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POISSON STATISTICS FOR EIGENVALUES OF CONTINUUM RANDOM SCHRÖDINGER OPERATORS

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We show absence of energy levels repulsion for the eigenvalues of random Schrödinger operators in the continuum. We prove that, in the localization region at the bottom of the spectrum, the properly rescaled eigenvalues of a continuum Anderson Hamiltonian are distributed as a Poisson point process with intensity measure given by the density of states. In addition, we prove that in this localization region the eigenvalues are simple.

These results rely on a Minami estimate for continuum Anderson Hamiltonians. We also give a simple, transparent proof of Minami's estimate for the (discrete) Anderson model.

1. Introduction

Local fluctuations of eigenvalues of random operators are believed to distinguish between localized and delocalized regimes, indicating an Anderson metal-insulator transition. Exponential decay of eigenfunctions implies that disjoint regions of space are uncorrelated and create almost independent eigenvalues, leading to the absence of energy levels repulsion, which is mathematically translated in terms of a Poisson point process. On the other hand, extended states imply that distant regions have mutual influence, and thus create some repulsion between energy levels.

Local fluctuations of eigenvalues have been studied within the context of random matrix theory, in particular Wigner matrices and GUE matrices [Bellissard 2004; Disertori et al. 2002; Erdős et al. 2009b; 2009a; Johansson 1998; 2001; Schenker and Schulz-Baldes 2007]. It is challenging to understand random hermitian band matrices from the perspective of their eigenvalues fluctuations, by proving a transition between Poisson statistics and a semi-circle law for the density of states (a signature of energy levels repulsion), and relate this to the (discrete) Anderson model [Bellissard 2004; Disertori et al. 2002]. CMV matrices are another class of random matrices for which Poisson statistics and a transition to energy levels repulsion have been proved been proved [Killip and Stoiciu 2009; Stoiciu 2006; 2007].

For random Schrödinger operators, Poisson statistics for eigenvalues were first proved by Molchanov [1980/81] for the same one-dimensional continuum random Schrödinger operator for which Anderson localization was first rigorously established [Gol'dsheĭd et al. 1977]. Molchanov's proof was based on a detailed analysis of localization in finite intervals for this particular random Schrödinger operator [Molchanov 1978].

Poisson statistics for eigenvalues of the Anderson model was established in [Minami 1996]. The Anderson model, a random Schrödinger operator on $\ell^2(\mathbb{Z}^d)$, is the discrete analogue of the Anderson

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Hamiltonian. A crucial ingredient in Minami's proof is an estimate of the probability of two or more eigenvalues in an interval. The key step in the proof of this estimate, namely [Minami 1996, Lemma 2], estimates the average of a determinant whose entries are matrix elements of the imaginary part of the resolvent. The more recent proofs of Minami's estimate by Bellissard et al. [2007] and Graf and Vaghi [2007] are variants of Minami's. Since those arguments do not seem to extend to the continuum, a Minami-type estimate and Poisson statistics for the eigenvalues have until now been challenging questions for continuum Anderson Hamiltonians.

Here we introduce a fundamentally new approach to Minami's estimate. Unlike the previous approach, ours relies on averaging spectral projections, a technique that does extend to the continuum. Combined with a property of rank-one perturbations, it provides a simple and transparent proof of Minami's estimate for the Anderson model, valid for single-site probability distributions with compact support and no atoms, which is presented here as an illustration of the method. On the continuum, our proof of Minami's estimate circumvents the unavailability of that rank-one property by averaging the spectral shift function, using refined bounds on the density of states not previously available.

Once we have Minami's estimate in the continuum, we prove Poisson statistics for eigenvalues of the Anderson Hamiltonian. We start by approximating the point process defined by the rescaled eigenvalues by superpositions of independent point processes, as in [Molchanov 1980/81; Minami 1996]. But our proof that these superpositions converge weakly to the desired Poisson point process differs from Minami's for the Anderson model, since his way of identifying the intensity measure of the Poisson process, which relies on complex analysis, is not readily applicable in the continuum. We identify this intensity measure using methods of real analysis.

Klein and Molchanov [2006] showed that Minami's estimate implies simplicity of eigenvalues for the Anderson model, a result previously obtained by Simon [1994] by different methods. Their arguments can also be applied in the continuum, so we also obtain simplicity of eigenvalues in the continuum. Previous results [Combes and Hislop 1994; Germinet and Klein 2006] proved only finite multiplicity of the eigenvalues in the localization region.

2. Main results

To state our results we introduce the following notation. We write

$$\Lambda_L(x) := x + \left[-\tfrac{L}{2}, \tfrac{L}{2}\right[^d \tag{2-1}$$

for the (half-open, half-closed) box of side $L > 0$ centered at $x \in \mathbb{R}^d$. By Λ_L we denote a box $\Lambda_L(x)$ for some $x \in \mathbb{R}^d$. Given a box $\Lambda = \Lambda_L(x)$, we set $\widetilde{\Lambda} = \Lambda \cap \mathbb{Z}^d$. If B is a set, we write χ_B for its characteristic function. We set $\chi_x^{(L)} := \chi_{\Lambda_L(x)}$. The Lebesgue measure of a Borel set $B \subset \mathbb{R}$ will be denoted by $|B|$. If $r > 0$, we denote by $[r]$ the largest integer less than equal to r, and by $[\![r]\!]$ the smallest integer bigger than r. By a constant we will always mean a finite constant. Constants such as $C_{a,b,\ldots}$ will be finite and depending only on the parameters or quantities $a, b, \ldots$; they will be independent of other parameters or quantities in the equation.

We consider random Schrödinger operators on $\mathrm{L}^2(\mathbb{R}^d)$ of the type

$$H_\omega := -\Delta + V_{\mathrm{per}} + V_\omega, \tag{2-2}$$

where Δ is the d-dimensional Laplacian operator, V_{per} is a bounded $\mathbb{Z}^d$-periodic potentia,; and $V_{\boldsymbol{\omega}}$ is an Anderson-type random potential, given by

$$V_{\boldsymbol{\omega}}(x) := \sum_{j \in \mathbb{Z}^d} \omega_j \, u_j(x), \quad \text{with } u_j(x) = u(x-j), \tag{2-3}$$

where the single-site potential u is a nonnegative bounded measurable function on $\mathbb{R}^d$ with compact support, uniformly bounded away from zero in a neighborhood of the origin, and $\boldsymbol{\omega} = \{\omega_j\}_{j \in \mathbb{Z}^d}$ is a family of independent identically distributed random variables, whose common probability distribution μ is nondegenerate with a bounded density ρ with compact support.

We normalize $H_{\boldsymbol{\omega}}$ as follows. We first require $\inf \operatorname{supp} \mu = 0$, which can always be realized by changing the periodic potential V_{per}. Next we assume $\|u\|_\infty = 1$, which can achieved by rescaling μ. We then adjust V_{per} by adding a constant so $\inf \sigma(-\Delta + V_{\mathrm{per}}) = 0$, in which case $[0, E_*] \subset \sigma(-\Delta + V_{\mathrm{per}})$ for some $E_* > 0$. Thus, without loss of generality, we will assume that the random Schrödinger operator H_ω given in (2-2)–(2-3) is normalized as follows:

(I) The free Hamiltonian $H_0 := -\Delta + V_{\mathrm{per}}$ has 0 as the bottom of its spectrum:

$$\inf \sigma(H_0) = 0. \tag{2-4}$$

(II) The single-site potential u is a measurable function on $\mathbb{R}^d$ such that

$$\|u\|_\infty = 1 \quad \text{and} \quad u_- \chi_{\Lambda_{\delta_-}(0)} \le u \le \chi_{\Lambda_{\delta_+}(0)} \quad \text{with } u_-, \delta_\pm \in \left]0, \infty\right[; \tag{2-5}$$

we set

$$U_+ := \left\| \textstyle\sum_{j \in \mathbb{Z}^d} u_j \right\|_\infty \le \max\{1, \delta_+^d\}. \tag{2-6}$$

(III) $\boldsymbol{\omega} = \{\omega_j\}_{j \in \mathbb{Z}^d}$ is a family of independent, identically distributed random variables, whose common probability distribution μ has a density ρ such that

$$\{0, M_\rho\} \in \operatorname{ess\,supp} \rho \subset [0, M_\rho] \quad \text{with } M_\rho \in \left]0, \infty\right[\text{ and } \rho_+ := \|\rho\|_\infty < \infty. \tag{2-7}$$

A random Schrödinger operator H_ω on $\mathrm{L}^2(\mathbb{R}^d)$ as in (2-2)–(2-3), normalized as in (I)-(III), will be called an *Anderson Hamiltonian*. The common probability distribution μ in (III) is said to be *uniform-like* if its density ρ also satisfies $\rho_- := \operatorname{ess\,inf} \rho \chi_{[0, M_\rho]} > 0$, in which case we have

$$\rho_- \chi_{[0, M_\rho]} \le \rho \le \rho_+ \chi_{[0, M_\rho]} \quad \text{with } \rho_\pm, M_\rho \in \left]0, \infty\right[. \tag{2-8}$$

An Anderson Hamiltonian $H_{\boldsymbol{\omega}}$ is a $\mathbb{Z}^d$-ergodic family of random self-adjoint operators. It follows from standard results [Klein and Molchanov 2006; Carmona and Lacroix 1990; Pastur and Figotin 1992] that there exist fixed subsets Σ, Σ_{pp}, Σ_{ac} and Σ_{sc} of $\mathbb{R}$ so that the spectrum $\sigma(H_{\boldsymbol{\omega}})$ of $H_{\boldsymbol{\omega}}$, as well as its pure point, absolutely continuous, and singular continuous components, are equal to these fixed sets with probability one. With our normalization, the nonrandom spectrum Σ of an Anderson Hamitonian $H_{\boldsymbol{\omega}}$ satisfies [Kirsch and Martinelli 1982]

$$\sigma(H_0) \subset \Sigma \subset \left[0, \infty\right[, \tag{2-9}$$

so $\inf \Sigma = 0$ and $[0, E_*] \subset \Sigma$ for some $E_* = E_*(V_{\mathrm{per}}) > 0$. Note that $\Sigma = \sigma(-\Delta) = \left[0, \infty\right[$ if $V_{\mathrm{per}} = 0$.

An Anderson Hamiltonian H_ω exhibits Anderson and dynamical localization at the bottom of the spectrum [Martinelli and Holden 1984; Combes and Hislop 1994; Klopp 1995; Kirsch et al. 1998; Germinet and De Bièvre 1998; Damanik and Stollmann 2001; Germinet and Klein 2001; 2003a; Aizenman et al. 2006]. More precisely, there exists an energy $E_1 > 0$ such that $[0, E_1] \subset \Xi^{\mathrm{CL}}$, where Ξ^{CL} is the region of complete localization for the random operator H_ω [Germinet and Klein 2004; 2006]. (See Appendix A for a discussion of localization. Note that $\mathbb{R} \setminus \Sigma \subset \Xi^{\mathrm{CL}}$ in our definition.) Similarly, given an energy $E_1 > 0$, we have $[0, E_1] \subset \Xi^{\mathrm{CL}}$ if ρ_+ in (2-7) is sufficiently small, corresponding to a large disorder regime.

Finite volume operators will be defined for finite boxes $\Lambda = \Lambda_L(j)$, where $j \in \mathbb{Z}^d$ and $L \in 2\mathbb{N}$, $L > \delta_+$. Given such Λ, we will consider the random Schrödinger operator $H_\omega^{(\Lambda)}$ on $\mathrm{L}^2(\Lambda)$ given by the restriction of the Anderson Hamiltonian H_ω to Λ with periodic boundary condition. To do so, we identify Λ with a torus in the usual way by identifying opposite edges, and define finite volume operators

$$H_\omega^{(\Lambda)} := H_0^{(\Lambda)} + V_\omega^{(\Lambda)} \quad \text{on} \quad \mathrm{L}^2(\Lambda). \tag{2-10}$$

The finite volume free Hamiltonian $H_0^{(\Lambda)}$ is given by

$$H_0^{(\Lambda)} := -\Delta^{(\Lambda)} + V_{\mathrm{per}}^{(\Lambda)} \quad \text{on} \quad \mathrm{L}^2(\Lambda), \tag{2-11}$$

where $\Delta^{(\Lambda)}$ is the Laplacian on Λ with periodic boundary condition and $V_{\mathrm{per}}^{(\Lambda)}$ is the restriction of V_{per} to Λ. The random potential $V_\omega^{(\Lambda)}$ is the restriction of $V_{\omega^{(\Lambda)}}$ to Λ, where, given $\boldsymbol{\omega} = \{\omega_i\}_{i \in \mathbb{Z}^d}$, $\boldsymbol{\omega}^{(\Lambda)} = \{\omega_i^{(\Lambda)}\}_{i \in \mathbb{Z}^d}$ is defined as follows:

$$\omega_i^{(\Lambda)} = \begin{cases} \omega_i & \text{if } i \in \Lambda, \\ \omega_k^{(\Lambda)} & \text{if } k - i \in L\mathbb{Z}^d. \end{cases} \tag{2-12}$$

The random finite volume operator $H_\omega^{(\Lambda)}$ is covariant with respect to translations in the torus. If $B \subset \mathbb{R}$ is a Borel set, we write $P_\omega^{(\Lambda)}(B) := \chi_B(H_\omega^{(\Lambda)})$ and $P_\omega(B) := \chi_B(H_\omega)$ for the spectral projections.

The finite volume operator $H_\omega^{(\Lambda)}$ has a compact resolvent, and hence its ($\boldsymbol{\omega}$-dependent) spectrum consists of isolated eigenvalues with finite multiplicity. It satisfies a Wegner estimate [Combes and Hislop 1994; Combes et al. 2007a]: Given $E_0 > 0$, there exists a constant K_W, independent of Λ, such that for all intervals $I \subset [0, E_0]$ we have

$$\mathbb{E}\{\operatorname{tr} P_\omega^{(\Lambda)}(I)\} \le K_W\, \rho_+ |I| |\Lambda|. \tag{2-13}$$

The constant K_W given in [Combes and Hislop 1994; Combes et al. 2007a] depends on E_0, d, u, V_{per}, M_ρ, but not on ρ_+.

The integrated density of states (IDS) for H_ω is given, for a.e. $E \in \mathbb{R}$, by

$$N(E) := \lim_{L \to \infty} |\Lambda_L(0)|^{-1} \operatorname{tr} P_\omega^{(\Lambda_L(0))}(]-\infty, E]) \quad \text{for } \mathbb{P}\text{-a.e. } \boldsymbol{\omega}, \tag{2-14}$$

in the sense that the limit exists and is the same for $\mathbb{P}$-a.e. $\boldsymbol{\omega}$ [Carmona and Lacroix 1990; Pastur and Figotin 1992]. It follows from (2-13) that the IDS $N(E)$ is locally Lipschitz, hence continuous, so (2-14) holds for all $E \in \mathbb{R}$. For all $E \in \mathbb{R}$ we have

$$N(E) = \lim_{L \to \infty} \mathbb{E}\{|\Lambda_L|^{-1} \operatorname{tr} P_\omega^{(\Lambda_L)}(]-\infty, E])\}. \tag{2-15}$$

$N(E)$ is a nondecreasing absolutely continuous function on $\mathbb{R}$, the cumulative distribution function of the density of states measure, given by

$$\eta(B) := \mathbb{E}\,\mathrm{tr}\{\chi_0^{(1)} P_\omega(B)\chi_0^{(1)}\} \quad \text{for a Borel set } B \subset \mathbb{R}. \tag{2-16}$$

In particular $N(E)$ is differentiable a.e. with respect to Lebesgue measure, with $n(E) := N'(E) \geq 0$ being the density of the measure η, so $n(E) > 0$ for η-a.e. E.

Given an energy $\mathscr{E} \in \Sigma$, using (2-13) we define a point process $\xi_{\mathscr{E},\omega}^{(\Lambda)}$ on the real line by the rescaled spectrum of the finite volume operator $H_\omega^{(\Lambda)}$ near $\mathscr{E}$:

$$\xi_{\mathscr{E},\omega}^{(\Lambda)}(B) := \mathrm{tr}\{\chi_B(|\Lambda|(H_\omega^{(\Lambda)} - \mathscr{E}))\} = \mathrm{tr}\{P_\omega^{(\Lambda)}(\mathscr{E} + |\Lambda|^{-1}B)\} \tag{2-17}$$

for a Borel set $B \subset \mathbb{R}$. (We refer to [Daley and Vere-Jones 1988] for definitions and results concerning random measures and point processes.)

Theorem 2.1. *Let H_ω be an Anderson Hamiltonian with $\delta_- \geq 2$ and a uniform-like distribution μ. Then there exists an energy $E_0 > 0$, such that*:

(a) *For all energies $\mathscr{E} \in \Xi^{\mathrm{CL}} \cap [0, E_0[$ such that the IDS $N(E)$ is differentiable at $\mathscr{E}$ with $n(\mathscr{E}) := N'(\mathscr{E})$ positive, the point process $\xi_{\mathscr{E},\omega}^{(\Lambda_L)}$ converges weakly, as $L \to \infty$, to the Poisson point process $\xi_{\mathscr{E}}$ on $\mathbb{R}$ with intensity measure $\nu_{\mathscr{E}}(B) := \mathbb{E}\,\xi_{\mathscr{E}}(B) = n(\mathscr{E})|B|$, that is, $\mathrm{d}\nu_{\mathscr{E}} = n(\mathscr{E})\mathrm{d}E$.*

(b) *With probability one, every eigenvalue of H_ω in $\Xi^{\mathrm{CL}} \cap [0, E_0[$ is simple.*

Similarly, given an energy $E_0 > 0$, (a) and (b) hold if the probability distribution μ in (2-8) has a density ρ with $(\rho_+/\rho_-)\rho_+^{2^d-1}$ sufficiently small. In fact, there exists a constant $Q_{d,V_{\mathrm{per}}} > 0$, such that (a) and (b) hold whenever

$$U_+ u_-^{-2^d} \frac{\rho_+}{\rho_-} \rho_+^{2^d-1} \gamma_d(E_0) \min\{1, E_0^{2^d-d-1}\} \max\{1, E_0^{2^{d+2}}\} \leq Q_{d,V_{\mathrm{per}}}, \tag{2-18}$$

where we have $\gamma_d(E_0) = 1$ if $d \geq 2$, and $\gamma_1(E_0) = \gamma_{1,V_{\mathrm{per}}}(E_0) \in\,]0, 1]$ with $\lim_{E_0 \to 0} \gamma_1(E_0) = 0$.

The next theorem gives our Minami estimate for the continuum Anderson Hamiltonian, a crucial ingredient for proving Theorem 2.1.

Theorem 2.2. *Let H_ω be an Anderson Hamiltonian with $\delta_- \geq 2$ and a uniform-like distribution μ. Then there exists a constant $Q_{d,V_{\mathrm{per}}} > 0$, such that whenever (2-18) holds for an energy $E_0 > 0$, we have the Minami estimate*

$$\mathbb{E}\{(\mathrm{tr}\, P_\omega^{(\Lambda)}(I))(\mathrm{tr}\, P_\omega^{(\Lambda)}(I) - 1)\} \leq K_M(\rho_+|I||\Lambda|)^2, \tag{2-19}$$

for all intervals $I \subset [0, E_0]$ and $\Lambda = \Lambda_L$ with $L \geq L(E_0)$, with a constant

$$K_M \leq C_{d,V_{\mathrm{per}},M_\rho}(1 + E_0)^{4[\![d/4]\!]}. \tag{2-20}$$

In more detail:

(i) *If H_ω is an Anderson Hamiltonian with $\delta_- \geq 2$, there exists a constant $C_{d,V_{\mathrm{per}}}$ such that, given an energy $E_0 > 0$, the Wegner estimate (2-13) holds for all intervals $I \subset [0, E_0]$ with a constant*

$$K_W \leq C_{d,V_{\mathrm{per}}} u_-^{-2^d} \rho_+^{2^d-1} \gamma_d(E_0) \min\{1, E_0^{2^d-d-1}\} \max\{1, E_0^{2^{d+2}}\}, \tag{2-21}$$

where we have $\gamma_d(E_0) = 1$ if $d \geq 2$, and $\gamma_1(E_0) = \gamma_{1,V_{\mathrm{per}}}(E_0) \in\,]0, 1]$ with $\lim_{E_0 \to 0} \gamma_1(E_0) = 0$.

(ii) *If H_ω is an Anderson Hamiltonian with a uniform-like distribution μ, and for a given $E_0 > 0$ the constant K_W in (2-13) satisfies*

$$2K_W U_+ \frac{\rho_+}{\rho_-} \leq 1, \tag{2-22}$$

then (2-19) holds for all intervals $I \subset [0, E_0]$ with a constant $K_M = C_{d,V_{\mathrm{per}},u,M_\rho,E_0} K_W$. If in addition $\delta_- \geq 2$, we have (2-20).

Our approach to Minami's estimate is discussed in Section 3, where it is illustrated by a proof of the estimate for the (discrete) Anderson model (Theorem 3.3). We also comment on the differences between the discrete and the continuum cases.

On the lattice (the Anderson model), the Wegner estimate (2-13) is a simple consequence of spectral averaging ((3-14)), and holds with $K_W = 1$ for all E_0 [Wegner 1981; Fröhlich and Spencer 1983; Carmona et al. 1987; Kirsch 2008]. On the continuum the Wegner estimate, which has not been as simple to prove, comes with an E_0 dependent constant K_W (which also depends on d, V_{per}, and u) [Combes and Hislop 1994; Combes et al. 2007a]. The proof given in [Combes and Hislop 1994] requires the covering condition $\delta_- \geq 1$. It allows estimates of the constant, but the estimates do not go to 0 as either E_0 or ρ_+ go to 0. The proof in [Combes et al. 2007a] does not require a covering condition, but it uses [Combes et al. 2003, Proposition 1.3] (cf. [Combes et al. 2007a, Theorem 2.1]), which relies on the unique continuation principle to show that some constant is strictly positive, giving no control on the constant in (2-13). To prove that (2-22) holds, so we have (2-19), we need suitable control of the constant K_W, as in (2-21). To obtain this control we introduce a double averaging procedure which uses the covering condition $\delta_- \geq 2$.

Note that the estimate (2-21) provides a bound on the differentiated density of states $n(E) := N'(E)$ in the interval $[0, E_0]$, whenever it exists, since it then follows from (2-13) and (2-21) that

$$n(E) \leq C_{d,V_{\mathrm{per}}} u_-^{-2^d} \rho_+^{2^d} \gamma_d(E) \min\left\{1, E^{2^d-d-1}\right\} \max\left\{1, E^{2^{d+2}}\right\}. \tag{2-23}$$

Once we have the Minami estimate (2-19), we may prove Poisson statistics and simplicity of eigenvalues. The next theorem is proven for arbitrary Anderson Hamiltonians.

Theorem 2.3. *Let H_ω be an Anderson Hamiltonian. Suppose there exists an open interval $\mathcal{I}$ such that for all large boxes Λ the estimate (2-19) holds for any interval $I \subset \mathcal{I}$ with $|I| \leq \delta_0$, for some $\delta_0 > 0$, with some constant K_M.*

(a) *For all energies $\mathcal{E} \in \mathcal{I} \cap \Xi^{\mathrm{CL}}$ such that the IDS $N(E)$ is differentiable at $\mathcal{E}$ with $n(\mathcal{E}) := N'(\mathcal{E}) > 0$, the point process $\xi_{\mathcal{E},\omega}^{(\Lambda_L)}$ converges weakly, as $L \to \infty$, to the Poisson point process $\xi_{\mathcal{E}}$ on $\mathbb{R}$ with intensity measure $\nu_{\mathcal{E}}(B) := \mathbb{E}\,\xi_{\mathcal{E}}(B) = n(\mathcal{E})|B|$, that is, $\mathrm{d}\nu_{\mathcal{E}} = n(\mathcal{E})\mathrm{d}E$.*

(b) *With probability one, every eigenvalue of H_ω in $\mathcal{I} \cap \Xi^{\mathrm{CL}}$ is simple.*

Theorem 2.3(a) is proven by approximating the point process $\xi_{\mathcal{E},\omega}^{(\Lambda_L)}$ by superpositions of independent point processes, as in [Molchanov 1980/81; Minami 1996], which are then shown to converge weakly to the desired Poisson point process. But here our proof diverges from Minami's, who used the connection, valid for the Anderson model, between the Borel transform of the density of states measure η and averages of the matrix elements of the imaginary part of the resolvent, to identify the intensity measure of the limit point process. Instead, we introduce the random measures

$$\theta_{\mathcal{E},\omega}^{(\Lambda)}(B) := \operatorname{tr}\left\{\chi_\Lambda P_\omega(\mathcal{E} + |\Lambda|^{-1}B)\chi_\Lambda\right\} \quad \text{for a Borel set } B \subset \mathbb{R}, \tag{2-24}$$

justified by (2-13)–(2-16), which we show to have the same weak limit as the point processes $\xi^{(\Lambda)}_{\mathscr{E},\boldsymbol{\omega}}$, and use them to show that, thanks to the Lebesgue Differentiation Theorem, the intensity measure $\nu_{\mathscr{E}}$ of the limit point process $\xi_{\mathscr{E}}$ satisfies $\mathrm{d}\nu_{\mathscr{E}} = n(\mathscr{E})\mathrm{d}E$.

Theorem 2.1 follows immediately by combining Theorem 2.2 and Theorem 2.3. Theorem 2.2 is proven in Sections 4 and 5. In Section 4 we prove Wegner estimates with control of the constant in Lemma 4.1, and a Wegner estimate with one random variable ω_j fixed in Lemma 4.2. Theorem 2.2(i) follows from Lemma 4.1(i). Section 5 contains the proof of Minami's estimate: Theorem 2.2(ii) is proven in Lemma 5.1(i), completing the proof of Theorem 2.2. Theorem 2.3 is proven in Sections 6 and 7. In Section 6 we prove Theorem 2.3(a), namely the convergence of the rescaled eigenvalues to a Poisson point process. Finally, in Section 7 we discuss how Theorem 2.3(b) follows from the Minami estimate (2-19) and [Klein and Molchanov 2006].

Some comments about our notation: Finite volumes will always be understood to be boxes $\Lambda = \Lambda_L(j_0)$ with $j_0 \in \mathbb{Z}^d$ and $L \in 2\mathbb{N}$, $L > \delta_+$. We will always identify such Λ with the torus $j_0 + \mathbb{R}^d/L\mathbb{Z}^d$. If $j \in \widetilde{\Lambda}$, we will consider subboxes $\Lambda_s^{(\Lambda)}(j)$ of Λ, where $0 < s \le L$, defined by

$$\Lambda_s^{(\Lambda)}(j) := \left\{\bigcup_{k\in L\mathbb{Z}^d} \Lambda_s(j+k)\right\} \cap \Lambda,$$

that is, $\chi_{\Lambda_s^{(\Lambda)}(j)} := \chi_\Lambda \sum_{k\in L\mathbb{Z}^d} \chi_{\Lambda_s(j+k)}$. Similarly, we define functions $u_j^{(\Lambda)}$ on the torus Λ by $u_j^{(\Lambda)} := \chi_\Lambda \sum_{k\in L\mathbb{Z}^d} u_{j+k}$, that is, the function u_j will be assumed to have been wrapped around the torus Λ. Note that we then have $V_{\boldsymbol{\omega}}^{(\Lambda)} = \sum_{j\in\widetilde{\Lambda}} \omega_j u_j^{(\Lambda)}$. We will abuse the notation and just write $\Lambda_s(j)$ for $\Lambda_s^{(\Lambda)}(j)$, u_j for $u_j^{(\Lambda)}$, and $V_{\boldsymbol{\omega}}^{(\Lambda)} = \sum_{j\in\widetilde{\Lambda}} \omega_j u_j$. In addition, given $j \in \Upsilon \cap \mathbb{Z}^d$, where $\Upsilon = \Lambda_L(0)$ or $\mathbb{R}^d$, we write $\boldsymbol{\omega} = (\boldsymbol{\omega}_j^{\perp}, \omega_j)$, and $H^{(\Upsilon)}_{(\omega_j^{\perp},\omega_j=s)} = H^{(\Upsilon)}_{(\omega_j^{\perp},s)}$, $P^{(\Upsilon)}_{(\omega_j^{\perp},\omega_j=s)}(I) = P^{(\Upsilon)}_{(\omega_j^{\perp},s)}(I)$ when we want to make explicit that $\omega_j = s$.

3. A new approach to Minami's estimate illustrated by a proof for the (discrete) Anderson Model

The starting point and key idea in our approach is contained in the following simple lemma.

Lemma 3.1. *Consider the self-adjoint operator $H_s = H_0 + sW$ on the Hilbert space $\mathscr{H}$, where H_0 and W are self-adjoint operators on $\mathscr{H}$, with $W \ge 0$ bounded, and $s \ge 0$. Let $P_s(J) = \chi_J(H_s)$ for an interval J, and suppose* $\operatorname{tr} P_s(]-\infty, c]) < \infty$ *for all $c \in \mathbb{R}$ and $s \ge 0$. Then, for all $a, b \in \mathbb{R}$ with $a < b$ we have*

$$\operatorname{tr} P_s(]a,b]) \le \left\{\operatorname{tr} P_0(]-\infty,b]) - \operatorname{tr} P_t(]-\infty,b])\right\} + \operatorname{tr} P_t(]a,b]) \quad \text{for} \quad 0 \le s \le t. \tag{3-1}$$

Proof. Let $a, b \in \mathbb{R}$ with $a < b$ and $0 \le s \le t$. Then, since $W \ge 0$, we have

$$\begin{aligned} \operatorname{tr} P_s(]a,b]) &= \operatorname{tr} P_s(]-\infty,b]) - \operatorname{tr} P_s(]-\infty,a]) \\ &\le \operatorname{tr} P_0(]-\infty,b]) - \operatorname{tr} P_t(]-\infty,a]) \\ &= \operatorname{tr} P_0(]-\infty,b]) - \operatorname{tr} P_t(]-\infty,b]) + \operatorname{tr} P_t(]a,b]), \end{aligned} \tag{3-2}$$

as required. □

We will also use the basic spectral averaging estimate: Let H_0 and W be self-adjoint operators on a Hilbert space $\mathscr{H}$, with $W \ge 0$ bounded. Consider the random operator $H_\xi := H_0 + \xi W$, where ξ is a random variable with a nondegenerate probability distribution μ with compact support. The basic spectral averaging estimate for such perturbations of self-adjoint operators says that, given $\varphi \in \mathscr{H}$ with

$\|\varphi\| = 1$, then for all bounded intervals $I \subset \mathbb{R}$ we have (see [Combes and Hislop 1994, Corollary 4.2], [Combes et al. 2007a, (3.16)])

$$\mathbb{E}_\xi\left\{\langle\varphi, \sqrt{W}\chi_I(H_\xi)\sqrt{W}\varphi\rangle\right\} := \int \mathrm{d}\mu(\xi)\left\langle\varphi, \sqrt{W}\chi_I(H_\xi)\sqrt{W}\varphi\right\rangle \le Q_\mu(|I|), \tag{3-3}$$

where

$$Q_\mu(s) := \begin{cases} \rho_\infty s & \text{if } \mu \text{ has a bounded density } \rho \text{ as in (2-7)}, \\ 8\sup_{a\in\mathbb{R}} \mu([a, a+s]) & \text{otherwise.} \end{cases} \tag{3-4}$$

As a consequence, given a trace class operator $S \ge 0$ on $\mathcal{H}$, we have

$$\mathbb{E}_\xi\left\{\operatorname{tr}\{\sqrt{W}\chi_I(H_\xi)\sqrt{W}S\}\right\} \le (\operatorname{tr} S)\, Q_\mu(|I|). \tag{3-5}$$

Note that the measure μ has no atoms if and only if $\lim_{s\downarrow 0} Q_\mu(s) = 0$.

Lemma 3.1 will allow the decoupling of random variables for the performance of two spectral averagings.

We will first illustrate our approach to Minami's estimate by giving a simple and transparent proof of the estimate for in the discrete case, that is, for the Anderson model. We will then comment on how to proceed in the continuum case, that is, for the Anderson Hamiltonian.

Minami's estimate for the (discrete) Anderson model. An Anderson model will be a discrete random Schrödinger operator of the form

$$H_\omega = H_0 + V_\omega \quad \text{on} \quad \ell^2(\mathbb{Z}^d), \tag{3-6}$$

where H_0 is a bounded self-adjoint operator and V_ω is the random potential given by $V_\omega(j) = \omega_j$ for $j \in \mathbb{Z}^d$, where $\boldsymbol{\omega} = \{\omega_j\}_{j\in\mathbb{Z}^d}$ is a family of independent, identically distributed random variables with common probability distribution μ. (The usual Anderson model has $H_0 = -\Delta$, where Δ is the discrete Laplacian.) We assume μ has compact support and no atoms. Adjusting H_0 and μ, we may assume

$$\{0, M\} \in \operatorname{supp}\mu \subset [0, M] \quad \text{with } M \in]0, \infty[. \tag{3-7}$$

Restrictions of H_ω to finite volumes $\Lambda \subset \mathbb{Z}^d$ are denoted by $H_\omega^{(\Lambda)}$, a self-adjoint operator of the form

$$H_\omega^{(\Lambda)} = H_0^{(\Lambda)} + V_\omega^{(\Lambda)} \quad \text{on} \quad \ell^2(\Lambda), \tag{3-8}$$

where $H_0^{(\Lambda)}$ is a self-adjoint restriction of H_0 to the finite-dimensional Hilbert space $\ell^2(\Lambda)$, and $V_\omega^{(\Lambda)}$ is the restriction of V_ω to Λ. (In the discrete case our results are not sensitive to the choice of $H_{0,\Lambda}$, they hold for any boundary condition.) Given a Borel set $J \subset \mathbb{R}$, we write $P_\omega^{(\Lambda)}(J) = P_{H_\omega}^{(\Lambda)}(J) = \chi_J(H_\omega^{(\Lambda)})$ for the associated spectral projection.

What makes the discrete case much easier than the continuum is that in the discrete case finite volume operators are finite-dimensional and each random variable couples a rank-one perturbation. Given a unit vector φ in a Hilbert space $\mathcal{H}$, we let Π_φ denote the orthogonal projection onto $\mathbb{C}\varphi$, the one-dimensional subspace spanned by φ. With this notation, the potentials in (3-6) and (3-8) are given by sums of rank-one perturbations:

$$V_\omega = \sum_{j\in\mathbb{Z}^d} \omega_j \Pi_j \quad \text{and} \quad V_\omega^{(\Lambda)} = \sum_{j\in\Lambda} \omega_j \Pi_j, \quad \text{with } \Pi_j = \Pi_{\delta_j}. \tag{3-9}$$

For rank-one perturbations Lemma 3.1 has the following consequence:

Lemma 3.2. *Let H_s be as in Lemma 3.1 with $W = \Pi_\varphi$ for some unit vector $\varphi \in \mathcal{H}$. Then, for all $a, b \in \mathbb{R}$ with $a < b$ we have*

$$\operatorname{tr} P_s(]a, b]) \leq 1 + \operatorname{tr} P_t(]a, b]) \quad \textit{for all } 0 \leq s \leq t. \tag{3-10}$$

Proof. Let $0 \leq s \leq t$. Recall that for any $c \in \mathbb{R}$ we always have

$$0 \leq \operatorname{tr} P_s(]-\infty, c]) - \operatorname{tr} P_t(]-\infty, c]) \leq 1, \tag{3-11}$$

the last inequality being a consequence of the min-max principle applied to rank-one perturbations, for example, [Kirsch 2008, Lemma 5.22]. Thus (3-10) follows immediately from (3-1). □

For rank-one perturbations the fundamental spectral averaging estimate (3-3) may be stated as follows: Consider the random self-adjoint operator

$$H_\xi = H_0 + \xi \Pi_\varphi \quad \text{on } \mathcal{H}, \tag{3-12}$$

where H_0 is a self-adjoint operator on the Hilbert space $\mathcal{H}$, $\varphi \in \mathcal{H}$ with $\|\varphi\| = 1$, and ξ is a random variable with a nondegenerate probability distribution μ with compact support. Let $P_\xi(J) = \chi_J(H_\xi)$ for a Borel set $J \subset \mathbb{R}$. Then for all bounded intervals $I \subset \mathbb{R}$ we have [Wegner 1981; Fröhlich and Spencer 1983; Carmona et al. 1987; Kirsch 2008; Combes and Hislop 1994; Combes et al. 2007a]

$$\mathbb{E}_\xi\left\{\langle\varphi, P_\xi(I)\varphi\rangle\right\} := \int \mathrm{d}\mu(\xi)\, \langle\varphi, P_\xi(I)\varphi\rangle \leq Q_\mu(|I|). \tag{3-13}$$

The Wegner estimate for an Anderson model [Wegner 1981; Fröhlich and Spencer 1983; Carmona et al. 1987; Kirsch 2008] is an immediate consequence of (3-13):

$$\mathbb{E}\left\{\operatorname{tr} P_{H_\omega}^{(\Lambda)}(I)\right\} = \sum_{j \in \Lambda} \mathbb{E}_{\omega_j^\perp}\left\{\mathbb{E}_{\omega_j}\{\langle\delta_j, P_{H_\omega}^{(\Lambda)}(I)\delta_j\rangle\}\right\} \leq Q_\mu(|I|)|\Lambda|. \tag{3-14}$$

We can now prove Minami's estimate for an Anderson model for arbitrary μ with compact support and no atoms, a result previously known only for μ with a bounded density [Minami 1996; Bellissard et al. 2007; Graf and Vaghi 2007].

Theorem 3.3. *Let H_ω be an Anderson model as in (3-6), with μ arbitrary except for compact support and no atoms. Let $\Lambda \subset \mathbb{Z}^d$ be a finite volume. For any bounded interval I we have*

$$\mathbb{E}\left\{(\operatorname{tr} P_\omega^{(\Lambda)}(I))(\operatorname{tr} P_\omega^{(\Lambda)}(I) - 1)\right\} \leq (Q_\mu(|I|)|\Lambda|)^2. \tag{3-15}$$

Theorem 3.3 is extended in [Combes et al. 2009], allowing for n arbitrary intervals and arbitrary single-site probability measure μ with no atoms. We also give applications of (3-15), deriving new results about the multiplicity of eigenvalues and Mott's formula for the ac-conductivity when the single-site probability distribution is Hölder continuous.

Proof of Theorem 3.3. Fix $\Lambda \subset \mathbb{Z}^d$ and let I be a bounded interval. Since the measure μ has no atoms, it follows from (3-14) that $\mathbb{E}_\omega\left\{\operatorname{tr} P_\omega^{(\Lambda)}(\{c\})\right\} = 0$ for any $c \in \mathbb{R}$. Thus we may take all intervals to be of the form $]a, b]$, and use Lemma 3.2 to decouple the random variable ω_j from the random variables $\boldsymbol{\omega}_j^{\perp}$. In view of (3-7), for all $\tau_j \geq M$, $j \in \mathbb{Z}^d$, we have

$$\begin{aligned}(\operatorname{tr} P_{\boldsymbol{\omega}}^{(\Lambda)}(I))(\operatorname{tr} P_{\boldsymbol{\omega}}^{(\Lambda)}(I)-1) &= \sum_{j\in\Lambda}\big\{\langle\delta_j, P_{\boldsymbol{\omega}}^{(\Lambda)}(I)\delta_j\rangle(\operatorname{tr} P_{\boldsymbol{\omega}}^{(\Lambda)}(I)-1)\big\}\\ &\leq \sum_{j\in\Lambda}\big\{\langle\delta_j, P_{(\boldsymbol{\omega}_j^{\perp},\omega_j)}^{(\Lambda)}(I)\delta_j\rangle(\operatorname{tr} P_{(\boldsymbol{\omega}_j^{\perp},\tau_j)}^{(\Lambda)}(I))\big\}. \qquad (3\text{-}16)\end{aligned}$$

We now average over the random variables $\boldsymbol{\omega} = \{\omega_j\}_{j\in\mathbb{Z}^d}$. Using (3-13), we get

$$\begin{aligned}\mathbb{E}_{\boldsymbol{\omega}}\big\{(\operatorname{tr} P_{\boldsymbol{\omega}}^{(\Lambda)}(I))(\operatorname{tr} P_{\boldsymbol{\omega}}^{(\Lambda)}(I)-1)\big\} &\leq \sum_{j\in\Lambda}\mathbb{E}_{\boldsymbol{\omega}_j^{\perp}}\big\{(\operatorname{tr} P_{(\boldsymbol{\omega}_j^{\perp},\tau_j)}^{(\Lambda)}(I))\big(\mathbb{E}_{\omega_j}\big\{\langle\delta_j, P_{(\boldsymbol{\omega}_j^{\perp},\omega_j)}^{(\Lambda)}(I)\delta_j\rangle\big\}\big)\big\}\\ &\leq Q_\mu(|I|)\sum_{j\in\Lambda}\mathbb{E}_{\boldsymbol{\omega}_j^{\perp}}\big\{\operatorname{tr} P_{(\boldsymbol{\omega}_j^{\perp},\tau_j)}^{(\Lambda)}(I)\big\}. \qquad (3\text{-}17)\end{aligned}$$

This holds for all $\tau_j \geq M$, $j \in \mathbb{Z}^d$, so we now take $\tau_j = M + \tilde{\omega}_j$, where $\tilde{\boldsymbol{\omega}} = \{\tilde{\omega}_j\}_{j\in\mathbb{Z}^d}$ and $\boldsymbol{\omega} = \{\omega_j\}_{j\in\mathbb{Z}^d}$ are two independent, identically distributed collections of random variables. Now $\boldsymbol{\tau} = \{\tau_j\}_{j\in\mathbb{Z}^d}$ are independent identically distributed random variables with a common probability distribution $\mu_{\boldsymbol{\tau}}$ such that $Q_{\mu_{\boldsymbol{\tau}}} = Q_\mu$. We get

$$\begin{aligned}\mathbb{E}_{\boldsymbol{\omega}}\big\{(\operatorname{tr} P_{\boldsymbol{\omega}}^{(\Lambda)}(I))(\operatorname{tr} P_{\boldsymbol{\omega}}^{(\Lambda)}(I)-1)\big\} &= \mathbb{E}_{\boldsymbol{\tau}}\big\{\mathbb{E}_{\boldsymbol{\omega}}\big\{(\operatorname{tr} P_{\boldsymbol{\omega}}^{(\Lambda)}(I))(\operatorname{tr} P_{\boldsymbol{\omega}}^{(\Lambda)}(I)-1)\big\}\big\}\\ &\leq Q_\mu(|I|)\sum_{j\in\Lambda}\mathbb{E}_{(\boldsymbol{\omega}_j^{\perp},\tau_j)}(\operatorname{tr} P_{(\boldsymbol{\omega}_j^{\perp},\tau_j)}^{(\Lambda)}(I)) \leq (Q_\mu(|I|)|\Lambda|)^2, \qquad (3\text{-}18)\end{aligned}$$

where we used the Wegner estimate (3-14). (More precisely, we estimate as in (3-14); the random variables do not need to be identically distributed.) □

Stepping up to the continuum. Unfortunately things are not so simple for the continuum Anderson Hamiltonian. The main reason is that the random potential $V_{\boldsymbol{\omega}}$ in (2-3) is a sum of independent random perturbations of infinite rank, not of rank one as in the discrete case, and thus the a priori bound in (3-11), and also Lemma 3.2, are not applicable anymore.

To prove Minami's estimate on the continuum we will use the fundamental spectral averaging estimate as in (3-5). The straightforward expansion of the trace in (3-14) and (3-17) cannot be used for the spectral averaging, even with u_j instead of δ_j, and will be replaced by a more sophisticated expansion in terms of trace class operators, as in [Combes and Hislop 1994; Combes et al. 2007a] ((4-1)–(4-5)). Lemma 3.1 will be modified, since the term in brackets in (3-1) does not satisfy an a priori bound as in (3-11) anymore. This term will be estimated using the Birman–Solomyak formula; see (5-3), (5-4). The bound in (3-11) is then replaced by averaging the resulting expression over all the other random variables and using the Wegner estimate (2-13); see (5-9). The resulting bound is useful if the constant K_W in (2-13) is not too big (we have $K_W = 1$ in the lattice, as can be seen in (3-14)). Since previous proofs of the Wegner estimate do not give the desired control of K_W, we must revisit the Wegner estimate. We introduce a double averaging procedure that provides the desired estimates on the constant K_W (Lemma 4.1). In addition, because of the way we use the Birman–Solomyak formula, we do not have freedom in the choice of τ_j as in (3-16), we have to take $\tau_j = M_\rho$. Thus we cannot average in $\boldsymbol{\tau}$ as in (3-18); this argument is replaced by a refinement of the Wegner estimate where one of the random variables is fixed (Lemma 4.2).

4. The Wegner estimate revisited

Let H_ω be the Anderson Hamiltonian, $E_0 > 0$, $I \subset [0, E_0]$ an interval, and Λ a finite box. To prove the Wegner estimate (2-13), it is shown in [Combes and Hislop 1994; Combes et al. 2007a] that

$$\operatorname{tr} P_\omega^{(\Lambda)}(I) \le Q_1 \sum_{j,k\in\widetilde{\Lambda}} \left|\operatorname{tr}\{\sqrt{u_k}\, P_\omega^{(\Lambda)}(I)\sqrt{u_j}\, T_{j,k}^{(\Lambda)}\}\right|, \tag{4-1}$$

where $\{T_{j,k}^{(\Lambda)}\}_{j,k\in\widetilde{\Lambda}}$ are (nonrandom) trace class operators in $\mathrm{L}^2(\Lambda)$ such that

$$\max_{j\in\widetilde{\Lambda}} \left\{\sum_{k\in\widetilde{\Lambda}} \|T_{j,k}^{(\Lambda)}\|_1\right\} \le Q_2, \tag{4-2}$$

the constants Q_1, Q_2 depending only on $E_0, d, u, V_{\mathrm{per}}, M_\rho$. Letting

$$T_{j,k}^{(\Lambda)} = U_{j,k}^{(\Lambda)} |T_{j,k}^{(\Lambda)}|$$

be the polar decomposition of the operator $T_{j,k}^{(\Lambda)}$, recalling that then $|T_{j,k}^{(\Lambda)*}| = U_{j,k}^{(\Lambda)} T_{j,k}^{(\Lambda)} U_{j,k}^{(\Lambda)*}$, and setting

$$S_j^{(\Lambda)} := \frac{1}{2}\sum_{k\in\widetilde{\Lambda}} \left(|T_{j,k}^{(\Lambda)*}| + |T_{k,j}^{(\Lambda)}|\right) \ge 0 \quad \text{for} \quad j \in \widetilde{\Lambda}, \tag{4-3}$$

we obtain

$$\operatorname{tr} P_\omega^{(\Lambda)}(I) \le Q_1 \sum_{j\in\widetilde{\Lambda}} \operatorname{tr}\left\{\sqrt{u_j}\, P_\omega^{(\Lambda)}(I)\sqrt{u_j}\, S_j^{(\Lambda)}\right\}, \tag{4-4}$$

with

$$\max_{j\in\widetilde{\Lambda}} \left\{\operatorname{tr} S_j^{(\Lambda)}\right\} \le Q_2. \tag{4-5}$$

If we now take the expectation in (4-4), use (3-5) and (4-5), we get the Wegner estimate (2-13) with $K_W = Q_1 Q_2$.

We will need control of the constant K_W and a Wegner estimate with one of the random variables, say ω_0, fixed. In the course of obtaining control over K_W we will derive (4-1) with estimates on the constants Q_1 and Q_2 in the case when $\delta_- \ge 1$.

A Wegner estimate with control of the constants.

Lemma 4.1. *Let H_ω be an Anderson Hamiltonian.*

(i) *Assume $\delta_- \ge 2$. Then there exists a constant $C_{d,V_{\mathrm{per}}}$ such that, given an energy $E_0 > 0$, (2-13) holds for all intervals $I \subset [0, E_0]$ with a constant*

$$K_W \le C_{d,V_{\mathrm{per}}} \left(\frac{\rho_+}{u_-}\right)^{2^d} \gamma_d(E_0) \min\left\{1, E_0^{2^d-d-1}\right\} \max\left\{1, E_0^{2^{d+2}}\right\}, \tag{4-6}$$

where we have $\gamma_d(E_0) = 1$ if $d \ge 2$, and $\gamma_1(E_0) = \gamma_{1,V_{\mathrm{per}}}(E_0) \in]0, 1]$ with $\lim_{E_0\to 0} \gamma_1(E_0) = 0$.

(ii) *Assume $\delta_- \ge 1$. Then, given an energy $E_0 > 0$, (4-1)–(4-5) hold for all intervals $I \subset [0, E_0]$ with constants*

$$Q_1 = (1 + E_0)^{2[\![d/4]\!]} \quad \textit{and} \quad Q_2 = C'_{d,V_{\mathrm{per}}}, \tag{4-7}$$

and hence (2-13) *holds for all intervals* $I \subset [0, E_0]$ *with a constant*

$$K_W \le C'_{d,V_{\text{per}}}(1+E_0)^{2[\![d/4]\!]}. \tag{4-8}$$

Proof. Assume $\delta_- \ge m$, where m is either 1 or 2. We set

$$\chi_j^{(m)} = \chi_{\Lambda_m(j)} \quad \text{for } j \in \widetilde{\Upsilon} := \Upsilon \cap \mathbb{Z}^d,$$

where Υ is either $\mathbb{R}^d$ or a finite box Λ (recall that in this case $\chi_{\Lambda_m(j)}$ denotes $\chi^{(\Lambda)}_{\Lambda_m(j)}$, a subbox in the torus). Note that for any $j_0 \in \widetilde{\Upsilon}$ we have

$$\sum_{j\in(j_0+m\mathbb{Z}^d)\cap\Upsilon} \chi_j^{(m)} = 1. \tag{4-9}$$

We also let $\hat{\chi}_j^{(m)} = u_j{}^{-1/2}\chi_j^{(m)}$ on $\Lambda_m(j)$, $\hat{\chi}_j^{(m)} = 0$ otherwise. It follows from (2-5) that

$$\hat{\chi}_j^{(m)} \le u_-^{-1/2}\chi_j^{(m)}.$$

(Recall we write u_j for $u_j^{(\Lambda)}$.)

To prove (i), assume $\delta_- \ge 2$. We write $\boldsymbol{\omega}' = \{\omega_j\}_{j\in 2\mathbb{Z}^d}$, $\boldsymbol{\omega}'' = \{\omega_j\}_{j\notin 2\mathbb{Z}^d}$. We set

$$H_{\boldsymbol{\omega}''} := H_0 + V_{\boldsymbol{\omega}''}, \qquad V_{\boldsymbol{\omega}''} := \sum_{j\notin 2\mathbb{Z}^d} \omega_j u_j. \tag{4-10}$$

Note that $H_{\boldsymbol{\omega}''}$ is a $2\mathbb{Z}^d$ ergodic family of random self-adjoint operators, and we have

$$H_{\boldsymbol{\omega}} \ge H_{\boldsymbol{\omega}''} \ge H_0, \qquad H_{\boldsymbol{\omega}''} \ge V_{\boldsymbol{\omega}''}. \tag{4-11}$$

Fix an energy $E_0 > 0$, a box Λ, and let $I =]a, b] \subset [0, E_0]$. Set $p = 2^{d+1}$. Given $t > 0$, the function $g_t(x) = (1+tx)^{-2p}$ is convex on the interval $]-1/t, \infty[$. Thus, using (4-11), we can proceed as in [Combes and Hislop 1994] using convexity and Jensen's inequality (see Lemma B.1 in Appendix B), and then (4-9) and (2-5), to get

$$\begin{aligned}
\operatorname{tr} P_{\boldsymbol{\omega}}^{(\Lambda)}(I) &\le (1+tE_0)^{2p} \operatorname{tr}\bigl\{P_{\boldsymbol{\omega}}^{(\Lambda)}(I)(1+tH_{\boldsymbol{\omega}}^{(\Lambda)})^{-2p} P_{\boldsymbol{\omega}}^{(\Lambda)}(I)\bigr\}\\
&\le (1+tE_0)^{2p} \operatorname{tr}\bigl\{P_{\boldsymbol{\omega}}^{(\Lambda)}(I)(1+tH_{\boldsymbol{\omega}''}^{(\Lambda)})^{-2p} P_{\boldsymbol{\omega}}^{(\Lambda)}(I)\bigr\}\\
&= (1+tE_0)^{2p} \operatorname{tr}\bigl\{P_{\boldsymbol{\omega}}^{(\Lambda)}(I)(1+tH_{\boldsymbol{\omega}''}^{(\Lambda)})^{-2p}\bigr\}\\
&= (1+tE_0)^{2p} \sum_{j,k\in\Lambda\cap 2\mathbb{Z}^d} \operatorname{tr}\bigl\{P_{\boldsymbol{\omega}}^{(\Lambda)}(I)\chi_j^{(2)}(1+tH_{\boldsymbol{\omega}''}^{(\Lambda)})^{-2p}\chi_k^{(2)}\bigr\}\\
&= (1+tE_0)^{2p} \sum_{j,k\in\Lambda\cap 2\mathbb{Z}^d} \operatorname{tr}\bigl\{\sqrt{u_k}P_{\boldsymbol{\omega}}^{(\Lambda)}(I)\sqrt{u_j}\hat{\chi}_j^{(2)}(1+tH_{\boldsymbol{\omega}''}^{(\Lambda)})^{-2p}\hat{\chi}_k^{(2)}\bigr\}.
\end{aligned} \tag{4-12}$$

It then follows from (3-5), proceeding as in (4-1)–(4-4) (see also [Combes et al. 2007a, Lemma 2.1]), that

$$\begin{aligned}\mathbb{E}_{\omega'} \operatorname{tr} P_{\omega}^{(\Lambda)}(I) &\le (1+tE_0)^{2p}\rho_+|I| \sum_{j,k\in\Lambda\cap 2\mathbb{Z}^d} \big\|\hat\chi_j^{(2)}(1+tH_{\omega''}^{(\Lambda)})^{-2p}\hat\chi_k^{(2)}\big\|_1\\ &\le (1+tE_0)^{2p}u_-^{-1}\rho_+|I| \sum_{j,k\in\Lambda\cap 2\mathbb{Z}^d} \big\|\chi_j^{(2)}(1+tH_{\omega''}^{(\Lambda)})^{-2p}\chi_k^{(2)}\big\|_1. \end{aligned} \tag{4-13}$$

We now use several deterministic estimates. First,

$$\big\|\chi_j^{(2)}(1+tH_{\omega''}^{(\Lambda)})^{-2p}\chi_k^{(2)}\big\|_1 \le \sum_{r\in\Lambda\cap 2\mathbb{Z}^d} \big\|\chi_j^{(2)}(1+tH_{\omega''}^{(\Lambda)})^{-p}\chi_r^{(2)}\big\|_2 \big\|\chi_r^{(2)}(1+tH_{\omega''}^{(\Lambda)})^{-p}\chi_k^{(2)}\big\|_2. \tag{4-14}$$

Second,

$$\big\|\chi_j^{(2)}(1+tH_{\omega''}^{(\Lambda)})^{-p}\chi_r^{(2)}\big\|_2^2 \le \big\|\chi_j^{(2)}(1+tH_{\omega''}^{(\Lambda)})^{-p}\chi_r^{(2)}\big\| \big\|\chi_j^{(2)}(1+tH_{\omega''}^{(\Lambda)})^{-p}\chi_r^{(2)}\big\|_1. \tag{4-15}$$

Third, we estimate

$$\big\|\chi_j^{(2)}(1+tH_{\omega''}^{(\Lambda)})^{-p}\chi_r^{(2)}\big\|$$

using the Combes–Thomas estimate. We use the precise estimate provided in [Germinet and Klein 2003b, (19) in Theorem 1] (with $\gamma=\frac{1}{2}$), modified for finite volume operators with periodic boundary condition as in [Figotin and Klein 1996, Lemma 18] and [Klein and Koines 2001, Theorem 3.6], plus the fact that we are using boxes of side 2. For $L\ge L_d$ we have, with $d_\Lambda(j,r)$ the distance on the torus Λ,

$$\begin{aligned}\big\|\chi_j^{(2)}(1+tH_{\omega''}^{(\Lambda)})^{-p}\chi_r^{(2)}\big\| &= t^{-p}\big\|\chi_j^{(2)}(t^{-1}+H_{\omega''}^{(\Lambda)})^{-p}\chi_r^{(2)}\big\| \le t^{-p}\big(\tfrac{4}{3}t\big)^p \exp\frac{1}{2\sqrt{t}}\exp\Big(-\frac{1}{8\sqrt{td}}d_\Lambda(j,r)\Big)\\ &= \big(\tfrac{4}{3}\big)^p\exp\frac{1}{2\sqrt{t}}\exp\Big(-\frac{1}{8\sqrt{td}}d_\Lambda(j,r)\Big). \end{aligned} \tag{4-16}$$

Fourth, note that

$$\begin{aligned}\big\|\chi_j^{(2)}(1+tH_{\omega''}^{(\Lambda)})^{-p}\chi_r^{(2)}\big\|_1 &\le \big\|\chi_j^{(2)}(1+tH_{\omega''}^{(\Lambda)})^{-p/2}\big\|_2\big\|\chi_r^{(2)}(1+tH_{\omega''}^{(\Lambda)})^{-p/2}\big\|_2\\ &= \big\|\chi_j^{(2)}(1+tH_{\omega''}^{(\Lambda)})^{-p}\chi_j^{(2)}\big\|_1^{1/2}\big\|\chi_r^{(2)}(1+tH_{\omega''}^{(\Lambda)})^{-p}\chi_r^{(2)}\big\|_1^{1/2}. \end{aligned} \tag{4-17}$$

We now average over ω''. Using (4-14)–(4-17), we have

$$\begin{aligned}&\mathbb{E}_{\omega''}\Big\{\big\|\chi_j^{(2)}(1+tH_{\omega''}^{(\Lambda)})^{-p}\chi_r^{(2)}\big\|_1^{1/2}\big\|\chi_r^{(2)}(1+tH_{\omega''}^{(\Lambda)})^{-p}\chi_k^{(2)}\big\|_1^{1/2}\Big\}\\ &\quad\le \mathbb{E}_{\omega''}\Big\{\big\|\chi_j^{(2)}(1+tH_{\omega''}^{(\Lambda)})^{-p}\chi_j^{(2)}\big\|_1^{1/4}\big\|\chi_r^{(2)}(1+tH_{\omega''}^{(\Lambda)})^{-p}\chi_r^{(2)}\big\|_1^{1/2}\times\big\|\chi_k^{(2)}(1+tH_{\omega''}^{(\Lambda)})^{-p}\chi_k^{(2)}\big\|_1^{1/4}\Big\}\\ &\quad\le \beta_t := \mathbb{E}_{\omega''}\Big\{\big\|\chi_0^{(2)}(1+tH_{\omega''}^{(\Lambda)})^{-p}\chi_0^{(2)}\big\|_1\Big\}, \end{aligned} \tag{4-18}$$

where we used Hölder's inequality plus translation invariance (in the torus) of the expectation.

It now follows from (4-14), (4-15), (4-16), (4-17), and (4-18) that

$$
\begin{aligned}
&\mathbb{E}_{\omega''}\Big\{\sum_{j,k\in\Lambda\cap 2\mathbb{Z}^d}\big\|\chi_j^{(2)}(1+tH_{\omega''}^{(\Lambda)})^{-2p}\chi_k^{(2)}\big\|_1\Big\}\\
&\qquad\le \beta_t\,\exp\frac{1}{2\sqrt{t}}\big(\tfrac{4}{3}\big)^p\sum_{j,k,r\in\Lambda\cap 2\mathbb{Z}^d}\exp\Big(-\frac{1}{16\sqrt{td}}d_\Lambda(j,r)\Big)\exp\Big(-\frac{1}{16\sqrt{td}}d_\Lambda(r,k)\Big)\\
&\qquad\le 2^{-d}\beta_t\,\exp\frac{1}{2\sqrt{t}}\big(\tfrac{4}{3}\big)^p|\Lambda|\Big(\sum_{r\in 2\mathbb{Z}^d}\exp\Big(-\frac{1}{16\sqrt{td}}|r|\Big)\Big)^2\\
&\qquad= 2^{-d}\beta_t\,\exp\frac{1}{2\sqrt{t}}\big(\tfrac{4}{3}\big)^p|\Lambda|\Big(\sum_{s\in\mathbb{Z}}\exp\Big(-\frac{1}{8d\sqrt{t}}|s|\Big)\Big)^{2d}\\
&\qquad\le 2^{-d}\beta_t\,\exp\frac{1}{2\sqrt{t}}\big(\tfrac{4}{3}\big)^p|\Lambda|\Big(1+2\int_0^\infty ds\,\exp\Big(-\frac{1}{8d\sqrt{t}}|s|\Big)\Big)^{2d}\\
&\qquad\le 2^{-d}\beta_t\exp\frac{1}{2\sqrt{t}}\big(\tfrac{4}{3}\big)^p|\Lambda|(1+16d\sqrt{t})^{2d},
\end{aligned}
\tag{4-19}
$$

so we conclude from (4-13) that

$$
\mathbb{E}_\omega\operatorname{tr}P_\omega^{(\Lambda)}(I)\le\big(\tfrac{4}{3}\big)^p\frac{1}{2u_-}(1+tE_0)^{2p}\beta_t\,\exp\frac{1}{2\sqrt{t}}(1+16d\sqrt{t})^{2d}\rho_+|I||\Lambda|. \tag{4-20}
$$

We now estimate β_t. We have, using periodicity, and again Lemma B.1 with $H_{\omega''}^{(\Lambda)}\ge V_{\omega''}$ and (2-5),

$$
\begin{aligned}
\beta_t &:= \mathbb{E}_{\omega''}\big\{\operatorname{tr}\{\chi_0^{(2)}(1+tH_{\omega''}^{(\Lambda)})^{-p}\chi_0^{(2)}\}\big\}=\frac{2^d}{|\Lambda|}\mathbb{E}_{\omega''}\big\{\operatorname{tr}\{(1+tH_{\omega''}^{(\Lambda)})^{-p}\}\big\}\\
&\le\frac{2^d}{|\Lambda|}\mathbb{E}_{\omega''}\big\{\operatorname{tr}\big\{(1+tH_{\omega''}^{(\Lambda)})^{-p/4}(1+tV_{\omega''})^{-p/2}(1+tH_{\omega''}^{(\Lambda)})^{-p/4}\big\}\big\}\\
&=\frac{2^d}{|\Lambda|}\mathbb{E}_{\omega''}\big\{\operatorname{tr}\big\{(1+tV_{\omega''})^{-p/4}(1+tH_{\omega''}^{(\Lambda)})^{-p/2}(1+tV_{\omega''})^{-p/4}\big\}\big\}\\
&=\mathbb{E}_{\omega''}\big\{\operatorname{tr}\big\{\chi_0^{(2)}(1+tV_{\omega''})^{-p/4}(1+tH_{\omega''}^{(\Lambda)})^{-p/2}(1+tV_{\omega''})^{-p/4}\chi_0^{(2)}\big\}\big\}\\
&=\mathbb{E}_{\omega''}\big\{\operatorname{tr}\big\{(1+tV_{\omega''})^{-p/4}\chi_0^{(2)}(1+tH_{\omega''}^{(\Lambda)})^{-p/2}\chi_0^{(2)}(1+tV_{\omega''})^{-p/4}\big\}\big\}\\
&\le\mathbb{E}_{\omega''}\big\{(1+tu_-\hat{\omega}_0)^{-p/2}\operatorname{tr}\{\chi_0^{(2)}(1+tH_{\omega''}^{(\Lambda)})^{-p/2}\chi_0^{(2)}\}\big\},
\end{aligned}
\tag{4-21}
$$

where we set, with $Q:=\{0,1\}^d\setminus\{0\}\subset\mathbb{Z}^d$,

$$
\hat{\omega}_0=\sum_{q\in Q}\hat{\omega}_{0,q}\quad\text{with }\hat{\omega}_{0,q}:=\min\{\omega_{q+i}:i\in 2\mathbb{Z}^d,\ |q+i|_\infty=1\}. \tag{4-22}
$$

Note that $|Q|=2^d-1$, and $(q+2\mathbb{Z}^d)\cap(q'+2\mathbb{Z}^d)=\varnothing$ if $q,q'\in Q$ with $q\ne q'$, so $\{\hat{\omega}_{0,q}\}_{q\in Q}$ are independent random variables.

Now, with $\Theta:=\max\{-\operatorname{ess\,inf}V_{\mathrm{per}},0\}$,

$$
\begin{aligned}
\operatorname{tr}\big\{\chi_0^{(2)}(1+tH_{\omega''}^{(\Lambda)})^{-p/2}\chi_0^{(2)}\big\}&\le\Big\{\sup_{E\ge 0}\Big(\frac{1+\Theta+E}{1+tE}\Big)^{p/2}\Big\}\operatorname{tr}\big\{\chi_0^{(2)}(H_{\omega''}^{(\Lambda)}+1+\Theta)^{-p/2}\chi_0^{(2)}\big\}\\
&\le C_{d,\Theta}\max\{1,t^{-p/2}\},
\end{aligned}
\tag{4-23}
$$

where (as in the proof of Lemma A.4 of [Germinet and Klein 2004], for example) we used the fact that $\operatorname{tr}\{\chi_0^{(2)}(H_{\omega''}^{(\Lambda)}+1+\Theta)^{-p/2}\chi_0^{(2)}\}$ is uniformly bounded, independently of Λ — itself a consequence of the inequality $p=2^{d+1}\geq 4[\![d/4]\!]$, where $[\![d/4]\!]$ is the smallest integer exceeding $d/4$.

Moreover, since $p=2^{d+1}>2(2^d-1)$,

$$
\begin{aligned}
\mathbb{E}_{\omega''}\{(1+tu_-\hat{\omega}_0)^{-p/2}\} &\leq \prod_{q\in Q}\mathbb{E}_{\omega''}\big\{(1+tu_-\hat{\omega}_{0,q})^{-p/(2(2^d-1))}\big\}\\
&=\prod_{q\in Q}\mathbb{E}_{\omega''}\big\{\max_{\substack{i\in 2\mathbb{Z}^d\\ |q+i|_\infty=1}}(1+tu_-\omega_{q+i})^{-p/(2(2^d-1))}\big\}\\
&\leq \big(2d\,\mathbb{E}_{\omega_0}\big\{(1+tu_-\omega_0)^{-p/(2(2^d-1))}\big\}\big)^{2^d-1}\\
&\leq \left(2d\,\rho_+\int_0^\infty \mathrm{d}\omega_0\,(1+tu_-\omega_0)^{-p/(2(2^d-1))}\right)^{2^d-1}\\
&\leq \left(\frac{2d(2^d-1)\,\rho_+}{(2^d-1-\frac{p}{2})tu_-}\right)^{2^d-1}=C'_d\Big(\frac{\rho_+}{tu_-}\Big)^{2^d-1}.
\end{aligned}\tag{4-24}
$$

Thus, we have

$$
\beta_t\leq C'_{d,\Theta}\max\{1,t^{-2^d}\}\Big(\frac{\rho_+}{tu_-}\Big)^{2^d-1},\tag{4-25}
$$

so it follows from (4-20) that

$$
\mathbb{E}_{\omega}\operatorname{tr}P_{\omega}^{(\Lambda)}(I)\leq \frac{C'_{d,\Theta}}{u_-}(1+tE_0)^{2^{d+2}}\ \exp\frac{1}{2\sqrt{t}}(1+16d\sqrt{t})^{2d}\max\{1,t^{-2^d}\}\Big(\frac{\rho_+}{tu_-}\Big)^{2^d-1}\rho_+|I||\Lambda|.\tag{4-26}
$$

If $E_0\leq 3$, we choose $t=1/E_0$, obtaining

$$
\mathbb{E}_{\omega}\operatorname{tr}P_{\omega}^{(\Lambda)}(I)\leq C''_{d,\Theta}\Big(\frac{\rho_+}{u_-}\Big)^{2^d}E_0^{2^d-d-1}|I||\Lambda|.\tag{4-27}
$$

If $E_0>3$, we take $t=1$, getting

$$
\mathbb{E}_{\omega}\operatorname{tr}P_{\omega}^{(\Lambda)}(I)\leq C'''_{d,\Theta}\Big(\frac{\rho_+}{u_-}\Big)^{2^d}E_0^{2^{d+2}}|I||\Lambda|.\tag{4-28}
$$

Thus, for all $E_0>0$ we have

$$
\mathbb{E}_{\omega}\operatorname{tr}P_{\omega}^{(\Lambda)}(I)\leq \frac{C_{d,\Theta}}{u_-}\Big(\frac{\rho_+}{u_-}\Big)^{2^d-1}\min\{1,E_0^{2^d-d-1}\}\max\{1,E_0^{2^{d+2}}\}\rho_+|I||\Lambda|.\tag{4-29}
$$

For $d=1$ we need to do a bit better. In this case we redo (4-23) as follows:

$$
\operatorname{tr}\{\chi_0^{(2)}(1+tH_{\omega''}^{(\Lambda)})^{-p/2}\chi_0^{(2)}\}\leq \operatorname{tr}\{\chi_0^{(2)}(1+tH_{\omega''}^{(\Lambda)})^{-1}\chi_0^{(2)}\}\leq\alpha_t:=\operatorname{tr}\{\chi_0^{(2)}(1+tH_0^{(\Lambda)})^{-1}\chi_0^{(2)}\}.\tag{4-30}
$$

For $d=1$ the estimate (4-26) now becomes

$$
\mathbb{E}_{\omega}\operatorname{tr}P_{\omega}^{(\Lambda)}(I)\leq \frac{C_{1,\theta}}{u_-}(1+tE_0)^{8}\ \exp\frac{1}{2\sqrt{t}}(1+16\sqrt{t})^2\alpha_t\Big(\frac{\rho_+}{tu_-}\Big)\rho_+|I||\Lambda|,\tag{4-31}
$$

and thus (4-29) becomes

$$\mathbb{E}_{\boldsymbol{\omega}} \operatorname{tr} P_{\boldsymbol{\omega}}^{(\Lambda)}(I) \le \frac{C_{1,\Theta}}{u_-} \frac{\rho_+}{u_-} \gamma_1(E_0) \max\{1, E_0^8\} \rho_+ |I||\Lambda|. \tag{4-32}$$

where $\gamma_1(E_0) \le 1$ and $\lim_{E_0 \to 0} \gamma_1(E_0) = 0$ uniformly in Λ large.

This proves (i). To prove (ii), we now assume $\delta_- \ge 1$. We proceed as in the proof of (i), with $\boldsymbol{\omega}' = \boldsymbol{\omega}$ and $\boldsymbol{\omega}'' = \{\omega_j\}_{j \notin \mathbb{Z}^d} = \emptyset$, that is $V_{\boldsymbol{\omega}''} = 0$ and $H_{\boldsymbol{\omega}''} = H_0$. We also now fix $p = 2[\![d/4]\!]$. Then (4-12) yields (4-1) with $Q_1 = (1 + tE_0)^{2p}$ and $T_{j,k}^{(\Lambda)} = \hat{\chi}_j^{(1)} (1 + tH_0^{(\Lambda)})^{-2p} \hat{\chi}_k^{(1)}$. Proceeding as in (4-14)–(4-19) gives (4-2) with

$$Q_2 = \beta_t^{(0)} \exp \frac{1}{4\sqrt{t}} \left(\tfrac{4}{3}\right)^p (1 + 32d\sqrt{t})^{2d}, \tag{4-33}$$

where, as in (4-23),

$$\beta_t^{(0)} := \left\| \chi_0^{(1)} (1 + tH_0^{(\Lambda)})^{-p} \chi_0^{(1)} \right\|_1 \le C_{d,\Theta} \max\{1, t^{-p}\} \le C_{d,\Theta}. \tag{4-34}$$

We now set $t = 1$, obtaining (4-7) and (4-8). □

A Wegner estimate with ω_0 fixed. Let $\Upsilon = \Lambda_L(0)$ or $\mathbb{R}^d$. Given $\tau \in \mathbb{R}$, we consider (recall $u_0 = u$)

$$H_{(\boldsymbol{\omega}^{(0)},\tau)}^{(\Upsilon)} = H_{(\boldsymbol{\omega}^{(0)},\omega_0=\tau)}^{(\Upsilon)} = H_{\boldsymbol{\omega}}^{(\Upsilon)} + (\tau - \omega_0)u. \tag{4-35}$$

Lemma 4.2. *Let $H_{\boldsymbol{\omega}}$ be an Anderson Hamiltonian, $E_0 > 0$. Given $\tau \in \mathbb{R}$, there exists a constant $\widetilde{K}_W = \widetilde{K}_W(d, u, V_{\text{per}}, E_0, M_\rho, \tau)$, such that for any interval $I \subset [0, E_0]$ and finite box $\Lambda = \Lambda_L(0)$ we have*

$$\mathbb{E}_{\boldsymbol{\omega}^{(0)}} \left\{ \operatorname{tr} P_{(\boldsymbol{\omega}^{(0)},\tau)}^{(\Lambda)}(I) \right\} \le \widetilde{K}_W \rho_+ |I||\Lambda|. \tag{4-36}$$

Moreover, if $\delta_- \ge 2$, we have

$$\widetilde{K}_W \le C_{d,V_{\text{per}},\tau} (1 + E_0)^{2[\![\frac{d}{4}]\!]}. \tag{4-37}$$

Proof. We will show that the proof of Theorem 1.3 of [Combes et al. 2007a] can be modified to yield the proposition. *All references of the form (2.N) in this proof will be to that paper unless otherwise stated.*

We introduce the background potential

$$H_1 := H_0 + \tau \sum_{j \in 2\mathbb{Z}^d} u_j = -\Delta + V_{\text{per}}^{(2)}, \tag{4-38}$$

where $V_{\text{per}}^{(2)} = V_{\text{per}} + \tau \sum_{j \in 2\mathbb{Z}^d} u_j$ is a $2\mathbb{Z}^d$-periodic potential. It follows that

$$H_{(\boldsymbol{\omega}^{(0)},\tau)} = H_1 + V_{\boldsymbol{\omega}^{(0)}}(\tau) \quad \text{with } V_{\boldsymbol{\omega}^{(0)}}(\tau) := \sum_{j \in (2\mathbb{Z})^d \setminus \{0\}} (\omega_j - \tau) u_j + \sum_{j \in \mathbb{Z} \setminus (2\mathbb{Z})^d} \omega_j u_j. \tag{4-39}$$

The main point is that the single-site potential $u_0 = u$ does not appear in the sum, but all the other u_j's appear with a random coefficient.

To prove (4-36) with no conditions on δ_-, we proceed as in Section 2 of [Combes et al. 2007a]. We take an interval $I \subset [0, E_0]$, write $\tilde{I} = [0, E_0{+}1]$; I and $\tilde{I}$ replace the intervals Δ and $\tilde{\Delta}$ in that paper. The potential V_Λ in equation (2.7) there is replaced by $V_{\boldsymbol{\omega}^{(0)}}^{(\Lambda)}(\tau)$, which only involves the random variables $\boldsymbol{\omega}^{(0)}$. As a consequence, the sum in (2.10) runs over indices $i, j \in \widetilde{\Lambda} \setminus \{0\}$. The spectral averaging in (2.13) can thus be performed with respect to the random variables $\boldsymbol{\omega}^{(0)}$. Similarly for (2.18), since $\tilde{K}(n)_{i_1,j_n}$

of (2.17) is now constructed only with the single-site potentials u_j's present in $V^{(\Lambda)}_{\omega^{(0)}}(\tau)$, that is, u_j with $j \in \widetilde{\Lambda} \setminus \{0\}$. We thus get the analog of (2.20), with $M_0 = M_\rho + |\tau|$, namely, with $P_1(B) = \chi_B(H_1)$,

$$\mathbb{E}_{\omega^{(0)}}\big\{\mathrm{tr}\big\{P^{(\Lambda)}_{(\omega^{(0)},\tau)}(I)P^{(\Lambda)}_1(\mathbb{R}\setminus\tilde{I})\big\}\big\} \le K_1\rho_+|I||\Lambda|, \tag{4-40}$$

for an appropriate constant K_1.

It remains to bound $\mathbb{E}_{\omega^{(0)}}\big\{\mathrm{tr}\big\{P^{(\Lambda)}_{(\omega^{(0)},\tau)}(I)P^{(\Lambda)}_1(\tilde{I})\big\}\big\}$. For this purpose, we set

$$\tilde{V}_1 = \sum_{j\in(\mathrm{e}_1+2\mathbb{Z}^d)} u_j, \tag{4-41}$$

where $\mathrm{e}_1 = (1, 0, 0, \dots, 0) \notin 2\mathbb{Z}^d$, and we use H_1 and $\tilde{V}^{(\Lambda)}_1$, the restriction of $\tilde{V}_1$ to Λ, instead of H_0 and $\tilde{V}_\Lambda = \sum_{j\in\mathbb{Z}^d\cap\Lambda} u_j$, in the crucial estimate (2.1) of [Combes et al. 2007a]. Since H_1 and $\tilde{V}_1$ are both $2\mathbb{Z}^d$-periodic, we have[1] the equivalent of (2.1),

$$P^{(\Lambda)}_1(\tilde{I})\tilde{V}^{(\Lambda)}_1 P^{(\Lambda)}_1(\tilde{I}) \ge C(E_0, u, V_{\mathrm{per}}, \tau)P^{(\Lambda)}_1(\tilde{I}), \tag{4-42}$$

with a constant $C(E_0, u, V_{\mathrm{per}}, \tau) > 0$. Since

$$\tilde{V}_1 \le \tilde{V}_{0^\perp} := \sum_{j\in\mathbb{Z}^d\setminus\{0\}} u_j, \tag{4-43}$$

it follows that

$$P^{(\Lambda)}_1(\tilde{I})\tilde{V}^{(\Lambda)}_{0^\perp} P^{(\Lambda)}_1(\tilde{I}) \ge C(E_0, u, V_{\mathrm{per}}, \tau)P^{(\Lambda)}_1(\tilde{I}). \tag{4-44}$$

As a consequence, we get (2.21) with $\tilde{V}_\Lambda$ replaced by $\tilde{V}^{(\Lambda)}_{0^\perp}$, and hence we obtain the analogue of (2.31):

$$\mathbb{E}_{\omega^{(0)}}\big\{\mathrm{tr}\big\{P^{(\Lambda)}_1(\tilde{I})\tilde{V}^{(\Lambda)}_{0^\perp}P^{(\Lambda)}_{(\omega^{(0)},\tau)}(I)\tilde{V}^{(\Lambda)}_{0^\perp}P^{(\Lambda)}_1(\tilde{I})\big\}\big\} \le K_2\rho_+|I||\Lambda|, \tag{4-45}$$

for an appropriate constant K_2.

The desired bound (4-36) now follows as the analogue of (2.32).

If $\delta_- \ge 2$, we have

$$\sum_{j\in((j_0+\mathbb{Z}^d)\setminus\{0\})\cap\Lambda} u_j \ge u_-\chi_\Lambda, \tag{4-46}$$

so we can apply the proof of Lemma 4.1(ii) to the random operator $H_{\omega^{(0)},\tau}$ getting (4-36) with (4-37). □

5. The Minami estimate

Theorem 2.2 follows by combining Lemma 4.1(i) and the following lemma:

Lemma 5.1. *Let H_ω be an Anderson Hamiltonian with a uniform-like distribution μ. Let $E_0 > 0$ and suppose the Wegner estimate (2-13) holds for all intervals $I \subset [0, E_0]$ with a constant K_W such that*

$$2K_W U_+ \frac{\rho_+}{\rho_-} \le 1. \tag{5-1}$$

Then there exists a constant $K_M = K_M(u, \rho_\pm, M_\rho, E_0, d)$ such that the Minami estimate (2-19) holds for all intervals $I \subset [0, E_0]$.

[1] by [Combes et al. 2003, Proposition 1.3]; see also [Combes et al. 2007a, Theorem 2.1].

If $\delta_- \geq 2$, we have the estimate

$$K_M \leq C_{d,V_{\mathrm{per}},M_\rho}(1+E_0)^{4[\![\frac{d}{4}]\!]}. \tag{5-2}$$

Proof. Let Λ be a finite box. It follows from (2-13) that $\mathbb{E}_\omega\{\operatorname{tr} P_\omega^{(\Lambda)}(\{c\})\} = 0$ for any $c \in \mathbb{R}$. Thus we may take all bounded intervals to be of the form $]a, b]$. For such an interval we modify Lemma 3.1 as follows: Given $\delta > 0$ small, we pick a nonincreasing function $h \in C^\infty(\mathbb{R})$, such that $h(t) = 1$ for $t \leq 0$ and $h(t) = 0$ for $t \geq \delta$. Note that $0 \leq h \leq 1$, $h' \leq 0$, $\operatorname{supp} h' \subset [0, \delta]$, $\int_{\mathbb{R}} \mathrm{d}t\, h'(t) = -1$, and we can choose h so $|h'| \leq \frac{2}{\delta}$. Given $c \in \mathbb{R}$, we set $h_c(t) = h(t-c)$, and note that $h_{c-\delta} \leq \chi_{]-\infty,c]} \leq h_c$. We let $I =]a, b]$, $I_\delta =]a-\delta, b+\delta]$. Using h, we rework (3-1) in the following way. Given $j \in \widetilde{\Lambda}$ and $\tau \geq M_\rho$, we have

$$\begin{aligned}
\operatorname{tr} P_\omega^{(\Lambda)}(I) &\leq \operatorname{tr} h_b(H_\omega^{(\Lambda)}) - \operatorname{tr} h_{a-\delta}(H_\omega^{(\Lambda)}) \\
&\leq \bigl\{\operatorname{tr} h_b\bigl(H^{(\Lambda)}_{(\omega_j^\perp,\omega_j=0)}\bigr) - \operatorname{tr} h_b\bigl(H^{(\Lambda)}_{(\omega_j^\perp,\omega_j=\tau)}\bigr)\bigr\} + \bigl\{\operatorname{tr} h_b\bigl(H^{(\Lambda)}_{(\omega_j^\perp,\omega_j=\tau)}\bigr) - \operatorname{tr} h_{a-\delta}\bigl(H^{(\Lambda)}_{(\omega_j^\perp,\omega_j=\tau)}\bigr)\bigr\} \\
&\leq \bigl\{\operatorname{tr} h_b\bigl(H^{(\Lambda)}_{(\omega_j^\perp,\omega_j=0)}\bigr) - \operatorname{tr} h_b\bigl(H^{(\Lambda)}_{(\omega_j^\perp,\omega_j=\tau)}\bigr)\bigr\} + \operatorname{tr} P^{(\Lambda)}_{(\omega_j^\perp,\omega_j=\tau)}(I_\delta).
\end{aligned} \tag{5-3}$$

We now fix $\tau = M_\rho$ and use the Birman–Solomyak formula [Simon 1998] as in [Combes et al. 2007b, (7)–(8)], plus the hypothesis (2-8), obtaining

$$\begin{aligned}
\xi^{(\Lambda)}_{b,\tau}(\omega_j^\perp) &:= \operatorname{tr} h_b\bigl(H^{(\Lambda)}_{(\omega_j^\perp,\omega_j=0)}\bigr) - \operatorname{tr} h_b\bigl(H^{(\Lambda)}_{(\omega_j^\perp,\omega_j=\tau)}\bigr) \\
&= -\int_0^\tau \mathrm{d}s\, \operatorname{tr}\bigl\{\sqrt{u_j}\, h_b'(H^{(\Lambda)}_{(\omega_j^\perp,\omega_j=s)})\sqrt{u_j}\bigr\} \\
&\leq \frac{2}{\delta}\int_0^\tau \mathrm{d}s\, \operatorname{tr}\bigl\{\sqrt{u_j}\, P^{(\Lambda)}_{(\omega_j^\perp,\omega_j=s)}(]b, b+\delta])\sqrt{u_j}\bigr\} \\
&\leq \frac{2}{\delta\rho_-}\int \mathrm{d}s\rho(s)\, \operatorname{tr}\bigl\{\sqrt{u_j}\, P^{(\Lambda)}_{(\omega_j^\perp,\omega_j=s)}(]b, b+\delta])\sqrt{u_j}\bigr\}.
\end{aligned} \tag{5-4}$$

Note that $\xi^{(\Lambda)}_{b,\tau}(\omega_j^\perp)$ is closely related to the spectral shift function associated to the pair $H^{(\Lambda)}_{(\omega_j^\perp,\omega_j=0)}$ and $H^{(\Lambda)}_{(\omega_j^\perp,\omega_j=\tau)}$.

Now fix $E_0 > 0$, let $I =]a, b] \subset [0, E_0[$, and consider $\delta > 0$ such that $b + \delta \leq E_0$, so $I_\delta \subset [0, E_0]$. If $\operatorname{tr} P_\omega^{(\Lambda)}(I) \geq 1$, it follows from (4-4) that

$$(\operatorname{tr} P_\omega^{(\Lambda)}(I))(\operatorname{tr} P_\omega^{(\Lambda)}(I) - 1) \leq Q_1 \sum_{j\in\widetilde{\Lambda}} \operatorname{tr}\bigl\{\sqrt{u_j}\, P_\omega^{(\Lambda)}(I)\sqrt{u_j}\, S_j^{(\Lambda)}\bigr\}(\operatorname{tr} P_\omega^{(\Lambda)}(I) - 1), \tag{5-5}$$

so, using (5-3) and (5-4), we get

$$(\operatorname{tr} P_\omega^{(\Lambda)}(I))(\operatorname{tr} P_\omega^{(\Lambda)}(I) - 1) \leq Q_1 \sum_{j\in\widetilde{\Lambda}} \bigl\{\bigl(\operatorname{tr}\bigl\{\sqrt{u_j}\, P_\omega^{(\Lambda)}(I)\sqrt{u_j}\, S_j^{(\Lambda)}\bigr\}\bigr)\Phi^{(\Lambda)}_{b,\tau}(\omega_j^\perp)\bigr\}, \tag{5-6}$$

where for each $j \in \widetilde{\Lambda}$

$$\Phi^{(\Lambda)}_{b,\tau}(\omega_j^\perp) := \bigl(\xi^{(\Lambda)}_{b,\tau}(\omega_j^\perp) - 1\bigr) + \operatorname{tr} P^{(\Lambda)}_{(\omega_j^\perp,\tau)}(I_\delta) \tag{5-7}$$

is independent of the random variable ω_j. If $\operatorname{tr} P_\omega^{(\Lambda)}(I) < 1$, we have $P_\omega^{(\Lambda)}(I) = 0$, and hence we also have (5-6).

Thus, if we now take the expectation in (5-6), use (3-5) and (4-5), we get

$$\begin{aligned}\mathbb{E}\big\{(\operatorname{tr} P_\omega^{(\Lambda)}(I))(\operatorname{tr} P_\omega^{(\Lambda)}(I)-1)\big\} &\le Q_1Q_2\rho_+|I| \sum_{j\in\widetilde{\Lambda}} \mathbb{E}_{\omega_j^\perp}\big\{\Phi_{b,\tau}^{(\Lambda)}(\omega_j^\perp)\big\}\\ &= Q_1Q_2\rho_+|I|\,|\Lambda|\mathbb{E}_{\omega_k^\perp}\big\{\Phi_{b,\tau}^{(\Lambda)}(\omega_k^\perp)\big\} \end{aligned}\tag{5-8}$$

for any $k\in\widetilde{\Lambda}$.

We will now estimate $\mathbb{E}_{\omega_k^\perp}\{\Phi_{b,\tau}^{(\Lambda)}(\omega_k^\perp)\}$. It follows from (5-4) and (2-13) that, if we have (5-1),

$$\begin{aligned}\mathbb{E}_{\omega_k^\perp}\{\xi_{b,\tau}^{(\Lambda)}(\omega_k^\perp)\} &\le \frac{2}{\delta\rho_-}\mathbb{E}_\omega\big\{\operatorname{tr}\big\{\sqrt{u_k}P_\omega^{(\Lambda)}(]b,b+\delta])\sqrt{u_k}\big\}\big\}\\ &= \frac{2}{\delta\rho_-|\Lambda|}\mathbb{E}_\omega\Big\{\sum_{j\in\widetilde{\Lambda}}\operatorname{tr}\big\{\sqrt{u_j}P_{(\omega)}^{(\Lambda)}(]b,b+\delta])\sqrt{u_j}\big\}\Big\}\\ &\le \frac{2U_+}{\delta\rho_-|\Lambda|}\mathbb{E}_\omega\big\{\operatorname{tr} P_\omega^{(\Lambda)}(]b,b+\delta])\big\} \le 2K_WU_+\frac{r}{r}\rho_+\rho_- \le 1. \end{aligned}\tag{5-9}$$

In this case, we have

$$\mathbb{E}_{\omega_k^\perp}\big\{\Phi_{b,\tau}^{(\Lambda)}(\omega_k^\perp)\big\} \le \mathbb{E}_{\omega_k^\perp}\big\{\operatorname{tr} P_{(\omega_k^\perp,\tau)}^{(\Lambda)}(I_\delta)\big\} \le \widetilde{K}_W\rho_+(|I|+2\delta)|\Lambda|, \tag{5-10}$$

where we used Lemma 4.2, where $\widetilde{K}_W=\widetilde{K}_W(d,u,V_{\text{per}},E_0,M_\rho)$.

Combining (5-8) and (5-10) we get

$$\mathbb{E}\big\{(\operatorname{tr} P_\omega^{(\Lambda)}(I))(\operatorname{tr} P_\omega^{(\Lambda)}(I)-1)\big\} \le Q_1Q_2\widetilde{K}_W|I|(|I|+2\delta)(\rho_+|\Lambda|)^2. \tag{5-11}$$

Letting $\delta\to 0$ we get (2-19) with $K_M=Q_1Q_2\widetilde{K}_W$.

If $\delta_-\ge 2$, the estimate (5-2) follows from (4-7) and (4-37). □

6. Poisson statistics

In this section we prove Theorem 2.3(a).

Let H_ω be an Anderson Hamiltonian, and suppose $\mathcal{I}$ is an open interval such that for all large boxes Λ the estimate (2-19) holds for any interval $I\subset\mathcal{I}$ with $|I|\le\delta_0$, for some $\delta_0>0$, with some constant K_M. (We will assume that a given Λ is large enough.) Recall we have (2-13) for these intervals with some constant K_W.

Let $\mathcal{E}\in\mathcal{I}\cap\Xi^{\text{CL}}$ be such that the IDS $N(E)$ is differentiable at $\mathcal{E}$ with $n(\mathcal{E}):=N'(\mathcal{E})>0$. It follows from (2-13) that we then have

$$0<n(\mathcal{E})\le K_W\rho_+. \tag{6-1}$$

We fix an open interval $\mathcal{I}_1$ such that $\mathcal{E}\in\mathcal{I}_1\subset\overline{\mathcal{I}}_1\subset\mathcal{I}\cap\Xi^{\text{CL}}$. Note that for each bounded Borel set $B\subset\mathbb{R}$ there exists a finite $c_B=c_{B,\mathcal{E},\mathcal{I}_1}$ such that $\mathcal{E}+|\Lambda|^{-1}B\subset\mathcal{I}_1$ and $|\mathcal{E}+|\Lambda|^{-1}B|\le\delta_0$ if $|\Lambda|\ge c_B$. The point process $\xi_\omega^{(\Lambda)}=\xi_{\mathcal{E},\omega}^{(\Lambda)}$ of (2-17) has an intensity measure given by $\nu^{(\Lambda)}(B):=\mathbb{E}\,\xi_\omega^{(\Lambda)}(B)$ for a Borel set $B\subset\mathbb{R}$; it follows from (2-13) that,

$$\nu^{(\Lambda)}(B)\le K_W\rho_+|B| \quad \text{for all } \Lambda \text{ with } |\Lambda|\ge c_B. \tag{6-2}$$

We start with the same general strategy used in [Molchanov 1980/81; Minami 1996]. We fix $a \in]0,1[$, and divide $\Lambda = \Lambda_L(0)$ into M_L boxes $\Lambda^{(m)} = \Lambda_\ell(k_m)$ of side $\ell \approx L^a$, $\ell \in 2\mathbb{N}$, centered at $k_m \in \Lambda \cap (2\mathbb{Z}^d)$; note $M_L = \frac{|\Lambda_L|}{|\Lambda_\ell|} \approx L^{(1-a)d}$. For each $m = 1, 2, \ldots, M_L$ we define point processes

$$\xi_\omega^{(\Lambda,m)}(B) := \operatorname{tr} P_\omega^{(\Lambda^{(m)})}(\mathscr{E} + |\Lambda|^{-1} B) \quad \text{for a Borel set } B \subset \mathbb{R}. \tag{6-3}$$

Note that $\{\xi_\omega^{(\Lambda,m)}\}_{m=1,2,\ldots,M_L}$ are independent, identically distributed point processes, each with intensity measure (using (2-13))

$$\nu^{(\Lambda,m)}(B) := \mathbb{E}\, \xi_\omega^{(\Lambda,m)}(B) \le K_W \rho_+ |B| M_L^{-1} \quad \text{for all } \Lambda \text{ with } |\Lambda| \ge c_B. \tag{6-4}$$

We consider their superposition, the point process

$$\widetilde{\xi}_\omega^{(\Lambda)} := \sum_{m=1}^{M_L} \xi_\omega^{(\Lambda,m)}, \tag{6-5}$$

with intensity measure

$$\widetilde{\nu}^{(\Lambda)}(B) := \mathbb{E}\, \widetilde{\xi}_\omega^{(\Lambda)}(B) \le K_W \rho_+ |B| \quad \text{for all } \Lambda \text{ with } |\Lambda| \ge c_B. \tag{6-6}$$

We will prove that $\widetilde{\xi}_\omega^{(\Lambda)} \approx \xi_\omega^{(\Lambda)}$ as $L \to \infty$, and that $\widetilde{\xi}_\omega^{(\Lambda)}$ converges weakly, as $L \to \infty$, to the Poisson point process ξ with intensity measure $\nu(B) := \mathbb{E}\, \xi(B) = n(\mathscr{E})|B|$. But here we must use different methods from [Molchanov 1980/81; Minami 1996].

So let $\theta_\omega^{(\Lambda)} = \theta_{\mathscr{E},\omega}^{(\Lambda)}$ be the random measure defined in (2-24); its intensity measure is

$$\eta^{(\Lambda)}(B) := \mathbb{E}\, \theta_\omega^{(\Lambda)}(B) = |\Lambda| \eta(\mathscr{E} + |\Lambda|^{-1} B), \tag{6-7}$$

where η is the density of states measure, given in (2-16). It again follows from (2-13) that

$$\eta^{(\Lambda)}(B) \le K_W \rho_+ |B| \quad \text{for all } \Lambda \text{ with } |\Lambda| \ge c_B. \tag{6-8}$$

We start with a lemma. Given a measure η on $\mathbb{R}$, we write $\eta(f) := \int_{\mathbb{R}} f \, d\eta$ for suitable functions f, say, $f \in \mathscr{F}_{b,K}$, the collection of bounded Borel functions on $\mathbb{R}$ vanishing outside a compact interval. It follows from (2-17) that for all $f \in \mathscr{F}_{b,K}$ we have

$$\xi_\omega^{(\Lambda)}(f) = \operatorname{tr} f_\Lambda(H_\omega^{(\Lambda)}), \quad \text{where} \quad f_\Lambda(E) := f(|\Lambda|(E - \mathscr{E})), \tag{6-9}$$

with similar expressions for $\widetilde{\xi}_\omega^{(\Lambda)}(f)$, $\xi_\omega^{(\Lambda,m)}(f)$, and $\theta_\omega^{(\Lambda)}(f)$.

Lemma 6.1. *For all $f \in \mathscr{F}_{b,K}$ we have*

$$\lim_{L\to\infty} \mathbb{E} \big| \xi_\omega^{(\Lambda)}(f) - \widetilde{\xi}_\omega^{(\Lambda)}(f) \big| = 0 \tag{6-10}$$

and

$$\lim_{L\to\infty} \mathbb{E} \big| \xi_\omega^{(\Lambda)}(f) - \theta_\omega^{(\Lambda)}(f) \big| = 0. \tag{6-11}$$

Proof. In view of (6-2), (6-6), and (6-8), it suffices to prove (6-10) and (6-11) for $f \in C_K^\infty(\mathbb{R})$, since $\{f \in C_K^\infty(\mathbb{R}) : \operatorname{supp} f \subset J\}$ is dense in $\mathrm{L}^1(J, \mathrm{d}E)$ for any interval J.

So let $f \in C_K^\infty(\mathbb{R})$. To prove (6-10), we set $\ell' \approx \ell - \sqrt{\ell}$, $\Lambda^{(m,\prime)} = \Lambda_{\ell'}(k_m)$, and $\Lambda^{(m,\prime\prime)} = \Lambda_\ell(k_m) \setminus \Lambda_{\ell'}(k_m)$. Using $\chi_\Lambda = \sum_{m=1}^{M_L} \chi_{\Lambda^{(m)}}$, we get

$$\begin{aligned}
\xi_\omega^{(\Lambda)}(f) - \widetilde{\xi}_\omega^{(\Lambda)}(f) &= \sum_{m=1}^{M_L} \Big(\operatorname{tr}\big\{\chi_{\Lambda^{(m)}} f_\Lambda(H_\omega^{(\Lambda)})\chi_{\Lambda^{(m)}}\big\} - \operatorname{tr} f_\Lambda(H_\omega^{(\Lambda^{(m)})})\Big) \\
&= \sum_{m=1}^{M_L} \Big(\operatorname{tr}\big\{\chi_{\Lambda^{(m,\prime)}} f_\Lambda(H_\omega^{(\Lambda)})\chi_{\Lambda^{(m,\prime)}}\big\} - \operatorname{tr}\{\chi_{\Lambda^{(m,\prime)}} f_\Lambda(H_\omega^{(\Lambda^{(m)})})\chi_{\Lambda^{(m,\prime)}}\}\Big) \\
&\qquad + \sum_{m=1}^{M_L} \Big(\operatorname{tr}\big\{\chi_{\Lambda^{(m,\prime\prime)}} f_\Lambda(H_\omega^{(\Lambda)})\chi_{\Lambda^{(m,\prime\prime)}}\big\} - \operatorname{tr}\big\{\chi_{\Lambda^{(m,\prime\prime)}} f_\Lambda(H_\omega^{(\Lambda^{(m)})})\chi_{\Lambda^{(m,\prime\prime)}}\big\}\Big). \qquad (6\text{-}12)
\end{aligned}$$

We now use the fact that the expectation is invariant under translations in the torus to get, for any m,

$$\mathbb{E}\big|\xi_\omega^{(\Lambda)}(f) - \widetilde{\xi}_\omega^{(\Lambda)}(f)\big| \le M_L \mathbb{E}\big|\operatorname{tr}\big\{\chi_{\Lambda^{(m,\prime)}} f_\Lambda(H_\omega^{(\Lambda)})\chi_{\Lambda^{(m,\prime)}}\big\} - \operatorname{tr}\big\{\chi_{\Lambda^{(m,\prime)}} f_\Lambda(H_\omega^{(\Lambda^{(m)})})\chi_{\Lambda^{(m,\prime)}}\big\}\big| \qquad (6\text{-}13)$$

$$+ M_L \mathbb{E}\big|\operatorname{tr}\big\{\chi_{\Lambda^{(m,\prime\prime)}} f_\Lambda(H_\omega^{(\Lambda)})\chi_{\Lambda^{(m,\prime\prime)}}\big\} - \operatorname{tr}\big\{\chi_{\Lambda^{(m,\prime\prime)}} f_\Lambda(H_\omega^{(\Lambda^{(m)})})\chi_{\Lambda^{(m,\prime\prime)}}\big\}\big|. \qquad (6\text{-}14)$$

It follows from the Wegner estimate (2-13) that

$$\begin{aligned}
M_L \mathbb{E}\big|\operatorname{tr}\big\{\chi_{\Lambda^{(m,\prime\prime)}} f_\Lambda(H_\omega^{(\Lambda)})\chi_{\Lambda^{(m,\prime\prime)}}\big\}\big| &\le M_L \frac{|\Lambda^{(m,\prime\prime)}|}{|\Lambda|} \mathbb{E}\operatorname{tr}\big\{|f_\Lambda|(H_\omega^{(\Lambda)})\big\} \\
&\le M_L \frac{|\Lambda^{(m,\prime\prime)}|}{|\Lambda|} K_W \rho_+ |\Lambda| \int_{\mathbb{R}} |f_\Lambda|(E)\,\mathrm{d}E \\
&= \frac{|\Lambda^{(m,\prime\prime)}|}{|\Lambda^{(m)}|} K_W \rho_+ \|f\|_1. \qquad (6\text{-}15)
\end{aligned}$$

Similarly,

$$\begin{aligned}
M_L \mathbb{E}\big|\operatorname{tr}\big\{\chi_{\Lambda^{(m,\prime\prime)}} f_\Lambda(H_\omega^{(\Lambda^{(m)})})\chi_{\Lambda^{(m,\prime\prime)}}\big\}\big| &\le M_L \frac{|\Lambda^{(m,\prime\prime)}|}{|\Lambda^{(m)}|} \mathbb{E}\operatorname{tr}\big\{|f_\Lambda|(H_\omega^{(\Lambda^{(m)})})\big\} \\
&\le M_L \frac{|\Lambda^{(m,\prime\prime)}|}{|\Lambda^{(m)}|} K_W \rho_+ |\Lambda^{(m)}| \int_{\mathbb{R}} |f_\Lambda|(E)\,\mathrm{d}E \\
&= \frac{|\Lambda^{(m,\prime\prime)}|}{|\Lambda^{(m)}|} K_W \rho_+ \|f\|_1. \qquad (6\text{-}16)
\end{aligned}$$

Since

$$\frac{|\Lambda^{(m,\prime\prime)}|}{|\Lambda^{(m)}|} \approx \frac{\ell^{d-1}\sqrt{\ell}}{\ell^d} = \frac{1}{\sqrt{\ell}} \approx \frac{1}{L^{\frac{a}{2}}} \to 0 \quad \text{as} \quad L \to \infty, \qquad (6\text{-}17)$$

the term in (6-14) goes to 0 as $L \to \infty$.

To finish the proof of (6-10) we need to show that the term in (6-13) also goes to 0 as $L \to \infty$. To do that we will use that $\overline{\mathcal{I}}_1 \subset \Xi^{\mathrm{CL}}$, the Helffer–Sjöstrand formula for smooth functions of self-adjoint operators, and estimates on Schrödinger operators.

Given a box Λ, we identify $\mathrm{L}^2(\Lambda)$ with the subspace of $\mathrm{L}^2(\mathbb{R}^d)$ consisting of functions vanishing outside Λ. Given a function $\phi \in C_K^\infty(\mathbb{R})$, we let $W(\phi)$ to be the closure of the local first order differential operator $[\Delta, \phi]$ on $C_K^\infty(\mathbb{R})$. We set

$$\chi_\phi := \chi_{\operatorname{supp}\phi}, \qquad \chi_{\nabla\phi} := \chi_{\operatorname{supp}\nabla\phi},$$

and note that $W(\phi) = \chi_{\nabla\phi} W(\phi) = W(\phi)\chi_{\nabla\phi} = \chi_{\nabla\phi} W(\phi)\chi_{\nabla\phi}$. We recall that if $\operatorname{supp}\phi \subset \Lambda^\circ$, the interior of Λ, which here may be either a finite box or $\mathbb{R}^d$, we have

$$\big\|(H_\omega^{(\Lambda)}+1)^{-1/2} W(\phi)\big\| = \big\|W(\phi)(H_\omega^{(\Lambda)}+1)^{-1/2}\big\| \le C_\phi := C_1\big(\|\Delta\phi\|_\infty + \|\nabla\phi\|_\infty\big), \tag{6-18}$$

where C_1 depends only on d. We also recall that for all $x \in \Lambda$ we have

$$\big\|\chi_{\Lambda_1(x)}(H_\omega^{(\Lambda)}+1)^{-1}\big\|_{p_d} \le C_2 < \infty \quad \text{with } p_d = [\tfrac{d}{2}]+1, \tag{6-19}$$

the constant C_2 being independent of x and Λ for $L \ge 2$ [Klein et al. 2002, (130)–(136)].

We now recall the Helffer–Sjöstrand formula; refer to [Hunziker and Sigal 2000, Appendix B] for details. Given $g \in C^\infty(\mathbb{R})$ and $m \in \mathbb{N}$, we set

$$\{\!\{g\}\!\}_m := \sum_{r=0}^{m} \int_{\mathbb{R}} \mathrm{d}u\, |g^{(r)}(u)|\, (1+|u|^2)^{(r-1)/2}. \tag{6-20}$$

If $\{\!\{g\}\!\}_m < \infty$ with $m \ge 2$, then for any self-adjoint operator K we have

$$f(K) = \int_{\mathbb{R}^2} \mathrm{d}\tilde{g}(z)\,(K-z)^{-1}, \tag{6-21}$$

where the integral converges absolutely in operator norm. Here $z = x+iy$, $\tilde{g}(z)$ is an *almost analytic extension* of g to the complex plane, $\mathrm{d}\tilde{g}(z) := \frac{1}{2\pi}\partial_{\bar z}\tilde{g}(z)\,\mathrm{d}x\,\mathrm{d}y$ with $\partial_{\bar z} = \partial_x + i\partial_y$, and $|\mathrm{d}\tilde{g}(z)| := (2\pi)^{-1}|\partial_{\bar z}\tilde{g}(z)|\,\mathrm{d}x\,\mathrm{d}y$. Moreover, for all $p \ge 0$ we have

$$\int_{\mathbb{R}^2} |\mathrm{d}\tilde{g}(z)| \frac{1}{|\Im z|^p} \le c_p\ \{\!\{g\}\!\}_m < \infty \quad \text{for} \quad m \ge p+1 \tag{6-22}$$

with a constant c_p.

Since $f \in C_K^\infty(\mathbb{R})$, we have, using the Helffer–Sjöstrand formula with $\Lambda = \Lambda_L$, $R_\omega^{(\Lambda)}(z) = (H_\omega^{(\Lambda)} - z)^{-1}$ and $R_\omega^{(\Lambda,m)}(z) = (H_\omega^{(\Lambda^{(m)})} - z)^{-1}$, and taking $\phi_0 \in C_K^\infty(\Lambda_{\ell-10d}(k_m))$ such that $\phi_0\chi_{\Lambda_{\ell-20d}(k_m)} = \chi_{\Lambda_{\ell-20d}(k_m)}$ and $0 \le \phi_0 \le 1$, that

$$\begin{aligned} T_\omega^{(\Lambda)} &:= \chi_{\Lambda^{(m,\prime)}} f_\Lambda(H_\omega^{(\Lambda)})\chi_{\Lambda^{(m,\prime)}} - \chi_{\Lambda^{(m,\prime)}} f_\Lambda(H_\omega^{(\Lambda^{(m)})})\chi_{\Lambda^{(m,\prime)}} &(6\text{-}23)\\ &= \int_{\mathbb{R}^2} \mathrm{d}\tilde{f}_\Lambda(z)\big\{\chi_{\Lambda^{(m,\prime)}} R_\omega^{(\Lambda)}(z)\chi_{\Lambda^{(m,\prime)}} - \chi_{\Lambda^{(m,\prime)}} R_\omega^{(\Lambda,m)}(z)\chi_{\Lambda^{(m,\prime)}}\big\}\\ &= \int_{\mathbb{R}^2} \mathrm{d}\tilde{f}_\Lambda(z)\big\{\chi_{\Lambda^{(m,\prime)}} R_\omega^{(\Lambda)}(z)\phi_0\chi_{\Lambda^{(m,\prime)}} - \chi_{\Lambda^{(m,\prime)}}\phi_0 R_\omega^{(\Lambda,m)}(z)\chi_{\Lambda^{(m,\prime)}}\big\}\\ &= \int_{\mathbb{R}^2} \mathrm{d}\tilde{f}_\Lambda(z)\big\{\chi_{\Lambda^{(m,\prime)}} R_\omega^{(\Lambda)}(z) W(\phi_0) R_\omega^{(\Lambda,m)}(z)\chi_{\Lambda^{(m,\prime)}}\big\}, &(6\text{-}24)\end{aligned}$$

where we used the geometric resolvent identity.

Now let us pick functions $\phi_i \in C_K^\infty(\mathbb{R})$, $i = 1, 2, \dots, 2p-1$, such that $0 \le \phi_i \le 1$, $\phi_i \chi_{\nabla\phi_{i-1}} = \chi_{\nabla\phi_{i-1}}$, and $\chi_{\phi_i}\chi_{\Lambda_{\ell-30d}(k_m)} = 0$ for $i = 1, 2, \dots, 2p-1$. Using the resolvent identity $2p-1$ times we get

$$
\begin{aligned}
&\chi_{\Lambda^{(m,\ell)}} R_\omega^{(\Lambda)}(z) W(\phi_0) \\
&\qquad = \chi_{\Lambda^{(m,\ell)}} R_\omega^{(\Lambda)}(z) W(\phi_{2p-1}) R_\omega^{(\Lambda)}(z) W(\phi_{2p-2}) \dots R_\omega^{(\Lambda)}(z) W(\phi_1) R_\omega^{(\Lambda)}(z) W(\phi_0) \\
&\qquad = \big\{\chi_{\Lambda^{(m,\ell)}} R_\omega^{(\Lambda)}(z)\big\}\big\{W(\phi_{2p-1}) R_\omega^{(\Lambda)}(z) W(\phi_{2p-2})\big\}\big\{\chi_{\nabla\phi_{2p-2}} R_\omega^{(\Lambda)}(z)\big\} \\
&\qquad\qquad\qquad \times \big\{W(\phi_{2p-3}) R_\omega^{(\Lambda)}(z) W(\phi_{2p-4})\big\} \dots \big\{\chi_{\nabla\phi_2} R_\omega^{(\Lambda)}(z)\big\}\big\{W(\phi_1) R_\omega^{(\Lambda)}(z) W(\phi_0)\big\}.
\end{aligned}
\tag{6-25}
$$

We now use that the integral in (6-24) is performed over a compact domain in $\mathbb{R}^2$, which depends only on the function f, so there is constant C_f such that for z in the region of integration we have

$$
\big\| (H_\omega^{(\Lambda)} + 1) R_\omega^{(\Lambda)}(z) \big\| \le \frac{C_f}{|\Im z|}, \tag{6-26}
$$

and hence, using (6-18) and (6-19), we have

$$
\big\| W(\phi_i) R_\omega^{(\Lambda)}(z) W(\phi_{i-1}) \big\| \le \frac{C_f C_{\phi_i} C_{\phi_{i-1}}}{|\Im z|} \tag{6-27}
$$

and, for $B \subset \Lambda_{L'} \subset \Lambda$,

$$
\big\| \chi_B R_\omega^{(\Lambda)}(z) \big\|_{p_d} \le \frac{C_f C_2}{|\Im z|} |\Lambda_{L'}|. \tag{6-28}
$$

We now choose $p = p_d$ as in (6-19), and note that we can choose the functions $\phi_i \in C_K^\infty(\mathbb{R})$, $i = 1, 2, \dots, 2p_d - 1$ so that the constants C_{ϕ_i} are independent of Λ, say all $C_{\phi_i} \le C_3$ From (6-25), (6-27), and (6-28), we get

$$
\begin{aligned}
\big\| \chi_{\Lambda^{(m,\ell)}} R_\omega^{(\Lambda)}(z) W(\phi_0) R_\omega^{(\Lambda,m)}(z) \chi_{\Lambda^{(m,\ell)}} \big\|_1 &\le \left(\frac{C_f C_2}{|\Im z|} |\Lambda^{(m)}| \right)^{p_d} \left(\frac{C_f C_3^2}{|\Im z|} \right)^{p_d} \big\| \chi_{\nabla\phi_0} R_\omega^{(\Lambda,m)}(z) \chi_{\Lambda^{(m,\ell)}} \big\| \\
&\le C_4 C_f' \ell^{p_d} |\Im z|^{-2p_d} \big\| \chi_{\nabla\phi_0} R_\omega^{(\Lambda,m)}(z) \chi_{\Lambda^{(m,\ell)}} \big\|.
\end{aligned}
\tag{6-29}
$$

We now use that $\overline{\mathcal{I}}_1 \subset \Xi^{\mathrm{CL}}$, the region of complete localization for H_ω. The term in (6-13) is $M_L \mathbb{E}\{T_\omega^{(\Lambda)}\}$, with $T_\omega^{(\Lambda)}$ as in (6-23). It follows from (6-24), (6-25), and (6-29) that for large L,

$$
\begin{aligned}
M_L \mathbb{E}\{T_\omega^{(\Lambda)}\} &\le M_L C_4 C_f' \ell^{p_d} \int_{\mathbb{R}^2} |\mathrm{d}\tilde{f}_\Lambda(z)|\, |\Im z|^{-2p_d} \mathbb{E}\big\{ \big\| \chi_{\nabla\phi_0} R_\omega^{(\Lambda,m)}(z) \chi_{\Lambda^{(m,\ell)}} \big\| \big\} \\
&\le M_L C_4 C_f' \ell^{p_d} \int_{\mathbb{R}^2} |\mathrm{d}\tilde{f}_\Lambda(z)|\, |\Im z|^{-2p_d - \frac{4}{5}} \mathbb{E}\big\{ \big\| \chi_{\nabla\phi_0} R_\omega^{(\Lambda,m)}(z) \chi_{\Lambda^{(m,\ell)}} \big\|^{1/5} \big\} \\
&\le M_L C_4 C_f' \ell^{p_d + 2d} (\rho_+ + \sqrt{\rho_+}) \int_{\mathbb{R}^2} |\mathrm{d}\tilde{f}_\Lambda(z)|\, |\Im z|^{-2p_d - \frac{4}{5}} \mathrm{e}^{-\ell^{1/4}} \\
&\le L^d \ell^{p_d + d} \mathrm{e}^{-\ell^{1/4}} c_{2p_d + \frac{4}{5}} C_4 C_f' (\rho_+ + \sqrt{\rho_+})\, \{\!\{ f_\Lambda \}\!\}_{2p_d + 2}\, .
\end{aligned}
\tag{6-30}
$$

where we used (A-4) and (6-22). Note that $2p_d \le d + 1$ and

$$
\{\!\{ f_\Lambda \}\!\}_m \le C_{E_0, f, m} |\Lambda|^{m-1} \quad \text{for all} \quad m = 2, 3, \dots . \tag{6-31}
$$

It follows that

$$M_L \mathbb{E}\{T_\omega^{(\Lambda)}\} \le L^{d^2+3d} \ell^{3d/2+1} \mathrm{e}^{-\ell^{1/4}} c_{2p_d+\frac{4}{5}} C_{f,E_0,d}(\rho_+ + \sqrt{\rho_+}) \to 0 \quad \text{as} \quad L \to \infty. \tag{6-32}$$

Thus (6-10) is proven.

The proof of (6-11) is similar. With $\Lambda = \Lambda_L(0)$, we set $L' \approx L - \sqrt{L}$, $\Lambda' = \Lambda_{L'}(0)$, and $\Lambda'' = \Lambda \setminus \Lambda'$. We have

$$\begin{aligned} \theta_\omega^{(\Lambda)}(f) - \xi_\omega^{(\Lambda)}(f) &= \operatorname{tr}\{\chi_\Lambda f_\Lambda(H_\omega)\chi_\Lambda\} - \operatorname{tr} f_\Lambda(H_\omega^{(\Lambda)}) \\ &= \big(\operatorname{tr}\{\chi_{\Lambda'} f_\Lambda(H_\omega)\chi_{\Lambda'}\} - \operatorname{tr}\{\chi_{\Lambda'} f_\Lambda(H_\omega^{(\Lambda)})\chi_{\Lambda'}\}\big) \\ &\quad + \big(\operatorname{tr}\{\chi_{\Lambda''} f_\Lambda(H_\omega)\chi_{\Lambda''}\} - \operatorname{tr}\{\chi_{\Lambda''} f_\Lambda(H_\omega^{(\Lambda)})\chi_{\Lambda''}\}\big), \end{aligned} \tag{6-33}$$

and hence

$$\mathbb{E}\big|\theta_\omega^{(\Lambda)}(f) - \xi_\omega^{(\Lambda)}(f)\big| \le \mathbb{E}\big|\operatorname{tr}\{\chi_{\Lambda'} f_\Lambda(H_\omega)\chi_{\Lambda'}\} - \operatorname{tr}\{\chi_{\Lambda'} f_\Lambda(H_\omega^{(\Lambda)})\chi_{\Lambda'}\}\big| \tag{6-34}$$

$$+ \mathbb{E}\big|\operatorname{tr}\{\chi_{\Lambda''} f_\Lambda(H_\omega)\chi_{\Lambda''}\} - \operatorname{tr}\{\chi_{\Lambda''} f_\Lambda(H_\omega^{(\Lambda)})\chi_{\Lambda''}\}\big|. \tag{6-35}$$

We now use the Wegner estimate (2-13) to obtain

$$\begin{aligned} \mathbb{E}\big|\operatorname{tr}\{\chi_{\Lambda''} f_\Lambda(H_\omega^{(\Lambda)})\chi_{\Lambda''}\}\big| &\le \frac{|\Lambda''|}{|\Lambda|} \mathbb{E}\operatorname{tr}\big\{|f_\Lambda|(H_\omega^{(\Lambda)})\big\} \\ &\le \frac{|\Lambda''|}{|\Lambda|} K_W \rho_+ |\Lambda| \int_{\mathbb{R}} |f_\Lambda|(E)\,\mathrm{d}E = \frac{|\Lambda''|}{|\Lambda|} K_W \rho_+ \|f\|_1, \end{aligned} \tag{6-36}$$

and

$$\begin{aligned} \mathbb{E}\big|\operatorname{tr}\{\chi_{\Lambda''} f_\Lambda(H_\omega)\chi_{\Lambda''}\}\big| &\le |\Lambda''| \mathbb{E}\operatorname{tr}\big\{\chi_0 |f_\Lambda|(H_\omega^{(\Lambda)})\chi_0\big\} = |\Lambda''| N(|f_\Lambda|) \\ &\le |\Lambda''| K_W \rho_+ \int_{\mathbb{R}} |f_\Lambda|(E)\,\mathrm{d}E = \frac{|\Lambda''|}{|\Lambda|} K_W \rho_+ \|f\|_1. \end{aligned} \tag{6-37}$$

Since $|\Lambda''|/|\Lambda| \approx 1/\sqrt{L}$, the term in (6-35) goes to 0 as $L \to \infty$.

To finish the proof of (6-11) , we need to show that the term in (6-34) also goes to 0 as $L \to \infty$. As before, we use the Helffer–Sjöstrand formula. We have, taking $\phi_0 \in C_K^\infty(\Lambda_{L-10d}(0))$ such that $0 \le \phi_0 \le 1$ and $\phi_0 \chi_{\Lambda_{L-20d}(0)} = \chi_{\Lambda_{L-20d}(0)}$, that

$$S_\omega^{(\Lambda)} := \chi_{\Lambda'} f_\Lambda(H_\omega)\chi_{\Lambda'} - \chi_{\Lambda'} f_\Lambda(H_\omega^{(\Lambda)})\chi_{\Lambda'} \tag{6-38}$$

$$\begin{aligned} &= \int_{\mathbb{R}^2} \mathrm{d}\tilde{f}_\Lambda(z)\big\{\chi_{\Lambda'} R_\omega(z)\chi_{\Lambda'} - \chi_{\Lambda'} R_\omega^{(\Lambda)}(z)\chi_{\Lambda'}\big\} \\ &= \int_{\mathbb{R}^2} \mathrm{d}\tilde{f}_\Lambda(z)\big\{\chi_{\Lambda'} R_\omega(z)\phi_0\chi_{\Lambda'} - \chi_{\Lambda'}\phi_0 R_\omega^{(\Lambda)}(z)\chi_{\Lambda'}\big\} \\ &= \int_{\mathbb{R}^2} \mathrm{d}\tilde{f}_\Lambda(z)\big\{\chi_{\Lambda'} R_\omega(z) W(\phi_0) R_\omega^{(\Lambda)}(z)\chi_{\Lambda'}\big\}. \end{aligned} \tag{6-39}$$

Proceeding as in (6-25)–(6-29), we get

$$\big\|\chi_{\Lambda'} R_\omega(z) W(\phi_0) R_\omega^{(\Lambda)}(z)\chi_{\Lambda'}\big\|_1 \le C_4 C_f' L^{p_d} |\Im z|^{-2p_d} \|\chi_{\nabla\phi_0} R_\omega^{(\Lambda)}(z)\chi_{\Lambda'}\|. \tag{6-40}$$

Recall that $\overline{\mathcal{I}}_1 \subset \Xi^{\mathrm{CL}}$. The term in (6-34) is $\mathbb{E}\{S_\omega^{(\Lambda)}\}$, with $S_\omega^{(\Lambda)}$ as in (6-38). It follows from (6-39) and (6-40) that for large L,

$$\begin{aligned}
\mathbb{E}\{S_\omega^{(\Lambda)}\} &\le C_4 C_f' L^{p_d} \int_{\mathbb{R}^2} |\mathrm{d}\tilde{f}_\Lambda(z)|\,|\Im z|^{-2p_d}\,\mathbb{E}\big\{\|\chi_{\nabla\phi_0} R_\omega^{(\Lambda)}(z)\chi_{\Lambda'}\|\big\}\\
&\le M_L C_4 C_f' L^{p_d} \int_{\mathbb{R}^2} |\mathrm{d}\tilde{f}_\Lambda(z)|\,|\Im z|^{-2p_d-\frac{4}{5}}\,\mathbb{E}\big\{\|\chi_{\nabla\phi_0} R_\omega^{(\Lambda)}(z)\chi_{\Lambda'}\|^{1/5}\big\}\\
&\le C_4 C_f' L^{p_d+2d}(\rho_+ + \sqrt{\rho_+}) \int_{\mathbb{R}^2} |\mathrm{d}\tilde{f}_\Lambda(z)|\,|\Im z|^{-2p_d-\frac{4}{5}}\mathrm{e}^{-L^{1/4}}\\
&\le L^{p_d+2d}\mathrm{e}^{-L^{1/4}} c_{2p_d+\frac{4}{5}} C_4 C_f' (\rho_+ + \sqrt{\rho_+})\,\{\!\{f_\Lambda\}\!\}_{2p_d+2}\\
&\le L^{d^2+5d}\mathrm{e}^{-L^{1/4}} c_{2p_d+\frac{4}{5}} C_{f,E_0,d}(\rho_+ + \sqrt{\rho_+}) \to 0 \quad \text{as} \quad L\to\infty,
\end{aligned} \tag{6-41}$$

where we used (A-4) and (6-22).

Thus (6-11) is proven, and with it the lemma. □

Given point processes $\{\zeta_n\}_{n\in\mathbb{N}}$ and ζ on $\mathbb{R}$, we let $\zeta_n \Rightarrow \zeta$ denote the weak convergence of ζ_n to ζ as $n\to\infty$. We recall [Daley and Vere-Jones 1988, Proposition 9.1.VII] that $\zeta_n \Rightarrow \zeta$ if and only if

$$\lim_{n\to\infty} \mathbb{E}\,\mathrm{e}^{-\zeta_n(f)} = \mathbb{E}\,\mathrm{e}^{-\zeta(f)} \quad \text{for all} \quad f\in C_{K,+}(\mathbb{R}). \tag{6-42}$$

The following lemma shows that it suffices to prove that $\widetilde{\xi}_\omega^{(\Lambda)} \Rightarrow \xi$ to prove Theorem 2.3(b).

Lemma 6.2. *$\xi_\omega^{(\Lambda)} \Rightarrow \xi$ if and only if $\widetilde{\xi}_\omega^{(\Lambda)} \Rightarrow \xi$.*

Proof. If ζ_i, $i=1,2$, are point processes on $\mathbb{R}$, defined on the same probability space, we have, for all $f\in C_{K,+}(\mathbb{R})$,

$$\big|\mathbb{E}\mathrm{e}^{-\zeta_1(f)} - \mathbb{E}\mathrm{e}^{-\zeta_2(f)}\big| \le \mathbb{E}\big|\zeta_1(f)-\zeta_2(f)\big|. \tag{6-43}$$

The lemma follows immediately from (6-42), (6-43), and Lemma 6.1. □

We are now ready to prove Theorem 2.3(a). In view of Lemma 6.2, it suffices to prove that $\widetilde{\xi}_\omega^{(\Lambda)} \Rightarrow \xi$. By standard results from the theory of point processes (cf. [Daley and Vere-Jones 1988, Theorem 9.2.V and subsequent remark]; see also [Kritchevski 2008, Theorem 2.3]), this is equivalent to verifying the following three conditions for all bounded intervals I (recall $\Lambda = \Lambda_L(0)$):

$$\lim_{L\to\infty}\ \max_{m=1,2,\dots,M_L} \mathbb{P}\{\xi_\omega^{(\Lambda,m)}(I)\ge 1\} = 0, \tag{6-44}$$

$$\lim_{L\to\infty} \sum_{m=1}^{M_L} \mathbb{P}\{\xi_\omega^{(\Lambda,m)}(I)\ge 1\} = n(\mathscr{E})|I|, \tag{6-45}$$

$$\lim_{L\to\infty} \sum_{m=1}^{M_L} \mathbb{P}\{\xi_\omega^{(\Lambda,m)}(I)\ge 2\} = 0. \tag{6-46}$$

Since $\mathbb{P}\{\xi_\omega^{(\Lambda,m)}(I)\ge 1\} \le \mathbb{E}\{\xi_\omega^{(\Lambda,m)}(I)\}$, (6-44) follows immediately from (6-4). In addition, it follows from the definition (6-3) and the estimate (2-19), that for all Λ with $|\Lambda|\ge c_I$ we have

$$\mathbb{P}\{\xi_\omega^{(\Lambda,m)}(I)\ge 2\} \le \tfrac{1}{2}\mathbb{E}\big\{(\xi_\omega^{(\Lambda,m)}(I))(\xi_\omega^{(\Lambda,m)}(I)-1)\big\} \le \tfrac{1}{2}K_M(\rho_+|I|M_L^{-1})^2, \tag{6-47}$$

so (6-46) follows.

Thus Theorem 2.3(a) is proved if we verify condition (6-45). To do so, we first notice that

$$\mathbb{E}\{\xi_\omega^{(\Lambda,m)}(I)\} = \sum_{k=1}^{\infty} \mathbb{P}\{\xi_\omega^{(\Lambda,m)}(I) \geq k\}, \tag{6-48}$$

and, as in [Kritchevski 2008],

$$\begin{aligned}\sum_{k=2}^{\infty} \mathbb{P}\{\xi_\omega^{(\Lambda,m)}(I) \geq k\} &= \sum_{k=2}^{\infty}(k-1)\mathbb{P}\{\xi_\omega^{(\Lambda,m)}(I) = k\} \\ &\leq \sum_{k=2}^{\infty} k(k-1)\mathbb{P}\{\xi_\omega^{(\Lambda,m)}(I) = k\} = \mathbb{E}\big\{(\xi_\omega^{(\Lambda,m)}(I))(\xi_\omega^{(\Lambda,m)}(I)-1)\big\}.\end{aligned} \tag{6-49}$$

It thus follows, as in (6-47), that

$$0 \leq \mathbb{E}\{\widetilde{\xi}_\omega^{(\Lambda)}(I)\} - \sum_{m=1}^{M_L} \mathbb{P}\{\xi_\omega^{(\Lambda,m)}(I) \geq 1\} \leq M_L K_M (\rho_+ |I| M_L^{-1})^2 \to 0 \quad \text{as} \quad L \to \infty. \tag{6-50}$$

We conclude that (6-45) is equivalent to

$$\lim_{L\to\infty} \mathbb{E}\{\widetilde{\xi}_\omega^{(\Lambda)}(I)\} = n(\mathscr{E})|I|, \tag{6-51}$$

and hence, by Lemma 6.1, equivalent to

$$\lim_{L\to\infty} \mathbb{E}\{\theta_\omega^{(\Lambda)}(I)\} = n(\mathscr{E})|I|. \tag{6-52}$$

But it follows from (6-7) that, for all Λ such that $|\Lambda| \geq c_I$

$$\mathbb{E}\{\theta_\omega^{(\Lambda)}(I)\} = |\Lambda|\eta(\mathscr{E} + |\Lambda|^{-1}I) = |\Lambda| \int_{\mathscr{E}+|\Lambda|^{-1}I} n(E)\,\mathrm{d}E. \tag{6-53}$$

Since by our hypothesis $\mathscr{E}$ is a Lebesgue point of the locally integrable function $n(E)$ (cf. [Yeh 2006, Definition 25.13]), and the sets $\mathscr{E}+|\Lambda|^{-1}I$ shrink nicely to $\mathscr{E}$ as $L \to \infty$ (cf. [Yeh 2006, Definition 25.16]), we can use the Lebesgue Differentiation Theorem (cf. [Yeh 2006, Theorem 25.17]) to conclude that

$$\lim_{L\to\infty} |\Lambda| \int_{\mathscr{E}+|\Lambda|^{-1}I} n(E)\,\mathrm{d}E = n(\mathscr{E})|I|. \tag{6-54}$$

Thus (6-52), and hence (6-45), is proven, completing the proof of Theorem 2.3(a).

7. Simplicity of eigenvalues

We prove Theorem 2.3(b) proceeding as in [Klein and Molchanov 2006]. Let H_ω be an Anderson Hamiltonian, and let $\mathscr{I}$ be an open interval such that for large boxes Λ the estimate (2-19) holds for any interval $I \subset \mathscr{I}$ with $|I| \leq \delta_0$, for some $\delta_0 > 0$, with some constant K_M. We call $\varphi \in \mathrm{L}^2(\mathbb{R}^d)$ fast decaying if it has β-decay for some $\beta > \frac{5}{2}d$, which in the continuum means that $\|\chi_x^{(1)}\varphi\| \leq C_\varphi \langle x \rangle^{-\beta}$ for some constant C_φ, where $\langle x \rangle := \sqrt{1+|x|^2}$. We will show that, with probability one, H_ω cannot have an eigenvalue in $\mathscr{I}$ with 2 linearly independent fast decaying eigenfunctions.

Let $I \subset \mathcal{I}$ be a closed interval, $q > 2d$, $L \in 2\mathbb{N}$ large, $\Lambda_L = \Lambda_L(0)$. We cover the interval I by $2\big([L^q/2|I|]+1\big) \leq L^q|I|+2$ intervals of length $2L^{-q}$, in such a way that any subinterval $J \subset I$ with length $|J| \leq L^{-q}$ will be contained in one of these intervals. ($[x]$ denotes the largest integer $\leq x$.) Let $\mathcal{B}_{L,I,q}$ denote the complement to the event that $\operatorname{tr} P_\omega^{(\Lambda_L)}(J) \leq 1$ for all subintervals $J \subset I$ with length $|J| \leq L^{-q}$. The probability of $\mathcal{B}_{L,I,q}$ can be estimated, using (2-19) and

$$\mathbb{P}\{\operatorname{tr} P_\omega^{(\Lambda)}(I) \geq 2\} \leq \tfrac{1}{2}\mathbb{E}\big\{(\operatorname{tr} P_\omega^{(\Lambda)}(I))(\operatorname{tr} P_\omega^{(\Lambda)}(I)-1)\big\}, \tag{7-1}$$

by

$$\mathbb{P}\{\mathcal{B}_{L,I,q}\} \leq \tfrac{1}{2}K_M\rho_+^2(L^q|I|+2)(2L^{-q})^2L^{2d} \leq 2K_M\rho_+^2(|I|+1)L^{-q+2d}. \tag{7-2}$$

Thus, taking scales $L_k = 2^k$, $k = 1, 2, \ldots$, it follows from the Borel–Cantelli Lemma that, with probability one, the event $\mathcal{B}_{L_k,I,q}$ eventually does not occur.

Let ω be in the set of probability one for which we have pure point spectrum with exponentially decaying eigenfunctions in the region of complete localization Ξ^{CL}. Suppose there exists $E \in \mathcal{I} \cap \Xi^{\mathrm{CL}}$ which is an eigenvalue of H_ω with 2 linearly independent eigenfunctions. In particular these eigenfunctions decay exponentially, so, if we fix $\beta > \frac{5}{2}d$, they both have β-decay. Pick an open interval $I \ni E$, such that $\bar{I} \subset \mathcal{I} \cap \Xi^{\mathrm{CL}}$. [Klein and Molchanov 2006, Lemma 1] can be adapted to the continuum by using smooth functions to localize the eigenfunctions in finite boxes. It then follows that for L large enough the finite volume operator $H_\omega^{(\Lambda_L)}$ has at least 2 eigenvalues in the interval $J_{E,L} = [E-\varepsilon_L, E+\varepsilon_L]$, where $\varepsilon_L = CL^{-\beta+\frac{d}{2}}$ for an appropriate constant C independent of L. Since $\beta > \frac{5d}{2}$ there exists $q > 2d$ such that $\beta - \frac{d}{2} > q$, and hence $\varepsilon_L < L^{-q}$ for all large L. But with probability one this is impossible since the event $\mathcal{B}_{L_k,\bar{I},q}$ does not occur for large L_k.

Theorem 2.3(b) is proven.

Appendix A. The region of complete localization

In this appendix we discuss localization for an Anderson Hamiltonian H_ω. Localization is most commonly taken to be *Anderson localization*: pure point spectrum with exponentially decaying eigenstates with probability one. It is also natural to consider *dynamical localization*, where the moments of a wave packet, initially localized both in space and in energy, should remain uniformly bounded under time evolution. For the multidimensional continuum Anderson Hamiltonian, localization has been proved by a multiscale analysis [Martinelli and Holden 1984; Combes and Hislop 1994; Klopp 1995; Kirsch et al. 1998; Germinet and De Bièvre 1998; Damanik and Stollmann 2001; Germinet and Klein 2001; 2003a], and, in the case when we have the covering condition $\delta_- \geq 1$, also by the fractional moment method [Aizenman et al. 2006]. These methods give more than just Anderson or dynamical localization, although they imply both. In the case when both methods are available, that is, $\delta_- \geq 1$, they have the same region of applicability [Germinet and Klein 2006; Klein 2008].

Thus, following [Germinet and Klein 2006], we consider *the region of complete localization* Ξ^{CL} for an Anderson Hamiltonian H_ω, defined as the set of energies $E \in \mathbb{R}$ where we have the conclusions of the bootstrap multiscale analysis of [Germinet and Klein 2001], that is, as the set of $E \in \mathbb{R}$ for which there exists some open interval $I \ni E$, such that given any ζ, $0 < \zeta < 1$, and α, $1 < \alpha < \zeta^{-1}$, there is a length scale $L_0 \in 2\mathbb{N}$ and a mass $m > 0$, so if we take $L_{k+1} \approx L_k^\alpha$ with $L_{k+1} \in 2\mathbb{N}$, $k = 0, 1, \ldots,$ we have

$$\mathbb{P}\{R(m, L_k, I, x, y)\} \geq 1 - \mathrm{e}^{-L_k^\zeta} \tag{A-1}$$

for all $k = 0, 1, \ldots,$ and $x, y \in \mathbb{Z}^d$ with $|x - y| > L_k + \varrho$, where $\varrho > 0$ is a constant depending only on $\operatorname{supp} u$, and

$$R(m, L, I, x, y) = \omega\{\text{; for every } E' \in I \text{ either } \Lambda_L(x) \text{ or } \Lambda_L(y) \text{ is } (\omega, m, E')\text{-regular}\}. \tag{A-2}$$

Given $E \in \mathbb{R}$, $x \in \mathbb{Z}^d$ and $L \in 6\mathbb{N}$, we say that the box $\Lambda_L(x)$ is (ω, m, E)-regular for a given $m > 0$ if $E \notin \sigma(H_\omega^{(\Lambda_L(x))})$ and

$$\left\|\Gamma_x^{(L)} R_\omega^{(\Lambda_L(x))}(E + i\delta)\chi_{\Lambda_{\frac{L}{3}}(x)}\right\| \leq \exp\Bigl(-m\frac{L}{2}\Bigr) \quad \text{for all } \delta \in \mathbb{R}, \tag{A-3}$$

where $R_\omega^{(\Lambda_L(x))}(E + i\delta) = (H_\omega^{(\Lambda_L(x))} - (E + i\delta))^{-1}$ and $\Gamma_x^{(L)}$ denotes the charateristic function of the belt $\overline{\Lambda}_{L-1}(x) \backslash \Lambda_{L-3}(x)$. (See [Germinet and Klein 2001; 2004; 2006; Klein 2008]; note that all the proofs work with the definition (A-3), that is, with the insertion of "for all $\delta \in \mathbb{R}$" They also work with the finite volume operators with periodic boundary condition used in this article.)

By construction Ξ^{CL} is an open set. It can be characterized in many different ways [Germinet and Klein 2004; 2006]. For convenience, our definition includes the complement of the spectrum of H_ω in the region of complete localization, that is, $\mathbb{R} \setminus \Sigma \subset \Xi^{\mathrm{CL}}$. The spectral region of complete localization, $\Xi^{\mathrm{CL}} \cap \Sigma$, is called the "strong insulator region" in [Germinet and Klein 2004].) If the conditions for the fractional moment method are satisfied, Ξ^{CL} coincides with the set of energies where the fractional moment method can be performed. (Minami [1996] proved Poisson statistics for the Anderson model in the region of validity of the fractional moment method, in other words, in the region of complete localization for the Anderson model.)

Proposition A.1. *Consider a closed bounded interval $I \subset \Xi^{\mathrm{CL}}$. Then for all $z \in \mathbb{C}$ with $\Re z \in I$, and boxes $\Lambda = \Lambda_L$, we have, for $s \in]0, \frac{1}{4}[$ and $\xi \in]0, 1[$, and $x, y \in \Lambda$ with $|x - y| \geq (\log L)^{(1/\xi)+}$,*

$$\mathbb{E}\Bigl\{\|\chi_x^{(1)} R_\omega^{(\Lambda)}(z)\chi_y^{(1)}\|^s\Bigr\} \leq C_{s,I,\zeta}(\rho_+ + \sqrt{\rho_+})\mathrm{e}^{-|x-y|^\xi} \tag{A-4}$$

for $L \geq L_1(\xi, I, s)$.

We will need the following consequence of the Wegner estimate (2-13).

Lemma A.2. *Let $I = [c, d]$ be such that (2-13) holds for any subinterval of $[c-1, d+1]$ with a constant K_W. Then for any $s \in \bigl]0, \frac{1}{2}\bigl[$, box Λ, and $z \in \mathbb{C}$ with $\Re z \in I$, we have*

$$\mathbb{E}\Bigl\{\|R_\omega^{(\Lambda)}(z)\|^s\Bigr\} \leq C_s K_W \rho_+ |\Lambda|. \tag{A-5}$$

Proof. Let $\Re z \in I$. It follows from (2-13) that for all $t \geq 1$

$$\mathbb{P}\Bigl\{\|R_\omega^{(\Lambda)}(z)\| \geq t\Bigr\} \leq \frac{2}{t} K_W \rho_+ |\Lambda| \tag{A-6}$$

Thus

$$\begin{aligned}\mathbb{E}\{\|R_\omega^{(\Lambda)}(z)\|^s\} = \int_0^\infty t\, \mathbb{P}\Bigl\{\|R_\omega^{(\Lambda)}(z)\|^s \geq t\Bigr\} \mathrm{d}t &\leq 1 + \int_1^\infty t\bigl(2t^{-1/s} K_W \rho_+ |\Lambda|\bigr)\, \mathrm{d}t \\ &\leq 1 + C_s' K_W \rho_+ |\Lambda|.\end{aligned}$$

□

If we have the covering condition $\delta_- \geq 1$, (A-5) holds without the volume factor in the right hand side [Aizenman et al. 2006].

Proof of Proposition A.1. Given $0 < \xi < 1$, we pick ζ such that $\zeta^2 < \xi < \zeta < 1$ (always possible) and set $\alpha = \zeta/\xi$, note $\alpha < \zeta^{-1}$. Since $I \subset \Xi^{\mathrm{CL}}$, there is a scale $L_0 \in 2\mathbb{N}$ and a mass $m_\zeta > 0$, such that, if we set $L_{k+1} \approx L_k^\alpha$, with $L_{k+1} \in 2\mathbb{N}$, $k = 0, 1, \ldots$, we have the estimate (A-1) for $x, y \in \mathbb{Z}^d$ such that $|x-y| > L_k + \varrho$.

Let us now fix $\Lambda = \Lambda_L$, $x, y \in \Lambda_L \cap \mathbb{Z}^d$ and pick k such that $L_{k+1} + \varrho \geq |x-y| > L_k + \varrho$. In this case, if $\boldsymbol{\omega} \in R\left(m_\zeta, L_k, I, x, y\right)$, then for $\Re z \in I$ either $\Lambda_{L_k}(x)$ or $\Lambda_{L_k}(y)$ is $(\omega, m, \Re z)$-regular; say $\Lambda_{L_k}(x)$ is $(\omega, m, \Re z)$-regular. (Note that we take the boxes of size L_k in the torus Λ.) Then, using (A-3) and [Germinet and Klein 2001, (2.9)], we reach

$$\begin{aligned} \left\| \chi_y^{(1)} R_{\boldsymbol{\omega}}^{(\Lambda)}(z) \chi_x^{(1)} \right\| &\leq \gamma_I \left\| \Gamma_x^{(L_k)} R_{\boldsymbol{\omega}}^{(\Lambda_{L_k}(x))}(z) \chi_x^{(1)} \right\| \left\| \chi_y^{(1)} R_{\boldsymbol{\omega}}^{(\Lambda)}(z) \Gamma_x^{(L_k)} \right\| \\ &\leq \gamma_I \exp\Big(-m_\zeta \frac{L_k}{2}\Big) \left\| R_{\boldsymbol{\omega}}^{(\Lambda)}(z) \right\|. \end{aligned} \tag{A-7}$$

Thus, with $s \in]0, \frac{1}{4}[$, using Lemma A.2,

$$\begin{aligned} &\mathbb{E}\Big\{ \left\| \chi_y^{(1)} R_{\boldsymbol{\omega}}^{(\Lambda)}(z) \chi_x^{(1)} \right\|^s : \boldsymbol{\omega} \in R(m_\zeta, L_k, I, x, y) \Big\} \\ &\qquad \leq \gamma_I^s \exp\Big(-s m_\zeta \frac{L_k}{2}\Big)\, \mathbb{E}\big\{ \| R_{\boldsymbol{\omega}}^{(\Lambda)}(z) \|^s \big\} \\ &\qquad \leq C_s K_W \rho_+ |\Lambda| \gamma_I^s \exp\Big(-s m_\zeta \frac{L_k}{2}\Big) \leq C_{s,I} \rho_+ |\Lambda| \exp\Big(-s m_\zeta \frac{L_k}{2}\Big), \end{aligned} \tag{A-8}$$

and

$$\begin{aligned} \mathbb{E}\Big\{ \left\| \chi_y^{(1)} R_{\boldsymbol{\omega}}^{(\Lambda)}(z) \chi_x^{(1)} \right\|^s : \boldsymbol{\omega} \notin R(m_\zeta, L_k, I, x, y) \Big\} &\leq \big(\mathbb{E}\{ \| R_{\boldsymbol{\omega}}^{(\Lambda)}(z) \|^{2s} \}\big)^{1/2} \big(\mathbb{P}\{ \boldsymbol{\omega} \notin R(m_\zeta, L_k, I, x, y) \}\big)^{1/2} \\ &\leq (C_{2s} K_W \rho_+ |\Lambda|)^{1/2} \exp\big(-\tfrac{1}{2} L_k^\zeta\big) \\ &\leq C'_{s,I} (\rho_+ |\Lambda|)^{1/2} \exp\big(-\tfrac{1}{2} L_k^\zeta\big). \end{aligned} \tag{A-9}$$

It follows that for L_k sufficiently large, that is, $|x-y|$ large, we have

$$\begin{aligned} \mathbb{E}\Big\{ \left\| \chi_y^{(1)} R_{\boldsymbol{\omega}}^{(\Lambda)}(z) \chi_x^{(1)} \right\|^s \Big\} &\leq C_{s,I,\zeta} (\rho_+ + \sqrt{\rho_+}) |\Lambda| \exp\big(-\tfrac{1}{2} L_k^\zeta\big) \\ &\leq C_{s,I,\zeta} (\rho_+ + \sqrt{\rho_+}) |\Lambda| \exp\big(-\tfrac{1}{2} L_{k+1}^\xi\big) \\ &\leq C'_{s,I,\zeta} (\rho_+ + \sqrt{\rho_+}) |\Lambda| \exp\big(-\tfrac{1}{2} |x-y|^\xi\big), \end{aligned} \tag{A-10}$$

so (A-4) follows for $|x-y| \geq (\log L)^{(1/\xi)+}$ (with a slightly smaller ξ). □

Appendix B. A convexity inequality for traces

The following inequality was used in [Combes and Hislop 1994, Proof of Proposition 4.5] and also in the derivation of (4-12) above.

Lemma B.1. *Let H_1 and H_2 be two self-adjoint operators on a Hilbert space $\mathcal{H}$, such that H_1 is diagonalizable and $H_1 \geq H_2$. Let f and g be bounded Borel functions on some open interval $I \supset \sigma(H_1)$, such that g is real-valued, nonincreasing, and convex on I. Then*

$$\operatorname{tr}\big\{ \bar{f}(H_1) g(H_1) f(H_1) \big\} \leq \operatorname{tr}\big\{ \bar{f}(H_1) g(H_2) f(H_1) \big\}. \tag{B-1}$$

Proof. Let $\varphi \in \mathcal{H}$ be an eigenvector of H_1 with eigenvalue λ and satisfying $\|\varphi\| = 1$. Then

$$\begin{aligned}\langle\varphi, \bar{f}(H_1)g(H_1)f(H_1)\varphi\rangle &= \bar{f}(\lambda)g(\lambda)f(\lambda) = \bar{f}(\lambda)g(\langle\varphi, H_1\varphi\rangle)f(\lambda) \le \bar{f}(\lambda)g(\langle\varphi, H_2\varphi\rangle)f(\lambda)\\ &\le \bar{f}(\lambda)\langle\varphi, g(H_2)\varphi\rangle f(\lambda) = \langle\varphi, \bar{f}(H_1)g(H_2)f(H_1)\varphi\rangle, \end{aligned} \tag{B-2}$$

where the first inequality follows from g nonincreasing and $H_1 \ge H_2$, and the second inequality used the convexity of the function g, Jensen's inequality (compare [Yeh 2006, Theorem 14.16]), and the spectral theorem.

Since H_1 is diagonalizable, (B-1) follows by expanding the trace on an orthonormal basis of eigenvalues for H_1 and using (B-2) for each term. □

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