency space (Theorem 3.3). So we deal with the last two theorems and just explain the modifications to prove them. The maximal version of the different arguments has been shown first by M. Lacey [2000] and then improved by C. Demeter, T. Tao and C. Thiele [2005]. In these articles, the authors study the behavior of the maximal averages (like in Theorem 3.4). [Demeter et al. 2005, Remark 1.6] specifies the similarity between the operators of Theorems 3.4 and 3.5. So in fact the previous three theorems are an illustration of the same ideas, and we will not detail them.

The reduction on page 6 is based on the decomposition of the bilinear operator by discrete models. For our maximal operators, the same reduction is shown in [Demeter et al. 2005, Theorem 4.4] and the important condition (2-2) for the tiles is always satisfied. Then the maximal version of Proposition

2.6 is given in [Demeter et al. 2005] too (there is a new factor in the different inequalities but it is not important). We have exactly the same version of Lemma 2.9 for maximal bilinear operators [Demeter et al. 2005, Proposition 6.2]. Using these technical modifications, we can prove Theorem 1.1 and obtain its maximal versions: Theorems 3.3, 3.4 and 3.5. □

Proof of Theorem 1.1 for x-dependent symbols. In this subsection, we prove the "off-diagonal estimates" of Theorem 1.1 in the case where the symbol σ depends on the spatial variable x and also we complete the proof of our main result.

Theorem 3.6. *Let Δ be a nondegenerate line of the frequency space. Let $\sigma \in C^\infty(\mathbb{R}^3)$ be a symbol satisfying for all $a, b, c \geq 0$,*

$$\left|\partial_x^a \partial_\alpha^b \partial_\beta^c \sigma(x, \alpha, \beta)\right| \lesssim \big(1 + d((\alpha, \beta), \Delta)\big)^{-b-c}.$$

Then the bilinear operator T_σ (defined on $\mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R})$ by (1-2)) verifies

$$T_\sigma \in \mathbb{O}_{1,\delta}(L^p \times L^q, L^r)$$

for any $\delta \geq 0$ and any exponents p, q, r such that

$$0 < \frac{1}{r} = \frac{1}{p} + \frac{1}{q} < \frac{3}{2} \quad \textit{and} \quad 1 < p, q \leq \infty.$$

Our assumptions for the symbol correspond to the class $BS^0_{1,0;\theta}$ of [Bényi et al. 2006], where the angle $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \setminus \{0, -\frac{\pi}{4}\}$ is given by the line

$$\Delta := \{(\alpha, \beta),\ \beta = \alpha \tan\theta\}.$$

For convenience, we will deal in the proof only with the case $\theta = \frac{\pi}{4}$. The important fact is that the singular quantity $(\beta - \alpha \tan\theta)$ does not correspond to the quantity $\alpha + \beta$, which appears in the exponential term of (1-2). The limit and particular case $\theta = -\frac{\pi}{4}$ is studied in [Bényi et al. 2006].

Proof. The proof is quite technical. We will also assume that $r \geq 1$ (which allows us to simplify a few arguments). Then we will explain in Remark 3.7 how to modify the proof to obtain the same result when $r < 1$.

So we fix an interval I of length $|I| = 1$. We use a decomposition of the symbol σ. Let Φ be a smooth function on $\mathbb{R}$ such that if $|x| \leq 1$ then

$$\Phi(x) = 1 \quad \text{and} \quad \operatorname{supp}(\Phi) \subset [-2, 2].$$

We also have

$$\begin{aligned}\sigma(x, \alpha, \beta) &= \sigma(x, \alpha, \beta)(1 - \Phi(\alpha - \beta)) + \sigma(x, \alpha, \beta)\Phi(\alpha, \beta) \\ &:= \sigma^\infty(x, \alpha, \beta) + \sigma^0(x, \alpha, \beta).\end{aligned}$$

(i) The case of the symbol σ^∞.

We have an operator associated to this symbol

$$T^\infty(f, g)(x) := \int_{\mathbb{R}^2} e^{ix(\alpha+\beta)} \widehat{f}(\alpha)\widehat{g}(\beta)\, \sigma(x, \alpha, \beta)\, (1 - \Phi(\alpha - \beta))\, d\alpha\, d\beta,$$

which can been written as

$$T^\infty(f,g)(x) = U_x(f,g)(x),$$

with U defined by

$$U_y(f,g)(x) := \int_{\mathbb{R}^2} e^{ix(\alpha+\beta)} \widehat{f}(\alpha)\widehat{g}(\beta)\sigma(y,\alpha,\beta)(1-\Phi(\alpha-\beta))\, d\alpha\, d\beta.$$

By using the Sobolev embedding

$$W^{1,r}(I) \hookrightarrow L^\infty(I)$$

because $r \geq 1$, we get

$$|T^\infty(f,g)(x)| \leq \|U_y(f,g)(x)\|_{\infty,y\in I} \lesssim \sum_{k=0}^{1} \|\partial_y^k U_y(f,g)(x)\mathbf{1}_I(y)\|_{r,dy}.$$

for all $x \in I$. Then by integrating for $x \in I$ and using Fubini's Theorem, we obtain

$$\|T^\infty(f,g)\|_{r,I} \lesssim \sum_{k=0}^{1} \|\|\partial_y^k U_y(f,g)\|_{r,I}\|_{r,I,dy}.$$

We can fix $k \in \{0,1\}$ and $y \in I$. Then we have

$$\|\partial_y^k U_y(f,g)\|_{r,dx} \lesssim \|V(f,g)\|_{r,I},$$

where V is the bilinear operator defined by

$$V(f,g)(x) := \int_{\mathbb{R}^2} e^{ix(\alpha+\beta)} \widehat{f}(\alpha)\widehat{g}(\beta)\partial_y^k\sigma(y,\alpha,\beta)(1-\Phi(\alpha-\beta))\, d\alpha\, d\beta.$$

So $V = T_\tau$ is the bilinear operator associated to the x-independent symbol

$$\tau(\alpha,\beta) := \partial_y^k \sigma(y,\alpha,\beta)(1-\Phi(\alpha-\beta)).$$

From the assumptions about σ, the symbol τ satisfies $\left|\partial_\alpha^b \partial_\beta^c \tau(\alpha,\beta)\right| \lesssim |\alpha-\beta|^{-n-p}$ for all $b, c \geq 0$. In addition, τ is supported in the domain $\{(\alpha,\beta), |\alpha-\beta| \geq 1\}$. We can also apply Theorem 1.1 proved in Section 2 for x-independent symbol. For all $\delta \geq 1$, we have an "off-diagonal estimate" at the scale 1,

$$\|V(f,g)\|_{r,I} \lesssim \left(\sum_{k_1\geq 0} 2^{-k_1(1/p+\delta)}\|f\|_{p,2^{k_1}I}\right)\left(\sum_{k_2\geq 0} 2^{-k_2(1/q+\delta)}\|g\|_{q,2^{k_2}I}\right).$$

All theses estimates are uniform with respect to $k \in \{0,1\}$ and $y \in I$, so we get

$$\|T^\infty(f,g)\|_{r,I} \lesssim \left(\sum_{k_1\geq 0} 2^{-k_1(1/p+\delta)}\|f\|_{p,2^{k_1}I}\right)\left(\sum_{k_2\geq 0} 2^{-k_2(1/q+\delta)}\|g\|_{q,2^{k_2}I}\right). \tag{3-1}$$

So we have shown the desired estimates for this first term.

(ii) The case of the symbol σ^0. The associated operator is given by

$$T^0(f,g)(x) := \int_{\mathbb{R}^2} e^{ix(\alpha+\beta)} \widehat{f}(\alpha)\widehat{g}(\beta)\sigma(x,\alpha,\beta)\Phi(\alpha,\beta)\, d\alpha\, d\beta.$$

We use the same arguments as for the first point. So we have to study the operator V defined by

$$V(f,g)(x) := \int_{\mathbb{R}^2} e^{ix(\alpha+\beta)} \widehat{f}(\alpha)\widehat{g}(\beta)\partial_y^k \sigma(y,\alpha,\beta)\Phi(\alpha-\beta)\, d\alpha\, d\beta.$$

The parameters $k \in \{0,1\}$ and $y \in I$ are fixed. The symbol associated to this operator is supported on

$$\{(\alpha,\beta),\ |\alpha-\beta| \leq 2\}.$$

That is why we use modulations to move this support:

$$\begin{aligned} V(f,g)(x) &= \int_{\mathbb{R}^2} e^{ix(\alpha+\beta)} \widehat{f}(\alpha+3)\widehat{g}(\beta-3)\partial_y^k \sigma(y,\alpha+3,\beta-3)\Phi(\alpha-\beta+6)\, d\alpha\, d\beta \\ &= \int_{\mathbb{R}^2} e^{ix(\alpha+\beta)} \widehat{e^{3i\cdot} f}(\alpha)\widehat{e^{-3i\cdot} g}(\beta)\partial_y^k \sigma(y,\alpha+3,\beta-3)\Phi(\alpha-\beta+6)\, d\alpha\, d\beta. \end{aligned}$$

Also V is now the bilinear operator, applied to the modulated functions $e^{3i\cdot} f$ and $e^{-3i\cdot} g$, whose (x-independent) symbol

$$\tau(\alpha,\beta) := \partial_y^k \sigma(y,\alpha+3,\beta-3)\Phi(\alpha-\beta+6)$$

is supported on

$$\{(\alpha,\beta),\ |\alpha-\beta+6| \leq 2\} \subset \{(\alpha,\beta),\ 1 \leq |\alpha-\beta| \leq 8\}$$

and satisfies for all $b, c \geq 0$,

$$\left|\partial_\alpha^b \partial_\beta^c \tau(\alpha,\beta)\right| \lesssim \max_{0\leq j\leq b} \max_{0\leq i\leq c} (1+|\alpha-\beta+6|)^{-i-j} \mathbf{1}_{1\leq|\alpha-\beta|\leq 8} \lesssim \mathbf{1}_{1\leq|\alpha-\beta|\leq 8} \lesssim \mathbf{1}_{1\leq|\alpha-\beta|\leq 8} |\alpha-\beta|^{-b-c}.$$

Also we can use Theorem 1.1 (proved in Section 2 for x-independent symbol) again and we obtain

$$\|V(f,g)\|_{r,I} \lesssim \left(\sum_{k_1\geq 0} 2^{-k_1(1/p+\delta)} \|f\|_{p,I}\right)\left(\sum_{k_2\geq 0} 2^{-k_2(1/q+\delta)} \|g\|_{q,I}\right). \qquad \square$$

Remark 3.7. We want to explain here how to modify the previous proof when $r < 1$. When we study bilinear operators with $r < 1$, we have to use the associated trilinear form and the concept of "restricted weak type" (see Definition 2.7). These two arguments allow us to get around the lack of the triangular inequality in the space L^r. Let

$$\Lambda(f,g,h) := \langle T(f,g), h\rangle.$$

We have

$$\Lambda(f,g,h) = \int_{\mathbb{R}} \int_{\mathbb{R}^2} e^{ix(\alpha+\beta)} \sigma(x,\alpha,\beta) \widehat{f}(\alpha)\widehat{g}(\beta) h(x)\, d\alpha\, d\beta dx.$$

We use the same decomposition of σ, getting the trilinear forms Λ^∞ and Λ^0. Let us study first Λ^∞ and fix an interval I of length $|I| = 1$. We take a function $h \in \mathcal{S}(\mathbb{R})$, which is supported on I. We use again the Sobolev embedding $W^{1,1}(I) \hookrightarrow L^\infty(I)$. By writing

$$|\Lambda^\infty(f,g,h)| \leq \int_{\mathbb{R}} \|U_y(f,g)(x)\mathbf{1}_I(y)\|_{\infty,I,dy} |h(x)| \mathbf{1}_I(x)\, dx,$$

we can also obtain

$$|\Lambda^\infty(f,g,h)| \lesssim \int_I \int_I |U_y(f,g)(x)|\,|h(x)|\,dx\,dy + \int_I \int_I |\partial_y U_y(f,g)(x)|\,|h(x)|\,dx\,dy.$$

Then when $y \in I$ and $k \in \{0,1\}$ are fixed, we find again the quantities

$$\int_I \left|\partial_y^k U_y(f,g)(x)\right| |h(x)|\,dx.$$

Now the bilinear operator $\partial_y^k U_y$ is associated to an x-independent symbol, which verifies the good assumptions. We can also use Theorem 2.10 in order to obtain the wished estimates (3-1) in a "restricted weak type sense" for the exponent r. We produce the same modifications to study Λ^0. By noticing that the way to construct the substantial subset (in the definition of restricted weak type) does not depend on the trilinear form, we can deduce that the trilinear form Λ satisfies (3-1) in a "restricted weak type sense" too. Then we use interpolation on the exponent r, to obtain exactly (3-1), which allows us to conclude.

4. Continuities for bilinear operators satisfying "off-diagonal estimates"

Recall that in the linear case, by using the maximal sharp function, we can prove weighted continuities for linear operator with the Muckenhoupt weights. In the bilinear case, we do not have a good substitute to the maximal sharp function. That is why we shall use the previous "off-diagonal estimates" to obtain weighted global continuities on Lebesgue spaces and in particular to prove Theorem 1.3.

First we want to give an application of these "off-diagonal estimates". Recall that in the previous sections, we have proved that our bilinear operators (and maximal bilinear operators) satisfy these "off-diagonal estimates" at any order. The time-frequency analysis does not work for functions in the L^∞ space. So we do not know if our operators T are bounded from $L^\infty \times L^\infty$ in *BMO*. However these local estimates give a weak result about the behavior of $T(f,g)$ when the two functions f and g belong to L^∞.

Proposition 4.1. *Let f, g be two functions of $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and fix $r \in (1,\infty)$. If there exist $L > 0$, $\delta \geq 1$ and $p, q > 1$ such that an operator*

$$T \in \mathbb{O}_{\delta,L}(L^p \times L^q, L^r),$$

then we have

$$\limsup_{|I|\to\infty} \left(\frac{1}{|I|}\int_I |T(f,g)|^r\right)^{1/r} \lesssim \|f\|_\infty \|g\|_\infty.$$

Here we take the limit when I is an interval with $|I| \to \infty$ and the implicit constant does not depend on the two functions f and g and on the parameter L.

Proof. We set

$$I_i := [iL, (i+1)L[$$

for all $i \in \mathbb{Z}$. Then for I with $|I| \gg L$, we get

$$\int_I |T(f,g)|^r \leq \sum_{\substack{i\in\mathbb{Z}\\ I_i\cap I\neq\emptyset}} \int_{I_i} |T(f,g)|^r.$$

However, the number of indices i which appears in the sum is bounded by $|I|/L$, so by using the local estimate we get

$$\int_I |T(f,g)|^r \lesssim \sum_{\substack{i\in\mathbb{Z}\\ I_i\cap I\neq\varnothing}} \frac{L}{|I_i|}\int_{I_i} |T(f,g)|^r \lesssim \sum_{\substack{i\in\mathbb{Z}\\ I_i\cap I\neq\varnothing}} L\|f\|_\infty^r\|g\|_\infty^r \lesssim |I|\|f\|_\infty^r\|g\|_\infty^r.$$

The second inequality is due to the fact that

$$|I_i|^{1/r}\|T(f,g)\|_{r,I_i} \lesssim \inf_{x\in I_i} M_{HL}(f)(x)\ \inf_{x\in I_i} M_{HL}(g)(x) \lesssim \|f\|_\infty\|g\|_\infty.$$

So we obtain

$$\Big(\frac{1}{|I|}\int_I |T_{max}(f,g)|^r\Big)^{1/r} \lesssim \|f\|_\infty\|g\|_\infty$$

uniformly with L for $|I|$ large enough. □

Let us now define our weights.

Definition 4.2. Let $\theta>0$ and $l>0$ be fixed. We set that a nonnegative function ω belongs to the *class* $\mathbb{P}_\theta(l)$ if there exists a constant C such that for all interval I of length $|I|=l$ and for all integer $k\geq 0$, we have

$$2^{-k\theta}\sup_{x\in I}\omega(x) \leq C\inf_{2^kI}\omega(x). \tag{4-1}$$

We claim that a function $\omega\in\mathbb{P}_\theta(l)$ is likely to be a polynomial function whose degree is less than θ and is almost constant at the scale l. We show in the next example that these classes are not empty.

Example 4.3. For all $\theta>0$ and $\alpha\in[0,\theta)$, the functions

$$x\mapsto 1, \qquad x\mapsto (1+|x|)^\alpha \qquad \text{and} \qquad x\mapsto(1+|x|)^{-\alpha}$$

belong to the class $\mathbb{P}_\theta(1)$. The proof is easy and is left to the reader.

Remark 4.4. In fact, it is easy to prove that a weight ω belongs to the class $\mathbb{P}_\theta(l)$ if and only if there exists a constant C such that for all $x,y\in\mathbb{R}$,

$$\omega(x)\leq C\Big(1+\frac{|x-y|}{l}\Big)^\theta\omega(y).$$

We cannot compare these weights with the Muckenhoupt weights, because for $\omega\in\mathbb{P}_\theta(l)$ we have information only at the scale l.

Theorem 4.5. *Let T be a bilinear operator and $p,q,r\in(0,\infty)$ be exponents satisfying*

$$\frac{1}{r}=\frac{1}{p}+\frac{1}{q} \qquad \textit{and} \qquad 1\leq p,q.$$

For $\delta>0$ and $l>0$, if T satisfies "off-diagonal estimates" at the order δ and at the scale l, then for all $\omega\in\mathbb{P}_\theta(l)$ with $0\leq\theta<\delta\max\{r,1\}$, the operator T is continuous from $L^p(\omega)\times L^q(\omega)$ into $L^r(\omega)$.

Proof. To check this, we recall that for all interval I of length $|I| = l$,

$$\left(\int_I |T(f,g)|^r\right)^{1/r} \lesssim \Big(\sum_{k\geq 0} 2^{-k(1/p+\delta)} \|f\|_{p,2^kI}\Big)\Big(\sum_{k\geq 0} 2^{-k(1/q+\delta)} \|g\|_{q,2^kI}\Big). \tag{4-2}$$

Then we decompose the whole space $\mathbb{R}$ with the disjoint intervals I_i defined by

$$I_i = [il, (i+1)l]$$

for $i \in \mathbb{Z}$. So we have

$$\|T(f,g)\|_{r,wdx} = \| \|T(f,g)\|_{r,wdx,I_i}\|_{r,i\in\mathbb{Z}}.$$

Let $i \in \mathbb{Z}$ be fixed. We use (4-1) and (4-2) to obtain

$$\begin{aligned}\|T(f,g)\|_{r,wdx,I_i} &\leq \|w\|_{\infty,I_i}^{1/r} \|T(f,g)\|_{r,I_i} \\ &\lesssim \|w\|_{\infty,I_i}^{1/r} \left(\sum_{k\geq 0} 2^{-k(1/p+\delta)} \|f\|_{p,2^kI_i}\right)\left(\sum_{k\geq 0} 2^{-k(1/q+\delta)} \|g\|_{q,2^kI_i}\right).\end{aligned}$$

We estimate the first sum with

$$\begin{aligned}\|w\|_{\infty,I_i}^{1/p}\left(\sum_{k\geq 0} 2^{-k(1/p+\delta)} \|f\|_{p,2^kI_i}\right) &\lesssim \sum_{k\geq 0} 2^{-k(1/p+\delta)} \|w\|_{\infty,I_i}^{1/p} \|f\|_{p,2^kI_i} \\ &\lesssim \sum_{k\geq 0} 2^{-k(1/p+\delta)} 2^{k\theta/p} \inf_{2^kI_i} \omega^{1/p} \|f\|_{p,2^kI_i} \\ &\lesssim \sum_{k\geq 0} 2^{-k(1/p+\delta-\theta/p)} \|f\|_{p,wdx,2^kI_i}.\end{aligned}$$

The second term is studied by the same way. By summing over $i \in \mathbb{Z}$, we get

$$\|T(f,g)\|_{r,wdx} \lesssim \left\| \left(\sum_{k\geq 0} 2^{-k(1/p+\delta-\theta/p)} \|f\|_{p,wdx,2^kI_i}\right)\left(\sum_{k\geq 0} 2^{-k(1/q+\delta-\theta/q)} \|g\|_{q,wdx,2^kI_i}\right)\right\|_{r,i\in\mathbb{Z}}.$$

With the help of Hölder's and Minkowski's inequalities, we obtain

$$\|T(f,g)\|_{r,wdx} \lesssim \left(\sum_{k\geq 0} 2^{-k(1/p+\delta-\theta/p)} \| \|f\|_{p,wdx,2^kI_i}\|_{p,i\in\mathbb{Z}}\right)\left(\sum_{k\geq 0} 2^{-k(1/q+\delta-\theta/q)} \| \|g\|_{q,wdx,2^kI_i}\|_{q,i\in\mathbb{Z}}\right).$$

However the collection of sets $(2^kI_i)_i$ is a 2^k-covering, so

$$\|T(f,g)\|_{r,wdx} \lesssim \left(\sum_{k\geq 0} 2^{-k(\delta-\theta/p)} \|f\|_{p,wdx}\right)\left(\sum_{k\geq 0} 2^{-k(\delta-\theta/q)} \|g\|_{q,wdx}\right).$$

Then we conclude with the fact that $p, q > 1$ and hence

$$\max\left\{\frac{\theta}{p}, \frac{\theta}{q}\right\} \leq \begin{cases} \dfrac{\theta}{r} < \delta & \text{if } r \geq 1, \\ \theta < \delta & \text{if } r \leq 1. \end{cases}$$

□

Remark 4.6. Since it is obvious that the weight $\omega(x)=1$ belongs to the class $\mathbb{P}_\theta(L)$, we have also proved that the operators of Theorem 1.1 and the maximal operators of Theorems 3.3, 3.4 and 3.5 are bounded in classical Lebesgue spaces.

Definition 4.7. Let ω be a weight on $\mathbb{R}$. For all $m \geq 0$ and $p \in (1,\infty)$, we set $W^{m,p}(\omega)$ for the Sobolev space on $\mathbb{R}$ with the weight ω, defined as the set of distributions $f \in \mathcal{S}'(\mathbb{R})$ such that

$$J_m(f) \in L^p(\omega),$$

where $J_m := (\mathrm{Id} - \Delta)^{m/2}$.

We complete this result with a proposition in Sobolev spaces:

Proposition 4.8. *Let Δ be a nondegenerate line, ω be a weight in $\bigcup_{\theta \geq 0} \mathbb{P}_\theta(1)$ and $\sigma \in C^\infty(\mathbb{R}^3)$ be a symbol satisfying*

$$\left|\partial_x^a \partial_\alpha^b \partial_\beta^c \sigma(x,\alpha,\beta)\right| \lesssim \left(1 + d((\alpha,\beta),\Delta)\right)^{-b-c},$$

for all $a,b,c \geq 0$. Let p, q and r be exponents satisfying

$$0 < \frac{1}{r} = \frac{1}{p} + \frac{1}{q} < \frac{3}{2} \quad \text{and} \quad 1 < p,q < \infty.$$

Then the bilinear operator T_σ (defined on $\mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R})$ by (1-2)) satisfies

$$\|D^{(n)} T_\sigma(f,g)\|_{L^r(\omega)} \lesssim \sum_{\substack{0 \leq i,j \leq n \\ i+j \leq n}} \|D^{(i)} f\|_{L^p(\omega)} \|D^{(j)} g\|_{L^q(\omega)}, \tag{4-3}$$

for all integer $n \geq 0$ and for all functions $f,g \in \mathcal{S}(\mathbb{R})$. Here we write $D^{(i)}$ for the differentiation operator of order i. Also T_σ is continuous from $W^{m,p}(\omega) \times W^{m,q}(\omega)$ into $W^{m,r}(\omega)$ for all real $m \geq 0$.

Proof. Let us begin to prove (4-3). The two functions f and g are smooth so we can differentiate the integral defining $T_\sigma(f,g)$. It is also easy to check that

$$D^{(1)} T_\sigma(f,g) = T_\sigma(D^{(1)} f, g) + T_\sigma(f, D^{(1)} g) + T_{\partial_x \sigma}(f,g).$$

Then for higher orders, we get

$$D^{(n)} T_\sigma(f,g) = \sum_{\substack{0 \leq i,j,k \leq n \\ i+j+k=n}} T_{\partial_x^k \sigma}(D^{(i)} f, D^{(j)} g).$$

By using the previous Theorems 1.1 and 4.5, we obtain (4-3). We can also deduce a weaker estimate

$$\|D^{(n)} T_\sigma(f,g)\|_{r,\omega} \lesssim \|f\|_{W^{n,p}(\omega)} \|g\|_{W^{n,q}(\omega)},$$

for all $f,g \in \mathcal{S}(\mathbb{R})$. By density (see Lemma 4.9), the operator T_σ can be continuously extended from $W^{n,p}(\omega) \times W^{n,q}(\omega)$ into $W^{n,r}(\omega)$. Then we will use interpolation to extend this result when n is not an integer. The exponents p, q and r are fixed and we study the bilinear operator T_σ. We have shown that T_σ is continuous from $W^{n,p}(\omega) \times W^{n,q}(\omega)$ into $W^{n,r}(\omega)$, for all integer n. By using bilinear interpolation (with Lemma 4.9) on n, we finish the proof. (The theory of multilinear interpolation is studied in [Lions and Peetre 1964, Chapter 4] for the real case and in [Bergh and Löfström 1976, Theorem 4.4.1] for the complex case.) □

Lemma 4.9. *For all weight*

$$\omega \in \bigcup_{\theta \geq 0} \mathbb{P}_\theta(1),$$

all exponent $p \in (1, \infty)$ *and all real* $s \geq 0$, *the space* $\mathcal{S}(\mathbb{R})$ *is a dense subspace in* $W^{s,p}(\omega)$. *In addition, the collection of Sobolev spaces* $(W^{s,p}(\omega))_{s\geq 0}$ *form an interpolation scale.*

Proof. Let ω be a fixed weight in $\bigcup_{\theta\geq 0} \mathbb{P}_\theta(1)$. We have seen in Remark 4.4 that ω has a polynomial growth. Since $J_s(\mathcal{S}(\mathbb{R})) = \mathcal{S}(\mathbb{R})$, we have the inclusion $\mathcal{S}(\mathbb{R}) \subset W^{s,p}(\omega)$. We recall that

$$J_s := (\mathrm{Id} - \Delta)^{s/2}.$$

In addition, we have that

$$L^p(\omega) \subset \mathcal{S}'(\mathbb{R}),$$

so we can compute the operator J_{-s} on the space $L^p(\omega)$. We finally obtain that J_s is an automorphism from $W^{s,p}(\omega)$ to $L^p(\omega)$ and an isomorphism on $\mathcal{S}(\mathbb{R})$. As $\mathcal{S}(\mathbb{R})$ is dense in $L^p(\omega)$, we get the density of $\mathcal{S}(\mathbb{R})$ into the Sobolev space $W^{s,p}(\omega)$.

For the interpolation claim, we omit the details. The classical proof for complex interpolation with $\omega = 1$ can easily be extended to the general case. $\square$

Remark 4.10. From the fact that the weight $\omega(x) = 1$ belongs to the class $\mathbb{P}_\theta(1)$, we have also proved that the operators of Theorem 1.3 satisfy an Hölder's inequality in Sobolev spaces.

Remark 4.11. Also with the notation of [Bényi et al. 2006], we have proved continuities for all operators associated to symbols $\sigma \in BS^0_{1,0;\theta}$. In addition, we have described the action of these operators on Sobolev spaces. This is an interesting improvement of the last article and it incites us to obtain new results in order to continue the construction of a bilinear pseudo-differential calculus. We will do it in a next paper [Bernicot 2008] by introducing new larger symbolic classes of bilinear symbols of order (m_1, m_2) and studying rules of a bilinear symbolic calculus.

About continuities in Lebesgue spaces, a question is still open: What about the classes $BS^0_{\rho,\delta;\theta}$ (defined in [Bényi et al. 2006])?

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CONSTRUCTION OF ONE-DIMENSIONAL SUBSETS OF THE REALS NOT CONTAINING SIMILAR COPIES OF GIVEN PATTERNS

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For any countable collection of sets of three points we construct a compact subset of the real line with Hausdorff dimension 1 that contains no similar copy of any of the given triplets.

1. Introduction

An old conjecture of Erdős [1974] (also known as the Erdős similarity problem) states that for any infinite set $A \subset \mathbb{R}$ there exists a set $E \subset \mathbb{R}$ of positive Lebesgue measure which does not contain any similar (that is, translated and rescaled) copy of A. It is known that slowly decaying sequences are not counterexamples [Falconer 1984; Bourgain 1987; Kolountzakis 1997] (see for example [Humke and Laczkovich 1998; Komjáth 1983; Svetic 2000] for other related results) but nothing is known about any infinite sequence that converges to zero at least exponentially. On the other hand, it follows easily from Lebesgue's density theorem that any set $E \subset \mathbb{R}$ of positive Lebesgue measure contains similar copies of every finite set.

Bisbas and Kolountzakis [2006] gave an incomplete proof of a related statement: For every infinite set $A \subset \mathbb{R}$ there exists a compact set $E \subset \mathbb{R}$ of Hausdorff dimension 1 such that E contains no similar copy of A. Kolountzakis asked whether the same holds for finite sets as well. Iosevich asked a similar question: if $A \subset \mathbb{R}$ is a finite set and $E \subset [0, 1]$ is a set of given Hausdorff dimension, must E contain a similar copy of A?

In this paper we answer these questions by showing that for any set $A \subset \mathbb{R}$ of at least 3 elements there exists a 1-dimensional set that contains no similar copy of A. In fact, we obtain a bit more by proving the following theorem, which immediately yields the two subsequent corollaries.

Theorem 1.1. *For any countable set $A \subset (1, \infty)$ there exists a compact set $E \subset \mathbb{R}$ with Hausdorff dimension 1 such that if $x < y < z$ and $x, y, z \in E$, then*

$$\frac{z-x}{z-y} \notin A.$$

Corollary 1.2. *For any sequence $B_1, B_2, \ldots \subset \mathbb{R}$ of sets of at least three elements there exists a compact set $E \subset \mathbb{R}$ with Hausdorff dimension 1 that contains no similar copy of any of $B_1, B_2, \ldots$.*

Corollary 1.3. *For any countable set $B \subset \mathbb{R}$ there exists a compact set $E \subset \mathbb{R}$ with Hausdorff dimension 1 that intersects any similar copy of B in at most two points.*

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The method of the construction is similar to the method used in [Keleti 1998], where a compact set A of Hausdorff dimension 1 is constructed such that A does not contain any set of the form

$$\{a, a+b, a+c, a+b+c\}$$

for any $a, b, c \in \mathbb{R}$, $b, c \neq 0$, so in particular A does not contain any nontrivial 3-term arithmetic progression.

Laba and Pramanik [2007] obtained a positive result by proving that if a compact set $E \subset \mathbb{R}$ has Hausdorff dimension sufficiently close to 1 and E supports a probability measure whose Fourier transform has appropriate decay at infinity then E must contain nontrivial 3-term arithmetic progressions. It would be interesting to know whether similar conditions could guarantee other finite patterns as well.

Perhaps one can even find conditions weaker than having positive measure that implies that a compact subset of $\mathbb{R}$ contains similar copies of all finite subsets. This is not impossible since Erdős and Kakutani [1957] constructed a compact set of measure zero with this property. The Erdős–Kakutani set has Hausdorff dimension 1 but, using the ideas from [Elekes and Steprāns 2004], Máthé [$\geq$ 2008] constructed such a set with Hausdorff dimension 0. However, the packing dimension of such a set must be 1, since the argument of the proof of [Darji and Keleti 2003, Theorem 2] gives that if a compact set $C \subset \mathbb{R}$ contains similar copies of all sets of n points then C has packing dimension at least $\frac{n-2}{n}$.

2. Proof of Theorem 1.1

Fix a sequence $\alpha_1, \alpha_2, \ldots \subset A$ so that each element of A appears infinitely many times in the sequence (α_k). Let

$$\beta_k = \max\Big(6\alpha_k, \frac{6\alpha_k}{\alpha_k - 1}\Big), \qquad (k \in \mathbb{N}). \tag{1}$$

Since $A \subset (1, \infty)$, the number β_k is defined and $\beta_k > 6$ for every k. We can clearly choose a sequence $m_1, m_2, \ldots \subset \{3, 4, 5, \ldots\}$ so that

$$\lim_{k\to\infty} \frac{\log(\beta_1 \cdots \beta_k)}{\log(m_1 \cdots m_{k-1})} = 0. \tag{2}$$

Let

$$\delta_k = \frac{1}{\beta_1 \cdots \beta_k \cdot m_1 \cdots m_k}. \tag{3}$$

By induction we shall define sets

$$E_0 \supset E_1 \supset E_2 \supset \ldots$$

such that for each $k \in \mathbb{N}$

(∗) E_k consists of $m_1 \cdots m_k$ closed intervals of length δ_k which are separated by gaps of at least δ_k and each interval of E_{k-1} contains m_k intervals of E_k.

We will denote by

$$I_1^k, I_2^k, \ldots, I_{m_1 \cdots m_k}^k$$

the intervals of E_k ordered from left to right, and by

$$(J_n, K_n, L_n)_{n\in\mathbb{Z}}$$

an enumeration of the set

$$\Gamma = \left\{(I_a^k, I_b^k, I_c^k) : a, b, c, k \in \mathbb{N}, a < b < c \le m_1 \cdots m_k\right\}$$

such that if $n > 1$ and $(J_n, K_n, L_n) = (I_a^k, I_b^k, I_c^k)$ then $n > k$. Since each element of A appears infinitely many times in the sequence (α_k), by repeating each element of Γ infinitely many times we can also guarantee that for all $a \in A$ and for all $(J, K, L) \in \Gamma$, there exists $n \in \mathbb{N}$ such that

$$\alpha_n = a, \quad \text{and} \quad (J_n, K_n, L_n) = (J, K, L). \tag{4}$$

Let $E_0 = [0, 1]$ and choose E_1 so that $(*)$ holds for $k = 1$. Suppose that $k \ge 2$ and $E_1, \ldots, E_{k-1}$ are already defined so that $(*)$ holds for $1, \ldots, k-1$. Then (J_k, K_k, L_k) is already defined and each interval of E_{k-1} is either contained in exactly one of J_k, K_k and L_k or disjoint from them.

We shall define E_k so that

$$x \in E_k \cap J_k, \quad y \in E_k \cap K_k \quad \text{and} \quad z \in E_k \cap L_k$$

will imply that

$$\frac{z-x}{z-y} \ne \alpha_k.$$

Let I be an interval of E_{k-1} which is contained in J_k. Since I has length δ_{k-1} and using (3) and (1) we have

$$\frac{\delta_{k-1}}{3\alpha_k\delta_k} = \frac{m_k\beta_k}{3\alpha_k} \ge 2m_k > m_k + 1,$$

and I contains more than m_k points of the form $3\alpha_k\delta_k i$ for $i \in \mathbb{Z}$. Hence we can choose the m_k intervals of E_k in I as segments of the form

$$\delta_k(3i\alpha_k + [0, 1]) \quad (i \in \mathbb{Z}).$$

If I is an interval of E_{k-1} which is contained in K_k, then similarly, since

$$\frac{\delta_{k-1}}{3\delta_k} = \frac{m_k\beta_k}{3} \ge 2m_k > m_k + 1,$$

we can choose the m_k intervals of E_k in I as segments of the form

$$\delta_k(3j + [0, 1]) \quad (j \in \mathbb{Z}).$$

If I is an interval of E_{k-1} which is contained in L_k, then, since by (3) and (1) we have

$$\frac{\delta_{k-1}}{\frac{3\alpha_k}{\alpha_k-1}\delta_k} = \frac{m_k\beta_k}{\frac{3\alpha_k}{\alpha_k-1}} \ge 2m_k > m_k + 1,$$

we can choose the m_k intervals of E_k in I as segments of the form

$$\delta_k\Big(\frac{3\alpha_k}{\alpha_k-1}(l + \tfrac{1}{2}) + [0, 1]\Big) \quad (l \in \mathbb{Z}).$$

In each of the rest of the intervals of E_{k-1} we define the m_k intervals of length δ_k of E_k arbitrarily so that they are separated by gaps of at least length δ_k.

This way we defined E_k so that $(*)$ holds. Let

$$E = \bigcap_{k=1}^{\infty} E_k.$$

Then E is clearly a compact subset of $\mathbb{R}$. Condition $(*)$ implies that the Hausdorff dimension of E is at least

$$\liminf_{k\to\infty} \frac{\log(m_1\cdots m_{k-1})}{-\log(m_k\delta_k)}$$

(see [Falconer 1990, Example 4.6]). On the other hand, using (3) and (2) we get that

$$\liminf_{k\to\infty} \frac{\log(m_1\cdots m_{k-1})}{-\log(m_k\delta_k)} = \liminf_{k\to\infty} \frac{\log(m_1\cdots m_{k-1})}{\log(\beta_1\cdots\beta_k)+\log(m_1\cdots m_{k-1})} = 1,$$

and therefore the Hausdorff dimension of E is 1.

Finally, to get a contradiction, suppose that

$$x, y, z \in E, \quad x < y < z, \quad \text{and} \quad \frac{z-x}{z-y} \in A.$$

Since $\delta_k \to 0$, there exists a $k \in \mathbb{N}$ such that x, y and z are in distinct intervals of E_k. Then, by (4) there exists an $n \in \mathbb{N}$ so that

$$x \in J_n, \quad y \in K_n, \quad z \in L_n \quad \text{and} \quad \frac{z-x}{z-y} = \alpha_n.$$

By the construction of E_n, there exists $i, j, l \in \mathbb{Z}$ such that

$$x \in \delta_n(3i\alpha_n + [0,1]),\ \ y \in \delta_n(3j+[0,1]),\ \text{and } z \in \delta_n\Big(\frac{3\alpha_n}{\alpha_n-1}(l+\tfrac{1}{2}) + [0,1]\Big).$$

Let

$$X = 3i\alpha_n + [0,1], \quad Y = 3j + [0,1], \quad \text{and} \quad Z = \frac{3\alpha_n}{\alpha_n - 1}(l+\tfrac{1}{2}) + [0,1].$$

Then $\frac{x}{\delta_n} \in X$, $\frac{y}{\delta_n} \in Y$ and $\frac{z}{\delta_n} \in Z$. On the other hand, $\frac{z-x}{z-y} = \alpha_n$ implies that $\alpha_n y = x + (\alpha_n - 1)z$, so (by using the notation $A + B = \{a+b : a \in A, b \in B\}$) we must have

$$\alpha_n Y \cap (X + (\alpha_n - 1)Z) \neq \varnothing. \tag{5}$$

By definition (and using that $\alpha_n > 1$),

$$\alpha_n Y = \alpha_n(3j + [0,1]) \tag{6}$$

and

$$\begin{aligned} X + (\alpha_n - 1)Z &= 3i\alpha_n + [0,1] + 3\alpha_n(l+\tfrac{1}{2}) + (\alpha_n - 1)[0,1] \\ &= 3(i+l)\alpha_n + \big[\tfrac{3}{2}\alpha_n, \tfrac{5}{2}\alpha_n\big] \\ &= \alpha_n(3(i+l) + \big[\tfrac{3}{2}, \tfrac{5}{2}\big]). \end{aligned} \tag{7}$$

Since $i, j, l \in \mathbb{Z}$, (6) and (7) contradict (5).

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VANISHING VISCOSITY PLANE PARALLEL CHANNEL FLOW AND RELATED SINGULAR PERTURBATION PROBLEMS

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We study a special class of solutions to the three-dimensional Navier–Stokes equations $\partial_t u^\nu + \nabla_{u^\nu} u^\nu + \nabla p^\nu = \nu \Delta u^\nu$, with no-slip boundary condition, on a domain of the form $\Omega = \{(x, y, z) : 0 \le z \le 1\}$, dealing with velocity fields of the form $u^\nu(t, x, y, z) = (v^\nu(t, z), w^\nu(t, x, z), 0)$, describing plane-parallel channel flows. We establish results on convergence $u^\nu \to u^0$ as $\nu \to 0$, where u^0 solves the associated Euler equations. These results go well beyond previously established L^2-norm convergence, and provide a much more detailed picture of the nature of this convergence. Carrying out this analysis also leads naturally to consideration of related singular perturbation problems on bounded domains.

1. Introduction

We look at a special class of solutions to the three-dimensional Navier–Stokes equations on a region $\Omega \subset \mathbb{R}^3$ with boundary:

$$\partial_t u^\nu + \nabla_{u^\nu} u^\nu + \nabla p^\nu = \nu \Delta u^\nu + F, \quad \operatorname{div} u^\nu = 0, \tag{1.0.1}$$

with no-slip boundary data

$$u^\nu(t, q) = B(t, q), \quad q \in \partial\Omega, \tag{1.0.2}$$

given $B(t, q)$ a vector field tangent to $\partial\Omega$. This class consists of what are called plane parallel channel flows. They involve a domain of the form

$$\Omega = \{(x, y, z) : 0 \le z \le 1\}, \tag{1.0.3}$$

velocity fields of the form

$$u^\nu(t, x, y, z) = (v^\nu(t, z), w^\nu(t, x, z), 0), \tag{1.0.4}$$

and external forces of the form

$$F = (f(t, z), g(t, x, z), 0). \tag{1.0.5}$$

This class is mentioned by X. Wang [2001] as a class to which his main theorem on $L^2(\Omega)$-convergence as $\nu \to 0$ (itself a refinement of earlier work of T. Kato [1984]) applies.

There is substantial motivation to obtain a much more detailed picture of the behavior as $\nu \to 0$, including convergence in much stronger topologies, especially away from the boundary, if the initial data and forces satisfy appropriate smoothness hypotheses, and also an analysis of the boundary layer

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on which the solution can make an abrupt transition. The goal of this paper is to establish such stronger results for this class of fluid flows, and to explore some related singular perturbation problems that arise in the course of the analysis.

To begin the analysis, we note that if u^ν has the form (1.0.4) then $\operatorname{div} u^\nu = 0$ and

$$\nabla_{u^\nu} u^\nu = (0, v^\nu(t,z)\partial_x w^\nu(t,x,z), 0), \tag{1.0.6}$$

and hence

$$\operatorname{div} \nabla_{u^\nu} u^\nu = 0. \tag{1.0.7}$$

Thus we can take $p^\nu \equiv 0$ in (1.0.1) and rewrite the system (1.0.1) as

$$\begin{aligned} \frac{\partial v^\nu}{\partial t} &= \nu \frac{\partial^2 v^\nu}{\partial z^2} + f(t,z), \\ \frac{\partial w^\nu}{\partial t} + v^\nu \frac{\partial w^\nu}{\partial x} &= \nu \Big(\frac{\partial^2 w^\nu}{\partial x^2} + \frac{\partial^2 w^\nu}{\partial z^2} \Big) + g(t,x,z). \end{aligned} \tag{1.0.8}$$

(Note: The equations stated on p. 228 of [Wang 2001] have two misprints.) The boundary conditions take the form

$$\begin{aligned} v^\nu(t,z) &= a(t,z), \qquad z = 0, 1, \\ w^\nu(t,x,z) &= b(t,x,z), \quad z = 0, 1. \end{aligned} \tag{1.0.9}$$

We take initial data independent of ν:

$$\begin{aligned} v^\nu(0,z) &= V(z), \\ w^\nu(0,x,z) &= W(x,z). \end{aligned} \tag{1.0.10}$$

One wants to establish convergence of u^ν to u^0, the solution to the Euler equation

$$\partial_t u^0 + \nabla_{u^0} u^0 + \nabla p^0 = F, \quad \operatorname{div} u^0 = 0, \tag{1.0.11}$$

with boundary condition

$$u^0(t,p) \parallel \partial\Omega, \tag{1.0.12}$$

for $p \in \partial\Omega$, and initial condition

$$u^0(0,x,y,z) = (V(z), W(x,z), 0). \tag{1.0.13}$$

We have

$$u^0(t,x,y,z) = (v^0(t,z), w^0(t,x,z), 0), \tag{1.0.14}$$

satisfying

$$\frac{\partial v^0}{\partial t} = f(t,z), \qquad \frac{\partial w^0}{\partial t} + v^0 \frac{\partial w^0}{\partial x} = g(t,x,z). \tag{1.0.15}$$

Initial data are as in (1.0.10).

We begin the analysis of the convergence of v^ν to v^0 and of w^ν to w^0 in Chapter 2. For simplicity we take vanishing forces and boundary velocity. We also take functions to be periodic in x and work on

$$\overline{\mathcal{O}} = \{(x,z) : x \in \mathbb{R}/\mathbb{Z},\ z \in [0,1]\}. \tag{1.0.16}$$

In Section 2.1 we take the particular case $V \equiv 1$ in (1.0.10) and in Section 2.2 we consider general initial velocities of the form (1.0.10). We see that while the convergence of v^ν to v^0 has a simple nature, with a boundary layer phenomenon easily treatable via the method of images, the nature of the convergence of w^ν to w^0 is much more subtle. One tool we use to analyze w^ν is to compare it with the solution to the analogue of the second equation in (1.0.8) with v^ν replaced by $V(z)$. To state the strategy more abstractly, we analyze the solution to

$$\frac{\partial w^\nu}{\partial t} = \nu \Delta w^\nu - X_\nu w^\nu, \quad w^\nu\big|_{\mathbb{R}\times\partial\mathbb{O}} = 0, \tag{1.0.17}$$

where $\Delta = \partial_x^2 + \partial_z^2$ and $X_\nu = v^\nu(t,z)\partial_x$, by considering the solution to

$$\frac{\partial w^\nu}{\partial t} = \nu \Delta w^\nu - X w^\nu + g^\nu, \quad w^\nu\big|_{\mathbb{R}^+\times\partial\mathbb{O}} = 0, \tag{1.0.18}$$

where $X = V(z)\partial_x$ and $g^\nu = (X - X_\nu)w^\nu$. To tackle (1.0.17), we use Duhamel's formula, which gives

$$w^\nu(t) = e^{t(\nu\Delta - X)} W + \int_0^t e^{(t-s)(\nu\Delta - X)} g^\nu(s)\, ds. \tag{1.0.19}$$

This leads to some successful estimates, produced in §Section 2.1–2.2, on the difference $R^\nu(t,x,z) = w^\nu(t) - e^{t(\nu\Delta - X)}W$. We show that for each $p \in [1,\infty)$, $t \in (0,T]$,

$$\|R^\nu(t,\cdot)\|_{L^p(\mathbb{O})} \le C_p \nu^{1/2p} t^{1+1/2p}, \tag{1.0.20}$$

and that, as $\nu \to 0$,

$$R^\nu(t,x,z) \to 0, \quad \text{uniformly for } t \in [0,T],\ (x,z,\nu) \in \mathbb{O}_\eta, \tag{1.0.21}$$

where $\mathbb{O}_\eta = \{(x,z,\nu) : \operatorname{dist}((x,z),\partial\mathbb{O}) \ge \eta(\nu)\}$, for each $\eta(\nu)$ satisfying $\eta(\nu)/\nu^{1/2} \to \infty$ as $\nu \to 0$.

Thus much information about w^ν is revealed by the behavior of $e^{t(\nu\Delta - X)}W$. In case $V \equiv 1$, the operators X and Δ commute, and the behavior of $e^{t(\nu\Delta - X)}W = e^{-tX}e^{t\nu\Delta}W$ is also quite accessible via the method of images. For general $V(z)$, the behavior of $e^{t(\nu\Delta - X)}$ requires further study.

Chapter 3 is devoted to the study of $e^{t(\nu\Delta - X)}$. It is natural to work in a more general setting than in Chapter 2. In place of (1.0.16), we take $\overline{\mathbb{O}}$ to be a compact Riemannian manifold with smooth boundary, with Laplace-Beltrami operator Δ, and we take a smooth vector field X on $\overline{\mathbb{O}}$ satisfying

$$X \parallel \partial\mathbb{O}, \quad \operatorname{div} X = 0. \tag{1.0.22}$$

We obtain convergence results

$$e^{t(\nu\Delta - X)} f \to e^{-tX} f \tag{1.0.23}$$

as $\nu \to 0$, in a number of function spaces, including L^q-Sobolev spaces $H^{\sigma,q}(\mathbb{O})$, for $q \in [2,\infty)$, $\sigma \in [0, 1/q)$, and also spaces

$$\mathcal{V}^k(\mathbb{O}) = \{f \in L^2(\mathbb{O}) : Y_1 \cdots Y_j f \in L^2(\mathbb{O}),\ \forall\, j \le k,\ Y_\ell \in \mathfrak{X}^1\}, \tag{1.0.24}$$

where $\mathfrak{X}^1$ consists of smooth vector fields on $\overline{\mathbb{O}}$ that are tangent to $\partial\mathbb{O}$.

We also produce a layer potential analysis of $e^{t(\nu\Delta - X)} f$, which provides a detailed picture of the boundary layer behavior as $\nu \to 0$. To do this, we find it convenient to work with

$$v^\nu(t) = e^{tX} e^{t(\nu\Delta - X)} f. \tag{1.0.25}$$

One of the main results is given in Proposition 3.7.4, that for $I = [0, T]$, $\delta > 0$,

$$\|v^\nu - (f - 2\mathcal{D}_\nu^0 f^b)\|_{L^\infty(I\times\mathcal{O})} \leq C(I)\nu^{1/2}\|f\|_{C^{1,\delta}(\overline{\mathcal{O}})}, \tag{1.0.26}$$

where $f^b(t, y) = \chi_{\mathbb{R}^+}(t) f(y)$ and $\mathcal{D}_\nu^0$ is a certain layer potential operator:

$$\mathcal{D}_\nu^0 f^b(t, x) = \nu \int_0^t \int_{\partial\mathcal{O}} f(y) \frac{\partial H_0}{\partial n_{s,y}}(\nu, s, t, x, y)\, dS_s(y)\, ds. \tag{1.0.27}$$

See Section 3.7 for more details, including the definitions of $dS_s(y)$, $\partial/\partial n_{s,y}$, and the Gaussian-type integral kernel $H_0(\nu, s, t, x, y)$.

In Chapter 4 we again consider solutions to (1.0.17). Here we work on a compact Riemannian manifold with boundary $\overline{\mathcal{O}}$ as in Chapter 3. We take X_ν to be a family of time dependent vector fields, suitably generalizing the class $X_\nu = v^\nu(t, z)\partial_x$ that arose in Chapter 2, converging to X in a similar way as $v^\nu(t, z)\partial_x$ converges to $V(z)\partial_x$. The main results are given in Propositions 4.2.1–4.2.4. We obtain convergence results

$$w^\nu(t) \to e^{-tX} f \tag{1.0.28}$$

as $\nu \to 0$, in $\mathcal{V}^k(\mathcal{O})$, and in $L^p(\mathcal{O})$, for $1 \leq p < \infty$. Analogues of (1.0.19) play a role in the analysis, and we make strong use of results of Chapter 3.

In Chapter 5 we return to the specific setting of plane parallel channel flow and draw further conclusions about the convergence of v^ν to v^0 and of w^ν to w^0. We extend the scope of Chapter 2 by allowing for some nonzero boundary velocity, arising from rigidly translating the flat boundary faces. We take boundary data $B(t, q)$ of the form

$$B(t, x, z) = (\alpha_j(t), \beta_j(t), 0), \qquad z = j \in \{0, 1\}, \tag{1.0.29}$$

and allow $\alpha_j(t)$ and $\beta_j(t)$ to be fairly rough. We start with the special case $(\alpha_j(t), 0, 0)$, giving motions of the boundary parallel to the x-axis.

The spaces $\mathcal{V}^k(\mathcal{O})$ in (1.0.24) are special cases of "weighted b-Sobolev spaces," introduced and studied in [Melrose 1993]. In Appendix A we discuss this point and use it to establish some complex interpolation results for these spaces, which are of use in Sections 3.3 and 4.2.

This paper is a companion to [Lopes Filho et al. 2007], whose goal was to give a precise analysis of the convergence of the solution of the Navier–Stokes equation, as the vorticity tends to zero, to a steady solution of the Euler equation for 2D circularly symmetric flow in a disk or annulus, sharpening L^2 analyses done in [Matsui 1994], [Bona and Wu 2002], and [Lopes Filho et al. 2008].

2. First results on plane parallel channel flows

Here we start our investigation of the convergence of v^ν and w^ν as $\nu \to 0$, when these functions are solutions to (1.0.8) (with $f = g = 0$ and vanishing boundary condition). The main result of this chapter is the estimate (2.2.11) on

$$w^\nu(t,x,z) - e^{t(\nu\Delta - X)} W(x,z), \tag{2.0.1}$$

together with some of its consequences. To carry on, we need to understand the second term in (2.0.1). This motivates the work of Chapter 3.

2.1. ***Particular case.*** Let us take $f \equiv g \equiv 0$ in (1.0.8) and in (1.0.15), and

$$V \equiv 1, \quad W = W(x,z) \tag{2.1.1}$$

in (1.0.10). Consequently we have

$$v^0(t,z) \equiv 1, \quad w^0(t,x,z) = W(x-t,z) \tag{2.1.2}$$

as the solution to the Euler equations. Let us also take $a \equiv b \equiv 0$ in (1.0.9), i.e., boundary conditions

$$v^\nu(t,z) = w^\nu(t,x,z) = 0, \quad z = 0, 1. \tag{2.1.3}$$

Consequently, for the solution $(v^\nu, w^\nu, 0)$ to the Navier–Stokes equation, we have first of all that

$$v^\nu(t,z) = e^{t\nu A} v_0(z) = e^{t\nu A} 1(z), \tag{2.1.4}$$

where A is the self-adjoint operator on $L^2([0,1])$ defined by

$$\mathfrak{D}(A) = H^2([0,1]) \cap H_0^1([0,1]), \quad A = \partial_z^2 \text{ on } \mathfrak{D}(A). \tag{2.1.5}$$

One can analyze (2.1.4) via the method of images to get a good picture of the boundary layer near $z = 0,\ 1$. Then the equation for w^ν becomes

$$\frac{\partial w^\nu}{\partial t} + v^\nu \frac{\partial w^\nu}{\partial x} = \nu\Big(\frac{\partial^2 w^\nu}{\partial x^2} + \frac{\partial^2 w^\nu}{\partial z^2}\Big), \tag{2.1.6}$$

with initial condition given in (2.1.1) and boundary condition given in (2.1.3).

Let us assume $W(x,z)$ in (2.1.1) is smooth and periodic of period 1 in x, so

$$W \in C^\infty(\overline{\mathbb{O}}), \quad \overline{\mathbb{O}} = \{(x,z) : x \in \mathbb{R}/\mathbb{Z},\ z \in [0,1]\}. \tag{2.1.7}$$

Elementary estimates imply

$$\|w^\nu(t)\|_{L^p(\mathbb{O})} \le \|W\|_{L^p(\mathbb{O})}, \quad 1 \le p \le \infty. \tag{2.1.8}$$

Note that for $k \in \mathbb{Z}^+$,

$$w_k^\nu = \partial_x^k w^\nu \tag{2.1.9}$$

satisfies

$$\frac{\partial w_k^\nu}{\partial t} + v^\nu \frac{\partial w_k^\nu}{\partial x} = \nu \Delta w_k^\nu, \tag{2.1.10}$$

where we have set

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}. \tag{2.1.11}$$

Also

$$w_k^\nu(t,x,z) = 0, \quad z = 0,\ 1. \tag{2.1.12}$$

Hence, parallel to (2.1.8), we have

$$\|w_k^\nu(t)\|_{L^p(\mathcal{O})} \le \|\partial_x^k W\|_{L^p(\mathcal{O})}, \quad 1 \le p \le \infty. \tag{2.1.13}$$

To obtain a finer analysis of $w^\nu(t,x,z)$, let us rewrite (2.1.6) as

$$\frac{\partial w^\nu}{\partial t} = -\partial_x w^\nu + \nu \Delta w^\nu + (1 - v^\nu)\frac{\partial w^\nu}{\partial x}. \tag{2.1.14}$$

Then Duhamel's formula gives

$$w^\nu(t,x,z) = e^{t(\nu\Delta - \partial_x)} W(x,z) + \int_0^t e^{(t-s)(\nu\Delta-\partial_x)} \Big[(1 - v^\nu(s,z))\frac{\partial w^\nu}{\partial x}(s,x,z)\Big]\, ds. \tag{2.1.15}$$

Here Δ stands for the self adjoint operator given by (2.1.11), with

$$\mathcal{D}(\Delta) = H^2(\mathcal{O}) \cap H_0^1(\mathcal{O}). \tag{2.1.16}$$

Note that $e^{t\nu\Delta}$ and $e^{-t\partial_x}$ are commuting semigroups, with $e^{-t\partial_x} f(x,z) = f(x-t,z)$. Hence we have

$$w^\nu(t,x,z) = e^{t\nu\Delta} W(x-t,z) + \int_0^t e^{(t-s)\nu\Delta}\big[(1 - v^\nu(s,z)) w_1^\nu(s, x-t+s, z)\big]\, ds, \tag{2.1.17}$$

where, as in (2.1.9), we have $w_1^\nu = \partial_x w^\nu$. Let us write (2.1.16) as

$$w^\nu(t,x,z) = e^{t\nu\Delta} W(x-t,z) + R^\nu(t,x,z). \tag{2.1.18}$$

By the method of images (or otherwise) we have a clear picture of the first term on the right side of (2.1.18). Let us estimate the remainder, $R^\nu(t,x,z)$. By (2.1.13) and the positivity of $e^{(t-s)\nu\Delta}$, we have

$$|R^\nu(t,x,z)| \le C \int_0^t e^{(t-s)\nu\Delta} |1 - v^\nu(s,z)|\, ds, \tag{2.1.19}$$

since $\partial_x W \in L^\infty(\mathcal{O})$. The analysis of (2.1.4) via the method of images gives

$$|1 - v^\nu(s,z)| \le C_T \varphi\big((s\nu)^{-1/2}\delta(z)\big), \tag{2.1.20}$$

for $s \in [0,T]$, where $\delta(z) = \operatorname{dist}(z, \{0,1\})$ and $\varphi(\zeta)$ is rapidly decreasing as $\zeta \to \infty$. Hence, for $p \in [1,\infty)$,

$$\|R^\nu(t,\cdot)\|_{L^p(\mathcal{O})} \le C \int_0^t \Big(\int_0^1 |1 - v^\nu(s,z)|^p\, dz\Big)^{1/p} ds \le C_p \nu^{1/2p} t^{1+1/2p}. \tag{2.1.21}$$

Furthermore we have, as $\nu \to 0$,

$$R^\nu(t,x,z) \to 0, \quad \text{uniformly for } t \in [0,T],\ \delta(z) \ge \delta_0, \tag{2.1.22}$$

given $\delta_0 > 0$. Indeed, given $\eta(\nu)$ such that

$$\frac{\eta(\nu)}{\nu^{1/2}} \to \infty \ \text{ as } \ \nu \to 0, \tag{2.1.23}$$

and

$$\mathcal{O}_\eta = \{(x,z,\nu) : x \in \mathbb{R}/\mathbb{Z},\ \delta(z) \ge \eta(\nu)\}, \tag{2.1.24}$$

we have

$$R^{\nu}(t,x,z) \to 0, \quad \text{uniformly for } t \in [0,T], \ (x,z,\nu) \in \mathbb{O}_{\eta}. \tag{2.1.25}$$

However, (2.1.15)–(2.1.19) do not reveal the fine structure of $w^{\nu}(t,x,z)$ on the boundary layer. Some other approach will be required for this.

2.2. ***More general case.*** As in Section 2.1, we take $f \equiv g \equiv 0$ in (1.0.8), but now we extend (2.1.1) to the more general case

$$v^{\nu}(0,z) = V(z) \in C^{\infty}(I), \quad w^{\nu}(0,x,z) = W(x,z) \in C^{\infty}(\overline{\mathbb{O}}), \tag{2.2.1}$$

with $\overline{\mathbb{O}}$ as in (2.1.7). Then (2.1.2) is modified to

$$v^{0}(t,z) = V(z), \quad w^{0}(t,x,z) = W(x - tV(z), z). \tag{2.2.2}$$

We retain the boundary conditions (2.1.3), i.e.,

$$v^{\nu}(t,z) = w^{\nu}(t,x,z) = 0, \quad z = 0, 1. \tag{2.2.3}$$

Thus, in place of (2.1.4), we have

$$v^{\nu}(t,z) = e^{t\nu A} V(z), \tag{2.2.4}$$

again with A as in (2.1.5). With these modifications, one still has the Equation (2.1.6) for w^{ν}. We continue to have the estimates (2.1.8) on $\|w^{\nu}(t)\|_{L^p(\mathbb{O})}$. We also have the estimates (2.1.13) on $\|w_k^{\nu}(t)\|_{L^p}$, where $w_k^{\nu} = \partial_x^k w^{\nu}$.

To obtain a finer analysis of $w^{\nu}(t,x,z)$, we use the following modification of (2.1.14):

$$\frac{\partial w^{\nu}}{\partial t} = -V(z)\partial_x w^{\nu} + \nu \Delta w^{\nu} + (V - v^{\nu})\frac{\partial w^{\nu}}{\partial x}. \tag{2.2.5}$$

Then Duhamel's formula gives the following variant of (2.1.15):

$$w^{\nu}(t,x,z) = e^{t(\nu\Delta - V\partial_x)} W(x,z) + \int_0^t e^{(t-s)(\nu\Delta - V\partial_x)} \Big[(V - v^{\nu}(s)) \frac{\partial w^{\nu}}{\partial x}(s) \Big] \, ds. \tag{2.2.6}$$

Here $\nu\Delta - V\partial_x$ generates a contraction semigroup on $L^2(\mathbb{O})$ with domain

$$\mathfrak{D}(\nu\Delta - V\partial_x) = H_0^1(\mathbb{O}) \cap H^2(\mathbb{O}). \tag{2.2.7}$$

It also generates a contraction semigroup on $L^p(\mathbb{O})$ for $1 \le p \le \infty$, strongly continuous for $p \in [1, \infty)$, but not for $p = \infty$. We mention that the Trotter product formula — for which see [Trotter 1959] or [Taylor 1996, Chapter 11, Appendix A] — holds here. Given $p \in [1, \infty)$ and $f \in L^p(\mathbb{O})$, we have

$$e^{t(\nu\Delta - V\partial_x)} f = \lim_{n\to\infty} \big(e^{(t/n)\nu\Delta} e^{-(t/n)V\partial_x}\big)^n f, \quad \text{in } L^p\text{-norm}. \tag{2.2.8}$$

Of course,

$$e^{-sV\partial_x} f(x,z) = f(x - sV(z), z). \tag{2.2.9}$$

To proceed, we have, parallel to (2.1.18)–(2.1.19),

$$w^{\nu}(t,x,z) = e^{t(\nu\Delta - V\partial_x)} W(x,z) + R^{\nu}(t,x,z), \tag{2.2.10}$$

with

$$|R^\nu(t,x,z)| \le C\int_0^t e^{(t-s)(\nu\Delta - V\partial_x)}|V - v^\nu(s)|\,ds = C\int_0^t e^{(t-s)\nu\Delta}|V(z) - v^\nu(s,z)|\,ds, \tag{2.2.11}$$

since $\partial_x W \in L^\infty(\mathbb{O})$. Again, to get this, one uses the estimate (2.1.13) with $k=1$, and the positivity of $e^{(t-s)(\nu\Delta - V\partial_x)}$. For the last identity in (2.2.11), one uses the fact that $V(z) - v^\nu(s,z)$ is independent of x. Once we have (2.2.11), we can again apply the method of images to estimate

$$|V(z) - v^\nu(s,z)| \le C_T\varphi\big((s\nu)^{-1/2}\delta(z)\big), \tag{2.2.12}$$

as in (2.1.20), except now we have only $\varphi(\zeta) \le C(1+\zeta^2)^{-1}$. This is enough for the estimates (2.1.21)–(2.1.25) on $R^\nu(t,x,z)$ continue to hold.

In the current setting, the term $e^{t(\nu\Delta - V\partial_x)}W$ requires a more vigorous investigation for general smooth $V(z)$ on $[0,1]$ than it did in the case $V\equiv 1$, considered in Section 2.1. We want to establish results of the form

$$e^{t(\nu\Delta - X)}f \to e^{-tX}f, \quad \text{as} \ \ \nu \to 0, \tag{2.2.13}$$

in L^p-norm, for all $f \in L^p(\mathbb{O})$, where

$$X = V(z)\partial_x. \tag{2.2.14}$$

We also want to investigate such convergence in other function spaces. We will obtain such results, in a more general context, in the chapters that follow.

3. Analysis of solutions to $u_t = \nu\Delta u - Xu$

We examine the solution operator $e^{t(\nu\Delta - X)}f = u(t)$, given by

$$\frac{\partial u}{\partial t} = \nu\Delta u - Xu, \quad u(0) = f, \quad u(t,x) = 0 \ \text{ for } \ x \in \partial\mathbb{O}. \tag{3.0.1}$$

We work in a more general context than in Section 2.2. Assume $\overline{\mathbb{O}}$ is a compact Riemannian manifold, with smooth boundary $\partial\mathbb{O}$, and with Laplace-Beltrami operator Δ, and X is a smooth, real vector field on $\overline{\mathbb{O}}$, satisfying

$$X \parallel \partial\mathbb{O}, \quad \operatorname{div} X = 0. \tag{3.0.2}$$

Under such hypotheses, for each $\nu \in (0,\infty)$, $e^{t(\nu\Delta - X)}$ is a strongly continuous contraction semigroup on $L^p(\mathbb{O})$ for each $p \in [1,\infty)$. Furthermore, the Trotter product formula holds; given $p \in [1,\infty)$, $f \in L^p(\mathbb{O})$,

$$e^{t(\nu\Delta - X)}f = \lim_{n\to\infty}\Big(e^{(t/n)\nu\Delta}e^{-(t/n)X}\Big)^n f, \quad \text{in} \ \ L^p\text{-norm}. \tag{3.0.3}$$

Our goal is to obtain precise results on convergence

$$e^{t(\nu\Delta - X)}f \to e^{-tX}f, \tag{3.0.4}$$

as $\nu \searrow 0$. In particular, we establish convergence in a variety of function spaces. In Section 3.1 we establish such convergence in the L^q-Sobolev space $H^{s,q}(\mathbb{O})$ for $q \in [2,\infty)$ and $s \in [0, 1/q)$. In

Section 3.2 we study local convergence. For this, it is convenient to work with

$$v^\nu(t) = e^{tX} e^{t(\nu\Delta - X)} f, \tag{3.0.5}$$

which solves

$$\frac{\partial v^\nu}{\partial t} = \nu L(t) v^\nu, \quad v^\nu(0) = f, \tag{3.0.6}$$

with boundary condition $v^\nu = 0$ on $\mathbb{R}^+ \times \partial\mathbb{O}$, where $L(t)$ is the smooth family of strongly elliptic differential operators given by $L(t) = e^{tX}\Delta e^{-tX}$. Given $\Omega_1 \Subset \Omega_0 \subset\subset \mathbb{O}$, we show that if $f \in L^2(\mathbb{O})$ and $f \in H^k(\Omega_0)$, then $v^\nu(t) \to f$ in $H^k(\Omega_1)$. In Section 3.3 we establish convergence in the space

$$\mathcal{V}^k(\mathbb{O}) = \{f \in L^2(\mathbb{O}) : Y_1 \cdots Y_j f \in L^2(\mathbb{O}),\ \forall\, j \le k,\ Y_\ell \in \mathfrak{X}^1\}, \tag{3.0.7}$$

where $\mathfrak{X}^1$ consists of all smooth vector fields on $\overline{\mathbb{O}}$ that are tangent to $\partial\mathbb{O}$. In Section 3.4 we show that the Laplace operator, with Dirichlet boundary condition, generates a holomorphic semigroup on $\mathcal{V}^k(\mathbb{O})$. This result is peripheral to the other results of this chapter, but it will prove useful in Section 4.1.

In Section 3.5 we extend the results of Section 3.1 to convergence in $H^{\sigma,q}$ for all $q \in [2, \infty)$, $\sigma \ge 0$, in case $\mathbb{O}$ is replaced by a compact manifold without boundary, M. These results are relatively easy, since it is only the presence of a boundary that causes a problem. They are recorded here to lay a foundation for the work in §Section 3.6–3.7. Section 3.6 is devoted to constructing a parametrix for the solution of (3.0.6) on $\mathbb{R}^+ \times M$, valid uniformly for $\nu \in (0, 1]$, and with increased precision as $\nu \searrow 0$. The construction here is parallel to, but somewhat more elaborate than the construction of a parametrix for the heat equation $(\partial_t - \Delta)u = 0$ on $\mathbb{R}^+ \times M$, yielding short time asymptotics. The parametrix constructed in Section 3.6 is used in Section 3.7 to produce a layer potential attack on solutions to (3.0.6) on $\mathbb{R}^+ \times \mathbb{O}$, yielding sharp results on convergence in (3.0.4), including a picture of the boundary layer behavior.

3.1. *L^q-Sobolev estimates on $e^{t(\nu\Delta - X)}$*. This section is devoted to L^q-Sobolev estimates. To begin, take $q = 2$. We have, for each $\nu > 0$,

$$\mathcal{D}(\nu\Delta - X) = \{f \in H^2(\mathbb{O}) : f|_{\partial\mathbb{O}} = 0\}, \tag{3.1.1}$$

$$\mathcal{D}((\nu\Delta - X)^2) = \{f \in H^4(\mathbb{O}) : f|_{\partial\mathbb{O}} = 0,\ \nu\Delta f - Xf|_{\partial\mathbb{O}} = 0\}, \tag{3.1.2}$$

and, for $k \ge 3$,

$$\mathcal{D}((\nu\Delta - X)^k) = \{f \in H^{2k}(\mathbb{O}) : f|_{\partial\mathbb{O}} = 0,\ (\nu\Delta - X)^j f|_{\partial\mathbb{O}} = 0 \text{ for } j < k\}. \tag{3.1.3}$$

Comparison with analogous formulas for $\mathcal{D}(\Delta^k)$ yields the following.

Proposition 3.1.1. *We have, for each $\nu > 0$,*

$$\mathcal{D}((\nu\Delta - X)^k) = \mathcal{D}(\Delta^k), \quad \textit{for } k = 1, 2. \tag{3.1.4}$$

Proof. The case $k = 1$ is immediate from (3.1.1). As for $k = 2$, note that if $f \in H^4(\mathbb{O})$ and $f|_{\partial\mathbb{O}} = 0$, then also $Xf|_{\partial\mathbb{O}} = 0$ (since $X \parallel \partial\mathbb{O}$), and hence $\Delta f|_{\partial\mathbb{O}} = 0 \Leftrightarrow (\nu\Delta - X)f|_{\partial\mathbb{O}} = 0$. □

As stated in Section 2.2, we want to establish results of the form

$$e^{t(\nu\Delta - X)} f \to e^{-tX} f, \quad \text{as } \nu \to 0, \text{ in } L^p\text{-norm}, \tag{3.1.5}$$

for all $f \in L^p(\mathbb{O})$, $p \in [1, \infty)$. Since we know $e^{t(\nu\Delta - X)}$ is a contraction semigroup on $L^p(\mathbb{O})$, if we can establish (3.1.5) for f in a dense linear subspace $\mathcal{V}$ of $L^p(\mathbb{O})$, we will have it for all $f \in L^p(\mathbb{O})$. This is the approach we will take for $p \in [1, 2]$, using

$$\mathcal{V} = \mathcal{D}(\Delta^2) = \mathcal{D}((\nu\Delta - X)^2), \quad \text{given by (3.1.2).} \tag{3.1.6}$$

Given such f, $u(t) = e^{t(\nu\Delta - X)} f$ satisfies

$$\frac{\partial u}{\partial t} = -Xu + \nu\Delta u, \quad u(0) = f, \tag{3.1.7}$$

and belongs to $C([0, \infty), \mathcal{D}(\Delta^2)) \cap C^1([0, \infty), \mathcal{D}(\Delta))$. Duhamel's formula yields

$$u(t) = e^{-tX} f + \nu \int_0^t e^{-(t-s)X} \Delta u(s)\, ds. \tag{3.1.8}$$

Thus

$$\|e^{t(\nu\Delta - X)} f - e^{-tX} f\|_{L^p} \le \nu \int_0^t \|\Delta u(s)\|_{L^p}\, ds, \tag{3.1.9}$$

so we have (3.1.5) whenever we can obtain a favorable estimate on the right side of (3.1.9). The following lemma provides a key, first for $p = 2$.

Lemma 3.1.2. *Take $f \in \mathcal{V}$, given by (3.1.6), and set $u(t) = e^{t(\nu\Delta - X)} f$, with $\nu > 0$. Then there exists $K \in (0, \infty)$, independent of ν, such that*

$$\|\Delta u(t)\|_{L^2}^2 \le e^{2Kt} \|\Delta f\|_{L^2}^2. \tag{3.1.10}$$

Proof. We have

$$\begin{aligned}
\frac{d}{dt}\|\Delta u(t)\|_{L^2}^2 &= 2\,\mathrm{Re}\,(\Delta \partial_t u, \Delta u)_{L^2} = 2\,\mathrm{Re}\,(\nu\Delta^2 u, \Delta u)_{L^2} - 2\,\mathrm{Re}\,(\Delta X u, \Delta u)_{L^2} \\
&\le -2\,\mathrm{Re}\,(\Delta X u, \Delta u)_{L^2} = -2\,\mathrm{Re}\,(X\Delta u, \Delta u)_{L^2} - 2\,\mathrm{Re}\,([\Delta, X]u, \Delta u)_{L^2} \\
&\le 2K\|\Delta u\|_{L^2}^2,
\end{aligned} \tag{3.1.11}$$

with K independent of ν. The last estimate holds because

$$g \in \mathcal{D}(\Delta) \implies |(Xg, g)_{L^2}| \le K_1 \|g\|_{L^2}^2, \tag{3.1.12}$$

and

$$\begin{aligned}
u(t) \in \mathcal{D}(\Delta^2) \;\Rightarrow\; [\Delta, X]u(t) \in L^2(\mathbb{O}) \;\text{ and}& \\
\|[\Delta, X]u(t)\|_{L^2} \le \widetilde{K}_2 \|u(t)\|_{H^2} \le K_2 \|\Delta u(t)\|_{L^2}.&
\end{aligned} \tag{3.1.13}$$

The asserted estimate (3.1.10) follows. ☐

Proposition 3.1.3. *Given $p \in [1, \infty)$ and $f \in L^p(\mathbb{O})$, we have (3.1.5), with convergence in L^p-norm.*

Proof. For $p \in [1, 2]$, this follows from the operator bound $\|e^{t(\nu\Delta - X)}\|_{\mathcal{L}(L^p)} \le 1$, the denseness of $\mathcal{V}$ in $L^p(\mathbb{O})$, and the application of (3.1.10) to (3.1.9), which gives convergence in L^2-norm, and a fortiori in L^p-norm, for each $f \in \mathcal{V}$.

Suppose now that $p \in (2, \infty)$, with dual exponent $p' \in (1, 2)$. All considerations above apply with X replaced by $-X$, so we have

$$e^{t(\nu\Delta + X)} g \to e^{tX} g, \quad \text{as } \nu \to 0, \tag{3.1.14}$$

in $L^{p'}$-norm, for each $g \in L^{p'}$. This implies that for each $f \in L^p(\mathcal{O})$, convergence in (3.1.5) holds in the weak* topology of $L^p(\mathcal{O})$. Now, since e^{-tX} is an *isometry* on $L^p(\mathcal{O})$, we have

$$\|e^{-tX} f\|_{L^p} \geq \limsup_{\nu \to 0} \|e^{t(\nu\Delta - X)} f\|_{L^p}, \tag{3.1.15}$$

for each $f \in L^p(\mathcal{O})$. Since $L^p(\mathcal{O})$ is a *uniformly convex* Banach space for such p, this yields L^p-norm convergence in (3.1.5). □

To continue, we have from (3.1.10) the estimate

$$\|e^{t(\nu\Delta - X)} f\|_{\mathcal{D}(\Delta)} \leq e^{Kt} \|f\|_{\mathcal{D}(\Delta)}, \tag{3.1.16}$$

first for each $f \in \mathcal{V}$, hence for each $f \in \mathcal{D}(\Delta)$. Interpolation with the L^2- estimate then yields

$$\|e^{t(\nu\Delta - X)} f\|_{\mathcal{D}((-\Delta)^{s/2})} \leq e^{Kt} \|f\|_{\mathcal{D}((-\Delta)^{s/2})}, \tag{3.1.17}$$

for each $s \in [0, 2]$, $f \in \mathcal{D}((-\Delta)^{s/2})$. Now

$$\mathcal{D}((-\Delta)^{s/2}) = H^s(\mathcal{O}), \quad \text{for } s \in \left[0, \tfrac{1}{2}\right), \tag{3.1.18}$$

so we have

$$\|e^{t(\nu\Delta - X)} f\|_{H^s(\mathcal{O})} \leq C e^{Kt} \|f\|_{H^s(\mathcal{O})}, \quad s \in \left[0, \tfrac{1}{2}\right), \tag{3.1.19}$$

where the factor of C might arise due to the choice of H^s-norm; the important fact is that C and K are independent of $\nu \in (0, \infty)$. We can interpolate the estimate (3.1.19) with

$$\|e^{t(\nu\Delta - X)} f\|_{L^p(\mathcal{O})} \leq \|f\|_{L^p(\mathcal{O})}, \quad 1 \leq p < \infty. \tag{3.1.20}$$

Using

$$[H^s(\mathcal{O}), L^p(\mathcal{O})]_\theta = H^{(1-\theta)s, q(\theta)}(\mathcal{O}), \quad \frac{1}{q(\theta)} = \frac{1-\theta}{2} + \frac{\theta}{p}, \tag{3.1.21}$$

we have

$$\|e^{t(\nu\Delta - X)} f\|_{H^{\sigma,q}(\mathcal{O})} \leq C_{\sigma,q}\, e^{Kt} \|f\|_{H^{\sigma,q}(\mathcal{O})}, \tag{3.1.22}$$

valid for

$$2 \leq q < \infty, \quad \sigma q \in [0, 1). \tag{3.1.23}$$

We mention that similar arguments give analogous operator bounds on e^{-tX}, and also on e^{tX}.

Remark. In the absence of further compatibility conditions between X and Δ, one does *not* have

$$e^{-tX} : \mathcal{D}(\Delta^2) \to \mathcal{D}(\Delta^2). \tag{3.1.24}$$

Hence, typically, for $f \in \mathcal{D}(\Delta^2)$,

$$\sup_{\nu \in (0,1]} \|e^{t(\nu\Delta - X)} f\|_{\mathcal{D}(\Delta^2)} = \infty. \tag{3.1.25}$$

In some cases one does have (3.1.24), for example when X and Δ commute. In such a case, $e^{t(\nu\Delta - X)} = e^{\nu t\Delta} e^{-tX}$. It is our goal here to analyze $e^{t(\nu\Delta - X)}$ when one does not have this extra compatibility.

From (3.1.22), we have the following convergence result.

Proposition 3.1.4. *Let q, σ satisfy (3.1.23). Then, for each $t \in (0, \infty)$,*

$$f \in H^{\sigma,q}(\mathcal{O}) \Longrightarrow \lim_{\nu \to 0} e^{t(\nu\Delta - X)} f = e^{-tX} f, \tag{3.1.26}$$

in $H^{\sigma,q}$-norm.

Proof. Given $f \in H^{\sigma,q}(\mathcal{O})$, (3.1.22) implies $\{e^{t(\nu\Delta-X)} f : \nu \in (0,1]\}$ is bounded in $H^{\sigma,q}(\mathcal{O})$, for each $t \in (0, \infty)$, so there are weak* limit points. But Proposition 3.1.3 yields convergence to $e^{-tX} f$ in L^q-norm, so $e^{-tX} f$ is the only possible weak* limit point. Norm convergence in $H^{\tau,q}(\mathcal{O})$, for each $\tau < \sigma$, then follows from the compactness of the inclusion $H^{\sigma,q}(\mathcal{O}) \hookrightarrow H^{\tau,q}(\mathcal{O})$. Now we can pick $\sigma' > \sigma$ so that $\sigma' q < 1$, and take $f_k \in H^{\sigma',q}(\mathcal{O})$ so that $f_k \to f$ in $H^{\sigma,q}$-norm. We deduce from the argument just made that as $\nu \to 0$, $e^{t(\nu\Delta-X)} f_k \to e^{-tX} f_k$ in $H^{\sigma,q}$-norm, for each k. Application of (3.1.22) with f replaced by $f - f_k$ then finishes the proof. □

We move on to some convergence results for classes of data f that vanish on $\partial\mathcal{O}$.

Proposition 3.1.5. *For each $t \in (0, \infty)$,*

$$f \in \mathcal{D}(\Delta) \Longrightarrow \lim_{\nu \to 0} e^{t(\nu\Delta - X)} f = e^{-tX} f \tag{3.1.27}$$

weak* in $\mathcal{D}(\Delta) = H^2(\mathcal{O}) \cap H^1_0(\mathcal{O})$, hence in H^s-norm for each $s < 2$.

Proof. Lemma 3.1.2 gives $\{e^{t(\nu\Delta-X)} f : \nu \in (0,1]\}$ bounded in $\mathcal{D}(\Delta)$ for each $f \in \mathcal{V}$, hence for each $f \in \mathcal{D}(\Delta)$, as noted in (3.1.16). Since we have convergence to $e^{-tX} f$ in L^2-norm, the weak* convergence in $\mathcal{D}(\Delta)$ follows. The norm convergence in $H^s(\mathcal{O})$ for each $s < 2$ then follows from compactness of the inclusion $H^2(\mathcal{O}) \hookrightarrow H^s(\mathcal{O})$. □

Proposition 3.1.6. *Let $C_b(\overline{\mathcal{O}}) = \{f \in C(\overline{\mathcal{O}}) : f|_{\partial\mathcal{O}} = 0\}$. Then for each $t \in (0, \infty)$,*

$$f \in C_b(\overline{\mathcal{O}}) \Longrightarrow \lim_{\nu \to 0} e^{t(\nu\Delta - X)} f = e^{-tX} f, \tag{3.1.28}$$

in the supremum norm, provided $\dim \mathcal{O} \le 3$.

Proof. For $\dim \mathcal{O} \le 3$, $\mathcal{D}(\Delta) \subset C(\overline{\mathcal{O}})$, and it is dense in $C_b(\overline{\mathcal{O}})$. Since $e^{t(\nu\Delta-X)}$ is a contraction on $C_b(\overline{\mathcal{O}})$, a standard argument yields (3.1.28) from (3.1.27). □

If the hypothesis in (3.1.28) is weakened to $f \in C(\overline{\mathcal{O}})$, results obtained above yield convergence, weak* in $L^\infty(\mathcal{O})$, but of course one does not have L^∞-norm convergence if f does not vanish on $\partial\mathcal{O}$. In Section 3.2 we will show that convergence does hold uniformly on compact subsets of $\mathcal{O}$.

3.2. *Local regularity and convergence results for $e^{t(\nu\Delta - X)}$.*

Given a function f on $\mathcal{O}$, consider

$$v(t) = e^{tX} e^{t(\nu\Delta - X)} f. \tag{3.2.1}$$

We have

$$\frac{\partial v}{\partial t} = e^{tX}[X + \nu\Delta - X] e^{t(\nu\Delta - X)} f = \nu e^{tX} \Delta e^{t(\nu\Delta - X)} f. \tag{3.2.2}$$

Now

$$L(t) = e^{tX} \Delta e^{-tX} \tag{3.2.3}$$

is a one-parameter family of strongly elliptic differential operators on $\mathcal{O}$, depending smoothly on t, and (3.2.2) yields

$$\frac{\partial v}{\partial t} = \nu L(t) e^{tX} e^{t(\nu\Delta - X)} f, \tag{3.2.4}$$

so $v(t)$ is uniquely characterized by

$$\frac{\partial v}{\partial t} = \nu L(t) v, \quad v(0) = f, \quad v\big|_{\mathbb{R}^+\times\partial\mathcal{O}} = 0. \tag{3.2.5}$$

We now prove the following local regularity result.

Proposition 3.2.1. *Let $f \in L^2(\mathcal{O})$ and assume Ω_j are smoothly bounded domains satisfying $\Omega_1 \Subset \Omega_0 \Subset \mathcal{O}$. Assume $k \in \mathbb{N}$ and $f \in H^k(\Omega_0)$. Then the solution $v = v^\nu$ to (3.2.5) belongs to $C([0,\infty), H^k(\Omega_1))$, and for each $T \in (0,\infty)$ we have*

$$\|v^\nu(t)\|^2_{H^k(\Omega_1)} + c_{Tk}\nu \int_0^t \|v^\nu(s)\|^2_{H^{k+1}(\Omega_1)}\, ds \le C_{Tk}\big(\|f\|^2_{H^k(\Omega_0)} + \|f\|^2_{L^2(\mathcal{O})}\big), \quad 0 \le t \le T, \tag{3.2.6}$$

with $c_{Tk},\ C_{Tk} \in (0,\infty)$, independent of $\nu \in \mathbb{R}^+$.

Proof. To start, note that

$$\frac{d}{dt}\|v\|^2_{L^2} = 2\nu(L(t)v, v) le - C\nu\|\nabla v\|^2_{L^2} + C'\nu\|v\|^2_{L^2}; \tag{3.2.7}$$

hence, for $0 \le t \le T$,

$$\|v(t)\|^2_{L^2(\mathcal{O})} + c_{T0}\nu \int_0^t \|\nabla v(s)\|^2_{L^2(\mathcal{O})}\, ds \le C_{T0}\|f\|^2_{L^2(\mathcal{O})}, \tag{3.2.8}$$

which contains (3.2.6) for $k = 0$. To proceed, take $\varphi \in C_0^\infty(\Omega_0)$ such that $\varphi = 1$ on a neighborhood of $\overline{\Omega}_1$. Then $w = \varphi v^\nu$ satisfies

$$\partial_t w = \nu L(t) w + \nu Y(t) v, \quad w(0) = \varphi f, \tag{3.2.9}$$

with

$$Y(t) = [\varphi, L(t)]. \tag{3.2.10}$$

Note that $Y(t)$ is a smooth family of differential operators of order 1. Now pick $m \in \{1, \dots, k\}$. We have, for $\|D^m w\|^2_{L^2} = \sum_{|\alpha|\le m} \|D^\alpha w\|^2_{L^2}$,

$$\begin{aligned} \frac{d}{dt}\|D^m w\|^2_{L^2} &= 2\nu(D^m[L(t)w + Y(t)v], D^m w)_{L^2} \\ &= 2\nu(L(t)D^m w, D^m w) + 2\nu([D^m, L(t)]w, D^m w) + 2\nu(D^m Y(t)v, D^m w) \\ &\le -C_1\nu\|D^{m+1}w\|^2_{L^2} + C_2\nu\|D^m w\|^2_{L^2} + C_3\nu\|D^{m-1}Y(t)v\|^2_{L^2}. \end{aligned} \tag{3.2.11}$$

(To get from the second line to the third, integrate by parts to put the term $2\nu(D^m Y(t)v, D^m w)$) in the form $2\nu(D^{m-1}Y(t)v, D^{m+1}w)$.) Hence we obtain, for $t \in [0, T]$,

$$\|D^m w(t)\|^2_{L^2} + c_{Tm}\nu \int_0^t \|D^{m+1}w(s)\|^2_{L^2}\, ds \le C_{Tm}\Big[\|D^m w(0)\|^2_{L^2} + \nu \int_0^t \|D^m v(s)\|^2_{L^2(\Omega_0)}\, ds\Big], \tag{3.2.12}$$

from which (3.2.6) follows inductively. ∎

We can deduce local convergence results from Proposition 3.2.1. Since

$$v^\nu(t) - f = \nu \int_0^t L(s) v(s)\, ds, \tag{3.2.13}$$

we see that under the hypotheses of Proposition 3.2.1,

$$\|v^\nu(t) - f\|_{H^{k-2}(\Omega_1)} \le C\nu^{1/2}\big(\|f\|_{H^k(\Omega_0)} + \|f\|_{L^2(\mathcal{O})}\big). \tag{3.2.14}$$

Interpolation with the bound on $\|v^\nu(t)\|_{H^k(\Omega_1)}$ in (3.2.6) then gives

$$\|v^\nu(t) - f\|_{H^{k-2\theta}(\Omega_1)} \le C\nu^{\theta/2}\big(\|f\|_{H^k(\Omega_0)} + \|f\|_{L^2(\mathcal{O})}\big), \tag{3.2.15}$$

for $\theta \in (0,1]$. Now if we take $f_j \in L^2(\mathcal{O})$ such that $f_j \in H^{k+1}(\Omega_0)$ and $f_j \to f$ in $L^2(\mathcal{O})$-norm and in $H^k(\Omega_0)$-norm, an argument such as used at the end of the proof of Proposition 3.1.4 gives:

Proposition 3.2.2. *Under the hypotheses of Proposition 3.2.1, as $\nu \to 0$,*

$$v^\nu(t) \to f \ \ in \ \ H^k(\Omega_1), \tag{3.2.16}$$

for each $t \ge 0$.

We can pass from Proposition 3.2.2 to other local convergence results. Here is one.

Proposition 3.2.3. *Let $f \in C(\overline{\mathcal{O}})$, and take Ω_j as in Proposition 3.2.1. Then the solution v^ν to (3.2.5) satisfies*

$$v^\nu(t) \to f, \ \ uniformly\ on \ \ \Omega_1, \tag{3.2.17}$$

as $\nu \to 0$. This holds uniformly in $t \in [0,T]$.

Proof. Take $k > n/2$ ($n = \dim \mathcal{O}$), and take $\varepsilon > 0$. Take $g_\varepsilon \in H^k(\mathcal{O})$ such that $\|f - g_\varepsilon\|_{L^\infty(\mathcal{O})} \le \varepsilon$. Let v^ν_ε satisfy

$$\frac{\partial v^\nu_\varepsilon}{\partial t} = \nu L(t) v^\nu_\varepsilon, \quad v^\nu_\varepsilon(0) = g_\varepsilon, \quad v\big|_{\mathbb{R}^+ \times \partial\mathcal{O}} = 0. \tag{3.2.18}$$

We have, by the maximum principle,

$$\|v^\nu_\varepsilon(t) - v^\nu(t)\|_{L^\infty(\mathcal{O})} \le \|f - g\|_{L^\infty(\mathcal{O})} \le \varepsilon. \tag{3.2.19}$$

Meanwhile, Proposition 3.2.2 gives

$$v^\nu_\varepsilon(t) \to g_\varepsilon \ \ \text{in} \ \ H^k(\Omega_1) \subset C(\overline{\Omega}_1), \tag{3.2.20}$$

as $\nu \to 0$, so (3.2.17) holds. ☐

3.3. *Conormal type estimates on $e^{t(\nu\Delta - X)}$.* Here we aim to show that $\{e^{t(\nu\Delta - X)} : \nu \in (0,1]\}$ is a strongly continuous semigroup, with norm bounds independent of $\nu \in (0,1]$, on spaces of the following form:

$$\mathcal{V}^k(\mathcal{O}) = \{u \in L^2(\mathcal{O}) : Y_1 \cdots Y_j u \in L^2(\mathcal{O}), \ \forall\, j \le k, Y_\ell \in \mathfrak{X}^1\}, \tag{3.3.1}$$

for $k \in \mathbb{Z}^+ = \{0, 1, 2, \dots\}$, where

$$\mathfrak{X}^1 = \{Y \text{ smooth vector field on } \overline{\mathcal{O}} : Y \parallel \partial\mathcal{O}\}. \tag{3.3.2}$$

See the Remark on page for a discussion of why $\mathcal{V}^k(\mathcal{O})$-norm estimates are called conormal estimates.

Before starting to produce estimates, we develop some notation and preliminary material.

Lemma 3.3.1. *There exists a finite set*

$$\{Y_j : 1 \le j \le M\} \subset \mathfrak{X}^1 \tag{3.3.3}$$

with the property that each element of $\mathfrak{X}^1$ is a linear combination, with coefficients in $C^\infty(\overline{\mathbb{O}})$, of these vector fields Y_j.

Proof. Routine. □

From here, take Y_j as in (3.3.3), let

$$J = (j_1, \dots, j_k), \tag{3.3.4}$$

and set

$$Y^J = Y_{j_1} \cdots Y_{j_k}, \quad |J| = k. \tag{3.3.5}$$

Also set

$$\mathfrak{X}^k = \operatorname{Span}\{Z_1 \cdots Z_j : j \le k,\ Z_\ell \in \mathfrak{X}^1\}. \tag{3.3.6}$$

We have

$$\mathfrak{X}^k = \text{Span over } C^\infty(\overline{\mathbb{O}}) \text{ of } \{Y^J : |J| \le k\}, \tag{3.3.7}$$

and

$$\begin{aligned}\mathcal{V}^k(\mathbb{O}) &= \{u \in L^2(\mathbb{O}) : Y^J u \in L^2(\mathbb{O}),\ \forall\, |J| \le k\} \\ &= \{u \in L^2(\mathbb{O}) : Lu \in L^2(\mathbb{O}),\ \forall\, L \in \mathfrak{X}^k\}.\end{aligned} \tag{3.3.8}$$

We have the following square-norm and norm on $\mathcal{V}^k(\mathbb{O})$:

$$N_k^2(u) = \sum_{|J| \le k} \|Y^J u\|_{L^2}^2, \quad N_k(u) = N_k^2(u)^{1/2}. \tag{3.3.9}$$

Also set

$$P_k^2(u) = \sum_{|J| = k} \|Y^J u\|_{L^2}^2. \tag{3.3.10}$$

We now estimate the rate of change of $P_k^2(u(t))$ for

$$u(t) = e^{t(\nu\Delta - X)} f, \quad f \in \mathcal{V}^k(\mathbb{O}), \tag{3.3.11}$$

starting with the case $k = 0$:

$$\frac{d}{dt}\|u\|_{L^2}^2 = 2(u_t, u)_{L^2} = 2\nu(\Delta u, u)_{L^2} - 2(Xu, u)_{L^2} = -2\nu\|\nabla u\|_{L^2}^2, \tag{3.3.12}$$

since, for $t > 0$, $u(t) \in \mathfrak{D}((\nu\Delta - X)^m)$ for all m, and hence $u(t) \in H^{2m}(\mathbb{O}) \cap H_0^1(\mathbb{O})$. Moving on to $k = 1$, we have

$$\begin{aligned}\frac{d}{dt}\|Y_j u\|_{L^2}^2 &= 2(Y_j u_t, Y_j u)_{L^2} \\ &= 2\nu(Y_j \Delta u, Y_j u)_{L^2} - 2(Y_j X u, Y_j u)_{L^2} \\ &= 2\nu(\Delta Y_j u, Y_j u)_{L^2} + 2\nu([Y_j, \Delta]u, Y_j u)_{L^2} - 2(X Y_j u, Y_j u)_{L^2} - 2([Y_j, X]u, Y_j u)L^2 \\ &= -2\nu\|\nabla Y_j u\|_{L^2}^2 + 2\nu([Y_j, \Delta]u, Y_j u)_{L^2} - 2([Y_j, X]u, Y_j u)_{L^2}.\end{aligned} \tag{3.3.13}$$

Of the three terms in the last line, the first has a clear significance. For the third, we have $[Y_j, X] \in \mathfrak{X}^1$, and hence

$$2([Y_j, X]u, Y_j u)_{L^2} \le C P_1^2(u). \tag{3.3.14}$$

It remains to estimate the second term. For this, write

$$[Y, \Delta] = \sum_\ell A_\ell B_\ell, \tag{3.3.15}$$

with A_ℓ, B_ℓ smooth vector fields on $\overline{\mathcal{O}}$. We have

$$2\nu([Y_j, \Delta]u, Y_j u)_{L^2} = 2\nu \sum_\ell (B_\ell u, A_\ell^* Y_j u)_{L^2} \le \nu\|\nabla Y_j u\|_{L^2}^2 + \nu\|Y_j u\|_{L^2}^2 + K_1\nu\|\nabla u\|_{L^2}^2. \tag{3.3.16}$$

Plugging (3.3.14) and (3.3.16) into (3.3.13) and summing over j gives

$$\frac{d}{dt} P_1^2(u) \le -\nu \sum_j \|\nabla Y_j u\|_{L^2}^2 + (MC + \nu) P_1^2(u) + MK_1\nu\|\nabla u\|^2. \tag{3.3.17}$$

The term $MK_1\nu\|\nabla u\|_{L^2}^2$ is tamed by bringing in (3.3.12), to obtain

$$\frac{d}{dt}\Big(P_1^2(u) + \frac{MK_1}{2} P_0^2(u)\Big) \le -\nu \sum_j \|\nabla Y_j u\|_{L^2}^2 + (MC + \nu) P_1^2(u). \tag{3.3.18}$$

Proceeding to general k, we take $|J| = k$ and look at

$$\begin{aligned}
\frac{d}{dt}\|Y^J u\|_{L^2}^2 &= 2(Y^J u_t, Y^J u)_{L^2} \\
&= 2\nu(Y^J \Delta u, Y^J u)_{L^2} - 2(Y^J X u, Y^J u)_{L^2} \\
&= 2\nu(\Delta Y^J u, Y^J u)_{L^2} + 2\nu([Y^J, \Delta]u, Y^J u)_{L^2} - 2(XY^J u, Y^J u)_{L^2} - 2([Y^J, X]u, Y^J u)_{L^2} \\
&= -2\nu\|\nabla Y^J u\|_{L^2}^2 + 2\nu([Y^J, \Delta]u, Y^J u)_{L^2} - 2([Y^J, X]u, Y^J u)_{L^2}.
\end{aligned} \tag{3.3.19}$$

As with (3.3.13), of the three terms in the last line of (3.3.19), the first has a clear significance. For the third, we have

$$[X, Y^J] = [X, Y_{j_1}]Y_{j_2} \cdots Y_{j_k} + \cdots + Y_{j_1} \cdots Y_{j_{k-1}}[X, Y_{j_k}] \in \mathfrak{X}^k, \tag{3.3.20}$$

and hence

$$|([Y^J, X]u, Y^J u)_{L^2}| \le C_k P_k^2(u). \tag{3.3.21}$$

It remains to estimate the second term in the last line of (3.3.19). For this, write

$$[\Delta, Y^J] = \sum_{\ell=1}^k Y_{j_1} \cdots Y_{j_{\ell-1}}[\Delta, Y_{j_\ell}]Y_{j_{\ell+1}} \cdots Y_{j_k} = \sum_{\ell=1}^k Y_{j_1} \cdots Y_{j_{\ell-1}} L_{j_\ell} Y_{j_{\ell+1}} \cdots Y_{j_k}, \tag{3.3.22}$$

where $L_{j_\ell} = [\Delta, Y_{j_\ell}]$ is a second order differential operator that annihilates constants. We say a product of k factors

$$Y_{j_1} \cdots Y_{j_{\ell-1}} L_{j_\ell} Y_{j_{\ell+1}} \cdots Y_{j_k} \tag{3.3.23}$$

is of type (k, ℓ), meaning it is a product of k factors, all being vector fields in $\mathfrak{X}^1$ except one, in position ℓ, which is a second order differential operator that annihilates constants. If $\ell \ge 2$, we can write (3.3.23) as

$$Y_{j_1} \cdots Y_{j_{\ell-2}} L_{j_\ell} \cdots Y_{j_k} + Y_{j_1} \cdots Y_{j_{\ell-2}}[Y_{j_{\ell-1}}, L_{j_\ell}] \cdots Y_{j_k}, \tag{3.3.24}$$

a sum of terms of type $(k, \ell-1)$ and of type $(k-1, \ell-1)$. Repeating this process, we convert (3.3.23) into a sum of terms of type $(j, 1)$, for $j \le k$. Hence we have

$$([Y^J, \Delta]u, Y^J u)_{L^2} = \sum_{|I|\le k-1} (L_I Y^I u, Y^J u)_{L^2}, \tag{3.3.25}$$

where the L_I are differential operators of order 2, annihilating constants; hence

$$L_I = \sum_j A_{Ij} B_{Ij}, \tag{3.3.26}$$

where A_{Ij} are first order differential operators and B_{Ij} are vector fields. We then have

$$\begin{aligned} 2\nu([Y^J, \Delta]u, Y^J u)_{L^2} &= 2\nu \sum_{|I|\le k-1} \sum_j (B_{Ij} Y^I u, A^*_{Ij} Y^J u)_{L^2} \\ &\le \tilde{C}\nu \sum_{|I|\le k-1} \|\nabla Y^I u\|_{L^2} \cdot \big(\|\nabla Y^J u\|_{L^2} + \|Y^J u\|_{L^2}\big) \\ &\le \nu \|\nabla Y^J u\|^2_{L^2} + \nu \|Y^J u\|^2_{L^2} + C_k \nu \sum_{|I|\le k-1} \|\nabla Y^I u\|^2_{L^2}. \end{aligned} \tag{3.3.27}$$

Inserting (3.3.21) and (3.3.27) into (3.3.19), we get

$$\frac{d}{dt}\|Y^J u\|^2_{L^2} \le -\nu \|\nabla Y^J u\|^2_{L^2} + (C_k + \nu) P^2_k(u) + C_k \nu \sum_{|I|\le k-1} \|\nabla Y^I u\|^2_{L^2}, \tag{3.3.28}$$

hence, for $\nu \in (0, 1]$, and with $C_k + 1$ re-notated as C_k,

$$\frac{d}{dt} P^2_k(u) \le -\nu \sum_{|J|=k} \|\nabla Y^J u\|^2_{L^2} + MC_k P^2_k(u) + MC_k \nu \sum_{|I|\le k-1} \|\nabla Y^I u\|^2_{L^2}. \tag{3.3.29}$$

It follows that there exist $A_{kj} \in (0, \infty)$ and $B_k \in (0, \infty)$ such that if we set

$$\widetilde{N}^2_k(u) = P^2_k(u) + \sum_{j=0}^{k-1} A_{kj} P^2_j(u), \tag{3.3.30}$$

then

$$\frac{d}{dt}\widetilde{N}^2_k(u) \le -\nu \sum_{|J|=k} \|\nabla Y^J u\|^2_{L^2} + 2B_k \widetilde{N}^2_k(u), \tag{3.3.31}$$

when $u = u(t)$ is given by (3.3.11). In particular, taking

$$\|u\|^2_{\mathcal{V}^k} = \widetilde{N}^2_k(u), \tag{3.3.32}$$

we obtain

$$\|u(t)\|_{\mathcal{V}^k} \le e^{(t-s)B_k} \|u(s)\|_{\mathcal{V}^k}, \tag{3.3.33}$$

for $0 < s < t < \infty$. The next result will allow us to pass to the limit $s \searrow 0$ for $f \in \mathcal{V}^k$.

Lemma 3.3.2. *For each $k \in \mathbb{Z}^+$, $C^\infty_0(\mathcal{O})$ is dense in $\mathcal{V}^k(\mathcal{O})$.*

Proof. Let $\psi \in C^\infty(\mathbb{R})$ satisfy

$$\begin{aligned}\psi(s) = 0 \quad &\text{for } s \le \tfrac{1}{2},\\ 1 \quad &\text{for } s \ge 1,\end{aligned} \tag{3.3.34}$$

and set

$$\varphi_\delta(x)(x) = \psi\big(\delta^{-1}\operatorname{dist}(x, \partial\mathcal{O})\big). \tag{3.3.35}$$

There exists $\delta_0 > 0$ such that $\varphi_\delta \in C_0^\infty(\mathcal{O})$ for $\delta \in (0, \delta_0)$. Given $f \in \mathcal{V}^k(\mathcal{O})$ and $|J| \le k$, we have

$$Y^J(\varphi_\delta f) = \varphi_\delta Y^J f + \sum_{(I_1,I_2)} (Y^{I_1}\varphi_\delta)(Y^{I_2} f), \tag{3.3.36}$$

where (I_1, I_2) runs over the partitions of the ordered set $\{j_1, \dots j_k\}$ into two subsets, such that $|I_1| \ge 1$ (hence $|I_2| \le k-1$). It is clear from (3.3.35) that $\varphi_\delta Y^J f \to Y^J f$ in L^2-norm as $\delta \searrow 0$. Meanwhile $Y^{I_1}\varphi_\delta = 0$ on $\{x \in \mathcal{O} : \operatorname{dist}(x, \partial\mathcal{O}) \ge \delta\}$, and

$$Y_j \in \mathfrak{X}^1 \Longrightarrow \|Y^{I_1}\varphi_\delta\|_{L^\infty} \le C_{I_1}, \text{ independent of } \delta \in (0, \delta_0/2), \tag{3.3.37}$$

so the sum over (I_1, I_2) in (3.3.36) tends to 0 in L^2-norm as $\delta \searrow 0$. Hence, whenever $f \in \mathcal{V}^k(\mathcal{O})$,

$$\varphi_\delta f \to f \quad \text{in } \mathcal{V}^k\text{-norm}. \tag{3.3.38}$$

From here the density of $C_0^\infty(\mathcal{O})$ in $\mathcal{V}^k(\mathcal{O})$ follows by a standard mollifier argument. $\square$

Since $C_0^\infty(\mathcal{O}) \subset \mathcal{D}((\nu\Delta - X)^m)$ for all m, we have $u \in C^\infty([0,\infty) \times \overline{\mathcal{O}})$ whenever $f \in C_0^\infty(\mathcal{O})$, and hence (3.3.31) holds for $t \ge 0$ and (3.3.33) holds for $s = 0$. That is to say, we have

$$\|e^{t(\nu\Delta - X)} f\|_{\mathcal{V}^k} \le e^{tB_k}\|f\|_{\mathcal{V}^k}, \tag{3.3.39}$$

for all f in the dense linear subspace $C_0^\infty(\mathcal{O})$ of $\mathcal{V}^k(\mathcal{O})$, and hence for all $f \in \mathcal{V}^k$. Also this density implies:

Proposition 3.3.3. *For each $k \in \mathbb{Z}^+$, $\nu > 0$, $e^{t(\nu\Delta - X)}$ is a strongly continuous semigroup on $\mathcal{V}^k(\mathcal{O})$, and (3.3.39) holds for each $f \in \mathcal{V}^k(\mathcal{O})$.*

We emphasize that (3.3.39) holds with B_k independent of $\nu \in (0, 1]$. From here we can obtain convergence results as $\nu \searrow 0$.

Proposition 3.3.4. *In the setting of Proposition 3.3.3,*

$$f \in \mathcal{V}^k(\mathcal{O}) \Longrightarrow \lim_{\nu \searrow 0} e^{t(\nu\Delta - X)} f = e^{-tX} f, \tag{3.3.40}$$

in norm, in $\mathcal{V}^k(\mathcal{O})$.

Proof. The estimate (3.3.39) implies $\{e^{t(\nu\Delta - X)} f : \nu \in (0, 1]\}$ has weak* limit points as $\nu \searrow 0$. By Proposition 3.1.3, (with $p = 2$), $e^{-tX} f$ is the only possible such limit point. This gives convergence in (3.3.40), weak* in $\mathcal{V}^k(\mathcal{O})$. We next aim to improve this to norm convergence. In view of the uniform bounds in (3.3.39), it suffices to establish norm convergence on a dense linear subspace of $\mathcal{V}^k(\mathcal{O})$. Take $f \in C_0^\infty(\mathcal{O})$. We use the complex interpolation identity

$$\mathcal{V}^k(\mathcal{O}) = [L^2(\mathcal{O}), \mathcal{V}^{2k}]_{1/2}. \tag{3.3.41}$$

See Proposition A.1.1 in the Appendix for a proof. This implies

$$\|g\|_{\mathcal{V}^k} \le \|g\|_{L^2}^{1/2} \|g\|_{\mathcal{V}^{2k}}^{1/2}, \tag{3.3.42}$$

for $g \in \mathcal{V}^{2k}(\mathbb{O})$. Hence, for $f \in \mathcal{V}^{2k}(\mathbb{O})$,

$$\left\|(e^{t(\nu\Delta - X)} - e^{-tX}) f\right\|_{\mathcal{V}^k} \le \left\|(e^{t(\nu\Delta - X)} - e^{-tX}) f\right\|_{L^2}^{1/2} \left\|(e^{t(\nu\Delta - X)} - e^{-tX}) f\right\|_{\mathcal{V}^{2k}}^{1/2}. \tag{3.3.43}$$

The first factor on the right side tends to zero as $\nu \searrow 0$, by Proposition 3.1.3, and the last factor is uniformly bounded as $\nu \searrow 0$, by (3.3.39), with k replaced by $2k$. This completes the proof. □

Remark. The class of differential operators $\mathfrak{X}^k$, $k \ge 1$, together with multiplications by smooth functions on $\overline{\mathbb{O}}$, is what is called the algebra of totally characteristic differential operators in [Melrose 1981; 1993]. These works also develop a related class of pseudodifferential operators; see also [Melrose 1996] and [Hörmander 1985, §18.3]. The spaces $\mathcal{V}^k(\mathbb{O})$ are special cases of "weighted b-Sobolev spaces," introduced in [Melrose 1993]. This is discussed further in Appendix A.

We briefly comment on why we call $\mathcal{V}^k(\mathbb{O})$-norm estimates "conormal estimates." The term "conormal distribution" was introduced in [Hörmander 1971]. In essence, if M is a smooth manifold, Σ a smooth submanifold and $\mathcal{L}$ a given Banach space of distributions on M (such as $L^2(M)$) and if f and $X_1 \cdots X_k f$ belong to $\mathcal{L}$ for all k and all smooth vector fields X_j on M that are tangent to Σ, then f is said to be conormal distribution with respect to Σ. See also [Hörmander 1985, §18.2] for a detailed treatment.

3.4. ***Holomorphy of the semigroup $e^{\zeta\Delta}$ on $\mathcal{V}^k(\mathbb{O})$.*** As usual, take $\mathcal{D}(\Delta) = H^2(\mathbb{O}) \cap H_0^1(\mathbb{O})$. The semigroup $e^{\zeta\Delta}$ is a holomorphic semigroup on $L^2(\mathbb{O})$, for $\operatorname{Re}\zeta > 0$. Here we show it has a bound

$$\|e^{\zeta\Delta} f\|_{\mathcal{V}^k} \le e^{B|\zeta|} \|f\|_{\mathcal{V}^k}, \tag{3.4.1}$$

uniformly for ζ in a wedge

$$\mathcal{W}_K = \{t + is : t > 0,\ |s| < Kt\}, \tag{3.4.2}$$

with $B = B(k, K)$. We then derive some useful consequences from this.

To start, take $\theta \in \mathbb{R}$ and set $s = \theta t$ and consider

$$u(t) = e^{t(1+i\theta)\Delta} f, \tag{3.4.3}$$

supressing θ in the notation on the left side of (3.4.3). We have

$$\frac{d}{dt}\|u\|_{L^2}^2 = 2\operatorname{Re}(u_t, u)_{L^2} = 2\operatorname{Re}((1 + i\theta)\Delta u, u)_{L^2} = -2\|\nabla u\|_{L^2}^2. \tag{3.4.4}$$

This is the standard result for $\mathcal{V}^0(\mathbb{O}) = L^2(\mathbb{O})$. Moving on to $\mathcal{V}^k(\mathbb{O})$ with $k = 1$, we have

$$\begin{aligned} \frac{d}{dt}\|Y_j u\|_{L^2}^2 &= 2\operatorname{Re}(Y_j u_t, Y_j u)_{L^2} \\ &= 2\operatorname{Re}(1 + i\theta)(Y_j \Delta u, Y_j u)_{L^2} \\ &= 2\operatorname{Re}(1 + i\theta)(\Delta Y_j u, Y_j u)_{L^2} + 2\operatorname{Re}(1 + i\theta)([Y_j, \Delta]u, Y_j u)_{L^2} \\ &\le -2\|\nabla Y_j u\|_{L^2}^2 + 2\Theta|([Y_j, \Delta]u, Y_j u)_{L^2}|, \end{aligned} \tag{3.4.5}$$

where we have set $\Theta = \sqrt{1+\theta^2}$. As in (3.3.15)–(3.3.16), we have

$$2\Theta|([Y_j, \Delta]u, Y_j u)_{L^2}| \le \|\nabla Y_j u\|^2_{L^2} + \|Y_j u\|^2_{L^2} + K_1\|\nabla u\|^2_{L^2}, \tag{3.4.6}$$

and hence, parallel to (3.3.17),

$$\frac{d}{dt} P_1^2(u) = -\sum_j \|\nabla Y_j u\|^2_{L^2} + K_2\|\nabla u\|^2_{L^2}. \tag{3.4.7}$$

Then, parallel to (3.3.18), we have

$$\frac{d}{dt}\big(P_1^2(u) + K_2 P_0^2(u)\big) \le -\sum_j \|\nabla Y_j u\|^2_{L^2}, \tag{3.4.8}$$

giving (3.4.1) for $k=1$, first for $f \in C_0^\infty(\mathcal{O})$, which is dense in $\mathcal{V}^1(\mathcal{O})$, then for general $f \in \mathcal{V}^1(\mathcal{O})$.

The passage to general k proceeds along the same lines, in parallel with estimates done in (3.3.19)–(3.3.31), but with the simplification that X is not involved.

We record some standard but significant consequences of the holomorphy of $e^{\zeta\Delta}$ and the estimates (3.4.1). First,

$$\Big\|\frac{d}{dt} e^{\zeta\Delta} f\Big\|_{\mathcal{V}^k} \le C|\zeta|^{-1} e^{B|\zeta|}\|f\|_{\mathcal{V}^k}, \tag{3.4.9}$$

for $\zeta \in \mathcal{W}_{K/2}$, as follows from the Cauchy integral formula applied to a circle of radius $\sim c|\zeta|$ centered about ζ. This estimate implies

$$\|\Delta e^{t\Delta} f\|_{\mathcal{V}^k} \le \frac{C}{t} e^{Bt}\|f\|_{\mathcal{V}^k}, \tag{3.4.10}$$

for $t > 0$, and hence

$$\|Y^J \Delta e^{t\Delta} f\|_{L^2} \le \frac{C}{t} e^{Bt}\|f\|_{\mathcal{V}^k}, \quad |J| = k. \tag{3.4.11}$$

Using this, we will establish the following.

Proposition 3.4.1. *Take $T_0 \in (0,\infty)$. Then, for $t \in [0, T_0]$, we have*

$$tY^J e^{t\Delta} : \mathcal{V}^k(\mathcal{O}) \to H^2(\mathcal{O}) \ \textit{bounded, for} \ |J| = k. \tag{3.4.12}$$

Proof. We use induction on k. For $k=0$, (3.4.12) follows from the $k=0$ case of (3.4.10). To establish (3.4.12) for $k \ge 1$, it suffices to show that

$$t\Delta Y^J e^{t\Delta} : \mathcal{V}^k(\mathcal{O}) \to L^2(\mathcal{O}) \ \text{ is bounded, for } \ |J| = k. \tag{3.4.13}$$

Using (3.3.22)–(3.3.25), we have

$$t\Delta Y^J e^{t\Delta} = tY^J \Delta e^{t\Delta} + t\sum_{|I|\le k-1} L_I Y^I e^{t\Delta}, \tag{3.4.14}$$

where each L_I is a second order differential operator. The bound on the first term on the right side of (3.4.14) in $\mathcal{L}(\mathcal{V}^k(\mathcal{O}), L^2(\mathcal{O}))$ follows from (3.4.11). The bound on the sum over $|I| \le k-1$ follows by the induction hypothesis. This proves (3.4.12). ☐

We can interpolate the bound

$$\|Y^J e^{t\Delta} f\|_{H^2(\mathcal{O})} \le \frac{C}{t}\|f\|_{\mathcal{V}^k} \tag{3.4.15}$$

with the bound

$$\|Y^J e^{t\Delta} f\|_{L^2(\mathcal{O})} \le C\|f\|_{\mathcal{V}^k}, \tag{3.4.16}$$

valid for $t \in [0, T_0]$ by (3.4.1), using

$$\|F\|_{H^1} \le C\|F\|_{L^2}^{1/2}\|F\|_{H^2}^{1/2}, \tag{3.4.17}$$

to obtain:

Corollary 3.4.2. *In the setting of Proposition 3.4.1,*

$$\|Y^J e^{t\Delta} f\|_{H^1(\mathcal{O})} \le \frac{C}{t^{1/2}}\|f\|_{\mathcal{V}^k}, \quad |J| = k. \tag{3.4.18}$$

Consequently

$$\|e^{t\Delta} f\|_{\mathcal{V}^{k+1}} \le \frac{C}{t^{1/2}}\|f\|_{\mathcal{V}^k}. \tag{3.4.19}$$

3.5. *Estimates on $e^{t(\nu\Delta - X)}$ in case of empty boundary.* Here we consider the family of semigroups $e^{t(\nu\Delta - X)}$ acting on functions on M, a compact, n-dimensional, Riemannian manifold without boundary. Again Δ is the Laplace-Beltrami operator. We assume X is a smooth vector field on M. This time we will not assume that $\operatorname{div} X = 0$. We will show that in this setting we have much stronger convergence results than obtained in Section 3.1. Ultimately it will be our goal to use the results obtained here to strengthen the results of Section 3.1.

To begin, let us note that in the current context, (3.1.4) is strengthened to

$$\mathcal{D}((\nu\Delta - X)^k) = \mathcal{D}(\Delta^k) = H^{2k}(M), \quad \forall\, k \in \mathbb{N}, \tag{3.5.1}$$

whenever $\nu > 0$. Because of this, we can improve Lemma 3.1.2 to the following.

Lemma 3.5.1. *Take $f \in C^\infty(M)$, and set $u(t) = e^{t(\nu\Delta - X)} f$, with $\nu > 0$. For each $k \in \mathbb{Z}^+$, there exists $K = K(k) \in (0, \infty)$, independent of ν, such that*

$$\|(1-\Delta)^k u(t)\|_{L^2}^2 \le e^{2Kt}\|(1-\Delta)^k f\|_{L^2}^2. \tag{3.5.2}$$

Proof. Straightforward analogue of the proof of Lemma 3.1.2. □

Corollary 3.5.2. *We have, for each $k \in \mathbb{Z}^+$,*

$$\|e^{t(\nu\Delta - X)} f\|_{\mathcal{D}(\Delta^k)} \le e^{Kt}\|f\|_{\mathcal{D}(\Delta^k)}, \tag{3.5.3}$$

for each $f \in C^\infty(M)$, hence for each $f \in \mathcal{D}(\Delta^k)$.

Remark. This contrasts with the possibility of (3.1.25), which can occur in case of nonempty boundary.

Note that the maximum principle holds, so, for each $\nu > 0$,

$$\|e^{t(\nu\Delta - X)} f\|_{L^\infty} \le \|f\|_{L^\infty}. \tag{3.5.4}$$

Interpolation with the case $k = 0$ of (3.5.3) implies

$$\|e^{t(\nu\Delta - X)} f\|_{L^p} \le e^{Kt}\|f\|_{L^p}, \tag{3.5.5}$$

for $f \in L^p(M)$, $p \in [2,\infty)$. We could also get this for $p \in [1,2)$, but we will not take the space to do this. We can further apply interpolation to (3.5.5) and the estimates

$$\|e^{t(\nu\Delta-X)} f\|_{H^{2k}} \le Ce^{Kt}\|f\|_{H^{2k}}, \quad k \in \mathbb{Z}^+, \tag{3.5.6}$$

which follow from (3.5.3) and (3.5.1). First, we have

$$\|e^{t(\nu\Delta-X)} f\|_{H^{s}} \le Ce^{Kt}\|f\|_{H^{s}}, \quad s \in \mathbb{R}^+, \tag{3.5.7}$$

with $C = C_s$, $K = K_s$, independent of ν. Then, in place of (3.1.21), we have

$$[H^s(M), L^p(M)]_\theta = H^{(1-\theta)s,q(\theta)}(M), \quad \frac{1}{q(\theta)} = \frac{1-\theta}{2} + \frac{\theta}{p}, \tag{3.5.8}$$

and hence

$$\|e^{t(\nu\Delta-X)} f\|_{H^{\sigma,q}(M)} \le C_{\sigma,q} e^{Kt}\|f\|_{H^{\sigma,q}(M)}, \tag{3.5.9}$$

valid for $q \in [2,\infty)$, $\sigma > 0$.

We next consider convergence results, as $\nu \to 0$. As in (3.1.8), we have for $u(t) = e^{t(\nu\Delta-X)} f$ the identity

$$u(t) = e^{-tX} f + \nu \int_0^t e^{(t-s)X} \Delta u(s)\, ds, \tag{3.5.10}$$

hence

$$\|u(t) - e^{-tX} f\|_{\mathfrak{D}(\Delta^k)} \le \nu \int_0^t \|e^{(t-s)X} \Delta u(s)\|_{\mathfrak{D}(\Delta^k)}\, ds. \tag{3.5.11}$$

We use (3.5.3) plus the analogous estimate on e^{-tX} to deduce that

$$\|e^{t(\nu\Delta-X)} f - e^{-tX} f\|_{\mathfrak{D}(\Delta^k)} \le C\nu\|f\|_{\mathfrak{D}(\Delta^{k+1})}, \tag{3.5.12}$$

for $f \in C^\infty(M)$. We hence have

$$e^{t(\nu\Delta-X)} f \to e^{-tX} f \tag{3.5.13}$$

in $\mathfrak{D}(\Delta^k)$-norm (hence in H^{2k}-norm), for each $f \in C^\infty(M)$, hence, via (3.5.3), for each $f \in \mathfrak{D}(\Delta^k)$. Then, using (3.5.9) and (3.5.4), and standard density arguments, we have:

Proposition 3.5.3. *Given $f \in H^{\sigma,q}(M)$, $\sigma \ge 0$, $q \in [2,\infty)$, convergence in (3.5.13) holds in $H^{\sigma,q}$-norm, as $\nu \to 0$. Given $f \in C(M)$, convergence in (3.5.13) holds uniformly, as $\nu \to 0$.*

3.6. ***Parametrix for $\partial_t - \nu L(t)$ on $\mathbb{R}^+ \times M$.*** As in Section 3.5, let M be a compact, n-dimensional, Riemannian manifold without boundary, with Laplace-Beltrami operator Δ, and let X be a smooth vector field on M. As in Section 3.2, let $L(t) = e^{tX}\Delta e^{-tX}$, so, for $f \in \mathfrak{D}'(M)$,

$$v(t) = e^{tX} e^{t(\nu\Delta-X)} f \tag{3.6.1}$$

solves

$$\frac{\partial v}{\partial t} = \nu L(t) v, \quad v(0) = f. \tag{3.6.2}$$

We denote the solution operator by S_ν^t:

$$S_\nu^t = e^{tX} e^{t(\nu\Delta-X)}. \tag{3.6.3}$$

Parallel to results of Section 3.5, we have

$$\|S_\nu^t f\|_{H^{s,p}} \le Ce^{Kt}\|f\|_{H^{s,p}}, \tag{3.6.4}$$

for $f \in H^{s,p}(M)$, with $C = C_{s,p},\ K = K_{s,p}$ independent of $\nu > 0$, given $p \ge 2,\ s \ge 0$. (With a little more work, we could take any $p \in (1,\infty),\ s \in \mathbb{R}$.) Our goal here is to construct a parametrix, revealing the fine structure of S_ν^t as $\nu \to 0$.

Preparatory to beginning this parametrix construction, it is also useful to note that Proposition 3.2.1 continues to hold in the current setting. In particular, given $\Omega_1 \Subset \Omega_0 \subset M,\ k \in \mathbb{N}$,

$$\|S_\nu^t f\|^2_{H^k(\Omega_1)} \le C_{Tk}\big(\|f\|^2_{H^k(\Omega_0)} + \|f\|^2_{L^2(M)}\big), \quad 0 \le t \le T, \tag{3.6.5}$$

with C_{Tk} independent of $\nu > 0$. Applying this and a partition of unity argument, we see it suffices to construct a parametrix for $S_\nu^t f$ when f is supported on a coordinate patch $\Omega \subset M$, and it suffices to analyze this approximation to $S_\nu^t f(x)$ for $(t,x) \in [0,T] \times \Omega$, uniformly in $\nu \in (0,1]$.

On a coordinate patch Ω, we have

$$L(t)u = \sum_{1\le|\alpha|\le 2} L_\alpha(t,x)\partial_x^\alpha. \tag{3.6.6}$$

(Note that $L(t)1 = 0$.) Let us set

$$L_k(t,x,\xi) = \sum_{|\alpha|=k} L_\alpha(t,x)(i\xi)^\alpha, \qquad k = 1,2. \tag{3.6.7}$$

Note that

$$L_2(t,x,\xi) = -G(t,x,\xi) = -\sum_{ij} g^{ij}(t,x)\xi_i\xi_j, \tag{3.6.8}$$

where $(g_{ij}(t,x)) = (g^{ij}(t,x))^{-1}$ is the metric tensor on M, pulled back via the flow generated by X.

We write our approximate solution to (3.6.2) on $\mathbb{R}^+ \times \Omega$ as

$$\mathfrak{S}_\nu^t f(x) = (2\pi)^{-n/2} \int a(\nu,t,x,\xi)e^{ix\cdot\xi}\hat{f}(\xi)\,d\xi, \tag{3.6.9}$$

where $\hat{f}(\xi)$ is the Fourier transform of f, given by

$$\hat{f}(\xi) = (2\pi)^{-n/2} \int f(x)e^{-ix\cdot\xi}\,dx,$$

and the *amplitude* $a(\nu,t,x,\xi)$ will take the form of an asymptotic series

$$a(\nu,t,x,\xi) \sim \sum_{j\ge 0} a_j(\nu,t,x,\xi), \tag{3.6.10}$$

whose terms a_j will be constructed below. In outline this construction is similar to that done in [Taylor 1996, Chapter 7, §13], constructing a parametrix for $e^{t\Delta}$ for small t, but here the set-up is more complicated.

We start with the following consequence of the Leibniz identity:

$$\nu L(t)(ae^{ix\cdot\xi}) = \Big[\nu L_2(t,x,\xi)a(\nu,t,x,\xi) + \nu\sum_{\ell=1}^{2} B_{2-\ell}(t,x,\xi,D_x)a(\nu,t,x,\xi)\Big]e^{ix\cdot\xi}, \tag{3.6.11}$$

where $B_{2-\ell}(t,x,\xi,D_x)$ is a differential operator of order ℓ, whose coefficients are polynomials of degree $2-\ell$ in ξ, and smooth in (t,x). To satisfy (3.6.2) formally, we require

$$\frac{\partial a}{\partial t} \sim \nu L_2(t,x,\xi)a + \nu \sum_{\ell=1}^{2} B_{2-\ell}(t,x,\xi,D_x)a, \quad a(\nu,0,x,\xi)=1. \tag{3.6.12}$$

This tells us how to construct the terms a_j. For starters, a_0 is defined by

$$\frac{\partial a_0}{\partial t} = -\nu G(t,x,\xi)a_0, \quad a_0(\nu,0,x,\xi)=1, \tag{3.6.13}$$

so

$$a_0(\nu,t,x,\xi) = e^{-\nu t H(t,x,\xi)}, \quad H(t,x,\xi) = \frac{1}{t}\int_0^t G(s,x,\xi)\,ds. \tag{3.6.14}$$

Note that $H(t,x,\xi)$ is a polynomial in ξ, homogeneous of degree 2, with coefficients smooth in (t,x), and

$$H(t,x,\xi) \ge C|\xi|^2, \tag{3.6.15}$$

for some $C>0$. For $j \ge 1$, a_j solves

$$\frac{\partial a_j}{\partial t} = -\nu G(t,x,\xi)a_j + \Omega_j(\nu,t,x,\xi), \quad a_j(\nu,0,x,\xi)=0, \tag{3.6.16}$$

where

$$\Omega_j(\nu,t,x,\xi) = \nu \sum_{\ell=1}^{2} B_{2-\ell}(t,x,\xi,D_x)a_{j-\ell}(\nu,t,x,\xi), \tag{3.6.17}$$

with the convention (operative for $j=1,\ \ell=2$) that $a_{-1}\equiv 0$. We hence have

$$a_j(\nu,t,x,\xi) = e^{-\nu t H(t,x,\xi)} \int_0^t e^{\nu s H(s,x,\xi)}\Omega_j(\nu,s,x,\xi)\,ds. \tag{3.6.18}$$

Another way to display these terms in the amplitude is to set

$$a_j(\nu,t,x,\xi) = A_j(\nu,t,x,\xi)e^{-\nu t H(t,x,\xi)}. \tag{3.6.19}$$

Also set

$$\Omega_j(\nu,t,x,\xi) = \Gamma_j(\nu,t,x,\xi)e^{-\nu t H(t,x,\xi)}, \tag{3.6.20}$$

so (3.6.17) becomes

$$\Gamma_j(\nu,t,x,\xi) = \nu e^{\nu t H(t,x,\xi)} \sum_{\ell=1}^{2} B_{2-\ell}(t,x,\xi,D_x)\big(A_{j-\ell}e^{-\nu t H(t,x,\xi)}\big), \tag{3.6.21}$$

and (3.6.18) becomes

$$A_j(\nu,t,x,\xi) = \int_0^t \Gamma_j(\nu,s,x,\xi)\,ds. \tag{3.6.22}$$

We next take an explicit look at the case $j=1$. In that case, (3.6.17) gives

$$\Omega_1 = \nu B_1(t,x,\xi,D_x)e^{-\nu t H(t,x,\xi)} = -\nu^2 t e^{-\nu t H(t,x,\xi)} B_1(t,x,\xi,D_x)H(t,x,\xi), \tag{3.6.23}$$

and recall that B_1 is a differential operator of order 1, whose coefficients are polynomials of degree 1 in ξ. A formula equivalent to (3.6.23) is

$$\Gamma_1 = -\nu^2 t B_1(t,x,\xi,D_x) H(t,x,\xi) = -\nu^2 t \sum_{|\alpha|\le 3} C_1^\alpha(t,x)\xi^\alpha, \tag{3.6.24}$$

with $C_1^\alpha(t,x)$ smooth. Then, by (3.6.22),

$$A_1(\nu,t,x,\xi) = -\nu^2 \sum_{|\alpha|\le 3} \Big(\int_0^t sC_1^\alpha(s,x)\,ds\Big)\xi^\alpha = -(\nu t)^2 \sum_{|\alpha|\le 3} D_1^\alpha(t,x)\xi^\alpha, \tag{3.6.25}$$

with $D_1^\alpha(t,x)$ smooth, and we have

$$a_1(\nu,t,x,\xi) = -(\nu t)^2 \sum_{|\alpha|\le 3} D_1^\alpha(t,x)\xi^\alpha\, e^{-\nu t H(t,x,\xi)}. \tag{3.6.26}$$

Let us now recall the definition of a symbol class, important in the theory of pseudodifferential operators. Given $m \in \mathbb{R}$, we say

$$p(x,\xi) \in S^m_{1,0} \iff |D_x^\beta D_\xi^\alpha p(x,\xi)| \le C_{\alpha\beta}(1+|\xi|)^{m-|\alpha|}, \tag{3.6.27}$$

and we say a family $\{p(\nu,t,x,\xi) : t\in[0,T],\ \nu\in(0,1]\}$ is bounded in $S^m_{1,0}$ provided such estimates hold with $C_{\alpha\beta}$ independent of ν and t. In follows from (3.6.14) that

$$\{a_0(\nu,t,x,\xi) : t\in[0,T], \nu\in(0,1]\} \ \text{ is bounded in } \ S^0_{1,0}, \tag{3.6.28}$$

or as we say for short, $a_0(\nu,t,x,\xi)$ is bounded in $S^0_{1,0}$. Similarly, from (3.6.26) we have

$$a_1(\nu,t,x,\xi) \ \text{ bounded in } S^{-1}_{1,0} \ \text{ and } \ O((\nu t)^{1/2}) \text{ in } S^0_{1,0}, \tag{3.6.29}$$

the latter property meaning that $(\nu t)^{-1/2} a_1(\nu,t,x,\xi)$ is bounded in $S^0_{1,0}$.

To extend (3.6.28)–(3.6.29) to a_j for larger j, it is convenient to have another presentation. Set

$$\zeta = (\nu t)^{1/2}\xi, \quad \omega = \nu t \xi. \tag{3.6.30}$$

Now (3.6.14) and (3.6.26) give

$$\begin{aligned} a_0(\nu,t,x,\xi) &= e^{-H(t,x,\zeta)},\\ a_1(\nu,t,x,\xi) &= \nu t \mathscr{A}_1(\nu t,t,x,\xi,\zeta) e^{-H(t,x,\zeta)}, \end{aligned} \tag{3.6.31}$$

where $\mathscr{A}_1(\tau,t,x,\xi,\zeta)$ is a polynomial in τ of degree 1, in ξ of degree 1 and in ζ of degree 2, with coefficients smooth in (t,x). It will be useful to have the following:

Definition. The space $\mathscr{P}_k$ is characterized by

$$\begin{aligned} F(\nu t,t,x,\xi,\zeta,\omega) \in \mathscr{P}_k \iff & F \ \text{ is a polynomial in } \ \nu t,\ \zeta,\ \omega, \ \text{ and } \ \xi, \ \text{ even in } \zeta,\\ & \text{of degree } \le k \text{ in } \xi, \text{ withcoefficients smooth in } (t,x). \end{aligned} \tag{3.6.32}$$

Without loss of generality, we can assume the degree in ω is ≤ 1.

Then a_1 satisfies (3.6.31) with

$$\mathcal{A}_1(\nu t, t, x, \xi, \zeta) \in \mathcal{P}_1. \tag{3.6.33}$$

(Actually $\mathcal{A}_1$ is independent of ω, but other amplitudes will have ω dependence.)

Theorem 3.6.1. *For each* $k = 0, 1, 2, \dots,$ *we have*

$$\begin{aligned} a_{2k}(\nu, t, x, \xi) &= (\nu t)^k \mathcal{A}_{2k} e^{-H(t,x,\zeta)}, \quad \mathcal{A}_{2k} \in \mathcal{P}_0, \\ a_{2k+1}(\nu, t, x, \xi) &= (\nu t)^{k+1} \mathcal{A}_{2k+1} e^{-H(t,x,\zeta)}, \quad \mathcal{A}_{2k+1} \in \mathcal{P}_1. \end{aligned} \tag{3.6.34}$$

Proof. The results in (3.6.31) give (3.6.34) for $k = 0$. We proceed by induction on k. To set this up, let us assume

$$a_j = (\nu t)^{\alpha_j} \mathcal{A}_j e^{-H(t,x,\zeta)}, \quad \mathcal{A}_j \in \mathcal{P}_{\beta_j}, \tag{3.6.35}$$

for $j \le 2k+1$, with indices α_j and β_j consistent with (3.6.34). Then (3.6.17) gives

$$\Omega_{j+1} = \Omega^1_{j+1} + \Omega^0_{j+1} \tag{3.6.36}$$

with

$$\begin{aligned} \Omega^1_{j+1} &= \nu (\nu t)^{\alpha_j} B_1(t, x, \xi, D_x)\big(\mathcal{A}_j e^{-H(t,x,\zeta)}\big) \\ &= \nu (\nu t)^{\alpha_j} \mathcal{B}^1_{j+1} e^{-H(t,x,\zeta)}, \quad \mathcal{B}^1_{j+1} \in \mathcal{P}_{\beta_j+1}, \end{aligned} \tag{3.6.37}$$

so $\Gamma^1_{j+1} = \nu(\nu t)^{\alpha_j} \mathcal{B}^1_{j+1}$ and

$$A^1_{j+1}(\nu, t, x, \xi) = \int_0^t \Gamma^1_{j+1}(\nu, s, x, \xi)\, ds \in (\nu t)^{\alpha_j+1} \cdot \mathcal{P}_{\beta_j+1}, \tag{3.6.38}$$

and furthermore

$$\begin{aligned} \Omega^0_{j+1} &= \nu (\nu t)^{\alpha_{j-1}} B_0(t, x, D_x)\big(\mathcal{A}_{j-1} e^{-H(t,x,\zeta)}\big) \\ &= \nu (\nu t)^{\alpha_{j-1}} \mathcal{B}^0_{j+1} e^{-H(t,x,\zeta)}, \quad \mathcal{B}^0_{j+1} \in \mathcal{P}_{\beta_{j-1}}, \end{aligned} \tag{3.6.39}$$

so $\Gamma^0_{j+1} = \nu(\nu t)^{\alpha_{j-1}} \mathcal{B}^0_{j+1}$ and

$$A^0_{j+1}(\nu, t, x, \xi) = \int_0^t \Gamma^0_{j+1}(\nu, s, x, \xi)\, ds \in (\nu t)^{\alpha_{j-1}+1} \cdot \mathcal{P}_{\beta_{j-1}}. \tag{3.6.40}$$

We are now ready to verify the induction step in the proof of Theorem 3.6.1. Suppose (3.6.34) holds for a given $k \in \mathbb{Z}^+$, i.e.,

$$A_{2k} \in (\nu t)^k \cdot \mathcal{P}_0, \quad A_{2k+1} \in (\nu t)^{k+1} \cdot \mathcal{P}_1. \tag{3.6.41}$$

(If $k \ge 1$, assume also the counterpart of (3.6.41) with k replaced by $k-1$.) Then, using the fact that (3.6.35) implies (3.6.38) and (3.6.40), we obtain

$$A_{2k+2} = A^1_{2k+2} + A^0_{2k+2} \in (\nu t)^{k+2} \cdot \mathcal{P}_2 + (\nu t)^{k+1} \cdot \mathcal{P}_0 \subset (\nu t)^{k+1} \cdot \mathcal{P}_0, \tag{3.6.42}$$

(upon noting that $(\nu t) \cdot \mathcal{P}_2 \subset \mathcal{P}_0$), and furthermore

$$A_{2k+3} = A^1_{2k+3} + A^0_{2k+3} \in (\nu t)^{k+2} \cdot \mathcal{P}_1. \tag{3.6.43}$$

This completes the proof. □

We can use Theorem 3.6.1 to extend (3.6.28)–(3.6.29), as follows.

Corollary 3.6.2. *In the setting of Theorem 3.6.1, we have*

$$a_{2k}(\nu,t,x,\xi)=O((\nu t)^k) \text{ in } S^0_{1,0}, \text{ bounded in } S^{-2k}_{1,0}, \tag{3.6.44}$$

and

$$a_{2k+1}(\nu,t,x,\xi)=O((\nu t)^{k+1}) \text{ in } S^1_{1,0}, \text{ bounded in } S^{-2k-1}_{1,0}, \tag{3.6.45}$$

hence, for $j\ge 0$,

$$a_j(\nu,t,x,\xi)=O((\nu t)^{j/2}) \text{ in } S^0_{1,0}, \text{ bounded in } S^{-j}_{1,0}. \tag{3.6.46}$$

Proof. The result (3.6.34) directly gives (3.6.44)–(3.6.45), and (3.6.46) follows from this plus the observation that the condition

$$p(\nu,t,x,\xi)=(\nu t)\mathcal{A}e^{-H(t,x,\zeta)},\quad \mathcal{A}\in\mathcal{P}_1 \tag{3.6.47}$$

implies $p(\nu,t,x,\xi)=O((\nu t)^{1/2})$ in $S^0_{1,0}$. □

Returning to (3.6.9)–(3.6.10), let us fix $N\in\mathbb{N}$ and set

$$a(\nu,t,x,\xi)=\sum_{j=0}^{N}a_j(\nu,t,x,\xi). \tag{3.6.48}$$

We use this to define $\mathfrak{S}^t_\nu f$ in (3.6.9). Then we have

$$(\partial_t-\nu L(t))\mathfrak{S}^t_\nu f(x)=(2\pi)^{-n/2}\int R_N(\nu,t,x,\xi)e^{ix\cdot\xi}\hat{f}(\xi)\,d\xi, \tag{3.6.49}$$

with

$$\begin{aligned}&R_N(\nu,t,x,\xi)\\&\quad=\nu B_1(t,x,\xi,D_x)a_N(\nu,t,x,\xi)+\nu B_0(t,x,D_x)\big[a_{N-1}(\nu,t,x,\xi)+a_N(\nu,t,x,\xi)\big].\end{aligned} \tag{3.6.50}$$

Arguments used in the proof of (3.6.34) and (3.6.45) give

$$\begin{aligned}\nu B_1(t,x,\xi,D_x)a_N(\nu,t,x,\xi)&=O(\nu(\nu t)^{N/2}) \;\text{ in }\; S^1_{1,0},\\&O(\nu(\nu t)^{(N-1)/2}) \;\text{ in }\; S^0_{1,0},\\&O(\nu) \;\text{ in }\; S^{-(N-1)}_{1,0},\end{aligned} \tag{3.6.51}$$

and

$$\begin{aligned}\nu B_0(t,x,D_x)[a_{N-1}+a_N]&=O(\nu(\nu t)^{(N-1)/2}) \;\text{ in }\; S^0_{1,0},\\&O(\nu) \;\text{ in }\; S^{-(N-1)}_{1,0}.\end{aligned} \tag{3.6.52}$$

In conclusion:

Proposition 3.6.3. *If* $N\in\mathbb{N}$ *is given,* a *is defined as in (3.6.48), and* $\mathfrak{S}^t_\nu$ *as in (3.6.9), then*

$$u^\nu(t)=\mathfrak{S}^t_\nu f \tag{3.6.53}$$

solves

$$\frac{\partial u^\nu}{\partial t}=\nu L(t)u+g^\nu,\quad u^\nu(0)=f, \tag{3.6.54}$$

with

$$g^\nu(t,x)=(2\pi)^{-n/2}\int R_N(\nu,t,x,\xi)e^{ix\cdot\xi}\hat{f}(\xi)\,d\xi, \tag{3.6.55}$$

where

$$R_N(\nu,t,x,\xi) = O(\nu(\nu t)^{(N-1)/2}) \ \text{in} \ S^0_{1,0},$$
$$O(\nu) \ \text{in} \ S^{-(N-1)}_{1,0}. \tag{3.6.56}$$

Using standard pseudodifferential operator estimates, we obtain:

Corollary 3.6.4. *In the setting of Proposition 3.6.3, if $p \in (1,\infty)$, $s \in \mathbb{R}$, then, for $t \in [0,T]$, $\nu \in (0,1]$,*

$$\|g^\nu(t)\|_{H^{s,p}(M)} \le C_T \nu^{(N+1)/2} \|f\|_{H^{s,p}(M)}, \tag{3.6.57}$$

and

$$\|g^\nu(t)\|_{H^{s+N-1,p}(M)} \le C_T \nu \|f\|_{H^{s,p}(M)}, \tag{3.6.58}$$

with C_T independent of ν.

We can compare the approximate solution $\mathfrak{S}^t_\nu f$ with the exact solution $S^t_\nu f$ to (3.6.2) by applying the Duhamel formula to (3.6.54), which gives

$$\mathfrak{S}^t_\nu f = S^t_\nu f + \int_0^t S^{s,t}_\nu g^\nu(s)\, ds, \tag{3.6.59}$$

where, for $0 \le s \le t$, $S^{s,t}_\nu$ is the solution operator to (3.6.2) defined by

$$v(t) = S^{s,t}_\nu v(s), \quad \text{equivalently,} \ \ S^{s,t}_\nu = e^{tX} e^{(t-s)(\nu\Delta - X)} e^{-sX}. \tag{3.6.60}$$

A straightforward analogue of (3.6.4) is

$$\|S^{s,t}_\nu f\|_{H^{\sigma,p}} \le C e^{K(t-s)} \|f\|_{H^{\sigma,p}}, \tag{3.6.61}$$

valid for $p \in [2,\infty)$, $\sigma \in [0,\infty)$, with $C = C_{\sigma,p}$ and $K = K_{\sigma,p}$ independent of $\nu \in (0,1]$. This gives:

Corollary 3.6.5. *In the setting of Proposition 3.6.3, if $p \in [2,\infty)$, $\sigma \ge 0$, then for $t \in [0,T]$, $\nu \in (0,1]$,*

$$\|\mathfrak{S}^t_\nu f - S^t_\nu f\|_{H^{\sigma,p}(M)} \le C_T \nu^{(N+1)/2} \|f\|_{H^{\sigma,p}(M)}, \tag{3.6.62}$$

and

$$\|\mathfrak{S}^t_\nu f - S^t_\nu f\|_{H^{\sigma+N-1,p}(M)} \le C_T \nu \|f\|_{H^{\sigma,p}(M)}, \tag{3.6.63}$$

with C_T independent of ν.

Remark. Applying Corollary 3.6.5 with N replaced by $N+2$ and taking into account the fact that this just adds $a_{N+1} + a_{N+2}$ to the amplitude in the formula for $\mathfrak{S}^t_\nu$, we obtain a complement to (3.6.62)–(3.6.63), namely

$$\|\mathfrak{S}^t_\nu f - S^t_\nu f\|_{H^{\sigma+N+1,p}(M)} \le C_T \|f\|_{H^{\sigma,p}(M)}. \tag{3.6.64}$$

The family of operators $S^{s,t}_\nu$ is as important as the family S^t_ν, and it is also of interest to have a parametrix for this family. This is obtained by a slight modification of the previous construction. Parallel to (3.6.9)–(3.6.10), this parametrix has the form

$$\mathfrak{S}^{s,t}_\nu f(x) = (2\pi)^{-n/2} \int a(\nu,s,t,x,\xi) e^{ix\cdot\xi} \hat{f}(\xi)\, d\xi, \tag{3.6.65}$$

with

$$a(\nu, s, t, x, \xi) \sim \sum_{j \geq 0} a_j(\nu, s, t, x, \xi), \tag{3.6.66}$$

given by equations similar to (3.6.12), except that the initial condition is

$$a(\nu, s, s, x, \xi) = 1. \tag{3.6.67}$$

Thus, in place of (3.6.14) we have

$$\begin{aligned} a_0(\nu, s, t, x, \xi) &= e^{-\nu(t-s)H(s,t,x,\xi)}, \\ H(s, t, x, \xi) &= \frac{1}{t-s} \int_s^t G(\sigma, x, \xi)\, d\sigma, \end{aligned} \tag{3.6.68}$$

and in place of (3.6.31) we have

$$a_1(\nu, s, t, x, \xi) = \nu(t-s)\mathcal{A}_1(\nu(t-s), s, t, x, \xi, \zeta)e^{-H(s,t,x,\zeta)}, \tag{3.6.69}$$

this time with

$$\zeta = (\nu(t-s))^{1/2}\xi, \quad \omega = \nu(t-s)\xi, \quad \mathcal{A}_1 \in \mathcal{P}_1, \tag{3.6.70}$$

where now $\mathcal{P}_k$ is defined to consist of functions $F(\nu(t-s), s, t, x, \xi, \zeta, \omega)$, polynomials in $\nu(t-s)$, ζ, ω, and ξ, even in ζ, of degree $\leq k$ in ξ and of degree ≤ 1 in ω, with coefficients smooth in (s, t, x), the obvious variant of (3.6.32). (As in (3.6.31), $\mathcal{A}_1$ does not actually depend on ω.) More generally, parallel to (3.6.34), we have

$$\begin{aligned} a_{2k}(\nu, s, t, x, \xi) &= (\nu(t-s))^k \mathcal{A}_{2k}\, e^{-H(s,t,x,\zeta)}, & \mathcal{A}_{2k} \in \mathcal{P}_0, \\ a_{2k+1}(\nu, s, t, x, \xi) &= (\nu(t-s))^{k+1} \mathcal{A}_{2k+1}\, e^{-H(s,t,x,\zeta)}, & \mathcal{A}_{2k+1} \in \mathcal{P}_1, \end{aligned} \tag{3.6.71}$$

except now with $\zeta = (\nu(t-s))^{1/2}\xi$ (as in (3.6.70)), with $\mathcal{A}_j = \mathcal{A}_j(\nu(t-s), s, t, x, \xi, \zeta, \omega)$, and with $\mathcal{P}_k$ as redefined above. In place of (3.6.46), we have

$$a_j(\nu, s, t, x, \xi) = O((\nu(t-s))^{j/2}) \text{ in } S^0_{1,0}, \text{ bounded in } S^{-j}_{1,0}. \tag{3.6.72}$$

The estimates recorded in Corollary 3.6.5 readily extend, to yield:

Proposition 3.6.6. *Given $N \in \mathbb{N}$, take*

$$a(\nu, s, t, x, \xi) = \sum_{j=0}^{N} a_j(\nu, s, t, x, \xi), \tag{3.6.73}$$

and define $\mathfrak{S}^{s,t}_\nu f$ by (3.6.65). Then for $p \in [2, \infty)$, $\sigma \geq 0$, $0 \leq s \leq t \leq T$, and $\nu \in (0, 1]$, we have

$$\begin{aligned} \|\mathfrak{S}^{s,t}_\nu f - S^{s,t}_\nu f\|_{H^{\sigma,p}(M)} &\leq C_T \nu^{(N+1)/2} \|f\|_{H^{\sigma,p}(M)}, \\ \|\mathfrak{S}^{s,t}_\nu f - S^{s,t}_\nu f\|_{H^{\sigma+N+1,p}(M)} &\leq C_T \|f\|_{H^{\sigma,p}(M)}, \end{aligned} \tag{3.6.74}$$

with C_T independent of ν.

The formula (3.6.65) represents the parametrix $\mathfrak{S}^{s,t}_\nu$ in Fourier integral form. We next obtain a more explicit representation of its integral kernel. We examine the individual terms

$$\mathfrak{S}^{s,t}_{\nu,j} f(x) = (2\pi)^{-n/2} \int a_j(\nu,s,t,x,\xi) e^{ix\cdot\xi} \hat{f}(\xi)\,d\xi = \int K_j(\nu,s,t,x,x-y) f(y)\,dy, \tag{3.6.75}$$

where

$$K_j(\nu,s,t,x,z) = (2\pi)^{-n} \int a_j(\nu,s,t,x,\xi) e^{iz\cdot\xi}\,d\xi, \quad z = x-y. \tag{3.6.76}$$

In case $j=0$, let us rewrite a_0 as

$$a_0(\nu,s,t,x,\xi) = e^{-\nu(t-s)\mathcal{H}(s,t,x)\xi\cdot\xi}, \tag{3.6.77}$$

where $\mathcal{H}(s,t,x)$ is a positive-definite $n\times n$ matrix. We have a standard Gaussian integral:

$$\begin{aligned} K_0(\nu,s,t,x,z) &= (2\pi)^{-n} \int e^{-\nu(t-s)\mathcal{H}(s,t,x)\xi\cdot\xi} e^{iz\cdot\xi}\,d\xi \\ &= \big(4\pi\nu(t-s)\big)^{-n/2} \det \mathcal{G}(s,t,x)^{1/2} e^{-\mathcal{G}(s,t,x)z\cdot z/4\nu(t-s)}, \end{aligned} \tag{3.6.78}$$

where

$$\mathcal{G}(s,t,x) = \mathcal{H}(s,t,x)^{-1}. \tag{3.6.79}$$

Note from (3.6.8) that

$$\mathcal{H}^{ij}(s,t,x) = \frac{1}{t-s}\int_s^t g^{ij}(\sigma,x)\,d\sigma, \tag{3.6.80}$$

where $(g^{ij}) = (g_{ij})^{-1}$, so in particular $\mathcal{H}^{ij}(s,s,x) = g^{ij}(s,x)$ and

$$\mathcal{G}_{ij}(s,s,x) = g_{ij}(s,x). \tag{3.6.81}$$

To compute K_j more generally, we use (3.6.71), which we restate as follows:

$$\begin{aligned} a_{2k}(\nu,s,t,x,\xi) = (\nu(t-s))^k \sum_{\substack{\alpha \text{ even} \\ |\beta|\le 1}} & F_{\alpha\beta}(\nu(t-s),s,t,x) \\ & \times \big((\nu(t-s))^{1/2}\xi\big)^\alpha \big(\nu(t-s)\xi\big)^\beta e^{-\nu(t-s)\mathcal{H}\xi\cdot\xi}, \end{aligned} \tag{3.6.82}$$

and

$$\begin{aligned} a_{2k+1}(\nu,s,t,x,\xi) = (\nu(t-s))^{k+1} \sum_{\substack{\alpha \text{ even} \\ |\beta|\le 1,\ell}} & F_{\alpha\beta\ell}(\nu(t-s),s,t,x) \\ & \times \xi_\ell \big((\nu(t-s))^{1/2}\xi\big)^\alpha \big(\nu(t-s)\xi\big)^\beta e^{-\nu(t-s)\mathcal{H}\xi\cdot\xi} \\ + (\nu(t-s))^{k+1} \sum_{\substack{\alpha \text{ even} \\ |\beta|\le 1}} & F^0_{\alpha\beta}(\nu(t-s),s,t,x) \\ & \times \big((\nu(t-s))^{1/2}\xi\big)^\alpha \big(\nu(t-s)\xi\big)^\beta e^{-\nu(t-s)\mathcal{H}\xi\cdot\xi}. \end{aligned} \tag{3.6.83}$$

Here $\mathcal{H} = \mathcal{H}(s,t,x)$ is as in (3.6.77), and $F_{\alpha\beta}$, $F_{\alpha\beta\ell}$, and $F^0_{\alpha\beta}$ are smooth functions of their arguments. All the sums are finite. To compute the integrals in (3.6.76), we use the following result:

$$(2\pi)^{-n} \int \xi^\alpha e^{-\mathcal{H}\xi\cdot\xi} e^{iz\cdot\xi}\,d\xi = \big(\det(4\pi\mathcal{H})\big)^{-1/2} D_z^\alpha e^{-\mathcal{G}z\cdot z/4} = p_\alpha(\mathcal{H},z) e^{-\mathcal{G}z\cdot z/4}, \tag{3.6.84}$$

where the last identity defines $p_\alpha(\mathcal{H}, z)$, which is a polynomial of degree $|\alpha|$ whose coefficients depend smoothly on $\mathcal{H}$, and $\mathcal{G} = \mathcal{H}^{-1}$. We note that

$$p_\alpha(\mathcal{H}, -z) = (-1)^{|\alpha|} p_\alpha(\mathcal{H}, z). \tag{3.6.85}$$

Taking

$$\mu = \nu(t-s), \tag{3.6.86}$$

we go from (3.6.82)–(3.6.83) to formulas for $K_j(\nu, s, t, x, z)$ via the identities

$$(2\pi)^{-n} \int (\mu^{1/2}\xi)^\alpha (\mu\xi)^\beta e^{-\mu\mathcal{H}\xi\cdot\xi} e^{iz\cdot\xi}\, d\xi = \mu^{(-n+|\beta|)/2} p_{\alpha+\beta}(\mathcal{H}, \mu^{-1/2}z) e^{-\mathcal{G}z\cdot z/4\mu}, \tag{3.6.87}$$

and

$$(2\pi)^{-n} \int \xi_\ell (\mu^{1/2}\xi)^\alpha (\mu\xi)^\beta e^{-\mu\mathcal{H}\xi\cdot\xi} e^{iz\cdot\xi}\, d\xi = \mu^{(-n+|\beta|-1)/2} p_{\alpha+\beta+\varepsilon_\ell}(\mathcal{H}, \mu^{-1/2}z) e^{-\mathcal{G}z\cdot z/4\mu}. \tag{3.6.88}$$

We obtain

$$\begin{aligned} K_{2k}(\nu, s, t, x, z) = (\nu(t-s))^{-n/2+k} \sum_{\substack{\alpha \text{ even} \\ |\beta|\le 1}} &(\nu(t-s))^{|\beta|/2} F_{\alpha\beta}(\nu(t-s), s, t, x) \\ &\times p_{\alpha+\beta}(\mathcal{H}, (\nu(t-s))^{-1/2}z) e^{-\mathcal{G}z\cdot z/4\nu(t-s)}, \end{aligned} \tag{3.6.89}$$

hence

$$\begin{aligned} K_{2k}(\nu, s, t, x, z) = (\nu(t-s))^{-n/2+k} \sum_{b=0}^{1} &(\nu(t-s))^{b/2} \Phi_{2k,b}(\nu(t-s), s, t, x, (\nu(t-s))^{-1/2}z) \\ &\times e^{-\mathcal{G}(s,t,x)z\cdot z/4\nu(t-s)}, \end{aligned} \tag{3.6.90}$$

where $\Phi_{2k,b}$ is a polynomial in $(\nu(t-s))^{-1/2}z = Z$, with coefficients smooth in $\nu(t-s)$, s, t, x, satisfying

$$\Phi_{2k,b}(\nu(t-s), s, t, x, -Z) = (-1)^b \Phi_{2k,b}(\nu(t-s), s, t, x, Z). \tag{3.6.91}$$

Similarly,

$$\begin{aligned} K_{2k+1}(\nu, s, t, x, z) = (\nu(t-s))^{-n/2+k+1/2} \sum_{\substack{\alpha \text{ even} \\ |\beta|\le 1,\ell}} &(\nu(t-s))^{|\beta|/2} F_{\alpha\beta\ell}(\nu(t-s), s, t, x) \\ &\times p_{\alpha+\beta+\varepsilon_\ell}(\mathcal{H}, (\nu(t-s))^{-1/2}z) e^{-\mathcal{G}z\cdot z/4\nu(t-s)} \\ + (\nu(t-s))^{-n/2+k+1} \sum_{\substack{\alpha \text{ even} \\ |\beta|\le 1}} &(\nu(t-s))^{|\beta|/2} F^0_{\alpha\beta}(\nu(t-s), s, t, x) \\ &\times p_{\alpha+\beta}(\mathcal{H}, (\nu(t-s))^{-1/2}z) e^{-\mathcal{G}z\cdot z/4\nu(t-s)}, \end{aligned} \tag{3.6.92}$$

hence

$$\begin{aligned} K_{2k+1}(\nu, s, t, x, z) = (\nu(t-s))^{-n/2+k+1/2} \sum_{b=0}^{1} &(\nu(t-s))^{b/2} \Phi_{2k+1,b}(\nu(t-s), s, t, x, (\nu(t-s))^{-1/2}z) \\ &\times e^{-\mathcal{G}(s,t,x)z\cdot z/4\nu(t-s)} \\ + (\nu(t-s))^{-n/2+k+1} \sum_{b=0}^{1} &(\nu(t-s))^{b/2} \Phi^0_{2k+1,b}(\nu(t-s), s, t, x, (\nu(t-s))^{-1/2}z) \\ &\times e^{-\mathcal{G}(s,t,x)z\cdot z/4\nu(t-s)}, \end{aligned} \tag{3.6.93}$$

where $\Phi_{2k+1,b}$ is a polynomial in $(\nu(t-s))^{-1/2}z = Z$, with coefficients smooth in $\nu(t-s), s, t, x$, satisfying

$$\Phi_{2k+1,b}(\nu(t-s), s, t, x, -Z) = (-1)^{b+1}\Phi_{2k+1,b}(\nu(t-s), s, t, x, Z), \tag{3.6.94}$$

and $\Phi^0_{2k+1,b}$ is a polynomial in $(\nu(t-s))^{-1/2}z$ with coefficients smooth in $\nu(t-s), s, t, x$, satisfying

$$\Phi^0_{2k+1,b}(\nu(t-s), s, t, x, -Z) = (-1)^b\Phi^0_{2k+1,b}(\nu(t-s), s, t, x, Z). \tag{3.6.95}$$

While the formulas (3.6.89)–(3.6.90) and (3.6.92)–(3.6.93) for the functions $K_j(\nu, s, t, x, z)$ are rather lengthy, they are not difficult to comprehend. The basic result to be gleaned from these calculations is that for $j \ge 1$, $K_j(\nu, s, t, x, z)$ is *smaller* and *smoother* than the dominant term $K_0(\nu, s, t, x, z)$, given by the comparatively simple formula (3.6.78).

3.7. *Boundary layer analysis of $e^{t(\nu\Delta - X)}$*. In this section we examine the fine behavior near $\partial\mathbb{O}$ as $\nu \searrow 0$ of $e^{t(\nu\Delta-X)}f$, with emphasis on the cases $f \in C(\overline{\mathbb{O}})$ and $f \in C^\infty(\overline{\mathbb{O}})$. As in Section 3.2, we work with solutions to

$$\frac{\partial v^\nu}{\partial t} = \nu L(t)v^\nu, \quad v^\nu\big|_{\mathbb{R}^+\times\partial\mathbb{O}} = 0, \quad v^\nu(0) = f, \tag{3.7.1}$$

where

$$L(t) = e^{tX}\Delta e^{-tX} \tag{3.7.2}$$

is a smooth family of strongly elliptic operators, as in (3.2.3) and (3.6.6). From this, the behavior of

$$e^{t(\nu\Delta-X)}f = e^{-tX}v^\nu(t) \tag{3.7.3}$$

is easily deduced.

We assume $\mathbb{O}$ is a smoothly bounded open subset of a compact Riemannian manifold M without boundary. To begin the analysis of (3.7.1), we extend f to $\tilde f$ on M, having the same degree of smoothness as f, e.g.,

$$f \in C(\overline{\mathbb{O}}) \Rightarrow \tilde f \in C(M), \quad f \in C^\infty(\overline{\mathbb{O}}) \Rightarrow \tilde f \in C^\infty(M), \quad \text{etc.} \tag{3.7.4}$$

We also extend X to be a smooth vector field on M (we need not assume $\operatorname{div} X = 0$ on M), and define V^ν on $\mathbb{R}^+ \times M$ by

$$\frac{\partial V^\nu}{\partial t} = \nu L(t)V^\nu \text{ on } \mathbb{R}^+\times M, \quad V^\nu(0, x) = \tilde f(x). \tag{3.7.5}$$

Here $L(t)$ is given by (3.7.2). The solution to (3.7.5) has the form

$$V^\nu(t, x) = \int_M \tilde f(y)H(\nu, 0, t, x, y)\,dV(y), \tag{3.7.6}$$

where dV is the Riemannian volume element on M. More generally, for $0 \le s < t$,

$$V^\nu(t, x) = \int_M V^\nu(s, y)H(\nu, s, t, x, y)\,dV_s(y), \tag{3.7.7}$$

where dV_s is the pull-back of dV via the flow generated by X, or equivalently the Riemannian volume element for g_s, the metric tensor g of $\mathbb{O}$ pulled back by this flow. In local coordinates, we have

$$H(\nu, s, t, x, y) = g(s, y)^{-1/2}K(\nu, s, t, x, x-y), \tag{3.7.8}$$

where $K(\nu, s, t, x, x-y)$ has the form

$$K(\nu, s, t, x, z) = \sum_{j=0}^{N} K_j(\nu, s, t, x, z) + R_N(\nu, s, t, x, z), \tag{3.7.9}$$

with R_N the kernel of an operator satisfying the results given in Proposition 3.6.6, i.e., negligible for N large. As seen in (3.6.78),

$$K_0(\nu, s, t, x, z) = (4\pi\nu(t-s))^{-n/2} \det \mathcal{G}(s, t, x)^{1/2} e^{-\mathcal{G}(s,t,x)z\cdot z/4\nu(t-s)}, \tag{3.7.10}$$

and for $j \geq 1$, $K_j(\nu, s, t, x, z)$ are given by (3.6.90) and (3.6.93), as integral kernels that are smaller and smoother then $K_0(\nu, s, t, x, z)$. As before, $n = \dim M = \dim \mathcal{O}$.

Having V^ν, we can write the solution to (3.7.1) as

$$v^\nu(t, x) = V^\nu(t, x) - u^\nu(t, x), \quad t \geq 0, \ x \in \mathcal{O}, \tag{3.7.11}$$

where $u^\nu(t, x)$ is defined by

$$\begin{aligned} \frac{\partial u^\nu}{\partial t} &= \nu L(t) u^\nu \quad \text{on } \mathbb{R} \times \mathcal{O}, \\ u^\nu &= g^\nu \qquad\quad\ \text{on } \mathbb{R} \times \partial\mathcal{O}, \\ u^\nu &= 0 \qquad\quad\ \ \text{on } (-\infty, 0) \times \mathcal{O}, \end{aligned} \tag{3.7.12}$$

where

$$g^\nu(t, x) = \chi_{\mathbb{R}^+}(t)\, V^\nu(t, x), \quad x \in \partial\mathcal{O}. \tag{3.7.13}$$

We now describe how to use the method of layer potentials to solve (3.7.12).

We start with the case $\nu = 1$ and then explain the modifications that work for $\nu \in (0, 1]$. With H as in (3.7.7)–(3.7.8), we set

$$\mathcal{D}_1 h(t, x) = \int_0^t \int_{\partial\mathcal{O}} h(s, y) \frac{\partial H}{\partial n_{s,y}}(1, s, t, x, y)\, dS_s(y)\, ds, \quad t \geq 0, \ x \in \mathcal{O}. \tag{3.7.14}$$

Here dS_s is the area element on $\partial\mathcal{O}$ induced by the metric tensor g_s, described as below (3.7.7), and $\partial/\partial n_{s,y}$ is the outward unit normal to $\partial\mathcal{O}$ at $y \in \partial\mathcal{O}$, determined by this metric tensor. The boundary trace relation for $\mathcal{D}_1$ is

$$\mathcal{D}_1 h\big|_{\mathbb{R}\times\partial\mathcal{O}} = \big(\tfrac{1}{2} I + N_1\big) h, \tag{3.7.15}$$

assuming $h(s, y) = 0$ for $s < 0$, where

$$N_1 h(t, x) = \int_0^t \int_{\partial\mathcal{O}} h(s, y) \frac{\partial H}{\partial n_{s,y}}(1, s, t, x, y)\, dS_s(y)\, ds, \quad t \geq 0, \ x \in \partial\mathcal{O}. \tag{3.7.16}$$

The integral formula on the right sides of (3.7.14) and (3.7.16) have an identical appearance, but in the former case $x \in \mathcal{O}$ and in the latter case $x \in \partial\mathcal{O}$. It follows that we can solve (3.7.12), in the case $\nu = 1$, as

$$u^1 = \mathcal{D}_1 h^1, \tag{3.7.17}$$

provided h^1 solves

$$\big(\tfrac{1}{2} I + N_1\big) h^1 = g^1. \tag{3.7.18}$$

For general $\nu > 0$, we have essentially the same situation, except that $\nu L(t)$ is the Laplace operator for the metric tensor $\nu^{-1} g_t$. One has the analogue of (3.7.16), with this scaled metric tensor. This rescaling requires that $\partial/\partial n_{s,y}$ be replaced by $\nu^{1/2}\partial/\partial n_{s,y}$ and that dS_s be replaced by $\nu^{-(n-1)/2}\, dS_s$. Also dV is replaced by $\nu^{-n/2}\, dV$, so we need to replace $H(1, s, t, x, y)$ by $\nu^{n/2} H(\nu, s, t, x, y)$. Since

$$\nu^{1/2}\nu^{-(n-1)/2}\nu^{n/2} = \nu, \tag{3.7.19}$$

we obtain

$$\mathcal{D}_\nu h(t, x) = \nu \int_0^t \int_{\partial\mathcal{O}} h(s, y) \frac{\partial H}{\partial n_{s,y}}(\nu, s, t, x, y)\, dS_s(y)\, ds. \tag{3.7.20}$$

The boundary trace result (3.7.15) becomes

$$\mathcal{D}_\nu h\big|_{\mathbb{R}\times\partial\mathcal{O}} = \big(\tfrac{1}{2} I + \nu N_\nu\big) h, \tag{3.7.21}$$

for supp $h \subset \mathbb{R}^+ \times \partial\mathcal{O}$, where

$$N_\nu h(t, x) = \int_0^t \int_{\partial\mathcal{O}} h(s, y) \frac{\partial H}{\partial n_{s,y}}(\nu, s, t, x, y)\, dS_s(y)\, ds. \tag{3.7.22}$$

Hence the solution to (3.7.12) has the form

$$u^\nu(t, x) = \mathcal{D}_\nu h^\nu(t, x), \tag{3.7.23}$$

provided H^ν solves

$$\big(\tfrac{1}{2} I + \nu N_\nu\big) h^\nu = g^\nu, \tag{3.7.24}$$

with $g^\nu(t, x)$ given by (3.7.13).

We now tackle the problem of inverting $((1/2)I + \nu N_\nu)$ in (3.7.24). The results (3.7.8)–(3.7.10) on H and related estimates on K_j established in Section 3.6 imply

$$\left\| \frac{\partial H}{\partial n_{s,y}}(\nu, s, t, x, \cdot) \right\|_{L^1(\partial\mathcal{O})} \le C(\nu(t-s))^{-1/2}, \quad x \in \partial\mathcal{O}, \tag{3.7.25}$$

and

$$\left\| \frac{\partial H}{\partial n_{s,y}}(\nu, s, t, x, \cdot) \right\|_{L^1(\partial\mathcal{O})} \le C(\nu(t-s))^{-1}, \quad x \in \mathcal{O}, \tag{3.7.26}$$

uniformly for $0 \le s < t \le T_0$. For the present analysis, the focus is on (3.7.25). It implies for $I = [0, T_0]$

$$\|\nu N_\nu h\|_{L^\infty(I\times\partial\mathcal{O})} \le C(T_0)\, \nu^{1/2}. \tag{3.7.27}$$

Hence, given $T_0 \in (0, \infty)$, as long as ν is so small that $C(T_0)\nu^{1/2} \le 1/2$, if $g^\nu \in L^\infty(I \times \partial\mathcal{O})$, Equation (3.7.24) is solved by

$$h^\nu = 2(I + 2\nu N_\nu)^{-1} g^\nu = 2(I - 2\nu N_\nu + 4\nu^2 N_\nu^2 - \cdots) g^\nu. \tag{3.7.28}$$

Note that

$$\|h^\nu - 2g^\nu\|_{L^\infty(I\times\partial\mathcal{O})} \le C\nu^{1/2} \|g^\nu\|_{L^\infty(I\times\partial\mathcal{O})}. \tag{3.7.29}$$

We are motivated to estimate $\mathcal{D}_\nu(h^\nu - 2g^\nu)$. The estimate (3.7.26) is not adequate for this; instead we argue as follows. Denote the solution to (3.7.12) by

$$u^\nu = \mathrm{PI}_\nu g^\nu. \tag{3.7.30}$$

The content of (3.7.21) and (3.7.28) is that

$$\mathrm{PI}_\nu g^\nu = \mathcal{D}_\nu h^\nu, \quad \big(\tfrac{1}{2}I + \nu N_\nu\big)h^\nu = g^\nu. \tag{3.7.31}$$

Hence

$$\mathcal{D}_\nu(h^\nu - 2g^\nu) = \mathrm{PI}_\nu\big(\tfrac{1}{2}I + \nu N_\nu\big)(h^\nu - 2g^\nu). \tag{3.7.32}$$

Now the maximum principle gives

$$\|\mathrm{PI}_\nu h\|_{L^\infty(I\times\mathcal{O})} \le \|h\|_{L^\infty(I\times\partial\mathcal{O})}, \tag{3.7.33}$$

so we have the general estimate

$$\|\mathcal{D}_\nu h\|_{L^\infty(I\times\mathcal{O})} \le C\|h\|_{L^\infty(I\times\partial\mathcal{O})}, \tag{3.7.34}$$

with C independent of $\nu \in (0,1]$, and in particular

$$\|\mathcal{D}_\nu(h^\nu - 2g^\nu)\|_{L^\infty(I\times\mathcal{O})} \le C\|h^\nu - 2g^\nu\|_{L^\infty(I\times\partial\mathcal{O})} \le C\nu^{1/2}\|g^\nu\|_{L^\infty(I\times\partial\mathcal{O})}, \tag{3.7.35}$$

the last inequality by (3.7.29).

Proposition 3.7.1. *The solution u^ν to (3.7.12) has the property*

$$\|u^\nu - 2\mathcal{D}_\nu g^\nu\|_{L^\infty(I\times\mathcal{O})} \le C(I)\nu^{1/2}\|g^\nu\|_{L^\infty(I\times\partial\mathcal{O})} \le C'(I)\nu^{1/2}\|\tilde{f}\|_{L^\infty(M)}. \tag{3.7.36}$$

Proof. The first inequality in (3.7.36) follows from (3.7.35) and the fact that $u^\nu = \mathcal{D}_\nu h^\nu$. The second follows from the identification of g^ν in (3.7.13) and the maximum principle, applied to (3.7.5). □

Recalling (3.7.11), we have:

Corollary 3.7.2. *The solution v^ν to (3.7.1) has the property*

$$\|v^\nu - (V^\nu - 2\mathcal{D}_\nu g^\nu)\|_{L^\infty(I\times\mathcal{O})} \le C(I)\nu^{1/2}\|\tilde{f}\|_{L^\infty(M)}. \tag{3.7.37}$$

We can obtain simpler approximations to u^ν and v^ν if we assume more regularity on f. Using (3.5.9), we have, for $q \in [2,\infty)$, $\sigma > 0$,

$$\|V^\nu(t,\cdot)\|_{H^{\sigma,q}(M)} \le C\|\tilde{f}\|_{H^{\sigma,q}(M)}, \quad 0 \le t \le T_0, \tag{3.7.38}$$

with C independent of $\nu \in (0,1]$. Taking $\sigma = 2+\varepsilon$ and q sufficiently large, we obtain

$$\|V^\nu(t,\cdot)\|_{C^2(M)} \le C\|\tilde{f}\|_{H^{2+\varepsilon,q}(M)}, \quad 0 \le t \le T_0, \tag{3.7.39}$$

for each $\varepsilon > 0$, $q > n/\varepsilon$, with C independent of ν. Hence the solution V^ν to (3.7.5) satisfies

$$\|V^\nu(t) - \tilde{f}\|_{L^\infty(M)} \le C\nu\|\tilde{f}\|_{H^{2+\varepsilon,q}(M)}, \quad 0 \le t \le T_0. \tag{3.7.40}$$

Interpolation with

$$\|V^\nu(t) - \tilde{f}\|_{L^\infty(M)} \le 2\|\tilde{f}\|_{L^\infty(M)} \le C\|\tilde{f}\|_{H^{\varepsilon,q}(M)} \tag{3.7.41}$$

gives

$$\|V^\nu(t) - \tilde{f}\|_{L^\infty(M)} \le C\nu^{1/2}\|\tilde{f}\|_{H^{1+\varepsilon,q}(M)} \le C'\nu^{1/2}\|\tilde{f}\|_{C^{1,\delta}(M)}, \tag{3.7.42}$$

the last inequality holding provided $\delta > \varepsilon$. We hence have the following.

Proposition 3.7.3. *In the setting of Proposition 3.7.1 and Corollary 3.7.2, we have, for each $\delta > 0$,*

$$\|u^\nu - 2\mathcal{D}_\nu f^b\|_{L^\infty(I\times\mathcal{O})} \le C(I)\nu^{1/2}\|\tilde{f}\|_{C^{1,\delta}(M)} \tag{3.7.43}$$

and

$$\|v^\nu - (f - 2\mathcal{D}_\nu f^b)\|_{L^\infty(I\times\mathcal{O})} \le C(I)\nu^{1/2}\|\tilde{f}\|_{C^{1,\delta}(M)}, \tag{3.7.44}$$

where

$$f^b = \chi_{\mathbb{R}^+}(t)\, f\big|_{\partial\mathcal{O}}. \tag{3.7.45}$$

Proof. From (3.7.42) we have

$$\|g^\nu - f^b\|_{L^\infty(I\times\partial\mathcal{O})} \le C\nu^{1/2}\|\tilde{f}\|_{C^{1,\delta}(M)}, \tag{3.7.46}$$

and then the estimate (3.7.34) applied to $h = g^\nu - f^b$ gives (3.7.43) from (3.7.36). The proof of (3.7.44) is similar. □

For a further simplification, we compare $\mathcal{D}_\nu$ with $\mathcal{D}_\nu^0$, defined by

$$\mathcal{D}_\nu^0 h(t,x) = \nu\int_0^t\int_{\partial\mathcal{O}} h(s,y)\frac{\partial H_0}{\partial n_{s,y}}(\nu,s,t,x,y)\,dS_s(y)\,ds, \tag{3.7.47}$$

where, parallel to (3.7.8), we set

$$H_0(\nu,s,t,x,y) = g(s,y)^{-1/2}K_0(\nu,s,t,x,x-y), \tag{3.7.48}$$

with K_0 given by (3.7.10). By (3.7.9) we have

$$K - K_0 = \sum_{j=1}^N K_j + R_N. \tag{3.7.49}$$

Parallel to (3.7.26) we have

$$\left\|\frac{\partial K_1}{\partial n_{s,y}}(\nu,s,t,x,\cdot)\right\|_{L^1(\partial\mathcal{O})} \le C(\nu(t-s))^{-1/2}, \quad x\in\mathcal{O}, \tag{3.7.50}$$

with better estimates on $\partial K_j/\partial n_{s,y}$ for $j \ge 2$ and on $\partial R_N/\partial n_{s,y}$. This leads to:

Proposition 3.7.4. *With $\mathcal{D}_\nu^0$ defined by (3.7.47)–(3.7.48), we have*

$$\|\mathcal{D}_\nu h - \mathcal{D}_\nu^0 h\|_{L^\infty(I\times\mathcal{O})} \le C(I)\nu^{1/2}\|h\|_{L^\infty(I\times\partial\mathcal{O})}. \tag{3.7.51}$$

Hence, in the setting of Proposition 3.7.3, we have, for each $\delta > 0$,

$$\|u^\nu - 2\mathcal{D}_\nu^0 f^b\|_{L^\infty(I\times\mathcal{O})} \le C(I)\nu^{1/2}\|\tilde{f}\|_{C^{1,\delta}(M)} \tag{3.7.52}$$

and

$$\|v^\nu - (f - 2\mathcal{D}_\nu^0 f^b)\|_{L^\infty(I\times\mathcal{O})} \le C(I)\nu^{1/2}\|\tilde{f}\|_{C^{1,\delta}(M)}. \tag{3.7.53}$$

4. Analysis of solutions to $u_t = \nu\Delta u - X_\nu u$

In this chapter, we extend some of the results of Chapter 3 from the setting of solutions to $u_t = \nu\Delta u - Xu$ to the more subtle setting of solutions to $u_t = \nu\Delta u - X_\nu u$, directly relevant to the equation for w^ν in (1.0.8). As in that chapter, we assume $\overline{\mathbb{O}}$ is a compact Riemannian manifold with boundary $\partial\mathbb{O}$, and with Laplace Beltrami operator Δ. We take X_ν, for $\nu \in (0, 1]$, to be a family of (time dependent) vector fields on $\mathbb{O}$ having certain properties that we will specify below, and take $u = u^\nu$ to solve

$$\frac{\partial u}{\partial t} = \nu\Delta u - X_\nu u, \quad u\big|_{\mathbb{R}^+\times\partial\mathbb{O}} = 0, \quad u(0) = f. \tag{4.0.1}$$

In Section 4.1 we estimate $u^\nu(t)$ in the spaces $\mathcal{V}^k(\mathbb{O})$, introduced in Section 3.3, given $f \in \mathcal{V}^k(\mathbb{O})$, extending the scope of the uniform boundedness results of Section 3.3. In Section 4.2 we establish convergence of $u^\nu(t)$ to $e^{-tX} f$ in $\mathcal{V}^k(\mathbb{O})$, for such f, when $\nu \searrow 0$ and $X_\nu \to X$ in an appropriate sense, specified there. We also obtain L^p-norm convergence results, for $p \in [1, \infty)$.

4.1. *Conormal type estimates.* We will find it useful to extend the class of function spaces $\mathcal{V}^k(\mathbb{O})$. Given $k \in \mathbb{Z}^+ = \{0, 1, 2, \dots\}$, $p \in [1, \infty]$, we define

$$\mathcal{V}^{k,p}(\mathbb{O}) = \{u \in L^p(\mathbb{O}) : Y_1 \cdots Y_j\, u \in L^p(\mathbb{O}),\ \forall j \le k, Y_\ell \in \mathfrak{X}^1\}, \tag{4.1.1}$$

with

$$\mathfrak{X}^1 = \{Y \text{ smooth vector field on } \overline{\mathbb{O}} : Y \parallel \partial\mathbb{O}\}. \tag{4.1.2}$$

Recall that the case $p = 2$ is defined in (3.3.1). As in (3.3.3), there exists a finite set

$$\{Y_j : 1 \le j \le M\} \subset \mathfrak{X}^1 \tag{4.1.3}$$

with the property that each element of $\mathfrak{X}^1$ is a linear combination, with coefficients in $C^\infty(\overline{\mathbb{O}})$ of these vector fields Y_j. We recall and generalize some further useful notation from Section 3.3. With Y_j as in (4.1.3), let $J = (j_1, \dots, j_k)$ and set

$$Y^J = Y_{j_1} \cdots Y_{j_k}, \quad |J| = k. \tag{4.1.4}$$

Also set

$$\mathfrak{X}^k = \mathrm{Span}\,\{Z_1 \cdots Z_j : j \le k,\ Z_\ell \in \mathfrak{X}^1\}. \tag{4.1.5}$$

We have

$$\mathfrak{X}^k = \text{Span over } C^\infty(\overline{\mathbb{O}}) \text{ of } \{Y^J : |J| \le k\}, \tag{4.1.6}$$

and

$$\begin{aligned}\mathcal{V}^{k,p}(\mathbb{O}) &= \{u \in L^p(\mathbb{O}) : Y^J u \in L^p(\mathbb{O}),\ \forall\, |J| \le k\} \\ &= \{u \in L^p(\mathbb{O}) : Lu \in L^p(\mathbb{O}) : \forall\, L \in \mathfrak{X}^k\}.\end{aligned} \tag{4.1.7}$$

Let us also set

$$\mathcal{V}^{\infty,p}(\mathbb{O}) = \bigcap_k \mathcal{V}^{k,p}(\mathbb{O}). \tag{4.1.8}$$

We now discuss conditions on X_ν. We require

$$X_\nu \in \widehat{\mathfrak{X}}^1, \tag{4.1.9}$$

a space of t-dependent vector fields on $\overline{\mathcal{O}}$, depending on the parameter $\nu \in (0,1]$, which we proceed to define. We want to include the example arising in (2.2.4)–(2.2.5), i.e.,

$$X_\nu = v^\nu(t,z)\frac{\partial}{\partial x}, \quad v^\nu(t,z) = e^{\nu t A}V(z). \tag{4.1.10}$$

In this case we have $\mathcal{O} = \mathbb{T}\times I$, $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, $I = [0,1]$, and A is given by (2.1.5).

Lemma 4.1.1. *Given $T_0 \in (0,\infty)$, we have*

$$v^\nu(t,\cdot) \in \mathcal{V}^{\infty,\infty}(\mathcal{O}), \tag{4.1.11}$$

with bounds independent of $t \in [0,T_0]$, $\nu \in (0,1]$.

Proof. Straightforward from the construction of $e^{\nu t A}V(z)$ via the method of images. There is no x-dependence, so the result is actually $v^\nu(t,\cdot) \in \mathcal{V}^{\infty,\infty}(I)$, with uniform bounds. In this setting, we mention that $\mathfrak{X}^1(I)$ consists of smooth vector fields on I that vanish at the endpoints. □

To define $\widehat{\mathfrak{X}}^1$ in general, we first specify that, on any compact $\overline{\Omega} \Subset \mathcal{O}$, an element $X_\nu(t)$ has uniform bounds in $C^k(\overline{\Omega})$ for all k. To complete the definition, we take a collar neighborhood U of $\partial\mathcal{O}$, diffeomorphic to $\partial\mathcal{O}\times I$, take coordinates $(x,z) \in \partial\mathcal{O}\times I$, and write

$$X_\nu = v^\nu(t,x,z)\frac{\partial}{\partial x} + w^\nu(t,x,z)\beta(x,z)\frac{\partial}{\partial z}. \tag{4.1.12}$$

Here $v^\nu\partial/\partial x$ is shorthand for $\sum_j v_j^\nu \partial/\partial x_j$. We require (with bounds uniform in $t \in [0,T_0]$, $\nu \in (0,1]$),

$$v^\nu, w^\nu \in \mathcal{V}^{\infty,\infty}(\mathcal{O}), \quad \beta \in C^\infty(\overline{\mathcal{O}}), \quad \beta\big|_{\partial\mathcal{O}} = 0. \tag{4.1.13}$$

These conditions define $\widehat{\mathfrak{X}}^1$.

Lemma 4.1.2. *We have*

$$X_\nu \in \widehat{\mathfrak{X}}^1,\ Y \in \mathfrak{X}^1 \implies [X_\nu, Y] \in \widehat{\mathfrak{X}}^1. \tag{4.1.14}$$

Proof. The bounds on $[X_\nu, Y]$ on any $\overline{\Omega} \Subset \mathcal{O}$ are clear. Near $\partial\mathcal{O}$, we represent X_ν as in (4.1.12) and set

$$Y = a(x,z)\frac{\partial}{\partial x} + b(x,z)\frac{\partial}{\partial z}, \quad a, b \in C^\infty(\overline{\mathcal{O}}), \quad b\big|_{\partial\mathcal{O}} = 0. \tag{4.1.15}$$

Then

$$[X_\nu, Y] = \xi^\nu(t,x,z)\frac{\partial}{\partial x} + \eta^\nu(t,x,z)\frac{\partial}{\partial z}, \tag{4.1.16}$$

with

$$\begin{aligned} \xi^\nu &= v^\nu(\partial_x a) + w^\nu\beta(\partial_z a) - a(\partial_x v^\nu) - b(\partial_z v^\nu), \\ \eta^\nu &= v^\nu(\partial_x b) + w^\nu\beta(\partial_z b) - a\partial_x(w^\nu\beta) - b\partial_z(w^\nu\beta). \end{aligned} \tag{4.1.17}$$

Comparison with the defining conditions in (4.1.12)–(4.1.13) gives $[X_\nu, Y] \in \widehat{\mathfrak{X}}^1$. □

Next we define

$$\widehat{\mathfrak{X}}^k = \mathrm{Span}\,\{X_\nu Y^J : X_\nu \in \widehat{\mathfrak{X}}^1,\ Y^J \in \mathfrak{X}^{k-1}\}. \tag{4.1.18}$$

Lemma 4.1.3. *We have*

$$P_\nu \in \widehat{\mathfrak{X}}^k,\ Y \in \mathfrak{X}^1 \implies Y P_\nu \in \widehat{\mathfrak{X}}^{k+1}; \tag{4.1.19}$$

hence

$$P_\nu \in \widehat{\mathfrak{X}}^k,\ Y^I \in \mathfrak{X}^\ell \implies Y^I P_\nu \in \widehat{\mathfrak{X}}^{k+\ell}. \tag{4.1.20}$$

Proof. To prove (4.1.19), note that for $X_\nu \in \widehat{\mathfrak{X}}^1,\ Y^J \in \mathfrak{X}^{k-1}$,

$$Y X_\nu Y^J = X_\nu Y Y^J + [Y, X_\nu] Y^J, \tag{4.1.21}$$

and apply Lemma 4.1.2 to the second term on the right side of (4.1.21). The result (4.1.20) follows directly from (4.1.19). □

Lemma 4.1.3 will prove useful in connection with the following. With Y_j as in (4.1.3), let us set

$$\|u\|_{\mathcal{V}^{k,p}} = \sum_{|J|\le k} \|Y^J u\|_{L^p}. \tag{4.1.22}$$

From the representation (4.1.12), we have

$$X_\nu \in \widehat{\mathfrak{X}}^1 \implies X_\nu = \sum a^j_{\nu,t} Y_j, \quad a^j_{\nu,t} \in L^\infty(\mathcal{O}), \tag{4.1.23}$$

with bounds independent of $\nu \in (0, 1],\ t \in [0, T_0]$, hence, given $X_\nu \in \widehat{\mathfrak{X}}^1$,

$$\|X_\nu u\|_{L^p} \le C \|u\|_{\mathcal{V}^{1,p}}, \tag{4.1.24}$$

and, by (4.1.20),

$$\|X_\nu u\|_{\mathcal{V}^{k,p}} \le C \|u\|_{\mathcal{V}^{k+1,p}}. \tag{4.1.25}$$

We also set

$$P_k^2(u) = \sum_{|J|=k} \|Y^J u\|^2_{L^2}, \tag{4.1.26}$$

so

$$\|u\|^2_{\mathcal{V}^{k,2}} \approx \sum_{j\le k} P_j^2(u). \tag{4.1.27}$$

We also denote $\mathcal{V}^{k,2}$ by $\mathcal{V}^k$.

We now estimate the rate of change of $P_k^2(u(t))$ for $u(t)$ satisfying (4.0.1). We assume

$$X_\nu \in \widehat{\mathfrak{X}}^1, \quad \operatorname{div} X_\nu = 0. \tag{4.1.28}$$

We also assume u is sufficiently smooth on $(0, \infty) \times \overline{\mathcal{O}}$ for the calculations made below to work. We will comment on how to verify this assumption later in this section.

We start with the case $k = 0$:

$$\frac{d}{dt} \|u\|^2_{L^2} = 2(u_t, u)_{L^2} = 2\nu(\Delta u, u)_{L^2} - 2(X_\nu u, u)_{L^2} = -2\nu \|\nabla u\|^2_{L^2}, \tag{4.1.29}$$

Moving on to $k = 1$, we have

$$\begin{aligned} \frac{d}{dt} \|Y_j u\|^2_{L^2} &= 2(Y_j u_t, Y_j u)_{L^2} = 2\nu(Y_j \Delta u, Y_j u)_{L^2} - 2(Y_j X_\nu u, Y_j u)_{L^2} \\ &= 2\nu(\Delta Y_j u, Y_j u)_{L^2} + 2\nu([Y_j, \Delta]u, Y_j u)_{L^2} - 2(X_\nu Y_j u, Y_j u)_{L^2} - 2([Y_j, X_\nu]u, Y_j u) L^2 \\ &= -2\nu \|\nabla Y_j u\|^2_{L^2} + 2\nu([Y_j, \Delta]u, Y_j u)_{L^2} - 2([Y_j, X_\nu]u, Y_j u)_{L^2}. \end{aligned} \tag{4.1.30}$$

Of the three terms in the last line, the first has a clear significance. For the third, we have $[Y_j, X_\nu] \in \widehat{\mathfrak{X}}^1$, by Lemma 4.1.2, and hence, by (4.1.23),

$$2([Y_j, X_\nu]u, Y_j u)_{L^2} \le C P_1^2(u). \tag{4.1.31}$$

It remains to estimate the second term. For this, write

$$[Y, \Delta] = \sum_\ell A_\ell B_\ell, \tag{4.1.32}$$

with A_ℓ, B_ℓ smooth vector fields on $\overline{\mathcal{O}}$. We have

$$2\nu([Y_j, \Delta]u, Y_j u)_{L^2} = 2\nu \sum_\ell (B_\ell u, A_\ell^* Y_j u)_{L^2} \le \nu\|\nabla Y_j u\|_{L^2}^2 + \nu\|Y_j u\|_{L^2}^2 + K_1\nu\|\nabla u\|_{L^2}^2. \tag{4.1.33}$$

Plugging (4.1.31) and (4.1.33) into (4.1.30) and summing over j gives

$$\frac{d}{dt} P_1^2(u) \le -\nu \sum_j \|\nabla Y_j u\|_{L^2}^2 + (MC+\nu)P_1^2(u) + MK_1\nu\|\nabla u\|^2. \tag{4.1.34}$$

The term $MK_1\nu\|\nabla u\|_{L^2}^2$ is tamed by bringing in (4.1.29), to obtain

$$\frac{d}{dt}\Big(P_1^2(u) + MK_1 P_0^2(u)\Big) \le -\nu \sum_j \|\nabla Y_j u\|_{L^2}^2 + (MC+\nu)P_1^2(u). \tag{4.1.35}$$

Proceeding to general k, we take $|J| = k$ and look at

$$\begin{aligned}
\frac{d}{dt}\|Y^J u\|_{L^2}^2 &= 2(Y^J u_t, Y^J u)_{L^2} = 2\nu(Y^J \Delta u, Y^J u)_{L^2} - 2(Y^J X_\nu u, Y^J u)_{L^2} \\
&= 2\nu(\Delta Y^J u, Y^J u)_{L^2} + 2\nu([Y^J, \Delta]u, Y^J u)_{L^2} - 2(X_\nu Y^J u, Y^J u)_{L^2} - 2([Y^J, X_\nu]u, Y^J u)_{L^2} \\
&= -2\nu\|\nabla Y^J u\|_{L^2}^2 + 2\nu([Y^J, \Delta]u, Y^J u)_{L^2} - 2([Y^J, X_\nu]u, Y^J u)_{L^2}.
\end{aligned} \tag{4.1.36}$$

As with (4.1.30), of the three terms in the last line of (4.1.36), the first has a clear significance. For the third, we have, by Lemmas 4.1.2–4.1.3,

$$[X_\nu, Y^J] = [X_\nu, Y_{j_1}]Y_{j_2} \cdots Y_{j_k} + \cdots + Y_{j_1} \cdots Y_{j_{k-1}}[X_\nu, Y_{j_k}] \in \widehat{\mathfrak{X}}^k, \tag{4.1.37}$$

and hence, by (4.1.25),

$$([Y^J, X_\nu]u, Y^J u)_{L^2} \le C_k \|u\|_{\mathcal{V}^k}^2. \tag{4.1.38}$$

It remains to estimate the second term in the last line of (4.1.36). For this, write

$$[\Delta, Y^J] = \sum_{\ell=1}^k Y_{j_1} \cdots Y_{j_{\ell-1}}[\Delta, Y_{j_\ell}]Y_{j_{\ell+1}} \cdots Y_{j_k} = \sum_{\ell=1}^k Y_{j_1} \cdots Y_{j_{\ell-1}} L_{j_\ell} Y_{j_{\ell+1}} \cdots Y_{j_k}, \tag{4.1.39}$$

where $L_{j_\ell} = [\Delta, Y_{j_\ell}]$ is a second order differential operator that annihilates constants.. We say a product of k factors

$$Y_{j_1} \cdots Y_{j_{\ell-1}} L_{j_\ell} Y_{j_{\ell+1}} \cdots Y_{j_k} \tag{4.1.40}$$

is of type (k, ℓ), meaning it is a product of k factors, all being vector fields in $\mathfrak{X}^1$ except one, in position ℓ, which is a second order differential operator that annihilates constants. If $\ell \ge 2$, we can write (4.1.40) as

$$Y_{j_1} \cdots Y_{j_\ell - 2} L_{j_\ell} \cdots Y_{j_k} + Y_{j_1} \cdots Y_{j_{\ell-2}}[Y_{j_{\ell-1}}, L_{j_\ell}] \cdots Y_{j_k}, \tag{4.1.41}$$

a sum of terms of type $(k, \ell-1)$ and of type $(k-1, \ell-1)$. Repeating this process, we convert (4.1.40) into a sum of terms of type $(j, 1)$, for $j \le k$. Hence we have

$$([Y^J, \Delta]u, Y^J u)_{L^2} = \sum_{|I| \le k-1} (L_I Y^I u, Y^J u)_{L^2}, \tag{4.1.42}$$

where the L_I are differential operators of order 2, annihilating constants; hence

$$L_I = \sum_j A_{Ij} B_{Ij}, \tag{4.1.43}$$

where A_{Ij} are first order differential operators and B_{Ij} are vector fields. We then have

$$\begin{aligned} 2\nu([Y^J, \Delta]u, Y^J u)_{L^2} &= 2\nu \sum_{|I| \le k-1} \sum_j (B_{Ij} Y^I u, A^*_{Ij} Y^J u)_{L^2} \\ &\le \tilde{C}\nu \sum_{|I| \le k-1} \|\nabla Y^I u\|_{L^2} \cdot \big(\|\nabla Y^J u\|_{L^2} + \|Y^J u\|_{L^2} \big) \\ &\le \nu \|\nabla Y^J u\|^2_{L^2} + \nu \|Y^J u\|^2_{L^2} + C_k \nu \sum_{|I| \le k-1} \|\nabla Y^I u\|^2_{L^2}. \end{aligned} \tag{4.1.44}$$

Inserting (4.1.38) and (4.1.44) into (4.1.36), we get

$$\frac{d}{dt} \|Y^J u\|^2_{L^2} \le -\nu \|\nabla Y^J u\|^2_{L^2} + (C_k + \nu) \|u\|^2_{\mathcal{V}^k} + C_k \nu \sum_{|I| \le k-1} \|\nabla Y^I u\|^2_{L^2}; \tag{4.1.45}$$

hence, for $\nu \in (0, 1]$, and with $C_k + 1$ re-notated as C_k,

$$\frac{d}{dt} P^2_k(u) \le -\nu \sum_{|J|=k} \|\nabla Y^J u\|^2_{L^2} + M C_k \|u\|^2_{\mathcal{V}^k} + M C_k \nu \sum_{|I| \le k-1} \|\nabla Y^I u\|^2_{L^2}. \tag{4.1.46}$$

It follows that there exist $A_{kj} \in (0, \infty)$ and $B_k \in (0, \infty)$ such that if we set

$$\widetilde{N}^2_k(u) = P^2_k(u) + \sum_{j=0}^{k-1} A_{kj} P^2_j(u), \tag{4.1.47}$$

then

$$\frac{d}{dt} \widetilde{N}^2_k(u) \le -\nu \sum_{|J|=k} \|\nabla Y^J u\|^2_{L^2} + 2B_k \widetilde{N}^2_k(u), \tag{4.1.48}$$

when $u = u(t)$ is given by (4.0.1). In particular, redefining $\|u\|^2_{\mathcal{V}^k}$ as

$$\|u\|^2_{\mathcal{V}^k} = \widetilde{N}^2_k(u), \tag{4.1.49}$$

we obtain

$$\|u(t)\|_{\mathcal{V}^k} \le e^{(t-s)B_k} \|u(s)\|_{\mathcal{V}^k}, \tag{4.1.50}$$

for $0 < s < t < \infty$.

The estimates (4.1.48)–(4.1.50) have been established under the assumption that $u(t) = u^\nu(t)$ is sufficiently smooth on $\overline{\mathbb{O}}$ for $t > 0$. For example, if we add the assumption

$$X_\nu \in C^\infty((0, \infty) \times \overline{\mathbb{O}}) \tag{4.1.51}$$

for each $\nu \in (0, 1]$, we have such estimates, since well known parabolic regularity results give $u \in C^\infty((0,\infty) \times \overline{\mathcal{O}})$. (We emphasize that we do *not* assume $X_\nu \in C([0,\infty) \times \overline{\mathcal{O}})$.) Let us record this result.

Proposition 4.1.4. *Let $u = u^\nu$ solve (4.0.1). Assume X_ν satisfies (4.1.9) and (4.1.51). Then the estimate (4.1.50) holds, for $0 < s < t < \infty$, with B_k and the $\mathcal{V}^k$-norm independent of $\nu \in (0, 1]$.*

Next we want to pass to the limit $s = 0$ in (4.1.50), obtaining

$$\|u(t)\|_{\mathcal{V}^k} \le e^{tB_k} \|f\|_{\mathcal{V}^k}. \tag{4.1.52}$$

It is clear that we can do this in the context of Proposition 4.1.4 if we also know that

$$u \in C([0,\infty), \mathcal{V}^k(\mathcal{O})). \tag{4.1.53}$$

In turn, since the hypotheses of Proposition 4.1.4 already imply the result $u \in C^\infty((0,\infty) \times \overline{\mathcal{O}})$, it remains to establish that

$$f \in \mathcal{V}^k(\mathcal{O}) \implies u \in C([0, T_\nu], \mathcal{V}^k(\mathcal{O})), \tag{4.1.54}$$

for some $T_\nu > 0$ (possibly depending on ν). We turn to this task.

We set

$$\mathfrak{X} = C([0, T_\nu], \mathcal{V}^k(\mathcal{O})), \tag{4.1.55}$$

and seek $u \in \mathfrak{X}$ as a unique solution to

$$u(t) = e^{t\nu\Delta} f - \int_0^t e^{(t-s)\nu\Delta} X_\nu(s) u(s)\, ds, \tag{4.1.56}$$

i.e., as a fixed point of $\Phi : \mathfrak{X} \to \mathfrak{X}$, defined by

$$\Phi u(t) = e^{t\nu\Delta} f - \int_0^t e^{(t-s)\nu\Delta} X_\nu(s) u(s)\, ds. \tag{4.1.57}$$

This will work if we are able to show $\Phi : \mathfrak{X} \to \mathfrak{X}$ is a contraction map for $T_\nu > 0$ sufficiently small. We have

$$\Phi u(t) - \Phi v(t) = -\int_0^t e^{(t-s)\nu\Delta} X_\nu(x)(u(s) - v(s))\, ds. \tag{4.1.58}$$

Note that, by (4.1.25),

$$\|X_\nu(s)(u(s) - v(s))\|_{\mathcal{V}^{k-1}} \le C\|u(s) - v(s)\|_{\mathcal{V}^k}. \tag{4.1.59}$$

Meanwhile, it follows from (3.4.19) that

$$\|e^{(t-s)\nu\Delta} g\|_{\mathcal{V}^k} \le \frac{C}{\nu^{1/2}(t-s)^{1/2}} \|g\|_{\mathcal{V}^{k-1}}. \tag{4.1.60}$$

Hence

$$\|\Phi u(t) - \Phi v(t)\|_{\mathcal{V}^k} \le C\frac{t^{1/2}}{\nu^{1/2}} \sup_{0\le s\le t} \|u(s) - v(s)\|_{\mathcal{V}^k}. \tag{4.1.61}$$

A similar estimate works on (4.1.57), and we deduce that Φ is a contraction map on $\mathfrak{X}$ provided $T_\nu \le \nu/2C^2$.

We summarize what has been accomplished.

Proposition 4.1.5. *In the setting of Proposition 4.1.4, given $f \in \mathcal{V}^k(\mathcal{O})$, there is a unique solution $u = u^\nu$ to (4.0.1), satisfying*

$$u \in C([0,\infty), \mathcal{V}^k(\mathcal{O})) \cap C^\infty((0,\infty) \times \overline{\mathcal{O}}), \tag{4.1.62}$$

and we have

$$\|u(t)\|_{\mathcal{V}^k} \le e^{tB_k} \|f\|_{\mathcal{V}^k}. \tag{4.1.63}$$

4.2. *Vanishing ν limits.* As in Section 4.1, we assume $u = u^\nu$ solves

$$\frac{\partial u^\nu}{\partial t} = \nu \Delta u^\nu - X_\nu u^\nu, \quad u^\nu\big|_{\mathbb{R}^+ \times \partial\mathcal{O}} = 0, \quad u(0) = f, \tag{4.2.1}$$

with $f \in \mathcal{V}^k(\mathcal{O})$. We assume, as in (4.1.28), that

$$X_\nu \in \widehat{\mathfrak{X}}^1, \quad \operatorname{div} X_\nu = 0, \tag{4.2.2}$$

and as in (4.1.51) that

$$X_\nu \in C^\infty((0,\infty) \times \overline{\mathcal{O}}). \tag{4.2.3}$$

We also assume

$$X \in \mathfrak{X}^1, \quad \operatorname{div} X = 0. \tag{4.2.4}$$

Here is our first convergence result.

Proposition 4.2.1. *Under these hypotheses, we have, as $\nu \searrow 0$,*

$$u^\nu(t) \to e^{-tX} f, \quad \textit{weak* in } \mathcal{V}^k(\mathcal{O}), \tag{4.2.5}$$

provided X_ν also satisfies the following: we can write

$$X_\nu = \sum_j a_j^\nu(t,x) Y_j, \quad X = \sum_j a_j(x) Y_j, \tag{4.2.6}$$

where, as in (4.1.3), the set $\{Y_j : 1 \le j \le M\} \subset \mathfrak{X}^1$ spans $\mathfrak{X}^1$ over $C^\infty(\overline{\mathcal{O}})$, and we have $\|a_j^\nu(t,\cdot)\|_{L^\infty(\mathcal{O})}$, $\|a_j\|_{L^\infty(\mathcal{O})} \le K$, and

$$\lim_{\nu \searrow 0} [a_j^\nu(t,x) - a_j(x)] = 0, \quad \textit{uniformly on compact subsets of } \mathcal{O}. \tag{4.2.7}$$

Remark. Looking at (4.1.10), we see that (4.2.6)–(4.2.7) hold when X_ν is the family arising in the plane-parallel chanel flow problem.

Proof. Rewrite (4.2.1) as

$$\frac{\partial u^\nu}{\partial t} = (\nu\Delta - X) u^\nu + (X - X_\nu) u^\nu, \tag{4.2.8}$$

so

$$u^\nu(t) = e^{t(\nu\Delta - X)} f + \int_0^t e^{(t-s)(\nu\Delta - X)} (X - X_\nu(s)) u^\nu(s)\, ds. \tag{4.2.9}$$

We have

$$(X - X_\nu(s)) u^\nu(s) = \sum_j [a_j(x) - a_j^\nu(s,x)] Y_j u^\nu(s), \tag{4.2.10}$$

and $u^\nu(s)$ is bounded in $\mathcal{V}^k(\mathcal{O})$. As long as $k \geq 1$, $Y_j u^\nu(s)$ is bounded in $L^2(\mathcal{O})$, and the hypotheses on a_j^ν give

$$\|(X - X_\nu(s))u^\nu(s)\|_{L^p(\mathcal{O})} \to 0, \quad \text{as } \nu \searrow 0, \ \forall\, p < 2, \tag{4.2.11}$$

with uniform bounds in $L^2(\mathcal{O})$. Now $e^{t(\nu\Delta - X)}$ is a contraction semigroup on each space $L^p(\mathcal{O})$, so from (4.2.9) we obtain

$$\lim_{\nu\searrow 0} \|u^\nu(t) - e^{t(\nu\Delta - X)} f\|_{L^p} = 0, \quad \forall\, p < 2. \tag{4.2.12}$$

This result together with the uniform bounds on $u^\nu(t)$ and on $e^{t(\nu\Delta - X)}$ in $\mathcal{V}^k(\mathcal{O})$, and in concert with the result that

$$e^{t(\nu\Delta - X)} f \to e^{-tX} f, \quad \text{weak}^* \text{ in } \mathcal{V}^k(\mathcal{O}), \tag{4.2.13}$$

given in Proposition 3.3.4, yield the asserted convergence (4.2.5), for $k \geq 1$. The case $k = 0$ then follows since $\mathcal{V}^1(\mathcal{O})$ is dense in $\mathcal{V}^0(\mathcal{O}) = L^2(\mathcal{O})$. □

We will improve weak* convergence in (4.2.5) to norm convergence. Here is a first step.

Proposition 4.2.2. *In the setting of Proposition 4.2.1,*

$$f \in L^2(\mathcal{O}) \implies u^\nu(t) \to e^{-tX} f, \ \text{ in } L^2\text{-norm}, \ \text{ as } \nu \searrow 0. \tag{4.2.14}$$

Proof. We already have weak* convergence in $L^2(\mathcal{O})$. Also, results of Section 4.1, involving (4.1.29), imply

$$\|u^\nu(t)\|_{L^2(\mathcal{O})} \leq \|f\|_{L^2(\mathcal{O})}, \quad \forall\, \nu, t > 0. \tag{4.2.15}$$

Since for $X \in \mathfrak{X}^1$ such that $\operatorname{div} X = 0$ we have $\|e^{-tX} f\|_{L^2} = \|f\|_{L^2}$, the conclusion in (4.2.14) follows from the weak* convergence. □

An alternative proof of a generalization of Proposition 4.2.2 will be provided in Proposition 4.2.3 below. We begin with the elementary inequality

$$\|u^\nu(t)\|_{L^p} \leq \|f\|_{L^p}, \quad 1 \leq p \leq \infty, \tag{4.2.16}$$

for solutions to (4.2.1) with $f \in L^p(\mathcal{O})$. If also $f \in \mathcal{V}^k(\mathcal{O})$ with $k > n/2$, the result that $u^\nu(t) \to e^{-tX} f$ weak* in $\mathcal{V}^k(\mathcal{O})$, proven in Proposition 4.2.1, implies

$$u^\nu(t) \to e^{-tX} f \ \text{ locally uniformly on } \ \mathcal{O}. \tag{4.2.17}$$

In particular,

$$f \in C^\infty(\overline{\mathcal{O}}) \implies u^\nu(t) \to e^{-tX} f, \ \text{ boundedly and locally uniformly.} \tag{4.2.18}$$

Combining (4.2.16) and (4.2.18) and using standard approximation arguments, we have:

Proposition 4.2.3. *In the setting of Proposition 4.2.1,*

$$f \in C(\overline{\mathcal{O}}) \implies u^\nu(t) \to e^{-tX} f, \ \textit{ boundedly and locally uniformly on } \ \mathcal{O}, \tag{4.2.19}$$

and, for $1 \leq p < \infty$,

$$f \in L^p(\mathcal{O}) \implies u^\nu(t) \to e^{-tX} f \ \text{ in } \ L^p\text{-norm}. \tag{4.2.20}$$

We now sharpen Proposition 4.2.1.

Proposition 4.2.4. *In the setting of Proposition 4.2.1, (4.2.5) can be sharpened to*

$$u^\nu(t) \to e^{-tX} f, \quad \text{in } \mathcal{V}^k\text{-norm.} \tag{4.2.21}$$

Proof. In view of uniform bounds on $\|u^\nu(t)\|_{\mathcal{V}^k}$ in (4.1.63), it suffices to establish (4.2.21) for f in a dense subspace of $\mathcal{V}^k(\mathcal{O})$, so take $f \in C_0^\infty(\mathcal{O})$. As in the proof of Proposition 3.3.4, we use the complex interpolation identity

$$\mathcal{V}^k(\mathcal{O}) = [L^2(\mathcal{O}), \mathcal{V}^{2k}(\mathcal{O})]_{1/2}, \tag{4.2.22}$$

established in Proposition A.1.1 of the Appendix, which yields, for $f \in \mathcal{V}^{2k}(\mathcal{O})$,

$$\|u^\nu(t) - e^{-tX} f\|_{\mathcal{V}^k} \le \|u^\nu(t) - e^{-tX} f\|_{L^2}^{1/2} \, \|u^\nu(t) - e^{-tX} f\|_{\mathcal{V}^{2k}}^{1/2}. \tag{4.2.23}$$

The first factor on the right side tends to zero as $\nu \searrow 0$, by Proposition 4.2.2 (or Proposition 4.2.3), and the last factor is uniformly bounded as $\nu \searrow 0$ by (4.1.63) (with k replaced by $2k$). This completes the proof. ☐

Let us tie these results more closely to estimates obtained in Section 2.2. In such a case we had additional structure to exploit. Namely, X and X_ν were given in (2.2.5) as $V(z)\partial_x$ and $v^\nu(t,z)\partial_x$, respectively, where $v^\nu(t,z) = e^{\nu t \partial_z^2} V(z)$ (see also (4.1.10)). To generalize a bit to our present context, we assume in addition to (4.2.2)–(4.2.4) that

$$X = vZ, \quad X_\nu = v^\nu Z, \quad Z \in \mathfrak{X}^1, \quad Z \text{ commutes with } \Delta \text{ and with } X \text{ and } X_\nu. \tag{4.2.24}$$

The last two conditions are equivalent to

$$Zv = Zv^\nu = 0. \tag{4.2.25}$$

In such a case, (4.2.9) becomes

$$u^\nu(t) = e^{t(\nu\Delta - X)} f + \int_0^t e^{(t-s)(\nu\Delta - X)} \big((v - v^\nu) Z u^\nu(s)\big)\, ds. \tag{4.2.26}$$

The commutation properties yield

$$w^\nu(t) = Zu^\nu(t) \implies \big(\partial_t w^\nu = (\nu\Delta - X_\nu) w^\nu, \quad w^\nu\big|_{\mathbb{R}^+ \times \partial\mathcal{O}} = 0, \quad w^\nu(0) = Zf\big). \tag{4.2.27}$$

Then the maximum principle gives

$$\|Zu^\nu(s)\|_{L^\infty} \le \|Zf\|_{L^\infty}. \tag{4.2.28}$$

Let us assume $Zf \in L^\infty(\mathcal{O})$ and set $\|Zf\|_{L^\infty} = K$. Since $e^{(t-s)(\nu\Delta - X)}$ is positivity preserving, we have from (4.2.26) that

$$\big|u^\nu(t,x) - e^{t(\nu\Delta - X)} f(x)\big| \le K \int_0^t e^{(t-s)(\nu\Delta - X)} |v - v^\nu(s)|\, ds. \tag{4.2.29}$$

Now (4.2.24)–(4.2.25) imply $Ze^{(t-s)(\nu\Delta - X)}|v - v^\nu(s)| = 0$, and hence

$$e^{(t-s)(\nu\Delta - X)}|v - v^\nu(s)| = e^{(t-s)\nu\Delta}|v - v^\nu(s)|, \tag{4.2.30}$$

so we have

$$\big|u^\nu(t,x) - e^{t(\nu\Delta - X)} f(x)\big| \le K \int_0^t e^{(t-s)\nu\Delta} |v - v^\nu(s)|\, ds, \tag{4.2.31}$$

which can be compared to (2.2.10)–(2.2.11). To be sure, results of Chapter 3 apply to the right side of (4.2.29), as we have seen in the analysis of (4.2.9), but the analysis of the right side of (4.2.31) is more elementary.

5. Further conclusions on plane parallel flows

This chapter contains further results pertaining to plane parallel flows in a channel. In Section 5.1 we generalize the analysis of the vanishing viscosity limit for plane parallel flows to include flows sheared by a moving boundary, translated at varying speed parallel to the x-axis. In Section 5.2 we consider more general boundary motions, parallel to the x-y-plane. We continue to assume (1.0.1)–(1.0.4) and we take the forcing $F = 0$.

5.1. ***Moving boundary, parallel to x-axis.*** We begin with the case in which both channel walls move with the same velocity $\alpha(t)$, that is, we take the vector field B in (1.0.2) of the form:

$$B(t, p) = (\alpha(t), 0, 0), \qquad p \in \partial\mathbb{O}. \tag{5.1.1}$$

Recall $\mathbb{O} = \mathbb{R}/\mathbb{Z} \times [0, 1]$. Since α is spatially constant, this is consistent with the assumption of periodicity in x. Later we extend the analysis to independent motion of the walls, in (5.1.47), and then extend it further in (5.2.1).

The goal is again to study the limit of vanishing viscosity and the corresponding boundary layer, assuming a rough boundary velocity α. The case of circularly symmetric flows in a rotating circle or annulus was studied in [Lopes Filho et al. 2007]. We follow the notation used there.

It is convenient to assume α is defined on the whole $\mathbb{R}$ but supported in $[0, \infty)$. If $\mathfrak{X}$ is a space of distributions on $\mathbb{R}$, we indicate with $\mathfrak{X}_b$ the space of elements of $\mathfrak{X}$ supported on $[0, \infty)$. We then take $\alpha \in \mathrm{BV}_b(\mathbb{R})$ or even $\alpha \in L^p_b(\mathbb{R})$. Since $C^\infty_b(\mathbb{R})$ is dense in these spaces ($p < \infty$), we can first pick $\alpha \in C^\infty_b$ and then use limiting arguments.

In order to highlight the effect of the moving boundary, we again take smooth initial data compatible with (1.0.4) and independent of ν, that is,

$$u^\nu(0, x, y, z) = (V(z), W(x, z), 0), \tag{5.1.2}$$

with $V \in C^\infty([0, 1])$ and $W \in C^\infty(\overline{\mathbb{O}})$. Here u^ν satisfies the system (1.0.8) with $f = g = 0$, which we repeat here for convenience:

$$\frac{\partial v^\nu}{\partial t} = \nu \frac{\partial^2 v^\nu}{\partial z^2}, \tag{5.1.3}$$

$$\frac{\partial w^\nu}{\partial t} + v^\nu \frac{\partial w^\nu}{\partial x} = \nu\Big(\frac{\partial^2 w^\nu}{\partial x^2} + \frac{\partial^2 w^\nu}{\partial z^2}\Big). \tag{5.1.4}$$

At the same time, since the inviscid flow does not see the moving boundary due to slip boundary conditions (see below), we do not impose compatibility of the initial data with the motion of the boundary, (i. e., in this context, we do not assume that $V(z) = \alpha(0)$ for $z = 0, 1$). Consequently, the viscous flow has an initial layer at $t = 0$.

As we will demonstrate, the vanishing viscosity limit in this context takes the form $u^\nu \to u^0$, where

$$u^0(t, x, y, z) = (v^0(t, z), w^0(t, x, z), 0), \tag{5.1.5}$$

is the solution of the Euler equations (1.0.15) again with $f = g = 0$, that is,

$$\frac{\partial v^0}{\partial t} = 0, \qquad \frac{\partial w^0}{\partial t} + v^0 \frac{\partial w^0}{\partial x} = 0. \tag{5.1.6}$$

Initial data are as in (5.1.2), so that

$$u^0(0, x, y, z) = (V(z), W(x, z), 0), \tag{5.1.7}$$

and the boundary conditions (1.0.12) are automatically satisfied in this case. In particular, the Euler flow is independent of the moving boundary and there is a boundary layer in the limit $\nu \to 0$.

As in [Lopes Filho et al. 2008; 2007], we pass to a frame moving with the boundary. Equivalently, we set

$$\bar{v}^\nu(t, z) = v^\nu(t, z) - \alpha(t), \qquad \bar{u}^\nu = (\bar{v}^\nu, w^\nu, 0). \tag{5.1.8}$$

We still assume $\alpha \in C_b^\infty(\mathbb{R})$, in particular $\alpha(0) = 0$. Then $\bar{u}^\nu$ must solve the following problem in $\mathbb{O}$:

$$\frac{\partial \bar{v}^\nu}{\partial t} = \nu \frac{\partial^2 \bar{v}^\nu}{\partial z^2} - \alpha'(t), \tag{5.1.9}$$

$$\frac{\partial w^\nu}{\partial t} + V \frac{\partial w^\nu}{\partial x} + (\bar{v}^\nu + \alpha(t) - V) \frac{\partial w^\nu}{\partial x} = \nu \Big(\frac{\partial^2 w^\nu}{\partial x^2} + \frac{\partial^2 w^\nu}{\partial z^2} \Big), \tag{5.1.10}$$

$$\bar{u}^\nu(t, x, z) = 0 \ \text{ on } \ \partial\mathbb{O}, \tag{5.1.11}$$

$$\bar{u}^\nu(0, x, z) = (V(z), W(x, z), 0). \tag{5.1.12}$$

By Duhamel's principle, the system above is equivalent to:

$$\bar{v}^\nu = e^{\nu t A} V(z) - \int_0^t [e^{\nu(t-s) A} \, 1] \, d\alpha(s), \tag{5.1.13}$$

$$w^\nu = e^{t(\nu \Delta - X)} W + \int_0^t e^{(t-s)(\nu \Delta - X)} [(V - \bar{v}^\nu - \alpha(s)) \, \partial_x w^\nu] \, ds. \tag{5.1.14}$$

The solution to the Euler system is given by

$$v^0(t, z) = V(z), \qquad t > 0, \ z \in [0, 1], \tag{5.1.15}$$

$$\begin{aligned} w^0(t, x, z) &= e^{-t X} \, W_0(x, z) \\ &= W(x - t\, V(z), z), \qquad t > 0, \ x \in \mathbb{R}/\mathbb{Z}, \ z \in [0, 1], \end{aligned} \tag{5.1.16}$$

as long as V and W are smooth enough.

We separate the contribution of the boundary conditions by writing (5.1.13) as

$$v^\nu(t) = \bar{v}^\nu(t) + \alpha(t) = e^{\nu t A} V(z) + \mathcal{S}^\nu \alpha(t), \quad \text{where } \mathcal{S}^\nu \alpha(t) := \int_0^t [(I - e^{\nu (t-s) A}) \, 1] \, d\alpha(s), \tag{5.1.17}$$

with the integral defined as a Bochner integral. As long as $\nu > 0$, we have

$$\mathcal{S}^\nu : C_b^\infty(\mathbb{R}) \to C_b^1(\mathbb{R}, C^\infty([0, 1])),$$

and in particular the boundary conditions are satisfied pointwise, since $e^{\nu s A} \, 1 \, \alpha(s)$ is continuous in s and vanishes at $z = 0, 1$ for $s \geq 0$. The trace at the boundary takes value in two copies of $C_b^1(\mathbb{R})$.

To treat less regular α, we observe that for α smooth (5.1.9) is equivalent to (5.1.3), so that $S^\nu\alpha$ is a classical solution of (5.1.3) with $V \equiv 0$. Therefore, the maximum principle for the heat equation gives

$$\mathcal{S}^\nu : C_b(\mathbb{R}) \to C_b\big(\mathbb{R}, C([0,1])\big) \subset L^2_{b,\mathrm{loc}}\big(\mathbb{R}, C([0,1])\big). \tag{5.1.18}$$

Next, we observe that if $\beta \in C_b^\infty(\mathbb{R})$ then

$$\alpha = \beta' \implies \mathcal{S}^\nu\alpha = \partial_t \mathcal{S}^\nu\beta.$$

so that

$$\mathcal{S}^\nu\partial_t = \partial_t\mathcal{S}^\nu : C_b^\infty(\mathbb{R}) \to C_b\big(\mathbb{R}, C^\infty([0,1])\big).$$

From (5.1.18) it follows that

$$\mathcal{S}^\nu\partial_t = \partial_t\mathcal{S}^\nu : C_b(\mathbb{R}) \to H^{-1}_{b,\mathrm{loc}}\big(\mathbb{R}, C([0,1])\big). \tag{5.1.19}$$

But each $\alpha \in L_b^{p'}(\mathbb{R})$, $p' \geq 1$, has the form $\alpha = \beta'$ with $\beta \in C_b(\mathbb{R})$, namely $\beta(t) = \int_{-\infty}^t \alpha(s)\,ds$. Hence

$$\mathcal{S}^\nu : L_b^{p'}(\mathbb{R}) \to H^{-1}_{b,\mathrm{loc}}\big(\mathbb{R}, C([0,1])\big), \tag{5.1.20}$$

for each $p' \geq 1$. Consequently we have the continuous linear map

$$\mathrm{Tr} \circ \mathcal{S}^\nu : L_b^{p'}(\mathbb{R}) \to \big(H^{-1}_{b,\mathrm{loc}}(\mathbb{R}) \oplus H^{-1}_{b,\mathrm{loc}}(\mathbb{R})\big), \tag{5.1.21}$$

By density, then, the boundary condition $v^\nu(t)|_{\partial\mathcal{O}} = \alpha$ in $H^{-1}_{b,\mathrm{loc}}(\mathbb{R})$ holds for any $\alpha \in L_b^{p'}(\mathbb{R})$, $p' \geq 1$ and also $\alpha \in \mathrm{BV}_b(\mathbb{R}) \subset L^1_b(\mathbb{R})$. The vanishing viscosity limit cannot hold in these spaces, which have good trace properties; in fact, we seek convergence as $\nu \to 0$ in $H^\sigma(\mathcal{O})$, $0 \leq \sigma < 1/2$, locally uniformly in t. Note that $L^2(\mathcal{O})$ is the energy space for solutions to the Euler system, but convergence in L^2-norm is relatively weak compared to the convergence results we are in a position to establish.

We first consider $\alpha \in \mathrm{BV}_b(\mathbb{R})$. Let X be a Banach space of functions on $[0,1]$ such that $1 \in X$ and $\{e^{tA} : t \geq 0\}$ is a strongly continuous semigroup on X. For example, $X = L^p([0,1])$, $1 \leq p < \infty$. More generally, we could take $X = H^{s,p}([0,1])$, with $p \in (1,\infty)$ and $s \in [0, 1/p)$. Recall that $\mathcal{S}^\nu\alpha$ is given explicitly in (5.1.17) for α smooth. By an approximation argument using mollifiers with support in $(0, 1/k)$, we can extend the validity of that expression to more singular α's (for details, we refer to [Lopes Filho et al. 2007], Proposition 2.1). We observe that the integral in (5.1.17) can be taken over $[0,t)$ or $[0,t]$, since the integrand vanishes at $s = t$.

Lemma 5.1.1. *If X is a space such as described in the previous paragraph, we have*

$$\mathcal{S}^\nu : \mathrm{BV}_b(\mathbb{R}) \to C_b(\mathbb{R}, X),$$

given by

$$\mathcal{S}^\nu\alpha(t) = \int_{I(t)} \big[(I - e^{\nu(t-s)A})1\big]\,d\alpha(s), \qquad I(t) = [0,t], \tag{5.1.22}$$

where the integral is a Lebesgue-Stieltjes-Bochner integral.

Formula (5.1.22) also implies the estimate

$$\|\mathcal{S}^\nu\alpha(t)\|_X \leq \|\alpha\|_{\mathrm{BV}([0,t])} \sup_{s\in[0,t]} \|e^{\nu s A} f_1 - f_1\|_X, \tag{5.1.23}$$

and, if $v^\nu(0) = V \in X$,

$$\|v^\nu(t) - V\|_X \le \|e^{\nu t A}V - V\|_X + \|\mathcal{S}^\nu \alpha(t)\|_X \to 0, \tag{5.1.24}$$

as $\nu \to 0$, which shows the zero-viscosity limit holds in the X-norm, for the v component of the velocity, in view of (5.1.15).

We next consider some rougher α, namely $\alpha \in L^{p'}$, for a certain range of p'. To begin, take $\alpha \in C_b^\infty(\mathbb{R})$, in particular $\alpha(0) = 0$, and integrate by parts in formula (5.1.22):

$$\int_0^t [e^{\nu(t-s)A} \cdot 1]\, d\alpha(s) = \alpha(t) - e^{\nu t A}\alpha(0) - \lim_{\epsilon \to 0} \int_0^{t-\epsilon} \nu\big(Ae^{\nu(t-s)A}\, 1\big)\, \alpha(s)\, ds,$$

using that $e^{\nu(t-s)A}\, 1 \in \mathfrak{D}(A)$, whenever $s < t$. The limit $\epsilon \to 0$ exists at least in $L^2([0, 1])$ and we write

$$\lim_{\epsilon \to 0} \int_0^{t-\epsilon} \nu\big(Ae^{\nu(t-s)A}\, 1\big)\, \alpha(s)\, ds = \int_0^t \nu(A\, e^{\nu(t-s)A}\, 1)\, \alpha(s)\, ds.$$

Equation (5.1.13) then becomes

$$\bar{v}^\nu = e^{\nu t A} V(z) - \alpha(t) + \int_0^t (\nu\, A\, e^{\nu(t-s)A}\, 1)\, \alpha(s)\, ds, \tag{5.1.25}$$

and

$$v^\nu = e^{\nu t A} V(z) + \int_0^t (\nu\, A\, e^{\nu(t-s)A}\, 1)\, \alpha(s)\, ds. \tag{5.1.26}$$

Consequently, to establish convergence of the v component of the velocity to the corresponding Euler solution in the limit $\nu \to 0$ it is enough to prove the last integral vanishes in the limit.

We observe that $e^{\nu t A}1$ and $\nu A\, e^{\nu t A}1$ can be explicitly computed using Fourier series. However, it is preferable to use Green's function methods as we are interested in the limit $\nu t \to 0$. To this end, we bring in the Sobolev spaces $H^\sigma([0, 1])$ with $0 \le \sigma < 1/2$. We recall the well-known interpolation estimate

$$[L^2(M), H_0^1(M)]_\sigma = \begin{cases} H_0^\sigma(M) & \text{if } \frac{1}{2} < \sigma \le 1, \\ H^\sigma(M) & \text{if } 0 \le \sigma < \frac{1}{2}, \end{cases} \tag{5.1.27}$$

where $M = [0, 1]$ or $M = \mathbb{O}$ here, which gives

$$\mathfrak{D}((-A)^{\sigma/2}) = H^\sigma([0, 1]) \quad \text{for } \sigma \in \big[0, \tfrac{1}{2}\big). \tag{5.1.28}$$

Hence, we first have uniformly in $t \in [0, T]$ for any $0 < T < \infty$,

$$e^{\nu t A}\, V \to V \text{ strongly in } H^\sigma([0, 1]), \quad \text{as } \nu \to 0. \tag{5.1.29}$$

We next observe as in [Lopes Filho et al. 2007, Equations 3.8–3.11] that

$$\begin{aligned}
\|\nu A e^{\nu s A}\, 1\|_{H^\sigma([0,1])} &\le C\|\nu(-A)^{1+\sigma/2} e^{\nu s A}\, 1\|_{L^2([0,1])} \\
&= C\|\nu(-A)^{1-(\tau-\sigma)/2} e^{\nu s A}(-A)^{\tau/2}\, 1\|_{L^2([0,1])} \\
&= C\nu^{(\tau-\sigma)/2} s^{(\tau-\sigma)/2-1}\|(-\nu s A)^{1-(\tau-\sigma)/2} e^{\nu s A}(-A)^{\tau/2}\, 1\|_{L^2([0,1])} \\
&\le C\nu^{(\tau-\sigma)/2} s^{(\tau-\sigma)/2-1}\|1\|_{H^\tau([0,1])}.
\end{aligned}$$

for $0 \le \sigma < \tau < 1/2$, so that by Hölder's inequality we have, with p' the conjugate exponent to p,

$$\begin{aligned}\int_0^t \|\nu A e^{\nu(t-s)A}\, 1\, \alpha(s)\|_{H^\sigma(D)}\, ds &\le \|\alpha\|_{L^{p'}([0,t])} \Big(\int_0^t \|\nu A e^{\nu s A}\, 1\|^p_{H^\sigma(D)}\, ds \Big)^{1/p} \\ &\le C_{p\sigma\tau}\, \nu^{(\tau-\sigma)/2}\, t^{(\tau-\sigma)/2-1+1/p} \|\alpha\|_{L^{p'}([0,t])} \|1\|_{H^\tau(D)}, \qquad (5.1.30)\end{aligned}$$

provided $1 \le p < 2/(2-(\tau-\sigma))$. For example, it is enough to have $p' > 4$. The same estimate holds for $\alpha \in L^{p'}_b(\mathbb{R})$ using a smooth approximation by convolutions.

Combining the estimates in (5.1.29) and (5.1.30), we obtain convergence of the v component of the velocity in the limit $\nu \to 0$ in the Sobolev space $H^\sigma([0,1])$. We record this result in a proposition.

Proposition 5.1.2. *Let $0 \le \sigma < \tau < 1/2$ and assume $\alpha \in L^{p'}_b(\mathbb{R})$ with $p' = \dfrac{p}{p-1}$ and $1 \le p < \dfrac{2}{2-(\tau-\sigma)}$. Then $\mathcal{S}^\nu \alpha(t) = \int_0^t (\nu\, A\, e^{\nu(t-s)\,A}\, 1)\, \alpha(s)\, ds$ defines a map*

$$\mathcal{S}^\nu : L^{p'}_b(\mathbb{R}) \to C_b(\mathbb{R}, H^\sigma([0,1])),$$

satisfying estimate (5.1.30). Furthermore, uniformly in $t \in [0,T]$ for any $0 < T < \infty$,

$$v^\nu \to v^0 \text{ strongly in } H^\sigma([0,1]), \quad \text{as } \nu \to 0. \qquad (5.1.31)$$

Having settled the analysis of the first Equation (5.1.3), we now turn to Equation (5.1.4) in its mild fomulation (5.1.14), which we solve as a fixed-point problem, but first we record some useful *a priori* estimates.

We denote again $\partial_x^k w^\nu$ by w_k^ν, $k \in \mathbb{Z}_+$. Since α depends only on t and $\bar{v}^\nu$ depend only on t, z, the same arguments as in (2.1.9) – (2.1.12) gives that w_k^ν also solves (5.1.4). Integrating by parts in that equation, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w^\nu\|^2_{L^2(\mathbb{O})} &+ \int_0^1\int_0^{2\pi} \Big[(\bar{v}^\nu(t,z)+\alpha(t))\frac{\partial}{\partial x}\frac{|w^\nu(t,x,z)|^2}{2}\Big]\, dx\, dz \\ &= \frac{1}{2}\frac{d}{dt}\|w^\nu\|^2_{L^2(\mathbb{O})} + \int_0^1\int_0^{2\pi} (\bar{v}^\nu(t,z)+\alpha(t))\Big[\frac{|w^\nu(t,x,z)|^2}{2}\Big]_0^{2\pi} dz \\ &= \frac{1}{2}\frac{d}{dt}\|w^\nu\|^2_{L^2(\mathbb{O})} = -\nu\|\nabla w^\nu\|^2_{L^2(\mathbb{O})} \le 0,\end{aligned}$$

using periodicity in x. Therefore

$$\|w_k^\nu(t)\|_{L^2(\mathbb{O})} \le \|\partial_x^k W\|_{L^2(\mathbb{O})}. \qquad (5.1.32)$$

On the other hand the maximum principle gives

$$\|w_k^\nu(t)\|_{L^\infty(\mathbb{O})} \le \|\partial_x^k W\|_{L^\infty(\mathbb{O})}. \qquad (5.1.33)$$

These estimates continue to hold for $\alpha \in \mathrm{BV}$ or $L^{p'}$ $(1 \le p' < +\infty)$ by approximation with smooth functions.

We write (5.1.14) as $w^\nu(t) = e^{t(\nu\Delta - X)} W(t) + \mathfrak{F}^\nu(w^\nu)(t)$, where

$$\mathfrak{F}^\nu(t, V, \alpha, v)(w) = \mathfrak{F}^\nu(w)(t) = \int_0^t e^{(t-s)(\nu\Delta - X)}[(V - \bar{v}^\nu - \alpha(s))\,\partial_x w(s)]\, ds. \qquad (5.1.34)$$

To establish the existence of a unique solution to (5.1.14), it is enough to prove that $\mathfrak{F}^\nu$ is a contraction in $L^\infty([0,T], L^2(\mathbb{O}))$, T small enough, since then continuation of the solution follows from the uniform estimate (5.1.32).

We observe first that Proposition 5.1.2 and the Sobolev embedding implies that

$$V - \bar{v} - \alpha \in L^{p'}([0,T], L^q([0,1]))$$

for any $1 \le q < \infty$. Furthermore, at fixed viscosity, given that V is smooth and bounded on $[0,1]$ with all its derivatives, a scaling argument gives

$$\|e^{t\,(\nu\Delta - X)} f\|_{H^1(\mathbb{O})} \le C_{\nu,V}\, t^{-(1/r-1/2)-1/2}\, \|f\|_{L^r(\mathbb{O})}, \tag{5.1.35}$$

if $1 \le r \le 2$, $0 < t \le 1$. We apply this estimate below with $1/r = 1/q + 1/2$, q large, so that $r > 1$.

Let $|||w||| = \|w\|_{L^\infty([0,T],L^2(\mathbb{O}))}$. Then, from (5.1.35),

$$\begin{aligned}|||\mathfrak{F}(w) - \mathfrak{F}(w')||| &\le C_{\nu,V} \int_0^T (t-s)^{-1/r} \|(V - \bar{v}^\nu(s) - \alpha(s))\,(w(s) - w'(s)\|_{L^r(\mathbb{O})}\, ds \\ &\le C_{\nu,V} \int_0^T (t-s)^{-1/r} \|V - \bar{v}^\nu(s) - \alpha(s)\|_{L^q([0,1])} \|w(s) - w'(s)\|_{L^2(\mathbb{O})}\, ds \\ &\le C_{\nu,V} T^{1/p-1/r} \|V - \bar{v} - \alpha\|_{L^{p'}([0,T],L^q([0,1]))}\, |||w - w'|||,\end{aligned} \tag{5.1.36}$$

using that $V - \bar{v}^\nu - \alpha$ commutes with ∂_x. This estimate holds provided $p < r$, where p is the conjugate exponent to p' and $1/r = 1/q + 1/2$. If $p' > 4$, we can find such an $r > p > 4/3$ by choosing $q > 4$ in (5.1.35). The estimate above gives that $\mathfrak{F}$ is a strict contraction on $L^\infty([0,T], L^2(\mathbb{O}))$ if T is sufficiently small. We therefore have existence and uniqueness of solutions to (5.1.4) in $L^\infty([0,T], L^2(\mathbb{O}))$, and hence in $L^\infty([0,\infty), L^2(\mathbb{O}))$ thanks to (5.1.32). Furthermore, since w_k^ν satisfies the same equation for all $k \in \mathbb{Z}_+$, w_k^ν is the unique solution to (5.1.14) in $L^\infty([0,\infty), L^2(\mathbb{O}))$ and we conclude that $w^\nu \in L^\infty([0,\infty), \mathcal{V}^k(\mathbb{O}))$ for all $k \in \mathbb{Z}_+$. Also w^ν is smooth in x, z for $t > 0$, and satifies the boundary condition $w^\nu \equiv 0$ on $\partial\mathbb{O}$ pointwise.

We now turn to the analysis of the vanishing viscosity limit $w^\nu \to w^0$ as $\nu \to 0$. For this analysis, we rely on the results in Section 3.1 on the behavior of the semigroup $e^{t(\nu\Delta - X)}$ as $\nu \to 0$. In view of (5.1.16), we can write

$$(w^\nu - w^0)(t,x,z) = [e^{t(\nu\Delta - X)} - e^{-tX}]W(x,z) + R^\nu(t,x,z),$$

where

$$R^\nu(t,x,z) = \int_0^t e^{(t-s)(\nu\,\Delta - X)}[(V(z) - \bar{v}^\nu(s,z) - \alpha(s))\,\partial_x w^\nu(s,x,z)]\,ds.$$

We estimate the easier term $R^\nu(t,x,z)$ first. This can be done exactly as in (2.2.11), using (5.1.33) and the positivity of the kernel of $e^{t(\nu\Delta - X)}$:

$$\begin{aligned}|R^\nu(t,x,z)| &\le C\,\|\partial_x W\|_{L^\infty(\mathbb{O})} \int_0^t e^{(t-s)(\nu\Delta - X)}\,|V(z) - \bar{v}^\nu(s,z) - \alpha(s)|\,ds \\ &= C\,\|\partial_x W\|_{L^\infty(\mathbb{O})} \int_0^t e^{(t-s)\nu\Delta}|V(z) - v^\nu(s,z)|\,ds,\end{aligned} \tag{5.1.37}$$

where the equality follows since $V - v^\nu$ is independent of x. Next, since $V - v^\nu \to 0$ strongly in $L^q([0, 1])$, $1 \le q < \infty$, uniformly in $t \in [0, T]$ from (5.1.30), and $e^{\nu(t-s)\Delta}$ is uniformly bounded in t and ν on $L^q(\mathbb{O})$, we conclude

$$R^\nu(x, z, t) \to 0 \text{ strongly in } L^q(\mathbb{O}) \text{ uniformly in } t \in [0, T], \text{ as } \nu \to 0. \tag{5.1.38}$$

In fact, when $q = 2$ and $V = 0$, the estimate (5.1.30) gives also an upper bound for the rate of convergence:

$$\sup_{0 \le t \le T} \|R^\nu(\cdot, t)\|_{L^2(\mathbb{O})} \le C_V\, \nu^{\tau/2}\, T^{\tau/2+2-1/p} \|1\|_{H^\tau([0,1])} \|\alpha\|_{L^{p'}([0,T])}, \tag{5.1.39}$$

with again $p = p'/(p' - 1)$, $0 < \tau < 1/2$. In the case $p = \infty$, we get a rate consistent with estimate (2.1.21) for $\alpha = 0$. We now turn to the more delicate term $[e^{t(\nu\Delta - X)} - e^{-tX}]W(x, z)$ for which we directly use Proposition 4.3 to conclude:

$$[e^{t(\nu\Delta - X)} - e^{-tX}]W \to 0 \text{ strongly in } L^q(\mathbb{O}) \text{ uniformly in } t \in [0, T], \tag{5.1.40}$$

as $\nu \to 0$. Putting together (5.1.40) and (5.1.38) we obtain convergence in $L^q(\mathbb{O})$ of the w component of the velocity in the vanishing viscosity limit, and hence of the Navier–Stokes solution to the Euler solution.

Proposition 5.1.3. *Let $\alpha \in L^{p'}_b(\mathbb{R})$, $p' > 4$. Let $u^\nu = (v^\nu, w^\nu)$ be the solution of the Navier–Stokes system (5.1.3)–(5.1.4) with initial condition (5.1.2) and boundary conditions (5.1.1). Let u^0 be the solution of the Euler system (5.1.6) with the same initial condition, given by formulas (5.1.15)-(5.1.16). Then, as $\nu \to 0$,*

$$u^\nu(t) \to u^0(t) \text{ strongly in } L^q(\mathbb{O}), \qquad \forall q \in [1, \infty),$$

locally uniformly in $t \in [0, \infty)$.

Exploiting the analysis of Section 3.2 yields convergence in higher norms in the interior. Recall that v^ν is given by formula (5.1.25), and w^ν by formula (5.1.14) respectively. Below, v^0 and w^0 are the components of the Euler solution, given respectively by (5.1.15) and (5.1.16). Let the set Ω_j be defined as in Proposition 3.2.1, i. e., $\Omega_1 \Subset \Omega_0 \Subset \mathbb{O}$. Projecting along the z-direction we then have two maximal intervals $I_1 \subset\subset I_0 \Subset [0, 1]$.

Lemma 5.1.4. *Let $k \in \mathbb{N}$ and fix $0 < T < \infty$. Then v^ν defined in (5.1.25) belongs to $C^\infty([0, T], H^k(I_1))$ and*

$$v^\nu \to V = v^0 \text{ in } L^\infty([0, T], H^k(I_1)), \quad \text{as } \nu \to 0. \tag{5.1.41}$$

Proof. The limit $e^{tA} f \to f$ as $t \to 0$ in $H^k(I_1) \cap L^2([0, 1])$ follows easily from the explicit formula for the Green's function. Since $V \in C^\infty(\bar{\mathbb{O}})$, we immediately have $e^{\nu t A} V \to V$ as $\nu \to 0$ in $H^k(I_1)$, $\forall k \in \mathbb{N}$. We also have $e^{\nu t A} 1 \to 1$ in $L^\infty([0, T], H^k(I_1))$ as $\nu \to 0$, so that

$$\lim_{\nu \to 0} \mathcal{S}^\nu(\alpha) = 0, \qquad \text{in } L^\infty([0, T], H^k(I_1)),$$

since $\mathcal{S}^\nu \alpha(t) = \displaystyle\int_0^t (\nu A\, e^{\nu(t-s)A}\, 1)\, \alpha(s)\, ds.$ ∎

From the Lemma, proceeding as in the proof of Proposition 3.2.3, we obtain

$$v^\nu \to V = v^0 \quad \text{as } \nu \to 0, \text{ uniformly on } I_1 \text{ for } t \in [0, T]. \tag{5.1.42}$$

The method of images yields more precise estimates. In fact, from (2.1.20), when $\alpha \in \mathrm{BV}_b(\mathbb{R})$

$$\left|\mathcal{S}^\nu \alpha(z,t)\right| = \left|\int_0^t [1 - e^{\nu(t-s)A}\, 1]\, d\alpha(s)\right| \le C^T\, \|\alpha\|_{TV([0,t])} \sup_{0\le s\le t} \varphi\big((\nu s)^{-1/2}\delta(z)\big), \tag{5.1.43}$$

for $t \in [0,T]$, where $\delta(z) = \mathrm{dist}(z,\{0,1\})$ and $\varphi(\zeta)$ is rapidly decreasing as $\zeta \to \infty$. Similarly, if $\alpha \in L^p_b(\mathbb{R})$, $1 \le p \le \infty$,

$$\left|\mathcal{S}^\nu \alpha(z,t)\right| = \left|\int_0^t (\nu\, \Delta\, e^{\nu(t-s)A}\, 1)\alpha(s)\, ds\right| \le C^T\, \|\alpha\|_{L^1(\mathbb{R})} \sup_{0\le s\le t} \psi\big((s\nu)^{-1/2}\delta(z)\big), \tag{5.1.44}$$

where $\psi(\zeta)$ vanishes at 0 and is rapidly decreasing as $\zeta \to \infty$.

Next, we address convergence of w^ν.

Lemma 5.1.5. *Fix $0 < T < \infty$. Then w^ν defined in (5.1.14) belongs to $C^\infty([0,T], C(\Omega_1))$ and*

$$w^\nu \to w^0 \quad \textit{as } \nu \to 0, \textit{ uniformly on } \Omega_1 \textit{ for } t \in [0,T]. \tag{5.1.45}$$

Proof. We first observe that, since $e^{\nu t\,\Delta}$ is uniformly bounded in $L^\infty(\mathbb{O})$ (though not strongly continuous), estimate (5.1.37) together with (5.1.42) implies

$$R^\nu(t,x,z) \to 0 \quad \text{as } \nu \to 0, \text{ uniformly on } \Omega_1 \text{ for } t \in [0,T]. \tag{5.1.46}$$

Therefore, it is enough to show that $[e^{t(\nu\Delta - X)} - e^{-tX}]W(x,z) \to 0$ uniformly as $\nu \to 0$. In fact, it is equivalent to show

$$e^{tX}\, e^{t(\nu\Delta - X)} W(x,z) \to W(x,z),$$

given that e^{tX} is an isometry. This result then follows from Proposition 3.2.3 (via (3.2.1)). □

We combine the two lemmas in a proposition (see also Proposition 4.2.3).

Proposition 5.1.6. *In the setting of Proposition 5.1.3, let $\Omega_1 \Subset \Omega_0 \Subset \mathbb{O}$. Then, as $\nu \to 0$,*

$$u^\nu(t,x,z) \to u^0(t,x,z) \quad \textit{uniformly in } (x,z) \in \Omega_1, \quad t \in [0,T].$$

If α is sufficiently regular, then it follows from (2.1.20) and (5.1.25) that $X_\nu = v^\nu(t,z)\partial_x \in \widehat{\mathfrak{X}}_1$ and hence the results in § 3.7 can be applied to w_ν to obtain a more detailed analysis in the boundary layer.

We now generalize the setting to allow for the two channel walls to move with different velocities, that is, we replace the boundary condition (5.1.1) with:

$$(v^\nu(t,j), w^\nu(t,x,j), 0) = (\alpha_j(t), 0, 0), \ x \in \mathbb{R}/\mathbb{Z},\ t > 0,\ j \in \{0,1\}. \tag{5.1.47}$$

It is straightforward to extend the results derived above to this case. We begin by replacing (5.1.8) with

$$\bar{v}^\nu(t,z) = v^\nu(t,z) - \Phi(t,z), \qquad \bar{u}^\nu = (\bar{v}^\nu, w^\nu, 0), \tag{5.1.48}$$

where Φ is given by

$$\Phi(t,z) = [\alpha_1(t) - \alpha_0(t)]z + \alpha_0(t). \tag{5.1.49}$$

Note that Φ solves

$$\begin{aligned} \partial_z^2 \Phi(t,\cdot) &= 0 \qquad \text{on } [0,1], \\ \Phi(t,0) &= \alpha_0(t), \quad t>0, \\ \Phi(t,1) &= \alpha_1(t), \quad t>0. \end{aligned}$$

Formula (5.1.17) is then replaced by

$$\begin{aligned} v^\nu(t) &= e^{\nu t A}V + \mathcal{S}^\nu(\alpha_0,\alpha_1)(t), \\ \mathcal{S}^\nu(\alpha_0,\alpha_1)(t,z) &= \int_{[0,t]} [(I - e^{\nu(t-s)A})\partial_s \Phi(s,z)]\,ds \\ &= \int_{[0,t]} [(I - e^{\nu(t-s)A})(1-z)]\,d\alpha_0(s) + \int_{[0,t]} [(I - e^{\nu(t-s)A})z]\,d\alpha_1(s). \end{aligned} \tag{5.1.50}$$

Integrating by parts we can obtain the analog of (5.1.25). Estimates analogous to those done above on $\mathcal{S}^\nu\alpha(t)$ are readily verified.

5.2. ***Moving boundary, parallel to the x-y-plane.*** In this section, we take a look at the following more general motion of $\partial\mathcal{O}$, namely

$$B(t,x,z) = (\alpha_j(t), \beta_j(t), 0), \quad z = j \in \{0,1\}. \tag{5.2.1}$$

Most of the techniques have been developed in Section 5.1, so we will be brief. First note that allowing β_j to be nonzero has no effect on the component $v^\nu(t,z)$, and (5.1.50) continues to hold.

Let us analyze the effect on $w^\nu(t,x,z)$. Take $\beta_j \in C_b^\infty(\mathbb{R})$ to start (though later we can extend to $\beta_j \in \mathrm{BV}_b(\mathbb{R})$). Set

$$\Psi(t,z) = [\beta_1(t) - \beta_0(t)]z + \beta_0(t). \tag{5.2.2}$$

We see that

$$\overline{w}^\nu(t,x,z) = w^\nu(t,x,z) - \Psi(t,z) \tag{5.2.3}$$

vanishes on $\partial\mathcal{O}$ and satisfies

$$\partial_t \overline{w}^\nu + v^\nu \partial_x \overline{w}^\nu = \nu\Delta\overline{w}^\nu - \partial_t\Psi, \quad \overline{w}^\nu(0,x,z) = W(x,z). \tag{5.2.4}$$

Hence, with $X = V\partial_x$,

$$\begin{aligned} \overline{w}^\nu(t,x,z) = e^{t(\nu\Delta - X)}W(x,z) &+ \int_0^t e^{(t-s)(\nu\Delta - X)}(V - v^\nu(s,z))\partial_x \overline{w}^\nu(s,x,z)\,ds \\ &- \int_0^t e^{(t-s)(\nu\Delta - X)}\partial_s\Psi(s,z)\,ds, \end{aligned} \tag{5.2.5}$$

so, making use of the fact that $\Psi(s,z)$ is independent of x, we obtain

$$\begin{aligned} w^\nu(t,x,z) = e^{t(\nu\Delta - X)}W(x,z) &+ \int_0^t e^{(t-s)(\nu\Delta - X)}(V - v^\nu(s,z))\partial_s w^\nu(s,x,z)\,ds \\ &+ \int_0^t (I - e^{(t-s)\nu\Delta})\partial_s\Psi(s,z)\,ds. \end{aligned} \tag{5.2.6}$$

One can write the last integral as

$$\int_0^t (I - e^{(t-s)\nu\Delta})(1-z)\,d\beta_0(s) + \int_0^t (I - e^{(t-s)\nu\Delta})z\,d\beta_1(s). \tag{5.2.7}$$

Previously developed techniques apply to (5.2.6)–(5.2.7).

Finally, we draw further conclusions when (5.2.1) is specialized to

$$B(t,x,z) = (0, \beta_j(t), 0), \quad z = j \in \{0,1\}. \tag{5.2.8}$$

In such a case, $v^\nu(t,z)$ is as in Chapters 3–4. Consequently, (5.2.4) is

$$\partial_t \overline{w}^\nu = (\nu\Delta - X_\nu)\overline{w}^\nu - \partial_t \Psi, \tag{5.2.9}$$

with initial data $w^\nu(0,x,z) = W(x,z)$, boundary data 0 on $\partial\mathbb{O}$, and with X_ν exactly as in Section 2.2. Hence the results of Chapter 4 apply. We have

$$\overline{w}^\nu(t,x,z) = \Sigma_\nu^{0,t} W(x,z) - \int_0^t \Sigma_\nu^{s,t}\, \partial_s \Psi(s,z)\,ds, \tag{5.2.10}$$

where $\Sigma_\nu^{s,t}$ is the solution operator to

$$\partial_t u = (\nu\Delta - X_\nu)u, \quad u\big|_{\mathbb{R}^+ \times \partial\mathbb{O}} = 0, \tag{5.2.11}$$

i.e., $u(t) = \Sigma_\nu^{s,t} u(s)$ for $0 \le s < t$. Hence

$$w^\nu(t,x,z) = \Sigma_\nu^{0,t} W(x,z) + \int_0^t (I - \Sigma_\nu^{s,t})\partial_s \Psi(s,z)\,ds, \tag{5.2.12}$$

and we can write the last integral as

$$\int_0^t (I - \Sigma_\nu^{s,t})(1-z)\,d\beta_0(s) + \int_0^t (I - \Sigma_\nu^{s,t})z\,d\beta_1(s). \tag{5.2.13}$$

Results of Chapter 4 then give convergence

$$w^\nu(t) \to w^0(t) \tag{5.2.14}$$

in various function spaces, including $\mathcal{V}^k(\mathbb{O})$.

Obtaining such convergence in the context of (5.2.1) would require some extra hypotheses on $\alpha_j(t)$, which we will not pursue here.

Appendix A. $\mathcal{V}^k(\mathbb{O})$ and b-Sobolev spaces

We take $\overline{\mathbb{O}}$ to be a compact Riemannian manifold with smooth boundary. Recall from (3.3.1) the definition

$$\mathcal{V}^k(\mathbb{O}) = \{u \in L^2(\mathbb{O}) : Y_1 \cdots Y_j u \in L^2(\mathbb{O}),\ \forall\, j \le k,\, Y_\ell \in \mathfrak{X}^1\}, \tag{A.0.1}$$

for $k \in \{0,1,2,\dots\}$, where

$$\mathfrak{X}^1 = \{Y \text{ smooth vector field on } \overline{\mathbb{O}} : Y \parallel \partial\mathbb{O}\}. \tag{A.0.2}$$

These spaces are special cases of weighted b-Sobolev spaces, introduced and studied in [Melrose 1993] (see also [Melrose 1996]). Here we discuss this matter and draw some conclusions that are useful in Sections 3.3 and 4.2.

The manifold $\mathbb{O}$ carries a complete Riemannian metric, called a "b-metric," which on a collar neighborhood of $\partial\mathbb{O}$, identified with $[0,1)\times\partial\mathbb{O}$ (with $\{0\}\times\partial\mathbb{O}$ identified with $\partial\mathbb{O}\subset\overline{\mathbb{O}}$) has the form

$$g=\left(\frac{dy}{y}\right)^2+h, \tag{A.0.3}$$

where h is a smooth metric tensor on $\partial\mathbb{O}$ and y the parameter on $[0,1)$. We use the symbol $\widetilde{\mathbb{O}}$ to denote $\mathbb{O}$ as a Riemannian manifold with such a Riemannian metric. The b-Sobolev spaces $H^k_b(\mathbb{O})$ are defined by

$$H^k_b(\mathbb{O})=\{u\in L^2_b(\mathbb{O}):Y_1\cdots Y_j u\in L^2_b(\mathbb{O}),\ \forall\, j\le k,\, Y_\ell\in\mathfrak{X}^1\}, \tag{A.0.4}$$

where $\mathfrak{X}^1$ is as in (A.0.2) and

$$L^2_b(\mathbb{O})=L^2(\widetilde{\mathbb{O}}). \tag{A.0.5}$$

Different choices of b-metrics on $\mathbb{O}$ give the same spaces, with equivalent norms. To define weighted b-Sobolev spaces, take a defining function ρ for $\partial\mathbb{O}$, i.e., $\rho\in C^\infty(\overline{\mathbb{O}})$, $\rho>0$ on $\mathbb{O}$, $\rho=0$ on $\partial\mathbb{O}$, $\nabla\rho(x)\ne 0$, $\forall\, x\in\partial\mathbb{O}$. Thus, for $s\in\mathbb{R}$, set

$$\rho^s H^k_b(\mathbb{O})=\{\rho^s u:u\in H^k_b(\mathbb{O})\}. \tag{A.0.6}$$

An inductive argument shows that

$$\rho^s H^k_b(\mathbb{O})=\{u\in\rho^s L^2_b(\mathbb{O}):Y_1\cdots Y_j u\in\rho^s L^2_b(\mathbb{O}),\ \forall\, j\le k,\, Y_\ell\in\mathfrak{X}^1\}. \tag{A.0.7}$$

We also have

$$L^2(\mathbb{O})=\rho^{-1/2}L^2_b(\mathbb{O}). \tag{A.0.8}$$

Hence

$$\mathcal{V}^k(\mathbb{O})=\rho^{-1/2}H^k_b(\mathbb{O}). \tag{A.0.9}$$

Remark. The use of "b" as a subscript in names of function spaces is different in this appendix than it was in Chapter 5. We trust this warning will forestall confusion.

A.1. *Interpolation identities.* This identity (A.0.9) is of use in establishing the following result, which is valuable in §Section 3.3 and 4.2.

Proposition A.1.1. *If $0<k<\ell$ and $k=\ell\theta$, then*

$$[L^2(\mathbb{O}),\mathcal{V}^\ell(\mathbb{O})]_\theta=\mathcal{V}^k(\mathbb{O}), \tag{A.1.1}$$

where the left side is the complex interpolation space.

In light of (A.0.9), this follows straight away from:

Proposition A.1.2. *If $0<k<\ell$ and $k=\ell\theta$, then*

$$[L^2_b(\mathbb{O}),H^\ell_b(\mathbb{O})]_\theta=H^k_b(\mathbb{O}). \tag{A.1.2}$$

In turn, Proposition A.1.2 can be proven by identifying $H^k_b(\mathbb{O})$ with a regular Sobolev space of functions on the complete Riemannian manifold $\widetilde{\mathbb{O}}$. (Thanks to R. Mazzeo for pointing this out.) In detail, we set

$$H^k(\widetilde{\mathbb{O}}) = \{u \in L^2(\widetilde{\mathbb{O}}) : \nabla^j u \in L^2(\widetilde{\mathbb{O}}),\ \forall\, j \le k\}, \tag{A.1.3}$$

where a priori $\nabla^j u$ is a distributional section of $\otimes^j T^*\widetilde{\mathbb{O}}$, whose fiber $\otimes^j T^*_x\widetilde{\mathbb{O}}$ inherits an inner product from that of $T_x\widetilde{\mathbb{O}}$ given by the complete Riemannian metric tensor on $\widetilde{\mathbb{O}}$ described above. Since the Riemannian manifold $\widetilde{\mathbb{O}}$ considered here, arising from $\overline{\mathbb{O}}$ via a b-metric, has special structure as a Riemannian manifold with bounded geometry, we can give a convenient alternative characterization of $H^k(\widetilde{\mathbb{O}})$, as follows. There exist $K \in \mathbb{N}$ and smooth maps from the closed unit ball $\overline{B}_1 \subset \mathbb{R}^n$ into $\widetilde{\mathbb{O}}$ ($n = \dim \widetilde{\mathbb{O}}$)

$$\varphi_\nu : \overline{B}_1 \to \widetilde{\mathbb{O}}, \tag{A.1.4}$$

with the following properties:

$$\begin{aligned}
&\varphi_\nu \ \text{ is a diffeomorphism of } \ \overline{B}_1 \ \text{ onto its image;}\\
&\{\varphi_\nu^* g\} \ \text{ is a } C^\infty \text{ bounded family of metric tensors on } \overline{B}_1;\\
&\{\varphi_\nu(B_{1/2})\} \text{ covers } \widetilde{\mathbb{O}};\\
&\text{each } p \in \widetilde{\mathbb{O}} \text{ is contained in at most } K \text{ sets } \varphi_\nu(\overline{B}_1).
\end{aligned} \tag{A.1.5}$$

Given a function $u \in L^1_{\mathrm{loc}}(\widetilde{\mathbb{O}})$, set

$$u_\nu = \varphi_\nu^* u \in L^1(B_1). \tag{A.1.6}$$

Then

$$H^k(\widetilde{\mathbb{O}}) = \Big\{u \in L^2(\widetilde{\mathbb{O}}) : \sum_\nu \sum_{|\alpha|\le k} \|D^\alpha u_\nu\|^2_{L^2(B_1)} < \infty\Big\}. \tag{A.1.7}$$

Note also that

$$u \in H^k(\widetilde{\mathbb{O}}) \Leftrightarrow \sum_\nu \sum_{|\alpha|\le k} \|D^\alpha u_\nu\|^2_{L^2(B_{1/2})} < \infty. \tag{A.1.8}$$

An examination of the behavior of elements of $\mathfrak{X}^1$ when pushed forward to B_1 via φ_ν establishes:

Proposition A.1.3. *For $k \in \mathbb{Z}^+$,*

$$H^k_b(\mathbb{O}) = H^k(\widetilde{\mathbb{O}}). \tag{A.1.9}$$

Hence (A.1.2) follows from the result that

$$[L^2(\widetilde{\mathbb{O}}), H^\ell(\widetilde{\mathbb{O}})]_\theta = H^k(\widetilde{\mathbb{O}}). \tag{A.1.10}$$

To establish this, it is convenient to bring in the Laplace-Beltrami operator of $\widetilde{\mathbb{O}}$, which we denote L. This is defined as an unbounded operator on $L^2(\widetilde{\mathbb{O}})$ via the Friedrichs construction:

$$u \in \mathfrak{D}(L) \ \text{ and } \ Lu = f \iff u \in H^1(\widetilde{\mathbb{O}}) \text{ and } (\nabla u, \nabla g)_{L^2(\widetilde{\mathbb{O}})} = -(f, g)_{L^2(\widetilde{\mathbb{O}})},\ \forall\, g \in H^1(\widetilde{\mathbb{O}}). \tag{A.1.11}$$

The fact that $\widetilde{\mathbb{O}}$ is complete implies L is a negative self adjoint operator and $C_0^\infty(\widetilde{\mathbb{O}})$ is dense in the domain of all powers of L, defined inductively by

$$u \in \mathfrak{D}(L^{k+1}) \implies u \in \mathfrak{D}(L) \text{ and } Lu \in \mathfrak{D}(L^k). \tag{A.1.12}$$

Compare [Chernoff 1973]. More generally, for each $s \in [0,\infty)$, $(-L)^s$ is defined via the spectral theorem as a positive self adjoint operator, and one has the classical interpolation identity

$$[L^2(\widetilde{\mathcal{O}}), \mathfrak{D}((-L)^s)]_\theta = \mathfrak{D}((-L)^{s\theta}). \tag{A.1.13}$$

Hence the identity (A.1.10) is a consequence of:

Proposition A.1.4. *For $k \in \mathbb{N}$,*

$$H^k(\widetilde{\mathcal{O}}) = \mathfrak{D}((-L)^{k/2}). \tag{A.1.14}$$

Proof. That

$$\mathfrak{D}((-L)^{1/2}) = H^1(\widetilde{\mathcal{O}}) \tag{A.1.15}$$

is a fundamental property of the Friedrichs construction. Next, from (A.1.11) we have

$$\mathfrak{D}(L) = \{u \in H^1(\widetilde{\mathcal{O}}) : Lu \in L^2(\widetilde{\mathcal{O}})\}, \tag{A.1.16}$$

where Lu is a priori a distribution on $\widetilde{\mathcal{O}}$. Clearly $H^2(\widetilde{\mathcal{O}}) \subset \mathfrak{D}(L)$. We can use the interior elliptic estimates

$$\sum_{|\alpha|\le 2} \|D^\alpha u\|^2_{L^2(B_{1/2})} \le C\Big(\|u\|^2_{L^2(B_1)} + \|L_\nu u\|^2_{L^2(B_1)}\Big), \tag{A.1.17}$$

with L_ν the image of L on B_1 via φ_ν. The estimate (A.1.17) holds with C independent of ν. We use this together with the equivalence of (A.1.7) and (A.1.8), to obtain the reverse inclusion, hence

$$\mathfrak{D}(L) = H^2(\widetilde{\mathcal{O}}). \tag{A.1.18}$$

To continue, we note that (A.1.17) extends to

$$\sum_{|\alpha|\le 2k} \|D^\alpha u\|^2_{L^2(B_{1/2})} \le C_k\Big(\|u\|^2_{L^2(B_1)} + \|L^k_\nu u\|^2_{L^2(B_1)}\Big), \tag{A.1.19}$$

again with C_k independent of ν, and this together with (A.1.7)–(A.1.8) gives

$$\{u \in H^1(\widetilde{\mathcal{O}}) : L^k u \in L^2(\widetilde{\mathcal{O}})\} \subset H^{2k}(\widetilde{\mathcal{O}}). \tag{A.1.20}$$

By comparison, the definition (A.1.12) says

$$\mathfrak{D}(L^k) = \{u \in H^1(\widetilde{\mathcal{O}}) : Lu \in \mathfrak{D}(L^{k-1})\}. \tag{A.1.21}$$

The right side of (A.1.21) is contained in the left side of (A.1.20). On the other hand, if we know that $\mathfrak{D}(L^{k-1}) = H^{2k-2}(\widetilde{\mathcal{O}})$, it readily follows that $H^{2k}(\widetilde{\mathcal{O}}) \subset \mathfrak{D}(L^k)$. Hence it follows inductively that

$$\mathfrak{D}(L^k) = H^{2k}(\widetilde{\mathcal{O}}). \tag{A.1.22}$$

To complete the proof of (A.1.14), we use

$$\mathfrak{D}((-L)^{k+1/2}) = \{u \in \mathfrak{D}(L^k) : L^k u \in \mathfrak{D}((-L)^{1/2})\} = \{u \in H^{2k}(\widetilde{\mathcal{O}}) : L^k u \in H^1(\widetilde{\mathcal{O}})\}, \tag{A.1.23}$$

and the interior regularity estimate

$$\sum_{|\alpha|\le 2k+1} \|D^\alpha u\|^2_{L^2(B_{1/2})} \le C_k\Big(\|u\|^2_{L^2(B_1)} + \|L^k_\nu u\|^2_{H^1(B_1)}\Big). \tag{A.1.24}$$

This proves Proposition A.1.4, and hence Propositions A.1.1–A.1.3. □

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NONCOMMUTATIVE VARIATIONS ON LAPLACE'S EQUATION

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As a first step toward developing a theory of noncommutative nonlinear elliptic partial differential equations, we analyze noncommutative analogues of Laplace's equation and its variants (some of them nonlinear) over noncommutative tori. Along the way we prove noncommutative analogues of many results in classical analysis, such as Wiener's Theorem on functions with absolutely convergent Fourier series, and standard existence and nonexistence theorems on elliptic functions. We show that many classical methods, including the maximum principle, the direct method of the calculus of variations, and the use of the Leray–Schauder Theorem, have analogues in the noncommutative setting.

1. Introduction

Gelfand's Theorem shows that $X \rightsquigarrow C_0(X)$ sets a contravariant equivalence of categories from the category of locally compact (Hausdorff) spaces and proper maps to the category of commutative C^*-algebras and $*$-homomorphisms. This observation is the key to the whole subject of noncommutative geometry, which is based on the following dictionary:

Classical		Noncommutative
locally compact space	$\leftrightarrow$	C^*-algebra
compact space	$\leftrightarrow$	unital C^*-algebra
vector bundle	$\leftrightarrow$	finitely generated projective module
smooth manifold	$\leftrightarrow$	C^*-algebra with smooth subalgebra
real-valued function	$\leftrightarrow$	self-adjoint element
partial derivative	$\leftrightarrow$	unbounded derivation
integral	$\leftrightarrow$	tracial state

The object of this paper is to begin to use this dictionary to set up a noncommutative theory of elliptic partial differential equations, both linear and nonlinear, along with corresponding aspects of the calculus of variations. Since the theory is still in its infancy, we begin with the very simplest case: Laplace's equation and PDEs closely connected to it, and concentrate on the simplest nontrivial example of a noncommutative manifold, the irrational rotation algebra (or noncommutative 2-torus) A_θ, for $\theta \in \mathbb{R} \setminus \mathbb{Q}$.

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A definition of elliptic partial differential operators, along with the study of one example associated with the irrational rotation algebra, was given in Connes' fundamental paper [1980], but there the emphasis was on pseudodifferential calculus and index theory. Here we focus on other things: variational methods, the maximum principle, an analogue of Wiener's Theorem, tools for treating *nonlinear* equations, the beginnings of a theory of harmonic unitaries, and some aspects of noncommutative complex analysis.

What is the motivation for a noncommutative theory of elliptic PDE? For the most part, it comes from physics. Many of the classical elliptic PDEs arise from variational problems in Riemannian geometry, and are also the field equations of physical theories. But the *uncertainty principle* forces quantum observables to be noncommutative. There is also increasing evidence, as in [Connes and Lott 1990; Chamseddine and Connes 1997; Connes et al. 1998; Seiberg and Witten 1999; Mathai and Rosenberg 2005; 2006], that quantum field theories should allow for the possibility of noncommutative space-times. *Noncommutative sigma-models*, for which the very earliest and simplest investigations are in [Dąbrowski et al. 2000; 2003], will require the noncommutative harmonic map equation, which generalizes the Laplace equation studied in this paper.

We use as our starting point the noncommutative differential geometry of Alain Connes [1980]. This theory only works well with highly symmetric noncommutative spaces, as the smooth elements are taken to be the C^∞ vectors for an action of a Lie group on a C^*-algebra, but this theory is well adapted to the case of the irrational rotation algebra, which carries an ergodic gauge action of the 2-torus $\mathbb{T}^2$.

The outline of this paper is as follows. We begin in Section 2 with the basic properties of the Laplacian on A_θ. Included are analogues of Wiener's theorem (Theorem 2.8) and the maximum principle (Proposition 2.9). In Section 3, we take up the basic properties of Sobolev spaces on A_θ, which are needed for a deeper analysis of some aspects of noncommutative PDEs. We should point out that some of the material of this section has already appeared in [Polishchuk 2006, §3] and in [Luef 2006]. The heart of this paper is contained in Sections 4 and 5, which begin to develop a theory of *nonlinear* elliptic partial differential equations, using methods analogous to those traditional in the theory of nonlinear elliptic PDE. Finally, Section 6 deals with noncommutative complex analysis.

We should mention that another example of noncommutative elliptic PDE and an associated variational problem on noncommutative tori, namely, noncommutative Yang–Mills theory, has already been studied by Connes and Rieffel [Connes and Rieffel 1987; Rieffel 1990]. Furthermore, Theorem 2.8 was previously proved by Gröchenig and Leinert [Gröchenig and Leinert 2004] by another method, and variations on the Gröchenig–Leinert work can be found in [Luef 2006]. In their paper, Gröchenig and Leinert point out some applications to harmonic analysis and wavelet theory, which go off in a somewhat different direction than the applications to mathematical physics which we envisage, though obviously there is some overlap between the two.

2. The linear Laplacian

We will be studying the C^*-algebra A_θ generated by two unitaries U, V satisfying

$$UV = e^{2\pi i\theta} VU.$$

A_θ is simple with unique trace τ if $\theta \in \mathbb{R} \setminus \mathbb{Q}$. (See for example [Rieffel 1981] for a review of the basic facts about A_θ.) The torus $G = \mathbb{T}^2$ acts by

$$\alpha_{(z_1,z_2)}U = z_1 U, \quad \alpha_{(z_1,z_2)}V = z_2 V, \quad |z_1| = |z_2| = 1.$$

The space of C^∞ vectors for the action α is the *smooth irrational rotation algebra*

$$A_\theta^\infty = \left\{ \sum_{m,n} c_{m,n} U^m V^n : c_{m,n} \text{ rapidly decreasing} \right\}.$$

This should be viewed as a noncommutative deformation of the algebra $C^\infty(\mathbb{T}^2)$ of smooth functions on an ordinary 2-torus, and the decomposition of an element of this algebra in terms of multiples of $U^m V^n$ should be viewed as a sort of noncommutative Fourier series decomposition, with $c_{m,n}$ as a sort of Fourier coefficient. For $a \in A_\theta$ but not necessarily in A_θ^∞, the Fourier coefficients $c_{m,n}$ are well defined and satisfy $|c_{m,n}| \le \|a\|$, since $c_{m,n} = \tau(V^{-n}U^{-m}a)$, but the Fourier series expansion of a is only a formal expansion, and need not converge in the topology of A_θ, just as one has functions in $C(\mathbb{T}^2)$ whose Fourier series do not converge absolutely or even pointwise.

We denote by δ_1 and δ_2 the infinitesimal generators of the actions of the two $\mathbb{T}$ factors in $\mathbb{T}^2$ under α. These are unbounded derivations on A_θ, and map A_θ^∞ to itself. They are given by

$$\delta_1(U) = 2\pi i U, \quad \delta_2(V) = 2\pi i V, \quad \delta_2(U) = \delta_1(V) = 0.$$

These derivations δ_j obviously commute with the adjoint operation $*$, and play the roles of the partial derivatives $\partial/\partial x_j$ in classical analysis on the 2-torus. Since the action α of $\mathbb{T}^2$ preserves the tracial state τ, $\tau \circ \delta_j = 0$, $j = 1, 2$. This fact is the basis for the following Lemma, which we will use many times in the future.

Lemma 2.1 (Integration by parts). *If $a, b \in A_\theta^\infty$, then $\tau(\delta_j(a)b) = -\tau(\delta_j(b)a)$, $j = 1, 2$.*

Proof. We have $0 = \tau(\delta_j(ab)) = \tau(\delta_j(a)b) + \tau(a\delta_j(b))$. □

Definition 2.2. In analogy with the usual notation in analysis, we let

$$\Delta = \delta_1^2 + \delta_2^2.$$

This should be viewed as a noncommutative elliptic partial differential operator. (The notion of ellipticity was defined rigorously in [Connes 1980, p. 602].) Clearly, Δ is a "negative" operator, and its spectrum consists of the numbers $-4\pi^2(m^2+n^2)$, $m, n \in \mathbb{Z}$, with eigenfunctions $U^m V^n$. Via the noncommutative Fourier expansion discussed earlier, the pair $(A_\theta^\infty, \Delta)$ is isomorphic to $C^\infty(\mathbb{T}^2)$ with the usual Laplacian Δ, provided one looks just at the linear structure and forgets the noncommutativity of the multiplication. (This was already observed in [Connes 1980, p. 602].)

Proposition 2.3. *For any $\lambda > 0$ (or not of the form $-4\pi^2 n$ with $n \in \mathbb{N}$), the map $-\Delta + \lambda \colon A_\theta^\infty \to A_\theta^\infty$ is bijective.*

Proof. We have

$$(-\Delta + \lambda)\left(\sum_{m,n} c_{m,n} U^m V^n \right) = \sum_{m,n} \left(4\pi^2(m^2+n^2) + \lambda\right) c_{m,n} U^m V^n.$$

It is immediate that $-\Delta+\lambda$ has no kernel and has an inverse given by the formula

$$\sum_{m,n} c_{m,n} U^m V^n \mapsto \sum_{m,n} \frac{1}{4\pi^2(m^2+n^2)+\lambda} c_{m,n} U^m V^n,$$

since if $c_{m,n}$ is rapidly decreasing, so are the coefficients on the right. □

It is also easy to characterize the image of Δ.

Proposition 2.4. *The image of* $\Delta\colon A_\theta^\infty \to A_\theta^\infty$ *is precisely* $A_\theta^\infty \cap \ker\tau$, *the smooth elements with zero trace.*

Proof. We have $\Delta(\sum_{m,n} c_{m,n}U^mV^n) = -4\pi^2\sum_{m,n}(m^2+n^2)c_{m,n}U^mV^n$, and the factor (m^2+n^2) kills the term with $m=n=0$. Thus the image of Δ is contained in the kernel of τ. Conversely, suppose $a=\sum_{m,n} d_{m,n}U^mV^n$ is an arbitrary element of $A_\theta^\infty \cap \ker\tau$. That means $d_{m,n}$ is rapidly decreasing and $d_{0,0}=0$. Then $d_{m,n}/(m^2+n^2)$ is also rapidly decreasing, and

$$\sum_{m,n}{}' \frac{-d_{m,n}}{4\pi^2\,(m^2+n^2)} U^m V^n,$$

where the $'$ indicates we omit the term with $m=n=0$, converges to an element b of A_θ^∞ with $\Delta b = a$. □

The following consequence is an analogue of a well-known fact about subharmonic functions on compact manifolds.

Corollary 2.5. *If* $a\in A_\theta^\infty$ *is subharmonic* (*i.e., if* $\Delta a\ge 0$), *then* a *is constant.*

Proof. Suppose $a\in A_\theta^\infty$ and $\Delta a \ge 0$. By Proposition 2.4, $\tau(\Delta a)=0$. But τ is a faithful trace, which means that if $b\ge 0$ and $\tau(b)=0$, then $b=0$. Apply this with $b=\Delta a$ and we see that $\Delta a=0$. This implies a is a scalar multiple of 1. □

For future use, we are also going to want to study other "function spaces" on the noncommutative torus. For example, we have the analogue of the Fourier algebra of functions with absolutely convergent Fourier series.

Definition 2.6. Fix $\theta\in\mathbb{R}\setminus\mathbb{Q}$, and let

$$\mathcal{B}_\theta = \left\{ \sum_{m,n} c_{m,n}U^mV^n : \sum_{m,n} \left|c_{m,n}\right| < \infty \right\}.$$

This is obviously a Banach subspace of A_θ with norm $\|\cdot\|_{\ell^1}$ given by the ℓ^1 norm of the coefficients $c_{m,n}$. We also obviously have $\|a\|_{\ell^1}\ge\|a\|$ for $a\in\mathcal{B}_\theta$. ($\|\cdot\|$ will for us always denote the C^*-algebra norm.)

The following lemma, related in spirit to the Sobolev Embedding Theorem [Kazdan 1983, Theorem 1.1], relates the topology of $\mathcal{B}_\theta$ to the subject of Propositions 2.3 and 2.4. More details of noncommutative Sobolev space theory will be taken up in Section 3 below.

Lemma 2.7. *Let* $f\in A_\theta^\infty$. *Then there is a constant* $C>0$ *such that* (*in the notation of Definition* 2.6) $\|f\|_{\ell^1}\le C\|(-\Delta+1)f\|$. *In particular, the domain of* Δ, *as an unbounded operator on* A_θ, *is contained in* $\mathcal{B}_\theta$.

Proof. Suppose $f = \sum_{m,n} c_{m,n} U^m V^n \in A_\theta^\infty$. Then

$$\|f\|_{\ell^1} = \sum_{m,n} |c_{m,n}| = \sum_{m,n} \bigl(1 + 4\pi^2(m^2+n^2)\bigr)\, c_{m,n} \cdot \frac{a_{m,n}}{1+4\pi^2(m^2+n^2)},$$

where $|a_{m,n}| = 1$. View this as an ℓ^2 inner product and estimate it by Cauchy–Schwarz. We obtain

$$\|f\|_{\ell^1} \le C \|(-1+\Delta) f\|_{\ell^2},$$

where $\|\cdot\|_{\ell^2}$ is the ℓ^2 norm of the sequence of Fourier coefficients (this can also be defined by $\|c\|_{\ell^2} = \tau(c^*c)^{1/2}$) and where

$$C = \Bigl\| \bigl\{ \bigl(1+4\pi^2(m^2+n^2)\bigr)^{-1} \bigr\}_{m,n} \Bigr\|_{\ell^2} = \left(\sum_{m,n} \frac{1}{\bigl(1+4\pi^2(m^2+n^2)\bigr)^2} \right)^{1/2} < \infty.$$

Since the ℓ^2 norm is bounded by the C^*-algebra norm, as $\|c\|_{\ell^2} = \tau(c^*c)^{1/2} \le \|c^*c\|^{1/2} = \|c\|$, the result follows. □

The next result was proved in [Gröchenig and Leinert 2004], using the theory of symmetric L^1-algebras as developed by Leptin, Ludwig, Hulanicki, *et al.* We include a brief proof for the sake of completeness.

Theorem 2.8 (Wiener's Theorem). *The Banach space $\mathcal{B}_\theta$ is a Banach $*$-algebra and is closed under the holomorphic functional calculus of A_θ. Thus if $a \in \mathcal{B}_\theta$ and a is invertible in A_θ, $a^{-1} \in \mathcal{B}_\theta$.*

Proof. Suppose $a = \sum c_{m,n} U^m V^n$ with the sum absolutely convergent. Then

$$a^* = \sum_{m,n} \overline{c_{m,n}} V^{-n} U^{-m} = \sum_{m,n} \overline{c_{m,n}} e^{-2\pi i m n \theta} U^{-m} V^{-n}$$

so $a^* \in \mathcal{B}_\theta$. Similarly, if also $b = \sum d_{m,n} U^m V^n$ (absolutely convergent sum), then ab has Fourier coefficients given by twisted convolution of the Fourier coefficients of a and b, and since the twisting only involves scalars of absolute value 1, the Fourier coefficients of ab are absolutely convergent. More precisely,

$$\begin{aligned} ab = \Bigl(\sum_{m,n} c_{m,n} U^m V^n\Bigr)\Bigl(\sum_{k,l} d_{k,l} U^k V^l\Bigr) &= \sum_{m,n,k,l} c_{m,n} d_{k,l} U^m V^n U^k V^l \\ &= \sum_{m,n,k,l} c_{m,n} d_{k,l} e^{-2\pi i k n \theta} U^{m+k} V^{n+l} = \sum_{p,q} f_{p,q} U^p V^q, \end{aligned}$$

where

$$f_{p,q} = \sum_{m,n} c_{m,n} d_{p-m,q-n} e^{-2\pi i (p-m) n \theta}, \quad \text{so that} \quad |f_{p,q}| \le \sum_{m,n} |c_{m,n}|\, |d_{p-m,q-n}| \le \|c\|_{\ell^1} \|d\|_{\ell^1}.$$

This confirms that $\mathcal{B}_\theta$ is a Banach $*$-algebra and of course a $*$-subalgebra of A_θ.

To prove the analogue of Wiener's Theorem, we unfortunately cannot use the cute proof using the Gelfand transform, since $\mathcal{B}_\theta$ is not commutative. We also cannot use another very elementary proof from [Newman 1975] since this also relies on commutativity. However, Newman's proof is related to the fact — implicit in [Connes 1980, Lemma 1] — that A_θ^∞ is closed under the holomorphic functional calculus of A_θ. To prove this one has to show that if $b \in A_\theta^\infty$ with b invertible in A_θ, then b^{-1} also lies in

A_θ^∞. To prove this fact, iterate the identity $\delta_j(b^{-1}) = -b^{-1}\,\delta_j(b)\,b^{-1}$ to see that b^{-1} lies in the domain of all monomials in δ_1 and δ_2. One might think that since A_θ^∞ is dense in $\mathcal{B}_\theta$, this should be enough to prove Wiener's Theorem for the latter, but this doesn't work, since in general the spectrum and spectral radius functions are only upper semicontinuous, not continuous, on a noncommutative Banach algebra [Newburgh 1951].

To prove the theorem, we rely on an observation of Hulanicki [1972, Proposition 2.5], based on [Raikov 1946, Theorem 5]: if a Banach $*$-algebra B (with isometric involution and a faithful $*$-representation on a Hilbert space) is embedded in its enveloping C^*-algebra A, then the spectra of self-adjoint elements of B are the same whether computed in B or in A if and only if B is symmetric (i.e., for $x \in B$, the spectrum in B of x^*x is contained in $[0, \infty)$). We will apply this with $B = \mathcal{B}_\theta$ and with $A = A_\theta$. Hulanicki also showed [Hulanicki 1970] that the L^1 algebras of discrete nilpotent groups are symmetric. In particular, the L^1 algebra of the discrete Heisenberg group H (with generators a, b, c, where c is central and $aba^{-1}b^{-1} = c$) is symmetric. Thus $\mathcal{B}_\theta$, which is the quotient of $L^1(H)$ by the (self-adjoint) ideal generated by $c - e^{2\pi i\theta}$, is also symmetric. (If B is a symmetric Banach $*$-algebra and J is a closed self-adjoint ideal, then B/J is also symmetric, since if $\dot{x} \in B/J$ is the image of $x \in B$, then the spectrum of $\dot{x}^*\dot{x}$ in B/J is contained in the spectrum of x^*x in B, hence is contained in $[0, \infty)$.) So for $x = x^* \in \mathcal{B}_\theta$, by Hulanicki's observation, if x is invertible in A_θ, $x^{-1} \in \mathcal{B}_\theta$. Suppose $a \in \mathcal{B}_\theta$ and a is invertible in A_θ. Then a^* is also invertible in A_θ, so $x = a^*a \in \mathcal{B}_\theta$ and x is invertible in A_θ. Hence $x^{-1} = a^{-1}a^{*-1} \in \mathcal{B}_\theta$ and $a^{-1} = x^{-1}a^* \in \mathcal{B}_\theta$. □

In the classical theory of the Laplacian, one of the most useful tools is the maximum principle — see, for example, [Kazdan 1983, p. 20]. The following is a noncommutative analogue.

Proposition 2.9 (Maximum principle). *Let $h = h^* \in A_\theta^\infty$, and let $[t_0, t_1]$ be the smallest closed interval containing the spectrum $\sigma(h)$ of h in A_θ, so that $t_1 = \max\{t : t \in \sigma(h)\}$ and $t_0 = \min\{t : t \in \sigma(h)\}$. Then there exists a state φ of A_θ with $\varphi(h) = t_1$, and for such a state, $\varphi(\Delta h) \le 0$. Similarly, there exists a state ψ of A_θ with $\psi(h) = t_0$, and for such a state, $\psi(\Delta h) \ge 0$.*

Proof. The commutative C^*-algebra $C^*(h)$ must have pure states $\widetilde{\varphi}$ and $\widetilde{\psi}$ with $\widetilde{\varphi}(h) = t_1$, $\widetilde{\psi}(h) = t_0$, since t_0, $t_1 \in \sigma(h)$. Extend these to states φ, ψ of the larger C^*-algebra A_θ. Then for $s \in G = \mathbb{T}^2$, the functions $s \mapsto \varphi(\alpha_s(h))$ and $s \mapsto \psi(\alpha_s(h))$ must have a maximum and a minimum, respectively, at the identity element of $\mathbb{T}^2$. (Recall that α is the gauge action by $*$-automorphisms.) Differentiate twice and the result follows by the second derivative test. □

Just as in the classical setting, Laplace's equation arises as the Euler–Lagrange equation of a variational problem.

Definition 2.10. For $a \in A_\theta^\infty$, let

$$E(a) = \tfrac{1}{2}\tau\big(\delta_1(a)^2 + \delta_2(a)^2\big).$$

This is clearly the noncommutative analogue of the classical energy functional

$$f \mapsto \frac{1}{2}\int_M \|\nabla f\|^2\, d\mathrm{vol}$$

on a compact manifold M.

Proposition 2.11. *The Euler–Lagrange equation for critical points of the energy functional E of Definition 2.10, restricted to self-adjoint elements of A_θ^∞, is just Laplace's equation $\Delta a = 0$. Thus the only critical points are the scalar multiples of the identity, which are the points where $E(a) = 0$ and are strict minima for E.*

Proof. This works very much like the classical case. If $a = a^*$ and $h = h^*$, then

$$\left.\frac{d}{dt}\right|_{t=0} E(a+th) = \tfrac{1}{2}\tau\big(\delta_1(a)\delta_1(h) + \delta_1(h)\delta_1(a) + \delta_2(a)\delta_2(h) + \delta_2(h)\delta_2(a)\big).$$

Because of the trace property, we can write this as $\tau(\delta_1(a)\delta_1(h)+\delta_2(a)\delta_2(h))$. For a to be a critical point of E, this must vanish for all choices of h. Integrating by parts using Lemma 2.1, we obtain $\tau(h\Delta(a))=0$ for all h, and since the trace pairing is nondegenerate, we get the Euler–Lagrange equation $\Delta a = 0$. Since Δ has pure point spectrum with eigenvalues $-4\pi^2(m^2+n^2)$ and eigenfunctions U^mV^n, the equation has the unique solution $a = \lambda 1$, $\lambda \in \mathbb{R}$. These are also the points where E takes its minimum value of 0. □

3. Sobolev spaces

In the treatment of Laplace's equation above, we alluded to the theory of Sobolev spaces. One can develop this theory in the noncommutative setting in complete analogy with the classical case. To simplify the treatment, we deal here only with the L^2 theory, which gives rise to Hilbert spaces. These spaces are convenient for applications to nonlinear elliptic PDE, as we will see in the next section.

Definition 3.1. For $a \in A_\theta$, we define its L^2 norm[1] by

$$\|a\|_{\ell^2} = \tau(a^*a)^{1/2}.$$

We let L^2 or H^0 (this is the Sobolev space of "functions" with 0 derivatives in L^2) be the completion of A_θ in this norm. Obviously this is a Hilbert space, with inner product extending

$$\langle a,\, b\rangle = \tau(b^*a)$$

on A_θ. Also note that the norm of L^2 is simply the ℓ^2 norm for the Fourier coefficients, since if $a \in A_\theta^\infty$ has the Fourier expansion $\sum_{m,n} c_{m,n}U^mV^n$, then

$$\|a\|_{\ell^2}^2 = \tau(a^*a) = \tau\bigg(\sum_{k,l,m,n} \big(c_{m,n}U^mV^n\big)^* c_{k,l}U^kV^l\bigg) = \tau\bigg(\sum_{k,l,m,n} \overline{c_{m,n}}\, c_{k,l}V^{-n}U^{-m}U^kV^l\bigg) = \sum_{m,n} |c_{m,n}|^2.$$

Now let $n \in \mathbb{N}$. We define the Sobolev space[2] H^n of "functions" with n derivatives in L^2 to be the completion of A_θ^∞ in the norm

$$\|a\|_{H^n}^2 = \sum_{0\le|\beta|\le n} \|\delta_\beta(a)\|_{\ell^2}^2.$$

(These spaces are also defined, with slightly different notation, in [Polishchuk 2006, §3].) Here $\beta = \beta_1\beta_2\cdots\beta_{|\beta|}$ runs over sequences with $\beta_j = 1$ or 2 and δ_β means $\delta_{\beta_1}\cdots\delta_{\beta_{|\beta|}}$, a partial derivative of order

[1] This is really the norm for the Hilbert space of the II_1 factor representation of A_θ determined by the trace τ.

[2] Usually this would be called $H^{n,2}$, but we are trying to simplify notation.

$|\beta|$. For example,

$$\|a\|_{H^1}^2 = \|a\|_{\ell^2}^2 + \|\delta_1(a)\|_{\ell^2}^2 + \|\delta_2(a)\|_{\ell^2}^2.$$

The Sobolev space H^n is clearly a Hilbert space, and we obviously have norm-decreasing inclusions $H^n \hookrightarrow H^{n-1}$. Furthermore, it is clear that the Sobolev norms are invariant under taking adjoints and can easily be expressed in terms of the Fourier coefficients; for example, if $a \in A_\theta^\infty$ has the Fourier expansion $\sum_{m,n} c_{m,n} U^m V^n$, then

$$\|a\|_{H^1}^2 = \sum_{m,n} \bigl(1 + 4\pi^2(m^2+n^2)\bigr)|c_{m,n}|^2.$$

The next result is the exact analogue of the classical Sobolev embedding theorem [Kazdan 1983, Theorem 1.1] for $\mathbb{T}^2$.

Theorem 3.2 (Sobolev embedding). *The inclusion $H^n \hookrightarrow H^{n-1}$ is compact. The space H^1 is not contained in A_θ, but H^2 has a compact inclusion into $\mathcal{B}_\theta$ (and thus into A_θ).*

Proof. Since the Sobolev norms just depend on the decay of the Fourier coefficients, this follows immediately from the classical Sobolev Embedding Theorem in dimension 2. The inclusion of H^2 into $\mathcal{B}_\theta$ also follows from the estimate

$$\|f\|_{\ell^1} \le C\|(-1+\Delta)f\|_{\ell^2},$$

in the proof of Lemma 2.7, with the compactness coming from the fact that we can approximate by the finite rank operators that truncate the Fourier series after finitely many terms. □

4. Nonlinear problems involving the Laplacian

Somewhat more interesting, and certainly more difficult to treat than the situation of Proposition 2.11, are certain nonlinear problems involving the Laplacian, of the general form $\Delta u = f(u)$. Such problems arise classically from the problem of prescribing the scalar curvature of a metric $e^u g$ obtained by conformally deforming the original metric g on a Riemannian manifold M [Kazdan 1983, Chapters 5, 7]. For example, if g is the usual flat metric on $\mathbb{T}^2$, then the scalar curvature h of the pointwise conformal metric $e^u g$ solves the equation $\Delta u = -he^u$. (This equation is studied in detail in [Kazdan and Warner 1974, §5].) Because of the Gauss–Bonnet theorem on the torus, h must integrate out to 0, so there are no solutions with h a constant unless $h = 0$ and u is a constant. This fact has an exact analogue in our noncommutative setting.

Proposition 4.1. *If $\lambda \in \mathbb{R}$, the equation $\Delta u = -\lambda e^u$ has no solution $u = u^* \in A_\theta^\infty$ unless $\lambda = 0$ and u is a scalar multiple of* 1.

Proof. Suppose $u = u^* \in A_\theta^\infty$. Then $e^u \ge 0$, so if $\lambda \ne 0$, either $\lambda e^u \ge 0$ or $-\lambda e^u \ge 0$. Thus if $\Delta u = -\lambda e^u$, either u or $-u$ is subharmonic. The result now follows from Corollary 2.5. □

Alternative proof. Use the maximum principle, Proposition 2.9. Let $[a, b]$ be the smallest closed interval containing the spectrum of u. Then for any state φ of A_θ, $a \le \varphi(u) \le b$ and $\varphi(e^u) \ge e^a > 0$. If $\Delta u = -\lambda e^u$ and $\lambda > 0$, then by Proposition 2.9, there is a state φ with $\varphi(u) = a$ and $\varphi(\Delta u) \ge 0$, while $\varphi(-\lambda e^u) < 0$, a contradiction. Similarly, if $\lambda < 0$ and $\Delta u = -\lambda e^u$, there is a state φ with $\varphi(u) = b$ and $\varphi(\Delta u) \le 0$, while $\varphi(-\lambda e^u) > 0$, a contradiction. □

Proposition 4.1 suggests that we consider the equation $\Delta u = -\frac{1}{2}\left(he^u + e^u h\right)$ with $h = h^*$ not a scalar. (Note that we have symmetrized the right-hand side to make it self-adjoint, since $u = u^*$ implies Δu is self-adjoint.) Once again, a slight variation on the argument of Proposition 4.1 shows that there is no solution if $h \geq 0$ or if $h \leq 0$; again this is not surprising since one gets the same result in the classical case as a consequence of Gauss–Bonnet.

Proposition 4.2. *If $h \geq 0$ or $h \leq 0$ in A_θ^∞, the equation $\Delta u = -\frac{1}{2}\left(he^u + e^u h\right)$ has no solution $u = u^* \in A_\theta^\infty$ unless $h = 0$ and u is a scalar multiple of* 1.

Proof. This is just like the proof of Proposition 4.1. If $h \geq 0$ and $\Delta u = -\frac{1}{2}\left(he^u + e^u h\right)$, then applying τ to both sides, we get

$$0 = \tau(\Delta u) = -\tau(he^u) = -\tau\left(h^{1/2} e^u h^{1/2}\right). \tag{4-1}$$

Since

$$h^{1/2} e^u h^{1/2} = \left(e^{u/2} h^{1/2}\right)^* \left(e^{u/2} h^{1/2}\right) \geq 0$$

and τ is faithful, that implies $e^{u/2} h^{1/2} = 0$. Since $e^{u/2}$ is invertible, it follows that $h^{1/2} = 0$ and $h = 0$. The case where $h \leq 0$ is almost identical; just replace h by $-h$ and change the sign of the right-hand side of (4-1). □

Unfortunately, the rest of the treatment in [Kazdan and Warner 1974, §5] doesn't extend to our setting, since from the calculation

$$\tau(h) = \tfrac{1}{2}\tau\left(e^{-u} h e^u + h\right) = -\tau(e^{-u}\Delta u),$$

it is not clear if $\tau(h) < 0$ follows. (The problem is that we can't commute the various factors that arise from expanding $\delta_j(e^{-u})$ after integration by parts.) But since the main purpose of this section is just to test various techniques and see to what extent they apply to nonlinear noncommutative elliptic PDEs, we will consider instead the following more tractable equation from [Kazdan 1983, Chapter 5]:

$$\Delta u = \mu\, e^u - \lambda, \qquad \lambda,\ \mu \in \mathbb{R},\ \lambda,\ \mu > 0. \tag{4-2}$$

Theorem 4.3. *The equation (4-2) has the unique solution $t_0 = \ln(\lambda/\mu)$ in $\left(A_\theta^\infty\right)_{s.a.}$.*

Proof. Let

$$\mathcal{L}(u) = E(u) + \tau(\mu\, e^u - \lambda u).$$

Note that for $t \in \mathbb{R}$, $\mu\, e^t - \lambda t$ has an absolute minimum at $t = t_0$, so $\mu\, e^u - \lambda u \geq \lambda(1 - t_0)$ for $u = u^*$ and so $\mathcal{L}(u) \geq \lambda(1 - t_0)$ for $u = u^*$. Furthermore, the Euler–Lagrange equation for a critical point of $\mathcal{L}$ is precisely (4-2), since

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{L}(u + th) = \tau(\delta_1(u)\delta_1(h) + \delta_2(u)\delta_2(h) - \lambda h) + \left.\frac{d}{dt}\right|_{t=0} \tau\left(\mu\, e^{u+th}\right),$$

via the calculation in the proof of Proposition 2.11. Now

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=0} \tau\left(e^{u+th}\right) &= \left.\frac{d}{dt}\right|_{t=0} \sum_{n=0}^{\infty} \frac{1}{n!}\tau\left((u+th)^n\right) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}\tau\left(u^{n-1}h + u^{n-2}hu + \cdots + uhu^{n-2} + hu^{n-1}\right) = \sum_{n=0}^{\infty} \frac{n}{n!}\tau\left(hu^{n-1}\right) = \tau(he^u)\end{aligned}$$

by the invariance of the trace under cyclic permutations of the factors. So applying Lemma 2.1, we see that

$$\left.\frac{d}{dt}\right|_{t=0} \mathscr{L}(u+th) = \tau(-h\Delta(u) - \lambda h + \mu\, he^u) = -\tau\big(h\cdot(\Delta u + \lambda - \mu\, e^u)\big).$$

So nondegeneracy of the trace pairing gives (4-2) as the Euler–Lagrange equation for a critical point of $\mathscr{L}$. It is also clear that t_0 is an absolute minimum for $\mathscr{L}$ and a solution of (4-2). It remains to prove the uniqueness. Suppose u is a solution of (4-2) and write $u = t_0 + v$. Then v satisfies the equation $\Delta v = \lambda(e^v - 1)$, and we need to show $v = 0$. Multiply both sides by v and apply τ. We obtain (using Lemma 2.1)

$$-2E(v) = \tau(v\Delta v) = \lambda\tau(v(e^v - 1)).$$

The left-hand side is ≤ 0, while since $\lambda > 0$ and $t(e^t - 1) \ge 0$ with equality only at $t = 0$, the right-hand side is ≥ 0. Thus $E(v) = 0$, which implies v is a scalar with $v(e^v - 1) = 0$, i.e., $v = 0$. □

With techniques reminiscent of [Kazdan 1983, Chapter 5] we can study a slightly more complicated variant of (4-2).

Theorem 4.4. *Let $a \ge 0$ be invertible in A_θ^∞. Then the equation*

$$\Delta u = \mu\, e^u - a, \quad \mu \in \mathbb{R},\ \mu > 0 \tag{4-3}$$

has a solution $u \in \left(A_\theta^\infty\right)_{s.a.}$.

Without loss of generality (as a result of replacing u by $u - \ln \mu$) we can take $\mu = 1$; that simplifies the calculations and we make this simplification from now on. Some condition on a beyond the fact that $a \ge 0$, for example at least $a \ne 0$, is necessary because of Proposition 4.1, and we see that any solution of (4-3) must satisfy $\tau(e^u) = \tau(a) > 0$.

Proof. Several methods are available for proving existence, but the simplest seems to be to apply the Leray–Schauder Theorem ([Leray and Schauder 1934], [Kazdan 1983, Theorem 5.5]). Consider the family of equations

$$\Delta u = (1-t)\, u + t\, e^u - a, \quad 0 \le t \le 1. \tag{4-4}$$

When $t = 0$ this reduces to $\Delta u = u - a$, or $(-\Delta + 1)\, u = a$, which by Proposition 2.3 has the unique solution $u = (-\Delta + 1)^{-1} a$. When $t = 1$, (4-4) reduces to (4-3). We begin by using the maximum principle, Proposition 2.9, which implies an *a priori* bound on solutions of (4-4). (Compare the argument in [Kazdan 1983, pp. 56–57].) Indeed, suppose u satisfies (4-4) for some $0 \le t \le 1$, and let $[c, d]$ be the smallest closed interval containing $\sigma(u)$. We may choose a state φ of A_θ with $\varphi(u) = d$, $\varphi(e^u) = e^d$, and by Proposition 2.9, $\varphi(\Delta u) \le 0$. Since

$$\varphi\big((1-t)\, u + t\, e^u - a\big) = (1-t)\, d + t\, e^d - \varphi(a) \ge (1-t)\, d + t\, e^d - \|a\|,$$

we get a contradiction if $(1-t)\, d + t\, e^d - \|a\| > 0$, which is the case if $d > \|a\|$. So $d \le \|a\|$. Similarly, we may choose a state ψ of A_θ with $\psi(u) = c$, $\psi(e^u) = e^c$, and by Proposition 2.9, $\psi(\Delta u) \ge 0$. Since

$$\psi\big((1-t)\, u + t\, e^u - a\big) = (1-t)\, c + t\, e^c - \psi(a) \le (1-t)\, c + t\, e^c - \frac{1}{\|a^{-1}\|},$$

we get a contradiction if $e^c - 1/\|a^{-1}\| < 0$. Thus $e^c - 1/\|a^{-1}\| \geq 0$ and $c \geq -\ln \|a^{-1}\|$. In other words, any solution of (4-4), for any $0 \leq t \leq 1$, satisfies the *a priori* estimate

$$-\ln \|a^{-1}\| \leq u \leq \|a\|. \tag{4-5}$$

Now rewrite (4-4) in the form

$$u = (-\Delta + 1)^{-1}\big(a + t\,u - t\,e^u\big).$$

The right-hand side is well-defined and continuous in the C^*-algebra norm topology for $u = (A_\theta)_{\text{s.a.}}$, since $(-\Delta+1)^{-1}$ is bounded by Lemma 2.7. In fact, this Lemma also shows $(-\Delta+1)^{-1}$ is bounded as a map $A_\theta \to \mathcal{B}_\theta$, so as a map $A_\theta \to A_\theta$, it is a limit of operators of finite rank, namely the restrictions of the operator to the span of $\{U^m V^n : m^2 + n^2 \leq N\}$, as $N \to \infty$. Thus $(-\Delta+1)^{-1}$ is not only bounded, but also compact. Together with the *a priori* estimate (4-5) and the fact that there is a solution for $t = 0$, this shows that (4-4) satisfies the hypotheses of the Leray–Schauder Theorem. Hence (4-4) has a solution for all $t \in [0,\ 1]$. Thus (4-3) (which is the special case of (4-4) for $t = 1$) has a solution in $\operatorname{dom} \Delta \subseteq A_\theta$, and thus in $\mathcal{B}_\theta$ by Lemma 2.7.

The last step of the proof is *elliptic regularity*. In other words, we need to show that a solution to (4-3), so far only known to be in $\mathcal{B}_\theta$, lies in A_θ^∞. Since $a \in A_\theta^\infty$ and $\mathcal{B}_\theta$ is closed under holomorphic functional calculus (by Theorem 2.8), the right-hand side of (4-3) lies in $\mathcal{B}_\theta$, i.e., has absolutely summable Fourier coefficients. Then (4-3) implies that the Fourier coefficients $c_{m,n}$ of u have even faster decay, namely,

$$\sum_{m,n} (1 + m^2 + n^2)|c_{m,n}| < \infty.$$

Now one can iterate this argument. This is a bit tricky, as at each step one needs a new Banach subalgebra of A_θ to replace $\mathcal{B}$ (we drop the subscript θ for simplicity of notation), so we indicate how this works at the next step, and then sketch how to proceed further. For $u \in \mathcal{B}$ with Fourier coefficients $c_{m,n}$, let

$$\|u\|_1 = \sum_{m,n} (2 + m^2 + n^2)|c_{m,n}|,$$

assuming this converges. We have seen that we know $\|u\|_1 < \infty$. We claim that $\|\cdot\|_1$ is a Banach $*$-algebra norm. This will follow by the argument in the proof of Theorem 2.8 if we can show that

$$\sum_{p,q} (2 + p^2 + q^2) \sum_{m,n} |c_{m,n}|\,|d_{p-m,q-n}| \leq \left(\sum_{m,n} (2 + m^2 + n^2)|c_{m,n}| \right) \left(\sum_{l,k} (2 + l^2 + k^2)|d_{l,k}| \right).$$

Comparing the two sides of this inequality, one sees it is equivalent to proving that

$$(2 + p^2 + q^2) \leq (2 + m^2 + n^2)(2 + (p-m)^2 + (q-n)^2),$$

or with $\vec{v} = (m, n)$ and $\vec{w} = (p - m, q - n)$ vectors in Euclidean 2-space, that

$$\big(2 + \|\vec{v} + \vec{w}\|^2\big) \leq \big(2 + \|\vec{v}\|^2\big)\big(2 + \|\vec{w}\|^2\big).$$

This inequality in turn follows from the standard inequality

$$\|\vec{v} + \vec{w}\|^2 \leq \|\vec{v}\|^2 + \|\vec{w}\|^2 + 2\|\vec{v}\| \cdot \|\vec{w}\| \leq 2\big(\|\vec{v}\|^2 + \|\vec{w}\|^2\big).$$

This shows the completion of A_θ^∞ in the norm $\|\cdot\|_1$ is a Banach $*$-algebra $\mathcal{B}_1$. Since u and a are in $\mathcal{B}_1$, so is $e^u - a$. By (4-3) again, u has still more rapid decay; its Fourier coefficients satisfy

$$\sum_{m,n}(m^2+n^2)^2|c_{m,n}| < \infty.$$

Now we iterate again using still another Banach $*$-algebra $\mathcal{B}_2$ with the norm

$$\|u\|_2 = \sum_{m,n}\bigl(8+(m^2+n^2)^2\bigr)|c_{m,n}|.$$

Again one has to check that this is a Banach algebra norm, which will follow from the inequalities

$$\begin{aligned}8+\|\vec{v}+\vec{w}\|^4 &= 8+\bigl(\|\vec{v}+\vec{w}\|^2\bigr)^2\\ &\le 8+\bigl(2(\|\vec{v}\|^2+\|\vec{w}\|^2)\bigr)^2 \le 8+4\bigl(\|\vec{v}\|^4+\|\vec{w}\|^4+2\|\vec{v}\|^2\cdot\|\vec{v}\|^2\bigr)\\ &\le 8+4\bigl(2(\|\vec{v}\|^4+\|\vec{w}\|^4)\bigr) \le \bigl(8+\|\vec{v}\|^4\bigr)\bigl(8+\|\vec{w}\|^4\bigr).\end{aligned}$$

Thus $\mathcal{B}_2$ is a Banach algebra and $e^u - a \in \mathcal{B}_2$, so that $\Delta u \in \mathcal{B}_2$ and the Fourier coefficients of u decay faster than $(m^2+n^2)^3$, etc. Repeating in this way, we show by induction that $c_{m,n}$ is rapidly decreasing, and thus that $u \in A_\theta^\infty$. □

Sketch of a second proof. One could also approach this problem using "variational methods." By the argument at the beginning of the proof of Theorem 4.3, (4-3) is the Euler–Lagrange equation for critical points of

$$\mathcal{L}(u) = E(u) + \tau(e^u - u\,a) = E(u) + \tau(e^u - a^{1/2}ua^{1/2}).$$

This functional is bounded below since $E(u) \ge 0$ and $\tau(e^u - a^{1/2}ua^{1/2})$ is bounded below (by a constant depending only on a). Indeed, for t and $\lambda > 0$ real, $e^t - \lambda t$ has a global minimum at $t = \ln\lambda$, so $e^t - \lambda t \ge \lambda(1-\ln\lambda)$. If we write $u = u_+ - u_-$ with $u_+u_- = u_-u_+ = 0$ and $u_+,\ u_- \ge 0$, then

$$-\tau(u\,a) = \tau(u_-a) - \tau(u_+a) = -\tau\bigl(u_+^{1/2}au_+^{1/2}\bigr) + \tau\bigl(u_-^{1/2}au_-^{1/2}\bigr) \ge -\tau\bigl(u_+^{1/2}\|a\|u_+^{1/2}\bigr) + 0 = -\|a\|\,\tau(u_+).$$

On the other hand,

$$\tau(e^u) = \tau\bigl(e^{u_+} + e^{-u_-} - 1\bigr) \ge \tau\bigl(e^{u_+}\bigr) - 1,$$

and thus

$$\tau(e^u - u\,a) \ge \tau\bigl(e^{u_+}\bigr) - \|a\|\,\tau(u_+) - 1 = \tau\bigl(e^{u_+} - \|a\|\,u_+\bigr) - 1 \ge \|a\|\,\bigl(1-\ln\|a\|\bigr) - 1.$$

So we will show that $\mathcal{L}$ must have a minimum point, which will be a solution of (4-3).

Choose $u_n = u_n^* \in A_\theta^\infty$ with $\mathcal{L}(u_n)$ decreasing to $\inf\bigl\{\mathcal{L}(u) : u \in \bigl(A_\theta^\infty\bigr)_{\text{s.a.}}\bigr\}$. Since E and $\tau(e^u - a^{1/2}ua^{1/2})$ are separately bounded below, $E(u_n)$ must remain bounded. That means that $\|\delta_j(u_n)\|_{\ell^2}$ remains bounded for $j = 1,\ 2$.

We can also assume that $\|u_n\|_{\ell^2}$ remains bounded. To see this, it is easiest to use a trick (cf. [Kazdan 1983, pp. 56–57]). Because of the *a priori* bound on solutions of (4-3) coming from the maximum principle (see the first proof above), we can modify the function e^u on the right-hand side of the equation and replace it by some C^∞ function that grows linearly for $u \ge \|a\| + 1$ and decays linearly for $u \le -1 - \ln\|a\|$. (This does not affect the maximum principle argument, so the solutions of the modified equation are the same as for the original one.) This has the effect of changing the term $\tau(e^u)$ in the formula for $\mathcal{L}$ to something that outside of a finite interval behaves like a constant times $\tau(u^2)$, which is $\|u\|_{\ell^2}^2$.

Thus we can assume our minimizing sequence u_n is bounded in the Sobolev space H^1. Since the unit ball of a Hilbert space is weakly compact, after passing to a subsequence, we can assume that u_n converges weakly in the Hilbert space H^1, and by Theorem 3.2, strongly in $H^0 = L^2$, to some $u \in H^1$ which is a minimizer for $\mathcal{L}$. (Compare the argument in [Kazdan 1983, Theorem 5.2].) This u is a "weak solution" of our equation and we just need to show it is smooth, i.e., corresponds to a genuine element of A_θ^∞. This requires an elliptic regularity argument similar to the one in the first proof. □

5. Harmonic unitaries

In this section, we discuss the noncommutative analogue of the classical problem of studying harmonic maps $M \to S^1$, where M is a compact Riemannian manifold and S^1 is given its usual metric. This problem was studied and solved in [Eells and Sampson 1964, pp. 128–129]. The homotopy classes of maps $M \to S^1$ are classified by $H^1(M, \mathbb{Z})$. For each homotopy class in $H^1(M, \mathbb{Z})$, we can think of it as an integral class in $H^1(M, \mathbb{R})$, and represent it (by the de Rham and Hodge Theorems) by a unique harmonic 1-form with integral periods. Integrating this 1-form gives a harmonic map $M \to S^1$ in the given homotopy class. This map is not quite unique since we can compose with an isometry (rotation) of the circle, but except for this we have uniqueness. (This follows from [Eells and Sampson 1964, Proposition, p. 123].)

If we dualize a map $M \to S^1$, we obtain a unital $*$-homomorphism $C(S^1) \to C(M)$, which since $C(S^1)$ is the universal C^*-algebra on a single unitary generator, is basically the same as a choice of a unitary element $u \in C(M)$. This analysis suggests that the noncommutative analogue of a harmonic map to S^1 should be a "harmonic" unitary in a noncommutative C^*-algebra A. Each unitary in A defines a class in the topological K-theory group $K_1(A)$, and for A a unital C^*-algebra, every K_1 class is represented by a unitary in $M_n(A)$ for some n, so since we can replace A by $M_n(A)$, the natural problem is to search for a harmonic representative in a given connected component of $U(A)$ (or, passing to the stable limit, in a given K_1 class).

The next level of complexity up from the case where $A = C(M)$ is commutative is the case where $A = C(M, M_n(\mathbb{C}))$ for some n. In this case, a unitary in $U(A)$ is the same thing as a map $M \to U(n)$, and a harmonic unitary should be the same thing as a harmonic map $M \to U(n)$. For example, suppose $M = S^3$ and $n = 2$. Since there are no maps $M \to S^1$ which are not homotopic to a constant, it is natural to look first at smooth maps $f\colon S^3 \to U(2)$ with $\det \circ f\colon S^3 \to \mathbb{T}$ identically equal to 1, i.e., to look at maps $f\colon S^3 \to SU(2) = S^3$, with both copies of S^3 equipped with the standard round metric. This problem is treated in [Eells and Sampson 1964, Proposition, pp. 129–131]. For example, the identity map $S^3 \to S^3 = SU(2) \hookrightarrow U(2)$ is a harmonic map representing the generator of $K_1(A) = K^{-1}(S^3)$. The study of harmonic maps in other homotopy classes, even just in the simple case of $S^3 \to S^3$, is a complicated issue (see, e.g., [Eells and Sampson 1964, Proposition, pp. 129–131] and [Schoen and Uhlenbeck 1984]); however, this is quite tangential to the main theme of this article, so we won't consider it further.

Instead, we consider now the notion of harmonic unitaries in the case of A_θ. Recall first that $K_1(A_\theta) \cong \mathbb{Z}^2$, with U and V as generators [Pimsner and Voiculescu 1980, Corollary 2.5], and that the canonical map $U(A_\theta)/U(A_\theta)_0 \to K_1(A_\theta)$ is an isomorphism [Rieffel 1987].

Definition 5.1. If $u \in A_\theta^\infty$ is unitary, we define the *energy* of u to be

$$E(u) = \tfrac{1}{2}\,\tau\big((\delta_1(u))^*\delta_1(u) + (\delta_2(u))^*\delta_2(u)\big).$$

Obviously this is constructed so as to be ≥ 0. This definition also coincides with the energy defined in Definition 2.10, provided we insert the appropriate $*$'s in the latter (which we can do without changing anything since there we were taking u to be self-adjoint). The unitary u is called *harmonic* if it is a critical point for $E\colon\ U(A_\theta^\infty) \to [0,\infty)$. By the discussion above, a harmonic unitary is the noncommutative analogue of a harmonic circle-valued function on a manifold.

Remark 5.2. Note that in Definition 5.1, $E(u)$ is invariant under multiplication of u by a scalar $\lambda \in \mathbb{T}$. Thus E descends to a functional on the *projective* unitary group $PU(A_\theta^\infty)$ and any sort of uniqueness result for harmonic unitaries can only be up to multiplication of u by a scalar $\lambda \in \mathbb{T}$. This is analogous to what happens in the case of harmonic maps $M \to \mathbb{T}$, where the associated harmonic 1-form is unique but the map itself is only defined up to a constant of integration.

Theorem 5.3. *If $u \in A_\theta^\infty$ is unitary, then u is harmonic if and only if it satisfies the Euler–Lagrange equation*

$$u^*(\Delta u) + (\delta_1(u))^*\,\delta_1(u) + (\delta_2(u))^*\,\delta_2(u) = 0. \tag{5-1}$$

Note that this equation is elliptic (if we drop lower-order terms, it reduces to Laplace's equation $\Delta u = 0$), but highly nonlinear.

Proof. First note that for u unitary, since $u\,u^* = u^*u = 1$, we have

$$\delta_j(u)\,u^* + u\,(\delta_j(u))^* = (\delta_j(u))^*\,u + u^*\,\delta_j(u) = 0,$$

$j = 1,\ 2$. If u is unitary, then any nearby unitary is of the form ue^{ith}, $h = h^*$, and

$$\frac{d}{dt}\bigg|_{t=0} E(ue^{ith}) = \tfrac{1}{2}\,\tau\Big(-i\delta_1(h)u^*\delta_1(u) + i\delta_1(u)^*u\delta_1(h)\ +\ \text{similar expression with } \delta_2\Big).$$

We can use the trace property to move all the $\delta_j(h)$'s to the front. So u is a critical point if and only if for all $h = h^*$,

$$\tau\Big(\delta_1(h)\operatorname{Im}\big(u^*\delta_1(u)\big) + \delta_2(h)\operatorname{Im}\big(u^*\delta_2(u)\big)\Big) = 0. \tag{5-2}$$

In (5-2), the Im's can be omitted since we have seen that u unitary $\Rightarrow \delta_j(u)^*u$ skew-adjoint. Thus u is harmonic if and only if

$$\tau\Big(\delta_1(h)\,\big(u^*\delta_1(u)\big) + \delta_2(h)\,\big(u^*\delta_2(u)\big)\Big) = 0$$

for all $h = h^*$ in A_θ^∞. Now apply integration by parts (Lemma 2.1). We see that u is harmonic if and only if

$$\tau\Big(h\,\delta_1\big(u^*\delta_1(u)\big) + h\,\delta_2\big(u^*\delta_2(u)\big)\Big) = 0$$

for all $h = h^*$ in A_θ^∞. Since the trace pairing is nondegenerate, the Theorem follows. $\square$

It seems natural to make the following conjecture:

Conjecture 5.4. *In each connected component of $PU(A_\theta^\infty)$, the functional E has a unique minimum, given by scalar multiples of U^nV^m. These are the only harmonic unitaries in this component.*

Unfortunately, because of the complicated nonlinearity in (5-1), plus complications coming from noncommutativity, we have not been able to prove the Conjecture 5.4. However, we have the following partial result. In particular, we see that every connected component in $U(A_\theta^\infty)$ contains a harmonic unitary which is energy-minimizing.

Theorem 5.5. *The scalar multiples of $U^m V^n$ are harmonic and are strict local minima for E. Any harmonic unitary u depending on U alone is a scalar multiple of a power of U. Similarly, any harmonic unitary u depending on V alone is a scalar multiple of a power of V.*

Proof. First suppose u depends on U alone. Then $\delta_2(u) = 0$. So by the proof of Theorem 5.3, if u is harmonic, then $\tau\big(\delta_1(h)\cdot\delta_1(u)^*u\big) = 0$ for all $h = h^*$. This must also hold for general h (not necessarily self-adjoint) since we can split h into its self-adjoint and skew-adjoint parts. Since the range of δ_1 contains U^m unless $m = 0$, $\tau(\delta_1(u)^*u\,U^m)$=0 for $m \neq 0$, which means (since $\delta_1(u)^*u$ depends only on U) that $\delta_1(u)^*u$ is a scalar. Thus u is an eigenfunction for δ_1 and so $u = e^{i\lambda}U^m$ for some m. The case where u depends on V alone is obviously similar.

Next let's examine $u = U^m V^n$. Since $E(U^m V^n) = 2\pi^2(m^2+n^2)$ while

$$(U^m V^n)^*\Delta(U^m V^n) = -4\pi^2(m^2+n^2),$$

u satisfies (5-1) and is therefore harmonic. We show it is a local minimum for E; in fact, the minimum is strict once we pass to $PU(A_\theta^\infty)$. We expand $\delta_j(ue^{ith})$, with $h = h^*$, out to second order in t. Note that with $\delta = \delta_1$ or δ_2,

$$\delta(ue^{ith}) = \delta(u) + it\,\big(\delta(u)h + u\delta(h)\big) - \tfrac{1}{2}t^2\,\big(\delta(u)\,h^2 + u\,\delta(h)\,h + u\,h\,\delta(h)\big) + O(t^3).$$

We substitute this into the formula for $E(ue^{ith})$. The terms linear in t cancel since u is harmonic, and we find that

$$\begin{aligned}E\big(u\,e^{ith}\big) &= 2\pi^2(m^2+n^2)\\ &\quad + t^2\,\tau\Big(\big(\delta_1(u)h + u\delta_1(h)\big)^*\big(\delta_1(u)h + u\delta_1(h)\big) - \tfrac{1}{2}\delta_1(u)^*\big(\delta_1(u)\,h^2 + u\,\delta_1(h)\,h + u\,h\,\delta_1(h)\big)\\ &\qquad - \tfrac{1}{2}(h^2\,\delta_1(u)^* + h\,\delta_1(h)\,u^* + \delta_1(h)\,h\,u^*)\,\delta_1(u) + \text{similar expressions with } \delta_2\Big) + O(t^3).\end{aligned}$$

This actually simplifies considerably since u is an eigenvector for both δ_1 and δ_2, so that $\delta_j(u)^*\delta_j(u)$, $\delta_j(u)^*u$, and $u^*\delta_j(u)$ are all scalars. It turns out that almost everything cancels and one gets

$$\begin{aligned}E(ue^{iht}) &= 2\pi^2(m^2+n^2) + \tfrac{1}{2}t^2\tau\big(\delta_1(h)^2 + \delta_2(h)^2\big) + O(t^3)\\ &= 2\pi^2(m^2+n^2) + t^2\,E(h) + O(t^3).\end{aligned}$$

By Proposition 2.11, the term in t^2 vanishes exactly when h is a constant, and in that case $E(ue^{iht}) = E(u) = 2\pi^2(m^2+n^2)$ (exactly). Otherwise, the coefficient of t^2 is strictly positive and $E(ue^{iht})$ has a strict local minimum at $t = 0$. □

6. The Laplacian and holomorphic geometry

As we have seen, Δ on A_θ behaves very much like the classical Laplacian on $\mathbb{T}^2$. But the Laplacian in (real) dimension 2 is very closely related to holomorphic geometry in complex dimension 1. That suggests that the theory we have developed should be closely related to the Cauchy–Riemann operators ∂ and $\bar\partial$ on noncommutative elliptic curves, as developed in references like [Polishchuk and Schwarz 2003; Polishchuk 2004].

In classical analysis (in one complex variable), one usually sets $\bar\partial = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\,\frac{\partial}{\partial x_2}\right)$, the Cauchy–Riemann operator, with ∂ its complex conjugate. Then $\Delta = 4\,\partial\,\bar\partial$. In our situation, the obvious analogue is to set $\bar\partial = \frac{1}{2}\left(\delta_1 + i\,\delta_2\right)$.[3] Comparable to Proposition 2.4 is:

Proposition 6.1. *The operator* $\bar\partial\colon\ A_\theta^\infty \to A_\theta^\infty$ *has kernel given by scalar multiples of the identity, and restricts to a bijection on* $\ker\tau$.

Proof. Immediate from the fact that if $a = \sum_{m,n} c_{m,n} U^m V^n$, then

$$\bar\partial a = \pi\, i \sum_{m,n} (m + in)\, c_{m,n} U^m V^n,$$

together with the characterization of elements of A_θ^∞ in terms of rapidly decreasing Fourier series. □

Thus the noncommutative torus admits no nontrivial global holomorphic functions. This is not surprising since a compact complex manifold admits no nonconstant global holomorphic functions. However, assuming $\tau(f) = 0$, we can solve the *inhomogeneous Cauchy–Riemann equation* $\bar\partial u = f$, which in the classical case is related to the proof of the Mittag-Leffler Theorem (see, for example, [Hörmander 1990, Chapter 1]).

In some situations, one is led to the more complicated equation $(\bar\partial u)\, u^{-1} = f$, (similar to the one above but with $\bar\partial$ replaced by the logarithmic Cauchy–Riemann operator. This equation can be rewritten as $\bar\partial u = f u$. Is was already studied (under an alternative convention about whether one should multiply on the left or the right) in a (different) noncommutative context in [Bost 1990], and then by Polishchuk:

Theorem 6.2 [Polishchuk 2006]. *Let* $f \in A_\theta$. *Then the equation* $\bar\partial u = f u$ *has a nonzero solution if and only if* $\tau(f) \in \pi i(\mathbb{Z} + i\mathbb{Z})$.

(A slightly different convention is used in the given reference, and in [Polishchuk and Schwarz 2003]: in those works, $\bar\partial$ is taken as $(x + iy)\delta_1 + \delta_2$, with $y < 0$. When $x = 0$ and $y = -1$, this is what we have here, up to a constant factor of $-2i$. This constant explains why the result looks different. With our convention, $u = U^m V^n$ solves $\bar\partial u = f u$ with $f = \pi i(m + in)$.)

The relevance of Theorem 6.2 concerns the theory of noncommutative *meromorphic* functions. While a compact complex manifold admits no nonconstant global holomorphic functions, it can admit nonconstant meromorphic functions, such as (in the case of an elliptic curve) elliptic functions like the Weierstraß $\wp$ function. There are two ways we can view meromorphic functions on a Riemann surface M. On the one hand, they can be considered as ratios of holomorphic sections of holomorphic line bundles $\mathscr{L}$ of M. On the other hand, they can be considered as formal quotients of functions that satisfy the Cauchy–Riemann equation.

These points of view, applied to a noncommutative torus, are equivalent via the following reasoning. A holomorphic vector bundle is defined via its module of (smooth) sections, which is a finitely generated

[3] We could also study different conformal structures on the torus, by changing the i here to another complex number in the upper half-plane, but for the problems we will study here, this makes no essential difference.

projective (right) A_θ^∞-module. This module must be equipped with an operator $\overline{\nabla}$ satisfying the basic axiom

$$\overline{\nabla}(s \cdot a) = \overline{\nabla}(s) \cdot a + s \cdot \bar{\partial}(a).$$

If we assume the module is A_θ^∞ itself (i.e., the vector bundle is of dimension 1, i.e., a line bundle), then this operator is determined by $f = \overline{\nabla}(1)$, in that for any s,

$$\overline{\nabla}(s) = \overline{\nabla}(1 \cdot s) = f \cdot s + 1\, \bar{\partial}(s) = \bar{\partial}(s) + fs.$$

A holomorphic section of the bundle is then a solution s of $\bar{\partial}(s) + fs = 0$.

On the other hand, the natural definition of meromorphic functions is the following.

Definition 6.3. A *meromorphic function on the noncommutative torus* A_θ is a formal quotient $u^{-1}v$, with $u,\ v \in \mathrm{dom}(\bar{\partial}) \subset A_\theta$, satisfying the Cauchy–Riemann equation (in the sense to be made precise below). Here we don't want to require that u be invertible in A_θ (otherwise $u^{-1}v$ would be holomorphic, hence constant), so we simply want u to be regular (in the sense of not being either a left or right zero divisor), and the inverse is to be interpreted in a formal sense (or in the maximal ring of quotients [Berberian 1982], the algebra of unbounded operators affiliated to the hyperfinite II_1 factor obtained by completing A_θ in its trace representation). Then the condition that $u^{-1}v$ be meromorphic is that

$$0 = \bar{\partial}(u^{-1}v) = \bar{\partial}(u^{-1})v + u^{-1}\bar{\partial}v = -u^{-1}\bar{\partial}(u)u^{-1}v + u^{-1}\bar{\partial}v,$$

or (via multiplication by u on the left) that

$$\bar{\partial}v = fv, \quad \bar{\partial}u = fu, \tag{6-1}$$

which says precisely that our meromorphic function is a quotient of two holomorphic sections of a holomorphic line bundle with $\overline{\nabla} = \bar{\partial} + f$. In the other direction, if u and v satisfy (6-1) and u is regular, so that the formal expression $u^{-1}v$ makes sense, then we formally have

$$\begin{aligned}\bar{\partial}(u^{-1}v) &= \bar{\partial}(u^{-1})v + u^{-1}\bar{\partial}v = -u^{-1}\bar{\partial}(u)u^{-1}v + u^{-1}\bar{\partial}v \\ &= -u^{-1}fuu^{-1}v + u^{-1}fv = -u^{-1}fv + u^{-1}fv = 0,\end{aligned}$$

and $u^{-1}v$ is meromorphic.

In accordance with the classical existence theorem of Weierstraß for elliptic functions, we have:

Proposition 6.4. *There exist nonconstant meromorphic functions on the noncommutative torus A_θ, in the sense of Definition 6.3.*

Proof. This follows immediately from the discussion in [Polishchuk 2006, §3], which shows that there are choices for f for which the holomorphic connection $\overline{\nabla}$ is reducible, with a space of holomorphic sections of dimension bigger than 1, and thus there are solutions of (6-1) with u and v not linearly dependent. Note that if this is the case, u cannot be invertible ([Polishchuk 2006, Lemma 3.14]—we also know this independently from Proposition 6.1). But we do require u to be regular, so we need to check that this can be achieved. For example, suppose e is a proper projection in A_θ^∞ ("proper" means $0 < \tau(e) < 1$) of trace $m + n\theta$ with n relatively prime to both m and $1 - m$. The trivial rank-one right A_θ^∞ module splits as $eA_\theta^\infty \oplus (1 - e)A_\theta^\infty$, and we can arrange to choose a holomorphic connection on A_θ^∞ that is reducible in a way compatible with this splitting, so that there are 1-dimensional spaces of holomorphic sections on each of eA_θ^∞ and $(1-e)A_\theta^\infty$. By the explicit formulas in [Polishchuk and Schwarz 2003, Proposition 2.5], these come from real-analytic functions in $\mathcal{S}(\mathbb{R})$, and so it's evident that the u that results from putting these together is regular, as by [Berberian 1982], it's enough to observe that its left and right support projections are equal to 1. □

On the other hand, there is also a nonexistence result for meromorphic functions on the (classical) torus: no such nonconstant function exists with only a single simple pole [Ahlfors 1978, Corollary to Theorem 4, p. 271]. We can find an analogue of this in the noncommutative situation also. To explain it, first note that in the sense of distributions on the complex plane, $\bar\partial\left(\frac{1}{z}\right)$ is not zero (if it were, $\frac{1}{z}$ would have a removable singularity, by elliptic regularity), but rather is equal to $\pi\,\delta$, where δ is the Dirac δ-distribution at 0. Suppose there were a meromorphic function f on $\mathbb{T}^2 = \mathbb{C}/(\mathbb{Z}+i\mathbb{Z})$ with at worst one simple pole and no other poles. Then f would be locally integrable and, after translation to move the pole to 0, would define a distribution on $\mathbb{T}^2$ with $\bar\partial(f)$ a multiple of δ. Thus the Fourier series of $\bar\partial(f)$ would be a multiple of the Fourier series of δ, which is $\sum_{m,n} U^m V^n$. And in fact Fourier analysis gives another proof of the nonexistence theorem not using residue calculus. Suppose f were nonconstant. Since a compact complex manifold admits no nonconstant holomorphic functions, f cannot be holomorphic, which means that $\bar\partial f$ must be nonzero in the sense of distributions. Since $\bar\partial(f)$ is a multiple of $\sum_{m,n} U^m V^n$, the proportionality constant, which is also the $(0,0)$ Fourier coefficient of $\bar\partial f$, must be nonzero. But this is impossible since the Fourier series of any distribution in the image on $\bar\partial$ must have zero constant term. The noncommutative analogue of all this is the following:

Proposition 6.5. *Let f be a distribution in the dual of A_θ^∞. (The distributions consist of formal Fourier series $\sum_{m,n} c_{m,n} U^m V^n$ with $\{c_{m,n}\}$ of tempered growth.) Suppose $\bar\partial f$ is a multiple of $\sum_{m,n} U^m V^n$. Then f is a constant.*

Proof. This follows exactly the lines as the argument above for the classical theorem. If $\bar\partial f$ has formal Fourier expansion $c\sum_{m,n} U^m V^n$, then the (m,n) coefficient, c, must be divisible by $m+in$ for all (m,n). Because of the $(0,0)$ coefficient, this is only possible if $c=0$. But if $c=0$, then f is in the distributional kernel of $\bar\partial$, which forces all the Fourier coefficients of f to vanish except for the constant term. □

In fact, essentially the same proof proves a slightly more general statement, which in the classical case is equivalent to [Ahlfors 1978, Theorem 4, p. 271]. For the analysis above shows that the sum of the residues of a meromorphic function f on $\mathbb{T}^2$, when the function is considered as a distribution[4], is precisely the constant term in the Fourier series of $\bar\partial f$, up to a factor of π. The analogue of the sum of the residues theorem in the noncommutative world is this:

Proposition 6.6. *Let f be a distribution in the dual of A_θ^∞. Then the constant term in the (formal) Fourier series of $\bar\partial f$ is zero.*

Proof. Essentially the same as before. □

The connection with the main subject of this paper is of course that meromorphic functions w as studied in this section are singular solutions of Laplace's equation $\Delta w = 0$, since $\Delta = 4\,\partial\,\bar\partial$. More precisely, "singular solution" means classically that as a distribution, Δw is not necessarily 0, but has

[4]This requires a comment. A meromorphic function with simple poles is locally integrable, thus defines a distribution in the obvious way. A meromorphic function with higher-order poles is not locally integrable, but can be made into a distribution of principal value integral type. This distribution is not a measure.

countable support. In the noncommutative setting, we do not have a notion of support for a distribution, but the same basic idea applies.

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AN INVERSE SOURCE PROBLEM IN OPTICAL MOLECULAR IMAGING

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We study the direct and an inverse source problem for the radiative transfer equation arising in optical molecular imaging. We show that for generic absorption and scattering coefficients, the direct problem is well-posed and the inverse one is uniquely solvable, with a stability estimate.

1. Introduction

We consider an inverse source problem arising in optical molecular imaging (OMI) which is currently undergoing a rapid expansion. The design of new biochemical markers that can detect faulty genes and other molecular processes allows us to detect diseases before macroscopic symptoms appear. This has been studied extensively in the bioengineering literature. See for instance [Chang et al. 1997; Contag et al. 1998; Jang et al. 2000]. Unlike higher-energetic markers used in classical nuclear imaging techniques such as single photon emission computed tomography (SPECT), positron emission tomography (PET), as well as magnetic resonance imaging (MRI), optical markers emit relatively low-frequency photons. The objective of OMI is to reconstruct the concentration of such markers from their radiations measured at the boundary of the domain. The radiations in OMI are governed by the equations of radiative transfer and the inverse problem in OMI is thus an inverse transport source problem, at least once the optical properties of the underlying medium are known. We now describe more precisely the mathematical problem.

We assume that Ω is a bounded domain of $\mathbb{R}^n$ with smooth boundary. We will assume also that Ω is strictly convex. This is not an essential assumption since for the problem that we study, one can always push the boundary away and make it strictly convex, without losing generality. In our main result Theorem 3.1, we assume that the data is given on the boundary of a larger $\Omega_1 \Supset \Omega$. This is not essential for the uniqueness result but it is essential for the stability estimate (9).

The radiative transport equation is given by

$$\theta \cdot \nabla_x u(x,\theta) + \sigma(x,\theta)u(x,\theta) - \int_{S^{n-1}} k(x,\theta,\theta')u(x,\theta')\,\mathrm{d}\theta' = f(x), \qquad u|_{\partial_- S\Omega} = 0, \tag{1}$$

where the *absorption* σ and the *collision kernel* k are functions with a regularity that will be specified below. The source term f is assumed to depend on x only.

In Section 2 we study the direct problem. We show that for an open and dense set of absorption and scattering coefficients the direct problem (1) is well-posed. See Theorem 2.1 for details.

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The boundary measurements are modeled by

$$Xf(x,\theta) = u|_{\partial_+ S\Omega}, \quad (x,\theta) \in \partial_+ S\Omega,$$

where $u(x,\theta)$ is a solution of (1), and $\partial_+ S\Omega$ denotes the points $x \in \partial\Omega$ with direction θ pointing outwards.

In Section 3 we consider the inverse source problem, that consists in determining the source term f from measuring Xf. Notice that in the case $\sigma = k = 0$ the linear operator X is the standard X-ray transform and when $k = 0$, X is a weighted X-ray transform (see Section 2).

This inverse problem has been considered in several papers in the mathematical and engineering community [Bal and Tamasan 2007; Larsen 1975; Panchenko 1993; Sharafutdinov 1997; Siewert 1993; Yi et al. 1992]. In particular in [Bal and Tamasan 2007] it is shown that one can prove uniqueness when $k = k(x, \theta\cdot\theta')$, and k is small enough in a suitable norm. We show that for the absorption and scattering in an dense and open subset we can uniquely determine the source f from the boundary measurements. We also prove a stability estimate. See Theorem 3.1 for details.

2. The direct problem

Set

$$T_0 = \theta\cdot\nabla_x, \quad T_1 = T_0 + \sigma, \quad T = T_0 + \sigma - K,$$

where σ is viewed as the operator of multiplication by $\sigma(x,\theta)$, and K is the integral operator in (1).

Let u solve

$$Tu = f, \quad u|_{\partial_- S\Omega} = 0. \tag{2}$$

As mentioned in the introduction the operator X is the X-ray transform, if $\sigma = k = 0$,

$$Xf(x,\theta) = If(x,\theta) := \int_{\tau_-(x,\theta)}^{0} f(x+t\theta)\,\mathrm{d}t, \quad (x,\theta) \in \partial_+ S\Omega,$$

where $\pm\tau_\pm(x,\theta) \geq 0$ are defined by $(x, x+\tau_\pm(x,\theta)) \in \partial_\pm S\Omega$. We will always extend f as 0 outside Ω so we can assume that we integrate above over $\mathbb{R}$. If $k = 0$, then X reduces to the following weighted X-ray transform

$$Xf(x,\theta) = I_\sigma f(x,\theta) := \int E(x+t\theta,\theta) f(x+t\theta)\,\mathrm{d}t, \quad (x,\theta) \in \partial_+ S\Omega, \tag{3}$$

where

$$E(x,\theta) = \exp\biggl(-\int_0^\infty \sigma(x+s\theta,\theta)\,\mathrm{d}s\biggr).$$

If $\sigma > 0$ depends on x only, this is known as the attenuated X-ray transform, that is injective, and there is an explicit inversion formula (see [Novikov 2002; Arbuzov et al. 1997]).

We define the adjoint X^* of X with respect to the measure $\mathrm{d}\Sigma$ defined above. We will view X as a perturbation of I_σ, and our goal is to show that X^*X is a relatively compact perturbation of $I_\sigma^* I_\sigma$.

First we will analyze the direct problem. Some conditions are needed for its well-posedness, that usually involve smallness of k with respect to σ; see, for example, [Dautray and Lions 1993; Reed and Simon 1979] and [Sharafutdinov 1997] for the Riemannian case. In the next theorem, f is allowed to depend on θ as well and we show that the direct problem is generically solvable.

Theorem 2.1. *There exists an open and dense set of pairs* $(\sigma, k) \in C^2(\bar{\Omega} \times S^{n-1}) \times C^2(\bar{\Omega} \times S^{n-1} \times S^{n-1})$, *including a neighborhood of* $(0, 0)$, *so that for each* (σ, k) *in that set,*

(a) *the direct problem* (2) *has a unique solution* $u \in L^2(\Omega \times S^{n-1})$ *for any* $f \in L^2(\Omega \times S^{n-1})$ *depending both on* x *and* θ;

(b) X *extends to a bounded operator*

$$X : L^2(\Omega \times S^{n-1}) \to L^2(\partial_+ S\Omega,\ \mathrm{d}\Sigma).$$

Proof. We start with the analysis of the direct problem (2). In what follows, let T_0, T_1 and T denote the operators given by (1) in $L^2(\Omega \times S^{n-1})$ with domain

$$D(T_0) = D(T_1) = D(T) = \big\{ f \in L^2(\Omega \times S^{n-1});\ \theta \cdot \nabla_x u \in L^2(\Omega \times S^{n-1}),\ u|_{\partial_- S\Omega} = 0 \big\}.$$

We will assume here that f depends both on x and θ. Note first that the solution to the problem (2) with $k = 0$ is given by $u = T_1^{-1} f$, where

$$[T_1^{-1} f](x, \theta) = \int_{-\infty}^{0} \exp\Big(- \int_s^0 \sigma(x + \tau\theta, \theta)\, \mathrm{d}\tau \Big) f(x + s\theta, \theta)\, \mathrm{d}s.$$

This follows easily from the fact that E is an integrating factor, that is, $T_0 = E^{-1} T_1 E$.

Apply T_1^{-1} to both sides of (2) to get

$$u = T_1^{-1}(Ku + f).$$

We therefore see that (2) is equivalent to the integral equation

$$(\mathrm{Id} - T_1^{-1} K)u = T_1^{-1} f.$$

Therefore, if $(\mathrm{Id} - T_1^{-1} K)$ is invertible, (2) is uniquely solvable, and the solution is given by

$$u = T^{-1} f = (\mathrm{Id} - T_1^{-1} K)^{-1} T_1^{-1} f. \tag{4}$$

When f depends on x only, set

$$[Jf](x, \theta) := f(x).$$

Then

$$u = T^{-1} J f = (\mathrm{Id} - T_1^{-1} K)^{-1} T_1^{-1} J f.$$

Lemma 2.2. *The operator* $K T_1^{-1} J : L^2(\Omega) \to L^2(\Omega \times S^{n-1})$ *is compact.*

Proof. Let first f depend both on x and θ. Then

$$\begin{aligned} [K T_1^{-1} f](x, \theta) &= \int_{S^{n-1}} k(x, \theta, \theta') \int_{-\infty}^{0} \exp\Big(- \int_s^0 \sigma(x + \tau\theta', \theta')\, \mathrm{d}\tau \Big) f(x + s\theta', \theta')\, \mathrm{d}s\, \mathrm{d}\theta' \\ &= \int \frac{\Sigma\big(x, |x-y|, \frac{x-y}{|x-y|}\big) k\big(x, \theta, \frac{x-y}{|x-y|}\big)}{|x-y|^{n-1}} f\Big(y, \frac{x-y}{|x-y|}\Big)\, \mathrm{d}y, \end{aligned} \tag{5}$$

where

$$\Sigma(x, s, \theta') = \exp\Big(- \int_{-s}^{0} \sigma(x + \tau\theta', \theta')\, \mathrm{d}\tau \Big)$$

(we replaced s by $-s$ and then made the change $x - s\theta' = y$).

Assume now that f depends on x only, that is, we have Jf above with such an f. Then

$$[KT_1^{-1}J]f(x,\theta) = \int_\Omega \frac{\Sigma\big(x, |x-y|, \frac{x-y}{|x-y|}\big)k\big(x,\theta,\frac{x-y}{|x-y|}\big)}{|x-y|^{n-1}} f(y)\,\mathrm{d}y. \tag{6}$$

The integral above is a typical singular operator with a weakly singular kernel, and an additional parameter θ; see [Michlin and Prössdorf 1980; Stein 1970]. Under the smoothness assumptions on σ and k, it is easy to see that $\partial_\theta KT_1^{-1}$ and $\partial_x KT_1^{-1}$ are bounded operators; see Proposition 3.4 below. This completes the proof of the lemma. $\square$

Remark 2.3. The arguments above do not prove that KT_1^{-1} is compact in $L^2(\Omega\times S^{n-1})$ because there are no enough integrations in this case to apply the same arguments. Its square however is compact, as the next lemma shows. On the other hand, under appropriate smoothness assumptions on k, similar to those in Theorem 3.1 (see (9)), the operator KT_1^{-1} is compact, indeed. This is a consequence of the velocity averaging lemma that states that if $k = k(\theta')$ with k of appropriate regularity, then KT^{-1} is compact in $L^2(\Omega\times S^{n-1})$. The gained regularity then is $\frac{1}{2}$ only, not 1. Now, for $k = k(x,\theta',\theta)$ smooth enough, one can approximate K uniformly with finite sums of operators with kernels $\kappa(x)\Theta'(\theta')\Theta(\theta)$, each one of which is compact. For more details, we refer to [Mokhtar-Kharroubi 1997] and the references there.

Lemma 2.4. *The operator $KT_1^{-1}K : L^2(\Omega\times S^{n-1}) \to L^2(\Omega\times S^{n-1})$ is compact.*

Proof. Replace $f\big(y, \frac{x-y}{|x-y|}\big)$ in (5) by

$$[Kf]\Big(y, \frac{x-y}{|x-y|}\Big) = \int_{S^{n-1}} k\Big(y, \frac{x-y}{|x-y|}, \theta'\Big) f(y,\theta')\,\mathrm{d}\theta'.$$

Then the compactness follows from the same arguments as in Lemma 2.2. Indeed, we have

$$[KT_1^{-1}Kf](x,\theta) = \iint_{\Omega\times S^{n-1}} \frac{\alpha\big(x, y, |x-y|, \frac{x-y}{|x-y|}, \theta, \theta'\big)}{|x-y|^{n-1}} f(y,\theta')\,\mathrm{d}y\,\mathrm{d}\theta',$$

with an obvious definition of α. In particular, all second order derivatives of α are bounded. Let $g(x,\theta,\theta')$ be the y-integral above, that is, the right hand side above becomes $\int g(x,\theta,\theta')\,\mathrm{d}\theta'$. Then by Proposition 3.4 below,

$$\int_\Omega |\partial_x g(x,\theta,\theta')|^2\,\mathrm{d}x \le C\int_\Omega |f(x,\theta')|^2\,\mathrm{d}x$$

for any θ, θ'. In particular,

$$\iint_{\Omega\times S^{n-1}} |\partial_x g(x,\theta,\theta')|^2\,\mathrm{d}x\,\mathrm{d}\theta' \le C\|f\|_{L^2}^2.$$

Then

$$\begin{aligned}\|\partial_x KT_1^{-1}Kf\|^2 &= \iint_{\Omega\times S^{n-1}} \Big|\int_{S^{n-1}} \partial_x g(x,\theta,\theta')\,\mathrm{d}\theta'\Big|^2\,\mathrm{d}x\,\mathrm{d}\theta\\ &\le C\iint_{\Omega\times S^{n-1}}\int_{S^{n-1}} |\partial_x g(x,\theta,\theta')|^2\,\mathrm{d}\theta'\mathrm{d}x\,\mathrm{d}\theta\\ &\le C'\|f\|_{L^2}^2.\end{aligned}$$

It is easy to see that $\partial_\theta K T_1^{-1} K f \in L^2$ as well. This, and the estimate above, imply the compactness of $K T_1^{-1} K$. □

We proceed with the proof of part (a) of the theorem. We are looking for k so that T^{-1} exists. Consider

$$A(\lambda) = \big(\mathrm{Id} - (\lambda K T_1^{-1})^2\big)^{-1}$$

in $L^2(\Omega \times S^{n-1})$. The operator $(K T_1^{-1})^2$ is compact, and for $\lambda = 0$, the resolvent above exists. By the analytic Fredholm theorem [Reed and Simon 1980], $A(\lambda)$ is a meromorphic family of bounded operators. In particular, it exists for all but a discrete set of λ's. Thus for the those λ's, the resolvent $(\mathrm{Id} - \lambda K T_1^{-1})^{-1}$ exists and is given by

$$(\mathrm{Id} - \lambda K T_1^{-1})^{-1} = (\mathrm{Id} + \lambda K T_1^{-1}) A(\lambda). \tag{7}$$

Indeed, it is obvious that the operator on the right hand side above is a right inverse to $\mathrm{Id} - \lambda K T_1^{-1}$. For $|\lambda| \ll 1$, one can use Neumann series to show that it is left inverse as well. One can use analytic continuation around the poles to show that this remains true for all λ that are not poles.

By (4), then T^{-1} exists for such λ's and k replaced by λk. In particular, this shows that the set of such (k, σ) is dense. Standard perturbation arguments show that the set of k's for which $\mathrm{Id} - \lambda K T_1^{-1}$ is invertible, is open in C^0 for a fixed σ and the set of pairs $(\sigma, k) \in C^0 \times C^0$ with the same property is open, too. Since we just showed that it is dense as well in $C^0 \times C^0$, this completes the proof of (a).

We proceed with the proof of (b). For X we get (see (4)),

$$Xf = R_+ T^{-1} f = R_+ (\mathrm{Id} - T_1^{-1} K)^{-1} T_1^{-1} f,$$

where $R_+ h = h|_{\partial_+ S\Omega}$. If f depends on x only, then

$$Xf = R_+ T^{-1} J f = R_+ (\mathrm{Id} - T_1^{-1} K)^{-1} T_1^{-1} J f. \tag{8}$$

Notice first that

$$(\mathrm{Id} - T_1^{-1} K)^{-1} T_1^{-1} = T_1^{-1} (\mathrm{Id} - K T_1^{-1})^{-1},$$

and in particular, the resolvent on the left exists if and only if the resolvent in the right hand side does. We therefore have

$$Xf = R_+ T_1^{-1} (\mathrm{Id} - K T_1^{-1})^{-1} J f.$$

To prove (b), it is enough to show that

$$R_+ T_1^{-1} : L^2(\Omega \times S^{n-1}) \to L^2(\partial_+ S\Omega,\ \mathrm{d}\Sigma)$$

is bounded. A straightforward computation (see also [Choulli and Stefanov 1999]) shows that

$$\int_{\partial_+ S\Omega} \int_{\tau_-(x,\theta)}^{0} f(x - t\theta, \theta)\, \mathrm{d}t\, \mathrm{d}\Sigma = \int_{\Omega \times S^{n-1}} f(x, \theta)\, \mathrm{d}x\, \mathrm{d}\theta$$

for any $f \in L^1(\Omega \times S^{n-1})$. Therefore,

$$\begin{aligned}\|R_+T_1^{-1}f\|^2_{L^2(\partial_+S\Omega,\,\mathrm{d}\Sigma)} &= \int_{\partial_+S\Omega} |R_+T_1^{-1}f(x,\theta)|^2 \mathrm{d}\Sigma \le \int_{\partial_+S\Omega} \left| \int_{\tau_-(x,\theta)}^0 f(x+t\theta,\theta)\,\mathrm{d}t \right|^2 \mathrm{d}\Sigma \\ &\le \int_{\partial_+S\Omega} \left(|\tau_-(x,\theta)| \int_{\tau_-(x,\theta)}^0 |f(x+t\theta,\theta)|^2\,\mathrm{d}t \right) \mathrm{d}\Sigma \\ &\le \operatorname{diam}(\Omega)\, \|f\|^2_{L^2(\Omega\times S^{n-1})}. \end{aligned}$$

□

3. The inverse source problem

In this section we consider the inverse source problem. The next theorem shows that for generic (σ, k) there is uniqueness and stability. As mentioned in the introduction a similar result has been proven in [Bal and Tamasan 2007] in the case where $k = k(x, \theta \cdot \theta')$, and k is small enough in a suitable norm.

Fix another strictly convex bounded domain Ω_1 so that $\Omega_1 \Supset \Omega$. Extend (σ, k) with regularity as below to functions in Ω_1 with the same regularity. We chose and fix that extension as a continuous operator in those spaces. Define the operator $X_1 : L^2(\Omega_1) \to L^2(\partial_+SM_1)$ in the same way as X. We will be interested in the restriction of X_1 to functions f supported in $\bar\Omega$. We always extend such f as zero to $\Omega_1 \setminus \Omega$. This corresponds to taking measurements on $\partial\Omega_1$ instead of $\partial\Omega$.

Theorem 3.1. *There exists an open and dense set of pairs*

$$(\sigma, k) \in C^2(\bar\Omega \times S^{n-1}) \times C^2\big(\bar\Omega \times S^{n-1}_{\theta'};\ C^{n+1}(S^{n-1}_\theta)\big), \tag{9}$$

including a neighborhood of $(0,0)$, *so that for each* (σ, k) *in that set, the conclusions of Theorem 2.1 hold in* Ω_1, *and*

(a) *the map* X_1 *is injective on* $L^2(\Omega)$,

(b) *the following stability estimate holds*:

$$\|f\|_{L^2(\Omega)} \le C\|X_1^*X_1 f\|_{H^1(\Omega_1)}, \tag{10}$$

for all $f \in L^2(\Omega)$, *with a constant* $C > 0$ *locally uniform in* (σ, k).

Remark 3.2. The smoothness requirement on k can be reduced to $k \in C^2$ if k is of a special form, like $k = \Theta(\theta)\kappa(x, \theta')$ or a finite sum of such; see (15), (16).

From now on, we will drop the subscript 1, and all operators below are as defined before but in the domain Ω_1. We assume that (σ, k) are such that T^{-1} exists. We assume now that X is applied to f that depends on x only. For now, it is not important that f is supported in $\bar\Omega$; that will be needed in (20) and after that; so we apply X to functions in $L^2(\Omega_1)$. By (8),

$$X = I_\sigma + L, \quad L := R_+\big(-\mathrm{Id} + (\mathrm{Id} - T_1^{-1}K)^{-1}\big)T_1^{-1}J \tag{11}$$

(see also (3)). Then

$$X^*X = I_\sigma^* I_\sigma + \mathscr{L}, \quad \mathscr{L} := I_\sigma^* L + L^* I_\sigma + L^* L. \tag{12}$$

In our analysis, we will apply a parametrix of $I_\sigma^* I_\sigma$ to X^*X. That parametrix is a first order operator. For this reason, we study $\partial_x I_\sigma^* L$.

Lemma 3.3. *The operators*

$$\partial_x I_\sigma^* L, \quad \partial_x L^* I_\sigma, \quad \partial_x L^* L$$

are compact as operators mapping $L^2(\Omega_1)$ *into* $L^2(\Omega_1)$.

Proof. To analyze $I_\sigma^* L$, note that L also admits the following representation

$$L = R_+ T_1^{-1} K T_1^{-1} (\mathrm{Id} - K T_1^{-1})^{-1} J. \tag{13}$$

We need to study $I_\sigma^* R_+ T_1^{-1} K T_1^{-1} h$, where $h = h(x, \theta)$. Notice first that

$$[I_\sigma^* h](x) = \int_{S^{n-1}} \overline{E}(x, \theta) h^\sharp(x, \theta)\, \mathrm{d}\theta,$$

where $\overline{E}$ denotes complex conjugate, and $h^\sharp$ is the extension of $h \in C(\partial_+ S\Omega_1)$ as a constant along the lines originating from x in the direction $-\theta$; see, for example, [Frigyik et al. 2008, Section 4]. In other words,

$$h^\sharp(x, \theta) = h\big(x + \tau_+(x, \theta), \theta\big).$$

Next, $R_+ T_1^{-1} h$ looks just like I_σ (see (3)) but with f there depending on θ as well. Therefore,

$$[I_\sigma^* R_+ T_1^{-1} g](x) = \int_{S^{n-1}} \overline{E}(x, \theta) \left[\int_{-\infty}^0 E(x + t\theta, \theta) g(x + t\theta, \theta)\, \mathrm{d}t \right]^\sharp \mathrm{d}\theta.$$

This yields (see [Frigyik et al. 2008] again):

$$\begin{aligned} [I_\sigma^* R_+ T_1^{-1} g](x) &= \iint_{S^{n-1}} \overline{E}(x, \theta)(Eg)(x + t\theta, \theta)\, \mathrm{d}\theta\, \mathrm{d}t \\ &= 2 \int_{\Omega_1} \frac{\big[\overline{E}\big(x, \frac{y-x}{|y-x|}\big)(Eg)\big(y, \frac{y-x}{|y-x|}\big)\big]_{\mathrm{even}}}{|y - x|^{n-1}}\, \mathrm{d}y, \end{aligned} \tag{14}$$

where $F_{\mathrm{even}}(x, \theta)$ is the even part of F as a function of θ. If we set $g = K T_1^{-1} h$, that would give us $I_\sigma^* R_+ T_1^{-1} K T_1^{-1} h$.

Instead of assuming (9), we will make the following weaker assumption at this point: k can be written as the infinite sum

$$k(x, \theta, \theta') = \sum_{j=1}^{\infty} \Theta_j(\theta) \kappa_j(x, \theta') \tag{15}$$

with some functions Θ_j and κ_j so that

$$\sum_{j=1}^{\infty} \|\Theta_j\|_{H^1(S^{n-1})} \|\kappa_j\|_{L^\infty(\Omega_1 \times S^{n-1})} < \infty. \tag{16}$$

One such way to do this is to choose Θ_j to be the spherical harmonics Y_j; then κ_j are the corresponding Fourier coefficients. Then $\|Y_j\|_{H^1(S^{n-1})} \le C(1 + \lambda_j)$, where λ_j^2 are the eigenvalues of the positive Laplacian on S^{n-1}. Since $\lambda_j = O(j^{1/(n-1)})$, for the uniform convergence of (15) it is enough to have $\|\kappa_j\|_{L^\infty} \le C(1 + \lambda_j)^{-n-\varepsilon}$ with $\varepsilon > 0$. This would be guaranteed if $k \in L^\infty\big(\Omega_1 \times S_{\theta'}^{n-1};\ C_\theta^{n+1}(S^{n-1})\big)$ by standard integration by parts arguments. Therefore, the hypothesis (9) of the theorem implies (15) and (16).

Under this assumption, for $K_j T_1^{-1} h$, where K_j has kennel $\Theta_j \kappa_j$, we have (see (5)):

$$
\begin{aligned}
&[K_j T_1^{-1} h](x,\theta) = \Theta_j(\theta)[B_j h](x),\\
&B_j h(x) := \int_{\Omega_1} \frac{\Sigma\big(x, |x-y|, \frac{x-y}{|x-y|}\big)\kappa_j\big(x, \frac{x-y}{|x-y|}\big)}{|x-y|^{n-1}} h\Big(y, \frac{x-y}{|x-y|}\Big)\, \mathrm{d}y.
\end{aligned}
\tag{17}
$$

We claim now that $B_j(\mathrm{Id} - KT_1^{-1})^{-1} J : L^2(\Omega_1) \to L^2(\Omega_1)$ is compact. We have

$$
(\mathrm{Id} - KT_1^{-1})^{-1} J = J + (\mathrm{Id} - KT_1^{-1})^{-1} K T_1^{-1} J.
$$

By Lemma 2.2, the second term on the right is compact. Therefore, it remains to show that $B_j J$ is compact. Observe that $B_j J h$ is given by (17) with $h = h(x)$. The compactness then follows from Proposition 3.4, assuming that $\kappa_j \in C^2$. On the other hand, $B_j J$ is compact under the assumption that $\kappa_j \in L^\infty$ only, by [Michlin and Prössdorf 1980, Theorem VII.3.3]. Moreover, its norm is bounded by $C\|\kappa_j\|_{L^\infty}$.

We can now write

$$
\begin{aligned}
\partial_x I_\sigma^* L &= \partial_x I_\sigma^* R_+ T_1^{-1} K T_1^{-1} (\mathrm{Id} - KT_1^{-1})^{-1} J\\
&= \sum_{j=1}^{\infty} (\partial_x I_\sigma^* R_+ T_1^{-1} \Theta_j J)\big(B_j (\mathrm{Id} - KT_1^{-1})^{-1} J\big).
\end{aligned}
\tag{18}
$$

We notice first that $\partial_x I_\sigma^* R_+ T_1^{-1} \Theta_j J : L^2(\Omega_1) \to L^2(\Omega_1)$ is bounded by Proposition 3.4 (b), compare to (14), with a norm bounded by $C\|\sigma\|_{C^2}\|\Theta_j\|_{H^1}$. The operator $B_j(\mathrm{Id} - KT_1^{-1})^{-1}J$ on the right is compact, as we have just seen. Therefore, each summand in the right hand side of (18) is a compact operator with a norm not exceeding $C\|\Theta_j\|_{H^1}\|\kappa_j\|_{L^\infty}$, where C depends on σ as well. Then the series in (18) converges uniformly by (16). Under this condition, $\partial_x I_\sigma^* L$ is compact.

To analyze $\partial_x L^* L$, we will follow the proof above. It is enough to show that $\partial_x L^* R_+ T_1^{-1} \Theta_j J : L^2(\Omega) \to L^2(\Omega_1)$ is bounded. We have (see (13)):

$$
\begin{aligned}
\partial_x L^* R_+ T_1^{-1} \Theta_j J &= \partial_x \big(R_+ T_1^{-1} (\mathrm{Id} - KT_1^{-1})^{-1} K T_1^{-1} J\big)^* R_+ T_1^{-1} \Theta_j J\\
&= \partial_x (K T_1^{-1} J)^* \big(R_+ T_1^{-1} (\mathrm{Id} - KT_1^{-1})^{-1}\big)^* R_+ T_1^{-1} \Theta_j J.
\end{aligned}
\tag{19}
$$

Since $R_+ T_1^{-1}$ is bounded, it remains to show that the operator $\partial_x (K T_1^{-1} J)^* : L^2(\Omega_1 \times S^{n-1}) \to L^2(\Omega)$ is bounded, as well. The kernel of the latter is (see (6))

$$
(x, (y,\theta)) \quad \mapsto \quad \partial_x \frac{\Sigma\big(y, |y-x|, \frac{y-x}{|y-x|}\big) k\big(y, \theta, \frac{y-x}{|y-x|}\big)}{|y-x|^{n-1}}.
$$

Then the boundedness of $\partial_x (K T_1^{-1} J)^*$ then follows as in Lemma 2.4.

Finally, the fact that $\partial_x L^* I_\sigma$ is bounded follows from the proof for $\partial_x L^* L$. Indeed,

$$
\partial_x L^* I_\sigma = \partial_x L^* R_+ T_1^{-1} J,
$$

compare with (19), where we can set $\Theta_j = 1$.

This completes the proof of Lemma 3.3. ∎

Proof of Theorem 3.1. We return to the analysis of the operator X^*X; see (12). We showed in Lemma 3.3 that, up to a relative compact operator, X^*X coincides with $I_\sigma^* I_\sigma$. Assume that σ and k are C^∞. Let Q be a parametrix (of order 1) to the elliptic ΨDO $I_\sigma^* I_\sigma$ in Ω_1. We restrict the image of Q to $L^2(\Omega)$, that is, we view Q as an operator $Q : H^1(\Omega_1) \to L^2(\Omega)$. Then for any f supported in $\bar{\Omega}$, we have

$$Q I_\sigma^* I_\sigma f = f + K_1 f, \tag{20}$$

where K_1 is of order -1 near Ω. Apply Q to X^*X to get

$$QX^*Xf = f + K_2 f, \quad K_2 := K_1 + Q\mathscr{L}. \tag{21}$$

Then $K_2 : L^2(\Omega) \to L^2(\Omega)$ is compact. We get that the problem of inverting X^*X is reduced to a Fredholm equation. We will show that it is generically solvable, as in the theorem.

We show first that the set of pairs for which X is injective is dense.

By the results of [Frigyik et al. 2008, Theorems 1 and 2], if σ is real analytic in a $\bar{\Omega}_1$, then I_σ is injective, and therefore $I_\sigma^* I_\sigma$, is injective as well. Moreover, for a small $C^2(\bar{\Omega})$, perturbation preserves that property. Actually, the remark after [Frigyik et al. 2008, Theorem 2] shows that this is true even for small enough C^1 perturbations. Fix σ real analytic in $\bar{\Omega}_1$. Fix k as well so that (σ, k) belongs to the generic set in Theorem 2.1, related to Ω_1, and the regularity assumption (9) is satisfied. That can be done for an open dense set of k's by the proof of Theorem 2.1. Consider X related to $(\sigma, \lambda k)$ with λ belonging to some complex neighborhood $\mathscr{C}$ of $[0, 1]$. The operator K_2 in (21) depends meromorphically on $\lambda \in \mathscr{C}$. Indeed, K_1 is related to $(\sigma, 0)$ (that is, to $\lambda = 0$), and is therefore independent of λ. The parametrix Q is also independent of λ. The analysis above shows that $\mathscr{L}$ is a meromorphic function of λ because L has that property; see (7) and (11). For $\lambda = 0$, we have $\mathscr{L} = 0$, and then $K_2 = K_1$. By adding a finite rank operator to Q, we can arrange that $\mathrm{Id} + K_1$ (see (20)) is injective; see also the proof of [Stefanov and Uhlmann 2005, Proposition 4]. We can then apply the analytic Fredholm theorem again in $\mathscr{C}$ with the poles of $(\mathrm{Id} - \lambda K)^{-1} T_1^{-1}$ removed. The latter is a connected set, containing $\lambda = 0$ and $\lambda = 1$. The analytic Fredholm theorem then implies that QX^*X is invertible for all λ in that set with the possible exception of a discrete set. In particular, there are λ's as close to $\lambda = 1$ as needed with that property. For those λ's, X^*X and X are injective as well. This shows that there is a dense set of pairs (σ, k) in the space (9) so that X is injective. Let us call that set $\mathscr{U}$.

We show next that for (σ, k) in some neighborhood of $\mathscr{U}$, X is still injective.

Let $(k, \sigma) \in \mathscr{U}$. Then $X : L^2(\Omega) \to L^2(\partial\Omega_1, \mathrm{d}\Sigma)$ is injective . Then $X^*X : L^2(\Omega) \to H^1(\Omega_1)$ is injective as well, as an integration by parts shows. By adding a finite rank operator to Q, we can arrange that $\mathrm{Id} + K_1$ (see (20)) is injective, as above. Then $\mathrm{Id} + K_1$ is invertible on $L^2(\Omega)$, and we deduce that (10) holds.

The analysis above implies that the norm $\|X^*X\|_{L^2(\Omega)\to H^1(\Omega_1)}$ depends continuously on (σ, k) as in (9). Therefore, we can perturb (σ, k), and (10) would remain true because the perturbation of the right hand side will be absorbed by the left hand side. On the other hand, injectivity of X^*X implies injectivity of X.

This proves that the set of pairs (σ, k), for which X is injective, is open subset of the (generic) set of pairs, for which the direct problem is guaranteed to be uniquely solvable by Theorem 2.1. Moreover, (10) holds with C locally uniform.

This completes the proof of Theorem 3.1. □

In the proof of the theorem, we used the following proposition about singular operators.

Proposition 3.4. *Let A be the operator*

$$[Af](x) = \int \frac{\alpha\big(x, y, |x-y|, \frac{x-y}{|x-y|}\big)}{|x-y|^{n-1}} f(y)\,\mathrm{d}y$$

with $\alpha(x, y, r, \theta)$ compactly supported in x, y. Then

(a) *if $\alpha \in C^2$, then $A : L^2 \to H^1$ is continuous with a norm not exceeding $C\|\alpha\|_{C^2}$;*

(b) *let $\alpha(x, y, r, \theta) = \alpha'(x, y, r, \theta)\phi(\theta)$ then*

$$\|A\|_{L^2 \to H^1} \le C\|\alpha'\|_{C^2}\|\phi\|_{H^1(S^{n-1})}.$$

Proof. We recall some facts about the Calderón–Zygmund theory of singular operators; see [Michlin and Prössdorf 1980]. First, if K is an integral operator with singular kernel $k(x, y) = \phi(x, \theta)r^{-n}$, where $\theta = \frac{x-y}{|x-y|}$, $r = |x-y|$, and if the "characteristic" ϕ has a mean value 0 as a function of θ, for any x, then K is a well-defined operator on test functions, where the integral has to be understood in the principle value sense. Moreover, K extends to a bounded operator to L^2 with a norm not exceeding $C \sup_x \|\phi(x, \cdot)\|_{L^2(S^{n-1})}$; see [Michlin and Prössdorf 1980, Theorem XI.3.1]. The characteristic ϕ does not need to have zero mean value in θ but then the integral has to be considered as a convolution in distribution sense. The latter is well defined because the Fourier transform of the kernel with respect to the variable $z = r\theta$ is homogeneous of order 0, thus bounded.

Also, see [Michlin and Prössdorf 1980, Theorem XI.11.1]; if B is an operator with a weakly singular kernel $\psi(x, \theta)r^{-n+1}$, then $\partial_x B$ is an integral operator with singular kernel $\partial_x[\beta(x, \theta)r^{-n+1}]$. The latter, up to a weakly singular operator, has a singular kernel of the type ϕr^{-n}, and the integration is again understood in the principle value sense; see the next paragraph. In particular, the zero mean value condition is automatically satisfied.

In our case, $\beta = \alpha$ depends on y and r as well. Assume first that it does not, that is, B is as above. Extend β as a homogeneous function of θ of order 0 near S^{n-1}. Then

$$\begin{aligned}
\partial_{x_i} \frac{\beta(x, \theta)}{r^{n-1}} &= (1-n)\frac{\theta_i}{r^n}\beta + \sum_j \frac{\frac{\partial\beta}{\partial\theta_j}}{r^{n-1}} \frac{\partial\theta_j}{\partial x_i} + \frac{\beta_{x_i}(x, \theta)}{r^{n-1}} \\
&= (1-n)\frac{\theta_i}{r^n}\beta + \sum_j \frac{\frac{\partial\beta}{\partial\theta_j}}{r^n}(\delta_{ij} - \theta_i\theta_j) + \frac{\beta_{x_i}(x, \theta)}{r^{n-1}} \\
&= \frac{(1-n)\theta_i\beta + \frac{\partial\beta}{\partial\theta_i}}{r^n} + \frac{\beta_{x_i}(x, \theta)}{r^{n-1}}.
\end{aligned} \tag{22}$$

We used the fact that $\sum_j \theta_j \frac{\partial\beta}{\partial\theta_j} = 0$ because β is homogeneous of order 0 in θ. It is not hard to show that the "characteristic"

$$\phi(x, \theta) = (1-n)\theta_i\beta + \frac{\partial\beta}{\partial\theta_i}$$

has zero mean over S_θ^{n-1}; see [Michlin and Prössdorf 1980, p. 243]. In this particular case where $\alpha(x, y, \theta) = \beta(x, \theta)$, independent of y and r, statement (a) can be proven as follows. Choose a finite atlas of charts for S^{n-1} so that for each chart, $n-1$ of the θ coordinates (that we keep fixed in $\mathbb{R}^n$) can be chosen as local coordinates. By rearranging the x, and respectively, the θ coordinates, in each fixed chart, we can assume that they are $\theta' = (\theta_1, \dots, \theta_{n-1})$. Then $\frac{\partial \beta}{\partial \theta_n} = -\sum_{i=1}^{n-1} \frac{\partial \beta}{\partial \theta_i}$. Then in (22), we have derivatives of β with respect to θ' (and x) with smooth coefficients. The contribution of the first term then can be estimated by the Calderón–Zygmund theorem. The second term is a kernel of a weakly singular operator. The following criterion can be applied to it: If K has an integral kernel $k(x, y)$ with the property

$$\sup_x \int |k(x,y)| \mathrm{d}x \le M, \quad \sup_y \int |k(x,y)| \mathrm{d}y \le M, \tag{23}$$

then K is bounded in L^2 with a norm not exceeding M [Taylor 1996, Proposition A.5.1].

This proves (a) for $\alpha = \beta$.

To replace $\beta(x, \theta)$ above by $\alpha(x, y, \theta)$, write $\alpha(x, y, r, \theta) = \alpha(x, x, 0, \theta) + r\gamma(x, y, r, \theta)$.

To prove (b), write first as above,

$$\alpha(x, y, r, \theta) = \beta'(x, \theta)\phi(\theta) + r\gamma(x, y, r, \theta)\phi(\theta), \quad \beta'(x, \theta) := \alpha_1(x, x, 0, \theta),$$

where $\gamma \in C^1$. Notice then that in (22), with $\beta = \beta'\phi$, we have

$$(1-n)\theta_i \beta + \frac{\partial \beta}{\partial \theta_i} = (1-n)\theta_i \beta' \phi + \phi \frac{\partial \beta'}{\partial \theta_i} + \beta' \frac{\partial \phi}{\partial \theta_i}.$$

Choosing local coordinates as above, and applying the Calderón–Zygmund theorem again, we get that the first term above contributes a singular operator with a norm not exceeding $\|\alpha_1\|_{C^1} \|\phi\|_{H^1}$. The second term $r\gamma$ generates an operator with a kernel $\gamma(x, y, r, \theta)\phi(\theta) r^{-n+2}$. Differentiate with respect to x, and we still get a weakly singular operator whose norm can be estimated as in (23) to give a norm not exceeding $\|\gamma_1\|_{C^1} \|\phi\|_{H^1}$. □

Remark 3.5. The only second order derivatives of α that were needed in the proof of (a) were $\partial_{(x,\theta)} \partial_{(t,r)} \alpha$.

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