Analysis & PDE

Volume 2 No. 3 2009

mathematical sciences publishers

Analysis & PDE

pjm.math.berkeley.edu/apde

See inside back cover or pjm.math.berkeley.edu/apde for submission instructions.

The subscription price for 2009 is US $120/year for the electronic version, and $180/year for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to Mathematical Sciences Publishers, Department of Mathematics, University of California, Berkeley, CA 94720-3840, USA.

Analysis & PDE, at Mathematical Sciences Publishers, Department of Mathematics, University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFLOW™ from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

http://www.mathscipub.org

A NON-PROFIT CORPORATION

Typeset in LaTeX

ANALYSIS AND PDE
Vol. 2, No. 3, 2009

GLOBAL EXISTENCE OF SMOOTH SOLUTIONS OF A 3D LOG-LOG ENERGY-SUPERCRITICAL WAVE EQUATION

TRISTAN ROY

We prove global existence of smooth solutions of the 3D log-log energy-supercritical wave equation

$$\partial_{tt} u - \triangle u = -u^5 \log^c(\log(10+u^2))$$

with $0 < c < 8/225$ and smooth initial data ($u(0) = u_0,\ \partial_t u(0) = u_1$). First we control the $L_t^4 L_x^{12}$ norm of the solution on an arbitrary size time interval by an expression depending on the energy and an a priori upper bound of its $L_t^\infty \tilde{H}^2(\mathbb{R}^3)$ norm, with $\tilde{H}^2(\mathbb{R}^3) := \dot{H}^2(\mathbb{R}^3) \cap \dot{H}^1(\mathbb{R}^3)$. The proof of this long time estimate relies upon the use of some potential decay estimates and a modification of an argument by Tao. Then we find an a posteriori upper bound of the $L_t^\infty \tilde{H}^2(\mathbb{R}^3)$ norm of the solution by combining the long time estimate with an induction on time of the Strichartz estimates.

1. Introduction

We shall consider the defocusing log-log energy-supercritical wave equation

$$\partial_{tt} u - \triangle u = -f(u) \tag{1-1}$$

where $u : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ is a real-valued scalar field and $f(u) := u^5 g(u)$ with $g(u) := \log^c(\log(10+u^2))$, $0 < c < 8/225$. *Classical solutions* of (1-1) are solutions that are infinitely differentiable and compactly supported in space for each fixed time t. It is not difficult to see that classical solutions of (1-1) satisfy the energy conservation law

$$E := \frac{1}{2}\int_{\mathbb{R}^3} (\partial_t u(t,x))^2\, dx + \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u(t,x)|^2\, dx + \int_{\mathbb{R}^3} F(u(t,x))\, dx \tag{1-2}$$

where $F(u) := \int_0^u f(v)\, dv$. Classical solutions of (1-1) enjoy three symmetry properties that we use throughout this paper:

- *time translation invariance*: if u is a solution of (1-1) and t_0 is a fixed time then $\tilde{u}(t,x) := u(t-t_0, x)$ is also a solution of (1-1);
- *space translation invariance*: if u is a solution of (1-1) and x_0 is a fixed point lying in $\mathbb{R}^3$ then $\tilde{u}(t,x) := u(t, x-x_0)$ is also a solution of (1-1);
- *time reversal invariance*: if u is a solution to (1-1) then $\tilde{u}(t,x) := u(-t,x)$ is also a solution.

MSC2000: 35Q55.
Keywords: global regularity, log-log energy supercritical wave equation.

The defocusing log-log energy-supercritical wave equation (1-1) is closely related to the power-type defocusing wave equations, namely,

$$\partial_{tt} u - \triangle u = -|u|^{p-1} u. \tag{1-3}$$

Solutions of (1-3) have an invariant scaling

$$u(t,x) \to u^{\lambda}(t,x) := \frac{1}{\lambda^{2/(p-1)}} u\Big(\frac{t}{\lambda}, \frac{x}{\lambda}\Big) \tag{1-4}$$

and (1-3) is s_c-critical, where $s_c := \frac{3}{2} - \frac{2}{p-1}$. Thus the $\dot{H}^{s_c}(\mathbb{R}^3) \times \dot{H}^{s_c-1}(\mathbb{R}^3)$ norm of $(u(0), \partial_t u(0))$ is invariant under scaling, i.e.,

$$\begin{aligned} \|u^{\lambda}(0)\|_{\dot{H}^{s_c}(\mathbb{R}^3)} &= \|u(0)\|_{\dot{H}^{s_c}(\mathbb{R}^3)}, \\ \|\partial_t u^{\lambda}(0)\|_{\dot{H}^{s_c-1}(\mathbb{R}^3)} &= \|\partial_t u(0)\|_{\dot{H}^{s_c-1}(\mathbb{R}^3)}. \end{aligned}$$

If $p = 5$, then $s_c = 1$ and this is why the quintic defocusing cubic wave equation

$$\partial_{tt} u - \triangle u = -u^5 \tag{1-5}$$

is called the energy-critical equation. If $1 < p < 5$ then $s_c < 1$ and (1-3) is energy-subcritical while if $p > 5$ then $s_c > 1$ and (1-3) is energy-supercritical. Notice that for every $p > 5$ there exists two positive constant $\lambda_1(p)$, $\lambda_2(p)$ such that

$$\lambda_1(p)|u|^5 \le |f(u)| \le \lambda_2(p) \max(1, |u|^p). \tag{1-6}$$

This is why (1-1) is said to belong to the group of barely supercritical equations. There is another way to see that. Notice that a simple integration by part shows that

$$F(u) \sim \frac{u^6}{6} g(u), \tag{1-7}$$

and consequently the nonlinear potential term of the energy $\int_{\mathbb{R}^3} F(u)\,dx \sim \int_{\mathbb{R}^3} u^6 g(u)\,dx$ just barely fails to be controlled by the linear component, in contrast to (1-5).

The energy-critical wave equation (1-5) has received a great deal of attention. Grillakis [1990; 1992] established global existence of smooth solutions (global regularity) of this equation with smooth initial data $u(0) = u_0$, $\partial_t u(0) = u_1$. His work followed that of Rauch [1981, part I] for small data and that of Struwe [1988] on the spherically symmetric case. Later Shatah and Struwe [1993] gave a simplified proof of this result. Kapitanski [1994] and, independently, Shatah and Struwe [1994] proved global existence of solutions with data (u_0, u_1) in the energy class.

We are interested in proving global regularity of (1-1) with smooth initial data (u_0, u_1). By standard persistence of regularity results it suffices to prove global existence of solutions

$$u \in \mathscr{C}\big([0,\ T],\ \tilde{H}^2(\mathbb{R}^3)\big) \cap \mathscr{C}^1\big([0,\ T], H^1(\mathbb{R}^3)\big),$$

with data $(u_0, u_1) \in \tilde{H}^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$. Here the following space

$$\tilde{H}^2(\mathbb{R}^3) := \dot{H}^2(\mathbb{R}^3) \cap \dot{H}^1(\mathbb{R}^3). \tag{1-8}$$

In view of the local well-posedness theory [Lindblad and Sogge 1995], standard limit arguments and the finite speed of propagation it suffices to find an a priori upper bound of the form

$$\big\|(u(T), \partial_t u(T))\big\|_{\tilde{H}^2(\mathbb{R}^3)\times H^1(\mathbb{R}^3)} \leq C_1\big(\|u_0\|_{\tilde{H}^2(\mathbb{R}^3)}, \ \|u_1\|_{H^1(\mathbb{R}^3)}, \ T\big) \tag{1-9}$$

for all times $T > 0$ and for classical solutions u of (1-1) with smooth and compactly supported data (u_0, u_1). Here C_1 is a constant depending only on $\|u_0\|_{\tilde{H}^2(\mathbb{R}^3)}$, $\|u_1\|_{H^1(\mathbb{R}^3)}$ and the time T.

The global behavior of the solutions of the supercritical wave equations is poorly understood, mostly because of the lack of conservation laws in $\tilde{H}^2(\mathbb{R}^3)$. Nevertheless Tao [2007] was able to prove global regularity for another barely supercritical equation, namely

$$\partial_{tt} u - \triangle u = -u^5 \log(2 + u^2), \tag{1-10}$$

with radial data. The main result of this paper is:

Theorem 1. *The solution of* (1-1) *with smooth data* (u_0, u_1) *exists for all time. Moreover there exists a nonnegative constant* $M_0 = M_0(\|u_0\|_{\tilde{H}^2(\mathbb{R}^3)}, \|u_1\|_{H^1(\mathbb{R}^3)})$ *depending only on* $\|u_0\|_{\tilde{H}^2(\mathbb{R}^3)}$ *and* $\|u_1\|_{H^1(\mathbb{R}^3)}$ *such that*

$$\|u\|_{L_t^\infty \tilde{H}^2(\mathbb{R}\times\mathbb{R}^3)} + \|\partial_t u\|_{L_t^\infty H^1(\mathbb{R}\times\mathbb{R}^3)} \leq M_0. \tag{1-11}$$

We recall some basic properties and estimates. Let Q be a function, let J be an interval and let $t_0 \in J$ be a fixed time. If u is a *classical solution* of the more general problem $\partial_{tt} u - \triangle u = Q$ then u satisfies the Duhamel formula

$$u(t) = u_{l,t_0}(t) + u_{nl,t_0}(t), \quad t \in J, \tag{1-12}$$

with u_{l,t_0}, u_{nl,t_0} denoting the linear part and the nonlinear part respectively of the solution starting from t_0. Recall that

$$u_{l,t_0}(t) = \cos(t - t_0)D u(t_0) + \frac{\sin(t - t_0)D}{D}\partial_t u(t_0) \tag{1-13}$$

and

$$u_{nl,t_0}(t) = -\int_{t_0}^{t} \frac{\sin(t - t')D}{D} Q(t')\, dt', \tag{1-14}$$

with D the multiplier defined by $\widehat{Df}(\xi) := |\xi|\widehat{f}(\xi)$. An explicit formula for $((\sin(t - t')D)/D)Q(t')$ and $t \neq t'$ is

$$\Big[\frac{\sin(t - t')D}{D} Q(t')\Big](x) = \frac{1}{4\pi|t - t'|}\int_{|x - x'| = |t - t'|} Q(t', x')\, dS(x'). \tag{1-15}$$

For a proof see [Sogge 1995]. We recall that u_{l,t_0} satisfies

$$\partial_{tt} u_{l,t_0} - \triangle u_{l,t_0} = 0, \quad u_{l,t_0}(t_0) = u(t_0), \quad \partial_t u_{l,t_0}(t_0) = \partial_t u(t_0),$$

while u_{nl,t_0} is the solution of

$$\partial_{tt} u_{nl,t_0} - \triangle u_{nl,t_0} = Q, \quad u_{nl,t_0}(t_0) = 0, \quad \partial_t u_{nl,t_0}(t_0) = 0.$$

We recall the Strichartz estimate [Ginibre and Velo 1995; Keel and Tao 1998; Lindblad and Sogge 1995; Sogge 1995]

$$\|u\|_{L_t^q L_x^r(J\times\mathbb{R}^3)} \lesssim \|\partial_t u(t_0)\|_{L_x^2(\mathbb{R}^3)} + \|\nabla u(t_0)\|_{L_x^2(\mathbb{R}^3)} + \|Q\|_{L_t^1 L_x^2(J\times\mathbb{R}^3)}, \tag{1-16}$$

if (q, r) is wave admissible, that is, $(q, r) \in (2, \infty] \times [2, \infty]$ and $1/q + 3/r = 1/2$.

We set some notation that appears throughout the paper. We write $C = C(a_1, \dots, a_n)$ if C only depends on the parameters $a_1, \dots, a_n$. We write $A \lesssim B$ if there exists a universal nonnegative constant $C' > 0$ such that $A \leq C'B$. $A = O(B)$ means $A \lesssim B$. More generally we write $A \lesssim_{a_1,\dots,a_n} B$ if there exists a nonnegative constant $C' = C(a_1, \dots, a_n)$ such that $A \leq C'B$. We say that C'' is the constant determined by $\lesssim$ in $A \lesssim_{a_1,\dots,a_n} B$ if C'' is the smallest constant among the C's such that $A \leq C'B$. We write $A \ll_{a_1,\dots,a_n} B$ if there exists a universal nonnegative small constant $c = c(a_1, \dots, a_n)$ such that $A \leq cB$. Similar notions are defined for $A \gtrsim B$, $A \gtrsim_{a_1,\dots,a_n} B$ and $A \gg B$. In particular we say that C'' is the constant determined by $\gtrsim$ in $A \gtrsim B$ if C'' is the largest constant among the C's such that $A \geq C'B$. If x is number then $x+$ and $x-$ are slight variations of x: $x+ := x + \alpha\epsilon$ and $x- := x - \beta\epsilon$ for some $\alpha > 0$, $\beta > 0$ and $0 < \epsilon \ll 1$.

Let Γ_+ denote the forward light cone

$$\Gamma_+ = \{(t, x) : t > |x|\}, \tag{1-17}$$

and if $J = [a, b]$ is an interval, let $\Gamma_+(J)$ denote the light cone truncated to J, that is,

$$\Gamma_+(J) := \Gamma_+ \cap (J \times \mathbb{R}^3). \tag{1-18}$$

Let $e(t)$ denote the local energy, that is,

$$e(t) := \frac{1}{2}\int_{|x|\leq t} (\partial_t u(t, x))^2 \, dx + \frac{1}{2}\int_{|x|\leq t} |\nabla u(t, x)|^2 \, dx + \int_{|x|\leq t} F(u(t, x))\, dx. \tag{1-19}$$

If u is a solution of (1-1) then by using the finite speed of propagation and the Strichartz estimates we have

$$\|u\|_{L_t^q L_x^r(\Gamma_+(J))} \lesssim \|\nabla u(b)\|_{L_x^2(\mathbb{R}^3)} + \|\partial_t u(b)\|_{L_x^2(\mathbb{R}^3)} + \|Q\|_{L_t^1 L_x^2(\Gamma_+(J))} \tag{1-20}$$

if (q, r) is wave admissible. If $J_1 := [a_1, a_2]$ and $J_2 := [a_2, a_3]$ then we also have

$$\|u\|_{L_t^q L_x^r(\Gamma_+(J_1))} \lesssim \|\nabla u(a_3)\|_{L_x^2(\mathbb{R}^3)} + \|\partial_t u(a_3)\|_{L_x^2(\mathbb{R}^3)} + \|Q\|_{L_t^1 L_x^2(\Gamma_+(J_1\cup J_2))}. \tag{1-21}$$

We recall also the well-known Sobolev embeddings. If h is a smooth function then

$$\|h\|_{L^\infty(\mathbb{R}^3)} \lesssim \|h\|_{\tilde{H}^2(\mathbb{R}^3)} \tag{1-22}$$

and

$$\|h\|_{L^6(\mathbb{R}^3)} \lesssim \|\nabla h\|_{L^2(\mathbb{R}^3)}. \tag{1-23}$$

If u is the solution of (1-1) with data $(u_0, u_1) \in \tilde{H}^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$, then we get from (1-22)

$$E \lesssim \|u_0\|^2_{\tilde{H}^2(\mathbb{R}^3)} \max\big(1, \|u_0\|^4_{\tilde{H}^2(\mathbb{R}^3)} g(\|u_0\|_{\tilde{H}^2(\mathbb{R}^3)})\big). \tag{1-24}$$

We shall use the Paley–Littlewood technology. Let $\phi(\xi)$ be a bump function adapted to $\{\xi \in \mathbb{R}^3 : |\xi| \leq 2\}$ and equal to one on $\{\xi \in \mathbb{R}^3 : |\xi| \leq 1\}$. If $(M, N) \in 2^{\mathbb{Z}} \times 2^{\mathbb{Z}}$ are dyadic numbers then the Paley–Littlewood projection operators P_M, $P_{<N}$ and $P_{\geq N}$ are defined in the Fourier domain by

$$\widehat{P_M f}(\xi) := \left(\phi\Big(\frac{\xi}{M}\Big) - \phi\Big(\frac{\xi}{2M}\Big)\right)\hat{f}(\xi), \quad \widehat{P_{<N} f}(\xi) := \sum_{M<N} \widehat{P_M f}(\xi), \quad \widehat{P_{\geq N} f}(\xi) := \sum_{M\geq N} \widehat{P_M f}(\xi).$$

The inverse Sobolev inequality can be stated as follows:

Proposition 2 (Inverse Sobolev inequality [Tao 2006]). *Let g be a smooth function such that*

$$\|g\|_{\dot{H}^1(\mathbb{R}^3)} \lesssim E^{1/2} l, \qquad \|P_{\geq N} g\|_{L_x^6(\mathbb{R}^3)} \gtrsim \eta,$$

for some real number $\eta > 0$ and for some dyadic number $N > 0$. Then there exists a ball $B(x,r) \subset \mathbb{R}^3$ with $r = O(1/N)$ such that we have the mass concentration estimate

$$\int_{B(x,r)} |g(y)|^2 \, dy \gtrsim \eta^3 E^{-1/2} r^2. \tag{1-25}$$

We also recall a result that shows that the mass of solutions of (1-1) can be locally in time controlled.

Proposition 3 (Local mass is locally stable [Tao 2006]). *Let J be a time interval, let t, $t' \in J$ and let $B(x,r)$ be a ball. Let u be a solution of* (1-1). *Then*

$$\left(\int_{B(x,r)} |u(t',y)|^2 \, dy \right)^{1/2} = \left(\int_{B(x,r)} |u(t,y)|^2 \, dy \right)^{1/2} + O\big(E^{1/2} |t-t'|\big). \tag{1-26}$$

This result, proved for (1-5) in [Tao 2006], is also true for (1-1). Indeed the proof relied upon the fact that the $L^2(\mathbb{R}^3)$ norm of the velocity of the solution of (1-5) at time t is bounded by the square root of its energy, which is also true for the solution of (1-1) (by (1-2) and (1-7)).

Now we make some comments with respect to Theorem 1. If the function g were a positive constant, it would be easy to prove that the solution of (1-1) with data (u_0, u_1) lies in $\tilde{H}^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$, since we have a good global theory for (1-5). Therefore we can hope to prove global well-posedness for g slowly increasing to infinity, by extending the technology to prove global well-posedness for (1-5). Notice also that Tao [2006] found that the solution u of (1-5) satisfies

$$\|u\|_{L_t^4 L_x^{12}(\mathbb{R} \times \mathbb{R}^3)} \lesssim \tilde{E}^{\tilde{E}^{O(1)}}, \tag{1-27}$$

with $\tilde{E}$ the energy of u. The structure of g is a double log: it is, roughly speaking, the inverse function of the towel exponential bound in (1-27).

Now we explain the main ideas of this paper.

Tao [2006] was able to bound on arbitrary long time intervals the $L_t^4 L_x^{12}$ norm of solutions of the energy-critical equation (1-5) by a quantity that depends exponentially on their energy. This estimate can be viewed as a long time estimate. Unfortunately we cannot expect to prove a similar result for (1-1) since we are not in the energy-critical regime. However we shall prove the following proposition:

Proposition 4 (Long time estimate). *Let $J = [t_1, t_2]$ be a time interval. Let u be a classical solution of* (1-1). *Assume that*

$$\|u\|_{L_t^\infty \tilde{H}^2(J \times \mathbb{R}^3)} \leq M \tag{1-28}$$

for some $M \geq 0$. Then there exist three constants $C_{L,0} > 0$, $C_{L,1} > 0$ and $C_{L,2} > 0$ such that

- *if $E \ll \dfrac{1}{g^{1/2}(M)}$ (small energy regime) then*

$$\|u\|^4_{L_t^4 L_x^{12}(J \times \mathbb{R}^3)} \leq C_{L,0}; \tag{1-29}$$

- *if $E \gtrsim \dfrac{1}{g^{1/2}(M)}$ (large energy regime) then*

$$\|u\|^4_{L_t^4 L_x^{12}(J \times \mathbb{R}^3)} \leq \big(C_{L,1}(E g(M))\big)^{C_{L,2}(E^{193/4+} g^{225/8+}(M))}. \tag{1-30}$$

This proposition shows that we can control the $L_t^4 L_x^{12}(J \times \mathbb{R}^3)$ norm of solutions of (1-1) by their energy and an a priori bound of their $L_t^\infty \tilde{H}^2(J \times \mathbb{R}^3)$ norm. We would like to control the pointwise-in-time $\tilde{H}^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$ norm of u on an interval $[0, T]$, with T arbitrarily large. This is done by an induction on time. We assume that this norm is controlled on $[0, T]$ by a number M_0. Then by continuity we can find a slightly larger interval $[0, T']$ such that this norm is bounded by (say) $2M_0$ on $[0, T']$. This is our a priori bound. We subdivide $[0, T']$ into subintervals where the $L_t^4 L_x^{12}$ norm of u is small and we control the pointwise-in-time $\tilde{H}^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$ norm of u on each of these subintervals (see Lemma 6). Since g varies slowly we can estimate the number of intervals of this partition by using Proposition 4 and we can prove a posteriori that $\|u(t)\|_{\tilde{H}^2(\mathbb{R}^3)} + \|\partial_t u(t)\|_{\tilde{H}^1(\mathbb{R}^3)}$ is bounded on $[0, T']$ by M_0, provided that M_0 is large enough; see Section 2.

The proof of Proposition 4 is a modification of the argument used in [Tao 2006] to establish a tower-exponential bound of the $L_t^4 L_x^{12}(J \times \mathbb{R}^3)$ norm of v, the solution of (1-5). We divide J into subintervals J_i where the $L_t^4 L_x^{12}$ norm of u, the solution of (1-1), is "substantial". Then by using the Strichartz estimates and the Sobolev embedding (1-22) we notice that the $L_t^\infty L_x^6(J_i \times \mathbb{R}^3)$ norm of u is also substantial, more precisely, we find a lower bound that depends on the energy E and $g(M)$. Then by Proposition 2 we can localize a bubble where the mass concentrates and we prove that the size of these subintervals is also substantially large. Tao [2006] used the mass concentration to construct a solution $\tilde{v}$ of (1-5) that has a smaller energy than v and that coincides with v outside a cone. The idea behind that is to use an induction on the levels of energy, due to Bourgain [1999], and the small energy theory following from the Strichartz estimates in order to control the $L_t^4 L_x^{12}$ norm of v outside a cone. Unfortunately it seems almost impossible to apply this procedure to our problem. Indeed the energy of the constructed solution $\tilde{u}$ is smaller than the energy E of u by an amount that depends on E but also on $g(M)$ and therefore an induction on the levels of the energy is possible if the $L_t^\infty \tilde{H}^2(J \times \mathbb{R}^3)$ norm of $\tilde{u}$ can be controlled by M, which is far from being trivial. It turns out that we do not need to use the Bourgain induction method. Indeed since we know that the size of the subintervals J_i s is substantially large and since we have a good control of the $L_t^4 L_x^{12}$ norm on these subintervals it suffices to find an upper bound of the size of their union in order to conclude. To this end we divide a cone containing the ball where the mass concentrates and the J_is into truncated-in-time cones where the $L_t^4 L_x^{12}$ norm of u is substantial. Let $\tilde{J}_1, \tilde{J}_2, \dots$ be the sequence of time intervals resulting from this partition. The mass concentration helps us to control the size of the first time interval $\tilde{J}_1$. By using an asymptotic stability result we can prove, roughly speaking, that if we consider two successive subintervals $\tilde{J}_j$, $\tilde{J}_{j+1}$ resulting from this partition of the cone then the size of $\tilde{J}_{j+1}$ can be controlled by the size of $\tilde{J}_j$; see (3-34). But a potential energy decay estimate shows that if the size of the union of the J_i s is too large then we can find a large subinterval $[t_1', t_2']$ such that the $L_t^4 L_x^{12}$ norm of u on the cone truncated to $[t_1', t_2']$ is small. Therefore $[t_1', t_2']$ cannot be covered by many $\tilde{J}_j$s and one of them is very large in comparison with its predecessor, which contradicts (3-34). At the end of the process we can find an upper bound of the size of the union of the subintervals J_i s and consequently we can control the $L_t^4 L_x^{12}$ norm of u on the interval J.

Remark 5. We will frequently use the $x+$ and $x-$ notations. Indeed the point $(2, \infty)$ is not wave admissible. Therefore we will work with the point $(2+, \infty-)$: see (5-6) and (7-9). This generates slight variations of many quantities throughout this paper. Sometimes we might deal with quantities like $z := x+/y-$. We cannot conclude directly that $z = (x/y)+$. In this case we create a variation of y so small (compared to that of x) that we have $z = (x/y)+$. These details have been omitted for the sake of readability. We strongly recommend that the reader ignores these slight variations at the first reading.

2. Proof of Theorem 1

The proof relies upon Proposition 4 and the following lemma, which we prove on page 268.

Lemma 6 (Local boundedness). *Let $J = [t_1, t_2]$ be an interval. Assume that u is a classical solution of* (1-1). *Let $Z(t) := \|(u(t), \partial_t u(t))\|_{\tilde{H}^2(\mathbb{R}^3)\times H^1(\mathbb{R}^3)}$. There exists $0 < \epsilon \ll$ constant such that if*

$$\|u\|_{L_t^4 L_x^{12}(J\times\mathbb{R}^3)} \leq \frac{\epsilon}{g^{1/4}(Z(t_1))}, \tag{2-1}$$

then there exists $C_l > 0$ such that

$$Z(t) \leq 2C_l Z(t_1) \quad \textit{for } t \in J. \tag{2-2}$$

We claim that the set

$$\mathcal{F} := \left\{T \in [0,\ \infty) : \sup_{t\in[0,T]} \left\|(u(t), \partial_t u(t))\right\|_{\tilde{H}^2(\mathbb{R}^3)\times H^1(\mathbb{R}^3)} \leq M_0\right\} \tag{2-3}$$

is equal to $[0,\ \infty)$ for some constant $M_0 := M_0(\|u_0\|_{\tilde{H}^2(\mathbb{R}^3)}, \|u_1\|_{H^1(\mathbb{R}^3)})$ large enough. Indeed, $0 \in \mathcal{F}$ (this is clear); $\mathcal{F}$ is closed, by continuity; and $\mathcal{F}$ is open. To see this last fact, let $T \in \mathcal{F}$. Then by continuity there exists $\delta > 0$ such that

$$\sup_{t\in[0,T']} \|\ (u(t), \partial_t u(t))\ \|_{H^2(\mathbb{R}^3)\times H^1(\mathbb{R}^3)} \leq 2M_0 \tag{2-4}$$

for every $T' \in [0, T+\delta)$. By (1-29) and (1-30) we have

$$\|u\|^4_{L_t^4 L_x^{12}([0,T']\times\mathbb{R}^3)} \leq \max\left(C_{L,0},\, (C_{L,1} E\, g(2M_0))^{C_{L,2}(E^{(193/4)+} g^{(225/8)+}(2M_0))}\right). \tag{2-5}$$

Let $N \geq 1$ and let $\underline{Z}(0) := \max(Z(0),\ 1)$. Without loss of generality we can assume that $C_l \gg 1$ so that $2C_l\underline{Z}(0) \gg 1$ and $\log^c\left(2C_l\underline{Z}(0)\right) \gg 1$. We have, by the elementary rules of the logarithm and the inequality $\log^c(2nx) \leq \log^c((2n)^x)$ for $n \geq 1$ and $x \gg 1$:

$$\begin{aligned}
\sum_{n=1}^{N} \frac{\epsilon^4}{g\left((2C_l)^n Z_0\right)} &\geq \sum_{n=1}^{N} \frac{\epsilon^4}{\log^c\left(\log((2C_l)^{2n}\underline{Z}^{2n}(0)+10)\right)} \gtrsim \sum_{n=1}^{N} \frac{1}{\log^c\left(2n\log\left(2C_l\underline{Z}(0)\right)\right)} \\
&\gtrsim \frac{1}{\log^c\left(2C_l\underline{Z}(0)\right)} \sum_{n=1}^{N} \frac{1}{\log^c(2n)} \gtrsim \frac{1}{\log^c\left(2C_l\underline{Z}(0)\right)} \int_1^{N+1} \frac{1}{\log^c(2t)}\, dt \\
&\gtrsim \frac{1}{\log^c\left(2C_l\underline{Z}(0)\right)} \int_1^{N+1} \frac{1}{t^{1/2}}\, dt \gtrsim \frac{N^{1/2}}{\log^c\left(2C_l\underline{Z}(0)\right)}.
\end{aligned} \tag{2-6}$$

By Lemma 6, (2-5) and (2-6) we can construct a partition $(J_n)_{1\le n\le N}$ of $[0, T']$ such that

$$\|u\|_{L_t^4 L_x^{12}(J_n\times\mathbb{R}^3)} = \frac{\epsilon}{g^{1/4}\left((2C_l)^n Z_0\right)}, \quad 1\le n<N,$$
$$\|u\|_{L_t^4 L_x^{12}(J_N\times\mathbb{R}^3)} \le \frac{\epsilon}{g^{1/4}\left((2C_l)^N Z_0\right)}, \quad Z(t)\le (2C_l)^n Z(0),$$

for $t\in J_1\cup\cdots\cup J_n$ and

$$\frac{N^{1/2}}{\log^c(2C_l\underline{Z}(0))} \le \max\left(C_{L,0},\, (C_{L,1}E\, g(2M_0))^{C_{L,2}(E^{193/4+}g^{225/8+}(2M_0))}\right). \tag{2-7}$$

Since $c<8/225$ we have by (1-24)

$$\begin{aligned}\log N &\lesssim \log^c(2C_l\underline{Z}(0)) + \log(C_{L,0})\\ &\qquad + C_{L,2}E^{(193/4)+}\log^{(225c/8)+}\log(10+4M_0^2)\log\left(C_{L,1}E\log^c\log(10+4M_0^2)\right)\\ &\le \log\left(\frac{\log(M_0/Z(0))}{\log(2C_l)}\right),\end{aligned} \tag{2-8}$$

if $M_0 = M_0(\|u_0\|_{\tilde{H}^2(\mathbb{R}^3)}, \|u_1\|_{H^1(\mathbb{R})})$ is large enough. To prove the last inequality in (2-8) it is enough, by using (1-24), to notice that $\lim_{M_0\to\infty} f(M_0)=0$ with

$$f(M_0) := \frac{\log^c(2C_l\underline{Z}(0)) + \log(C_{L,0}) + C_{L,2}E^{(193/4)+}\log^{(225c/8)+}\log(10+4M_0^2)\log(C_{L,1}E\log^c\log(10+4M_0^2))}{\log\left(\frac{\log(M_0/Z(0))}{\log(2C_l)}\right)}. \tag{2-9}$$

Therefore we conclude that

$$\sup_{t\in[0,T']}\|(u(t),\partial_t u(t))\|_{H^2(\mathbb{R}^3)\times H^1(\mathbb{R}^3)} \le (2C_l)^N Z(0)\le M_0. \tag{2-10}$$

Proof of Lemma 6. By the Strichartz estimates (1-16), the Sobolev embeddings (1-22) and (1-23) and the elementary estimate $|u^5\nabla(g(u))|\lesssim |u^4\nabla u g(u)|$, we have

$$\begin{aligned} Z(t) &\lesssim Z(t_1) + \|u^5 g(u)\|_{L_t^1L_x^2([t_1,t]\times\mathbb{R}^3)} + \|u^4\nabla u g(u)\|_{L_t^1L_x^2([t_1,t]\times\mathbb{R}^3)} + \|u^5\nabla(g(u))\|_{L_t^1L_x^2([t_1,t]\times\mathbb{R}^3)}\\ &\lesssim Z(t_1) + \|u^5 g(u)\|_{L_t^1L_x^2([t_1,t]\times\mathbb{R}^3)} + \|u^4\nabla u g(u)\|_{L_t^1L_x^2([t_1,t]\times\mathbb{R}^3)}\\ &\lesssim Z(t_1) + \|u\|^4_{L_t^4L_x^{12}([t_1,t]\times\mathbb{R}^3)}\|u\|_{L_t^\infty L_x^6([t_1,t]\times\mathbb{R}^3)} g\left(\|u\|_{L_t^\infty L_x^\infty([t_1,t]\times\mathbb{R}^3)}\right)\\ &\qquad\qquad + \|u\|^4_{L_t^4L_x^{12}([t_1,t]\times\mathbb{R}^3)}\|\nabla u\|_{L_t^\infty L_x^6([t_1,t]\times\mathbb{R}^3)} g\left(\|u\|_{L_t^\infty L_x^\infty([t_1,t]\times\mathbb{R}^3)}\right)\\ &\lesssim Z(t_1) + \|u\|^4_{L_t^4L_x^{12}([t_1,t]\times\mathbb{R}^3)} Z(t) g(Z(t)).\end{aligned} \tag{2-11}$$

Let C_l be the constant determined by the last inequality in (2-11). From (2-1), (2-11) and a continuity argument, we have (2-2). □

3. Proof of Proposition 4

The proof relies upon five lemmas, which we state here and then prove in subsequent sections, after seeing how they imply the proposition.

Lemma 7 (Long time estimate if energy small). *Let $J=[t_1,t_2]$ be a time interval. Let u be a classical solution of* (1-1). *Assume that* (1-28) *holds. If*

$$E \ll \frac{1}{g^{1/2}(M)}, \tag{3-1}$$

then

$$\|u\|_{L_t^4 L_x^{12}(J\times\mathbb{R}^3)} \lesssim 1. \tag{3-2}$$

Lemma 8 (If $\|u\|_{L_t^4 L_x^{12}(J\times\mathbb{R}^3)}$ is nonnegligible a mass concentration bubble exists and the size of J is bounded from below). *Let u be a classical solution of* (1-1). *Let J be a time interval. Assume that* (1-28) *holds. Let η be a positive number such that*

$$\eta \leq \frac{E^{1/12}}{g^{5/24}(M)}. \tag{3-3}$$

If $\|u\|_{L_t^4 L_x^{12}(J\times\mathbb{R}^3)} \geq \eta$, then

$$\|u\|_{L_t^\infty L_x^6(J\times\mathbb{R}^3)} \gtrsim \eta^{2+} E^{-((1/2)+)}. \tag{3-4}$$

Moreover, there exist a point $x_0\in\mathbb{R}^3$, a time $t_0\in J$ and a positive number r such that we have the mass concentration estimate in the ball $B(x_0,r)$

$$\int_{B(x_0,r)} |u(t_0,y)|^2\,dy \gtrsim \eta^{6+} E^{-(2+)} r^2, \tag{3-5}$$

and the following lower bound on the size of J:

$$|J| \gtrsim \eta^4 E^{-2/3} r. \tag{3-6}$$

Lemma 9 (Potential energy decay estimate). *Let u be a classical solution of* (1-1). *Let $[a,b]$ be an interval. Then we have the potential energy decay estimate*

$$\int_{|x|\leq b} F(u(b,x))\,dx \lesssim \frac{a}{b}\left(e(a)+e^{1/3}(a)\right) + e(b)-e(a) + (e(b)-e(a))^{1/3}. \tag{3-7}$$

Lemma 10 ($L_t^4 L_x^{12}$ norm of u is small on a large truncation of the forward light cone). *Let $J=[t_1,\ t_2]$ be an interval. Let u be a classical solution of* (1-1). *Assume that* (1-28) *holds. Let η be a positive number such that*

$$\eta \ll \min\Bigl(E^{1/4}, E^{5/18}, \frac{E^{1/12}}{g^{5/24}(M)}\Bigr). \tag{3-8}$$

Assume also that there exists $C_2 \gg 1$ such that

$$\bigl[t_1,\ (C_2 E^{10+}\eta^{-(36+)})^{4C_2 E^{10+}\eta^{-(36+)}} t_1\bigr] \subset J. \tag{3-9}$$

Then there exists a subinterval $J'=[t_1',t_2']$ such that $\left|t_2'/t_1'\right| \sim E^{10+}\eta^{-(36+)}$ and

$$\|u\|_{L_t^4 L_x^{12}(\Gamma_+(J'))} \leq \eta. \tag{3-10}$$

Lemma 11 (Asymptotic stability). *Let $J=[t_1,t_2]$ be a time interval. Let $J'=[t_1',\ t_2']\subset J$ and let $t\in J/J'$. Let u be a classical solution of* (1-1). *Assume that* (1-28) *holds. Then*

$$\|u_{l,t_2'}(t)-u_{l,t_1'}(t)\|_{L_x^\infty(\mathbb{R}^3)} \lesssim \frac{E^{5/6} g^{1/6}(M)}{\operatorname{dist}^{1/2}(t,\ J')}. \tag{3-11}$$

We are ready to prove Proposition 4. We assume that we have an a priori bound M of the $L_t^\infty \tilde{H}^2(J\times\mathbb{R}^3)$ norm of the solution u. There are two steps:

- If $E \ll 1/g^{1/2}(M)$, then we know from Lemma 7 that (1-29) holds.
- Therefore we assume that the energy is large, that is,

$$E \gtrsim \frac{1}{g^{1/2}(M)}. \tag{3-12}$$

We can assume without loss of generality that

$$\|u\|_{L_t^4 L_x^{12}(J\times\mathbb{R}^3)} \geq \frac{E^{1/12}}{g^{5/24}(M)}. \tag{3-13}$$

From (3-13) we can partition J into subintervals $J_1, \dots, J_l$ such that for $i = 1, \dots, l-1$,

$$\|u\|_{L_t^4 L_x^{12}(J_i\times\mathbb{R}^3)} = \frac{E^{1/12}}{g^{5/24}(M)} \quad \text{and} \quad \|u\|_{L_t^4 L_x^{12}(J_l\times\mathbb{R}^3)} \leq \frac{E^{1/12}}{g^{5/24}(M)}. \tag{3-14}$$

Before moving forward we say that an interval J_i is *exceptional* if

$$\|u_{l,t_1}\|_{L_t^4 L_x^{12}(J_i\times\mathbb{R}^3)} + \|u_{l,t_2}\|_{L_t^4 L_x^{12}(J_i\times\mathbb{R}^3)} \geq \frac{1}{(C_3 E g(M))^{C_4(E^{(193/4)+} g^{(225/8)+}(M))}}, \tag{3-15}$$

for some $C_3 \gg 1$, $C_4 \gg 1$ to be chosen later. (The numbers 193/4 and 225/8 will play an important role in (3-44).) Otherwise J_i is *unexceptional*. Let $\mathscr{E}$ denote the set of J_i' s that are exceptional and let $\overline{\mathscr{E}^c}$ denote the set of nonempty sequences of consecutive unexceptional intervals J_i. By (1-16), (3-12) and (3-15),

$$\operatorname{card}(\mathscr{E}) \lesssim E^2 \left[O(Eg(M))\right]^{O(E^{(193/4)+} g^{(225/8)+}(M))} \lesssim \left[O(Eg(M))\right]^{O(E^{(193/4)+} g^{(225/8)+}(M))}. \tag{3-16}$$

Since $\operatorname{card}(\overline{\mathscr{E}^c}) \lesssim \operatorname{card}(\mathscr{E})$ we have

$$\|u\|^4_{L_t^4 L_x^{12}(J\times\mathbb{R}^3)} \lesssim \left[O(Eg(M))\right]^{O(E^{(193/4)+} g^{(225/8)+}(M))} \left(\frac{E^{1/3}}{g^{5/6}(M)} + \sup_{K\in\overline{\mathscr{E}^c}} \|u\|^4_{L_t^4 L_x^{12}(K\times\mathbb{R}^3)}\right). \tag{3-17}$$

Let $K = J_{i_0} \cup \dots \cup J_{i_1}$ be a sequence of consecutive unexceptional intervals. If $N(K)$ is the number of J_i s making K then by (3-12), (3-14) and (3-17) we have

$$\|u\|_{L_t^4 L_x^{12}(J\times\mathbb{R}^3)} \lesssim \left(\sup_{K\in\overline{\mathscr{E}^c}} N(K)\right) \left[O(Eg(M))\right]^{O(E^{(193/4)+} g^{(225/8)+}(M))}. \tag{3-18}$$

Therefore it suffices to estimate $N(K)$ for every $K = J_{i_0} \cup \dots \cup J_{i_1}$. We will do that by first determining a lower bound for the size of the elements J_is and then by estimating the size of K. By (3-12), (3-14) and Lemma 8, there exists for $i \in [i_0, \dots i_1]$ a $(t_i, r_i, x_i) \in \left(J_i \times (0,\infty) \times \mathbb{R}^3\right)$ such that

$$\frac{1}{r_i^2} \int_{B(x_i,r_i)} |u(t_i, y)|^2 \, dy \gtrsim \frac{E^{-(3/2+)}}{g^{5/4+}(M)} \tag{3-19}$$

and

$$|J_i| \gtrsim \frac{E^{-1/3} r_i}{g^{5/6}(M)}. \tag{3-20}$$

Let $k \in [i_0, \dots, i_1]$ be such that $r_k = \min_{i \in [i_0, i_1]} r_i$; let $f(t,r,x) := \frac{1}{r^2} \int_{B(x,r)} |u(t,y)|^2 \, dy$; let C_5 be the constant determined by (3-19); and let $r_0 = r_0(M)$ be defined by

$$r_0 M^2 = \frac{C_5 E^{-((3/2)+)}}{4 g^{(5/4)+}(M)}.$$

Since $f(t,r,x) \le r M^2$ we have

$$f(t, r_0, x) \le \frac{C_5 E^{-((3/2)+)}}{4 g^{(5/4)+}(M)}.$$

The set $A := \{(t,r,x) : t \in K,\ r_0 \le r \le r_k,\ x \in \mathbb{R}^3\}$ is connected. Therefore its image is connected by f and there exists $(\tilde{t}, \tilde{r}, \tilde{x}) \in K \times [r_0, r_k] \times \mathbb{R}^3$ such that $f(\tilde{t}, \tilde{r}, \tilde{x}) = (C_5 E^{-((3/2)+)})/(2 g^{(5/4)+}(M))$. In other words we have the following mass concentration

$$\frac{1}{\tilde{r}^2} \int_{B(\tilde{x},\tilde{r})} u^2(\tilde{t}, x) \, dx = \frac{C_5 E^{-(3/2+)}}{2 g^{(5/4)+}(M)}. \tag{3-21}$$

Moreover we have the useful lower bound for the size of J_i,[1] $i_0 \le i \le i_1$:

$$|J_i| \gtrsim \tilde{r} \frac{E^{-1/3}}{g^{5/6}(M)}. \tag{3-22}$$

At this point we need to use the following lemma, which gives information about the size of K.

Lemma 12. *Let K be a sequence of unexceptional intervals. Assume there exist $\bar{t} \in K$, $\bar{x} \in \mathbb{R}^3$ and $\bar{r} \in (0, \infty)$ such that*

$$\frac{1}{\bar{r}^2} \int_{B(\bar{x},\bar{r})} u^2(\bar{t}, y) \, dy \gtrsim E^{-((3/2)+)} g^{(5/4)+}(M). \tag{3-23}$$

Then there exist two constants $C_6 \gg 1$, $C_7 \gg 1$ such that

$$|K| \le (C_6 E g(M))^{C_7 E^{(193/4)+} g^{(225/8)+}(M)} \bar{r}. \tag{3-24}$$

If we combine the lemma with (3-22) we can estimate $N(K)$. More precisely, by Lemma 12, (3-22) and (3-12) we have

$$N(K) \lesssim \frac{(C_6 E g(M))^{C_7 E^{(193/4)+} g^{(225/8)+}(M)} \tilde{r}}{\tilde{r} \frac{E^{-(1/3)}}{g^{5/6}(M)}} \lesssim \big(O(E g(M))\big)^{O(E^{(193/4)+} g^{(225/8)+}(M))}. \tag{3-25}$$

Plugging this upper bound for $N(K)$ into (3-18) we get (1-30), completing the proof of the proposition (modulo the lemmas).

Proof of Lemma 12. By using the space translation invariance of (1-1) we can reduce to the case where $\bar{x}$ vanishes.[2] By using the time reversal invariance and the time translation invariance[3] it suffices to estimate $|K \cap [\bar{t}, \infty)|$. By using the time translation invariance again[4] we can assume that $\bar{t} = \bar{r}$ and

[1] Notice that we have the lower bound $\tilde{r} \ge C_5 E^{-((3/2)+)}/(4 M^2 g^{(5/4)+}(M))$. One might think that the presence of $\tilde{r}$ in (3-22) is annoying since this lower bound is crude. However we will see that $\tilde{r}$ disappears at the end of the process: see (3-25). Therefore a sharp lower bound is not required.

[2] We consider the function $u_1(t,x) = u(t, x - \bar{x})$ and we abuse notation in the sequel by writing u_1 for u.

[3] We consider the function $u_2(t,x) := u(2\bar{t} - t, x)$ and we abuse notation in the sequel by writing u_2 for u.

[4] We consider the function $u_3(t,x) := u(t + (\bar{t} - \bar{r}), x)$ and we abuse notation in the sequel by writing u_3 for u.

therefore $\bar{r} \in K$. Let $K_+ := K \cap [\bar{r}, \infty)$. We are interested in estimating $|K_+|$. We would like to use Lemma 10. Therefore, we consider the set $\Gamma_+(K_+)$. We have

$$\frac{1}{\bar{r}^2} \int_{B(0,\bar{r})} |u(\bar{r}, y)|^2 \, dy \gtrsim \frac{E^{-((3/2)+)}}{g^{(5/4)+}(M)}. \tag{3-26}$$

Therefore by Proposition 3 and (3-26) we have

$$\int_{B(0,\bar{r})} |u(t, y)|^2 \, dy \gtrsim \frac{E^{-((3/2)+)} \bar{r}^2}{g^{(5/4)+}(M)} \tag{3-27}$$

if $(t-\bar{r})E^{1/2} \leq (c_0 E^{-((3/4)+)} \bar{r} / g^{(5/8)+}(M))$ for some $c_0 \ll 1$. Therefore by Hölder there exists $0 < c_1 \ll 1$ small enough such that

$$\|u\|_{L_t^4 L_x^{12}\left(\Gamma_+\left(\left[\bar{r}, \bar{r} + \frac{c_0 E^{-((5/4)+)} \bar{r}}{g^{(5/8)+}(M)}\right]\right)\right)} \geq c_1 \frac{E^{-17/16}}{g^{25/32}(M)}. \tag{3-28}$$

Suppose first that $\|u\|_{L_t^4 L_x^{12}(\Gamma_+(K_+))} \leq c_1 \dfrac{E^{-(17/16)}}{g^{(25/32)}(M)}$. In this case we get from (3-28)

$$K_+ \subset \left[\bar{r}, \, \bar{r} + \frac{c_0 E^{-((5/4)+)} \bar{r}}{g^{(5/8)+}(M)}\right], \tag{3-29}$$

and, using also (3-12), we get (3-24).

Now suppose instead that $\|u\|_{L_t^4 L_x^{12}(\Gamma_+(K_+))} \geq c_1 \dfrac{E^{-((17/16)+)}}{g^{(25/32)+}(M)}$. Define

$$\tilde{\eta} := \frac{c_1}{4} \frac{E^{-((17/16))+}}{g^{(25/32)+}(M)}, \tag{3-30}$$

and divide $\Gamma_+(K_+)$ into consecutive cone truncations $\Gamma_+(\tilde{J}_1), \dots, \Gamma_+(\tilde{J}_k)$ such that, for $j = 1, \dots, k-1$,

$$\|u\|_{L_t^4 L_x^{12}(\Gamma_+(\tilde{J}_j))} = \tilde{\eta} \tag{3-31}$$

and

$$\|u\|_{L_t^4 L_x^{12}(\Gamma_+(\tilde{J}_k))} \leq \tilde{\eta}. \tag{3-32}$$

We get from (3-28)

$$\tilde{J}_1 \subset \left[\bar{r}, \bar{r} + \frac{c_0 E^{-((5/4)+)} \bar{r}}{g^{(5/8)+}(M)}\right]. \tag{3-33}$$

Result 13. *If $j \in [1, \dots, k-1]$ we either have*

$$|\tilde{J}_{j+1}| \lesssim |\tilde{J}_j| \tilde{\eta}^{-4} E^{8/3} g^{1/3}(M) \tag{3-34}$$

or

$$|\tilde{J}_j| \geq (C_6 E g(M))^{C_7 E^{(193/4)+} g^{(225/8)+}(M)} \bar{r} \tag{3-35}$$

for some constants $C_6 \gg 1$, $C_7 \gg 1$.

Proof. We get from (1-21), (3-12) and (3-30)

$$\begin{aligned}\|u-u_{l,t_{j+1}}\|_{L_t^4L_x^{12}(\Gamma_+(\tilde{J}_j))} &\lesssim \|u^5g(u)\|_{L_t^1L_x^2(\Gamma_+(\tilde{J}_j\cup\tilde{J_{j+1}}))}\\ &\lesssim \|u^4\|_{L_t^1L_x^3(\Gamma_+(\tilde{J}_j\cup\tilde{J_{j+1}}))}\|ug^{1/6}(u)\|_{L_t^\infty L_x^6(\Gamma_+(\tilde{J}_j\cup\tilde{J_{j+1}}))}g^{5/6}(M)\\ &\lesssim \tilde{\eta}^4E^{1/6}g^{5/6}(M)\\ &\ll \tilde{\eta},\end{aligned} \tag{3-36}$$

with $J_j=[t_{j-1},t_j]$. Therefore by (3-31) we have $\|u_{l,t_{j+1}}\|_{L_t^4L_x^{12}(\Gamma_+(\tilde{J}_j))}\sim\tilde{\eta}$. This implies that

$$\|u_{l,t_{j+1}}-u_{l,t_2}\|_{L_t^4L_x^{12}(\Gamma_+(\tilde{J}_j))}\gtrsim\tilde{\eta} \tag{3-37}$$

or

$$\|u_{l,t_2}\|_{L_t^4L_x^{12}(\Gamma_+(\tilde{J}_j))}\gtrsim\tilde{\eta}. \tag{3-38}$$

Case 1. $\|u_{l,t_{j+1}}-u_{l,t_2}\|_{L_t^4L_x^{12}(\Gamma_+(\tilde{J}_i))}\gtrsim\tilde{\eta}$. By Lemma 11 and Hölder we have

$$\begin{aligned}\|u_{l,t_{j+1}}-u_{l,t_2}\|_{L_t^4L_x^{12}(\Gamma_+(\tilde{J}_j))} &\lesssim |\tilde{J}_j|^{1/4}\|u_{l,t_{j+1}}-u_{l,t_2}\|_{L_t^\infty L_x^{12}(\Gamma_+(\tilde{J}_j))}\\ &\lesssim |\tilde{J}_j|^{1/4}\|u_{l,t_{j+1}}-u_{l,t_2}\|^{1/2}_{L_t^\infty L_x^\infty(\Gamma_+(\tilde{J}_j))}\|u_{l,t_{j+1}}-u_{l,t_2}\|^{1/2}_{L_t^\infty L_x^6(\Gamma_+(\tilde{J}_j))}\\ &\lesssim \frac{|\tilde{J}_j|^{1/4}E^{2/3}g^{1/12}(M)}{|\tilde{J}_{j+1}|^{1/4}}.\end{aligned} \tag{3-39}$$

We get (3-34) from (3-37) and (3-39).

Case 2. $\|u_{l,t_2}\|_{L_t^4L_x^{12}(\Gamma_+(\tilde{J}_j))}\gtrsim\tilde{\eta}$. In this case $\|u_{l,t_2}\|_{L_t^4L_x^{12}(\tilde{J}_j)}\gtrsim\tilde{\eta}$. Recall that K_+ is a subinterval of $K=J_{i_0}\cup\cdots\cup J_{i_1}$, sequence of unexceptional intervals J_i, $i_0\le i\le i_1$. Consequently there are at least $\sim\tilde{\eta}(C_3Eg(M))^{C_4E^{(193/4)+}g^{(225/8)+}(M)}$ intervals J_j that cover $\tilde{J}_i$. Therefore we get (3-35) from (3-22) and (3-12). □

Using Result 13 and Lemma 10 we can get an upper bound on the size $|K_+|$:

Result 14. *We have*

$$|K_+|\le(C_6Eg(M))^{C_7(E^{(193/4)+}g^{(225/8)+}(M))}\bar{r}. \tag{3-40}$$

Proof. Let $B:=(C_6Eg(M))^{C_7(E^{(193/4)+}g^{(225/8)+}(M))}$. Assume that (3-40) fails. Let $\tilde{J}_{j_1}$ be the first interval for which $|\tilde{J}_1\cup\cdots\cup\tilde{J}_{j_1}|$ exceeds $B\bar{r}$. Then $j_1\ne1$, $|\tilde{J}_{j_1}|\lesssim|\tilde{J}_{j_1-1}|\tilde{\eta}^{-4}E^{8/3}g^{1/3}(M)$ and we have

$$\frac{c_1E^{-5/4}\bar{r}}{g^{(5/8)}(M)}+T_2-T_1+(T_2-T_1)\tilde{\eta}^{-4}E^{8/3}g^{1/3}(M)\gtrsim|\tilde{J}_1|+\cdots+|\tilde{J}_{j_1}|\ge B\bar{r}, \tag{3-41}$$

if $[T_1,T_2]:=\tilde{J}_2\cup\cdots\cup\tilde{J}_{j_1-1}$. Therefore by (3-12) and (3-41) we have

$$T_2-T_1\gtrsim\frac{\tilde{\eta}^4E^{-(8/3)}B\bar{r}}{g^{1/3}(M)}. \tag{3-42}$$

Moreover $T_1\le\bar{r}+(c_1E^{-((5/4)+)}\bar{r})/(g^{(5/8)+}(M))$. Therefore by (3-12) we have

$$T_1=O(\bar{r}). \tag{3-43}$$

By (3-42) and (3-43) we have

$$\frac{T_2}{T_1} \geq \left(C_2 E^{10+}\left(\frac{\tilde{\eta}}{4}\right)^{-(36+)}\right)^{4C_2 E^{10+}(\tilde{\eta}/4)^{-(36+)}}, \tag{3-44}$$

with C_2 defined in Lemma 10, provided that $C_6,\ C_7 \gg \max(c_1, C_2)$. Therefore we can apply Lemma 10 and find a subinterval $[t_1', t_2'] \subset \tilde{J}_2 \cup \cdots \cup \tilde{J}_{j_1-1}$ with $|t_2'/t_1'| \sim E^{10+}\tilde{\eta}^{-(36+)}$ and $\|u\|_{L_t^4 L_x^{12}([t_1',t_2'])} \leq \tilde{\eta}/4$. This means that $[t_1', t_2'] \subset [T_1, T_2]$ is covered by at most two consecutive intervals. It is convenient to introduce $[t_1', t_2']_g$, the geometric mean of t_1' and t_2'. We have $[t_1', t_2']_g \sim \tilde{\eta}^{-18}E^5 t_1'$. There are two cases.

Case 1. $[t_1', t_2']$ is covered by one interval $\tilde{J}_{\bar{j}} = [a_{\bar{j}}, b_{\bar{j}}]$, $2 \leq \bar{j} \leq j_1 - 1$. Then $|\tilde{J}_{\bar{j}}| \gtrsim \tilde{\eta}^{-(36+)}E^{10+}t_1'$ and $|\tilde{J}_{\bar{j}-1}| \leq t_1'$. Therefore $|\tilde{J}_{\bar{j}}| \gtrsim \tilde{\eta}^{-(36+)}E^{10+}|\tilde{J}_{\bar{j}-1}|$. Contradiction with (3-12) and (3-34).

Case 2. $[t_1', t_2']$ is covered by two intervals $\tilde{J}_{\bar{j}} = [a_{\bar{j}}, b_{\bar{j}}]$ and $\tilde{J}_{\bar{j}+1} = [a_{\bar{j}+1}, b_{\bar{j}+1}]$ for some $2 \leq \bar{j} \leq j_1 - 2$. Then there are two subcases.

Case 2a. $b_{\bar{j}} \leq [t_1', t_2']_g$. In this case $|\tilde{J}_{\bar{j}+1}| \gtrsim \tilde{\eta}^{-(36+)}E^{10+}t_1'$ and $|\tilde{J}_{\bar{j}}| \leq \tilde{\eta}^{-(18+)}E^{5+}t_1'$. Therefore by (3-12) we have $|\tilde{J}_{\bar{j}+1}| \gtrsim \tilde{\eta}^{-(18+)}E^{5+}|\tilde{J}_{\bar{j}}|$. Contradiction with (3-12) and (3-34).

Case 2b. $b_{\bar{j}} \geq [t_1', t_2']_g$. In this case by (3-12) $|\tilde{J}_{\bar{j}}| \gtrsim \tilde{\eta}^{-(18+)}E^{5+}t_1'$ and $|\tilde{J}_{\bar{j}-1}| \leq t_1'$. Therefore $|\tilde{J}_{\bar{j}}| \gtrsim \tilde{\eta}^{-(18+)}E^{5+}|\tilde{J}_{\bar{j}-1}|$. Contradiction with (3-12) and (3-34).

This exhausts all cases. Thus we have proved Result 14 and so also Lemma 12. □

Remark 15. It seems likely that we can find a better upper bound for $|K_+|$ than (3-40) by exploiting Lemma 11 in a better way. For instance we can consider k successive time intervals $\tilde{J}_{j+1}, \ldots, \tilde{J}_{j+k}$, $k > 1$ and prove an estimate like

$$|\tilde{J}_{j+1}| + \cdots |\tilde{J}_{j+k}| \lesssim |\tilde{J}_j| \tilde{\eta}^{-4} E^{8/3} g^{1/3}(M). \tag{3-45}$$

This estimate is stronger than (3-34). We can probably find a smaller B such that (3-44) holds with $\tilde{\eta}$ substituted for something like $k\tilde{\eta}$ and, by modifying the argument above, find a contradiction with (3-45). At the end of the process we can probably prove global existence of smooth solutions to (1-1) for $0 < c < c_0$, with $c_0 > 8/225$ to be determined. We will not pursue these matters.

4. Proof of Lemma 7

Applying the Strichartz estimates and the Hölder inequality,

$$\begin{aligned} \|u\|_{L_t^4 L_x^{12}(J\times\mathbb{R}^3)} &\lesssim E^{1/2} + \|u^4\|_{L_t^1 L_x^2(J\times\mathbb{R}^3)} \|u g^{1/6}(u)\|_{L_t^\infty L_x^6(J\times\mathbb{R}^3)} \|g^{5/6}(u)\|_{L_t^\infty L_x^\infty(J\times\mathbb{R}^3)} \\ &\lesssim E^{1/2} + E^{1/6} g^{5/6}(M) \|u\|^4_{L_t^4 L_x^{12}(J\times\mathbb{R}^3)}. \end{aligned} \tag{4-1}$$

Hence (3-2) by (3-1) and a continuity argument.

5. Proof of Lemma 8

Let $J' = [t_1', t_2'] \subset J$ be such that $\|u\|_{L_t^4 L_x^{12}(J'\times\mathbb{R}^3)} = \eta$. Then by (1-22) and (3-3)

$$\begin{aligned}\|f(u)\|_{L_t^1 L_x^2(J'\times\mathbb{R}^3)} &\lesssim \|ug^{1/6}(u)\|_{L_t^\infty L_x^6(J'\times\mathbb{R}^3)} \|u\|^4_{L_t^4 L_x^{12}(J'\times\mathbb{R}^3)} \|g^{5/6}(u)\|_{L_t^\infty L_x^\infty(J'\times\mathbb{R}^3)} \\ &\lesssim E^{1/6}\eta^4 g^{5/6}(M) \lesssim E^{1/2}.\end{aligned} \tag{5-1}$$

It is slightly unfortunate that $(2,\infty)$ is not wave admissible. Therefore we consider the admissible pair $(2+\epsilon,\ 6(2+\epsilon)/\epsilon)$ with $\epsilon \ll 1$. By the Strichartz estimates and (5-1), we have

$$\|u\|_{L_t^{2+\epsilon} L_x^{(6(2+\epsilon))/\epsilon}(J'\times\mathbb{R}^3)} \lesssim \|\nabla u(t_1')\|_{L^2(\mathbb{R}^3)} + \|u(t_1')\|_{L^2(\mathbb{R}^3)} + \|f(u)\|_{L_t^1 L_x^2(J'\times\mathbb{R}^3)} \lesssim E^{1/2}. \tag{5-2}$$

Let N be a frequency to be chosen later. By the Bernstein inequality and (1-7) we have

$$\|P_{<N}u\|_{L_t^4 L_x^{12}(J'\times\mathbb{R}^3)} \lesssim N^{1/4}|J'|^{1/4}\|u\|_{L_t^\infty L_x^6(J'\times\mathbb{R}^3)} \lesssim N^{1/4}|J'|^{1/4}E^{1/6}. \tag{5-3}$$

Therefore

$$\|P_{<N}u\|_{L_t^4 L_x^{12}(J'\times\mathbb{R}^3)} \lesssim |J'|^{1/4}N^{1/4}E^{1/6}. \tag{5-4}$$

Let $c_2 \ll 1$. Then if $N = c_2^4(\eta^4/(|J'|E^{2/3}))$ we have

$$\|P_{\geq N}u\|_{L_t^4 L_x^{12}(J'\times\mathbb{R}^3)} \gtrsim \eta \quad \text{and} \quad \|u\|_{L_t^4 L_x^{12}(J'\times\mathbb{R}^3)} \sim \|P_{\geq N}u\|_{L_t^4 L_x^{12}(J'\times\mathbb{R}^3)}. \tag{5-5}$$

By (5-2) and (5-5) we have

$$\begin{aligned}\eta &\sim \|P_{\geq N}u\|_{L_t^4 L_x^{12}(J'\times\mathbb{R}^3)} \\ &\lesssim \|P_{\geq N}u\|^{(2+\epsilon)/4}_{L_t^{2+\epsilon} L_x^{(6(2+\epsilon))/\epsilon}(J'\times\mathbb{R}^3)} \|P_{\geq N}u\|^{1-(2+\epsilon)/4}_{L_t^\infty L_x^6(J'\times\mathbb{R}^3)} \\ &\lesssim E^{(2+\epsilon)/8}\|P_{\geq N}u\|^{1-((2+\epsilon)/4)}_{L_t^\infty L_x^6(J'\times\mathbb{R}^3)}.\end{aligned} \tag{5-6}$$

Therefore we conclude that $\|P_{\geq N}\|_{L_t^\infty L_x^6(J'\times\mathbb{R}^3)} \gtrsim \eta^{2+}E^{-((1/2)+)}$. Applying Proposition 2 we get (3-5).

6. Proof of Lemma 9

Bahouri and Gerard [1999, page 171] used arguments from Grillakis [1990; 1992] and Shatah–Struwe [1993] to derive an a priori estimate of the solution u to the 3D quintic defocusing wave equation, that is, $\partial_{tt}u - \triangle u + u^5 = 0$. More precisely they were able to prove

$$\int_{|x|\leq b} |u(b,x)|^6\,dx \lesssim \frac{a}{b}(\tilde{e}(a) + \tilde{e}^{1/3}(a)) + \tilde{e}(b) - \tilde{e}(a) + (\tilde{e}(b) - \tilde{e}(a))^{1/3}, \tag{6-1}$$

with

$$\tilde{e}(t) := \frac{1}{2}\int_{|x|\leq t} (\partial_t u)^2\,dx + \frac{1}{2}\int_{|x|\leq t} |\nabla u|^2\,dx + \frac{1}{6}\int_{|x|\leq t} u^6\,dx. \tag{6-2}$$

Since we apply their ideas to the potential f we just sketch the proof. Given the cone $\Gamma_+([a,b])$ we denote by $\partial\Gamma_+([a,b])$ the mantle of the cone $\Gamma_+([a,b])$, that is,

$$\partial\Gamma_+([a,b]) := \{(t',x) \in [a,b]\times\mathbb{R}^3,\ t = |x|\}. \tag{6-3}$$

The local energy identity

$$e(b)-e(a)=\frac{1}{2\sqrt{2}}\int_{\partial\Gamma_{+}([a,b])}\left|\frac{x\partial_t u}{t}+\nabla u\right|^2+\frac{1}{\sqrt{2}}\int_{\partial\Gamma_{+}([a,b])}F(u) \tag{6-4}$$

results from the integration of the identity $\partial_t u\,(\partial_{tt}u-\triangle u+f(u))=0$ on the cone $\Gamma_{+}([a,b])$. We have [Shatah and Struwe 1998]

$$\partial_t\Big(\frac{t}{2}(\partial_t u)^2+\frac{t}{2}|\nabla u|^2+(x.\nabla u)\partial_t u+tF(u)+u\partial_t u\Big) \\ -\operatorname{div}\Big(t\nabla u\partial_t u+(x.\nabla u)\nabla u-\frac{|\nabla u|^2 x}{2}+\frac{(\partial_t u)^2 x}{2}-xF(u)+u\nabla u\Big)+uf(u)-4F(u)=0. \tag{6-5}$$

Integrating this identity on $\Gamma_{+}([a,b])$, we have

$$X(b)-X(a)+Y(a,b)=\int_{\Gamma_{+}([a,b])}4F(u)-uf(u), \tag{6-6}$$

with

$$X(t):=\int_{|x|\le t}\frac{t}{2}(\partial_t u)^2+\frac{t}{2}|\nabla u|^2+(x.\nabla u)\partial_t u+tF(u)+u\partial_t u \tag{6-7}$$

and

$$Y(a,b):= \\ -\frac{1}{\sqrt{2}}\int_{\partial\Gamma_{+}([a,b])}\Big(\frac{t}{2}(\partial_t u)^2+\frac{t}{2}|\nabla u|^2+(x.\nabla u)\partial_t u+tF(u)+u\partial_t u+t\frac{\nabla u.x}{|x|}\partial_t u+\frac{|x.\nabla u|^2}{|x|} \\ -\frac{|\nabla u|^2}{2}|x|+\frac{(\partial_t u)^2|x|}{2}-|x|F(u)+u\frac{\nabla u.x}{|x|}\Big). \tag{6-8}$$

In fact we have [Shatah and Struwe 1993]

$$X(t)=\int_{|x|\le t}t\Big[\frac{1}{2}(\partial_t u)^2+\frac{1}{2}\Big|\nabla u+\frac{ux}{|x|^2}\Big|^2\Big]+\partial_t u(x.\nabla u+u)+tF(u)-\int_{|x|=t}\frac{u^2}{2}. \tag{6-9}$$

Since $t=|x|$ on $\partial\Gamma_{+}([a,b])$ we have

$$Y(a,b)=-\frac{1}{\sqrt{2}}\int_{\partial\Gamma_{+}([a,b])}|x|(\partial_t u)^2+2(x.\nabla u)\partial_t u+u\partial_t u+\frac{(x.\nabla u)^2}{|x|}+u\frac{\nabla u.x}{|x|}, \tag{6-10}$$

and after some computations [Shatah and Struwe 1993], we get

$$Y(a,b)=-\frac{1}{\sqrt{2}}\int_{\partial\Gamma_{+}([a,b])}\frac{1}{t}(t\partial_t u+(\nabla u.x)+u)^2+\int_{|x|=b}\frac{u^2}{2}-\int_{|x|=a}\frac{u^2}{2}. \tag{6-11}$$

Therefore, if

$$H(t):=\int_{|x|\le t}t\Big[\frac{1}{2}(\partial_t u)^2+\frac{1}{2}\Big|\nabla u+\frac{ux}{|x|^2}\Big|^2\Big]+\partial_t u(x.\nabla u+u)+tF(u), \tag{6-12}$$

then

$$H(b)-H(a)=\frac{1}{\sqrt{2}}\int_{\partial\Gamma_{+}([a,b])}\frac{1}{t}(t\partial_t u+\nabla u.x+u)^2+\int_{\Gamma_{+}([a,b])}4F(u)-uf(u). \tag{6-13}$$

We estimate $H(t)$, following [Bahouri and Gérard 1999]. We have

$$|\partial_t u(x.\nabla u+u)| \leq \frac{t}{2}\Big((\partial_t u)^2+\Big|\nabla u+\frac{ux}{|x|^2}\Big|^2\Big) \lesssim t\Big((\partial_t u)^2+|\nabla u|^2+\frac{u^2}{|x|^2}\Big). \tag{6-14}$$

Therefore by (6-14), the Hölder inequality and (1-7), we have

$$H(t) \lesssim t\Big(e(t)+\int_{|x|\leq t}\frac{u^2}{|x|^2}\Big) \lesssim t\Big(e(t)+\Big(\int_{|x|\leq t}u^6\Big)^{1/3}\Big) \lesssim t\big(e(t)+e^{1/3}(t)\big). \tag{6-15}$$

Moreover by (6-4), the Hölder inequality and (1-7), we have

$$\begin{aligned}\frac{1}{\sqrt{2}}\int_{\partial\Gamma_+([a,b])}\frac{1}{t}\big(t\partial_t u+\nabla u.x+u\big)^2 &\lesssim \frac{b}{2\sqrt{2}}\int_{\partial\Gamma_+([a,b])}\Big(\frac{\nabla u\cdot x}{t}+\partial_t u\Big)^2+\frac{1}{2\sqrt{2}}\int_{\partial\Gamma_+([a,b])}\frac{u^2}{t^2}\\ &\lesssim b\int_{\partial\Gamma_+([a,b])}\Big|\frac{x}{t}\partial_t u+\nabla u\Big|^2+\frac{1}{2\sqrt{2}}\Big(\int_{\partial\Gamma_+([a,b])}u^6\Big)^{1/3}\\ &\lesssim b\big((e(b)-e(a))+(e(b)-e(a))^{1/3}\big).\end{aligned} \tag{6-16}$$

We get from (1-7)

$$4F(u)-uf(u)\leq 0. \tag{6-17}$$

By (6-13), and (6-15)–(6-17), we have

$$\begin{aligned}\int_{|x|\leq b}F(u) \lesssim \frac{H(b)}{b} &\lesssim \frac{H(a)+\frac{1}{\sqrt{2}}\int_{\partial\Gamma_+([a,b])}\frac{1}{t}\,(t\partial_t u+\nabla u.x+u)^2}{b}\\ &\lesssim \frac{a}{b}\,\big(e(a)+e^{1/3}(a)\big)+e(b)-e(a)+(e(b)-e(a))^{1/3}.\end{aligned} \tag{6-18}$$

7. Proof of Lemma 10

The proof relies upon two results that we prove in the subsections.

Result 16. *Let u be a classical solution of* (1-1). *Assume that* (1-28) *holds. Let η be a positive number such that* (3-3) *holds. If $\|u\|_{L_t^4 L_x^{12}(\Gamma_+(J))} \geq \eta$ then*

$$\|u\|_{L_t^\infty L_x^6(\Gamma_+(J))} \gtrsim \eta^{2+}E^{-((1/2)+)}. \tag{7-1}$$

Result 17. *Let u be a smooth solution to* (1-1). *Assume that* (1-28) *holds. Let η be a positive number such that*

$$\eta \leq \min(1,\,E^{1/18}). \tag{7-2}$$

Let $J=[t_1,t_2]$ be an interval such that $[t_1, t_1(E\eta^{-18})^{4E\eta^{-18}}] \subset J$. Then there exists a subinterval $J'=[t_1',t_2']$ such that $\big|t_2'/t_1'\big| = E\eta^{-18}$ and

$$\|u\|_{L_t^\infty L_x^6(\Gamma_+(J'))} \lesssim \eta. \tag{7-3}$$

Let C_9 be the constant determined by $\gtrsim$ in (7-1). Let C_{10} be the constant determined by $\lesssim$ in (7-3). We get from (3-9):

$$\left[t_1,\ t_1\left(E\left(\frac{C_9\eta^{2+}E^{-(1/2)+}}{2C_{10}}\right)^{-18}\right)^{4E\left(\frac{C_9\eta^{2+}E^{-(1/2)+}}{2C_{10}}\right)^{-18}}\right] \subset \left[t_1,\ C_2(E^{10+}\eta^{-(36+)})^{4C_2E^{10+}\eta^{-(36+)}}t_1\right] \subset J, \tag{7-4}$$

if $C_2 \gg \max(C_9, C_{10})$. Therefore, since $(C_9\eta^{2+}E^{-(1/2+)})/(2C_{10})$ satisfies (7-2) by (3-8), we can use Result 17 and show that there exists a subinterval $J' = [t'_1,\ t'_2]$ such that $|t'_2/t'_1| \sim E^{10+}\eta^{-(36+)}$ and

$$\|u\|_{L_t^\infty L_x^6(\Gamma_+(J'))} \le \frac{C_9\eta^{2+}E^{-(1/2+)}C_{10}}{2C_{10}} \le C_9\frac{\eta^{2+}E^{-(1/2+)}}{2}. \tag{7-5}$$

Now we claim that $\|u\|_{L_t^4L_x^{12}(\Gamma_+(J'))} \le \eta$. If not by (3-8) and Result 16 we have

$$\|u\|_{L_t^\infty L_x^6(\Gamma_+(J'))} \ge C_9\eta^{2+}E^{-(1/2+)}. \tag{7-6}$$

Contradiction with (7-5).

Proof of Result 16. We substitute J' for $\Gamma_+(J')$ in (5-1) to get

$$\|f(u)\|_{L_t^1L_x^2(\Gamma_+(J'))} \lesssim E^{1/2}. \tag{7-7}$$

By the Strichartz estimates (1-20) on the truncated cone $\Gamma_+(J')$ we have

$$\|u\|_{L_t^{2+\epsilon}L_x^{(6(2+\epsilon))/\epsilon}(\Gamma_+(J'))} \lesssim E^{1/2}, \tag{7-8}$$

after following similar steps to prove (5-2). Therefore

$$\eta = \|u\|_{L_t^4L_x^{12}(\Gamma_+(J))} \lesssim \|u\|^{(2+\epsilon)/4}_{L_t^{2+\epsilon}L_x^{(6(2+\epsilon))/\epsilon}(\Gamma_+(J'))}\|u\|^{1-((2+\epsilon)/4)}_{L_t^\infty L_x^6(\Gamma_+(J'))} \lesssim E^{(2+\epsilon)/8}\|u\|^{1-((2+\epsilon)/4)}_{L_t^\infty L_x^6(\Gamma_+(J'))}. \tag{7-9}$$

Therefore (7-1) holds. □

Proof of Result 17. By (7-2) we have $E\eta^{-18} \ge 1$. Let n be the largest integer such that $2n \le 4E\eta^{-18}$. This implies that $n \ge E\eta^{-18}$. Let $A := E\eta^{-18}$. Now we consider the interval $[t_1, A^{2n}t_1] \subset J$. We write $[t_1, A^{2n}t_1] = [t_1,\ A^2t_1] \cup \cdots \cup [A^{2(n-1)}t_1,\ A^{2n}t_1]$. We have

$$\sum_{i=1}^{n} e(A^{2i}t_1) - e(A^{2(i-1)}t_1) \le 2E, \tag{7-10}$$

and by the pigeonhole principle there exists $i_0 \in [1, n]$ such that

$$e(A^{2i_0}t_1) - e(A^{2(i_0-1)}t_1) \lesssim \eta^{18}. \tag{7-11}$$

Now we choose $a := A^{2(i_0-1)}t_1$ and $b \in [A^{2i_0-1}t_1,\ A^{2i_0}t_1]$. Let $t'_1 := A^{2(i_0-1)}t_1,\ t'_2 := A^{2i_0-1}t_1$ and $J' := [t'_1, t'_2]$. We apply (3-7) and (7-2) to get

$$\|u\|_{L_t^\infty L_x^6(\Gamma_+([t'_1,t'_2]))} \lesssim \|F(u)\|_{L_t^\infty L_x^1(\Gamma_+([t'_1,t'_2]))} \lesssim (E^{-1}\eta^{18}(E+E^{1/3}) + \eta^{18} + \eta^6)^{1/6} \lesssim \eta. \qquad \square$$

Proof of Lemma 11

We have after computation of the derivative of $e(t)$

$$\partial_t e(t) \geq \int_{|x|=t} F(u)\, dS, \tag{7-12}$$

and integrating with respect of time

$$\int_I \int_{|x|\leq t} g(u)u^6(t', x')\, dS\, dt' \lesssim E. \tag{7-13}$$

By using the space and time translation invariance

$$\int_J \int_{|x'-x|=|t'-t|} g(u)u^6(t', x')\, dS\, dt' \lesssim E. \tag{7-14}$$

Therefore (1-15), (1-22), (7-14) and the Hölder inequality give us

$$\begin{aligned}
\left| -\int_{J'} \frac{\sin(t-t')D}{D} g(u)u^5\, dt' \right| &= \left| \frac{1}{4\pi\,|t-t'|} \int_{|x'-x|=|t'-t|} g^{5/6}(u)u^5 g^{1/6}(u)\, dS dt' \right| \\
&\lesssim \int_{J'} \frac{1}{|t-t'|} \Big(\int_{|x'-x|=|t'-t|} u^6 g(u) dS \Big)^{5/6} \Big(\int_{|x'-x|=|t'-t|} g(u)\, dS \Big)^{1/6} dt' \\
&\lesssim g^{1/6}(M) \int_{J'} \frac{1}{|t-t'|^{2/3}} \Big(\int_{|x'-x|=|t'-t|} u^6 g(u)\, dS \Big)^{5/6} dt' \\
&\lesssim g^{1/6}(M) E^{5/6} \Big(\int_{J'} \frac{1}{|t-t'|^4} \Big)^{1/6} \lesssim g^{1/6}(M) \frac{E^{5/6}}{\operatorname{dist}^{1/2}(t, J')}.
\end{aligned} \tag{7-15}$$

Notice that

$$u(t) = u_{l,t_i}(t) - \int_{t_i}^{t} \frac{\sin(t-t')D}{D} u^5(t') g(u(t'))\, dt', \tag{7-16}$$

for $i = 1, 2$. We get (3-11) from (7-15) and (7-16).

Acknowledgements

The author thanks Terence Tao for suggesting him this problem.

References

[Bahouri and Gérard 1999] H. Bahouri and P. Gérard, "High frequency approximation of solutions to critical nonlinear wave equations", *Amer. J. Math.* **121**:1 (1999), 131–175. MR 2000i:35123 Zbl 0919.35089

[Bourgain 1999] J. Bourgain, "Global wellposedness of defocusing critical nonlinear Schrödinger equation in the radial case", *J. Amer. Math. Soc.* **12**:1 (1999), 145–171. MR 99e:35208

[Ginibre and Velo 1995] J. Ginibre and G. Velo, "Generalized Strichartz inequalities for the wave equation", *J. Funct. Anal.* **133**:1 (1995), 50–68. MR 97a:46047 Zbl 0849.35064

[Grillakis 1990] M. G. Grillakis, "Regularity and asymptotic behaviour of the wave equation with a critical nonlinearity", *Ann. of Math.* (2) **132**:3 (1990), 485–509. MR 92c:35080 Zbl 0736.35067

[Grillakis 1992] M. G. Grillakis, "Regularity for the wave equation with a critical nonlinearity", *Comm. Pure Appl. Math.* **45**:6 (1992), 749–774. MR 93e:35073 Zbl 0785.35065

[Kapitanski 1994] L. Kapitanski, "Global and unique weak solutions of nonlinear wave equations", *Math. Res. Lett.* **1**:2 (1994), 211–223. MR 95f:35158 Zbl 0841.35067

[Keel and Tao 1998] M. Keel and T. Tao, "Endpoint Strichartz estimates", *Amer. J. Math.* **120**:5 (1998), 955–980. MR 2000d:35018 Zbl 0922.35028

[Lindblad and Sogge 1995] H. Lindblad and C. D. Sogge, "On existence and scattering with minimal regularity for semilinear wave equations", *J. Funct. Anal.* **130**:2 (1995), 357–426. MR 96i:35087 Zbl 0846.35085

[Rauch 1981] J. Rauch, "I: The u^5 Klein–Gordon equation; II: Anomalous singularities for semilinear wave equations", pp. 335–364 in *Nonlinear partial differential equations and their applications* (Paris, 1978/1979), vol. 1, edited by H. Brezis and J. L. Lions, Res. Notes in Math. **53**, Pitman, Boston, MA, 1981. MR 83a:35066 Zbl 0473.35055

[Shatah and Struwe 1993] J. Shatah and M. Struwe, "Regularity results for nonlinear wave equations", *Ann. of Math.* (2) **138**:3 (1993), 503–518. MR 95f:35164 Zbl 0836.35096

[Shatah and Struwe 1994] J. Shatah and M. Struwe, "Well-posedness in the energy space for semilinear wave equations with critical growth", *Internat. Math. Res. Notices* 7 (1994), 303–309. MR 95e:35132 Zbl 0830.35086

[Shatah and Struwe 1998] J. Shatah and M. Struwe, *Geometric wave equations*, Courant Lecture Notes in Mathematics **2**, Courant Institute of Mathematical Sciences, New York, 1998. MR 2000i:35135 Zbl 0993.35001

[Sogge 1995] C. D. Sogge, *Lectures on nonlinear wave equations*, Monographs in Analysis **2**, International Press, Boston, MA, 1995. MR 2000g:35153 Zbl 1089.35500

[Struwe 1988] M. Struwe, "Globally regular solutions to the u^5 Klein-Gordon equation", *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) **15**:3 (1988), 495–513. MR 90j:35142 Zbl 0728.35072

[Tao 2006] T. Tao, "Spacetime bounds for the energy-critical nonlinear wave equation in three spatial dimensions", *Dyn. Partial Differ. Equ.* **3**:2 (2006), 93–110. MR 2007c:35116

[Tao 2007] T. Tao, "Global regularity for a logarithmically supercritical defocusing nonlinear wave equation for spherically symmetric data", *J. Hyperbolic Differ. Equ.* **4**:2 (2007), 259–265. MR 2009b:35294 Zbl 1124.35043

Received 4 Nov 2008. Revised 7 Jun 2009. Accepted 21 Jul 2009.

TRISTAN ROY: `triroy@math.ucla.edu`
Department of Mathematics, University of California, Los Angeles, CA 90095, United States

ANALYSIS AND PDE
Vol. 2, No. 3, 2009

PERIODIC STOCHASTIC KORTEWEG–DE VRIES EQUATION WITH ADDITIVE SPACE-TIME WHITE NOISE

TADAHIRO OH

We prove the local well-posedness of the periodic stochastic Korteweg–de Vries equation with the additive space-time white noise. To treat low regularity of the white noise in space, we consider the Cauchy problem in the Besov-type space $\hat{b}^s_{p,\infty}(\mathbb{T})$ for $s = -\frac{1}{2}+,\ p = 2+$ such that $sp < -1$. In establishing local well-posedness, we use a variant of the Bourgain space adapted to $\hat{b}^s_{p,\infty}(\mathbb{T})$ and establish a nonlinear estimate on the second iteration on the integral formulation. The deterministic part of the nonlinear estimate also yields the local well-posedness of the deterministic KdV in $M(\mathbb{T})$, the space of finite Borel measures on $\mathbb{T}$.

1. Introduction

In this paper, we prove the local well-posedness of the periodic stochastic Korteweg–de Vries (SKdV) equation with additive space-time white noise:

$$\begin{cases} du + (\partial_x^3 u + u\partial_x u)dt = dW, \\ u(x,0) = u_0(x), \end{cases} \tag{1}$$

where u is a real-valued function, $(x,t) \in \mathbb{T} \times \mathbb{R}^+$ with $\mathbb{T} = [0, 2\pi)$, and $W(t) = \partial B/\partial x$ is a cylindrical Wiener process on $L^2(\mathbb{T})$. With $e_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}$, we have $W(t) = \beta_0(t)e_0 + \sum_{n\neq 0} \frac{1}{\sqrt{2}} \beta_n(t) e_n(x)$, where $\{\beta_n\}_{n\geq 0}$ is a family of mutually independent complex-valued Brownian motions (here we take β_0 to be real-valued) in a fixed probability space $(\Omega, \mathcal{F}, P)$ associated with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and $\beta_{-n}(t) = \overline{\beta_n}(t)$ for $n \geq 1$. Note that $\mathrm{Var}(\beta_n(1)) = 2$ for $n \geq 1$.

De Bouard et al. [2004] considered

$$\begin{cases} du + (\partial_x^3 u + u\partial_x u)dt = \phi dW, \\ u(x,0) = u_0(x), \end{cases} \tag{2}$$

where ϕ is a bounded linear operator in $L^2(\mathbb{T})$. They showed that (2) is locally well posed when ϕ is a Hilbert–Schmidt operator from $L^2(\mathbb{T})$ to $H^s(\mathbb{T})$ for $s > -\frac{1}{2}$. See the references in their paper for earlier work in the periodic and nonperiodic settings.

In this work, we consider the case when ϕ is the identity operator on $L^2(\mathbb{T})$, that is, we take the additive noise to be the space-time white noise $\partial^2 B/\partial t\partial x$, where $B(x,t)$ is a two parameter Brownian motion on $\mathbb{T} \times \mathbb{R}^+$. Note that ϕ is a Hilbert–Schmidt operator from $L^2(\mathbb{T})$ to $H^s(\mathbb{T})$ for $s < -\frac{1}{2}$ but not for $s \geq -\frac{1}{2}$.

MSC2000: 35Q53, 60H15.
Keywords: stochastic KdV, white noise, local well-posedness.

Suppose that u is the solution to (1), or equivalently to (2) with $\phi = \mathrm{Id}$, the identity operator on $L^2(\mathbb{T})$. Let $v_1(x,t) = u(x+\alpha_0 t, t) - \alpha_0$, where α_0 is the mean of u_0. Then, v_1 satisfies (1) with the mean-zero initial condition $u_0 - \alpha_0$. Now, let $\mathbb{P}_0$ be the projection onto the spatial frequency 0, and $\mathbb{P}_{n\neq 0} = \mathrm{Id} - \mathbb{P}_0$. Note that $\mathbb{P}_0 W(t) = \beta_0(t) e_0(x) = \frac{1}{\sqrt{2\pi}}\beta_0(t)$. By letting $v_2 = v_1 - \frac{1}{\sqrt{2\pi}}\beta_0(t)$, we see that u satisfies (1) if and only if v_2 satisfies

$$\begin{cases} dv_2 + \big(\partial_x^3 v_2 + (v_2 + \frac{1}{\sqrt{2\pi}}\beta_0(t))\partial_x v_2\big)dt = \mathbb{P}_{n\neq 0}\, dW, \\ v_2(x,0) = u_0(x) - \alpha_0, \end{cases}$$

almost surely since $\beta_0(0) = 0$ a.s. By setting $v_3(x,t) = v_2(x + c_\omega(t), t)$ with $c_\omega(t) = \int_0^t \frac{1}{\sqrt{2\pi}}\beta_0(t')dt'$, it follows that v_3 satisfies

$$\begin{cases} dv_3 + (\partial_x^3 v_3 + v_3 \partial_x v_3)dt = d\widetilde{W}, \\ v_3(x,0) = u_0(x) - \alpha_0, \end{cases}$$

where

$$\widetilde{W}(x,t) = \sum_{n\neq 0} \frac{1}{\sqrt{2}}\beta_n(t) e_n(x + c_\omega(t)) = \sum_{n\neq 0} \frac{1}{\sqrt{2}}\beta_n(t) e^{inc_\omega(t)} e_n(x);$$

that is, v_3 solves (2), where

$$\phi = \mathrm{diag}(\phi_n; n \neq 0) \qquad \text{with} \quad \phi_n(t) = e^{inc_\omega(t)} \quad \text{and} \quad c_\omega(t) = \int_0^t \frac{1}{\sqrt{2\pi}}\beta_0(t')dt' \tag{3}$$

(with respect to the basis $\{e_n\}_{n\in\mathbb{Z}}$). Moreover, v_3 has spatial mean 0 (as long as it exists) since e_0 does not lie in the range of ϕ. Therefore, in the remainder of the paper, we concentrate on studying the local well-posedness of (2) with ϕ given by (3) and the mean-zero initial condition u_0 (which implies that u has spatial mean 0 as long as it exists).

Recall that u is called a (local-in-time) mild solution to (2) if u satisfies

$$u(t) = S(t)u_0 - \frac{1}{2}\int_0^t S(t-t')\partial_x u^2(t')dt' + \int_0^t S(t-t')\phi(t')dW(t') \tag{4}$$

at least for $t \in [0,T]$ for some $T > 0$, where $S(t) = e^{-t\partial_x^3}$.

Note that the first two terms in (4) also appear in the deterministic KdV theory. Thus, we briefly review recent well-posedness results of the periodic (deterministic) KdV.

$$\begin{cases} u_t + u_{xxx} + uu_x = 0, \\ u\big|_{t=0} = u_0, \end{cases} \qquad (x,t) \in \mathbb{T}\times\mathbb{R}. \tag{5}$$

Bourgain [1993] introduced a new weighted space-time Sobolev space $X^{s,b}$ whose norm is given by

$$\|u\|_{X^{s,b}(\mathbb{T}\times\mathbb{R})} = \|\langle n\rangle^s \langle \tau - n^3\rangle^b \hat{u}(n,\tau)\|_{L^2_{n,\tau}(\mathbb{Z}\times\mathbb{R})}, \tag{6}$$

where $\langle\,\cdot\,\rangle = 1 + |\cdot|$. He proved the local well-posedness of (5) in $L^2(\mathbb{T})$ via the fixed point argument, immediately yielding the global well-posedness in $L^2(\mathbb{T})$ thanks to the conservation of the L^2 norm. Kenig et al. [1996] improved Bourgain's result and established the local well-posedness in $H^{-\frac{1}{2}}(\mathbb{T})$ by establishing the bilinear estimate

$$\|\partial_x(uv)\|_{X^{s,-\frac{1}{2}}} \lesssim \|u\|_{X^{s,\frac{1}{2}}}\|v\|_{X^{s,\frac{1}{2}}} \tag{7}$$

for $s \geq -\frac{1}{2}$ under the mean-zero assumption on u and v. Colliander et al. [2003] proved the corresponding global well-posedness result via the I-method.

There are also results on (5) which exploit its complete integrability. Bourgain [1997] proved the global well-posedness of (5) in the class $M(\mathbb{T})$ of measures μ, assuming that its total variation $\|\mu\|$ is sufficiently small. His proof is based on the trilinear estimate on the second iteration of the integral formulation of (5), assuming an a priori uniform bound on the Fourier coefficients of the solution u of the form

$$\sup_{n \in \mathbb{Z}} |\hat{u}(n,t)| < C \quad \text{for all } t \in \mathbb{R}. \tag{8}$$

He then established (8) using complete integrability. More recently, Kappeler and Topalov [2006] proved the global well-posedness of the KdV in $H^{-1}(\mathbb{T})$ via the inverse spectral method.

There are also results on the necessary conditions on the regularity with respect to smoothness or uniform continuity of the solution map : $u_0 \in H^s(\mathbb{T}) \to u(t) \in H^s(\mathbb{T})$. Bourgain [1997] showed that if the solution map is C^3, then $s \geq -\frac{1}{2}$. Christ et al. [2003] proved that if the solution map is uniformly continuous, then $s \geq -\frac{1}{2}$. (See also [Kenig et al. 2001].) These results, in particular, imply that we cannot hope to have a local-in-time solution of KdV via the fixed point argument in H^s, $s < -\frac{1}{2}$. Recall that, for each fixed t, the space-time white noise $\partial^2 B/\partial t \partial x$ lies in $\bigcap_{s<-1/2} H^s \setminus H^{-\frac{1}{2}}$ almost surely. Hence, these results for KdV cannot be applied to study the local well-posedness of (1).

Now, let us discuss the spaces which capture the regularities of the spatial and space-time white noise. Recently, we proved the invariance of the (spatial) white noise for the (deterministic) KdV in [Oh 2009a] (also see [Oh 2009b]) by first establishing the local well-posedness in an appropriate Banach space containing the support of the (spatial) white noise. Define the Besov-type space via the norm

$$\|f\|_{\hat{b}^s_{p,\infty}} := \|\hat{f}\|_{b^s_{p,\infty}} = \sup_j \big\| \langle n \rangle^s \hat{f}(n) \big\|_{L^p_{|n|\sim 2^j}} = \sup_j \bigg(\sum_{|n|\sim 2^j} \langle n \rangle^{sp} |\hat{f}(n)|^p \bigg)^{1/p}. \tag{9}$$

In [Oh 2009a], using the theory of abstract Wiener spaces, we showed that $\hat{b}^s_{p,\infty}$ contains the full support of the (spatial) white noise for $sp < -1$. (The statement also holds true for $sp = -1$.)

Let's consider the stochastic convolution $\Phi(t)$ given by

$$\Phi(t) = \int_0^t S(t-t')\phi(t')dW(t'), \tag{10}$$

where ϕ is given by (3). Define a variant of the $X^{s,b}$ space adjusted to $\hat{b}^s_{p,\infty}(\mathbb{T})$. Let $X^{s,b}_{p,q}$ be the completion of the Schwartz class $\mathcal{S}(\mathbb{T} \times \mathbb{R})$ under the norm

$$\|u\|_{X^{s,b}_{p,q}} = \big\| \langle n \rangle^s \langle \tau - n^3 \rangle^b \hat{u}(n,\tau) \big\|_{b^0_{p,\infty} L^q_\tau}. \tag{11}$$

Note that $X^{s,b}_{p,q}$ defined in (11) is the space of functions u such that $S(-t)u(\,\cdot\,,t) \in (\hat{b}^s_{p,\infty})_x (\mathcal{F}L^{b,q})_t$, where $\mathcal{F}L^{b,q}$ is defined via the norm

$$\|f\|_{\mathcal{F}L^{b,q}} := \|\langle \tau \rangle^b \hat{f}(\tau)\|_{L^q}. \tag{12}$$

In the same paper we also showed that the local-in-time white noise is supported on $\mathcal{F}L^{c,q}$ for $cq < -1$. This implies that the Brownian motion belongs locally in time to $\mathcal{F}L^{b,q}$ for $(b-1)q < -1$. Hence, with

$b < \frac{1}{2}$ and $q = 2$, we see that the local-in-time stochastic convolution $\eta(t)\Phi(t)$ lies in $X^{s,b}_{p,q}$ almost surely, with $sp < -1$, $b < \frac{1}{2}$ and $q = 2$, where $\eta(t)$ is a smooth cutoff supported on $[-1, 2]$ with $\eta(t) \equiv 1$ on $[0, 1]$.

The argument in [De Bouard et al. 2004] is based on the result by Roynette [1993] on the endpoint regularity of the Brownian motion, which states that the Brownian motion $\beta(t)$ belongs to the Besov space $B^{1/2}_{p,q}$ if and only if $q = \infty$ (with $1 \le p < \infty$). The authors then proved a variant of the bilinear estimate (7) by Kenig, Ponce and Vega, adjusted to their Besov space setting, establishing the local well-posedness via the fixed point theorem. The use of a variant of the bilinear estimate (7) required a slight regularization of the noise in space via ϕ so that the smoothed noise has the spatial regularity $s > -\frac{1}{2}$. Thus, they could not treat the space-time white noise, that is, $\phi = \mathrm{Id}$.

Our result is based on two observations. First, our l^p_n-based function spaces $\hat{b}^s_{p,\infty}$ in (9) and $X^{s,b}_{p,q}$ in (11) capture the regularity of the spatial and space-time white noise for $sp < -1$, $b < \frac{1}{2}$ and $q = 2$. The second is that we can indeed carry out the argument in [Bourgain 1997], a nonlinear estimate on the second iteration, *without* assuming the a priori bound (8), if we take the initial data $u_0 \in \hat{b}^s_{p,\infty}$ for $s > -\frac{1}{2}$ with $p > 2$. Then, we construct a solution u as a strong limit of the smooth solutions u^N (with smooth u^N_0 and ϕ^N) of (2). Note that our nonlinear estimate on the second iteration in Section 5 depends on the stochastic term, whereas the bilinear estimate in [De Bouard et al. 2004] is entirely deterministic.

Finally, we present our main results.

Theorem 1. *Let ϕ be as in* (3) *and $p = 2+$. Let $s = -\frac{1}{2} + \delta$ with $(p-2)/(4p) < \delta < (p-2)/(2p)$, so $sp < -1$. Also, let u_0 be $\mathcal{F}_0$-measurable such that it has mean* 0 *and belongs to $\hat{b}^s_{p,\infty}(\mathbb{T})$ almost surely. Then, there exists a stopping time $T_\omega > 0$ and a unique process $u \in C\big([0, T_\omega]; \hat{b}^s_{p,\infty}(\mathbb{T})\big)$ satisfying* (2) *on $[0, T_\omega]$ almost surely.*

As a corollary, we obtain:

Theorem 2. *The stochastic KdV* (1) *with the additive space-time white noise is locally well posed almost surely* (*with the prescribed mean on u_0*).

Remark 1.1. Our argument provides an answer to the question posed in [Bourgain 1997, remark on p. 120], at least in the local-in-time setting. The deterministic part of the nonlinear estimate in Section 5 can be used to establish the local well-posedness of (5) for a finite Borel measure $u_0 = \mu \in M(\mathbb{T})$ with $\|\mu\| < \infty$ *without* the complete integrability or the smallness assumption on μ. Note that $\mu \in \hat{b}^s_{p,\infty}$ for $sp \le -1$ since $\sup_n |\widehat{\mu}(n)| < \|\mu\| < \infty$. Hence, it can be used to study the Cauchy problem on $M(\mathbb{T})$ for nonintegrable KdV-variants. Also, see [Oh 2009b].

Remark 1.2. Let $\mathcal{F}L^{s,p}(\mathbb{T})$ be the space of functions on $\mathbb{T}$ defined via the norm $\|f\|_{\mathcal{F}L^{s,p}} = \|\langle n\rangle^s \hat{f}(n)\|_{L^p_n}$. Recall from [Oh 2009a] that $\mathcal{F}L^{s,p}(\mathbb{T})$ contains the support of the (spatial) white noise when $sp < -1$. Then, Theorems 1 and 2 can also be established in $\mathcal{F}L^{s,p}(\mathbb{T})$ for $s = -\frac{1}{2}+$, $p = 2+$ with $sp < -1$. The modification is straightforward once we note that $\|f\|_{\mathcal{F}L^{s-\varepsilon,p}} \lesssim \|f\|_{\hat{b}^s_{p,\infty}}$ for any $\varepsilon > 0$, and thus we omit the details.

This paper is organized as follows: In Section 2, we introduce some notations. In Section 3, we introduce function spaces along with their embeddings and state deterministic linear estimates from [Bourgain 1993] and [Oh 2009a]. In Section 4, we study some basic properties of the stochastic convolution. In Section 5, we prove Theorem 1 by establishing the nonlinear estimate on the second iteration of the integral formulation (4).

2. Notation

In the periodic setting on $\mathbb{T}$, the spatial Fourier domain is $\mathbb{Z}$. Let dn be the normalized counting measure on $\mathbb{Z}$. We say that $f \in L^p(\mathbb{Z})$, where $1 \le p < \infty$, if

$$\|f\|_{L^p(\mathbb{Z})} = \left(\int_{\mathbb{Z}} |f(n)|^p dn \right)^{1/p} := \left(\frac{1}{2\pi} \sum_{n \in \mathbb{Z}} |f(n)|^p \right)^{1/p} < \infty.$$

If $p = \infty$, we have the obvious definition involving the supremum. We often drop 2π for simplicity. If a function depends on both x and t, we use $\wedge_x$ (and $\wedge_t$) to denote the spatial (and temporal) Fourier transform, respectively. However, when there is no confusion, we simply use $\wedge$ to denote the spatial Fourier transform, the temporal Fourier transform, and the space-time Fourier transform, depending on the context.

For a Banach space $X \subset \mathcal{S}'(\mathbb{T} \times \mathbb{R})$, we use $\hat{X}$ to denote the space of the Fourier transforms of the functions in X, which is a Banach space with the norm $\|f\|_{\hat{X}} = \|\mathcal{F}^{-1}_{n,\tau} f\|_X$, where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform (in n and τ). Also, for a space Y of functions on $\mathbb{Z}$, we use $\hat{Y}$ to denote the space of the inverse Fourier transforms of the functions in Y with the norm $\|f\|_{\hat{Y}} = \|\mathcal{F} f\|_Y$. Now, define $\hat{b}^s_{p,q}(\mathbb{T})$ by the norm

$$\|f\|_{\hat{b}^s_{p,q}(\mathbb{T})} = \|\hat{f}\|_{b^s_{p,q}(\mathbb{Z})} := \Big\| \|\langle n \rangle^s \hat{f}(n)\|_{L^p_{|n|\sim 2^j}} \Big\|_{l^q_j} = \left(\sum_{j=0}^{\infty} \left(\sum_{|n| \sim 2^j} \langle n \rangle^{sp} |\hat{f}(n)|^p \right)^{q/p} \right)^{1/q} \tag{13}$$

for $q < \infty$ and by (9) when $q = \infty$.

Throughout the paper, $\eta(t)$ denotes a smooth cutoff supported on $[-1, 2]$ with $\eta(t) \equiv 1$ on $[0, 1]$, and let $\eta_T(t) = \eta(T^{-1}t)$. We use c, C to denote various constants, usually depending only on s, p, and δ. If a constant depends on other quantities, we make it explicit. We use $A \lesssim B$ to denote an estimate of the form $A \le CB$. Similarly, we use $A \sim B$ to denote $A \lesssim B$ and $B \lesssim A$ and use $A \ll B$ when there is no general constant C such that $B \le CA$. We also use $a+$ and $a-$ to denote $a + \varepsilon$ and $a - \varepsilon$, respectively, for arbitrarily small $\varepsilon \ll 1$.

3. Function spaces and basic embeddings

Let $X^{s,b}$ denote the usual periodic Bourgain space defined in (6). We often use the shorthand notation $\|\cdot\|_{s,b}$ to denote the $X^{s,b}$ norm. Now, define $X^{s,b}_{p,q}$, the Bourgain space adapted to $\hat{b}^s_{p,\infty}$, to be the completion of the Schwartz functions on $\mathbb{T} \times \mathbb{R}$ with respect to the norm given by

$$\|u\|_{X^{s,b}_{p,q}} = \big\| \langle n \rangle^s \langle \tau - n^3 \rangle^b \hat{u}(n, \tau) \big\|_{b^0_{p,\infty} L^q_\tau} = \sup_j \big\| \langle n \rangle^s \langle \tau - n^3 \rangle^b \hat{u}(n, \tau) \big\|_{L^p_{|n|\sim 2^j} L^q_\tau}. \tag{14}$$

In the following, we take $p = 2+$ and $s = -\frac{1}{2}+ = -\frac{1}{2} + \delta$ with $\delta < (p-2)/2p$ (and $\delta > (p-2)/4p$) such that $sp < -1$. Lastly, given $T > 0$, we define $X^{s,b,T}_{p,q}$ as a restriction of $X^{s,b}_{p,q}$ on $[0, T]$ by

$$\|u\|_{X^{s,b,T}_{p,q}} = \|u\|_{X^{s,b}_{p,q}[0,T]} = \inf\big\{ \|\widetilde{u}\|_{X^{s,b}_{p,q}} : \widetilde{u}|_{[0,T]} = u \big\}.$$

We define the local-in-time versions of the other function spaces analogously.

Now, we discuss the basic embeddings. For $p \geq 2$, we have $\|a_n\|_{L^p_n} \leq \|a_n\|_{L^2_n}$. Thus, we have $\|f\|_{\hat{b}^s_{p,\infty}} \leq \|f\|_{H^s}$, and thus

$$\|u\|_{X^{s,b}_{p,2}} \leq \|u\|_{X^{s,b}}. \tag{15}$$

By the Hölder inequality, we have

$$\begin{aligned}\|f\|_{H^{-\frac{1}{2}-\delta}} &= \Big(\sum_j (2^j)^{0-} \|\langle n\rangle^{-\frac{1}{2}-\delta+} \hat{f}(n)\|^2_{|n|\sim 2^j}\Big)^{1/2} \\ &\leq \sup_j \|\langle n\rangle^{-2\delta+}\|_{L^{2p/p-2}} \|\langle n\rangle^{-\frac{1}{2}+\delta} \hat{f}(n)\|_{L^p_n} \leq \|f\|_{\hat{b}^s_{p,\infty}}\end{aligned} \tag{16}$$

for $s = -\frac{1}{2}+\delta$ with $\delta > (p-2)/4p$. Hence, for $s = -\frac{1}{2}+\delta$ with $\delta > (p-2)/4p$, we have

$$\|u\|_{X^{-\frac{1}{2}-\delta,b}} \lesssim \|u\|_{X^{s,b}_{p,2}}. \tag{17}$$

Now, we briefly go over the linear estimates. Let $S(t) = e^{-t\partial_x^3}$ and $T \leq 1$ in the following. We first present the homogeneous and nonhomogeneous linear estimates. See [Bourgain 1993; Kenig et al. 1993; Oh 2009a] for details of the proofs.

Lemma 3.1. *For any $s \in \mathbb{R}$ and $b < \frac{1}{2}$, we have $\|S(t)u_0\|_{X^{s,b,T}_{p,2}} \lesssim T^{(1/2)-b}\|u_0\|_{\hat{b}^s_{p,\infty}}$.*

Lemma 3.2. *For any $s \in \mathbb{R}$ and $b \leq \frac{1}{2}$, we have*

$$\Big\|\int_0^t S(t-t')F(x,t')dt'\Big\|_{X^{s,b,T}_{p,2}} \lesssim \|F\|_{X^{s,b-1}_{p,2}} + \|F\|_{X^{s,-1}_{p,1}}.$$

Also, we have $\big\|\int_0^t S(t-t')F(x,t')dt'\big\|_{X^{s,b,T}_{p,2}} \lesssim \|F\|_{X^{s,b-1}_{p,2}}$ for $b > \frac{1}{2}$.

The next lemma is the periodic L^4 Strichartz estimate due to Bourgain [1993].

Lemma 3.3. *Let u be a function on $\mathbb{T}\times\mathbb{R}$. Then, we have $\|u\|_{L^4_{x,t}} \lesssim \|u\|_{X^{0,\frac{1}{3}}}$.*

Lastly, recall that by restricting the Bourgain spaces onto a small time interval $[0, T]$, we can gain a small power of T. See [Colliander and Oh 2009] for the proof.

Lemma 3.4. *For $0 \leq b' < b \leq \frac{1}{2}$, we have*

$$\|u\|_{X^{s,b',T}} = \|\eta_T u\|_{X^{s,b',T}} \lesssim T^{b-b'-}\|u\|_{X^{s,b}}.$$

4. Stochastic convolution

In this section, we study basic properties of the stochastic convolution $\Phi(t)$ defined in (10). In particular, we prove that $\eta\Phi$ belongs to $X^{s,b,T}_{p,2}$ and is continuous from $[0, T]$ into $\hat{b}^s_{p,\infty}$ for $T \leq 1$ almost surely for $sp < -1$ and $(b-1)\cdot 2 < -1$, where $\eta(t)$ is a smooth cutoff supported on $[-1, 2]$ with $\eta(t) \equiv 1$ on $[0, 1]$.

Before stating the main results, we point out the following. Let ϕ be the identity operator on $L^2(\mathbb{T})$ or be as in (3). Then, we know that such ϕ is Hilbert–Schmidt from $L^2(\mathbb{T})$ into $H^s(\mathbb{T})$ if and only if $s < -\frac{1}{2}$. In other words, with a slight abuse of notation, define

$$\phi := \sum_{n\in\mathbb{Z}} \phi e_n = \sum_{n\in\mathbb{Z}} \phi_n e_n \tag{18}$$

in view of $\phi = \operatorname{diag}(\phi_n; n \neq 0)$. Then, we have $\phi \in H^s(\mathbb{T})$ if and only if $s < -\frac{1}{2}$. Moreover, we have $\|\phi\|_{HS(L^2;H^s)} = \|\phi\|_{H^s}$, where $\|\cdot\|_{HS(L^2;H^s)}$ denotes the Hilbert–Schmidt norm from $L^2(\mathbb{T})$ to $H^s(\mathbb{T})$. For such ϕ, we also have $\phi \in \hat{b}^s_{p,\infty}(\mathbb{T})$ if and only if $sp \le -1$, and we can use $\|\phi\|_{\hat{b}^s_{p,\infty}}$ to discuss the regularity of ϕ in place of the Hilbert–Schmidt norm. This is one of the reasons for using this space. (We need only $sp < -1$ for our purpose since the nonlinear estimate in Section 5 holds for $s = -\frac{1}{2}$ and $p = 2+$ with $sp < -1$.)

Proposition 4.1. *Let $0 < T \le 1$ and $p = 2+$. Let $s = -\frac{1}{2} + \delta$ and $b = \frac{1}{2} - \delta$, with $(p-2)/4p < \delta < (p-2)/2p$ such that $sp < -1$ and $(b-1)\cdot 2 < -1$. Then, for the stochastic convolution $\Phi(t)$ defined in (10) with ϕ as in (3), we have*

$$\mathbb{E}\big[\|\eta\Phi\|_{X^{s,b,T}_{p,2}}\big] \le C(\eta, s, p) < \infty. \tag{19}$$

In particular, $\Phi \in X^{-\frac{1}{2}+\delta,\frac{1}{2}-\delta,T}_{p,2}$ almost surely.

Before going into the proof of Proposition 4.1, recall the following. Let β_1 and β_2 be independent real-valued Brownian motions on $(\Omega, \mathcal{F}, P)$, and $f_1(t,\omega)$ and $f_2(t,\omega)$ be real-valued stochastic processes independent of β_1 and β_2. Then, we can regard β_j and f_j as $\beta_j(t,\omega) = \beta_j(t,\omega_1)$ and $f_j(t,\omega) = f_j(t,\omega_2)$, where $\omega = (\omega_1,\omega_2) \in \Omega_1 \times \Omega_2 = \Omega$. Thus, in taking an expectation, we can first integrate over $\omega_1 \in \Omega_1$. Then, for $m \in \mathbb{N}$, we have

$$\begin{aligned}
\mathbb{E}\bigg[\bigg|\int_a^b f_1(t)d\beta_1(t) + \int_a^b f_2(t)d\beta_2(t)\bigg|^{2m}\bigg]
&= \mathbb{E}\bigg[\sum_{k=0}^{2m}\binom{2m}{k}\Big(\int_a^b f_1(t)d\beta_1(t)\Big)^k\Big(\int_a^b f_2(t)d\beta_2(t)\Big)^{2m-k}\bigg]\\
&= \mathbb{E}_{\Omega_2}\bigg[\sum_{n=0}^{m}\binom{2m}{2n}\frac{(2n)!}{2^n n!}\|f_1(\cdot,\omega_2)\|^{2n}_{L^2(a,b)}\frac{(2(m-n))!}{2^{m-n}(m-n)!}\|f_2(\cdot,\omega_2)\|^{2(m-n)}_{L^2(a,b)}\bigg].
\end{aligned}\tag{20}$$

In the computation above, we used the fact that, for each fixed ω_2, $\int_a^b f_j(t,\omega_2)d\beta_j(t,\omega_1)$ is a Gaussian random variable on Ω_1 with variance $\|f_j(\cdot,\omega_2)\|^2_{L^2(a,b)}$.

Proof. By the Hölder inequality, we have

$$\big\|\langle\tau - n^3\rangle^{(1/2)-\delta}\hat{u}(n,\tau)\big\|_{L^2_\tau} \le \|\langle\tau - n^3\rangle^{-2\delta}\|_{L^{2p/p-2}_\tau}\big\|\langle\tau - n^3\rangle^{(1/2)+\delta}\hat{u}(n,\tau)\big\|_{L^p_\tau},$$

that is, we have $\|\eta\Phi\|_{X^{s,1/2-\delta}_{p,2}} \lesssim \|\eta\Phi\|_{X^{s,1/2+\delta}_{p,p}}$ as long as $\delta > (p-2)/4p$. Thus, we will work in $X^{s,1/2+\delta}_{p,p}$ in the following.

Define $g(t) = \eta(t)\int_0^t S(-r)\phi(r)dW(r)$ such that $\eta(t)\Phi(\cdot,t) = S(t)g(\cdot,t)$. Assume that each β_n is extended to a Brownian motion on $\mathbb{R}$ in such a way that the family $\{\beta_n\}_{n\ge 0}$ is still independent. Note that for $t \in [0,T]$, we have

$$\hat{g}(n,t) = \eta(t)\int_0^t \eta(r)e^{-irn^3}\phi_n(r)\chi_{[0,T]}(r)\tfrac{1}{\sqrt{2}}d\beta_n(r). \tag{21}$$

We have inserted $\eta(r)$ and $\chi_{[0,T]}(r)$ in the integrand since $\eta(r)\chi_{[0,T]}(r) \equiv 1$ for $r \in [0,t] \subset [0,T]$. For notational simplicity, we use $\phi_n(r)$ to denote $\phi_n(r)\chi_{[0,T]}(r)$ in the following, that is, we assume that ϕ_n is supported on $[0,T]$. By (3), we have $|\phi_n(r)| \le 1$ for $r \in \mathbb{R}$.

Now, we write the left-hand side of (19) as

$$\mathbb{E}\Big[\|\eta\Phi\|_{X^{s,\frac{1}{2}+\delta,T}_{p,p}}\Big]\lesssim\mathbb{E}\bigg[\sup_j 2^{js}\bigg(\sum_{|n|\sim 2^j}\sum_{k=1}^{\infty}2^{kp(\frac{1}{2}+\delta)}\int_{|\tau|\sim 2^k}|\hat{g}(n,\tau)|^p d\tau\bigg)^{1/p}\bigg]$$
$$+\mathbb{E}\bigg[\sup_j 2^{js}\bigg(\sum_{|n|\sim 2^j}\int_{|\tau|\le 2}|\hat{g}(n,\tau)|^p d\tau\bigg)^{1/p}\bigg]. \quad (22)$$

Part 1. First, we estimate the second term in (22). Let

$$G_n(r,\tau)=\eta(r)e^{-irn^3}\phi_n(r)\int_r^{\infty}\eta(t)e^{-it\tau}dt. \quad (23)$$

Also write $\beta_n=\beta_n^{(r)}+i\beta_n^{(i)}$ where $\beta_n^{(r)}=\operatorname{Re}\beta_n$ and $\beta_n^{(i)}=\operatorname{Im}\beta_n$. Then, by the stochastic Fubini Theorem, we have, for $m\in\mathbb{N}$,

$$\begin{aligned}\mathbb{E}\big[|\hat{g}(n,\tau)|^{2m}\big]&=\mathbb{E}\bigg[\bigg|\int_{\mathbb{R}}\eta(t)e^{-it\tau}\int_{-\infty}^{t}\eta(r)e^{-irn^3}\phi_n(r)\tfrac{1}{\sqrt{2}}d\beta_n(r)dt\bigg|^{2m}\bigg]\\&=2^{-m}\mathbb{E}\bigg[\bigg|\int_{-1}^{2}G_n(r,\tau)d\beta_n(r)\bigg|^{2m}\bigg]\\&\lesssim\mathbb{E}\bigg[\bigg|\int_{-1}^{2}\operatorname{Re}G_n(r,\tau)d\beta_n^{(r)}(r)-\int_{-1}^{2}\operatorname{Im}G_n(r,\tau)d\beta_n^{(i)}(r)\bigg|^{2m}\bigg]\\&\qquad+\mathbb{E}\bigg[\bigg|\int_{-1}^{2}\operatorname{Im}G_n(r,\tau)d\beta_n^{(r)}(r)+\int_{-1}^{2}\operatorname{Re}G_n(r,\tau)d\beta_n^{(i)}(r)\bigg|^{2m}\bigg]. \quad (24)\end{aligned}$$

Note that $|\operatorname{Re}G_n(r,\tau)|,\ |\operatorname{Im}G_n(r,\tau)|\le|G_n(r,\tau)|\le\|\eta\|_{L^1}|\phi_n(r)|\lesssim\|\eta\|_{L^1}\chi_{[0,T]}(r)$. Thus, we have

$$\big\|\operatorname{Re}G_n(r,\tau)\big\|_{L^2_r}^{2k}\big\|\operatorname{Im}G_n(r,\tau)\big\|_{L^2_r}^{2(m-k)}\lesssim\|\eta\|_{L^1}^{2m}$$

for $k=0,\dots,m$. Then, by (20) along with the independence of ϕ_n, $\beta_n^{(r)}$, and $\beta_n^{(i)}$, we have

$$\|\hat{g}(n,\tau)\|_{L^{2m}(\Omega)}\le C=C(\eta,m)$$

independent of n and τ. Hence, for $p\in(2,4)$, we have

$$\big(\mathbb{E}\big[|\hat{g}(n,\tau)|^p\big]\big)^{1/p}\le\|\hat{g}(n,\tau)\|_{L^2(\Omega)}^{\theta}\|\hat{g}(n,\tau)\|_{L^4(\Omega)}^{1-\theta}\lesssim 1 \quad (25)$$

by interpolation, where $\theta\in(0,1)$ such that $\dfrac{1}{p}=\dfrac{\theta}{2}+\dfrac{1-\theta}{4}$. Then, the second term in (22) is estimated by

$$\begin{aligned}(22)&\le\bigg(\sum_{j=0}^{\infty}2^{jsp}\sum_{|n|\sim 2^j}\int_{|\tau|\le 2}\mathbb{E}\big[|\hat{g}(n,\tau)|^p\big]d\tau\bigg)^{1/p}\\&\lesssim\bigg(\sum_{j=0}^{\infty}2^{jsp}\sum_{|n|\sim 2^j}1\bigg)^{1/p}\sim\bigg(\sum_{j=0}^{\infty}2^{(sp+1)j}\bigg)^{1/p}\le C<\infty, \quad (26)\end{aligned}$$

since $sp<-1$.

Part 2. Next, we estimate the first term in (22). Let

$$\begin{cases} G_n^{(1)}(r,\tau) = \eta(r)e^{-irn^3}\phi_n(r)\int_r^\infty \eta'(t)(e^{-it\tau}/i\tau)dt, \\ G_n^{(2)}(r,\tau) = \eta^2(r)e^{-irn^3}\phi_n(r)(e^{-ir\tau}/i\tau). \end{cases} \tag{27}$$

Then, by the stochastic Fubini theorem and integration by parts, we have

$$\begin{aligned} \sqrt{2}\hat{g}(n,\tau) &= \int_{-1}^2 G_n(r,\tau)d\beta_n(r) = \int_{-1}^2 G_n^{(1)}(r,\tau)d\beta_n(r) + \int_{-1}^2 G_n^{(2)}(r,\tau)d\beta_n(r) \\ &=: I_n^{(1)}(\tau) + I_n^{(2)}(\tau). \end{aligned} \tag{28}$$

Thus $|\hat{g}(n,\tau)|^p \lesssim \big|I_n^{(1)}(\tau)\big|^p + \big|I_n^{(2)}(\tau)\big|^p$.

First, we estimate the contribution from $G_n^{(1)}$. For $|\tau| \sim 2^k$, we have

$$\left|\int_r^\infty \eta'(t)\frac{e^{-it\tau}}{i\tau}dt\right| \le |\tau^{-2}\eta'(r)| + \left|\int_r^\infty \eta''(t)\frac{e^{-it\tau}}{\tau^2}dt\right| \le C_\eta 2^{-2k} \tag{29}$$

by partial integration. Thus, we have $|G_n^{(1)}(r,\tau)| \lesssim 2^{-2k}$. Then, repeating a similar computation as in Part 1, we obtain

$$\big(\mathbb{E}\big[|I_n^{(1)}(\tau)|^p\big]\big)^{1/p} \le \|I_n^{(1)}(\tau)\|_{L^2(\Omega)}^\theta \|I_n^{(1)}(\tau)\|_{L^4(\Omega)}^{1-\theta} \lesssim 2^{-2k}, \tag{30}$$

by (20) and interpolation. Hence, the contribution to (22) is estimated by

$$\begin{aligned} (22) &\le \bigg(\sum_{j=0}^\infty 2^{jsp}\sum_{|n|\sim 2^j}\sum_{k=1}^\infty 2^{kp(\frac12+\delta)}\int_{|\tau|\sim 2^k}\mathbb{E}\big[|I_n^{(1)}(\tau)|^p\big]d\tau\bigg)^{1/p} \\ &\lesssim \bigg(\sum_{j=0}^\infty 2^{j(sp+1)}\sum_{k=1}^\infty 2^{k(-(3p/2)+\delta p+1)}\bigg)^{1/p} \le C < \infty, \end{aligned} \tag{31}$$

since $sp < -1$ and $-\frac32 p + \delta p + 1 < 0$.

Now, we consider the contribution from $I_n^{(2)}(\tau)$. With $\beta_n = \beta_n^{(r)} + i\beta_n^{(i)}$, we have $|I_n^{(2)}(\tau)|^2 \lesssim \big|\int_{-1}^2 G_n^{(2)}(r,\tau)d\beta_n^{(r)}(r)\big|^2 + \big|\int_{-1}^2 G_n^{(2)}(r,\tau)d\beta_n^{(i)}(r)\big|^2$. We only estimate the first term since the second term is estimated in the same way. By the Ito formula (see [De Bouard et al. 2004]), we have

$$\begin{aligned} \left|\int_{-1}^2 G_n^{(2)}(r,\tau)d\beta_n^{(r)}(r)\right|^2 &= \int_{-1}^2 \eta^4(t)\frac{|\phi_n(t)|^2}{\tau^2}dt + 2\,\mathrm{Re}\int_{-1}^2\int_{-\infty}^t G_n^{(2)}(r,\tau)d\beta_n^{(r)}(r)\overline{G_n^{(2)}}(t,\tau)d\beta_n^{(r)}(t) \\ &=: I_n'(\tau) + I_n''(\tau). \end{aligned}$$

The contribution from $I_n'(\tau)$ is at most

$$\begin{aligned} (22) &\lesssim \bigg(\sum_{j=0}^\infty 2^{jsp}\sum_{|n|\sim 2^j}\sum_{k=1}^\infty 2^{kp(\frac12+\delta)}\int_{|\tau|\sim 2^k}|\tau|^{-p}d\tau\Big(\int_{-1}^2\eta^4(t)dt\Big)^{p/2}\bigg)^{1/p} \\ &\lesssim \|\eta\|_{L^4}^2\bigg(\sum_{j=0}^\infty 2^{j(sp+1)}\sum_{k=1}^\infty 2^{k(-\frac{p}{2}+\delta p+1)}\bigg)^{1/p} \le C < \infty, \end{aligned} \tag{32}$$

since $sp < -1$ and $\delta < (p-2)/2p$.

We finally estimate the contribution from $I''_n(\tau)$. Write

$$I''_n(\tau) = \int_{-1}^{2} H_n(t) d\beta_n^{(r)}(t),$$

where $H_n(t) = \int_{-\infty}^{t} \widetilde{H}_n(r,t) d\beta_n^{(r)}(r)$ with

$$\widetilde{H}_n(r,t) = 2\tau^{-2} \operatorname{Re}\big(\eta^2(r)\eta^2(t) e^{i(t-r)n^3} \phi_n(r)\overline{\phi_n(t)} e^{i(t-r)\tau}\big). \tag{33}$$

Then, by the Ito isometry and $|\phi_n(\omega,t)| \le 1$ for all $(\omega,t) \in \Omega \times \mathbb{R}$, we have

$$\begin{aligned}
\mathbb{E}\big[|I''_n(\tau)|^2\big] &= \mathbb{E}\bigg[\Big(\int_{-1}^{2} H_n(t) d\beta_n^{(r)}(t)\Big)^2\bigg] \sim \int_{-1}^{2} \mathbb{E}\big[H_n^2(t)\big] dt \\
&= \int_{-1}^{2} \mathbb{E}\bigg[\Big(\int_{-\infty}^{t} \widetilde{H}_n(r,t) d\beta_n^{(r)}(r)\Big)^2\bigg] dt = \int_{-1}^{2}\int_{-1}^{t} \mathbb{E}\big[|\widetilde{H}_n(r,t)|^2\big] dr dt \\
&\lesssim \tau^{-4} \int_{-1}^{2}\int_{-1}^{t} \eta^4(r)\eta^4(t) dr dt \lesssim \tau^{-4}.
\end{aligned} \tag{34}$$

Hence, the contribution from $I''_n(\tau)$ is at most

$$\begin{aligned}
(22) &\lesssim \bigg(\sum_{j=0}^{\infty} 2^{jsp} \sum_{|n|\sim 2^j} \sum_{k=1}^{\infty} 2^{kp(\frac{1}{2}+\delta)} \int_{|\tau|\sim 2^k} \mathbb{E}\big[|I''_n(\tau)|^{p/2}\big] d\tau\bigg)^{1/p} \\
&\lesssim \bigg(\sum_{j=0}^{\infty} 2^{jsp} \sum_{|n|\sim 2^j} \sum_{k=1}^{\infty} 2^{kp(\frac{1}{2}+\delta)} \int_{|\tau|\sim 2^k} \big(\mathbb{E}\big[|I''_n(\tau)|^2\big]\big)^{p/4} d\tau\bigg)^{1/p} \\
&\lesssim \bigg(\sum_{j=0}^{\infty} 2^{j(sp+1)} \sum_{k=1}^{\infty} 2^{k(-\frac{p}{2}+\delta p+1)}\bigg)^{1/p} \le C < \infty,
\end{aligned} \tag{35}$$

for $p \le 4$, $sp < -1$, and $\delta < (p-2)/2p$. □

We state a corollary to the proof of Proposition 4.1 for a general diagonal covariance operator $\phi(t,\omega) = \operatorname{diag}(\phi_n(t,\omega); n \in \mathbb{Z})$, which is independent of $\{\beta_n\}_{n\ge 1}$.

Corollary 4.2. *Let $0 < T \le 1$, $p = 2+$, and $s, s' \in \mathbb{R}$ with $s < s'$. Moreover, let $b = \frac{1}{2} - \delta$ with $(p-2)/4p < \delta < (p-2)/2p$, so $(b-1)\cdot 2 < -1$. Then, for the stochastic convolution $\Phi(t)$ defined in* (10) *with $\phi \in L^p([0,T]\times\Omega; \hat{b}^{s'}_{p,\infty})$, independent of $\{\beta_n\}_{n\ge1}$, we have*

$$\mathbb{E}\big[\|\eta\Phi\|_{X^{s,b,T}_{p,2}}\big] \le C(\eta, s, s', p)\|\phi\|_{L^p([0,T]\times\Omega;\hat{b}^{s'}_{p,\infty})}. \tag{36}$$

In particular, $\Phi \in X^{s,\frac{1}{2}-\delta,T}_{p,2}$ almost surely.

Proof. In the proof of Proposition 4.1, we used $|\phi_n(t)| \le 1$ whenever $\phi_n(t)$ appeared. Now, we briefly go through that proof, keeping track of $\phi_n(t)$. Since ϕ is independent of $\{\beta_n\}_{n\ge1}$, we regard β_n and ϕ_n as $\beta_n(t,\omega) = \beta_n(t,\omega_1)$ and $\phi_n(t,\omega) = \phi_n(t,\omega_2)$, where $\omega = (\omega_1,\omega_2) \in \Omega_1 \times \Omega_2 = \Omega$.

In (25), we have $\mathbb{E}\big[|\hat{g}(n,\tau)|^p\big] \lesssim \mathbb{E}_{\Omega_2}\|\phi_n(\cdot,\omega_2)\|^p_{L^2[0,T]}$. Then, in (26), we have

$$
\begin{aligned}
(22) &\le \bigg(\sum_{j=0}^{\infty} 2^{jsp} \sum_{|n|\sim 2^j} \int_{|\tau|\le 2} \mathbb{E}_{\Omega_2}\|\phi_n(\cdot,\omega_2)\|^p_{L^2[0,T]} d\tau\bigg)^{1/p} \\
&\le \bigg(\sum_{j=0}^{\infty} 2^{j(s-s')p} 2^{js'p} \sum_{|n|\sim 2^j} \|\phi_n(\cdot,\omega_2)\|^p_{L^p([0,T]\times\Omega_2)}\bigg)^{1/p} \lesssim \|\phi\|_{L^p([0,T]\times\Omega;\hat{b}^{s'}_{p,\infty})},
\end{aligned}
$$

since $s-s'<0$. A similar modification in (30) and (31) (alternatively, (32)) takes care of the contribution from $I_n^{(1)}(\tau)$ (alternatively, $I_n'(\tau)$). Now, as for $I_n''(\tau)$, we first integrate only over Ω_1 in (34) and obtain

$$
\mathbb{E}_{\Omega_1}\big[|I_n''(\tau)|^2\big] \lesssim \tau^{-4}\int_{-1}^{2}\int_{-1}^{t} \eta^4(r)\eta^4(t)|\phi_n(r)|^2|\phi_n(t)|^2 dr\,dt \lesssim \tau^{-4}\|\phi_n\|^4_{L^2[0,T]}.
$$

Then, in (35), we have

$$
\mathbb{E}\big[|I_n''(\tau)|^{p/2}\big] = \mathbb{E}_{\Omega_2}\big[\|I_n''(\tau)\|^{p/2}_{L^{p/2}(\Omega_1)}\big] \le \mathbb{E}_{\Omega_2}\big[\|I_n''(\tau)\|^{p/2}_{L^2(\Omega_1)}\big] \lesssim \tau^{-p}\,\mathbb{E}_{\Omega_2}\|\phi_n(\,\cdot\,,\omega_2)\|^p_{L^2[0,T]}
$$

for $p\in[2,4]$. The rest follows as before. □

Now, we discuss the continuity of the stochastic convolution. In the remaining of this section, we show that the stochastic convolution $\Phi(t)$ defined in (10) belongs to $C\big([0,T];\hat{b}^s_{p,\infty}(\mathbb{T})\big)$ almost surely. With $\beta_n = \beta_n^{(r)} + i\beta_n^{(i)}$, we have

$$
\Phi(t) = \frac{1}{\sqrt{2}}\sum_{n\ne 0}\int_0^t S(t-r)\phi_n(r)e_n d\beta_n^{(r)}(r) + i\frac{1}{\sqrt{2}}\sum_{n\ne 0}\int_0^t S(t-r)\phi_n(r)e_n d\beta_n^{(i)}(r), \tag{37}
$$

since $\phi e_0 = 0$ and $\phi e_n = \phi_n e_n$, $n\ne 0$. In the following, we only show the continuity of the first stochastic convolution in (37), which we denote by $\Phi^{(r)}(t)$. Also, let $W^{(r)}(t) = \frac{1}{\sqrt{2}}\sum_n \beta_n^{(r)}(t)e_n$. As in [Da Prato 2004], we use the factorization method based on the elementary identity

$$
\int_r^t (t-t')^{\alpha-1}(t'-r)^{-\alpha}dt' = \frac{\pi}{\sin\pi\alpha} \tag{38}
$$

with $\alpha\in(0,1)$ for $0\le r\le t'\le t$. Using (38), we can write the first term in (37) as

$$
\Phi^{(r)}(t) = \frac{\sin\pi\alpha}{\pi}\int_0^t S(t-t')(t-t')^{\alpha-1}Y(t')dt', \tag{39}
$$

where

$$
Y(t') = \int_0^{t'} S(t'-r)(t'-r)^{-\alpha}\phi(r)dW^{(r)}(r). \tag{40}
$$

First, we present a lemma that provides a criterion for the continuity of (39) in terms of the L^{2m}-integrability of $Y(t')$.

Lemma 4.3 [Da Prato 2004, Lemma 2.7]. *Let $T>0$, $\alpha\in(0,1)$, and $m>\frac{1}{2\alpha}$. For $f\in L^{2m}\big([0,T];\hat{b}^s_{p,\infty}(\mathbb{T})\big)$, let*

$$
F(t) = \int_0^t S(t-t')(t-t')^{\alpha-1}f(t')dt', \quad 0\le t\le T.
$$

Then, $F \in C\big([0,T]; \hat{b}^s_{p,\infty}(\mathbb{T})\big)$. *Moreover, there exists* $C = C(m,T)$ *such that*

$$\|F(t)\|_{\hat{b}^s_{p,\infty}} \le C\|f\|_{L^{2m}([0,T];\hat{b}^s_{p,\infty})}, \quad 0 \le t \le T.$$

Remark 4.4. Although Da Prato states his Lemma 2.7 for a Hilbert space H, his proof makes no use of the Hilbert space structure of H. Thus the same result holds for $\hat{b}^s_{p,\infty}(\mathbb{T})$ as well.

In view of Lemma 4.3, it suffices to show that $Y(t') \in L^{2m}\big([0,T]; \hat{b}^s_{p,\infty}(\mathbb{T})\big)$ a.s.

Proposition 4.5. *Let* $T > 0$, $m \ge 2$, $s = -\frac{1}{2}+$, *and* $p = 2+$ *such that* $sp < -1$. *Let* ϕ *be as in* (3). *Then, the stochastic convolution* $\Phi^{(r)}(t)$ *is continuous from* $[0,T]$ *into* $\hat{b}^s_{p,\infty}$ *almost surely. Moreover, there exists*

$$\mathbb{E}\Big(\sup_{t\in[0,T]} \|\Phi^{(r)}(t)\|^{2m}_{\hat{b}^s_{p,\infty}}\Big) \le C(m,T,s,p) < \infty.$$

Proof. Let $\alpha \in \big(\frac{1}{2m}, \frac{1}{2}\big)$ and Y as in (40). First, note that Y is real-valued since $\phi_{-n}(s)e_{-n} = \overline{\phi_n(s)e_n}$ and $\beta^{(r)}_{-n} = \beta^{(r)}_n$. Note that $\{\beta^{(r)}_n\}_{n\ne 0}$ and ϕ are independent since ϕ depends only on β_0. Thus, we can regard $\beta^{(r)}_n$ and ϕ as $\beta^{(r)}_n(\omega) = \beta^{(r)}_n(\omega_1)$ and $\phi(\omega) = \phi(\omega_2)$, where $\omega = (\omega_1,\omega_2) \in \mathbf{\Omega}_1 \times \mathbf{\Omega}_2 = \mathbf{\Omega}$. Then, for each fixed ω_2 and $t' \in [0,t]$, $\widehat{Y(t')}(n)$ is a Gaussian random variable on $\mathbf{\Omega}_1$ with $\mathrm{Var}_{\Omega_1}\big(\widehat{Y(t')}(n)\big) = \mathbb{E}_{\Omega_1}\big[|\widehat{Y(t')}(n)|^2\big]$.

Let $G_n(r,\omega_2) = (t'-r)^{-\alpha}e^{i(t'-r)n^3}\phi_n(r,\omega_2)$. Note that $|G_n(r,\omega_2)| = (t'-r)^{-\alpha}$ for $0 < r < t'$ and $n \ne 0$. By the Ito isometry, we have

$$\mathbb{E}_{\Omega_1}\big[|\widehat{Y(t')}(n)|^2\big] = \tfrac{1}{2}\,\mathbb{E}_{\Omega_1}\bigg[\bigg|\int_0^{t'} G_n(r,\omega_2)d\beta(r,\omega_1)\bigg|^2\bigg] = \frac{1}{2}\int_0^{t'} |G_n(r,\omega_2)|^2 dr \sim \int_0^{t'} (t'-r)^{-2\alpha}dr.$$

By the Minkowski integral inequality (with $p = 2+ < 2m$) after replacing $\sup_j$ by $\sum_j$, we have

$$\begin{aligned}
\mathbb{E}_{\Omega_1}\big(\|Y(t',\cdot,\omega_2)\|^{2m}_{\hat{b}^s_{p,\infty}}\big) &= \mathbb{E}_{\Omega_1}\bigg[\bigg(\sup_j \sum_{|n|\sim 2^j} \langle n\rangle^{sp}|\widehat{Y(t')}(n)|^p\bigg)^{2m/p}\bigg] \\
&\lesssim \bigg(\sum_{j=0}^\infty \sum_{|n|\sim 2^j} 2^{jsp}\Big(\mathbb{E}_{\Omega_1}\big[|\widehat{Y(t')}(n)|^{2m}\big]\Big)^{p/2m}\bigg)^{2m/p} \\
&\sim \bigg(\sum_{j=0}^\infty 2^{j(sp+1)}\bigg)^{2m/p}\bigg(\int_0^{t'}(t'-r)^{-2\alpha}dr\bigg)^m \lesssim \bigg(\frac{(t')^{1-2\alpha}}{1-2\alpha}\bigg)^m,
\end{aligned}$$

since $sp < -1$. Therefore

$$\begin{aligned}
\int_0^T \mathbb{E}\big(\|Y(t')\|^{2m}_{\hat{b}^s_{p,\infty}}\big)dt' &= \int_0^T \mathbb{E}_{\Omega_2}\mathbb{E}_{\Omega_1}\big(\|Y(t')\|^{2m}_{\hat{b}^s_{p,\infty}}\big)dt' \\
&\lesssim \int_0^T \bigg(\frac{(t')^{1-2\alpha}}{1-2\alpha}\bigg)^m dt' \lesssim T^{(1-2\alpha)m+1} < C(m,T,s,p) < \infty.
\end{aligned}$$

In particular, it follows that $Y(\cdot,\omega) \in L^{2m}\big([0,T]; \hat{b}^s_{p,\infty}\big)$ almost surely. Then, the desired result follows from Lemma 4.3. □

5. Nonlinear estimate on the second iteration

Now, we present the crucial nonlinear analysis. First, we briefly go over Bourgain's argument [1997]. By writing the integral equation, the deterministic KdV (5) is equivalent to

$$u(t) = S(t)u_0 - \tfrac{1}{2}\mathcal{N}(u,u)(t), \tag{41}$$

where $\mathcal{N}(\,\cdot\,,\cdot\,)$ is given by

$$\mathcal{N}(u_1,u_2)(t) := \int_0^t S(t-t')\partial_x(u_1u_2)(t')dt'. \tag{42}$$

In the following, we assume that the initial condition u_0 has mean 0, which implies that $u(t)$ has spatial mean 0 for each $t \in \mathbb{R}$. We use (n,τ), (n_1,τ_1), and (n_2,τ_2) to denote the Fourier variables for uu, the first factor, and the second factor u of uu in $\mathcal{N}(u,u)$, respectively, thus we have $n = n_1+n_2$ and $\tau = \tau_1+\tau_2$. By the mean-zero assumption on u and since we have $\partial_x(uu)$ in the definition of $\mathcal{N}(u,u)$, we assume $n, n_1, n_2 \neq 0$. We also use the following notation:

$$\sigma_0 := \langle \tau - n^3\rangle \text{ and } \sigma_j := \langle \tau_j - n_j^3\rangle.$$

One of the main ingredients is the observation due to Bourgain [1993]:

$$n^3 - n_1^3 - n_2^3 = 3nn_1n_2 \quad \text{for } n = n_1+n_2, \tag{43}$$

which in turn implies that

$$\text{MAX} := \max(\sigma_0,\sigma_1,\sigma_2) \gtrsim \langle nn_1n_2\rangle. \tag{44}$$

Now, define

$$A_j = \{(n,n_1,n_2,\tau,\tau_1,\tau_2) \in \mathbb{Z}^3\times\mathbb{R}^3 : \sigma_j = \text{MAX}\}, \tag{45}$$

and let $\mathcal{N}_j(u,u)$ denote the contribution of $\mathcal{N}(u,u)$ on A_j. By the standard bilinear estimate as in [Bourgain 1993; Kenig et al. 1996], we have

$$\|\mathcal{N}_0(u,u)\|_{-\frac{1}{2}+\delta,\frac{1}{2}-\delta} \le o(1)\|u\|^2_{-\frac{1}{2}-\delta,\frac{1}{2}-\delta}, \tag{46}$$

where $o(1) = T^\theta$ with some $\theta > 0$ by considering the estimate on a short time interval $[-T,T]$ (for example, Lemma 3.4). See (2.17), (2.26), and (2.68) in [Bourgain 1997]. Here, we abuse the notation and use $\|\cdot\|_{s,b} = \|\cdot\|_{X^{s,b}}$ to denote the local-in-time version as well. Note that the temporal regularity b equals $\frac{1}{2}-\delta < \frac{1}{2}$. This allows us to improve the spatial regularity by 2δ. Clearly, we cannot expect to do the same for $\mathcal{N}_1(u,u)$. (By symmetry, we do not consider $\mathcal{N}_2(u,u)$ in the following.) The bilinear estimate (7) is known to fail for any $s \in \mathbb{R}$ if $b < \frac{1}{2}$ due to the contribution from $\mathcal{N}_1(u,u)$ [Kenig et al. 1996]. Following the notation in [Bourgain 1997], let

$$I_{s,b} = \|\mathcal{N}_1(u,u)\|_{X^{s,b}} \qquad \text{and} \qquad \alpha := \tfrac{1}{2}-\delta < \tfrac{1}{2}. \tag{47}$$

Then, by Lemma 3.2 and duality with $\|d(n,\tau)\|_{L^2_{n,\tau}} \le 1$, we have

$$I_{-\alpha,1-\alpha} = \|\mathcal{N}_1(u,u)\|_{-\alpha,1-\alpha} \lesssim \sum_{\substack{n,n_1\\ n=n_1+n_2}} \int_{\tau=\tau_1+\tau_2} d\tau d\tau_1 \frac{\langle n\rangle^{1-\alpha}d(n,\tau)}{\sigma_0^\alpha}\hat{u}(n_1,\tau_1)\frac{\langle n_2\rangle^{1-\alpha}c(n_2,\tau_2)}{\sigma_2^\alpha}, \tag{48}$$

where

$$c(n_2,\tau_2)=\langle n_2\rangle^{-(1-\alpha)}\sigma_2^{\alpha}\,\hat{u}(n_2,\tau_2)\quad\text{so that } \|c\|_{L^2_{n,\tau}}=\|u\|_{-(1-\alpha),\alpha}=\|u\|_{-\frac{1}{2}-\delta,\frac{1}{2}-\delta}. \tag{49}$$

The main idea here is to consider the second iteration, that is, substitute (41) for $\hat{u}(n_1,\tau_1)$ in (48), thus leading to a trilinear expression. Since $\sigma_1=\text{MAX}\gtrsim\langle nn_1n_2\rangle\gg 1$ on A_1, we can assume that

$$\hat{u}(n_1,\tau_1)=(\mathcal{N}(u,u))^\wedge(n_1,\tau_1)\sim\frac{|n_1|}{\sigma_1}\sum_{n_1=n_3+n_4}\int_{\tau_1=\tau_3+\tau_4}\hat{u}(n_3,\tau_3)\hat{u}(n_4,\tau_4)d\tau_4. \tag{50}$$

Note that $\hat{u}(n_1,\tau_1)$ cannot come from $S(t)u_0$ of (41) since we have $\sigma_1\sim 1$ for the linear part. Moreover, by the standard computation [Bourgain 1993], we have

$$\begin{aligned}\mathcal{N}(u,u)(x,t)&=-i\sum_{k=1}^{\infty}\frac{i^kt^k}{k!}\sum_{n\neq 0}e^{i(nx+n^3t)}\int\eta(\lambda-n^3)\widehat{\partial_x u^2}(n,\lambda)d\lambda\\&\quad+i\sum_{n\neq 0}e^{inx}\int\frac{(1-\eta)(\tau-n^3)}{\tau-n^3}\widehat{\partial_x u^2}(n,\tau)e^{i\tau t}d\tau\\&\quad+i\sum_{n\neq 0}e^{i(nx+n^3t)}\int\frac{(1-\eta)(\lambda-n^3)}{\lambda-n^3}\widehat{\partial_x u^2}(n,\lambda)d\lambda\\&=:\mathcal{M}_1(u,u)(x,t)+\mathcal{M}_2(u,u)(x,t)+\mathcal{M}_3(u,u)(x,t).\end{aligned} \tag{51}$$

Note that $(\mathcal{M}_1(u,u))^\wedge(n_1,\tau_1)$ and $(\mathcal{M}_3(u,u))^\wedge(n_1,\tau_1)$ are distributions supported on $\{\tau_1-n_1^3=0\}$, so $\sigma_1\sim 1$. Hence, the only contribution for the second iteration on A_1 comes from $\mathcal{M}_2(u,u)$ whose Fourier transform is given in (50). This shows the validity of the assumption (50).

The σ_1 appearing in the denominator allows us to cancel $\langle n\rangle^{1-\alpha}$ and $\langle n_2\rangle^{1-\alpha}$ in the numerator in (48). Then, $I_{-\alpha,1-\alpha}$ can be estimated by

$$\lesssim\sum_{\substack{n=n_1+n_2\\n_1=n_3+n_4}}\int_{\substack{\tau=\tau_1+\tau_2\\\tau_1=\tau_3+\tau_4}}\frac{\langle n\rangle^{1-\alpha}d(n,\tau)}{\sigma_0^{\alpha}}\frac{|n_1|}{\sigma_1}\hat{u}(n_3,\tau_3)\hat{u}(n_4,\tau_4)\frac{\langle n_2\rangle^{1-\alpha}c(n_2,\tau_2)}{\sigma_2^{\alpha}}. \tag{52}$$

Bourgain then divided the argument into several cases, depending on the sizes of $\sigma_0,\dots,\sigma_4$. Here, the key algebraic relation is

$$n^3-n_2^3-n_3^3-n_4^3=3(n_2+n_3)(n_3+n_4)(n_4+n_2)\quad\text{with } n=n_2+n_3+n_4. \tag{53}$$

Then, Bourgain proved [1997, (2.69)]

$$I_{-\alpha,1-\alpha}\le o(1)\|u\|_{-(1-\alpha),\alpha}I_{-\alpha,1-\alpha}+o(1)\|u\|^3_{-(1-\alpha),\alpha}+o(1)\|u\|_{-(1-\alpha),\alpha}, \tag{54}$$

assuming the a priori estimate (8): $|\hat{u}(n,t)|<C$ for all $n\in\mathbb{Z}$, $t\in\mathbb{R}$. Indeed, the estimates involving the first two terms on the right-hand side of (54) were obtained without (8), and *only* the last term in (54) required (8) [Bourgain 1997, "Estimation of (2.62)"], which was then used to deduce

$$\|\hat{u}(n,\cdot)\|_{L^2_\tau}<C. \tag{55}$$

The a priori estimate (8) is derived via the isospectral property of the KdV flow and is false for a general function in $X^{-(1-\alpha),\alpha}$. (It is here that the smallness of the total variation $\|\mu\|$ is used.)

Our goal is to carry out a similar analysis for SKdV (2) on the second iteration *without* the a priori estimates (8) and (55) coming from the complete integrability of KdV. We achieve this goal by considering the estimate in

$$X_{p,2}^{-\alpha,\alpha} = X_{p,2}^{-\frac{1}{2}+\delta,\frac{1}{2}-\delta},$$

where $p = 2+$ and $\dfrac{p-2}{4p} < \delta < \dfrac{p-2}{2p}$. By (15) and (17) (recall $-\alpha = -\frac{1}{2}+\delta$ and $-(1-\alpha) = -\frac{1}{2}-\delta$), we have

$$\|u\|_{X_{p,2}^{-\alpha,\alpha}} \leq \|u\|_{X^{-\alpha,\alpha}} \qquad \text{and} \qquad \|u\|_{X^{-(1-\alpha),\alpha}} \lesssim \|u\|_{X_{p,2}^{-\alpha,\alpha}}. \tag{56}$$

Then, it follows from (46) and (56) that

$$\|\mathcal{N}_0(u,u)\|_{X_{p,2}^{-\alpha,\alpha}} \leq o(1)\|u\|^2_{X_{p,2}^{-\alpha,\alpha}}. \tag{57}$$

Now, we consider the estimate on $\|\mathcal{N}_1(u,u)\|_{X_{p,2}^{-\alpha,\alpha}}$. From (56) and $\alpha < 1-\alpha$, it suffices to control $I_{-\alpha,1-\alpha}$. As in the deterministic case, we consider the second iteration, and substitute (4) for $\hat{u}(n_1,\tau_1)$ in (48). As before, there is no contribution from $S(t)u_0$, or $\mathcal{M}_1(u,u)$, $\mathcal{M}_3(u,u)$ defined in (51). There are two contributions:

(i) $\mathcal{N}_1(\mathcal{M}_2(u,u),u)$ from the deterministic nonlinear part: In this case, we can use the estimates from [Bourgain 1997] *except* when the a priori bound (8) was assumed; that is, we need to estimate the contribution from [Bourgain 1997, (2.62)]:

$$R_\alpha := \sum_n \int_{\tau=\tau_2+\tau_3+\tau_4} \chi_B \frac{d(n,\tau)}{\langle n\rangle^{1+\alpha}\sigma_0^\alpha}\hat{u}(-n,\tau_2)\hat{u}(n,\tau_3)\hat{u}(n,\tau_4)d\tau_2 d\tau_3 d\tau_4, \tag{58}$$

where $\|d(n,\tau)\|_{L^2_{n,\tau}} \leq 1$ and $B = \{\sigma_0,\sigma_2,\sigma_3,\sigma_4 < |n|^\gamma\}$ with some small parameter $\gamma > 0$. Note that this corresponds to the case $n_2 = -n$ and $n_3 = n_4 = n$ in (52) after some reduction. In our analysis, we directly estimate R_α in terms of $\|u\|_{X_{p,2}^{-\alpha,\alpha}}$. The key observation is that we can take the spatial regularity $s = -\alpha$ to be greater than $-\frac{1}{2}$ by choosing $p > 2$.

(ii) $\mathcal{N}_1(\Phi,u)$ from the stochastic convolution Φ in (10): In view of (56), we estimate

$$\mathbb{E}\big[\|\mathcal{N}_1(\eta\Phi,u)\|_{X^{-\alpha,1-\alpha}}\big] \tag{59}$$

via the stochastic analysis from Section 4.

Remark 5.1. In fact, we do not need to take an expectation in (59) since we establish local well-posedness pathwise in ω, that is, for almost every *fixed* ω. Nonetheless, we estimate (59) with the expectation since it shows how F_1^N and F_2^N defined in (70) arise along with their estimates.

Estimate on (i). In [Bourgain 1997], the parameter $\gamma = \gamma(\alpha)$, subject to the conditions (2.43) and (2.60) therein, played a certain role in estimating R_α along with the a priori bound (8). However, it plays no role in our analysis. By the Cauchy–Schwarz and Young's inequalities, we have

$$(58) \leq \sum_n \|d(n,\cdot)\|_{L^2_\tau}\langle n\rangle^{-1-\alpha}\|\hat{u}(-n,\tau_2)\|_{L^{6/5}_{\tau_2}}\|\hat{u}(n,\tau_3)\|_{L^{6/5}_{\tau_3}}\|\hat{u}(n,\tau_4)\|_{L^{6/5}_{\tau_4}}.$$

By the Hölder inequality (with appropriate $\pm$ signs) and the fact that $-1-\alpha < -3\alpha$,

$$(58) \leq \sum_n \|d(n,\cdot)\|_{L^2_\tau} \prod_{j=2}^{4} \langle n\rangle^{-\alpha-} \|\sigma_j^{-\alpha}\|_{L^3_{\tau_j}} \|\sigma_j^\alpha \hat{u}(\pm n, \tau_j)\|_{L^2_{\tau_j}} \leq \|d(\cdot,\cdot)\|_{L^2_{n,\tau}} \|u\|^3_{X^{-\alpha,\alpha}_{6,2}} \leq \|u\|^3_{X^{-\alpha,\alpha}_{p,2}},$$

where the last two inequalities follow by choosing $\alpha > \frac{1}{3}$ and $p = 2+ < 6$.

Estimate on (ii). We use the notation from the proof of Proposition 4.1. It follows from (28) and $\eta(t)\Phi(\cdot,t) = S(t)g(\cdot,t)$ that

$$(\eta\Phi)^\wedge(n_1,\tau_1) = \hat{g}(n_1, \tau_1 - n_1^3) = \tfrac{1}{\sqrt{2}} \mathrm{I}^{(1)}_{n_1}(\tau_1 - n_1^3) + \tfrac{1}{\sqrt{2}} \mathrm{I}^{(2)}_{n_1}(\tau_1 - n_1^3).$$

Recall that $\sigma_1 = \langle \tau_1 - n_1^3\rangle \gtrsim \langle n n_1 n_2\rangle$. Also, recall from the proof of Proposition 4.1 that $|\phi_{n_1}(r)| = \chi_{[0,T]}(r)$ is independent of ω.

- Contribution from $\mathrm{I}^{(1)}_{n_1}(\tau_1 - n_1^3)$: From (48) with (27), (28), and (29), we estimate (59) by

$$(59) \lesssim \mathbb{E}\Bigg[\sum_{\substack{n,n_1\\ n=n_1+n_2}} \int_{\tau=\tau_1+\tau_2} d\tau d\tau_1 \frac{\langle n\rangle^{1-\alpha} d(n,\tau)}{\sigma_0^\alpha} \frac{1}{\sigma_1^2} \int_0^T |\phi_{n_1}(r)| d\beta_{n_1}(r) \frac{\langle n_2\rangle^{1-\alpha} c(n_2,\tau_2)}{\sigma_2^\alpha}\Bigg]. \tag{60}$$

By the Cauchy–Schwarz inequality in ω and the Ito isometry,

$$(59) \lesssim \sum_{\substack{n,n_1\\ n=n_1+n_2}} \int_{\tau=\tau_1+\tau_2} d\tau d\tau_1 \frac{d(n,\tau)}{\sigma_0^\alpha} \frac{\|\phi_{n_1}\|_{L^2[0,T]}}{\sigma_1^{(3/2)-\delta} \langle n_1\rangle^{(1/2)+\delta}} \frac{\|c(n_2,\tau_2)\|_{L^2(\Omega)}}{\sigma_2^\alpha}. \tag{61}$$

By the $L^4_{x,t}$, $L^2_{x,t}$, $L^4_{x,t}$-Hölder inequality along with Lemma 3.3, (16), (18), (49), and (56), this leads to

$$\begin{aligned}(59) &\lesssim T^\theta \|d\|_{L^2_{n,\tau}} \|\phi\|_{L^2([0,T];H^{-\frac{1}{2}-\delta})} \|c\|_{L^2(\Omega;L^2_{n,\tau})} \leq T^\theta \|\phi\|_{L^p([0,T];\hat{b}^{-\alpha}_{p,\infty})} \|u\|_{L^2(\Omega;X^{-(1-\alpha),\alpha})}\\ &\lesssim T^\theta \|\phi\|_{L^p([0,T];\hat{b}^{-\alpha}_{p,\infty})} \|u\|_{L^2(\Omega;X^{-\alpha,\alpha}_{p,2})}.\end{aligned}$$

Remark 5.2. Strictly speaking, we need to take the supremum over $\{\|d\|_{L^2_{n,\tau}} = 1\}$ *inside* the expectation in (60). However, we do not worry about this issue to simplify the presentation, since we have

$$\begin{aligned}(59) &\leq \|\mathcal{N}_1(\eta\Phi, u)\|_{L^2(\Omega;X^{-\alpha,1-\alpha})}\\ &\leq \Bigg(\sum_n \int \frac{\langle n\rangle^{2-2\alpha}}{\sigma_0^{2\alpha}} \mathbb{E}\Bigg[\Bigg|\int_0^T |\phi_{n_1}(r)| \sum_{n=n_1+n_2} \int_{\tau=\tau_1+\tau_2} \frac{\langle n_2\rangle^{1-\alpha} c(n_2,\tau_2)}{\sigma_1^2 \sigma_2^\alpha} d\tau_1 d\beta_{n_1}(r)\Bigg|^2\Bigg] d\tau\Bigg)^{1/2}\\ &= \sup_{\|d\|_{L^2_{n,\tau}}=1} (61)\end{aligned}$$

by the Ito isometry. Also, recall that we have $\mathrm{I}^{(1)}_{n_1}(\tau_1 - n_1^3) = \int_0^T G^{(1)}_{n_1}(r, \tau_1 - n_1^3) d\beta_{n_1}(r)$ where $G^{(1)}_n(r,\tau)$ is defined in (27). Hence, strictly speaking, we should replace $G^{(1)}_{n_1}(r, \tau_1 - n_1^3)$ by $\sigma_1^{-2}|\phi_{n_1}(r)|$ in (60) only after the application of the Ito isometry. Once again, we do not worry about this issue to simplify the presentation. The same remark applies to the following as well.

- Contribution from $\mathrm{I}^{(2)}_{n_1}(\tau_1 - n_1^3)$: Suppose $\max(\sigma_0, \sigma_2) \gtrsim \langle nn_1n_2\rangle^{1/100}$; say $\sigma_0 \geq \langle nn_1n_2\rangle^{1/100}$. Then

$$(59) \lesssim \mathbb{E}\Bigg[\sum_{\substack{n,n_1\\ n=n_1+n_2}} \int_{\tau=\tau_1+\tau_2} d\tau d\tau_1 \frac{\langle n\rangle^{1-\alpha} d(n,\tau)}{\sigma_0^\alpha} \frac{1}{\sigma_1} \int_0^T |\phi_{n_1}(r)| d\beta_{n_1}(r) \frac{\langle n_2\rangle^{1-\alpha} c(n_2,\tau_2)}{\sigma_2^\alpha}\Bigg]$$
$$\lesssim \sum_{\substack{n,n_1\\ n=n_1+n_2}} \int_{\tau=\tau_1+\tau_2} d\tau d\tau_1 \frac{d(n,\tau)}{\sigma_0^{\alpha-200\delta}} \frac{\|\phi_{n_1}\|_{L^2[0,T]}}{\sigma_1^{(1/2)+\delta}\langle n_1\rangle^{(1/2)+\delta}} \frac{\|c(n_2,\tau_2)\|_{L^2(\Omega)}}{\sigma_2^\alpha}. \tag{62}$$

Then we can conclude this case as before by the $L^4_{x,t}$, $L^2_{x,t}$, $L^4_{x,t}$-Hölder inequality as long as $\alpha - 200\delta > \frac{1}{3}$, which can be guaranteed by taking $\delta > 0$ sufficiently small, or equivalently, taking $p > 2$ sufficiently close to 2.

Now assume instead $\max(\sigma_0, \sigma_2) \ll \langle nn_1n_2\rangle^{1/100}$. We invoke a result contained in [Colliander et al. 2003, (7.50) and Lemma 7.4]. The conclusion there is stated with -1 as the exponent of $\langle \tau - n^3\rangle$, instead of $-\frac{3}{4}$; but by examining the proof, one sees that it will work with any exponent more negative than $-\big(\frac{2}{3} + \frac{1}{100}\big)$.

Lemma 5.3. *For* $\Omega(n) = \{\eta \in \mathbb{R} : \eta = -3nn_1n_2 + o(\langle nn_1n_2\rangle^{1/100})$ *for some* $n_1 \in \mathbb{Z}$ *with* $n = n_1 + n_2\}$,

$$\int \langle \tau - n^3\rangle^{-3/4} \chi_{\Omega(n)}(\tau - n^3)\, d\tau \lesssim 1. \tag{63}$$

We have

$$(59) \lesssim \mathbb{E}\Bigg[\sum_{\substack{n,n_1\\ n=n_1+n_2}} \int_{\tau=\tau_1+\tau_2} d\tau d\tau_1 \frac{\langle n\rangle^{1-\alpha} d(n,\tau)}{\sigma_0^\alpha} \frac{\chi_{\Omega(n_1)}(\tau_1 - n_1^3)}{\sigma_1} \int_0^T |\phi_{n_1}(r)| d\beta_{n_1}(r) \frac{\langle n_2\rangle^{1-\alpha} c(n_2,\tau_2)}{\sigma_2^\alpha}\Bigg].$$

By the Cauchy–Schwarz inequality and the Ito isometry, this yields

$$(59) \lesssim \sum_{\substack{n,n_1\\ n=n_1+n_2}} \int_{\tau=\tau_1+\tau_2} d\tau d\tau_1 \frac{d(n,\tau)}{\sigma_0^{\alpha}} \frac{\chi_{\Omega(n_1)}(\tau_1 - n_1^3)\|\phi_{n_1}\|_{L^2[0,T]}}{\sigma_1^{(1/2)-\delta}\langle n_1\rangle^{(1/2)+\delta}} \frac{\|c(n_2,\tau_2)\|_{L^2(\Omega)}}{\sigma_2^\alpha}. \tag{64}$$

By the $L^4_{x,t}$, $L^2_{x,t}$, $L^4_{x,t}$-Hölder inequality along with Lemma 3.3, Lemma 5.3, and Equations (16), (18), (49), and (56), we get

$$(59) \lesssim T^\theta \|d\|_{L^2_{n,\tau}} \Big\| \langle n_1\rangle^{-\frac{1}{2}-\delta} \|\phi_{n_1}\|_{L^2[0,T]} \|\chi_{\Omega(n_1)}(\tau_1 - n_1^3)\sigma_1^{-\frac{1}{2}+\delta}\|_{L^2_\tau}\Big\|_{L^2_n} \|c\|_{L^2(\Omega; L^2_{n,\tau})}$$
$$\leq T^\theta \|\phi\|_{L^2([0,T]; H^{-\frac{1}{2}-\delta})} \|u\|_{L^2(\Omega; X^{-(1-\alpha),\alpha})} \lesssim T^\theta \|\phi\|_{L^p([0,T]; \hat{b}^{-\alpha}_{p,\infty})} \|u\|_{L^2(\Omega; X^{-\alpha,\alpha}_{p,2})}.$$

Proof of Theorem 1. Fix a mean-zero $u_0 \in \hat{b}^{-\alpha'}_{p,\infty}(\mathbb{T})$ and ϕ as in (3), where $\alpha' = \frac{1}{2} - \delta -$ with $\frac{p-2}{4p} < \delta < \frac{p-2}{2p}$ such that $(-\alpha')p < -1$. Consider sequences of initial data $u_0^N \in L^2(\mathbb{T})$ and diagonal covariance operator $\phi^N \in HS(L^2; L^2)$, given by

$$u_0^N = \mathbb{P}_{\leq N} u_0 = \sum_{|n|\leq N} \hat{u}_0(n) e^{inx} \qquad \text{and} \qquad \phi^N(t,\omega) := \operatorname{diag}(\phi_n(t,\omega); 0 < |n| \leq N), \tag{65}$$

where ϕ_n is given in (3). Now, fix $\alpha = \frac{1}{2} - \delta > \alpha'$ as in (47). Note that such u_0^N converges to u_0 in $\mathcal{F}L^{-\alpha,p}(\mathbb{T})$, and thus in $\hat{b}^{-\alpha}_{p,\infty}(\mathbb{T})$. Also, ϕ^N converges to ϕ in $\mathcal{F}L^{-\frac{1}{2}-,p}(\mathbb{T})$ for each t and ω, and thus in $\hat{b}_{p,\infty}^{-\frac{1}{2}-}(\mathbb{T})$. Then, by the monotone convergence theorem, ϕ^N converges to ϕ in $L^p\big([0,1]\times\Omega;\hat{b}_{p,\infty}^{-\frac{1}{2}-}\big)$. (Indeed, the convergence is in $L^\infty\big([0,1]\times\Omega;\hat{b}_{p,\infty}^{-\frac{1}{2}-}\big)$, since we have $|\phi_n(t,\omega)|=1$ for all n, independent of $t\in\mathbb{R}$ and $\omega\in\Omega$.) Note that a slight loss of the regularity $-\alpha<-\alpha'$ was necessary since u_0^N defined in (65) does not necessarily converge to u_0 in $\hat{b}_{p,\infty}^{-\alpha'}(\mathbb{T})$ due to the L^∞ nature of the norm over the dyadic blocks. We can avoid such a loss of the regularity if we start with $u_0\in\mathcal{F}L^{s,p}(\mathbb{T})$.

Now, let $\Gamma^N=\Gamma^N_{u_0^N}$ be the map defined by

$$\Gamma^N v=\Gamma^N_{u_0^N}v:=S(t)u_0^N-\tfrac{1}{2}\mathcal{N}(v,v)+\eta\Phi^N, \tag{66}$$

where Φ^N is the stochastic convolution defined in (10) with the covariance operator ϕ^N. By the well-posedness result in [De Bouard et al. 2004], there exists a unique global solution $u^N\in L^\infty(\mathbb{R}^+;L^2(\mathbb{T}))\cap C(\mathbb{R}^+;B^{0-}_{2,1}(\mathbb{T}))$ a.s. to (66) for each N since $\phi^N\in HS(L^2;L^2)$.

Now, we put all the estimates together. Note that all the implicit constants are independent of N. Also, when there is no superscript N, it means that $N=\infty$. From Lemma 3.1, we have

$$\|S(t)u_0^N\|_{X^{s,b,T}_{p,2}}\le C_1\|u_0^N\|_{\hat{b}^s_{p,\infty}} \tag{67}$$

for any $s,b\in\mathbb{R}$ with $C_1=C_1(b)$. In particular, by taking $b>\frac{1}{2}$, we see that $S(t)u_0$ is continuous on $[0,T]$ with values in $\hat{b}^s_{p,\infty}$. Also, by taking $b<\frac{1}{2}$, we gain a power of T. From the definition of $\mathcal{N}_j(\cdot,\cdot)$ and (57), we have

$$\|\mathcal{N}(u^N,u^N)\|_{X^{-\alpha,\alpha,T}_{p,2}}\le C_2T^{\theta_1}\|u^N\|^2_{X^{-\alpha,\alpha,T}_{p,2}}+2\|\mathcal{N}_1(u^N,u^N)\|_{X^{-\alpha,\alpha,T}_{p,2}}. \tag{68}$$

Also, from (47) and (56), we have

$$\|\mathcal{N}_1(u^N,u^N)\|_{X^{-\alpha,1-\alpha,T}_{p,2}}\le I^N_{-\alpha,1-\alpha}. \tag{69}$$

Recall that $\eta\Phi\in X^{-\alpha,\alpha}_{p,2}$ a.s. from Proposition 4.1. Moreover, by defining F_1^N and F_2^N on $\mathbb{T}\times\mathbb{R}\times\Omega$ via their Fourier transforms

$$\begin{aligned}\widehat{F_1^N}(n,\tau)&=\langle n\rangle^{-\frac{1}{2}-\delta}\big(\sigma_0^{-\frac{3}{2}+\delta}+\sigma_0^{-\frac{1}{2}-\delta}\big)\int_0^T|\phi_n(r)|d\beta_n(r),\\ \widehat{F_2^N}(n,\tau)&=\langle n\rangle^{-\frac{1}{2}-\delta}\chi_{\Omega(n)}(\tau-n^3)\sigma_0^{-\frac{1}{2}+\delta}\int_0^T|\phi_n(r)|d\beta_n(r)\end{aligned} \tag{70}$$

for $|n|\le N$, we have $F_1^N,F_2^N\in L^2(\Omega;L^2_{x,t})$ by the Ito isometry and Lemma 5.3, which is basically shown in the estimate on (ii). See (61) and (64). Then, from (54) and the estimates on (i) and (ii), we have

$$I^N_{-\alpha,1-\alpha}\le C_3\big(T^{\theta_2}\|u^N\|_{X^{-\alpha,\alpha,T}_{p,2}}I^N_{-\alpha,1-\alpha}+T^{\theta_3}\|u^N\|^3_{X^{-\alpha,\alpha,T}_{p,2}}+T^{\theta_4}L^N_\omega\|u^N\|_{X^{-\alpha,\alpha,T}_{p,2}}\big), \tag{71}$$

where $L^N_\omega=L^N(F_1^N,F_2^N)(\omega):=\|F_1^N(\omega)\|_{L^2_{x,t}}+\|F_2^N(\omega)\|_{L^2_{x,t}}<\infty$ a.s. Moreover, L^N_ω is nondecreasing in N.

For fixed $R > 0$, choose $T > 0$ small such that $C_3 T^{\theta_2} R \le \frac{1}{2}$. Then, from (71), we have

$$I^N_{-\alpha,1-\alpha} \le 2C_3\big(T^{\theta_3}\|u^N\|^3_{X^{-\alpha,\alpha,T}_{p,2}} + T^{\theta_4} L^N_\omega \|u^N\|_{X^{-\alpha,\alpha,T}_{p,2}}\big) \tag{72}$$

for $\|u^N\|_{X^{-\alpha,\alpha,T}_{p,2}} \le R$. From (66)–(72), we have

$$\begin{aligned}\|u^N\|_{X^{-\alpha,\alpha,T}_{p,2}} = \|\Gamma^N u^N\|_{X^{-\alpha,\alpha,T}_{p,2}} &\le C_1\|u^N_0\|_{\hat{b}^{-\alpha}_{p,\infty}} + \tfrac{1}{2}C_2 T^{\theta_1}\|u^N\|^2_{X^{-\alpha,\alpha,T}_{p,2}} \\ &\quad + 2C_3\big(T^{\theta_3}\|u^N\|^3_{X^{-\alpha,\alpha,T}_{p,2}} + T^{\theta_4} L^N_\omega\|u^N\|_{X^{-\alpha,\alpha,T}_{p,2}}\big) + C_4\|\eta\Phi^N(\omega)\|_{X^{-\alpha,\alpha}_{p,2}},\end{aligned} \tag{73}$$

and

$$\begin{aligned}\|u^N - u^M\|_{X^{-\alpha,\alpha,T}_{p,2}} &= \|\Gamma^N u^N - \Gamma^M u^M\|_{X^{-\alpha,\alpha,T}_{p,2}} \\ &\le C_1\|u^N_0 - u^M_0\|_{\hat{b}^{-\alpha}_{p,\infty}} + \tfrac{1}{2}C_2T^{\theta_1}\big(\|u^N\|_{X^{-\alpha,\alpha,T}_{p,2}} + \|u^M\|_{X^{-\alpha,\alpha,T}_{p,2}}\big)\|u^N - u^M\|_{X^{-\alpha,\alpha,T}_{p,2}} \\ &\qquad + C_5T^{\theta_3}\big(\|u^N\|^2_{X^{-\alpha,\alpha,T}_{p,2}} + \|u^M\|^2_{X^{-\alpha,\alpha,T}_{p,2}}\big)\|u^N - u^M\|_{X^{-\alpha,\alpha,T}_{p,2}} \\ &\qquad + 2C_3T^{\theta_4}L^N_\omega\|u^N - u^M\|_{X^{-\alpha,\alpha,T}_{p,2}} + 2C_3T^{\theta_4}\widetilde{L}^{N,M}_\omega\|u^M\|_{X^{-\alpha,\alpha,T}_{p,2}} \\ &\qquad + C_4\|\eta(\Phi^N - \Phi^M)\|_{X^{-\alpha,\alpha}_{p,2}},\end{aligned} \tag{74}$$

where

$$\widetilde{L}^{N,M}_\omega := \|F^N_1 - F^M_1\|_{L^2_{x,t}} + \|F^N_2 - F^M_2\|_{L^2_{x,t}}. \tag{75}$$

Note that in estimating the difference $\Gamma^N u^N - \Gamma^M u^M$ on A_1, one needs to consider

$$\widetilde{I}_{-\alpha,1-\alpha} := \|\mathcal{N}_1(u^N, u^N) - \mathcal{N}_1(u^M, u^M)\|_{-\alpha,1-\alpha} \tag{76}$$

as in [Bourgain 1997]. We can follow the argument on pages 135–136 in that reference, except for R_α defined in (58), which yields the third term on the right-hand side of (74). As for R_α, we can write

$$\mathcal{N}(\mathcal{N}(u,u),u) - \mathcal{N}(\mathcal{N}(v,v),v) = \mathcal{N}(\mathcal{N}(u+v,u-v),u) + \mathcal{N}(\mathcal{N}(v,v),u-v) \tag{77}$$

as in [Bourgain 1997, (3.4)], and then we can repeat the computation done for R_α in the estimate on (i), also yielding the third term on the right-hand side of (74).

By the definition of u^N_0, we have $2C_1\|u^N_0\|_{\hat{b}^{-\alpha}_{p,\infty}} \le 2C_1\|u_0\|_{\hat{b}^{-\alpha}_{p,\infty}} + \frac{1}{2}$ for N sufficiently large. And since ϕ^N converges to ϕ in $L^p\big([0,1]\times\Omega; \hat{b}^{-\alpha+}_{p,\infty}\big)$, it follows from Corollary 4.2 and the estimate on (ii), see (61), (62), and (64), that $\mathbb{E}[\|\eta(\Phi^N - \Phi)\|_{X^{-\alpha,\alpha}_{p,2}}]$ and $\mathbb{E}[\widetilde{L}^{N,\infty}_\omega]$ defined in (75) converge to 0. Hence, $\|\eta(\Phi^N - \Phi)\|_{X^{-\alpha,\alpha}_{p,2}} + \widetilde{L}^{N,\infty}_\omega \to 0$ a.s. after selecting a subsequence (which we still denote with the index N.) Then, by Egoroff's theorem, given $\varepsilon > 0$, there exists a set Ω_ε with $\mathbb{P}(\Omega^c_\varepsilon) < 2^{-1}\varepsilon$ such that $\|\eta(\Phi^N - \Phi)\|_{X^{-\alpha,\alpha}_{p,2}} + \widetilde{L}^{N,\infty}_\omega \to 0$ uniformly in Ω_ε. In particular, $2C_4\|\eta\Phi^N\|_{X^{-\alpha,\alpha}_{p,2}} \le 2C_4\|\eta\Phi\|_{X^{-\alpha,\alpha}_{p,2}} + \frac{1}{2}$ for large N uniformly on Ω_ε. In the following, we will work on Ω_ε.

Now, let $R_\omega = 2(C_1\|u_0\|_{\hat{b}^{-\alpha}_{p,\infty}} + C_4\|\eta\Phi(\omega)\|_{X^{-\alpha,\alpha}_{p,2}}) + 1$, and define the stopping time T_ω by

$$T_\omega = \inf\big\{T > 0 : \max(C_3T^{\theta_2}R_\omega, P_1(T,R_\omega,\omega), P_2(T,R_\omega,\omega) \ge \tfrac{1}{2}\big\}, \tag{78}$$

where

$$\begin{aligned}P_1(T,R_\omega,\omega) &= \tfrac{1}{2}C_2T^{\theta_1}R_\omega + 2C_3T^{\theta_3}(R_\omega)^2 + 2C_3T^{\theta_4}L_\omega \quad \text{from (73)},\\ P_2(T,R_\omega,\omega) &= C_2T^{\theta_1}R_\omega + 2C_5T^{\theta_3}(R_\omega)^2 + 2C_3T^{\theta_4}L_\omega \quad \text{from (74)}.\end{aligned}$$

The first condition in the definition of T_ω guarantees (72), and hence (73) and (74), for $\|u^N\|_{X^{-\alpha,\alpha,T}_{p,2}} \le R_\omega$. The second condition along with (73) indeed guarantees that

$$\|u^N\|_{X^{-\alpha,\alpha,T}_{p,2}} \le R_\omega \tag{79}$$

for $T \le T_\omega$, for the following reason. Because of the temporal regularity $b=\alpha<\frac{1}{2}$, we have $\|u^N\|_{X^{-\alpha,\alpha,T}_{p,2}} = \|\chi_{[0,T]}u^N\|_{X^{-\alpha,\alpha}_{p,2}}$, where $\chi_{[0,T]}$ denotes the characteristic function of the time interval $[0,T]$ [Bourgain 1999]. Hence, $\|u^N\|_{X^{-\alpha,\alpha,T}_{p,2}}$ is continuous in T since

$$\big|\|u^N\|_{X^{-\alpha,\alpha,T+\delta}_{p,2}} - \|u^N\|_{X^{-\alpha,\alpha,T}_{p,2}}\big| \le \|u^N\|_{X^{-\alpha,\alpha}_{p,2}[T,T+\delta]} \lesssim \delta^\theta \|u^N\|_{X^{0-,\frac{1}{2}}[T,T+\delta]} \tag{80}$$

for sufficiently small $\delta>0$. Note that the last term in (80) is finite for small δ since the local-in-time solutions constructed in [De Bouard et al. 2004] are controlled in this norm (indeed in a stronger norm adapted to the Besov space $B^{0-}_{2,1}$.) Then, (79) follows from (73), the second condition in (78), and the continuity of the norm in T since (79) clearly holds at $T=0$.

From (74) along with the third condition in (78), we have

$$\|u^N - u^M\|_{X^{-\alpha,\alpha,T_\omega}_{p,2}} \le 2C_1\|u_0^N - u_0^M\|_{\hat{b}^{-\alpha}_{p,\infty}} + 4C_3 T^{\theta_4} R_\omega \widetilde{L}^{N,M}_\omega + 2C_4\|\eta(\Phi^N - \Phi^M)\|_{X^{-\alpha,\alpha}_{p,2}}. \tag{81}$$

The right-hand side of (81) goes to 0 as $N, M\to\infty$ since u_0^N is Cauchy in $\hat{b}^{-\alpha}_{p,\infty}$ and

$$\|\eta(\Phi^N - \Phi^M)\|_{X^{-\alpha,\alpha}_{p,2}} + \widetilde{L}^{N,M}_\omega \to 0$$

on Ω_ε uniformly in N, M. Let u denote the limit in $X^{-\alpha,\alpha,T_\omega}_{p,2}$.

In the following, we give a brief discussion to show that the limit u is a solution to (4). Clearly, $S(t)u_0^N$ and $\eta\Phi^N$ converge to $S(t)u_0$ and $\eta\Phi$ in $X^{-\alpha,\alpha,T_\omega}_{p,2}$. It follows from (57) that $\mathcal{N}_0(u^N,u^N)$ converges to $\mathcal{N}_0(u,u)$ in $X^{-\alpha,\alpha,T_\omega}_{p,2}$. In view of (72), (74), and (76), we see that $\mathcal{N}_j(u^N,u^N)$ is Cauchy in a slightly stronger space $X^{-\alpha,1-\alpha,T_\omega}_{p,2}$, $j=1,2$. Let v_j denote the corresponding limit. Thus, from (66), we have

$$u = S(t)u_0 - \tfrac{1}{2}\mathcal{N}_0(u,u) - \tfrac{1}{2}(v_1+v_2) + \eta\Phi. \tag{82}$$

Now, we need to show that $\mathcal{N}_j(u^N,u^N)$ indeed converges to $\mathcal{N}_j(u,u)$, $j=1,2$. By symmetry, we only consider $\mathcal{N}_1(u,u) - \mathcal{N}_1(u^N,u^N)$. As before, we substitute (82) (and (66)) in the first factor u (and u^N) of $\mathcal{N}_1(\,\cdot\,,\,\cdot\,)$, respectively. There are three contributions to consider.

(A) Contribution from the stochastic terms: We have

$$\mathcal{N}_1(\eta\Phi,u) - \mathcal{N}_1(\eta\Phi^N,u^N) = \mathcal{N}_1(\eta(\Phi-\Phi^N),u) + \mathcal{N}_1(\eta\Phi^N, u-u^N). \tag{83}$$

From the estimate on (ii), we have

$$\|(83)\|_{X^{-\alpha,\alpha,T_\omega}_{p,2}} \lesssim \widetilde{L}^{N,\infty}_\omega \|u\|_{X^{-\alpha,\alpha,T_\omega}_{p,2}} + L^N_\omega\|u^N - u\|_{X^{-\alpha,\alpha,T_\omega}_{p,2}} \to 0$$

as $N\to\infty$, since $\|u\|_{X^{-\alpha,\alpha,T}_{p,2}} \le R_\omega$ and $\widetilde{L}^{N,\infty}_\omega \to 0$ uniformly on Ω_ε.

(B) Contribution from $\mathcal{N}_0(\,\cdot\,,\,\cdot\,)$: In this case, we consider

$$\mathcal{N}_1(\mathcal{N}_0(u,u),u) - \mathcal{N}_1(\mathcal{N}_0(u^N,u^N),u^N). \tag{84}$$

Note that we have $\sigma_1 \geq \sigma_0, \sigma_2, \sigma_3, \sigma_4$ from the definition of $\mathcal{N}_1(\,\cdot\,,\cdot\,)$ and $\mathcal{N}_0(\,\cdot\,,\cdot\,)$, see (50) and (52). Indeed, we have $\sigma_1 \geq \sigma_0, \sigma_2$ since we are on A_1 defined in (45), and also $\sigma_1 \geq \sigma_3, \sigma_4$ since we are on the support of $\mathcal{N}_0(\,\cdot\,,\cdot\,)$ in the first factor of $\mathcal{N}_1(\,\cdot\,,\cdot\,)$. Once again, one can easily follow the argument in [Bourgain 1997, page 136] and show

$$\|(84)\|_{X_{p,2}^{-\alpha,\alpha,T_\omega}} \lesssim \big(\|u^N\|^2_{X_{p,2}^{-\alpha,\alpha,T_\omega}} + \|u\|^2_{X_{p,2}^{-\alpha,\alpha,T_\omega}}\big)\|u^N - u\|_{X_{p,2}^{-\alpha,\alpha,T_\omega}} \to 0.$$

In treating $R_\alpha - R_\alpha^N$ defined in (58), one needs to proceed as before, using (77) and the estimate on (i).

(C) Contribution from v_j and $\mathcal{N}_j(u^N, u^N)$, $j = 1$ or 2: By symmetry, assume $j = 1$. In this case, we have $\sigma_1 \geq \sigma_0, \sigma_2$ but $\sigma_3 \geq \sigma_1, \sigma_4$, thus we control (54) by the first term on the right-hand side [Bourgain 1997, (II.1) on page 126]. Now, we need to estimate

$$\mathcal{N}_1(v_1, u) - \mathcal{N}_1(\mathcal{N}_1(u^N, u^N), u^N) = \mathcal{N}_1(v_1 - \mathcal{N}_1(u^N, u^N), u) + \mathcal{N}_1(\mathcal{N}_1(u^N, u^N), u - u^N) =: \mathrm{I} + \mathrm{II}. \quad (85)$$

Then, by proceeding as in [Bourgain 1997] with (56) and (72), we have

$$\|\,\mathrm{II}\,\|_{X_{p,2}^{-\alpha,1-\alpha,T_\omega}} \lesssim I^N_{-\alpha,1-\alpha}\|u - u^N\|_{X^{-(1-\alpha),\alpha,T_\omega}} \lesssim \|u - u^N\|_{X_{p,2}^{-\alpha,\alpha,T_\omega}} \to 0.$$

By proceeding as in [Bourgain 1997, (II.1)] with $|n_1|^\alpha$ replaced by $|n_1|^{1-\alpha}$, followed by (56), we have

$$\begin{aligned}\|\,\mathrm{I}\,\|_{X_{p,2}^{-\alpha,1-\alpha,T_\omega}} &\lesssim \|v_1 - \mathcal{N}_1(u^N, u^N)\|_{-(1-\alpha),1-\alpha}\|u\|_{-(1-\alpha),\alpha} \\ &\lesssim \|v_1 - \mathcal{N}_1(u^N, u^N)\|_{X_{p,2}^{-\alpha,1-\alpha,T_\omega}}\|u\|_{X_{p,2}^{-\alpha,\alpha,T_\omega}} \to 0,\end{aligned}$$

since $v_1 = \lim_{N\to\infty}\mathcal{N}_1(u^N, u^N)$ in $X_{p,2}^{-\alpha,1-\alpha,T_\omega}$ by definition.

Hence, we have $u = \Gamma_{u_0}u$ for each $\omega \in \Omega_\varepsilon$, so u is a mild solution to (2) on $[0, T_\omega]$. Let $\Omega^{(1)} = \Omega_\varepsilon$. Now, we can recursively construct

$$\Omega^{(j+1)} \subset \Omega \setminus \bigcup_{k=1}^{j} \Omega^{(k)}$$

for $j = 1, 2, \ldots$ with $\mathbb{P}(\Omega \setminus \bigcup_{k=1}^{j}\Omega^{(k)}) < 2^{-j}\varepsilon$ such that $\|\eta(\Phi^N - \Phi)\|_{X_{p,2}^{-\alpha,\alpha}}$ and $\widetilde{L}_\omega^{N,\infty}$ converge to 0 uniformly in each $\Omega^{(j)}$. Then, by repeating the argument, we can construct a solution u on $\bigcup_{j=1}^{\infty}\Omega^{(j)}$. Note that $\mathbb{P}(\Omega \setminus \bigcup_{j=1}^{\infty}\Omega^{(j)}) = 0$.

We have constructed a solution u to (2) in $X_{p,2}^{-\alpha,\alpha,T_\omega}$ with $u_0 \in \hat{b}^{-\alpha'}_{p,\infty}$. Since u is a solution, the a priori estimate (73) holds with the regularity $(s, b) = (-\alpha', \alpha')$ in place of $(-\alpha, \alpha)$. Then, we easily see that $u \in X_{p,2}^{-\alpha',\alpha',T_\omega}$ by redefining R_ω and T_ω with this regularity. In the remaining of the paper, we work only with the spatial regularity $s = -\alpha'$, that is, there is no approximating sequences any more. Hence, for notational simplicity, we will use $-\alpha$ in place of $-\alpha'$ to denote the spatial regularity of the solution in the following.

We still need to take care of several issues. Note that the temporal regularity $b = \alpha = \frac{1}{2} - \delta$ of the solution u is less than $\frac{1}{2}$. In particular, we need to show that the solution u is continuous from $[0, T_\omega]$ into $\hat{b}^{-\alpha}_{p,\infty}$. We also need to show its uniqueness and continuous dependence on the initial data.

From Proposition 4.5, $\eta\Phi \in C([0, T_\omega]; \hat{b}^{-\alpha}_{p,\infty})$ a.s. Also, it follows from (67) with $b = \frac{1}{2}+\delta$, (69), (72), and symmetry on σ_1 and σ_2, that

$$S(t)u_0 + \mathcal{N}_1(u,u) + \mathcal{N}_2(u,u) \in X^{-\alpha,\frac{1}{2}+\delta,T_\omega}_{p,2} \subset C([0,T_\omega]; \hat{b}^{-\alpha}_{p,\infty})$$

almost surely. Now, we consider $\mathcal{N}_0(u,u)$, that is, when $\sigma_0 = \text{MAX}$. Note that the contribution comes only from $\mathcal{M}_2(u,u)$ defined in (51). Define

$$\begin{aligned}\mathcal{N}_3(u,u) &= \text{the contribution of } \mathcal{N}_0(u,u) \text{ on } \{\max(\sigma_1,\sigma_2) \gtrsim \langle nn_1n_2\rangle^{1/100}\},\\ \mathcal{N}_4(u,u) &= \mathcal{N}_0(u,u) - \mathcal{N}_3(u,u).\end{aligned}$$

(a) First, we consider $\mathcal{N}_3(u,u)$, so $\max(\sigma_1,\sigma_2) \gtrsim \langle nn_1n_2\rangle^{1/100}$; say $\sigma_1 \gtrsim \langle nn_1n_2\rangle^{1/100}$. Then, by (15) and Lemma 3.2, we have

$$\|\mathcal{N}_3(u,u)\|_{X^{-\alpha,\frac{1}{2}+\delta,T_\omega}_{p,2}} \lesssim \|\partial_x(u^2)\|_{X^{-\alpha,-\frac{1}{2}+\delta,T_\omega}_{p,2}} \lesssim \|\partial_x(u^2)\|_{X^{-\alpha,-\frac{1}{2}+\delta,T_\omega}}.$$

By duality and (44), the right-hand side equals

$$\begin{aligned}&\sup_{\|d\|_{L^2_{n,\tau}}=1} \sum_{\substack{n,n_1\\ n=n_1+n_2}} \int_{\tau=\tau_1+\tau_2} \frac{\langle n\rangle^{1-\alpha} d(n,\tau)}{\sigma_0^{(1/2)-\delta}} \prod_{j=1}^{2} \frac{\langle n_j\rangle^{1-\alpha} c(n_j,\tau_j)}{\sigma_j^\alpha} d\tau d\tau_1\\ &\qquad\lesssim \sup_{\|d\|_{L^2_{n,\tau}}=1} \sum_{\substack{n,n_1\\ n=n_1+n_2}} \int_{\tau=\tau_1+\tau_2} d(n,\tau) \frac{c(n_1,\tau_1)}{\sigma_1^{\alpha-200\delta}} \frac{c(n_2,\tau_2)}{\sigma_2^\alpha} d\tau d\tau_1,\end{aligned}$$

where $c(n,\tau)$ is defined in (49). Thus, by the $L^2_{x,t}$, $L^4_{x,t}$, $L^4_{x,t}$-Hölder inequality along with Lemma 3.3, (49), and (56), we conclude that

$$\|\mathcal{N}_3(u,u)\|_{X^{-\alpha,\frac{1}{2}+\delta,T_\omega}_{p,2}} \lesssim \|c\|^2_{L^2_{n,\tau}} \le \|u\|^2_{X^{-(1-\alpha),\alpha}} \lesssim \|u\|^2_{X^{-\alpha,\alpha}_{p,2}} < \infty.$$

(b) Now, consider $\mathcal{N}_4(u,u)$, so $\max(\sigma_1,\sigma_2) \ll \langle nn_1n_2\rangle^{1/100}$. It suffices to show that $\mathcal{N}_0(u,u) \in X^{-\alpha,0,T_\omega}_{p,1}$, since $X^{-\alpha,0,T_\omega}_{p,1} \subset C([0,T_\omega]; \hat{b}^{-\alpha}_{p,\infty})$. Then, by Cauchy–Schwarz inequality, Lemma 5.3 and duality, we have

$$\begin{aligned}\|\mathcal{N}_4(u,u)\|_{X^{-\alpha,0,T_\omega}_{p,1}} &\le \|\partial_x(u^2)\|_{X^{-\alpha,-1,T_\omega}_{2,1}} \le \Big\| \|\langle n\rangle^{-\alpha} \langle \tau-n^3\rangle^{-1} \chi_{\Omega(n)}(\tau-n^3)\widehat{\partial_x(u^2)}(n,\tau)\|_{L^1_\tau}\Big\|_{L^2_n}\\ &\le \Big\|\langle\tau-n^3\rangle^{-\frac{1}{2}+\delta}\chi_{\Omega(n)}(\tau-n^3)\Big\|_{L^2_\tau} \|\partial_x(u^2)\|_{-\alpha,-\frac{1}{2}-\delta}\\ &\lesssim \sup_{\|d\|_{L^2_{n,\tau}}=1} \sum_{\substack{n,n_1\\ n=n_1+n_2}} \int_{\tau=\tau_1+\tau_2} \frac{\langle n\rangle^{1-\alpha} d(n,\tau)}{\sigma_0^{(1/2)+\delta}} \prod_{j=1}^{2} \frac{\langle n_j\rangle^{1-\alpha} c(n_j,\tau_j)}{\sigma_j^\alpha} d\tau d\tau_1\\ &\lesssim \sup_{\|d\|_{L^2_{n,\tau}}=1} \sum_{\substack{n,n_1\\ n=n_1+n_2}} \int_{\tau=\tau_1+\tau_2} d(n,\tau) \frac{c(n_1,\tau_1)}{\sigma_1^\alpha} \frac{c(n_2,\tau_2)}{\sigma_2^\alpha} d\tau d\tau_1.\end{aligned}$$

The rest follows as before. Hence, the solution u is continuous from $[0,T_\omega]$ to $\hat{b}^{-\alpha}_{p,\infty}$.

Lastly, we show the uniqueness and the continuous dependence of the solutions on the initial data. Let u and v be the mild solutions of (2) on $[0, T_\omega]$ with initial data u_0 and v_0; then

$$u - v = \Gamma_{u_0} u - \Gamma_{v_0} v = S(t)(u_0 - v_0) - \tfrac{1}{2}\big(\mathcal{N}(u,u) - \mathcal{N}(v,v)\big), \tag{86}$$

where Γ is defined in (66). Moreover, assume that

$$\|u_0\|_{\hat{b}^{-\alpha}_{p,\infty}} \le R, \qquad \|v_0\|_{\hat{b}^{-\alpha}_{p,\infty}} \le R, \qquad \|u\|_{X^{-\alpha,\alpha,T_\omega}_{p,2}} \le R, \qquad \|v\|_{X^{-\alpha,\alpha,T_\omega}_{p,2}} \le R. \tag{87}$$

Let $\widetilde{\mathcal{N}}_j(u,v) := -\frac{1}{2}\big(\mathcal{N}_j(u,u) - \mathcal{N}_j(v,v)\big)$ for $j = 1, \dots, 4$. First, note that $\|\widetilde{\mathcal{N}}_4(u,v)\|_{X^{-\alpha,\varepsilon,T_\omega}_{p,1}} \lesssim R^2 < \infty$ from (a slight variation of) Case (b), and we have

$$\|(u-v) - \widetilde{\mathcal{N}}_4(u,v)\|_{X^{-\alpha,\varepsilon,T_\omega}_{p,1}} \le \Big\| S(t)(u_0 - v_0) + \sum_{j=1}^{3} \widetilde{\mathcal{N}}_j(u,v) \Big\|_{X^{-\alpha,\frac{1}{2}+\delta,T_\omega}_{p,2}} \lesssim C_1(R) < \infty$$

by Cauchy–Schwarz inequality with $\varepsilon < \delta$, followed by (67), (69), (72), Case (a), and (87). Then, by interpolation and Cauchy–Schwarz inequality, we have

$$\|u-v\|_{C([0,T_\omega];\hat{b}^{-\alpha}_{p,\infty})} \lesssim \|u-v\|_{X^{-\alpha,0,T_\omega}_{p,1}} \lesssim \|u-v\|^{\beta}_{X^{-\alpha,-\delta-,T_\omega}_{p,1}} \|u-v\|^{1-\beta}_{X^{-\alpha,\varepsilon,T_\omega}_{p,1}} \lesssim C_2(R) \|u-v\|^{\beta}_{X^{-\alpha,\frac{1}{2}-\delta,T_\omega}_{p,2}} \tag{88}$$

with $\beta = \dfrac{\varepsilon}{\varepsilon + \delta +} \in (0,1)$. From (67) and the nonlinear estimates (see (68), (72), (74), (76)), we have

$$\|u-v\|_{X^{-\alpha,\frac{1}{2}-\delta,T_\omega}_{p,2}} \lesssim \|u_0 - v_0\|_{\hat{b}^{-\alpha}_{p,\infty}} + C_3(R) T_\omega^{\theta} \|u-v\|_{X^{-\alpha,\frac{1}{2}-\delta,T_\omega}_{p,2}}.$$

Hence, for sufficiently small $T > 0$, we have

$$\|u-v\|_{X^{-\alpha,\frac{1}{2}-\delta,T_\omega}_{p,2}} \lesssim \|u_0 - v_0\|_{\hat{b}^{-\alpha}_{p,\infty}}. \tag{89}$$

Therefore, it follows from (88) and (89) that the solution map is Hölder continuous with the bound

$$\|u-v\|_{C([0,T_\omega];\hat{b}^{-\alpha}_{p,\infty})} \le C_4(R) \|u_0 - v_0\|^{\beta}_{\hat{b}^{-\alpha}_{p,\infty}}.$$

In particular, the solution is unique. This completes the proof of Theorem 1. □

Acknowledgments

The author thanks Prof. Jeremy Quastel and Prof. Catherine Sulem for suggesting this problem.

References

[Bourgain 1993] J. Bourgain, "Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations, II: The KdV-equation", *Geom. Funct. Anal.* **3**:3 (1993), 209–262. MR 95d:35160b Zbl 0787.35098

[Bourgain 1997] J. Bourgain, "Periodic Korteweg–de Vries equation with measures as initial data", *Selecta Math.* (*N.S.*) **3**:2 (1997), 115–159. MR 2000i:35173 Zbl 0891.35138

[Bourgain 1999] J. Bourgain, "Nonlinear Schrödinger equations", pp. 3–157 in *Hyperbolic equations and frequency interactions (Park City, UT, 1995)*, edited by L. Caffarelli and E. Weinan, IAS/Park City Math. Ser. **5**, Amer. Math. Soc., Providence, RI, 1999. MR 2000c:35216 Zbl 0952.35127

[Christ et al. 2003] M. Christ, J. Colliander, and T. Tao, "Asymptotics, frequency modulation, and low regularity ill-posedness for canonical defocusing equations", *Amer. J. Math.* **125**:6 (2003), 1235–1293. MR 2005d:35223 Zbl 1048.35101

[Colliander and Oh 2009] J. Colliander and T. Oh, "Almost sure well-posedness of the cubic nonlinear Schrödinger equation below $L^2(\mathbb{T})$", preprint, 2009.

[Colliander et al. 2003] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, "Sharp global well-posedness for KdV and modified KdV on $\mathbb{R}$ and $\mathbb{T}$", *J. Amer. Math. Soc.* **16**:3 (2003), 705–749. MR 2004c:35352 Zbl 1025.35025

[Da Prato 2004] G. Da Prato, *Kolmogorov equations for stochastic PDEs*, Birkhäuser Verlag, Basel, 2004. MR 2005m:60002 Zbl 1066.60061

[De Bouard et al. 2004] A. De Bouard, A. Debussche, and Y. Tsutsumi, "Periodic solutions of the Korteweg–de Vries equation driven by white noise", *SIAM J. Math. Anal.* **36**:3 (2004), 815–855. MR 2005k:60194 Zbl 1075.35121

[Kappeler and Topalov 2006] T. Kappeler and P. Topalov, "Global wellposedness of KdV in $H^{-1}(\mathbb{T}, \mathbb{R})$", *Duke Math. J.* **135**:2 (2006), 327–360. MR 2007i:35199 Zbl 1106.35081

[Kenig et al. 1993] C. E. Kenig, G. Ponce, and L. Vega, "The Cauchy problem for the Korteweg–de Vries equation in Sobolev spaces of negative indices", *Duke Math. J.* **71**:1 (1993), 1–21. MR 94g:35196 Zbl 0787.35090

[Kenig et al. 1996] C. E. Kenig, G. Ponce, and L. Vega, "A bilinear estimate with applications to the KdV equation", *J. Amer. Math. Soc.* **9**:2 (1996), 573–603. MR 96k:35159 Zbl 0848.35114

[Kenig et al. 2001] C. E. Kenig, G. Ponce, and L. Vega, "On the ill-posedness of some canonical dispersive equations", *Duke Math. J.* **106**:3 (2001), 617–633. MR 2002c:35265 Zbl 1034.35145

[Oh 2009a] T. Oh, "Invariance of the white noise for KdV", *Comm. Math. Phys.* **292**:1 (2009), 217–236.

[Oh 2009b] T. Oh, "White noise for KdV and mKdV on the circle", preprint, 2009. To appear in RIMS Kôkyûroku Bessatsu.

[Roynette 1993] B. Roynette, "Mouvement brownien et espaces de Besov", *Stochastics Stochastics Rep.* **43**:3-4 (1993), 221–260. MR 95h:60042 Zbl 0808.60071

Received 2 Jan 2009. Revised 28 Aug 2009. Accepted 19 Oct 2009.

TADAHIRO OH: oh@math.toronto.edu
Department of Mathematics, University of Toronto, 40 St. George Street, Rm 6290, Toronto, ON M5S 2E4, Canada

ANALYSIS AND PDE
Vol. 2, No. 3, 2009

STABILITY FOR STRONGLY COUPLED CRITICAL ELLIPTIC SYSTEMS IN A FULLY INHOMOGENEOUS MEDIUM

OLIVIER DRUET AND EMMANUEL HEBEY

We investigate and prove analytic stability for strongly coupled critical elliptic systems in the inhomogeneous context of a compact Riemannian manifold.

Coupled systems of nonlinear Schrödinger equations are now a part of several important branches of mathematical physics. They appear in the Hartree–Fock theory for Bose–Einstein double condensates, in fiber-optic theory, in the theory of Langmuir waves in plasma physics, and in the behavior of deep water waves and freak waves in the ocean. A general reference book on such systems and their role in physics has been written by Ablowitz et al. [2004]. We focus here on coupled Gross–Pitaevskii type equations. These systems of equations are strongly related to two branches of mathematical physics. They arise [Burke et al. 1997] in the Hartree–Fock theory for double condensates, which are binary mixtures of Bose–Einstein condensates in two different hyperfine states. They also arise in the study of incoherent solitons in nonlinear optics, as describe in [Akhmediev and Ankiewicz 1998; Christodoulides et al. 1997; Hioe 1999; Hioe and Salter 2002; Kanna and Lakshmanan 2001]. Looking for standing wave solutions for these time evolution systems gives rise to their elliptic analogues that we investigate here. We consider these elliptic systems of equations in arbitrary dimensions $n \geq 3$, in the critical energy regime, and in a fully inhomogeneous medium that we model by an arbitrary compact Riemannian manifold, thus breaking the various symmetries that these systems enjoy in the Euclidean setting.

In what follows we let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 3$. For $p \geq 1$ an integer, we let $M_p^s(\mathbb{R})$ denote the vector space of symmetrical $p \times p$ real matrices, and A be a C^1 map from M to $M_p^s(\mathbb{R})$. We can write $A = (A_{ij})_{i,j}$, where the A_{ij}'s are C^1 real valued functions in M. Let $\Delta_g = -\operatorname{div}_g \nabla$ be the Laplace–Beltrami operator on M. Let also $H^1(M)$ be the Sobolev space of functions in $L^2(M)$ with one derivative in $L^2(M)$. A p-map $\mathcal{U} = (u_1, \dots, u_p)$ from M to $\mathbb{R}^p$ is said to be nonnegative if $u_i \geq 0$ for all i. The coupled system of nonlinear Schrödinger equations we consider here is written as

$$\Delta_g u_i + \sum_{j=1}^{p} A_{ij}(x) u_j = |\mathcal{U}|^{2^\star - 2} u_i \tag{0-1}$$

in M for all i, where $|\mathcal{U}|^2 = \sum_{i=1}^{p} u_i^2$, and $2^\star = 2n/(n-2)$ is the critical Sobolev exponent for the embeddings of the Sobolev space $H^1(M)$ into Lebesgue's spaces. The systems (0-1) are weakly coupled by the linear matrix A, and strongly coupled by the Gross–Pitaevskii type nonlinearity in the right hand side of (0-1). Besides, (0-1) is critical for Sobolev embeddings. From the viewpoint of conformal

MSC2000: primary 35C20, 58J37; secondary 35Q51, 35Q55, 35Q60.
Keywords: Critical equations, elliptic systems, Riemannian manifolds, stability, strong coupling.
The authors were partially supported by the ANR grant ANR-08-BLAN-0335-01.

geometry, our systems are pure extensions of Yamabe type equations in the strongly coupled regime. As a by-product (0-1) inherits a conformal structure.

Our aim in this paper is to discuss stability for systems like (0-1). Contrary to time evolution equations, where perturbations of the initial data together with perturbations of the equations are used to measure stability, stability for elliptic equations has to do solely with perturbations of the equations. In the framework of systems such as (0-1), stability is naturally measured with respect to perturbations of the map A. In what follows, a system like (0-1) is said to be analytically stable if for any sequence $(A_\alpha)_\alpha$ of maps from M to $M_p^s(\mathbb{R})$, $\alpha \in \mathbb{N}$, and for any bounded sequence in H^1 of nonnegative nontrivial solutions $\mathcal{U}_\alpha$ of the associated systems (0-1), if $A_\alpha \to A$ in C^1 as $\alpha \to +\infty$, then, up to a subsequence, $\mathcal{U}_\alpha \to \mathcal{U}$ in C^2 as $\alpha \to +\infty$ for some nonnegative nontrivial solution $\mathcal{U}$ of (0-1). When the strong convergence in C^2 is replaced by a weak convergence $\mathcal{U}_\alpha \rightharpoonup \mathcal{U}$ in H^1, the system (0-1) is said to be weakly stable. We refer to Section 1 for more precise definitions.

Before stating our theorem we need to introduce two assumptions. Let $\mathbf{\Delta}_g$ be the Laplace–Beltrami operator acting on p-maps by acting on each of the components of the map, and let $\mathrm{Vect}_+(\mathbb{R}^p)$ be the set of vectors in $\mathbb{R}^p$ with nonnegative components. The first assumption we may impose is

$$\mathrm{Ker}(\mathbf{\Delta}_g + A) \cap L^2(M, \mathrm{Vect}_+(\mathbb{R}^p)) = \{0\}, \tag{H}$$

where $\mathrm{Ker}(\mathbf{\Delta}_g + A)$ is the kernel of $\mathbf{\Delta}_g + A$, and $L^2(M, \mathrm{Vect}_+(\mathbb{R}^p))$ stands for the set of L^2 maps from M to $\mathrm{Vect}_+(\mathbb{R}^p)$. In order to introduce our second assumption we let $A_n = A_n(A)$ be given by

$$A_n = A - \frac{n-2}{4(n-1)} S_g \ \mathrm{Id}_p, \tag{0-2}$$

where S_g is the scalar curvature of g, and Id_p is the identity $p \times p$ matrix. For $x \in M$, let also $\mathrm{Is}_{A_n(x)}$ be the set consisting of the isotropic vectors for $A_n(x)$, namely of the vectors $X \in \mathbb{R}^p$ which are such that $\langle A_n(x).X, X\rangle_{\mathbb{R}^p} = 0$, where $\langle \cdot, \cdot \rangle_{\mathbb{R}^p}$ is the Euclidean scalar product in $\mathbb{R}^p$. The second assumption we introduce is that for any $x \in M$, $A_n(x)$ should not possess stable subspaces with an orthonormal basis consisting of isotropic nonnegative vectors. More precisely, it is this:

> For any $x \in M$ and any $k \in \{1, \dots, p\}$, there does not exist an orthonormal family $(e_1, \dots, e_k)$ of vectors in $\mathrm{Is}_{A_n(x)} \cap \mathrm{Vect}_+(\mathbb{R}^p)$ such that $A_n(x)V \subset V$, where V is the k-dimensional subspace of $\mathbb{R}^p$ with basis $(e_1, \dots, e_k)$. (H′)

The case $k = 1$ in (H′) reduces to the nonexistence of a nontrivial vector in $\mathrm{Vect}_+(\mathbb{R}^p) \cap \mathrm{Ker}\, A_n(x)$, where $\mathrm{Ker}\, A_n(x)$ is the kernel of $A_n(x)$. An assumption like (H′) is automatically satisfied in several simple situations. This is the case if we prevent the existence of isotropic vectors for A_n. In particular, (H′) holds true if $A_n(x) > 0$ or $A_n(x) < 0$ for all x in the sense of bilinear forms. Clearly there are other cases where (H′) holds true. Assumption (H) is analytic in nature. Assumption (H′) is algebraic in nature and related to the underlying geometric conformal structure of the equations. Our main result, establishing analytic stability for (0-1), is stated as follows.

Theorem 0.1. *Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 4$ and $p \geq 1$ be an integer. For any C^1-map $A : M \to M_p^s(\mathbb{R})$ satisfying* (H) *and* (H′), *the system* (0-1) *is analytically stable when $n \neq 6$, and weakly stable when $n = 6$. Besides, there are examples of six-dimensional manifolds and C^1-maps A satisfying* (H) *and* (H′) *for which* (0-1) *is analytically unstable.*

A closely related notion to stability is that of compactness. A system like (0-1) is said to be compact if bounded sequences in H^1 of nonnegative solutions of (0-1) converge, up to a subsequence, in the C^2-topology. As is easily seen, analytic stability implies compactness. In particular, as a direct consequence of the analytic stability part in our theorem, we get that systems like (0-1) are compact when $n \neq 6$ as soon as (H) and (H′) are satisfied. Assumptions (H) and (H′) in Theorem 0.1 are sharp, as discussed in Section 1 below.

Most of the asymptotic analysis developed in this paper carries over to the $n = 3$ case. However, the concluding argument needs to be changed when $n = 3$. In this dimension the mass of the Green's expansion of the Schrödinger operator $\Delta_g + A$ leads over A_n. We can conclude when the mass is positive. The analysis is developed in [Druet et al. 2009].

The paper is organized as follows. We discuss general properties of stability and compactness in Section 1. We prove the $n = 6$ part of Theorem 0.1 in Section 2. We provide a complete classification of H^1-nonnegative solutions of the strongly coupled critical limit Euclidean system $\Delta u_i = |\mathcal{U}|^{2^\star-2} u_i$, $i = 1, \ldots, p$, in Section 3. We prove Theorem 0.1 in its $n \neq 6$ part in Sections 4 to 10. In the process we establish in Sections 5, 6, and 8 the full C^0-theory for the blow-up of arbitrary sequences of solutions of strongly coupled systems like (0-1).

1. General considerations on stability and compactness

We start with the precise definition of elliptic stability we use for our systems (0-1). As already mentioned stability is here measured with respect to perturbations of the parameter A in (0-1). In doing so we preserve the conformal structure of the equation. Historically speaking such type of perturbations were first considered in the early work of Aubin [1976] on the Yamabe equation. Given $(A_\alpha)_\alpha$ a sequence of C^1 maps from M to $M_p^s(\mathbb{R})$, with $A_\alpha = (A_{ij}^\alpha)_{i,j}$ for all α integer, we consider the systems

$$\Delta_g u_i + \sum_{j=1}^{p} A_{ij}^\alpha(x) u_j = |\mathcal{U}|^{2^\star-2} u_i. \tag{1-1}$$

A sequence $(\mathcal{U}_\alpha)_\alpha$ of C^2 maps from M to $\mathbb{R}^p$ is said to be a sequence of nonnegative solutions of (1-1) if for any $\alpha \in \mathbb{N}$, $\mathcal{U}_\alpha = (u_1^\alpha, \ldots, u_p^\alpha)$ solves (0-1) and $u_i^\alpha \geq 0$ for all i. The sequence is said to be bounded in $H^1(M)$, or to have finite energy, if its components u_α^i are all bounded in $H^1(M)$ with respect to α. Given $\Lambda > 0$, we define the slice $\mathcal{S}_A^\Lambda$ to be the set of p-maps $\mathcal{U} \in H^1$ such that $\mathcal{U}$ solves (0-1), $\mathcal{U}$ is nonnegative and the H^1-norm of $\mathcal{U}$ is less than or equal to Λ. By standard regularity, adapting classical arguments from Trudinger [1968], weak solutions in H^1 of systems like (0-1) are always of class C^2. In particular, $\mathcal{S}_A^\Lambda \subset C^2$ for all $\Lambda > 0$. For $X, Y \subset C^2$ we let $d_{C^2}^{\hookrightarrow}(X; Y)$ be the C^2-pointed distance from X to Y defined by

$$d_{C^2}^{\hookrightarrow}(X; Y) = \sup_{\mathcal{U} \in X} \inf_{\mathcal{V} \in Y} \|\mathcal{V} - \mathcal{U}\|_{C^2}, \tag{1-2}$$

where $\|\mathcal{V} - \mathcal{U}\|_{C^2} = \sum_i \|v_i - u_i\|_{C^2}$ and the u_i's and v_i's are the components of $\mathcal{U}$ and $\mathcal{V}$. Stability in the elliptic regime is defined in Definition 1.1 below. The C^1 convergence $A_\alpha \to A$ in Definition 1.1 refers to the C^1 convergence of the components A_{ij}^α of A_α to the components A_{ij} of A. Similarly, the C^2 convergences, and the weak convergences in H^1, of the $\mathcal{U}_\alpha$'s in Definition 1.1 refer to the convergences of the components of the maps.

Definition 1.1. Let (M, g) be a smooth compact Riemannian manifold, let $p \geq 1$ be an integer, and let $A : M \to M_p^s(\mathbb{R})$ be a C^1 map. The system (0-1) is said to be

(i) analytically stable if for any sequence $(A_\alpha)_\alpha$ of C^1 maps from M to $M_p^s(\mathbb{R})$ such that $A_\alpha \to A$ in $C^1(M)$ as $\alpha \to +\infty$, and for any bounded sequence $(\mathcal{U}_\alpha)_\alpha$ in $H^1(M)$ of nonnegative nontrivial solutions of (1-1), there exists a nonnegative nontrivial solution $\mathcal{U}$ of (0-1) such that, up to a subsequence, the $\mathcal{U}_\alpha$'s converge strongly to $\mathcal{U}$ in $C^2(M)$ as $\alpha \to +\infty$, and

(ii) weakly stable if for any sequence $(A_\alpha)_\alpha$ of C^1 maps from M to $M_p^s(\mathbb{R})$ such that $A_\alpha \to A$ in $C^1(M)$ as $\alpha \to +\infty$ and for any bounded sequence $(\mathcal{U}_\alpha)_\alpha$ in $H^1(M)$ of nonnegative nontrivial solutions of (1-1), there exists a nonnegative nontrivial solution $\mathcal{U}$ of (0-1) such that, up to a subsequence, the $\mathcal{U}_\alpha$'s converge weakly to $\mathcal{U}$ in $H^1(M)$ as $\alpha \to +\infty$.

The system is said to be geometrically stable, if the slices $\mathcal{S}_A^\Lambda$ are stable for all $\Lambda > 0$, where $\mathcal{S}_A^\Lambda$ is said to be stable, if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any C^1 map A' from M to $M_p^s(\mathbb{R})$, we have $d_{C^2}^{\hookrightarrow}(\mathcal{S}_{A'}^\Lambda; \mathcal{S}_A^\Lambda) < \varepsilon$ when $\|A' - A\|_{C^1} < \delta$.

As already mentioned, a classical notion in the study of critical elliptic equations is that of compactness. A system like (0-1) is said to be *compact* if any bounded sequence $(\mathcal{U}_\alpha)_\alpha$ in $H^1(M)$ of nonnegative nontrivial solutions of (0-1) converges in $C^2(M)$ as $\alpha \to +\infty$ to a nonnegative nontrivial solution $\mathcal{U}$ of (0-1). This corresponds to the particular situation where $A_\alpha = A$ for all α in (i). Analytic stability as defined in (i) implies weak stability, geometric stability, and compactness. More precisely:

Proposition 1.2. *Assume* (H). *If the system* (0-1) *is analytically stable, it is weakly stable, geometrically stable, and compact. A compact system is analytically stable if and only if it is geometrically stable.*

Proof. It is obvious that analytic stability implies weak stability, geometric stability, and compactness. The only assertion, which deserves to be proved, is that a compact geometrically stable system like (0-1) is analytically stable. Let $(A_\alpha)_\alpha$ be a sequence of C^1 maps from M to $M_p^s(\mathbb{R})$ such that $A_\alpha \to A$ in $C^1(M)$ as $\alpha \to +\infty$, and let $(\mathcal{U}_\alpha)_\alpha$ be a bounded sequence in H^1 of nonnegative nontrivial solutions of (1-1). Since (0-1) is geometrically stable there exists $(\mathcal{V}_\alpha)_\alpha$, a bounded sequence in H^1 of nonnegative solutions of (0-1), such that, up to a subsequence, $\mathcal{U}_\alpha - \mathcal{V}_\alpha$ converges to zero in C^2 as $\alpha \to +\infty$. Since (0-1) is compact, up to a subsequence, $\mathcal{V}_\alpha \to \mathcal{V}$ in C^2 as $\alpha \to +\infty$, where $\mathcal{V}$ is a nonnegative solution of (0-1). In particular, up to a subsequence, $\mathcal{U}_\alpha \to \mathcal{V}$ in C^2 as $\alpha \to +\infty$. It remains to prove that $\mathcal{V}$ is nontrivial, and this is given by Lemma 1.3 below. Proposition 1.2 is proved. □

The following lemma, which we derive as a direct consequence of (H), was used in the proof of Proposition 1.2. By standard elliptic theory, moreover, when A satisfies (H), we have $\mathcal{U}_\alpha \not\to 0$ in H^1 as $\alpha \to +\infty$.

Lemma 1.3. *Let* (M, g) *be a smooth compact Riemannian manifold, let* $p \geq 1$ *be an integer, and let* $A : M \to M_p^s(\mathbb{R})$ *be a* C^1 *map satisfying* (H). *Let* $(A_\alpha)_\alpha$ *be a sequence of* C^1 *maps from* M *to* $M_p^s(\mathbb{R})$ *such that* $A_\alpha \to A$ *in* $C^1(M)$ *as* $\alpha \to +\infty$, *and let* $(\mathcal{U}_\alpha)_\alpha$ *be a bounded sequence in* H^1 *of nonnegative nontrivial solutions of* (1-1). *Then* $\mathcal{U}_\alpha \not\to 0$ *in* $L^\infty(M)$ *as* $\alpha \to +\infty$.

Proof. By contradiction we assume that there exists $(\mathcal{U}_\alpha)_\alpha$, a bounded sequence in H^1 of nonnegative nontrivial solutions of (1-1), such that $\max_M |\mathcal{U}_\alpha|_\Sigma \to 0$ as $\alpha \to +\infty$, where $|\mathcal{U}_\alpha|_\Sigma = \sum_i u_{i,\alpha}$ is the sum of the components of the $\mathcal{U}_\alpha$'s. Let $\varepsilon_\alpha = |\mathcal{U}_\alpha|_\Sigma$ and define $v_{i,\alpha}$ by $v_{i,\alpha} = \varepsilon_\alpha^{-1} u_{i,\alpha}$ for all i and α. Then

$$\Delta_g v_{i,\alpha} + \sum_{j=1}^{p} A_{ij}^{\alpha}(x) v_{j,\alpha} = \varepsilon_\alpha^{4/(n-2)} |\mathcal{V}_\alpha|^{2^\star-2} v_{i,\alpha} \tag{1-3}$$

for all i and α, where $\mathcal{V}_\alpha$ is the p-map whose components are the $v_{i,\alpha}$'s for $i = 1, \dots, p$. By construction the $v_{i,\alpha}$'s are bounded in $L^\infty(M)$. By standard elliptic theory it follows that, up to a subsequence, they converge in $C^2(M)$ to v_i's as $\alpha \to +\infty$. Let $\mathcal{V}$ be the p-map with components v_i for $i = 1, \dots, p$. By construction $\mathcal{V}$ is nonnegative and nontrivial, since there is one point, where $|\mathcal{V}|_\Sigma$ equals one. Letting $\alpha \to +\infty$ in (1-3) it follows that $\mathcal{V} \in \mathrm{Ker}(\mathbf{\Delta}_g + A)$, and we get a contradiction with (H). □

Proposition 1.2 leaves open the question of whether or not there exist geometrically stable noncompact systems like (0-1). However, we can have noncompact systems with geometrically stable specific slices as discussed below. The most well-known example of a noncompact critical system like (0-1) is given by the Yamabe equation on the sphere. The Yamabe equation on the n-sphere possesses a $(n+1)$-parameter noncompact family of solutions and it turns out that it is also geometrically unstable. This is a direct consequence of the constructions in [Druet and Hebey 2005a], where arbitrarily high energy solutions of approximated equations are constructed, together with the property that all nonnegative nontrivial solutions of the Yamabe equation on the sphere have the same energy. On the other hand, the first blow-up slice for this equation is geometrically H^1-stable in the sense of Definition 1.1 when we replace the C^2-pointed distance and the C^2-norm in (1-2) by a H^1-pointed distance and a H^1-norm, where the first blow-up slice is given by $\Lambda = K_n^2$, and K_n is as in (3-8). This geometric H^1-stability of the first blow-up slice follows from H^1-decompositions as in Proposition 4.2. As a direct consequence, noncompact equations may have stable slices.

In the subcritical regime, compactness goes back to [Gidas and Spruck 1981]. In the more involved critical regime, it goes back to Schoen's conjecture [Schoen 1989; 1991] that compactness holds true for the geometric Yamabe equation as soon as the background manifold is distinct from the sphere. His conjecture has been a source of motivations for several years. The conjecture was proved to be true for conformally flat manifolds by Schoen [1989; 1991]. The nonconformally flat case turned out to be more intricate. The case of low-dimensional manifolds was recently addressed in [Druet 2004; Marques 2005; Li and Zhu 1999; Li and Zhang 2004; 2005], and compactness up to dimension 24 was finally proved recently [Khuri et al. 2009]. On the other hand, Brendle [2008a] and Brendle and Marques [2009] exhibited counterexamples to the conjecture in dimensions $n \geq 25$. For any $n \geq 25$ they constructed examples of nonconformally flat n-manifolds with the striking property that their associated Yamabe equations possess sequences of solutions with minimal type energy and unbounded L^∞-norms. In particular, they proved the very surprising result that the compactness conjecture is false for nonconformally flat manifolds in any dimension $n \geq 25$. A very interesting survey on the subject is [Brendle 2008b]. We refer also to [Druet and Hebey 2005b].

An easy remark is that if u is a solution of a scalar Yamabe type equation with linear term h, that is, an equation of the form

$$\Delta_g u + h(x) u = u^{2^\star-1}, \tag{1-4}$$

then $\mathcal{U} = \left(\frac{1}{\sqrt{p}}u, \dots, \frac{1}{\sqrt{p}}u\right)$ is a solution of (0-1) when $A_{ij} = h a_{ij}$ for all i, j, and $\sum_{j=1}^p a_{ij} = 1$ for all i. In what follows we let $(a_{ij})_{i,j}$ be a symmetrical matrix of C^1 functions $a_{ij} : M \to \mathbb{R}$ such that $\sum_{j=1}^p a_{ij}(x) = 1$ for all $i = 1, \dots, p$ and all $x \in M$. A possible choice is $a_{ij} = \delta_{ij}$ for all i, j. Then we define $A(g)$ and $A'(g)$ to be the C^1 maps from M to $M_p^s(\mathbb{R})$ given by

$$A(g)_{ij} = \frac{n-2}{4(n-1)} S_g a_{ij} \quad \text{and} \quad A'(g)_{ij} = \frac{n-2}{4(n-1)} (\max_M S_g) a_{ij}$$

for all $i, j = 1, \dots, p$, where S_g is the scalar curvature of g. By combining results in [Brendle 2008a; Brendle and Marques 2009], where noncompactness of the Yamabe equation in the nonconformally flat case is investigated, and those in [Druet and Hebey 2005a; Hebey and Vaugon 2001], where unstability of Yamabe type equations in the conformally flat case is investigated, we obtain the following theorem, in view of the remark above.

Theorem 1.4. *The system* (0-1) *associated with $A(g)$ is analytically unstable when posed on spherical spaces forms in any dimension $n \geq 6$, and even noncompact when posed on the sphere in any dimension $n \geq 3$. For any conformally flat manifold (M, g) of dimension $n \geq 4$ there exists a conformal metric $\tilde{g}$ to g of nonconstant scalar curvature having one and only one maximum point such that the system* (0-1) *associated with $A'(\tilde{g})$ is analytically unstable. In any dimension $n \geq 25$ there are examples of nonconformally flat manifolds such that the system* (0-1) *associated with $A(g)$ is noncompact, and thus also analytically unstable.*

The examples in Theorem 1.4 do not satisfy (H′). This can be checked by noting that $(1, \dots, 1) \in \operatorname{Ker} A_n(x)$ for all x, where A_n is as in (0-2). Theorem 0.1 and Theorem 1.4 complement each other. As a remark, the Yamabe equation on quotients of the sphere is obviously compact since it possesses a unique solution. In particular, there are compact equations which are neither analytically nor geometrically stable. Compactness does not imply stability. We concentrate in the rest of this section on the subcritical regime for systems like (0-1) and prove that analytic stability holds true in the subcritical regime without assuming (H′). Let $q \in (2, 2^\star)$ and let us consider the subcritical system

$$\Delta_g u_i + \sum_{j=1}^{p} A_{ij}(x) u_j = |\mathcal{U}|^{q-2} u_i \tag{1-5}$$

in M for all i, where $A = (A_{ij})_{i,j}$ is a C^1 map from M to $M_p^s(\mathbb{R})$. We define the notions of analytic stability, weak stability, and geometric stability for (1-5) as in Definition 1.1.

Proposition 1.5. *Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 3$, $p \geq 1$ be an integer, and $A : M \to M_p^s(\mathbb{R})$ be a C^1 map satisfying* (H). *For any $q \in (2, 2^\star)$ the subcritical system* (1-5) *is analytically stable.*

Proof. Let $(A_\alpha)_\alpha$ be a sequence of C^1 maps from M to $M_p^s(\mathbb{R})$ such that $A_\alpha \to A$ in $C^1(M)$ as $\alpha \to +\infty$, and let $(\mathcal{U}_\alpha)_\alpha$ be an arbitrary bounded sequence in H^1 of nonnegative nontrivial solutions of

$$\Delta_g u_{i,\alpha} + \sum_{j=1}^{p} A_{ij}^\alpha(x) u_{j,\alpha} = |\mathcal{U}_\alpha|^{q-2} u_{i,\alpha} \tag{1-6}$$

for all i and all α. We aim in proving that a subsequence of $(\mathcal{U}_\alpha)_\alpha$ converges in C^2 to a nonnegative nontrivial solution of (1-6). The nontriviality of any strong limit follows from (H) mimicking the proof of Lemma 1.3. Then, as is easily checked, it suffices to prove that the $\mathcal{U}_\alpha$'s are L^∞-bounded in M. By contradiction we assume that there exists a sequence $(x_\alpha)_\alpha$ of points, where the $|\mathcal{U}_\alpha|$'s are maximum such that, up to a subsequence, $|\mathcal{U}_\alpha(x_\alpha)| \to +\infty$ as $\alpha \to +\infty$. Let $\mu_\alpha = |\mathcal{U}_\alpha(x_\alpha)|^{-(q-2)/2}$. Then $\mu_\alpha \to 0$ as $\alpha \to +\infty$. Let $\tilde{\mathcal{U}}_\alpha$ be given for $x \in \mathbb{R}^n$ by

$$\tilde{\mathcal{U}}_\alpha(x) = \mu_\alpha^{2/(q-2)} \mathcal{U}_\alpha\big(\exp_{x_\alpha}(\mu_\alpha x)\big)$$

and let g_α be the metric given by $g_\alpha(x) = (\exp_{x_\alpha}^\star g)(\mu_\alpha x)$. We have $g_\alpha \to \xi$ in $C^2_{\mathrm{loc}}(\mathbb{R}^n)$ as $\alpha \to +\infty$, where ξ is the Euclidean metric. Noting that

$$\Delta_{g_\alpha} \tilde{u}_{i,\alpha} + \mu_\alpha^2 \sum_{j=1}^p A_{ij}^\alpha\big(\exp_{x_\alpha}(\mu_\alpha x)\big)\tilde{u}_{j,\alpha} = |\tilde{\mathcal{U}}_\alpha|^{q-2} \tilde{u}_{i,\alpha}$$

for all i and α, since $|\tilde{\mathcal{U}}_\alpha| \le 1$ for all α by construction, it follows from standard elliptic theory that there exists $\tilde{\mathcal{U}} \in C^2(\mathbb{R}^n)$ such that, up to a subsequence, $\tilde{\mathcal{U}}_\alpha \to \tilde{\mathcal{U}}$ in $C^2_{\mathrm{loc}}(\mathbb{R}^n)$. We have $|\tilde{\mathcal{U}}_\alpha(0)| = 1$ for all α. Hence, $|\tilde{\mathcal{U}}(0)| = 1$. Moreover, for any $R > 0$, and for α sufficiently large,

$$\begin{aligned}\int_{B_0(R)} |\tilde{\mathcal{U}}|^q \, dx &\le C \int_{B_0(R)} |\tilde{\mathcal{U}}_\alpha|^q \, dv_{g_\alpha} \\ &\le C \int_{B_0(1/\mu_\alpha)} |\tilde{\mathcal{U}}_\alpha|^q \, dv_{g_\alpha} = C \mu_\alpha^{2q/(q-2)-n} \int_{B_{x_\alpha}(1)} |\mathcal{U}_\alpha|^q \, dv_g \le C \mu_\alpha^{2q/(q-2)-n},\end{aligned}$$

since the $\mathcal{U}_\alpha$'s have bounded energy. Noting that $2q/(q-2) > n$ as soon as $q < 2^\star$ and letting $\alpha \to +\infty$ in the inequality above, we get $\int_{B_0(R)} |\tilde{\mathcal{U}}|^q \, dx = 0$. This is in contradiction with $|\tilde{\mathcal{U}}(0)| = 1$. The proposition is proved. □

Analytic stability for critical equations like (1-4) has been investigated in [Druet 2003]. The case $p = 1$ in Theorem 0.1, in its $n \neq 6$ part and when considering $C^{2,\theta}$-perturbations of h, was proved in the same paper. The proof we propose here extends to the case of systems, allows us to consider $C^{0,\eta}$-perturbations of h, see the remark at the end of Section 10, and is more direct. At the time of [Druet 2003], analytic stability was still referred to as compactness. The confusion in the terminology has been the source of several misunderstandings.

2. The six-dimensional case

We discuss and prove the six-dimensional last assertion in Theorem 0.1 concerning the existence of systems like (0-1) in dimension $n = 6$, which satisfy (H) and (H′), but which, contrary to what happens when $n \neq 6$, are not analytically stable. We restrict ourselves to a very explicit construction in the case of the unit sphere (S^6, g_0). A more general discussion could have been developed. We let $(a_{ij})_{i,j}$ be a symmetrical matrix of C^1 functions $a_{ij} : S^6 \to \mathbb{R}$ such that $\sum_{j=1}^p a_{ij}(x) = 1$ for all $i = 1, \dots, p$ and all $x \in S^6$. If $h : S^6 \to \mathbb{R}$ is of class C^1, we define $A(h)$ to be the C^1 map from S^6 to $M_p^s(\mathbb{R})$ with components $A(h)_{ij}$ given by $A(h)_{ij} = h a_{ij}$ for all $i, j = 1, \dots, p$. When $n = 6$, we have $2^\star = 3$. For the unit sphere (S^6, g_0), we also have

$$\frac{n-2}{4(n-1)} S_{g_0} = 6.$$

Proposition 2.1. *Let (S^6, g_0) be the unit six-dimensional sphere in $\mathbb{R}^7$. There exists $h : S^6 \to \mathbb{R}$, $h > 6$ everywhere and of class C^1, such that the system* (0-1) *associated with $A = A(h)$ is analytically unstable.*

Proof. We fix $x_0 \in S^6$ and let r be the distance to x_0. We let also θ be given by $\theta = \cos r$. First we claim that there exist smooth positive functions h and u in S^6, which we write into the form $h(x) = \hat{h}(\theta)$ and $u(x) = \hat{u}(\theta)$, such that

$$\Delta_{g_0} u + hu = u^2 \quad \text{and} \quad h > 6 \tag{2-1}$$

in S^6, and such that

$$\hat{h}(1) = 3\hat{u}(1), \quad \hat{u}(1) = 6 \quad \text{and} \quad \hat{h}'(1) = 2\hat{u}'(1). \tag{2-2}$$

To prove the claim we let $\hat{u}$ be given by

$$\hat{u}(\theta) = 6\bigl(1 - 2(\theta - 1) + 3(\theta - 1)^2\bigr). \tag{2-3}$$

Clearly, $\hat{u}(1) = 6$ and $\hat{u}'(1) = -12$. Since $\Delta_{g_0}\theta = 6\theta$ and $|\nabla\theta|^2 = 1 - \theta^2$, we get

$$\tfrac{1}{6}\Delta_{g_0} u = 6(7\theta^2 - 8\theta - 1).$$

In particular, the first equation in (2-1) is satisfied if we let $\hat{h}$ be given by

$$\tfrac{1}{6}\hat{h}(\theta) = 3\theta^2 - 8\theta + 6 - \frac{7\theta^2 - 8\theta - 1}{3\theta^2 - 8\theta + 6}. \tag{2-4}$$

As is easily checked from (2-4), $\hat{h}(1) = 3\hat{u}(1)$ and $\hat{h}'(1) = 2\hat{u}'(1)$. In particular, (2-2) holds true. Noting that $\hat{h} > 6$ for all $\theta \in [-1, +1]$, we get two explicit smooth positive functions h and u in S^6, given by (2-3) and (2-4), such that (2-1) and (2-2) hold true. This proves the above claim. Now, for $\beta > 1$, we define B_β by $B_\beta(x) = \hat{B}_\beta(\theta)$, where

$$\hat{B}_\beta(\theta) = 6(\beta^2 - 1)(\beta - \theta)^{-2}.$$

We have $\Delta_{g_0} B_\beta + 6B_\beta = B_\beta^2$ in S^6. Let

$$u_\beta = u + B_\beta \tag{2-5}$$

and $\hat{u}_\beta = \hat{u} + \hat{B}_\beta$, where u and $\hat{u}$ are as in (2-1) and (2-2). As is easily checked from (2-1) and the equation satisfied by B_β, we have

$$\Delta_{g_0} u_\beta + h_\beta u_\beta = u_\beta^2 \tag{2-6}$$

in S^6 for all $\beta > 1$, where $h_\beta = \hat{h}_\beta(\theta)$ is given by

$$\hat{h}_\beta = \hat{h} - \frac{(12\hat{u} + 6 - \hat{h})\hat{B}_\beta}{\hat{u} + \hat{B}_\beta}. \tag{2-7}$$

Noting, thanks to (2-2), that $h_\beta \to h$ in $C^0_{\text{loc}}(S^6)$ as $\beta \to 1$, while $\hat{h}'_\beta \to \hat{h}'$ in $L^\infty\bigl([-1, +1]\bigr)$ as $\beta \to 1$, we conclude that $h_\beta \to h$ in $C^1(S^6)$. Now we let $(\beta_\alpha)_\alpha$ be a sequence of positive real numbers such that $\beta_\alpha > 1$ for all α and $\beta_\alpha \to 1$ as $\alpha \to +\infty$. We let $\mathcal{U}_\alpha = p^{-1/2}(u_{\beta_\alpha}, \dots, u_{\beta_\alpha})$, $A_\alpha = A(h_{\beta_\alpha})$, and $A = A(h)$ where u_β is given by (2-5), $h_\beta = \hat{h}_\beta(\theta)$ is given by (2-7), and $h = \hat{h}(\theta)$ is given by (2-4). The $\mathcal{U}_\alpha$'s solve (1-1), they have bounded energy, and $A_\alpha \to A$ in C^1. Noting that $\|\mathcal{U}_\alpha\|_\infty \to +\infty$ as $\alpha \to +\infty$, this proves the proposition. □

It is easily checked that $A = A(h)$ satisfies (H). If $\mathcal{U} \in L^2(M, \text{Vect}_+(\mathbb{R}^p))$ is in the kernel of the vector Schrödinger operator associated with $A(h)$, we conclude by summing over the components that $|\mathcal{U}|_\Sigma = \sum_{i=1}^p u_i$ belongs to $\text{Ker}(\Delta_{g_0} + h)$. This is impossible unless $\mathcal{U} \equiv 0$ since $h > 0$. It is also easily seen that, at least for small perturbations a_{ij} of δ_{ij}, the map $A = A(h)$ satisfies (H′).

3. The limit system

Of importance in blow-up theory, when discussing critical equations, is the classification of the solutions of the critical limit Euclidean system we get by blowing up the original equations. In our case, we need to classify the $\dot{H}^1$-nonnegative solutions of the limit system

$$\Delta u_i = |\mathcal{U}|^{2^\star-2} u_i, \tag{3-1}$$

where $|\mathcal{U}|^2 = \sum_{i=1}^p u_i^2$, and $\Delta = -\sum_{i=1}^n \partial^2/\partial x_i^2$ is the Euclidean Laplace–Beltrami operator . Depending on the context, we let $\dot{H}^1(\mathbb{R}^n)$ be the homogeneous Sobolev space defined as the completion of functions with compact supports, or of p-maps with compact supports, with respect to the L^2-norm of their gradient. The classification result we prove here is stated as follows.

Proposition 3.1. *Let $p \geq 1$ and $\mathcal{U} \in \dot{H}^1(\mathbb{R}^n)$ be a nonnegative solution of* (3-1)*. Then there exist $a \in \mathbb{R}^n$, $\lambda > 0$, and $\Lambda \in S_+^{p-1}$, such that*

$$\mathcal{U}(x) = \left(\frac{\lambda}{\lambda^2 + \frac{|x-a|^2}{n(n-2)}} \right)^{(n-2)/2} \Lambda \tag{3-2}$$

for all $x \in \mathbb{R}^n$, where S_+^{p-1} consists of those elements $(\Lambda_1, \ldots, \Lambda_p)$ in the unit sphere S^{p-1} (in $\mathbb{R}^p$) that satisfy $\Lambda_i \geq 0$ for all i.

We prove Proposition 3.1 in several steps. Let $\mathcal{U} \in \dot{H}^1(\mathbb{R}^n)$ be a nonnegative solution of (3-1). Regularity theory and the maximum principle apply to (3-1). In particular, $\mathcal{U}$ is necessarily smooth with the property that for any i, either $u_i \equiv 0$ or $u_i > 0$ in $\mathbb{R}^n$. We may therefore assume that there exists $p' \leq p$ such that $u_i > 0$ in $\mathbb{R}^n$ for all $i = 1, \ldots, p'$. A first step in the proof of Proposition 3.1 is as follows.

Step 1. *Let $\mathcal{U} \in \dot{H}^1(\mathbb{R}^n)$ be a nonnegative solution of* (3-1) *such that $u_i > 0$ in $\mathbb{R}^n$ for all $i = 1, \ldots, p'$, $p' \leq p$. Then, for any $R > 0$,*

$$\min_{\partial B_0(R)} \frac{u_i}{u_j} \leq \frac{u_i}{u_j} \leq \max_{\partial B_0(R)} \frac{u_i}{u_j} \tag{3-3}$$

in $B_0(R)$ for all $i, j \in \{1, \ldots, p'\}$.

Proof of Step 1. By (3-1),

$$\Delta\left(\frac{u_i}{u_j}\right) = 2\left(\nabla\left(\frac{u_i}{u_j}\right), \nabla u_j\right) u_j^{-1}.$$

Applying the maximum principle we get (3-3). □

The main objective now is to prove that

$$\min_{\partial B_0(R)} \frac{u_i}{u_j} \to \lambda_{i,j} \quad \text{and} \quad \max_{\partial B_0(R)} \frac{u_i}{u_j} \to \lambda_{i,j} \tag{3-4}$$

as $R \to +\infty$ for some $\lambda_{i,j} > 0$ so that, together with Step 1, we obtain $u_i = \lambda_{i,j} u_j$ in $\mathbb{R}^n$ for all $i, j = 1, \ldots p'$. To prove (3-4) we first observe that

$$|x|^{(n-2)/2} u_i(x) \to 0 \tag{3-5}$$

as $|x| \to +\infty$ for all $i \in \{1, \dots, p'\}$. Indeed, let $r > 0$, and $V_r = r^{(n-2)/2}U(rx)$. We have $\Delta V_r = |V_r|^{2^\star-2}V_r$ and

$$\int_{B_0(2)\setminus B_0(1/2)} |V_r|^{2^\star}\, dx \to 0 \quad \text{as } r \to +\infty,$$

since $u_i \in L^{2^\star}(\mathbb{R})$ for all i. Then $v_r^i \to 0$ in $C^0_{\mathrm{loc}}\big(B_0\big(\frac{3}{2}\big) \setminus B_0\big(\frac{3}{4}\big)\big)$ as $r \to +\infty$ for all i, where the v_r^i's are the components of V_r. This proves (3-5). Now, in order to prove (3-4), we prove that the following step holds true.

Step 2. *Let $\mathcal{U} \in \dot{H}^1(\mathbb{R}^n)$ be a nonnegative solution of* (3-1) *such that $u_i > 0$ in $\mathbb{R}^n$ for all $i = 1, \dots, p'$, $p' \le p$. For any $0 < \varepsilon < \frac{1}{2}$, there exists $C_\varepsilon > 0$ such that*

$$u_i(x) \le C_\varepsilon |x|^{(2-n)(1-\varepsilon)}$$

for all $x \in \mathbb{R}^n$ and all $i \in \{1, \dots, p'\}$.

Proof of Step 2. Let $0 < \varepsilon < \frac{1}{2}$ and $R_\varepsilon > 0$ be such that

$$\sup_{x \in \mathbb{R}^n \setminus B_0(R_\varepsilon)} |x|^2 |U(x)|^{2^\star-2} \le \frac{(n-2)^2}{2}\varepsilon(1-\varepsilon).$$

It is always possible to find such a R_ε thanks to (3-5). For $R \ge R_\varepsilon$, we let

$$\eta(R) = \max_{i=1,\dots,p'} \max_{\partial B_0(R)} u_i$$

and

$$G_\varepsilon(x) = \eta(R_\varepsilon)\Big(\frac{|x|}{R_\varepsilon}\Big)^{(2-n)(1-\varepsilon)} + \eta(R)\Big(\frac{|x|}{R}\Big)^{(2-n)\varepsilon}.$$

It is clear that $u_i \le G_\varepsilon$ on $\partial B_0(R_\varepsilon) \bigcup \partial B_0(R)$. Let us assume that $\frac{u_i}{G_\varepsilon}$ possesses a local maximum at $x \in B_0(R) \setminus \overline{B_0(R_\varepsilon)}$. Then

$$\frac{\Delta u_i(x)}{u_i(x)} \ge \frac{\Delta G_\varepsilon(x)}{G_\varepsilon(x))}.$$

Since

$$\frac{\Delta G_\varepsilon(x)}{G_\varepsilon(x)} = \varepsilon(1-\varepsilon)(n-2)^2|x|^{-2},$$

we get

$$|x|^2|U(x)|^{2^\star-2} \ge \varepsilon(1-\varepsilon)(n-2)^2.$$

But this is absurd by the choice of R_ε we made. Thus we can write, for any $R > R_\varepsilon$ and any $i \in \{1, \dots, p'\}$,

$$u_i(x) \le \eta(R_\varepsilon)\Big(\frac{|x|}{R_\varepsilon}\Big)^{(2-n)(1-\varepsilon)} + \eta(R)\Big(\frac{|x|}{R}\Big)^{(2-n)\varepsilon} \tag{3-6}$$

in $B_0(R) \setminus B_0(R_\varepsilon)$. Fix $x \in \mathbb{R}^n \setminus B_0(R_\varepsilon)$. Passing to the limit as $R \to +\infty$ in (3-6), since, by (3-5), $R^{(n-2)/2}\eta(R) \to 0$ as $R \to +\infty$, we get

$$u_i(x) \le \eta(R_\varepsilon)\Big(\frac{|x|}{R_\varepsilon}\Big)^{(2-n)(1-\varepsilon)}.$$

This ends the proof of Step 2. $\square$

Step 3. *Let $\mathcal{U} \in \dot{H}^1(\mathbb{R}^n)$ be a nonnegative solution of* (3-1) *such that $u_i > 0$ in $\mathbb{R}^n$ for all $i = 1, \dots, p'$, $p' \le p$. Then $u_i \in L^{2^\star-1}(\mathbb{R}^n)$ and*

$$\lim_{|x|\to+\infty} |x|^{n-2} u_i(x) = \frac{1}{(n-2)\omega_{n-1}} \int_{\mathbb{R}^n} |U|^{2^\star-2} u_i \, dx$$

for all $i \in \{1, \dots, p'\}$.

Proof of Step 3. We apply Green's representation formula in $B_x(R)$ and get

$$u_i(x) = \frac{1}{(n-2)\omega_{n-1}} \int_{B_x(R)} \big(|x-y|^{2-n} - R^{2-n}\big) |U(y)|^{2^\star-2} u_i(y) \, dy + \frac{1}{\omega_{n-1} R^{n-1}} \int_{\partial B_x(R)} u_i d\sigma.$$

Thanks to the estimate of Step 2 with $0 < \varepsilon < 2/(n+2)$, we have $u_i \in L^{2^\star-1}(\mathbb{R}^n)$ for all i. Passing to the limit as $R \to +\infty$ we obtain

$$u_i(x) = \frac{1}{(n-2)\omega_{n-1}} \int_{\mathbb{R}^n} |x-y|^{2-n} |U(y)|^{2^\star-2} u_i(y) \, dy.$$

Thus

$$\begin{aligned} &|x|^{n-2} u_i(x) \\ &\quad = \frac{1}{(n-2)\omega_{n-1}} \int_{\mathbb{R}^n} \frac{|x|^{n-2}}{|x-y|^{n-2}} |U(y)|^{2^\star-2} u_i(y) \, dy \\ &\quad = \frac{1}{(n-2)\omega_{n-1}} \bigg(\int_{B_0(R)} |U(y)|^{2^\star-2} u_i(y) \, dy + o_R(1) + \int_{\mathbb{R}^n \setminus B_0(R)} \frac{|x|^{n-2}}{|x-y|^{n-2}} |U(y)|^{2^\star-2} u_i(y) \, dy \bigg), \end{aligned}$$

where $o_R(1) \to 0$ as $|x| \to +\infty$. Now, using Step 2, we write

$$\begin{aligned} &\int_{\mathbb{R}^n \setminus B_0(R)} \frac{|x|^{n-2}}{|x-y|^{n-2}} |U(y)|^{2^\star-2} u_i(y) \, dy \\ &\quad \le N^{(2^\star-2)/2} C_\varepsilon^{2^\star-1} \int_{B_x(|x|/2)} \frac{|x|^{n-2}}{|x-y|^{n-2}} dy \Big(\frac{|x|}{2}\Big)^{-(n+2)(1-\varepsilon)} + 2^{n-2} \int_{\mathbb{R}^n \setminus B_0(R)} |U(y)|^{2^\star-2} u_i(y) \, dy \\ &\quad \le N^{(2^\star-2)/2} C_\varepsilon^{2^\star-1} 2^{(n+2)(1-\varepsilon)-2} \omega_{n-1} |x|^{n-(n+2)(1-\varepsilon)} + 2^{n-2} \int_{\mathbb{R}^n \setminus B_0(R)} |U(y)|^{2^\star-2} u_i(y) \, dy. \end{aligned}$$

Choosing $0 < \varepsilon < \dfrac{2}{n+2}$, we thus obtain that

$$\lim_{R\to+\infty} \limsup_{|x|\to+\infty} \int_{\mathbb{R}^n \setminus B_0(R)} \frac{|x|^{n-2}}{|x-y|^{n-2}} |U(y)|^{2^\star-2} u_i(y) \, dy = 0.$$

This ends the proof of Step 3. □

Using Steps 1 and 3 we are now in a position to prove (3-4), and then Proposition 3.1.

Proof of Proposition 3.1. Let $\mathcal{U} \in \dot{H}^1(\mathbb{R}^n)$ be a nonnegative solution of (3-1) such that $u_i > 0$ in $\mathbb{R}^n$ for all $i = 1, \dots, p'$, $p' \le p$. Since the u_i's are all positive for $i = 1, \dots, p'$, we get from Step 3 that

$$\min_{\partial B_0(R)} \frac{u_i}{u_j}, \quad \max_{\partial B_0(R)} \frac{u_i}{u_j} \to \lambda_{i,j}$$

as $R \to +\infty$, where $\lambda_{i,j} > 0$ is given by

$$\lambda_{i,j} = \frac{\int_{\mathbb{R}^n} |U|^{2^\star-2} u_i \, dx}{\int_{\mathbb{R}^n} |U|^{2^\star-2} u_j \, dx}.$$

In particular, (3-4) holds true. Thanks to Step 1, we thus get

$$u_i = \lambda_i u_1$$

for all $i \in \{1, \dots, p'\}$ where $\lambda_i = \lambda_{i,1}$. By (3-1) we then get

$$\Delta u_1 = |\Lambda'|^{2^\star-2} u_1^{2^\star-1}$$

in $\mathbb{R}^n$ where $\Lambda' = (\lambda_i)_{i=1,\dots,p'}$. By [Caffarelli et al. 1989] we can write

$$u_1(x) = |\Lambda'|^{-1} \left(\frac{\mu}{\mu^2 + \frac{|x-x_0|^2}{n(n-2)}} \right)^{(n-2)/2} \tag{3-7}$$

for some $x_0 \in \mathbb{R}^n$ and some $\mu > 0$. In particular, since $u_i = \lambda_i u_1$, we get with (3-7) that (3-2) holds true with $\Lambda = (\Lambda_i)_i$, where $\Lambda_i = |\Lambda'|^{-1} \lambda_i$ for all $i = 1, \dots, p'$, and $\lambda_i = 0$ for all $i > p'$. Clearly, $|\Lambda| = 1$. This ends the proof of Proposition 3.1. □

Let K_n be the sharp constant for the Sobolev inequality $\|u\|_{2^\star} \le K \|\nabla u\|_2$ corresponding to the embedding $\dot{H}^1(\mathbb{R}^n) \subset L^{2^\star}(\mathbb{R}^n)$. Then, as is well known,

$$K_n = \sqrt{\frac{4}{n(n-2)\omega_n^{2/n}}}, \tag{3-8}$$

where ω_n is the volume of the unit sphere. The multipliers in (3-2), which we get by taking the Euclidean norm $|\mathcal{U}|$ of $\mathcal{U}$ in (3-2), turn out to be extremal functions for the sharp Euclidean Sobolev inequality $\|u\|_{2^\star} \le K_n \|\nabla u\|_2$. As a direct consequence of Proposition 3.1 we then get

$$\int_{\mathbb{R}^n} |\mathcal{U}|^{2^\star} dx = K_n^{-n} \tag{3-9}$$

for all nonnegative solutions $\mathcal{U} \in \dot{H}^1(\mathbb{R}^n)$ of (3-1), where K_n is as in (3-8). Proposition 3.1, combined with the moving sphere approach, gives the full classification of nonnegative solutions of (3-1), namely without the requirement that $\mathcal{U} \in \dot{H}^1$. This is carried out in [Druet et al. 2009].

4. Weak pointwise estimates

Let (M, g) be a smooth compact Riemannian manifold of dimension $n \ge 3$, $p \ge 1$ be an integer, and $(x_\alpha)_\alpha$ be a converging sequence of points in M. Let also $(\lambda_\alpha)_\alpha$ be a sequence of positive real numbers. For $\mathcal{U} : M \to \mathbb{R}^p$ and $\mathcal{V} : \mathbb{R}^n \to \mathbb{R}^p$, we define the direct $\hat{R}_{x_\alpha}^{\lambda_\alpha}$-rescalings and the inverse $\check{R}_{x_\alpha}^{\lambda_\alpha}$-rescalings by

$$\big(\hat{R}_{x_\alpha}^{\lambda_\alpha} \mathcal{U}\big)(x) = \lambda_\alpha^{(n-2)/2} \mathcal{U}(\exp_{x_\alpha}(\lambda_\alpha x)) \quad \text{and} \quad \big(\check{R}_{x_\alpha}^{\lambda_\alpha} \mathcal{V}\big)(x) = \lambda_\alpha^{(n-2)/2} \mathcal{V}(\lambda_\alpha \exp_{x_\alpha}^{-1}(x)), \tag{4-1}$$

where x in the first equation is a variable in $\mathbb{R}^n$, x in the second equation is a variable in M, localized around the limit of the x_α's, and $\exp_{x_\alpha}$ is the exponential map at x_α.

Definition 4.1. Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 3$ and $p \geq 1$ be an integer. A p-vector bubble is a sequence $(\mathcal{B}_\alpha)_\alpha$ of p-maps from M to $\mathbb{R}^p$ given by

$$\mathcal{B}_\alpha(x) = \left(\frac{\mu_\alpha}{\mu_\alpha^2 + \dfrac{d_g(x_\alpha, x)^2}{n(n-2)}}\right)^{(n-2)/2} \Lambda \tag{4-2}$$

for all $x \in M$ and all α, where $(x_\alpha)_\alpha$ is a converging sequence of points in M, $(\mu_\alpha)_\alpha$ is a sequence of positive real numbers converging to 0, and $\Lambda \in S_+^{p-1}$. The x_α's are the centers of the bubble, the μ_α's are the weights of the bubble, and Λ is the S^{p-1}-projection of the bubble.

The right-hand side in (4-2) can be seen as the Riemannian extension of the right-hand side in (3-2). At last we let $u_0 : \mathbb{R}^n \to \mathbb{R}$ be the function given by

$$u_0(x) = \left(1 + \frac{|x|^2}{n(n-2)}\right)^{-(n-2)/2} \tag{4-3}$$

for all $x \in \mathbb{R}^n$. Another possible definition of u_0 is that it is the unique nonnegative solution of $\Delta u = u^{2^\star - 1}$ which achieves its maximum at 0 and which is such that $u_0(0) = 1$. The result we prove in this section provides a complete description of the blow-up in Sobolev spaces and very useful pointwise estimates.

Proposition 4.2. *Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 3$, $p \geq 1$ be an integer, and $(A_\alpha)_\alpha$ be a sequence of C^1 maps from M to $M_p^s(\mathbb{R})$ such that $A_\alpha \to A$ in $C^1(M)$ as $\alpha \to +\infty$ for some C^1 map A from M to $M_p^s(\mathbb{R})$. Let also $(\mathcal{U}_\alpha)_\alpha$ be an arbitrary bounded sequence in $H^1(M)$ of nonnegative solutions of* (1-1) *such that $\|\mathcal{U}_\alpha\|_\infty \to +\infty$ as $\alpha \to +\infty$. Then there exist $N \in \mathbb{N}^\star$, a nonnegative solution U_∞ of* (0-1), *and vector bubbles $(\mathcal{B}_\alpha^i)_\alpha$ as in* (4-2) *for $i = 1, \dots, N$, such that, up to a subsequence,*

$$\begin{aligned} &\mathcal{U}_\alpha = \mathcal{U}_\infty + \sum_{i=1}^N \mathcal{B}_\alpha^i + \mathcal{R}_\alpha && \text{for all } \alpha, \\ &\int_M |\mathcal{U}_\alpha|^{2^\star} dv_g = \int_M |\mathcal{U}_\infty|^{2^\star} dv_g + N K_n^{-n} + o(1) && \text{for all } \alpha, \text{ and} \\ &\mathcal{D}_\alpha^{(n-2)/2} \left| \mathcal{U}_\alpha - \mathcal{U}_\infty - \sum_{i=1}^N \mathcal{B}_\alpha^i \right| \to 0 \quad \text{in } L^\infty(M) && \text{as } \alpha \to +\infty, \end{aligned} \tag{4-4}$$

where $\mathcal{R}_\alpha \to 0$ in $H^1(M)$ as $\alpha \to +\infty$, $o(1) \to 0$ as $\alpha \to +\infty$, K_n is as in (3-8), *$\mathcal{D}_\alpha : M \to \mathbb{R}^+$ is given by*

$$\mathcal{D}_\alpha(x) = \min_{i=1,\dots,N} \big(d_g(x_{i,\alpha}, x) + \mu_{i,\alpha}\big),$$

and the $x_{i,\alpha}$'s and $\mu_{i,\alpha}$'s are the centers and weights of the vector bubbles $(\mathcal{B}_\alpha^i)_\alpha$. Moreover, as $\alpha \to +\infty$,

$$\begin{aligned} &\frac{d_g(x_{i,\alpha}, x_{j,\alpha})^2}{\mu_{i,\alpha}\mu_{j,\alpha}} + \frac{\mu_{i,\alpha}}{\mu_{j,\alpha}} + \frac{\mu_{j,\alpha}}{\mu_{i,\alpha}} \to +\infty && \text{for all } i \neq j \quad \text{and} \\ &\hat{R}_{x_{i,\alpha}}^{\mu_{i,\alpha}} \mathcal{U}_\alpha \to u_0 \Lambda_i \text{ in } C^2_{\text{loc}}(\mathbb{R}^n \backslash \mathcal{S}_i) && \text{for all } i, \end{aligned} \tag{4-5}$$

where the $\hat{R}^{\mu_\alpha}_{x_\alpha}$-rescaling procedure is defined in (4-1), u_0 *is as in* (4-3), *the* Λ_i*'s are the* S^{p-1}*-projections of the* $(\mathcal{B}^i_\alpha)_\alpha$*'s,*

$$\mathcal{S}_i = \Big\{ \lim_{\alpha\to+\infty} \mu_{i,\alpha}^{-1} \exp_{x_{i,\alpha}}^{-1}(x_{j,\alpha}),\ j \in I_i \Big\},$$

the limits in the definition of $\mathcal{S}_i$ *are as* $\alpha \to +\infty$, *and* I_i *consists of the* j*'s such that* $d_g(x_{i,\alpha}, x_{j,\alpha}) = O(\mu_{i,\alpha})$ *and* $\mu_{j,\alpha} = o(\mu_{i,\alpha})$ *for all* α.

Proof. Let I_α be the free functionals associated with (0-1). They are defined for $\mathcal{U} \in H^1(M)$ by

$$I_\alpha(\mathcal{U}) = \frac{1}{2}\int_M \big(|\nabla\mathcal{U}|^2 + A_\alpha(\mathcal{U},\mathcal{U})\big)\,dv_g - \frac{1}{2^\star}\int_M |\mathcal{U}|^{2^\star} dv_g.$$

The $\mathcal{U}_\alpha$'s in Proposition 4.2 solve (0-1) and are bounded in H^1. In particular, $(\mathcal{U}_\alpha)_\alpha$ is a Palais–Smale sequence for the I_α's in the sense that the sequence $(I_\alpha(\mathcal{U}_\alpha))_\alpha$ is bounded and $DI_\alpha(\mathcal{U}_\alpha) \to 0$ in $H^1(M)'$ as $\alpha \to +\infty$. Let η be a smooth cutoff function in $\mathbb{R}^n$ with small support around 0. Mimicking the proof in [Struwe 1984] (see also [Druet et al. 2004] for its Riemannian analogue), we get that there exist $N \in \mathbb{N}^\star$, a nonnegative solution U_∞ of (0-1), converging sequences $(x_{i,\alpha})_\alpha$ in M, sequences $(\mu_{i,\alpha})_\alpha$ of positive real numbers converging to 0, and nonnegative solutions $\mathcal{U}_i \in \dot{H}^1(\mathbb{R}^n)$ of (3-1) in $\mathbb{R}^n$, $i = 1, \dots, N$, such that, up to a subsequence, the first equation in (4-5) holds true, such that

$$\mathcal{U}_\alpha = \mathcal{U}_\infty + \sum_{i=1}^N \eta^i_\alpha \check{R}^{1/\mu_{i,\alpha}}_{x_{i,\alpha}} \mathcal{U}_i + \mathcal{R}_\alpha \tag{4-6}$$

for all α, and such that

$$\int_M |\mathcal{U}_\alpha|^{2^\star} dv_g = \int_M |\mathcal{U}_\infty|^{2^\star} dv_g + \sum_{i=1}^N \int_{\mathbb{R}^n} |\mathcal{U}_i|^{2^\star} dx + o(1) \tag{4-7}$$

for all α, where $\eta^i_\alpha(x) = \eta(\exp^{-1}_{x_{i,\alpha}}(x))$, the $\check{R}^{1/\mu_{i,\alpha}}_{x_{i,\alpha}}$-rescalings are defined in (4-1), $\mathcal{R}_\alpha \to 0$ in $H^1(M)$ as $\alpha \to +\infty$, and $o(1) \to 0$ as $\alpha \to +\infty$. By Proposition 3.1,

$$\mathcal{U}_i(x) = \left(\frac{\lambda_i}{\lambda_i^2 + \dfrac{|x-a_i|}{n(n-2)}} \right)^{(n-2)/2} \Lambda_i \tag{4-8}$$

for some $a_i \in \mathbb{R}^n$, $\lambda_i > 0$, $\Lambda_i \in S^{p-1}_+$, and all $x \in \mathbb{R}^n$. Up to changing the $x_{i,\alpha}$'s and $\mu_{i,\alpha}$'s, letting $\tilde{x}_{i,\alpha} = \exp_{x_{i,\alpha}}(\mu_{i,\alpha} a_i)$ and $\tilde{\mu}_{i,\alpha} = \lambda_i \mu_{i,\alpha}$, we can write, as in [Druet and Hebey 2005b], that

$$\eta^i_\alpha \check{R}^{1/\mu_{i,\alpha}}_{x_{i,\alpha}} \mathcal{U}_i = \mathcal{B}^i_\alpha + \mathcal{R}_\alpha \tag{4-9}$$

for all α, where $\mathcal{U}_i$ is as in (4-8), $\mathcal{R}_\alpha \to 0$ in $H^1(M)$ as $\alpha \to +\infty$, and $(\mathcal{B}^i_\alpha)_\alpha$ is the vector bubble with center $\tilde{x}_{i,\alpha}$, weight $\tilde{\mu}_{i,\alpha}$, and S^{p-1}-projection Λ_i. Noting that the changes $x_{i,\alpha} \to \tilde{x}_{i,\alpha}$ and $\mu_{i,\alpha} \to \tilde{\mu}_{i,\alpha}$ do not affect the first equation in (4-5), it follows from the above discussion, from (3-9), and from (4-6), (4-7), and (4-9), that the two first equations in (4-4) and the first equation in (4-5) hold true. Now we forget about the tilde notation for the centers and weights of the bubbles and, for $i = 1, \dots, N$, we let

$\mathcal{S}_i$ be as in Proposition 4.2. As one can check from the first equations in (4-4) and (4-5), for any i,

$$\hat{R}^{\mu_{i,\alpha}}_{x_{i,\alpha}} \sum_{j=1}^{N} \mathcal{B}^j_\alpha \to u_0 \Lambda_i \text{ in } L^\infty_{\text{loc}}(\mathbb{R}^n \backslash \mathcal{S}_i) \quad \text{and} \quad \hat{R}^{\mu_{i,\alpha}}_{x_{i,\alpha}} \mathcal{U}_\alpha - \hat{R}^{\mu_{i,\alpha}}_{x_{i,\alpha}} \sum_{j=1}^{N} \mathcal{B}^j_\alpha \to 0 \text{ in } L^{2^\star}_{\text{loc}}(\mathbb{R}^n) \tag{4-10}$$

as $\alpha \to +\infty$, where the $\hat{R}^{\mu_{i,\alpha}}_{x_{i,\alpha}}$-rescalings are defined in (4-1), Λ_i is the S^{p-1}-projection of $(\mathcal{B}^i_\alpha)_\alpha$, and u_0 is as in (4-3). Moreover, in any compact subset of $\mathbb{R}^n$, and for α sufficiently large,

$$\Delta_{g_\alpha} \tilde{u}_{i,\alpha} + \mu^2_{i,\alpha} \sum_{j=1}^{p} \tilde{A}^\alpha_{ij}(x) \tilde{u}_{j,\alpha} = |\tilde{\mathcal{U}}_\alpha|^{2^\star-2} \tilde{u}_{i,\alpha} \tag{4-11}$$

for all α and all i, where the $\tilde{u}_{i,\alpha}$'s are the components of $\tilde{\mathcal{U}}_\alpha = \hat{R}^{\mu_{i,\alpha}}_{x_{i,\alpha}} \mathcal{U}_\alpha$,

$$\tilde{A}^\alpha_{ij}(x) = A^\alpha_{ij}(\exp_{x_{i,\alpha}}(\mu_{i,\alpha} x)),$$

and g_α is the Riemannian metric in $\mathbb{R}^n$ given by $g_\alpha(x) = (\exp^\star_{x_{i,\alpha}} g)(\mu_{i,\alpha} x)$. Since $\mu_{i,\alpha} \to 0$ as $\alpha \to +\infty$, we get that $g_\alpha \to \xi$ in $C^2_{\text{loc}}(\mathbb{R}^n)$ as $\alpha \to +\infty$, where ξ is the Euclidean metric. By (4-10), for any $x \in \mathbb{R}^n \backslash \mathcal{S}_i$,

$$\lim_{\delta \to 0} \limsup_{\alpha \to +\infty} \int_{B_x(\delta)} |\hat{R}^{\mu_{i,\alpha}}_{x_{i,\alpha}} \mathcal{U}_\alpha|^{2^\star} dx = 0. \tag{4-12}$$

In particular, the $L^{2^\star}$-norm of $\hat{R}^{\mu_{i,\alpha}}_{x_{i,\alpha}} \mathcal{U}_\alpha$ can be made uniformly arbitrarily small in small regions of $\mathbb{R}^n \backslash \mathcal{S}_i$, and by adapting and transposing the classical regularity argument [Trudinger 1968] to the present situation (see also [Struwe 1990]) we get from (4-11) and (4-12) that the $\tilde{\mathcal{U}}_\alpha$'s are uniformly bounded in $C^{2,\theta}_{\text{loc}}(\mathbb{R}^n \backslash \mathcal{S}_i)$. It easily follows that, up to a subsequence, the second equation in (4-5) also holds true. Now it remains to prove that the third equation in (4-4) holds true. We proceed by contradiction and assume that there exists $\varepsilon_0 > 0$ and a sequence $(x_\alpha)_\alpha$ in M such that, up to a subsequence,

$$\mathcal{D}_\alpha(x_\alpha)^2 \left| \mathcal{U}_\alpha(x_\alpha) - \mathcal{U}_\infty(x_\alpha) - \sum_{i=1}^{N} \mathcal{B}^i_\alpha(x_\alpha) \right|^{2^\star-2} = \max_M \mathcal{D}^2_\alpha \left| \mathcal{U}_\alpha - \mathcal{U}_\infty - \sum_{i=1}^{N} \mathcal{B}^i_\alpha \right|^{2^\star-2} \geq 4\varepsilon_0 \tag{4-13}$$

for all α. First we claim that

$$\mathcal{D}_\alpha(x_\alpha)^2 |\mathcal{B}^i_\alpha(x_\alpha)|^{2^\star-2} \to 0 \tag{4-14}$$

as $\alpha \to +\infty$, for all $i = 1, \dots, N$. In order to prove (4-14) we proceed by contradiction and assume that there exists $i = 1, \dots, N$ and $\varepsilon_1 > 0$ such that, up to a subsequence,

$$\mathcal{D}_\alpha(x_\alpha)^2 |\mathcal{B}^i_\alpha(x_\alpha)|^{2^\star-2} \geq \varepsilon_1 \tag{4-15}$$

for all α. Up to passing to another subsequence we may then assume that there is $\lambda \in [0, +\infty)$ such that

$$\frac{d_g(x_{i,\alpha}, x_\alpha)}{\mu_{i,\alpha}} \to \lambda \text{ as } \alpha \to +\infty, \quad \text{and} \quad \frac{\mu_{j,\alpha}}{\mu_{i,\alpha}} + \frac{d_g(x_{j,\alpha}, x_\alpha)}{\mu_{i,\alpha}} \geq \sqrt{\varepsilon_1} \text{ for all } \alpha \text{ and } j. \tag{4-16}$$

Then, letting $y_\alpha = \mu^{-1}_{i,\alpha} \exp^{-1}_{x_{i,\alpha}}(x_\alpha)$, we get from the second equation in (4-16) that there exists $\varepsilon > 0$ such that $d(y_\alpha, \mathcal{S}_i) \geq \varepsilon$ for all α, and it follows from the second equation in (4-5) that

$$\mathcal{D}_\alpha(x_\alpha)^2 \left| \mathcal{U}_\alpha(x_\alpha) - \mathcal{U}_\infty(x_\alpha) - \mathcal{B}^i_\alpha(x_\alpha) \right|^{2^\star-2} \to 0 \quad \text{as } \alpha \to +\infty. \tag{4-17}$$

By the first equation in (4-5), and by (4-16), we can also write

$$\mathcal{D}_\alpha(x_\alpha)^2\left|\mathcal{B}_\alpha^j(x_\alpha)\right|^{2^\star-2}\to 0 \tag{4-18}$$

as $\alpha\to+\infty$, for all $j\neq i$. Combining (4-17) and (4-18) we get a contradiction with (4-13). It follows that (4-14) holds true. Next we claim that

$$|\mathcal{U}_\alpha(x_\alpha)|\to+\infty \quad \text{as } \alpha\to+\infty. \tag{4-19}$$

By (4-13) and (4-14), we see that (4-19) holds if $\mathcal{D}_\alpha(x_\alpha)\to 0$ as $\alpha\to+\infty$. Suppose on the contrary that, up to a subsequence, $\mathcal{D}_\alpha(x_\alpha)\to\delta$ as $\alpha\to+\infty$ for some $\delta>0$. Then, by (4-13) and (4-14),

$$\left|\mathcal{U}_\alpha(x)-\mathcal{U}_\infty(x)\right|^{2^\star-2}+o(1)\le 8\left|\mathcal{U}_\alpha(x_\alpha)-\mathcal{U}_\infty(x_\alpha)\right|^{2^\star-2}+o(1) \tag{4-20}$$

for all $x\in B_{x_\alpha}(\delta/2)$ and all α sufficiently large. Now, if we assume that (4-19) is false, then we get from (4-20) that the $\mathcal{U}_\alpha$'s are bounded in a neighbourhood of the x_α's, and it follows from standard elliptic theory that $\mathcal{U}_\alpha(x_\alpha)-\mathcal{U}_\infty(x_\alpha)\to 0$ as $\alpha\to+\infty$. Noting that this convergence of the $(\mathcal{U}_\alpha-\mathcal{U}_\infty)(x_\alpha)$'s is in contradiction with (4-13) and (4-14), we obtain (4-19).

Now let the μ_α's be given by $\mu_\alpha^{1-(n/2)}=|\mathcal{U}_\alpha(x_\alpha)|$ for all α, and define the $\mathcal{V}_\alpha$'s by $\mathcal{V}_\alpha=\hat{R}_{x_\alpha}^{\mu_\alpha}\mathcal{U}_\alpha$, where the $\hat{R}_{x_\alpha}^{\mu_\alpha}$-rescalings are defined in (4-1). Then,

$$\Delta_{g_\alpha}v_{i,\alpha}+\mu_\alpha^2\sum_{j=1}^p\hat{A}_{ij}^\alpha(x)v_{j,\alpha}=|\mathcal{V}_\alpha|^{2^\star-2}v_{i,\alpha} \tag{4-21}$$

in $B_0(\delta/\mu_\alpha)$ for all α, where the $v_{i,\alpha}$'s are the components of $\mathcal{V}_\alpha$, the $\hat{A}_{ij}^\alpha$'s are given by $\hat{A}_{ij}^\alpha(x)=A_{ij}^\alpha(\exp_{x_\alpha}(\mu_\alpha x))$, and g_α is given by $g_\alpha(x)=(\exp_{x_\alpha}^\star g)(\mu_\alpha x)$. From (4-19) we have $\mu_\alpha\to 0$ as $\alpha\to+\infty$. In particular, $g_\alpha\to\xi$ in $C^2_{\mathrm{loc}}(\mathbb{R}^n)$ as $\alpha\to+\infty$. We also have $|\mathcal{V}_\alpha(0)|=1$ for all α. Noting that the $\mathcal{V}_\alpha$'s are bounded in $\dot{H}^1(\mathbb{R}^n)$, we may assume that, up to a subsequence, $\mathcal{V}_\alpha\rightharpoonup\mathcal{V}_\infty$ weakly in $H^1_{\mathrm{loc}}(\mathbb{R}^n)$ as $\alpha\to+\infty$ for some $\mathcal{V}_\infty\in H^1(\mathbb{R}^n)$ that solves (3-1). Let $\tilde{\mathcal{S}}$ be given by

$$\tilde{\mathcal{S}}=\left\{\lim_{\alpha\to+\infty}\frac{1}{\mu_\alpha}\exp_{x_\alpha}^{-1}(x_{i,\alpha}) : i\in J\right\},$$

where J consists of the $i=1,\dots,N$ which are such that $d_g(x_{i,\alpha},x_\alpha)=O(\mu_\alpha)$ and $\mu_{i,\alpha}=o(\mu_\alpha)$ for all α. In what follows we let $K\Subset\mathbb{R}^n\backslash\tilde{\mathcal{S}}$ be a compact subset of $\mathbb{R}^n\backslash\tilde{\mathcal{S}}$, and let $x\in K$. By (4-13) and (4-14) we have

$$\left|\mathcal{V}_\alpha(x)-\mu_\alpha^{(n-2)/2}\mathcal{U}_\infty(y_\alpha)-\mu_\alpha^{(n-2)/2}\sum_{i=1}^N\Lambda_iB_{i,\alpha}(y_\alpha)\right|^{2^\star-2}\le\left(\frac{\mathcal{D}_\alpha(x_\alpha)}{\mathcal{D}_\alpha(y_\alpha)}\right)^2(1+o(1))+o(1), \tag{4-22}$$

where $y_\alpha=\exp_{x_\alpha}(\mu_\alpha x)$ for all α, Λ_i is the S^{p-1}-projection of $(\mathcal{B}_\alpha^i)_\alpha$ for all i, and $B_{i,\alpha}=|\mathcal{B}_\alpha^i|$ for all α and i. Now we claim that

$$\mu_\alpha^{(n-2)/2}|B_{i,\alpha}(y_\alpha)|\to 0 \tag{4-23}$$

as $\alpha\to+\infty$, for all $i=1,\dots,N$. Equation (4-23) is obvious if $\mu_\alpha=o(\mu_{i,\alpha})$. On the other hand, if we assume that $\mu_{i,\alpha}=o(\mu_\alpha)$, then, since $d_\xi(x,\tilde{\mathcal{S}})>0$, we get $\mu_\alpha=O(d_g(x_{i,\alpha},y_\alpha))$. Here again, (4-23) holds true. At last we may assume that there exists $C>0$ such that $C^{-1}\mu_\alpha\le\mu_{i,\alpha}\le C\mu_\alpha$ for all α.

Then (4-23) holds true unless $d_g(x_{i,\alpha}, y_\alpha) = O(\mu_{i,\alpha})$. In this case we have $d_g(x_{i,\alpha}, x_\alpha) = O(\mu_{i,\alpha})$, and it follows that $|B_{i,\alpha}(x_\alpha)|/|\mathcal{U}_\alpha(x_\alpha)| \not\to 0$ as $\alpha \to +\infty$. Combining (4-13) and (4-14) we get a contradiction, and it follows that (4-23) holds true. In particular, by (4-19), (4-22), and (4-23), we can write

$$|\mathcal{V}_\alpha(x)|^{2^\star-2} \le \left(\frac{\mathcal{D}_\alpha(x_\alpha)}{\mathcal{D}_\alpha(y_\alpha)}\right)^2 (1+o(1)) + o(1). \tag{4-24}$$

At this point we claim that

$$\mathcal{D}_\alpha(x_\alpha) = O(\mathcal{D}_\alpha(y_\alpha)). \tag{4-25}$$

We prove (4-25) by contradiction and assume that

$$d_g(x_{i,\alpha}, y_\alpha) + \mu_{i,\alpha} = o(\mathcal{D}_\alpha(x_\alpha)). \tag{4-26}$$

If $d_g(x_{i,\alpha}, x_\alpha)/\mu_\alpha \to +\infty$ as $\alpha \to +\infty$, then

$$d_g(x_{i,\alpha}, y_\alpha) + \mu_{i,\alpha} \ge (1+o(1))d_g(x_{i,\alpha}, x_\alpha) + \mu_{i,\alpha} \ge (1+o(1))\mathcal{D}_\alpha(x_\alpha),$$

and this contradicts (4-26). Hence, $d_g(x_{i,\alpha}, x_\alpha) = O(\mu_\alpha)$. Then, by (4-26),

$$d_g(x_{i,\alpha}, y_\alpha) + \mu_{i,\alpha} = o(\mu_\alpha) + o(\mu_{i,\alpha}). \tag{4-27}$$

In particular, $d_g(x_{i,\alpha}, y_\alpha) = o(\mu_\alpha)$. Since $x \in K$, this implies in turn that $\mu_\alpha = O(\mu_{i,\alpha})$, and we get with (4-27) that $\mu_{i,\alpha} + o(\mu_{i,\alpha}) = 0$, another contradiction. This proves (4-25). By (4-24) and (4-25), for any compact subset $K \Subset \mathbb{R}^n \backslash \tilde{\mathcal{S}}$, there exists $C_K > 0$ such that $|\mathcal{V}_\alpha| \le C_K$ in K. In particular, by standard elliptic theory and (4-21), we get

$$\mathcal{V}_\alpha \to \mathcal{V}_\infty \text{ in } C^2_{\mathrm{loc}}(\mathbb{R}^n \backslash \tilde{\mathcal{S}}) \quad \text{as } \alpha \to +\infty. \tag{4-28}$$

Clearly $0 \notin \tilde{\mathcal{S}}$ since, if not the case, $\mathcal{D}_\alpha(x_\alpha) = o(\mu_\alpha)$ and we get a contradiction with (4-13). Thus, since $|\mathcal{V}_\alpha(0)| = 1$ for all α, we see that $|\mathcal{V}_\infty(0)| = 1$ and $\mathcal{V}_\infty \not\equiv 0$ is not identically zero. By Proposition 3.1 it follows that there exists $a \in \mathbb{R}^n$, $\lambda > 0$, and $\Lambda \in S^{p-1}_+$, such that

$$\mathcal{V}_\infty(x) = \left(\frac{\lambda}{\lambda^2 + \dfrac{|x-a|}{n(n-2)}}\right)^{(n-2)/2} \Lambda \tag{4-29}$$

for all $x \in \mathbb{R}^n$. Let $K \Subset \mathbb{R}^n \backslash \tilde{\mathcal{S}}$ be a nonempty compact subset of $\mathbb{R}^n \backslash \tilde{\mathcal{S}}$. By the first equation in (4-4) and by (4-23), we can write $\mathcal{V}_\alpha \to 0$ in $L^{2^\star}(K)$ as $\alpha \to +\infty$. Then, by (4-28), we get $\int_K |\mathcal{V}_\infty|^{2^\star} dx = 0$, a contradiction with (4-29). Proposition 4.2 is proved. □

5. A first strong pointwise estimate

We prove pointwise estimates on the $\mathcal{U}_\alpha$'s which we use as the initial step in the induction argument we develop in the next section. First we fix some notations. We let $(\mathcal{U}_\alpha)_\alpha$ be an arbitrary bounded sequence in $H^1(M)$ of nonnegative solutions of (1-1) such that $\|\mathcal{U}_\alpha\|_\infty \to +\infty$ as $\alpha \to +\infty$. Proposition 4.2 applies to the $\mathcal{U}_\alpha$'s. We let $\mathcal{S}$ be the set of the geometrical points of the $\mathcal{U}_\alpha$'s. Then,

$$\mathcal{S} = \left\{ \lim_{\alpha \to +\infty} x_{i,\alpha} : i = 1, \dots, N \right\}, \tag{5-1}$$

where all the limits do exist, up to a subsequence. For $\delta > 0$ small enough, we let

$$\eta_\alpha(\delta) = \max_{M \backslash \bigcup_{i=1}^N B_{x_{i,\alpha}}(\delta)} |\mathcal{U}_\alpha|. \tag{5-2}$$

Thanks to the last equation in (4-4) of Proposition 4.2

$$\limsup_{\alpha \to +\infty} \eta_\alpha(\delta) \le \|\mathcal{U}_\infty\|_\infty. \tag{5-3}$$

Moreover, by standard elliptic theory, for any $\delta' > \delta$,

$$\max_{M \backslash \bigcup_{i=1}^N B_{x_{i,\alpha}}(\delta')} |\nabla \mathcal{U}_\alpha|_g = O(\eta_\alpha(\delta)). \tag{5-4}$$

In what follows we let $R_0 > 0$ be such that for any $i = 1, \dots, N$,

$$|x| \le \frac{R_0}{2} \tag{5-5}$$

for all $x \in \mathcal{S}_i$, where $\mathcal{S}_i$ is as in Proposition 4.2. We also set

$$\mu_\alpha = \max_{i \in \{1,\dots,N\}} \mu_{i,\alpha}, \quad \text{and} \quad r_\alpha(x) = \min_{i \in \{1,\dots,N\}} d_g(x_{i,\alpha}, x). \tag{5-6}$$

The pointwise estimate we prove in this section is stated as follows.

Proposition 5.1. *Let (M, g) be a smooth compact Riemannian manifold of dimension $n \ge 3$, $p \ge 1$ be an integer, and $(A_\alpha)_\alpha$ be a sequence of C^1 maps from M to $M_p^s(\mathbb{R})$ such that $A_\alpha \to A$ in $C^1(M)$ as $\alpha \to +\infty$ for some C^1 map A from M to $M_p^s(\mathbb{R})$ satisfying* (H). *Let also $(\mathcal{U}_\alpha)_\alpha$ be an arbitrary bounded sequence in $H^1(M)$ of nonnegative solutions of* (1-1) *such that $\|\mathcal{U}_\alpha\|_\infty \to +\infty$ as $\alpha \to +\infty$. There exists $C_1 > 0$ such that, up to passing to a subsequence on the $\mathcal{U}_\alpha$'s, there holds that for any sequence $(x_\alpha)_\alpha$ of points in M,*

$$\left|\mathcal{U}_\alpha(x_\alpha) - \mathcal{U}_\infty(x_\alpha)\right| \le C_1 \mu_\alpha^{(n-2)/2} \mathcal{D}_\alpha(x_\alpha)^{2-n} + \varepsilon_\alpha \|\mathcal{U}_\infty\|_\infty, \tag{5-7}$$

where $\mathcal{D}_\alpha$ and $\mathcal{U}_\infty$ are as in Proposition 4.2, μ_α is as in (5-6), *and $\varepsilon_\alpha \to 0$ as $\alpha \to +\infty$.*

We divide the proof of Proposition 5.1 into two steps.

Step 1. *For any $0 < \varepsilon < \frac{1}{2}$, there exist $R_\varepsilon > 0$, $\delta_\varepsilon > 0$ and $C_\varepsilon > 0$ such that*

$$|\mathcal{U}_\alpha(x)| \le C_\varepsilon \left(\mu_\alpha^{(1-2\varepsilon)(n-2)/2} r_\alpha(x)^{(2-n)(1-\varepsilon)} + \eta_\alpha(\delta_\varepsilon) r_\alpha(x)^{(2-n)\varepsilon}\right)$$

for all α and all $x \in M \setminus \bigcup_{i=1}^N B_{x_{i,\alpha}}(R_\varepsilon \mu_{i,\alpha})$.

Proof of Step 1. Let $0 < \varepsilon < \frac{1}{2}$. Consider $\mathcal{G}$ the Green's function of the operator $u \mapsto \Delta_g u + u$. We know (see [Druet et al. 2004], for example) that there exist $\gamma_1 > 1$, $\gamma_2 > 0$ and $\gamma_3 > 0$ such that for any distinct $x, y \in M$,

$$\frac{1}{\gamma_1} \le d_g(x, y)^{n-2} \mathcal{G}(x, y) \le \gamma_1 \tag{5-8}$$

and

$$\frac{|\nabla \mathcal{G}(x, y)|^2}{\mathcal{G}(x, y)^2} \ge \gamma_2 d_g(x, y)^{-2} - \gamma_3, \tag{5-9}$$

where ∇ in (5-9) is with respect to one of the two variables, for instance y. We let

$$\Psi_{\alpha,\varepsilon}(x)=\mu_\alpha^{(1-2\varepsilon)(n-2)/2}\sum_{i=1}^N \mathcal{G}(x_{i,\alpha},x)^{1-\varepsilon}+\eta_\alpha(\delta)\sum_{i=1}^N \mathcal{G}(x_{i,\alpha},x)^{\varepsilon},$$

and let $y_\alpha\in M\setminus\bigcup_{i=1}^N B_{x_{i,\alpha}}(R\mu_{i,\alpha})$ be such that

$$\max_{M\setminus\bigcup_{i=1}^p B_{x_{i,\alpha}}(R\mu_{i,\alpha})}\frac{\sum_{i=1}^p u_{i,\alpha}}{\Psi_{\alpha,\varepsilon}}=\frac{\sum_{i=1}^N u_{i,\alpha}}{\Psi_{\alpha,\varepsilon}}(y_\alpha) \tag{5-10}$$

for all α. We claim that, if $\delta>0$ is chosen sufficiently small and $R>0$ sufficiently large, then

$$y_\alpha\in\partial\Big(M\setminus\bigcup_{i=1}^N B_{x_{i,\alpha}}(R\mu_{i,\alpha})\Big)\quad\text{or}\quad r_\alpha(y_\alpha)>\delta \tag{5-11}$$

for α large. We prove the claim by contradiction. Indeed, assume that (5-11) fails for all α. We can write

$$\frac{\Delta_g\big(\sum_{i=1}^p u_{i,\alpha}\big)}{\sum_{i=1}^p u_{i,\alpha}}(y_\alpha)\geq\frac{\Delta_g\Psi_{\alpha,\varepsilon}}{\Psi_{\alpha,\varepsilon}}(y_\alpha). \tag{5-12}$$

Thanks to (1-1),

$$\frac{\Delta_g\big(\sum_{i=1}^p u_{i,\alpha}\big)}{\sum_{i=1}^p u_{i,\alpha}}(y_\alpha)\leq|\mathcal{U}_\alpha(y_\alpha)|^{2^\star-2}+p\|A_\alpha\|_\infty,$$

where $\|A_\alpha\|_\infty=\max_{i,j}\|A_{ij}^\alpha\|_\infty$ for all α. By (5-12) we then get $\frac{\Delta_g\Psi_{\alpha,\varepsilon}}{\Psi_{\alpha,\varepsilon}}(y_\alpha)\leq|\mathcal{U}_\alpha(y_\alpha)|^{2^\star-2}+p\|A_\alpha\|_\infty$. Since $r_\alpha(y_\alpha)\leq\delta$, this yields

$$r_\alpha(y_\alpha)^2\frac{\Delta_g\Psi_{\alpha,\varepsilon}}{\Psi_{\alpha,\varepsilon}}(y_\alpha)\leq r_\alpha(y_\alpha)^2|\mathcal{U}_\alpha(y_\alpha)|^{2^\star-2}+\delta^2p\|A\|_\infty+o(1). \tag{5-13}$$

Now we write

$$\begin{aligned}\Delta_g\Psi_{\alpha,\varepsilon}(y_\alpha)=\varepsilon(1-\varepsilon)\mu_\alpha^{(1-2\varepsilon)(n-2)/2}&\sum_{i=1}^N\frac{\left|\nabla\mathcal{G}(x_{i,\alpha},y_\alpha)\right|_g^2}{\mathcal{G}(x_{i,\alpha},y_\alpha)^2}\mathcal{G}(x_{i,\alpha},y_\alpha)^{1-\varepsilon}\\ +\varepsilon(1-\varepsilon)\eta_\alpha(\delta)&\sum_{i=1}^N\frac{\left|\nabla\mathcal{G}(x_{i,\alpha},y_\alpha)\right|_g^2}{\mathcal{G}(x_{i,\alpha},y_\alpha)^2}\mathcal{G}(x_{i,\alpha},y_\alpha)^{\varepsilon}\\ &-\varepsilon\eta_\alpha(\delta)\sum_{i=1}^N\mathcal{G}(x_{i,\alpha},y_\alpha)^{\varepsilon}-(1-\varepsilon)\mu_\alpha^{(1-2\varepsilon)(n-2)/2}\sum_{i=1}^N\mathcal{G}(x_{i,\alpha},y_\alpha)^{1-\varepsilon}.\end{aligned}$$

Using (5-8) and (5-9), and since $0<\varepsilon<\frac{1}{2}$, it follows that

$$\begin{aligned}&\Delta_g\Psi_{\alpha,\varepsilon}(y_\alpha)\\ &\quad\geq-(1-\varepsilon)\Psi_{\alpha,\varepsilon}(y_\alpha)-\gamma_3\varepsilon(1-\varepsilon)\Psi_{\alpha,\varepsilon}(y_\alpha)+\varepsilon(1-\varepsilon)\gamma_2\mu_\alpha^{(1-2\varepsilon)(n-2)/2}\sum_{i=1}^N d_g(x_{i,\alpha},y_\alpha)^{-2}\mathcal{G}(x_{i,\alpha},y_\alpha)^{1-\varepsilon}\\ &\qquad+\varepsilon(1-\varepsilon)\gamma_2\eta_\alpha(\delta)\sum_{i=1}^N d_g(x_{i,\alpha},y_\alpha)^{-2}\mathcal{G}(x_{i,\alpha},y_\alpha)^{\varepsilon}\\ &\quad\geq-(1-\varepsilon)(1+\gamma_3\varepsilon)\Psi_{\alpha,\varepsilon}(y_\alpha)+\varepsilon(1-\varepsilon)\gamma_2\gamma_1^{\varepsilon-1}\mu_\alpha^{(1-2\varepsilon)(n-2)/2}r_\alpha(y_\alpha)^{-2-(n-2)(1-\varepsilon)}\\ &\qquad+\varepsilon(1-\varepsilon)\gamma_2\gamma_1^{-\varepsilon}\eta_\alpha(\delta)r_\alpha(y_\alpha)^{-2-(n-2)\varepsilon}.\end{aligned}$$

From (5-8) we obtain $\Psi_{\alpha,\varepsilon}(y_\alpha) \le N\gamma_1^{1-\varepsilon}\mu_\alpha^{(1-2\varepsilon)(n-2)/2} r_\alpha(y_\alpha)^{-(n-2)(1-\varepsilon)} + N\gamma_1^{\varepsilon}\eta_\alpha(\delta) r_\alpha(y_\alpha)^{-(n-2)\varepsilon}$, and we can write

$$r_\alpha(y_\alpha)^2 \Delta_g \Psi_{\alpha,\varepsilon}(y_\alpha) \ge -(1-\varepsilon)(1+\gamma_3\varepsilon) r_\alpha(y_\alpha)^2 \Psi_{\alpha,\varepsilon}(y_\alpha) + \frac{1}{N}\varepsilon(1-\varepsilon)\gamma_2\gamma_1^{2(\varepsilon-1)} \Psi_{\alpha,\varepsilon}(y_\alpha).$$

Coming back to (5-13), we thus get

$$\frac{1}{N}\varepsilon(1-\varepsilon)\gamma_2\gamma_1^{2(\varepsilon-1)} \le r_\alpha(y_\alpha)^2 |\mathcal{U}_\alpha(y_\alpha)|^{2^\star-2} + \delta^2 p\|A\|_\infty + o(1) + (1-\varepsilon)(1+\gamma_3\varepsilon)\delta^2,$$

since we assumed that $r_\alpha(y_\alpha) \le \delta$. By the last equation in (4-4) of Proposition 4.2 we can choose $\delta > 0$ and $R > 0$ so as to get a contradiction. Thus (5-11) is proved. Up to choosing R a little bit larger, we deduce from the second equation in (4-5) of Proposition 4.2, and the definitions of μ_α and $\eta_\alpha(\delta)$, that there exists $C > 0$ such that

$$\sup_{M\setminus\bigcup_{i=1}^N B_{x_{i,\alpha}}(R\mu_{i,\alpha})} \frac{\sum_{i=1}^p u_{i,\alpha}}{\Psi_{\alpha,\varepsilon}} \le C_\varepsilon.$$

Using (5-8), we obtain the existence of $\delta_\varepsilon > 0$, $R_\varepsilon > 0$ and $C_\varepsilon > 0$ such that

$$\sum_{i=1}^p u_{i,\alpha}(x) \le C_\varepsilon\Big(\mu_\alpha^{(1-2\varepsilon)(n-2)/2} r_\alpha(x)^{(2-n)(1-\varepsilon)} + \eta_\alpha(\delta_\varepsilon) r_\alpha(x)^{(2-n)\varepsilon}\Big)$$

for all α and all $x \in M \setminus \bigcup_{i=1}^N B_{x_{i,\alpha}}(R_\varepsilon\mu_{i,\alpha})$. This proves Step 1. □

Step 2. *There exists $C_0 > 0$ such that $|\mathcal{U}_\alpha(x)| \le C_0\big(\mu_\alpha^{(n-2)/2}\mathcal{D}_\alpha(x)^{2-n} + \|\mathcal{U}_\infty\|_\infty\big)$ for all α and all $x \in M$.*

Proof of Step 2. First we prove that there is $\delta > 0$ small such that for any sequence (y_α) of points in M,

$$\limsup_{\alpha\to+\infty} \frac{|\mathcal{U}_\alpha(y_\alpha)|}{\mu_\alpha^{(n-2)/2}\mathcal{D}_\alpha(y_\alpha)^{2-n} + \eta_\alpha(\delta)} < +\infty. \tag{5-14}$$

By the definition of $\eta_\alpha(\delta)$, it is clear that (5-14) holds if $r_\alpha(y_\alpha) \ge \delta$. Now assume that $r_\alpha(y_\alpha) = O(\mu_\alpha)$. Then $\mathcal{D}_\alpha(y_\alpha) = O(\mu_\alpha)$. We can use the last equation in (4-4) of Proposition 4.2 to obtain

$$\begin{aligned}\mathcal{D}_\alpha(y_\alpha)^2\mu_\alpha^{-1}|\mathcal{U}_\alpha(y_\alpha)|^{2/n-2} &= O(\mathcal{D}_\alpha(y_\alpha)\mu_\alpha^{-1}) + O\Bigg(\sum_{i=1}^N \mathcal{D}_\alpha(y_\alpha)^2\mu_\alpha^{-1}\mu_{i,\alpha}^{-1}\bigg(1+\frac{d_g(x_{i,\alpha},y_\alpha)^2}{n(n-2)\mu_{i,\alpha}^2}\bigg)^{-1}\Bigg)\\ &= O(1),\end{aligned}$$

since $\mathcal{D}_\alpha(y_\alpha) \le d_g(x_{i,\alpha}, y_\alpha) + \mu_{i,\alpha}$ for all $i \in \{1,\dots,N\}$. In particular, (5-14) holds true also in this case. Thus we may assume from now on that

$$r_\alpha(y_\alpha) \le \delta \quad\text{and}\quad \frac{r_\alpha(y_\alpha)}{\mu_\alpha} \to +\infty \quad\text{as } \alpha \to +\infty. \tag{5-15}$$

We let $\lambda > 1$ be such that $\lambda p\|A\|_\infty \notin Sp(\Delta_g)$ and we let G be the Green's function of $\Delta_g - \lambda p\|A\|_\infty$. Here again, there exist $C_1 > 1$, $C_2 > 0$ and $C_3 > 0$ such that

$$\frac{1}{C_1} d_g(x,y)^{2-n} - C_2 \le G(x,y) \le C_1 d_g(x,y)^{2-n} \tag{5-16}$$

and

$$|\nabla G(x,y)|_g \le C_3 d_g(x,y)^{1-n} \tag{5-17}$$

for all $x, y \in M$, $x \neq y$. We let $x_0 \in \mathcal{S}$ be such that $d_g(y_\alpha, x_0) \le \delta + o(1)$; such an x_0 does exist thanks to (5-15). We choose $\delta > 0$ such that

$$d_g(x,y) \ge 4\delta \tag{5-18}$$

for all $x, y \in \mathcal{S}$, $x \neq y$, and such that

$$\delta \le \tfrac{1}{4}(C_1C_2)^{-1/(n-2)}, \tag{5-19}$$

where C_1 and C_2 are as in (5-16). We write with Green's representation formula that

$$\begin{aligned}\sum_{i=1}^p u_{i,\alpha}(y_\alpha) &= \int_{B_{x_0}(2\delta)} G(y_\alpha,x)\left(\Delta_g\left(\sum_{i=1}^p u_{i,\alpha}\right) - \lambda p\|A\|_\infty \sum_{i=1}^p u_i^\alpha\right)(x)\,dv_g(x)\\ &\quad + \int_{\partial B_{x_0}(2\delta)} G(y_\alpha,x)\partial_\nu\left(\sum_{i=1}^p u_{i,\alpha}\right)(x)\,d\sigma_g(x) - \int_{\partial B_{x_0}(2\delta)} \partial_\nu G(y_\alpha,x)\left(\sum_{i=1}^p u_{i,\alpha}\right)(x)\,d\sigma_g(x). \end{aligned}\tag{5-20}$$

Since $\lambda > 1$, we get with (1-1) that

$$\Delta_g\left(\sum_{i=1}^p u_{i,\alpha}\right) - \lambda p\|A\|_\infty \sum_{i=1}^p u_{i,\alpha} \le |\mathcal{U}_\alpha|^{2^\star-2}\sum_{i=1}^p u_{i,\alpha}.$$

We have $G(y_\alpha, x) \ge 0$ in $B_{x_0}(2\delta)$ for α large, thanks to (5-16) and (5-19). Thus we can write

$$\begin{aligned}\int_{B_{x_0}(2\delta)} G(y_\alpha,x)&\left(\Delta_g\left(\sum_{i=1}^p u_{i,\alpha}\right) - \lambda p\|A\|_\infty \sum_{i=1}^p u_{i,\alpha}\right)(x)\,dv_g(x)\\ &\le C_1\int_M d_g(y_\alpha,x)^{2-n}|\mathcal{U}_\alpha(x)|^{2^\star-2}\sum_{i=1}^p u_{i,\alpha}(x)\,dv_g(x).\end{aligned}\tag{5-21}$$

From (5-18), we also know that $d_g\big(x_{i,\alpha}, \partial B_{x_0}(2\delta)\big) \ge \delta$ for all $i = 1, \dots, N$ and for α large so that we can control the boundary terms in (5-20) thanks to (5-4), (5-16) and (5-17). We thus obtain that

$$|\mathcal{U}_\alpha(y_\alpha)| = O(\eta_\alpha(\delta)) + O\left(\int_M d_g(y_\alpha,x)^{2-n}|\mathcal{U}_\alpha(x)|^{2^\star-1}dv_g(x)\right). \tag{5-22}$$

We fix $0 < \varepsilon < \dfrac{1}{n+2}$ and we let $R_\varepsilon > 0$, $\delta_\varepsilon > 0$ and $C_\varepsilon > 0$ be given by Step 1. We write

$$\begin{aligned}\int_M & d_g(y_\alpha,x)^{2-n}|\mathcal{U}_\alpha(x)|^{2^\star-1}dv_g(x)\\ &\le \int_{M_{\alpha,\varepsilon}} d_g(y_\alpha,x)^{2-n}|\mathcal{U}_\alpha(x)|^{2^\star-1}dv_g(x) + \sum_{i=1}^p\int_{B_{x_{i,\alpha}}(R_\varepsilon\mu_\alpha)} d_g(y_\alpha,x)^{2-n}|\mathcal{U}_\alpha(x)|^{2^\star-1}dv_g(x),\end{aligned}$$

where $M_{\alpha,\varepsilon} = M \setminus \bigcup_{i=1}^N B_{x_{i,\alpha}}(R_\varepsilon\mu_\alpha)$. From (5-15) and Hölder's inequalities we obtain

$$\int_{B_{x_{i,\alpha}}(R_\varepsilon\mu_\alpha)} d_g(y_\alpha,x)^{2-n}|\mathcal{U}_\alpha(x)|^{2^\star-1}dv_g(x) = O\big(\mu_\alpha^{(n-2)/2} d_g(x_{i,\alpha},y_\alpha)^{2-n}\big)$$

for all $i \in \{1, \dots, N\}$. Thus we get

$$\int_M d_g(y_\alpha, x)^{2-n} |\mathcal{U}_\alpha(x)|^{2^\star-1} dv_g(x) \leq \int_{M_{\alpha,\varepsilon}} d_g(y_\alpha, x)^{2-n} |\mathcal{U}_\alpha(x)|^{2^\star-1} dv_g(x) + O\big(\mu_\alpha^{(n-2)/2} r_\alpha(y_\alpha)^{2-n}\big). \quad (5\text{-}23)$$

Using Step 1, we know that for any $x \in M_{\alpha,\varepsilon}$,

$$|\mathcal{U}_\alpha(x)|^{2^\star-1} \leq 2^{2^\star-2} C_\varepsilon^{2^\star-1} \left(\frac{\mu_\alpha^{(1-2\varepsilon)(n+2)/2}}{r_\alpha(x)^{(n+2)(1-\varepsilon)}} + \frac{\eta_\alpha(\delta_\varepsilon)^{2^\star-1}}{r_\alpha(x)^{(n+2)\varepsilon}} \right)$$

so that

$$\begin{aligned}
\int_{M_{\alpha,\varepsilon}} & d_g(y_\alpha, x)^{2-n} |\mathcal{U}_\alpha(x)|^{2^\star-1} dv_g(x) \\
&\leq 2^{2^\star-2} C_\varepsilon^{2^\star-1} \mu_\alpha^{(1-2\varepsilon)(n+2)/2} \int_{M_{\alpha,\varepsilon}} d_g(y_\alpha, x)^{2-n} r_\alpha(x)^{-(n+2)(1-\varepsilon)} dv_g(x) \\
&\quad + 2^{2^\star-2} C_\varepsilon^{2^\star-1} \eta_\alpha(\delta_\varepsilon)^{2^\star-1} \int_{M_{\alpha,\varepsilon}} d_g(y_\alpha, x)^{2-n} r_\alpha(x)^{-(n+2)\varepsilon} dv_g(x) \\
&\leq 2^{2^\star-2} C_\varepsilon^{2^\star-1} \mu_\alpha^{(1-2\varepsilon)(n+2)/2} \sum_{i=1}^N \int_{M \setminus B_{x_{i,\alpha}}(R_\varepsilon \mu_\alpha)} d_g(y_\alpha, x)^{2-n} d_g(x_{i,\alpha}, x)^{-(n+2)(1-\varepsilon)} dv_g(x) \\
&\quad + 2^{2^\star-2} C_\varepsilon^{2^\star-1} \eta_\alpha(\delta_\varepsilon)^{2^\star-1} \sum_{i=1}^N \int_{M \setminus B_{x_{i,\alpha}}(R_\varepsilon \mu_\alpha)} d_g(y_\alpha, x)^{2-n} d_g(x_{i,\alpha}, x)^{-(n+2)\varepsilon} dv_g(x).
\end{aligned}$$

From (5-15), straightforward computations yield

$$\int_{M_{\alpha,\varepsilon}} d_g(y_\alpha, x)^{2-n} |\mathcal{U}_\alpha(x)|^{2^\star-1} dv_g(x) = O\big(\mu_\alpha^{(n-2)/2} r_\alpha(y_\alpha)^{2-n}\big) + O\big(\eta_\alpha(\delta_\varepsilon)^{2^\star-1}\big).$$

Coming back to (5-23), using (5-3), we finally obtain that

$$\int_M d_g(y_\alpha, x)^{2-n} |\mathcal{U}_\alpha(x)|^{2^\star-1} dv_g(x) = O\big(\mu_\alpha^{(n-2)/2} r_\alpha(y_\alpha)^{2-n}\big) + O(\eta_\alpha(\delta_\varepsilon)).$$

Coming back to (5-22), taking $0 < \delta < \delta_\varepsilon$ such that (5-18) and (5-19) hold, we get (5-14) under assumption (5-15). In particular, if δ is chosen sufficiently small, (5-14) holds. Now we claim that if $\mathcal{U}_\infty \equiv 0$, then

$$\eta_\alpha(\delta) = O\big(\mu_\alpha^{(n-2)/2}\big). \quad (5\text{-}24)$$

As a consequence of (5-14), there exists $C_0 > 0$ such that in any compact subset K of $M \setminus \mathcal{S}$,

$$|\mathcal{U}_\alpha(x)| \leq C_0 \Big(\mu_\alpha^{(n-2)/2} C_K + \eta_\alpha(\delta) \Big)$$

for some $C_K > 0$. If (5-24) were false, we would get by standard elliptic theory that

$$\frac{\mathcal{U}_\alpha}{\eta_\alpha(\delta)} \to H \text{ in } C^2_{\mathrm{loc}}(M \setminus \mathcal{S}) \quad \text{as } \alpha \to +\infty,$$

where H satisfies $\Delta_g H + AH = 0$ in $M \setminus \mathcal{S}$ and $|H| \le C_0$ in $M \setminus \mathcal{S}$. This implies that H is in the kernel of $\Delta_g + A$. Since all the components of H are nonnegative and H is not identically zero by the definition of $\eta_\alpha(\delta)$, this would contradict assumption (H). In particular, (5-24) is proved. Noting that if $\mathcal{U}_\infty \not\equiv 0$, then, by (5-3), $\eta_\alpha(\delta) = O(\|\mathcal{U}_\infty\|_\infty)$, we get with (5-14) that Step 2 holds true. □

Conclusion of the proof of Proposition 5.1. If $\mathcal{U}_\infty \equiv 0$, the proposition is a direct consequence of Step 2. Assume now that $\mathcal{U}_\infty \not\equiv 0$. We let $\mathcal{H}$ be the Green's function of the Laplacian on M normalized such that $\mathcal{H}(x, y) \ge 1$ for all $x, y \in M$, $x \ne y$. There exists $\Theta_1 > 1$ such that

$$\frac{1}{\Theta_1} d_g(x, y)^{2-n} \le \mathcal{H}(x, y) \le \Theta_1 d_g(x, y)^{2-n} \tag{5-25}$$

for all $x, y \in M$, $x \ne y$. We let (x_α) be a sequence of points in M and prove that

$$|\mathcal{U}_\alpha(x_\alpha) - \mathcal{U}_\infty(x_\alpha)| = O\left(\mu_\alpha^{(n-2)/2} \mathcal{D}_\alpha(x_\alpha)^{2-n}\right) + o(1). \tag{5-26}$$

If $\mathcal{D}_\alpha(x_\alpha) = O(\mu_\alpha)$, then (5-26) is a direct consequence of the last equation in (4-4) of Proposition 4.2. We may therefore assume that

$$\frac{\mathcal{D}_\alpha(x_\alpha)}{\mu_\alpha} \to +\infty \quad \text{as } \alpha \to +\infty. \tag{5-27}$$

By standard elliptic theory,

$$\mathcal{U}_\alpha \to \mathcal{U}_\infty \text{ in } C^2_{\text{loc}}(M \setminus \mathcal{S}) \quad \text{as } \alpha \to +\infty, \tag{5-28}$$

where $\mathcal{S}$ is as in (5-1). We write using Green's representation formula that

$$\begin{aligned}\sum_{i=1}^p u_{i,\alpha}(x_\alpha) - \sum_{i=1}^p u_{i,\infty}(x_\alpha) = \frac{1}{V_g} \sum_{i=1}^p \int_M \big(u_{i,\alpha}(x_\alpha) - u_{i,\infty}(x_\alpha)\big)\, dv_g \\ + \sum_{i=1}^p \int_M \mathcal{H}(x_\alpha, x) \Delta_g (u_{i,\alpha} - u_{i,\infty})(x)\, dv_g(x),\end{aligned}$$

where V_g is the volume of (M, g), and the $u_{i,\infty}$'s are the components of $\mathcal{U}_\infty$. Then we get

$$\sum_{i=1}^p u_{i,\alpha}(x_\alpha) - \sum_{i=1}^p u_{i,\infty}(x_\alpha) = \sum_{i=1}^p \int_M \mathcal{H}(x_\alpha, x) \Delta_g (u_{i,\alpha} - u_{i,\infty})(x)\, dv_g(x) + o(1). \tag{5-29}$$

Thanks to (5-28) there exists $\delta_\alpha > 0$, $\delta_\alpha \to 0$ as $\alpha \to +\infty$, such that, up to a subsequence,

$$\|\mathcal{U}_\alpha - \mathcal{U}_\infty\|_{C^2(\{\mathcal{D}_\alpha > \delta_\alpha\})} = o(1), \tag{5-30}$$

where $\|\mathcal{U}\|_{C^2} = \sum_{i=1}^p \|u_i\|_{C^2}$, and $\{\mathcal{D}_\alpha > \delta_\alpha\}$ is the subset of M consisting of the $x \in M$ such that $\mathcal{D}_\alpha(x) > \delta_\alpha$. In particular, it follows from (5-25), (5-29) and (5-30) that

$$\sum_{i=1}^p u_{i,\alpha}(x_\alpha) - \sum_{i=1}^p u_{i,\infty}(x_\alpha) = \sum_{i=1}^p \int_{\{\mathcal{D}_\alpha(x) \le \delta_\alpha\}} \mathcal{H}(x_\alpha, x) \Delta_g u_{i,\alpha}(x)\, dv_g(x) + o(1).$$

The proof of (5-26) then follows the lines of the proof of Step 2, using the estimate of that step. This ends the proof of Proposition 5.1. □

6. Strong pointwise estimates and sharp asymptotics

We now turn to pointwise estimates and sharp asymptotics. Our main result in this section is this:

Proposition 6.1. *Let (M, g) be a smooth compact Riemannian manifold of dimension $n \ge 3$, $p \ge 1$ be an integer, and $(A_\alpha)_\alpha$ be a sequence of C^1 maps from M to $M_p^s(\mathbb{R})$ such that $A_\alpha \to A$ in $C^1(M)$ as $\alpha \to +\infty$ for some C^1 map A from M to $M_p^s(\mathbb{R})$ satisfying* (H). *Let also $(\mathcal{U}_\alpha)_\alpha$ be an arbitrary bounded sequence in $H^1(M)$ of nonnegative solutions of* (1-1) *such that $\|\mathcal{U}_\alpha\|_\infty \to +\infty$ as $\alpha \to +\infty$. Up to passing to a subsequence on the $\mathcal{U}_\alpha$'s, there holds that for any sequence $(x_\alpha)_\alpha$ of points in M,*

$$\left|\mathcal{U}_\alpha(x_\alpha) - \mathcal{U}_\infty(x_\alpha) - \sum_{i=1}^N B_{i,\alpha}(x_\alpha)\Lambda_i\right| = \varepsilon_\alpha \|\mathcal{U}_\infty\|_\infty + O\big(\mu_\alpha^{(n-2)/2}\big) + o\Bigg(\sum_{i=1}^N B_{i,\alpha}(x_\alpha)\Bigg), \tag{6-1}$$

where $\mathcal{B}_\alpha^i = B_{i,\alpha}\Lambda_i$ for all α and all i, where $\mathcal{U}_\infty$, N, and the $\mathcal{B}_\alpha^i$'s are as in Proposition 4.2, where $\varepsilon_\alpha \to 0$ as $\alpha \to +\infty$, and $\mu_\alpha = \max_i \mu_{i,\alpha}$ is the maximum weight of the weights of the $B_{i,\alpha}$'s as in (5-6).

We prove Proposition 6.1 in several steps, based on induction on the following statement, defined for $1 \le k \le N+1$:

> There exists $C_k > 0$ such that, up to a subsequence on the $\mathcal{U}_\alpha$'s, for any sequence (x_α) of points in M,
> $$\begin{aligned}&\left|\mathcal{U}_\alpha(x_\alpha) - \mathcal{U}_\infty(x_\alpha) - \sum_{i=1}^{k-1} B_{i,\alpha}(x_\alpha)\Lambda_i\right| \\ &\qquad\le C_k\big(\mu_\alpha^{(n-2)/2} + \chi_k \mu_{k,\alpha}^{(n-2)/2} R_{k,\alpha}(x_\alpha)^{2-n}\big) + \varepsilon_\alpha \|\mathcal{U}_\infty\|_\infty + o\Bigg(\sum_{i=1}^{k-1} B_{i,\alpha}(x_\alpha)\Bigg),\end{aligned} \tag{$\mathcal{I}_k$}$$
> where $\varepsilon_\alpha \to 0$ as $\alpha \to +\infty$, $\chi_k = 1$ if $k \le N$, and $\chi_{N+1} = 0$.

Here we have reordered the blow-up points in such a way that $\mu_\alpha = \mu_{1,\alpha} \ge \mu_{2,\alpha} \ge \cdots \ge \mu_{N,\alpha}$, and we have defined

$$r_{i,\alpha}(x) = \min_{i \le j \le N} d_g(x_{j,\alpha}, x), \tag{6-2}$$

$$R_{i,\alpha}(x) = \min_{i \le j \le N} \big(d_g(x_{j,\alpha}, x) + \mu_{j,\alpha}\big). \tag{6-3}$$

We have $R_{1,\alpha}(x) = \mathcal{D}_\alpha(x)$ and $r_{1,\alpha}(x) = r_\alpha(x)$, where $\mathcal{D}_\alpha$ is as in Proposition 4.2 and r_α is as in (5-6).

We will refer to the whole indented statement above as $(\mathcal{I}_k)$, as well as the inequality so labeled. Clearly, Proposition 6.1 is equivalent to $(\mathcal{I}_{N+1})$, while Proposition 5.1 implies $(\mathcal{I}_1)$.

We apply induction on k to pass from $(\mathcal{I}_1)$ to $(\mathcal{I}_N)$, and then we use a slightly distinct argument to pass from $(\mathcal{I}_N)$ to $(\mathcal{I}_{N+1})$. In the following, we fix $1 \le \kappa \le N-1$ and assume that $(\mathcal{I}_\kappa)$ holds true. We proceed in several steps, but first we fix some notation. As in the preceding section we let $\mathcal{G}$ be the Green's function of the operator $u \mapsto \Delta_g u + u$. Then (5-8) and (5-9) hold. We fix $0 < \varepsilon < 1/(n+2)$ and fix R_0 as in (5-5). For any $1 \le i \le \kappa$, we define

$$\Phi_{i,\alpha}^\varepsilon(x) = \min\big\{\mu_{i,\alpha}^{(1-2\varepsilon)(n-2)/2}\mathcal{G}(x_{i,\alpha}, x)^{1-\varepsilon};\ D_0\mu_{i,\alpha}^{-(1-2\varepsilon)(n-2)/2}\mathcal{G}(x_{i,\alpha}, x)^{\varepsilon}\big\}, \tag{6-4}$$

where $x \in M \setminus \{x_{i,\alpha}\}$,

$$D_0 = \gamma_1^{2\varepsilon-1}(4R_0)^{(2-n)(1-2\varepsilon)},$$

and γ_1 is as in (5-8). This choice of D_0, together with (5-8), implies that

$$\Phi^\varepsilon_{i,\alpha}(x) = D_0 \mu_{i,\alpha}^{-(1-2\varepsilon)(n-2)/2} \mathcal{G}(x_{i,\alpha}, x)^\varepsilon,$$

if $d_g(x_{i,\alpha}, x) \le 2R_0 \mu_{i,\alpha}$. We also let

$$\varphi^\varepsilon_\alpha(x) = \max\left\{\|\mathcal{U}_\infty\|_\infty;\ \mu_\alpha^{(1-2\varepsilon)(n-2)/2}\right\} \sum_{i=1}^N \mathcal{G}(x_{i,\alpha}, x)^\varepsilon \tag{6-5}$$

and

$$\Psi^\varepsilon_\alpha(x) = \sum_{i=\kappa+1}^N \mathcal{G}(x_{i,\alpha}, x)^{1-\varepsilon}. \tag{6-6}$$

For $1 \le i \le \kappa$, we set

$$\Omega^\varepsilon_{i,\alpha} = \left\{x \in M \text{ s.t. } \Phi^\varepsilon_{i,\alpha}(x) \ge \Phi^\varepsilon_{j,\alpha}(x) \text{ for all } 1 \le j \le \kappa\right\}. \tag{6-7}$$

We also set

$$D(\varepsilon) = \frac{\gamma_2 \varepsilon (1-\varepsilon)}{2N}, \tag{6-8}$$

where γ_2 is as in (5-8), and we define $\nu_{\kappa,\alpha}$ by

$$\nu_{\kappa,\alpha}^{(1-2\varepsilon)(n-2)/2} = \max\left\{\mu_{\kappa+1,\alpha}^{(1-2\varepsilon)(n-2)/2};\ \max_{1\le i\le\kappa} \sup_{\tilde{\Omega}^\varepsilon_{i,\alpha}} \frac{\Phi^\varepsilon_{i,\alpha}(x)}{\Psi^\varepsilon_\alpha(x)}\right\}, \tag{6-9}$$

where

$$\tilde{\Omega}^\varepsilon_{i,\alpha} = \left\{x \in \Omega^\varepsilon_{i,\alpha} \text{ s.t. } d_g(x_{i,\alpha}, x)^2 \mathcal{Q}_{\kappa,\alpha}(x)^{2^\star-2} \ge D(\varepsilon)\right\}, \tag{6-10}$$

and $\mathcal{Q}_{\kappa,\alpha}(x) = |\mathcal{U}_\alpha(x) - \mathcal{U}_\infty(x) - \sum_{i=1}^\kappa B_{i,\alpha}(x)\Lambda_i|$. By convention, the suprema in (6-9) are $-\infty$ if the sets $\tilde{\Omega}^\varepsilon_{i,\alpha}$ are empty. We can now start the proof of Proposition 6.1.

Step 1. $\nu_{\kappa,\alpha} = O(\mu_{\kappa,\alpha})$.

Proof of Step 1. We let $y_\alpha \in \tilde{\Omega}^\varepsilon_{i,\alpha}$ and assume that

$$\nu_{\kappa,\alpha}^{1-2\varepsilon} \Psi^\varepsilon_\alpha(y_\alpha)^{2/(n-2)} = \Phi^\varepsilon_{i,\alpha}(y_\alpha)^{2/(n-2)}.$$

This and (5-8) imply that

$$\nu_{\kappa,\alpha}^{1-2\varepsilon} = O\left(R_{\kappa+1,\alpha}(y_\alpha)^{2(1-\varepsilon)} \Phi^\varepsilon_{i,\alpha}(y_\alpha)^{2/(n-2)}\right). \tag{6-11}$$

Since $(\mathcal{I}_k)$ holds and $y_\alpha \in \tilde{\Omega}^\varepsilon_{i,\alpha}$, we also have

$$D(\varepsilon) \le o(1) + o\left(\sum_{j=1}^{\kappa-1} d_g(x_{i,\alpha}, y_\alpha)^2 B_{j,\alpha}(y_\alpha)^{2^\star-2}\right) + O\left(\mu_{\kappa,\alpha}^2 d_g(x_{i,\alpha}, y_\alpha)^2 R_{\kappa,\alpha}(y_\alpha)^{-4}\right).$$

Since $y_\alpha \in \tilde{\Omega}^\varepsilon_{i,\alpha}$ and $\tilde{\Omega}^\varepsilon_{i,\alpha} \subset \Omega^\varepsilon_{i,\alpha}$, we can write

$$\sum_{j=1}^{\kappa-1} d_g(x_{i,\alpha}, y_\alpha)^2 B_{j,\alpha}(y_\alpha)^{2^\star-2} = O(1),$$

and we thus get

$$R_{\kappa,\alpha}(y_\alpha)^2 = O\left(\mu_{\kappa,\alpha} d_g(x_{i,\alpha}, y_\alpha)\right). \tag{6-12}$$

If $R_{\kappa+1,\alpha}(y_\alpha) = O(R_{\kappa,\alpha}(y_\alpha))$, we get from (6-11) and (6-12) that

$$\nu_{\kappa,\alpha}^{1-2\varepsilon} = O\left(\mu_{\kappa,\alpha}^{1-\varepsilon} d_g(x_{i,\alpha}, y_\alpha)^{1-\varepsilon} \Phi^\varepsilon_{i,\alpha}(y_\alpha)^{2/(n-2)}\right) = O(\mu_{\kappa,\alpha}^{1-\varepsilon} \mu_{i,\alpha}^{-\varepsilon}) = O(\mu_{\kappa,\alpha}^{1-2\varepsilon}),$$

and Step 1 is proved. Assume now that $R_{\kappa,\alpha}(y_\alpha) = o(R_{\kappa+1,\alpha}(y_\alpha))$. Then (6-12) becomes

$$\big(d_g(x_{\kappa,\alpha}, y_\alpha) + \mu_{\kappa,\alpha}\big)^2 = O(\mu_{\kappa,\alpha} d_g(x_{i,\alpha}, y_\alpha)). \tag{6-13}$$

If $i = \kappa$ we obtain $d_g(x_{i,\alpha}, y_\alpha) = O(\mu_{i,\alpha})$; using the last equation in (4-4), and since $\mu_{i,\alpha} = o(R_{\kappa+1,\alpha}(y_\alpha))$, we obtain that

$$d_g(x_{i,\alpha}, y_\alpha)^2 \left| \mathcal{U}_\alpha(y_\alpha) - \mathcal{U}_\infty(y_\alpha) - \sum_{j=1}^{\kappa} B_{j,\alpha}(y_\alpha)\Lambda_j \right|^{2^\star - 2} \to 0 \quad \text{as } \alpha \to +\infty.$$

This contradicts the fact that $y_\alpha \in \tilde{\Omega}^\varepsilon_{i,\alpha}$. Thus we must have $1 \le i \le \kappa - 1$. Since $\Phi^\varepsilon_{i,\alpha}(y_\alpha) \ge \Phi^\varepsilon_{\kappa,\alpha}(y_\alpha)$, because of (5-8), we can write

$$\mu_{\kappa,\alpha}^{1-2\varepsilon} d_g(x_{\kappa,\alpha}, y_\alpha)^{-2\varepsilon} \big(\mu_{\kappa,\alpha} + d_g(x_{\kappa,\alpha}, y_\alpha)\big)^{-2(1-2\varepsilon)} = O\Big(\mu_{i,\alpha}^{1-2\varepsilon} d_g(x_{i,\alpha}, y_\alpha)^{-2\varepsilon} \big(\mu_{i,\alpha} + d_g(x_{i,\alpha}, y_\alpha)\big)^{-2(1-2\varepsilon)}\Big).$$

In particular we obtain with (6-13) that

$$\big(\mu_{i,\alpha} + d_g(x_{i,\alpha}, y_\alpha)\big)^{1-\varepsilon} = O(\mu_{\kappa,\alpha}^{\varepsilon} \mu_{i,\alpha}^{1-2\varepsilon}).$$

Since $\mu_{k,\alpha} \le \mu_{i,\alpha}$, this implies that $d_g(x_{i,\alpha}, y_\alpha) = O(\mu_{i,\alpha})$. We also get $\mu_{i,\alpha} = O(\mu_{\kappa,\alpha})$. Then we obtain with (6-13) that $d_g(x_{\kappa,\alpha}, y_\alpha) = O(\mu_{i,\alpha})$, and this contradicts the first equation in (4-5) of Proposition 4.2. Step 1 is proved. □

Step 2. *There exists $C_\varepsilon > 0$ such that*

$$|\mathcal{U}_\alpha(x)| \le C_\varepsilon \left(\sum_{i=1}^{\kappa} \Phi^\varepsilon_{i,\alpha}(x) + \nu_{\kappa,\alpha}^{(1-2\varepsilon)(n-2)/2} r_{\kappa+1,\alpha}(x)^{(2-n)(1-\varepsilon)} + \max\big\{\|\mathcal{U}_\infty\|_\infty ; \mu_\alpha^{(1-2\varepsilon)(n-2)/2}\big\} r_\alpha(x)^{(2-n)\varepsilon} \right)$$

for all $x \in M \setminus \bigcup_{i=\kappa+1}^N B_{x_{i,\alpha}}(R_0 \mu_{i,\alpha})$.

Proof of Step 2. We let $y_\alpha \in M \setminus \bigcup_{i=\kappa+1}^N B_{x_{i,\alpha}}(R_0 \mu_{i,\alpha})$ be such that

$$\frac{\sum_{i=1}^p u_{i,\alpha}}{\sum_{i=1}^\kappa \Phi^\varepsilon_{i,\alpha} + \nu_{\kappa,\alpha}^{(1-2\varepsilon)(n-2)/2} \Psi_{\alpha,\varepsilon} + \varphi^\varepsilon_\alpha}(y_\alpha) = \sup_{M \setminus \bigcup_{i=\kappa+1}^N B_{x_{i,\alpha}}(R_\varepsilon \mu_{i,\alpha})} \frac{\sum_{i=1}^p u_{i,\alpha}}{\sum_{i=1}^\kappa \Phi^\varepsilon_{i,\alpha} + \nu_{\kappa,\alpha}^{(1-2\varepsilon)(n-2)/2} \Psi_{\alpha,\varepsilon} + \varphi^\varepsilon_\alpha}, \tag{6-14}$$

and we assume by contradiction that

$$\frac{\sum_{i=1}^p u_i^\alpha}{\sum_{i=1}^\kappa \Phi^\varepsilon_{i,\alpha} + \nu_{\kappa,\alpha}^{(1-2\varepsilon)(n-2)/2} \Psi_{\alpha,\varepsilon} + \varphi^\varepsilon_\alpha}(y_\alpha) \to +\infty \quad \text{as } \alpha \to +\infty. \tag{6-15}$$

From $(\mathcal{I}_\kappa)$ and (6-15) we get

$$r_\alpha(y_\alpha) \to 0 \quad \text{as } \alpha \to +\infty. \tag{6-16}$$

We also have, using the second equation in (4-5),

$$\frac{d_g(x_{j,\alpha}, y_\alpha)}{\mu_{j,\alpha}} \to +\infty \tag{6-17}$$

as $\alpha \to +\infty$ for all $\kappa + 1 \le j \le N$. Here we used the fact that, by (6-9), $\nu_{\kappa,\alpha} \ge \mu_{\kappa+1,\alpha}$. Thanks to (6-15) and the second equation in (4-5), we also know that, for any $1 \le j \le \kappa$, either

$$d_g(x_{j,\alpha}, y_\alpha) \le R_0 \mu_{j,\alpha} \quad \text{or} \quad \frac{d_g(x_{j,\alpha}, y_\alpha)}{\mu_{j,\alpha}} \to +\infty \quad \text{as } \alpha \to +\infty. \tag{6-18}$$

In particular, thanks to (6-14) we can write

$$\frac{\Delta_g \sum_{i=1}^{p} u_{i,\alpha}}{\sum_{i=1}^{p} u_{i,\alpha}}(y_\alpha) \geq \frac{\Delta_g \Big(\sum_{i=1}^{\kappa} \Phi_{i,\alpha}^{\varepsilon} + \nu_{\kappa,\alpha}^{(1-2\varepsilon)(n-2)/2} \Psi_{\alpha,\varepsilon} + \varphi_{\alpha}^{\varepsilon} \Big)}{\sum_{i=1}^{\kappa} \Phi_{i,\alpha}^{\varepsilon} + \nu_{\kappa,\alpha}^{(1-2\varepsilon)(n-2)/2} \Psi_{\alpha,\varepsilon} + \varphi_{\alpha}^{\varepsilon}}(y_\alpha). \tag{6-19}$$

From (1-1), (5-8), and (5-9), we then get

$$\begin{aligned} 0 \geq \sum_{i=1}^{\kappa} \bigg(d_g(x_{i,\alpha}, y_\alpha)^{-2} - \frac{|\mathcal{U}_\alpha(y_\alpha)|^{2^\star-2}}{\gamma_2 \varepsilon(1-\varepsilon)} - A_\varepsilon \bigg) \Phi_{i,\alpha}^{\varepsilon}(y_\alpha) \\ + \bigg(\frac{r_{\kappa+1,\alpha}(y_\alpha)^{-2}}{N \gamma_1^{2(1-\varepsilon)}} - \frac{|\mathcal{U}_\alpha(y_\alpha)|^{2^\star-2}}{\gamma_2 \varepsilon(1-\varepsilon)} - A_\varepsilon \bigg) \nu_{\kappa,\alpha}^{(1-2\varepsilon)(n-2)/2} \Psi_{\alpha,\varepsilon}(y_\alpha) \\ + \bigg(\frac{r_\alpha(y_\alpha)^{-2}}{N \gamma_1^{2\varepsilon}} - \frac{|\mathcal{U}_\alpha(y_\alpha)|^{2^\star-2}}{\gamma_2 \varepsilon(1-\varepsilon)} - A_\varepsilon \bigg) \varphi_{\alpha,\varepsilon}(y_\alpha), \end{aligned} \tag{6-20}$$

where

$$A_\varepsilon = \frac{p \|A_\alpha\|_\infty + (1+\gamma_3 \varepsilon)(1-\varepsilon)}{\gamma_2 \varepsilon (1-\varepsilon)}.$$

We let in the following $1 \leq i \leq \kappa$ be such that $y_\alpha \in \Omega_{i,\alpha}^{\varepsilon}$. Then we deduce from (6-20) that

$$\begin{aligned} 0 \geq \bigg(d_g(x_{i,\alpha}, y_\alpha)^{-2} - \frac{\kappa}{\gamma_2 \varepsilon(1-\varepsilon)} |\mathcal{U}_\alpha(y_\alpha)|^{2^\star-2} - \kappa A_\varepsilon \bigg) \Phi_{i,\alpha}^{\varepsilon}(y_\alpha) \\ + \bigg(\frac{r_{\kappa+1,\alpha}(y_\alpha)^{-2}}{N \gamma_1^{2(1-\varepsilon)}} - \frac{|\mathcal{U}_\alpha(y_\alpha)|^{2^\star-2}}{\gamma_2 \varepsilon(1-\varepsilon)} - A_\varepsilon \bigg) \nu_{\kappa,\alpha}^{(1-2\varepsilon)(n-2)/2} \Psi_{\alpha,\varepsilon}(y_\alpha) \\ + \bigg(\frac{r_\alpha(y_\alpha)^{-2}}{N \gamma_1^{2\varepsilon}} - \frac{|\mathcal{U}_\alpha(y_\alpha)|^{2^\star-2}}{\gamma_2 \varepsilon(1-\varepsilon)} - A_\varepsilon \bigg) \varphi_{\alpha,\varepsilon}(y_\alpha). \end{aligned} \tag{6-21}$$

From (6-15), we know that

$$\|\mathcal{U}_\infty\|_\infty = o(|\mathcal{U}_\alpha(y_\alpha)|) \quad \text{and} \quad \mu_\alpha^{(n-2)/2} = o(|\mathcal{U}_\alpha(y_\alpha)|), \tag{6-22}$$

and that

$$B_{j,\alpha}(y_\alpha) = o(|\mathcal{U}_\alpha(y_\alpha)|) \tag{6-23}$$

for all $1 \leq j \leq \kappa$ since

$$B_{j,\alpha}(y_\alpha) = O(\Phi_{j,\alpha}^{\varepsilon}(y_\alpha)) \tag{6-24}$$

for all $1 \leq j \leq \kappa$. From (6-17), we have

$$R_{\kappa+1,\alpha}(y_\alpha)^2 B_{j,\alpha}(y_\alpha)^{2^\star-2} = o(1) \tag{6-25}$$

for all $\kappa+1 \leq j \leq N$. Thus we can deduce from the last equation in (4-4) of Proposition 4.2 together with (6-22), (6-23), and (6-25), that

$$\mathcal{D}_\alpha(y_\alpha)^2 |\mathcal{U}_\alpha(y_\alpha)|^{2^\star-2} = o(1). \tag{6-26}$$

Using (6-16) and (6-26), we can transform (6-21) into

$$\begin{aligned}0 \geq \left(d_g(x_{i,\alpha}, y_\alpha)^{-2} - \frac{\kappa}{\gamma_2\varepsilon(1-\varepsilon)}|\mathcal{U}_\alpha(y_\alpha)|^{2^\star-2} - \kappa A_\varepsilon\right)\Phi_{i,\alpha}^\varepsilon(y_\alpha) \\ + \left(\frac{r_{\kappa+1,\alpha}(y_\alpha)^{-2}}{N\gamma_1^{2(1-\varepsilon)}} - \frac{|\mathcal{U}_\alpha(y_\alpha)|^{2^\star-2}}{\gamma_2\varepsilon(1-\varepsilon)} - A_\varepsilon\right)\nu_{\kappa,\alpha}^{(1-2\varepsilon)(n-2)/2}\Psi_{\alpha,\varepsilon}(y_\alpha) \\ + \left(\frac{1}{N\gamma_1^{2\varepsilon}} + o(1)\right)r_\alpha(y_\alpha)^{-2}\varphi_{\alpha,\varepsilon}(y_\alpha). \qquad (6\text{-}27)\end{aligned}$$

Since $(\mathcal{I}_\kappa)$ holds true, we can prove with (6-22) and (6-23) that

$$|\mathcal{U}_\alpha(y_\alpha)|^{2^\star-2} = O\big(\mu_{\kappa,\alpha}^2 R_{\kappa+1,\alpha}(y_\alpha)^{-4}\big). \qquad (6\text{-}28)$$

This implies that

$$R_{\kappa+1,\alpha}(y_\alpha)^2|\mathcal{U}_\alpha(y_\alpha)|^{2^\star-2} \to 0 \quad \text{as } \alpha \to +\infty. \qquad (6\text{-}29)$$

Indeed, if it is not the case, we would have from (6-28) that

$$R_{\kappa+1,\alpha}(y_\alpha) = O(\mu_{k,\alpha})$$

and thanks to (6-26) that there exists $j \in \{1, \dots, \kappa\}$ such that

$$d_g(x_{j,\alpha}, y_\alpha) + \mu_{j,\alpha} = o(R_{\kappa+1,\alpha}(y_\alpha)).$$

In particular, we get a contradiction since $\mu_{j,\alpha} \geq \mu_{\kappa,\alpha}$. As a remark, (6-28) also implies that

$$R_{\kappa+1,\alpha}(y_\alpha) \to 0 \quad \text{as } \alpha \to +\infty, \qquad (6\text{-}30)$$

due to (6-23). Now, thanks to (6-29) and (6-30), we deduce from (6-27) that

$$\begin{aligned}0 \geq \left(d_g(x_{i,\alpha}, y_\alpha)^{-2} - \frac{\kappa}{\gamma_2\varepsilon(1-\varepsilon)}|\mathcal{U}_\alpha(y_\alpha)|^{2^\star-2} - \kappa A_\varepsilon\right)\Phi_{i,\alpha}^\varepsilon(y_\alpha) \\ + \left(\frac{1}{N\gamma_1^{2(1-\varepsilon)}} + o(1)\right)\nu_{\kappa,\alpha}^{(1-2\varepsilon)(n-2)/2}r_{\kappa+1,\alpha}(y_\alpha)^{-2}\Psi_{\alpha,\varepsilon}(y_\alpha) \\ + \left(\frac{1}{N\gamma_1^{2\varepsilon}} + o(1)\right)r_\alpha(y_\alpha)^{-2}\varphi_{\alpha,\varepsilon}(y_\alpha). \qquad (6\text{-}31)\end{aligned}$$

If $y_\alpha \notin \tilde{\Omega}_{i,\alpha}^\varepsilon$, we transform (6-31) into

$$\begin{aligned}0 \geq \left(1 + o(1) - \frac{\kappa D(\varepsilon)}{\gamma_2\varepsilon(1-\varepsilon)} - \kappa A_\varepsilon d_g(x_{i,\alpha}, y_\alpha)^2\right)d_g(x_{i,\alpha}, y_\alpha)^{-2}\Phi_{i,\alpha}^\varepsilon(y_\alpha) \\ + \left(\frac{1}{N\gamma_1^{2(1-\varepsilon)}} + o(1)\right)\nu_{\kappa,\alpha}^{(1-2\varepsilon)(n-2)/2}r_{\kappa+1,\alpha}(y_\alpha)^{-2}\Psi_{\alpha,\varepsilon}(y_\alpha) + \left(\frac{1}{N\gamma_1^{2\varepsilon}} + o(1)\right)r_\alpha(y_\alpha)^{-2}\varphi_{\alpha,\varepsilon}(y_\alpha)\end{aligned}$$

by using (6-22) and (6-23). This leads to

$$\left(\frac{1}{N\gamma_1^{2\varepsilon}} + o(1)\right)r_\alpha(y_\alpha)^{-2}\varphi_{\alpha,\varepsilon}(y_\alpha) = O\big(\mu_{i,\alpha}^{(1-2\varepsilon)(n-2)/2}\big),$$

thanks to our choice of $D(\varepsilon)$. From (5-8), (6-16), and the definition of $\varphi_{\alpha,\varepsilon}$, we clearly get a contradiction.

Thus $y_\alpha \in \tilde{\Omega}^\varepsilon_{i,\alpha}$. Coming back to (6-31), we obtain in this situation that

$$\left(\frac{1}{N\gamma_1^{2(1-\varepsilon)}}+o(1)\right)\nu_{\kappa,\alpha}^{(1-2\varepsilon)(n-2)/2}\Psi_{\alpha,\varepsilon}(y_\alpha) \le \left(\frac{\kappa}{\gamma_2\varepsilon(1-\varepsilon)}|\mathcal{U}_\alpha(y_\alpha)|^{2^\star-2}+\kappa A_\varepsilon\right) r_{\kappa+1,\alpha}(y_\alpha)^2\Phi^\varepsilon_{i,\alpha}(y_\alpha).$$

Using (6-29), (6-30), and the definition of $\nu_{\kappa,\alpha}$, this leads to

$$\left(\frac{1}{N\gamma_1^{2(1-\varepsilon)}}+o(1)\right)\nu_{\kappa,\alpha}^{(1-2\varepsilon)(n-2)/2}\Psi_{\alpha,\varepsilon}(y_\alpha) = o(\Phi^\varepsilon_{i,\alpha}(y_\alpha)) = o\big(\nu_{\kappa,\alpha}^{(1-2\varepsilon)(n-2)/2}\Psi_{\alpha,\varepsilon}(y_\alpha)\big)$$

and this is again a contradiction. Thus (6-15) cannot hold true and we get the equation in Step 2 from (5-8). This ends the proof of Step 2. ☐

Step 3. *There exists $C_0 > 0$ such that*

$$|\mathcal{U}_\alpha(x)| \le C_0\left(\sum_{i=1}^{\kappa} B_{i,\alpha}(x) + \|\mathcal{U}_\infty\|_\infty + \nu_{\kappa,\alpha}^{(n-2)/2} R_{\kappa+1,\alpha}(x)^{2-n}\right) \quad \textit{for all } x \in M \textit{ and all } \alpha > 0.$$

Proof of Step 3. We let (y_α) be a sequence of points in M and we aim to prove that

$$\limsup_{\alpha\to+\infty} \frac{|\mathcal{U}_\alpha(y_\alpha)|}{\sum_{i=1}^{\kappa} B_{i,\alpha}(y_\alpha) + \|\mathcal{U}_\infty\|_\infty + \nu_{\kappa,\alpha}^{(n-2)/2} R_{\kappa+1,\alpha}(y_\alpha)^{2-n}} < +\infty. \tag{6-32}$$

Since $(\mathcal{I}_\kappa)$ holds true, it is clear that (6-32) also holds true as soon as

$$\mu_{\kappa,\alpha}^{(n-2)/2} R_{\kappa+1,\alpha}(y_\alpha)^{2-n} = O(B_{i,\alpha}(y_\alpha))$$

for some $1 \le i \le \kappa$. By contradiction we assume in what follows that (6-32) does not hold true. Thus we assume from now on that

$$R_{\kappa+1,\alpha}(y_\alpha)^2 = o(\mu_{i,\alpha}\mu_{\kappa,\alpha}) + o\left(\frac{\mu_{\kappa,\alpha}}{\mu_{i,\alpha}} d_g(x_{i,\alpha}, y_\alpha)^2\right) \tag{6-33}$$

for all $1 \le i \le \kappa$. This implies in particular that

$$R_{\kappa+1,\alpha}(y_\alpha) \to 0 \quad \text{as } \alpha \to +\infty. \tag{6-34}$$

Thanks to the last equation in (4-4) and to (6-33), we can assume that

$$R_\alpha(y_\alpha) = R_{\kappa+1,\alpha}(y_\alpha) \quad \text{and} \quad \frac{R_{\kappa+1,\alpha}(y_\alpha)}{\nu_{\kappa,\alpha}} \to +\infty \quad \text{as } \alpha \to +\infty. \tag{6-35}$$

Indeed, otherwise, (6-32) holds true. We let $\lambda > 1$ be such that $\lambda p\|A\|_\infty \notin \mathrm{Sp}(\Delta_g)$, where $\mathrm{Sp}(\Delta_g)$ is the spectrum of Δ_g, and let G be the Green's function of $\mathfrak{L}_g = \Delta_g - \lambda p\|A\|_\infty$. There exist $C_1 > 1$, $C_2 > 0$ and $C_3 > 0$ such that

$$\frac{1}{C_1} d_g(x,y)^{2-n} - C_2 \le G(x,y) \le C_1 d_g(x,y)^{2-n} \tag{6-36}$$

and

$$|\nabla G(x,y)|_g \le C_3 d_g(x,y)^{1-n} \tag{6-37}$$

for all $x, y \in M$, $x \ne y$. We let $x_0 \in \mathcal{S}$ be such that $d_g(y_\alpha, x_0) \le \delta + o(1)$; such an x_0 does exist thanks to (6-34). We choose $\delta > 0$ such that

$$d_g(x,y) \ge 4\delta \tag{6-38}$$

for all distinct $x, y \in \mathcal{S}$, and such that

$$\delta \le \tfrac{1}{4}(C_1C_2)^{-1/(n-2)}, \tag{6-39}$$

where C_1 and C_2 are as in (6-36). We write with Green's representation formula that

$$\begin{aligned}\sum_{i=1}^{p} u_{i,\alpha}(y_\alpha) &= \int_{B_{x_0}(2\delta)} G(y_\alpha, x)\mathfrak{L}_g\Big(\sum_{i=1}^{p} u_{i,\alpha}\Big)(x)\,dv_g(x)\\ &\quad+\int_{\partial B_{x_0}(2\delta)} G(y_\alpha, x)\partial_\nu\Big(\sum_{i=1}^{p} u_{i,\alpha}\Big)(x)\,d\sigma_g(x) - \int_{\partial B_{x_0}(2\delta)} \partial_\nu G(y_\alpha, x)\Big(\sum_{i=1}^{p} u_{i,\alpha}\Big)(x)\,d\sigma_g(x).\end{aligned} \tag{6-40}$$

Since $\lambda > 1$, we get with (1-1) that

$$\mathfrak{L}_g\Big(\sum_{i=1}^{p} u_{i,\alpha}\Big) \le |\mathcal{U}_\alpha|^{2^\star-2}\sum_{i=1}^{p} u_{i,\alpha}.$$

We have $G(y_\alpha, x) \ge 0$ in $B_{x_0}(2\delta)$ for α large by (6-36) and (6-39). Thus we can write

$$\int_{B_{x_0}(2\delta)} G(y_\alpha, x)\mathfrak{L}_g\Big(\sum_{i=1}^{p} u_{i,\alpha}\Big)(x)\,dv_g(x) \le C_1\int_M d_g(y_\alpha, x)^{2-n}|\mathcal{U}_\alpha(x)|^{2^\star-2}\sum_{i=1}^{p} u_{i,\alpha}(x)\,dv_g(x). \tag{6-41}$$

From (6-38), we also know that

$$d_g\big(x_{i,\alpha}, \partial B_{x_0}(2\delta)\big) \ge \delta$$

for α large. In particular, we can control the boundary terms in (6-40) thanks to Proposition 5.1 and standard elliptic theory. We thus obtain that

$$|\mathcal{U}_\alpha(y_\alpha)| = O\Big(\max\big\{\mu_\alpha^{(n-2)/2};\ \|\mathcal{U}_\infty\|_\infty\big\}\Big) + O\Big(\int_M d_g(y_\alpha, x)^{2-n}|\mathcal{U}_\alpha(x)|^{2^\star-1}\,dv_g(x)\Big). \tag{6-42}$$

We can now write thanks to Step 2 that

$$\begin{aligned}\int_M d_g(y_\alpha, x)^{2-n}|\mathcal{U}_\alpha(x)|^{2^\star-1}dv_g(x) &= O\Big(\sum_{i=1}^{\kappa}\int_M d_g(y_\alpha, x)^{2-n}\Phi_{i,\alpha}^{\varepsilon}(x)^{2^\star-1}dv_g(x)\Big)\\ &\quad+ O\Big(\max\big\{\|\mathcal{U}_\infty\|_\infty^{2^\star-1};\ \mu_\alpha^{(1-2\varepsilon)(n+2)/2}\big\}\int_M \frac{d_g(y_\alpha, x)^{2-n}}{r_\alpha(x)^{(n+2)\varepsilon}}dv_g(x)\Big)\\ &\quad+ O\Big(\nu_{\kappa,\alpha}^{(1-2\varepsilon)(n+2)/2}\int_{\{r_{\kappa+1,\alpha}(x)\ge R_0\nu_{\kappa,\alpha}\}} \frac{d_g(y_\alpha, x)^{2-n}}{r_{\kappa+1,\alpha}(x)^{(n+2)(1-\varepsilon)}}dv_g(x)\Big)\\ &\quad+ O\Big(\int_{\{r_{\kappa+1,\alpha}(x)\le R_0\nu_{\kappa,\alpha}\}} d_g(y_\alpha, x)^{2-n}|\mathcal{U}^\alpha|^{2^\star-1}dv_g(x)\Big).\end{aligned} \tag{6-43}$$

Since $0 < \varepsilon < \dfrac{1}{n+2}$, it follows from Giraud's lemma that

$$\int_M d_g(y_\alpha, x)^{2-n} r_\alpha(x)^{-(n+2)\varepsilon}\,dv_g(x) = O(1). \tag{6-44}$$

We can also write, for $1 \le i \le \kappa$,

$$\begin{aligned}\int_M d_g(y_\alpha, x)^{2-n} \Phi_{i,\alpha}^{\varepsilon}(x)^{2^\star-1} dv_g(x) &= O\left(\mu_{i,\alpha}^{-(1-2\varepsilon)(n+2)/2} \int_{\{d_g(x_{i,\alpha},x)\le \mu_{i,\alpha}\}} d_g(y_\alpha, x)^{2-n} d_g(x_{i,\alpha}, x)^{-(n+2)\varepsilon} dv_g(x)\right) \\ &\quad + O\left(\mu_{i,\alpha}^{(1-2\varepsilon)(n+2)/2} \int_{\{d_g(x_{i,\alpha},x)\ge \mu_{i,\alpha}\}} d_g(y_\alpha, x)^{2-n} d_g(x_{i,\alpha}, x)^{-(n+2)(1-\varepsilon)} dv_g(x)\right)\end{aligned}$$

thanks to (5-8) and (6-4). Direct computations, using Giraud's lemma and the inequalities $0 < \varepsilon < \dfrac{1}{n+2}$, lead then to

$$\int_M d_g(y_\alpha, x)^{2-n} \Phi_{i,\alpha}^{\varepsilon}(x)^{2^\star-1}\, dv_g(x) = O(B_{i,\alpha}(y_\alpha)). \tag{6-45}$$

By direct computations, using Giraud's lemma, the inequalities $0 < \varepsilon < \dfrac{1}{n+2}$ and (6-35), we also get

$$\nu_{\kappa,\alpha}^{(1-2\varepsilon)(n+2)/2} \int_{\{r_{\kappa+1,\alpha}(x)\ge R_0\nu_{\kappa,\alpha}\}} d_g(y_\alpha, x)^{2-n} r_{\kappa+1,\alpha}(x)^{-(n+2)(1-\varepsilon)} dv_g(x) = O\big(\nu_{\kappa,\alpha}^{(n-2)/2} R_{\kappa+1,\alpha}(y_\alpha)^{2-n}\big), \tag{6-46}$$

while, using (6-35), the fact that $\nu_{\kappa,\alpha} \ge \mu_{\kappa+1,\alpha}$, and Hölder's inequalities, we also have

$$\int_{\{r_{\kappa+1,\alpha}(x)\le R_0\nu_{\kappa,\alpha}\}} d_g(y_\alpha, x)^{2-n} |\mathcal{U}_\alpha|^{2^\star-1} dv_g(x) = O\big(\nu_{\kappa,\alpha}^{(n-2)/2} R_{\kappa+1,\alpha}(y_\alpha)^{2-n}\big). \tag{6-47}$$

Coming back to (6-42) with (6-43)-(6-47), we obtain a contradiction with the assumption that (6-32) does not hold true. This proves Step 3. □

The fourth step in the proof of Proposition 6.1 is as follows. The constants $C > 0$ in the statement of this step and its proof are independent of α and built on C_κ. They may change from line to line.

Step 4. *There exists $C > 0$ such that for any sequence (y_α) of points in M,*

$$\begin{aligned}|\mathcal{U}_\alpha(y_\alpha) - \mathcal{U}_\infty(y_\alpha) - \sum_{i=1}^{\kappa} B_{i,\alpha}(y_\alpha)\Lambda_i| \\ \le \varepsilon_\alpha \|\mathcal{U}_\infty\|_\infty + o\left(\sum_{i=1}^{\kappa} B_{i,\alpha}(y_\alpha)\right) + C\big(\mu_\alpha^{(n-2)/2} + \nu_{\kappa,\alpha}^{(n-2)/2} R_{\kappa+1,\alpha}(y_\alpha)^{2-n}\big),\end{aligned}$$

where $\varepsilon_\alpha \to 0$ as $\alpha \to +\infty$.

Proof of Step 4. Let (y_α) be a sequence of points in M. Assume first that

$$R_{\kappa+1,\alpha}(y_\alpha) = O(\nu_{\kappa,\alpha}). \tag{6-48}$$

If $R_{\kappa+1,\alpha}(y_\alpha) = \mathfrak{D}_\alpha(y_\alpha)$, we can apply the last equation in (4-4) of Proposition 4.2 to obtain

$$\left|\mathcal{U}_\alpha(y_\alpha) - \mathcal{U}_\infty(y_\alpha) - \sum_{i=1}^{\kappa} B_{i,\alpha}(y_\alpha)\Lambda_i\right| \le C\nu_{\kappa,\alpha}^{(n-2)/2} R_{\kappa+1,\alpha}(y_\alpha)^{2-n}.$$

In particular, the estimate of Step 4 holds true. If $\mathfrak{D}_\alpha(y_\alpha) < R_{\kappa+1,\alpha}(y_\alpha)$, then from Step 1 and (6-48) we obtain the existence of some $1 \le i \le \kappa$ such that

$$d_g(x_{i,\alpha}, y_\alpha) + \mu_{i,\alpha} < R_{\kappa+1,\alpha}(y_\alpha) = O(\mu_{\kappa,\alpha}).$$

This implies the following facts:

$$\mu_{i,\alpha} = O(\mu_{\kappa,\alpha}),$$
$$d_g(x_{i,\alpha}, y_\alpha) = O(\mu_{i,\alpha}),$$
$$R_{\kappa+1,\alpha}(y_\alpha) \ge \mu_{i,\alpha}.$$

Using (4-5) in Proposition 4.2 we get

$$\left| \mathcal{U}_\alpha(y_\alpha) - \mathcal{U}_\infty(y_\alpha) - \sum_{i=1}^{\kappa} B_{i,\alpha}(y_\alpha)\Lambda_i \right| = o(B_{i,\alpha}(y_\alpha)),$$

and the estimate of Step 4 holds also in this case. As a consequence, we may assume below that

$$\frac{R_{\kappa+1,\alpha}(y_\alpha)}{\nu_{\kappa,\alpha}} \to +\infty \quad \text{as } \alpha \to +\infty. \tag{6-49}$$

The rest of the proof is based on controlling the different terms we get from Green's representation formula. We let $\mathcal{H}$ be the Green's function of the Laplacian on M normalized such that $\mathcal{H}(x, y) \ge 1$ for all $x, y \in M$, $x \ne y$. Then (5-25) holds and moreover

$$(x, y) \overset{\Phi}{\mapsto} d_g(x, y)^{n-2}\mathcal{H}(x, y)$$

extends to a continuous function in $M \times M$ whose value on the diagonal is

$$\Phi(x, x) = \frac{1}{(n-2)\omega_{n-1}}$$

for all x. Now we write, for any $i \in \{1, \dots, p\}$,

$$u_{i,\alpha}(y_\alpha) - u_{i,\infty}(y_\alpha) = \frac{1}{V_g}\int_M (u_{i,\alpha} - u_{i,\infty})\, dv_g + \int_M \mathcal{H}(x, y_\alpha)\Delta_g(u_{i,\alpha} - u_{i,\infty})(x)\, dv_g(x). \tag{6-50}$$

Since $(\mathcal{I}_\kappa)$ holds true, we can write

$$\left| \int_M (u_{i,\alpha} - u_{i,\infty})\, dv_g \right| \le C\mu_\alpha^{(n-2)/2} + \varepsilon_\alpha \|\mathcal{U}_\infty\|_\infty,$$

where $\varepsilon_\alpha \to 0$ as $\alpha \to +\infty$. Thus we can transform (6-50) into

$$\left| u_{i,\alpha}(y_\alpha) - u_{i,\infty}(y_\alpha) - \int_M \mathcal{H}(x, y_\alpha)\Delta_g(u_{i,\alpha} - u_{i,\infty})(x)\, dv_g(x) \right| \le C\mu_\alpha^{(n-2)/2} + \varepsilon_\alpha \|\mathcal{U}_\infty\|_\infty. \tag{6-51}$$

In view of the equations satisfied by the $\mathcal{U}_\alpha$'s and $\mathcal{U}_\infty$, we can now write

$$\begin{aligned}
\Delta_g(u_{i,\alpha} - u_{i,\infty}) &= |\mathcal{U}_\alpha|^{2^\star-2}u_{i,\alpha} - |\mathcal{U}_\infty|^{2^\star-2}u_{i,\infty} - \sum_{j=1}^{p} A_{ij}^\alpha u_{j,\alpha} + \sum_{j=1}^{p} A_{ij}u_{j,\infty} \\
&= |\mathcal{U}_\alpha - \mathcal{U}_\infty|^{2^\star-2}(u_{i,\alpha} - u_{i,\infty}) + \left(|\mathcal{U}_\alpha|^{2^\star-2} - |\mathcal{U}_\alpha - \mathcal{U}_\infty|^{2^\star-2}\right)(u_{i,\alpha} - u_{i,\infty}) \\
&\qquad + \left(|\mathcal{U}_\alpha|^{2^\star-2} - |\mathcal{U}_\infty|^{2^\star-2}\right)u_{i,\infty} - \sum_{j=1}^{p} A_{ij}^\alpha(u_{j,\alpha} - u_{j,\infty}) + \sum_{j=1}^{p}(A_{ij} - A_{ij}^\alpha)u_{j,\infty}.
\end{aligned}$$

Thus we obtain

$$\begin{aligned}\Big|u_{i,\alpha}(y_\alpha)-u_{i,\infty}(y_\alpha)-\int_M \mathcal{H}_{y_\alpha}|\mathcal{U}_\alpha-\mathcal{U}_\infty|^{2^\star-2}(u_{i,\alpha}-u_{i,\infty})\,dv_g\Big| \\ \le \Big|\int_M \mathcal{H}_{y_\alpha}\Big(|\mathcal{U}_\alpha|^{2^\star-2}-|\mathcal{U}_\alpha-\mathcal{U}_\infty|^{2^\star-2}\Big)(u_{i,\alpha}-u_{i,\infty})\,dv_g\Big| \\ +\Big|\int_M \mathcal{H}_{y_\alpha}\Big(|\mathcal{U}_\alpha|^{2^\star-2}-|\mathcal{U}_\infty|^{2^\star-2}\Big)u_{i,\infty}dv_g\Big|+\Big|\sum_{j=1}^p\int_M \mathcal{H}_{y_\alpha}\Big(A_{ij}-A_{ij}^\alpha\Big)(x)u_{j,\infty}dv_g\Big| \\ +\Big|\sum_{j=1}^p\int_M \mathcal{H}_{y_\alpha}A_{ij}^\alpha\big(u_{j,\alpha}-u_{j,\infty}\big)\,dv_g\Big|+C\mu_\alpha^{(n-2)/2}+\varepsilon_\alpha|\mathcal{U}_\infty|_\infty,\end{aligned}\tag{6-52}$$

where $\mathcal{H}_{y_\alpha}(x)=\mathcal{H}(y_\alpha,x)$ for all x. The convergence of the A_α's to A, together with (5-25), implies that

$$\sum_{j=1}^p\int_M \mathcal{H}_{y_\alpha}(A_{ij}-A_{ij}^\alpha)u_{j,\infty}dv_g=\varepsilon_\alpha\|\mathcal{U}_\infty\|_\infty,\tag{6-53}$$

where $\varepsilon_\alpha\to 0$ as $\alpha\to+\infty$. Now we get with (5-25) that

$$\Big|\sum_{j=1}^p\int_M \mathcal{H}_{y_\alpha}A_{ij}^\alpha\,\big(u_{j,\alpha}-u_{j,\infty}\big)\,dv_g\Big|\le pC\|A_\alpha\|_\infty\int_M d_g(y_\alpha,x)^{2-n}|\mathcal{U}_\alpha(x)-\mathcal{U}_\infty(x)|\,dv_g(x).$$

Thanks to $(\mathcal{I}_\kappa)$, we can write

$$|\mathcal{U}_\alpha(x)-\mathcal{U}_\infty(x)|\le D_1\Big(\sum_{j=1}^\kappa B_{j,\alpha}(x)+\mu_{\kappa,\alpha}^{(n-2)/2}R_{\kappa+1,\alpha}(x)^{2-n}+\mu_\alpha^{(n-2)/2}\Big)+\varepsilon_\alpha\|\mathcal{U}_\infty\|_\infty$$

for some $D_1>0$, where $\varepsilon_\alpha\to 0$ as $\alpha\to+\infty$, while, thanks to Step 3, we have

$$|\mathcal{U}_\alpha(x)-\mathcal{U}_\infty(x)|\le D_2\Big(\sum_{j=1}^\kappa B_{j,\alpha}(x)+\nu_{\kappa,\alpha}^{(n-2)/2}R_{\kappa+1,\alpha}(x)^{2-n}+\|\mathcal{U}_\infty\|_\infty\Big)$$

for some $D_2>0$. Thus we can write

$$\begin{aligned}\Big|\sum_{j=1}^p\int_M \mathcal{H}_{y_\alpha}A_{ij}^\alpha\,\big(u_{j,\alpha}-u_{j,\infty}\big)\,dv_g\Big|\le C\Big(\sum_{j=1}^\kappa\int_M d_g(y_\alpha,x)^{2-n}B_{j,\alpha}(x)\,dv_g(x)\Big)+\varepsilon_\alpha\|\mathcal{U}_\infty\|_\infty+C\mu_\alpha^{(n-2)/2} \\ +C\mu_{\kappa,\alpha}^{(n-2)/2}\int_{\{R_{\kappa+1,\alpha}(x)\ge\eta_\alpha\}}d_g(y_\alpha,x)^{2-n}R_{\kappa+1,\alpha}(x)^{2-n}dv_g(x) \\ +C\nu_{\kappa,\alpha}^{(n-2)/2}\int_{\{R_{\kappa+1,\alpha}(x)\le\eta_\alpha\}}d_g(y_\alpha,x)^{2-n}R_{\kappa+1,\alpha}(x)^{2-n}dv_g(x),\end{aligned}$$

where $\eta_\alpha=2\mathrm{diam}_gM$ if $\mathcal{U}_\infty\equiv 0$, $\eta_\alpha=\mu_{\kappa,\alpha}^{1/2}$ otherwise, and diam_gM is the diameter of M with respect to g. Simple computations, using Giraud's lemma, then lead to the estimate

$$\begin{aligned}\Big|\sum_{j=1}^p\int_M \mathcal{H}_{y_\alpha}A_{ij}^\alpha(u_{j,\alpha}-u_{j,\infty})\,dv_g\Big| \\ \le o\Big(\sum_{j=1}^\kappa|B_j^\alpha(y_\alpha)|\Big)+C\mu_\alpha^{(n-2)/2}+\varepsilon_\alpha\|\mathcal{U}_\infty\|_\infty+C\nu_{\kappa,\alpha}^{(n-2)/2}R_{\kappa+1,\alpha}(y_\alpha)^{2-n}.\end{aligned}\tag{6-54}$$

If $\mathcal{U}_\infty \equiv 0$, we have

$$\int_M \mathcal{H}_{y_\alpha}\big(|\mathcal{U}_\alpha|^{2^\star-2} - |\mathcal{U}_\infty|^{2^\star-2}\big)u_{i,\infty}dv_g = 0,$$

while, if $\mathcal{U}_\infty \not\equiv 0$, we write, thanks to Proposition 5.1,

$$\begin{aligned}\int_M \mathcal{H}_{y_\alpha}\big(|\mathcal{U}_\alpha|^{2^\star-2}-|\mathcal{U}_\infty|^{2^\star-2}\big)u_{i,\infty}dv_g &= o(1) + \int_{\{\mathcal{D}_\alpha \le \mu_\alpha^{1/4}\}} \mathcal{H}_{y_\alpha}\big(|\mathcal{U}_\alpha|^{2^\star-2} - |\mathcal{U}_\infty|^{2^\star-2}\big)u_{i,\infty}dv_g \\ &= o(1) + O\bigg(\int_{\{\mathcal{D}_\alpha \le \mu_\alpha^{1/4}\}} d_g(x, y_\alpha)^{2-n}|\mathcal{U}_\alpha(x)|^{2^\star-2}dv_g(x)\bigg). \quad (6\text{-}55)\end{aligned}$$

Now we use Step 3 and we briefly distinguish the $n = 3, 4, 5$, and $n \ge 6$ cases in the forthcoming computations. We let $(R_\alpha)_\alpha$ be suitably chosen such that $R_\alpha \to +\infty$ as $\alpha \to +\infty$. Assuming that $n = 3, 4, 5$, we write with (6-55) that

$$\begin{aligned}&\int_M \mathcal{H}_{y_\alpha}\big(|\mathcal{U}_\alpha|^{2^\star-2} - |\mathcal{U}_\infty|^{2^\star-2}\big)u_{i,\infty}dv_g \\ &\qquad = o(1) + O\bigg(\int_{\{R_{\kappa+1,\alpha} \le R_{\kappa+1,\alpha}(y_\alpha)/R_\alpha\}} d_g(x, y_\alpha)^{2-n}|\mathcal{U}_\alpha(x)|^{2^\star-2}\, dv_g(x)\bigg) \\ &\qquad\quad + O\bigg(\int_{\{R_{\kappa+1,\alpha} \ge R_{\kappa+1,\alpha}(y_\alpha)/R_\alpha\}\cap\{\mathcal{D}_\alpha \le \mu_\alpha^{1/4}\}} d_g(x, y_\alpha)^{2-n}|\mathcal{U}_\alpha(x)|^{2^\star-2}dv_g(x)\bigg) \\ &\qquad = o(1) + O\bigg(\sum_{j=1}^{\kappa}\int_M d_g(x, y_\alpha)^{2-n}B_{j,\alpha}(x)^{2^\star-2}dv_g(x)\bigg) \\ &\qquad\quad + O\bigg(\nu_{\kappa,\alpha}^2\int_{\{R_{\kappa+1,\alpha} \ge R_{\kappa+1,\alpha}(y_\alpha)/R_\alpha\}} d_g(x, y_\alpha)^{2-n}R_{\kappa+1,\alpha}(x)^{-4}dv_g(x)\bigg) \\ &\qquad = o(1) + o\bigg(\sum_{j=1}^{\kappa}B_{j,\alpha}(y_\alpha)\bigg) + o\big(\nu_{\kappa,\alpha}^{(n-2)/2}R_{\kappa+1,\alpha}(y_\alpha)^{2-n}\big),\end{aligned}$$

and, assuming that $n \ge 6$, since $2^\star - 2 \in (0, 1]$ in this case, we get from (6-55) that

$$\begin{aligned}\int_M \mathcal{H}_{y_\alpha}\big(|\mathcal{U}_\alpha|^{2^\star-2}-|\mathcal{U}_\infty|^{2^\star-2}\big)u_{i,\infty}dv_g &= o(1) + O\bigg(\int_{\{\mathcal{D}_\alpha \le \mu_\alpha^{1/4}\}} d_g(x, y_\alpha)^{2-n}|\mathcal{U}_\alpha(x)|dv_g(x)\bigg) \\ &= o(1) + O\bigg(\sum_{j=1}^{\kappa}\int_M d_g(x, y_\alpha)^{2-n}B_{j,\alpha}(x)\, dv_g(x)\bigg) \\ &\quad + O\bigg(\nu_{\kappa,\alpha}^{(n-2)/2}\int_M d_g\,(x, y_\alpha)^{2-n}\, R_{\kappa+1,\alpha}(x)^{2-n}dv_g(x)\bigg) \\ &= o(1) + o\bigg(\sum_{j=1}^{\kappa}B_{j,\alpha}(y_\alpha)\bigg) + o\Big(\nu_{\kappa,\alpha}^{(n-2)/2}R_{\kappa+1,\alpha}(y_\alpha)^{2-n}\Big).\end{aligned}$$

Thus, in all cases,

$$\int_M \mathcal{H}_{y_\alpha}\big(|\mathcal{U}_\alpha|^{2^\star-2} - |\mathcal{U}_\infty|^{2^\star-2}\big)u_{i,\infty}dv_g \\ = o\bigg(\sum_{j=1}^{\kappa} B_{j,\alpha}(y_\alpha)\bigg) + o\big(\nu_{\kappa,\alpha}^{(n-2)/2} R_{\kappa+1,\alpha}(y_\alpha)^{2-n}\big) + \varepsilon_\alpha \|\mathcal{U}_\infty\|_\infty. \quad (6\text{-}56)$$

Similarly, if $\mathcal{U}_\infty \equiv 0$, then

$$\int_M \mathcal{H}_{y_\alpha}\big(|\mathcal{U}_\alpha|^{2^\star-2} - |\mathcal{U}_\alpha - \mathcal{U}_\infty|^{2^\star-2}\big)(u_{i,\alpha} - u_{i,\infty})\,dv_g = 0,$$

while, if $\mathcal{U}^\infty \not\equiv 0$, we can write

$$\begin{aligned}\int_M \mathcal{H}_{y_\alpha}\big(|\mathcal{U}_\alpha|^{2^\star-2} - |\mathcal{U}_\alpha - \mathcal{U}_\infty|^{2^\star-2}\big)(u_{i,\alpha} - u_{i,\infty})\,dv_g & \\ &= o(1) + O\bigg(\int_{\{R_\alpha(x)\le \mu_\alpha^{1/4}\}} d_g(y_\alpha, x)^{2-n}|\mathcal{U}_\alpha(x)|^{2^\star-2}\,dv_g(x)\bigg) \\ &= o(1) + o\bigg(\sum_{j=1}^{\kappa} B_{j,\alpha}(y_\alpha)\bigg) + o\big(\nu_{\kappa,\alpha}^{(n-2)/2} R_{\kappa+1,\alpha}(y_\alpha)^{2-n}\big).\end{aligned}$$

Thus we have obtained

$$\int_M \mathcal{H}_{y_\alpha}\big(|\mathcal{U}_\alpha|^{2^\star-2} - |\mathcal{U}_\alpha - \mathcal{U}_\infty|^{2^\star-2}\big)(u_{i,\alpha} - u_{i,\infty})\,dv_g \\ = o\bigg(\sum_{j=1}^{\kappa} B_{j,\alpha}(y_\alpha)\bigg) + o\Big(\nu_{\kappa,\alpha}^{(n-2)/2} R_{\kappa+1,\alpha}(y_\alpha)^{2-n}\Big) + \varepsilon_\alpha \|\mathcal{U}_\infty\|_\infty. \quad (6\text{-}57)$$

Coming back to (6-52): thanks to (6-53)–(6-57), we now obtain

$$\bigg|\mathcal{U}_\alpha(y_\alpha) - \mathcal{U}_\infty(y_\alpha) - \int_M \mathcal{H}_{y_\alpha}|\mathcal{U}_\alpha - \mathcal{U}_\infty|^{2^\star-2}\,(\mathcal{U}_\alpha - \mathcal{U}_\infty)\;dv_g\bigg| \\ \le o\bigg(\sum_{j=1}^{\kappa} B_{j,\alpha}(y_\alpha)\bigg) + o\Big(\nu_{\kappa,\alpha}^{(n-2)/2} R_{\kappa+1,\alpha}(y_\alpha)^{2-n}\Big) + \varepsilon_\alpha \|\mathcal{U}_\infty\|_\infty + C\mu_\alpha^{(n-2)/2}. \quad (6\text{-}58)$$

Using (4-5), (5-25), and the extension property of $d_g(x, y)^{n-2}\mathcal{H}(x, y)$ mentioned above, we can find a sequence $(R_\alpha)_\alpha$ such that $R_\alpha \to +\infty$ as $\alpha \to +\infty$, that

$$\bigg|\int_{M_{i,\alpha}} \mathcal{H}_{y_\alpha}|\mathcal{U}_\alpha - \mathcal{U}_\infty|^{2^\star-2}(\mathcal{U}_\alpha - \mathcal{U}_\infty)\,dv_g - B_{i,\alpha}(y_\alpha)\Lambda_i\bigg| \le C\mu_\alpha^{(n-2)/2} + o(B_{i,\alpha}(y_\alpha))$$

for all $i \in \{1, \dots, \kappa\}$, and that the sets

$$M_{i,\alpha} = B_{x_{i,\alpha}}(R_\alpha \mu_{i,\alpha}) \setminus \bigcup_{i+1\le j\le N} B_{x_{j,\alpha}}(R_\alpha^{-1}\mu_{i,\alpha})$$

are disjoint.

Then we can write thanks to Proposition 5.1, Step 3, and (6-49), that

$$\begin{aligned}\int_{M\setminus\bigcup_{1\le i\le\kappa}M_{i,\alpha}} & d_g(y_\alpha,x)^{2-n}|\mathcal{U}_\alpha(x)-\mathcal{U}_\infty(x)|^{2^\star-1}dv_g(x)\\ &\le \varepsilon_\alpha\|\mathcal{U}_\infty\|_\infty + C\mu_\alpha^{(n-2)/2} + C\sum_{j=1}^{\kappa}\int_{M_\alpha} d_g(y_\alpha,x)^{2-n}B_{j,\alpha}(x)^{2^\star-1}dv_g(x)\\ &\qquad + C\nu_{\kappa,\alpha}^{(n+2)/2}\int_{\{R_{\kappa+1,\alpha}\ge\nu_{\kappa,\alpha}\}}\frac{d_g(y_\alpha,x)^{2-n}}{R_{\kappa+1,\alpha}(x)^{(n+2)}}dv_g(x) + C\nu_{\kappa,\alpha}^{(n-2)/2}R_{\kappa+1,\alpha}(y_\alpha)^{2-n}\\ &\le \varepsilon_\alpha\|\mathcal{U}_\infty\|_\infty + C\mu_\alpha^{(n-2)/2} + o\Bigg(\sum_{j=1}^{\kappa}B_{j,\alpha}(y_\alpha)\Bigg) + C\nu_{\kappa,\alpha}^{(n-2)/2}R_{\kappa+1,\alpha}(y_\alpha)^{2-n}.\end{aligned}$$

Coming back to (6-58), this ends the proof of Step 4. □

Step 5. $\nu_{\kappa,\alpha}=\mu_{\kappa+1,\alpha}$.

Proof of Step 5. We proceed by contradiction and thus assume that there exists $i\in\{1,\dots,\kappa\}$ and a sequence (y_α) of points in $\tilde{\Omega}^\varepsilon_{i,\alpha}$ such that

$$\nu_{\kappa,\alpha}^{(1-2\varepsilon)(n-2)/2}\Psi^\varepsilon_\alpha(y_\alpha)=\Phi^\varepsilon_{i,\alpha}(y_\alpha). \tag{6-59}$$

Since $y_\alpha\in\tilde{\Omega}^\varepsilon_{i,\alpha}$, we know that

$$\Phi^\varepsilon_{i,\alpha}(y_\alpha)\ge\Phi^\varepsilon_{j,\alpha}(y_\alpha) \tag{6-60}$$

for all $1\le j\le\kappa$ and that

$$d_g(x_{i,\alpha},y_\alpha)^2\left|\mathcal{U}_\alpha(y_\alpha)-\mathcal{U}_\infty(y_\alpha)-\sum_{j=1}^{\kappa}B_{j,\alpha}(y_\alpha)\Lambda_j\right|^{2^\star-2}\ge D(\varepsilon). \tag{6-61}$$

Clearly,

$$d_g(x_{i,\alpha},x)^2B_{i,\alpha}(x)^{2^\star-2}=O\left(1\right). \tag{6-62}$$

We now claim that

$$d_g(x_{i,\alpha},y_\alpha)^2B_{j,\alpha}(y_\alpha)^{2^\star-2}\to 0 \quad \text{as } \alpha\to+\infty \tag{6-63}$$

for all $1\le j\le\kappa$, $j\ne i$. In order to prove (6-63), we proceed by contradiction once again and assume that there exists $1\le j\le\kappa$, $j\ne i$, such that

$$\big(d_g(x_{j,\alpha},y_\alpha)+\mu_{j,\alpha}\big)^2=O\big(\mu_{j,\alpha}d_g(x_{i,\alpha},y_\alpha)\big). \tag{6-64}$$

Since $\Phi^\varepsilon_{i,\alpha}(y_\alpha)\ge\Phi^\varepsilon_{j,\alpha}(y_\alpha)$, we then get

$$\big(d_g(x_{i,\alpha},y_\alpha)+\mu_{i,\alpha}\big)^{1-\varepsilon}=O(\mu^\varepsilon_{j,\alpha}\mu^{1-2\varepsilon}_{i,\alpha}),$$

so $\mu_{i,\alpha}=O(\mu_{j,\alpha})$ and $d_g(x_{i,\alpha},y_\alpha)^{1-\varepsilon}=O(\mu^\varepsilon_{j,\alpha}\mu^{1-2\varepsilon}_{i,\alpha})$. Coming back to (6-64), we also obtain $\mu_{j,\alpha}=O(\mu_{i,\alpha})$ and $d_g(x_{j,\alpha},y_\alpha)=O(\mu_{i,\alpha})$. This contradicts the first equation in (4-5). Thus (6-63) is proved.

Applying Step 4, we get from (6-61), (6-62), and (6-63) that

$$R_{\kappa+1,\alpha}(y_\alpha)^2=O\big(\nu_{\kappa,\alpha}d_g(x_{i,\alpha},y_\alpha)\big). \tag{6-65}$$

Using (5-8) and (6-59), we also have

$$\left(d_g(x_{i,\alpha}, y_\alpha) + \mu_{i,\alpha}\right)^{2(1-\varepsilon)} = O\left(\mu_{i,\alpha}^{1-2\varepsilon} \nu_{\kappa,\alpha}^{2\varepsilon-1} R_{\kappa+1,\alpha}(y_\alpha)^{2(1-\varepsilon)}\right),$$

so that, with (6-65) and Step 1, we get that $\mu_{i,\alpha} = O(\mu_{\kappa,\alpha})$, that $d_g(x_{i,\alpha}, y_\alpha) = O(\mu_{i,\alpha})$ and that $\mu_{i,\alpha} = O\left(R_{\kappa+1,\alpha}(y_\alpha)\right)$. Using the second equation in (4-5) of Proposition 4.2 we then obtain

$$d_g(x_{i,\alpha}, y_\alpha)^2 |\mathcal{U}_\alpha(y_\alpha) - B_{i,\alpha}\Lambda_i(y_\alpha)|^{2^\star-2} \to 0 \quad \text{as } \alpha \to +\infty.$$

This contradicts (6-61) thanks to (6-63). Step 5 is proved. □

Conclusion of the proof of Proposition 6.1. By Proposition 5.1 we know that $(\mathcal{I}_1)$ holds true. By Steps 4 and 5, and by induction, it follows that $(\mathcal{I}_N)$ holds true. It remains to prove that $(\mathcal{I}_{N+1})$ also holds true. For this we proceed with similar arguments to those developed in the proof of Step 4. We let (y_α) be a sequence of points in M and write, for any $i = 1, \dots, p$,

$$u_{i,\alpha}(y_\alpha) - u_{i,\infty}(y_\alpha) = \frac{1}{V_g}\int_M (u_{i,\alpha} - u_{i,\infty})\, dv_g + \int_M \mathcal{H}_{y_\alpha} \Delta_g(u_{i,\alpha} - u_{i,\infty})\, dv_g, \tag{6-66}$$

where $\mathcal{H}_{y_\alpha}(\,\cdot\,) = \mathcal{H}(\,\cdot\,, y_\alpha)$ and $\mathcal{H}$ is the Green's function of Δ_g normalized so that $\mathcal{H} \geq 1$. Since $(\mathcal{I}_N)$ holds true,

$$\int_M |u_{i,\alpha} - u_{i,\infty}| dv_g \leq C\mu_\alpha^{(n-2)/2} + \varepsilon_\alpha \|\mathcal{U}_\infty\|_\infty,$$

where $C > 0$ is independent of α, and $\varepsilon_\alpha \to 0$ as $\alpha \to +\infty$. Using the equations satisfied by the $\mathcal{U}_\alpha$'s and $\mathcal{U}_\infty$, but also $(\mathcal{I}_N)$, mimicking what was done in the proof of Step 4, we get with (6-66) that

$$\begin{aligned}\left|\mathcal{U}_\alpha(y_\alpha) - \mathcal{U}_\infty(y_\alpha) - \int_M \mathcal{H}_{y_\alpha} |\mathcal{U}_\alpha - \mathcal{U}_\infty|^{2^\star-2}(u_{i,\alpha} - u_{i,\infty})\, dv_g\right| \\ \leq C\mu_\alpha^{(n-2)/2} + o\left(\sum_{i=1}^N B_{i,\alpha}(y_\alpha)\right) + \varepsilon_\alpha \|\mathcal{U}_\infty\|_\infty.\end{aligned} \tag{6-67}$$

We also have

$$\begin{aligned}\left|\int_M \mathcal{H}_{y_\alpha} |\mathcal{U}_\alpha - \mathcal{U}_\infty|^{2^\star-2}(u_{i,\alpha} - u_{i,\infty})\, dv_g - \sum_{i=1}^N B_{i,\alpha}(y_\alpha)\Lambda_i\right| \\ \leq C\mu_\alpha^{(n-2)/2} + o\left(\sum_{i=1}^N B_{i,\alpha}(y_\alpha)\right) + \varepsilon_\alpha \|\mathcal{U}_\infty\|_\infty,\end{aligned} \tag{6-68}$$

where $C > 0$ in (6-67), (6-68) is independent of α, and $\varepsilon_\alpha \to 0$ as $\alpha \to +\infty$. Combining (6-67) and (6-68), we get $(\mathcal{I}_{N+1})$. This ends the proof of Proposition 6.1. □

7. A Pohozaev identity for systems

Let (M, g) be a smooth compact Riemannian manifold. Let also X be a smooth 1-form over M and $\mathcal{U} : M \to \mathbb{R}^p$ be a C^2-map. We define $X(\nabla\mathcal{U})$ by $X(\nabla\mathcal{U}) = (\nabla\mathcal{U}, X)$. This is a p-map with components $X(\nabla\mathcal{U})_i = (\nabla u_i, X)$ where the u_i's are the components of $\mathcal{U}$. We define also $|\nabla\mathcal{U}|$ and $(T_\nabla X)_\mathcal{U}$ by $|\nabla\mathcal{U}|^2 = \sum_{i=1}^p |\nabla u_i|^2$ and $(T_\nabla X)_\mathcal{U} = \sum_{i=1}^p S_X^\sharp(\nabla u_i, \nabla u_i)$, where $S_X^\sharp$ is the $(0, 2)$-tensor field we obtain from the $(2,0)$-tensor field S_X by the musical isomorphism, and

$$S_X = \nabla X - \frac{1}{n}(\mathrm{div}_g X) g. \tag{7-1}$$

For Ω a smooth bounded domain in M we let ν be the unit outer normal to $\partial\Omega$. The Pohozaev-type identity for systems we prove is stated as follows.

Proposition 7.1. *Let (M,g) be a smooth compact Riemannian n-manifold, Ω be a smooth bounded domain in M, and $A : M \to M_p^s(\mathbb{R})$ be a C^1-map. Let X be a smooth 1-form over M and $\mathcal{U}$ be a solution of* (0-1). *Then*

$$\begin{aligned}\int_\Omega \langle A\mathcal{U}, X(\nabla\mathcal{U})\rangle_{\mathbb{R}^p} dv_g + \frac{n-2}{4n}\int_\Omega (\Delta_g(\mathrm{div}_g X))|\mathcal{U}|^2 dv_g + \frac{n-2}{2n}\int_\Omega (\mathrm{div}_g X)\langle A\mathcal{U}, \mathcal{U}\rangle_{\mathbb{R}^p} dv_g \\ = -\int_\Omega (T_\nabla X)_{\mathcal{U}} dv_g + \frac{n-2}{2n}\int_{\partial\Omega} X(\nu)|\mathcal{U}|^{2^\star} d\sigma_g - \frac{n-2}{4n}\int_{\partial\Omega}\partial_\nu(\mathrm{div}_g X)|\mathcal{U}|^2 d\sigma_g \\ + \frac{n-2}{2n}\int_{\partial\Omega}(\mathrm{div}_g X)\langle\partial_\nu\mathcal{U}, \mathcal{U}\rangle_{\mathbb{R}^p} d\sigma_g - \int_{\partial\Omega} B_{\partial\Omega}(\mathcal{U})\, d\sigma_g,\end{aligned} \tag{7-2}$$

where $X(\nabla\mathcal{U})$ and $(T_\nabla X)_{\mathcal{U}}$ are as above, $B_{\partial\Omega}(\mathcal{U}) = \frac{1}{2}X(\nu)|\nabla\mathcal{U}|^2 - \langle X(\nabla\mathcal{U}), \partial_\nu\mathcal{U}\rangle_{\mathbb{R}^p}$ on $\partial\Omega$, and $\langle\cdot,\cdot\rangle_{\mathbb{R}^p}$ is the scalar product in $\mathbb{R}^p$.

Proof. Integrating by parts we easily see that for $u : M \to \mathbb{R}$ of class C^2,

$$\int_\Omega (\nabla u, X)\,\Delta_g u dv_g = \int_\Omega S^\sharp_{X,2}(\nabla u, \nabla u)\, dv_g + \int_{\partial\Omega}\left(\frac{1}{2}\int_\Omega X(\nu)|\nabla u|^2 - (\nabla u, X)\partial_\nu u\right) d\sigma_g, \tag{7-3}$$

where $S_{X,2} = \nabla X - \frac{1}{2}(\mathrm{div}_g X)g$. If we assume now that $\mathcal{U}$ is a p-map, applying (7-3) to the components u_i of $\mathcal{U}$, and summing over i, we obtain

$$\int_\Omega \langle X(\nabla\mathcal{U}), \Delta_g\mathcal{U}\rangle_{\mathbb{R}^p} dv_g = \int_\Omega S^\sharp_{X,2}(\nabla\mathcal{U}, \nabla\mathcal{U})\, dv_g + \int_{\partial\Omega} B_{\partial\Omega}(\mathcal{U})\, d\sigma_g.$$

We assume now that $\mathcal{U}$ solves (0-1) and we use the equations satisfied by $\mathcal{U}$ to explicit the left-hand side in the preceding equation. We can write

$$\begin{aligned}\int_\Omega \langle X(\nabla\mathcal{U}), \Delta_g\mathcal{U}\rangle_{\mathbb{R}^p} dv_g &= \int_\Omega |\mathcal{U}|^{2^\star-2}\langle X(\nabla\mathcal{U}), \mathcal{U}\rangle_{\mathbb{R}^p} dv_g - \int_\Omega \langle A\mathcal{U}, X(\nabla\mathcal{U})\rangle_{\mathbb{R}^p} dv_g \\ &= \frac{1}{2^\star}\int_\Omega (\nabla|\mathcal{U}|^{2^\star}, X)\, dv_g - \int_\Omega \langle A\mathcal{U}, X(\nabla\mathcal{U})\rangle_{\mathbb{R}^p} dv_g \\ &= -\frac{1}{2^\star}\int_\Omega (\mathrm{div}_g X)|\mathcal{U}|^{2^\star} dv_g - \int_\Omega \langle A\mathcal{U}, X(\nabla\mathcal{U})\rangle_{\mathbb{R}^p} dv_g + \frac{1}{2^\star}\int_{\partial\Omega} X(\nu)|\mathcal{U}|^{2^\star} d\sigma_g.\end{aligned}$$

Then we get

$$\begin{aligned}\int_\Omega \langle A\mathcal{U}, X(\nabla\mathcal{U})\rangle_{\mathbb{R}^p} dv_g + \frac{1}{2^\star}\int_\Omega (\mathrm{div}_g X)|\mathcal{U}|^{2^\star} dv_g + \int_\Omega S^\sharp_{X,2}(\nabla\mathcal{U}, \nabla\mathcal{U})\, dv_g \\ = \frac{1}{2^\star}\int_{\partial\Omega} X(\nu)|\mathcal{U}|^{2^\star} d\sigma_g - \int_{\partial\Omega} B_{\partial\Omega}(\mathcal{U})\, d\sigma_g.\end{aligned}$$

Using once again (0-1), we obtain

$$\begin{aligned}\int_\Omega S^\sharp_{X,2}(\nabla\mathcal{U},\nabla\mathcal{U})\,dv_g &= \int_\Omega S^\sharp_X(\nabla\mathcal{U},\nabla\mathcal{U})\,dv_g - \frac{1}{2^\star}\int_\Omega(\mathrm{div}_g X)|\nabla\mathcal{U}|^2 dv_g\\ &= \int_\Omega S^\sharp_X(\nabla\mathcal{U},\nabla\mathcal{U})\,dv_g - \frac{1}{2^\star}\int_{\partial\Omega}(\mathrm{div}_g X)\langle\partial_\nu\mathcal{U},\mathcal{U}\rangle_{\mathbb{R}^p}d\sigma_g\\ &\qquad + \frac{n-2}{4n}\int_{\partial\Omega}\partial_\nu(\mathrm{div}_g X)|\mathcal{U}|^2 d\sigma_g + \frac{n-2}{4n}\int_\Omega(\Delta_g(\mathrm{div}_g X))|\mathcal{U}|^2 dv_g\\ &\qquad - \frac{1}{2^\star}\int_\Omega(\mathrm{div}_g X)|\mathcal{U}|^{2^\star}dv_g + \frac{1}{2^\star}\int_\Omega(\mathrm{div}_g X)\langle A\mathcal{U},\mathcal{U}\rangle_{\mathbb{R}^p}dv_g,\end{aligned}$$

and (7-2) easily follows. This ends the proof of the proposition. □

The Pohozaev-type identity (7-2) is used repeatedly, with different choices of X, in the next section.

8. The range of influence of blow-up points

We start with notations and the definition of the range of influence of blow-up points. The blow-up points $x_{i,\alpha}$ of Proposition 4.2 come with vector bubbles $(\mathcal{B}^i_\alpha)_\alpha$ as in the same proposition. We let Λ_i be the S^{p-1} projection of $(\mathcal{B}^i_\alpha)_\alpha$, and $B_{i,\alpha}=|\mathcal{B}^i_\alpha|$ for all i and all α. As above, $(A_\alpha)_\alpha$ is a sequence of C^1 maps from M to $M^s_p(\mathbb{R})$ such that $A_\alpha\to A$ in $C^1(M)$ as $\alpha\to+\infty$ for some C^1 map A from M to $M^s_p(\mathbb{R})$ satisfying (H), and we order the blow-up points in such a way that

$$\mu_\alpha=\mu_{1,\alpha}\ge\cdots\ge\mu_{N,\alpha}, \tag{8-1}$$

where the $\mu_{i,\alpha}$'s are the weights of the vector bubble $(\mathcal{B}^i_\alpha)_\alpha$. Given $i,j\in\{1,\dots,N\}$, $i\ne j$, we let $s_{i,j,\alpha}$ be given by

$$s^2_{i,j,\alpha}=\frac{\mu_{i,\alpha}}{\mu_{j,\alpha}}\frac{d_g(x_{i,\alpha},x_{j,\alpha})^2}{n(n-2)}+\mu_{i,\alpha}\mu_{j,\alpha}=\mu_{i,\alpha}B_{j,\alpha}(x_{i,\alpha})^{-2/(n-2)} \tag{8-2}$$

and we define the range of influence of the blow-up point $x_{i,\alpha}$ by

$$r_{i,\alpha}=\begin{cases}\min_{j\in\mathcal{A}_i}s_{i,j,\alpha} & \text{if } \mathcal{U}_\infty\equiv 0,\\ \min\{\min_{j\in\mathcal{A}_i}s_{i,j,\alpha};\sqrt{\mu_{i,\alpha}}\} & \text{if } \mathcal{U}_\infty\not\equiv 0.\end{cases} \tag{8-3}$$

where

$$\mathcal{A}_i=\{j\in\{1,\dots,N\},\ j\ne i \text{ s.t. } \mu_{i,\alpha}=O(\mu_{j,\alpha})\}. \tag{8-4}$$

If $\mathcal{A}_i=\varnothing$ (so that, in particular, $i=1$) and $\mathcal{U}_\infty\equiv 0$, we let by definition $r_{i,\alpha}=\frac{1}{2}i_g$, where i_g is the injectivity radius of (M,g). Using the first equation in (4-5) it is easily checked that

$$\frac{s_{i,j,\alpha}}{\mu_{i,\alpha}}\to+\infty \text{ as } \alpha\to+\infty \text{ for all } i,\ j\in\{1,\dots,N\} \text{ and all } j\in\mathcal{A}_i. \tag{8-5}$$

This implies in particular that

$$\frac{r_{i,\alpha}}{\mu_{i,\alpha}}\to+\infty\quad\text{as } \alpha\to+\infty. \tag{8-6}$$

If $j\in\mathcal{A}_i$ and $i\in\mathcal{A}_j$, we let $\lambda_{i,j}\ge 0$ be given by

$$\lambda_{i,j}=\left(\lim_{\alpha\to+\infty}\frac{\mu_{j,\alpha}}{\mu_{i,\alpha}}\right)^{(n-2)/2}. \tag{8-7}$$

Given $i \in \{1, \dots, N\}$, we also let

$$\mathcal{B}_i = \begin{cases} \big\{ j \in \{1, \dots, N\},\ j \neq i \text{ s.t. } d_g(x_{i,\alpha}, x_{j,\alpha}) = O(r_{i,\alpha}) \big\} & \text{if } r_{i,\alpha} \to 0, \\ \big\{ j \in \{1, \dots, N\},\ j \neq i \text{ s.t. } x_j \in B_{x_i}(\frac{1}{2} i_g) \big\} & \text{if } r_{i,\alpha} \not\to 0. \end{cases} \tag{8-8}$$

and, for $j \in \mathcal{B}_i$,

$$z_{i,j} = \lim_{\alpha \to +\infty} r_{i,\alpha}^{-1} \exp_{x_{i,\alpha}}^{-1}(x_{j,\alpha}). \tag{8-9}$$

Up to a subsequence, all these limits exist. We let $\delta_i > 0$ be such that for any i and any $j \in \mathcal{B}_i$,

$$|z_{i,j}| \neq 0 \Rightarrow |z_{i,j}| \geq 10\delta_i. \tag{8-10}$$

We also define $\mathcal{C}_i$ to be the subset of $\mathcal{B}_i$ given by

$$\mathcal{C}_i = \big\{ j \in \mathcal{B}_i \text{ s.t. } z_{i,j} = 0 \big\} \cap \mathcal{A}_i^c. \tag{8-11}$$

It can be proved that there exists a subset $\mathcal{D}_i$ of $\mathcal{C}_i$ and a family $(R_{i,j})_{j \in \mathcal{D}_i}$ of positive real numbers such that the two following assertions hold true: for any $j, k \in \mathcal{D}_i$, $j \neq k$,

$$\frac{d_g(x_{j,\alpha}, x_{k,\alpha})}{s_{j,i,\alpha}} \to +\infty \tag{8-12}$$

as $\alpha \to +\infty$, and for any $j \in \mathcal{C}_i$ there exists a unique $k \in \mathcal{D}_i$ such that

$$\limsup_{\alpha \to +\infty} \frac{d_g(x_{j,\alpha}, x_{k,\alpha})}{s_{k,i,\alpha}} \leq \frac{R_{i,k}}{20} \quad \text{and} \quad \limsup_{\alpha \to +\infty} \frac{s_{j,i,\alpha}}{s_{k,i,\alpha}} \leq \frac{R_{i,k}}{20}. \tag{8-13}$$

We also introduce the subsets

$$\Omega_{i,\alpha} = B_{x_{i,\alpha}}(\delta_i r_{i,\alpha}) \setminus \bigcup_{j \in \mathcal{D}_i} \Omega_{i,j,\alpha} \tag{8-14}$$

of M, where

$$\Omega_{i,j,\alpha} = B_{x_{j,\alpha}}(R_{i,j} s_{j,i,\alpha}) \tag{8-15}$$

for all $j \in \mathcal{D}_i$. The $\Omega_{i,j,\alpha}$'s are disjoint for α sufficiently large.

We now prove two lemmas to be used in the proof of Theorem 0.1.

Lemma 8.1. *Let $i \in \{1, \dots, N\}$. Up to passing to a subsequence,*

$$|\mathcal{U}_\alpha - \Lambda_i B_{i,\alpha}| = o(B_{i,\alpha}) + O\big(\mu_{i,\alpha}^{(n-2)/2} r_{i,\alpha}^{2-n}\big) + O\Bigg(\sum_{j \in \mathcal{D}_i} B_{j,\alpha} \Bigg) = O(B_{i,\alpha})$$

in $B_{x_{i,\alpha}}\big(4\delta_i r_{i,\alpha}\big) \setminus \bigcup_{j \in \mathcal{D}_i} B_{x_{j,\alpha}}\big(\frac{1}{10} R_{i,j} s_{j,i,\alpha}\big)$, and so, in particular, in $\Omega_{i,\alpha}$.

Proof. Let $x_\alpha \in B_{x_{i,\alpha}}\big(4\delta_i r_{i,\alpha}\big) \setminus \bigcup_{j \in \mathcal{D}_i} B_{x_{j,\alpha}}\big(\frac{1}{10} R_{i,j} s_{j,i,\alpha}\big)$. Thanks to Proposition 6.1 we can write

$$\mathcal{U}_\alpha(x_\alpha) = \mathcal{U}_\infty(x_\alpha) + \varepsilon_\alpha \|\mathcal{U}_\infty\|_\infty + O\big(\mu_\alpha^{(n-2)/2}\big) + \sum_{j=1}^{N} \big(\Lambda_j + o(1)\big) B_{j,\alpha}(x_\alpha). \tag{8-16}$$

By the definition of $r_{i,\alpha}$, we know that $r_{i,\alpha}^2 \leq \mu_{i,\alpha}$ if $\mathcal{U}_\infty \not\equiv 0$ so that

$$\mathcal{U}_\infty(x_\alpha) + \varepsilon_\alpha \|\mathcal{U}_\infty\|_\infty = \mu_{i,\alpha}^{(n-2)/2} r_{i,\alpha}^{2-n} \begin{cases} 0 & \text{if } \mathcal{U}_\infty \equiv 0, \\ \big(\lim_{\alpha \to +\infty} r_{i,\alpha}^{n-2} \mu_{i,\alpha}^{1-(n/2)}\big) \mathcal{U}_\infty(x_i) + o(1) & \text{if } \mathcal{U}_\infty \not\equiv 0. \end{cases} \tag{8-17}$$

We now estimate $B_{j,\alpha}(x_\alpha)$. Assume first that $j \notin \mathscr{C}_i$ and $j \neq i$. As one can check with a little bit of work from the above definitions, if $r_{i,\alpha} \to 0$ as $\alpha \to +\infty$, then

$$B_{j,\alpha}(x_\alpha) = \mu_{i,\alpha}^{(n-2)/2} r_{i,\alpha}^{2-n} \Lambda_{i,j,\alpha}, \tag{8-18}$$

where

$$\Lambda_{i,j,\alpha} = \begin{cases} \left(\dfrac{n(n-2)}{|z-z_{i,j}|^2}\,\dfrac{\mu_{j,\alpha}}{\mu_{i,\alpha}}\right)^{(n-2)/2} + o(1) & \text{if } j \in \mathscr{A}_i \cap \mathscr{B}_i \text{ and } i \in \mathscr{A}_j, \\ r_{i,\alpha}^{n-2}/s_{i,j,\alpha}^{n-2} + o(1) & \text{if } j \in \mathscr{A}_i \setminus \mathscr{B}_i \text{ or } j \in \mathscr{A}_i \cap \mathscr{B}_i \text{ and } i \notin \mathscr{A}_j, \\ o(1) & \text{if } j \in \mathscr{A}_i^c \setminus \mathscr{C}_i, \end{cases}$$

where, up to a subsequence,

$$z = \lim_{\alpha \to +\infty} r_{i,\alpha}^{-1} \exp_{x_{i,\alpha}}^{-1}(x_\alpha).$$

Note that $z_{i,j} \neq 0$ if $j \in \mathscr{A}_i \cap \mathscr{B}_i$ and $i \in \mathscr{A}_j$. This is a direct consequence of the definition of the $s_{i,j,\alpha}$'s and (8-6). Moreover, $|z - z_{i,j}| \geq 6\delta_i$ in this case. As a consequence we have proved that

$$\begin{aligned} \mathscr{U}_\alpha(x_\alpha) = O\Big(\mu_\alpha^{(n-2)/2}\Big) + o\Big(\mu_{i,\alpha}^{(n-2)/2} r_{i,\alpha}^{2-n}\Big) + \big(\Lambda_i + o(1)\big) B_{i,\alpha}(x_\alpha) \\ + \mu_{i,\alpha}^{(n-2)/2} r_{i,\alpha}^{2-n} \Lambda(1)_{i,\alpha} + \mu_{i,\alpha}^{(n-2)/2} r_{i,\alpha}^{2-n} \sum_{j \in \mathscr{A}_i} \Lambda(2)_{i,j,\alpha} \Lambda_j + \sum_{j \in \mathscr{C}_i} \big(\Lambda_j + o(1)\big) B_{j,\alpha}(x_\alpha), \end{aligned} \tag{8-19}$$

where

$$\Lambda(1)_{i,\alpha} = \begin{cases} 0 & \text{if } \mathscr{U}_\infty \equiv 0, \\ \big(\lim_{\alpha \to +\infty} r_{i,\alpha}^{n-2} \mu_{i,\alpha}^{1-(n/2)}\big) \mathscr{U}_\infty(x_i) & \text{if } \mathscr{U}_\infty \not\equiv 0, \end{cases}$$

and

$$\Lambda(2)_{i,j,\alpha} = \begin{cases} \big(n(n-2)\big)^{(n-2)/2} \lambda_{i,j}/|z - z_{i,j}|^{n-2} & \text{if } j \in \mathscr{B}_i \text{ and } i \in \mathscr{A}_j, \\ \lim_{\alpha \to +\infty} r_{i,\alpha}^{n-2}/s_{i,j,\alpha}^{n-2} & \text{if } j \notin \mathscr{B}_i \text{ or } i \notin \mathscr{A}_j. \end{cases}$$

Let $j \in \mathscr{C}_i$. We claim that, up to a subsequence,

$$\lim_{\alpha \to +\infty} \left(\frac{B_{j,\alpha}(x_\alpha)}{B_{i,\alpha}(x_\alpha)}\right)^{2/(n-2)} = n(n-2) \lim_{\alpha \to +\infty} \frac{s_{j,i,\alpha}^2}{d_g(x_{j,\alpha}, x_\alpha)^2}. \tag{8-20}$$

To prove (8-20), we first remark that $i \in \mathscr{A}_j$ since $j \in \mathscr{C}_i$ (and in particular $j \notin \mathscr{A}_i$). Thus, using (8-5), we obtain that

$$\begin{aligned} \left(\frac{B_{j,\alpha}(x_\alpha)}{B_{i,\alpha}(x_\alpha)}\right)^{2/(n-2)} &= \big(1 + o(1)\big) \mu_{j,\alpha} \mu_{i,\alpha}^{-1} d_g(x_{j,\alpha}, x_\alpha)^{-2} \big(n(n-2)\mu_{i,\alpha}^2 + d_g(x_{i,\alpha}, x_\alpha)^2\big) \\ &= n(n-2) \frac{s_{j,i,\alpha}^2}{d_g(x_{j,\alpha}, x_\alpha)^2} + o(1) + O\left(\frac{\mu_{j,\alpha}}{\mu_{i,\alpha}} \frac{\big|d_g(x_{i,\alpha}, x_\alpha)^2 - d_g(x_{i,\alpha}, x_{j,\alpha})^2\big|}{d_g(x_{j,\alpha}, x_\alpha)^2}\right). \end{aligned}$$

From the triangle inequality, we easily get

$$\frac{\big|d_g(x_{i,\alpha}, x_\alpha)^2 - d_g(x_{i,\alpha}, x_{j,\alpha})^2\big|}{d_g(x_{j,\alpha}, x_\alpha)^2} \leq 1 + 2\frac{d_g(x_{i,\alpha}, x_{j,\alpha})}{d_g(x_{j,\alpha}, x_\alpha)} = O\,(1) + O\left(\frac{s_{j,i,\alpha}}{d_g(x_{j,\alpha}, x_\alpha)} \sqrt{\frac{\mu_{i,\alpha}}{\mu_{j,\alpha}}}\right),$$

hence the estimate (8-20). Now, for $j \in \mathscr{C}_i$, we let $k \in \mathscr{D}_i$ be given by (8-13). By (8-20) it is easily checked that

$$B_{j,\alpha}(x_\alpha) = O(B_{k,\alpha}(x_\alpha)). \tag{8-21}$$

Since

$$\mu_{i,\alpha}^{(n-2)/2} r_{i,\alpha}^{2-n} = O(B_{i,\alpha}(x_\alpha)), \tag{8-22}$$

the first estimate in the lemma clearly holds true thanks to (8-19) and (8-21). Here it can be noted that

$$\mu_\alpha^{(n-2)/2} = O\big(\mu_{i,\alpha}^{(n-2)/2} r_{i,\alpha}^{2-n}\big)$$

for all i. Applying (8-20) again we easily obtain the second estimate in the lemma. This ends the proof of Lemma 8.1. □

Now we prove that the following elliptic type lemma holds true. Lemma 8.2 provides estimates on the $\mathcal{U}_\alpha$'s and $\nabla\mathcal{U}_\alpha$'s in small regions around the blow-up points $x_{i,\alpha}$.

Lemma 8.2. *There exists $C > 0$ such that, up to a subsequence,*

$$|\mathcal{U}_\alpha| \le C\mu_{i,\alpha}^{(n-2)/2} r_{i,\alpha}^{2-n} \quad \text{and} \quad |\nabla\mathcal{U}_\alpha| \le C\mu_{i,\alpha}^{(n-2)/2} r_{i,\alpha}^{1-n}$$

in $B_{x_{i,\alpha}}(2\delta_i r_{i,\alpha}) \setminus B_{x_{i,\alpha}}\big(\frac{1}{2}\delta_i r_{i,\alpha}\big)$. There also exists $C > 0$ such that, up to a subsequence, for any $j \in \mathcal{D}_i$,

$$|\mathcal{U}_\alpha| \le C\mu_{j,\alpha}^{(n-2)/2} s_{j,i,\alpha}^{2-n} \quad \text{and} \quad |\nabla\mathcal{U}_\alpha| \le C\mu_{j,\alpha}^{(n-2)/2} s_{j,i,\alpha}^{1-n}$$

in $B_{x_{j,\alpha}}(5R_{i,j}s_{j,i,\alpha}) \setminus B_{x_{j,\alpha}}\big(\frac{1}{5}R_{i,j}s_{j,i,\alpha}\big)$.

Proof. The lemma follows from standard elliptic theory and the estimates we proved in Lemma 8.1. Assuming first that $x_\alpha \in B_{x_{i,\alpha}}(4\delta_i r_{i,\alpha}) \setminus B_{x_{i,\alpha}}\big(\frac{1}{4}\delta_i r_{i,\alpha}\big)$, we easily get from Lemma 8.1 that

$$|\mathcal{U}_\alpha(x_\alpha)| = O\big(\mu_{i,\alpha}^{(n-2)/2} r_{i,\alpha}^{2-n}\big). \tag{8-23}$$

On the other hand, if we let $\tilde{\mathcal{U}}_\alpha$ be given by $\tilde{\mathcal{U}}_\alpha(x) = r_{i,\alpha}^{(n-2)/2}\mathcal{U}_\alpha\big(\exp_{x_{i,\alpha}}(r_{i,\alpha}x)\big)$, then

$$\Delta_{\tilde{g}_\alpha}\tilde{\mathcal{U}}_\alpha + r_{i,\alpha}^2\tilde{A}_\alpha\tilde{\mathcal{U}}_\alpha = |\tilde{\mathcal{U}}_\alpha|^{2^\star-2}\tilde{\mathcal{U}}_\alpha, \tag{8-24}$$

where $\tilde{g}_\alpha = \big(\exp_{x_{i,\alpha}}^\star g\big)(r_{i,\alpha}x)$ and $\tilde{A}_\alpha(x) = A_\alpha\big(\exp_{x_{i,\alpha}}(r_{i,\alpha}x)\big)$. The first two estimates in Lemma 8.2 follow from (8-23) and (8-24) by standard elliptic theory. Similarly, if we assume that

$$x_\alpha \in B_{x_{j,\alpha}}(10R_{i,j}s_{j,i,\alpha}) \setminus B_{x_{j,\alpha}}\big(\tfrac{1}{10}R_{i,j}s_{j,i,\alpha}\big),$$

noting that $s_{j,i,\alpha} = o(r_{i,\alpha})$ in this case, we get from Lemma 8.1 and (8-20) that

$$|\mathcal{U}_\alpha(x_\alpha)| = O\big(\mu_{j,\alpha}^{(n-2)/2} s_{j,i,\alpha}^{2-n}\big). \tag{8-25}$$

Letting $\hat{\mathcal{U}}_\alpha$ be given by $\hat{\mathcal{U}}_\alpha(x) = s_{j,i,\alpha}^{(n-2)/2}\mathcal{U}_\alpha\big(\exp_{x_{j,\alpha}}(s_{j,i,\alpha}x)\big)$, we also have

$$\Delta_{\hat{g}_\alpha}\hat{\mathcal{U}}_\alpha + s_{j,i,\alpha}^2\hat{A}_\alpha\hat{\mathcal{U}}_\alpha = |\hat{\mathcal{U}}_\alpha|^{2^\star-2}\hat{\mathcal{U}}_\alpha, \tag{8-26}$$

where $\hat{g}_\alpha = (\exp_{x_{j,\alpha}}^\star g)(s_{j,i,\alpha}x)$ and $\hat{A}_\alpha(x) = A_\alpha\big(\exp_{x_{j,\alpha}}(s_{j,i,\alpha}x)\big)$. The last two estimates in Lemma 8.2 follow from (8-25) and (8-26) here again by standard elliptic theory. This proves Lemma 8.2. □

9. Sharp asymptotics for the range of influence

Our goal now is to prove the sharp asymptotics connecting the range of influence $r_{i,\alpha}$ of the blow-up points with the weights $\mu_{i,\alpha}$ of the bubbles in the decomposition of Proposition 4.2. This is the subject of Proposition 9.2. We adopt here the notations of the preceding section. In particular, $(A_\alpha)_\alpha$ is a sequence of C^1 maps from M to $M_p^s(\mathbb{R})$ such that $A_\alpha \to A$ in $C^1(M)$ as $\alpha \to +\infty$ for some C^1 map A from M to $M_p^s(\mathbb{R})$ satisfying (H), and we order the blow-up points in such a way that

$$\mu_\alpha = \mu_{1,\alpha} \geq \cdots \geq \mu_{N,\alpha},$$

where the $\mu_{i,\alpha}$'s are the weights of the vector bubble $(\mathcal{B}_\alpha^i)_\alpha$ in Proposition 4.2, and let the $r_{i,\alpha}$'s be given by (8-3). First we prove:

Lemma 9.1. *If $r_{i,\alpha} = o\big(\sqrt{\mu_{i,\alpha}/\mu_\alpha}\big)$, then, up to a subsequence,*

$$r_{i,\alpha}^{n-2}\mu_{i,\alpha}^{1-(n/2)}\mathcal{U}_\alpha\big(\exp_{x_{i,\alpha}}(r_{i,\alpha}z)\big) \to \big(n(n-2)\big)^{(n-2)/2}\big(\Lambda_i|z|^{2-n} + \mathcal{H}_i(z)\big)$$

in $C^2_{\mathrm{loc}}\big(B_0(2\delta_i)\setminus\{0\}\big)^p$ as $\alpha \to +\infty$, where

$$\mathcal{H}_i(z) = \sum_{\substack{j\in\mathcal{A}_i\cap\mathcal{B}_i,\\ i\in\mathcal{A}_j}} \frac{\lambda_{i,j}\Lambda_j}{|z-z_{i,j}|^{n-2}} + X_i$$

is a smooth function in $B_0\,(2\delta_i)$ satisfying that $\mathcal{H}_i(0) \neq 0$, the $\lambda_{i,j}$'s are as in (8-7), *δ_i is as in* (8-10), *and the X_i's are nonnegative vectors in $\mathbb{R}^p$.*

Proof. Let $z \in B_0(3\delta_i)\setminus\{0\}$ and set $x_\alpha = \exp_{x_{i,\alpha}}(r_{i,\alpha}z)$. Let also $\mathcal{W}_\alpha$ be given by

$$\overline{\mathcal{U}}_\alpha(x) = r_{i,\alpha}^{n-2}\mu_{i,\alpha}^{-(n-2)/2}\mathcal{U}_\alpha\big(\exp_{x_{i,\alpha}}(r_{i,\alpha}x)\big).$$

Then

$$\Delta_{g_\alpha}\overline{\mathcal{U}}_\alpha + r_{i,\alpha}^2\tilde{A}_\alpha\overline{\mathcal{U}}_\alpha = \Big(\frac{\mu_{i,\alpha}}{r_{i,\alpha}}\Big)^2|\overline{\mathcal{U}}_\alpha|^{2^\star-2}\overline{\mathcal{U}}_\alpha, \tag{9-1}$$

where $g_\alpha = (\exp^\star_{x_{i,\alpha}} g)(r_{i,\alpha}x)$ and $\tilde{A}_\alpha(x) = A_\alpha\big(\exp_{x_{i,\alpha}}(r_{i,\alpha}x)\big)$. In particular, we get by (8-6), (8-19) and (8-20) that, if $r_{i,\alpha}^2 = o\big(\mu_{i,\alpha}/\mu_\alpha\big)$, then

$$\lim_{\alpha\to+\infty} r_{i,\alpha}^{n-2}\mu_{i,\alpha}^{1-\frac{n}{2}}\mathcal{U}_\alpha(x_\alpha) = \big(n(n-2)\big)^{(n-2)/2}\big(\Lambda_i|z|^{2-n} + \mathcal{H}_i(z)\big), \tag{9-2}$$

where $\mathcal{H}_i(z)$ is the sum of two terms:

$$\begin{cases} 0 & \text{if } \mathcal{U}_\infty \equiv 0 \\ \big(\lim_{\alpha\to+\infty} r_{i,\alpha}^{n-2}\mu_{i,\alpha}^{1-(n/2)}\big)\mathcal{U}_\infty(x_i) & \text{if } \mathcal{U}_\infty \not\equiv 0 \end{cases}$$

and

$$\sum_{j\in\mathcal{A}_i}\Lambda_j\begin{cases} \big(n(n-2)\big)^{(n-2)/2}\lambda_{i,j}/|z-z_{i,j}|^{n-2} & \text{if } j\in\mathcal{B}_i \text{ and } i\in\mathcal{A}_j, \\ \lim_{\alpha\to+\infty} r_{i,\alpha}^{n-2}/s_{i,j,\alpha}^{n-2} & \text{if } j\notin\mathcal{B}_i \text{ or } i\notin\mathcal{A}_j. \end{cases}$$

As a remark, if $j \in \mathcal{A}_i$ and $i \in \mathcal{A}_j$, then $\mu_{i,\alpha} \sim \mu_{j,\alpha}$. In particular, $z_{i,j} \neq 0$ since, if not the case, we would get from the inequality $r_{i,\alpha} \leq s_{i,j,\alpha}$ that $r_{i,\alpha} = o(\mu_{i,\alpha})$ and then that $d_g(x_{i,\alpha}, x_{j,\alpha}) = o(\mu_{i,\alpha})$, a contradiction with the first equation in (4-5) of Proposition 4.2. By (9-1) and (9-2), standard elliptic theory gives the lemma, up to the proof that $\mathcal{H}_i(0) \neq 0$. Assume first that there exists $j \in \mathcal{A}_i$ such that $s_{i,j,\alpha} = r_{i,\alpha}$. Then in the term involving this j in the above sum over $\mathcal{A}_i$ there is at least one line which is positive. Since all the other terms are nonnegative, this proves that $\mathcal{H}_i(0) \neq 0$. The other possibility is that $\mathcal{U}_\infty \not\equiv 0$ and that $r_{i,\alpha}^2 = \mu_{i,\alpha}$ so the first term in the definition of $\mathcal{H}_i$ is nonzero. Indeed, by the maximum principle, since

$$\Delta_g |\mathcal{U}_\infty|_\Sigma + \Lambda |\mathcal{U}_\infty|_\Sigma \geq 0$$

for some $\Lambda > 0$, where $|\mathcal{U}_\infty|_\Sigma = \sum_i u_{i,\infty}$ is the sum of the components of $\mathcal{U}_\infty$, we get that $|\mathcal{U}_\infty|_\Sigma > 0$ in M if $\mathcal{U}_\infty \not\equiv 0$. Then, here again, $\mathcal{H}_i(0) \neq 0$. Noting that the above two possibilities are the only two possibilities since our assumption on $r_{i,\alpha}$ clearly implies that $r_{i,\alpha} \to 0$ as $\alpha \to +\infty$, this ends the proof of Lemma 9.1. □

As it can be checked from the above proof, we have an explicit formula for the X_i's in Lemma 9.1. We get that

$$X_i = \Big(\lim_{\alpha\to+\infty} r_{i,\alpha}^{n-2} \mu_{i,\alpha}^{1-(n/2)}\Big)\mathcal{U}_\infty(x_i) + \sum_{j\in(\mathcal{A}_i\backslash\mathcal{B}_i)\cup\Theta_i} \Big(\lim_{\alpha\to+\infty} \frac{r_{i,\alpha}}{s_{i,j,\alpha}}\Big)^{n-2} \Lambda_j, \tag{9-3}$$

where we adopt the convention that the first term in the right-hand side of (9-3) is zero if $\mathcal{U}_\infty \equiv 0$, that the second term is zero if $(\mathcal{A}_i\backslash\mathcal{B}_i)\cup\Theta_i = \varnothing$, and where $\Theta_i = \{j \in \mathcal{A}_i \text{ s.t. } i \notin \mathcal{A}_j\}$. Now, at this point, we can state Proposition 9.2 which establishes sharp asymptotics connecting the range of influence $r_{i,\alpha}$ of the blow-up points $x_{i,\alpha}$ to the weights $\mu_{i,\alpha}$ of the bubbles in the decomposition of Proposition 4.2.

Proposition 9.2. *Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 4$, $p \geq 1$ be an integer, and $(A_\alpha)_\alpha$ be a sequence of C^1 maps from M to $M_p^s(\mathbb{R})$ such that $A_\alpha \to A$ in $C^1(M)$ as $\alpha \to +\infty$ for some C^1 map A from M to $M_p^s(\mathbb{R})$ satisfying* (H). *Let also $(\mathcal{U}_\alpha)_\alpha$ be an arbitrary bounded sequence in $H^1(M)$ of nonnegative solutions of* (1-1) *such that $\|\mathcal{U}_\alpha\|_\infty \to +\infty$ as $\alpha \to +\infty$. Let $i \in \{1, \dots, N\}$ and assume that, up to a subsequence, $r_{i,\alpha} = o\big(\sqrt{\mu_{i,\alpha}/\mu_\alpha}\big)$. Then*

$$\big(\big(A(x_i) - \tfrac{1}{6} S_g(x_i)\ \mathrm{Id}_p\big)\Lambda_i + o(1)\big) r_{i,\alpha}^2 \ln \frac{1}{\mu_{i,\alpha}} = 2\mathcal{H}_i(0) + o(1) \tag{9-4}$$

if $n = 4$, and

$$\begin{aligned}\bigg(\Big(A(x_i) - \frac{n-2}{4(n-1)} S_g(x_i)\,\mathrm{Id}_p\Big)\Lambda_i + o(1)\bigg)\mu_{i,\alpha}^{4-n} r_{i,\alpha}^{n-2} \\ = \frac{n^{n-2}(n-2)^{n-1}\omega_{n-1}}{\int_{\mathbb{R}^n} u_0^2\,dx}\Big(\mathcal{H}_i(0) + \frac{n-4}{2}\langle\Lambda_i, \mathcal{H}_i(0)\rangle_{\mathbb{R}^p}\Lambda_i\Big) + o(1)\end{aligned} \tag{9-5}$$

if $n \geq 5$, where $\mathcal{H}_i$ is as in Lemma 9.1, the $r_{i,\alpha}$'s are as in (8-3), *and u_0 is given by* (4-3). *Moreover, $\langle\Lambda_i, \nabla\mathcal{H}_i(0)\rangle_{\mathbb{R}^p} \equiv 0$.*

We prove Proposition 9.2 by reverse induction on i. We let $i \in \{1, \dots, N\}$ be such that $\sqrt{\mu_\alpha} r_{i,\alpha} = o\big(\sqrt{\mu_{i,\alpha}}\big)$ and, in case $i < N$, we assume that

$$\text{for any } j = i+1, \dots, N, \text{ (9-4) and (9-5) hold for } j \text{ if } \sqrt{\mu_\alpha} r_{j,\alpha} = o\big(\sqrt{\mu_{j,\alpha}}\big). \tag{H$_i$}$$

If $i = N$ we do not assume anything. Then we aim to prove that (9-4) and (9-5) hold true for i. As a remark it should be noted that we always have

$$\mathcal{H}_i(0) + \frac{n-4}{2}\langle\Lambda_i, \mathcal{H}_i(0)\rangle_{\mathbb{R}^p}\Lambda_i \neq 0. \tag{9-6}$$

Let $i \in \{1, \dots, N\}$, $i < N$, be arbitrary. Assuming (H_i), we get that for any $j \in \mathcal{D}_i$,

$$s_{j,i,\alpha}^{2-n} = \begin{cases} O\big(-\ln \mu_{j,\alpha}\big) & \text{if } n = 4, \\ O\big(\mu_{j,\alpha}^{4-n}\big) & \text{if } n \geq 5. \end{cases} \tag{9-7}$$

Indeed, if $j \in \mathcal{D}_i$, then $j > i$. Moreover, for any $j \in \mathcal{D}_i$, we have $i \in \mathcal{A}_j$, so $s_{j,i,\alpha} \geq r_{j,\alpha}$, and clearly $s_{j,i,\alpha}^2 = o(\mu_{j,\alpha}\mu_{i,\alpha}^{-1}) = o(\mu_{j,\alpha}\mu_\alpha^{-1})$. In particular, $\sqrt{\mu_\alpha} r_{j,\alpha} = o\big(\sqrt{\mu_{j,\alpha}}\big)$, and (9-7) is a direct consequence of (H_i), thanks to (9-6). Now we prove Proposition 9.2 in several steps. In the sequel we let $R_t(\alpha)$ represent any quantity such that

$$R_t(\alpha) = \begin{cases} o\big(-\mu_{i,\alpha}^2 \ln \mu_{i,\alpha}\big) & \text{if } n = 4, \\ o(\mu_{i,\alpha}^2) & \text{if } n \geq 5. \end{cases} \tag{9-8}$$

The first step in the proof of Proposition 9.2 is as follows.

Step 1. *Let $i \in \{1, \dots, N\}$ be arbitrary. In case $i < N$, assume that* (H_i) *holds true. Let*

$$\mathcal{F}_\alpha = \Big(64\omega_3\big(\langle A(x_i)\Lambda_i, \Lambda_i\rangle_{\mathbb{R}^p} - \tfrac{1}{6} S_g(x_i)\big) + o(1)\Big)\mu_{i,\alpha}^2 \ln \frac{r_{i,\alpha}}{\mu_{i,\alpha}} + o\big(-\mu_{i,\alpha}^2 \ln \mu_{i,\alpha}\big)$$

if $n = 4$, and

$$\mathcal{F}_\alpha = \left(\left(\langle A(x_i)\Lambda_i, \Lambda_i\rangle_{\mathbb{R}^p} - \frac{n-2}{4(n-1)} S_g(x_i)\right) \int_{\mathbb{R}^n} u_0^2 dx + o(1)\right)\mu_{i,\alpha}^2$$

if $n \geq 5$. Then we have, up to passing to a subsequence,

$$\mathcal{F}_\alpha = \Big(\tfrac{1}{2} n^{n-2}(n-2)^n \omega_{n-1}\langle \Lambda_i, \mathcal{H}_i(0)\rangle_{\mathbb{R}^p} + o(1)\Big)\mu_{i,\alpha}^{n-2} r_{i,\alpha}^{2-n}$$

if $\sqrt{\mu_\alpha} r_{i,\alpha} = o\big(\sqrt{\mu_{i,\alpha}}\big)$, and $\mathcal{F}_\alpha = O\big(\mu_{i,\alpha}^{(n-2)/2}\mu_\alpha^{(n-2)/2}\big)$ otherwise, where $\mathcal{H}_i$ is as in Lemma 9.1, the $r_{i,\alpha}$'s are as in (8-3), *and u_0 is as in* (4-3).

Proof of Step 1. We apply the Pohozaev identity (7-2) of Proposition 7.1 in Section 7 to $\mathcal{U}_\alpha$ in $\Omega_{i,\alpha}$ with $X = X^\alpha$ given by

$$X^\alpha(x) = \left(1 - \frac{1}{6(n-1)} Rc_g^\sharp(x)\big(\nabla f_\alpha(x), \nabla f_\alpha(x)\big)\right)\nabla f_\alpha(x), \tag{9-9}$$

where $f_\alpha(x) = \frac{1}{2} d_g(x_{i,\alpha}, x)^2$, and $Rc_g^\sharp$ is the $(0,2)$-tensor field we get from the $(2,0)$-Ricci tensor Rc_g due to the musical isomorphism. We obtain

$$\begin{aligned} &\int_{\Omega_{i,\alpha}} \langle A_\alpha \mathcal{U}_\alpha, X^\alpha(\nabla \mathcal{U}_\alpha)\rangle_{\mathbb{R}^p} dv_g \\ &\qquad + \frac{n-2}{4n}\int_{\Omega_{i,\alpha}} (\Delta_g(\operatorname{div}_g X^\alpha))|\mathcal{U}_\alpha|^2 dv_g + \frac{n-2}{2n}\int_{\Omega_{i,\alpha}} (\operatorname{div}_g X^\alpha)\langle A_\alpha \mathcal{U}_\alpha, \mathcal{U}_\alpha\rangle_{\mathbb{R}^p} dv_g \\ &\qquad\qquad = Q_\alpha - \sum_{j \in \mathcal{D}_i} Q_\alpha^j + R_{1,\alpha} + R_{2,\alpha} - \sum_{j \in \mathcal{D}_i} R_{2,\alpha}^j, \end{aligned} \tag{9-10}$$

where, if $\nu = \nu_\alpha$ stands for the unit outer normal to $\partial B_{x_{i,\alpha}}(\delta_i r_{i,\alpha})$, the Q_α's are given by

$$Q_\alpha = \frac{n-2}{2n}\int_{\partial B_{x_{i,\alpha}}(\delta_i r_{i,\alpha})} (\mathrm{div}_g X^\alpha)\langle \partial_\nu \mathcal{U}_\alpha, \mathcal{U}_\alpha\rangle_{\mathbb{R}^p} d\sigma_g \\ - \int_{\partial B_{x_{i,\alpha}}(\delta_i r_{i,\alpha})} \left(\tfrac{1}{2} X^\alpha(\nu)|\nabla \mathcal{U}_\alpha|^2 - \langle X^\alpha(\nabla \mathcal{U}_\alpha), \partial_\nu \mathcal{U}_\alpha\rangle_{\mathbb{R}^p}\right) d\sigma_g, \quad (9\text{-}11)$$

the Q_α^j's are given by

$$Q_\alpha^j = \frac{n-2}{2n}\int_{\partial \Omega_{i,j,\alpha}} (\mathrm{div}_g X^\alpha)\langle \partial_\nu \mathcal{U}_\alpha, \mathcal{U}_\alpha\rangle_{\mathbb{R}^p} d\sigma_g \\ - \int_{\partial \Omega_{i,j,\alpha}} \left(\tfrac{1}{2} X^\alpha(\nu)|\nabla \mathcal{U}_\alpha|^2 - \langle X^\alpha(\nabla \mathcal{U}_\alpha), \partial_\nu \mathcal{U}_\alpha\rangle_{\mathbb{R}^p}\right) d\sigma_g, \quad (9\text{-}12)$$

where $\Omega_{i,j,\alpha}$ is as in (8-15), the $R_{1,\alpha}$'s are given by

$$R_{1,\alpha} = -\int_{\Omega_{i,\alpha}} (T_\nabla X^\alpha)_{\mathcal{U}_\alpha} dv_g, \quad (9\text{-}13)$$

where $(T_\nabla X)_{\mathcal{U}} = \sum_{i=1}^p S_X^\sharp(\nabla u_i, \nabla u_i)$ and S_X is as in (7-1), the $R_{2,\alpha}$'s are given by

$$R_{2,\alpha} = \frac{n-2}{2n}\int_{\partial B_{x_{i,\alpha}}(\delta_i r_{i,\alpha})} X^\alpha(\nu)|\mathcal{U}_\alpha|^{2^\star} d\sigma_g - \frac{n-2}{4n}\int_{\partial B_{x_{i,\alpha}}(\delta_i r_{i,\alpha})} (\partial_\nu(\mathrm{div}_g X^\alpha))|\mathcal{U}_\alpha|^2 d\sigma_g, \quad (9\text{-}14)$$

and the $R_{2,\alpha}^j$'s are given by

$$R_{2,\alpha}^j = \frac{n-2}{2n}\int_{\partial \Omega_{i,j,\alpha}} X^\alpha(\nu)|\mathcal{U}_\alpha|^{2^\star} d\sigma_g - \frac{n-2}{4n}\int_{\partial \Omega_{i,j,\alpha}} (\partial_\nu(\mathrm{div}_g X^\alpha))|\mathcal{U}_\alpha|^2 d\sigma_g. \quad (9\text{-}15)$$

Note that $\mathfrak{D}_i = \varnothing$ if $i = N$. Thanks to the expression of the X^α's in (9-9) we have the estimates

$$\begin{aligned} |X^\alpha(x)| &= O(d_g(x_{i,\alpha}, x)), \\ \mathrm{div}_g X^\alpha(x) - n &= O(d_g(x_{i,\alpha}, x)^2), \\ |\nabla(\mathrm{div}_g X^\alpha)(x)| &= O(d_g(x_{i,\alpha}, x)), \\ \Delta_g(\mathrm{div}_g X^\alpha)(x) &= \frac{n}{n-1} S_g(x_{i,\alpha}) + O(d_g(x_{i,\alpha}, x)). \end{aligned} \quad (9\text{-}16)$$

In what follows we estimate the different terms involved in (9-10). We start with estimates on the Q_α^j's and $R_{2,\alpha}^j$'s in (9-12) and (9-15). Since

$$d_g(x_{i,\alpha}, x) \le d_g(x_{i,\alpha}, x_{j,\alpha}) + R_{i,j} s_{j,i,\alpha} = O\left(\sqrt{\frac{\mu_{i,\alpha}}{\mu_{j,\alpha}}} s_{j,i,\alpha}\right)$$

on $\partial\Omega_{i,j,\alpha}$, we obtain from Lemma 8.2, (9-7) and (9-16) that

$$Q_\alpha^j + R_{2,\alpha}^j = O\left(\sqrt{\frac{\mu_{i,\alpha}}{\mu_{j,\alpha}}} \mu_{j,\alpha}^{n-2} s_{j,i,\alpha}^{2-n}\right) = R_t(\alpha), \quad (9\text{-}17)$$

where $R_t(\alpha)$ is as in (9-8). Now we estimate the $R_{2,\alpha}$'s in (9-14). Still from Lemma 8.2, we obtain by direct computations, using (8-6) and (9-16), that

$$R_{2,\alpha} = O(\mu_{i,\alpha}^{n-2}) + o\big(\mu_{i,\alpha}^{n-2} r_{i,\alpha}^{2-n}\big). \quad (9\text{-}18)$$

Concerning the right-hand side of (9-10) it remains to estimate the Q_α's in (9-11) and the $R_{1,\alpha}$'s in (9-13). We start with estimates for the $R_{1,\alpha}$'s. We remark that $S^\sharp_{X^\alpha} = O(d_g(x_{i,\alpha}, x)^2)$ and that

$$(T_\nabla X^\alpha)_{\mathcal{B}^i_\alpha} = O\big(d_g(x_{i,\alpha}, x)^3 |\nabla B_{i,\alpha}|^2\big).$$

In particular, we can write

$$\begin{aligned} R_{1,\alpha} = O\bigg(\int_{\Omega_{i,\alpha}} d_g(x_{i,\alpha}, x)^3 |\nabla B_{i,\alpha}|^2 dv_g\bigg) + O\bigg(\int_{\Omega_{i,\alpha}} d_g(x_{i,\alpha}, x)^2 |\nabla B_{i,\alpha}| |\nabla \left(\mathcal{U}^\alpha - B_{i,\alpha}\Lambda_i\right)| dv_g\bigg) \\ + O\bigg(\int_{\Omega_{i,\alpha}} d_g(x_{i,\alpha}, x)^2 |\nabla \left(\mathcal{U}^\alpha - B_{i,\alpha}\Lambda_i\right)|^2 dv_g\bigg). \end{aligned}$$

Direct computations lead to

$$\int_{\Omega_{i,\alpha}} d_g(x_{i,\alpha}, x)^{2+\kappa} |\nabla B_{i,\alpha}|^2 dv_g = \begin{cases} O_\kappa\big(-\mu_{i,\alpha}^2 \ln \mu_{i,\alpha}\big) & \text{if } n = 4, \\ O_\kappa(\mu_{i,\alpha}^2) & \text{if } n \geq 5, \end{cases}$$

where $O_\kappa = O$ if $\kappa = 0$, and $O_\kappa = o$ if $\kappa = 1$. Integrating by parts and using Lemma 8.1, Lemma 8.2, and (9-7), we can write

$$\begin{aligned} &\int_{\Omega_{i,\alpha}} d_g(x_{i,\alpha}, x)^2 |\nabla(\mathcal{U}_\alpha - B_{i,\alpha}\Lambda_i)|^2 dv_g \\ &= O\bigg(\int_{\partial\Omega_{i,\alpha}} |\mathcal{U}_\alpha - B_{i,\alpha}\Lambda_i| d_g(x_{i,\alpha}, x)^2 |\nabla(\mathcal{U}_\alpha - B_{i,\alpha}\Lambda_i)| d\sigma_g\bigg) + O\bigg(\int_{\partial\Omega_{i,\alpha}} d_g(x_{i,\alpha}, x) |\mathcal{U}_\alpha - B_{i,\alpha}\Lambda_i|^2 d\sigma_g\bigg) \\ &\qquad + O\bigg(\int_{\Omega_{i,\alpha}} |\mathcal{U}_\alpha - B_{i,\alpha}\Lambda_i|^2 dv_g\bigg) + \int_{\Omega_{i,\alpha}} d_g(x_{i,\alpha}, x)^2 \big\langle \mathcal{U}_\alpha - B_{i,\alpha}\Lambda_i, \Delta_g(\mathcal{U}_\alpha - B_{i,\alpha}\Lambda_i)\big\rangle_{\mathbb{R}^p} dv_g, \end{aligned}$$

and then

$$\begin{aligned} &\int_{\Omega_{i,\alpha}} d_g(x_{i,\alpha}, x)^2 |\nabla(\mathcal{U}_\alpha - B_{i,\alpha}\Lambda_i)|^2 dv_g \\ &\qquad = \int_{\Omega_{i,\alpha}} d_g(x_{i,\alpha}, x)^2 \big\langle \mathcal{U}_\alpha - B_{i,\alpha}\Lambda_i, \Delta_g(\mathcal{U}_\alpha - B_{i,\alpha}\Lambda_i)\big\rangle_{\mathbb{R}^p} dv_g + o\big(\mu_{i,\alpha}^{n-2} r_{i,\alpha}^{2-n}\big) + R_t(\alpha), \end{aligned}$$

where $R_t(\alpha)$ is as in (9-8). It remains to remark that thanks to the equations satisfied by the $\mathcal{U}_\alpha$'s, and the expression of Δ_g in geodesic polar coordinates, we have

$$\begin{aligned} &\int_{\Omega_{i,\alpha}} d_g(x_{i,\alpha}, x)^2 \big\langle \mathcal{U}_\alpha - B_{i,\alpha}\Lambda_i, \Delta_g(\mathcal{U}_\alpha - B_{i,\alpha}\Lambda_i)\big\rangle_{\mathbb{R}^p} dv_g \\ &\quad = O\bigg(\int_{\Omega_{i,\alpha}} d_g(x_{i,\alpha}, x)^2 |\mathcal{U}_\alpha - B_{i,\alpha}\Lambda_i| \big(|\mathcal{U}_\alpha|^{2^\star - 1} + B_{i,\alpha}^{2^\star - 1}\big) dv_g\bigg) \\ &\qquad + O\bigg(\int_{\Omega_{i,\alpha}} d_g(x_{i,\alpha}, x)^2 |\mathcal{U}_\alpha - B_{i,\alpha}\Lambda_i| |\mathcal{U}_\alpha| dv_g\bigg) + O\bigg(\int_{\Omega_{i,\alpha}} d_g(x_{i,\alpha}, x)^3 |\mathcal{U}_\alpha - B_{i,\alpha}\Lambda_i| |\nabla B_{i,\alpha}| dv_g\bigg), \end{aligned}$$

so that, by Lemma 8.1, using Hölder's inequalities,

$$R_{1,\alpha} = o\big(\mu_{i,\alpha}^{n-2} r_{i,\alpha}^{2-n}\big) + R_t(\alpha), \tag{9-19}$$

where $R_t(\alpha)$ is as in (9-8). Still concerning the right-hand side of (9-10) it remains to estimate the Q_α's in (9-11). Thanks to Lemma 8.2 and Lemma 9.1, we get by simple computations that

$$Q_\alpha = \left(-\tfrac{1}{2}n^{n-2}(n-2)^n\omega_{n-1}\langle\Lambda_i, \mathcal{H}_i(0)\rangle_{\mathbb{R}^p} + o(1)\right)\mu_{i,\alpha}^{n-2}r_{i,\alpha}^{2-n} + O\left(\mu_{i,\alpha}^{n-2}\right), \tag{9-20}$$

if $r_{i,\alpha} = o\left(\sqrt{\mu_{i,\alpha}/\mu_\alpha}\right)$, and $Q_\alpha = O\left(\mu_{i,\alpha}^{(n-2)/2}\mu_\alpha^{(n-2)/2}\right)$ otherwise. Now we concentrate on the left-hand side of (9-10). Writing $A_\alpha(x) = A_\alpha(x_{i,\alpha}) + O(d_g(x_{i,\alpha}, x))$, we get

$$\int_{\Omega_{i,\alpha}}\left\langle A_\alpha\mathcal{U}_\alpha, X^\alpha(\nabla\mathcal{U}_\alpha)\right\rangle_{\mathbb{R}^p}dv_g = \sum_{j,k=1}^{p}A_{jk}^\alpha(x_{i,\alpha})\int_{\Omega_{i,\alpha}}u_{j,\alpha}X^\alpha(\nabla\mathcal{U}_\alpha)_k dv_g + O\left(\int_{\Omega_{i,\alpha}}d_g(x_{i,\alpha},x)^2|\mathcal{U}_\alpha||\nabla\mathcal{U}_\alpha|dv_g\right).$$

Using the Cauchy–Schwarz inequality, we can write

$$\int_{\Omega_{i,\alpha}} d_g(x_{i,\alpha}, x)^2|\mathcal{U}_\alpha||\nabla\mathcal{U}_\alpha|dv_g \le \Pi_{j=0}^1\left(\int_{\Omega_{i,\alpha}} d_g(x_{i,\alpha}, x)^{3-2j}|\nabla^{1-j}\mathcal{U}_\alpha|^2dv_g\right)^{1/2}.$$

Using Lemma 8.1 it is easily checked that

$$\int_{\Omega_{i,\alpha}} d_g(x_{i,\alpha}, x)|\mathcal{U}_\alpha|^2dv_g = R_t(\alpha), \tag{9-21}$$

where $R_t(\alpha)$ is as in (9-8). We integrate by parts and use the equations satisfied by the $\mathcal{U}_\alpha$'s, together with Lemma 8.1, Lemma 8.2, and (9-7), to obtain

$$\begin{aligned}\int_{\Omega_{i,\alpha}} d_g(x_{i,\alpha}, x)^3|\nabla\mathcal{U}_\alpha|^2dv_g = O\left(\int_{\partial\Omega_{i,\alpha}} d_g(x_{i,\alpha}, x)^3|\nabla\mathcal{U}_\alpha||\mathcal{U}_\alpha|d\sigma_g\right)&\\ + O\left(\int_{\partial\Omega_{i,\alpha}} d_g(x_{i,\alpha}, x)^2|\mathcal{U}_\alpha|^2d\sigma_g\right) + O\left(\int_{\Omega_{i,\alpha}} d_g(x_{i,\alpha}, x)^3|\mathcal{U}_\alpha|^{2^\star}dv_g\right)&\\ + O\left(\int_{\Omega_{i,\alpha}} d_g(x_{i,\alpha}, x)|\mathcal{U}_\alpha|^2dv_g\right),&\end{aligned}$$

and then that

$$\int_{\Omega_{i,\alpha}} d_g(x_{i,\alpha}, x)^3|\nabla\mathcal{U}_\alpha|^2dv_g = o\left(\mu_{i,\alpha}^{n-2}r_{i,\alpha}^{2-n}\right) + R_t(\alpha),$$

where $R_t(\alpha)$ is as in (9-8). Thus we get that

$$\int_{\Omega_{i,\alpha}}\langle A_\alpha\mathcal{U}_\alpha, X^\alpha(\nabla\mathcal{U}_\alpha)\rangle_{\mathbb{R}^p}dv_g = \sum_{j,k=1}^{p}A_{jk}^\alpha(x_{i,\alpha})\int_{\Omega_{i,\alpha}}u_{j,\alpha}X^\alpha(\nabla\mathcal{U}_\alpha)_k dv_g + o\left(\mu_{i,\alpha}^{n-2}r_{i,\alpha}^{2-n}\right) + R_t(\alpha).$$

Integrating by parts again, and estimating the different terms as above, it is easily checked that we actually have

$$\int_{\Omega_{i,\alpha}}\langle A_\alpha\mathcal{U}_\alpha, X^\alpha(\nabla\mathcal{U}_\alpha)\rangle_{\mathbb{R}^p}dv_g = -\frac{n}{2}\sum_{j,k=1}^{p}A_{jk}^\alpha(x_{i,\alpha})\int_{\Omega_{i,\alpha}}u_{j,\alpha}u_{k,\alpha}dv_g + o\left(\mu_{i,\alpha}^{n-2}r_{i,\alpha}^{2-n}\right) + R_t(\alpha),$$

where $R_t(\alpha)$ is as in (9-8). Proceeding as above, thanks to (9-16), one finally gets that

$$\begin{aligned}\int_{\Omega_{i,\alpha}} \big\langle A_\alpha \mathcal{U}_\alpha, X^\alpha(\nabla \mathcal{U}_\alpha)\big\rangle_{\mathbb{R}^p} dv_g + \frac{n-2}{4n}\int_{\Omega_{i,\alpha}} (\Delta_g(\operatorname{div}_g X^\alpha))|\mathcal{U}_\alpha|^2 dv_g \\ + \frac{n-2}{2n}\int_{\Omega_{i,\alpha}} (\operatorname{div}_g X^\alpha)\langle A_\alpha \mathcal{U}_\alpha, \mathcal{U}_\alpha\rangle_{\mathbb{R}^p} dv_g \\ = -\sum_{j,k=1}^{p} A^\alpha_{jk}(x_{i,\alpha})\int_{\Omega_{i,\alpha}} u_{j,\alpha}u_{k,\alpha}dv_g + \frac{n-2}{4(n-1)}S_g(x_{i,\alpha})\int_{\Omega_{i,\alpha}} |\mathcal{U}_\alpha|^2 dv_g + o\big(\mu_{i,\alpha}^{n-2} r_{i,\alpha}^{2-n}\big) + R_t(\alpha),\end{aligned} \tag{9-22}$$

where $R_t(\alpha)$ is as in (9-8). We have

$$\int_{\Omega_{i,j,\alpha}} B_{i,\alpha}^2 dv_g = R_t(\alpha) \quad \text{for all } j \in \mathcal{D}_i. \tag{9-23}$$

Indeed, if $d_g(x_{i,\alpha}, x_{j,\alpha})/s_{j,i,\alpha} \to +\infty$ as $\alpha \to +\infty$, then

$$\int_{B_{x_{j,\alpha}}(R_{i,j}s_{j,i,\alpha})} B_{i,\alpha}^2 dv_g = O\big(s_{j,i,\alpha}^n B_{i,\alpha}(x_{j,\alpha})^2\big) = O\big(\mu_{j,\alpha}^{n-2} s_{j,i,\alpha}^{4-n}\big) = R_t(\alpha),$$

thanks to (9-7), and if $d_g(x_{i,\alpha}, x_{j,\alpha}) = O(s_{j,i,\alpha})$, then $s_{j,i,\alpha} = o(\mu_{i,\alpha})$ and

$$\int_{B_{x_{j,\alpha}}(R_{i,j}s_{j,i,\alpha})} B_{i,\alpha}^2 dv_g = O\big(\mu_{i,\alpha}^{2-n} s_{j,i,\alpha}^n\big) = o(\mu_{i,\alpha}^2).$$

Clearly, (9-23) follows from these two equations. Plugging (9-23) into (9-22), we get from Lemma 8.1 that

$$\begin{aligned}\int_{\Omega_{i,\alpha}} \big\langle A_\alpha \mathcal{U}_\alpha, X^\alpha(\nabla \mathcal{U}_\alpha)\big\rangle_{\mathbb{R}^p} dv_g + \frac{n-2}{4n}\int_{\Omega_{i,\alpha}} (\Delta_g(\operatorname{div}_g X^\alpha))|\mathcal{U}_\alpha|^2 dv_g + \frac{n-2}{2n}\int_{\Omega_{i,\alpha}} (\operatorname{div}_g X^\alpha)\langle A_\alpha \mathcal{U}_\alpha, \mathcal{U}_\alpha\rangle_{\mathbb{R}^p} dv_g \\ = -\Big(\big\langle A(x_i)\Lambda_i, \Lambda_i\big\rangle_{\mathbb{R}^p} - \frac{n-2}{4(n-1)}S_g(x_i) + o(1)\Big)\int_{B_{x_{i,\alpha}}(\delta_i r_{i,\alpha})} B_{i,\alpha}^2 dv_g + o\big(\mu_{i,\alpha}^{n-2} r_{i,\alpha}^{2-n}\big) + R_t(\alpha).\end{aligned} \tag{9-24}$$

We have

$$\int_{B_{x_{i,\alpha}}(\delta_i r_{i,\alpha})} B_{i,\alpha}^2 dv_g = \begin{cases} 64\omega_3 \mu_{i,\alpha}^2 \ln(r_{i,\alpha}/\mu_{i,\alpha}) + o\big(-\mu_{i,\alpha}^2 \ln \mu_{i,\alpha}\big) & \text{if } n = 4, \\ \big(\int_{\mathbb{R}^n} u_0^2 dx\big)\mu_{i,\alpha}^2 + o(\mu_{i,\alpha}^2) & \text{if } n \geq 5, \end{cases} \tag{9-25}$$

where u_0 is given by (4-3). Combining (9-10), (9-17)–(9-20), (9-24), and (9-25) yields the proof of Step 1. $\square$

Step 2. *Let $i \in \{1, \dots, N\}$ be arbitrary. In case $i < N$, assume that* (H_i) *holds. Let $\mathcal{K}_\alpha$ be given by*

$$\mathcal{K}_\alpha = \Big(64\omega_3\big(A(x_i)\Lambda_i - \tfrac{1}{6}S_g(x_i)\Lambda_i\big) + o(1)\Big)\mu_{i,\alpha}^2 \ln \frac{r_{i,\alpha}}{\mu_{i,\alpha}} + o\big(-\mu_{i,\alpha}^2 \ln \mu_{i,\alpha}\big)$$

in case $n = 4$, and

$$\mathcal{K}_\alpha = \bigg(\Big(A(x_i)\Lambda_i - \frac{n-2}{4(n-1)}S_g(x_i)\Lambda_i\Big)\int_{\mathbb{R}^n} u_0^2 dx + o(1)\bigg)\mu_{i,\alpha}^2$$

in case $n \ge 5$. Then, up to passing to a subsequence, we have

$$\mathcal{K}_\alpha = \Big(n^{n-2}(n-2)^{n-1}\omega_{n-1}\Big(\mathcal{H}_i(0) + \frac{n-4}{2}\langle\mathcal{H}_i(0), \Lambda_i\rangle\Lambda_i\Big) + o(1)\Big)\mu_{i,\alpha}^{n-2} r_{i,\alpha}^{2-n},$$

if $\sqrt{\mu_\alpha} r_{i,\alpha} = o\big(\sqrt{\mu_{i,\alpha}}\big)$, and $\mathcal{K}_\alpha = O\big(\mu_{i,\alpha}^{(n-2)/2}\mu_\alpha^{(n-2)/2}\big)$ otherwise, where $\mathcal{H}_i$ is as in Lemma 9.1, the $r_{i,\alpha}$'s are as in (8-3), *and u_0 is as in* (4-3).

Proof of Step 2. We multiply the line k of the system (1-1) by $u_{l,\alpha}$ and integrate over $\Omega_{i,\alpha}$. This leads to

$$\int_{\Omega_{i,\alpha}} u_{l,\alpha}\Delta_g u_{k,\alpha} dv_g + \sum_{m=1}^p \int_{\Omega_{i,\alpha}} A_{km}^\alpha u_{l,\alpha} u_{m,\alpha} dv_g = \int_{\Omega_{i,\alpha}} |\mathcal{U}_\alpha|^{2^\star-2} u_{k_\alpha} u_{l,\alpha} dv_g. \tag{9-26}$$

Let the $\Lambda_{i,k}$'s, $k = 1, \dots, p$, be the components of Λ_i, and the $\mathcal{H}_{i,k}$'s be the components of $\mathcal{H}_i$. We define $S_{k,l}^\alpha$ by

$$S_{k,l}^\alpha = \Big(n^{n-2}(n-2)^{n-1}\omega_{n-1}\big(\Lambda_{i,k}\mathcal{H}_{i,l}(0) - \Lambda_{i,l}\mathcal{H}_{i,k}(0)\big) + o(1)\Big)\mu_{i,\alpha}^{n-2} r_{i,\alpha}^{2-n},$$

if $r_{i,\alpha} = o\big(\sqrt{\mu_{i,\alpha}/\mu_\alpha}\big)$, and $S_{k,l}^\alpha = O\Big(\mu_{i,\alpha}^{(n-2)/2}\mu_\alpha^{(n-2)/2}\Big)$ otherwise. We also define $T_{k,l}^\alpha$ by

$$T_{k,l}^\alpha = \begin{cases} \big(64\omega_3 W_{k,l} + o(1)\big)\mu_{i,\alpha}^2 \ln(r_{i,\alpha}/\mu_{i,\alpha}) + o\big(-\mu_{i,\alpha}^2 \ln \mu_{i,\alpha}\big) & \text{if } n = 4, \\ \big(W_{k,l}\int_{\mathbb{R}^n} u_0^2 dx + o(1)\big)\mu_{i,\alpha}^2 & \text{if } n \ge 5, \end{cases}$$

where

$$W_{k,l} = \sum_{m=1}^p \big(A(x_i)_{lm}\Lambda_{i,k}\Lambda_{i,m} - A(x_i)_{km}\Lambda_{i,l}\Lambda_{i,m}\big),$$

and u_0 is given by (4-3). Integrating by parts, thanks to Lemma 8.2 and Lemma 9.1, we have

$$\begin{aligned}\int_{\Omega_{i,\alpha}} u_{l,\alpha}\Delta_g u_{k,\alpha} dv_g &= \int_{\Omega_{i,\alpha}} u_{k,\alpha}\Delta_g u_{l,\alpha} dv_g + \int_{\partial\Omega_{i,\alpha}} (u_{k,\alpha}\partial_\nu u_{l,\alpha} - u_{l,\alpha}\partial_\nu u_{k,\alpha})\, d\sigma_g \\ &= \int_{\Omega_{i,\alpha}} |\mathcal{U}_\alpha|^{2^\star-2} u_{k_\alpha} u_{l,\alpha} dv_g - \sum_{m=1}^p \int_{\Omega_{i,\alpha}} A_{lm}^\alpha u_{k,\alpha} u_{m,\alpha} dv_g + O\Big(\sum_{j\in\mathfrak{D}_i} \mu_{j,\alpha}^{n-2} s_{j,i,\alpha}^{2-n}\Big) + S_{k,l}^\alpha.\end{aligned}$$

Now we write $A_\alpha(x) = A_\alpha(x_{i,\alpha}) + O(d_g(x_{i,\alpha}, x))$. With similar estimates as in the proof of Step 1, thanks to (9-21), we get that

$$\sum_{m=1}^p \int_{\Omega_{i,\alpha}} A_{lm}^\alpha u_{k,\alpha} u_{m,\alpha} dv_g - \sum_{m=1}^p \int_{\Omega_{i,\alpha}} A_{km}^\alpha u_{l,\alpha} u_{m,\alpha} dv_g = o\big(\mu_{i,\alpha}^{n-2} r_{i,\alpha}^{2-n}\big) + T_{k,l}^\alpha.$$

Coming back to (9-26) with all these estimates, thanks to (9-7), we obtain that $S_{k,l}^\alpha = T_{k,l}^\alpha$. In particular, $\sum_k S_{k,l}^\alpha \Lambda_{i,k} = \sum_k T_{k,l}^\alpha \Lambda_{i,k}$ and Step 2 follows from Step 1. This ends the proof of Step 2. □

Conclusion of the proof of Proposition 9.2. Equations (9-4) and (9-5) follow from Step 2. It remains to prove that $\langle\Lambda_i, \nabla\mathcal{H}_i(0)\rangle_{\mathbb{R}^p} \equiv 0$. We assume here that $\sqrt{\mu_\alpha} r_{i,\alpha} = o\big(\sqrt{\mu_{i,\alpha}}\big)$. In particular, $r_{i,\alpha} \to 0$ as $\alpha \to +\infty$. Let Y be an arbitrary 1-form in $\mathbb{R}^n$. We apply once more the Pohozaev identity (7-2) to $\mathcal{U}_\alpha$ in $\Omega_{i,\alpha}$. However, here we choose $X = X^\alpha$ to be given in the exponential chart at $x_{i,\alpha}$ by

$$X_\kappa^\alpha = Y_\kappa - \tfrac{2}{3} R_{\kappa jkl}(x_{i,\alpha}) x^j x^k Y^l,$$

where $Y^l = Y_l$ for all l and the $R_{\kappa jkl}$ are the components of the Riemann tensor Rm_g at $x_{i,\alpha}$ in the exponential chart. As is easily checked, still in geodesic normal coordinates at $x_{i,\alpha}$,

$$(\nabla X^\alpha)_{\kappa j} = -\mathcal{R}_{\kappa jkl}(x_{i,\alpha})x^k Y^l + O(|x|^2),$$

so that $\operatorname{div}_g(X^\alpha) = O\big(|x|^2\big)$. Then, thanks to the symmetries of the Riemann tensor, we obtain with the Pohozaev identity that

$$\begin{aligned}\int_{\partial\Omega_{i,\alpha}} &\big(\tfrac{1}{2}X^\alpha(\nu)|\nabla\mathcal{U}_\alpha|^2 - \langle X^\alpha(\nabla\mathcal{U}_\alpha), \partial_\nu\mathcal{U}_\alpha\rangle_{\mathbb{R}^p}\big)\,d\sigma_g + \int_{\Omega_{i,\alpha}}\langle A_\alpha\mathcal{U}_\alpha, X^\alpha(\nabla\mathcal{U}_\alpha)\rangle_{\mathbb{R}^p}dv_g \\ &= O\bigg(\int_{\Omega_{i,\alpha}}|\mathcal{U}_\alpha|^2dv_g\bigg) + O\bigg(\int_{\Omega_{i,\alpha}}d_g(x_{i,\alpha},x)^2|\nabla\mathcal{U}_\alpha|^2dv_g\bigg) + O\bigg(\int_{\partial\Omega_{i,\alpha}}|\mathcal{U}_\alpha|^{2^\star}d\sigma_g\bigg) \qquad (9\text{-}27)\\ &\quad + O\bigg(\int_{\partial\Omega_{i,\alpha}}|\mathcal{U}_\alpha|^2d\sigma_g\bigg) + O\bigg(\int_{\partial\Omega_{i,\alpha}}|\partial_\nu\mathcal{U}_\alpha||\mathcal{U}_\alpha|d\sigma_g\bigg),\end{aligned}$$

Estimating the right-hand side of (9-27) via (9-7) and using Lemma 8.1 and Lemma 8.2, we get

$$\int_{\partial\Omega_{i,\alpha}}\big(\tfrac{1}{2}X^\alpha(\nu)|\nabla\mathcal{U}_\alpha|^2 - \langle X^\alpha(\nabla\mathcal{U}_\alpha), \partial_\nu\mathcal{U}_\alpha\rangle_{\mathbb{R}^p}\big)\,d\sigma_g + \int_{\Omega_{i,\alpha}}\langle A_\alpha\mathcal{U}_\alpha, X^\alpha(\nabla\mathcal{U}_\alpha)\rangle_{\mathbb{R}^p}dv_g = \hat{R}_t(\alpha), \quad (9\text{-}28)$$

where $\hat{R}_t(\alpha)$ is such that

$$\hat{R}_t(\alpha) = \begin{cases} O\big(-\mu_{i,\alpha}^2\ln\mu_{i,\alpha}\big) + O\big(\mu_{i,\alpha}^{n-2}r_{i,\alpha}^{2-n}\big) & \text{if } n=4,\\ O(\mu_{i,\alpha}^2) + O\big(\mu_{i,\alpha}^{n-2}r_{i,\alpha}^{2-n}\big) & \text{if } n\ge 5.\end{cases} \qquad (9\text{-}29)$$

Now we can write

$$\int_{\Omega_{i,\alpha}}\langle A_\alpha\mathcal{U}_\alpha, X^\alpha(\nabla\mathcal{U}_\alpha)\rangle_{\mathbb{R}^p}dv_g = \sum_{k,l=1}^p A_{kl}^\alpha(x_{i,\alpha})\int_{\Omega_{i,\alpha}}u_{k,\alpha}X^\alpha(\nabla\mathcal{U}_\alpha)_l dv_g + O(T_\alpha),$$

where

$$T_\alpha = \int_{\Omega_{i,\alpha}}d_g(x_{i,\alpha},x)|\nabla\mathcal{U}_\alpha||\mathcal{U}_\alpha|dv_g,$$

obtaining

$$\begin{aligned}\int_{\Omega_{i,\alpha}}&\langle A_\alpha\mathcal{U}_\alpha, X^\alpha(\nabla\mathcal{U}_\alpha)\rangle_{\mathbb{R}^p}dv_g \\ &= \frac{1}{2}\sum_{k,l=1}^p A_{kl}^\alpha(x_{i,\alpha})\int_{\partial\Omega_{i,\alpha}}u_{k,\alpha}u_{l,\alpha}X^\alpha(\nu)\,d\sigma_g - \frac{1}{2}\sum_{k,l=1}^p A_{kl}^\alpha(x_{i,\alpha})\int_{\Omega_{i,\alpha}}u_{k,\alpha}u_{l,\alpha}\operatorname{div}_g\big(X^\alpha\big)\,dv_g + O(T_\alpha).\end{aligned}$$

As above, estimating the various terms in this equation, it follows that

$$\int_{\Omega_{i,\alpha}}\big\langle A_\alpha\mathcal{U}_\alpha, X^\alpha(\nabla\mathcal{U}_\alpha)\big\rangle_{\mathbb{R}^p}dv_g = \hat{R}_t(\alpha), \qquad (9\text{-}30)$$

where $\hat{R}_t(\alpha)$ is as in (9-29). As a consequence, coming back to (9-28), thanks to (9-30), we get

$$\int_{\partial\Omega_{i,\alpha}}\big(\tfrac{1}{2}X^\alpha(\nu)|\nabla\mathcal{U}_\alpha|^2 - \big\langle X^\alpha(\nabla\mathcal{U}_\alpha), \partial_\nu\mathcal{U}_\alpha\big\rangle_{\mathbb{R}^p}\big)\,d\sigma_g = \hat{R}_t(\alpha), \qquad (9\text{-}31)$$

where $\hat{R}_t(\alpha)$ is as in (9-29). By Lemmas 8.2 and 9.1, together with (9-7), we have

$$\int_{\partial\Omega_{i,\alpha}} \left(\tfrac{1}{2}X^\alpha(\nu)|\nabla\mathcal{U}_\alpha|^2 - \langle X^\alpha(\nabla\mathcal{U}_\alpha), \partial_\nu\mathcal{U}_\alpha\rangle_{\mathbb{R}^p}\right) d\sigma_g = \left(n^{n-2}(n-2)^{n-1}\omega_{n-1}\langle\Lambda_i, (Y(\nabla\mathcal{H}_i))_0\rangle_{\mathbb{R}^p} + o(1)\right)\mu_{i,\alpha}^{n-2} r_{i,\alpha}^{1-n} + \hat{W}_\alpha, \quad (9\text{-}32)$$

where $(Y(\nabla\mathcal{H}_i))_0 \in \mathbb{R}^p$ is such that $(Y(\nabla\mathcal{H}_i))_0^l = \sum_{\kappa=1}^n Y_\kappa(\nabla_\kappa\mathcal{H}_{i,l})(0)$ for all $l = 1, \dots, p$, and

$$\hat{W}_\alpha = \begin{cases} o\left(\mu_{i,\alpha}^2(-\ln\mu_{i,\alpha})^{3/2}\right) & \text{if } n = 4, \\ o(\mu_{i,\alpha}^2) & \text{if } n \geq 5. \end{cases}$$

As a consequence of Step 2 we have

$$r_{i,\alpha} = \begin{cases} O\left((-\ln\mu_{i,\alpha})^{-1/2}\right) & \text{if } n = 4, \\ O\left(\mu_{i,\alpha}^{(n-4)/(n-2)}\right) & \text{if } n \geq 5. \end{cases}$$

Coming back to (9-31)–(9-32), it follows that $\langle\Lambda_i, (Y(\nabla\mathcal{H}_i))_0\rangle_{\mathbb{R}^p} = 0$, and since Y is arbitrary, we get $\langle\Lambda_i, \nabla\mathcal{H}_i(0)\rangle_{\mathbb{R}^p} \equiv 0$. □

10. Proof of Theorem 0.1

We prove Theorem 0.1 using Proposition 9.2. We let $(A_\alpha)_\alpha$ be a sequence of C^1 maps from M to $M_p^s(\mathbb{R})$ such that $A_\alpha \to A$ in $C^1(M)$ as $\alpha \to +\infty$ for some C^1 map A from M to $M_p^s(\mathbb{R})$ satisfying (H) and (H′). We also let $(\mathcal{U}_\alpha)_\alpha$ be an arbitrary bounded sequence in $H^1(M)$ of nonnegative solutions of (1-1) and we assume by contradiction that $\|\mathcal{U}_\alpha\|_\infty \to +\infty$ as $\alpha \to +\infty$. We order the blow-up points of the $\mathcal{U}_\alpha$'s in such a way that

$$\mu_\alpha = \mu_{1,\alpha} \geq \cdots \geq \mu_{N,\alpha},$$

where the $\mu_{i,\alpha}$'s are the weights of the vector bubble $(\mathcal{B}_\alpha^i)_\alpha$ in Proposition 4.2, and we let $\mathcal{A}_i$ be as in (8-4). We consider $\mathcal{A}_1$. By (H′), $\operatorname{Ker} A_n(x) \cap \operatorname{Vect}_+(\mathbb{R}^p) = \{0\}$ for all $x \in M$, where A_n is as in (0-2). In particular, if the $r_{i,\alpha}$'s are as in (8-3), it follows from Step 2 in Section 9 that $r_{1,\alpha} \to 0$ as $\alpha \to +\infty$. As a direct consequence, $\mathcal{A}_1 \neq \varnothing$. Let $i \in \mathcal{A}_1$. Still by Step 2 in Section 9, we have $r_{i,\alpha} \to 0$ as $\alpha \to +\infty$. By Proposition 9.2, since $\operatorname{Ker} A_n(x) \cap \operatorname{Vect}_+(\mathbb{R}^p) = \{0\}$ for all $x \in M$, for any $i \in \mathcal{A}_1 \cup \{1\}$, there exists $C_i > 0$ such that

$$r_{i,\alpha}^2 \ln\frac{1}{\mu_{i,\alpha}} \to C_i \text{ if } n = 4 \quad\text{and}\quad r_{i,\alpha}^{n-2}\mu_{i,\alpha}^{4-n} \to C_i \text{ if } n \geq 5 \quad (10\text{-}1)$$

as $\alpha \to +\infty$. By (10-1), $\mu_{i,\alpha} = o(r_{i,\alpha})$ for all $i \in \mathcal{A}_1 \cup \{1\}$. We also get from (10-1) that for any $i \in \mathcal{A}_1 \cup \{1\}$,

$$\mu_{i,\alpha} = o(r_{i,\alpha}^2) \text{ if } n = 4, 5 \quad\text{and}\quad r_{i,\alpha}^2 = o(\mu_{i,\alpha}) \text{ if } n \geq 7. \quad (10\text{-}2)$$

As a remark, it follows from (10-2) that $\mathcal{U}_\infty \equiv 0$ when $n = 4, 5$ since, if not the case, $r_{i,\alpha}^2 \leq \mu_{i,\alpha}$. It also follows from (10-2) that for any $i \in \mathcal{A}_1 \cup \{1\}$, $\mathcal{A}_i \cap \mathcal{B}_i \neq \varnothing$, where the $\mathcal{B}_i$'s are as in (8-8). By (9-3), we get with (10-2) that

$$\mathcal{H}_i(z) = \sum_{j\in\mathcal{A}_i\cap\mathcal{B}_i} \frac{\lambda_{i,j}\Lambda_j}{|z - z_{i,j}|^{n-2}}, \quad (10\text{-}3)$$

where $\mathcal{H}_i$ is as in Lemma 9.1. In particular, the Λ_i's are the S^{p-1} projections of the bubbles $(\mathcal{B}^i_\alpha)_\alpha$. Let $\mathcal{E}_1 = (\mathcal{A}_1 \cap \mathcal{B}_1) \cup \{1\}$. For any $i \in \mathcal{A}_1 \cap \mathcal{B}_1$, we have $\mathcal{A}_i \cap \mathcal{B}_i = \mathcal{E}_1 \backslash \{i\}$. We pick up some $i \in \mathcal{E}_1$ such that $d_g(x_{1,\alpha}, x_{i,\alpha}) \geq d_g(x_{1,\alpha}, x_{j,\alpha})$ for all $j \in \mathcal{E}_1$. By Proposition 9.2 we have $\langle \Lambda_i, \nabla \mathcal{H}_i(0) \rangle_{\mathbb{R}^p} = 0$. Together with (10-3), this implies that $\langle \Lambda_i, \Lambda_j \rangle_{\mathbb{R}^p} = 0$ for all $j \in \mathcal{E}_1 \setminus \{i\}$. Repeating the operation with $\mathcal{E}_1 \setminus \{i\}$, and so on up to exhaust all the indices in $\mathcal{E}_1$, we obtain that $\langle \Lambda_i, \Lambda_j \rangle_{\mathbb{R}^p} = \delta_{ij}$ for all $i, j \in \mathcal{E}_1$. Moreover, it follows from (9-4) and (9-5) in Proposition 9.2 that $V = \mathrm{Vect}\{\Lambda_i,\ i \in \mathcal{E}_1\}$ is a stable vector space of $A_n(x_1)$. Noting that $\langle \Lambda_i, \mathcal{H}_i(0) \rangle_{\mathbb{R}^p} = 0$ for all $i \in \mathcal{E}_1$, we also get with (9-4) and (9-5) in Proposition 9.2 that the Λ_i's are isotropic vectors for $A_n(x_1)$ for all $i \in \mathcal{E}_1$. In particular, we get a contradiction with (H$'$). This proves Theorem 0.1 when $n \neq 6$. When $n = 6$, thanks to Proposition 2.1, it remains to prove that our systems are weakly stable, and thus that we necessarily have $\mathcal{U}_\infty \not\equiv 0$ if we assume (H$'$). When $n = 6$, it follows from (10-1) that $r^2_{i,\alpha} \sim \mu_{i,\alpha}$. Then, by (9-3),

$$\mathcal{H}_i(z) = \sum_{j \in \mathcal{A}_i \cap \mathcal{B}_i} \frac{\lambda_{i,j} \Lambda_j}{|z - z_{i,j}|^{n-2}} + C\mathcal{U}_\infty(x_1), \tag{10-4}$$

where $r^{-4}_{i,\alpha} \mu^2_{i,\alpha} \to C$ as $\alpha \to +\infty$. As above, $\langle \Lambda_i, \Lambda_j \rangle_{\mathbb{R}^p} = \delta_{ij}$ for all $i, j \in \mathcal{E}_1$, but we may have $\mathcal{E}_1 = \{1\}$. By Proposition 9.2, $V = \mathrm{Vect}\{\Lambda_i,\ i \in \mathcal{E}_1\}$ is a stable vector space of $A_6(x_1)$ and the Λ_i's are isotropic vectors for $A_6(x_1)$ for all $i \in \mathcal{E}_1$ if $\mathcal{U}_\infty(x_1) = 0$. In particular, we do get a contradiction with (H$'$) if $\mathcal{U}_\infty(x_1) = 0$. This proves Theorem 0.1 when $n = 6$.

As a remark, if $n = 6$ and $A_6 < 0$ in M in the sense of bilinear forms, where A_6 is as in (0-2), then we also get a contradiction by (9-5) in Proposition 9.2 since $r^2_{i,\alpha} \sim \mu_{i,\alpha}$ and $\mathcal{H}_i(0), \langle \Lambda_i, \mathcal{H}_i(0) \rangle_{\mathbb{R}^p} \Lambda_i \in \mathrm{Vect}_+(\mathbb{R}^p)$. In particular, we recover analytic stability for our systems if we assume that $A_6 < 0$ in M in the sense of bilinear forms. More precisely, letting (M, g) be a smooth compact six-dimensional Riemannian manifold, $p \geq 1$ be an integer, and $A : M \to M^s_p(\mathbb{R})$ be a C^1-map such that A satisfies (H), the system (0-1) associated with A is analytically stable if $A_6(x) < 0$ in the sense of bilinear forms for all x.

As another remark, it is easily seen from (9-4) and (9-5) in Proposition 9.2 that for any $n \geq 4$, and any $i \in \mathcal{A}_1 \cup \{1\}$, $A_n(x_1)\Lambda_i \in \mathrm{Vect}_+(\mathbb{R}^p)$. In particular, we can replace (H$'$) in Theorem 0.1 by the slightly more general condition that for any $x \in M$, and any $k \in \{1, \dots, p\}$, there does not exist an orthonormal family $(e_1, \dots, e_k)$ of vectors in $\mathrm{Is}_{A_n(x)} \cap \mathrm{Vect}_+(\mathbb{R}^p)$ such that $A_n(x)V \subset V$ and $A_n(x)e_i \in \mathrm{Vect}_+(\mathbb{R}^p)$ for all i, where V is the k-dimensional subspace of $\mathbb{R}^p$ with basis $(e_1, \dots, e_k)$.

As a final remark we mention that Theorem 0.1 still holds true, and can be proved with only slight modifications in the arguments of Section 9, if the C^1 convergence of the A_α's is replaced by a $C^{0,\theta}$-convergence of the A_α's with $\theta = 1$ when $n = 4$, and $\theta > 2/(n-2)$ when $n \geq 5$.

References

[Ablowitz et al. 2004] M. J. Ablowitz, B. Prinari, and A. D. Trubatch, *Discrete and continuous nonlinear Schrödinger systems*, London Mathematical Society Lecture Note Series **302**, Cambridge University Press, 2004. MR 2005c:37117 Zbl 1057.35058

[Akhmediev and Ankiewicz 1998] N. Akhmediev and A. Ankiewicz, "Partially coherent solitons on a finite background", *Phys. Rev. Lett.* **82** (1998), 2661–2664.

[Aubin 1976] T. Aubin, "Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire", *J. Math. Pures Appl.* (9) **55**:3 (1976), 269–296. MR 55 #4288 Zbl 0336.53033

[Brendle 2008a] S. Brendle, "Blow-up phenomena for the Yamabe equation", *J. Amer. Math. Soc.* **21**:4 (2008), 951–979. MR 2425176

[Brendle 2008b] S. Brendle, "On the conformal scalar curvature equation and related problems", pp. 1–19 in *Surveys in differential geometry, XII: Geometric flows*, edited by H.-D. Cao and S.-T. Yau, Int. Press, Somerville, MA, 2008. MR 2488950 Zbl 05504035

[Brendle and Marques 2009] S. Brendle and F. C. Marques, "Blow-up phenomena for the Yamabe equation. II", *J. Differential Geom.* **81**:2 (2009), 225–250. MR 2472174 Zbl 1166.53025

[Burke et al. 1997] J. Burke, J. Bohn, B. Esry, and C. Greene, "Hartree-Fock theory for double condensates", *Phys. Rev. Lett.* **78** (1997), 3594–3597.

[Caffarelli et al. 1989] L. A. Caffarelli, B. Gidas, and J. Spruck, "Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth", *Comm. Pure Appl. Math.* **42**:3 (1989), 271–297. MR 90c:35075 Zbl 0702.35085

[Christodoulides et al. 1997] D. Christodoulides, T. Coskun, M. Mitchell, and M. Segev, "Theory of incoherent self-focusing in biased photorefractive media", *Phys. Rev. Lett.* **78** (1997), 646–649.

[Druet 2003] O. Druet, "From one bubble to several bubbles: the low-dimensional case", *J. Differential Geom.* **63**:3 (2003), 399–473. MR 2004h:53051 Zbl 1070.53017

[Druet 2004] O. Druet, "Compactness for Yamabe metrics in low dimensions", *Int. Math. Res. Not.* **23** (2004), 1143–1191. MR 2005b:53056 Zbl 1085.53029

[Druet and Hebey 2005a] O. Druet and E. Hebey, "Blow-up examples for second order elliptic PDEs of critical Sobolev growth", *Trans. Amer. Math. Soc.* **357**:5 (2005), 1915–1929. MR 2005i:58023 Zbl 1061.58017

[Druet and Hebey 2005b] O. Druet and E. Hebey, "Elliptic equations of Yamabe type", *IMRS Int. Math. Res. Surv.* **1** (2005), 1–113. MR 2006b:53046 Zbl 1081.53034

[Druet et al. 2004] O. Druet, E. Hebey, and F. Robert, *Blow-up theory for elliptic PDEs in Riemannian geometry*, Mathematical Notes **45**, Princeton University Press, 2004. MR 2005g:53058 Zbl 1059.58017

[Druet et al. 2009] O. Druet, E. Hebey, and J. Vétois, "Bounded stability for strongly coupled critical elliptic systems below the geometric threshold of the conformal laplacian", 2009. To appear in *J. Funct. Anal.*

[Gidas and Spruck 1981] B. Gidas and J. Spruck, "A priori bounds for positive solutions of nonlinear elliptic equations", *Comm. Partial Differential Equations* **6**:8 (1981), 883–901. MR 82h:35033 Zbl 0462.35041

[Hebey and Vaugon 2001] E. Hebey and M. Vaugon, "From best constants to critical functions", *Math. Z.* **237**:4 (2001), 737–767. MR 2002h:58061 Zbl 0992.58016

[Hioe 1999] F. Hioe, "Solitary waves for *N* coupled nonlinear Schrödinger equations", *Phys. Rev. Lett.* **82** (1999), 1152–1155.

[Hioe and Salter 2002] F. T. Hioe and T. S. Salter, "Special set and solutions of coupled nonlinear Schrödinger equations", *J. Phys. A* **35**:42 (2002), 8913–8928. MR 2003k:35230 Zbl 1040.35115

[Kanna and Lakshmanan 2001] T. Kanna and M. Lakshmanan, "Exact soliton solutions, shape changing collisions, and partially coherent solitons in coupled nonlinear Schrödinger equations", *Phys. Rev. Lett.* **86** (2001), 5043–5046.

[Khuri et al. 2009] M. A. Khuri, F. C. Marques, and R. M. Schoen, "A compactness theorem for the Yamabe problem", *J. Differential Geom.* **81**:1 (2009), 143–196. MR 2477893 Zbl 1162.53029

[Li and Zhang 2004] Y. Y. Li and L. Zhang, "A Harnack type inequality for the Yamabe equation in low dimensions", *Calc. Var. Partial Differential Equations* **20**:2 (2004), 133–151. MR 2005f:53053 Zbl 1113.35070

[Li and Zhang 2005] Y. Y. Li and L. Zhang, "Compactness of solutions to the Yamabe problem, II", *Calc. Var. Partial Differential Equations* **24**:2 (2005), 185–237. MR 2006f:53049 Zbl 02220860

[Li and Zhu 1999] Y. Li and M. Zhu, "Yamabe type equations on three-dimensional Riemannian manifolds", *Commun. Contemp. Math.* **1**:1 (1999), 1–50. MR 2000m:53051 Zbl 0973.53029

[Marques 2005] F. C. Marques, "A priori estimates for the Yamabe problem in the non-locally conformally flat case", *J. Differential Geom.* **71**:2 (2005), 315–346. MR 2006i:53046 Zbl 1101.53019

[Schoen 1989] R. M. Schoen, "Variational theory for the total scalar curvature functional for Riemannian metrics and related topics", pp. 120–154 in *Topics in calculus of variations (Montecatini Terme, 1987)*, edited by M. Giaquinta, Lecture Notes in Math. **1365**, Springer, Berlin, 1989. MR 90g:58023 Zbl 0702.49038

[Schoen 1991] R. M. Schoen, "On the number of constant scalar curvature metrics in a conformal class", pp. 311–320 in *Differential geometry*, edited by B. Lawson and K. Tenenblat, Pitman Monogr. Surveys Pure Appl. Math. **52**, Longman Sci. Tech., Harlow, 1991. MR 94e:53035 Zbl 0733.53021

[Struwe 1984] M. Struwe, "A global compactness result for elliptic boundary value problems involving limiting nonlinearities", *Math. Z.* **187**:4 (1984), 511–517. MR 86k:35046 Zbl 0535.35025

[Struwe 1990] M. Struwe, *Variational methods*, Springer, Berlin, 1990. MR 92b:49002 Zbl 0746.49010

[Trudinger 1968] N. S. Trudinger, "Remarks concerning the conformal deformation of Riemannian structures on compact manifolds", *Ann. Scuola Norm. Sup. Pisa* (3) **22** (1968), 265–274. MR 39 #2093 Zbl 0159.23801

Received 29 Jan 2009. Revised 22 Jun 2009. Accepted 21 Jul 2009.

OLIVIER DRUET: Olivier.Druet@umpa.ens-lyon.fr
Ecole normale supérieure de Lyon, Département de Mathématiques - UMPA, 46 allée d'Italie, 69364 Lyon Cedex 07, France

EMMANUEL HEBEY: hebey@u-cergy.fr
Université de Cergy-Pontoise, Département de Mathématiques, Site de Saint Martin, 2 avenue Adolphe Chauvin, 95302 Cergy-Pontoise Cedex, France

ANALYSIS AND PDE
Vol. 2, No. 3, 2009

GLOBAL REGULARITY FOR A LOGARITHMICALLY SUPERCRITICAL HYPERDISSIPATIVE NAVIER–STOKES EQUATION

TERENCE TAO

Let $d \geq 3$. We consider the global Cauchy problem for the generalized Navier–Stokes system

$$\partial_t u + (u \cdot \nabla) u = -D^2 u - \nabla p, \quad \nabla \cdot u = 0, \quad u(0, x) = u_0(x)$$

for $u : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^d$ and $p : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}$, where $u_0 : \mathbb{R}^d \to \mathbb{R}^d$ is smooth and divergence free, and D is a Fourier multiplier whose symbol $m : \mathbb{R}^d \to \mathbb{R}^+$ is nonnegative; the case $m(\xi) = |\xi|$ is essentially Navier–Stokes. It is folklore that one has global regularity in the critical and subcritical hyperdissipation regimes $m(\xi) = |\xi|^\alpha$ for $\alpha \geq (d+2)/4$. We improve this slightly by establishing global regularity under the slightly weaker condition that $m(\xi) \geq |\xi|^{(d+2)/4}/g(|\xi|)$ for all sufficiently large ξ and some nondecreasing function $g : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\int_1^\infty ds/(sg(s)^4) = +\infty$. In particular, the results apply for the logarithmically supercritical dissipation $m(\xi) := |\xi|^{(d+2)/4}/\log(2 + |\xi|^2)^{1/4}$.

1. Introduction

Let $d \geq 3$. This note is concerned with solutions to the generalised Navier–Stokes system

$$\begin{aligned} \partial_t u + (u \cdot \nabla) u &= -D^2 u - \nabla p, \\ \nabla \cdot u &= 0, \\ u(0, x) &= u_0(x), \end{aligned} \tag{1}$$

where $u : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^d$, $p : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}$ are smooth, and $u_0 : \mathbb{R}^d \to \mathbb{R}^d$ is smooth, compactly supported, and divergence-free, and D is a Fourier multiplier[1] whose symbol $m : \mathbb{R}^d \to \mathbb{R}^+$ is nonnegative; the case $m(\xi) = |\xi|$ is essentially the Navier–Stokes system, while the case $m = 0$ is the Euler system.

For $d \geq 3$, the global regularity of the Navier–Stokes system is of course a notoriously difficult unsolved problem, due in large part to the supercritical nature of the equation with respect to the energy $E(u(t)) := \int_{\mathbb{R}^d} |u(t, x)|^2 \, dx$. This supercriticality can be avoided by strengthening the dissipative symbol $m(\xi)$, for instance setting $m(\xi) := |\xi|^\alpha$ for some $\alpha > 1$. This *hyper-dissipative* variant of the Navier–Stokes equation becomes subcritical for $\alpha > (d+2)/4$ (and critical for $\alpha = (d+2)/4$), and it is known that global regularity can be recovered in these cases; see [Katz and Pavlović 2002] for further discussion. For $1 \leq \alpha < (d+2)/4$, only partial regularity results are known; see [Caffarelli et al. 1982] for the case $\alpha = 1$ and [Katz and Pavlović 2002] for the case $\alpha > 1$.

MSC2000: 35Q30.
Keywords: Navier–Stokes, energy method.
The author is supported by NSF Research Award DMS-0649473, the NSF Waterman award and a grant from the MacArthur Foundation.

[1] The exact definition of the Fourier transform is inessential here, but for concreteness, take $\hat{f}(\xi) := \int_{\mathbb{R}^d} f(x) e^{-ix\cdot\xi} dx$.

The purpose of this note is to extend the global regularity result very slightly into the supercritical regime.

Theorem 1.1. *Suppose that m obeys the lower bound*

$$m(\xi) \geq |\xi|^{(d+2)/4}/g(|\xi|) \tag{2}$$

for all sufficiently large $|\xi|$, where $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing function such that

$$\int_1^\infty \frac{ds}{sg(s)^4} = \infty. \tag{3}$$

Then for any smooth, compactly supported initial data u_0, one has a global smooth solution to (1).

Note that the hypotheses are for instance satisfied when

$$m(\xi) := |\xi|^{(d+2)/4}/\log^{1/4}(2+|\xi|^2); \tag{4}$$

thus

$$|D|^2 = \frac{(-\Delta)^{(d+2)/4}}{\log^{1/2}(2-\Delta)}.$$

Analogous "barely supercritical" global regularity results were established for the nonlinear wave equation recently [Tao 2007; Roy 2008; 2009].

The argument is quite simple, being based on the classical energy method and Sobolev embedding. The basic point is that whereas in the critical and subcritical cases one can get an energy inequality of the form

$$\partial_t \|u(t)\|^2_{H^k(\mathbb{R}^d)} \leq Ca(t)\|u(t)\|^2_{H^k(\mathbb{R}^d)}$$

for some locally integrable function $a(t)$ of time, a constant C, and some large k, which by Gronwall's inequality is sufficient to establish a suitable *a priori* bound, in the logarithmically supercritical case (4) one instead obtains the slightly weaker inequality

$$\partial_t \|u(t)\|^2_{H^k(\mathbb{R}^d)} \leq Ca(t)\|u(t)\|^2_{H^k(\mathbb{R}^d)} \log(2+\|u(t)\|_{H^k(\mathbb{R}^d)})$$

(thanks to an endpoint version of Sobolev embedding, closely related to an inequality of Brézis and Wainger [1980]), which is still sufficient to obtain an *a priori* bound (though one which is now double-exponential rather than single-exponential; compare [Beale et al. 1984]).

Remark 1.2. It may well be that the condition (3) can be relaxed further by a more sophisticated argument. Indeed, the following heuristic suggests that one should be able to weaken (3) to

$$\int_1^\infty \frac{ds}{sg(s)^2} = \infty,$$

thus allowing one to increase the $\frac{1}{4}$ exponent in (4) to $\frac{1}{2}$. Consider a blowup scenario in which the solution blows up at some finite time T_*, and is concentrated on a ball of radius $1/N(t)$ for times $0 < t < T_*$, where $N(t) \to \infty$ as $t \to T_*$. As the energy of the fluid must stay bounded, we obtain the heuristic bound $u(t) = O(N(t)^{d/2})$ for times $0 < t < T_*$. In particular, we expect the fluid to propagate at speeds $O(N(t)^{d/2})$, leading to the heuristic ODE

$$\frac{d}{dt}\frac{1}{N(t)} = O(N(t)^{d/2})$$

for the radius $1/N(t)$ of the fluid. Solving this ODE, we are led to a heuristic upper bound $N(t) = O\big((T_* - t)^{2/(d+2)}\big)$ on the blowup rate. On the other hand, from the energy inequality

$$2\int_0^{T_*}\int_{\mathbb{R}^d} |Du(t,x)|^2\,dxdt \le \int_{\mathbb{R}^d} |u_0(x)|^2\,dx$$

one is led to the heuristic bound

$$\int_0^{T_*}\frac{1}{N(t)^{(d+2)/2}g(N(t))^2}\,dt < \infty.$$

This is incompatible with the upper bound $N(t) = O((T_* - t)^{2/(d+2)})$ if $\int_1^{T_*} ds/(sg(s)^2) = \infty$. Unfortunately the author was not able to make this argument precise, as there appear to be multiple and inequivalent ways to rigorously define an analogue of the "frequency scale" $N(t)$, and all attempts of the author to equate different versions of these analogues lost one or more powers of $g(s)$.

To go beyond the barrier $\int_1^{\infty} ds/(sg(s)^2) = \infty$ (with the aim of getting closer to the Navier–Stokes regime, in which $g(s) = s^{1/4}$ in three dimensions), the heuristic analysis above suggests that one would need to force the energy to not concentrate into small balls, but instead to exhibit turbulent behaviour.

2. Proof of theorem

We now prove Theorem 1.1. Let k be a large integer (for example, $k := 100d$ will suffice).

Standard energy method arguments (see, for example, [Kato 1985]) show that if the initial data is smooth and compactly supported, then either a smooth H^∞ solution exists for all time, or there exists a smooth solution up to some blowup time $0 < T_* < \infty$, and $\|u(t)\|_{H^k(\mathbb{R}^d)} \to \infty$ as $t \to T_*$. Thus, to establish global regularity, it suffices to prove an *a priori bound* of the form

$$\|u(t)\|_{H^k(\mathbb{R}^d)} \le C(k, d, \|u_0\|_{H^k(\mathbb{R}^d)}, T, g)$$

for all $0 \le t \le T < \infty$ and all smooth H^∞ solutions $u : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ to (1), where the quantity $C(k, d, \|u_0\|_{H^k(\mathbb{R}^d)}, T, g)$ only depends on k, d, $\|u_0\|_{H^k(\mathbb{R}^d)}$, T, and g.

We now fix u_0, u, T, and let C denote any constant depending on k, d, $\|u_0\|_{H^k(\mathbb{R}^d)}$, T, and g (whose value can vary from line to line). Multiplying the Navier–Stokes equation by u and integrating by parts, we obtain the well-known energy identity

$$\partial_t \int_{\mathbb{R}^d} |u(t,x)|^2\,dx = -2a(t),$$

where

$$a(t) := \|Du\|^2_{L^2(\mathbb{R}^d)} \tag{5}$$

(note that the pressure term ∇p disappears thanks to the divergence free condition); integrating this in time, we obtain the energy dissipation bound

$$\int_0^T a(t)\,dt \le C. \tag{6}$$

Now, we consider the higher energy

$$E_k(t) := \sum_{j=0}^{k} \int_{\mathbb{R}^d} |\nabla^j u(t,x)|^2\,dx. \tag{7}$$

Differentiating (7) in time and integrating by parts, we obtain

$$\partial_t E_k(t) = -2\sum_{j=0}^{k} \|\nabla^j Du(t)\|_{L^2(\mathbb{R}^d)}^2 - 2\sum_{j=0}^{k}\int_{\mathbb{R}^d} \nabla^j u(t,x)\cdot \nabla^j((u\cdot\nabla)u)(t,x)\,dx;$$

again, the pressure term disappears thanks to the divergence-free condition. For brevity we shall now drop explicit mention of the t and x variables.

We apply the Leibniz rule to $\nabla^j((u\cdot\nabla)u)$. There is one term involving $(j+1)$-st derivatives of u, but the contribution of that term vanishes by integration by parts and the divergence free property. The remaining terms give contributions of the form

$$\sum_{j=0}^{k} \sum_{\substack{1\le j_1,j_2\le j\\ j_1+j_2=j+1}} \int_{\mathbb{R}^d} \mathbb{O}(\nabla^j u \nabla^{j_1} u \nabla^{j_2} u)\,dx,$$

where $\mathbb{O}(\nabla^j u \nabla^{j_1} u \nabla^{j_2} u)$ denotes some constant-coefficient trilinear combination of the components of $\nabla^j u$, $\nabla^{j_1} u$, and $\nabla^{j_2} u$ whose explicit form is easily computed, but is not of importance to our argument. We can integrate by parts using D and D^{-1} and then use Cauchy–Schwarz to obtain the bound

$$\int_{\mathbb{R}^d} \mathbb{O}(\nabla^j u \nabla^{j_1} u \nabla^{j_2} u)\,dx \le \|(1+D)\nabla^j u\|_{L^2(\mathbb{R}^d)} \big\|(1+D)^{-1}(\mathbb{O}(\nabla^{j_1} u \nabla^{j_2} u))\big\|_{L^2(\mathbb{R}^d)}.$$

By the arithmetic mean-geometric mean inequality we then have

$$\int_{\mathbb{R}^d} \mathbb{O}(\nabla^j u \nabla^{j_1} u \nabla^{j_2} u)\,dx \le c\|(1+D)\nabla^j u\|_{L^2(\mathbb{R}^d)}^2 + \frac{1}{c}\big\|(1+D)^{-1}(\mathbb{O}(\nabla^{j_1} u \nabla^{j_2} u))\big\|_{L^2(\mathbb{R}^d)}^2$$

for any $c > 0$. Finally, from the triangle inequality, (7), and the fact that D commutes with ∇^j, we have

$$\|(1+D)\nabla^j u\|_{L^2(\mathbb{R}^d)}^2 \le C\big(\|\nabla^j Du\|_{L^2(\mathbb{R}^d)}^2 + E_k\big).$$

Putting this all together and choosing c small enough, we conclude that

$$\partial_t E_k \le CE_k + C \sum_{\substack{1\le j_1\le j_2\le k\\ j_1+j_2\le k+1}} \big\|(1+D)^{-1}(\mathbb{O}(\nabla^{j_1} u \nabla^{j_2} u))\big\|_{L^2(\mathbb{R}^d)}^2. \tag{8}$$

To estimate this expression, we introduce a parameter $N > 1$ (depending on t) to be optimised later, and write

$$(1+D)^{-1} = (1+D)^{-1}P_{\le N} + (1+D)^{-1}P_{>N},$$

where $P_{\leq N}$ and $P_{>N}$ are the Fourier projections to the regions $\{\xi : |\xi| \leq N\}$ and $\{\xi : |\xi| > N\}$.

We first deal with the low-frequency contribution to (8). From Plancherel's theorem and (2) we obtain

$$\left\| (1+D)^{-1} P_{\leq N}(\mathbb{O}(\nabla^{j_1} u \nabla^{j_2} u)) \right\|_{L^2(\mathbb{R}^d)} \leq C g(N) \left\| \langle \nabla \rangle^{-(d+2)/4} \mathbb{O}(\nabla^{j_1} u \nabla^{j_2} u) \right\|_{L^2(\mathbb{R}^d)},$$

where $\langle \nabla \rangle^{-(d+2)/4}$ is the Fourier multiplier with symbol $\langle \xi \rangle^{-(d+2)/4}$, where $\langle \xi \rangle := (1+|\xi|^2)^{1/2}$. Applying Sobolev embedding, we can bound the right-hand side by

$$\leq C g(N) \left\| |\nabla^{j_1} u| \, |\nabla^{j_2} u| \right\|_{L^{4d/(3d+2)}(\mathbb{R}^d)}.$$

By Hölder's inequality and the Gagliardo–Nirenberg inequality, we can bound this by

$$\leq C g(N) \|\nabla u\|_{L^{4d/(d+2)}(\mathbb{R}^d)} \|\nabla^{j_1+j_2-1} u\|_{L^2(\mathbb{R}^d)},$$

which by (7) is bounded by

$$\leq C g(N) \|\nabla u\|_{L^{4d/(d+2)}(\mathbb{R}^d)} E_k^{1/2}.$$

Next, we partition

$$\|\nabla u\|_{L^{4d/(d+2)}(\mathbb{R}^d)} \leq \|\nabla P_{\leq N} u\|_{L^{4d/(d+2)}(\mathbb{R}^d)} + \|\nabla P_{>N} u\|_{L^{4d/(d+2)}(\mathbb{R}^d)}.$$

From Sobolev embedding and Plancherel, together with (2) and (5), we have

$$\|\nabla P_{\leq N} u\|_{L^{4d/(d+2)}(\mathbb{R}^d)} \leq C \|\langle \nabla \rangle^{(d+2)/4} P_{\leq N} u\|_{L^2(\mathbb{R}^d)} \leq C g(N)(1+a(t))^{1/2}.$$

Meanwhile, from Sobolev embedding we have

$$\|\nabla P_{>N} u\|_{L^{4d/(d+2)}(\mathbb{R}^d)} \leq \frac{1}{N} E_k^{1/2},$$

(say) if k is large enough. Putting this all together, we see that the low-frequency contribution to (8) is

$$\leq C g(N)^2 E_k \Big[g(N)^2 (1+a(t)) + \frac{1}{N^2} E_k \Big].$$

Next, we turn to the high-frequency contribution to (8). From Plancherel, Hölder's inequality, and (7) we have

$$\begin{aligned} \left\| (1+D)^{-1} P_{\geq N}(\mathbb{O}(\nabla^{j_1} u \nabla^{j_2} u)) \right\|_{L^2(\mathbb{R}^d)} &\leq C g(N) N^{-(d+2)/4} \left\| |\nabla^{j_1} u| \, |\nabla^{j_2} u| \right\|_{L^2(\mathbb{R}^d)} \\ &\leq C g(N) N^{-(d+2)/4} \|\nabla^{j_1} u\|_{L^\infty(\mathbb{R}^d)} E_k^{1/2}, \end{aligned}$$

while from Sobolev embedding and (7) we see (for k large enough) that

$$\|\nabla^{j_1} u\|_{L^\infty(\mathbb{R}^d)} \leq C E_k^{1/2}.$$

Thus the high-frequency contribution to (8) is at most $C g(N)^2 N^{-(d+2)/2} E_k^2$.

Putting this all together, we conclude that

$$\partial_t E_k \leq C g(N)^2 E_k \Big[g(N)^2 (1+a(t)) + \frac{1}{N} E_k \Big].$$

We now optimize in N, setting $N := 1 + E_k$, to obtain

$$\partial_t E_k \leq C g(1+E_k)^4 E_k (1+a(t)).$$

From (6), (3) and separation of variables we see that the ODE

$$\partial_t E = Cg(1+E)^4 E(1+a(t))$$

with initial data $E(0) \geq 0$ does not blow up in time. Also, from (7) we have $E_k(0) \leq C$. A standard ODE comparison (or continuity) argument then shows that $E_k(t) \leq C(T)$ for all $0 \leq t \leq T$, and the claim follows.

Remark 2.1. It should be clear to the experts that the domain $\mathbb{R}^d$ here could be replaced by any other sufficiently smooth domain, for example, the torus $\mathbb{R}^d/\mathbb{Z}^d$, using standard substitutes for the Littlewood–Paley type operators $P_{\leq N}$, $P_{>N}$ (one could use spectral projections of the Laplacian). We omit the details.

References

[Beale et al. 1984] J. T. Beale, T. Kato, and A. Majda, "Remarks on the breakdown of smooth solutions for the 3-D Euler equations", *Comm. Math. Phys.* **94**:1 (1984), 61–66. MR 85j:35154 Zbl 0573.76029

[Brézis and Wainger 1980] H. Brézis and S. Wainger, "A note on limiting cases of Sobolev embeddings and convolution inequalities", *Comm. Partial Differential Equations* **5**:7 (1980), 773–789. MR 81k:46028 Zbl 0437.35071

[Caffarelli et al. 1982] L. Caffarelli, R. Kohn, and L. Nirenberg, "Partial regularity of suitable weak solutions of the Navier–Stokes equations", *Comm. Pure Appl. Math.* **35**:6 (1982), 771–831. MR 84m:35097 Zbl 0509.35067

[Kato 1985] T. Kato, *Abstract differential equations and nonlinear mixed problems*, Scuola Normale Superiore, Pisa, 1985. MR 88m:34058

[Katz and Pavlović 2002] N. H. Katz and N. Pavlović, "A cheap Caffarelli–Kohn–Nirenberg inequality for the Navier–Stokes equation with hyper-dissipation", *Geom. Funct. Anal.* **12**:2 (2002), 355–379. MR 2003e:35243 Zbl 0999.35069

[Roy 2008] T. Roy, "Global existence of smooth solutions of a 3D loglog energy-supercritical wave equation", preprint, 2008. arXiv 0810.5175

[Roy 2009] T. Roy, "One remark on barely $\dot{H}^{s_p}$ supercritical wave equations", preprint, 2009. arXiv 0906.0044

[Tao 2007] T. Tao, "Global regularity for a logarithmically supercritical defocusing nonlinear wave equation for spherically symmetric data", *J. Hyperbolic Differ. Equ.* **4**:2 (2007), 259–265. MR 2009b:35294 Zbl 1124.35043

Received 16 Jun 2009. Revised 22 Sep 2009. Accepted 23 Oct 2009.

TERENCE TAO: tao@math.ucla.edu
University of California, Department of Mathematics, Los Angeles, CA 90095-1555, United States
http://www.math.ucla.edu/~tao/

Guidelines for Authors

Authors may submit manuscripts in PDF format on-line at the Submission page at pjm.math.berkeley.edu/apde.

Originality. Submission of a manuscript acknowledges that the manuscript is original and and is not, in whole or in part, published or under consideration for publication elsewhere. It is understood also that the manuscript will not be submitted elsewhere while under consideration for publication in this journal.

Language. Articles in APDE are usually in English, but articles written in other languages are welcome.

Required items. A brief abstract of about 150 words or less must be included. It should be self-contained and not make any reference to the bibliography. If the article is not in English, two versions of the abstract must be included, one in the language of the article and one in English. Also required are keywords and subject classifications for the article, and, for each author, postal address, affiliation (if appropriate), and email address.

Format. Authors are encouraged to use LaTeX but submissions in other varieties of TeX, and exceptionally in other formats, are acceptable. Initial uploads should be in PDF format; after the refereeing process we will ask you to submit all source material.

References. Bibliographical references should be complete, including article titles and page ranges. All references in the bibliography should be cited in the text. The use of BibTeX is preferred but not required. Tags will be converted to the house format, however, for submission you may use the format of your choice. Links will be provided to all literature with known web locations and authors are encouraged to provide their own links in addition to those supplied in the editorial process.

Figures. Figures must be of publication quality. After acceptance, you will need to submit the original source files in vector graphics format for all diagrams in your manuscript: vector EPS or vector PDF files are the most useful.

Most drawing and graphing packages (Mathematica, Adobe Illustrator, Corel Draw, MATLAB, etc.) allow the user to save files in one of these formats. Make sure that what you are saving is vector graphics and not a bitmap. If you need help, please write to graphics@mathscipub.org with details about how your graphics were generated.

White space. Forced line breaks or page breaks should not be inserted in the document. There is no point in your trying to optimize line and page breaks in the original manuscript. The manuscript will be reformatted to use the journal's preferred fonts and layout.

Proofs. Page proofs will be made available to authors (or to the designated corresponding author) at a Web site in PDF format. Failure to acknowledge the receipt of proofs or to return corrections within the requested deadline may cause publication to be postponed.

Analysis & PDE

Volume 2 No. 3 2009