ANALYSIS & PDE

Volume 2 No. 1 2009

mathematical sciences publishers

Analysis & PDE

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Analysis & PDE, at Mathematical Sciences Publishers, Department of Mathematics, University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFLOW™ from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

http://www.mathscipub.org

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Typeset in LaTeX

ANALYSIS AND PDE
Vol. 2, No. 1, 2009

UNIQUENESS OF GROUND STATES FOR PSEUDORELATIVISTIC HARTREE EQUATIONS

ENNO LENZMANN

We prove uniqueness of ground states $Q \in H^{1/2}(\mathbb{R}^3)$ for the pseudorelativistic Hartree equation,

$$\sqrt{-\Delta + m^2}\, Q - \big(|x|^{-1} * |Q|^2\big) Q = -\mu Q,$$

in the regime of Q with sufficiently small L^2-mass. This result shows that a uniqueness conjecture by Lieb and Yau [1987] holds true at least for $N = \int |Q|^2 \ll 1$ except for at most countably many N.

Our proof combines variational arguments with a nonrelativistic limit, leading to a certain Hartree-type equation (also known as the Choquard–Pekard or Schrödinger–Newton equation). Uniqueness of ground states for this limiting Hartree equation is well-known. Here, as a key ingredient, we prove the so-called nondegeneracy of its linearization. This nondegeneracy result is also of independent interest, for it proves a key spectral assumption in a series of papers on effective solitary wave motion and classical limits for nonrelativistic Hartree equations.

1. Introduction

The pseudorelativistic Hartree energy functional, given (in appropriate units) by

$$\mathscr{E}(\psi) = \int_{\mathbb{R}^3} \overline{\psi}\sqrt{-\Delta + m^2}\, \psi - \frac{1}{2}\int_{\mathbb{R}^3} \big(|x|^{-1} * |\psi|^2\big)|\psi|^2, \tag{1-1}$$

arises in the mean-field limit of a quantum system describing many self-gravitating, relativistic bosons with rest mass $m > 0$. Such a physical system is often referred to as a *boson star*, and various models for these — at least theoretical — objects have attracted a great deal of attention in theoretical and numerical astrophysics over the past years.

In order to gain some rigorous insight into the theory of boson stars, it is of particular interest to study ground states (that is, minimizers) for the variational problem

$$E(N) = \inf\Big\{\mathscr{E}(\psi) : \psi \in H^{1/2}(\mathbb{R}^3) \text{ and } \int_{\mathbb{R}^3} |\psi|^2 = N\Big\}, \tag{1-2}$$

where the parameter $N > 0$ plays the role of the stellar mass. Provided that problem (1-2) has indeed a ground state $Q \in H^{1/2}(\mathbb{R}^3)$, one readily finds that it satisfies the *pseudorelativistic Hartree equation*,

$$\sqrt{-\Delta + m^2}\, Q - \big(|x|^{-1} * |Q|^2\big) Q = -\mu Q, \tag{1-3}$$

with $\mu = \mu(Q) \in \mathbb{R}$ being some Lagrange multiplier.

MSC2000: 35Q55.
Keywords: pseudorelativistic Hartree equation, ground state, uniqueness, boson stars.
Partly supported by NSF Grant DMS–0702492.

In fact, the existence of symmetric-decreasing ground states $Q = Q^*(|x|) \geq 0$ minimizing (1-2) was first proven by Lieb and Yau [1987], where the authors also conjectured that uniqueness holds true in the following sense. For each $N > 0$, the variational problem (1-2) has at most one symmetric-decreasing ground state. If true, this result further implies, by strict rearrangement inequalities, that we have indeed uniqueness of all the ground states of (1-2) for each $N > 0$, up to phase and translation.

However, the nonlocality of $\sqrt{-\Delta + m^2}$ as well as the convolution-type nonlinearity both complicate the analysis of the pseudorelativistic Hartree equation (1-3) in a substantial way. In particular, the set of its radial solutions is not amenable to ODE techniques (for example, shooting arguments and comparison principles) which are key arguments for proving uniqueness of ground states for nonlinear Schrödinger equations (NLS) with local nonlinearities; see [Peletier and Serrin 1983; McLeod and Serrin 1987; Kwong 1989; McLeod 1993].

A further complication in the analysis of (1-3) stems from the fact that there are no simple scaling arguments that relate ground states with different N, due to the presence of $m > 0$. Indeed, this lack of a simple scaling mechanism is essential for the existence of a critical stellar mass $N_* > 0$; see Theorem 1.

As a first step towards proving uniqueness of ground states for (1-2), we present Theorem 2 below, which shows that ground states for problem are indeed unique (modulo translation and phase) for all sufficiently small $N > 0$ except for at most countably many. Our proof uses variational arguments combined with a nonrelativistic limit, leading to the nonlinear Hartree equation (also called Choquard–Pekar or Schrödinger–Newton equation) given by

$$-\frac{1}{2m}\Delta Q_\infty - \big(|x|^{-1} * |Q_\infty|^2\big) Q_\infty = -\lambda Q_\infty. \tag{1-4}$$

It is known this equation has a unique radial, positive solution $Q_\infty \in H^1(\mathbb{R}^3)$ for $\lambda > 0$ given; see [Lieb 1977] and Appendix A.

In the present paper, we prove (as a key ingredient) that $Q_\infty \in H^1(\mathbb{R}^3)$ has a *nondegenerate linearization*. By this we mean that the linearization of (1-4) around Q_∞ has a nullspace that is entirely due to the equation's invariance under phase and translation transformation; see Theorem 4 below and its remarks for a precise statement. In particular, we show that the linear operator L_+ given by

$$L_+\xi = -\frac{1}{2m}\Delta\xi + \lambda\xi - \big(|x|^{-1} * |Q_\infty|^2\big)\xi - 2Q_\infty\big(|x|^{-1} * (Q_\infty\xi)\big) \tag{1-5}$$

satisfies

$$\ker L_+ = \operatorname{span}\{\partial_{x_1} Q_\infty, \partial_{x_2} Q_\infty, \partial_{x_3} Q_\infty\}. \tag{1-6}$$

Furthermore, by a perturbation argument, we conclude an analogous nondegeneracy result for ground states of the pseudorelativistic Hartree equation (1-3) with sufficiently small L^2-mass; see Theorem 3 below.

In addition to being a mere technical key fact proven in this paper, the nondegeneracy result for (1-4) is also of independent interest. For example, it proves a key spectral assumption in a series of papers on effective solitary wave motion and classical limits for Hartree equations; see [Fröhlich et al. 2002; 2004; Jonsson et al. 2006; Abou Salem 2007] and also the remark following Theorem 4. Another very recent application of the nondegeneracy result (1-6) is presented in [Krieger et al. 2008], where two soliton solutions to the time-dependent version of (1-4) are constructed.

In the context of ground states for NLS with *local nonlinearities*, the nondegeneracy of linearizations is a well-known fact (see [Weinstein 1985; Chang et al. 2007]) and it plays a central role in the stability analysis of solitary waves for NLS. However, the arguments for NLS with local nonlinearities make use of Sturm–Liouville theory, which, by contrast, is not applicable to L_+ given by (1-5) due to its nonlocal character. For more details, we refer to Section 7 below.

Apart from their minimizing property, the ground states for (1-2) also play an important role for the *time-dependent pseudorelativistic Hartree equation*,

$$i\partial_t \psi = \sqrt{-\Delta + m^2}\, \psi - \left(|x|^{-1} * |\psi|^2\right)\psi, \tag{1-7}$$

with the wave field $\psi : [0, T) \times \mathbb{R}^3 \to \mathbb{C}$. Clearly, Equation (1-7) has solitary wave solutions

$$\psi(t, x) = e^{it\mu} Q(x), \tag{1-8}$$

whenever $Q \in H^{1/2}(\mathbb{R}^3)$ is a nontrivial solution to (1-3). Let us also mention that the dispersive nonlinear PDE (1-7) exhibits a rich variety of phenomena, such as stable and unstable traveling solitary waves, as well as finite-time blowup solutions indicating the "gravitational collapse" of a boson star; see [Fröhlich et al. 2007a; 2007b; Fröhlich and Lenzmann 2007]. For well-posedness results concerning (1-7) and its rigorous derivation from many-body quantum mechanics, we refer to [Cho and Ozawa 2006; Lenzmann 2007] and [Elgart and Schlein 2007], respectively.

For the reader's convenience, we conclude our introduction by summarizing the existence result about ground states for problem (1-2) along with a list of their basic properties.

Theorem 1 (Existence and properties of ground states). *Suppose that $m > 0$ holds in* (1-1). *Then there exists a universal constant $N_* > 4/\pi$* (*independent of m*) *such that the following holds.*

(i) (Existence) *There exists a ground state $Q \in H^{1/2}(\mathbb{R}^3)$ for problem* (1-2) *if and only if*

$$0 < N < N_*.$$

Moreover, the function Q satisfies the pseudorelativistic Hartree equation (1-3) *in the sense of distributions with some Lagrange multiplier $\mu \in \mathbb{R}$.*

(ii) (Smoothness and exponential decay) *Any ground state Q belongs to $H^s(\mathbb{R}^d)$ for all $s \geq 0$ and $e^{+\delta|x|} Q \in L^\infty(\mathbb{R}^3)$ for some $\delta = \delta(Q) > 0$.*

(iii) (Radiality and strict positivity) *Any ground state Q is equal to its spherical-symmetric rearrangement $Q^*(|x|)$ up to phase and translation. Moreover, we have $Q^*(|x|) > 0$ for all $x \in \mathbb{R}^3$.*

Remark. For the proofs of (i) and (ii)–(iii), we refer to [Lieb and Yau 1987] and [Lenzmann 2006; Fröhlich et al. 2007a], respectively. In physical terms, the constant $N_* > 0$ can be regarded as the "*Chandrasekhar limit mass*" of a pseudorelativistic boson star.

2. Main results

We now state our first main result concerning the uniqueness of ground states for the pseudorelativistic Hartree equation (1-3).

Theorem 2 (Uniqueness of ground states for $N \ll 1$). *Assume that* $m > 0$ *holds in* (1-1). *Then*, *for* $0 < N \ll 1$, *we have uniqueness of ground states for problem* (1-2) *up to phase and translation whenever* $E'(N)$ *exists. In particular*, *the symmetric-decreasing ground state* $Q = Q^* \in H^{1/2}(\mathbb{R}^3)$ *minimizing* (1-2) *is unique for such* $N > 0$.

Remarks. (1) Since it is known from [Lieb and Yau 1987] that the ground state energy $E(N)$ is strictly concave, the derivative $E'(N)$ exists for all $N \in (0, N_*)$, except on a subset Σ which is at most countable. In particular, it is easy to see that the Lagrange multiplier μ is unique for such $N \in (0, N_*) \setminus \Sigma$, in the sense that μ only depends on Q through $N = \int |Q|^2$. Our argument to prove Theorem 2 has to avoid the "exceptional" set Σ. A natural conjecture would be that $\Sigma = \varnothing$ holds.

(2) It would be desirable to extend this uniqueness result (whose proof partly relies on perturbative arguments) to the whole range $0 < N < N_*$ of existence; or, more interestingly, to disprove uniqueness for some $N > 0$ sufficiently large.

(3) By definition, ground states for the pseudorelativistic Hartree equation (1-2) are always minimizers for the variational problem (1-2). In principle, we cannot exclude the possibility that (1-3) has a positive solution without being a minimizer for (1-2).

(4) To the author's knowledge, this is the first uniqueness result for ground states that solve a nonlinear pseudo-differential equation in space dimension $n > 1$. In fact, apart from a very special case arising in $n = 1$ dimensions for solitary waves solving Benjamin–Ono-type equations (see [Amick and Toland 1991; Albert 1995]), nothing seems to be known, for instance, about uniqueness of ground states $\varphi \in H^s(\mathbb{R}^n)$ for nonlinear equations involving the fractional Laplacian $(-\Delta)^{s/2}\varphi + f(\varphi) = -\mu\varphi$,where $f(\varphi)$ denotes some nonlinearity and $\mu \in \mathbb{R}$ is given. The author plans to pursue this question in future work.

(5) If $m = 0$ vanishes, we have existence of ground states for problem (1-2) if and only if $N = N_*$. In what follows, we shall exclusively deal with the physically relevant case where $m > 0$ holds. Nevertheless, it remains an interesting open question whether uniqueness of ground states also holds for $m = 0$, since the methods developed here are clearly not applicable to this limiting case.

Our next result proves a so-called nondegeneracy condition, which was introduced in [Fröhlich et al. 2007b] as a spectral assumption supported by numerical evidence. There, the effective motion of solitary waves for (1-7) with an slowly varying external potential was studied. Furthermore, the following nondegeneracy result allows us to give an unconditional proof for the *cylindrical symmetry of traveling solitary waves* for the time-dependent pseudorelativistic Hartree equation (1-7); see [Fröhlich et al. 2007b] for more details. The precise nondegeneracy statement reads as follows.

Theorem 3 (Nondegeneracy of ground states for $N \ll 1$). *Let* $m > 0$ *in* (1-1) *and suppose that* $Q = Q^*$ *is a symmetric-decreasing ground state for problem* (1-2) *with Lagrange multiplier* $\mu \in \mathbb{R}$. *Furthermore*, *we consider the linear operator* L_+ *given by*

$$L_+\xi = \bigl(\sqrt{-\Delta + m^2} + \mu\bigr)\xi - \bigl(|x|^{-1} * |Q|^2\bigr)\xi - 2Q\bigl(|x|^{-1} * (Q\xi)\bigr),$$

acting on $L^2(\mathbb{R}^3)$ *with domain* $H^1(\mathbb{R}^3)$. *Then*, *for* $0 < N \ll 1$, *the operator* L_+ *is nondegenerate*, *that is*, *its kernel satisfies*

$$\ker L_+ = \operatorname{span}\bigl\{\partial_{x_1} Q, \partial_{x_2} Q, \partial_{x_3} Q\bigr\}.$$

Remarks. (1) This completely characterizes the kernel of the linearization of the pseudorelativistic Hartree equation (1-3) around ground state $Q = Q^*$ with $\int |Q|^2 \ll 1$. Note that, due to the presence of $|Q|^2$ in the nonlinearity, the linearized operator is not $\mathbb{C}$-linear. See also the remark following Theorem 4 below for more details on the analogous statement for the nonrelativistic equation (1-4).

(2) The nondegeneracy of L_+ holds for all $N = \int |Q|^2 \ll 1$. The extra condition that $E'(N)$ exists, which is present in Theorem 2, is not needed here.

In order to prove Theorem 3, we first have to show the nondegeneracy for the linearization around the ground state $Q_\infty \in H^1(\mathbb{R}^3)$ solving the nonrelativistic Hartree equation (1-4). As mentioned before, this spectral result is of independent interest, since it proves a key assumption in [Fröhlich et al. 2002; Fröhlich et al. 2004; Jonsson et al. 2006; Abou Salem 2007]. See also [Krieger et al. 2008], where the following nondegeneracy result is needed. Hence we record this fact about (1-4) as one of our main results.

Theorem 4 (Nondegeneracy for Q_∞). *Let $m > 0$ and $\lambda > 0$ be given. Furthermore, suppose that $Q_\infty \in H^1(\mathbb{R}^3)$ is the unique radial, positive solution to the nonrelativistic Hartree equation* (1-4). *Then the linear operator L_+ given by*

$$L_+\xi = -\frac{1}{2m}\Delta\xi + \lambda\xi - \big(|x|^{-1} * |Q_\infty|^2\big)\xi - 2Q_\infty\big(|x|^{-1} * (Q_\infty\xi)\big) \tag{2-1}$$

acting on $L^2(\mathbb{R}^3)$ with domain $H^2(\mathbb{R}^3)$, satisfies

$$\ker L_+ = \operatorname{span}\big\{\partial_{x_1} Q_\infty, \partial_{x_2} Q_\infty, \partial_{x_3} Q_\infty\big\}. \tag{2-2}$$

Remarks. (1) The linearized operator L for (1-4) at Q_∞ is found to be

$$Lh = -\frac{1}{2m}\Delta h + \lambda h - \big(|x|^{-1} * |Q_\infty|^2\big)h - Q_\infty\big(|x|^{-1} * (Q_\infty(h+\bar{h}))\big).$$

It is convenient to view the operator L (which is not $\mathbb{C}$-linear) as acting on

$$\begin{pmatrix} \operatorname{Re} h \\ \operatorname{Im} h \end{pmatrix},$$

so that it can be written as

$$L = \begin{pmatrix} L_+ & 0 \\ 0 & L_- \end{pmatrix}.$$

Here L_+ is as in Theorem 4 above, and L_- is the (local) operator

$$L_- = -\frac{1}{2m}\Delta + \lambda - \big(|x|^{-1} * |Q_\infty|^2\big).$$

It is easy to see that $\ker L_- = \operatorname{span}\{Q_\infty\}$ holds. Hence, by Theorem 4, we obtain

$$\ker L = \operatorname{span}\left\{ \begin{pmatrix} \partial_{x_1} Q_\infty \\ 0 \end{pmatrix}, \begin{pmatrix} \partial_{x_2} Q_\infty \\ 0 \end{pmatrix}, \begin{pmatrix} \partial_{x_2} Q_\infty \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ Q_\infty \end{pmatrix} \right\}.$$

(2) The precise knowledge of $\ker L$ implies, by well-known arguments along the lines for NLS with local nonlinearities (given in [Weinstein 1985]), the following coercivity estimate: There is a constant

$\delta > 0$ such that

$$\langle f, L_+ f\rangle + \langle g, L_- g\rangle \geq \delta(\|f\|_{H^1}^2 + \|g\|_{H^1}^2),$$

when $f \perp \operatorname{span}\{Q_\infty, x_i Q_\infty\}_{i=1}^3$ and $g \perp \operatorname{span}\{2Q_\infty + r\partial_r Q_\infty, \partial_{x_i} Q_\infty\}_{i=1}^3$, which means that (f, g) is *symplectically orthogonal* to the "soliton manifold" generated by Q_∞; see, for example, [Fröhlich et al. 2004]. This coercivity estimate plays a central role in the stability analysis of solitary waves for NLS-type equations and their effective motion in an external potential; see, for example, [Weinstein 1985; Bronski and Jerrard 2000; Fröhlich et al. 2004; 2007b; Jonsson et al. 2006; Abou Salem 2007; Holmer and Zworski 2008].

Organization of the paper. This paper is structured as follows. In Section 3, we study the nonrelativistic limit of ground states for a dimensionalized version of the variational problem (1-2). In Section 4, we prove a nondegeneracy result for the nonrelativistic ground state $Q_\infty \in H^1(\mathbb{R}^3)$ in the radial setting. Then, in Section 5, we establish a local uniqueness result around $Q_\infty \in H^1(\mathbb{R}^3)$ by means of an implicit-function-type argument.

We prove Theorem 2 in Section 6, and Theorems 3 and 4 in Section 7. Appendices A and B collect some auxiliary results and we also give a uniqueness proof for the ground state $Q_\infty \in H^1(\mathbb{R}^3)$, which differs from [Lieb 1977] in certain ways.

Notation and conventions. As usual $H^s(\mathbb{R}^n)$ stands for the inhomogeneous Sobolev space of order $s \in \mathbb{R}$, equipped with norm $\|f\|_{H^s} = \|\langle\nabla\rangle^s f\|_{L^2}$, where $\langle\nabla\rangle$ is defined via its multiplier $\langle\xi\rangle = (1+\xi^2)^{1/2}$ in the Fourier domain. Also, we shall make use of the space of radial and real-valued functions that belong to $H^1(\mathbb{R}^3)$, which we denote by

$$H^1_{\mathrm{r}}(\mathbb{R}^3) = \{f : f \in H^1(\mathbb{R}^3),\ f \text{ is radial and real-valued}\}.$$

With the usual abuse of notation we shall write both $f(x)$ and $f(r)$, with $r = |x|$, for radial functions f on $\mathbb{R}^n$. For any measurable function $f : \mathbb{R}^n \to \mathbb{C}$ that vanishes at infinity, we denote its symmetric-decreasing rearrangement by $f^* = f^*(r) \geq 0$.

Throughout this paper, we assume that the mass parameter $m > 0$ in (1-1) is strictly positive, which is the physically relevant case.

For the reader's orientation, we mention that our definition of $\mathscr{E}(\psi)$ in (1-1) differs from the conventions in [Lieb and Yau 1987; Fröhlich et al. 2007a] by an inessential factor of 2 and by the fact that we use $\sqrt{-\Delta + m^2}$ instead of $\sqrt{-\Delta + m^2} - m$. Obviously, these slight alterations in our definition of $\mathscr{E}(\psi)$ do not affect any results on (1-2) that are derived or quoted in the present paper.

Finally, we point out that the function $Q_\infty \in H^1_r(\mathbb{R}^3)$, which denotes the unique ground state for (1-4), appears throughout the paper. However, for the sake of simple notation, we shall also denote all its rescaled copies $aQ_\infty(b\cdot)$, with $a > 0$ and $b > 0$, simply by Q_∞, whenever there is no source of confusion.

3. Nonrelativistic limit

As a preliminary step towards the proof of Theorems 2 and 3, we study the nonrelativistic limit of ground states for the pseudorelativistic Hartree energy functional. More precisely, we reinstall the speed of light

$c > 0$ into $\mathscr{E}(\psi)$ defined in (1-1), which yields the c-depending Hartree energy functional

$$\mathscr{E}_c(\psi) = \int_{\mathbb{R}^3} \overline{\psi}\sqrt{-c^2\Delta + m^2c^4}\,\psi - \frac{1}{2}\int_{\mathbb{R}^3} \big(|x|^{-1} * |\psi|^2\big)|\psi|^2. \tag{3-1}$$

An elementary calculation shows that, for any $\psi \in H^{1/2}(\mathbb{R}^3)$,

$$\mathscr{E}(\psi) = c^{-3}\mathscr{E}_c(\widetilde{\psi}), \quad \text{with } \psi(x) = c^{-2}\widetilde{\psi}(c^{-1}x). \tag{3-2}$$

Thus we immediately find the following equivalence.

Lemma 1. *Let $c > 0$ and $N > 0$. Then $\widetilde{Q} \in H^{1/2}(\mathbb{R}^3)$ minimizes $\mathscr{E}_c(\psi)$ subject to $\int |\psi|^2 = N$ if and only if $Q = c^{-2}\widetilde{Q}(c^{-1}\cdot)$ minimizes $\mathscr{E}(\psi)$ subject to $\int |\psi|^2 = c^{-1}N$.*

In particular, we have existence of ground states for $\mathscr{E}_c(\psi)$ subject to $\int |\psi|^2 = N$ if and only if $0 < N < cN_*$ holds, where $N_* > 4/\pi$ denotes the same universal constant as in Theorem 1.

We now study the behavior of ground states Q_c for $\mathscr{E}_c(\psi)$ as $c \to \infty$ with $\int_{\mathbb{R}^3} |Q_c|^2 = N$ being fixed. By Lemma 1, this is equivalent (after a suitable rescaling) to studying ground states for $\mathscr{E}(\psi)$ with $\int |\psi|^2 = N$ as $N \to 0$. However, the following analysis turns out to be more transparent when working with $c > 0$ as a parameter and sending c to infinity. Concerning the nonrelativistic limit $c \to \infty$ of ground states for $\mathscr{E}_c(\psi)$, we have the following result.

Proposition 1. *Let $m > 0$ and $N > 0$ be given, and suppose that $c_n \to \infty$ as $n \to \infty$. Furthermore, we assume that $\{Q_{c_n}\}_{n=1}^{\infty}$ is a sequence of symmetric-decreasing ground states such that $\int_{\mathbb{R}^3} |Q_{c_n}|^2 = N$ for all $n \geq 1$, and each $Q_{c_n} \in H^{1/2}(\mathbb{R}^3)$ minimizes $\mathscr{E}_{c_n}(\psi)$ subject to $\int_{\mathbb{R}^3} |\psi|^2 = N$. Finally, let $\{\mu_{c_n}\}_{n=1}^{\infty}$ denote the sequence of Lagrange multipliers corresponding to $\{Q_{c_n}\}_{n=1}^{\infty}$.*

Then the following holds:

$$\begin{aligned} Q_{c_n} &\to Q_\infty \quad \textit{in} \quad H^1(\mathbb{R}^3) && \textit{as} \quad n \to \infty, \\ -\mu_{c_n} - mc_n^2 &\to -\lambda && \textit{as} \quad n \to \infty, \end{aligned}$$

where $Q_\infty \in H^1(\mathbb{R}^3)$ is the unique radial, positive solution to

$$-\frac{1}{2m}\Delta Q_\infty - \big(|x|^{-1} * |Q_\infty|^2\big)Q_\infty = -\lambda Q_\infty, \tag{3-3}$$

such that $\int_{\mathbb{R}^3} |Q_\infty|^2 = N$. Here $\lambda > 0$ is determined through $Q_\infty = Q_\infty^* \in H^1(\mathbb{R}^3)$, which is the unique symmetric-decreasing minimizer of the variational problem

$$E_{\mathrm{nr}}(N) = \inf\Big\{\mathscr{E}_{\mathrm{nr}}(\psi) : \psi \in H^1(\mathbb{R}^3) \text{ and } \int_{\mathbb{R}^3} |\psi|^2 = N\Big\}, \tag{3-4}$$

where

$$\mathscr{E}_{\mathrm{nr}}(\psi) = \frac{1}{2m}\int_{\mathbb{R}^3} |\nabla\psi|^2 - \frac{1}{2}\int_{\mathbb{R}^3} \big(|x|^{-1} * |\psi|^2\big)|\psi|^2. \tag{3-5}$$

Remarks. (1) A similar result for the nonrelativistic limit of ground states (and excited states) solving the Dirac–Fock equations can be found in [Esteban and Séré 2001]. However, unlike the Dirac–Fock and Hartree–Fock energy functionals in atomic physics treated in [Esteban and Séré 2001], the energy functional in (3-1) is not weakly lower semicontinuous due to its attractive potential term. Therefore, an a priori bound on the sequence of Lagrange multipliers μ_{c_n} (away from the essential spectrum of the limiting equation) is not sufficient to conclude strong convergence. To deal with this, we also have to use the radial symmetry of the Q_{c_n} in order to prove strong convergence.

(2) The uniqueness of the symmetric-decreasing ground state for problem (3-4) was proven by Lieb [1977]. For the reader's convenience, we provide a (partly different) proof of this fact in Appendix A.

3.1. ***Proof of Proposition 1.*** We begin with some auxiliary results.

Lemma 2. *Let $\{\mu_{c_n}\}_{n=1}^{\infty}$ be as in Proposition 1. Then there exist constants $\delta_1 > 0$ and $\delta_2 > 0$ such that*

$$mc_n^2 - \delta_1 \le -\mu_{c_n} \le mc_n^2 - \delta_2, \quad \text{for all } n \ge n_0,$$

where $n_0 \gg 1$ is some number.

Proof. The existence of $\delta_2 > 0$ can be deduced as follows. The Euler–Lagrange equation for Q_{c_n} reads

$$\sqrt{-c_n^2\Delta + m^2c_n^4}\, Q_{c_n} - \big(|x|^{-1} * |Q_{c_n}|^2\big) Q_{c_n} = -\mu_{c_n} Q_{c_n}, \tag{3-6}$$

which upon multiplication with Q_{c_n} and integration gives us

$$\mathscr{E}_{c_n}(Q_{c_n}) - \frac{1}{2}\int_{\mathbb{R}^3} \big(|x|^{-1} * |Q_{c_n}|^2\big)|Q_{c_n}|^2 = -\mu_{c_n} N. \tag{3-7}$$

Next, we recall the operator inequality

$$\sqrt{-c^2\Delta + m^2c^4} \le -\frac{1}{2m}\Delta + mc^2,$$

which directly follows in the Fourier domain and the fact that $\sqrt{1+t} \le t/2+1$ holds for all $t \ge 0$. Hence we have that $\mathscr{E}_{c_n}(Q_{c_n}) \le \mathscr{E}_{\mathrm{nr}}(Q_{c_n}) + Nmc_n^2$. Furthermore, since Q_{c_n} is a ground state for $\mathscr{E}_{c_n}(\psi)$, we deduce

$$\mathscr{E}_{c_n}(Q_{c_n}) \le E_{\mathrm{nr}}(N) + Nmc_n^2,$$

with $E_{\mathrm{nr}}(N)$ defined in (3-4), so that (3-7) gives us

$$-\mu_{c_n} N \le E_{\mathrm{nr}}(N) + Nmc_n^2.$$

From [Lieb 1977] we know that $E_{\mathrm{nr}}(N) < 0$ and thus $\delta_2 = -E_{\mathrm{nr}}(N)/N > 0$ is a legitimate choice.

To prove the existence of $\delta_1 > 0$, we observe that each $Q_{c_n} \ge 0$ is the ground state of the "relativistic" Schrödinger operator

$$H_{c_n} = \sqrt{-c_n^2\Delta + m^2c_n^4} - \big(|x|^{-1} * |Q_{c_n}|^2\big).$$

Since all Q_{c_n} are radial functions with $\|Q_{c_n}\|_{L^2}^2 = N$ for all $n \ge 1$, we can invoke Newton's theorem to find

$$\int_{\mathbb{R}^3} \frac{|Q_{c_n}(y)|^2}{|x-y|}\, dy \le \frac{N}{|x|}.$$

By the min-max principle, we infer the lower bound

$$-\mu_{c_n} \ge \inf \sigma(\overline{H}_{c_n})$$

where

$$\overline{H}_{c_n} = \sqrt{-c_n^2\Delta + m^2c_n^4} - \frac{N}{|x|}.$$

From [Herbst 1977] and reinstalling the speed of light $c > 0$ there, we recall that we have $\inf \sigma(\overline{H}_{c_n}) > -\infty$ if and only if $N < (2/\pi)c_n$. Thus $\overline{H}_{c_n}$ is bounded below for $n \gg 1$ and, moreover, we have an explicit lower bound (see [Herbst 1977] again) given by

$$\inf \sigma(\overline{H}_{c_n}) \geq mc_n^2 \sqrt{1 - \left(\frac{\pi N}{2c_n}\right)^2}.$$

Since $\sqrt{1-x^2} \geq 1 - x^2$ for $|x| \leq 1$, we conclude

$$-\mu_{c_n} \geq mc_n^2\left(1 - \left(\frac{\pi N}{2c_n}\right)^2\right) = mc_n^2 - \frac{1}{4}m\pi^2 N^2, \quad \text{for all } n \geq n_0,$$

provided that $n_0 \gg 1$. By choosing $\delta_1 = \frac{1}{4}m\pi^2 N^2 > 0$, we complete the proof of Lemma 2. □

Next, we derive an a priori bound on the sequence of ground states.

Lemma 3. *Let $\{Q_{c_n}\}_{n=1}^\infty$ be as in Proposition 1. Then there exists a constant $M > 0$ such that*

$$\|Q_{c_n}\|_{H^1} \leq M, \quad \textit{for all } n \geq 1.$$

Proof. Since $\|Q_{c_n}\|_{L^2}^2 = N$ for all $n \geq 1$, we only have to derive a uniform bound for $\|\nabla Q_{c_n}\|_{L^2}$ which can be done as follows. From (3-6) we obtain

$$\begin{aligned} &c_n^2\|\nabla Q_{c_n}\|_{L^2}^2 + m^2c_n^4\|Q_{c_n}\|_{L^2}^2 \\ &\quad = \left\langle \sqrt{-c_n^2\Delta + m^2c_n^4}\, Q_{c_n}, \sqrt{-c_n^2\Delta + m^2c_n^4}\, Q_{c_n}\right\rangle \\ &\quad \leq \mu_{c_n}^2\langle Q_{c_n}, Q_{c_n}\rangle + 2|\mu_{c_n}|\left\langle Q_{c_n}, (|x|^{-1} * |Q_{c_n}|^2)Q_{c_n}\right\rangle + \left\langle Q_{c_n}(|x|^{-1} * |Q_{c_n}|^2), (|x|^{-1} * |Q_{c_n}|^2)Q_{c_n}\right\rangle. \end{aligned}$$

To bound the terms on the right, we notice that Kato's inequality $|x|^{-1} \lesssim |\nabla|$ implies

$$\||x|^{-1} * |Q_{c_n}|^2\|_{L^\infty} \lesssim \langle Q_{c_n}, |\nabla| Q_{c_n}\rangle \lesssim \|Q_{c_n}\|_{L^2}\|\nabla Q_{c_n}\|_{L^2}.$$

Using this bound, Hölder's inequality, and the bound $|\mu_{c_n}| \leq mc_n^2$ for $n \gg 1$ from Lemma 2, we obtain

$$c_n^2\|\nabla Q_{c_n}\|_{L^2}^2 \lesssim mc_n^2 N^{3/2}\|\nabla Q_{c_n}\|_{L^2} + N^2\|\nabla Q_{c_n}\|_{L^2}^2,$$

for $n \gg 1$. Since $c_n \to \infty$ and N is fixed, we conclude that there exists $M > 0$ such that

$$\|\nabla Q_{c_n}\|_{L^2} \leq M$$

for $n \gg 1$. By choosing $M > 0$ possibly larger, we extend this bound to all $n \geq 1$. □

We now come the proof of Proposition 1 itself. By the a priori bound in Lemma 3, we have (after possibly passing to a subsequence) that

$$Q_{c_n} \rightharpoonup Q_\infty \text{ in } H^1(\mathbb{R}^3) \text{ and } Q_{c_n}(x) \to Q_\infty(x) \text{ for a.e. } x \in \mathbb{R}^3 \text{ as } n \to \infty,$$

for some $Q_\infty \in H^1(\mathbb{R}^3)$. By radiality and strict positivity of all the Q_{c_n}, it follows that $Q_\infty(|x|) \geq 0$ is a radial and nonnegative function. Furthermore, since $\{Q_{c_n}\}_{n=1}^\infty$ forms a sequence of radial functions on $\mathbb{R}^3$ with a uniform H^1-bound, a classical result (see [Strauss 1977]) yields that

$$Q_{c_n} \to Q_\infty \text{ in } L^p(\mathbb{R}^3) \text{ as } n \to \infty \text{ for any } 2 < p < 6. \tag{3-8}$$

By Lemma 2, we have that $\{-\mu_{c_n} - mc_n^2\}_{n=1}^\infty$ is a bounded sequence, which is also uniformly bounded away from 0. Hence extracting a suitable subsequence yields

$$\lim_{n\to\infty} (-\mu_{c_n} - mc_n^2) = -\lambda < 0, \tag{3-9}$$

for some $\lambda > 0$.

Using that $Q_{c_n} \rightharpoonup Q_\infty$ in H^1 and the strong convergence (3-8), we can pass to the limit in (3-6) and find that the radial, nonnegative function $Q_\infty \in H^1(\mathbb{R}^3)$ satisfies

$$-\frac{1}{2m}\Delta Q_\infty - \big(|x|^{-1} * |Q_\infty|^2\big) Q_\infty = -\lambda Q_\infty \quad \text{in } H^{-1}(\mathbb{R}^3). \tag{3-10}$$

When taking this limit, we use the fact that

$$\lim_{n\to\infty} \left\langle f, \Big(\sqrt{-c_n^2\Delta + m^2 c_n^4} - mc_n^2 + \frac{1}{2m}\Delta\Big) Q_{c_n} \right\rangle = 0 \quad \text{for all } f \in H^1(\mathbb{R}^3),$$

which is easy to verify for test functions $f \in C_0^\infty(\mathbb{R}^3)$ by taking the Fourier transform and using that

$$\sqrt{c_n^2\xi^2 + m^2c_n^4} - mc_n^2 - \frac{\xi^2}{2m} \to 0 \quad \text{for every } \xi \in \mathbb{R}^3 \text{ as } c_n \to \infty.$$

The claim above extends to all $f \in H^1(\mathbb{R}^3)$ by a simple density argument.

Next we prove that in fact $\int |Q_\infty|^2 = N$ holds, which a-posteriori would show that $Q_{c_n} \to Q_\infty$ strongly in $L^2(\mathbb{R}^3)$. To prove this claim, we note that Equation (3-6) and its limit (3-10) give us

$$(-\mu_{c_n} - mc_n^2)N = \frac{1}{2m}\int_{\mathbb{R}^3} |\nabla Q_{c_n}|^2 - \int_{\mathbb{R}^3} \big(|x|^{-1} * |Q_{c_n}|^2\big)|Q_{c_n}|^2 + r_n, \tag{3-11}$$

with $r_n \to 0$ as $n \to \infty$. Note that the right-hand side is not weakly lower semicontinuous (with respect to weak H^1-convergence), unlike the case of atomic Hartree and Hartree–Fock energy functionals. To deal with the non weakly lower semicontinuous part given by the potential energy term, we use (3-8) again and the Hardy–Littlewood–Sobolev inequality. Then, by the weak lower semicontinuity of the kinetic energy term in (3-11), we deduce from (3-11) and (3-10) that

$$-\lambda N \geq \frac{1}{2m}\int_{\mathbb{R}^3} |\nabla Q_\infty|^2 - \int_{\mathbb{R}^3} \big(|x|^{-1} * |Q_\infty|^2\big)|Q_\infty|^2 = -\lambda \int_{\mathbb{R}^3} |Q_\infty|^2.$$

Because of $\lambda > 0$, we see that $\int |Q_\infty|^2 \geq N$ must hold. On the other hand, we have $N \geq \int |Q_\infty|^2$ by the weak L^2-convergence. Thus we have $\int |Q_\infty|^2 = N$ and, consequently,

$$Q_{c_n} \to Q_\infty \text{ in } L^2(\mathbb{R}^3) \text{ as } n \to \infty. \tag{3-12}$$

By Lemma 9 and a simple scaling argument, we see that Q_∞ is the unique radial, nonnegative solution to (3-10) with $\int |Q_\infty|^2 = N$. Here $\lambda > 0$ is determined through Q_∞, and Q_∞ is in fact strictly positive.

It remains to show that

$$Q_{c_n} \to Q_\infty \text{ in } H^1(\mathbb{R}^3) \text{ as } n \to \infty. \tag{3-13}$$

To see this, we verify that $\{Q_{c_n}\}_{n=1}^\infty$ with $\int |Q_{c_n}|^2 = N$ furnishes a minimizing sequence for the nonrelativistic Hartree energy $\mathscr{E}_{\mathrm{nr}}(\psi)$ subject to $\int |\psi|^2 = N$, that is, for problem (3-4). Indeed, using (3-11) and (3-9) as well as the strong convergence (3-8) to pass to the limit in the potential energy, we deduce that

$$\mathscr{E}_{\mathrm{nr}}(Q_{c_n}) \to -\lambda N + \frac{1}{2}\int_{\mathbb{R}^3} \big(|x|^{-1} * |Q_\infty|^2\big)|Q_\infty|^2 \quad \text{as} \quad n \to \infty.$$

However, this limit for $\mathscr{E}_{\mathrm{nr}}(Q_{c_n})$ is equal to $\mathscr{E}_{\mathrm{nr}}(Q_\infty)$, as can be seen by multiplying (3-10) with Q_∞ and integrating. Hence $\{Q_{c_n}\}_{n=1}^{\infty}$ is a minimizing sequence for problem (3-4). Next, we notice that standard concentration-compactness methods yield relative compactness in $H^1(\mathbb{R}^3)$ for any radial minimizing sequence for problem (3-4), which has a unique radial, nonnegative minimizer Q_∞. Therefore (after possibly passing to another subsequence) we deduce that (3-13) holds.

To conclude the proof of Proposition 1, we note that we have convergence along every subsequence because of the uniqueness of the limit point $Q_\infty \in H^1(\mathbb{R}^3)$. □

4. Radial nondegeneracy of nonrelativistic ground states

We consider the linear operator

$$L_+\xi = -\frac{1}{2m}\Delta\xi + \lambda\xi - \big(|x|^{-1} * |Q_\infty|^2\big)\xi - 2Q_\infty\big(|x|^{-1} * (Q_\infty\xi)\big), \tag{4-1}$$

where $Q_\infty \in H^1(\mathbb{R}^3)$ is the radial, positive solution taken from Proposition 1. By standard arguments, it follows that L_+ is a self-adjoint operator acting on $L^2(\mathbb{R}^3)$ with domain $H^2(\mathbb{R}^3)$. In this section, we study the restriction of L_+ acting on $L^2_{\mathrm{rad}}(\mathbb{R}^3)$ (that is, the radial L^2-functions on $\mathbb{R}^3$).

As a main result, we prove the so-called nondegeneracy of L_+ on $L^2_{\mathrm{rad}}(\mathbb{R}^3)$; that is, the triviality of its kernel.

Proposition 2. *For the linear operator L_+ be given by* (4-1), *we have*

$$\ker L_+ = \{0\} \quad \textit{when } L_+ \textit{ is restricted to } L^2_{\mathrm{rad}}(\mathbb{R}^3).$$

Remark. (1) As shown in Section 7 below, we will see that the triviality of the kernel of L_+ on $L^2_{\mathrm{rad}}(\mathbb{R}^3)$ implies

$$\ker L_+ = \operatorname{span}\big\{\partial_{x_1}Q_\infty, \partial_{x_2}Q_\infty, \partial_{x_3}Q_\infty\big\}. \tag{4-2}$$

For linearized operators arising from ground states for NLS with *local nonlinearities*, this fact is well-known; see [Chang et al. 2007; Weinstein 1985]. However, the proof given there *cannot be adapted* to L_+ given by (4-1) due to its nonlocal component. We refer to Section 7 for further details.

(2) Numerical evidence indicating that 0 is not an eigenvalue of L_+ when restricted to radial functions can be found in [Harrison et al. 2003].

4.1. ***Proof of Proposition 2.*** Suppose that $Q_\infty \in H^1(\mathbb{R}^3)$ is the unique radial, positive solution to (3-3) with $\int |Q_\infty|^2 = N$ for some $N > 0$ given. In what follows, it will be convenient and without loss of generality to assume that Q_∞ satisfies

$$-\Delta Q_\infty - \big(|x|^{-1} * |Q_\infty|^2\big)Q_\infty = -Q_\infty, \tag{4-3}$$

which amounts to rescaling $Q_\infty(x) \mapsto aQ_\infty(bx)$ with suitable $a > 0$ and $b > 0$. Likewise, the linear operator L_+ then reads

$$L_+\xi = -\Delta\xi + \xi - \big(|x|^{-1} * |Q_\infty|^2\big)\xi - 2Q_\infty\big(|x|^{-1} * (Q_\infty\xi)\big). \tag{4-4}$$

Recall that we restrict ourselves to radial $\xi \in L^2_{\mathrm{rad}}(\mathbb{R}^3)$. Therefore, we can rewrite the nonlocal term in L_+ by invoking Newton's theorem in $\mathbb{R}^3$ (see [Lieb and Loss 2001, Theorem 9.7]): For any radial function $\rho = \rho(|x|)$ such that $\rho \in L^1(\mathbb{R}^3, (1+|x|)^{-1}dx)$, we have

$$-(|x|^{-1} * \rho)(r) = \int_0^r K(r,s)\rho(s)\,ds - \int_{\mathbb{R}^3} \frac{\rho(|x|)}{|x|}, \tag{4-5}$$

for $r = |x| \geq 0$, where $K(r,s)$ is given by

$$K(r,s) = 4\pi s\big(1 - \frac{s}{r}\big) \geq 0, \quad \text{for } r \geq s. \tag{4-6}$$

Since the ground state Q_∞ is exponentially decaying, we can apply Newton's theorem to $\rho = Q_\infty\xi$ for any $\xi \in L^2_{\mathrm{rad}}(\mathbb{R}^3)$ and obtain the following result.

Lemma 4. *For any $\xi \in L^2_{\mathrm{rad}}(\mathbb{R}^3)$, we have*

$$L_+\xi = \mathscr{L}_+\xi - 2Q_\infty\bigg(\int_{\mathbb{R}^3} \frac{Q_\infty\xi}{|x|}\bigg), \tag{4-7}$$

where $\mathscr{L}_+$ is given by

$$\mathscr{L}_+\xi = -\Delta\xi + \xi - \big(|x|^{-1} * |Q_\infty|^2\big)\xi + W\xi, \tag{4-8}$$

with

$$(W\xi)(r) = 2Q_\infty(r)\int_0^r K(r,s)Q_\infty(s)\xi(s)\,ds. \tag{4-9}$$

The following auxiliary result shows exponential growth of solutions v to the linear equation $\mathscr{L}_+v = 0$.

Lemma 5. *Suppose the radial function $v = v(r)$ solves $\mathscr{L}_+v = 0$ with $v(0) \neq 0$ and $v'(0) = 0$. Then the function $v(r)$ has no sign change and $v(r)$ grows exponentially as $r \to \infty$. More precisely, for any $0 < \delta < 1$, there exist constants $C > 0$ and $R > 0$ such that*

$$|v(r)| \geq Ce^{+\delta r}, \quad \textit{for all } r \geq R.$$

In particular, we have that $v \notin L^2_{\mathrm{rad}}(\mathbb{R}^3)$.

Proof. Since $\mathscr{L}_+v = 0$ is a linear equation, we can assume without loss of generality that $v(0) > 0$; and moreover it is convenient to assume that $v(0) > Q_\infty(0)$ holds. Next, we write $\mathscr{L}_+v = 0$ as

$$v''(r) + \frac{2}{r}v'(r) = V(r)v(r) + W(r), \tag{4-10}$$

with

$$V(r) = 1 - (|x|^{-1} * |Q_\infty|^2)(r), \tag{4-11}$$

$$W(r) = 2Q_\infty(r)\int_0^r K(r,s)Q_\infty(s)v(s)\,ds. \tag{4-12}$$

Note that $Q_\infty(r)$ satisfies (4-10) with $W(r)$ being removed, that is,

$$Q''_\infty(r) + \frac{2}{r} Q'_\infty(r) = V(r) Q_\infty(r). \tag{4-13}$$

We now compare $\upsilon(r)$ and $Q_\infty(r)$ as follows. An elementary calculation, using equations (4-10) and (4-13), leads to the "Wronskian-type" identity

$$\big(r^2(Q_\infty \upsilon' - Q'_\infty \upsilon)\big)' = r^2 Q_\infty W, \tag{4-14}$$

which, by integration, gives us

$$r^2(Q_\infty \upsilon' - Q'_\infty \upsilon)(r) = \int_0^r s^2 Q_\infty(s) W(s)\, ds. \tag{4-15}$$

Hence, while keeping in mind that $Q_\infty(r) > 0$, we find

$$r^2 \left(\frac{\upsilon(r)}{Q_\infty(r)} \right)' = \frac{1}{Q_\infty(r)^2} \int_0^r s^2 Q_\infty(s) W(s)\, ds. \tag{4-16}$$

From this identity we now claim that

$$\upsilon(r) > Q_\infty(r), \quad \text{for all } r \geq 0. \tag{4-17}$$

To see this, recall that $\upsilon(0) > Q_\infty(0)$ and, by continuity, we have that $\upsilon(r) > Q(r)$ for $r > 0$ sufficiently small. Suppose now, on the contrary to (4-17), that there is a first intersection at some positive $r = r_*$, say, so that $\upsilon(r_*) = Q_\infty(r_*)$. It is easy to see that the left-hand side of (4-16) (or equivalently (4-15)) has to be ≤ 0 at $r = r_*$. On the other hand, since $\upsilon(r) > Q_\infty(r) > 0$ on $[0, r_*)$, we conclude that the integral on right-hand side of (4-16) at $r = r_*$ must be strictly positive. This contradiction shows that (4-17) must hold. In particular, the function $\upsilon(r)$ never changes its sign.

Next, we insert the estimate (4-17) back into (4-16), which yields

$$r^2 \left(\frac{\upsilon(r)}{Q(r)} \right)'(r) \geq \frac{2}{Q_\infty(r)^2} \int_0^r s^2 Q_\infty(s)^2 \int_0^s K(s,t) Q_\infty(t)^2 \, dt\, ds. \tag{4-18}$$

We notice that $Q_\infty(r) > 0$ is the unique ground state for the Schrödinger operator

$$H = -\Delta + \widetilde{V}, \quad \text{with } \widetilde{V} = -|x|^{-1} * |Q_\infty|^2. \tag{4-19}$$

Since $H Q_\infty = -Q_\infty$ and $\widetilde{V}$ is a continuous function with $\widetilde{V} \to 0$ as $|x| \to \infty$, standard arguments show that, for any $\varepsilon > 0$, there exists a constant $A_\varepsilon > 0$ such that

$$Q_\infty(r) \leq A_\varepsilon e^{-(1-\varepsilon)r}, \quad \text{for all } r \geq 0. \tag{4-20}$$

Furthermore, since $Q_\infty(r) > 0$ is the ground state of H, we can obtain the following lower bound: For any $\varepsilon > 0$, there exists a constant $B_\varepsilon > 0$ such that

$$Q_\infty(r) \geq B_\varepsilon e^{-(1+\varepsilon)r}, \quad \text{for all } r \geq 0. \tag{4-21}$$

For this classical result on ground states for Schrödinger operators. See, for example, [Carmona and Simon 1981, Theorem 3.2] where a probabilistic proof is given.

Now let $0 < \varepsilon < 1$ be given. Inserting the bounds (4-20) and (4-21) into Equation (4-18), we obtain

$$r^2 \left(\frac{v(r)}{Q(r)} \right)'(r) \geq C e^{(2-2\varepsilon)r} \int_0^r s^2 e^{-(2+2\varepsilon)s} \int_0^s K(s,t) e^{-(2+2\varepsilon)t} \, dt \, ds, \tag{4-22}$$

with some constant $C = C_\varepsilon > 0$ (we drop its dependence on ε henceforth). Since the double integral on the right-hand side converges as $r \to \infty$ to some finite positive value, there exists some $a > 0$ such that

$$r^2 \left(\frac{v(r)}{Q(r)} \right)'(r) \geq C e^{(2-2\varepsilon)r}, \quad \text{for all } r \geq a, \tag{4-23}$$

with some constant $C > 0$. Integrating this lower bound and using (4-21) again, we find that

$$v(r) \geq C \frac{e^{(1-3\varepsilon)r}}{r^2}, \quad \text{for all } r \geq R, \tag{4-24}$$

with some constants $C > 0$ and $R \gg 1$. Thus, for any $0 < \delta < 1$, we arrive at the claim of Lemma 5 by taking $0 < \varepsilon < \frac{1}{3}(1-\delta)$ and choosing $C > 0$ appropriately. □

With the help of Lemma 5 we are now able to prove the triviality of the kernel of L_+ in the radial sector.

Lemma 6. *For L_+ be given by* (4-1), *we have that $L_+\xi = 0$ with $\xi \in L^2_{\mathrm{rad}}(\mathbb{R}^3)$ implies that $\xi \equiv 0$.*

Proof. Suppose there exists $\xi \in L^2_{\mathrm{rad}}(\mathbb{R}^3)$ with $\xi \not\equiv 0$ such that $L_+\xi = 0$. Then, by Lemma 4, the function ξ solves the inhomogeneous problem

$$\mathscr{L}_+\xi = 2\sigma Q_\infty, \quad \text{with } \sigma = \int_{\mathbb{R}^3} \frac{Q_\infty \xi}{|x|}. \tag{4-25}$$

Therefore,

$$\xi = v + w, \tag{4-26}$$

where w is any particular solution to (4-25) and v is some function such that $\mathscr{L}_+ v = 0$. As shown below, it suffices to restrict ourselves to smooth v and w.

We shall now construct a smooth $w \in L^2_{\mathrm{rad}}(\mathbb{R}^3)$ as follows. We define the smooth radial function

$$R = 2Q_\infty + r\partial_r Q_\infty \in L^2_{\mathrm{rad}}(\mathbb{R}^3), \tag{4-27}$$

where a calculation shows that

$$L_+ R = -2Q_\infty. \tag{4-28}$$

Furthermore, by applying Lemma 4 to R, we find

$$\mathscr{L}_+ R = 2(\tau - 1) Q_\infty, \quad \text{with } \tau = \int_{\mathbb{R}^3} \frac{Q_\infty R}{|x|}. \tag{4-29}$$

Note that $\tau \neq 1$ must hold, for otherwise Lemma 5 with $v = R$ (and $v(0) = R(0) = Q(0) > 0$ and $v'(0) = R'(0) = 0$) would yield that $R \notin L^2_{\mathrm{rad}}(\mathbb{R}^3)$, which is a contradiction. Thus we have found a smooth particular solution to (4-25) given by

$$w = \frac{\sigma}{\tau - 1} R \in L^2_{\mathrm{rad}}(\mathbb{R}^3). \tag{4-30}$$

Further, we notice that $\xi \in L^2_{\mathrm{rad}}(\mathbb{R}^3)$ with $L_+\xi = 0$ is smooth by bootstrapping this equation. Therefore, by (4-26), we conclude that v has to be smooth as well. Suppose that $v \equiv 0$. Then we have $\xi = w$ and $\sigma \neq 0$ (since otherwise $w = 0 \neq \xi$). This, however, contradicts that $L_+\xi = 0$ and $L_+ w = -2(\sigma/(\tau-1))Q_\infty \neq 0$.

Thus we see that $v \not\equiv 0$ in (4-26), where $v'(0) = 0$ by smoothness of v. Suppose now that $v(0) \neq 0$. Then Lemma 5 yields that $v \notin L^2_{\mathrm{rad}}(\mathbb{R}^3)$, which contradicts (4-26) together with the fact that ξ and w both belong to $L^2_{\mathrm{rad}}(\mathbb{R}^3)$. Finally, suppose that $v(0) = 0$ holds. Then v solves the equation $\mathscr{L}_+ v = 0$ with initial data $v(0) = 0$ and $v'(0) = 0$. However, by a standard fixed point argument, we see that the linear integro-differential equation $\mathscr{L}_+ v = 0$ with given initial data $v(0) \in \mathbb{R}$ and $v'(0) = 0$ has a unique solution. So $v(0) = 0$ and $v'(0) = 0$ implies that $v \equiv 0$. Again, we arrive at a contradiction as above. □

Clearly, Lemma 6 completes the proof of Proposition 2. □

5. Local uniqueness around Q_∞

Recall that $H^1_{\mathrm{r}}(\mathbb{R}^3)$ denotes space of radial and real-valued functions that belong to $H^1(\mathbb{R}^3)$. By using Proposition 2, we can now prove the following local uniqueness result for a small neighborhood around Q_∞ in $H^1_{\mathrm{r}}(\mathbb{R}^3)$.

Proposition 3. *Let $m > 0$ and $N > 0$ be given. Furthermore, suppose that $Q_\infty \in H^1_{\mathrm{r}}(\mathbb{R}^3)$ is the unique radial, positive solution to*

$$-\frac{1}{2m}\Delta Q_\infty - \big(|x|^{-1} * |Q_\infty|^2\big) Q_\infty = -\lambda Q_\infty, \tag{5-1}$$

with $\int |Q_\infty|^2 = N$, where $\lambda > 0$ is determined through Q_∞. Then there exist constants $c_0 \gg 1$, $\varepsilon > 0$, and $\delta > 0$ such that the following holds. For any (c, μ) with

$$c \geq c_0, \quad -\lambda - \varepsilon \leq -\mu - mc^2 \leq -\lambda + \varepsilon,$$

the equation

$$\sqrt{-c^2\Delta + m^2c^4}\, Q - \big(|x|^{-1} * |Q|^2\big) Q = -\mu Q \tag{5-2}$$

has a unique solution $Q \in H^1_{\mathrm{r}}(\mathbb{R}^3)$, provided that $\|Q - Q_\infty\|_{H^1} \leq \delta$.

5.1. ***Proof of Proposition 3.*** For $\beta \geq 0$ and $z > 0$, we define the map

$$G(u, \beta, z) = u + \mathscr{R}(\beta, z) g(u), \tag{5-3}$$

where we set

$$g(u) = -\big(|x|^{-1} * |u|^2\big) u, \tag{5-4}$$

and, for $\beta \geq 0$ and $z > 0$, we define the family of resolvents

$$\mathscr{R}(\beta, z) = \begin{cases} \big(-(1/2m)\Delta + z\big)^{-1} & \text{if } \beta = 0, \\ \big(\sqrt{-\beta^{-2}\Delta + m^2\beta^{-4}} - m\beta^{-2} + z\big)^{-1} & \text{if } \beta > 0. \end{cases} \tag{5-5}$$

By an elementary calculation, we verify the following equivalences:

$$Q \in H^1_{\mathrm{r}}(\mathbb{R}^3) \text{ solves (5-1) if and only if } G(Q, 0, \lambda) = 0; \tag{5-6}$$

$$Q \in H^1_{\mathrm{r}}(\mathbb{R}^3) \text{ solves (5-2) if and only if } G(Q, c^{-1}, \mu + mc^2) = 0. \tag{5-7}$$

To prove Proposition 3, we now construct an implicit function-type argument for the map

$$G : H_{\mathrm{r}}^1(\mathbb{R}^3) \times [0, \beta_0] \times [\lambda - \varepsilon, \lambda + \varepsilon] \to H_{\mathrm{r}}^1(\mathbb{R}^3), \tag{5-8}$$

where $\beta_0 > 0$ and $\varepsilon > 0$ are small constants. To see that indeed $G(u, \beta, z) \in H_{\mathrm{r}}^1(\mathbb{R}^3)$ for $u \in H_{\mathrm{r}}^1(\mathbb{R}^3)$, we notice that $\mathcal{R}(\beta, z) : H_{\mathrm{r}}^1(\mathbb{R}^3) \to H_{\mathrm{r}}^1(\mathbb{R}^3)$, as can be seen by using the Fourier transform. That $g(u)$ maps $H_{\mathrm{r}}^1(\mathbb{R}^3)$ into itself follows readily from the Hardy–Littlewood–Sobolev inequality and Sobolev embeddings. Hence (5-8) is indeed well-defined.

Next, we show that the derivative

$$\partial_u G(u, \beta, z) = 1 + \mathcal{R}(\beta, z)\partial_u g(u) : H_{\mathrm{r}}^1(\mathbb{R}^3) \to H_{\mathrm{r}}^1(\mathbb{R}^3) \tag{5-9}$$

depends continuously on (u, β, z). Here $\partial_u g(u)$ acting on $\xi \in H_{\mathrm{r}}^1(\mathbb{R}^3)$ is found to be

$$\partial_u g(u)\xi = -\big(|x|^{-1} * |u|^2\big)\xi - 2u\big(|x|^{-1} * (u\xi)\big). \tag{5-10}$$

By using the Hardy–Littlewood–Sobolev inequality and Sobolev embeddings, we obtain that

$$\|(\partial_u g(u_1) - \partial_u g(u_2))\xi\|_{H^1} \lesssim (\|u_1\|_{H^1} + \|u_2\|_{H^1})\|u_1 - u_2\|_{H^1}\|\xi\|_{H^1}; \tag{5-11}$$

see, for example, [Lenzmann 2007] for similar estimates proving Lipschitz continuity of $g(u)$. Using this estimate, we find for $u_1, u_2, \xi \in H_{\mathrm{r}}^1(\mathbb{R}^3)$, $\beta_1, \beta_2 \in [0, \beta_0]$, and $z_1, z_2 > 0$,

$$\begin{aligned}
&\|(\partial_u G(u_1, \beta_1, z_1) - \partial_u G(u_2, \beta_2, z_2))\xi\|_{H^1} \\
&\quad \le \|(\mathcal{R}(\beta_1, z_1) - \mathcal{R}(\beta_2, z_2))\partial_u g(u_1)\xi\|_{H^1} + \|\mathcal{R}(\beta_2, z_2)(\partial_u g(u_1) - \partial_u g(u_2))\xi\|_{H^1} \\
&\quad \lesssim \|\mathcal{R}(\beta_1, z_1) - \mathcal{R}(\beta_2, z_2)\|_{L^2 \to L^2}\|u_1\|_{H^1}^2\|\xi\|_{H^1} \\
&\qquad\qquad + \|\mathcal{R}(\beta_2, z_2)\|_{L^2 \to L^2}(\|u_1\|_{H^1} + \|u_2\|_{H^1})\|u_1 - u_2\|_{H^1}\|\xi\|_{H^1},
\end{aligned} \tag{5-12}$$

where we also use the fact that $\|\mathcal{R}(\beta, z)\|_{H^s \to H^s} = \|\mathcal{R}(\beta, z)\|_{L^2 \to L^2}$ for any $s \in \mathbb{R}$, since $\mathcal{R}(\beta, z)$ commutes with $\langle \nabla \rangle$. Moreover, by using the Fourier transform, one verifies

$$\|\mathcal{R}(\beta_1, z_1) - \mathcal{R}(\beta_2, z_2)\|_{L^2 \to L^2} \to 0 \quad \text{as} \quad (\beta_1, z_1) \to (\beta_2, z_2), \tag{5-13}$$

for any $\beta_1, \beta_2 \ge 0$ and $z_1, z_2 > 0$. (For later use, we record that (5-13) also holds for complex $z_1, z_2 \in \mathbb{C} \setminus [0, \infty)$.) Going back to (5-12), we thus find

$$\|\partial_u G(u_1, \beta_1, z_1) - \partial_u G(u_2, \beta_2, z_2)\|_{H^1 \to H^1} \to 0$$

as $\|u_1 - u_2\|_{H^1} \to 0$ and $(\beta_1, z_1) \to (\beta_2, z_2)$. Hence $\partial_u G(u, \beta, z)$ depends continuously on (u, β, z).

By Proposition 2 and its following remark, we have that the radial restriction of the linearized operator L_+ around Q_∞ has trivial kernel. This implies that the compact operator $(-(1/(2m))\Delta + \lambda)^{-1}\partial_u g(Q_\infty)$ does not have -1 in its spectrum. Hence the inverse operator

$$\big(\partial_u G(Q_\infty, 0, \lambda)\big)^{-1} : H_{\mathrm{r}}^1(\mathbb{R}^3) \to H_{\mathrm{r}}^1(\mathbb{R}^3) \tag{5-14}$$

exists. By the continuity of $\partial_u G(u, \beta, z)$ shown above, an appropriate version of an implicit function theorem (see, for example, [Chang 2005]) implies that, for $\beta_0 > 0$ and $\varepsilon > 0$ sufficiently small, there exists a unique solution $Q = Q(\beta, z) \in H^1_{\mathrm{r}}(\mathbb{R}^3)$ such that

$$G(Q(\beta, z), \beta, z) = 0 \quad \text{for } \beta \in [0, \beta_0] \text{ and } z \in [\lambda - \varepsilon, \lambda + \varepsilon] \tag{5-15}$$

with

$$\|Q(\beta, z) - Q_\infty\|_{H^1} \le \delta \quad \text{for some } \delta > 0. \tag{5-16}$$

Moreover, the map $(\beta, z) \mapsto Q(\beta, z) \in H^1_{\mathrm{r}}(\mathbb{R}^3)$ is continuous.

By setting $c_0 = \beta_0^{-1}$ and recalling the equivalence (5-7), we complete the proof of Proposition 3. □

6. Proof of Theorem 2

First, we notice that it is sufficient to prove uniqueness of symmetric-decreasing ground states for the variational problem (1-2), thanks to Theorem 1(iii). Next, we make use of the rescaling correspondence formulated in Lemma 1, which relates ground states for the dimensionalized and de-dimensionalized Hartree energy functionals $\mathscr{E}_c(\psi)$ and $\mathscr{E}(\psi)$ defined in (3-1) and (1-1), respectively.

In what follows, we fix $\int |Q_c|^2 = 1$ and we suppose that $Q_c = Q_c^* \in H^{1/2}(\mathbb{R}^3)$ is a symmetric-decreasing ground state for $\mathscr{E}_c(\psi)$ subject to $\int |\psi|^2 = 1$. Recall from Lemma 1 that Q_c indeed exists for $c \ge c_0$ with c_0 being a sufficiently large constant. Let $\mu(Q_c)$ denote the Lagrange multiplier associated to Q_c for $c \ge c_0$. We now claim that μ only depends on c except for some countable set, that is, we have

$$\mu(Q_c) = \mu(c), \quad \text{for } c \in (c_0, \infty) \setminus \Xi, \tag{6-1}$$

where Ξ is some countable set. To prove (6-1), we argue as follows. By Lemma 1, we see that $Q = c^{-2} Q_c(c^{-1}\cdot)$ is a symmetric-decreasing ground state for $\mathscr{E}(\psi)$ subject to $\int |\psi|^2 = N = c^{-1}$; and moreover the Lagrange multiplier $\mu(Q)$ for Q is found to be

$$\mu(Q) = c^{-2} \mu(Q_c). \tag{6-2}$$

Next, we consider the ground state energy $E(N)$ given by (1-2) for $0 < N < c_0^{-1}$. From [Lieb and Yau 1987; Fröhlich et al. 2007b] we know that $E(N)$ is strictly concave. Hence $E'(N)$ exists for all $N \in (0, c_0^{-1}) \setminus \Sigma$, where is Σ is some countable set, and we readily find that

$$E'(N) = -\mu(Q), \quad \text{for } N \in (0, c_0^{-1}) \setminus \Sigma. \tag{6-3}$$

Therefore the left-hand side of (6-2) only depends on $N = c^{-1}$ except when $N \in \Sigma$, which proves (6-1) with the countable set $\Xi = \{c : c > c_0 \text{ and } c^{-1} \in \Sigma\}$.

Suppose $\{c_n\}_{n=1}^\infty$ is a sequence with such that $c_n \to \infty$ and values in $c_n \in (c_0, \infty) \setminus \Xi$. Correspondingly, let $\{Q_{c_n}\}_{n=1}^\infty$ be a sequence of symmetric-decreasing ground states for $\mathscr{E}_c(\psi)$ with $\int |Q_{c_n}|^2 = 1$ for all $n \ge 1$. By Proposition 1, for any such sequence $\{Q_{c_n}\}$, we have that Q_{c_n} and its corresponding Lagrange multipliers μ_{c_n} satisfy the assumption of Proposition 3, provided that $n \gg 1$. By the local uniqueness result stated in Proposition 3 and the fact μ_{c_n} only depends on c_n, we conclude that the symmetric-decreasing ground state Q_c for $\mathscr{E}_c(\psi)$ subject to $\int |\psi|^2 = 1$ is unique, provided that $c \in (c_0, \infty) \setminus \Xi$ holds, where $c_0 \gg 1$ is sufficiently large and Ξ is some countable set.

Finally, by Lemma 1, we deduce uniqueness of symmetric-decreasing ground states Q for $\mathscr{E}(\psi)$ subject to $\int |\psi|^2 = N$, provided that $N \in (0, N_0) \setminus \Sigma$ holds, where $N_0 = c_0^{-1} \ll 1$ is sufficiently small and Σ denotes some countable set. □

7. Proof of Theorems 3 and 4

We first prove Theorem 4. By rescaling $Q_\infty(r) \mapsto aQ_\infty(br)$ with suitable $a > 0$ and $b > 0$, we can assume without loss of generality that $Q_\infty \in H^1_{\mathrm{r}}(\mathbb{R}^3)$ satisfies the normalized equation

$$-\Delta Q_\infty - \big(|x|^{-1} * |Q_\infty|^2\big) Q_\infty = -Q_\infty. \tag{7-1}$$

To complete the proof of Theorem 4, it suffices to prove the following result.

Proposition 4. *Let $Q_\infty \in H^1_{\mathrm{r}}(\mathbb{R}^3)$ be the unique radial and positive solution to Equation* (7-1). *Then the linearized operator L_+ given by*

$$L_+\xi = -\Delta\xi + \xi - \big(|x|^{-1} * |Q_\infty|^2\big)\xi - 2Q_\infty\big(|x|^{-1} * (Q_\infty\xi)\big),$$

acting on $L^2(\mathbb{R}^3)$ with domain $H^1(\mathbb{R}^3)$, has the kernel

$$\ker L_+ = \operatorname{span}\big\{\partial_{x_1} Q_\infty, \partial_{x_2} Q_\infty, \partial_{x_3} Q_\infty\big\}.$$

Remark. For linearized operators L_+ arising from ground states Q for NLS with local nonlinearities, it is a well-known fact that $\ker L_+ = \{0\}$ when L_+ is restricted to radial functions implies that $\ker L_+$ is spanned by $\{\partial_{x_i} Q\}_{i=1}^3$.

The proof, however, involves some Sturm–Liouville theory which is not applicable to L_+ given above, due to the presence of the nonlocal term. (Also, recall that Newton's theorem is not at our disposal, since we do not restrict ourselves to radial functions anymore.) To overcome this difficulty, we have to develop Perron–Frobenius-type arguments for the action of L_+ with respect to decomposition into spherical harmonics.

7.1. ***Proof of Proposition 4.*** Since $Q_\infty(r)$ and $|x|^{-1}$ are radial functions, the operator L_+ commutes with rotations in $\mathbb{R}^3$; that is, we have that $(L_+\xi(R\cdot))(x) = (L_+\xi)(Rx)$ for all $R \in O(3)$. Therefore, we decompose any $\xi \in L^2(\mathbb{R}^3)$ using spherical harmonics according to

$$\xi(x) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} f_{\ell m}(r) Y_{\ell m}(\Omega), \tag{7-2}$$

where $x = r\Omega$ with $r = |x|$ and $\Omega \in \mathbb{S}^2$. This gives us the direct decomposition

$$L^2(\mathbb{R}^3) = \bigoplus_{\ell=0}^{\infty} \mathcal{H}_{(\ell)}\,, \tag{7-3}$$

so that L_+ acts invariantly on each

$$\mathcal{H}_{(\ell)} = L^2(\mathbb{R}_+, r^2 dr) \otimes \mathcal{Y}_{(\ell)}. \tag{7-4}$$

Here $\mathcal{Y}_{(\ell)} = \operatorname{span}\{Y_{\ell m}\}_{m=-\ell}^{+\ell}$ denotes the $(2\ell+1)$-dimensional eigenspace corresponding to the eigenvalue $\kappa_\ell = -\ell(\ell+1)$ of the spherical Laplacian $\Delta_{\mathbb{S}^2}$ acting on $L^2(\mathbb{S}^2)$.

Let us now find an explicit formula for the action of L_+ on each $\mathcal{H}_{(\ell)}$. To this end, we recall the well-known the fact that

$$-\Delta = -\partial_r^2 - \frac{2}{r}\partial_r + \frac{\ell(\ell+1)}{r^2} \quad \text{on } \mathcal{H}_{(\ell)}, \tag{7-5}$$

as well as the multipole expansion

$$\frac{1}{|x-x'|} = 4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} \frac{1}{2\ell+1} \frac{r_<^\ell}{r_>^{\ell+1}} Y_{\ell m}(\Omega) Y_{\ell m}^*(\Omega'), \tag{7-6}$$

where $r_< = \min(|x|, |x'|)$ and $r_> = \max(|x|, |x'|)$. An elementary calculation leads to the following equivalence: We have that $L_+\xi = 0$ if and only if

$$L_{+,(\ell)} f_{\ell m} = 0, \quad \text{for } \ell = 0, 1, 2, \dots \text{ and } m = -\ell, \dots, +\ell, \tag{7-7}$$

with ξ given by (7-2). Here the operator $L_{+,(\ell)}$ acting on $L^2(\mathbb{R}_+, r^2 dr)$ is (formally) given by

$$(L_{+,(\ell)} f)(r) = -f''(r) - \frac{2}{r} f'(r) + \frac{\ell(\ell+1)}{r^2} f(r) + V(r) f(r) + (W_{(\ell)} f)(r), \tag{7-8}$$

with the local potential

$$V(r) = -\big(|x|^{-1} * |Q_\infty|^2\big)(r), \tag{7-9}$$

and the nonlocal linear operator

$$(W_{(\ell)} f)(r) = -\frac{8\pi}{2\ell+1} Q_\infty(r) \int_0^\infty \frac{r_<^\ell}{r_>^{\ell+1}} Q_\infty(s) f(s)\, s^2\, ds, \tag{7-10}$$

where $r_< = \min(r, s)$ and $r_> = \max(r, s)$.

To prove Proposition 4, it suffices to assume henceforth that $\ell \geq 1$ holds, since $L_{+,(0)} f = 0$ implies that $f \equiv 0$ holds, by Proposition 2 above. Hence any nontrivial elements in the kernel of L_+ can only belong to $\mathcal{H}_{(\ell)}$ with $\ell \geq 1$. Before we proceed, we show that each $L_{+,(\ell)}$ enjoys a Perron–Frobenius property as follows.

Lemma 7. *For each $\ell \geq 1$, the operator $L_{+,(\ell)}$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}_+) \subset L^2(\mathbb{R}_+, r^2 dr)$ and bounded below. Moreover, each $L_{+,(\ell)}$ has the Perron–Frobenius property. That is, if $e_{0,(\ell)}$ denotes the lowest eigenvalue of $L_{+,(\ell)}$, then $e_{0,(\ell)}$ is simple and the corresponding eigenfunction $\phi_{0,(\ell)}(r) > 0$ is strictly positive.*

Remarks. (1) We have indeed the lower bound $L_{+,(\ell)} \geq 0$ for all $\ell \geq 1$. This follows from $\mathcal{H}_{(\ell)} \perp Q_\infty$ for $\ell \geq 1$ and the fact that $L_+|_{Q_\infty^\perp} \geq 0$, which can be proven in the same way as for ground states for local NLS; see, for example, [Chang et al. 2007; Weinstein 1985].

(2) It is easy to see that $L_{+,(\ell)}$ has in fact infinitely many eigenvalues between 0 and 1. Indeed, the lower bound $Q_\infty(r) \geq B_\varepsilon e^{-(1+\varepsilon)r}$ (see the proof of Lemma 5) leads, by using Newton's theorem, to the upper bound $V(r) \leq -\alpha r^{-1}$ with some $\alpha > 0$. Furthermore, one finds that $\langle f, W^{(\ell)} f\rangle < 0$ for $f \not\equiv 0$. Hence, we conclude

$$L_{+,(\ell)} \leq -\partial_r^2 - \frac{2}{r}\partial_r + 1 + \frac{\ell(\ell+1)}{r^2} - \frac{\alpha}{r}$$

on $L^2(\mathbb{R}_+, r^2 dr)$. From the well-known spectral properties of the hydrogen atom Hamiltonian, we infer that the operator on the right has infinitely many eigenvalues below 1, and so does $L_{+,(\ell)}$ by the min-max principle.

Proof of Lemma 7. Since $Q_\infty(r)$ is exponentially decaying, it is straightforward to verify that $W_{(\ell)}$ is a bounded operator. Also, we have that $V \in L^\infty$ holds. Thus $L_{+,(\ell)}$ is bounded below (see also the remark following Lemma 7). Furthermore, it is well-known that

$$-\Delta_{(\ell)} = -\partial_r^2 - \frac{2}{r}\partial_r + \frac{\ell(\ell+1)}{r^2} \tag{7-11}$$

is essentially self-adjoint on $C_0^\infty(\mathbb{R}_+)$ provided that $\ell \geq 1$. In fact, this follows from [Reed and Simon 1980, Theorem X.10 and Example 4] which shows that $-\partial_r^2 - (2/r)\partial_r + \ell(\ell+1)/r^2$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}_+)$ if $\ell(\ell+1)/r^2 \geq 3/4r^2$. Furthermore, by the Kato–Rellich theorem and the fact that V and $W_{(\ell)}$ are bounded and self-adjoint, we deduce that $L_{+,(\ell)} = -\Delta_{(\ell)} + V + W_{(\ell)}$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}_+)$ as well.

The Perron–Frobenius property of $L_{+,(\ell)}$ can be shown as follows. First, we consider the kinetic energy part in $L_{+,(\ell)}$, where we find that

$$e^{t\Delta_{(\ell)}} \text{ is positivity improving on } L^2(\mathbb{R}_+, r^2 dr) \text{ for all } t > 0. \tag{7-12}$$

(Recall that, by definition, this means that $e^{t\Delta_{(\ell)}} f > 0$ when $f \geq 0$ with $f \not\equiv 0$.) Indeed, an argument given in Appendix B shows that the integral kernel of $e^{t\Delta_{(\ell)}}$ is strictly positive:

$$e^{t\Delta_{(\ell)}}(r,s) = \frac{1}{2t}\sqrt{\frac{1}{rs}}\, e^{-\frac{r^2+s^2}{4t}} I_{\ell+1/2}\left(\frac{rs}{2t}\right) > 0, \quad \text{for } r, s > 0. \tag{7-13}$$

Here $I_k(z)$ denotes the modified Bessel function of the first kind of order k. For later use, we record that (7-12) and the formula (by functional calculus)

$$(-\Delta_{(\ell)} + \mu)^{-1} = \int_0^\infty e^{-t\mu} e^{t\Delta_{(\ell)}}\, dt, \quad \text{for } \mu > 0, \tag{7-14}$$

immediately show that

$$(-\Delta_{(\ell)} + \mu)^{-1} \text{ is positivity improving on } L^2(\mathbb{R}_+, r^2 dr) \text{ for all } \mu > 0. \tag{7-15}$$

Next, let $A_{(\ell)}$ denote the bounded self-adjoint operator

$$A_{(\ell)} = V + W_{(\ell)}, \tag{7-16}$$

where V and $W_{(\ell)}$ are defined in (7-9) and (7-10), respectively. Note that $A_{(\ell)}$ is nonlocal. Using that $Q_\infty(r)$ is strictly positive, we readily find that

$$-A_{(\ell)} \text{ is positivity improving on } L^2(\mathbb{R}_+, r^2 dr). \tag{7-17}$$

This leads to the following auxiliary result.

Lemma 8. *For $\mu \gg 1$, the resolvent*

$$\big(L_{+,(\ell)} + \mu\big)^{-1} = \big(-\Delta_{(\ell)} + A_{(\ell)} + \mu\big)^{-1}$$

is positivity improving on $L^2(\mathbb{R}_+, r^2 dr)$.

Proof. For $\mu \gg 1$, we have

$$\frac{1}{L_{+,(\ell)}+\mu} = \frac{1}{-\Delta_{(\ell)}+\mu}\frac{1}{1+A_{(\ell)}(-\Delta_{(\ell)}+\mu)^{-1}}.$$

Since $A_{(\ell)}$ is bounded, we conclude that $\|A_{(\ell)}(-\Delta_{(\ell)}+\mu)^{-1}\|_{L^2\to L^2} < 1$ for $\mu \gg 1$. Thus a Neumann expansion yields

$$\frac{1}{L_{+,(\ell)}+\mu} = \frac{1}{-\Delta_{(\ell)}+\mu}\sum_{\nu=0}^{\infty}\big(-A_{(\ell)}(-\Delta_{(\ell)}+\mu)^{-1}\big)^{\nu}, \tag{7-18}$$

provided that $\mu \gg 1$. Next, we recall from (7-15) that $(-\Delta_{(\ell)}+\mu)^{-1}$ is positivity improving. By this fact and (7-17), we deduce from (7-18) that $(L_{+,(\ell)}+\mu)^{-1}$ must be positivity improving for $\mu \gg 1$. This completes the proof of Lemma 8. □

We now return to the proof of Lemma 7, which we complete as follows. Let $\ell \geq 1$ be fixed and suppose $e_{0,(\ell)} = \inf\sigma(L_{+,(\ell)})$ is the lowest eigenvalue. Furthermore, we choose $\mu \gg 1$ such that, by Lemma 8,

$$B = \big(L_{+,(\ell)}+\mu\big)^{-1} \tag{7-19}$$

is positivity improving on $L^2(\mathbb{R}_+, r^2dr)$. Clearly, the operator B is bounded and self-adjoint, and its largest eigenvalue $\lambda_0 = \sup\sigma(B)$ is given by $\lambda_0 = (e_{(\ell),0}+\mu)^{-1}$. Also, the corresponding eigenspaces of $L_{+,(\ell)}$ and B coincide. Since B is positivity improving (and hence ergodic), we can invoke [Reed and Simon 1978, Theorem XIII.43] to conclude that λ_0 is simple and that the corresponding eigenfunction $\phi_{(\ell),0}(r)$ is strictly positive on $\mathbb{R}_+$. This proof of Lemma 7 is therefore complete. □

Let us now come back to the proof of Proposition 4, stating that $\ker L_+$ is spanned by $\{\partial_{x_i}Q_\infty\}_{i=1}^3$. By differentiating the nonlinear equation satisfied by Q_∞, we readily obtain that $L_+\partial_{x_i}Q_\infty = 0$ for $i = 1, 2, 3$. Since $\partial_{x_i}Q_\infty(r) = Q'_\infty(r)(x_i/r) \in \mathcal{H}_{(1)}$, this show that

$$L_{+,(1)}Q'_\infty = 0. \tag{7-20}$$

Furthermore, by monotonicity of $Q_\infty(r)$, we have that $Q'_\infty(r) \leq 0$. Since $L_{+,(1)}$ is self-adjoint and Q'_∞ is an eigenfunction that does not change its sign, Lemma 7 shows that in fact $Q'_\infty(r) = -\phi_{0,(1)}(r)$ holds, where $\phi_{0,(1)} > 0$ is the strictly positive ground state of $L_{+,(1)}$, with $e_{0,(1)} = 0$ being its corresponding eigenvalue. Therefore any $\xi \in \mathcal{H}_{(1)}$ such that $L_+\xi = 0$ must be some linear combination of $\{\partial_{x_i}Q_\infty\}_{i=1}^3$.

To complete the proof of Proposition 4, we now claim that

$$L_{+,(\ell)} > 0, \quad \text{for } \ell \geq 2, \tag{7-21}$$

which in particular shows that $L_+\xi = 0$ with $\xi \in \mathcal{H}_{(\ell)}$ for some $\ell \geq 2$ implies that $\xi \equiv 0$. To prove (7-21), let $\ell \geq 2$ be fixed and set

$$e_{0,(\ell)} = \inf\sigma(L_{+,(\ell)}). \tag{7-22}$$

Indeed, by the remark following Lemma 7, we know that $e_{0,(\ell)} < 1$ is attained. (If $e_{0,(\ell)}$ was not attained, then $e_{0,(\ell)} = \inf\sigma_{\text{ess}}(L_{+,(\ell)}) = 1$ and (7-21) follows immediately.) By Lemma 7, the eigenvalue $e_{0,(\ell)}$ is simple and its corresponding eigenfunction $\phi_{0,(\ell)}(r) > 0$ is strictly positive. Next, we notice that

$$e_0 = \langle\phi_{0,(\ell)}, L_{+,(\ell)}\phi_{0,(\ell)}\rangle = \langle\phi_{0,(\ell)}, L_{+,(1)}\phi_{0,(\ell)}\rangle + K_{(\ell)}, \tag{7-23}$$

where

$$K_{(\ell)} = \int_0^\infty \frac{(\ell(\ell+1)-2)}{r^2} \phi_{0,(\ell)}(r)^2\, r^2\, dr + 8\pi \int_0^\infty \int_0^\infty Q_\infty(r)\phi_{0,(\ell)}(r)\Big(\frac{1}{3}\frac{r_<}{r_>^2} - \frac{1}{2\ell+1}\frac{r_<^\ell}{r_>^{\ell+1}}\Big) Q_\infty(s)\phi_{0,(\ell)}(s)\, r^2 s^2\, dr\, ds,$$

with $r_< = \min(r,s)$ and $r_> = \max(r,s)$. Using the strict positivity of $Q_\infty(r)$ and $\phi_{0,(\ell)}(r)$, we see that $K_{(\ell)} > 0$ holds because of $\ell \geq 2$ and $(r_</r_>) \leq 1$. Moreover, we recall from the preceding discussion that $L_{+,(1)} \geq e_{0,(1)} = 0$. Therefore, by (7-23),

$$e_{0,(\ell)} \geq K_{(\ell)} > 0, \quad \text{for all } \ell \geq 2, \tag{7-24}$$

which proves (7-21), completing the proof of Proposition 4, whence the proof of Theorem 4 follows. □

7.2. ***Proof of Theorem 3.*** As in the proof of Theorem 2 above, it is convenient to fix $N > 0$ and to consider symmetric-decreasing ground state $Q_c \in H_r^1(\mathbb{R}^3)$ minimizing $\mathscr{E}_c(\psi)$ with $\int |Q_c|^2 = N$, where we take $c > 0$ sufficiently large. In what follows, let μ_c denote the Lagrange multiplier associated to Q_c. (It is possible that μ_c depends on Q_c and not just on c.)

Recall from Proposition 1 that

$$\|Q_c - Q_\infty\|_{H^1} \leq \delta_1 \quad \text{and} \quad |-\mu_c - mc^2 + \lambda| \leq \delta_2, \tag{7-25}$$

where $\delta_1 \to 0$ and $\delta_2 \to 0$ as $c \to \infty$. Here $Q_\infty \in H_r^1(\mathbb{R}^3)$ is the unique radial positive solution to (3-3) with $\int |Q_\infty|^2 = N$, where $\lambda > 0$ is determined through Q_∞. By Theorem 4, the linear operator L_+ given by

$$L_+\xi = -\frac{1}{2m}\Delta + \lambda - \big(|x|^{-1} * |Q_\infty|^2\big)\xi - 2Q_\infty\big(|x|^{-1} * (Q_\infty\xi)\big) \tag{7-26}$$

has the kernel

$$\ker L_+ = \operatorname{span}\{\partial_{x_1} Q_\infty, \partial_{x_2} Q_\infty, \partial_{x_3} Q_\infty\}. \tag{7-27}$$

Next, let $L_{+,c}$ denote the linear operators defined as

$$L_{+,c}\xi = \sqrt{-c^2\Delta + m^2c^4}\,\xi + \mu_c\xi - \big(|x|^{-1} * |Q_c|^2\big)\xi - 2Q_c\big(|x|^{-1} * (Q_c\xi)\big). \tag{7-28}$$

Again, upon differentiating the Euler–Lagrange equation satisfied by Q_c, we see that $L_{+,c}\partial_{x_i} Q_c = 0$ for $i = 1, 2, 3$. Hence

$$\operatorname{span}\{\partial_{x_1} Q_c, \partial_{x_2} Q_c, \partial_{x_3} Q_c\big\} \subseteq \ker L_{+,c}. \tag{7-29}$$

By the following perturbation argument, we show that in fact equality holds for $c \gg 1$. By standard arguments, we see that $0 \in \sigma(L_+)$ is an isolated eigenvalue. Thus we can construct the Riesz projection P_0 onto $\ker L_+$ by

$$P_0 = \frac{1}{2\pi i} \oint_{\Gamma_r} (L_+ - z)^{-1}\, dz, \tag{7-30}$$

where the curve Γ_r parametrizes the circle $\{z \in \mathbb{C} : |z| = r\}$. Here $r > 0$ is chosen sufficiently small such that 0 is the only eigenvalue of L_+ inside $|z| \leq r$. Next, we claim that the projection

$$P_{0,c} = \frac{1}{2\pi i} \oint_{\Gamma_r} (L_{+,c} - z)^{-1}\, dz \tag{7-31}$$

exists for $c \gg 1$ and satisfies

$$\|P_{0,c} - P_0\|_{L^2 \to L^2} \to 0 \quad \text{as} \quad c \to \infty. \tag{7-32}$$

Indeed, by using (7-25) and similar arguments as in the proof of Proposition 3 (see, for example, the resolvent estimate (5-13)), we conclude that

$$\|(L_{+,c} - z)^{-1}\|_{L^2 \to L^2} \leq C \|(L_+ - z)^{-1}\|_{L^2 \to L^2}, \tag{7-33}$$

for all $c \gg 1$ and $z \in \Gamma_r$, where $C > 0$ is some constant. Furthermore, we have

$$\|(L_{+,c} - z)^{-1} - (L_+ - z)^{-1}\|_{L^2 \to L^2} \to 0 \quad \text{as} \quad c \to \infty, \tag{7-34}$$

for all $z \in \Gamma_r$. This shows that $P_{0,c}$ exists for $c \gg 1$ and that (7-32) holds. Since rank $P_0 = 3$ and the rank of $P_{0,c}$ remains constant for $c \gg 1$, by (7-32), we infer that $P_{0,c}$ has at most 3 eigenvalues (counted with their multiplicity) inside $|z| \leq r$, provided that $c \gg 1$. In particular, we conclude that $\dim \ker L_{+,c} \leq 3$ for $c \gg 1$. Therefore equality must hold in (7-29) whenever $c \gg 1$.

Thus we have found that $L_{+,c}$ has the desired kernel property if $c \gg 1$. By a rescaling argument formulated in Lemma 1, we conclude the analogous statement for the linear operator L_+ arising from the unique symmetric-decreasing ground state Q minimizing $\mathscr{E}(\psi)$ subject to $\int |\psi|^2 = N$ with $N \ll 1$. The proof of Theorem 3 is now complete. ☐

Appendix A. Uniqueness of Q_∞

Suppose that $Q_\infty \in H^1(\mathbb{R}^3)$ solves

$$-\frac{1}{2m} \Delta Q_\infty - \big(|x|^{-1} * |Q_\infty|^2\big) Q_\infty = -\lambda Q_\infty, \tag{A-1}$$

with $m > 0$ and $\lambda > 0$ given. By rescaling $Q_\infty(r) \mapsto a Q_\infty(br)$ with suitable $a > 0$ and $b > 0$, we can consider without loss of generality solutions $Q_\infty \in H^1(\mathbb{R}^3)$ to the "normalized" equation

$$-\Delta Q_\infty - \big(|x|^{-1} * |Q_\infty|^2\big) Q_\infty = -Q_\infty. \tag{A-2}$$

The following result is due to [Lieb 1977]; see also [Tod and Moroz 1999]. Here we provide a partly different proof, which is directly based on a comparison argument.

Lemma 9. *Equation* (A-2) *has a unique radial, nonnegative solution* $Q \in H^1_{\mathrm{r}}(\mathbb{R}^3)$ *with* $Q \not\equiv 0$. *Moreover, we have that* $Q(r)$ *is in fact strictly positive.*

Proof. Existence of a nonnegative, nontrivial solution $Q_\infty \in H^1_{\mathrm{r}}(\mathbb{R}^3)$ of (A-2) follows from variational arguments; see [Lieb 1977].

To prove that any nonnegative $Q \in H^1(\mathbb{R}^3)$, with $Q \not\equiv 0$, solving (A-2) is strictly positive, we can simply argue as follows. We rewrite (A-2) as

$$Q(x) = \big((-\Delta + 1)^{-1}(VQ)\big)(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{e^{-|x-y|}}{|x-y|} V(y) Q(y)\, dy \tag{A-3}$$

with $V = |x|^{-1} * |Q|^2$. Since $V \geq 0$ and $Q \geq 0$ (with $V \not\equiv 0$ and $Q \not\equiv 0$), Equation (A-3) shows that Q is strictly positive.

Let us now prove the claimed uniqueness. Suppose $Q \in H^1_{\mathrm{r}}(\mathbb{R}^3)$, with $Q \not\equiv 0$, is a solution to (A-2). Using Newton's theorem, we find that $Q(r)$ solves (after a suitable rescaling $Q(r) \mapsto a^2 Q(ar)$ for some $a > 0$; see [Lieb 1977]) the initial-value problem

$$-v''(r) - \frac{2}{r}v'(r) - v(r) + \left(\int_0^r K(r,s)v(s)^2\,ds\right)v(r) = 0, \quad v(0) = v_0, \quad v'(0) = 0, \tag{A-4}$$

with $v_0 = Q(0) \in \mathbb{R}$. (Recall that $K(r,s) \geq 0$ is given by (4-6) above.) By standard fixed point arguments, we deduce that (A-4) has a unique local C^2-solution for given initial data $v(0) \in \mathbb{R}$ and $v'(0) = 0$, and $v(r)$ exists up to some maximal radius $R \in (0, \infty]$.

Suppose now that $Q \in H^1_{\mathrm{r}}(\mathbb{R}^3)$ and $\widetilde{Q} \in H^1_{\mathrm{r}}(\mathbb{R}^3)$ are two radial, nonnegative (and nontrivial) solutions to (A-2) with $Q \not\equiv \widetilde{Q}$. From the preceding discussion we know that Q and $\widetilde{Q}$ are in fact strictly positive, and (after appropriate rescaling) both satisfy (A-4) with $v_0 = Q(0) > 0$ and $v_0 = \widetilde{Q}(0) > 0$, respectively. By uniqueness for (A-4), we conclude that $Q(0) \neq \widetilde{Q}(0)$ holds, since otherwise $Q \equiv \widetilde{Q}$. Therefore, we can henceforth assume that

$$\widetilde{Q}(0) > Q(0). \tag{A-5}$$

Next, we notice that a calculation (similar to the one in the proof of Lemma 5) yields the integrated "Wronskian-type" identity

$$r^2\big(Q(r)\widetilde{Q}'(r) - Q'(r)\widetilde{Q}(r)\big) = \int_0^r s^2 Q(s)\widetilde{Q}(s)\big(\widetilde{V}(s) - V(s)\big)\,ds. \tag{A-6}$$

Here,

$$V(r) = \int_0^r K(r,s)Q(s)^2\,ds \quad \text{and} \quad \widetilde{V}(r) = \int_0^r K(r,s)\widetilde{Q}(s)^2\,ds. \tag{A-7}$$

By continuity and (A-5), we have $\widetilde{Q}(r) > Q(r)$ at least initially for $r \geq 0$. Next, we conclude, by (A-6), that in fact

$$\widetilde{Q}(r) > Q(r), \quad \text{for all } r \geq 0. \tag{A-8}$$

To see this, suppose on the contrary that $\widetilde{Q}(r) > 0$ intersects $Q(r) > 0$ for the first time at $r = r_* > 0$, say. Then the left-hand side of (A-6) is found to be nonnegative at $r = r_*$, whereas the right-hand side must be strictly positive at $r = r_*$ since $\widetilde{V}(r) > V(r)$ on $(0, r_*)$. This contradiction shows that (A-8) holds.

Finally, we show that (A-8) leads to a contradiction (along the lines of [Lieb 1977]) as follows. To this end, we consider the Schrödinger operators

$$H = -\Delta + V \quad \text{and} \quad \widetilde{H} = -\Delta + \widetilde{V}, \tag{A-9}$$

so that $HQ = Q$ and $\widetilde{H}\widetilde{Q} = \widetilde{Q}$. By standard theory of Schrödinger operators, we conclude that Q and $\widetilde{Q}$ are (up to a normalization factor) the unique positive ground states (with eigenvalue $e = 1$) for H and $\widetilde{H}$, respectively. Therefore,

$$\langle \phi, H\phi\rangle \geq \|\phi\|^2_{L^2} \quad \text{and} \quad \langle \phi, \widetilde{H}\phi\rangle \geq \|\phi\|^2_{L^2}, \quad \text{for } \phi \in H^1(\mathbb{R}^3), \tag{A-10}$$

where equality holds if and only if $\phi = \lambda Q$ or $\phi = \lambda\widetilde{Q}$ for some $\lambda \in \mathbb{C}$, respectively.

Going back to (A-8), we find that $\widetilde{V}(r) > V(r)$ for all $r > 0$, which leads to

$$\|\widetilde{Q}\|^2_{L^2} \leq \langle \widetilde{Q}, H\widetilde{Q}\rangle = \langle \widetilde{Q}, \widetilde{H}\widetilde{Q}\rangle - \langle \widetilde{Q}, (\widetilde{V} - V)\widetilde{Q}\rangle = \|\widetilde{Q}\|^2_{L^2} - \delta,$$

for some $\delta > 0$, which is a contradiction.

Hence (A-2) does not admit two different radial and nonnegative (and nontrivial) solutions $Q \in H^1_{\mathrm{r}}(\mathbb{R}^3)$ and $\widetilde{Q} \in H^1_{\mathrm{r}}(\mathbb{R}^3)$. $\square$

Appendix B. Decomposition of $e^{t\Delta}$ using spherical harmonics

Recall the explicit formula for the heat kernel of the Laplacian Δ on $\mathbb{R}^3$:

$$e^{t\Delta}(x, y) = \frac{1}{(4\pi t)^{3/2}} e^{-|x-y|^2/(4t)} = \frac{1}{(4\pi t)^{3/2}} e^{-(x^2+y^2)/(4t)} e^{(x\cdot y)/(2t)}. \tag{B-1}$$

Moreover, we have the well-known identity

$$e^{ax\cdot y} = 4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} i_\ell(a|x||y|) Y_{\ell m}(\Omega) Y^*_{\ell m}(\Omega') \tag{B-2}$$

for $a > 0$, $x = |x|\Omega$ and $y = |y|\Omega'$ where $\Omega, \Omega' \in \mathbb{S}^2$. Here

$$i_\ell(z) = \sqrt{\frac{\pi}{2z}} I_{\ell+1/2}(z) \tag{B-3}$$

is the modified spherical Bessel function of the first kind of order ℓ; whereas $I_k(z)$ denotes the modified Bessel function of the first kind of order k.

Let $\Delta_{(\ell)}$ denote the restriction of Δ acting on $\mathscr{H}_{(\ell)}$ (that is, the space of $L^2(\mathbb{R}^3)$ functions whose "angular momentum" is $\ell \geq 0$). From (B-1) and (B-2) we deduce that the integral kernel of $e^{t\Delta_{(\ell)}}$ acting on $L^2(\mathbb{R}_+, r^2 dr)$ is given by

$$e^{t\Delta_{(\ell)}}(r, s) = \frac{1}{2t} \sqrt{\frac{1}{rs}} e^{-(r^2+s^2)/(4t)} I_{\ell+1/2}\Big(\frac{rs}{2t}\Big). \tag{B-4}$$

An explicit integral representation for $I_k(z)$ shows that $I_{\ell+1/2}(z) > 0$ for all $z > 0$ and $\ell \geq 0$.

Acknowledgments

It is a pleasure to thank Joachim Krieger and Maciej Zworski for helpful discussions, as well as Mathieu Lewin for pointing out to results on the nonrelativistic limit for Dirac–Fock equations. The author is also indebted to Mohammed Lemou and Pierre Raphaël, who found a gap in the previous version of this paper. This work was partially supported by the National Science Foundation Grant DMS-0702492.

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Received 25 Jan 2008. Revised 17 Sep 2008. Accepted 11 Jan 2009.

ENNO LENZMANN: `lenzmann@math.mit.edu`
Massachusetts Institute of Technology, Department of Mathematics, Room 2-230, Cambridge, MA 02139, United States

ANALYSIS AND PDE
Vol. 2, No. 1, 2009

RESONANCES FOR NONANALYTIC POTENTIALS

ANDRÉ MARTINEZ, THIERRY RAMOND AND JOHANNES SJÖSTRAND

We consider semiclassical Schrödinger operators on $\mathbb{R}^n$, with C^∞ potentials decaying polynomially at infinity. The usual theories of resonances do not apply in such a nonanalytic framework. Here, under some additional conditions, we show that resonances are invariantly defined up to any power of their imaginary part. The theory is based on resolvent estimates for families of approximating distorted operators with potentials that are holomorphic in narrow complex sectors around $\mathbb{R}^n$.

1. Introduction

The notion of quantum resonance was born around the same time as quantum mechanics itself. Its introduction was motivated by the behavior of various quantities related to scattering experiments, such as the scattering cross-section. At certain energies, these quantities present peaks (nowadays called Breit–Wigner peaks), which were modeled by a Lorentzian-shaped function

$$w_{a,b} : \lambda \mapsto \frac{1}{\pi} \frac{b/2}{(\lambda - a)^2 + (b/2)^2}.$$

The real numbers a and $2/(\pi b) > 0$ stand for the location of the maximum of the peak and its height. The number b is the width of the peak (more precisely its width at half its height). Of course for $\rho = a - ib/2 \in \mathbb{C}$, one has

$$w_{a,b}(\lambda) = \frac{1}{\pi} \frac{\operatorname{Im} \rho}{|\lambda - \rho|^2},$$

and the complex number ρ was called a resonance. Such complex values for energies had also appeared for example in [Gamow 1928], to explain α-radioactivity.

There is a standard discussion in physics textbooks that may help understand the normalization chosen for $w_{a,b}(\lambda)$. Suppose ψ_0 is a resonant state (not in L^2) corresponding to the resonance $\rho = a - ib/2$. Its time evolution should be written

$$\psi(t) = e^{-ita - tb/2} \psi_0,$$

so that the probability of survival beyond time t is

$$p(t) = \frac{|\psi(t)|^2}{|\psi_0|^2} = e^{-bt},$$

MSC2000: 35B34, 35P99, 47A10, 81Q20.
Keywords: resonances, Schroedinger operators, Breit–Wigner peaks.
The first author was partly supported by Università di Bologna, Funds for Selected Research Topics and Founds for Agreements with Foreign Universities.

and b is the decay rate of that probability. Moreover, the resonant state $\psi(t)$ has an associated energy space representation

$$\hat{\psi}(\lambda)=\frac{1}{\sqrt{2\pi}}\int_0^{+\infty} e^{itE}\,\psi(t)\,dt=\frac{1}{i\sqrt{2\pi}}\,\frac{\psi_0}{(a-\lambda)-ib/2},$$

which is interpreted saying that the probability density $d\sigma(\lambda)$ of the resonant state is proportional to $|\hat{\psi}(\lambda)|^2$ and leads to the following formula if one requires that the total probability is 1:

$$d\sigma(\lambda)=\frac{1}{2\pi}\,\frac{b}{(a-\lambda)^2+(b/2)^2}\,d\lambda=w_{a,b}(\lambda)\,d\lambda.$$

However, these complex numbers $\rho=a-ib/2$ are not defined in a completely exact way, in the sense that the peaks in the scattering cross section or the above probability distribution do not perceivably change if these numbers are modified by a quantity much smaller than their imaginary part. Indeed, a straightforward computation shows that the relative difference between such two peaks $w_{a,b}$ and $w_{a',b'}$ satisfies

$$\sup_{\lambda\in\mathbb{R}}\left|\frac{w_{a,b}(\lambda)-w_{a',b'}(\lambda)}{w_{a',b'}(\lambda)}\right|\le 3\frac{|\rho-\rho'|}{|\mathrm{Im}\,\rho|}+\frac{|\rho-\rho'|^2}{|\mathrm{Im}\,\rho|^2}$$

where we have also set $\rho'=a'-ib'/2$ and chosen $|\mathrm{Im}\,\rho|\le|\mathrm{Im}\,\rho'|$ to make the formula simpler. As a consequence, the two peaks become indistinguishable if $|\rho-\rho'|\ll|\mathrm{Im}\,\rho|$, that is, there is no physical relevance to associate the resonance $\rho=a-ib$ to $w_{a,b}$ rather than any other ρ' satisfying $|\rho-\rho'|\ll|\mathrm{Im}\,\rho|$. Notice also that the more the resonance is far from the real line, the more irrelevant this precision becomes.

On the mathematical side, the more recent theory of resonances for Schrödinger operators has made it possible to create a rigorous framework and obtain very precise results, in particular on the location of resonances in relation with the geometry of the underlying classical flow. However, it is based on the notion of complex scaling, in more and more sophisticated versions that all require analyticity assumptions on the potential or its Fourier transform; see, for example, [Aguilar and Combes 1971; Balslev and Combes 1971; Simon 1979; Sigal 1984; Cycon 1985; Helffer and Sjöstrand 1986; Hunziker 1986; Nakamura 1989; 1990; Sjöstrand and Zworski 1991]. It is important to notice that these different definitions coincide when their domain of validity overlap [Helffer and Martinez 1987]. In this mathematical framework, the Breit–Wigner formula for the scattering phase has now been studied by many authors in different situations, as shape resonances, clouds of resonances, or barrier-top resonances; see for example [Gérard et al. 1989; Petkov and Zworski 1999; Bruneau and Petkov 2003; Fujiié and Ramond 2003].

There are a small number of works about the definition of resonances for nonanalytic potentials, for example, [Orth 1990; Gérard and Sigal 1992; Soffer and Weinstein 1998; Cancelier et al. 2005; Jensen and Nenciu 2006]. In [Orth 1990; Gérard and Sigal 1992; Soffer and Weinstein 1998; Jensen and Nenciu 2006], the point of view is quite different from ours, while in [Cancelier et al. 2005], the definition is based on the use of an almost-analytic extension of the potential and seems to strongly depend both on the choice of this extension and on the complex distortion.

Here our purpose is to give a definition that fulfills both the mathematical requirement of being invariant with respect to the choices one has to make and the physical requirement of being more accurate as the resonance become closer to the real (or, equivalently, as the Breit–Wigner peak becomes narrower). Dropping the physically irrelevant precision for the definition of resonances, we can also drop the spurious assumption on the analyticity of the potential.

More precisely, we associate to a Schrödinger operator P a discrete set $\Lambda \subset \mathbb{C}$ with certain properties, such that for any other set Λ' with the same properties, there exists a bijection $B : \Lambda' \to \Lambda$ with $B(\rho) - \rho = \mathbb{O}(|\operatorname{Im} \rho|^\infty)$ uniformly. The set of resonances of P is the corresponding equivalence class of Λ. Of course, when the potential is dilation analytic at infinity, we recover the usual set of resonances up to the same error $\mathbb{O}(|\operatorname{Im} \rho|^\infty)$.

The properties characterizing Λ basically involve the resonances of a (essentially arbitrary) family of dilation-analytic operators $(P^\mu)_{(0<\mu\leq\mu_0)}$, such that

$$\begin{aligned} &P^\mu \text{ is dilation-analytic in a complex sector of angle } \mu \text{ around } \mathbb{R}^n, \\ &\|P^\mu - P\| = \mathbb{O}(\mu^\infty) \text{ uniformly as } \mu \to 0_+, \end{aligned}$$

and the constructive proof of the existence of the set Λ mainly consists in studying such a family and in particular, in obtaining resolvent estimates uniform in μ.

In this paper, we address the case of an isolated cluster of resonances whose cardinality is bounded (with respect to h). We hope to treat the general case elsewhere, as well as to give a detailed description of the quantum evolution $e^{itP/h} = e^{itP^\mu/h} + \mathbb{O}(|t|h^{-1}\mu^\infty)$ in terms of the resonances in Λ.

The paper is organized as follows. We give our assumptions and state our main results in Section 2. Then in Section 3, we give two paradigmatic situations where our constructions apply: the nontrapping case and the shape resonances case. In Section 4 we present a suitable notion of analytic approximation of a C^∞ function through which we define the operator P^μ. In Section 5 we show that a properly defined analytic distorted operator P_θ^μ of the latter satisfies a nice resolvent estimate in the upper half complex plane even very near to the real axis. Sections 6, 7 and 8 are devoted to the proof of Theorem 2.1, Theorem 2.2 and Theorem 2.5 respectively. We construct the set of resonances Λ and prove Theorem 2.6 in Section 9. In the last Section 10, we prove our statements concerning the shape resonances. Finally, we have placed in the Appendix the proofs of two technical lemmas.

2. Notations and main results

We consider the semiclassical Schrödinger operator

$$P = -h^2\Delta + V,$$

where $V = V(x)$ is a real smooth function of $x \in \mathbb{R}^n$, such that

$$\partial^\alpha V(x) = \mathbb{O}(\langle x\rangle^{-\nu-|\alpha|}), \tag{2-1}$$

for some $\nu > 0$ and for all $\alpha \in \mathbb{Z}_+^n$. We also fix $\tilde{\nu} \in (0, \nu)$ once for all, and for any $\mu > 0$ small enough, we denote by V^μ a $|x|$-analytic $(\mu, \tilde{\nu})$-approximation of V in the sense of Section 4. In particular, V^μ is analytic with respect to $r = |x|$ in $\{r \geq 1\}$, it can be extended into a holomorphic function of r in the sector $\Sigma := \{\operatorname{Re} r \geq 1\,,\ |\operatorname{Im} r| \leq 2\mu \operatorname{Re} r\}$, and it satisfies

$$V^\mu(x) - V(x) = \mathbb{O}(\mu^\infty\langle x\rangle^{-\tilde{\nu}}), \tag{2-2}$$

uniformly on $\mathbb{R}^n$. (See Section 4 for more properties of V^μ.)

Then for any $\theta \in (0, \mu]$, the operator

$$P^\mu := -h^2\Delta + V^\mu, \tag{2-3}$$

can be distorted analytically into

$$P^\mu_\theta := U_\theta P^\mu U_\theta^{-1}, \tag{2-4}$$

where U_θ is any transformation of the type

$$U_\theta\varphi(x) := \varphi(x + i\theta A(x)), \tag{2-5}$$

with $A(x) := a(|x|)x$, $a \in C^\infty(\mathbb{R}_+)$, $a = 0$ near 0, $0 \le a \le 1$ everywhere, $a(|x|) = 1$ for $|x|$ large enough. The essential spectrum of P^μ_θ is $e^{-2i\theta}\mathbb{R}$, and its discrete spectrum $\sigma_{\rm disc}(P^\mu_\theta)$ is included in the lower half-plane and does not depend on the choice of the function a. Moreover, it does not depend on θ, in the sense that for any $\theta_0 \in (0, \mu]$, and any $\theta \in [\theta_0, \mu]$, one has

$$\sigma_{\rm disc}(P^\mu_\theta) \cap \Sigma_{\theta_0} = \sigma_{\rm disc}(P^\mu_{\theta_0}) \cap \Sigma_{\theta_0},$$

where $\Sigma_{\theta_0} := \{z \in \mathbb{C}\ ;\ -2\theta_0 < \arg z \le 0\}$ (observe that one also has $\sigma_{\rm disc}(P^\mu_\theta) = \sigma_{\rm disc}(\widetilde{U}_\theta P^\mu \widetilde{U}_\theta^{-1})$, where $\widetilde{U}_\theta\varphi(x) := \sqrt{\det(\mathrm{Id} + i\theta\, {}^t dA(x))}\,\varphi(x + i\theta A(x))$ is an analytic distortion more widely used in the literature).

We denote by

$$\Gamma(P^\mu) := \sigma_{\rm disc}(P^\mu_\mu) \cap \Sigma_\mu,$$

the set of resonances of P^μ counted with their multiplicity. In what follows, we also use the following notation: If E and E' are two h-dependent subsets of $\mathbb{C}$, and $\alpha = \alpha(h)$ is a h-dependent positive quantity that tends to 0 as h tends to 0_+, we write

$$E' = E + \mathbb{O}(\alpha),$$

when there exists a constant $C > 0$ (uniform with respect to all other parameters) and a bijection

$$b : E' \to E,$$

such that

$$|b(\lambda) - \lambda| \le C\alpha$$

for all $h > 0$ small enough.

Now, we fix some energy level $\lambda_0 > 0$, and a constant $\delta > 0$. For any h-dependent numbers $\tilde\mu(h)$, $\mu(h)$, and any h-dependent bounded intervals $I(h)$, $J(h)$, satisfying

$$0 < \tilde\mu(h) \le \mu(h) \le h^\delta, \tag{2-6}$$

$$I(h) \subset J(h), \qquad \mathrm{diam}(J \cup \{\lambda_0\}) \le h^\delta, \tag{2-7}$$

we consider the following property (see Figure 1):

$$\mathcal{P}(\tilde\mu, \mu; I, J): \qquad \begin{cases} \mathrm{Re}(\Gamma(P^\mu) \cap (J - i[0, \lambda_0\tilde\mu])) \subset I, \\ \#(\Gamma(P^\mu) \cap (J - i[0, \lambda_0\tilde\mu])) \le \delta^{-1}, \\ \mathrm{dist}(I, \mathbb{R}\setminus J) \ge h^{-\delta}\omega_h(\tilde\mu), \end{cases}$$

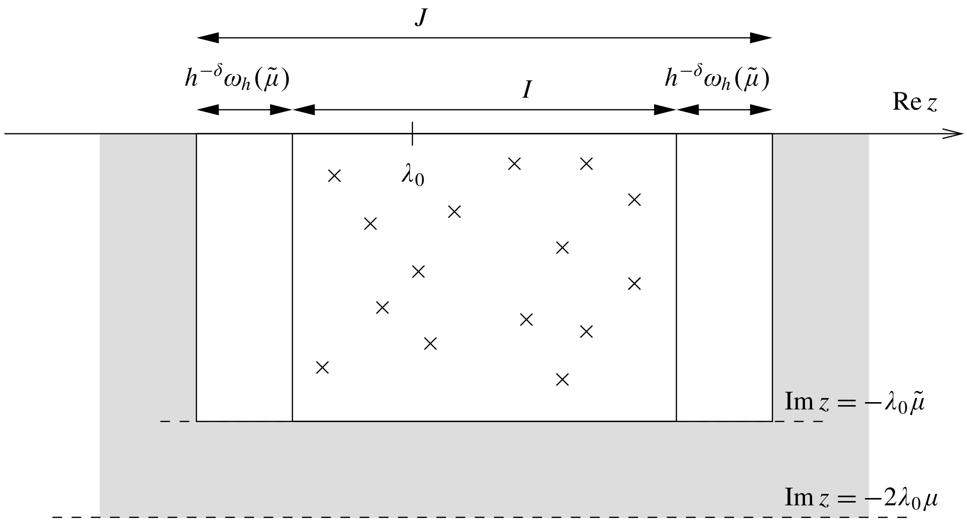

Figure 1. The property $\mathcal{P}(\tilde\mu, \mu; I, J)$.

where, for $\theta > 0$, we have set

$$\omega_h(\theta) := \theta\left(\ln\frac{1}{\theta} + h^{-n}\left(\ln\frac{1}{h}\right)^{n+1}\right)^{1/2}.$$

Notice that by (2-7), the property $\mathcal{P}(\tilde\mu, \mu; I, J)$ implies $\omega_h(\tilde\mu) \le h^{2\delta}$.

In the applications, it will be necessary to check that $\mathcal{P}(\tilde\mu, \mu; I, J)$ holds for values of $\tilde\mu$ that are essentially of order h^ν for some $\nu > n$. In that case, of course, the order of the quantity $\omega_h(\tilde\mu)$ can be simplified into $h^{\nu-n/2}(\ln(1/h))^{(n+1)/2}$. However, in the proof of our results, $\mathcal{P}(\tilde\mu, \mu; I, J)$ will be also used as an inductive condition that will permit us to consider arbitrarily small values of $\tilde\mu$ (including exponentially small values), and this is why we have to keep the somewhat intriguing above expression for $\omega_h(\theta)$.

Theorem 2.1. *Suppose $\mathcal{P}(\tilde\mu, \mu; I, J)$ holds for some $\tilde\mu$, μ, I and J satisfying* (2-6)–(2-7). *Then for all $\theta \in]0, \tilde\mu]$, there exists an interval*

$$J' = J + \mathcal{O}(\omega_h(\theta)),$$

such that

$$\|(P_\theta^\mu - z)^{-1}\| \le C\theta^{-C} \prod_{\rho\in\Gamma(\tilde\mu,\mu,J)} |z-\rho|^{-1},$$

for all $z \in J' + i[-C\theta h^{n_1}, C\theta h^{n_1}]$. Here we have set $n_1 := n + \delta$ and

$$\Gamma(\tilde\mu, \mu, J) := \Gamma(P^\mu) \cap (J - i[0, \lambda_0\tilde\mu]),$$

and $C > 0$ is a constant independent of $\tilde\mu$, μ, θ, I and J.

Thanks to this result, one can compare the resonances of the operators P^μ for different values of μ:

Theorem 2.2. *Let $N_0 \geq 1$ be a constant. Suppose $\mathcal{P}(\tilde{\mu}, \mu; I, J)$ holds for some $\tilde{\mu}$, μ, I and J satisfying* (2-6)–(2-7), *and that $\tilde{\mu} > \mu^{N_0}$. Then for any $\theta \in [\mu^{N_0}, \tilde{\mu}]$, there exist an interval*

$$J' = J + \mathcal{O}(\omega_h(\theta))$$

and $\tau \in [h^{n_1}\theta, 2h^{n_1}\theta]$, such that for any constant $N_1 \geq 1$ and any $\mu' \in [\mu^{N_1}, \mu^{1/N_1}]$ with $\theta \leq \mu'$, one has

$$\Gamma(P^{\mu'}) \cap (J' - i[0, \tau]) = \Gamma(P^\mu) \cap (J' - i[0, \tau]) + \mathcal{O}(\mu^\infty).$$

Remark 2.3. The only properties of V^μ used in the proof of this result are that V^μ is a holomorphic function of r in the sector $\Sigma := \{\operatorname{Re} r \geq 1\,,\ |\operatorname{Im} r| \leq 2\mu \operatorname{Re} r\}$, and it satisfies (2-2) and (4-2) for some $\tilde{\nu} > 0$. In particular, the proof also shows that, up to $\mathcal{O}(\mu^\infty)$, the set $\Gamma(P^\mu)$ does not depend on any particular choice of V^μ.

Remark 2.4. As we will see in the proof, the condition $\tau \in [h^{n_1}\theta, 2h^{n_1}\theta]$ can actually be replaced by $\tau \in [h^{n_1}\theta, h^{n_1}\theta + (h^{n_1}\theta)^M]$, for any fixed $M \geq 1$.

We also show that the validity of $\mathcal{P}(\tilde{\mu}, \mu; I, J)$ persists when decreasing $\tilde{\mu}$ and μ suitably, up to a small change of I and J.

Theorem 2.5. *Suppose $\mathcal{P}(\tilde{\mu}, \mu; I, J)$ holds for some $\tilde{\mu}$, μ, I and J satisfying* (2-6)–(2-7). *Assume furthermore that there is a constant $N_0 \geq 1$ with $\tilde{\mu} \geq \mu^{N_0}$. Then there exist two intervals*

$$I' = I + \mathcal{O}(\mu^\infty),$$
$$J' = J + \mathcal{O}(\omega_h(\tilde{\mu})),$$

such that $\mathcal{P}(h^{n_1}\mu', \mu'; I', J')$ holds, for any $\mu' \in (0, \tilde{\mu}]$.

Finally, the next result gives a definition of resonances for P, up to any power of their imaginary part.

Theorem 2.6. *Suppose $\mathcal{P}(\tilde{\mu}, \mu; I, J)$ holds for some $\tilde{\mu}$, μ, I and J satisfying* (2-6)–(2-7). *Assume furthermore that there is a constant $N_0 \geq 1$ with $\tilde{\mu} \geq \mu^{N_0}$. Then there exist*

$$\begin{aligned} &\text{an interval} \quad I' = I + \mathcal{O}(\mu^\infty),\\ &\text{an interval} \quad J' = J + \mathcal{O}(\omega_h(\tilde{\mu})),\\ &\text{a discrete set } \Lambda \subset I' - i[0, 2h^{2n_1}\tilde{\mu}], \end{aligned}$$

such that

$$\left.\begin{aligned} &\text{for any } \mu' \in (0, \tilde{\mu}], \text{ there exists } \tau \in [h^{2n_1}\mu', 2h^{2n_1}\mu'] \text{ with}\\ &\Gamma(P^{\mu'}) \cap (J' - i[0, \tau]) = \Lambda \cap (J' - i[0, \tau]) + \mathcal{O}((\mu')^\infty). \end{aligned}\right\} \qquad (\star)$$

Moreover, any other set $\widetilde{\Lambda} \subset I' - i[0, 2h^{2n_1}\tilde{\mu}]$ satisfying $(\star)$, possibly with some other choice of V^μ, is such that there exist $\tau' \in [\frac{1}{2}h^{2n_1}\tilde{\mu}, h^{2n_1}\tilde{\mu}]$ and a bijection

$$B : \Lambda \cap (J' - i[0, \tau']) \to \widetilde{\Lambda} \cap (J' - i[0, \tau']), \quad \text{with } B(\lambda) - \lambda = \mathcal{O}(|\operatorname{Im} \lambda|^\infty).$$

The set Λ will be called the set of resonances of P in $J' - i[0, \frac{1}{2}h^{2n_1}\tilde{\mu}]$. Here we adopt the convention that real elements of Λ are counted with a positive integer multiplicity in the natural way (see Section 9).

Remark 2.7. The main shortcoming of our condition $\mathcal{P}(\tilde{\mu}, \mu; I, J)$ is that the number of resonances in the corresponding box has to be bounded. It might be that this restriction could be eliminated by a finer analysis, based for example on results by P. Stefanov [2003]. We plan to come back to this point in a forthcoming work.

3. Two examples

Here, we describe two explicit situations where the previous results apply.

3.1. ***The nontrapping case.*** We suppose first that the energy λ_0 is nontrapping, that is, for any $(x, \xi) \in p^{-1}(\lambda_0)$ we have

$$|\exp tH_p(x, \xi)| \to \infty \text{ as } |t| \to \infty,$$

where $p(x, \xi) := \xi^2 + V(x)$ is the principal symbol of P, and $H_p = \partial_\xi p \partial_x - \partial_x p \partial_\xi$ is the Hamilton field of p.

Then the result of [Martinez 2002b] can be applied to P^μ with $\mu = Ch\ln(h^{-1})$ for any arbitrary constant $C > 0$, and tells us that P^μ has no resonances in $[\lambda_0 - 2\varepsilon, \lambda_0 + 2\varepsilon] - i[0, \lambda_0\mu]$ with some $\varepsilon > 0$ constant. In that case, for any $\delta > 0$, $\mathcal{P}(h^{n_1}\mu, \mu; I, J)$ is satisfied with $I = [\lambda_0 - h^\delta, \lambda_0 + h^\delta]$ and $J = [\lambda_0 - 2h^\delta, \lambda_0 + 2h^\delta]$, and the previous results tell us that P has no resonances in $I - i[0, \frac{1}{2}h^{3n_1}\mu]$ in the sense of Theorem 2.6.

3.2. ***The shape resonances.*** Now we assume instead that, in addition to (2-1), the potential V presents the geometric configuration of the so-called "point-well in an island", as described in [Helffer and Sjöstrand 1986]. More precisely, we suppose

$$\left.\begin{array}{l}\text{There exist a connected bounded open set } \ddot{O} \subset \mathbb{R}^n \text{ and } x_0 \in \ddot{O}, \text{ such that}\\ \quad\bullet\ \lambda_0 := V(x_0) > 0\,;\ V > \lambda_0 \text{ on } \ddot{O}\backslash\{x_0\}\,;\ \nabla^2 V(x_0) > 0\,;\ V = \lambda_0 \text{ on } \partial\ddot{O};\\ \quad\bullet\ \text{any point of } \{(x, \xi) \in \mathbb{R}^{2n}\,;\ x \in \mathbb{R}^n\backslash\ddot{O}\,,\ \xi^2 + V(x) = \lambda_0\} \text{ is nontrapping.}\end{array}\right\}\qquad \text{(H)}$$

We denote by $(e_k)_{k\geq 1}$ the increasing sequence of (possibly multiple) eigenvalues of the harmonic oscillator $H_0 = -\Delta + \frac{1}{2}\langle V''(x_0)x, x\rangle$. We have:

Theorem 3.1. *Assume* (2-1) *and* (*H*). *Then for any* $k_0 \geq 1$ *and any* $\delta > 0$, $\mathcal{P}(\tilde{\mu}, \mu; I, J)$ *holds with*

$$\mu = h^\delta, \quad \tilde{\mu} = h^{\max(n/2,1)+1+\delta}, I = [\lambda_0 + (e_1 - \varepsilon)h, \quad \lambda_0 + (e_{k_0} + \varepsilon)h], \quad J = [\lambda_0, \lambda_0 + (e_{k_0+1} - \varepsilon)h],$$

where $\varepsilon > 0$ *is any fixed number in* $(0, \min(e_1/2, (e_{k_0+1} - e_{k_0})/3)]$.

Actually, we prove in Section 10 that any resonance ρ of P^μ in $J - i[0, \tilde{\mu}]$ is such that there exists $k \leq k_0$ with

$$\operatorname{Re}\rho - (\lambda_0 + e_k h) = \mathbb{O}(h^{3/2}),$$
$$\operatorname{Im}\rho = \mathbb{O}(e^{-2S_1/h}),$$

where $S_1 > 0$ is any number less than the Agmon distance between x_0 and $\partial\ddot{O}$. Recall that the Agmon distance is the pseudo-distance associated to the degenerate metric $(V(x) - \lambda_0)_+ dx^2$.

More generally, if $\mu' \in [e^{-\eta/h}, \mu]$ with $\eta > 0$ small enough, we prove that any resonance ρ of $P^{\mu'}$ in $J - i[0, \lambda_0 \min(\mu', h^{2+\delta})]$, satisfies

$$\operatorname{Re}\rho - (\lambda_0 + e_k h) = \mathbb{O}(h^{3/2}),$$

for some $k \leq k_0$, and

$$\operatorname{Im}\rho = \mathbb{O}(e^{-2(S_0-\eta)/h}).$$

Applying Theorem 2.6 with $\mu' = e^{-\eta/h}$ $(0 < \eta < S_0)$, we deduce that the resonances of P in

$$[\lambda_0, \lambda_0 + Ch] - i[0, \tfrac{1}{2}h^{2n+\max(n/2,1)+1+3\delta}]$$

satisfy the same estimates.

4. Preliminaries

In this section, following an idea of [Fujiié et al. 2008], we define and study the notion of analytic $(\mu, \tilde{\nu})$-approximations.

Definition 4.1. For any $\mu > 0$ and $\tilde{\nu} \in (0, \nu)$, we say that a real smooth function V^μ on $\mathbb{R}^n$ is a $|x|$-analytic $(\mu, \tilde{\nu})$-approximation of V, if V^μ is analytic with respect to $r = |x|$ in $\{r \geq 1\}$, V^μ can be extended into a holomorphic function of r in the sector $\Sigma(2\mu) := \{\operatorname{Re} r \geq 1\,,\ |\operatorname{Im} r| < 2\mu \operatorname{Re} r\}$, and for any multi-index α, it satisfies

$$\partial^\alpha (V^\mu(x) - V(x)) = \mathbb{O}(\mu^\infty \langle x\rangle^{-\tilde{\nu}-|\alpha|}), \tag{4-1}$$

uniformly with respect to $x \in \mathbb{R}^n$ and $\mu > 0$ small enough, and

$$\partial^\alpha V^\mu(x) = \mathbb{O}(\langle \operatorname{Re} x\rangle^{-\tilde{\nu}-|\alpha|}), \tag{4-2}$$

uniformly with respect to $x \in \Sigma(2\mu)$ and $\mu > 0$ small enough.

Proposition 4.2. *Let $V = V(x)$ be a real smooth function of $x \in \mathbb{R}^n$ satisfying* (2-1).

(i) *For any $\mu > 0$ and $\tilde{\nu} \in (0, \nu)$, there exists a $|x|$-analytic $(\mu, \tilde{\nu})$-approximation of V.*

(ii) *If V^μ and W^μ are two $|x|$-analytic $(\mu, \tilde{\nu})$-approximations of V, then for all $\alpha \in \mathbb{N}^n$, one has*

$$\partial^\alpha (V^\mu(x) - W^\mu(x)) = \mathbb{O}(\mu^\infty \langle \operatorname{Re} x\rangle^{-\tilde{\nu}-|\alpha|}),$$

uniformly with respect to $x \in \Sigma(\mu)$ and $\mu > 0$ small enough.

Proof. We denote by $\widetilde{V}$ a smooth function on $\mathbb{C}^n$ satisfying the following:

- $\widetilde{V} = V$ on $\mathbb{R}^n$.
- $\bar{\partial}\widetilde{V} = \mathbb{O}\big((|\operatorname{Im} x|/\langle \operatorname{Re} x\rangle)^\infty \langle \operatorname{Re} x\rangle^{-\nu}\big)$, uniformly on $\{|\operatorname{Im} x| \leq C\langle \operatorname{Re} x\rangle\}$, for any $C > 0$.
- $\partial^\alpha \widetilde{V} = \mathbb{O}\big(\langle \operatorname{Re} x\rangle^{-\nu-|\alpha|}\big)$, uniformly on $\{|\operatorname{Im} x| \leq C\langle \operatorname{Re} x\rangle\}$, for any $C > 0$ and $\alpha \in \mathbb{N}^n$.

Note that such a function $\widetilde{V}$ (called an "almost-analytic" extension of V: See, for example, [Melin and Sjöstrand 1975]) can easily be obtained by taking a resummation of the formal series

$$\widetilde{V}(x) \sim \sum_{\alpha\in\mathbb{N}^n} \frac{i^{|\alpha|}(\operatorname{Im} x)^\alpha}{\alpha!} \partial^\alpha V(\operatorname{Re} x). \tag{4-3}$$

Indeed, since we have $\partial^\alpha V(\operatorname{Re} x) = \mathbb{O}(\langle \operatorname{Re} x\rangle^{-\nu-|\alpha|})$, the resummation is well defined up to

$$\mathbb{O}\big((|\operatorname{Im} x|/\langle \operatorname{Re} x\rangle)^\infty \langle \operatorname{Re} x\rangle^{-\nu}\big),$$

and the standard procedure of resummation (see, for example, [Dimassi and Sjöstrand 1999; Martinez 2002a]) also gives the required estimates on the derivatives of $\widetilde{V}$. Conversely, by a Taylor expansion, we see that any $\widetilde{V}$ satisfying the required conditions is necessarily a resummation of the series (4-3).

Now, if V^μ is a $|x|$-analytic $(\mu, \tilde{\nu})$-approximation of V, then for any $x = r\omega \in \Sigma(\mu)$ ($\omega \in S^{n-1}$) and $N \geq 0$, we have

$$\begin{aligned}
V^\mu(x) - \widetilde{V}(x) &= \sum_{k=0}^{N} \frac{i^k(\operatorname{Im} r)^k}{k!} \partial_r^k V^\mu(\operatorname{Re} r\cdot\omega) + \frac{(i\operatorname{Im} r)^{N+1}}{(N+1)!}\int_0^1 \partial_r^{N+1}\big(V^\mu((\operatorname{Re} r + it\operatorname{Im} r)\cdot\omega)\big)\,dt - \widetilde{V}(x) \\
&= \sum_{k=0}^{N} \frac{i^k(\operatorname{Im} r)^k}{k!} \partial_r^k\big(V^\mu(\operatorname{Re} x) - V(\operatorname{Re} x)\big) + \mathbb{O}(\mu^{N+1}\langle \operatorname{Re} x\rangle^{-\tilde{\nu}}) \\
&= \mathbb{O}(\mu^\infty \langle \operatorname{Re} x\rangle^{-\tilde{\nu}}) + \mathbb{O}(\mu^{N+1}\langle \operatorname{Re} x\rangle^{-\tilde{\nu}}),
\end{aligned}$$

and similarly, for any $\alpha \in \mathbb{N}^n$,

$$\partial^\alpha(V^\mu(x) - \widetilde{V}(x)) = \mathbb{O}(\mu^\infty \langle \operatorname{Re} x\rangle^{-\tilde{\nu}-|\alpha|}).$$

In particular, we have proved (ii).

Now, we proceed with the construction of such a V^μ.

For $x \in \mathbb{R}^n\backslash 0$, we set $\omega = x/|x|$, $r = |x|$, and $s = \ln r$. In particular, for any t real small enough, the dilation $x \mapsto e^t x$ becomes $(s, \omega) \mapsto (s + t, \omega)$ in the new coordinates (s, ω).

For $\omega \in S^{n-1}$ and $s \in \mathbb{C}$ with $|\operatorname{Im} s|$ small enough, we set $\widetilde{V}_1(s, \omega) := \widetilde{V}(e^s\omega)$, where $\widetilde{V}$ is an almost-analytic extension of V as before. Then for $|\operatorname{Im} s| < 2\mu$ and $\operatorname{Re} s \geq -\mu$, we define

$$V_1^\mu(s, \omega) := \frac{e^{-\tilde{\nu}s}}{2i\pi}\int_\gamma \frac{e^{\tilde{\nu}s'}\widetilde{V}_1(s', \omega)}{s - s'}\,ds', \tag{4-4}$$

where γ is the oriented complex contour

$$\gamma := ((+\infty, -2\mu] + 2i\mu) \cup (-2\mu + 2i[\mu, -\mu]) \cup ([-2\mu, +\infty) - 2i\mu). \tag{4-5}$$

By construction, $\widetilde{V}_1(s', \omega) = \mathbb{O}(e^{-\nu \operatorname{Re} s'})$, so that the previous integral is indeed absolutely convergent. Therefore, the (s, ω)-smoothness and s-holomorphy of V_1^μ are obvious consequences of Lebesgue's dominated convergence theorem. Since γ is symmetric with respect to $\mathbb{R}$, we also have that $V_1^\mu(s, \omega)$ is real for s real. Moreover, since $|s - s'| \geq \mu$ on γ, we see that

$$V_1^\mu(s, \omega) = \frac{e^{-\tilde{\nu}s}}{2i\pi}\int_{\gamma(s)} \frac{e^{\tilde{\nu}s'}\widetilde{V}_1(s', \omega)}{s - s'}\,ds' + \mathbb{O}(e^{-(\nu-\tilde{\nu})/(2\mu) - \tilde{\nu}\operatorname{Re} s}),$$

where

$$\gamma(s) := \Big(\gamma \cap \Big\{\operatorname{Re} s' \leq \operatorname{Re} s + \frac{1}{\mu}\Big\}\Big) \cup \Big(\operatorname{Re} s + \frac{1}{\mu} + 2i[-\mu, \mu]\Big).$$

In particular, $\gamma(s)$ is a simple oriented loop around s, and therefore, one obtains

$$V_1^\mu(s,\omega)-\widetilde{V}_1(s,\omega)=\frac{e^{-\tilde{\nu}s}}{2i\pi}\int_{\gamma(s)}\frac{e^{\tilde{\nu}s'}\widetilde{V}_1(s',\omega)-e^{\tilde{\nu}s}\widetilde{V}_1(s,\omega)}{s-s'}\,ds'+\mathcal{O}(e^{-(\nu-\tilde{\nu})/(2\mu)-\tilde{\nu}\operatorname{Re}s}). \tag{4-6}$$

Then writing

$$e^{\tilde{\nu}s'}\widetilde{V}_1(s',\omega)-e^{\tilde{\nu}s}\widetilde{V}_1(s,\omega)=(s-s')f(s,s',\omega)+\overline{(s-s')}g(s,s',\omega), \tag{4-7}$$

with $|\bar{\partial}_{s'}f|+|g|=\mathcal{O}(\mu^\infty)$, by Stokes' formula, we see that, for $\operatorname{Re}s\le 2/\mu$ and $|\operatorname{Im}s|\le\mu$, we have

$$V_1^\mu(s,\omega)-\widetilde{V}_1(s,\omega)=\mathcal{O}(\mu^\infty e^{-\tilde{\nu}\operatorname{Re}s}).$$

When $\operatorname{Re}s>2/\mu$ and $|\operatorname{Im}s|\le\mu$, setting

$$\gamma_1(s):=\left(\gamma\cap\left\{\operatorname{Re}s'\le\frac{1}{\mu}\right\}\right)\cup\left(\frac{1}{\mu}+2i[-\mu,\mu]\right),$$

Stokes' formula directly gives

$$\int_{\gamma_1(s)}\frac{e^{\tilde{\nu}s'}\widetilde{V}_1(s',\omega)}{s-s'}\,ds'=\mathcal{O}(\mu^\infty),$$

and thus, using again that $\widetilde{V}_1(s',\omega)=\mathcal{O}(e^{-\nu\operatorname{Re}s'})$, in that case we obtain

$$|V_1^\mu(s,\omega)|+|\widetilde{V}_1(s,\omega)|=\mathcal{O}(\mu^\infty e^{-\tilde{\nu}\operatorname{Re}s}).$$

In particular, in both cases we obtain

$$V_1^\mu(s,\omega)-\widetilde{V}_1(s,\omega)=\mathcal{O}(\mu^\infty e^{-\tilde{\nu}\operatorname{Re}s}), \tag{4-8}$$

uniformly for $\operatorname{Re}s\ge-\mu$, $|\operatorname{Im}s|\le\mu$ and $\mu>0$ small enough.

Then for $\alpha\in\mathbb{N}^n$ arbitrary, by differentiating (4-4) and observing that

$$e^{\tilde{\nu}s'}\widetilde{V}_1(s',\omega)-\sum_{k=0}^{N}\frac{1}{k!}(s'-s)^k\partial_s^k\big(e^{\tilde{\nu}s}\widetilde{V}_1(s,\omega)\big)=(s'-s)^{N+1}f_N(s,s',\omega)+g_N(s,s',\omega),$$

with $|\bar{\partial}_{s'}f_N|+|g_N|=\mathcal{O}(\mu^\infty)$, the same procedure gives

$$\partial^\alpha(V_1^\mu(s,\omega)-\widetilde{V}_1(s,\omega))=\mathcal{O}(\mu^\infty e^{-\tilde{\nu}\operatorname{Re}s}), \tag{4-9}$$

uniformly for $\operatorname{Re}s\ge-\mu$, $|\operatorname{Im}s|\le\mu$ and $\mu>0$ small enough. In particular, using the properties of $\widetilde{V}_1$, on the same set we also obtain

$$\partial^\alpha V_1^\mu(s,\omega)=\mathcal{O}(e^{-\tilde{\nu}\operatorname{Re}s}), \tag{4-10}$$

uniformly.

Now, let $\chi_1\in C^\infty(\mathbb{R};[0,1])$ be such that $\chi_1=1$ on $(-\infty,-1]$, and $\chi_1=0$ on $\mathbb{R}_+$. We set

$$V_2^\mu(s,\omega):=\chi_1(s/\mu)\widetilde{V}_1(s,\omega)+(1-\chi_1(s/\mu))V_1^\mu(s,\omega). \tag{4-11}$$

In particular, V_2^μ is well defined and smooth on $\mathbb{R}_- \cup (\mathbb{R}_+ + i[-\mu, \mu])$, and one has

$$\begin{aligned} V_2^\mu &= \widetilde{V}_1 && \text{for } s \in (-\infty, -\mu], \\ V_2^\mu &= V_1^\mu && \text{for } s \in \mathbb{R}_+ + i[-\mu, \mu], \\ \partial^\alpha (V_2^\mu - \widetilde{V}_1) &= \mathbb{O}(\mu^\infty) && \text{for } s \in [-\mu, \mu]. \end{aligned}$$

Finally, setting

$$V^\mu(x) := V_2^\mu\left(\ln|x|, \frac{x}{|x|}\right), \tag{4-12}$$

for $x \neq 0$, and $V^\mu(0) = \widetilde{V}(0)$, we easily deduce from the previous discussion (in particular (4-8), (4-9) and (4-10), and the fact that $\partial_r = r^{-1}\partial_s$), that V^μ is a $|x|$-analytic $(\mu, \tilde{\nu})$-approximation of V. □

5. The analytic distortion

In this section, for any $\theta > 0$ small enough, we construct a suitable distortion $x \mapsto x + i\theta A(x)$ satisfying $A(x) = x$ for $|x|$ large enough, and such that for $\mu \geq \theta$, the resolvent $(P_\theta^\mu - z)^{-1}$ of the corresponding distorted Hamiltonian P_θ^μ, admits sufficiently good estimates when $\operatorname{Im} z \geq h^{n_1}\theta$.

We fix $R_0 \geq 1$ arbitrarily. In the Appendix we will justify the following lemma by constructing the function announced in it:

Lemma 5.1. *For any $\lambda > 1$ large enough, there exists $f_\lambda \in C^\infty(\mathbb{R}_+)$ such that*

(i) $\operatorname{supp} f_\lambda \subset [R_0, +\infty)$;

(ii) $f_\lambda(r) = \lambda r$ *for* $r \geq 2\ln\lambda$;

(iii) $0 \leq f_\lambda(r) \leq r f_\lambda'(r) \leq 2\lambda r$ *everywhere*;

(iv) $f_\lambda' + |f_\lambda''| = \mathbb{O}(1 + f_\lambda)$ *uniformly*;

(v) *for any* $k \geq 1$, $f_\lambda^{(k)} = \mathbb{O}(\lambda)$ *uniformly.*

Now, we take $\lambda := h^{-n_1}$, and we set

$$b(r) := \frac{1}{\lambda} f_\lambda(r). \tag{5-1}$$

By the lemma, b satisfies

- $\operatorname{supp} b \subset [R_0, +\infty)$;
- $b(r) = r$ for $r \geq 2n_1 \ln(1/h)$;
- $0 \leq b(r) \leq r b'(r) \leq 2r$ everywhere;
- $b' + |b''| = \mathbb{O}(h^{n_1} + b)$ uniformly;
- For any $k \geq 1$, $b^{(k)} = \mathbb{O}(1)$ uniformly.

We set

$$A(x) := b(|x|)\frac{x}{|x|} = a(|x|)x,$$

where $a(r) := r^{-1}b(r) \in C^\infty(\mathbb{R}_+)$. For $\mu \geq \theta$ (both small enough), we can define the distorted operator P_θ^μ as in (2-4) obtained from P^μ by using the distortion

$$\Phi_\theta : \mathbb{R}^n \ni x \mapsto x + i\theta A(x) \in \mathbb{C}^n. \tag{5-2}$$

Here we use the fact that $|A(x)| \leq 2|x|$, and we also observe that, for any $\alpha \in \mathbb{N}^n$ with $|\alpha| \geq 1$, one has $\partial^\alpha \Phi_\theta(x) = \mathcal{O}(\theta\langle x\rangle^{1-|\alpha|})$ uniformly.

Proposition 5.2. *If R_0 is fixed sufficiently large, then for $0 < \theta \leq \mu$ both small enough, $h > 0$ small enough, $u \in H^2(\mathbb{R}^n)$, and $z \in \mathbb{C}$ such that $\operatorname{Re} z \in [\lambda_0/2, 2\lambda_0]$ and $\operatorname{Im} z \geq h^{n_1}\theta$, one has*

$$|\langle (P_\theta^\mu - z)u, u\rangle_{L^2}| \geq \frac{\operatorname{Im} z}{2}\|u\|_{L^2}^2.$$

Proof. Setting $F := {}^t dA(x) = dA(x)$, and $V_\theta^\mu(x) := V^\mu(x + i\theta A(x))$, we have

$$\begin{aligned}\langle P_\theta^\mu u, u\rangle &= \big\langle \big((I + i\theta F(x))^{-1} hD_x\big)^2 u, u\big\rangle + \langle V_\theta^\mu u, u\rangle \\ &= \langle (1 + i\theta F(x))^{-2} hD_x u, hD_x u\rangle \\ &\qquad + ih\big\langle \big(({}^t\nabla_x)(I + i\theta F(x))^{-1}\big)(I + i\theta F(x))^{-1} h\nabla_x u, u\big\rangle + \langle V_\theta^\mu u, u\rangle.\end{aligned}$$

Therefore, using Lemma A.1 and the equality

$$|\operatorname{Im} V^\mu(x)| = \mathcal{O}(|\operatorname{Im} x|\langle \operatorname{Re} x\rangle^{-\nu-1}),$$

valid for x complex, we find

$$\operatorname{Im}\langle P_\theta^\mu u, u\rangle \leq -\theta\|\sqrt{a(|x|)}hD_x u\|^2 + Ch\theta \int \left(|b''| + \frac{b'}{|x|} + \frac{b}{|x|^2}\right)|hD_x u|\cdot|u|\,dx + C_0\theta\left\|\frac{\sqrt{b}}{|x|^{(\nu+1)/2}}u\right\|^2$$

for some constants $C, C_0 > 0$; moreover C_0 is independent of the choice of R_0.

Thus, using the properties of b listed after (5-1), we obtain (with some other constant $C > 0$)

$$\begin{aligned}\operatorname{Im}\langle P_\theta^\mu u, u\rangle \leq -\theta\|\sqrt{a(|x|)}hD_x u\|^2 + Ch\theta \int \left(|b''| + \frac{b}{|x|} + h^{n_1}\right)|hD_x u|\cdot|u|\,dx \\ + C_0 R_0^{-\nu}\theta\left\|\sqrt{a}u\right\|^2.\end{aligned} \tag{5-3}$$

On the other hand for $z \in \mathbb{C}$, a similar computation gives

$$\begin{aligned}&\operatorname{Re}\langle \sqrt{a}(P_\theta^\mu - z)u, \sqrt{a}u\rangle \\ &\quad = -(\operatorname{Re} z)\|\sqrt{a}u\|^2 + \operatorname{Re}\big\langle \sqrt{a}\big((I + i\theta F(x))^{-1}hD_x\big)^2 u, \sqrt{a}u\big\rangle + \operatorname{Re}\langle \sqrt{a}V_\theta^\mu u, \sqrt{a}u\rangle \\ &\quad \leq -(\operatorname{Re} z)\|\sqrt{a}u\|^2 + (1 - 2\theta)^{-2}\|\sqrt{a}hD_x u\|^2 \\ &\qquad\qquad + Ch\int\left(|b''| + \frac{b}{|x|} + h^{n_1}\right)|hD_x u|\cdot|u|\,dx + C_0 R_0^{-\nu}\left\|\sqrt{a}u\right\|^2,\end{aligned}$$

still with C, C_0 positive constants, and C_0 independent of the choice of R_0. Therefore, if $\operatorname{Re} z \geq \lambda_0/2 > 0$ and R_0 is chosen sufficiently large, then for θ small enough, we obtain

$$\|\sqrt{a}u\|^2 \leq 4\lambda_0^{-1}\|\sqrt{a}hD_xu\|^2 + 4C\lambda_0^{-1}h\int\Big(|b''| + \frac{b}{|x|} + h^{n_1}\Big)|hD_xu|\cdot|u|\,dx \\ + 4\lambda_0^{-1}|\langle\sqrt{a}(P_\theta^\mu - z)u, \sqrt{a}u\rangle|. \tag{5-4}$$

The insertion of this estimate into (5-3) gives

$$\operatorname{Im}\langle P_\theta^\mu u, u\rangle \leq -(1 - 4C_0\lambda_0^{-1}R_0^{-\nu})\theta\|\sqrt{a}hD_xu\|^2 + C'h\theta\int\Big(|b''| + \frac{b}{|x|} + h^{n_1}\Big)|hD_xu|\cdot|u|\,dx \\ + C'\theta|\langle\sqrt{a}(P_\theta^\mu - z)u, \sqrt{a}u\rangle|, \tag{5-5}$$

with $C' > 0$ a constant.

Now, for $r \geq 2n_1\ln(1/h)$, by construction we have $b''(r) = 0$, while, for $r \leq 2n_1\ln(1/h)$, we have

$$|b''(r)| = \mathcal{O}(h^{n_1} + b) = \mathcal{O}(h^{n_1} + (\ln(1/h))a). \tag{5-6}$$

Then we deduce from (5-5)

$$\operatorname{Im}\langle P_\theta^\mu u, u\rangle \leq -(1 - 4C_0\lambda_0^{-1}R_0^{-\nu})\theta\|\sqrt{a}hD_xu\|^2 + C'h\theta\ln(1/h)\|\sqrt{a}hD_xu\|\cdot\|\sqrt{a}u\| \\ + C'h^{n_1+1}\theta\|hD_xu\|\cdot\|u\| + C'\theta|\langle\sqrt{a}(P_\theta^\mu - z)u, \sqrt{a}u\rangle|, \tag{5-7}$$

with some other constant $C' > 0$. Using again (5-6), we also deduce from (5-4) that

$$\|\sqrt{a}u\|^2 = \mathcal{O}\big(\|\sqrt{a}hD_xu\|^2 + |\langle\sqrt{a}(P_\theta^\mu - z)u, \sqrt{a}u\rangle| + h^{n_1+1}\|hD_xu\|\cdot\|u\|\big),$$

uniformly for $h > 0$ small enough, and thus, by (5-7),

$$\operatorname{Im}\langle P_\theta^\mu u, u\rangle \leq -\big(1 - 4C_0\lambda_0^{-1}R_0^{-\nu} - Ch\ln(1/h)\big)\theta\|\sqrt{a}hD_xu\|^2 \\ + Ch^{n_1+1}\theta\|hD_xu\|\cdot\|u\| + C\theta|\langle\sqrt{a}(P_\theta^\mu - z)u, \sqrt{a}u\rangle|. \tag{5-8}$$

Finally, for $\operatorname{Re} z \leq 2\lambda_0$, we use the (standard) ellipticity of the second-order partial differential operator $\operatorname{Re} P_\theta^\mu$, and the properties of V^μ, to obtain

$$\operatorname{Re}\langle(P_\theta^\mu - z)u, u\rangle \geq \tfrac{1}{C}\|hD_xu\|^2 - C\|u\|^2,$$

where C is again a new positive constant, independent of μ and θ. Combining with (5-8), and possibly increasing the value of R_0, this leads to

$$\operatorname{Im}\langle(P_\theta^\mu - z)u, u\rangle \leq (Ch^{n_1+1}\theta - \operatorname{Im} z)\|u\|^2 \\ + Ch^{n_1+1}\theta|\langle(P_\theta^\mu - z)u, u\rangle|^{1/2}\|u\| + C\theta|\langle(P_\theta^\mu - z)u, u\rangle|, \tag{5-9}$$

and thus, for $\operatorname{Im} z \geq h^{n_1}\theta$, and for $h, \theta > 0$ small enough, we can deduce

$$|\langle(P_\theta^\mu - z)u, u\rangle| \geq \frac{3\operatorname{Im} z}{4}\|u\|^2 - Ch^{n_1+1}\theta|\langle(P_\theta^\mu - z)u, u\rangle|^{1/2}\|u\|. \tag{5-10}$$

Then the result easily follows by solving this second-order inequality where the unknown variable is $|\langle (P_\theta^\mu - z)u, u\rangle|^{1/2}$, and by using again that $\operatorname{Im} z \gg h^{n_1+1}\theta$. $\square$

6. Proof of Theorem 2.1

6.1. *The invertible reference operator.* The purpose of this section is to introduce an operator without eigenvalues near λ_0, obtained as a finite-rank perturbation of P_θ^μ, $0 < \theta \le \mu$, and for which we have a nice estimate for the resolvent in the lower half plane. This operator will be used in the next section to construct a convenient Grushin problem.

Let $\chi_0 \in C_0^\infty(\mathbb{R}_+; [0,1])$ be equal to 1 on $[0, 1 + 2\lambda_0 + \sup|V|]$, and let $C_0 > \sup|\nabla V|$. We set

$$R = R(h) := 2n_1 \ln(1/h);$$
$$\widetilde{P}_\theta^\mu := P_\theta^\mu - iC_0\theta\chi_0(h^2D_x^2 + R^{-2}x^2).$$

Observe that $h^2D_x^2 + R^{-2}x^2$ is unitarily equivalent to $hR^{-1}(D_x^2 + x^2)$, so the rank of $\chi_0(h^2D_x^2 + R^{-2}x^2)$ is $\mathcal{O}(R^nh^{-n})$.

For $m \in \mathbb{R}$, we denote by $S(\langle\xi\rangle^m)$ the set of functions $a \in C^\infty(\mathbb{R}^{2n})$ such that for all $\alpha \in \mathbb{N}^{2n}$, one has

$$\partial_{x,\xi}^\alpha a(x,\xi) = \mathcal{O}(\langle\xi\rangle^m) \text{ uniformly.}$$

We also denote

$$\operatorname{Op}_h^W(a)u(x) = \frac{1}{(2\pi h)^n} \iint e^{i(x-y)\xi/h} a\Big(\frac{x+y}{2}, \xi\Big) u(y)\, dy\, d\xi, \tag{6-1}$$

the semiclassical Weyl quantization of such a symbol a.

Denoting by $\tilde{p}_\theta^\mu \in S(\langle\xi\rangle^2)$ the Weyl symbol of $\widetilde{P}_\theta^\mu$, we see that

$$\tilde{p}_\theta^\mu(x,\xi) = \big(({}^t d\Phi_\theta(x))^{-1}\xi\big)^2 + V^\mu(\Phi_\theta(x)) - iC_0\theta\chi_0(\xi^2 + R^{-2}x^2) + \mathcal{O}(h\theta\langle\xi\rangle), \tag{6-2}$$

uniformly with respect to (x,ξ), μ, θ, and h, and where the estimate on the remainder is in the sense of symbols (that is, one has the same estimate for all the derivatives). In particular, we have

$$\operatorname{Re} \tilde{p}_\theta^\mu(x,\xi) = \xi^2 + V(x) + \mathcal{O}(\theta\langle\xi\rangle^2). \tag{6-3}$$

Moreover,

- if $|x| \ge R$ and $|\xi|^2 \ge \lambda_0/2$, then
$$\operatorname{Im} \tilde{p}_\theta^\mu(x,\xi) \le -\frac{\theta}{C}\langle\xi\rangle^2 + \mathcal{O}(\theta R^{-\nu}) \le -\frac{\theta}{2C}\langle\xi\rangle^2; \tag{6-4}$$
- if $|x| \le R$ and $|\xi|^2 \le 2\lambda_0 + \sup|V|$, then
$$\operatorname{Im} \tilde{p}_\theta^\mu \le -C_0\theta + \theta\sup|\nabla V| + \mathcal{O}(h\theta) \le -\frac{\theta}{2C}, \tag{6-5}$$

where $C > 0$ is a constant, and the estimates are valid for h small enough. (For (6-5) we used the inequality $\operatorname{Im}\big(({}^t d\Phi_\theta(x))^{-1}\xi\big)^2 \le 0$, due to the particular form of $\Phi_\theta(x)$. See Lemma A.1 in the Appendix.)

Proposition 6.1. *There exists a constant $\widetilde{C} \geq 1$ such that for all $\mu > 0$, for all $\theta \in (0, \mu]$, for all z satisfying $|\operatorname{Re} z - \lambda_0| + \theta^{-1}|\operatorname{Im} z| \leq 4/\widetilde{C}$, and for all $h \in (0, 1/\widetilde{C}]$, one has*

$$\|(z - \widetilde{P}_\theta^\mu)^{-1}\| \leq \widetilde{C}\theta^{-1}.$$

Proof. We take two functions $\varphi_1, \varphi_2 \in C_b^\infty(\mathbb{R}^{2n}; [0, 1])$ (the space of smooth functions bounded with all their derivatives), such that

- $\varphi_1^2 + \varphi_2^2 = 1$ on $\mathbb{R}^{2n}$;
- $\operatorname{supp} \varphi_1$ is included in a small enough neighborhood of $\{\xi^2 + V(x) = \lambda_0\}$;
- $\varphi_1 = 1$ near $\{\xi^2 + V(x) = \lambda_0\}$.

In particular, φ_1 can be chosen in such a way that, on $\operatorname{supp} \varphi_1$, one has either $|x| \geq R$ together with $|\xi|^2 \geq \lambda_0/2$, or $|x| \leq R$ together with $|\xi|^2 \leq 2\lambda_0 + \sup|V|$. Therefore, we deduce from (6-4)–(6-5)

$$\frac{1}{\theta} \operatorname{Im} \tilde{p}_\theta^\mu \leq -\frac{1}{2C} \text{ on } \operatorname{supp} \varphi_1,$$

and thus,

$$\varphi_1^2 \frac{1}{\theta} \operatorname{Im} \tilde{p}_\theta^\mu + \frac{1}{2C}\varphi_1^2 \leq 0 \text{ on } \mathbb{R}^{2n}. \tag{6-6}$$

Moreover, it is easy to check that the function $(x, \xi) \mapsto \theta^{-1} \operatorname{Im} \tilde{p}_\theta^\mu$ is a nice symbol in $S(\langle\xi\rangle^2)$, uniformly with respect to μ and θ, that is, for all $\alpha \in \mathbb{N}^{2n}$, one has

$$\partial_{x,\xi}^\alpha (\theta^{-1} \operatorname{Im} \tilde{p}_\theta^\mu)(x, \xi) = \mathcal{O}(\langle\xi\rangle^2) \text{ uniformly,}$$

and we see from (6-2) that

$$\theta^{-1} \operatorname{Im} \tilde{p}_\theta^\mu \leq CR^{-\nu} + Ch\langle\xi\rangle,$$

with some new uniform constant $C > 0$.

Then setting $\phi_j := \operatorname{Op}_h^W(\varphi_j)$, writing $I = \phi_1^2 u + \phi_2^2 u + hQ$ where Q is a uniformly bounded pseudo-differential operator, and using the sharp Gårding inequality, we obtain

$$\begin{aligned}\langle \theta^{-1} \operatorname{Im} \widetilde{P}_\theta^\mu u, u\rangle &= \langle \phi_1 \theta^{-1} \operatorname{Im} \widetilde{P}_\theta^\mu \phi_1 u, u\rangle + \langle \theta^{-1} \operatorname{Im} \widetilde{P}_\theta^\mu \phi_2 u, \phi_2 u\rangle + \mathcal{O}(h\|u\|_{H^1}^2)\\ &\leq -\frac{1}{2C}\|\phi_1 u\|^2 + CR^{-\nu}\|\phi_2 u\|^2 + Ch\|\langle hD_x\rangle u\|^2,\end{aligned}$$

for all $u \in H^2(\mathbb{R}^n)$, and still for some new uniform constant $C > 0$. Hence,

$$|\operatorname{Im}\langle \widetilde{P}_\theta^\mu u, u\rangle| \geq \frac{\theta}{2C}\|\phi_1 u\|^2 - C\theta R^{-\nu}\|\phi_2 u\|^2 - Ch\theta\|\langle hD_x\rangle u\|^2. \tag{6-7}$$

On the other hand since $\operatorname{Re} \tilde{p}_\theta^\mu - \lambda_0 \in S(\langle\xi\rangle^2)$ is uniformly elliptic on $\operatorname{supp} \varphi_2$, the symbolic calculus permits us to construct $a \in S(\langle\xi\rangle^{-2})$ (still depending on μ and θ, but with uniform estimates), such that

$$a \,\sharp\, (\tilde{p}_{k,\theta} - \lambda_0) = \varphi_2 \,\sharp\, \varphi_2 + \mathcal{O}(h^\infty) \text{ in } S(1),$$

where $\sharp$ stands for the Weyl composition of symbols. As a consequence, denoting by A the Weyl quantization of a, we obtain

$$\|\langle hD_x\rangle \phi_2 u\|^2 = \big\langle \langle hD_x\rangle^2 A(\widetilde{P}_\theta^\mu - \lambda_0)u, u\big\rangle + \mathcal{O}(h)\|u\|^2,$$

and thus

$$\|(\widetilde{P}_\theta^\mu-\lambda_0)u\|\cdot\|u\|\geq\frac{1}{C}\|\langle hD_x\rangle\phi_2 u\|^2-Ch\|u\|^2. \tag{6-8}$$

Now, if $z\in\mathbb{C}$ is such that $|\operatorname{Re} z-\lambda_0|\leq\varepsilon$ and $|\operatorname{Im} z|\leq\varepsilon\theta$ ($\varepsilon>0$ fixed), we deduce from (6-7)–(6-8) that

$$\|(\widetilde{P}_\theta^\mu-z)u\|\cdot\|u\|\geq|\operatorname{Im}\langle(\widetilde{P}_\theta^\mu-z)u,u\rangle|\geq\frac{\theta}{2C}\|\phi_1u\|^2-C\theta R^{-\nu}\|\phi_2u\|^2-Ch\theta\|\langle hD_x\rangle u\|^2-\varepsilon\theta\|u\|^2,$$

$$\theta\|(\widetilde{P}_\theta^\mu-z)u\|\cdot\|u\|\geq\frac{\theta}{C}\|\langle hD_x\rangle\phi_2u\|^2-Ch\theta\|u\|^2-2\varepsilon\theta\|u\|^2,$$

which yields

$$(1+\theta)\|(\widetilde{P}_\theta^\mu-z)u\|\cdot\|u\|\geq\frac{\theta}{2C}\left(\|\phi_1u\|^2+\|\langle hD_x\rangle\phi_2u\|^2\right)-\theta(2Ch+CR^{-\nu}+3\varepsilon)\|\langle hD_x\rangle u\|^2. \tag{6-9}$$

Moreover, since ξ remains bounded on $\operatorname{supp}\varphi_1$, the norms $\|\langle hD_x\rangle u\|$ and $\|\phi_1u\|+\|\langle hD_x\rangle\phi_2u\|$ are uniformly equivalent, and thus, for ε and h small enough, we deduce from (6-9) that

$$\|(\widetilde{P}_\theta^\mu-z)u\|\cdot\|u\|\geq\frac{\theta}{4C}\|\langle hD_x\rangle u\|^2,$$

and the result follows. □

6.2. ***The Grushin problem.*** In this section, we reduce the estimate on $(P_\theta^\mu-z)^{-1}$ in Theorem 2.1, to that of a finite matrix, by means of some convenient Grushin problem (see for example [Sjöstrand 1997]).

Denote by $(e_1,\dots,e_M)$ an orthonormal basis of the range of the operator

$$K:=C_0\chi_0(h^2D_x^2+R^{-2}x^2).$$

In particular, $M=M(h)$ satisfies

$$M=\mathbb{O}(R^nh^{-n}). \tag{6-10}$$

Let $z\in\mathbb{C}$, and consider the two operators

$$\begin{aligned}R_+:\ L^2(\mathbb{R}^n)&\to\mathbb{C}^M\\ u&\mapsto(\langle u,e_j\rangle)_{1\leq j\leq M},\end{aligned}\qquad\begin{aligned}R_-(z):\ \mathbb{C}^M&\to L^2(\mathbb{R}^n)\\ u^-&\mapsto\textstyle\sum_{j=1}^M u_j^-(\widetilde{P}_\theta^\mu-z)e_j.\end{aligned}$$

Then the Grushin operator

$$\mathcal{G}(z):=\begin{pmatrix}P_\theta^\mu-z & R_-(z)\\ R_+ & 0\end{pmatrix}$$

is well defined from $H^2(\mathbb{R}^n)\times\mathbb{C}^M$ to $L^2(\mathbb{R}^n)\times\mathbb{C}^M$, and for z as in Proposition 6.1, it is easy to show that $\mathcal{G}(z)$ is invertible, and its inverse is given by

$$\mathcal{G}(z)^{-1}:=\begin{pmatrix}E(z) & E^+(z)\\ E^-(z) & E^{-+}(z)\end{pmatrix},$$

where

$$E(z) = (1 - T_M)(\widetilde{P}_\theta^\mu - z)^{-1}, \quad \text{with } T_M v := \sum_{j=1}^{M} \langle v, e_j \rangle e_j \ (v \in L^2),$$

$$E^+(z)v^+ = \sum_{j=1}^{M} v_j^+ \big(e_j + i\theta(1 - T_M)(\widetilde{P}_\theta^\mu - z)^{-1} K e_j\big), \quad (v_+ = (v_j^+)_{1\le j\le M} \in \mathbb{C}^M),$$

$$E^-(z)v = \big(\langle (\widetilde{P}_\theta^\mu - z)^{-1} v, e_j \rangle\big)_{1\le j\le M},$$

$$E^{-+}(z)v^+ = -v^+ + i\theta \Bigg(\sum_{\ell=1}^{M} v_\ell^+ \langle (\widetilde{P}_\theta^\mu - z)^{-1} K e_\ell, e_j \rangle \Bigg)_{1\le j\le M}.$$

Proposition 6.1 gives

$$\|E(z)\| + \|E^-(z)\| = \mathbb{O}(\theta^{-1}), \tag{6-11}$$

$$\|E^+(z)\| + \|E^{-+}(z)\| = \mathbb{O}(1), \tag{6-12}$$

uniformly for $\mu > 0$, $\theta \in (0, \mu]$, $h > 0$ small enough, and $|\operatorname{Re} z - \lambda_0| + \theta^{-1}|\operatorname{Im} z|$ small enough.

Hence, using the algebraic identity

$$(P_\theta^\mu - z)^{-1} = E(z) - E^+(z)E^{-+}(z)^{-1}E^-(z), \tag{6-13}$$

we finally obtain:

Proposition 6.2. *If $z \in \mathbb{C}$ is such that $|\operatorname{Re} z - \lambda_0| \le \widetilde{C}^{-1}$ and $|\operatorname{Im} z| \le 2\widetilde{C}^{-1}\theta$, and $E^{-+}(z)$ is invertible, then so is $P_\theta^\mu - z$, and one has*

$$\|(P_\theta^\mu - z)^{-1}\| = \mathbb{O}\big(\theta^{-1}(1 + \|E^{-+}(z)^{-1}\|)\big),$$

uniformly with respect to $\mu > 0$, $\theta \in (0, \mu]$, $h > 0$ small enough, and z such that $|\operatorname{Re} z - \lambda_0| \le \widetilde{C}^{-1}$ and $|\operatorname{Im} z| \le \widetilde{C}^{-1}\theta$.

Therefore, we have reduced the study of $(P_\theta^\mu - z)^{-1}$ to that of the $M \times M$ matrix $E^{-+}(z)^{-1}$.

6.3. ***Using the Maximum Principle.*** For $z \in J + i[-\theta/\widetilde{C}, 2\theta/\widetilde{C}]$, we define

$$D(z) := \det E^{-+}(z).$$

Then $z \mapsto D(z)$ is holomorphic in $J + i[-\theta/\widetilde{C}, 2\theta/\widetilde{C}]$. Setting $N := \#(\sigma(P_\theta^\mu) \cap (J + i[-\theta/\widetilde{C}, 2\theta/\widetilde{C}])$ and using (6-13), we see that $D(z)$ can be written as

$$D(z) = G(z) \prod_{\ell=1}^{N} (z - \rho_\ell),$$

with G holomorphic in $J + i[-\theta/\widetilde{C}, 2\theta/\widetilde{C}]$, $G(z) \ne 0$ for all $z \in J - i[0, \widetilde{C}^{-1}\theta]$.

Moreover, using (6-12) and (6-10), we obtain that for some uniform constant $C_1 > 0$,

$$|D(z)| = \prod_{\lambda \in \sigma(E^{-+}(z))} |\lambda| \le \|E^{-+}(z)\|^M \le C_1 e^{C_1 R^n h^{-n}}. \tag{6-14}$$

Lemma 6.3. *For every $\theta \in [0, \mu]$, there exists $r_\theta \in [\theta/(2\widetilde{C}), \theta/\widetilde{C}]$, such that for all $z \in J - ir_\theta$, and for all $\ell = 1, \dots, N$, one has*

$$|z - \rho_\ell| \geq \frac{\theta}{8\widetilde{C}N}.$$

Proof. By contradiction, if it was not the case, then for all t in $[-\theta/2\widetilde{C}, -\theta/\widetilde{C}]$, there should exist ℓ such that

$$|t - \operatorname{Im} \rho_\ell| < \frac{\theta}{8\widetilde{C}N}.$$

Therefore, the interval $[-\theta/2\widetilde{C}, -\theta/\widetilde{C}]$ would be included in $\bigcup_{\ell=1}^N [\operatorname{Im} \rho_\ell - \theta/(8\widetilde{C}N), \operatorname{Im} \rho_\ell + \theta/(8\widetilde{C}N)]$, which is impossible because of their respective sizes. □

From now on, we assume $\mathcal{P}(\tilde{\mu}, \mu; I, J)$ and setting

$$\mathcal{W}_\theta(J) := J + i[-r_\theta, 2\theta/\widetilde{C}],$$

we deduce from Lemma 6.3 that, for $\theta \in (0, \tilde{\mu}]$, z on the boundary of $\mathcal{W}_\theta(J)$, and for all $\ell = 1, \dots, N$,

$$|z - \rho_\ell| \geq \frac{1}{C_2}\theta,$$

for some constant $C_2 > 0$. As a consequence, using (6-14), on this set we obtain

$$|G(z)| \leq \theta^{-C_3} e^{C_3 R^n h^{-n}},$$

with some other uniform constant $C_3 > 0$. Then the maximum principle tells us that this estimate remains valid in the interior of $\mathcal{W}_\theta(J)$:

Proposition 6.4. *There exists a constant $C_3 > 0$ such that for all $\tilde{\mu}$, μ, I and J satisfying* (2-6)–(2-7) *such that $\mathcal{P}(\tilde{\mu}, \mu; I, J)$ holds, one has*

$$|G(z)| \leq \theta^{-C_3} e^{C_3 R^n h^{-n}},$$

for all $\theta \in (0, \tilde{\mu}]$, $z \in \mathcal{W}_\theta(J)$, and $h \in (0, 1/C_3]$.

6.4. ***Using the Harnack Inequality.*** Since $G(z) \neq 0$ on $\mathcal{W}_\theta(J)$, we can consider the function

$$H(z) := C_3 R^n h^{-n} - C_3 \ln \theta - \ln |G(z)|.$$

Then H is harmonic in $\mathcal{W}_\theta(J)$, and by Proposition 6.4, it is also nonnegative. Using the algebraic formula

$$E^{-+}(z)^{-1} = -R_+(P_\theta^\mu - z)^{-1} R_-(z)$$

and the fact that $(P_\theta^\mu - z)^{-1} R_-(z) u^- = \sum_{j=1}^M u_j (I - i\theta(P_\theta^\mu - z)^{-1} K) e_j$, we deduce from Proposition 5.2 that, for $z \in [\lambda_0/2, 2\lambda_0] + i[\theta h^{n_1}, 1]$, one has

$$\|E^{-+}(z)^{-1}\| \leq 1 + 2C_0 h^{-n_1}.$$

As a consequence, for such values of z, we obtain

$$\frac{1}{D(z)} = \det E^{-+}(z)^{-1} \leq (1 + 2C_0 h^{-n_1})^M,$$

and thus

$$|G(z)| = |D(z)| \prod_{\ell=1}^{N} |z-\rho_\ell|^{-1} \geq \frac{1}{C_4} h^{n_1 M},$$

with some uniform constant $C_4 > 0$. In particular, for any $\lambda \in \mathbb{R}$ such that $\lambda + i\theta h^{n_1} \in \mathcal{W}_\theta(J)$, this gives

$$H(\lambda + i\theta h^{n_1}) \leq C_3 R^n h^{-n} - C_3 \ln\theta + \ln C_4 - n_1 M \ln h. \tag{6-15}$$

Now, the Harnack inequality tells us that, for any λ, r, such that

$$\operatorname{dist}(\lambda, \mathbb{R}\setminus J) \geq \widetilde{C}^{-1}\theta, \quad r \in [0, \widetilde{C}^{-1}\theta)$$

and for any $\alpha \in \mathbb{R}$, one has

$$H(\lambda + ih^{n_1}\theta + re^{i\alpha}) \leq \frac{\widetilde{C}^{-2}\theta^2}{(\widetilde{C}^{-1}\theta - r)^2} H(\lambda + ih^{n_1}\theta).$$

In particular, setting

$$\widetilde{\mathcal{W}}_\theta(J) := \left\{ z \in \mathbb{C};\ \operatorname{dist}(\operatorname{Re} z, \mathbb{R}\setminus J) \geq \widetilde{C}^{-1}\theta\,,\ |\operatorname{Im} z| \leq (2\widetilde{C})^{-1}\theta \right\}$$

and using (6-15), we find

$$H(z) \leq 5C_3 R^n h^{-n} - 5C_3 \ln\theta + 5\ln C_4 - 5n_1 M \ln h,$$

for all $z \in \widetilde{\mathcal{W}}_\theta(J)$, that is,

$$\ln|G(z)| \geq -4C_3 R^n h^{-n} + 4C_3 \ln\theta - 5\ln C_4 + 5n_1 M \ln h,$$

or, equivalently,

$$|G(z)| \geq C_4^{-5} \theta^{4C_3} h^{5n_1 M} e^{-4C_3 R^n h^{-n}}. \tag{6-16}$$

Finally, writing $E^{-+}(z)^{-1} = D(z)^{-1}\widetilde{E}^{-+}(z)$, where $\widetilde{E}^{-+}(z)$ stands for the transposed of the comatrix of $E^{-+}(z)$, we see that

$$\|E^{-+}(z)^{-1}\| \leq e^{CM} |G(z)|^{-1} \prod_{\ell=1}^{N} |z-\rho_\ell|^{-1},$$

and therefore we deduce from (6-16) and (6-10) that

$$\|E^{-+}(z)^{-1}\| \leq \theta^{-C} h^{-CR^n h^{-n}} \prod_{\ell=1}^{N} |z-\rho_\ell|^{-1},$$

with some new uniform constant $C \geq 1$. Thus, using Proposition 6.2, and the fact that $R = \mathbb{O}(|\ln h|)$, we have proved:

Proposition 6.5. *There exists a constant $\check{C} > 0$ such that for all $\tilde{\mu}$, μ, I and J satisfying* (2-6)–(2-7) *such that $\mathcal{P}(\tilde{\mu}, \mu; I, J)$ holds, one has*

$$\|(P_\theta^\mu - z)^{-1}\| \leq \theta^{-\check{C}} h^{-\check{C}|\ln h|^n h^{-n}} \prod_{\ell=1}^{N} |z-\rho_\ell|^{-1},$$

for all $\theta \in (0, \tilde{\mu}]$, $z \in \widetilde{\mathcal{W}}_\theta(J)$, and $h \in (0, 1/\check{C}]$.

6.5. ***Using the 3-lines theorem.*** Now, following an idea of [Tang and Zworski 1998], we define

$$\Psi(z) := \int_a^b e^{-(z-\lambda)^2/\theta^2}\, d\lambda,$$

where

$$[a,b] := \big\{\lambda \in \mathbb{R}\,;\ \operatorname{dist}(\lambda, \mathbb{R}\setminus J) \geq \widetilde{C}^{-1}\theta + \check{C}^{1/2}\omega_h(\theta)\big\}.$$

We have the following:

- If $\operatorname{Im} z = 2\theta h^{n_1}$, then
$$|\Psi(z)| \leq (b-a)e^{4h^{2n_1}} = \mathbb{O}(h^\delta) \leq 1.$$
- If $\operatorname{Im} z = -\theta/(2\widetilde{C})$, then
$$|\Psi(z)| \leq (b-a)e^{1/4\widetilde{C}^2} = \mathbb{O}(h^\delta) \leq 1.$$
- If $\operatorname{Re} z \in \big\{a - \check{C}^{1/2}\omega_h(\theta), b + \check{C}^{1/2}\omega_h(\theta)\big\}$ and $\operatorname{Im} z \in [-\theta/(2\widetilde{C}), 2\theta h^{n_1}]$, then
$$|\Psi(z)| \leq (b-a)e^{1/4\widetilde{C}^2} e^{-\check{C}\omega_h(\theta)^2/\theta^2} = \mathbb{O}(h^\delta)\theta^{\check{C}} h^{\check{C}|\ln h|^n h^{-n}} \leq \theta^{\check{C}} h^{\check{C}|\ln h|^n h^{-n}}.$$

Then for $z \in \widetilde{\mathcal{W}}_\theta(J)$, we consider the operator-valued function

$$Q(z) := \Psi(z) \prod_{\ell=1}^{N} (z - \rho_\ell)(P_\theta^\mu - z)^{-1}$$

that is holomorphic on $\widetilde{\mathcal{W}}_\theta(J)$ (this can be seen, for example, from (6-13)). Using, Proposition 5.2, Proposition 6.5, and the previous properties of $\Psi(z)$, we see that $Q(z)$ satisfies:

- If $\operatorname{Im} z = 2\theta h^{n_1}$, then
$$\|Q(z)\| \leq \theta^{-1} h^{-n_1}.$$
- If $\operatorname{Im} z = -\theta/(2\widetilde{C})$, then
$$\|Q(z)\| \leq \theta^{-\check{C}} h^{-\check{C}|\ln h|^n h^{-n}}.$$
- If $\operatorname{Re} z \in \{a - \check{C}^{1/2}\omega_h(\theta), b + \check{C}^{1/2}\omega_h(\theta)\}$ and $\operatorname{Im} z \in [-\theta/(2\widetilde{C}), 2\theta h^{n_1}]$, then
$$\|Q(z)\| \leq 1.$$

Therefore, setting

$$\check{\mathcal{W}}_\theta(J) := [a - \check{C}^{1/2}\omega_h(\theta), b + \check{C}^{1/2}\omega_h(\theta)] + i[-\theta/(2\widetilde{C}), 2\theta h^{n_1}],$$

(that is included in $\widetilde{\mathcal{W}}_\theta(J)$), we see that the subharmonic function $z \mapsto \ln \|Q(z)\|$ satisfies

$$\ln \|Q(z)\| \leq \psi(z) \quad \text{on } \partial\check{\mathcal{W}}_\theta(J),$$

where ψ is the harmonic function defined by

$$\psi(z) := \frac{2\theta h^{n_1} - \operatorname{Im} z}{2\theta h^{n_1} + \theta/(2\widetilde{C})}\, \check{C}\big(|\ln h|^{n+1}h^{-n} + |\ln\theta|\big) + \frac{\operatorname{Im} z + \theta/(2\widetilde{C})}{2\theta h^{n_1} + \theta/(2\widetilde{C})}\, |\ln(\theta h^{n_1})|.$$

As a consequence, by the properties of subharmonic functions, we deduce that $\ln\|Q(z)\| \le \psi(z)$ everywhere in $\check{\mathcal{W}}_\theta(J)$, and in particular, for $|\mathrm{Im}\, z| \le 2\theta h^{n_1}$, we obtain

$$\ln\|Q(z)\| \le 8\widetilde{C}\check{C}h^{n_1}\bigl(|\ln h|^{n+1}h^{-n} + |\ln\theta|\bigr) + |\ln(\theta h^{n_1})|$$

Hence, since $n_1 > n$, we have proved the existence of some uniform constant $C \ge 1$, such that

$$\ln\|Q(z)\| \le \ln C + C|\ln(\theta h^{n_1})| \quad \text{ for } z \in \check{\mathcal{W}}_\theta(J) \text{ and } h \in (0, 1/C].$$

Coming back to P_θ^μ, this means that, for $z \in \check{\mathcal{W}}_\theta(J)$ different from $\rho_1, \dots, \rho_N$, we have

$$|\Psi(z)|\,\|(P_\theta^\mu - z)^{-1}\| \le C(\theta h^{n_1})^{-C}\prod_{\ell=1}^{N}|z-\rho_\ell|^{-1}.$$

On the other hand if $\mathrm{dist}(\mathrm{Re}\, z, \mathbb{R}\setminus J) \ge 2\check{C}^{1/2}\omega_h(\theta)$, and $|\mathrm{Im}\, z| \le 2\theta h^{n_1}$, then writing $z = s + it$, we have

$$\Psi(z) = \theta e^{t^2/\theta^2}\int_{(a-s)/\theta}^{(b-s)/\theta} e^{-u^2+2i(t/\theta)u}\,du.$$

Now, $|t/\theta| \le 2h^{n_1} \to 0$ uniformly, and we see that

$$(a-s)/\theta \le \widetilde{C}^{-1} - \check{C}^{1/2}\omega_h(\theta)/\theta \le \widetilde{C}^{-1} - (h^{-n}|\ln h|)^{1/2} \to -\infty \text{ uniformly.}$$

In the same way, we have $(b-s)/\theta \to +\infty$ uniformly as $h \to 0_+$. Therefore, we easily conclude that

$$|\Psi(z)| \ge \frac{\theta}{C},$$

when $h \in (0, 1/C]$, with some new uniform constant $C > 0$.

As a consequence, using also that $\theta \le h^\delta$, we finally obtain:

Proposition 6.6. *There exists a constant $C_0 \ge 1$, such that for all $\tilde{\mu}$, μ, I and J satisfying* (2-6)–(2-7), *the property $\mathcal{P}(\tilde{\mu},\ mu; I, J)$ implies*

$$\|(P_\theta^\mu - z)^{-1}\| \le C_0\theta^{-C_0}\prod_{\ell=1}^{N}|z-\rho_\ell|^{-1}, \tag{6-17}$$

for all $z \in J' + i[-2\theta h^{n_1}, 2\theta h^{n_1}]$, and for all $h \in (0, 1/C_0]$, where

$$J' = \bigl\{\lambda \in \mathbb{R}\,;\ \mathrm{dist}(\lambda, \mathbb{R}\setminus J) \ge C_0\omega_h(\theta)\bigr\}.$$

Since $J' = J + \mathcal{O}(\omega_h(\theta))$, Theorem 2.1 is proved.

7. Proof of Theorem 2.2

Suppose $\mathcal{P}(\tilde{\mu}, \mu; I, J)$ holds, and $\tilde{\mu} \ge \mu^{N_0}$ for some constant $N_0 \ge 1$. Then for any $\theta \in [\mu^{N_0}, \tilde{\mu}]$, any constant $N_1 \ge 1$, and any $\mu' \in [\max(\theta, \mu^{N_1}), \mu^{1/N_1}]$, we can write

$$z - P_\theta^{\mu'} = (z - P_\theta^\mu)(1 + (z - P_\theta^\mu)^{-1}W), \tag{7-1}$$

with

$$W := P_\theta^\mu - P_\theta^{\mu'} = V^\mu(x + iA_\theta(x)) - V^{\mu'}(x + iA_\theta(x)) = \mathcal{O}(\mu^\infty \langle x\rangle^{-\nu}), \tag{7-2}$$

uniformly (see Section 4). Moreover, taking J' as in Proposition 6.6, we have:

Lemma 7.1. *Let $N \geq 1$ be a constant, such that $N \geq \#\Gamma(\tilde{\mu}, \mu, J)$ for all h small enough. Then for any $\theta \in [\mu^{N_0}, \tilde{\mu}]$, there exists $\tau \in [\theta h^{n_1}, 2\theta h^{n_1}]$, such that*

$$\operatorname{dist}\big(\partial(J' + i[-\tau, \tau]), \Gamma(\tilde{\mu}, \mu, J)\big) \geq \frac{\theta h^{n_1}}{4N}. \tag{7-3}$$

Here, $\partial(J' + i[-\tau, \tau])$ stands for the boundary of $J' + i[-\tau, \tau]$.

Proof. If it were not the case, using $\mathcal{P}(\tilde{\mu}, \mu; I, J)$, we see that, for all $t \in [-2\theta h^{n_1}, -\theta h^{n_1}]$, there should exist $\rho \in \Gamma(\tilde{\mu}, \mu, J)$, such that

$$|t - \operatorname{Im} \rho| \leq \frac{\theta h^{n_1}}{4N}.$$

That is, we would have

$$[-2\theta h^{n_1}, -\theta h^{n_1}] \subset \bigcup_{\rho \in \Gamma(\tilde{\mu}, \mu, J)} \left[\rho - \frac{\theta h^{n_1}}{4N}, \rho + \frac{\theta h^{n_1}}{4N}\right],$$

which, again, is not possible because of the respective size of these two sets. □

Remark 7.2. With a similar proof, we see that the result of Lemma 7.1 remains valid if one replaces the interval $[\theta h^{n_1}, 2\theta h^{n_1}]$ by $[\theta h^{n_1}, \theta h^{n_1} + (\theta h^{n_1})^M]$, and one substitutes $(\theta h^{n_1})^M$ to θh^{n_1} in (7-3), where $M \geq 1$ is any arbitrary fixed number.

Using Lemma 7.1 and Theorem 2.1, we see that, for any $z \in \partial(J' + i[-\tau, \tau])$, we have

$$\|(P_\theta^\mu - z)^{-1}\| \leq C_1 \theta^{-C_1} \leq C_1 \mu^{-C_1 N_0},$$

with some new uniform constant C_1, and thus, by (7-1) and (7-2), $z - P_\theta^{\mu'}$ is invertible, too, for $z \in \partial(J' + i[-\tau, \tau])$, and the two spectral projectors

$$\Pi := \frac{1}{2i\pi} \oint_{\partial(J'+i[-\tau,\tau])} (z - P_\theta^\mu)^{-1}\, dz, \qquad \Pi' := \frac{1}{2i\pi} \oint_{\partial(J'+i[-\tau,\tau])} (z - P_\theta^{\mu'})^{-1}\, dz, \tag{7-4}$$

are well-defined and satisfy

$$\|\Pi - \Pi'\| = \mathcal{O}(\mu^\infty). \tag{7-5}$$

In particular, Π and Π' have the same rank ($\leq N$), and one has

$$\|P_\theta^\mu \Pi - P_\theta^{\mu'} \Pi'\| = \mathcal{O}(\mu^\infty). \tag{7-6}$$

Therefore, the two sets $\sigma(P_\theta^{\mu'}) \cap (J' + i[-\tau, \tau])$ and $\sigma(P_\theta^\mu) \cap (J' + i[-\tau, \tau])$ coincide up to $\mathcal{O}(\mu^\infty)$ uniformly by standard finite dimensional arguments, and Theorem 2.2 follows.

8. Proof of Theorem 2.5

Now, for any integer $k \geq 0$, we set

$$\mu_k := h^{kn_1} \tilde{\mu}.$$

Since $\mathcal{P}(\tilde{\mu}, \mu; I, J)$ holds, we can apply Theorem 2.2 with $\mu' \in [\mu_1, \mu_0]$, and deduce the existence of $J_1 \subset J$, with $J_1 = J + \mathcal{O}(\omega_h(\mu_0))$ and $I_1 \supset I$ with $I_1 = I + \mathcal{O}(\mu_0^\infty)$, independent of μ', such that $\mathcal{P}(h^{n_1}\mu', \mu'; I_1, J_1)$ holds. In particular, $\mathcal{P}(\mu_1, \mu_0; I_1, J_1)$ holds, and we can apply Theorem 2.2 again, this time with $\mu' \in [\mu_2, \mu_1]$. Iterating the procedure, we see that, for any $k \geq 0$, there exists

$$I_{k+1} = I_k + \mathcal{O}(\mu_k^\infty), \quad J_{k+1} = J_k + \mathcal{O}(\omega_h(\mu_k)),$$

hence,

$$I_{k+1} = I + \mathcal{O}(\mu_0^\infty + \cdots + \mu_k^\infty), \quad J_{k+1} = J + \mathcal{O}(\omega_h(\mu_0) + \cdots + \omega_h(\mu_k)),$$

where the $\mathcal{O}$'s are also uniform with respect to k, such that $\mathcal{P}(h^{n_1}\mu', \mu'; I_{k+1}, J_{k+1})$ holds for all $\mu' \in [\mu_{k+1}, \mu_k]$.

Since the two series $\sum_k \omega_h(\mu_k) = \mathcal{O}(\omega_h(\tilde{\mu}))$ and $\sum_k \mu_k^M = \mathcal{O}(\mu^M)$ ($M \geq 1$ arbitrary) are convergent, one can find $I' = I + \mathcal{O}(\mu^\infty)$ and $J' = J + \mathcal{O}(\omega_h(\tilde{\mu}))$, such that

$$I' \supset \bigcup_{k\geq 0} I_k, \quad J' \subset \bigcap_{k \geq 0} J_k.$$

Then by construction, $\mathcal{P}(h^{n_1}\mu', \mu'; I', J')$ holds for all $\mu' \in (0, \tilde{\mu}]$, and Theorem 2.5 is proved.

9. Proof of Theorem 2.6: the set of resonances

From the proof of Theorem 2.5 (and with the same notation) we learn that, for all $k \geq 0$, property $\mathcal{P}(\mu_{k+1}, \mu_k; I_{k+1}, J_{k+1})$ holds. Therefore, applying Theorem 2.2 with $\theta = \mu' = \mu_{k+1}$, we see that there exist $\tau_{k+2} \in [\mu_{k+2}, 2\mu_{k+2}]$, $J'_{k+1} = J_{k+1} + \mathcal{O}(\omega_h(\mu_{k+1}))$, and a bijection

$$b_k : \Gamma(P^{\mu_k}) \cap (J'_{k+1} - i[0, \tau_{k+2}]) \to \Gamma(P^{\mu_{k+1}}) \cap (J'_{k+1} - i[0, \tau_{k+2}])$$

such that

$$b_k(\lambda) - \lambda = \mathcal{O}(\mu_k^\infty) \text{ uniformly.} \tag{9-1}$$

In addition, we deduce from Lemma 7.1 that the τ_k can be chosen in such a way that

$$\operatorname{dist}\big(\partial(J'_{k+1} + i[-\tau_{k+2}, \tau_{k+2}]), \Gamma(P^{\mu_k})\big) \geq \frac{\mu_k^C}{C}, \tag{9-2}$$

for some constant $C > 0$. We set

$$\Lambda_k := \Gamma(P^{\mu_k}) \cap (J'_{k+1} - i[0, \tau_{k+2}]),$$

where the elements are repeated according to their multiplicity (see Figure 2).

Starting from an arbitrary element λ_j of Λ_0 ($1 \leq j \leq N := \# \Lambda_0 = \mathcal{O}(1)$), we distinguish two cases.

Case A. For all $k \geq 0$, $b_k \circ b_{k-1} \circ \cdots \circ b_0(\lambda_j) \in \Lambda_{k+1}$.

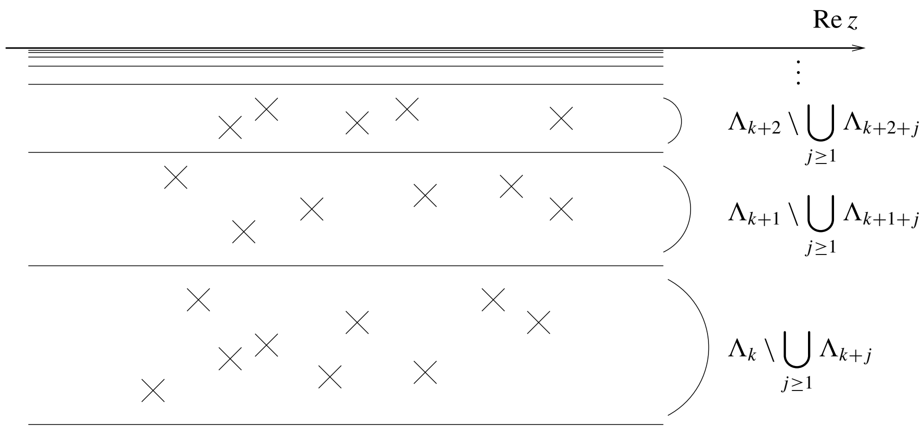

Figure 2. Construction of the set of resonances.

In that case, we can consider the sequence defined by

$$\lambda_{j,k} := b_k \circ b_{k-1} \circ \cdots \circ b_0(\lambda_j), \quad k \geq 0.$$

Using (9-1), we see that, for any $k > \ell \geq 0$, we have

$$|\lambda_{j,k} - \lambda_{j,\ell}| \leq \sum_{m=\ell}^{k-1} |\lambda_{j,m+1} - \lambda_{j,m}| \leq C_1 \sum_{m=\ell}^{k-1} \mu_{m+1} \leq C_1 \mu_0 \frac{h^{n_1 \ell}}{1-h^{n_1}},$$

so that $(\lambda_{j,k})_{k\geq 1}$ is a Cauchy sequence, and we set

$$\rho_j := \lim_{k\to+\infty} \lambda_{j,k}.$$

Notice that according to this definition, we have a natural notion of multiplicity of a resonance ρ, namely the number of sequences ρ_j converging to ρ.

Case B. There exists $k_j \geq 0$ such that $b_{k-1} \circ \cdots \circ b_0(\lambda_j) \in \Lambda_k$ for all $k \leq k_j$, while $b_{k_j} \circ \cdots \circ b_0(\lambda_j) \notin \Lambda_{k_j+1}$.

(Here, and in the sequel, we use the convention of notation: $b_{-1} \circ b_0 := \mathrm{Id}$.) We set

$$\rho_j := b_{k_j} \circ \cdots \circ b_0(\lambda_j).$$

In particular, since, by construction, $\operatorname{Re} \rho_j \in I_{k_j+2} \subset J_{k_j+1}$, and $\rho_j \notin \Lambda_{k_j+1}$, we see that, necessarily, $\operatorname{Im} \rho_j \in [-\tau_{k_j+2}, -\tau_{k_j+3})$.

Moreover, if in Case A, we set $k_j := +\infty$, then for any $j = 1, \dots, \#\Lambda_0$ and $k \geq 0$, in both cases we have the equivalence

$$|\operatorname{Im} \rho_j| \leq \tau_{k+2} \iff k \leq k_j. \tag{9-3}$$

Now, if $\mu' \in (0, \tilde{\mu}]$, then $\mu' \in (\mu_{k+1}, \mu_k]$ for some $k \geq 0$, and Theorem 2.2 tells us that the intersection $\Gamma(P^{\mu'}) \cap (J'_{k+1} - i[0, \tau_{k+2}])$ coincides with Λ_k up to $\mathbb{O}(\mu_k^\infty)$ $(= \mathbb{O}((\mu')^\infty))$. Therefore, setting

$$\Lambda := \{\rho_1, \dots, \rho_N\},$$

the first part of Theorem 2.6 will be proved if we can show the existence, for any $k \geq 0$, of a bijection

$$\tilde{b}_k : \Lambda \cap (J'_{k+1} - i[0, \tau_{k+2}]) \to \Lambda_k,$$

such that $\tilde{b}_k(\rho) - \rho = \mathcal{O}(\mu_k^\infty)$ uniformly. (Actually, we do not necessarily have $\tau_{k+2} \in [h^{2n_1}\mu', 2h^{2n_1}\mu']$, but rather, $\tau_{k+2} \in [h^{2n_1}\mu', 2h^{n_1}\mu')$. However, if $\tau_{k+2} \geq 2h^{2n_1}\mu'$, an argument similar to that of Lemma 6.3 or Lemma 7.1 gives the result stated in Theorem 2.6.)

By construction, we have

$$\Lambda_k = \{b_{k-1} \circ \cdots \circ b_0(\lambda_j)\,;\ j = 1, \dots, N \text{ such that } k_j \geq k\}.$$

while, by (9-3),

$$\Lambda \cap (J'_{k+1} - i[0, \tau_{k+2}]) = \{\rho_j\,;\ j = 1, \dots, N \text{ such that } k_j \geq k\}.$$

Then for all j satisfying $k_j \geq k$, we set

$$\tilde{b}_k(\rho_j) := b_{k-1} \circ \cdots \circ b_0(\lambda_j),$$

so that $\tilde{b}_k$ defines a bijection between $\Lambda \cap (J'_{k+1} - i[0, \tau_{k+2}])$ and Λ_k. Moreover, in Case A, for any $M \geq 1$, we have

$$|\tilde{b}_k(\rho_j) - \rho_j| = \lim_{\ell \to +\infty} |b_\ell \circ \cdots \circ b_k(\tilde{b}_k(\lambda_j)) - \tilde{b}_k(\lambda_j)| \leq \sum_{m=k}^{+\infty} C_M \mu_m^M = \frac{C_M \mu_k^M}{1 - h^{n_1}},$$

while, in Case B, we obtain

$$|\tilde{b}_k(\rho_j) - \rho_j| = |b_{k_j} \circ \cdots \circ b_k(\tilde{b}_k(\lambda_j)) - \tilde{b}_k(\lambda_j)| \leq \sum_{k \leq m \leq k_j} C_M \mu_m^M \leq \frac{C_M \mu_k^M}{1 - h^{n_1}},$$

(with the usual convention $\sum_{m \in \varnothing} := 0$). Therefore, in both cases, for $h > 0$ small enough, we find

$$|\tilde{b}_k(\rho_j) - \rho_j| \leq 2C_M \mu_k^M,$$

and this gives the first part of Theorem 2.6.

For the second part of Theorem 2.6, let $\widetilde{\Lambda}$ be another set satisfying ($\star$). In particular, for any $k \geq 0$, there exist $\tau_{k+2}, \widetilde{\tau}_{k+2} \in [\mu_{k+2}, 2\mu_{k+2}]$, such that $\widetilde{\Lambda} \cap (J'_{k+1} - i[0, \widetilde{\tau}_{k+2}])$ (respectively $\Lambda \cap (J'_{k+1} - i[0, \tau_{k+2}])$) coincides with $\widetilde{\Lambda}_k := \Gamma(P^{\mu_k}) \cap (J'_{k+1} - i[0, \widetilde{\tau}_{k+2}])$, (respectively Λ_k), up to $\mathcal{O}(\mu_k^\infty)$.

Therefore, taking $k = 0$ and using again an argument similar to that of Lemma 6.3 or Lemma 7.1 that gives the existence of $\tau' \in [\frac{1}{2}\mu_2, \mu_2]$ and $C > 0$ constant such that

$$\operatorname{dist}\big(\partial(J'_1 + i[-\tau', \tau']), \Gamma(P^{\mu_0})\big) \geq \frac{\mu_0^C}{C}, \tag{9-4}$$

we obtain that the two sets $\Lambda \cap (J'_1 - i[0, \tau'])$ and $\widetilde{\Lambda} \cap (J'_1 - i[0, \tau'])$ coincide up to $\mathcal{O}(\mu_0^\infty)$, and thus have same cardinality. For $k \geq 0$, we denote by

$$B_k : \Lambda_k \to \Lambda \cap (J'_{k+1} - i[0, \tau_{k+2}]),$$
$$\widetilde{B}_k : \widetilde{\Lambda}_k \to \widetilde{\Lambda} \cap (J'_{k+1} - i[0, \widetilde{\tau}_{k+2}])$$

the corresponding bijections. Then, thanks to (9-4), we can consider the bijection

$$\varphi_0 = \widetilde{B}_0 \circ B_0^{-1}|_{\Lambda\cap(J_1'-i[0,\tau'])} : \Lambda \cap (J_1' - i[0,\tau']) \to \widetilde{\Lambda} \cap (J_1' - i[0,\tau']).$$

Using (9-2) and the fact that $\widetilde{B}_k$ differ from the identity by $\mathcal{O}(\mu_k^\infty)$, we see that, for $k \geq 1$,

$$\operatorname{dist}\left(\partial(J_{k+1}' + i[-\tau_{k+2}, \tau_{k+2}]), \widetilde{\Lambda}\right) \geq \frac{\mu_k^C}{C}, \tag{9-5}$$

for some other constant $C > 0$.

Then setting

$$\mathscr{E}_0 := \Lambda \cap \{-\tau' \leq \operatorname{Im} z < -\tau_3\},$$

and for $k \geq 1$,

$$\mathscr{E}_k := \Lambda \cap \{-\tau_{k+2} \leq \operatorname{Im} z < -\tau_{k+3}\},$$

we see that, for all $k \geq 1$, the map

$$\widetilde{B}_k \circ B_k^{-1}|_{\mathscr{E}_k} : \mathscr{E}_k \to \widetilde{\Lambda} \cap \{-\tau_{k+2} \leq \operatorname{Im} z < -\tau_{k+3}\} \tag{9-6}$$

is a bijection.

Finally, for $\rho \in \Lambda \cap (J_1' - i[0,\tau'])$, we define

- $B(\rho) = \widetilde{B}_k \circ B_k^{-1}(\rho)$, if $\rho \in \mathscr{E}_k$ for some $k \geq 0$;
- $B(\rho) = \rho$, if $\rho \in \mathbb{R}$.

We first show:

Lemma 9.1. $\Lambda \cap \mathbb{R} = \widetilde{\Lambda} \cap \mathbb{R}$.

Proof. We only show that any ρ in $\Lambda \cap \mathbb{R}$ is also in $\widetilde{\Lambda}$, the proof of the other inclusion being similar. For such a ρ, $B_k^{-1}(\rho) \in \Lambda_k$ is well defined for all $k \geq 1$, and since B_k^{-1} differs from the identity by $\mathcal{O}(\mu_k^\infty)$, we obtain

$$\alpha_k := B_k^{-1}(\rho) \to \rho \quad \text{as } k \to +\infty.$$

On the other hand since $\Lambda_{k+1} \subset \widetilde{\Lambda}_k = \widetilde{B}_k^{-1}(\widetilde{\Lambda})$, there exists some $\widetilde{\rho}_k \in \widetilde{\Lambda}$ such that $\alpha_{k+1} = \widetilde{B}_k^{-1}(\widetilde{\rho}_k)$. By taking a subsequence, we can assume that $\tilde{\rho}_k$ admits a limit $\tilde{\rho} \in \widetilde{\Lambda}$ as $k \to +\infty$. Then using that $\widetilde{B}_k^{-1}$ differs from the identity by $\mathcal{O}(\mu_k^\infty)$, we also obtain

$$\alpha_{k+1} \to \tilde{\rho} \quad \text{as } k \to +\infty.$$

Therefore, we deduce that $\rho = \widetilde{\rho} \in \widetilde{\Lambda}$ and the lemma is proved. □

Using Lemma 9.1, we see that the map B is well defined from $\Lambda \cap (J_1' - i[0,\tau'])$ to $\widetilde{\Lambda} \cap (J_1' - i[0,\tau'])$. Moreover, if $\rho \in \mathscr{E}_k$ for some $k \geq 0$, we have

$$|B(\rho) - \rho| = |\widetilde{B}_k \circ B_k^{-1}(\rho) - \rho| = \mathcal{O}(\mu_k^\infty),$$

and since $\tau_{k+3} \leq |\operatorname{Im} \rho| \leq \tau_{k+2} = \mathcal{O}(h^{2n_1})$, we also have

$$\mu_k \leq h^{-3n_1}\tau_{k+3} \leq h^{-3n_1}|\operatorname{Im} \rho| \leq C|\operatorname{Im} \rho|^{1/C},$$

where $C > 0$ is a large enough constant. Thus, we always have

$$|B(\rho) - \rho| = \mathbb{O}(|\operatorname{Im}\rho|^{\infty}).$$

Therefore, it just remains to see that B is a bijection, but this is an obvious consequence of (9-6), Lemma 9.1, and the definition of B. Thus Theorem 2.6 is proved.

10. Shape resonances

Here we prove Theorem 3.1. Under the assumptions of Section 3, one can construct, as in [Gérard and Martinez 1988], a function $G_1 \in C^{\infty}(\mathbb{R}^{2n})$, supported near $p^{-1}([\lambda_0 - 2\varepsilon, \lambda_0 + 2\varepsilon])\backslash\{x_0\}$ for some $\varepsilon > 0$, such that

$$G_1(x,\xi) = x\cdot\xi \text{ for } x \text{ large enough and } |p(x,\xi) - \lambda_0| \le \varepsilon, \tag{10-1}$$

$$H_p G_1(x,\xi) \ge \varepsilon \qquad \text{for } x \in \mathbb{R}^n\backslash\ddot{\mathrm{O}} \text{ and } |p(x,\xi) - \lambda_0| \le \varepsilon. \tag{10-2}$$

We also set

$$\widetilde{P} := P + W,$$

where $W = W(x) \in C^{\infty}(\mathbb{R}^n)$ is a nonnegative function, supported in a small enough neighborhood of x_0, and such that $W(x_0) > 0$. In particular, denoting by $\tilde{p}(x,\xi) = \xi^2 + V(x) + W(x)$ the principal symbol of $\widetilde{P}$, we have $\tilde{p}^{-1}(\lambda_0) \subset (\mathbb{R}^n\backslash\ddot{\mathrm{O}}) \times \mathbb{R}^n$, and thus λ_0 is a nontrapping energy for $\widetilde{P}$.

Now, we take μ and $\tilde{\mu}$ such that

$$\mu \le h^{\delta}, \quad \tilde{\mu} \le \min(\mu, h^{2+\delta})$$

with $\delta > 0$ arbitrary (so that μ, $\tilde{\mu}$ satisfy (2-6)), and we denote by V^{μ} a $|x|$-analytic $(\mu, \tilde{\nu})$-approximation of V as before. We also set

$$P^{\mu} = -h^2\Delta + V^{\mu}, \quad \widetilde{P}^{\mu} = P^{\mu} + W,$$

and if in (2-5) we take A supported away from $\operatorname{supp} W$, we see that the distorted operators P^{μ}_{θ} and $\widetilde{P}^{\mu}_{\theta}$ are well defined for $0 < \theta \le \tilde{\mu}$. Then we set

$$G(x,\xi) := G_1(x,\xi) - A(x)\cdot\xi,$$

that, by (10-1), is in $C_0^{\infty}(\mathbb{R}^n;\mathbb{R})$, and we consider its semiclassical Weyl-quantization $G^W = \operatorname{Op}_h^W(G)$ (see (6-1)).

Since $\theta/h^2 \le \tilde{\mu}/h^2 \le h^{\delta}$, a straightforward computation shows that the operator

$$R^{\mu}_{\theta} := \frac{1}{\theta}\operatorname{Im}\left(e^{\theta G^W/h}\widetilde{P}^{\mu}_{\theta}e^{-\theta G^W/h}\right)$$

is a semiclassical pseudodifferential operator, with symbol r^{μ}_{θ} satisfying

$$\partial^{\alpha} r^{\mu}_{\theta} = \mathbb{O}(\langle\xi\rangle^2) \text{ for all } \alpha \in \mathbb{N}^{2n},$$

$$r^{\mu}_{\theta}(x,\xi) = -H_{\tilde{p}^{\tilde{\mu}}}(A(x)\cdot\xi + G) + \mathbb{O}(h^{\delta}) = -H_p G_1(x,\xi) + \mathbb{O}(h^{\delta}),$$

uniformly with respect to $\theta \in (0, \tilde{\mu}]$ and $h > 0$ small enough. As a consequence, using (10-2), we see that R^{μ}_{θ} is elliptic in a neighborhood of $\{p(x,\xi) + W(x) = \lambda_0\}$ (uniformly with respect to θ and μ). Then by arguments similar to those of Section 6.1, we deduce that the operator

$$Q_\theta^\mu := e^{\theta G^W/h} \widetilde{P}_\theta^\mu e^{-\theta G^W/h}$$

satisfies

$$\|(Q_\theta^\mu - z)^{-1}\| = \mathbb{O}(\theta^{-1}),$$

uniformly for $|\operatorname{Re} z - \lambda_0| + \theta^{-1}|\operatorname{Im} z|$ small enough, $\theta \in (0, \tilde{\mu}]$, and $h > 0$ small enough. Since $\|\theta G^W/h\| \to 0$ uniformly as $h \to 0$, this also gives

$$\|(\widetilde{P}_\theta^\mu - z)^{-1}\| = \mathbb{O}(\theta^{-1}),$$

and from this point, one can follow all the procedure used in [Helffer and Sjöstrand 1986, Sections 9 and 10]. In particular, using the same notation as in that paper, by Agmon-type inequalities we see that the distribution kernel $K_{(\widetilde{P}_\theta^\mu - z)^{-1}}$ of $(\widetilde{P}_\theta^\mu - z)^{-1}$ satisfies

$$K_{(\widetilde{P}_\theta^\mu - z)^{-1}}(x, y) = \widetilde{\mathbb{O}}(\theta^{-1} e^{-d(x,y)/h})$$

where $d(x, y)$ is the Agmon distance between x and y (see [Helffer and Sjöstrand 1986, Lemma 9.4]). Then, assuming $\theta = \tilde{\mu} \geq e^{-\eta/h}$ for some $\eta > 0$ constant small enough and performing a suitable Grushin problem as in [Helffer and Sjöstrand 1986], we deduce that the resonances of P^μ in $[\lambda_0, \lambda_0 + Ch] - i[0, \lambda_0 \min(\mu, h^{2+\delta})]$ ($C > 0$ an arbitrary constant) are close to the eigenvalues of the Dirichlet realization of P on $\{d(x, \mathbb{R}^n \backslash \ddot{O}) \geq \eta/3)\}$, up to $\mathbb{O}(e^{-2(S_0-\eta)/h})$. Since these eigenvalues are real and admit semiclassical asymptotic expansions of the form

$$\lambda_k \sim \lambda_0 + e_k h + \sum_{\ell \geq 1} \lambda_{k,\ell} h^{1+\ell/2}$$

(where the e_k's are as in Theorem 3.1), we obtain for the corresponding resonances ρ_k of P^μ

$$\operatorname{Re} \rho_k \sim \lambda_0 + e_k h + \sum_{\ell \geq 1} \lambda_{k,\ell} h^{1+\ell/2}, \quad \operatorname{Im} \rho_k = \mathbb{O}(e^{-2(S_0-\eta)/h}), \tag{10-3}$$

uniformly. In particular, taking μ and $\tilde{\mu}$ as in Theorem 3.1, the result easily follows. Moreover, since the previous discussion can be applied to any $\mu' \in [e^{-\eta/h}, h^\delta]$, application of Theorem 2.6 tells us that the resonances of P in

$$[\lambda_0, \lambda_0 + Ch] - i[0, \tfrac{1}{2} h^{2n+\max(n/2,1)+1+3\delta}]$$

satisfy the same estimates (10-3).

Appendix

Proof of Lemma 5.1. We denote by χ_0 a real smooth function on $\mathbb{R}$ satisfying

- $\chi_0(s) = 0$ for $s \leq 0$;
- $\chi_0(s) = 1$ for $s \geq \ln 2$;
- χ_0 is nondecreasing.

Then for $r \geq 0$, we set

$$G(r) := \chi_0(r - R_0)\big(1 - \chi_0(r - \ln\lambda)\big)e^r + 2\lambda\chi_0(r - \ln\lambda), \quad g(r) := \int_0^r G(s)\,ds.$$

In particular, g satisfies Condition (i) of Lemma 5.1, and we have

- $G(r) = \chi_0(r - R_0)e^r$ for $r \in [R_0, \ln\lambda]$;
- $G(r) = (1 - \chi_0(r - \ln\lambda))e^r + 2\lambda\chi_0(r - \ln\lambda)$ for $r \in [\ln\lambda, \ln 2\lambda]$;
- $G(r) = 2\lambda$ for $r \in [\ln 2\lambda, +\infty)$.

Thus, $g' = G \leq 2\lambda$ and $g''(r) = G'(r) \geq 0$ on $\mathbb{R}_+$ (this is immediate on $[R_0, \ln\lambda] \cup [\ln 2\lambda, +\infty)$, while, on $[\ln\lambda, \ln 2\lambda]$, we compute, $G'(r) = (1 - \chi_0(r - \ln\lambda))e^r + \chi_0'(r - \ln\lambda)(2\lambda - e^r) \geq 0$).

Therefore, g is convex on $\mathbb{R}_+$, so that Condition (iii) of Lemma 5.1 is satisfied by g, too, while Condition (v) is obvious.

As for condition (iv), we observe the following:

- On $[0, R_0 + \ln 2]$, one has $g' + |g''| = \mathbb{O}(1)$.
- On $[R_0 + \ln 2, \ln\lambda]$, one has $g(r) \geq \int_{R_0+\ln 2}^r e^s\,ds = e^r - 2e^{R_0}$, while $g'(r) = g''(r) = e^r \leq g(r) + 2e^{R_0}$.
- On $[\ln\lambda, +\infty)$, one has $g(r) \geq g(\ln\lambda) = \lambda$, and thus $g' + |g''| = \mathbb{O}(g)$.

So, g satisfies Conditions (ii)–(v) of Lemma 5.1.

For $r \in [\ln 2\lambda, +\infty)$, we have

$$g(r) = g(\ln 2\lambda) + 2\lambda(r - \ln 2\lambda) = 2\lambda r - \alpha_\lambda, \tag{A-1}$$

where $\alpha_\lambda := 2\lambda \ln 2\lambda - g(\ln 2\lambda)$, and since

$$g(\ln 2\lambda) \leq \int_0^{\ln\lambda} e^r\,dr + \int_{\ln\lambda}^{\ln 2\lambda} 2\lambda\,dr = (1 + 2\ln 2)\lambda,$$
$$g(\ln 2\lambda) \geq \int_{R_0+\ln 2}^{\ln 2\lambda} e^r\,dr \geq 2\lambda - 2e^{R_0}.$$

we see that

$$2\lambda\ln 2\lambda - (1 + 2\ln 2)\lambda \leq \alpha_\lambda \leq 2\lambda\ln 2\lambda - 2\lambda + 2e^{R_0}.$$

Therefore, for λ large enough, the unique point r_λ, solution of $g(r_\lambda) = \lambda r_\lambda$, is given by

$$r_\lambda = \frac{\alpha_\lambda}{\lambda} \in [2\ln\lambda - 1, 2\ln\lambda - 2 + 2\ln 2 + 2\lambda^{-1}e^{R_0}] \subset [2\ln\lambda - 1, 2\ln\lambda - \varepsilon_0], \tag{A-2}$$

where $\varepsilon_0 := 1 - \ln 2 > 0$.

Now, we fix some real-valued function $\varphi_0 \in C^\infty(\mathbb{R})$, such that

- $\varphi_0(s) = 2s$ for $s \leq -\varepsilon_0$;
- $\varphi_0(s) = s$ for $s \geq \varepsilon_0$;
- $1 \leq \varphi_0' \leq 2$ everywhere.

Then using (A-1)–(A-2), we see that the function f_λ defined by

- $f_\lambda(r) := g(r)$ for $r \in [0, \ln 2\lambda]$;
- $f_\lambda(r) := \lambda\varphi_0(r - r_\lambda) + \alpha_\lambda$ for $r \geq \ln 2\lambda$,

is smooth on $\mathbb{R}_+$, and satisfies all the conditions required in Lemma 5.1. □

The distorted Laplacian.

Lemma A.1. *If* $\theta > 0$ *is small enough, the function* Φ_θ *defined in* (5-2) *satisfies*

$$\operatorname{Im}\left(({}^t d\Phi_\theta(x))^{-1}\xi\right)^2 \leq -\theta a(|x|)|\xi|^2.$$

for all $(x, \xi) \in \mathbb{R}^{2n}$.

Proof. Let $F := {}^t dA = dA = (F_{i,j})_{1\leq i,j\leq n}$. We compute

$$F_{i,j}(x) = a(x)\delta_{i,j} + a'(|x|)\frac{x_i x_j}{|x|},$$

that is, denoting by $\pi_x := |x|^{-2}x \cdot {}^t x$ the orthogonal projection onto $\mathbb{R}x$, and recalling the notation $b(r) = ra(r)$,

$$F(x) = a(|x|)I + a'(|x|)|x|\pi_x = b'(|x|)\pi_x + a(|x|)(I - \pi_x).$$

In particular, using Lemma 5.1, we obtain

$$0 \leq a(|x|) \leq F(x) \leq 2,$$

in the sense of self-adjoint matrices. On the other hand we have

$$({}^t d\Phi_\theta(x))^2 = (I + i\theta F(x))^2 = S_\theta + iT_\theta,$$

with $S_\theta = I - \theta^2 F(x)^2$ and $T_\theta = 2\theta F(x)$. Hence, $T_\theta \geq 0$, and since S_θ, T_θ and F commute, an easy computation gives

$$\operatorname{Im}\left(({}^t d\Phi_\theta(x))^{-1}\xi\right)^2 = -T_\theta(S_\theta^2 + T_\theta^2)^{-1}\xi \cdot \xi = -2\theta F(1 + \theta^2 F^2)^{-2}\xi \cdot \xi.$$

As a consequence, for θ small enough, we find

$$\operatorname{Im}\left[({}^t d\Phi_\theta(x))^{-1}\xi\right]^2 \leq -\theta F(x)\xi \cdot \xi \leq -\theta a(|x|)|\xi|^2.$$ □

Acknowledgments

We thank the anonymous referee for useful comments, which helped us improve significantly the presentation of this paper. We also thank the Dipartimento di Matematica at Università di Bologna and the Département de Mathématiques d'Orsay at Université Paris Sud 11, for providing us the opportunity to come together and complete this work.

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Received 11 May 2008. Revised 18 Dec 2008. Accepted 11 Jan 2009.

ANDRÉ MARTINEZ: `martinez@dm.unibo.it`
Università di Bologna, Dipartimento di Matematica, Piazza di Porta San Donato 5, 40127 Bologna, Italy
http://www.dm.unibo.it/~martinez/

THIERRY RAMOND: `thierry.ramond@math.u-psud.fr`
Département de Mathématiques, Université Paris-Sud 11, UMR CNRS 8628, 91405 Orsay, France
www.math.u-psud.fr/~ramond

JOHANNES SJÖSTRAND: `johannes.sjostrand@u-bourgogne.fr`
IMB (UMR CNRS 5584), Université de Bourgogne, 9 av. A. Savary, BP 47870, 21078 Dijon Cedex, France

ANALYSIS AND PDE
Vol. 2, No. 1, 2009

GLOBAL EXISTENCE AND UNIQUENESS RESULTS FOR WEAK SOLUTIONS OF THE FOCUSING MASS-CRITICAL NONLINEAR SCHRÖDINGER EQUATION

TERENCE TAO

We consider the focusing mass-critical NLS $iu_t + \Delta u = -|u|^{4/d}u$ in high dimensions $d \geq 4$, with initial data $u(0) = u_0$ having finite mass $M(u_0) = \int_{\mathbb{R}^d} |u_0(x)|^2\, dx < \infty$. It is well known that this problem admits unique (but not global) strong solutions in the Strichartz class $C^0_{t,\mathrm{loc}}L^2_x \cap L^2_{t,\mathrm{loc}}L^{2d/(d-2)}_x$, and also admits global (but not unique) weak solutions in $L^\infty_t L^2_x$. In this paper we introduce an intermediate class of solution, which we call a *semi-Strichartz class solution*, for which one does have global existence and uniqueness in dimensions $d \geq 4$. In dimensions $d \geq 5$ and assuming spherical symmetry, we also show the equivalence of the Strichartz class and the strong solution class (and also of the semi-Strichartz class and the semi-strong solution class), thus establishing *unconditional* uniqueness results in the strong and semi-strong classes. With these assumptions we also characterise these solutions in terms of the continuity properties of the mass function $t \mapsto M(u(t))$.

1. Introduction

1.1. *The focusing mass-critical NLS.* This paper deals with low regularity solutions $u : I \times \mathbb{R}^d \to \mathbb{C}$ to the initial value problem to the focusing mass-critical nonlinear Schrödinger equation (NLS)

$$\begin{aligned} iu_t + \Delta u &= F(u), \\ u(t_0) &= u_0, \end{aligned} \tag{1}$$

in high dimensions $d \geq 4$, where $I \subset \mathbb{R}$ is a time interval containing a time $t_0 \in \mathbb{R}$, $F : \mathbb{C} \to \mathbb{C}$ is the nonlinearity $F(z) := -|z|^{4/d}z$, and we assume u_0 to merely lie in the class $L^2_x(\mathbb{R}^d)$ of functions of finite mass $M(u_0) := \int_{\mathbb{R}^2} |u_0(x)|^2\, dx$. The exponent $1 + 4/d$ in the nonlinearity makes the equation *mass-critical*, so that the mass $M(u)$ is invariant under the scaling $u(t,x) \mapsto (1/\lambda^{d/2})u(t/\lambda^2, x/\lambda)$ of the equation. The mass is also *formally* conserved by the flow, thus we formally have $M(u(t)) = M(u_0)$ for all t, though it will be important in this paper to bear in mind that this formal mass conservation can break down if the solution is too weak in nature.

Remark 1.2. The condition $d \geq 4$ is assumed in order to ensure that the nonlinearity $F(u)$ is locally integrable in space for $u \in L^2_x(\mathbb{R}^d)$, so that (1) makes sense distributionally[1] for $u \in L^\infty_{t,\mathrm{loc}}L^2_x(I \times \mathbb{R}^d)$. It

MSC2000: 35Q30.
Keywords: Strichartz estimates, nonlinear Schrodinger equation, weak solutions, unconditional uniqueness.
The author is supported by NSF grant DMS-0649473 and a grant from the Macarthur Foundation.

[1] Here and in the sequel, we use the subscript loc to denote boundedness of norms on compact sets, thus for instance $u \in L^\infty_{t,\mathrm{loc}}L^2_x(I \times \mathbb{R}^d)$ if and only if $u \in L^\infty_t L^2_x(J \times \mathbb{R}^d)$ for all compact $J \subset I$, with the function space $L^\infty_{t,\mathrm{loc}}L^2_x(I \times \mathbb{R}^d)$ then being given the induced Frechet space topology.

will be clear from our arguments that our results would also apply if F were replaced by any other nonlinearity of growth $1+4/d$, whose derivative grew like $|z|^{4/d}$ and which enjoyed the Galilean invariance $F(e^{i\theta}z) = e^{i\theta}F(z)$ (in order to formally conserve mass), though in this more general setting, the mass $M(Q)$ of the ground state would need to be replaced by some unspecified constant $\varepsilon_{F,d} > 0$ depending on the nonlinearity F and the dimension d.

The notion of a distributional solution, by itself, is too weak for applications; for instance, one has difficulty interpreting what the initial data condition $u(0) = u_0$ means for a distributional solution in $L^\infty_{t,\mathrm{loc}} L^2_x$. In practice, one strengthens the notion of solution at this regularity by working with the integral formulation[2]

$$u(t) = e^{i(t-t_0)\Delta}u_0 + i\int_{t_0}^{t} e^{i(t-t')\Delta}F(u(t'))\,dt' \tag{2}$$

of the equation, where $e^{it\Delta}$ is defined via the Fourier transform $\hat{u}(\xi) := \int_{\mathbb{R}^d} e^{-ix\cdot\xi}u(x)\,dx$ as

$$\widehat{e^{it\Delta}u}(\xi) := e^{-it|\xi|^2}\hat{u}(\xi),$$

which is well-defined in the class of tempered distributions.

Remark 1.3. If $u_0 \in L^2_x(\mathbb{R}^d)$ and $u \in L^\infty_{t,\mathrm{loc}} L^2_x(I \times \mathbb{R}^d)$, then $F(u) \in L^\infty_{t,\mathrm{loc}} L^1_{x,\mathrm{loc}}(I \times \mathbb{R}^d)$, and the right side of Equation (2) makes sense as a tempered distribution in x for each time t. Furthermore, it is easy to verify (by the standard duality argument) that the right side of (2) is continuous in t in the topology $\mathcal{S}(\mathbb{R}^d)^*$ of tempered distributions.

1.4. *Weak, strong, and Strichartz class solutions.* With these preparations, we can now introduce the three standard solution classes for this problem in $L^2_x(\mathbb{R}^d)$.

Definition 1.5 (Weak, strong, Strichartz solutions). Fix a dimension $d \geq 4$, an initial data $u_0 \in L^2_x(\mathbb{R}^d)$ and a time interval $I \subset \mathbb{R}$ containing a time $t_0 \in \mathbb{R}$.

- A *weak solution* (or *mild solution*) to Equation (1) is a function $u \in L^\infty_{t,\mathrm{loc}} L^2_x(I \times \mathbb{R}^d)$ which obeys (2) in the sense of tempered distributions for almost every[3] time t.
- A *strong solution* to (1) is a weak solution u such that $t \mapsto u(t)$ is continuous in the L^2_x topology, thus u lies in $C^0_{t,\mathrm{loc}} L^2_x(I \times \mathbb{R}^d)$.
- A *Strichartz-class solution* to (1) is a strong solution which also lies in $L^2_{t,\mathrm{loc}} L^{2d/(d-2)}_x(I \times \mathbb{R}^d)$; thus u lies in $C^0_{t,\mathrm{loc}} L^2_x(I \times \mathbb{R}^d) \cap L^2_{t,\mathrm{loc}} L^{2d/(d-2)}_x(I \times \mathbb{R}^d)$.

Remark 1.6 (Shifting initial data). Because the right side of Equation (2) is continuous in the distributional topology for any of the above three notions of solutions, we observe that if u is a solution to (1) in any of the above classes on an interval I, and $t_1 \in I$, then u is also a solution to (1) in the same class with initial time t_1 and initial data $u(t_1)$ (as defined using the right side of (2)). Thus one may legitimately discuss solutions to NLS in one of the above three classes without reference to an initial time or initial data.

[2] We adopt the usual convention $\int_b^a = -\int_a^b$ when $a < b$.

[3] By definition of L^∞_t, weak solutions are only defined for almost every time t, though from Remark 1.3, one can canonically define $u(t)$ for all $t \in I$.

Remark 1.7. For future reference, we make the trivial remark that if one restricts a solution in any of the above classes to a subinterval $J \subset I$, then one still obtains a solution in the same class. Conversely, if one has a family of solutions in the same class on different time intervals I_n, such that $\bigcap_n I_n \neq \varnothing$ and any two solutions agree on their common domain of definition, then one can glue them together to form a solution in the same class on the union $\bigcup_n I_n$.

Remark 1.8. From Remark 1.3 we make the important observation that if $u \in L_t^\infty L_x^2(I \times \mathbb{R}^d)$ is a weak solution to Equation (1), then the map $t \mapsto u(t)$ is continuous in the weak topology of $L_x^2(\mathbb{R}^d)$. In particular we have the convergence property

$$\lim_{t' \to t} \langle u(t'), u(t) \rangle_{L_x^2(\mathbb{R}^d)} = M(u(t)), \tag{3}$$

for all $t \in I$, which by the cosine rule implies the asymptotic mass decoupling identity

$$\lim_{t' \to t} M(u(t')) - M(u(t') - u(t)) - M(u(t)) = 0. \tag{4}$$

Thus any L^2 discontinuity of u at t can be detected and quantified by the mass function $t \mapsto M(u(t))$; in particular, the solution $t \mapsto u(t)$ is continuous in L^2 at precisely those points for which the mass function $t \mapsto M(u(t))$ is continuous.

In the Strichartz class, one has a satisfactory local existence and uniqueness theory:

Proposition 1.9 (Local existence and uniqueness in the Strichartz class). *Let $d \geq 4$, $u_0 \in L_x^2(\mathbb{R}^d)$, and $t_0 \in \mathbb{R}$.*

(i) (*Local existence*) *There exists an open interval I containing t_0 and a Strichartz class solution $u \in C^0_{t,\mathrm{loc}} L_x^2(I \times \mathbb{R}^2) \cap L^2_{t,\mathrm{loc}} L_x^{2d/(d-2)}(I \times \mathbb{R}^d)$.*

(ii) (*Uniqueness*) *If I is an interval containing* 0, *and $u, u' \in C^0_{t,\mathrm{loc}} L_x^2(I \times \mathbb{R}^2) \cap L^2_{t,\mathrm{loc}} L_x^{2d/(d-2)}(I \times \mathbb{R}^d)$ are Strichartz class solutions to* (1) *on I, then $u = u'$.*

(iii) (*Mass conservation*) *If $u \in C^0_{t,\mathrm{loc}} L_x^2(I \times \mathbb{R}^2) \cap L^2_{t,\mathrm{loc}} L_x^{2d/(d-2)}(I \times \mathbb{R}^d)$ is a Strichartz solution, then the function $t \mapsto M(u(t))$ is constant.*

Proof. This is a standard consequence of the endpoint Strichartz estimate[4]

$$\|u\|_{L_t^2 L_x^{2d/(d-2)}(I \times \mathbb{R}^d)} + \|u\|_{C_t^0 L_x^2(I \times \mathbb{R}^d)} \lesssim_d \|u(t_0)\|_{L_x^2(\mathbb{R}^d)} + \|iu_t + \Delta u\|_{L_t^2 L_x^{2d/(d+2)}(I \times \mathbb{R}^d)}, \tag{5}$$

from [Keel and Tao 1998]; see [Cazenave and Weissler 1989; Cazenave 2003]. Mass conservation is obtained in these references by first regularising the data and nonlinearity so that the solution is smooth (and the formal conservation of mass can be rigorously justified), and then taking limits using (5). □

Because of this proposition (and Remark 1.7), every initial data $u_0 \in L_x^2(\mathbb{R}^d)$ and initial time $t_0 \in \mathbb{R}$ admits a unique *maximal Strichartz-class Cauchy development*

$$u \in C^0_{t,\mathrm{loc}} L_x^2(I \times \mathbb{R}^2) \cap L^2_{t,\mathrm{loc}} L_x^{2d/(d-2)}(I \times \mathbb{R}^d),$$

[4]Here and in the sequel we use the usual notation $X \lesssim Y$ or $X = O(Y)$ to denote the estimate $|X| \leq CY$ for some absolute constant $C > 0$; if the implied constant C depends on a parameter (such as d), we will indicate this by subscripts, for example, $X \lesssim_d Y$ or $X = O_d(Y)$.

where I is an open interval containing t_0, and u is a Strichartz-class solution to (1) which cannot be extended to any larger time interval.

Unfortunately, the lifespan I of this maximal Strichartz-class Cauchy development need not be global if the mass $M(u_0)$ is large. For instance, if Q is a nontrivial Schwartz-class solution to the ground state equation

$$\Delta Q + |Q|^{4/d} Q = Q, \tag{6}$$

then as is well known, the function

$$u(t,x) := \frac{1}{|t|^{d/2}} e^{-i/t} e^{i|x|^2/4t} Q(x/t) \tag{7}$$

is a Strichartz-class solution on $(0,+\infty)\times\mathbb{R}^d$ or $(-\infty,0)\times\mathbb{R}^d$ but cannot be extended in this class across the time $t=0$. One can also use Glassey's virial identity [Glassey 1977] to infer indirectly the nonglobal nature of maximal Strichartz-class Cauchy developments for suitably smooth and decaying data with negative energy.

Remark 1.10. In the defocusing case $F(z) = +|z|^{4/d} z$, it is conjectured that all maximal Strichartz-class Cauchy developments are global. This has recently been established in the spherically symmetric case [Tao et al. 2006], and is also known for data with additional regularity (for example, energy class) or decay (for example, $xu_0 \in L^2_x(\mathbb{R}^d)$), or with small mass; see [Tao et al. 2006] and the references therein for further discussion. In the focusing case, the results of [Killip et al. 2007] give global existence for spherically symmetric data when the mass $M(u_0)$ is strictly less than the mass $M(Q)$ of the ground state; see [Killip et al. 2008a] for a treatment of the endpoint case $M(u_0) = M(Q)$. Again, it is conjectured that the same results hold without the spherical symmetry assumption, but this remains open.

On the other hand, it is possible to continue solutions in a weak sense beyond the time for which Strichartz-class solutions blow up. In particular, we have the following standard result:

Proposition 1.11 (Global existence in the weak class). *Let $d \geq 4$, $u_0 \in L^2_x(\mathbb{R}^d)$, and $t_0 \in \mathbb{R}$. Then there exists a global weak solution $u \in L^\infty_t L^2_x(\mathbb{R}\times\mathbb{R}^d)$ to* (1). *Furthermore we have $M(u(t)) \leq M(u_0)$ for all $t \in \mathbb{R}$.*

Proof. We will prove a stronger result than this shortly, so we only give a sketch of proof here. By Remark 1.7 and time reversal symmetry, it suffices to build a solution on $[t_0,+\infty)$. For each $\varepsilon > 0$, one can easily use parabolic theory to construct a global (strong) solution to the damped NLS $iu^{(\varepsilon)}_t + \Delta u^{(\varepsilon)} = i\varepsilon\Delta u^{(\varepsilon)} + F_\varepsilon(u^{(\varepsilon)})$ on $[t_0,+\infty)$, whose mass is bounded above by $M(u_0)$, where F_ε is a suitably damped version of F (for example, $F_\varepsilon(z) := -\max(|z|, 1/\varepsilon)^{4/d} z$); extracting a weakly convergent subsequence and taking weak limits we obtain the claim. $\square$

Unfortunately, while these weak solutions are global, they are nonunique, as the following standard example shows.

Example 1.12. Consider the function given by Equation (7) for $t \in (0,+\infty)$ and by zero for $t \in (-\infty,0]$. This is a global weak solution in the sense of the above proposition (taking t_0 to be any positive time, and setting $u_0 = u(t_0)$), but is not unique; if, for instance, one takes u to equal (7) for $t \in (-\infty,0)$ rather than equal to zero, then the new solution is still a global weak solution with the same initial data. Note that a modification of this example shows that uniqueness of weak solutions can break down even if the

$$\begin{tikzcd} \text{Strichartz} \arrow[r] \arrow[d] & \text{strong} \arrow[d] & \\ \text{semi-Strichartz} \arrow[r] & \text{semi-strong} \arrow[r] & \text{weak} \end{tikzcd}$$

Figure 1. Inclusions between solution classes. In dimensions $d \geq 5$ and assuming spherical symmetry, we will show that two horizontal inclusions on the left are in fact equivalences.

initial data is zero, and so one cannot hope to recover uniqueness purely by strengthening the hypotheses on the initial data.

Remark 1.13. Example 1.12 also shows that mass is not necessarily conserved for weak solutions. On the other hand, from Equation (4) we see that the function $t \mapsto M(u(t))$ is lower semi-continuous, at least.

1.14. ***Semi-Strichartz solutions.*** To summarise the discussion so far, the Strichartz class of solutions has uniqueness but no global existence, while the class of weak solutions has global existence but no uniqueness. It is thus natural to ask whether there is an intermediate class of solutions for which one has both global existence and uniqueness. To answer this we define some further solution classes.

Definition 1.15 (Semi-strong and semi-Strichartz solutions). Fix a dimension $d \geq 4$, an initial data $u_0 \in L^2_x(\mathbb{R}^d)$ and a time interval $I \subset \mathbb{R}$ containing a time $t_0 \in \mathbb{R}$. A *semi-strong solution* (resp. *semi-Strichartz class solution*) to Equation (1) is a weak solution u such that for every $t \in I \cap [t_0, +\infty)$ there exists $\varepsilon > 0$ such that u is a strong solution (resp. Strichartz class solution) when restricted to $I \cap [t, t+\varepsilon)$, and for every $t \in I \cap (-\infty, t_0]$ there exists $\varepsilon > 0$ such that u is a strong solution (resp. Strichartz class solution) when restricted to $I \cap (t - \varepsilon, t]$.

We summarise the obvious inclusions between the five classes of solution in Figure 1. Note that unlike the weak, strong, and Strichartz classes, the semi-strong and semi-Strichartz classes of solution depend on the choice of initial time t_0.

Example 1.16. Consider the weak solution $u \in L^\infty_t L^2_x(\mathbb{R} \times \mathbb{R}^d)$ which is given by Equation (7) for $t > 0$ and is zero for $t \leq 0$, let $t_0 > 0$, and set $u_0 := u(t_0)$. Then u is a semi-Strichartz class solution (and thus semi-strong solution) to (1), but is not strong or Strichartz-class. If one redefines u for $t < 0$ by (7), then u remains a weak solution, but is no longer semi-strong or semi-Strichartz.

Remark 1.17. The constructions in [Bourgain and Wang 1997], in our notation, yield semi-Strichartz class solutions which blow up in the Strichartz class at a specified finite set of points in time, and are equal to a prescribed state in $L^2_x(\mathbb{R}^d)$ at the final blowup time, in dimensions $d = 1, 2$.

Our first main result is that the semi-Strichartz solution class enjoys global existence and uniqueness:

Theorem 1.18 (Global existence and uniqueness in the semi-Strichartz solution class). *Suppose $d \geq 4$, $u_0 \in L^2_x(\mathbb{R}^d)$, and $t_0 \in \mathbb{R}$. Then there exists a global semi-Strichartz class solution $u \in L^\infty_t L^2_x(\mathbb{R} \times \mathbb{R}^d)$ to* (1). *Furthermore, this solution is unique in the sense that any other semi-Strichartz solution to* (1) *on a time interval I containing t_0 is the restriction of u to I. Finally, $M(u(t))$ is monotone nonincreasing* *for $t \geq t_0$ and monotone nondecreasing for $t \leq t_0$ (in particular, the only possible discontinuities are jump discontinuities), and has a jump discontinuity exactly at those times t for which u is not locally a Strichartz class solution.*

Remark 1.19. Informally, the unique semi-Strichartz class solution is formed by solving the equation in the Strichartz class whenever possible, and deleting any mass that escapes to spatial or frequency infinity when the solution leaves the Strichartz class. The relationship between this class of solution and Strichartz class solutions is analogous to the relationship between Ricci flow with surgery and Ricci flow in the work of Perelman [2007; 2003], though of course the situation here is massively simpler than with Ricci flow on account of the semilinear and flat nature of our equation. On the other hand, the *entropy*-type solutions constructed in Proposition 1.11 do not necessarily converge to the solution in Theorem 1.18. For instance, the arguments in [Merle 1992] can be adapted to show that if one starts with the initial data of Equation (7) at time $t = -1$, say, and evolves a parabolically regularised version of Equation (1) using some viscosity parameter ε, then the solution at $t = +1$ can converge to an arbitrary phase rotation of the solution (7) along a subsequence of ε, and in particular these solutions do not converge to the semi-Strichartz solution (which vanishes after the singularity time). However, it is conceivable that the entropy solutions do converge to the semi-Strichartz solutions for *generic* data, although the author does not know how one would prove this.

Remark 1.20. One can push the global existence result further, to obtain scattering at $t = \pm\infty$, and can in fact even push the solution "beyond" $t = +\infty$ and $t = -\infty$ by using the pseudoconformal transform or lens transform, in the spirit of [Tao 2006]. We omit the details.

Remark 1.21. While the semi-Strichartz class enjoys global existence and uniqueness, it does not enjoy continuous dependence on the data and is thus not a well-posed class of solutions. Indeed, if one considers the solution in Example 1.16 for the spherically symmetric ground state Q, and then perturbs the initial data $u_0 = u(t_0)$ to have slightly smaller mass (while staying spherically symmetric), then from the results in [Killip et al. 2007] we know that the perturbed solution exists globally in the Strichartz class, and in particular has mass close to $M(Q)$ for all negative times, in contrast to the original solution in Example 1.16 which has zero mass for all negative times, thus contradicting continuous dependence on the data in any reasonable topology. Indeed this argument strongly suggests that there is no solution class for this equation which is globally well-posed in the sense that one simultaneously has global existence, uniqueness, and continuous dependence of the data, and which is compatible with the Strichartz class of solutions.

Remark 1.22. In [Merle and Raphaël 2005; Fibich et al. 2006], solutions to Equation (1) are constructed which are initially in $H^1_x(\mathbb{R}^d)$, but at the first blowup time develop a single point of concentration, plus a residual component u^* which is not in $L^p_x(\mathbb{R}^d)$ for any $p > 2$, and in particular has left $H^1_x(\mathbb{R}^d)$. The semi-Strichartz solution would continue the evolution from u^* at this time. Thus, we do not have persistence of regularity for the semi-Strichartz class: a semi-Strichartz solution can exit the space in finite time. (A similar phenomenon for the supercritical focusing NLS was also obtained in [Merle and Raphaël 2008]. In contrast, the solution in Example 1.16 has H^1 norm going to infinity as $t \to 0^+$, but never actually leaves $H^1_x(\mathbb{R}^d)$; similarly for the solutions in [Bourgain and Wang 1997]).

Theorem 1.18 is in fact an easy consequence of Proposition 1.9 and is proven in Section 2. One can be somewhat more precise about the jump discontinuities:

Theorem 1.23 (Quantisation of mass loss). *Let $d \geq 4$, $u_0 \in L_x^2(\mathbb{R}^d)$, and $t_0 \in \mathbb{R}$. Let $u \in L_t^\infty L_x^2(\mathbb{R} \times \mathbb{R}^d)$ be the unique global semi-Strichartz class solution to Equation* (1) *given by Theorem 1.18. Then there exists an absolute constant $\varepsilon_d > 0$ (depending only on d) such that every jump discontinuity of the function $t \mapsto M(u(t))$ has jump at least ε_d. If u_0 is spherically symmetric, one can take ε_d to be the mass $M(Q)$ of the ground state.*

Remark 1.24. A closely related result in the spherically symmetric case was established in [Killip et al. 2008b, Corollary 1.12], in which it was shown that any blowup of a spherically symmetric Strichartz class solution in two dimensions must concentrate an amount of mass at least equal to the ground state $M(Q)$; the same result in higher dimensions follows by the same argument together with the results in [Killip et al. 2007]. Indeed, we will use the results in [Killip et al. 2007] to establish the spherically symmetric case of this theorem. From Example 1.12 we see that $M(Q)$ cannot be replaced by any larger quantity in the above theorem.

Theorem 1.23 is of course consistent with the existence of a lower bound ε_d for mass concentration at a point [Bourgain 1999; Merle and Vega 1998; Keraani 2006], although neither result seems to directly imply the other. (The proof of Theorem 1.23 uses global-in-space Strichartz estimates, whereas the mass concentration result requires more localised tools.)

Theorem 1.23, combined with Theorem 1.18 and Proposition 1.9(iii), has an immediate corollary:

Corollary 1.25. *If u is a global semi-Strichartz class solution to Equation* (1), *the function $t \mapsto M(u(t))$ is piecewise constant with at most finitely many jump discontinuities, with u being a Strichartz class solution on each of the piecewise constant intervals.*

We prove Theorem 1.23 in Section 3.

1.26. ***The spherically symmetric case.*** Now we turn to the question of whether strong (resp. semi-strong) solutions are necessarily in the Strichartz class (resp. semi-Strichartz class), which would imply (by Proposition 1.9 and Theorem 1.18) that they are unique. These type of results are known as *unconditional uniqueness* (or *unconditional well-posedness*) results in the literature. For solutions in higher regularities, such as the energy class, one can obtain unconditional uniqueness by exploiting Sobolev embedding to obtain additional integrability of the strong solution u [Kato 1995; 1996; Furioli and Terraneo 2003a; 2003b; Cazenave 2003; Tsutsumi 2007]. Unfortunately at the $L_x^2(\mathbb{R}^d)$ level of regularity, for which Sobolev embedding is not available[5], it appears to be rather difficult to establish such an unconditional uniqueness result, although the author tentatively conjectures it to be true. On the other hand, we were able to establish this uniqueness under the additional simplifying assumption of spherical symmetry (and assuming very high dimension $d \geq 5$), thus replacing the data space $L_x^2(\mathbb{R}^d)$ by the subspace $L_{\mathrm{rad}}^2(\mathbb{R}^d)$ of spherically symmetric functions:

Theorem 1.27 (Unconditional uniqueness for spherically symmetric solutions). *Let $d \geq 5$, $u_0 \in L_{\mathrm{rad}}^2(\mathbb{R}^d)$, I be an interval, and $t_0 \in \mathbb{R}$. Let $u \in L_t^\infty L_x^2(I \times \mathbb{R}^d)$ be a spherically symmetric weak solution to Equation* (1). *Then the following are equivalent*:

(i) *u is a Strichartz class solution.*

[5]Related to this difficulty is the Galilean invariance of the NLS equation at $L_x^2(\mathbb{R}^d)$, which strongly suggests that direct application of Sobolev or Littlewood–Paley theory is unlikely to be helpful.

(ii) *u is a strong solution.*

(iii) *The function $t \mapsto M(u(t))$ is constant.*

(iv) *The function $t \mapsto M(u(t))$ is continuous.*

(v) *One has $M(u(t)) \geq \limsup_{t' \to t} M(u(t')) - \varepsilon_d$ for all $t \in I$, where $\varepsilon_d > 0$ is a suitably small absolute constant depending only on d. (Note from lower semi-continuity that we automatically have $M(u(t)) \leq \limsup_{t' \to t} M(u(t'))$.)*

Example 1.28. If u is given by Equation (7) for $t \neq 0$ and a spherically symmetric Q and vanishes for $t = 0$, then u is a spherically symmetric weak solution but fails to conserve mass at $t = 0$, and is thus not in the Strichartz class in a neighbourhood of $t = 0$.

Remark 1.29. From Theorem 1.27 and Proposition 1.9(ii), we see that spherically symmetric strong solutions to (1) are unique. Another quick corollary of Theorem 1.27 is that any spherically symmetric weak solution whose mass is always strictly smaller than ε_d is necessarily a Strichartz class solution (and hence strong solution also), and thus also unique. In view of Theorem 1.23, it is natural to conjecture that one can take ε_d to be the mass $M(Q)$ of the ground state, which is the limit of weak uniqueness thanks to Example 1.12, but our methods do not give this.

Remark 1.30. The above theorem shows that if a weak solution fails to be in the Strichartz class, then at some time t it must lose at least a fixed amount ε_d of mass, though it is possible that this mass is then instantly recovered (consider for instance the solution given by (7) for $t \neq 0$ and zero for $t = 0$). On the other hand, it is conceivable that there exist weak solutions in which the mass function $t \mapsto M(u(t))$ exhibits oscillating singularities rather than jump discontinuities, in which the mass oscillates infinitely often as one approaches a given time; the above theorem implies that the net oscillation is at least ε_d but does not otherwise control the behaviour of this function. If for instance there existed a nontrivial weak solution on a compact interval I which vanished at both endpoints of the interval [Scheffer 1993], then one could achieve such an oscillating behaviour by gluing together rescaled, time-translated versions of this solution. However, we do not know if such weak solutions exist; solutions such as (7) constructed using the pseudo-conformal transformation only exhibit vanishing at a single time t.

Remark 1.31. Note that we need to assume the solution is spherically symmetric, and not just the initial data. In the category of weak solutions, at least, it is not necessarily the case that spherically symmetric data leads to spherically symmetric solutions: consider for instance the weak solution which is equal to a time-translated version of (7) for $t \neq 0$ and vanishes for $t = 0$; this solution is spherically symmetric at time zero but not at other times.

We prove Theorem 1.27 in Section 6.1, after establishing an important preliminary smoothing estimate for weak solutions in Section 4. Our main tool here is the in/out decomposition of waves used in [Tao 2004; Killip et al. 2008b], which is particularly powerful for understanding the dispersion of spherically symmetric waves, and upon which we will rely heavily in order to establish a substantial gain of regularity for weak solutions. Our arguments only barely fail at $d = 4$ and it is quite likely that a refinement of the methods here can be extended to that case, but we do not pursue this matter here.

There is an analogue of Theorem 1.27 for semi-strong and semi-Strichartz class solutions:

Theorem 1.32 (Characterisation of spherically symmetric semi-Strichartz solution). *Suppose $d \geq 5$, $u_0 \in L^2_{\text{rad}}(\mathbb{R}^d)$, I be an interval, i and $t_0 \in \mathbb{R}$. Let $u \in L^\infty_t L^2_x(I \times \mathbb{R}^d)$ be a spherically symmetric weak solution to Equation* (1). *Then the following are equivalent*:

(i) *u is the unique semi-Strichartz solution given by Theorem 1.18* (*restricted to I, of course*).

(ii) *u is a semi-strong solution.*

(iii) *The function $t \mapsto M(u(t))$ is right-continuous for $t \geq t_0$ and left-continuous for $t \leq t_0$, and is piecewise constant with only finitely many jump discontinuities, with each jump being at least $M(Q)$ in size.*

(iv) *The function $t \mapsto M(u(t))$ is right-continuous for $t \geq t_0$ and left-continuous for $t \leq t_0$.*

(v) *$M(u(t)) \geq \limsup_{t' \to t^+} M(u(t')) - \varepsilon_d$ for all $t \geq t_0$ and $M(u(t)) \geq \limsup_{t' \to t^-} M(u(t')) - \varepsilon_d$ for all $t \leq t_0$, where $\varepsilon_d > 0$ is a suitably small absolute constant depending only on d.*

We prove Theorem 1.32 in Section 6.2, using a minor modification of the argument used to prove Theorem 1.27.

2. Proof of Theorem 1.18

We first establish uniqueness. Suppose we have two semi-Strichartz class solutions $u, u' \in L^\infty_t L^2(I \times \mathbb{R}^2)$ to (1). Let J be the connected component of $\{t \in I : u(t) = u(t')\}$ that contains t_0. Since u, u' are weak solutions, we see from Remark 1.8 that J is closed. From the uniqueness component of Proposition 1.9, and Definition 1.15, we also see that J is right-open in $I \cap [t_0, +\infty)$ (that is, for each $t \in J \cap [t_0, +\infty)$ there exists $\varepsilon > 0$ such that $I \cap [t, t + \varepsilon) \subset J$) and left-open in $I \cap (-\infty, t_0]$; by connectedness we conclude that $J = I$, establishing uniqueness.

Now we establish global existence. It suffices to establish a semi-Strichartz class solution on $[t_0, +\infty)$, as by time reversal symmetry we may then obtain a semi-Strichartz class solution and $(-\infty, t_0]$, and glue them together to obtain the desired global solution on $\mathbb{R}$.

Let J denote the set of all times $T \in [t_0, +\infty)$ for which there exists a semi-Strichartz class solution u on $[t_0, T]$ with $M(u(t))$ monotone nonincreasing on $[t_0, T]$; thus J is a connected subset of $[t_0, +\infty)$ containing t_0. By the existence and mass conservation component of Proposition 1.9, we see that J is right-open. Now we establish that J is closed. If t_n is a sequence of times in J increasing to a limit t_*, then by gluing together all the associated semi-Strichartz class solutions (using uniqueness) we obtain a semi-Strichartz solution u on $[t_0, t_*)$ with $M(u(t))$ monotone nonincreasing on $[t_0, t_*)$; in particular u lies in $L^\infty_t L^2_x([t_0, t_*) \times \mathbb{R}^d)$, and $F(u)$ lies in $L^\infty_t L^{2d/(d+4)}_x([t_0, t_*) \times \mathbb{R}^d)$. From this we see that the right side of Equation (2) is continuous all the way up to t_* in the space of tempered distributions, and so we can extend u as a weak solution to $[t_0, t_*]$. This is still a semi-Strichartz solution, and by Fatou's lemma we see that $M(u(t))$ is still nondecreasing on $[t_0, t_*]$, and so $t_* \in J$, thus establishing that J is closed. By connectedness we conclude that $J = [t_0, +\infty)$, and so we can obtain semi-Strichartz class solutions on $[t_0, T]$ for any $t_0 \leq T < \infty$. Gluing these solutions together we obtain the desired solution on $[t_0, +\infty)$, establishing global existence.

The above argument has also established monotonicity of mass. Whenever u is a Strichartz class solution in a neighbourhood of a time t_1, it follows from Proposition 1.9 that mass is constant near t_1, so the only remaining task is to show that whenever mass is continuous at a time t_1, then u is a Strichartz class solution in a neighbourhood of t.

The claim is obvious for $t_1 = t_0$, so without loss of generality we may take $t_1 > t_0$. By hypothesis, $M(u(t))$ converges to $M(u(t_1))$ as $t \to t_1$. By Equation (4), we conclude that $u(t)$ converges strongly to $u(t_1)$ in $L^2_x(\mathbb{R}^d)$.

Let $\varepsilon > 0$ be a small number. By the endpoint Strichartz estimate (5), we have

$$\|e^{i(t-t_1)\Delta}u(t_1)\|_{L^2_t L^{2d/(d-2)}_x(\mathbb{R}\times\mathbb{R}^d)} < \infty,$$

so by the monotone convergence theorem we have

$$\|e^{i(t-t_1)\Delta}u(t_1)\|_{L^2_t L^{2d/(d-2)}_x(I\times\mathbb{R}^d)} < \varepsilon,$$

when I is a sufficiently small neighbourhood of t_1.

Fix I. Let t_2 converge to t_1, then by the previous discussion $u(t_2)$ converges strongly to $u(t_1)$ in $L^2_x(\mathbb{R}^d)$. By the continuity (and unitarity) of the Schrödinger propagator, this implies that $e^{i(t_2-t_1)\Delta}u(t_2)$ converges to $u(t_1)$. Applying the endpoint Strichartz estimate (5), we conclude that $e^{i(\cdot-t_2)\Delta}u(t_2)$ converges in $L^2_t L^{2d/(d-2)}_x(I\times\mathbb{R}^d)$ to $e^{i(\cdot-t_1)\Delta}u(t_1)$. In particular, we have

$$\|e^{i(t-t_2)\Delta}u(t_2)\|_{L^2_t L^{2d/(d-2)}_x(I\times\mathbb{R}^d)} < \varepsilon,$$

for all t_2 sufficiently close to t_1. On the other hand, we have $M(u(t_2)) \le M(u_0)$. Thus if ε is chosen to be sufficiently small depending on $M(u_0)$, we may apply the standard Picard iteration argument based on the endpoint Strichartz estimate (5) and construct a Strichartz-class solution to NLS on I which equals $u(t_2)$ at t_2. Applying this with t_2 slightly smaller than t_1 and using the uniqueness of semi-Strichartz class solutions, we see that u is equal to this Strichartz-class solution on I, and the claim follows.

3. Proof of Theorem 1.23

It is convenient here to use the original nonendpoint Strichartz estimate [Strichartz 1977]:

$$\|u\|_{L^{2(d+2)/d}_{t,x}(I\times\mathbb{R}^d)} + \|u\|_{C^0_t L^2_x(I\times\mathbb{R}^d)} \lesssim_d \|u(t_0)\|_{L^2_x(\mathbb{R}^d)} + \|iu_t + \Delta u\|_{L^2_t L^{2(d+2)/(d+4)}_x(I\times\mathbb{R}^d)}. \tag{8}$$

Let $\varepsilon_d > 0$ be chosen later, and let u be a semi-Strichartz class solution. Suppose for contradiction that we had a jump discontinuity at some time t_1 of jump less than ε_d. As before we may assume without loss of generality that $t_1 > t_0$.

Let t approach t_1 from below, then $M(u(t)) - M(u(t_1))$ converges to a limit less than ε_d. By Equation (4), we conclude that $\|u(t) - u(t_1)\|_{L^2_x(\mathbb{R}^d)}$ converges to a limit less than ε_d.

By (8) and monotone convergence as before, we can find a small neighbourhood I of t_1 such that

$$\|e^{i(t-t_1)\Delta}u(t_1)\|_{L^{2(d+2)/d}_{t,x}(I\times\mathbb{R}^d)} < \varepsilon_d^{1/2}.$$

If we let t_2 approach t_1 from below, then for t_2 sufficiently close to t_1 we thus have

$$\|e^{i(t-t_2)\Delta}u(t_1)\|_{L^{2(d+2)/d}_{t,x}(I\times\mathbb{R}^d)} < \varepsilon_d^{1/2}.$$

On the other hand, from Equation (8) we have (for t_2 sufficiently close to t_1) that

$$\|e^{i(t-t_2)\Delta}(u(t_1)-u(t_2))\|_{L^{2(d+2)/d}_{t,x}(I\times\mathbb{R}^d)} \lesssim_d \|u(t_2)-u(t_1)\|_{L^2_x(\mathbb{R}^d)} \lesssim \varepsilon_d^{1/2},$$

and thus by the triangle inequality

$$\|e^{i(t-t_2)\Delta}u(t_2)\|_{L^{2(d+2)/d}_{t,x}(I\times\mathbb{R}^d)} \lesssim_d \varepsilon_d^{1/2}.$$

If ε_d is sufficiently small depending on d, we can then perform a Picard iteration, using (8) to control the nonlinear portion $u(t)-e^{i(t-t_2)\Delta}u(t_2)$ of the solution, to construct a solution in the class

$$C^0_t L^2_x \cap L^{2(d+2)/d}_{t,x}(I\times\mathbb{R}^d)$$

that equals $u(t_2)$ on t_2. Applying Strichartz estimates once more, we see that this solution is a Strichartz solution on I. By uniqueness of semi-Strichartz solutions, we conclude that u is a Strichartz solution on I and thus has no jump discontinuity at t_1, a contradiction.

Now we handle the spherically symmetric case. We will need the following result from [Killip et al. 2007]:

Theorem 3.1 (Scattering below the ground state). *Let $d\geq 3$. Then for every $0<m<M(Q)$ there exists a quantity $A(m)<\infty$ such that whenever $t_0\in\mathbb{R}$ and $u_0\in L^2_x(\mathbb{R}^d)$ with $M(u_0)\leq m$, then there exists a global Strichartz-class solution u to* (1) *with $\|u\|_{L^2_t L^{2d/(d-2)}_x(\mathbb{R}\times\mathbb{R}^d)}\leq A(m)$.*

Proof. See [Killip et al. 2007, Theorem 1.5]. □

Now suppose for contradiction that we have a global semi-Strichartz class solution from spherically symmetric initial data u_0 which has a mass jump discontinuity of less than $M(Q)$ at some time t_1; we can assume $t_1>t_0$ as before.

Since u_0 is spherically symmetric, we see from rotation invariance and uniqueness that u is spherically symmetric. By arguing as before, we see that as t_2 approaches t_1 from below, $M(u(t_2)-u(t_1))$ converges to a limit less than $M(Q)$. In particular this limit is less than m for some $0<m<M(Q)$.

Let $\varepsilon>0$ be a small number depending on m and $M(u_0)$ to be chosen later. By endpoint Strichartz (5) and monotone convergence as before, we can find a small neighbourhood I of t_1 such that

$$\|e^{i(t-t_1)\Delta}u(t_1)\|_{L^2_t L^{2d/(d-2)}_x(I\times\mathbb{R}^d)} < \varepsilon,$$

and thus for t_2 sufficiently close to t_1

$$\|e^{i(t-t_2)\Delta}u(t_1)\|_{L^2_t L^{2d/(d-2)}_x(I\times\mathbb{R}^d)} < \varepsilon. \tag{9}$$

On the other hand, we also have

$$M(u(t_2)-u(t_1)) < m,$$

for t_2 sufficiently close to (and below) t_1. By Theorem 3.1, we may thus find a Strichartz-class solution v on I of mass at most m with $v(t_2)=u(t_2)-u(t_1)$ and

$$\|v\|_{L^2_t L^{2d/(d-2)}_x(I\times\mathbb{R}^d)} \leq A(m).$$

From this and Equation (9) and standard perturbation theory [Tao et al. 2008, Lemma 3.1], we may thus find a Strichartz-class solution on I which equals $u(t_2)$ at t_2. Arguing as before we conclude that u has no jump discontinuity at t_1, a contradiction.

Remark 3.2. It is conjectured that the spherical symmetry assumption can be removed from Theorem 3.1. If this conjecture is true, then it is clear that one can take $\varepsilon_d = M(Q)$ in the nonspherically-symmetric case of Theorem 1.23 as well.

4. A smoothing effect for spherically symmetric weak solutions

In this section, we establish a preliminary smoothing effect for spherically symmetric weak solutions that will be needed to prove Theorems 1.27 and 1.32. More precisely, we show

Theorem 4.1 (Smoothing effect). *Let $d \geq 4$, let I be a compact interval, and let $u \in L_t^\infty L_x^2(I \times \mathbb{R}^d)$ be a spherically symmetric weak solution to NLS with $M(u(t)) \leq m$ for all $t \in I$. Then for every $R > 0$ one has the bound*

$$\|u\|_{L_t^2 L_x^{2d/(d-2)}(I \times (\mathbb{R}^d \backslash B(0,R)))} \lesssim_{I,m,d} R^{-1} + 1, \tag{10}$$

where $B(0, R)$ is the ball of radius R centred at the origin.

Remark 4.2. Theorem 4.1 asserts that a spherically symmetric weak solution behaves like a Strichartz-class solution away from the spatial origin. The R^{-1} term on the right side is sharp, as can be seen by considering a rescaled stationary solution $u(t, x) = R^{-d/2} e^{it/R^2} Q(x/R)$, where Q is a nontrivial spherically symmetric solution to Equation (6).

We shall prove this theorem using the method of in/out projections, as used in [Tao 2004; Killip et al. 2008b; 2007]. We first recall some Littlewood–Paley notation.

Let $\varphi(\xi)$ be a radial bump function supported in the ball $\{\xi \in \mathbb{R}^d : |\xi| \leq 11/10\}$ and equal to 1 on the ball $\{\xi \in \mathbb{R}^d : |\xi| \leq 1\}$. For each number $N > 0$, we define the Fourier multipliers

$$\begin{aligned} \widehat{P_{\leq N} f}(\xi) &:= \varphi(\xi/N)\hat{f}(\xi), \\ \widehat{P_{>N} f}(\xi) &:= (1 - \varphi(\xi/N))\hat{f}(\xi), \\ \widehat{P_N f}(\xi) &:= \psi(\xi/N)\hat{f}(\xi) := (\varphi(\xi/N) - \varphi(2\xi/N))\hat{f}(\xi). \end{aligned}$$

We similarly define $P_{<N}$ and $P_{\geq N}$. All sums over N will be over integer powers of two unless otherwise stated.

We now subdivide the Littlewood–Paley projections P_N on the spherically symmetric space $L^2(\mathbb{R}^d)_{\mathrm{rad}}$ into two components, an *outgoing projection* $P_+ P_N$ and *incoming projection* $P_- P_N$, as described in the following lemma:

Proposition 4.3 (In/out decomposition). *Let $d \geq 1$. Then there exist bounded linear operators $P_+, P_- : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ with the following properties*:

(i) *P_+, P_- extend to bounded linear operators on $L^p(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$ for every $1 < p < \infty$.*

(ii) *$P_+ + P_-$ is the orthogonal linear projection from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)_{\mathrm{rad}}$.*

(iii) *For any $N > 0$, $|x| \gtrsim N^{-1}$, $t \gtrsim N^{-2}$, and choice of sign $\pm$, the integral kernel*[6] $[P_\pm P_N e^{\mp it\Delta}](x, y)$ *obeys the estimate*

$$|[P_\pm P_N e^{\mp it\Delta}](x, y)| \lesssim_d (|x||y|)^{-(d-1)/2}|t|^{-1/2},$$

when $|y| - |x| \sim N|t|$, and

$$|[P_\pm P_N e^{\mp it\Delta}](x, y)| \lesssim_{d,m} \frac{N^d}{(N|x|)^{(d-1)/2}\langle N|y|\rangle^{(d-1)/2}}\langle N^2 t + N|x| - N|y|\rangle^{-m},$$

for any $m \geq 0$ otherwise.

(iii) *For any $N > 0$, $|x| \gtrsim N^{-1}$, $|t| \lesssim N^{-2}$, and choice of sign $\pm$, we have*

$$|[P_\pm P_N e^{\mp it\Delta}](x, y)| \lesssim_{d,m} \frac{N^d}{(N|x|)^{(d-1)/2}\langle N|y|\rangle^{(d-1)/2}}\langle N|x| - N|y|\rangle^{-m},$$

for any $m \geq 0$.

Proof. See [Killip et al. 2008b, Proposition 6.2] (for the $d = 2$ case) or [Killip et al. 2007, Lemma 4.1, Lemma 4.2] (for the higher d case). □

Remark 4.4. Heuristically, $P_- P_N e^{it\Delta}$ and $P_+ P_N e^{-it\Delta}$ for $t > 0$ both propagate away from the origin at speeds $\sim N$. The decay $(|x||y|)^{-(d-1)/2}|t|^{-1/2}$ is superior to the standard decay $|t|^{-d/2}$, which reflects the additional averaging away from the origin caused by the spherical symmetry. (In the proof of [Killip et al. 2008b, Proposition 6.2], this additional averaging is captured using the standard asymptotics of Bessel and Hankel functions.)

Now we prove Theorem 4.1. Fix d, I, u, m, R; we allow implied constants to depend on d, I, m. We may take R to be a power of 2. By the triangle inequality, we have

$$\|u\|_{L_t^2 L_x^{2d/(d-2)}(I \times (\mathbb{R}^d \backslash B(0,R)))} \lesssim \|P_{\leq 1/R} u\|_{L_t^2 L_x^{2d/(d-2)}(I \times \mathbb{R}^d)} + \sum_{N > 1/R} \sum_{\pm} \|P_\pm P_N u\|_{L_t^2 L_x^{2d/(d-2)}(I \times (\mathbb{R}^d \backslash B(0,R)))}.$$

For the first term, we use Bernstein's inequality to estimate

$$\|P_{\leq 1/R} u(t)\|_{L_x^{2d/(d-2)}(\mathbb{R}^d)} \lesssim R^{-1}\|u(t)\|_{L_x^2(\mathbb{R}^d)} \lesssim R^{-1},$$

which is acceptable, so we turn to the latter terms. For ease of notation we shall just deal with the incoming terms $\pm = -$, as the outgoing terms $\pm = -$ terms are handled similarly (but using Duhamel backwards in time instead of forwards).

Write $I = [t_0, t_1]$, then by Duhamel's formula we have

$$P_- P_N u(t) = P_- P_N e^{i(t-t_0)\Delta} u(t_0) - i \int_{t_0}^{t} P_- P_N e^{i(t-t')\Delta} F(u(t'))\, dt'.$$

[6]The integral kernel $T(x, y)$ of a linear operator T is the function for which $Tf(x) = \int_{\mathbb{R}^d} K(x, y) f(y)\, dy$ for all test functions f.

The contribution of the linear term $P_- P_N e^{i(t-t_0)\Delta} u(t_0)$ is bounded by

$$\begin{aligned}\Big\| \sum_{N>1/R} P_- P_N e^{i(t-t_0)\Delta} u(t_0) \Big\|_{L^2_t L^{2d/(d-2)}_x (I \times \mathbb{R}^d)} &\lesssim \Big\| \sum_{N>1/R} P_N e^{i(t-t_0)\Delta} u(t_0) \Big\|_{L^2_t L^{2d/(d-2)}_x (I \times \mathbb{R}^d)} \\ &\lesssim \| e^{i(t-t_0)\Delta} u(t_0) \|_{L^2_t L^{2d/(d-2)}_x (I \times \mathbb{R}^d)} \\ &\lesssim \| u(t_0) \|_{L^2_x(\mathbb{R}^d)} \\ &\lesssim 1,\end{aligned}$$

thanks to Proposition 4.3(i), the boundedness of the Littlewood–Paley projection $P_{>1/R}$, and the endpoint Strichartz estimate (5). Thus this contribution is acceptable, and it remains to show that

$$\sum_{N>1/R} \Big\| \int_{t_0}^t P_- P_N e^{i(t-t')\Delta} F(u(t'))\, dt' \Big\|_{L^2_t L^{2d/(d-2)}_x (I \times (\mathbb{R}^d \backslash B(0,R)))} \lesssim R^{-1}. \tag{11}$$

As we are allowed to let implied constants depend on I, it suffices to show that

$$\int_{t_0}^t \| P_- P_N e^{i(t-t')\Delta} F(u(t')) \|_{L^{2d/(d-2)}_x (\mathbb{R}^d \backslash B(0,R))}\, dt' \lesssim (NR)^{-c} R^{-1},$$

for some absolute constant $c > 0$ and all $t \in I$ and $N > 1/R$. By dyadic decomposition it suffices to show that

$$\int_{t_0}^t \| P_- P_N e^{i(t-t')\Delta} F(u(t'))\, dt' \|_{L^{2d/(d-2)}_x (B(0,2^{m+1}R) \backslash B(0,2^m R))}\, dt' \lesssim (2^m NR)^{-c} R^{-1},$$

for all $m \geq 0$. Replacing R by $2^m R$, we thus see that it suffices to show that

$$\int_{t_0}^t \| P_- P_N e^{i(t-t')\Delta} F(u(t'))\, dt' \|_{L^{2d/(d-2)}_x (B(0,2R)) \backslash B(0,R)}\, dt' \lesssim (NR)^{-c} R^{-1},$$

whenever $R > 0$, $N > 1/R$, and $t \in I$.

From Proposition 4.3 we see that

$$\| P_- P_N e^{it\Delta} f \|_{L^\infty_x (B(0,2R) \backslash B(0,R))} \lesssim (R(R + N|t|))^{-(d-1)/2} |t|^{-1/2} \| f \|_{L^1_x(\mathbb{R}^d)},$$

for $t \gtrsim N^{-2}$, and

$$\| P_- P_N e^{it\Delta} f \|_{L^\infty_x (B(0,2R) \backslash B(0,R))} \lesssim R^{-(d-1)/2} N \| f \|_{L^1_x(\mathbb{R}^d)},$$

for $0 < t \lesssim N^{-2}$; we unify these two estimates as

$$\| P_- P_N e^{it\Delta} f \|_{L^\infty_x (B(0,2R) \backslash B(0,R))} \lesssim (R(R + N|t|))^{-(d-1)/2} N \langle N^2 t \rangle^{-1/2} \| f \|_{L^1_x(\mathbb{R}^d)},$$

for $t > 0$. On the other hand, as P_-, P_N, $e^{it\Delta}$ are bounded on L^2 we have

$$\| P_- P_N e^{it\Delta} f \|_{L^2_x (B(0,2R) \backslash B(0,R))} \lesssim \| f \|_{L^2_x(\mathbb{R}^d)},$$

and hence by interpolation

$$\| P_- P_N e^{it\Delta} f \|_{L^{2d/(d-4)}_x (B(0,2R) \backslash B(0,R))} \lesssim [(R(R + N|t|))^{-(d-1)/2} N \langle N^2 t \rangle^{-1/2}]^{4/d} \| f \|_{L^{2d/(d+4)}_x(\mathbb{R}^d)}.$$

Since

$$\|F(u(t'))\|_{L_x^{2d/(d+4)}(\mathbb{R}^d)} \lesssim \|u(t')\|_{L_x^2(\mathbb{R}^d)}^{1+4/d} \lesssim 1$$

for all $t' \in I$, we thus have

$$\|P_- P_N e^{i(t-t')\Delta} F(u(t'))\|_{L_x^{2d/(d-4)}(B(0,2R)\backslash B(0,R))} \lesssim [(R(R+N|t-t'|))^{-(d-1)/2} N\langle N^2(t-t')\rangle^{-1/2}]^{4/d},$$

and hence by Hölder's inequality

$$\|P_- P_N e^{i(t-t')\Delta} F(u(t'))\|_{L_x^{2d/(d-2)}(B(0,2R)\backslash B(0,R))} \lesssim R[(R(R+N|t-t'|))^{-(d-1)/2} N\langle N^2(t-t')\rangle^{-1/2}]^{4/d}.$$

We can thus bound the left side of Equation (11) by

$$\int_{-\infty}^{t} R[(R(R+N|t-t'|))^{-(d-1)/2} N\langle N^2(t-t')\rangle^{-1/2}]^{4/d}\, dt'.$$

The dominant contribution of this integral occurs in the region when $|t-t'| \sim R/N$, and so we obtain a total contribution of

$$\lesssim R(R/N)(R^{-(d-1)}(R/N)^{-1/2})^{4/d} = R^{-1}(RN)^{-(d-2)/d},$$

which is acceptable. This proves Theorem 4.1. □

Remark 4.5. One can improve the 1 term on the right side of (10) to R^{-c} for some $c > 0$, by using the improved Strichartz estimates in [Shao $\geq$ 2009] that are available in the spherically symmetric case. However, we will not need this improvement here.

5. Nearly continuous solutions are Strichartz class

Theorem 4.1 gives Strichartz norm control of a solution away from the spatial origin. When the solution is sufficiently close in $L_t^\infty L_x^2$ to a Strichartz class solution, we can bootstrap Theorem 4.1 to in fact obtain Strichartz control all the way up to the origin. More precisely, we now show:

Theorem 5.1 (Strichartz class criterion). *Let $d \geq 5$, let I be a compact interval, and let $u \in L_t^\infty L_x^2(I \times \mathbb{R}^d)$ be a spherically symmetric weak solution to NLS. Suppose also that there exists a Strichartz-class solution $v \in C_t^0 L_x^2 \cap L_t^2 L_x^{2d/(d-2)}(I \times \mathbb{R}^2)$ such that $\|u - v\|_{L_t^\infty L_x^2(\mathbb{R}^d)} \leq \varepsilon$. If ε is sufficiently small depending on d, then $u \in L_t^2 L_x^{2d/(d-2)}(I \times \mathbb{R}^d)$.*

Remark 5.2. The theorem fails if ε is large, as one can see from the weak solution defined by Equation (7) for $t \neq 0$ and vanishing for $t = 0$. The arguments in fact give an effective upper bound for the $L_t^2 L_x^{2d/(d-2)}$ norm of u in terms of the corresponding norm of v. Heuristically, the point is that when u (or $u - v$) has small mass, then there are not enough nonlinear effects in play to support persistent mass concentration (as in the example in Remark 4.2) that would cause the $L_t^2 L^{2d/(d-2)}$ norm to become large.

We now prove Theorem 5.1. We fix d, I, u, v, ε and allow all implied constants to depend on d. By shrinking the interval I and using compactness we may assume that

$$\|v\|_{L_t^2 L_x^{2d/(d-2)}(I \times \mathbb{R}^2)} \leq \varepsilon. \tag{12}$$

We write $w := u - v$, thus

$$\|w\|_{L_t^\infty L_x^2(I \times \mathbb{R}^d)} \lesssim \varepsilon \tag{13}$$

and w solves the difference equation

$$i w_t + \Delta w = F(v+w) - F(v) \tag{14}$$

in the integral sense. From the fundamental theorem of calculus (or mean-value theorem) we have the elementary inequality

$$F(v+w) - F(v) = O(|w|(|v|+|w|)^{4/d}). \tag{15}$$

For each integer k, let c_k denote the quantity

$$c_k := \|w\|_{L^2_t L^{2d/(d-2)}_x (I \times (\mathbb{R}^d \backslash B(0,2^k)))}. \tag{16}$$

From Theorem 4.1 and the triangle inequality we have

$$c_k \lesssim_I 2^{-k} + 1, \tag{17}$$

for all k. To prove the theorem, it suffices by the monotone convergence theorem to show that $\sup_k c_k$ is finite. For this we use the following inequality:

Proposition 5.3 (Key inequality). *Let $d \geq 4$. For every k we have*

$$c_k \lesssim \varepsilon + \varepsilon^{4/d} \sum_{j \leq k} 2^{-\frac{d-2}{2}(k-j)} (c_j + c_j^{1-4/d}).$$

Proof. Fix k. By the triangle inequality, we have

$$c_k \lesssim \|P_{\leq 2^{-k}} w(t)\|_{L^2_t L^{2d/(d-2)}_x (I \times \mathbb{R}^d)} + \sum_{\pm} \|P_\pm P_{>2^{-k}} w(t)\|_{L^2_t L^{2d/(d-2)}_x (I \times (\mathbb{R}^d \backslash B(0,2^k)))}. \tag{18}$$

Consider the first term on the right side. By (13)–(15) and (5) we have

$$\|P_{\leq 2^{-k}} w(t)\|_{L^2_t L^{2d/(d-2)}_x (I \times \mathbb{R}^d)} \lesssim \varepsilon + \|P_{\leq 2^{-k}} O(|w|(|v|+|w|)^{4/d})\|_{L^2_t L^{2d/(d+2)}_x (I \times \mathbb{R}^d)},$$

so to show that the contribution of this case is acceptable, it suffices to show that

$$\left\| P_{\leq 2^{-k}} O(|w|(|v|+|w|)^{4/d}) \right\|_{L^2_t L^{2d/(d+2)}_x (I \times \mathbb{R}^d)} \lesssim \varepsilon^{4/d} \sum_{j \leq k} 2^{-\frac{d-2}{2}(k-j)} (c_j + c_j^{4/d}).$$

By the triangle inequality, we can bound the left side by

$$\begin{aligned} &\left\| P_{\leq 2^{-k}} O(|w| 1_{\mathbb{R}^d \backslash B(0,2^{-k})} (|v|+|w|)^{4/d}) \right\|_{L^2_t L^{2d/(d+2)}_x (I \times \mathbb{R}^d)} \\ &\qquad + \sum_{j<k} \left\| P_{\leq 2^{-k}} O(|w| 1_{B(0,2^{-j+1}) \backslash B(0,2^{-j})} (|v|+|w|)^{4/d}) \right\|_{L^2_t L^{2d/(d+2)}_x (I \times \mathbb{R}^d)}. \end{aligned} \tag{19}$$

For the first term of (19), we discard the $P_{\leq 2^{-k}}$ projection and use Hölder's inequality to bound this by

$$\begin{aligned} &\lesssim \|w\|^{1-4/d}_{L^2_t L^{2d/(d+2)}_x (I \times (\mathbb{R}^d \backslash B(0,2^{-k})))} \|v\|^{4/d}_{L^2_t L^{2d/(d+2)}_x (I \times \mathbb{R}^d)} \|w\|^{4/d}_{L^\infty_t L^2_x (I \times \mathbb{R}^d)} \\ &\quad + \|w\|_{L^2_t L^{2d/(d+2)}_x (I \times (\mathbb{R}^d \backslash B(0,2^{-k})))} \|w\|^{4/d}_{L^\infty_t L^2_x (I \times \mathbb{R}^d)}, \end{aligned}$$

which by (13), (12), and (16) is bounded by

$$\lesssim \varepsilon^{4/d} c_k^{1-4/d} + \varepsilon^{4/d} c_k,$$

which is acceptable.

For the second term of (19), we observe from the Hölder and Bernstein inequalities that

$$\begin{aligned}\| P_{\leq 2^{-k}}(f 1_{B(0,2^{-j+1})\backslash B(0,2^{-j})}) \|_{L_x^{2d/(d+2)}(\mathbb{R}^d)} &\lesssim 2^{-\frac{d-2}{2}k} \| f 1_{B(0,2^{-j+1})\backslash B(0,2^{-j})} \|_{L_x^1(\mathbb{R}^d)} \\ &\lesssim 2^{-\frac{d-2}{2}(k-j)} \| f \|_{L_x^{2d/(d+2)}(B(0,2^{-j+1})\backslash B(0,2^{-j}))},\end{aligned}$$

for any f. Using this inequality and arguing as before, we see that the second term of (19) is bounded by

$$\lesssim \sum_{j<k} 2^{-\frac{d-2}{2}(k-j)} (\varepsilon^{4/d} c_j^{1-4/d} + \varepsilon^{4/d} c_j),$$

which is acceptable.

Since we have dealt with the first term of (18), it now suffices by the triangle inequality to show that

$$\| P_\pm P_{>2^{-k}} w(t) \|_{L_t^2 L_x^{2d/(d-2)}(I\times(\mathbb{R}^d\backslash B(0,2^m)))} \lesssim \varepsilon + \varepsilon^{4/d} \sum_{j\leq k} 2^{-\frac{d-2}{2}(k-j)} (c_j + c_j^{1-4/d}),$$

for either choice of sign $\pm$. We shall just do this for the incoming case $\pm = -$: the outgoing case $\pm = +$ is similar but requires one to apply Duhamel's formula backwards in time.

Write $I = [t_0, t_1]$. By Duhamel's formula and (14), we have

$$P_- P_{>2^{-k}} w(t) = P_- P_{>2^{-k}} e^{i(t-t_0)\Delta} w(t_0) - i \int_{t_0}^t P_- P_{>2^{-k}} e^{i(t-t')\Delta} (F(v+w) - F(v))(t')\, dt'.$$

The contribution of the first term is $O(\varepsilon)$ by Proposition 4.3(i), Equations (5) and (13), so it suffices to show that

$$\Big\| \int_{t_0}^t P_- P_{>2^{-k}} e^{i(t-t')\Delta} (F(v+w) - F(v))(t')\, dt' \Big\|_{L_t^2 L_x^{2d/(d-2)}(I\times(\mathbb{R}^d\backslash B(0,2^k)))} \lesssim \varepsilon^{4/d} \sum_{j\leq k} 2^{-\frac{d-2}{2}(k-j)} (c_j + c_j^{1-4/d}).$$

We split

$$F(v+w) - F(v) = (F(v+w) - F(v)) 1_{\mathbb{R}^d\backslash B(0,2^{k-1})} + \sum_{j<k-1} (F(v+w) - F(v)) 1_{B(0,2^{j+1})\backslash B(0,2^j)}.$$

The contribution of the first term can be estimated using Proposition 4.3(i), Equations (5) and (15) to be

$$\lesssim \| |w| (|v|^{4/d} + |w|^{4/d}) \|_{L_t^2 L_x^{2d/(d+2)}(I\times(\mathbb{R}^d\backslash B(0,2^{k-1})))}.$$

By a slight modification of the calculation used to bound the first term of (19), we can control this expression by

$$\lesssim \varepsilon^{4/d} c_{k-1}^{1-4/d} + \varepsilon^{4/d} c_{k-1},$$

and so by the triangle inequality it suffices to show that

$$\sum_{N>2^{-k}} \left\| \int_{t_0}^{t} P_- P_N e^{i(t-t')\Delta}[(F(v+w)-F(v))(t') 1_{B(0,2^{j+1})\backslash B(0,2^j)}]\, dt' \right\|_{L_t^2 L_x^{2d/(d-2)}(I\times(\mathbb{R}^d\backslash B(0,2^k)))} \lesssim \varepsilon^{4/d} 2^{-\frac{d-2}{2}(k-j)}(c_j + c_j^{1-4/d}), \quad (20)$$

for each $j < k-1$.

Fix j. By Proposition 4.3(ii), (iii), the integral kernel $(P_- P_N e^{i(t-t')\Delta})(x,y)$ for $x \in \mathbb{R}^d\backslash B(0,2^k)$, $t' < t$, $N > 2^{-k}$, and $y \in B(0,2^{j+1})\backslash B(0,2^j)$ obeys the bounds

$$\begin{aligned} |[P_- P_N e^{i(t-t')\Delta}](x,y)| &\lesssim \frac{N^d}{(N|x|)^{(d-1)/2}\langle 2^j N\rangle^{(d-1)/2}} \langle N^2(t-t') + N|x|\rangle^{-100d} \\ &\lesssim N^d (N|x|)^{-50d} \langle N^2(t-t')\rangle^{-50d}, \end{aligned}$$

say. From this we obtain the pointwise bound

$$|P_- P_N e^{i(t-t')\Delta}(f 1_{B(0,2^{j+1})\backslash B(0,2^j)})(x)| \lesssim N^d (N|x|)^{-50d} \langle N^2(t-t')\rangle^{-50d} \|f\|_{L_x^1(B(0,2^{j+1})\backslash B(0,2^j))},$$

for $x \in \mathbb{R}^d\backslash B(0,2^k)$ and any f, which by Hölder's inequality implies the bounds

$$\begin{aligned} &\| P_- P_N e^{i(t-t')\Delta}(f 1_{B(0,2^{j+1})\backslash B(0,2^j)}) \|_{L_x^{2d/(d-2)}(\mathbb{R}^d\backslash B(0,2^k))} \\ &\qquad\lesssim 2^{\frac{d-2}{2}k} 2^{\frac{d-2}{2}j} N^d (2^k N)^{-50d} \langle N^2(t-t')\rangle^{-50d} \|f\|_{L_x^{2d/(d+2)}(B(0,2^{j+1})\backslash B(0,2^j))}. \end{aligned}$$

By Young's inequality we conclude that the left side of (20) is bounded by

$$\lesssim \sum_{N>2^{-k}} 2^{\frac{d-2}{2}k} 2^{\frac{d-2}{2}j} N^{d-2} (2^k N)^{-50d} \|F(v+w) - F(v)\|_{L_t^2 L_x^{2d/(d+2)}(I\times B(0,2^{j+1})\backslash B(0,2^j))}.$$

Modifying the computation used to bound the first term of (19), this expression can be controlled by

$$\lesssim \sum_{N>2^{-k}} 2^{\frac{d-2}{2}k} 2^{\frac{d-2}{2}j} N^{d-2} (2^k N)^{-50d} (\varepsilon^{4/d} c_j^{1-4/d} + \varepsilon^{4/d} c_j),$$

and on performing the summation in N one obtains the claim (20), and Proposition 5.3 follows. □

From Proposition 5.3 (and using the hypothesis $d \geq 5$ to make the decay $2^{-\frac{d-2}{2}(k-j)}$ faster than the blowup of 2^{-j}), we see that if we have any bound of the form

$$c_k \leq A + B 2^{-k},$$

for all k and some $A, B > 0$, then (if ε is sufficiently small, and A is sufficiently large depending on ε), one can conclude a bound of the form

$$c_k \leq A + \tfrac{1}{2} B 2^{-k},$$

for all k. Iterating this and taking limits, we conclude that

$$c_k \leq A,$$

for all k. Applying this argument starting from Equation (17) we conclude that $c_k \lesssim 1$ for all k, as desired, and Theorem 5.1 follows.

6. Proofs of theorems

With Theorem 5.1 in hand, it is now an easy matter to establish Theorems 1.27 and 1.32.

6.1. ***Proof of Theorem 1.27.*** It is clear that (i) implies (ii), and that (iii) implies (iv) implies (v). From Proposition 1.9 we also see that (ii) implies (iii). So the only remaining task is to show that (v) implies (i). It suffices to do this locally, that is, to show that for any time t for which (v) holds, that u is a Strichartz class solution in some neighbourhood of t in I.

By the hypothesis (v), one can find a connected neighbourhood J of t in I such that

$$M(u(t')) \leq M(u(t)) + \varepsilon_d,$$

for all $t' \in J$. By Equation (4) (and shrinking J if necessary) we conclude that

$$\|u(t') - u(t)\|^2_{L^2_x(\mathbb{R}^d)} \leq 2\varepsilon_d,$$

say, for all $t' \in J$.

By shrinking J some more, we may apply Proposition 1.9 to find a Strichartz class solution

$$v \in C^0_t L^2_x \cap L^2_t L^{2d/(d-2)}_x (J \times \mathbb{R}^d)$$

on J with $v(t) = u(t)$. Since v is a strong solution, by shrinking J some more we may assume that $\|v(t') - v(t)\|_{L^2_x(\mathbb{R}^d)} \leq \varepsilon_d^{1/2}$ for all $t' \in J$. By the triangle inequality we thus see that

$$\|u - v\|_{L^\infty_t L^2_x(J \times \mathbb{R}^d)} \lesssim \varepsilon_d^{1/2}.$$

Applying Theorem 5.1 and taking ε_d sufficiently small, we conclude that u is a Strichartz class solution on J as required, and Theorem 1.27 follows.

6.2. ***Proof of Theorem 1.32.*** It is clear that (i) implies (ii) and that (iii) implies (iv) implies (v). From Corollary 1.25 we know that (i) implies (iii), while from Proposition 1.9(iii) and Definition 1.15 we see that (ii) implies (iv). Thus, as before, the only remaining task is to show that (v) implies (i). Again, it suffices to establish the local claim that if $t \geq t_0$ is such that (v) holds, then u is in the Strichartz class for some $[t, t+\varepsilon) \cap I$, and similarly for $t \leq t_0$ and $(t-\varepsilon, t] \cap I$. But this follows by a routine modification of the arguments in Section 6.1. □

Acknowledgement

The author thanks Jim Colliander for helpful discussions, and Tim Candy, Fabrice Planchon and Monica Visan for corrections and references, and the anonymous referee for valuable comments.

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Received 16 Jul 2008. Accepted 17 Feb 2009.

TERENCE TAO: `tao@math.ucla.edu`
University of California, Los Angeles, Mathematics Department, Los Angeles CA 90095-1555, United States
http://ftp.math.ucla.edu/~tao/

ANALYSIS AND PDE
Vol. 2, No. 1, 2009

THE LINEAR PROFILE DECOMPOSITION FOR THE AIRY EQUATION AND THE EXISTENCE OF MAXIMIZERS FOR THE AIRY STRICHARTZ INEQUALITY

SHUANGLIN SHAO

We establish the linear profile decomposition for the Airy equation with complex or real initial data in L^2. As an application, we obtain a dichotomy result on the existence of maximizers for the symmetric Airy Strichartz inequality.

1. Introduction

In this paper, we consider the problem of the linear profile decomposition for the Airy equation with the L^2 initial data

$$\begin{cases} \partial_t u + \partial_x^3 u = 0, t \in \mathbb{R}, \ x \in \mathbb{R}, \\ u(0,x) = u_0(x) \in L^2, \end{cases} \tag{1}$$

where $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ or $\mathbb{C}$. Roughly speaking, the profile decomposition is to investigate the general structure of a sequence of solutions to the Airy equation with bounded initial data in L^2. We expect that it can be expressed, up to a subsequence, as a sum of a superposition of concentrating waves — profiles — and a reminder term. The profiles are "almost orthogonal" in the Strichartz space and in L^2 while the remainder term is small in the same Strichartz norm and can be negligible in practice. The profile decomposition is also referred to as the *bubble decomposition* in the literature; see [Killip and Visan 2008b, p.35] for an interesting historical discussion.

The same problem in the context of the wave or Schrödinger equations has been intensively studied recently. For the wave equations, Bahouri and Gérard [1999] established a linear profile decomposition for the energy critical wave equation in $\mathbb{R}^3$ (their argument can be generalized to higher dimensions). Following [Bahouri and Gérard 1999], Keraani [2001] obtained a linear profile decomposition for energy critical Schrödinger equations; see also [Shao 2009]. For the mass critical Schrödinger equations, when $d = 2$, Merle and Vega [1998] established a linear profile decomposition, similar in spirit to that in [Bourgain 1998]; Carles and Keraani [2007] treated the $d = 1$ case, while the higher-dimensional analogue was obtained by Bégout and Vargas [2007]. In general, a nonlinear profile decomposition can be achieved from the linear case via a perturbation argument. The first ingredient of the proof of linear profile decompositions is to start with some refined inequality: the refined Sobolev embedding or the refined Strichartz inequality. Usually establishing such refinements needs some nontrivial work. For instance, in the Schrödinger case, the two-dimensional improvement is due to Moyua et al. [1999] involving the X_p^q spaces; the one-dimensional improvement due to Carles and Keraani [2007] using the

MSC2000: 35Q53.
Keywords: gKdV, mass-critical, profile decomposition, maximizer.

Hausdorff–Young inequality and the weighted Fefferman–Phong inequality [Fefferman 1983], which Kenig et al. [2000] first introduced to prove their refined Strichartz inequality (5) for the Airy equation; the higher-dimensional refinement is due to Bégout and Vargas [2007] based on a new bilinear restriction estimate for paraboloids by Tao [2003]. Another important ingredient of the arguments is the idea of the concentration-compactness principle which aims to compensate for the defect of compactness of the Strichartz inequality, and was exploited in [Bahouri and Gérard 1999; Merle and Vega 1998; Carles and Keraani 2007; Bégout and Vargas 2007]; also see [Schindler and Tintarev 2001] for an abstract version of this principle in the Hilbert space. The profile decompositions turn out to be quite useful in nonlinear dispersive equations. For instance, they can be used to analyze the mass concentration phenomena near the blow up time for the mass critical Schrödinger equation; see [Merle and Vega 1998; Carles and Keraani 2007; Bégout and Vargas 2007]. They were also used to show the existence of minimal mass or energy blow-up solutions for the Schrödinger or wave equations at critical regularity, which is an important step in establishing the global well-posedness and scattering results for such equations; see [Kenig and Merle 2006; 2007; Killip et al. 2007; Tao et al. 2007; Killip and Visan 2008a. Shao [2009] used it to establish the existence of maximizers for the nonendpoint Strichartz and Sobolev–Strichartz inequalities for the Schrödinger equation.

The discussion above motivates the question of profile decompositions for the Airy equation, which is the free form of the mass critical generalized Korteweg–de Vries (gKdV) equation

$$\begin{cases} \partial_t u + \partial_x^3 u \pm u^4 \partial_x u = 0, t \in \mathbb{R},\ x \in \mathbb{R}, \\ u(0,x) = u_0(x). \end{cases} \tag{2}$$

This is one of the (generalized) KdV equations [Tao 2006b] and is the natural analogy to the mass critical nonlinear Schrödinger equation in one spatial dimension. The KdV equations arise from describing the waves on shallow water surfaces, and turn out to have connections to many other physical problems. As is well known, the class of solutions to (1) enjoys a number of symmetries that preserve the mass $\int |u|^2 dx$. We will employ the notations from [Killip et al. 2007] and first discuss the symmetries at the initial time $t = 0$.

Definition 1.1 (Mass-preserving symmetry group). For any phase $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, position $x_0 \in \mathbb{R}$ and scaling parameter $h_0 > 0$, we define the unitary transform $g_{\theta,x_0,h_0} : L^2 \to L^2$ by the formula

$$[g_{\theta,x_0,h_0} f](x) := \frac{1}{h_0^{1/2}} e^{i\theta} f\Big(\frac{x - x_0}{h_0}\Big).$$

We let G be the collection of such transformations. It is easy to see that G is a group.

Unlike the free Schrödinger equation

$$\begin{cases} i\partial_t u - \triangle u = 0, t \in \mathbb{R},\ x \in \mathbb{R}^d, \\ u(0,x) = u_0(x), \end{cases} \tag{3}$$

two important symmetries are missing for (1), namely, the Galilean symmetry

$$u(t,x) \mapsto e^{ix\xi_0 + it|\xi_0|^2} u(t, x + 2t\xi_0),$$

and the pseudo-conformal symmetry

$$u(t,x) \mapsto |t|^{-d/2} e^{-i|x|^2/(4t)} u(-1/t, x/t).$$

This lack of symmetries causes difficulties if we try to mimic the existing argument of profile decompositions for the Schrödinger equations. In this paper, we will show how to compensate for the lack of the Galilean symmetry when developing the analogous version of linear profile decompositions for the Airy Equation (1).

Like Schrödinger equations, an important family of inequalities, the Airy Strichartz inequality [Kenig et al. 1991, Theorem 2.1], is associated with the Airy equation (1). It is invariant under the symmetry group and asserts that

$$\|D^\alpha e^{-t\partial_x^3} u_0\|_{L_t^q L_x^r} \lesssim \|u_0\|_{L^2}, \tag{4}$$

if and only if $-\alpha + 3/q + 1/r = 1/2$ and $-1/2 < \alpha \le 1/q$, where $e^{-t\partial_x^3} u_0$ and D^α are defined in Section 2. When $q = r = 6$ and $\alpha = 1/6$, we also have the following refined Strichartz estimate due to Kenig–Ponce–Vega, which is the key to establishing the profile decomposition results for the Airy equation in this paper.

Lemma 1.2 (KPV's refined Strichartz [Kenig et al. 2000]). *Let $p > 1$. Then*

$$\left\| D^{1/6} e^{-t\partial_x^3} u_0 \right\|_{L^6_{t,x}} \le C \big(\sup_\tau |\tau|^{\frac{1}{2} - \frac{1}{p}} \|\widehat{u_0}\|_{L^p(\tau)} \big)^{\frac{1}{3}} \|u_0\|_{L^2}^{\frac{2}{3}}, \tag{5}$$

where τ denotes an interval of the real line with length $|\tau|$.

In Section 3, we will present a new proof suggested by Terence Tao by using the Whitney decomposition.

As in the Schrödinger case, the Airy Strichartz inequality (4) cannot guarantee the solution map from the L^2 space to the Strichartz space to be compact, namely, every L^2-bounded sequence will produce a convergent subsequence of solutions in the Strichartz space. The particular Strichartz space we are interested in is equipped with the norm $\|D^{1/6} u\|_{L^6_{t,x}}$. The failure of compactness can be seen explicitly from creating counter-examples by considering the symmetries in L^2 such as the space and time translations, or scaling symmetry or frequency modulation. Indeed, given $x_0 \in \mathbb{R}$, $t_0 \in \mathbb{R}$ and $h_0 \in (0, \infty)$, we denote by τ_{x_0}, S_{h_0} and R_{t_0} the operators defined by

$$\tau_{x_0}\phi(x) := \phi(x - x_0), \quad S_{h_0}\phi(x) := \frac{1}{h_0^{1/2}} \phi\Big(\frac{x}{h_0}\Big), \quad R_{t_0}\phi(x) := e^{-t_0\partial_x^3}\phi(x).$$

Let $(x_n)_{n\ge1}$, $(t_n)_{n\ge1}$ be sequences both going to infinity, and $(h_n)_{n\ge1}$ be a sequence going to zero as n goes to infinity. Then for any nontrivial $\phi \in \mathcal{S}$, $(\tau_{x_n}\phi)_{n\ge1}$, $(S_{h_n}\phi)_{n\ge1}$ and $(R_{t_n}\phi)_{n\ge1}$ weakly converge to zero in L^2. However, their Strichartz norms are all equal to $\|D^{1/6} e^{-t\partial_x^3}\phi\|_{L^6_{t,x}}$, which is nonzero. Hence these sequences are not relatively compact in the Strichartz spaces. Moreover, the frequency modulation also exhibits the defect of compactness: for $\xi_0 \in \mathbb{R}$, we define M_{ξ_0} via

$$M_{\xi_0}\phi(x) := e^{ix\xi_0}\phi(x).$$

Choosing $(\xi_n)_{n\ge1}$ to be a sequence going to infinity as n goes to infinity, we see that $(M_{\xi_n}\phi)_{n\ge1}$ converges weakly to zero. However, from Remark 1.7, $\|D^{1/6} e^{-t\partial_x^3}(e^{i(\cdot)\xi_n}\phi)\|_{L^6_{t,x}}$ converges to $3^{-1/6}\|e^{-it\partial_x^2}\phi\|_{L^6_{t,x}}$, which is not zero. This shows that the modulation operator M_{ξ_0} is not compact either.

It will be clear from the statements of Theorem 1.5 and Theorem 1.6 that these four symmetries in L^2 above are the only obstructions to the compactness of the solution map. Hence the parameter (h_0, ξ_0, x_0, t_0) plays a special role in characterizing this defect of compactness; moreover, a sequence of such parameters needs to satisfy some *orthogonality* constraint (the term is used in the sense of Lemma 5.2).

Definition 1.3 (Orthogonality). For $j \neq k$, two sequences

$$\Gamma_n^j := (h_n^j, \xi_n^j, x_n^j, t_n^j)_{n\geq 1} \quad \text{and} \quad \Gamma_n^k := (h_n^k, \xi_n^k, x_n^k, t_n^k)_{n\geq 1}$$

in $(0, \infty) \times \mathbb{R}^3$ are orthogonal if one of the following holds:

- $\lim_{n\to\infty}\left(\dfrac{h_n^j}{h_n^k} + \dfrac{h_n^k}{h_n^j} + h_n^j|\xi_n^j - \xi_n^k|\right) = \infty,$
- $(h_n^j, \xi_n^j) = (h_n^k, \xi_n^k)$ and

$$\lim_{n\to\infty}\left(\frac{|t_n^k - t_n^j|}{(h_n^j)^3} + \frac{3|(t_n^k - t_n^j)\xi_n^j|}{(h_n^j)^2} + \frac{|x_n^j - x_n^k + 3(t_n^j - t_n^k)(\xi_n^j)^2|}{h_n^j}\right) = \infty.$$

Remark 1.4. For any $\Gamma_n^j = (h_n^j, \xi_n^j, x_n^j, t_n^j)_{n\geq 1}$, it is clear that, up to a subsequence, $\lim_{n\to\infty} |h_n^j\xi_n^j|$ is either finite or infinite. For the former, we can reduce to $\xi_n^j \equiv 0$ for all n by changing profiles; see Remark 3.6. For the latter, the corresponding profiles exhibit a Schrödinger behavior in some sense; see Remark 1.7. In view of this, we will group the decompositions accordingly in the statements of our main theorems below.

Now we are able to state the main theorems. When the initial data to Equation (1) is complex, the following theorem on the linear Airy profile decomposition is proven in Section 5.

Theorem 1.5 (Complex version). *Let $(u_n)_{n\geq 1}$ be a sequence of complex-valued functions satisfying $\|u_n\|_{L^2} \leq 1$. Then up to a subsequence, there exists a sequence of L^2 functions $(\phi^j)_{j\geq 1} : \mathbb{R} \to \mathbb{C}$ and a family of pair-wise orthogonal sequences $\Gamma_n^j = (h_n^j, \xi_n^j, x_n^j, t_n^j) \in (0, \infty) \times \mathbb{R}^3$ such that, for any $l \geq 1$, there exists an L^2 function $w_n^l : \mathbb{R} \to \mathbb{C}$ satisfying*

$$u_n = \sum_{\substack{1\leq j\leq l,\ \xi_n^j\equiv 0 \\ \text{or } |h_n^j\xi_n^j|\to\infty}} e^{t_n^j\partial_x^3} g_n^j\big[e^{i(\cdot)h_n^j\xi_n^j}\phi^j\big] + w_n^l, \tag{6}$$

where $g_n^j := g_{0,x_n^j,h_n^j} \in G$ and

$$\lim_{l\to\infty}\lim_{n\to\infty}\big\|D^{1/6}e^{-t\partial_x^3}w_n^l\big\|_{L^6_{t,x}} = 0. \tag{7}$$

Moreover, for every $l \geq 1$,

$$\lim_{n\to\infty}\left(\|u_n\|_{L^2}^2 - \left(\sum_{j=1}^{l}\|\phi^j\|_{L^2}^2 + \|w_n^l\|_{L^2}^2\right)\right) = 0. \tag{8}$$

When the initial sequence is of real-value, we analogously obtain the following real-version profile decomposition. Note that we can restrict the frequency parameter ξ_n^j to be nonnegative.

Theorem 1.6 (Real version). *Let $(u_n)_{n\geq 1}$ be a sequence of real-valued functions satisfying $\|u_n\|_{L^2} \leq 1$. Then up to a subsequence there exists a sequence of L^2 functions, $(\phi^j)_{j\geq 1}$: $\mathbb{R} \to \mathbb{C}$, and a family of orthogonal sequences $\Gamma_n^j = (h_n^j, \xi_n^j, x_n^j, t_n^j) \in (0,\infty)\times[0,\infty)\times\mathbb{R}^2$ such that, for any $l \geq 1$, there exists an L^2 function w_n^l: $\mathbb{R}\to\mathbb{R}$ satisfying*

$$u_n = \sum_{\substack{1\leq j\leq l,\, \xi_n^j\equiv 0 \\ \text{or } |h_n^j\xi_n^j|\to\infty}} e^{t_n^j\partial_x^3} g_n^j\big[\mathrm{Re}(e^{i(\cdot)h_n^j\xi_n^j}\phi^j)\big] + w_n^l, \tag{9}$$

where $g_n^j := g_{0,x_n^j,h_n^j} \in G$ and

$$\lim_{l\to\infty}\lim_{n\to\infty}\big\|D^{1/6}e^{-t\partial_x^3}w_n^l(x)\big\|_{L^6_{t,x}} = 0. \tag{10}$$

Moreover, for every $l \geq 1$,

$$\lim_{n\to\infty}\Bigg(\|u_n\|_{L^2}^2 - \Big(\sum_{\substack{1\leq j\leq l,\, \xi_n^j\equiv 0 \\ \text{or } |h_n^j\xi_n^j|\to\infty}} \big\|\mathrm{Re}(e^{i(\cdot)h_n^j\xi_n^j}\phi^j)\big\|_{L^2}^2 + \|w_n^l\|_{L^2}^2\Big)\Bigg) = 0. \tag{11}$$

When $\lim_{n\to\infty}|h_n^j\xi_n^j| = \infty$ for some $1\leq j\leq l$, the profile will exhibit asymptotic "Schrödinger" behavior. For simplicity, we just look at the complex case.

Remark 1.7 (Asymptotic Schrödinger behavior). Without loss of generality, we assume $\phi^j \in \mathcal{S}$ with the compact Fourier support $[-1, 1]$. Then

$$\begin{aligned} D^{1/6}e^{-(t-t_n^j)\partial_x^3} g_n^j\big[e^{i(\cdot)h_n^j\xi_n^j}\phi^j\big](x) &= \int e^{i(x-x_n^j)\xi + i(t-t_n^j)\xi^3}|\xi|^{1/6}(h_n^j)^{1/2}\widehat{\phi^j}(h_n^j(\xi-\xi_n^j))\,d\xi \\ &= (h_n^j)^{-1/2}|\xi_n^j|^{1/6}e^{i(x-x_n^j)\xi_n^j + i(t-t_n^j)(\xi_n^j)^3} \\ &\qquad \times \int e^{i\big[\frac{\eta(x-x_n^j+3(t-t_n^j)(\xi_n^j)^2)}{h_n^j} + \frac{\eta^3(t-t_n^j)}{(h_n^j)^3} + \frac{3\eta^2(t-t_n^j)\xi_n^j}{(h_n^j)^2}\big]}\left|1+\frac{\eta}{h_n^j\xi_n^j}\right|^{1/6}\widehat{\phi^j}(\eta)\,d\eta. \end{aligned}$$

Set $x' := \dfrac{x - x_n^j + 3(t-t_n^j)(\xi_n^j)^2}{h_n^j}$ and $t' := \dfrac{3(t-t_n^j)\xi_n^j}{(h_n^j)^2}$. Then the dominated convergence theorem yields

$$\begin{aligned} \big\|D^{1/6}e^{-(t-t_n^j)\partial_x^3}g_n^j[e^{i(\cdot)h_n^j\xi_n^j}\phi^j]\big\|_{L^6_{t,x}} &= 3^{-1/6}\left\|\int e^{ix'\eta+it'\eta^2}e^{it'\frac{\eta^3}{3h_n^j\xi_n^j}}\left|1+\frac{\eta}{h_n^j\xi_n^j}\right|^{1/6}\widehat{\phi^j}\,d\eta\right\|_{L^6_{t',x'}} \\ &\to_{n\to\infty} 3^{-1/6}\|e^{-it'\partial_x^2}\phi^j\|_{L^6_{t',x'}}, \end{aligned}$$

where $e^{-it\partial_x^2}$ denotes the Schrödinger evolution operator defined via

$$e^{-it\partial_x^2}f(x) := \int_{\mathbb{R}} e^{ix\xi+it|\xi|^2}\widehat{f}(\xi)\,d\xi.$$

Indeed,

$$\int e^{ix'\eta+it'\eta^2}e^{it'\frac{\eta^3}{3h_n^j\xi_n^j}}\left|1+\frac{\eta}{h_n^j\xi_n^j}\right|^{1/6}\widehat{\phi^j}\,d\eta \to e^{-it'\partial_x^2}\phi^j(x') \quad \text{a.e.},$$

and by using [Stein 1993, Corollary, p. 334] or integration by parts,

$$\left| \int e^{ix'\eta + it'\eta^2} e^{it'\frac{\eta^3}{3h_n^j \xi_n^j}} |1 + \frac{\eta}{h_n^j \xi_n^j}|^{1/6} \widehat{\phi^j}\, d\eta \right| \le C_{\phi^j} B(t', x')$$

for n large enough but still uniform in n. Here

$$B(t', x') = \begin{cases} (1+|t'|)^{-1/2} \le C\big[(1+|x'|)(1+|t'|)\big]^{-1/4} & \text{for } |x'| \le 6|t'|, \\ (1+|x'|)^{-1} \le C\big[(1+|x'|)(1+|t'|)\big]^{-1/2} & \text{for } |x'| > 6|t'|. \end{cases}$$

It is easy to observe that $B \in L^6_{t',x'}$.

In the next three paragraphs, we outline the proof of Theorem 1.5 in three steps; Theorem 1.6 follows similarly. Given an L^2-bounded sequence $(u_n)_{n\ge1}$, at the first step, we use the refined Strichartz inequality (5) and an iteration argument to obtain a preliminary decomposition for $(u_n)_{n\ge1}$: up to a subsequence

$$u_n = \sum_{j=1}^{N} f_n^j + q_n^N,$$

where $\widehat{f_n^j}$ is supported on an interval $(\xi_n^j - \rho_n^j, \xi_n^j + \rho_n^j)$ and $|\widehat{f_n^j}| \le C(\rho_n^j)^{-1/2}$, and $e^{-t\partial_x^3} q_n^N$ is small in the Strichartz norm. Then we impose the orthogonality condition on (ρ_n^j, ξ_n^j): for $j \ne k$,

$$\lim_{n\to\infty} \Big(\frac{\rho_n^j}{\rho_n^k} + \frac{\rho_n^k}{\rho_n^j} + \frac{|\xi_n^j - \xi_n^k|}{\rho_n^j} \Big) = \infty,$$

to regroup the decomposition.

At the second step, for each $j \in [1, N]$, we will perform a further decomposition to f_n^j to extract the space and time parameters. For simplicity, we suppress all the superscripts j and rescale $(f_n)_{n\ge1}$ to obtain $P = (P_n)_{n\ge1}$ by setting

$$\widehat{P_n}(\cdot) := \rho_n^{1/2} \widehat{f_n} \left(\rho_n (\cdot + \rho_n^{-1} \xi_n) \right),$$

from which we can infer that each $\widehat{P_n}$ is bounded and supported on a finite interval centered at the origin. We apply the concentration-compactness argument to $(P_n)_{n\ge1}$ to extract (y_n^α, s_n^α): for any $A \ge 1$, up to a subsequence,

$$P_n(x) = \sum_{\alpha=1}^{A} e^{-ix\rho_n^{-1}\xi_n} e^{s_n^\alpha \partial_x^3} \big[e^{i(\cdot)\rho_n^{-1}\xi_n} \phi^\alpha(\cdot) \big](x - y_n^\alpha) + P_n^A(x). \tag{12}$$

More precisely, we will investigate the set of weak limits,

$$\mathcal{W}(P) := \big\{ w\text{-}\lim_{n\to\infty} e^{-ix\rho_n^{-1}\xi_n} e^{-s_n \partial_x^3} \big[e^{i(\cdot)\rho_n^{-1}\xi_n} P_n(\cdot) \big](x + y_n) \text{ in } L^2 : (y_n, s_n) \in \mathbb{R}^2 \big\},$$

where the notion $w\text{-}\lim_{n\to\infty} f_n$ denotes, up to a subsequence, the weak limit of $(f_n)_{n\ge1}$ in L^2. Note that due to the lack of Galilean transform and the additional multiplier weight in the current Strichartz norm, it is a slight but necessary modification to the Schrödinger case [Carles and Keraani 2007], where $\mathcal{W}(P)$ is the set

$$\{w\text{-}\lim_{n\to\infty} e^{is_n\partial_x^2} P_n(x+y_n) \text{ in } L^2 : (y_n, s_n) \in \mathbb{R}^2\}.$$

In (12), we impose the orthogonality condition on (y_n^α, s_n^α): for $\alpha \neq \beta$,

$$\lim_{n\to\infty}\left(\left|y_n^\beta - y_n^\alpha + \frac{3(s_n^\beta - s_n^\alpha)(\xi_n)^2}{(\rho_n)^2}\right| + \left|\frac{3(s_n^\beta - s_n^\alpha)\xi_n}{\rho_n}\right| + \left|s_n^\beta - s_n^\alpha\right|\right) = \infty. \tag{13}$$

The error term $P^A := (P_n^A)_{n\geq 1}$ is small in the weak sense that

$$\lim_{A\to\infty} \mu(P^A) := \lim_{A\to\infty} \sup\{\|\phi\|_{L^2} : \phi \in \mathcal{W}(P^A)\} = 0. \tag{14}$$

Since $f_n(x) = \sqrt{\rho_n} e^{ix\xi_n} P_n(\rho_n x)$,

$$f_n(x) = \sum_{\alpha=1}^{A} \sqrt{\rho_n} e^{s_n^\alpha \partial_x^3}\big[e^{i(\cdot)\rho_n^{-1}\xi_n}\phi^\alpha(\cdot)\big](\rho_n x - y_n^\alpha) + \sqrt{\rho_n} e^{ix\xi_n} P_n^A(\rho_n x).$$

Let $e_n^A := \sqrt{\rho_n} e^{ix\xi_n} P_n^A(\rho_n x)$. Now the major task is to upgrading this weak convergence in (14) to

$$\lim_{A\to\infty}\lim_{n\to\infty} \|D^{1/6} e^{-t\partial_x^3} e_n^A\|_{L^6_{t,x}} = 0.$$

To achieve this, we will interpolate $L^6_{t,x}$ between $L^q_{t,x}$ and $L^\infty_{t,x}$ for some $4 \leq q < 6$. The $L^q_{t,x}$ norm is controlled by some localized restriction estimates and the $L^\infty_{t,x}$ norm is expected to be controlled by $\mu(P^A)$. Unlike the Schrödinger case, we will distinguish the case $\lim_{n\to\infty} |\rho_n^{-1}\xi_n| = +\infty$ from $\lim_{n\to\infty} |\rho_n^{-1}\xi_n| < +\infty$ due to the additional multiplier weight in the current Strichartz norm.

The final decomposition is obtained by setting

$$(h_n^j, \xi_n^j, x_n^j, t_n^j) := \big((\rho_n^j)^{-1}, \xi_n^j, (\rho_n^j)^{-1} y_n^j, (\rho_n^j)^{-3} s_n^j\big),$$

and showing two orthogonality results for the profiles.

1.8. The second part of this paper is devoted to applying the linear profile decomposition result to the problem of the existence of maximizers for the Airy Strichartz inequality. As a corollary of Theorems 1.5 and 1.6, we will establish a dichotomy result. Denote

$$S_{\text{airy}}^{\mathbb{C}} := \sup\big\{\big\|D^{1/6} e^{-t\partial_x^3} u_0\big\|_{L^6_{t,x}} : \|u_0\|_{L^2} = 1\big\}, \tag{15}$$

when u_0 is complex-valued; similarly we define $S_{\text{airy}}^{\mathbb{R}}$ for real-valued initial data. We are interested in determining whether there exists a maximizing function u_0 with $\|u_0\|_{L^2} = 1$ for which

$$\|D^{1/6} e^{-t\partial_x^3} u_0\|_{L^6_{t,x}} = S_{\text{airy}} \|u_0\|_{L^2},$$

where S_{airy} represents either $S_{\text{airy}}^{\mathbb{C}}$ or $S_{\text{airy}}^{\mathbb{R}}$. The analogous question to the Schrödinger Strichartz inequalities was studied by Kunze [2003], Foschi [2007], Hundertmark and Zharnitsky [2006], Carneiro [2008], Bennett et al. [2008] and Shao [2009]. We set

$$S^{\mathbb{C}}_{\text{schr}} := \sup\left\{ \left\| e^{-it\Delta} u_0 \right\|_{L^6_{t,x}(\mathbb{R}\times\mathbb{R}^d)} : \|u_0\|_{L^2(\mathbb{R}^d)} = 1 \right\}. \tag{16}$$

The fact $S^{\mathbb{C}}_{\text{schr}} < \infty$ is due to Strichartz [1977] which in turn had precursors in [Tomas 1975]. For the problem of existence of such optimal $S^{\mathbb{C}}_{\text{schr}}$ and explicitly characterizing the maximizers, Kunze [2003] treated the $d = 1$ case and showed that maximizers exist by an elaborate concentration-compactness method. Foschi [2007] explicitly determined the best constants when $d = 1, 2$, and showed that the only maximizers are Gaussians up to the natural symmetries associated to the Strichartz inequality by using the sharp Cauchy–Schwarz inequality and the space-time Fourier transform. Hundertmark and Zharnitsky [2006] independently obtained this result by an interesting representation formula of the Strichartz inequalities in lower dimensions. Recently, Carneiro [2008] proved a sharp Strichartz-type inequality by following the arguments in [Hundertmark and Zharnitsky 2006] and found its maximizers, which derives the same results in [Hundertmark and Zharnitsky 2006] as a corollary when $d = 1, 2$. Very recently, Bennett et al. [2008] offered a new proof to determine the best constants by using the method of heat-flow. Shao [2009] showed that a maximizer exists for all nonendpoint Strichartz inequalities and in all dimensions by relying on the recent linear profile decomposition results for the Schrödinger equations. We will continue this approach for (15). Additionally, we will use a simple but beautiful idea of asymptotic embedding of a NLS solution to an approximate gKdV solution, which was previously exploited in [Christ et al. 2003; Tao 2007]. This gives that in the complex case, $S^{\mathbb{C}}_{\text{schr}} \le 3^{1/6} S^{\mathbb{C}}_{\text{airy}}$ while in the real case, $S^{\mathbb{C}}_{\text{schr}} \le 2^{1/2} 3^{1/6} S^{\mathbb{R}}_{\text{airy}}$.

Theorem 1.9. *We have the following dichotomy on the existence of maximizers for* (15) *with the complex- or real-valued initial data*, *respectively*:

- *In the complex case*, *either a maximizer is attained for* (15), *or there exists ϕ of complex value satisfying*

$$\|\phi\|_{L^2} = 1 \quad \textit{and} \quad S^{\mathbb{C}}_{\text{schr}} = \|e^{-it\partial_x^2}\phi\|_{L^6_{t,x}},$$

  and a sequence $(a_n)_{n\ge1}$ satisfying $\lim_{n\to\infty} |a_n| = \infty$ *such that*

$$\lim_{n\to\infty} \left\| D^{1/6} e^{-t\partial_x^3} [e^{i(\cdot)a_n}\phi] \right\|_{L^6_{t,x}} = S^{\mathbb{C}}_{\text{airy}}, \qquad S^{\mathbb{C}}_{\text{schr}} = 3^{1/6} S^{\mathbb{C}}_{\text{airy}}.$$

- *In the real case*, *a similar statement holds*; *more precisely*, *either a maximizer is attained for* (15), *or there exists ϕ of complex value satisfying*

$$S^{\mathbb{C}}_{\text{schr}} = \frac{\|e^{-it\partial_x^2}\phi\|_{L^6_{t,x}}}{\|\phi\|_{L^2}},$$

  and a positive sequence $(a_n)_{n\ge1}$ satisfying $\lim_{n\to\infty} a_n = \infty$ *and* $\lim_{n\to\infty} \|\operatorname{Re}(e^{i(\cdot)a_n}\phi)\|_{L^2} = 1$ *such that*

$$\lim_{n\to\infty} \left\| D^{1/6} e^{-t\partial_x^3} \operatorname{Re}(e^{i(\cdot)a_n}\phi) \right\|_{L^6_{t,x}} = S^{\mathbb{R}}_{\text{airy}}, \qquad S^{\mathbb{C}}_{\text{schr}} = 2^{1/2} 3^{1/6} S^{\mathbb{R}}_{\text{airy}}.$$

Remark 1.10. Note that when $S^{\mathbb{C}}_{\text{schr}} = 3^{1/6} S^{\mathbb{C}}_{\text{airy}}$ or $S^{\mathbb{C}}_{\text{schr}} = 2^{1/2} 3^{1/6} S^{\mathbb{R}}_{\text{airy}}$, the explicit ϕ had been uniquely determined by Foschi [2007] and Hundertmark and Zharnitsky [2006] independently: they are Gaussians up to the natural symmetries enjoyed by the Strichartz inequality for the Schrödinger equation.

This paper is organized as follows: in Section 2 we establish some notation. In Section 3, we make a preliminary decomposition for an L^2-bounded sequence $(u_n)_{n\geq 1}$ of complex value. In Section 4, we obtain similar results for a real sequence. In Section 5, we prove Theorems 1.5 and 1.6. In Section 6, we prove Theorem 1.9.

2. Notation

We use $X \lesssim Y$, $Y \gtrsim X$, or $X = O(Y)$ to denote the estimate $|X| \leq CY$ for some constant $0 < C < \infty$, which might depend on the dimension but not on the functions. If $X \lesssim Y$ and $Y \lesssim X$ we will write $X \sim Y$. If the constant C depends on a special parameter, we shall denote it explicitly by subscripts.

We define the space-time norm $L_t^q L_x^r$ of f on $\mathbb{R} \times \mathbb{R}$ by

$$\|f\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R})} := \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(t,x)|^r \, dx\right)^{q/r} dt\right)^{1/q},$$

with the usual modifications when q or r are equal to infinity, or when the domain $\mathbb{R} \times \mathbb{R}$ is replaced by a small space-time region. When $q = r$, we abbreviate it by $L_{t,x}^q$. Unless specified, all the space-time integrations are taken over $\mathbb{R} \times \mathbb{R}$, and all the spatial integrations over $\mathbb{R}$.

We fix the notation that $\lim_{n\to\infty}$ should be understood as $\limsup_{n\to\infty}$ throughout this paper.

The spatial Fourier transform is defined via

$$\widehat{u_0}(\xi) := \int_{\mathbb{R}} e^{-ix\xi} u_0(x) \, dx;$$

the space-time Fourier transform is defined analogously.

The Airy evolution operator $e^{-t\partial_x^3}$ is defined via

$$e^{-t\partial_x^3} u_0(x) := \int_{\mathbb{R}} e^{ix\xi + it\xi^3} \widehat{u_0}(\xi) \, d\xi.$$

The spatial derivative ∂_x^k, for k a positive integer, is defined via the Fourier transform

$$\widehat{\partial_x^k}(\xi) = (i\xi)^k.$$

The fractional differentiation operator D^α, $\alpha \in \mathbb{R}$, is defined via

$$D^\alpha f(x) := \int_{\mathbb{R}} e^{ix\xi} |\xi|^\alpha \widehat{f}(\xi) \, d\xi.$$

The inner product $\langle \cdot, \cdot \rangle_{L^2}$ in the Hilbert space L^2 is defined via

$$\langle f, g \rangle_{L^2} := \int_{\mathbb{R}} f(x) \overline{g}(x) \, dx,$$

where $\overline{g}$ denotes the usual complex conjugate of g in the complex plane $\mathbb{C}$.

3. Preliminary decomposition: complex version

To begin proving Theorems 1.5 and 1.6, we present a new proof of the refined Strichartz inequality (5) based on the Whitney decomposition. The following notation is taken from [Killip and Visan 2008b].

Definition 3.1. Given $j \in \mathbb{Z}$, we denote by $\mathscr{D}_j$ the set of all dyadic intervals in $\mathbb{R}$ of length 2^j:

$$\mathscr{D}_j := \{2^j[k, k+1) : k \in \mathbb{Z}\}.$$

We also write $\mathscr{D} := \cup_{j\in\mathbb{Z}}\mathscr{D}_j$. Given $I \in \mathscr{D}$, we define f_I by $\widehat{f_I} := \widehat{f} 1_I$ where 1_I denotes the indicator function of I.

Then the Whitney decomposition we need is as follows: Given two distinct $\xi, \xi' \in \mathbb{R}$, there is a unique maximal pair of dyadic intervals $I \in \mathscr{D}$ and $I' \in \mathscr{D}$ such that

$$|I| = |I'|, \operatorname{dist}(I, I') \geq 4|I|, \tag{17}$$

where $\operatorname{dist}(I, I')$ denotes the distance between I and I', and $|I|$ denotes the length of the dyadic interval I. Let $\mathscr{F}$ denote all such pairs as $\xi \neq \xi'$ varies over $\mathbb{R} \times \mathbb{R}$. Then we have

$$\sum_{(I,I')\in\mathscr{F}} 1_I(\xi)1_{I'}(\xi') = 1, \text{ for a.e. } (\xi, \xi') \in \mathbb{R} \times \mathbb{R}. \tag{18}$$

Since I and I' are maximal, $\operatorname{dist}(I, I') \leq 10|I|$. This shows that for a given $I \in \mathscr{D}$, there exists a bounded number of I' so that $(I, I') \in \mathscr{F}$, that is,

$$\#\{I' : (I, I') \in \mathscr{F}\} \lesssim 1 \quad \text{for all } I \in \mathscr{D}. \tag{19}$$

Proof of Lemma 1.2. Given $p > 1$, we normalize $\sup_{\tau\in\mathbb{R}} |\tau|^{1/2-1/p}\|\widehat{f}\|_{L^p(\tau)} = 1$. Then for all dyadic intervals $I \in \mathscr{D}$,

$$\int_I |\widehat{f}|^p d\xi \leq |I|^{1-p/2}. \tag{20}$$

We square the left side of (5) and reduce to proving

$$\left\| \iint e^{ix(\xi-\eta)+it(\xi^3-\eta^3)}|\xi\eta|^{1/6}\widehat{f}(\xi)\, \overline{\widehat{f}}(\eta)\, d\xi\, d\eta \right\|_{L^3_{t,x}} \lesssim \|\widehat{f}\|^{4/3}_{L^2}. \tag{21}$$

We change variables $a := \xi - \eta$ and $b := \xi^3 - \eta^3$ and use the Hausdorff–Young inequality in both t and x, we need to show

$$\iint \frac{|\xi\eta|^{1/4}|\widehat{f}(\xi)\widehat{f}(\eta)|^{3/2}}{|\xi+\eta|^{1/2}|\xi-\eta|^{1/2}}\, d\xi\, d\eta \lesssim \int |\widehat{f}|^2 d\xi. \tag{22}$$

By symmetries of this expression, it is sufficient to work in the region $\{(\xi, \eta) : \xi \geq 0, \eta \geq 0\}$. In this case, $|\xi\eta|^{1/4} \lesssim |\xi+\eta|^{1/2}$; so we reduce to proving

$$\iint \frac{|\widehat{f}(\xi)\widehat{f}(\eta)|^{3/2}}{|\xi-\eta|^{1/2}} d\xi d\eta \lesssim \int |\widehat{f}|^2 d\xi. \tag{23}$$

In view of (23), we assume $\widehat{f} \geq 0$ from now on. Then we apply the Whitney decomposition to obtain

$$\widehat{f}(\xi)\widehat{f}(\eta) = \sum_{(I,I')\in\mathscr{F}} \widehat{f_I}(\xi)\widehat{f_{I'}}(\eta), \text{ for a. e. } (\xi, \eta) \in \mathbb{R} \times \mathbb{R}, \tag{24}$$

and

$$\text{for all } (\xi, \eta) \in I \times I' \text{ with } (I, I') \in \mathscr{F}, |\xi - \eta| \sim |I|. \tag{25}$$

If we choose a slightly larger dyadic interval containing both I and I' but still of length comparable to I, still denoted by I, we reduce to proving

$$\sum_{I\in\mathcal{D}} \frac{\left(\int \widehat{f_I}^{3/2} d\xi\right)^2}{|I|^{1/2}} \lesssim \int \widehat{f}^{\,2} d\xi. \tag{26}$$

To prove (26) we will make a further decomposition to $f_I = \sum_{n\in\mathbb{Z}} f_{n,I}$: for any $n \in \mathbb{Z}$, define $f_{n,I}$ via

$$\widehat{f_{n,I}} := \widehat{f} 1_{\{\xi:\, 2^n|I|^{-1/2} \le \widehat{f}(\xi) \le 2^{n+1}|I|^{-1/2}\}}.$$

By the Cauchy–Schwarz inequality, for any $\varepsilon > 0$,

$$\left(\int \widehat{f_I}^{3/2} d\xi\right)^2 = \left(\sum_{n\in\mathbb{Z}} \int \widehat{f_{n,I}}^{3/2} d\xi\right)^2 \lesssim_\varepsilon \sum_{n\in\mathbb{Z}} 2^{|n|\varepsilon} \left(\int \widehat{f_{n,I}}^{3/2} d\xi\right)^2. \tag{27}$$

Now (26) is an easy consequence of the following claim:

$$\sum_{I\in\mathcal{D}} \frac{\left(\int \widehat{f_{n,I}}^{3/2} d\xi\right)^2}{|I|^{1/2}} \lesssim 2^{-|n|\varepsilon} \int \widehat{f}^{\,2} d\xi, \text{ for some } \varepsilon > 0. \tag{28}$$

By the Cauchy–Schwarz inequality,

$$\left(\int \widehat{f_{n,I}}^{3/2} d\xi\right)^2 \lesssim \int \widehat{f_{n,I}}^{2} d\xi \int \widehat{f_{n,I}} d\xi. \tag{29}$$

On the one hand, when $n \ge 0$, by the Chebyshev's inequality and (20),

$$\begin{aligned}
\int \widehat{f_{n,I}} d\xi &\lesssim 2^n |I|^{-1/2} |\{\xi \in I : \widehat{f}(\xi) \ge 2^n |I|^{-1/2}\}| \\
&\lesssim 2^n |I|^{-1/2} \frac{\int_I \widehat{f}^p d\xi}{2^{np} |I|^{-p/2}} \\
&\lesssim 2^{n(1-p)} |I|^{-1/2} |I|^{p/2} |I|^{1-p/2} \\
&= 2^{-|n|(p-1)} |I|^{1/2},
\end{aligned}$$

for any $p > 1$. On the other hand, when $n < 0$,

$$\int \widehat{f_{n,I}} d\xi \lesssim 2^n |I|^{-1/2} |I| = 2^{-|n|} |I|^{1/2}.$$

Combining these estimates, there exists an $\varepsilon > 0$ such that

$$\sum_{I\in\mathcal{D}} \frac{\left(\int \widehat{f_{n,I}}^{3/2} d\xi\right)^2}{|I|^{1/2}} \lesssim 2^{-|n|\varepsilon} \sum_{I\in\mathcal{D}} \int \widehat{f_{n,I}}^{2} d\xi. \tag{30}$$

Interchanging the summation order, we have

$$\sum_{I\in\mathcal{D}} \int \widehat{f_{n,I}}^2 \, d\xi = \sum_{j\in\mathbb{Z}} \sum_{I\in\mathcal{D}_j} \int \widehat{f}^{\,2} 1_{\{\xi\in I:\widehat{f}\sim 2^{n-j/2}\}} \, d\xi = \int_{\mathbb{R}} \sum_{j:\widehat{f}\sim 2^{n-j/2}} \widehat{f}^{\,2} \, d\xi \lesssim \int \widehat{f}^{\,2} \, d\xi. \tag{31}$$

Then the claim (28) follows from (30) and (31). Hence the proof of Lemma 1.2 is complete. □

By using this refined Airy Strichartz inequality (5), we extract the scaling and frequency parameters ρ_n^j and ξ_n^j following the approach in [Carles and Keraani 2007].

Lemma 3.2 (Complex version: extraction of ρ_n^j and ξ_n^j). *Let $(u_n)_{n\geq 1}$ be a sequence of complex valued functions with $\|u_n\|_{L^2} \leq 1$. Then up to a subsequence, for any $\delta > 0$, there exists $N := N(\delta)$, a family $(\rho_n^j, \xi_n^j)_{\substack{1\leq j\leq N \\ n\geq 1}} \in (0,\infty)\times\mathbb{R}$ and a family $(f_n^j)_{\substack{1\leq j\leq N \\ n\geq 1}}$ of L^2-bounded sequences such that, if $j \neq k$,*

$$\lim_{n\to\infty} \left(\frac{\rho_n^j}{\rho_n^k} + \frac{\rho_n^k}{\rho_n^j} + \frac{|\xi_n^j - \xi_n^k|}{\rho_n^j} \right) = \infty, \tag{32}$$

for every $1 \leq j \leq N$, there exists a compact K in $\mathbb{R}$ such that

$$\sqrt{\rho_n^j} |\widehat{f_n^j}(\rho_n^j \xi + \xi_n^j)| \leq C_\delta 1_K(\xi), \tag{33}$$

and

$$u_n = \sum_{j=1}^{N} f_n^j + q_n^N, \tag{34}$$

which satisfies

$$\| D^{\frac{1}{6}} e^{-t\partial_x^3} q_n^N \|_{L^6_{t,x}} \leq \delta, \tag{35}$$

and

$$\lim_{n\to\infty} \left(\|u_n\|_{L^2}^2 - \left(\sum_{j=1}^{N} \|f_n^j\|_{L^2}^2 + \|q_n^N\|_{L^2}^2 \right) \right) = 0. \tag{36}$$

Proof. For $\gamma_n = (\rho_n, \xi_n) \in (0,\infty)\times\mathbb{R}$, we define $G_n : L^2 \to L^2$ by setting

$$G_n(f)(\xi) := \sqrt{\rho_n} f(\rho_n \xi + \xi_n).$$

We will induct on the Strichartz norm. If $\|D^{\frac{1}{6}} e^{-t\partial_x^3} u_n\|_{L^6_{t,x}} \leq \delta$, then there is nothing to prove. Otherwise, up to a subsequence, we have

$$\|D^{\frac{1}{6}} e^{-t\partial_x^3} u_n\|_{L^6_{t,x}} > \delta.$$

On the one hand, applying Lemma 1.2 with $p = 4/3$, we see that there exists a family of intervals $I_n^1 := [\xi_n^1 - \rho_n^1, \xi_n^1 + \rho_n^1]$ such that

$$\int_{I_n^1} |\widehat{u_n}|^{4/3} d\xi \geq C_1 \delta^4 (\rho_n^1)^{\frac{1}{3}},$$

where C_1 only depends on C, the constant in Lemma 1.2; note that we used $\|u_n\|_{L^2} \le 1$ here. On the other hand, for any $A > 0$,

$$\int_{I_n^1 \cap \{|\widehat{u_n}| > A\}} |\widehat{u_n}|^{4/3} d\xi \le A^{-\frac{2}{3}} \|\widehat{u_n}\|_{L^2}^2 \le A^{-\frac{2}{3}}.$$

Let $C_\delta := (C_1/2)^{-3/2}\delta^{-6}$. Then from the two considerations above, we have

$$\int_{I_n^1 \cap \{|\widehat{u_n}| \le C_\delta (\rho_n^1)^{-1/2}\}} |\widehat{u_n}|^{4/3} d\xi \ge \frac{C_1}{2} \delta^4 (\rho_n^1)^{1/3}.$$

From the Hölder inequality, we have

$$\int_{I_n^1 \cap \{|\widehat{u_n}| \le C_\delta (\rho_n^1)^{-1/2}\}} |\widehat{u_n}|^{\frac{4}{3}} d\xi \le C_2 \Big(\int_{I_n^1 \cap \{|\widehat{u_n}| \le C_\delta (\rho_n^1)^{-1/2}\}} |\widehat{u_n}|^2 d\xi \Big)^{2/3} (|I_n^1|)^{1/3}.$$

This yields

$$\int_{I_n^1 \cap \{|\widehat{u_n}| \le C_\delta (\rho_n^1)^{-1/2}\}} |\widehat{u_n}|^2 d\xi \ge C' \delta^6,$$

where $C' > 0$ is some constant depending only on C_1 and C_2. Define v_n^1 and γ_n^1 by

$$\widehat{v_n^1} := \widehat{u_n} 1_{I_n^1 \cap \{|\widehat{u_n}| \le C_\delta (\rho_n^1)^{-1/2}\}}, \ \gamma_n^1 := (\rho_n^1, \xi_n^1).$$

Then $\|v_n^1\|_{L^2} \ge (C')^{1/2}\delta^3$. Also by the definition of G, we have

$$|G_n^1(\widehat{v_n^1})(\xi)| = |(\rho_n^1)^{1/2} \widehat{v_n^1}(\rho_n^1 \xi + \xi_n^1)| \le C_\delta 1_{[-1,1]}(\xi).$$

Moreover, since the supports are disjoint on the Fourier side, we have

$$\|u_n\|_{L^2}^2 = \|u_n - v_n^1\|_{L^2}^2 + \|v_n^1\|_{L^2}^2.$$

We repeat the same argument with $u_n - v_n^1$ in place of u_n. At each step, the L^2-norm decreases by at least $(C')^{1/2}\delta^3$. Hence after $N := N(\delta)$ steps, we obtain $(v_n^j)_{1 \le j \le N}$ and $(\gamma_n^j)_{1 \le j \le N}$, so

$$u_n = \sum_{j=1}^N v_n^j + q_n^N, \qquad \|u_n\|_{L^2}^2 = \sum_{j=1}^N \|v_n^j\|_{L^2}^2 + \|q_n^N\|_{L^2}^2,$$

where the latter equality is due to the disjoint Fourier supports. We have the error term estimate

$$\|D^{\frac{1}{6}} e^{-t\partial_x^3} q_n^N\|_{L^6_{t,x}} \le \delta,$$

which gives (35). The properties we obtain now are almost the case except for the first point of this lemma (32). To obtain it, we will reorganize the decomposition. We impose the following condition on $\gamma_n^j := (\rho_n^j, \xi_n^j)$: γ_n^j and γ_n^k are orthogonal if

$$\lim_{n\to\infty} \Big(\frac{\rho_n^j}{\rho_n^k} + \frac{\rho_n^k}{\rho_n^j} + \frac{|\xi_n^j - \xi_n^k|}{\rho_n^j} \Big) = \infty.$$

Then we define f_n^1 to be a sum of those v_n^j whose γ_n^j are not orthogonal to γ_n^1. Then taking the least $j_0 \in [2, N]$ such that $\gamma_n^{j_0}$ is orthogonal to γ_n^1, we can define f_n^2 to be a sum of those v_n^j whose γ_n^j are orthogonal to γ_n^1 but not to $\gamma_n^{j_0}$. Repeating this argument a finite number of times, we obtain (34). This decomposition automatically gives (32). Since the supports of the functions are disjoint on the Fourier side, we also have (36). Finally we want to make sure that, up to a subsequence, (33) holds.

By construction, those v_n^j kept in the definition of f_n^1 are such that the γ_n^j are not orthogonal to γ_n^1, that is, for those j, we have

$$\lim_{n\to\infty}\frac{\rho_n^j}{\rho_n^1}+\frac{\rho_n^1}{\rho_n^j}<\infty,\qquad \lim_{n\to\infty}\frac{|\xi_n^j-\xi_n^1|}{\rho_n^j}<\infty. \tag{37}$$

To show (33), it is sufficient to show that, up to a subsequence, $G_n^1(\widehat{v_n^j})$ is bounded by a compactly supported and bounded function, which will imply (33) with $j=1$. On the one hand, by construction,

$$|G_n^j(\widehat{v_n^j})|\le C_\delta 1_{[-1,1]}.$$

On the other hand, we observe that

$$G_n^1(\widehat{v_n^j})=G_n^1(G_n^j)^{-1}G_n^j(\widehat{v_n^j}),\qquad G_n^1(G_n^j)^{-1}f(\xi)=\sqrt{\frac{\rho_n^1}{\rho_n^j}}\,f\left(\frac{\rho_n^1}{\rho_n^j}\xi+\frac{\xi_n^1-\xi_n^j}{\rho_n^j}\right),$$

which yields the desired estimate for $G_n^1(\widehat{v_n^j})$ by (37). Inductively we obtain (32). Hence the proof of Lemma 3.2 is complete. □

The following lemma is useful in upgrading the weak convergence of error terms to the strong convergence in the Strichartz norm in Lemma 3.5.

Lemma 3.3. *We have the following two localized restriction estimates: for $9/2<q<6$ and $\widehat{G}\in L^\infty(B(0,R))$ for some $R>0$,*

$$\left\|D^{1/6}e^{-t\partial_x^3}G\right\|_{L^q_{t,x}}\le C_{q,R}\|\widehat{G}\|_{L^\infty}. \tag{38}$$

For the same G, $4\le q<6$ and $|\xi_0|\ge 10R$,

$$\left\|e^{-t\partial_x^3}(e^{i(\cdot)\xi_0}G)\right\|_{L^q_{t,x}}\le C_{q,R}|\xi_0|^{-1/q}\|\widehat{G}\|_{L^\infty}. \tag{39}$$

Proof. Let us start with the proof of (38). Let $q=2r$ with $9/4<r<3$. After squaring, we are reduced to proving

$$\left\|\int_{B(0,R)}\int_{B(0,R)}e^{ix(\xi_1-\xi_2)+it(\xi_1^3-\xi_2^3)}|\xi_1\xi_2|^{1/6}\widehat{G}(\xi_1)\,\overline{\widehat{G}}(\xi_2)\,d\xi_1\,d\xi_2\right\|_{L^r_{t,x}}\le C_{q,R}\|\widehat{G}\|^2_{L^\infty(B(0,R))}.$$

Let $s_1:=\xi_1-\xi_2$ and $s_2:=\xi_1^3-\xi_2^3$ and denote the resulting image of $B(0,R)\times B(0,R)$ by Ω under this change of variables. Then by using the Hausdorff–Young inequality since $r>2$, we see that the left side of the inequality above is bounded by

$$C\left(\int_\Omega\left||\xi_1\xi_2|^{1/6}\frac{\widehat{G}(\xi_1)\widehat{G}(\xi_2)}{|\xi_1^2-\xi_2^2|}\right|^{r'}ds_1ds_2\right)^{\frac{1}{r'}}.$$

Then if we change variables back, we obtain

$$C\bigg(\int_{B(0,R)\times B(0,R)}\frac{|\xi_1\xi_2|^{r'/6}}{|\xi_1+\xi_2|^{r'-1}|\xi_1-\xi_2|^{r'-1}}|\widehat{G}(\xi_1)\widehat{G}(\xi_2)|^{r'}\,d\xi_1 d\xi_2\bigg)^{\frac{1}{r'}}.$$

As in the proof of Lemma 1.2, we may assume that $\xi_1,\xi_2\geq 0$. So we have $|\xi_1\xi_2|^{\frac{1}{2}}\lesssim|\xi_1+\xi_2|$, which leads to $(\xi_1\xi_2)^{r'/6}\lesssim(\xi_1+\xi_2)^{r'/3}$ and thus

$$\frac{|\xi_1\xi_2|^{r'/6}}{|\xi_1+\xi_2|^{r'-1}|\xi_1-\xi_2|^{r'-1}}\lesssim\frac{1}{|\xi_1-\xi_2|^{\frac{5}{3}r'-2}}+\frac{1}{|\xi_1+\xi_2|^{\frac{5}{3}r'-2}}.$$

Then since $|\xi|^{-\frac{5}{3}r'+2}$ is locally integrable when $3/2<r'<9/5$ and $\widehat{G}\in L^\infty$, we obtain (38).

The proof of (39) is similar. Setting $q=2r$ with $2\leq r<3$ and following the same procedure as above, we have

$$\begin{aligned}
\|e^{-t\partial_x^3}(e^{i(\cdot)\xi_0}G)\|^2_{L^q_{t,x}} &= \big\|e^{-t\partial_x^3}(e^{i(\cdot)\xi_0}G)\overline{e^{-t\partial_x^3}(e^{i(\cdot)\xi_0}G)}\big\|_{L^r_{t,x}}\\
&= \bigg\|\int e^{ix(\xi-\eta)+it[(\xi+\xi_0)^3-(\eta+\xi_0)^3]}\widehat{G}(\xi)\,\overline{\widehat{G}}(\eta)\,d\xi d\eta\bigg\|_{L^r_{t,x}}\\
&\lesssim\bigg(\int\frac{|\widehat{G}(\xi)|^{r'}|\widehat{G}(\eta)|^{r'}}{|\xi-\eta|^{r'-1}|\xi+\eta+2\xi_0|^{r'-1}}d\xi d\eta\bigg)^{1/r'}\\
&\lesssim\bigg(\int\frac{|\widehat{G}(\xi)|^{r'}|\widehat{G}(\eta)|^{r'}}{|\xi-\eta|^{r'-1}|\xi_0|^{r'-1}}d\xi d\eta\bigg)^{1/r'}\\
&\leq C_{q,R}|\xi_0|^{-1+1/r'}\|\widehat{G}\|^2_{L^\infty}\leq C_{q,R}|\xi_0|^{-2/q}\|\widehat{G}\|^2_{L^\infty},
\end{aligned}$$

where we have used $|\xi+\eta+2\xi_0|\sim|\xi_0|$ since $\xi,\eta\in B(0,R)$ and $|\xi_0|\geq 10R$. □

In Lemma 3.2, we have determined the scaling and frequency parameters. Recall that from Section 1, we are left with extracting the space and time translation parameters. For this purpose, we will apply the concentration–compactness argument. For simplicity, we present the following lemma of this kind adapted to Airy evolution but not involving the frequency and scaling parameters. The general case is similar and will be done in the next lemma.

Lemma 3.4 (Concentration–compactness). *Suppose $P:=(P_n)_{n\geq 1}$ with $\|P_n\|_{L^2}\leq 1$. Then up to a subsequence, there exists a sequence $(\phi^\alpha)_{\alpha\geq 1}\in L^2$ and a family $(y_n^\alpha,s_n^\alpha)\in\mathbb{R}^2$ such that they satisfy the following constraints: for $\alpha\neq\beta$,*

$$\lim_{n\to\infty}\big(|y_n^\alpha-y_n^\beta|+|s_n^\alpha-s_n^\beta|\big)=\infty, \tag{40}$$

and for $A\geq 1$, there exists $P_n^A\in L^2$ so that

$$P_n(x)=\sum_{\alpha=1}^{A}e^{s_n^\alpha\partial_x^3}\phi^\alpha(x-y_n^\alpha)+P_n^A(x), \tag{41}$$

and

$$\lim_{A\to\infty} \mu(P^A)=0,$$

where $\mu(P^A)$ *is defined in the argument below*; *moreover we have the following almost orthogonality identity*: *for any* $A\geq 1$,

$$\lim_{n\to\infty}\Bigl(\|P_n\|_{L^2}^2-\Bigl(\sum_{\alpha=1}^{A}\|\phi^\alpha\|_{L^2}^2+\|P_n^A\|_{L^2}^2\Bigr)\Bigr)=0. \tag{42}$$

Proof. Let $\mathcal{W}(P)$ be the set of weak limits of subsequences of P in L^2 after the space and time translations:

$$\mathcal{W}(P):=\{w\text{-}\lim_{n\to\infty} e^{-s_n\partial_x^3}P_n(x+y_n) \text{ in } L^2:(y_n,s_n)\in\mathbb{R}^2)\}.$$

We set $\mu(P):=\sup\{\|\phi\|_{L^2}:\phi\in\mathcal{W}(P)\}$. Clearly we have

$$\mu(P)\leq \lim_{n\to\infty}\|P_n\|_{L^2}.$$

If $\mu(P)=0$, then there is nothing to prove. Otherwise $\mu(P)>0$, then up to a subsequence, there exists a $\phi^1\in L^2$ and a sequence $(y_n^1,s_n^1)_{n\geq 1}\in\mathbb{R}^2$ such that

$$\phi^1(x)=w\text{-}\lim_{n\to\infty} e^{-s_n^1\partial_x^3}P_n(x+y_n^1) \text{ in } L^2, \tag{43}$$

and $\|\phi^1\|_{L^2}\geq \frac{1}{2}\mu(P)$. We set $P_n^1:=P_n-e^{s_n^1\partial_x^3}\phi^1(x-y_n^1)$. Then since $e^{-t\partial_x^3}$ is an unitary operator on L^2, we have

$$\begin{aligned}
\|P_n^1\|_{L^2}^2&=\langle P_n^1,P_n^1\rangle_{L^2}\\
&=\langle P_n-e^{s_n^1\partial_x^3}\phi^1(x-y_n^1),P_n-e^{s_n^1\partial_x^3}\phi^1(x-y_n^1)\rangle_{L^2}\\
&=\langle e^{-s_n^1\partial_x^3}\bigl(P_n-e^{s_n^1\partial_x^3}\phi^1(x-y_n^1)\bigr),e^{-s_n^1\partial_x^3}\bigl(P_n-e^{s_n^1\partial_x^3}\phi^1(x-y_n^1)\bigr)\rangle_{L^2}\\
&=\langle e^{-s_n^1\partial_x^3}P_n-\phi^1(x-y_n^1),e^{-s_n^1\partial_x^3}P_n-\phi^1(x-y_n^1)\rangle_{L^2}\\
&=\langle e^{-s_n^1\partial_x^3}P_n(x+y_n^1)-\phi^1(x),e^{-s_n^1\partial_x^3}P_n(x+y_n^1)-\phi^1(x)\rangle_{L^2}\\
&=\langle P_n,P_n\rangle_{L^2}+\langle\phi^1,\phi^1\rangle_{L^2}-\langle e^{-s_n^1\partial_x^3}P_n(x+y_n^1),\phi^1\rangle_{L^2}-\langle\phi^1,e^{-s_n^1\partial_x^3}P_n(x+y_n^1)\rangle_{L^2}.
\end{aligned}$$

Taking $n\to\infty$ and using (43), we see that

$$\lim_{n\to\infty}\bigl(\|P_n\|_{L^2}^2-(\|\phi^1\|_{L^2}^2+\|P_n^1\|_{L^2}^2)\bigr)=0, \qquad e^{-s_n^1\partial_x^3}P_n^1(x+y_n^1)\to 0, \text{ weakly in } L^2.$$

We replace P_n with P_n^1 and repeat the same process: if $\mu(P^1)>0$, we obtain ϕ^2 and $(y_n^2,s_n^2)_{n\geq 1}$ so that $\|\phi^2\|_{L^2}\geq\frac{1}{2}\mu(P^1)$ and

$$\phi^2(x)=w\text{-}\lim_{n\to\infty} e^{-s_n^2\partial_x^3}P_n^1(x+y_n^2) \text{ in } L^2.$$

Moreover, $(y_n^1,s_n^1)_{n\geq 1}$ and $(y_n^2,s_n^2)_{n\geq 1}$ satisfy (40). Otherwise, up to a subsequence, we may assume that

$$\lim_{n\to\infty} s_n^2-s_n^1=s_0,\ \lim_{n\to\infty} y_n^2-y_n^1=y_0,$$

where $(s_0, y_0) \in \mathbb{R}^2$. Then for any $\phi \in \mathcal{S}$,

$$\lim_{n\to\infty} \left\| e^{-(s_n^2 - s_n^1)\partial_x^3} \phi(x + (y_n^2 - y_n^1)) - e^{-s_0 \partial_x^3} \phi(x + y_0) \right\|_{L^2} = 0.$$

That is to say,

$$\left(e^{-(s_n^2 - s_n^1)\partial_x^3} \phi(x + (y_n^2 - y_n^1))\right)_{n\geq 1}$$

converges strongly in L^2. On the other hand, we rewrite,

$$e^{-s_n^2 \partial_x^3} P_n^1(x + y_n^2) = e^{-(s_n^2 - s_n^1)\partial_x^3} (e^{-s_n^1 \partial_x^3} P_n^1(x + y_n^1))(x + (y_n^2 - y_n^1)).$$

Now the strong convergence and weak convergence together yield $\phi^2 = 0$, hence $\mu(P^1) = 0$, a contradiction. Hence (40) holds.

Iterating this argument, a diagonal process produces a family of pairwise orthogonal sequences

$$(y_n^\alpha, s_n^\alpha)_{\alpha\geq 1} \quad \text{and} \quad (\phi^\alpha)_{\alpha\geq 1}$$

satisfying (41) and (42). From (42), $\sum_\alpha \|\phi^\alpha\|_{L^2}^2$ is convergent and hence $\lim_{\alpha\to\infty} \|\phi^\alpha\|_{L^2} = 0$. This gives

$$\lim_{A\to\infty} \mu(P^A) = 0,$$

since $\mu(P^A) \leq 2\|\phi^A\|_{L^2}$ by construction. □

We are ready to extract the space and time parameters of the profiles.

Lemma 3.5 (Complex version: extraction of $x_n^{j,\alpha}$ and $s_n^{j,\alpha}$). *Suppose an L^2-bounded sequence $(f_n)_{n\geq 1}$ satisfies*

$$\sqrt{\rho_n} |\widehat{f_n}(\rho_n(\xi + (\rho_n)^{-1}\xi_n))| \leq F(\xi),$$

with $F \in L^\infty(K)$ for some compact set K in $\mathbb{R}$ independent of n. Then up to a subsequence, there exists a family $(y_n^\alpha, s_n^\alpha) \in \mathbb{R} \times \mathbb{R}$ and a sequence $(\phi^\alpha)_{\alpha\geq 1}$ of L^2 functions such that, if $\alpha \neq \beta$,

$$\lim_{n\to\infty} \left(\left| y_n^\beta - y_n^\alpha + \frac{3(s_n^\beta - s_n^\alpha)(\xi_n)^2}{(\rho_n)^2} \right| + \left| \frac{3(s_n^\beta - s_n^\alpha)\xi_n}{\rho_n} \right| + \left| s_n^\beta - s_n^\alpha \right| \right) = \infty, \tag{44}$$

and for every $A \geq 1$, there exists $e_n^A \in L^2$,

$$f_n(x) = \sum_{\alpha=1}^{A} \sqrt{\rho_n} e^{s_n^\alpha \partial_x^3} \left[e^{i(\cdot)\rho_n^{-1}\xi_n} \phi^\alpha(\cdot) \right] (\rho_n x - y_n^\alpha) + e_n^A(x), \tag{45}$$

and

$$\lim_{A\to\infty} \lim_{n\to\infty} \|D^{\frac{1}{6}} e^{-t\partial_x^3} e_n^A\|_{L^6_{t,x}} = 0, \tag{46}$$

and for any $A \geq 1$,

$$\lim_{n\to\infty} \left(\|f_n\|_{L^2}^2 - \left(\sum_{\alpha=1}^{A} \|\phi^\alpha\|_{L^2}^2 + \|e_n^A\|_{L^2}^2 \right) \right) = 0. \tag{47}$$

Proof. Setting $P := (P_n)_{n\geq 1}$ with $\widehat{P_n}(\xi) := \sqrt{\rho_n}\widehat{f_n}(\rho_n(\xi + (\rho_n)^{-1}\xi_n))$. Then

$$\widehat{P_n} \in L^\infty(K).$$

Let $\mathcal{W}(P)$ be the set of weak limits in L^2 defined by

$$\mathcal{W}(P) := \big\{ w\text{-}\lim_{n\to\infty} e^{-ix\rho_n^{-1}\xi_n} e^{-s_n\partial_x^3}\big[e^{i(\cdot)\rho_n^{-1}\xi_n} P_n(\cdot)\big](x+y_n) \text{ in } L^2 : (y_n, s_n) \in \mathbb{R}^2\big\},$$

and $\mu(P)$ as in the previous lemma. Then a similar concentration-compactness argument shows that, up to a subsequence, there exists a family $(y_n^\alpha, s_n^\alpha)_{\substack{\alpha\geq 1\\ n\geq 1}}$ and $(\phi^\alpha)_{\alpha\geq 1} \in L^2$ such that (44) holds, and

$$P_n(x) = \sum_{\alpha=1}^{A} e^{-ix\rho_n^{-1}\xi_n} e^{s_n^\alpha\partial_x^3}\big[e^{i(\cdot)\rho_n^{-1}\xi_n}\phi^\alpha(\cdot)\big](x - y_n^\alpha) + P_n^A(x).$$

As weak limits, each $\widehat{\phi^\alpha}$ has the same support as $\widehat{P_n}$, so does $\widehat{P_n^A}$. Furthermore, we may assume that $\widehat{\phi^\alpha}, \widehat{P_n^A} \in L^\infty(K)$. Setting $P^A := (P_n^A)_{n\geq 1}$. Then the sequence $(P^A)_{A\geq 1}$ satisfies

$$\lim_{A\to\infty} \mu(P^A) = 0. \tag{48}$$

For any $A \geq 1$, we also have

$$\lim_{n\to\infty}\bigg(\|P_n\|_{L^2}^2 - \Big(\sum_{\alpha=1}^{A} \|\phi^\alpha\|_{L^2}^2 + \|P_n^A\|_{L^2}^2\Big)\bigg) = 0.$$

Since $f_n(x) = \sqrt{\rho_n} e^{ix\xi_n} P_n(\rho_n x)$, the decomposition (45) of f_n follows after setting

$$e_n^A(x) := \sqrt{\rho_n} e^{ix\xi_n} P_n^A(\rho_n x).$$

What remains to show is that

$$\lim_{A\to\infty}\lim_{n\to\infty} \big\| D^{\frac{1}{6}} e^{-t\partial_x^3}[\sqrt{\rho_n} e^{iy\xi_n} P_n^A(\rho_n y)]\big\|_{L^6_{t,x}} = 0,$$

which will follow from (48) and the restriction estimates in Lemma 3.3 by an interpolation argument. Indeed, by scaling, it is equivalent to showing that

$$\lim_{A\to\infty}\lim_{n\to\infty} \big\| D^{1/6} e^{-t\partial_x^3}[e^{i(\cdot)a_n} P_n^A]\big\|_{L^6_{t,x}} = 0, \tag{49}$$

where $a_n := (\rho_n)^{-1}\xi_n$. Up to a subsequence, we split into two cases according to whether $\lim_{n\to\infty}|a_n| = \infty$ or not.

Case 1. $\lim_{n\to\infty}|a_n| = \infty$. By using the Hörmander–Mikhlin multiplier theorem [Tao 2006a, Theorem 4.4], for sufficiently large n, we have

$$\big\| D^{1/6} e^{-t\partial_x^3}[e^{i(\cdot)a_n} P_n^A]\big\|_{L^6_{t,x}} \lesssim |a_n|^{1/6} \big\| e^{-t\partial_x^3}[e^{i(\cdot)a_n} P_n^A]\big\|_{L^6_{t,x}}.$$

We will show that, after taking limits in n, the right hand side is bounded by $C_q\mu(P^A)^{1-q/6}$ for some $4 \leq q < 6$. Then $\lim_{A\to\infty}\mu(P^A) = 0$ yields the result. We choose a cut-off $\chi_n(t,x) := \chi_{n,1}(t)\chi_{n,2}(x)$ satisfying

$$\chi_{n,2}(x) := \chi_2(x)e^{ixa_n}, \quad \chi_2 \in \mathcal{S},$$

where $\widehat{\chi_2}$ is compactly supported and $\widehat{\chi_2}(\xi) := 1$ on the common support K of $\widehat{P_n}$, and

$$\widehat{\chi}_{n,1}((\xi+a_n)^3) := \widehat{\chi}_1(\xi^3), \quad \chi_1 \in \mathcal{S},$$

where $\widehat{\chi}_1(\xi^3) := 1$ on $\operatorname{Supp}\widehat{\chi_2}$. Let $*$ denote space-time convolution; then

$$\chi_n * [e^{-t\partial_x^3}(e^{i(\cdot)a_n}P_n^A)] = e^{-t\partial_x^3}(e^{i(\cdot)a_n}P_n^A). \tag{50}$$

Indeed, the space-time Fourier transform of χ_n is equal to

$$\widehat{\chi_n}(\tau,\xi) := \int e^{-it\tau-ix\xi}\chi_n(t,x)\,dt\,dx = \widehat{\chi_2}(\xi-a_n)\widehat{\chi}_{n,1}(\tau).$$

On the support of the space-time Fourier transform of $e^{-t\partial_x^3}(e^{i(\cdot)a_n}P_n^A)$, we see that

$$\widehat{\chi_n}(\tau,\xi) \equiv 1.$$

This gives (50). Then by the Hölder inequality and the restriction estimate (39) in Lemma 3.3, for sufficiently large n,

$$\begin{aligned}
\|e^{-t\partial_x^3}(e^{i(\cdot)a_n}P_n^A)\|_{L^6_{t,x}} &= \|\chi_n * [e^{-t\partial_x^3}(e^{i(\cdot)a_n}P_n^A)]\|_{L^6_{t,x}} \\
&\lesssim \|\chi_n * \big[e^{-t\partial_x^3}(e^{i(\cdot)a_n}P_n^A)\big]\|_{L^q_{t,x}}^{q/6}\|\chi_n * \big[e^{-t\partial_x^3}(e^{i(\cdot)a_n}P_n^A)\big]\|_{L^\infty_{t,x}}^{1-q/6} \\
&\lesssim |a_n|^{-1/6}\|F\|_{L^\infty}^{q/6}\|\chi_n * \big[e^{-t\partial_x^3}(e^{i(\cdot)a_n}P_n^A)\big]\|_{L^\infty_{t,x}}^{1-q/6},
\end{aligned}$$

for some $4 < q < 6$. There exists $(t_n, y_n)_{n\ge 1}$ such that

$$\|\chi_n * [e^{-t\partial_x^3}(e^{i(\cdot)a_n}P_n^A)]\|_{L^\infty_{t,x}} \sim \Big|\chi_n * [e^{-t\partial_x^3}(e^{i(\cdot)a_n}P_n^A)](t_n,y_n)\Big|.$$

We expand the right side out,

$$\left|\iint \chi_{n,1}(-t)\chi_{n,2}(-x)e^{-t\partial_x^3}[e^{-t_n\partial_x^3}(e^{i(\cdot)a_n}P_n^A)(\cdot+y_n)](x)\,dx\,dt\right|.$$

Setting $p_n(x) = e^{-t_n\partial_x^3}(e^{i(\cdot)a_n}P_n^A)(x+y_n)$, then it equals

$$\left|\iint \widehat{\chi}_1(\eta^3)\widehat{\chi_2}(\eta)e^{-ix\eta}d\eta\, e^{-ixa_n}p_n(x)\,dx\right| = \left|\int \chi_2(-x)\,e^{-ixa_n}p_n(x)\,dx\right|.$$

Taking $n\to\infty$, and using the definition of $\mathcal{W}(P^A)$ followed by the Cauchy–Schwarz inequality, we obtain

$$\lim_{n\to\infty}\|\chi_n * [e^{-t\partial_x^3}(e^{i(\cdot)a_n}P_n^A)]\|_{L^\infty_{t,x}} \lesssim \|\chi_2\|_{L^2}\mu(P^A) \lesssim \chi_2\mu(P^A).$$

Hence the claim (49) follows.

Case 2. $\lim_{n\to\infty}|a_n| < \infty$. From the Hölder inequality, we have the $L^6_{t,x}$ norm in (49) is bounded by

$$\|D^{1/6}e^{-t\partial_x^3}[e^{i(\cdot)a_n}P_n^A]\|_{L^q_{t,x}}^{q/6}\,\|D^{1/6}e^{-t\partial_x^3}[e^{i(\cdot)a_n}P_n^A]\|_{L^\infty_{t,x}}^{1-q/6}$$

for some $4 < q < 6$. On the one hand, since $\lim_{n\to\infty} |a_n|$ is finite and $\widehat{P_n^A} \in L^\infty(K)$, there exists a large $R > 0$ so that

$$\operatorname{Supp} \mathscr{F}[e^{i(\cdot)a_n} P_n^A] \subset B(0, R),$$

where $\mathscr{F}(f)$ denotes the spatial Fourier transform of f. Then from (38) in Lemma 3.3, we see that

$$\big\| D^{1/6} e^{-t\partial_x^3} [e^{i(\cdot)a_n} P_n^A] \big\|_{L^q_{t,x}} \le C_{q,R} \|F\|_{L^\infty},$$

which is independent of n. On the other hand, from the Bernstein inequality, we have

$$\big\| D^{1/6} e^{-t\partial_x^3} [e^{i(\cdot)a_n} P_n^A] \big\|_{L^\infty_{t,x}} \le C_{q,R} \big\| e^{-t\partial_x^3} [e^{i(\cdot)a_n} P_n^A] \big\|_{L^\infty_{t,x}}.$$

Then a similar argument as in Case 1 shows that $\|e^{-t\partial_x^3}[e^{i(\cdot)a_n} P_n^A]\|_{L^\infty_{t,x}}$ is bounded by $\mu(P^A)^c$ for some $c > 0$. Hence (49) follows and the proof of Lemma 3.5 is complete. □

Remark 3.6. In view of the previous lemma, we will make a very useful reduction when $\lim_{n\to\infty} \rho_n^{-1}\xi_n = a$ is finite: we will take $\xi_n \equiv 0$. Indeed, we first replace $e^{i(\cdot)\rho_n^{-1}\xi_n}\phi^\alpha$ with $e^{i(\cdot)a}\phi^\alpha$ by putting the difference into the error term; then we can reduce it further by regarding $e^{i(\cdot)a}\phi^\alpha$ as a new ϕ^α.

Next we will show that the profiles obtained in (45) are strongly decoupled under the orthogonality condition (44); more general version is in Lemma 5.2. Abusing notation, we define

$$\widetilde{g_n^\alpha}(\phi^\alpha)(x) := \sqrt{\rho_n} e^{s_n^\alpha \partial_x^3} [e^{i(\cdot)\rho_n^{-1}\xi_n} \phi^\alpha(\cdot)](\rho_n x - y_n^\alpha),$$

where $\xi_n \equiv 0$ when $\lim_{n\to\infty} \rho_n^{-1}\xi_n$ is finite.

Corollary 3.7. *Under* (44), *for any* $\alpha \ne \beta$, *we have*

$$\lim_{n\to\infty} \big| \langle \widetilde{g_n^\alpha}(\phi^\alpha), \widetilde{g_n^\beta}(\phi^\beta) \rangle_{L^2} \big| = 0 \tag{51}$$

and for any $1 \le \alpha \le A$,

$$\lim_{n\to\infty} \big| \langle \widetilde{g_n^\alpha}(\phi^\alpha), e_n^A \rangle_{L^2} \big| = 0. \tag{52}$$

Proof. Without loss of generality, we assume that ϕ^α and ϕ^β are Schwartz functions with compact Fourier support. We first prove (51). By changing variables, we have

$$\begin{aligned}
\big| \langle \widetilde{g_n^\alpha}(\phi^\alpha), \widetilde{g_n^\beta}(\phi^\beta) \rangle_{L^2} \big| &= \Big| \Big\langle \sqrt{\rho_n} e^{s_n^\alpha \partial_x^3} \big[e^{i(\cdot)\rho_n^{-1}\xi_n} \phi^\alpha(\cdot) \big](\rho_n x - y_n^\alpha), \sqrt{\rho_n} e^{s_n^\beta \partial_x^3} \big[e^{i(\cdot)\rho_n^{-1}\xi_n} \phi^\beta(\cdot) \big](\rho_n x - y_n^\beta) \Big\rangle_{L^2} \Big| \\
&= \Big| \Big\langle e^{-(s_n^\beta - s_n^\alpha)\partial_x^3} \big[e^{i(\cdot)\rho_n^{-1}\xi_n} \phi^\alpha(\cdot) \big](x + y_n^\beta - y_n^\alpha), e^{ix\rho_n^{-1}\xi_n} \phi^\beta(x) \Big\rangle_{L^2} \Big| \\
&\le \Big\langle \Big| \int e^{i\xi\left(x + y_n^\beta - y_n^\alpha + 3\frac{(s_n^\beta - s_n^\alpha)\xi_n^2}{\rho_n^2}\right) + i\xi^3 (s_n^\beta - s_n^\alpha) + 3i\xi^2 \frac{(s_n^\beta - s_n^\alpha)\xi_n}{\rho_n}} \widehat{\phi^\alpha}(\xi)\, d\xi \Big|, \big|\phi^\beta\big| \Big\rangle_{L^2}.
\end{aligned}$$

Hence if (44) holds, by using [Stein 1993, Corollary, p. 334] or integration by parts combined with the dominated convergence theorem, we conclude that this expression goes to zero as n goes to infinity.

To prove (52), we write

$$e_n^A = \sum_{\beta=A+1}^{B} \widetilde{g_n^\beta}(\phi^\beta) + e_n^B,$$

for any $B > A$. Recall
$$e_n^B = \sqrt{\rho_n}\big(e^{i(\cdot)\rho_n^{-1}\xi_n} P_n^B\big)(\rho_n x).$$
Then
$$\big|\langle \widetilde{g_n^\alpha}(\phi^\alpha), e_n^A\rangle_{L^2}\big| \le \sum_{\beta=A+1}^{B} \Big|\langle \widetilde{g_n^\alpha}(\phi^\alpha), \widetilde{g_n^\beta}(\phi^\beta)\rangle_{L^2}\Big| + \Big|\big\langle \phi^\alpha, e^{-ix\rho_n^{-1}\xi_n} e^{-s_n^\alpha\partial_x^3}(e^{i(\cdot)\rho_n^{-1}\xi_n} P_n^B)(x+y_n^\alpha)\big\rangle_{L^2}\Big|.$$
When n goes to infinity, the first term goes to zero because of (51). The second term is less than $\|\phi^\alpha\|_{L^2}\mu(P^B)$ by the definitions of $\mathcal{W}(P^B)$ and $\mu(P^B)$, and the Cauchy–Schwarz inequality; so it can be made arbitrarily small if taking B large enough. Hence (52) is obtained by taking $B \to \infty$. □

4. Preliminary decomposition: real version

To prove Theorem 1.6, we need the corresponding real version of lemmas in the previous section, especially of Lemmas 3.2 and 3.5. To develop the real analogue of Lemma 3.2, we recall the following lemma due to Kenig et al. [2000].

Lemma 4.1. *Let $u_0 \in L^2$ be a real-valued function with $\|u_0\|_{L^2} = 1$. Then for any $\delta > 0$, there exists a positive integer $N = N(\delta)$, real-valued functions $f^1, \dots, f^N$ and e^N, intervals $\tau_1, \dots, \tau_N$, and a positive constant C_δ such that*
$$\overline{\widehat{f}}^j(\xi) = \widehat{f^j}(-\xi), \quad \operatorname{Supp}\widehat{f^j} \subset \tau_j \cup (-\tau_j), |\tau_j| = \rho_j, \quad |\widehat{f^j}| \le C_\delta \rho_j^{-1/2},$$
and
$$u_0 = \sum_{j=1}^{N} f^j + e^N,$$
with
$$\|u_0\|_{L^2}^2 = \sum_{j=1}^{N} \|f^j\|_{L^2}^2 + \|e^N\|_{L^2}^2, \qquad \big\|D^{1/6}e^{-t\partial_x^3}e^N\big\|_{L^6_{t,x}} < \delta.$$
The proof of this lemma is similar to that of the previous Lemma 3.2 with the help that, for real function f, $\overline{\widehat{f}} = \widehat{f}(-\xi)$. For our purpose, we will do a little more on the decomposition above. Indeed, from the proof in [Kenig et al. 2000] we know that $\widehat{f^j}(\xi) = 1_{\{\xi\in\tau_j\cup(-\tau_j):\ |\widehat{u_0}|\le C_\delta\rho_j^{-1/2}\}}\widehat{u_0}(\xi)$ and $\tau_j \subset (0,\infty)$. We can decompose f^j further by setting
$$\begin{aligned} f^j &:= f^{j,+} + f^{j,-},\\ \widehat{f^{j,+}} &:= 1_{\{\xi\in\tau_j:\ |\widehat{u_0}|\le C_\delta\rho_j^{-1/2}\}}\widehat{u_0},\\ \widehat{f^{j,-}} &:= 1_{\{\xi\in-\tau_j:\ |\widehat{u_0}|\le C_\delta\rho_j^{-1/2}\}}\widehat{u_0}. \end{aligned}$$
Since u_0 is real, we have $\overline{\widehat{u_0}}(\xi) = \widehat{u_0}(-\xi)$, which yields
$$\overline{\widehat{f^{j,+}}}(\xi) = \widehat{f^{j,-}}(-\xi), \quad \text{and} \quad f^{j,-} = \overline{f^{j,+}}.$$
Hence
$$f^j = 2\operatorname{Re} f^{j,+}.$$

Now we return to prove Theorem 1.6. We repeat the process above for each real-valued u_n to obtain $v_n^1, \dots, v_n^N$ and real-valued e_n^N such that

$$u_n = \sum_{j=1}^{N} 2\,\mathrm{Re}(v_n^j) + e_n^N, \tag{53}$$

with

$$\sqrt{\rho_n^j}\,|\widehat{v_n^j}(\rho_n^j\xi + \xi_n^j)| \le C_\delta 1_K(\xi), \qquad \text{with } \xi_n^j > 0, \text{ for some compact } K, \tag{54}$$

and

$$\|u_n\|_{L^2}^2 = \sum_{j=1}^{N} 4\big\|\mathrm{Re}(v_n^j)\big\|_{L^2}^2 + \big\|e_n^N\big\|_{L^2}^2. \tag{55}$$

Still we define the real version of the orthogonality condition on the sequence $(\rho_n^j, \xi_n^j)_{n\ge 1} \in (0,+\infty)^2$ as before: for $j \ne k$,

$$\lim_{n\to\infty}\Big(\frac{\rho_n^j}{\rho_n^k} + \frac{\rho_n^k}{\rho_n^j} + \frac{|\xi_n^j - \xi_n^k|}{\rho_n^j}\Big) = \infty. \tag{56}$$

Based on (53) and (54), the basic idea of obtaining the real version is to apply the procedure in the previous section to v_n^j, and then take the real part. The only issue here is to show that the error term is still small in the Strichartz norm, and the almost orthogonality in L^2 norm still holds. We omit the details.

Lemma 4.2 (Real version: extraction of ρ_n^j and ξ_n^j). *Let $(u_n)_{n\ge1}$ be a sequence of real-valued functions with $\|u_n\|_{L^2} \le 1$. Then up to a subsequence, for any $\delta > 0$, there exists $N = N(\delta)$, an orthogonal family $(\rho_n^j, \xi_n^j)_{\substack{1\le j\le N\\ n\ge1}} \in (0,\infty)^2$ satisfying (56) and a sequence $(f_n^j)_{\substack{1\le j\le N\\ n\ge1}} \in L^2$ such that, for every $1 \le j \le N$, there is a compact set K in $\mathbb{R}$ such that*

$$\sqrt{\rho_n^j}\,|\widehat{f_n^j}(\rho_n^j\xi + \xi_n^j)| \le C_\delta 1_K(\xi), \tag{57}$$

and for any $N \ge 1$, there exists a real-valued $q_n^N \in L^2$ such that

$$u_n = 2\sum_{j=1}^{N} \mathrm{Re}(f_n^j) + q_n^N, \tag{58}$$

with

$$\|D^{\frac{1}{6}} e^{-t\partial_x^3} q_n^N\|_{L^6_{t,x}} \le \delta, \tag{59}$$

and for any $N \ge 1$,

$$\lim_{n\to\infty}\bigg(\|u_n\|_{L^2}^2 - \Big(\sum_{j=1}^{N} 4\big\|\mathrm{Re}(f_n^j)\big\|_{L^2}^2 + \big\|q_n^N\big\|_{L^2}^2\Big)\bigg) = 0. \tag{60}$$

Then we focus on decomposing f_n^j further as in Lemma 3.5. Taking real parts automatically produces a decomposition for $\mathrm{Re}(f_n^j)$. We will be sketchy on how to resolve issues of the convergence of the error term and the almost L^2 orthogonality.

Lemma 4.3 (Real version: extraction of $x_n^{j,\alpha}$ and $s_n^{j,\alpha}$). *Let $(f_n)_{n\geq 1} \in L^2$ be a sequence of real-valued functions satisfying $\|f_n\|_{L^2} \leq 1$ and*

$$\sqrt{\rho_n}\,\big|\widehat{f_n}(\rho_n(\xi+(\rho_n)^{-1}\xi_n))\big| \leq F(\xi),$$

with $F \in L^\infty(K)$ for some compact set K and $\xi_n > 0$. Then up to a subsequence, there exists a family $(y_n^\alpha, s_n^\alpha) \in \mathbb{R}\times\mathbb{R}$ and a sequence of complex-valued functions $(\phi^\alpha)_{\alpha\geq 1} \in L^2$ such that, if $\alpha \neq \beta$,

$$\lim_{n\to\infty}\left(\Big|y_n^\beta - y_n^\alpha + \frac{3(s_n^\beta - s_n^\alpha)(\xi_n)^2}{(\rho_n)^2}\Big| + \Big|\frac{3(s_n^\beta - s_n^\alpha)\xi_n}{\rho_n}\Big| + \big|s_n^\beta - s_n^\alpha\big|\right) = \infty, \tag{61}$$

and for each $A \geq 1$, there exists $e_n^A \in L^2$ of complex-value such that

$$f_n(x) = \sum_{\alpha=1}^{A} \widetilde{g_n^\alpha}(\phi^\alpha)(x) + \mathrm{Re}(e_n^A)(x), \tag{62}$$

where

$$\widetilde{g_n^\alpha}(\phi^\alpha)(x) = \sqrt{\rho_n}\, e^{s_n^\alpha \partial_x^3}\big[\mathrm{Re}(e^{i(\cdot)\rho_n^{-1}\xi_n}\phi^\alpha)\big](\rho_n x - y_n^\alpha),$$

with $\xi_n \equiv 0$ when $\rho_n^{-1}\xi_n$ converges to some finite limit, and

$$\lim_{A\to\infty}\lim_{n\to\infty}\big\|D^{\frac{1}{6}} e^{-t\partial_x^3}\,\mathrm{Re}(e_n^A)\big\|_{L^6_{t,x}} = 0, \tag{63}$$

and for any $A \geq 1$,

$$\lim_{n\to\infty}\left(\|f_n\|_{L^2}^2 - \Big(\sum_{\alpha=1}^{A}\big\|\mathrm{Re}(e^{i(\cdot)\rho_n^{-1}\xi_n}\phi^\alpha)\big\|_{L^2}^2 + \big\|\mathrm{Re}(e_n^A)\big\|_{L^2}^2\Big)\right) = 0. \tag{64}$$

Moreover, for any $\alpha \neq \beta$,

$$\lim_{n\to\infty}\Big|\langle\widetilde{g_n^\alpha}(\phi^\alpha), \widetilde{g_n^\beta}(\phi^\beta)\rangle_{L^2}\Big| = 0, \tag{65}$$

and for any $1 \leq \alpha \leq A$,

$$\lim_{n\to\infty}\big|\langle\widetilde{g_n^\alpha}(\phi^\alpha), \mathrm{Re}(e_n^A)\rangle_{L^2}\big| = 0. \tag{66}$$

Proof. We briefly describe how to obtain these identities. Equations (61), (62) follow along similar lines as in Lemma 3.5. Equation (63) follows from (46) and the pointwise inequality

$$\big|D^{\frac{1}{6}} e^{-t\partial_x^3}\,\mathrm{Re}(e_n^A)(x)\big| = \big|\mathrm{Re}(D^{\frac{1}{6}} e^{-t\partial_x^3} e_n^A)(x)\big| \leq \big|D^{\frac{1}{6}} e^{-t\partial_x^3} e_n^A(x)\big|.$$

Equation (64) follows from (65) and (66), which are proven similarly as in Corollary 3.7. □

5. Final decomposition: proof of Theorems 1.5 and 1.6

In this section, we will only prove the complex version Theorem 1.5 by following the approach in [Keraani 2001]; the real version Theorem 1.6 can be obtained similarly. We go back to the decompositions (34), (45) and set

$$(h_n^j, \xi_n^j, x_n^{j,\alpha}, t_n^{j,\alpha}) := ((\rho_n^j)^{-1}, \xi_n^j, (\rho_n^j)^{-1} y_n^{j,\alpha}, (\rho_n^j)^{-3} s_n^{j,\alpha}).$$

Then we use Remark 3.6 and put all the error terms together,

$$u_n = \sum_{\substack{1\le j\le N,\ \xi_n^j\equiv 0\\ \text{or } |h_n^j\xi_n^j|\to\infty}} \sum_{\alpha=1}^{A_j} e^{t_n^{j,\alpha}\partial_x^3} g_n^{j,\alpha}\big[e^{i(\cdot)h_n^j\xi_n^j}\phi^{j,\alpha}\big] \;+\; w_n^{N,A_1,\dots,A_N}, \tag{67}$$

where $g_n^{j,\alpha} = g_{0,x_n^{j,\alpha},h_n^j} \in G$ and

$$w_n^{N,A_1,\dots,A_N} = \sum_{j=1}^{N} e_n^{j,A_j} + q_n^N. \tag{68}$$

We enumerate the pairs (j,α) by ω satisfying

$$\omega(j,\alpha) < \omega(k,\beta) \text{ if } j+\alpha < k+\beta \text{ or } j+\alpha = k+\beta \text{ and } j<k. \tag{69}$$

After relabeling, Equation (67) can be further rewritten as

$$u_n = \sum_{\substack{1\le j\le l,\ \xi_n^j\equiv 0\\ \text{or}|h_n^j\xi_n^j|\to\infty}} e^{t_n^j\partial_x^3} g_n^j\big[e^{i(\cdot)h_n^j\xi_n^j}\phi^j\big] + w_n^l, \tag{70}$$

where $w_n^l = w_n^{N,A_1,\dots,A_N}$ with $l=\sum_{j=1}^N A_j$. To establish Theorem 1.5, we are thus left with three points to investigate.

(i) The family $\Gamma_n^j = (h_n^j, \xi_n^j, t_n^j, x_n^j)$ is pairwise orthogonal, that is, it satisfies Definition 1.3. In fact, we have two possibilities:

(a) The two pairs are in the form $\Gamma_n^j = (h_n^i, \xi_n^i, t_n^{i,\alpha}, x_n^{i,\alpha})$ and $\Gamma_n^k = (h_n^m, \xi_n^m, t_n^{m,\beta}, x_n^{m,\beta})$ with $i\neq m$. In this case, the orthogonality follows from

$$\lim_{n\to\infty}\left(\frac{h_n^i}{h_n^m} + \frac{h_n^m}{h_n^i} + h_n^i|\xi_n^i - \xi_n^m|\right) = \infty,$$

which is (32) in Lemma 3.2.

(b) The two pairs are in form $\Gamma_n^j = (h_n^i, \xi_n^i, t_n^{i,\alpha}, x_n^{i,\alpha})$ and $\Gamma_n^k = (h_n^i, \xi_n^i, t_n^{i,\beta}, x_n^{i,\beta})$ with $\alpha\neq\beta$. In this case, the orthogonality follows from

$$\lim_{n\to\infty}\left(\frac{|t_n^{i,\beta}-t_n^{i,\alpha}|}{(h_n^i)^3} + \frac{3|t_n^{i,\beta}-t_n^{i,\alpha}||\xi_n^i|}{(h_n^i)^2} + \left|\frac{x_n^{i,\beta}-x_n^{i,\alpha}+3(t_n^{i,\beta}-t_n^{i,\alpha})(\xi_n^i)^2}{h_n^i}\right|\right) = \infty,$$

which is (44) in Lemma 3.5.

(ii) The almost orthogonality identity (8) is satisfied. In fact, combining (36) and (47), we obtain that for any $N \geq 1$,

$$\begin{aligned}\|u_n\|_{L^2}^2 &= \sum_{j=1}^{N}\Big(\sum_{\alpha=1}^{A_j}\|\phi^{j,\alpha}\|_{L^2}^2 + \|e_n^{j,A_j}\|_{L^2}^2\Big) + \|q_n^N\|_{L^2}^2 + o_n(1)\\ &= \sum_{j=1}^{N}\Big(\sum_{\alpha=1}^{A_j}\|\phi^{j,\alpha}\|_{L^2}^2\Big) + \|w_n^{N,A_1,\dots,A_N}\|_{L^2}^2 + o_n(1) = \sum_{j=1}^{l}\|\phi^j\|_{L^2}^2 + \|w_n^l\|_{L^2}^2 + o_n(1),\end{aligned}$$

where $\lim_{n\to\infty} o_n(1) = 0$. Note that we have used the fact that

$$\|w_n^l\|_{L^2}^2 = \|w_n^{N,A_1,\dots,A_N}\|_{L^2}^2 = \sum_{j=1}^{N}\|e_n^{j,A_j}\|_{L^2}^2 + \|q_n^N\|_{L^2}^2,$$

which is due to the disjoint supports on the Fourier side.

(iii) The remainder $e^{-t\partial_x^3}\omega_n^{N,A_1,\dots,A_N}$ converges to zero in the Strichartz norm. In view of the adapted enumeration, we have to prove that

$$\lim_{n\to\infty}\big\|D^{1/6}e^{-t\partial_x^3}\omega_n^{N,A_1,\dots,A_N}\big\|_{L^6_{t,x}} \to 0, \text{ as } \inf_{1\le j\le N}\{N, j+A_j\} \to \infty. \tag{71}$$

Let $\delta > 0$ be an arbitrarily small number. Take N_0 such that, for every $N \geq N_0$,

$$\lim_{n\to\infty}\big\|D^{1/6}e^{-t\partial_x^3}q_n^N\big\|_{L^6_{t,x}} \le \delta/3. \tag{72}$$

For every $N \geq N_0$, there exists B_N such that, whenever $A_j \geq B_N$,

$$\lim_{n\to\infty}\big\|D^{1/6}e^{-t\partial_x^3}e_n^{j,A_j}\big\|_{L^6_{t,x}} \le \delta/3N. \tag{73}$$

The remainder $w_n^{N,A_1,\dots,A_N}$ can be rewritten in the form

$$w_n^{N,A_1,\dots,A_N} = q_n^N + \sum_{1\le j\le N} w_n^{j,A_j\vee B_N} + S_n^{N,A_1,\dots,A_N},$$

where $A_j \vee B_N := \max\{A_j, B_N\}$ and

$$S_n^{N,A_1,\dots,A_N} = \sum_{\substack{1\le j\le N\\ A_j<B_N}}(w_n^{j,A_j} - w_n^{j,B_N}),$$

that is,

$$S_n^{N,A_1,\dots,A_N} = \sum_{\substack{1\le j\le N\\ A_j<B_N}}\ \sum_{A_j<\alpha\le B_N} e^{t_n^{j,\alpha}\partial_x^3}g_n^{j,\alpha}[e^{i(\cdot)h_n^j\xi_n^j}\phi^{j,\alpha}],$$

with $\xi_n^j \equiv 0$ when $\lim_{n\to\infty}|h_n^j\xi_n^j| < \infty$. From (72) and (73), it follows that

$$\lim_{n\to\infty}\big\|D^{1/6}e^{-t\partial_x^3}w_n^{N,A_1,\dots,A_N}\big\|_{L^6_{t,x}} \le 2\delta/3 + \lim_{n\to\infty}\big\|D^{1/6}e^{-t\partial_x^3}S_n^{N,A_1,\dots,A_N}\big\|_{L^6_{t,x}}. \tag{74}$$

Now we need the following almost-orthogonality result:

Lemma 5.1. *Let $\Gamma_n^j = (h_n^j, \xi_n^j, x_n^j, t_n^j)$ be a family of orthogonal sequences. Then for every $l \geq 1$,*

$$\lim_{n\to\infty}\left(\Big\|\sum_{j=1}^{l} D^{1/6}e^{-(t-t_n^j)\partial_x^3} g_n^j[e^{i(\cdot)h_n^j\xi_n^j}\phi^j]\Big\|_{L^6_{t,x}}^6 - \sum_{j=1}^{l}\Big\|D^{1/6}e^{-(t-t_n^j)\partial_x^3} g_n^j[e^{i(\cdot)h_n^j\xi_n^j}\phi^j]\Big\|_{L^6_{t,x}}^6\right) = 0, \tag{75}$$

with $\xi_n^j \equiv 0$ when $\lim_{n\to\infty}|h_n^j\xi_n^j| < \infty$.

Suppose this lemma were proven, we show how to conclude the proof of (71). From Lemma 5.1, it follows that

$$\lim_{n\to\infty}\|D^{1/6}e^{-t\partial_x^3}S_n^{N,A_1,\dots,A_N}\|_{L^6_{t,x}}^6 = \sum_{\substack{1\leq j\leq N\\ A_j<B_N}}\sum_{A_j<\alpha\leq B_N}\lim_{n\to\infty}\|D^{1/6}e^{-(t-t_n^{j,\alpha})\partial_x^3}g_n^{j,\alpha}[e^{i(\cdot)h_n^j\xi_n^j}\phi^{j,\alpha}]\|_{L^6_{t,x}}^6. \tag{76}$$

The Strichartz inequality gives

$$\begin{aligned}\sum_{\substack{1\leq j\leq N\\ A_j<B_N}}\sum_{A_j<\alpha\leq B_N}\|D^{1/6}e^{-(t-t_n^{j,\alpha})\partial_x^3}g_n^{j,\alpha}\big[e^{i(\cdot)h_n^j\xi_n^j}\phi^{j,\alpha}\big]\|_{L^6_{t,x}}^6 &\lesssim \sum_{\substack{1\leq j\leq N\\ A_j<B_N}}\sum_{A_j<\alpha\leq B_N}\|\phi^{j,\alpha}\|_{L^2}^6\\ &\leq \sum_{j,\alpha}\|\phi^{j,\alpha}\|_{L^2}^6.\end{aligned} \tag{77}$$

On the other hand, $\sum_{j,\alpha}\|\phi^{j,\alpha}\|_{L^2}^2$ is convergent; hence the right side of (77) is finite. This shows that

$$\left(\sum_{\substack{j,\alpha\\ \alpha>A_j}}\Big\|D^{1/6}e^{-(t-t_n^{j,\alpha})\partial_x^3}g_n^{j,\alpha}[e^{i(\cdot)h_n^j\xi_n^j}\phi^{j,\alpha}]\Big\|_{L^6_{t,x}}^6\right)^{1/6} \leq \delta/3 \tag{78}$$

provided that $\inf_{1\leq j\leq N}\{N,\, j+A_j\}$ is large enough. Combining (74), (76) and (78), we obtain

$$\lim_{n\to\infty}\|D^{1/6}e^{-t\partial_x^3}w_n^{N,A_1,\dots,A_N}\|_{L^6_{t,x}} = 0 \tag{79}$$

provided that $\inf_{1\leq j\leq N}\{N,\, j+A_j\}$ is large enough. Hence the proof of (71) is complete.

Proof of Lemma 5.1. By using the Hölder inequality, we need to show that for $j\neq k$, as n goes to infinity,

$$\|D^{1/6}e^{-(t-t_n^j)\partial_x^3}g_n^j\big[e^{i(\cdot)h_n^j\xi_n^j}\phi^j\big]D^{1/6}e^{-(t-t_n^k)\partial_x^3}g_n^k\big[e^{i(\cdot)h_n^k\xi_n^k}\phi^k\big]\|_{L^3_{t,x}} \to 0. \tag{80}$$

By the pigeonhole principle, we can assume that ξ_n^j and ξ_n^k are of the same sign if they are not zero; moreover by a density argument, we also assume that ϕ^j and ϕ^k are Schwartz functions with compact Fourier supports. Evidence in favor of (80) is that, if $\lim_{n\to\infty}|h_n\xi_n| = \infty$, $D^{1/6}e^{-(t-t_n)\partial_x^3}g_n[e^{i(\cdot)h_n\xi_n}\phi]$ is somehow a Schrödinger wave in the sense of Remark 1.7. For the pairwise orthogonal Schrödinger waves, however, the analogous result to (80) is true; see [Merle and Vega 1998; Carles and Keraani 2007; Bégout and Vargas 2007].

To prove (80) we will have two possibilities. First, the two pairs are in the form $\Gamma_n^j = (h_n^i, \xi_n^i, t_n^{i,\alpha}, x_n^{i,\alpha})$ and $\Gamma_n^k = (h_n^m, \xi_n^m, t_n^{m,\beta}, x_n^{m,\beta})$ with $i \neq m$. In this case, the orthogonality is given by

$$\lim_{n\to\infty}\Big(\frac{h_n^i}{h_n^m} + \frac{h_n^m}{h_n^i} + h_n^i|\xi_n^i - \xi_n^m|\Big) = \infty.$$

So we have two subcases. We begin with the case where $\lim_{n\to\infty} h_n^i|\xi_n^i - \xi_n^m| = \infty$; moreover, we may assume that $h_n^i = h_n^m$ for all n (when both limits are infinity, the reasoning is similar, using the argument below). By changing variables, we see that the left side of (80) equals

$$\Big\| D^{1/6}e^{-t\partial_x^3}\big(e^{i(\cdot)h_n^i\xi_n^i}\phi^{i,\alpha}\big) D^{1/6}e^{-\big(t+\frac{t_n^{i,\alpha}-t_n^{m,\beta}}{(h_n^i)^3}\big)\partial_x^3}\big(e^{i(\cdot)h_n^i\xi_n^m}\phi^{m,\beta}\big)\Big(x + \frac{x_n^{m,\alpha}-x_n^{i,\beta}}{h_n^i}\Big)\Big\|_{L^3_{t,x}}. \tag{81}$$

The integrand above equals

$$\iint e^{ix[(\xi+h_n^i\xi_n^i)+(\eta+h_n^i\xi_n^m)]+it[(\xi+h_n^i\xi_n^i)^3+(\eta+h_n^i\xi_n^m)^3]}\,|\xi+h_n^i\xi_n^i|^{1/6}\,|\eta+h_n^i\xi_n^m|^{1/6}$$
$$\times\, e^{i(\eta+h_n^i\xi_n^m)(x_n^{i,\alpha}-x_n^{m,\beta})/h_n^i + i(\eta+h_n^i\xi_n^m)^3(t_n^{i,\alpha}-t_n^{m,\beta})/(h_n^i)^3}\,\widehat{\phi^{i,\alpha}}(\xi)\widehat{\phi^{m,\beta}}(\eta)\,d\xi\,d\eta.$$

Applying the change of variables $a := (\xi + h_n^i\xi_n^i) + (\eta + h_n^i\xi_n^m)$ and $b := (\xi + h_n^i\xi_n^i)^3 + (\eta + h_n^i\xi_n^m)^3$, followed by the Hausdorff–Young inequality, we see that (81) is bounded by

$$C\bigg(\iint \frac{|\xi+h_n^i\xi_n^i|^{1/4}\,|\eta+h_n^i\xi_n^m|^{1/4}|\widehat{\phi^{i,\alpha}}(\xi)\widehat{\phi^{m,\beta}}(\eta)|^{3/2}}{|\xi+h_n^i\xi_n^i+\eta+h_n^i\xi_n^m|^{1/2}\,|\xi-\eta+h_n^i(\xi_n^i-\xi_n^m)|^{1/2}}d\xi d\eta\bigg)^{2/3}.$$

We consider two subcases according to the limits of $|h_n^i\xi_n^i|$ and $|h_n^m\xi_n^m|$. Note that $\lim_{n\to\infty} h_n^i|\xi_n^i - \xi_n^m| = \infty$, then either both are infinity or only one is.

- In the former case, since ξ_n^i and ξ_n^m are of the same sign, we have
$$\frac{|\xi+h_n^i\xi_n^i|^{1/4}|\eta+h_n^i\xi_n^m|^{1/4}}{|\xi+\eta+h_n^i(\xi_n^i+\xi_n^m)|^{1/2}} \sim \frac{|\xi_n^i\xi_n^m|^{1/4}}{|\xi_n^i+\xi_n^m|^{1/2}} \lesssim 1.$$
Then (81) is further bounded by $C_{\phi^{i,\alpha},\phi^{m,\beta}}(h_n^i|\xi_n^i-\xi_n^m|)^{-1/3}$, which goes to zero as n goes to infinity.
- In the latter case, say $\lim_{n\to\infty}|h_n^i\xi_n^i| = \infty$, we will have $\xi_n^m = 0$. Then
$$\frac{|\xi+h_n^i\xi_n^i|^{1/4}\,|\eta+h_n^i\xi_n^m|^{1/4}}{|\xi+\eta+h_n^i(\xi_n^i+\xi_n^m)|^{1/2}} \lesssim |h_n^i\xi_n^i|^{-1/4}.$$
Then (81) is further bounded by $C_{\phi^{i,\alpha},\phi^{m,\beta}}|h_n^i\xi_n^i|^{-1/2}$, which goes to zero as n goes to infinity.

Under the first possibility, we still need to consider the case when

$$\lim_{n\to\infty}\bigg(\frac{h_n^i}{h_n^m} + \frac{h_n^m}{h_n^i}\bigg) = \infty.$$

We may assume that $\lim_{n\to\infty}|h_n^i\xi_n^i - h_n^m\xi_n^m| < \infty$. It follows that $\lim_{n\to\infty}|h_n^i\xi_n^i|$ and $\lim_{n\to\infty}|h_n^m\xi_n^m|$ are finite or infinite simultaneously. We will consider the case where they are both infinite since the other follows similarly. Under this consideration, we deduce that

$$\left|\frac{h_n^m \xi_n^m}{h_n^i \xi_n^i}\right| \sim 1$$

for sufficiently large n. To prove (80), we will use the idea of regarding the profile term as a Schrödinger wave as in Remark 1.7. We recall that

$$D^{1/6} e^{-(t-t_n^j)\partial_x^3} g_n^j [e^{i(\cdot)h_n^j \xi_n^j} \phi^j] = (h_n^i)^{-1/2} |\xi_n^i|^{1/6} e^{i\xi_n^i (x - x_n^{i,\alpha}) + i(\xi_n^i)^3 (t - t_n^{i,\alpha})}$$
$$\times \int e^{i\xi\left[\frac{x - x_n^{i,\alpha}}{h_n^i} + 3(\xi_n^i)^2 \frac{t - t_n^{i,\alpha}}{h_n^i}\right] + i\xi^3 \frac{t - t_n^{i,\alpha}}{(h_n^i)^3} + 3i\xi^2 \xi_n^i \frac{t - t_n^{i,\alpha}}{(h_n^i)^2}} \left|1 + \frac{\xi}{h_n^i \xi_n^i}\right|^{1/6} \widehat{\phi^{i,\alpha}} d\xi,$$

Similarly for $D^{1/6} e^{-(t-t_n^k)\partial_x^3} g_n^k [e^{i(\cdot)h_n^k \xi_n^k} \phi^k]$. For any $R > 0$, we set

$$A_R^i := \left\{(t, x) \in \mathbb{R} \times \mathbb{R} : \left|3\xi_n^i \frac{t - t_n^{i,\alpha}}{(h_n^i)^2}\right| + \left|\frac{x - x_n^{i,\alpha}}{h_n^i} + 3(\xi_n^i)^2 \frac{t - t_n^{i,\alpha}}{h_n^i}\right| \le R\right\},$$
$$A_R^m := \left\{(t, x) \in \mathbb{R} \times \mathbb{R} : \left|3\xi_n^m \frac{t - t_n^{m,\beta}}{(h_n^m)^2}\right| + \left|\frac{x - x_n^{m,\beta}}{h_n^m} + 3(\xi_n^m)^2 \frac{t - t_n^{m,\beta}}{h_n^m}\right| \le R\right\}.$$

By the Hölder inequality, the Strichartz inequality and Remark 1.7, we only need to show, for a large $R > 0$,

$$\lim_{n\to\infty} \left\| D^{1/6} e^{-(t-t_n^i)\partial_x^3} g_n^j \left[e^{i(\cdot)h_n^i \xi_n^i} \phi^j\right] D^{1/6} e^{-(t-t_n^m)\partial_x^3} g_n^k \left[e^{i(\cdot)h_n^m \xi_n^m} \phi^k\right] \right\|_{L^3_{t,x}(A_R^i \cap A_R^m)} = 0. \tag{82}$$

Indeed, $\mathbb{R}^2 \setminus (A_R^i \cap A_R^m) \subset (\mathbb{R}^2 \setminus A_R^i) \cup (\mathbb{R}^2 \setminus A_R^m)$; here we only consider the integration over the region $\mathbb{R}^2 \setminus A_R^i$ since the other case is similar. By the Hölder inequality and the Strichartz inequality,

$$\left\| D^{1/6} e^{-(t-t_n^i)\partial_x^3} g_n^j [e^{i(\cdot)h_n^i \xi_n^i} \phi^j] D^{1/6} e^{-(t-t_n^m)\partial_x^3} g_n^k [e^{i(\cdot)h_n^m \xi_n^m} \phi^k] \right\|_{L^3_{t,x}(\mathbb{R}^2 \setminus A_R^i)}$$
$$\lesssim \left\| D^{1/6} e^{-(t-t_n^i)\partial_x^3} g_n^j [e^{i(\cdot)h_n^i \xi_n^i} \phi^j] \right\|_{L^6_{t,x}(\mathbb{R}^2 \setminus A_R^i)} \left\| D^{1/6} e^{-(t-t_n^m)\partial_x^3} g_n^k [e^{i(\cdot)h_n^m \xi_n^m} \phi^k] \right\|_{L^6_{t,x}}$$
$$\lesssim \|\phi^k\|_{L^2} \left\| D^{1/6} e^{-(t-t_n^i)\partial_x^3} g_n^j [e^{i(\cdot)h_n^i \xi_n^i} \phi^j] \right\|_{L^6_{t,x}(\mathbb{R}^2 \setminus A_R^i)}.$$

Let

$$x' := \frac{x - x_n^{i,\alpha} + 3(\xi_n^i)^2 (t - t_n^{i,\alpha})}{h_n^i} \quad \text{and} \quad t' := \frac{3\xi_n^i (t - t_n^{i,\alpha})}{(h_n^i)^2}.$$

Then a change of variables and similar computations as in Remark 1.7 show that

$$\left\| D^{1/6} e^{-(t-t_n^i)\partial_x^3} g_n^j [e^{i(\cdot)h_n^i \xi_n^i} \phi^j] \right\|_{L^6_{t,x}(\mathbb{R}^2 \setminus A_R^i)} \lesssim \left\| \int e^{i(x'\xi + t'\xi^2) + i\frac{\xi^3 t'}{3h_n^i \xi_n^i}} \left|1 + \frac{\xi}{h_n^i \xi_n^i}\right|^{1/6} \widehat{\phi^{i,\alpha}}(\xi)\, d\xi \right\|_{L^6_{t',x'}(|t'|+|x'| \ge R)}$$
$$\to \|e^{-it'\Delta} \phi^{i,\alpha}\|_{L^6_{t',x'}(|t'|+|x'| \ge R)} \to 0,$$

as $n \to \infty$ followed by $R \to \infty$. Returning to (82), using L^∞-bounds for the integrands, we see that it is bounded by

$$C \left\| D^{1/6} e^{-(t-t_n^i)\partial_x^3} g_n^j [e^{i(\cdot)h_n^i\xi_n^i}\phi^j] \right\|_{L^\infty} \left\| D^{1/6} e^{-(t-t_n^m)\partial_x^3} g_n^k [e^{i(\cdot)h_n^m\xi_n^m}\phi^k] \right\|_{L^\infty} \min\{|A_R^i|^{1/3}, |A_R^m|^{1/3}\}$$
$$\le C_{R,\phi^j,\phi^k} (h_n^i h_n^m)^{-1/2} |\xi_n^i \xi_n^m|^{1/6} \min\{[(h_n^i)^3|\xi_n^i|^{-1}]^{1/3},\ [(h_n^m)^3|\xi_n^m|^{-1}]^{1/3}\}$$
$$\le C_{R,\phi^j,\phi^k} \min\left\{ \left(\frac{h_n^i}{h_n^m}\right)^{2/3} \left| \frac{h_n^m \xi_n^m}{h_n^i \xi_n^i} \right|^{1/6}, \quad \left(\frac{h_n^m}{h_n^i}\right)^{2/3} \left| \frac{h_n^i \xi_n^i}{h_n^m \xi_n^m} \right|^{1/6} \right\}.$$

Hence (80) holds when $\lim_{n\to\infty}(h_n^i/h_n^m + h_n^m/h_n^i) = \infty$.

Secondly, the two pairs are in form $\Gamma_n^j = (h_n^i, \xi_n^i, t_n^{i,\alpha}, x_n^{i,\alpha})$ and $\Gamma_n^k = (h_n^i, \xi_n^i, t_n^{i,\beta}, x_n^{i,\beta})$, with $\alpha \neq \beta$. In this case, the orthogonality is given by

$$\lim_{n\to\infty} \left(\frac{|t_n^{i,\beta} - t_n^{i,\alpha}|}{(h_n^i)^3} + \frac{3|t_n^{i,\beta} - t_n^{i,\alpha}||\xi_n^i|}{(h_n^i)^2} + \frac{|x_n^{i,\beta} - x_n^{i,\alpha} + 3(t_n^{i,\beta} - t_n^{i,\alpha})(\xi_n^i)^2|}{h_n^i} \right) = \infty.$$

We assume $\lim_{n\to\infty} |h_n^i \xi_n^i| = \infty$ since the other case is similar. We expand the left-hand side of (80) out, which is equal to

$$(h_n^i)^{-\frac{4}{3}} \left\| D^{1/6} e^{-\frac{t-t_n^{i,\alpha}}{(h_n^i)^3}\partial_x^3} [e^{i(\cdot)h_n^i\xi_n^i}\phi^{i,\alpha}] \Big(\frac{x - x_n^{i,\alpha}}{h_n^i}\Big) D^{1/6} e^{-\frac{t-t_n^{m,\beta}}{(h_n^i)^3}\partial_x^3} [e^{i(\cdot)h_n^i\xi_n^i}\phi^{m,\beta}] \Big(\frac{x - x_n^{m,\beta}}{h_n^i}\Big) \right\|_{L^3_{t,x}}$$
$$= \frac{|\xi_n^i|^{1/3}}{h_n^i} \left\| \int e^{i\left[\frac{\eta(x-x_n^{i,\alpha}+3(t-t_n^{i,\alpha})(\xi_n^i)^2)}{h_n^i} + \frac{\eta^3(t-t_n^{i,\alpha})}{(h_n^i)^3} + \frac{3\eta^2(t-t_n^{i,\alpha})\xi_n^i}{(h_n^i)^2}\right]} \left| 1 + \frac{\eta}{h_n^i\xi_n^i} \right|^{1/6} \widehat{\phi^{i,\alpha}}(\eta)\, d\eta \right.$$
$$\left. \times \int e^{i\left[\frac{\eta(x-x_n^{i,\beta}+3(t-t_n^{i,\beta})(\xi_n^i)^2)}{h_n^i} + \frac{\eta^3(t-t_n^{i,\beta})}{(h_n^i)^3} + \frac{3\eta^2(t-t_n^{i,\beta})\xi_n^i}{(h_n^i)^2}\right]} \left| 1 + \frac{\eta}{h_n^i\xi_n^i} \right|^{1/6} \widehat{\phi^{i,\beta}}(\eta)\, d\eta \right\|_{L^3_{t,x}}$$

Through the change of variables $t' = \dfrac{3(t - t_n^{i,\beta})\xi_n^i}{(h_n^i)^2}$, $x' = \dfrac{x - x_n^{i,\beta} + 3(t - t_n^{i,\beta})(\xi_n^i)^2}{h_n^i}$, this reduces to

$$C \left\| \int e^{i\eta\left[x' + \frac{x_n^{i,\beta} - x_n^{i,\alpha} + 3(t_n^{i,\beta} - t_n^{i,\alpha})(\xi_n^i)^2}{h_n^i}\right] + i\eta^3\left[\frac{t_n^{i,\beta} - t_n^{i,\alpha}}{(h_n^i)^3} + \frac{t'}{3h_n^i\xi_n^i}\right] + i\eta^2\left[t' + \frac{3(t_n^{i,\beta} - t_n^{i,\alpha})\xi_n^i}{(h_n^i)^2}\right]} \right.$$
$$\left. \times \left| 1 + \frac{\eta}{h_n^i\xi_n^i} \right|^{1/6} \widehat{\phi^{i,\alpha}}(\eta)\, d\eta \int e^{ix'\eta + it'\eta^2} e^{i\eta^3 \frac{t'}{3h_n^i\xi_n^i}} \left| 1 + \frac{\eta}{h_n^i\xi_n^i} \right|^{1/6} \widehat{\phi^{i,\beta}}(\eta)\, d\eta \right\|_{L^3_{t',x'}}.$$

Using the Hölder inequality followed by the principle of the stationary phase or integration by parts, we see that (80) holds. □

Similarly, we can obtain the following generalization of Corollary 3.7 about the orthogonality of profiles in L^2 space. Its proof will be omitted.

Lemma 5.2. *Assume $\Gamma_n^j = (h_n^j, \xi_n^j, t_n^j, x_n^j)$ and $\Gamma_n^k = (h_n^k, \xi_n^k, t_n^k, x_n^k)$ are pairwise orthogonal. Then*

$$\lim_{n\to\infty} \langle e^{t_n^j\partial_x^3} g_n^j [e^{i(\cdot)h_n^j\xi_n^j}\phi^j],\, e^{t_n^k\partial_x^3} g_n^k [e^{i(\cdot)h_n^k\xi_n^k}\phi^k] \rangle_{L^2} = 0, \tag{83}$$

and for $1 \le j \le l$,

$$\lim_{n\to\infty} \langle e^{t_n^j\partial_x^3} g_n^j [e^{i(\cdot)h_n^j\xi_n^j}\phi^j],\, w_n^l \rangle_{L^2} = 0, \tag{84}$$

with $\xi_n^j \equiv 0$ when $\lim_{n\to\infty} |h_n^j \xi_n^j| < \infty$.

6. The existence of maximizers for the symmetric Airy Strichartz inequality

This section is devoted to establishing Theorem 1.9, a dichotomy result on the existence of maximizers for the symmetric Airy Strichartz inequality. First, we will exploit the idea of asymptotically embedding a Schrödinger solution into an approximate Airy solution. We will show that the best constant for the Airy Schrödinger Strichartz bounds that for the symmetric Schrödinger Strichartz inequality up to a constant. We will follow the approach in [Tao 2007], in which Tao shows that any qualitative scattering result on the mass critical gKdV equation $\partial_t u + \partial_x^3 u \pm |u|^4 \partial_x u = 0$ automatically implies an analogous scattering result for the mass critical nonlinear Schrödinger equation $i\partial_t u + \partial_x^2 u \pm |u|^4 u = 0$.

Lemma 6.1 (Asymptotic embedding of Schrödinger into Airy). *Corresponding to Theorems 1.5 and 1.6, we have, respectively,*

$$S^{\mathbb{C}}_{\text{schr}} \le 3^{1/6} S^{\mathbb{C}}_{\text{airy}}, \tag{85}$$

$$S^{\mathbb{C}}_{\text{schr}} \le 2^{1/2} 3^{1/6} S^{\mathbb{R}}_{\text{airy}}. \tag{86}$$

Proof. We first prove (86). Let u_0 to a maximizer to (16). Since d=1, from the work in [Foschi 2007], we can assume that u_0 is a standard Gaussian; hence it is even and its Fourier transform is another Gaussian. Denote

$$u_N(0,x) := \frac{1}{(3N)^{1/4}} \operatorname{Re}\Big(e^{ixN} u_0\Big(\frac{x}{\sqrt{3N}}\Big)\Big).$$

Let $u_N(t,x)$ solve the Airy Equation (1) with initial data $u_N(0,x)$. From the Airy Strichartz inequality,

$$\big\| D^{1/6} u_N \big\|_{L^6_{t,x}} \le S^{\mathbb{R}}_{\text{airy}} \big\| u_N(0,x) \big\|_{L^2}. \tag{87}$$

On the one hand, a computation shows that

$$\|u_N(0,x)\|^2_{L^2} = \frac{1}{2} \int |u_0(x)|^2 + \operatorname{Re}\big(e^{2\sqrt{3} i N^{3/2} x} u_0^2(x)\big)\, dx. \tag{88}$$

From the Riemann–Lebesgue lemma, we know the second term above rapidly goes to zero as $N \to \infty$. On the other hand,

$$\widehat{u_N}(0,\xi) = \frac{(3N)^{1/4}}{2} \big(\widehat{u_0}(\sqrt{3N}(\xi - N)) + \widehat{u_0}(\sqrt{3N}(\xi + N))\big),$$

which yields

$$\begin{aligned} D^{1/6} u_N(t,x) &= \int e^{ix\xi + it\xi^3} |\xi|^{1/6} \widehat{u_N}(0,\xi)\, d\xi \\ &= \frac{(3N)^{1/4}}{2} \int e^{ix\xi + it\xi^3} |\xi|^{1/6} \big(\widehat{u_0}(\sqrt{3N}(\xi - N)) + \widehat{u_0}(\sqrt{3N}(\xi + N))\big)\, d\xi \\ &= 2^{-1} 3^{-1/4} N^{-1/12} e^{ixN + itN^3} \int e^{i\left[\eta((3N)^{-1/2}x + \sqrt{3}N^{3/2}t) + t\eta^2 + t(3N)^{-3/2}\eta^3\right]} \\ &\qquad\qquad \times \Big|1 + \frac{\eta}{N\sqrt{3N}}\Big|^{1/6} \big(\widehat{u_0}(\eta) + \widehat{u_0}(\eta + 2N\sqrt{3N})\big)\, d\eta. \end{aligned}$$

Changing variables $x' = (3N)^{-1/2}x + \sqrt{3}N^{3/2}t$ and $t' = t$, we obtain

$$\begin{aligned}&\left\| D^{1/6}u_N(t,x)\right\|_{L^6_{t,x}}\\&\quad = 2^{-1}3^{-1/6}\left\| \int e^{i\left[x'\eta + t'\eta^2 + t'(3N)^{-3/2}\eta^3\right]} \times \left|1 + \frac{\eta}{N\sqrt{3N}}\right|^{1/6} \left(\widehat{u_0}(\eta) + \widehat{u_0}(\eta + 2N\sqrt{3N})\right) d\eta \right\|_{L^6_{t',x'}} \qquad (89)\end{aligned}$$

Comparing (87), (88), (89) and letting $N \to \infty$, as in Remark 1.7, we obtain,

$$2^{-1}3^{-1/6}\left\| \int e^{ix'\eta + it'\eta^2}\widehat{u_0}(\eta)\, d\eta \right\|_{L^6_{t',x'}} \le 2^{-1/2}S^{\mathbb{R}}_{\text{airy}}\|u_0\|_{L^2}. \qquad (90)$$

By the choice of u_0, we have

$$2^{-1}3^{-1/6}S^{\mathbb{C}}_{\text{schr}} \le 2^{-1/2}S^{\mathbb{R}}_{\text{airy}},$$

that is, $S^{\mathbb{C}}_{\text{schr}} \le 2^{1/2}3^{1/6}S^{\mathbb{R}}_{\text{airy}}$. Hence (86) follows. To show (85), we choose

$$\phi_N(x) := \frac{1}{(3N)^{1/4}} e^{ixN} u_0\Big(\frac{x}{\sqrt{3N}}\Big).$$

Then

$$\|\phi_N\|_{L^2} = \|u_0\|_{L^2},\ \left\| e^{-it\partial_x^2}\phi_N\right\|_{L^6_{t,x}(\mathbb{R}\times\mathbb{R})} = S^{\mathbb{C}}_{\text{schr}}\|u_0\|_{L^2}.$$

Also an easy computation shows that

$$\left\| D^{1/6}e^{-t\partial_x^3}\phi_N\right\|_{L^6_{t,x}} \to 3^{-1/6}\left\| e^{-it\partial_x^2}u_0\right\|_{L^6_{t,x}}, \text{ as } N \to \infty.$$

From the Airy Strichartz inequality,

$$\left\| D^{1/6}e^{-t\partial_x^3}\phi_N\right\|_{L^6_{t,x}} \le S^{\mathbb{C}}_{\text{airy}}\|\phi_N\|_{L^2} = S^{\mathbb{C}}_{\text{airy}}\|u_0\|_{L^2},$$

we conclude that (85) follows. □

Now we are ready to prove Theorem 1.9.

Proof of Theorem 1.9. We only prove the complex version by using Theorem 1.5. For the real version, we use Theorem 1.6 instead but its proof is similar.

We choose a maximizing sequence $(u_n)_{n\ge1}$ with $\|u_n\|_{L^2} = 1$, and decompose it into the linear profiles as in Theorem 1.5 to obtain

$$u_n = \sum_{\substack{1\le j\le l,\ \xi_n^j \equiv 0\\ \text{or } |h_n^j\xi_n^j|\to\infty}} e^{t_n^j\partial_x^3} g_n^j[e^{i(\cdot)h_n^j\xi_n^j}\phi^j] + w_n^l. \qquad (91)$$

Then from the asymptotically vanishing Strichartz norm (7) and the triangle inequality, we obtain that, up to a subsequence, for any given $\varepsilon > 0$, there exists n_0, for all $l \ge n_0$ and $n \ge n_0$,

$$\left\| \sum_{j=1}^{l} D^{1/6}e^{-(t-t_n^j)\partial_x^3} g_n^j\left[e^{i(\cdot)h_n^j\xi_n^j}\phi^j\right]\right\|_{L^6_{t,x}} \ge S^{\mathbb{C}}_{\text{airy}} - \varepsilon,$$

with $\xi_n^j \equiv 0$ when $\lim_{n\to\infty} |h_n^j \xi_n^j| < \infty$. On the other hand, Lemma 5.1 yields,

$$\Big\| \sum_{j=1}^{l} D^{1/6} e^{-(t-t_n^j)\partial_x^3} g_n^j \big[e^{i(\cdot)h_n^j \xi_n^j} \phi^j \big] \Big\|_{L^6_{t,x}}^6 \le \sum_{j=1}^{l} \Big\| D^{1/6} e^{-(t-t_n^j)\partial_x^3} g_n^j \big[e^{i(\cdot)h_n^j \xi_n^j} \phi^j \big] \Big\|_{L^6_{t,x}}^6 + o_n(1). \tag{92}$$

Then up to a subsequence, there exists n_1 such that, for large $n \ge n_1$ and $l \ge n_1$,

$$\sum_{j=1}^{l} \Big\| D^{1/6} e^{-(t-t_n^j)\partial_x^3} g_n^j \big[e^{i(\cdot)h_n^j \xi_n^j} \phi^j \big] \Big\|_{L^6_{t,x}}^6 \ge (S_{\mathrm{airy}}^{\mathbb{C}})^6 - 2\varepsilon. \tag{93}$$

Choosing j_0 such that

$$D^{1/6} e^{-(t-t_n^{j_0})\partial_x^3} g_n^{j_0} [e^{i(\cdot)h_n^{j_0} \xi_n^{j_0}} \phi^{j_0}]$$

has the biggest Strichartz norm among $1 \le j \le l$, we see that, by Strichartz and the almost orthogonal identity (8),

$$\begin{aligned}
(S_{\mathrm{airy}}^{\mathbb{C}})^6 - 2\varepsilon &\le \Big\| D^{1/6} e^{-(t-t_n^{j_0})\partial_x^3} g_n^{j_0} \big[e^{i(\cdot)h_n^{j_0} \xi_n^{j_0}} \phi^{j_0} \big] \Big\|_{L^6_{t,x}}^4 \sum_{j=1}^{l} \Big\| D^{1/6} e^{-(t-t_n^j)\partial_x^3} g_n^j \big[e^{i(\cdot)h_n^j \xi_n^j} \phi^j \big] \Big\|_{L^6_{t,x}}^2 \\
&\le \Big\| D^{1/6} e^{-(t-t_n^{j_0})\partial_x^3} g_n^{j_0} \big[e^{i(\cdot)h_n^{j_0} \xi_n^{j_0}} \phi^{j_0} \big] \Big\|_{L^6_{t,x}}^4 \sum_{j=1}^{l} \big(S_{\mathrm{airy}}^{\mathbb{C}} \|\phi^j\|_{L^2} \big)^2 \\
&\le (S_{\mathrm{airy}}^{\mathbb{C}})^2 \Big\| D^{1/6} e^{-(t-t_n^{j_0})\partial_x^3} g_n^{j_0} \big[e^{i(\cdot)h_n^{j_0} \xi_n^{j_0}} \phi^{j_0} \big] \Big\|_{L^6_{t,x}}^4 .
\end{aligned}$$

This yields,

$$\Big\| D^{1/6} e^{-(t-t_n^{j_0})\partial_x^3} g_n^{j_0} \big[e^{i(\cdot)h_n^{j_0} \xi_n^{j_0}} \phi^{j_0} \big] \Big\|_{L^6_{t,x}} \ge \Big((S_{\mathrm{airy}}^{\mathbb{C}})^{-2} \big[(S_{\mathrm{airy}}^{\mathbb{C}})^6 - 2\varepsilon \big] \Big)^{1/4} \ge S_{\mathrm{airy}}^{\mathbb{C}} - \varepsilon. \tag{94}$$

Moreover, (8) implies that there exists $J > 0$ such that

$$\|\phi^j\|_{L^2} \le 1/100 \quad \text{for all } j > J.$$

This, together with (94) and the Strichartz inequality

$$\Big\| D^{1/6} e^{-(t-t_n^{j_0})\partial_x^3} g_n^{j_0} \big[e^{i(\cdot)h_n^{j_0} \xi_n^{j_0}} \phi^{j_0} \big] \Big\|_{L^6_{t,x}} \le S_{\mathrm{airy}}^{\mathbb{C}} \|\phi^{j_0}\|_{L^2},$$

shows that, for ε small enough, j_0 is between 1 and J; otherwise $S_{\mathrm{airy}}^{\mathbb{C}}/2 \le S_{\mathrm{airy}}^{\mathbb{C}}/100$, a contradiction. Hence j_0 does not depend on l, n and ε. So we can freely take ε to zero without changing j_0. Now we consider two cases:

Case I. When $h_n^{j_0} \xi_n^{j_0} \to \xi^{j_0} \in \mathbb{R}$, we can take $\xi_n^{j_0} \equiv 0$. Then

$$\Big\| D^{1/6} e^{-(t-t_n^{j_0})\partial_x^3} g_n^{j_0} (\phi^{j_0}) \Big\|_{L^6_{t,x}} = \Big\| D^{1/6} e^{-t\partial_x^3} \phi^{j_0} \Big\|_{L^6_{t,x}}.$$

Then we take $\varepsilon \to 0$ in (94) to obtain

$$\|\phi^{j_0}\|_{L^2} = 1, \quad S^{\mathbb{C}}_{\text{airy}} = \big\| D^{1/6} e^{-t\partial_x^3} \phi^{j_0} \big\|_{L^6_{t,x}}.$$

This shows that ϕ^{j_0} is a maximizer for (15).

Case II. When $|h_n^{j_0} \xi_n^{j_0}| \to \infty$, we take $n \to \infty$ in (94) and use Remark 1.7,

$$\begin{aligned} S^{\mathbb{C}}_{\text{airy}} - \varepsilon &\le \lim_{n\to\infty} \big\| D^{1/6} e^{-(t-t_n^{j_0})\partial_x^3} g_n^{j_0} \big[e^{i(\cdot) h_n^{j_0} \xi_n^{j_0}} \phi^{j_0} \big] \big\|_{L^6_{t,x}} \\ &= \lim_{n\to\infty} \big\| D^{1/6} e^{-t\partial_x^3} \big[e^{i(\cdot) h_n^{j_0} \xi_n^{j_0}} \phi^{j_0} \big] \big\|_{L^6_{t,x}} \\ &= 3^{-1/6} \big\| e^{-it\partial_x^2} \phi^{j_0} \big\|_{L^6_{t,x}} \le 3^{-1/6} S^{\mathbb{C}}_{\text{schr}} \|\phi^{j_0}\|_{L^2} \\ &\le S^{\mathbb{C}}_{\text{airy}} \|\phi^{j_0}\|_{L^2}. \end{aligned}$$

Taking $\varepsilon \to 0$ forces all the inequality signs to be equal. Hence we obtain

$$\|\phi^{j_0}\|_{L^2} = 1, \; S^{\mathbb{C}}_{\text{airy}} = 3^{-1/6} S^{\mathbb{C}}_{\text{schr}}$$

and

$$S^{\mathbb{C}}_{\text{airy}} = \lim_{n\to\infty} \big\| D^{1/6} e^{-t\partial_x^3} \big[e^{i(\cdot) h_n^{j_0} \xi_n^{j_0}} \phi^{j_0} \big] \big\|_{L^6_{t,x}} = 3^{-1/6} \big\| e^{-it\partial_x^2} \phi^{j_0} \big\|_{L^6_{t,x}}.$$

This shows that $S^{\mathbb{C}}_{\text{schr}} = \big\| e^{-it\partial_x^2} \phi^{j_0} \big\|_{L^6_{t,x}}$; hence ϕ^{j_0} is a maximizer for (16). Set $a_n := h_n^{j_0} \xi_n^{j_0}$. Then the proof of Theorem 1.9 is complete. □

Acknowledgments

The author is grateful to Terence Tao for many helpful discussions. The author would like to thank Jincheng Jiang and Monica Visan for their comments. The author also thanks the anonymous referees and Silvio Levy, scientific editor at MSP, for their valuable comments and suggestions, which have been incorporated into this paper.

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Received 31 Aug 2008. Revised 17 Feb 2009. Accepted 17 Feb 2009.

SHUANGLIN SHAO: *Department of Mathematics, University of California, Los Angeles, Los Angeles, CA 90095, United States*

and

School of Mathematics, Institute for Advanced Study, Princeton, NJ 08540, United States

slshao@math.ucla.edu

slshao@math.ias.edu

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