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MICROLOCAL PROPAGATION NEAR RADIAL POINTS AND SCATTERING FOR SYMBOLIC POTENTIALS OF ORDER ZERO

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In this paper, the scattering and spectral theory of $H = \Delta_g + V$ is developed, where Δ_g is the Laplacian with respect to a scattering metric g on a compact manifold X with boundary and $V \in \mathscr{C}^\infty(X)$ is real; this extends our earlier results in the two-dimensional case. Included in this class of operators are perturbations of the Laplacian on Euclidean space by potentials homogeneous of degree zero near infinity. Much of the particular structure of geometric scattering theory can be traced to the occurrence of radial points for the underlying classical system. In this case the radial points correspond precisely to critical points of the restriction, V_0, of V to ∂X and under the additional assumption that V_0 is Morse a functional parameterization of the generalized eigenfunctions is obtained.

The main subtlety of the higher dimensional case arises from additional complexity of the radial points. A normal form near such points obtained by Guillemin and Schaeffer is extended and refined, allowing a microlocal description of the null space of $H-\sigma$ to be given for all but a finite set of "threshold" values of the energy; additional complications arise at the discrete set of "effectively resonant" energies. It is shown that each critical point at which the value of V_0 is less than σ is the source of solutions of $Hu = \sigma u$. The resulting description of the generalized eigenspaces is a rather precise, distributional, formulation of asymptotic completeness. We also derive the closely related L^2 and time-dependent forms of asymptotic completeness, including the absence of L^2 channels associated with the nonminimal critical points. This phenomenon, observed by Herbst and Skibsted, can be attributed to the fact that the eigenfunctions associated to the nonminimal critical points are "large" at infinity; in particular they are too large to lie in the range of the resolvent $R(\sigma \pm i0)$ applied to compactly supported functions.

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1. Introduction

In this paper, which is a continuation of [Hassell et al. 2004] (sometimes referred to as Part I) scattering theory is developed for symbolic potentials of order zero. The general setting is the same as in Part I, consisting of a compact manifold with boundary, X, equipped with a scattering metric, g, and a real potential, $V \in \mathscr{C}^\infty(X)$. Recall that such a scattering metric on X is a smooth metric in the interior of X taking the form

$$g = \frac{dx^2}{x^4} + \frac{h}{x^2} \tag{1–1}$$

near the boundary, where x is a boundary defining function and h is a smooth cotensor which restricts to a metric on $\{x = 0\} = \partial X$. This makes the interior, X°, of X a complete manifold which is asymptotically flat and is metrically asymptotic to the large end of a cone, since in terms of the singular normal coordinate $r = x^{-1}$, the leading part of the metric at the boundary takes the form $dr^2 + r^2 h(y, dy)$. In the compactification of X° to X, ∂X corresponds to the set of asymptotic directions of geodesics. In particular, this setting subsumes the case of the standard metric on Euclidean space, or a compactly supported perturbation of it, with a potential which is a classical symbol of order zero, hence not decaying at infinity but rather with leading term which is asymptotically homogeneous of degree zero. The study of the scattering theory for such potentials was initiated by Herbst [1991].

Let $V_0 \in \mathscr{C}^\infty(\partial X)$ be the restriction of V to ∂X, and denote by $\mathrm{Cv}(V)$ the set of critical values of V_0. It is shown in [Hassell et al. 2004] that the operator $H = \Delta_g + V$ (where the Laplacian is normalized to be positive) is essentially self-adjoint with continuous spectrum occupying $[\min V_0, \infty)$. There may be discrete spectrum of finite multiplicity in $(-\min_X V, \max V_0]$ with possible accumulation points only at $\mathrm{Cv}(V)$. To obtain finer results, it is natural to assume, as we do throughout this paper unless otherwise noted, that V_0 is a Morse function, that is, has only nondegenerate critical points; in particular $\mathrm{Cv}(V)$ is then a finite set; by definition this is the set of *threshold energies,* or *thresholds*.

From the microlocal point of view scattering theory is largely about the study of radial points, that is, the points in the cotangent bundle where the Hamilton vector field is a multiple of the radial vector field (that is, the vector field $A = \sum_i z_i \partial_{z_i}$ on Euclidean space, where $(z_1, \dots, z_n) \in \mathbb{R}^n$). These correspond in the classical dynamical system to the places where the particle is moving either in purely incoming or outgoing sense. In scattering theory for potentials decaying at infinity, there is a radial point for each point on the sphere at infinity; thus there is a manifold of radial points and the behaviour of the flow in a neighbourhood of these points is rather simple, either attracting (at the outgoing radial surface) or repelling (at the incoming radial surface) in the transverse direction. Estimates involving commutation with the radial vector field A multiplied by suitable powers of $|z|$ and perhaps additional microlocalizing operators, are usually sufficient to control the behaviour of generalized eigenfunctions. These are known as Mourre-type estimates and play a fundamental role in conventional scattering theory. In the present case, assuming V_0 is a Morse function, the radial points are isolated and occur in pairs, one pair (incoming/outgoing) for each critical point of V_0. The linearized Hamiltonian flow at the radial points is rather more complicated since it depends on the Hessian of V_0 at the critical point, which is arbitrary apart from being nondegenerate. This makes the higher dimensional case more intricate than the case $\dim X = 2$ which we treated in [Hassell et al. 2004]. Correspondingly one needs more elaborate commutator estimates in order to control the behaviour of generalized eigenfunctions. We give a rather general and complete analysis of the regularity of solutions of $Pu = 0$ in a microlocal neighbourhood of a radial point of P, using the concept of a test module of operators. This is a family of pseudodifferential operators which is a module over the zero-order operators, contains P, and is closed under commutation. By choosing a test module closely tailored to the Hamilton flow of P near the radial point we are able to produce enough positive-commutator estimates to parametrize the microlocal solutions of $Pu = 0$. The construction of appropriate test modules (which can be thought of as simply an effective bookkeeping device for keeping track of a rather intricate set of commutator estimates) to analyze general radial points is the main technical innovation of this paper.

The general study of radial points was initiated by Guillemin and Schaeffer [1977]. This was done in a slightly different context, where P is a standard pseudodifferential operator with homogeneous principal symbol and a radial point is one where the Hamilton vector field is a multiple of the vector field $\sum_i \xi_i \partial_{\xi_i}$ generating dilations in the cotangent space. This setting is completely equivalent to ours, via conjugation by a "local Fourier Transform" (see Section 3.1). They analyzed the situation in the nonresonant case. We refine their analysis by treating the resonant case, which is crucial in our application since we have a family of operators parametrized by the energy level, and the closure of the set of energies which give rise to resonant radial points may have nonempty interior. Moreover, we show that our parametrization of microlocal solutions is smooth except at a set of "effectively resonant" energies which is always discrete.

Bony, Fujiie, Ramond and Zerzeri [Bony et al. 2007] have studied the microlocal kernel of pseudodifferential operators at a hyperbolic fixed point, corresponding, in our setting, to a radial point associated to a local maximum of V_0. Their results partially overlap ours, being most closely related to [Hassell et al. 2004, Section 10] and [Hassell et al. 2001].

1.1. *Previous results.* The Euclidean setting described above was first studied by Herbst [1991], who showed that any finite energy solution of the time dependent Schrödinger equation, so $u = e^{-itH} f$ with $f \in L^2(\mathbb{R}^n)$, can concentrate, in an L^2 sense, asymptotically as $t \to \infty$ only in directions which are critical points of V_0. This was subsequently refined by Herbst and Skibsted [2008], who showed that such concentration can only occur near local minima of V_0. In contrast, solutions of the classical flow can concentrate near any critical point of V_0.

Asymptotic completeness has been studied by Agmon, Cruz and Herbst [1999], by Herbst and Skibsted [1999; 2008; 2004] and the present authors in [Hassell et al. 2004]. Agmon, Cruz and Herbst showed asymptotic completeness for sufficiently high energies, while Herbst and Skibsted extended this to all energies except for an explicitly given union of bounded intervals; in the two dimensional case, they showed asymptotic completeness for all energies. These results were obtained by time-dependent methods. On the other hand the principal result of [Hassell et al. 2004] involves a precise description of the generalized eigenspaces of H

$$E^{-\infty}(\sigma) = \{u \in \mathscr{C}^{-\infty}(X); (H - \sigma)u = 0\};$$

note that the space of "extendible distributions" $\mathscr{C}^{-\infty}(X)$ is the analogue of tempered distributions and reduces to it in case X is the radial compactification of $\mathbb{R}^n$. Thus we are studying all *tempered* eigenfunctions of H. Let us recall these results in more detail.

For any $\sigma \notin \mathrm{Cv}(V)$ the space $E_{\mathrm{pp}}(\sigma)$ of L^2 eigenfunctions is finite dimensional, and reduces to zero except for σ in a discrete (possibly empty) subset of $[\min_X V, \max V_0] \setminus \mathrm{Cv}(V)$. It is always the case that $E_{\mathrm{pp}}(\sigma) \subset \dot{\mathscr{C}}^\infty(X)$ consists of rapidly decreasing functions. Hence $E^{-\infty}_{\mathrm{ess}}(\sigma) \subset E^{-\infty}(\sigma)$, the orthocomplement of $E_{\mathrm{pp}}(\sigma)$, is well defined for $\sigma \notin \mathrm{Cv}(V)$. Furthermore, as shown in the Euclidean case by Herbst [1991], the resolvent, $R(\sigma)$ of H, acting on this orthocomplement, has a limit, $R(\sigma \pm i0)$, on $[\min V_0, \infty) \setminus \mathrm{Cv}(V)$ from above and below. The subspace of "smooth" eigenfunctions is then defined as

$$E^{\infty}_{\mathrm{ess}}(\sigma) = \mathrm{Sp}(\sigma)\left(\dot{C}^\infty(X) \ominus E_{\mathrm{pp}}(\sigma)\right) \subset E^{-\infty}(\sigma), \quad \mathrm{Sp}(\sigma) \equiv \frac{1}{2\pi i}\left(R(\sigma + i0) - R(\sigma - i0)\right). \tag{1–2}$$

In fact

$$E^{\infty}_{\mathrm{ess}}(\sigma) \subset \bigcap_{\epsilon > 0} x^{-1/2-\epsilon} L^2(X).$$

An alternative characterization of $E^{\infty}_{\mathrm{ess}}(\sigma)$ can be given in terms of the *scattering wavefront set* at the boundary of X.

The scattering cotangent bundle, ${}^{\mathrm{sc}}T^*X$, of X is naturally isomorphic to the cotangent bundle over the interior of X, and indeed globally isomorphic to T^*X by a nonnatural isomorphism; the natural identification exhibits both "compression" and "rescaling" at the boundary. If (x, y) are local coordinates near a boundary point of X, with x a boundary defining function, then linear coordinates (ν, μ) are defined on the scattering cotangent bundle by requiring that $q \in {}^{\mathrm{sc}}T^*X$ be written as

$$q = -\nu \frac{dx}{x^2} + \sum_i \mu_i \frac{dy_i}{x}, \qquad \nu \in \mathbb{R}, \qquad \mu \in \mathbb{R}^{n-1}. \tag{1–3}$$

This makes (ν, μ) dual to the basis $(-x^2\partial_x, x\partial_{y_i})$ of vector fields which form an approximately unit length basis, uniformly up to the boundary, for any scattering metric. In Euclidean space, ν is dual to ∂_r and μ_i is dual to the constant-length angular derivative $r^{-1}\partial_{y_i}$. In the analysis of the microlocal aspects of $H - \sigma$, in part for compatibility with [Guillemin and Schaeffer 1977], it is convenient pass to an operator "of first order" by multiplying $H - \sigma$ by x^{-1}, that is, to replace it by

$$P = P(\sigma) = x^{-1}(H - \sigma).$$

The classical dynamical system giving the behaviour of particles, asymptotically near ∂X, moving under the influence of the potential corresponds to "the bicharacteristic vector field," see (2–3), determined by the *boundary symbol, p*, of P. This vector field is defined on ${}^{\mathrm{sc}}T^*_{\partial X}X$, which is to say on ${}^{\mathrm{sc}}T^*X$ at, and tangent to, the boundary ${}^{\mathrm{sc}}T^*_{\partial X}X = {}^{\mathrm{sc}}T^*X \cap \{x = 0\}$. It has the property that ν is nondecreasing under the flow; we refer to points (y, ν, μ) where $\mu = 0$ as *incoming* if $\nu < 0$ and *outgoing* if $\nu > 0$. What is important in understanding the behaviour of the null space of P, that is, tempered distributions, u, satisfying $Pu = 0$, is bicharacteristic flow inside $\{p = 0,\ x = 0\}$, a submanifold to which it is tangent. The only critical points of the flow are at points $(y, \nu, 0)$ where y is a critical point of P and $\nu = \pm\sqrt{\sigma - V(y)}$. Thus, the only possible asymptotic escape directions of classical particles under the influence of the potential V are the finite number of critical points of V_0. Moreover, only the local minima are stable; the others have unstable directions according to the number of unstable directions as a critical point of $V_0 : \partial X \longrightarrow \mathbb{R}$.

The classical dynamics of p and the quantum dynamics of P are linked via the scattering wavefront set. Let $u \in C^{-\infty}(X)$ be a tempered distribution on X (that is, in the dual space of $\dot{C}^\infty(X;\Omega)$). The part of the scattering wavefront set, $\mathrm{WF}_{\mathrm{sc}}(u)$, of u lying over the boundary $\{x = 0\}$, which is all that is of interest here, is a closed subset of ${}^{\mathrm{sc}}T^*_{\partial X}X$ which measures the linear oscillations (Fourier modes, in the case of Euclidean space) present in u asymptotically near boundary points; see [Melrose 1994] for the precise definition. We shall also need to use the scattering wavefront set $\mathrm{WF}^s_{\mathrm{sc}}(u)$ with respect to the space $x^sL^2(X)$ which measures the microlocal regions where u fails to be in $x^sL^2(X)$. There is a propagation theorem for the scattering wavefront set in the style of the theorem of Hörmander in the standard setting; if $Pu \in \dot{C}^\infty(X)$, then the scattering wavefront set of u is contained in $\{p = 0\}$ and is invariant under the bicharacteristic flow of P; see [Melrose 1994]. In particular, generalized eigenfunctions of u have scattering wavefront set invariant under the bicharacteristic flow of P. Note that the elliptic part of this statement is already a uniform version of the smoothness of solutions.

In view of this propagation theorem, it is possible to consider where generalized eigenfunctions "originate", although the direction of propagation is fixed by convention. Let us say that a generalized eigenfunction *originates* at a radial point q, if $q \in \mathrm{WF}_{\mathrm{sc}}(u)$ and if $\mathrm{WF}_{\mathrm{sc}}(u)$ is contained in the forward flowout $\Phi_+(q)$ of q; thus each point in $\mathrm{WF}_{\mathrm{sc}}(u)$ can be reached from q by travelling along curves that are everywhere tangent to the flow and with ν nondecreasing along the curve, so allowing the possibility of passing through radial points, where the flow vanishes, on the way. In Part I of this paper we showed, in the two-dimensional case and provided the eigenvalue σ is a nonthreshold value:

- Every L^2 eigenfunction is in $\dot{C}^\infty(X)$.
- Every nontrivial generalized eigenfunction pairing to zero with the L^2 eigenspace fails to be in $x^{-1/2}L^2(X)$.
- There are generalized eigenfunctions originating at each of the incoming radial points in $\{p = 0\}$, that is, at each critical point of V_0 with value less than σ.
- There are fundamental differences between the behaviour of eigenfunctions near a local minimum and at other critical points. The radial point corresponding to a local minimum is always an isolated point of the scattering wavefront set for some nontrivial eigenfunction. For other critical points, the scattering wavefront set necessarily propagates and in generic situations each nontrivial generalized eigenfunction is singular at some minimal radial point.
- A generalized eigenfunction, u, with an isolated point in its scattering wavefront set, necessarily a radial point corresponding to a local minimum of V_0, has a complete asymptotic expansion there. The expansion is determined by its leading term, which is a Schwartz function of $n-1$ variables. The resulting map extends by continuity to an injective map from $E^\infty_{\mathrm{ess}}(\sigma)$ into $\bigoplus_q L^2(\mathbb{R}^{n-1})$, where the direct sum is over local minima of V_0 with value less than the energy σ.
- The space $E^0_{\mathrm{ess}}(\sigma)$, consisting of those generalized eigenfunctions which are in $x^{-1/2}L^2$ microlocally near $\{\nu = 0\}$, is a Hilbert space and the map above extends to a unitary isomorphism, $M_+(\sigma)$, from $E^0_{\mathrm{ess}}(\sigma)$ to $\bigoplus_q L^2(\mathbb{R}^{n-1})$. A similar map $M_-(\sigma)$ can be defined by reversal of sign or complex conjugation and the scattering matrix for $P = P(\sigma)$ at energy σ may be written
$$S(\sigma) = M_+(\sigma)M_-^{-1}(\sigma).$$

In this paper we extend these results to higher dimensions.

1.2. ***Results and structure of the paper.*** We treat this problem by microlocal methods. Thus, the "classical" system, consisting of the bicharacteristic vector field, plays a dominant role. The main step involves reducing this vector field to an appropriate normal form in a neighbourhood of each of its zeroes, which are just the radial points. Nondegeneracy of the critical points of V_0 implies nondegeneracy of the linearization of the bicharacteristic vector field at the corresponding radial points. If there are no resonances, Sternberg's Linearization Theorem, following an argument of Guillemin and Schaeffer, allows the bicharacteristic vector field to be reduced to its linearization by a contact transformation of ${}^{\mathrm{sc}}T^*_{\partial X}X$. At the quantum level this means that conjugation by a (scattering) Fourier integral operator, associated to this contact transformation, microlocally replaces P by an operator with principal symbol in normal form. For this normal form we construct "test modules" of pseudodifferential operators and analyze the commutators with the transformed operator. Modulo lower order terms, the operator itself becomes a quadratic combination of elements of the test module. Just as in Part I, we use the resulting system of regularity constraints to determine the microlocal structure of the eigenfunctions and ultimately show the existence of asymptotic expansions for eigenfunctions with some additional regularity.

However, the problem of resonances cannot be avoided. Even for a fixed operator and fixed critical point, the closure of the set of values of σ for which resonances occur may have nonempty interior. Such resonances prevent the reduction of the bicharacteristic vector field to its linearization, and hence of the symbol of P to an associated model, although partial reductions are still possible. In general it is necessary to allow many more terms in the model. Fortunately most of these terms are not relevant to the construction of the test modules and to the derivation of the asymptotic expansions. We distinguish between "effectively nonresonant" energies, where the additional resonant terms are such that the definition of the test modules, now only to finite order, proceeds much as before and the "effectively resonant" energies, where this is not the case. Ultimately, we analyze the regularity of solutions at all (nonthreshold) energies. Near effectively nonresonant energies, smoothness of families of eigenfunctions may still be readily shown. Effectively resonant energies are harder to analyze, but the set of these is shown to be *discrete*. In any case, the space of microlocal eigenfunctions is parameterized at all nonthreshold energies. At effectively resonant energies the problems arising from the failure of the direct analogue of Sternberg's linearization are overcome by showing that, to an appropriate finite order, the operator may be reduced to a nonquadratic function of the test module.

In outline, the discussion proceeds as follows. In Sections 2–4 we study radial points. This is a general microlocal study except that we work under the assumption that the symplectic map associated to the linearization of the flow at each radial point (see Lemma 2.5) has no 4-dimensional irreducible invariant subspaces; this assumption is always fulfilled in the case of our operator $\Delta + V - \sigma$. The main result is Theorem 3.11 in which the operator is microlocally conjugated to a linear vector field plus certain "error terms". In the nonresonant case the error terms can be made to vanish identically, while in the effectively nonresonant case the error terms have a good property with respect to a test module of pseudodifferential operators, namely they can be expressed as a positive power x^ϵ, $\epsilon > 0$, times a power of the module. In the effectively resonant case this is no longer possible and we must allow "genuinely" resonant terms, but the set of effectively resonant energies is discrete in the parameter σ in all dimensions.

We then turn in Sections 5–7 to studying microlocal eigenfunctions which are microlocally outgoing at a given radial point q. The main result here is Theorem 6.7 (or Theorem 7.3 in the effectively resonant case) which gives a parameterization of such microlocal eigenfunctions. For a minimal radial point, they are parameterized by $\mathcal{S}(\mathbb{R}^{n-1})$, Schwartz functions of $n-1$ variables, for a maximal radial point they are parameterized by formal power series in $n-1$ variables, and in the intermediate case of a saddle point with k positive directions, they are parameterized by formal power series in $n-1-k$ variables with values in $\mathcal{S}(\mathbb{R}^k)$. In all cases, the parameterizing data appear explicitly in the asymptotic expansion of the eigenfunction at the critical point.

We next investigate in Sections 8 and 9 the manner in which the various radial points interact, and prove, in Theorem 9.2, a "microlocal Morse decomposition." This shows that for each nonthreshold energy σ there are genuine eigenfunctions (as opposed to microlocal eigenfunctions) in $E^{\infty}_{\text{ess}}(\sigma)$ associated to each energy-permissible critical point.

Then we turn in Sections 10 and 11 to the spectral decomposition of P and prove several versions of asymptotic completeness. First this is established at a fixed, nonthreshold energy; see Theorem 10.1 which shows that the natural map from $E^{0}_{\text{ess}}(\sigma)$ to the leading term in its asymptotic expansion (that is, to its parameterizing data) is unitary. Next we prove a form valid uniformly over an interval of the spectrum, Theorem 10.10. In Section 11 a time-dependent formulation is derived, as Theorem 11.4. This is based on the behaviour at large times of solutions of the time-dependent Schrödinger equation $D_t u = Pu$ and is subsequently used to derive a result of Herbst and Skibsted's on the absence of L^2-channels corresponding to nonminimal critical points (Corollary 11.7).

1.3. *Results used from [Hassell et al. 2004].* Throughout this paper we state the specific location of results used from [Hassell et al. 2004] (Part I). For the convenience of the reader we summarize here the relevant locations. Sections 1–3 of Part I are used as the basic background (and Section 3 of Part I relies on Section 4 there). The present Section 4 is the analogue of Section 5 of Part I, although we restate many of the arguments due to the slightly different (more general) setting. The basic analytic technique using test modules in Section 5 comes from Section 6 of Part I. Certain results and methods from Sections 11 and 12 of Part I are used here in Sections 9 and 10. However, the results of the intermediate Sections 7–10 of Part I, while certainly of interest when comparing to the results of Sections 6 and 7 here, are never used in the present work directly or indirectly.

In addition, there was an error in the proof of Proposition 6.7 of Part I. While this error is minor and is easily remedied, we present the modified proof, together with some of the context, here in the Appendix since this proposition lies at the heart of the analysis in both papers.

1.4. *Notation.* The items listed below without a reference whose definition is not immediate from the stated brief description are defined in [Melrose 1994].

Notation	*Description or definition*	*Reference*
V_0	restriction of V to ∂X	
$\mathrm{Cv}(V)$	set of critical values of V_0	
${}^{\mathrm{sc}}T^*X$	scattering cotangent bundle over X	(1–3)
${}^{\mathrm{sc}}T^*_{\partial X}X$	restriction of ${}^{\mathrm{sc}}T^*X$ to ∂X	(1–3)
x	boundary defining function of X such that (1–1) holds	
y	coordinates on ∂X	
(ν, μ)	fibre coordinates on ${}^{\mathrm{sc}}T^*X$	(1–3)

Notation	*Description or definition*	*Reference*
$y = (y', y'', y''')$	decomposition of y variable	Lemma 2.7
$\mu = (\mu', \mu'', \mu''')$	dual decomposition of μ variable	Lemma 2.7
r'_i, r''_j, r'''_k	eigenvalues of the contact map A	Lemma 2.7
Y''_j	$y''_j / x^{r''_j}$	(5–18)
Y'''_k	$y'''_k / x^{1/2}$	(5–18)
Δ	(positive) Laplacian with respect to g	
P	$x^{-1}(\Delta + V - \sigma)$	Section 2
H	$\Delta + V$	
$R(\sigma)$	resolvent of H, $(H-\sigma)^{-1}$	
$R(\sigma \pm i0)$	limit of resolvent on real axis from above/below	
$\widetilde{V}$	modified potential	Lemma 8.5
$\mathrm{Sp}(\sigma)$	(generalized) spectral projection of H at energy σ	(1–2)
$\widetilde{R}(\sigma)$	resolvent of modified potential $(\Delta + \widetilde{V} - \sigma)^{-1}$	
$L^2_{\mathrm{sc}}(X)$	L^2 space with respect to Riemannian density of g	
$H^{m,0}_{\mathrm{sc}}(X)$	Sobolev space; image of $L^2_{\mathrm{sc}}(X)$ under $(1+\Delta)^{-m/2}$	
$H^{m,l}_{\mathrm{sc}}(X)$	$x^l H^{m,0}_{\mathrm{sc}}(X)$	
$\Psi^{m,0}_{\mathrm{sc}}(X)$	scattering pseudodiff. ops. of differential order m	
$\Psi^{m,l}_{\mathrm{sc}}(X)$	$x^l \Psi^{m,0}_{\mathrm{sc}}(X)$; maps $H^{m',l'}_{\mathrm{sc}}(X)$ to $H^{m'-m,l'+l}_{\mathrm{sc}}(X)$	
$\sigma_{\partial,l}(A)$	boundary symbol of $A \in \Psi^{m,l}_{\mathrm{sc}}(X)$; $\mathscr{C}^\infty$ fn. on ${}^{\mathrm{sc}}T^*_{\partial X}X$	
$\sigma_\partial(A)$	$\sigma_{\partial,0}(A)$	
$\mathrm{WF}_{\mathrm{sc}}(u)$	scattering wavefront set of u; closed subset of ${}^{\mathrm{sc}}T^*_{\partial X}X$	
$\mathrm{WF}^{m,l}_{\mathrm{sc}}(u)$	scattering wavefront set with respect to $H^{m,l}_{\mathrm{sc}}$	
$\mathrm{WF}'_{\mathrm{sc}}(A)$	operator scattering wave front set; in its complement A is microlocally in $\Psi^{*,\infty}_{\mathrm{sc}}(X)$, in other words, is trivial	
${}^{\mathrm{sc}}H_p$	scattering Hamilton vector field	Section 2
$\Phi_+(q)$	forward flowout from $q \in {}^{\mathrm{sc}}T^*_{\partial X}X$	Section 1.1
radial point	point in ${}^{\mathrm{sc}}T^*_{\partial X}X$ where p and ${}^{\mathrm{sc}}H_p$ vanish	Section 2
$\mathrm{RP}_\pm(\sigma)$	set of radial points of $H-\sigma$ where $\pm\nu > 0$	
$\mathrm{Min}_+(\sigma)$	subset of $\mathrm{RP}_+(\sigma)$ associated to local minima of V_0	
$\leq$	partial order on $\mathrm{RP}_+(\sigma)$ compatible with Φ_+	Definition 8.3
$\widetilde{E}_{\mathrm{mic},+}(O,P)$	microlocal solutions of $Pu = 0$ in the set O	(4–1)
$E_{\mathrm{mic},+}(q,\sigma)$	microlocal solutions of $(H-\sigma)u = 0$ near q	(4–4)
$E^s_{\mathrm{ess}}(\sigma)$	space of generalized σ-eigenfunctions of H	(9–1)
$E^s(\Gamma,\sigma)$	subset of $u \in E^s_{\mathrm{ess}}(\sigma)$ with $\mathrm{WF}_{\mathrm{sc}}(u) \cap \mathrm{RP}_+(\sigma) \subset \Gamma$	(9–4)
$E^s_{\mathrm{Min},+}(\sigma)$	$E^s(\Gamma,\sigma)$, with $\Gamma = \mathrm{Min}_+(\sigma)$	
$\mathcal{M}$	test module	Section 5
$I^{(s)}_{\mathrm{sc}}(O,\mathcal{M})$	space of iteratively-regular functions with respect to $\mathcal{M}$	(5–6)
τ	rescaled time variable; $\tau = xt$	Section 11
X_{Sch}	$X \times \overline{\mathbb{R}}_\tau$	(11–2)

2. Radial points

Let X be a compact n-dimensional manifold with smooth boundary. Recall that if (x, y) are local coordinates on X, with x a boundary defining function, then dual scattering coordinates (ν, μ) on the scattering cotangent bundle are determined. The restriction of the scattering cotangent bundle to ∂X is denoted ${}^{\mathrm{sc}}T^*_{\partial X}X$ and has a natural contact structure, the contact form at the boundary being

$$\alpha = -d\nu + \sum_i \mu_i dy_i \tag{2–1}$$

in local coordinates. Recall that a contact structure on a $2n-1$-dimensional manifold, here ${}^{\mathrm{sc}}T^*_{\partial X}X$, is given by a nondegenerate one-form, that is, a one-form α with $\alpha \wedge (d\alpha)^{n-1}$ everywhere nonzero; correspondingly its kernel is a maximally nonintegrable hyperplane field on ${}^{\mathrm{sc}}T^*_{\partial X}X$. One refers to either the line bundle given by the span of α, or the hyperplane field given by its kernel, as the contact structure.

Suppose that $P \in \Psi^{*,-1}_{\mathrm{sc}}(X)$ is a scattering pseudodifferential operator of order -1 at the boundary; for example, $P = x^{-1}(\Delta + V - \sigma)$. Then the boundary part of its principal symbol, $p = \sigma_\partial(xP)$, is a $\mathscr{C}^\infty$ function on ${}^{\mathrm{sc}}T^*_{\partial X}X$. In this, and the next, section we consider radial points of a general real-valued function, $p \in \mathscr{C}^\infty({}^{\mathrm{sc}}T^*_{\partial X}X)$, with only occasional references to the particular case, $p = |\zeta|^2 + V_0 - \sigma$, of direct interest in this paper. Although we discuss radial points in the context of boundary points in the scattering calculus this analysis applies directly (and could alternatively be done for) radial points in the usual microlocal picture, as described in the Introduction. Our objective in this section is to find a change of coordinates, preserving the contact structure, in which the form of p is simplified. In this section we consider the simplification of p up to second order, in a sense made precise below.

The basic nondegeneracy assumption we make is that

$$p = 0 \text{ implies } dp \neq 0; \tag{2–2}$$

this excludes true “thresholds” which however do occur for our problem, when σ is a critical value of V_0. It follows directly from (2–2) that the boundary part of the characteristic variety

$$\Sigma = \{q \in {}^{\mathrm{sc}}T^*_{\partial X}X;\ p(q) = 0\} \text{ is smooth;}$$

we shall assume that Σ is compact, corresponding to the ellipticity of P.

Definition 2.1. A *radial point* for a function p satisfying (2–2) is a point $q \in \Sigma$ such that $dp(q)$ is a (necessarily nonzero) multiple of the contact form α given by (2–1). Conversely, if $q \in \Sigma$ and dp and α are linearly independent at q then we say that p is of *principal type* at q.

We may extend p to a $\mathscr{C}^\infty$ function on ${}^{\mathrm{sc}}T^*X$, still denoted by p. Over the interior ${}^{\mathrm{sc}}T^*_{X^\circ}X$ is naturally identified with T^*X°, which is a symplectic manifold with canonical symplectic form ω. Near the boundary, expressed in terms of scattering-dual coordinates,

$$\omega = d\Big(-\nu\frac{dx}{x^2} + \sum_i \mu_i \frac{dy_i}{x}\Big) = (-d\nu + \sum_i \mu_i\, dy_i) \wedge \frac{dx}{x^2} + \sum_i d\mu_i \wedge \frac{dy_i}{x}.$$

Consider the Hamilton vector field, $H_{x^{-1}p}$, of $x^{-1}p$, which we shall denote ${}^{\mathrm{sc}}H_p$, fixed by the identity $\omega(\cdot\,, {}^{\mathrm{sc}}H_p) = dp$. Then ${}^{\mathrm{sc}}H_p$ extends to a vector field on ${}^{\mathrm{sc}}T^*X$ tangent to its boundary, so ${}^{\mathrm{sc}}H_p$ lies in

$\mathcal{V}_{\mathrm{b}}({}^{\mathrm{sc}}T^*X)$.[1] At the boundary ${}^{\mathrm{sc}}H_p$, as an element of $\mathcal{V}_{\mathrm{b}}({}^{\mathrm{sc}}T^*X)$, is independent of the extension of p. We denote the restriction of ${}^{\mathrm{sc}}H_p$ (as a vector field) to ${}^{\mathrm{sc}}T^*_{\partial X}X$ by W, so W is a vector field on ${}^{\mathrm{sc}}T^*_{\partial X}X$. Explicitly in local coordinates

$$\begin{aligned}{}^{\mathrm{sc}}H_p = &-(\partial_\nu p)(x\partial_x + \mu\cdot\partial_\mu) + (x\partial_x p - p + \mu\cdot\partial_\mu p)\partial_\nu \\ &+ \sum_j \left(\partial_{\mu_j} p\, \partial_{y_j} - \partial_{y_j} p\, \partial_{\mu_j}\right) + x\mathcal{V}_{\mathrm{b}}({}^{\mathrm{sc}}T^*X);\end{aligned} \tag{2–3}$$

since p is smooth up to the boundary, $x\partial_x p = 0$ at ${}^{\mathrm{sc}}T^*_{\partial X}X$. Thus,

$$W = -(\partial_\nu p)\mu\cdot\partial_\mu + (\mu\cdot\partial_\mu p - p)\partial_\nu + \sum_j \left(\partial_{\mu_j} p\, \partial_{y_j} - \partial_{y_j} p\, \partial_{\mu_j}\right). \tag{2–4}$$

Alternatively W may be described in terms of the contact structure on ${}^{\mathrm{sc}}T^*_{\partial X}X$. Namely W is the Legendre vector field of p, determined by

$$d\alpha(., W) + \gamma\alpha = dp,\ \ \alpha(W) = p \tag{2–5}$$

for some function γ. It follows that W is tangent to Σ, since $dp(W) = \gamma\alpha(W) = \gamma p = 0$ at any point at which p vanishes. An equivalent definition of $q \in \Sigma$ being a radial point is that the vector field W vanishes as q, as follows from (2–5) and the nondegeneracy of α.

Definition 2.2. A radial point $q \in \Sigma$ for a real-valued function $p \in \mathcal{C}^\infty({}^{\mathrm{sc}}T^*_{\partial X}X)$ satisfying (2–2) is said to be *nondegenerate* if the vector field W, restricted to $\Sigma = \{p = 0\}$, has a nondegenerate zero at q. Note that this implies that a nondegenerate radial point is necessarily isolated in the set of radial points.

Since the vector field W vanishes at a radial point q, its linearization is well defined as a linear map, A' on $T_q\,{}^{\mathrm{sc}}T^*_{\partial X}X$, (later we will use the transpose, A, as a map on differentials)

$$A'v = [V, W](q),$$

for any smooth vector field V with $V(q) = v$; it is independent of the choice of extension and can also be written in terms of the Lie derivative

$$A'v = -\mathcal{L}_W V(q). \tag{2–6}$$

Since $Wp = \gamma p$, A' preserves the subspace $T_q\Sigma$. Since α is normal to $T_q\Sigma$, the restriction of $d\alpha$ to $T_q\Sigma$ is a symplectic 2-form, ω_q.

Lemma 2.3. *At a nondegenerate radial point for p, where $dp = \lambda\alpha$, the linearization A' acting on $T_q\Sigma$ is such that*

$$S \equiv A' - \frac{1}{2}\lambda\,\mathrm{Id} \in \mathfrak{sp}(2(n-1))$$

is in the Lie algebra of the symplectic group with respect to ω_q:

$$\omega_q(Sv_1, v_2) + \omega_q(v_1, Sv_2) = 0,\ \ \forall\, v_1, v_2 \in T_q\Sigma.$$

[1]Here $\mathcal{V}_{\mathrm{b}}(M)$ denotes the space of smooth vector fields on the manifold with boundary M that are tangent to ∂M.

Proof. Observe that (2–5) implies that

$$L_W\alpha = (d\alpha)(W,\cdot\,) + d(\alpha(W)) = \gamma\alpha.$$

For two vector smooth vector fields V_i, defined near q,

$$\begin{aligned} W(d\alpha(V_1, V_2)) &= L_W(d\alpha(V_1, V_2)) \\ &= (L_W d\alpha)(V_1, V_2) + d\alpha(L_W V_1, V_2) + d\alpha(V_1, L_W V_2). \end{aligned}$$

The left side vanishes at q so using (2–6)

$$\omega_q(A'v_1, v_2) + \omega_q(v_1, A'v_2) = \lambda\omega_q(v_1, v_2)\ \forall\ v_1, v_2 \in T_q\Sigma. \qquad \square$$

It follows from Lemma 2.3 (see for example [Guillemin and Schaeffer 1977]) that A' is decomposable into invariant subspaces of dimension 2 and 4, with eigenvalues on the two-dimensional subspaces of the form λr, $\lambda(1-r)$, $r \le 1/2$ real or $\lambda(1/2 + ia)$, $\lambda(1/2 - ia)$, with $a > 0$.

Note that, by (2–5), $d_\nu p(q) = -\gamma(q) = -\lambda$, so from (2–3), the Hamilton vector field ${}^{\mathrm{sc}}H_p$ is equal to $\lambda x\partial_x$ modulo vector fields of the form $f \cdot W'$ where W is tangent to $\{x = 0\}$ and $f(q) = 0$. Therefore if $\lambda > 0$, then x is increasing along bicharacteristics of p in the interior of ${}^{\mathrm{sc}}T^*X$, that is, the bicharacteristics leave the boundary, that is, "come in from infinity" if ∂X is removed, while if $\lambda < 0$, the bicharacteristics approach the boundary, that is, "go out to infinity". Correspondingly we make the following definition.

Definition 2.4. We say that a nondegenerate radial point q for p with $dp(q) = \lambda\alpha(q)$ is outgoing if $\lambda < 0$, and we say that it is incoming if $\lambda > 0$.

For $p = |\zeta|^2 + V_0 - \sigma$, we have $\lambda = -\partial_\nu p = -2\nu$. Hence, radial points are outgoing for $\nu > 0$ and incoming for $\nu < 0$ in this case. We next discuss the form the linearization takes for $p = |\zeta|^2 + V_0 - \sigma$.

Lemma 2.5. *For the function $p = |\zeta|^2 + V_0 - \sigma$ with V_0 Morse, the radial points are all nondegenerate and the linear operator S associated with each has only two-dimensional invariant symplectic subspaces.*

Remark 2.6. In view of the nonoccurrence of nondecomposable invariant subspaces of dimension 4 in this case we will exclude them from further discussion below.

Proof. Choose Riemannian normal coordinates y_j on ∂X, so the metric function h satisfies $h - |\mu|^2 = \mathcal{O}(|y|^2)$. Since the Hessian of $V|_{\partial X}$ at a critical point is a symmetric matrix, it can be diagonalized by a linear change of coordinates on ∂X, given by a matrix in $\mathrm{SO}(n-1)$, which thus preserves the form of the metric. It follows that for each j, $(dy_j, d\mu_j)$ is an invariant subspace of A. $\square$

Let $\mathcal{I}$ denote the ideal of $\mathscr{C}^\infty$ functions on ${}^{\mathrm{sc}}T^*_{\partial X}X$ vanishing at a given radial point, q. The linearization of W then acts on $T^*_q\left({}^{\mathrm{sc}}T^*_{\partial X}X\right) = \mathcal{I}/\mathcal{I}^2$; $dp(q)$, or equivalently α_q, is necessarily an eigenvector of A with eigenvalue 0. Similarly, ${}^{\mathrm{sc}}H_p$ defines a linear map $\widetilde{A}$ on $T^*_q({}^{\mathrm{sc}}T^*X)$. By (2–3), $\widetilde{A}$ preserves the conormal line, $\operatorname{span} dx$ and the eigenvalue of $\widetilde{A}$ corresponding to the eigenvector dx is λ. Thus $\widetilde{A}$ acts on the quotient

$$T^*_q\left({}^{\mathrm{sc}}T^*_{\partial X}X\right) \equiv T^*_q\left({}^{\mathrm{sc}}T^*X\right) / \operatorname{span} dx,$$

and this action clearly reduces to A.

By Darboux's theorem we may make a local contact diffeomorphism of ${}^{\mathrm{sc}}T^*_{\partial X}X$ and arrange that $q = (0, 0, 0)$. Thus, as a module over $\mathscr{C}^\infty({}^{\mathrm{sc}}T^*_{\partial X}X)$ in terms of multiplication of functions, $\mathcal{I}$ is generated by ν, y_j and the μ_j, for $j = 1, \ldots, n-1$. Thus in general we have the following possibilities for the two-dimensional invariant subspaces of A.

(i) There are two independent real eigenvectors with eigenvalues in $\lambda(\mathbb{R} \setminus [0, 1])$.

(ii) There are two independent real eigenvectors with eigenvalues in $\lambda(0, 1)$.

(iii) There are no real eigenvectors and two complex eigenvectors with eigenvalues in $\lambda(\frac{1}{2} + i(\mathbb{R} \setminus \{0\}))$.

(iv) There is only one nonzero real eigenvector with eigenvalue $\frac{1}{2}\lambda$.

Case (iv) was called the "Hessian threshold" case in Part I. In all cases the sum of the two (generalized) eigenvalues is λ.

Lemma 2.7. *By making a change of contact coordinates, that is, a change of coordinates on* ${}^{\mathrm{sc}}T^*_{\partial X}X$ *preserving the contact structure, near a radial point* q *for* $p \in \mathscr{C}^\infty({}^{\mathrm{sc}}T^*_{\partial X}X)$ *for which the linearization has neither a Hessian threshold subspace, (iv), nor any nondecomposable* 4*-dimensional invariant subspace, coordinates* y *and* μ*, decomposed as* $y = (y', y'', y''')$ *and* $\mu = (\mu', \mu'', \mu''')$*, may be introduced so that*

(i)
$$(y', \mu') = (y_1, \ldots, y_{s-1}, \mu_1, \ldots, \mu_{s-1}),$$
where $e'_j = dy'_j$, $f'_j = d\mu'_j$ *are eigenvectors of* A *with eigenvalues* $\lambda r'_j$, $\lambda(1 - r'_j)$, $j = 1, \ldots, s-1$ *with* $r'_j < 0$ *real and negative.*

(ii) $(y'', \mu'') = (y_s, \ldots, y_{m-1}, \mu_s, \ldots, \mu_{m-1})$ *where* $e''_j = dy''_j$, $f''_j = d\mu''_j$ *are eigenvectors with eigenvalues* $\lambda r''_j$, $\lambda(1 - r''_j)$, $j = s, \ldots, m-1$ *where* $0 < r''_j \le 1/2$ *is real and positive.*

(iii) $(y''', \mu''') = (y_m, \ldots, y_{n-1}, \mu_m, \ldots, \mu_{n-1})$, *where some complex combination* e'''_j, f'''_j, *of* dy'''_j *and* $d\mu'''_j$, $m \le j \le n-1$, *are eigenvectors with eigenvalues* $\lambda r'''_j$ *and* $\lambda(1 - r'''_j)$ *with* $r'''_j = 1/2 + i\beta'''_j$, $\beta'''_j > 0$.

Thus if we set $e = (e', e'', e''')$, $f = (f', f'', f''')$ the eigenvectors of A are $d\nu, e_j$ and f_j, with respective eigenvalues $0, \lambda r_j$ and $\lambda(1 - r_j)$; we will take the coordinates so that the r_j are ordered by their real parts.

Remark 2.8. We emphasize that the change of coordinates here is on the contact space, ${}^{\mathrm{sc}}T^*_{\partial X}X$, and it is, in general, not induced by a change of coordinates on X. Analytically it is implemented by a scattering FIO (see Section 3.1).

In coordinates in which the eigenspaces take this form it can be seen directly that

$$p = \lambda\Big(-\nu + \sum_{j=1}^{m-1} r_j y_j \mu_j + \sum_{j=m}^{n-1} Q_j(y_j, \mu_j) + \nu g_1 + g_2\Big) \tag{2–7}$$

with the Q_j elliptic homogeneous polynomials of degree 2, g_1 vanishing at least linearly and g_2 to third order.

Remark 2.9. For the function $p = |\zeta|^2 + V_0 - \sigma$ with V_0 Morse, the eigenvalues of A at a radial point q are easily calculated in the coordinates used in the proof of Lemma 2.5. Indeed, since the 2-dimensional invariant subspaces decouple, the results of [Hassell et al. 2004, Proof of Proposition 1.2] can be used.

The eigenvalues corresponding to the 2-dimensional subspace in which the eigenvalue of the Hessian is $2a_j$ are thus

$$\lambda\left(\frac{1}{2}\pm\sqrt{\frac{1}{4}-\frac{a_j}{\sigma-V_0(0)}}\right),\quad \text{where } \lambda=-2\nu(q).$$

In fact, below we do not need the full power of Lemma 2.7. Essentially it suffices if we arrange that the eigenvectors corresponding to the (in absolute value) larger eigenvalues, namely $\lambda(1-r'_j)$, if $r'_j<0$, or $\lambda(1-r''_j)$, if $r''_j\in(0,\frac{1}{2})$, are in a model form on the two dimensional eigenspaces. The advantage of the weaker conclusion is that one has more freedom in choosing the contact change of coordinates.

Lemma 2.10 (Weaker version of Lemma 2.7). *Suppose that $\frac{1}{2}\lambda$ is not an eigenvalue of A. By making a change of contact coordinates, that is, a change of coordinates on ${}^{\mathrm{sc}}T^*_{\partial X}X$ preserving the contact structure, near a radial point q for $p\in\mathscr{C}^\infty({}^{\mathrm{sc}}T^*_{\partial X}X)$ for which the linearization has neither a Hessian threshold subspace, (iv), nor any nondecomposable 4-dimensional invariant subspace, coordinates y and μ, decomposed as $y=(y',y'',y''')$ and $\mu=(\mu',\mu'',\mu''')$, may be introduced so that:*

(i)
$$(y',\mu')=(y_1,\dots,y_{s-1},\mu_1,\dots,\mu_{s-1}),$$
where some real linear combinations e'_j of $d\mu'_j$ and dy'_j, respectively $f'_j=d\mu'_j$ are eigenvectors of A with eigenvalues $\lambda r'_j$, respectively, $\lambda(1-r'_j)$, $j=1,\dots,s-1$ with $r'_j<0$ real and negative.

(ii) $(y'',\mu'')=(y_s,\dots,y_{m-1},\mu_s,\dots,\mu_{m-1})$ *where some real linear combinations e''_j of $d\mu''_j$ and dy''_j, respectively, $f''_j=d\mu''_j$ are eigenvectors with eigenvalues $\lambda r''_j$, $\lambda(1-r''_j)$, $j=s,\dots,m-1$ where $0<r''_j<1/2$ is real and positive.*

(iii) $(y''',\mu''')=(y_m,\dots,y_{n-1},\mu_m,\dots,\mu_{n-1})$, *where some complex combination e'''_j, f'''_j, of dy'''_j and $d\mu'''_j$, $m\le j\le n-1$, are eigenvectors with eigenvalues $\lambda r'''_j$ and $\lambda(1-r'''_j)$ with $r'''_j=1/2+i\beta'''_j$, $\beta'''_j>0$.*

Again, if we set $e=(e',e'',e''')$, $f=(f',f'',f''')$ the eigenvectors of A are $d\nu$, e_j and f_j, with respective eigenvalues 0, λr_j and $\lambda(1-r_j)$; we will take the coordinates so that the r_j are ordered by their real parts. In these coordinates a version of (2–7) still holds, namely if a_j and b_j are any functions on ${}^{\mathrm{sc}}T^*_{\partial X}X$ vanishing at $(0,0,0)$ with differential e_j, respectively f_j, $j=1,\dots,m-1$ (so we may take $b_j=\mu_j$, and we may take a_j a $\mathbb{R}$-linear combination of y_j and μ_j) then

$$\begin{aligned}p&=\lambda\Big(-\nu+\sum_{j=1}^{m-1}r_ja_jb_j+\sum_{j=m}^{n-1}Q_j(y_j,\mu_j)+\nu g_1+g_2\Big)\\&=\lambda\Big(-\nu+\sum_{j=1}^{m-1}r_jy_j\mu_j+\sum_{j=1}^{m-1}c_j\mu_j^2+\sum_{j=m}^{n-1}Q_j(y_j,\mu_j)+\nu g_1+g_2\Big),\end{aligned}\tag{2–8}$$

where the c_j are real, the Q_j are elliptic homogeneous polynomials of degree 2, g_1 vanishes at least linearly and g_2 to third order.

As mentioned, Lemma 2.10 is weaker than, hence is an immediate consequence of, Lemma 2.7. Although it is by no means essential, this weaker result leaves more freedom in choosing the contact map which is useful in making the choice rather explicit, if this is desired. In fact, if $p=|\zeta|^2+V_0-\sigma$, as in Lemma 2.5, we immediately deduce the following.

Lemma 2.11. *For the function $p = |\zeta|^2 + V_0 - \sigma$ with V_0 Morse, the contact map in Lemma 2.10 can be taken as the composition of the contact map on ${}^{sc}T^*_{\partial X}X$ induced by a change of coordinates on X, with the canonical relation of multiplication by a function of the form $e^{i\phi/x}$, $\phi \in \mathscr{C}^\infty(X)$.*

Remark 2.12. The canonical relation of multiplication by $e^{i\phi/x}$ is given, in local coordinates (y, ν, μ), by the map

$$\Phi_\phi : (y, \nu, \mu) \mapsto (y, \nu + \phi(y), \mu + \partial_y \phi(y)),$$

that is, if we write $\Phi_\phi(y, \nu, \mu) = (\bar{y}, \bar{\nu}, \bar{\mu})$, then $\bar{\mu}_k = \mu_k + \partial_{y_k}\phi(y)$. Note that while ϕ is a function on X, the canonical relation only depends on $\phi|_{\partial X}$, which is why we simply regard ϕ as a function on ∂X and write $\phi(y)$ here.

Proof. As in the proof of Lemma 2.5 we may assume, by a change of coordinates on X, that the critical point of V_0 over which the radial point q lies is $y = 0$, that $h - |\mu|^2 = \mathbb{O}(|y|^2)$ and that the Hessian of V_0 at 0 is diagonal, so for each j, $(dy_j, d\mu_j)$ is an invariant subspace of A. Note that in the coordinates (y, ν, μ), $q = (0, \nu_0, 0)$. With the notation of Remark 2.9 above, if dy_j is an eigenvector of the Hessian with eigenvalue $2a_j$ then the eigenvectors of A of eigenvalue λr_j, respectively $\lambda(1 - r_j)$, are $\tilde{e}_j = (\lambda/2)(1 - r_j)dy_j + d\mu_j$, respectively $\tilde{f}_j = (\lambda/2)r_j dy_j + d\mu_j$; see Remark 1.3 of [Hassell et al. 2004]. In particular, if r_j is real, so is $\tilde{f}_j$.

Now, the contact map Φ_ϕ induced by multiplication by $e^{i\phi/x}$ as above acts on $T^*{}^{sc}T^*_{\partial X}X$ by pullbacks, namely

$$\Phi_\phi^*\Big(\sum_k \bar{y}_k^*\, d\bar{y}_k + \bar{\nu}^*\, d\bar{\nu} + \sum_k \bar{\mu}_k^*\, d\bar{\mu}_k\Big)$$
$$= \sum_k \bar{y}_k^*\, dy_k + \bar{\nu}^*(d\nu + \sum_j (\partial y_j \phi)\, dy_j) + \sum_k \bar{\mu}_k^*(d\mu_k + \sum_j \partial_{y_j}\partial_{y_k}\phi(y)\, dy_j).$$

Thus, by the above remark, Φ will map q to $(0, 0, 0)$ provided $\phi(0) = -\nu_0$, $\partial_{y_j}\phi(0) = 0$ for all j. In this case, moreover, the pullback Φ_ϕ^* will map dy_k to dy_k, $d\nu$ to $d\nu$ and $d\mu_k$ to $d\mu_k + \sum_j \partial_{y_j}\partial_{y_k}\phi(y)$. Correspondingly, by letting $\phi(y) = -\nu_0 + \sum_{j=1}^{m-1} b_j y_j^2$, $b_j = (\lambda/4)r_j$, $(\Phi_\phi^{-1})^*$ maps $\tilde{f}_j$ to $d\mu_j$, $j = 1, \dots, m-1$. Since the Legendre vector field W' of $(\Phi_\phi^{-1})^* p$ is the pushforward of the Legendre vector field W of p under Φ_ϕ, it follows that $d\mu_j$ is an eigenvector of the linearization of W' with eigenvalue $\lambda(1 - r_j)$. As Φ_ϕ^* also maps the 2-dimensional subspaces $(dy_j, d\mu_j)$ (at $(0, 0, 0)$) to the 2-dimensional subspaces $(dy_j, d\mu_j)$ (at q), and the latter are invariant under A, so are the former under the linearization of W'. This proves the lemma. □

3. Microlocal normal form

Let $P \in \Psi_{sc}^{*,-1}(X)$ be an operator with real principal symbol p obeying (2–2), as in the previous section, and assume that q is a nondegenerate radial point for p. In this section we shall reduce p to a normal form, via conjugation with a scattering Fourier integral operator. We first pause to define such operators.

3.1. ***Scattering Fourier integral operators.*** Scattering Fourier integral operators (FIOs) are defined in terms of conventional FIOs via the local Fourier transform, as defined in [Melrose and Zworski 1996]. Let X be a manifold of dimension n with boundary, and (x, y) local coordinates where x is a boundary defining function. We can always identify a neighbourhood $U \subset \partial X$ of $y_0 \in \partial X$ with an open set

$V \in S^{n-1}$, which we can think of as embedded in $\mathbb{R}^n$ in the standard way. Correspondingly we may identify the interior of a neighbourhood $[0, \epsilon)_x \times U \subset X$ of $(0, y_0) \in X$ with the an asymptotically conic open set $(\epsilon^{-1}, \infty) \times V \subset \mathbb{R}^n$ in $\mathbb{R}^n$. If we choose a function $\phi \in C^\infty(X)$ supported in $[0, \epsilon)_x \times U$ which is identically 1 in a neighbourhood of $(0, y_0)$, then the operator $\mathcal{F}$ with kernel

$$e^{iz\cdot y/x}\phi(x, y)\frac{d\omega(y)dx}{x^{n+1}}$$

is called a "local Fourier transform" on X. Here $z = (z_1, \dots, z_n) \in \mathbb{R}^n$, $z \cdot y$ denotes the inner product on $\mathbb{R}^n$ and $d\omega(y)$ denotes the standard measure on S^{n-1} (pulled back to ∂X and then to X via the identifications above). Of course, if X is the radial compactification of $\mathbb{R}^n$ and the identification between U and V is the identity, then $\mathcal{F}$ really *is* the Fourier transform premultiplied by the cutoff function ϕ.

It is shown in [Melrose and Zworski 1996] that $\mathcal{F}$ induces a local bijection between ${}^{\mathrm{sc}}T^*_{\partial X}X$ and the cosphere bundle of $\mathbb{R}^n$. In fact, using our identification between U and $V \subset S^{n-1}$ we may represent points in ${}^{\mathrm{sc}}T^*_U X$ as $(\hat{z}, \zeta)$ where $\hat{z} = z/|z| \in V$ represents a point in U and ζ represents the point in the fibre given by (ν, μ) where ν is the parallel component of ζ relative to $\hat{z}$ and μ is the orthogonal component. The identification is then given by the Legendre map

$$L(\hat{z}, \zeta) = (\zeta, -\hat{z}) \in S^*\mathbb{R}^n.$$

In other words, $\mathcal{F}$ sets up a bijection between scattering wavefront set and conventional wavefront set. Moreover, it is shown in [Melrose and Zworski 1996] that conjugation by $\mathcal{F}$ maps the scattering pseudodifferential operators $A \in \Psi^{*,l}_{\mathrm{sc}}(X)$ microsupported near (y_0, ν_0, μ_0) to the conventional pseudodifferential operators microsupported near $L(y_0, \nu_0, \mu_0)$, with principal symbols related by

$$\sigma^l(\mathcal{F}A\mathcal{F}^*)(L(q)) = a(q),$$

where a is the boundary symbol of A (of order l).

Definition 3.1. A scattering FIO is an operator E from $\dot{C}^\infty(X)$ to $C^{-\infty}(X)$ such that, for any local Fourier transforms $\mathcal{F}_1$, $\mathcal{F}_2$ on X, $\mathcal{F}_2 E \mathcal{F}_1^*$ is a conventional FIO on $\mathbb{R}^n$.

A simple example of a scattering FIO is multiplication by an oscillatory factor $e^{i\psi(y)/x}$. Under conjugation by a local Fourier transform this becomes a conventional FIO given by an oscillatory integral with phase function $(z - z') \cdot \zeta + |\zeta|\psi(\zeta/|\zeta|)$. The scattering resolvent kernel constructed by the Hassell and Vasy [1999; 2001], microlocalized to the interior of the "propagating Legendrian", is another example.

It follows then that we can find a scattering FIO quantizing any given contact transformation from a neighbourhood of a point $q \in {}^{\mathrm{sc}}T^*_{\partial X}X$ to itself, since we may conjugate by a local Fourier transform and reduce the problem to finding a conventional FIO quantizing a homogeneous canonical transformation from a conic neighbourhood of $L(q) \in S^*\mathbb{R}^n$ to itself. We can also use the local Fourier transform to import Egorov's theorem into the scattering calculus. Namely, if $B \in \Psi^{*,-1}_{\mathrm{sc}}(X)$ is a scattering pseudodifferential operator of order -1, with real principal symbol, and $P \in \Psi^{*,-1}_{\mathrm{sc}}(X)$ then also $e^{-iB}Pe^{iB} \in \Psi^{*,-1}_{\mathrm{sc}}(X)$ is a scattering pseudodifferential operator of order -1, whose symbol p' is related to that of P by the time 1 flow of the Hamilton vector field of B. This indeed is how we shall conjugate the principal symbol p of our operator to normal form.

3.2. ***Normal form.*** In this section we put the principal symbol of P into a normal form p_{norm}. For later purposes we shall also need the subprincipal symbol of P in a normal form, but only along the "flow-out", that is, the unstable manifold, of q, which can be done via conjugation by a function; this is accomplished in Lemma 6.1. (The model form of the subprincipal symbol only plays a role in the polyhomogeneous, as opposed to just conormal, analysis, which is the reason it is postponed to Section 6.)

For this purpose, we only need to construct the principal symbol $\sigma(B)$ of B as in the first subsection. This in turn can be written as $x^{-1}\tilde{b}$, $\tilde{b} \in \mathscr{C}^\infty({}^{\mathrm{sc}}T^*X)$, so we only need to construct a function b on ${}^{\mathrm{sc}}T^*_{\partial X}X$ such that the pullback $\Phi^* p$ of p by the time 1 flow Φ of $H_{x^{-1}\tilde{b}}$ is the desired model form p_{norm}, where $\tilde{b}$ is some extension of b to ${}^{\mathrm{sc}}T^*X$; this property is independent of the chosen extension. Thus *any* B with $\sigma(B) = \tilde{b}$ will conjugate P to an operator with principal symbol p_{norm}. This construction is accomplished in two steps, following Guillemin and Schaeffer [1977] in the nonresonant setting. First we construct the Taylor series of b at $q = (0, 0, 0)$, which puts p into a model form modulo terms vanishing to infinite order at q. Next, we remove this error *along the unstable manifold* of q by modifying an argument due to Nelson [1969].

Rather than using powers of $\mathscr{I}$ to filter the Taylor series of b, we proceed as in [Guillemin and Schaeffer 1977] and assign degree 1 to y and μ but degree two to ν in local coordinates as discussed above. Thus, let $\mathfrak{h}^j$ denote the space of functions

$$\mathfrak{h}^j = \sum_{2a+|\alpha|+|\beta|-2=j} \nu^a y^\alpha \mu^\beta \mathscr{C}^\infty({}^{\mathrm{sc}}T^*_{\partial X}X)$$

Note that this is well-defined, independently of our choice of local coordinates, since $-d\nu$ is the contact form α at q, so ν is well-defined up to quadratic terms. The Poisson bracket preserves this filtration of $\mathscr{I}$ in the following sense. If $\tilde{a}$, $\tilde{b}$ are some smooth extensions to ${}^{\mathrm{sc}}T^*X$ of elements $a \in \mathfrak{h}^i$, $b \in \mathfrak{h}^j$ then

$$x^{-1}\tilde{c} = \{x^{-1}\tilde{a}, x^{-1}\tilde{b}\} \Longrightarrow c = \tilde{c}|_{{}^{\mathrm{sc}}T^*_{\partial X}X} \in \mathfrak{h}^{i+j}.$$

When this holds we write $c = \{\{a, b\}\}$; explicitly,

$$\{\{a, b\}\} = W_a(b) + \frac{\partial a}{\partial \nu} b - \frac{\partial b}{\partial \nu} a, \tag{3–1}$$

with W given by (2–4). Thus

$$\{\{.,.\}\} : \mathfrak{h}^i \times \mathfrak{h}^j \mapsto \mathfrak{h}^{i+j}.$$

We then consider the quotient

$$\mathfrak{g}^j = \mathfrak{h}^j / \mathfrak{h}^{j+1},$$

so the bracket $\{\{.,.\}\}$ descends to

$$\mathfrak{g}^i \times \mathfrak{g}^j \to \mathfrak{g}^{i+j}.$$

Remark 3.2. These statements remain true with $\mathfrak{h}^j$ replaced by $\mathscr{I}^j$. However, note that $p = -\nu$ in $\mathscr{I}/\mathscr{I}^2$, since $dp = -d\nu$ at q, but it is *not the case* that $p = -\nu$ in $\mathfrak{g}^0$. In fact, p is given by (3–2) below in $\mathfrak{g}^0$.

Using contact coordinates as discussed above, $\mathfrak{g}^j$ may be freely identified with the space of homogeneous functions of ν, y, μ of degree $j + 2$ where the degree of ν is 2. Now let p_0 be the part of p of homogeneity degree two. In order to use Lemmas 2.7 and 2.10, *we assume throughout the paper from here on that case (iv) above Lemma 2.7 does not apply.* Hence from (2–7)

$$p_0 = \lambda\Big(-\nu + \sum_{j=1}^{m-1} r_j y_j \mu_j + \sum_{j=m}^{n-1} Q_j(y_j, \mu_j)\Big), \;\; p - p_0 \in \mathfrak{h}^1. \tag{3–2}$$

If we take $b \in \mathfrak{h}^l$, $l \geq 1$ and let Φ be the time 1 flow of $H_{x^{-1}b}$ then

$$x\Phi^*(x^{-1}p) = p + \{\{p, b\}\} = p + \{\{p_0, b\}\}, \;\text{modulo } \mathfrak{h}^{l+1}.$$

This allows us to remove higher order term in the Taylor series of the symbol successively provided we can solve the "homological equation"

$$\{\{p_0, b\}\} = e \in \mathfrak{h}^l, \;\text{modulo } \mathfrak{h}^{l+1}.$$

Thus we need to consider the range of this linear map; its eigenfunctions are easily found from the eigenfunctions of the linearization of W.

Lemma 3.3. *The (equivalence classes of the) monomials $p_0^a e^\alpha f^\beta$ with $2a + |\alpha| + |\beta| = l + 2$ satisfy*

$$\{\{p_0, p_0^a e^\alpha f^\beta\}\} = R_{a,\alpha,\beta}\, p_0^a e^\alpha f^\beta$$

with eigenvalues

$$R_{a,\alpha,\beta} = \lambda\Big(a - 1 + \sum_{j=1}^{n-1} \alpha_j r_j + \sum_{j=1}^{n-1} \beta_j (1 - r_j)\Big) \tag{3–3}$$

and give a basis of eigenvectors for $\{\{p_0, .\}\}$ acting on $\mathfrak{g}^l$.

Here we identify the differentials e_j and f_j with linear functions with these differentials.

Remark 3.4. In fact, the contact coordinates given by Lemma 2.10 suffice for the proof of this lemma; the additional information in Lemma 2.7 is not needed. In this case, by (2–8),

$$p_0 = \lambda\Big(-\nu + \sum_{j=1}^{m-1} r_j e_j f_j + \sum_{j=m}^{n-1} Q_j(y_j, \mu_j)\Big).$$

We also remark that we could equally well use the eigenvector basis for $\{\{p_0, .\}\}$ acting on $\mathfrak{g}^l$ given by $\nu^a e^\alpha f^\beta$ with $2a + |\alpha| + |\beta| = l + 2$. This follows from the lemma using that

$$\nu = \sum_{j=1}^{m-1} r_j y_j \mu_j + \sum_{j=m}^{n-1} Q_j(y_j, \mu_j) - \lambda^{-1} p_0$$

in $\mathfrak{g}^0$, and $y_j \mu_j$ as well as $Q_j(y_j, \mu_j)$ are eigenvectors with eigenvalue $\lambda(r_j + (1 - r_j)) = \lambda$, and so is p_0.

Proof. Taking into account the eigenvalues and eigenvectors of A, all eigenvalues and eigenvectors of $\{\{p_0, .\}\}$ can be calculated iteratively using the derivation property of the original Poisson bracket. This implies

$$\begin{aligned}\{\{p_0, ab\}\} &= x\{x^{-1}p_0, x(x^{-1}a)(x^{-1}b)\}\\ &= x^{-1}\{x^{-1}p_0, x\}ab + x\{x^{-1}p_0, x^{-1}a\}b + xa\{x^{-1}p_0, x^{-1}b\}\\ &= \lambda ab + \{\{p_0, a\}\}b + a\{\{p_0, b\}\},\end{aligned}$$

where each term within $\{.,.\}$ really uses a $\mathcal{C}^\infty$ extensions of the a, b, p_0 to ${}^{\mathrm{sc}}T^*X$, followed by evaluation of the bracket and then restriction to ${}^{\mathrm{sc}}T^*_{\partial X}X$. Since

$$\{\{p_0, a\}\} = x\{x^{-1}p_0, x^{-1}a\} = x\{x^{-1}p_0, x^{-1}\}a + \{x^{-1}p_0, a\} = -\lambda a + \{x^{-1}p_0, a\},$$

on $\mathfrak{g}^{-1}$ the eigenvectors of $\{\{p_0, .\}\}$ are the eigenvectors e_j and f_j of A with eigenvalues $-\lambda + \lambda r_j$ and $-\lambda + \lambda(1 - r_j)$. Moreover, in $\mathfrak{g}^0$, p_0 is an eigenvector of $\{\{p_0, .\}\}$ with eigenvalue 0. Thus, e_j, f_j and p_0 satisfy the claim of the lemma. Since the other generators of $\mathfrak{g}^0$, as well as generators of $\mathfrak{g}^j$, $j \geq 1$, can be written as a products of the e_j, f_j and p_0, the conclusion of the lemma follows by induction. □

Definition 3.5. We call the multiindices in the set

$$I = \left\{(a, \alpha, \beta);\ R_{a,\alpha,\beta} = 0 \text{ and } 2a + |\alpha| + |\beta| \geq 3\right\}, \tag{3–4}$$

with $R_{a,\alpha,\beta}$ given by (3–3), *resonant*.

Conjugation therefore allows us to remove, by iteration, all terms except those with indices in I. Expanding p_0^a using (3–2) we deduce the following.

Proposition 3.6. *If P is as above and the leading term of $p = \sigma_{\partial,-1}(P)$ is given by (3–2) near a given radial point q then there exists a local contact diffeomorphism Φ near q such that*

$$\Phi^* p = \lambda\Big(-\nu + \sum_{j=1}^{m} r_j y_j \mu_j + \sum_{j=m+1}^{n-1} Q_j(y_j, \mu_j) + \sum_{(a,\alpha,\beta)\in I} c_{a,\alpha,\beta}\nu^a e^\alpha f^\beta\Big) \text{ modulo } \mathcal{I}^\infty = \mathfrak{h}^\infty \text{ at } q \tag{3–5}$$

with I given by (3–4).

Proof. The Taylor series of Φ at q can be constructed inductively over the filtration $\mathfrak{h}^j$ as indicated above. At the j–th stage, the terms of weighted homogeneity j can be removed from p except for those in the null space of $\{\{p_0, \cdot\}\}$, that is, the resonant terms with $R_{a,\alpha,\beta} = 0$. This leads to (3–5) in the sense of formal power series. However, by use of Borel's Lemma a local contact diffeomorphism can be found giving (3–5). □

Now a small extension of Nelson's proof of Sternberg's linearization theorem can be used to remove the infinite order vanishing error along the unstable manifold, that is, at $\nu = 0$, $\mu = 0$, $y'' = 0$, $y''' = 0$.

Proposition 3.7. *Suppose that X and X_0 are $\mathcal{C}^\infty$ vector fields on $\mathbb{R}^N$ with $X_0(0) = 0$ and $X_1 = X - X_0$ vanishing to infinite order at 0. Suppose also that they are both linear outside a compact set and equal there to their common linearization, $DX(0)$, at 0 which is assumed to have no pure imaginary eigenvalue. Let $U(t)$, $U_0(t)$ be the flows generated by X and X_0. If E is a linear submanifold invariant under X_0 such that*

$$\lim_{t\to\infty} U_0(t)x = 0 \quad \text{for all } x \in E \tag{3–6}$$

then for all $j = 0, 1, 2, \dots$ *and* $x \in E$

$$\lim_{t\to\infty} D^j(U(-t)U_0(t))x \tag{3–7}$$

exists, and is continuous in $x \in E$, *and*

$$W_- x = \lim_{t\to\infty} U(-t)U_0(t)x,\ x \in E$$

has a $\mathscr{C}^\infty$ *extension*, G, *to* $\mathbb{R}^N$ *which is the identity to infinite order at* 0 *and such that* $(G^{-1})_* X = X_0$ *to infinite order along* E *in a neighbourhood of* 0.

Remark 3.8. Note that the derivatives D^j in (3–7) refer to the ambient space $\mathbb{R}^N$, and *not* merely to E. This is useful in producing the Taylor series of G for the last part of the conclusion.

Also, the limit $t \to \infty$ means $t \to +\infty$, as in Nelson's book.

Proof. We follow the proof of Theorem 8 in [Nelson 1969]. Indeed, if X_0 was assumed to be linear then Nelson's theorem would apply directly. Dropping this assumption has little effect on the proof; the main difference is that a little more work is required to show the exponential contraction property, (3–8) below.

Since the real part of every eigenvalue of $DX(0)$ is nonzero, $\mathbb{R}^N = E_+ \oplus E_-$ where E_+, respectively E_-, is the direct sum of the generalized eigenspaces of $DX(0)$ with eigenvalues with positive, respectively negative, real parts. Since E is invariant under X_0, and hence under $DX(0)$, necessarily $E \subset E_-$. We actually apply the theorem with $E = E_-$, but, as in Nelson's discussion, the more general case is useful for the inductive argument for the derivatives.

Let e_j denote a basis of E_- consisting of generalized eigenvectors of $DX(0)$ with corresponding eigenvalue σ_j; we shall consider the e_j as differentials of linear functions f_j on $\mathbb{R}^N$. For $x \in \mathbb{R}^N$, let $x(t) = U_0(t)x$, $F_j(t) = f_j(x(t))$. Then $dF_j/dt|_{t=t_0} = (X_0 f_j)(x(t_0))$ where

$$X_0 f_j(y) = DX(0) f_j(y) + \mathbb{O}(\|y\|^2).$$

Moreover, for $y \in E_-$, $\|y\|^2 \le C_1 \sum_j f_j^2$ for some $C_1 > 0$. So, setting $\rho = \sum f_j^2$, we deduce that

$$X_0 \rho(y) = \sum_j 2\sigma_j f_j^2(y) + \mathbb{O}(\rho(y)^{3/2}),$$

hence with $R(t) = \rho(x(t))$, $c_0 \in (\sup \sigma_j, 0)$, there exists $\delta > 0$ such that for $\|R(t)\| \le \delta$,

$$\frac{dR}{dt} - 2c_0 R \le 0,$$

and hence $R(t) \le e^{-2c_0 t}\|x\|$ for $t \ge 0$, $\|r(x)\| \le \delta$, $x \in E_-$. A corresponding estimate also holds outside a compact set, as X_0 is given by $DX(0)$ there, so a patching argument and (3–6) yield the estimate $R(t) \le C_0 e^{-2c_0 t}\|x\|$ for all $x \in E_-$. Since $R(t)^{1/2}$ is equivalent to $\|.\|$, we deduce that there are constants $C, c > 0$ such that

$$\|U_0(t)x\| \le Ce^{-ct}\|x\| \ \forall\ x \in E \text{ and } t \ge 0. \tag{3–8}$$

For the remainder of the argument we can follow Nelson's proof even more closely. Thus, let κ be a Lipschitz constant for X and X_0, and choose m such that $cm > \kappa$. Note that there exists $c_0 > 0$ such that for all $x \in \mathbb{R}^N$,

$$\|X_1(x)\| \le c_0\|x\|^m, \ \ X_1 = X - X_0. \tag{3–9}$$

For $t_1 \ge t_2 \ge 0$, $t_1 = t_2 + t$, $x \in E$,

$$\begin{aligned} I = \|U(-t_1)U_0(t_1)x - U(-t_2)U_0(t_2)x\| &= \|U(-t_2)\,(U(-t)U_0(t) - \mathrm{Id})\,U_0(t_2)x\| \\ &\le e^{\kappa t_2}\|(U(-t)U_0(t) - \mathrm{Id})U_0(t_2)x\| \end{aligned}$$

by the Lipschitz condition (see [Nelson 1969, Theorem 5]). But with $X = X_0 + X_1$, by [Nelson 1969, Proof of Theorem 6, (5)]

$$\|U(-t)U_0(t)y - y\| \le \int_0^t e^{\kappa s}\|X_1(U_0(s)y)\|\,ds.$$

Applying this with $y = U_0(t_2)x$, we deduce that

$$I \le e^{\kappa t_2}\int_0^t e^{\kappa s}\|X_1(U_0(s+t_2)x)\|\,ds. \tag{3–10}$$

Thus, by (3–9) and (3–8),

$$I \le e^{\kappa t_2}\int_0^t e^{\kappa s}c_0C^m e^{-cm(s+t_2)}\|x\|^m\,ds \le e^{\kappa t_2}\int_0^\infty e^{\kappa s}c_0C^m e^{-cm(s+t_2)}\|x\|^m\,ds = \frac{c_0C^m e^{-(cm-\kappa)t_2}\|x\|^m}{cm-\kappa}.$$

Letting $t_2 \to \infty$ shows that $W_- x = \lim_{t\to\infty} U(-t)U_0(t)x$ exists, with convergence uniform on compact sets, hence W_- is continuous in $x \in E$. Moreover, applying the estimate with $t_2 = 0$ shows that $W_-(x) - x = \mathcal{O}(\|x\|^m)$. Since m is arbitrary, as long as it is sufficiently large, this shows that W_- is the identity to infinite order at 0, provided it is smooth, as we proceed to show.

Smoothness can be seen by a similar argument, although we need to put a slight twist into Nelson's argument. Namely, first consider the first derivatives, or rather the 1-jet. Thus, we work on $\mathbb{R}^N \oplus \mathcal{L}(\mathbb{R}^N)$. Let (x, ξ) denote the components with respect to this decomposition. These evolve under the flow $U'(t)$, respectively $U_0'(t)$, given by

$$X'(x,\xi) = (X(x),\, DX(x)\cdot\xi), \ \ X_0'(x,\xi) = (X_0(x),\, DX_0(x)\cdot\xi),$$

where $DX(x)$ and ξ are considered as elements of $\mathcal{L}(\mathbb{R}^N)$, and $\cdot$ is composition of operators. Note that the second, $\mathcal{L}(\mathbb{R}^N)$, component of these vector fields is a homogeneous degree zero vector field, that is, it is invariant under pushforward by the natural $\mathbb{R}^+$-action (by dilations).

The twist, as compared to Nelson's work, is that we identify $\mathcal{L}(\mathbb{R}^N)$ with $\mathbb{R}^{N^2}$, which we radially compactify to a (closed) ball B^{N^2}, which we further embed as the closed unit ball in $\mathbb{R}^{N^2}$ in such a fashion that the smooth structure of the ball agrees with the restriction of the smooth structure from $\mathbb{R}^{N^2}$. Let $\iota : \mathbb{R}^{N^2} \to \mathbb{R}^{N^2}$ be this map with range the interior of B^{N^2}. Then the pushforward under ι of a homogeneous degree zero vector field, such as $DX(x)\cdot\xi$ is for each $x \in \mathbb{R}^N$, extends to a $\mathscr{C}^\infty$ vector field on the closed ball B^{N^2}, which by homogeneity is tangent to the boundary. Furthermore, if $\iota_1 = \mathrm{id}_{\mathbb{R}^N} \times \iota$, then $(\iota_1)_*X'$ and $(\iota_1)_*X_0'$ extend to $\mathscr{C}^\infty$ vector field on $\mathbb{R}^N \times B^{N^2}$ tangent to the boundary and their difference, $(\iota_1)_*X_1'$, in addition vanishes to infinite order at $\{0\} \times B^{N^2}$. Thus $(\iota_1)_*X'$ and $(\iota_1)_*X_0'$ are Lipschitz with some Lipschitz constant κ': this is automatic over a compact subset of $\mathbb{R}^N \times B^{N^2}$, which in fact suffices here, but in fact holds on all of $\mathbb{R}^N \times B^{N^2}$ since outside the inverse image of a compact subset of $\mathbb{R}^N \times B^{N^2}$, X' and X'_0 are linear, so in particular their B^{N^2} component is independent of x.

To minimize confusion about the "change of coordinates", we write the coordinates on $\mathbb{R}^N \times B^{N^2}$ as (x, η) below. With c as in (3–8), choose m such that $cm > \kappa'$. Then the infinite order vanishing of $(\iota_1)_* X'_1$ at $x = 0$ yields

$$\|((\iota_1)_* X'_1)(x, \eta)\| \le c'_0 \|x\|^m$$

for all (x, η). Let $U'(t)$, $U'_0(t)$ denote the evolution groups generated by $(\iota_1)_* X'$ and $(\iota_1)_* X'_0$, respectively. Thus, for all real t,

$$\|U'(t)(x, \eta)\| \le e^{\kappa' t} \|(x, \eta)\|, \tag{3–11}$$

see [Nelson 1969, Theorem 5]. So (3–10) still applies, with X_1 replaced by $(\iota_1)_* X'_1$, κ replaced by κ', etc. Thus, by (3–8) and (3–11),

$$\begin{aligned} I' &= \|U'(-t_1)U'_0(t_1)(x, \eta) - U'(-t_2)U'_0(t_2)(x, \eta)\| \\ &\le e^{\kappa' t_2} \int_0^t e^{\kappa' s} \|((\iota_1)_* X'_1)(U'_0(s + t_2)(x, \eta))\| \, ds \\ &\le e^{\kappa' t_2} \int_0^t e^{\kappa' s} c'_0 C^m e^{-cm(s+t_2)} \|x\|^m \, ds \\ &\le e^{\kappa' t_2} \int_0^\infty e^{\kappa' s} c'_0 C^m e^{-cm(s+t_2)} \|x\|^m \, ds = \frac{c'_0 C^m e^{-(cm-\kappa')t_2} \|x\|^m}{cm - \kappa'}. \end{aligned}$$

Thus, $\lim_{t\to\infty} U'(-t)U'_0(t)x$ exists, with convergence uniform on compact sets, so the limit depends continuously on (x, ξ) for $x \in E$.

The higher derivatives can be handled similarly. The resulting Taylor series about E can be summed asymptotically, giving G: this part of the argument of Nelson is unchanged. □

3.3. ***Effective resonance and nonresonance.*** Next we apply this general result to the symbol p. Following Lemma 2.7, when resonances occur we cannot remove all error terms even in the sense of formal power series. Consequently we do not attempt to get a full normal form in a neighbourhood of the critical point, but only along the submanifold

$$S = \{\nu = 0,\ y'' = 0,\ y''' = 0,\ \mu = 0\}, \tag{3–12}$$

which is the unstable manifold for W_0. After reduction to normal form, errors which are polynomial in the normal directions to S will remain. For later purposes, we divide these into two parts.

Definition 3.9. With I as in Definition 3.5, let

$$\begin{aligned} &I_{\mathrm{er}} = I'_{\mathrm{er}} \cup I''_{\mathrm{er}}, \\ &I'_{\mathrm{er}} = \{(a, \alpha, \beta) \in I : \alpha = (\alpha', \alpha'', \alpha'''),\ \beta = (\beta', \beta'', \beta'''),\ a = 0,\ \alpha''' = 0,\ \beta''' = 0,\ \alpha'' = 0,\ \beta'' = 0,\ |\beta'| = 1\}, \\ &I''_{\mathrm{er}} = \{(a, \alpha, \beta) \in I : \alpha = (\alpha', \alpha'', \alpha'''),\ \beta = (\beta', \beta'', \beta'''),\ a = 0,\ \alpha''' = 0,\ \beta''' = 0,\ \alpha' = 0,\ \beta' = 0\}. \end{aligned} \tag{3–13}$$

An *effectively resonant* function is a polynomial of the form

$$r_{\mathrm{er}} = \sum_{(a,\alpha,\beta)\in I_{\mathrm{er}}} c_{a,\alpha,\beta}\, p_0^a e^\alpha f^\beta,$$

or equivalently

$$r_{\mathrm{er}} = \sum_{(a,\alpha,\beta)\in I_{\mathrm{er}}} c_{a,\alpha,\beta}\, \nu^a e^\alpha f^\beta.$$

Thus, elements of I_{er} satisfy $(0, \alpha, \beta) \in I$ (that is, are resonant; see Definition 3.5), with $\alpha = (\alpha', \alpha'', 0)$, $\beta = (\beta', \beta'', 0)$, and either $\alpha'' = 0$, $\beta'' = 0$, $|\beta'| = 1$, or $\alpha' = 0$, $\beta' = 0$.

Moreover, an effectively resonant function has the form

$$\sum_{\alpha',|\beta'|=1} c_{\alpha'\beta'} (e')^{\alpha'} (f')^{\beta'} + \sum_{\alpha'',\beta''} c_{\alpha''\beta''} (e'')^{\alpha''} (f'')^{\beta''}. \tag{3–14}$$

For a fixed critical point of a fixed operator P (for example, $P = x^{-1}(\Delta + V - \sigma)$ for a fixed σ), the set I_{er} is finite. Thus, only a finite number of terms can occur in (3–14), and hence restricting to polynomials in the definition of effectively resonant functions (rather than infinite formal sums) is in fact not a restriction. To see this, note that in the expression for $R_{a,\alpha,\beta}$ in (3–4), we have $a = 0$, $\alpha''' = \beta''' = 0$ and either (i) $\alpha'' = \beta'' = 0$ and $|\beta'| = 1$ or (ii) $\alpha' = \beta' = 0$. In case (i), if $\beta'_j = 1$ then to have $R_{a,\alpha,\beta} = 0$ we need $\sum \alpha'_k r'_k = r'_j$, which is only possible for $|\alpha'| \le |r'_j| / \min_k |r'_k|$. In case (ii), we need $\sum \alpha''_j r''_j + \sum \beta''_j (1 - r''_j) = 1$, which is only possible for $|\alpha''| \le 1/\min r''_k$ and $|\beta''| \le 2$. (Actually in case (ii) we must have $|\beta''| \le 1$ in order to satisfy the condition $2a + |\alpha| + |\beta| \ge 3$ in (3–4).)

Definition 3.10. Let $\mathcal{J}_S$ denote the ideal of $\mathcal{C}^\infty$ functions on ${}^{\mathrm{sc}}T^*_{\partial X} X$ which vanish on S and set

$$J'' = \Big\{ (\alpha'', \beta'');\ \sum_{j=s}^{m-1} r''_j \alpha''_j + (1 - r''_j)\beta''_j \in (1, 2) \Big\}.$$

An *effectively nonresonant* function is an element of $\mathcal{J}_S$ of the form

$$\begin{aligned} r_{\mathrm{enr}} = \sum_{j=1}^{s-1} h_j f'_j + \sum_{(\alpha'',\beta'')\in I''} h''_{\alpha'',\beta''} e^{\alpha''} f^{\beta''} + \sum_{j,k} h'''_{jk} e'''_j f'''_k \\ h_j \in \mathcal{J}_S,\ j = 0, 1, \dots, s,\ h''_{\alpha'',\beta''} \in \mathcal{C}^\infty({}^{\mathrm{sc}}T^*_{\partial X} X),\ (\alpha'', \beta'') \in I'', \\ h'''_{jk} \in \mathcal{J}_S,\ j, k = m, \dots, n-1. \end{aligned} \tag{3–15}$$

Note that J'' is finite, hence all sums in the definition are finite.

Theorem 3.11. *Using the notation of Lemma 2.7 for coordinates near a radial point of q of p there is a local contact diffeomorphism Φ from a neighbourhood of $(0, 0, \dots, 0)$ to a neighbourhood of q such that $\Phi^* p = p_{\mathrm{norm}}$ such that*

$$\lambda^{-1} p_{\mathrm{norm}} = -\nu + \sum_j r_j y_j \mu_j + \sum_{j=m}^{n-1} Q_j(y_j, \mu_j) + r_{\mathrm{enr}} + r_{\mathrm{er}}, \tag{3–16}$$

with r_{enr} of the form (3–15) and r_{er} of the form (3–14); in addition at a nonresonant critical point, that is, if $I = \varnothing$, then we may take $r_{\mathrm{enr}} = r_{\mathrm{er}} = 0$ near q.

Remark 3.12. If F is an elliptic Fourier integral operator with canonical relation Φ then $\widetilde{P} = F^{-1} P F$ satisfies $\sigma_{\partial,-1}(\widetilde{P}) = p_{\mathrm{norm}}$.

Remark 3.13. As will be seen below, of the two error terms, only r_{er} has any effect on the leading asymptotics of microlocal solutions. The construction below shows that modulo $\mathcal{I}^\infty$, r_{enr} may be chosen to consist of resonant terms only, that is, to be an asymptotic sum of resonant terms. However, this plays no role in the paper; all the relevant information is contained in the statement of the theorem.

Remark 3.14. We do not need the full power of Lemma 2.7 to find Φ as in this theorem; Lemma 2.10 suffices. Indeed, the terms $\sum_{j=1}^{m-1} c_j \mu_j^2$ in (2–8) can be absorbed in r_{enr}.

Similarly any term $\nu^a \mu^\beta y^\alpha$ with $a + |\beta| \geq 2$ and $a \neq 0$, or with $|\beta| \geq 3$ can be included in r_{er} or r_{enr}. The same is true for any term with $|\beta| \geq 2$ such that $\beta_j \neq 0$ for some j with $\operatorname{Re} r_j \neq \frac{1}{2}$. In particular, if $\operatorname{Re} r_j \neq \frac{1}{2}$ for any j, the only terms which need to be removed have $a + |\beta| \leq 1$. The conjugating Fourier integral operator can therefore also be arranged to have such terms only and thus to be of the form e^{iB}, with $B = Z + (f/x)$ where Z is a vector field on X tangent to its boundary and f is a real valued smooth function on X. Correspondingly, the normal form may be achieved by conjugation of P by an oscillatory function, $e^{if/x}$, followed by pullback by a local diffeomorphism of X, that is, a change of coordinates. However, if $\operatorname{Re} r_j = \frac{1}{2}$ for some j, some quadratic terms in μ would also need to be removed for the model form, but since they play a role analogous to r_{er}, the arguments of Section 5, giving conormality, are unaffected, and only the polyhomogeneous statements of Section 6 would need alterations. However, the contact diffeomorphism (that is, FIO conjugation) approach we present here is both more unified and more concise.

If $p = |\zeta|^2 + V_0 - \sigma$, the model form of Lemma 2.10 also only required a change of coordinates and multiplication by an oscillatory function (see Lemma 2.11), the model form of this theorem can be obtained by these two operations, starting from the original operator P with symbol p.

Proof. First we apply Proposition 3.6. Next we need to show that r_{er} as in (3–14) and r_{enr} as in (3–14) can be chosen to have Taylor series at 0 given exactly by the error term in (3–5).

So, consider a monomial $\nu^a e^\alpha f^\beta$ with $(a, \alpha, \beta) \in I$. If $\alpha''' \neq 0$ then $\beta''' \neq 0$ since $\operatorname{Im} r_j''' > 0$, and only the eigenvalues of f_j''' have negative imaginary parts, and conversely. In addition, $2a + |\alpha| + |\beta| \geq 3$ implies that a monomial with $\alpha''' \neq 0$ or $\beta''' \neq 0$ has the form $\nu^a e^{\tilde\alpha} f^{\tilde\beta} e_j''' f_k'''$ for some j, k with $2a + |\tilde\alpha| + |\tilde\beta| \geq 1$ and

$$\operatorname{Re}\Big(a + \sum r_l \tilde\alpha_l + \sum (1 - r_l)\tilde\beta_l\Big) = 0.$$

Since $\operatorname{Re}(1 - r_l) > 0$ for all l and $\operatorname{Re} r_l > 0$ for $l \geq s$, while $r_l < 0$ for $l \leq s-1$, we must have $\tilde\alpha' \neq 0$ (that is, $\tilde\alpha_l \neq 0$ for some $l \leq s-1$) and correspondingly $a + |\tilde\alpha''| + |\tilde\alpha'''| + |\tilde\beta| > 0$. Due to the latter, $\nu^a e^{\tilde\alpha} f^{\tilde\beta}$ vanishes on S, so the terms with $\alpha''' \neq 0$ or $\beta''' \neq 0$ appear in r_{enr}.

So we may assume that $\alpha''' = \beta''' = 0$. If $a \neq 0$, the monomial is of the form $\nu^{\tilde a} e^{\tilde\alpha} f^{\tilde\beta} \nu$, $\tilde a = a - 1$, $2\tilde a + |\tilde\alpha| + |\tilde\beta| \geq 1$ with

$$\tilde a + \sum r_j \tilde\alpha_j + \sum (1 - r_j)\tilde\beta_j = 0.$$

Arguing as in the previous paragraph we deduce that the terms with $a \neq 0$ also appear in r_{enr}.

So we may now assume that $a = 0$, $\alpha''' = \beta''' = 0$. If $\beta' \neq 0$, the monomial is of the form $\nu^a e^{\tilde\alpha} f^{\tilde\beta} f_j$ for some j, and $2a + |\tilde\alpha| + |\tilde\beta| \geq 2$,

$$a + \sum r_l \tilde\alpha_l + \sum (1 - r_l)\tilde\beta_l = r_j < 0.$$

We can still conclude that $\tilde\alpha' \neq 0$, but it is not automatic that $a+|\tilde\alpha''|+|\tilde\beta|>0$. However, if $a+|\tilde\alpha''|+|\tilde\beta|>0$ then $\nu^a e^{\tilde\alpha} f^{\tilde\beta} f_j$ is again included in r_{enr}, while if $a+|\tilde\alpha''|+|\tilde\beta|=0$, then the monomial is included in r_{er}.

Finally then, we may assume that $a=0$, $\beta'=0$, $\alpha'''=\beta'''=0$. Since $r'_j<0$ for all $j=1,\dots,s-1$

$$\sum (r''_j\alpha''_j+(1-r''_j)\beta''_j) \geq \sum r'_j\alpha'_j + \sum (r''_j\alpha''_j+(1-r''_j)\beta''_j)=1.$$

Moreover, the equality holds if and only if $\alpha'=0$, in which case this term is included in r_{er}. The terms with $\alpha'\neq 0$ can be included in $h''_{\tilde\alpha'',\tilde\beta''}e^{\tilde\alpha''}f^{\tilde\beta''}$ for some $\tilde\alpha''\leq\alpha''$, $\tilde\beta''\leq\beta''$, chosen by reducing α'' and/or β'' to make

$$\sum (r''_j\tilde\alpha''_j+(1-r''_j)\tilde\beta''_j)\in(1,2).$$

This can be done since r''_j, $1-r''_j\in(0,1)$.

It follows that $\lambda^{-1}p$ can be conjugated to the form

$$-\nu+\sum_j r_j y_j\mu_j+\sum_{j=m}^{n-1} Q_j(y_j,\mu_j)+r_{\text{enr}}+r_{\text{er}}+r_\infty, \tag{3–17}$$

where $r_{\text{enr}}, r_{\text{er}}$ are as in (3–15), (3–14), with both vanishing if q is nonresonant, and r_∞ vanishes to infinite order at $(0,0,0)$. Thus, it remains to show that we can remove the r_∞ term in a neighbourhood of the origin.

To do this we apply Proposition 3.7. Let X' be the Legendre vector field of (3–17), and let X'_1 be the Legendre vector field of r_∞, while $X'_0=X'-X'_1$. Let $\widetilde X$ be the linear vector field with differential equal to $DX(0)$, let χ be compactly supported, identically 1 near 0, and let $X=-(\chi X'+(1-\chi)\widetilde X)$, $X_0=-(\chi X'_0+(1-\chi)\widetilde X)$. The overall minus sign is due to S being the unstable manifold of X'_0 near the origin, hence the stable manifold of $-X'_0$. Let E be the subspace S of $\mathbb{R}^{2n-1}$, defined by (3–12). Then Proposition 3.7 is applicable, and G given by it may be chosen as a contact diffeomorphism since $U(t)$, $U_0(t)$ are such; see [Guillemin and Schaeffer 1977, Section 3, Theorem 4]. □

3.4. ***Parameter-dependent normal form.*** We also need a parameter-dependent version of this theorem. Namely, suppose that p depends smoothly on a parameter σ, can we make the normal form depend smoothly on σ as well? This problem can be approached in at least two different ways. One can consider σ simply as a parameter, so $p\in\mathscr{C}^\infty((\partial^{\text{sc}}T^*X)\times I)=\mathscr{C}^\infty(({}^{\text{sc}}T^*_{\partial X}X)\times I)$ and then try to carry out the reduction to normal form uniformly. Alternatively, one identify p with the function p' on the larger space $\partial^{\text{sc}}T^*(X\times I)$ arising by the pullback under the natural projection

$$p'=\pi^*p,\ \ \pi:{}^{\text{sc}}T^*_{\partial X\times I}(X\times I)\to({}^{\text{sc}}T^*_{\partial X}X)\times I$$

and then carry out the reduction to a model on the larger space. Whilst the second approach may be more natural from a geometric stance, we will adopt the first, since it is closer to the point of view of spectral theory of [Hassell et al. 2004]. Clearly the difficulty in obtaining a uniform normal form is particularly acute near a value of σ at which the effectively resonant terms do not vanish. Fortunately in the case of central interest here, and in other cases too, the set of points at which such problems arise is discrete.

Lemma 3.15. *If $P=P(\sigma)=x^{-1}(\Delta+V-\sigma)$, $q=q(\sigma)$ is a radial point of P lying over the critical point $z=\pi(q)$ of V_0 and $I(\sigma)$, respectively $I_{\text{er}}(\sigma)$, are the sets (3–4), respectively (3–13), for $p(\sigma)$ then* *the set* $\mathcal{R}_z = \mathcal{R}_{\mathrm{Ht},z} \cup \mathcal{R}_{\mathrm{er},z}$, *defined by*

$$\mathcal{R}_{\mathrm{Ht},z} = \{\sigma \in (V_0(z), +\infty) \mid \exists j \text{ such that } r_j = \tfrac{1}{2}\},$$
$$\mathcal{R}_{\mathrm{er},z} = \{\sigma \in (V_0(z), +\infty) \mid I_{\mathrm{er}}(\sigma) \neq \varnothing\},$$

that is, the set of energies σ *which are either a Hessian threshold (see Lemma 2.7) or such that* $q(\sigma)$ *has a nontrivial effectively resonant error term (see Definition 3.9), is discrete in* $(V_0(\pi(q)), +\infty)$.

Remark 3.16. It follows that if $K \subset (V_0(z), +\infty)$ is compact then $K \cap \mathcal{R}_z$ is finite. Thus, to prove properties such as asymptotic completeness, one can ignore all $\sigma \in K$ which are Hessian thresholds or effectively resonant.

Note also that by the definition of $I_{\mathrm{er}}(\sigma)$,

$$\mathcal{R}_{\mathrm{er},z} = \{\sigma \in (V_0(z), +\infty) \mid \text{either } \exists\ (0, (\alpha', 0, 0), (\beta', 0, 0)) \in I(\sigma) \text{ with } |\beta'| = 1 \\ \text{or } \exists\ (0, (0, \alpha'', 0), (0, \beta'', 0)) \in I(\sigma)\}.$$

Proof. Using Remark 2.9, the set $\mathcal{R}_{\mathrm{Ht},z}$ of Hessian thresholds is given by $\{V_0(z) + 4a_j\}$ where a_j is an eigenvalue of the Hessian of V_0 at z and hence has cardinality at most $n-1$, so this set is trivially discrete.

Let K be a compact subset of $(V_0(z), +\infty)$. The set $K \cap \mathcal{R}_{\mathrm{er},z}$ of effectively resonant energies in K is the union of zeros of a finite number of analytic functions (none of which are identically zero). Indeed, $\mathcal{R}_{\mathrm{er},z}$ is given by the union of the set of zeros of the countable collection of functions

$$-1 + \sum_{j=s}^{m-1} \alpha''_j r''_j(\sigma) + \beta''_j(1 - r''_j(\sigma)), \qquad -1 + (1 - r_k) + \sum_{j=1}^{s-1} \alpha'_j r'_j(\sigma)$$

as $k = 1, \dots, s-1$, while α', α'', β'' are multiindices. But if $c > 0$ is large enough then $c^{-1} > |r_j(\sigma)| > c$ for all j and for all $\sigma \in K$ as K is compact and the r_j do not vanish there. Correspondingly, for $|\alpha'| > 2/c^2$,

$$-1 + (1 - r_k) + \sum_{j=1}^{s-1} \alpha'_j r'_j(\sigma) < -r_k - |\alpha'|c < -c^{-1},$$

and analogously for $|\alpha''| + |\beta''| > 2/c$,

$$-1 + \sum_{j=s}^{m-1} \alpha''_j r''_j(\sigma) + \beta''_j(1 - r''_j(\sigma)) > -1 + (|\alpha''| + |\beta''|)c > 1.$$

Thus, there are only a finite number of these analytic functions that may vanish in K, as claimed. □

If $q(\sigma)$ are the radial points corresponding to $z \in \mathrm{Cv}(V)$, and $\sigma \notin \mathcal{R}_{\mathrm{er},z}$, then we will say that $q(\sigma)$ is effectively nonresonant, or that σ is an effectively nonresonant energy for z. We now prove that, away from effectively resonant energies and Hessian thresholds, we have a normal form for $p(\sigma)$ of the form (3–16) with $r_{\mathrm{er}} = 0$ and depending smoothly on σ. Thus, for a given critical point z of V_0, consider an open interval $O \subset (V_0(z), +\infty) \setminus \mathcal{R}_z$. Apart from the coefficients h_j, $h''_{\alpha'',\beta''}$, etc., in (3–15) the only part of the model form depending on σ is

$$J''(\sigma) = \Big\{(\alpha'', \beta'');\ \sum_{j=s}^{m-1} r_j''(\sigma)\alpha_j'' + (1 - r_j''(\sigma))\beta_j'' \in (1, 2)\Big\}.$$

We note that on compact subsets K of O, there is a $c > 0$ such that $r_j''(\sigma) > c$ for $\sigma \in K$, and then for $|\alpha''| + |\beta''| > 2c^{-1}$,

$$\mathfrak{s}_{\alpha''\beta''}(\sigma) = \sum_{j=s}^{m-1} r_j''(\sigma)\alpha_j'' + (1 - r_j''(\sigma))\beta_j'' > 2,$$

so if we let

$$J_K = \bigcup_{\sigma\in K} J''(\sigma),$$

then J_K is a finite set of multiindices. For each multiindex (α'', β'') we let

$$O_{\alpha'',\beta''} = \mathfrak{s}^{-1}_{\alpha''\beta''}((1, 2)), \tag{3–18}$$

which is thus an open subset of O.

For the parameter dependent version of the Theorem 3.11 we introduce

$$\mathcal{S} = \{(y, \nu, \mu, \sigma);\ \nu = 0,\ \ y'' = 0,\ \ y''' = 0,\ \ \mu = 0,\ \ \sigma \in O\},$$

in place of S (3–12).

Theorem 3.17. *Suppose that $p \in \mathscr{C}^\infty({}^{\mathrm{sc}}T^*_{\partial X}X \times O)$, $O \subset (V_0(z), +\infty) \setminus \mathcal{R}_z$ is open, that the symplectic map S induced by the linearization A' of p at $q(\sigma)$ (see Lemma 2.3) can be smoothly decomposed (as a function of $\sigma \in O$) into two-dimensional invariant symplectic subspaces and that there exists $c > 0$ such that $r_j''(\sigma) > c$ for $\sigma \in O$. Then $\Phi(\sigma)$ and $F(\sigma)$ can be chosen smoothly in σ so that $p_{\mathrm{norm}}(\sigma) = \sigma_1(\widetilde{P}(\sigma))$, $\widetilde{P}(\sigma) = F(\sigma)^{-1}P(\sigma)F(\sigma)$, is of the form in Theorem 3.11, with $r_{\mathrm{er}} \equiv 0$, with the sum over J'' replaced by a locally finite sum (the sum is over J_K over compact subsets $K \subset O$), the h_j, etc., in (3–15) depending smoothly on σ, that is, they are in $\mathscr{C}^\infty({}^{\mathrm{sc}}T^*_{\partial X}X \times O)$, vanishing at $\mathcal{S}$ as in Theorem 3.11, and with the $h''_{\alpha''\beta''}$ supported in ${}^{\mathrm{sc}}T^*_{\partial X}X \times O_{\alpha''\beta''}$ in terms of (3–18).*

Remark 3.18. For $P = x^{-1}(\Delta + V - \sigma)$ the conditions of the theorem are satisfied for any bounded $O = I$ disjoint from the discrete set of effectively resonant σ, since in local coordinates (y, μ) on $\Sigma(\sigma)$, the eigenspaces of S are independent of σ as shown in the proof of Lemma 2.5, and the r_j'' are bounded below by Remark 2.9.

Proof. Since the invariant subspaces depend smoothly on σ by assumption, so do the eigenvalues of the linearization, and there is smooth family of local contact diffeomorphisms, that is, coordinate changes, under which $p(\sigma)$ takes the form (2–7), that is,

$$p(\sigma) = \lambda(\sigma)\Big(- \nu + \sum_{j=1}^{m-1} r_j(\sigma) y_j \mu_j + \sum_{j=m}^{n-1} Q_j(\sigma, y_j, \mu_j) + \nu g_1 + g_2\Big)$$

the $Q_j(\sigma, .)$, are homogeneous polynomials of degree 2, g_1 vanishes at least linearly and g_2 to third order, all depending smoothly on σ.

For the rest of the argument it is convenient to reduce the size of the parameter set O as follows. For $\sigma \in O$, let

$$\widehat{O}(\sigma) = \Bigg(\bigcap_{\substack{(\alpha'',\beta''):\\ \mathfrak{s}_{\alpha'',\beta''}(\sigma)\in(1,2)}} \mathfrak{s}^{-1}_{\alpha'',\beta''}((1,2)) \Bigg) \cap \Bigg(\bigcap_{\substack{(\alpha'',\beta''):\\ \mathfrak{s}_{\alpha'',\beta''}(\sigma)\in(-\infty,1)}} \mathfrak{s}^{-1}_{\alpha'',\beta''}((-\infty,1)) \Bigg), \tag{3–19}$$

an open set (as it is a finite intersection of open sets) that includes σ. Thus, $\{\widehat{O}(\sigma) : \sigma \in O\}$ is an open cover of O. Take a locally finite subcover and a partition of unity subordinate to it. It suffices now to show the theorem for each element $\widehat{O}(\sigma_0)$ of the subcover in place of O, for we can then paste together the models p_{norm} we thus obtain using the partition of unity. Thus, we may assume that $O = \widehat{O}(\sigma_0)$ for some $\sigma_0 \in O$, and prove the theorem with the sum over J'' replaced by a sum over $J''(\sigma_0)$. Hence, on O, for any (α'', β'') either

(a) $\mathfrak{s}_{\alpha''\beta''}(\sigma_0) > 1$, and then for some $(\tilde\alpha'', \tilde\beta'') \in J''(\sigma_0)$, $(\alpha'', \beta'') \geq (\tilde\alpha'', \tilde\beta'')$ (reduce $|\alpha''| + |\beta''|$ until $\mathfrak{s}_{\tilde\alpha'',\tilde\beta''} \in (1, 2)$ – this will happen as $r_j \in (0, 1/2)$) hence $\mathfrak{s}_{\alpha''\beta''}(\sigma) \geq \mathfrak{s}_{\tilde\alpha''\tilde\beta''}(\sigma) > 1$ for all $\sigma \in O$ by the definition of $\widehat{O}(\sigma_0)$, or

(b) $\mathfrak{s}_{\alpha''\beta''}(\sigma_0) < 1$, and then $\mathfrak{s}_{\alpha''\beta''}(\sigma) < 1$ for all $\sigma \in O$ by the definition of $\widehat{O}(\sigma_0)$.

In order to make $\Phi(\sigma)$ smooth in σ, we slightly modify the construction of the local contact diffeomorphism $\Phi_1(\sigma)$ in Proposition 3.6 so that for any given σ we do not necessarily remove every term we can (that is, which are nonresonant for that particular σ). Namely, we choose the set I' of multiindices (a, α, β) *which we do not remove by* $\Phi_1(\sigma)$ so that I' is independent of σ, and such that I' contains every multiindex which is resonant for *some* $\sigma \in O$, that is, $I' \supset \bigcup_{\sigma\in O} I(\sigma)$, with $I(\sigma)$ denoting the set of multiindices corresponding to resonant terms for $p(\sigma)$, as in Proposition 3.6. With any such choice of I', the local contact diffeomorphism of Proposition 3.6, $\Phi_1(\sigma)$, can be chosen smoothly in σ such that $\lambda^{-1}\Phi_1^* p$ is of the form

$$-\nu + \sum_{j=1}^{m} r_j(\sigma) y_j \mu_j + \sum_{j=m+1}^{n-1} Q_j(\sigma, y_j, \mu_j) + \sum_{I'} c_{a\alpha\beta}(\sigma) \nu^a e^\alpha f^\beta \text{ modulo } \mathcal{I}^\infty = \mathfrak{h}^\infty \text{ at } q,$$

with $c_{a\alpha\beta}$ depending smoothly on σ.

The requirement $I' \supset \bigcup_{\sigma\in O} I(\sigma)$ means that for $(a, \alpha, \beta) \notin I'$, $R_{a,\alpha,\beta}(\sigma)$ must not vanish for $\sigma \in O$. Here we recall that $R_{a,\alpha,\beta}(\sigma)$ is the eigenvalue of $\{\{p_0, .\}\}$ defined by (3–3), namely

$$R_{a,\alpha,\beta}(\sigma) = \lambda\Big(a - 1 + \sum_{j=1}^{n-1} \alpha_j r_j(\sigma) + \sum_{j=1}^{n-1} \beta_j (1 - r_j(\sigma))\Big). \tag{3–20}$$

Keeping this in mind, we choose I' by defining its complement $(I')^c$ to consist of multiindices (a, α, β) with $2a + |\alpha| + |\beta| \geq 3$ such that either

(i) $a + |\beta'| = 1$ and $\alpha'' = 0$, $\alpha''' = 0$, $\beta'' = 0$, $\beta''' = 0$, or

(ii) $|\alpha'''| \geq 1$, $\beta''' = 0$, or

(iii) $|\beta'''| \geq 1$, $\alpha''' = 0$, or

(iv) $a = 0$, $\beta' = 0$, $|\alpha'''| + |\beta'''| = 2$, $\alpha'' = 0$, $\beta'' = 0$, or

(v) $a = 0$, $\beta' = 0$, $\alpha''' = \beta''' = 0$, $\mathfrak{s}_{\alpha''\beta''}(\sigma) < 1$ (for one, hence all, $\sigma \in O$, as remarked above).

We next show that multiindices in $(I')^c$ are indeed nonresonant. In cases (ii)–(iii), $\operatorname{Im} R_{a,\alpha,\beta}(\sigma) \neq 0$ since the imaginary part of all terms in (3–20) (with nonzero imaginary part) has the same sign, and there is at least one term with nonzero imaginary part, so (a, α, β) is nonresonant.

In case (v), the nonresonance follows from

$$\lambda^{-1} R_{a,\alpha,\beta}(\sigma) \leq -1 + \mathfrak{s}_{\alpha''\beta''}(\sigma) < 0,$$

since $\lambda^{-1} R_{a,\alpha,\beta}(\sigma) = -1 + \mathfrak{s}_{\alpha''\beta''}(\sigma) + \sum_{j=1}^{s-1} \alpha_j r_j$, and each term in the last summation is nonpositive.

In case (i), if $a = 1$, $\beta' = 0$ then

$$\lambda^{-1} R_{a,\alpha,\beta}(\sigma) = \sum_{j=1}^{s-1} r_j \alpha_j < 0$$

since $|\alpha'| \geq 1$ due to $2a + |\alpha| + |\beta| \geq 3$. Also in case (i), if $a = 0$, $|\beta'| = 1$, with say $\beta_l = 1$, then

$$\lambda^{-1} R_{a,\alpha,\beta}(\sigma) = -r_l + \sum_{j=1}^{s-1} \alpha_j r_j$$

which does not vanish since otherwise (a, α, β) would be effectively resonant – it would correspond to one of the terms in the first summation in (3–14).

Finally, in case (iv),

$$\lambda^{-1} \operatorname{Re} R_{a,\alpha,\beta}(\sigma) = \sum_{j=1}^{s-1} \alpha_j r_j < 0$$

since $\alpha' \neq 0$ due to $2a + |\alpha| + |\beta| \geq 3$.

Thus, all terms corresponding to multiindices in $(I')^c$ can be removed from $p(\sigma)$ by a local contact diffeomorphism $\Phi_1(\sigma)$ that is $\mathscr{C}^\infty$ in σ. So we only need to remark that any term corresponding to a multiindex in I' can be absorbed into $r_{\mathrm{enr}}(\sigma)$. In fact, such a multiindex has either

(i) $a + |\beta'| \geq 2$, or

(ii) $a + |\beta'| = 1$ and $|\alpha''| + |\alpha'''| + |\beta''| + |\beta'''| \geq 1$, or

(iii) $|\alpha'''| + |\beta'''| \geq 3$ (with neither α''' nor β''' zero), or

(iv) $a = 0$, $\beta' = 0$, $|\alpha'''| = 1$, $|\beta'''| = 1$, $|\alpha''| + |\beta''| \geq 1$, or

(v) $a = 0$, $\beta' = 0$, $\alpha''' = 0$, $\beta''' = 0$, $\mathfrak{s}_{\alpha''\beta''} > 1$.

The first two cases can be incorporated into the h_0 or h_j terms in (3–15). The third and fourth ones can be incorporated into the h'''_{jk} term. Finally, in the fifth case, any infinite linear combination of these monomials can be written as

$$\sum_{(\tilde{\alpha}'',\tilde{\beta}'') \in J''(\sigma_0)} h''_{\tilde{\alpha}'',\tilde{\beta}''} (e'')^{\tilde{\alpha}''} (f'')^{\tilde{\beta}''},$$

as remarked in (a) after (3–19).

We thus obtain

$$\lambda(\sigma)\Big(-\nu + \sum_j r_j(\sigma) y_j \mu_j + \sum_{j=m}^{n-1} Q_j(y_j, \mu_j) + r_{\mathrm{enr}}(\sigma) + r_\infty \Big),$$

with r_{enr} as in (3–15), and r_∞ vanishes to infinite order at (0, 0, 0). Finally, we can remove the r_∞ term in a neighbourhood of the origin, smoothly in σ, using Proposition 3.7 as in the proof of Theorem 3.11, thus completing the proof of this theorem. □

4. Microlocal solutions

In [Hassell et al. 2004, Equation (0.15)] microlocally outgoing solutions were defined using the global function ν on ${}^{\text{sc}}T^*_{\partial X}X$. This is increasing along W and plays the role of a time function; microlocally incoming and outgoing solution are then determined by requiring the wave front set to lie on one side of a level surface of ν. In the present study of microlocal operators, no such global function is available. However there are always microlocal analogues, denoted here by ρ, defined in appropriate neighbourhoods of a critical point.

Lemma 4.1. *There is a neighbourhood $\mathbb{O}_1$ of q in ${}^{\text{sc}}T^*_{\partial X}X$ and a function $\rho \in \mathscr{C}^\infty(\mathbb{O}_1)$ such that $\mathbb{O}_1$ contains no radial point of P except q, $\rho(q) = 0$, and $W\rho \geq 0$ on $\Sigma \cap \mathbb{O}$ with $W\rho > 0$ on $\Sigma \cap \mathbb{O}_1 \setminus \{q\}$.*

Proof. This follows by considering the linearization of W. Namely, if P is conjugated to the form (2–7), then for outgoing radial points q take $\rho = |y'|^2 - (|y''|^2 + |y'''|^2 + |\mu|^2)$, defined in a coordinate neighbourhood $\mathbb{O}_0$, for incoming radial points take its negative. On Σ, $W\rho \geq c(|y|^2 + |\mu|^2) + h$ for some $c > 0$ and $h \in \mathcal{I}^3$. As (y, μ) form a coordinate system on Σ near q, it follows that $W\rho \geq (c/2)(|y|^2 + |\mu|^2)$ on a neighbourhood $\mathbb{O}'$ of q in Σ. Now let $\mathbb{O}_1 \subset \mathbb{O}_0$ be such that $\mathbb{O} \cap \Sigma = \mathbb{O}'$. Then $W\rho(p) = 0$, $p \in \mathbb{O}_1$, implies $p = q$, so there are indeed no other radial points in $\mathbb{O}_1$, finishing the proof. □

Remark 4.2. Below it is convenient to replace $\mathbb{O}_1$ by a smaller neighbourhood $\mathbb{O}$ of q with $\overline{\mathbb{O}} \subset \mathbb{O}_1$, so ρ is defined and increasing on a neighbourhood of $\overline{\mathbb{O}}$.

Consider the structure of the dynamics of W in $\mathbb{O}$. First, ρ is increasing (that is, "nondecreasing") along integral curves γ of W, and it is strictly increasing unless γ reduces to q. Moreover, W has no nontrivial periodic orbits and

Lemma 4.3. *Let $\mathbb{O}$ be as in Remark 4.2. If $\gamma : [0, T) \to \mathbb{O}$ or $\gamma : [0, +\infty) \to \mathbb{O}$ is a maximally forward-extended bicharacteristic, then either γ is defined on $[0, +\infty)$ and $\lim_{t\to+\infty} \gamma(t) = q$, or γ is defined on $[0, T)$ and leaves every compact subset K of $\mathbb{O}$, that is, there is $T_0 < T$ such that for $t > T_0$, $\gamma(t) \notin K$.*

An analogous conclusion holds for maximally backward-extended bicharacteristics.

Proof. If $\gamma : [0, +\infty) \to \mathbb{O}$ then $\lim_{t\to+\infty} \rho(\gamma(t)) = \rho_+$ exists by the monotonicity of ρ, and any sequence $\gamma_k : [0, 1] \to \Sigma$, $\gamma_k(t) = \gamma(t_k + t)$, $t_k \to +\infty$, has a uniformly convergent subsequence, which is then an integral curve $\tilde{\gamma}$ of W in Σ with image in $\overline{\mathbb{O}}$, hence in $\mathbb{O}_1$ along which ρ is constant. The only such bicharacteristic segment is the one with image $\{q\}$, so $\lim_{t\to+\infty} \gamma(t) = q$. The claim for γ defined on $[0, T)$ is standard. □

As in [Hassell et al. 2004] we make use of open neighbourhoods of the critical points which are well-behaved in terms of W.

Definition 4.4. By a W-balanced neighbourhood of a nondegenerate radial point q we shall mean a neighbourhood, O, of q in ${}^{\text{sc}}T^*_{\partial X}X$ with $\overline{O} \subset \mathbb{O}$ (in which ρ is defined) such that O contains no other radial point, meets $\Sigma(\sigma) \cap O$ in a W-convex set (that is, each integral curve of W meets $\Sigma(\sigma)$ in a single interval, possibly empty) and is such that the closure of each integral curve of W in O meets $\rho = \rho(q)$.

The existence of W-balanced neighbourhoods follows as in [Hassell et al. 2004, Lemma 1.8].

If q is a radial point for P and O a W-balanced neighbourhood of q we set

$$\widetilde{E}_{\mathrm{mic},+}(O, P) = \big\{u \in \mathscr{C}^{-\infty}(X);\ O \cap \mathrm{WF}_{\mathrm{sc}}(Pu) = \varnothing, \text{ and } \mathrm{WF}_{\mathrm{sc}}(u) \cap O \subset \{\rho \geq \rho(q)\}\big\}, \quad (4\text{–}1)$$

with $\widetilde{E}_{\mathrm{mic},-}(O, P)$ defined by reversing the inequality.

Lemma 4.5. *If $O \ni q$ is a W-balanced neighbourhood then every $u \in \tilde{E}_{\mathrm{mic},\pm}(O, P)$ satisfies $\mathrm{WF}_{sc}(u) \cap O \subset \Phi_{\pm}(\{q\})$; furthermore, for $u \in \widetilde{E}_{\mathrm{mic},\pm}(O, P)$*

$$\mathrm{WF}_{sc}(u) \cap O = \varnothing \iff q \notin \mathrm{WF}_{sc}(u).$$

Thus, we could have defined $\widetilde{E}_{\mathrm{mic},\pm}(O, P)$ by strengthening the restriction on the wavefront set to $\mathrm{WF}_{\mathrm{sc}}(u) \cap O \subset \Phi_{\pm}(\{q\})$. With such a definition there is no need for O to be W-balanced; the only relevant bicharacteristics would be those contained in $\Phi_{\pm}(\{q\})$. Moreover, with this definition ρ does not play any role in the definition, so it is clearly independent of the choice of ρ.

Proof. For the sake of definiteness consider $u \in \tilde{E}_{\mathrm{mic},+}(O, P)$; the other case follows similarly. Suppose $\zeta \in O \setminus \{q\}$. If $\rho(\zeta) < \rho(q)$, then $\zeta \notin \mathrm{WF}_{\mathrm{sc}}(u)$ by the definition of $\widetilde{E}_{\mathrm{mic},+}(O, P)$, so we may suppose that $\rho(\zeta) \geq \rho(q)$. Since $q \in \Phi_{+}(\{q\})$ we may also suppose that $\zeta \neq q$.

Let $\gamma : \mathbb{R} \to \Sigma$ be the bicharacteristic through ζ with $\gamma(0) = \zeta$. As O is W-convex, and $\mathrm{WF}_{\mathrm{sc}}(Pu) \cap O = \varnothing$, the analogue here of Hörmander's theorem on the propagation of singularities shows that

$$\zeta \in \mathrm{WF}_{\mathrm{sc}}(u) \Rightarrow \gamma(\mathbb{R}) \cap O \subset \mathrm{WF}_{\mathrm{sc}}(u).$$

As O is W-balanced, there exists $\zeta' \in \overline{\gamma(\mathbb{R})} \cap O$ such that $\rho(\zeta') = \rho(q)$. If $\rho(\zeta) = \rho(q) = 0$, we may assume that $\zeta' = \zeta$. From this assumption, and the fact that ρ is increasing along the segment of γ in $\mathbb{O}$, and O is W-convex, we conclude that $\zeta' \in \overline{\gamma((-\infty, 0])} \cap O$.

If $\zeta' = \gamma(t_0)$ for some $t_0 \in \mathbb{R}$, then for $t < t_0$, $\rho(\gamma(t)) < \rho(\gamma(t_0)) = \rho(q)$, and for sufficiently small $|t - t_0|$, $\gamma(t) \in O$ as O is open. Thus, $\gamma(t) \notin \mathrm{WF}_{\mathrm{sc}}(u)$ by the definition of $\widetilde{E}_{\mathrm{mic},+}(O, P)$, and hence we deduce that $\zeta \notin \mathrm{WF}_{\mathrm{sc}}(u)$.

On the other hand, if $\zeta' \notin \gamma(\mathbb{R})$, then as O is open $\gamma(t_k) \in O$ for a sequence $t_k \to -\infty$, and as O is W-convex, $\gamma|_{(-\infty,0]} \subset O$. Then, again from the propagation of singularities and Lemma 4.3, $\zeta' = q$. □

We may consider $\widetilde{E}_{\mathrm{mic},\pm}(O, P)$ as a space of microfunctions, $E_{\mathrm{mic},+}(q, P)$, by identifying elements which differ by functions with wavefront set not meeting O:

$$E_{\mathrm{mic},\pm}(q, P) = \widetilde{E}_{\mathrm{mic},\pm}(O, P)/\{u \in \mathscr{C}^{-\infty}(X);\ \mathrm{WF}_{\mathrm{sc}}(u) \cap O = \varnothing\}.$$

The result is then independent of the choice of O, as we show presently.

If O_1 and O_2 are two W-balanced neighbourhoods of q then

$$O_1 \subset O_2 \Longrightarrow \widetilde{E}_{\mathrm{mic},\pm}(O_2, P) \subset \widetilde{E}_{\mathrm{mic},\pm}(O_1, P). \quad (4\text{–}2)$$

Since $\{u \in \mathscr{C}^{-\infty}(X);\ \mathrm{WF}_{\mathrm{sc}}(u) \cap O = \varnothing\} \subset \tilde{E}_{\mathrm{mic},\pm}(O, P)$ for all O and this linear space decreases with O, the inclusions (4–2) induce similar maps on the quotients

$$\begin{gathered} E_{\mathrm{mic},\pm}(O, P) = \widetilde{E}_{\mathrm{mic},\pm}(O, P)/\{u \in \mathscr{C}^{-\infty}(X);\ \mathrm{WF}_{\mathrm{sc}}(u) \cap O = \varnothing\}, \\ O_1 \subset O_2 \Longrightarrow E_{\mathrm{mic},\pm}(O_2, P) \longrightarrow E_{\mathrm{mic},\pm}(O_1, P). \end{gathered} \quad (4\text{–}3)$$

Lemma 4.6. *Provided O_i, for $i = 1, 2$, are W-balanced neighbourhoods of q, the map in (4–3) is an isomorphism.*

Proof. We work with $E_{\text{mic},+}$ for the sake of definiteness.

The map in (4–3) is injective since any element u of its kernel has a representative $\tilde{u} \in \widetilde{E}_{\text{mic},+}(O_2, \sigma)$ which satisfies $q \notin \text{WF}_{\text{sc}}(\tilde{u})$, hence $\text{WF}_{\text{sc}}(\tilde{u}) \cap O_2 = \varnothing$ by Lemma 4.5, so $u = 0$ in $E_{\text{mic},+}(O_2, \sigma)$.

The surjectivity follows from Hörmander's existence theorem in the real principal type region [1971]. First, note that

$$R = \inf\{\rho(p) : p \in \Phi_+(\{q\}) \cap (\mathbb{O} \setminus O_1)\} > 0 = \rho(q)$$

since in $\mathbb{O}$, ρ is increasing along integral curves of W, and strictly increasing away from q. Let U be a neighbourhood of $\Phi_+(\{q\}) \cap \overline{O_1}$ such that $\overline{U} \subset \mathbb{O}$, and $\rho > R_0 = R/2$ on $U \setminus O_1$. Let $A \in \Psi_{\text{sc}}^{-\infty,0}(\mathbb{O})$ be such that $\text{WF}'_{\text{sc}}(\text{Id} - A) \cap \overline{O_1} \cap \Phi_+(\{q\}) = \varnothing$ and $\text{WF}'_{\text{sc}}(A) \subset U$. Thus, $\text{WF}_{\text{sc}}(Au) \subset U$ and $\text{WF}_{\text{sc}}(PAu) \subset U \setminus O_1$, so in particular $\rho > R_0$ on $\text{WF}_{\text{sc}}(PAu)$. We have thus found an element, namely $\tilde{u} = Au$, of the equivalence class of u with wave front set in $\mathbb{O}$ and such that $\rho > R_0 > 0 = \rho(q)$ on the wave front set of the "error", $P\tilde{u}$.

The forward bicharacteristic segments from $U \setminus O_1$ inside $\mathbb{O}$ leave $\overline{O_2}$ by the remark after Lemma 4.1; since $\overline{O_2} \setminus O_1$ is compact, there is an upper bound $T > 0$ for when this happens. Thus, Hörmander's existence theorem allows us to solve $Pv = P\tilde{u}$ on O_2 with $\text{WF}_{\text{sc}}(v)$ a subset of the forward bicharacteristic segments emanating from $U \setminus \overline{O_1}$. Then $u' = \tilde{u} - v$ satisfies $\text{WF}_{\text{sc}}(u') \subset \mathbb{O} \cap \{\rho \geq 0 = \rho(q)\}$, $\text{WF}_{\text{sc}}(Pu') \cap O_2 = \varnothing$, so $u' \in E_{\text{mic},+}(O_2, P)$, and $q \notin \text{WF}_{\text{sc}}(u' - u)$. Thus $\text{WF}_{\text{sc}}(u' - u) \cap O_1 = \varnothing$, hence u and u' are equivalent in $\widetilde{E}_{\text{mic},+}(O_1, P)$. This shows surjectivity. □

It follows from this Lemma that the quotient space $E_{\text{mic},\pm}(q, P)$ in (4–3) is well-defined, as the notation already indicates, and each element is determined by the behaviour microlocally "at" q. When P is the operator $x^{-1}(\Delta + V - \sigma)$, then we will denote this space

$$E_{\text{mic},\pm}(q, \sigma). \tag{4–4}$$

Definition 4.7. By a *microlocally outgoing solution* to $Pu = 0$ at a radial point q we shall mean either an element of $\widetilde{E}_{\text{mic},+}(O, P)$, where O is a W-balanced neighborhood of q, or of $E_{\text{mic},+}(q, P)$.

5. Test modules

Following Part I, [Hassell et al. 2004], we use test modules of pseudodifferential operators to analyze the regularity of microlocally incoming solutions near radial points. This involves microlocalizing near the critical point with errors which are well placed relative to the flow. For readers comparing this discussion to Part I, we mention that the microlocalizer Q in the following definition corresponds to the microlocalizer Q in Equation (6.27) of Part I; the orders in the commutator are different as now we are working with $P \in \Psi_{\text{sc}}^{*,-1}(X)$.

Definition 5.1. An element $Q \in \Psi_{\text{sc}}^{*,0}(X)$ is a *forward microlocalizer* in a neighbourhood $O \ni q$ of a radial point $q \in {}^{\text{sc}}T^*_{\partial X}X$ for $P \in \Psi_{\text{sc}}^{*,-1}(X)$ if it is elliptic at q and there exist $B, F \in \Psi_{\text{sc}}^{0,0}(O)$ and $G \in \Psi_{\text{sc}}^{0,1}(X)$ such that

$$i[Q^*Q, P] = (B^*B + G) + F \text{ and } \text{WF}'_{\text{sc}}(F) \cap \Phi_+(\{q\}) = \varnothing. \tag{5–1}$$

Using the normal form established earlier we can show that such forward microlocalizers exist under our standing assumption that

$$\text{the linearization has neither a Hessian threshold subspace, (iv),} \quad \text{nor any nondecomposable 4-dimensional invariant subspace.} \tag{5–2}$$

Proposition 5.2. *A forward microlocalizer exists in any neighbourhood of any nondegenerate outgoing radial point $q \in {}^{sc}T^*_{\partial X}X$ for $P \in \Psi^{*,-1}_{sc}(X)$ at which the linearization satisfies (5–2).*

Proof. Since the conditions (5–1) are microlocal and invariant under conjugation with an elliptic Fourier integral operator, it suffices to consider the model form in Theorem 3.11 which holds under the same conditions (5–2).

Let $R = |\mu'|^2 + |y''|^2 + |y'''|^2 + |\mu''|^2 + |\mu'''|^2$, and

$$S = \{p_{\text{norm}} = 0,\ \ R = 0\},$$

so S is the flow-out of q. We shall choose $Q \in \Psi^{-\infty,0}_{sc}(X)$ such that

$$\sigma_\partial(Q) = q = \chi_1(|y'|^2)\chi_2(R)\psi(p_{\text{norm}}),$$

where $\chi_1, \chi_2, \psi \in \mathscr{C}^\infty_c(\mathbb{R})$, $\chi_1, \chi_2 \geq 0$ are supported near 0, ψ is supported near 0, $\chi_1, \chi_2 \equiv 1$ near 0 and $\chi_1' \leq 0$ in $[0, \infty)$. Choosing all supports sufficiently small ensures that $Q \in \Psi^{-\infty,0}_{sc}(O)$. Note that $\operatorname{supp} d(\chi_2 \circ R) \cap S = \varnothing$. On the other hand,

$${}^{sc}H_p\chi_1\Big(\sum_j (y_j')\Big)^2) = 2\sum_j y_j'({}^{sc}H_p y_j')\chi_1'(|y'|^2) = 2\lambda y_j'(r_j' y_j' + h_j)\chi_1'(|y'|^2),$$

with h_j vanishing quadratically at q. Moreover, on $\operatorname{supp}\chi_1' \circ (|.|^2)$, y' is bounded away from 0. Since $r_j' < 0$, $-\sum_j r_j'(y_j')^2 > 0$ on $\operatorname{supp}\chi_1' \circ (|.|^2)$. The error terms h_j can be estimated in terms of $|y'|^2$, R and p^2_{norm}, so, given any $C > 0$, there exists $\delta > 0$ such that the $-\sum_j y_j'(r_j' y_j' + h_j) > 0$ if $\operatorname{supp}\chi_1 \subset (-\delta, \delta)$, $R/|y'|^2 < C$ and $|p_{\text{norm}}|/|y'| < C$. In particular, taking $C = 2$, $-\sum_j y_j'(r_j' y_j' + h_j) > 0$ on $S \cap \operatorname{supp}\chi_1' \circ (|.|^2)$, for $R = p_{\text{norm}} = 0$ on S. Thus (5–1) is satisfied (with B appropriately specified, microsupported near S), provided that χ_1 is chosen so that $(-\chi_1\chi_1')^{1/2}$ is smooth.

More explicitly, letting $\chi \in \mathscr{C}^\infty_c(\mathbb{R})$ be supported in $(-1, 1)$ be identically equal to 1 in $(-\frac{1}{2}, \frac{1}{2})$ with $\chi' \leq 0$ on $[0, \infty)$, $\chi \geq 0$, $\chi_1 = \chi_2 = \psi = \chi(./\delta)$. Indeed, for any choice of $\delta \in (0, 1)$, $|y'|^2 \geq \delta/2$ on $\operatorname{supp}\chi_1' \circ |.|^2$, hence $R/|y'|^2 < 2$, $|p_{\text{norm}}|/|y'| < 2$ on $\operatorname{supp} q \cap \operatorname{supp}\chi_1' \circ |.|^2$. With $C = 2$, choosing $\delta \in (0, 1)$ as above, we can write

$$\sigma_\partial(i[Q^*Q, P]) = -\,{}^{sc}H_p q^2 = -4\lambda\tilde{b}^2 + \tilde{f},$$

$$\tilde{b} = \Big(\sum_j y_j'(r_j' y_j' + h_j)\chi_1'(|y'|^2)\chi_1(|y'|^2)\Big)^{1/2}\chi_2(R)\psi(p_{\text{norm}}),\ \operatorname{supp}\tilde{f} \cap S = \varnothing,$$

which finishes the proof since $\lambda < 0$ for an outgoing radial point. □

A test module in an open set $O \subset {}^{sc}T^*_{\partial X}X$ is, by definition, a linear subspace $\mathcal{M} \subset \Psi^{*,-1}_{sc}(X)$ consisting of operators microsupported in O which contains and is a module over $\Psi^{*,0}_{sc}(X)$, is closed under commutators, and is algebraically finitely generated. To deduce regularity results we need extra conditions relating the module to the operator P.

Definition 5.3. If $P \in \Psi_{sc}^{*,-1}(X)$ has real principal symbol near a nondegenerate outgoing radial point q then a test module $\mathcal{M}$ is said to be P-positive at q if it is supported in a W-balanced neighbourhood of q and

(i) $\mathcal{M}$ is generated by $A_0 = \mathrm{Id}, A_1, \dots, A_N = P$ over $\Psi_{sc}^{*,0}(X)$,

(ii) for $1 \le i \le N-1$, $0 \le j \le N$ there exists $C_{ij} \in \Psi_{sc}^{*,0}(X)$, such that

$$i[A_i, xP] = \sum_{j=0}^{N} xC_{ij}A_j \tag{5–3}$$

where $\sigma_\partial(C_{ij})(\tilde{q}) = 0$, for all $0 \neq j < i$, and $\mathrm{Re}\,\sigma_\partial(C_{jj})(\tilde{q}) \ge 0$.

As shown in [Hassell et al. 2004], microlocal regularity of solutions of a pseudodifferential equation can be deduced by combining such a P-positive test module with a microlocalizing operator as discussed above. We recall and slightly modify this result.

Proposition 5.4 (Essentially Proposition 6.7 of [Hassell et al. 2004]; see Proposition A.1 below for a slightly modified statement and a corrected proof). *Suppose that* $P \in \Psi_{sc}^{*,-1}(X)$ *has real principal symbol*, q *is a nondegenerate outgoing radial point for* P,

$$\sigma_{\partial,1}(xP - (xP)^*)(q) = 0, \tag{5–4}$$

$\mathcal{M}$ *is a* P*-positive test module at* q, $Q, Q' \in \Psi_{sc}^{*,0}(X)$ *are forward microlocalizers for* P *at* q *with* $\mathrm{WF}'_{sc}(Q')$ *being a subset of the elliptic set of* Q. *Finally suppose that for some* $s < -\frac{1}{2}$, $u \in H_{sc}^{\infty,s}(X)$ *satisfies*

$$\mathrm{WF}_{sc}(u) \cap O \subset \Phi_+(\{q\}) \text{ and } Pu \in \dot{C}^\infty(X). \tag{5–5}$$

Then $u \in I_{sc}^{(s)}(O', \mathcal{M})$ *where* O' *is the elliptic set of* Q'.

Proof. As already noted this is essentially Proposition 6.7 of [Hassell et al. 2004], with a small change to the statement and the proof given in Proposition A.1 below. However, there are some small differences to be noted. In Part I (and here in the Appendix), the condition in (5–3) was $j > i$; here we changed to $j < i$ for a more convenient ordering. Since the labelling is arbitrary, this does not affect the proof of the Proposition.

Also, in Part I the proposition was stated for the 0–th order operators such as $\Delta + V - \sigma$, which are formally self-adjoint with respect to a scattering metric. This explains the appearance of xP both in (5–4) and in (5–3) here, even though in the applications below, $[A_i, x]$ could be absorbed in the C_{i0} term. In particular, $s < -1/2$ in (5–5) arises from a pairing argument that uses the formal self-adjointness of xP, modulo terms that can be estimated by $[x^s A^\alpha, xP]$, $s > 0$, A^α a product of the A_j.

The proposition in Part I is proved with (5–4) replaced by $(xP) = (xP)^*$, but (5–4) is sufficient for all arguments to go through, since $B = (xP) - (xP)^*$ would contribute error terms of the form $x^s A^\alpha B$ with $\sigma_{\partial,1}(B)(q) = 0$, which can thus be handled exactly the same way as the C_{jj} term in (5–3).

In fact (5–4) can always be arranged for any $P_0 \in \Psi_{sc}^{*,-1}(X)$ with a nondegenerate radial point and real principal symbol. Indeed, we only need to conjugate by x^k giving

$$P = x^k P_0 x^{-k}, \;\; k = \frac{-\sigma_{\partial,1}(B)(q)}{2i\lambda} \in \mathbb{R}$$

satisfies (5–4); here $dp|_q = \lambda\alpha|_q$, with α the contact form. Microlocal solutions $P_0u_0 = 0$, correspond to microlocal solutions $Pu = 0$ via $u = x^k u_0$, so $u \in H^{\infty,s}_{\mathrm{sc}}(X)$ is replaced by $u_0 \in H^{\infty,s-k}_{\mathrm{sc}}(X)$. □

Thus, iterative regularity with respect to the module essentially reduces to showing that the positive commutator estimates (5–3) hold. For each critical point q satisfying (5–2) a suitable (essentially maximal) module is constructed below, so microlocally outgoing solutions to $Pu = 0$ have iterative regularity under the module; that is, that

$$u \in I^{(s)}_{\mathrm{sc}}(O, \mathcal{M}) = \{u;\ \mathcal{M}^m u \subset H^{\infty,s}_{\mathrm{sc}}(X) \text{ for all } m\}. \tag{5–6}$$

The test modules are elliptic off the forward flow out $\Phi_+(q)$ which is an isotropic submanifold of Σ. Thus, it is natural to expect that u is some sort of an isotropic distribution. In fact the flow out (in the model setting just the submanifold S) has nonstandard homogeneity structure, so these distributions are more reasonably called "anisotropic".

First we construct a test module for the model operator when there are no resonant terms. Thus, we can assume that the principal symbol is

$$p_0 = \lambda\Big(-\nu + \sum_{j=1}^{m-1} r_j y_j \mu_j + \sum_{j=m}^{n-1} Q_j(y_j, \mu_j)\Big).$$

Then let $\mathcal{M}$ be the test module generated by Id and operators with principal symbols

$$x^{-1} f_j',\quad x^{-r_j''} e_j'',\quad x^{-(1-r_j'')} f_j'',\quad x^{-1/2} e_j''',\quad x^{-1/2} f_j''' \quad \text{and} \quad x^{-1} p_0 \tag{5–7}$$

over $\Psi^{*,0}_{\mathrm{sc}}(X)$.

Note that the order of the generators is given by the negative of the normalized eigenvalue (that is, the eigenvalue in Lemma 2.7 divided by λ) subject to the conditions that if the order would be < -1, it is adjusted to -1, and if it would be > 0, it is omitted. The latter restrictions conform to our definition of a test module, in which all terms of order 0 are included and there are no terms of order less than -1. These orders can be seen to be optimal (that is, most negative) by a principal symbol calculation) of the commutator with A in which the corresponding eigenvalue arises.

Lemma 5.5. *Suppose P is nonresonant at q. Then the module $\mathcal{M}$ generated by (5–7) is closed under commutators and satisfies condition (5–3).*

Proof. It suffices to check the commutators of generators to show that $\mathcal{M}$ is closed. In view of (2–3) (applied with a in place of p), $\{a, b\} = {}^{\mathrm{sc}}H_a b$, this can be easily done. Property (5–3) follows readily from (3–1). Indeed, we have the stronger property

$$i[A_i, P(\sigma)] = c_i A_i + G_i,\ \ G_i \in \Psi^{*,0}(X),\ \ \mathrm{Re}\, c_i \geq 0$$

where A_i is any of the generators of $\mathcal{M}$ listed in (5–7). □

Remark 5.6. We may take generators of $\mathcal{M}$ to be the operators

$$\begin{gathered} D_{y_j'},\quad x^{-r_j''} y_j'',\quad x^{r_j''} D_{y_j''},\ x^{-1/2} y_j''',\quad x^{1/2} D_{y_j'''} \text{ and} \\ x D_x + \sum_{j=1}^{m-1} r_j y_j D_{y_j} + \sum_{j=m}^{n-1} Q_j(x^{-1/2} y_j, x^{1/2} D_{y_j}). \end{gathered} \tag{5–8}$$

Combining this with Proposition 5.4 proves that, in the nonresonant case, if u is a microlocal solution at q, and if $\mathrm{WF}^s_{\mathrm{sc}}(u)$ is a subset of the W-flowout of q, then $u \in I^{(s)}_{\mathrm{sc}}(O, \mathcal{M})$ for all $s < -1/2$.

The discrepancy between the "resonance order" of polynomials in $\nu^a e^\beta f^\gamma$, given by $a + \sum_j \beta_j r_j + \sum_k \gamma_k(1 - r_k)$ and the "module order" given by the sum of the orders of the corresponding module elements is closely related to arguments which allow us to regard most resonant terms as "effectively nonresonant". To give an explicit example, take a resonant term of the form $y'_i \mu'_j (y'')^{\beta''}$, corresponding to a term like $x^{-1} y'_i (y'')^{\beta''} (x D_{y'_j})$ in P. Resonance requires that $r'_i + (1 - r'_j) + \sum_k \beta''_k r''_k = 1$ and $|\beta''| > 0$. In the module, this corresponds to a product of module elements with an additional factor of x^ϵ *with* $\epsilon > 0$, since we can write it

$$x^\epsilon y'_i \prod_k (x^{-r''_k} y''_k)^{\beta''_k} D_{y'_i}, \quad \epsilon = \sum_k \beta''_k r''_k > 0.$$

Since, by Proposition 5.4, the eigenfunction u remains in $x^s L^2(X)$, for all $s < -1/2$, under application of products of elements of $\mathcal{M}$, this term applied to u yields a factor x^ϵ, and therefore it can be treated as an error term in determining the asymptotic expansion of u; see the proof of Theorem 6.7. Only the terms with the module order equal to the resonance order affect the expansion of u to leading order, and it is these we have labelled "effectively resonant".

Next we consider the general resonant case. To do so, we need to enlarge the module $\mathcal{M}$ so that certain products of the generators of $\mathcal{M}$, such as those in the resonant terms of Theorem 3.11, are also included in the larger module $\widetilde{\mathcal{M}}$. For a simple example, see [Hassell et al. 2004, Section 8]. It is convenient to replace P_0 by $x D_x$ as the last generator of $\mathcal{M}$ listed in (5–8), though this is not necessary; all arguments below can be easily modified if this is not done. Let us denote the generators of $\mathcal{M}$ by

$$A_0 = \mathrm{Id},\ A_1 = x^{-s_1} B_1, \dots, A_{N-1} = x^{-s_{N-1}} B_{N-1},\ A_N = x D_x = x^{-1} B_N,$$
$$s_i = -\operatorname{order}(A_i),\ B_i \in \Psi^{-\infty,0}_{\mathrm{sc}}(O).$$

Note that for each $i = 1, \dots, N$, $d\sigma_{\partial,0}(B_i)$ is an eigenvector of the linearization of W; we denote the eigenvalue by σ_i. Thus,

$$s_i = \min(1, \sigma_i) > 0 \text{ for } i = 1, \dots, N.$$

For any multiindex $\alpha \in \mathbb{N}^N$ (with $\mathbb{N} = \{1, 2, \dots\}$) let

$$s(\alpha) = \min\Big(\sum s_i \alpha_i, 1\Big),\ \tilde{s}(\alpha) = \sum_i s_i \alpha_i - s(\alpha) = \max\Big(0, \sum_i s_i \alpha_i - 1\Big),$$

and let

$$A^\alpha = A_1^{\alpha_1} A_2^{\alpha_2} \dots A_N^{\alpha_N}.$$

Let e_i be the multiindex $e_i = (0, \dots, 0, 1, 0, \dots, 0)$, where the 1 is in the i–th slot, if $i = 1, \dots, N$, and let $e_0 = (0, \dots, 0)$.

To deal with resonant terms, we define a module $\mathcal{M}_k$ generated (over $\Psi^{-\infty,0}_{\mathrm{sc}}(O)$) by the operators

$$x^{\tilde{s}(\alpha)} A^\alpha \in \Psi^{-\infty,-s(\alpha)}_{\mathrm{sc}}(O),\ |\alpha| \le k. \tag{5–9}$$

Note that $\alpha = 0$ gives Id as one of the generators. Thus, the order of the generators in (5–9) is "truncated" so that it is always between 0 and -1; in particular $\mathcal{M}_k \subset \Psi^{-\infty,-1}_{\mathrm{sc}}(O)$. Since in computations below we will think of $\Psi_{sc}^{-\infty,0}(O)$ as the submodule of $\mathcal{M}_k$ consisting of trivial elements, it is convenient to work modulo such terms, so we use what is essentially the principal symbol equivalence relation on $\mathcal{M}_k$ where $P \sim Q$ if $P - Q \in \Psi_{sc}^{-\infty,0}(O)$.

While it appears that the ordering in the factors in the product A^α matters, this is not the case. Indeed, if σ is a permutation of $\{1, \dots, |\alpha|\}$, and $j : \{1, \dots, |\alpha|\} \mapsto \{1, \dots, N\}$ which takes α_m-times the value m, $m = 1, \dots, N$, then

$$x^{\tilde{s}(\alpha)} A_{j(1)} \dots A_{j(|\alpha|)} \sim x^{\tilde{s}(\alpha)} A_{j(\sigma(1))} \dots A_{j(\sigma(|\alpha|))},$$

for this is clear if σ interchanges n and $n+1$, as

$$x^{\tilde{s}(\alpha)} A_{j(1)} \dots A_{j(n-1)}[A_{j(n)}, A_{j(n+1)}] A_{j(n+2)} \dots A_{j(|\alpha|)} \\ \in \Psi_{sc}^{-\infty,\tilde{s}(\alpha)+1-\sum s_i\alpha_i}(O) \subset \Psi_{sc}^{-\infty,0}(O)$$

since $\tilde{s}(\alpha) + 1 - \sum_i s_i\alpha_i = 1 - s(\alpha) \geq 0$.

In addition, for $Q \in \Psi_{sc}^{-\infty,0}(O)$,

$$x^{\tilde{s}(\alpha)} Q A_{j(1)} \dots A_{j(|\alpha|)} \sim x^{\tilde{s}(\alpha)} A_{j(1)} \dots A_{j(m)} Q A_{j(m+1)} \dots A_{j(|\alpha|)}.$$

Similarly, one can shift powers of x from in front of the product to in between factors, so in fact the generators can be written equivalently, modulo $\Psi_{sc}^{-\infty,0}(O)$, as

$$x^{s(\alpha)} B^\alpha \in \Psi_{sc}^{-\infty,-s(\alpha)}(O), \ |\alpha| \leq k, \tag{5–10}$$

where $B^\alpha = B_1^{\alpha_1} \dots B_N^{\alpha_N}$.

Moreover, there is an integer J such that $\mathcal{M}_k = \mathcal{M}_J$ if $k \geq J$; indeed this is true for any $J \geq 2(r_s'')^{-1}$, where r_s'' is the smallest positive eigenvalue of the operator in Lemma 2.5 (or $J \geq 4$ if no eigenvalue lies in $(0, \frac{1}{2}]$), since then adding new elements to the product simply has the effect of multiplying by an element of $\Psi_{sc}^{*,0}(X)$.

In particular, note that the generators in (5–9) or (5–10) are usually not linearly independent: some B_{α_j} may be absorbable into a $\Psi_{sc}^{*,0}(O)$ factor without affecting $s(\alpha)$. We could easily give a linearly independent (over $\Psi_{sc}^{*,0}(O)$) subset of the generators, but this is of no importance here.

Suppose that $\widetilde{P}$, the normal operator for $P(\sigma)$ at q, contains resonant terms. Then Lemma 5.5 is replaced by:

Lemma 5.7. *Let $>$ be a total order on multiindices α satisfying*

(i) $|\alpha'| > |\alpha|$ *implies* $\alpha' > \alpha$;

(ii) $|\alpha'| = |\alpha|$ *and* $\sum_k s_k\alpha_k' > \sum_k s_k\alpha_k$ *imply* $\alpha' > \alpha$;

(iii) $|\alpha'| = |\alpha| = 1$, $\alpha' = e_i$, $\alpha = e_j$, $s_i = s_j = 1$, $\sigma_i > \sigma_j$ *imply that* $\alpha' > \alpha$.

With the corresponding ordering of the generators $x^{-\tilde{s}(\alpha)}A^\alpha$, the module $\mathcal{M}_J$ is a test module for $\widetilde{P}$ at q satisfying (5–3).

Remark 5.8. (ii) and (iii) could be replaced by (ii)': $|\alpha'| = |\alpha|$ and $\sum_k \sigma_k\alpha_k' > \sum_k \sigma_k\alpha_k$ imply $\alpha' > \alpha$, which would simplify the statement of the lemma. However, the proof is slightly simpler with the present statement. Note that (ii)+(iii) is not equivalent to (ii)', that is, the ordering of the generators may be different, but either ordering gives (5–3).

Proof. We first observe that $\mathcal{M}_J$ is closed under commutators. Indeed, not only is $\mathcal{M}$ closed under commutators, but the commutators $[A_i, A_j]$ can be written as $\sum_{l=0}^{N} C_l A_l$ with $C_l \in \Psi_{\mathrm{sc}}^{-\infty,0}(X)$ and $C_l = 0$ unless $s_l \le s_i + s_j - 1$. Expanding

$$[x^{\tilde{s}(\alpha)} Q_\alpha A^\alpha, x^{\tilde{s}(\beta)} Q_\beta A^\beta],\ \ Q_\alpha, Q_\beta \in \Psi_{\mathrm{sc}}^{-\infty,0}(O),$$

and ignoring momentarily the commutators with powers of x and with Q_α and Q_β, gives a sum of terms of the form

$$x^{\tilde{s}(\alpha)+\tilde{s}(\beta)} Q_\alpha Q_\beta A^{\alpha'} A^{\beta'} [A_i, A_j] A^{\alpha''} A^{\beta''}$$

with $\alpha = \alpha' + \alpha'' + e_i$, and similarly for β. Substituting in $[A_i, A_j] = \sum_{l=0}^{N} C_l A_l$ shows that this is an element of the module and is indeed equivalent, modulo $\Psi_{\mathrm{sc}}^{-\infty,0}(O)$, to

$$\begin{gathered}\sum_{l:s_l \le s_i+s_j-1} \big(C_l x^{\tilde{s}(\alpha)+\tilde{s}(\beta)-\tilde{s}(\gamma^{(l)})}\big) x^{\tilde{s}(\gamma^{(l)})} A^{\gamma^{(l)}},\\ \gamma^{(l)} = \alpha' + \alpha'' + \beta' + \beta'' + e_l = \alpha + \beta - e_i - e_j + e_l,\end{gathered} \tag{5–11}$$

provided that

$$\tilde{s}(\gamma^{(l)}) \le \tilde{s}(\alpha) + \tilde{s}(\beta). \tag{5–12}$$

But $\tilde{s}(\alpha)+\tilde{s}(\beta) \ge (\sum s_i\alpha_i - 1) + (\sum s_i\beta_i - 1) = \sum s_i\gamma_i^{(l)} + s_i + s_j - s_l - 2 \ge \sum s_i\gamma_i^{(l)} - 1$ as $s_i + s_j - s_l \ge 1$. Moreover, $\tilde{s}(\alpha) + \tilde{s}(\beta) \ge 0$, so

$$\tilde{s}(\alpha) + \tilde{s}(\beta) \ge \max\Big(\sum s_k\gamma_k^{(l)} - 1, 0) = \tilde{s}(\gamma^{(l)})\Big),$$

proving (5–12).

The commutators

$$x^{\tilde{s}(\beta)} Q_\beta [x^{\tilde{s}(\alpha)} Q_\alpha, A^\beta] A^\alpha,\ \ x^{\tilde{s}(\alpha)} Q_\alpha [A^\alpha, x^{\tilde{s}(\beta)} Q_\beta] A^\beta \tag{5–13}$$

also lie in $\mathcal{M}_J$. Indeed, $[A_i, x^\rho Q] = x^{\rho - s_i + 1} Q'$ for some $Q' \in \Psi_{\mathrm{sc}}^{-\infty,0}(O)$, so they are sums of terms of the form $x^{\tilde{s}(\alpha)+\tilde{s}(\beta)-s_i+1} Q' A^\gamma$ with $\gamma = \alpha + \beta - e_i$. Now,

$$\tilde{s}(\gamma) \le \tilde{s}(\alpha) + \tilde{s}(\beta) - s_i + 1$$

since $\tilde{s}(\alpha) + \tilde{s}(\beta) - s_i + 1 \ge 0$ as $1 \ge s_i$ as well as $\tilde{s}(\alpha) + \tilde{s}(\beta) - s_i + 1 \ge (\sum_k s_k\alpha_k - 1) + (\sum_k s_k\beta_k - 1) - s_i + 1 = \sum_k s_k\gamma_k - 1$, so $\tilde{s}(\alpha) + \tilde{s}(\beta) - s_i + 1 \ge \max(\sum_k s_k\gamma_k - 1, 0) = \tilde{s}(\gamma)$ indeed, proving that (5–13) is in $\mathcal{M}_J$. The commutators

$$[x^{\tilde{s}(\alpha)} Q_\alpha, x^{\tilde{s}(\beta)} Q_\beta] A^\alpha A^\beta \tag{5–14}$$

can be shown to lie in $\mathcal{M}_J$ by a similar argument, this time using $\gamma = \alpha + \beta$, and $\tilde{s}(\gamma) \le \tilde{s}(\alpha) + \tilde{s}(\beta) + 1$. Thus, we conclude that $[x^{\tilde{s}(\alpha)} Q_\alpha A^\alpha, x^{\tilde{s}(\beta)} Q_\beta A^\beta] \in \mathcal{M}_J$, and hence $\mathcal{M}_J = \mathcal{M}_{J+1} = \dots$ is closed under commutators.

Modulo $\Psi_{\mathrm{sc}}^{-\infty,0}(O)$, $x^{\tilde{s}(\gamma^{(l)}))} A^{\gamma^{(l)}}$ may be replaced by $x^{-s(\gamma^{(l)})} B^{\gamma^{(l)}}$. If $|\gamma^{(l)}| > J$ in (5–11), then this is written in terms of one of the generators listed in (5–10) (or equivalently, modulo $\Psi_{\mathrm{sc}}^{-\infty,0}(O)$, in (5–9)), only after some of the factors in $B^{\gamma^{(l)}}$, which we may always take from $B_l B^{\beta'} B^{\beta''}$, are moved to the front and are incorporated in C_l, that is, they are simply regarded as 0–th order operators and C_l is replaced by $\widetilde{C}_l = C_l B_l B^{\beta'} B^{\beta''}$. Notice the principal symbol of $\widetilde{C}_l$ always vanishes at q in this case. Analogous conclusions hold for the terms in (5–13) and (5–14).

On the other hand, if $|\gamma^{(l)}| \le J$, then $x^{\tilde{s}(\gamma^{(l)})} A^{\gamma^{(l)}}$ is one of the generators in (5–9), and $|\gamma^{(l)}| = |\alpha| + |\beta| - 1$ if $l \ge 1$, and $|\gamma^{(l)}| = |\alpha| + |\beta| - 2$ if $l = 0$. Moreover, if $\sum_k s_k \beta_k > 1$ then

$$\sum_k s_k \gamma_k^{(l)} = \sum_k s_k \alpha_k + \sum_k s_k \beta_k - s_i - s_j + s_l \ge \sum_k s_k \alpha_k + \sum_k s_k \beta_k - 1 > \sum_k s_k \alpha_k. \tag{5–15}$$

For the terms in (5–13) and (5–14), if $|\gamma| \le J$, we always get $|\gamma| \ge |\alpha| + |\beta| - 1$ since $\gamma = \alpha + \beta$ or $\gamma = \alpha + \beta - e_i$ for some i.

Now we turn to (5–3). First, with $\widetilde{P}$ replaced by P_0, (5–3) is certainly satisfied, exactly as in the nonresonant case, since the $\sigma_{\partial,0}(B^\alpha)$ are eigenvectors of the linearization of W with eigenvalue given in Section 3. Thus,

$$i[A^\alpha, x^{-1} P_0] \sum_\gamma C'_\gamma A^\gamma, \;\; C'_\gamma \in \Psi_{\mathrm{sc}}^{-\infty,0}(O), \tag{5–16}$$

with $\sigma_{\partial,0}(C'_\gamma(q)) = 0$ if $\alpha \ne \gamma$ and $\operatorname{Re} \sigma_{\partial,0}(C'_\alpha(q)) \ge 0$. So it remains to show that it also holds for the resonant terms. If $x^{-s(\beta)} Q_\beta B^\beta$ is a resonant term, then $s(\beta) = 1$. Moreover,

(i) if $|\beta| = 1$, then $x^{-1} Q_\beta B^\beta = \sum_{\mu'} (y')^{\mu'} D_{y'_k}$ for some μ' and some k; in particular it is a summand of r_{er};

(ii) if $|\beta| = 2$, then either $x^{-1} Q_\beta B^\beta = B_j D_{y'_k}$ for some $j > 0$, k, or $x^{-1} Q_\beta B^\beta$ is associated to the sum over J'' in (3–15); in either case $\sum s_k \beta_k > 1$.

We claim that for a resonant term $x^{-s(\beta)} Q_\beta B^\beta$,

$$[x^{-s(\alpha)} B^\alpha, x^{-s(\beta)} Q_\beta B^\beta] \sim \sum_\gamma \widetilde{C}_\gamma x^{-s(\gamma)} B^\gamma, \;\; \widetilde{C}_\gamma \in \Psi_{\mathrm{sc}}^{-\infty,0}(X), \tag{5–17}$$

and each term on the right hand side has the following property:

(i) Either $\sigma_{\partial,0}(\widetilde{C}_\gamma)(q) = 0$, or

(ii) $|\gamma| > |\alpha|$, or

(iii) $|\gamma| = |\alpha|$, $\sum_k s_k \gamma_k > \sum_k s_k \alpha_k$, or

(iv) $|\gamma| = |\alpha| = 1$, $\gamma = e_k$, $\alpha = e_j$, $s_j = s_k = 1$ and $\sigma_k > \sigma_j$.

Indeed, if $|\beta| \ge 3$, then either (i) or (ii) holds, depending on whether any factors A_k had to be cancelled to write the commutator in terms of the generators in (5–9). If $|\beta| = 2$, then $\sum s_k \beta_k > 1$. Thus, again, either (i) or (ii) holds, or $|\gamma| = |\alpha|$ and $\sum_k s_k \gamma_k > \sum_k s_k \alpha_k$ by (5–15), so (iii) holds. Finally, if $|\beta| = 1$, then $x^{-1} Q_\beta B^\beta = \sum_{\mu'} (y')^{\mu'} D_{y'_k}$ for some μ' and some k. Since $r_1 \le r_2 \le \ldots \le r_{s-1} < 0$, and the resonance condition is $\sum_{l=1}^{s-1} \mu'_l r_l + (1 - r_k) = 1$ with $|\mu'| + 1 \ge 3$, we immediately deduce that $\mu'_l = 0$ for $l \le k$. Thus, not only do powers of x commute with $x^{-1} Q_\beta B^\beta$, but all A_i commute with $D_{y'_k}$ and $[A_i, (y')^{\mu'}] = 0$ unless $A_i = D_{y'_j}$ and $\mu'_j \ne 0$ for some j, which in turn implies that $j > k$, so $1 - r_k > 1 - r_j$, hence (iv) holds. This completes the proof of (5–17).

By the assumption on the ordering of the multiindices α, we deduce that for all resonant terms $x^{-s(\beta)}B^\beta$,

$$i[A^\alpha, x^{-s(\beta)}B^\beta] = \sum_\gamma C_\gamma A^\gamma,\ C_\gamma \in \Psi_{\mathrm{sc}}^{-\infty,0}(O),$$

and either $\sigma_{\partial,0}(C_\gamma)(q) = 0$, or $\gamma > \alpha$. Combining this with (5–16), we deduce that $\mathcal{M}_J$ satisfies (5–3). This establishes the lemma. □

Corollary 5.9. *Let $\mathcal{M} = \mathcal{M}_J$ be as in the previous lemma. Suppose that*

$$s < -\tfrac{1}{2}, \quad u \in H_{sc}^{\infty,s}(X), \quad \widetilde{P}u \in \dot{C}^\infty(X), \quad \mathrm{WF}_{sc}(u) \cap O \subset \Phi_+(\{q\}).$$

Then $u \in I_{\mathrm{sc}}^{(s)}(O, \mathcal{M})$.

Regularity with respect to $\mathcal{M}$ can be understood more geometrically as follows. Suppose $\delta > 0$ is sufficiently small so that (x, y', y'', y'') define local coordinates on the region U given by $0 \le x < \delta$, $|y_j| < \delta$ for all j. Let

$$\Phi : U^\circ \to \mathbb{R}^n_+,\ \Phi(x, y', y'', y''') = (x, y', Y'', Y'''),\ Y''_j = \frac{y''_j}{x^{r_j}},\ Y'''_j = \frac{y'''_j}{x^{1/2}}. \tag{5–18}$$

Thus, Φ is a diffeomorphism onto its range $\Phi(U^\circ)$ with

$$\Phi^{-1}(x, y', Y'', Y''') = (x, y'_j, x^{r_j}Y''_j, x^{1/2}Y''').$$

Note that $\overline{\Phi(U^\circ)}$ is not compact; Y'' and Y''' are "global" variables. Thus Φ^{-1} is actually continuous on $\overline{\Phi(U^\circ)}$ since $r''_j > 0$. Thus, Φ is a blow-up and Φ^{-1} is a somewhat singular blow-down map. In the coordinates (x, y', Y'', Y''') the Riemannian density takes the form

$$ax^{-n-1}\,dx\,dy = ax^{-n+\sum r''_j+(n-m)/2-1}\,dx\,dy'\,dY''\,dY''',$$

$a > 0$, $a \in \mathscr{C}^\infty(X)$. We thus conclude that (for O small) $u \in I_{\mathrm{sc}}^{(s)}(O, \mathcal{M})$ if and only if for any $Q \in \Psi_{\mathrm{sc}}^{-\infty,0}(O)$ with Schwartz kernel supported in $U \times U$, its microlocalization Qu satisfies

$$(Y'')^{\gamma''}(Y''')^{\gamma'''}(xD_x)^a D_{y'}^{\beta'} D_{Y''}^{\beta''} D_{Y'''}^{\beta'''} Qu \in x^{s+n/2-\sum r''_j/2-(n-m)/4} L^2(x^{-1}\,dx\,dy'\,dY''\,dY'''), \tag{5–19}$$

for every a, β, γ'' and γ''', that is, if and only if microlocally u is conormal in (x, y') with values in Schwartz functions in (Y'', Y'''), with the weight given by $s + n/2 - \sum r''_j/2 - (n-m)/4$.

We also recall that for conormal functions, the L^2 and the L^∞ spaces are very close, namely they are included in each other with a loss of x^ϵ. Thus, $u \in I_{\mathrm{sc}}^{(s)}(O, \mathcal{M})$ implies that

$$(Y'')^{\gamma''}(Y''')^{\gamma'''}(xD_x)^a D_{y'}^{\beta'} D_{Y''}^{\beta''} D_{Y'''}^{\beta'''} Qu \in x^{s+n/2-\sum r''_j/2-(n-m)/4-\epsilon} L^\infty(x^{-1}\,dx\,dy'\,dY''\,dY'''),$$

for every $\epsilon > 0$.

6. Effectively nonresonant operators

We now assume that the normal form p_{norm} for $\sigma_1(P(\sigma))$ at q is such that the term r_{er} in Theorem 3.11 vanishes. If this is true, we shall call p_{norm} *effectively nonresonant*, and σ an *effectively nonresonant energy* for q. The significance of the notion of effective resonance in general is that the form of the asymptotics of microlocally outgoing solutions of $Pu = f$, $f \in \dot{\mathscr{C}}^\infty(X)$, is independent of r_{enr}; only r_{er} changes this form slightly. Moreover, effective nonresonance is a more typical condition than nonresonance. We deal with the effectively nonresonant case in this section and treat the effectively resonant case in the following section. In both cases, it is convenient to reduce P, and not only its principal symbol, to model form. This is accomplished in the following lemma. We recall here our ongoing assumption (5–2).

Lemma 6.1. *Let p_{norm} be as in Theorem 3.11 and $\widetilde{P}$ as in Remark 3.12, that is, $\sigma_{\partial,-1}(\widetilde{P}) = p_{\mathrm{norm}}$. Then $\widetilde{P}$ can be conjugated by a smooth function to the form*

$$
\begin{aligned}
P_{\mathrm{norm}} &= \lambda\Big(xD_x + \sum_{j=1}^{m-1} r_j y_j D_{y_j} + \sum_{j=m}^{n-1} Q_j(x^{-1/2}y_j, x^{1/2}D_{y_j}) + R_{\mathrm{er}} + b + R\Big)\\
R_{\mathrm{er}} &= \sum_{j=1}^{s-1} \mathcal{P}_j(y')D_{y_j} + \sum_{j=s}^{m-1} \mathcal{P}_j(y'')D_{y_j} + \mathcal{P}_0(y''),
\end{aligned}
\tag{6–1}
$$

where b is a constant, Q_j is a real elliptic homogeneous quadratic polynomial (that is, a harmonic oscillator), $\mathcal{P}_j$ and $\mathcal{P}_0$ are homogeneous polynomials of degree r_j, respectively 1, when y_k is assigned degree r_k, and $R \in x^\epsilon(\mathcal{M})^j$ for some $j \in \mathbb{N}$ and $\epsilon > 0$. In addition, for $s \le j \le m-1$, $\mathcal{P}_j$ is actually a polynomial in $y_s, \dots, y_{j-1}$ (that is, is independent of $y_j, \dots, y_{m-1}$) without constant or linear terms, while for $j \le s-1$, $\mathcal{P}_j$ is a polynomial in $y_{j+1}, \dots, y_{s-1}$.

We call P_{norm} a normal form for P. If p_{norm} is effectively nonresonant then $R_{\mathrm{er}} = 0$.

Remark 6.2. Note that $Q_j(x^{-1/2}y_j, x^{1/2}D_{y_j})$ is not completely well-defined since Q_j is a homogeneous quadratic polynomial, and y_j and D_{y_j} do not commute. However, any two choices for the quantization Q_j differ by a constant multiple of the commutator $[x^{-1/2}y_j, x^{1/2}D_{y_j}] = [y_j, D_{y_j}]$, hence by a constant.

In particular, with the notation of the previous section, $Q_j(Y_j, D_{Y_j})$ may be arranged to be self-adjoint with respect to dY_j, by symmetrizing if necessary, which changes Q_j at most by a constant.

Proof. With the notation of Lemma 5.7, any effectively resonant monomial (defined in Definition 3.9) gives rise to a term of the form $x^{-1}Q_\beta B^\beta$ with $\sum_k s_k\beta_k = 1$, while the effectively nonresonant terms (defined in Definition 3.10) are of the form $x^{-1}Q_\beta B^\beta$ with $\sum_k s_k\beta_k > 1$. This is indeed the key point in categorizing resonant terms as effectively resonant or nonresonant; see the proof of Theorem 6.7. But if $\epsilon = \sum s_k\beta_k - 1 > 0$, we can rewrite $x^{-1}Q_\beta B^\beta \sim x^\epsilon Q_\beta A^\beta$ (that is, the difference of the two sides is in $\Psi_{\mathrm{sc}}^{-\infty,0}(X)$), and $Q_\beta A^\beta \in \mathcal{M}^{|\beta|}$. Since there are only finitely many effectively nonresonant terms in (3–15), we deduce that any $\widetilde{P}$ with $\sigma_1(\widetilde{P}) = p_{\mathrm{norm}}$ may be written

$$
\lambda^{-1}\widetilde{P} = xD_x + \sum_{j=1}^{m-1} r_j y_j D_{y_j} + \sum_{j=m}^{n-1} Q_j(x^{-1/2}y_j, x^{1/2}D_{y_j}) + R_{\mathrm{er}} + B + \widetilde{R},
$$

where R_{er} is as in (6–1), $\widetilde{R} \in x^\epsilon \mathcal{M}_J$ for some $\epsilon > 0$, and $B \in \Psi_{\mathrm{sc}}^{*,0}(X)$. Note that $\mathcal{P}_j$ and $\mathcal{P}_0$ are polynomials, and the homogeneity claim is the meaning of the resonance condition Proposition 3.6. For $s \le j \le m-1$, $\mathcal{P}_j$ is independent of $y_j, \dots, y_{m-1}$ since $0 < r_s \le r_{s+1} \le \dots \le r_{m-1}$; y_j itself cannot appear in $\mathcal{P}_j$ due to the restriction $2a + |\beta| + |\gamma| \ge 3$ in Proposition 3.6. Similarly, for $j \le s-1$, $\mathcal{P}_j$ is independent of $y_1, \dots, y_j$ as $r_1 \le r_2 \le \dots \le r_{s-1} < 0$. This also shows that the polynomials $\mathcal{P}_j$, $j \neq 0$, have no constant or linear terms.

Let B have symbol $b(\nu, y, \mu)$. This can be reduced to the symbol $b'(0, (y', 0, 0), 0)$, modulo terms in $x^\epsilon \mathcal{M}^j$. Finally, by conjugating P_{norm} by a function $e^{if(y')}$, we can remove the y'-dependence of b'. Indeed, the Taylor series of f can be constructed iteratively. Let $\mathcal{I}'$ denote the ideal of functions of y' that vanish at 0. Conjugating $\widetilde{P}$ by e^{if} produces the terms $\sum_{j=1}^{s-1} r'_j y'_j D_{y'_j} f$, as well as terms from R_{er}, which map $(\mathcal{I}')^k \to (\mathcal{I}')^{k+1}$. For $k \ge 1$, $f \mapsto \sum_{j=1}^{s-1} r'_j y'_j \partial_{y'_j} f$ defines a linear map on $(\mathcal{I}')^k$, $k \ge 1$, with all eigenvalues negative since $r'_j < 0$ for $j = 1, \dots, s-1$. Thus, this map is invertible, and this shows that $b' - b'(0)$ can be conjugated away in Taylor series. Then it is straightforward to check that the infinite order vanishing error can also be removed. □

Later in this section we show that if p_{norm} is effectively nonresonant, the leading asymptotics of microlocally outgoing solutions for (6–1) and for the completely explicit operator

$$P_0 = x D_x + \sum_{j=1}^{m-1} r_j y_j D_{y_j} + \sum_{j=m}^{n-1} Q_j(x^{-1/2} y_j, x^{1/2} D_{y_j}) + b, \quad b \text{ constant} \tag{6–2}$$

are the same, if $R \in x^{1+\epsilon}\mathcal{M}^j$ for some $\epsilon > 0$, that is, R is indeed an "error term". An analogous conclusion holds in the effectively resonant case, with R_{er} included in the right hand side of (6–2).

First, however, we study the asymptotics of approximate solutions of $P_0 u = 0$. The constant b simply introduces a power x^{-ib} into the asymptotics, as can be seen by conjugation of P_0 by x^{-ib}. Here it is convenient to have the asymptotics for the ultimately relevant case, where the operator xP is self-adjoint, stated explicitly, so we assume that xP_0 is formally self-adjoint on $L^2_{\mathrm{sc}}(X)$, which amounts to

$$\operatorname{Im} b = \frac{n-1}{2} - \frac{1}{2}\Big(\sum_{j=1}^{s-1} r'_j + \sum_{j=s}^{m-1} r''_j\Big) - \frac{n-m}{2}, \tag{6–3}$$

provided that we have already made Q_j self-adjoint as stated in Remark 6.2. Note that

$$\frac{n-m}{2} = \sum_{j=m}^{n-1} \operatorname{Re} r'''_j.$$

For convenience, we separate the case where q is a source/sink of W, hence of the contact vector field of P_0. Recall from the previous section that

$$Y''_j = x^{-r''_j} y''_j, \quad Y''' = x^{-1/2} y''', \tag{6–4}$$

and define the exponents

$$\tilde{b} = b - i\frac{n-m}{4}, \; a_{\beta'} = -\sum_{j=1}^{s-1} r_j \beta'_j - i\tilde{b}. \tag{6–5}$$

Notice that $\operatorname{Re} a_{\beta'} \to \infty$ as $|\beta'| \to \infty$.

Proposition 6.3. *Suppose that the radial point q is a source/sink of W, and (6–3) holds. Suppose that $u \in I^{(s)}(O, \mathcal{M})$, and $P_0 u \in I^{(s')}(O, \mathcal{M})$ where $s < -1/2 < s'$. Then u takes the form*

$$u = \sum_k x^{-i\tilde{b} - i\kappa_k} w_k(Y'') v_k(Y''') + u' \tag{6–6}$$

where the sum is over $k \in \mathbb{N}$, $v_k(Y)$ is an L^2-normalized eigenfunction of the harmonic oscillator

$$\sum_{j=m}^{n-1} \widetilde{Q}_j(Y_j, D_{Y_j}),\ \ \widetilde{Q}_j(Y_j, D_{Y_j}) = Q_j(Y_j, D_{Y_j}) - \frac{1}{4}(Y_j D_{Y_j} + D_{Y_j} Y_j),\ \ Y_j = \frac{y_j}{x^{1/2}}, \tag{6–7}$$

with eigenvalue κ_k, w_k are Schwartz functions with each seminorm rapidly decreasing in k, and $u' \in I^{(s'-\epsilon)}(O, \mathcal{M})$ for every $\epsilon > 0$.

Conversely, given any rapidly decreasing Schwartz sequence, w_k, in Y'', meaning one for which all seminorm rapidly decreasing in k, and given any $f \in I_{\mathrm{sc}}^{(s')}(O, \mathcal{M})$, there exists $u \in \bigcap_{s<-1/2} I_{\mathrm{sc}}^{(s)}(O, \mathcal{M})$ of the form (6–6) with $\mathrm{WF}_{sc}(P_0 u - f) \cap O = \varnothing$.

Remark 6.4. The result is true if we only assume $s < s'$. However, if $s \geq -1/2$, we can replace s by $\tilde{s} > -1/2$, apply the proposition with $\tilde{s}$ in place of s, and then use $u \in I_{\mathrm{sc}}^{(s)}(O, \mathcal{M})$ to show that each w_k vanishes. On the other hand, if $s' \geq -1/2$, the proof of the proposition shows that $u \in I_{\mathrm{sc}}^{(s)}(O, \mathcal{M})$ implies $u \in I_{\mathrm{sc}}^{(s'-\epsilon)}(O, \mathcal{M})$ for every $\epsilon > 0$.

Proposition 6.5. *Suppose that q is a saddle point of W, and (6–3) holds. Suppose $u \in I^{(s)}(O, \mathcal{M})$, and $P_0 u \in I^{(s')}(O, \mathcal{M})$ for some $s < s' < \infty$. Then u takes the form*

$$u = \sum_{\beta', k} x^{a_{\beta'} - i\kappa_k} (y')^{\beta'} w_{\beta', k}(Y'') v_k(Y''') + u' \tag{6–8}$$

where the sum is over $k \in \mathbb{N}$ and a finite set of multiindices β', $v_k(Y)$ and κ_k are as above, $w_{\beta',k}$ is a rapidly decreasing Schwartz sequence and $u' \in I^{(s'-\epsilon)}(O, \mathcal{M})$ for every $\epsilon > 0$.

Conversely, given any rapidly decreasing sequence of Schwartz functions $w_{\beta',k}$, finite in β' and any $f \in I_{\mathrm{sc}}^{(s')}(O, \mathcal{M})$ there exists $u \in \bigcap_{s<-1/2} I_{\mathrm{sc}}^{(s)}(O, \mathcal{M})$ of the form (6–8) with $\mathrm{WF}_{sc}(P_0 u - f) \cap O = \varnothing$.

Remark 6.6. As shown later, $x^2 D_x$ gives rise to the terms in $\widetilde{Q} - Q$ after the change of variables $(x, y_j) \mapsto (x, y_j / x^{1/2})$. If Q_j is self-adjoint on $L^2(\mathbb{R}, dY_j)$ then $\widetilde{Q}_j$ has the same property.

Also, with

$$B = \frac{n-1}{2} - \frac{1}{2} \sum_j r_j'' - \frac{n-m}{4},$$

the (β', k) summand in (6–8) is in

$$I_{\mathrm{sc}}^{(\operatorname{Re} a_{\beta'} - B - 1/2 - \epsilon)}(O, \mathcal{M})$$

for every $\epsilon > 0$. We show below that $\operatorname{Im} \tilde{b} = B + d$, $d = -\frac{1}{2} \sum r_j' > 0$, so the (β', k) summand is in

$$I_{\mathrm{sc}}^{(d - \sum r_j \beta_j' - 1/2 - \epsilon)}(O, \mathcal{M})$$

for every $\epsilon > 0$, and in view of the rapid decay in k, the same is true after the k summation. Thus, for u as in (6–8), $u \in I_{\mathrm{sc}}^{(d-1/2-\epsilon)}(O, \mathcal{M})$ provided $s' > d - \frac{1}{2}$, that is, decays by a factor x^d faster than the microlocal solutions at sources/sinks of W.

Proof of Proposition 6.3. Suppose that $P_0 u = f \in I^{(s')}(O, \mathcal{M})$ for some $s' > -1/2$. Let O' be a W-balanced neighbourhood of q with $\overline{O'} \subset O$, and let $Q \in \Psi_{\mathrm{sc}}^{-\infty,0}(X)$ satisfy $\mathrm{WF}'_{\mathrm{sc}}(Q) \subset O$ (that is, $Q \in \Psi_{\mathrm{sc}}^{-\infty,0}(O)$) and $\mathrm{WF}'_{\mathrm{sc}}(\mathrm{Id} - Q) \cap \overline{O'} = \varnothing$, with Schwartz kernel supported in $U \times U$,

$$U = \{0 \le x < \delta,\ |y_j| < \delta \text{ for all } j\}.$$

(See (5–18) for the definition of the diffeomorphism Φ, the coordinates Y_j, etc.) Then, as noted in (5–19), by the definition of $I_{\mathrm{sc}}^{(s)}(O, \mathcal{M})$, $\tilde{u} = Qu$ satisfies

$$(Y'')^{\gamma''}(Y''')^{\gamma'''}(xD_x)^a D_{Y''}^{\beta''} D_{Y'''}^{\beta'''} \tilde{u} \in x^s L^2_{\mathrm{sc}}(X)$$

for all a, β'', β''', γ'' and γ'''. Here $\tilde{u}$ is a microlocalization of u since $\mathrm{WF}_{\mathrm{sc}}(u - Qu) \subset \mathrm{WF}'_{\mathrm{sc}}(\mathrm{Id} - Q)$, so $\mathrm{WF}_{\mathrm{sc}}(u - Qu) \cap O' = \varnothing$. Moreover,

$$P_0(Qu) = QP_0 u + [P_0, Q]u = Qf + f',\quad f' \in \dot{\mathcal{C}}^\infty(X),$$

since $\mathrm{WF}_{\mathrm{sc}}(u) \cap O \subset \{q\}$, while $\mathrm{WF}'_{\mathrm{sc}}([P_0, Q]) \subset \mathrm{WF}'_{\mathrm{sc}}(Q) \cap \mathrm{WF}'_{\mathrm{sc}}(\mathrm{Id} - Q) \subset O \setminus \overline{O'}$, so $\mathrm{WF}_{\mathrm{sc}}(u) \cap \mathrm{WF}'_{\mathrm{sc}}([P_0, Q]) = \varnothing$. Thus, with $\tilde{f} = Qf + f'$,

$$\begin{aligned} &P_0 \tilde{u} = \tilde{f}, \\ &(Y'')^{\gamma''}(Y''')^{\gamma'''}(xD_x)^a D_{Y''}^{\beta''} D_{Y'''}^{\beta'''} \tilde{f} \in x^{s'} L^2_{\mathrm{sc}}(X), \end{aligned} \tag{6–9}$$

for all a, β'', β''', γ'' and γ'''.

To prove first part of the proposition, it thus suffices to show that, with the notation of (6–6),

$$\tilde{u} = \sum_k x^{-i\tilde{b} - i\kappa_k} w_k(Y'') v_k(Y''') + u'.$$

Writing the operator P_0 in the coordinates x, Y'', Y''' we have

$$P_0 = xD_x|_Y + \sum_j \widetilde{Q}_j(Y_j''', D_{Y_j'''}) + \tilde{b} \tag{6–10}$$

with $\tilde{b} = b - i\frac{n-m}{4}$ as in (6–5). Formal self-adjointness of xP_0, that is, (6–3), requires that

$$\operatorname{Im} \tilde{b} = \frac{n-1}{2} - \frac{1}{2}\sum_j r_j'' - \frac{n-m}{4} \equiv B. \tag{6–11}$$

As already remarked, (6–9), which states that $\tilde{f}$ is conormal in x, and Schwartz in Y'', Y''', and belongs to $x^{s'} L^2(dx\,dy/x^{n+1})$, or in terms of the Y coordinates, to $x^{s' + n/2 - \sum r_j''/2 - (n-m)/4}\ L^2(dx\,dY/x)$, implies (by conormality) that

$$\tilde{f} \in x^{s' + 1/2 + B - \epsilon} L^\infty$$

for every $\epsilon > 0$, where B is defined by (6–11). More precisely, for all a, β, γ'' and γ''',

$$(Y'')^{\gamma''}(Y''')^{\gamma'''}(xD_x)^a D_{Y''}^{\beta''} D_{Y'''}^{\beta'''} \tilde{f} \in x^{s' + 1/2 + B - \epsilon} L^\infty$$

for every $\epsilon > 0$. Conversely these conditions imply that $\tilde{f}$ satisfies (6–9) with s' replaced by $s' - \epsilon$ for every $\epsilon > 0$.

Writing $\tilde{f}$ in the form

$$\tilde{f}(x, Y'', Y''') = \sum_k f_k(x, Y'')v_k(Y'''),$$

where f_k is conormal in x, rapidly decreasing as a Schwartz sequence in Y'', a particular solution to $P_0\tilde{u} = \tilde{f}$, is given by

$$\begin{aligned}\tilde{u} &= \sum_k u_k(x, Y'')v_k(Y'''),\\ u_k &= -ix^{-i\tilde{b}-i\kappa_k} \int_0^x f_k(t, Y'')t^{i\tilde{b}+i\kappa_k}\,\frac{dt}{t}.\end{aligned} \tag{6–12}$$

Since $s' + 1/2 > 0$, this integral is convergent and $\tilde{u} \in I^{(s'-\epsilon)}(O, \mathcal{M})$ for every $\epsilon > 0$.

On the other hand, the general solution to $P_0\tilde{u} = 0$ with $\tilde{u}$ Schwartz in Y'' and Y''' is given by

$$\tilde{u} = \sum_k x^{-i\tilde{b}-i\kappa_k} w_k(Y'')v_k(Y'''),$$

where w_k is rapidly decreasing in k. Since any solution is the sum of the particular solution (6–12) and some homogeneous solution, the first half of the proposition follows.

In fact, the second half also follows by defining

$$\tilde{u} = \sum_k u_k(x, Y'')v_k(Y''') + \sum_k x^{-i\tilde{b}-i\kappa_k} w_k(Y'')v_k(Y'''),$$

with u_k as in (6–12). Multiplying by a cutoff function $\phi \in \mathcal{C}^\infty(X)$ which is identically 1 near $(0, 0, \dots, 0)$, it follows that $u = \phi\tilde{u}$ satisfies all requirements. □

Proof of Proposition 6.5. We use a similar argument to prove this result. Let O', Q, etc., be as in the previous proof. With $\tilde{u} = Qu$, as noted in (5–19),

$$(Y'')^{\gamma''}(Y''')^{\gamma'''}(xD_x)^a D_{y'}^{\beta'} D_{Y''}^{\beta''} D_{Y'''}^{\beta'''}\tilde{u} \in x^s L^2_{\text{sc}}(X), \tag{6–13}$$

for all a, β, γ'' and γ'''. One of the main differences with the proof of Proposition 6.3 is that microlocalization introduces a nontrivial error, that is, $P_0\tilde{u}$ is not globally well-behaved (not as good as f was microlocally). However, the error is supported away from $y' = 0$. Indeed, now $\mathrm{WF}_{\text{sc}}(u) \cap O \subset S$, and

$$\tilde{f} = P_0\tilde{u} = Qf + f', \quad f' = [P_0, Q]u.$$

Here $\mathrm{WF}'_{\text{sc}}([P_0, Q]) \cap S \subset \{|y'| > \delta_0\}$ for some $\delta_0 > 0$, so $f' \in I^{(s)}_{\text{sc}}(O, \mathcal{M})$ in fact satisfies

$$(Y'')^{\gamma''}(Y''')^{\gamma'''}(xD_x)^a D_{y'}^{\beta'} D_{Y''}^{\beta''} D_{Y'''}^{\beta'''} f' \in x^s L^2_{\text{sc}}(X)$$

for all a, β', β'' and β''', γ'' and γ''', with the improved conclusion

$$\phi(y')(Y'')^{\gamma''}(Y''')^{\gamma'''}(xD_x)^a D_{y'}^{\beta'} D_{Y''}^{\beta''} D_{Y'''}^{\beta'''} f' \in \dot{\mathcal{C}}^\infty(X)$$

if ϕ is supported in $|y'| < \delta_0$. Correspondingly,

$$\phi(y')(Y'')^{\gamma''}(Y''')^{\gamma'''}(xD_x)^a D_{y'}^{\beta'} D_{Y''}^{\beta''} D_{Y'''}^{\beta'''}\tilde{f} \in x^{s'} L^2_{\text{sc}}(X). \tag{6–14}$$

The operator P_0 in the coordinates x, y', Y'', Y''' now takes the form

$$P_0 = x D_x|_{y',Y'',Y'''} + \sum_j r'_j y'_j D_{y'_j} + \sum_j \widetilde{Q}_j(Y'''_j, D_{Y'''_j}) + \tilde{b}, \tag{6–15}$$

with $\tilde{b} = b - i\frac{n-m}{4}$ as in (6–5). Again, (6–14) implies that $\tilde{f}$ is conormal in x, smooth in y', and Schwartz in Y'', Y''', and belongs to $x^{s'+1/2+B-\epsilon}L^\infty$ for every $\epsilon > 0$, where B is defined by (6–11), in the precise sense that for all a, β, γ'' and γ''',

$$\phi(y')(Y'')^{\gamma''}(Y''')^{\gamma'''}(xD_x)^a D^{\beta''}_{Y''} D^{\beta'''}_{Y'''} \tilde{f} \in x^{s'+1/2+B-\epsilon} L^\infty$$

for every $\epsilon > 0$. However, now formal self-adjointness of xP_0 requires that

$$\operatorname{Im} \tilde{b} = B + d, \quad d = -\frac{1}{2}\sum_j r'_j > 0,$$

so there is a discrepancy of d compared with the previous proposition. Write $\tilde{f}$ in the form

$$\tilde{f}(x, Y'', Y''') = \sum_k f_k(x, y', Y'')v_k(Y'''),$$

where f_k is rapidly decreasing sequence which is conormal in x, smooth in y' and Schwartz in Y''.

We start by describing solutions of the homogeneous equation $P_0\tilde{u} = 0$ in U which in addition satisfy (6–13). Decomposing $\tilde{u}$ in terms of the v_k, and factoring out a power of x for convenience, that is, writing $\tilde{u} = \sum_k x^{-i\tilde{b}-i\kappa_k} u_k(x, y', Y'')v_k(Y''')$, we see that the coefficients u_k satisfy

$$\Big(x\partial_x|_{y',Y'',Y'''} + \sum_j r'_j y'_j \partial_{y'_j}\Big)u_k = 0.$$

Since $\tilde{u}$ is smooth in the interior of U, $P_0\tilde{u} = 0$ amounts to demanding that u_k be constant along each integral curve segment of the vector field $x\partial_x + \sum_j r'_j y'_j \partial_{y'_j}$, with the value of $\tilde{u}$ depending smoothly on the choice of the integral curve. (We remark that U is convex for this vector field; $|y'|$ is increasing as $x \to 0$.) Thus, $u_k(x, y', Y'') = \hat{u}_k(Y', Y'')$ with $\hat{u}_k$ smooth in Y' and Schwartz in Y''. Here $Y'_j = y'_j/x^{r'_j}$; note that $r'_j < 0$. Expanding $\hat{u}_k$ in Taylor series around $Y' = 0$ to order N, we see that

$$u_k(x, y', Y'') - \sum_{|\beta'| \le N-1} x^{-\sum_j r'_j \beta'_j}(y')^{\beta'} w_{\beta',k}(Y'')$$

is a finite sum of terms of the form $x^{-\sum_j r'_j\beta'_j}(y')^{\beta'}\hat{u}_{k,\beta'}(Y', Y'')$ with $\hat{u}_{k,\beta'}$ smooth (Schwartz in Y'''), where the sum runs over β' with $|\beta'| = N$. Thus, given any s'' (for example, $s'' = s'$), we can choose N sufficiently large so this difference lies in $I^{(s'')}_{\mathrm{sc}}(O, \mathcal{M})$, which means it is ignorable for our purposes. Thus, the general solution to $P_0\tilde{u} = 0$ in U which satisfies (6–13) is given by

$$\tilde{u} = \sum_{\beta',k} x^{a_{\beta'}-i\kappa_k}(y')^{\beta'} w_{\beta',k}(Y'')v_k(Y'''),$$

modulo any $I^{(s'')}_{\mathrm{sc}}(O, \mathcal{M})$ (where the sum is understood as a finite one, due to the remark above), where the seminorms of $w_{\beta',k}$ are rapidly decreasing in k for each β'.

In expressing a particular solution $\tilde u$ of $P_0\tilde u = f$ in terms of f, we need to integrate along integral curves of the vector field $x\partial_x + \sum_j r'_j y'_j \partial_{y'_j}$, and since $r'_j < 0$, $|y'| \to \infty$ as $x \to 0$ along such curves (unless $y' = 0$); in fact $|y'|$ is increasing as $x \to 0$ as mentioned above. So we cannot integrate down to $x = 0$. Instead we fix an $x_0 > 0$ and use the formula

$$u_k(x, y', Y'') = \left(\frac{x}{x_0}\right)^{-i\tilde b - i\kappa_k} u_k\left(x_0, \left(\frac{x}{x_0}\right)^{-r'_j} y'_j, Y''\right) + i x^{-i\tilde b - i\kappa_k} \int_{x_0}^{x} f_k\left(t, \left(\frac{x}{t}\right)^{-r'_j} y'_j, Y''\right) t^{i\tilde b + i\kappa_k} \frac{dt}{t}. \tag{6–16}$$

Notice that $u_k(x_\sharp, y'_\sharp, Y''_\sharp)$ depends only on f_k evaluated at points (x, y', Y'') with $|y'| \le |y'_\sharp|$. Thus, (6–14) can be used to deduce properties of u_k, hence of $\tilde u$, in $|y'| < \delta_0$.

If $s' < -1/2 + d$, then (6–16) gives $\phi(y')\tilde u \in I^{(s'-\epsilon)}(O, \mathcal{M})$ for every $\epsilon > 0$, with ϕ as in (6–14). If $s' \ge -1/2 + d$, then $\phi(y')\tilde u \in I^{(-1/2+d-\epsilon)}(O, \mathcal{M})$ for every $\epsilon > 0$. However, this is actually a sum of terms solving the homogeneous equation, plus a function in $I^{(s'-\epsilon)}(O, \mathcal{M})$ for every $\epsilon > 0$. For simplicity we show this only in the case that $-1/2 + d < s' < -1/2 + d + |r'_{s-1}|$. Then we observe that $(x/x_0)^{-i\tilde b - i\kappa_k}\tilde u(x_0, 0, Y'')$ is a solution of the homogeneous equation, while the difference

$$\left(\frac{x}{x_0}\right)^{-i\tilde b - i\kappa_k} \tilde u\left(x_0, \left(\frac{x}{x_0}\right)^{-r'_j} y'_j, Y''\right) - \left(\frac{x}{x_0}\right)^{-i\tilde b - i\kappa_k} \tilde u(x_0, 0, Y'') = \sum_j \left(\frac{x}{x_0}\right)^{-r'_j} \int_0^1 y'_j \partial_{y'_j}\left(\tilde u\left(x_0, \tau\left(\frac{x}{x_0}\right)^{-r'_j} y'_j, Y''\right)\right) d\tau$$

has decay at least $x^{-r'_{s-1}}$ better, hence yields a term in $I^{(s'-\epsilon)}(O, \mathcal{M})$ for every $\epsilon > 0$. Similarly, if we replace $f_k(t, (\frac{x}{t})^{-r'_j} y'_j, Y'')$ in the integral by $f_k(t, 0, Y'')$ then we get a homogeneous term, while the difference gives a term in $I^{(s'-\epsilon)}(O, \mathcal{M})$ for every $\epsilon > 0$. The argument can be repeated, removing more and more terms in the Taylor series for $\tilde u$ and $\tilde f$, for larger values of s'. Since any solution is the sum of the particular solution above and the general solution, the first half of the proposition follows with O replaced by a smaller neighbourhood O'' of q. However, we recover the original statement by using the real principal type parametrix construction of Duistermaat and Hörmander [1972].

The second half can be proved as in the previous proposition. Fix some $x_0 > 0$, and let u_k be given by the second term on the right hand side of (6–16), and let $\hat u = \sum_k u_k(x, Y'')v_k(Y''')$. Then $P_0\hat u = f$, and as shown above, $\hat u$ has the form

$$\hat u = \sum_{\beta', k} x^{a_{\beta'} - i\kappa_k} (y')^{\beta'} \hat w_{\beta', k}(Y'') v_k(Y''') + \hat u',$$

with $\hat u' \in I^{(s'-\epsilon)}_{\mathrm{sc}}(O, \mathcal{M})$ for all $\epsilon > 0$. Then with

$$\tilde u = \sum_k u_k(x, Y'')v_k(Y''') + \sum_{\beta', k} x^{a_{\beta'} - i\kappa_k}(w_{\beta', k}(Y'') - \hat w_{\beta', k}(Y''))v_k(Y'''),$$

$u = \phi\tilde u$, $\phi \in \mathcal{C}^\infty(X)$ identically 1 near $(0, \dots, 0)$, u satisfies all requirements. □

These results on the explicit normal form P_0 then allow us to parameterize microlocally outgoing solutions for every effectively nonresonant critical point.

Theorem 6.7. *Suppose that $P(\sigma)$ is effectively nonresonant at q, with normal form P_{norm} near q as in Lemma 6.1, and (6–3) holds.*

(i) *If in addition q is a source/sink of W, then any microlocally outgoing solution u of P_{norm} has the form (6–6), and conversely given any Schwartz sequence of Schwartz functions w_k there is a microlocally outgoing solution u of P_{norm} which has the form (6–6). Thus, microlocal solutions at a source/sink of W are parameterized by Schwartz functions of the variables (Y'', Y''').*

(ii) *If q is a saddle point of W, then all microlocally outgoing solutions are in $x^{-1/2+\epsilon}L^2$ for some $\epsilon > 0$. For each monomial $(y')^\beta$ in the variables y', each $k \in \mathbb{N}$ and each Schwartz function $w(Y'')$ there is a microlocally outgoing solution of the form*

$$u = \sum_k x^{a_{\beta'} - i\kappa_k} (y')^{\beta'} w(Y'') v_k(Y''') + u', \tag{6–17}$$

where u' is in a strictly smaller weighted L^2 space than u, and every microlocally outgoing solution is a sum of such solutions, with the $w = w_{k,\beta'}$ rapidly decreasing as $k \to \infty$ in every seminorm.

Proof. First, $P_{\text{norm}} = \lambda(P_0 + R)$, $R \in x^\epsilon \mathcal{M}^j$, $\epsilon > 0$. Thus, if O is a neighbourhood of q as above, $\mathrm{WF}_{\mathrm{sc}}(P_{\text{norm}} u) \cap O = \varnothing$, then $u \in I^{(s)}_{\mathrm{sc}}(O, \mathcal{M})$ for all $s < -1/2$, so $Ru \in I^{(s')}_{\mathrm{sc}}(O, \mathcal{M})$ for some $s' > 1/2$. Hence $P_0 u = \lambda^{-1} P_{\text{norm}} u - Ru \in I^{(s')}_{\mathrm{sc}}(O, \mathcal{M})$.

If q is a source/sink of W, then Proposition 6.3 is applicable, and we deduce that u is microlocally of the form (6–6). Moreover, if q is a source/sink of W, then given any Schwartz sequence of Schwartz functions w_k, let $u_0 \in \bigcap_{s<-1/2} I^{(s)}_{\mathrm{sc}}(O, \mathcal{M})$ be of the form (6–6) with $P_0 u_0 \in \dot{\mathscr{C}}^\infty(X)$. We construct $u_k \in \bigcap_{r<-1/2-k\epsilon} I^{(r)}_{\mathrm{sc}}(O, \mathcal{M})$, $k \ge 1$, inductively so that $P_0 u_k + R u_{k-1} \in \dot{\mathscr{C}}^\infty(X)$ for $k \ge 1$; this can be done by the second half of Proposition 6.3. Asymptotically summing $\sum_k u_k$ to some $u \in \bigcap_{s<-1/2} I^{(s)}_{\mathrm{sc}}(O, \mathcal{M})$ gives a microlocally outgoing solution with the prescribed asymptotics, completing the proof of the theorem in this case.

If q is a saddle point of W, we apply Proposition 6.5 with $s' > -1/2$ as in the first paragraph of the proof. If $\epsilon' > 0$ is sufficiently small, all of the terms in (6–8) are in $I^{(-1/2+\epsilon')}_{\mathrm{sc}}(O, \mathcal{M})$ proving the first claim. To show the next, let $u_0 = x^{a_{\beta'} - i\kappa_k} (y')^{\beta'} w(Y'') v_k(Y''')$, so $P_0 u_0 = 0$ and $u_0 \in I^{(s)}_{\mathrm{sc}}(O, \mathcal{M})$ for any $s < -1/2 + d$. We construct u_k inductively as above, using Proposition 6.5, to obtain u. □

Remark 6.8. From (6–6) or (6–17) it is not hard to derive the asymptotic expansion of eigenfunctions of the original operator $\Delta + V - \sigma$; we need only apply the Fourier integral operator F^{-1} arising by composing any Fourier integral operators with canonical relation given by the contact maps in Lemma 2.7 and Theorem 3.11 to these expansions. In fact, as mentioned in Remark 3.14, this Fourier integral operator can be taken to be a composition of a change of coordinates with multiplication by an oscillatory function if q is either a source/sink (so $q \in \mathrm{Min}_+(\sigma)$) or the linearization of W has no nonreal eigenvalues (so there are no y''' variables).

In the case of a radial point $q \in \mathrm{Min}_+(\sigma)$, *in appropriate coordinates y on ∂X*, the expansion takes the form

$$u = e^{i\Phi(y)/x} \sum_k x^{-i\tilde{b}-i\kappa_k} w_k(Y'')v_k(Y''') + u',\ \ u' \in I^{-\frac{1}{2}+\epsilon}(O, \mathcal{M}) \text{ for some } \epsilon > 0 \tag{6–18}$$

where Φ is a smooth function (it parameterizes the Legendrian submanifold which is the image of the zero section under the canonical relation of F^{-1}). For a given σ, only a finite number of terms in the Taylor series for Φ are relevant. Similarly in the case of radial points $q \in \mathrm{RP}_+(\sigma) \setminus \mathrm{Min}_+(\sigma)$, the expansion (6–19) takes the form

$$u = e^{i\Phi(y)/x} \sum_k x^{a_{\beta'}-i\kappa_k} (y')^{\beta'} w(Y'')v_k(Y''') + u', \tag{6–19}$$

with Φ smooth. Again it parameterizes the image of the zero section under the canonical relation of F^{-1}. In this case, the value of Φ on the unstable manifold $\{y'' = y''' = 0\}$ is essential, but only a finite number of terms in the Taylor series for Φ about this unstable manifold are relevant.

These expansions were obtained directly in Part I (that is, without going via a normal form) in the two dimensional case.

7. Effectively resonant operators

If P is effectively resonant, the simple expressions (6–6) and (6–8) need to be replaced by slightly more complicated ones in which positive integral powers of $\log x$ also appear. Essentially, instead of powers, or Schwartz functions, of y_j/x^{r_j}, factors of $\log x$ also arise in the expressions for the Y_l.

First define a change of coordinates inductively that simplifies the vector field

$$V = (xD_x) + \sum_{j=1}^{m-1} (r_j y_j + \mathcal{P}_j(y_s, \dots, y_{j-1}))D_{y_j} \tag{7–1}$$

that appears in (6–1) as the combinations of the linear terms $\sum r_j y_j D_{y_j}$ and the effectively resonant vector fields in R_{er}. (Note that $r_j y_j$ and $\mathcal{P}_j(y_s, \dots, y_{j-1})$ are both homogeneous of degree r_j.) We do this in two steps to clarify the argument, first only dealing with the y'' terms, that is, $j = s, \dots, m-1$.

The coordinates Y_j, $j = s, \dots, m-1$, are a modification of the coordinates y_j/x^{r_j} that appear in (6–4), so that $Y_j - y_j/x^{r_j}$ are polynomials $\mathcal{P}^\sharp_j$ in $Y_s, \dots, Y_{j-1}, t = \log x$. Thus, we let

$$Y_s = \frac{y_s}{x^{r_s}},\ \ \mathcal{P}^\sharp_s = 0,\ \ \bar{Y}_s(Y_s, \log x) = Y_s + \mathcal{P}^\sharp_s(\log x)$$

and provided that $Y_s, \dots, Y_{j-1}, \mathcal{P}^\sharp_s, \dots, \mathcal{P}^\sharp_{j-1}$ have been defined, we let

$$\begin{aligned}
\mathcal{P}^\sharp_j(Y_s, \dots, Y_{j-1}, t) &= \int_0^t \mathcal{P}_j(\bar{Y}_s(Y_s, t'), \dots, \bar{Y}_{j-1}(Y_s, \dots, Y_{j-1}, t'))\, dt', \\
Y_j &= \frac{y_j}{x^{r_j}} - \mathcal{P}^\sharp_j(Y_s, \dots, Y_{j-1}, \log x), \\
\bar{Y}_j &= Y_j + \mathcal{P}^\sharp_j(Y_s, \dots, Y_{j-1}, \log x),\ \ j = s, \dots, m-1.
\end{aligned}$$

The point of the construction is that V annihilates Y_j for all j. This can be seen iteratively: for Y_s this is straightforward, and if $VY_s = \dots = VY_{j-1} = 0$ then (with $\partial_t \mathcal{P}_j^\sharp$ denoting the derivative with respect to the last variable, $t = \log x$)

$$\begin{aligned} VY_j &= -r_j \frac{y_j}{x^{r_j}} + (r_j y_j + \mathcal{P}_j(y_s, \dots, y_{j-1}))x^{-r_j} - (\partial_t \mathcal{P}_j^\sharp)(Y_s, \dots, Y_{j-1}, \log x) \\ &= \mathcal{P}_j(y_s x^{-r_s}, \dots, y_{j-1} x^{-r_{j-1}}) - \mathcal{P}_j(\overline{Y}_s(Y_s, \log x), \dots, \overline{Y}_{j-1}(Y_s, \dots, Y_{j-1}, \log x)) \\ &= 0 \end{aligned}$$

in view of the definition of $Y_s, \dots, Y_{j-1}$ and $\overline{Y}_s, \dots, \overline{Y}_{j-1}$.

One can deal with the $j = 1, \dots, s-1$ terms similarly. We define $\mathcal{P}_j^\sharp$, Y_j and $\overline{Y}_j$ inductively as above, starting with Y_{s-1}. Thus, we let

$$Y_{s-1} = \frac{y_{s-1}}{x^{r_{s-1}}}, \ \mathcal{P}_{s-1}^\sharp = 0, \ \overline{Y}_{s-1}(Y_{s-1}, \log x) = Y_{s-1} + \mathcal{P}_{s-1}^\sharp(\log x)$$

and provided that $Y_{j+1}, \dots, Y_{s-1}, \mathcal{P}_{j+1}^\sharp, \dots, \mathcal{P}_{s-1}^\sharp$ have been defined, we let

$$\begin{aligned} &\mathcal{P}_j^\sharp(Y_{j+1}, \dots, Y_{s-1}, t) = \int_0^t \mathcal{P}_j(\overline{Y}_{j+1}(Y_{j+1}, \dots, Y_{s-1}, t'), \dots, \overline{Y}_{s-1}(Y_{s-1}, t'))\, dt', \\ &Y_j = \frac{y_j}{x^{r_j}} - \mathcal{P}_j^\sharp(Y_{j+1}, \dots, Y_{s-1}, \log x), \\ &\overline{Y}_j = Y_j + \mathcal{P}_j^\sharp(Y_{j+1}, \dots, Y_{s-1}, \log x), \ \ j = 1, \dots, s-1. \end{aligned}$$

With these definitions, in the coordinates $X = x, Y_1, \dots, Y_{m-1}, y_m, \dots, y_{n-1}$, that is, (X, Y', Y'', y'''), which correspond to a blow-up of $x = y_s = \dots = y_{m-1} = 0$, $V = X^2 D_X$.

The zeroth order term is a polynomial $\mathcal{P}_0$ in $y_s, \dots, y_{m-1}$ which is homogeneous of degree 1 (where y_j has degree r_j). Thus,

$$x^{-1}\mathcal{P}_0(y_s, \dots, y_{m-1}) = \mathcal{P}_0(\overline{Y}_s(Y_s, \log x), \dots, \overline{Y}_{m-1}(Y_s, \dots, Y_{m-1}, \log x)).$$

Let

$$\mathcal{P}_0^\sharp(Y_s, \dots, Y_{j-1}, t) = \int_0^t \mathcal{P}_0(\overline{Y}_s(Y_s, t'), \dots, \overline{Y}_{j-1}(Y_s, \dots, Y_{j-1}, t'))\, dt',$$

which is thus a polynomial in $Y_s, \dots, Y_{j-1}, t$. Then $e^{i\mathcal{P}_0^\sharp(Y_s, \dots, Y_{j-1}, \log x)}$ can be used as an integrating factor, conjugating $\widetilde{P}$, to remove the zeroth order term in R_{er}.

Finally, to put the quadratic terms in a convenient form, we let

$$Y_j = \frac{y_j}{x^{1/2}}, \ \ j = m, \dots, n-1$$

as before.

Suppose first that $\mathcal{P}_0 = 0$. With our definition of the Y_j, (6–10), respectively (6–15), holds if q is a source/sink, respectively saddle point, of V_0. Thus, the statement and the proof of Proposition 6.3 holds without any changes, while the statement and the proof of Proposition 6.5 carry over provided $x^{a_{\beta'}}(y')^{\beta'}$ is replaced by $x^{-i\tilde{b}}(Y')^{\beta'}$. A minor difference is that slightly more effort is required to show that $|y'|$ decreases on the integral curves of the vector field (7–1) inside $|y'| < \delta_1$ for $\delta_1 > 0$ small. Namely we need to use that, as $\mathcal{P}_j$, $j = 1, \ldots, s-1$ have no linear or constant terms by Lemma 6.1, $V|y'|^2 = \sum_{j=1}^{s-1} r_j y_j^2 + \mathcal{O}(|y'|^3) \le r_{s-1}|y'|^2 + \mathcal{O}(|y'|^3)$, $r_{s-1} < 0$, to conclude that $V|y'|^2 \le 0$ for $|y'| < \delta_1$, $\delta_1 > 0$ small.

In general, with $\tilde{b} = b - \frac{1}{4}i(n-m)$ as in (6–5), Equations (6–10) and (6–15) are replaced by

$$P_0 = xD_x|_Y + \sum_j \widetilde{Q}_j(Y_j''', D_{Y_j'''}) + \mathcal{P}_0 + \tilde{b},$$

$$P_0 = xD_x|_{y',Y'',Y'''} + \sum_{j=1}^{s-1}(r_j' y_j' + \mathcal{P}_j)D_{y_j'} + \sum_j \widetilde{Q}_j(Y_j''', D_{Y_j'''}) + \mathcal{P}_0 + \tilde{b},$$

respectively. Thus we obtain,

$$e^{i\mathcal{P}_0^\sharp} P_0 e^{-i\mathcal{P}_0^\sharp} = xD_x|_Y + \sum_j \widetilde{Q}_j(Y_j''', D_{Y_j'''}) + \tilde{b},$$

$$e^{i\mathcal{P}_0^\sharp} P_0 e^{-i\mathcal{P}_0^\sharp} = xD_x|_{y',Y'',Y'''} + \sum_{j=1}^{s-1}(r_j' y_j' + \mathcal{P}_j)D_{y_j'} + \sum_j \widetilde{Q}_j(Y_j''', D_{Y_j'''}) + \tilde{b},$$

respectively. Since multiplication by $e^{\pm i\mathcal{P}_0^\sharp}$ preserves $I_{\mathrm{sc}}^{(s)}(O, \mathcal{M})$, the rest of the proof of the propositions is applicable with u replaced by $e^{i\mathcal{P}_0^\sharp}u$, $f = P_0 u$ replaced by $e^{i\mathcal{P}_0^\sharp}f$. We thus deduce the following analogues of Propositions 6.3–6.5 in the effectively resonant case.

Proposition 7.1. *Suppose that the radial point q is a source/sink of W, and (6–3) holds, that $u \in I^{(s)}(O, \mathcal{M})$, and $P_0 u \in I^{(s')}(O, \mathcal{M})$ where $s < -1/2 < s'$. Then u takes the form*

$$u = \sum_k x^{-i\tilde{b} - i\kappa_k} e^{-i\mathcal{P}_0^\sharp} w_k(Y'') v_k(Y''') + u' \tag{7–2}$$

where the sum is over $k \in \mathbb{N}$, $v_k(Y)$ is an L^2-normalized eigenfunction of the harmonic oscillator

$$\sum_{j=m}^{n-1} \widetilde{Q}_j(Y_j, D_{Y_j}),\ \ \widetilde{Q}_j(Y_j, D_{Y_j}) = Q_j(Y_j, D_{Y_j}) - \frac{1}{4}(Y_j D_{Y_j} + D_{Y_j} Y_j),\ \ Y_j = \frac{y_j}{x^{1/2}},$$

with eigenvalue κ_k, w_k are Schwartz functions with each seminorm rapidly decreasing in k, and $u' \in I^{(s'-\epsilon)}(O, \mathcal{M})$ for every $\epsilon > 0$.

Conversely, given any sequence w_k of Schwartz functions in Y'' with each seminorm rapidly decreasing in k, and given any $f \in I_{\mathrm{sc}}^{(s')}(O, \mathcal{M})$, there exists $u \in \bigcap_{s<-1/2} I_{\mathrm{sc}}^{(s)}(O, \mathcal{M})$ of the form (7–2) with $\mathrm{WF}_{sc}(P_0 u - f) \cap O = \varnothing$.

Proposition 7.2. *Suppose that q is a saddle point of W, and (6–3) holds, that $u \in I^{(s)}(O, \mathcal{M})$, and $P_0 u \in I^{(s')}(O, \mathcal{M})$ for some $s < s' < \infty$. Then u takes the form*

$$u = \sum_{\beta', k} x^{-i\tilde{b} - i\kappa_k} (Y')^{\beta'} e^{-i\mathcal{P}_0^\sharp} w_{\beta',k}(Y'') v_k(Y''') + u' \tag{7–3}$$

where the sum is over $k \in \mathbb{N}$ and a finite set of multiindices β', $v_k(Y)$ and κ_k are as above, $w_{\beta',k}$ are Schwartz functions with each seminorm rapidly decreasing in k, and $u' \in I^{(s'-\epsilon)}(O, \mathcal{M})$ for every $\epsilon > 0$.

Conversely, given any sequence of Schwartz functions $w_{\beta',k}$, finite in β' with each seminorm rapidly decreasing in k, and any $f \in I_{\mathrm{sc}}^{(s')}(O, \mathcal{M})$ there exists $u \in \bigcap_{s<-1/2} I_{\mathrm{sc}}^{(s)}(O, \mathcal{M})$ of the form (7–3) with $\mathrm{WF}_{sc}(P_0 u - f) \cap O = \varnothing$.

We thus deduce the following analogue of Theorem 6.7, with a similar proof.

Theorem 7.3. *Suppose that $P(\sigma)$ is effectively resonant at q, with normal form P_{norm} near q as in Lemma 6.1, and (6–3) holds.*

(i) *If in addition q is a source/sink of W, then any microlocal solution u of P_{norm} has the form (7–2), and conversely given any rapidly Schwartz sequence of functions w_k there is a microlocally outgoing solution u of P_{norm} which has the form (7–2). Thus, microlocal eigenfunctions at a source/sink are parameterized by Schwartz functions of the variables (Y'', Y''').*

(ii) *If q is a saddle point of W, then all microlocal solutions are in $x^{-1/2+\epsilon} L^2$ for some $\epsilon > 0$. For each monomial in the variables Y', each $k \in \mathbb{N}$ and each Schwartz function $w(Y'')$ there is a microlocally outgoing solution of the form*

$$u = x^{-i\tilde{b}-i\kappa_k} e^{-i\mathcal{P}_0^\sharp} (Y')^{\beta'} w(Y'') v_k(Y''') + u',$$

where u' is in a strictly faster decaying weighted L^2 space than u, and every microlocally outgoing solution is a sum of such solutions, with the $w = w_{k,\beta'}$ rapidly decreasing as $k \to \infty$ in every seminorm.

8. From microlocal to approximate eigenfunctions

We are interested in the structure of (global) eigenfunctions of $\Delta + V$. While in the first half of the paper a rather general element $P \in \Psi_{\mathrm{sc}}^{*,-1}(X)$ was considered, from now on attention is limited to

$$H = \Delta + V \in \Psi_{\mathrm{sc}}^{*,0}(X), \;\; H(\sigma) = H - \sigma,$$

in particular *the order of H at ∂X is* 0.

In the next section we obtain an iterative description of the "smooth" eigenfunctions in terms of the microlocal eigenspaces. As the first step, we show that if q is a radial point for $H(\sigma) = H - \sigma$, then elements of $E_{\mathrm{mic},+}(q, \sigma)$, which are the microlocally outgoing eigenfunctions near q, have representatives satisfying $(H - \sigma)u \in \dot{\mathscr{C}}^\infty(X)$, that is, they extend to approximate eigenfunctions, with $\mathrm{WF}_{\mathrm{sc}}(u)$ a subset of the forward flow-out of q. Stated explicitly this is:

Proposition 8.1. *If $q \in \mathrm{RP}_+(\sigma)$ then every element of $E_{\mathrm{mic},+}(q, \sigma)$ has a representative u such that $(H - \sigma)u \in \dot{\mathscr{C}}^\infty(X)$, and $\mathrm{WF}_{sc}(u) \subset \Phi_+(\{q\})$.*

Remark 8.2. From this result, given u as in Proposition 8.1 it is easy to produce an exact eigenfunction v such that $\mathrm{WF}_{\mathrm{sc}}(v) \cap \{\nu \geq 0\} \subset \Phi_+(\{q\})$: we simply take $v = u - R(\sigma - i0)(H - \sigma)u$.

The key ingredient of the proof, as in the two-dimensional case studied in [Hassell et al. 2004], is the microlocal solvability of the eigenequation through radial points. To avoid a microlocal construction along the lines of Hörmander [1971], we introduce, as in Lemma 5.3 of Part I, an operator $\widetilde{H}$ which arises from H by altering V appropriately. This is chosen to be equal to H near the radial point in question but to have no other radial points in $\mathrm{RP}_+(\sigma)$ at which ν takes a smaller value. One may then assume, in any argument concerning $q \in \mathrm{RP}_+(\sigma)$, that there is no $q' \in \mathrm{RP}_+(\sigma)$ with $\nu(q') < \nu(q)$.

As in Definition 11.3 of Part I, we introduce a partial order on $\mathrm{RP}_+(\sigma)$ corresponding to the flow-out under W.

Definition 8.3. If $q, q' \in \mathrm{RP}_+(\sigma)$ we say that $q \le q'$ if $q' \in \Phi_+(\{q\})$ and $q < q'$ if $q \le q'$ but $q' \neq q$. A subset $\Gamma \subset \mathrm{RP}_+(\sigma)$ is closed under $\le$ if, for all $q \in \Gamma$, $\{q' \in \mathrm{RP}_+(\sigma); q \le q'\} \subset \Gamma$. We call the set $\{q' \in \mathrm{RP}_+(\sigma); q \le q'\}$ the string generated by q.

Remark 8.4. This partial order relation between two radial points in $\mathrm{RP}_+(\sigma)$ corresponds to the existence of a sequence $q_j \in \mathrm{RP}_+(\sigma)$, $j = 0, \dots, k$, $k \ge 1$, with $q_0 = q$, $q_k = q'$ and such that for every $j = 0, \dots, k-1$, there is a bicharacteristic γ_j with $\lim_{t\to-\infty} \gamma_j = q_j$ and $\lim_{t\to+\infty} \gamma_j = q_{j+1}$.

Lemma 8.5. *Given $\sigma > \min V_0$ and $\tilde\nu > 0$, set $K = V_0^{-1}((-\infty, \sigma - \tilde\nu^2]) \subset \partial X$ then there exists a potential function $\widetilde{V} \in \mathscr{C}^\infty(X)$ with $\tilde V_0$ Morse such that*

(i) $\tilde V_0 \ge V_0$,

(ii) $\tilde V_0 = V_0$ *on a neighbourhood of* K,

(iii) *no critical value of $\widetilde V$ lies in the interval $(\sigma - \tilde\nu^2, \sigma]$,*

(iv) *if $\widetilde\Sigma(\sigma)$ is the characteristic variety at energy σ of $\widetilde H = \Delta + \widetilde V$ then*

$$\Sigma(\sigma) \cap \{\nu \ge \tilde\nu\} = \widetilde\Sigma(\sigma) \cap \{\nu \ge \tilde\nu\},$$

(v) *$\widetilde H - \sigma$ has no L^2 null space.*

Proof. Choose a smooth function f on the real line so that $f' > 0$, $f(t) = t$ if $t \le \sigma - \tilde\nu^2$ and $f(t) > \sigma$ for $t \ge \min\{V(q); dV(q) = 0 \text{ and } V(q) > \sigma - \tilde\nu^2\} > \sigma - \tilde\nu^2$. Then let $\widetilde V = f \circ V$, so the critical points of V_0 and $\tilde V_0$ are the same and are nondegenerate.

On $\Sigma(\sigma) \cap \{\nu \ge \tilde\nu\}$, we have $\nu^2 + |\mu|_y^2 + V_0 = \sigma$, hence $V_0 \le \sigma - \tilde\nu^2$, so $V_0 = \tilde V_0$, and therefore $\Sigma(\sigma) \cap \{\nu \ge \tilde\nu\} \subset \widetilde\Sigma(\sigma)$. With the converse direction proved similarly, (i)–(iv) follow. Property (v) can be arranged by a suitable perturbation of $\widetilde V$ with compact support in the interior. □

These properties of $\widetilde H$ are exploited in the proof of the following continuation result.

Lemma 8.6 (Lemma 5.5 of [Hassell et al. 2004]). *Suppose $u \in \mathscr{C}^{-\infty}(X)$ satisfies*

$$\mathrm{WF}_{sc}(u) \subset \{\nu \ge \nu_1\} \text{ and } \mathrm{WF}_{sc}((H-\sigma)u) \subset \{\nu \ge \nu_2\},$$

for some $0 < \nu_1 < \nu_2$, then there exists $\tilde u \in \mathscr{C}^{-\infty}(X)$ with $\mathrm{WF}_{sc}(u - \tilde u) \subset \{\nu \ge \nu_2\}$ and $(H - \sigma)\tilde u \in \dot{\mathscr{C}}^\infty(X)$.

Proof. We just sketch the proof here; for full details, see [Hassell et al. 2004]. The obvious idea of subtracting $R(\sigma + i0)((H - \sigma)u)$ from u does not quite work, since the forward flowout of other critical points $q' \in \mathrm{RP}_+(\sigma)$ with $\nu(q')$ less than $\nu(q)$ may strike q. To avoid this problem, choose $\tilde\nu$ with $\nu_1 < \tilde\nu < \nu_2$, sufficiently close to ν_2 so that there are no radial points q with $\nu(q) \in [\tilde\nu, \nu_2)$, and a corresponding $\widetilde V$ as in Lemma 8.5. Then consider the function $\widetilde R(\sigma + i0)(H - \sigma)Au$, where A is equal to the identity microlocally on $\{\nu \le \tilde\nu\} \cap \Sigma(\sigma)$ and vanishes microlocally in $\{\nu \ge \nu_2\}$. Since $\widetilde V_0$ has no critical points q with $0 < \nu(q) < \nu_2$ it follows readily $\tilde u = Au - \widetilde R(\sigma + i0)(H - \sigma)Au$ satisfies the desired conditions. □

From this we can readily deduce:

Lemma 8.7. *If $q \in \mathrm{RP}_+(\sigma)$ then every element of $E_{\mathrm{mic},+}(q,\sigma)$ has a representative $\tilde{u}$ such that $(H-\sigma)\tilde{u} \in \dot{\mathscr{C}}^\infty(X)$ and $\mathrm{WF}_{sc}(\tilde{u})$ is contained in the union of $\Phi_+(\{q\})$ and the $\Phi_+(\{q'\})$ for those $q' \in \mathrm{RP}_+(\sigma)$ with $\nu(q') > \nu(q)$.*

Proof. Let O be a W-balanced neighbourhood of q (see Definition 4.4). Let $A \in \Psi_{\mathrm{sc}}^{-\infty,0}(X)$ be microlocally equal to the identity on $\Phi_+(\{q\}) \cap \overline{O}$ and supported in a small neighbourhood of $\Phi_+(\{q\}) \cap \overline{O}$. Then there exists $\nu_2 > \nu(q)$ such that $\nu > \nu_2$ on $\Phi_+(\{q\}) \setminus O$, and $\mathrm{WF}'_{\mathrm{sc}}(A) \setminus O \subset \{\nu \geq \nu_2\}$. (Here $\mathrm{WF}'_{\mathrm{sc}}(A)$ is the operator wavefront set of A, that is, the complement in ${}^{\mathrm{sc}}T^*_{\partial X}X$ of the set where A is microlocally trivial; see [Melrose 1994].) Now let u be any representative. Since $\mathrm{WF}_{\mathrm{sc}}(u) \cap O \subset \Phi_+(\{q\})$, $\mathrm{WF}_{\mathrm{sc}}(Au-u) \cap O = \varnothing$. In addition, $\mathrm{WF}_{\mathrm{sc}}(Au) \subset \mathrm{WF}'_{\mathrm{sc}}(A) \cap \mathrm{WF}_{\mathrm{sc}}(u)$, hence $\nu \geq \nu(q)$ on $\mathrm{WF}_{\mathrm{sc}}(Au)$. Moreover, $\mathrm{WF}_{\mathrm{sc}}(Au-u) \cap O = \varnothing$ implies that

$$\mathrm{WF}_{\mathrm{sc}}((H-\sigma)Au) \cap O = \mathrm{WF}_{\mathrm{sc}}((H-\sigma)Au - (H-\sigma)u) \cap O = \varnothing,$$

so $\mathrm{WF}_{\mathrm{sc}}((H-\sigma)Au) \subset \mathrm{WF}'_{\mathrm{sc}}(A) \setminus O$, hence is contained in $\{\nu \geq \nu_2\}$. Then, by Lemma 8.6, there exists $\tilde{u} \in C^{-\infty}(X)$ such that $\nu \geq \nu_2$ on $\mathrm{WF}_{\mathrm{sc}}(\tilde{u} - Au)$ and $(H-\sigma)\tilde{u} \in \dot{\mathscr{C}}^\infty(X)$. In particular, $\nu \geq \nu(q)$ in $\mathrm{WF}_{\mathrm{sc}}(\tilde{u})$. Moreover, $\nu \geq \nu_2$ on $\mathrm{WF}_{\mathrm{sc}}(\tilde{u}-u) \cap O$, hence by Lemma 4.5, $\mathrm{WF}_{\mathrm{sc}}(\tilde{u}-u) \cap O = \varnothing$, so $\tilde{u}$ and u have the same image in $E_{\mathrm{mic},+}(O,\sigma)$. □

Finally, we can show that each microlocally outgoing eigenfunction is represented by an approximate eigenfunction.

Proof of Proposition 8.1. Let $\tilde{u}$ be a representative as in Lemma 8.7. If we choose q' from the set

$$\bigl\{q' \in \mathrm{RP}_+(\sigma) \cap \mathrm{WF}_{\mathrm{sc}}(\tilde{u});\ \nu(q') > \nu(q),\ \ q' \notin \Phi_+(\{q\})\bigr\}, \tag{8–1}$$

with $\nu(q')$ minimal, then, localizing $\tilde{u}$ near q', gives an element v of $E_{\mathrm{mic},+}(q')$. By subtracting from $\tilde{u}$ a representative of v given by Lemma 8.7, we remove the wavefront set near q'. Inductively choosing radial points from (8–1) and performing this procedure repeatedly, all wavefront set may be removed from $\tilde{u}$ except that contained in $\Phi_+(\{q\})$. □

9. Microlocal Morse decomposition

Next we show that global smooth eigenfunctions can, in an appropriate sense, be decomposed into components originating, in the sense of the Introduction, at a single radial point. We do this by defining subspaces of $E^\infty_{\mathrm{ess}}(\sigma)$ corresponding to the location of scattering wavefront set in $\{\nu > 0\}$ and showing that suitable quotients of these spaces are isomorphic to the spaces of microlocal eigenfunctions $E^\infty_{\mathrm{mic},+}(q,\sigma)$, $q \in \mathrm{RP}_+(\sigma)$, analyzed in Sections 6 and 7. Since each of the spaces $E^\infty_{\mathrm{mic},+}(q,\sigma)$, $q \in \mathrm{RP}_+(\sigma)$, is nontrivial this shows that each such radial point gives rise to eigenfunctions. However, as noted previously in [Herbst and Skibsted 1999; 2004; 2008] and [Hassell et al. 2004] in some special cases, there is a qualitative difference between the radial points corresponding to local minima of V_0 and the others. This is expressed by Proposition 10.3 where we show that the eigenfunctions $u \in E^\infty_{\mathrm{Min},+}(\sigma)$ originating only at minimum radial points are dense in $E^0_{\mathrm{ess}}(\sigma)$ (definitions of these spaces are given below).

Recall from [Hassell et al. 2004, Equation (3.14)] the spaces of eigenfunctions of fixed growth

$$E^s_{\mathrm{ess}}(\sigma) = \{u \in E^{-\infty}_{\mathrm{ess}}(\sigma);\ \mathrm{WF}^{0,s-1/2}_{\mathrm{sc}}(u) \cap \{\nu = 0\} = \varnothing\}. \tag{9–1}$$

This condition is equivalent to requiring that

$$Bu \in x^{s-1/2}L^2(X) \tag{9–2}$$

for some pseudodifferential operator $B \in \Psi_{\mathrm{sc}}^{0,0}(X)$ with boundary symbol which is elliptic on $\Sigma(\sigma)\cap\{\nu = 0\}$ and microsupported in $\{|\nu| < a(\sigma)\}$, where

$$a(\sigma) = \min\{|\nu(q)|;\, q \in \mathrm{RP}(\sigma)\}.$$

The space $E^0_{\mathrm{ess}}(\sigma)$ is of particular interest. Choose an operator $A \in \Psi_{\mathrm{sc}}^{0,0}(X)$ whose boundary symbol is 0 for $\nu \le -a(\sigma)$ and 1 for $\nu \ge a(\sigma)$. The space $E^0_{\mathrm{ess}}(\sigma)$ is a Hilbert space with norm

$$\|u\|^2_{E^0_{\mathrm{ess}}(\sigma)} = \langle i[H, A]u, u\rangle. \tag{9–3}$$

The positive-definiteness of this form, and its independence of the choice of operator A, was shown in [Hassell et al. 2004, Section 12]. An equivalent norm is

$$\|Bu\|_{x^{-1/2}L^2} + \|u\|_{x^{-1/2-\epsilon}L^2}$$

where $\epsilon > 0$ and B is as in (9–2); see [Hassell et al. 2004, Section 3].

We now define subspaces of $E^s_{\mathrm{ess}}(\sigma)$ depending on the location of the scattering wavefront set inside $\{\nu > 0\}$. Given any $\le$-closed subset Γ of $\mathrm{RP}_+(\sigma)$, we define

$$E^s_{\mathrm{ess}}(\sigma, \Gamma) = \{u \in E^s_{\mathrm{ess}}(\sigma);\, \mathrm{WF}_{\mathrm{sc}}(u) \cap \mathrm{RP}_+(\sigma) \subset \Gamma\}. \tag{9–4}$$

The set of radial points $q \in \mathrm{RP}_+(\sigma)$ lying above local minima of V is an example of a $\le$-closed subspace and will be denoted $\mathrm{Min}_+(\sigma)$. In this case we use the notation

$$E^s_{\mathrm{Min},+}(\sigma) \equiv E^s_{\mathrm{ess}}(\sigma, \mathrm{Min}_+(\sigma)) = \{u \in E^s_{\mathrm{ess}}(\sigma);\, \mathrm{WF}_{\mathrm{sc}}(u) \cap \mathrm{RP}_+(\sigma) \subset \mathrm{Min}_+(\sigma)\}$$

to be consistent with [Hassell et al. 2004].

Proposition 9.1. *Suppose that $\Gamma \subset \mathrm{RP}_+(\sigma)$ is $\le$-closed and q is a $\le$-minimal element of Γ. Then with $\Gamma' = \Gamma \setminus \{q\}$,*

$$0 \longrightarrow E^\infty_{\mathrm{ess}}(\sigma, \Gamma') \xrightarrow{\iota} E^\infty_{\mathrm{ess}}(\sigma, \Gamma) \xrightarrow{r_q} E_{\mathrm{mic},+}(\sigma, q) \longrightarrow 0$$

is a short exact sequence, where ι is the inclusion map and r_q is the microlocal restriction map.

Proof. The injectivity of ι follows from the definitions. The null space of the microlocal restriction map r_q, which can be viewed as restriction to a W-balanced neighbourhood of q, is precisely the subset of $E^\infty_{\mathrm{ess}}(\sigma, \Gamma)$ with wave front set disjoint from $\{q\}$, and this subset is $E^\infty_{\mathrm{ess}}(\sigma, \Gamma')$. Thus it only remains to check the surjectivity of r_q.

We do so first for the strings generated by $q \in \mathrm{RP}_+(\sigma)$. For $q \in \mathrm{Min}_+(\sigma)$, the string just consists of q itself and the result follows trivially. So consider the string $S(q)$ generated by $q \in \mathrm{RP}_+(\sigma) \setminus \mathrm{Min}_+(\sigma)$. By Proposition 8.1 any element of $E_{\mathrm{mic},+}(q, \sigma)$ has a representative $\tilde u$ satisfying $(H - \sigma)\tilde u \in \dot{\mathscr{C}}^\infty(X)$ with $\mathrm{WF}_{\mathrm{sc}}(\tilde u) \subset \Phi_+(\{q\})$. Then $u = \tilde u - R(\sigma - i0)(H - \sigma)\tilde u \in E^\infty_{\mathrm{ess}}(\sigma, \Gamma)$, which gives surjectivity in this case.

For any $\le$-closed set Γ and $\le$-minimal element q, the string $S(q)$ is contained in Γ, so the surjectivity of r_q follows in general. □

Notice that we can always find a sequence $\varnothing = \Gamma_0 \subset \Gamma_1 \subset \ldots \subset \Gamma_n = \mathrm{RP}_+(\sigma)$, of $\leq$-closed sets with $\Gamma_j \setminus \Gamma_{j-1}$ consisting of a single point q_j which is $\leq$-minimal in Γ_j: we simply order the $q_i \in \mathrm{RP}_+(\sigma)$ so that $\nu(q_1) \geq \nu(q_2) \geq \ldots$, and set $\Gamma_i = \{q_1, \ldots, q_i\}$. Then Proposition 9.1 implies the following:

Theorem 9.2 (Microlocal Morse decomposition). *Suppose that $\varnothing = \Gamma_0 \subset \Gamma_1 \subset \ldots \subset \Gamma_n = \mathrm{RP}_+(\sigma)$, is as described in the previous paragraph. Then*

$$\{0\} \longrightarrow E^\infty_{\mathrm{ess}}(\sigma, \Gamma_1) \hookrightarrow \ldots \hookrightarrow E^\infty_{\mathrm{ess}}(\sigma, \Gamma_{n-1}) \hookrightarrow E^\infty_{\mathrm{ess}}(\sigma),$$

with

$$E^\infty_{\mathrm{ess}}(\sigma, \Gamma_j)/E^\infty_{\mathrm{ess}}(\sigma, \Gamma_{j-1}) \simeq E_{\mathrm{mic},+}(q_j, \sigma), \;\; j = 1, 2, \ldots, n.$$

10. L^2-parameterization of the generalized eigenspaces

Recall from Theorem 6.7, or Theorem 7.3 in the effectively resonant case, that there is a surjective map

$$M_+(\sigma) : E^\infty_{\mathrm{Min},+}(\sigma) \to \bigoplus_{q \in \mathrm{Min}_+(\sigma)} \mathcal{S}(\mathbb{R}^{n-1}), \sigma \in (\min V_0, \infty) \setminus \Big(\mathrm{Cv}(V) \cup \bigcup_{z \in \mathrm{Cv}(V)} \mathcal{R}_{\mathrm{Ht},z}\Big), \tag{10–1}$$

given by taking $u \in E^\infty_{\mathrm{Min},+}(\sigma)$, microlocally restricting u to a neighbourhood of each q giving $u_q \in E^\infty_{\mathrm{mic},+}(\sigma, q)$ and sending u to the sum of the leading coefficients $\sum_k w_k(Y'')v_k(Y''')$, $(Y'', Y''') \in \mathbb{R}^{n-1}$, of each of the u_q. Since the v_k are normalized eigenfunctions of a harmonic oscillator and the w_k are Schwartz functions of Y'' with seminorms rapidly decreasing in k, the sum is a Schwartz function of (Y'', Y''').

Let us regard $\bigoplus_q \mathcal{S}(\mathbb{R}^{n-1})$ as a subspace of $\bigoplus_q L^2(\mathbb{R}^{n-1})$, endowed with the norm

$$\|(w_q)_{q \in \mathrm{Min}_+(\sigma)}\|^2 = \sum_q \int_{\mathbb{R}^{n-1}} |w_q(Y)|^2 d\omega_{q,\sigma}, \;\; d\omega_{q,\sigma} = 2\sqrt{\sigma - V(\pi(q))}\, d\omega_q, \tag{10–2}$$

where ω_q is the measure induced by Riemannian measure, namely the measure

$$x^{n-(n-m)/2-\sum_j r''_j} dg$$

divided by dx/x and restricted to $x = 0$. (It takes the form $dY''\, dY'''$ provided that the y are normal coordinates, centred at the critical point, for the metric $h(0, y, dy)$.)

The next result is the main content of this section.

Theorem 10.1. *The map $M_+(\sigma)$ in (10–1) has a unique extension to an unitary isomorphism*

$$M_+(\sigma) : E^0_{\mathrm{ess}}(\sigma) \to \bigoplus_{q \in \mathrm{Min}_+(\sigma)} L^2(\mathbb{R}^{n-1}).$$

Remark 10.2. Here, and throughout this section, we take $\sigma \in (\min V_0, \infty) \setminus \mathrm{Cv}(V)$.

To prove the theorem, we establish several intermediate results. First we show:

Proposition 10.3. *The space $E^\infty_{\mathrm{Min},+}(\sigma)$ is dense in $E^\infty_{\mathrm{ess}}(\sigma)$ in the topology of $E^0_{\mathrm{ess}}(\sigma)$.*

Proof. The proof is by induction. We consider a sequence $\Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_n = \mathrm{RP}_+(\sigma)$ as in the previous section, but with the additional condition that the radial points are ordered so that, among the points with equal values of ν, those corresponding to local minima of V_0 are placed last. We shall prove by induction that

$$E^\infty_{\mathrm{ess}}(\sigma, \Gamma_i \cap \mathrm{Min}_+(\sigma)) \text{ is dense in } E^\infty_{\mathrm{ess}}(\sigma, \Gamma_i) \text{ in the topology of } E^0_{\mathrm{ess}}(\sigma). \tag{10–3}$$

For $i = 1$ there is nothing to prove. Assume that (10–3) is true for $i = k$. Let $\Gamma_{k+1} \setminus \Gamma_k = \{q\}$. If q arises from a local minimum of V_0, then using a microlocal decomposition, any $u \in E^\infty_{\mathrm{ess}}(\sigma, \Gamma_{k+1})$ can be written as the sum of $u_1 \in E^\infty_{\mathrm{ess}}(\sigma, \{q\})$ and $u_2 \in E^\infty_{\mathrm{ess}}(\sigma, \Gamma_k)$. A similar statement is true for $u \in E^\infty_{\mathrm{ess}}(\sigma, \Gamma_{k+1} \cap \mathrm{Min}_+(\sigma))$, which proves (10–3) for $i = k+1$.

Next suppose that q does not arise from a local minimum of V_0. Then we adapt the argument of Proposition 11.6 of [Hassell et al. 2004] to prove (10–3) for $i = k+1$. We first make the assumption that σ is not in the point spectrum of H. Using our inductive assumption, it is enough to show that $E^\infty_{\mathrm{ess}}(\sigma, \Gamma_k)$ is dense in $E^\infty_{\mathrm{ess}}(\sigma, \Gamma_{k+1})$. Let $u \in E^\infty_{\mathrm{ess}}(\sigma, \Gamma_{k+1})$. Let $Q \in \Psi^{0,0}_{\mathrm{sc}}(X)$ be microlocally equal to the identity near $\Gamma_k \cap \mathrm{Min}_+(\sigma)$, and microsupported sufficiently close to $\Gamma_k \cap \mathrm{Min}_+(\sigma)$. Then away from $\mathrm{Min}_+(\sigma)$, $u \in x^{-1/2+\epsilon} L^2$ by (ii) of Theorem 6.7 and thus $(H - \sigma) Q u = [H, Q] u \in x^{1/2+\epsilon} L^2$ for some $\epsilon > 0$. This is also true near $\mathrm{Min}_+(\sigma)$ since Q is microlocally the identity there, so we have $(H - \sigma) Q u \in x^{1/2+\epsilon} L^2$ everywhere. This implies that

$$u = Qu - R(\sigma - i0)(H - \sigma) Qu, \tag{10–4}$$

since $v = u - (Qu - R(\sigma - i0)(H - \sigma) Qu)$ satisfies $(H - \sigma) v = 0$ and $v \in x^{-1/2+\epsilon} L^2$ microlocally for $\nu > 0$.

Now choose a modified potential function $\widetilde{V}$ as in Lemma 8.5, where we choose $\tilde{\nu}$ larger than $\nu(q)$ but smaller than $\nu(q')$ for every $q' \in \Gamma_k \cap \mathrm{Min}_+(\sigma)$. (This is possible because of the way we ordered the q_i.) Since $\mathrm{WF}_{\mathrm{sc}}(Qu)$ lies in $\{\nu > \tilde{\nu}\}$, we have

$$Qu = \widetilde{R}(\sigma + i0)(\widetilde{H} - \sigma) Qu. \tag{10–5}$$

Now take $u'_j = \phi(x/r_j) u$, where $\phi \in \mathscr{C}^\infty(\mathbb{R})$, $\phi(t) = 1$ for $t \geq 2$, $\phi(t) = 0$ for $t \leq 1$ and $r_j \to 0$ as $j \to \infty$. Then $u'_j \in \dot{C}^\infty(X)$, and w_j defined by

$$w_j = \widetilde{R}(\sigma + i0)(\widetilde{H} - \sigma) Q u'_j$$

converge to Qu in $x^{-1/2-\epsilon} L^2$. Our choice of $\widetilde{V}$ ensures that

$$\mathrm{WF}_{\mathrm{sc}}(w_j) \cap \mathrm{RP}_+(\sigma) \subset \Gamma_k.$$

Moreover,

$$(H - \sigma) w_j \text{ converges to } (H - \sigma) Qu \text{ in } x^{1/2+\epsilon} L^2. \tag{10–6}$$

Now define

$$u_j = w_j - R(\sigma - i0)(H - \sigma) w_j.$$

Then $u_j \in E^\infty_{\mathrm{ess}}(\sigma, \Gamma_k)$. We claim that $u_j \to u$ in the topology of $E^0_{\mathrm{ess}}(\sigma)$. Certainly, $u_j \to u$ in $x^{-1/2-\epsilon}L^2$. We must also show that $Bu_j \to Bu$ in $x^{-1/2}L^2$, where B is as in (9–2). To do this we write

$$\begin{aligned} Bu_j - Bu &= B\big(w_j - R(\sigma - i0)(H-\sigma)w_j\big) - B\big((\mathrm{Id}-Q)u + Qu\big) \\ &= B\big(\widetilde{R}(\sigma+i0)(\widetilde{H}-\sigma)Qu'_j - R(\sigma-i0)(H-\sigma)w_j \\ &\qquad\qquad + R(\sigma-i0)(H-\sigma)Qu - \widetilde{R}(\sigma+i0)(\widetilde{H}-\sigma)Qu\big), \end{aligned}$$

using (10–4) and (10–5), and this goes to zero in $x^{-1/2}L^2$ by (10–6) and propagation of singularities, Theorem 3.1 of [Hassell et al. 2004], as in the proof of [Hassell et al. 2004, Proposition 11.6].

If σ is in the point spectrum of H, then Equation (10–4) must be replaced by

$$u = \Pi\Big(Qu - R(\sigma - i0)(H-\sigma)Qu\Big),$$

where Π is projection off the L^2 σ-eigenspace. Consequently we must define w_j by $\Pi\widetilde{R}(\sigma+i0)(\widetilde{H}-\sigma)Qu'_j$, and then the rest of the proof goes through. □

The second intermediate result we need is:

Proposition 10.4. *The Hilbert norm (9–3) on the subspace $E^\infty_{\mathrm{Min},+}(\sigma) \subset E^0_{\mathrm{ess}}(\sigma)$ is given by the formula*

$$\|u\|^2_{E^0_{\mathrm{ess}}(\sigma)} = \sum_{q\in \mathrm{Min}_+(\sigma)} 2\sqrt{\sigma - V(\pi(q))}\int_{\mathbb{R}^{n-1}} \big|M^+(q,\sigma)u\big|^2\, d\omega_q. \tag{10–7}$$

Proof. The proof is the same as the one dimensional case, which is proved in Proposition 12.6 of [Hassell et al. 2004], so we just give a sketch here.

Let ϕ be as in the proof of Proposition 10.3. Then we can write the natural norm (9–3) on $E^0_{\mathrm{ess}}(\sigma)$ as a limit

$$\lim_{r\to 0} i\langle (H-\sigma)Au, \phi(x/r)u\rangle = \lim_{r\to 0} i\langle Au, [H,\phi(x/r)]u\rangle.$$

Since $u \in x^{-1/2-\epsilon}L^2$, the only term in $[H,\phi(x/r)]$ contributing in the limit is $2(x^2D_x)\phi(x/r)(x^2D_x)$. The cutoff operator A restricts attention to $\{\nu > 0\}$, and the limit vanishes when localized to any region where $u \in x^{-1/2+\epsilon}L^2$, so we can substitute for u a sum of expressions u_q as in (6–18) in the effectively nonresonant case, or its analogue in the effectively resonant setting arising from (7–2) (namely $e^{i\Phi(y)/x}$ times an expression as in (7–2); see Remark 6.8), one for each $q \in \mathrm{Min}_+(\sigma)$. A straightforward computation then gives (10–7). □

Proof of Theorem 10.1. Proposition 10.4 shows that $M_+(\sigma)$ maps $E^\infty_{\mathrm{Min},+}(\sigma)$ into a dense subspace of $\bigoplus_q L^2(\mathbb{R}^{n-1})$, with the Hilbert norm of $M_+(\sigma)u$, $u \in E^\infty_{\mathrm{Min},+}(\sigma)$, equal to that of u. By Proposition 10.3, $E^\infty_{\mathrm{Min},+}(\sigma)$ is dense in $E^\infty_{\mathrm{ess}}(\sigma)$, and by Corollary 3.13 of [Hassell et al. 2004], $E^\infty_{\mathrm{ess}}(\sigma)$ is dense in $E^0_{\mathrm{ess}}(\sigma)$. The result follows. □

So far we have only considered the microlocal restriction of eigenfunctions near radial points q satisfying $\nu(q) > 0$. For each critical point of V_0, there are two corresponding radial points with opposite signs of ν, and we can equally well consider microlocal restriction near radial points with $\nu(q) < 0$. This leads to an operator

$$M_-(\sigma) : E^0_{\mathrm{ess}}(\sigma) \to \bigoplus_{q\in \mathrm{Min}_-(\sigma)} L^2(\mathbb{R}^{n-1})$$

and the analogue of Theorem 10.1 holds also for $M_-(\sigma)$.

Definition 10.5. The inverses of $M_\pm(\sigma)$, $P_\pm(\sigma) : \bigoplus_{q\in \mathrm{Min}_\pm(\sigma)} L^2(\mathbb{R}^{n-1}) \to E^0_{\mathrm{ess}}(\sigma)$ of $M_\pm(\sigma)$ are called the *Poisson operators at energy* σ.

We can identify $\bigoplus_{q\in \mathrm{Min}_+(\sigma)} L^2(\mathbb{R}^{n-1})$ and $\bigoplus_{q\in \mathrm{Min}_-(\sigma)} L^2(\mathbb{R}^{n-1})$ in the obvious way, and may therefore assume that the $M_\pm(\sigma)$ have the same range, identified with the domain of $P_\pm(\sigma)$.

Corollary 10.6. *For $\sigma \notin \mathrm{Cv}(V)$, the S-matrix may be identified as the unitary operator*

$$S(\sigma) = M_+(\sigma) P_-(\sigma)$$

on $\bigoplus_{z\in \mathrm{Min}} L^2(\mathbb{R}^{n-1})$.

Remark 10.7. For $n = 2$, the structure of $S(\sigma)$ was described rather precisely in [Hassell et al. 2001] as an anisotropic Fourier integral operator.

Theorem 10.1 is essentially a pointwise version of asymptotic completeness in σ. Integrating gives a version of the usual statement, but some uniformity in σ is required for this. So we proceed to discuss an extension of part (i) of Theorem 6.7 that is valid in an interval rather than just at one value. For this purpose, let $I \subset (\min V_0, \infty)$ be a compact interval disjoint from the set of effectively resonant energies, the set of Hessian thresholds and $\mathrm{Cv}(V)$. Then for each $\sigma \in I$, the sets $\mathrm{Min}_+(\sigma) \subset \mathrm{RP}_+(\sigma)$ can be identified; we write $\mathrm{Min}_+(I)$ for this set. Each element of $\mathrm{Min}_+(I)$ is thus a continuous family $q(\sigma)$ of minimal radial points, with $q(\sigma) \in \mathrm{Min}_+(\sigma)$.

Proposition 10.8. *Let $I \subset (\min V_0, \infty)$ be as above, and let the $q(\sigma) \in \mathrm{Min}_+(I)$ be an outgoing radial point associated to a minimum point z of V_0, with Y'', Y''' the associated coordinates given by (5–18). For any $h(\sigma, \cdot) \in \mathscr{C}^\infty(I; \mathscr{S}(\overline{\mathbb{R}^{n-1}}))$ there is $\phi \in \dot{\mathscr{C}}^\infty(X)$ orthogonal to $E_{\mathrm{pp}}(I)$ such that for every $\sigma \in I$,*

$$\begin{aligned} F(\sigma)^{-1} R(\sigma + i0)\phi &= \sum_j x^{-i\tilde{b} - i\kappa_j} w_j(Y'', \sigma) v_j(Y''', \sigma) + u', \\ h(\sigma, Y'', Y''') &= \sum_j w_j(Y'', \sigma) v_j(Y''', \sigma), \end{aligned} \tag{10–8}$$

where w_j, v_j, κ_j and $\tilde{b}$ are as in Proposition 6.3, where $u' \in \mathscr{C}^\infty(I; I^{(l)}_{\mathrm{sc}}(X, \mathcal{M}))$ for some $l > -\frac{1}{2}$, and $F(\sigma)$ is as in Theorem 3.17.

Remark 10.9. The statement $u' \in \mathscr{C}^\infty(I; I^{(l)}_{\mathrm{sc}}(X, \mathcal{M}))$ is meant to underline that this is a global claim, namely $u' \in \mathscr{C}^\infty(I; I^{(l)}_{\mathrm{sc}}(O, \mathcal{M}))$ and that this is $\mathscr{C}^\infty$ with values in $\dot{\mathscr{C}}^\infty(X)$ microlocally away from $\{q(\sigma); \sigma \in I\}$, that is, for all $A \in \Psi_{\mathrm{sc}}(X)$ with $\mathrm{WF}'_{\mathrm{sc}}(A) \cap \{q(\sigma); \sigma \in I\} = \varnothing$, $Au' \in \mathscr{C}^\infty(I; \dot{\mathscr{C}}^\infty(X))$.

Proof. By the construction of Section 6, for each $\sigma \in I$ there is an approximate microlocally outgoing solution u_σ with $f_\sigma = (H - \sigma)u_\sigma \in \dot{\mathscr{C}}^\infty(X)$ and $F(\sigma)^{-1} u_\sigma$ of the same form as the right hand side of (10–8). Indeed, the construction is smooth in σ, in the sense that $(d/d\sigma)^k u \in I^s(O, \mathcal{M})$ for each k and each $s < -1/2$, so that with $f(\sigma, .) = f_\sigma(.)$, we have $f \in \mathscr{C}^\infty(I; \dot{\mathscr{C}}^\infty(X))$. Notice that there is no need to "globalize" using Proposition 8.1, since microlocally outgoing solutions over sources/sinks (that is, minima of V_0) are localized at $q(\sigma)$.

Let $\tilde{f} \in \dot{\mathscr{C}}_c^\infty(\mathbb{C} \times X)$ be an almost analytic extension of f with compact support, so $\bar{\partial}_\sigma f$ vanishes to infinite order at $\mathbb{R} \times X$, and let

$$\phi = \frac{-1}{2\pi i} \int_{\mathbb{C}} R(\sigma)\bar{\partial}_\sigma \tilde{f}\, d\sigma \wedge d\bar{\sigma}.$$

Thus, $\phi \in \dot{\mathscr{C}}^\infty(X)$ since $\bar{\partial}_\sigma \tilde{f}$ vanishes to infinite order on the real axis.

We also claim that (10–8) holds. Indeed, let $\sigma_0 \in \mathbb{R}$, $\chi \in \mathscr{C}_c^\infty(\mathbb{R})$, χ identically 1 near σ_0, let $\tilde{\chi}$ be an almost analytic extension of χ of compact support. Thus,

$$f(\sigma, .) = f(\sigma_0, .)\chi(\sigma) + (\sigma - \sigma_0)g(\sigma, .), \;\; \tilde{f}(\sigma, .) = f(\sigma_0, .)\tilde{\chi}(\sigma) + (\sigma - \sigma_0)\tilde{g}(\sigma, .)$$

with $g \in \dot{\mathscr{C}}_c^\infty(\mathbb{R} \times X)$, $\tilde{g} \in \dot{\mathscr{C}}_c^\infty(\mathbb{C} \times X)$. Then, writing $\sigma - \sigma_0 = (H - \sigma_0) - (H - \sigma)$,

$$\begin{aligned}\phi =& \frac{-1}{2\pi i}\Big(\int_{\mathbb{C}} R(\sigma)\bar{\partial}_\sigma \tilde{\chi}\, d\sigma \wedge d\bar{\sigma}\Big) f(\sigma_0, .)\\ &- \frac{1}{2\pi i}(H - \sigma_0)\int_{\mathbb{C}} R(\sigma)\bar{\partial}_\sigma \tilde{g}\, d\sigma \wedge d\bar{\sigma} + \frac{1}{2\pi i}\int_{\mathbb{C}} \bar{\partial}_\sigma \tilde{g}\, d\sigma \wedge d\bar{\sigma},\end{aligned}$$

where in the last term the identity $(H - \sigma)R(\sigma) = \mathrm{Id}$ is used. Since the last term vanishes (as $\tilde{g}$ is smooth), and the integral in the second term is in $\dot{\mathscr{C}}^\infty(X)$, while the integral in the first term is $\chi(H)$, we deduce that

$$\phi = f_{\sigma_0} + (H - \sigma_0) f'_{\sigma_0} = (H - \sigma_0)(u_{\sigma_0} + f'_{\sigma_0})$$

for some $f' \in \dot{\mathscr{C}}^\infty(I \times X)$. Then if $v \in E_{\mathrm{pp}}(I)$, $(H - \sigma_0)v = 0$, we have $v \in \dot{\mathscr{C}}^\infty(X)$, so $\langle \phi, v\rangle = \langle u_{\sigma_0} + f'_{\sigma_0}, (H - \sigma_0)v\rangle = 0$. Also $R(\sigma_0 + i0)\phi - R(\sigma_0 + i0) f_{\sigma_0} = f'_{\sigma_0} \in \dot{\mathscr{C}}^\infty(X)$, so $R(\sigma_0 + i0)\phi$ and $R(\sigma_0 + i0) f_{\sigma_0}$ indeed have the same asymptotics. In particular, (10–8) holds for every $\sigma_0 \in \mathbb{R}$. □

Now we state asymptotic completeness in a more standard form.

Theorem 10.10 (Asymptotic completeness). *Let $I \subset (\min V_0, \infty)$ be a compact interval as above. Then*

$$M_+(\cdot) \circ \mathrm{Sp}(\cdot) : \mathrm{Ran}(\Pi_I) \ominus E_{\mathrm{pp}}(I) \to \bigoplus_{q \in \mathrm{Min}_+(I)} L^2(I \times \mathbb{R}_q^{n-1};\; 2\pi\, d\sigma\, d\omega_{q,\sigma})$$

is unitary. Here, as before, $d\omega_{q,\sigma} = 2\sqrt{\sigma - V(\pi(q))}\, d\omega_q$.

Proof. For $f \in \dot{\mathscr{C}}^\infty(X)$ orthogonal to $E_{\mathrm{pp}}(I)$, let

$$u = u(\sigma) = (2\pi i)^{-1}(R(\sigma + i0) f - R(\sigma - i0) f) = \mathrm{Sp}(\sigma) f, \qquad \mathrm{Sp}(\sigma) = (2\pi i)^{-1}(R(\sigma + i0) - R(\sigma - i0))$$

is the spectral measure. The norm of u in $E^0_{\mathrm{ess}}(\sigma)$ is given by $\langle i(H - \sigma)Au, u\rangle$, where A is as in (9–3). Notice that

$$\begin{aligned}2\pi i(H - \sigma)Au - f &= (H - \sigma)A\big(R(\sigma + i0) - R(\sigma - i0)\big) f - (H - \sigma)R(\sigma + i0) f\\ &= (H - \sigma)\Big((A - \mathrm{Id})R(\sigma + i0) f - AR(\sigma - i0) f\Big) = (H - \sigma)v, \quad v \in \dot{C}^\infty(X),\end{aligned}$$

since

$$\mathrm{WF}'_{\mathrm{sc}}(A) \cap \mathrm{WF}_{\mathrm{sc}}(R(\sigma - i0) f) = \varnothing \text{ and } \mathrm{WF}'_{\mathrm{sc}}(A - \mathrm{Id}) \cap \mathrm{WF}_{\mathrm{sc}}(R(\sigma + i0) f) = \varnothing.$$

Hence

$$2\pi \|u\|^2_{E^0_{\mathrm{ess}}(\sigma)} = 2\pi i\langle (H - \sigma)Au, u\rangle = \langle f + (H - \sigma)v, u\rangle = \langle f, \mathrm{Sp}(\sigma) f\rangle.$$

The right hand side is continuous, hence so is the left hand side.

Integrating over σ in I, denoting the spectral projection of H to I by Π_I, and using Proposition 10.4, we deduce that $M_+(\sigma)\,\mathrm{Sp}(\sigma) f$ is continuous with values in L^2 and

$$\|\Pi_I f\|^2 = 2\pi \int_I \|M_+(\sigma)\,\mathrm{Sp}(\sigma) f\|^2\, d\sigma,$$

so $M_+(\cdot)\circ \mathrm{Sp}(\cdot)$ is an isometry on the orthocomplement of the finite dimensional space $E_{\mathrm{pp}}(I)$ in the range of Π_I.

It remains to prove that the range is dense in $\bigoplus_{q\in \mathrm{Min}} L^2(I\times\mathbb{R}^{n-1})$. It suffices to show that if $h \in \bigoplus_{q\in\mathrm{Min}} \dot{\mathscr{C}}^\infty(I\times\overline{\mathbb{R}^{n-1}})$, then there is a $f\in\dot{\mathscr{C}}^\infty(X)$ with $M_+(\sigma)\,\mathrm{Sp}(\sigma) f = h(\sigma, .)$. But this was proved in Proposition 10.8, so the proof of the theorem is complete. □

Remark 10.11. The results of this section can be related more closely with Theorem 9.2 by considering the closure of $E^\infty_{\mathrm{Min},+}(\sigma)$ as a subset of $E^\infty_{\mathrm{ess}}(\sigma)$ in the topology of $E^s_{\mathrm{ess}}(\sigma)$ for varying values of s. We have seen in Proposition 10.3 that $E^\infty_{\mathrm{Min},+}(\sigma)$ is dense, in the topology of $E^0_{\mathrm{ess}}(\sigma)$. In fact the proof of Proposition 10.3 shows that this is true in the topology of $E^s_{\mathrm{ess}}(\sigma)$ for $0\le s < s_0$, where s_0 is the smallest number such that every $u\in E^\infty_{\mathrm{mic}}(q)$, for every $q\in \mathrm{RP}_+(\sigma)\setminus \mathrm{Min}_+(\sigma)$, is in $x^{-1/2+s_0}L^2$ locally near $\pi(q)$; that s_0 is strictly positive follows from (ii) of Theorem 6.7. By contrast, $E^\infty_{\mathrm{Min},+}(\sigma)$ is closed in the $E^\infty_{\mathrm{ess}}(\sigma)$ topology. What happens as s increases is that the closure of $E^\infty_{\mathrm{Min},+}(\sigma)$ in the $E^s_{\mathrm{ess}}(\sigma)$ topology changes discretely, as s crosses certain values determined by the structure of eigenfunctions at the nonminimal critical points.

One way to understand this is in terms of microlocally *incoming* eigenfunctions at the outgoing radial points, that is, microlocal eigenfunctions u with scattering wavefront set near q is contained in $\Phi_-(q)$ as opposed to $\Phi_+(q)$. In Part I we showed (in all dimensions) that there are nondegenerate pairings

$$E_{\mathrm{mic},+}(q,\sigma)\times E_{\mathrm{mic},-}(q,\sigma)\to\mathbb{C},$$
$$E^s_{\mathrm{ess}}(\sigma)\times E^{-s}_{\mathrm{ess}}(\sigma)\to\mathbb{C}$$

(Lemma 12.2 and Proposition 12.3 of [Hassell et al. 2004]). The closure of $E^\infty_{\mathrm{Min},+}(\sigma)$, in the topology of $E^s_{\mathrm{ess}}(\sigma)$, may be identified with the annihilator, in $E^\infty_{\mathrm{ess}}(\sigma)$, of the eigenfunctions which are in $E^{-s}_{\mathrm{ess}}(\sigma)$ and have scattering wavefront set contained in

$$\bigcup_{q\in \mathrm{RP}_+(\sigma)\setminus \mathrm{Min}_+(\sigma)} \Phi_-(q)\cup\{\nu<0\}.$$

This set is trivial for $s<s_0$, and nontrivial for $s>s_0$. The fact that this set of eigenfunctions jumps discretely with s in shown in the two dimensional case in Section 10 of Part I.

11. Time-dependent Schrödinger equation

11.1. ***Long-time asymptotics.*** In this final section we apply the earlier results to deduce the long-time asymptotics for solutions of the initial value problem

$$(D_t+H)u=0,\ \ u|_{t=0}=u_0,\ \ u_0\in\dot{\mathscr{C}}^\infty(X), \tag{11–1}$$

for a dense set (in $L^2\ominus E_{\mathrm{pp}}(H)$) of initial data.

Our approach is to use the spectral resolution of u_0 and the functional calculus. In this way, we deduce the long-time asymptotics of u from the asymptotics of generalized eigenfunctions of H using the stationary phase lemma.

We first define the space X_{Sch} on which the asymptotics of the solution u of (11–1) will be described. Let us first choose a global boundary defining function x satisfying (1–1); we can specify, for example, that $x \equiv 1$ outside a collar neighbourhood of ∂X. We then introduce the variable $\tau = tx$, where t is time. Let us compactify the real τ-line $\mathbb{R}$ to an interval $\overline{\mathbb{R}}$ using τ^{-1} as a boundary defining function near $\tau = \infty$, and $-\tau^{-1}$ as a boundary defining function near $\tau = -\infty$. Then we define

$$X_{\mathrm{Sch}} = X \times \overline{\mathbb{R}}_\tau \tag{11–2}$$

Thus X_{Sch} is a compact manifold with corners, with boundary hypersurfaces if (the "infinity face") at $\tau = \pm\infty$ (or $t = \pm\infty$), naturally diffeomorphic to two copies of X (one at $t = +\infty$, one at $t = -\infty$), and a boundary hypersurface af (the "asymptotic face") diffeomorphic to $\partial X \times \overline{\mathbb{R}}_\tau$. At af, every point with $\tau > 0$ corresponds to $t = +\infty$ and every point with $\tau < 0$ corresponds to $t = -\infty$, so this is the place to look for long-time (and large-distance) asymptotics of the Schrödinger wave u. The variable τ has an interpretation of inverse speed; a particle travelling asymptotically radially at speed τ_0^{-1} will end up at af after infinite time at $\tau = \tau_0$.

We now specify a good subset of L^2 initial data u_0, for which the asymptotics as $t \to +\infty$ of the solution, u, to (11–1) are particularly simple. Let $I \subset (\min V_0, \infty)$ be a compact interval disjoint from $\mathrm{Cv}(V)$ and from the set of effectively resonant energies and Hessian thresholds. Let $(h(\sigma, \cdot))q \in \mathscr{C}^\infty(I; \mathscr{S}(\mathbb{R}^{n-1}))$ be a collection of smooth functions from I into Schwartz functions of $n-1$ variables, one for each $q \in \mathrm{Min}_+(I)$, and let $\phi = \phi(I, h) = \sum_q \phi(I, h_q) \in \dot{C}^\infty(X)$ be the function constructed in Proposition 10.8. Let

$$\mathscr{A}_I = \{\phi(I, h); h(\sigma, \cdot) \in \mathscr{C}^\infty(I; \mathscr{S}(\mathbb{R}^{n-1}))\} \quad \text{and} \quad \mathscr{A} = \sum_I \mathscr{A}_I$$

be the (algebraic) vector space sum of $\mathscr{A}_I$ over all such I as above. It is clear from Theorem 10.10 that $\mathscr{A}_I$ is dense in $\operatorname{Ran} \Pi_I(H) \ominus E_{\mathrm{pp}}(I)$, and hence that $\mathscr{A}$ is dense in $L^2 \ominus E_{\mathrm{pp}}(H) = H_{\mathrm{ac}}(H)$. To give the asymptotics of (11–1) with initial data from $\mathscr{A}$ it suffices to give the asymptotics starting from $u_0 = \phi(I, h)$ for some h as above.

Theorem 11.1. *Suppose that I is as above and that $\phi = \phi(I, h) \in \mathscr{A}_I$. Let $u(\cdot, t)$ be the solution of (11–1) with initial data $u_0 = \phi$, regarded as a function on X_{Sch}. Then u has trivial asymptotics (that is, u and all its derivatives are $O(t^{-\infty})$) at if. Also, if $w \in \partial X$ is not a local minimum of V_0, and $\tau > 0$, then u has trivial asymptotics in a neighbourhood of $(w, \tau) \in af$.*

Let z be a local minimum of V_0, and let (Y'', Y''') be the coordinates given by (5–18), where σ is determined in terms of τ by (11–4). Then, in a neighbourhood of $(z, \tau) \in af$, u takes the form

$$u(x, Y'', Y''', \tau) = c\tau^{-3/2} \sum_j x^{-i\tilde{b} - i\kappa_j + 1/2} e^{i\Psi(y,\tau)/x} w_j(Y'', \sigma(\tau)) v_j(Y''', \sigma(\tau)) + u', \tag{11–3}$$

where

$$h(\sigma(\tau), Y'', Y''') = \sum_j w_j(Y'', \sigma(\tau)) v_j(Y''', \sigma(\tau)), \quad c = \frac{1}{2\sqrt{\pi}} e^{-3i\pi/4}, \quad \sigma(\tau) = V_0(z) + \frac{1}{4\tau^2}, \tag{11–4}$$

$\tilde{b}$ is as in (6–5) and (6–3), κ_j is as in (6–6), Ψ is a smooth function of y and τ, h is decomposed as in Proposition 10.8, and u' decays faster than the leading term.

Proof. Let $v(\sigma) = \mathrm{Sp}(\sigma)\phi = (2\pi i)^{-1}(R(\sigma + i0) - R(\sigma - i0))\phi$. Then

$$u(t, \cdot) = \frac{1}{2\pi i} \int_I e^{-it\sigma} (R(\sigma + i0) - R(\sigma - i0))\phi \, d\sigma.$$

Shifting the contour of integration shows that, as $t \to \infty$, $R(\sigma - i0)\phi$ has trivial asymptotics. Hence it is enough to consider

$$u(t, \cdot) = \frac{1}{2\pi i} \int_I e^{-it\sigma} R(\sigma + i0)\phi \, d\sigma. \tag{11–5}$$

Let $F(\sigma)$ be the FIO constructed in Theorem 3.17, which conjugates $x^{-1}(H - \sigma)$ to normal form microlocally near the point $q \in \mathrm{RP}_+(\sigma)$ with $\pi(q) = z$. By construction, $F(\sigma)^{-1} R(\sigma + i0)\phi$ has asymptotics (10–8) for every σ. Since $F(\sigma)$ is a smooth family of FIOs, and (10–8) is a Legendre distribution associated to the zero section (that is, it is conormal at $x = 0$ with no oscillatory factor), it follows that $R(\sigma + i0)\phi$ itself has asymptotics

$$R(\sigma + i0)\phi = x^{-i\tilde{b} - i\kappa_j} e^{i\Phi(y,\sigma)/x} a(Y'', Y''', x, \sigma) + v', \tag{11–6}$$

where $\Phi(\cdot, \sigma)$ is a smooth function, parametrizing the image of the zero section under the canonical relation of $F(\sigma)$ (as in (6–18)). By assumption, a is smooth in σ, conormal in x and Schwartz in (Y'', Y'''). At the critical point z we have

$$\Phi(z, \sigma) = \sqrt{\sigma - V_0(z)}, \quad z = \pi(q), \quad q \in \mathrm{Min}_+(\sigma). \tag{11–7}$$

We may substitute (11–6) into (11–5) and compute

$$u(t, \cdot) = \frac{1}{2\pi i} \int_I e^{-it\sigma} \Big(e^{i\Phi(y,\sigma)/x} a(Y'', Y''', x, \sigma) + v' \Big) \, d\sigma, \tag{11–8}$$

exploiting the smoothness of Φ and a in σ.

Let $p \in X$ be an interior point. Then $(R(\sigma \pm i0)\phi)(p)$ is a smooth function of σ by Proposition 10.8. It follows that for a fixed interior point p the integral (11–8) is rapidly decreasing as $t \to \infty$, being the Fourier transform of a smooth, compactly supported function. Hence the asymptotics of u are trivial at af.

To investigate asymptotics at af, where $x \to 0$, we rewrite (11–8) as

$$u(\tau, x, Y'', Y''') = \frac{1}{2\pi i} \int e^{i(-\tau\sigma + \Phi(y,\sigma))/x} \Big(a(Y'', Y''', x, \sigma) + v'(Y'', Y''', x, \sigma) \Big) \, d\sigma, \tag{11–9}$$

and apply stationary phase to the integral. We first note that for any $w \in \partial X$ which is not a local minimum of V_0, the integrand is rapidly decreasing as $x \to 0$ in a neighbourhood of $(w, \sigma) \in \partial X \times I$, uniformly in

σ, so u is rapidly decreasing as $x \to 0$ in a neighbourhood of $(w, \tau) \in$ af. So we may restrict attention to a neighbourhood of $(z, \tau) \in$ af, where z is a local minimum of V_0.

To do this we apply stationary phase to (11–9). The phase is critical when $\tau = d_\sigma \Phi(y, \sigma)$. Since Φ is smooth in y, this gives

$$\tau_{\text{critical}} = d_\sigma \Phi(z, \sigma) + O(Y_i x^{r_i})$$

and, since a is Schwartz in Y, to compute the expansion of u to leading order we may drop the $O(Y_i x^{r_i})$ terms when we substitute $\tau = \tau_{\text{critical}}$ into a in (11–8). Since $\Phi(z, \sigma)$ is given by (11–7), we may therefore take τ in the argument of a to be given by $\tau = d_\sigma \Phi(z, \sigma)$ which implies (11–4). Moreover, the Hessian of the phase function at the critical point is $(4x)^{-1}(\sigma - V_0(z))^{-3/2} = 2x^{-1}\tau^3$. The stationary phase lemma then gives (11–3), with $\Psi(y, \tau) = -\tau\sigma(\tau) + \Phi(y, \sigma(\tau))$. □

Remark 11.2. Equation (11–4) is just the energy equation "total energy = potential energy+ kinetic energy" at infinity, since $1/\tau$ is the asymptotic speed. The factor $1/4$ comes from the fact that in writing our Hamiltonian as $\Delta + V$, we have taken the value of mass to be $1/2$ in our units.

Remark 11.3. We may *not* replace $\Psi(y, \tau)$ with $\Psi(z, \tau)$ in (11–3), due to the singular factor $1/x$ in the phase. In fact, if we expand $\Psi(y, \tau)$ in a Taylor series about $y = z$, written in terms of the variables $Y_i = y_i / x^{r_i}$, then we get an asymptotic expansion involving polynomials y^α in the variables y_i multiplied by nonnegative powers x^r, where $r = \sum \alpha_i r_i$. We may discard all terms in the Taylor series of Ψ with $r > 1$ since these will only contribute to the term u' decaying faster than the leading term, but we must keep all terms with $r \le 1$. The number of such terms is always finite, but depends on $\sigma - V_0(z)$ and the eigenvalues of the Hessian of V_0 at z.

11.2. *Asymptotic completeness: time dependent formulation.* We see that solutions of the time dependent Schrödinger equation (at least those with initial data in $\mathcal{A}$) have expansions at af which are equivalent to first spectrally resolving the initial data and looking at the expansion of the corresponding family of generalized eigenfunctions; the variable σ in the time-independent setting, and τ in the time-dependent setting, play equivalent roles and are linked by (11–4). In view of this, we can recast Theorem 10.10 in time-dependent terms as follows:

Theorem 11.4. *Let I and $\mathcal{A}_I$ be as in Theorem 11.1, let $u_0 \in \mathcal{A}_I$ and let u be the solution of the time dependent Schrödinger Equation (11–1) with initial data u_0. For a given local minimum z of V_0, let* $\mathrm{Min}_+(I)$ *be the associated family of outgoing radial points, and let $\widetilde{Q} = \sum_j \widetilde{Q}_j$, $\tilde{b}$ and κ_j be as in Proposition 6.3. The map*

$$\mathcal{A}_I \ni u_0 \mapsto \bigoplus_{q \in \mathrm{Min}_+(I)} \Big(e^{i \log x \widetilde{Q}} x^{i\tilde{b} - 1/2} e^{-i\Psi(y,\tau)/x} u(x, \tau, Y'', Y''') \Big)\Big|_{x=0} \tag{11–10}$$

whose existence is guaranteed by Theorem 11.1 extends uniquely by linearity and continuity to a unitary isomorphism

$$L^2 \ominus E_{\mathrm{pp}}(H) \to \bigoplus_q L^2\Big(\mathbb{R}^+_\tau \times \mathbb{R}^{n-1}_q; \frac{d\tau}{2\tau^4} \otimes \omega_{q,\tau}\Big). \tag{11–11}$$

Here $\omega_{q,\tau}$ is the measure in (10–2) and $\tau = \tau(q, \sigma)$ is given by (11–4).

Remark 11.5. The operator $e^{i \log x \widetilde{Q}}$ simply removes the factors of $x^{-i\sigma_k}$ in the expansion (11–3), so that we can take a limit as $x \to 0$.

Remark 11.6. The measure in (11–11) should be thought of as the product of $\tau^{-1}\omega_{q,\tau}$, which is the measure in Proposition 10.4, tensored with the measure $d\sigma = \tau^{-3}d\tau/2$.

Proof. Let $u_0 = \phi(I, h)$ be as in Theorem 11.1. We may take the L^2 norm of (11–3) for a fixed t, and take the limit as $t \to \infty$. To do this, we write $x = \tau/t$ and integrate with respect to the measure on X which is given by $t\tau^{-2}x^{-1-2\,\mathrm{Im}\,\tilde{b}}dY\,d\tau$. If we just look at the principal term in (11–3) then the powers of t cancel exactly and we get

$$\sum_q \int \frac{1}{2\pi}\Big|\sum_j w_j(Y'', \sigma(\tau))v_j(Y''', \sigma(\tau))\Big|^2 dY\,\frac{d\tau}{2\tau^4}. \tag{11–12}$$

Since $\omega_{q,\sigma} = \tau^{-1}dY$, and $d\sigma = \tau^{-3}d\tau/2$ using (11–4), this is given by

$$\frac{1}{2\pi}\sum_q \int_I 2\sqrt{\sigma - V_0(z)}\Big|\sum_j w_j(Y'', \sigma)v_j(Y''', \sigma)\Big|^2 d\omega_{q,\sigma}\,d\sigma.$$

The expression $\sum_j w_j v_j$ is equal to

$$M_+(q, \sigma)((R(\sigma + i0) - R(\sigma - i0))u_0,$$

or equivalently $2\pi i M_+(q, \sigma)\,\mathrm{Sp}(\sigma)u_0$. Also, the norm on $\bigoplus_q L^2(\mathbb{R}^{n-1})$ is given by (10–2). So we get, using Theorems 10.1 and 10.10,

$$(11\text{–}12) = 2\pi \sum_q \int_I \|M_+(q, \sigma)\,\mathrm{Sp}(\sigma)u_0\|^2\,d\sigma = \|u_0\|^2_{L^2}.$$

But (11–12) is precisely the square of the norm of the right hand side of (11–10). So we have established the conclusion of the theorem for the principal term in the asymptotic expansion in (11–3). Since the remainder term u' decays faster than the principal term, the L^2 norm of $u'(\cdot, t)$ goes to zero as $t \to \infty$, so the proof is complete. □

From Theorem 11.4 we can deduce the following result first proved by Herbst and Skibsted, using a direct method involving the uncertainty principle, rather than proceeding via the structure of generalized eigenfunctions as here.

Corollary 11.7 (Absence of L^2 channels at nonminimal critical points). *Let $\chi \in \mathscr{C}^\infty(X)$ vanish in a neighbourhood of the local minima of V_0 on ∂X. Let u be the solution of (11–1) on $X \times \mathbb{R}$ with initial value $u_0 \in L^2(X) \ominus E_{\mathrm{pp}}(H)$. Then*

$$\lim_{t\to\infty} \|\chi u(t, \cdot)\|_{L^2(X)} \to 0. \tag{11–13}$$

Proof. We may assume that $0 \le \chi \le 1$ without loss of generality.

Let $\epsilon > 0$ be given. Then by density of $\mathscr{A}$ in $L^2 \ominus E_{\mathrm{pp}}(H)$, we can find $\phi \in \mathscr{A}$, with ϕ equal to a sum of a finite number of $\phi_j(I_j, h_j) \in \mathscr{A}_{I_j}$, such that $\|u_0 - \phi\|_{L^2} < \epsilon$. Without loss of generality we may assume that all the I_j are disjoint. Let u' be the solution with initial condition ϕ. By direct calculation from (11–3) we find that

$$\lim_{t\to\infty} \|(1-\chi)u'(t, \cdot)\|^2_{L^2} = \sum_j \int_I \sqrt{\sigma - V_0(\pi(q))}\,\|h_j\|^2_{L^2(R^{n-1})}\,d\sigma,$$

which by Theorem 10.10 is equal to $\|\phi\|^2_{L^2}$. But by unitarity of e^{-itH}, we have

$$\|u'(t,\cdot)\|^2_{L^2} = \|\phi\|^2_{L^2} \text{ for each } t.$$

Since $0 \le \chi \le 1$, an elementary calculation shows that $\|(1-\chi)u'\|^2_{L^2} + \|\chi u'\|^2_{L^2} \le \|u'\|^2_{L^2}$, which implies

$$\lim_{t\to\infty} \|\chi u'(t,\cdot)\|^2_{L^2} = 0.$$

So (11–13) is true for u'. On the other hand,

$$\limsup_{t\to\infty} \|\chi(u(t,\cdot) - u'(t,\cdot))\|_{L^2} \le \epsilon,$$

so $\limsup_{t\to\infty} \|\chi u(t,\cdot)\|_{L^2} \le \epsilon$. Since this is true for every $\epsilon > 0$, the result follows. □

11.3. *Comparison with results of Herbst–Skibsted.* We first show that our results on the asymptotics of the solutions to the time-dependent Equation (11–1) are consistent with the comparison dynamics of Herbst–Skibsted [2004]. Herbst and Skibsted define comparison dynamics, that is, a family of unitary operators $U_0(t)$ for a given local minimum of V_0 and for either a "low energy" range or a "high energy" range which depends on the behaviour of the $r_i(\sigma)$ from Lemma 2.7. It has the property that the strong limit

$$\lim_{t\to\infty} e^{itH} U_0(t)$$

exists in $L^2(X)$ and defines a unitary wave operator.

Let us compare their results on long-time asymptotics with ours. For simplicity, we consider the "very low energy" energy interval in which all of the exponents r_i are complex, with real part $1/2$ (this is "below the Hessian threshold", in our terminology). For simplicity we also assume, as do Herbst and Skibsted, that $V_0(z) = 0$. In this case, the exponent $-i\tilde{b}$ in (11–3) is equal to $(n-1)/4$, and there are no Y'' variables. Moreover, the function $\Phi(y,\sigma)$ is equal to $\sqrt{\sigma}(1-|y|^2/4)$ (see [Hassell et al. 2004, Section 7], particularly (7.22) and (7.23) for the case $n=2$), which implies that $\Psi(y,\tau) = (1-|y|^2/4)/\tau$. If we substitute $x = \tau/t$ into (11–3) then we get

$$c\sum_j t^{-(n-1)/4-1/2+i\kappa_j}\tau^{(n-1)/4+1-i\kappa_j} e^{it(1-|y|^2/4)/\tau^2} w_j(\tau) v_j(Y''', \tau).$$

To compare this with Herbst and Skibsted's comparison dynamics, we adopt their notation: we decompose the variable $\underline{x} \in \mathbb{R}^n$ as $\underline{x} = (x_1, x^\perp)$ where $(1, 0, \dots, 0)$ is the point on the sphere at infinity where V_0 has a local minimum, and $x^\perp$ are $n-1$ orthogonal linear coordinates. We can identify our boundary defining function x with $1/x_1$. Thus $\tau = t/x_1$ and $y = x^\perp/x_1$, and we can write the expression above as

$$c\sum_j t^{-(n-1)/4-1/2+i\kappa_j}\tau^{(n-1)/4+1-i\kappa_j} e^{it/\tau^2} e^{i|x^\perp|^2/4t} w_j(\tau) v_j(x^\perp/\sqrt{x_1}, \tau). \tag{11–14}$$

In this very low-energy case the Herbst–Skibsted comparison dynamics is given explicitly by

$$U_0(t) = S_{t^{-1/2}} e^{i|x^\perp|^2/4} e^{-itp_1^2/2} e^{-i(\log t)H_2} \widehat{U}_0,$$

where the operator p_i stands for $D_{x_i} = -i\partial_{x_i}$, $S_{t^{-1/2}}$ is the scaling

$$S_{t^{-1/2}} f(x_1, x^\perp) = t^{-(n-1)/4} f(x_1, t^{-1/2}x^\perp),$$

the operator H_2 is given by

$$H_2 = \tfrac{1}{2}|p^\perp|^2 + \tfrac{1}{2}\langle x^\perp, \big(p_1^{-2}V^{(2)} - \operatorname{Id}/4\big)x^\perp\rangle$$

(where $V^{(2)}$ is the Hessian of V_0 at the critical point), and finally $\widehat{U}_0$ is an arbitrary unitary operator.

To compare this to our long-time asymptotic expansion (11–14), it is convenient to take $\widehat{U}_0$ to be inverse Fourier transform mapping functions of p_1 to functions of x_1. Then H_2 is a family of harmonic oscillators parametrized by p_1. The operator $e^{-itp_1^2/2}$ acting on $W(p_1, x^\perp)$ then takes the form

$$(2\pi)^{-1/2}\int e^{ix_1p_1}e^{-itp_1^2/2}W(p_1, x^\perp)\,dp_1$$

and by stationary phase we see that the large t asymptotics of this operation is given by

$$W(p_1, x^\perp) \mapsto t^{-1/2}e^{ix_1^2/2t}W\Big(\frac{x_1}{t}, x^\perp\Big).$$

Let us expand $W(p_1, x^\perp)$ in eigenfunctions of the operator $H_2 = H_2(p_1)$ as

$$W(p_1, x^\perp) = \sum_j \omega_j(p_1)\chi_j(x^\perp, p_1),$$

and write τ for t/x_1. A computation shows that $S_{t^{-1/2}}H_2S_{t^{1/2}}$ is equal to $p_1^{-1}\widetilde{Q}$ where $\widetilde{Q}$ is the operator from (6–7). The comparison dynamics therefore maps W to

$$t^{-(n-1)/4-1/2}e^{it/2\tau^2}e^{-i|x^\perp|^2/4t}\sum_j t^{i\kappa_j}\omega_j(\tau^{-1})\chi_j(x^\perp/\sqrt{t}, \tau^{-1}).$$

This agrees with (11–14), if we identify $w_j(\tau)$ with $\tau^{-1}\omega_j(\tau^{-1})$, and $v_j(Y''', \tau)$ with $\chi_j(Y'''\tau^{-1/2}, \tau^{-1})$. (The imaginary powers of τ simply amount to a different choice of normalized eigenfunction v_j. Also there are some discrepancies of factors of $1/2$ since Herbst–Skibsted's operator is *half* the Laplacian plus V.) Thus, the two expansions are consistent.

In the high energy regime, it is easier to check the agreement of the two expansions. In this case, there are no Y''' variables. The Herbst–Skibsted comparison dynamics takes the form

$$\widetilde{U}_0(t)f(\underline{x}) = e^{iS(t,\underline{x})}J(t,\underline{x})^{1/2}f(k(t,\underline{x}), w(t,\underline{x})),$$

where $S(t,\underline{x})$ is a solution to the eikonal equation

$$\partial_t S(t,\underline{x}) + \tfrac{1}{2}|\nabla_{\underline{x}}S(t,\underline{x})|^2 + V(\underline{x}) = 0$$

and $k(t,\underline{x})$ is the energy function

$$\tfrac{1}{2}k^2 = \tfrac{1}{2}|\nabla_{\underline{x}}S(t,\underline{x})|^2 + V(\underline{x}).$$

To make the link with our long time expansion (11–3), we begin by showing that S corresponds to our phase function Ψ/x. Indeed, Ψ is obtained from Φ, the phase function in (11–6) by performing stationary phase as in (11–9). The phase function Φ/x parametrizes a Legendrian submanifold which is the image of the zero section under the FIO F in Remark 6.8. Since the zero section is the flowout from the critical point in the eigendirections (of the linearized flow) with eigenvalues λr_i (as opposed to $\lambda(1-r_i)$), as can be computed easily from (2–7), the same is true of the Legendrian submanifold parametrized by Φ/x;

in particular, it corresponds precisely to Herbst–Skibsted's Lagrangian submanifold $\mathcal{M}_k$ parametrized by $\bar{S}$ (using the correspondence between conic Lagrangian submanifolds and Legendre manifolds "at infinity"); see [Herbst and Skibsted 2004, Theorem 2.1]. Then the way Ψ/x is obtained from Φ/x is exactly the same as the Legendre transform by which Herbst–Skibsted obtain S from $\bar{S}$ (see [Herbst and Skibsted 2004, page 559]), with k^2 corresponding to our σ and S corresponds to our Ψ. Moreover, from [Herbst and Skibsted 2004, page 561], we have

$$w_j = t^{-\beta_j(k)} k^{1-\beta_j(k)}(1+2\beta_j(k))u_j + O(u_j|u|).$$

In our notation, $\beta_j(k) = -r_j$, $u_j = y_j$ and $t = \tau/x$. Setting $Y''_j = y_j x^{-r_j}$ as above, we get

$$w_j = g_j(k) Y''_j + O(x^{\min r_j}).$$

Thus, up to an energy-dependent factor $g_j(k)$, the coordinate w in Herbst–Skibsted is equivalent to our Y''. The asymptotics (11–3) in this regime (where now there are no Y''' variables, hence no sum over j) thus take the form

$$c x^{-i\tilde{b}+1/2} \tau^{-3/2} e^{i\Psi(y,\sigma(\tau))/x} w(Y'', \sigma)$$

which is consistent with the Herbst–Skibsted comparison dynamics.

It is a little more difficult to make the link between our asymptotics and Herbst–Skibsted's in the low-energy regime where not all the r_i have real part equal to $1/2$, but the real parts are all at least $1/3$. We can, however, offer some explanation as to why the low-energy comparison dynamics fails to work *above* this energy level. Referring to Remark 11.3, above this energy we cannot approximate the function $\Psi(y, \sigma(\tau))$ by its quadratic approximation; we need to include at least cubic terms in the Taylor series of Ψ at $y = z$. These in turn depend on the cubic terms in the Taylor series for V_0 at z. The Herbst–Skibsted low energy comparison dynamics neglects these terms. It cannot therefore be expected to provide an accurate approximation to the long-time asymptotics of solutions to (11–1), since we have seen that in (11–3) that one *cannot* replace Ψ by its quadratic approximation.

We emphasize that our long time asymptotic formula (11–3) works for all energies (except for the discrete set of eigenvalues, effectively resonant energies and Hessian thresholds), whereas in Herbst and Skibsted's results there is a gap of "intermediate energies" in which they do not give any comparison dynamics. The formula (11–3) correctly interpolates between low energies, below the Hessian threshold, and high energies, where all the exponents r_i are real.

Appendix: Errata for [Hassell et al. 2004]

A.1. ***Correction to the proof of Proposition 6.7.*** With the stated assumptions, the proof of Proposition 6.7 in Part I needs to be two-step, and the conclusion is slightly modified, although this does not affect any of its applications, in particular Proposition 6.9 of Part I, which is its only use in that paper. Below equation numbers of the form (6.xx) refer to Part I, while equation numbers of the form (A.xx) refer to this appendix.

The error in the proof arises from the microlocalizers $Q \in \Psi_{sc}^{-\infty,0}(X)$ considered there, in (6.27), so we recall the assumptions on it. With O_m a neighborhood of q as (6.24) or (6.25), we assume that

$$\mathrm{WF}'_{\mathrm{sc}}(Q) \subset O_m \; q \notin \mathrm{WF}'_{\mathrm{sc}}(\mathrm{Id}-Q),$$

$$i[Q^*Q, P-\sigma] = x^{1/2}(\widetilde{B}^*\widetilde{B}+\widetilde{G})x^{1/2} + x^{1/2}\widetilde{F}x^{1/2},$$
$$\text{where } \widetilde{B},\ \widetilde{F} \in \Psi^{0,0}_{\mathrm{sc}}(O),\ \widetilde{G} \in \Psi^{0,1}_{\mathrm{sc}}(X),\ q \notin \mathrm{WF}'_{\mathrm{sc}}(\widetilde{F}), \tag{A.1}$$

and in addition, $\widetilde{F}$ satisfies $\mathrm{WF}'_{\mathrm{sc}}(\widetilde{F}) \subset \{\nu < \nu(q)\}$. (This condition on $\widetilde{F}$ ensures that $\mathrm{WF}'_{\mathrm{sc}}(\widetilde{F}) \cap \mathrm{WF}_{\mathrm{sc}}(u) = \varnothing$ for the application in Section 9 of Part I.)

In fact, due to the two step nature of the proof below, we also need another microlocalizer $Q' \in \Psi^{-\infty,0}_{\mathrm{sc}}(X)$ satisfying analogous assumptions with $\widetilde{B}$, etc., replaced by $\widetilde{B}'$, etc.,

$$i[(Q')^*Q', P-\sigma] = x^{1/2}((\widetilde{B}')^*\widetilde{B}'+\widetilde{G}')x^{1/2} + x^{1/2}\widetilde{F}'x^{1/2}, \tag{A.2}$$

with properties analogous to (A.1), except that $\mathrm{WF}'_{\mathrm{sc}}(Q') \subset O'_m$, etc., where O'_m is the elliptic set of Q.

The following is a slightly modified version of Proposition A.1, in that we need to assume the existence of Q' as above, and that the conclusion is on the elliptic set of Q' rather than that of Q.

Proposition A.1 (Modified version of [Hassell et al. 2004, Proposition 6.7]). *Suppose that* $m > 0$, $s < -1/2$, $q \in \mathrm{RP}_+(\sigma)$, $\sigma \notin \mathrm{Cv}(V)$, *either* (6.14) *or* (6.15) *hold, and let* O_m *be as in* (6.24) (*or* (6.25)). *Suppose that* $u \in I^{(s),m-1}_{\mathrm{sc}}(O_m, \mathcal{M})$, $\mathrm{WF}_{sc}((P-\sigma)u) \cap O_m = \varnothing$ *and that there exists* $Q, Q' \in \Psi^{-\infty,0}_{\mathrm{sc}}(O_m)$ *elliptic at* q *that satisfies* (A.1)–(A.2) *with* $\mathrm{WF}'_{sc}(\widetilde{F}) \cap \mathrm{WF}_{sc}(u) = \varnothing$, $\mathrm{WF}'_{sc}(\widetilde{F}') \cap \mathrm{WF}_{sc}(u) = \varnothing$. *Then* $u \in I^{(s),m}_{\mathrm{sc}}(O'', \mathcal{M})$ *where* O'' *is the elliptic set of* Q'.

The issue with the argument presented in the proof of Proposition 6.7 is that it gains a whole extra factor in the module at once: $u \in I^{(s),m-1}_{\mathrm{sc}}(O_m, \mathcal{M})$, is assumed, and $u \in I^{(s),m}_{\mathrm{sc}}(O', \mathcal{M})$ is concluded. Now, the novel part of such a statement, corresponding to the terms arising from factors from the module $\mathcal{M} \subset \Psi^{-\infty,-1}_{\mathrm{sc}}(X)$, is properly dealt with in the (erroneous) proof presented in Part I. However, there is a problem with the microlocalizer Q unless (6.27) is strengthened to make the error term $\widetilde{G}$ have two orders higher decay than the main term, that is, to make it order $(0, 2)$. This is of course the same issue as what makes one gain $1/2$ order at a time usually in positive commutator proofs for the propagation of singularities for operators of real principal type. Factors from the module $\mathcal{M}$ are fine because they essentially get reproduced by the commutator with $P-\sigma$. The problem is that $\widetilde{G}$ cannot be written as a multiple of Q, in general. Technically, this shows up in (6.29) where $\epsilon\|A_{\alpha,s}u'\|^2$ cannot be absorbed in the left hand side for it does not have a factor of Q. (One needs to remember that Au' is the vector of $QA_{\alpha,s}u'$, so all terms arising by commutators with the module generators are OK, the only issue is the microlocalizer Q.)

This error is easily remedied by a two-step argument. The cost of this is that the open set on which we conclude regularity is shrunk slightly from the elliptic set of Q to that of Q', although in relevant situations one can usually recover the original statement of Proposition 6.7 easily as in Proposition 6.9. First, the argument given in the proof of Proposition 6.7 proves the following lemma.

Lemma A.2. *Suppose that* $m > 0$, $r < -1/2$, $q \in \mathrm{RP}_+(\sigma)$, $\sigma \notin \mathrm{Cv}(V)$, *either* (6.14) *or* (6.15) *hold, and let* O_m *be as in* (6.24) (*or* (6.25)). *Suppose that* $u \in I^{(r),m-1}_{\mathrm{sc}}(O_m, \mathcal{M})$, $\mathrm{WF}_{sc}((P-\sigma)u) \cap O_m = \varnothing$ *and that there exists* $Q \in \Psi^{-\infty,0}_{\mathrm{sc}}(O_m)$ *elliptic at* q *that satisfies* (A.1) *with* $\mathrm{WF}'_{sc}(\widetilde{F}) \cap \mathrm{WF}_{sc}(u) = \varnothing$. *Then* $u \in I^{(r-1/2),m}_{\mathrm{sc}}(O', \mathcal{M})$ *where* O' *is the elliptic set of* Q.

Notice that under the same hypothesis as Proposition A.1, this lemma proves regularity under $\mathcal{M}^m$ (as Proposition A.1), but does so at the cost of losing half an order of decay: $u \in I_{\mathrm{sc}}^{(r-1/2),m}(O', \mathcal{M})$ rather than $u \in I_{\mathrm{sc}}^{(r),m}(O', \mathcal{M})$.

Proof of Lemma A.2. With the notation of the proof of Proposition 6.7 of Part I, let $s = r - 1/2$ (so in particular $s < -1/2$), let $A_{\alpha,s}$, etc., be as there. Then the pairing $\langle A_{\alpha,s}u', \widetilde{G}A_{\alpha,s}u'\rangle$ (where u' will be regularizations of u) is controlled by the a priori control of u' in $I_{\mathrm{sc}}^{(s+1/2),m-1}(O_m, \mathcal{M}) = I_{\mathrm{sc}}^{(r),m-1}(O_m, \mathcal{M})$. Indeed, $x^{1/2}A_{\alpha,s}$ and $x^{-1/2}\widetilde{G}A_{\alpha,s}$ are both the product of an element of $\Psi_{\mathrm{sc}}^{(0,-s+1/2)}(O_m)$ and m factors in the module $\mathcal{M} \subset \Psi_{\mathrm{sc}}^{0,-1}(O_m)$, hence in particular can be thought of (by combining the factor from $\Psi_{\mathrm{sc}}^{(0,-s+1/2)}(O_m)$ with a factor from $\mathcal{M}$) as the product of an element of $\Psi_{\mathrm{sc}}^{(0,-s-1/2)}(O_m)$ with $m-1$ factors in $\mathcal{M}$. So this gives $u \in I_{\mathrm{sc}}^{(s),m}(O', \mathcal{M}) = I_{\mathrm{sc}}^{(r-1/2),m}(O', \mathcal{M})$, proving the lemma. □

Proof of Proposition A.1. Lemma A.2 shows that $u \in I_{\mathrm{sc}}^{(s-1/2),m}(O', \mathcal{M})$ with O' as in Lemma A.2. With this additional knowledge, the argument stated in the proof of Proposition 6.7 of Part I, applied with the same s, goes through. (But now we apply it with Q replaced by Q', etc!) Indeed, the pairing $\langle A_{\alpha,s}u', \widetilde{G}'A_{\alpha,s}u'\rangle$ is controlled by the a priori information, as $x^{1/2}A_{\alpha,s}u' = A_{\alpha,s-1/2}u'$, so it is controlled in L^2 if u' is a priori controlled in $I_{\mathrm{sc}}^{(s-1/2),m}(O', \mathcal{M})$ (which we just have proved), and a similar conclusion holds for $x^{-1/2}\widetilde{G}'A_{\alpha,s}u'$ as $x^{-1/2}\widetilde{G}' \in \Psi_{\mathrm{sc}}^{0,1/2}(X)$ just like $x^{1/2}$ is. Thus, $u \in I_{\mathrm{sc}}^{(s),m}(O'', \mathcal{M})$, with O'' the elliptic set of Q', as desired. This finishes the proof. □

A.2. ***Correction to Proposition 9.4.*** The proof of Proposition 9.4 in Part I contains the statement " Since $r_1 < 0$, the vector field $x\partial_x + r_1 y\partial_y$ is nonresonant", which is false. To correct the proof, that statement should be deleted and the sentence following it replaced by: "By a change of coordinates $x' = a(y)x$, $y' = b(y)y$, where $a, b \in C^\infty$ near $y = 0$ satisfy the ODEs

$$a'(y) = -\frac{a(y)F(y)}{r_1 + yG(y)}, \quad b'(y) = -\frac{b(y)}{r_1 + yG(y)}, \quad a(0) = b(0) = 1$$

the F and G terms are eliminated and the vector field becomes

$$-\frac{2\tilde{\nu}}{a(y)}\big(((x')^2 D_{x'}) + r_1 y(x' D_{y'})\big),$$

modulo terms in $x^2\widetilde{\mathcal{M}}^2$ and subprincipal terms." This proves the proposition apart from the prefactor of $a(y)^{-1}$ in front of $\widetilde{P}_0$ which is irrelevant for the application of this proposition.

Of course, Proposition 9.4 also follows by applying the results of the present paper, noting that the case considered there is effectively nonresonant.

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References

[Agmon et al. 1999] S. Agmon, J. Cruz-Sampedro, and I. Herbst, "Generalized Fourier transform for Schrödinger operators with potentials of order zero", *J. Funct. Anal.* **167**:2 (1999), 345–369. MR 2000h:35032 Zbl 0937.35033

[Bony et al. 2007] J.-F. Bony, S. Fujiié, T. Ramond, and M. Zerzeri, "Microlocal kernel of pseudodifferential operators at a hyperbolic fixed point", *J. Funct. Anal.* **252**:1 (2007), 68–125. MR 2357351 Zbl 05214703

[Duistermaat and Hörmander 1972] J. J. Duistermaat and L. Hörmander, "Fourier integral operators. II", *Acta Math.* **128**:3-4 (1972), 183–269. MR 52 #9300 Zbl 0246.47059

[Guillemin and Schaeffer 1977] V. Guillemin and D. Schaeffer, "On a certain class of Fuchsian partial differential equations", *Duke Math. J.* **44**:1 (1977), 157–199. MR 55 #3504 Zbl 0356.35080

[Hassell and Vasy 1999] A. Hassell and A. Vasy, "The spectral projections and the resolvent for scattering metrics", *J. Anal. Math.* **79** (1999), 241–298. MR 2001d:58034 Zbl 0981.58025

[Hassell and Vasy 2001] A. Hassell and A. Vasy, "The resolvent for Laplace-type operators on asymptotically conic spaces", *Ann. Inst. Fourier* (*Grenoble*) **51**:5 (2001), 1299–1346. MR 2002i:58037

[Hassell et al. 2001] A. Hassell, R. Melrose, and A. Vasy, "Spectral and scattering theory for symbolic potentials of order zero", pp. XIII–1–21 in *Séminaire Équations aux Dérivées Partielles, 2000–2001*, École Polytech., Palaiseau, 2001. MR 2002i:35142 Zbl 1063.35126

[Hassell et al. 2004] A. Hassell, R. Melrose, and A. Vasy, "Spectral and scattering theory for symbolic potentials of order zero", *Adv. Math.* **181**:1 (2004), 1–87. MR 2005b:58048 Zbl 1038.35085

[Herbst 1991] I. W. Herbst, "Spectral and scattering theory for Schrödinger operators with potentials independent of $|x|$", *Amer. J. Math.* **113**:3 (1991), 509–565. MR 92k:35069 Zbl 0732.35063

[Herbst and Skibsted 1999] I. Herbst and E. Skibsted, "Quantum scattering for homogeneous of degree zero potentials: Absence of channels at local maxima and saddle points", Technical report, Center for Math. Physics and Stocastics, Århus, 1999, Available at http://www.maphysto.dk/cgi-bin/gp.cgi?publ=126.

[Herbst and Skibsted 2004] I. Herbst and E. Skibsted, "Quantum scattering for potentials independent of $|x|$: asymptotic completeness for high and low energies", *Comm. Partial Differential Equations* **29**:3-4 (2004), 547–610. MR 2005m:81331 Zbl 1064.35128

[Herbst and Skibsted 2008] I. Herbst and E. Skibsted, "Absence of quantum states corresponding to unstable classical channels", *Ann. Henri Poincaré* **9**:3 (2008), 509–552. MR 2419770 Zbl 1141.81029

[Hörmander 1971] L. Hörmander, "On the existence and the regularity of solutions of linear pseudo-differential equations", *Enseignement Math.* (2) **17** (1971), 99–163. MR 48 #9458 Zbl 0224.35085

[Melrose 1994] R. B. Melrose, "Spectral and scattering theory for the Laplacian on asymptotically Euclidian spaces", pp. 85–130 in *Spectral and scattering theory* (Sanda, 1992), edited by M. Ikawa, Lecture Notes in Pure and Appl. Math. **161**, Dekker, New York, 1994. MR 95k:58168 Zbl 0837.35107

[Melrose and Zworski 1996] R. Melrose and M. Zworski, "Scattering metrics and geodesic flow at infinity", *Invent. Math.* **124**:1-3 (1996), 389–436. MR 96k:58230 Zbl 0855.58058

[Nelson 1969] E. Nelson, *Topics in dynamics, I: Flows*, Princeton University Press, Princeton, N.J., 1969. MR 43 #8091 Zbl 0197.10702

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CR-INVARIANTS AND THE SCATTERING OPERATOR FOR COMPLEX MANIFOLDS WITH BOUNDARY

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Suppose that M is a strictly pseudoconvex CR manifold bounding a compact complex manifold X of complex dimension m. Under appropriate geometric conditions on M, the manifold X admits an approximate Kähler–Einstein metric g which makes the interior of X a complete Riemannian manifold. We identify certain residues of the scattering operator on X as conformally covariant differential operators on M and obtain the CR Q-curvature of M from the scattering operator as well. In order to construct the Kähler–Einstein metric on X, we construct a global approximate solution of the complex Monge–Ampère equation on X, using Fefferman's local construction for pseudoconvex domains in $\mathbb{C}^m$. Our results for the scattering operator on a CR-manifold are the analogue in CR-geometry of Graham and Zworski's result on the scattering operator on a real conformal manifold.

1. Introduction

The purpose of this paper is to describe certain CR-covariant differential operators on a strictly pseudoconvex CR manifold M as residues of the scattering operator for the Laplacian on an ambient complex Kähler manifold X having M as a "CR-infinity". We also characterize the CR Q-curvature in terms of the scattering operator. Our results parallel earlier results of Graham and Zworski [2003], who showed that if X is an asymptotically hyperbolic manifold carrying a Poincaré–Einstein metric, the Q-curvature and certain conformally covariant differential operators on the "conformal infinity" M of X can be recovered from the scattering operator on X. The results in this paper were announced in [Hislop et al. 2006].

To describe our results, we first recall some basic notions of CR geometry and recent results [Fefferman and Hirachi 2003; Gover and Graham 2005] concerning CR-covariant differential operators and CR-analogues of Q-curvature. If M is a smooth, orientable manifold of real dimension $(2n+1)$, a *CR-structure* on M is a real hyperplane bundle H on TM together with a smooth bundle map $J : H \to H$

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with $J^2 = -1$ that determines an almost complex structure on H. We denote by $T_{1,0}$ the eigenspace of J on $H \otimes \mathbb{C}$ with eigenvalue $+i$; we will always assume that the CR-structure on M is integrable in the sense that $[T_{1,0}, T_{1,0}] \subset T_{1,0}$. We will assume that M is orientable, so that the line bundle $H^\perp \subset T^*M$ admits a nonvanishing global section. A *pseudo-Hermitian structure* on M is smooth, nonvanishing section θ of $H^\perp$. The *Levi form* of θ is the Hermitian form

$$L_\theta(v, w) = d\theta(v, Jw)$$

on H. The CR-structure on M is called *strictly pseudoconvex* if the Levi form is positive definite. Note that this condition is actually independent of the choice of θ compatible with a given orientation of M. We will always assume that M is strictly pseudoconvex in what follows. It follows from strict pseudoconvexity that θ is a contact form, and the form $\theta \wedge (d\theta)^n$ is a volume form that defines a natural inner product on $\mathscr{C}^\infty(M)$ by integration. The pseudo-Hermitian structure on M also determines a connection on TM, the Tanaka–Webster connection ∇_θ; the basic data of pseudo-Hermitian geometry are the curvature and torsion of this connection (see [Tanaka 1975; Webster 1978]).

Given a fixed CR-structure (H, J) on M, any nonvanishing section $\bar\theta$ of $H^\perp$ compatible with a given orientation takes the form $e^{2\Upsilon}\theta$ for a fixed section θ of $H^\perp$ and some function $\Upsilon \in \mathscr{C}^\infty(M)$. The corresponding Levi form is given by

$$L_{\bar\theta} = e^{2\Upsilon} L_\theta.$$

In this sense the CR-structure determines a conformal class of pseudo-Hermitian structures on M.

For strictly pseudoconvex CR manifolds, Fefferman and Hirachi [2003] proved the existence of CR-covariant differential operators P_k of order $2k$, $k = 1, 2, \ldots, n+1$, whose principal parts are Δ_θ^k, where Δ_θ is the positive sub-Laplacian on M with respect to the pseudo-Hermitian structure θ. They exploit Fefferman's construction [1976] (formulated intrinsically by Lee [1986]) of a circle bundle $\mathscr{C}$ over M with a natural conformal structure and a mapping $\theta \mapsto g_\theta$ from pseudo-Hermitian structures on M to Lorentz metrics on $\mathscr{C}$ that respects conformal classes. They then construct the conformally covariant differential operators found in [Graham et al. 1992] (referred to here as GJMS operators) on $\mathscr{C}$, and show that these operators pull back to CR-covariant differential operators on M. The CR Q-curvature may be similarly defined as a pullback to M of Branson's Q-curvature (see [Branson 1993] and see also [Chang et al. 2008] for a review and further references) on the circle bundle $\mathscr{C}$. Here we will show that the operators P_k on M occur as residues for the scattering operator associated to a natural scattering problem with M as the boundary at infinity, and that the CR Q-curvature Q_θ^{CR} can be computed from the scattering operator.

To describe the scattering problem, we first discuss its geometric setting. Recall that if M is an integrable, strictly pseudoconvex CR manifold of dimension $(2n+1)$ with $n \geq 2$, there is a complex manifold X of complex dimension $m = n+1$ having M as its boundary so that the CR-structure on M is induced from the complex structure on X (this result is false, in general, when $n = 1$; see [Harvey and Lawson 1975]). Let φ be a defining function for M and denote by $\mathring{X}$ the interior of X (we take $\varphi < 0$ in $\mathring{X}$). The associated Kähler metric g on $\mathring{X}$ is the Kähler metric with Kähler form

$$\omega_\varphi = -\frac{i}{2}\partial\bar\partial \log(-\varphi) \tag{1-1}$$

in a neighborhood of M, extended smoothly to all of X. The metric has the form

$$g_\varphi = -\frac{\eta}{\varphi} + (1 - r\varphi)\Big(\frac{d\varphi^2}{\varphi^2} + \frac{\Theta^2}{\varphi^2}\Big). \tag{1-2}$$

in a neighborhood of M, where η and Θ have Taylor series to all orders in φ at $\varphi = 0$. The boundary values $\Theta|_M = \theta$, and $\eta|_H = h$ induce respectively a contact form on M and a Hermitian metric on H, where H is a *subbundle* of TM. The function r is a smooth function, the transverse curvature, which depends on the choice of φ (see [Graham and Lee 1988]). Thus, the conformal class of a Hermitian metric h on H, is a kind of "Dirichlet datum at infinity" for the metric g_φ, that is $(-\varphi)g_\varphi|_H = h$.

A motivating example for our work is the case of a strictly pseudoconvex domain $X \subset \mathbb{C}^m$ with Hermitian metric

$$g = \sum_{j,k=1}^{m} \frac{\partial^2}{\partial z_j \partial z_{\bar{k}}} \log\Big(-\frac{1}{\varphi}\Big) dz_j \otimes dz_{\bar{k}},$$

where φ is a defining function for the boundary of X with $\varphi < 0$ in the interior of X. In this example, observe that if

$$\Theta = \frac{i}{2}(\bar{\partial}\varphi - \partial\varphi)$$

and $\iota : M \to X$ is the natural inclusion, then $\theta = \iota^*\Theta$ is a contact form on M that defines the CR-structure $H = \ker\theta$. The form $d\theta$ induces the Levi form on M and so defines a pseudo-Hermitian structure on M. Denote by J the almost complex structure on H; the two-form $h = d\theta(\cdot\,, J\cdot)$ is a pseudo-Hermitian metric on M. It is not difficult to see that the conformal class of the pseudo-Hermitian structure on M, that is, its CR-structure, is independent of the choice of defining function φ.

It is natural to consider scattering theory for the positive Laplacian, Δ_g, on $(\mathring{X}, g)$, where X is a complex manifold with boundary M. As discussed in what follows, the metric g belongs to the class of Θ-metrics considered by Epstein, Melrose, and Mendoza [1991]; see also the recent paper of Guillarmou and Sá Barreto [2008] where scattering theory for asymptotically complex hyperbolic manifolds (a class which includes those considered here) is analyzed in depth. Thus, the full power of the Epstein–Melrose–Mendoza analysis of the resolvent

$$R(s) = (\Delta_g - s(m-s))^{-1}$$

of Δ_g is available to study scattering theory on $(\mathring{X}, g)$.

For $f \in \mathscr{C}^\infty(M)$, $\Re(s) = m/2$, and $s \neq m/2$, there is a unique solution u of the "Dirichlet problem"

$$(\Delta_g - s(m-s))u = 0, \quad u = (-\varphi)^{m-s}F + (-\varphi)^s G, \quad F|_M = f, \tag{1-3}$$

where $F, G \in \mathscr{C}^\infty(X)$. The uniqueness follows from the absence of L^2 solutions of the eigenvalue problem for $\Re(s) = m/2$; this may be proved, for example, using [Vasy and Wunsch 2005] (see the comments in [Guillarmou and Sá Barreto 2008]). Here we will use the explicit formulas for the Kähler form and Laplacian obtained in [Graham and Lee 1988] to obtain the asymptotic expansions of solutions to the generalized eigenvalue problem.

Unicity for the "Dirichlet problem" (1-3) implies that the Poisson map

$$\mathscr{P}(s) : \mathscr{C}^\infty(M) \to \mathscr{C}^\infty(\mathring{X}), \quad f \mapsto u \tag{1-4}$$

and the scattering operator

$$S_X(s) : \mathscr{C}^\infty(M) \to \mathscr{C}^\infty(M), \quad f \mapsto G|_M$$

are well defined. The operator $S_X(s)$ depends *a priori* on the boundary defining function φ for M. If $\bar{\varphi} = e^\upsilon \varphi$ is another defining function for M and $\upsilon|_M = \Upsilon$, the corresponding scattering operator $\bar{S}_X(s)$ is given by

$$\bar{S}_X(s) = e^{-s\Upsilon} S_X(s) e^{(s-m)\Upsilon}.$$

The operator $S_X(s)$ admits a meromorphic continuation to the complex plane, possibly with singularities at $s = 0, -1, -2, \ldots$; see [Melrose 1999] where the scattering operator is described and the problem of studying its poles and residues is posed, and see [Guillarmou and Sá Barreto 2008] for a detailed analysis of the scattering operator. The scattering operator is self-adjoint for s real. We will show that, with a geometrically natural choice of the boundary defining function φ, the residues of certain poles of $S_X(s)$ are CR-covariant differential operators.

To describe the setting for this result, recall that for strictly pseudoconvex domains Ω in $\mathbb{C}^m$, Fefferman [1976] proved the existence of a defining function φ for $\partial\Omega$ which is an approximate solution of the complex Monge–Ampère equation.

The complex Monge–Ampère equation for a function $\varphi \in \mathscr{C}^\infty(\Omega)$ is the equation

$$J[\varphi] = 1, \quad \varphi|_{\partial\Omega} = 0,$$

where J is the complex Monge–Ampère operator

$$J[\varphi] = \det \begin{pmatrix} \varphi & \varphi_j \\ \varphi_{\bar{k}} & \varphi_{j\bar{k}} \end{pmatrix}.$$

We say that $\varphi \in \mathscr{C}^\infty(\Omega)$ is an approximate solution of the complex Monge–Ampère equation if

$$J[\varphi] = 1 + \mathbb{O}(\varphi^{m+1}), \quad \varphi|_{\partial\Omega} = 0.$$

The Kähler metric g associated to such an approximate solution φ is an approximate Kähler–Einstein metric on Ω, that is, g obeys

$$\mathrm{Ric}(g) = -(m+1)\omega + \mathbb{O}(\varphi^{m-1}), \tag{1-5}$$

where ω is the Kähler form associated to φ, and Ric is the Ricci form.

Under certain conditions, Fefferman's result can be "globalized" to the setting of complex manifolds X with strictly pseudoconvex boundary M, as we discuss below. It follows that $\mathring{X}$ carries an approximate Kähler–Einstein metric g in the sense that (1-5) holds.

We will call a smooth function φ defined in a neighborhood of M a *globally defined approximate solution* of the Monge–Ampère equation on X if for each $p \in M$ there is a neighborhood U of p in X and a holomorphic coordinate system in U for which φ is an approximate solution of the Monge–Ampère equation. As we will show, such a solution exists if and only if M admits a pseudo-Hermitian structure θ which is volume-normalized with respect to some locally defined, closed $(n+1, 0)$-form in a neighborhood of any point $p \in M$ (see Section 2D.2 where we defined "volume-normalized", and see Burns–Epstein [1990] where a similar condition is used to construct a global solution of the Monge–Ampére equation when $\dim M = 3$). If $\dim M \geq 5$, we can give a more geometric formulation of this condition. Recall that a CR manifold is *pseudo-Einstein* if there is a pseudo-Hermitian structure θ for which the Webster Ricci curvature is a multiple of the Levi form. Lee [1988] introduced and studied this geometric notion; he proved that if $\dim M \geq 5$, then M admits a pseudo-Einstein, pseudo-Hermitian structure θ if and only if θ is volume-normalized with respect to a closed $(n+1,0)$-form in a neighborhood of any point $p \in M$. If $\dim M = 3$, the pseudo-Einstein condition is vacuous and must be replaced by a more stringent condition; see Section 2D.2 in what follows. If X is a pseudoconvex domain in $\mathbb{C}^m$, this condition is trivially satisfied since the pseudo-Hermitian structure induced by the Fefferman approximate solution is volume-normalized with respect to the restriction of $\zeta = dz^1 \wedge \cdots \wedge dz^m$ to M.

Theorem 1.1. *Let X be a complex manifold of complex dimension $m = n+1$ with strictly pseudoconvex boundary M. Let g be the Kähler metric on X associated to the Kähler form (1-1), and let $S_X(s)$ be the scattering operator for Δ_φ. Finally, suppose that Δ_φ has no L^2-eigenvalues. Then $S_X(s)$ has simple poles at the points $s = (m+k)/2$, $k \in \mathbb{N}$, and*

$$\operatorname*{Res}_{s=(m+k)/2} S_X(s) = c_k P_k,$$

where the P_k are differential operators of order $2k$, and

$$c_k = \frac{(-1)^k}{2^k k!(k-1)!}. \tag{1-6}$$

If g is an approximate Kähler–Einstein metric given by a globally defined approximate solution of the Monge–Ampère equation, then for $1 \leq k \leq m$, the operators P_k are CR-covariant differential operators.

Remark 1.2. It is not difficult to show that, for generic compactly supported perturbations of the metric, L^2-eigenvalues are absent. Our analysis applies if only the metric g has the form (1-2) in a neighborhood of M.

Remark 1.3. We view the operators P_k as operators on $\mathscr{C}^\infty(M)$; if one instead views these operators as acting on appropriate density bundles over M they are actually invariant operators. Gover and Graham [2005] showed that the CR-covariant differential operators P_k are logarithmic obstructions to the solution of the Dirichlet problem (1-3) when X is a pseudoconvex domain in $\mathbb{C}^m$ with a metric of Bergman type, but did not identify them as residues of the scattering operator.

It follows from the self-adjointness (s real) and conformal covariance of $S_X(s)$ that the operators P_k are self-adjoint and conformally covariant. As in [Graham and Zworski 2003], the analysis centers on the Poisson map $\mathscr{P}(s)$ defined in (1-4). As shown in [Epstein et al. 1991], the Poisson map is analytic in s for $\operatorname{Re}(s) > m/2$. Moreover, at the points $s = (m+k)/2$, $k = 1, 2, \ldots$, the Poisson operator takes the form

$$\mathscr{P}(s) f = (-\varphi)^{(m-k)/2} F + (-\varphi)^{(m+k)/2} \log(-\varphi)\, G$$

for functions $F, G \in \mathscr{C}^\infty(X)$ with

$$F|_M = f, \;\; G|_M = c_k P_k f.$$

Here P_k are differential operators determined by a formal power series expansion of the Laplacian (see Lemma 3.4), and are the same operators that appear as residues of the scattering operator at points $s = (m+k)/2$. An important ingredient in the analysis is the asymptotic form of the Laplacian due to Lee and Melrose [1982] and refined by Graham and Lee [1988].

If the defining function φ is an approximate solution of the complex Monge–Ampère equation, the differential operators P_k, $1 \le k \le m$, can be identified with the GJMS operators owing to the characterization of $\mathcal{P}(s)f$ described above (see [Gover and Graham 2005, Proposition 5.4]; the argument given there for pseudoconvex domains easily generalizes to the present setting).

Explicit computation shows that, for an approximate Kähler–Einstein metric g, the first operator has the form

$$P_1 = c_1\big(\Delta_b + n(2(n+1))^{-1}R\big),$$

where Δ_b is the sub-Laplacian on X and R is the Webster scalar curvature, that is, P_1 is the CR-Yamabe operator of Jerison and Lee [1984].

The CR Q-curvature is a pseudo-Hermitian invariant realized as the pullback to M of the Q-curvature of the circle bundle $\mathcal{C}$.

Theorem 1.4. *Suppose that X is a complex manifold with strictly pseudoconvex boundary M, and suppose that g is an approximate Kähler–Einstein metric given by a globally defined approximate solution of the Monge–Ampère equation. Let $S_X(s)$ be the associated scattering operator. The formula*

$$c_m Q_\theta^{CR} = \lim_{s\to m} S_X(s)1$$

holds, where c_m is given by (1-6).

It follows from Theorem 1.1 and the conformal covariance of $S_X(s)$ that if $\bar\theta = e^{2\Upsilon}\theta$, then

$$e^{2m\Upsilon} Q_{\bar\theta}^{CR} = Q_\theta^{CR} + P_m\Upsilon$$

as was already shown in [Fefferman and Hirachi 2003]. From this it follows that the integral

$$\int_M Q_\theta^{CR}\psi$$

is a CR-invariant (recall that ψ is the natural volume form on M defined by the contact form θ). We remark that the integral of Q_θ^{CR} vanishes for any three-dimensional CR manifold because the integrand is a total divergence (see [Fefferman and Hirachi 2003, Proposition 3.2] and comments below), while under the condition of our Theorem 1.4, there is a pseudo-Hermitian structure for which $Q_\theta^{CR} = 0$ (see [Fefferman and Hirachi 2003, Proposition 3.1]). In our case, if φ is a globally defined approximate solution of the Monge–Ampère equation, the induced contact form

$$\theta = \frac{i}{2}(\bar\partial\varphi - \partial\varphi)$$

on M is an "invariant contact form" in the language of [Fefferman and Hirachi 2003], and they show in Proposition 3.1 that $Q_\theta^{CR} = 0$ for an invariant contact form. Thus it is not clear at present under what circumstances this invariant is nontrivial for a general, strictly pseudoconvex manifold.

Finally, we prove a CR-analogue of [Graham and Zworski 2003, Theorem 3] using scattering theory.

Theorem 1.5. *Suppose that X is a compact complex manifold with strictly pseudoconvex boundary M, and g is an approximate Kähler–Einstein metric given by a globally defined approximate solution of the Monge–Ampère equation. Then*

$$\mathrm{vol}_g\{-\varphi > \varepsilon\} = c_0\varepsilon^{-n-1} + c_1\varepsilon^{-n} + \cdots + c_n\varepsilon^{-1} + L\log(-\varepsilon) + V + o(1).$$

where

$$L = c_m \int_M Q_\theta^{CR} \psi = 0.$$

We remark that Seshadri [2007] already showed that L is, up to a constant, the integral of Q_θ^{CR}. It is worth noting that our choice of defining function differs from Seshadri's.

2. Geometric preliminaries

2A. *CR manifolds.* Suppose that M is a smooth orientable manifold of real dimension $2n+1$, and let $\mathbb{C}TM = TM \otimes_{\mathbb{R}} \mathbb{C}$ be the complexified tangent bundle on M. A *CR-structure* on M is a complex n-dimensional subbundle $\mathcal{H}$ of $\mathbb{C}TM$ with the property that $\mathcal{H} \cap \overline{\mathcal{H}} = \{0\}$. If, also, $[\mathcal{H}, \mathcal{H}] \subseteq \mathcal{H}$, we say that the CR-structure is *integrable*. If we set $H = \operatorname{Re} \mathcal{H}$, then the bundle H has real codimension one in TM. The map

$$J : H \to H, \quad V + \overline{V} \mapsto i(V - \overline{V})$$

satisfies $J^2 = -I$ and gives H a natural complex structure.

Since M is orientable, there is a nonvanishing one-form θ on M with $\ker \theta = H$. This form is unique up to multiplication by a positive, nonvanishing function $f \in \mathscr{C}^\infty(M)$. A choice of such a one-form θ is called a *pseudo-Hermitian structure* on M. The *Levi form* is given by

$$L_\theta(V, \overline{W}) = -i d\theta(V, \overline{W}) \tag{2-1}$$

for $V, W \in \mathcal{H}$ (here $d\theta$ is extended to $\mathcal{H}$ by complex linearity). Note that

$$L_{f\theta} = f L_\theta$$

since θ annihilates $\mathcal{H}$. If $d\theta$ is nondegenerate, then there is a unique real vector field T on M, the *characteristic vector field* T, with the properties that $\theta(T) = 1$ and $T \lrcorner\, d\theta = 0$. If $\{W_\alpha\}$ is a local frame for $\mathcal{H}$ (here α ranges from 1 to n), then the vector fields $\{W_\alpha, W_{\bar{\alpha}}, T\}$ form a local frame for $\mathbb{C}TM$. If we choose $(1,0)$-forms θ^α dual to the W_α then $\{\theta^\alpha, \theta^{\bar{\alpha}}, \theta\}$ forms a dual coframe for $\mathbb{C}TM$. We say that $\{\theta^\alpha\}$ forms an admissible coframe dual to $\{W^\alpha\}$ if $\theta^\alpha(T) = 0$ for all α. The integrability condition is equivalent to the condition that

$$d\theta = d\theta^\alpha = 0 \mod \{\theta, \theta^\alpha\}.$$

The Levi form is then given by

$$L_\theta = h_{\alpha\bar{\beta}}\, \theta^\alpha \wedge \theta^{\bar{\beta}} \tag{2-2}$$

for a Hermitian matrix-valued function $h_{\alpha\bar{\beta}}$. We will use $h_{\alpha\bar{\beta}}$ to raise and lower indices in this article.

We will say that a given CR-structure is *strictly pseudoconvex* if L_θ is positive definite. Note that (up to sign) this condition is independent of the choice of pseudo-Hermitian structure θ.

In what follows, we will always suppose that M is orientable and that M carries a strictly pseudoconvex, integrable CR-structure. In this case, the pseudo-Hermitian geometry of M can be understood in terms of the *Tanaka–Webster connection* on M (see Tanaka [1975] and Webster [1978]). With respect to the frame discussed above, the Tanaka–Webster connection is given by

$$\nabla W_\alpha = \omega_\alpha{}^\beta \otimes W_\beta, \quad \nabla T = 0$$

for connection one-forms ${\omega_\alpha}^\beta$ obeying the structure equations

$$d\theta^\alpha = \theta^\beta \wedge {\omega_\alpha}^\beta + \theta \wedge \tau^\alpha,$$
$$d\theta = i h_{\alpha\bar\beta}\, \theta^\alpha \wedge \theta^{\bar\beta},$$

where the torsion one-forms are given by

$$\tau^\alpha = {A^\alpha}_{\bar\beta}\, \theta^{\bar\beta},$$

with $A_{\alpha\beta} = A_{\beta\alpha}$. The connection obeys the compatibility condition

$$dh_{\alpha\bar\beta} = \omega_{\alpha\bar\beta} + \omega_{\bar\beta\alpha}$$

with the Levi form described in (2-1) and (2-2).

2B. ***Complex manifolds with CR boundary.*** Now suppose that X is a compact complex manifold of dimension $m = n+1$ with boundary $\partial X = M$. We will denote by $\mathring{X}$ the interior of X. The manifold M inherits a natural CR-structure from the complex structure of the ambient manifold. We will suppose that M is strictly pseudoconvex; such a structure, induced by the complex structure of the ambient manifold, is always integrable.

We will suppose that $\varphi \in \mathscr{C}^\infty(X)$ is a defining function for M, that is, $\varphi < 0$ in $\mathring{X}$, $\varphi = 0$ on M, and $d\varphi(p) \neq 0$ for all $p \in M$. We will further suppose that φ has no critical points in a collar neighborhood of M so that the level sets $M^\varepsilon = \varphi^{-1}(-\varepsilon)$ are smooth manifolds for all ε sufficiently small.

Associated to the defining function φ is the Kähler form

$$\omega_\varphi = -\frac{i}{2}\partial\bar\partial \log(-\varphi) = \frac{i}{2}\left(\frac{\partial\bar\partial\varphi}{-\varphi} + \frac{\partial\varphi \wedge \bar\partial\varphi}{\varphi^2}\right).$$

We will study scattering on X with the metric induced by the Kähler form ω_φ. Since we can cover a neighborhood of M in X by coordinate charts, it suffices to consider the situation where U is an open subset of $\mathbb{C}^m$ and $\varphi : U \to \mathbb{R}$ is a smooth function with no critical points in U, the set $\{\varphi < 0\}$ is biholomorphically equivalent to a boundary neighborhood in X, and $\{\varphi = 0\}$ is diffeomorphic to the corresponding boundary neighborhood in M. We will now describe the asymptotic geometry near M, recalling the ambient metric of [Graham and Lee 1988] and computing the asymptotic form of the metric and volume form.

The manifolds M^ε inherit a natural CR-structure from the ambient manifold X with

$$\mathscr{H}^\varepsilon = \mathbb{C}TM^\varepsilon \cap T^{1,0}U.$$

Given a defining function φ, we define a one-form

$$\Theta = \frac{i}{2}(\bar\partial - \partial)\varphi$$

and let

$$\theta_\varepsilon = \iota_\varepsilon^* \Theta,$$

where $\iota_\varepsilon : M^\varepsilon \to U$ is the natural embedding. The contact form θ_ε gives M^ε a pseudo-Hermitian structure. We will denote by $\mathscr{H}$ the subbundle of $T^{1,0}U$ whose fibre over M^ε is $\mathscr{H}^\varepsilon$. Note that

$$d\Theta = i\partial\bar{\partial}\varphi$$

and the Levi form on M^ε is given by

$$L_{\theta_\varepsilon} = -i\,d\theta_\varepsilon.$$

We will assume that each M^ε is strictly pseudoconvex, that is, L_{θ_ε} is positive definite for all sufficiently small $\varepsilon > 0$. To simplify notation, we will write θ for θ_ε, suppressing the ε, as the meaning will be clear from the context.

2B.1. *Ambient connection.* In order to describe the asymptotic geometry of X, we recall the ambient connection defined by Graham and Lee [1988] that extends the Tanaka–Webster connection on each M^ε to $\mathbb{C}TU$. First we recall the following simple lemma (see [Lee and Melrose 1982, §2]).

Lemma 2.1. *There exists a unique* $(1,0)$*-vector field* ξ *on* U *so that*:

$$\partial\varphi(\xi) = 1 \quad \textit{and} \quad \xi \lrcorner\, \partial\bar{\partial}\varphi = r\,\bar{\partial}\varphi$$

for some $r \in \mathscr{C}^\infty(U)$.

The smooth function r is called the *transverse curvature*. We decompose ξ into real and imaginary parts as

$$\xi = \tfrac{1}{2}(N - iT),$$

where N and T are real vector fields on U. It easily follows that

$$d\varphi(N) = 2, \quad \theta(N) = 0, \quad \theta(T) = 1, \quad T \lrcorner\, d\theta = 0.$$

Thus T is the characteristic vector field for each M^ε, and N is normal to each M^ε.

Let $\{W_\alpha\}$ be a frame for $\mathscr{H}$. It follows from Lemma 2.1 that $\{W_\alpha, W_{\bar{\alpha}}, T\}$ forms a local frame for $\mathbb{C}TM^\varepsilon$, while $\{W_\alpha, W_{\bar{\alpha}}, \xi, \bar{\xi}\}$ forms a local frame for $\mathbb{C}TU$. If $\{\theta^\alpha\}$ is a dual coframe for $\{W_\alpha\}$, then $\{\theta^\alpha, \theta^{\bar{\alpha}}, \theta\}$ is a dual coframe for $\mathbb{C}TM^\varepsilon$, while $\{\theta^\alpha, \theta^{\bar{\alpha}}, \partial\varphi, \bar{\partial}\varphi\}$ is a dual coframe for $\mathbb{C}TU$. The Levi form on each $\mathscr{H}^\varepsilon$ is given by

$$L_\theta = h_{\alpha\bar{\beta}}\,\theta^\alpha \wedge \theta^{\bar{\beta}}$$

for a Hermitian matrix-valued function $h_{\alpha\bar{\beta}}$. We will use $h_{\alpha\bar{\beta}}$ to raise and lower indices. We will set

$$W_m = \xi, \quad W_{\bar{m}} = \bar{\xi}, \quad \theta^m = \partial\varphi, \quad \theta^{\bar{m}} = \bar{\partial}\varphi.$$

In what follows, repeated Greek indices are summed from 1 to n and repeated Latin indices are summed from 1 to $m = n+1$.

The following important lemma decomposes the form $d\Theta$ into "tangential" and "transverse" components.

Lemma 2.2. *We have*

$$\partial\bar{\partial}\varphi = h_{\alpha\bar{\beta}}\,\theta^\alpha \wedge \theta^{\bar{\beta}} + r\,\partial\varphi \wedge \bar{\partial}\varphi.$$

Proposition 2.3 [Graham and Lee 1988, Proposition 1.1]. *There exists a unique linear connection* ∇ *on* U *so that*:

(a) *For any vector fields* X *and* Y *on* U *tangent to some* M^{ε},

$$\nabla_X Y = \nabla^{\varepsilon}_X Y$$

where ∇^{ε} *is the pseudo-Hermitian connection on* M^{ε}.

(b) ∇ *preserves* $\mathcal{H}$, N, T *and* L_θ; *that is*, $\nabla_X \mathcal{H} \subset \mathcal{H}$ *for any* $X \in \mathbb{C}TU$, *and* $\nabla T = \nabla N = \nabla L_\theta = 0$.

(c) *If* $\{W_\alpha\}$ *is a frame for* $\mathcal{H}$, *and* $\{\theta^\alpha, \partial\varphi\}$ *is the dual* $(1,0)$-*coframe on* U, *then*

$$d\theta^\alpha = \theta^\beta \wedge \varphi_\beta{}^\alpha - i\partial\varphi \wedge \tau^\alpha + i(W^\alpha r)\,d\varphi \wedge \theta + \tfrac{1}{2} r\, d\varphi \wedge \theta^\alpha.$$

The connection ∇ is called the *ambient connection*.

2B.2. *Kähler metric.* Using Lemma 2.2, we can also compute the Kähler form

$$\omega_\varphi = \frac{i}{2}\Big(\frac{1}{-\varphi} h_{\alpha\bar\beta}\theta^\alpha \wedge \theta^{\bar\beta} + \frac{1 - r\varphi}{\varphi^2}\partial\varphi \wedge \bar\partial\varphi\Big).$$

The induced Hermitian metric is

$$g_\varphi = \frac{1}{-\varphi} h_{\alpha\bar\beta}\,\theta^\alpha \otimes \theta^{\bar\beta} + \frac{1 - r\varphi}{\varphi^2}\partial\varphi \otimes \bar\partial\varphi.$$

It is easy to compute that

$$g_\varphi(N, N) = 4\frac{1 - r\varphi}{\varphi^2},$$

so the outward unit normal field associated to the surfaces M^{ε} is

$$\nu = \frac{-\varphi}{2\sqrt{1 - r\varphi}} N.$$

We note for later use that the induced volume form ω_φ^m is given by

$$\omega_\varphi^m = \Big(\frac{i}{2}\Big)^m \Big(\frac{1 - r\varphi}{(-\varphi)^{m+1}} \det(h_{\alpha\bar\beta})\,\theta^1 \wedge \theta^{\bar 1} \wedge \cdots \wedge \theta^m \wedge \theta^{\bar m}\Big),$$

while

$$\nu \lrcorner\, \omega_\varphi^m|_{M^\varepsilon} = \frac{m}{2^m}\frac{\sqrt{1 + r\varepsilon}}{\varepsilon^m}(d\theta_\varepsilon)^n \wedge \theta_\varepsilon. \tag{2-3}$$

We will set

$$\psi = \frac{m}{2^m}(d\theta)^n \wedge \theta. \tag{2-4}$$

We also note for later use that if $u \in \mathscr{C}^\infty(X)$ and

$$du = u_\alpha \theta^\alpha + u_{\bar\alpha}\theta^{\bar\alpha} + u_m \partial\varphi + u_{\bar m}\bar\partial\varphi,$$

then

$$|du|^2_{g_\varphi} = -\varphi h^{\alpha\bar\beta} u_\alpha u_{\bar\beta} + \frac{\varphi^2}{1 - r\varphi} u_m u_{\bar m}.$$

2C. ***The Laplacian on X.*** The Laplacian on the Kähler manifold (X, ω_φ) is the positive operator[1]

$$\Delta_\varphi u = \mathrm{Tr}(i\partial\bar\partial u) = g^{j\bar k} u_{j\bar k},$$

for $u \in \mathscr{C}^\infty(X)$, where we now write Δ_φ rather than Δ_g to emphasize the dependence of Δ on the boundary defining function φ.

Graham and Lee [1988] computed the Laplacian in a collar neighborhood of M, separating "normal" and "tangential" parts. To state their results, recall that the sub-Laplacian is defined on each M^ε by

$$\Delta_b u = \left(u_\alpha{}^\alpha + u_{\bar\beta}{}^{\bar\beta}\right),$$

where covariant derivatives are taken with respect to the Tanaka–Webster connection on M^ε.

Theorem 2.4 [Graham and Lee 1988].

$$\Delta_\varphi = \frac{\varphi}{4}\left(\frac{-\varphi}{1-r\varphi}(N^2+T^2+2rN+2X_r) - 2\Delta_b + 2nN\right), \tag{2-5}$$

where $X_r = r^\alpha W_\alpha + r^{\bar\alpha} W_{\bar\alpha}$.

It will be useful to recast (2-5) for $\Delta_\varphi u$ in terms of $x = -\varphi$. Note that $N = 2\,\partial/\partial\varphi = -2\,\partial/\partial x$, so

$$-\Delta_\varphi u = \left(\frac{1}{1+rx}\right)\left(x\frac{\partial}{\partial x}\right)^2 u - (n+1)\left(x\frac{\partial}{\partial x}\right)u + \tfrac14\left(\frac{x^2}{1+rx}\right)(T^2u - 2ru_x + 2X_r u) + \tfrac14 x(-2\Delta_b u).$$

We think of Δ_φ as a variable-coefficient differential operator with respect to vector fields $x\,\partial/\partial_x$ and vector fields tangent to the boundary M. In a neighborhood of M we have

$$\Delta_\varphi \sim \sum_{k\ge 0} x^k L_k, \tag{2-6}$$

for differential operators $L_k = L_k(y;\, \partial_y, x\partial_x)$, where the indicial operator L_0 is

$$L_0 = -\left(x\frac{\partial}{\partial x}\right)^2 + mx\frac{\partial}{\partial x}, \tag{2-7}$$

and the operator L_1 is

$$L_1 = \tfrac12\Delta_b + r_0\left(x\frac{\partial}{\partial x}\right)^2,$$

where $r = r_0 + \mathbb{O}(x)$.

2D. ***The complex Monge–Ampère equation.***

2D.1. *Local theory.* Let Ω be a domain in $\mathbb{C}^m$ with smooth boundary. The complex Monge–Ampère equation is the nonlinear equation

$$J[u] = 1, \quad u|_{\partial\Omega} = 0,$$

for a function $u \in \mathscr{C}^\infty(\Omega)$, $u > 0$ on Ω, where $J[u]$ is the Monge–Ampère operator:

$$J[u] = (-1)^m \det\begin{pmatrix} u & u_{\bar j} \\ u_i & u_{i\bar j}\end{pmatrix}. \tag{2-8}$$

[1] Note that our definition differs from that of Graham and Lee by an overall factor of $-\frac14$.

If u solves the complex Monge–Ampère equation then

$$-\left(\log\left(\frac{1}{u}\right)\right)_{j\bar{k}} dz^j \otimes dz^{\bar{k}}$$

is a Kähler–Einstein metric.

Fefferman [1976] showed that there is a smooth function $\psi \in \mathscr{C}^\infty(\Omega)$ that satisfies

$$J[\varphi] = 1 + \mathbb{O}(\varphi^{m+1}), \quad \varphi|_{\partial\Omega} = 0,$$

and that ψ is uniquely determined up to order $m+1$. Cheng and Yau [1980] showed the existence of an exact solution belonging to $\mathscr{C}^\infty(\Omega) \cap C^{m+3/2-\varepsilon}(\overline{\Omega})$, while Lee and Melrose [1982] showed that the exact solution has an asymptotic expansion with logarithmic terms beginning at order $m+2$.

We will show that Fefferman's local approximate solution of the Monge–Ampère equation [Fefferman 1976] can be globalized to an approximate solution of the Monge–Ampère equation near the boundary of a complex manifold X. We will see later that, to globalize Fefferman's construction, we need to impose a geometric condition on the CR-structure of M inherited from the complex structure of X. For the convenience of the reader, we review the properties of the operator J under a holomorphic coordinate change and the connection between solutions of the Monge–Ampére equation and Kähler–Einstein metrics.

If $f : \Omega \subset \mathbb{C}^m \to \mathbb{C}^m$ is holomorphic, then f' denotes the matrix

$$(f')_{jk} = \frac{\partial f_j}{\partial z_k}.$$

Lemma 2.5. *Let f be a local biholomorphism. Then, for any local smooth function u on Ω,*

$$J\left[|\det(f')|^{-2/(m+1)}(u \circ f)\right] = J[u] \circ f.$$

A proof was given by Fefferman [1976]. Here we give an alternative proof using the following identity.

Lemma 2.6.

$$J[u] = u^{m+1} \det\left(\left(\log\left(\frac{1}{u}\right)\right)_{j\bar{k}}\right).$$

Proof. Using row-column operations, one proves that

$$\det\begin{pmatrix} u & u_{\bar{k}} \\ u_j & u_{j\bar{k}} \end{pmatrix} = u \det\left(u_{j\bar{k}} - \frac{u_j u_{\bar{k}}}{u}\right). \tag{2-9}$$

On the other hand, the identity

$$\left(\log\left(\frac{1}{u}\right)\right)_{j\bar{k}} = -\frac{u_{j\bar{k}}}{u} + \frac{u_j u_{\bar{k}}}{u^2}$$

shows that

$$J[u] = (-1)^m u \det\left(u_{j\bar{k}} - \frac{u_j u_{\bar{k}}}{u}\right) = u^{m+1} \det\left(\left(\log\left(\frac{1}{u}\right)\right)_{j\bar{k}}\right). \tag{2-10}$$

The lemma follows (2-9) and (2-10). □

We can use Lemma 2.6 to show that if u solves the Monge–Ampère equation, then u is the Kähler potential of a Kähler–Einstein metric. Recall that if

$$g = v_{j\bar{k}} dz^j \otimes dz^{\bar{k}},$$

then the Ricci curvature is

$$R_{a\bar{b}} = -\big(\log \det(v_{j\bar{k}})\big)_{a\bar{b}}.$$

Now let $v = \log(\frac{1}{u})$ where $J[u] = 1$. Then

$$R_{a\bar{b}} = -\big(\log \det(v_{j\bar{k}})\big)_{a\bar{b}} = -\big(\log(u^{-(m+1)})\big)_{a\bar{b}} = -(m+1)\Big(\log\Big(\frac{1}{u}\Big)\Big)_{a\bar{b}} = -(m+1) g_{a\bar{b}},$$

which is the Einstein equation.

Proof of Lemma 2.5. First, we compute

$$\big(\log(|\det f'|^{-2/(m+1)}\, u \circ f)\big)_{j\bar{k}} = \frac{-1}{m+1}\big(\log(|\det f'|^2)\big)_{j\bar{k}} + (\log(u \circ f))_{j\bar{k}},$$

where the first term on the right-hand side vanishes because $|\det f'|^2 = (\det f')\overline{(\det f')}$ and $\det f'$ is holomorphic. We note that the vanishing of the first term also shows that the Kähler metric with Kähler potential u (when u solves the Monge–Ampère equation) is invariant whether u is considered as a scalar function or a density. To compute the nonzero term on the right-hand side we first note that if f is a holomorphic map then we have the identity

$$(v \circ f)_{j\bar{k}} = \big((f')^t (v_{a\bar{b}} \circ f)\overline{f'}\big)_{j\bar{k}}.$$

Thus, using Lemma 2.6, we compute

$$\begin{aligned} J\big[|\det(f')|^{-2/(m+1)} u \circ f\big] &= |\det(f')|^{-2} (u \circ f)^{m+1} \cdot \det\big((f')^t\big) \det\Big(\log\Big(\frac{1}{u}\Big)_{a\bar{b}} \circ f\Big) \det(\overline{f'}) \\ &= (u \circ f)^{m+1} \det\Big(\log\Big(\frac{1}{u}\Big)_{a\bar{b}}\Big) \circ f = J[u] \circ f. \end{aligned}$$ □

It is essential for our globalization argument that an approximate solution to the Monge–Ampere equation be determined uniquely up to a certain order. This proof was given by Fefferman [1976] and we repeat it for the reader's convenience.

Lemma 2.7. *Any smooth, local, approximate solution $\psi \in \mathscr{C}^\infty(\Omega)$ to the Monge–Ampère equation is uniquely determined up to order $m+1$.*

Proof. Suppose that ρ is a smooth function on Ω defined in a neighborhood of $\partial\Omega$ with $\rho = 0$ on $\partial\Omega$ and $\rho'(p) \neq 0$ for all $p \in \partial\Omega$. We recall Fefferman's iterative construction of an approximate solution u to the Monge–Ampère equation, that is, a function $u \in \mathscr{C}^\infty$ with $u|_{\partial\Omega} = 0$ and $J[u] = 1 + \mathbb{O}(u^{m+2})$. To obtain a first approximation, note that for ρ as above, and for any smooth function η, we have

$$J[\eta\rho] = \eta^{m+1} J[\rho], \tag{2-11}$$

when $\rho = 0$, so the function

$$\psi^{(1)} = \rho \cdot J[\rho]^{-1/(m+1)}$$

satisfies $J[\psi^{(1)}] = 1$ on $\partial\Omega$, and $J[\psi^{(1)}] = 1 + \mathbb{O}(\psi^{(1)})$. The fact that $J[\rho]$ is nonzero on $\partial\Omega$ follows from pseudoconvexity that implies that $\rho_{j\bar{k}}$ is positive definite on $\ker \partial\rho$ on $\partial\Omega$, and that $\rho' \neq 0$ on $\partial\Omega$. Note that if φ and ψ are two functions vanishing on $\partial\Omega$, it follows that $\varphi = \eta\psi$ for some smooth function η. Thus, by (2-11), $J[\varphi] = \eta^{m+1} J[\psi]$. From this computation it follows that any approximate solution u is uniquely determined up to first order.

We now iterate this construction. Suppose that for an integer $s \geq 2$, we have an approximate solution to the Monge–Ampère equation to order $s-1$. That is, we have a smooth function ψ with $\psi = 0$ on $\partial\Omega$, $\psi'(p) \neq 0$ for all $p \in \partial\Omega$, and $J[\psi] = 1 + \mathbb{O}(\psi^{s-1})$. We seek a function of the form $v = \psi + \eta\psi^s$, where $\eta \in \mathscr{C}^\infty$ is chosen so that $J[v] = 1 + \mathbb{O}(\psi^s)$. The iteration is based on formula

$$J[\psi + \eta\psi^s] = J[\psi] + s(m+2-s)\eta\psi^{s-1} + \mathbb{O}(\psi^s),$$

for smooth functions ψ and η, again with the property that ψ vanishes on $\partial\Omega$. This formula is a straightforward computation using the formula (2-8). From this formula it follows that the desired function v is given by

$$v = \psi + \frac{1 - J[\psi]}{s(m+2-s)}\psi^s.$$

The iteration clearly works up to $s = m+1$ and produces an approximate solution with the desired properties. It also follows that any function $\tilde{u}$ with $u - \tilde{u} = \mathbb{O}(\psi^{m+2})$ satisfies $J[\tilde{u}] = J[u] + \mathbb{O}(\psi^{m+2})$. Thus, in particular, any smooth function having the same $(m+1)$-jet on $\partial\Omega$ as an approximate solution is also an approximate solution.

On the other hand, it is clear that any two approximate solutions *must* have the same $(m+1)$-jet on $\partial\Omega$. If ψ and $\widetilde{\psi}$ satisfy $\psi - \widetilde{\psi} = \eta\psi^s$ then

$$J[\psi] - J[\widetilde{\psi}] = s(m+2-s)\eta\psi^{s-1} + \mathbb{O}(\psi^s).$$

In particular, if $s < m+2$ and $J[\psi] - J[\widetilde{\psi}] = \mathbb{O}(\psi^{m+2})$ then ψ and $\widetilde{\psi}$ are approximate solutions uniquely determined up to order $m+2$. □

2D.2. *Global theory.* Now suppose X is a compact complex manifold of dimension $m = n+1$ with boundary $M = \partial X$. Note that M has real dimension $2n+1$ and inherits an integrable CR-structure from X. As always, we assume that M with this CR-structure is strictly pseudoconvex. We first say what it means for a single smooth function φ defined in a neighborhood of M to be an approximate solution of the complex Monge–Ampère equation. We denote by $\mathscr{C}^\infty(X)$ the smooth functions on X.

Definition 2.8. We will say that a function $\varphi \in \mathscr{C}^\infty(X)$ is a globally defined approximate solution of the complex Monge–Ampère equation near $M = \partial X$ if for any $p \in M$, there is a neighborhood V of p in X and holomorphic coordinates z on V so that φ is an approximate solution of the complex Monge–Ampère equation in the chosen coordinates.

As we will see later, we will need such a globally defined approximate solution in order to identify the residues of the scattering operator on X with CR-covariant differential operators.

If φ is a defining function for M with $\varphi < 0$ in the interior of X, we associate to φ a Kähler form

$$\omega_\varphi = \frac{i}{2}\partial\bar{\partial}\log\Big(-\frac{1}{\varphi}\Big)$$

and a pseudo-Hermitian structure

$$\theta = \frac{i}{2}(\bar\partial - \partial)\varphi|_M. \tag{2-12}$$

Observe that two defining functions φ and ρ generate the same Kähler metric if and only if $\rho = e^F\varphi$ for a pluriharmonic function F, that is, $\partial\bar\partial F = 0$. It is known that a pluriharmonic function F is uniquely determined by its boundary values (see, for example, Bedford [1980]). If θ_ρ and θ_φ are the corresponding pseudo-Hermitian structures on M then $\theta_\rho = e^f\theta_\varphi$, where $f = F|_M$.

We give a necessary and sufficient condition on M for a globally defined approximate solution of the Monge–Ampère equation to exist. Recall that the canonical bundle of M is the bundle generated by forms $f\theta^1\wedge\cdots\wedge\theta^n\wedge\theta$ where f is smooth, θ is a contact form, and $\{\theta^\alpha\}_{\alpha=1}^n$ is an admissible coframe. If M is the boundary of a strictly pseudoconvex domain in $\mathbb{C}^m$, the canonical bundle is generated by restrictions of forms $f\,dz^1\wedge\cdots\wedge dz^m$ to M. The sections of the canonical bundle are $(n+1,0)$-forms ζ on M.

If θ is a contact form, T is the characteristic vector field, and ζ is *any* nonvanishing section of the canonical bundle, it is not difficult to see that

$$\theta\wedge(T\lrcorner\zeta)\wedge(T\lrcorner\bar\zeta) = \lambda\theta\wedge(d\theta)^n$$

for a smooth positive function λ. We say that the contact form θ is *volume-normalized* with respect to a nonvanishing section ζ of the canonical bundle if

$$\theta\wedge(d\theta)^n = i^{n^2}n!\,\theta\wedge(T\lrcorner\zeta)\wedge(T\lrcorner\bar\zeta),$$

where T is the characteristic vector field. The following criterion will be useful.

Lemma 2.9. *The contact form θ given by (2-12) is volume-normalized with respect to the form*

$$\zeta = (dz^1\wedge\cdots\wedge dz^m)|_M$$

if and only if

$$J[\varphi] = 1 + \mathbb{O}(\varphi)$$

in the coordinates $(z_1,\ldots,z_m)$.

For the proof see [Farris 1986, Proposition 5.2]. Using Lemma 2.9 we can prove:

Proposition 2.10. *Suppose that X is a compact complex manifold with boundary $M = \partial X$. There is a globally defined approximate solution φ of the Monge–Ampère equation in a neighborhood of M if and only if M admits a pseudo-Hermitian structure θ with the following property: In a neighborhood of any point $p\in M$, there is a local, closed $(n+1,0)$ form ζ such that θ is volume-normalized with respect to ζ.*

Proof. (i) First, suppose that X admits a globally defined approximate solution φ of the Monge–Ampère equation. Let θ be the associated contact form on X, that is, θ is given by (2-12). Pick $p\in M$ and let $z\equiv(z_1,\ldots,z_m)$ be holomorphic coordinates near p so chosen that φ is an approximate solution of the Monge–Ampère equation near p in these coordinates. Let

$$\zeta = (dz^1\wedge\cdots\wedge dz^m)|_M.$$

Then θ is volume-normalized with respect to ζ by Lemma 2.9.

(ii) Suppose that θ is a given contact form on M with the property that, for each point $p \in M$, there is a neighborhood of p and a closed, locally defined section ζ of the canonical bundle with respect to which θ is volume-normalized. Write

$$\zeta = f(dz^1 \wedge \cdots \wedge dz^m)|_M$$

for holomorphic coordinates $\{z_1, \ldots, z_m\}$ defined in a neighborhood of p and a smooth function f. The condition $d\zeta = 0$ is equivalent to the condition

$$\bar{\partial}_b f = 0,$$

that is, f is a CR-holomorphic function. By the strict pseudoconvexity of M, there is a holomorphic extension F to a neighborhood V of p in X, that is, there is an F defined near p with $\bar{\partial} F = 0$ and $F|_{M\cap V} = f$ (see [Kohn and Rossi 1965]). We claim that we can find new holomorphic coordinates $w \equiv (w_1, \ldots, w_m)$ near p with the property that

$$\frac{\partial(w_1, \ldots, w_m)}{\partial(z_1, \ldots, z_m)} = F(z), \tag{2-13}$$

where the left-hand side is the determinant of the holomorphic Jacobian. If so then

$$\zeta = dw^1 \wedge \cdots \wedge dw^m|_M.$$

Constructing in V an approximate solution ψ_V of the Monge–Ampère equation in the w-coordinates (as in Lemma 2.7, following Fefferman [1976]), we conclude from Lemma 2.9 that the induced contact form

$$\theta_V = \frac{i}{2}(\bar{\partial} - \partial)\psi_V|_{M\cap V}$$

on $M \cap V$ is volume-normalized with respect to ζ, and thus coincides with θ.

We now claim that the local approximate solutions ψ_V can be glued together to form a globally defined approximate solution to the Monge–Ampère equation in the sense of Definition 2.8. We first note an important property of the transition map for two local coordinate systems. Let V_1 and V_2 be neighborhoods of M in X with nonempty intersection, let z and w be holomorphic coordinates on V_1 and V_2, and suppose that ψ_1 and ψ_2 are approximate solutions of the complex Monge–Ampère equation in these coordinates respectively. More precisely, $u_1 = \psi_1 \circ z$ and $u_2 = \psi_2 \circ w$ are approximate solutions to the Monge–Ampère equation on coordinate patches U_1 and U_2 in $\mathbb{C}^m$, and there is a biholomorphic map

$$g : U_2 \cap w^{-1}(V_1 \cap V_2) \to U_1 \cap z^{-1}(V_1 \cap V_2).$$

The function $u_2 = |g'|^{2/(m+1)} u_1 \circ g$ is also an approximate solution of the complex Monge–Ampere equation in $U_2 \cap w^{-1}(V_1 \cap V_2)$ by Lemma 2.5, so by uniqueness we have $u_2 = e^F u_1 \circ g$, up to order $m+1$, where $F = \frac{2}{m+1} \log|g'|$ is pluriharmonic. Moreover, since u_1 and u_2 both induce the contact form θ it follows that

$$(\bar{\partial} - \partial)u_2|_{U_2\cap w^{-1}(M\cap V_1\cap V_2)} = ((\bar{\partial} - \partial)u_1) \circ f|_{U_2\cap w^{-1}(M\cap V_1\cap V_2)}$$

from which we deduce that $F|_{U_2\cap w^{-1}(M\cap V_1\cap V_2)} = 0$, and hence $F = 0$ by the uniqueness of pluriharmonic extensions. In particular, the map g is unimodular, $|g'| = 1$. Thus $u_2 = u_1 \circ g$ on $U_2 \cap w^{-1}(V_1 \cap V_2)$ up to order $m+1$.

We now fix a boundary defining function ρ. Suppose that $\{U_i\}$ is a finite cover of a neighborhood of the boundary by holomorphic charts. Denote by F_i the map from $\mathbb{C}^m$ into U_i and set $F_{ij} = F_i^{-1} \circ F_j$. As proved above, the cover and holomorphic coordinates (U_i, F_i) may be chosen so that the transition maps are unimodular, that is, $|F'_{ij}| = 1$. Using Fefferman's construction, we can produce in each U_i an approximate solution u_i in the sense that

$$J[u_i] = 1 + \mathbb{O}(\rho^{m+1}).$$

Now suppose that $\{\chi_i\}$ is a $\mathscr{C}^\infty$ partition of unity subordinate to the cover $\{U_i\}$. We claim that the smooth function $u = \sum_i \chi_i u_i$ is an approximate solution of the Monge–Ampère equation in the sense of Definition 2.8. Choose U_i so that $p \in U$. We may write

$$u = \sum_j (\chi_j \circ F_i)(u_j \circ F_i).$$

Since $u_j \circ F_i = (u_j \circ F_j) \circ F_{ji}$, we see that $u_j \circ F_i$ is also an approximate solution to the Monge–Ampère equation in the F_i-coordinates. Thus, there is a smooth function η_{ji} so that

$$(u_j \circ F_i)(z) - (u_i \circ F_i)(z) = \eta_{ji}(z)(\rho \circ F_i)^{m+2}(z)$$

where η_{ji} is smooth. We conclude that

$$u(z) - u_i(z) = \mathbb{O}((\rho \circ F_i)^{m+2}).$$

This shows that u is also an approximate solution of the Monge–Ampère equation in the F_i-coordinates as claimed.

To finish the proof it suffices to establish that such a holomorphic coordinate change $z \mapsto w$, as in (2-13), exists. We consider a coordinate transformation given by

$$w(z) = (h(z), z_2, \dots, z_m), \tag{2-14}$$

where $h(z)$ is the unknown holomorphic function. Condition (2-13) is equivalent to

$$\frac{\partial h}{\partial z_1}(z_1, \dots, z_m) = F(z_1, z_2, \dots, z_m).$$

Here, F is the holomorphic extension of the CR-function f. We solve this equation for h as follows.

We set the convention that a boundary chart in $\mathbb{C}^m$ is the intersection of an open ball about 0 with the (real) half-space $\operatorname{Im} z_m \geq 0$. We assume that the boundary point p corresponds to $0 \in \partial\mathbb{C}^m$. The unknown function h is a complex-valued function defined in a neighborhood V of $0 \in \mathbb{C}^m$, is holomorphic in $V \cap \{\operatorname{Im} z_m > 0\}$, has CR boundary values, and satisfies $h(0) = 0$. Thus, the map $w(z)$, defined in (2-14), preserves the boundary $\operatorname{Im}(z_n) = 0$.

Consequently, the desired change of coordinates is obtained by solving the initial value problem

$$\begin{aligned} &\frac{\partial h}{\partial z_1}(z_1, \dots, z_m) = F(z_1, z_2, \dots, z_m), \\ &h(0, z_2, \dots, z_m) = 0, \end{aligned}$$

by simple integration. □

We can also express the basic criterion in Proposition 2.10 in geometric terms. Recall that the contact form θ defines a pseudo-Hermitian, pseudo-Einstein structure on M if the Webster Ricci tensor is a multiple of the Levi form. Lee [1988] proved:

Theorem 2.11. *Suppose that M is a CR manifold of dimension ≥ 5. A contact form θ on M is pseudo-Einstein if and only if for each $p \in M$ there is a neighborhood of p in M and a locally defined closed section ζ of the canonical bundle with respect to which θ is volume-normalized.*

As an immediate consequence of Theorem 2.11, we have:

Theorem 2.12. *Suppose that M is a CR manifold of dimension ≥ 5. There is a globally defined approximate solution φ of the complex Monge–Ampère equation in a neighborhood of M if and only if M carries a contact form θ for which the corresponding pseudo-Hermitian structure is pseudo-Einstein. In this case, the contact form θ is induced by the globally defined approximate solution to the Monge–Ampère equation φ.*

Remark 2.13. If φ is a global approximate solution to the Monge–Ampère equation, then so is $e^F\varphi$ where F is any pluriharmonic function. The effect of the factor F is simply to change the choice of local coordinates needed to obtain a local approximate solution of the Monge–Ampère equation in any chart, as the argument in the proof of Proposition 2.10 easily shows. As observed above, the Kähler form ω_φ is invariant under the change $\varphi \mapsto e^F\varphi$.

3. Poisson operator and scattering operator

In this section we study the Dirichlet problem (1-3) following a standard technique in geometric scattering theory (see, for example, Melrose [1995]; we follow closely the analysis of the Poisson operator and scattering operator on conformally compact manifolds by Graham and Zworski [2003]). Note that Epstein, Melrose and Mendoza [1991] had previously studied the Poisson operator for a class of manifolds that includes compact complex manifolds with strictly pseudoconvex boundaries. More recently, Guillarmou and Sá Barreto [2008] studied scattering theory and radiation fields for asymptotically complex hyperbolic manifolds, a class which also includes that studied here.

We will set $x = -\varphi$ and we will denote by $\mathscr{C}^\infty(X)$ the set of smooth functions on X having Taylor series to all orders at $x = 0$, and by $\dot{\mathscr{C}}^\infty(X)$ the space of functions vanishing to all orders at $x = 0$. The space $\mathscr{C}^\infty(\mathring{X})$ consists of smooth functions on $\mathring{X}$ with no restriction on boundary behavior. We will denote by $x^s\mathscr{C}^\infty(X)$ the set of functions in $\mathscr{C}^\infty(\mathring{X})$ having the form x^sF for $F \in \mathscr{C}^\infty(X)$.

Since

$$N = -2\frac{\partial}{\partial x}$$

it follows that

$$\nu = -\frac{x}{\sqrt{1+rx}}\frac{\partial}{\partial x} \tag{3-1}$$

is the outward normal to the hypersurface $x = \varepsilon$. Green's theorem implies that

$$\int_{x>\varepsilon} (u_1\Delta_\varphi u_2 - u_2\Delta_\varphi u_1)\,\omega^m = \int_{x=\varepsilon} (u_1\nu u_2 - u_2\nu u_1)\,\nu \lrcorner\,\omega^m. \tag{3-2}$$

We first note the "boundary pairing formula" (recall the definition (2-4)).

Proposition 3.1. *Suppose* $\operatorname{Re}(s) = m/2$, *that* u_1 *and* u_2 *belong to* $\mathscr{C}^\infty(\mathring{X})$ *and there are functions* $F_i, G_i \in \mathscr{C}^\infty(X)$ *so that* $u_i = x^{m-s}F_i + x^s G_i$, $i = 1, 2$. *Finally, suppose that* $(\Delta_\varphi - s(m-s))u_i = r_i \in \dot{\mathscr{C}}^\infty(X)$, $i = 1, 2$. *Then,*

$$\int_X (u_1 r_2 - u_2 r_1)\,\omega^m = (2s - m)\int_M (F_1 G_2 - F_2 G_1)\,\psi.$$

Proof. A standard computation using (3-2) and (3-1) together with (2-3) and (2-5). □

Remark 3.2. For $\operatorname{Re}(s) = m/2$ complex conjugation reverses the roles of s and $m - s$. Thus we obtain the formula

$$\int_X (u_1 \bar{r}_2 - \bar{u}_2 r_1)\,\omega^m = (2s - m)\int_M (F_1 \overline{F}_2 - G_1 \overline{G}_2)\,\psi. \tag{3-3}$$

For later use, we note an extension of the boundary pairing formula analogous to [Graham and Zworski 2003, Proposition 3.3].

Proposition 3.3. *Suppose that* $\operatorname{Re}(s) > m/2$ *and* $2s - m \notin \mathbb{N}$. *Suppose that* $u_i \in \mathscr{C}^\infty(\mathring{X})$ *takes the form* $u_i = x^{m-s}F_i + x^s G_i$ *and* $(\Delta_\varphi - s(m-s))u_i = 0$, *for* $i = 1, 2$. *Then*

$$\operatorname*{FP}_{\varepsilon\downarrow 0}\left(\int_{x>\varepsilon} (\langle \nabla u_1, \nabla u_2\rangle - s(m-s)u_1 u_2)\,\omega^m\right) = -m\int_M G_1 F_2\,\psi = -m\int_M F_1 G_2\,\psi,$$

where FP *denotes the Hadamard finite part of the integral as* $\varepsilon \downarrow 0$.

Proof. Green's formula (3-2) for the operator $(\Delta_\varphi - s(m-s))$ gives

$$\int_{x>\varepsilon} (\langle \nabla u_1, \nabla u_2\rangle - s(m-s)u_1 u_2)\,\omega^m = \int_{x=\varepsilon} u_1(\nu u_2)\nu \lrcorner\, \omega^m,$$

from which the claimed formula follows. □

3A. ***The Poisson map.*** We will now prove that the Dirichlet problem (1-3) has a unique solution if $\operatorname{Re}(s) \ge m/2$, $2s - m \notin \mathbb{Z}$, and $s(m-s)$ is not an eigenvalue of Δ_φ. Most of the formal arguments are almost identical to the case of even asymptotically hyperbolic manifolds considered in [Graham and Zworski 2003] since the form of the indicial operator (2-7) for the Laplacian is the same.

Lemma 3.4. *Suppose that* $u \in \mathscr{C}^\infty(\mathring{X})$ *satisfies* $u = x^{m-s}F + x^s G$ *for functions* F *and* G *belonging to* $\mathscr{C}^\infty(X)$, *and that*

$$(\Delta_\varphi - s(m-s))u \in \dot{\mathscr{C}}^\infty(X), \tag{3-4}$$

for $s \in \mathbb{C}$ *with* $2s - m \notin \mathbb{Z}$. *Then the Taylor expansions of* F *and* G *at* $x = 0$ *are formally determined respectively by* $F|_M$ *and* $G|_M$. *In particular, we have*

$$F \sim \sum_{k\ge 0} x^k f_k \quad \text{and} \quad G \sim \sum_{k\ge 0} x^k g_k$$

where

$$f_k = \frac{(-1)^k}{2^k k!}\frac{\Gamma(2s-m-k)}{\Gamma(2s-m)} P_{k,s} f_0 \quad \text{and} \quad g_k = \frac{(-1)^k}{2^k k!}\frac{\Gamma(m-2s-k)}{\Gamma(m-2s)} P_{k,m-s} g_0, \tag{3-5}$$

with $P_{k,s}$ the differential operators of order $2k$ holomorphic in s with leading symbol[2]

$$\sigma(P_{k,s}) = \sigma(\Delta_b^k).$$

Proof. Recall the asymptotic development (2-6) for the Laplacian which we use to derive a recurrence for the Taylor coefficients f_k and g_k of F and G. For $2s - m \notin \mathbb{Z}$, we may consider the terms involving F and G separately. We first consider F. Observe that

$$(L_0 - s(m-s))(x^{m-s+k} f) = k(2s-m-k)x^{m-s+k} f$$

for $f \in \mathscr{C}^\infty(M)$. Since $L_k = P(x\partial_x, \partial_y)$ for a defining function x and boundary coordinates y where P is a polynomial of degree at most two with smooth coefficients, the operators

$$Q_{k,\ell}(s) = x^{-m+s-\ell} L_{k-\ell} x^{m-s+\ell}$$

are differential operators of order at most two depending holomorphically on s. If $u \sim \sum_{k=0}^{\infty} x^{m-s+k} f_k$, it follows from (3-4) and (2-6) that for any $k \geq 1$,

$$f_k = -\frac{1}{k(2s-m-k)} \sum_{\ell=0}^{k-1} Q_{k,\ell}(s) f_\ell. \tag{3-6}$$

Similarly, if $u \sim \sum_{k \geq 0} x^{s+k} g_k$ for $g_k \in \mathscr{C}^\infty(M)$, we have

$$g_k = -\frac{1}{k(m-2s-k)} \sum_{\ell=0}^{k-1} Q_{k,\ell}(m-s) g_\ell.$$

The formulas for f_k, g_k and $P_{k,s}$ follow easily from these formulas and the fact that

$$Q_{k,k-1}(s) = \tfrac{1}{2}\Delta_b + r_0(m-s+k-1)^2. \qquad \square$$

Remark 3.5. We will write $p_{k,s}$ for the operator satisfying $f_k = p_{k,s} f_0$, so that

$$p_{k,s} = \frac{(-1)^k}{2^k k!} \frac{\Gamma(2s-m-k)}{\Gamma(2s-m)} P_{k,s},$$

where $P_{k,s}$ is described in Lemma 3.4. The operator $p_{k,s}$ is meromorphic with poles at

$$s = \tfrac{m}{2} + \tfrac{k}{2}, \dots, \tfrac{m}{2} + \tfrac{1}{2}.$$

We will denote

$$p_\ell = \operatorname*{Res}_{s=(m+\ell)/2} p_{\ell,s}.$$

The operator p_ℓ is a differential operator of order at most 2ℓ with principal symbol

$$\sigma(p_\ell) = \frac{(-1)^\ell}{2^{\ell+1}\ell!(\ell-1)!} \sigma(\Delta_b^\ell).$$

[2]Here in the sense of the ordinary (rather than the Heisenberg) calculus on M.

For $\mathrm{Re}(s) > m/2$, let

$$R(s) = (\Delta_\varphi - s(m-s))^{-1}$$

be the $L^2(X)$ resolvent, let $\sigma_p(\Delta_\varphi)$ denote the set of L^2-eigenvalues of Δ_φ, and let

$$\Sigma = \{s : \mathrm{Re}(s) > \tfrac{m}{2},\ s(m-s) \in \sigma_p(\Delta_\varphi)\}.$$

We will now solve the Dirichlet problem (1-3) for $\mathrm{Re}(s) \geq m/2$ and $s \notin \Sigma$.

The following result is an easy consequence of the work of Epstein, Melrose, and Mendoza [1991]. Note that in our case the Kähler metric is an even metric, that is, depends smoothly on the defining function φ (and not simply on its square root).

Proposition 3.6. *The set Σ contains at most finitely many points, and the resolvent operator $R(s)$ is a meromorphic operator-valued function for* $\mathrm{Re}(s) > m/2$ *having at most finitely many, finite-rank poles at $s \in \Sigma$. Moreover, for $s \notin \Sigma$ and* $\mathrm{Re}(s) > m/2$,

$$R(s) : \dot{\mathscr{C}}^\infty(X) \to x^s \mathscr{C}^\infty(X).$$

First, we prove uniqueness of the solutions to the Dirichlet problem (1-3) for s with $\mathrm{Re}(s) \geq m/2$, $s \notin \Sigma$, and $2s - m \notin \mathbb{Z}$.

Proposition 3.7. *Suppose that* $\mathrm{Re}(s) \geq m/2$, *$s \notin \Sigma$, and $2s-m \notin \mathbb{Z}$. Suppose that $u \in \mathscr{C}^\infty(\mathring{X})$ with $(\Delta_\varphi - s(m-s))u = 0$, and that $u = x^{m-s}F + x^s G$ with $F|_M = 0$. Then $u = 0$.*

Proof. First, suppose that $\mathrm{Re}(s) > m/2$ and $s \notin \Sigma$. It follows from Lemma 3.4 that $u = x^s G$ for $G \in \mathscr{C}^\infty(X)$. Since $\mathrm{Re}(s) > m/2$ it is clear that

$$\int_X |u|^2\, \omega^m < \infty,$$

hence $u \in L^2(X)$, hence $u = 0$.

If $\mathrm{Re}(s) = m/2$ but $s \neq m/2$, we may again assume that $u = x^s G$ for $G \in \mathscr{C}^\infty(X)$. Next, we set $u_1 = u_2 = u$ in (3-3) to conclude that

$$\int_M |G|^2 \psi = 0$$

so that $G|_M = 0$. Using Lemma 3.4 again we conclude that $G \in \dot{\mathscr{C}}^\infty(X)$, hence $u \in \dot{\mathscr{C}}^\infty(X)$. As in [Guillarmou and Sá Barreto 2008], we can now deduce from [Vasy and Wunsch 2005] that $u = 0$. □

To prove the existence of a solution of the Dirichlet problem (1-3), we follow the method of Graham and Zworski [2003]. Given $f \in \mathscr{C}^\infty(M)$ we can construct a formal power series solution $u = x^{m-s}F$ modulo $\dot{\mathscr{C}}^\infty(X)$, and then use the resolvent to correct this approximate solution to an exact solution. Using Borel's lemma we can sum the asymptotic series $\sum_{j\geq 0} f_j x^j$ (where f_j is computed via (3-6) with $f_0 = f$) to a function $F \in \mathscr{C}^\infty(X)$. As in [Graham and Zworski 2003], we obtain:

Lemma 3.8. *There is an operator $\Phi(s) : \mathscr{C}^\infty(M) \to x^{m-s}\mathscr{C}^\infty(X)$ with*

$$(\Delta_\varphi - s(m-s)) \circ \Phi : \mathscr{C}^\infty(M) \to \dot{\mathscr{C}}^\infty(X)$$

so that $\Gamma(m-2s)^{-1}\Phi(s)$ is holomorphic in s.

Note that $\Phi(s)$ need not be linear as the construction of F depends on the choice of cutoff functions in the application of Borel's lemma. As noted in [Graham and Zworski 2003], an expansion to finite order in x suffices for the construction. This guarantees the continuity of the map $\Phi(s)$ in the data f. Now define an operator

$$\mathcal{P}(s) : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(\mathring{X})$$

for s with $\mathrm{Re}(s) \geq m/2$, $s \neq m/2$ and $s \notin \Sigma$ by

$$\mathcal{P}(s) = \big(I - R(s)(\Delta_\varphi - s(m-s))\big) \circ \Phi(s).$$

Lemma 3.9. *For any $f \in \mathcal{C}^\infty(M)$, the function $u = \mathcal{P}(s) f$ solves the Dirichlet problem (1-3), and $f \mapsto \mathcal{P}(s) f$ is a linear operator.*

Proof. The linearity of $\mathcal{P}(s)$ will follow from the unicity of the solution to (1-3). It is immediate from the definition that $(\Delta_\varphi - s(m-s))u = 0$, and from the mapping property in Proposition 3.6, $u = x^{m-s}F + x^s G$ with

$$F = x^{s-m}\Phi(s) f \quad \text{and} \quad G = -x^{-s}R(s)\big((\Delta_\varphi - s(m-s))\Phi(s) f\big). \qquad \square$$

Theorem 3.10. *For $\mathrm{Re}(s) \geq m/2$, $2s - m \notin \mathbb{Z}$, and $s \notin \Sigma$, there exists a unique solution of the Dirichlet problem (1-3).*

3B. *The scattering operator.* The *scattering operator* for Δ_φ is the linear mapping

$$S_X(s) : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M), \qquad f \mapsto G|_M,$$

where $u = x^{m-s}F + x^s G$ solves (1-3). It is well defined by Theorem 3.10.

The scattering operator has infinite-rank poles when $\mathrm{Re}(s) > m/2$ and $2s - m \in \mathbb{Z}$ owing to the crossing of indicial roots for the normal operator L_0. At the exceptional points $s = (m+k)/2$ one expects solutions of the eigenvalue equation $(\Delta_\varphi - s(m-s))u = 0$ having the form

$$u = x^{m/2-k}F + (x^{m/2+k}\log x)G.$$

In order to study the singularities of the scattering operator at these points we modify the construction of the Poisson operator following the lines of [Graham and Zworski 2003, Section 4].

Let f_1 and f_2 belong to $\mathcal{C}^\infty(M)$ and let u_1 and u_2 solve the corresponding Dirichlet problems for some s with $\mathrm{Re}(s) > m/2$ and $2s - m \notin \mathbb{N}$. Applying the generalized boundary pairing formula (see Proposition 3.3) to u_1 and $\bar{u}_2$ for s real, we conclude that

$$\int_M f_1 \overline{S_X(s) f_2}\, \psi = \int_M (S_X(s) f_1)\overline{f_2}\, \psi$$

so $S_X(s)$ is self-adjoint in the natural inner product on $\mathcal{C}^\infty(M)$.

Now we study the scattering operator near the exceptional points. The arguments used here are exactly those of [Graham and Zworski 2003, Section 3] but we summarize them here for the reader's convenience.

Recall the operators $p_{k,s}$ and p_ℓ defined in Remark 3.5. First, we prove:

Lemma 3.11. *At the points $s = (m+\ell)/2$ for $\ell = 1, 2, \ldots$, $s \notin \Sigma$, the Poisson map takes the form*

$$\mathcal{P}(\tfrac{m}{2} + \tfrac{\ell}{2}) f = x^{m/2-\ell/2}F + (x^{m/2+\ell/2}\log x)G,$$

where $F|_M = f$, $G|_M = -2p_\ell f$ *and*

$$p_\ell = \operatorname*{Res}_{s=(m+\ell)/2} p_{\ell,s} \tag{3-7}$$

is a differential operator of order 2ℓ *with*

$$\sigma(p_\ell) = \frac{(-1)^\ell}{2^{\ell+1}\ell!(\ell-1)!}\sigma(\Delta_b^\ell).$$

Proof. We first show that the Poisson map $\mathcal{P}(s)$ is also regular at $s=(m+\ell)/2$, $\ell = 1, 2, \ldots$ so long as these points do not belong to Σ. As in [Graham and Zworski 2003] we introduce the operator

$$\Phi_\ell(s) = \Phi(s) - \Phi(m-s)\circ p_{\ell,s},$$

where $p_{\ell,s}$ is a differential operator of order 2ℓ defined in Remark 3.5, each term on the right-hand side has at most a first-order pole at $s=(m+\ell)/2$, the operators $p_{j,s}$ occurring in the definition of $\Phi(s)$ have at most first-order poles and $\Phi(m-s)$ is analytic in s for $\operatorname{Re}(s) > m/2$. For given $f \in \mathscr{C}^\infty(M)$, we compute the residue of $\Phi_\ell(s)f$ at $s=(m+\ell)/2$. First

$$\lim_{s\to(m+\ell)/2}\Big(s-\frac{m}{2}-\frac{\ell}{2}\Big)\Phi(s)f = x^{(m+\ell)/2}\operatorname*{Res}_{s=(m+\ell)/2}(p_{\ell,s}f) + \mathbb{O}(x^{m/2+\ell/2+1}),$$

since the remaining terms in the asymptotic expansion for $\Phi(s)f$ are holomorphic near $s=(m+\ell)/2$. Second,

$$\lim_{s\to(m+\ell)/2}\Big(s-\frac{m}{2}-\frac{\ell}{2}\Big)\Phi(m-s)(p_{\ell,s}f) = x^{m/2+\ell/2}\operatorname*{Res}_{s=(m+\ell)/2}(p_{\ell,s}f) + \mathbb{O}(x^{m/2+\ell/2+1}).$$

It follows that

$$\operatorname*{Res}_{s=(m+\ell)/2}\Phi_\ell(s)f = \mathbb{O}(x^{m/2+\ell/2+1}) \tag{3-8}$$

so that, by Lemma 3.4,

$$\operatorname*{Res}_{s=(m+\ell)/2}\Phi_\ell(s)f \in \dot{\mathscr{C}}^\infty(X).$$

Now let us define

$$\mathcal{P}_\ell(s) = \big(I - R(s)(\Delta_\varphi - s(m-s))\big)\circ\Phi_\ell(s).$$

Clearly, $\mathcal{P}_\ell(s)$ is holomorphic in a deleted neighborhood of $s=(m+\ell)/2$ (with at most a first-order pole at $s=(m+\ell)/2$) and maps $\mathscr{C}^\infty(M)$ into $\mathscr{C}^\infty(\mathring{X})$. If $s\notin\Sigma$, it follows from the definition of $\mathcal{P}_\ell(s)$, (3-8) and Proposition 3.6 that

$$\operatorname*{Res}_{s=(m+\ell)/2}\mathcal{P}_\ell(s)f \in x^s\mathscr{C}^\infty(X).$$

Hence the residue is an $L^2(X)$ function, and hence is zero. Thus $\mathcal{P}_\ell(s)$ is holomorphic at $s=(m+\ell)/2$. It follows from the uniqueness of solutions to the Dirichlet problem that $\mathcal{P}_\ell(s) = \mathcal{P}(s)$ wherever the former is defined. Exactly as in [Graham and Zworski 2003] we can compute $\mathcal{P}((m+\ell)/2)f$ by using $\mathcal{P}_\ell(s)$, the formula

$$\lim_{t\to 0}\frac{x^{-t}-x^t}{t} = -2\log x$$

and the fact that the $p_{k,s}$ have at most simple poles at $s=(m+\ell)/2$. This computation shows that $\mathcal{P}((m+\ell)/2)$ has the stated form. □

Proposition 3.12. *Suppose that Δ_X has no eigenvalues of the form $s(m-s)$ with $s=(m+\ell)/2$, $\ell=1,2,\ldots$. Then, the scattering operator $S_X(s)$ has a first-order pole at $s=(m+\ell)/2$, $\ell=1,2,\ldots$ with*

$$\operatorname*{Res}_{s=(m+\ell)/2} S_X(s) = -p_\ell,$$

where p_ℓ is the differential operator given by (3-7).

Proof. From the formula for the $\mathcal{P}(s)$, it is clear that for $2s-m\notin\mathbb{N}$, we can compute the scattering operator from

$$S_X(s)f = \big(-x^{-s}R(s)(\Delta_\varphi - s(m-s))\Phi(s)f\big)\big|_{x=0}.$$

Since $\mathcal{P}(s)$ is holomorphic at $s=(m+\ell)/2$, it follows that

$$\operatorname*{Res}_{s=(m+\ell)/2}(S_X(s)f) = -\operatorname*{Res}_{s=(m+\ell)/2}(x^{-s}\Phi(s)f)|_{x=0}.$$

But

$$\operatorname*{Res}_{s=(m+\ell)/2}(x^{-s}\Phi(s)f)|_{x=0} = \operatorname*{Res}_{s=(m+\ell)/2}\big((x^{-s}\Phi(m-s)p_{\ell,s}f)|_{x=0}\big) = \operatorname*{Res}_{s=(m+\ell)/2}(p_{\ell,s}f)$$

and the claimed formula holds. □

To connect the scattering operator and the CR Q-curvature, we will also need the following result about the pole of the scattering operator at $s=m$; this result is a direct analogue of [Graham and Zworski 2003, Proposition 3.7] but we give the short proof for the reader's convenience.

Proposition 3.13. *Let 1 denote the constant function on M. Then,*

$$S_X(m)1 = -\lim_{s\to m} p_{m,s}(1).$$

Proof. As $s\to m$ we have $\mathcal{P}(s)1\to 1$. On the other hand, for s with $|s-m|<\frac{1}{2}$,

$$\mathcal{P}(s)1 = \sum_{k=0}^{m} x^{m-s+k}p_{k,s}(1) + x^s S_X(s)1 + \mathbb{O}(x^{m+1/2}).$$

This implies that

$$\lim_{s\to m}(x^{2m-s}p_{m,s}(1) + x^m S_X(s)1) = 0$$

from which the claimed formula follows. □

Remark 3.14. Note that, although $p_{m,s}$ has a pole at $s=m$, the limit $\lim_{s\to m} p_{m,s}(1)$ exists. This implies that $P_{m,s}1$ (see (3-5)) has a first-order zero at $s=m$, that is,

$$P_{m,s}1 = (m-s)Q_{m,s}$$

for a scalar function $Q_{m,s}$. The CR Q-curvature is then given by $Q_{m,m}$ [Fefferman and Hirachi 2003].

4. CR-covariant operators

In this section we show that if φ is an approximate solution of the complex Monge–Ampère equation in the sense discussed above, then the residues of the scattering operator at $s = (m+\ell)/2$ for $\ell = 1, \ldots, m$ are the CR-covariant differential operators P_k defined in [Fefferman and Hirachi 2003]. In order to do this we first recall Fefferman and Graham's [1985] set-up for studying conformal invariants of compact manifolds and the construction of the GJMS operators [Graham et al. 1992]. We then recall its application to CR manifolds taking care that the arguments carry over from pseudoconvex domains in $\mathbb{C}^m$ to the manifold setting studied here.

4A. ***The GJMS construction.*** We begin by recalling Fefferman and Graham's construction of the ambient metric and ambient space for a conformal manifold and the GJMS conformally covariant operators on $\mathscr{C}$ obtained from this construction. Suppose that $(\mathscr{C}, [g])$ is a conformal manifold of signature (p, q), that is, a smooth manifold of dimension $N = p+q$ together with a conformal class of pseudo-Riemannian metrics of signature (p, q) on $\mathscr{C}$. Fix a conformal representative g_0. The *metric bundle* $\mathscr{G} \subset S^2T^*\mathscr{C}$ is a bundle on $\mathscr{C}$ with fibres

$$\mathscr{G}_p = \{t^2 g_0(p) : t > 0\}.$$

We denote by $\pi : \mathscr{G} \to M$ the natural projection. The *tautological metric* G on $\mathscr{G}$ is given by

$$G(X, Y) = g(\pi_* X, \pi_* Y)$$

for tangent vectors X and Y to $(p, g) \in \mathscr{G}$. There is a natural $\mathbb{R}^+$-action δ_s on $\mathscr{G}$ given by

$$\delta_s(p, g) = (p, s^2 g).$$

The *ambient space* over $\mathscr{C}$ is the space $\widetilde{\mathscr{G}} = \mathscr{G} \times (-1, 1)$. Note that the map $i : g \mapsto (g, 0)$ imbeds $\mathscr{G}$ in $\widetilde{\mathscr{G}}$.

Fefferman and Graham proved the existence of a unique metric $\widetilde{g}$ of signature $(p+1, q+1)$ on $\widetilde{\mathscr{G}}$, the *ambient metric* on $\widetilde{\mathscr{G}}$ having the following three properties:

(a) $i^*\widetilde{g} = G$;

(b) $\delta_s^*\widetilde{g} = s^2\widetilde{g}$;

(c) $\mathrm{Ric}(\widetilde{g}) = 0$ along $\mathscr{G}$ to infinite order if N is odd, and up to order $N/2$ if N is even.

Here the uniqueness is meant in the sense of formal power series.

To define the GJMS operators, we first define spaces of homogeneous functions on $\mathscr{G}$. For $w \in \mathbb{R}$ let $\mathscr{E}(w)$ denote the functions f on $\mathscr{G}$ homogeneous of degree w with respect to δ_s and smooth away from 0. The GJMS operators $\mathscr{P}_k$ may be defined in two ways:

(1) Given $f \in \mathscr{E}(-N/2+k)$, extend f to a function $\widetilde{f}$ homogeneous of the same degree on $\widetilde{\mathscr{G}}$, and set

$$\mathscr{P}_k f = \widetilde{\Delta}^k \widetilde{f}\,\big|_{\mathscr{G}}, \tag{4-1}$$

where $\widetilde{\Delta}$ is the Laplacian for the ambient metric $\widetilde{g}$ on $\widetilde{\mathscr{G}}$.

(2) Given $f \in \mathscr{E}(-N/2+k)$, $\mathscr{P}_k$ is the normalized obstruction to extending f to a smooth function $\widetilde{f}$ on $\widetilde{\mathscr{G}}$ having the same homogeneity and satisfying $\widetilde{\Delta}\widetilde{f} = 0$.

The existence of GJMS operators was proven in [Graham et al. 1992] for $k = 1, 2, \ldots$ if N is odd, and for $k = 1, 2, \ldots, N/2$ if N is even.

4B. ***Application to CR manifolds.*** Following [Gover and Graham 2005] we describe how the GJMS construction [Graham et al. 1992] can be used to prove the existence of CR-covariant differential operators. We begin with a CR manifold M of dimension $2n+1$ and show how to construct a conformal manifold $\mathscr{C}$ of dimension $2n+2$ and a conformal class of metrics with signature $(2n+1, 1)$ to which the GJMS construction may be applied. One then "pulls back" the GJMS operators on $\mathscr{C}$ to M.

Recall that the *canonical bundle* K over M is the bundle of holomorphic $(n+1)$-forms generated by holomorphic forms of the type $\theta \wedge \theta^1 \wedge \cdots \wedge \theta^n$ where θ is a contact form and $\{\theta^\alpha\}$ is a basis for $\mathscr{H}$ of admissible $(1,0)$-forms. We denote by K^* the canonical bundle of M with the zero section removed. The *circle bundle* $\mathscr{C}$ over M is the bundle

$$\mathscr{C} = (K^*)^{1/(n+2)}/\mathbb{R}^+.$$

The circle bundle is an S^1-bundle over M, having real dimension $2m$ if $m = n+1$. If we fix a contact form θ on M (and hence a pseudo-Hermitian structure on M), there is a corresponding section ζ of K^* chosen so that θ is volume-normalized with respect to ζ. We denote by ψ the angle determined by $\zeta(p)$ in each fibre of $\mathscr{C}$ and define a fibre variable

$$\gamma = \frac{\psi}{n+2}.$$

Note that γ is canonically determined by θ. Following Lee [1986], let us define a canonical one-form σ on $\mathscr{C}$ by

$$(n+2)\,\sigma = (n+2)\,d\gamma + i\omega_\alpha{}^\alpha - \frac{1}{2(n+1)} R\theta, \tag{4-2}$$

where $\omega_\alpha{}^\beta$ is the connection one-form and R is the Webster scalar curvature of the pseudo-Hermitian structure θ. The mapping $\theta \mapsto g_\theta$ given by

$$g_\theta = h_{\alpha\beta}\,\theta^\alpha \cdot \theta^{\bar\beta} + 2\theta \cdot \sigma, \tag{4-3}$$

where $\cdot$ denotes the symmetric product, defines a mapping of pseudo-Hermitian structures to Lorenz metrics which respects conformal classes. One can now obtain GJMS operators on $\mathscr{C}$ using the Fefferman–Graham construction.

Remark 4.1. It is immediate from formulas (4-2) and (4-3) that

$$g_\theta(T, T) = -\frac{1}{(n+1)(n+2)} R,$$

where R is the Webster scalar curvature, a pseudo-Hermitian invariant. On the other hand, Farris [1986] computed that, if θ is the contact form induced by an approximate solution of the complex Monge–Ampère equation, then

$$g_\theta(T, T) = 2r,$$

where r is the transverse curvature. It follows that the transverse curvature is, in this case, an intrinsic pseudo-Hermitian invariant.

To compute their pullbacks to M, we first note that the metric bundle $\mathcal{G}$ of $(\mathcal{C}, [g])$ is diffeomorphic to $(K^*)^{1/(n+2)}$ and $\widetilde{\mathcal{G}} \simeq (K^*)^{1/(n+2)} \times (-1, 1)$. We define spaces of functions

$$\begin{aligned}\mathcal{E}(w, w') &= \left\{ f \in \mathcal{C}^\infty\big((K^*)^{1/(n+2)}\big) : f(\lambda\xi) = \lambda^w \bar{\lambda}^{w'} f(\xi) \text{ for } \lambda \in \mathbb{C}^* \right\} \\ &= \left\{ f \in \mathcal{E}(w + w') : (e^{i\phi})^* f(\xi) = e^{i\phi(w-w')} f(\xi) \right\}.\end{aligned}$$

We will primarily be concerned with functions in

$$\mathcal{E}(w, w) = \{ f \in \mathcal{E}(2w) : (e^{i\phi})^* f(\xi) = f(\xi) \},$$

which descend to smooth functions on M.

For $k \in \mathbb{Z}$, we define

$$P_{w,w'} : \mathcal{E}(w, w') \to \mathcal{E}(w - k, w' - k), \qquad f \mapsto 2^{-k} \mathcal{P}_k f,$$

where $\mathcal{P}_k$ is defined in (4-1). Then choosing $w = w' = k - (n+1)/2$, we get operators P_k defined on $\mathcal{E}(-N/2 + k)$ (recall that $N/2 = n + 1$) which are invariant under the circle action $(e^{i\phi})^*$ and hence may be viewed as smooth sections of a density bundle over M. These operators P_k are the CR-covariant differential operators which we will connect to poles of the scattering operator.

If X admits a globally defined approximate solution φ of the Monge–Ampère equation, then for each $p \in M = \partial X$ there is a neighborhood U of p and holomorphic coordinates $(z_1, \dots, z_m)$ near p so that φ is an approximate solution of the Monge–Ampère equation in U. Let

$$\theta = \frac{i}{2}(\bar{\partial} - \partial)\varphi \,\big|_M$$

be the induced pseudo-Hermitian structure on M, and let $\zeta = dz^1 \wedge \cdots \wedge dz^m|_M$. Then θ is volume-normalized with respect to ζ.

Let us denote by z_0 the induced fibre coordinate of $(K^*)^{1/(n+2)}$ and let

$$Q = |z_0|^2 \varphi.$$

Then Q is a globally defined smooth function on $\widetilde{\mathcal{G}}$ (which is diffeomorphic to $\mathbb{C} \times N$ for a collar neighborhood N of M in X) and the ambient metric on $\widetilde{\mathcal{G}}$ is the Kähler metric associated to the Kähler form

$$\omega = i \partial\bar{\partial} Q$$

where the corresponding metric g_θ on $\mathcal{C}$ is given by (4-3). The key computation linking the GJMS operators to the Laplacian is given in [Gover and Graham 2005, Proposition 5.4] and clearly generalizes to our situation. Thus we have:

Proposition 4.2. *If u is a smooth function on X then*

$$\widetilde{\Delta}(|z_0|^{2w} \varphi^w u) = (|z_0|^{2w} \varphi^w)(\Delta_\varphi + w(n + 1 + w))u,$$

where Δ_φ is the Laplacian associated to the Kähler form

$$\omega_\varphi = \frac{i}{2} \partial\bar{\partial} \log\Big(-\frac{1}{\varphi}\Big).$$

5. Proofs of the main theorems

Finally, we prove Theorems 1.1, 1.4 and 1.5. We are grateful to the referee for suggesting the proof of Theorem 1.5 in what follows, based on ideas of Graham and Fefferman [2002]; see especially the proofs of Theorems 3.1 and 4.1 there.

Proof of Theorem 1.1. The statement about the poles of $S_X(s)$ and $s = (m+k)/2$ is proved in Proposition 3.12. If g is a metric on X associated to the Kähler form $\omega = i\bar{\partial}\partial \log(-\frac{1}{\varphi})$ for a globally defined approximate solution of the Monge–Ampère equation, then the identification of the residues of $S_X(s)$ with the CR-covariant differential operators of Fefferman and Hirachi is a consequence of Proposition 4.2 and the second characterization of the GJMS operators given in Section 4A. □

Proof of Theorem 1.4. Owing to Proposition 3.13, it suffices to identify $\lim_{s\to m} p_{m,s}1$ with the CR Q-curvature. This is a consequence of Remark 3.14. □

Proof of Theorem 1.5. To begin with we note that if 1 denotes the constant function with value 1 on M, then the mapping $s \mapsto \mathcal{P}(s)1$ is a holomorphic mapping into $\mathscr{C}^\infty(X)$ and that, moreover,

$$\mathcal{P}(s)1 = x^{m-s}F(s) + x^s G(s), \tag{5-1}$$

where F and G are smooth functions on X with Taylor series to all orders at the boundary and depend holomorphically on s (this is not true for $\mathcal{P}(s)f$ for general f, but does hold true when $f = 1$ since 1 lies in the kernel of the differential operators occurring in the logarithmic term). For $s \neq m$ we have

$$F(s)|_M = 1, \quad G(s)|_M = S_X(s)1,$$

and by holomorphy the same is true when $s = m$. By uniqueness we also have $\mathcal{P}(m)1 = 1$ so that

$$F(m) = 1 - x^m G(m). \tag{5-2}$$

Let $U = -\frac{d}{ds}\big|_{s=m} \mathcal{P}(s)1$. It is easy to see that

$$\Delta_\varphi U = m. \tag{5-3}$$

It follows from (5-1)–(5-2) that

$$U = \log x + Ax^m \log x + B,$$

where A and B are smooth functions having Taylor series to all orders at ∂X and

$$A|_{\partial X} = S_X(m)1.$$

By Proposition 3.13 and Remark 3.14 we have:

$$S_X(m)1 = c_m Q_\theta^{CR}.$$

On the other hand, we have from (5-3) that for x_0 sufficiently small,

$$\begin{aligned} m\,\mathrm{vol}(\varepsilon < x < x_0) &= \int_{\varepsilon<x<x_0} \Delta_g U\,\omega \\ &= -\varepsilon^{-m}\int_{x=\varepsilon}(1+r\varepsilon)\frac{\partial U}{\partial \nu}(\nu\rfloor\omega^m) + x_0^{-m}\int_{x=x_0}(1+rx_0)\frac{\partial U}{\partial \nu}(\nu\rfloor\omega^m) \\ &= -\varepsilon^{-m}\int_{x=\varepsilon}(1+r\varepsilon)\frac{\partial U}{\partial \nu}(\nu\rfloor\omega^m) + \mathbb{O}(1) \end{aligned}$$

where we have used Green's formula and (2-3). Recalling that

$$\frac{\partial}{\partial \nu} = -\frac{1}{\sqrt{1+rx}}\,x\frac{\partial}{\partial x}$$

(see (3-1)), it is clear that the coefficient of $\log\varepsilon$ in the expansion of $m\,\mathrm{vol}(\varepsilon < x < x_0)$ is

$$mL = m\int_M S_X(s)1|_{s=m}\,\psi$$

from which we conclude that

$$L = c_m \int_M Q_\theta^{CR}\,\psi. \qquad \square$$

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References

[Bedford 1980] E. Bedford, "$(\partial\bar{\partial})_b$ and the real parts of CR functions", *Indiana Univ. Math. J.* **29**:3 (1980), 333–340. MR 82a:32024 Zbl 0441.32008

[Branson 1993] T. P. Branson, *The functional determinant*, Lecture Notes Series **4**, Seoul National University Research Institute of Mathematics Global Analysis Research Center, Seoul, 1993. MR 96g:58203 Zbl 0827.58057

[Burns and Epstein 1990] D. Burns and C. L. Epstein, "Characteristic numbers of bounded domains", *Acta Math.* **164**:1-2 (1990), 29–71. MR 90k:32063 Zbl 0704.32005

[Chang et al. 2008] S.-Y. A. Chang, M. Eastwood, B. Ørsted, and P. C. Yang, "What is Q-curvature?", *Acta Appl. Math.* **102**:2-3 (2008), 119–125. MR 2407525 Zbl 05306233

[Cheng and Yau 1980] S. Y. Cheng and S. T. Yau, "On the existence of a complete Kähler metric on noncompact complex manifolds and the regularity of Fefferman's equation", *Comm. Pure Appl. Math.* **33**:4 (1980), 507–544. MR 82f:53074 Zbl 0506.53031

[Epstein et al. 1991] C. L. Epstein, R. B. Melrose, and G. A. Mendoza, "Resolvent of the Laplacian on strictly pseudoconvex domains", *Acta Math.* **167**:1-2 (1991), 1–106. MR 92i:32016 Zbl 0758.32010

[Farris 1986] F. A. Farris, "An intrinsic construction of Fefferman's CR metric", *Pacific J. Math.* **123**:1 (1986), 33–45. MR 87f:53068 Zbl 0599.32018

[Fefferman 1976] C. L. Fefferman, "Monge–Ampère equations, the Bergman kernel, and geometry of pseudoconvex domains", *Ann. of Math.* (2) **103**:2 (1976), 395–416. Correction in **104**:2 (1976), 393–394. MR 53 #11097a Zbl 0322.32012

[Fefferman and Graham 1985] C. Fefferman and C. R. Graham, "Conformal invariants", *Astérisque* Numero Hors Serie (1985), 95–116. MR 87g:53060 Zbl 0602.53007

[Fefferman and Graham 2002] C. Fefferman and C. R. Graham, "Q-curvature and Poincaré metrics", *Math. Res. Lett.* **9**:2-3 (2002), 139–151. MR 2003f:53053 Zbl 1016.53031

[Fefferman and Hirachi 2003] C. Fefferman and K. Hirachi, "Ambient metric construction of Q-curvature in conformal and CR geometries", *Math. Res. Lett.* **10**:5-6 (2003), 819–831. MR 2005d:53044 Zbl 02064736

[Gover and Graham 2005] A. R. Gover and C. R. Graham, "CR invariant powers of the sub-Laplacian", *J. Reine Angew. Math.* **583** (2005), 1–27. MR 2006f:32049 Zbl 1076.53048

[Graham and Lee 1988] C. R. Graham and J. M. Lee, "Smooth solutions of degenerate Laplacians on strictly pseudoconvex domains", *Duke Math. J.* **57**:3 (1988), 697–720. MR 90c:32031 Zbl 0699.35112

[Graham and Zworski 2003] C. R. Graham and M. Zworski, "Scattering matrix in conformal geometry", *Invent. Math.* **152**:1 (2003), 89–118. MR 2004c:58064 Zbl 1030.58022

[Graham et al. 1992] C. R. Graham, R. Jenne, L. J. Mason, and G. A. J. Sparling, "Conformally invariant powers of the Laplacian. I. Existence", *J. London Math. Soc.* (2) **46**:3 (1992), 557–565. MR 94c:58226 Zbl 0726.53010

[Guillarmou and Sá Barreto 2008] C. Guillarmou and A. Sá Barreto, "Scattering and inverse scattering on ACH Manifolds", *J. Reine. Angew. Math.* **622** (2008), 1–55. Zbl 05344063

[Harvey and Lawson 1975] F. R. Harvey and H. B. Lawson, Jr., "On boundaries of complex analytic varieties. I", *Ann. of Math.* (2) **102**:2 (1975), 223–290. MR 54 #13130 Zbl 0317.32017

[Hislop et al. 2006] P. D. Hislop, P. A. Perry, and S.-H. Tang, "CR-invariants and the scattering operator for complex manifolds with CR-boundary", *C. R. Math. Acad. Sci. Paris* **342**:9 (2006), 651–654. MR 2007g:32024 Zbl 1096.58009

[Jerison and Lee 1984] D. Jerison and J. M. Lee, "A subelliptic, nonlinear eigenvalue problem and scalar curvature on CR manifolds", pp. 57–63 in *Microlocal analysis* (Boulder, CO, 1983), edited by M. S. Baouendi et al., Contemp. Math. **27**, Amer. Math. Soc., Providence, RI, 1984. MR 85i:58122 Zbl 0577.53035

[Kohn and Rossi 1965] J. J. Kohn and H. Rossi, "On the extension of holomorphic functions from the boundary of a complex manifold", *Ann. of Math.* (2) **81** (1965), 451–472. MR 31 #1399 Zbl 0166.33802

[Lee 1986] J. M. Lee, "The Fefferman metric and pseudo-Hermitian invariants", *Trans. Amer. Math. Soc.* **296**:1 (1986), 411–429. MR 87j:32063 Zbl 0595.32026

[Lee 1988] J. M. Lee, "Pseudo-Einstein structures on CR manifolds", *Amer. J. Math.* **110**:1 (1988), 157–178. MR 89f:32034 Zbl 0638.32019

[Lee and Melrose 1982] J. M. Lee and R. Melrose, "Boundary behaviour of the complex Monge–Ampère equation", *Acta Math.* **148** (1982), 159–192. MR 84e:58078 Zbl 0496.35042

[Melrose 1995] R. B. Melrose, *Geometric scattering theory*, Cambridge University Press, Cambridge, 1995. MR 96k:35129 Zbl 0849.58071

[Melrose 1999] R. Melrose, "Scattering theory for strictly pseudoconvex domains", pp. 161–168 in *Differential equations: La Pietra 1996* (Florence, 1996), edited by M. Giaquinta et al., Proc. Sympos. Pure Math. **65**, Amer. Math. Soc., Providence, RI, 1999. MR 2000a:58081 Zbl 0927.37034

[Seshadri 2007] N. Seshadri, "Volume renormalization for complete Einstein–Kähler metrics", *Differential Geom. Appl.* **25**:4 (2007), 356–379. MR 2345277

[Tanaka 1975] N. Tanaka, *A differential geometric study on strongly pseudo-convex manifolds*, Lectures in Mathematics, Department of Mathematics, Kyoto University **9**, Kinokuniya Book-Store Co. Ltd., Tokyo, 1975. MR 53 #3361 Zbl 0331.53025

[Vasy and Wunsch 2005] A. Vasy and J. Wunsch, "Absence of super-exponentially decaying eigenfunctions of Riemannian manifolds with pinched negative curvature", *Math. Res. Lett.* **12**:5-6 (2005), 673–684. MR 2006m:58048 Zbl 1121.58025

[Webster 1978] S. M. Webster, "Pseudo-Hermitian structures on a real hypersurface", *J. Differential Geom.* **13**:1 (1978), 25–41. MR 80e:32015 Zbl 0379.53016

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THE MASS-CRITICAL NONLINEAR SCHRÖDINGER EQUATION WITH RADIAL DATA IN DIMENSIONS THREE AND HIGHER

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We establish global well-posedness and scattering for solutions to the mass-critical nonlinear Schrödinger equation $iu_t + \Delta u = \pm |u|^{4/d} u$ for large spherically symmetric $L_x^2(\mathbb{R}^d)$ initial data in dimensions $d \geq 3$. In the focusing case we require that the mass is strictly less than that of the ground state. As a consequence, we obtain that in the focusing case, any spherically symmetric blowup solution must concentrate at least the mass of the ground state at the blowup time.

1. Introduction

The d-dimensional mass-critical nonlinear Schrödinger equation is given by

$$iu_t + \Delta u = F(u) \quad \text{with } F(u) := \mu |u|^{\frac{4}{d}} u \tag{1-1}$$

where u is a complex-valued function of spacetime $\mathbb{R} \times \mathbb{R}^d$. Here $\mu = \pm 1$, with $\mu = 1$ known as the defocusing equation and $\mu = -1$ as the focusing equation.

The name "mass-critical" refers to the fact that the scaling symmetry

$$u(t,x) \mapsto u_\lambda(t,x) := \lambda^{-\frac{d}{2}} u(\lambda^{-2} t, \lambda^{-1} x)$$

leaves both the equation and the mass invariant. The mass of a solution is

$$M(u(t)) := \int_{\mathbb{R}^d} |u(t,x)|^2 \, dx$$

and is conserved under the flow.

In this paper, we investigate the Cauchy problem for (1-1) for spherically symmetric $L_x^2(\mathbb{R}^d)$ initial data in dimensions $d \geq 3$ by adapting the recent argument from [Killip et al. 2007], which treated the case $d = 2$. Before describing our results, we need to review some background material. We begin by making the notion of a solution more precise:

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Definition 1.1 (Solution). A function $u : I \times \mathbb{R}^d \to \mathbb{C}$ on a nonempty time interval $I \subset \mathbb{R}$ is a *solution* (more precisely, a strong $L_x^2(\mathbb{R}^d)$ solution) to (1-1) if it lies in the class

$$C_t^0 L_x^2(K \times \mathbb{R}^d) \cap L_{t,x}^{2(d+2)/d}(K \times \mathbb{R}^d)$$

for all compact $K \subset I$, and obeys the Duhamel formula

$$u(t_1) = e^{i(t_1-t_0)\Delta} u(t_0) - i \int_{t_0}^{t_1} e^{i(t_1-t)\Delta} F(u(t))\, dt \tag{1-2}$$

for all $t_0, t_1 \in I$. Note that by Lemma 2.7 below, the condition $u \in L_{t,x}^{2(d+2)/d}$ locally in time guarantees that the integral converges, at least in a weak-L_x^2 sense.

Remark. The condition that u is in $L_{t,x}^{2(d+2)/d}$ locally in time is natural. This space appears in the Strichartz inequality (Lemma 2.7); consequently, all solutions to the linear problem lie in this space. Existence of solutions to (1-1) in this space is guaranteed by the local theory discussed below; it is also necessary in order to ensure uniqueness of solutions in this local theory. Solutions to (1-1) in this class have been intensively studied; see for example [Bégout and Vargas 2007; Bourgain 1998; Carles and Keraani 2007; Cazenave and Weissler 1989; Cazenave 2003; Keraani 2006; Merle and Vega 1998; Tao 2006; Tao et al. 2006; 2007; Tsutsumi 1985].

Associated to this notion of solution is a corresponding notion of blowup. As we will see in Theorem 1.3 below, this precisely corresponds to the impossibility of continuing the solution.

Definition 1.2 (Blowup). We say that a solution u to (1-1) *blows up forward in time* if there exists a time $t_0 \in I$ such that

$$\int_{t_0}^{\sup I} \int_{\mathbb{R}^d} |u(t,x)|^{2(d+2)/d}\, dx\, dt = \infty$$

and that u *blows up backward in time* if there exists a time $t_0 \in I$ such that

$$\int_{\inf I}^{t_0} \int_{\mathbb{R}^d} |u(t,x)|^{2(d+2)/d}\, dx\, dt = \infty.$$

The local theory for (1-1) was worked out by Cazenave and Weissler [1989]. They constructed local-in-time solutions for arbitrary initial data in $L_x^2(\mathbb{R}^d)$; however, due to the critical nature of the equation, the resulting time of existence depends on the profile of the initial data and not merely on its L_x^2-norm. Cazenave and Weissler also constructed global solutions for small initial data. We summarize their results in the theorem below.

Theorem 1.3 (Local well-posedness [Cazenave and Weissler 1989; Cazenave 2003]). *Given $t_0 \in \mathbb{R}$ and $u_0 \in L_x^2(\mathbb{R}^d)$, there exists a unique maximal-lifespan solution u to (1-1) with $u(t_0) = u_0$. We will write I for the maximal lifespan. This solution also has the following properties*:

- (Local existence) *I is an open neighbourhood of t_0.*
- (Mass conservation) *The solution u obeys mass conservation*: *$M(u(t)) = M(u_0)$ for all $t \in I$.*
- (Blowup criterion) *If $\sup(I)$ or $\inf(I)$ is finite, then u blows up in the corresponding time direction.*

- (Continuous dependence) *The map that takes initial data to the corresponding strong solution is uniformly continuous on compact time intervals for bounded sets of initial data.*
- (Scattering) *If* $\sup(I) = +\infty$ *and* u *does not blow up forward in time, then* u *scatters forward in time, that is, there exists a unique* $u_+ \in L_x^2(\mathbb{R}^d)$ *such that*

$$\lim_{t\to+\infty} \|u(t) - e^{it\Delta}u_+\|_{L_x^2(\mathbb{R}^d)} = 0.$$

Similarly, if $\inf(I) = -\infty$ *and* u *does not blow up backward in time, then* u *scatters backward in time, that is, there is a unique* $u_- \in L_x^2(\mathbb{R}^d)$ *such that*

$$\lim_{t\to-\infty} \|u(t) - e^{it\Delta}u_-\|_{L_x^2(\mathbb{R}^d)} = 0.$$

- (Small data global existence) *If* $M(u_0)$ *is sufficiently small depending on* d, *then* u *is a global solution with finite* $L_{t,x}^{2(d+2)/d}$ *norm.*

It is widely believed that in the defocusing case, all L_x^2 initial data lead to a global solution with finite $L_{t,x}^{2(d+2)/d}$ spacetime norm (and hence also scattering).

In the focusing case, the general consensus is more subtle. Let Q denote the *ground state*, that is, the unique positive radial solution to

$$\Delta Q + Q^{1+4/d} = Q.$$

(The existence and uniqueness of Q were established in [Berestycki and Lions 1979] and [Kwong 1989] respectively.) Then

$$u(t,x) := e^{it}Q(x)$$

is a solution to (1-1), which is global but blows up both forward and backward in time (in the sense of Definition 1.2). More dramatically, by applying the pseudoconformal transformation to u, we obtain a solution

$$v(t,x) := |t|^{-d/2} e^{i\frac{|x|^2-4}{4t}} Q\Big(\frac{x}{t}\Big)$$

with the same mass that blows up in finite time. It is widely believed that this ground state example is the minimal-mass obstruction to global well-posedness and scattering in the focusing case.

To summarize, we subscribe to:

Conjecture 1.4 (Global existence and scattering). *Let* $d \geq 1$ *and* $\mu = \pm 1$. *In the defocusing case* $\mu = +1$, *all maximal-lifespan solutions to* (1-1) *are global and do not blow up either forward or backward in time. In the focusing case* $\mu = -1$, *all maximal-lifespan solutions* u *to* (1-1) *with* $M(u) < M(Q)$ *are global and do not blow up either forward or backward in time.*

Remark. While this conjecture is phrased for $L_x^2(\mathbb{R}^d)$ solutions, it is equivalent to a scattering claim for smooth solutions; see [Bégout and Vargas 2007; Carles 2002; Keraani 2006; Tao 2006]. In [Blue and Colliander 2006; Tao 2006], it is also shown that the global existence and the scattering claims are equivalent in the $L_x^2(\mathbb{R}^d)$ category.

The contribution of this paper toward settling this conjecture is:

Theorem 1.5. *Let* $d \geq 3$. *Then Conjecture 1.4 is true in the class of spherically symmetric initial data (for either choice of sign* μ*).*

Conjecture 1.4 has been the focus of much intensive study and several partial results for various choices of d, μ, and sometimes with the additional assumption of spherical symmetry. The most compelling evidence in favour of this conjecture stems from results obtained under the assumption that u_0 has additional regularity. For the defocusing equation, it is easy to prove global well-posedness for initial data in H^1_x; this follows from the usual contraction mapping argument combined with the conservation of mass and energy; see, for example, [Cazenave 2003]. Recall that the energy is given by

$$E(u(t)) := \int_{\mathbb{R}^d} \Big(\tfrac{1}{2} |\nabla u(t,x)|^2 + \mu \frac{d}{2(d+2)} |u(t,x)|^{\frac{2(d+2)}{d}} \Big)\, dx. \tag{1-3}$$

Note that for general L^2_x initial data, the energy need not be finite.

The focusing equation with data in H^1_x was treated by Weinstein. A key ingredient was his proof of the following result:

Theorem 1.6 (Sharp Gagliardo–Nirenberg inequality [Weinstein 1983]).

$$\int_{\mathbb{R}^d} |f(x)|^{\frac{2(d+2)}{d}}\, dx \le \frac{d+2}{d} \Big(\frac{\|f\|^2_{L^2}}{\|Q\|^2_{L^2}} \Big)^{\frac{2}{d}} \int_{\mathbb{R}^d} |\nabla f(x)|^2\, dx.$$

As noticed by Weinstein, this inequality implies that the energy (1-3) is positive once $M(u_0) < M(Q)$; indeed, it gives an upper bound on the $\dot H^1_x$-norm of the solution at all times of existence. Combining this with a contraction mapping argument and the conservation of mass and energy, Weinstein proved global well-posedness for the focusing equation with initial data in H^1_x and mass smaller than that of the ground state.

Note that the iterative procedure used to obtain a global solution both for the defocusing and the focusing equations with initial data in H^1_x does not yield finite spacetime norms; in particular, scattering does not follow even for more regular initial data.

In dimensions one and two, there has been much work [Bourgain 1998; Colliander et al. 2002; 2008b; Colliander et al. 2005; Colliander et al. 2007; De Silva et al. 2007a; Fang and Grillakis 2007; Tzirakis 2005] devoted to lowering the regularity of the initial data from H^1_x toward $L^2_x(\mathbb{R}^d)$ and thus toward establishing the conjecture. For analogous results in higher dimensions, see [De Silva et al. 2007b; Visan and Zhang 2007].

In the case of spherically symmetric solutions, Conjecture 1.4 was recently settled in the high-dimensional defocusing case $\mu = +1$, $d \ge 3$ in [Tao et al. 2007]; thus, only the $\mu = -1$ case of Theorem 1.5 is new. However, the techniques used in that reference do not seem to be applicable to the focusing problem, primarily because the Morawetz inequality is no longer coercive in that case. Instead, our argument is based on the recent preprint by Killip, Tao and Visan [Killip et al. 2007], which resolved the conjecture for $\mu = \pm 1$, $d = 2$, and spherically symmetric data. That work, in turn, used techniques developed to treat the analogous conjecture for the energy-critical problem, such as [Bourgain 1999b; Colliander et al. 2008a; Ryckman and Visan 2007; Tao 2005; Visan 2006; 2007] and particularly [Kenig and Merle 2006a]. We will give a more thorough discussion of the relation of the current work to these predecessors later, when we outline the argument.

Mass concentration in the focusing problem. Neither Theorem 1.5 nor Conjecture 1.4 addresses the focusing problem for masses greater than or equal to that of the ground state. In this case, blowup solutions exist and attention has been focused on describing their properties. For instance, finite-time blowup solutions with finite energy and mass equal to that of the ground state have been completely characterized by Merle [1993]; they are precisely the ground state solution up to symmetries of the equation.

Several works have shown that finite-time blowup solutions must concentrate a positive amount of mass around the blowup time T^*. For finite energy data, see [Merle and Tsutsumi 1990; Nawa 1999; Weinstein 1989] where it is shown that there exists $x(t) \in \mathbb{R}^d$ so that

$$\liminf_{t \nearrow T^*} \int_{|x-x(t)| \leq R} |u(t,x)|^2 \, dx \geq M(Q)$$

for any $R > 0$. For merely $L_x^2(\mathbb{R}^2)$ initial data, Bourgain [1998] proved that some small amount of mass must concentrate in parabolic windows (at least along a subsequence):

$$\limsup_{t \nearrow T^*} \sup_{x_0 \in \mathbb{R}^2} \int_{|x-x_0| \leq (T^*-t)^{1/2}} |u(t,x)|^2 \, dx \geq c,$$

where c is a small constant depending on the mass of u. This result was extended to other dimensions in [Bégout and Vargas 2007; Keraani 2006].

Combining Theorem 1.5 with the argument in [Killip et al. 2007, §10], one obtains the following concentration result.

Corollary 1.7 (Blowup solutions concentrate the mass of the ground state). *Let $d \geq 3$ and $\mu = -1$. Let u be a spherically symmetric solution to (1-1) that blows up at time $0 < T^* \leq \infty$. If $T^* < \infty$, there exists a sequence $t_n \nearrow T^*$ such that for any sequence $R_n \in (0, \infty)$ obeying $(T^* - t_n)^{-1/2} R_n \to \infty$,*

$$\limsup_{n \to \infty} \int_{|x| \leq R_n} |u(t_n, x)|^2 \, dx \geq M(Q).$$

If $T^* = \infty$, there exists a sequence $t_n \to \infty$ such that for any sequence R_n with $t_n^{-1/2} R_n \to \infty$ in $(0, \infty)$

$$\limsup_{n \to \infty} \int_{|x| \leq R_n} |u(t_n, x)|^2 \, dx \geq M(Q).$$

The analogous statement holds in the negative time direction.

Outline of the proof. Beginning with Bourgain's seminal work [1999b] on the energy-critical NLS, it has become apparent that in order to prove spacetime bounds for general solutions, it is sufficient to treat a special class of solutions, namely, those that are simultaneously localized in both frequency and space. For further developments, see [Colliander et al. 2008a; Ryckman and Visan 2007; Tao 2005; Visan 2006; 2007].

A new and much more efficient alternative to Bourgain's induction on mass (or energy) method has recently been developed. It uses a (concentration) compactness technique to isolate minimal-mass/energy blowup solutions as opposed to the almost-blowup solutions of the induction method. Building on earlier developments in [Bégout and Vargas 2007; Bourgain 1998; Keraani 2001; 2006; Merle and Vega 1998], Kenig and Merle [2006a] introduced this method to treat the energy-critical focusing problem with radial data in dimensions three, four, and five; see also [Kenig and Merle 2006b; Killip et al. 2007; Killip and

Visan 2008; Tao 2008a; 2008b; 2008c; Tao et al. 2007] for subsequent applications/developments of this method.

To explain what the concentration compactness argument gives in our context, we need to introduce the following important notion:

Definition 1.8 (Almost periodicity modulo scaling). Given $d \geq 1$ and $\mu = \pm 1$, a solution u with lifespan I is said to be *almost periodic modulo scaling* if there exists a (possibly discontinuous) function $N : I \to \mathbb{R}^+$ and a function $C : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\int_{|x| \geq C(\eta)/N(t)} |u(t,x)|^2 \, dx \leq \eta \quad \text{and} \quad \int_{|\xi| \geq C(\eta)N(t)} |\hat{u}(t,\xi)|^2 \, d\xi \leq \eta$$

for all $t \in I$ and $\eta > 0$. We refer to the function N as the *frequency scale function* and to C as the *compactness modulus function*.

Remarks. (1) The parameter $N(t)$ measures the frequency scale of the solution at time t, and $1/N(t)$ measures the spatial scale; see [Tao et al. 2006; 2007] for further discussion. We have the freedom to modify $N(t)$ by any bounded function of t, provided that we also modify the compactness modulus function C accordingly. In particular, one could restrict $N(t)$ to be a power of 2 if one wished, although we will not do so here. Alternatively, the fact that the solution trajectory $t \mapsto u(t)$ is continuous in $L^2_x(\mathbb{R}^d)$ can be used to show that the function N may be chosen to depend continuously on t.

(2) By the Ascoli–Arzelà Theorem, a family of functions is precompact in $L^2_x(\mathbb{R}^d)$ if and only if it is norm-bounded and there exists a compactness modulus function C so that

$$\int_{|x| \geq C(\eta)} |f(x)|^2 \, dx + \int_{|\xi| \geq C(\eta)} |\hat{f}(\xi)|^2 \, d\xi \leq \eta$$

for all functions f in the family. Thus, an equivalent formulation of Definition 1.8 is as follows: u is almost periodic modulo scaling if and only if

$$\{u(t) : t \in I\} \subseteq \{\lambda^{-d/2} f(x/\lambda) : \lambda \in (0,\infty) \text{ and } f \in K\}.$$

for some compact subset K of $L^2_x(\mathbb{R}^d)$.

In [Tao et al. 2006, Theorems 1.13 and 7.2] the following result was established (see also [Bégout and Vargas 2007; Keraani 2006]), showing that any failure of Conjecture 1.4 must be "caused" by a very special type of solution. For simplicity we state it only in the spherically symmetric case.

Theorem 1.9 (Reduction to almost periodic solutions). *Fix μ and $d \geq 2$. Suppose that Conjecture 1.4 fails for spherically symmetric data. Then there exists a spherically symmetric maximal-lifespan solution u which is almost periodic modulo scaling and which blows up both forward and backward in time, and in the focusing case we also have $M(u) < M(Q)$.*

In [Killip et al. 2007], this result was further refined so as to identify three specific enemies. Once again, we state it only in the spherically symmetric case.

Theorem 1.10 (Three special scenarios for blowup [Killip et al. 2007]). *Fix μ and $d \geq 2$ and suppose that Conjecture 1.4 fails for spherically symmetric data. Then there exists a spherically symmetric maximal-lifespan solution u which is almost periodic modulo scaling, blows up both forward and backward in* *time, and in the focusing case also obeys* $M(u) < M(Q)$. *Moreover, the solution* u *may be chosen to match one of the following three scenarios*:

- (Soliton-like solution) *We have* $I = \mathbb{R}$ *and* $N(t) = 1$ *for all* $t \in \mathbb{R}$ (*thus the solution stays in a bounded space/frequency range for all time*).
- (Double high-to-low frequency cascade) *We have* $I = \mathbb{R}$, $\liminf_{t\to-\infty} N(t) = \liminf_{t\to+\infty} N(t) = 0$, *and* $\sup_{t\in\mathbb{R}} N(t) < \infty$ *for all* $t \in I$.
- (Self-similar solution) *We have* $I = (0, +\infty)$ *and*
$$N(t) = t^{-1/2} \tag{1-4}$$
for all $t \in I$.

In light of this result, the proof of Theorem 1.5 is reduced to showing that none of these three scenarios can occur. In doing this, we follow the model set forth in [Killip et al. 2007]. In all cases, the key step is to prove that u has additional regularity. Indeed, to treat the first two scenarios, we need more than one derivative in L^2_x; for the self-similar scenario, H^1_x suffices. The possibility of showing such additional regularity stems from the fact that u is both frequency and space localized; this in turn is an expression of the fact that u has minimal mass among all blowup solutions.

A further manifestation of this minimality is the absence of a scattered wave at the endpoints of the lifespan I. More formally:

Lemma 1.11 [Tao et al. 2006, Section 6]. *Let* u *be a solution to* (1-1) *which is almost periodic modulo scaling on its maximal-lifespan* I. *Then, for all* $t \in I$,

$$u(t) = \lim_{T\nearrow \sup I} i\int_t^T e^{i(t-t')\Delta} F(u(t'))\,dt' = -\lim_{T\searrow \inf I} i\int_T^t e^{i(t-t')\Delta} F(u(t'))\,dt', \tag{1-5}$$

as weak limits in L^2_x.

Another important property of solutions that are almost periodic modulo scaling is that the behaviour of the spacetime norm is governed by that of $N(t)$. More precisely:

Lemma 1.12 (Spacetime bound [Killip et al. 2007]). *Let* u *be a nonzero solution to* (1-1) *with lifespan* I, *which is almost periodic modulo scaling with frequency scale function* $N : I \to \mathbb{R}^+$. *If* J *is any subinterval of* I, *then*

$$\int_J N(t)^2\,dt \lesssim_u \int_J\int_{\mathbb{R}^d} |u(t,x)|^{\frac{2(d+2)}{d}}\,dx\,dt \lesssim_u 1 + \int_J N(t)^2\,dt.$$

The nonexistence of self-similar solutions is proved in Section 3. We first prove that any such solution would belong to $C^0_t H^1_x$ and then observe that H^1_x solutions are global (see the discussion after Theorem 1.5), while self-similar solutions are not.

For the remaining two cases, higher regularity is proved in Section 5. In order to best take advantage of Lemma 1.11, we exploit a decomposition of spherically symmetric functions into incoming and outgoing waves; this is discussed in Section 4.

In Section 6, we use the additional regularity together with the conservation of energy to preclude the double high-to-low frequency cascade. In Section 7, we disprove the existence of soliton-like solutions using a truncated virial identity in much the same manner as [Kenig and Merle 2006a].

As noted earlier, the argument just described is closely modelled on [Killip et al. 2007], which treated the same equation in two dimensions. The main obstacle in extending that work to higher dimensions is the fractional power appearing in the nonlinearity. This problem presents itself when we prove additional regularity, which is already the most demanding part of [Killip et al. 2007]. Additional regularity is proved via a bootstrap argument using Duhamel's formula. However, fractional powers can downgrade regularity (a fractional power of a smooth function need not be smooth); in particular, they preclude the simple Littlewood–Paley arithmetic that is usually used in the case of polynomial nonlinearities.

The remedy is twofold: first we use fractional chain rules (see Lemmas 2.3 and 2.4) that allow us to take more than one derivative of a nonlinearity that is merely $C^{1+4/d}$ in u. Secondly, we push through the resulting complexities in the bootstrap argument. An important role is played by Lemma 2.1 (a Gronwall-type result), which we use to untangle the intricate relationship between frequencies in u and those in $|u|^{4/d}u$.

2. Notation and linear estimates

This section contains the basic linear estimates we use repeatedly in the paper.

Some notation. We use $X \lesssim Y$ or $Y \gtrsim X$ whenever $X \le CY$ for some constant $C > 0$. We use $O(Y)$ to denote any quantity X such that $|X| \lesssim Y$. We use the notation $X \sim Y$ whenever $X \lesssim Y \lesssim X$. The fact that these constants depend upon the dimension d will be suppressed. If C depends upon some additional parameters, we will indicate this with subscripts; for example, $X \lesssim_u Y$ denotes the assertion that $X \le C_u Y$ for some C_u depending on u.

We use the "Japanese bracket" convention

$$\langle x\rangle := (1+|x|^2)^{1/2}.$$

We write $L_t^q L_x^r$ to denote the Banach space with norm

$$\|u\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^d)} := \left(\int_{\mathbb{R}}\left(\int_{\mathbb{R}^d} |u(t,x)|^r\,dx\right)^{q/r} dt\right)^{1/q},$$

with the usual modifications when q or r is equal to infinity, or when the domain $\mathbb{R}\times\mathbb{R}^d$ is replaced by a smaller region of spacetime such as $I \times \mathbb{R}^d$. When $q = r$ we abbreviate $L_t^q L_x^q$ as $L_{t,x}^q$.

The next lemma is a variant of Gronwall's inequality that we will use to handle some bootstrap arguments below. The proof given is a standard application of techniques from the theories of Volterra and Toeplitz operators.

Lemma 2.1 (A Gronwall inequality). *Fix $r \in (0,1)$ and $K \ge 4$. Let b_k be a bounded sequence of nonnegative numbers and x_k a sequence obeying $0 \le x_k \le b_k$ for $0 \le k < K$ and*

$$0 \le x_k \le b_k + \sum_{l=0}^{k-K} r^{k-l} x_l, \tag{2-1}$$

for all $k \geq K$. *Then*

$$0 \leq x_k \lesssim \sum_{l=0}^{k} r^{k-l} \exp\Big(\frac{\log(K-1)}{K-1}(k-l)\Big) b_l \tag{2-2}$$

for all $k \geq 0$. *In particular, if* $b_k = O(2^{-k\sigma})$ *and* $2^\sigma r (K-1)^{1/(K-1)} < 1$, *then* $x_k = O(2^{-k\sigma})$.

Proof. Elementary monotonicity arguments show that we need only obtain the bound for the case of equality, namely, where

$$(1-A)x = b. \tag{2-3}$$

Here x and b denote the semiinfinite vectors built from the corresponding sequences, while A is the matrix with entries

$$A_{k,l} = \begin{cases} r^{k-l} & \text{if } k-l \geq K, \\ 0 & \text{otherwise.} \end{cases}$$

The triangular structure of A guarantees that (2-3) can be solved (though not *a priori* in ℓ^∞); more precisely, it guarantees that the geometric series for $(1-A)^{-1}$ converges entry-wise. To obtain bounds for the entries of this inverse matrix, it is simplest to use a functional model: under the mapping of sequences to functions

$$x_k \mapsto \sum_{k=0}^{\infty} x_k z^k \quad \text{and} \quad b_k \mapsto \sum_{k=0}^{\infty} b_k z^k,$$

the matrix A becomes multiplication by $r^K z^K (1-rz)^{-1}$. In the same way, the entries of $(1-A)^{-1}$ come from the Taylor coefficients of

$$a(z) := \frac{1-rz}{1-rz-r^K z^K}.$$

Using $e^x \geq 1+x$ with $x = -\log|rz|$, we see that

$$|1-rz| \geq \Big(\frac{1}{r|z|} - 1\Big) r|z| \geq \frac{\log(K-1)}{K-1} r|z| \geq \log(K-1)\, r^K |z|^K$$

on the disk $|z| \leq r^{-1}(K-1)^{-1/(K-1)}$. This shows that $a(z)$ is bounded and analytic on this disk. (Note that the hypothesis $K \geq 4$ implies that $\log(K-1) > 1$.) The inequality (2-2) now follows from the standard Cauchy estimates. □

Basic harmonic analysis. Consider a radial bump function $\varphi : \mathbb{R}^d \to \mathbb{R}$ such that

$$\varphi(\xi) = 1 \quad \text{if } |\xi| \leq 1 \qquad \text{and} \qquad \varphi(\xi) = 0 \quad \text{if } |\xi| \geq \tfrac{11}{10}. \tag{2-4}$$

For each number $N > 0$, define the Fourier multipliers

$$\widehat{P_{\leq N} f}(\xi) := \varphi(\xi/N)\hat{f}(\xi), \quad \widehat{P_{>N} f}(\xi) := \big(1-\varphi(\xi/N)\big)\hat{f}(\xi),$$
$$\widehat{P_N f}(\xi) := \psi(\xi/N)\hat{f}(\xi) := \big(\varphi(\xi/N) - \varphi(2\xi/N)\big)\hat{f}(\xi),$$

and similarly $P_{<N}$ and $P_{\geq N}$. We also define

$$P_{M<\cdot\leq N} := P_{\leq N} - P_{\leq M} = \sum_{M<N'\leq N} P_{N'}$$

whenever $M < N$. We will usually use these multipliers when M and N are *dyadic numbers* (that is, of the form 2^n for some integer n); in particular, all summations over N or M are understood to be over dyadic numbers. Nevertheless, it will occasionally be convenient to allow M and N to not be a power of 2. Note that P_N is not truly a projection; to get around this, we will occasionally need to use fattened Littlewood–Paley operators:

$$\tilde{P}_N := P_{N/2} + P_N + P_{2N}. \tag{2-5}$$

These obey $P_N \tilde{P}_N = \tilde{P}_N P_N = P_N$.

As with all Fourier multipliers, the Littlewood–Paley operators commute with the propagator $e^{it\Delta}$, as well as with differential operators such as $i\partial_t + \Delta$. We will use basic properties of these operators many times, including

Lemma 2.2 (Bernstein estimates). *For* $1 \le p \le q \le \infty$,

$$\begin{aligned} \big\| |\nabla|^{\pm s} P_N f \big\|_{L_x^p(\mathbb{R}^d)} &\sim N^{\pm s} \| P_N f \|_{L_x^p(\mathbb{R}^d)}, \\ \| P_{\le N} f \|_{L_x^q(\mathbb{R}^d)} &\lesssim N^{\frac{d}{p} - \frac{d}{q}} \| P_{\le N} f \|_{L_x^p(\mathbb{R}^d)}, \\ \| P_N f \|_{L_x^q(\mathbb{R}^d)} &\lesssim N^{\frac{d}{p} - \frac{d}{q}} \| P_N f \|_{L_x^p(\mathbb{R}^d)}. \end{aligned}$$

The next few results provide important tools for dealing with the fractional power appearing in the nonlinearity.

Lemma 2.3 (Fractional chain rule for a C^1 function [Christ and Weinstein 1991]). *Suppose* $G \in C^1(\mathbb{C})$, $s \in (0, 1]$, *and* $1 < p, p_1, p_2 < \infty$ *such that* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. *Then*

$$\big\| |\nabla|^s G(u) \big\|_p \lesssim \| G'(u) \|_{p_1} \big\| |\nabla|^s u \big\|_{p_2}.$$

When the function G is no longer C^1, but merely Hölder continuous, we have the following useful chain rule:

Lemma 2.4 (Fractional chain rule for a Hölder continuous function [Visan 2007]). *Let G be a Hölder continuous function of order* $0 < \alpha < 1$. *Then, for every* $0 < s < \alpha$, $1 < p < \infty$, *and* $\frac{s}{\alpha} < \sigma < 1$ *we have*

$$\big\| |\nabla|^s G(u) \big\|_p \lesssim \big\| |u|^{\alpha - \frac{s}{\sigma}} \big\|_{p_1} \big\| |\nabla|^\sigma u \big\|^{\frac{s}{\sigma}}_{\frac{s}{\sigma} p_2},$$

provided $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ *and* $(1 - s/\alpha\sigma) p_1 > 1$. *The implicit constant depends upon s.*

Corollary 2.5. *Let* $0 \le s < 1 + \frac{4}{d}$. *Then on any spacetime slab* $I \times \mathbb{R}^d$ *we have*

$$\begin{aligned} \big\| |\nabla|^s F(u) \big\|_{L_{t,x}^{2(d+2)/(d+4)}} &\lesssim \big\| |\nabla|^s u \big\|_{L_{t,x}^{2(d+2)/d}} \| u \|^{\frac{4}{d}}_{L_{t,x}^{2(d+2)/d}}, \\ \big\| |\nabla|^s F(u) \big\|_{L_t^\infty L_x^{\frac{2r}{r+4}}} &\lesssim \big\| |\nabla|^s u \big\|_{L_t^\infty L_x^2} \| u \|^{\frac{4}{d}}_{L_t^\infty L_x^{\frac{2r}{d}}}, \end{aligned}$$

for any $\max\{d, 4\} \le r \le \infty$. *The implicit constants depend upon s.*

Proof. Fix a compact interval I. Throughout the proof, all spacetime estimates will be on $I \times \mathbb{R}^d$.

For $0 < s \le 1$, both claims are easy consequences of Lemma 2.3. We now address the case $1 < s < 1 + \frac{4}{d}$ for $d \ge 5$; a few remarks on $d = 3, 4$ are given at the end of the proof. We start with the first claim.

By the chain rule and the fractional product rule, we estimate

$$\begin{aligned}
&\big\| |\nabla|^s F(u) \big\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}} \\
&\lesssim \big\| |\nabla|^{s-1} (\nabla u F_z(u) + \nabla \bar{u} F_{\bar{z}}(u)) \big\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}} \\
&\lesssim \big\| |\nabla|^s u \big\|_{L_{t,x}^{2(d+2)/d}} \|u\|_{L_{t,x}^{2(d+2)/d}}^{\frac{4}{d}} + \|\nabla u\|_{L_{t,x}^{2(d+2)/d}} \Big(\big\| |\nabla|^{s-1} F_z(u) \big\|_{L_{t,x}^{(d+2)/2}} + \big\| |\nabla|^{s-1} F_{\bar{z}}(u) \big\|_{L_{t,x}^{(d+2)/2}} \Big).
\end{aligned}$$

The claim will follow from this, once we establish

$$\big\| |\nabla|^{s-1} F_z(u) \big\|_{L_{t,x}^{(d+2)/2}} + \big\| |\nabla|^{s-1} F_{\bar{z}}(u) \big\|_{L_{t,x}^{(d+2)/2}} \lesssim \big\| |\nabla|^{\sigma} u \big\|_{L_{t,x}^{2(d+2)/d}}^{\frac{s-1}{\sigma}} \|u\|_{L_{t,x}^{2(d+2)/d}}^{\frac{4}{d} - \frac{s-1}{\sigma}} \tag{2-6}$$

for some σ such that $\frac{1}{4} d(s-1) < \sigma < 1$. Indeed, one simply has to note that by interpolation we have

$$\begin{aligned}
\big\| |\nabla|^{\sigma} u \big\|_{L_{t,x}^{2(d+2)/d}} &\lesssim \big\| |\nabla|^s u \big\|_{L_{t,x}^{2(d+2)/d}}^{\frac{\sigma}{s}} \|u\|_{L_{t,x}^{2(d+2)/d}}^{1-\frac{\sigma}{s}}, \\
\|\nabla u\|_{L_{t,x}^{2(d+2)/d}} &\lesssim \big\| |\nabla|^s u \big\|_{L_{t,x}^{2(d+2)/d}}^{\frac{1}{s}} \|u\|_{L_{t,x}^{2(d+2)/d}}^{1-\frac{1}{s}}.
\end{aligned}$$

To derive (2-6), we remark that F_z and $F_{\bar{z}}$ are Hölder continuous functions of order $4/d$ and use Corollary 2.5 (with $\alpha := 4/d$ and $s := s - 1$).

We now turn to the second claim. Note that the condition $r \ge 4$ simply insures that $\dfrac{2r}{r+4} \ge 1$. By the chain rule and the fractional product rule,

$$\begin{aligned}
&\big\| |\nabla|^s F(u) \big\|_{L_t^\infty L_x^{\frac{2r}{r+4}}} \\
&\qquad \lesssim \big\| |\nabla|^{s-1} \big(\nabla u F_z(u) + \nabla \bar{u} F_{\bar{z}}(u) \big) \big\|_{L_t^\infty L_x^{\frac{2r}{r+4}}} \\
&\qquad \lesssim \big\| |\nabla|^s u \big\|_{L_t^\infty L_x^2} \|u\|_{L_t^\infty L_x^{\frac{2r}{d}}}^{\frac{4}{d}} + \|\nabla u\|_{L_t^\infty L_x^{\frac{2rs}{r+(s-1)d}}} \big\| |\nabla|^{s-1} O(|u|^{\frac{4}{d}}) \big\|_{L_t^\infty L_x^{\frac{2rs}{(r-d)(s-1)+4s}}}.
\end{aligned}$$

By interpolation,

$$\|\nabla u\|_{L_t^\infty L_x^{\frac{2rs}{r+(s-1)d}}} \lesssim \big\| |\nabla|^s u \big\|_{L_t^\infty L_x^2}^{\frac{1}{s}} \|u\|_{L_t^\infty L_x^{\frac{2r}{d}}}^{1-\frac{1}{s}}.$$

Thus, the claim will follow once we establish that

$$\big\| |\nabla|^{s-1} O(|u|^{\frac{4}{d}}) \big\|_{L_t^\infty L_x^{\frac{2rs}{(r-d)(s-1)+4s}}} \lesssim \big\| |\nabla|^s u \big\|_{L_t^\infty L_x^2}^{1-\frac{1}{s}} \|u\|_{L_t^\infty L_x^{2r/d}}^{\frac{4}{d}+\frac{1}{s}-1}. \tag{2-7}$$

Applying Lemma 2.4, we obtain

$$\big\| |\nabla|^{s-1} O(|u|^{\frac{4}{d}}) \big\|_{L_t^\infty L_x^{\frac{2rs}{(r-d)(s-1)+4s}}} \lesssim \big\| |\nabla|^{\sigma} u \big\|_{L_t^\infty L_x^{\frac{2rs}{sd+\sigma(r-d)}}}^{\frac{s-1}{\sigma}} \|u\|_{L_t^\infty L_x^{2r/d}}^{\frac{4}{d}-\frac{s-1}{\sigma}}$$

for any σ such that $\frac{1}{4}d(s-1) < \sigma < 1$. The inequality (2-7) now follows from

$$\big\| |\nabla|^\sigma u \big\|_{L_t^\infty L_x^{\frac{2rs}{sd+\sigma(r-d)}}} \lesssim \big\| |\nabla|^s u \big\|_{L_t^\infty L_x^2}^{\frac{\sigma}{s}} \|u\|_{L_t^\infty L_x^{\frac{2r}{d}}}^{1-\frac{\sigma}{s}},$$

which is a consequence of interpolation.

Note that the restriction $r \geq d$ guarantees that certain Lebesgue exponents appearing above lie in the range $[1, \infty]$. In fact, one may relax this restriction a little, but we will not need this here.

The treatment of the two claims in the case $d = 4$ requires the bound

$$\big\| |\nabla|^\sigma |u| \big\|_{L_x^p} \lesssim \big\| |\nabla|^\sigma u \big\|_{L_x^p},$$

which holds for $1 < p < \infty$ and $0 < \sigma < 1$, in place of Lemma 2.4. Proofs of this slight variant of Lemma 2.3 can be found in [Kato 1995; Staffilani 1997; Taylor 2000].

We now discuss the case $d = 3$. When $1 < s < 2$, one may use the argument presented above. For $2 \leq s < \frac{7}{3}$, one first takes the Laplacian of $F(u)$ and then applies Lemma 2.4 to deal with the remaining fractional derivatives. Terms where the derivatives distribute themselves between several copies of u are dealt with by interpolation, as above. □

Strichartz estimates. Naturally, everything that we do for the nonlinear Schrödinger equation builds on basic properties of the linear propagator $e^{it\Delta}$.

From the explicit formula

$$e^{it\Delta} f(x) = \frac{1}{(4\pi i t)^{d/2}} \int_{\mathbb{R}^d} e^{i|x-y|^2/4t} f(y)\, dy,$$

we deduce the standard dispersive inequality

$$\|e^{it\Delta} f\|_{L^\infty(\mathbb{R}^d)} \lesssim \frac{1}{|t|^{d/2}} \|f\|_{L^1(\mathbb{R}^d)}$$

for all $t \neq 0$. Interpolating between this and the conservation of mass gives

$$\|e^{it\Delta} f\|_{L^p(\mathbb{R}^d)} \lesssim |t|^{\frac{d}{p}-\frac{d}{2}} \|f\|_{L^{p'}(\mathbb{R}^d)} \tag{2-8}$$

for all $t \neq 0$ and $2 \leq p \leq \infty$. Here p' is the dual of p, that is, $\frac{1}{p} + \frac{1}{p'} = 1$.

Finer bounds on the (frequency localized) linear propagator can be derived using stationary phase:

Lemma 2.6 (Kernel estimates). *For any $m \geq 0$, the kernel of the linear propagator obeys the estimates*

$$|(P_N e^{it\Delta})(x,y)| \lesssim_m \begin{cases} N^d \langle N|x-y|\rangle^{-m} & \text{if } |t| \leq N^{-2}, \\ |t|^{-d/2} & \text{if } |t| \geq N^{-2} \text{ and } |x-y| \sim N|t|, \\ \dfrac{N^d}{|N^2 t|^m \langle N|x-y|\rangle^m} & \text{otherwise.} \end{cases}$$

We also record the following standard Strichartz estimates:

Lemma 2.7 (Strichartz). *Let I be an interval, let $t_0 \in I$, and let*

$$u_0 \in L_x^2(\mathbb{R}^d) \quad \text{and} \quad f \in L_{t,x}^{2(d+2)/(d+4)}(I \times \mathbb{R}^d),$$

with $d \geq 3$. The function u defined by

$$u(t) := e^{i(t-t_0)\Delta} u_0 - i \int_{t_0}^{t} e^{i(t-t')\Delta} f(t')\, dt'$$

obeys the estimate

$$\|u\|_{C_t^0 L_x^2} + \|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}} + \|u\|_{L_t^2 L_x^{\frac{2d}{d-2}}} \lesssim \|u_0\|_{L_x^2} + \|f\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}},$$

where all spacetime norms are over $I \times \mathbb{R}^d$.

Proof. See, for example, [Ginibre and Velo 1992; Strichartz 1977]. For the endpoint see [Keel and Tao 1998]. □

We will also need three variants of the Strichartz inequality. First, we observe a weighted Strichartz estimate, which exploits the spherical symmetry heavily in order to obtain spatial decay. It is very useful in regions of space far from the origin $x = 0$.

Lemma 2.8 (Weighted Strichartz). *Let I be an interval, let $t_0 \in I$, and let*

$$u_0 \in L_x^2(\mathbb{R}^d) \quad \text{and} \quad f \in L_{t,x}^{2(d+2)/(d+4)}(I \times \mathbb{R}^d)$$

be spherically symmetric. The function u defined by

$$u(t) := e^{i(t-t_0)\Delta} u_0 - i \int_{t_0}^{t} e^{i(t-t')\Delta} f(t')\, dt'$$

obeys the estimate

$$\big\| |x|^{\frac{2(d-1)}{q}} u \big\|_{L_t^q L_x^{\frac{2q}{q-4}}(I \times \mathbb{R}^d)} \lesssim \|u_0\|_{L_x^2(\mathbb{R}^d)} + \|f\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}(I \times \mathbb{R}^d)}$$

if $4 \leq q \leq \infty$.

Proof. For $q = \infty$, this corresponds to the trivial endpoint in Strichartz inequality. We will only prove the result for the $q = 4$ endpoint, since the remaining cases then follow by interpolation.

As in the usual proof of Strichartz inequality, the method of TT^* together with the Christ–Kiselev lemma and Hardy–Littlewood–Sobolev inequality reduce matters to proving that

$$\big\| |x|^{\frac{d-1}{2}} e^{it\Delta} |x|^{\frac{d-1}{2}} g \big\|_{L_x^\infty(\mathbb{R}^d)} \lesssim |t|^{-\frac{1}{2}} \|g\|_{L_x^1(\mathbb{R}^d)} \tag{2-9}$$

for all radial functions g.

Let P_{rad} denote the projection onto radial functions. Then

$$[e^{it\Delta} P_{\text{rad}}](x, y) = (4\pi i t)^{-\frac{d}{2}} e^{i\frac{|x|^2+|y|^2}{4t}} \int_{S^{d-1}} e^{-i\frac{|y|\omega\cdot x}{2t}}\, d\sigma(\omega),$$

where $d\sigma$ denotes the uniform probability measure on the unit sphere S^{d-1}. This integral can be evaluated exactly in terms the Bessel function $J_{\frac{d-2}{2}}$. Using this, or simple stationary phase arguments, one sees that

$$\big|[e^{it\Delta} P_{\text{rad}}](x, y)\big| \lesssim |t|^{-\frac{d}{2}} \Big(\frac{|y||x|}{|t|}\Big)^{-\frac{d-1}{2}} \lesssim |t|^{-\frac{1}{2}} |x|^{-\frac{d-1}{2}} |y|^{-\frac{d-1}{2}}.$$

The radial dispersive estimate (2-9) now follows easily. □

We will rely crucially on a slightly different type of improvement to the Strichartz inequality in the spherically symmetric case due to Shao [2007], which improves the spacetime decay of the solution after localizing in frequency. The important thing for us will be the fact that this estimate can give decay as $N \to \infty$; this is not possible without the radial assumption.

Lemma 2.9 (Shao's Strichartz estimate [Shao 2007, Corollary 6.2]). *For $f \in L^2_{\mathrm{rad}}(\mathbb{R}^d)$ we have*

$$\|P_N e^{it\Delta} f\|_{L^q_{t,x}(\mathbb{R}\times\mathbb{R}^d)} \lesssim_q N^{\frac{d}{2}-\frac{d+2}{q}} \|f\|_{L^2_x(\mathbb{R}^d)},$$

provided $q > \frac{4d+2}{2d-1}$.

The last result is a bilinear estimate from [Visan 2006], which builds on earlier versions in [Bourgain 1999a; Colliander et al. 2008a]. It will be useful for controlling interactions between widely separated frequencies.

Lemma 2.10 (Bilinear Strichartz [Visan 2006, Lemma 2.5]). *For any spacetime slab $I \times \mathbb{R}^d$, any $t_0 \in I$, and any $M, N > 0$, we have*

$$\big\|(P_{\geq N}u)(P_{\leq M}v)\big\|_{L^2_{t,x}(I\times\mathbb{R}^d)} \lesssim N^{-\frac{1}{2}} M^{\frac{d-1}{2}} \Big(\|P_{\geq N}u(t_0)\|_{L^2} + \|(i\partial_t+\Delta)P_{\geq N}u\|_{L^{\frac{2(d+2)}{d+4}}_{t,x}(I\times\mathbb{R}^d)}\Big) \times\Big(\|P_{\leq M}v(t_0)\|_{L^2} + \|(i\partial_t+\Delta)P_{\leq M}v\|_{L^{\frac{2(d+2)}{d+4}}_{t,x}(I\times\mathbb{R}^d)}\Big),$$

for all functions u, v on I.

3. The self-similar solution

In this section we preclude self-similar solutions. As mentioned in Section 1, the key ingredient is additional regularity.

Theorem 3.1 (Regularity in the self-similar case). *Let $d \geq 3$ and let u be a spherically symmetric solution to (1-1) that is almost periodic modulo scaling and that is self-similar in the sense of Theorem 1.10. Then $u(t) \in H^s_x(\mathbb{R}^d)$ for all $t \in (0,\infty)$ and all $0 \leq s < 1 + \frac{4}{d}$.*

Corollary 3.2 (Absence of self-similar solutions). *For $d \geq 3$ there are no nonzero spherically symmetric solutions to (1-1) that are self-similar in the sense of Theorem 1.10.*

Proof. By Theorem 3.1, any such solution would obey $u(t) \in H^1_x(\mathbb{R}^d)$ for all $t \in (0,\infty)$. Then, by the H^1_x global well-posedness theory described after Theorem 1.5, there exists a global solution with initial data $u(t_0)$ at any time $t_0 \in (0,\infty)$; recall that we assume $M(u) < M(Q)$ in the focusing case. On the other hand, self-similar solutions blow up at time $t = 0$. These two facts (combined with the uniqueness statement in Theorem 1.3) yield a contradiction. □

The remainder of this section is devoted to proving Theorem 3.1. We will regard s as fixed and will allow constants to implicitly depend on s.

Let u be as in Theorem 3.1. For any $A > 0$, we define

$$\begin{aligned}\mathcal{M}(A) &:= \sup_{T>0} \|u_{>AT^{-\frac{1}{2}}}(T)\|_{L^2_x(\mathbb{R}^d)},\\ \mathcal{S}(A) &:= \sup_{T>0} \|u_{>AT^{-\frac{1}{2}}}\|_{L^{2(d+2)/d}_{t,x}([T,2T]\times\mathbb{R}^d)},\\ \mathcal{N}(A) &:= \sup_{T>0} \|P_{>AT^{-\frac{1}{2}}}F(u)\|_{L^{2(d+2)/(d+4)}_{t,x}([T,2T]\times\mathbb{R}^d)}.\end{aligned} \tag{3-1}$$

The notation chosen indicates the quantity being measured, namely, the mass, the symmetric Strichartz norm, and the nonlinearity in the adjoint Strichartz norm, respectively. Since u is self-similar, $N(t)$ is comparable to $T^{-\frac{1}{2}}$ for t in the interval $[T, 2T]$. Thus, the Littlewood–Paley projections are adapted to the natural frequency scale on each dyadic time interval.

To prove Theorem 3.1 it suffices to show that for every $0 < s < 1 + \frac{4}{d}$ we have

$$\mathcal{M}(A) \lesssim_{s,u} A^{-s},$$

whenever A is sufficiently large depending on u and s. To establish this, we need a variety of estimates linking $\mathcal{M}$, $\mathcal{S}$, and $\mathcal{N}$. From mass conservation, Lemma 1.12, self-similarity, and Hölder's inequality, we see that

$$\mathcal{M}(A) + \mathcal{S}(A) + \mathcal{N}(A) \lesssim_u 1 \tag{3-2}$$

for all $A > 0$. From the Strichartz inequality (Lemma 2.7), we also see that

$$\mathcal{S}(A) \lesssim \mathcal{M}(A) + \mathcal{N}(A) \tag{3-3}$$

for all $A > 0$. Another application of Strichartz combined with Lemma 1.12 and (1-4) shows that

$$\|u\|_{L^2_t L^{\frac{2d}{d-2}}_x([T,2T]\times\mathbb{R}^d)} \lesssim_u 1. \tag{3-4}$$

Next, we obtain a deeper connection between these quantities.

Lemma 3.3 (Nonlinear estimate). *Let $\eta > 0$ and $0 < s < 1 + \frac{4}{d}$. If $A > 100$ and $0 < \beta \le 1$, we have*

$$\mathcal{N}(A) \lesssim_u \sum_{N \le \eta A^\beta} \Big(\frac{N}{A}\Big)^s \mathcal{S}(N) + \big(\mathcal{S}(\eta A^{\frac{\beta}{2(d-1)}}) + \mathcal{S}(\eta A^\beta)\big)^{\frac{4}{d}} \mathcal{S}(\eta A^\beta) + A^{-\frac{2\beta}{d^2}}\big(\mathcal{M}(\eta A^\beta) + \mathcal{N}(\eta A^\beta)\big). \tag{3-5}$$

Proof. Fix $\eta > 0$ and $0 < s < 1 + \frac{4}{d}$. It suffices to bound

$$\big\|P_{>AT^{-\frac{1}{2}}}F(u)\big\|_{L^{2(d+2)/(d+4)}_{t,x}([T,2T]\times\mathbb{R}^d)}$$

by the right-hand side of (3-5) for arbitrary $T > 0$ and all $A > 100$ and $0 < \beta \le 1$.

To achieve this, we decompose

$$\begin{aligned}F(u) = F\big(u_{\le \eta A^\beta T^{-\frac{1}{2}}}\big) + O\big(|u_{\le \eta A^\alpha T^{-\frac{1}{2}}}|^{\frac{4}{d}}\, |u_{>\eta A^\beta T^{-\frac{1}{2}}}|\big)&\\ + O\big(|u_{\eta A^\alpha T^{-\frac{1}{2}} < \cdot \le \eta A^\beta T^{-\frac{1}{2}}}|^{\frac{4}{d}}\, |u_{>\eta A^\beta T^{-\frac{1}{2}}}|\big) + O\big(|u_{>\eta A^\beta T^{-\frac{1}{2}}}|^{1+\frac{4}{d}}\big),&\end{aligned} \tag{3-6}$$

where
$$\alpha=\frac{\beta}{2(d-1)}.$$
To estimate the contribution from the last two terms in the expansion above, we discard the projection to high frequencies and then use Hölder's inequality and (3-1):
$$\begin{aligned}\big\|\,|u_{\eta A^\alpha T^{-\frac12}<\cdot\le\eta A^\beta T^{-\frac12}}|^{\frac4d}\,u_{>\eta A^\beta T^{-\frac12}}\big\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}([T,2T]\times\mathbb{R}^d)}&\lesssim \mathcal{S}(\eta A^\alpha)^{\frac4d}\mathcal{S}(\eta A^\beta),\\ \big\|\,|u_{>\eta A^\beta T^{-\frac12}}|^{1+\frac4d}\big\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}([T,2T]\times\mathbb{R}^d)}&\lesssim \mathcal{S}(\eta A^\beta)^{1+\frac4d}.\end{aligned}$$
To estimate the contribution coming from second term on the right-hand side of (3-6), we discard the projection to high frequencies and then use Hölder's inequality, Lemmas 2.2 and 2.10, and (3-3):
$$\begin{aligned}&\big\|P_{>AT^{-\frac12}}O\big(|u_{\le\eta A^\alpha T^{-\frac12}}|^{\frac4d}\,|u_{>\eta A^\beta T^{-\frac12}}|\big)\big\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}([T,2T]\times\mathbb{R}^d)}\\ &\quad\lesssim \big\|u_{\le\eta A^\alpha T^{-\frac12}}u_{>\eta A^\beta T^{-\frac12}}\big\|_{L^2_{t,x}([T,2T]\times\mathbb{R}^d)}^{\frac{8}{d^2}}\\ &\qquad\qquad\times\big\|u_{>\eta A^\beta T^{-\frac12}}\big\|_{L_{t,x}^{2(d+2)/d}([T,2T]\times\mathbb{R}^d)}^{1-\frac{8}{d^2}}\big\|u_{\le\eta A^\alpha T^{-\frac12}}\big\|_{L^2_{t,x}([T,2T]\times\mathbb{R}^d)}^{\frac4d-\frac{8}{d^2}}\\ &\quad\lesssim_u \big((\eta A^\beta T^{-\frac12})^{-\frac12}(\eta A^\alpha T^{-\frac12})^{\frac{d-1}{2}}\big)^{\frac{8}{d^2}}\big(\mathcal{M}(\eta A^\beta)+\mathcal{N}(\eta A^\beta)\big)^{\frac{8}{d^2}}\mathcal{S}(\eta A^\beta)^{1-\frac{8}{d^2}}T^{\frac2d-\frac{4}{d^2}}\\ &\quad\lesssim_u A^{-\frac{2\beta}{d^2}}\big(\mathcal{M}(\eta A^\beta)+\mathcal{N}(\eta A^\beta)\big).\end{aligned}$$
We now turn to the first term on the right-hand side of (3-6). By Lemma 2.2 and Corollary 2.5 combined with (3-2), we estimate
$$\begin{aligned}\big\|P_{>AT^{-\frac12}}F\big(u_{\le\eta A^\beta T^{-\frac12}}\big)\big\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}([T,2T]\times\mathbb{R}^d)}&\lesssim (AT^{-\frac12})^{-s}\big\||\nabla|^sF\big(u_{\le\eta A^\beta T^{-\frac12}}\big)\big\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}([T,2T]\times\mathbb{R}^d)}\\ &\lesssim_u (AT^{-\frac12})^{-s}\big\||\nabla|^s u_{\le\eta A^\beta T^{-\frac12}}\big\|_{L_{t,x}^{\frac{2(d+2)}{d}}([T,2T]\times\mathbb{R}^d)}\\ &\lesssim_u \sum_{N\le\eta A^\beta}\Big(\frac NA\Big)^s\mathcal{S}(N),\end{aligned}$$
which is acceptable. This finishes the proof of the lemma. □

We have some decay as $A\to\infty$:

Lemma 3.4 (Qualitative decay).
$$\lim_{A\to\infty}\mathcal{M}(A)=\lim_{A\to\infty}\mathcal{S}(A)=\lim_{A\to\infty}\mathcal{N}(A)=0.$$
Proof. The vanishing of the first limit follows from Definition 1.8, self-similarity, and (3-1). By interpolation, (3-1), and (3-4),
$$\mathcal{S}(A)\lesssim\mathcal{M}(A)^{\frac{2}{d+2}}\big\|u_{\ge AT^{-\frac12}}\big\|_{L^2_tL_x^{\frac{2d}{d-2}}([T,2T]\times\mathbb{R}^d)}^{\frac{d}{d+2}}\lesssim_u\mathcal{M}(A)^{\frac{2}{d+2}}.$$

Thus, as the first limit in Lemma 3.4 vanishes, we obtain that the second limit vanishes. The vanishing of the third limit follows from that of the second and Lemma 3.3. □

We have now gathered enough tools to prove some regularity, albeit in the symmetric Strichartz space. As such, the next result is the crux of this section.

Proposition 3.5 (Quantitative decay estimate). *Let $0<\eta<1$ and $0<s<1+\frac{4}{d}$. If η is sufficiently small depending on u and s, and A is sufficiently large depending on u, s, and η, then*

$$\mathcal{S}(A) \le \sum_{N\le \eta A} \left(\frac{N}{A}\right)^{s} \mathcal{S}(N) + A^{-\frac{1}{d^2}}. \tag{3-7}$$

In particular,

$$\mathcal{S}(A) \lesssim_u A^{-\frac{1}{d^2}}, \tag{3-8}$$

for all $A>0$.

Proof. Fix $\eta\in(0,1)$ and $0<s<1+\frac{4}{d}$. To establish (3-7), it suffices to show

$$\|u_{>AT^{-\frac12}}\|_{L_{t,x}^{\frac{2(d+2)}{d}}([T,2T]\times\mathbb{R}^d)} \lesssim_{u,\varepsilon} \sum_{N\le\eta A}\left(\frac{N}{A}\right)^{s+\varepsilon}\mathcal{S}(N) + A^{-\frac{3}{2d^2}} \tag{3-9}$$

for all $T>0$ and some small $\varepsilon=\varepsilon(d,s)>0$, since then (3-7) follows by requiring η to be small and A to be large, both depending upon u and also ε.

Fix $T>0$. By writing the Duhamel formula (1-2) beginning at $T/2$ and then using Lemma 2.7, we obtain

$$\begin{aligned}\|u_{>AT^{-\frac12}}\|_{L_{t,x}^{\frac{2(d+2)}{d}}([T,2T]\times\mathbb{R}^d)} \\ \lesssim \|P_{>AT^{-\frac12}}e^{i(t-T/2)\Delta}u(T/2)\|_{L_{t,x}^{\frac{2(d+2)}{d}}([T,2T]\times\mathbb{R}^d)} + \|P_{>AT^{-\frac12}}F(u)\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}([\frac{T}{2},2T]\times\mathbb{R}^d)}.\end{aligned}$$

We first consider the second term. By (3-1), we have

$$\|P_{>AT^{-\frac12}}F(u)\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}([\frac{T}{2},2T]\times\mathbb{R}^d)} \lesssim \mathcal{N}(A/2).$$

Using Lemma 3.3 (with $\beta=1$ and s replaced by $s+\varepsilon$ for some $0<\varepsilon<1+\frac{4}{d}-s$) combined with Lemma 3.4 (choosing A sufficiently large depending on u, s, and η), and (3-2), we derive

$$\|P_{>AT^{-\frac12}}F(u)\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}([\frac{T}{2},2T]\times\mathbb{R}^d)} \lesssim_{u,\varepsilon} \sum_{N\le\eta A}\left(\frac{N}{A}\right)^{s+\varepsilon}\mathcal{S}(N) + A^{-\frac{3}{2d^2}},$$

whose right-hand side is that of (3-9). Thus, the second term is acceptable.

We now consider the first term. It suffices to show

$$\|P_{>AT^{-\frac12}}e^{i(t-T/2)\Delta}u(T/2)\|_{L_{t,x}^{\frac{2(d+2)}{d}}([T,2T]\times\mathbb{R}^d)} \lesssim_u A^{-\frac{3}{2d^2}}, \tag{3-10}$$

which we will deduce by first proving two estimates at a single frequency scale, interpolating between them, and then summing.

From Lemma 2.9 and mass conservation, we have

$$\big\| P_{BT^{-\frac12}} e^{i(t-T/2)\Delta} u(T/2) \big\|_{L^q_{t,x}([T,2T]\times\mathbb{R}^d)} \lesssim_{u,q} (BT^{-\frac12})^{\frac{d}{2}-\frac{d+2}{q}} \tag{3-11}$$

whenever

$$\frac{4d+2}{2d-1} < q \le \frac{2(d+2)}{d}$$

and $B > 0$. This is our first estimate.

Using the Duhamel formula (1-2), we write

$$P_{BT^{-\frac12}} e^{i(t-\frac{T}{2})\Delta} u\Big(\frac{T}{2}\Big) = P_{BT^{-\frac12}} e^{i(t-\varepsilon)\Delta} u(\varepsilon) - i \int_\varepsilon^{\frac{T}{2}} P_{BT^{-\frac12}} e^{i(t-t')\Delta} F(u(t'))\, dt'$$

for any $\varepsilon > 0$. By self-similarity, the former term converges strongly to zero in L^2_x as $\varepsilon \to 0$. Convergence to zero in $L_x^{2d/(d-2)}$ then follows from Lemma 2.2. Thus, using Hölder's inequality followed by the dispersive estimate (2-8), and then (3-4), we estimate

$$\begin{aligned}
&\big\| P_{BT^{-\frac12}} e^{i(t-T/2)\Delta} u(T/2) \big\|_{L_{t,x}^{\frac{2d}{d-2}}([T,2T]\times\mathbb{R}^d)} \\
&\qquad \lesssim T^{\frac{d-2}{2d}} \Big\| \int_0^{\frac{T}{2}} \frac{1}{t-t'} \| F(u(t')) \|_{L_x^{\frac{2d}{d+2}}}\, dt' \Big\|_{L_t^\infty([T,2T])} \lesssim T^{-\frac{d+2}{2d}} \| F(u) \|_{L_t^1 L_x^{\frac{2d}{d+2}}((0,\frac{T}{2}]\times\mathbb{R}^d)} \\
&\qquad \lesssim T^{-\frac{d+2}{2d}} \sum_{0<\tau\le\frac{T}{4}} \| F(u) \|_{L_t^1 L_x^{\frac{2d}{d+2}}([\tau,2\tau]\times\mathbb{R}^d)} \\
&\qquad \lesssim T^{-\frac{d+2}{2d}} \sum_{0<\tau\le\frac{T}{4}} \tau^{1/2} \| u \|_{L_t^2 L_x^{\frac{2d}{d-2}}([\tau,2\tau]\times\mathbb{R}^d)} \| u \|^{\frac{4}{d}}_{L_t^\infty L_x^2([\tau,2\tau]\times\mathbb{R}^d)} \\
&\qquad \lesssim_u T^{-1/d}.
\end{aligned}$$

Interpolating between the estimate just proved and (3-11) with

$$q = \frac{2d(d+2)(4d-3)}{4d^3-3d^2+12},$$

we obtain

$$\big\| P_{BT^{-\frac12}} e^{i(t-T/2)\Delta} u(T/2) \big\|_{L_{t,x}^{\frac{2(d+2)}{d}}([T,2T]\times\mathbb{R}^d)} \lesssim_u B^{-\frac{3}{2d^2}}.$$

Summing this over dyadic $B \ge A$ yields (3-10) and hence (3-9).

We now justify (3-8). Given an integer $K \ge 4$, we set $\eta = 2^{-K}$. Then there exists A_0, depending on u and K, so that (3-7) holds for $A \ge A_0$. By (3-2), we need only bound $\mathcal{S}(A)$ for $A \ge A_0$.

Let $k \geq 0$ and set $A = 2^k A_0$ in (3-7). Then, writing $N = 2^l A_0$ and using (3-2), we have

$$\mathcal{S}(2^k A_0) \leq \sum_{l \leq k-K} 2^{-(k-l)s} \mathcal{S}(2^l A_0) + (2^k A_0)^{-\beta} \leq \sum_{l=0}^{k-K} 2^{-(k-l)s} \mathcal{S}(2^l A_0) + \frac{2^{-ks}}{1-2^{-s}} \mathcal{S}(0) + 2^{-k\beta} A_0^{-\beta},$$

where $\beta := d^{-2}$. Setting $s = 1$ and applying Lemma 2.1 with $x_k = \mathcal{S}(2^k A_0)$ and $b_k = O_u(2^{-k\beta})$, we deduce

$$\mathcal{S}(2^k A_0) \lesssim_u 2^{-k/d^2},$$

provided K is chosen sufficiently large. This gives the necessary bound on $\mathcal{S}$. □

Corollary 3.6. *For any $A > 0$ we have*

$$\mathcal{M}(A) + \mathcal{S}(A) + \mathcal{N}(A) \lesssim_u A^{-1/d^2}.$$

Proof. The bound on $\mathcal{S}$ was proved in the previous proposition. The bound on $\mathcal{N}$ follows from this, Lemma 3.3 with $\beta = 1$, and (3-2).

We now turn to the bound on $\mathcal{M}$. By Lemma 1.11,

$$\|P_{>AT^{-\frac{1}{2}}} u(T)\|_2 \lesssim \sum_{k=0}^{\infty} \left\| \int_{2^k T}^{2^{k+1} T} e^{i(T-t')\Delta} P_{>AT^{-\frac{1}{2}}} F(u(t'))\, dt' \right\|_2, \tag{3-12}$$

where weak convergence has become strong convergence because of the frequency projection and the fact that $N(t) = t^{-1/2} \to 0$ as $t \to \infty$. Intuitively, the reason for using (1-5) forward in time is that the solution becomes smoother as $N(t) \to 0$.

Combining (3-12) with Lemma 2.7 and (3-1), we get

$$\mathcal{M}(A) = \sup_{T>0} \|P_{>AT^{-\frac{1}{2}}} u(T)\|_2 \lesssim \sum_{k=0}^{\infty} \mathcal{N}(2^{k/2} A). \tag{3-13}$$

The desired bound on $\mathcal{M}$ now follows from that on $\mathcal{N}$. □

Proof of Theorem 3.1. Let $0 < s < 1 + \frac{4}{d}$. Combining Lemma 3.3 (with $\beta = 1 - \frac{1}{2d^2}$), (3-3), and (3-13), we deduce that if

$$\mathcal{S}(A) + \mathcal{M}(A) + \mathcal{N}(A) \lesssim_u A^{-\sigma}$$

for some $0 < \sigma < s$, then

$$\mathcal{S}(A) + \mathcal{M}(A) + \mathcal{N}(A) \lesssim_u A^{-\sigma}\big(A^{-\frac{s-\sigma}{2d^2}} + A^{-\frac{(d+1)(3d-2)\sigma}{2d^3(d-1)}} + A^{-\frac{3-\sigma}{2d^2} - \frac{d^2-2}{2d^4}}\big).$$

More precisely, Lemma 3.3 provides the bound on $\mathcal{N}(A)$, then (3-13) gives the bound on $\mathcal{M}(A)$ and then finally (3-3) gives the bound on $\mathcal{S}(A)$.

Iterating this statement shows that $u(t) \in H^s_x(\mathbb{R}^d)$ for all $0 < s < 1 + \frac{4}{d}$. Note that Corollary 3.6 allows us to begin the iteration with $\sigma = d^{-2}$. □

4. An in/out decomposition

In this section, we will often write radial functions on $\mathbb{R}^d$ just in terms of the radial variable. With this convention,

$$f(r) = r^{\frac{2-d}{2}} \int_0^\infty J_{\frac{d-2}{2}}(kr) \hat{f}(k)\, k^{\frac{d}{2}}\, dk \quad \text{and} \quad \hat{f}(k) = k^{\frac{2-d}{2}} \int_0^\infty J_{\frac{d-2}{2}}(kr) f(r)\, r^{\frac{d}{2}}\, dr,$$

as can be seen from [Stein and Weiss 1971, Theorem IV.3.3]. Here J_ν denotes the Bessel function of order ν. In particular,

$$g(k,r) := r^{\frac{2-d}{2}} J_{\frac{d-2}{2}}(kr)$$

solves the radial Helmholtz equation

$$-g_{rr} - \tfrac{d-1}{r} g_r = k^2 g, \tag{4-1}$$

which corresponds to the fact that $g(k,r)$ represents a spherical standing wave of frequency $k^2/(2\pi)$. Incoming and outgoing spherical waves are represented by two further solutions of (4-1), namely,

$$g_-(k,r) := r^{\frac{2-d}{2}} H^{(2)}_{\frac{d-2}{2}}(kr) \quad \text{and} \quad g_+(k,r) := r^{\frac{2-d}{2}} H^{(1)}_{\frac{d-2}{2}}(kr),$$

respectively. Note that

$$g = \tfrac{1}{2} g_+ + \tfrac{1}{2} g_-.$$

This leads us to define the projection onto outgoing spherical waves by

$$\begin{aligned}
[P^+ f](r) &= \tfrac{1}{2} \int_0^\infty r^{\frac{2-d}{2}} H^{(1)}_{\frac{d-2}{2}}(kr) \hat{f}(k)\, k^{\frac{d}{2}}\, dk \\
&= \tfrac{1}{2} r^{\frac{2-d}{2}} \int_0^\infty \biggl(\int_0^\infty H^{(1)}_{\frac{d-2}{2}}(kr) J_{\frac{d-2}{2}}(k\rho)\, k\, dk \biggr) f(\rho)\, \rho^{\frac{d}{2}}\, d\rho \\
&= \tfrac{1}{2} f(r) + \tfrac{i}{\pi} \int_0^\infty \frac{r^{2-d} f(\rho)\, \rho^{d-1}\, d\rho}{r^2 - \rho^2}.
\end{aligned} \tag{4-2}$$

In order to derive the last equality we used [Gradshteyn and Ryzhik 2000, §6.521.2] together with analytic continuation. Similarly, we define the projection onto incoming waves by

$$[P^- f](r) = \frac{1}{2} \int_0^\infty r^{\frac{2-d}{2}} H^{(2)}_{\frac{d-2}{2}}(kr) \hat{f}(k)\, k^{\frac{d}{2}}\, dk = \frac{1}{2} f(r) - \frac{i}{\pi} \int_0^\infty \frac{r^{2-d} f(\rho)\, \rho^{d-1}\, d\rho}{r^2 - \rho^2}.$$

Note that the kernel of P^- is the complex conjugate of that belonging to P^+, as is required by time-reversal symmetry.

We will write $P^\pm_N$ for the product $P^\pm P_N$.

Remark. For $f(\rho) \in L^2(\rho^{d-1}\, d\rho)$,

$$\int_0^\infty |f(\rho)|^2 \rho^{d-1}\, d\rho = \frac{1}{2} \int |s^{\frac{d-2}{4}} f(\sqrt{s})|^2\, ds$$

and with $t = r^2$,

$$\int_0^\infty \frac{r^{2-d} f(\rho)\,\rho^{d-1}\,d\rho}{r^2-\rho^2} = \tfrac{1}{2} t^{-\frac{d-2}{4}} \int_0^\infty \Big(\frac{s}{t}\Big)^{\frac{d-2}{4}} \frac{s^{\frac{d-2}{4}} f(\sqrt{s})\,ds}{t-s}.$$

Thus $P^+ : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is bounded if and only if the Hilbert transform is bounded in the weighted space $L^2([0,\infty), t^{-(d-2)/2}\,dt)$. Thus P^+ is unbounded on $L^2(\mathbb{R}^d)$ for $d \geq 4$.

Lemma 4.1 (Kernel estimates). *For $|x| \gtrsim N^{-1}$ and $t \gtrsim N^{-2}$, the integral kernel obeys*

$$\big|[P_N^{\pm} e^{\mp it\Delta}](x,y)\big| \lesssim \begin{cases} (|x||y|)^{-\frac{d-1}{2}} |t|^{-\frac{1}{2}} & \text{if } |y|-|x| \sim Nt, \\ \dfrac{N^d}{(N|x|)^{\frac{d-1}{2}} \langle N|y|\rangle^{\frac{d-1}{2}}} \langle N^2 t + N|x| - N|y|\rangle^{-m} & \text{otherwise}, \end{cases}$$

for any $m \geq 0$. For $|x| \gtrsim N^{-1}$ and $|t| \lesssim N^{-2}$, the integral kernel obeys

$$|[P_N^{\pm} e^{\mp it\Delta}](x,y)| \lesssim \frac{N^d}{(N|x|)^{\frac{d-1}{2}} \langle N|y|\rangle^{\frac{d-1}{2}}} \langle N|x| - N|y|\rangle^{-m}$$

for any $m \geq 0$.

Proof. The proof is an exercise in stationary phase. We will only provide the details for $P_N^+ e^{-it\Delta}$, the other kernel being its complex conjugate. By (4-2) we have the following formula for the kernel:

$$[P_N^+ e^{-it\Delta}](x,y) = \tfrac{1}{2}(|x||y|)^{-\frac{d-2}{2}} \int_0^\infty H^{(1)}_{\frac{d-2}{2}}(k|x|) J_{\frac{d-2}{2}}(k|y|) e^{itk^2} \psi\Big(\frac{k}{N}\Big)\, k\,dk \tag{4-3}$$

where ψ is the multiplier from the Littlewood–Paley projection. To proceed, we use the following information about Bessel/Hankel functions:

$$J_{\frac{d-2}{2}}(r) = \frac{a(r)e^{ir}}{\langle r\rangle^{1/2}} + \frac{\bar{a}(r)e^{-ir}}{\langle r\rangle^{1/2}}, \tag{4-4}$$

where $a(r)$ obeys the symbol estimates

$$\left|\frac{\partial^m a(r)}{\partial r^m}\right| \lesssim_m \langle r\rangle^{-m} \quad \text{for all } m \geq 0. \tag{4-5}$$

The Hankel function $H^{(1)}_{\frac{d-2}{2}}(r)$ has a singularity at $r = 0$; however, for $r \gtrsim 1$,

$$H^{(1)}_{\frac{d-2}{2}}(r) = \frac{b(r)e^{ir}}{r^{1/2}} \tag{4-6}$$

for a smooth function $b(r)$ obeying (4-5). Since we assumed $|x| \gtrsim N^{-1}$, the singularity does not enter into our considerations.

Substituting (4-4) and (4-6) into (4-3), we see that a stationary phase point can only occur in the term containing $\bar{a}(r)$ and even then only if $|y| - |x| \sim Nt$. In this case, stationary phase yields the first estimate. In all other cases, integration by parts yields the second estimate.

The short-time estimate is also a consequence of (4-3) and stationary phase techniques. Since t is so small, e^{ik^2t} shows no appreciable oscillation and can be incorporated into $\psi(\frac{k}{N})$. For $\big||y|-|x|\big| \le N^{-1}$, the result follows from the naive L^1 estimate. For larger $|x|-|y|$, one integrates by parts m times. □

Lemma 4.2 (Properties of $P^{\pm}$). (i) *$P^+ + P^-$ acts as the identity on $L^2_{\mathrm{rad}}(\mathbb{R}^d)$.*

(ii) *Fix $N>0$. For any spherically symmetric function $f \in L^2_x(\mathbb{R}^d)$,*

$$\|P^{\pm}P_{\ge N}f\|_{L^2_x(|x|\ge\frac{1}{100}N^{-1})} \lesssim \|f\|_{L^2_x(\mathbb{R}^d)}$$

with an N-independent constant.

Proof. Part (i) is immediate from the definition.

We now turn to part (ii). We only prove the inequality for P^+, as the result for P^- can be deduced from this. Let χ be a nonnegative smooth function on $\mathbb{R}^+$ vanishing in a neighborhood of the origin and obeying $\chi(r)=1$ for $r \ge \frac{1}{100}$. With this definition and (4-2),

$$\begin{aligned}\|P^{\pm}P_{\ge N}f\|^2_{L^2_x(|x|\ge N^{-1})} &\le \|\chi(N|x|)P^{\pm}P_{\ge N}f\|^2_{L^2_x(\mathbb{R}^d)}\\ &= \int_0^\infty \Big|\int_0^\infty H^{(1)}_{\frac{d-2}{2}}(kr)\hat f(k)k^{\frac{d}{2}}\Big(1-\varphi\Big(\frac{k}{N}\Big)\Big)\,dk\Big|^2\chi(Nr)^2 r\,dr,\end{aligned}$$

where φ is a cutoff function as in (2-4). Note that, by scaling, it suffices to treat the case $N=1$. Because of the cutoffs, the only nonzero contribution comes from the region $kr \gtrsim 1$. This allows us to use the following information about Hankel functions: for $\rho \gtrsim 1$,

$$H^{(1)}_{\frac{d-2}{2}}(\rho) = \Big(\frac{2}{\pi\rho}\Big)^{\frac12}(1+b(\rho))e^{i\rho-i(d-1)\frac{\pi}{4}}$$

where b is a symbol of order -1, that is,

$$\Big|\frac{\partial^m b(\rho)}{\partial\rho^m}\Big| \lesssim_m \langle\rho\rangle^{-m-1},$$

for all $m \ge 0$; see for example [Gradshteyn and Ryzhik 2000]. Note that this is more refined than formula (4-6) used in the previous proof. With these observations, our goal has been reduced to showing that

$$\int_0^\infty \Big|\int_0^\infty e^{ikr}(1+b(kr))(1-\varphi(k))g(k)\,dk\Big|^2\chi(r)^2\,dr \lesssim \int_0^\infty |g(k)|^2\,dk$$

or, equivalently, that

$$K(k,k') := (1-\varphi(k))(1-\varphi(k'))\int_0^\infty e^{i(k-k')r}(1+b(kr))(1+\bar b(k'r))\chi(r)^2\,dr$$

is the kernel of a bounded operator on $L^2_k([0,\infty))$. To this end, we will decompose K as the sum of two kernels, each of which we can estimate.

First, we consider

$$K_1(k,k') := (1-\varphi(k))(1-\varphi(k'))\int_0^\infty e^{i(k-k')r}\chi(r)^2\,dr.$$

Without the prefactors, the integral is the kernel of a bounded Fourier multiplier and so a bounded operator on L_k^2. As φ is a bounded function, we may then deduce that K_1 is itself the kernel of a bounded operator.

Our second kernel is

$$K_2(k,k') := (1-\varphi(k))(1-\varphi(k')) \int_0^\infty e^{i(k-k')r}\big(b(kr)+\bar{b}(k'r)+b(kr)\bar{b}(k'r)\big)\chi(r)^2\,dr,$$

which we will show to be bounded using Schur's test. Note that the factors in front of the integral ensure that the kernel is zero unless $k \gtrsim 1$ and $k' \gtrsim 1$. By integration by parts, we see that

$$K_2(k,k') \lesssim_m |k-k'|^{-m}$$

for any $m \geq 1$, which offers ample control away from the diagonal. To obtain a good estimate near the diagonal, we need to break the integral into two pieces. We do this by writing

$$1 = \chi\Big(\frac{r}{R}\Big) + \Big(1-\chi\Big(\frac{r}{R}\Big)\Big),$$

with $R \gg 1$. Integrating by parts once when r is large and not at all when r is small leads to

$$\begin{aligned} K_2(k,k') \lesssim \frac{1}{|k-k'|}\int \Big(\Big(\frac{1}{kr^2}+\frac{1}{k'r^2}+\frac{1}{kk'r^3}\Big)\chi\Big(\frac{r}{R}\Big) + \Big(\frac{1}{kr}+\frac{1}{k'r}+\frac{1}{kk'r^2}\Big)\frac{1}{R}\chi'\Big(\frac{r}{R}\Big)\Big)\,dr \\ + \int\Big(\frac{1}{kr}+\frac{1}{k'r}+\frac{1}{kk'r^2}\Big)\chi(r)^2\Big(1-\chi\Big(\frac{r}{R}\Big)\Big)\,dr \\ \lesssim \frac{1}{R|k-k'|} + \log R. \end{aligned}$$

Choosing $R = |k-k'|^{-1}$ provides sufficient control near the diagonal to complete the application of Schur's test. □

5. Additional regularity

This section is devoted to prove:

Theorem 5.1 (Regularity in the global case). *Let $d \geq 3$ and let u be a global spherically symmetric solution to (1-1) that is almost periodic modulo scaling. Suppose also that $N(t) \lesssim 1$ for all $t \in \mathbb{R}$. Then $u \in L_t^\infty H_x^s(\mathbb{R}\times\mathbb{R}^d)$ for all $0 \leq s < 1 + \frac{4}{d}$.*

The argument mimics that in [Killip et al. 2007], though the nonpolynomial nature of the nonlinearity introduces several technical complications. That $u(t)$ is moderately smooth will follow from a careful study of the Duhamel formulae (1-5). Near t, we use the fact that there is little mass at high frequencies, as is implied by the definition of almost periodicity and the boundedness of the frequency scale function $N(t)$. Far from t, we use the spherical symmetry of the solution. As this symmetry is only valuable at large radii, we are only able to exploit it by using the in/out decomposition described in Section 4.

Let us now begin the proof. For the remainder of the section, u will denote a solution to (1-1) that obeys the hypotheses of Theorem 5.1. Once again, we will regard s as fixed and suppress the dependence of implicit constants upon this parameter.

We first record some basic local estimates. From mass conservation we have

$$\|u\|_{L_t^\infty L_x^2(\mathbb{R}\times\mathbb{R}^d)} \lesssim_u 1,$$

while from Definition 1.8 and the fact that $N(t)$ is bounded we have

$$\lim_{N\to\infty} \|u_{\geq N}\|_{L_t^\infty L_x^2(\mathbb{R}\times\mathbb{R}^d)} = 0.$$

From Lemma 1.12 and $N(t) \lesssim 1$, we have

$$\|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}(J\times\mathbb{R}^d)} \lesssim_u \langle |J|\rangle^{\frac{d}{2(d+2)}} \tag{5-1}$$

for all intervals $J \subset \mathbb{R}$. By Hölder's inequality, this implies

$$\|F(u)\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}(J\times\mathbb{R}^d)} \lesssim_u \langle |J|\rangle^{\frac{d+4}{2(d+2)}}$$

and then, by the (endpoint) Strichartz inequality (Lemma 2.7),

$$\|u\|_{L_t^2 L_x^{\frac{2d}{d-2}}(J\times\mathbb{R}^d)} \lesssim_u \langle |J|\rangle^{\frac{1}{2}}. \tag{5-2}$$

More precisely, one first treats the case $|J| = O(1)$ using (5-1) and then larger intervals by subdivision. Similarly, from the weighted Strichartz inequality (Lemma 2.8),

$$\big\| |x|^{\frac{d-1}{2}} u_{N_1\leq\cdot\leq N_2} \big\|_{L_t^4 L_x^\infty(J\times\mathbb{R}^d)} \lesssim_u \langle |J|\rangle^{\frac{1}{4}} \tag{5-3}$$

uniformly in $0 < N_1 \leq N_2 < \infty$.

Now, for any dyadic number N, define

$$\mathcal{M}(N) := \|u_{\geq N}\|_{L_t^\infty L_x^2(\mathbb{R}\times\mathbb{R}^d)}.$$

From the discussion above, we see that $\mathcal{M}(N) \lesssim_u 1$ and

$$\lim_{N\to\infty} \mathcal{M}(N) = 0. \tag{5-4}$$

To prove Theorem 5.1, it suffices to show $\mathcal{M}(N) \lesssim_{u,s} N^{-s}$ for any $0 < s < 1 + \frac{4}{d}$ and all N sufficiently large depending on u and s. As we will explain momentarily, this will follow from Lemma 2.1 and the following

Proposition 5.2 (Regularity). *Let u be as in Theorem 5.1, let $0 < s < 1 + \frac{4}{d}$, and let $\eta > 0$ be a small number. Then*

$$\mathcal{M}(N) \leq N^{-s} + \sum_{M\leq \eta N} \Big(\frac{M}{N}\Big)^s \mathcal{M}(M),$$

whenever N is sufficiently large depending on u, s, and η.

Indeed, given $\varepsilon > 0$, set $\eta = 2^{-K}$, where K is so large that

$$2\log(K-1) < \varepsilon(K-1).$$

Let N_0 be sufficiently large depending on u, s, and K so that the inequality in Proposition 5.2 holds for $N \geq N_0$. If we write $r = 2^{-s}$, $x_k = \mathcal{M}(2^k N_0)$, and

$$b_k = 2^{-ks} N_0^{-s} + \sum_{l \leq -1} 2^{-s(k-l)} \mathcal{M}(2^l N_0) \lesssim_u 2^{-ks} \lesssim_u 2^{-k(s-\varepsilon)},$$

then (2-1) holds. Therefore, $\mathcal{M}(N) \lesssim_{u,s} N^{\varepsilon - s}$ by the last sentence in Lemma 2.1.

The rest of this section is devoted to proving Proposition 5.2. Fix $0 < s < 1 + \frac{4}{d}$ and $\eta > 0$. Our task is to show that

$$\|u_{\geq N}(t_0)\|_{L_x^2(\mathbb{R}^d)} \leq N^{-s} + \sum_{M \leq \eta N} \Big(\frac{M}{N}\Big)^s \mathcal{M}(M)$$

for all times t_0 and all N sufficiently large (depending on u, s, and η). By time translation symmetry, we may assume $t_0 = 0$. As noted above, one of the keys to obtaining additional regularity is Lemma 1.11. Specifically, we have

$$\begin{aligned} u_{\geq N}(0) &= (P^+ + P^-) u_{\geq N}(0) \\ &= \lim_{T\to\infty} i \int_0^T P^+ e^{-it\Delta} P_{\geq N} F(u(t))\, dt - \lim_{T\to\infty} i \int_{-T}^0 P^- e^{-it\Delta} P_{\geq N} F(u(t))\, dt, \end{aligned} \tag{5-5}$$

where the limit is to be interpreted as a weak limit in L^2. However, this representation is not useful for $|x|$ small because the kernels of $P^{\pm}$ have a strong singularity at $x = 0$. To this end, we introduce the cutoff

$$\chi_N(x) := \chi(N|x|),$$

where χ is the characteristic function of $[1, \infty)$. As short times and large times will be treated differently, we rewrite (5-5) as

$$\begin{aligned} \chi_N(x) u_{\geq N}(0, x) &= i \int_0^\delta \chi_N(x) P^+ e^{-it\Delta} P_{\geq N} F(u(t))\, dt - i \int_{-\delta}^0 \chi_N(x) P^- e^{-it\Delta} P_{\geq N} F(u(t))\, dt \\ &\quad + \lim_{T\to\infty} \sum_{M \geq N} i \int_\delta^T \int_{\mathbb{R}^d} \chi_N(x) [P_M^+ e^{-it\Delta}](x,y) [\tilde{P}_M F(u(t))](y)\, dy\, dt \\ &\quad - \lim_{T\to\infty} \sum_{M \geq N} i \int_{-T}^{-\delta} \int_{\mathbb{R}^d} \chi_N(x) [P_M^- e^{-it\Delta}](x,y) [\tilde{P}_M F(u(t))](y)\, dy\, dt, \end{aligned} \tag{5-6}$$

as weak limits in L_x^2. We have used the identity

$$P_{\geq N} = \sum_{M \geq N} P_M \tilde{P}_M,$$

where $\tilde{P}_M := P_{M/2} + P_M + P_{2M}$, because of the way we will estimate the large-time integrals.

The analogous representation for treating small x is

$$\begin{aligned}(1-\chi_N(x))u_{\geq N}(0,x) &= \lim_{T\to\infty} i\int_0^T (1-\chi_N(x))e^{-it\Delta}P_{\geq N}F(u(t))\,dt \\ &= i\int_0^\delta (1-\chi_N(x))e^{-it\Delta}P_{\geq N}F(u(t))\,dt \\ &\qquad + \lim_{T\to\infty}\sum_{M\geq N} i\int_\delta^T\int_{\mathbb{R}^d}(1-\chi_N(x))[P_M e^{-it\Delta}](x,y)[\tilde{P}_M F(u(t))](y)\,dy\,dt, \end{aligned} \tag{5-7}$$

also as weak limits.

To deal with the poor nature of the limits in (5-6) and (5-7), we note that

$$f_T \to f \text{ weakly} \quad \Longrightarrow \quad \|f\| \leq \limsup_{T\to\infty}\|f_T\|, \tag{5-8}$$

or equivalently, that the unit ball is weakly closed.

Despite the fact that different representations will be used depending on the size of $|x|$, some estimates can be dealt with in a uniform manner. The first such example is a bound on integrals over short times.

Lemma 5.3 (Local estimate). *Let $0 < s < 1 + \frac{4}{d}$. For any sufficiently small $\eta > 0$, there exists $\delta = \delta(u,\eta) > 0$ such that*

$$\left\|\int_0^\delta e^{-it\Delta}P_{\geq N}F(u(t))\,dt\right\|_{L_x^2} \leq N^{-s} + \tfrac{1}{10}\sum_{M\leq \eta N}\left(\frac{M}{N}\right)^s \mathcal{M}(M),$$

provided N is sufficiently large depending on u, s, and η. An analogous estimate holds for integration over $[-\delta, 0]$ and after premultiplication by $\chi_N P^{\pm}$.

Proof. By Lemma 2.7, it suffices to prove

$$\mathcal{N}(N) := \|P_{\geq N}F(u)\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}(J\times\mathbb{R}^d)} \lesssim_u N^{-s-\varepsilon} + \sum_{M\leq\eta N}\left(\frac{M}{N}\right)^{s+\varepsilon}\mathcal{M}(M) \tag{5-9}$$

for some small $\varepsilon = \varepsilon(d,s) > 0$, any interval J of length $|J| \leq \delta$, and all sufficiently large N depending on u, s, and η, since the claim would follow by requiring η small and N large, both depending on u.

From (5-4), there exists $N_0 = N_0(u,\eta)$ such that

$$\|u_{\geq N_0}\|_{L_t^\infty L_x^2(\mathbb{R}\times\mathbb{R}^d)} \leq \eta^{100d^2}. \tag{5-10}$$

Let $N > N_1 := \eta^{-1}N_0$. We decompose

$$F(u) = F(u_{\leq\eta N}) + O\big(|u_{\leq N_0}|^{\frac{4}{d}}\,|u_{>\eta N}|\big) + O\big(|u_{N_0\leq\cdot\leq\eta N}|^{\frac{4}{d}}\,|u_{>\eta N}|\big) + O\big(|u_{>\eta N}|^{1+\frac{4}{d}}\big). \tag{5-11}$$

Using Lemma 2.2, Corollary 2.5 together with (5-1), and Lemma 2.7, we estimate the contribution of the first term on the right-hand side of (5-11) as follows:

$$\begin{aligned}
\|P_{\geq N}F(u_{\leq \eta N})\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}(J\times\mathbb{R}^d)} &\lesssim N^{-s-3\varepsilon}\big\||\nabla|^{s+3\varepsilon}F(u_{\leq\eta N})\big\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}(J\times\mathbb{R}^d)} \\
&\lesssim_u \langle\delta\rangle^{\frac{2}{d+2}} N^{-s-3\varepsilon}\big\||\nabla|^{s+3\varepsilon}u_{\leq\eta N}\big\|_{L_{t,x}^{\frac{2(d+2)}{d}}(J\times\mathbb{R}^d)} \\
&\lesssim_u \langle\delta\rangle^{\frac{2}{d+2}} \sum_{M\leq\eta N}\Big(\frac{M}{N}\Big)^{s+3\varepsilon}(\mathcal{M}(M)+\mathcal{N}(M)) \\
&\lesssim_u \eta^{\varepsilon}\langle\delta\rangle^{\frac{2}{d+2}} \sum_{M\leq\eta N}\Big(\frac{M}{N}\Big)^{s+2\varepsilon}(\mathcal{M}(M)+\mathcal{N}(M)),
\end{aligned}$$

for any positive $\varepsilon < \frac{1}{3}(1+\frac{4}{d}-s)$.

To estimate the contribution of the second term on the right-hand side of (5-11), we use Hölder's inequality, Lemma 2.2, and (5-1):

$$\begin{aligned}
&\big\|O(|u_{\leq N_0}|^{\frac{4}{d}}\,|u_{>\eta N}|)\big\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}(J\times\mathbb{R}^d)} \\
&\qquad\lesssim \delta^{\frac12}\,\|u_{\leq N_0}\|_{L_{t,x}^{2(d+2)/d}(J\times\mathbb{R}^d)}^{\frac{2}{d}}\,\|u_{\leq N_0}\|_{L_{t,x}^{\infty}(J\times\mathbb{R}^d)}^{\frac{2}{d}}\,\|u_{>\eta N}\|_{L_t^\infty L_x^2(J\times\mathbb{R}^d)} \\
&\qquad\lesssim_u \delta^{\frac12}\langle\delta\rangle^{\frac{1}{d+2}} N_0\mathcal{M}(\eta N).
\end{aligned}$$

Finally, to estimate the contribution of the last two terms on the right-hand side of (5-11), we use Hölder's inequality, interpolation combined with (5-2) and (5-10), and then Lemma 2.7 to obtain

$$\begin{aligned}
&\big\|O(|u_{N_0\leq\cdot\leq\eta N}|^{\frac{4}{d}}|u_{>\eta N}|)\big\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}(J\times\mathbb{R}^d)} \\
&\qquad\lesssim \|u_{N_0\leq\cdot\leq\eta N}\|_{L_{t,x}^{2(d+2)/d}(J\times\mathbb{R}^d)}^{\frac{4}{d}}\|u_{>\eta N}\|_{L_{t,x}^{2(d+2)/d}(J\times\mathbb{R}^d)} \\
&\qquad\lesssim \|u_{N_0\leq\cdot\leq\eta N}\|_{L_t^\infty L_x^2(J\times\mathbb{R}^d)}^{\frac{8}{d(d+2)}}\|u_{N_0\leq\cdot\leq\eta N}\|_{L_t^2L_x^{2d/(d-2)}(J\times\mathbb{R}^d)}^{\frac{4}{d+2}}\big(\mathcal{M}(\eta N)+\mathcal{N}(\eta N)\big) \\
&\qquad\lesssim_u \eta^8\langle\delta\rangle^{\frac{2}{d+2}}\big(\mathcal{M}(\eta N)+\mathcal{N}(\eta N)\big);
\end{aligned}$$

similarly,

$$\big\|O(|u_{>\eta N}|^{1+\frac{4}{d}})\big\|_{L^{\frac{2(d+2)}{d+4}}(J\times\mathbb{R}^d)} \lesssim_u \eta^8\langle\delta\rangle^{\frac{2}{d+2}}\big(\mathcal{M}(\eta N)+\mathcal{N}(\eta N)\big).$$

Putting everything together and taking η sufficiently small depending on u and s, then δ sufficiently small depending upon N_0 and η, we derive

$$\mathcal{N}(N) \leq \sum_{M\leq\eta N}\Big(\frac{M}{N}\Big)^{s+2\varepsilon}\big(\mathcal{M}(M)+\mathcal{N}(M)\big) \tag{5-12}$$

for all $N > N_1$ and $\varepsilon > 0$ as above. The claim (5-9) follows from this and Lemma 2.1. More precisely, let $\eta = 2^{-K}$ where K is sufficiently large so that $2\log(K-1) < \varepsilon(K-1)$. If we write $r = 2^{-s-2\varepsilon}$, $x_k = \mathcal{N}(2^k N_1)$, and

$$b_k = \sum_{l\le k-K} 2^{-(s+2\varepsilon)(k-l)}\mathcal{M}(2^l N_1) + \sum_{l\le -1} 2^{-(s+2\varepsilon)(k-l)}\mathcal{N}(2^l N_1)$$
$$\lesssim_u \sum_{l\le k-K} 2^{-(s+2\varepsilon)(k-l)}\mathcal{M}(2^l N_1) + 2^{-(s+2\varepsilon)k},$$

then (5-12) implies (2-1). With a few elementary manipulations, (2-2) implies (5-9).

The last claim follows from Lemma 4.2 after employing $P_{\ge N} = P_{\ge N/2}P_{\ge N}$. □

To estimate the integrals where $|t| \ge \delta$, we break the region of (t, y) integration into two pieces, namely, where $|y| \gtrsim M|t|$ and $|y| \ll M|t|$. The former is the more significant region; it contains the points where the integral kernels $P_M e^{-it\Delta}(x, y)$ and $P_M^{\pm} e^{-it\Delta}(x, y)$ are large (see Lemmas 2.6 and 4.1). More precisely, when $|x| \le N^{-1}$, we use (5-7); in this case $|y-x| \sim M|t|$ implies $|y| \gtrsim M|t|$ for $|t| \ge \delta \ge N^{-2}$. (This last condition can be subsumed under our hypothesis N sufficiently large depending on u and η.) When $|x| \ge N^{-1}$, we use (5-6); in this case $|y|-|x| \sim M|t|$ implies $|y| \gtrsim M|t|$.

The next lemma bounds the integrals over the significant region $|y| \gtrsim M|t|$. Let χ_k denote the characteristic function of the set

$$\{(t, y) : 2^k\delta \le |t| \le 2^{k+1}\delta,\ |y| \gtrsim M|t|\}.$$

Lemma 5.4 (Main contribution). *Let $0 < s < 1 + \frac{4}{d}$, let $\eta > 0$ be a small number, and let δ be as in Lemma 5.3. Then*

$$\sum_{M\ge N}\sum_{k=0}^{\infty}\left\|\int_{\mathbb{R}}\int_{\mathbb{R}^d}[P_M e^{-it\Delta}](x, y)\,\chi_k(t, y)\,[\tilde P_M F(u(t))](y)\,dy\,dt\right\|_{L_x^2} \le \frac{1}{10}\sum_{L\le\eta N}\left(\frac{L}{N}\right)^s\mathcal{M}(L)$$

for all N sufficiently large depending on u, s, and η. An analogous estimate holds with P_M replaced by $\chi_N P_M^+$ or $\chi_N P_M^-$; moreover, the time integrals may be taken over $[-2^{k+1}\delta, -2^k\delta]$.

Proof. We decompose

$$F(u) = F(u_{\le\eta M}) + O(|u_{>\eta M}|^{1+\frac{4}{d}}) + O(|u_{\le\eta M}|^{\frac{4}{d}}|u_{>\eta M}|). \tag{5-13}$$

We first consider the contribution coming from the last two terms in the decomposition above. By the adjoint Strichartz inequality and Hölder's inequality,

$$\left\|\int_{\mathbb{R}}\int_{\mathbb{R}^d}[P_M e^{-it\Delta}](x, y)\chi_k(t, y)\tilde P_M\big[O(|u_{>\eta M}|^{\frac{d+4}{d}}) + O(|u_{\le\eta M}|^{\frac{4}{d}}|u_{>\eta M}|)\big](y)\,dy\,dt\right\|_{L_x^2}$$
$$\lesssim \big\|\chi_k\tilde P_M\big[O(|u_{>\eta M}|^{\frac{d+4}{d}}) + O(|u_{\le\eta M}|^{\frac{4}{d}}|u_{>\eta M}|)\big]\big\|_{L_t^1L_y^2}$$
$$\lesssim (M2^k\delta)^{-\frac{2(d-1)}{d}}(2^k\delta)^{\frac{d-1}{d}}\Big(\big\||y|^{\frac{2(d-1)}{d}}\tilde P_M O(|u_{>\eta M}|^{\frac{d+4}{d}})\big\|_{L_t^dL_y^2([2^k\delta,2^{k+1}\delta]\times\mathbb{R}^d)}$$
$$+\big\||y|^{\frac{2(d-1)}{d}}\tilde P_M O(|u_{\le\eta M}|^{\frac{4}{d}}|u_{>\eta M}|)\big\|_{L_t^dL_y^2([2^k\delta,2^{k+1}\delta]\times\mathbb{R}^d)}\Big).$$

As $\tilde{P}_M$ is a Mihlin multiplier and $|y|^{\frac{4(d-1)}{d}}$ is an A_2 weight, $\tilde{P}_M$ is bounded on $L^2(|y|^{\frac{4(d-1)}{d}}\,dy)$; see [Stein 1993, Chapter V]. Thus, by Hölder's inequality and (5-3),

$$\begin{aligned}
&\Big\| \int_{\mathbb{R}} \int_{\mathbb{R}^d} [P_M e^{-it\Delta}](x,y)\chi_k(t,y)\tilde{P}_M\big[O(|u_{>\eta M}|^{\frac{d+4}{d}}) + O(|u_{\le \eta M}|^{\frac{4}{d}}|u_{>\eta M}|)\big](y)\,dy\,dt \Big\|_{L^2_x} \\
&\quad\lesssim (M2^k\delta)^{-\frac{2(d-1)}{d}}(2^k\delta)^{\frac{d-1}{d}} \Big(\big\| |y|^{\frac{2(d-1)}{d}} |u_{>\eta M}|^{\frac{d+4}{d}} \big\|_{L^d_t L^2_y([2^k\delta,2^{k+1}\delta]\times\mathbb{R}^d)} \\
&\qquad\qquad + \big\| |y|^{\frac{2(d-1)}{d}} |u_{\le \eta M}|^{\frac{4}{d}} |u_{>\eta M}| \big\|_{L^d_t L^2_y([2^k\delta,2^{k+1}\delta]\times\mathbb{R}^d)} \Big) \\
&\quad\lesssim (M2^k\delta)^{-\frac{2(d-1)}{d}}(2^k\delta)^{\frac{d-1}{d}} \|u_{>\eta M}\|_{L^\infty_t L^2_y} \Big(\big\| |y|^{\frac{d-1}{2}} u_{>\eta M} \big\|^{\frac{4}{d}}_{L^4_t L^\infty_y([2^k\delta,2^{k+1}\delta]\times\mathbb{R}^d)} \\
&\qquad\qquad + \| |y|^{\frac{d-1}{2}} u_{\le \eta M} \|^{\frac{4}{d}}_{L^4_t L^\infty_y([2^k\delta,2^{k+1}\delta]\times\mathbb{R}^d)} \Big) \\
&\quad\lesssim_u (M2^k\delta)^{-\frac{2(d-1)}{d}}(2^k\delta)^{\frac{d-1}{d}} \mathcal{M}(\eta N)\langle 2^k\delta\rangle^{\frac{1}{d}}.
\end{aligned}$$

Summing first in $k \ge 0$ and then in $M \ge N$, we estimate the contribution of the last two terms on the right-hand side of (5-13) by

$$(N^2\delta)^{-1+\frac{1}{d}}\mathcal{M}(\eta N).$$

Next we consider the contribution coming from the first term on the right-hand side of (5-13). By the adjoint of the weighted Strichartz inequality in Lemma 2.8, Hölder's inequality, Corollary 2.5, and Lemma 2.2,

$$\begin{aligned}
&\Big\| \int_{\mathbb{R}} \int_{\mathbb{R}^d} [P_M e^{-it\Delta}](x,y)\,\chi_k(t,y)\,[\tilde{P}_M F(u_{\le \eta M}(t))](y)\,dy\,dt \Big\|_{L^2_x} \\
&\qquad\lesssim (M2^k\delta)^{-\frac{2(d-1)}{q}} \big\| \chi_k \tilde{P}_M F(u_{\le \eta M}) \big\|_{L_t^{\frac{q}{q-1}} L_y^{\frac{2q}{q+4}}} \\
&\qquad\lesssim (M2^k\delta)^{-\frac{2(d-1)}{q}} (2^k\delta)^{\frac{q-1}{q}} M^{-s} \big\| |\nabla|^s F(u_{\le \eta M}) \big\|_{L^\infty_t L_y^{\frac{2q}{q+4}}} \\
&\qquad\lesssim (M2^k\delta)^{-\frac{2(d-1)}{q}} (2^k\delta)^{\frac{q-1}{q}} M^{-s} \|u_{\le \eta M}\|^{\frac{4}{d}}_{L^\infty_t L_y^{\frac{2q}{d}}} \big\| |\nabla|^s u_{\le \eta M} \big\|_{L^\infty_t L^2_y} \\
&\qquad\lesssim_u (M2^k\delta)^{-\frac{2(d-1)}{q}} (2^k\delta)^{\frac{q-1}{q}} (\eta M)^{\frac{2(q-d)}{q}} \sum_{L \le \eta M} \Big(\frac{L}{M}\Big)^s \mathcal{M}(L) \\
&\qquad\lesssim_u (M^2 2^k\delta)^{-\frac{2d-q-1}{q}} \sum_{L \le \eta N} \Big(\frac{L}{N}\Big)^s \mathcal{M}(L)
\end{aligned}$$

provided $q \ge \max\{d,4\}$ and $M \ge N$. In order to deduce the last inequality, we used the fact that, for $M \ge N$,

$$\sum_{L\le\eta M}\Big(\frac{L}{M}\Big)^s\mathcal{M}(L)\le\sum_{L\le\eta N}\Big(\frac{L}{N}\Big)^s\mathcal{M}(L)+\sum_{\eta N\le L\le\eta M}\Big(\frac{L}{M}\Big)^s\mathcal{M}(L)$$
$$\lesssim\sum_{L\le\eta N}\Big(\frac{L}{N}\Big)^s\mathcal{M}(L)+\eta^s\mathcal{M}(\eta N)\lesssim\sum_{L\le\eta N}\Big(\frac{L}{N}\Big)^s\mathcal{M}(L).\tag{5-14}$$

Therefore, choosing $q=d+1$,

$$\sum_{M\ge N}\sum_{k=0}^{\infty}\bigg\|\int_{\mathbb{R}}\int_{\mathbb{R}^d}[P_Me^{-it\Delta}](x,y)\,\chi_k(t,y)[\tilde P_M F(u_{\le\eta M}(t))](y)\,dy\,dt\bigg\|_{L_x^2}$$
$$\lesssim(N^2\delta)^{-\frac{d-2}{d+1}}\sum_{L\le\eta N}\Big(\frac{L}{N}\Big)^s\mathcal{M}(L).$$

Putting everything together we obtain

$$\sum_{M\ge N}\sum_{k=0}^{\infty}\bigg\|\int_{\mathbb{R}}\int_{\mathbb{R}^d}[P_Me^{-it\Delta}](x,y)\,\chi_k(t,y)\,[\tilde P_M F(u(t))](y)\,dy\,dt\bigg\|_{L_x^2}$$
$$\lesssim_u\Big((N^2\delta)^{-1+\frac{2}{d}}+(N^2\delta)^{-1+\frac{1}{d}}+(N^2\delta)^{-\frac{d-2}{d+1}}\Big)\sum_{L\le\eta N}\Big(\frac{L}{N}\Big)^s\mathcal{M}(L).$$

Choosing N sufficiently large depending on u, δ, and s (and hence only on u, η, and s), we obtain the desired bound.

The last claim follows from the L_x^2-boundedness of $\chi_N P^{\pm}P_M$ (see Lemma 4.2) and the time-reversal symmetry of the argument just presented. □

We turn now to the region of (t,y) integration where $|y|\ll M|t|$. First, we describe the bounds that we will use for the kernels of the propagators. For $|x|\le N^{-1}$, $|y|\ll M|t|$, and $|t|\ge\delta\gg N^{-2}$, we have

$$|P_Me^{-it\Delta}(x,y)|\lesssim\frac{1}{(M^2|t|)^{50d}}\frac{M^d}{\langle M(x-y)\rangle^{50d}};\tag{5-15}$$

this follows from Lemma 2.6 since under these constraints, $|y-x|\ll M|t|$. For $|x|\ge N^{-1}$ and y and t as above,

$$|P_M^{\pm}e^{-it\Delta}(x,y)|\lesssim\frac{1}{(M^2|t|)^{50d}}\frac{M^d}{\langle Mx\rangle^{\frac{d-1}{2}}\langle My\rangle^{\frac{d-1}{2}}\langle M|x|-M|y|\rangle^{50d}},\tag{5-16}$$

by Lemma 4.1. To simplify the bound in (5-16) we used the inequalities $|y|-|x|\ll M|t|$ and

$$\langle M^2|t|+M|x|-M|y|\rangle^{-100d}\lesssim(M^2|t|)^{-50d}\langle M|x|-M|y|\rangle^{-50d}.$$

From (5-15) and (5-16) we see that under the hypotheses set out above,

$$|P_Me^{-it\Delta}(x,y)|+|P_M^{\pm}e^{-it\Delta}(x,y)|\lesssim\frac{1}{(M^2|t|)^{50d}}K_M(x,y),\tag{5-17}$$

where

$$K_M(x,y) := \frac{M^d}{\langle M(x-y)\rangle^{50d}} + \frac{M^d}{\langle Mx\rangle^{\frac{d-1}{2}}\langle My\rangle^{\frac{d-1}{2}}\langle M|x|-M|y|\rangle^{50d}}.$$

Note that by Schur's test, this is the kernel of a bounded operator on $L^2_x(\mathbb{R}^d)$.

Let $\tilde\chi_k$ denote the characteristic function of the set

$$\{(t,y):\ 2^k\delta \le |t| \le 2^{k+1}\delta,\ |y| \ll M|t|\}.$$

Lemma 5.5 (The tail). *Let* $0 < s < 1+\frac{4}{d}$, *let* $\eta > 0$ *be a small number, and let* δ *be as in Lemma 5.3. Then*

$$\sum_{M\ge N}\sum_{k=0}^{\infty}\Big\|\int_{\mathbb{R}}\int_{\mathbb{R}^d}\frac{K_M(x,y)}{(M^2|t|)^{50d}}\,\tilde\chi_k(t,y)\,|\tilde P_M F(u(t))|(y)\,dy\,dt\Big\|_{L^2_x} \le \tfrac{1}{10}\sum_{L\le\eta N}\Big(\frac{L}{N}\Big)^s \mathcal{M}(L)$$

for all N *sufficiently large depending on* u, s, *and* η (*in particular, we require* $N \gg \delta^{-1/2}$).

Proof. Using Hölder's inequality, the L^2-boundedness of the operator with kernel K_M, and Lemma 2.2,

$$\begin{aligned}
&\Big\|\int_{\mathbb{R}}\int_{\mathbb{R}^d}\frac{K_M(x,y)}{(M^2|t|)^{50d}}\,\tilde\chi_k(t,y)\,|\tilde P_M F(u(t))|(y)\,dy\,dt\Big\|_{L^2_x}\\
&\qquad\lesssim (M^2 2^k\delta)^{-50d}(2^k\delta)^{\frac{d-2}{d}}M^{\frac{2(d-2)}{d}}\big\|\tilde P_M F(u)\big\|_{L_t^{\frac{d}{2}}L_x^{\frac{2d^2}{d^2+4d-8}}([2^k\delta,2^{k+1}\delta]\times\mathbb{R}^d)}\\
&\qquad\lesssim (M^2 2^k\delta)^{-49d}\big\|\tilde P_M F(u)\big\|_{L_t^{\frac{d}{2}}L_x^{\frac{2d^2}{d^2+4d-8}}([2^k\delta,2^{k+1}\delta]\times\mathbb{R}^d)}.
\end{aligned}$$

We decompose

$$F(u) = F(u_{\le\eta M}) + O\big(|u_{\le\eta M}|^{\frac{4}{d}}\,|u_{>\eta M}|\big) + O\big(|u_{>\eta M}|^{1+\frac{4}{d}}\big). \tag{5-18}$$

Discarding the projection $\tilde P_M$, we use Hölder and (5-2) to estimate

$$\begin{aligned}
&\big\|\tilde P_M O(|u_{\le\eta M}|^{\frac{4}{d}}\,|u_{>\eta M}|)\big\|_{L_t^{\frac{d}{2}}L_x^{\frac{2d^2}{d^2+4d-8}}([2^k\delta,2^{k+1}\delta]\times\mathbb{R}^d)}\\
&\qquad\lesssim \big\|u_{\le\eta M}\big\|^{\frac{4}{d}}_{L_t^2L_x^{\frac{2d}{d-2}}([2^k\delta,2^{k+1}\delta]\times\mathbb{R}^d)}\big\|u_{>\eta M}\big\|_{L_t^\infty L_x^2} \lesssim_u \langle 2^k\delta\rangle^{\frac{2}{d}}\mathcal{M}(\eta N),\\
&\big\|\tilde P_M O(|u_{>\eta M}|^{1+\frac{4}{d}})\big\|_{L_t^{\frac{d}{2}}L_x^{\frac{2d^2}{d^2+4d-8}}([2^k\delta,2^{k+1}\delta]\times\mathbb{R}^d)}\\
&\qquad\lesssim \big\|u_{>\eta M}\big\|^{\frac{4}{d}}_{L_t^2L_x^{\frac{2d}{d-2}}([2^k\delta,2^{k+1}\delta]\times\mathbb{R}^d)}\big\|u_{>\eta M}\big\|_{L_t^\infty L_x^2} \lesssim_u \langle 2^k\delta\rangle^{\frac{2}{d}}\mathcal{M}(\eta N).
\end{aligned}$$

To estimate the contribution coming from the first term on the right-hand side of (5-18), we use Lemma 2.2, Corollary 2.5 (with $r = d^2/(d-2)$) combined with Hölder's inequality in the time variable, (5-2), and (5-14), to estimate

$$\begin{aligned}\big\|\tilde P_M F(u_{\le\eta M})\big\|_{L_t^{\frac d2}L_x^{\frac{2d^2}{d^2+4d-8}}([2^k\delta,2^{k+1}\delta]\times\mathbb{R}^d)} &\lesssim M^{-s}\big\||\nabla|^s F(u_{\le\eta M})\big\|_{L_t^{\frac d2}L_x^{\frac{2d^2}{d^2+4d-8}}([2^k\delta,2^{k+1}\delta]\times\mathbb{R}^d)}\\ &\lesssim M^{-s}\big\||\nabla|^s u_{\le\eta M}\big\|_{L_t^\infty L_x^2}\big\|u_{\le\eta M}\big\|^{\frac4d}_{L_t^2L_x^{\frac{2d}{d-2}}([2^k\delta,2^{k+1}\delta]\times\mathbb{R}^d)}\\ &\lesssim_u \langle 2^k\delta\rangle^{\frac2d}\sum_{L\le\eta M}\Big(\frac LM\Big)^s\mathcal{M}(L)\lesssim_u \langle 2^k\delta\rangle^{\frac2d}\sum_{L\le\eta N}\Big(\frac LN\Big)^s\mathcal{M}(L)\end{aligned}$$

for any $M\ge N$.

Putting everything together, we deduce

$$\bigg\|\int_{\mathbb{R}}\int_{\mathbb{R}^d}\frac{K_M(x,y)}{(M^2|t|)^{50d}}\,\tilde\chi_k(t,y)\,[\tilde P_M F(u(t))](y)\,dy\,dt\bigg\|_{L_x^2}\lesssim_u (M^2 2^k\delta)^{-49d}\langle 2^k\delta\rangle^{\frac2d}\sum_{L\le\eta N}\Big(\frac LN\Big)^s\mathcal{M}(L).$$

Summing over $k\ge 0$ and $M\ge N$, we obtain

$$\sum_{M\ge N}\sum_{k=0}^\infty\bigg\|\int_{\mathbb{R}}\int_{\mathbb{R}^d}\frac{K_M(x,y)}{(M^2|t|)^{50d}}\,\tilde\chi_k(t,y)[\tilde P_M F(u(t))](y)\,dy\,dt\bigg\|_{L_x^2}\lesssim_u (N^2\delta)^{-49d}\sum_{L\le\eta N}\Big(\frac LN\Big)^s\mathcal{M}(L).$$

The claim follows by choosing N sufficiently large depending on δ, η, and s (and hence only on u, s, and η). □

Combining it all together:

Proof of Proposition 5.2. Naturally, we may bound $\|u_{\ge N}\|_{L^2}$ by separately bounding the L^2 norm on the ball $\{|x|\le N^{-1}\}$ and on its complement. On the ball, we use (5-7), while outside the ball we use (5-6). Invoking (5-8) and the triangle inequality, we reduce the proof to bounding certain integrals. The integrals over short times were estimated in Lemma 5.3. For $|t|\ge\delta$, we further partition the region of integration into two pieces. The first piece, where $|y|\gtrsim M|t|$, was dealt with in Lemma 5.4. To estimate the remaining piece, $|y|\ll M|t|$, one combines (5-17) and Lemma 5.5. □

6. The double high-to-low frequency cascade

In this section, we use the additional regularity provided by Theorem 5.1 to preclude double high-to-low frequency cascade solutions. We argue as in [Killip et al. 2007].

Proposition 6.1 (Absence of double cascades). *Let $d\ge 3$. There are no nonzero global spherically symmetric solutions to (1-1) that are double high-to-low frequency cascades in the sense of Theorem 1.10.*

Proof. Suppose to the contrary that there is such a solution u. By Theorem 5.1, u lies in $C_t^0H_x^1(\mathbb{R}\times\mathbb{R}^d)$. Hence the energy

$$E(u)=E(u(t))=\int_{\mathbb{R}^d}\tfrac12|\nabla u(t,x)|^2+\mu\frac{d}{2(d+2)}|u(t,x)|^{2(d+2)/d}\,dx$$

is finite and conserved (see [Cazenave 2003], for example). Since $M(u) < M(Q)$ in the focusing case, the sharp Gagliardo–Nirenberg inequality (Theorem 1.6) gives

$$\|\nabla u(t)\|^2_{L^2_x(\mathbb{R}^d)} \sim_u E(u) \sim_u 1 \tag{6-1}$$

for all $t \in \mathbb{R}$. We will now reach a contradiction by proving that $\|\nabla u(t)\|_2 \to 0$ along any sequence where $N(t) \to 0$. The existence of two such time sequences is guaranteed by the fact that u is a double high-to-low frequency cascade.

Let $\eta > 0$ be arbitrary. By Definition 1.8, we can find $C = C(\eta, u) > 0$ such that

$$\int_{|\xi| \geq CN(t)} |\hat{u}(t,\xi)|^2 \, d\xi \leq \eta^2$$

for all t. Meanwhile, by Theorem 5.1, $u \in C^0_t H^s_x(\mathbb{R} \times \mathbb{R}^d)$ for some $s > 1$. Thus,

$$\int_{|\xi| \geq CN(t)} |\xi|^{2s} |\hat{u}(t,\xi)|^2 \, d\xi \lesssim_u 1$$

for all t and some $s > 1$. Thus, by Hölder's inequality,

$$\int_{|\xi| \geq CN(t)} |\xi|^2 |\hat{u}(t,\xi)|^2 \, d\xi \lesssim_u \eta^{2(s-1)/s}.$$

On the other hand, from mass conservation and Plancherel's theorem we have

$$\int_{|\xi| \leq CN(t)} |\xi|^2 |\hat{u}(t,\xi)|^2 \, d\xi \lesssim_u C^2 N(t)^2.$$

Summing these last two bounds and using Plancherel's theorem again, we obtain

$$\|\nabla u(t)\|_{L^2_x(\mathbb{R}^d)} \lesssim_u \eta^{(s-1)/s} + CN(t)$$

for all t. As $\eta > 0$ is arbitrary and there exists a sequence of times $t_n \to \infty$ such that $N(t_n) \to 0$ (u is a double high-to-low frequency cascade), we conclude $\|\nabla u(t_n)\|_2 \to 0$. This contradicts (6-1). □

Remark. As mentioned in [Killip et al. 2007], the argument presented can be used to rule out nonradial single-sided cascade solutions that lie in $C^0_t H^s_x$ for some $s > 1$. (By a single-sided cascade we mean a solution with $N(t)$ bounded on a semiinfinite interval, say $[T, \infty)$, with $\liminf_{t\to\infty} N(t) = 0$.) For such regular solutions u, we may define the total momentum $\int_{\mathbb{R}^d} \operatorname{Im}(\bar{u}\nabla u)$, which is conserved. By a Galilean transformation, we can set this momentum equal to zero; thus

$$\int_{\mathbb{R}^d} \xi |\hat{u}(t,\xi)|^2 \, d\xi = 0.$$

From this, mass conservation, and the uniform H^s_x bound for some $s > 1$, one can show that $\xi(t) \to 0$ whenever $N(t) \to 0$. On the other hand, a modification of the above argument gives

$$1 \sim_u \|\nabla u(t)\|_2 \lesssim \eta^{(s-1)/s} + C\big(N(t) + |\xi(t)|\big),$$

which is absurd.

7. Death of a soliton

In this section, we use the additional regularity proved in Theorem 5.1 to rule out the third and final enemy, the soliton-like solution. Once again, we follow [Killip et al. 2007]; the method is similar to that in [Kenig and Merle 2006a]. Let

$$M_R(t) := 2\,\mathrm{Im} \int_{\mathbb{R}^d} \psi\Big(\frac{|x|}{R}\Big)\bar{u}(t,x)x\cdot\nabla u(t,x)\,dx,$$

where ψ is a smooth function obeying

$$\psi(r) = \begin{cases} 1 & \text{if } r \le 1, \\ 0 & \text{if } r \ge 2, \end{cases}$$

and R denotes a radius to be chosen momentarily. For solutions u to (1-1) belonging to $C_t^0 H_x^1$, $M_R(t)$ is a well-defined function. Indeed,

$$|M_R(t)| \lesssim R\|u(t)\|_2\|\nabla u(t)\|_2 \lesssim_u R.$$

An oft-repeated calculation (essentially that in the derivation of the Morawetz and virial identities) gives:

Lemma 7.1.

$$\partial_t M_R(t) = 8E(u(t))$$

$$-\int_{\mathbb{R}^d}\Big(\frac{d^2-1}{R|x|}\psi'\Big(\frac{|x|}{R}\Big)+\frac{2d+1}{R^2}\psi''\Big(\frac{|x|}{R}\Big)+\frac{|x|}{R^3}\psi'''\Big(\frac{|x|}{R}\Big)\Big)|u(t,x)|^2\,dx \tag{7-1}$$

$$+4\int_{\mathbb{R}^d}\Big(\psi\Big(\frac{|x|}{R}\Big)-1+\frac{|x|}{R}\psi'\Big(\frac{|x|}{R}\Big)\Big)|\nabla u(t,x)|^2\,dx \tag{7-2}$$

$$+\frac{4\mu}{d+2}\int_{\mathbb{R}^d}\Big(d\Big(\psi\Big(\frac{|x|}{R}\Big)-1\Big)+\frac{|x|}{R}\psi'\Big(\frac{|x|}{R}\Big)\Big)|u(t,x)|^{\frac{2(d+2)}{d}}\,dx, \tag{7-3}$$

where $E(u)$ is the energy of u as defined in (1-3).

Proposition 7.2 (Absence of solitons). *Let $d \ge 3$. There are no nonzero global spherically symmetric solutions to (1-1) that are soliton-like in the sense of Theorem 1.10.*

Proof. Assume to the contrary that there is such a solution u. Then, by Theorem 5.1, $u \in C_t^0 H_x^s$ for some $s > 1$. In particular,

$$|M_R(t)| \lesssim_u R. \tag{7-4}$$

Recall that in the focusing case, $M(u) < M(Q)$. As a consequence, the sharp Gagliardo–Nirenberg inequality (Theorem 1.6) implies that the energy is a positive quantity in the focusing case as well as in the defocusing case. Indeed,

$$E(u) \gtrsim_u \int_{\mathbb{R}^d} |\nabla u(t,x)|^2\,dx > 0.$$

We will show that for R sufficiently large, (7-1) through (7-3) are small terms compared with $E(u)$. Combining this fact with Lemma 7.1, we conclude $\partial_t M_R(t) \gtrsim E(u) > 0$, which contradicts (7-4).

We first turn our attention to (7-1). This is trivially bounded as

$$|(7\text{-}1)| \lesssim_u R^{-2}. \tag{7-5}$$

We now study (7-2) and (7-3). Let $\eta > 0$ be a small number to be chosen later. By Definition 1.8 and the fact that $N(t) = 1$ for all $t \in \mathbb{R}$, if R is sufficiently large depending on u and η, then

$$\int_{|x|\geq \frac{R}{4}} |u(t,x)|^2\, dx \leq \eta \tag{7-6}$$

for all $t \in \mathbb{R}$. Let χ denote a smooth cutoff to the region $|x| \geq R/2$, chosen so that $\nabla\chi$ is bounded by R^{-1} and supported where $|x| \sim R$. As $u \in C_t^0 H_x^s$ for some $s > 1$, using interpolation and (7-6), we estimate

$$|(7\text{-}2)| \lesssim \|\chi\nabla u(t)\|_2^2 \lesssim \|\nabla(\chi u(t))\|_2^2 + \|u(t)\nabla\chi\|_2^2 \lesssim \|\chi u(t)\|_2^{\frac{2(s-1)}{s}} \|u(t)\|_{H_x^s}^{\frac{2}{s}} + \eta \lesssim_u \eta^{\frac{s-1}{s}} + \eta. \tag{7-7}$$

Finally, we are left to consider (7-3). Using the same χ as above together with the Gagliardo–Nirenberg inequality and (7-6),

$$|(7\text{-}3)| \lesssim \|\chi u(t)\|_{\frac{2(d+2)}{d}}^{\frac{2(d+2)}{d}} \lesssim \|\chi u(t)\|_2^{\frac{4}{d}} \|\nabla(\chi u(t))\|_2^2 \lesssim_u \eta^{\frac{2}{d}}. \tag{7-8}$$

Combining (7-5), (7-7), and (7-8) and choosing η sufficiently small depending on u and R sufficiently large depending on u and η, we obtain

$$|(7\text{-}1)| + |(7\text{-}2)| + |(7\text{-}3)| \leq \tfrac{1}{100} E(u).$$

This completes the proof of the proposition for the reasons explained in the third paragraph. □

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References

[Bégout and Vargas 2007] P. Bégout and A. Vargas, "Mass concentration phenomena for the L^2-critical nonlinear Schrödinger equation", *Trans. Amer. Math. Soc.* **359**:11 (2007), 5257–5282. MR 2008g:35190 Zbl 05178492

[Berestycki and Lions 1979] H. Berestycki and P.-L. Lions, "Existence d'ondes solitaires dans des problèmes nonlinéaires du type Klein–Gordon", *C. R. Acad. Sci. Paris Sér. A-B* **288**:7 (1979), A395–A398. MR 80i:35076 Zbl 0397.35024

[Blue and Colliander 2006] P. Blue and J. Colliander, "Global well-posedness in Sobolev space implies global existence for weighted L^2 initial data for L^2-critical NLS", *Commun. Pure Appl. Anal.* **5**:4 (2006), 691–708. MR 2007f:35265 Zbl 1137.35066

[Bourgain 1998] J. Bourgain, "Refinements of Strichartz' inequality and applications to 2D-NLS with critical nonlinearity", *Internat. Math. Res. Not.* **1998**:5 (1998), 253–283. MR 99f:35184 Zbl 0917.35126

[Bourgain 1999a] J. Bourgain, *Global solutions of nonlinear Schrödinger equations*, American Mathematical Society Colloquium Publications **46**, American Mathematical Society, Providence, RI, 1999. MR 2000h:35147 Zbl 0933.35178

[Bourgain 1999b] J. Bourgain, "Global wellposedness of defocusing critical nonlinear Schrödinger equation in the radial case", *J. Amer. Math. Soc.* **12**:1 (1999), 145–171. MR 99e:35208 Zbl 0958.35126

[Carles 2002] R. Carles, "Critical nonlinear Schrödinger equations with and without harmonic potential", *Math. Models Methods Appl. Sci.* **12**:10 (2002), 1513–1523. MR 2004b:35298 Zbl 1029.35208

[Carles and Keraani 2007] R. Carles and S. Keraani, "On the role of quadratic oscillations in nonlinear Schrödinger equations. II. The L^2-critical case", *Trans. Amer. Math. Soc.* **359**:1 (2007), 33–62. MR 2008a:35260 Zbl 1115.35119

[Cazenave 2003] T. Cazenave, *Semilinear Schrödinger equations*, Courant Lecture Notes in Mathematics **10**, New York University Courant Institute of Mathematical Sciences, New York, 2003. MR 2004j:35266 Zbl 1055.35003

[Cazenave and Weissler 1989] T. Cazenave and F. B. Weissler, "Some remarks on the nonlinear Schrödinger equation in the subcritical case", pp. 59–69 in *New methods and results in nonlinear field equations* (Bielefeld, 1987), edited by P. Blanchard et al., Lecture Notes in Phys. **347**, Springer, Berlin, 1989. MR 91c:35136 Zbl 0699.35217

[Christ and Weinstein 1991] F. M. Christ and M. I. Weinstein, "Dispersion of small amplitude solutions of the generalized Korteweg-de Vries equation", *J. Funct. Anal.* **100**:1 (1991), 87–109. MR 92h:35203 Zbl 0743.35067

[Colliander et al. 2002] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, "Almost conservation laws and global rough solutions to a nonlinear Schrödinger equation", *Math. Res. Lett.* **9**:5-6 (2002), 659–682. MR 2003j:35294 Zbl 01930988

[Colliander et al. 2005] J. Colliander, S. Raynor, C. Sulem, and J. D. Wright, "Ground state mass concentration in the L^2-critical nonlinear Schrödinger equation below H^1", *Math. Res. Lett.* **12**:2-3 (2005), 357–375. MR 2006e:35300 Zbl 1084.35088

[Colliander et al. 2007] J. Colliander, M. Grillakis, and N. Tzirakis, "Improved interaction Morawetz inequalities for the cubic nonlinear Schrödinger equation on $\mathbb{R}^2$", *Int. Math. Res. Not.* **2007**:23 (2007), Art. ID rnm090. Zbl 1142.35085 arXiv math/0703606

[Colliander et al. 2008a] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, "Global well-posedness and scattering for the energy-critical nonlinear Schrödinger equation in $\mathbb{R}^3$", *Ann. of Math.* (2) **167**:3 (2008), 767–865.

[Colliander et al. 2008b] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, "Resonant decompositions and the I-method for the cubic nonlinear Schrödinger equation on $\mathbb{R}^2$", *Discrete Contin. Dyn. Syst.* **21**:3 (2008), 665–686. MR 2399431 Zbl 05304859

[De Silva et al. 2007a] D. De Silva, N. Pavlović, G. Staffilani, and N. Tzirakis, "Global well-posedness and polynomial bounds for the defocusing L^2-critical nonlinear Schrödinger equation in $\mathbb{R}$", preprint, 2007. arXiv math/0702707

[De Silva et al. 2007b] D. De Silva, N. Pavlović, G. Staffilani, and N. Tzirakis, "Global well-posedness for the L^2 critical nonlinear Schrödinger equation in higher dimensions", *Commun. Pure Appl. Anal.* **6**:4 (2007), 1023–1041. MR 2341818 Zbl 05257774

[Fang and Grillakis 2007] Y. F. Fang and M. G. Grillakis, "On the global existence of rough solutions of the cubic defocusing Schrödinger equation in $\mathbf{R}^{2+1}$", *J. Hyperbolic Differ. Equ.* **4**:2 (2007), 233–257. MR 2008c:35303 Zbl 1122.35132

[Ginibre and Velo 1992] J. Ginibre and G. Velo, "Smoothing properties and retarded estimates for some dispersive evolution equations", *Comm. Math. Phys.* **144**:1 (1992), 163–188. MR 93a:35065 Zbl 0762.35008

[Gradshteyn and Ryzhik 2000] I. S. Gradshteyn and I. M. Ryzhik, *Table of integrals, series, and products*, 6th ed., Academic Press, San Diego, CA, 2000. Translated from the Russian. MR 2001c:00002 Zbl 0981.65001

[Kato 1995] T. Kato, "On nonlinear Schrödinger equations. II. H^s-solutions and unconditional well-posedness", *J. Anal. Math.* **67** (1995), 281–306. MR 98a:35124a Zbl 0848.35124

[Keel and Tao 1998] M. Keel and T. Tao, "Endpoint Strichartz estimates", *Amer. J. Math.* **120**:5 (1998), 955–980. MR 2000d:35018 Zbl 0922.35028

[Kenig and Merle 2006a] C. E. Kenig and F. Merle, "Global well-posedness, scattering and blow-up for the energy-critical, focusing, non-linear Schrödinger equation in the radial case", *Invent. Math.* **166**:3 (2006), 645–675. MR 2007g:35232 Zbl 1115.35125

[Kenig and Merle 2006b] C. E. Kenig and F. Merle, "Global well-posedness, scattering and blow-up for the energy critical focusing non-linear wave equation", preprint, 2006. arXiv math/0610801

[Keraani 2001] S. Keraani, "On the defect of compactness for the Strichartz estimates of the Schrödinger equations", *J. Differential Equations* **175**:2 (2001), 353–392. MR 2002j:35281 Zbl 1038.35119

[Keraani 2006] S. Keraani, "On the blow up phenomenon of the critical nonlinear Schrödinger equation", *J. Funct. Anal.* **235**:1 (2006), 171–192. MR 2007e:35260 Zbl 1099.35132

[Killip and Visan 2008] R. Killip and M. Visan, "The focusing energy-critical nonlinear Schrödinger equation in dimensions five and higher", preprint, 2008. arXiv 0804.1018

[Killip et al. 2007] R. Killip, T. Tao, and M. Visan, "The cubic nonlinear Schrödinger equation in two dimensions with radial data", preprint, 2007. arXiv 0707.3188

[Kwong 1989] M. K. Kwong, "Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in $\mathbf{R}^n$", *Arch. Rational Mech. Anal.* **105**:3 (1989), 243–266. MR 90d:35015 Zbl 0676.35032

[Merle 1993] F. Merle, "Determination of blow-up solutions with minimal mass for nonlinear Schrödinger equations with critical power", *Duke Math. J.* **69**:2 (1993), 427–454. MR 94b:35262 Zbl 0808.35141

[Merle and Tsutsumi 1990] F. Merle and Y. Tsutsumi, "L^2 concentration of blow-up solutions for the nonlinear Schrödinger equation with critical power nonlinearity", *J. Differential Equations* **84**:2 (1990), 205–214. MR 91e:35194 Zbl 0722.35047

[Merle and Vega 1998] F. Merle and L. Vega, "Compactness at blow-up time for L^2 solutions of the critical nonlinear Schrödinger equation in 2D", *Internat. Math. Res. Not.* **1998**:8 (1998), 399–425. MR 99d:35156 Zbl 0913.35126

[Nawa 1999] H. Nawa, "Asymptotic and limiting profiles of blowup solutions of the nonlinear Schrödinger equation with critical power", *Comm. Pure Appl. Math.* **52**:2 (1999), 193–270. MR 99m:35235 Zbl 0964.37014

[Ryckman and Visan 2007] E. Ryckman and M. Visan, "Global well-posedness and scattering for the defocusing energy-critical nonlinear Schrödinger equation in $\mathbb{R}^{1+4}$", *Amer. J. Math.* **129**:1 (2007), 1–60. MR 2007k:35474

[Shao 2007] S. Shao, "Sharp linear and bilinear restriction estimates for paraboloid in the cylindrically symmetric case", preprint, 2007. arXiv 0706.3759

[Staffilani 1997] G. Staffilani, "On the generalized Korteweg-de Vries-type equations", *Differential Integral Equations* **10**:4 (1997), 777–796. MR 2001a:35005 Zbl 0891.35135

[Stein 1993] E. M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series **43**, Princeton University Press, Princeton, NJ, 1993. MR 95c:42002 Zbl 0821.42001

[Stein and Weiss 1971] E. M. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton Mathematical Series **32**, Princeton University Press, Princeton, N.J., 1971. MR 46 #4102 Zbl 0232.42007

[Strichartz 1977] R. S. Strichartz, "Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations", *Duke Math. J.* **44**:3 (1977), 705–714. MR 58 #23577 Zbl 0372.35001

[Tao 2005] T. Tao, "Global well-posedness and scattering for the higher-dimensional energy-critical nonlinear Schrödinger equation for radial data", *New York J. Math.* **11** (2005), 57–80. MR 2006e:35308 Zbl 1119.35092

[Tao 2006] T. Tao, "A pseudoconformal compactification of the nonlinear Schrödinger equation and applications", preprint, 2006. arXiv math/0606254

[Tao 2008a] T. Tao, "Global regularity of wave maps III", preprint, 2008. arXiv 0805.4666

[Tao 2008b] T. Tao, "Global regularity of wave maps IV", preprint, 2008. arXiv 0806.3592

[Tao 2008c] T. Tao, "Global regularity of wave maps V", preprint, 2008. arXiv 0808.0368

[Tao et al. 2006] T. Tao, M. Visan, and X. Zhang, "Minimal-mass blowup solutions of the mass-critical NLS", preprint, 2006. To appear in Forum Math. arXiv math/0609690

[Tao et al. 2007] T. Tao, M. Visan, and X. Zhang, "Global well-posedness and scattering for the defocusing mass-critical nonlinear Schrödinger equation for radial data in high dimensions", *Duke Math. J.* **140**:1 (2007), 165–202. MR 2355070 Zbl 05208403

[Taylor 2000] M. E. Taylor, *Tools for PDE. Pseudodifferential operators, paradifferential operators, and layer potentials*, Mathematical Surveys and Monographs **81**, American Mathematical Society, Providence, RI, 2000. MR 2001g:35004 Zbl 0963.35211

[Tsutsumi 1985] Y. Tsutsumi, "Scattering problem for nonlinear Schrödinger equations", *Ann. Inst. H. Poincaré Phys. Théor.* **43**:3 (1985), 321–347. MR 87g:35221 Zbl 0612.35104

[Tzirakis 2005] N. Tzirakis, "The Cauchy problem for the semilinear quintic Schrödinger equation in one dimension", *Differential Integral Equations* **18**:8 (2005), 947–960. MR 2006b:35306 Zbl 1075.35082

[Visan 2006] M. Visan, *The defocusing energy-critical nonlinear Schrödinger equation in dimensions five and higher*, PhD Thesis, University of California, Los Angeles, 2006.

[Visan 2007] M. Visan, "The defocusing energy-critical nonlinear Schrödinger equation in higher dimensions", *Duke Math. J.* **138**:2 (2007), 281–374. MR 2008f:35387 Zbl 1131.35081

[Visan and Zhang 2007] M. Visan and X. Zhang, "On the blowup for the L^2-critical focusing nonlinear Schrödinger equation in higher dimensions below the energy class", *SIAM J. Math. Anal.* **39**:1 (2007), 34–56. MR 2008b:35284 Zbl 1135.35079

[Weinstein 1983] M. I. Weinstein, "Nonlinear Schrödinger equations and sharp interpolation estimates", *Comm. Math. Phys.* **87**:4 (1983), 567–576. MR 84d:35140 Zbl 0527.35023

[Weinstein 1989] M. I. Weinstein, "The nonlinear Schrödinger equation—singularity formation, stability and dispersion", pp. 213–232 in *The connection between infinite-dimensional and finite-dimensional dynamical systems* (Boulder, CO, 1987), edited by B. Nicolaenko et al., Contemp. Math. **99**, Amer. Math. Soc., Providence, RI, 1989. MR 90m:35188 Zbl 0703.35159

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