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CR-INVARIANTS AND THE SCATTERING OPERATOR FOR COMPLEX MANIFOLDS WITH BOUNDARY

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Suppose that M is a strictly pseudoconvex CR manifold bounding a compact complex manifold X of complex dimension m. Under appropriate geometric conditions on M, the manifold X admits an approximate Kähler–Einstein metric g which makes the interior of X a complete Riemannian manifold. We identify certain residues of the scattering operator on X as conformally covariant differential operators on M and obtain the CR Q-curvature of M from the scattering operator as well. In order to construct the Kähler–Einstein metric on X, we construct a global approximate solution of the complex Monge–Ampère equation on X, using Fefferman's local construction for pseudoconvex domains in $\mathbb{C}^m$. Our results for the scattering operator on a CR-manifold are the analogue in CR-geometry of Graham and Zworski's result on the scattering operator on a real conformal manifold.

1. Introduction

The purpose of this paper is to describe certain CR-covariant differential operators on a strictly pseudoconvex CR manifold M as residues of the scattering operator for the Laplacian on an ambient complex Kähler manifold X having M as a "CR-infinity". We also characterize the CR Q-curvature in terms of the scattering operator. Our results parallel earlier results of Graham and Zworski [2003], who showed that if X is an asymptotically hyperbolic manifold carrying a Poincaré–Einstein metric, the Q-curvature and certain conformally covariant differential operators on the "conformal infinity" M of X can be recovered from the scattering operator on X. The results in this paper were announced in [Hislop et al. 2006].

To describe our results, we first recall some basic notions of CR geometry and recent results [Fefferman and Hirachi 2003; Gover and Graham 2005] concerning CR-covariant differential operators and CR-analogues of Q-curvature. If M is a smooth, orientable manifold of real dimension $(2n+1)$, a *CR-structure* on M is a real hyperplane bundle H on TM together with a smooth bundle map $J : H \to H$

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with $J^2 = -1$ that determines an almost complex structure on H. We denote by $T_{1,0}$ the eigenspace of J on $H \otimes \mathbb{C}$ with eigenvalue $+i$; we will always assume that the CR-structure on M is integrable in the sense that $[T_{1,0}, T_{1,0}] \subset T_{1,0}$. We will assume that M is orientable, so that the line bundle $H^\perp \subset T^*M$ admits a nonvanishing global section. A *pseudo-Hermitian structure* on M is smooth, nonvanishing section θ of $H^\perp$. The *Levi form* of θ is the Hermitian form

$$L_\theta(v, w) = d\theta(v, Jw)$$

on H. The CR-structure on M is called *strictly pseudoconvex* if the Levi form is positive definite. Note that this condition is actually independent of the choice of θ compatible with a given orientation of M. We will always assume that M is strictly pseudoconvex in what follows. It follows from strict pseudoconvexity that θ is a contact form, and the form $\theta \wedge (d\theta)^n$ is a volume form that defines a natural inner product on $\mathscr{C}^\infty(M)$ by integration. The pseudo-Hermitian structure on M also determines a connection on TM, the Tanaka–Webster connection ∇_θ; the basic data of pseudo-Hermitian geometry are the curvature and torsion of this connection (see [Tanaka 1975; Webster 1978]).

Given a fixed CR-structure (H, J) on M, any nonvanishing section $\overline{\theta}$ of $H^\perp$ compatible with a given orientation takes the form $e^{2\Upsilon}\theta$ for a fixed section θ of $H^\perp$ and some function $\Upsilon \in \mathscr{C}^\infty(M)$. The corresponding Levi form is given by

$$L_{\overline{\theta}} = e^{2\Upsilon} L_\theta.$$

In this sense the CR-structure determines a conformal class of pseudo-Hermitian structures on M.

For strictly pseudoconvex CR manifolds, Fefferman and Hirachi [2003] proved the existence of CR-covariant differential operators P_k of order $2k$, $k = 1, 2, \ldots, n+1$, whose principal parts are Δ_θ^k, where Δ_θ is the positive sub-Laplacian on M with respect to the pseudo-Hermitian structure θ. They exploit Fefferman's construction [1976] (formulated intrinsically by Lee [1986]) of a circle bundle $\mathscr{C}$ over M with a natural conformal structure and a mapping $\theta \mapsto g_\theta$ from pseudo-Hermitian structures on M to Lorentz metrics on $\mathscr{C}$ that respects conformal classes. They then construct the conformally covariant differential operators found in [Graham et al. 1992] (referred to here as GJMS operators) on $\mathscr{C}$, and show that these operators pull back to CR-covariant differential operators on M. The CR Q-curvature may be similarly defined as a pullback to M of Branson's Q-curvature (see [Branson 1993] and see also [Chang et al. 2008] for a review and further references) on the circle bundle $\mathscr{C}$. Here we will show that the operators P_k on M occur as residues for the scattering operator associated to a natural scattering problem with M as the boundary at infinity, and that the CR Q-curvature Q_θ^{CR} can be computed from the scattering operator.

To describe the scattering problem, we first discuss its geometric setting. Recall that if M is an integrable, strictly pseudoconvex CR manifold of dimension $(2n+1)$ with $n \geq 2$, there is a complex manifold X of complex dimension $m = n+1$ having M as its boundary so that the CR-structure on M is induced from the complex structure on X (this result is false, in general, when $n = 1$; see [Harvey and Lawson 1975]). Let φ be a defining function for M and denote by $\mathring{X}$ the interior of X (we take $\varphi < 0$ in $\mathring{X}$). The associated Kähler metric g on $\mathring{X}$ is the Kähler metric with Kähler form

$$\omega_\varphi = -\frac{i}{2}\partial\bar{\partial}\log(-\varphi) \tag{1-1}$$

in a neighborhood of M, extended smoothly to all of X. The metric has the form

$$g_\varphi = -\frac{\eta}{\varphi} + (1 - r\varphi)\Big(\frac{d\varphi^2}{\varphi^2} + \frac{\Theta^2}{\varphi^2}\Big). \tag{1-2}$$

in a neighborhood of M, where η and Θ have Taylor series to all orders in φ at $\varphi = 0$. The boundary values $\Theta|_M = \theta$, and $\eta|_H = h$ induce respectively a contact form on M and a Hermitian metric on H, where H is a *subbundle* of TM. The function r is a smooth function, the transverse curvature, which depends on the choice of φ (see [Graham and Lee 1988]). Thus, the conformal class of a Hermitian metric h on H, is a kind of "Dirichlet datum at infinity" for the metric g_φ, that is $(-\varphi)g_\varphi|_H = h$.

A motivating example for our work is the case of a strictly pseudoconvex domain $X \subset \mathbb{C}^m$ with Hermitian metric

$$g = \sum_{j,k=1}^{m} \frac{\partial^2}{\partial z_j \partial z_{\bar{k}}} \log\Big(-\frac{1}{\varphi}\Big) dz_j \otimes dz_{\bar{k}},$$

where φ is a defining function for the boundary of X with $\varphi < 0$ in the interior of X. In this example, observe that if

$$\Theta = \frac{i}{2}(\bar{\partial}\varphi - \partial\varphi)$$

and $\iota : M \to X$ is the natural inclusion, then $\theta = \iota^*\Theta$ is a contact form on M that defines the CR-structure $H = \ker\theta$. The form $d\theta$ induces the Levi form on M and so defines a pseudo-Hermitian structure on M. Denote by J the almost complex structure on H; the two-form $h = d\theta(\,\cdot\,, J\cdot\,)$ is a pseudo-Hermitian metric on M. It is not difficult to see that the conformal class of the pseudo-Hermitian structure on M, that is, its CR-structure, is independent of the choice of defining function φ.

It is natural to consider scattering theory for the positive Laplacian, Δ_g, on $(\mathring{X}, g)$, where X is a complex manifold with boundary M. As discussed in what follows, the metric g belongs to the class of Θ-metrics considered by Epstein, Melrose, and Mendoza [1991]; see also the recent paper of Guillarmou and Sá Barreto [2008] where scattering theory for asymptotically complex hyperbolic manifolds (a class which includes those considered here) is analyzed in depth. Thus, the full power of the Epstein–Melrose–Mendoza analysis of the resolvent

$$R(s) = (\Delta_g - s(m-s))^{-1}$$

of Δ_g is available to study scattering theory on $(\mathring{X}, g)$.

For $f \in \mathscr{C}^\infty(M)$, $\Re(s) = m/2$, and $s \neq m/2$, there is a unique solution u of the "Dirichlet problem"

$$(\Delta_g - s(m-s))u = 0, \quad u = (-\varphi)^{m-s}F + (-\varphi)^s G, \quad F|_M = f, \tag{1-3}$$

where $F, G \in \mathscr{C}^\infty(X)$. The uniqueness follows from the absence of L^2 solutions of the eigenvalue problem for $\Re(s) = m/2$; this may be proved, for example, using [Vasy and Wunsch 2005] (see the comments in [Guillarmou and Sá Barreto 2008]). Here we will use the explicit formulas for the Kähler form and Laplacian obtained in [Graham and Lee 1988] to obtain the asymptotic expansions of solutions to the generalized eigenvalue problem.

Unicity for the "Dirichlet problem" (1-3) implies that the Poisson map

$$\mathscr{P}(s) : \mathscr{C}^\infty(M) \to \mathscr{C}^\infty(\mathring{X}), \quad f \mapsto u \tag{1-4}$$

and the scattering operator

$$S_X(s) : \mathscr{C}^\infty(M) \to \mathscr{C}^\infty(M), \quad f \mapsto G|_M$$

are well defined. The operator $S_X(s)$ depends *a priori* on the boundary defining function φ for M. If $\overline{\varphi} = e^{\upsilon}\varphi$ is another defining function for M and $\upsilon|_M = \Upsilon$, the corresponding scattering operator $\overline{S}_X(s)$ is given by

$$\overline{S}_X(s) = e^{-s\Upsilon} S_X(s) e^{(s-m)\Upsilon}.$$

The operator $S_X(s)$ admits a meromorphic continuation to the complex plane, possibly with singularities at $s = 0, -1, -2, \ldots$; see [Melrose 1999] where the scattering operator is described and the problem of studying its poles and residues is posed, and see [Guillarmou and Sá Barreto 2008] for a detailed analysis of the scattering operator. The scattering operator is self-adjoint for s real. We will show that, with a geometrically natural choice of the boundary defining function φ, the residues of certain poles of $S_X(s)$ are CR-covariant differential operators.

To describe the setting for this result, recall that for strictly pseudoconvex domains Ω in $\mathbb{C}^m$, Fefferman [1976] proved the existence of a defining function φ for $\partial\Omega$ which is an approximate solution of the complex Monge–Ampère equation.

The complex Monge–Ampère equation for a function $\varphi \in \mathscr{C}^\infty(\Omega)$ is the equation

$$J[\varphi] = 1, \quad \varphi|_{\partial\Omega} = 0,$$

where J is the complex Monge–Ampère operator

$$J[\varphi] = \det \begin{pmatrix} \varphi & \varphi_j \\ \varphi_{\bar{k}} & \varphi_{j\bar{k}} \end{pmatrix}.$$

We say that $\varphi \in \mathscr{C}^\infty(\Omega)$ is an approximate solution of the complex Monge–Ampère equation if

$$J[\varphi] = 1 + \mathbb{O}(\varphi^{m+1}), \quad \varphi|_{\partial\Omega} = 0.$$

The Kähler metric g associated to such an approximate solution φ is an approximate Kähler–Einstein metric on Ω, that is, g obeys

$$\mathrm{Ric}(g) = -(m+1)\omega + \mathbb{O}(\varphi^{m-1}), \tag{1-5}$$

where ω is the Kähler form associated to φ, and Ric is the Ricci form.

Under certain conditions, Fefferman's result can be "globalized" to the setting of complex manifolds X with strictly pseudoconvex boundary M, as we discuss below. It follows that $\mathring{X}$ carries an approximate Kähler–Einstein metric g in the sense that (1-5) holds.

We will call a smooth function φ defined in a neighborhood of M a *globally defined approximate solution* of the Monge–Ampère equation on X if for each $p \in M$ there is a neighborhood U of p in X and a holomorphic coordinate system in U for which φ is an approximate solution of the Monge–Ampère equation. As we will show, such a solution exists if and only if M admits a pseudo-Hermitian structure θ which is volume-normalized with respect to some locally defined, closed $(n+1, 0)$-form in a neighborhood of any point $p \in M$ (see Section 2D.2 where we defined "volume-normalized", and see Burns–Epstein [1990] where a similar condition is used to construct a global solution of the Monge–Ampére equation when $\dim M = 3$). If $\dim M \geq 5$, we can give a more geometric formulation of this condition. Recall that a CR manifold is *pseudo-Einstein* if there is a pseudo-Hermitian structure θ for which the Webster Ricci curvature is a multiple of the Levi form. Lee [1988] introduced and studied this geometric notion; he proved that if $\dim M \geq 5$, then M admits a pseudo-Einstein, pseudo-Hermitian structure θ if and only if θ is volume-normalized with respect to a closed $(n+1,0)$-form in a neighborhood of any point $p \in M$. If $\dim M = 3$, the pseudo-Einstein condition is vacuous and must be replaced by a more stringent condition; see Section 2D.2 in what follows. If X is a pseudoconvex domain in $\mathbb{C}^m$, this condition is trivially satisfied since the pseudo-Hermitian structure induced by the Fefferman approximate solution is volume-normalized with respect to the restriction of $\zeta = dz^1 \wedge \cdots \wedge dz^m$ to M.

Theorem 1.1. *Let X be a complex manifold of complex dimension $m = n+1$ with strictly pseudoconvex boundary M. Let g be the Kähler metric on X associated to the Kähler form* (1-1), *and let $S_X(s)$ be the scattering operator for Δ_φ. Finally, suppose that Δ_φ has no L^2-eigenvalues. Then $S_X(s)$ has simple poles at the points $s = (m+k)/2$, $k \in \mathbb{N}$, and*

$$\operatorname*{Res}_{s=(m+k)/2} S_X(s) = c_k P_k,$$

where the P_k are differential operators of order $2k$, and

$$c_k = \frac{(-1)^k}{2^k k!(k-1)!}. \tag{1-6}$$

If g is an approximate Kähler–Einstein metric given by a globally defined approximate solution of the Monge–Ampère equation, then for $1 \leq k \leq m$, the operators P_k are CR-covariant differential operators.

Remark 1.2. It is not difficult to show that, for generic compactly supported perturbations of the metric, L^2-eigenvalues are absent. Our analysis applies if only the metric g has the form (1-2) in a neighborhood of M.

Remark 1.3. We view the operators P_k as operators on $\mathscr{C}^\infty(M)$; if one instead views these operators as acting on appropriate density bundles over M they are actually invariant operators. Gover and Graham [2005] showed that the CR-covariant differential operators P_k are logarithmic obstructions to the solution of the Dirichlet problem (1-3) when X is a pseudoconvex domain in $\mathbb{C}^m$ with a metric of Bergman type, but did not identify them as residues of the scattering operator.

It follows from the self-adjointness (s real) and conformal covariance of $S_X(s)$ that the operators P_k are self-adjoint and conformally covariant. As in [Graham and Zworski 2003], the analysis centers on the Poisson map $\mathscr{P}(s)$ defined in (1-4). As shown in [Epstein et al. 1991], the Poisson map is analytic in s for $\mathrm{Re}(s) > m/2$. Moreover, at the points $s = (m+k)/2$, $k = 1, 2, \ldots$, the Poisson operator takes the form

$$\mathscr{P}(s)f = (-\varphi)^{(m-k)/2} F + (-\varphi)^{(m+k)/2} \log(-\varphi)\, G$$

for functions $F, G \in \mathscr{C}^\infty(X)$ with

$$F|_M = f, \;\; G|_M = c_k P_k f.$$

Here P_k are differential operators determined by a formal power series expansion of the Laplacian (see Lemma 3.4), and are the same operators that appear as residues of the scattering operator at points $s = (m+k)/2$. An important ingredient in the analysis is the asymptotic form of the Laplacian due to Lee and Melrose [1982] and refined by Graham and Lee [1988].

If the defining function φ is an approximate solution of the complex Monge–Ampère equation, the differential operators P_k, $1 \le k \le m$, can be identified with the GJMS operators owing to the characterization of $\mathcal{P}(s)f$ described above (see [Gover and Graham 2005, Proposition 5.4]; the argument given there for pseudoconvex domains easily generalizes to the present setting).

Explicit computation shows that, for an approximate Kähler–Einstein metric g, the first operator has the form

$$P_1 = c_1\big(\Delta_b + n(2(n+1))^{-1} R\big),$$

where Δ_b is the sub-Laplacian on X and R is the Webster scalar curvature, that is, P_1 is the CR-Yamabe operator of Jerison and Lee [1984].

The CR Q-curvature is a pseudo-Hermitian invariant realized as the pullback to M of the Q-curvature of the circle bundle $\mathscr{C}$.

Theorem 1.4. *Suppose that X is a complex manifold with strictly pseudoconvex boundary M, and suppose that g is an approximate Kähler–Einstein metric given by a globally defined approximate solution of the Monge–Ampère equation. Let $S_X(s)$ be the associated scattering operator. The formula*

$$c_m Q_\theta^{CR} = \lim_{s \to m} S_X(s) 1$$

holds, where c_m is given by (1-6).

It follows from Theorem 1.1 and the conformal covariance of $S_X(s)$ that if $\bar{\theta} = e^{2\Upsilon}\theta$, then

$$e^{2m\Upsilon} Q_{\bar{\theta}}^{CR} = Q_\theta^{CR} + P_m \Upsilon$$

as was already shown in [Fefferman and Hirachi 2003]. From this it follows that the integral

$$\int_M Q_\theta^{CR} \psi$$

is a CR-invariant (recall that ψ is the natural volume form on M defined by the contact form θ). We remark that the integral of Q_θ^{CR} vanishes for any three-dimensional CR manifold because the integrand is a total divergence (see [Fefferman and Hirachi 2003, Proposition 3.2] and comments below), while under the condition of our Theorem 1.4, there is a pseudo-Hermitian structure for which $Q_\theta^{CR} = 0$ (see [Fefferman and Hirachi 2003, Proposition 3.1]). In our case, if φ is a globally defined approximate solution of the Monge–Ampère equation, the induced contact form

$$\theta = \frac{i}{2}(\bar{\partial}\varphi - \partial\varphi)$$

on M is an "invariant contact form" in the language of [Fefferman and Hirachi 2003], and they show in Proposition 3.1 that $Q_\theta^{CR} = 0$ for an invariant contact form. Thus it is not clear at present under what circumstances this invariant is nontrivial for a general, strictly pseudoconvex manifold.

Finally, we prove a CR-analogue of [Graham and Zworski 2003, Theorem 3] using scattering theory.

Theorem 1.5. *Suppose that X is a compact complex manifold with strictly pseudoconvex boundary M, and g is an approximate Kähler–Einstein metric given by a globally defined approximate solution of the Monge–Ampère equation. Then*

$$\operatorname{vol}_g\{-\varphi > \varepsilon\} = c_0 \varepsilon^{-n-1} + c_1 \varepsilon^{-n} + \cdots + c_n \varepsilon^{-1} + L \log(-\varepsilon) + V + o(1).$$

where

$$L = c_m \int_M Q_\theta^{CR} \psi = 0.$$

We remark that Seshadri [2007] already showed that L is, up to a constant, the integral of Q_θ^{CR}. It is worth noting that our choice of defining function differs from Seshadri's.

2. Geometric preliminaries

2A. ***CR manifolds.*** Suppose that M is a smooth orientable manifold of real dimension $2n+1$, and let $\mathbb{C}TM = TM \otimes_{\mathbb{R}} \mathbb{C}$ be the complexified tangent bundle on M. A *CR-structure* on M is a complex n-dimensional subbundle $\mathcal{H}$ of $\mathbb{C}TM$ with the property that $\mathcal{H} \cap \overline{\mathcal{H}} = \{0\}$. If, also, $[\mathcal{H}, \mathcal{H}] \subseteq \mathcal{H}$, we say that the CR-structure is *integrable*. If we set $H = \operatorname{Re} \mathcal{H}$, then the bundle H has real codimension one in TM. The map

$$J : H \to H, \quad V + \overline{V} \mapsto i(V - \overline{V})$$

satisfies $J^2 = -I$ and gives H a natural complex structure.

Since M is orientable, there is a nonvanishing one-form θ on M with $\ker \theta = H$. This form is unique up to multiplication by a positive, nonvanishing function $f \in \mathscr{C}^\infty(M)$. A choice of such a one-form θ is called a *pseudo-Hermitian structure* on M. The *Levi form* is given by

$$L_\theta(V, \overline{W}) = -i\, d\theta(V, \overline{W}) \tag{2-1}$$

for $V, W \in \mathcal{H}$ (here $d\theta$ is extended to $\mathcal{H}$ by complex linearity). Note that

$$L_{f\theta} = f L_\theta$$

since θ annihilates $\mathcal{H}$. If $d\theta$ is nondegenerate, then there is a unique real vector field T on M, the *characteristic vector field* T, with the properties that $\theta(T) = 1$ and $T \lrcorner\, d\theta = 0$. If $\{W_\alpha\}$ is a local frame for $\mathcal{H}$ (here α ranges from 1 to n), then the vector fields $\{W_\alpha, W_{\overline{\alpha}}, T\}$ form a local frame for $\mathbb{C}TM$. If we choose $(1,0)$-forms θ^α dual to the W_α then $\{\theta^\alpha, \theta^{\overline{\alpha}}, \theta\}$ forms a dual coframe for $\mathbb{C}TM$. We say that $\{\theta^\alpha\}$ forms an admissible coframe dual to $\{W^\alpha\}$ if $\theta^\alpha(T) = 0$ for all α. The integrability condition is equivalent to the condition that

$$d\theta = d\theta^\alpha = 0 \mod \{\theta, \theta^\alpha\}.$$

The Levi form is then given by

$$L_\theta = h_{\alpha\overline{\beta}}\, \theta^\alpha \wedge \theta^{\overline{\beta}} \tag{2-2}$$

for a Hermitian matrix-valued function $h_{\alpha\overline{\beta}}$. We will use $h_{\alpha\overline{\beta}}$ to raise and lower indices in this article.

We will say that a given CR-structure is *strictly pseudoconvex* if L_θ is positive definite. Note that (up to sign) this condition is independent of the choice of pseudo-Hermitian structure θ.

In what follows, we will always suppose that M is orientable and that M carries a strictly pseudoconvex, integrable CR-structure. In this case, the pseudo-Hermitian geometry of M can be understood in terms of the *Tanaka–Webster connection* on M (see Tanaka [1975] and Webster [1978]). With respect to the frame discussed above, the Tanaka–Webster connection is given by

$$\nabla W_\alpha = {\omega_\alpha}^\beta \otimes W_\beta, \quad \nabla T = 0$$

for connection one-forms ${\omega_\alpha}^\beta$ obeying the structure equations

$$d\theta^\alpha = \theta^\beta \wedge {\omega_\alpha}^\beta + \theta \wedge \tau^\alpha,$$
$$d\theta = i h_{\alpha\bar\beta}\, \theta^\alpha \wedge \theta^{\bar\beta},$$

where the torsion one-forms are given by

$$\tau^\alpha = {A^\alpha}_{\bar\beta}\, \theta^{\bar\beta},$$

with $A_{\alpha\beta} = A_{\beta\alpha}$. The connection obeys the compatibility condition

$$dh_{\alpha\bar\beta} = \omega_{\alpha\bar\beta} + \omega_{\bar\beta\alpha}$$

with the Levi form described in (2-1) and (2-2).

2B. ***Complex manifolds with CR boundary.*** Now suppose that X is a compact complex manifold of dimension $m = n+1$ with boundary $\partial X = M$. We will denote by $\mathring{X}$ the interior of X. The manifold M inherits a natural CR-structure from the complex structure of the ambient manifold. We will suppose that M is strictly pseudoconvex; such a structure, induced by the complex structure of the ambient manifold, is always integrable.

We will suppose that $\varphi \in \mathscr{C}^\infty(X)$ is a defining function for M, that is, $\varphi < 0$ in $\mathring{X}$, $\varphi = 0$ on M, and $d\varphi(p) \neq 0$ for all $p \in M$. We will further suppose that φ has no critical points in a collar neighborhood of M so that the level sets $M^\varepsilon = \varphi^{-1}(-\varepsilon)$ are smooth manifolds for all ε sufficiently small.

Associated to the defining function φ is the Kähler form

$$\omega_\varphi = -\frac{i}{2}\partial\bar\partial \log(-\varphi) = \frac{i}{2}\Big(\frac{\partial\bar\partial\varphi}{-\varphi} + \frac{\partial\varphi \wedge \bar\partial\varphi}{\varphi^2}\Big).$$

We will study scattering on X with the metric induced by the Kähler form ω_φ. Since we can cover a neighborhood of M in X by coordinate charts, it suffices to consider the situation where U is an open subset of $\mathbb{C}^m$ and $\varphi : U \to \mathbb{R}$ is a smooth function with no critical points in U, the set $\{\varphi < 0\}$ is biholomorphically equivalent to a boundary neighborhood in X, and $\{\varphi = 0\}$ is diffeomorphic to the corresponding boundary neighborhood in M. We will now describe the asymptotic geometry near M, recalling the ambient metric of [Graham and Lee 1988] and computing the asymptotic form of the metric and volume form.

The manifolds M^ε inherit a natural CR-structure from the ambient manifold X with

$$\mathscr{H}^\varepsilon = \mathbb{C}TM^\varepsilon \cap T^{1,0}U.$$

Given a defining function φ, we define a one-form

$$\Theta = \frac{i}{2}(\bar\partial - \partial)\varphi$$

and let

$$\theta_\varepsilon = \iota_\varepsilon^* \Theta,$$

where $\iota_\varepsilon : M^\varepsilon \to U$ is the natural embedding. The contact form θ_ε gives M^ε a pseudo-Hermitian structure. We will denote by $\mathcal{H}$ the subbundle of $T^{1,0}U$ whose fibre over M^ε is $\mathcal{H}^\varepsilon$. Note that

$$d\Theta = i\partial\bar{\partial}\varphi$$

and the Levi form on M^ε is given by

$$L_{\theta_\varepsilon} = -i\,d\theta_\varepsilon.$$

We will assume that each M^ε is strictly pseudoconvex, that is, L_{θ_ε} is positive definite for all sufficiently small $\varepsilon > 0$. To simplify notation, we will write θ for θ_ε, suppressing the ε, as the meaning will be clear from the context.

2B.1. *Ambient connection.* In order to describe the asymptotic geometry of X, we recall the ambient connection defined by Graham and Lee [1988] that extends the Tanaka–Webster connection on each M^ε to $\mathbb{C}TU$. First we recall the following simple lemma (see [Lee and Melrose 1982, §2]).

Lemma 2.1. *There exists a unique* $(1,0)$*-vector field* ξ *on* U *so that*:

$$\partial\varphi(\xi) = 1 \quad \textit{and} \quad \xi \lrcorner\, \partial\bar{\partial}\varphi = r\,\bar{\partial}\varphi$$

for some $r \in \mathscr{C}^\infty(U)$.

The smooth function r is called the *transverse curvature*. We decompose ξ into real and imaginary parts as

$$\xi = \tfrac{1}{2}(N - iT),$$

where N and T are real vector fields on U. It easily follows that

$$d\varphi(N) = 2, \quad \theta(N) = 0, \quad \theta(T) = 1, \quad T \lrcorner\, d\theta = 0.$$

Thus T is the characteristic vector field for each M^ε, and N is normal to each M^ε.

Let $\{W_\alpha\}$ be a frame for $\mathcal{H}$. It follows from Lemma 2.1 that $\{W_\alpha, W_{\bar{\alpha}}, T\}$ forms a local frame for $\mathbb{C}TM^\varepsilon$, while $\{W_\alpha, W_{\bar{\alpha}}, \xi, \bar{\xi}\}$ forms a local frame for $\mathbb{C}TU$. If $\{\theta^\alpha\}$ is a dual coframe for $\{W_\alpha\}$, then $\{\theta^\alpha, \theta^{\bar{\alpha}}, \theta\}$ is a dual coframe for $\mathbb{C}TM^\varepsilon$, while $\{\theta^\alpha, \theta^{\bar{\alpha}}, \partial\varphi, \bar{\partial}\varphi\}$ is a dual coframe for $\mathbb{C}TU$. The Levi form on each $\mathcal{H}^\varepsilon$ is given by

$$L_\theta = h_{\alpha\bar{\beta}}\,\theta^\alpha \wedge \theta^{\bar{\beta}}$$

for a Hermitian matrix-valued function $h_{\alpha\bar{\beta}}$. We will use $h_{\alpha\bar{\beta}}$ to raise and lower indices. We will set

$$W_m = \xi, \quad W_{\bar{m}} = \bar{\xi}, \quad \theta^m = \partial\varphi, \quad \theta^{\bar{m}} = \bar{\partial}\varphi.$$

In what follows, repeated Greek indices are summed from 1 to n and repeated Latin indices are summed from 1 to $m = n+1$.

The following important lemma decomposes the form $d\Theta$ into "tangential" and "transverse" components.

Lemma 2.2. *We have*

$$\partial\bar{\partial}\varphi = h_{\alpha\bar{\beta}}\,\theta^\alpha \wedge \theta^{\bar{\beta}} + r\,\partial\varphi \wedge \bar{\partial}\varphi.$$

Proposition 2.3 [Graham and Lee 1988, Proposition 1.1]. *There exists a unique linear connection* ∇ *on* U *so that*:

(a) *For any vector fields* X *and* Y *on* U *tangent to some* M^{ε},

$$\nabla_X Y = \nabla^{\varepsilon}_X Y$$

where ∇^{ε} *is the pseudo-Hermitian connection on* M^{ε}.

(b) ∇ *preserves* $\mathcal{H}$, N, T *and* L_θ; *that is,* $\nabla_X \mathcal{H} \subset \mathcal{H}$ *for any* $X \in \mathbb{C}TU$, *and* $\nabla T = \nabla N = \nabla L_\theta = 0$.

(c) *If* $\{W_\alpha\}$ *is a frame for* $\mathcal{H}$, *and* $\{\theta^\alpha, \partial\varphi\}$ *is the dual* $(1,0)$*-coframe on* U, *then*

$$d\theta^\alpha = \theta^\beta \wedge \varphi_\beta{}^\alpha - i\partial\varphi \wedge \tau^\alpha + i(W^\alpha r)\, d\varphi \wedge \theta + \tfrac{1}{2} r\, d\varphi \wedge \theta^\alpha.$$

The connection ∇ is called the *ambient connection*.

2B.2. *Kähler metric.* Using Lemma 2.2, we can also compute the Kähler form

$$\omega_\varphi = \frac{i}{2}\Big(\frac{1}{-\varphi} h_{\alpha\bar\beta}\theta^\alpha \wedge \theta^{\bar\beta} + \frac{1-r\varphi}{\varphi^2}\partial\varphi \wedge \bar\partial\varphi\Big).$$

The induced Hermitian metric is

$$g_\varphi = \frac{1}{-\varphi} h_{\alpha\bar\beta}\,\theta^\alpha \otimes \theta^{\bar\beta} + \frac{1-r\varphi}{\varphi^2}\partial\varphi \otimes \bar\partial\varphi.$$

It is easy to compute that

$$g_\varphi(N, N) = 4\frac{1-r\varphi}{\varphi^2},$$

so the outward unit normal field associated to the surfaces M^ε is

$$\nu = \frac{-\varphi}{2\sqrt{1-r\varphi}} N.$$

We note for later use that the induced volume form ω_φ^m is given by

$$\omega_\varphi^m = \Big(\frac{i}{2}\Big)^m \bigg(\frac{1-r\varphi}{(-\varphi)^{m+1}} \det(h_{\alpha\bar\beta})\, \theta^1 \wedge \theta^{\bar 1} \wedge \cdots \wedge \theta^m \wedge \theta^{\bar m}\bigg),$$

while

$$\nu \lrcorner\, \omega_\varphi^m|_{M^\varepsilon} = \frac{m}{2^m}\frac{\sqrt{1+r\varepsilon}}{\varepsilon^m}(d\theta_\varepsilon)^n \wedge \theta_\varepsilon. \tag{2-3}$$

We will set

$$\psi = \frac{m}{2^m}(d\theta)^n \wedge \theta. \tag{2-4}$$

We also note for later use that if $u \in \mathscr{C}^\infty(X)$ and

$$du = u_\alpha \theta^\alpha + u_{\bar\alpha}\theta^{\bar\alpha} + u_m \partial\varphi + u_{\bar m}\bar\partial\varphi,$$

then

$$|du|^2_{g_\varphi} = -\varphi h^{\alpha\bar\beta} u_\alpha u_{\bar\beta} + \frac{\varphi^2}{1-r\varphi} u_m u_{\bar m}.$$

2C. ***The Laplacian on X.*** The Laplacian on the Kähler manifold (X, ω_φ) is the positive operator[1]

$$\Delta_\varphi u = \mathrm{Tr}(i\partial\bar\partial u) = g^{j\bar k} u_{j\bar k},$$

for $u \in \mathscr{C}^\infty(X)$, where we now write Δ_φ rather than Δ_g to emphasize the dependence of Δ on the boundary defining function φ.

Graham and Lee [1988] computed the Laplacian in a collar neighborhood of M, separating "normal" and "tangential" parts. To state their results, recall that the sub-Laplacian is defined on each M^ε by

$$\Delta_b u = \left(u_\alpha{}^\alpha + u_{\bar\beta}{}^{\bar\beta}\right),$$

where covariant derivatives are taken with respect to the Tanaka–Webster connection on M^ε.

Theorem 2.4 [Graham and Lee 1988].

$$\Delta_\varphi = \frac{\varphi}{4}\Big(\frac{-\varphi}{1-r\varphi}(N^2+T^2+2rN+2X_r) - 2\Delta_b + 2nN\Big), \tag{2-5}$$

where $X_r = r^\alpha W_\alpha + r^{\bar\alpha} W_{\bar\alpha}$.

It will be useful to recast (2-5) for $\Delta_\varphi u$ in terms of $x = -\varphi$. Note that $N = 2\,\partial/\partial\varphi = -2\,\partial/\partial x$, so

$$-\Delta_\varphi u = \Big(\frac{1}{1+rx}\Big)\Big(x\frac{\partial}{\partial x}\Big)^2 u - (n+1)\Big(x\frac{\partial}{\partial x}\Big)u + \tfrac14\Big(\frac{x^2}{1+rx}\Big)(T^2u - 2ru_x + 2X_r u) + \tfrac14 x(-2\Delta_b u).$$

We think of Δ_φ as a variable-coefficient differential operator with respect to vector fields $x\,\partial/\partial_x$ and vector fields tangent to the boundary M. In a neighborhood of M we have

$$\Delta_\varphi \sim \sum_{k\ge 0} x^k L_k, \tag{2-6}$$

for differential operators $L_k = L_k(y;\partial_y, x\partial_x)$, where the indicial operator L_0 is

$$L_0 = -\Big(x\frac{\partial}{\partial x}\Big)^2 + mx\frac{\partial}{\partial x}, \tag{2-7}$$

and the operator L_1 is

$$L_1 = \tfrac12 \Delta_b + r_0\Big(x\frac{\partial}{\partial x}\Big)^2,$$

where $r = r_0 + \mathbb{O}(x)$.

2D. ***The complex Monge–Ampère equation.***

2D.1. *Local theory.* Let Ω be a domain in $\mathbb{C}^m$ with smooth boundary. The complex Monge–Ampère equation is the nonlinear equation

$$J[u] = 1, \quad u|_{\partial\Omega} = 0,$$

for a function $u \in \mathscr{C}^\infty(\Omega)$, $u > 0$ on Ω, where $J[u]$ is the Monge–Ampère operator:

$$J[u] = (-1)^m \det\begin{pmatrix} u & u_{\bar j} \\ u_i & u_{i\bar j}\end{pmatrix}. \tag{2-8}$$

[1]Note that our definition differs from that of Graham and Lee by an overall factor of $-\frac14$.

If u solves the complex Monge–Ampère equation then

$$-\Big(\log\Big(\frac{1}{u}\Big)\Big)_{j\bar{k}}\, dz^j \otimes dz^{\bar{k}}$$

is a Kähler–Einstein metric.

Fefferman [1976] showed that there is a smooth function $\psi \in \mathscr{C}^\infty(\Omega)$ that satisfies

$$J[\varphi] = 1 + \mathbb{O}(\varphi^{m+1}), \quad \varphi|_{\partial\Omega} = 0,$$

and that ψ is uniquely determined up to order $m+1$. Cheng and Yau [1980] showed the existence of an exact solution belonging to $\mathscr{C}^\infty(\Omega) \cap C^{m+3/2-\varepsilon}(\overline{\Omega})$, while Lee and Melrose [1982] showed that the exact solution has an asymptotic expansion with logarithmic terms beginning at order $m+2$.

We will show that Fefferman's local approximate solution of the Monge–Ampère equation [Fefferman 1976] can be globalized to an approximate solution of the Monge–Ampère equation near the boundary of a complex manifold X. We will see later that, to globalize Fefferman's construction, we need to impose a geometric condition on the CR-structure of M inherited from the complex structure of X. For the convenience of the reader, we review the properties of the operator J under a holomorphic coordinate change and the connection between solutions of the Monge–Ampére equation and Kähler–Einstein metrics.

If $f : \Omega \subset \mathbb{C}^m \to \mathbb{C}^m$ is holomorphic, then f' denotes the matrix

$$(f')_{jk} = \frac{\partial f_j}{\partial z_k}.$$

Lemma 2.5. *Let f be a local biholomorphism. Then, for any local smooth function u on Ω,*

$$J\big[|\det(f')|^{-2/(m+1)}(u \circ f)\big] = J[u] \circ f.$$

A proof was given by Fefferman [1976]. Here we give an alternative proof using the following identity.

Lemma 2.6.

$$J[u] = u^{m+1} \det\Big(\Big(\log\Big(\frac{1}{u}\Big)\Big)_{j\bar{k}}\Big).$$

Proof. Using row-column operations, one proves that

$$\det\begin{pmatrix} u & u_{\bar{k}} \\ u_j & u_{j\bar{k}} \end{pmatrix} = u \det\Big(u_{j\bar{k}} - \frac{u_j u_{\bar{k}}}{u}\Big). \tag{2-9}$$

On the other hand, the identity

$$\Big(\log\Big(\frac{1}{u}\Big)\Big)_{j\bar{k}} = -\frac{u_{j\bar{k}}}{u} + \frac{u_j u_{\bar{k}}}{u^2}$$

shows that

$$J[u] = (-1)^m u \det\Big(u_{j\bar{k}} - \frac{u_j u_{\bar{k}}}{u}\Big) = u^{m+1} \det\Big(\Big(\log\Big(\frac{1}{u}\Big)\Big)_{j\bar{k}}\Big). \tag{2-10}$$

The lemma follows (2-9) and (2-10). $\square$

We can use Lemma 2.6 to show that if u solves the Monge–Ampère equation, then u is the Kähler potential of a Kähler–Einstein metric. Recall that if

$$g = v_{j\bar{k}}\, dz^j \otimes dz^{\bar{k}},$$

then the Ricci curvature is

$$R_{a\bar{b}} = -\big(\log\det(v_{j\bar{k}})\big)_{a\bar{b}}.$$

Now let $v = \log(\frac{1}{u})$ where $J[u] = 1$. Then

$$R_{a\bar{b}} = -\big(\log\det(v_{j\bar{k}})\big)_{a\bar{b}} = -\big(\log(u^{-(m+1)})\big)_{a\bar{b}} = -(m+1)\Big(\log\Big(\frac{1}{u}\Big)\Big)_{a\bar{b}} = -(m+1)g_{a\bar{b}},$$

which is the Einstein equation.

Proof of Lemma 2.5. First, we compute

$$\big(\log(|\det f'|^{-2/(m+1)}\, u \circ f)\big)_{j\bar{k}} = \frac{-1}{m+1}\big(\log(|\det f'|^2)\big)_{j\bar{k}} + (\log(u \circ f))_{j\bar{k}},$$

where the first term on the right-hand side vanishes because $|\det f'|^2 = (\det f')\overline{(\det f')}$ and $\det f'$ is holomorphic. We note that the vanishing of the first term also shows that the Kähler metric with Kähler potential u (when u solves the Monge–Ampère equation) is invariant whether u is considered as a scalar function or a density. To compute the nonzero term on the right-hand side we first note that if f is a holomorphic map then we have the identity

$$(v \circ f)_{j\bar{k}} = \big((f')^t (v_{a\bar{b}} \circ f)\overline{f'}\big)_{j\bar{k}}.$$

Thus, using Lemma 2.6, we compute

$$\begin{aligned} J\big[|\det(f')|^{-2/(m+1)} u \circ f\big] &= |\det(f')|^{-2}(u \circ f)^{m+1} \cdot \det\big((f')^t\big)\det\Big(\log\Big(\frac{1}{u}\Big)_{a\bar{b}} \circ f\Big)\det(\overline{f'}) \\ &= (u \circ f)^{m+1}\det\Big(\log\Big(\frac{1}{u}\Big)_{a\bar{b}}\Big) \circ f = J[u] \circ f. \end{aligned}$$

□

It is essential for our globalization argument that an approximate solution to the Monge–Ampere equation be determined uniquely up to a certain order. This proof was given by Fefferman [1976] and we repeat it for the reader's convenience.

Lemma 2.7. *Any smooth, local, approximate solution $\psi \in \mathscr{C}^\infty(\Omega)$ to the Monge–Ampère equation is uniquely determined up to order $m+1$.*

Proof. Suppose that ρ is a smooth function on Ω defined in a neighborhood of $\partial\Omega$ with $\rho = 0$ on $\partial\Omega$ and $\rho'(p) \neq 0$ for all $p \in \partial\Omega$. We recall Fefferman's iterative construction of an approximate solution u to the Monge–Ampère equation, that is, a function $u \in \mathscr{C}^\infty$ with $u|_{\partial\Omega} = 0$ and $J[u] = 1 + \mathscr{O}(u^{m+2})$. To obtain a first approximation, note that for ρ as above, and for any smooth function η, we have

$$J[\eta\rho] = \eta^{m+1} J[\rho], \tag{2-11}$$

when $\rho = 0$, so the function

$$\psi^{(1)} = \rho \cdot J[\rho]^{-1/(m+1)}$$

satisfies $J[\psi^{(1)}]=1$ on $\partial\Omega$, and $J[\psi^{(1)}]=1+\mathbb{O}(\psi^{(1)})$. The fact that $J[\rho]$ is nonzero on $\partial\Omega$ follows from pseudoconvexity that implies that $\rho_{j\bar{k}}$ is positive definite on $\ker\partial\rho$ on $\partial\Omega$, and that $\rho'\neq 0$ on $\partial\Omega$. Note that if φ and ψ are two functions vanishing on $\partial\Omega$, it follows that $\varphi=\eta\psi$ for some smooth function η. Thus, by (2-11), $J[\varphi]=\eta^{m+1}J[\psi]$. From this computation it follows that any approximate solution u is uniquely determined up to first order.

We now iterate this construction. Suppose that for an integer $s\geq 2$, we have an approximate solution to the Monge–Ampère equation to order $s-1$. That is, we have a smooth function ψ with $\psi=0$ on $\partial\Omega$, $\psi'(p)\neq 0$ for all $p\in\partial\Omega$, and $J[\psi]=1+\mathbb{O}(\psi^{s-1})$. We seek a function of the form $v=\psi+\eta\psi^s$, where $\eta\in\mathscr{C}^\infty$ is chosen so that $J[v]=1+\mathbb{O}(\psi^s)$. The iteration is based on formula

$$J[\psi+\eta\psi^s]=J[\psi]+s(m+2-s)\eta\psi^{s-1}+\mathbb{O}(\psi^s),$$

for smooth functions ψ and η, again with the property that ψ vanishes on $\partial\Omega$. This formula is a straightforward computation using the formula (2-8). From this formula it follows that the desired function v is given by

$$v=\psi+\frac{1-J[\psi]}{s(m+2-s)}\psi^s.$$

The iteration clearly works up to $s=m+1$ and produces an approximate solution with the desired properties. It also follows that any function $\widetilde{u}$ with $u-\widetilde{u}=\mathbb{O}(\psi^{m+2})$ satisfies $J[\widetilde{u}]=J[u]+\mathbb{O}(\psi^{m+2})$. Thus, in particular, any smooth function having the same $(m{+}1)$-jet on $\partial\Omega$ as an approximate solution is also an approximate solution.

On the other hand, it is clear that any two approximate solutions *must* have the same $(m{+}1)$-jet on $\partial\Omega$. If ψ and $\widetilde{\psi}$ satisfy $\psi-\widetilde{\psi}=\eta\psi^s$ then

$$J[\psi]-J[\widetilde{\psi}]=s(m+2-s)\eta\psi^{s-1}+\mathbb{O}(\psi^s).$$

In particular, if $s<m+2$ and $J[\psi]-J[\widetilde{\psi}]=\mathbb{O}(\psi^{m+2})$ then ψ and $\widetilde{\psi}$ are approximate solutions uniquely determined up to order $m+2$. □

2D.2. *Global theory.* Now suppose X is a compact complex manifold of dimension $m=n+1$ with boundary $M=\partial X$. Note that M has real dimension $2n+1$ and inherits an integrable CR-structure from X. As always, we assume that M with this CR-structure is strictly pseudoconvex. We first say what it means for a single smooth function φ defined in a neighborhood of M to be an approximate solution of the complex Monge–Ampère equation. We denote by $\mathscr{C}^\infty(X)$ the smooth functions on X.

Definition 2.8. We will say that a function $\varphi\in\mathscr{C}^\infty(X)$ is a globally defined approximate solution of the complex Monge–Ampère equation near $M=\partial X$ if for any $p\in M$, there is a neighborhood V of p in X and holomorphic coordinates z on V so that φ is an approximate solution of the complex Monge–Ampère equation in the chosen coordinates.

As we will see later, we will need such a globally defined approximate solution in order to identify the residues of the scattering operator on X with CR-covariant differential operators.

If φ is a defining function for M with $\varphi<0$ in the interior of X, we associate to φ a Kähler form

$$\omega_\varphi=\frac{i}{2}\partial\bar{\partial}\log\Big(-\frac{1}{\varphi}\Big)$$

and a pseudo-Hermitian structure

$$\theta = \frac{i}{2}(\bar\partial - \partial)\varphi|_M. \tag{2-12}$$

Observe that two defining functions φ and ρ generate the same Kähler metric if and only if $\rho = e^F\varphi$ for a pluriharmonic function F, that is, $\partial\bar\partial F = 0$. It is known that a pluriharmonic function F is uniquely determined by its boundary values (see, for example, Bedford [1980]). If θ_ρ and θ_φ are the corresponding pseudo-Hermitian structures on M then $\theta_\rho = e^f\theta_\varphi$, where $f = F|_M$.

We give a necessary and sufficient condition on M for a globally defined approximate solution of the Monge–Ampère equation to exist. Recall that the canonical bundle of M is the bundle generated by forms $f\theta^1\wedge\cdots\wedge\theta^n\wedge\theta$ where f is smooth, θ is a contact form, and $\{\theta^\alpha\}_{\alpha=1}^n$ is an admissible coframe. If M is the boundary of a strictly pseudoconvex domain in $\mathbb{C}^m$, the canonical bundle is generated by restrictions of forms $f\,dz^1\wedge\cdots\wedge dz^m$ to M. The sections of the canonical bundle are $(n+1,0)$-forms ζ on M.

If θ is a contact form, T is the characteristic vector field, and ζ is *any* nonvanishing section of the canonical bundle, it is not difficult to see that

$$\theta\wedge(T\lrcorner\zeta)\wedge(T\lrcorner\bar\zeta) = \lambda\theta\wedge(d\theta)^n$$

for a smooth positive function λ. We say that the contact form θ is *volume-normalized* with respect to a nonvanishing section ζ of the canonical bundle if

$$\theta\wedge(d\theta)^n = i^{n^2}n!\,\theta\wedge(T\lrcorner\zeta)\wedge(T\lrcorner\bar\zeta),$$

where T is the characteristic vector field. The following criterion will be useful.

Lemma 2.9. *The contact form θ given by* (2-12) *is volume-normalized with respect to the form*

$$\zeta = (dz^1\wedge\cdots\wedge dz^m)|_M$$

if and only if

$$J[\varphi] = 1 + \mathbb{O}(\varphi)$$

in the coordinates $(z_1, \dots, z_m)$.

For the proof see [Farris 1986, Proposition 5.2]. Using Lemma 2.9 we can prove:

Proposition 2.10. *Suppose that X is a compact complex manifold with boundary $M = \partial X$. There is a globally defined approximate solution φ of the Monge–Ampère equation in a neighborhood of M if and only if M admits a pseudo-Hermitian structure θ with the following property*: *In a neighborhood of any point $p\in M$, there is a local, closed $(n+1,0)$ form ζ such that θ is volume-normalized with respect to ζ.*

Proof. (i) First, suppose that X admits a globally defined approximate solution φ of the Monge–Ampère equation. Let θ be the associated contact form on X, that is, θ is given by (2-12). Pick $p\in M$ and let $z\equiv(z_1,\dots,z_m)$ be holomorphic coordinates near p so chosen that φ is an approximate solution of the Monge–Ampère equation near p in these coordinates. Let

$$\zeta = (dz^1\wedge\cdots\wedge dz^m)|_M.$$

Then θ is volume-normalized with respect to ζ by Lemma 2.9.

(ii) Suppose that θ is a given contact form on M with the property that, for each point $p \in M$, there is a neighborhood of p and a closed, locally defined section ζ of the canonical bundle with respect to which θ is volume-normalized. Write

$$\zeta = f(dz^1 \wedge \cdots \wedge dz^m)|_M$$

for holomorphic coordinates $\{z_1, \ldots, z_m\}$ defined in a neighborhood of p and a smooth function f. The condition $d\zeta = 0$ is equivalent to the condition

$$\bar{\partial}_b f = 0,$$

that is, f is a CR-holomorphic function. By the strict pseudoconvexity of M, there is a holomorphic extension F to a neighborhood V of p in X, that is, there is an F defined near p with $\bar{\partial} F = 0$ and $F|_{M\cap V} = f$ (see [Kohn and Rossi 1965]). We claim that we can find new holomorphic coordinates $w \equiv (w_1, \ldots, w_m)$ near p with the property that

$$\frac{\partial(w_1, \ldots, w_m)}{\partial(z_1, \ldots, z_m)} = F(z), \tag{2-13}$$

where the left-hand side is the determinant of the holomorphic Jacobian. If so then

$$\zeta = dw^1 \wedge \cdots \wedge dw^m|_M.$$

Constructing in V an approximate solution ψ_V of the Monge–Ampère equation in the w-coordinates (as in Lemma 2.7, following Fefferman [1976]), we conclude from Lemma 2.9 that the induced contact form

$$\theta_V = \frac{i}{2}(\bar{\partial} - \partial)\psi_V|_{M\cap V}$$

on $M \cap V$ is volume-normalized with respect to ζ, and thus coincides with θ.

We now claim that the local approximate solutions ψ_V can be glued together to form a globally defined approximate solution to the Monge–Ampère equation in the sense of Definition 2.8. We first note an important property of the transition map for two local coordinate systems. Let V_1 and V_2 be neighborhoods of M in X with nonempty intersection, let z and w be holomorphic coordinates on V_1 and V_2, and suppose that ψ_1 and ψ_2 are approximate solutions of the complex Monge–Ampère equation in these coordinates respectively. More precisely, $u_1 = \psi_1 \circ z$ and $u_2 = \psi_2 \circ w$ are approximate solutions to the Monge–Ampère equation on coordinate patches U_1 and U_2 in $\mathbb{C}^m$, and there is a biholomorphic map

$$g : U_2 \cap w^{-1}(V_1 \cap V_2) \to U_1 \cap z^{-1}(V_1 \cap V_2).$$

The function $u_2 = |g'|^{2/(m+1)} u_1 \circ g$ is also an approximate solution of the complex Monge–Ampere equation in $U_2 \cap w^{-1}(V_1 \cap V_2)$ by Lemma 2.5, so by uniqueness we have $u_2 = e^F u_1 \circ g$, up to order $m+1$, where $F = \frac{2}{m+1} \log |g'|$ is pluriharmonic. Moreover, since u_1 and u_2 both induce the contact form θ it follows that

$$(\bar{\partial} - \partial)u_2|_{U_2\cap w^{-1}(M\cap V_1\cap V_2)} = ((\bar{\partial} - \partial)u_1) \circ f|_{U_2\cap w^{-1}(M\cap V_1\cap V_2)}$$

from which we deduce that $F|_{U_2\cap w^{-1}(M\cap V_1\cap V_2)} = 0$, and hence $F = 0$ by the uniqueness of pluriharmonic extensions. In particular, the map g is unimodular, $|g'| = 1$. Thus $u_2 = u_1 \circ g$ on $U_2 \cap w^{-1}(V_1 \cap V_2)$ up to order $m+1$.

We now fix a boundary defining function ρ. Suppose that $\{U_i\}$ is a finite cover of a neighborhood of the boundary by holomorphic charts. Denote by F_i the map from $\mathbb{C}^m$ into U_i and set $F_{ij} = F_i^{-1} \circ F_j$. As proved above, the cover and holomorphic coordinates (U_i, F_i) may be chosen so that the transition maps are unimodular, that is, $|F'_{ij}| = 1$. Using Fefferman's construction, we can produce in each U_i an approximate solution u_i in the sense that

$$J[u_i] = 1 + \mathbb{O}(\rho^{m+1}).$$

Now suppose that $\{\chi_i\}$ is a $\mathscr{C}^\infty$ partition of unity subordinate to the cover $\{U_i\}$. We claim that the smooth function $u = \sum_i \chi_i u_i$ is an approximate solution of the Monge–Ampère equation in the sense of Definition 2.8. Choose U_i so that $p \in U$. We may write

$$u = \sum_j (\chi_j \circ F_i)(u_j \circ F_i).$$

Since $u_j \circ F_i = (u_j \circ F_j) \circ F_{ji}$, we see that $u_j \circ F_i$ is also an approximate solution to the Monge–Ampère equation in the F_i-coordinates. Thus, there is a smooth function η_{ji} so that

$$(u_j \circ F_i)(z) - (u_i \circ F_i)(z) = \eta_{ji}(z)(\rho \circ F_i)^{m+2}(z)$$

where η_{ji} is smooth. We conclude that

$$u(z) - u_i(z) = \mathbb{O}((\rho \circ F_i)^{m+2}).$$

This shows that u is also an approximate solution of the Monge–Ampère equation in the F_i-coordinates as claimed.

To finish the proof it suffices to establish that such a holomorphic coordinate change $z \mapsto w$, as in (2-13), exists. We consider a coordinate transformation given by

$$w(z) = (h(z), z_2, \dots, z_m), \tag{2-14}$$

where $h(z)$ is the unknown holomorphic function. Condition (2-13) is equivalent to

$$\frac{\partial h}{\partial z_1}(z_1, \dots, z_m) = F(z_1, z_2, \dots, z_m).$$

Here, F is the holomorphic extension of the CR-function f. We solve this equation for h as follows.

We set the convention that a boundary chart in $\mathbb{C}^m$ is the intersection of an open ball about 0 with the (real) half-space $\operatorname{Im} z_m \geq 0$. We assume that the boundary point p corresponds to $0 \in \partial\mathbb{C}^m$. The unknown function h is a complex-valued function defined in a neighborhood V of $0 \in \mathbb{C}^m$, is holomorphic in $V \cap \{\operatorname{Im} z_m > 0\}$, has CR boundary values, and satisfies $h(0) = 0$. Thus, the map $w(z)$, defined in (2-14), preserves the boundary $\operatorname{Im}(z_n) = 0$.

Consequently, the desired change of coordinates is obtained by solving the initial value problem

$$\begin{aligned} &\frac{\partial h}{\partial z_1}(z_1, \dots, z_m) = F(z_1, z_2, \dots, z_m), \\ &h(0, z_2, \dots, z_m) = 0, \end{aligned}$$

by simple integration. $\square$

We can also express the basic criterion in Proposition 2.10 in geometric terms. Recall that the contact form θ defines a pseudo-Hermitian, pseudo-Einstein structure on M if the Webster Ricci tensor is a multiple of the Levi form. Lee [1988] proved:

Theorem 2.11. *Suppose that M is a CR manifold of dimension ≥ 5. A contact form θ on M is pseudo-Einstein if and only if for each $p \in M$ there is a neighborhood of p in M and a locally defined closed section ζ of the canonical bundle with respect to which θ is volume-normalized.*

As an immediate consequence of Theorem 2.11, we have:

Theorem 2.12. *Suppose that M is a CR manifold of dimension ≥ 5. There is a globally defined approximate solution φ of the complex Monge–Ampère equation in a neighborhood of M if and only if M carries a contact form θ for which the corresponding pseudo-Hermitian structure is pseudo-Einstein. In this case, the contact form θ is induced by the globally defined approximate solution to the Monge–Ampère equation φ.*

Remark 2.13. If φ is a global approximate solution to the Monge–Ampère equation, then so is $e^F\varphi$ where F is any pluriharmonic function. The effect of the factor F is simply to change the choice of local coordinates needed to obtain a local approximate solution of the Monge–Ampère equation in any chart, as the argument in the proof of Proposition 2.10 easily shows. As observed above, the Kähler form ω_φ is invariant under the change $\varphi \mapsto e^F\varphi$.

3. Poisson operator and scattering operator

In this section we study the Dirichlet problem (1-3) following a standard technique in geometric scattering theory (see, for example, Melrose [1995]; we follow closely the analysis of the Poisson operator and scattering operator on conformally compact manifolds by Graham and Zworski [2003]). Note that Epstein, Melrose and Mendoza [1991] had previously studied the Poisson operator for a class of manifolds that includes compact complex manifolds with strictly pseudoconvex boundaries. More recently, Guillarmou and Sá Barreto [2008] studied scattering theory and radiation fields for asymptotically complex hyperbolic manifolds, a class which also includes that studied here.

We will set $x = -\varphi$ and we will denote by $\mathscr{C}^\infty(X)$ the set of smooth functions on X having Taylor series to all orders at $x = 0$, and by $\dot{\mathscr{C}}^\infty(X)$ the space of functions vanishing to all orders at $x = 0$. The space $\mathscr{C}^\infty(\mathring{X})$ consists of smooth functions on $\mathring{X}$ with no restriction on boundary behavior. We will denote by $x^s\mathscr{C}^\infty(X)$ the set of functions in $\mathscr{C}^\infty(\mathring{X})$ having the form x^sF for $F \in \mathscr{C}^\infty(X)$.

Since

$$N = -2\frac{\partial}{\partial x}$$

it follows that

$$\nu = -\frac{x}{\sqrt{1+rx}}\frac{\partial}{\partial x} \tag{3-1}$$

is the outward normal to the hypersurface $x = \varepsilon$. Green's theorem implies that

$$\int_{x>\varepsilon}(u_1\Delta_\varphi u_2 - u_2\Delta_\varphi u_1)\,\omega^m = \int_{x=\varepsilon}(u_1\nu u_2 - u_2\nu u_1)\,\nu\lrcorner\,\omega^m. \tag{3-2}$$

We first note the "boundary pairing formula" (recall the definition (2-4)).

Proposition 3.1. *Suppose* $\mathrm{Re}(s) = m/2$, *that* u_1 *and* u_2 *belong to* $\mathscr{C}^\infty(\mathring{X})$ *and there are functions* $F_i, G_i \in \mathscr{C}^\infty(X)$ *so that* $u_i = x^{m-s} F_i + x^s G_i$, $i = 1, 2$. *Finally, suppose that* $(\Delta_\varphi - s(m-s))u_i = r_i \in \dot{\mathscr{C}}^\infty(X)$, $i = 1, 2$. *Then,*

$$\int_X (u_1 r_2 - u_2 r_1)\, \omega^m = (2s - m) \int_M (F_1 G_2 - F_2 G_1)\, \psi.$$

Proof. A standard computation using (3-2) and (3-1) together with (2-3) and (2-5). □

Remark 3.2. For $\mathrm{Re}(s) = m/2$ complex conjugation reverses the roles of s and $m - s$. Thus we obtain the formula

$$\int_X (u_1 \bar{r}_2 - \bar{u}_2 r_1)\, \omega^m = (2s - m) \int_M (F_1 \overline{F}_2 - G_1 \overline{G}_2)\, \psi. \tag{3-3}$$

For later use, we note an extension of the boundary pairing formula analogous to [Graham and Zworski 2003, Proposition 3.3].

Proposition 3.3. *Suppose that* $\mathrm{Re}(s) > m/2$ *and* $2s - m \notin \mathbb{N}$. *Suppose that* $u_i \in \mathscr{C}^\infty(\mathring{X})$ *takes the form* $u_i = x^{m-s} F_i + x^s G_i$ *and* $(\Delta_\varphi - s(m-s))u_i = 0$, *for* $i = 1, 2$. *Then*

$$\underset{\varepsilon \downarrow 0}{\mathrm{FP}} \left(\int_{x > \varepsilon} (\langle \nabla u_1, \nabla u_2 \rangle - s(m-s) u_1 u_2)\, \omega^m \right) = -m \int_M G_1 F_2\, \psi = -m \int_M F_1 G_2\, \psi,$$

where FP *denotes the Hadamard finite part of the integral as* $\varepsilon \downarrow 0$.

Proof. Green's formula (3-2) for the operator $(\Delta_\varphi - s(m-s))$ gives

$$\int_{x > \varepsilon} (\langle \nabla u_1, \nabla u_2 \rangle - s(m-s) u_1 u_2)\, \omega^m = \int_{x = \varepsilon} u_1 (\nu u_2) \nu \lrcorner\, \omega^m,$$

from which the claimed formula follows. □

3A. ***The Poisson map.*** We will now prove that the Dirichlet problem (1-3) has a unique solution if $\mathrm{Re}(s) \geq m/2$, $2s - m \notin \mathbb{Z}$, and $s(m-s)$ is not an eigenvalue of Δ_φ. Most of the formal arguments are almost identical to the case of even asymptotically hyperbolic manifolds considered in [Graham and Zworski 2003] since the form of the indicial operator (2-7) for the Laplacian is the same.

Lemma 3.4. *Suppose that* $u \in \mathscr{C}^\infty(\mathring{X})$ *satisfies* $u = x^{m-s} F + x^s G$ *for functions* F *and* G *belonging to* $\mathscr{C}^\infty(X)$, *and that*

$$(\Delta_\varphi - s(m-s))u \in \dot{\mathscr{C}}^\infty(X), \tag{3-4}$$

for $s \in \mathbb{C}$ *with* $2s - m \notin \mathbb{Z}$. *Then the Taylor expansions of* F *and* G *at* $x = 0$ *are formally determined respectively by* $F|_M$ *and* $G|_M$. *In particular, we have*

$$F \sim \sum_{k \geq 0} x^k f_k \quad \text{and} \quad G \sim \sum_{k \geq 0} x^k g_k$$

where

$$f_k = \frac{(-1)^k}{2^k k!} \frac{\Gamma(2s - m - k)}{\Gamma(2s - m)} P_{k,s} f_0 \quad \text{and} \quad g_k = \frac{(-1)^k}{2^k k!} \frac{\Gamma(m - 2s - k)}{\Gamma(m - 2s)} P_{k,m-s} g_0, \tag{3-5}$$

with $P_{k,s}$ the differential operators of order $2k$ holomorphic in s with leading symbol [2]

$$\sigma(P_{k,s}) = \sigma(\Delta_b^k).$$

Proof. Recall the asymptotic development (2-6) for the Laplacian which we use to derive a recurrence for the Taylor coefficients f_k and g_k of F and G. For $2s-m \notin \mathbb{Z}$, we may consider the terms involving F and G separately. We first consider F. Observe that

$$(L_0 - s(m-s))(x^{m-s+k} f) = k(2s-m-k)x^{m-s+k} f$$

for $f \in \mathscr{C}^\infty(M)$. Since $L_k = P(x\partial_x, \partial_y)$ for a defining function x and boundary coordinates y where P is a polynomial of degree at most two with smooth coefficients, the operators

$$Q_{k,\ell}(s) = x^{-m+s-\ell} L_{k-\ell} x^{m-s+\ell}$$

are differential operators of order at most two depending holomorphically on s. If $u \sim \sum_{k=0}^\infty x^{m-s+k} f_k$, it follows from (3-4) and (2-6) that for any $k \geq 1$,

$$f_k = -\frac{1}{k(2s-m-k)} \sum_{\ell=0}^{k-1} Q_{k,\ell}(s) f_\ell. \tag{3-6}$$

Similarly, if $u \sim \sum_{k\geq 0} x^{s+k} g_k$ for $g_k \in \mathscr{C}^\infty(M)$, we have

$$g_k = -\frac{1}{k(m-2s-k)} \sum_{\ell=0}^{k-1} Q_{k,\ell}(m-s) g_\ell.$$

The formulas for f_k, g_k and $P_{k,s}$ follow easily from these formulas and the fact that

$$Q_{k,k-1}(s) = \tfrac{1}{2}\Delta_b + r_0(m-s+k-1)^2. \qquad \square$$

Remark 3.5. We will write $p_{k,s}$ for the operator satisfying $f_k = p_{k,s} f_0$, so that

$$p_{k,s} = \frac{(-1)^k}{2^k k!} \frac{\Gamma(2s-m-k)}{\Gamma(2s-m)} P_{k,s},$$

where $P_{k,s}$ is described in Lemma 3.4. The operator $p_{k,s}$ is meromorphic with poles at

$$s = \tfrac{m}{2} + \tfrac{k}{2}, \ldots, \tfrac{m}{2} + \tfrac{1}{2}.$$

We will denote

$$p_\ell = \operatorname*{Res}_{s=(m+\ell)/2} p_{\ell,s}.$$

The operator p_ℓ is a differential operator of order at most 2ℓ with principal symbol

$$\sigma(p_\ell) = \frac{(-1)^\ell}{2^{\ell+1}\ell!(\ell-1)!} \sigma(\Delta_b^\ell).$$

[2] Here in the sense of the ordinary (rather than the Heisenberg) calculus on M.

For $\mathrm{Re}(s) > m/2$, let

$$R(s) = (\Delta_\varphi - s(m-s))^{-1}$$

be the $L^2(X)$ resolvent, let $\sigma_p(\Delta_\varphi)$ denote the set of L^2-eigenvalues of Δ_φ, and let

$$\Sigma = \{s : \mathrm{Re}(s) > \tfrac{m}{2},\ s(m-s) \in \sigma_p(\Delta_\varphi)\}.$$

We will now solve the Dirichlet problem (1-3) for $\mathrm{Re}(s) \geq m/2$ and $s \notin \Sigma$.

The following result is an easy consequence of the work of Epstein, Melrose, and Mendoza [1991]. Note that in our case the Kähler metric is an even metric, that is, depends smoothly on the defining function φ (and not simply on its square root).

Proposition 3.6. *The set Σ contains at most finitely many points, and the resolvent operator $R(s)$ is a meromorphic operator-valued function for* $\mathrm{Re}(s) > m/2$ *having at most finitely many, finite-rank poles at $s \in \Sigma$. Moreover, for $s \notin \Sigma$ and* $\mathrm{Re}(s) > m/2$,

$$R(s) : \dot{\mathscr{C}}^\infty(X) \to x^s \mathscr{C}^\infty(X).$$

First, we prove uniqueness of the solutions to the Dirichlet problem (1-3) for s with $\mathrm{Re}(s) \geq m/2$, $s \notin \Sigma$, and $2s - m \notin \mathbb{Z}$.

Proposition 3.7. *Suppose that* $\mathrm{Re}(s) \geq m/2$, $s \notin \Sigma$, *and* $2s - m \notin \mathbb{Z}$. *Suppose that $u \in \mathscr{C}^\infty(\mathring{X})$ with* $(\Delta_\varphi - s(m-s))u = 0$, *and that $u = x^{m-s}F + x^s G$ with $F|_M = 0$. Then $u = 0$.*

Proof. First, suppose that $\mathrm{Re}(s) > m/2$ and $s \notin \Sigma$. It follows from Lemma 3.4 that $u = x^s G$ for $G \in \mathscr{C}^\infty(X)$. Since $\mathrm{Re}(s) > m/2$ it is clear that

$$\int_X |u|^2\, \omega^m < \infty,$$

hence $u \in L^2(X)$, hence $u = 0$.

If $\mathrm{Re}(s) = m/2$ but $s \neq m/2$, we may again assume that $u = x^s G$ for $G \in \mathscr{C}^\infty(X)$. Next, we set $u_1 = u_2 = u$ in (3-3) to conclude that

$$\int_M |G|^2 \psi = 0$$

so that $G|_M = 0$. Using Lemma 3.4 again we conclude that $G \in \dot{\mathscr{C}}^\infty(X)$, hence $u \in \dot{\mathscr{C}}^\infty(X)$. As in [Guillarmou and Sá Barreto 2008], we can now deduce from [Vasy and Wunsch 2005] that $u = 0$. □

To prove the existence of a solution of the Dirichlet problem (1-3), we follow the method of Graham and Zworski [2003]. Given $f \in \mathscr{C}^\infty(M)$ we can construct a formal power series solution $u = x^{m-s}F$ modulo $\dot{\mathscr{C}}^\infty(X)$, and then use the resolvent to correct this approximate solution to an exact solution. Using Borel's lemma we can sum the asymptotic series $\sum_{j\geq 0} f_j x^j$ (where f_j is computed via (3-6) with $f_0 = f$) to a function $F \in \mathscr{C}^\infty(X)$. As in [Graham and Zworski 2003], we obtain:

Lemma 3.8. *There is an operator $\Phi(s) : \mathscr{C}^\infty(M) \to x^{m-s}\mathscr{C}^\infty(X)$ with*

$$(\Delta_\varphi - s(m-s)) \circ \Phi : \mathscr{C}^\infty(M) \to \dot{\mathscr{C}}^\infty(X)$$

so that $\Gamma(m-2s)^{-1}\Phi(s)$ is holomorphic in s.

Note that $\Phi(s)$ need not be linear as the construction of F depends on the choice of cutoff functions in the application of Borel's lemma. As noted in [Graham and Zworski 2003], an expansion to finite order in x suffices for the construction. This guarantees the continuity of the map $\Phi(s)$ in the data f. Now define an operator

$$\mathcal{P}(s) : \mathscr{C}^\infty(M) \to \mathscr{C}^\infty(\mathring{X})$$

for s with $\mathrm{Re}(s) \geq m/2$, $s \neq m/2$ and $s \notin \Sigma$ by

$$\mathcal{P}(s) = \big(I - R(s)(\Delta_\varphi - s(m-s))\big) \circ \Phi(s).$$

Lemma 3.9. *For any* $f \in \mathscr{C}^\infty(M)$, *the function* $u = \mathcal{P}(s) f$ *solves the Dirichlet problem* (1-3), *and* $f \mapsto \mathcal{P}(s) f$ *is a linear operator.*

Proof. The linearity of $\mathcal{P}(s)$ will follow from the unicity of the solution to (1-3). It is immediate from the definition that $(\Delta_\varphi - s(m-s))u = 0$, and from the mapping property in Proposition 3.6, $u = x^{m-s} F + x^s G$ with

$$F = x^{s-m} \Phi(s) f \quad \text{and} \quad G = -x^{-s} R(s)\big((\Delta_\varphi - s(m-s))\Phi(s) f\big). \qquad \square$$

Theorem 3.10. *For* $\mathrm{Re}(s) \geq m/2$, $2s - m \notin \mathbb{Z}$, *and* $s \notin \Sigma$, *there exists a unique solution of the Dirichlet problem* (1-3).

3B. *The scattering operator.* The *scattering operator* for Δ_φ is the linear mapping

$$S_X(s) : \mathscr{C}^\infty(M) \to \mathscr{C}^\infty(M), \quad f \mapsto G|_M,$$

where $u = x^{m-s} F + x^s G$ solves (1-3). It is well defined by Theorem 3.10.

The scattering operator has infinite-rank poles when $\mathrm{Re}(s) > m/2$ and $2s - m \in \mathbb{Z}$ owing to the crossing of indicial roots for the normal operator L_0. At the exceptional points $s = (m+k)/2$ one expects solutions of the eigenvalue equation $(\Delta_\varphi - s(m-s))u = 0$ having the form

$$u = x^{m/2-k} F + (x^{m/2+k} \log x) G.$$

In order to study the singularities of the scattering operator at these points we modify the construction of the Poisson operator following the lines of [Graham and Zworski 2003, Section 4].

Let f_1 and f_2 belong to $\mathscr{C}^\infty(M)$ and let u_1 and u_2 solve the corresponding Dirichlet problems for some s with $\mathrm{Re}(s) > m/2$ and $2s - m \notin \mathbb{N}$. Applying the generalized boundary pairing formula (see Proposition 3.3) to u_1 and $\bar{u}_2$ for s real, we conclude that

$$\int_M f_1 \overline{S_X(s) f_2}\, \psi = \int_M (S_X(s) f_1) \overline{f_2}\, \psi$$

so $S_X(s)$ is self-adjoint in the natural inner product on $\mathscr{C}^\infty(M)$.

Now we study the scattering operator near the exceptional points. The arguments used here are exactly those of [Graham and Zworski 2003, Section 3] but we summarize them here for the reader's convenience.

Recall the operators $p_{k,s}$ and p_ℓ defined in Remark 3.5. First, we prove:

Lemma 3.11. *At the points* $s = (m+\ell)/2$ *for* $\ell = 1, 2, \ldots$, $s \notin \Sigma$, *the Poisson map takes the form*

$$\mathcal{P}\left(\tfrac{m}{2} + \tfrac{\ell}{2}\right) f = x^{m/2-\ell/2} F + (x^{m/2+\ell/2} \log x) G,$$

where $F|_M = f$, $G|_M = -2p_\ell f$ and

$$p_\ell = \operatorname*{Res}_{s=(m+\ell)/2} p_{\ell,s} \tag{3-7}$$

is a differential operator of order 2ℓ with

$$\sigma(p_\ell) = \frac{(-1)^\ell}{2^{\ell+1}\ell!(\ell-1)!}\sigma(\Delta_b^\ell).$$

Proof. We first show that the Poisson map $\mathcal{P}(s)$ is also regular at $s = (m+\ell)/2$, $\ell = 1, 2, \ldots$ so long as these points do not belong to Σ. As in [Graham and Zworski 2003] we introduce the operator

$$\Phi_\ell(s) = \Phi(s) - \Phi(m-s) \circ p_{\ell,s},$$

where $p_{\ell,s}$ is a differential operator of order 2ℓ defined in Remark 3.5, each term on the right-hand side has at most a first-order pole at $s = (m+\ell)/2$, the operators $p_{j,s}$ occurring in the definition of $\Phi(s)$ have at most first-order poles and $\Phi(m-s)$ is analytic in s for $\operatorname{Re}(s) > m/2$. For given $f \in \mathscr{C}^\infty(M)$, we compute the residue of $\Phi_\ell(s)f$ at $s = (m+\ell)/2$. First

$$\lim_{s\to(m+\ell)/2}\Big(s - \frac{m}{2} - \frac{\ell}{2}\Big)\Phi(s)f = x^{(m+\ell)/2}\operatorname*{Res}_{s=(m+\ell)/2}(p_{\ell,s}f) + \mathbb{O}(x^{m/2+\ell/2+1}),$$

since the remaining terms in the asymptotic expansion for $\Phi(s)f$ are holomorphic near $s = (m+\ell)/2$. Second,

$$\lim_{s\to(m+\ell)/2}\Big(s - \frac{m}{2} - \frac{\ell}{2}\Big)\Phi(m-s)(p_{\ell,s}f) = x^{m/2+\ell/2}\operatorname*{Res}_{s=(m+\ell)/2}(p_{\ell,s}f) + \mathbb{O}(x^{m/2+\ell/2+1}).$$

It follows that

$$\operatorname*{Res}_{s=(m+\ell)/2}\Phi_\ell(s)f = \mathbb{O}(x^{m/2+\ell/2+1}) \tag{3-8}$$

so that, by Lemma 3.4,

$$\operatorname*{Res}_{s=(m+\ell)/2}\Phi_\ell(s)f \in \dot{\mathscr{C}}^\infty(X).$$

Now let us define

$$\mathcal{P}_\ell(s) = \big(I - R(s)(\Delta_\varphi - s(m-s))\big) \circ \Phi_\ell(s).$$

Clearly, $\mathcal{P}_\ell(s)$ is holomorphic in a deleted neighborhood of $s = (m+\ell)/2$ (with at most a first-order pole at $s = (m+\ell)/2$) and maps $\mathscr{C}^\infty(M)$ into $\mathscr{C}^\infty(\mathring{X})$. If $s \notin \Sigma$, it follows from the definition of $\mathcal{P}_\ell(s)$, (3-8) and Proposition 3.6 that

$$\operatorname*{Res}_{s=(m+\ell)/2}\mathcal{P}_\ell(s)f \in x^s\mathscr{C}^\infty(X).$$

Hence the residue is an $L^2(X)$ function, and hence is zero. Thus $\mathcal{P}_\ell(s)$ is holomorphic at $s = (m+\ell)/2$. It follows from the uniqueness of solutions to the Dirichlet problem that $\mathcal{P}_\ell(s) = \mathcal{P}(s)$ wherever the former is defined. Exactly as in [Graham and Zworski 2003] we can compute $\mathcal{P}((m+\ell)/2)f$ by using $\mathcal{P}_\ell(s)$, the formula

$$\lim_{t\to 0}\frac{x^{-t} - x^t}{t} = -2\log x$$

and the fact that the $p_{k,s}$ have at most simple poles at $s = (m+\ell)/2$. This computation shows that $\mathcal{P}((m+\ell)/2)$ has the stated form. $\square$

Proposition 3.12. *Suppose that Δ_X has no eigenvalues of the form $s(m-s)$ with $s=(m+\ell)/2$, $\ell=1,2,\ldots$. Then, the scattering operator $S_X(s)$ has a first-order pole at $s=(m+\ell)/2$, $\ell=1,2,\ldots$ with*

$$\operatorname*{Res}_{s=(m+\ell)/2} S_X(s) = -p_\ell,$$

where p_ℓ is the differential operator given by (3-7).

Proof. From the formula for the $\mathcal{P}(s)$, it is clear that for $2s-m \notin \mathbb{N}$, we can compute the scattering operator from

$$S_X(s)f = \left(-x^{-s}R(s)(\Delta_\varphi - s(m-s))\Phi(s)f\right)\big|_{x=0}.$$

Since $\mathcal{P}(s)$ is holomorphic at $s=(m+\ell)/2$, it follows that

$$\operatorname*{Res}_{s=(m+\ell)/2}(S_X(s)f) = -\operatorname*{Res}_{s=(m+\ell)/2}(x^{-s}\Phi(s)f)|_{x=0}.$$

But

$$\operatorname*{Res}_{s=(m+\ell)/2}(x^{-s}\Phi(s)f)|_{x=0} = \operatorname*{Res}_{s=(m+\ell)/2}\left((x^{-s}\Phi(m-s)p_{\ell,s}f)|_{x=0}\right) = \operatorname*{Res}_{s=(m+\ell)/2}(p_{\ell,s}f)$$

and the claimed formula holds. □

To connect the scattering operator and the CR Q-curvature, we will also need the following result about the pole of the scattering operator at $s=m$; this result is a direct analogue of [Graham and Zworski 2003, Proposition 3.7] but we give the short proof for the reader's convenience.

Proposition 3.13. *Let* 1 *denote the constant function on M. Then,*

$$S_X(m)1 = -\lim_{s\to m} p_{m,s}(1).$$

Proof. As $s\to m$ we have $\mathcal{P}(s)1 \to 1$. On the other hand, for s with $|s-m|<\frac{1}{2}$,

$$\mathcal{P}(s)1 = \sum_{k=0}^{m} x^{m-s+k}p_{k,s}(1) + x^s S_X(s)1 + \mathbb{O}(x^{m+1/2}).$$

This implies that

$$\lim_{s\to m}(x^{2m-s}p_{m,s}(1) + x^m S_X(s)1) = 0$$

from which the claimed formula follows. □

Remark 3.14. Note that, although $p_{m,s}$ has a pole at $s=m$, the limit $\lim_{s\to m} p_{m,s}(1)$ exists. This implies that $P_{m,s}1$ (see (3-5)) has a first-order zero at $s=m$, that is,

$$P_{m,s}1 = (m-s)Q_{m,s}$$

for a scalar function $Q_{m,s}$. The CR Q-curvature is then given by $Q_{m,m}$ [Fefferman and Hirachi 2003].

4. CR-covariant operators

In this section we show that if φ is an approximate solution of the complex Monge–Ampère equation in the sense discussed above, then the residues of the scattering operator at $s = (m + \ell)/2$ for $\ell = 1, \ldots, m$ are the CR-covariant differential operators P_k defined in [Fefferman and Hirachi 2003]. In order to do this we first recall Fefferman and Graham's [1985] set-up for studying conformal invariants of compact manifolds and the construction of the GJMS operators [Graham et al. 1992]. We then recall its application to CR manifolds taking care that the arguments carry over from pseudoconvex domains in $\mathbb{C}^m$ to the manifold setting studied here.

4A. ***The GJMS construction.*** We begin by recalling Fefferman and Graham's construction of the ambient metric and ambient space for a conformal manifold and the GJMS conformally covariant operators on $\mathscr{C}$ obtained from this construction. Suppose that $(\mathscr{C}, [g])$ is a conformal manifold of signature (p, q), that is, a smooth manifold of dimension $N = p+q$ together with a conformal class of pseudo-Riemannian metrics of signature (p, q) on $\mathscr{C}$. Fix a conformal representative g_0. The *metric bundle* $\mathscr{G} \subset S^2T^*\mathscr{C}$ is a bundle on $\mathscr{C}$ with fibres

$$\mathscr{G}_p = \{t^2 g_0(p) : t > 0\}.$$

We denote by $\pi : \mathscr{G} \to M$ the natural projection. The *tautological metric* G on $\mathscr{G}$ is given by

$$G(X, Y) = g(\pi_* X, \pi_* Y)$$

for tangent vectors X and Y to $(p, g) \in \mathscr{G}$. There is a natural $\mathbb{R}^+$-action δ_s on $\mathscr{G}$ given by

$$\delta_s(p, g) = (p, s^2 g).$$

The *ambient space* over $\mathscr{C}$ is the space $\widetilde{\mathscr{G}} = \mathscr{G} \times (-1, 1)$. Note that the map $i : g \mapsto (g, 0)$ imbeds $\mathscr{G}$ in $\widetilde{\mathscr{G}}$.

Fefferman and Graham proved the existence of a unique metric $\widetilde{g}$ of signature $(p + 1, q + 1)$ on $\widetilde{\mathscr{G}}$, the *ambient metric* on $\widetilde{\mathscr{G}}$ having the following three properties:

(a) $i^*\widetilde{g} = G$;

(b) $\delta_s^*\widetilde{g} = s^2\widetilde{g}$;

(c) $\mathrm{Ric}(\widetilde{g}) = 0$ along $\mathscr{G}$ to infinite order if N is odd, and up to order $N/2$ if N is even.

Here the uniqueness is meant in the sense of formal power series.

To define the GJMS operators, we first define spaces of homogeneous functions on $\mathscr{G}$. For $w \in \mathbb{R}$ let $\mathscr{E}(w)$ denote the functions f on $\mathscr{G}$ homogeneous of degree w with respect to δ_s and smooth away from 0. The GJMS operators $\mathscr{P}_k$ may be defined in two ways:

(1) Given $f \in \mathscr{E}(-N/2 + k)$, extend f to a function $\widetilde{f}$ homogeneous of the same degree on $\widetilde{\mathscr{G}}$, and set

$$\mathscr{P}_k f = \widetilde{\Delta}^k \widetilde{f}\,\big|_{\mathscr{G}}, \tag{4-1}$$

where $\widetilde{\Delta}$ is the Laplacian for the ambient metric $\widetilde{g}$ on $\widetilde{\mathscr{G}}$.

(2) Given $f \in \mathscr{E}(-N/2 + k)$, $\mathscr{P}_k$ is the normalized obstruction to extending f to a smooth function $\widetilde{f}$ on $\widetilde{\mathscr{G}}$ having the same homogeneity and satisfying $\widetilde{\Delta}\widetilde{f} = 0$.

The existence of GJMS operators was proven in [Graham et al. 1992] for $k = 1, 2, \ldots$ if N is odd, and for $k = 1, 2, \ldots, N/2$ if N is even.

4B. ***Application to CR manifolds.*** Following [Gover and Graham 2005] we describe how the GJMS construction [Graham et al. 1992] can be used to prove the existence of CR-covariant differential operators. We begin with a CR manifold M of dimension $2n+1$ and show how to construct a conformal manifold $\mathscr{C}$ of dimension $2n+2$ and a conformal class of metrics with signature $(2n+1, 1)$ to which the GJMS construction may be applied. One then "pulls back" the GJMS operators on $\mathscr{C}$ to M.

Recall that the *canonical bundle* K over M is the bundle of holomorphic $(n+1)$-forms generated by holomorphic forms of the type $\theta \wedge \theta^1 \wedge \cdots \wedge \theta^n$ where θ is a contact form and $\{\theta^\alpha\}$ is a basis for $\mathscr{H}$ of admissible $(1,0)$-forms. We denote by K^* the canonical bundle of M with the zero section removed. The *circle bundle* $\mathscr{C}$ over M is the bundle

$$\mathscr{C} = (K^*)^{1/(n+2)}/\mathbb{R}^+.$$

The circle bundle is an S^1-bundle over M, having real dimension $2m$ if $m = n+1$. If we fix a contact form θ on M (and hence a pseudo-Hermitian structure on M), there is a corresponding section ζ of K^* chosen so that θ is volume-normalized with respect to ζ. We denote by ψ the angle determined by $\zeta(p)$ in each fibre of $\mathscr{C}$ and define a fibre variable

$$\gamma = \frac{\psi}{n+2}.$$

Note that γ is canonically determined by θ. Following Lee [1986], let us define a canonical one-form σ on $\mathscr{C}$ by

$$(n+2)\,\sigma = (n+2)\,d\gamma + i\omega_\alpha{}^\alpha - \frac{1}{2(n+1)} R\theta, \tag{4-2}$$

where $\omega_\alpha{}^\beta$ is the connection one-form and R is the Webster scalar curvature of the pseudo-Hermitian structure θ. The mapping $\theta \mapsto g_\theta$ given by

$$g_\theta = h_{\alpha\beta}\theta^\alpha \cdot \theta^{\overline{\beta}} + 2\theta \cdot \sigma, \tag{4-3}$$

where $\cdot$ denotes the symmetric product, defines a mapping of pseudo-Hermitian structures to Lorenz metrics which respects conformal classes. One can now obtain GJMS operators on $\mathscr{C}$ using the Fefferman–Graham construction.

Remark 4.1. It is immediate from formulas (4-2) and (4-3) that

$$g_\theta(T, T) = -\frac{1}{(n+1)(n+2)} R,$$

where R is the Webster scalar curvature, a pseudo-Hermitian invariant. On the other hand, Farris [1986] computed that, if θ is the contact form induced by an approximate solution of the complex Monge–Ampère equation, then

$$g_\theta(T, T) = 2r,$$

where r is the transverse curvature. It follows that the transverse curvature is, in this case, an intrinsic pseudo-Hermitian invariant.

To compute their pullbacks to M, we first note that the metric bundle $\mathcal{G}$ of $(\mathcal{C}, [g])$ is diffeomorphic to $(K^*)^{1/(n+2)}$ and $\widetilde{\mathcal{G}} \simeq (K^*)^{1/(n+2)} \times (-1, 1)$. We define spaces of functions

$$\begin{aligned}\mathcal{E}(w, w') &= \big\{ f \in \mathcal{C}^\infty\big((K^*)^{1/(n+2)}\big) : f(\lambda\xi) = \lambda^w \bar{\lambda}^{w'} f(\xi) \text{ for } \lambda \in \mathbb{C}^* \big\} \\ &= \big\{ f \in \mathcal{E}(w + w') : (e^{i\phi})^* f(\xi) = e^{i\phi(w-w')} f(\xi) \big\}.\end{aligned}$$

We will primarily be concerned with functions in

$$\mathcal{E}(w, w) = \{ f \in \mathcal{E}(2w) : (e^{i\phi})^* f(\xi) = f(\xi) \},$$

which descend to smooth functions on M.

For $k \in \mathbb{Z}$, we define

$$P_{w,w'} : \mathcal{E}(w, w') \to \mathcal{E}(w - k, w' - k), \quad f \mapsto 2^{-k} \mathcal{P}_k f,$$

where $\mathcal{P}_k$ is defined in (4-1). Then choosing $w = w' = k - (n+1)/2$, we get operators P_k defined on $\mathcal{E}(-N/2 + k)$ (recall that $N/2 = n + 1$) which are invariant under the circle action $(e^{i\phi})^*$ and hence may be viewed as smooth sections of a density bundle over M. These operators P_k are the CR-covariant differential operators which we will connect to poles of the scattering operator.

If X admits a globally defined approximate solution φ of the Monge–Ampère equation, then for each $p \in M = \partial X$ there is a neighborhood U of p and holomorphic coordinates $(z_1, \dots, z_m)$ near p so that φ is an approximate solution of the Monge–Ampère equation in U. Let

$$\theta = \frac{i}{2}(\bar{\partial} - \partial)\varphi \,\big|_M$$

be the induced pseudo-Hermitian structure on M, and let $\zeta = dz^1 \wedge \cdots \wedge dz^m|_M$. Then θ is volume-normalized with respect to ζ.

Let us denote by z_0 the induced fibre coordinate of $(K^*)^{1/(n+2)}$ and let

$$Q = |z_0|^2 \varphi.$$

Then Q is a globally defined smooth function on $\widetilde{\mathcal{G}}$ (which is diffeomorphic to $\mathbb{C} \times N$ for a collar neighborhood N of M in X) and the ambient metric on $\widetilde{\mathcal{G}}$ is the Kähler metric associated to the Kähler form

$$\omega = i \partial\bar{\partial} Q$$

where the corresponding metric g_θ on $\mathcal{C}$ is given by (4-3). The key computation linking the GJMS operators to the Laplacian is given in [Gover and Graham 2005, Proposition 5.4] and clearly generalizes to our situation. Thus we have:

Proposition 4.2. *If u is a smooth function on X then*

$$\widetilde{\Delta}(|z_0|^{2w} \varphi^w u) = (|z_0|^{2w} \varphi^w)(\Delta_\varphi + w(n + 1 + w))u,$$

where Δ_φ is the Laplacian associated to the Kähler form

$$\omega_\varphi = \frac{i}{2} \partial\bar{\partial} \log\Big(-\frac{1}{\varphi}\Big).$$

5. Proofs of the main theorems

Finally, we prove Theorems 1.1, 1.4 and 1.5. We are grateful to the referee for suggesting the proof of Theorem 1.5 in what follows, based on ideas of Graham and Fefferman [2002]; see especially the proofs of Theorems 3.1 and 4.1 there.

Proof of Theorem 1.1. The statement about the poles of $S_X(s)$ and $s=(m+k)/2$ is proved in Proposition 3.12. If g is a metric on X associated to the Kähler form $\omega = i\bar{\partial}\partial\log(-\frac{1}{\varphi})$ for a globally defined approximate solution of the Monge–Ampère equation, then the identification of the residues of $S_X(s)$ with the CR-covariant differential operators of Fefferman and Hirachi is a consequence of Proposition 4.2 and the second characterization of the GJMS operators given in Section 4A. □

Proof of Theorem 1.4. Owing to Proposition 3.13, it suffices to identify $\lim_{s\to m} p_{m,s}1$ with the CR Q-curvature. This is a consequence of Remark 3.14. □

Proof of Theorem 1.5. To begin with we note that if 1 denotes the constant function with value 1 on M, then the mapping $s \mapsto \mathcal{P}(s)1$ is a holomorphic mapping into $\mathscr{C}^\infty(X)$ and that, moreover,

$$\mathcal{P}(s)1 = x^{m-s}F(s) + x^s G(s), \tag{5-1}$$

where F and G are smooth functions on X with Taylor series to all orders at the boundary and depend holomorphically on s (this is not true for $\mathcal{P}(s)f$ for general f, but does hold true when $f=1$ since 1 lies in the kernel of the differential operators occurring in the logarithmic term). For $s \neq m$ we have

$$F(s)|_M = 1, \quad G(s)|_M = S_X(s)1,$$

and by holomorphy the same is true when $s=m$. By uniqueness we also have $\mathcal{P}(m)1 = 1$ so that

$$F(m) = 1 - x^m G(m). \tag{5-2}$$

Let $U = -\frac{d}{ds}\big|_{s=m}\mathcal{P}(s)1$. It is easy to see that

$$\Delta_\varphi U = m. \tag{5-3}$$

It follows from (5-1)–(5-2) that

$$U = \log x + Ax^m \log x + B,$$

where A and B are smooth functions having Taylor series to all orders at ∂X and

$$A|_{\partial X} = S_X(m)1.$$

By Proposition 3.13 and Remark 3.14 we have:

$$S_X(m)1 = c_m Q_\theta^{CR}.$$

On the other hand, we have from (5-3) that for x_0 sufficiently small,

$$\begin{aligned} m\,\mathrm{vol}(\varepsilon < x < x_0) &= \int_{\varepsilon<x<x_0} \Delta_g U\,\omega \\ &= -\varepsilon^{-m} \int_{x=\varepsilon} (1+r\varepsilon)\frac{\partial U}{\partial \nu}(\nu \rfloor \omega^m) + x_0^{-m} \int_{x=x_0} (1+rx_0)\frac{\partial U}{\partial \nu}(\nu \rfloor \omega^m) \\ &= -\varepsilon^{-m} \int_{x=\varepsilon} (1+r\varepsilon)\frac{\partial U}{\partial \nu}(\nu \rfloor \omega^m) + \mathbb{O}(1) \end{aligned}$$

where we have used Green's formula and (2-3). Recalling that

$$\frac{\partial}{\partial \nu} = -\frac{1}{\sqrt{1+rx}}\,x\frac{\partial}{\partial x}$$

(see (3-1)), it is clear that the coefficient of $\log \varepsilon$ in the expansion of $m\mathrm{vol}(\varepsilon < x < x_0)$ is

$$mL = m\int_M S_X(s)1|_{s=m}\,\psi$$

from which we conclude that

$$L = c_m \int_M Q_\theta^{CR}\,\psi. \qquad \square$$

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