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DYNAMICS OF NONLINEAR SCHRÖDINGER/GROSS–PITAEVSKII EQUATIONS: MASS TRANSFER IN SYSTEMS WITH SOLITONS AND DEGENERATE NEUTRAL MODES

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Nonlinear Schrödinger/Gross–Pitaevskii equations play a central role in the understanding of nonlinear optical and macroscopic quantum systems. The large time dynamics of such systems is governed by interactions of the nonlinear ground state manifold, discrete neutral modes ("excited states") and dispersive radiation. Systems with symmetry, in spatial dimensions larger than one, typically have degenerate neutral modes. Thus, we study the large time dynamics of systems with degenerate neutral modes. This requires a new normal form (nonlinear matrix Fermi Golden Rule) governing the system's large time asymptotic relaxation to the ground state (soliton) manifold.

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1. Introduction

Nonlinear Schrödinger/Gross–Pitaevskii (NLS/GP) equations are a class of dispersive Hamiltonian partial differential equations (PDEs) of the form:

$$i\partial_t\psi(x,t) = -\Delta\psi(x,t) + \big(V(x) - f(|\psi(x,t)|^2)\big)\psi(x,t). \tag{1-1}$$

Here, $\psi = \psi(x,t)$ is a scalar complex-valued function of position, $x \in \mathbb{R}^d$ and time, $t \in \mathbb{R}$. The function $V : \mathbb{R}^d \to \mathbb{R}$ denotes a linear *potential* and $f : \mathbb{R}_+ \to \mathbb{R}$, a nonlinear potential. For example, V can be taken to be a smooth, nonpositive potential well, with rapid decay as $|x| \to \infty$ and $f(|\psi|^2) = -g|\psi|^2$, g constant. For $g > 0$, the nonlinearity is called *repulsive* or *defocusing*. For $g < 0$ it is called *attractive* or *focusing*. In this paper, we focus on spatial dimensions $d \geq 3$. Precise hypotheses on V and f are given below. We are interested in the initial value problem (IVP) for (1-1) with finite energy data $\psi(x,0)$ and solutions $\psi(x,t)$, which are sufficiently regular and decaying to zero as $|x| \to \infty$. A precise well-posedness result is cited below; see Theorem 3.1.

NLS/GP equations play a central role in the understanding of nonlinear optical [Moloney and Newell 2004; Boyd 2008; Sulem and Sulem 1999] and macroscopic quantum systems [Erdős and Yau 2001]. A striking and important feature of NLS/GP is that it can have localized standing waves or *nonlinear bound state* solutions, some of which are stable and play a central role in the general dynamics. In particular, for a wide variety of potentials and nonlinearities there exists an interval $\mathcal{I} \subset \mathbb{R}$ such that for any $\lambda \in \mathcal{I}$, (1-1) has *nonlinear ground state* solutions. These are solutions of the form

$$\psi(x,t) = e^{i\lambda t}\phi^\lambda(x),$$

where

$$-\Delta\phi^\lambda + \big(V - f(|\phi^\lambda|^2)\big)\phi^\lambda = -\lambda\phi^\lambda \tag{1-2}$$

with $\phi^\lambda \in H^1$ and $\phi^\lambda > 0$.

The gauge (phase-translation) invariance of (1-1),

$$\psi \mapsto e^{i\gamma}\psi, \quad \gamma \in [0, 2\pi),$$

generates a *nonlinear ground state* or *“soliton”* [1] *manifold*:

$$\mathcal{M}_{\mathcal{I}} := \{e^{i\gamma}\phi^\lambda,\ \lambda \in \mathcal{I},\ \gamma \in [0,2\pi)\}. \tag{1-3}$$

If V is identically zero, then NLS/GP admits a larger group of symmetries and the definition of soliton manifold (which exists in the focusing case, $g < 0$) is naturally extended to incorporate these additional symmetries; see, for example, [Weinstein 1986; Grillakis et al. 1987].

Orbital stability. The soliton manifold $\mathcal{M}_{\mathcal{I}}$ is said to be *orbitally stable* if any initial condition ψ_0, which is close to $\mathcal{M}_{\mathcal{I}}$ in H^1, gives rise to a solution $\psi(t)$, which is H^1 close for $t \neq 0$. There is an extensive literature on the *orbital stability* of the soliton manifold. For the case $V \equiv 0$, orbital stability (stability modulo spatial and phase translations) of global energy minimizers was proved in [Cazenave and Lions 1982] by compactness arguments. In [Weinstein 1985; 1986], it is shown that positive solutions,

[1] The term *soliton* sometimes refers, more specifically, to particle-like solutions of completely integrable PDEs.

which are index one critical points (Hessian with one strictly negative eigenvalue) and satisfy the slope condition[2]:

$$\frac{d}{d\lambda}\int_{\mathbb{R}^d}|\phi^\lambda(x)|^2\,dx>0, \tag{1-4}$$

are H^1 orbitally stable. For the focusing case

$$V\equiv 0,\quad f(|\psi|^2)=-g|\psi|^2,\quad g<0,$$

(1-4) is equivalent to $\sigma<2/d$. Orbital stability of solitary waves of NLS/GP for a class of potentials V was studied by Rose and Weinstein [1988] and, for a semiclassical setting, by Oh [1988]. A general formulation of a stability/instability theory is presented in [Grillakis et al. 1987].

Asymptotic stability. We say the soliton manifold $\mathcal{M}_{\mathcal{G}}$ is *asymptotically stable* if ψ_0 close to $\mathcal{M}_{\mathcal{G}}$ in a suitable norm implies that $\psi(t)$ remains close to and *converges to* $\mathcal{M}_{\mathcal{G}}$ (in a possibly different norm), as t tends to infinity.

Are solitary waves *asymptotically stable*? This is a local variant of the problem of *asymptotic resolution* [Tao 2008], that is, whether general initial conditions resolve into stable nonlinear bound states of the system plus dispersive radiation. A great deal of progress has been made on this problem in recent years. The study of asymptotic stability of solitary waves was initiated in [Soffer and Weinstein 1990; 1992]; see also [Buslaev and Perel′man 1992; Pillet and Wayne 1997; Gustafson et al. 2004; Weder 2000]. In the translation invariant case, asymptotic stability was then investigated by [Buslaev and Perel′man 1995]. Asymptotic stability analysis requires two new analytical features: one dynamical systems and the other harmonic analysis / spectral theory.

First, since we do not know in advance which nonlinear ground state in $\mathcal{M}_{\mathcal{G}}$ emerges in the large time limit, a decomposition with flexibility allowing for the asymptotic soliton to dynamically emerge is required[3]. To this end, the solution is decomposed in terms of a motion along the soliton manifold and components symplectic orthogonal or biorthogonal to it. Dynamics along the soliton manifold, $\mathcal{M}_{\mathcal{G}}$, are governed by *modulation equations*; see, for example, [Weinstein 1985; Fröhlich et al. 2004; Holmer and Zworski 2007].

Secondly, in order to prove convergence to the soliton manifold $\mathcal{M}_{\mathcal{G}}$, we need to show that the deviation of the solution from $\mathcal{M}_{\mathcal{G}}$ decays with advancing time. This requires time-decay estimates (L^p, weighted $L^2(\mathbb{R}^d)$ or space-time norms) for the linearized (about the soliton) propagator on the subspace symplectic orthogonal or biorthogonal to the discrete spectral subspace. The discrete subspace is the union of a *zero frequency mode subspace* spanned by infinitesimal generators of the NLS/GP symmetries (translation, gauge) acting on ϕ^λ, and often a subspace of *neutral modes* (sometimes called internal modes) with nonzero frequencies.

Since a typical perturbation of the ground state solitary wave in $\mathcal{M}_{\mathcal{G}}$ excites all discrete spectral components, one must understand the mechanisms, due to which these do not interfere with the asymptotic convergence of $\psi(x,t)$ to $\mathcal{M}_{\mathcal{G}}$. In brief: Concerning the zero modes, the choice of modulation equations

[2] $\mu=-\lambda$ is the typical definition of soliton frequency. Therefore the slope condition (1-4) often appears as a rate of change with respect to μ being negative.

[3] The case of integrable systems, such as one-dimensional NLS $V=0$, $f(|\psi|^2)\psi=|\psi|^2\psi$ is an important class for which it *is* possible to determine the emerging coherent structures from the *scattering transform* of the initial data.

"quotients out" the zero modes; perturbations exciting these induce motion *along* the soliton manifold. And concerning the nonzero frequency neutral modes, these are shown to damp to zero, as $t \to \infty$, due to the resonant nonlinear coupling of discrete to radiation modes. Related to this is a further dynamical systems aspect of the analysis. The neutral mode amplitudes are governed by nonlinear oscillator equations, coupled to a dispersive wave field. Near-identity changes of variables are used to put the system in an appropriate normal form, wherein the mechanism of energy transfer from the neutral modes to the evolving soliton and propagating radiation is made explicit. Energy transfer shows up as an explicit (nonlinear) damping term in the normal form; see the discussion below. The positive damping coefficient (matrix, in the present work) is a nonlinear variant of Fermi Golden Rule [Cohen-Tannoudji et al. 1992]. See [Buslaev and Perel′man 1995] regarding the dynamics near solitary waves of the translation invariant NLS equations and [Soffer and Weinstein 1999] for "breathers" of a class of nonlinear wave equations. In [Soffer and Weinstein 2004] this mechanism was proved to be responsible for *ground state selection* in NLS/GP equations; see also [Weinstein 2006]. Experimental verification of the prediction in [Soffer and Weinstein 2004; 2005] is reported in [Mandelik et al. 2005]. Related work on resonant radiation damping appears in [Tsai and Yau 2002b; 2002c; Buslaev and Sulem 2003; Tsai 2003; Cuccagna et al. 2006; Cuccagna and Mizumachi 2008]. The role of the Fermi Golden Rule in the nonpersistence of coherent structures for nonlinear wave equations was first demonstrated, via Floquet analysis, in [Sigal 1993]. There is a close relation to the perturbation theory of embedded eigenvalues for linear problems [Reed and Simon 1979; Soffer and Weinstein 1998; Cuccagna et al. 2005].

The above works on nonlinear resonance required that the neutral modes frequencies (a) lie sufficiently close to the essential spectrum and (b) are of geometric multiplicity one. For example, for the cubic nonlinearity, $f(|\psi|^2) = -g|\psi|^2$, *close* means that coupling to radiation modes occurs at order $|g|^2$. The situation where simple neutral modes are with a large spectral gap has been studied in [Gang and Sigal 2006; 2007; Gang 2007; Cuccagna and Mizumachi 2008; Cuccagna 2008]. Here, coupling of the discrete to continuous modes occurs at some high order in g. Thus, the normal form expansion gives a damping term at some even order $|g|^{2k}$ with $k \geq 2$.

Results of this paper — systems with degenerate neutral modes. An important situation, not covered by previous results, is the dynamics in the presence of degenerate neutral modes. This case arises naturally in systems of spatial dimensions $d \geq 2$ with symmetry. For example, if the potential is spherically symmetric, $V = V(|x|)$, then the first and higher excited states are degenerate, with the degree of degeneracy related to the order of the associated spherical harmonics. Another interesting class of examples is a class of multiwell potentials; see Appendix A.

In this paper we prove the asymptotic stability of the ground state / soliton manifold, $\mathcal{M}_{\mathcal{G}}$, of NLS/GP when the linearized spectrum has degenerate neutral modes. We show that the solution has three *interacting* parts:

(i) a modulating soliton, parametrized by the motion along $\mathcal{M}_{\mathcal{G}}$,

(ii) oscillatory, spatially localized, neutral modes, which decay with time and

(iii) a dispersive part, which decays in a local energy norm.

The neutral modes and dispersive waves decay via transferring their mass to the soliton manifold or to spatial infinity. Additionally, degenerate neutral modes are coupled and exchange mass among themselves in addition to with the soliton and radiation. These degenerate modes cannot be viewed as very weakly coupled "oscillators" [Tsai 2003]. We require instead a new normal form expansion. This is related to ideas developed in [Kirr and Weinstein 2001], where a parametrically forced linear Hamiltonian PDE was considered, and a normal form, uniform in discrete eigenvalue spacing, was required.

We outline the perspective we take and give a rough form of the main theorem, Theorem 7.1. Consider NLS/GP, where $-\Delta + V$ has a ground state $\xi_0(x) > 0$, whose energy $e_0 < 0$, and a *degenerate* excited state, whose energy e_1 with $e_0 < e_1 < 0$ is assumed sufficiently close to zero. Typical solutions of the linear Schrödinger equation, evolving from localized initial data ψ_0,

$$\psi(t) = \exp(-i(-\Delta + V)\,t)\psi_0$$

will be a time-quasiperiodic superposition of spatially localized ground state and time-periodic excited states, plus a part which disperses to zero, that is, tends to zero as t advances in L^2_{loc}. This picture emerges from the spectral decomposition of $-\Delta + V$ in L^2, with respect to which the bound state projections of the solution evolve as independent *oscillators* and the continuous spectral part of the solution has a character, qualitatively like a solution to the free Schrödinger equation.

For NLS/GP, for example $-g|\psi|^2\psi$ with $g \neq 0$, the dynamics of discrete and continuous modes are coupled. We consider an appropriate open set of initial conditions near the soliton manifold. *In contrast to the linear Schrödinger equation*, we show that the solution converges to a nonlinear ground state. To see this, we view NLS/GP as a infinite-dimensional Hamiltonian system comprising two subsystems: (i) a finite-dimensional system governing dynamics on the soliton manifold $\mathcal{M}_{\mathcal{I}}$, parametrized by $(\lambda(t), \gamma(t))$, the zero modes amplitudes (a_1, a_2) and the neutral mode amplitudes $z = (z_1, z_2, \ldots, z_n)^T$; (ii) an infinite-dimensional dispersive Schrödinger wave equation. A very detailed analysis of this coupled system (the bulk of this paper) yields the following (rough) form for the asymptotic behavior of small amplitude solutions of NLS/GP:

Main Theorem. *Consider the initial value problem for NLS/GP. Assume arbitrary localized initial data, which are sufficiently near a small amplitude nonlinear bound state ϕ^{λ_0}. Then the solution of NLS/GP evolves as a modulated soliton plus decaying error in the following form*:

$$\psi(t) = \exp\Big(i\int_0^t \lambda(s)\,ds\Big)\cdot\exp\big(i\big(\gamma(t) + a_2(z(t),\bar z(t))\big)\big)\cdot\big(\phi^{\lambda(t)+a_1(z(t),\bar z(t))} + \mathbb{O}(|z(t)|) + R(t)\big),$$

where $\lambda(t) \to \lambda_\infty$, $\mathbb{O}(|z(t)|)$ represents a localized nonspreading decaying part satisfying

$$|z(t)| \le C\langle t\rangle^{-1/2},$$

$a_j = a_j(z,\bar z) = \mathbb{O}(|z|^2)$ and $R(t)$ represents a spreading dispersively decaying part and tends to zero as $t \to \infty$ in L^2_{loc}, more precisely $\|\langle x\rangle^{-\nu} R(t)\|_2 \to 0$ with $\nu > 0$.

For the precise statement, see Theorem 7.1.

A key part of the proof of this theorem is to show that $|z(t)|$ tends to zero and that $\lambda(t)$ has a limiting value $\lambda_\infty \in \mathcal{I}$ as t tends to infinity. We prove the latter by showing $\partial_t\lambda(t) \in L^1(\mathbb{R}^+)$. We have two comments on the approach of this article to these issues:

New normal form. We show that there exist a *nonnegative symmetric matrix* $\Gamma(z,\bar z) = \mathbb{O}(|z|^2)$ and a skew symmetric matrix $\Lambda(z,\bar z) = \mathbb{O}(|z|^2)$ (see (7-3) below) such that

$$\partial_t z = -iE(\lambda)z - \Gamma(z,\bar z)z + \Lambda(z,\bar z)z + \mathbb{O}\big((1+t)^{-\frac{3}{2}-\delta}\big), \tag{1-5}$$

with $\delta > 0$. The matrix Γ is defined in terms of the spectral decomposition of the $L(\lambda) = JH(\lambda)$, the generator of the linearized flow about the nonlinear bound state ϕ^λ; see Section 5. Our analysis requires that $\Gamma = \Gamma(z,\bar z;\lambda)$ is positive-definite for an open λ-interval. A variant of this hypothesis appears in [Soffer and Weinstein 2004; Tsai and Yau 2002b; 2002c; Buslaev and Sulem 2003; Tsai 2003; Gang and Sigal 2006; 2007; Cuccagna et al. 2006; Cuccagna and Mizumachi 2008]. It is expected to hold, in some sense, generically. In Section 6 we state a hypothesis under which positive-definiteness holds for a class of potentials of multiwell type, constructed in Appendix A. This hypothesis, denoted (FGR) (see also Theorem 6.1), is a nonlinear variant of the Fermi Golden Rule [Cohen-Tannoudji et al. 1992; Reed and Simon 1979; Soffer and Weinstein 1998]. We note that for finite-dimensional Hamiltonian systems a damping term is absent; it would violate phase-volume conservation. This term arises due to nonlinearity induced by the coupling between discrete and continuous (radiational) spectral modes, a phenomenon associated with continuous spectra, arising in PDEs on spatially infinite domains; see [Soffer and Weinstein 1999; Weinstein 2006]. We show that (1-5) and (FGR) imply the bound $|z(t)| = \mathbb{O}(t^{-1/2})$. For the case of multiple simple bound states with well-separated frequencies, a system of type (1-5) holds with Γ, a *diagonal* matrix [Tsai 2003]. Equation (1-5) can be viewed as a new normal form, a special case of one valid uniformly in neutral mode eigenfrequency-separation.

Choice of basis for the neutral mode subspace. We prove that $\lambda(t)$ approaches some λ_∞ as $t \to \infty$, by proving that $\partial_t\lambda(t)$ is integrable. If there are n simple well-separated neutral modes, one initially finds

$$\partial_t\lambda(t) = \sum_{m=1}^{n} a_m|z_m|^2 + \mathbb{O}(t^{-3/2}).$$

Since we expect $|z_m| = \mathbb{O}(t^{-1/2})$ we can not conclude integrability of $\partial_t\lambda(t)$. However, it can be shown that, after near identity change of variables $z \mapsto z + \mathbb{O}(|z|^2)$, we can take $a_m = 0$; see the normal form expansion in [Gang and Sigal 2006; 2007; Soffer and Weinstein 2004]. In the degenerate (similarly, not well-separated) case, $\lambda(t)$ satisfies:

$$\partial_t\lambda(t) = \sum_{m,k} a_{m,k} z_m \bar z_k + \mathbb{O}(t^{-3/2}).$$

In the present paper we show very generally that, by appropriate choice of neutral subspace basis, we can take $a_{m,k} = 0$.

Finally, we expect that our techniques can be extended to more complicated situations, for example, where coupling of neutral to continuum modes occurs at higher order in the nonlinearity.

Outline of the paper. The paper is organized as follows. Section 2 displays notation which is often used. Section 3 is a brief section outlining structural properties of NLS/GP and gives a statement of a basic well-posedness result. Section 4 introduces solitary waves (solitons) in the regime of weak nonlinearity. Section 5 has a detailed discussion of the spectral properties of $L(\lambda) = JH(\lambda)$, the generator of the linearized dynamics about the soliton: zero energy subspace, degenerate neutral subspace and continuous spectral subspace. Projections associated with theses subspaces are defined and decay estimates of the linearized evolution on the continuous spectral subspace are recalled. In Section 6 the Fermi Golden Rule matrix Γ is introduced explicitly in Theorem 6.1. The detailed calculations, proving the symmetry and nonnegativity, are given in Appendix B. Note that the main theorem requires positive-definiteness of Γ. Proposition 6.2 is a result reducing the required positive-definiteness to a condition involving the spectral properties of $-\Delta + V$. Section 7 contains a statement of the main theorem, Theorem 7.1. In Section 8 we give a more precise formulation of Theorem 7.1. This formulation makes explicit the dynamical (modulation) equations for the solitary wave parameters, the neutral mode amplitudes and the dispersive part. These are proved via normal form methods in Sections 9 and 10. In Section 11 we prove the reformulated Theorem 7.1 in the setting of Theorem 8.1. Appendix contains some important calculations used in the body of the paper. Of particular interest is Appendix A, where a class of multiwell three-dimensional potentials is constructed, to which we apply Theorem 7.1.

2. Notation

(1) $\alpha_+ = \max\{\alpha, 0\}$ and $[\tau] = \max_{\tilde{\tau}\in Z}\{\tilde{\tau} \le \tau\}$.

(2) $\Re z$ denotes the real part of z and $\Im z$ the imaginary part of z.

(3) Multiindices:

$$w = (w_1, \ldots, w_N), \quad \bar{w} = (\bar{w}_1, \ldots, \bar{w}_N) \in \mathbb{C}^N,$$
$$a = (a_1, \ldots, a_N) \in \mathbb{N}^N, \quad z^a = z_1^{a_1} \cdots z_N^{a_N}, \quad |a| = |a_1| + \cdots + |a_N|,$$

where z denotes the vector of *neutral mode* amplitudes, ξ denotes the vector whose j-th entry ξ_j is the j-th neutral vector-mode of $JL(\lambda)$.

(4) $Q_{m,n}$ denotes an expression of the form

$$Q_{m,n} = \sum_{\substack{|a|=m\\|b|=n}} q_{a,b} z^a \bar{z}^b = \sum_{\substack{|a|=m\\|b|=n}} q_{a,b} \prod_{k=1}^{N} z_k^{a_k} \bar{z}_k^{b_k}.$$

(5)
$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} L_+ & 0 \\ 0 & L_- \end{pmatrix}, \quad L = JH = \begin{pmatrix} 0 & L_- \\ -L_+ & 0 \end{pmatrix}.$$

(6) $\sigma_{\mathrm{ess}}(L) = \sigma_c(L)$ is the essential (continuous) spectrum and $\sigma_{\mathrm{disc}}(L)$ the discrete spectrum of L.

(7) Riesz projections:

$$P_c(L) = I - P_{\mathrm{disc}}(L),$$

where $P_{\mathrm{disc}}(L)$ projects to the discrete spectral of L and $P_c(L)$ to the continuous spectral of L.

(8)
$$\langle f, g\rangle = \int f(x)\overline{g(x)}\,dx.$$

(9)
$$\|f\|_p^p = \int_{\mathbb{R}^d} |f(x)|^p\,dx, \qquad 1 \le p \le \infty.$$

(10)
$$\|f\|_{H^{s,\nu}}^2 = \int_{\mathbb{R}^d} \left|\langle x\rangle^{\nu}(I-\Delta)^{s/2} f(x)\right|^2 dx.$$

3. Hamiltonian structure

NLS/GP can be expressed as a Hamiltonian system

$$i\partial_t \psi = \frac{\delta \mathscr{E}[\psi, \bar{\psi}]}{\delta \bar{\psi}}$$

where the Hamiltonian energy $\mathscr{E}[\cdot]$ is defined by

$$\mathscr{E}[\psi] = \mathscr{E}[\psi, \bar{\psi}] = \int \left(\tfrac{1}{2}\nabla\psi \cdot \nabla\bar{\psi} + \tfrac{1}{2}V(x)\psi\bar{\psi} - F(\psi\bar{\psi})\right) dx$$

with

$$F(u) = \tfrac{1}{2}\int_0^u f(\xi)\, d\xi.$$

Equation (1-1) is a Hamiltonian system on Sobolev space $H^1(\mathbb{R}^d, \mathbb{C})$ viewed as a real space

$$H^1(\mathbb{R}^d, \mathbb{R}) \oplus H^1(\mathbb{R}^d, \mathbb{R}),$$

that is,

$$H^1(\mathbb{R}^d, \mathbb{C}) \ni f \leftrightarrow (\Re f, \Im f) \in H^1(\mathbb{R}^d, \mathbb{R}) \oplus H^1(\mathbb{R}^d, \mathbb{R}),$$

with the symplectic form

$$\omega(\psi, \phi) = \Im \int_{\mathbb{R}^d} \psi\bar{\phi}\, dx.$$

Equation (1-1) is invariant under time-translation and gauge-translation (phase-translation):

$$t \mapsto t + t_0, \qquad \phi \mapsto \phi e^{i\gamma}$$

with $\gamma \in \mathbb{R}$, yielding, by Noether's Theorem, the conservation laws of energy

$$\mathscr{E}[\psi(t)] = \mathscr{E}[\psi(0)]$$

and of particle number (optical power)

$$\mathcal{N}[\psi(t)] = \mathcal{N}[\psi(0)]$$

where

$$\mathcal{N}[\psi] = \int |\psi|^2\, dx.$$

Assumptions on the potential V and nonlinearity f

(fA) $f(\tau)$ is a smooth function satisfying $f(\tau) = \mathbb{O}(\tau)$ for $|x|$ is small. Thus, the nonlinearity in NLS is cubic at small amplitudes, that is, $f(|\psi|^2)\psi \sim g|\psi|^2\psi$.

(VA) V is smooth and decays exponentially as $|x|$ tends to ∞.

To ensure the global well-posedness of the initial value problem for (1-1) we impose:

(fB) Subcritical nonlinearity for large amplitudes

$$|f(\xi)| \le c(1+|\xi|^{\beta})$$

for some $\beta \in [0, 2/d)$ and

$$|f'(\xi)| \le c(1+|\xi|^{\alpha-1})$$

for some $\alpha \in [0, 2/(d-2)_+)$ where $s_+ = \max\{s, 0\}$.

The following well-posedness theorem can be found in [Cazenave 2003; Sulem and Sulem 1999].

Theorem 3.1. *Assume that the nonlinearity f satisfies the condition (fB), and the potential V satisfies (VA). Then (1-1) is globally well-posed in H^1, that is, the Cauchy problem for (1-1) with the initial data $\psi(0) \in H^1$ has a unique solution $\psi(t)$ in the space H^1, which depends continuously on $\psi(0)$. Moreover, the solution $\psi(t)$ satisfies conservation of energy and conservation of particle number.*

4. Bifurcation and Lyapunov stability of solitons in the weakly nonlinear regime

In this section we discuss the existence of solitons in the weakly nonlinear regime. The following arguments are similar to those in [Rose and Weinstein 1988; Tsai and Yau 2002c] except that the excited states are degenerate. We assume that the linear operator $-\Delta+V$ has the following properties:

(Eig$_V$) The linear operator $-\Delta+V$ has two eigenvalues $e_0 < e_1 < 0$ with $2e_1 > e_0$. e_0 is the lowest eigenvalue with ground state $\phi_{\text{lin}} > 0$. The eigenvalue e_1 is degenerate with multiplicity N and eigenfunctions $\xi_1^{\text{lin}}, \xi_2^{\text{lin}}, \dots, \xi_N^{\text{lin}}$.

Remark. In Appendix A we construct a class of double-well examples V for $d = 3$ and with multiplicity $N = 2$.

The following result shows that nonlinear bound state solutions (ϕ^λ, λ) of NLS/GP (1-2) bifurcate from the zero state and the linear ground state energy $(0, \lambda = -e_0)$.

Proposition 4.1. *Suppose $-\Delta+V$ satisfies the conditions in (Eig$_V$). Then there exists a constant $\delta_0 > 0$ and a nonempty interval $\mathcal{I}_{\delta_0} \subset [-e_0-\delta_0, -e_0+\delta_0]$ such that for any $\lambda \in \mathcal{I}_{\delta_0}$ (1-1) has solutions of the form*

$$\psi(x,t) = e^{i\lambda t}\phi^\lambda \in L^2$$

with

$$\phi^\lambda = \delta(\lambda)\cdot\big(\phi_{\text{lin}} + \mathcal{O}(\delta(\lambda))\big), \qquad \delta(\lambda) = \mathcal{O}\Big(\big|\lambda-|e_0|\big|^{1/2}\Big)$$

for $\big|\lambda-|e_0|\big|$ small. Moreover, for some $c > 0$ independent of λ,

$$|\phi^\lambda(x)| \le ce^{-c|x|}, \qquad |\partial_\lambda\phi^\lambda(x)| \le ce^{-c|x|},$$

and similarly for the spatial derivatives of ϕ^λ and $\partial_\lambda\phi^\lambda$.

Remark. Suppose $f(|\psi|^2)\psi = -g|\psi|^2 + o(|\psi|^2)$. Then for $g > 0$ (repulsive case) we have

$$\mathcal{I}_{\delta_0} = (-e_0, -e_0 + \delta_0)$$

and for $g < 0$ (attractive case) we have

$$\mathcal{I}_{\delta_0} = (-e_0 - \delta_0, -e_0).$$

Finally, we conclude this section by noting that for $\delta' \leq \delta_0$ sufficiently small that soliton manifold $\mathcal{M}_{\delta'}$ (see (1-3)) is H^1 orbitally stable; see the discussions in the introduction and [Weinstein 1986; Rose and Weinstein 1988; Grillakis et al. 1987].

5. $L(\lambda) = JH(\lambda)$, the linearized operator about ϕ^λ

We now turn to a discussion of the operator obtained by linearization around the soliton and the existence of neutral modes with nonzero frequencies. Rewrite (1-1) as

$$\frac{\partial \psi}{\partial t} = G(\psi),$$

where the nonlinear map $G(\psi)$ is defined by

$$G(\psi) = -i(-\Delta + \lambda + V)\psi + if(|\psi|^2)\psi.$$

Then the linearization of (1-1) can be written as

$$\frac{\partial \chi}{\partial t} = dG(\phi^\lambda)\chi,$$

where $dG(\phi^\lambda)$ is the Fréchet derivative of $G(\psi)$ at ϕ^λ. It is computed to be

$$dG(\phi^\lambda)\chi = -i(-\Delta + \lambda + V)\chi + if[(\phi^\lambda)^2]\chi + if'[(\phi^\lambda)^2](\phi^\lambda)^2(\chi + \bar{\chi}).$$

This operator is real linear but not complex linear. To convert it to a complex linear operator we pass from complex functions to real vector-functions

$$\chi \longleftrightarrow \vec{\chi} = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}$$

where $\chi_1 = \Re\chi$ and $\chi_2 = \Im\chi$. Then $dG(\phi^\lambda)\chi \longleftrightarrow L(\lambda)\vec{\chi}$ where the operator $L(\lambda)$ is given by

$$L(\lambda) = JH(\lambda) \tag{5-1}$$

where J is a skew-symmetric matrix

$$J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and $H(\lambda)$ is a selfadjoint matrix

$$H(\lambda) := \begin{pmatrix} L_+(\lambda) & 0 \\ 0 & L_-(\lambda) \end{pmatrix}$$

with

$$L_-(\lambda) := -\Delta + \lambda + V - f[(\phi^\lambda)^2]$$

and
$$L_+(\lambda) := -\Delta + \lambda + V - f[(\phi^\lambda)^2] - 2f'[(\phi^\lambda)^2](\phi^\lambda)^2.$$
We extend the operator $L(\lambda)$ to the complex space $H^2(\mathbb{R}^d, \mathbb{C}) \oplus H^2(\mathbb{R}^d, \mathbb{C})$.

5A. ***The spectrum of $L(\lambda)$.*** The operator $L(\lambda)$ has neutral modes.

Proposition 5.1. *Let $L(\lambda)$, or more explicitly, $L(\lambda(\delta), \delta)$ denote the linearized operator about the bifurcating state ϕ^λ, $\lambda = \lambda(\delta)$. Note that $\lambda(0) = -e_0$. Corresponding to the degenerate energy value e_1 of $-\Delta + V$, the matrix operator*
$$L(\lambda = -e_0, \delta = 0)$$
has degenerate eigenvalues $\pm i E(-e_0) = \pm i(e_1 - e_0)$, each with multiplicity N. For $\delta > 0$ and small, these bifurcate to (possibly degenerate) eigenvalues $\pm i E_1(\lambda), \dots, \pm i E_N(\lambda)$ with eigenfunctions
$$\begin{pmatrix} \xi_1 \\ \pm i\eta_1 \end{pmatrix}, \begin{pmatrix} \xi_2 \\ \pm i\eta_2 \end{pmatrix}, \dots, \begin{pmatrix} \xi_N \\ \pm i\eta_N \end{pmatrix}$$
with
$$\langle \xi_m, \eta_n \rangle = \delta_{m,n}$$
and
$$0 \neq \lim_{\lambda \to e_0} \xi_j = \lim_{\lambda \to e_0} \eta_j \in \operatorname{span}\{\xi_j^{\text{lin}}, j = 1, 2, \dots, N\} \subset H^k \text{ for any } k > 0.$$
Moreover, for δ sufficiently small $2E_j(\lambda) > \lambda$ for $j = 1, 2, \dots, N$ (nonlinear coupling of discrete to continuous spectrum at second order).

For the case of a radial potential $V = V(|x|)$, the neutral modes have the following structure:

Proposition 5.2. *If the potential is radial $V = V(|x|)$, then ϕ^λ, hence $\partial_\lambda \phi^\lambda$, is spherically symmetric. If the degenerate linear excited states ξ_n^{lin} are of the form $\xi_j^{\text{lin}} = \frac{x_j}{|x|}\xi^{\text{lin}}(|x|)$ for some function ξ^{lin}, then $E_j = E_1$ for any $j = 1, 2, \dots, N = d$ and we can choose ξ_j and η_j such that $\xi_j = \frac{x_j}{|x|}\xi(|x|)$ and $\eta_j = \frac{x_j}{|x|}\eta(|x|)$ for some real functions ξ and η.*

Remark. For $d = 3$, the hypothesis on the linear excited states says that they are proportional to $\xi^{\text{lin}}(|x|)Y_1^m(\theta, \phi)$ for $m = -1, 0, 1$, where Y_1^m are the spherical harmonics of degree one.

Sketch of proof. If V is spherically symmetric then by the uniqueness of the ground state and the fact $-\Delta + V$ is invariant under unitary transformations we have ϕ^λ, hence $\partial_\lambda \phi^\lambda$ is spherically symmetric.

We now outline a proof of the existence of ξ_j and η_j with desired structure. Define a linear space
$$\mathcal{Y}^k = \left\{ J \in H^k, J(x) = \frac{x_1}{|x|} g(|x|) \right\}.$$
By definition $L(\lambda) : \mathcal{Y}^2 \to \mathcal{Y}^0$. Note that, restricted to $\mathcal{Y}^2$, $\frac{x_1}{|x|}\xi^{\text{lin}}(|x|)$ is an eigenfunction of $-\Delta + V$ of multiplicity one. Applying the bifurcation theory to $\mathcal{Y}^2$, we prove there exists an eigenfunction $(\xi_1, i\eta_1)^T \in \mathcal{Y}^2$ with eigenvalue E_1. The other eigenfunctions with the same eigenvalue are obtained by noting that this computation can be carried out for any x_j with $j = 1, \dots, d$. □

Based on the above discussion, we assume:

(SA) Structure of the discrete spectrum of $L(\lambda) = JH(\lambda)$.

(1) $\sigma_d(L(\lambda))$ consists of an eigenvalue at 0 and complex conjugate eigenvalues at $\pm iE(\lambda)$.

(2) The discrete subspace, corresponding to the eigenvalue 0, is spanned by the associated eigenfunctions
$$\begin{pmatrix} 0 \\ \phi^\lambda \end{pmatrix}, \qquad \begin{pmatrix} \partial_\lambda \phi^\lambda \\ 0 \end{pmatrix}.$$

(3) The discrete subspace, corresponding to the eigenvalue $iE(\lambda)$ with $E(\lambda) > 0$, is N-dimensional and is spanned by the (complex) eigenfunctions $v_1, v_2, \ldots, v_N$.

(4) Thus, $\overline{v_1}, \overline{v_2}, \ldots, \overline{v_N}$ are the eigenfunctions which span the discrete subspace corresponding to the eigenvalue $-iE(\lambda)$.

(5) Moreover we observe that Jv_n are eigenfunctions of the adjoint operator $L(\lambda)^*$ with eigenvalue $-iE(\lambda)$:
$$L(\lambda)^* Jv_n = -JL(\lambda)v_n = -iE(\lambda)Jv_n.$$

Concerning the continuous spectrum of $L(\lambda)$, we apply Weyl's Theorem to the stability of the essential spectrum for localized perturbations of $J(-\Delta)$ [Hislop and Sigal 1996; Reed and Simon 1979] to obtain
$$\sigma_{\text{ess}}(L(\lambda)) = (-i\infty, -i\lambda] \cup [i\lambda, i\infty)$$
if the potential V in (1-1) decays sufficiently rapidly as $|x|$ tends to infinity.

The end points of the essential spectrum are called threshold energies.

Definition 5.3. Let $d \geq 3$. A function h is called a *threshold resonance function* of $L(\lambda)$ at $\mu = \pm i\lambda$, the endpoints of the essential spectrum, if $h \notin L^2$, $|h(x)| \leq c\langle x\rangle^{-(d-2)_+}$ and h is C^2 and solves the equation
$$(L(\lambda) - \mu)h = 0.$$

In this paper we make the following spectral assumption on the thresholds $\pm i\lambda$:

(Thresh$_\lambda$) There exists δ' with $0 < \delta' \leq \delta_0$ (see Proposition 4.1) such that for $\lambda \in \mathcal{I}_{\delta'}$, $L(\lambda)$ has no threshold resonances at $\pm i\lambda$.

In the weak amplitude limit, property (Thresh$_\lambda$) can be referred to the question of whether the scalar operator $-\Delta + V(x)$ has a threshold (zero energy) resonance. In [Jensen and Kato 1979] it was shown that $-\Delta + V$ has a zero energy resonance or eigenvector if and only if the operator
$$I + (-\Delta + i0)^{-1}V : \langle x\rangle^2 L^2 \to \langle x\rangle^2 L^2$$
is not invertible. Moreover, this operator is generically invertible. That is, if we replace V by qV where q is a real number, then we have noninvertibility for only a discrete set of q values [Rauch 1978; Jensen and Kato 1979].

The reduction from the properties of $L(\lambda)$ to those of $-\Delta + V$ is seen as follows. Let
$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix},$$
so $U^*U = I$. Then,
$$\sigma_3 \mathcal{H}(\lambda) = -iU^* L(\lambda) U. \tag{5-2}$$
It follows that $\pm i\lambda$ are threshold resonances of $L(\lambda)$ if and only if $\pm\lambda$ are threshold resonances of $\sigma_3\mathcal{H}(\lambda)$.

We next observe that $\sigma_3\mathcal{H}$ is a small perturbation of $\sigma_3(-\Delta+V+\lambda)$. Indeed, a computation of $\sigma_3\mathcal{H}(\lambda)$ yields

$$\sigma_3\mathcal{H} = \sigma_3(\mathcal{H}_0 + V) + V_{\text{small}}$$

where

$$\sigma_3\mathcal{H}_0 := (-\Delta+\lambda)\sigma_3, \qquad |V_{\text{small}}| \le e^{-c|x|}o(1)$$

for some $c > 0$, where $o(1) \to 0$ as $\big|\lambda - |e_0|\big| \to 0$.

Therefore, the generic validity of (Thresh$_\lambda$) follows from the generic absence of zero energy threshold resonances for $-\Delta + V$ by the following result proved for $d = 3$ using the results in [Cuccagna et al. 2005]. The proof for general dimensions is similar.

Proposition 5.4. *Let $d = 3$. If the operator*

$$I + (-\Delta + i0)^{-1}V : \langle x\rangle^2 L^2 \to \langle x\rangle^2 L^2$$

is invertible, then (Thresh$_\lambda$) holds when $\big|\lambda - |e_0|\big|$ is sufficiently small.

Proof. We begin by proving that the operator

$$I + (\sigma_3\mathcal{H}_0 \pm \lambda + i0)^{-1}(\sigma_3 V + V_{\text{small}}) : \langle x\rangle^2 L^2 \to \langle x\rangle^2 L^2$$

is invertible. Observe that $-2\lambda \approx 2e_0$ is not an eigenvalue of the operator $-\Delta+V$ so $I+(-\Delta+2\lambda)^{-1}V$ is invertible. This, together with the hypothesis, implies that $I + (\sigma_3\mathcal{H}_0 \pm \lambda + i0)^{-1}\sigma_3 V$ is invertible with a uniformly bounded inverse. On the other hand the norm of the operator $(\sigma_3\mathcal{H}_0 \pm \lambda + i0)^{-1}V_{\text{small}}$ is small when $|e_0 + \lambda|$ is small. Hence

$$I+(\sigma_3\mathcal{H}_0\pm\lambda+i0)^{-1}(\sigma_3 V+V_{\text{small}}) = (I+(\sigma_3\mathcal{H}_0\pm\lambda+i0)^{-1}\sigma_3 V)\big(1+(1+(\sigma_3\mathcal{H}_0\pm\lambda+i0)^{-1}\sigma_3 V)^{-1}V_{\text{small}}\big)$$

is invertible when $\big|\lambda - |e_0|\big|$ is small. Moreover in [Cuccagna et al. 2005] it is proved that the operator $L(\lambda)$ has no threshold resonance functions if the operator

$$I + (\sigma_3\mathcal{H}_0 \pm \lambda + i0)^{-1}(\sigma_3 V + V_{\text{small}}) : \langle x\rangle^2 L^2 \to \langle x\rangle^2 L^2$$

is invertible. This completes the proof. □

Choice of basis for degenerate subspaces. In our analysis, it is important that we choose an appropriate basis of the degenerate eigenspaces corresponding to $\pm iE(\lambda)$. We present this choice of basis and its construction here.

Proposition 5.5. *There exist real functions ξ_n, η_n for $n = 1, 2, \ldots, N$ such that*

$$\operatorname{span}\left\{\begin{pmatrix}\xi_n\\ i\eta_n\end{pmatrix}\right\} = \operatorname{span}\{v_1, v_2, \ldots, v_N\}$$

and for any $m, n \in \{1, 2, \ldots, N\}$,

$$\int f'[(\phi^\lambda)^2](\phi^\lambda)^2(\xi_m\eta_n - \xi_n\eta_m)\,dx = 0 \tag{5-3}$$

and

$$\langle\phi^\lambda, \xi_n\rangle = \langle\partial_\lambda\phi^\lambda, \eta_n\rangle = 0, \qquad \langle\xi_n, \eta_m\rangle = \delta_{m,n}. \tag{5-4}$$

The proof is given in Appendix D.

Remark. If ϕ^λ is spherically symmetric, then

$$\xi_n = \frac{x_n}{|x|}\xi(|x|), \qquad \eta_n = \frac{x_n}{|x|}\eta(|x|)$$

for $n \in \{1, 2, \dots, N = d\}$; see Proposition 5.2). Therefore (5-3) trivially holds because $\xi_m\eta_n - \xi_n\eta_m = 0$.

We conclude this section with the explicit form of the projection P_{disc}, whose proof for dimension one can be found in [Gang and Sigal 2005]. The proof for general dimensions is similar, and hence omitted. Recall that $\langle \xi_m, \eta_n\rangle = \delta_{m,n}$.

Proposition 5.6. *For the nonselfadjoint operator $L(\lambda)$, the (Riesz) projection onto the discrete spectrum subspace of $L(\lambda)$, $P_{\text{disc}} = P_{\text{disc}}(L(\lambda)) = P^\lambda_{\text{disc}}$, is given by*

$$P_{\text{disc}} = \frac{2}{\partial_\lambda \|\phi^\lambda\|^2}\left(\left| \begin{matrix} 0 \\ \phi^\lambda \end{matrix} \right\rangle \left\langle \begin{matrix} 0 \\ \partial_\lambda\phi^\lambda \end{matrix} \right| + \left| \begin{matrix} \partial_\lambda\phi^\lambda \\ 0 \end{matrix} \right\rangle \left\langle \begin{matrix} \phi^\lambda \\ 0 \end{matrix} \right| \right) - \frac{i}{2}\sum_{n=1}^{N}\left(\left| \begin{matrix} \xi_n \\ i\eta_n \end{matrix} \right\rangle \left\langle \begin{matrix} -i\eta_n \\ \xi_n \end{matrix} \right| - \left| \begin{matrix} \xi_n \\ -i\eta_n \end{matrix} \right\rangle \left\langle \begin{matrix} i\eta_n \\ \xi_n \end{matrix} \right| \right).$$

We define the projection onto the continuous spectral subspace of $L(\lambda)$ by

$$P_c = P_c(L(\lambda)) = P_c^\lambda \equiv I - P_{\text{disc}}. \tag{5-5}$$

5B. ***Estimates of the propagator.*** We will need estimates of the evolution operator $U(t) := e^{tL(\lambda_1)}$ for $\lambda_1 \in \mathcal{I}$. Recall that $L(\lambda_1)$ has two branches of essential spectrum: $[i\lambda_1, i\infty)$ and $(-i\infty, -i\lambda_1]$. We denote by $P_+ = P_+^{\lambda_1}$ and $P_- = P_-^{\lambda_1}$ the spectral projections associated with these two branches of the essential spectrum. Hence, $P_c^{\lambda_1} = P_+ + P_-$.

Theorem 5.7. *Let $d \geq 3$ and define $k := [\frac{d}{2}] + 1$ and $\nu := \frac{5+d}{2}$. Assume that $2E(\lambda_1) > \lambda$ so that $\pm 2iE(\lambda_1) \in \sigma_{\text{ess}}(L(\lambda_1))$. Then, for any time $t \geq 0$ and $\lambda_1 \in \mathcal{I}$ there exists a constant c such that*

$$\left\| \langle x\rangle^{-\nu}(-\Delta+1)^{k/2}U(t)\big(L(\lambda_1) \pm 2iE(\lambda_1) - 0\big)^{-n} P_\pm h\right\|_2 \leq c(1+t)^{-d/2}\|\langle x\rangle^{\nu}(-\Delta+1)^{k/2}h\|_2 \tag{5-6}$$

with $n = 0, 1, 2$. For any time $t \in (-\infty, \infty)$ and $\lambda_1 \in \mathcal{I}$ there exists a constant $C_{\mathcal{I}}$ such that

$$\|\langle x\rangle^{-\nu}(-\Delta+1)^{k/2}U(t)P_\pm h\|_2 \leq C_{\mathcal{I}}(1+|t|)^{-d/2}\|\langle x\rangle^{\nu}(-\Delta+1)^{k/2}h\|_2, \tag{5-7}$$

$$\|U(t)P_\pm h\|_\infty \leq C_{\mathcal{I}}|t|^{-d/2}\|h\|_1, \tag{5-8}$$

$$\|U(t)P_\pm h\|_\infty \leq C_{\mathcal{I}}(1+|t|)^{-d/2}(\|h\|_{H^k} + \|h\|_1), \tag{5-9}$$

$$\|U(t)P_\pm h\|_3 \leq C_{\mathcal{I}}(1+|t|)^{-d/6}(\|h\|_{H^k} + \|h\|_1), \tag{5-10}$$

$$\|\langle x\rangle^{-\nu}U(t)P_\pm h\|_2 \leq C_{\mathcal{I}}(1+|t|)^{-d/2}(\|h\|_1 + \|h\|_2). \tag{5-11}$$

We refer the proof of the estimates to [Soffer and Weinstein 1999; Gang and Sigal 2007; Tsai and Yau 2002a; Goldberg and Schlag 2004]. For the constant $C_{\mathcal{I}}$ can be taken uniformly for $\lambda_1 \in \mathcal{I}$, see [Cuccagna 2001; 2003].

6. Matrix Fermi Golden Rule

As highlighted in the introduction, the decay of neutral mode components, associated with the linearized NLS/GP equation, is necessary for asymptotic stability of the soliton manifold $\mathcal{M}_{\mathcal{G}}$. We shall prove that, after near-identity transformations, the system governing these neutral mode amplitudes is (1-5):

$$\partial_t z = -iE(\lambda)z - \Gamma(z,\bar{z})z + \Lambda(z,\bar{z})z + \mathbb{O}((1+t)^{-3/2-\delta}), \qquad \delta > 0,$$

where $\pm iE(\lambda)$ are complex conjugate N-fold degenerate neutral eigenfrequencies of $L(\lambda) = JH(\lambda)$, Γ is symmetric and Λ is skew symmetric. It follows that

$$\partial_t |z(t)|^2 = -2z^*\Gamma(z,\bar{z})z + \cdots . \tag{6-1}$$

Our strategy to show that $|z(t)|$ tends to zero is based on proving that Γ is positive-definite and that the corrections to (6-1) decay sufficiently rapidly as t tends to infinity. If $L(\lambda)$ has a complex conjugate pair of simple neutral eigenvalues, then Γ reduces to a nonnegative scalar. If $L(\lambda)$ has multiple, well-separated pairs of neutral modes, then Γ reduces to a diagonal matrix [Soffer and Weinstein 1999; 2004; Tsai and Yau 2002a; 2002b; 2002c; Tsai 2003; Buslaev and Sulem 2003]. The present case of problem of degenerate neutral modes is more involved due to coupling among the various discrete modes and with the continuous spectrum. Our computation yields a nondiagonal FGR matrix, Γ. In this section, we display the expression for Γ and state a result on its general properties. The detailed derivation of the expression for Γ is carried out in Section 10.

***The FGR matrix* $\boldsymbol{\Gamma(z,\bar{z})}$.** To construct Γ we must first introduce some notation.

Define vector functions G_k for $k = 1, 2, \ldots, N$ as

$$G_k(z,x) := \begin{pmatrix} B(k) \\ D(k) \end{pmatrix} \tag{6-2}$$

with the functions $B(k)$ and $D(k)$ defined as

$$\begin{aligned} B(k) &:= -if'[(\phi^\lambda)^2]\phi^\lambda((z\cdot\xi)\eta_k + (z\cdot\eta)\xi_k), \\ D(k) &:= -f'[(\phi^\lambda)^2]\phi^\lambda(3(z\cdot\xi)\xi_k - (z\cdot\eta)\eta_k) - 2f''[(\phi^\lambda)^2](\phi^\lambda)^3(z\cdot\xi)\xi_k, \end{aligned}$$

where

$$z\cdot\xi := \sum_{n=1}^{N} z_n\xi_n, \qquad z\cdot\eta := \sum_{n=1}^{N} z_n\eta_n.$$

In terms of the column 2-vector G_k, we define a $N \times N$ matrix $Z(z,\bar{z})$ as

$$Z(z,\bar{z}) = \big(Z^{(k,l)}(z,\bar{z})\big), \tag{6-3}$$

for $1 \le k, l \le N$, where

$$Z^{(k,l)}(z,\bar{z}) \equiv -\big\langle (L(\lambda) + 2iE(\lambda) - 0)^{-1} P_c G_l, iJG_k \big\rangle.$$

Since $P_c(L)^*J = JP_c(L)$, a consequence of $L = JH$ and $H^* = H$ (see (5-1) and Proposition E.1), we have the more symmetric expression for $Z^{(k,l)}$:

$$Z^{(k,l)} = -\big\langle (L(\lambda) + 2iE(\lambda) - 0)^{-1} P_c G_l, iJP_cG_k \big\rangle.$$

Finally, we define $\Gamma(z,\bar z)$ as

$$\Gamma(z,\bar z) := \tfrac{1}{2}(Z(z,\bar z) + Z^*(z,\bar z)).$$

Thus,

$$\big(\Gamma(z,\bar z)\big)_{kl} = -\Re\big\langle (L(\lambda) + 2iE(\lambda) - 0)^{-1} P_c G_l, iJP_cG_k\big\rangle.$$

Concerning the properties of Γ, we have the following general result:

Theorem 6.1 (Matrix nonlinear Fermi Golden Rule). (1) $\Gamma(z,\bar z) = \Gamma(z,\bar z;\lambda)$ *is a nonnegative symmetric $N\times N$ matrix.*

(2) *Define*

$$K(\lambda,\vec G) := \min_{s,z\neq 0} \frac{s^*\Gamma(z,\bar z)s}{|s|^2|z|^2},$$

where $\vec G = (G_1,\dots,G_N)$ defined in (6-2). Then, $K(\lambda,\vec G)$ depends continuously on λ and $\vec G$ (in the space $\langle x\rangle^3 L^\infty$).

We shall require the following *Fermi Golden Rule resonance condition*:

(FGR) There exists δ' with $0 < \delta' \le \delta$ (see Proposition 4.1) and a constant $C > 0$ such that for any $s = (s_1,\dots,s_N)^T$ and $z = (z_1,\dots,z_N)^T \in \mathbb{C}^N$, we have

$$s^*\Gamma(z,\bar z;\lambda)s \ge C|s|^2|z|^2,$$

where $\lambda \in \mathcal{I}_{\delta'}$.

Remark. In the weakly nonlinear regime (see Section 5A) $E(\lambda) \sim e_1 - e_0$, $\lambda \sim -e_0$ and therefore the condition for resonance with the continuous spectrum at second order is

$$2E(\lambda) - \lambda \sim 2(e_1 - e_0) + e_0 = 2e_1 - e_0 > 0.$$

Our next result is a reduction of the condition (FGR) for the class of multiwell potentials discussed in Appendix A to an explicit condition on the operator V.

Proposition 6.2. *Let V denote the multiwell potential satisfying condition (Eig$_V$) and constructed according to Appendix A. Thus, $-\Delta + V$ has two negative eigenvalues $e_0 < e_1 < 0$ with $2e_1 - e_0 > 0$. The excited state eigenvalue e_2 is degenerate of multiplicity $N = 2$ with spanning eigenfunctions $\{\xi_1^{\text{lin}}, \xi_2^{\text{lin}}\}$.*

Let $f(|\psi|^2) = -g|\psi|^2$. Assume the nonnegative matrix

$$\Big(\Im\big\langle(-\Delta + V - (2e_1 - e_0) - i0)^{-1} P_c\phi_{\text{lin}}\xi_m^{\text{lin}}\xi_n^{\text{lin}}, \phi_{\text{lin}}\xi_m^{\text{lin}}\xi_n^{\text{lin}}\big\rangle\Big)_{1\le m,n\le 2} \tag{6-4}$$

is positive-definite. Then there exists $\delta' > 0$ such that, for ϕ^λ denotes the soliton of Proposition 4.1, if $\big|\lambda - |e_0|\big| < \delta'$ then $K(\lambda,\vec G) > 0$. And the Fermi Golden Rule condition holds by taking

$$C = \inf_{\lambda\in\overline{\mathcal{I}_{\delta'}}} K(\lambda,\vec G(\lambda)) > 0$$

in (FGR). Here $\mathcal{I}_{\delta'}$ denotes a sufficiently small subinterval of the range of λ-values for which the soliton exists; see Proposition 4.1.

Remark. Positive-definiteness of the matrix in (6-4) is equivalent to

$$\Im\big\langle(-\Delta+V-(2e_1-e_0)-i0)^{-1}P_c\phi_{\rm lin}(z_1\xi_1^{\rm lin}+z_2\xi_2^{\rm lin})^2,\,\phi_{\rm lin}(z_1\xi_1^{\rm lin}+z_2\xi_2^{\rm lin})^2\big\rangle\ge C|z|^2,$$

for all $z_1, z_2 \in \mathbb{C}$.

Proof of Proposition 6.2. In what follows we sketch the proof, which is very similar to the case $N=1$ (see [Soffer and Weinstein 1999; Tsai and Yau 2002c]).

Recall the transformation of $L(\lambda)$ in (5-2):

$$\begin{aligned}
&(L(\lambda)+2iE(\lambda)-0)^{-1}\\
&\quad=\big(iU\sigma_3\mathscr{H}U^*+2iE(\lambda)-0\big)^{-1}=-iU(\sigma_3\mathscr{H}+2E(\lambda)+i0)^{-1}U^*\\
&\quad=-iU\big(\sigma_3(\mathscr{H}_0+V)+2E(\lambda)+i0\big)^{-1}U^*\\
&\qquad\qquad+iU\big(\sigma_3\mathscr{H}+2E(\lambda)+i0\big)^{-1}V_{\rm small}\big(\sigma_3(\mathscr{H}_0+V)+2E(\lambda)+i0\big)^{-1}U^*
\end{aligned}$$

and

$$\big(\sigma_3(\mathscr{H}_0+V)+2E(\lambda)+i0\big)^{-1}=\begin{pmatrix}\big(-\Delta+V-(-\lambda-2E(\lambda))\big)^{-1} & 0\\ 0 & -\big(-\Delta-(2E(\lambda)-\lambda)-i0\big)^{-1}\end{pmatrix}. \tag{6-5}$$

Furthermore, by Propositions 4.1 and 5.1 we have, in the space H^2, that

$$\frac{1}{\|\phi^\lambda\|_{H^2}}\phi^\lambda\to\frac{1}{\|\phi_{\rm lin}\|_{H^2}}\phi_{\rm lin},\qquad\Big(\frac{1}{\|\xi_n\|_{H^2}}\xi_n,\,\frac{1}{\|\eta_n\|_{H^2}}\eta_n\Big)\to\frac{1}{\|\xi_n^{\rm lin}\|_{H^2}}(\xi_n^{\rm lin},\xi_n^{\rm lin})$$

for some $\xi_n^{\rm lin}$ as $\big|\lambda-|e_0|\big|\to 0$. If the nonlinearity $f(\tau)=\tau^\sigma$ with $\sigma\ge 1$, we have

$$U^*P_c\sum_l z_lG_l=C\,\|\phi_{\rm lin}\|_{H^2}^{2\sigma-1}\begin{pmatrix} * \\ P_c\phi_{\rm lin}^{2\sigma-1}(z_1\xi_1^{\rm lin}+z_2\xi_2^{\rm lin})^2\end{pmatrix}(1+o(1))$$

for some constant $\tilde c\in\mathbb{C}$.

In considering (6-5), note that $-\lambda-2E(\lambda)\sim e_0-2(e_1-e_0)<0$ and $2E(\lambda)-\lambda\sim 2e_1-e_0<0$. Thus

$$\Im\big\langle(-\Delta+V+\lambda+2E(\lambda))^{-1}F,\,F\big\rangle=0$$

for any F. Furthermore, $\|e^{-\tau|x|}V_{\rm small}\|_{L^\infty}$ is small for some $\tau>0$, we have

$$K(\lambda,\vec G)=|\tilde c|^2\|\phi_{\rm lin}\|_{H^2}^{4\sigma-2}K_0\big(1+o(1)\big)$$

with

$$\begin{aligned}
K_0&:=(1+o(1))\\
&\qquad\times\Im\big\langle(-\Delta+V+e_0-2e_1-i0)^{-1}P_c(\phi_{\rm lin})^{2\sigma-1}(z_1\xi_1^{\rm lin}+z_2\xi_2^{\rm lin})^2,\,(\phi_{\rm lin})^{2\sigma-1}(z_1\xi_1^{\rm lin}+z_2\xi_2^{\rm lin})^2\big\rangle.
\end{aligned}$$

The proof is complete. In Appendix C we have a simpler formula for (FGR) when the potential V is spherical symmetric. □

The proof of Theorem 6.1 is deferred to Appendix B.

7. Main theorem

In this section we state precisely the main theorem of this paper. Recall the notations $\xi = (\xi_1, \dots, \xi_N)$ and $\eta = (\eta_1, \dots, \eta_N)$ for components of the neutrally stable modes of frequencies $\pm i E(\lambda)$ of the linearized operator. Recall the definition of the interval $\mathcal{I}$ in (1-4).

Theorem 7.1. *Assume (fA), (fB) on the nonlinearity $f(|\psi|^2)$* (page 274), *(VA) on the potential $V(x)$* (p. 274), *(SA) on the structure of the discrete spectral subspace of the linearization about ϕ^λ* (page 278), *(Thresh$_\lambda$) on the absence of threshold resonances* (page 278), *and (FGR), the nonlinear Fermi Golden Rule resonance condition* (page 282). *Fix $\nu > 0$ sufficiently large and let $k = [\frac{d}{2}]+1$ where $d \geq 3$ denotes the spatial dimension. Then there exist constants $c, \epsilon_0 > 0$ such that, if for some $\lambda_0 \in \mathcal{I}$*

$$\inf_{\gamma \in \mathbb{R}} \left\| \psi_0 - e^{i\gamma}(\phi^{\lambda_0} + (\Re z^{(0)}) \cdot \xi + i(\Im z^{(0)}) \cdot \eta) \right\|_{H^{k,\nu}} \leq c|z^{(0)}| \leq \epsilon_0, \tag{7-1}$$

then there exist smooth functions

$$\lambda(t) : \mathbb{R}^+ \to \mathcal{I}, \quad \gamma(t) : \mathbb{R}^+ \to \mathbb{R}, \quad z(t) : \mathbb{R}^+ \to \mathbb{C}^N, \quad R(x,t) : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{C}$$

such that the solution of NLS evolves in the form:

$$\psi(x,t) = e^{i\int_0^t \lambda(s)\,ds} e^{i\gamma(t)} \big(\phi^\lambda + a_1(z,\bar{z})\,\partial_\lambda \phi^\lambda + i a_2(z,\bar{z})\phi^\lambda + \Re\tilde{z}\cdot\xi + i\Im\tilde{z}\cdot\eta + R\big) \tag{7-2}$$

where $\lim_{t\to\infty} \lambda(t) = \lambda_\infty$ for some $\lambda_\infty \in \mathcal{I}$, $a_1(z,\bar{z})$, $a_2(z,\bar{z}) : \mathbb{C}^N \times \mathbb{C}^N \to \mathbb{R}$ and $\tilde{z} - z : \mathbb{C}^N \times \mathbb{C}^N \to \mathbb{C}^N$ are polynomials of z and $\bar{z}$, beginning with terms of order $|z|^2$. Moreover,

(A) $|z(t)| \leq c(1+t)^{-1/2}$ *and z satisfies the initial value problem*

$$\partial_t z = -iE(\lambda)z - \Gamma(z,\bar{z})z + \Lambda(z,\bar{z})z + \mathcal{O}\big((1+t)^{-19/5}\big) \tag{7-3}$$

where $\Gamma(z,\bar{z})$ is a symmetric, positive-definite matrix defined in (6-3) and $\Lambda(z,\bar{z})$ is a skew symmetric matrix.

(B) $\vec{R}(t) = (\Re R(t), \Im R(t))^T$ *lies in the essential spectral part of $L(\lambda(t))$. Equivalently, $R(\cdot,t)$ satisfies the symplectic orthogonality conditions*:

$$\omega\langle R, i\phi^\lambda\rangle = \omega\langle R, \partial_\lambda\phi^\lambda\rangle = 0,$$
$$\omega\langle R, i\eta_n\rangle = \omega\langle R, \xi_n\rangle = 0, \quad \text{for } n = 1, 2, \dots, N,$$

where $\omega\langle X, Y\rangle = \Im \int X\overline{Y}\,dx$ and satisfies the decay estimate:

$$\|(1+x^2)^{-\nu}\vec{R}(t)\|_2 \leq c(1+t)^{-1}.$$

Remark. We conclude this section by stating that all the hypotheses except (FGR) in our main result apply to the multiwell example of Appendix A; see Proposition 6.2 for a reduction of (FGR) is an explicit condition on the spectral condition on $-\Delta + V$. We expect (FGR) to hold generically in an appropriate sense.

8. Reformulation of the main theorem

In proving Theorem 7.1 we establish more detailed characterization of the perturbation about $\mathcal{M}_{\mathcal{J}}$.

First, we introduce the following simplying

Notation. We always use z to stand for a complex N-dimensional vector $z=(z_1,z_2,\dots,z_N)$ and an upper case letter or a Greek letter with two subindices, for example, $Q_{m,n}$ to stand for

$$Q_{m,n}(\lambda)=\sum_{\substack{a,b\in\mathbb{N}^N\\|a|=m,|b|=n}} q_{a,b}(\lambda)\prod_{k=1}^{N} z_k^{a_k}\bar{z}_k^{b_k},$$

where $|a|:=\sum_{k=1}^{N}a_k$. We refer to this kind term as (m,n) term.

Theorem 8.1. *The following more precise decomposition of the solution in Theorem 7.1 holds: The perturbation $\vec{R}$ in (7-2) can be decomposed as*

$$\vec{R}=\sum_{m+n=2} R_{m,n}(\lambda)+\tilde{R} \tag{8-1}$$

where $R_{m,n}$ are functions of the form

$$R_{m,n}=(L(\lambda)+iE(\lambda)(m-n)-0)^{-1}\phi_{m,n},$$

$\phi_{m,n}$ are polynomials of z and $\bar{z}$ with coefficients being smooth, exponentially decaying functions. The function $\tilde{R}$ satisfies

$$\partial_t\tilde{R}=L(\lambda)\tilde{R}+M_2(z,\bar{z})\tilde{R}+P_cN_2(\vec{R},z)+P_cS_2(z,\bar{z}). \tag{8-2}$$

In this formula, $S_2(z,\bar{z})=\mathbb{O}(|z|^3)$ is a polynomial in z and $\bar{z}$ with λ-dependent coefficients, and each coefficient can be written as the sum of functions of either of the following two forms:

$$(L(\lambda)+2iE(\lambda)-0)^{-k}P_c\phi_{+k}(\lambda),\qquad (L(\lambda)-2iE(\lambda)-0)^{-k}P_c\phi_{-k}(\lambda), \tag{8-3}$$

where $k=0,1,2$ and the functions $\phi_{\pm k}(\lambda)$ are smooth and decay exponentially fast at ∞. $M_2(z,\bar{z})$ is an operator defined by

$$M_2(z,\bar{z}):=\dot{\gamma}P_cJ+\dot{\lambda}P_{c\lambda}+X, \tag{8-4}$$

where X is a 2×2 matrix satisfying the bound $|X|\le c|z|e^{-\epsilon_0|x|}$. $N_2(\vec{R},z)$ can be separated into localized term and nonlocal term

$$N_2=\mathrm{Loc}+\mathrm{NonLoc} \tag{8-5}$$

where Loc *consists of terms spatially localized (exponentially) function of $x\in\mathbb{R}^d$ as a factor and satisfies the estimate*

$$\|\langle x\rangle^{\nu}(-\Delta+1)\mathrm{Loc}\|_2+\|\mathrm{Loc}\|_1+\|\mathrm{Loc}\|_{4/3}\le c\big(|z(t)|^3+|z(t)|\,\|\langle x\rangle^{-\nu}(-\Delta+1)\vec{R}\|_2\big) \tag{8-6}$$

and NonLoc *is given by*

$$\mathrm{NonLoc}:=f(R_1^2+R_2^2)J\vec{R} \tag{8-7}$$

and consists of purely nonlinear terms in $\vec{R}$ *with no spatially localized factors.* (Here ν is the same as in Theorem 7.1.)

Denote by Remainder(t) *any quantity which satisfies the estimate*

$$|\mathrm{Remainder}(t)| \lesssim |z(t)|^4 + \|\langle x\rangle^{-\nu}(-\Delta+1)\vec{R}(t)\|_2^2 + \|\vec{R}(t)\|_\infty^2 + |z(t)|\,\|\langle x\rangle^{-\nu}\tilde{R}(t)\|_2. \tag{8-8}$$

The functions λ, γ, z *have the following properties*:

$$\dot{\lambda} = \mathrm{Remainder}(t), \tag{8-9}$$

$$\dot{\gamma} = \Upsilon_{1,1} + \mathrm{Remainder}(t), \tag{8-10}$$

$$\partial_t z = -iE(\lambda)z - \Gamma(z,\bar{z})z + \Lambda(z,\bar{z})z + \mathrm{Remainder}(t) \tag{8-11}$$

where

$$\Upsilon_{1,1} := \frac{\left\langle \phi^\lambda\left(\frac{3}{2}f'[(\phi^\lambda)^2] + f''[(\phi^\lambda)^2](\phi^\lambda)^2\right)|z\cdot\xi|^2 + \frac{1}{2}f'[(\phi^\lambda)^2]|z\cdot\eta|^2,\ \partial_\lambda\phi^\lambda\right\rangle}{\langle\phi^\lambda,\,\partial_\lambda\phi^\lambda\rangle}, \tag{8-12}$$

$\Gamma(z,\bar{z})$ *is the* $N\times N$ *positive-definite matrix used in* (FGR) *and* $\Lambda(z,\bar{z})$ *is skew symmetric.*

9. Modulation equations for $z(t)$, $\lambda(t)$, $\gamma(t)$ and the dispersive part, $R(\cdot,t)$

In this section we derive evolution equations for z, λ, γ and R.

We decompose the solution as

$$\begin{aligned}\psi(x,t) &= e^{i\int_0^t\lambda(s)\,ds}e^{i\gamma(t)}\left(\phi^\lambda + a_1\phi^\lambda_\lambda + ia_2\phi^\lambda + \sum_{n=1}^N(\alpha_n+p_n)\xi_n + i\sum_{n=1}^N(\beta_n+q_n)\eta_n + R\right)\\ &= e^{i\int_0^t\lambda(s)\,ds}e^{i\gamma(t)}(\phi^\lambda + a_1\phi^\lambda_\lambda + ia_2\phi^\lambda + (\alpha+p)\cdot\xi + (\beta+q)\cdot\eta + R).\end{aligned} \tag{9-1}$$

Here and going forward let

$$\alpha = (\alpha_1,\dots,\alpha_N)^T,\ \beta = (\beta_1,\dots,\beta_N)^T,\ \xi = (\xi_1,\dots,\xi_N)^T,\ \eta = (\eta_1,\dots,\eta_N)^T.$$

Let $z = \alpha + i\beta$ then

$$\alpha = \tfrac{1}{2}(z+\bar{z}),\qquad \beta = \tfrac{1}{2i}(z-\bar{z}).$$

We seek polynomials in z and $\bar{z}$, which are of degree two or higher:

$$a_j = a_j(z,\bar{z}) = \mathbb{O}(|z|^2),\quad p_n = p_n(z,\bar{z}) = \mathbb{O}(|z|^2),\quad q_n = q_n(z,\bar{z}) = \mathbb{O}(|z|^2)$$

where $j = 1, 2$ and $n = 1,\dots,N$. Substituting Ansatz (9-1) into NLS (1-1), we have the system of equations

$$\begin{aligned}\partial_t\vec{R} = L(\lambda)\vec{R} + \dot{\gamma}J\vec{R} - J\vec{N}(\vec{R},z) - \begin{pmatrix}\partial_\lambda\phi^\lambda(\dot{\lambda}+\partial_t a_1)\\ \phi^\lambda(\dot{\gamma}+\partial_t a_2 - a_1)\end{pmatrix} + \begin{pmatrix}\xi\cdot\big(E(\lambda)(\beta+q) - \partial_t(\alpha+p)\big)\\ -\eta\cdot\big(E(\lambda)(\alpha+p) + \partial_t(\beta+q)\big)\end{pmatrix}\\ + \dot{\gamma}\begin{pmatrix}(\beta+q)\cdot\eta\\ -(\alpha+p)\cdot\xi\end{pmatrix} - \dot{\lambda}\begin{pmatrix}a_1\partial_\lambda^2\phi^\lambda + (\alpha+p)\cdot\partial_\lambda\xi\\ a_2\partial_\lambda\phi^\lambda + (\beta+q)\cdot\partial_\lambda\eta\end{pmatrix},\end{aligned} \tag{9-2}$$

where

$$\vec{R} \equiv \begin{pmatrix} R_1 \\ R_2 \end{pmatrix}, \quad \vec{N} = \begin{pmatrix} \Re N(\vec{R}, z) \\ \Im N(\vec{R}, z) \end{pmatrix}, \quad J\vec{N}(\vec{R}, z) = \begin{pmatrix} \Im N(\vec{R}, z) \\ -\Re N(\vec{R}, z) \end{pmatrix} \tag{9-3}$$

with $R_1 \equiv \Re R$, $R_2 \equiv \Im R$ and

$$\Im N(\vec{R}, z) := f[|\phi^\lambda + I_1 + iI_2|^2]I_2 - f[(\phi^\lambda)^2]I_2,$$
$$\Re N(\vec{R}, z) := (f[|\phi^\lambda + I_1 + iI_2|^2] - f[(\phi^\lambda)^2])(\phi^\lambda + I_1) - 2f'[(\phi^\lambda)^2](\phi^\lambda)^2 I_1,$$

in which

$$I_1 = A_1 + A_2 + R_1, \quad I_2 = B_1 + B_2 + R_2$$
$$A_1 = \alpha \cdot \xi, \quad A_2 = a_1 \partial_\lambda \phi^\lambda + p \cdot \xi,$$
$$B_1 = \beta \cdot \eta, \quad B_2 = a_2 \phi^\lambda + q \cdot \eta.$$

From (9-2) and the orthogonality conditions (5-4) we obtain equations for $\dot{\lambda}$, $\dot{\gamma}$ and $z_n = \alpha_n + i\beta_n$ with $n = 1, \dots, N$:

$$\partial_t(\alpha_n + p_n) - E(\lambda)(\beta_n + q_n) + \langle \Im N(\vec{R}, z), \eta_n \rangle = F_{1n}, \tag{9-4}$$

$$\partial_t(\beta_n + q_n) + E(\lambda)(\alpha_n + p_n) - \langle \Re N(\vec{R}, z), \xi_n \rangle = F_{2n}, \tag{9-5}$$

$$\dot{\gamma} + \partial_t a_2 - a_1 - \frac{1}{\langle \phi^\lambda, \partial_\lambda \phi^\lambda \rangle} \langle \Re N(\vec{R}, z), \partial_\lambda \phi^\lambda \rangle = F_3, \tag{9-6}$$

$$\dot{\lambda} + \partial_t a_1 + \frac{1}{\langle \phi^\lambda, \partial_\lambda \phi^\lambda \rangle} \langle \Im N(\vec{R}, z), \phi^\lambda \rangle = F_4, \tag{9-7}$$

where the scalar functions F_{1n}, F_{2n}, F_3 and F_4 are defined as

$$F_{1n} = \dot{\gamma}\langle (\beta + q) \cdot \eta, \eta_n \rangle - \dot{\lambda} a_1 \langle \partial_\lambda^2 \phi^\lambda, \eta_n \rangle - \dot{\lambda}\langle (\alpha + p) \cdot \partial_\lambda \xi, \eta_n \rangle - \dot{\gamma}\langle R_2, \eta_n \rangle + \dot{\lambda}\langle R_1, \partial_\lambda \eta_n \rangle,$$

$$F_{2n} = -\dot{\gamma}\langle (\alpha + p) \cdot \xi, \xi_n \rangle - \dot{\lambda} a_2 \langle \phi_\lambda^\lambda, \xi_n \rangle - \dot{\lambda}\langle (\beta + q) \cdot \partial_\lambda \eta, \xi_n \rangle + \dot{\gamma}\langle R_1, \xi_n \rangle + \dot{\lambda}\langle R_2, \partial_\lambda \xi_n \rangle,$$

$$F_3 = \frac{1}{\langle \phi^\lambda, \phi_\lambda^\lambda \rangle} \big(\dot{\lambda}\langle R_2, \phi_{\lambda\lambda}^\lambda \rangle - \dot{\gamma}\langle R_1, \phi_\lambda^\lambda \rangle - \langle \dot{\gamma}(\alpha + p) \cdot \xi + \dot{\lambda} a_2 \phi_\lambda^\lambda + \dot{\lambda}(\beta + q) \cdot \partial_\lambda \eta, \phi_\lambda^\lambda \rangle \big),$$

$$F_4 = \frac{1}{\langle \phi^\lambda, \phi_\lambda^\lambda \rangle} \big(\dot{\lambda}\langle R_1, \phi_\lambda^\lambda \rangle + \dot{\gamma}\langle R_2, \phi^\lambda \rangle + \langle \dot{\gamma}(\beta + q) \cdot \eta - \dot{\lambda} a_1 \partial_\lambda^2 \phi^\lambda - \dot{\lambda}(\alpha + p) \cdot \partial_\lambda \xi, \phi^\lambda \rangle \big).$$

Remarks. (a) Recall the estimate of Remainder in (8-8). By (9-4)–(9-7) we have

$$\dot{\lambda}, \dot{\gamma}, \partial_t z_n + iE(\lambda) z_n = \mathbb{O}(|z|^2) + \text{Remainder}. \tag{9-8}$$

(b) The functions $a_j(z, \bar{z})$, $p_n(z, \bar{z})$ and $q_n(z, \bar{z})$ for $j = 1, 2$ and $n = 1, \dots, N$ will be chosen to eliminate "nonresonant" terms $z^m \bar{z}^n$ with $2 \le |m| + |n| \le 3$.

Finally, we derive an equation for

$$\vec{R} = P_c^{\lambda(t)} \vec{R} = P_c \vec{R},$$

the continuous spectral part of the solution, relative to the operator $L(\lambda(t))$. Applying $P_c = P_c^{\lambda(t)}$ to (9-2) and using the commutator identity:

$$P_c \partial_t \vec{R} = \partial_t \vec{R} - \dot{\lambda} \partial_\lambda P_c \vec{R},$$

we obtain

$$\partial_t \vec{R} = L(\lambda(t))\vec{R} - P_c^{\lambda(t)} J\vec{N}(\vec{R}, z) + L_{(\dot{\lambda},\dot{\gamma})}\vec{R} + \mathcal{G}. \tag{9-9}$$

The operator $L_{(\dot{\lambda},\dot{\gamma})}$ and the vector function $\mathcal{G}$ are defined as

$$L_{(\dot{\lambda},\dot{\gamma})} = \dot{\lambda}(\partial_\lambda P_c^{\lambda(t)}) + \dot{\gamma} P_c^{\lambda(t)} J, \tag{9-10}$$

$$\mathcal{G} = P_c^{\lambda(t)} \begin{pmatrix} \dot{\gamma}(\beta+q)\cdot\eta - \dot{\lambda} a_1 \partial_\lambda^2 \phi^\lambda - \dot{\lambda}(\alpha+p)\cdot\partial_\lambda \xi \\ -\dot{\gamma}(\alpha+p)\cdot\xi - \dot{\lambda} a_2 \phi_\lambda^\lambda - \dot{\lambda}(\beta+q)\cdot\partial_\lambda \eta \end{pmatrix}. \tag{9-11}$$

We now summarize the preceding calculation in

Proposition 9.1 (Reformulation of NLS). *Using the Ansatz* (9-1)

$$\psi(x,t) = e^{i\int_0^t \lambda(s)\,ds} e^{i\gamma(t)}\big(\phi^\lambda + a_1\phi_\lambda^\lambda + i a_2 \phi^\lambda + (\alpha+p)\cdot\xi + (\beta+q)\cdot\eta + R\big),$$

NLS can be equivalently expressed as a coupled system of equations (9-4)–(9-7) *for modulating solitary wave parameters $\lambda(t)$ and $\gamma(t)$, neutral mode amplitudes $z_n(t) = \alpha_n(t) + i\beta_n(t)$ for $n = 1,\dots,N$, together with Equation* (9-9) *governing "dispersive part" $\vec{R}(t)$ which evolves in the continuous spectral subspace of $L(\lambda(t))$, that is, $P_c^{\lambda(t)}\vec{R}(t) = \vec{R}(t)$; see* (5-5). *Moreover, the functions $a_j = a_j(z,\bar{z})$ for $j = 1, 2$, $\big(p(z,\bar{z}), q(z,\bar{z})\big) = (p_n, q_n)_{n=1,\dots,N}$ are $\mathcal{O}(|z|^2)$ polynomials chosen (in what follows) to eliminate "nonresonant" terms of the form $z^a\bar{z}^b$ with $2 \le |a| + |b| \le 3$.*

***Extracting the $\mathcal{O}(|z|^2)$ part of $\vec{R}(t)$; proof of* (8-2).** For fixed $z(t) \in \mathbb{C}^N$, the equation for $\vec{R}(t)$ is forced by terms of order $|z(t)|^2$; linear terms are removed due to the equations satisfied by $z(t) = \alpha(t) + i\beta(t)$. In our analysis, we need to explicitly extract the quadratic part in $z, \bar{z}$ of $\vec{R}(t)$.

Thus, we consider the quadratic terms generated by the nonlinearity:

$$\sum_{m+n=2} J\vec{N}_{m,n} = J\vec{N}_{2,0} + J\vec{N}_{1,1} + J\vec{N}_{0,2} = \begin{pmatrix} 2f'[(\phi^\lambda)^2]\phi^\lambda A_1 B_1 \\ -\big(3f'[(\phi^\lambda)^2]\phi^\lambda + 2f''[(\phi^\lambda)^2](\phi^\lambda)^3\big)A_1^2 - f'[(\phi^\lambda)^2]\phi^\lambda B_1^2 \end{pmatrix}, \tag{9-12}$$

where $A_1 = \alpha\cdot\xi$, $B_1 = \beta\cdot\eta$.

Theorem 9.2. *Define*

$$\vec{R}_{m,n} := (L(\lambda) + iE(\lambda)(m-n) - 0)^{-1} P_c J\vec{N}_{m,n} \tag{9-13}$$

and decompose $\vec{R}(t)$ as

$$\vec{R} = \sum_{m+n=2} \vec{R}_{m,n} + \tilde{R}. \tag{9-14}$$

The function $\tilde{R}(x,t)$ satisfies (8-2).

Proof. $\tilde{R}$, defined in Equation (9-14), satisfies the equation:

$$\partial_t \tilde{R} = L(\lambda)\tilde{R} + L_{(\dot{\lambda},\dot{\gamma})}\tilde{R} + \sum_{m+n=2} L_{(\dot{\lambda},\dot{\gamma})} R_{m,n} + \mathcal{G} - \sum_{m+n=2} (\partial_t \vec{R}_{m,n} + iE(\lambda)(m-n)\vec{R}_{m,n}) - P_c J\vec{N}_{>2}$$

where, recall the definitions of $\vec{R}_{m,n}$ in (9-13), the definitions of the operator $L_{\dot{\lambda},\dot{\gamma}}$ and the term $\mathcal{G}$ in (9-9), and we define

$$J\vec{N}_{>2} := J\vec{N}(\vec{R}, z) - \sum_{m+n=2} J\vec{N}_{m,n}.$$

Next we further decompose $J\vec{N}_{>2}$ and find M_2, S_2 and N_2 in (8-2). We consider the functions $JN_{m,n}$ with $m+n=3$, the third order terms of $J\vec{N}_{>2}$:

$$\sum_{m+n=3} J\vec{N}_{m,n} = X\Big(\sum_{m+n=2}\vec{R}_{m,n}+\begin{pmatrix}A_2\\ B_2\end{pmatrix}\Big)+\begin{pmatrix}G_1(A_1^2,B_1^2)B_1\\ -G_2(A_1^2,B_1^2)A_1\end{pmatrix} \tag{9-15}$$

where, recall the definitions of A_1, B_1, A_2 and B_2 from (9-3),

$$G_1(A_1^2,B_1^2) := f'[(\phi^\lambda)^2](A_1^2+B_1^2)+2f''[(\phi^\lambda)^2](\phi^\lambda)^2A_1^2,$$
$$G_2(A_1^2,B_1^2) := \big(f'[(\phi^\lambda)^2]+2f''[(\phi^\lambda)^2](\phi^\lambda)^2\big)(A_1^2+B_1^2)+\big(2f''[(\phi^\lambda)^2](\phi^\lambda)^2+\tfrac{4}{3}f'''[(\phi^\lambda)^2](\phi^\lambda)^4\big)A_1^2$$

and X is a 2×2 matrix of order $|z|$ defined as

$$X = X_{0,1}+X_{1,0} = \begin{pmatrix} 2f'[(\phi^\lambda)^2]\phi^\lambda B_1 & 2f'[(\phi^\lambda)^2]\phi^\lambda A_1\\ -\big(6f'[(\phi^\lambda)^2]\phi^\lambda+4f''[(\phi^\lambda)^2](\phi^\lambda)^3\big)A_1 & -2f'[(\phi^\lambda)^2]\phi^\lambda B_1\end{pmatrix}. \tag{9-16}$$

We define the linear operator $M_2(z,\bar{z})$ as

$$M_2(z,\bar{z}) := X+L_{(\dot{\lambda},\dot{\gamma})}$$

which satisfies (8-4).

The function S_2 in the statement of Theorem 7.1 is defined as

$$S_2(z,\bar{z}) := \sum_{m+n=2} L_{(\dot{\lambda},\dot{\gamma})}R_{m,n}+\mathcal{G}-\sum_{m+n=2}\big(\partial_t R_{m,n}+iE(\lambda)(m-n)R_{m,n}\big)+\sum_{m+n=3} JN_{m,n}.$$

By (9-8) and

$$\begin{aligned}[\partial_t,(L(\lambda)\pm iE(\lambda)-0)^{-1}] &\equiv \partial_t(L(\lambda)\pm iE(\lambda)-0)^{-1}-(L(\lambda)\pm iE(\lambda)-0)^{-1}\partial_t\\ &= \dot{\lambda}(L(\lambda)\pm iE(\lambda)-0)^{-1}\partial_\lambda(L(\lambda)\pm iE(\lambda))(L(\lambda)\pm iE(\lambda)-0)^{-1}\end{aligned}$$

we have that $S_2(z,\bar{z})$ satisfies the estimate in the first part of Theorem 8.1. For the details we refer to [Gang and Sigal 2007].

Lastly, we define the nonlinear term

$$\vec{N}_2(\vec{R},z) := -\Big(J\vec{N}(\vec{R},z)-\sum_{m+n=2,3} JN_{m,n}\Big). \tag{9-17}$$

Using the smoothness of the nonlinearity $f[\cdot]$ and removing $\mathbb{O}(|z|^2)$ and $\mathbb{O}(|z|^3)$ terms, we have that $N_2(\vec{R},z)=\mathrm{Loc}+\mathrm{NonLoc}$ (see (8-5)) satisfying (8-6) and (8-7). The computation is straightforward but tedious and is therefore omitted.

Collecting the various definitions and estimates above we have (8-2). □

$z(t)$ dependence of equations for $\lambda(t)$ and $\gamma(t)$. In this subsection we present the proofs of (8-9) and (8-10), crucial to controlling the large time behavior.

Here's the idea. Central to our claim about the large time dynamics of NLS is that the solution settles into an asymptotic solitary wave ϕ^{λ_∞} where $\lambda(t)\to\lambda_\infty$. We show this by establishing the integrability and uniform smallness of $\dot{\lambda}$. Since we expect the neutral mode amplitudes $z(t)$ to decay with a rate $t^{-1/2}$, we require that there be no $\mathbb{O}(|z(t)|^2)$ terms in the (9-7): $\dot{\lambda}(t)+\partial_t a_1(z,\bar{z})=\dots$. The strategy is to choose the quadratic part of the polynomial $a_1(z,\bar z)$ so as to eliminate all quadratic nonresonant terms. The latter are terms whose z-behavior is like $(z_k)^2$ or $(\bar z_k)^2$ and are oscillatory with frequencies $\sim \pm 2iE(\lambda)$. But what about the terms of the form $z_k\bar z_m$, which are resonant (nonoscillatory)? This is where we use the choice of basis for the degenerate subspace; see Appendix D. A consequence of this choice is that there are *no* resonant quadratic terms appearing in the equation for λ! The calculation is carried out below; see Lemma 9.4.

In what follows we use the notations $N^{\Im}_{m,n}$ and $N^{\Re}_{m,n}$ to denote functions satisfying

$$\begin{pmatrix} N^{\Im}_{m,n} \\ -N^{\Re}_{m,n}\end{pmatrix} = JN_{m,n}.$$

We define the polynomials a_1, a_2 and p_k, q_k for $k=1,2,\dots,N$ in (9-1) (see also (7-2)) as

$$a_j(z,\bar z) := \sum_{\substack{m+n=2,3\\ m\neq n}} A^{(j)}_{m,n}(\lambda),\qquad p_k(z,\bar z) := \sum_{m+n=2,3} P^{(k)}_{m,n}(\lambda),\qquad q_k(z,\bar z) := \sum_{m+n=2,3} Q^{(k)}_{m,n}(\lambda), \tag{9-18}$$

with $j=1,2$, $k=1,2,\dots,N$, and the explicit forms

$$\begin{aligned} 2iE(\lambda)A^{(1)}_{2,0} &:= \frac{1}{\langle\phi^\lambda,\partial_\lambda\phi^\lambda\rangle}\langle N^{\Im}_{2,0},\phi^\lambda\rangle, \qquad 3iE(\lambda)A^{(1)}_{3,0} := \frac{1}{\langle\phi^\lambda,\partial_\lambda\phi^\lambda\rangle}\langle N^{\Im}_{3,0},\phi^\lambda\rangle,\\ iE(\lambda)A^{(1)}_{2,1} &:= \frac{1}{\langle\phi^\lambda,\partial_\lambda\phi^\lambda\rangle}\big(\langle N^{\Im}_{2,1},\phi^\lambda\rangle - \tfrac{i}{2}\Upsilon_{1,1}\,\langle z\cdot\eta,\phi^\lambda\rangle\big)\end{aligned} \tag{9-19}$$

where $\Upsilon_{1,1}$ is given in (8-12); similarly

$$\begin{aligned} -2iE(\lambda)A^{(2)}_{2,0} + A^{(1)}_{2,0} &:= \frac{1}{\langle\phi^\lambda,\partial_\lambda\phi^\lambda\rangle}\langle N^{\Re}_{2,0},\partial_\lambda\phi^\lambda\rangle, \qquad -3iE(\lambda)A^{(2)}_{3,0} + A^{(1)}_{3,0} := \frac{1}{\langle\phi^\lambda,\partial_\lambda\phi^\lambda\rangle}\langle N^{\Re}_{3,0},\partial_\lambda\phi^\lambda\rangle,\\ -iE(\lambda)A^{(2)}_{2,1} + A^{(1)}_{2,1} &:= \frac{1}{\langle\phi^\lambda,\partial_\lambda\phi^\lambda\rangle}\big(\langle N^{\Re}_{2,1},\partial_\lambda\phi^\lambda\rangle - \tfrac{1}{2}\Upsilon_{1,1}\langle z\cdot\xi,\partial_\lambda\phi^\lambda\rangle\big)\end{aligned} \tag{9-20}$$

and

$$\begin{aligned} -2iE(\lambda)P^{(n)}_{2,0} - E(\lambda)Q^{(n)}_{2,0} &:= -\langle N^{\Im}_{2,0},\eta_n\rangle, \qquad -2iE(\lambda)Q^{(n)}_{2,0} + E(\lambda)P^{(n)}_{2,0} := \langle N^{\Re}_{2,0},\xi_n\rangle,\\ -3iE(\lambda)P^{(n)}_{3,0} - E(\lambda)Q^{(n)}_{3,0} &:= -\langle N^{\Im}_{3,0},\eta_n\rangle, \qquad -3iE(\lambda)Q^{(n)}_{3,0} + E(\lambda)P^{(n)}_{3,0} := \langle N^{\Re}_{3,0},\xi_n\rangle,\\ 2iE(\lambda)P^{(n)}_{1,2} - 2E(\lambda)Q^{(n)}_{1,2} &:= -\langle N^{\Im}_{1,2},\eta_n\rangle + i\langle N^{\Re}_{1,2},\xi_n\rangle + i\Upsilon_{1,1}\sum_{k=1}^N \bar z_k(\langle\eta_k,\eta_n\rangle - \langle\xi_k,\xi_n\rangle),\\ E(\lambda)Q^{(n)}_{1,1} &:= \langle N^{\Im}_{1,1},\eta_n\rangle, \qquad E(\lambda)P^{(n)}_{1,1} := \langle N^{\Re}_{1,1},\xi_n\rangle\end{aligned} \tag{9-21}$$

$$A^{(j)}_{a,b} := \overline{A^{(j)}_{a,b}},\qquad P^{(n)}_{a,b} := \overline{P^{(n)}_{a,b}},\qquad Q^{(n)}_{a,b} := \overline{Q^{(n)}_{a,b}}$$

for $j=1,2$, $a+b=2,3$, $a\neq b$.

The following is the main result.

Proposition 9.3. *Define the polynomials $a_1(z,\bar z)$, $a_2(z,\bar z)$, $p_n(z,\bar z)$, $q_n(z,\bar z)$ as above. Then,* (8-9)–(8-10) *hold and*

$$\partial_t\lambda = \text{Remainder}(t), \qquad \partial_t\gamma = \Upsilon_{1,1} + \text{Remainder}(t),$$
$$\partial_t z_n + iE(\lambda)z_n = -\left\langle JN_{2,1}, \begin{pmatrix}\eta_n\\ -i\xi_n\end{pmatrix}\right\rangle + \tfrac{1}{2}\Upsilon_{1,1}\sum_{m=1}^{N} z_m\left\langle\begin{pmatrix}-i\eta_m\\ \xi_m\end{pmatrix}, \begin{pmatrix}\eta_n\\ i\xi_n\end{pmatrix}\right\rangle + \text{Remainder}(t), \tag{9-22}$$

where $\Upsilon_{1,1}$ is defined in (8-12), *and moreover,*

$$|\text{Remainder}(t)| \lesssim |z(t)|^4 + \|\langle x\rangle^{-\nu}(-\Delta+1)\vec R(t)\|_2^2 + \|\vec R(t)\|_\infty^2 + |z(t)|\cdot\|\langle x\rangle^{-\nu}\tilde R(t)\|_2.$$

Before proving the proposition we state the following key observation.

Lemma 9.4. $$\langle N_{1,1}^{\Im}, \phi^\lambda\rangle = 0.$$

Proof. Recall that

$$A_1 = \alpha\cdot\xi = \tfrac{1}{2}(z\cdot\xi + \bar z\cdot\xi), \qquad B_1 = \beta\cdot\eta = \tfrac{1}{2i}(z\cdot\eta - \bar z\cdot\eta).$$

The explicit form of

$$JN_{2,0} + JN_{1,1} + JN_{0,2}$$

in (9-12) implies that

$$\begin{aligned} N_{2,0}^{\Im} + N_{1,1}^{\Im} + N_{0,2}^{\Im} &= 2f'[(\phi^\lambda)^2]\phi^\lambda A_1 B_1 \\ &= \tfrac{1}{2i} f'[(\phi^\lambda)^2]\phi^\lambda\left(\sum_{n=1}^{N} z_n\xi_n + \sum_{n=1}^{N}\bar z_n\xi_n\right)\left(\sum_{m=1}^{N} z_m\eta_m - \sum_{m=1}^{N}\bar z_m\eta_m\right). \end{aligned}$$

By taking the relevant terms we have

$$\begin{aligned} N_{1,1}^{\Im} &= \frac{1}{2i} f'[(\phi^\lambda)^2]\phi^\lambda\left(\sum_{n=1}^{N}\bar z_n\xi_n\sum_{m=1}^{N} z_m\eta_m - \sum_{n=1}^{N} z_n\xi_n\sum_{m=1}^{N}\bar z_m\eta_m\right) \\ &= \frac{1}{2i}\sum_{n=1}^{N}\sum_{m=1}^{N}\bar z_n z_m f'[(\phi^\lambda)^2]\phi^\lambda(\xi_n\eta_m - \xi_m\eta_n), \end{aligned}$$

which, together with (5-3), yields

$$\langle N_{1,1}^{\Im}, \phi^\lambda\rangle = \frac{1}{2i}\sum_{n=1}^{N}\sum_{m=1}^{N}\bar z_n z_m\int f'[(\phi^\lambda)^2](\phi^\lambda)^2(\xi_n\eta_m - \xi_m\eta_n) = 0. \qquad \square$$

Proof of Proposition 9.3. Recall the estimate of any term denoted Remainder in (8-8). We put (9-6) and (9-7) in the matrix form

$$(\mathrm{Id} + M(z,\vec R,p,q))\begin{pmatrix}\dot\lambda\\ \dot\gamma - \Upsilon_{1,1}\end{pmatrix} = \Omega + \text{Remainder}, \tag{9-23}$$

where the matrix Ω is defined as

$$\Omega := \begin{pmatrix} -\frac{1}{\langle \phi^\lambda, \partial_\lambda \phi^\lambda \rangle} \big(\langle \Im N, \phi^\lambda \rangle + \frac{i}{2} \Upsilon_{1,1} \langle (z-\bar z)\cdot \eta, \phi^\lambda \rangle \big) - \partial_t a_1 \\ \frac{1}{\langle \phi^\lambda, \partial_\lambda \phi^\lambda \rangle} \big(\langle \Re N, \partial_\lambda \phi^\lambda \rangle - \frac{1}{2} \Upsilon_{1,1} \langle (z+\bar z)\cdot \xi, \partial_\lambda \phi^\lambda \rangle \big) - \Upsilon_{1,1} - \partial_t a_2 + a_1 \end{pmatrix}, \tag{9-24}$$

the term Remainder is produced by

$$\frac{\Upsilon_{1,1}}{\langle \phi^\lambda, \partial_\lambda \phi^\lambda \rangle} \begin{pmatrix} -\langle R_1, \partial_\lambda \phi^\lambda \rangle + p \langle \xi, \partial_\lambda \phi^\lambda \rangle \\ \langle R_2, \phi^\lambda \rangle + q \langle \eta, \phi^\lambda \rangle \end{pmatrix},$$

Id is the 2×2 identity matrix, $M(z, \vec R, p, q)$ is a vector depending on z, $\vec R$, p and q and satisfies the estimate

$$\| M(z, \vec R, p, q) \| = \mathbb{O}(|z|) + \text{Remainder}. \tag{9-25}$$

Now by the definitions of a_1 and a_2 in (9-18), we remove the lower order terms in $z, \bar z$ from

$$\langle \Im N, \phi^\lambda \rangle - \tfrac{i}{2} \Upsilon_{1,1} \langle (z-\bar z)\cdot \eta, \phi^\lambda \rangle \quad \text{and} \quad \langle \Re N, \partial_\lambda \phi^\lambda \rangle + \tfrac{1}{2} \Upsilon_{1,1} \langle (z+\bar z)\cdot \xi, \partial_\lambda \phi^\lambda \rangle$$

to get

$$\Omega = D_1 + D_2 \tag{9-26}$$

with

$$D_1 := \frac{1}{\langle \phi^\lambda, \partial_\lambda \phi^\lambda \rangle} \begin{pmatrix} -\big\langle \Im N - \sum_{m+n=2,3} N^{\Im}_{m,n}, \phi^\lambda \big\rangle \\ \big\langle \Re N - \sum_{m+n=2,3} N^{\Re}_{m,n}, \partial_\lambda \phi^\lambda \big\rangle \end{pmatrix}$$

by Lemma 9.4 and

$$D_2 := -\sum_{m+n=2,3} \begin{pmatrix} \partial_t A^{(1)}_{m,n} + iE(\lambda)(m-n) A^{(1)}_{m,n} \\ \partial_t A^{(2)}_{m,n} + iE(\lambda)(m-n) A^{(2)}_{m,n} \end{pmatrix}.$$

We claim that

$$D_1, D_2 = \text{Remainder}. \tag{9-27}$$

If the claim holds then estimates (8-9) and (8-10) follow from (9-26) and the estimates (9-23), (9-25). Next we prove the claim (9-27) together with (9-22).

Since we removed all the second and third order terms of $J\vec N$ we obtain $D_1 = \text{Remainder}$. Recall the estimate of Remainder in (8-8). To estimate D_2 we have to start with studying the equation for z. By the fact that

$$\partial_t z_n + iE(\lambda) z_n = \mathbb{O}(|z|^2) + \text{Remainder}$$

in (9-8) we obtain $D_2 = \mathbb{O}(|z|^3) + \text{Remainder}$. Hence,

$$\dot\lambda = \mathbb{O}(|z|^3) + \text{Remainder}, \qquad \dot\gamma - \Upsilon_{1,1} = \mathbb{O}(|z|^3) + \text{Remainder}, \tag{9-28}$$

which, together with the expansion of $J\vec N$ in (9-17), yields

$$\partial_t(\alpha_n + p_n) - E(\lambda)(\beta_n + q_n) + \sum_{k+l=2,3} \langle N^{\Im}_{k,l}, \eta_n \rangle = -\tfrac{i}{2} \Upsilon_{1,1} \langle (z-\bar z)\cdot \eta, \eta_n \rangle + \text{Remainder},$$

$$\partial_t(\beta_n + q_n) + E(\lambda)(\alpha_n + p_n) - \sum_{k+l=2,3} \langle N^{\Re}_{k,l}, \xi_n \rangle = -\tfrac{1}{2} \Upsilon_{1,1} \langle (z+\bar z)\cdot \xi, \xi_n \rangle + \text{Remainder},$$

where the real function $\Upsilon_{1,1}$ is defined in (8-10). Choose p_n and q_n as in (9-21) to remove the lower order terms as in the equations for $\dot{\lambda}$ and $\dot{\gamma}$, which, together with the definition $z_n = \alpha_n + i\beta_n$, enables us to obtain

$$\partial_t z_n + iE(\lambda) z_n = -\left\langle JN_{2,1} + \tfrac{1}{2}\Upsilon_{1,1}\begin{pmatrix} iz\cdot\eta \\ z\cdot\xi \end{pmatrix}, \begin{pmatrix} \eta_n \\ -i\xi_n \end{pmatrix}\right\rangle + D_3(n) + \text{Remainder}$$

with $D_3(n)$ defined as

$$D_3(n) := -\sum_{k+l=2,3}\big(\partial_t P_{k,l}^{(n)} + i(k-l)E(\lambda)P_{k,l}^{(n)}\big) - i\sum_{k+l=2,3}\big(\partial_t Q_{k,l}^{(n)} + i(k-l)E(\lambda)Q_{k,l}^{(n)}\big).$$

We claim that this together with the equations for $\dot{\lambda}$ in (9-23) implies that

$$|D_2|, |D_3(n)| = \text{Remainder}. \tag{9-29}$$

Indeed, by (9-8), we have

$$\partial_t z_n + iE(\lambda) z_n = \mathcal{O}(|z|^2) + \text{Remainder},$$

which, together with the equation for $\dot{\lambda}$ in (9-28), implies $D_3 = \mathcal{O}(|z|^3) + \text{Remainder}$. In turn we have an improved equation for z_n as

$$\partial_t z_n = -iE(\lambda) z_n + \mathcal{O}(|z|^3) + \text{Remainder}.$$

Using this and repeating the analysis we find there is no $\mathcal{O}(|z|^3)$ term in D_2 and D_3. Hence (9-29) holds which leads to (9-22) and (9-27). □

10. Proof of the normal form equation (8-11)

Recall the definitions of the functions $B(n)$ and $D(n)$ after (6-2). Then the function $JN_{2,0}$ in (9-12) admits the form

$$JN_{2,0} = \sum_{n=1}^{N} z_n \begin{pmatrix} B(n) \\ D(n) \end{pmatrix}. \tag{10-1}$$

The following is the result establishing the desired normal form of the differential equation for the neutral mode amplitudes $z(t)$.

Theorem 10.1. *With polynomials a_1, a_2, p_n and q_n for $n = 1, 2, \ldots, N$ defined in (9-18)–(9-21), (8-11) holds.*

Proof. Recall the definitions of $JN_{m,n}$ with $m+n=3$ in (9-15). The first two terms on the right-hand side of (9-22) admit the expansion

$$\sum_{k=1}^{5} K_k(n)$$

where

$$K_1(n) := -\left\langle X_{0,1}R_{2,0}, \begin{pmatrix} \eta_n \\ -i\xi_n \end{pmatrix}\right\rangle = -\left\langle R_{2,0}, X_{0,1}^* \begin{pmatrix} \eta_n \\ -i\xi_n \end{pmatrix}\right\rangle,$$

$$K_2(n) := -\left\langle X_{1,0}\begin{pmatrix} \sum_{k=1}^N P_{1,1}^{(k)}\xi_k \\ \sum_{k=1}^N Q_{1,1}^{(k)}\eta_k \end{pmatrix} + X_{0,1}\begin{pmatrix} \sum_{k=1}^N P_{2,0}^{(k)}\xi_k + A_{2,0}^{(1)}\partial_\lambda\phi^\lambda \\ \sum_{k=1}^N Q_{2,0}^{(k)}\eta_k + A_{2,0}^{(2)}\phi^\lambda \end{pmatrix}, \begin{pmatrix} \eta_n \\ -i\xi_n \end{pmatrix}\right\rangle,$$

$$\begin{aligned} K_3(n) := &-\tfrac{1}{8}\left\langle \big(f'[(\phi^\lambda)^2] + 2f''[(\phi^\lambda)^2](\phi^\lambda)^2\big)\big((z\cdot\xi)^2 - (z\cdot\eta)^2\big)\begin{pmatrix} i\bar{z}\cdot\eta \\ -\bar{z}\cdot\xi \end{pmatrix}, \begin{pmatrix} \eta_n \\ -i\xi_n \end{pmatrix}\right\rangle \\ &+\tfrac{1}{4}\left\langle \big(f'[(\phi^\lambda)^2] + 2f''[(\phi^\lambda)^2](\phi^\lambda)^2\big)\big(|z\cdot\xi|^2 + |z\cdot\eta|^2\big)\begin{pmatrix} iz\cdot\eta \\ z\cdot\xi \end{pmatrix}, \begin{pmatrix} \eta_n \\ -i\xi_n \end{pmatrix}\right\rangle \\ &-\tfrac{3}{4}i\left\langle f''[(\phi^\lambda)^2](\phi^\lambda)^2(z\cdot\eta)^2\begin{pmatrix} \bar{z}\cdot\eta \\ 0 \end{pmatrix}, \begin{pmatrix} \eta_n \\ -i\xi_n \end{pmatrix}\right\rangle \\ &-\left\langle \big(\tfrac{3}{4}f''[(\phi^\lambda)^2](\phi^\lambda)^2 + \tfrac{1}{2}f'''[(\phi^\lambda)^2](\phi^\lambda)^4\big)(z\cdot\xi)^2\begin{pmatrix} 0 \\ -\bar{z}\cdot\xi \end{pmatrix}, \begin{pmatrix} \eta_n \\ -i\xi_n \end{pmatrix}\right\rangle, \end{aligned}$$

$$K_4(n) := \tfrac{1}{2}\Upsilon_{1,1}\left\langle \begin{pmatrix} -iz\cdot\eta \\ z\cdot\xi \end{pmatrix}, \begin{pmatrix} \eta_n \\ i\xi_n \end{pmatrix}\right\rangle,$$

$$K_5(n) := -\left\langle R_{1,1}, X_{1,0}^* \begin{pmatrix} \eta_n \\ -i\xi_n \end{pmatrix}\right\rangle$$

with X defined in (9-16) divided into two terms $X = X_{1,0} + X_{0,1}$:

$$\begin{aligned} X_{1,0} &:= \begin{pmatrix} -if'[(\phi^\lambda)^2]\phi^\lambda z\cdot\eta & f'[(\phi^\lambda)^2]\phi^\lambda z\cdot\xi \\ -\big(3f'[(\phi^\lambda)^2]\phi^\lambda + 2f''[(\phi^\lambda)^2](\phi^\lambda)^3\big)z\cdot\xi & if'[(\phi^\lambda)^2]\phi^\lambda z\cdot\eta \end{pmatrix}, \\ X_{0,1} &= \overline{X_{1,0}} \end{aligned}$$

and the real function $\Upsilon_{1,1}$ given in (8-10).

Next we study $K_j(n)$ for $j = 1, 2, 3, 4, 5$. We start with the important term, $K_1(n)$. Recall the definition of G_n in (6-2). By direct computation we obtain

$$X_{0,1}^* \begin{pmatrix} \eta_n \\ -i\xi_n \end{pmatrix} = -iJ\begin{pmatrix} B(n) \\ D(n) \end{pmatrix} = -iJG_n \tag{10-2}$$

which, together with (9-13) and (10-1), implies that

$$K_1(n) = \sum_{k=1}^N z_k\big\langle (L(\lambda) + 2iE(\lambda) - 0)^{-1}P_cG_k, iJG_n\big\rangle.$$

Define

$$Z(k,n) := -\big\langle (L(\lambda) + 2iE(\lambda) - 0)^{-1}P_cG_n, iJG_k\big\rangle$$

and a $N \times N$ matrix

$$\Gamma(z,\bar{z}) := [A(k,l)] \tag{10-3}$$

with $A(k,l) := \frac{1}{2}(Z(k,l) + \overline{Z(l,k)})$ for $1 \le k, l \le d$.

For the sum of $K_2(n)$ through $K_5(n)$ we claim that it can decomposed into the matrix form

$$\sum_{j=2}^{5} K_j(n) = (S(n,1), S(n,2), \dots, S(n,d))z \tag{10-4}$$

with $S(k,l) + \overline{S(l,k)} = 0$. Define a $N \times N$ skew symmetric matrix

$$\Lambda(z,\bar{z}) := [\Lambda(j,k)]$$

with $\Lambda(j,k) := S_{k,l} + \frac{1}{2}(Z(k,l) - \overline{Z(l,k)})$. This together with (9-22) and (10-3) yields the equation for z in (8-11)

What is left is to prove (10-4). To avoid the tedious but simple computations, we only analyze $K_2(n)$ and $K_3(n)$.

(A) Consider the part of $K_2(n)$ given by

$$\Psi_{2,1}(n) := -\left\langle X_{0,1}\begin{pmatrix} A_{2,0}^{(1)}\partial_\lambda\phi^\lambda \\ A_{2,0}^{(2)}\phi^\lambda \end{pmatrix}, \begin{pmatrix} \eta_n \\ -i\xi_n \end{pmatrix}\right\rangle.$$

The analysis of the other terms is similar. By (10-2) we rewrite $\Psi_{2,1}(n)$ as

$$\Psi_{2,1}(n) = \left\langle \begin{pmatrix} A_{2,0}^{(1)}\partial_\lambda\phi^\lambda \\ A_{2,0}^{(2)}\phi^\lambda \end{pmatrix}, 4i\begin{pmatrix} D(n) \\ -B(n) \end{pmatrix}\right\rangle = -4iA_{2,0}^{(1)}\langle\partial_\lambda\phi^\lambda, D(n)\rangle + 4iA_{2,0}^{(2)}\langle\phi^\lambda, B(n)\rangle.$$

Equation (9-19) relates $\langle N_{2,0}^{\Re}, \phi_\lambda^\lambda\rangle$ and $\langle N_{2,0}^{\Im}, \phi^\lambda\rangle$ to $A_{2,0}^{(1)}$ and $A_{2,0}^{(2)}$, which, together with the expression of $JN_{2,0}$ in (10-1), yields

$$\Psi_{2,1}(n) = \sum_{k=1}^{N} \Psi(n,k) z_k \tag{10-5}$$

with

$$\Psi(n,k) := \frac{2}{E(\lambda)\langle\phi^\lambda, \partial_\lambda\phi^\lambda\rangle}\big(\langle B(k),\phi^\lambda\rangle\langle\partial_\lambda\phi^\lambda, D(n)\rangle - \langle D(k), D_\lambda\phi^\lambda\rangle\langle\phi^\lambda, B(n)\rangle\big) \\ + \frac{i}{E^2(\lambda)\langle\phi^\lambda, \partial_\lambda\phi^\lambda\rangle}\langle B(k),\phi^\lambda\rangle\langle\phi^\lambda, B(n)\rangle.$$

By straightforward computation we have

$$\Psi(n,k) + \overline{\Psi(k,n)} = 0. \tag{10-6}$$

By (10-5) and (10-6) we complete the proof for $\Psi_{2,1}(n)$.

(B) To simplify the notation we introduce

$$\rho := \tfrac{1}{2}(z\cdot\xi) = \tfrac{1}{2}\sum_{n=1}^{N} z_n\xi_n, \quad \omega := \tfrac{1}{2}(z\cdot\eta) = \tfrac{1}{2}\sum_{n=1}^{N} z_n\eta_n.$$

This implies that

$$\rho^2=\tfrac{1}{2}\sum_{n=1}^{N} z_n\xi_n\rho,\ \ \omega^2=\tfrac{1}{2}\sum_{n=1}^{N} z_n\eta_n\omega,\ \ \rho\bar{\rho}+\omega\bar{\omega}=\tfrac{1}{2}\sum_{n=1}^{N} z_n(\xi_n\bar{\rho}+\eta_n\bar{\omega}),\ \ \rho^2-\omega^2=\tfrac{1}{2}\sum_{n=1}^{N} z_n(\rho\xi_n-\omega\eta_n).$$

By the definition of $K_3(n)$ it is not hard to get

$$K_3(n)=\sum_{k=1}^{N} z_k\Phi(n,k) \tag{10-7}$$

where

$$\begin{aligned}\Phi(n,k) := {}&\tfrac{i}{2}\big\langle\big(f'[(\phi^\lambda)^2]+2f''[(\phi^\lambda)^2](\phi^\lambda)^2\big)(\rho\xi_k-\omega\eta_k),(\rho\xi_n-\omega\eta_n)\big\rangle\\ &+i\big\langle\big(f'[(\phi^\lambda)^2]+2f''[(\phi^\lambda)^2](\phi^\lambda)^2\big)(\bar{\omega}\eta_k+\bar{\rho}\xi_k),(\bar{\omega}\eta_n+\bar{\rho}\xi_n)\big\rangle\\ &+i\big\langle\big(3f''[(\phi^\lambda)^2](\phi^\lambda)^2+2f'''[(\phi^\lambda)^2](\phi^\lambda)^4\big)\rho\xi_k,\rho\xi_n\big\rangle\\ &-i\langle 3f''[(\phi^\lambda)^2](\phi^\lambda)^2\omega\eta_k,\omega\eta_n\rangle.\end{aligned}$$

Immediately we have

$$\Phi(n,k)+\overline{\Phi(k,n)}=0.$$

This together with (10-7) completes the proof for $K_3(n)$. □

11. Proof of Theorem 7.1

For simplicity, we present the proof of Theorem 7.1 for the case $d=3$; the proof can be easily modified to cover $d\geq 3$. The main difference is that, in controlling $\|\vec{R}(t)\|_{L^\infty(\mathbb{R}^d)}$ by $\|\vec{R}(t)\|_{H^k(\mathbb{R}^d)}$ for $d=3$ we take $k=2$ while in general we need $k=[d/2]+1$; see Section 5B.

Estimation strategy. This subsection discusses our strategy for studying the large time behavior of solutions.

We begin by introducing a family of space-time norms for measuring the decay of $z(t)$ and $\vec{R}(t)$ for $0\leq t\leq T$ with arbitrary T. We then prove that this family of norms satisfies a set of coupled inequalities, from which we can infer the desired large time asymptotic behavior.

We claim that

$$T_0 := \left|z^{(0)}\right|^{-1} \tag{11-1}$$

where $z^{(0)}$ is defined in Theorem 7.1.

Family of Norms.

$$\begin{aligned}Z(T) &:= \max_{t\leq T}(T_0+t)^{1/2}|z(t)|, & \mathcal{R}_1(T) &:= \max_{t\leq T}(T_0+t)\|\langle x\rangle^{-\nu}\vec{R}(t)\|_{H^2},\\ \mathcal{R}_2(T) &:= \max_{t\leq T}(T_0+t)\|\vec{R}(t)\|_\infty, & \mathcal{R}_3(T) &:= \max_{t\leq T}(T_0+t)^{7/5}\|\langle x\rangle^{-\nu}\tilde{R}(t)\|_2,\\ \mathcal{R}_4(T) &:= \max_{t\leq T}\|\vec{R}(t)\|_{H^2}, & \mathcal{R}_5(T) &:= \max_{t\leq T}\frac{(T_0+t)^{1/2}}{\log(T_0+t)}\|\vec{R}(t)\|_3\end{aligned} \tag{11-2}$$

where the constant ν is defined in Theorem 7.1.

Remark on choice of norms. It is clear that a combination of H^2, spatially weighted H^2 and L^∞ norms of $\vec{R}(t)$, as well as a bound on $|z(t)|$, are plausible choices of norms to control the large time behavior. This accounts for the definitions of $Z(T)$, $\mathcal{R}_1(T)$, $\mathcal{R}_2(T)$ and $\mathcal{R}_4(T)$. Our list of norms also includes estimation of the time decay of $\|\vec{R}(t)\|_3$, that is, $\mathcal{R}_5$, and the local L^2 norm of an auxiliary function $\tilde{R}(t)$, that is, $\mathcal{R}_3$. Why these two additional norms? As will be seen, $\zeta(t) = |z(t)|$ satisfies an equation of the form

$$\dot{\zeta} \sim -\kappa^2 \zeta^3 + c(t),$$

where $c(t)$ consists of various coupling terms (products) involving neutral mode amplitudes $z(t)$, the ground state $\phi^{\lambda(t)}$ and dispersive terms $\vec{R}(t)$. First, neglecting $c(t)$, we observe that $\zeta(t) \sim t^{-1/2}$. To treat $c(t)$ as a small perturbation for large t, it is necessary that it decays more rapidly than the term $\zeta^3(t) \sim t^{-3/2}$. Without any further decomposition of $\vec{R}(t)$, we find among the coupling terms one is of order $|z(t)| \cdot \|\langle x\rangle^{-\nu}\ \vec{R}(t)\|_2$. The expected decay rate of each factor implies this term is of order $t^{-3/2}$ for large t, which is of the *same* order as $\zeta^3(t)$. The resolution is to expand $\vec{R}(t)$ as a leading order part consisting of terms $R_{m,n} = z^m \bar{z}^n$ with $m + n = 2$ plus a more rapidly decaying correction $\tilde{R}(t)$ with $\|\langle x\rangle^{-\nu} \tilde{R}(t)\|_2 = \mathcal{O}(t^{-1-\delta})$ $(\delta > 0)$; see (8-1). This modification yields an equation with an improved correction term of order $|z(t)| \cdot \|\langle x\rangle^{-\nu} \tilde{R}(t)\|_2 = t^{-\frac{3}{2}-\delta}$ $(\delta > 0)$, which can be treated as a small perturbation in the large time dynamics.

Remark on the estimation strategy. See also [Buslaev and Sulem 2003; Soffer and Weinstein 2004]. Estimation of the norms $\mathcal{R}_j(T)$ proceeds as follows. We first express $\vec{R}$, the solution to Equation (9-2), in terms of the Duhamel integral equation, relative to the linear operator, $L(\lambda_1)$. Here, $\lambda_1 = \lambda(T)$, $T > 0$ is fixed and arbitrary. Namely,

$$\begin{aligned} \partial_t \vec{R} = L(\lambda)\vec{R} + \cdots \implies \partial_t P_c^{\lambda_1} \vec{R} &= L(\lambda_1) P_c^{\lambda_1} \vec{R} + P_c^{\lambda_1}\big(L(\lambda(t)) - L(\lambda_1)\big)\vec{R} + \cdots \\ \implies P_c^{\lambda_1} \vec{R}(t) &= e^{L(\lambda_1)t} \vec{R}(0) + \int_0^t e^{L(\lambda_1)(t-s)}(\cdots)\, ds. \end{aligned}$$

We can therefore apply the time-decay estimates of Theorem 5.7 to obtain bounds on local decay and L^∞ norms of $P_c^{\lambda_1} \vec{R}(t)$. However, we need bounds on $\vec{R}(t) = P_c^{\lambda(t)} \vec{R}(t)$. Since

$$\vec{R}(t) = P_c^{\lambda_1} \vec{R}(t) + P_{\text{disc}}^{\lambda_1} \vec{R}(t),$$

it suffices to bound $P_{\text{disc}}^{\lambda_1} \vec{R}(t)$. This is done as follows.

$$\begin{aligned} P_{\text{disc}}^{\lambda_1} R &= (P_{\text{disc}}^{\lambda_1} - P_{\text{disc}}^{\lambda(t)}) R(t) + P_{\text{disc}}^{\lambda(t)} R(t) = (P_{\text{disc}}^{\lambda_1} - P_{\text{disc}}^{\lambda(t)}) R(t) \qquad (\text{because } P_{\text{disc}}^{\lambda(t)} R(t) = 0) \\ &= (P_{\text{disc}}^{\lambda_1} - P_{\text{disc}}^{\lambda(t)}) P_{\text{disc}}^{\lambda_1} R(t) + (P_{\text{disc}}^{\lambda_1} - P_{\text{disc}}^{\lambda(t)}) P_c^{\lambda_1} R(t), \end{aligned}$$

which implies

$$\big(I - (P_{\text{disc}}^{\lambda_1} - P_{\text{disc}}^{\lambda(t)})\big) P_{\text{disc}}^{\lambda_1} R = (P_{\text{disc}}^{\lambda_1} - P_{\text{disc}}^{\lambda(t)}) P_c^{\lambda_1} R(t).$$

Therefore,

$$P_{\text{disc}}^{\lambda_1} R(t) = (I - \delta(\lambda, \lambda_1))^{-1} \delta(\lambda, \lambda_1) P_c^{\lambda_1} R(t)$$

and we estimate $R(t)$ in either a local energy $H^2(\mathbb{R}^d; \langle x\rangle^{-\sigma}\, dx)$ or $L^\infty(\mathbb{R}^d)$ via

$$\|R(t)\|_X \le \|P_c^{\lambda_1} R(t)\|_X + \|P_{\text{disc}}^{\lambda_1} R(t)\|_X \le \|P_c^{\lambda_1} R(t)\|_X + \|P_c^{\lambda_1} R(t)\|_X.$$

Here $\delta(\lambda,\lambda_1)=P^{\lambda_1}_{\text{disc}}-P^{\lambda(t)}_{\text{disc}}$ is of finite rank and of small norm proportional to $\int_t^T|\dot\lambda(s)|\,ds$.

We now derive the integral equation for $P_c^{\lambda_1}\vec R$, which is the basis for our time-decay estimates. If we write

$$L(\lambda(t))=L(\lambda_1)+L(\lambda(t))-L(\lambda_1),$$

then (9-9) for $P_c^{\lambda_1}\vec R$, which takes the form

$$\partial_t P_c^{\lambda_1}\vec R=L(\lambda_1)P_c^{\lambda_1}\vec R+(\lambda-\lambda_1+\dot\gamma)P_c^{\lambda_1}J\vec R+\cdots.$$

Recall that $L(\lambda)$ has two branches of essential spectrum $[i\lambda,i\infty)$ and $(-i\infty,-i\lambda]$. We use P_+ and P_- to denote the projection operators onto these two branches of the essential spectrum of $L(\lambda_1)$.

Lemma 11.1. *For any function h and any large constant $\nu>0$, we have*

$$\big\|\langle x\rangle^{\nu}(-\Delta+1)(P_c^{\lambda_1}J-i(P_+-P_-))h\big\|_2\le c\|\langle x\rangle^{-\nu}(-\Delta+1)h\|_2.$$

For $d=1$ the proof of this lemma can be found in [Buslaev and Sulem 2003]; the proof for $d\ge 3$ is similar, hence omitted here.

Equation (9-9) can be rewritten as

$$\partial_t P_c^{\lambda_1}\vec R=L(\lambda_1)P_c^{\lambda_1}\vec R+i(\dot\gamma+\lambda-\lambda_1)(P_+-P_-)\vec R+P_c^{\lambda_1}O_1\vec R+P_c^{\lambda_1}P_c^{\lambda(t)}\mathcal{G}-P_c^{\lambda_1}P_c^{\lambda(t)}JN(\vec R,z),\tag{11-3}$$

where O_1 is the operator defined by

$$O_1:=\dot\lambda P_{c\lambda}+L(\lambda)-L(\lambda_1)+\dot\gamma P_c^{\lambda}J-i(\dot\gamma+\lambda-\lambda_1)(P_+-P_-).\tag{11-4}$$

By (11-3) and the observation that the operators P_+, P_- and $L(\lambda_1)$ commute with each other, we have

$$P_c^{\lambda_1}\vec R=e^{tL(\lambda_1)+a(t,0)(P_+-P_-)}P_c^{\lambda_1}\vec R(0)+\int_0^t e^{(t-s)L(\lambda_1)+a(t,s)(P_+-P_-)}P_c^{\lambda_1}\big(O_1\vec R+P_c^{\lambda}\mathcal{G}-P_c^{\lambda}JN(\vec R,z)\big)\,ds\tag{11-5}$$

with $a(t,s)=i\int_s^t(\dot\gamma(\tau)+\lambda(\tau)-\lambda_1)\,d\tau$. We observe that $P_+P_-=P_-P_+=0$ and for any $t_1\le t_2$ the operator

$$e^{a(t_2,t_1)(P_+-P_-)}=e^{a(t_2,t_1)}P_++e^{-a(t_2,t_1)}P_-:\ H^2\to H^2$$

is uniformly bounded.

The following result, whose proof is given in the Appendix F, will be used repeatedly in our estimates:

Proposition 11.2. *Let $T_0\ge 2$. There exists a constant $c>0$ such that*

$$\int_0^t\frac{1}{(1+t-s)^{3/2}}\frac{1}{(T_0+s)^{\sigma}}\,ds\le\frac{c}{(T_0+t)^{\sigma}},\qquad \sigma\in[0,\tfrac32],\tag{11-6}$$

$$\int_0^t(t-s)^{-1/2}(T_0+s)^{-1}\,ds\le c(T_0+t)^{-1/2}\log(T_0+t).\tag{11-7}$$

Similar versions can be found in many literature, for example [Soffer and Weinstein 1999; Buslaev and Sulem 2003].

Estimate for $\mathcal{R}_1(T) := \max_{t\le T}(T_0+t)\|\langle x\rangle^{-\nu}\vec{R}(t)\|_{H^k}$.

Proposition 11.3. $\mathcal{R}_1 \le c\big(T_0\|\langle x\rangle^{\nu}\vec{R}(0)\|_{H^2}+\mathcal{R}_4^2\mathcal{R}_2+Z^2+T_0^{-1/2}(Z^3+Z\mathcal{R}_1+\mathcal{R}_4\mathcal{R}_2^2)\big).$

With a view toward proving the time decay estimate of Proposition 11.3, we now first give appropriate norm-estimates of the latter terms in (11-3).

First from the norm definitions (11-2) and Lemma 11.1, we estimate the $O_1\vec{R}$ and $\mathcal{G}$ terms

$$\begin{aligned}\|\langle x\rangle^{\nu}(-\Delta+1)O_1\vec{R}\|_2 &\le c(T_0+t)^{-3/2}Z\mathcal{R}_1,\\ \|\langle x\rangle^{\nu}(-\Delta+1)\mathcal{G}\|_2 &\le c(T_0+t)^{-3/2}Z^3.\end{aligned}\tag{11-8}$$

Next, we estimate the nonlinear term JN:

Lemma 11.4.

$$\|(-\Delta+1)JN(\vec{R},z)\|_1+\|(-\Delta+1)JN(\vec{R},z)\|_2\le c(T_0+t)^{-1}(\mathcal{R}_4^2\mathcal{R}_2+Z^2)+c(T_0+t)^{-3/2}(Z\mathcal{R}_1+\mathcal{R}_4\mathcal{R}_2^2).$$

Proof. Recall the definition

$$N_2(\vec{R},z) := -J\vec{N}(\vec{R},z)+\sum_{m+n=2,3} JN_{m,n}$$

in (9-17) and the decomposition N_2 as the sum of Loc and NonLoc in (8-5). By the fact $JN_{m,n}$ for $m+n=2,3$ are localized functions we have the estimate

$$\begin{aligned}&\|(-\Delta+1)(JN(\vec{R},z)-\mathrm{NonLoc})\|_1+\|(-\Delta+1)(JN(\vec{R},z)-\mathrm{NonLoc})\|_2\\ &\qquad\le c|z|(|z|+\|\langle x\rangle^{-\nu}\vec{R}\|_2)\le c((T_0+t)^{-1}Z^2+(T_0+t)^{-3/2}Z\mathcal{R}_1).\end{aligned}$$

More challenging is the term $NonLoc$ defined in (8-7), which is purely nonlinear, having no spatially localized factors. We use the estimate

$$\begin{aligned}&\|(-\Delta+1)\mathrm{NonLoc}\|_1+\|(-\Delta+1)\mathrm{NonLoc}\|_2\\ &\qquad\le c(\|\vec{R}\|_{H^2}^2\|\vec{R}\|_\infty+\|\vec{R}\|_{H^2}\|\vec{R}\|_\infty^2)\le c(T_0+t)^{-1}\mathcal{R}_4^2\mathcal{R}_2+c(T_0+t)^{-3/2}\mathcal{R}_4\mathcal{R}_2^2\end{aligned}$$

by the fact $f(x^2)x$ is of the order x^3 around $x=0$ for $d=3$. □

Proof of Proposition 11.3. By (11-5) and estimates (5-7), (5-11) for $d=3$ we have

$$\begin{aligned}&\|\langle x\rangle^{-\nu}(-\Delta+1)P_c^{\lambda_1}\vec{R}(t)\|_2\\ &\quad\le\|\langle x\rangle^{-\nu}(-\Delta+1)e^{tL(\lambda_1)}P_c^{\lambda_1}\vec{R}(0)\|_2\\ &\qquad+\left\|\int_0^t\langle x\rangle^{-\nu}(-\Delta+1)e^{(t-s)L(\lambda_1)}P_c^{\lambda_1}(O_1(s)\vec{R}+P_c^{\lambda}\mathcal{G}-P_c^{\lambda}JN(\vec{R},z))\,ds\right\|_2\\ &\quad\le c(1+t)^{-3/2}\|\langle x\rangle^{\nu}(-\Delta+1)\vec{R}(0)\|_2+\int_0^t(1+t-s)^{-3/2}\|\langle x\rangle^{\nu}(-\Delta+1)(O_1\vec{R}+P_c^{\lambda}\mathcal{G})\,ds\|_2\\ &\qquad+\int_0^t(1+t-s)^{-3/2}\big(\|(-\Delta+1)P_c^{\lambda}JN(\vec{R}(s),z)\|_1+\|(-\Delta+1)P_c^{\lambda}JN(\vec{R}(s),z)\|_2\big)\,ds.\end{aligned}$$

Therefore, by the estimates (11-8) and Lemma 11.4 we have

$$\|\langle x\rangle^{-\nu}(-\Delta+1)P_c^{\lambda_1}\vec{R}\|_2 \le c(1+t)^{-3/2}\|\langle x\rangle^{\nu}(-\Delta+1)\vec{R}(0)\|_2 + \big(\mathcal{R}_4^2\mathcal{R}_2+Z^2+T_0^{-1/2}(Z^3+Z\mathcal{R}_1+\mathcal{R}_4\mathcal{R}_2^2)\big)\int_0^t (1+t-s)^{-3/2}(T_0+s)^{-1}\,ds.$$

Using the time convolution estimate (11-6) we obtain

$$\|\langle x\rangle^{-\nu}(-\Delta+1)P_c^{\lambda_1}\vec{R}\|_2 \le c(T_0+t)^{-1}\big(T_0\|\langle x\rangle^{\nu}\vec{R}(0)\|_2+\mathcal{R}_4^2\mathcal{R}_2+Z^2+T_0^{-1/2}(Z^3+Z\mathcal{R}_1+\mathcal{R}_4\mathcal{R}_2^2)\big).$$

This implies Proposition 11.3. □

Estimate for $\mathcal{R}_2(T) := \max_{t\le T}(T_0+t)\|\vec{R}(t)\|_\infty$.

Proposition 11.5. $\mathcal{R}_2 \le c\big(T_0(\|\vec{R}(0)\|_1+\|\vec{R}(0)\|_{H^2})+Z^2+\mathcal{R}_4^2\mathcal{R}_2+T_0^{-1/2}(Z^3+Z\mathcal{R}_1+\mathcal{R}_1^2)\big)$.

To prove this we use the following result whose proof is very similar to that of Lemma 11.4.

Lemma 11.6.

$$\|P_c^{\lambda}JN(\vec{R},z)\|_1+\|P_c^{\lambda}JN(\vec{R},z)\|_{H^2} \le c(T_0+t)^{-1}(Z^2+\mathcal{R}_4^2\mathcal{R}_2)+c(T_0+t)^{-3/2}(Z^3+Z\mathcal{R}_1+\mathcal{R}_1^2).$$

Proof of Proposition 11.5. By estimate (5-9) for $d=3$ and (11-3) we have that

$$\begin{aligned}\|P_c^{\lambda_1}\vec{R}(t)\|_\infty &\le \|e^{tL(\lambda_1)}P_c^{\lambda_1}\vec{R}(0)\|_\infty\\ &\le c(1+t)^{-3/2}(\|\vec{R}(0)\|_1+\|\vec{R}(0)\|_{H^2})\\ &\quad + c\int_0^t (1+t-s)^{-3/2}(\|O_1(s)\vec{R}+P_c^{\lambda}\mathcal{G}\|_1+\|O_1(s)\vec{R}+P_c^{\lambda}\mathcal{G}\|_{H^2})\,ds\\ &\quad + c\int_0^t (1+t-s)^{-3/2}(\|P_c^{\lambda}JN(\vec{R},z)\|_1+\|P_c^{\lambda}JN(\vec{R},z)\|_{H^2})\,ds.\end{aligned}$$

By the properties of O_1 (see (11-4)) and $\mathcal{G}$ (see (9-9)) we have

$$\|O_1(s)\vec{R}+P_c^{\lambda}\mathcal{G}\|_1+\|O_1(s)\vec{R}+P_c^{\lambda}\mathcal{G}\|_{H^2} \le c(T_0+t)^{-3/2}(Z\mathcal{R}_1+Z^3).$$

This, together with Lemma 11.6, yields

$$\|P_c^{\lambda_1}\vec{R}(t)\|_\infty \le c(T_0+t)^{-1}\big(T_0\|\vec{R}(0)\|_1+T_0\|\vec{R}(0)\|_{H^2}+Z^2+\mathcal{R}_4^2\mathcal{R}_2+T_0^{-1/2}(Z^3+Z\mathcal{R}_1+\mathcal{R}_1^2)\big). \quad □$$

Estimate for $\mathcal{R}_5(T) := \max_{t\le T}\dfrac{(T_0+t)^{1/2}}{\log(T_0+t)}\|\vec{R}(t)\|_3$@.

Proposition 11.7. $\mathcal{R}_5 \le c\big(T_0(\|\vec{R}(0)\|_1+\|\vec{R}(0)\|_{H^2})+Z^2+T_0^{-1/2}(\mathcal{R}_5^2\mathcal{R}_2+Z^3+Z\mathcal{R}_1+\mathcal{R}_1^2+\mathcal{R}_2^2)\big)$.

Lemma 11.8. $\|JN(\vec{R},z)\|_{3/2} \le c(T_0+t)^{-1}Z^2+c(T_0+t)^{-3/2}(\mathcal{R}_5^2\mathcal{R}_2+Z^3+Z\mathcal{R}_1+\mathcal{R}_1^2+\mathcal{R}_2^2)$.

Proof. As in the proof of Lemma 11.4 we decompose JN into the localized term

$$JN(\vec{R},z)-\text{NonLoc}$$

and nonlocalized term NonLoc. The estimate of the first term is similar to that of Lemma 11.4 and hence omitted. The nonlocal term NonLoc defined in (8-7) admits the estimate

$$\|\mathrm{NonLoc}\|_{3/2} \le c\Big(\int |\vec{R}|^{5/4}\Big)^{2/3} \le c\|\vec{R}\|_3^2\|\vec{R}\|_\infty.$$

By using the definitions of estimating functions on all the terms above we have the lemma. □

Proof of Proposition 11.7. By estimate (5-10) for $d=3$ and Lemma 11.4 we have that

$$\begin{aligned}\|P_c^{\lambda_1}\vec{R}(t)\|_3 &\le \|e^{tL(\lambda_1)}P_c^{\lambda_1}\vec{R}(0)\|_3 + \int_0^t \|e^{(t-s)L(\lambda_1)}P_c^{\lambda_1}(O_1(s)\vec{R}+P_c^{\lambda}\mathcal{G}-P_c^{\lambda}JN(\vec{R},z))\|_3\,ds\\ &\le c(1+t)^{-1/2}(\|\vec{R}(0)\|_1+\|\vec{R}(0)\|_{H^2})\\ &\qquad + c\int_0^t (t-s)^{-1/2}\|O_1(s)\vec{R}+P_c^{\lambda}\mathcal{G}\|_{3/2}\,ds + \int_0^t (t-s)^{-1/2}\|P_c^{\lambda}JN(\vec{R},z)\|_{3/2}\,ds.\end{aligned}$$

By the properties of O_1 (see (11-4)) and $\mathcal{G}$ (see (9-9)) we have

$$\|O_1(s)\vec{R}+P_c^{\lambda}\mathcal{G}\|_{3/2} \le c(T_0+t)^{-3/2}(Z\mathcal{R}_1+Z^3).$$

This, together with Lemma 11.6 and (11-7), implies

$$\begin{aligned}&\|P_c^{\lambda_1}\vec{R}(t)\|_3\\ &\le c(T_0+t)^{-1/2}\log(T_0+t)\big(T_0^{1/2}\|\vec{R}(0)\|_1+T_0\|\vec{R}(0)\|_{H^2}+Z^2+T_0^{-1/2}(\mathcal{R}_5^2\mathcal{R}_2+Z^3+Z\mathcal{R}_1+\mathcal{R}_1^2+\mathcal{R}_2^2)\big).\end{aligned}$$

This estimate and the definition of $\mathcal{R}_5$ yield the proposition. □

Estimate for $\mathcal{R}_3(T) := \max_{t\le T}(T_0+t)^{7/5}\|\langle x\rangle^{-\nu}\tilde{R}(t)\|_2$.

Proposition 11.9. *Let the constant ν the same as that in (5-6) and (5-7) with $d=3$. Then*

$$\mathcal{R}_3 \le c\big(T_0^{3/2}(\|\langle x\rangle^{\nu}\vec{R}(0)\|_2+|z(0)|^2)\big)+cT_0^{-1/20}(Z^3+Z\mathcal{R}_3+Z\mathcal{R}_1+\mathcal{R}_5^3+R_2^2\mathcal{R}_4).$$

As usual we estimate the nonlinear term $N_2(\vec{R},z)$.

Lemma 11.10. $\displaystyle\int_0^t \|\langle x\rangle^{-\nu}e^{(t-s)L(\lambda)}P_c^{\lambda_1}P_c^{\lambda}N_2(\vec{R},z)\|_2\,ds \le c(T_0+t)^{-7/5}T_0^{-1/20}(Z^3+Z\mathcal{R}_1+\mathcal{R}_5^3+\mathcal{R}_2^2\mathcal{R}_4).$

Proof. We start with the function N_2. Recall that $N_2 = \mathrm{Loc}+\mathrm{NonLoc}$ in (8-5) and the estimate of Loc after that. The nonlocal term NonLoc defined in (8-7) admits the estimate

$$\|\mathrm{NonLoc}\|_1+\|\mathrm{NonLoc}\|_2 \le c(\|\vec{R}\|_3^3+\|\vec{R}\|_6^3) \le c(\|\vec{R}\|_3^3+\|\vec{R}\|_\infty^2\|\vec{R}\|_2).$$

By the definition of estimating function we have

$$\begin{aligned}\|\mathrm{NonLoc}\|_1+\|\mathrm{NonLoc}\|_2 &\le c(T_0+t)^{-3/2}(\log(T_0+t))^{3/2}\mathcal{R}_5^3+(T_0+t)^{-2}\mathcal{R}_2^2\mathcal{R}_4\\ &\le c(T_0+t)^{-7/5}T_0^{-1/20}(\mathcal{R}_5^3+\mathcal{R}_2^2\mathcal{R}_4).\end{aligned}$$

Finally, by the propagator estimates (5-9) and (5-11), we have

$$\int_0^t \|\langle x\rangle^{-\nu} e^{(t-s)L(\lambda)} P_c^{\lambda_1} P_c^{\lambda} N_2(\vec{R}, z)\|_2\, ds \\ \leq c \int_0^t (1+t-s)^{-3/2} \big(\|\mathrm{NonLoc}(\vec{R}, z)\|_1 + \|\mathrm{NonLoc}(\vec{R}, z)\|_2 + \|\langle x\rangle^{\nu} \mathrm{Loc}\|_2 \big)\, ds.$$

This together with the estimates of Loc and NonLoc above yields the lemma. □

Proof of Proposition 11.9. By the same techniques as in deriving (11-3) we have

$$\partial_t P_c^{\lambda_1} \tilde{R} = L(\lambda_1) P_c^{\lambda_1} \tilde{R} + i(\dot{\gamma} + \lambda - \lambda_1)(P_+ - P_-)\tilde{R} + P(z, \bar{z})\tilde{R} + P_c^{\lambda_1} S_2(z, \bar{z}) + P_c^{\lambda_1} P_c^{\lambda} N_2(\vec{R}, z), \quad (11\text{-}9)$$

where the operator $P(z, \bar{z})$ is defined as

$$P(z, \bar{z}) := P_c^{\lambda_1} M_2(z, \bar{z}) - i(\dot{\gamma} + \lambda - \lambda_1)(P_+ - P_-) + P_c^{\lambda_1}(L(\lambda) - L(\lambda_1))$$

and the terms $P_c^{\lambda} N_2(\vec{R}, z)$, $S_2(z, \bar{z})$, $M_2(z, \bar{z})$ are defined in Theorem 8.1.

Rewrite (11-9) in the integral form by the Duhamel principle to obtain

$$\|\langle x\rangle^{-\nu} P_c^{\lambda_1} \tilde{R}(t)\|_2 \leq \|\langle x\rangle^{-\nu} e^{tL(\lambda_1)} P_c^{\lambda_1} \tilde{R}(0)\|_2 \\ + \int_0^t \big\| \langle x\rangle^{-\nu} e^{(t-s)L(\lambda_1)} \big(P(z, \bar{z})\tilde{R} + P_c^{\lambda_1} S_2(z, \bar{z}) + P_c^{\lambda_1} P_c^{\lambda} N_2(\vec{R}, z) \big) \big\|_2\, ds. \quad (11\text{-}10)$$

For the left-hand side we claim that

$$\|\langle x\rangle^{-\nu} e^{tL(\lambda_1)} P_c^{\lambda_1} \tilde{R}(0)\|_2 \leq c(1+t)^{-3/2} (\|\langle x\rangle^{\nu} \vec{R}(0)\|_2 + |z(0)|^2). \quad (11\text{-}11)$$

Indeed recall that

$$\tilde{R} = \vec{R} - \sum_{m+n=2} R_{m,n}$$

with $R_{m,n}$ defined in (9-13). Therefore, with the time-dependent of $\tilde{R}$, λ and z, we have

$$\|\langle x\rangle^{-\nu} e^{tL(\lambda_1)} P_c^{\lambda_1} \tilde{R}(0)\|_2 \leq \|\langle x\rangle^{-\nu} e^{tL(\lambda_1)} P_c^{\lambda_1} \vec{R}(0)\|_2 + \sum_{m+n=2} \|\langle x\rangle^{-\nu} e^{tL(\lambda_1)} P_c^{\lambda_1} R_{m,n}(0)\|_2.$$

By (5-6) and the fact that $R_{m,n}$ is the summation of terms of order $|z|^2$ we have

$$\|\langle x\rangle^{-\nu} e^{tL(\lambda_1)} P_c^{\lambda_1} R_{m,n}(0)\|_2 \leq c|z(0)|^2 (1+t)^{-3/2}.$$

This, together with the estimate

$$\|\langle x\rangle^{-\nu} e^{tL(\lambda_1)} P_c^{\lambda_1} \vec{R}(0)\|_2 \leq c(1+t)^{-3/2} \|\langle x\rangle^{\nu} \vec{R}(0)\|_2,$$

implies (11-11).

Use (5-6) on the right-hand side of (11-10) to obtain

$$\int_0^t \big\|\langle x\rangle^{-\nu} e^{(t-s)L(\lambda_1)}\big(P(z,\bar z)\tilde R + P_c^{\lambda_1} S_2(z,\bar z) + P_c^{\lambda_1} P_c^{\lambda} N_2(\vec R, z)\big)\big\|_2\, ds$$
$$\le \int_0^t (1+t-s)^{-3/2}\big(\|\langle x\rangle^{\nu} P(z,\bar z)\tilde R\|_2 + \|N_2(\vec R,z)\|_1 + \|N_2(\vec R,z)\|_2\big)\, ds$$
$$+ \int_0^t \|\langle x\rangle^{-\nu} e^{(t-s)L(\lambda_1)} P_c^{\lambda_1} S_2(z,\bar z)\|_2\, ds.$$

We estimate these terms in detail:

(A) By the definition of $S_2(z,\bar z)$ in (8-3) and estimate (5-6) with $d=3$ we have that

$$\left|\int_0^t \|\langle x\rangle^{-\nu} e^{(t-s)L(\lambda_1)} P_c^{\lambda_1} S_2(z,\bar z)\|_2\, ds\right| \le cZ^3 \int_0^t (1+t-s)^{-3/2}(T_0+s)^{-3/2}\, ds \le cZ^3 (T_0+t)^{-3/2}.$$

(B) By the definition of $P(z,\bar z)$ and the estimate of $M_2(z,\bar z)$ in (8-4),

$$\|\langle x\rangle^{\nu} P(z(s),\bar z(s))\tilde R(s)\|_2 \le c|z|\cdot \|\langle x\rangle^{-\nu}\tilde R(s)\|_2 \le c(T_0+s)^{-19/20} Z\mathcal{R}_3.$$

Hence by (11-6),

$$\int_0^t (1+t-s)^{-3/2}\|\langle x\rangle^{\nu} P(z(s),\bar z(s))\tilde R\|_2\, ds$$
$$\le cT_0^{-1/20} Z\mathcal{R}_3 \int_0^t (1+t-s)^{-3/2}(T_0+s)^{-7/5}\, ds \le cT_0^{-1/20} Z\mathcal{R}_3 (T_0+t)^{-7/5}.$$

These, together with Lemma 11.10, implies

$$\|\langle x\rangle^{-\nu} P_c^{\lambda_1}\tilde R\|_2$$
$$\le c(1+t)^{-3/2}(\|\langle x\rangle^{\nu}\vec R(0)\|_2 + |z(0)|^2) + c(T_0+t)^{-7/5} T_0^{-1/20}(Z\mathcal{R}_1 + Z\mathcal{R}_3 + Z^3 + \mathcal{R}_5^3 + \mathcal{R}_2^2\mathcal{R}_4)$$
$$\le c(T_0+t)^{-7/5}\big(T_0^{7/5}\|\langle x\rangle^{\nu}\vec R(0)\|_2 + T_0^{7/5}|z(0)|^2 + T_0^{-1/20}(Z\mathcal{R}_1 + Z\mathcal{R}_3 + Z^3 + \mathcal{R}_5^3 + \mathcal{R}_2^2\mathcal{R}_4)\big),$$

which implies the proposition. □

Proposition 11.11. $\mathcal{R}_4^2 \le \|\vec R(0)\|_{H^2}^2 + cT_0^{-1}(\mathcal{R}_1^2 + Z^2\mathcal{R}_1 + Z^2\mathcal{R}_1^2 + \mathcal{R}_4^2\mathcal{R}_2^2).$

Before the proof we estimate the nonlinear terms.

Lemma 11.12. $\big|\big\langle(-\Delta+1)P_c^{\lambda} JN(\vec R,z), (-\Delta+1)\vec R\big\rangle\big| \le c(T_0+t)^{-2}(Z^2\mathcal{R}_1 + \mathcal{R}_4^2\mathcal{R}_2^2).$

Proof. As in Lemma 11.4 we decompose $J\vec N$ into the localized term Loc and the nonlocalized NonLoc defined in (8-7). The Localized part satisfies the estimate

$$|\langle(-\Delta+1)\mathrm{Loc}, (-\Delta+1)\vec R\rangle| \le c(|z|^2 + \|\langle x\rangle^{-\nu}\vec R\|_2^2).$$

By the definition of NonLoc in (8-7) we obtain

$$|\langle(-\Delta+1)\mathrm{NonLoc}, (-\Delta+1)\vec R\rangle| \le c\|(-\Delta+1)\vec R\|_2^2\|\vec R\|_\infty^2.$$

This together with the definitions of estimating functions implies the lemma. □

Proof of Proposition 11.11. By (9-9) we have

$$\partial_t\langle(-\Delta+1)\vec{R},(-\Delta+1)\vec{R}\rangle=\Big\langle(-\Delta+1)\frac{d}{dt}\vec{R},(-\Delta+1)\vec{R}\Big\rangle+\Big\langle(-\Delta+1)\vec{R},(-\Delta+1)\frac{d}{dt}\vec{R}\Big\rangle=\sum_{n=1}^{4}K_n,$$

with

$$\begin{aligned}
K_1&:=\langle(-\Delta+1)(L(\lambda)+\dot{\gamma}J)\vec{R},(-\Delta+1)\vec{R}\rangle+\langle(-\Delta+1)\vec{R},(-\Delta+1)(L(\lambda)+\dot{\gamma}J)\vec{R}\rangle,\\
K_2&:=\dot{\lambda}\langle(-\Delta+1)P_{c\lambda}\vec{R},(-\Delta+1)\vec{R}\rangle+\dot{\lambda}\langle(-\Delta+1)\vec{R},(-\Delta+1)P_{c\lambda}\vec{R}\rangle,\\
K_3&:=-\langle(-\Delta+1)P_c^{\lambda}JN(\vec{R},z),(-\Delta+1)\vec{R}\rangle-\langle(-\Delta+1)\vec{R},(-\Delta+1)P_c^{\lambda}JN(\vec{R},z)\rangle,\\
K_4&:=\langle(-\Delta+1)P_c^{\lambda}\mathcal{G},(-\Delta+1)\vec{R}\rangle+\langle(-\Delta+1)\vec{R},(-\Delta+1)P_c^{\lambda}\mathcal{G}\rangle.
\end{aligned}$$

Recall the definition of the operator $L(\lambda)$ in (5-1). By the observation $J^*=-J$ and the fact that $JL(\lambda)$ is selfadjoint we cancel all the nonlocal terms in K_1:

$$|K_1|\le c\|\langle x\rangle^{-\nu}\vec{R}\|_{H^2}^2\le c(T_0+t)^{-2}\mathcal{R}_1^2.$$

By observing that $|\dot{\lambda}|=\mathcal{O}(|z|^2)$ and $P_{c\lambda}\vec{R}$ is localized we have that

$$|K_2|\le c|z(t)|^2\|\langle x\rangle^{-\nu}\vec{R}(t)\|_{H^2}^2\le c(T_0+t)^{-2}Z^2(t)\mathcal{R}_1^2(t).$$

By the lemma we just prove, we have

$$|K_3|\le c(T_0+t)^{-2}(Z^2\mathcal{R}_1^2+\mathcal{R}_4^2\mathcal{R}_2^2).$$

By the property of $P_c^{\lambda}\mathcal{G}$ in (9-9) we have

$$|K_4|\le c|z|^2\|\langle x\rangle^{-\nu}\vec{R}\|_{H^2}\le c(T_0+t)^{-2}Z^2\mathcal{R}_1.$$

Collecting all the estimates above, we obtain

$$\Big|\frac{d}{dt}\langle(-\Delta+1)\vec{R},(-\Delta+1)\vec{R}\rangle\Big|\le c(T_0+t)^{-2}(\mathcal{R}_1^2+Z^2\mathcal{R}_1+Z^2\mathcal{R}_1^2+\mathcal{R}_4^2\mathcal{R}_2^2).$$

After integrating the equation above from 0 to t we have proposition. □

Estimate for $Z(T)=\max_{t\le T}(T_0+t)^{1/2}|z(t)|$. Recall that by (FGR)

$$z^*(Z(z,\bar{z})+Z^*(z,\bar{z}))z\ge C|z|^4.$$

Proposition 11.13. *There exists an order one constant $m>0$ such that if $m<T_0<|z(0)|^{-2}$ then*

$$Z(T)\le 1+\frac{K}{T_0^{2/5}}Z(T)(Z(T)+\mathcal{R}_1^2(T)+\mathcal{R}_2^2(T)+Z(T)\mathcal{R}_3(T)).$$

Proof. By (8-11) we have

$$\frac{d}{dt}|z|^2=-z^*(Z(z,\bar{z})+Z^*(z,\bar{z}))z+\Re(\bar{z}\text{Remainder}(t))$$

which can be transformed into a Riccati inequality:

$$\partial_t|z(t)|^2\le -C|z(t)|^4+2|z(t)||\text{Remainder}(t)|.$$

By (8-8),

$$|z(t)|\,|\mathrm{Remainder}(t)| \le \frac{c}{(T_0+t)^{2+\delta}} Z(T)(Z(T)+\mathcal{R}_1^2(T)+\mathcal{R}_2^2(T)+Z(T)\mathcal{R}_3(T)),$$

where $\delta = 2/5$.

Lemma 11.14. *Suppose that $z(t)$ is any function satisfying the equation*

$$\partial_t |z(t)|^2 \le -|z(t)|^4 + g(t), \qquad z(0) = z_0, \tag{11-12}$$

where $g(t)$ is a function satisfying the estimate

$$|g(t)| \le c_\#(T_0+t)^{-2-\delta} \tag{11-13}$$

with the constants $c_\#, \delta > 0$. Then there exists $K > 0$ independent of T_0 and $c_\#$ such that if $c_\# T_0^{-\delta}$ is sufficiently small then the function $z(t)$ in (11-12) admits the bound

$$|z(t)| \le \frac{1+Kc_\# T_0^{-\delta}}{(\kappa+t)^{1/2}} \tag{11-14}$$

where $\kappa = \min\{T_0, |z_0|^{-2}\}$.

The proof of this lemma is in Appendix G.

We now chose

$$m < T_0 < |z(0)|^{-2}$$

where m is an order one positive constant. Then,

$$Z(T) \le 1 + \frac{K}{T_0^{2/5}} Z(T)(Z(T)+\mathcal{R}_1^2(T)+\mathcal{R}_2^2(T)+Z(T)\mathcal{R}_3(T)). \qquad \square$$

Closing the estimates.

Proof of Theorem 7.1. We seek to obtain T-independent bounds on $\mathcal{R}_j(T)$ and $Z(T)$ defined in (11-2). This will be achieved by choosing the parameter T_0 in the norm definitions sufficiently large and the data $R(0)$ sufficiently small with T_0 and $R(0)$ related in a manner to be specified.

Define

$$M(T) := \sum_{n\neq 4} \mathcal{R}_n(T), \qquad S := T_0^{3/2}(\|\vec{R}(0)\|_{H^2} + \|\langle x\rangle^{\nu}\vec{R}(0)\|_2)$$

where T_0 is defined in (11-1). By the conditions in (7-1) we have that $\mathcal{R}_4(0)$ is small and $M(0)$ and $Z(0)$ are bounded.

Recall the estimates of $\mathcal{R}_n$ for $n = 1, 2, 3, 4, 5$ and Z in Propositions 11.3, 11.5, 11.9, 11.11, 11.7 and 11.13. By plugging the estimate of Z and $\mathcal{R}_4$ in Propositions 11.13, 11.11 into Propositions 11.3, 11.5 and 11.7, we obtain

$$\begin{aligned} M(T) &\le c(S+1) + (R_4(T) + T_0^{-1/20}) P(M(T), Z(T)), \\ Z(T) &\le 1 + T_0^{-1/20} P(M(T), Z(T)), \\ \mathcal{R}_4^2(T) &\le \|\vec{R}(0)\|_{H^2}^2 + T_0^{-1} P(M(T), Z(T)) \end{aligned} \tag{11-15}$$

where $P(x, y) > 0$ is a polynomial in x and y. Using an implicit-function-theorem type argument (see below) we have that if S and $M(0)$ are bounded then

$$M(T) + Z(T) \leq \mu(S) \quad \text{and} \quad \mathcal{R}_4 \ll 1 \tag{11-16}$$

where μ is a bounded function for S bounded. By the definitions of $\mathcal{R}_j(T)$ and $Z(T)$ there exists some constant c such that

$$\|\langle x\rangle^{-\nu}\vec{R}(t)\|_2, \|\vec{R}(t)\|_\infty \leq c(T_0+t)^{-1}, \qquad |z(t)| \leq c(T_0+t)^{-1/2}, \tag{11-17}$$

which is statement (B) in Theorem 7.1.

By the bound of Remainder in (8-8) and the estimates (11-17) we have

$$|\text{Remainder}| \leq c(T_0+t)^{-19/5}$$

which, together with (8-11), implies statement (A).

The convergence of λ comes from (8-9) and the fact that Remainder is integrable at ∞. □

In the following we prove (11-15) implies (11-16) by using implicit function theorem. For the other methods we refer to [Soffer and Weinstein 1999; 2004; Tsai and Yau 2002b; 2002c; Buslaev and Sulem 2003; Tsai 2003; Cuccagna and Mizumachi 2008]. First we transform the inequalities by taking square root of the third equation of (11-15) and plugging it into the first one, then

$$\begin{aligned}
M(T) &\leq c(S+1) + (\|\vec{R}(0)\|_{H^2} + T_0^{-1/20})P(M(T), Z(T)),\\
Z(T) &\leq 1 + T_0^{-1/20}P(M(T), Z(T)),\\
\mathcal{R}_4(T) &\leq \|\vec{R}(0)\|_{H^2} + T_0^{-1/20}P(M(T), Z(T)).
\end{aligned}$$

In what follows we use this equation instead of (11-15). Define a vector function $F_{\epsilon,\delta}(\tilde{M}, \tilde{Z})$ as

$$F_{\epsilon,\delta}(\tilde{M}, \tilde{Z}) := (F^{(1)}_{\epsilon,\delta}(\tilde{M}, \tilde{Z}), F^{(2)}_{\epsilon}(\tilde{M}, \tilde{Z}), F^{(3)}_{\epsilon,\delta}(\tilde{M}, \tilde{Z}))$$

with

$$F^{(1)}_{\epsilon,\delta}(\tilde{M}, \tilde{Z}) := c(S+1) + (\delta+\epsilon)P(\tilde{M}, \tilde{Z}),\;\; F^{(2)}_{\epsilon}(\tilde{M}, \tilde{Z}) := 1 + \epsilon P(\tilde{M}, \tilde{Z}),\;\; F^{(3)}_{\epsilon,\delta} := \delta + \epsilon P(\tilde{M}, \tilde{Z}).$$

Immediately we can see that

$$M_0 = c(1+S), \quad Z_0 = 1, \quad R_0 = 0$$

is a solution to the equation

$$(M_0, Z_0, R_0) = F_{0,0}(M_0, Z_0).$$

Define a closed set

$$\Sigma := [0, 2c(S+1)] \times [0, 2] \times [0, 1].$$

Lemma 11.15. *There exists $\delta_0 \geq 0$ such that if $\epsilon, \delta \in [0, \delta_0]$ then*

$$(\tilde{M}, \tilde{Z}, \tilde{R}) = F_{\epsilon,\delta}(\tilde{M}, \tilde{Z}) \tag{11-18}$$

has a unique solution in Σ. Moreover, for any continuous functions $M, Z, \mathcal{R} : \mathbb{R}^+ \to \mathbb{R}^+$, satisfying

$$\big(M(0), Z(0), \mathcal{R}(0)\big) \leq (\tilde{M}, \tilde{Z}, \tilde{R}) \quad \textit{and} \quad \big(M(t), Z(t), R(t)\big) \leq F_{\epsilon,\delta}\big(M(t), Z(t)\big),$$

we have

$$\big(M(t), Z(t), \mathfrak{R}(t)\big) \le (\tilde{M}, \tilde{Z}, \tilde{R}). \tag{11-19}$$

for any time t.

Proof. The proof of existence and uniqueness of the solution is not difficult by observing

$$\big\|\big(\partial_M F_{\epsilon,\delta}(M, Z), \partial_Z F_{\epsilon,\delta}(M, Z), \partial_{\mathfrak{R}} F_{\epsilon,\delta}(M, Z)\big)\big\| \le c(\delta+\epsilon)$$

if $(M, Z, \mathfrak{R}) \in \Sigma$. Hence by implicit function theorem we have that if $c(\epsilon+\delta) \le 1/2$ there exists a unique solution to (11-18).

We next prove (11-19) by contradiction. Suppose that (11-19) fails at time t. Since $(M(t), Z(t), \mathfrak{R}(t))$ is continuous there exists a time $t_1 \le t$ such that $(M(t_1), Z(t_1), \mathfrak{R}(t_1)) \in \Sigma$ and (11-19) does not hold. Without loss of generality we assume $t = t_1$. Then by subtracting the inequality for $(M(t), Z(t), \mathfrak{R}(t))$ by (11-18) we get

$$M(t) - \tilde{M} \le (\delta+\epsilon)\big(K_1(M(t)-\tilde{M}) + K_2(Z(t)-\tilde{Z})\big)$$

and

$$Z(t) - \tilde{Z}, \mathfrak{R}(t) - \tilde{R} \le \epsilon\big(K_3(M(t)-\tilde{M}) + K_4(Z(t)-\tilde{Z})\big)$$

for some K_n with $n = 1, 2, 3, 4$ depending on $(M(t), Z(t), \mathfrak{R}(t))$ and $(\tilde{M}, \tilde{Z}, \tilde{R})$. By the fact that

$$(\tilde{M}, \tilde{Z}, \tilde{R}), (M(t), Z(t), \mathfrak{R}(t)) \in \Sigma$$

and $P(x, y)$ is a polynomial with positive coefficient, we have that K_n are *positive* and bounded. By these inequalities and the fact $0 \le \epsilon, \delta \ll 1$ we derive (11-19). This contradicts our assumption. Thus (11-19) holds for any time $t \ge 0$. □

12. Summary and discussion

We have extended the asymptotic stability / scattering theory of solitary waves of the nonlinear Schrödinger/Gross–Pitaevskii (NLS/GP) equation to the important case where the linearized dynamics about the Lyapunov stable bound state has degenerate neutral modes. This is the prevalent case in situation where the equation is invariant under a nontrivial symmetry. We construct a class of multiwell potentials to which the theory applies. The current theory, as all previous work on soliton scattering in systems with nontrivial neutral modes, requires a Fermi Golden Rule (FGR) nondegeneracy hypothesis. The analytical verification of this hypothesis for either specific or generic NLS/GP systems is an open question. Numerical experiments for the time-dependent NLS/GP equations, in which decay rates of neutral modes are measured, are consistent with the generic validity of the (FGR) nondegeneracy hypothesis.

We conclude by mentioning an interesting direction for further exploration:

Semiclassical limits and higher order nonlinear Fermi Golden Rule. A problem of great interest is NLS/GP on $\mathbb{R}^d$ in the semiclassical limit:

$$i\partial_t\psi = -\Delta\psi + V(hx)\psi - f(|\psi|^2)\psi, \qquad \psi(x, 0) = \psi_0(x)$$

where $0<h\ll 1$. The nonlinearity is taken to be focusing (attractive) but subcritical. Using the Lyapunov–Schmidt method it has been shown in [Floer and Weinstein 1986; Oh 1988; Ambrosetti et al. 1996] that for h sufficiently small a soliton concentrated at a nondegenerate critical point of V can be constructed.

The soliton, constructed in this manner, is soliton of the translation invariant nonlinear Schrödinger equation, scaled to be highly concentrated about the critical point of V. Therefore, the linearized operator $JH^h(\lambda)$ is expected to have spectrum, quite closely related to the linearization about the translation invariant NLS soliton. If the soliton is concentrated near a minimum of V, then it is Lyapunov stable [Oh 1988], and therefore the spectrum of $JH^h(\lambda)$ is a subset of the imaginary axis. As we have seen for NLS/GP, there is a two-dimensional generalized eigenspace corresponding to an eigenvalue zero. h being small implies that the $2 \times d$ *zero modes* associated with the translation symmetry

$$\psi(x,t) \mapsto \psi(x+x_0,t)$$

and Galilean symmetry

$$\psi(x,t) \mapsto e^{iv\cdot(x-vt)}\psi(x-2vt,t)$$

perturb to d complex conjugate pairs of eigenvalues. Although we expect semiclassical, highly localized solitons to be asymptotically stable and for the degenerate neutral modes to damp by resonant radiation damping, as elucidated in this article, we note that for h very small, the complex conjugate neutral modes of $JH^h(\lambda)$ are very close to zero and the condition $2E(\lambda)-\lambda > 0$, which is necessary (although not sufficient) for the Fermi Golden Rule resonance condition (FGR) to hold, fails. It remains an open question to derive the normal form when resonance of discrete modes with the continuum occurs at some arbitrary order in the coupling parameter g (recall $f(|\psi|^2)\psi = -g|\psi|^2\psi$ and see also the discussion in Section 1). For results in this direction, see [Gang 2007; Cuccagna and Mizumachi 2008].

Appendix A. A class of multiwell potentials for which $-\Delta + V$ satisfies condition (Eig_V) and $L(\lambda)$ satisfies (SA) and (Thresh_λ)

In this section we find an example $-\Delta+V$ in a subspace of $L^2(\mathbb{R}^3)$ satisfying condition (**Eig_V**), motivated by the study of double well potentials. Define

$$\mathcal{A} := \{f : \mathbb{R}^3 \to \mathbb{C} \mid f(-x) = f(x) \text{ for any } \ x\}.$$

Observe that $\mathcal{A}$ is a self-closed subspace, that is, if $f_1, f_2 \in \mathcal{A}$ then $f_1+f_2, f_1 f_2, \Delta f_1 \in \mathcal{A}$. Hence we can study (1-1) in the space $\mathcal{A} \cap L^2(\mathbb{R}^3)$ and obtain all the results. The following is the main result

Proposition A.1. *There exists a potential V such that the linear operator $-\Delta+V$ acting on the subspace $\mathcal{A} \cap L^2(\mathbb{R}^3)$ has two eigenvalues $e_0 < e_1 < 0$ with $2e_1 > e_0$. e_0, the lowest eigenvalue, is simple, and eigenvalue e_1 is degenerate with multiplicity 2. Moreover the operator*

$$1 + (-\Delta + i0)^{-1} V : \langle x\rangle^2 L^2 \to \langle x\rangle^2 L^2$$

is invertible.

If the nonlinearity $f(x) = x$ and if $|\lambda - |e_0||$ is sufficiently small and ϕ^λ is the ground state satisfying

$$-\Delta\phi^\lambda + V\phi^\lambda + \lambda\phi^\lambda - (\phi^\lambda)^3 = 0$$

then we have the following results for the linearized operator $L(\lambda)$ defined in (5-1).

Proposition A.2. *The operator $L(\lambda)$ satisfies the spectral conditions (SA) and (Thresh_λ).*

Proposition A.1 is implied by Proposition A.5 below. Proposition A.2 will be proved at the end of this section.

As proved in [Albeverio et al. 2005, Theorem 1.1.4 on page 116] the operator $-\Delta - q\delta(x)$ has only one eigenfunction, that is, the ground state, for any $q > 0$. By this observation we have:

Lemma A.3. *For any $q > 0$, there exists a constant $\lambda \in (0, \infty)$ such that the operators*

$$-\Delta - q\lambda^{-\frac{3}{2}} e^{-\frac{|x|^2}{\lambda}}, \qquad -\Delta - \frac{1}{3} q\lambda^{-\frac{3}{2}} e^{-\frac{|x|^2}{\lambda}}$$

each have only one eigenfunction in $\mathcal{A}$.

To facilitate later discussions we define

$$W := q\lambda^{-3/2} e^{-|x|^2/\lambda}.$$

We start with constructing a family of operators. Define

$$M_1 := (m, 0, 0), \quad M_2 := (0, m, 0), \quad M_3 := (0, 0, m),$$

and

$$W_{M_k}(x) := \tfrac{1}{2}(W(x + M_k) + W(x - M_k)) \quad \text{for } k = 1, 2, 3.$$

Lemma A.4. *If m is sufficiently large then in the subspace $\mathcal{A} \cap L^2(\mathbb{R}^3)$ each of the operators $-\Delta - W_{M_k}$ and $-\Delta - \frac{1}{3} W_{M_k}$, for $k = 1, 2, 3$, has only one eigenfunction.*

Proof. We only prove the result for $-\Delta - W_{M_1}$. The proof of the other cases is similar, hence omitted.

First we have that if m is sufficiently large then

$$\big\langle (-\Delta - W_{M_1})\big(\phi(\cdot + M_1) + \phi(\cdot - M_1)\big),\ \phi(\cdot + M_1) + \phi(\cdot - M_1)\big\rangle < 0.$$

This principle [Reed and Simon 1979] implies that the operator $-\Delta - W_{M_1}$ has at least one eigenstate.

Second the min-max principle implies that any function $f \perp \phi(\cdot + M_1), \phi(\cdot - M_1)$ satisfies

$$\langle (-\Delta - W_{M_1}) f, f\rangle = \tfrac{1}{2}\big(\langle (-\Delta - W(\cdot + M)) f, f\rangle + \langle (-\Delta - W(\cdot - M)) f, f\rangle\big) \geq 0.$$

This, together with the facts

$$\phi(\cdot + M_1) - \phi(\cdot - M_1) \perp L^2(\mathbb{R}^3) \cap \mathcal{A} \quad \text{and} \quad \operatorname{span}\{\phi(\cdot - M_1), \phi(\cdot + M_1)\} = \operatorname{span}\{\phi(\cdot - M_1) \pm \phi(\cdot + M_1)\},$$

yields that

$$\langle (-\Delta - W_{M_1}) f, f\rangle \geq 0$$

for any $f \in \mathcal{A} \cap L^2(\mathbb{R}^3)$ and $f \perp \phi(\cdot + M_1) + \phi(\cdot - M_1)$.

Collecting what was proved we have that the operator $-\Delta - W_{M_1}$ has only one eigenfunction, its ground state. □

To prove the main result we define

$$V_m := \tfrac{1}{3}(W_{M_1} + W_{M_2} + W_{M_3}).$$

Proposition A.5. *There exists at least one $m \in [0, \infty)$ such that $-\Delta - V_m$ has all the properties in Proposition A.1.*

Proof. We need the following facts:

(A) *For any $m \in [0, \infty)$ the operator $-\Delta - V_m$ has at most three eigenfunctions in $\mathcal{A} \cap L^2$.* Recall in Lemma A.4 we proved that if $f \perp \phi(\cdot + M_k) + \phi(\cdot - M_k)$ for $k = 1, 2, 3$ and $f \in \mathcal{A} \cap L^2$ then $\langle (-\Delta - W_{M_k}) f, f \rangle \geq 0$. Consequently if $f \perp \phi(\cdot + M_k) + \phi(\cdot - M_k)$ for $k = 1, 2, 3$ then

$$\langle (-\Delta - V_m) f, f \rangle = \tfrac{1}{3}\big(\langle (-\Delta - W_{M_1}) f, f \rangle + \langle (-\Delta - W_{M_2}) f, f \rangle + \langle (-\Delta - W_{M_3}) f, f \rangle\big) \geq 0.$$

The min-max principle [Reed and Simon 1979] implies that there are at most three eigenfunctions.

(B) *If m is sufficiently large then in the space $L^2 \cap \mathcal{A}$ the operator $-\Delta - V_m$ has three eigenfunctions and two eigenvalues: one ground state and two degenerate eigenstates.* The fact that $-\Delta + V_m$ has three eigenfunctions follows from the min-max principle. The proof is similar to the case of double-well potentials [Harrell 1980] and is omitted. We need to prove that these eigenstates are degenerate. Indeed, as $m \to \infty$ the three eigenfunctions converge to a linear combination of the functions:

$$\phi(\cdot + M_k) + \phi(\cdot - M_k) \qquad \text{for } k = 1, 2, 3.$$

In particular, the ground state converges to

$$\sum_{k=1}^{3} \phi(\cdot + M_k) + \phi(\cdot - M_k).$$

Moreover, the ground state is simple and orthogonal to the excited eigenstates. The excited eigenstates are not invariant under a permutation: $(x_1, x_2, x_3) \mapsto (x_{n(1)}, x_{n(2)}, x_{n(3)})$. Since V_m is invariant under permutation, a second, linearly independent, eigenstate may be obtained from a particular choice via permutation.

(C) *When $m = 0$, $-\Delta - V_m$ has only one eigenfunction, the ground state.* This is clear since $V_m = W$ when $m = 0$.

(D) *For any $m \geq 0$, $-\Delta - V_m$ has at least one eigenfunction with eigenvalue less than some $-c_0 < 0$.* Let ϕ_2 be the normalized ground state of $-\Delta - \frac{1}{3} W_{M_2}$ with eigenvalue $-c_0 < 0$. Then we have

$$\langle (-\Delta - V_m)\phi_2, \phi_2 \rangle < -c_0$$

by the facts $\phi_2 > 0$ and $W > 0$. By the min-max principle $-\Delta - V_m$ has a ground state with eigenvalue $< -c_0$.

The definition of W implies that $(-\Delta + k)^{-1} W(\cdot + z)$ is analytic in z if $k \in \mathbb{C} \backslash \mathbb{R}^+$. By [Reed and Simon 1979] we have that the eigenvalues are analytic functions of z in a suitable subset of $\mathbb{C}$. Since the eigenvalues of the excited states are degenerate for sufficiently large m (see (B)), they are degenerate for any m before the excited states disappear into the essential spectrum. Hence there exists at least one m such that $-\Delta - V_m$ has one eigenvalue, e_0, less than $-c_0$ (defined in (D)) and two degenerate excited states with eigenvalue e_1, sufficiently close to the essential spectrum (see (A), (C)); $e_1 - e_0 > -e_1$ or $2e_1 - e_0 > 0$.

In the final step we find m and q such that the operator

$$1 + (-\Delta + i0)^{-1} V_m : \langle x \rangle^2 L^2 \to \langle x \rangle^2 L^2$$

is invertible. Recall that $V_m = qV_2(m)$, with $V_2(m)$ independent of q by its definition. For a fixed q_0 we proved that there exists at least one $m = m_0$ such that the eigenvalues of $-\Delta - q_0 V_2(m_0)$ have the desired properties. We now consider the family of operators

$$X(q) := 1 - q(-\Delta + i0)^{-1} V_2(m_0)$$

which is analytic in q. Moreover, the operator $q(-\Delta+i0)^{-1}V_2(m_0) : \langle x\rangle^2 L^2 \to \langle x\rangle^2 L^2$ is compact. By [Reed and Simon 1979] the operators $X(q) : \langle x\rangle^2 L^2 \to \langle x\rangle^2 L^2$ are either invertible everywhere (that is, no threshold resonance) except for some discrete points or not invertible anywhere. The first case holds because the operator is invertible when $q = 0$.

Now we consider $-\Delta - qV_2(m_0)$ with $q \in [q_0 - \epsilon, q_0 + \epsilon]$. Choose ϵ sufficiently small such that for every q the operator $-\Delta - qV_2(m_0)$ has at least three eigenvectors. On the other hand by what we proved above it has at most three eigenvectors and two of them must be degenerate. Since $1 - q(-\Delta + i0)^{-1}V_2(m_0)$ is not invertible only at discrete points we obtain the desired result. □

Proof of Proposition A.2. The fact $L(\lambda)$ has no resonances at $\pm i\lambda$ follows from the invertibility of $I + (-\Delta + i0)^{-1}V$ and $|\,\lambda - |e_0|\,|$ being small.

Next we prove the neutral modes are degenerate. Recall that the potential we constructed is of the form $V = V_{m_0}$ for some m_0. For each $m > 0$ there are $\lambda = \lambda_m$ and $\phi^\lambda = \phi^{\lambda,m}$ satisfying

$$-\Delta\phi^{\lambda,m} + \lambda_m \phi^{\lambda,m} + V_m \phi^{\lambda,m} - (\phi^{\lambda,m})^3 = 0$$

with λ_m and ϕ_m analytic in m in some proper neighborhood of positive real axis.

Recall that when m is sufficiently large the neutral modes of $-\Delta + V_m$ can be generated by permuting one of them. Hence the neutral modes of $L(\lambda) = L(\lambda, m)$ are degenerate when m is large. Moreover the eigenvalues of $L(\lambda, m)$ are analytic in m, thus the neutral modes must be degenerate. □

Appendix B. The Fermi Golden Rule

The proof of Theorem 6.1, given at the end of this section, requires the following:

Proposition B.1. *Given smooth functions $\mathcal{F}, \mathcal{G} : \mathbb{R}^d \to \mathbb{C}^2$, there exists $\tilde{\mathcal{F}} = (\tilde{\mathcal{F}}_1, \tilde{\mathcal{F}}_2)^T$ and $\tilde{\mathcal{G}} = (\tilde{\mathcal{G}}_1, \tilde{\mathcal{G}}_2)^T$ (see the definitions below) such that*

$$-\Re\big\langle (L(\lambda) + 2iE(\lambda) - 0)^{-1} P_c \mathcal{F}, iJP_c\mathcal{G}\big\rangle = \pi\big\langle \delta(-\Delta - (2E(\lambda) - \lambda))\tilde{\mathcal{F}}_2, \tilde{\mathcal{G}}_2\big\rangle \tag{B-1}$$

Proof of Proposition B.1. The entries of Γ are expressions of the form

$$-\Re\big\langle (L(\lambda) + 2iE(\lambda) - 0)^{-1} P_c \mathcal{F},\ iJP_c\mathcal{G}\big\rangle \tag{B-2}$$

which we now proceed to simplify. Recall $L(\lambda)$ is of the form

$$L(\lambda) = (-\Delta + \lambda)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & V_1 \\ V_2 & 0 \end{pmatrix}$$

where V_1 and V_2 are real-valued and exponentially decaying as $|x|$ tends to infinity. Introduce the unitary matrix

$$U = \tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}.$$

Note that

$$L(\lambda) = iU\sigma_3\mathcal{H}(\lambda)U^*, \qquad \mathcal{H}^* = \mathcal{H}$$

where

$$\mathcal{H} := \mathcal{H}_0 + \tilde{V},\ \mathcal{H}_0 := (-\Delta+\lambda)\,\mathrm{Id},\ \tilde{V} := \begin{pmatrix} V_1 - V_2 & -i(V_1+V_2) \\ i(V_1+V_2) & V_1 - V_2 \end{pmatrix},\ \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We now use the unitary transformation U to obtain an expression in terms of the operator $\sigma_3\mathcal{H}$:

$$\begin{aligned} -\big\langle (L(\lambda)+2iE(\lambda)-0)^{-1}P_c\mathcal{F},\ iJP_c\mathcal{G}\big\rangle &= -\big\langle \big(iU(\sigma_3\mathcal{H}(\lambda)+2E(\lambda)+i0)U^*\big)^{-1}P_c\mathcal{F},\ iJP_c\mathcal{G}\big\rangle \\ &= \big\langle (\sigma_3\mathcal{H}(\lambda)+2E(\lambda)+i0)^{-1}U^*P_c\mathcal{F}, U^*JP_c\mathcal{G}\big\rangle \\ &= \big\langle (\sigma_3\mathcal{H}(\lambda)+2E(\lambda)+i0)^{-1}U^*P_c\mathcal{F}, (U^*JU)U^*P_c\mathcal{G}\big\rangle \\ &= -i\big\langle (\sigma_3\mathcal{H}(\lambda)+2E(\lambda)+i0)^{-1}U^*P_c\mathcal{F}, \sigma_3U^*P_c\mathcal{G}\big\rangle \end{aligned} \tag{B-3}$$

where we have used that $U^*JU = i\sigma_3$.

Next we introduce $P_c(\sigma_3\mathcal{H})$, the projection onto the continuous spectral of $\sigma_3\mathcal{H}$ and wave operators $W : L^2 \to P_c(\sigma_3\mathcal{H})L^2$ and $Z : P_c(\sigma_3\mathcal{H})L^2 \to L^2$ (see [Cuccagna et al. 2005]), which satisfy

$$P_c(\sigma_3\mathcal{H})^*\sigma_3 = \sigma_3P_c(\sigma_3\mathcal{H}),\ \ W^*\sigma_3 = \sigma_3Z,\ \ Z^*\sigma_3 = \sigma_3W,\ \ Z\sigma_3\mathcal{H} = \sigma_3\mathcal{H}_0Z. \tag{B-4}$$

Now we use the wave operators W and Z to transform the previous expression into one in terms of the "free operator" $\sigma_3(-\Delta+\lambda)$. First note that $U^*P_c\mathcal{F}$ lies in the range of $P_c(\sigma_3\mathcal{H})$ and therefore there exists $\tilde{\mathcal{F}} = (\tilde{\mathcal{F}}_1, \tilde{\mathcal{F}}_2)^T$ such that $W\tilde{\mathcal{F}} = U^*P_c\mathcal{F}$. Similarly, there exists $\tilde{\mathcal{G}} = (\tilde{\mathcal{G}}_1, \tilde{\mathcal{G}}_2)^T$ such that $W\tilde{\mathcal{G}} = U^*P_c\mathcal{G}$. Substituting into the final expression in (B-3) and using of the properties (B-4) we have

$$\begin{aligned} &i\big\langle (\sigma_3\mathcal{H}(\lambda)+2E(\lambda)+i0)^{-1}U^*P_c\mathcal{F}, \sigma_3U^*P_c\mathcal{G}\big\rangle \\ &\qquad = i\big\langle (\sigma_3\mathcal{H}(\lambda)+2E(\lambda)+i0)^{-1}W\tilde{\mathcal{F}}, \sigma_3W\tilde{\mathcal{G}}\big\rangle = i\big\langle (\sigma_3\mathcal{H}(\lambda)+2E(\lambda)+i0)^{-1}W\tilde{\mathcal{F}}, Z^*\sigma_3\tilde{\mathcal{G}}\big\rangle \\ &\qquad = i\big\langle Z(\sigma_3\mathcal{H}(\lambda)+2E(\lambda)+i0)^{-1}W\tilde{\mathcal{F}}, \sigma_3\tilde{\mathcal{G}}\big\rangle = i\big\langle (\sigma_3(-\Delta+\lambda)+2E(\lambda)+i0)^{-1}ZW\tilde{\mathcal{F}}, \sigma_3\tilde{\mathcal{G}}\big\rangle \\ &\qquad = i\big\langle (\sigma_3(-\Delta+\lambda)+2E(\lambda)+i0)^{-1}\tilde{\mathcal{F}}, \sigma_3\tilde{\mathcal{G}}\big\rangle. \end{aligned}$$

Referring back to (B-2) we recall that we are interested in the real part of this expression:

$$\begin{aligned} &-\Re\, i\big\langle (\sigma_3(-\Delta+\lambda)+2E(\lambda)+i0)^{-1}\tilde{\mathcal{F}},\ \sigma_3\tilde{\mathcal{G}}\big\rangle \\ &\qquad = \Im\,\big\langle (\sigma_3(-\Delta+\lambda)+2E(\lambda)+i0)^{-1}\tilde{\mathcal{F}}, \sigma_3\tilde{\mathcal{G}}\big\rangle \\ &\qquad = \Im\left\langle \begin{pmatrix} (-\Delta+\lambda+2E(\lambda)+i0)^{-1} & 0 \\ 0 & -(-\Delta+\lambda-2E(\lambda)-i0)^{-1} \end{pmatrix}\tilde{\mathcal{F}},\ \sigma_3\tilde{\mathcal{G}}\right\rangle \\ &\qquad = \Im\left\langle \begin{pmatrix} (-\Delta+\lambda+2E(\lambda)+i0)^{-1} & 0 \\ 0 & (-\Delta+\lambda-2E(\lambda)-i0)^{-1}) \end{pmatrix}\tilde{\mathcal{F}},\ \tilde{\mathcal{G}}\right\rangle \\ &\qquad = \Im\,\big\langle (-\Delta-(2E(\lambda)-\lambda)-i0)^{-1}\tilde{\mathcal{F}}_2, \tilde{\mathcal{G}}_2\big\rangle \\ &\qquad = \pi\big\langle \delta(-\Delta-(2E(\lambda)-\lambda))\tilde{\mathcal{F}}_2, \tilde{\mathcal{G}}_2\big\rangle. \end{aligned}$$

We uses that

$$0 < 2E(\lambda) - \lambda \in \sigma_c(-\Delta), \qquad -2E(\lambda) - \lambda \notin \sigma_c(-\Delta)$$

and the distributional (Plemelj) identity:

$$\Im(x - i0)^{-1} = \lim_{\varepsilon \downarrow 0} \Im(x - i\varepsilon)^{-1} = \pi\delta(x)$$

to get the last equality.

Summarizing, we have shown

$$-\Re\big\langle (L(\lambda) + 2iE(\lambda) - 0)^{-1} P_c \mathcal{F},\, iJP_c\mathcal{G}\big\rangle = \pi\big\langle \delta(-\Delta - (2E(\lambda) - \lambda))\tilde{\mathcal{F}}_2,\, \tilde{\mathcal{G}}_2\big\rangle. \qquad \square$$

Proof of Theorem 6.1. We use Proposition B.1 with $\mathcal{F} = G_k$ and $\mathcal{G} = G_l$, $\tilde{\mathcal{F}} = \tilde{G}_k$ and $\tilde{\mathcal{G}} = \tilde{G}_l$. By (B-1) we have

$$\Gamma_{k,l} = \pi\big\langle \delta(-\Delta - (2E(\lambda) - \lambda))\tilde{G}_{l,2},\, \tilde{G}_{k,2}\big\rangle.$$

To see that $\Gamma_{k,l}$ is nonnegative, observe that for any $s \in \mathbb{C}^N$ we have

$$s^*\Gamma s = \sum_{k,l=1}^{N} \Gamma_{k,l} s_k \bar{s}_l = \pi\big\langle \delta(-\Delta - (2E(\lambda) - \lambda))\tilde{\mathcal{G}},\, \tilde{\mathcal{G}}\big\rangle \geq 0$$

where $\tilde{\mathcal{G}} = \sum_{k=1}^{N} s_k \tilde{G}_{k,2}$.

For the second statement we only sketch the proof. Recall the transformation of $L(\lambda)$ in (5-2). Then for any 2×1 vector functions $\vec{F}$ and $\vec{G}$ we have

$$\begin{aligned}\big\langle (L(\lambda) + 2iE(\lambda) - 0)^{-1} P_c\vec{F},\, P_c\vec{G}\big\rangle &= -i\big\langle (\sigma_3\mathcal{H}(\lambda) + 2E(\lambda) + i0)^{-1} U^*P_c\vec{F},\, U^*P_c\vec{G}\big\rangle \\ &= -i\big\langle K(\lambda)\big(\sigma_3(\mathcal{H}_0 + V) + 2E(\lambda) + i0\big)^{-1} U^*P_c\vec{F},\, U^*P_c\vec{G}\big\rangle\end{aligned}$$

where $K(\lambda)$ is the operator defined as $(1 + K_{\text{small}})^{-1}$ with

$$K_{\text{small}} := (\sigma_3(\mathcal{H}_0 + V) + 2E(\lambda) + i0)^{-1} V_{\text{small}}.$$

The operator $\big(\sigma_3(\mathcal{H}_0 + V) + 2E(\lambda) + i0\big)^{-1}$ is well defined since

$$-\lambda - 2E(\lambda) = e_0 - 2(e_1 - e_0)$$

is not an eigenvalue of $-\Delta + V$ and the operator $-\Delta + V$ has no embedded eigenvalues in the essential spectrum.

Since the operator $K_{\text{small}} : \langle x\rangle^2 L^\infty \to \langle x\rangle^2 L^\infty$ has a small norm and is continuous in λ we have

$$(1 + K_{\text{small}})^{-1} = \sum_{n=0}^{\infty} (-K_{\text{small}})^n$$

is continuous in λ. This, together with the fact

$$(\sigma_3(\mathcal{H}_0 + V) + 2E(\lambda) + i0)^{-1} U^*P_c\vec{F} \in \langle x\rangle^2 L^\infty$$

is continuous in λ, implies that $\big\langle (L(\lambda) + 2iE(\lambda) - 0)^{-1} P_c\vec{F},\, P_c\vec{G}\big\rangle$ is continuous in λ. $\square$

Appendix C. Fermi Golden Rule for symmetric potentials

In this section we derive the simpler form of the FGR matrix and condition for positivity in the case where the potential $V(x)$ is a function of $|x|$. In fact, it is proved in Proposition 5.2 that if the potential V, hence ϕ^{λ}, is spherically symmetric then the functions ξ_n, η_n satisfy

$$\xi_n = \frac{x_n}{|x|}\xi(|x|), \qquad \eta_n = \frac{x_n}{|x|}\eta(|x|)$$

for some functions $\xi(|x|)$ and $\eta(|x|)$. By the assumptions on V, ϕ^{λ}, ξ_k and η_k with $k = 1, 2, \dots, N = d$ we have

$$G_k(z, x) = x_k(z \cdot x)G(|x|)$$

for some radial vector function $G(|x|)$.

Before stating the results we define two constants

$$\begin{aligned}\Re Z_0^{(1,1)} &= -\Re\big\langle (L(\lambda) + 2iE(\lambda) - 0)^{-1} P_c x_1^2 G(|x|), iJx_1^2 G(|x|)\big\rangle,\\ \Re Z_0^{(2,2)} &= -\Re\big\langle (L(\lambda) + 2iE(\lambda) - 0)^{-1} P_c x_1 x_2 G(|x|), iJx_1 x_2 G(|x|)\big\rangle.\end{aligned}$$

Proposition C.1. *(i) Suppose that V, ξ_n, η_n satisfy the conditions above. Then the assumption (FGR) holds provided that*

$$\Re Z_0^{(1,1)} > 0, \qquad \Re Z_0^{(2,2)} > 0. \tag{C-1}$$

(ii) From Proposition B.1, it follows that

$$\Re Z_0^{(1,1)} \geq 0, \qquad \Re Z_0^{(2,2)} \geq 0.$$

And, generically, the strict positivity in (C-1) holds.

Proof. For any vectors $s, \beta, z \in \mathbb{C}^N$, we define

$$\mathfrak{Q}(s, \beta; z) := -\Re\big\langle (L(\lambda) + 2iE(\lambda) - 0)^{-1} P_c (z \cdot x)(s \cdot x)G(|x|), iJ(z \cdot x)(\beta \cdot x)G(|x|)\big\rangle.$$

Note that

$$\mathfrak{Q}(s, s; z) = \tfrac{1}{2}s^*(Z(z, \bar{z}) + Z^*(z, \bar{z}))s = \Re s^* Z(z, \bar{z})s.$$

Therefore, verifying (FGR) is equivalent to checking that there is a constant $C > 0$ for which

$$\mathfrak{Q}(s, s; z) \geq C|s|^2|z|^2$$

with $s, z \in \mathbb{C}^d$.

To simplify $\mathfrak{Q}(s, s; z)$, first note that since operator $L(\lambda)$ and $G(|x|)$ are invariant under $x \mapsto T^*x$, where T is a unitary transformations, the value of $\mathfrak{Q}(s, \beta; z)$ is unchanged when x is replaced by T^*x. Therefore,

$$\mathfrak{Q}(s, \beta; z) = \mathfrak{Q}(Ts, T\beta; Tz). \tag{C-2}$$

Now choose T to be a unitary matrix such that

$$Tz = |z|e_1 = |z|(1, 0, \dots, 0)^T.$$

With this choice of T, we have by (C-2) with $\beta = s$,

$$\mathfrak{Q}(s, s; z) = -\Re\big\langle (L(\lambda) + 2i E(\lambda) - 0)^{-1} P_c |z| x_1 (Ts \cdot x) G(|x|), i|z| x_1 (Ts \cdot x) J G(|x|) \big\rangle. \tag{C-3}$$

The following argument will show that

$$\mathfrak{Q}(s, s; z) \geq C|Ts|^2|z|^2 = C|s|^2|z|^2,$$

the latter holding since T is unitary. Therefore, without any loss of generality, consider (C-3) with T set equal to the identity. Explicitly writing out the inner products and using bilinearity and symmetry, we have

$$\begin{aligned}
\mathfrak{Q}(s, s; z) &= -|z|^2\Re\Big(\sum_{p,q=1}^{d} \big\langle (L(\lambda) + 2i E(\lambda) - 0)^{-1} P_c x_1 x_p G(|x|), i x_1 x_q J G(|x|) \big\rangle s_p \overline{s_q} \Big) \\
&= -|z|^2\Re\Big(\sum_{p=1}^{d} \big\langle (L(\lambda) + 2i E(\lambda) - 0)^{-1} P_c x_1 x_p G(|x|), i x_1 x_p J G(|x|) \big\rangle |s_p|^2 \Big) \\
&= -|z|^2|s_1|^2\Re\big\langle (L(\lambda) + 2i E(\lambda) - 0)^{-1} P_c x_1^2 G(|x|), i x_1^2 J G(|x|) \big\rangle \\
&\qquad - |z|^2 \sum_{q=2}^{d} |s_q|^2\Re\big\langle (L(\lambda) + 2i E(\lambda) - 0)^{-1} P_c x_1 x_2 G(|x|), i x_1 x_2 J G(|x|) \big\rangle \\
&= |z|^2 \Big(|s_1|^2 \, \Re Z_0^{(1,1)} + \sum_{q=2}^{d} |s_q|^2 \, \Re Z_0^{(2,2)} \Big) \\
&\geq |s|^2|z|^2 \min\{\Re Z_0^{(1,1)}, \Re Z_0^{(2,2)}\} \equiv C|s|^2|z|^2 > 0. \qquad \square
\end{aligned}$$

Appendix D. Choice of basis for the degenerate subspace

In the proof of Proposition 5.5 we need the following lemma.

Lemma D.1. *If* $u = \begin{pmatrix} u_1 \\ i u_2 \end{pmatrix} \neq 0$ *is an eigenfunction of* $L(\lambda)$ *with eigenvalue* $i E(\lambda)$, $E(\lambda) > 0$ *then*

$$\langle u_1, u_2 \rangle > 0. \tag{D-1}$$

Proof. The fact $L(\lambda) u = i E(\lambda) u$ yields

$$L_-(\lambda) u_2 = E(\lambda) u_1, \qquad L_+(\lambda) u_1 = E(\lambda) u_2. \tag{D-2}$$

Therefore,

$$\langle u_1, u_2 \rangle = \frac{1}{E(\lambda)} \langle L_-(\lambda) u_2, u_2 \rangle.$$

Equation (D-1) follows from the two claims that $L_-(\lambda)$ is a positive-definite selfadjoint operator on the space $\{v \mid v \perp \phi^\lambda\}$ and $u_2 \notin \operatorname{span}\{\phi^\lambda\}$. The first fact is well known; see for example [Weinstein 1986]. We prove the second by contradiction. Suppose that $u_2 = c\phi^\lambda$ for some constant c then we have $L_-(\lambda) u_2 = 0$, which, together with (D-2) and the fact $E(\lambda) \neq 0$, implies $u_1 = u_2 = 0$, that is, $u = 0$. This contradicts to the fact $u \neq 0$. Thus $u_2 \notin \operatorname{span}\{\phi^\lambda\}$. $\square$

Proof of Proposition 5.5. We start by constructing N independent vectors $u_n \in \mathrm{span}\{v_1,\ v_2, \dots, v_N\}$ for $n = 1, 2, \dots, N$ such that the vector

$$\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} u_n$$

is real. Suppose that

$$v_n = \begin{pmatrix} v_1^{(n)} \\ v_2^{(n)} \end{pmatrix}.$$

Then the definition of $L(\lambda)$ in (5-1) implies

$$\begin{pmatrix} \Re v_1^{(n)} \\ i\Im v_2^{(n)} \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} \Im v_1^{(n)} \\ -i\Re v_2^{(n)} \end{pmatrix}$$

are also eigenfunctions of $L(\lambda)$ with eigenvalues $iE(\lambda)$. This, together with the fact

$$\left\{ \begin{pmatrix} \Re v_1^{(n)} \\ i\Im v_2^{(n)} \end{pmatrix}, \begin{pmatrix} \Im v_1^{(n)} \\ -i\Re v_2^{(n)} \end{pmatrix},\ n = 1, 2, \dots, N \right\} = \{v_n,\ n = 1, 2, \dots, N\},$$

enables us to choose N independent eigenfunctions u_n with $n = 1, 2, \dots, N$ for $iE(\lambda)$ such that

$$\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} u_n$$

are real vectors.

Using (D-1) and a standard Gram–Schmidt procedure in linear algebra, one can find N pairs of real functions (ξ_n, η_n) for $n = 1, 2, \dots, N$ such that

$$\mathrm{span}\left\{ \begin{pmatrix} \xi_n \\ i\eta_n \end{pmatrix},\ n = 1, 2, \dots, N \right\} = \mathrm{span}\{v_n, n = 1, 2, \dots, N\} \quad \text{and} \quad \langle \xi_n, \eta_m \rangle = \delta_{n,m}.$$

We now turn to the verification of (5-3). The observations

$$L_-(\lambda) - L_+(\lambda) = 2f'[(\phi^\lambda)^2](\phi^\lambda)^2$$

and

$$L_-(\lambda)\eta_n = E(\lambda)\xi_n, \qquad L_+(\lambda)\xi_n = E(\lambda)\eta_n$$

for $n = 1, 2, \dots, N$ yield

$$\int f'[(\phi^\lambda)^2](\phi^\lambda)^2(\xi_m\eta_n - \xi_n\eta_m)\,dx$$
$$= \tfrac{1}{2}\big(\langle \xi_m, L_-(\lambda)\eta_n\rangle - \langle L_+(\lambda)\xi_m, \eta_n\rangle - \langle \xi_n, L_-(\lambda)\eta_m\rangle + \langle L_+(\lambda)\xi_n, \eta_m\rangle\big) = 0.$$

Finally (5-4) is seen as follows:

$$\langle \phi^\lambda, \xi_n\rangle = \frac{1}{E(\lambda)}\langle \phi^\lambda, L_-(\lambda)\eta_n\rangle = \frac{1}{E(\lambda)}\langle L_-(\lambda)\phi^\lambda, \eta_n\rangle = 0,$$
$$\langle \partial_\lambda\phi^\lambda, \eta_n\rangle = \frac{1}{E(\lambda)}\langle \partial_\lambda\phi^\lambda, L_+(\lambda)\xi_n\rangle = -\frac{1}{E(\lambda)}\langle \phi^\lambda, \xi_n\rangle = 0$$

□

Appendix E. The identity $P_c(JH)^*J = JP_c(JH)$

Proposition E.1. *$L = JH$ and $H = H^*$ imply*

$$P_c(L)^*J = JP_c(L).$$

Proof. Represent $P_c(L)$ as a Riesz projection

$$P_c(L) = \frac{1}{2\pi i}\oint (zI - JH)^{-1}\,dz$$

where the contour of the integration is counterclockwise. Moreover, the essential spectrum of L is

$$(-i\infty, -i\lambda] \cup [i\lambda, i\infty).$$

The spectrum associated with the upper branch $[i\lambda, i\infty)$ is given by

$$P^+(JH) = \tfrac{1}{2\pi}(A - B),$$

where

$$A = \int_\lambda^\infty (i\tau + 0 - JH)^{-1}\,d\tau, \qquad B = \int_\lambda^\infty (i\tau - 0 - JH)^{-1}\,d\tau.$$

We claim that

$$A^*J = -JB, \qquad B^*J = -JA. \tag{E-1}$$

This implies

$$(P^+(JH))^*J = JP^+(JH), \qquad (P_c(JH))^*J = JP_c(JH).$$

To complete the proof of the proposition, we now prove (E-1). By direct computation using $J^* = -J$ we have

$$A^* = \int_\lambda^\infty (-i\tau + 0 + HJ)^{-1}\,d\tau.$$

Therefore,

$$A^*J = \int_\lambda^\infty (J(i\tau)J - J0J - JJHJ)^{-1}\,d\tau\, J = \int_\lambda^\infty (-J)(i\tau - 0 - JH)^{-1}(-J)\,d\tau\, J = -JB,$$

thus proving the first identity in (E-1). The second can be proved similarly. □

Appendix F. Time convolution lemmas

Proof of Proposition 11.2. In what follows we only prove the case $\sigma = 1$ of (11-6); the other cases and (11-7) are similar.

$$I(t) := \int_0^t \frac{1}{(1+t-s)^{3/2}}\frac{1}{T_0+s}\,ds \le \frac{1}{(1+\frac{t}{2})^{3/2}}\int_0^{t/2}\frac{1}{T_0+s}\,ds + \frac{1}{T_0+\frac{t}{2}}\int_{t/2}^t \frac{1}{(1+t-s)^{3/2}}\,ds$$
$$\le \frac{\log(1+\frac{t}{2T_0})}{(1+\frac{t}{2})^{3/2}} + \frac{2}{T_0+\frac{t}{2}}.$$

On the other hand, we also have

$$\int_0^t \frac{1}{(1+t-s)^{3/2}} \frac{1}{T_0+s}\, ds \le \frac{2}{T_0}.$$

Thus,

$$I(t) \le c_1 \min\left\{\frac{1}{T_0}, \frac{1}{1+t}\right\}.$$

We now claim that for some constant $c > 0$,

$$I(t) \le \frac{c}{T_0+t}.$$

It sufficies to find a constant c independent of T_0 and t such that

$$m(t) := (T_0+t) \min\left\{\frac{1}{T_0}, \frac{1}{1+t}\right\} \le c.$$

If t is such that the above minimum is T_0^{-1} then $T_0^{-1} \le (1+t)^{-1}$, that is, $t \le T_0 - 1$. Therefore,

$$m(t) \le \frac{2T_0-1}{T_0} \le \frac{3}{2}.$$

If t is such that the above minimum is $(1+t)^{-1}$ then $t \ge T_0 - 1$. Therefore,

$$m(t) \le \frac{2T_0-1}{T_0}$$

since $m(t)$ is decreasing with t. Since $T \ge 2$, $m(t) \le 3/2$. This completes the proof. □

Appendix G. Bounds on solutions to a weakly perturbed ODE

Proof of Lemma 11.14. Let β denote the solution to the differential equation

$$\partial_t |\beta_\rho|^2 = -|\beta_\rho|^4 + g, \qquad |\beta_\rho|^2(0) = |z(0)|^2 - \rho$$

for $\rho > 0$. Since

$$\partial_t(|z(t)|^2 - |\beta_\rho(t)|^2) \le -|z(t)|^4 + |\beta_\rho(t)|^4 = -(|z(t)|^2 + |\beta_\rho(t)|^2)\,(|z(t)|^2 - |\beta_\rho(t)|^2)$$

with the initial condition

$$|z(0)|^2 - |\beta_\rho(0)|^2 = \rho > 0.$$

Thus $|z(t)|^2 \le |\beta_\rho(t)|^2$ for all $t \ge 0$. Letting ρ tend to zero, we have

$$|z(t)|^2 \le |\beta(t)|^2$$

so it suffices to prove the bound:

$$|\beta(t)| \le (1 + K c_\# T_0^{-\delta})\,(\kappa + t)^{-1/2},$$

where $\kappa = \min\{T_0, |w_0|^{-2}\}$ and $\beta(t)$ solves the initial value problem

$$\partial_t |\beta|^2 = -|\beta|^4 + g, \qquad |\beta(0)|^2 = |w_0|^2. \tag{G-1}$$

The proof of (11-14) for β is divided into two cases:

Case $|w_0| \geq T_0^{-1/2}$. By local existence of the solutions for the initial value problem (G-1), we have that for some $t_1 > 0$

$$\frac{1}{2(T_0+t)^{1/2}} \leq |\beta(t)| \tag{G-2}$$

with $t \in [0, t_1]$. Then using the assumed bound on $g(t)$ in (11-13) we have

$$|g(t)| \leq \frac{c_\#}{(T_0+t)^{2+\delta}} = \frac{c_\#}{(T_0+t)^2} \frac{1}{(T_0+t)^\delta} \leq 2^4 c_\# |\beta(t)|^4 \cdot \frac{1}{T_0^\delta} = c_{1\#} T_0^{-\delta} |\beta(t)|^4$$

where $c_{1\#} := 2^4 c_\#$. It follows from (G-1) that

$$\partial_t |\beta(t)|^2 \leq -(1 - c_{1\#} T_0^{-\delta}) |\beta(t)|^4$$

or

$$\partial_t |\beta(t)|^{-2} \geq 1 - c_{1\#} T_0^{-\delta}.$$

Integration over the interval $[0, t]$ for $t \leq t_1$ yields

$$|\beta(t)| \leq \frac{1 + c_{2\#} T_0^{-\delta}}{(|w_0|^{-2} + t)^{1/2}} \tag{G-3}$$

where $c_{2\#} \sim c_{1\#} \sim c_\#$ and we use that $c_\# T_0^{-\delta}$ is sufficiently small. Now set $\kappa = \min\{|w_0|^{-2}, T_0\}$ and we have

$$|z(t)| \leq |\beta(t)| \leq \frac{1 + c_{2\#} T_0^{-\delta}}{(\kappa + t)^{1/2}}$$

for $0 \leq t \leq t_1$. Now let $[0, \Xi)$ denote the maximal subset of $\mathbb{R}_+$, on which the upper bound in (G-3) holds. If $\Xi < \infty$ then by continuity and the assumption that $|w_0| \geq T_0^{-1/2}$ we have

$$|\beta(\Xi)| = \frac{1 + c_{2\#} T_0^{-\delta}}{(|w_0|^{-2} + \Xi)^{1/2}} \geq \frac{1}{(|w_0|^{-2} + \Xi)^{1/2}} \geq \frac{3}{4} \frac{1}{(T_0 + \Xi)^{1/2}},$$

implying (see (G-2)) that the above argument can be applied beyond $t = \Xi$, contradicting its maximality.

Case $|w_0| < T_0^{-1/2}$. Denote by $\beta_1(t)$ the solution to (11-12) with the initial condition $\beta_1(0) = T_0^{-1/2}$. As shown in the previous case

$$|\beta_1(t)| \leq (1 + K\ c_\# T_0^{-\delta})(T_0 + t)^{-1/2}.$$

Observing that

$$\partial_t (|\beta|^2 - |\beta_1|^2) = -(|\beta|^2 + |\beta_1|^2)(|\beta|^2 - |\beta_1|^2), \qquad |\beta(0)|^2 - |\beta_1(0)|^2 < 0,$$

we have $|\beta(t)|^2 \leq |\beta_1(t)|^2$ for any time t. This, together with the estimate of β_1, completes the proof of the second case. □

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THE PSEUDOSPECTRUM OF SYSTEMS OF SEMICLASSICAL OPERATORS

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The pseudospectrum (or spectral instability) of non-self-adjoint operators is a topic of current interest in applied mathematics. In fact, for non-self-adjoint operators the resolvent could be very large outside the spectrum, making numerical computation of the complex eigenvalues very hard. This has importance, for example, in quantum mechanics, random matrix theory and fluid dynamics.

The occurrence of false eigenvalues (or pseudospectrum) of non-self-adjoint semiclassical differential operators is due to the existence of quasimodes, that is, approximate local solutions to the eigenvalue problem. For scalar operators, the quasimodes appear generically since the bracket condition on the principal symbol is not satisfied for topological reasons.

In this paper we shall investigate how these results can be generalized to square systems of semiclassical differential operators of principal type. These are the systems whose principal symbol vanishes of first order on its kernel. We show that the resolvent blows up as in the scalar case, except in a nowhere dense set of degenerate values. We also define quasisymmetrizable systems and systems of subelliptic type, for which we prove estimates on the resolvent.

1. Introduction

In this paper we shall study the pseudospectrum or spectral instability of square non-self-adjoint semiclassical systems of principal type. Spectral instability of non-self-adjoint operators is currently a topic of interest in applied mathematics; see [Davies 2002] and [Trefethen and Embree 2005]. It arises from the fact that, for non-self-adjoint operators, the resolvent could be very large in an open set containing the spectrum. For semiclassical differential operators, this is due to the bracket condition and is connected to the problem of solvability. In applications where one needs to compute the spectrum, the spectral instability has the consequence that discretization and round-off errors give false spectral values, so-called pseudospectra; see [Trefethen and Embree 2005] and references there.

We shall consider bounded systems $P(h)$ of semiclassical operators given by (2.2), and we shall generalize the results of the scalar case in [Dencker et al. 2004]. Actually, the study of unbounded operators can in many cases be reduced to the bounded case; see Proposition 2.20 and Remark 2.21. We shall also study semiclassical operators with analytic symbols in the case when the symbols can be extended analytically to a tubular neighborhood of the phase space satisfying (2.3). The operators we study will be of principal type, which means that the principal symbol vanishes of first order on the kernel; see Definition 3.1.

The definition of *semiclassical* pseudospectrum in [Dencker et al. 2004] is essentially the bracket condition, which is suitable for symbols of principal type. By instead using the definition of (injectivity)

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pseudospectrum in [Pravda-Starov 2006a] we obtain a more refined view of the spectral instability; see Definition 2.27. For example, z is in the pseudospectrum of infinite index for $P(h)$ if for any N the resolvent norm blows up faster than any power of the semiclassical parameter:

$$\|(P(h) - z\,\mathrm{Id})^{-1}\| \geq C_N h^{-N} \qquad 0 < h \ll 1 \tag{1.1}$$

In [Dencker et al. 2004] it was proved that (1.1) holds almost everywhere in the *semiclassical* pseudospectrum. We shall generalize this to systems and prove that for systems of principal type, except for a nowhere dense set of degenerate values, the resolvent blows up as in the scalar case; see Theorem 3.10. The complication is that the eigenvalues don't have constant multiplicity in general, only almost everywhere.

At the boundary of the semiclassical pseudospectrum, we obtained in [Dencker et al. 2004] a bound on the norm of the semiclassical resolvent, under the additional condition of having no unbounded (or closed) bicharacteristics. In the systems case, the picture is more complicated and it seems to be difficult to get an estimate on the norm of the resolvent using only information about the eigenvalues, even in the principal type case; see Example 4.1. In fact, the norm is essentially preserved under multiplication with elliptic systems, but the eigenvalues are changed. Also, the multiplicities of the eigenvalues could be changing at all points on the boundary of the eigenvalues; see Example 3.9. We shall instead introduce *quasisymmetrizable* systems, which generalize the normal forms of the scalar symbols at the boundary of the eigenvalues; see Definition 4.5. Quasisymmetrizable systems are of principal type and we obtain estimates on the resolvent as in the scalar case; see Theorem 4.15.

For boundary points of *finite type*, we obtained in [Dencker et al. 2004] subelliptic types of estimates on the semiclassical resolvent. This is the case when one has nonvanishing higher order brackets. For systems the situation is less clear; there seems to be no general results on the subellipticity for systems. In fact, the real and imaginary parts do not commute in general, making the bracket condition meaningless. Even when they do, Example 5.2 shows that the bracket condition is not sufficient for subelliptic types of estimates. Instead we shall introduce invariant conditions on the order of vanishing of the symbol along the bicharacteristics of the eigenvalues. For systems, there could be several (limit) bicharacteristics of the eigenvalues going through a characteristic point; see Example 5.9. Therefore we introduce the *approximation* property in Definition 5.10 which gives that the all (limit) bicharacteristics of the eigenvalues are parallel at the characteristics; see Remark 5.11. The general case presently looks too complicated to handle. We shall generalize the property of being of finite type to systems, introducing systems of *subelliptic type*. These are quasisymmetrizable systems satisfying the approximation property, such that the imaginary part on the kernel vanishes of finite order along the bicharacteristics of the real part of the eigenvalues. This definition is invariant under multiplication with invertible systems and taking adjoints, and for these systems we obtain subelliptic types of estimates on the resolvent; see Theorem 5.20.

As an example, we may look at

$$P(h) = h^2 \Delta\, \mathrm{Id}_N + i K(x)$$

where $\Delta = -\sum_{j=1}^{n} \partial_{x_j}^2$ is the positive Laplacian, and $K(x) \in C^\infty(\mathbb{R}^n)$ is a symmetric $N \times N$ system. If we assume some conditions of ellipticity at infinity for $K(x)$, we may reduce to the case of bounded symbols by Proposition 2.20 and Remark 2.21; see Example 2.22. Then we obtain that $P(h)$ has discrete spectrum in the right half plane $\{z : \operatorname{Re} z \geq 0\}$, and in the first quadrant if $K(x) \geq 0$, by Proposition 2.19.

We obtain from Theorem 3.10 that the L^2 operator norm of the resolvent grows faster than any power of h as $h \to 0$, thus (1.1) holds for almost all values z such that $\operatorname{Re} z > 0$ and $\operatorname{Im} z$ is an eigenvalue of K; see Example 3.12.

For $\operatorname{Re} z = 0$ and almost all eigenvalues $\operatorname{Im} z$ of K, we find from Theorem 5.20 that the norm of the resolvent is bounded by $Ch^{-2/3}$; see Example 5.22. In the case $K(x) \geq 0$ and $K(x)$ is invertible at infinity, we find from Theorem 4.15 that the norm of the resolvent is bounded by Ch^{-1} for $\operatorname{Re} z > 0$ and $\operatorname{Im} z = 0$ by Example 4.17. The results in this paper are formulated for operators acting on the trivial bundle over $\mathbb{R}^n$. But since our results are mainly local, they can be applied to operators on sections of fiber bundles.

2. The definitions

We shall consider $N \times N$ systems of semiclassical pseudo-differential operators, and use the Weyl quantization:

$$P^w(x, hD_x)u = \frac{1}{(2\pi)^n} \iint_{T^*\mathbb{R}^n} P\Big(\frac{x+y}{2}, h\xi\Big) e^{i\langle x-y,\xi\rangle} u(y)\, dy\, d\xi \tag{2.1}$$

for matrix valued $P \in C^\infty(T^*\mathbb{R}^n, \mathscr{L}(\mathbb{C}^N, \mathbb{C}^N))$. We shall also consider the semiclassical operators

$$P(h) \sim \sum_{j=0}^{\infty} h^j P_j^w(x, hD) \tag{2.2}$$

with $P_j \in C_{\mathrm{b}}^\infty(T^*\mathbb{R}^n, \mathscr{L}(\mathbb{C}^N, \mathbb{C}^N))$. Here C_{b}^∞ is the set of C^∞ functions having all derivatives in L^∞ and $P_0 = \sigma(P(h))$ is the principal symbol of $P(h)$. The operator is said to be elliptic if the principal symbol P_0 is invertible, and of principal type if P_0 vanishes of first order on the kernel; see Definition 3.1. Since the results in the paper only depend on the principal symbol, one could also have used the Kohn–Nirenberg quantization because the different quantizations only differ in the lower order terms. We shall also consider operators with analytic symbols; then we shall assume that $P_j(w)$ are bounded and holomorphic in a tubular neighborhood of $T^*\mathbb{R}^n$ satisfying

$$\|P_j(z,\zeta)\| \leq C_0 C^j j^j \qquad |\operatorname{Im}(z,\zeta)| \leq 1/C \quad \forall\, j \geq 0 \tag{2.3}$$

which will give exponentially small errors in the calculus, here $\|A\|$ is the norm of the matrix A. But the results hold for more general analytic symbols; see Remarks 3.11 and 4.19. In the following, we shall use the notation $w = (x, \xi) \in T^*\mathbb{R}^n$.

We shall consider the spectrum $\operatorname{Spec} P(h)$ which is the set of values λ such that the resolvent $(P(h) - \lambda \operatorname{Id}_N)^{-1}$ is a bounded operator, here Id_N is the identity in $\mathbb{C}^N$. The spectrum of $P(h)$ is essentially contained in the spectrum of the principal symbol $P(w)$, which is given by

$$|P(w) - \lambda \operatorname{Id}_N| = 0$$

where $|A|$ is the determinant of the matrix A. For example, if $P(w) = \sigma(P(h))$ is bounded and z_1 is not an eigenvalue of $P(w)$ for any $w = (x, \xi)$ (or a limit eigenvalue at infinity) then $P(h) - z_1 \operatorname{Id}_N$ is invertible by Proposition 2.19. When $P(w)$ is an unbounded symbol one needs additional conditions; see for example Proposition 2.20. We shall mostly restrict our study to bounded symbols, but we can reduce to this case if $P(h) - z_1 \operatorname{Id}_N$ is invertible, by considering

$$(P(h) - z_1 \operatorname{Id}_N)^{-1}(P(h) - z_2 \operatorname{Id}_N) \qquad z_2 \neq z_1$$

see Remark 2.21. But unless we have conditions on the eigenvalues at infinity, this does not always give a bounded operator.

Example 2.1. Let

$$P(\xi) = \begin{pmatrix} 0 & \xi \\ 0 & 0 \end{pmatrix}, \qquad \xi \in \mathbb{R}.$$

Then 0 is the only eigenvalue of $P(\xi)$ but

$$(P(\xi) - z \operatorname{Id}_N)^{-1} = -1/z \begin{pmatrix} 1 & \xi/z \\ 0 & 1 \end{pmatrix}$$

and $(P^w - z \operatorname{Id}_N)^{-1} P^w = -z^{-1} P^w$ is unbounded for any $z \neq 0$.

Definition 2.2. Let $P \in C^\infty(T^*\mathbb{R}^n)$ be an $N \times N$ system. We denote the closure of the set of eigenvalues of P by

$$\Sigma(P) = \overline{\{\lambda \in \mathbb{C} : \exists\, w \in T^*\mathbb{R}^n,\ |P(w) - \lambda \operatorname{Id}_N| = 0\}}$$

and the eigenvalues at infinity:

$$\Sigma_\infty(P) = \left\{\lambda \in \mathbb{C} : \exists\, w_j \to \infty\ \exists\, u_j \in \mathbb{C}^N \setminus 0;\ |P(w_j)u_j - \lambda u_j|/|u_j| \to 0,\ j \to \infty\right\}$$

which is closed in $\mathbb{C}$.

In fact, that $\Sigma_\infty(P)$ is closed follows by taking a suitable diagonal sequence. Observe that as in the scalar case, we could have $\Sigma_\infty(P) = \Sigma(P)$, for example if $P(w)$ is constant in one direction. It follows from the definition that $\lambda \notin \Sigma_\infty(P)$ if and only if the resolvent is defined and bounded when $|w|$ is large enough:

$$\|(P(w) - \lambda \operatorname{Id}_N)^{-1}\| \le C \qquad |w| \gg 1 \tag{2.4}$$

In fact, if (2.4) does not hold there would exist $w_j \to \infty$ such that $\|(P(w_j) - \lambda \operatorname{Id}_N)^{-1}\| \to \infty$, $j \to \infty$. Thus, there would exist $u_j \in \mathbb{C}^N$ such that $|u_j| = 1$ and $P(w_j)u_j - \lambda u_j \to 0$. On the contrary, if (2.4) holds then $|P(w)u - \lambda u| \ge |u|/C$ for any $u \in \mathbb{C}^N$ and $|w| \gg 1$.

It is clear from the definition that $\Sigma_\infty(P)$ contains all finite limits of eigenvalues of P at infinity. In fact, if $P(w_j)u_j = \lambda_j u_j$, $|u_j| = 1$, $w_j \to \infty$ and $\lambda_j \to \lambda$ then

$$P(w_j)u_j - \lambda u_j = (\lambda_j - \lambda)u_j \to 0.$$

Example 2.1 shows that in general $\Sigma_\infty(P)$ could be a larger set.

Example 2.3. Let $P(\xi)$ be given by Example 2.1; then $\Sigma(P) = \{0\}$ but $\Sigma_\infty(P) = \mathbb{C}$. In fact, for any $\lambda \in \mathbb{C}$ we find

$$|P(\xi)u_\xi - \lambda u_\xi| = \lambda^2 \quad \text{when} \quad u_\xi = {}^t(\xi, \lambda).$$

We have that $|u_\xi| = \sqrt{|\lambda|^2 + \xi^2} \to \infty$ so $|P(\xi)u_\xi - \lambda u_\xi|/|u_\xi| \to 0$ when $|\xi| \to \infty$.

For bounded symbols we get equality according to the following proposition.

Proposition 2.4. *If $P \in C_b^\infty(T^*\mathbb{R}^n)$ is an $N \times N$ system then $\Sigma_\infty(P)$ is the set of all limits of the eigenvalues of P at infinity.*

Proof. Since $\Sigma_\infty(P)$ contains all limits of eigenvalues of P at infinity, we only have to prove the opposite inclusion. Let $\lambda \in \Sigma_\infty(P)$ then by the definition there exist $w_j \to \infty$ and $u_j \in \mathbb{C}^N$ such that $|u_j| = 1$ and $|P(w_j)u_j - \lambda u_j| = \varepsilon_j \to 0$. Then we may choose $N \times N$ matrix A_j such that $\|A_j\| = \varepsilon_j$ and $P(w_j)u_j = \lambda u_j + A_j u_j$ thus λ is an eigenvalue of $P(w_j) - A_j$. Now if A and B are $N \times N$ matrices and $d(\mathrm{Eig}(A), \mathrm{Eig}(B))$ is the minimal distance between the sets of eigenvalues of A and B under permutations, then we have that $d(\mathrm{Eig}(A), \mathrm{Eig}(B)) \to 0$ when $\|A - B\| \to 0$. In fact, a theorem of Elsner [1985] gives

$$d(\mathrm{Eig}(A), \mathrm{Eig}(B)) \le N(2\max(\|A\|, \|B\|))^{1-1/N}\|A - B\|^{1/N}.$$

Since the matrices $P(w_j)$ are uniformly bounded we find that they have an eigenvalue μ_j such that $|\mu_j - \lambda| \le C_N \varepsilon_j^{1/N} \to 0$ as $j \to \infty$, thus $\lambda = \lim_{j\to\infty} \mu_j$ is a limit of eigenvalues of $P(w)$ at infinity. $\square$

One problem with studying systems $P(w)$, is that the eigenvalues are not very regular in the parameter w, generally they depend only continuously (and eigenvectors measurably) on w.

Definition 2.5. For an $N \times N$ system $P \in C^\infty(T^*\mathbb{R}^n)$ we define

$$\kappa_P(w, \lambda) = \mathrm{Dim}\,\mathrm{Ker}(P(w) - \lambda\,\mathrm{Id}_N)$$
$$K_P(w, \lambda) = \max\left\{k : \partial_\lambda^j p(w, \lambda) = 0 \text{ for } j < k\right\}$$

where $p(w, \lambda) = |P(w) - \lambda\,\mathrm{Id}_N|$ is the characteristic polynomial. We have $\kappa_P \le K_P$ with equality for symmetric systems but in general we need not have equality; see Example 2.7. If

$$\Omega_k(P) = \left\{(w, \lambda) \in T^*\mathbb{R}^n \times \mathbb{C} : K_P(w, \lambda) \ge k\right\} \qquad k \ge 1,$$

then $\varnothing = \Omega_{N+1}(P) \subseteq \Omega_N(P) \subseteq \cdots \subseteq \Omega_1(P)$ and we may define

$$\Xi(P) = \bigcup_{j>1} \partial\Omega_j(P)$$

where $\partial\Omega_j(P)$ is the boundary of $\Omega_j(P)$ in the relative topology of $\Omega_1(P)$.

Clearly, $\Omega_j(P)$ is a closed set for any $j \ge 1$. By definition we find that the multiplicity K_P of the zeros of $|P(w) - \lambda\,\mathrm{Id}_N|$ is locally constant on $\Omega_1(P) \setminus \Xi(P)$. If $P(w)$ is symmetric then $\kappa_P = \mathrm{Dim}\,\mathrm{Ker}(P(w) - \lambda\,\mathrm{Id}_N)$ also is constant on $\Omega_1(P) \setminus \Xi(P)$. This is not true in general; see Example 3.9.

Remark 2.6. We find that $\Xi(P)$ is closed and nowhere dense in $\Omega_1(P)$ since it is the union of boundaries of closed sets. We also find that

$$(w, \lambda) \in \Xi(P) \Leftrightarrow (w, \bar{\lambda}) \in \Xi(P^*)$$

since $|P^* - \bar{\lambda}\,\mathrm{Id}_N| = \overline{|P - \lambda\,\mathrm{Id}_N|}$.

Example 2.7. If

$$P(w) = \begin{pmatrix} \lambda_1(w) & 1 \\ 0 & \lambda_2(w) \end{pmatrix}$$

where $\lambda_j(w) \in C^\infty$, $j = 1, 2$, then

$$\Omega_1(P) = \{(w, \lambda) : \lambda = \lambda_j(w),\ \ j = 1,\ 2\}$$
$$\Omega_2(P) = \{(w, \lambda) : \lambda = \lambda_1(w) = \lambda_2(w)\},$$

but $\kappa_P \equiv 1$ on $\Omega_1(P)$.

Example 2.8. Let

$$P(t) = \begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix} \qquad t \in \mathbb{R}.$$

Then $P(t)$ has the eigenvalues $\pm\sqrt{t}$, and $\kappa_P \equiv 1$ on $\Omega_1(P)$.

Example 2.9. If

$$P = \begin{pmatrix} w_1 + w_2 & w_3 \\ w_3 & w_1 - w_2 \end{pmatrix}$$

then

$$\Omega_1(P) = \left\{(w; \lambda_j) : \lambda_j = w_1 + (-1)^j\sqrt{w_2^2 + w_3^2},\ \ j = 1,\ 2\right\}.$$

We have that $\Omega_2(P) = \{(w_1, 0, 0; w_1) : w_1 \in \mathbb{R}\}$ and $\kappa_P = 2$ on $\Omega_2(P)$.

Definition 2.10. Let π_j be the projections

$$\pi_1(w, \lambda) = w \quad \text{and} \quad \pi_2(w, \lambda) = \lambda.$$

Then we define for $\lambda \in \mathbb{C}$ the closed sets

$$\Sigma_\lambda(P) = \pi_1\big(\Omega_1(P) \cap \pi_2^{-1}(\lambda)\big) = \{w : |P(w) - \lambda\,\mathrm{Id}_N| = 0\}$$
$$X(P) = \pi_1\left(\Xi(P)\right) \subseteq T^*\mathbb{R}^n.$$

Remark 2.11. Observe that $X(P)$ is nowhere dense in $T^*\mathbb{R}^n$ and $P(w)$ has constant characteristics near $w_0 \notin X(P)$. This means that $|P(w) - \lambda\,\mathrm{Id}_N| = 0$ if and only if $\lambda = \lambda_j(w)$ for $j = 1, \dots k$, where the eigenvalues $\lambda_j(w) \neq \lambda_k(w)$ for $j \neq k$ when $|w - w_0| \ll 1$.

In fact, $\pi_1^{-1}(w)$ is a finite set for any $w \in T^*\mathbb{R}^n$ and since the eigenvalues are continuous functions of the parameters, the relative topology on $\Omega_1(P)$ is generated by $\pi_1^{-1}(\omega) \cap \Omega_1(P)$ for open sets $\omega \subset T^*\mathbb{R}^n$.

Definition 2.12. For an $N \times N$ system $P \in C^\infty(T^*\mathbb{R}^n)$ we define the *weakly singular eigenvalue set*

$$\Sigma_{\mathrm{ws}}(P) = \pi_2\left(\Xi(P)\right) \subseteq \mathbb{C}$$

and the *strongly singular eigenvalue set*

$$\Sigma_{\mathrm{ss}}(P) = \left\{\lambda : \pi_2^{-1}(\lambda) \cap \Omega_1(P) \subseteq \Xi(P)\right\}.$$

Remark 2.13. It is clear from the definition that $\Sigma_{\mathrm{ss}}(P) \subseteq \Sigma_{\mathrm{ws}}(P)$. We have that $\Sigma_{\mathrm{ws}}(P) \cup \Sigma_\infty(P)$ and $\Sigma_{\mathrm{ss}}(P) \cup \Sigma_\infty(P)$ are closed, and $\Sigma_{\mathrm{ss}}(P)$ is nowhere dense.

In fact, if $\lambda_j \to \lambda \notin \Sigma_\infty(P)$, then $\pi_2^{-1}(\lambda_j) \cap \Omega_1(P)$ is contained in a compact set for $j \gg 1$, which then either intersects $\Xi(P)$ or is contained in $\Xi(P)$. Since $\Xi(P)$ is closed, these properties are preserved in the limit.

Also, if $\lambda \in \overline{\Sigma_{\mathrm{ss}}(P)}$, then there exists $(w_j, \lambda_j) \in \Xi(P)$ such that $\lambda_j \to \lambda$ as $j \to \infty$. Since $\Xi(P)$ is nowhere dense in $\Omega_1(P)$, there exists $(w_{jk}, \lambda_{jk}) \in \Omega_1(P) \setminus \Xi(P)$ converging to (w_j, λ_j) as $k \to \infty$. Then $\Sigma(P) \setminus \Sigma_{\mathrm{ss}}(P) \ni \lambda_{jj} \to \lambda$, so $\Sigma_{\mathrm{ss}}(P)$ is nowhere dense. On the other hand, it is possible that $\Sigma_{\mathrm{ws}}(P) = \Sigma(P)$ by the following example.

Example 2.14. Let $P(w)$ be the system in Example 2.9; then we have

$$\Sigma_{\mathrm{ws}}(P) = \Sigma(P) = \mathbb{R}$$

and $\Sigma_{\mathrm{ss}}(P) = \varnothing$. In fact, the eigenvalues coincide only when $w_2 = w_3 = 0$ and the eigenvalue $\lambda = w_1$ is also attained at some point where $w_2 \neq 0$. If we multiply $P(w)$ with $w_4 + i w_5$, we obtain that $\Sigma_{\mathrm{ws}}(P) = \Sigma(P) = \mathbb{C}$. If we set $\widetilde{P}(w_1, w_2) = P(0, w_1, w_2)$ we find that $\Sigma_{\mathrm{ss}}(\widetilde{P}) = \Sigma_{\mathrm{ws}}(\widetilde{P}) = \{0\}$.

Lemma 2.15. *Let $P \in C^\infty(T^*\mathbb{R}^n)$ be an $N \times N$ system. If $(w_0, \lambda_0) \in \Omega_1(P) \setminus \Xi(P)$ then there exists a unique C^∞ function $\lambda(w)$ so that $(w, \lambda) \in \Omega_1(P)$ if and only if $\lambda = \lambda(w)$ in a neighborhood of (w_0, λ_0). If $\lambda_0 \in \Sigma(P) \setminus (\Sigma_{\mathrm{ws}}(P) \cup \Sigma_\infty(P))$ then there is $\lambda(w) \in C^\infty$ such that $(w, \lambda) \in \Omega_1(P)$ if and only if $\lambda = \lambda(w)$ in a neighborhood of $\Sigma_{\lambda_0}(P)$.*

We find from Lemma 2.15 that $\Omega_1(P) \setminus \Xi(P)$ is locally given as a C^∞ manifold over $T^*\mathbb{R}^n$, and that the eigenvalues $\lambda_j(w) \in C^\infty$ outside $X(P)$. This is not true if we instead assume that κ_P is constant on $\Omega_1(P)$; see Example 2.8.

Proof. If $(w_0, \lambda_0) \in \Omega_1(P) \setminus \Xi(P)$, then

$$\lambda \to |P(w) - \lambda \, \mathrm{Id}_N|$$

vanishes of exactly order $k > 0$ on $\Omega_1(P)$ in a neighborhood of (w_0, λ_0), so

$$\partial_\lambda^k |P(w_0) - \lambda \, \mathrm{Id}_N| \neq 0 \qquad \text{for } \lambda = \lambda_0.$$

Thus $\lambda = \lambda(w)$ is the unique solution to $\partial_\lambda^{k-1} |P(w) - \lambda \, \mathrm{Id}_N| = 0$ near w_0 which is C^∞ by the Implicit Function Theorem.

If $\lambda_0 \in \Sigma(P) \setminus (\Sigma_{\mathrm{ws}}(P) \cup \Sigma_\infty(P))$ then we obtain this in a neighborhood of any $w_0 \in \Sigma_{\lambda_0}(P) \Subset T^*\mathbb{R}^n$. Using a C^∞ partition of unity we find by uniqueness that $\lambda(w) \in C^\infty$ in a neighborhood of $\Sigma_{\lambda_0}(P)$. □

Remark 2.16. Observe that if $\lambda_0 \in \Sigma(P) \setminus (\Sigma_{\mathrm{ws}}(P) \cup \Sigma_\infty(P))$ and $\lambda(w) \in C^\infty$ satisfies $|P(w) - \lambda(w) \, \mathrm{Id}_N| \equiv 0$ near $\Sigma_{\lambda_0}(P)$ and $\lambda|_{\Sigma_{\lambda_0}(P)} = \lambda_0$, then we find by Sard's Theorem that $d \operatorname{Re} \lambda$ and $d \operatorname{Im} \lambda$ are linearly independent on the codimension 2 manifold $\Sigma_\mu(P)$ for almost all values μ close to λ_0. Thus for $n = 1$ we find that $\Sigma_\mu(P)$ is a discrete set for almost all values μ close to λ_0.

In fact, since $\lambda_0 \notin \Sigma_\infty(P)$ we find that $\Sigma_\mu(P) \to \Sigma_{\lambda_0}(P)$ when $\mu \to \lambda_0$ so $\Sigma_\mu(P) = \{w : \lambda(w) = \mu\}$ for $|\mu - \lambda_0| \ll 1$.

Definition 2.17. A C^∞ function $\lambda(w)$ is called a *germ of eigenvalues* at w_0 for the $N \times N$ system $P \in C^\infty(T^*\mathbb{R}^n)$ if

$$|P(w) - \lambda(w) \, \mathrm{Id}_N| \equiv 0 \qquad \text{in a neighborhood of } w_0.$$

If this holds in a neighborhood of every point in $\omega \Subset T^*\mathbb{R}^n$ then we say that $\lambda(w)$ is a germ of eigenvalues for P on ω.

Remark 2.18. If $\lambda_0 \in \Sigma(P) \setminus (\Sigma_{ss}(P) \cup \Sigma_\infty(P))$ then there exists $w_0 \in \Sigma_{\lambda_0}(P)$ so that $(w_0, \lambda_0) \in \Omega_1(P) \setminus \Xi(P)$. By Lemma 2.15 there exists a C^∞ germ $\lambda(w)$ of eigenvalues at w_0 for P such that $\lambda(w_0) = \lambda_0$. If $\lambda_0 \in \Sigma(P) \setminus (\Sigma_{ws}(P) \cup \Sigma_\infty(P))$ then there exists a C^∞ germ $\lambda(w)$ of eigenvalues on $\Sigma_{\lambda_0}(P)$.

As in the scalar case we obtain that the spectrum is essentially discrete outside $\Sigma_\infty(P)$.

Proposition 2.19. *Assume that the $N \times N$ system $P(h)$ is given by (2.2) with principal symbol $P \in C_b^\infty(T^*\mathbb{R}^n)$. Let Ω be an open connected set, satisfying*

$$\overline{\Omega} \cap \Sigma_\infty(P) = \varnothing \quad \textit{and} \quad \Omega \cap \complement\Sigma(P) \neq \varnothing.$$

Then $(P(h) - z\,\mathrm{Id}_N)^{-1}$, $0 < h \ll 1$, $z \in \Omega$, is a meromorphic family of operators with poles of finite rank. In particular, for h sufficiently small, the spectrum of $P(h)$ is discrete in any such set. When $\Omega \cap \Sigma(P) = \varnothing$ we find that Ω contains no spectrum of $P^w(x, hD)$.

Proof. We shall follow the proof of Proposition 3.3 in [Dencker et al. 2004]. If Ω satisfies the assumptions of the proposition then there exists $C > 0$ such that

$$|(P(w) - z\,\mathrm{Id}_N)^{-1}| \le C \qquad \text{if } z \in \Omega \text{ and } |w| > C\ . \tag{2.5}$$

In fact, otherwise there would exist $w_j \to \infty$ and $z_j \in \Omega$ such that $|(P(w_j) - z_j\,\mathrm{Id}_N)^{-1}| \to \infty$, $j \to \infty$. Thus, there exists $u_j \in \mathbb{C}^N$ such that $|u_j| = 1$ and $P(w_j)u_j - z_j u_j \to 0$. Since $\Sigma(P) \Subset \mathbb{C}$ we obtain after picking a subsequence that $z_j \to z \in \overline{\Omega} \cap \Sigma_\infty(P) = \varnothing$. The assumption that $\Omega \cap \complement\Sigma(p) \neq \varnothing$ implies that for some $z_0 \in \Omega$ we have $(P(w) - z_0\,\mathrm{Id}_N)^{-1} \in C_b^\infty$. Let $\chi \in C_0^\infty(T^*\mathbb{R}^n)$, $0 \le \chi(w) \le 1$ and $\chi(w) = 1$ when $|w| \le C$, where C is given by (2.5). Let

$$R(w, z) = \chi(w)(P(w) - z_0\,\mathrm{Id}_N)^{-1} + (1 - \chi(w))(P(w) - z\,\mathrm{Id}_N)^{-1} \in C_b^\infty$$

for $z \in \Omega$. The symbolic calculus then gives

$$R^w(x, hD, z)(P(h) - z\,\mathrm{Id}_N) = I + hB_1(h, z) + K_1(h, z)$$
$$(P(h) - z\,\mathrm{Id}_N)R^w(x, hD, z) = I + hB_2(h, z) + K_2(h, z)$$

where $K_j(h, z)$ are compact operators on $L^2(\mathbb{R}^n)$ depending holomorphically on z, vanishing for $z = z_0$, and $B_j(h, z)$ are bounded on $L^2(\mathbb{R}^n)$, $j = 1, 2$. By the analytic Fredholm theory we conclude that $(P(h) - z\,\mathrm{Id}_N)^{-1}$ is meromorphic in $z \in \Omega$ for h sufficiently small. When $\Omega \cap \Sigma(P) = \varnothing$ we can choose $R(w, z) = (P(w) - z\,\mathrm{Id}_N)^{-1}$, then $K_j \equiv 0$ for $j = 1, 2$, and $P(h) - z\,\mathrm{Id}_N$ is invertible for small enough h. □

We shall show how the reduction to the case of bounded operator can be done in the systems case, following [Dencker et al. 2004]. Let $m(w)$ be a positive function on $T^*\mathbb{R}^n$ satisfying

$$1 \le m(w) \le C\langle w - w_0\rangle^N m(w_0)\,, \qquad \forall\, w,\ w_0 \in T^*\mathbb{R}^n$$

for some fixed C and N, where $\langle w\rangle = 1+|w|$. Then m is an admissible weight function and we can define the symbol classes $P \in S(m)$ by

$$\|\partial_w^\alpha P(w)\| \le C_\alpha m(w) \qquad \forall\alpha.$$

Following [Dimassi and Sjöstrand 1999] we then define the semiclassical operator $P(h) = P^w(x, hD)$. In the analytic case we require that the symbol estimates hold in a tubular neighborhood of $T^*\mathbb{R}^n$:

$$\|\partial_w^\alpha P(w)\| \le C_\alpha m(\operatorname{Re} w) \qquad \text{for} \quad |\operatorname{Im} w| \le 1/C \qquad \forall\alpha \tag{2.6}$$

One typical example of an admissible weight function is $m(x,\xi) = (\langle\xi\rangle^2 + \langle x\rangle^p)$.

Now we make the ellipticity assumption

$$\|P^{-1}(w)\| \le C_0 m^{-1}(w) \qquad |w| \gg 1 \tag{2.7}$$

and in the analytic case we assume this in a tubular neighborhood of $T^*\mathbb{R}^n$ as in (2.6). By Leibniz' rule we obtain that $P^{-1} \in S(m^{-1})$ at infinity, that is,

$$\|\partial_w^\alpha P^{-1}(w)\| \le C'_\alpha m^{-1}(w) \qquad |w| \gg 1.$$

When $z \notin \Sigma(P) \cup \Sigma_\infty(P)$ we find as before that

$$\|(P(w) - z\operatorname{Id}_N)^{-1}\| \le C \qquad \forall\, w$$

since the resolvent is uniformly bounded at infinity. This implies that $P(w)(P(w) - z\operatorname{Id}_N)^{-1}$ and $(P(w) - z\operatorname{Id}_N)^{-1}P(w)$ are bounded. Again by Leibniz' rule, (2.7) holds with P replaced by $P - z\operatorname{Id}_N$ thus $(P(w)-z\operatorname{Id}_N)^{-1} \in S(m^{-1})$ and we may define the semiclassical operator $((P-z\operatorname{Id}_N)^{-1})^w(x, hD)$. Since $m \ge 1$ we find that $P(w) - z\operatorname{Id}_N \in S(m)$, so by using the calculus we obtain that

$$\begin{aligned}(P^w - z\operatorname{Id}_N)((P - z\operatorname{Id}_N)^{-1})^w &= 1 + hR_1^w\\ ((P - z\operatorname{Id}_N)^{-1})^w(P^w - z\operatorname{Id}_N) &= 1 + hR_2^w\end{aligned}$$

where $R_j \in S(1)$, $j = 1, 2$. For small enough h we get invertibility and the following result.

Proposition 2.20. *Assume that $P \in S(m)$ is an $N\times N$ system satisfying (2.7) and that $z \notin \Sigma(P)\cup\Sigma_\infty(P)$. Then we find that $P^w - z\operatorname{Id}_N$ is invertible for small enough h.*

This makes it possible to reduce to the case of operators with bounded symbols.

Remark 2.21. If $z_1 \notin \operatorname{Spec}(P)$ we may define the operator

$$Q = (P - z_1\operatorname{Id}_N)^{-1}(P - z_2\operatorname{Id}_N) \qquad z_2 \ne z_1.$$

The resolvents of Q and P are related by

$$(Q - \zeta\operatorname{Id}_N)^{-1} = (1-\zeta)^{-1}(P - z_1\operatorname{Id}_N)\Big(P - \frac{\zeta z_1 - z_2}{\zeta - 1}\operatorname{Id}_N\Big)^{-1} \qquad \zeta \ne 1$$

when $(\zeta z_1 - z_2)/(\zeta - 1) \notin \operatorname{Spec}(P)$.

Example 2.22. Let
$$P(x,\xi) = |\xi|^2 \operatorname{Id}_N + iK(x)$$
where $0 \le K(x) \in C_b^\infty$; then we find that $P \in S(m)$ with $m(x,\xi) = |\xi|^2 + 1$. If $0 \notin \Sigma_\infty(K)$ then $K(x)$ is invertible for $|x| \gg 1$, so $P^{-1} \in S(m^{-1})$ at infinity. Since $\operatorname{Re} z \ge 0$ in $\Sigma(P)$ we find from Proposition 2.20 that $P^w(x,hD) + \operatorname{Id}_N$ is invertible for small enough h and $P^w(x,hD)(P^w(x,hD)+\operatorname{Id}_N)^{-1}$ is bounded in L^2 with principal symbol $P(w)(P(w)+\operatorname{Id}_N)^{-1} \in C_b^\infty$.

In order to measure the singularities of the solutions, we shall introduce the semiclassical wave front sets.

Definition 2.23. For $u \in L^2(\mathbb{R}^n)$ we say that $w_0 \notin \mathrm{WF}_h(u)$ if there exists $a \in C_0^\infty(T^*\mathbb{R}^n)$ such that $a(w_0) \ne 0$ and the L^2 norm
$$\|a^w(x,hD)u\| \le C_k h^k \qquad \forall k. \tag{2.8}$$
We call $\mathrm{WF}_h(u)$ the semiclassical wave front set of u.

Observe that this definition is equivalent to Definition (2.5) in [Dencker et al. 2004] which use the FBI transform T given by (4.26): $w_0 \notin \mathrm{WF}_h(u)$ if $\|Tu(w)\| = \mathbb{O}(h^\infty)$ when $|w - w_0| \ll 1$. We may also define the *analytic* semiclassical wave front set by the condition that $\|Tu(w)\| = \mathbb{O}(e^{-c/h})$ in a neighborhood of w_0 for some $c > 0$; see (2.6) in [Dencker et al. 2004].

Observe that if $u = (u_1, \dots, u_N) \in L^2(\mathbb{R}^n, \mathbb{C}^N)$ we may define $\mathrm{WF}_h(u) = \bigcap_j \mathrm{WF}_h(u_j)$ but this gives no information about which components of u that are singular. Therefore we shall define the corresponding vector valued polarization sets.

Definition 2.24. For $u \in L^2(\mathbb{R}^n, \mathbb{C}^N)$, we say that $(w_0, z_0) \notin \mathrm{WF}_h^{\mathrm{pol}}(u) \subseteq T^*\mathbb{R}^n \times \mathbb{C}^N$ if there exists $A(h)$ given by (2.2) with principal symbol $A(w)$ such that $A(w_0)z_0 \ne 0$ and $A(h)u$ satisfies (2.8). We call $\mathrm{WF}_h^{\mathrm{pol}}(u)$ the semiclassical polarization set of u.

We could similarly define the *analytic* semiclassical polarization set by using the FBI transform and analytic pseudodifferential operators.

Remark 2.25. The semiclassical polarization sets are closed, linear in the fiber and has the functorial properties of the C^∞ polarization sets in [Dencker 1982]. In particular, we find that
$$\pi(\mathrm{WF}_h^{\mathrm{pol}}(u) \setminus 0) = \mathrm{WF}_h(u) = \bigcup_j \mathrm{WF}_h(u_j)$$
if π is the projection along the fiber variables: $\pi : T^*\mathbb{R}^n \times \mathbb{C}^N \mapsto T^*\mathbb{R}^n$. We also find that
$$A(\mathrm{WF}_h^{\mathrm{pol}}(u)) = \left\{(w, A(w)z) : (w,z) \in \mathrm{WF}_h^{\mathrm{pol}}(u)\right\} \subseteq \mathrm{WF}_h^{\mathrm{pol}}(A(h)u)$$
if $A(w)$ is the principal symbol of $A(h)$, which implies that $\mathrm{WF}_h^{\mathrm{pol}}(Au) = A(\mathrm{WF}_h^{\mathrm{pol}}(u))$ when $A(h)$ is elliptic.

This follows from the proofs of Propositions 2.5 and 2.7 in [Dencker 1982].

Example 2.26. Let $u = (u_1, \dots, u_N) \in L^2(T^*\mathbb{R}^n, \mathbb{C}^N)$ where $\mathrm{WF}_h(u_1) = \{w_0\}$ and $\mathrm{WF}_h(u_j) = \varnothing$ for $j > 1$. Then
$$\mathrm{WF}_h^{\mathrm{pol}}(u) = \{(w_0, (z, 0, \dots)) : z \in \mathbb{C}\}$$

since $\|A^w(x,hD)u\| = \mathcal{O}(h^\infty)$ if $A^w u = \sum_{j>1} A_j^w u_j$ and $w_0 \in \mathrm{WF}_h(u)$. By taking a suitable invertible E we obtain

$$\mathrm{WF}_h^{\mathrm{pol}}(Eu) = \{(w_0, zv) : z \in \mathbb{C}\}$$

for any $v \in \mathbb{C}^N$.

We shall use the following definitions from [Pravda-Starov 2006a], here and in the following $\|P(h)\|$ will denote the L^2 operator norm of $P(h)$.

Definition 2.27. Let $P(h)$, $0 < h \le 1$, be a semiclassical family of operators on $L^2(\mathbb{R}^n)$ with domain D. For $\mu > 0$ we define the *pseudospectrum of index* μ as the set

$$\Lambda_\mu^{\mathrm{sc}}(P(h)) = \left\{z \in \mathbb{C} : \forall\, C > 0,\ \forall\, h_0 > 0, \exists\, 0 < h < h_0, \|(P(h) - z\,\mathrm{Id}_N)^{-1}\| \ge C h^{-\mu}\right\}$$

and the *injectivity pseudospectrum of index* μ as

$$\lambda_\mu^{\mathrm{sc}}(P(h)) = \left\{z \in \mathbb{C} : \forall\, C > 0,\ \forall\, h_0 > 0, \exists\, 0 < h < h_0,\ \ \exists\, u \in D,\ \|u\| = 1,\ \|(P(h) - z\,\mathrm{Id}_N)u\| \le C h^{\mu}\right\}.$$

We define the *pseudospectrum of infinite index* as $\Lambda_\infty^{\mathrm{sc}}(P(h)) = \bigcap_\mu \Lambda_\mu^{\mathrm{sc}}(P(h))$ and correspondingly the *injectivity pseudospectrum of infinite index*.

Here we use the convention that $\|(P(h) - \lambda\,\mathrm{Id}_N)^{-1}\| = \infty$ when λ is in the spectrum $\mathrm{Spec}(P(h))$. Observe that we have the obvious inclusion $\lambda_\mu^{\mathrm{sc}}(P(h)) \subseteq \Lambda_\mu^{\mathrm{sc}}(P(h))$ for all μ. We get equality if, for example, $P(h)$ is Fredholm of index ≥ 0.

3. The interior case

Recall that the scalar symbol $p(x,\xi) \in C^\infty(T^*\mathbb{R}^n)$ is of *principal type* if $dp \ne 0$ when $p = 0$. In the following we let $\partial_\nu P(w) = \langle \nu, dP(w)\rangle$ for $P \in C^1(T^*\mathbb{R}^n)$ and $\nu \in T^*\mathbb{R}^n$. We shall use the following definition of systems of principal type, in fact, most of the systems we consider will be of this type. We shall denote $\operatorname{Ker} P$ and $\operatorname{Ran} P$ the kernel and range of P.

Definition 3.1. The $N \times N$ system $P(w) \in C^\infty(T^*\mathbb{R}^n)$ is of *principal type* at w_0 if

$$\operatorname{Ker} P(w_0) \ni u \mapsto \partial_\nu P(w_0)u \in \operatorname{Coker} P(w_0) = \mathbb{C}^N / \operatorname{Ran} P(w_0) \tag{3.1}$$

is bijective for some $\nu \in T_{w_0}(T^*\mathbb{R}^n)$. The operator $P(h)$ given by (2.2) is of principal type if the principal symbol $P = \sigma(P(h))$ is of principal type.

Remark 3.2. If $P(w) \in C^\infty$ is of principal type and $A(w)$, $B(w) \in C^\infty$ are invertible then APB is of principal type. We have that $P(w)$ is of principal type if and only if the adjoint P^* is of principal type.

In fact, by Leibniz' rule we have

$$\partial(APB) = (\partial A)PB + A(\partial P)B + AP\partial B \tag{3.2}$$

and $\operatorname{Ran}(APB) = A(\operatorname{Ran} P)$ and $\operatorname{Ker}(APB) = B^{-1}(\operatorname{Ker} P)$ when A and B are invertible, which gives the invariance under left and right multiplication. Since $\operatorname{Ker} P^*(w_0) = \operatorname{Ran} P(w_0)^\perp$ we find that P satisfies (3.1) if and only if

$$\operatorname{Ker} P(w_0) \times \operatorname{Ker} P^*(w_0) \ni (u, v) \mapsto \langle \partial_\nu P(w_0)u, v\rangle$$

is a nondegenerate bilinear form. Since $\langle \partial_\nu P^* v, u\rangle = \overline{\langle \partial_\nu P u, v\rangle}$ we find that P^* is of principal type if and only if P is.

Observe that if P only has one vanishing eigenvalue λ (with multiplicity one) then the condition that P is of principal type reduces to the condition in the scalar case: $d\lambda \neq 0$. In fact, by using the spectral projection one can find invertible systems A and B so that

$$APB = \begin{pmatrix} \lambda & 0 \\ 0 & E \end{pmatrix}$$

with E invertible $(N-1)\times(N-1)$ system, and this system is obviously of principal type.

Example 3.3. Consider the system in Example 2.7

$$P(w) = \begin{pmatrix} \lambda_1(w) & 1 \\ 0 & \lambda_2(w) \end{pmatrix}$$

where $\lambda_j(w) \in C^\infty$, $j = 1, 2$. We find that $P(w) - \lambda \operatorname{Id}_2$ is not of principal type when $\lambda = \lambda_1(w) = \lambda_2(w)$ since $\operatorname{Ker}(P(w) - \lambda \operatorname{Id}_2) = \operatorname{Ran}(P(w) - \lambda \operatorname{Id}_2) = \mathbb{C} \times \{0\}$ is preserved by ∂P.

Observe that the property of being of principal type is not stable under C^1 perturbation, not even when $P = P^*$ is symmetric, by the following example.

Example 3.4. The system

$$P(w) = \begin{pmatrix} w_1 - w_2 & w_2 \\ w_2 & -w_1 - w_2 \end{pmatrix} = P^*(w) \qquad w = (w_1, w_2)$$

is of principal type when $w_1 = w_2 = 0$, but *not* of principal type when $w_2 \neq 0$ and $w_1 = 0$. In fact,

$$\partial_{w_1} P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

is invertible, and when $w_2 \neq 0$ we have that

$$\operatorname{Ker} P(0, w_2) = \operatorname{Ker} \partial_{w_2} P(0, w_2) = \{z(1, 1) : z \in \mathbb{C}\}$$

which is mapped to $\operatorname{Ran} P(0, w_2) = \{z(1, -1) : z \in \mathbb{C}\}$ by $\partial_{w_1} P$.

We shall obtain a simple characterization of systems of principal type. Recall κ_P, K_P and $\Xi(P)$ given by Definition 2.5.

Proposition 3.5. *Assume $P(w) \in C^\infty$ is an $N \times N$ system and that $(w_0, \lambda_0) \in \Omega_1(P) \setminus \Xi(P)$; then $P(w) - \lambda_0 \operatorname{Id}_N$ is of principal type at w_0 if and only if $\kappa_P \equiv K_P$ at (w_0, λ_0) and $d\lambda(w_0) \neq 0$ for the C^∞ germ of eigenvalues $\lambda(w)$ for P at w_0 satisfying $\lambda(w_0) = \lambda_0$.*

Thus, in the case $\lambda_0 = 0 \notin \Sigma_{\mathrm{ws}}(P)$ we find that $P(w)$ is of principal type if and only if λ is of principal type and we have no nontrivial Jordan boxes in the normal form. Observe that by the proof of Lemma 2.15 the C^∞ germ $\lambda(w)$ is the unique solution to $\partial_\lambda^k p(w, \lambda) = 0$ for $k = K_P(w, \lambda) - 1$ where $p(w, \lambda) = |P(w) - \lambda \operatorname{Id}_N|$ is the characteristic equation. Thus we find that $d\lambda(w) \neq 0$ if and only if $\partial_w \partial_\lambda^k p(w, \lambda) \neq 0$. For symmetric operators we have $\kappa_P \equiv K_P$ and we only need this condition when $(w_0, \lambda_0) \notin \Xi(P)$.

Example 3.6. The system $P(w)$ in Example 3.4 has eigenvalues $-w_2 \pm \sqrt{w_1^2 + w_2^2}$ which are equal if and only if $w_1 = w_2 = 0$, so $\{0\} = \Sigma_{ws}(P)$. When $w_2 \neq 0$ and $w_1 \approx 0$ the eigenvalue close to zero is $w_1^2/2w_2 + \mathcal{O}(w_1^4)$ which has vanishing differential at $w_1 = 0$. The characteristic equation is $p(w, \lambda) = \lambda^2 + 2\lambda w_2 - w_1^2$, so $d_w p = 0$ when $w_1 = \lambda = 0$.

Proof of Proposition 3.5. Of course, it is no restriction to assume $\lambda_0 = 0$. First we note that $P(w)$ is of principal type at w_0 if and only if

$$\partial_\nu^k |P(w_0)| \neq 0 \qquad k = \kappa_P(w_0, 0) \tag{3.3}$$

for some $\nu \in T(T^*\mathbb{R}^n)$. Observe that $\partial^j |P(w_0)| = 0$ for $j < k$. In fact, by choosing bases for $\operatorname{Ker} P(w_0)$ and $\operatorname{Im} P(w_0)$ respectively, and extending to bases of $\mathbb{C}^N$, we obtain matrices A and B so that

$$AP(w)B = \begin{pmatrix} P_{11}(w) & P_{12}(w) \\ P_{21}(w) & P_{22}(w) \end{pmatrix}$$

where $|P_{22}(w_0)| \neq 0$ and P_{11}, P_{12} and P_{21} all vanish at w_0. By the invariance, P is of principal type if and only if $\partial_\nu P_{11}$ is invertible for some ν, so by expanding the determinant we obtain (3.3).

Since $(w_0, 0) \in \Omega_1(P) \setminus \Xi(P)$ we find from Lemma 2.15 that we may choose a neighborhood ω of $(w_0, 0)$ such that $(w, \lambda) \in \Omega_1(P) \cap \omega$ if and only if $\lambda = \lambda(w) \in C^\infty$. Then

$$|P(w) - \lambda \operatorname{Id}_N| = (\lambda(w) - \lambda)^m e(w, \lambda)$$

near w_0, where $e(w, \lambda) \neq 0$ and $m = K_P(w_0, 0) \geq \kappa_P(w_0, 0)$. Letting $\lambda = 0$ we obtain that $\partial_\nu^j |P(w_0)| = 0$ if $j < m$ and $\partial_\nu^m |P(w_0)| = (\partial_\nu \lambda(w_0))^m e(w_0, 0)$. □

Remark 3.7. Proposition 3.5 shows that for a *symmetric* system the property to be of principal type is stable outside $\Xi(P)$: if the symmetric system $P(w) - \lambda \operatorname{Id}_N$ is of principal type at a point $(w_0, \lambda_0) \notin \Xi(P)$ then it is in a neighborhood. It follows from Sard's Theorem that symmetric systems $P(w) - \lambda \operatorname{Id}_N$ are of principal type almost everywhere on $\Omega_1(P)$.

In fact, for symmetric systems we have $\kappa_P \equiv K_P$ and the differential $d\lambda \neq 0$ almost everywhere on $\Omega_1(P) \setminus \Xi(P)$. For C^∞ germs of eigenvalues we can define the following bracket condition.

Definition 3.8. Let $P \in C^\infty(T^*\mathbb{R}^n)$ be an $N \times N$ system; then we define

$$\Lambda(P) = \overline{\Lambda_-(P) \cup \Lambda_+(P)}$$

where $\Lambda_\pm(P)$ is the set of $\lambda_0 \in \Sigma(P)$ such that there exists $w_0 \in \Sigma_{\lambda_0}(P)$ so that $(w_0, \lambda_0) \notin \Xi(P)$ and

$$\pm \{\operatorname{Re} \lambda, \operatorname{Im} \lambda\}(w_0) > 0 \tag{3.4}$$

for the unique C^∞ germ $\lambda(w)$ of eigenvalues at w_0 for P such that $\lambda(w_0) = \lambda_0$.

Observe that $\Lambda_\pm(P) \cap \Sigma_{ss}(P) = \emptyset$, and it follows from Proposition 3.5 that $P(w) - \lambda_0 \operatorname{Id}_N$ is of principal type at $w_0 \in \Lambda_\pm(P)$ if and only if $\kappa_P = K_P$ at (w_0, λ_0), since $d\lambda(w_0) \neq 0$ when (3.4) holds. Because of the bracket condition (3.4) we find that $\Lambda_\pm(P)$ is contained in the interior of the values $\Sigma(P)$.

Example 3.9. Let

$$P(x, \xi) = \begin{pmatrix} q(x, \xi) & \chi(x) \\ 0 & q(x, \xi) \end{pmatrix} \qquad (x, \xi) \in T^*\mathbb{R}$$

where $q(x, \xi) = \xi + ix^2$ and $0 \le \chi(x) \in C^\infty(\mathbb{R})$ such that $\chi(x) = 0$ when $x \le 0$ and $\chi(x) > 0$ when $x > 0$. Then $\Sigma(P) = \{\operatorname{Im} z \ge 0\}$, $\Lambda_\pm(P) = \{\operatorname{Im} z > 0\}$ and $\Xi(P) = \varnothing$. For $\operatorname{Im}\lambda > 0$ we find $\Sigma_\lambda(P) = \{(\pm\sqrt{\operatorname{Im}\lambda}, \operatorname{Re}\lambda)\}$ and $P - \lambda \operatorname{Id}_2$ is of principal type at $\Sigma_\lambda(P)$ only when $x < 0$.

Theorem 3.10. *Let $P \in C^\infty(T^*\mathbb{R}^n)$ be an $N \times N$ system; then we have that*

$$\Lambda(P) \setminus \big(\Sigma_{\mathrm{ws}}(P) \cup \Sigma_\infty(P)\big) \subseteq \overline{\Lambda_-(P)} \tag{3.5}$$

when $n \ge 2$. Assume that $P(h)$ is given by (2.2) with principal symbol $P \in C_{\mathrm{b}}^\infty(T^*\mathbb{R}^n)$, and that $\lambda_0 \in \Lambda_-(P)$, $0 \ne u_0 \in \operatorname{Ker}(P(w_0) - \lambda_0 \operatorname{Id}_N)$ and $P(w) - \lambda \operatorname{Id}_N$ is of principal type on $\Sigma_\lambda(P)$ near w_0 for $|\lambda - \lambda_0| \ll 1$, for the $w_0 \in \Sigma_{\lambda_0}(P)$ in Definition 3.8. Then there exists $h_0 > 0$ and $u(h) \in L^2(\mathbb{R}^n)$, $0 < h \le h_0$, so that $\|u(h)\| \le 1$,

$$\|(P(h) - \lambda_0 \operatorname{Id}_N)u(h)\| \le C_N h^N \qquad \forall N \qquad 0 < h \le h_0 \tag{3.6}$$

and $\mathrm{WF}_h^{\mathrm{pol}}(u(h)) = \{(w_0, u_0)\}$. There also exists a dense subset of values $\lambda_0 \in \Lambda(P)$ so that

$$\|(P(h) - \lambda_0 \operatorname{Id}_N)^{-1}\| \ge C'_N h^{-N} \qquad \forall N. \tag{3.7}$$

If all the terms P_j in the expansion (2.2) are analytic satisfying (2.3) then $h^{\pm N}$ may be replaced by $\exp(\mp c/h)$ in (3.6)–(3.7).

Here we use the convention that $\|(P(h) - \lambda \operatorname{Id}_N)^{-1}\| = \infty$ when λ is in the spectrum $\operatorname{Spec}(P(h))$. Condition (3.6) means that λ_0 is in the *injectivity* pseudospectrum $\lambda_\infty^{\mathrm{sc}}(P)$, and (3.7) means that λ_0 is in the pseudospectrum $\Lambda_\infty^{\mathrm{sc}}(P)$.

Remark 3.11. If $P(h)$ is Fredholm of nonnegative index then (3.6) holds for λ_0 in a dense subset of $\Lambda(P)$. In the analytic case, it follows from the proof that it suffices that $P_j(w)$ is analytic satisfying (2.3) in a fixed complex neighborhood of $w_0 \in \Sigma_\lambda(P)$ for all j.

In fact, if $P(h)$ is Fredholm of nonnegative index and $\lambda_0 \in \operatorname{Spec}(P(h))$ then the dimension of $\operatorname{Ker}(P(h) - \lambda_0 \operatorname{Id}_N)$ is positive and (3.6) holds.

Example 3.12. Let

$$P(x, \xi) = |\xi|^2 \operatorname{Id} + iK(x) \qquad (x, \xi) \in T^*\mathbb{R}^n$$

where $K(x) \in C^\infty(\mathbb{R}^n)$ is symmetric for all x. Then we find that

$$\overline{\Lambda_-(P)} = \Lambda(P) = \Big\{\operatorname{Re} z \ge 0 \wedge \operatorname{Im} z \in \overline{\Sigma(K) \setminus \big(\Sigma_{\mathrm{ss}}(K) \cup \Sigma_\infty(K)\big)}\Big\}.$$

In fact, for any $\operatorname{Im} z \in \Sigma(K) \setminus (\Sigma_{\mathrm{ss}}(K) \bigcup \Sigma_\infty(K))$ there exists a germ of eigenvalues $\lambda(x) \in C^\infty(\omega)$ for $K(x)$ in an open set $\omega \subset \mathbb{R}^n$ so that $\lambda(x_0) = \operatorname{Im} z$ for some $x_0 \in \omega$. By Sard's Theorem, we find that almost all values of $\lambda(x)$ in ω are nonsingular, and if $d\lambda \ne 0$ and $\operatorname{Re} z > 0$ we may choose $\xi_0 \in \mathbb{R}^n$ so that $|\xi_0|^2 = \operatorname{Re} z$ and $\langle \xi_0, \partial_x \lambda \rangle < 0$. Then the C^∞ germ of eigenvalues $|\xi|^2 + i\lambda(x)$ for P satisfies (3.4) at (x_0, ξ_0) with the minus sign. Since $K(x)$ is symmetric, we find that $P(w) - z \operatorname{Id}_N$ is of principal type.

Proof of Theorem 3.10. First we are going to prove (3.5) in the case $n \ge 2$. Let

$$W = \Sigma_{\mathrm{ws}}(P) \cup \Sigma_\infty(P)$$

which is a closed set by Remark 2.13; then we find that every point in $\Lambda(P) \setminus W$ is a limit point of

$$\big(\Lambda_-(P) \cup \Lambda_+(P)\big) \setminus W = (\Lambda_-(P) \setminus W) \cup (\Lambda_+(P) \setminus W).$$

Thus, we only have to show that $\lambda_0 \in \overline{\Lambda_-(P)}$ if

$$\lambda_0 \in \Lambda_+(P) \setminus W = \Lambda_+(P) \setminus \big(\Sigma_{\mathrm{ws}}(P) \cup \Sigma_\infty(P)\big). \tag{3.8}$$

By Lemma 2.15 and Remark 2.16 we find from (3.8) that there exists a C^∞ germ of eigenvalues $\lambda(w) \in C^\infty$ so that $\Sigma_\mu(P)$ is equal to the level sets $\{w : \lambda(w) = \mu\}$ for $|\mu - \lambda_0| \ll 1$. By definition we find that the Poisson bracket $\{\operatorname{Re}\lambda, \operatorname{Im}\lambda\}$ does not vanish identically on $\Sigma_{\lambda_0}(P)$. Now by Remark 2.16, $d\operatorname{Re}\lambda$ and $d\operatorname{Im}\lambda$ are linearly independent on $\Sigma_\mu(P)$ for almost all μ close to λ_0, and then $\Sigma_\mu(P)$ is a C^∞ manifold of codimension 2. By using Lemma 3.1 of [Dencker et al. 2004] we obtain that $\{\operatorname{Re}\lambda, \operatorname{Im}\lambda\}$ changes sign on $\Sigma_\mu(P)$ for almost all values μ near λ_0, so we find that those values also are in $\Lambda_-(P)$. By taking the closure we obtain (3.5).

Next, assume that $\lambda \in \Lambda_-(P)$, it is no restriction to assume $\lambda = 0$. By the assumptions there exists $w_0 \in \Sigma_0(P)$ and $\lambda(w) \in C^\infty$ such that $\lambda(w_0) = 0$, $\{\operatorname{Re}\lambda, \operatorname{Im}\lambda\} < 0$ at w_0, $(w_0, 0) \notin \Xi(P)$, and $P(w) - \lambda \operatorname{Id}_N$ is of principal type on $\Sigma_\lambda(P)$ near w_0 when $|\lambda| \ll 1$. Then Proposition 3.5 gives that $\kappa_P \equiv K_P$ is constant on $\Omega_1(P)$ near (w_0, λ_0), so

$$\operatorname{Dim}\operatorname{Ker}(P(w) - \lambda(w)\operatorname{Id}_N) \equiv K > 0$$

in a neighborhood of w_0. Since the dimension is constant we can construct a base $\{u_1(w), \dots, u_K(w)\} \in C^\infty$ for $\operatorname{Ker}(P(w) - \lambda(w)\operatorname{Id}_N)$ in a neighborhood of w_0. By orthonormalizing it and extending to $\mathbb{C}^N$ we obtain orthogonal $E(w) \in C^\infty$ so that

$$E^*(w)P(w)E(w) = \begin{pmatrix} \lambda(w)\operatorname{Id}_K & P_{12} \\ 0 & P_{22} \end{pmatrix} = P_0(w). \tag{3.9}$$

If $P(w)$ is analytic in a tubular neighborhood of $T^*\mathbb{R}^n$ then $E(w)$ can be chosen analytic in that neighborhood. Since P_0 is of principal type at w_0 by Remark 3.2 and $\partial P_0(w_0)$ maps $\operatorname{Ker} P_0(w_0)$ into itself, we find that $\operatorname{Ran} P_0(w_0) \cap \operatorname{Ker} P_0(w_0) = \{0\}$ and thus $|P_{22}(w_0)| \neq 0$. In fact, if there exists $u'' \neq 0$ such that $P_{22}(w_0)u'' = 0$, then by applying $P(w_0)$ on $u = (0, u'') \notin \operatorname{Ker} P_0(w_0)$ we obtain

$$0 \neq P_0(w_0)u = (P_{12}(w_0)u'', 0) \in \operatorname{Ker} P_0(w_0) \cap \operatorname{Ran} P_0(w_0)$$

which gives a contradiction. Clearly, the norm of the resolvent $P(h)^{-1}$ only changes with a multiplicative constant under left and right multiplication of $P(h)$ by invertible systems. Now $E^w(x, hD)$ and $(E^*)^w(x, hD)$ are invertible in L^2 for small enough h, and

$$(E^*)^w P(h) E^w = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$$

where $\sigma(P_{11}) = \lambda\operatorname{Id}_N$, $P_{21} = \mathbb{O}(h)$ and $P_{22}(h)$ is invertible for small h. By multiplying from the right by

$$\begin{pmatrix} \operatorname{Id}_K & 0 \\ -P_{22}(h)^{-1}P_{21}(h) & \operatorname{Id}_{N-K} \end{pmatrix}$$

for small h, we obtain that $P_{21}(h) \equiv 0$ and this only changes lower order terms in $P_{11}(h)$. Then by multiplying from the left by

$$\begin{pmatrix} \mathrm{Id}_K & -P_{12}(h)P_{22}(h)^{-1} \\ 0 & \mathrm{Id}_{N-K} \end{pmatrix}$$

we obtain that $P_{12}(h) \equiv 0$ without changing $P_{11}(h)$ or $P_{22}(h)$.

Thus, in order to prove (3.6) we may assume $N = K$ and $P(w) = \lambda(w)\,\mathrm{Id}_K$. By conjugating similarly as in the scalar case (see the proof of Proposition 26.3.1 in Volume IV of [Hörmander 1983–1985]), we can reduce to the case when $P(h) = \lambda^w(x, hD)\,\mathrm{Id}_K$. In fact, let

$$P(h) = \lambda^w(x, hD)\,\mathrm{Id}_K + \sum_{j\ge 1} h^j P_j^w(x, hD) \tag{3.10}$$

$A(h) = \sum_{j\ge 0} h^j A_j^w(x, hD)$ and $B(h) = \sum_{j\ge 0} h^j B_j^w(x, hD)$ with $B_0(w) \equiv A_0(w)$. Then the calculus gives

$$P(h)A(h) - B(h)\lambda^w(x, hD) = \sum_{j\ge 1} h^j E_j^w(x, hD)$$

with

$$E_k = \frac{1}{2i} H_\lambda(A_{k-1} + B_{k-1}) + P_1 A_{k-1} + \lambda(A_k - B_k) + R_k \qquad k \ge 1.$$

Here H_λ is the Hamilton vector field of λ, R_k only depends on A_j and B_j for $j < k-1$ and $R_1 \equiv 0$. Now we can choose A_0 so that $A_0 = \mathrm{Id}_K$ on $V_0 = \{w : \operatorname{Im}\lambda(w) = 0\}$ and $\frac{1}{i} H_\lambda A_0 + P_1 A_0$ vanishes of infinite order on V_0 near w_0. In fact, since $\{\operatorname{Re}\lambda, \operatorname{Im}\lambda\} \ne 0$ we find $d\operatorname{Im}\lambda \ne 0$ on V_0, and V_0 is noncharacteristic for $H_{\operatorname{Re}\lambda}$. Thus, the equation determines all derivatives of A_0 on V_0, and we may use the Borel Theorem to obtain a solution. Then, by taking

$$B_1 - A_1 = \Big(\frac{1}{i} H_\lambda A_0 + P_1 A_0\Big)\lambda^{-1} \in C^\infty$$

we obtain $E_0 \equiv 0$. Lower order terms are eliminated similarly, by making

$$\frac{1}{2i} H_\lambda(A_{j-1} + B_{j-1}) + P_1 A_{j-1} + R_j$$

vanish of infinite order on V_0. Observe that only the difference $A_{j-1} - B_{j-1}$ is determined in the previous step. Thus we can reduce to the case $P = \lambda^w(x, hD)\,\mathrm{Id}$ and then the C^∞ result follows from the scalar case (see Theorem 1.2 in [Dencker et al. 2004]) by using Remark 2.25 and Example 2.26.

The analytic case follows as in the proof of Theorem 1.2$'$ in [Dencker et al. 2004] by applying a holomorphic WKB construction to $P = P_{11}$ on the form

$$u(z, h) \sim e^{i\phi(z)/h} \sum_{j=0}^{\infty} A_j(z) h^j \qquad z = x + iy \in \mathbb{C}^n$$

which will be an approximate solution to $P(h)u(z, h) = 0$. Here $P(h)$ is given by (2.2) with $P_0(w) = \lambda(w)\,\mathrm{Id}$, P_j satisfying (2.3) and $P_j^w(z, hD_z)$ given by the formula (2.1) where the integration may be deformed to a suitable chosen contour instead of $T^*\mathbb{R}^n$ (see [Sjöstrand 1982, Section 4]). The holomorphic phase function $\phi(z)$ satisfying $\lambda(z, d_z\phi) = 0$ is constructed as in [Dencker et al. 2004] so that

$d_z\phi(x_0) = \xi_0$ and $\operatorname{Im}\phi(x) \geq c|x - x_0|^2$, $c > 0$, and $w_0 = (x_0, \xi_0)$. The holomorphic amplitude $A_0(z)$ satisfies the transport equation

$$\sum_j \partial_{\zeta_j}\lambda(z, d_z\phi(z))D_{z_j}A_0(z) + P_1(z, d_z\phi(z))A_0(z) = 0$$

with $A_0(x_0) \neq 0$. The lower order terms in the expansion solve

$$\sum_j \partial_{\zeta_j}\lambda(z, d_z\phi(z))D_{z_j}A_k(z) + P_1(z, d_z\phi(z))A_k(z) = S_k(z)$$

where $S_k(z)$ only depends on A_j and P_{j+1} for $j < k$. As in the scalar case, we find from (2.3) that the solutions satisfy $\|A_k(z)\| \leq C_0C^kk^k$ see Theorem 9.3 in [Sjöstrand 1982]. By solving up to $k < c/h$, cutting of near x_0 and restricting to $\mathbb{R}^n$ we obtain that $P(h)u = \mathbb{O}(e^{-c/h})$. The details are left to the reader; see the proof of Theorem 1.2′ in [Dencker et al. 2004].

For the last result, we observe that $\{\operatorname{Re}\bar\lambda, \operatorname{Im}\bar\lambda\} = -\{\operatorname{Re}\lambda, \operatorname{Im}\lambda\}$, $\lambda \in \Sigma(P) \Leftrightarrow \bar\lambda \in \Sigma(P^*)$, P^* is of principal type if and only if P is, and Remark 2.6 gives $(w, \lambda) \in \Xi(P) \Leftrightarrow (w, \bar\lambda) \in \Xi(P^*)$. Thus, $\lambda \in \Lambda_+(P)$ if and only if $\bar\lambda \in \Lambda_-(P^*)$ and

$$\|(P(h) - \lambda \operatorname{Id}_N)^{-1}\| = \|(P^*(h) - \bar\lambda \operatorname{Id}_N)^{-1}\|.$$

From the definition, we find that any $\lambda_0 \in \Lambda(P)$ is an accumulation point of $\Lambda_\pm(P)$, so we obtain the result from (3.6). □

Remark 3.13. In order to get the estimate (3.6) it suffices that there exists a semibicharacteristic Γ of $\lambda - \lambda_0$ through w_0 such that $\Gamma \times \{\lambda_0\} \cap \Xi(P) = \emptyset$, $P(w) - \lambda \operatorname{Id}_N$ is of principal type near Γ for λ near λ_0 and that condition $(\overline{\Psi})$ is not satisfied on Γ; see [Hörmander 1983–1985, Definition 26.4.6, Volume IV]. This means that there exists $0 \neq q \in C^\infty$ such that Γ is a bicharacteristic of $\operatorname{Re} q(\lambda - \lambda_0)$ through w_0 and $\operatorname{Im} q(\lambda - \lambda_0)$ changes sign from $+$ to $-$ when going in the positive direction on Γ.

In fact, once we have reduced to the normal form (3.10), the construction of approximate local solutions in the proof of [Hörmander 1983–1985, Theorem 26.4.7, Volume IV] can be adapted to this case, since the principal part is scalar. See also Theorem 1.3 in [Pravda-Starov 2006b, Section 3.2] for a similar scalar semiclassical estimate.

When $P(w)$ is not of principal type, the reduction in the proof of Theorem 3.10 may not be possible since P_{22} in (3.9) needs not be invertible by the following example.

Example 3.14. Let

$$P(h) = \begin{pmatrix} \lambda^w(x, hD) & 1 \\ h & \lambda^w(x, hD) \end{pmatrix}$$

where $\lambda \in C^\infty$ satisfies the bracket condition (3.4). The principal symbol is

$$P(w) = \begin{pmatrix} \lambda(w) & 1 \\ 0 & \lambda(w) \end{pmatrix}$$

with eigenvalue $\lambda(w)$ and we have

$$\operatorname{Ker}(P(w) - \lambda(w)\operatorname{Id}_2) = \operatorname{Ran}(P(w) - \lambda(w)\operatorname{Id}_2) = \{(z, 0) : z \in \mathbb{C}\} \qquad \forall\, w.$$

We find that P is *not* of principal type since $dP = d\lambda\,\mathrm{Id}_2$. Observe that $\Xi(P) = \emptyset$ since K_P is constant on $\Omega_1(P)$.

When the dimension is equal to one, we have to add some conditions in order to get the inclusion (3.5).

Lemma 3.15. *Let $P(w) \in C^\infty(T^*\mathbb{R})$ be an $N \times N$ system. Then for every component Ω of $\mathbb{C} \setminus (\Sigma_{\mathrm{ws}}(P) \cup \Sigma_\infty(P))$ which has nonempty intersection with $\complement\Sigma(P)$ we find that*

$$\Omega \subseteq \overline{\Lambda_-(P)}. \tag{3.11}$$

The condition of having nonempty intersection with the complement is necessary even in the scalar case; see the remark and Lemma 3.2' on page 394 in [Dencker et al. 2004].

Proof. If $\mu \notin \Sigma_\infty(P)$ we find that the index

$$i = \operatorname{var}\arg_\gamma |P(w) - \mu\,\mathrm{Id}_N| \tag{3.12}$$

is well-defined and continuous when γ is a positively oriented circle $\{w : |w| = R\}$ for $R \gg 1$. If $\lambda_0 \notin \Sigma_{\mathrm{ws}}(P) \cup \Sigma_\infty(P)$ then we find from Lemma 2.15 that the characteristic polynomial is equal to

$$|P(w) - \mu\,\mathrm{Id}_N| = (\lambda(w) - \mu)^k e(w, \mu)$$

near $w_0 \in \Sigma_{\lambda_0}(P)$, here $\lambda, e \in C^\infty$, $e \neq 0$ and $k = K_P(w_0)$. By Remark 2.16 we find for almost all μ close to λ_0 that $d\operatorname{Re}\lambda \wedge d\operatorname{Im}\lambda \neq 0$ on $\lambda^{-1}(\mu) = \Sigma_\mu(P)$, which is then a finite set of points on which the Poisson bracket is nonvanishing. If $\mu \notin \Sigma(P)$ we find that the index (3.12) vanishes, since one can then let $R \to 0$. Thus, if a component Ω of $\mathbb{C} \setminus (\Sigma_{\mathrm{ws}}(P) \cup \Sigma_\infty(P))$ has nonempty intersection with $\complement\Sigma(P)$, we obtain that $i = 0$ in Ω. When $\mu_0 \in \Omega \cap \Lambda(P)$ we find from the definition that the Poisson bracket $\{\operatorname{Re}\lambda, \operatorname{Im}\lambda\}$ cannot vanish identically on $\Sigma_\mu(P)$ for all μ close to μ_0. Since the index is equal to the sum of positive multiples of the values of the Poisson brackets at $\Sigma_\mu(P)$, we find that the bracket must be negative at some point $w_0 \in \Sigma_\mu(P)$, for almost all μ near λ_0, which gives (3.11). □

4. The quasisymmetrizable case

First we note that if the system $P(w) - z\,\mathrm{Id}_N$ is of principal type near $\Sigma_z(P)$ for z close to $\lambda \in \partial\Sigma(P) \setminus (\Sigma_{\mathrm{ws}}(P) \cup \Sigma_\infty(P))$ and $\Sigma_\lambda(P)$ has no closed semibicharacteristics, then one can generalize Theorem 1.3 in [Dencker et al. 2004] to obtain

$$\|(P(h) - \lambda\,\mathrm{Id}_N)^{-1}\| \leq C/h \qquad h \to 0. \tag{4.1}$$

In fact, by using the reduction in the proof of Theorem 3.10 this follows from the scalar case; see Example 4.12. But then the eigenvalues close to λ have constant multiplicity.

Generically, we have that the eigenvalues of the principal symbol P have constant multiplicity almost everywhere since $\Xi(P)$ is nowhere dense. But at the boundary $\partial\Sigma(P)$ this needs not be the case. For example, if

$$P(t, \tau) = \tau\,\mathrm{Id} + iK(t)$$

where $C^\infty \ni K \geq 0$ is unbounded and $0 \in \Sigma_{\mathrm{ss}}(K)$, then $\mathbb{R} = \partial\Sigma(P) \subseteq \Sigma_{\mathrm{ss}}(P)$.

When the multiplicity of the eigenvalues of the principal symbol is not constant the situation is more complicated. The following example shows that then it is not sufficient to have conditions only on the eigenvalues in order to obtain the estimate (4.1), not even in the principal type case.

Example 4.1. Let $a_1(t)$, $a_2(t) \in C^\infty(\mathbb{R})$ be real valued, $a_2(0)=0$, $a_2'(0)>0$ and let

$$P^w(t,hD_t)=\begin{pmatrix} hD_t+a_1(t) & a_2(t)-ia_1(t) \\ a_2(t)+ia_1(t) & -hD_t+a_1(t)\end{pmatrix}=P^w(t,hD_t)^*.$$

Then the eigenvalues of $P(t,\tau)$ are

$$\lambda=a_1(t)\pm\sqrt{\tau^2+a_1^2(t)+a_2^2(t)}\in\mathbb{R}$$

which coincide if and only if $\tau=a_1(t)=a_2(t)=0$. We have that

$$\frac{1}{2}\begin{pmatrix}1 & i\\ 1 & -i\end{pmatrix}P\begin{pmatrix}1 & 1\\ i & -i\end{pmatrix}=\begin{pmatrix}hD_t+ia_2(t) & 0\\ 2a_1(t) & hD_t-ia_2(t)\end{pmatrix}=\widetilde{P}(h).$$

Thus we can construct $u_h(t)={}^t(0,u_2(t))$ so that $\|u_h\|=1$ and $\widetilde{P}(h)u_h=\mathbb{O}(h^N)$ for $h\to 0$; see Theorem 1.2 in [Dencker et al. 2004]. When a_2 is analytic we may obtain that $\widetilde{P}(h)u_h=\mathbb{O}(\exp(-c/h))$ by Theorem 1.2′ in [Dencker et al. 2004]. By the invariance, we see that P is of principal type at $t=\tau=0$ if and only if $a_1(0)=0$. If $a_1(0)=0$ then $\Sigma_{\mathrm{ws}}(P)=\{0\}$ and when $a_1\neq 0$ we have that P^w is a self-adjoint diagonalizable system. In the case $a_1(t)\equiv 0$ and $a_2(t)\equiv t$ the eigenvalues of $P(t,hD_t)$ are $\pm\sqrt{2nh}$, $n\in\mathbb{N}$; see the proof of Proposition 3.6.1 in [Helffer and Sjöstrand 1990].

Of course, the problem is that the eigenvalues are not invariant under multiplication with elliptic systems. To obtain the estimate (4.1) for operators that are *not* of principal type, it is not even sufficient that the eigenvalues are real having constant multiplicity.

Example 4.2. Let $a(t)\in C^\infty(\mathbb{R})$ be real valued, $a(0)=0$, $a'(0)>0$ and

$$P^w(t,hD_t)=\begin{pmatrix} hD_t & a(t)\\ -ha(t) & hD_t\end{pmatrix}.$$

Then the principal symbol is

$$P(t,\tau)=\begin{pmatrix}\tau & a(t)\\ 0 & \tau\end{pmatrix}$$

so the only eigenvalue is τ. Thus $\Xi(P)=\varnothing$ but the principal symbol is not diagonalizable, and when $a(t)\neq 0$ the system is not of principal type. We have

$$\begin{pmatrix}h^{1/2} & 0\\ 0 & -1\end{pmatrix}P\begin{pmatrix}h^{-1/2} & 0\\ 0 & 1\end{pmatrix}=\sqrt{h}\begin{pmatrix}\sqrt{h}D_t & a(t)\\ a(t) & -\sqrt{h}D_t\end{pmatrix}$$

thus we obtain that $\|P^w(t,hD_t)^{-1}\|\geq C_N h^{-N}$ for all N, when $h\to 0$ by using Example 4.1 with $a_1\equiv 0$ and $a_2\equiv a$. When a is analytic we obtain $\|P(t,hD_t)^{-1}\|\geq \exp(c/\sqrt{h})$.

For nonprincipal type operators, to obtain the estimate (4.1) it is not even sufficient that the principal symbol has real eigenvalues of multiplicity one.

Example 4.3. Let $a(t) \in C^\infty(\mathbb{R})$, $a(0) = 0$, $a'(0) > 0$ and

$$P(h) = \begin{pmatrix} 1 & hD_t \\ h & iha(t) \end{pmatrix}$$

with principal symbol

$$\begin{pmatrix} 1 & \tau \\ 0 & 0 \end{pmatrix}$$

thus the eigenvalues are 0 and 1, so $\Xi(P) = \emptyset$. Since

$$\begin{pmatrix} 1 & 0 \\ -h & 1 \end{pmatrix} P(h) \begin{pmatrix} 1 & -hD_t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -h \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & hD_t - ia(t) \end{pmatrix}$$

we obtain as in Example 4.1 that $\|P(h)^{-1}\| \geq C_N h^{-N}$ when $h \to 0$ for all N, and for analytic $a(t)$ we obtain $\|P(h)^{-1}\| \geq Ce^{c/h}$, $h \to 0$. Now $\partial_\tau P$ maps $\operatorname{Ker} P(0)$ into $\operatorname{Ran} P(0)$ so the system is not of principal type. Observe that this property is not preserved under the multiplications above, since the systems are not elliptic.

Instead of using properties of the eigenvalues of the principal symbol, we shall use properties that are invariant under multiplication with invertible systems. First we consider the scalar case, recall that a scalar $p \in C^\infty$ is of *principal type* if $dp \neq 0$ when $p = 0$. We have the following normal form for scalar principal type operators near the boundary $\partial\Sigma(P)$. Recall that a *semibicharacteristic* of p is a nontrivial bicharacteristic of $\operatorname{Re} qp$, for $q \neq 0$.

Example 4.4. Assume that $p(x, \xi) \in C^\infty(T^*\mathbb{R}^n)$ is of principal type and $0 \in \partial\Sigma(p) \setminus \Sigma_\infty(p)$. Then we find from the proof of Lemma 4.1 in [Dencker et al. 2004] that there exists $0 \neq q \in C^\infty$ so that

$$\operatorname{Im} qp \geq 0 \quad \text{and} \quad d \operatorname{Re} qp \neq 0$$

in a neighborhood of $w_0 \in \Sigma_0(p)$. In fact, condition (1.7) in that lemma is not needed to obtain a local preparation. By making a symplectic change of variables and using the Malgrange preparation theorem we then find that

$$p(x, \xi) = e(x, \xi)(\xi_1 + if(x, \xi')) \qquad \xi = (\xi_1, \xi') \tag{4.2}$$

in a neighborhood of $w_0 \in \Sigma_0(p)$, where $e \neq 0$ and $f \geq 0$. If there are no closed semibicharacteristics of p then we obtain this in a neighborhood of $\Sigma_0(p)$ by a partition of unity.

This normal form in the scalar case motivates the following definition.

Definition 4.5. We say that the $N \times N$ system $P(w) \in C^\infty(T^*\mathbb{R}^n)$ is *quasisymmetrizable* with respect to the real C^∞ vector field V in $\Omega \subseteq T^*\mathbb{R}^n$ if $\exists$ $N \times N$ system $M(w) \in C^\infty(T^*\mathbb{R}^n)$ so that in Ω we have

$$\operatorname{Re}\langle M(VP)u, u\rangle \geq c\|u\|^2 - C\|Pu\|^2 \qquad c > 0 \tag{4.3}$$

$$\operatorname{Im}\langle MPu, u\rangle \geq -C\|Pu\|^2 \tag{4.4}$$

for any $u \in \mathbb{C}^N$, the system M is called a *symmetrizer* for P.

The definition is clearly independent of the choice of coordinates in $T^*\mathbb{R}^n$ and choice of base in $\mathbb{C}^N$. When P is elliptic, we may take $M = iP^*$ as multiplier; then P is quasisymmetrizable with respect to any vector field because $\|Pu\| \cong \|u\|$. Observe that for a *fixed* vector field V the set of multipliers M satisfying (4.3)–(4.4) is a convex cone, a sum of two multipliers is also a multiplier. Thus, given a vector field V it suffices to make a local choice of multiplier M and then use a partition of unity to get a global one.

Taylor has studied *symmetrizable* systems of the type $D_t \operatorname{Id} + iK$, for which there exists $R > 0$ making RK symmetric (see Definition 4.3.2 in [Taylor 1981]). These systems are quasisymmetrizable with respect to ∂_τ with symmetrizer R. We see from Example 4.4 that the scalar symbol p of principal type is quasisymmetrizable in neighborhood of any point at $\partial\Sigma(p) \setminus \Sigma_\infty(p)$.

The invariance properties of quasisymmetrizable systems are partly due to the following simple and probably well-known result on semibounded matrices. In the following, we shall denote $\operatorname{Re} A = \frac{1}{2}(A + A^*)$ and $i \operatorname{Im} A = \frac{1}{2}(A - A^*)$ the symmetric and antisymmetric parts of the matrix A. Also, if U and V are linear subspaces of $\mathbb{C}^N$, then we let $U + V = \{u + v : u \in U \ \wedge \ v \in V\}$.

Lemma 4.6. *Assume that Q is an $N \times N$ matrix such that $\operatorname{Im} zQ \ge 0$ for some $0 \ne z \in \mathbb{C}$. Then we find*

$$\operatorname{Ker} Q = \operatorname{Ker} Q^* = \operatorname{Ker}(\operatorname{Re} Q) \cap \operatorname{Ker}(\operatorname{Im} Q) \tag{4.5}$$

and $\operatorname{Ran} Q = \operatorname{Ran}(\operatorname{Re} Q) + \operatorname{Ran}(\operatorname{Im} Q)$ *is orthogonal to* $\operatorname{Ker} Q$.

Proof. By multiplying with z we may assume that $\operatorname{Im} Q \ge 0$, clearly the conclusions are invariant under multiplication with complex numbers. If $u \in \operatorname{Ker} Q$, then we have $\langle \operatorname{Im} Qu, u\rangle = \operatorname{Im}\langle Qu, u\rangle = 0$. By using the Cauchy–Schwarz inequality on $\operatorname{Im} Q \ge 0$ we find that $\langle \operatorname{Im} Qu, v\rangle = 0$ for any v. Thus $u \in \operatorname{Ker}(\operatorname{Im} Q)$ so $\operatorname{Ker} Q \subseteq \operatorname{Ker} Q^*$. We get equality and (4.5) by the rank theorem, since $\operatorname{Ker} Q^* = \operatorname{Ran} Q^\perp$.

For the last statement we observe that $\operatorname{Ran} Q \subseteq \operatorname{Ran}(\operatorname{Re} Q) + \operatorname{Ran}(\operatorname{Im} Q) = (\operatorname{Ker} Q)^\perp$ by (4.5) where we also get equality by the rank theorem. □

Proposition 4.7. *Assume that $P(w) \in C^\infty(T^*\mathbb{R}^n)$ is a quasisymmetrizable system; then we find that P is of principal type. Also, the symmetrizer M is invertible if $\operatorname{Im} MP \ge cP^*P$ for some $c > 0$.*

Observe that by adding $i\varrho P^*$ to M we may assume that $Q = MP$ satisfies

$$\operatorname{Im} Q \ge (\varrho - C)P^*P \ge P^*P \ge cQ^*Q \qquad c > 0 \tag{4.6}$$

for $\varrho \ge C + 1$, and then the symmetrizer is invertible by Proposition 4.7.

Proof. Assume that (4.3)–(4.4) hold at w_0, $\operatorname{Ker} P(w_0) \ne \{0\}$ but (3.1) is not a bijection. Then there exists $0 \ne u \in \operatorname{Ker} P(w_0)$ and $v \in \mathbb{C}^N$ such that $VP(w_0)u = P(w_0)v$, so (4.3) gives

$$\operatorname{Re}\langle MP(w_0)v, u\rangle = \operatorname{Re}\langle MVP(w_0)u, u\rangle \ge c\|u\|^2 > 0.$$

This means that

$$\operatorname{Ran} MP(w_0) \not\subseteq \operatorname{Ker} P(w_0)^\perp. \tag{4.7}$$

Let $M_\varrho = M + i\varrho P^*$ then we have that

$$\operatorname{Im}\langle M_\varrho Pu, u\rangle \ge (\varrho - C)\|Pu\|^2$$

so for large enough ϱ we have $\operatorname{Im} M_\varrho P \geq 0$. By Lemma 4.6 we find

$$\operatorname{Ran} M_\varrho P \perp \operatorname{Ker} M_\varrho P.$$

Since $\operatorname{Ker} P \subseteq \operatorname{Ker} M_\varrho P$ and $\operatorname{Ran} P^* P \subseteq \operatorname{Ran} P^* \perp \operatorname{Ker} P$ we find that $\operatorname{Ran} MP \perp \operatorname{Ker} P$ for any ϱ. This gives a contradiction to (4.7), thus P is of principal type.

Next, we shall show that M is invertible at w_0 if $\operatorname{Im} MP \geq cP^*P$ at w_0 for some $c > 0$. Then we find as before from Lemma 4.6 that $\operatorname{Ran} MP(w_0) \perp \operatorname{Ker} MP(w_0)$. By choosing a base for $\operatorname{Ker} P(w_0)$ and completing it to a base of $\mathbb{C}^N$ we may assume that

$$P(w_0) = \begin{pmatrix} 0 & P_{12}(w_0) \\ 0 & P_{22}(w_0) \end{pmatrix}$$

where P_{22} is $(N-K) \times (N-K)$ system, $K = \operatorname{Dim} \operatorname{Ker} P(w_0)$. Now, by multiplying P from the left with an orthogonal matrix E we may assume that $P_{12}(w_0) = 0$. In fact, this only amounts to choosing an orthonormal base for $\operatorname{Ran} P(w_0)^\perp$ and completing to an orthonormal base for $\mathbb{C}^N$. Observe that MP is unchanged if we replace M with ME^{-1}, which is invertible if and only if M is. Since $\operatorname{Dim} \operatorname{Ker} P(w_0) = K$ we obtain $|P_{22}(w_0)| \neq 0$. Letting

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix},$$

we find

$$MP = \begin{pmatrix} 0 & 0 \\ 0 & M_{22}P_{22} \end{pmatrix} \qquad \text{at } w_0.$$

In fact, $(MP)_{12}(w_0) = M_{12}(w_0)P_{22}(w_0) = 0$ since $\operatorname{Ran} MP(w_0) = \operatorname{Ker} MP(w_0)^\perp$. We obtain that $M_{12}(w_0) = 0$, and by assumption

$$\operatorname{Im} M_{22}P_{22} \geq cP_{22}^*P_{22} \qquad \text{at } w_0,$$

which gives $|M_{22}(w_0)| \neq 0$. Since P_{11}, P_{21} and M_{12} vanish at w_0 we find

$$\operatorname{Re} V(MP)_{11}(w_0) = \operatorname{Re} M_{11}(w_0)VP_{11}(w_0) > c$$

which gives $|M_{11}(w_0)| \neq 0$. Since $M_{12}(w_0) = 0$ and $|M_{22}(w_0)| \neq 0$ we obtain that $M(w_0)$ is invertible. □

Remark 4.8. The $N \times N$ system $P \in C^\infty(T^*\mathbb{R}^n)$ is quasisymmetrizable with respect to V if and only if there exists an invertible symmetrizer M such that $Q = MP$ satisfies

$$\operatorname{Re}\langle (VQ)u, u\rangle \geq c\|u\|^2 - C\|Qu\|^2 \qquad c > 0 \tag{4.8}$$

$$\operatorname{Im}\langle Qu, u\rangle \geq 0 \tag{4.9}$$

for any $u \in \mathbb{C}^N$.

In fact, by the Cauchy–Schwarz inequality we find

$$|\langle (VM)Pu, u\rangle| \leq \varepsilon\|u\|^2 + C_\varepsilon\|Pu\|^2 \qquad \forall \varepsilon > 0 \quad \forall u \in \mathbb{C}^N.$$

Since M is invertible, we also have that $\|Pu\| \cong \|Qu\|$.

Definition 4.9. If the $N \times N$ system $Q \in C^\infty(T^*\mathbb{R}^n)$ satisfies (4.8)–(4.9) then Q is *quasisymmetric* with respect to the real C^∞ vector field V.

Proposition 4.10. *Let $P(w) \in C^\infty(T^*\mathbb{R}^n)$ be an $N \times N$ quasisymmetrizable system; then P^* is quasisymmetrizable. If $A(w)$ and $B(w) \in C^\infty(T^*\mathbb{R}^n)$ are invertible $N \times N$ systems then BPA is quasisymmetrizable.*

Proof. Clearly (4.8)–(4.9) are invariant under *left* multiplication of P with invertible systems E, just replace M with ME^{-1}. Since we may write $BPA = B(A^*)^{-1}A^*PA$ it suffices to show that E^*PE is quasisymmetrizable if E is invertible. By Remark 4.8 there exists a symmetrizer M so that $Q = MP$ is quasisymmetric, that is, satisfies (4.8)–(4.9). It then follows from Proposition 4.11 that

$$Q_E = E^*QE = E^*M(E^*)^{-1}E^*PE$$

also satisfies (4.8) and (4.9), so E^*PE is quasisymmetrizable.

Finally, we shall prove that P^* is quasisymmetrizable if P is. Since $Q = MP$ is quasisymmetric, we find from Proposition 4.11 that $Q^* = P^*M^*$ is quasisymmetric. By multiplying with $(M^*)^{-1}$ from the right, we find from the first part of the proof that P^* is quasisymmetrizable. □

Proposition 4.11. *If $Q \in C^\infty(T^*\mathbb{R}^n)$ is quasisymmetric, then Q^* is quasisymmetric. If $E \in C^\infty(T^*\mathbb{R}^n)$ is invertible, then E^*QE are quasisymmetric.*

Proof. First we note that (4.8) holds if and only if

$$\mathrm{Re}\langle (VQ)u, u\rangle \ge c\|u\|^2 \qquad \forall\, u \in \mathrm{Ker}\, Q \tag{4.10}$$

for some $c > 0$. In fact, Q^*Q has a positive lower bound on the orthogonal complement $\mathrm{Ker}\, Q^\perp$ so that

$$\|u\| \le C\|Qu\| \qquad \text{for } u \in \mathrm{Ker}\, Q^\perp.$$

Thus, if $u = u' + u''$ with $u' \in \mathrm{Ker}\, Q$ and $u'' \in \mathrm{Ker}\, Q^\perp$ we find that $Qu = Qu''$,

$$\mathrm{Re}\langle (VQ)u', u''\rangle \ge -\varepsilon\|u'\|^2 - C_\varepsilon\|u''\|^2 \ge -\varepsilon\|u'\|^2 - C'_\varepsilon\|Qu\|^2 \qquad \forall\, \varepsilon > 0$$

and $\mathrm{Re}\langle (VQ)u'', u''\rangle \ge -C\|u''\|^2 \ge -C'\|Qu\|^2$. By choosing ε small enough we obtain (4.8) by using (4.10) on u'.

Next, we note that $\mathrm{Im}\, Q^* = -\mathrm{Im}\, Q$ and $\mathrm{Re}\, Q^* = \mathrm{Re}\, Q$, so $-Q^*$ satisfies (4.9) and (4.10) with V replaced by $-V$, and thus it is quasisymmetric. Finally, we shall show that $Q_E = E^*QE$ is quasisymmetric when E is invertible. We obtain from (4.9) that

$$\mathrm{Im}\langle Q_E u, u\rangle = \mathrm{Im}\langle QEu, Eu\rangle \ge 0 \qquad \forall\, u \in \mathbb{C}^N.$$

Next, we shall show that Q_E satisfies (4.10) on $\mathrm{Ker}\, Q_E = E^{-1}\,\mathrm{Ker}\, Q$, which will give (4.8). We find from Leibniz' rule that $VQ_E = (VE^*)QE + E^*(VQ)E + E^*Q(VE)$ where (4.10) gives

$$\mathrm{Re}\langle E^*(VQ)Eu, u\rangle \ge c\|Eu\|^2 \ge c'\|u\|^2 \qquad u \in \mathrm{Ker}\, Q_E$$

since then $Eu \in \mathrm{Ker}\, Q$. Similarly we obtain that $\langle (VE^*)QEu, u\rangle = 0$ when $u \in \mathrm{Ker}\, Q_E$. Now since $\mathrm{Im}\, Q_E \ge 0$ we find from Lemma 4.6 that

$$\mathrm{Ker}\, Q_E^* = \mathrm{Ker}\, Q_E$$

which gives $\langle E^* Q(VE)u, u\rangle = \langle E^{-1}(VE)u, Q_E^* u\rangle = 0$ when $u \in \operatorname{Ker} Q_E = \operatorname{Ker} Q_E^*$. Thus Q_E satisfies (4.10) so it is quasisymmetric. □

Example 4.12. Assume that $P(w) \in C^\infty$ is an $N \times N$ system such that $z \in \Sigma(P) \setminus (\Sigma_{\mathrm{ws}}(P) \cap \Sigma_\infty(P))$ and that $P(w) - \lambda \operatorname{Id}_N$ is of principal type when $|\lambda - z| \ll 1$. By Lemma 2.15 and Proposition 3.5 there exists a C^∞ germ of eigenvalues $\lambda(w) \in C^\infty$ for P so that $\operatorname{Dim}\operatorname{Ker}(P(w) - \lambda(w)\operatorname{Id}_N)$ is constant near $\Sigma_z(P)$. By using the spectral projection as in the proof of Proposition 3.5 and making a base change $B(w) \in C^\infty$ we obtain

$$P(w) = B^{-1}(w)\begin{pmatrix} \lambda(w)\operatorname{Id}_K & 0 \\ 0 & P_{22}(w)\end{pmatrix} B(w) \tag{4.11}$$

in a neighborhood of $\Sigma_z(P)$, here $|P_{22} - \lambda(w)\operatorname{Id}| \neq 0$. We find from Proposition 3.5 that $d\lambda \neq 0$ when $\lambda = z$, so $\lambda - z$ is of principal type. Proposition 4.10 gives that $P - z\operatorname{Id}_N$ is quasisymmetrizable near any $w_0 \in \Sigma_z(P)$ if $z \in \partial\Sigma(\lambda)$. In fact, by Example 4.4 there exists $q(w) \in C^\infty$ so that

$$|d \operatorname{Re} q(\lambda - z)| \neq 0 \tag{4.12}$$

$$\operatorname{Im} q(\lambda - z) \geq 0 \tag{4.13}$$

and we get the normal form (4.2) for λ near $\Sigma_z(P) = \{\lambda(w) = z\}$. One can then take V normal to $\Sigma = \{\operatorname{Re} q(\lambda - z) = 0\}$ at $\Sigma_z(P)$ and use

$$M = B^*\begin{pmatrix} q\operatorname{Id}_K & 0 \\ 0 & M_{22}\end{pmatrix} B$$

with $M_{22}(w) = (P_{22}(w) - z\operatorname{Id})^{-1}$ for example. Then

$$Q = M(P - z\operatorname{Id}_N) = B^*\begin{pmatrix} q(\lambda - z)\operatorname{Id}_K & 0 \\ 0 & \operatorname{Id}_{N-K}\end{pmatrix} B. \tag{4.14}$$

If there are no closed semibicharacteristics of $\lambda - z$ then we also find from Example 4.4 that $P - z\operatorname{Id}_N$ is quasisymmetrizable in a neighborhood of $\Sigma_z(P)$; see the proof of Lemma 4.1 in [Dencker et al. 2004].

Example 4.13. Let

$$P(x, \xi) = |\xi|^2 \operatorname{Id}_N + iK(x)$$

where $0 \leq K(x) \in C^\infty$. When $z > 0$ we find that $P - z\operatorname{Id}_N$ is quasisymmetric in a neighborhood of $\Sigma_z(P)$ with respect to the exterior normal $\langle \xi, \partial_\xi\rangle$ to $\Sigma_z(P) = \{|\xi|^2 = z\}$.

For scalar symbols, we find that $0 \in \partial\Sigma(p)$ if and only if p is quasisymmetrizable, see Example 4.4. But in the system case, this needs not be the case according to the following example.

Example 4.14. Let

$$P(w) = \begin{pmatrix} w_2 + iw_3 & w_1 \\ w_1 & w_2 - iw_3\end{pmatrix} \qquad w = (w_1, w_2, w_3),$$

which is quasisymmetrizable with respect to ∂_{w_1} with symmetrizer

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}.$$

In fact, $\partial_{w_1} MP = \mathrm{Id}_2$ and

$$MP(w) = \begin{pmatrix} w_1 & w_2 - iw_3 \\ w_2 + iw_3 & w_1 \end{pmatrix} = (MP(w))^*$$

so $\operatorname{Im} MP \equiv 0$. Since eigenvalues of $P(w)$ are $w_2 \pm \sqrt{w_1^2 - w_3^2}$ we find that $\Sigma(P) = \mathbb{C}$ so $0 \in \mathring{\Sigma}(P)$ is not a boundary point of the eigenvalues.

For quasisymmetrizable systems we have the following result.

Theorem 4.15. *Let the $N \times N$ system $P(h)$ be given by (2.2) with principal symbol $P \in C_{\mathrm{b}}^\infty(T^*\mathbb{R}^n)$. Assume that $z \notin \Sigma_\infty(P)$ and there exists a real valued function $T(w) \in C^\infty$ such that $P(w) - z\,\mathrm{Id}_N$ is quasisymmetrizable with respect to the Hamilton vector field $H_T(w)$ in a neighborhood of $\Sigma_z(P)$. Then for any $K > 0$ we have*

$$\left\{\zeta \in \mathbb{C} : |\zeta - z| < Kh\log(1/h)\right\} \cap \operatorname{Spec}(P(h)) = \varnothing \tag{4.15}$$

for $0 < h \ll 1$, and

$$|(P(h) - z)^{-1}| \le C/h \quad 0 < h \ll 1. \tag{4.16}$$

If P is analytic in a tubular neighborhood of $T^*\mathbb{R}^n$ then there exists $c_0 > 0$ such that

$$\left\{\zeta \in \mathbb{C} : |\zeta - z| < c_0\right\} \cap \operatorname{Spec}(P(h)) = \varnothing.$$

Condition (4.16) means that $\lambda \notin \Lambda_1^{\mathrm{sc}}(P)$, which is the pseudospectrum of index 1 by Definition 2.27. The reason for the difference between (4.15) and (4.16) is that we make a change of norm in the proof that is not uniform in h. The conditions in Theorem 4.15 give some geometrical information on the bicharacteristic flow of the eigenvalues according to the following result.

Remark 4.16. The conditions in Theorem 4.15 imply that the limit set at $\Sigma_z(P)$ of the nontrivial semibicharacteristics of the eigenvalues close to zero of $Q = M(P - z\,\mathrm{Id}_N)$ is a union of compact curves on which T is strictly monotone, thus they cannot form closed orbits.

In fact, locally $(w, \lambda) \in \Omega_1(P) \setminus \Xi(P)$ if and only if $\lambda = \lambda(w) \in C^\infty$ by Lemma 2.15. Since $P(w) - \lambda\,\mathrm{Id}_N$ is of principal type by Proposition 4.7, we find that $\operatorname{Dim}\operatorname{Ker}(P(w) - \lambda(w)\,\mathrm{Id}_N)$ is constant by Proposition 3.5. Thus we obtain the normal form (4.14) as in Example 4.12. This shows that the Hamilton vector field H_λ of an eigenvalue is determined by $\langle dQu, u\rangle$ with $0 \ne u \in \operatorname{Ker}(P - \nu\,\mathrm{Id}_N)$ for ν close to $z = \lambda(w)$ by the invariance property given by (3.2). Now $\langle (H_T \operatorname{Re} Q)u, u\rangle > 0$ for $0 \ne u \in \operatorname{Ker}(P - z\,\mathrm{Id}_N)$, and $d\langle \operatorname{Im} Qu, u\rangle = 0$ for $u \in \operatorname{Ker} M(P - z\,\mathrm{Id}_N)$ by (4.9). Thus by picking subsequences we find that the limits of nontrivial semibicharacteristics of eigenvalues λ of Q close to 0 give curves on which T is strictly monotone. Since $z \notin \Sigma_\infty(P)$ these limit bicharacteristics are compact and cannot form closed orbits.

Example 4.17. Consider the system in Example 4.13

$$P(x, \xi) = |\xi|^2\,\mathrm{Id}_N + iK(x)$$

where $0 \le K(x) \in C^\infty$. Then for $z > 0$ we find that $P - z\,\mathrm{Id}_N$ is quasisymmetric in a neighborhood of $\Sigma_z(P)$ with respect to $V = H_T$, for $T(x, \xi) = -\langle \xi, x\rangle$. If $K(x) \in C_{\mathrm{b}}^\infty$ and $0 \notin \Sigma_\infty(K)$ then we obtain from Proposition 2.20, Remark 2.21, Example 2.22 and Theorem 4.15 that

$$\|(P^w(x,hD)-z)^{-1}\| \le C/h \qquad 0 < h \ll 1$$

since $z \notin \Sigma_\infty(P)$.

Proof of Theorem 4.15. We shall first consider the C_{b}^∞ case. We may assume without loss of generality that $z = 0$, and we shall follow the proof of Theorem 1.3 in [Dencker et al. 2004]. By the conditions, we find from Definition 4.5, Remark 4.8 and (4.6) that there exists a function $T(w) \in C_0^\infty$ and a multiplier $M(w) \in C_{\mathrm{b}}^\infty(T^*\mathbb{R}^n)$ so that $Q = MP$ satisfies

$$\operatorname{Re} H_T Q \ge c - C \operatorname{Im} Q \tag{4.17}$$

$$\operatorname{Im} Q \ge c\, Q^* Q \tag{4.18}$$

for some $c > 0$ and then M is invertible by Proposition 4.7. In fact, outside a neighborhood of $\Sigma_0(P)$ we have $P^*P \ge c_0$; then we may choose $M = iP^*$ so that $Q = iP^*P$ and use a partition of unity to get a global multiplier. Let

$$C_1 h \le \varepsilon \le C_2 h \log \frac{1}{h} \tag{4.19}$$

where $C_1 \gg 1$ will be chosen large. Let $T = T^w(x,hD)$

$$Q(h) = M^w(x,hD)P(h) = Q^w(x,hD) + \mathcal{O}(h) \tag{4.20}$$

$$Q_\varepsilon(h) = e^{\varepsilon T/h} Q(h) e^{-\varepsilon T/h} = e^{\frac{\varepsilon}{h}\mathrm{ad}_T} Q(h) \sim \sum_{k=0}^{\infty} \frac{\varepsilon^k}{h^k k!} (\mathrm{ad}_T)^k (Q(h))$$

where $\mathrm{ad}_T\, Q(h) = [T(h), Q(h)] = \mathcal{O}(h)$. By the assumption on ε and the boundedness of ad_T/h we find that the asymptotic expansion makes sense. Since $\varepsilon^2 = \mathcal{O}(h)$ we see that the symbol of $Q_\varepsilon(h)$ is equal to

$$Q_\varepsilon = Q + i\varepsilon\{T, Q\} + \mathcal{O}(h).$$

Since T is a scalar function, we obtain

$$\operatorname{Im} Q_\varepsilon = \operatorname{Im} Q + \varepsilon \operatorname{Re} H_T Q + \mathcal{O}(h). \tag{4.21}$$

Now to simplify notation, we drop the parameter h in the operators $Q(h)$ and $P(h)$, and we shall use the same letters for operators and the corresponding symbols. Using (4.17) and (4.18) in (4.21), we obtain for small enough ε that

$$\operatorname{Im} Q_\varepsilon \ge (1 - C\varepsilon) \operatorname{Im} Q + c\varepsilon - Ch \ge c\varepsilon - Ch \tag{4.22}$$

Since the symbol of $\frac{1}{2i}(Q_\varepsilon - (Q_\varepsilon)^*)$ is equal to the expression (4.22) modulo $\mathcal{O}(h)$, the sharp Gårding inequality for systems in Proposition A.5 gives

$$\operatorname{Im}\langle Q_\varepsilon u, u\rangle \ge (c\varepsilon - C_0 h)\|u\|^2 \ge \frac{\varepsilon c}{2}\|u\|^2$$

for $h \ll \varepsilon \ll 1$. By using the Cauchy–Schwarz inequality, we obtain

$$\frac{\varepsilon c}{2}\|u\| \le \|Q_\varepsilon u\|. \tag{4.23}$$

Since $Q = MP$ the calculus gives

$$Q_\varepsilon = M_\varepsilon P_\varepsilon + \mathcal{O}(h) \tag{4.24}$$

where $P_\varepsilon = e^{-\varepsilon T/h} P e^{\varepsilon T/h}$ and $M_\varepsilon = e^{-\varepsilon T/h} M e^{\varepsilon T/h} = M + \mathcal{O}(\varepsilon)$ is bounded and invertible for small enough ε. For $h \ll \varepsilon$ we obtain from (4.23)–(4.24) that

$$\|u\| \le \frac{C}{\varepsilon} \|P_\varepsilon u\| \tag{4.25}$$

so P_ε is injective with closed range. Now $-Q^*$ satisfies the conditions (4.3)–(4.4), with T replaced by $-T$. Thus we also obtain the estimate (4.23) for $Q_\varepsilon^* = P_\varepsilon^* M_\varepsilon^* + \mathcal{O}(h)$. Since M_ε^* is invertible for small enough h we obtain the estimate (4.25) for P_ε^*, thus P_ε is surjective. Because the conjugation by $e^{\varepsilon T/h}$ is uniformly bounded on L^2 when $\varepsilon \le Ch$ we obtain the estimate (4.16) from (4.25).

Now conjugation with $e^{\varepsilon T/h}$ is bounded in L^2 (but not uniformly) also when (4.19) holds. By taking C_2 arbitrarily large in (4.19) we find from the estimate (4.25) for P_ε and P_ε^* that

$$D\left(0, Kh \log \frac{1}{h}\right) \cap \operatorname{Spec}(P) = \varnothing$$

for any $K > 0$ when $h > 0$ is sufficiently small.

The analytic case. We assume as before that $z = 0$ and

$$P(h) \sim \sum_{j \ge 0} h^j P_j^w(x, hD) \qquad P_0 = P$$

where the P_j are bounded and holomorphic in a tubular neighborhood of $T^*\mathbb{R}^n$, satisfy (2.3), and $P_j^w(z, hD_z)$ is defined by the formula (2.1), where we may change the integration to a suitable chosen contour instead of $T^*\mathbb{R}^n$ (see [Sjöstrand 1982, Section 4]). As before, we shall follow the proof of Theorem 1.3 in [Dencker et al. 2004] and use the theory of the weighted spaces $H(\Lambda_{\varrho T})$ developed in [Helffer and Sjöstrand 1990] (see also [Martinez 2002]).

The complexification $T^*\mathbb{C}^n$ of the symplectic manifold $T^*\mathbb{R}^n$ is equipped with a complex symplectic form $\omega_{\mathbb{C}}$ giving two natural real symplectic forms $\operatorname{Im} \omega_{\mathbb{C}}$ and $\operatorname{Re} \omega_{\mathbb{C}}$. We find that $T^*\mathbb{R}^n$ is Lagrangian with respect to the first form and symplectic with respect to the second. In general, a submanifold satisfying these two conditions is called an *IR-manifold*.

Assume that $T \in C_0^\infty(T^*\mathbb{R}^n)$; then we may associate to it a natural family of IR-manifolds:

$$\Lambda_{\varrho T} = \{ w + i\varrho H_T(w) : \ w \in T^*\mathbb{R}^n \} \subset T^*\mathbb{C}^n \quad \text{with } \varrho \in \mathbb{R} \text{ and } |\varrho| \text{ small}$$

where as before we identify $T(T^*\mathbb{R}^n)$ with $T^*\mathbb{R}^n$; see [Dencker et al. 2004, page 391]. Since $\operatorname{Im}(\zeta dz)$ is closed on $\Lambda_{\varrho T}$,we find that there exists a function G_ϱ on $\Lambda_{\varrho T}$ such that

$$dG_\varrho = -\operatorname{Im}(\zeta dz)|_{\Lambda_{\varrho T}}$$

In fact, we can write it down explicitly by parametrizing $\Lambda_{\varrho T}$ by $T^*\mathbb{R}^n$:

$$G_\varrho(z, \zeta) = -\langle \xi, \varrho \nabla_\xi T(x, \xi) \rangle + \varrho T(x, \xi) \quad \text{for} \quad (z, \zeta) = (x, \xi) + i\varrho H_T(x, \xi)$$

The associated spaces $H(\Lambda_{\varrho T})$ are going to be defined by using the FBI transform:

$$T : L^2(\mathbb{R}^n) \to L^2(T^*\mathbb{R}^n)$$

given by

$$Tu(x,\xi) = c_n h^{-3n/4} \int_{\mathbb{R}^n} e^{i(\langle x-y,\xi\rangle + i|x-y|^2)/(2h)} u(y)\,dy. \tag{4.26}$$

The FBI transform may be continued analytically to $\Lambda_{\varrho T}$ so that $T_{\Lambda_{\varrho T}} u \in C^\infty(\Lambda_{\varrho T})$. Since $\Lambda_{\varrho T}$ differs from $T^*\mathbb{R}^n$ on a compact set only, we find that $T_{\Lambda_{\varrho T}} u$ is square integrable on $\Lambda_{\varrho T}$. The FBI transform can of course also be defined on $u \in L^2(\mathbb{R}^n)$ having values in $\mathbb{C}^N$, and the spaces $H(\Lambda_{\varrho T})$ are defined by putting h dependent norms on $L^2(\mathbb{R}^n)$:

$$\|u\|^2_{H(\Lambda_{\varrho T})} = \int_{\Lambda_{\varrho T}} |T_{\Lambda_{\varrho T}} u(z,\zeta)|^2 e^{-2G_\varrho(z,\zeta)/h} (\omega|_{\Lambda_{\varrho T}})^n/n! = \|T_{\Lambda_{\varrho T}} u\|^2_{L^2(\varrho,h)}$$

Suppose that P_1 and P_2 are bounded and holomorphic $N \times N$ systems in a neighbourhood of $T^*\mathbb{R}^n$ in $T^*\mathbb{C}^n$ and that $u \in L^2(\mathbb{R}^n, \mathbb{C}^N)$. Then we find for $\varrho > 0$ small enough

$$\begin{aligned}\langle P_1^w(x,hD)u,\, &P_2^w(x,hD)v\rangle_{H(\Lambda_{\varrho T})} \\ &= \langle (P_1|_{\Lambda_{\varrho T}}) T_{\Lambda_{\varrho T}} u,\, (P_2|_{\Lambda_{\varrho T}}) T_{\Lambda_{\varrho T}} v\rangle_{L^2(\varrho,h)} + \mathcal{O}(h)\|u\|_{H(\Lambda_{\varrho T})}\|v\|_{H(\Lambda_{\varrho T})}.\end{aligned}$$

By taking $P_1 = P_2 = P$ and $u = v$ we obtain

$$\|P^w(x,hD)u\|^2_{H(\Lambda_{\varrho T})} = \|(P|_{\Lambda_{\varrho T}}) T_{\Lambda_{\varrho T}} u\|^2_{L^2(\varrho,h)} + \mathcal{O}(h)\|u\|^2_{H(\Lambda_{\varrho T})} \tag{4.27}$$

as in the scalar case; see [Helffer and Sjöstrand 1990] or [Martinez 2002].

By Remark 4.8 we may assume that $MP = Q$ satisfies (4.8)–(4.9), with invertible M. The analyticity of P gives

$$P(w + i\varrho H_T) = P(w) + i\varrho H_T P(w) + \mathcal{O}(\varrho^2) \qquad |\varrho| \ll 1$$

by Taylor's formula; thus

$$\operatorname{Im} M(w)P(w + i\varrho H_T(w)) = \operatorname{Im} Q(w) + \varrho \operatorname{Re} M(w) H_T P(w) + \mathcal{O}(\varrho^2).$$

Since we have $\operatorname{Re} M H_T P > c - C \operatorname{Im} Q$, $c > 0$, by (4.8) and $\operatorname{Im} Q \ge 0$ by (4.9), we obtain for sufficiently small $\varrho > 0$ that

$$\operatorname{Im} M(w)P(w + i\varrho H_T(w)) \ge (1 - C\varrho) \operatorname{Im} Q(w) + c\varrho + \mathcal{O}(\varrho^2) \ge c\varrho/2 \tag{4.28}$$

which gives by the Cauchy–Schwarz inequality that $\|P \restriction_{\Lambda_{\varrho T}} u\| \ge c'\varrho\|u\|$. Thus

$$\|P^{-1} \restriction_{\Lambda_{\varrho T}}\| \le C/\varrho. \tag{4.29}$$

Now recall that $H(\Lambda_{\varrho T})$ is equal to L^2 as a space and that the norms are equivalent for every fixed h (but not uniformly). Thus the spectrum of $P(h)$ does not depend on whether the operator is realized on L^2 or on $H(\Lambda_{\varrho T})$. We conclude from (4.27) and (4.29) that 0 has an h-independent neighbourhood which is disjoint from the spectrum of $P(h)$, when h is small enough. $\square$

Summing up, we have proved the following result.

Proposition 4.18. *Assume that $P(h)$ is an $N \times N$ system on the form given by (2.2) with analytic principal symbol $P(w)$, and that there exists a real valued function $T(w) \in C^\infty(T^*\mathbb{R}^n)$ such that $P(w) - z\,\mathrm{Id}_N$ is quasisymmetrizable with respect to H_T in a neighborhood of $\Sigma_z(P)$. Define the IR-manifold*

$$\Lambda_{\varrho T} = \{w + i\varrho H_T(w);\ w \in T^*\mathbb{R}^n\}$$

for $\varrho > 0$ small enough. Then

$$P(h) - z : H(\Lambda_{\varrho T}) \longrightarrow H(\Lambda_{\varrho T})$$

has a bounded inverse for h small enough, which gives

$$\mathrm{Spec}(P(h)) \cap D(z,\delta) = \emptyset \qquad 0 < h < h_0$$

for δ small enough.

Remark 4.19. It is clear from the proof of Theorem 4.15 that in the analytic case it suffices that P_j is analytic in a fixed complex neighborhood of $\Sigma_z(P) \Subset T^*\mathbb{R}^n$, $j \geq 0$.

5. The subelliptic case

We shall investigate when we have an estimate of the resolvent which is better than the quasisymmetric estimate, for example the subelliptic type of estimate

$$\|(P(h) - \lambda\,\mathrm{Id}_N)^{-1}\| \leq Ch^{-\mu} \qquad h \to 0$$

with $\mu < 1$, which we obtain in the scalar case under the bracket condition; see Theorem 1.4 in [Dencker et al. 2004].

Example 5.1. Consider the scalar operator $p^w = hD_t + if^w(t,x,hD_x)$ where $0 \leq f(t,x,\xi) \in C_{\mathrm{b}}^\infty$, $(t,x) \in \mathbb{R} \times \mathbb{R}^n$, and $0 \in \partial\Sigma(f)$. Then we obtain from Theorem 1.4 in [Dencker et al. 2004] the estimate

$$h^{k/k+1}\|u\| \leq C\|p^w u\| \qquad h \ll 1 \qquad \forall u \in C_0^\infty \tag{5.1}$$

if $0 \notin \Sigma_\infty(f)$ and

$$\sum_{j \leq k} |\partial_t^j f| \neq 0. \tag{5.2}$$

These conditions are also necessary. For example, if $|f(t)| \leq C|t|^k$ then an easy computation gives $\|hD_t u + ifu\|/\|u\| \leq ch^{k/k+1}$ if $u(t) = \phi(th^{-1/k+1})$ with $0 \neq \phi(t) \in C_0^\infty(\mathbb{R})$.

The following example shows that condition (5.2) is not sufficient for systems.

Example 5.2. Let $P = hD_t\,\mathrm{Id}_2 + iF(t)$ where

$$F(t) = \begin{pmatrix} t^2 & t^3 \\ t^3 & t^4 \end{pmatrix}.$$

Then we have

$$F^{(3)}(0) = \begin{pmatrix} 0 & 6 \\ 6 & 0 \end{pmatrix}$$

which gives that

$$\bigcap_{j\le 3} \operatorname{Ker} F^{(j)}(0) = \{0\}\,.$$

But by taking $u(t)=\chi(t)(t,-1)^t$ with $0\ne\chi(t)\in C_0^\infty(\mathbb{R})$, we obtain $F(t)u(t)\equiv 0$ so we find $\|Pu\|/\|u\|\le ch$. Observe that

$$F(t)=\begin{pmatrix}1 & -t\\ t & 1\end{pmatrix}\begin{pmatrix}t^2 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}1 & t\\ -t & 1\end{pmatrix}.$$

Thus $F(t)=t^2B^*(t)\Pi(t)B(t)$ where $B(t)$ is invertible and $\Pi(t)$ is a projection of rank one.

Example 5.3. Let $P=hD_t\operatorname{Id}_2+iF(t)$ where

$$F(t)=\begin{pmatrix}t^2+t^8 & t^3-t^7\\ t^3-t^7 & t^4+t^6\end{pmatrix}=\begin{pmatrix}1 & -t\\ t & 1\end{pmatrix}\begin{pmatrix}t^2 & 0\\ 0 & t^6\end{pmatrix}\begin{pmatrix}1 & t\\ -t & 1\end{pmatrix}.$$

Then we have that

$$P=(1+t^2)^{-1}\begin{pmatrix}1 & -t\\ t & 1\end{pmatrix}\begin{pmatrix}hD_t+i(t^2+t^4) & 0\\ 0 & hD_t+i(t^6+t^8)\end{pmatrix}\begin{pmatrix}1 & t\\ -t & 1\end{pmatrix}+\mathcal{O}(h).$$

Thus we find from the scalar case that $h^{6/7}\|u\|\le C\|Pu\|$ for $h\ll 1$; see [Dencker et al. 2004, Theorem 1.4]. Observe that this operator is, element for element, a higher order perturbation of the operator of Example 5.2.

Definition 5.4. Let $0\le F(t)\in L^\infty_{\mathrm{loc}}(\mathbb{R})$ be an $N\times N$ system; then we define

$$\Omega_\delta(F)=\Big\{t:\min_{\|u\|=1}\langle F(t)u,u\rangle\le\delta\Big\}\qquad 0<\delta\le 1$$

which is well-defined almost everywhere and contains $\Sigma_0(F)=|F|^{-1}(0)$.

Observe that one can also use this definition in the scalar case, then $\Omega_\delta(f)=f^{-1}([0,\delta])$ for nonnegative functions f.

Remark 5.5. Observe that if $F\ge 0$ and E is invertible then we find that

$$\Omega_\delta(E^*FE)\subseteq\Omega_{C\delta}(F)$$

where $C=\|E^{-1}\|^2$.

Example 5.6. For the scalar symbols $p(x,\xi)=\tau+if(t,x,\xi)$ in Example 5.1 we find from Proposition A.1 that (5.2) is equivalent to

$$|\{t:f(t,x,\xi)\le\delta\}|=|\Omega_\delta(f_{x,\xi})|\le C\delta^{1/k}\qquad 0<\delta\ll 1\quad\forall x,\xi,$$

where $f_{x,\xi}(t)=f(t,x,\xi)$.

Example 5.7. For the matrix $F(t)$ in Example 5.3 we find from Remark 5.5 that $|\Omega_\delta(F)|\le C\delta^{1/6}$, and for the matrix in Example 5.2 we find that $|\Omega_\delta(F)|=\infty$.

We also have examples when the semidefinite imaginary part vanishes of infinite order.

Example 5.8. Let $p(x, \xi) = \tau + if(t, x, \xi)$ where $0 \le f(t, x, \xi) \le Ce^{-1/|t|^\sigma}$, $\sigma > 0$, then we obtain that

$$|\Omega_\delta(f_{x,\xi})| \le C_0 |\log \delta|^{-1/\sigma} \qquad 0 < \delta \ll 1 \quad \forall x, \xi.$$

(We owe this example to Y. Morimoto.)

The following example shows that for subelliptic type of estimates it is not sufficient to have conditions only on the vanishing of the symbol, we also need conditions on the semibicharacteristics of the eigenvalues.

Example 5.9. Let

$$P = hD_t \operatorname{Id}_2 + \alpha h \begin{pmatrix} D_x & 0 \\ 0 & -D_x \end{pmatrix} + i(t - \beta x)^2 \operatorname{Id}_2 \qquad (t, x) \in \mathbb{R}^2$$

with α, $\beta \in \mathbb{R}$. Then we see from the scalar case that P satisfies the estimate (5.1) with $\mu = 2/3$ if and only either $\alpha = 0$ or $\alpha \neq 0$ and $\beta \neq \pm 1/\alpha$.

Definition 5.10. Let $Q(w) \in C^\infty(T^*\mathbb{R}^n)$ be an $N \times N$ system and let $w_0 \in \Sigma \subset T^*\mathbb{R}^n$. We say that Q satisfies the *approximation property* on Σ near w_0 if there exists a Q invariant C^∞ subbundle $\mathcal{V}$ of $\mathbb{C}^N$ over $T^*\mathbb{R}^n$ such that $\mathcal{V}(w_0) = \operatorname{Ker} Q^N(w_0)$ and

$$\operatorname{Re}\langle Q(w)v, v\rangle = 0 \qquad v \in \mathcal{V}(w) \qquad w \in \Sigma \tag{5.3}$$

near w_0. That $\mathcal{V}$ is Q invariant means that $Q(w)v \in \mathcal{V}(w)$ for $v \in \mathcal{V}(w)$.

Here $\operatorname{Ker} Q^N(w_0)$ is the space of the generalized eigenvectors corresponding to the zero eigenvalue. The symbol of the system in Example 5.9 satisfies the approximation property on $\Sigma = \{\tau = 0\}$ if and only if $\alpha = 0$.

Let $\widetilde{Q} = Q|_{\mathcal{V}}$ then since $\operatorname{Im} i\widetilde{Q} = \operatorname{Re} \widetilde{Q}$ we obtain from Lemma 4.6 that $\operatorname{Ran} \widetilde{Q} \perp \operatorname{Ker} \widetilde{Q}$ on Σ. Thus $\operatorname{Ker} \widetilde{Q}^N = \operatorname{Ker} \widetilde{Q}$ on Σ, and since $\operatorname{Ker} \widetilde{Q}^N(w_0) = \mathcal{V}(w_0)$ we find that $\operatorname{Ker} Q^N(w_0) = \mathcal{V}(w_0) = \operatorname{Ker} Q(w_0)$. It follows from Example 5.13 that $\operatorname{Ker} Q \subseteq \mathcal{V}$ near w_0.

Remark 5.11. Assume that Q satisfies the approximation property on the C^∞ hypersurface Σ and is quasisymmetric with respect to $V \notin T\Sigma$. Then the limits of the nontrivial semibicharacteristics of the eigenvalues of Q close to zero coincide with the bicharacteristics of Σ.

In fact, the approximation property in Definition 5.10 and Example 5.13 give that $\langle \operatorname{Re} Qu, u\rangle = 0$ for $u \in \operatorname{Ker} Q \subseteq \mathcal{V}$ on Σ. Since $\operatorname{Im} Q \ge 0$ we find that

$$\langle dQu, u\rangle = 0 \qquad \forall u \in \operatorname{Ker} Q \qquad \text{on } T\Sigma. \tag{5.4}$$

By Remark 4.16 the limits of the nontrivial semibicharacteristics of the eigenvalues close to zero of Q are curves with tangents determined by $\langle dQu, u\rangle$ for $u \in \operatorname{Ker} Q$. Since $V \operatorname{Re} Q \neq 0$ on $\operatorname{Ker} Q$ we find from (5.4) that the limit curves coincide with the bicharacteristics of Σ, which are the flow-outs of the Hamilton vector field.

Example 5.12. Observe that Definition 5.10 is empty if $\operatorname{Dim} \operatorname{Ker} Q^N(w_0) = 0$. If $\operatorname{Dim} \operatorname{Ker} Q^N(w_0) > 0$, then there exists $\varepsilon > 0$ and a neighborhood ω to w_0 so that

$$\Pi(w) = \frac{1}{2\pi i} \int_{|z|=\varepsilon} (z \operatorname{Id}_N - Q(w))^{-1} \, dz \in C^\infty(\omega) \tag{5.5}$$

is the spectral projection on the (generalized) eigenvectors with eigenvalues having absolute value less than ε. Then $\operatorname{Ran}\Pi$ is a Q invariant bundle over ω so that $\operatorname{Ran}\Pi(w_0) = \operatorname{Ker} Q^N(w_0)$. Condition (5.3) with $\mathscr{V} = \operatorname{Ran}\Pi$ means that $\Pi^* \operatorname{Re} Q\Pi \equiv 0$ in ω. When $\operatorname{Im} Q(w_0) \ge 0$ we find that $\Pi^* Q\Pi(w_0) = 0$; then Q satisfies the approximation property on Σ near w_0 with $\mathscr{V} = \operatorname{Ran}\Pi$ if and only if

$$d(\Pi^*(\operatorname{Re} Q)\Pi)|_{T\Sigma} \equiv 0 \qquad \text{near } w_0.$$

Example 5.13. If Q satisfies the approximation property on Σ, then by choosing an orthonormal base for $\mathscr{V}$ and extending it to an orthonormal base for $\mathbb{C}^N$ we obtain the system on the form

$$Q = \begin{pmatrix} Q_{11} & Q_{12} \\ 0 & Q_{22} \end{pmatrix} \tag{5.6}$$

where Q_{11} is $K \times K$ system such that $Q_{11}^N(w_0) = 0$, $\operatorname{Re} Q_{11} = 0$ on Σ and $|Q_{22}| \ne 0$. By multiplying from the left with

$$\begin{pmatrix} \mathrm{Id}_K & -Q_{12}Q_{22}^{-1} \\ 0 & \mathrm{Id}_{N-K} \end{pmatrix}$$

we obtain that $Q_{12} \equiv 0$ without changing Q_{11} or Q_{22}.

In fact, the eigenvalues of Q are then eigenvalues of either Q_{11} or Q_{22}. Since $\mathscr{V}(w_0)$ are the (generalized) eigenvectors corresponding to the zero eigenvalue of $Q(w_0)$ we find that all eigenvalues of $Q_{22}(w_0)$ are nonvanishing, thus Q_{22} is invertible near w_0,

Remark 5.14. If Q satisfies the approximation property on Σ near w_0, then it satisfies the approximation property on Σ near w_1, for w_1 sufficiently close to w_0.

In fact, let Q_{11} be the restriction of Q to $\mathscr{V}$ as in Example 5.13, then since $\operatorname{Re} Q_{11} = \operatorname{Im} i Q_{11} = 0$ on Σ we find from Lemma 4.6 that $\operatorname{Ran} Q_{11} \perp \operatorname{Ker} Q_{11}$ and $\operatorname{Ker} Q_{11} = \operatorname{Ker} Q_{11}^N$ on Σ. Since Q_{22} is invertible in (5.6), we find that $\operatorname{Ker} Q \subseteq \mathscr{V}$. Thus, by using the spectral projection (5.5) of Q_{11} near $w_1 \in \Sigma$ for small enough ε we obtain an invariant subbundle $\widetilde{\mathscr{V}} \subseteq \mathscr{V}$ so that $\widetilde{\mathscr{V}}(w_1) = \operatorname{Ker} Q_{11}(w_1) = \operatorname{Ker} Q^N(w_1)$.

If $Q \in C^\infty$ satisfies the approximation property and $Q_E = E^* Q E$ with invertible $E \in C^\infty$, then it follows from the proof of Proposition 5.18 below that there exist invertible $A, B \in C^\infty$ such that AQ_E and Q^*B satisfy the approximation property.

Definition 5.15. Let $P \in C^\infty(T^*\mathbb{R}^n)$ be an $N \times N$ system and let $\phi(r)$ be a positive nondecreasing function on $\mathbb{R}_+$. We say that P is of *subelliptic type* ϕ if for any $w_0 \in \Sigma_0(P)$ there exists a neighborhood ω of w_0, a C^∞ hypersurface $\Sigma \ni w_0$, a real C^∞ vector field $V \notin T\Sigma$ and an invertible symmetrizer $M \in C^\infty$ so that $Q = MP$ is quasisymmetric with respect to V in ω and satisfies the approximation property on $\Sigma \cap \omega$. Also, for every bicharacteristic γ of Σ the arc length

$$\left|\gamma \cap \Omega_\delta(\operatorname{Im} Q) \cap \omega\right| \le C\phi(\delta) \qquad 0 < \delta \ll 1 \tag{5.7}$$

We say that z is of subelliptic type ϕ for $P \in C^\infty$ if $P - z\,\mathrm{Id}_N$ is of subelliptic type ϕ. If $\phi(\delta) = \delta^\mu$ then we say that the system is of finite type of order $\mu \ge 0$, which generalizes the definition of finite type for scalar operators in [Dencker et al. 2004].

Recall that the bicharacteristics of a hypersurface in T^*X are the flow-outs of the Hamilton vector field of Σ. Of course, if P is elliptic then by choosing $M = iP^{-1}$ we obtain $Q = i\,\mathrm{Id}_N$, so P is trivially of subelliptic type. If P is of subelliptic type, then it is quasisymmetrizable by definition and thus of principal type.

Remark 5.16. Observe that we may assume that

$$\operatorname{Im}\langle Qu, u\rangle \geq c\|Qu\|^2 \qquad \forall u \in \mathbb{C}^N \tag{5.8}$$

in Definition 5.15.

In fact, by adding $i\varrho P^*$ to M we obtain (5.8) for large enough ϱ by (4.6), and this only increases $\operatorname{Im} Q$.

Since Q is in C^∞ the estimate (5.7) cannot be satisfied for any $\phi(\delta) \ll \delta$ (unless Q is elliptic) and it is trivially satisfied with $\phi \equiv 1$, thus we shall only consider $c\delta \leq \phi(\delta) \ll 1$ (or finite type of order $0 < \mu \leq 1$). Actually, for C^∞ symbols of finite type, the only relevant values in (5.7) are $\mu = 1/k$ for even $k > 0$; see Proposition A.2 in the Appendix.

Actually, the condition that ϕ is nondecreasing is unnecessary, since the left-hand side in (5.7) is nondecreasing (and upper semicontinuous) in δ, we can replace $\phi(\delta)$ by $\inf_{\varepsilon>\delta} \phi(\varepsilon)$ to make it nondecreasing (and upper semicontinuous).

Example 5.17. Assume that Q is quasisymmetric with respect to the real vector field V, satisfying (5.7) and the approximation property on Σ. Then by choosing an orthonormal base as in Example 5.13 we obtain the system on the form

$$Q = \begin{pmatrix} Q_{11} & Q_{12} \\ 0 & Q_{22} \end{pmatrix}$$

where Q_{11} is $K \times K$ system such that $Q_{11}^N(w_0) = 0$, $\operatorname{Re} Q_{11} = 0$ on Σ and $|Q_{22}| \neq 0$. Since Q is quasisymmetric with respect to V we also obtain that $Q_{11}(w_0) = 0$, $\operatorname{Re} VQ_{11} > 0$, $\operatorname{Im} Q_{jj} \geq 0$ for $j = 1$, 2. In fact, then Lemma 4.6 gives that $\operatorname{Im} Q \perp \operatorname{Ker} Q$ so $\operatorname{Ker} Q^N = \operatorname{Ker} Q$. Since Q satisfies (5.7) and $\Omega_\delta(\operatorname{Im} Q_{11}) \subseteq \Omega_\delta(\operatorname{Im} Q)$ we find that Q_{11} satisfies (5.7). By multiplying from the left as in Example 5.13 we obtain that $Q_{12} \equiv 0$ without changing Q_{11} or Q_{22}.

Proposition 5.18. *If the $N \times N$ system $P(w) \in C^\infty(T^*\mathbb{R}^n)$ is of subelliptic type ϕ then P^* is of subelliptic type ϕ. If $A(w)$ and $B(w) \in C^\infty(T^*\mathbb{R}^n)$ are invertible $N \times N$ systems, then APB is of subelliptic type ϕ.*

Proof. Let M be the symmetrizer in Definition 5.15 so that $Q = MP$ is quasisymmetric with respect to V. By choosing a suitable base and changing the symmetrizer as in Example 5.17, we may write

$$Q = \begin{pmatrix} Q_{11} & 0 \\ 0 & Q_{22} \end{pmatrix} \tag{5.9}$$

where Q_{11} is $K \times K$ system such that $Q_{11}(w_0) = 0$, $V \operatorname{Re} Q_{11} > 0$, $\operatorname{Re} Q_{11} = 0$ on Σ and that Q_{22} is invertible. We also have $\operatorname{Im} Q \geq 0$ and that Q satisfies (5.7). Let $\mathcal{V}_1 = \{u \in \mathbb{C}^N : u_j = 0 \text{ for } j > K\}$ and $\mathcal{V}_2 = \{u \in \mathbb{C}^N : u_j = 0 \text{ for } j \leq K\}$, these are Q invariant bundles such that $\mathcal{V}_1 \oplus \mathcal{V}_2 = \mathbb{C}^N$.

First we are going to show that $\widetilde{P} = APB$ is of subelliptic type. By taking $\widetilde{M} = B^{-1}MA^{-1}$ we find that

$$\widetilde{M}\widetilde{P} = \widetilde{Q} = B^{-1}QB$$

and it is clear that $B^{-1}\mathscr{V}_j$ are $\widetilde{Q}$ invariant bundles, $j = 1, 2$. By choosing bases in $B^{-1}\mathscr{V}_j$ for $j = 1, 2$, we obtain a base for $\mathbb{C}^N$ in which $\widetilde{Q}$ has a block form:

$$\widetilde{Q} = \begin{pmatrix} \widetilde{Q}_{11} & 0 \\ 0 & \widetilde{Q}_{22} \end{pmatrix}$$

Here $\widetilde{Q}_{jj} : B^{-1}\mathscr{V}_j \mapsto B^{-1}\mathscr{V}_j$, is given by $\widetilde{Q}_{jj} = B_j^{-1} Q_{jj} B_j$ with

$$B_j : B^{-1}\mathscr{V}_j \ni u \mapsto Bu \in \mathscr{V}_j \qquad j = 1,\ 2.$$

By multiplying $\widetilde{Q}$ from the left with

$$\mathscr{B} = \begin{pmatrix} B_1^* B_1 & 0 \\ 0 & B_2^* B_2 \end{pmatrix}$$

we obtain that

$$\overline{Q} = \mathscr{B}\widetilde{Q} = \mathscr{B}\widetilde{M}\widetilde{P} = \begin{pmatrix} B_1^* Q_{11} B_1 & 0 \\ 0 & B_2^* Q_{22} B_2 \end{pmatrix} = \begin{pmatrix} \overline{Q}_{11} & 0 \\ 0 & \overline{Q}_{22} \end{pmatrix}.$$

It is clear that $\operatorname{Im} \overline{Q} \geq 0$, $Q_{11}(w_0) = 0$, $\operatorname{Re} \overline{Q}_{11} = 0$ on Σ, $|\overline{Q}_{22}| \neq 0$ and $V \operatorname{Re} \overline{Q}_{11} > 0$ by Proposition 4.11. Finally, we obtain from Remark 5.5 that

$$\Omega_\delta(\operatorname{Im} \overline{Q}) \subseteq \Omega_{C\delta}(\operatorname{Im} Q)$$

for some $C > 0$, which proves that $\widetilde{P} = APB$ is of subelliptic type. Observe that $\overline{Q} = AQ_B$, where $Q_B = B^* Q B$ and $A = \mathscr{B} B^{-1}(B^*)^{-1}$.

To show that P^* also is of subelliptic type, we may assume as before that $Q = MP$ is on the form (5.9) with $Q_{11}(w_0) = 0$, $V \operatorname{Re} Q_{11} > 0$, $\operatorname{Re} Q_{11} = 0$ on Σ, Q_{22} is invertible, $\operatorname{Im} Q \geq 0$ and Q satisfies (5.7). Then we find that

$$-P^* M^* = -Q^* = \begin{pmatrix} -Q_{11}^* & 0 \\ 0 & -Q_{22}^* \end{pmatrix}$$

satisfies the same conditions with respect to $-V$, so it is of subelliptic type with multiplier Id_N. By the first part of the proof we obtain that P^* is of subelliptic type. $\square$

Example 5.19. In the scalar case, $p \in C^\infty(T^*\mathbb{R}^n)$ is quasisymmetrizable with respect to $H_t = \partial_\tau$ near w_0 if and only if

$$p(t, x; \tau, \xi) = q(t, x; \tau, \xi)(\tau + i f(t, x, \xi)) \qquad \text{near } w_0 \tag{5.10}$$

with $f \geq 0$ and $q \neq 0$; see Example 4.4. If $0 \notin \Sigma_\infty(p)$ we find by taking q^{-1} as symmetrizer that p in (5.10) is of finite type of order μ if and only if $\mu = 1/k$ for an even k such that

$$\sum_{j \leq k} |\partial_t^k f| > 0$$

by Proposition A.1. In fact, the approximation property is trivial since f is real. Thus we obtain the case in [Dencker et al. 2004, Theorem 1.4]; see Example 5.1.

Theorem 5.20. *Assume that the $N \times N$ system $P(h)$ is given by the expansion (2.2) with principal symbol $P \in C_b^\infty(T^*\mathbb{R}^n)$. Assume that $z \in \Sigma(P) \setminus \Sigma_\infty(P)$ is of subelliptic type ϕ for P, where $\phi > 0$ is nondecreasing on $\mathbb{R}_+$. Then there exists $h_0 > 0$ so that*

$$\|(P(h) - z\,\mathrm{Id}_N)^{-1}\| \le C/\psi(h) \qquad 0 < h \le h_0 \tag{5.11}$$

where $\psi(h) = \delta$ is the inverse to $h = \delta\phi(\delta)$. It follows that there exists $c_0 > 0$ such that

$$\{w : |w - z| \le c_0\psi(h)\} \cap \sigma(P(h)) = \varnothing \qquad 0 < h \le h_0.$$

Theorem 5.20 will be proved in Section 6. Observe that if $\phi(\delta) \to c > 0$ as $\delta \to 0$ then $\psi(h) = \mathbb{O}(h)$ and Theorem 5.20 follows from Theorem 4.15. Thus we shall assume that $\phi(\delta) \to 0$ as $\delta \to 0$, then we find that $h = \delta\phi(\delta) = o(\delta)$ so $\psi(h) \gg h$ when $h \to 0$. In the finite type case: $\phi(\delta) = \delta^\mu$ we find that $\delta\phi(\delta) = \delta^{1+\mu}$ and $\psi(h) = h^{1/\mu+1}$. When $\mu = 1/k$ we find that $1 + \mu = (k+1)/k$ and $\psi(h) = h^{k/k+1}$. Thus Theorem 5.20 generalizes Theorem 1.4 in [Dencker et al. 2004] by Example 5.19. Condition (5.11) with $\psi(h) = h^{1/\mu+1}$ means that $\lambda \notin \Lambda^{\mathrm{sc}}_{1/\mu+1}(P)$, which is the pseudospectrum of index $1/\mu + 1$.

Example 5.21. Assume that $P(w) \in C^\infty$ is $N \times N$ and $z \in \Sigma(P) \setminus (\Sigma_{\mathrm{ws}}(P) \cup \Sigma_\infty(P))$. Then $\Sigma_\mu(P) = \{\lambda(w) = \mu\}$ for μ close to z, where $\lambda \in C^\infty$ is a germ of eigenvalues for P at $\Sigma_z(P)$; see Lemma 2.15. If $z \in \partial\Sigma(\lambda)$ we find from Example 4.12 that $P(w) - z\,\mathrm{Id}_N$ is quasisymmetrizable near $w_0 \in \Sigma_z(P)$ if $P(w) - \lambda\,\mathrm{Id}_N$ is of principal type when $|\lambda - z| \ll 1$. Then P is on the form (4.11) and there exists $q(w) \in C^\infty$ so that (4.12)–(4.13) hold near $\Sigma_z(P)$. We can then choose the multiplier M so that Q is on the form (4.14). By taking $\Sigma = \{\mathrm{Re}\, q(\lambda - z) = 0\}$ we obtain that $P - z\,\mathrm{Id}_N$ is of subelliptic type ϕ if (5.7) is satisfied for $\mathrm{Im}\, q(\lambda - z)$. In fact, by the invariance we find that the approximation property is trivially satisfied since $\mathrm{Re}\, q\lambda \equiv 0$ on Σ.

Example 5.22. Let

$$P(x, \xi) = |\xi|^2\,\mathrm{Id}_N + iK(x) \qquad (x, \xi) \in T^*\mathbb{R}^n$$

where $K(x) \in C^\infty(\mathbb{R}^n)$ is symmetric as in Example 3.12. We find that $P - z\,\mathrm{Id}_N$ is of finite type of order $1/2$ when $z = i\lambda$ for almost all $\lambda \in \Sigma(K) \setminus (\Sigma_{\mathrm{ws}}(K) \cup \Sigma_\infty(K))$ by Example 5.21. In fact, then $z \in \Sigma(P) \setminus (\Sigma_{\mathrm{ws}}(P) \cap \Sigma_\infty(P))$ and the C^∞ germ of eigenvalues for P near $\Sigma_z(P)$ is $\lambda(x, \xi) = |\xi|^2 + i\kappa(x)$, where $\kappa(x)$ is a C^∞ germ of eigenvalues for $K(x)$ near $\Sigma_\lambda(K) = \{\kappa(x) = \lambda\}$. For almost all values λ we have $d\kappa(x) \ne 0$ on $\Sigma_\lambda(K)$. By taking $q = i$ we obtain for such values that (5.7) is satisfied for $\mathrm{Im}\, i(\lambda(w) - i\lambda) = |\xi|^2$ with $\phi(\delta) = \delta^{1/2}$, since $\mathrm{Re}\, i(\lambda(w) - i\lambda) = \lambda - \kappa(x) = 0$ on $\Sigma = \Sigma_\lambda(K)$. If $K(x) \in C_b^\infty$ and $0 \notin \Sigma_\infty(K)$ then we may use Theorem 5.20, Proposition 2.20, Remark 2.21 and Example 2.22 to obtain the estimate

$$\|(P^w(x, hD) - z\,\mathrm{Id}_N)^{-1}\| \le Ch^{-2/3} \qquad 0 < h \ll 1$$

on the resolvent.

Example 5.23. Let

$$P(t, x; \tau, \xi) = \tau M(t, x, \xi) + iF(t, x, \xi) \in C^\infty$$

where $M \ge c_0 > 0$ and $F \ge 0$ satisfies

$$\left|\left\{t : \inf_{|u|=1} \langle F(t, x, \xi)u, u\rangle \le \delta\right\}\right| \le C\phi(\delta) \qquad \forall\, x, \xi. \tag{5.12}$$

Then P is quasisymmetrizable with respect to ∂_τ with symmetrizer Id_N. When $\tau = 0$ we obtain that $\operatorname{Re} P = 0$, so by taking $\mathscr{V} = \operatorname{Ran} \Pi$ for the spectral projection Π given by (5.5) for F, we find that P satisfies the approximation property with respect to $\Sigma = \{\tau = 0\}$. Since $\Omega_\delta(\operatorname{Im} P) = \Omega_\delta(F)$ we find from (5.12) that P is of subelliptic type ϕ. Observe that if $0 \notin \Sigma_\infty(F)$ we obtain from Proposition A.2 that (5.12) is satisfied for $\phi(\delta) = \delta^\mu$ if and only if $\mu \le 1/k$ for an even $k \ge 0$ so that

$$\sum_{j\le k} |\partial_t^j \langle F(t,x,\xi)u(t), u(t)\rangle| > 0 \qquad \forall\, t, x, \xi$$

for any $0 \ne u(t) \in C^\infty(\mathbb{R})$.

6. Proof of Theorem 5.20

By subtracting $z\,\mathrm{Id}_N$ we may assume $z = 0$. Let $\widetilde{w}_0 \in \Sigma_0(P)$; then by Definition 5.15 and Remark 5.16 there exist a C^∞ hypersurface Σ and a real C^∞ vector field $V \notin T\Sigma$, an invertible symmetrizer $M \in C^\infty$ so that $Q = MP$ satisfies (5.7), the approximation property on Σ, and

$$V \operatorname{Re} Q \ge c - C \operatorname{Im} Q \tag{6.1}$$

$$\operatorname{Im} Q \ge c\, Q^* Q \tag{6.2}$$

in a neighborhood ω of $\widetilde{w}_0$, here $c > 0$.

Since (6.1) is stable under small perturbations in V we can replace V with H_t for some real $t \in C^\infty$ after shrinking ω. By solving the initial value problem $H_t \tau \equiv -1$, $\tau|_\Sigma = 0$, and completing to a symplectic C^∞ coordinate system (t, τ, x, ξ) we obtain that $\Sigma = \{\tau = 0\}$ in a neighborhood of $\widetilde{w}_0 = (0, 0, w_0)$. We obtain from Definition 5.15 that

$$\operatorname{Re}\langle Qu, u\rangle = 0 \qquad \text{when } \tau = 0 \text{ and } u \in \mathscr{V} \tag{6.3}$$

near $\widetilde{w}_0$. Here $\mathscr{V}$ is a Q invariant C^∞ subbundle of $\mathbb{C}^N$ such that $\mathscr{V}(\widetilde{w}_0) = \operatorname{Ker} Q^N(\widetilde{w}_0) = \operatorname{Ker} Q(\widetilde{w}_0)$ by Lemma 4.6. By condition (5.7) we have that

$$\left|\Omega_\delta(\operatorname{Im} Q_w) \cap \{|t| < c\}\right| \le C\phi(\delta) \qquad 0 < \delta \ll 1 \tag{6.4}$$

when $|w - w_0| < c$, here $Q_w(t) = Q(t, 0, w)$. Since these are all local conditions, we may assume that M and $Q \in C_{\mathrm{b}}^\infty$. We shall obtain Theorem 5.20 from the following estimate.

Proposition 6.1. *Assume that $Q \in C_{\mathrm{b}}^\infty(T^*\mathbb{R}^n)$ is an $N \times N$ system satisfying (6.1)–(6.4) in a neighborhood of $\widetilde{w}_0 = (0, 0, w_0)$ with $V = \partial_\tau$ and nondecreasing $\phi(\delta) \to 0$ as $\delta \to 0$. Then there exist $h_0 > 0$ and $R \in C_{\mathrm{b}}^\infty(T^*\mathbb{R}^n)$ so that $\widetilde{w}_0 \notin \operatorname{supp} R$ and*

$$\psi(h)\|u\| \le C(\|Q^w(t,x,hD_{t,x})u\| + \|R^w(t,x,hD_{t,x})u\| + h\|u\|) \qquad 0 < h \le h_0 \tag{6.5}$$

for any $u \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^N)$. Here $\psi(h) = \delta \gg h$ is the inverse to $h = \delta\phi(\delta)$.

Let ω be a neighborhood of $\widetilde{w}_0$ such that $\operatorname{supp} R \cap \omega = \emptyset$, where R is given by Proposition 6.1. Take $\varphi \in C_0^\infty(\omega)$ such that $0 \le \varphi \le 1$ and $\varphi = 1$ in a neighborhood of $\widetilde{w}_0$. By substituting $\varphi^w(t,x,hD_{t,x})u$ in (6.5) we obtain from the calculus that for any N we have

$$\psi(h)\|\varphi^w(t,x,hD_{t,x})u\| \le C_N(\|Q^w(t,x,hD_{t,x})\varphi^w(t,x,hD_{t,x})u\| + h^N\|u\|) \qquad \forall\, u \in C_0^\infty \tag{6.6}$$

for small enough h since $R\varphi \equiv 0$. Now the commutator

$$\|[Q^w(t,x,hD_{t,x}), \varphi^w(t,x,hD_{t,x})]u\| \le Ch\|u\| \qquad u \in C_0^\infty.$$

Since $Q = MP$ the calculus gives

$$\|Q^w(t,x,hD_{t,x})u\| \le \|M^w(t,x,hD_{t,x})P(h)u\| + Ch\|u\| \le C'(\|P(h)u\| + h\|u\|) \quad u \in C_0^\infty. \tag{6.7}$$

The estimates (6.6)–(6.7) give

$$\psi(h)\|\varphi^w(t,x,hD_{t,x})u\| \le C(\|P(h)u\| + h\|u\|). \tag{6.8}$$

Since $0 \notin \Sigma_\infty(P)$ we obtain by using the Borel Theorem finitely many functions $\phi_j \in C_0^\infty$, $j = 1, \dots, N$, such that $0 \le \phi_j \le 1$, $\sum_j \phi_j = 1$ on $\Sigma_0(P)$ and the estimate (6.8) holds with $\phi = \phi_j$. Let $\phi_0 = 1 - \sum_{j\ge 1}\phi_j$; then since $0 \notin \Sigma_\infty(P)$ we find that $\|P^{-1}\| \le C$ on $\operatorname{supp}\phi_0$. Thus $\phi_0 = \phi_0 P^{-1}P$ and the calculus gives

$$\|\phi_0^w(t,x,hD_{t,x})u\| \le C(\|P(h)u\| + h\|u\|) \qquad u \in C_0^\infty.$$

By summing up, we obtain

$$\psi(h)\|u\| \le C(\|P(h)u\| + h\|u\|) \qquad u \in C_0^\infty. \tag{6.9}$$

Since $h = \delta\phi(\delta) \ll \delta$ we find $\psi(h) = \delta \gg h$ when $h \to 0$. Thus, we find for small enough h that the last term in the right hand side of (6.9) can be cancelled by changing the constant; then $P(h)$ is injective with closed range. Since $P^*(h)$ also is of subelliptic type ϕ by Proposition 5.18 we obtain the estimate (6.9) for $P^*(h)$. Thus $P^*(h)$ is injective making $P(h)$ is surjective, which together with (6.9) gives Theorem 5.20.

Proof of Proposition 6.1. First we shall prepare the symbol Q locally near $\widetilde{w}_0 = (0, 0, w_0)$. Since $\operatorname{Im} Q \ge 0$ we obtain from Lemma 4.6 that $\operatorname{Ran} Q(\widetilde{w}_0) \perp \operatorname{Ker} Q(\widetilde{w}_0)$ which gives $\operatorname{Ker} Q^N(\widetilde{w}_0) = \operatorname{Ker} Q(\widetilde{w}_0)$. Let $\operatorname{Dim}\operatorname{Ker} Q(\widetilde{w}_0) = K$ then by choosing an orthonormal base and multiplying from the left as in Example 5.17, we may assume that

$$Q = \begin{pmatrix} Q_{11} & 0 \\ 0 & Q_{22} \end{pmatrix}$$

where Q_{11} is $K \times K$ matrix, $Q_{11}(\widetilde{w}_0) = 0$ and $|Q_{22}(\widetilde{w}_0)| \ne 0$. Also, we find that Q_{11} satisfies the conditions (6.1)–(6.4) with $\mathscr{V} = \mathbb{C}^K$ near $\widetilde{w}_0$.

Now it suffices to prove the estimate with Q replaced by Q_{11}. In fact, by using the ellipticity of Q_{22} at $\widetilde{w}_0$ we find

$$\|u''\| \le C(\|Q_{22}^w u''\| + \|R_1^w u''\| + h\|u''\| \qquad u'' \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^{N-K})$$

where $u = (u', u'')$ and $\widetilde{w}_0 \notin \operatorname{supp} R_1$. Thus, if we have the estimate (6.5) for Q_{11}^w with $R = R_2$, then since $\psi(h)$ is bounded we obtain the estimate for Q^w:

$$\psi(h)\|u\| \le C_0(\|Q_{11}^w u'\| + \|Q_{22}^w u''\| + \|R^w u\| + h\|u\|) \le C_1(\|Q^w u\| + \|R^w u\| + h\|u\|)$$

where $\widetilde{w}_0 \notin \operatorname{supp} R$, $R = (R_1, R_2)$.

Thus, in the following we may assume that $Q = Q_{11}$ is $K \times K$ system satisfying the conditions (6.1)–(6.4) with $\mathscr{V} = \mathbb{C}^K$ near $\widetilde{w}_0$. Since $\partial_\tau \operatorname{Re} Q > 0$ at $\widetilde{w}_0$ by (6.1), we find from the matrix version of the Malgrange Preparation Theorem in [Dencker 1993, Theorem 4.3] that

$$Q(t, \tau, w) = E(t, \tau, w)(\tau \operatorname{Id} + K_0(t, w)) \qquad \text{near } \widetilde{w}_0$$

where $E, K_0 \in C^\infty$, and $\operatorname{Re} E > 0$ at $\widetilde{w}_0$. By taking $M(t, w) = E(t, 0, w)$ we find $\operatorname{Re} M > 0$ and

$$Q(t, \tau, w) = E_0(t, \tau, w)(\tau M(t, w) + i K(t, w)) = E_0(t, \tau, w) Q_0(t, \tau, w)$$

where $E_0(t, 0, w) \equiv \operatorname{Id}$. Thus we find that Q_0 satisfies (6.2), (6.3) and (6.4) when $\tau = 0$ near $\widetilde{w}_0$. Since $K(0, w_0) = 0$ we obtain that $\operatorname{Im} K \equiv 0$ and $K \ge c K^2 \ge 0$ near $(0, w_0)$. We have $\operatorname{Re} M > 0$ and

$$|\langle \operatorname{Im} M u, u\rangle| \le C \langle K u, u\rangle^{1/2} \|u\| \qquad \text{near } (0, w_0). \tag{6.10}$$

In fact, we have

$$0 \le \operatorname{Im} Q \le K + \tau(\operatorname{Im} M + \operatorname{Re}(E_1 K)) + C\tau^2$$

where $E_1(t, w) = \partial_\tau E(t, 0, w)$. Lemma A.7 gives

$$|\langle \operatorname{Im} M u, u\rangle + \operatorname{Re}\langle E_1 K u, u\rangle| \le C \langle K u, u\rangle^{1/2} \|u\|$$

and since $K^2 \le C K$ we obtain

$$|\operatorname{Re}\langle E_1 K u, u\rangle| \le C \|K u\| \|u\| \le C_0 \langle K u, u\rangle^{1/2} \|u\|$$

which gives (6.10). Now by cutting off when $|\tau| \ge c > 0$ we obtain that

$$Q^w = E_0^w Q_0^w + R_0^w + h R_1^w$$

where $R_j \in C_b^\infty$ and $\widetilde{w}_0 \notin \operatorname{supp} R_0$. Thus, it suffices to prove the estimate (6.5) for Q_0^w. We may now reduce to the case when $\operatorname{Re} M \equiv \operatorname{Id}$. In fact,

$$Q_0^w \cong M_0^w((\operatorname{Id} + i M_1^w) h D_t + i K_1^w) M_0^w \qquad \text{modulo } \mathbb{O}(h)$$

where $M_0 = (\operatorname{Re} M)^{1/2}$ is invertible, $M_1^* = M_1$ and $K_1 = M_0^{-1} K M_0^{-1} \ge 0$. By changing M_1 and K_1 and making $K_1 > 0$ outside a neighborhood of $(0, w_0)$ we may assume that $M_1, K_1 \in C_b^\infty$ and $i K_1$ satisfies (6.4) for all $c > 0$ and any w, by the invariance given by Remark 5.5. Observe that condition (6.10) also is invariant under the mapping $Q_0 \mapsto E^* Q_0 E$.

We shall use the symbol classes $f \in S(m, g)$ defined by

$$|\partial_{\nu_1} \dots \partial_{\nu_k} f| \le C_k m \prod_{j=1}^{k} g(\nu_j)^{1/2} \qquad \forall\, \nu_1, \dots, \nu_k \quad \forall\, k$$

for constant weight m and metric g, and $\operatorname{Op} S(m, g)$ the corresponding Weyl operators f^w. We shall need the following estimate for the model operator Q_0^w.

Proposition 6.2. *Assume that*

$$Q = M^w(t, x, h D_x) h D_t + i K^w(t, x, h D_x),$$

where $M(t,w)$ *and* $0 \le K(t,w) \in L^\infty(\mathbb{R}, C_b^\infty(T^*\mathbb{R}^n))$ *are* $N \times N$ *system such that* $\operatorname{Re} M \equiv \mathrm{Id}$, $\operatorname{Im} M$ *satisfies* (6.10) *and* iK *satisfies* (6.4) *for all* w *and* $c > 0$ *with non-decreasing* $0 < \phi(\delta) \to 0$ *as* $\delta \to 0$. *Then there exists a real valued* $B(t,w) \in L^\infty(\mathbb{R}, S(1, H|dw|^2/h))$ *such that* $hB(t,w)/\psi(h) \in \operatorname{Lip}(\mathbb{R}, S(1, H|dw|^2/h))$, *and*

$$\psi(h)\|u\|^2 \le \operatorname{Im}\langle Qu, B^w(t,x,hD_x)u\rangle + Ch^2\|D_t u\|^2 \qquad 0 < h \ll 1 \tag{6.11}$$

for any $u \in C_0^\infty(\mathbb{R}^{n+1}, \mathbb{C}^N)$. *Here the bounds on* $B(t,w)$ *are uniform,* $\psi(h) = \delta \gg h$ *is the inverse to* $h = \delta\phi(\delta)$ *so* $0 < H = \sqrt{h/\psi(h)} \ll 1$ *as* $h \to 0$.

Observe that $H^2 = h/\psi(h) = \phi(\psi(h)) \to 0$ and $h/H = \sqrt{\psi(h)h} \ll \psi(h) \to 0$ as $h \to 0$, since $0 < \phi(\delta)$ is non-decreasing.

To prove Proposition 6.1 we shall cut off where $|\tau| \gtrsim \varepsilon\sqrt{\psi}/h$. Take $\chi_0(r) \in C_0^\infty(\mathbb{R})$ such that $0 \le \chi_0 \le 1$, $\chi_0(r) = 1$ when $|r| \le 1$ and $|r| \le 2$ in $\operatorname{supp}\chi_0$. Then $1 - \chi_0 = \chi_1$ where $0 \le \chi_1 \le 1$ is supported where $|r| \ge 1$. Let $\phi_{j,\varepsilon}(r) = \chi_j(hr/\varepsilon\sqrt{\psi})$, $j = 0, 1$, for $\varepsilon > 0$; then $\phi_{0,\varepsilon}$ is supported where $|r| \le 2\varepsilon\sqrt{\psi}/h$ and $\phi_{1,\varepsilon}$ is supported where $|r| \ge \varepsilon\sqrt{\psi}/h$. We have that $\phi_{j,\varepsilon}(\tau) \in S(1, h^2 d\tau^2/\psi)$, $j = 0$, 1, and $u = \phi_{0,\varepsilon}(D_t)u + \phi_{1,\varepsilon}(D_t)u$, where we shall estimate each term separately. Observe that we shall use the ordinary quantization and not the semiclassical for these operators.

To estimate the first term, we substitute $\phi_{0,\varepsilon}(D_t)u$ in (6.11). We find that

$$\begin{aligned}\psi(h)\|\phi_{0,\varepsilon}(D_t)u\|^2 \le{}& \operatorname{Im}\langle Qu, \phi_{0,\varepsilon}(D_t)B^w(t,x,hD_x)\phi_{0,\varepsilon}(D_t)u\rangle \\ &+ \operatorname{Im}\langle [Q, \phi_{0,\varepsilon}(D_t)]u, B^w(t,x,hD_x)\phi_{0,\varepsilon}(D_t)u\rangle + 4C\varepsilon^2\psi\|u\|^2\end{aligned} \tag{6.12}$$

In fact, $h\|D_t\phi_{0,\varepsilon}(D_t)u\| \le 2\varepsilon\sqrt{\psi}\|u\|$ since it is a Fourier multiplier and $|h\tau\phi_{0,\varepsilon}(\tau)| \le 2\varepsilon\sqrt{\psi}$. Next we shall estimate the commutator term. Since $\operatorname{Re} Q = hD_t \mathrm{Id} - h\partial_t \operatorname{Im} M^w/2$ and $\operatorname{Im} Q = h \operatorname{Im} M^w D_t + K^w + h\partial_t \operatorname{Im} M^w/2i$ we find that $[\operatorname{Re} Q, \phi_{0,\varepsilon}(D_t)] \in \operatorname{Op} S(h, \mathscr{G})$ and

$$[Q, \phi_{0,\varepsilon}(D_t)] = i[\operatorname{Im} Q, \phi_{0,\varepsilon}(D_t)] = i[K^w, \phi_{0,\varepsilon}(D_t)] = -h\partial_t K^w\phi_{2,\varepsilon}(D_t)/\varepsilon\sqrt{\psi}$$

is a symmetric operator modulo $\operatorname{Op} S(h, \mathscr{G})$, where $\mathscr{G} = dt^2 + h^2 d\tau^2/\psi + |dx|^2 + h^2|d\xi|^2$ and $\phi_{2,\varepsilon}(\tau) = \chi_0'(h\tau/\varepsilon\sqrt{\psi})$. In fact, we have that $h^2/\psi(h) \le Ch$, $h[\partial_t \operatorname{Im} M^w, \phi_{0,\varepsilon}(D_t)]$ and $[\operatorname{Im} M^w, \phi_{0,\varepsilon}(D_t)]hD_t \in \operatorname{Op} S(h, \mathscr{G})$, since $|\tau| \le \varepsilon\sqrt{\psi}/h$ in $\operatorname{supp}\phi_{0,\varepsilon}(\tau)$. Thus, we find that

$$\begin{aligned}-2i \operatorname{Im}\big(\phi_{0,\varepsilon}(D_t)B^w[Q, \phi_{0,\varepsilon}(D_t)]\big) &= 2ih\varepsilon^{-1}\psi^{-1/2}\operatorname{Im}\big(\phi_{0,\varepsilon}(D_t)B^w\partial_t K^w\phi_{2,\varepsilon}(D_t)\big) \\ &= h\varepsilon^{-1}\psi^{-1/2}\Big(\phi_{0,\varepsilon}(D_t)B^w[\partial_t K^w, \phi_{2,\varepsilon}(D_t)] + \phi_{0,\varepsilon}(D_t)[B^w, \phi_{2,\varepsilon}(D_t)]\partial_t K^w \\ &\qquad + \phi_{2,\varepsilon}(D_t)[\phi_{0,\varepsilon}(D_t), B^w]\partial_t K^w + \phi_{2,\varepsilon}(D_t)B^w[\phi_{0,\varepsilon}(D_t), \partial_t K^w]\Big)\end{aligned} \tag{6.13}$$

modulo $\operatorname{Op} S(h, \mathscr{G})$. As before, the calculus gives that $[\phi_{j,\varepsilon}(D_t), \partial_t K^w] \in \operatorname{Op} S(h\psi^{-1/2}, \mathscr{G})$ for any j. Since $t \to hB^w/\psi \in \operatorname{Lip}(\mathbb{R}, \operatorname{Op} S(1, \mathscr{G}))$ uniformly and $\phi_{j,\varepsilon}(\tau) = \chi_j(h\tau/\varepsilon\sqrt{\psi})$ with $\chi_j' \in C_0^\infty(\mathbb{R})$, Lemma A.4 with $\kappa = \varepsilon\sqrt{\psi}/h$ gives that

$$\big\|[\phi_{j,\varepsilon}(D_t), B^w]\big\|_{\mathscr{L}(L^2(\mathbb{R}^{n+1}))} \le C\sqrt{\psi}/\varepsilon$$

uniformly. If we combine the estimates above we can estimate the commutator term:

$$|\operatorname{Im}\langle [Q, \phi_{0,\varepsilon}(D_t)]u, B^w(t,x,hD_x)\phi_{0,\varepsilon}(D_t)u\rangle| \le Ch\|u\|^2 \ll \psi(h)\|u\|^2 \qquad h \ll 1 \tag{6.14}$$

which together with (6.12) will give the estimate for the first term for small enough ε and h.

We also have to estimate $\phi_{1,\varepsilon}(D_t)u$; then we shall use that Q is elliptic when $|\tau| \ne 0$. We have

$$\|\phi_{1,\varepsilon}(D_t)u\|^2 = \langle \chi^w(D_t)u, u\rangle$$

where $\chi(\tau) = \phi_{1,\varepsilon}^2(\tau) \in S(1, h^2 d\tau^2/\psi)$ is real with support where $|\tau| \ge \varepsilon\sqrt{\psi}/h$. Thus, we may write $\chi(D_t) = \varrho(D_t)hD_t$ where $\varrho(\tau) = \chi(\tau)/h\tau \in S(\psi^{-1/2}, h^2 d\tau^2/\psi)$ by Leibniz' rule since $|\tau|^{-1} \le h/\varepsilon\sqrt{\psi}$ in $\operatorname{supp}\varrho$. Now $hD_t \operatorname{Id} = \operatorname{Re} Q + h\partial_t \operatorname{Im} M^w/2$ so we find

$$\langle \chi(D_t)u, u\rangle = \operatorname{Re}\langle \varrho(D_t)Qu, u\rangle + \frac{h}{2}\operatorname{Re}\langle \varrho(D_t)(\partial_t \operatorname{Im} M^w)u, u\rangle + \operatorname{Im}\langle \varrho(D_t)\operatorname{Im} Qu, u\rangle$$

where $|h\operatorname{Re}\langle \varrho(D_t)(\partial_t \operatorname{Im} M^w)u, u\rangle| \le Ch\|u\|^2/\varepsilon\sqrt{\psi}$ and

$$|\operatorname{Re}\langle \varrho(D_t)Qu, u\rangle| \le \|Qu\|\|\varrho(D_t)u\| \le \|Qu\|\|u\|/\varepsilon\sqrt{\psi}$$

since $\varrho(D_t)$ is a Fourier multiplier and $|\varrho(\tau)| \le 1/\varepsilon\sqrt{\psi}$. We have that

$$\operatorname{Im} Q = K^w(t,x,hD_x) + hD_t \operatorname{Im} M^w(t,x,hD_x) - \frac{h}{2i}\partial_t \operatorname{Im} M^w(t,x,hD_x)$$

where $\operatorname{Im} M^w(t,x,hD_x)$ and $K^w(t,x,hD_x) \in \operatorname{Op} S(1,\mathcal{G})$ are symmetric. Since $\varrho = \chi/h\tau \in S(\psi^{-1/2},\mathcal{G})$ is real we find that

$$\begin{aligned}\operatorname{Im}(\varrho(D_t)\operatorname{Im} Q) &= \operatorname{Im}\varrho(D_t)K^w + \operatorname{Im}\chi(D_t)\operatorname{Im} M^w \\ &= \frac{1}{2i}\big([\varrho(D_t), K^w(t,x,hD_x)] + [\chi(D_t), \operatorname{Im} M^w(t,x,hD_x)]\big)\end{aligned}$$

modulo terms in $\operatorname{Op} S(h/\sqrt{\psi},\mathcal{G}) \subseteq \operatorname{Op} S(h/\psi,\mathcal{G})$. Here the calculus gives

$$[\varrho(D_t), K^w(t,x,hD_x)] \in \operatorname{Op} S(h/\psi,\mathcal{G})$$

and similarly we have that

$$[\chi(D_t), \operatorname{Im} M^w(t,x,hD_x)] \in \operatorname{Op} S(h/\sqrt{\psi},\mathcal{G}) \subseteq \operatorname{Op} S(h/\psi,\mathcal{G})$$

which gives that $|\operatorname{Im}\langle \varrho(D_t)\operatorname{Im} Qu, u\rangle| \le Ch\|u\|^2/\psi$. In fact, since the metric $\mathcal{G}$ is constant, it is uniformly σ temperate for all $h > 0$. We obtain that

$$\psi(h)\|\phi_{1,\varepsilon}(D_t)u\|^2 \le C_\varepsilon(\sqrt{\psi}\|Qu\|\|u\| + h\|u\|^2)$$

which together with (6.12) and (6.14) gives the estimate (6.5) for small enough ε and h, since $h/\psi(h) \to 0$ as $h \to 0$. □

Proof of Proposition 6.2. We shall do a second microlocalization in $w = (x,\xi)$. By making a linear symplectic change of coordinates $(x,\xi) \mapsto (h^{1/2}x, h^{-1/2}\xi)$ we see that $Q(t,\tau,x,h\xi)$ is changed into

$$Q(t,\tau,h^{1/2}w) \in S(1, dt^2 + d\tau^2 + h|dw|^2) \qquad \text{when } |\tau| \le c.$$

In these coordinates we find $B(h^{1/2}w) \in S(1, G)$, $G = H|dw|^2$, if $B(w) \in S(1, H|dw|^2/h)$. In the following, we shall use ordinary Weyl quantization in the w variables.

We shall follow an approach similar to the one of [Dencker et al. 2004, Section 5]. To localize the estimate we take $\{\phi_j(w)\}_j$, $\{\psi_j(w)\}_j \in S(1, G)$ with values in ℓ^2, such that $0 \le \phi_j$, $0 \le \psi_j$, $\sum_j \phi_j^2 \equiv 1$ and $\phi_j \psi_j = \phi_j$ for all j. We may also assume that ψ_j is supported in a G neighborhood of w_j. This can be done uniformly in H, by taking $\phi_j(w) = \Phi_j(H^{1/2}w)$ and $\psi_j(w) = \Psi_j(H^{1/2}w)$, with $\{\Phi_j(w)\}_j$ and $\{\Psi_j(w)\}_j \in S(1, |dw|^2)$. Since $\sum \phi_j^2 = 1$ and $G = H|dw|^2$ the calculus gives

$$\sum_j \|\phi_j^w(x, D_x)u\|^2 - CH^2\|u\|^2 \le \|u\|^2 \le \sum_j \|\phi_j^w(x, D_x)u\|^2 + CH^2\|u\|^2$$

for $u \in C_0^\infty(\mathbb{R}^n)$, thus for small enough H we find

$$\sum_j \|\phi_j^w(x, D_x)u\|^2 \le 2\|u\|^2 \le 4\sum_j \|\phi_j^w(x, D_x)u\|^2 \qquad \text{for } u \in C_0^\infty(\mathbb{R}^n). \tag{6.15}$$

Observe that since ϕ_j has values in ℓ^2 we find that $\{\phi_j^w R_j^w\}_j \in \operatorname{Op} S(H^\nu, G)$ also has values in ℓ^2 if $R_j \in S(H^\nu, G)$ uniformly. Such terms will be summable:

$$\sum_j \|r_j^w u\|^2 \le CH^{2\nu}\|u\|^2 \tag{6.16}$$

for $\{r_j\}_j \in S(H^\nu, G)$ with values in ℓ^2; see [Hörmander 1983–1985, Volume III, page 169]. Now we fix j and let

$$Q_j(t, \tau) = Q(t, \tau, h^{1/2}w_j) = M_j(t)\tau + iK_j(t)$$

where $M_j(t) = M(t, h^{1/2}w_j)$ and $K_j(t) = K(t, h^{1/2}w_j) \in L^\infty(\mathbb{R})$. Since $K(t, w) \ge 0$ we find from Lemma A.7 and (6.10) that

$$|\langle \operatorname{Im} M_j(t)u, u\rangle| + |\langle d_w K(t, h^{1/2}w_j)u, u\rangle| \le C\langle K_j(t)u, u\rangle^{1/2}\|u\| \qquad \forall u \in \mathbb{C}^N \qquad \forall t \tag{6.17}$$

and condition (6.4) means that

$$\left|\left\{t : \inf_{|u|=1} \langle K_j(t)u, u\rangle \le \delta\right\}\right| \le C\phi(\delta). \tag{6.18}$$

We shall prove an estimate for the corresponding one-dimensional operator

$$Q_j(t, hD_t) = M_j(t)hD_t + iK_j(t)$$

by using the following result.

Lemma 6.3. *Assume that*

$$Q(t, hD_t) = M(t)hD_t + iK(t)$$

where $M(t)$ and $0 \le K(t)$ are $N \times N$ systems, which are uniformly bounded in $L^\infty(\mathbb{R})$, such that $\operatorname{Re} M \equiv \mathrm{Id}$, $\operatorname{Im} M$ *satisfies (6.10) for almost all t and iK satisfies (6.4) for any $c > 0$ with non-decreasing $\phi(\delta) \to 0$ as $\delta \to 0$. Then there exists a uniformly bounded real $B(t) \in L^\infty(\mathbb{R})$ so that $hB(t)/\psi(h) \in$* $\operatorname{Lip}(\mathbb{R})$ *uniformly and*

$$\psi(h)\|u\|^2 + \langle Ku, u\rangle \le \operatorname{Im}\langle Qu, Bu\rangle + Ch^2\|D_t u\|^2 \qquad 0 < h \ll 1 \tag{6.19}$$

for any $u \in C_0^\infty(\mathbb{R}, \mathbb{C}^N)$. Here $\psi(h) = \delta \gg h$ is the inverse to $h = \delta\phi(\delta)$.

Proof. Let $0 \le \Phi_h(t) \le 1$ be the characteristic function of the set $\Omega_\delta(K)$ with $\delta = \psi(h)$. Since $\delta = \psi(h)$ is the inverse of $h = \delta\phi(\delta)$ we find that $\phi(\psi(h)) = h/\delta = h/\psi(h)$. Thus, we obtain from (6.18) that

$$\int \Phi_h(t)\,dt = |\Omega_\delta(K)| \le Ch/\psi(h)$$

Letting

$$E(t) = \exp\Big(\frac{\psi(h)}{h}\int_0^t \Phi_h(s)\,ds\Big),$$

we find that E and $E^{-1} \in L^\infty(\mathbb{R}) \cap C^0(\mathbb{R})$ uniformly and $E' = \psi(h)h^{-1}\Phi_h E$ in $\mathscr{D}'(\mathbb{R})$. We have

$$\begin{aligned} E(t)Q(t, hD_t)E^{-1}(t) &= Q(t, hD_t) + E(t)h[M(t)D_t, E^{-1}(t)]\operatorname{Id}_N \\ &= Q(t, hD_t) + i\psi(h)\Phi_h(t)\operatorname{Id}_N - \psi(h)\Phi_h(t)\operatorname{Im} M(t) \end{aligned} \tag{6.20}$$

since $(E^{-1})' = -E'E^{-2}$. In the following, we let

$$F(t) = K(t) + \psi(h)\operatorname{Id}_N \ge \psi(h)\operatorname{Id}_N.$$

By definition we have $\Phi_h(t) < 1 \implies K(t) \ge \psi(h)\operatorname{Id}_N$, so

$$K(t) + \psi(h)\Phi_h(t)\operatorname{Id}_N \ge \tfrac{1}{2}F(t).$$

Thus by taking the inner product in $L^2(\mathbb{R})$ we find from (6.20) that

$$\operatorname{Im}\langle E(t)Q(t, hD_t)E^{-1}(t)u, u\rangle \ge \tfrac{1}{2}\langle F(t)u, u\rangle + \langle \operatorname{Im} M(t)hD_t u, u\rangle - ch\|u\|^2 \qquad u \in C_0^\infty(\mathbb{R}, \mathbb{C}^N)$$

since $\operatorname{Im} Q(t, hD_t) = K(t) + \operatorname{Im} M(t)hD_t + \frac{h}{2i}\partial_t \operatorname{Im} M(t)$. Now we may use (6.10) to estimate for any $\varepsilon > 0$

$$|\langle \operatorname{Im} MhD_t u, u\rangle| \le \varepsilon\langle Ku, u\rangle + C_\varepsilon(h^2\|D_t u\|^2 + h\|u\|^2) \qquad \forall u \in C_0^\infty(\mathbb{R}, \mathbb{C}^N). \tag{6.21}$$

In fact, $u = \chi_0(hD_t)u + \chi_1(hD_t)u$ where $\chi_0(r) \in C_0^\infty(\mathbb{R})$ and $|r| \ge 1$ in $\operatorname{supp}\chi_1$. We obtain from (6.10) for any $\varepsilon > 0$ that

$$|\langle \operatorname{Im} M(t)\chi_0(h\tau)h\tau u, u\rangle| \le C\langle K(t)u, u\rangle^{1/2}|\chi_0(h\tau)h\tau|\|u\| \le \varepsilon\langle K(t)u, u\rangle + C_\varepsilon\|\chi_0(h\tau)h\tau u\|^2$$

so by using Gårdings inequality in Proposition A.5 on

$$\varepsilon K(t) + C_\varepsilon\chi_0^2(hD_t)h^2D_t^2 \pm \operatorname{Im} M(t)\chi_0(hD_t)hD_t$$

we obtain

$$|\langle \operatorname{Im} M(t)\chi_0(hD_t)hD_t u, u\rangle| \le \varepsilon\langle K(t)u, u\rangle + C_\varepsilon h^2\|D_t u\|^2 + C_0 h\|u\|^2 \quad \forall u \in C_0^\infty(\mathbb{R}, \mathbb{C}^N)$$

since $\|\chi_0(hD_t)hD_t u\| \le C\|hD_t u\|$. The other term is easier to estimate:

$$|\langle \operatorname{Im} M(t)\chi_1(hD_t)hD_t u, u\rangle| \le C\|hD_t u\|\|\chi_1(hD_t)u\| \le C_1h^2\|D_t u\|^2$$

since $|\chi_1(h\tau)| \le C|h\tau|$. By taking $\varepsilon = 1/6$ in (6.21) we obtain

$$\langle F(t)u, u\rangle \le 3\operatorname{Im}\langle E(t)Q(t, hD_t)E^{-1}(t)u, u\rangle + C(h^2\|D_t u\|^2 + h\|u\|^2).$$

Now $hD_tEu = EhD_tu - i\psi(h)\Phi_h Eu$ so we find by substituting $E(t)u$ that

$$\psi(h)\|E(t)u\|^2 + \langle KE(t)u, E(t)u\rangle \le 3\,\mathrm{Im}\langle Q(t,hD_t)u, E^2(t)u\rangle + C(h^2\|D_tu\|^2 + h\|u\|^2 + \psi^2(h)\|E(t)u\|^2)$$

for $u \in C_0^\infty(\mathbb{R}, \mathbb{C}^N)$. Since $E \ge c$, $K \ge 0$ and $h \ll \psi(h) \ll 1$ when $h \to 0$ we obtain (6.19) with scalar $B = \varrho E^2$ for $\varrho \gg 1$ and $h \ll 1$. □

To finish the proof of Proposition 6.2, we substitute $\phi_j^w u$ in the estimate (6.19) with $Q = Q_j$ to obtain

$$\psi(h)\|\phi_j^w u\|^2 + \langle K_j\phi_j^w u, \phi_j^w u\rangle \le \mathrm{Im}\langle \phi_j^w Q_j(t,hD_t)u, B_j(t)\phi_j^w u\rangle + Ch^2\|\phi_j^w D_tu\|^2 \tag{6.22}$$

for $u \in C_0^\infty(\mathbb{R}^{n+1}, \mathbb{C}^N)$, since $\phi_j^w(x, D_x)$ and $Q_j(t,hD_t)$ commute. Next, we shall replace the approximation Q_j by the original operator Q. In a G neighborhood of $\operatorname{supp}\phi_j$ we may use the Taylor expansion in w to write for almost all t

$$Q(t,\tau,h^{1/2}w) - Q_j(t,\tau) = i(K(t,h^{1/2}w) - K_j(t)) + (M(t,h^{1/2}w) - M_j(t))\tau. \tag{6.23}$$

We shall start by estimating the last term in (6.23). Since $M(t,w) \in C_{\mathrm{b}}^\infty$ we have

$$|M(t,h^{1/2}w) - M_j(t)| \le Ch^{1/2}H^{-1/2} \qquad \text{in } \operatorname{supp}\phi_j \tag{6.24}$$

because then $|w - w_j| \le cH^{-1/2}$. Since $M(t,h^{1/2}w) \in S(1, h|dw|^2)$ and $h \ll H$ we find from (6.24) that $M(t,h^{1/2}w) - M_j(t) \in S(h^{1/2}H^{-1/2}, G)$ in $\operatorname{supp}\phi_j$ uniformly in t. By the Cauchy–Schwarz inequality we find

$$|\langle \phi_j^w(M^w - M_j)hD_tu, B_j(t)\phi_j^w u\rangle| \le C(\|\chi_j^w hD_tu\|^2 + hH^{-1}\|\phi_j^w u\|^2) \tag{6.25}$$

for $u \in C_0^\infty(\mathbb{R}^{n+1}, \mathbb{C}^N)$ where $\chi_j^w = h^{-1/2}H^{1/2}\phi_j^w(M^w - M_j) \in \operatorname{Op} S(1, G)$ uniformly in t with values in ℓ^2. Thus we find from (6.16) that

$$\sum_j \|\chi_j^w hD_tu\|^2 \le C\|hD_tu\|^2 \qquad u \in C_0^\infty(\mathbb{R}^{n+1})$$

and for the last terms in (6.25) we have

$$hH^{-1}\sum_j \|\phi_j^w u\|^2 \le 2hH^{-1}\|u\|^2 \ll \psi(h)\|u\|^2 \qquad h \to 0 \qquad u \in C_0^\infty(\mathbb{R}^{n+1})$$

by (6.15). For the first term in the right hand side of (6.23) we find from Taylor's formula

$$K(t,h^{1/2}w) - K_j(t) = h^{1/2}\langle S_j(t), W_j(w)\rangle + R_j(t,\tau,w) \qquad \text{in } \operatorname{supp}\phi_j$$

where $S_j(t) = \partial_w K(t,h^{1/2}w_j) \in L^\infty(\mathbb{R})$, $R_j \in S(hH^{-1}, G)$ uniformly for almost all t and $W_j \in S(h^{-1/2}, h|dw|^2)$ such that $\phi_j(w)W_j(w) = \phi_j(w)(w - w_j) = \mathcal{O}(H^{-1/2})$. Here we could take $W_j(w) = \chi(h^{1/2}(w - w_j))(w - w_j)$ for a suitable cut-off function $\chi \in C_0^\infty$. We obtain from the calculus that

$$\phi_j^w K_j(t) = \phi_j^w K^w(t, h^{1/2}x, h^{1/2}D_x) - h^{1/2}\phi_j^w\langle S_j(t), W_j^w\rangle + \widetilde{R}_j^w,$$

where $\{\widetilde{R}_j\}_j \in S(hH^{-1}, G)$ with values in ℓ^2 for almost all t. Thus we may estimate the sum of these error terms by (6.16) to obtain

$$\sum_j |\langle \widetilde{R}_j^w u, B_j(t)\phi_j^w u\rangle| \le ChH^{-1}\|u\|^2 \ll \psi(h)\|u\|^2 \quad \text{as } h \to 0 \quad \text{for } u \in C_0^\infty(\mathbb{R}^{n+1}, \mathbb{C}^N). \tag{6.26}$$

Observe that it follows from (6.17) for any $\kappa > 0$ and almost all t that

$$|\langle S_j(t)u, u\rangle| \le C\langle K_j(t)u, u\rangle^{1/2}\|u\| \le \kappa\langle K_j(t)u, u\rangle + C\|u\|^2/\kappa \qquad \forall u \in \mathbb{C}^N.$$

Let $F_j(t) = F(t, h^{1/2}w_j) = K_j(t) + \psi(h)\,\mathrm{Id}_N$; then by taking $\kappa = \varrho H^{1/2}h^{-1/2}$ we find that for any $\varrho > 0$ there exists $h_\varrho > 0$ so that

$$h^{1/2}H^{-1/2}|\langle S_j u, u\rangle| \le \varrho\langle K_j u, u\rangle + ChH^{-1}\|u\|^2/\varrho \le \varrho\langle F_j u, u\rangle \quad \forall u \in \mathbb{C}^N \quad 0 < h \le h_\varrho \tag{6.27}$$

since $hH^{-1} \ll \psi(h)$ when $h \ll 1$. Now F_j and S_j only depend on t, so by (6.27) we may use Remark A.6 in the Appendix for fixed t with $A = h^{1/2}H^{-1/2}S_j$, $B = \varrho F_j$, u replaced with $\phi_j^w u$ and v with $B_j H^{1/2}\phi_j^w W_j^w u$. Integration then gives

$$h^{1/2}|\langle B_j\phi_j^w\langle S_j(t), W_j^w\rangle u, \phi_j^w u\rangle| \le \frac{3\varrho}{2}(\langle F_j(t)\phi_j^w u, \phi_j^w u\rangle + \langle F_j(t)\psi_j^w u, \psi_j^w u\rangle) \tag{6.28}$$

for $u \in C_0^\infty(\mathbb{R}^{n+1}, \mathbb{C}^N)$, $0 < h \le h_\varrho$, where

$$\psi_j^w = B_j H^{1/2}\phi_j^w W_j^w \in \operatorname{Op} S(1, G) \qquad \text{with values in } \ell^2.$$

In fact, since $\phi_j \in S(1, G)$ and $W_j \in S(h^{-1/2}, h|dw|^2)$ we find that

$$\phi_j^w W_j^w = (\phi_j W_j)^w \text{ modulo } \operatorname{Op} S(H^{1/2}, G).$$

Also, since $|\phi_j W_j| \le CH^{-1/2}$ we find from Leibniz' rule that $\phi_j W_j \in S(H^{-1/2}, G)$. Now $F \ge \psi(h)\,\mathrm{Id}_N \gg hH^{-1}\,\mathrm{Id}_N$ so by using Proposition A.9 in the Appendix and then integrating in t we find that

$$\sum_j \langle F_j(t)\psi_j^w u, \psi_j^w u\rangle \le C\sum_j \langle F_j(t)\phi_j^w u, \phi_j^w u\rangle \qquad u \in C_0^\infty(\mathbb{R}^{n+1}, \mathbb{C}^N).$$

We obtain from (6.15) that

$$\psi(h)\|u\|^2 \le 2\sum_j \langle F_j(t)\phi_j^w u, \phi_j^w u\rangle \qquad u \in C_0^\infty(\mathbb{R}^{n+1}, \mathbb{C}^N).$$

Thus, for any $\varrho > 0$ we obtain from (6.22) and (6.25)–(6.28) that

$$(1 - C_0\varrho)\sum_j \langle F_j(t)\phi_j^w u, \phi_j^w u\rangle \le \sum_j \operatorname{Im}\langle \phi_j^w Q u, B_j(t)\phi_j^w u\rangle + C_\varrho h^2\|D_t u\|^2 \qquad 0 < h \le h_\varrho.$$

We have that $\sum_j B_j\phi_j^w\phi_j^w \in S(1, G)$ is a scalar symmetric operator uniformly in t. When $\varrho = 1/2C_0$ we obtain the estimate (6.11) with $B^w = 4\sum_j B_j\phi_j^w\phi_j^w$, which finishes the proof of Proposition 6.2. □

Appendix

We shall first study the condition for the one-dimensional model operator

$$hD_t \operatorname{Id}_N + iF(t) \qquad 0 \le F(t) \in C^\infty(\mathbb{R})$$

to be of finite type of order μ:

$$|\Omega_\delta(F)| \le C\delta^\mu \qquad 0 < \delta \ll 1 \tag{A.1}$$

and we shall assume that $0 \notin \Sigma_\infty(P)$. When $F(t) \notin C^\infty(\mathbb{R})$ we may have any $\mu > 0$ in (A.1), for example with $F(t) = |t|^{1/\mu} \operatorname{Id}_N$. But when $F \in C_b^1$ the estimate cannot hold with $\mu > 1$, and since it trivially holds for $\mu = 0$ the only interesting cases are $0 < \mu \le 1$.

When $0 \le F(t)$ is diagonalizable for any t with eigenvalues $\lambda_j(t) \in C^\infty$, $j = 1, \dots, N$, then condition (A.1) is equivalent to

$$\left|\Omega_\delta(\lambda_j)\right| \le C\delta^\mu \qquad \forall j \quad 0 < \delta \ll 1$$

since $\Omega_\delta(F) = \bigcup_j \Omega_\delta(\lambda_j)$. Thus we shall start by studying the scalar case.

Proposition A.1. *Assume that $0 \le f(t) \in C^\infty(\mathbb{R})$ such that $f(t) \ge c > 0$ when $|t| \gg 1$, that is, $0 \notin \Sigma_\infty(f)$. We find that f satisfies (A.1) with $\mu > 0$ if and only if $\mu \le 1/k$ for an even $k \ge 0$ so that*

$$\sum_{j \le k} |\partial_t^j f(t)| > 0 \qquad \forall t. \tag{A.2}$$

Simple examples as $f(t) = e^{-t^2}$ show that the condition that $0 \notin \Sigma_\infty(f)$ is necessary for the conclusion of Proposition A.1.

Proof. Assume that (A.2) does not hold with $k \le 1/\mu$; then there exists t_0 such that $f^{(j)}(t_0) = 0$ for all integer $j \le 1/\mu$. Then Taylor's formula gives that $f(t) \le c|t - t_0|^k$ and $|\Omega_\delta(f)| \ge c\delta^{1/k}$ where $k = [1/\mu] + 1 > 1/\mu$, which contradicts condition (A.1).

Assume now that condition (A.2) holds for some k, then $f^{-1}(0)$ consists of finitely many points. In fact, else there would exist t_0 where f vanishes of infinite order since $f(t) \ne 0$ when $|t| \gg 1$. Also note that $\bigcap_{\delta > 0} \Omega_\delta(f) = f^{-1}(0)$, in fact f must have a positive infimum outside any neighborhood of $f^{-1}(0)$. Thus, in order to estimate $|\Omega_\delta(f)|$ for $\delta \ll 1$ we only have to consider a small neighborhood ω of $t_0 \in f^{-1}(0)$. Assume that

$$f(t_0) = f'(t_0) = \cdots = f^{(j-1)}(t_0) = 0 \text{ and } f^{(j)}(t_0) \ne 0$$

for some $j \le k$. Since $f \ge 0$ we find that j must be even and $f^{(j)}(t_0) > 0$. Taylor's formula gives as before $f(t) \ge c|t - t_0|^j$ for $|t - t_0| \ll 1$ and thus we find that

$$|\Omega_\delta(f) \cap \omega| \le C\delta^{1/j} \le C\delta^{1/k} \qquad 0 < \delta \ll 1$$

if ω is a small neighborhood of t_0. Since $f^{-1}(0)$ consists of finitely many points we find that (A.1) is satisfied with $\mu = 1/k$ for an even k. $\square$

So if $0 \le F \in C^\infty(\mathbb{R})$ is C^∞ diagonalizable system and $0 \notin \Sigma_\infty(P)$, condition (A.1) is equivalent to

$$\sum_{j \le k} |\partial_t^j \langle F(t)u(t), u(t)\rangle| / \|u(t)\|^2 > 0 \qquad \forall t$$

for any $0 \neq u(t) \in C^\infty(\mathbb{R})$, since this holds for diagonal matrices and is invariant. This is true also in the general case by the following proposition.

Proposition A.2. *Assume that $0 \le F(t) \in C^\infty(\mathbb{R})$ is an $N \times N$ system such that $0 \notin \Sigma_\infty(F)$. We find that F satisfies (A.1) with $\mu > 0$ if and only if $\mu \le 1/k$ for an even $k \ge 0$ so that*

$$\sum_{j \le k} |\partial_t^j \langle F(t)u(t), u(t)\rangle| / \|u(t)\|^2 > 0 \qquad \forall t \tag{A.3}$$

for any $0 \neq u(t) \in C^\infty(\mathbb{R})$.

Observe that since $0 \notin \Sigma_\infty(F)$ it suffices to check condition (A.3) on a compact interval.

Proof. First we assume that (A.1) holds with $\mu > 0$, let $u(t) \in C^\infty(\mathbb{R}, \mathbb{C}^N)$ such that $|u(t)| \equiv 1$, and $f(t) = \langle F(t)u(t), u(t)\rangle \in C^\infty(\mathbb{R})$. Then we have $\Omega_\delta(f) \subset \Omega_\delta(F)$ so (A.1) gives

$$|\Omega_\delta(f)| \le |\Omega_\delta(F)| \le C\delta^\mu \qquad 0 < \delta \ll 1.$$

The first part of the proof of Proposition A.1 then gives (A.3) for some $k \le 1/\mu$.

For the proof of the sufficiency of (A.3) we need the following simple lemma.

Lemma A.3. *Assume that $F(t) = F^*(t) \in C^k(\mathbb{R})$ is an $N \times N$ system with eigenvalues $\lambda_j(t) \in \mathbb{R}$, $j = 1, \dots, N$. Then, for any $t_0 \in \mathbb{R}$, there exist analytic $v_j(t) \in \mathbb{C}^N$, $j = 1, \dots, N$, so that $\{v_j(t_0)\}$ is a base for $\mathbb{C}^N$ and*

$$\left|\lambda_j(t) - \langle F(t)v_j(t), v_j(t)\rangle\right| \le C|t - t_0|^k \qquad \text{for } |t - t_0| \le 1$$

after a renumbering of the eigenvalues.

By a well-known theorem of Rellich, the eigenvalues $\lambda(t) \in C^1(\mathbb{R})$ for symmetric $F(t) \in C^1(\mathbb{R})$; see [Kato 1966, Theorem II.6.8].

Proof. It is no restriction to assume $t_0 = 0$. By Taylor's formula

$$F(t) = F_k(t) + R_k(t)$$

where F_k and R_k are symmetric, $F_k(t)$ is a polynomial of degree $k-1$ and $R_k(t) = \mathbb{O}(|t|^k)$. Since $F_k(t)$ is symmetric and holomorphic, it has a base of normalized holomorphic eigenvectors $v_j(t)$ with real holomorphic eigenvalues $\widetilde{\lambda}_j(t)$ by [Kato 1966, Theorem II.6.1]. Thus $\widetilde{\lambda}_j(t) = \langle F_k(t)v_j(t), v_j(t)\rangle$ and by the minimax principle we may renumber the eigenvalues so that

$$|\lambda_j(t) - \widetilde{\lambda}_j(t)| \le \|R_k(t)\| \le C|t|^k \qquad \forall j.$$

The result then follows since

$$|\langle (F(t) - F_k(t))v_j(t), v_j(t)\rangle| = |\langle R_k(t)v_j(t), v_j(t)\rangle| \le C|t|^k \qquad \forall j. \qquad \square$$

Assume now that Equation (A.3) holds for some k. As in the scalar case, we have that k is even and $\bigcap_{\delta>0} \Omega_\delta(F) = \Sigma_0(F) = |F|^{-1}(0)$. Thus, for small δ we only have to consider a small neighborhood of $t_0 \in \Sigma_0(F)$. Then by using Lemma A.3 we have after renumbering that for each eigenvalue $\lambda(t)$ of $F(t)$ there exists $v(t) \in C^\infty$ so that $|v(t)| \ge c > 0$ and

$$|\lambda(t) - \langle F(t)v(t), v(t)\rangle| \le C|t - t_0|^{k+1} \qquad \text{when } |t - t_0| \le c. \tag{A.4}$$

Now if $\Sigma_0(F) \ni t_j \to t_0$ is an accumulation point, then after choosing a subsequence we obtain that for some eigenvalue λ_k we have $\lambda_k(t_j) = 0$ for all j. Then λ_k vanishes of infinite order at t_0, contradicting (A.3) by (A.4). Thus, we find that $\Sigma_0(F)$ is a finite collection of points. By using (A.3) with $u(t) = v(t)$ we find as in the second part of the proof of Proposition A.1 that

$$\langle F(t)v(t), v(t)\rangle \geq c|t-t_0|^j \qquad |t-t_0| \ll 1$$

for some even $j \leq k$, which by (A.4) gives that

$$\lambda(t) \geq c|t-t_0|^j - C|t-t_0|^{k+1} \geq c'|t-t_0|^j \qquad |t-t_0| \ll 1.$$

Thus $|\Omega_\delta(\lambda) \cap \omega| \leq c\delta^{1/j}$ if ω for $\delta \ll 1$ if ω is a small neighborhood of $t_0 \in \Sigma_0(F)$. Since $\Omega_\delta(F) = \bigcup_j \Omega_\delta(\lambda_j)$, where $\{\lambda_j(t)\}_j$ are the eigenvalues of $F(t)$, we find by adding up that $|\Omega_\delta(F)| \leq C\delta^{1/k}$. Thus the largest μ satisfying (A.1) must be $\geq 1/k$. $\square$

Let $A(t) \in \mathrm{Lip}(\mathbb{R}, \mathscr{L}(L^2(\mathbb{R}^n)))$ be the $L^2(\mathbb{R}^n)$ bounded operators which are Lipschitz continuous in the parameter $t \in \mathbb{R}$. This means that

$$\frac{A(s)-A(t)}{s-t} = B(s,t) \in \mathscr{L}(L^2(\mathbb{R}^n)) \qquad \text{uniformly in } s \text{ and } t. \tag{A.5}$$

One example is $A(t) = a^w(t,x,D_x)$ where $a(t,x,\xi) \in \mathrm{Lip}(\mathbb{R}, S(1,G))$ for a σ temperate metric G which is constant in t such that $G/G^\sigma \leq 1$.

Lemma A.4. *Assume that* $A(t) \in \mathrm{Lip}(\mathbb{R}, \mathscr{L}(L^2(\mathbb{R}^n)))$ *and* $\phi(\tau) \in C^\infty(\mathbb{R})$ *such that* $\phi'(\tau) \in C_0^\infty(\mathbb{R})$. *Then for* $\kappa > 0$ *we can estimate the commutator*

$$\big\|\big[\phi(D_t/\kappa), A(t)\big]\big\|_{\mathscr{L}(L^2(\mathbb{R}^{n+1}))} \leq C\kappa^{-1},$$

where the constant only depends on ϕ *and the bound on* $A(t)$ *in* $\mathrm{Lip}(\mathbb{R}, \mathscr{L}(L^2(\mathbb{R}^n)))$.

Proof. In the following, we shall denote by $A(t,x,y)$ the distribution kernel of $A(t)$. Then we find from (A.5) that

$$A(s,x,y) - A(t,x,y) = (s-t)B(s,t,x,y), \tag{A.6}$$

where $B(s,t,x,y)$ is the kernel for $B(s,t)$ for $s,t \in \mathbb{R}$. Then

$$\begin{aligned}&\big\langle\big[\phi(D_t/\kappa), A(t)\big]u, v\big\rangle \\ &\qquad = (2\pi)^{-1}\int e^{i(t-s)\tau}\phi(\tau/\kappa)(A(s,x,y)-A(t,x,y))u(s,x)\overline{v(t,y)}\,d\tau\,ds\,dt\,dx\,dy\end{aligned} \tag{A.7}$$

for $u, v \in C_0^\infty(\mathbb{R}^{n+1})$, and by using (A.6) we obtain that the commutator has kernel

$$\frac{1}{2\pi}\int e^{i(t-s)\tau}\phi(\tau/\kappa)(s-t)B(s,t,x,y)\,d\tau = \frac{1}{\kappa}\int e^{i(t-s)\tau}\rho(\tau/\kappa)B(s,t,x,y)\,d\tau = \widehat{\rho}(\kappa(s-t))B(s,t,x,y)$$

in $\mathscr{D}(\mathbb{R}^{2n+2})$, where $\rho \in C_0^\infty(\mathbb{R})$. Thus, we may estimate (A.7) by using Cauchy–Schwarz:

$$\int |\widehat{\varrho}(\kappa s)\langle B(s+t,t)u(s+t), v(t)\rangle_{L^2(\mathbb{R}^n)}|\,dt\,ds \leq C\kappa^{-1}\|u\|\|v\|$$

where the norms are in $\mathscr{L}(L^2(\mathbb{R}^{n+1}))$. $\square$

We shall need some results about the lower bounds of systems, and we shall use the following version of the Gårding inequality for systems. A convenient way for proving the inequality is to use the Wick quantization of $a \in L^\infty(T^*\mathbb{R}^n)$ given by

$$a^{\mathrm{Wick}}(x, D_x)u(x) = \int_{T^*\mathbb{R}^n} a(y,\eta)\Sigma^w_{y,\eta}(x, D_x)u(x)\,dy\,d\eta \qquad u \in \mathscr{S}(\mathbb{R}^n)$$

using the rank one orthogonal projections $\Sigma^w_{y,\eta}(x, D_x)$ in $L^2(\mathbb{R}^n)$ with Weyl symbol

$$\Sigma_{y,\eta}(x,\xi) = \pi^{-n}\exp\left(-|x-y|^2 - |\xi-\eta|^2\right)$$

(see [Dencker 1999, Appendix B] or [Lerner 1997, Section 4]). We find that $a^{\mathrm{Wick}}\colon \mathscr{S}(\mathbb{R}^n) \mapsto \mathscr{S}'(\mathbb{R}^n)$ is symmetric on $\mathscr{S}(\mathbb{R}^n)$ if a is real-valued,

$$a \ge 0 \implies \left(a^{\mathrm{Wick}}(x, D_x)u, u\right) \ge 0 \qquad u \in \mathscr{S}(\mathbb{R}^n), \tag{A.8}$$
$$\|a^{\mathrm{Wick}}(x, D_x)\|_{\mathscr{L}(L^2(\mathbb{R}^n))} \le \|a\|_{L^\infty(T^*\mathbb{R}^n)},$$

which is the main advantage with the Wick quantization. If $a \in S(1, h|dw|^2)$ we find that

$$a^{\mathrm{Wick}} = a^w + r^w \tag{A.9}$$

where $r \in S(h, h|dw|^2)$. For a reference; see [Lerner 1997, Proposition 4.2].

Proposition A.5. *Let $0 \le A \in C^\infty_{\mathrm{b}}(T^*\mathbb{R}^n)$ be an $N \times N$ system, then we find that*

$$\langle A^w(x, hD)u, u\rangle \ge -Ch\|u\|^2 \qquad \forall\, u \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^N).$$

This result is well known (see for example Theorem 18.6.14 in Volume III of [Hörmander 1983–1985]) but we shall give a short and direct proof.

Proof. By making a L^2 preserving linear symplectic change of coordinates: $(x,\xi) \mapsto (h^{1/2}x, h^{-1/2}\xi)$ we may assume that $0 \le A \in S(1, h|dw|^2)$. Then we find from (A.9) that $A^w = A^{\mathrm{Wick}} + R^w$ where $R \in S(h, h|dw|^2)$. Since $A \ge 0$ we obtain from (A.8) that

$$\langle A^w u, u\rangle \ge \langle R^w u, u\rangle \ge -Ch\|u\|^2 \qquad \forall\, u \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^N). \qquad \square$$

Remark A.6. Assume that A and B are $N \times N$ matrices such that $\pm A \le B$. Then we find

$$|\langle Au, v\rangle| \le \tfrac{3}{2}\left(\langle Bu, u\rangle + \langle Bv, v\rangle\right) \qquad \forall\, u,\ v \in \mathbb{C}^n.$$

In fact, since $B \pm A \ge 0$ we find by the Cauchy–Schwarz inequality that

$$2\,|\langle (B \pm A)u, v\rangle| \le \langle (B \pm A)u, u\rangle + \langle (B \pm A)v, v\rangle \qquad \forall\, u,\ v \in \mathbb{C}^n$$

and $2\,|\langle Bu, v\rangle| \le \langle Bu, u\rangle + \langle Bv, v\rangle$. The estimate can then be expanded to give the inequality, since

$$|\langle Au, u\rangle| \le \langle Bu, u\rangle \qquad \forall\, u \in \mathbb{C}^n$$

by the assumption.

Lemma A.7. *Assume that $0 \le F(t) \in C^2(\mathbb{R})$ is an $N \times N$ system such that $F'' \in L^\infty(\mathbb{R})$. Then we have*

$$|\langle F'(0)u, u\rangle|^2 \le C\|F''\|_{L^\infty}\langle F(0)u, u\rangle\|u\|^2 \qquad \forall\, u \in \mathbb{C}^N.$$

Proof. Take $u \in \mathbb{C}^N$ with $|u| = 1$ and let $0 \le f(t) = \langle F(t)u, u\rangle \in C^2(\mathbb{R})$. Then $|f''| \le \|F''\|_{L^\infty}$ so Lemma 7.7.2 in Volume I of [Hörmander 1983–1985] gives

$$|f'(0)|^2 = |\langle F'(0)u, u\rangle|^2 \le C\|F''\|_{L^\infty} f(0) = C\|F''\|_{L^\infty}\langle F(0)u, u\rangle. \qquad \square$$

Lemma A.8. *Assume that $F \ge 0$ is an $N \times N$ matrix and that A is a L^2 bounded scalar operator. Then*

$$0 \le \langle FAu, Au\rangle \le \|A\|^2\langle Fu, u\rangle$$

for any $u \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^N)$.

Proof. Since $F \ge 0$ we can choose an orthonormal base for $\mathbb{C}^N$ such that $\langle Fu, u\rangle = \sum_{j=1}^N f_j|u_j|^2$ for $u = (u_1, u_2, \dots) \in \mathbb{C}^N$, where $f_j \ge 0$ are the eigenvalues of F. In this base we find

$$0 \le \langle FAu, Au\rangle = \sum_j f_j\|Au_j\|^2 \le \|A\|^2 \sum_j f_j\|u_j\|^2 = \|A\|^2\langle Fu, u\rangle$$

for $u \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^N)$. $\square$

Proposition A.9. *Assume that $h/H \le F \in S(1, g)$ is an $N \times N$ system, $\{\phi_j\}$ and $\{\psi_j\} \in S(1, G)$ with values in ℓ^2 such that $\sum_j |\phi_j|^2 \ge c > 0$ and ψ_j is supported in a fixed G neighborhood of $w_j \in \operatorname{supp}\phi_j$ for all j. Here $g = h|dw|^2$ and $G = H|dw|^2$ are constant metrics, $0 < h \le H \le 1$. If $F_j = F(w_j)$ we find for $H \ll 1$ that*

$$\sum_j \langle F_j\psi_j^w(x, D_x)u, \psi_j^w(x, D_x)u\rangle \le C\sum_j \langle F_j\phi_j^w(x, D_x)u, \phi_j^w(x, D_x)u\rangle \tag{A.10}$$

for any $u \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^N)$.

Proof. Since $\chi = \sum_j |\phi_j|^2 \ge c > 0$ we find that $\chi^{-1} \in S(1, G)$. The calculus gives

$$(\chi^{-1})^w \sum_j \overline{\phi}_j^w\phi_j^w = 1 + r^w$$

where $r \in S(H, G)$ uniformly in H. Thus, the mapping $u \mapsto (\chi^{-1})^w \sum_j \overline{\phi}_j^w\phi_j^w u$ is a homeomorphism on $L^2(\mathbb{R}^n)$ for small enough H. Now the constant metric $G = H|dw|^2$ is trivially *strongly σ temperate* according to Definition 7.1 in [Bony and Chemin 1994], so Theorem 7.6 in the same reference gives $B \in S(1, G)$ such that

$$B^w(\chi^{-1})^w \sum_j \overline{\phi}_j^w\phi_j^w = \sum_j B_j^w\phi_j^w = 1$$

where $B_j^w = B^w(\chi^{-1})^w\overline{\phi}_j^w \in \operatorname{Op} S(1, G)$ uniformly, which gives $1 = \sum_j \overline{\phi}_j^w\overline{B}_j^w$ since $(B_j^w)^* = \overline{B}_j^w$. Now we shall put

$$\widetilde{\mathcal{F}}^w(x, D_x) = \sum_j \overline{\psi}_j^w(x, D_x)F_j\psi_j^w(x, D_x).$$

Then

$$\widetilde{\mathcal{F}}^w = \sum_{jk} \overline{\phi}_j^w\overline{B}_j^w\widetilde{\mathcal{F}}^w B_k^w\phi_k^w = \sum_{jkl} \overline{\phi}_j^w\overline{B}_j^w\overline{\psi}_l^w F_l\psi_l^w B_k^w\phi_k^w. \tag{A.11}$$

Let $C^w_{jkl} = \bar{B}^w_j \Psi^w_l \psi^w_l B^w_k$; then we find from (A.11) that

$$\langle \widetilde{\mathcal{F}}^w u, u\rangle = \sum_{jkl} \langle F_l C^w_{jkl} \phi^w_k u, \phi^w_j u\rangle.$$

Let d_{jk} be the $H^{-1}|dw|^2$ distance between the G neighborhoods in which ψ_j and ψ_k are supported. The usual calculus estimates (see [Hörmander 1983–1985, Volume III, page 168] or [Bony and Chemin 1994, Theorem 2.6]) gives that the L^2 operator norm of C^w_{jkl} can be estimated by

$$\|C^w_{jkl}\| \le C_N(1 + d_{jl} + d_{lk})^{-N}$$

for any N. We find by Taylor's formula, Lemma A.7 and the Cauchy–Schwarz inequality that

$$|\langle (F_j - F_k)u, u\rangle| \le C_1|w_j - w_k|\langle F_k u, u\rangle^{1/2} h^{1/2}\|u\| + C_2 h|w_j - w_k|^2\|u\|^2 \le C\langle F_k u, u\rangle(1 + d_{jk})^2$$

since $|w_j - w_k| \le C(d_{jk} + H^{-1/2})$ and $h \le hH^{-1} \le F_k$. Since $F_l \ge 0$ we obtain that

$$2|\langle F_l u, v\rangle| \le \langle F_l u, u\rangle^{1/2}\langle F_l v, v\rangle^{1/2} \le C\langle F_j u, u\rangle^{1/2}\langle F_k v, v\rangle^{1/2}(1 + d_{jl})(1 + d_{lk})$$

and Lemma A.8 gives

$$\langle F_k C^w_{jkl}\phi^w_k u, F_k C^w_{jkl}\phi^w_k u\rangle \le \|C_{jkl}\|^2\langle F_k \phi^w_k u, \phi^w_k u\rangle.$$

Thus we find that

$$\begin{aligned}\sum_{jkl}\langle F_l C^w_{jkl}\phi^w_k u, \phi^w_j u\rangle &\le C_N \sum_{jkl}(1 + d_{jl} + d_{lk})^{2-N}\langle F_k \phi^w_k u, \phi^w_k u\rangle^{1/2}\langle F_j \phi^w_j u, \phi^w_j u\rangle^{1/2} \\ &\le C_N \sum_{jkl}(1 + d_{jl})^{1-N/2}(1 + d_{lk})^{1-N/2}\big(\langle F_j \phi^w_j u, \phi^w_j u\rangle + \langle F_k \phi^w_k u, \phi^w_k u\rangle\big)\end{aligned}$$

Since $\sum_j (1 + d_{jk})^{-N} \le C$ for all k for N large enough by [Hörmander 1983–1985, Volume III, page 168]), we obtain the estimate (A.10) and the result. $\square$

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AN IMPROVED LOWER BOUND ON THE SIZE OF KAKEYA SETS OVER FINITE FIELDS

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In a recent breakthrough, Dvir showed that every Kakeya set in $\mathbb{F}^n$ must have cardinality at least $c_n|\mathbb{F}|^n$, where $c_n \approx 1/n!$. We improve this lower bound to $\beta^n|\mathbb{F}|^n$ for a constant $\beta > 0$. This pins down the correct growth of the constant c_n as a function of n (up to the determination of β).

Let $\mathbb{F}$ be a finite field with q elements. A set $K \subseteq \mathbb{F}^n$ is said to be a *Kakeya set* in $\mathbb{F}^n$ if, for every $\boldsymbol{b} \in \mathbb{F}^n$, there exists a point $\boldsymbol{a} \in \mathbb{F}^n$ such that, for every $t \in \mathbb{F}$, the point $\boldsymbol{a} + t\boldsymbol{b}$ lies in K. In other words, K contains an affine line in every direction.

The question of establishing lower bounds on the size of Kakeya sets was posed in [Wolff 1999]. Till recently, the best known lower bound on the size of Kakeya sets was of the form $q^{\alpha n}$ for some $\alpha < 1$. In a recent breakthrough Dvir [2008] showed that every Kakeya set must have cardinality at least $c_n q^n$ for $c_n = (n!)^{-1}$. (Dvir originally achieved a weaker lower bound of $c_n q^{n-1}$, but the paper cited includes the stronger bound of $c_n q^n$, the improvements being attributed to Alon and Tao.) We show:

Theorem 1. *There exist constants $c_0, c_1 > 0$ such that for all n, if K is a Kakeya set in $\mathbb{F}^n$ then $|K| \geq c_0(c_1 q)^n$.*

Remark. Our proofs give some tradeoffs on the constants c_0, c_1 that are achievable. We comment on the constants at the end of the paper.

Our improvement shows that c_n remains bounded from below by β^n for some fixed $\beta > 0$. While this improvement in the lower bound on the size of Kakeya sets is quantitatively small (say, compared to the improvement of Alon and Tao over Dvir's original bound), it is qualitatively significant in that it does determine the growth of the leading constant c_n, up to the determination of the right constant β. In particular, it compares well with known upper bounds. Previously, it was known there exists a constant $\beta < 1$ such that there are Kakeya sets of cardinality at most $\beta^n q^n$, for every odd q. A bound of $\beta \leq 1/\sqrt{2}$ follows from [Mockenhaupt and Tao 2004] and the fact that products of Kakeya sets are Kakeya sets (in higher dimension). The best known constant has $\beta \to \frac{1}{2}$ due to Dvir (personal communication, 2008). We include his proof here (see Section 3), complementing it with a similar construction and bound for the case of even q as well (so now the upper bounds work for all large fields).

Our proof follows the one in [Dvir 2008]. Given a Kakeya set K in $\mathbb{F}^n$, we show that there exists an n-variate polynomial, whose degree is bounded from above by some function of $|K|$, that vanishes at all of K. Looking at restrictions of this polynomial to lines yields that this polynomial has too many zeroes, which in turn yields a lower bound on the size of K. Our main difference is that we look for polynomials that vanish with high multiplicity at each point in K. The requirement of high multiplicity forces the

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degree of the n-variate polynomial to go up slightly, but yields more zeroes when this polynomial is restricted to lines. The resulting tradeoff turns out to yield an improved bound. (We note that this is similar to the techniques used for the improved method of list decoding of Reed-Solomon codes created by Guruswami and Sudan [1999].)

In the next section we give the preliminaries that will be needed for the proof of Theorem 1. The actual proof of the theorem appears in Section 2. In Section 3, we give Dvir's proof for the upper bound for the size of Kakeya sets.

1. Preliminaries

For $\boldsymbol{x} = \langle x_1, \ldots, x_n \rangle$, let $\mathbb{F}[\boldsymbol{x}]$ denote the ring of polynomials in $x_1, \ldots, x_n$ with coefficients in $\mathbb{F}$. We recall the following basic fact on polynomials.

Fact 2. *Let $P \in \mathbb{F}[\boldsymbol{x}]$ be a polynomial of degree at most $q-1$ in each variable. If $P(\boldsymbol{a}) = 0$ for all $\boldsymbol{a} \in \mathbb{F}^n$, then $P \equiv 0$.*

For integer $m \geq 0$, let $N_q(n, m)$ denote the number of monomials in n variables of total degree less than mq and of individual degree at most $q-1$ in each variable.

We say that a polynomial $g \in \mathbb{F}[\boldsymbol{x}]$ has a zero of *multiplicity* m at a point $\boldsymbol{a} \in \mathbb{F}^n$ if the polynomial $g_{\boldsymbol{a}}(\boldsymbol{x}) = g(\boldsymbol{x} + \boldsymbol{a})$ has no support on monomials of degree strictly less than m. Note that the coefficients of $g_{\boldsymbol{a}}$ are (homogeneous) linear forms in the coefficients of g and thus the constraint g has a zero of multiplicity m at $\boldsymbol{a}$ yields $\binom{m+n-1}{n}$ homogeneous linear constraints on the coefficients of g. As a result we conclude:

Proposition 3. *Given a set $S \subseteq \mathbb{F}^n$ satisfying $\binom{m+n-1}{n} |S| < N_q(n, m)$, there exists a nonzero polynomial $g \in \mathbb{F}[\boldsymbol{x}]$ of total degree less than mq and degree at most $q-1$ in each variable such that g has a zero of multiplicity m at every point $\boldsymbol{a} \in S$.*

Proof. The number of possible coefficients for g is $N_q(n, m)$ and the number of (homogeneous) linear constraints is $\binom{m+n-1}{n} |S| < N_q(n, m)$. Since the number of constraints is strictly smaller than the number of unknowns, there is a nontrivial solution. $\square$

For $g \in \mathbb{F}[\boldsymbol{x}]$ we let $g_{\boldsymbol{a},\boldsymbol{b}}(t) = g(\boldsymbol{a} + t\boldsymbol{b})$ denote its restriction to the line $\{\boldsymbol{a} + t\boldsymbol{b} \mid t \in \mathbb{F}\}$. We note the following facts on the restrictions of polynomials to lines.

Proposition 4. *If $g \in \mathbb{F}[\boldsymbol{x}]$ has a root of multiplicity m at some point $\boldsymbol{a} + t_0\boldsymbol{b}$ then $g_{\boldsymbol{a},\boldsymbol{b}}$ has a root of multiplicity m at t_0.*

Proof. By definition, the fact that g has a zero of multiplicity m at $\boldsymbol{a} + t_0\boldsymbol{b}$ implies that the polynomial $g(\boldsymbol{x} + (\boldsymbol{a} + t_0\boldsymbol{b}))$ has no support on monomials of degree less than m. Thus, under the homogeneous substitution $\boldsymbol{x} \leftarrow t\boldsymbol{b}$, we get no monomials of degree less than m either, and thus we have t^m divides $g(t\boldsymbol{b} + (\boldsymbol{a} + t_0\boldsymbol{b})) = g(\boldsymbol{a} + (t + t_0)\boldsymbol{b}) = g_{\boldsymbol{a},\boldsymbol{b}}(t + t_0)$. The final form implies that $g_{\boldsymbol{a},\boldsymbol{b}}$ has a zero of multiplicity m at t_0. $\square$

Proposition 5 [Dvir 2008]. *Let $g \in \mathbb{F}[\boldsymbol{x}]$ be a nonzero polynomial of total degree d and let g_0 be the (unique, nonzero) homogeneous polynomial of degree d such that $g = g_0 + g_1$ for some polynomial g_1 of degree strictly less than d. Then $g_{\boldsymbol{a},\boldsymbol{b}}(t) = g_0(\boldsymbol{b})t^d + h(t)$ where h is a polynomial of degree strictly less than d.*

2. Proof of Theorem 1

Lemma 6. *If K is a Kakeya set in $\mathbb{F}^n$, then for every integer $m \geq 0$, $|K| \geq \binom{m+n-1}{n}^{-1} N_q(n, m)$.*

Proof. Assume for a contradiction that $|K| < \binom{m+n-1}{n}^{-1} N_q(n, m)$. Let $g \in \mathbb{F}[\boldsymbol{x}]$ be a nonzero polynomial of total degree less than mq and degree at most $q-1$ in each variable that has a zero of multiplicity m for each $x \in K$. (Such a polynomial exists by Proposition 3.) Let $d < mq$ denote the total degree of g and let $g = g_0 + g_1$ where g_0 is homogeneous of degree d and g_1 has degree less than d. Note that g_0 is also nonzero and has degree at most $q-1$ in every variable.

Now fix a direction $\boldsymbol{b} \in \mathbb{F}^n$. Since K is a Kakeya set, there exists $\boldsymbol{a} \in \mathbb{F}^n$ such that $\boldsymbol{a} + t\boldsymbol{b} \in K$ for every $t \in \mathbb{F}$. Now consider the restriction $g_{\boldsymbol{a},\boldsymbol{b}}$ of g to the line through $\boldsymbol{a}$ in direction $\boldsymbol{b}$; it is a univariate polynomial of degree at most $d < mq$. At every point $t_0 \in \mathbb{F}$ we have that $g_{\boldsymbol{a},\boldsymbol{b}}$ has a zero of multiplicity m (Proposition 4). Thus counting up the zeroes of $g_{\boldsymbol{a},\boldsymbol{b}}$ we find it has mq zeroes (m at every $t_0 \in \mathbb{F}$) which is more than its degree. Thus $g_{\boldsymbol{a},\boldsymbol{b}}$ must be identically zero. In particular its leading coefficient must be zero. By Proposition 5 this leading coefficient is $g_0(\boldsymbol{b})$ and so we conclude $g_0(\boldsymbol{b}) = 0$.

We conclude that g_0 is zero on all of $\mathbb{F}^n$ which contradicts the fact (Fact 2) that it is a nonzero polynomial of degree at most $q-1$ in each of its variables. □

Proof of Theorem 1. The theorem now follows by choosing m appropriately. Using for instance $m = n$, we obtain $|K| \geq \binom{2n-1}{n}^{-1} N_q(n, n)$. It easily follows by the definition of $N_q(n, m)$ that $N_q(n, n) = q^n$, since there are q choices for the individual degree of every variable in an n variate monomial, and this already forces total degree to be at most nq. Hence $|K| \geq \binom{2n-1}{n}^{-1} q^n \geq (q/4)^n$, establishing the theorem for $c_0 = 1$ and $c_1 = \frac{1}{4}$.

A better choice is with $m = \lceil n/2 \rceil \leq (n+1)/2$. In this case $N_q(n, m) \geq \frac{1}{2} q^n$ (since at least half the monomials of individual degree at most $q-1$ have degree at most $nq/2$). This leads to a bound of $|K| \geq \frac{1}{2}\binom{3n/2}{n}^{-1} q^n \geq \frac{1}{2}(q/2.6)^n$, yielding the theorem for $c_0 = 1/2$ and $c_1 = 1/2.6$. □

To improve the constant c_1 further, one could study the asymptotics of $N_q(n, m)$ closer. Let τ_α denote the quantity $\liminf_{n\to\infty}\{\liminf_{q\to\infty}(1/q)N_q(n, \alpha n)^{1/n}\}$. That is, for sufficiently large n and sufficiently larger q, $N_q(n, \alpha n) \to \tau_\alpha^n q^n$. Lemma 6 can be reinterpreted in these terms as saying that for every $\alpha \in [0, 1]$, every Kakeya set has size at least $c_0(c_\alpha q)^n - o(q^n)$ for some $c_0 > 0$, where $c_\alpha \to \tau_\alpha / 2^{(1+\alpha)H(1/(1+\alpha))}$ (where $H(x) = -x \log_2 x - (1-x)\log_2(1-x)$ is the binary entropy function). The best estimate on τ_α we were able to obtain does not have a simple closed form expression. As $q \to \infty$, τ_α^n equals the volume of the following region in $\mathbb{R}^n$: $\{(x_1, x_2, \ldots x_n) \in [0, 1]^n \mid \sum_{i=1}^n x_i \leq \alpha n\}$. This volume can be expressed in terms of Eulerian numbers (See [Marichal and Mossinghoff 2008], §4.3). [Giladi and Keller 1994, §6] gives some asymptotics for Eulerian numbers and using their estimates $\alpha = 0.398$, it seems one can reduce c_α to something like $\frac{1}{2.46}$. This still remains bounded away from the best known upper bound which has $c_1 \to 1/2$.

Remark. While the main theorem only gives the limiting behavior of Kakeya sets for large n and q, Lemma 6 can still be applied to specific choices and get improvements over [Dvir 2008]. For example, for $n = 3$, using $m = 2$ we get a lower bound of $\frac{5}{24}q^3$ as opposed to the bound of $\frac{1}{6}q^3$ obtainable from [Dvir 2008].

3. An upper bound on Kakeya sets

We include here Dvir's proof (personal communication, 2008) giving a nontrivial upper bound on the size of Kakeya sets in fields of odd characteristic. The proof is based on the construction of Mockenhaupt and Tao [2004]. For the case of even characteristic we complement their results by using a variation (obtained with Swastik Kopparty) of their construction.

Theorem 7 (Dvir). *For every $n \geq 2$, and field $\mathbb{F}$, there exists a Kakeya set in $\mathbb{F}^n$ of cardinality at most $2^{-(n-1)} q^n + O(q^{n-1})$.*

Proof. We consider two cases depending on whether $\mathbb{F}$ is of odd or even characteristic.
Odd characteristic: Let

$$D_n = \{\langle \alpha_1, \dots, \alpha_{n-1}, \beta\rangle | \alpha_i, \beta \in \mathbb{F}, \alpha_i + \beta^2 \text{is a square}\}.$$

Now let $K_n = D_n \cup (\mathbb{F}^{n-1} \times \{0\})$ where $\mathbb{F}^{n-1} \times \{0\}$ denotes the set $\{\langle \boldsymbol{a}, 0\rangle | \boldsymbol{a} \in \mathbb{F}^{n-1}\}$. We claim that K_n is a Kakeya set of the appropriate size.

Consider a direction $\boldsymbol{b} = \langle b_1, \dots, b_n\rangle$. If $b_n = 0$, for $\boldsymbol{a} = \langle 0, \dots, 0\rangle$ we have that $\boldsymbol{a} + t\boldsymbol{b} \in \mathbb{F}^{n-1} \times \{0\} \subseteq K_n$. The more interesting case is when $b_n \neq 0$. In this case let

$$\boldsymbol{a} = \langle (b_1/(2b_n))^2, \dots, (b_{n-1}/(2b_n))^2, 0\rangle.$$

The point $\boldsymbol{a} + t\boldsymbol{b}$ has coordinates $\langle \alpha_1, \dots, \alpha_{n-1}, \beta\rangle$ where $\alpha_i = (b_i/(2b_n))^2 + tb_i$ and $\beta = tb_n$. We have

$$\alpha_i + \beta^2 = (b_i/(2b_n) + tb_n)^2$$

which is a square for every i and so $\boldsymbol{a} + t\boldsymbol{b} \in D_n \subseteq K_n$. This proves that K_n is indeed a Kakeya set.

Finally we verify that the size of K_n is as claimed. First note that the size of D_n is exactly

$$|D_n| = q\,((q+1)/2)^{n-1} = 2^{-(n-1)} q^n + O(q^{n-1})$$

(q choices for β and $(q+1)/2$ choices for each $\alpha_i + \beta^2$).

Hence, as claimed, the size of K_n is at most

$$|K_n| = |D_n| + q^{n-1} = 2^{-(n-1)} q^n + O(q^{n-1}).$$

Even characteristic: This case is handled similarly with minor variations in the definition of K_n. Specifically, we let

$$K_n = E_n = \{\langle \alpha_1, \dots, \alpha_{n-1}, \beta\rangle | \alpha_i, \beta \in \mathbb{F}, \exists \gamma_i \in \mathbb{F} \text{ such that } \alpha_i = \gamma_i^2 + \gamma_i \beta\}.$$

(As we see below E_n contains $\mathbb{F}^{n-1} \times \{0\}$ and so there is no need to set $K_n = E_n \cup \mathbb{F}^{n-1} \times \{0\}$.)

Now consider direction $\boldsymbol{b} = \langle b_1, \dots, b_n\rangle$. If $b_n = 0$, then let $\boldsymbol{a} = 0$. We note that

$$\boldsymbol{a} + t\boldsymbol{b} = \langle tb_1, \dots, tb_{n-1}, 0\rangle = \langle \gamma_1^2 + \beta\gamma_1, \dots \gamma_{n-1}^2 + \beta\gamma_{n-1}, \beta\rangle$$

for $\beta = 0$ and $\gamma_i = \sqrt{tb_i} = (tb_i)^{q/2}$. We conclude that $\boldsymbol{a} + t\boldsymbol{b} \in E_n$ for every $t \in \mathbb{F}$ in this case.

Now consider the case where $b_n \neq 0$. Let

$$\boldsymbol{a} = \langle (b_1/b_n)^2, \dots, (b_{n-1}/b_n)^2, 0\rangle.$$

The point $\boldsymbol{a}+t\boldsymbol{b}$ has coordinates $\langle \alpha_1, \ldots, \alpha_{n-1}, \beta \rangle$ where $\alpha_i = (b_i/b_n)^2 + tb_i$ and $\beta = tb_n$.

For $\gamma_i = (b_i/b_n)$,

$$\gamma_i^2 + \gamma_i\beta = (b_1/b_n)^2 + tb_i = \alpha_i.$$

Hence $\boldsymbol{a}+t\boldsymbol{b} \in E_n = K_n$.

It remains to compute the size of E_n. The number of points of the form $\langle \alpha_1, \ldots, \alpha_{n-1}, 0\} \in E_n$ is exactly q^{n-1}. We now determine the size of $\langle \alpha_1, \ldots, \alpha_{n-1}, \beta\} \in E_n$ for fixed $\beta \neq 0$. We first claim that the set $\{\gamma^2+\beta\gamma | \gamma \in \mathbb{F}\}$ has size exactly $q/2$. This is so since for every $\gamma \in \mathbb{F}$, we have $\gamma^2+\beta\gamma = \tau^2+\beta\tau$ for $\tau = \gamma+\beta \neq \gamma$, and so the map $\gamma \mapsto \gamma^2+\beta\gamma$ is a 2-to-1 map on its image. Thus, for $\beta \neq 0$, the number of points of the form $\langle \alpha_1, \ldots, \alpha_{n-1}, \beta\}$ in E_n is exactly $(q/2)^{n-1}$. We conclude that E_n has cardinality

$$|E_n| = (q-1)(q/2)^{n-1} + q^{n-1} = 2^{-(n-1)}q^n + O(q^{n-1}). \qquad \square$$

We remark that for the case of odd characteristic, one can also use a recursive construction, replacing the set $\mathbb{F}^{n-1} \times \{0\}$ by $K_{n-1} \times \{0\}$. This would reduce the constant in the $O(q^{n-1})$ term, but not alter the leading term. Also we note that the construction used in the even case essentially also works in the odd characteristic case. Specifically the set $E_n \cup \mathbb{F}^{n-1} \times \{0\}$ is a Kakeya set also for odd characteristic. Its size can also be argued to be $2^{-(n-1)}q^n + O(q^{n-1})$.

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