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DYNAMICS OF VORTICES FOR THE COMPLEX GINZBURG–LANDAU EQUATION

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We study a complex Ginzburg–Landau equation in the plane, which has the form of a Gross–Pitaevskii equation with some dissipation added. We focus on the regime corresponding to well-prepared unitary vortices and derive their asymptotic motion law.

1. Introduction

We study the dynamics of vortices for a complex Ginzburg–Landau equation on the plane, namely

$$\frac{\delta}{|\log \varepsilon|}\partial_t u_\varepsilon + \alpha i \partial_t u_\varepsilon = \Delta u_\varepsilon + \frac{1}{\varepsilon^2} u_\varepsilon (1 - |u_\varepsilon|^2), \qquad \text{(CGL)}_\varepsilon$$

where $u_\varepsilon : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}^2$ is a complex-valued map. Here δ, α, and ε denote positive real parameters, and we will mainly focus on the asymptotics as ε tends to zero while δ and α are kept fixed. Up to a change of scale, we may further assume that $\alpha = 1$, and we set $k_\varepsilon = \delta/|\log \varepsilon|$. The complex Ginzburg–Landau equation (CGL)_ε reduces to the Gross–Pitaevskii equation when $\delta = 0$ and to the parabolic Ginzburg–Landau equation when $\alpha = 0$. Both the Gross–Pitaevskii and the Ginzburg–Landau equations have been widely investigated in the regime we will consider (see, for example, [Colliander and Jerrard 1998; Lin and Xin 1999; Jerrard and Spirn 2008; Bethuel et al. 2008] for the Gross–Pitaevskii equation and [Jerrard and Soner 1998; Serfaty 2007; Bethuel et al. 2007] and references therein for the parabolic Ginzburg–Landau equation). Typical functions u_ε in this regime are given explicitly by

$$u_\varepsilon^*(a_i, d_i)(z) := \prod_{i=1}^{l} u_{\varepsilon,d_i}(z - a_i) = \prod_{i=1}^{l} f_{1,d_i}\Big(\frac{|z-a_i|}{\varepsilon}\Big)\Big(\frac{z-a_i}{|z-a_i|}\Big)^{d_i},$$

where the points $a_i \in \mathbb{R}^2$, $d_i = \pm 1$, and the functions $f_{1,d_i} : \mathbb{R}^+ \mapsto [0, 1]$, which satisfy $f_{1,d_i}(0) = 0$, $f_{1,d_i}(+\infty) = 1$, are in some sense optimal profiles. The points a_i are called the vortices of the fields u_ε^* and the d_i their degrees. This class of functions u_ε^* is, of course, not invariant by any of the flows corresponding to these equations, but it is not far from it (see the notion of well-preparedness in Definition 1.2). In particular, it is possible to define notions of point vortices for solutions of (CGL)_ε, at least in an asymptotic way as $\varepsilon \to 0$, and to study their dynamics. This dynamics is eventually governed by a system of ordinary differential equations, at least before collisions.

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Two relevant quantities in the study of vortex dynamics are the Ginzburg–Landau energy

$$E_\varepsilon(u) = \int_{\mathbb{R}^2} e_\varepsilon(u)\,dx = \int_{\mathbb{R}^2} \frac{|\nabla u|^2}{2} + \frac{(1-|u|^2)^2}{4\varepsilon^2}\,dx$$

through its energy density $e_\varepsilon(u)$, and the Jacobian

$$Ju = \tfrac{1}{2}\operatorname{curl}(u \times \nabla u)$$

through its primitive $j(u) = u \times \nabla u$. In the regime we will consider, one has

$$\frac{e_\varepsilon(u_\varepsilon)}{|\log\varepsilon|}\,dx \rightharpoonup \pi \sum_{i=1}^{l} \delta_{a_i} \quad \text{and} \quad Ju_\varepsilon\,dx \rightharpoonup \pi \sum_{i=1}^{l} d_i \delta_{a_i}$$

as $\varepsilon \to 0$, which describes asymptotically the positions and degrees of the vortices. The quantity $e_\varepsilon(u_\varepsilon)$ has been especially used in the study of the parabolic Ginzburg–Landau equation, while $j(u_\varepsilon)$ has been used in the study of the Gross–Pitaevskii equation. Here, we will rely on both of them.

In the case of the domain being the entire plane $\mathbb{R}^2$, which we consider here, the reference fields $u_\varepsilon^*(a_i, d_i)$ have infinite Ginzburg–Landau energy E_ε whenever $d = \sum d_i \neq 0$. In [Bethuel and Smets 2007], a notion of renormalized energy for such data — not to be confused with the one in [Bethuel et al. 1994] — was introduced to solve the Cauchy problem for the Gross–Pitaevskii equation. This notion was later used in [Bethuel et al. 2008] to study the dynamics of vortices for the Gross–Pitaevskii equation in the plane. Our definition of well-prepared data below and part of the subsequent analysis is borrowed from this last reference.

The complex Ginzburg–Landau equation $(\mathrm{CGL})_\varepsilon$, either in the plane or on the real line, has been widely considered in the literature, especially as a model for amplitude oscillation in weakly nonlinear systems undergoing a Hopf bifurcation (see [Aranson and Kramer 2002] for a survey). The mathematical analysis of vortices for $(\mathrm{CGL})_\varepsilon$ was initiated in [Lin and Xin 1999], where it was presented as an alternative approach (a regularized version) for the study of the Gross–Pitaevskii equation. We believe, however, that the conclusion regarding the dynamics of vortices for $(\mathrm{CGL})_\varepsilon$ in [Lin and Xin 1999] is erroneous, and that Theorem 1.3 represents the correct version.

After the completion of this work we were informed that Kurzke, Melcher, Moser, and Spirn [Kurzke et al. 2008] independently obtained similar results concerning the dynamics of vortices for $(\mathrm{CGL})_\varepsilon$ in bounded and simply connected domains.

Renormalized energy and the Cauchy problem. As mentioned, for $d = \sum d_i \neq 0$ the Ginzburg–Landau energy of $u_\varepsilon^*(a_i, d_i)$ is infinite. It can actually be computed that

$$\int_{\mathbb{R}^2} \frac{|\nabla |u_\varepsilon^*(a_i, d_i)||^2}{2} + \frac{(1-|u_\varepsilon^*(a_i, d_i)|^2)^2}{4\varepsilon^2}\,dz < +\infty,$$

whereas as $|z| \to +\infty$,

$$|\nabla u_\varepsilon^*(a_i, d_i)|^2(z) \sim \frac{d^2}{|z|^2},$$

so that

$$\int_{\mathbb{R}^2} \frac{|\nabla u_\varepsilon^*(a_i, d_i)|^2}{2} = +\infty.$$

The renormalized energy introduced in [Bethuel and Smets 2007] is obtained by subtracting the divergent part of the gradient at infinity. More precisely, given a smooth map U_d such that

$$U_d = \left(\frac{z}{|z|}\right)^d \quad \text{on } \mathbb{R}^2 \setminus B(0,1),$$

we have as $|z| \to +\infty$

$$|\nabla u_\varepsilon^*(a_i, d_i)|^2 \sim |\nabla U_d|^2$$

and one may define

$$\mathscr{E}_{\varepsilon,U_d}(u_\varepsilon^*(a_i,d_i)) := \lim_{R\to+\infty} \int_{B(R)} \big(e_\varepsilon(u_\varepsilon^*(a_i,d_i)) - \tfrac{1}{2}|\nabla U_d|^2\big) < +\infty. \tag{1-1}$$

This definition extends to a larger class of functions, and is a useful ingredient in solving the Cauchy problem. Following Bethuel and Smets, we define

$$\mathscr{V} = \big\{U \in L^\infty(\mathbb{R}^2, \mathbb{C}) : \nabla^k U \in L^2 \text{ for all } k \geq 2,\ (1-|U|^2) \in L^2,\ \nabla|U| \in L^2\big\}.$$

In particular, the space $\mathscr{V}$ contains all the maps u_ε^* as well as the reference maps U_d. Our first result, which we prove in the Appendix, establishes global well-posedness in the class $\mathscr{V} + H^1(\mathbb{R}^2)$. (In passing, we mention that Ginibre and Velo [1997] investigated the Cauchy problem in local spaces for a more general class of complex Ginzburg–Landau equations.)

Theorem 1.1. *Let $u_0 = U + w_0$ be in $\mathscr{V} + H^1(\mathbb{R}^2)$. There exists a unique global solution u to (CGL)$_\varepsilon$ such that $u \in C^0(\{U\} + H^1(\mathbb{R}^2))$. If we write $u(t) = U + w(t)$, then w is the unique solution in $C^0(\mathbb{R}_+, H^1(\mathbb{R}^2))$ to*

$$\begin{cases} (k_\varepsilon + i)\partial_t w = \Delta w + f_U(w), \\ w(0) = w_0, \end{cases} \tag{1-2}$$

where

$$f_U(w) = \Delta U + \varepsilon^{-2}(U + w)(1 - |U + w|^2).$$

In addition, w satisfies

$$w \in L^1_{\text{loc}}\big(\mathbb{R}_+, H^2(\mathbb{R}^2)\big) \cap L^\infty_{\text{loc}}\big(\mathbb{R}_+^*, L^\infty(\mathbb{R}^2)\big), \quad \partial_t w \in L^1_{\text{loc}}\big(\mathbb{R}_+, L^2(\mathbb{R}^2)\big), \quad w \in C^\infty\big(\mathbb{R}_+^*, C^\infty(\mathbb{R}^2)\big).$$

Finally, the functional $E_{\varepsilon,U}(u) := E_{\varepsilon,U}(w)$ defined by

$$E_{\varepsilon,U}(u) = \int_{\mathbb{R}^2} \frac{|\nabla w|^2}{2} - \int_{\mathbb{R}^2} \Delta U \cdot w + \int_{\mathbb{R}^2} \frac{(1 - |U + w|^2)^2}{4\varepsilon^2}$$

satisfies

$$\frac{d}{dt} E_{\varepsilon,U}(u) = -k_\varepsilon \int_{\mathbb{R}^2} |\partial_t w|^2 \, dx \quad \textit{for all } t \geq 0.$$

As a matter of fact, it follows from integration by parts that if $u \in \{U\} + H^1(\mathbb{R}^2)$ is as in Theorem 1.1 and if U satisfies in addition $|\nabla U(x)| \leq C/\sqrt{|x|}$, then

$$E_{\varepsilon,U}(u(t)) \equiv \mathscr{E}_{\varepsilon,U}(u(t)) = \lim_{R\to+\infty} \int_{B(R)} \big(e_\varepsilon(u(t)) - \tfrac{1}{2}|\nabla U|^2\big)\, dx.$$

The functions $u_\varepsilon^*(a_i, d_i)$ are not H^1 perturbations of one another, even for fixed $d = \sum d_i$, unless algebraic relations connect the a_i and d_i. To handle a class of functions containing all the u_ε^*, it is useful to introduce on the set $\mathcal{V}$ the equivalence relation defined, for $U, U' \in \mathcal{V}$, by

$$U \sim U' \quad \Longleftrightarrow \quad \deg_\infty(U) = \deg_\infty(U') \text{ and } |\nabla U|^2 - |\nabla U'|^2 \in L^1(\mathbb{R}^2).$$

Denoting by $[U]$ the corresponding equivalence class of U, we observe that $u_\varepsilon^*(a_i, d_i) \in [U_d]$ for any configuration (a_i, d_i) such that $\sum d_i = d$. Therefore the space $[U_d] + H^1(\mathbb{R}^2)$ contains all H^1 perturbations of reference maps u_ε^* of degree d at infinity.

For a map u in $[U_d] + H^1(\mathbb{R}^2)$, we may now define

$$\mathscr{E}_{\varepsilon,[U_d]}(u) := \lim_{R\to+\infty} \int_{B(R)} \left(e_\varepsilon(u) - \tfrac{1}{2}|\nabla U_d|^2\right),$$

which is a finite quantity. Moreover, for any solution $u \in C^0(\{U\} + H^1(\mathbb{R}^2))$ with $U \in [U_d]$, we infer from Theorem 1.1 that

$$\frac{d}{dt}\mathscr{E}_{\varepsilon,[U_d]}(u) = \frac{d}{dt}\mathscr{E}_{\varepsilon,U}(u) = -k_\varepsilon \int_{\mathbb{R}^2} |\partial_t u|^2,$$

which means that the renormalized energy has the same dissipation rate as the Ginzburg–Landau energy in the finite energy case $d = 0$.

Statement of the main result. In the sequel, A_n denotes the annulus $B(2^{n+1}) \setminus B(2^n)$ for $n \in \mathbb{N}$, so that $\mathbb{R}^2 = B(2^{n_0}) \cup \left(\bigcup_{n\geq n_0} A_n\right)$.

Definition 1.2. Let $a_1, \dots, a_l$ be l distinct points in $\mathbb{R}^2$, $d_i \in \{-1, +1\}$ for $i = 1, \dots, l$ and set $d = \sum d_i$. Let $(u_\varepsilon)_{0<\varepsilon<1}$ be a family of maps in $[U_d] + H^1(\mathbb{R}^2)$. We say that $(u_\varepsilon)_{0<\varepsilon<1}$ is *well-prepared* with respect to the configuration (a_i, d_i) if there exist $R = 2^{n_0} > \max |a_i|$ and a constant $K_0 > 0$ such that, with $E_\varepsilon(u, B) \equiv \int_B e_\varepsilon(u)$, the following conditions are satisfied:

$$\lim_{\varepsilon\to 0} \Big\| Ju_\varepsilon - \pi \sum_{i=1}^{l} d_i \delta_{a_i} \Big\|_{W_0^{1,\infty}(B(R))^*} = 0, \tag{WP$_1$}$$

$$\sup_{0<\varepsilon<1} E_\varepsilon(u_\varepsilon, A_n) \leq K_0 \quad \text{for all } n \geq n_0, \tag{WP$_2$}$$

$$\lim_{\varepsilon\to 0} \left(\mathscr{E}_{\varepsilon,[U_d]}(u_\varepsilon) - \mathscr{E}_{\varepsilon,[U_d]}(u_\varepsilon^*(a_i, d_i))\right) = 0. \tag{WP$_3$}$$

Theorem 1.3. *Let $(u_\varepsilon^0)_{0<\varepsilon<1}$ in $[U_d]+H^1(\mathbb{R}^2)$ be a family of well-prepared initial data with respect to the configuration (a_i^0, d_i) with $d_i = \pm 1$, and let $(u_\varepsilon(t))_{0<\varepsilon<1}$ in $C(\mathbb{R}_+, [U_d]+H^1(\mathbb{R}^2))$ be the corresponding solution of (CGL)$_\varepsilon$. Let $\{a_i(t)\}_{\{i=1,\dots,l\}}$ denote the solution of the ordinary differential equation*

$$\begin{cases} \pi \dot{a}_i(t) = C_i(\delta d_i \mathbb{I}_2 - \mathbb{J}_2)\nabla_{a_i} W, \\ a_i(0) = a_i, \end{cases} \qquad i = 1, \dots, l, \tag{1-5}$$

where $C_i = -d_i/(1+\delta^2)$, $\mathbb{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\mathbb{J}_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and W is the Kirchhoff–Onsager functional defined by $W(a_i, d_i) = -\pi \sum_{i\neq j} d_i d_j \log |a_i - a_j|$. Denote by $[0, T^*)$ its maximal interval of existence. Then, for every $t \in [0, T^*)$, the family $(u_\varepsilon(t))_{0<\varepsilon<1}$ is well-prepared with respect to the configuration $(a_i(t), d_i)$.

2. Evolution formula for u_ε

We now recall or derive a number of evolution formulae involving quantities related to u_ε which we introduce now.

Notation. For $x=(x_1,x_2)\in\mathbb{R}^2$, we set $x^\perp=\mathbb{J}_2 x=(-x_2,x_1)$, or $x^\perp=ix$ under the standard identification of $\mathbb{R}^2$ with $\mathbb{C}$. For z and $z'\in\mathbb{C}$, we denote by $z\cdot z'=\mathrm{Re}(z\bar{z}')$ the scalar product and $z\times z'=z^\perp\cdot z'=-\mathrm{Im}(z\bar{z}')$ the exterior product of z and z' in $\mathbb{R}^2$. For $\vec{a}:\mathbb{R}^2\to\mathbb{R}^2$, we define $\mathrm{curl}(\vec{a})=\partial_1 a_2-\partial_2 a_1$. If $u:\mathbb{R}^2\to\mathbb{C}$, we denote by

$$j(u)=u\times\nabla u=iu\cdot\nabla u=u^\perp\cdot\nabla u$$

the linear momentum and by

$$J(u)=\partial_1 u\times\partial_2 u=\det(\nabla u)$$

the Jacobian of u. For $u\in H^1_{\mathrm{loc}}(\mathbb{R}^2)$, it can be checked that $J(u)=\frac{1}{2}\,\mathrm{curl}\,j(u)$ in the distribution sense. On the set where u does not vanish, we have for $k=1,2$

$$\partial_k u=\partial_k u\cdot\frac{u}{|u|}\frac{u}{|u|}+\partial_k u\cdot\frac{iu}{|u|}\frac{iu}{|u|}.$$

This yields

$$\partial_k u=\partial_k|u|\frac{u}{|u|}+\frac{j_k(u)}{|u|}\frac{u^\perp}{|u|};\tag{2-1}$$

hence

$$\partial_k u\cdot\partial_l u=\partial_k|u|\,\partial_l|u|+\frac{j_k(u)j_l(u)}{|u|^2},\tag{2-2}$$

and it follows that

$$|\nabla u|^2=\big|\nabla|u|\big|^2+\frac{|j(u)|^2}{|u|^2}.\tag{2-3}$$

The Hopf differential of u is defined as

$$\omega(u)=|\partial_1 u|^2-|\partial_2 u|^2-2i\partial_1 u\cdot\partial_2 u=4\partial_z u\overline{\partial_{\bar{z}} u}.$$

It follows from (2-2) that $\omega(u)$ may be rewritten in terms of the components of $\nabla|u|$ and $j(u)$ as

$$\omega(u)=\partial_1|u|^2-\partial_2|u|^2-2i\partial_1|u|\,\partial_2|u|+\frac{1}{|u|^2}\big(j_1^2(u)-j_2^2(u)-2ij_1(u)j_2(u)\big).\tag{2-4}$$

We recall that the Ginzburg–Landau energy density is defined by

$$e_\varepsilon(u)=\frac{|\nabla u|^2}{2}+\frac{(1-|u|^2)^2}{4\varepsilon^2}=\frac{|\nabla u|^2}{2}+V(u),$$

and we set

$$\mu_\varepsilon(u)=\frac{e_\varepsilon(u)}{|\log\varepsilon|}.$$

In view of (2-3), we then have

$$e_\varepsilon(u)=e_\varepsilon(|u|)+\frac{|j(u)|^2}{|u|^2}.\tag{2-5}$$

Finally, we write the right-hand side in (CGL)$_\varepsilon$ as

$$\nabla E(u) = \nabla E_\varepsilon(u) = \Delta u + \frac{1}{\varepsilon^2} u(1-|u|^2).$$

Evolution formulae involving the Jacobian and the energy density. For a smooth map u in space-time, direct computations by integration by parts yield for the energy

$$\frac{d}{dt}\int_{\mathbb{R}^2} e_\varepsilon(u)\varphi\,dx = -\int_{\mathbb{R}^2} \partial_t u\cdot\nabla E(u)\varphi\,dx - \int_{\mathbb{R}^2} \nabla\varphi\cdot(\partial_t u\cdot\nabla u)\,dx \tag{2-6}$$

and for the Jacobian

$$\frac{d}{dt}\int_{\mathbb{R}^2} J(u)\chi\,dx = -\int_{\mathbb{R}^2} \nabla^\perp\chi\cdot(\partial_t u^\perp\cdot\nabla u)\,dx, \tag{2-7}$$

where $\chi,\varphi\in\mathfrak{D}(\mathbb{R}^2)$.

At the same time, the Pohozaev identity (see [Bethuel et al. 2005], for example) yields, for any vector field $\vec{X}\in\mathfrak{D}(\mathbb{R}^2,\mathbb{C})$,

$$\int_{\mathbb{R}^2} \vec{X}\cdot(\nabla E(u)\cdot\nabla u)\,dx = -\int_{\mathbb{R}^2} \mathrm{Re}\Big(\omega(u)\frac{\partial\vec{X}}{\partial\bar{z}}\Big)\,dz + \int_{\mathbb{R}^2} V(u)\nabla\cdot\vec{X}\,dx.$$

In particular, the choice of $\vec{X}=\nabla\varphi$, for which $\partial_{\bar{z}}X = 2\dfrac{\partial^2\varphi}{\partial\bar{z}^2}$, or $\vec{X}=\nabla^\perp\chi$, for which $\partial_{\bar{z}}X = 2i\dfrac{\partial^2\chi}{\partial\bar{z}^2}$, leads to

$$\int_{\mathbb{R}^2} \nabla\varphi\cdot(\nabla E(u)\cdot\nabla u)\,dx = -2\int_{\mathbb{R}^2} \mathrm{Re}\Big(\omega(u)\frac{\partial^2\varphi}{\partial\bar{z}^2}\Big)\,dz + \int_{\mathbb{R}^2} V(u)\Delta\varphi\,dx$$

and

$$\int_{\mathbb{R}^2} \nabla^\perp\chi\cdot(\nabla E(u)\cdot\nabla u)\,dx = 2\int_{\mathbb{R}^2} \mathrm{Im}\Big(\omega(u)\frac{\partial^2\chi}{\partial\bar{z}^2}\Big)\,dz. \tag{2-8}$$

We next consider a solution u of (CGL)$_\varepsilon$, which is smooth in view of Theorem 1.1. In this case, $\nabla E(u)$ and $\partial_t u$ are related by

$$\partial_t u = \frac{1}{\alpha_\varepsilon}\nabla E(u) = \beta_\varepsilon\nabla E(u), \tag{2-9}$$

where $\alpha_\varepsilon = \dfrac{\delta}{|\log\varepsilon|} + i = k_\varepsilon + i$. Using (2-9) in (2-6) and (2-7), we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^2} e_\varepsilon(u)\varphi\,dx = -k_\varepsilon\int_{\mathbb{R}^2} |\partial_t u|^2\varphi\,dx - \int_{\mathbb{R}^2}\nabla\varphi\cdot(\beta_\varepsilon\nabla E(u)\cdot\nabla u)\,dx,$$

$$\frac{d}{dt}\int_{\mathbb{R}^2} J(u)\chi\,dx = -\int_{\mathbb{R}^2}\nabla^\perp\chi\cdot(i\beta_\varepsilon\nabla E(u)\cdot\nabla u)\,dx.$$

To get rid of the terms of the form $\displaystyle\int_{\mathbb{R}^2}\vec{X}\cdot(i\nabla E(u)\cdot\nabla u)$, we compute

$$\frac{d}{dt}\int_{\mathbb{R}^2}[bJ(u)\chi - a e_\varepsilon(u)\varphi],$$

where $\beta_\varepsilon = a + ib$. This yields

$$\frac{d}{dt}\int_{\mathbb{R}^2}[bJ(u)\chi - ae_\varepsilon(u)]\varphi$$
$$= (b^2+a^2)\int_{\mathbb{R}^2}\nabla^\perp\chi\cdot(\nabla E\cdot\nabla u) + ak_\varepsilon\int_{\mathbb{R}^2}|\partial_t u|^2\,dx + \int_{\mathbb{R}^2}(\nabla\varphi-\nabla^\perp\chi)\cdot(a(a+ib)\nabla E\cdot\nabla u). \quad (2\text{-}10)$$

Since $a = k_\varepsilon/(k_\varepsilon^2+1)$ and $b = -1/(k_\varepsilon^2+1)$, we finally infer from this relation and (2-8) the following:

Proposition 2.1. *Let u solve (CGL)$_\varepsilon$. Then for all $\varphi, \chi \in \mathfrak{D}(\mathbb{R}^2)$,*

$$\frac{d}{dt}\int_{\mathbb{R}^2}[J(u)\chi + k_\varepsilon e_\varepsilon(u)\varphi] = -k_\varepsilon^2\int_{\mathbb{R}^2}|\partial_t u|^2\varphi - 2\int_{\mathbb{R}^2}\operatorname{Im}\left(\omega(u)\frac{\partial^2\chi}{\partial\bar{z}^2}\right) + R_\varepsilon(t,\varphi,\chi,u),$$

where the remainder R_ε is defined by either of the equivalent relations

$$R_\varepsilon(t,\varphi,\chi,u) = -k_\varepsilon\int_{\mathbb{R}^2}(\nabla\varphi-\nabla^\perp\chi)\cdot(\beta_\varepsilon\nabla E(u)\cdot\nabla u),$$
$$R_\varepsilon(t,\varphi,\chi,u) = -k_\varepsilon\int_{\mathbb{R}^2}(\nabla\varphi-\nabla^\perp\chi)\cdot(\partial_t u\cdot\nabla u).$$

Proposition 2.1 allows us to derive formally the motion law for the vortices. Indeed, assume that we have

$$Ju_\varepsilon(t) \to \pi\sum_{i=1}^{l} d_i\delta_{a_i(t)}, \qquad \mu_\varepsilon(u_\varepsilon)(t) \to \pi\sum_{i=1}^{l}\delta_{a_i(t)},$$

and $u_\varepsilon(t)$ is close in some sense to $u_\varepsilon^*(a_i(t), d_i)$ and therefore to $u^*(a_i(t), d_i)$, where

$$u^*(a_i, d_i) = \prod_{i=1}^{l}\left(\frac{z-a_i}{|z-a_i|}\right)^{d_i}.$$

We use Proposition 2.1 with u formally replaced by $u^*(a_i(t), d_i)$ and with choices of test functions φ and χ which are localized and affine near each point $a_i(t)$ and satisfy $\nabla\varphi = \nabla^\perp\chi$ there, so that both terms $k_\varepsilon^2\int_{\mathbb{R}^2}|\partial_t u|^2\varphi$ and $R_\varepsilon(t,\varphi,\chi,u_\varepsilon)$ vanish in the limit $\varepsilon\to 0$. Using the formula

$$2\int_{\mathbb{R}^2}\operatorname{Im}\left(\omega\big(u^*(a_i,d_i)\big)\frac{\partial^2\chi}{\partial\bar{z}^2}\right) = 2\pi\sum_{j\neq i} d_i d_j\frac{(a_i-a_j)^\perp}{|a_i-a_j|^2}\cdot\nabla\chi(a_i)$$

from [Bethuel et al. 2005, (7.2)], we then obtain that for each i

$$\pi d_i\dot{a}_i(t)\cdot\nabla\chi(a_i) + \delta\pi\dot{a}_i(t)\cdot\nabla\varphi(a_i) = -2\pi\sum_{j:j\neq i} d_i d_j\frac{(a_i-a_j)^\perp}{|a_i-a_j|^2}\cdot\nabla\chi(a_i).$$

Taking into account that $\nabla\varphi(a_i) = \nabla^\perp\chi(a_i)$, we infer that

$$\pi\left(d_i\dot{a}_i(t) - \delta\dot{a}_i^\perp(t)\right)\cdot\nabla\chi(a_i) = -2\pi\sum_{j\neq i} d_i d_j\frac{(a_i-a_j)^\perp}{|a_i-a_j|^2}\cdot\nabla\chi(a_i),$$

which yields the ODE (1-5).

In Sections 4 and 5, in order to give a rigorous meaning to the previous computations, we will prove the convergence of the Jacobians and of the energy densities to the weighted sums of Dirac masses mentioned above, and then show that both the energy dissipation $k_\varepsilon^2 \int_{\mathbb{R}^2} |\partial_t u_\varepsilon|^2$ and the remainder $R_\varepsilon(t, \varphi, \chi, u_\varepsilon)$ vanish when ε tends to zero. In Section 6, we will establish some asymptotic control of $\omega(u_\varepsilon) - \omega(u^*(a_i), d_i)$ away from the vortices in terms of the excess energy $\mathscr{E}_{\varepsilon,[U_d]}(u_\varepsilon) - \mathscr{E}_{\varepsilon,[U_d]}(u_\varepsilon^*(a_i))$, and finally prove that this excess energy converges to zero by mean of a Grönwall inequality.

3. Some results on the renormalized energy

Degree and energy at infinity. In this paragraph, we collect some results from [Bethuel et al. 2008] related to the energy at infinity, which require the notion of degree at infinity.

Let A be the annulus $B(2) \setminus B(1)$. We define

$$T_d = \left\{u \in H^1(A) : \text{some } B \subset B(u) \text{ satisfies } |B| \geq \tfrac{3}{4} \text{ and } \deg(u, \partial B(r)) = d \text{ for all } r \in B\right\},$$

where $B(u)$ is the set of radii $r \in [1, 2]$ such that the restriction $u_{|\partial B(r)}$ is continuous and does not vanish, and we define the sublevel sets

$$E_\varepsilon^\Lambda = \{u \in H^1(A) : E_\varepsilon(u, A) < \Lambda\}.$$

The topological sector of degree d is then defined as

$$S_{d,\varepsilon}^\Lambda = E_\varepsilon^\Lambda \cap T_d.$$

Theorem 3.1 [Almeida 1999]. *For all* $\Lambda > 0$, *there exists* $\varepsilon_\Lambda > 0$ *such that for every* $0 < \varepsilon < \varepsilon_\Lambda$, *we have*

$$E_\varepsilon^\Lambda = \bigcup_{d \in \mathbb{Z}} S_{d,\varepsilon}^\Lambda.$$

The map $\deg : E_\varepsilon^\Lambda \to \mathbb{Z}$, $u \in S_{d,\varepsilon}^\Lambda \mapsto d$ *is continuous.*

For the rest of this section, we fix $\Lambda > \Lambda_d = 2\pi d^2 \log 2$ and we set

$$S_d \equiv S_{d,\varepsilon_\Lambda}^\Lambda,$$

so in particular the map U_d belongs to S_d, since $|U_d| \equiv 1$ on A and $\int_A \frac{1}{2}|\nabla U_d|^2 = \pi d^2 \log 2$.

One easily infers from Theorem 3.1 that if $u \in [U_d] + H^1(\mathbb{R}^2)$, then for any sufficiently large k the map $u(2^k \cdot)$ belongs to some $S_{d(k)}$. In fact, one can find a radius from which $d(k) \equiv d$, that is, u has well defined and constant degree d at infinity.

Proposition 3.2 [Bethuel et al. 2008]. *Let* $d \in \mathbb{Z}$, $\Lambda > \Lambda_d$ *and* $u \in [U_d] + H^1(\mathbb{R}^2)$. *There exists an integer* $n \in \mathbb{N}^*$ *such that for any* $k \geq n$, *the map* $u_k : z \in A \mapsto u(2^k z)$ *belongs to the topological sector* S_d. *We denote by* $n(u)$ *the smallest integer with this property.*

For maps $u \in [U_d] + H^1(\mathbb{R}^2)$ satisfying in addition a uniform energy bound on large annuli one can characterize $n(u)$ as follows (see, for example, the proof of Lemma 7.1 in [Bethuel et al. 2008]).

Lemma 3.3. *Let* $\Lambda > \Lambda_d$ *be given and* $0 < \varepsilon < \varepsilon_\Lambda$. *Let* $u \in [U_d] + H^1(\mathbb{R}^2)$ *and assume that there exists some* $n_0 \in \mathbb{N}^*$ *such that* $E_\varepsilon(u, A_n) < \Lambda$ *for all* $n \geq n_0$. *Then* $n(u) \leq n_0$.

The next lemma provides a lower bound for the energy on large annuli.

Lemma 3.4 [Bethuel et al. 2008]. *Let $d \in \mathbb{Z}$ and $u \in [U_d] + H^1(\mathbb{R}^2)$. Then, for any $k \geq n(u)$, we have for $0 < \varepsilon < \varepsilon_\Lambda$*

$$\int_{A_k} \bigl(e_\varepsilon(u) - \tfrac{1}{2}|\nabla U_d|^2\bigr) \geq -C2^{-2k}\varepsilon^2.$$

One can then derive from Lemma 3.4 an upper bound for $E_\varepsilon(u, B) - E_\varepsilon(u^*_\varepsilon, B)$ on large balls B in terms of the excess energy $\mathscr{E}_{\varepsilon,[U_d]}(u) - \mathscr{E}_{\varepsilon,[U_d]}(u^*_\varepsilon)$. We will therefore be able to rely on properties of the Ginzburg–Landau energy on bounded domains in the course of the proof of Theorem 1.3.

Lemma 3.5 [Bethuel et al. 2008]. *Let $d \in \mathbb{Z}$, $u \in [U_d] + H^1(\mathbb{R}^2)$, $a_1, \ldots, a_l \in \mathbb{R}^2$ and $d_1, \ldots, d_l \in \mathbb{Z}^*$ such that $d = \sum d_i$. Let $k \geq 1 + \max\{\log_2 |a_1|, \ldots, \log_2 |a_l|, n(u)\}$ and $R = 2^k$. Then, we have*

$$\int_{B(R)} \bigl[e_\varepsilon(u) - e_\varepsilon(u^*_\varepsilon(a_i, d_i))\bigr] \leq \mathscr{E}_{\varepsilon,[U_d]}(u) - \mathscr{E}_{\varepsilon,[U_d]}(u^*_\varepsilon(a_i, d_i)) + \frac{C}{R},$$

where C depends only on l and d.

Explicit identities for the reference map u^*_ε. We present here an account of some classical identities for the energy of u^*_ε, borrowed from [Bethuel et al. 2008].

We consider a configuration (a_i, d_i) with $d_i \in \mathbb{Z}^*$ and we set $d = \sum d_i$. We begin with an explicit expansion near each vortex a_j.

Lemma 3.6. *For $j \in \{1, \ldots, l\}$ and $0 < \varepsilon < 1$,*

$$\int_{B(a_j,r)} e_\varepsilon(u^*_\varepsilon(a_i, d_i)) = \pi d_j^2 \log\Bigl(\frac{r}{\varepsilon}\Bigr) + \gamma(|d_j|) + O\Bigl(\frac{r}{r_a}\Bigr)^2 + O\Bigl(\frac{\varepsilon}{r}\Bigr)^2,$$

where $\gamma(|d_j|)$ is some universal constant.

On the other hand, $u^*_\varepsilon(a_i, d_i)$ behaves as $u^*(a_i, d_i)$ away from the vortices, so its energy on $\Omega_{R,r} = B(R) \setminus \bigcup_{j=1}^l B(a_j, r)$ is close to the energy of $u^*(a_i, d_i)$ on $\Omega_{R,r}$ which we can compute explicitly [Bethuel et al. 1994]. Combining the previous expansions, we obtain:

Proposition 3.7. *Let*

$$r_a = \tfrac{1}{8} \min_{i \neq j}\{|a_i - a_j|\}, \qquad R_a = \max\{|a_i|\}.$$

Then for $R > R_a + 1$, we have as $\varepsilon \to 0$

$$\int_{B(R)} e_\varepsilon(u^*_\varepsilon(a_i, d_i)) = \pi \sum_{i=1}^{l} d_i^2 |\log \varepsilon| + W(a_i, d_i) + \sum_{i=1}^{l} \gamma(|d_i|) + \pi d^2 \log R + O\Bigl(\frac{R_a}{R}\Bigr) + o_\varepsilon(1).$$

Observe that $\pi d^2 \log R = \int_{B(R)\setminus B(1)} \frac{1}{2}|\nabla U_d|^2$. This yields an expansion for the renormalized energy:

Corollary 3.8. *When $\varepsilon \to 0$,*

$$\mathscr{E}_{\varepsilon,[U_d]}(u^*_\varepsilon(a_i, d_i)) = \pi \sum_{i=1}^{l} d_i^2 |\log \varepsilon| + W(a_i, d_i) + \sum_{i=1}^{l} \gamma(|d_i|) - \int_{B(1)} \frac{|\nabla U_d|^2}{2} + o_\varepsilon(1).$$

Concerning the energy on annuli, we finally quote the following:

Lemma 3.9. *For $R > R_a$, we have*

$$\int_{B(2R)\setminus B(R)} e_\varepsilon(u^*_\varepsilon(a_i,d_i)) = \pi d^2 \log 2 + O\Big(\frac{R_a}{R}\Big)$$

or, in view of the properties of U_d at infinity,

$$\int_{B(2R)\setminus B(R)} e_\varepsilon(u^*_\varepsilon(a_i,d_i)) = \int_{B(2R)\setminus B(R)} \frac{|\nabla U_d|^2}{2} + O\Big(\frac{R_a}{R}\Big).$$

4. Coercivity for the renormalized energy

In this section, we supplement some results from [Bethuel et al. 2008] and [Jerrard and Spirn 2007] with estimates to be used later. These results establish precise estimates in various norms for maps u being close to $u^*_\varepsilon(a_i,d_i)$ in terms of the excess energy with respect to the configuration (a_i,d_i). For a map $u \in [U_d]+H^1(\mathbb{R}^2)$ and a given configuration (a_i,d_i) with $d_i=\pm 1$, we define this excess energy Σ_ε as

$$\Sigma_\varepsilon = \Sigma_\varepsilon(a_i,d_i) = \mathscr{E}_{\varepsilon,[U_d]}(u) - \mathscr{E}_{\varepsilon,[U_d]}(u^*_\varepsilon(a_i,d_i)).$$

We also set

$$r_a = \tfrac{1}{8}\min_{i\neq j}\{|a_i-a_j|\}, \qquad R_a = \max_{i=1,\dots,l}\{|a_i|\}.$$

Theorem 4.1. *Let $r \le r_a$ and let $2^{n_0} = R_0 > R_a$ be such that $\bigcup_{i=1}^l B(a_i,r) \subset B(R_0)$. There exist ε_0 and η_0, depending only on l, r, r_a, R_a, and R_0, such that for all $u \in [U_d]+H^1(\mathbb{R}^2)$ satisfying*

$$\eta = \|Ju - \pi\sum_{i=1}^l d_i\delta_{a_i}\|_{W_0^{1,\infty}(B(R_0))^*} \le \eta_0 \quad \textit{and} \quad 2^{n(u)} \le R_0, \tag{4-1}$$

we have

$$\int_{B(R_0)\setminus\cup B(a_i,r)} e_\varepsilon(|u|) + \frac{1}{8}\left|\frac{j(u)}{|u|} - j(u^*(a_i,d_i))\right|^2 \le \Sigma_\varepsilon + C\Big(\eta,\varepsilon,\frac{1}{R_0}\Big) \quad \textit{for } \varepsilon \le \varepsilon_0, \tag{4-2}$$

where C is a continuous function on $\mathbb{R}^3$ that vanishes at the origin. Furthermore, there exist points $b_i \in B(a_i, r/2)$ such that, for some continuous functions f on $\mathbb{R}^2$ and g on $\mathbb{R}^4$, we have

$$\Big\|Ju - \pi\sum_{i=1}^l d_i\delta_{b_i}\Big\|_{W_0^{1,\infty}(B(R_0))^*} \le f(R_0,\Sigma_\varepsilon)\varepsilon|\log\varepsilon|, \tag{4-3}$$

$$\Big\|\mu_\varepsilon(u) - \pi\sum_{i=1}^l \delta_{b_i}\Big\|_{W_0^{1,\infty}(B(R_0))^*} \le \frac{g(R_0,r,r_a,\Sigma_\varepsilon)}{|\log\varepsilon|}. \tag{4-4}$$

Proof. Except for the energy concentration (4-4), each of the statements is proved in [Bethuel et al. 2008, Theorem 6.1]. We first infer from (4-1) that

$$\|Ju - \pi d_i\delta_{a_i}\|_{W_0^{1,\infty}(B(a_i,r))^*} \le \eta_0 \quad \text{for all } i.$$

If η_0 is small enough with respect to r this gives in view of [Jerrard and Spirn 2007, Theorem 3] that $K_0^i \ge C(r)$, where K_0^i is the local excess energy near the vortex i defined by $K_0^i = \int_{B(a_i,r)} e_\varepsilon(u) - \pi\log\big(r/\varepsilon\big)$.

It follows that

$$\int_{B(a_i,r)} e_\varepsilon(u) \le \int_{B(R_0)} e_\varepsilon(u) - \pi(l-1)|\log\varepsilon| - C(r).$$

At the same time, since $n(u) \le n_0$, we have according to Lemma 3.5 and Proposition 3.7

$$\int_{B(R_0)} e_\varepsilon(u) \le \int_{B(R_0)} e_\varepsilon(u_\varepsilon^*(a_i,d_i)) + \Sigma_\varepsilon + \frac{C}{R_0} \le \pi l|\log\varepsilon| + \Sigma_\varepsilon + C.$$

This first implies that $K_0^i \le C + \Sigma_\varepsilon$. Also, replacing r by $3r/4$ we see that

$$\int_{B(R_0)\setminus\cup B(a_i,3r/4)} \mu_\varepsilon(u) \le \frac{C+\Sigma_\varepsilon}{|\log\varepsilon|},$$

where C only depends on R_0, r, r_a, R_a.

Now, according to [Jerrard and Spirn 2007, Theorem 2′], the energy density $\mu_\varepsilon(u)$ on $B(a_i,r)$ is concentrated at the point $b_i \in B(a_i,r/2)$ where $J(u)$ concentrates. From [Colliander and Jerrard 1999, Theorem 3.2.1] and the estimate for K_0^i it follows that

$$\|\mu_\varepsilon(u) - \pi\delta_{b_i}\|_{W_0^{1,\infty}(B(a_i,r))^*} \le \frac{f(\Sigma_\varepsilon, C)}{|\log\varepsilon|}.$$

Combining this and the upper bound for the energy density outside the vortex balls yields (4-4). □

5. Convergence to Lipschitz vortex paths

In this section, we establish compactness for the Jacobians and the energy densities in a more general situation, replacing assumption (WP$_3$) in Theorem 1.3 by a uniform bound on the initial excess energy.

Theorem 5.1. *Let (a_i^0, d_i) with $d_i = \pm 1$ be a configuration of vortices. Let $R = 2^{n_0}$ and $(u_\varepsilon^0)_{0<\varepsilon<1}$ in $[U_d] + H^1(\mathbb{R}^2)$ such that*

$$\lim_{\varepsilon\to 0}\Big\| Ju_\varepsilon^0 - \pi\sum_{i=1}^{l} d_i\delta_{a_i^0}\Big\|_{W_0^{1,\infty}(B(R))^*} = 0, \tag{WP$_1$}$$

$$\sup_{0<\varepsilon<1} E_\varepsilon(u_\varepsilon^0, A_n) \le K_0 \quad \textit{for all } n \ge n_0, \tag{WP$_2$}$$

$$\sup_{0<\varepsilon<1} \big(\mathscr{E}_{\varepsilon,[U_d]}(u_\varepsilon^0) - \mathscr{E}_{\varepsilon,[U_d]}(u_\varepsilon^*(a_i^0,d_i))\big) \le K_1. \tag{WP$_{3'}$}$$

Then there exist $R' = 2^{n_1}$ and $T > 0$ depending only on K_1, R, r_{a^0} and R_{a^0}, a sequence $\varepsilon_k \to 0$ and l Lipschitz paths $b_i : [0,T] \to \mathbb{R}^2$ starting from a_i^0 such that

$$\sup_{t\in[0,T]}\Big\| Ju_{\varepsilon_k}(t) - \pi\sum_{i=1}^{l} d_i\delta_{b_i(t)}\Big\|_{W_0^{1,\infty}(B(R'))^*} \to 0, \qquad k\to+\infty, \tag{5-1}$$

$$\sup_{t\in[0,T]}\Big\| \mu_{\varepsilon_k}(u_{\varepsilon_k})(t) - \pi\sum_{i=1}^{l} \delta_{b_i(t)}\Big\|_{W^{1,\infty}(B(R'))^*} \to 0, \qquad k\to+\infty. \tag{5-2}$$

Moreover, there exist a constant $C_0 > 0$ depending only on r_{a^0}, R, K_1 and K_0 and a constant $C_1 > 0$ depending on r_{a^0}, R and K_1 such that for all $t \in [0, T]$ and $k \in \mathbb{N}$, we have

$$E_{\varepsilon_k}(u_{\varepsilon_k}(t), A_n) \leq C_0 \quad \textit{for all } n \geq n_1, \tag{5-3}$$

$$\mathscr{E}_{\varepsilon_k,[U_d]}(u_{\varepsilon_k}(t)) - \mathscr{E}_{\varepsilon_k,[U_d]}(u^*_{\varepsilon_k}(b_i(t), d_i)) \leq C_1. \tag{5-4}$$

Proof. The proof is very similar to that of [Bethuel et al. 2008, Theorem 4]. In the sequel, C will be a constant depending only on r_{a^0}, R, R_{a^0}, and K_1. To simplify the notations further we will set $r_a = r_{a^0}$ and $R_a = R_{a^0}$.

We first consider $\Lambda > \max(K_0, \Lambda_d)$. Thanks to Lemma 3.3 and (WP$_2$), there exists $\varepsilon_\Lambda > 0$ such that $n(u^0_\varepsilon) \leq n_0$ for all $0 < \varepsilon < \varepsilon_\Lambda$. We fix such a Λ and from now on only consider $0 < \varepsilon < \varepsilon_\Lambda$.

We next introduce the smallest integer $n_1 \geq n_0$ such that $2^{n_1} \geq \max(R, R_a + r_a)$ and define $R' = 2^{n_1}$. In the remainder of the proof, we write $\|\cdot\|$ instead of $\|\cdot\|_{W_0^{1,\infty}(B(R'))^*}$. Our aim is to apply Theorem 4.1 to each $u_\varepsilon(t)$ for the choice $r = r_a$ and $R_0 = R'$. Let η_0 and ε_0 be the constants provided by Theorem 4.1 for this choice. First, thanks to (WP$_2$) and (WP$_{3'}$), the convergence in (WP$_1$) still holds on the larger ball $B(R')$ (see the proof of Lemma 7.3 in [Bethuel et al. 2008]). Therefore, since $t \mapsto Ju_\varepsilon(t) \in L^1(B(R'))$ is continuous for each ε, there exists a time $T_\varepsilon > 0$ such that

$$\Big\| Ju_\varepsilon(s) - \pi \sum_{i=1}^{l} d_i \delta_{a_i^0} \Big\| < \eta_0, \qquad \forall s \in [0, T_\varepsilon).$$

We take T_ε as the maximum time smaller than T^* having this property, where T^* is as in Theorem 1.3.

Meanwhile, since $t \mapsto E_\varepsilon(u_\varepsilon(t), A_n)$ is uniformly continuous with respect to n and $\Lambda > K_0$, we infer from (WP$_2$) that there exists $T'_\varepsilon > 0$ such that for $s \in [0, T'_\varepsilon]$

$$E_\varepsilon(u_\varepsilon(s), A_n) < \Lambda \quad \text{for all } n \geq n_1,$$

so according to Lemma 3.3 we have $n(u_\varepsilon(s)) \leq n_1$ for $s \in [0, T'_\varepsilon]$. We take $T'_\varepsilon \leq T^*$ maximal with this property.

We claim that there exists a constant D depending on K_1, r_a, R, and K_0 such that for all $s \in [0, \min(T_\varepsilon, T'_\varepsilon))$,

$$E_\varepsilon(u_\varepsilon(s), A_n) \leq D \quad \text{for all } n \geq n_1. \tag{5-5}$$

Consequently, if we assume from the beginning that $\Lambda > \max(K_0, \Lambda_d, D)$, then $T'_\varepsilon \geq T_\varepsilon$, and it follows from Lemma 3.3 that $n(u_\varepsilon(s)) \leq n_1$ on $[0, T_\varepsilon]$.

Proof of (5-5). As in [Bethuel et al. 2008], we decompose each $E_\varepsilon(u_\varepsilon(s), A_n) - E_\varepsilon(u^*_\varepsilon(a_i^0, d_i), A_n)$, for $n \geq n_1$, as

$$\sum_{\substack{k=n_1 \\ k \neq n}}^{+\infty} \big(E_\varepsilon(u^*_\varepsilon(a_i^0, d_i), A_k) - E_\varepsilon(u_\varepsilon(t), A_k)\big) \\ + E_\varepsilon(u^*_\varepsilon(a_i^0, d_i), B(R')) - E_\varepsilon(u_\varepsilon(s), B(R')) + \mathscr{E}_{\varepsilon,[U_d]}(u_\varepsilon(s)) - \mathscr{E}_{\varepsilon,[U_d]}(u^*_\varepsilon(a_i^0, d_i)).$$

We first handle each term of the sum in the right. In view of Lemmas 3.4 and 3.9, we have for $k \geq n_1$

$$E_\varepsilon(u_\varepsilon(s), A_k) \geq -C\varepsilon^2 2^{-2k} + \int_{A_k} \frac{|\nabla U_d|^2}{2} \geq E_\varepsilon(u^*_\varepsilon(a_i^0, d_i), A_k) - C(R_a)2^{-k} - C\varepsilon^2 2^{-2k},$$

so we deduce that

$$\sum_{\substack{k=n_1\\ k\neq n}}^{+\infty} \big(E_\varepsilon(u_\varepsilon^*(a_i^0,d_i),A_k)-E_\varepsilon(u_\varepsilon(s),A_k)\big)\le C.$$

Next, we infer from the definition of T_ε and [Jerrard and Spirn 2007, Theorem 3] that

$$\int_{B(a_i^0,r_a)} e_\varepsilon(u_\varepsilon(s))\ge \pi|\log\varepsilon|-C.$$

Observe that R' is chosen so that $\cup B(a_i^0,r_a)\subset B(R')$, so this leads to

$$E_\varepsilon(u_\varepsilon(s),B(R'))\ge \pi l|\log\varepsilon|-C.$$

Using Proposition 3.7, we thus find

$$E_\varepsilon(u_\varepsilon^*(a_i^0,d_i),B(R'))-E_\varepsilon(u_\varepsilon(s),B(R'))\le C. \tag{5-6}$$

Finally, we define $\Sigma_\varepsilon^0(s):=\mathscr{E}_{\varepsilon,[U_d]}(u_\varepsilon(s))-\mathscr{E}_{\varepsilon,[U_d]}(u_\varepsilon^*(a_i^0,d_i))$. Since $\mathscr{E}_{\varepsilon,[U_d]}(u_\varepsilon(t))$ is nonincreasing, ($\mathrm{WP}_{3'}$) yields $\Sigma_\varepsilon^0(s)\le\mathscr{E}_{\varepsilon,[U_d]}(u_\varepsilon^0)-\mathscr{E}_{\varepsilon,[U_d]}(u_\varepsilon^*(a_i^0,d_i))\le K_1$, and (5-5) follows. □

We can now apply Theorem 4.1 to each $u_\varepsilon(t)$ on $[0,T_\varepsilon]$. This provides points $b_i^\varepsilon(s)\in B\big(a_i^0,r_a/2\big)$ for $0\le s\le T_\varepsilon$. Since $\Sigma_\varepsilon^0(s)\le K_1$, the estimate (4-2) turns into

$$\int_{\Omega_{R',r_a}} e_\varepsilon(|u_\varepsilon(s)|)+\frac18\left|\frac{j(u_\varepsilon(s))}{|u_\varepsilon(s)|}-j(u^*(a_i^0,d_i))\right|^2\le C,$$

where $\Omega_{R',r_a}=B(R')\setminus\bigcup B(a_i^0,r_a)$. Also, we have by (2-4) and (2-5)

$$\int_{\Omega_{R',r_a}} e_\varepsilon(u_\varepsilon(s))\le C \tag{5-7}$$

and

$$\|\omega(u_\varepsilon(s))\|_{L^1(\Omega_{R',r_a})}\le C, \tag{5-8}$$

where $C=C(R,r_a,K_1)$. For convenience, we will now write μ_ε instead of $\mu_\varepsilon(u_\varepsilon)$.

Given any configuration (a_i,d_i), we denote by $\mathscr{H}(a_i)$ the set of functions $\chi,\varphi\in\mathfrak{D}(\mathbb{R}^2)$ such that

$$\chi=\sum_{i=1}^{l}\chi_i,\qquad \varphi=\sum_{i=1}^{l}\varphi_i,$$

where for all i

$$\chi_i,\varphi_i\in\mathfrak{D}\Big(B\Big(a_i,\frac{3r_a}{2}\Big)\Big),\qquad \nabla\varphi_i=\nabla^\perp\chi_i \text{ on } B(a_i,r_a),$$

and χ_i (hence φ_i) is affine on $B(a_i,r_a)$ with $|\nabla\chi_i(a_i)|=|\nabla\varphi_i(a_i)|\le 1$.

By definition of r_a such functions χ and φ always exist, and we can moreover estimate their L^∞ norms by

$$\|D\varphi\|_\infty,\|D\chi\|_\infty\le\frac{C}{r_a},\qquad \|D^2\varphi\|_\infty,\|D^2\chi\|_\infty\le\frac{C}{r_a^2}.$$

We next control the remainder terms appearing in Proposition 2.1.

Lemma 5.2. *There exists a constant $C = C(r_a, R, K_1, T^*)$ such that*

$$\int_0^{T_\varepsilon}\int_{\mathbb{R}^2} \frac{|\partial_t u_\varepsilon|^2}{|\log\varepsilon|^2}\,ds \le \frac{C}{|\log\varepsilon|}$$

and for all $\chi, \varphi \in \mathcal{H}(a_i^0)$

$$\left|\int_0^{T_\varepsilon}\int_{\mathbb{R}^2} (\nabla^\perp\chi - \nabla\varphi)\cdot\frac{\partial_t u_\varepsilon\cdot\nabla u_\varepsilon}{|\log\varepsilon|}\,ds\right| \le \frac{C}{|\log\varepsilon|^{1/2}}.$$

Proof. To prove the first inequality, we use Theorem 1.1 and obtain

$$\frac{\delta}{|\log\varepsilon|}\int_0^{T_\varepsilon}\int_{\mathbb{R}^2}|\partial_t u_\varepsilon|^2 = \mathcal{E}_{\varepsilon,[U_d]}(u_\varepsilon^0) - \mathcal{E}_{\varepsilon,[U_d]}(u_\varepsilon(T_\varepsilon)) \le K_1 + \mathcal{E}_{\varepsilon,[U_d]}(u_\varepsilon^*(a_i^0, d_i)) - \mathcal{E}_{\varepsilon,[U_d]}(u_\varepsilon(T_\varepsilon)).$$

Since $n(u_\varepsilon(T_\varepsilon)) \le n_1$ we have by Lemma 3.5

$$\mathcal{E}_{\varepsilon,[U_d]}(u_\varepsilon^*(a_i^0, d_i)) - \mathcal{E}_{\varepsilon,[U_d]}(u_\varepsilon(T_\varepsilon)) \le \int_{B(R')} e_\varepsilon(u_\varepsilon^*(a_i^0, d_i)) - \int_{B(R')} e_\varepsilon(u_\varepsilon(T_\varepsilon)) + \frac{C}{R'},$$

which is bounded in view of (5-6). It then suffices to divide all terms by $|\log\varepsilon|$.

For the second assertion, we set $\xi = \nabla^\perp\chi - \nabla\varphi$, which has compact support in $A = \bigcup A_i$, where $A_i = B(a_i^0, 3r_a/2)\setminus B(a_i^0, r_a)$, and we apply the Cauchy–Schwarz inequality. We obtain

$$\left(\int_0^{T_\varepsilon}\int_{\mathbb{R}^2}(\nabla^\perp\chi - \nabla\varphi)\cdot\frac{\partial_t u_\varepsilon\cdot\nabla u_\varepsilon}{|\log\varepsilon|}\right)^2 \le \left(\int_0^{T_\varepsilon}\int_{\mathbb{R}^2}\frac{|\partial_t u_\varepsilon|^2}{|\log\varepsilon|^2}\right)\cdot\left(\int_0^{T_\varepsilon}\int_A |\nabla u_\varepsilon|^2|\xi|^2\right).$$

Since $A \subset \Omega_{R', r_a}$ and $\sup_{0<\varepsilon<1} T_\varepsilon \le T^*$, we infer from (5-7) that

$$\int_0^{T_\varepsilon}\int_A |\nabla u_\varepsilon|^2|\xi|^2 \le \|\xi\|_\infty^2\int_0^{T_\varepsilon}\int_A|\nabla u_\varepsilon|^2 \le CT^*\|\xi\|_\infty^2,$$

and the conclusion finally follows from the first part of the proof. □

Lemma 5.3. *There exists $T = T(r_a, R_a, R, K_1) > 0$ such that*

$$\liminf_{\varepsilon\to 0} T_\varepsilon \ge T.$$

Proof. We first show that for $(\chi, \varphi) \in \mathcal{H}(a_i^0)$, for $s, t \in [0, T_\varepsilon]$ and $i = 1, \dots, l$ we have

$$\big|\langle\chi_i, Ju_\varepsilon(t) - Ju_\varepsilon(s)\rangle + \delta\langle\varphi_i, \mu_\varepsilon(t) - \mu_\varepsilon(s)\rangle\big| \le C|t-s| + \frac{C}{|\log\varepsilon|^{1/2}}. \tag{5-9}$$

Indeed, we fix i and we invoke Proposition 2.1 for $u \equiv u_\varepsilon$ and the choice of test functions (χ_i, φ_i). Integrating the formula in that proposition over $[s, t]$ yields

$$\begin{aligned}&\big|\langle\chi_i, Ju_\varepsilon(t) - Ju_\varepsilon(s)\rangle + \delta\langle\varphi_i, \mu_\varepsilon(t) - \mu_\varepsilon(s)\rangle\big| \\ &\qquad \le 2\int_s^t\int\left|\mathrm{Im}\left(\omega(u_\varepsilon)\frac{\partial^2\chi_i}{\partial\bar z^2}\right)\right| + \int_s^t\int\left|\frac{|\partial_t u_\varepsilon|^2}{|\log\varepsilon|^2}\varphi_i + (\nabla^\perp\chi_i - \nabla\varphi_i)\cdot\frac{\partial_t u_\varepsilon\cdot\nabla u_\varepsilon}{|\log\varepsilon|}\right|,\end{aligned}$$

where $\partial^2\chi_i/\partial\bar z^2$ has support in $C_i \subset \Omega_{R', r_a}$, and it finally suffices to use (5-8) and Lemma 5.2.

In a second step, we take advantage of the equality $\left\| J u_\varepsilon(T_\varepsilon) - \pi \sum_{i=1}^{l} d_i \delta_{a_i^0} \right\| \equiv \eta_0$. We set

$$\nu_{i,\varepsilon} = d_i \frac{b_i^\varepsilon(T_\varepsilon) - a_i^0}{|b_i^\varepsilon(T_\varepsilon) - a_i^0|}, \qquad i = 1, \dots, l$$

and we define $\chi_{i,\varepsilon}$, $\varphi_{i,\varepsilon}$ so that for $x \in B(a_i^0, r_a)$,

$$\chi_{i,\varepsilon}(x) = \nu_{i,\varepsilon} \cdot x, \qquad \varphi_{i,\varepsilon}(x) = \nu_{i,\varepsilon}^\perp \cdot x.$$

We require additionally that $\chi = \sum \chi_{i,\varepsilon}$ and $\varphi = \sum \varphi_{i,\varepsilon}$ belong to $\mathcal{H}(a_i^0)$; we can moreover choose $\varphi_{i,\varepsilon}$ and $\chi_{i,\varepsilon}$ so that their norms in $C^2(B(R))$ remain bounded uniformly in ε. Since $b_i^\varepsilon(T_\varepsilon) \in B(a_i^0, r_a/2)$, we have $|d_i| \left| b_i^\varepsilon(T_\varepsilon) - a_i^0 \right| = d_i \chi\left(b_i^\varepsilon(T_\varepsilon) - a_i^0\right) + \delta\varphi\left(b_i^\varepsilon(T_\varepsilon) - a_i^0\right)$, so

$$\left\| \pi \sum_{i=1}^{l} d_i (\delta_{b_i^\varepsilon(T_\varepsilon)} - \delta_{a_i^0}) \right\| = \left\langle \pi \sum_{i=1}^{l} d_i (\delta_{b_i^\varepsilon(T_\varepsilon)} - \delta_{a_i^0}), \chi \right\rangle + \delta \left\langle \pi \sum_{i=1}^{l} (\delta_{b_i^\varepsilon(T_\varepsilon)} - \delta_{a_i^0}), \varphi \right\rangle.$$

On the other hand, we have

$$\left\| J u_\varepsilon(T_\varepsilon) - \pi \sum_{i=1}^{l} d_i \delta_{a_i^0} \right\| \le \left\| J u_\varepsilon(T_\varepsilon) - \pi \sum_{i=1}^{l} d_i \delta_{b_i^\varepsilon(T_\varepsilon)} \right\| + \left\| \pi \sum_{i=1}^{l} d_i (\delta_{b_i^\varepsilon(T_\varepsilon)} - \delta_{a_i^0}) \right\|.$$

The second term in the right-hand side may be rewritten as

$$\left\langle \pi \sum_{i=1}^{l} d_i (\delta_{b_i^\varepsilon(T_\varepsilon)} - \delta_{a_i^0}), \chi \right\rangle + \delta \left\langle \pi \sum_{i=1}^{l} (\delta_{b_i^\varepsilon(T_\varepsilon)} - \delta_{a_i^0}), \varphi \right\rangle = A + B + C,$$

where

$$\begin{aligned} A &= \left\langle \pi \sum_{i=1}^{l} d_i \delta_{b_i^\varepsilon(T_\varepsilon)} - J u_\varepsilon(T_\varepsilon), \chi \right\rangle + \delta \left\langle \pi \sum_{i=1}^{l} \delta_{b_i^\varepsilon(T_\varepsilon)} - \mu_\varepsilon(T_\varepsilon), \varphi \right\rangle \\ &\le C \left(\left\| J u_\varepsilon(T_\varepsilon) - \sum_{i=1}^{l} d_i \delta_{b_i^\varepsilon(T_\varepsilon)} \right\| + \delta \left\| \mu_\varepsilon(T_\varepsilon) - \sum_{i=1}^{l} \delta_{b_i^\varepsilon(T_\varepsilon)} \right\| \right), \\ B &= \langle J u_\varepsilon(T_\varepsilon) - J u_\varepsilon(0), \chi \rangle + \delta \langle \mu_\varepsilon(T_\varepsilon) - \mu_\varepsilon(0), \varphi \rangle, \\ C &= \left\langle J u_\varepsilon^0 - \pi \sum_{i=1}^{l} d_i \delta_{a_i^0}, \chi \right\rangle + \delta \left\langle \mu_\varepsilon(u_\varepsilon^0) - \pi \sum_{i=1}^{l} \delta_{a_i^0}, \varphi \right\rangle \\ &\le C \left(\left\| J u_\varepsilon^0 - \sum_{i=1}^{l} d_i \delta_{a_i^0} \right\| + \delta \left\| \mu_\varepsilon(u_\varepsilon^0) - \sum_{i=1}^{l} \delta_{a_i^0} \right\| \right). \end{aligned}$$

In view of the bound provided by (5-9) for B, estimates (4-3)–(4-4) and the fact that $\Sigma_\varepsilon^0(s) \le K_1$ for $0 \le s \le T_\varepsilon$, this implies

$$\eta_0 = \left\| J u_\varepsilon(T_\varepsilon) - \pi \sum_{i=1}^{l} d_i \delta_{a_i^0} \right\| \le C \left(\varepsilon |\log \varepsilon| + |\log \varepsilon|^{-1} + |\log \varepsilon|^{-\frac{1}{2}} \right) + C T_\varepsilon,$$

and letting $\varepsilon \to 0$ yields the conclusion. Lemma 5.3 is proved. □

Conclusion of the proof of Theorem 5.1. We consider $t, s \in [0, T]$. Arguing as in the proof of Lemma 5.3 (with T_ε and 0 replaced by t and s), we find that for all χ, φ belonging to $\mathcal{K}(a_i^0)$,

$$\Big|\sum_{i=1}^{l} d_i\big[\chi(b_i^\varepsilon(t)) - \chi(b_i^\varepsilon(s))\big] + \delta\big[\varphi(b_i^\varepsilon(t)) - \varphi(b_i^\varepsilon(s))\big]\Big|$$
$$\leq C \sup_{\tau\in[0,T]} \Big(\Big\| Ju_\varepsilon(\tau) - \sum_{i=1}^{l} d_i \delta_{b_i^\varepsilon(\tau)} \Big\| + \delta \Big\| \mu_\varepsilon(\tau) - \sum_{i=1}^{l} \delta_{b_i^\varepsilon(\tau)} \Big\| \Big)$$
$$+ \Big| \langle Ju_\varepsilon(t) - Ju_\varepsilon(s), \chi\rangle + \delta\langle \mu_\varepsilon(t) - \mu_\varepsilon(s), \varphi\rangle \Big|,$$

which is bounded by $o_\varepsilon(1) + c|t-s|$ because of (4-3), (4-4) and (5-9). Considering successively $\chi(x) = e_1 \cdot x$ and $\chi(x) = e_2 \cdot x$ on each $B(a_i^0, r_a)$, we obtain

$$|b_i^\varepsilon(t) - b_i^\varepsilon(s)| \leq c|t-s| + o_\varepsilon(1). \tag{5-10}$$

Next, using that $b_i^\varepsilon \in B(a_i^0, r_a)$ and a standard diagonal argument, we may construct a sequence $(\varepsilon_k) \to 0$ and paths $b_i(t)$ such that $b_i^{\varepsilon_k}(t)$ converges to $b_i(t)$ for all $t \in \mathbb{Q} \cap [0, T]$. We infer then from (4-3)–(4-4) that the convergence statements (5-1)–(5-2) in Theorem 5.1 hold for these times. Moreover, in view of (5-10) these paths are Lipschitz on $[0, T] \cap \mathbb{Q}$, so that they can be extended in a unique way to Lipschitz paths (still denoted by $b_i(t)$) on the whole of $[0, T]$. We can finally establish that the convergence (5-1)–(5-2) holds uniformly with respect to $t \in [0, T]$ by again using (5-10) and (4-3)–(4-4).

Finally, we already know from (5-5) that the estimate (5-3) holds for the full family $(u_\varepsilon)_{\varepsilon<\varepsilon_\Lambda}$. To show (5-4), we now recall the uniform bound $\mathcal{E}_{\varepsilon,[U_d]}(u_\varepsilon(t)) - \mathcal{E}_{\varepsilon,[U_d]}(u_\varepsilon^*(a_i^0, d_i)) \leq K_1$, and observe also that Corollary 3.8 gives

$$\mathcal{E}_{\varepsilon,[U_d]}(u_\varepsilon^*(a_i^0, d_i)) - \mathcal{E}_{\varepsilon,[U_d]}(u_\varepsilon^*(b_i(t), d_i)) = W(a_i^0, d_i) - W(b_i(t), d_i) \leq C,$$

since the b_i are continuous and remain separated on $[0, T]$. This yields the bound (5-4) and concludes the proof of Theorem 5.1. □

As mentioned early in the proof of Theorem 5.1, the convergence of the initial data in (WP_1) actually holds on every large ball $B(L)$, $L = 2^n \geq R$, so we find the same conclusions replacing R by L.

Lemma 5.4 [Bethuel et al. 2008, Lemma 7.3]. *There exists a subsequence, still denoted by ε_k, such that for all $L \geq 2^{n_1}$,*

$$\eta_k := \sup_{[0,T]} \Big\| Ju_{\varepsilon_k}(t) - \pi \sum_{i=1}^{l} d_i \delta_{b_i(t)} \Big\|_{W_0^{1,\infty}(B(L))^*} \to 0, \qquad k \to +\infty.$$

For $t \in [0, T]$ and sufficiently large $k \in \mathbb{N}$, we may therefore apply Theorem 4.1 to $u_{\varepsilon_k}(t)$ with respect to the configuration $(b_i(t), d_i)$ and with the choice $R_0 = L = 2^n$ for each $n \geq n_1$. We are led to introduce the excess energy at time t with respect to the configuration $(b_i(t), d_i)$ by

$$\Sigma_{\varepsilon_k}(t) = \mathcal{E}_{\varepsilon_k, U_d}(u_{\varepsilon_k}(t)) - \mathcal{E}_{\varepsilon_k, U_d}(u_{\varepsilon_k}^*(b_i(t), d_i)),$$

which is uniformly bounded on $[0, T]$ in view of (5-4). Letting first k, then n tend to $+\infty$, we can get rid of the dependence on R in (4-2).

Lemma 5.5. *For all $r \le r_a/2$ and $K \ge 2^{n_1}$, we have for sufficiently large k and $t, t_1, t_2 \in [0, T]$*

$$\int_{B(K)\setminus\cup B(b_i(t),r)} e_{\varepsilon_k}\big(|u_{\varepsilon_k}(t)|\big) + \frac{1}{8}\left|\frac{j(u_{\varepsilon_k}(t))}{|u_{\varepsilon_k}(t)|} - j(u^*(b_i(t), d_i))\right|^2 \le \Sigma_{\varepsilon_k}(t) + C\Big(\varepsilon_k, \eta_k, \frac{1}{K}\Big).$$

Therefore, we have as $k \to +\infty$

$$\limsup_{k\to+\infty} \int_{t_1}^{t_2}\int_{B(K)\setminus\cup B(b_i(t),r)} e_{\varepsilon_k}\big(|u_{\varepsilon_k}(t)|\big) + \frac{1}{8}\left|\frac{j(u_{\varepsilon_k})(t)}{|u_{\varepsilon_k}(t)|} - j(u^*(b_i(t), d_i))\right|^2 \le \limsup_{k\to+\infty}\int_{t_1}^{t_2} \Sigma_{\varepsilon_k}(t).$$

Thus, the distance between $u_{\varepsilon_k}(t)$ and $u^*(b_i(t), d_i)$ can be asymptotically controlled by $\limsup \Sigma_{\varepsilon_k}(t)$.

We now define the trajectory set

$$\mathcal{T} = \{(t, b_i(t)),\ t \in [0, T],\ i = 1, \dots, l\}$$

and its complement

$$\mathcal{G} = [0, T] \times \mathbb{R}^2 \setminus \mathcal{T}.$$

Thanks to the uniform bounds in $L^2_{\mathrm{loc}}(\mathcal{G})$ provided by Lemma 5.5, we establish:

Proposition 5.6. *There exists a subsequence, still denoted ε_k, such that*

$$\frac{j(u_{\varepsilon_k})}{|u_{\varepsilon_k}|} \rightharpoonup j(u^*(b_i(\cdot), d_i))$$

weakly in $L^2_{\mathrm{loc}}(\mathcal{G})$ as $k \to +\infty$.

Proof. Let B be any bounded subset of $\mathbb{R}^2$. According to Lemma 5.4,

$$\operatorname{curl}(j(u_{\varepsilon_k})) = 2Ju_{\varepsilon_k} \to 2\pi \sum_{i=1}^{l} d_i \delta_{b_i(\cdot)} = \operatorname{curl}\big(j(u^*(b_i(\cdot), d_i))\big) \quad \text{in } \mathcal{D}'([0, T] \times B). \tag{5-11}$$

At the same time, we have

$$\operatorname{div}(j(u_{\varepsilon_k})) \to 0 = \operatorname{div}\big(j(u^*(b_i(\cdot), d_i))\big) \quad \text{in } \mathcal{D}'([0, T] \times B). \tag{5-12}$$

Indeed, since u_{ε_k} solves (CGL)$_\varepsilon$, we obtain by considering the exterior product

$$k_{\varepsilon_k} u_{\varepsilon_k} \times \partial_t u_{\varepsilon_k} + u_{\varepsilon_k} \cdot \partial_t u_{\varepsilon_k} = u_{\varepsilon_k} \times \Delta u_{\varepsilon_k} = \operatorname{div}(j(u_{\varepsilon_k})),$$

so we are led to

$$\operatorname{div}\big(j(u_{\varepsilon_k})\big) = k_{\varepsilon_k} u_{\varepsilon_k} \times \partial_t u_{\varepsilon_k} + \tfrac{1}{2}\varepsilon_k \frac{d}{dt}\left(\frac{|u_{\varepsilon_k}|^2 - 1}{\varepsilon_k}\right). \tag{5-13}$$

Now, applying Lemma 3.5 to u_{ε_k}, we find

$$\sup_{[0,T]} E_{\varepsilon_k}(u_{\varepsilon_k}(t), B) \le \pi l |\log \varepsilon| + \Sigma_{\varepsilon_k}(t) + C \le \pi l |\log \varepsilon_k| + C, \tag{5-14}$$

where the second inequality is itself a consequence of (5-4). This implies that $|u_{\varepsilon_k}| \to 1$ in $L^2([0, T] \times B)$. Moreover, we infer that the second term on the right-hand side of (5-13) converges to zero in the distribution sense on $[0, T] \times B$. For the first one, it suffices to use the Cauchy–Schwarz inequality combined with the L^2 bound provided by Lemma 5.2 and the already mentioned uniform bounds of $|u_{\varepsilon_k}|$ in L^2_{loc}.

We then infer from Lemma 5.4 and (5-14) that $j(u_{\varepsilon_k})$ is uniformly bounded in $L^p_{\text{loc}}([0,T]\times\mathbb{R}^2)$ for all $p<2$. This is a consequence of, for example, [Colliander and Jerrard 1999, Theorem 3.2.1 and subsequent remarks]. We deduce from (5-11) and (5-12) that up to a subsequence, we have

$$j(u_{\varepsilon_k}) \rightharpoonup j_1 = j(u^*(b_i(\cdot), d_i)) + H \tag{5-15}$$

weakly in $L^p_{\text{loc}}([0,T]\times\mathbb{R}^2)$, where H is harmonic in x on $[0,T]\times\mathbb{R}^2$.

On the other hand, it follows from the first part of Lemma 5.5 that there exists j_2 such that, taking subsequences if necessary, $j(u_{\varepsilon_k})/|u_{\varepsilon_k}| \rightharpoonup j_2$ weakly in $L^2_{\text{loc}}(\mathcal{G})$.

Taking into account the strong convergence $|u_{\varepsilon_k}|\to 1$ in $L^2_{\text{loc}}([0,T]\times\mathbb{R}^2)$, we obtain $j_1=j_2\in L^2_{\text{loc}}(\mathcal{G})$. The second part of Lemma 5.5 combined with (5-15) then yields

$$\|H\|_{L^2_{\text{loc}}(\mathcal{G})} \le \liminf_{k\to+\infty} \left\| \frac{j(u_{\varepsilon_k})}{|u_{\varepsilon_k}|} - j(u^*(b_i,d_i)) \right\|_{L^2_{\text{loc}}(\mathcal{G})} \le CT,$$

where C depends only on K_1, R, and r_a, so finally $\|H\|_{L^2([0,T]\times\mathbb{R}^2)} \le CT$. Since H is harmonic in x, we find that $H(t,\cdot)$ is bounded on $\mathbb{R}^2$ for almost every t and therefore is identically zero. We end up with $j_1=j_2=j(u^*(b_i(\cdot),d_i))$ in $\mathcal{G}$, and the conclusion follows. □

6. Proof of Theorem 1.3

Let $\{b_i(t)\}$ be the l Lipschitz paths on $[0,T]$ provided by Theorem 5.1, and $\{a_i(t)\}$ the unique maximal solution defined on $I=[0,T^*)$ to (1-5) with initial conditions a_i^0. Our aim is to show that $a_i(t)\equiv b_i(t)$ on I. We prove this first on $[0,T]$. By Rademacher's Theorem, the time derivatives $\dot b_i(t)$ exist and are bounded almost everywhere on $[0,T]$. Without loss of generality, we may assume $T<T^*$, so

$$|\dot a_i(t)| \le C,\ |\dot b_i(t)| \le C \qquad \text{a.e. on } [0,T]. \tag{6-1}$$

Moreover, we may assume, possibly after decreasing T, that $|a_i(t)-b_i(t)|\le r_a/2$ for all i. Hence, the trajectories $a_i(t)$ remain in $B(a_i^0, r_a)$ on $[0,T]$. We introduce

$$h(t)=\sum_{i=1}^{l}\int_0^t \left|\dot a_i(s)-\dot b_i(s)\right| ds, \qquad \sigma(t)=\sum_{i=1}^{l}\left|a_i(t)-b_i(t)\right|.$$

Then h is Lipschitz on $[0,T]$ and for almost every $t\in[0,T]$ we have $h'(t)=\sum_{i=1}^{l}\left|\dot a_i(t)-\dot b_i(t)\right|$. Note that since σ is absolutely continuous and $\sigma(0)=0$, we have for all $t\in[0,T]$

$$\sigma(t)=\int_0^t \sigma'(s)\,ds \le h(t).$$

Therefore it suffices to show that h is identically zero on $[0,T]$. This will be done by means of Grönwall's Lemma.

Lemma 6.1. *For all $t_1, t_2, t\in[0,T]$, we have*

$$\limsup_{k\to+\infty} \Sigma_{\varepsilon_k}(t) \le Ch(t) \quad \textit{and} \quad \limsup_{k\to+\infty} \int_{t_1}^{t_2} \Sigma_{\varepsilon_k}(s)\,ds \le C\int_{t_1}^{t_2} h(s)\,ds,$$

where C only depends on r_a, K_0, and R_a.

Proof. For $t \in [0, T]$, we decompose $\Sigma_{\varepsilon_k}(t)$ as

$$\Sigma_{\varepsilon_k}(t) = \mathscr{E}_{\varepsilon_k,[U_d]}(u_{\varepsilon_k}(t)) - \mathscr{E}_{\varepsilon_k,[U_d]}(u^0_{\varepsilon_k}) + \Sigma_{\varepsilon_k}(0) + \mathscr{E}_{\varepsilon_k,[U_d]}(u^*_{\varepsilon_k}(a^0_i, d_i)) - \mathscr{E}_{\varepsilon_k,[U_d]}(u^*_{\varepsilon_k}(b_i(t), d_i)).$$

Appealing to Corollary 3.8 and Theorem 1.1, we obtain

$$\Sigma_{\varepsilon_k}(t) = -\delta \int_0^t \int_{\mathbb{R}^2} \frac{|\partial_t u_{\varepsilon_k}|^2}{|\log \varepsilon_k|} + \Sigma_{\varepsilon_k}(0) + W(a^0_i, d_i) - W(b_i(t), d_i) + o_{\varepsilon_k}(1).$$

Using that W is Lipschitz away from zero, we estimate the difference on the right by

$$\begin{aligned} W(a^0_i, d_i) - W(b_i(t), d_i) &= W(a^0_i, d_i) - W(a_i(t), d_i) + W(a_i(t), d_i) - W(b_i(t), d_i) \\ &\le - \int_0^t \sum_{i=1}^{l} \dot{a}_i(s) \cdot \nabla_{a_i} W(s)\, ds + C\sigma(t). \end{aligned}$$

Since the a_i solve the Cauchy problem (1-5), an explicit computation gives

$$\dot{a}_i(s) \cdot \nabla_{a_i} W(s) = \frac{\delta}{\pi} C_i d_i |\nabla_{a_i} W|^2 = -\delta\pi\, |\dot{a}_i(s)|^2,$$

so

$$\Sigma_{\varepsilon_k}(t) \le \Sigma_{\varepsilon_k}(0) + \delta\pi \int_0^t \sum_{i=1}^{l} |\dot{a}_i(s)|^2\, ds - \delta \int_0^t \int_{\mathbb{R}^2} \frac{|\partial_t u_{\varepsilon_k}|^2}{|\log \varepsilon_k|} + C\sigma(t) + o_{\varepsilon_k}(1).$$

For the energy dissipation on the right-hand side, we need a lower bound as ε_k tends to zero. In view of the convergence of the Jacobians (5-1) and the upper bound for the energy

$$\sup_{t\in[0,T]} E_{\varepsilon_k}(u_{\varepsilon_k}(t), B(R')) \le \pi l |\log \varepsilon_k| + C$$

stated in (5-14), Proposition 3 in [Jerrard 1999] (see also Corollary 7 in [Sandier and Serfaty 2004]) provides the lower mobility bound

$$\liminf_{k\to+\infty} \int_0^t \int_{\mathbb{R}^2} \frac{|\partial_t u_{\varepsilon_k}|^2}{|\log \varepsilon_k|} \ge \pi \sum_{i=1}^{l} \int_0^t \left|\dot{b}_i(t)\right|^2 ds. \tag{6-2}$$

Now, thanks to (6-1), we have

$$\sum_{i=1}^{l} \int_0^t \left(|\dot{a}_i(s)|^2 - |\dot{b}_i(s)|^2\right) \le C \sum_{i=1}^{l} \int_0^t \left|\dot{a}_i(s) - \dot{b}_i(s)\right| ds = Ch(t),$$

whereas $\Sigma_{\varepsilon_k}(0) \to 0$ by assumption; hence we get

$$\limsup_{k\to+\infty} \Sigma_{\varepsilon_k}(t) \le C\,(\sigma(t) + h(t)).$$

Applying Fatou's Lemma in (6-2) yields the corresponding integral version as well. We conclude by using that $\sigma \le h$. □

As suggested in the introduction, the map $u^*(a_i(t), d_i)$ solves the evolution formula provided by Proposition 2.1 in the asymptotic limit where $\varepsilon \to 0$.

Lemma 6.2. *For $t \in [0,T]$ and $\chi, \varphi \in \mathcal{H}(a_i^0)$,*

$$\pi \frac{d}{dt} \sum_{i=1}^{l} \big[d_i \chi(a_i(t)) + \delta \varphi(a_i(t))\big] = -2 \int_{\mathbb{R}^2} \operatorname{Im}\left(\omega(u^*(a_i(t), d_i)) \frac{\partial^2 \chi}{\partial \bar{z}^2}\right).$$

Proof. We use the following formula, proved in [Bethuel et al. 2005] and valid for any configuration (a_i, d_i) and any test function χ that is affine near the point vortices:

$$-2 \int_{\mathbb{R}^2} \operatorname{Im}\left(\omega(u^*(a_i(t), d_i)) \frac{\partial^2 \chi}{\partial \bar{z}^2}\right) = -2\pi \sum_{i \neq j} d_i d_j \frac{(a_i(t) - a_j(t))^\perp}{|a_i(t) - a_j(t)|^2} \cdot \nabla \chi(a_i(t)).$$

We also compute

$$\frac{d}{dt} \sum_{i=1}^{l} \big[d_i \chi(a_i) + \delta \varphi(a_i)\big] = \sum_{i=1}^{l} \big[d_i \nabla \chi(a_i^0) \cdot \dot{a}_i(t) + \delta \nabla \varphi(a_i^0) \cdot \dot{a}_i(t)\big] = \sum_{i=1}^{l} d_i \nabla \chi(a_i^0) \cdot \big(\dot{a}_i(t) - \delta d_i \dot{a}_i^\perp(t)\big),$$

where the second equality follows from the relation $\nabla \varphi(a_i^0) = \nabla^\perp \chi(a_i^0)$. Next, we deduce from (1-5)

$$\pi \big(\dot{a}_i(t) - \delta d_i \dot{a}_i^\perp(t)\big) = -C_i (1 + \delta^2 d_i^2) \nabla_{a_i}^\perp W = d_i \nabla_{a_i}^\perp W,$$

and we obtain

$$\pi \frac{d}{dt} \sum_{i=1}^{l} \big[d_i \chi(a_i) + \delta \varphi(a_i)\big] = \sum_{i=1}^{l} \nabla \chi(a_i) \cdot \nabla_{a_i}^\perp W = -2\pi \sum_{i \neq j} d_i d_j \frac{(a_i - a_j)^\perp}{|a_i - a_j|^2} \cdot \nabla \chi(a_i),$$

which yields the conclusion. □

Lemma 6.3. *Set $A = \bigcup_{i=1}^{l} B(a_i^0, 2r_a) \setminus B(a_i^0, r_a)$ and let $t_1, t_2 \in [0,T]$. For all $\varphi \in \mathcal{D}(A)$, we have*

$$\limsup_{k \to +\infty} \left| \int_{t_1}^{t_2} \int_A \big[\omega(u_{\varepsilon_k}(s)) - \omega(u^*(b_i(s), d_i))\big] \varphi \right| \leq C \|\varphi\|_\infty \int_{t_1}^{t_2} h(s)\, ds.$$

Proof. We apply the pointwise equality (2-4) to $u \equiv u_{\varepsilon_k}(t)$ and $u^* \equiv u^*(b_i(t), d_i)$ for all t. Since $|u^*(b_i(t), d_i)| = 1$, this gives

$$\omega(u) - \omega(u^*) = \sum_{k,l=1}^{2} \left(a_{k,l} \partial_l |u| \partial_k |u| + b_{k,l} \left[\frac{j_k(u)}{|u|} \frac{j_l(u)}{|u|} - j_k(u^*) j_l(u^*)\right]\right),$$

where $a_{k,l}, b_{k,l} \in \mathbb{C}$. We rewrite the terms involving the components of j as

$$\frac{j_k(u)}{|u|} \frac{j_l(u)}{|u|} - j_k(u^*) j_l(u^*) = \left(\frac{j_k(u)}{|u|} - j_k(u^*)\right)\left(\frac{j_l(u)}{|u|} - j_l(u^*)\right) + j_k(u^*)\left(\frac{j_l(u)}{|u|} - j_l(u^*)\right) + j_l(u^*)\left(\frac{j_k(u)}{|u|} - j_k(u^*)\right).$$

We multiply the previous equality by φ, integrate on $[t_1, t_2] \times A$ and let k go to $+\infty$. Using the weak convergence in L^2 of $j(u_{\varepsilon_k})/|u_{\varepsilon_k}|$ to $j(u^*(b_i(.), d_i))$ on $[0, T] \times A \subset \mathcal{G}$, we deduce that

$$\limsup_{k\to+\infty} \left| \int_{t_1}^{t_2} \int_A \big[\omega(u_{\varepsilon_k}(s)) - \omega(u^*(b_i(s), d_i))\big]\varphi \right| \leq C\|\varphi\|_\infty \limsup_{k\to+\infty} \int_{t_1}^{t_2} \int_A \left(\big|\nabla |u_{\varepsilon_k}|\big|^2 + \left| \frac{j(u_{\varepsilon_k})}{|u_{\varepsilon_k}|} - j(u^*(b_i, d_i)) \right|^2 \right).$$

The conclusion finally follows from Lemmas 5.5 and 6.1. □

We are now in a position to complete the proof of Theorem 1.3. We consider arbitrary χ, φ belonging to $\mathcal{K}(a_i^0)$, we fix $0 \leq s \leq t \leq T$ and we integrate the evolution formula given by Proposition 2.1 on $[s, t]$. We obtain

$$\int_s^t \frac{d}{d\tau} \int_{\mathbb{R}^2} Ju_{\varepsilon_k}(\tau)\chi + \delta \int_{\mathbb{R}^2} \mu_{\varepsilon_k}(\tau)\varphi = \int_s^t g_k^1(\tau) + \int_s^t g_k^2(\tau),$$

where

$$g_k^1(\tau) = -\delta \int_{\mathbb{R}^2} \frac{|\partial_t u_{\varepsilon_k}|^2}{|\log \varepsilon_k|^2} + R_{\varepsilon_k}(\tau, \chi, \varphi, u_{\varepsilon_k}), \qquad g_k^2(\tau) = -2 \int_{\mathbb{R}^2} \mathrm{Im}\left(\omega(u_{\varepsilon_k}(\tau)) \frac{\partial^2 \chi}{\partial \bar{z}^2} \right).$$

We decompose the latter as

$$g_k^2 = -2 \int_{\mathbb{R}^2} \mathrm{Im}\left(\big[\omega(u_{\varepsilon_k}) - \omega(u^*(b_i, d_i))\big] \frac{\partial^2 \chi}{\partial \bar{z}^2} \right) - 2 \int_{\mathbb{R}^2} \mathrm{Im}\left(\big[\omega(u^*(b_i, d_i)) - \omega(u^*(a_i, d_i))\big] \frac{\partial^2 \chi}{\partial \bar{z}^2} \right) - 2 \int_{\mathbb{R}^2} \mathrm{Im}\left(\omega(u^*(a_i, d_i)) \frac{\partial^2 \chi}{\partial \bar{z}^2} \right) = A_k(\tau) + B_k(\tau) + C_k(\tau).$$

We next substitute the expression given in Lemma 6.2 for C_k in the previous equalities. Setting

$$f_{k,\chi,\varphi}(\tau) = \int_{\mathbb{R}^2} Ju_{\varepsilon_k}(\tau)\chi + \delta \int_{\mathbb{R}^2} \mu_{\varepsilon_k}(\tau)\varphi - \pi \sum_{i=1}^{l} \big[d_i \chi(a_i(\tau)) + \delta\varphi(a_i(\tau))\big],$$

we obtain

$$f_{k,\chi,\varphi}(t) - f_{k,\chi,\varphi}(s) = \int_s^t g_k^1 + \int_s^t A_k + \int_s^t B_k.$$

Lemma 5.2 with $T_\varepsilon = T$ first gives $\left|\int_s^t g_k^1(\tau)\, d\tau\right| \leq C|\log \varepsilon_k|^{-1/2}$ for all k. Moreover, it follows from Lemma 6.3 and inclusion $\operatorname{supp} \partial^2\chi/\partial\bar{z}^2 \subset A$ that

$$\limsup_{k\to+\infty} \left| \int_s^t A_k(\tau)\, d\tau \right| \leq C \int_s^t h(\tau)\, d\tau.$$

Finally, the regularity of $\omega(u^*)$ away from the vortices gives

$$\int_s^t |B_k(\tau)|\, d\tau \leq C \int_s^t \sigma(\tau)\, d\tau \leq C \int_s^t h(\tau)\, d\tau.$$

Letting k go to $+\infty$, we deduce from the convergence statements in Theorem 5.1 that for $0 \le s \le t \le T$,

$$|f_{\chi,\varphi}(t) - f_{\chi,\varphi}(s)| \le C \int_s^t h(\tau)\,d\tau, \tag{6-3}$$

where the constant C depends only on χ, φ and the initial conditions, and $f_{\chi,\varphi}$ is defined by

$$f_{\chi,\varphi} = \pi \sum_{i=1}^{l} \big[d_i(\chi(b_i) - \chi(a_i)) + \delta(\varphi(b_i) - \varphi(a_i))\big].$$

We now fix a time $t \in [0, T]$ at which all the vortices b_i have a time derivative. Since the a_i are C^1, it follows that $f_{\chi,\varphi}$ is differentiable at t with time derivative given by

$$f'_{\chi,\varphi}(t) = \pi \sum_{i=1}^{l} \big(d_i \nabla\chi(a_i^0) + \delta\nabla^{\perp}\chi(a_i^0)\big) \cdot \big(\dot{b}_i(t) - \dot{a}_i(t)\big).$$

Dividing by $t - s$ in (6-3) and letting $s \to t$ then gives

$$\left|\pi \sum_{i=1}^{l} \big(d_i \nabla\chi(a_i^0) + \delta\nabla^{\perp}\chi(a_i^0)\big) \cdot \big(\dot{b}_i(t) - \dot{a}_i(t)\big)\right| \le C\,h(t).$$

So, considering in particular $\chi, \varphi \in \mathcal{H}(a_i^0)$ such that χ and φ vanish near each point a_i^0 except for one, we obtain for all $i = 1, \dots, l$

$$\left|\pi\,\big(d_i \nabla\chi(a_i^0) + \delta\nabla^{\perp}\chi(a_i^0)\big) \cdot \big(\dot{b}_i(t) - \dot{a}_i(t)\big)\right| \le C\,h(t).$$

Choosing then successively $\chi(x) = x_1$ and $\chi(x) = x_2$ near a_i^0 we end up with $|\dot{b}_i(t) - \dot{a}_i(t)| \le Ch(t)$, and it follows by summation that $h'(t) \le Ch(t)$ for a.e. $t \in [0, T]$. Since $h(0) = 0$, this implies that $h = 0$ on $[0, T]$, and hence $\sigma = 0$ on $[0, T]$. Applying Lemma 6.1, we infer that $\limsup_{k\to+\infty} \Sigma_{\varepsilon_k}(t) \le 0$. Besides, Lemma 3.5 yields for all $L \ge 2^{n_1}$

$$\liminf_{k\to+\infty} \Sigma_{\varepsilon_k}(t) \ge \liminf_{k\to+\infty} \int_{B(L)} \big[e_{\varepsilon_k}(u_{\varepsilon_k}(t)) - e_{\varepsilon_k}(u^*_{\varepsilon_k}(a_i(t), d_i))\big] - \frac{C}{L} \ge -\frac{C}{L},$$

where the second inequality follows from the convergence of Jacobians on $B(L)$ stated in Lemma 5.4; see [Jerrard and Spirn 2007; Lin and Xin 1999]. Letting L tend to $+\infty$, we obtain $\liminf_{k\to+\infty} \Sigma_{\varepsilon_k}(t) \ge 0$, so we deduce from (5-3) that $(u_{\varepsilon_k}(t))_{k\in\mathbb{N}}$ is well-prepared with respect to the configuration $(a_i(t), d_i)$. By the uniqueness of the limit, this holds for the full family $(u_\varepsilon(t))_{0<\varepsilon<1}$ on $[0, T]$.

In conclusion, we observe that in our definition T only depends on K_1, r_a and $\max(R, R_a + r_a)$, so we can extend our results to the whole of $[0, T^*)$ by repeating the previous arguments.

Appendix: The Cauchy problem for (CGL)$_\epsilon$

We present here the proof of Theorem 1.1. We omit the dependence on ε and rewrite (1-2) in the form

$$\begin{cases} \partial_t w = (a + ib)(\Delta w + f_{U_0}(w)), \\ w(0) = w_0 \in H^1(\mathbb{R}^2), \end{cases} \tag{CGL}$$

where a is positive, b is real and

$$f_{U_0}(w) = \Delta U_0 + (U_0 + w)(1 - |U_0 + w|^2).$$

We denote by $S = S(t, x)$ the semigroup operator associated to the corresponding homogeneous linear equation. Every solution $w \in C^0\big([0, T], H^1(\mathbb{R}^2)\big)$ to (CGL) satisfies the Duhamel formula

$$w(t, \cdot) = S(t, \cdot) * w_0 + \int_0^t S(t - s, \cdot) * g_{U_0}(w(s), \cdot)\, ds, \qquad s \in [0, T],$$

where $g_{U_0} = (a + ib) f_{U_0}$. The kernel S is explicitly given by

$$S(t, x) = \frac{1}{4\pi (a + ib)t} \exp\Big(\frac{-|x|^2}{4(a + ib)t}\Big).$$

Since a is positive, S decays at infinity like the standard heat kernel; therefore (CGL) enjoys the same smoothing properties as the parabolic Ginzburg–Landau equation. In particular, we have for all $t > 0$ and for all $1 \le r \le +\infty$

$$\big\| S(t, \cdot)\big\|_{L^r(\mathbb{R}^2)} \le \frac{C_{a,b}}{t^{1-1/r}} \tag{A-1}$$

and concerning the space derivatives of $S(t)$,

$$\big\| D^k S(t, \cdot)\big\|_{L^r(\mathbb{R}^2)} \le \frac{C_{a,b}}{t^{(|k|/2)+1-1/r}}. \tag{A-2}$$

We will often use Young's inequality, which states that, if $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ and $f \in L^p(\mathbb{R}^2)$, $g \in L^q(\mathbb{R}^2)$, then

$$\|f * g\|_{L^r(\mathbb{R}^2)} \le \|f\|_{L^p(\mathbb{R}^2)} \|g\|_{L^q(\mathbb{R}^2)}.$$

We first state a local well-posedness result for (CGL).

Proposition A.4. *Let $w_0 \in H^1(\mathbb{R}^2)$. There exists a positive time T^* depending on $\|w_0\|_{H^1}$ and a unique solution $w \in C^0\big([0, T^*), H^1(\mathbb{R}^2)\big)$ to (CGL).*

Proof. We intend to apply the fixed point theorem to the map $\psi : w \in H^1(\mathbb{R}^2) \mapsto \psi(w)$, where

$$\psi(w)(t) = S(t) * w_0 + \int_0^t S(t - s) * g_{U_0}(w(s))\, ds.$$

To this aim, we introduce $R = \|w_0\|_{H^1(\mathbb{R}^2)}$ and for $T > 0$

$$B(T, R) = \big\{w \in L^\infty\big([0, T], H^1(\mathbb{R}^2)\big) : \|w\|_{L^\infty(H^1)} \le (2C_{a,b} + 1)R\big\},$$

where $C_{a,b}$ is the constant appearing in (A-1)–(A-2). We next show that we can choose $T = T(R)$ so that ψ maps $B(T(R), R)$ into itself and is a contraction on this ball.

For $T > 0$, we let $w \in B(T, R)$ and expand $f_{U_0}(w)$. Using that $H^1(\mathbb{R}^2)$ is continuously embedded in $L^p(\mathbb{R}^2)$ for all $2 \le p < +\infty$ and that U_0 belongs to $\mathcal{V}$, it can be shown that

$$\big\| f_{U_0}\big\|_{L^\infty([0,T],L^2)} \le C(U_0, R) \tag{A-3}$$

(see [Bethuel and Smets 2007, Lemma 1]), and that for $w_1, w_2 \in B(T, R)$

$$\| f_{U_0}(w_1) - f_{U_0}(w_2)\|_{L^\infty([0,T],L^2)} \le C(U_0, R)\|w_1 - w_2\|_{L^\infty([0,T],H^1)}. \tag{A-4}$$

We next apply Young's inequality to obtain

$$\begin{aligned}\|\psi(w)(t)\|_{H^1} &\le \|\psi(w)(t)\|_{L^2} + \|\nabla\psi(w)(t)\|_{L^2}\\ &\le 2\|S(t)\|_{L^1}\|w_0\|_{H^1} + \int_0^t \|S(t-s)+\nabla S(t-s)\|_{L^1}\|g_{U_0}(s)\|_{L^2}\,ds\\ &\le 2C_{a,b}\|w_0\|_{H^1} + C\int_0^t \bigl(1+(t-s)^{-1/2}\bigr)\|g_{U_0}(w(s))\|_{L^2}\,ds,\end{aligned}$$

where the last inequality is a consequence of (A-1) and (A-2) with the choice $r=1$. This yields, by (A-3) and (A-4),

$$\sup_{t\in[0,T]}\|\psi(w)(t)\|_{H^1} \le 2C_{a,b}\|w_0\|_{H^1} + C(U_0,R)(T+\sqrt{T})$$

and similarly,

$$\sup_{t\in[0,T]}\|\psi(w_1)(t)-\psi(w_2)(t)\|_{H^1} \le C'(U_0,R)(T+\sqrt{T})\sup_{t\in[0,T]}\|w_1(t)-w_2(t)\|_{H^1}.$$

The conclusion follows by choosing $T=T(R)$ sufficiently small so that $C(U_0,R)(T+\sqrt{T})\le R$ and $C'(U_0,R)(T+\sqrt{T})<1$. □

We next show additional regularity for a solution to (CGL).

Lemma A.5. *Let $w\in C^0\bigl([0,T],H^1(\mathbb{R}^2)\bigr)$ be a solution to (CGL). Then w belongs to*

$$L^1_{\mathrm{loc}}\bigl([0,T],H^2(\mathbb{R}^2)\bigr)\cap C^0\bigl((0,T],H^2(\mathbb{R}^2)\bigr),$$

and therefore to $L^1_{\mathrm{loc}}\bigl([0,T],L^\infty(\mathbb{R}^2)\bigr)$.

Proof. We first differentiate $f_{U_0}(w)$ and use [Bethuel and Smets 2007, Lemma 2] which states by means of various Sobolev embeddings, Hölder and Gagliardo–Nirenberg inequalities that

$$\partial_i f_{U_0}(w) = g_1(w)+g_2(w)\in L^\infty\bigl([0,T],L^2(\mathbb{R}^2)\bigr)+L^\infty\bigl([0,T],L^r(\mathbb{R}^2)\bigr)$$

for all $1<r<2$. Moreover, we have $\sup_{s\in[0,T]}\bigl(\|g_1(w)(s)\|_{L^2(\mathbb{R}^2)}+\|g_2(w)(s)\|_{L^r(\mathbb{R}^2)}\bigr)\le C(U_0,A(T),r)$, where $A(T)=\sup_{s\in[0,T]}\|w(s)\|_{H^1(\mathbb{R}^2)}$. Next, differentiating twice, Duhamel's formula gives

$$\partial_{ij}w(t) = \partial_j S(t)*\partial_i w_0 + \int_0^t \partial_j S(t-s)*\partial_i f_{U_0}(s)\,ds,$$

so taking into account the decomposition $\partial_i f_{U_0}=g_1+g_2$ we get

$$\|\partial_{ij}w(t)\|_{L^2}\le \|\nabla S(t)\|_{L^1}\|\nabla w_0\|_{L^2}+\int_0^t \|\nabla S(t-s)\|_{L^1}\|g_1(s)\|_{L^2}\,ds+\int_0^t \|\nabla S(t-s)\|_{L^\alpha}\|g_2(s)\|_{L^r}\,ds,$$

where α is chosen so that $1+\frac{1}{2}=\frac{1}{\alpha}+\frac{1}{r}$. This finally yields, in view of (A-2),

$$\|\partial_{ij}w(t)\|_{L^2}\le Ct^{-1/2}\|w_0\|_{H^1}+C(U_0,A(T),r)\int_0^t \Bigl((t-s)^{-\frac{1}{2}}+(t-s)^{-\frac{1}{2}-1+\frac{1}{\alpha}}\Bigr)\,ds.$$

Since $\frac{1}{2}+1-\frac{1}{\alpha}=\frac{1}{r}<1$, the right-hand side is finite, so $\partial_{ij}w\in L^1_{\mathrm{loc}}\bigl([0,T],L^2(\mathbb{R}^2)\bigr)$. □

Lemma A.5 enables us to show that the renormalized energy is nonincreasing and hence to control $\|w(t)\|_{H^1(\mathbb{R}^2)}$. For (CGL), this energy is given by

$$E_{U_0}(w)(t) = \int_{\mathbb{R}^2} \frac{|\nabla w|^2}{2} - \int_{\mathbb{R}^2} \Delta U_0 \cdot w + \int_{\mathbb{R}^2} \frac{(1-|U_0+w|^2)^2}{4}.$$

It is well-defined and continuous in time for $w \in C^0(H^1(\mathbb{R}^2))$.

Lemma A.6. *Let $w \in C^0\big([0,T), H^1(\mathbb{R}^2)\big)$ be a solution to (CGL). Then for all $t \in (0,T)$ we have*

$$\frac{d}{dt} E_{U_0}(w)(t) \le 0.$$

Moreover, there exists C_{U_0,w_0} depending only on U_0 and $\|w_0\|_{H^1}$ such that

$$\|w(t)\|_{H^1} \le C_{U_0,w_0} \exp(C_{U_0,w_0} t) \quad \text{for all } t \in [0,T). \tag{A-5}$$

Proof. We infer from (CGL) and Lemma A.5 that $\partial_t w$ belongs to $L^\infty_{\text{loc}}\big((0,T], L^2(\mathbb{R}^2)\big)$, so we can compute

$$\begin{aligned}\frac{d}{dt} E_{U_0}(w(t)) &= \int_{\mathbb{R}^2} \nabla w \cdot \nabla \partial_t w - \Delta U_0 \cdot \partial_t w - \partial_t w \cdot (U_0+w)(1-|U_0+w|^2) \\ &= -\int_{\mathbb{R}^2} \partial_t w \cdot (\Delta w + f_{U_0}(w)) = -\int_{\mathbb{R}^2} \partial_t w \cdot \Big(\frac{1}{a+ib}\partial_t w\Big) = \frac{-a}{a^2+b^2}\int_{\mathbb{R}^2} |\partial_t w|^2 \le 0.\end{aligned}$$

We now turn to (A-5). We compute, for $t \in (0,T)$,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w(t)\|^2_{L^2(\mathbb{R}^2)} &= \int_{\mathbb{R}^2} w \cdot \partial_t w = \int_{\mathbb{R}^2} w \cdot [(a+ib)\Delta w] + \int_{\mathbb{R}^2} w \cdot [(a+ib) f_{U_0}(w)] \\ &= -a\int_{\mathbb{R}^2} |\nabla w|^2 + \int_{\mathbb{R}^2} w \cdot (a+ib)\Delta U_0 + \int_{\mathbb{R}^2} w \cdot \big[(a+ib)(U_0+w)(1-|U_0+w|^2)\big].\end{aligned}$$

We then split the last term in the previous equality as

$$\int_{\mathbb{R}^2} w \cdot \big[(a+ib)(U_0+w)(1-|U_0+w|^2)\big] = \int_{\mathbb{R}^2} w \cdot \big[(a+ib)U_0(1-|U_0+w|^2)\big] + a\int_{\mathbb{R}^2} |w|^2(1-|U_0+w|^2).$$

The last term on the right is clearly bounded by $a\|w(t)\|^2_{L^2(\mathbb{R}^2)}$. Using the Cauchy–Schwarz inequality for the first term, we obtain

$$\int_{\mathbb{R}^2} w \cdot \big[(a+ib)(U_0+w)(1-|U_0+w|^2)\big] \le C(U_0)\|w(t)\|_{L^2} V(t)^{1/2} + a\|w(t)\|^2_{L^2},$$

where $V(t) = \int_{\mathbb{R}^2}(1-|U_0+w(t)|^2)^2$. We are led to

$$\frac{d}{dt}\|w(t)\|^2_{L^2(\mathbb{R}^2)} \le C(U_0)\big(\|w(t)\|^2_{L^2} + 1 + V(t)\big). \tag{A-6}$$

On the other hand, Cauchy–Schwarz inequality gives

$$E_{U_0}(w)(t) \ge \int_{\mathbb{R}^2} \frac{|\nabla w|^2}{2}\,dx - C(U_0)\|w(t)\|_{L^2} + \frac{V(t)}{4},$$

which yields, since $E_{U_0}(w)$ is nonincreasing,

$$\frac{V(t)}{4}+\int_{\mathbb{R}^2}\frac{|\nabla w|^2}{2}\le E_{U_0}(w_0)+C(U_0)\|w(t)\|_{L^2}. \tag{A-7}$$

We infer from (A-6) and (A-7)

$$\|w(t)\|_{L^2}\le(1+\|w_0\|_{H^1})\exp(Ct)$$

and finally deduce (A-5) by using (A-7) once more. □

Lemma A.6 provides global well-posedness for (CGL).

Proposition A.7. *Let $w_0\in H^1(\mathbb{R}^2)$. Then there exists a unique and global solution $w\in C^0\big(\mathbb{R}_+,H^1(\mathbb{R}^2)\big)$ to* (CGL).

Proof. Let $w\in C^0\big([0,T^*),H^1(\mathbb{R}^2)\big)$ be the unique maximal solution with initial condition w_0. If T^* is finite, we have according to (A-5)

$$\limsup_{t\to T^*}\|w(t)\|_{H^1(\mathbb{R}^2)}\le C(U_0,T^*,w_0)<+\infty,$$

so that we can extend w to a solution $\overline{w}$ on $[0,T^*+\delta]$ for some positive δ. This yields a contradiction. □

We conclude this section with the following

Proposition A.8. *Let $w\in C^0\big(\mathbb{R}_+,H^1(\mathbb{R}^2)\big)$ be the solution to* (CGL). *Then $w\in C^\infty\big(\mathbb{R}_+^*,C^\infty(\mathbb{R}^2)\big)$.*

Proof. Step 1. Let $p\ge 2$ and $v\in H^p(\mathbb{R}^2)$. We show that $D^k f_{U_0}(v)\in L^2(\mathbb{R}^2)+L^{4/3}(\mathbb{R}^2)$ for all $|k|\le p$.

We may assume in view of the proof of Lemma A.5 that $|k|\ge 2$. We decompose $f_{U_0}(v)$ as $f_{U_0}(v)=\Delta U_0+h_{U_0}(v)$, where

$$h_{U_0}(v)=(U_0+v)(1-|U_0+v|^2).$$

Since $U_0\in\mathcal{V}$, it suffices to show that $D^k h_{U_0}(v)\in L^2(\mathbb{R}^2)+L^{4/3}(\mathbb{R}^2)$. Applying Leibniz's formula to $h_{U_0}(v)$, we obtain

$$\begin{aligned}D^k h_{U_0}(v)&=\sum_{m\le k}\binom{k}{m}D^{k-m}(U_0+v)D^m(1-|U_0+v|^2)\\&=D^k(U_0+v)-\sum_{\substack{m\le k\\ n\le m}}\binom{k}{m}\binom{m}{n}D^{k-m}(U_0+v)D^n(U_0+v)\cdot D^{m-n}(U_0+v).\end{aligned}$$

Since $2\le|k|\le p$, $v\in H^p(\mathbb{R}^2)$ and $U_0\in\mathcal{V}$, we clearly have $D^k(U_0+v)\in L^2(\mathbb{R}^2)$.

For the second term in the right-hand side, we write each product inside the sum as

$$D^a(U_0+v)D^b(U_0+v)\cdot D^c(U_0+v)$$

with $|a|+|b|+|c|=|k|\ge 2$, and we examine all cases. We observe that $D^a(v+U_0)$ belongs to $H^1(\mathbb{R}^2)$ whenever $1\le|a|\le p-1$ and hence to $L^4(\mathbb{R}^2)$, whereas $D^a(v+U_0)$ belongs to $L^2(\mathbb{R}^2)$ for $2\le|a|\le p$. Since $U_0+v\in L^\infty$, we finally obtain

$$D^a(U_0+v)D^b(U_0+v)\cdot D^c(U_0+v)\in L^2(\mathbb{R}^2)+L^{4/3}(\mathbb{R}^2),$$

which yields the conclusion.

Step 2: regularity in space for a solution to (CGL). Let $w \in C^0\big(\mathbb{R}_+, H^1(\mathbb{R}^2)\big)$ be the solution to (CGL). We show that $w \in C^0(\mathbb{R}_+^*, H^p(\mathbb{R}^2))$ for all $p \geq 1$.

We proceed by induction on p. The case $p = 2$ has already been treated in Lemma A.5, so we assume $w \in C^0\big(\mathbb{R}_+^*, H^p(\mathbb{R}^2)\big)$ for some $p \geq 2$. For $|k| \leq p+1$, we differentiate $w(t)$ and we find

$$D^k w(t) = D^k(S(t) * w_0) + D^k \int_0^t S(t-s) * g_{U_0}(s)\, ds$$

which we rewrite as

$$D^k w(t) = D^k S(t) * w_0 + \int_0^{t/2} (D^k S(t-s)) * g_{U_0}(s)\, ds + \int_{t/2}^t D^m S(t-s) * D^{k-m} g_{U_0}(s)\, ds,$$

where m is a multiindex such that $|m| = 1$.

It follows from (A-2) that $t \mapsto D^k S(t) * w_0 \in C^0(\mathbb{R}_+^*, L^2(\mathbb{R}^2))$. Next, arguing that $g_{U_0} \in C^0\big(\mathbb{R}_+, L^2(\mathbb{R}^2)\big)$ and using (A-2) with $r = 1$, we find

$$\left\| \int_0^{t/2} (D^k S(t-s)) * g_{U_0}(s)\, ds \right\|_{L^2} \leq C \int_0^{t/2} \frac{ds}{(t-s)^{|k|/2}} \leq \frac{C}{t^{(|k|/2)-1}}.$$

Also, since $|k-m| = |k|-1 \leq p$ and $w(s) \in H^p(\mathbb{R}^2)$ by assumption, Step 1 provides the decomposition

$$D^{k-m} g_{U_0}(s) = d^1(s) + d^2(s),$$

where d^1 belongs to $C^0(\mathbb{R}_+^*, L^2(\mathbb{R}^2))$ and d^2 to $C^0\big(\mathbb{R}_+^*, L^{4/3}(\mathbb{R}^2)\big)$. It follows from (A-2) that

$$\begin{aligned}\left\| \int_{t/2}^t D^m S(t-s) * D^{k-m} g_{U_0}(s)\, ds \right\|_{L^2} &\leq \int_{t/2}^t \big(\|\nabla S(t-s)\|_{L^1} \|d^1(s)\|_{L^2} + \|\nabla S(t-s)\|_{L^r} \|d^2(s)\|_{L^{4/3}} \big)\, ds \\ &\leq C(t) \int_{t/2}^t \big((t-s)^{-\frac{1}{2}} + (t-s)^{-\frac{1}{2}-1+\frac{1}{r}} \big)\, ds,\end{aligned}$$

where r satisfies $1 + \frac{1}{2} = \frac{1}{r} + \frac{3}{4}$. The last term is finite since $\frac{1}{2} + 1 - \frac{1}{r} = \frac{3}{4} < 1$, so we infer that $w \in C^0\big(\mathbb{R}_+^*, H^{p+1}(\mathbb{R}^2)\big)$, as we wanted.

Step 3. Let $w \in C^0(\mathbb{R}_+, H^1(\mathbb{R}^2))$ be the solution to (CGL). We show that $w \in C^k\big(\mathbb{R}_+^*, C^l(\mathbb{R}^2)\big)$ for all $k, l \in \mathbb{N}$.

Fix $k, l \in \mathbb{N}$. we show by induction on $0 \leq j \leq k$ that $w \in C^j\big(\mathbb{R}_+^*, C^{l+2k-2j}(\mathbb{R}^2)\big)$. This holds for $j = 0$ according to Step 2 and since H^p is embedded in C^{l+2k} for large enough p. We next assume that $w \in C^j\big(\mathbb{R}_+^*, C^{l+2k-2j}(\mathbb{R}^2)\big)$ for some $0 \leq j \leq k-1$, and it follows that

$$\Delta w,\ f_{U_0}(w) \in C^j\big(\mathbb{R}_+^*, C^{l+2k-2j-2}(\mathbb{R}^2)\big).$$

Going back to Equation (CGL), we obtain

$$w \in C^{j+1}\big(\mathbb{R}_+^*, C^{l+2k-2j-2}(\mathbb{R}^2)\big).$$

This concludes the proof of Proposition A.8. □

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