Analysis & PDE

Volume 3 No. 3 2010

mathematical sciences publishers

Analysis & PDE

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Analysis & PDE, at Mathematical Sciences Publishers, Department of Mathematics, University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFLOW™ from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

http://www.mathscipub.org

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Typeset in LaTeX

ANALYSIS AND PDE
Vol. 3, No. 3, 2010

LOCAL WKB CONSTRUCTION FOR WITTEN LAPLACIANS ON MANIFOLDS WITH BOUNDARY

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WKB p-forms are constructed as approximate solutions to boundary value problems associated with semiclassical Witten Laplacians. Naturally distorted Neumann or Dirichlet boundary conditions are considered.

1. Introduction

Motivation. In order to compute accurately the small eigenvalues, that is, of order $\mathbb{O}(e^{-C/h})$ with $C > 0$, of a self-adjoint Witten Laplacian acting on 0-forms,

$$\Delta_{f,h}^{(0)} = -h^2\Delta + |\nabla f(x)|^2 - h\Delta f(x)\ ,$$

as the small parameter $h > 0$ tends to 0, we need Wentzel–Kramers–Brillouin (WKB) approximations of the 1-eigenforms associated with the small eigenvalues of $\Delta_{f,h}^{(1)}$, the self-adjoint Witten Laplacian acting on 1-forms. The function f is assumed to be a Morse function on some bounded domain $\overline{\Omega}$ with or without boundary.

In [Helffer et al. 2004], which improves the previous works [Bovier et al. 2004; 2005] done in a probabilistic point of view, the authors compute accurately the small eigenvalues of $\Delta_{f,h}^{(0)}$ in the case of a manifold without boundary. In this case, the WKB approximations of 1-eigenforms are the one provided in the work by Helffer and Sjöstrand [1985], where the analysis is done for general p-forms.

In the case without boundary, it is moreover well known, since the article by Witten [1982], that the dimension of the spectral subspace associated with the small eigenvalues (i.e., smaller than h) of $\Delta_{f,h}^{(p)}$, the self-adjoint Witten Laplacian acting on p-forms, is $m_p(f)$, the number of critical points of f with index p. Furthermore, the corresponding eigenvectors are concentrated around these critical points (see also [Helffer and Sjöstrand 1985; Helffer et al. 2004; Helffer 1988]).

According to [Chang and Liu 1995; Helffer and Nier 2006; Koldan et al. 2009; Le Peutrec 2008], in the case of a compact manifold with boundary, these last statements require the introduction of *generalized* critical points of f with index p (see Definition 2.6). For a self-adjoint Witten Laplacian $\Delta_{f,h}^{(p)}$ with Neumann or Dirichlet type boundary conditions, $\Delta_{f,h}^{(p)}$ admits $m_p(f)$ eigenvalues, where $m_p(f)$ is the number of *generalized* critical points of f with index p. Moreover, the corresponding p-eigenforms are concentrated around these *generalized* critical points, which can belong to the boundary. The proper definition of *generalized* critical point of f relies on the additional assumption that f has no critical point on the boundary $\partial\Omega$ and that $f|_{\partial\Omega}$ is also a Morse function (see Assumption 2.5). This definition is

MSC2000: 58J10, 58J32, 58J37, 81Q20.
Keywords: WKB expansion, boundary value problem, Witten Laplacian.

different for Neumann or Dirichlet type boundary conditions, but, in both cases, the interior *generalized* critical points of f with index p are the usual critical points with index p (see again Definition 2.6).

Hence, in the case of a manifold with boundary, some WKB approximations of 1-eigenforms have to be constructed near some *generalized* critical points which lie on the boundary. This was done in [Helffer and Nier 2006] for Dirichlet type boundary conditions. Nevertheless, the construction there relies on some specific trick which cannot be extended to the construction of local WKB 1-forms in the Neumann case. In order to treat this last case (see [Le Peutrec 2008]), a finer treatment of the three geometries involved in the boundary problem (boundary, metric, Morse function) is carried out.

It happens that the Neumann case for 1-forms contains all the technical obstructions for a general WKB ansatz for p-eigenforms. Moreover, this construction can be extended to the Dirichlet case, for general p-forms, using "dual" computations.

Therefore we show in this paper how to construct local WKB p-forms localized near the boundary in both Neumann and Dirichlet cases. However, only the construction of local WKB p-forms is considered here and the comparison with the corresponding p-eigenforms has only be treated in the case $p = 1$, in [Helffer and Nier 2006; Le Peutrec 2008].

Main results. Before enunciating our results, let us introduce some notation used in their statements. We refer in particular the reader to Definition 2.3 and connected material behind.

The operators $\boldsymbol{n}$ and $\boldsymbol{t}$ denote the normal and tangential components, and j^* the canonical pull-back associated with the embedding $j : \partial\Omega \to \overline{\Omega}$. They are defined in the next section.

The function Φ is the degenerate Agmon distance to the generalized critical point U associated with the function f. This is the only nonnegative solution to $|\nabla\Phi|^2 = |\nabla f|^2$ around U (Sections 4A and 4D).

Recall also that for a p-form b_h, the notation $b_h = \mathbb{O}(h^\infty)$ means that, for each N in $\mathbb{N}$, we have $b_h = \mathbb{O}(h^N)$ in the sense that $\|b_h\| \le C_N h^N$ for some $C_N > 0$. Here $\|\cdot\|$ is the L^2-norm over the p-forms inherited from the Riemannian structure.

Lastly, for $A \in \mathcal{L}(T_x^*\overline{\Omega})$, $x \in \overline{\Omega}$ ($T_x^*\overline{\Omega}$ denoting the cotangential space at x), and a p-form $\omega_1 \wedge \cdots \wedge \omega_p$, $A^{(p)}(\omega_1 \wedge \cdots \wedge \omega_p)(x)$ denotes the following p-form (see also Definition B.1):

$$(A\omega_1 \wedge \cdots \wedge \omega_p) + \cdots + (\omega_1 \wedge \cdots \wedge A\omega_p).$$

Theorem 1.1 (Neumann case). *Let U be a generalized critical point of f with index p on the boundary, for Neumann type boundary conditions. There exists locally, in a neighborhood of U, a $\mathscr{C}^\infty$ solution u_p^{WKB} to*

$$\Delta_{f,h}^{(p)} u_p^{\mathrm{WKB}} = e^{-\Phi/h}\mathbb{O}(h^\infty), \tag{1-1}$$

$$\boldsymbol{n} u_p^{\mathrm{WKB}} = 0 \quad \text{on } \partial\Omega, \tag{1-2}$$

$$\boldsymbol{n} d_{f,h} u_p^{\mathrm{WKB}} = 0 \quad \text{on } \partial\Omega, \tag{1-3}$$

where u_p^{WKB} has the form

$$u_p^{\mathrm{WKB}} = a_h e^{-\Phi/h},$$

with $a_h \sim \sum_k a^k h^k$, $a^0(U) = \boldsymbol{t} a^0(U) \neq 0$, and

$$a^0(U) \in \operatorname{Ker}\bigl(2(\operatorname{Hess}(f|_{\partial\Omega})(U) j^*)^{(p)} - \operatorname{Tr}(\operatorname{Hess}(f|_{\partial\Omega} - \Phi|_{\partial\Omega})(U))\bigr).$$

When restricted to tangential p-forms, $2(\mathrm{Hess}(f|_{\partial\Omega})(U)j^*)^{(p)} - \mathrm{Tr}(\mathrm{Hess}(f|_{\partial\Omega} - \Phi|_{\partial\Omega})(U))$ *has a one-dimensional kernel. The tangential form* a^0 *is then unique up to multiplication by a constant.*

Theorem 1.2 (Dirichlet case). *Let U be a generalized critical point of f with index p on the boundary, for Dirichlet type boundary conditions. There exists locally, in a neighborhood of U, a $\mathscr{C}^\infty$ solution u_p^{WKB} to*

$$\Delta_{f,h}^{(p)} u_p^{\mathrm{WKB}} = e^{-\Phi/h}\mathbb{O}(h^\infty) \tag{1-4}$$

$$\boldsymbol{t} u_p^{\mathrm{WKB}} = 0 \text{ on } \partial\Omega \tag{1-5}$$

$$\boldsymbol{t} d_{f,h}^* u_p^{\mathrm{WKB}} = 0 \text{ on } \partial\Omega\,, \tag{1-6}$$

where u_p^{WKB} has the form

$$u_p^{\mathrm{WKB}} = a_h e^{-\Phi/h},$$

with $a_h \sim \sum_k a^k h^k$, $a^0(U) = \boldsymbol{n} a^0(U) \neq 0$, and

$$a^0(U) \in \mathrm{Ker}\big(2(\mathrm{Hess}(f|_{\partial\Omega})(U)j^*)^{(p)} - \mathrm{Tr}(\mathrm{Hess}(f|_{\partial\Omega} - \Phi|_{\partial\Omega})(U))\big).$$

When restricted to normal p-forms, $2(\mathrm{Hess}(f|_{\partial\Omega})(U)j^*)^{(p)} - \mathrm{Tr}(\mathrm{Hess}(f|_{\partial\Omega} - \Phi|_{\partial\Omega})(U))$ *has a one-dimensional kernel. The normal form* a^0 *is then unique up to multiplication by a constant.*

Remark 1.3. When $B \in \mathscr{L}(T_x^*\partial\Omega)$, $x \in \partial\Omega$, note that $Bj^* = Bj^*\boldsymbol{t} \in \mathscr{L}(T_x^*\overline{\Omega}; T_x^*\partial\Omega) \subset \mathscr{L}(T_x^*\overline{\Omega})$ and $(Bj^*)^{(p)} \neq (B)^{(p)}j^*$. For example, if $\vec{n}$ is the outgoing normal at the boundary and $\vec{n}^*$ its dual for the Riemannian scalar product, then for $\omega \wedge \vec{n}^*$ with $\omega = \boldsymbol{t}\omega$,

$$(Bj^*)^{(p)}(\omega \wedge \vec{n}^*) = ((Bj^*)^{(p-1)}\omega) \wedge \vec{n}^* \quad (= (B^{(p-1)}(j^*\omega)) \wedge \vec{n}^* \,).$$

To prove these results and make some explicit computations, we are going to work in local coordinates. To carry out properly the analysis, we need to choose suitably these local coordinates with respect to the geometry of the problem. Some "adapted coordinates" will then be defined in Section 3A. They will be more finely specified in Sections 4A and 4D; see (4-6) and (4-30). The last statements of Theorems 1.1 and 1.2 simply specify the polarization of $a^0(U)$ which is imposed, while solving degenerate transport equations (see Sections 4C and 4F). Again, this is more explicit later, choosing the suitable coordinate system. In particular, with the coordinate formulation, the fact that $a^0(U)$ lies in a given one-dimensional space appears clearly in (4-25) after Proposition 4.1 for the Neumann case and in (4-49) after Proposition 4.4 for the Dirichlet case. These theorems are respectively proved in Sections 4C and 4F.

When the metric is Euclidean, $g = \sum_{i=1}^n (dx^i)^2$, the manifold Ω is locally $\mathbb{R}^n_- = \mathbb{R}^{n-1} \times (-\infty, 0)$, the boundary $\partial\Omega$ is locally $\partial\Omega = \{x^n = 0\}$, and the function f is of the form

$$f(x) = -x^n - \tfrac{1}{2}|\lambda_1|(x^1)^2 - \cdots - \tfrac{1}{2}|\lambda_p|(x^p)^2 + \tfrac{1}{2}|\lambda_{p+1}|(x^{p+1})^2 + \tfrac{1}{2}|\lambda_{n-1}|(x^{n-1})^2$$

in the Neumann case, or

$$f(x) = +x^n - \tfrac{1}{2}|\lambda_1|(x^1)^2 - \cdots - \tfrac{1}{2}|\lambda_p|(x^p)^2 + \tfrac{1}{2}|\lambda_{p+1}|(x^{p+1})^2 + \tfrac{1}{2}|\lambda_{n-1}|(x^{n-1})^2$$

in the Dirichlet case, the "adapted coordinates" are simply $(x^1, \ldots, x^n)$. The general case is more involved because the three geometries of the boundary, of the metric (curvature), and of the level sets of the function f do not match.

Our goal consists in reducing the analysis to a problem on the boundary, hence to a problem in a manifold without boundary. Once this is done, we will be able to apply the results of [Helffer and Sjöstrand 1985], obtained in the case of a manifold without boundary, to this reduced problem.

2. Generalities about Witten Laplacians

On both manifolds with or without boundary. Let $\overline{\Omega}$ be a $\mathscr{C}^\infty$ connected compact oriented Riemannian manifold with dimension $n \in \mathbb{N}^*$. We will denote by g_0 the given Riemannian metric on $\overline{\Omega}$; Ω and $\partial\Omega$ will denote respectively its interior and its boundary.

The cotangent and tangent bundles on Ω are denoted by $T^*\Omega$ and $T\Omega$, respectively, and the corresponding exterior fiber bundles by $\Lambda T^*\Omega = \bigoplus_{p=0}^n \Lambda^p T^*\Omega$ and $\Lambda T\Omega = \bigoplus_{p=0}^n \Lambda^p T\Omega$. The fiber bundles $\Lambda T\partial\Omega = \bigoplus_{p=0}^{n-1} \Lambda^p T\partial\Omega$ and $\Lambda T^*\partial\Omega = \bigoplus_{p=0}^{n-1} \Lambda^p T^*\partial\Omega$ are defined similarly. The space of $\mathscr{C}^\infty$, $\mathscr{C}_0^\infty$, L^2, H^s, etc. sections in any of these fiber bundles, E, on $O = \Omega$ or $O = \partial\Omega$, will be denoted respectively by $\mathscr{C}^\infty(O; E)$, $\mathscr{C}_0^\infty(O; E)$, $L^2(O; E)$, $H^s(O; E)$, etc.

When no confusion is possible we will simply use the short notation $\Lambda^p\mathscr{C}^\infty$, $\Lambda^p\mathscr{C}_0^\infty$, $\Lambda^p L^2$ and $\Lambda^p H^s$ for $E = \Lambda^p T^*\Omega$ or $E = \Lambda^p T^*\partial\Omega$.

Note that the L^2 spaces are those associated with the unit volume form for the Riemannian structure on Ω or $\partial\Omega$ (Ω and $\partial\Omega$ are oriented).

The notation $\mathscr{C}^\infty(\overline{\Omega}; E)$ is used for the set of $\mathscr{C}^\infty$ sections up to the boundary.

Let d be the exterior differential on $\mathscr{C}_0^\infty(\Omega; \Lambda T^*\Omega)$,

$$d^{(p)} : \mathscr{C}_0^\infty(\Omega; \Lambda^p T^*\Omega) \to \mathscr{C}_0^\infty(\Omega; \Lambda^{p+1} T^*\Omega),$$

and d^* its formal adjoint with respect to the L^2-scalar product inherited from the Riemannian structure,

$$d^{(p),*} : \mathscr{C}_0^\infty(\Omega; \Lambda^{p+1} T^*\Omega) \to \mathscr{C}_0^\infty(\Omega; \Lambda^p T^*\Omega).$$

Remark 2.1. Note that d and d^* are both well defined on $\mathscr{C}^\infty(\overline{\Omega}; \Lambda T^*\Omega)$.

For a function $f \in \mathscr{C}^\infty(\overline{\Omega}; \mathbb{R})$ and $h > 0$, we introduce distorted operators defined on $\mathscr{C}^\infty(\overline{\Omega}; \Lambda T^*\Omega)$:

$$d_{f,h} = e^{-f(x)/h}\,(hd)\,e^{f(x)/h} \quad \text{and} \quad d_{f,h}^* = e^{f(x)/h}\,(hd^*)\,e^{-f(x)/h}.$$

The Witten Laplacian is the differential operator defined on $\mathscr{C}^\infty(\overline{\Omega}; \Lambda T^*\Omega)$ by

$$\Delta_{f,h} = d_{f,h}^* d_{f,h} + d_{f,h} d_{f,h}^* = (d_{f,h} + d_{f,h}^*)^2. \tag{2-1}$$

The last equality follows from the property $dd = d^*d^* = 0$ which implies

$$d_{f,h} d_{f,h} = d_{f,h}^* d_{f,h}^* = 0. \tag{2-2}$$

This means, by restriction to the p-forms in $\mathscr{C}^\infty(\overline{\Omega}; \Lambda^p T^*\Omega)$, that

$$\Delta_{f,h}^{(p)} = d_{f,h}^{(p),*} d_{f,h}^{(p)} + d_{f,h}^{(p-1)} d_{f,h}^{(p-1),*}\ .$$

We next give some uselful relations involving exterior and interior products (denoted respectively by $\wedge$ and $\boldsymbol{i}$), gradients (denoted by ∇) and Lie derivatives (denoted by $\mathscr{L}$):

$$(df\wedge)^* = \boldsymbol{i}_{\nabla f} \quad (\text{in } L^2(\overline{\Omega};\Lambda^p T^*\Omega)), \tag{2-3}$$
$$d_{f,h} = hd + df\wedge, \tag{2-4}$$
$$d^*_{f,h} = hd^* + \boldsymbol{i}_{\nabla f}, \tag{2-5}$$
$$d\circ \boldsymbol{i}_X + \boldsymbol{i}_X\circ d = \mathscr{L}_X, \tag{2-6}$$
$$\Delta_{f,h} = h^2(d+d^*)^2 + |\nabla f|^2 + h(\mathscr{L}_{\nabla f} + \mathscr{L}^*_{\nabla f}), \tag{2-7}$$

where X denotes a vector field on Ω or $\overline{\Omega}$.

Remark 2.2. The operators introduced depend on the Riemannian metric g_0 but we omit this dependence for conciseness.

On manifolds with boundary.

Definition 2.3. We denote by $\vec{n}_\sigma$ the outgoing normal at $\sigma\in\partial\Omega$ and by $\vec{n}^*_\sigma$ the 1-form dual to $\vec{n}_\sigma$ for the Riemannian scalar product.

For any $\omega\in\mathscr{C}^\infty(\overline{\Omega};\Lambda^p T^*\Omega)$, the form $\boldsymbol{t}\omega$ is the element of $\mathscr{C}^\infty(\partial\Omega;\Lambda^p T^*\Omega)$ defined by

$$(\boldsymbol{t}\omega)_\sigma(X_1,\dots,X_p) = \omega_\sigma(X_1^T,\dots,X_p^T) \quad \text{for all } \sigma\in\partial\Omega,$$

with the decomposition into the tangential and normal components to $\partial\Omega$ at σ; i.e., $X_i = X_i^T \oplus x_i^\perp \vec{n}_\sigma$. Moreover,

$$(\boldsymbol{t}\omega)_\sigma = \boldsymbol{i}_{\vec{n}_\sigma}(\vec{n}^*_\sigma\wedge\omega_\sigma)\,.$$

The projected form $\boldsymbol{t}\omega$, which depends on the choice of $\vec{n}_\sigma$ (hence on g_0), can be compared with the canonical pull-back $j^*\omega$ associated with the embedding $j:\partial\Omega\to\overline{\Omega}$. Actually, the exact relationship is $j^*\omega = j^*(\boldsymbol{t}\omega)$.

The normal part of ω on $\partial\Omega$ is defined by

$$\boldsymbol{n}\omega = \omega|_{\partial\Omega} - \boldsymbol{t}\omega \quad \in\mathscr{C}^\infty(\partial\Omega;\Lambda^p T^*\Omega).$$

In the sequel, the form $\omega\in\mathscr{C}^\infty(\overline{\Omega};\Lambda^p T^*\Omega)$ will be said tangential or normal if $\omega=\boldsymbol{t}\omega$ or $\omega=\boldsymbol{n}\omega$, respectively, at any point of the boundary.

Definition 2.4. We denote by $\dfrac{\partial f}{\partial n}(\sigma)$ or $\partial_n f(\sigma)$ the normal derivative of f at σ:

$$\frac{\partial f}{\partial n}(\sigma) = \partial_n f(\sigma) := \langle \nabla f(\sigma)\,|\,\vec{n}_\sigma\rangle.$$

Assumption 2.5. The functions $f\in\mathscr{C}^\infty(\overline{\Omega},\mathbb{R})$ and $f|_{\partial\Omega}\in\mathscr{C}^\infty(\partial\Omega,\mathbb{R})$ are Morse functions. Moreover, the function f has no critical point on $\partial\Omega$.

The Neumann realization of the Witten Laplacian, denoted by $\Delta^N_{f,h}$, is the self-adjoint realization of $\Delta_{f,h}$ whose domain is

$$D(\Delta^N_{f,h}) = \{\omega\in\Lambda H^2(\Omega) : \boldsymbol{n}\omega = 0,\ \boldsymbol{n}d_{f,h}\omega = 0\}.$$

An analogous statement holds for the Dirichlet realization $\Delta^D_{f,h}$, the domain now being

$$D(\Delta^D_{f,h}) = \{\omega\in\Lambda H^2(\Omega) : \boldsymbol{t}\omega = 0,\ \boldsymbol{t}d^*_{f,h}\omega = 0\}.$$

See [Chang and Liu 1995; Helffer and Nier 2006; Le Peutrec 2008] for these results.

Definition 2.6. A point $U \in \overline{\Omega}$ is called a generalized critical point of f with index p if either $U \in \Omega$ and U is a critical point of f with index p, or $U \in \partial\Omega$ and

- in the Neumann case, U is a critical point with index p of $f|_{\partial\Omega}$ such that $\partial_n f(U) < 0$;
- in the Dirichlet case, U is a critical point with index $p-1$ of $f|_{\partial\Omega}$ such that $\partial_n f(U) > 0$.

Remark 2.7. This convention implies that the index p of a generalized critical point U on the boundary satisfies $p \in \{0, \dots, n-1\}$ in the Neumann case and $p \in \{1, \dots, n\}$ in the Dirichlet case.

We end this section by giving the statement extending to the case of a manifold with boundary the analysis done by Witten [1982]; see [Chang and Liu 1995; Helffer and Nier 2006; Le Peutrec 2008].

Theorem 2.8. *Under Assumption 2.5, there exists $h_0 > 0$ such that $\Delta^N_{f,h}$ and $\Delta^D_{f,h}$ have, for $h \in (0, h_0]$, the following property: For any $p \in \{0, \dots, n\}$, the spectral subspaces*

$$\operatorname{Ran} 1_{[0,h^{3/2})}(\Delta^{N,(p)}_{f,h}) \quad \text{or} \quad \operatorname{Ran} 1_{[0,h^{3/2})}(\Delta^{D,(p)}_{f,h})$$

have rank $m_p(f)$, the number of generalized critical points of f with index p in the respective cases (Neumann or Dirichlet).

The proofs in [Helffer and Nier 2006; Le Peutrec 2008] in fact show that the corresponding eigenvectors are concentrated around these critical points.

3. Preliminaries, coordinate systems

Since more than two geometries overlap around a generalized critical point of f with index p on the boundary and since systems of PDE are considered, the choice of the proper coordinate systems is a crucial point for making the analysis possible.

3A. *Existence of an adapted local coordinate system.*

Definition 3.1. Let σ be a point on the boundary $\partial\Omega$. An adapted local coordinate system around σ is a local coordinate system $(x^1, \dots, x^n) = (x', x^n)$ centered at σ satisfying the following properties:

(i) $dx^1, \dots, dx^n$ is an orthonormal positively oriented basis of $T^*_\sigma(\overline{\Omega})$, the cotangent space at σ.

(ii) The boundary $\partial\Omega$ corresponds locally to $x^n = 0$ and the interior Ω to $x^n < 0$.

(iii) $(\partial/\partial x^n)|_{\partial\Omega} = \vec{n}$, the outgoing normal at the boundary. Moreover, $(\partial/\partial x^n)$ is unitary and normal to $\{x^n = \text{Constant}\}$.

Such a coordinate system is more specific than the one provided by the collar theorem in [Schwarz 1995; Duff 1952; Duff and Spencer 1952]. Moreover, owing to the analysis done in [Petersen 1998, 117–122], it can be proven that such a system always exists. This is the aim of the next result.

Proposition 3.2. *A local coordinate system satisfying Definition 3.1 always exists.*

Proof. As in [Petersen 1998, 119–120], we look at

$$T\partial\Omega^{\perp} = \{v \in T_\sigma\overline{\Omega} : \sigma \in \partial\Omega, v \in (T_\sigma\partial\Omega)^{\perp} \subset T_\sigma\overline{\Omega}\},$$

where $(T_\sigma\partial\Omega)^\perp$ is the orthogonal complement of $T\partial\Omega$ in $T_\sigma\overline{\Omega}$ (so $T_\sigma\overline{\Omega} = T_\sigma\partial\Omega \oplus^\perp (T_\sigma\partial\Omega)^\perp$ for each $\sigma \in \partial\Omega$). Then, the map $\exp^\perp$ introduced in [Petersen 1998] is a diffeomorphism from an open neighborhood of the zero section in $T\partial\Omega^\perp$ onto its image in $\overline{\Omega}$. It means, choosing a point σ near the boundary $\partial\Omega$, that there exists an unique geodesic ν joining σ to a point σ_b on the boundary which satisfies $\dot\nu(\sigma_b) \in T\partial\Omega^\perp$. It is equivalent to say that there exists an unique geodesic ν joining σ to σ_b with $\dot\nu(\sigma_b) = \vec{n}_{\sigma_b}$.

Now let $-x^n$ be the geodesic distance to $\partial\Omega$ and take x' such that $x'|_{\partial\Omega}$ is a coordinate system on the boundary and x' is constant along the geodesics parametrized by x^n. The second point of the definition is then satisfied and $\partial/\partial x^n$ is unitary. Moreover, the choice of $x'|_{\partial\Omega}$ is arbitrary and we can choose it centered at U such that $dx^1, \dots, dx^n$ is an orthonormal basis of $T^*_U(\overline{\Omega})$ positively oriented. Then the first point of the definition is also satisfied.

We now verify that the third point of the definition is fulfilled. Write

$$\begin{aligned}\frac{\partial}{\partial x^n}\Big\langle\frac{\partial}{\partial x^n}\,\Big|\,\frac{\partial}{\partial x^i}\Big\rangle_\sigma &= \Big\langle\nabla_{\partial/\partial x^n}\frac{\partial}{\partial x^n}\,\Big|\,\frac{\partial}{\partial x^i}\Big\rangle_\sigma + \Big\langle\frac{\partial}{\partial x^n}\,\Big|\,\nabla_{\partial/\partial x^n}\frac{\partial}{\partial x^i}\Big\rangle_\sigma = 0 + \Big\langle\frac{\partial}{\partial x^n}\,\Big|\,\nabla_{\partial/\partial x^n}\frac{\partial}{\partial x^i}\Big\rangle_\sigma \\ &= \Big\langle\frac{\partial}{\partial x^n}\,\Big|\,\nabla_{\partial/\partial x^i}\frac{\partial}{\partial x^n}\Big\rangle_\sigma = \frac{1}{2}\frac{\partial}{\partial x^i}\Big\langle\frac{\partial}{\partial x^n}\,\Big|\,\frac{\partial}{\partial x^n}\Big\rangle_\sigma = 0,\end{aligned}$$

where we used the fact that ∇ is the Levi-Civita connection and $\nabla_{\partial/\partial x^n}\partial/\partial x^n = 0$ since x^n is a geodesic curve. Hence,

$$\Big\langle\frac{\partial}{\partial x^n}\,\Big|\,\frac{\partial}{\partial x^i}\Big\rangle_\sigma = \Big\langle\frac{\partial}{\partial x^n}\,\Big|\,\frac{\partial}{\partial x^i}\Big\rangle_{\sigma_b} = \Big\langle\vec{n}_{\sigma_b}\,\Big|\,\frac{\partial}{\partial x^i}\Big\rangle_{\sigma_b} = 0,$$

which gives the third point of the definition. □

Remark 3.3. In an adapted local coordinate system (x', x^n) around σ, remark that the metric g_0 can be written as

$$g_0(x) = (dx^n)^2 + \sum_{1\le i,j<n} g_{ij}(x)\,dx^i\,dx^j.$$

Moreover, it can be convenient to work with matrices and we write $G_0(x) = (g_{ij}(x))_{ij}$, $G_0^{-1}(x) = (g^{ij}(x))_{ij}$. Remember that $g_{ij} = \langle(\partial/\partial x^i) \mid (\partial/\partial x^j)\rangle$, $g^{ij} = \langle dx^i \mid dx^j\rangle$, and $dx^i(\partial/\partial x^j) = \delta_{ij}$.

Hence, in the (x', x^n) coordinate system, $G_0^{\pm1}(x)$ has the form

$$G_0^{\pm1}(x) = \begin{pmatrix} & & 0 \\ & G_0^{\pm1'}(x) & \vdots \\ & & 0 \\ 0 & \cdots & 0\ \ 1\end{pmatrix}, \quad \text{with } G_0^{\pm1}(0) = \mathrm{Id}_n\,.$$

3B. *Separating the x^n-variable.*

Lemma 3.4. (1) *Let f_1 belong to $\mathscr{C}^\infty(\overline{\Omega}, \mathbb{R})$ and let $U \in \partial\Omega$ be a critical point of $f_1|_{\partial\Omega}$ such that*

$$\frac{\partial f_1}{\partial n}(U) \neq 0. \tag{3-1}$$

Assume also that $\alpha \in \mathscr{C}^\infty(\partial\Omega, \mathbb{R})$ is a local solution to $|\nabla_T\alpha|^2 = |\nabla_T f_1|^2$ around U.

Then there exists a neighborhood $\mathscr{V}$ *of* U *in* $\overline{\Omega}$ *such that the eikonal equation*

$$|\nabla\Phi_\pm|^2 = |\nabla f_1|^2 \tag{3-2}$$

with boundary conditions

$$\Phi_\pm|_{\partial\Omega\cap\mathscr{V}} = \alpha, \quad \partial_n\Phi_\pm|_{\partial\Omega\cap\mathscr{V}} = \pm\frac{\partial f_1}{\partial n}\Big|_{\partial\Omega\cap\mathscr{V}}$$

admits a unique local smooth real-valued solution. (On the boundary, (3-2) is to be interpreted as saying that $|\nabla\Phi_\pm|^2 = |\partial_n\Phi_\pm|^2 + |\nabla_T\Phi_\pm|^2$; see details in the proof.)

(2) *There exist local coordinates* $(\bar{x}^1, \dots, \bar{x}^n) = (\bar{x}', \bar{x}^n)$ *in a neighborhood of* U *in* $\overline{\Omega}$ *with*

$$(\bar{x}', \bar{x}^n)(U) = 0$$

where the function $\Phi_\pm$ *and the metric* g_0 *have the form*

$$\Phi_\pm = \mp\bar{x}^n + \alpha(\bar{x}') \quad \textit{and} \quad g_0 = g_{nn}(\bar{x})\,(d\bar{x}^n)^2 + \sum_{i,j=1}^{n-1} g_{ij}(\bar{x})\,d\bar{x}^i\,d\bar{x}^j.$$

Moreover, the boundary $\partial\Omega$ *is locally defined by* $\{\bar{x}^n = 0\}$ *and* Ω *corresponds to*

$$\Big\{\operatorname{sgn}\Big(\frac{\partial f_1}{\partial n}(U)\Big)\,\bar{x}^n > 0\Big\}. \tag{3-3}$$

Proof. (1) Take an adapted local coordinate system (x', x^n) around U in order to write (3-2) as

$$|\partial_{x^n}\Phi_\pm|^2 + |\nabla_T\Phi_\pm|^2 = |\partial_{x^n} f_1|^2 + |\nabla_T f_1|^2$$

(see Appendix A for the exact meaning of ∇_T in the interior).

In particular, we obtain on the boundary

$$|\partial_n\Phi_\pm|^2 + |\nabla_T\Phi_\pm|^2 = |\partial_n f_1|^2 + |\nabla_T\alpha|^2.$$

The first point is then a direct consequence of the Hamilton–Jacobi theorem, due to the condition

$$\frac{\partial f_1}{\partial n}(U) \neq 0.$$

(2) As in [Helffer and Sjöstrand 1985], set

$$f_+ = \Phi_+ - \Phi_- \quad \text{and} \quad f_- = \Phi_+ + \Phi_-,$$

and note the relations

$$\Phi_- = -\tfrac{1}{2}f_+ + \tfrac{1}{2}f_-, \quad \Phi_+ = \tfrac{1}{2}f_+ + \tfrac{1}{2}f_-, \tag{3-4}$$

$$\nabla f_+ \cdot \nabla f_- = 0, \tag{3-5}$$

$$f_+|_{\partial\Omega\cap\mathscr{V}} = 0, \quad f_-|_{\partial\Omega\cap\mathscr{V}} = 2\alpha, \tag{3-6}$$

$$\frac{\partial f_+}{\partial n}\Big|_{\partial\Omega\cap\mathscr{V}} = 2\frac{\partial f_1}{\partial n}\Big|_{\partial\Omega\cap\mathscr{V}} \neq 0, \quad \frac{\partial f_-}{\partial n}\Big|_{\partial\Omega\cap\mathscr{V}} = 0. \tag{3-7}$$

Let $(\bar{x}^1, \dots, \bar{x}^{n-1}) = \bar{x}'$ denote a set of coordinates on $\partial\Omega$ in a neighborhood of U (then contained in $\mathscr{V}$) and such that $\bar{x}^j(U) = 0$. We extend them to a neighborhood of U in $\overline{\Omega}$ as constant along the integral curve of the vector field ∇f_+. Then we take $\bar{x}^n = -\frac{1}{2} f_+(x)$ for the last coordinate.

In these coordinates, the functions $\Phi_\pm$ and the metric g_0 have the forms announced in the lemma.

Further, by (3-6), (3-7), and (3-1), the boundary $\partial\Omega$ is locally defined by $\{\bar{x}^n = 0\}$ and Ω corresponds to the set in (3-3). □

In the sequel, we will apply part (1) of this lemma in the Neumann and Dirichlet cases in order to specify the Agmon distance, associated with the function f, to a generalized critical point U with index p on the boundary.

Then, using part (2) of the lemma and Proposition 3.2.11 of [Le Peutrec 2008] (in the Neumann case) or Proposition 3.3.9 of [Helffer and Nier 2006] (in the Dirichlet case), we view $\Delta_{f,h}^{(p),N}$ and $\Delta_{f,h}^{(p),D}$ locally in $\mathscr{V}$ around $U \in \partial\Omega$ as the restrictions to $\mathscr{V}$ of $\mathscr{A}_N^{(p)}$ and $\mathscr{A}_D^{(p)}$, the latter being the self-adjoint Witten Laplacian operators on $\mathbb{R}^n_- = \mathbb{R}^{n-1} \times (-\infty, 0)$ (possibly after choosing $-\bar{x}^n$ instead of $\bar{x}^n$) whose domains are

$$D(\mathscr{A}_N) = \{\omega \in \Lambda H^2(\mathbb{R}^n_-) : \boldsymbol{n}\omega = \boldsymbol{n} d_{f,h}\omega = 0\}, \quad D(\mathscr{A}_D) = \{\omega \in \Lambda H^2(\mathbb{R}^n_-) : \boldsymbol{t}\omega = \boldsymbol{t} d^*_{f,h}\omega = 0\}$$

(see also [Koldan et al. 2009]), and which satisfy

$$\dim \operatorname{Ker} \mathscr{A}_N^{(p)} = 1, \quad \sigma(\mathscr{A}_N^{(p)}) \setminus \{0\} \subset [Ch^{6/5}, +\infty), \tag{3-8}$$

$$\dim \operatorname{Ker} \mathscr{A}_D^{(p)} = 1, \quad \sigma(\mathscr{A}_D^{(p)}) \setminus \{0\} \subset [Ch^{6/5}, +\infty). \tag{3-9}$$

4. WKB construction near the boundary for $\Delta_{f,h}^{(p)}$, with p in $\{0, \dots, n\}$

4A. ***Local WKB construction in the Neumann case.*** Let U be a generalized critical point of f with index p in the Neumann case, that is, a critical point with index $p \in \{0, \dots, n-1\}$ of $f|_{\partial\Omega}$ satisfying $\frac{\partial f}{\partial n}(U) < 0$, and take an adapted local coordinate system (x', x^n) around U.

Let Φ and φ be respectively the Agmon distance to U associated with the function f and its restriction to the boundary. The Agmon distance associated with f, that is, with the metric $|\nabla f(x)|^2\, dx^2$, is denoted by d_{Ag}: $\Phi(x) = d_{\mathrm{Ag}}(x, U)$. Recall that, locally,

$$|\nabla f|^2 = |\nabla\Phi|^2$$

and that Φ is smooth near U; see [Helffer and Sjöstrand 1984]. Moreover, φ is nothing but the Agmon distance to U on the boundary and satisfies locally, on the boundary,

$$|\nabla_T f|^2 = |\nabla\varphi|^2.$$

We now use Lemma 3.4(1) with $f_1 = f$ and $\alpha = \varphi$. The function Φ_+ of the lemma is consequently Φ and we have locally

$$|\partial_n\Phi|^2 + |\nabla_T\Phi|^2 = |\nabla\Phi|^2 = |\nabla f|^2, \tag{4-1}$$

$$\Phi|_{\partial\Omega} = \varphi, \tag{4-2}$$

$$\partial_n\Phi|_{\partial\Omega} = \frac{\partial f}{\partial n}\Big|_{\partial\Omega}. \tag{4-3}$$

Moreover,

$$\partial^2_{x^n x^n}(f-\Phi)(0) = \partial^2_{nn}(f-\Phi)(0) = 0. \tag{4-4}$$

Indeed, we can write in the coordinates (x', x^n), for the metric g_0,

$$|\partial_{x^n}\Phi|^2 + |\nabla_T \Phi|^2_{g_0} = |\partial_{x^n} f|^2 + |\nabla_T f|^2_{g_0},$$

where $|\nabla_T \Phi|^2_{g_0} = \mathbb{O}(|x|^2)$ and $|\nabla_T f|^2_{g_0} = \mathbb{O}(|x|^2)$ because 0 is a critical point of $f|_{\partial\Omega}$ in the coordinates (x', x^n) (see for example Appendix A). Then apply ∂_{x^n} to the last equation:

$$\partial_{x^n}|\partial_{x^n}\Phi|^2 + \mathbb{O}(|x|) = \partial_{x^n}|\partial_{x^n} f|^2 + \mathbb{O}(|x|),$$

that is, using (4-3),

$$2\partial^2_{x^n x^n}(f-\Phi)\partial_{x^n} f = \mathbb{O}(|x|),$$

which yields the result. According to [Helffer and Sjöstrand 1985, 279–280], there exist local coordinates $(\bar{x}', \bar{x}^n)$ centered at U, where $\bar{x}' = (\bar{x}^1, \dots, \bar{x}^{n-1})$ are Morse coordinates for $f|_{\partial\Omega}$ around U, such that $d\bar{x}^1, \dots, d\bar{x}^{n-1}, dx^n$ is orthonormal at U, and

$$\begin{aligned} f(\bar{x}', 0) &= \tfrac{1}{2}\lambda_1(\bar{x}^1)^2 + \cdots + \tfrac{1}{2}\lambda_{n-1}(\bar{x}^{n-1})^2 + f(U), \\ \varphi(\bar{x}') &= \tfrac{1}{2}|\lambda_1|(\bar{x}^1)^2 + \cdots + \tfrac{1}{2}|\lambda_{n-1}|(\bar{x}^{n-1})^2, \end{aligned} \tag{4-5}$$

with $\lambda_i < 0$ for $i \in \{1, \dots, p\}$ and $\lambda_i > 0$ for $i \in \{p+1, \dots, n-1\}$. Furthermore, the coordinates (x', x^n) can be chosen such that $dx^1, \dots, dx^{n-1}$ and $d\bar{x}^1, \dots, d\bar{x}^{n-1}$ coincide at U, and even such that $x'|_{\partial\Omega} = \bar{x}'|_{\partial\Omega}$ since $x'|_{\partial\Omega}$ can be chosen freely.

Specification of the coordinate system for Theorem 1.1. In the rest of the paper we are going to work in an adapted local coordinate system $x = (x', x^n)$ around U such that

$$dx^i = d\bar{x}^i \text{ at } U \quad \text{for all } i \in \{1, \dots, n-1\}. \tag{4-6}$$

4B. ***First boundary conditions in the Neumann case.*** We first write out the function $a_h(x) = a(x, h)$ in our coordinate system:

$$a(x,h) = \sum_{I \in \mathcal{I}} a_I(x,h)\, dx^I = \sum_{I' \in \mathcal{I}'} a_{I'}(x,h)\, dx^{I'} + \sum_{I_n \in \mathcal{I}_n} a_{I_n}(x,h)\, dx^{I_n}, \tag{4-7}$$

where

$$\begin{aligned} \mathcal{I} &:= \left\{(i_1, \dots, i_p) \in \{1, \dots, n\}^p,\ i_1 < \cdots < i_p\right\}, \\ \mathcal{I}' &:= \left\{(i_1, \dots, i_p) \in \{1, \dots, n\}^p,\ i_1 < \cdots < i_p < n\right\}, \\ \mathcal{I}_n &:= \left\{(i_1, \dots, i_p) \in \{1, \dots, n\}^p,\ i_1 < \cdots < i_p = n\right\}, \end{aligned}$$

and $dx^{(i_1,\dots,i_p)} = dx^{i_1} \wedge \cdots \wedge dx^{i_p}$. We will use in the sequel the Einstein summation convention to write (4-7) without the summation symbols:

$$a(x,h) = a_I(x,h)\, dx^I = a_{I'}(x,h)\, dx^{I'} + a_{I_n}(x,h)\, dx^{I_n}.$$

The first boundary condition (1-2) simply says that

$$a_{I_n}((x',0),h) \sim \sum_k a^k_{I_n}(x',0)h^k \equiv 0 \quad \text{for all } I_n \in \mathcal{I}_n, \tag{4-8}$$

which is equivalent to

$$a^k_{I_n}(x',0) \equiv 0 \quad \text{for all } k \in \mathbb{N} \text{ and } I_n \in \mathcal{I}_n. \tag{4-9}$$

The rest of this subsection specifies some consequence of these conditions. These consequences will be used in the next subsection to prove Theorem 1.1.

Proposition 4.1. *In the notation of Appendices A and B, the following relations are satisfied for every tangential p-form $b(x) = b_I(x)\,dx^I$, that is, every p-form $b(x)$ satisfying $b_{I_n}(x',0) \equiv 0$ for all $I_n \in \mathcal{I}_n$:*

$$\boldsymbol{t}((2\mathcal{L}_{\nabla\Phi} + \mathcal{R}_1)b) = (2\mathcal{L}_{\nabla\tilde{\Phi}} \otimes \mathrm{Id} + \mathcal{R}^T_{\mathrm{Neu}})b_{I'}dx^{I'} + 2\frac{\partial\Phi}{\partial x^n}\boldsymbol{i}_{\partial/\partial x^n}db,$$

$$\boldsymbol{n}((2\mathcal{L}_{\nabla\Phi} + \mathcal{R}_1)b) = 2\Big(\frac{\partial b_{I_n}}{\partial x^n}\frac{\partial\Phi}{\partial x^n} + \ell_{I_n}(x',0)\Big)\,dx^{I_n},$$

where the ℓ_{I_n} are $\mathscr{C}^\infty(\partial\Omega)$-linear combinations of the $b_{I'}$ (for I' in $\mathcal{I}'$) that do not depend on the b_{I_n} (for I_n in $\mathcal{I}_n$) and $\mathcal{R}^T_{\mathrm{Neu}}$ is an order-zero differential operator given on the boundary, in the coordinates (x',x^n), by the matrix

$$\mathcal{R}^T_{\mathrm{Neu}}(x',0) = \begin{pmatrix} & & 0 \\ & \mathcal{R}^{T'}_{\mathrm{Neu}}(x') & \vdots \\ & & 0 \\ 0 & \cdots & 0 \;\; \beta(x') \end{pmatrix}^{(p)} - \gamma(x')\,\mathrm{Id},$$

where

$$\beta(0) = 0, \quad \gamma(0) = \mathrm{Tr}\,(\mathrm{Hess}(f|_{\partial\Omega} - \varphi)(0)), \quad \textit{and} \quad \mathcal{R}^{T'}_{\mathrm{Neu}}(0) = 2\,\mathrm{Hess}(f|_{\partial\Omega})(0).$$

In particular, this is true for a^k for k in $\mathbb{N}$ when (4-9) *is fulfilled.*

The following elementary result is important to notice here and also while verifying the final compatibility conditions (see pages 242–244).

Lemma 4.2. *Let $b(x)$ be a* tangential *p-form. The p-form*

$$\boldsymbol{i}_{\vec{n}}(db)$$

is then tangential and the equivalence

$$\boldsymbol{i}_{\vec{n}}(db) = 0 \iff \boldsymbol{n}db = 0$$

is locally valid on the boundary $\partial\Omega$. In particular, this is true for a^k for k in $\mathbb{N}$ when (4-9) *is fulfilled.*

Proof. On the boundary $\partial\Omega$, we have, in the coordinate system (x',x^n),

$$\boldsymbol{i}_{\partial/\partial x^n}(db) = \boldsymbol{i}_{\partial/\partial x^n}\boldsymbol{n}\,db + \boldsymbol{i}_{\partial/\partial x^n}\boldsymbol{t}\,db = \boldsymbol{i}_{\partial/\partial x^n}\boldsymbol{n}\,db + 0 = \boldsymbol{i}_{\partial/\partial x^n}(db)_{I_n}\,dx^{I_n} = (-1)^p(db)_{I_n}\,dx^{I_n\setminus\{n\}},$$

which leads to the result. □

Lemma 4.3. *For every tangential p-form $b(x)$, we have*

$$\boldsymbol{t}((\mathcal{L}_{\nabla\Phi} - \mathcal{L}_{\nabla\tilde{\Phi}})b) = \boldsymbol{t}((\mathcal{L}_{\nabla_T\Phi} - \mathcal{L}_{\nabla\tilde{\Phi}})b) = \frac{\partial\Phi}{\partial x^n}\boldsymbol{i}_{\partial/\partial x^n}db,$$

$$\boldsymbol{n}((\mathcal{L}_{\nabla\Phi} - \mathcal{L}_{\nabla\tilde{\Phi}})b) = \Big(\frac{\partial b_{I_n}}{\partial x^n}\frac{\partial\Phi}{\partial x^n} + \tilde{\ell}_{I_n}(x',0)\Big)\,dx^{I_n},$$

where the $\tilde{\ell}_{I_n}$ are $\mathscr{C}^\infty(\partial\Omega)$-linear combinations of the $b_{I'}$ (for I' in $\mathscr{I}'$) that do not depend on the b_{I_n} (for I_n in $\mathscr{I}_n$).

Proof. On the boundary $\partial\Omega$, we have the decomposition

$$(\mathscr{L}_{\nabla\Phi} - \mathscr{L}_{\nabla\tilde{\Phi}})b = \mathscr{L}_{(\partial\Phi/\partial x^n)(\partial/\partial x^n)}b + (\mathscr{L}_{\nabla_T\Phi} - \mathscr{L}_{\nabla\tilde{\Phi}})b. \tag{4-10}$$

Thanks to Cartan's formula (2-6), we can rewrite (4-10) as

$$(\mathscr{L}_{\nabla\Phi}-\mathscr{L}_{\nabla\tilde{\Phi}})b=\boldsymbol{i}_{(\partial\Phi/\partial x^n)(\partial/\partial x^n)}db+d(\boldsymbol{i}_{(\partial\Phi/\partial x^n)(\partial/\partial x^n)}b)+\boldsymbol{i}_{(\nabla_T\Phi-\nabla\tilde{\Phi})}db+d(\boldsymbol{i}_{(\nabla_T\Phi-\nabla\tilde{\Phi})}b). \tag{4-11}$$

Using Lemma 4.2, the first term on the right side of (4-11) is tangential:

$$\boldsymbol{i}_{(\partial\Phi/\partial x^n)(\partial/\partial x^n)}\, db = \frac{\partial\Phi}{\partial x^n}\boldsymbol{i}_{\partial/\partial x^n}\, db.$$

Moreover, since $\nabla_T\Phi = \nabla\tilde{\Phi}$ on the boundary (see Appendix A), the term $\boldsymbol{i}_{(\nabla_T\Phi-\nabla\tilde{\Phi})}db$ of the right side equals 0 on $\partial\Omega$. Hence, we can write on $\partial\Omega$

$$(\mathscr{L}_{\nabla\Phi} - \mathscr{L}_{\nabla\tilde{\Phi}})b = \boldsymbol{i}_{(\partial\Phi/\partial x^n)(\partial/\partial x^n)}db + d(\boldsymbol{i}_{(\partial\Phi/\partial x^n)(\partial/\partial x^n)}b) + d(\boldsymbol{i}_{(\nabla_T\Phi-\nabla\tilde{\Phi})}b). \tag{4-12}$$

Let us study in a first time the term $d(\boldsymbol{i}_{(\partial\Phi/\partial x^n)(\partial/\partial x^n)}b)$. Writing

$$b = b_I\, dx^I = b_{I'}dx^{I'} + b_{I_n}dx^{I_n},$$

we deduce (in $\overline{\Omega}$) that

$$\boldsymbol{i}_{(\partial\Phi/\partial x^n)(\partial/\partial x^n)}b = b_{I_n}\boldsymbol{i}_{(\partial\Phi/\partial x^n)(\partial/\partial x^n)}dx^{I_n} = (-1)^{p-1}b_{I_n}\frac{\partial\Phi}{\partial x^n}dx^{I_n\setminus\{n\}},$$

and, applying d to this last relation, we obtain on $\partial\Omega$ (remembering that $b_{I_n} = 0$ on $\partial\Omega$)

$$\begin{aligned} d(\boldsymbol{i}_{(\partial\Phi/\partial x^n)(\partial/\partial x^n)}b) &= (-1)^{p-1}\sum_{i=1}^{n}\frac{\partial}{\partial x^i}\Big(b_{I_n}\frac{\partial\Phi}{\partial x^n}\Big)\, dx^i \wedge dx^{I_n\setminus\{n\}} \\ &= (-1)^{p-1}\frac{\partial b_{I_n}}{\partial x^n}\frac{\partial\Phi}{\partial x^n}\, dx^n \wedge dx^{I_n\setminus\{n\}} + 0 = \frac{\partial b_{I_n}}{\partial x^n}\frac{\partial\Phi}{\partial x^n}\, dx^{I_n}. \end{aligned} \tag{4-13}$$

Now recall that $\mathscr{I} \ni I = (i_1,\dots,i_p)$ with $1 \le i_1 \le \cdots \le i_p \le n$, and denote by $\operatorname{ind}(i_k)$ the integer k. Looking at the third term of the right side of (4-12), we write

$$\begin{aligned} \boldsymbol{i}_{(\nabla_T\Phi-\nabla\tilde{\Phi})}b_I\, dx^I &= b_I\, dx^I(\nabla_T\Phi - \nabla\tilde{\Phi}) = b_I\sum_{j\in I}(-1)^{\operatorname{ind}(j)+1}(\nabla_T\Phi - \nabla\tilde{\Phi})_j\, dx^{I\setminus\{j\}} \\ &= b_I\sum_{j\in I}(-1)^{\operatorname{ind}(j)+1}\alpha_j\, dx^{I\setminus\{j\}}, \end{aligned}$$

where, due to (A-2) and (A-3), for all j in $\{1,\dots,n\}$,

$$\alpha_j = (\nabla_T\Phi - \nabla\tilde{\Phi})_j = \sum_{i=1}^{n} g^{ij}\Big(\frac{\partial\Phi}{\partial x^i}(x) - \frac{\partial\Phi}{\partial x^i}(x',0)\Big).$$

Moreover, due to the block diagonal form of G_0^{-1}, for all j in $\{1,\dots,n\}$, α_j satisfies, again by (A-2) and (A-3),

$$\alpha_n(x)\equiv 0 \qquad \text{and} \qquad \alpha_j(x',0)\equiv 0 \quad \text{for all } j\in\{1,\dots,n-1\},$$

Hence, we obtain on $\partial\Omega$

$$\begin{aligned} d(\boldsymbol{i}_{(\nabla_T\Phi-\nabla\tilde{\Phi})}b_I\,dx^I)(x',0) &= \sum_{l=1}^{n}\sum_{j\in I}(-1)^{\operatorname{ind}(j)+1}\frac{\partial}{\partial x^l}(b_I\alpha_j)(x',0)\,dx^l\wedge dx^{I\setminus\{j\}}\\ &= 0+\sum_{j\in I}(-1)^{\operatorname{ind}(j)+1}\frac{\partial}{\partial x^n}(b_I\alpha_j)(x',0)\,dx^n\wedge dx^{I\setminus\{j\}}\\ &= \sum_{j\in I'}(-1)^{\operatorname{ind}(j)+1}\frac{\partial}{\partial x^n}(b_{I'}\alpha_j)(x',0)\,dx^n\wedge dx^{I'\setminus\{j\}}\\ &\qquad +\sum_{j\in I_n\setminus\{n\}}(-1)^{\operatorname{ind}(j)+1}\frac{\partial}{\partial x^n}(b_{I_n}\alpha_j)(x',0)\,dx^n\wedge dx^{I_n\setminus\{j\}}\\ &= \sum_{j\in I'}(-1)^{\operatorname{ind}(j)+1}\frac{\partial}{\partial x^n}(b_{I'}\alpha_j)(x',0)\,dx^n\wedge dx^{I'\setminus\{j\}}, \end{aligned}$$

where we used $\alpha_j(x',0)\equiv 0$ at the second line and $\alpha_n(x)\equiv 0$ at the second to last line. Using again $\alpha_j(x',0)\equiv 0$ allows us to write on $\partial\Omega$

$$\begin{aligned} d(\boldsymbol{i}_{(\nabla_T\Phi-\nabla\tilde{\Phi})}b_I\,dx^I)(x',0) &= b_{I'}\sum_{j\in I'}(-1)^{\operatorname{ind}(j)+1}\frac{\partial\alpha_j}{\partial x^n}(x',0)\,dx^n\wedge dx^{I'\setminus\{j\}}\\ &= b_{I'}\sum_{j\in I'}(-1)^{\operatorname{ind}(j)+p}\frac{\partial\alpha_j}{\partial x^n}(x',0)\,dx^{I'\setminus\{j\}}\wedge dx^n\\ &=: \tilde{\ell}_{I_n}(x',0)\,dx^{I_n}, \end{aligned} \tag{4-14}$$

where the $\tilde{\ell}_{I_n}$ are $\mathscr{C}^\infty(\partial\Omega)$-linear combinations of the $b_{I'}$ (for I' in $\mathscr{I}'$) that do not depend on the b_{I_n} (for I_n in $\mathscr{I}_n$). Combining (4-12), (4-13), and (4-14) leads to the result announced in Lemma 4.3. □

Proof of Proposition 4.1. From Section B2 we have

$$\mathscr{L}_{\nabla\Phi}-\mathscr{L}^*_{\nabla\Phi}+\mathscr{L}_{\nabla f}+\mathscr{L}^*_{\nabla f}=2\mathscr{L}_{\nabla\Phi}+\mathscr{R}_1,$$

where $\mathscr{R}_1$ is an order-zero differential operator. Writing $\mathscr{R}_1=\mathscr{R}_1^T+\mathscr{R}_1^N$, we deduce from (B-1), since $b_I\,dx^I=b_{I'}dx^{I'}$ on the boundary, that

$$\begin{aligned} \boldsymbol{t}(\mathscr{R}_1(b_I\,dx^I)) &= b_{I'}(x',0)\mathscr{R}_1^T(dx^{I'}),\\ \boldsymbol{n}(\mathscr{R}_1(b_I\,dx^I)) &= b_{I'}(x',0)\mathscr{R}_1^N(dx^{I'})=\tilde{\ell}'_{I_n}(x',0)\,dx^{I_n}, \end{aligned}$$

where the $\tilde{\ell}'_{I_n}$ are $\mathscr{C}^\infty(\partial\Omega)$-linear combinations of the $b_{I'}$ (for I' in $\mathscr{I}'$) that do not depend on the b_{I_n} (for I_n in $\mathscr{I}_n$).

Moreover, from (4-1)–(4-4), $f-\Phi$ satisfies the assumptions of Corollary B.5; thus $\mathscr{R}_1^T$ is given on the boundary, in the coordinates (x',x^n), by

$$\mathscr{R}_1^T(x',0)=\begin{pmatrix} & & 0\\ & \mathscr{R}_1^{T'}(x') & \vdots\\ & & 0\\ 0 & \cdots & 0\ \ \beta(x')\end{pmatrix}^{(p)} - \gamma(x')\,\mathrm{Id},$$

where β and γ are $\mathscr{C}^\infty$ functions that satisfy

$$\beta(0)=0,\quad \gamma(0)=\mathrm{Tr}(\mathrm{Hess}(f|_{\partial\Omega}-\varphi)(0)),\quad \text{and}\quad \mathscr{R}_1^{T'}(0)=2\,\mathrm{Hess}(f|_{\partial\Omega}-\varphi)(0).$$

Having in mind Lemma 4.3, we now look at the term $2\mathscr{L}_{\nabla\tilde{\Phi}}+\mathscr{R}_1$. From Proposition B.3, we write

$$2\mathscr{L}_{\nabla\tilde{\Phi}}=2\mathscr{L}_{\nabla\tilde{\Phi}}\otimes\mathrm{Id}+\mathscr{R}_3,$$

where $\mathscr{R}_3=\mathscr{R}_3^T+\mathscr{R}_3^N$ is an order-zero differential operator such that, since $\tilde{\Phi}$ satisfies the assumptions of Corollary B.5,

$$\begin{aligned}\boldsymbol{t}(\mathscr{R}_3(b_I\,dx^I))&=b_{I'}(x',0)\mathscr{R}_3^T(dx^{I'}),\\ \boldsymbol{n}(\mathscr{R}_3(b_I\,dx^I))&=b_{I'}(x',0)\mathscr{R}_3^N(dx^{I'})=\tilde{\ell}''_{I_n}(x',0)\,dx^{I_n},\end{aligned}$$

where the $\tilde{\ell}''_{I_n}$ are $\mathscr{C}^\infty(\partial\Omega)$-linear combinations of the $b_{I'}$ (for I' in $\mathscr{I}'$) that do not depend on the b_{I_n}, and $\mathscr{R}_3^T$ is given on the boundary, in the coordinates (x',x^n), by

$$\mathscr{R}_3^T(x',0)=\begin{pmatrix} & & 0\\ & \mathscr{R}_3^{T'}(x') & \vdots\\ & & 0\\ 0 & \cdots & 0\ \ 0\end{pmatrix}^{(p)},$$

with

$$\mathscr{R}_3^{T'}(0)=2\,\mathrm{Hess}(\tilde{\Phi}|_{\partial\Omega})(0)=2\,\mathrm{Hess}(\varphi)(0).$$

Note that, according to Remark B.4, the (n,n)-entry of the matrix is indeed 0 since $\partial^2\tilde{\Phi}/(\partial x^n)^2\equiv 0$.

Set $\mathscr{R}_{\mathrm{Neu}}=\mathscr{R}_1+\mathscr{R}_3$ and $\tilde{\ell}^{(3)}_{I_n}=\tilde{\ell}'_{I_n}+\tilde{\ell}''_{I_n}$ for I_n in $\mathscr{I}_n$. Then $\mathscr{R}_{\mathrm{Neu}}$ is an order-zero differential operator satisfying

$$2\mathscr{L}_{\nabla\tilde{\Phi}}+\mathscr{R}_1=2\mathscr{L}_{\nabla\tilde{\Phi}}\otimes\mathrm{Id}+\mathscr{R}_{\mathrm{Neu}} \tag{4-15}$$

and

$$\begin{aligned}\boldsymbol{t}(\mathscr{R}_{\mathrm{Neu}}(b_I\,dx^I))&=b_{I'}(x',0)\mathscr{R}_{\mathrm{Neu}}^T(dx^{I'}),\\ \boldsymbol{n}(\mathscr{R}_{\mathrm{Neu}}(b_I\,dx^I))&=b_{I'}(x',0)\mathscr{R}_{\mathrm{Neu}}^N(dx^{I'})=\tilde{\ell}^{(3)}_{I_n}(x',0)\,dx^{I_n},\end{aligned} \tag{4-16}$$

where the $\tilde{\ell}^{(3)}_{I_n}$ are $\mathscr{C}^\infty(\partial\Omega)$-linear combinations of the $b_{I'}$ (for I' in $\mathscr{I}'$) that do not depend on the b_{I_n} (for I_n in $\mathscr{I}_n$). Moreover, $\mathscr{R}_{\mathrm{Neu}}^T$ is given on the boundary, in the coordinates (x',x^n), by

$$\mathscr{R}_{\mathrm{Neu}}^T(x',0)=\begin{pmatrix} & & 0\\ & \mathscr{R}_1^{T'}(x',0)+\mathscr{R}_3^{T'}(x',0) & \vdots\\ & & 0\\ 0 & \cdots & 0\ \ \beta(x')\end{pmatrix}^{(p)} - \gamma(x')\,\mathrm{Id},$$

where

$$\beta(0)=0,\quad \gamma(0)=\mathrm{Tr}\,(\mathrm{Hess}(f|_{\partial\Omega}-\varphi)(0))\,,\quad \text{and}\quad \mathcal{R}_1^{T'}(0)+\mathcal{R}_3^{T'}(0)=2\,\mathrm{Hess}(f|_{\partial\Omega})(0).$$

Now look at the term $2\mathcal{L}_{\nabla\tilde{\Phi}}\otimes\mathrm{Id}$. By Cartan's formula (2-6), we have

$$(2\mathcal{L}_{\nabla\tilde{\Phi}}\otimes\mathrm{Id})b=db_{I'}(\nabla\tilde{\Phi})\,dx^{I'}+db_{I_n}(\nabla\tilde{\Phi})\,dx^{I_n},$$

and, using the boundary condition satisfied by the b_{I_n} (for I_n in $\mathcal{I}_n$) and the fact that $\nabla\tilde{\Phi}$ is a tangential vector field, we obtain

$$(2\mathcal{L}_{\nabla\tilde{\Phi}}\otimes\mathrm{Id})b=\sum_{i=1}^{n-1}\frac{\partial b_{I'}}{\partial x^i}(\nabla\tilde{\Phi})_i\,dx^{I'}=(2\mathcal{L}_{\nabla\tilde{\Phi}}\otimes\mathrm{Id})b_{I'}\,dx^{I'}. \tag{4-17}$$

Set $\ell_{I_n}=\tilde{\ell}_{I_n}+\frac{1}{2}\tilde{\ell}_{I_n}^{(3)}$ for I_n in $\mathcal{I}_n$. Writing

$$(2\mathcal{L}_{\nabla\Phi}+\mathcal{R}_1)b=2(\mathcal{L}_{\nabla\Phi}-\mathcal{L}_{\nabla\tilde{\Phi}})b+(2\mathcal{L}_{\nabla\tilde{\Phi}}+\mathcal{R}_1)b$$

and using (4-15)–(4-17), we obtain Proposition 4.1 after the application of Lemma 4.3. □

4C. ***Proof of Theorem 1.1.*** We shall first consider a WKB-approximation for

$$(\Delta_{f,h}^{(p)}-E(h))u_p^{\mathrm{WKB}}=e^{-\Phi/h}\mathbb{O}(h^\infty) \tag{4-18}$$

with $E(h)=O(h^2)$ and the boundary conditions (1-2) and (1-3) and then check $E(h)=O(h^\infty)$.

Writing

$$d_{f,h}(e^{-\Phi/h}a^k)=e^{-\Phi/h}\big(h\,da^k+d(f-\Phi)\wedge a^k\big)\quad \text{for all } k\in\mathbb{N},$$

where, due to (1-2) and (4-3), a^k and $d(f-\Phi)$ are tangential forms, the second boundary condition (1-3) corresponds to

$$\boldsymbol{n}(da^k)=0\quad\text{for all } k\in\mathbb{N}. \tag{4-19}$$

We now recall a relation that will be very useful; see [Helffer and Sjöstrand 1985] for a complete proof:

$$\begin{aligned} e^{\Phi/h}\Delta_{f,h}e^{-\Phi/h}&=h^2(d+d^*)^2+|\nabla f|^2-|\nabla\Phi|^2+h(\mathcal{L}_{\nabla\Phi}-\mathcal{L}_{\nabla\Phi}^*+\mathcal{L}_{\nabla f}+\mathcal{L}_{\nabla f}^*)\\ &=h^2(d+d^*)^2+h(\mathcal{L}_{\nabla\Phi}-\mathcal{L}_{\nabla\Phi}^*+\mathcal{L}_{\nabla f}+\mathcal{L}_{\nabla f}^*). \end{aligned} \tag{4-20}$$

We then write, in the notation of Section B2,

$$\mathcal{L}_{\nabla\Phi}-\mathcal{L}_{\nabla\Phi}^*+\mathcal{L}_{\nabla f}+\mathcal{L}_{\nabla f}^*=2\mathcal{L}_{\nabla\Phi}+\mathcal{R}_1=2\mathcal{L}_{\nabla\Phi}\otimes\mathrm{Id}+\mathcal{R},$$

where $\mathcal{R}$ and $\mathcal{R}_1$ are order-zero differential operators defined in Section B2.

By looking for $E(h)\sim\sum_{k=1}^{\infty}h^{k+1}E_k$, the interior equation (4-18) reads

$$e^{\Phi/h}(\Delta_{f,h}-E(h))e^{-\Phi/h}=h^2\big((d+d^*)^2-h^{-2}E(h)\big)+h\,(2\mathcal{L}_{\nabla\Phi}\otimes\mathrm{Id}+\mathcal{R})\,.$$

We now verify that it is possible to construct a solution u_p^{WKB} to (4-18) in Ω which can be extended to $\overline{\Omega}$ and satisfying the boundary conditions (1-2) and (1-3). The construction of an interior WKB solution in Ω is standard as an inductive Cauchy problem, once the a^k are known on $\partial\Omega$; see [Dimassi and Sjöstrand 1999; Helffer 1988]. Actually the noncharacteristic Cauchy problems

$$(2\mathcal{L}_{\nabla\Phi}\otimes \mathrm{Id}+\mathcal{R})a^k=-(d+d^*)^2a^{k-1}+\sum_{\ell=1}^{k}E_\ell a^{k-\ell}\quad \text{in } \overline{\Omega} \tag{4-21}$$

are solved by induction, with the convention $a_{-1}=0$.

Hence the problem is reduced to the solving of the system made of the boundary conditions (4-9), (4-19) and of the compatibility equation on the boundary (see Section B2 for the meaning of the notation):

$$(2\mathcal{L}_{\nabla\Phi}+\mathcal{R}_1)\,a^k=-(d+d^*)^2a^{k-1}+\sum_{\ell=1}^{k}E_\ell a^{k-\ell}\quad \text{on } \partial\Omega. \tag{4-22}$$

Owing to Proposition 4.1 (with the notation of Section 4B) and to (4-3), the system (4-22), (4-9), (4-19) is equivalent to the following differential system on $\partial\Omega$:

$$\begin{aligned}
-\boldsymbol{t}(d+d^*)^2a^{k-1}+\sum_{\ell=1}^{k}E_\ell a^{k-\ell}&=(2\mathcal{L}_{\nabla\tilde{\Phi}}\otimes\mathrm{Id}+\mathcal{R}^T_{\mathrm{Neu}})a^k_{I'}\,dx^{I'}+2\frac{\partial f}{\partial x^n}\boldsymbol{i}_{\partial/\partial x^n}da^k,\\
-\boldsymbol{n}(d+d^*)^2a^{k-1}-2\ell_{I_n}(x',0)\,dx^{I_n}&=2\frac{\partial f}{\partial n}\frac{\partial a^k_{I_n}}{\partial x^n}\,dx^{I_n},\\
a^k_{I_n}|_{\partial\Omega}\equiv 0\quad \text{and}\quad \boldsymbol{n}(da^k)&=0\quad \text{for all } k\in\mathbb{N},
\end{aligned}$$

where the ℓ_{I_n} are $\mathscr{C}^\infty(\partial\Omega)$-linear combinations of the $a^k_{I'}$ (for I' in $\mathcal{I}'$) that do not depend on the $a^k_{I_n}$ (for I_n in $\mathcal{I}_n$). Note also, owing to Lemma 4.2, that the first line of this system simply reads

$$-\boldsymbol{t}(d+d^*)^2a^{k-1}+\sum_{\ell=1}^{k}E_\ell a^{k-\ell}=(2\mathcal{L}_{\nabla\tilde{\Phi}}\otimes\mathrm{Id}+\mathcal{R}^T_{\mathrm{Neu}})a^k_{I'}\,dx^{I'}. \tag{4-23}$$

Moreover, since $dx^i=d\bar{x}^i$ for $i\in\{1,\dots,n-1\}$ at the point U, it follows from Corollary B.5, (4-5), and the results in [Helffer and Sjöstrand 1985, 271–275] that $\mathcal{R}^T_{\mathrm{Neu}}(0)$ restricted to tangential forms is symmetric with the one-dimensional kernel $\mathbb{R}dx^1\wedge\cdots\wedge dx^p$.

Since $a^k_{I'}dx^{I'}$ is tangential and $2\mathcal{L}_{\nabla\tilde{\Phi}}\otimes\mathrm{Id}$ only differentiates the $a^k_{I'}$ tangentially, because

$$(2\mathcal{L}_{\nabla\tilde{\Phi}}\otimes\mathrm{Id})a^k_{I'}\,dx^{I'}=\sum_{i=1}^{n-1}\frac{\partial a^k_{I'}}{\partial x^i}(\nabla\tilde{\Phi})_i\,dx^{I'},$$

it turns out that (4-23) can be rewritten as a tangential system that can be solved according to the analysis of the boundaryless case done in [Helffer and Sjöstrand 1985]. Here are the details: thanks to Lemma 4.2, the complete system (4-21), (4-22), (4-9) and (4-19) becomes equivalent to the system

$$\left.\begin{aligned}
(2\mathcal{L}_{\nabla\tilde{\Phi}}\otimes\mathrm{Id}+\mathcal{R}^T_{\mathrm{Neu}})a^k_{I'}dx^{I'}&=-\boldsymbol{t}(d+d^*)^2a^{k-1}+\sum_{\ell=1}^{k-1}E_\ell a^{k-\ell}+E_ka^0 &&\text{on } \partial\Omega,\\
(2\mathcal{L}_{\nabla\Phi}\otimes\mathrm{Id}+\mathcal{R})a^k&=-(d+d^*)^2a^{k-1}+\sum_{\ell=1}^{k}E_\ell a^{k-\ell} &&\text{on } \overline{\Omega},\\
a_{I_n}|_{\partial\Omega}&\equiv 0 &&\text{for } I_n\in\mathcal{I}_n.
\end{aligned}\right\}\quad (\mathcal{S}_{\mathrm{Neu}})$$

The first line is a degenerate matrix transport equation, which can be solved following [Helffer and Sjöstrand 1985, page 275] and [Helffer 1988, pages 13–14]: for $k = 0$, the homogeneous boundary equation

$$(2\mathscr{L}_{\nabla\tilde{\Phi}} \otimes \mathrm{Id} + \mathscr{R}^T_{\mathrm{Neu}}) a^0_{I'} dx^{I'} = 0$$

admits some solution if and only if

$$a^0_{I'}(0)\, dx^{I'} \in \mathrm{Ker}(\mathscr{R}^T_{\mathrm{Neu}}(0)), \tag{4-25}$$

and the solution is unique once $a^0_{I'}(0)\, dx^{I'}$ has been chosen. This shows the uniqueness of a^0 up to multiplication by a constant. Note also that the formulation of Theorem 1.1 is a coordinate-free rewriting of this condition for $a^0(U)$. Indeed, it has already been mentioned that, when restricted to tangential p-forms, the kernel of $\mathscr{R}^T_{\mathrm{Neu}}(0)$ is one-dimensional. Recall moreover that, *in our coordinate system* (see Proposition 4.1), at $U \cong 0$,

$$\begin{aligned}\mathscr{R}^T_{\mathrm{Neu}}(U) &= 2\begin{pmatrix} & & 0 \\ & \mathrm{Hess}(f|_{\partial\Omega})(U) & \vdots \\ 0 & \cdots & 0\end{pmatrix}^{(p)} - \mathrm{Tr}\,(\mathrm{Hess}(f|_{\partial\Omega} - \Phi|_{\partial\Omega})(U)) \\ &= 2A^{(p)} - \mathrm{Tr}\,(\mathrm{Hess}(f|_{\partial\Omega} - \Phi|_{\partial\Omega})(U))\,,\end{aligned}$$

and that, for a tangential p-form $dx^{I'} = dx^{i_1} \wedge \cdots \wedge dx^{i_p}$,

$$A^{(p)} dx^{I'} = (A\, dx^{i_1}) \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_p} + \cdots + dx^{i_1} \wedge \cdots \wedge dx^{i_{p-1}} \wedge (A\, dx^{i_p}),$$

where, for $\ell \in \{1, \dots, p\}$,

$$A\, dx^{i_\ell} = (\mathrm{Hess}(f|_{\partial\Omega})(U))\, dx^{i_\ell} + 0.dx^n = (\mathrm{Hess}(f|_{\partial\Omega})(U)\, j^*)\, dx^{i_\ell}.$$

Lastly, note that in the previous equation, we wrote $(\mathrm{Hess}(f|_{\partial\Omega})(U))\, dx^{i_\ell}$ with a slight abuse of notation, since $\mathrm{Hess}(f|_{\partial\Omega})(U) \in \mathscr{L}(T^*_U\partial\Omega)$ and $dx^{i_\ell}(U) \in T^*_U\overline{\Omega}$. Indeed, the proper notation would be $(\mathrm{Hess}(f|_{\partial\Omega})(U)\, j^*)\, dx^{i_\ell}$, where

$$(\mathrm{Hess}(f|_{\partial\Omega})(U)\, j^*)\, dx^{i_\ell} \in \mathscr{L}(T^*_U\overline{\Omega}; T^*_U\partial\Omega) \subset \mathscr{L}(T^*_U\overline{\Omega}; T^*_U\overline{\Omega}).$$

Now take $a^0(0) = dx^1 \wedge \cdots \wedge dx^p \in \mathrm{Ker}(\mathscr{R}^T_{\mathrm{Neu}}(0))$. For $k = 1$, we have to solve the boundary equation

$$(2\mathscr{L}_{\nabla\tilde{\Phi}} \otimes \mathrm{Id} + \mathscr{R}^T_{\mathrm{Neu}}) a^1_{I'} dx^{I'} = -\boldsymbol{t}(d + d^*)^2 a^0 + E_1 a^0.$$

Choose then E_1 such that

$$-\boldsymbol{t}(d + d^*)^2 a^0(0) + E_1 a^0(0) \in \mathrm{Ran}(\mathscr{R}^T_{\mathrm{Neu}}(0)) = \big(\mathrm{Ker}(\mathscr{R}^T_{\mathrm{Neu}}(0))\big)^{\perp},$$

where the last equality follows from both the symmetry of $\mathscr{R}^T_{\mathrm{Neu}}(0)$ and $G_0(0) = \mathrm{Id}_n$. This is equivalent to choosing E_1 such that

$$E_1 = \frac{\langle \boldsymbol{t}(d + d^*)^2 a^0(0) \mid a^0(0)\rangle_{g_0(0)}}{\|a^0(0)\|^2_{g_0(0)}},$$

and this is indeed possible since $\operatorname{Ker}(\mathcal{R}^T_{\mathrm{Neu}}(0)) = \mathbb{R}a^0(0) \neq \{0\}$. Next take $a^1_{I'}(0)\,dx^{I'}$ such that

$$\mathcal{R}^T_{\mathrm{Neu}}(0)(a^1_{I'}(0)\,dx^{I'}) = -\boldsymbol{t}(d+d^*)^2a^0(0) + E_1a^0(0).$$

Then, for each $k > 2$, choose E_k such that the compatibility condition

$$-\boldsymbol{t}(d+d^*)^2a^{k-1}(0) + \sum_{\ell=1}^{k-1} E_\ell a^{k-\ell}(0) + E_k a^0(0) \in \big(\operatorname{Ker}(\mathcal{R}^T_{\mathrm{Neu}}(0))\big)^{\perp}$$

is satisfied, or, more precisely, such that

$$E_k = \frac{\langle \boldsymbol{t}(d+d^*)^2a^{k-1}(0) - \sum_{\ell=1}^{k-1} E_\ell a^{k-\ell}(0) \mid a^0(0)\rangle_{g_0(0)}}{\|a^0(0)\|^2_{g_0(0)}},$$

and take $a^{k-1}_{I'}(0)\,dx^{I'}$ in

$$(\mathcal{R}^T_{\mathrm{Neu}}(0))^{-1}\Big(-\boldsymbol{t}(d+d^*)^2a^{k-1}(0) + \sum_{\ell=1}^{k-1} E_\ell a^{k-\ell}(0) + E_k a^0(0)\Big).$$

Thus, at every step $k \in \mathbb{N}$, the first and third lines of the system $(\mathcal{S}_{\mathrm{Neu}})$ fully determine the Cauchy data $a^k(x',0)$ and the number E_k. The first line fully determines the restrictions of the $a_{I'}$ to $\partial\Omega$. The second line solves the interior problem with these Cauchy data and contains, with the two other lines, thanks to Lemma 4.2, the second trace condition (4-19).

We now check that $E(h) = \mathcal{O}(h^\infty)$. We prove this by comparing with the half-space problem, for which we know by (3-8) that the first eigenvalue is 0 with multiplicity one and that the second one is larger than $Ch^{6/5}$. Take a cut-off function $\chi \in \mathscr{C}^\infty_0(\overline{\Omega})$ satisfying $\chi = 1$ in a neighborhood of U and $\partial\chi/\partial n|_{\partial\Omega} = 0$. Then set

$$u^K_p = \chi e^{-\Phi/h} \sum_{k=0}^{K} a^k h^k = \chi e^{-\Phi/h} A^K_h.$$

From $\partial\chi/\partial n|_{\partial\Omega} \equiv 0$ and

$$d_{f,h}(\chi A^K_h) = (hd + df\wedge)\chi A^K_h = hd\chi \wedge A^K_h + \chi d_{f,h} A^K_h,$$

the form $u^K_p \in \Lambda^1 H^2(\mathbb{R}^n_-)$ belongs to the domain of $\mathcal{A}^{(p)}_N$ and the approximations u^K_p and $E^K(h) = \sum_{k=1}^K E_k h^{k+1}$ satisfy

$$\begin{aligned}
\big(\mathcal{A}^{(p)}_N - E^K(h)\big)u^K_p &= h^{K+2}\rho^K e^{-\Phi/h} - h^2[\Delta,\chi]u^K_p = \mathcal{O}(h^{K+2}) &&\text{in } \overline{\mathbb{R}^n_-},\\
\boldsymbol{n}u^K_p &= 0 &&\text{on } \mathbb{R}^{n-1}\times\{0\},\\
\boldsymbol{n}d_{f,h}u^K_p &= 0 &&\text{on } \mathbb{R}^{n-1}\times\{0\},
\end{aligned}$$

for some $\mathscr{C}^\infty$ 1-form ρ^K defined in a neighborhood of U and independent of h. From a direct Laplace method we obtain

$$\|u^K_p\| \sim ch^{(n+1)/4},$$

and the spectral theorem then implies that there exists an eigenvalue $\lambda(h)$ of $\mathcal{A}_N^{(p)}$ such that

$$|E^K(h) - \lambda(h)| = \mathcal{O}(h^{K+2-(n+1)/4}).$$

Choosing the integer K large enough, we deduce from the inclusion

$$\sigma(\mathcal{A}_N^{(p)}) \setminus \{0\} \subset [Ch^{6/5}, +\infty)$$

combined with the estimate $E^K(h) = \mathcal{O}(h^2)$ that $\lambda(h) = 0$. The number K being arbitrary, the construction of the previous quasimode is then possible only if $E_k = 0$ for all $k \in \mathbb{N}^*$. □

4D. ***Local WKB construction in the Dirichlet case.*** Let U be a generalized critical point of f with index p in the Dirichlet case, i.e., a critical point of index $p-1$, with $p \in \{1, \dots, n\}$, of $f|_{\partial\Omega}$ satisfying $(\partial f/\partial n)(U) > 0$, and again take an adapted local coordinate system (x', x^n) around U, as in Section 4A.

Let φ be the Agmon distance to U on the boundary and use Lemma 3.4(1) with $f_1 = f$ and $\alpha = \varphi$. Denoting by Φ the function Φ_- of the lemma, Φ is then the Agmon distance to U and we have locally

$$|\partial_n \Phi|^2 + |\nabla_T \Phi|^2 = |\nabla \Phi|^2 = |\nabla f|^2, \tag{4-26}$$

$$\Phi|_{\partial\Omega} = \varphi, \tag{4-27}$$

$$\partial_n \Phi|_{\partial\Omega} = -\frac{\partial f}{\partial n}\Big|_{\partial\Omega}. \tag{4-28}$$

Moreover, the following relation is satisfied (see the proof of (4-4) and replace $\partial_n \Phi|_{\partial\Omega} = \partial_n f|_{\partial\Omega}$ by $\partial_n \Phi|_{\partial\Omega} = -\partial_n f|_{\partial\Omega}$):

$$\partial^2_{x^n x^n}(f + \Phi)(0) = \partial^2_{nn}(f + \Phi)(0) = 0. \tag{4-29}$$

As in Section 4A, there exist other local coordinates $(\bar{x}', \bar{x}^n)$ centered at U, with $\bar{x}' = (\bar{x}^1, \dots, \bar{x}^{n-1})$ and $d\bar{x}^1, \dots, d\bar{x}^{n-1}, dx^n$ orthonormal at U, such that (4-5) is satisfied with $\lambda_i < 0$ for $i \in \{1, \dots, p-1\}$ and $\lambda_i > 0$ for $i \in \{p, \dots, n-1\}$. Furthermore, the coordinates (x', x^n) can be chosen in such a way that $dx^1, \dots, dx^{n-1}$ and $d\bar{x}^1, \dots, d\bar{x}^{n-1}$ coincide at U and even such that $x'|_{\partial\Omega} = \bar{x}'|_{\partial\Omega}$.

Specification of the coordinate system for Theorem 1.2. In the rest of this section, we are again going to work in an adapted local coordinate system $x = (x', x^n)$ around U such that

$$dx^i = d\bar{x}^i \quad \text{at } U \qquad \text{for all } i \in \{1, \dots, n-1\}. \tag{4-30}$$

The proof is quite close to the one for the Neumann case, but here it turns out to be more natural to make "dual computations". In particular, we will work with d^* where we worked with d in the Neumann case. This leads to somewhat more complicated computations.

4E. ***First boundary conditions in the Dirichlet case.*** Writing

$$a_h(x) = a(x,h) = a_I(x,h)\, dx^I = a_{I'}(x,h)\, dx^{I'} + a_{I_n}(x,h)\, dx^{I_n},$$

the first boundary condition (1-5) is equivalent to

$$a^k_{I'}(x', 0) \equiv 0 \quad \text{for all } k \in \mathbb{N} \text{ and } I' \in \mathcal{I}'. \tag{4-31}$$

The rest of this subsection specifies some consequences of these conditions, in the same spirit as those specified in the Section 4B concerning the Neumann case.

4E1. *About $\mathscr{L}+\mathscr{L}^*$.* The relation

$$\mathscr{L}_{\nabla\Phi}-\mathscr{L}^*_{\nabla\Phi}+\mathscr{L}_{\nabla f}+\mathscr{L}^*_{\nabla f}=-2\mathscr{L}^*_{\nabla\Phi}+\mathscr{L}_{\nabla(f+\Phi)}+\mathscr{L}^*_{\nabla(f+\Phi)}$$

is obviously satisfied, and using again Proposition B.3, we can write

$$\mathscr{L}^*_{\nabla(f+\Phi)}+\mathscr{L}_{\nabla(f+\Phi)}=\mathscr{R}_4,$$

where $\mathscr{R}_4$ is an order-zero differential operator.

Writing $\mathscr{R}_4=\mathscr{R}_4^T+\mathscr{R}_4^N$, we deduce from (B-2), since $a_I^k\,dx^I=a_{I_n}^k\,dx^{I_n}$ on the boundary, that

$$\begin{aligned}\boldsymbol{t}(\mathscr{R}_4(a_I^k\,dx^I))&=a_{I_n}^k(x',0)\mathscr{R}_4^N(dx^{I_n})=\tilde{\ell}'_{I'}(x',0)\,dx^{I'},\\ \boldsymbol{n}(\mathscr{R}_4(a_I^k\,dx^I))&=a_{I_n}^k(x',0)\mathscr{R}_4^T(dx^{I_n}),\end{aligned}$$

where the $\tilde{\ell}'_{I'}$ are $\mathscr{C}^\infty(\partial\Omega)$-linear combinations of the $a_{I_n}^k$ (for I_n in $\mathscr{I}_n$) that do not depend on the $a_{I'}^k$ (for I' in $\mathscr{I}'$).

Moreover, by (4-26)–(4-29), here $f+\Phi$ satisfies the assumptions of Corollary B.5; thus $\mathscr{R}_4^T$ is given on the boundary, in the coordinates (x',x^n), by

$$\mathscr{R}_4^T(x',0)=\begin{pmatrix} & & 0\\ & \mathscr{R}_4^{T'}(x') & \vdots\\ & & 0\\ 0 & \cdots & 0\ \ \delta(x')\end{pmatrix}^{(p)}-\kappa(x')\,\mathrm{Id},$$

where δ, κ are $\mathscr{C}^\infty$ functions which satisfy

$$\delta(0)=0,\quad \kappa(0)=\mathrm{Tr}(\mathrm{Hess}(f|_{\partial\Omega}+\varphi)(0)),\quad \mathscr{R}_4^{T'}(0)=2\,\mathrm{Hess}(f|_{\partial\Omega}+\varphi)(0).$$

4E2. *Expression of the codifferential d^*.* As already mentioned, to make a study similar to the one done in Section 4B for the Neumann case, we need to work with d^*, so we must have a handy expression for this operator.

For a differential form ω we set, in the coordinate system (x',x^n),

$$\nabla_i=\nabla_{x^i},\quad \boldsymbol{a}_i^*\omega=dx^i\wedge\omega,\quad \boldsymbol{a}_i\omega=\boldsymbol{i}_{\nabla x^i}\omega.$$

Then d and d^* have the following form (see [Cycon et al. 1987, pages 238–247]):

$$d=\sum_{i=1}^n\boldsymbol{a}_i^*\nabla_i=-\sum_{i=1}^n(\nabla_i)^*\boldsymbol{a}_i^*,\tag{4-32}$$

$$d^*=-\sum_{i=1}^n\boldsymbol{a}_i\nabla_i.\tag{4-33}$$

Recall also the characteristic relations

$$\boldsymbol{a}_i^*\boldsymbol{a}_j^*+\boldsymbol{a}_j^*\boldsymbol{a}_i^*=0,\quad \boldsymbol{a}_i\boldsymbol{a}_j+\boldsymbol{a}_j\boldsymbol{a}_i=0,\quad \boldsymbol{a}_i^*\boldsymbol{a}_j+\boldsymbol{a}_j\boldsymbol{a}_i^*=g^{ij},\quad \text{for all } i,j\in\{1,\dots,n\}.$$

Denoting by ∂_i the operator defined by components with differentiation in a fixed coordinate system,

$$\partial_i(\omega_I\,dx^I) = \frac{\partial\omega_I}{\partial x^i}\,dx^I,$$

∇_i becomes (see again [Cycon et al. 1987, pages 238–247])

$$\nabla_i = \partial_i - \sum_{j,l,m} \Gamma^j_{il} g_{jm} \boldsymbol{a}^*_l \boldsymbol{a}_m, \tag{4-34}$$

where the Γ^j_{il} are the Christoffel symbols. Then d^* becomes

$$\begin{aligned} d^* &= -\sum_i \boldsymbol{a}_i\partial_i + \sum_{i,j,l,m} \Gamma^j_{il} g_{jm} \boldsymbol{a}_i \boldsymbol{a}^*_l \boldsymbol{a}_m \\ &= -\sum_i \boldsymbol{a}_i\partial_i + \sum_{i,j,l,m} \Gamma^j_{il} g_{jm} \left(\boldsymbol{a}_i \boldsymbol{a}^*_l + \boldsymbol{a}^*_l \boldsymbol{a}_i\right) \boldsymbol{a}_m - \sum_{i,j,l,m} \Gamma^j_{il} g_{jm} \boldsymbol{a}^*_l \boldsymbol{a}_i \boldsymbol{a}_m \\ &= -\sum_i \boldsymbol{a}_i\partial_i + \sum_{i,j,l,m} \Gamma^j_{il} g_{jm} g^{il} \boldsymbol{a}_m - \sum_{i,j,l,m} \Gamma^j_{il} g_{jm} \boldsymbol{a}^*_l \boldsymbol{a}_i \boldsymbol{a}_m. \end{aligned} \tag{4-35}$$

4E3. *Results.*

Proposition 4.4. *In the notation of Appendix A and Section 4E1, the following relations are satisfied for every normal p-form $b(x) = b_I(x)\,dx^I$ (that is, every p-form $b(x)$ satisfying $b_{I'}(x',0) \equiv 0$ for all $I' \in \mathcal{I}'$):*

$$\boldsymbol{t}((-2\mathcal{L}^*_{\nabla\Phi} + \mathcal{R}_4)b) = 2\left(\frac{\partial b_{I'}}{\partial x^n}\frac{\partial\Phi}{\partial x^n} + \ell_{I'}(x',0)\right) dx^{I'},$$

$$\boldsymbol{n}((-2\mathcal{L}^*_{\nabla\Phi} + \mathcal{R}_4)b) = (2\mathcal{L}_{\nabla\tilde{\Phi}} \otimes \mathrm{Id} + \mathcal{R}^T_{\mathrm{Dir}})b_{I_n}\,dx^{I_n} - 2\frac{\partial\Phi}{\partial x^n}\,dx^n \wedge d^*b,$$

where the $\ell_{I'}$ are $\mathcal{C}^\infty(\partial\Omega)$-linear combinations of the b_{I_n} (for I_n in $\mathcal{I}_n$) that do not depend on the $b_{I'}$ (for I' in $\mathcal{I}'$) and $\mathcal{R}^T_{\mathrm{Dir}}$ is an order-zero differential operator given in the coordinates (x',x^n), on the boundary by the following matrix, by

$$\mathcal{R}^T_{\mathrm{Dir}}(x',0) = \begin{pmatrix} & & 0 \\ & \mathcal{R}^{T'}_{\mathrm{Dir}}(x') & \vdots \\ & & 0 \\ 0 & \cdots & 0 \;\; \delta(x') \end{pmatrix}^{(p)} - \kappa_2(x')\,\mathrm{Id},$$

where

$$\delta(0) = 0, \qquad \kappa_2(0) = \mathrm{Tr}(\mathrm{Hess}(f|_{\partial\Omega} - \varphi)(0)), \qquad \mathcal{R}^{T'}_{\mathrm{Dir}}(0) = 2\,\mathrm{Hess}(f|_{\partial\Omega})(0).$$

In particular, this is true for a^k for k in $\mathbb{N}$, when (4-31) *is fulfilled.*

Lemma 4.5. *Let $b(x)$ be a* normal *p-form. The p-form $\vec{n}^* \wedge d^*b$ is then normal and the following equivalence is locally valid on the boundary $\partial\Omega$:*

$$\vec{n}^* \wedge d^*b = 0 \iff \boldsymbol{t}\,d^*b = 0.$$

In particular, this is true for a^k for k in $\mathbb{N}$, when (4-31) *is fulfilled.*

Proof. On the boundary $\partial\Omega$, we can write, in the coordinate system (x', x^n),

$$\begin{aligned} dx^n \wedge d^* b &= dx^n \wedge \boldsymbol{n} d^* b + dx^n \wedge \boldsymbol{t} d^* b = 0 + dx^n \wedge \boldsymbol{t} d^* b = dx^n \wedge (d^* b)_{I'} dx^{I'} \\ &= (-1)^{p-1} (d^* b)_{I'} dx^{I'} \wedge dx^n. \end{aligned}$$

Since $dx^n = \vec{n}^*$, this leads to the result. □

Lemma 4.6. *For every tangential p-form $b(x)$,*

$$\begin{aligned} \boldsymbol{n}((\mathscr{L}^*_{\nabla\Phi} - \mathscr{L}^*_{\nabla\tilde{\Phi}})b) &= \frac{\partial\Phi}{\partial x^n}\, dx^n \wedge d^* b, \\ \boldsymbol{t}((\mathscr{L}^*_{\nabla\Phi} - \mathscr{L}^*_{\nabla\tilde{\Phi}})b) &= \Big(-\frac{\partial b_{I'}}{\partial x^n}\frac{\partial\Phi}{\partial x^n} + \tilde{\ell}_{I'}(x', 0)\Big)\, dx^{I'}, \end{aligned}$$

where the $\tilde{\ell}_{I'}$ are $\mathscr{C}^\infty(\partial\Omega)$-linear combinations of the b_{I_n} (for I_n in $\mathscr{I}_n$) that do not depend on the $b_{I'}$ (for I' in $\mathscr{I}_n$).

Proof. Owing to (2-3) and to Cartan's formula (2-6), we write, in the coordinates (x', x^n),

$$\begin{aligned} (\mathscr{L}^*_{\nabla\Phi} &- \mathscr{L}^*_{\nabla\tilde{\Phi}})b \\ &= d^*(d\Phi\wedge b) + d\Phi\wedge d^* b + d^*(d\tilde{\Phi}\wedge b) + d\tilde{\Phi}\wedge d^* b \\ &= d^*\Big(\frac{\partial\Phi}{\partial x^n}\, dx^n\wedge b\Big) + \frac{\partial\Phi}{\partial x^n}\, dx^n\wedge d^* b + d^*((d_T\Phi - d\tilde{\Phi})\wedge b) + (d_T\Phi - d\tilde{\Phi})\wedge d^* b, \end{aligned} \tag{4-36}$$

where the function $\tilde{\Phi}$ is defined in Appendix A.

The second summand on the last line of (4-36) is normal by Lemma 4.5. Moreover, since $d_T\Phi = d\tilde{\Phi}$ on the boundary, the last summand also equals 0 on $\partial\Omega$. Hence, on $\partial\Omega$,

$$(\mathscr{L}^*_{\nabla\Phi} - \mathscr{L}^*_{\nabla\tilde{\Phi}})b = \frac{\partial\Phi}{\partial x^n} dx^n \wedge d^* b + d^*\Big(\frac{\partial\Phi}{\partial x^n} dx^n \wedge b\Big) + d^*((d_T\Phi - d\tilde{\Phi})\wedge b). \tag{4-37}$$

We study first the second summand on the right-hand side. Writing

$$b = b_I\, dx^I = b_{I'} dx^{I'} + b_{I_n} dx^{I_n},$$

we deduce that, in $\overline{\Omega}$,

$$\frac{\partial\Phi}{\partial x^n}\, dx^n \wedge b = \frac{\partial\Phi}{\partial x^n} b_{I'} dx^n \wedge dx^{I'}.$$

Applying d^* to this last relation (see (4-35)) and recalling that $b_{I'} = 0$ on $\partial\Omega$, we obtain on $\partial\Omega$

$$\begin{aligned} d^*\Big(\frac{\partial\Phi}{\partial x^n}\, dx^n \wedge b\Big) &= -\sum_i \boldsymbol{a}_i \partial_i\Big(\frac{\partial\Phi}{\partial x^n} b_{I'}\, dx^n \wedge\, dx^{I'}\Big) + \sum_{i,j,l,m} \Gamma^j_{il} g_{jm} g^{il} \boldsymbol{a}_m \Big(\frac{\partial\Phi}{\partial x^n} b_{I'}\, dx^n \wedge\, dx^{I'}\Big) \\ &\qquad - \sum_{i,j,l,m} \Gamma^j_{il} g_{jm} \boldsymbol{a}^*_l \boldsymbol{a}_i \boldsymbol{a}_m \Big(\frac{\partial\Phi}{\partial x^n} b_{I'}\, dx^n \wedge\, dx^{I'}\Big) \\ &= -\sum_i \boldsymbol{a}_i \partial_i\Big(\frac{\partial\Phi}{\partial x^n} b_{I'}\, dx^n \wedge\, dx^{I'}\Big) + 0 \\ &= -\boldsymbol{i}_{\nabla x^n} \frac{\partial\Phi}{\partial x^n}\frac{\partial b_{I'}}{\partial x^n}\, dx^n \wedge\, dx^{I'} = -\frac{\partial\Phi}{\partial x^n}\frac{\partial b_{I'}}{\partial x^n}\, dx^{I'}. \end{aligned} \tag{4-38}$$

We used on the last line the fact that G_0^{-1} is block diagonal with $g^{nn} \equiv 1$.

Now look at the third term of the right-hand side of (4-37) and write, in view of (A-5),

$$(d_T\Phi - d\tilde{\Phi}) \wedge b_I\, dx^I = \sum_{i=1}^{n-1} \left(\frac{\partial \Phi}{\partial x^i}(x) - \frac{\partial \Phi}{\partial x^i}(x',0) \right) b_I\, dx^i \wedge dx^I =: \sum_{i=1}^{n-1} \alpha_i b_I\, dx^i \wedge dx^I,$$

where, $\alpha_i = \dfrac{\partial \Phi}{\partial x^i}(x) - \dfrac{\partial \Phi}{\partial x^i}(x',0)$ for i in $\{1,\dots,n-1\}$. Hence we have

$$\alpha_j(x',0) \equiv 0 \quad \text{for all } j \in \{1,\dots,n-1\}.$$

Taking (4-35) again into account, we therefore obtain, on $\partial\Omega$,

$$\begin{aligned} d^*((d_T\Phi - d\tilde{\Phi}) \wedge b_I\, dx^I)(x',0) & \\ = -\sum_i \boldsymbol{a}_i \partial_i \sum_{j=1}^{n-1} \alpha_j b_I\, dx^j \wedge dx^I & \\ + \Big(\sum_{i,j,l,m} \Gamma_{il}^j g_{jm} g^{il} \boldsymbol{a}_m - \sum_{i,j,l,m} \Gamma_{il}^j g_{jm} \boldsymbol{a}_l^* \boldsymbol{a}_i \boldsymbol{a}_m \Big) \sum_{j=1}^{n-1} \alpha_j b_I\, dx^j \wedge dx^I & \\ = -\sum_i \boldsymbol{a}_i \partial_i \sum_{j=1}^{n-1} \alpha_j b_I\, dx^j \wedge dx^I = -\boldsymbol{a}_n \sum_{j=1}^{n-1} \frac{\partial}{\partial x^n}(\alpha_j b_I)\, dx^j \wedge dx^I, & \end{aligned}$$

where we used $\alpha_j(x',0) \equiv 0$ twice on the last line. Now, since $g^{ni} = g^{in} = 0$ for i in $\{1,\dots,n-1\}$, we can write, for all $I' \in \mathcal{I}'$,

$$\boldsymbol{a}_n dx^{I'} = \boldsymbol{i}_{\nabla x^n} dx^{I'} = 0.$$

This implies

$$\begin{aligned} d^*((d_T\Phi - d\tilde{\Phi}) \wedge b_I\, dx^I)(x',0) &= -\boldsymbol{a}_n \sum_{j=1}^{n-1} \frac{\partial}{\partial x^n}(\alpha_j b_I)\, dx^j \wedge dx^I = -\boldsymbol{a}_n \sum_{j=1}^{n-1} \frac{\partial}{\partial x^n}(\alpha_j b_{I_n})\, dx^j \wedge dx^{I_n} \\ &= (-1)^{p+1} \sum_{j=1}^{n-1} \frac{\partial}{\partial x^n}(\alpha_j b_{I_n})\, dx^j \wedge dx^{I_n \setminus \{n\}} \\ &= (-1)^{p+1} \sum_{j=1}^{n-1} b_{I_n} \frac{\partial \alpha_j}{\partial x^n}(x',0)\, dx^j \wedge dx^{I_n \setminus \{n\}} \\ &=: \tilde{\ell}_{I'}(x',0)\, dx^{I'}, \end{aligned} \tag{4-39}$$

where the $\tilde{\ell}_{I'}$ are $\mathscr{C}^\infty(\partial\Omega)$-linear combinations of the b_{I_n} (for I_n in $\mathcal{I}_n$) that do not depend on the $b_{I'}$ (for I' in $\mathcal{I}'$).

Combining (4-37), (4-38), and (4-39) leads to the result announced in Lemma 4.6. $\square$

Proof of Proposition 4.4. Having in mind Lemma 4.6, we now look at the term $-2\mathcal{L}^*_{\nabla\tilde{\Phi}} + \mathcal{R}_4$. Again by Proposition B.3, we can write

$$-2\mathcal{L}^*_{\nabla\tilde{\Phi}} = 2\mathcal{L}_{\nabla\tilde{\Phi}} + \mathcal{R}_5 = 2\mathcal{L}_{\nabla\tilde{\Phi}} \otimes \mathrm{Id} + \mathcal{R}_5 + \mathcal{R}_6,$$

where $\mathcal{R}_5 = \mathcal{R}_5^T + \mathcal{R}_5^N$ and $\mathcal{R}_6 = \mathcal{R}_6^T + \mathcal{R}_6^N$ are order-zero differential operators satisfying, for $i \in \{5, 6\}$ (since $b_I\, dx^I = b_{I_n} dx^{I_n}$ on the boundary),

$$\begin{aligned} \boldsymbol{t}(\mathcal{R}_i(b_I\, dx^I)) &= b_{I_n}(x', 0)\mathcal{R}_i^N(dx^{I_n}) = \tilde{\ell}^{i\prime}_{I'}(x', 0)\, dx^{I'}, \\ \boldsymbol{n}(\mathcal{R}_i(b_I\, dx^I)) &= b_{I_n}(x', 0)\mathcal{R}_i^T(dx^{I_n}). \end{aligned}$$

Here the $\tilde{\ell}^{i\prime}_{I'}(x', 0)$ are $\mathscr{C}^\infty(\partial\Omega)$-linear combinations of the b_{I_n} (for I_n in $\mathcal{I}_n$) that do not depend on the $b_{I'}$ (for I' in $\mathcal{I}'$). Moreover, since $\tilde{\Phi}$ satisfies the assumptions of Corollary B.5, $\mathcal{R}_5^T$ and $\mathcal{R}_6^T$ are given on the boundary, in the coordinates (x', x^n), by

$$\mathcal{R}_5^T = \begin{pmatrix} & & 0 \\ & \mathcal{R}_5^{T'} & \vdots \\ & & 0 \\ 0 & \cdots \;\; 0 & 0 \end{pmatrix}^{(p)} - \zeta(x')\,\mathrm{Id} \quad \text{and} \quad \mathcal{R}_6^T = \begin{pmatrix} & & 0 \\ & \mathcal{R}_6^{T'} & \vdots \\ & & 0 \\ 0 & \cdots \;\; 0 & 0 \end{pmatrix}^{(p)},$$

where

$$\zeta(0) = -2\,\mathrm{Tr}(\mathrm{Hess}(\tilde{\Phi}|_{\partial\Omega})(0)) = -2\,\mathrm{Tr}(\mathrm{Hess}(\varphi)(0)), \;\; \mathcal{R}_5^{T'}(0) = -4\,\mathrm{Hess}(\varphi)(0), \;\; \mathcal{R}_6^{T'}(0) = 2\,\mathrm{Hess}(\varphi)(0).$$

Set $\mathcal{R}_{\mathrm{Dir}} = \mathcal{R}_4 + \mathcal{R}_5 + \mathcal{R}_6$ and $\tilde{\ell}^{(3)}_{I'} = \tilde{\ell}'_{I'} + \tilde{\ell}^{5\prime}_{I'} + \tilde{\ell}^{6\prime}_{I'}$ for I' in $\mathcal{I}'$. Then $\mathcal{R}_{\mathrm{Dir}}$ is an order-zero differential operator satisfying

$$-2\mathcal{L}_{\nabla\tilde{\Phi}} + \mathcal{R}_4 = 2\mathcal{L}_{\nabla\tilde{\Phi}} \otimes \mathrm{Id} + \mathcal{R}_{\mathrm{Dir}} \tag{4-40}$$

and

$$\begin{aligned} \boldsymbol{t}(\mathcal{R}_{\mathrm{Dir}}(b_I\, dx^I)) &= b_{I_n}(x', 0)\mathcal{R}_{\mathrm{Dir}}^N(dx^{I_n}) = \tilde{\ell}^{(3)}_{I'}(x', 0)\, dx^{I'}, \\ \boldsymbol{n}(\mathcal{R}_{\mathrm{Dir}}(b_I\, dx^I)) &= b_{I_n}(x', 0)\mathcal{R}_{\mathrm{Dir}}^T(dx^{I_n}), \end{aligned} \tag{4-41}$$

where the $\tilde{\ell}^{(3)}_{I'}$ are $\mathscr{C}^\infty(\partial\Omega)$-linear combinations of the b_{I_n} (for I_n in $\mathcal{I}_n$) that do not depend on the $b_{I'}$ (for I' in $\mathcal{I}'$). Moreover, $\mathcal{R}_{\mathrm{Dir}}^T$ is given on the boundary, in the coordinates (x', x^n), by

$$\mathcal{R}_{\mathrm{Dir}}^T(x', 0) = \begin{pmatrix} & & 0 \\ & \mathcal{R}_{\mathrm{Dir}}^{T'}(x', 0) & \vdots \\ & & 0 \\ 0 & \cdots \;\; 0 & \delta(x') \end{pmatrix}^{(p)} - \kappa_2(x')\,\mathrm{Id},$$

where

$$\begin{aligned} &\delta(0) = 0, \\ &\kappa_2(0) = \kappa(0) + \zeta(0) = \mathrm{Tr}\,(\mathrm{Hess}(f|_{\partial\Omega} + \varphi)(0)) - 2\,\mathrm{Tr}\,(\mathrm{Hess}(\varphi)(0)) = \mathrm{Tr}\,(\mathrm{Hess}(f|_{\partial\Omega} - \varphi)(0)), \\ &\mathcal{R}_{\mathrm{Dir}}^{T'}(0) = \mathcal{R}_4^{T'}(0) + \mathcal{R}_5^{T'}(0) + \mathcal{R}_6^{T'}(0) = 2\,\mathrm{Hess}(f|_{\partial\Omega} + \varphi)(0) - 2\,\mathrm{Hess}(\varphi)(0) = 2\,\mathrm{Hess}(f|_{\partial\Omega})(0). \end{aligned}$$

We now look at the term $2\mathcal{L}_{\nabla\tilde{\Phi}} \otimes \mathrm{Id}$. By Cartan's formula (2-6),

$$(2\mathcal{L}_{\nabla\tilde{\Phi}} \otimes \mathrm{Id})b = db_{I'}(\nabla\tilde{\Phi})\, dx^{I'} + db_{I_n}(\nabla\tilde{\Phi})\, dx^{I_n},$$

and, using the boundary conditions satisfied by the b_I (for I in $\mathcal{I}$) and the fact that $\nabla\tilde{\Phi}$ is a tangential vector field, we obtain

$$(2\mathcal{L}_{\nabla\tilde{\Phi}} \otimes \mathrm{Id})b = \sum_{i=1}^{n-1} \frac{\partial b_{I_n}}{\partial x^i}(\nabla\tilde{\Phi})_i \, dx^{I_n} = 2\mathcal{L}_{\nabla\tilde{\Phi}} \otimes \mathrm{Id}\, b_{I_n}\, dx^{I_n}. \tag{4-42}$$

Set $\ell_{I'} = -\tilde{\ell}_{I'} + \frac{1}{2}\tilde{\ell}^{(3)}_{I'}$ and write

$$(-2\mathcal{L}^*_{\nabla\Phi} + \mathcal{R}_3)b = -2(\mathcal{L}^*_{\nabla\Phi} - \mathcal{L}^*_{\nabla\tilde{\Phi}})b + (-2\mathcal{L}^*_{\nabla\tilde{\Phi}} + \mathcal{R}_3)b.$$

Using (4-40)–(4-42), Proposition 4.4 is then a direct consequence of Lemma 4.6. □

4F. ***Proof of Theorem 1.2.*** Although the calculations are different, the scheme of the proof is the same as for Theorem 1.1. Consider first a WKB-approximation for

$$(\Delta^{(p)}_{f,h} - E(h))u^{\mathrm{WKB}}_p = e^{-\Phi/h}\mathbb{O}(h^\infty), \tag{4-43}$$

with $E(h) = O(h^2)$ and the boundary conditions (1-5) and (1-6).

From

$$d^*_{f,h}(e^{-\Phi/h}a^k) = e^{-\Phi/h}\big(hd^*a^k + \boldsymbol{i}_{\nabla(f+\Phi)}a^k\big) \quad \text{for all } k \in \mathbb{N},$$

where, due to (1-5) and (4-28), a^k is a normal form and $\nabla(f+\Phi)$ is a tangential vector field, the second boundary condition (1-6) corresponds to

$$\boldsymbol{t}(d^*a^k) = 0 \quad \text{for all } k \in \mathbb{N}, \tag{4-44}$$

We now recall that, in the notation of Section B2 and Section 4E1,

$$e^{\Phi/h}\Delta_{f,h}e^{-\Phi/h} = h^2(d+d^*)^2 + h(2\mathcal{L}_{\nabla\Phi} \otimes \mathrm{Id} + \mathcal{R}) = h^2(d+d^*)^2 + h(-2\mathcal{L}^*_{\nabla\Phi} + \mathcal{R}_4).$$

By looking for $E(h) \sim \sum_{k=1}^{\infty} h^{k+1}E_k$, the interior equation (4-43) reads, as in Section 4C,

$$e^{\Phi/h}(\Delta_{f,h} - E(h))e^{-\Phi/h} = h^2\big((d+d^*)^2 - h^{-2}E(h)\big) + h(2\mathcal{L}_{\nabla\Phi} \otimes \mathrm{Id} + \mathcal{R}).$$

Hence, as in Section 4C, the construction of an interior WKB solution in Ω is standard as an inductive Cauchy problem, once the a^k are known on $\partial\Omega$, since the noncharacteristic Cauchy problems

$$\big(2\mathcal{L}_{\nabla\Phi} \otimes \mathrm{Id} + \mathcal{R}\big)a^k = -(d+d^*)^2a^{k-1} + \sum_{\ell=1}^{k} E_\ell a^{k-\ell} \quad \text{in } \overline{\Omega} \tag{4-45}$$

are solved by induction with the convention $a_{-1} = 0$.

The problem is then reduced to solving the system made of the boundary conditions (4-31) and (4-44) and of the compatibility equation

$$\big(-2\mathcal{L}^*_{\nabla\Phi} + \mathcal{R}_4\big)\, a^k = -(d+d^*)^2a^{k-1} + \sum_{\ell=1}^{k} E_\ell a^{k-\ell} \quad \text{on } \partial\Omega \tag{4-46}$$

(see Section 4E1 for the notation).

Owing to Proposition 4.4 (with the notation of Section 4E3) and to (4-28), the system (4-46), (4-31), (4-44) is equivalent to the following differential system on $\partial\Omega$:

$$\begin{aligned}
-\boldsymbol{n}(d+d^*)^2 a^{k-1} + \sum_{\ell=1}^{k} E_\ell a^{k-\ell} &= (2\mathcal{L}_{\nabla\tilde{\Phi}} \otimes \mathrm{Id} + \mathcal{R}^T_{\mathrm{Dir}}) a^k_{I_n}\, dx^{I_n} + 2\frac{\partial f}{\partial x^n}\, dx^n \wedge d^* a^k,\\
-\boldsymbol{t}(d+d^*)^2 a^{k-1} - 2\ell_{I'}(x',0)\, dx^{I'} &= -2\frac{\partial f}{\partial n}\frac{\partial a^k_{I'}}{\partial x^n}\, dx^{I'},\\
a^k_{I'}|_{\partial\Omega} \equiv 0 \quad \text{and} \quad \boldsymbol{t}(d^* a^k) &= 0 \quad \text{for all } k \in \mathbb{N},
\end{aligned}$$

where the $\ell_{I'}$ are $\mathscr{C}^\infty(\partial\Omega)$-linear combinations of the $a^k_{I_n}$ (for I_n in $\mathcal{I}_n$) which do not depend on the $a^k_{I'}$ (for I' in $\mathcal{I}'$). Note also, according to Lemma 4.5, that the first line of the last system reads

$$-\boldsymbol{n}(d+d^*)^2 a^{k-1} + \sum_{\ell=1}^{k} E_\ell a^{k-\ell} = (2\mathcal{L}_{\nabla\tilde{\Phi}} \otimes \mathrm{Id} + \mathcal{R}^T_{\mathrm{Dir}}) a^k_{I_n}\, dx^{I_n}. \tag{4-47}$$

Moreover, since $dx^i = d\bar{x}^i$ for $i \in \{1,\dots,n-1\}$ at the point U, it follows from Corollary B.5, (4-5), and the results in [Helffer and Sjöstrand 1985, pages 271–275] that $\mathcal{R}^T_{\mathrm{Dir}}(0)$ restricted to normal forms is symmetric with the one-dimensional kernel $\mathbb{R}dx^1 \wedge \cdots \wedge dx^{p-1} \wedge dx^n$.

Since $a^k_{I_n} dx^{I_n}$ is normal and $2\mathcal{L}_{\nabla\tilde{\Phi}} \otimes \mathrm{Id}$ only differentiates the $a^k_{I_n}$ tangentially, because

$$(2\mathcal{L}_{\nabla\tilde{\Phi}} \otimes \mathrm{Id}) a^k_{I_n} dx^{I_n} = \sum_{i=1}^{n-1} \frac{\partial a^k_{I_n}}{\partial x^i} (\nabla\tilde{\Phi})_i\, dx^{I_n},$$

it turns out (4-47) can be rewritten as a tangential system that can be solved according to the analysis of the boundaryless case done in [Helffer and Sjöstrand 1985]. Here are the details: thanks to Lemma 4.5, the complete system becomes equivalent to

$$\left.\begin{aligned}
(2\mathcal{L}_{\nabla\tilde{\Phi}} \otimes \mathrm{Id} + \mathcal{R}^T_{\mathrm{Dir}}) a^k_{I_n} dx^{I_n} &= -\boldsymbol{n}(d+d^*)^2 a^{k-1} + \sum_{\ell=1}^{k-1} E_\ell a^{k-\ell} + E_k a^0 \text{on } \partial\Omega,\\
(2\mathcal{L}_{\nabla\Phi} \otimes \mathrm{Id} + \mathcal{R}) a^k &= -(d+d^*)^2 a^{k-1} + \sum_{\ell=1}^{k} E_\ell a^{k-\ell} \qquad \text{on } \overline{\Omega},\\
a_{I'}|_{\partial\Omega} &\equiv 0 \qquad \text{for all } I' \in \mathcal{I}'.
\end{aligned}\right\} \tag{$\mathcal{S}_{\mathrm{Dir}}$}$$

The first line is again a homogeneous degenerate matrix transport equation which can be solved following [Helffer and Sjöstrand 1985; Helffer 1988]: for $k = 0$, take

$$a^0(0) = dx^1 \wedge \cdots \wedge dx^{p-1} \wedge dx^n \in \mathrm{Ker}(\mathcal{R}^T_{\mathrm{Dir}}(0)). \tag{4-49}$$

The formulation of Theorem 1.2 is just a coordinate-free rewriting of this condition for $a^0(U)$. Recall that, *in our coordinate system* (see Proposition 4.4), at $U \cong 0$,

$$\begin{aligned}
\mathcal{R}^T_{\mathrm{Dir}}(U) = 2\begin{pmatrix} & & 0\\ & \mathrm{Hess}(f|_{\partial\Omega})(U) & \vdots\\ 0 & \cdots & 0\end{pmatrix}^{(p)} &- \mathrm{Tr}(\mathrm{Hess}(f|_{\partial\Omega} - \Phi|_{\partial\Omega})(U))\\
&= 2A^{(p)} - \mathrm{Tr}(\mathrm{Hess}(f|_{\partial\Omega} - \Phi|_{\partial\Omega})(U)),
\end{aligned}$$

and that, for a normal p-form $dx^{I_n} = dx^{i_1} \wedge \cdots \wedge dx^{i_{p-1}} \wedge dx^n$,

$$A^{(p)} dx^{I_n} = (A\, dx^{i_1}) \wedge dx^{i_2} \cdots \wedge dx^n + \cdots + dx^{i_1} \wedge \cdots \wedge (A\, dx^{i_{p-1}}) \wedge dx^n + 0,$$

where, for $\ell \in \{1, \dots, p-1\}$,

$$A\, dx^{i_\ell} = (\mathrm{Hess}(f|_{\partial\Omega})(U))\ dx^{i_\ell} + 0.dx^n = \big(\mathrm{Hess}(f|_{\partial\Omega})(U)\, j^*\big)\ dx^{i_\ell}.$$

Finally, as in the analogous part of the proof in the Neumann case, the writing $(\mathrm{Hess}(f|_{\partial\Omega})(U))\, dx^{i_\ell}$ is a slight abuse of notation, the proper one being $(\mathrm{Hess}(f|_{\partial\Omega})(U)\, j^*)\, dx^{i_\ell}$.

Then, for $k > 0$, choose E_k such that the compatibility condition

$$-\boldsymbol{n}(d + d^*)^2 a^{k-1}(0) + \sum_{\ell=1}^{k-1} E_\ell a^{k-\ell}(0) + E_k a^0(0) \in \big(\mathrm{Ker}(\mathcal{R}^T_{\mathrm{Dir}}(0))\big)^\perp$$

is satisfied and take $a^k_{I_n}(0)\, dx^{I_n}$ in

$$(\mathcal{R}^T_{\mathrm{Dir}}(0))^{-1}\Big(-\boldsymbol{n}(d + d^*)^2 a^{k-1}(0) + \sum_{\ell=1}^{k-1} E_\ell a^{k-\ell}(0) + E_k a^0(0)\Big).$$

Thus, at every step $k \in \mathbb{N}$, the first and the third line of $(\mathcal{S}_{\mathrm{Dir}})$ fully determine the Cauchy data $a^k(x', 0)$ and the number E_k. The second line solves the interior problem with these Cauchy data and contains, with the two other lines, thanks to Lemma 4.5, the second trace condition (4-44). Checking $E(h) = O(h^\infty)$ is then identical to the end of the proof of Theorem 1.1 done in Section 4C after choosing a cut-off function χ which satisfies $\nabla\chi = \nabla_T \chi$ on the boundary $\partial\Omega$. □

Appendices: Computations in adapted local coordinate systems

In the two appendices below we work in an adapted local coordinate system (x', x^n) around $U \in \partial\Omega$ so as to be able to apply the results both to the Neumann and Dirichlet cases.

Appendix A. A modified Agmon distance

Define $\tilde{\Phi}$ around U in the coordinates (x', x^n) by

$$\tilde{\Phi}(x', x^n) = \Phi(x', 0) \quad \text{for all } x = (x', x^n), \tag{A-1}$$

and note the following relation satisfied for all x around U, in the coordinates (x', x^n), due to the form of $G_0^{\pm 1}$ (see Remark 3.3):

$$d\tilde{\Phi}(x) = d_T \tilde{\Phi}(x) + \frac{\partial\tilde{\Phi}}{\partial x^n}(x) dx^n = d_T \tilde{\Phi}(x),$$

$$\nabla\tilde{\Phi}(x) = \nabla_T \tilde{\Phi}(x) + \frac{\partial\tilde{\Phi}}{\partial x^n}(x) \frac{\partial}{\partial x^n} = \nabla_T \tilde{\Phi}(x).$$

For a vector (or a vector field) $X = \sum_{i=1}^{n} X_i \dfrac{\partial}{\partial x^i}$, with the identification $X = \begin{pmatrix} X_1 \\ \vdots \\ X_N \end{pmatrix}$, the tangential and normal parts of X are defined as

$$X_T = \begin{pmatrix} X_1 \\ \vdots \\ X_{n-1} \\ 0 \end{pmatrix}, \qquad X_N = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ X_n \end{pmatrix}.$$

Similarly, for a (n,n)-matrix $A(x) = (a_{ij}(x))_{i,j}$, define $A_T(x)$ and $A_N(x)$ by

$$A_T = \begin{pmatrix} & & & 0 \\ & A' & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & a_{nn} \end{pmatrix}, \qquad A_N = \begin{pmatrix} & & & a_{1n} \\ & [0] & & \vdots \\ & & & a_{n-1n} \\ a_{n1} & \cdots & a_{nn-1} & 0 \end{pmatrix}.$$

Recall moreover that, for a vector (or a vector field) X and a $\mathscr{C}^\infty$ function ψ, the identification $\langle \nabla\psi \mid X\rangle_{g_0} = d\psi(X)$ leads to

$$\nabla\psi = G_0^{-1} \begin{pmatrix} \partial\psi/\partial x^1 \\ \vdots \\ \partial\psi/\partial x^n \end{pmatrix}.$$

Hence, due to the form of G_0^{-1} (see Remark 3.3), the following relations are satisfied:

$$(\nabla\psi)_T = \nabla_T\psi = G_0^{-1} \begin{pmatrix} \partial\psi/\partial x^1 \\ \vdots \\ \partial\psi/\partial x^{n-1} \\ 0 \end{pmatrix}, \qquad (\nabla\psi)_N = \frac{\partial\psi}{\partial x^n}\frac{\partial}{\partial x^n} = G_0^{-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \partial\psi/\partial x^n \end{pmatrix}.$$

In the Neumann case, we will compare $\mathscr{L}_{\nabla\Phi}$ and $\mathscr{L}_{\nabla\tilde{\Phi}}$ and the following relations can sometimes be convenient:

$$\nabla\Phi - \nabla\tilde{\Phi} = G_0^{-1} \begin{pmatrix} \dfrac{\partial\Phi}{\partial x^1}(x) - \dfrac{\partial\Phi}{\partial x^1}(x',0) \\ \vdots \\ \dfrac{\partial\Phi}{\partial x^{n-1}}(x) - \dfrac{\partial\Phi}{\partial x^{n-1}}(x',0) \\ \dfrac{\partial\Phi}{\partial x^n}(x) \end{pmatrix}, \tag{A-2}$$

$$\nabla_T\Phi - \nabla\tilde{\Phi} = G_0^{-1} \begin{pmatrix} \dfrac{\partial\Phi}{\partial x^1}(x) - \dfrac{\partial\Phi}{\partial x^1}(x',0) \\ \vdots \\ \dfrac{\partial\Phi}{\partial x^{n-1}}(x) - \dfrac{\partial\Phi}{\partial x^{n-1}}(x',0) \\ 0 \end{pmatrix}. \tag{A-3}$$

We will compare $\mathscr{L}^*_{\nabla\Phi}$ and $\mathscr{L}^*_{\nabla\tilde{\Phi}}$ in the Dirichlet case and the following relations are also convenient:

$$d\Phi - d\tilde{\Phi} = \sum_{i=1}^{n-1} \Big(\frac{\partial \Phi}{\partial x^i}(x) - \frac{\partial \Phi}{\partial x^i}(x',0) \Big)\, dx^i + \frac{\partial \Phi}{\partial x^n}(x)\, dx^n, \tag{A-4}$$

$$d_T\Phi - d\tilde{\Phi} = \sum_{i=1}^{n-1} \Big(\frac{\partial \Phi}{\partial x^i}(x) - \frac{\partial \Phi}{\partial x^i}(x',0) \Big)\, dx^i. \tag{A-5}$$

Appendix B. About $\mathscr{L} + \mathscr{L}^*$

B1. ***For a general $\mathscr{C}^\infty$ function h.*** Here we give similar results to those found in [Helffer and Sjöstrand 1985, Appendix A].

Let h be a $\mathscr{C}^\infty$ function from $\overline{\Omega}$ to $\mathbb{R}$ and write

$$\nabla h = \sum_{i=1}^{n} (\nabla h)_i \frac{\partial}{\partial x^i}.$$

Following [Helffer and Sjöstrand 1985], we make the following algebraic definition:

Definition B.1. For a Euclidean space $(E, \langle \cdot \mid \cdot \rangle)$ and $A \in \mathscr{L}(E)$, $A^{(p)}$ and $\Gamma^{(p)}(A)$ denote respectively the linear application $A^{(p)} \in \mathscr{L}(\Lambda^p E)$ and the application $\Gamma^{(p)}(A) = A \otimes \cdots \otimes A$:

$$\begin{aligned} A^{(p)}(\omega_1 \wedge \cdots \wedge \omega_p) &= (A\omega_1 \wedge \cdots \wedge \omega_p) + \cdots + (\omega_1 \wedge \cdots \wedge A\omega_p), \\ \Gamma^{(p)}(A)(\omega_1 \wedge \cdots \wedge \omega_p) &= (A\omega_1) \wedge \cdots \wedge (A\omega_p), \end{aligned}$$

with the obvious convention $A^{(0)} = 0$ and $\Gamma^{(0)}(A) = 1$.

Remark B.2. Under the canonical identification $\Lambda^1 E = E$, note that $A^{(1)} = A$. Moreover, if A^* denotes the adjoint of A according to the scalar product on E, the adjoint of $A^{(p)}$ is simply $(A^{(p)})^* = (A^*)^{(p)} =: A^{(p),*}$. Recall that $\Lambda^p E$ is a Euclidean space with the scalar product $\langle \cdot \mid \cdot \rangle_p$:

$$\langle \omega_1 \wedge \cdots \wedge \omega_p \mid \mu_1 \wedge \cdots \wedge \mu_p \rangle_p = \det\big(\langle \omega_i \mid \mu_j \rangle \big)_{i,j}.$$

We also remark that for a p-form $a_I^k\, dx^I = a_{I'}^k\, dx^{I'} + a_{I_n}^k\, dx^{I_n}$ we have, in the notation of Appendix A,

$$A^{(p)} = A_T^{(p)} + A_N^{(p)}$$

and

$$\begin{aligned} \boldsymbol{t}(A^{(p)}(a_I^k\, dx^I)) &= a_{I'}^k(x',0) A_T^{(p)}(dx^{I'}) + a_{I_n}^k(x',0) A_N^{(p)}(dx^{I_n}), \\ \boldsymbol{n}(A^{(p)}(a_I^k\, dx^I)) &= a_{I_n}^k(x',0) A_T^{(p)}(dx^{I_n}) + a_{I'}^k(x',0) A_N^{(p)}(dx^{I'}). \end{aligned}$$

For any order-zero differential operator $\mathscr{A} = A^{(p)} + \psi\,\mathrm{Id}$, where ψ is a $\mathscr{C}^\infty$ function, we define the order-zero differential operators

$$\mathscr{A}^T = A_T^{(p)} + \psi\,\mathrm{Id} \quad \text{and} \quad \mathscr{A}^N = A_N^{(p)}.$$

(If $\psi \equiv 0$ then $\mathscr{A}^T$ coincides with $A_T^{(p)}$ and $\mathscr{A}^N$ with $A_N^{(p)}$.) Our aim is to work with tangential forms in the Neumann case (i.e., $a_I^k\,dx^I = a_{I'}^k dx^{I'}$ on $\partial\Omega$) and with normal forms in the Dirichlet case (i.e., $a_I^k\,dx^I = a_{I_n}^k\,dx^{I_n}$ on $\partial\Omega$). Hence, for any tangential form in the Neumann case we write

$$\begin{aligned} \boldsymbol{t}(\mathscr{A}(a_I^k\,dx^I)) &= a_{I'}^k(x',0)A_T^{(p)}(dx^{I'}) + \psi(x',0)a_{I'}^k(x',0)\,dx^{I'} = \boldsymbol{t}(\mathscr{A}^T(a_I^k\,dx^I)),\\ \boldsymbol{n}(\mathscr{A}(a_I^k\,dx^I)) &= a_{I'}^k(x',0)A_N^{(p)}(dx^{I'}) = \boldsymbol{n}(\mathscr{A}^N(a_I^k\,dx^I)), \end{aligned} \tag{B-1}$$

and for any normal form in the Dirichlet case, we write

$$\begin{aligned} \boldsymbol{t}(\mathscr{A}(a_I^k\,dx^I)) &= a_{I_n}^k(x',0)A_N^{(p)}(dx^{I'}) = \boldsymbol{t}(\mathscr{A}^N(a_I^k\,dx^I)),\\ \boldsymbol{n}(\mathscr{A}(a_I^k\,dx^I)) &= a_{I_n}^k(x',0)A_T^{(p)}(dx^{I_n}) + \psi(x',0)a_{I_n}^k(x',0)\,dx^{I_n} = \boldsymbol{n}(\mathscr{A}^T(a_I^k\,dx^I)). \end{aligned} \tag{B-2}$$

Proposition B.3. *In the coordinates (x',x^n), we have $\mathscr{L}_{\nabla h} = \mathscr{L}_{\nabla h}\otimes \mathrm{Id} + \mathscr{R}_h$ and*

$$\mathscr{L}_{\nabla h} + \mathscr{L}_{\nabla h}^* = \mathscr{R}_h + \mathscr{R}_h^* - \left(\sum_{i=1}^n \left(\frac{\partial(\nabla h)_i}{\partial x^i} + \frac{1}{2}(\nabla h)_i \frac{\partial \ln \det G_0}{\partial x^i}\right)\right)\mathrm{Id} - \sum_{i=1}^n (\nabla h)_i \Big(G_0 \frac{\partial (G_0^{-1})}{\partial x^i}\Big)^{(p)},$$

where $(\mathscr{L}_{\nabla h}\otimes\mathrm{Id})a_I^k\,dx^I = (\mathscr{L}_{\nabla h}(a_I^k))\,dx^I$, $\mathscr{R}_h$ is the order-zero differential operator given by the matrix

$$\mathscr{R}_h(x) = \Big(\frac{\partial(\nabla h)_j}{\partial x^i}\Big)_{i,j}^{(p)} =: A_h^{(p)},$$

and $\Big(\frac{\partial(\nabla h)_j}{\partial x^i}\Big)_{i,j}$ and $G_0\frac{\partial(G_0^{-1})}{\partial x^i}$ are viewed as endomorphisms of $T_x^*\overline{\Omega}$. Further, the matrix of $\mathscr{R}_h^*$ is

$$\mathscr{R}_h^* := A_h^{(p),*} = (G_0{}^t\!A_h G_0^{-1})^{(p)}.$$

Remark B.4. According to the computations in Appendix A, $(\nabla h)_n = \partial h/\partial x^n$. Moreover, due to the form of $G_0^{\pm 1}$, note that

$$\mathscr{R}_h + \mathscr{R}_h^* - \sum_{i=1}^n (\nabla h)_i \Big(G_0 \frac{\partial[G_0^{-1}]}{\partial x^i}\Big)^{(p)}$$

is given by the matrix

$$\begin{pmatrix} A_h' + G_0'{}^t A_h' G_0^{-1'} - \sum_{i=1}^n (\nabla h)_i G_0' \dfrac{\partial[G_0^{-1'}]}{\partial x^i} & \Big(\dfrac{\partial^2 h}{\partial x^n \partial x^i}\Big)_{i,1} + G_0'\Big(\dfrac{\partial(\nabla h)_i}{\partial x^n}\Big)_{i,1} \\ \Big(\dfrac{\partial(\nabla h)_j}{\partial x^n}\Big)_{1,j} + \Big(\dfrac{\partial^2 h}{\partial x^n \partial x^j}\Big)_{1,j} G_0^{-1'} & \dfrac{\partial^2 h}{(\partial x^n)^2} \end{pmatrix}^{(p)}.$$

Corollary B.5. *In the coordinates (x',x^n), assume that the function h admits a critical point at 0, that $\partial h/\partial x^n \equiv 0$ on the boundary $\partial\Omega$, and that $((\partial^2 h)/(\partial x^n)^2)(0) = 0$. Then the following relations are true*:

$$\mathscr{R}_h(0) = \mathscr{R}_h^*(0) = \begin{pmatrix} & & 0 \\ & \mathrm{Hess}(h|_{\partial\Omega})(0) & \vdots \\ 0 & \cdots & 0 \end{pmatrix}^{(p)}$$

and

$$(\mathscr{L}_{\nabla h}+\mathscr{L}^*_{\nabla h})(0)=2\mathscr{R}_h(0)-\mathrm{Tr}(\mathrm{Hess}(h|_{\partial\Omega})(0))\,\mathrm{Id}\,.$$

Proof. Since (x',x^n) are adapted local coordinates around $U\cong 0$ and 0 is a critical point of h, note first that, for all i in $\{1,\dots,n\}$,

$$(\nabla h)_i=\sum_{j=1}^n g^{ij}\frac{\partial h}{\partial x^j}=\frac{\partial h}{\partial x^i}+\mathscr{O}(|x|^2).$$

This implies

$$\mathscr{R}_h(x)=\Big(\frac{\partial(\nabla h)_j}{\partial x^i}\Big)^{(p)}_{i,j}=(\mathrm{Hess}(h))^{(p)}+\mathscr{O}(|x|).$$

At 0, in particular, since $\partial h/\partial x^n\equiv 0$ on the boundary and $((\partial^2 h)/(\partial x^n)^2)(0)=0$, we have

$$\mathscr{R}_h(0)=\begin{pmatrix} & & 0\\ & \mathrm{Hess}(h|_{\partial\Omega})(0) & \vdots\\ 0 & \cdots & 0\end{pmatrix}^{(p)}.$$

Moreover, we deduce from $G_0^{\pm1}(0)=\mathrm{Id}_n$ and the symmetry of $\mathrm{Hess}(h|_{\partial\Omega})(0)$,

$$\mathscr{R}_h^*(0)=\mathscr{R}_h(0).$$

At last, we obtain from $\dfrac{\partial^2 h}{(\partial x^n)^2}(0)=0$ that

$$-\Big(\sum_{i=1}^n\frac{\partial(\nabla h)_i}{\partial x^i}\Big)\mathrm{Id}=-\,\mathrm{Tr}\,(\mathrm{Hess}(h|_{\partial\Omega})(0))\quad\text{ at } 0,$$

which leads to the end of the proof, using that, for all i in $\{1,\dots,n\}$,

$$(\nabla h)_i(0)=\frac{\partial h}{\partial x^i}(0)=0.\qquad\square$$

Proof of Proposition B.3. The first equality is proved in [Helffer and Sjöstrand 1985, pages 334–336]. There is also a proof of the second equality in the same paper, but we need to be more precise here. From the first equality, let us deduce

$$\mathscr{L}^*_{\nabla h}=(\mathscr{L}_{\nabla h}\otimes\mathrm{Id})^*+\mathscr{R}^*_h.$$

Remarking that the scalar product of two p-forms ω and η is given by

$$\langle\omega\mid\eta\rangle_{g_0}=\langle\omega\mid\Gamma^{(p)}(G_0^{-1})\eta\rangle_{g_e},$$

where g_e is the Euclidean metric $\sum_{i=1}^n(dx^i)^2$, we obtain

$$\mathscr{R}^*_h=\Gamma^{(p)}(G_0)({}^t\!A_h)^{(p)}\Gamma^{(p)}(G_0^{-1})=(G_0{}^t\!A_hG_0^{-1})^{(p)}.$$

Now look at the term $(\mathscr{L}_{\nabla h}\otimes\mathrm{Id})^*$. Take first two p-forms $\alpha\omega$ and $\beta\eta$ where α, β are $\mathscr{C}^\infty_0(\Omega,\mathbb{R})$ functions, and ω, η are two p-forms dx^I and dx^J. Denoting by $V_{g_0}(dx)$ the normalized volume form,

$V_{g_0}(dx)$ satisfies

$$V_{g_0}(dx) = (\det G_0(x))^{1/2} dx^1 \wedge \cdots \wedge dx^n =: \nu(x)\, dx^1 \wedge \cdots \wedge dx^n.$$

Hence we deduce

$$\langle \alpha\omega \mid (\mathcal{L}_{\nabla h} \otimes \mathrm{Id})^* \beta\eta \rangle_{g_0} = \langle \mathcal{L}_{\nabla h}(\alpha)\omega \mid \eta \rangle_{g_0} = \int (\mathcal{L}_{\nabla h}(\alpha))\beta \langle \omega \mid \eta \rangle_{g_0(x)} (\det G_0(x))^{1/2} dx.$$

Using Cartan's formula (2-6), $\mathcal{L}_{\nabla h}(\alpha) = d\alpha(\nabla h) = \sum_{i=1}^n \frac{\partial \alpha}{\partial x^i} (\nabla h)_i$ and we obtain

$$\begin{aligned} \int (\mathcal{L}_{\nabla h}(\alpha))\beta \langle \omega \mid \eta \rangle_{g_0(x)} (\det G_0(x))^{1/2} dx &= \int \Big(\sum_{i=1}^n \frac{\partial \alpha}{\partial x^i} (\nabla h)_i \beta \Big) \langle \omega \mid \eta \rangle_{g_0(x)} \nu\, dx \\ &= -\int \alpha \sum_{i=1}^n \frac{\partial}{\partial x^i} \big((\nabla h)_i \beta \langle \omega \mid \eta \rangle_{g_0(x)} \nu \big)\, dx. \end{aligned}$$

Now write

$$\begin{aligned} &\int \alpha \sum_{i=1}^n \frac{\partial}{\partial x^i} \big((\nabla h)_i \beta \langle \omega \mid \eta \rangle_{g_0(x)} \nu \big)\, dx \\ &\quad = -\int \alpha \sum_{i=1}^n \Big(\frac{\partial (\nabla h)_i}{\partial x^i} \beta \langle \omega \mid \eta \rangle_{g_0(x)} \nu \Big)\, dx - \int \alpha \sum_{i=1}^n \Big((\nabla h)_i \frac{\partial \beta}{\partial x^i} \langle \omega \mid \eta \rangle_{g_0(x)} \nu \Big)\, dx \\ &\qquad\qquad - \int \alpha \sum_{i=1}^n \Big((\nabla h)_i \beta \frac{\partial}{\partial x^i} (\langle \omega \mid \eta \rangle_{g_0(x)}) \nu \Big)\, dx - \int \alpha \sum_{i=1}^n \Big((\nabla h)_i \beta \langle \omega \mid \eta \rangle_{g_0(x)} \frac{\partial \nu}{\partial x^i} \Big)\, dx \\ &\quad = -\int \alpha \sum_{i=1}^n \Big(\frac{\partial (\nabla h)_i}{\partial x^i} \beta \langle \omega \mid \eta \rangle_{g_0(x)} \nu \Big)\, dx - \int \alpha (\mathcal{L}_{\nabla h}(\beta)) \langle \omega \mid \eta \rangle_{g_0(x)} \nu\, dx \\ &\qquad\qquad - \int \alpha \sum_{i=1}^n \Big((\nabla h)_i \beta \frac{\partial}{\partial x^i} (\langle \omega \mid \eta \rangle_{g_0(x)}) \nu \Big)\, dx - \int \alpha \sum_{i=1}^n \Big((\nabla h)_i \beta \langle \omega \mid \eta \rangle_{g_0(x)} \frac{\partial \nu}{\partial x^i} \Big)\, dx. \end{aligned}$$

Noting that, for all i in $\{1, \dots, n\}$,

$$\begin{aligned} \frac{\partial}{\partial x^i} \Gamma^{(p)}(G_0^{-1}) &= \Big(\frac{\partial G_0^{-1}}{\partial x^i} \otimes G_0^{-1} \otimes \cdots \otimes G_0^{-1} \Big) + \cdots + \Big(G_0^{-1} \otimes \cdots \otimes G_0^{-1} \otimes \frac{\partial G_0^{-1}}{\partial x^i} \Big) \\ &= \Gamma^{(p)}(G_0^{-1}) \Big(G_0 \frac{\partial [G_0^{-1}]}{\partial x^i} \Big)^{(p)}, \end{aligned}$$

we deduce that, for all i in $\{1, \dots, n\}$,

$$\frac{\partial}{\partial x^i} \langle \omega \mid \eta \rangle_{g_0(x)} = \Big\langle \omega \,\Big|\, \Big(G_0 \frac{\partial [G_0^{-1}]}{\partial x^i} \Big)^{(p)} \eta \Big\rangle_{g_0(x)}.$$

Consequently,

$$(\mathcal{L}_{\nabla h} \otimes \mathrm{Id})^* = -\mathcal{L}_{\nabla h} \otimes \mathrm{Id} - \Big(\sum_{i=1}^n \Big(\frac{\partial (\nabla h)_i}{\partial x^i} + \frac{(\nabla h)_i}{\nu} \frac{\partial \nu}{\partial x^i} \Big) \Big) \mathrm{Id} - \sum_{i=1}^n (\nabla h)_i \Big(G_0 \frac{\partial [G_0^{-1}]}{\partial x^i} \Big)^{(p)},$$

which leads to the second result of Proposition B.3. □

B2. ***Application to*** $\mathscr{L}_{\nabla\Phi} - \mathscr{L}^*_{\nabla\Phi} + \mathscr{L}_{\nabla f} + \mathscr{L}^*_{\nabla f}$. Let us first write

$$\mathscr{L}_{\nabla\Phi} - \mathscr{L}^*_{\nabla\Phi} + \mathscr{L}_{\nabla f} + \mathscr{L}^*_{\nabla f} = 2\mathscr{L}_{\nabla\Phi} + \mathscr{L}_{\nabla(f-\Phi)} + \mathscr{L}^*_{\nabla(f-\Phi)}.$$

By Proposition B.3, we deduce

$$\mathscr{L}^*_{\nabla(f-\Phi)} + \mathscr{L}_{\nabla(f-\Phi)} = \mathscr{R}_1,$$

where $\mathscr{R}_1$ is an order-zero differential operator.

Next, using the first equality of Proposition B.3, we get

$$2\mathscr{L}_{\nabla\Phi} = 2\mathscr{L}_{\nabla\Phi} \otimes \mathrm{Id} + \mathscr{R}_2,$$

where $\mathscr{R}_2$ is an order-zero differential operator too.

Consequently, setting $\mathscr{R} = \mathscr{R}_1 + \mathscr{R}_2$, we obtain

$$\mathscr{L}_{\nabla\Phi} - \mathscr{L}^*_{\nabla\Phi} + \mathscr{L}_{\nabla f} + \mathscr{L}^*_{\nabla f} = 2\mathscr{L}_{\nabla\Phi} \otimes \mathrm{Id} + \mathscr{R},$$

where $\mathscr{R}$ is an order-zero differential operator.

Acknowledgement

The author thanks F. Nier and T. Jecko for profitable discussions, and the anonymous referee for suggesting various improvements.

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Received 3 Nov 2008. Revised 19 Nov 2009. Accepted 20 Dec 2009.

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ANALYSIS AND PDE
Vol. 3, No. 3, 2010

ON THE SCHRÖDINGER EQUATION OUTSIDE STRICTLY CONVEX OBSTACLES

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We prove sharp Strichartz estimates for the semiclassical Schrödinger equation on a compact Riemannian manifold with a smooth, strictly geodesically concave boundary. We deduce classical Strichartz estimates for the Schrödinger equation outside a strictly convex obstacle, local existence for the H^1-critical (quintic) Schrödinger equation, and scattering for the subcritical Schrödinger equation in three dimensions.

1. Introduction

Let (M, g) be a Riemannian manifold of dimension $n \geq 2$. Strichartz estimates are a family of dispersive estimates on solutions $u(x, t) : M \times [-T, T] \to \mathbb{C}$ to the Schrödinger equation

$$i\partial_t u + \Delta_g u = 0, \quad u(x, 0) = u_0(x), \tag{1-1}$$

where Δ_g denotes the Laplace–Beltrami operator on (M, g). In their most general form, local Strichartz estimates state that

$$\|u\|_{L^q([-T,T],L^r(M))} \leq C\|u_0\|_{H^s(M)}, \tag{1-2}$$

where $H^s(M)$ denotes the Sobolev space over M and $2 \leq q, r \leq \infty$ satisfy $(q, r, n) \neq (2, \infty, 2)$ (for the case $q = 2$ see [Keel and Tao 1998]) and are given by the scaling admissibility condition

$$\frac{2}{q} + \frac{n}{r} = \frac{n}{2}. \tag{1-3}$$

In $\mathbb{R}^n$ and for $g_{ij} = \delta_{ij}$, Strichartz estimates in the context of the wave and Schrödinger equations have a long history, beginning with the pioneering work [Strichartz 1977], where the particular case $q = r$ for the wave and (classical) Schrödinger equations was proved. This was later generalized to mixed $L^q_t L^r_x$ norms by Ginibre and Velo [1985] for Schrödinger equations, where (q, r) is sharp admissible and $q > 2$; the wave estimates were obtained independently by the same authors [1995] and by Lindblad and Sogge [1995], following [Kapitanskiĭ 1989]. The remaining endpoints for both equations were finally settled by Keel and Tao [1998]. In that case $s = 0$ and $T = \infty$; see also [Kato 1987; Cazenave and Weissler 1990]. Estimates for the flat 2-torus were shown by Bourgain [2003] to hold for $q = r = 4$ and any $s > 0$.

In the variable coefficients case, even without boundaries, the situation is much more complicated: we simply recall the pioneering work of Staffilani and Tataru [2002], dealing with compact, nontrapping perturbations of the flat metric, the works by Hassell et al. [2006], Robbiano and Zuily [2005], and Bouclet and Tzvetkov [2008] which considerably weakens the decay of the perturbation (retaining the

MSC2000: 35Q55, 37K05, 37K50.
Keywords: Schrödinger equation, Strichartz estimates, exterior domain.
The author was supported by grant A.N.R.-07-BLAN-0250.

nontrapping character at spatial infinity). On compact manifolds without boundaries, Burq et al. [2004b] established Strichartz estimates with $s = 1/p$, hence with a loss of derivatives when compared to the case of flat geometries. Recently, Blair et al. [2008] improved on the current results for compact (M, g) where either $\partial M \neq \varnothing$, or $\partial M = \varnothing$ and g Lipschitz, by showing that Strichartz estimates hold with a loss of $s = 4/3p$ derivatives. This appears to be the natural analog of the estimates of Burq et al. for the general boundaryless case.

In this paper we prove that Strichartz estimates for the semiclassical Schrödinger equation also hold on Riemannian manifolds with smooth, strictly geodesically concave boundaries. By the last condition we understand that the second fundamental form on the boundary of the manifold is strictly positive definite. moreover the manifold to be flat at infinity; i.e., the metric coincides with the Euclidean one outside a compact set (though presumably one may use [Bouclet and Tzvetkov 2008] result to combine both situations). We have two main examples of such manifolds in mind: first, we consider the case of a compact manifold with strictly concave boundary, which we shall denote S in the rest of the paper. The second example is the exterior of the strictly convex obstacle in $\mathbb{R}^n$, which will be denoted by Ω.

Assumption 1.1. *Let (S, g) be a smooth n-dimensional compact Riemannian manifold with C^∞ boundary. Assume ∂S is strictly geodesically concave. Let Δ_g be the Laplace–Beltrami operator associated to g.*

Let $0 < \alpha_0 \le \frac{1}{2}$, $2 \le \beta_0$, $\Psi \in C_0^\infty(\mathbb{R} \setminus \{0\})$ be compactly supported in the interval (α_0, β_0). We introduce the operator $\Psi(-h^2\Delta_g)$ using the Dynkin–Helffer–Sjöstrand formula [Davies 1995] and refer to [Nier 1993], [Davies 1995], or [Ivanovici and Planchon 2008] for a complete overview of its properties. See also [Burq et al. 2004b] for compact manifolds without boundaries.

Definition 1.2. Given $\Psi \in C_0^\infty(\mathbb{R})$, we have

$$\Psi(-h^2\Delta_g) = -\frac{1}{\pi}\int_{\mathbb{C}} \bar\partial\tilde\Psi(z)(z + h^2\Delta_g)^{-1}\,dL(z),$$

where $dL(z)$ denotes the Lebesque measure on $\mathbb{C}$ and $\tilde\Psi$ is an almost analytic extension of Ψ, for example, with $\langle z\rangle = (1+|z|^2)^{1/2}$, $N \ge 0$,

$$\tilde\Psi(z) = \Big(\sum_{m=0}^{N} \frac{\partial^m\Psi(\operatorname{Re} z)(i\operatorname{Im} z)^m}{m!}\Big)\,\tau\Big(\frac{\operatorname{Im} z}{\langle \operatorname{Re} z\rangle}\Big),$$

where τ is a nonnegative C^∞ function such that $\tau(s) = 1$ if $|s| \le 1$ and $\tau(s) = 0$ if $|s| \ge 2$.

Our main result is this:

Theorem 1.3. *Under Assumption 1.1, given (q, r) satisfying the scaling condition* (1-3), *$q > 2$, and $T > 0$ sufficiently small, there exists a constant $C = C(T) > 0$ such that the solution $v(x, t)$ of the semiclassical Schrödinger equation on $S \times \mathbb{R}$ with Dirichlet boundary conditions*

$$\begin{cases} ih\partial_t v + h^2\Delta_g v = 0 \quad \text{on } S\times\mathbb{R}, \\ v(x,0) = \Psi(-h^2\Delta_g)v_0(x), \\ v|_{\partial S} = 0 \end{cases} \tag{1-4}$$

satisfies

$$\|v\|_{L^q((-T,T),L^r(S))} \le Ch^{-1/q}\left\|\Psi(-h^2\Delta_g)v_0\right\|_{L^2(S)}. \tag{1-5}$$

Remark 1.4. An example of a compact manifold with smooth, strictly concave boundary is given by the Sinai billiard (defined as the complementary of a strictly convex obstacle on a cube of $\mathbb{R}^n$ with periodic boundary conditions).

We deduce from Theorem 1.3 and [Ivanovici and Planchon 2008, Theorem 1.1] (see also Lemma 3.7), as in [Burq et al. 2004b], the following Strichartz estimates with derivative loss:

Corollary 1.5. *Under Assumption 1.1, given* (q, r) *satisfying the scaling condition* (1-3), $q > 2$, *and* I *any finite time interval, there exists a constant* $C = C(I) > 0$ *such that the solution* $u(x, t)$ *of the* (*classical*) *Schrödinger equation on* $S \times \mathbb{R}$ *with Dirichlet boundary conditions*

$$\begin{cases} i\partial_t u + \Delta_g u = 0 & \text{on } S \times \mathbb{R}, \\ u(x,0) = u_0(x), & u|_{\partial S} = 0 \end{cases} \tag{1-6}$$

satisfies

$$\|u\|_{L^q((I,L^r(S)))} \leq C(I)\|u_0\|_{H^{1/q}(S)}. \tag{1-7}$$

The proof of Theorem 1.3 is based on the finite speed of propagation of the semiclassical flow [Lebeau 1992] and the energy conservation which allow us to use the arguments of Smith and Sogge [1995] for the wave equation: using the Melrose and Taylor parametrix [1985; 1986] for the stationary wave (see also [Zworski 1990]) we obtain, by Fourier transform in time, a parametrix for the Schrödinger operator near a "glancing" point. Since in the elliptic and hyperbolic regions the solution of (1-8) will clearly satisfy the same Strichartz estimates as on a manifold without boundary (in which case we refer to [Burq et al. 2004b]), we need to restrict our attention only on the glancing region.

As an application of Theorem 1.3 we prove classical, global Strichartz estimates for the Schrödinger equation outside a strictly convex domain in $\mathbb{R}^n$.

Assumption 1.6. *Let* $\Omega = \mathbb{R}^n \setminus \Theta$, *where* Θ *is a compact with smooth boundary. Assume that* $n \geq 2$ *and that* $\partial\Omega$ *is strictly geodesically concave throughout. Let* $\Delta_D = \sum_{j=1}^n \partial_j^2$ *denote the Dirichlet Laplace operator* (*with constant coefficients*) *on* Ω.

Theorem 1.7. *Under Assumption 1.6, given* (q, r) *satisfying the scaling condition* (1-3), $q > 2$ *and* $u_0 \in L^2(\Omega)$, *there exists a constant* $C > 0$ *such that the solution* $u(x, t)$ *of the Schrödinger equation on* $\Omega \times \mathbb{R}$ *with Dirichlet boundary conditions*

$$\begin{cases} i\partial_t u + \Delta_D u = 0 & \text{on } \Omega \times \mathbb{R}, \\ u(x,0) = u_0(x), \\ u|_{\partial\Omega} = 0 \end{cases} \tag{1-8}$$

satisfies

$$\|u\|_{L^q(\mathbb{R},L^r(\Omega))} \leq C\|u_0\|_{L^2(\Omega)}. \tag{1-9}$$

The proof of Theorem 1.7 combines several arguments. First, we perform a time rescaling, first used by Lebeau [1992] in the context of control theory, which transforms the equation into a semiclassical problem for which we can use the time-local semiclassical Strichartz estimates proved in Theorem 1.3. Second, we adapt a result of Burq [2002], which provides Strichartz estimates without loss for a nontrapping problem, with a metric that equals the identity outside a compact set. The proof relies on a local smoothing effect for the free evolution $\exp(it\Delta_D)$, first observed independently by Constantin and Saut [1989], Sjölin [1987], and Vega [1988] in the flat case, and then by Doi [1996] on nontrapping manifolds and by Burq et al. [2004a] on exterior domains. Following a strategy suggested by Staffilani and Tataru [2002], we prove that away from the obstacle the free evolution enjoys the Strichartz estimates exactly as for the free space.

We give two applications of Theorem 1.7. The first is a local existence result for the quintic Schrödinger equation in three dimensions, while the second is a scattering result for the subcritical (subquintic) Schrödinger equation in three-dimensional domains.

Theorem 1.8 (local existence for the quintic Schrödinger equation). *Let Ω be a three dimensional Riemannian manifold satisfying Assumption 1.6. Let $T > 0$ and $u_0 \in H_0^1(\Omega)$. Then there exists a unique solution $u \in C\big([0,T], H_0^1(\Omega)\big) \cap L^5\big((0,T], W^{1,30/11}(\Omega)\big)$ of the quintic nonlinear equation*

$$i\partial_t u + \Delta_D u = \pm|u|^4 u \text{ on } \Omega \times \mathbb{R}, \qquad u|_{t=0} = u_0 \text{ on } \Omega, \qquad u|_{\partial\Omega} = 0. \tag{1-10}$$

Moreover, for any $T > 0$, the flow $u_0 \to u$ is Lipschitz continuous from any bounded set of $H_0^1(\Omega)$ to $C\big([-T,T), H_0^1(\Omega)\big)$. If the initial data u_0 has sufficiently small H^1 norm, then the solution is global in time.

Theorem 1.9 (scattering for subcritical Schrödinger equation). *Let Ω be a three dimensional Riemannian manifold satisfying Assumption 1.6. Let $1 + \frac{4}{3} \le p < 5$ and $u_0 \in H_0^1(\Omega)$. Then the time-global solution of the defocusing Schrödinger equation*

$$i\partial_t u + \Delta_D u = |u|^{p-1} u, \qquad u|_{t=0} = u_0 \text{ on } \Omega, \qquad u|_{\partial\Omega} = 0 \tag{1-11}$$

scatters in $H_0^1(\Omega)$. If $p = 5$ and the gradient ∇u_0 of the initial data has sufficiently small L^2 norm, then the global solution of the critical Schrödinger equation scatters in $H_0^1(\Omega)$.

Results for the Cauchy problem associated to the critical wave equation outside a strictly convex obstacle were obtained by Smith and Sogge [1995]. Their result was a consequence of the fact that the Strichartz estimates for the Euclidean wave equation also hold on Riemannian manifolds with smooth, compact, and strictly concave boundaries.

Burq et al. [2008] proved that the defocusing quintic wave equation with Dirichlet boundary conditions is globally wellposed on $H^1(M) \times L^2(M)$ for any smooth, compact domain $M \subset \mathbb{R}^3$. Their proof relies on L^p estimates for the spectral projector obtained by Smith and Sogge [2007]. A similar result for the defocusing critical wave equation with Neumann boundary conditions was obtained in [Burq and Planchon 2009].

In the case of Schrödinger equation in $\mathbb{R}^3 \times \mathbb{R}_t$, Colliander et al. [2008] established global well-posedness and scattering for energy-class solutions to the quintic defocusing Schrödinger equation (1-10), which is energy-critical. When the domain is the complementary of an obstacle in $\mathbb{R}^3$, nontrapping but not convex, the counterexamples constructed in [Ivanovici 2010] for the wave equation suggest that losses are likely to occur in the Strichartz estimates for the Schrödinger equation too. In this case Burq et al. [2004a] proved global existence for subcubic defocusing nonlinearities while Anton [2008] proved it for the cubic case. Recently, Planchon and Vega [2009] improved the local well-posedness theory to H^1-subcritical (subquintic) nonlinearities for $n = 3$. Theorem 1.9 is proved in [Planchon and Vega 2009] in the case of the exterior of a star-shaped domain for the particular case $p = 3$, using the estimate

$$\|u\|_{L^4_{t,x}}^4 \lesssim \|u_0\|_{L^2}^3 \|\nabla u_0\|_{L^2}$$

on the solution to the linear problem, but with no control of the $L^4_t L^\infty_x$ norm one has to use local smoothing estimates close to the boundary, and Strichartz estimates for the usual Laplacian on $\mathbb{R}^3$ away from it. Here we give a simpler proof on the exterior of a strictly convex obstacle and for every $1+\frac{4}{3} < p < 5$ using the Strichartz estimates (1-9).

2. Estimates for the semiclassical Schrödinger equation in a compact domain with strictly concave boundary

In this section we prove Theorem 1.3. In what follows Assumption 1.1 are supposed to hold. We may assume that the metric g is extended smoothly across the boundary, so that S is a geodesically concave subset of a complete, compact Riemannian manifold $\tilde{S}$. By the free semiclassical Schrödinger equation we mean the semiclassical Schrödinger equation on $\tilde{S}$, where the data v_0 has been extended to $\tilde{S}$ by an extension operator preserving the Sobolev spaces. By a broken geodesic in S we mean a geodesic that is allowed to reflect off ∂S according to the reflection law for the metric g.

Restriction in a small neighborhood of the boundary: Elliptic and hyperbolic regions. We consider $\delta > 0$ a small positive number and for $T > 0$ small enough we set

$$S(\delta, T) := \big\{(x,t) \in S \times [-T,T] : \operatorname{dist}(x, \partial S) < \delta\big\}.$$

On the complement of $S(\delta,T)$ in $S \times [-T,T]$, the solution $v(x,t)$ equals, in the semiclassical regime and modulo $O_{L^2}(h^\infty)$ errors, the solution of the semiclassical Schrödinger equation on a manifold without boundary for which sharp semiclassical Strichartz estimates follow by the work of Burq et al. [2004b], thus it suffices to establish Strichartz estimates for the norm of v over $S(\delta,T)$.

We show that in order to prove Theorem 1.3 it will be sufficient to consider only data v_0 supported outside a small neighborhood of the boundary. Recall that Lebeau [1992] proved that if Ψ is supported in an interval $[\alpha_0, \beta_0]$ and if $\varphi \in C_0^\infty(\mathbb{R})$ is equal to 1 near the interval $[-\beta_0, -\alpha_0]$, then for t in a bounded set (and for $D_t = i^{-1}\partial_t$) one has

$$\forall N \geq 1, \quad \exists C_N > 0 \quad \big|(1-\varphi)(hD_t)\exp(ith\Delta_g)\Psi(-h^2\Delta_g)v_0\big| \leq C_N h^N. \tag{2-1}$$

For δ and T sufficiently small, let $\chi(x,t) \in C_0^\infty$ be compactly supported and be equal to 1 on $S(\delta,T)$. Let $t_0 > 0$ be such that $T = t_0/4$ and let $A \in C^\infty(\mathbb{R}^n)$, $A = 0$ near ∂S, $A = 1$ outside a neighborhood of the boundary be such that every broken bicharacteristic γ starting at $t = 0$ from the support of $\chi(x,t)$ and for $-\tau \in [\alpha_0, \beta_0]$ (where τ denotes the dual time variable), satisfies

$$\operatorname{dist}\big(\gamma(t), \operatorname{supp}(1-A)\big) > 0 \quad \text{for all } t \in [-2t_0, -t_0]. \tag{2-2}$$

Let $\psi \in C^\infty(\mathbb{R})$, $\psi(t) = 0$ for $t \leq -2t_0$, $\psi(t) = 1$ for $t > -t_0$ and set

$$w(x,t) = \psi(t)\exp(ith\Delta_g)\Psi(-h^2\Delta_g)v_0.$$

Then w satisfies

$$\begin{cases} ih\partial_t w + h^2\Delta_g w = ih\psi'(t)e^{ith\Delta_g}\Psi(-h^2\Delta_g)v_0, \\ w|_{\partial S\times\mathbb{R}} = 0, \quad w|_{t\leq -2t_0} = 0, \end{cases}$$

and writing Duhamel's formula we have

$$w(x,t)=\int_{-2t_0}^{t} e^{i(t-s)h\Delta_g}\psi'(s)e^{ish\Delta_g}\Psi(-h^2\Delta_g)v_0\,ds.$$

Notice that $w(x,t)=v(x,t)$ if $t\geq -t_0$, hence for $t\in[-t_0,T]$ we can write

$$v(x,t)=\int_{-2t_0}^{-t_0} e^{i(t-s)h\Delta_g}\psi'(s)e^{ish\Delta_g}\Psi(-h^2\Delta_g)v_0\,ds. \tag{2-3}$$

In particular, for $t\in[-T,T]$, $T=t_0/4$, $v(x,t)=w(x,t)$ is given by (2-3). We want to estimate the $L^q_tL^r_x$ norms of $v(x,t)$ for (x,t) on $S(\delta,T)$ where $v=\chi v$. Let

$$v_Q(x,t)=\int_{-2t_0}^{-t_0} e^{i(t-s)h\Delta_g}\psi'(s)Q(x)e^{ish\Delta_g}\Psi(-h^2\Delta_g)v_0\,ds, \quad \text{where } Q\in\{A,1-A\}.$$

Then $v=v_A+v_{1-A}$, where v_{1-A} solves

$$\begin{cases} ih\partial_t v_{1-A}+h^2\Delta_g v_{1-A}=ih\psi'(t)(1-A)e^{ith\Delta_g}\Psi(-h^2\Delta_g)v_0,\\ v_{1-A}|_{\partial S\times\mathbb{R}}=0, \quad v_{1-A}|_{t<-2t_0}=0.\end{cases}$$

We apply Proposition A.8 from Appendix A with $Q=1-A$, $\tilde{\psi}=\psi'$ to deduce that if $\rho_0\in WF_b(v_{1-A})$ then the broken bicharacteristic starting from ρ_0 must intersect the wave front set

$$WF_b\big((1-A)v\big)\cap\big\{t\in[-2t_0,-t_0]\big\}.$$

Since we are interested in estimating the norm of v on $S(\delta,T)$ it is enough to consider only $\rho_0\in WF_b(\chi v_{1-A})$. Thus, if γ is a broken bicharacteristic starting at $t=0$ from ρ_0, $-\tau\in[\alpha_0,\beta_0]$, then Proposition A.8 implies that for some $t\in[-2t_0,-t_0]$, $\gamma(t)$ must intersect $WF_b((1-A)v)$. On the other hand from (2-2) this implies (see Definition A.2) that for every $\sigma\geq 0$

$$\forall N\geq 0 \quad \exists C_N>0 \quad \|\chi v_{1-A}\|_{H^\sigma(S\times\mathbb{R})}\leq C_N h^N. \tag{2-4}$$

We are thus reduced to estimating $v(x,t)$ for initial data supported outside a small neighborhood of the boundary. Indeed, suppose that the estimates (1-5) hold true for any initial data compactly supported where $A\neq 0$. It follows from (2-3) and (2-4) that

$$\begin{aligned}\|\chi v_A\|_{L^q((-T,T),L^r(S))} &\leq \big\|\psi'(s)A(x)e^{ish\Delta_g}\Psi(-h^2\Delta_g)v_0\big\|_{L^1(s\in(-2t_0,-t_0),L^2(S))}\\ &\lesssim \left(\int_{-2t_0}^{-t_0}|\Psi'(s)|\,ds\right)\big\|\Psi(-h^2\Delta_g)v_0\big\|_{L^2(S)}\\ &= \big\|\Psi(-h^2\Delta_g)v_0\big\|_{L^2(S)},\end{aligned}$$

where we used the fact that the semiclassical Schrödinger flow $\exp(ihs\Delta_g)\Psi(-h^2\Delta_g)$, which maps data at time 0 to data at time s, is an isomorphism on $H^\sigma(S)$ for every $\sigma\geq 0$.

Remark 2.1. When dealing with the wave equation, since the speed of propagation is exact, one can take $\psi(t)=1_{\{t\geq -t_0\}}$ for some small $t_0\geq 0$ and reduce the problem to proving Strichartz estimates for the flow $\exp(ih(t_0+.)\Delta_g)\Psi(-h^2\Delta_g)$ and initial data compactly supported outside a small neighborhood of ∂S. This was precisely the strategy followed by Smith and Sogge [1995].

Let Δ_0 denote the Laplacian on $\tilde{S}$ coming from extending the metric g smoothly across the boundary ∂S. We let $\mathcal{M}$ denote the outgoing solution to the Dirichlet problem for the semiclassical Schrödinger operator on $S \times \mathbb{R}$. Thus, if g is a function on $\partial S \times \mathbb{R}$ which vanishes for $t \leq -2t_0$, then $\mathcal{M}g$ is the solution on $S \times \mathbb{R}$ to

$$\begin{cases} ih\partial_t \mathcal{M}g + h^2 \Delta_g \mathcal{M}g = 0, \\ \mathcal{M}g|_{\partial S \times \mathbb{R}} = g. \end{cases} \tag{2-5}$$

Then, for $t \in [-t_0, T]$ and data f supported outside a small neighborhood of the boundary and localized at frequency $1/h$ (that is, such that $f = \Psi(-h^2\Delta_g) f$), we have

$$\begin{aligned} \chi v_A(x,t) &= \chi \int_{-2t_0}^{-t_0} e^{i(t-s)h\Delta_g} \psi'(s) A(x) e^{ish\Delta_g} f\, ds \\ &= \chi \int_{-2t_0}^{-t_0} e^{i(t-s)h\Delta_0} \psi'(s) A(x) e^{ish\Delta_0} f\, ds - \mathcal{M}\Big(\chi \int_{-2t_0}^{-t_0} e^{i(t-s)h\Delta_0} \psi'(s) A(x) e^{ish\Delta_0} f\, ds|_{\partial S \times \mathbb{R}}\Big). \end{aligned}$$

The cotangent bundle of $\partial S \times \mathbb{R}$ is divided into three disjoint sets: the hyperbolic and elliptic regions, where the Dirichlet problem is respectively hyperbolic and elliptic, and the glancing region, which is the boundary between the two.

Let local coordinates be chosen such that $S = \{(x', x_n) : x_n > 0\}$ and $\Delta_g = \partial_{x_n}^2 - r(x, D_{x'})$. A point $(x', t, \eta', \tau) \in T^*(\partial S \times \mathbb{R})$ is classified as one of three distinct types. It is said to be *hyperbolic* if $-\tau + r(x', 0, \eta') > 0$, so that there are two distinct nonzero real solutions η_n to $\tau - r(x', 0, \eta') = \eta_n^2$. These two solutions yield two distinct bicharacteristics, one of which enters S as t increases (the *incoming ray*) and one which exits S as t increases (the *outgoing ray*). The point is *elliptic* if $-\tau + r(x', 0, \eta') < 0$, so there are no real solutions η_n to $\tau - r(x', 0, \eta') = \eta_n^2$. In the remaining case $-\tau + r(x', 0, \eta') = 0$, there is a unique solution which yields a glancing ray, and the point is said to be a *glancing point*. We decompose the identity operator into

$$\mathrm{Id}(x,t) = \frac{1}{(2\pi h)^n} \int e^{(i/h)((x'-y')\eta' + (t-s)\tau)} (\chi_h + \chi_e + \chi_{gl})(y', \eta', \tau)\, d\eta' d\tau,$$

where at (y', η', τ) we have

$$\chi_h := 1_{\{-\tau + r(y',0,\eta') \geq c\}}, \qquad \chi_e := 1_{\{-\tau + r(y',0,\eta') \leq -c\}}, \qquad \chi_{gl} := 1_{\{-\tau + r(y',0,\eta') \in [-c,c]\}},$$

for some $c > 0$ sufficiently small. The corresponding operators with symbols χ_h, χ_e, denoted Π_h, Π_e, respectively, are pseudodifferential cutoffs essentially supported inside the hyperbolic and elliptic regions, while the operator with symbol χ_{gl}, denoted Π_{gl}, is essentially supported in a small set around the glancing region. Thus, on $S(\delta, T)$ we can write χv_A as the sum of four terms:

$$\begin{aligned} \chi \int_{-2t_0}^{-t_0} e^{i(t-s)h\Delta_g} \psi'(s) A(x) e^{ish\Delta_g} f\, ds &= \chi \int_{-2t_0}^{-t_0} e^{i(t-s)h\Delta_0} \psi'(s) A(x) e^{ish\Delta_0} f\, ds \\ &\quad - \sum_{\Pi \in \{\Pi_e, \Pi_h, \Pi_{gl}\}} \mathcal{M}\Pi\Big(\chi \int_{-2t_0}^{-t_0} e^{i(t-s)h\Delta_0} \psi'(s) A(x) e^{ish\Delta_0} f\, ds \,\Big|_{\partial S \times \mathbb{R}}\Big). \end{aligned} \tag{2-6}$$

Remark 2.2. For the first term in the right, $\chi \int_{-2t_0}^{-t_0} e^{i(t-s)h\Delta_0} \psi'(s) A(x) e^{ish\Delta_0} f\, ds$, the desired estimates follow as in the boundaryless case by the results of Staffilani and Tataru [2002] (since we considered the extension of the metric g across the boundary to be smooth).

Elliptic region. From Proposition A.3 in Appendix A there follows the inclusion

$$WF_b\bigg(\chi\int_{-2t_0}^{-t_0} e^{i(t-s)h\Delta_0}\psi'(s)A(x)e^{ish\Delta_0}f\,ds\,\big|_{\partial S\times\mathbb{R}}\bigg)\subset\mathcal{H}\cup\mathcal{G},$$

where $\mathcal{H}$ and $\mathcal{G}$ denote the hyperbolic and the glancing regions, respectively. Together with the compactness argument from the proof of Proposition A.7, this implies that the elliptic part satisfies, for all $\sigma\geq 0$,

$$\mathcal{M}\Pi_e\bigg(\chi\int_{-2t_0}^{-t_0} e^{i(t-s)h\Delta_0}\psi'(s)A(x)e^{ish\Delta_0}f\,ds\,\big|_{\partial S\times\mathbb{R}}\bigg)=O(h^\infty)\|f\|_{H^\sigma(S)}.$$

For the definition and properties of the b-wave front set see Appendix A.

Hyperbolic region. If local coordinates are chosen such that $S=\{(x',x_n):x_n>0\}$, on the essential support of Π_h the forward Dirichlet problem can be solved locally, modulo smoothing kernels, on an open set in $\tilde{S}\times\mathbb{R}$ around ∂S. Precisely, microlocally near a hyperbolic point, the solution v to (1-4) can be decomposed modulo smoothing operators into an incoming part v_- and an outgoing part v_+ where

$$v_\pm(x,t)=\frac{1}{(2\pi h)^d}\int e^{(i/h)\varphi_\pm(x,t,\xi)}\sigma_\pm(x,t,\xi,h)\,d\xi,$$

where the phases $\varphi_\pm$ satisfy the eikonal equations

$$\begin{cases}\partial_s\varphi_\pm+\langle d\varphi_\pm,d\varphi_\pm\rangle_g=0,\\ \varphi_+|_{\partial S}=\varphi_-|_{\partial S},\quad \partial_{x_n}\varphi_+|_{\partial S}=-\partial_{x_n}\varphi_-|_{\partial S},\end{cases}$$

where $\langle\cdot,\cdot\rangle_g$ denotes the inner product induced by the metric g. The symbols are asymptotic expansions in h and write $\sigma_\pm(\cdot,h)=\sum_{k\geq 0}h^k\sigma_{\pm,k}$, where σ_0 solves the linear transport equation

$$\partial_s\sigma_{\pm,0}+(\Delta_g\varphi_\pm)\sigma_{\pm,0}+\langle d\varphi_\pm,d\sigma_{\pm,0}\rangle_g=0,$$

while for $k\geq 1$, $\sigma_{\pm,k}$ satisfies the nonhomogeneous transport equations

$$\partial_s\sigma_{\pm,k}+(\Delta_g\varphi_\pm)\sigma_{\pm,k}+\langle d\varphi_\pm,d\sigma_{\pm,k}\rangle_g=i\,\Delta_g\sigma_{\pm,k-1}.$$

A direct computation shows that

$$\Big\|\sum_\pm v_\pm\Big\|^2_{H^\sigma(S\times\mathbb{R})}\simeq\sum_\pm\|v_\pm\|^2_{H^\sigma(S\times\mathbb{R})}\simeq\|v\|^2_{H^\sigma(S\times\mathbb{R})}\simeq\|v\|^2_{L^\infty(\mathbb{R})H^\sigma(S)}.$$

Each component $v_\pm$ is a solution of linear Schrödinger equation (without boundary) and consequently satisfies the usual Strichartz estimates [Burq et al. 2004b].

Note that $\sum_\pm v_\pm$ contains the contribution from

$$\mathcal{M}\Pi_h\bigg(\chi\int_{-2t_0}^{-t_0} e^{i(t-s)h\Delta_0}\psi'(s)A(x)e^{ish\Delta_0}\Psi(-h^2\Delta_g)v_0\,ds\,\big|_{\partial S\times\mathbb{R}}\bigg)$$

and a contribution from $\chi\int_{-2t_0}^{-t_0} e^{i(t-s)h\Delta_0}\psi'(s)A(x)e^{ish\Delta_0}\Psi(-h^2\Delta_g)v_0\,ds$.

Glancing region. Near a diffractive point we use the Melrose and Taylor construction for the wave equation in order to write, following Zworski [1990], the solution to the wave equation as a finite sum of pseudodifferential cutoffs, each essentially supported in a suitably small neighborhood of a glancing ray. Using the Fourier transform in time we obtain a parametrix for the semiclassical Schrödinger equation (1-4) microlocally near a glancing direction and modulo smoothing operators.

Preliminaries: Parametrix for the wave equation near the glancing region. We start by recalling the results by Melrose and Taylor [1985; 1986] and Zworski [1990, Proposition 4.1] for the wave equation near the glancing region. Let w solve the (semiclassical) wave equation on S with Dirichlet boundary conditions

$$\begin{cases} h^2D_t^2 w + h^2\Delta_g w = 0, \quad S\times\mathbb{R}, \quad w|_{\partial S\times\mathbb{R}} = 0, \\ w(x,0) = f(x), \quad D_t w(x,0) = g(x), \end{cases} \tag{2-7}$$

where f, g are compactly supported in S and localized at spatial frequency $1/h$, and where $D_t = i^{-1}\partial_t$.

Proposition 2.3. *Microlocally near a glancing direction the solution to* (2-7) *can be written, modulo smoothing operators, as*

$$w(x,t) = \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n} e^{(i/h)(\theta(x,\xi)+it\xi_1)}\biggl[a(x,\xi/h)\biggl(A_-(\zeta(x,\xi/h)) - A_+(\zeta(x,\xi/h))\frac{A_-(\zeta_0(\xi/h))}{A_+(\zeta_0(\xi/h))}\biggr) \\ + b(x,\xi/h)\biggl(A'_-(\zeta(x,\xi/h)) - A'_+(\zeta(x,\xi/h))\frac{A_-(\zeta_0(\xi/h))}{A_+(\zeta_0(\xi/h))}\biggr)\biggr]\times\widehat{K(f,g)}\Bigl(\frac{\xi}{h}\Bigr)\,d\xi, \tag{2-8}$$

where the symbols a, b, and the phases θ, ζ have the following properties: a and b are symbols of type $(1,0)$ *and order* $\frac{1}{6}$ *and* $-\frac{1}{6}$, *respectively, both of which are supported in a small conic neighborhood of the ξ_1 axis, and the phases θ and ζ are real, smooth and homogeneous of degree* 1 *and* $\frac{2}{3}$, *respectively. Further, K is a classical Fourier integral operator of order* 0 *in f and order* -1 *in g, compactly supported on both sides. The $A_\pm$ are defined by* $A_\pm(z) = \mathrm{Ai}(e^{\mp 2\pi i/3}z)$, *where* Ai *denotes the Airy function.*

Remark 2.4. If local coordinates are chosen so that Ω is given by $x_n > 0$, the phase functions θ, ζ satisfy the eikonal equations

$$\begin{cases} \xi_1^2 - \langle d\theta, d\theta\rangle_g + \zeta\langle d\zeta, d\zeta\rangle_g = 0, \\ \langle d\theta, d\zeta\rangle_g = 0, \\ \zeta(x',0,\xi) = \zeta_0(\xi) = -\xi_1^{-1/3}\xi_n, \end{cases} \tag{2-9}$$

in the region $\zeta\le 0$. Here $x' = (x_1,\dots,x_{n-1})$ and $\langle\,\cdot\,,\cdot\,\rangle_g$ denotes the inner product given by the metric g. The phases also satisfy the eikonal equations (2-9) to infinite order at $x_n = 0$ in the region $\zeta > 0$.

Remark 2.5. One can think of $A_-(\zeta)$ (at least away from the boundary $x_n = 0$) as the incoming contribution and of $A_+(\zeta)A_-(\zeta_0)/A_+(\zeta_0)$ as the outgoing one. From [Zworski 1990, Section 2] we have

$$\frac{A_-}{A_+}(z) \simeq \begin{cases} -e^{i\pi/3} + O(z^{-\infty}), & z\to\infty, \\ e^{i(4/3)(-z)^{3/2}}\sum_{j\ge 0}\beta_j z^{-3j/2}, & z\to-\infty, \end{cases}$$

where the part $z \to \infty$ corresponds to the free wave, while the oscillatory one to the billiard ball map shift corresponding to reflection. Using $\mathrm{Ai}(\zeta) = e^{i\pi/3}A_+(\zeta) + e^{-i\pi/3}A_-(\zeta)$, we write

$$A_-(\zeta) - A_+(\zeta)\frac{A_-(\zeta_0)}{A_+(\zeta_0)} = e^{i\pi/3}\left(\mathrm{Ai}(\zeta) - A_+(\zeta)\frac{\mathrm{Ai}(\zeta_0)}{A_+(\zeta_0)}\right).$$

Parametrix for the solution to the semiclassical Schrödinger equation near the glancing region. Let $v(x,t)$ be the solution of the semiclassical Schrödinger equation (1-4) where the initial data $v_0 \in L^2(S)$ is spectrally localized at spatial frequency $1/h$; that is, $v_0(x) = \Psi(-h^2\Delta_g)v_0(x)$. From the discussion at the beginning of this section we see that it will be enough to consider v_0 compactly supported outside some small neighborhood of ∂S. Under this assumption $\Psi(-h^2\Delta_g)v_0$ is a well-defined pseudodifferential operator for which the results of [Burq et al. 2004b, Section 2] apply.

Let $(e_\lambda(x))_{\lambda\geq 0}$ be the eigenbasis of $L^2(S)$ consisting in eigenfunctions of $-\Delta_g$ associated to the eigenvalues (λ^2), so that $-\Delta_g e_\lambda = \lambda^2 e_\lambda$. We write

$$\Psi(-h^2\Delta_g)v_0(x) = \sum_{h^2\lambda^2\in[\alpha_0,\beta_0]} \Psi(h^2\lambda^2)v_\lambda e_\lambda(x), \tag{2-10}$$

and hence

$$e^{ith\Delta_g}\Psi(-h^2\Delta_g)v_0(x) = \sum_{h^2\lambda^2\in[\alpha_0,\beta_0]} \Psi(h^2\lambda^2)e^{-ith\lambda^2}v_\lambda e_\lambda(x). \tag{2-11}$$

If δ denotes the Dirac function, the Fourier transform of $v(x,t)$ can be written as

$$\hat{v}\left(x, \frac{\tau}{h}\right) = h \sum_{h^2\lambda^2\in[\alpha_0,\beta_0]} \Psi(h^2\lambda^2)\delta_{\{-\tau=h^2\lambda^2\}}v_\lambda e_\lambda(x). \tag{2-12}$$

For $t \in \mathbb{R}$ we can define (since $\hat{v}$ has compact support away from 0)

$$\begin{aligned} w(x,t) &:= \frac{1}{2\pi h}\int_0^\infty e^{it\sigma/h}\hat{v}\left(x, -\frac{\sigma^2}{h}\right)d\sigma = -\frac{1}{4\pi h}\int_{-\infty}^0 e^{it\sqrt{-\tau}/h}\frac{1}{\sqrt{-\tau}}\hat{v}\left(x,\frac{\tau}{h}\right)d\tau \\ &= -\frac{1}{2}\sum_{h^2\lambda^2\in[\alpha_0,\beta_0]} \Psi(h^2\lambda^2)\left(\frac{1}{2\pi}\int_{-\infty}^0 e^{it\sqrt{-\tau}/h}\frac{1}{\sqrt{-\tau}}\delta_{\{-\tau=h^2\lambda^2\}}d\tau\right)v_\lambda e_\lambda(x) \\ &= -\frac{1}{2}\sum_{h^2\lambda^2\in[\alpha_0,\beta_0]} \frac{1}{h\lambda}\Psi(h^2\lambda^2)e^{it\lambda}v_\lambda e_\lambda(x). \end{aligned} \tag{2-13}$$

Then $w(x,t)$ solves the wave equation

$$\begin{cases} h^2D_t^2 w + h^2\Delta_g w = 0 \quad \text{on } S\times\mathbb{R}, \quad w|_{\partial S\times\mathbb{R}} = 0, \\ w(x,0) = f_h(x), \quad D_t w(x,0) = g_h(x), \end{cases} \tag{2-14}$$

where the initial data are given by

$$f_h(x) = -\frac{1}{2}\sum_{h^2\lambda^2\in[\alpha_0,\beta_0]} \frac{1}{h\lambda}\Psi(h^2\lambda^2)v_\lambda e_\lambda(x), \tag{2-15}$$

$$g_h(x) = -\frac{1}{2h}\sum_{h^2\lambda^2\in[\alpha_0,\beta_0]} \Psi(h^2\lambda^2)v_\lambda e_\lambda(x) = -\frac{1}{2h}\Psi(-h^2\Delta_g)v_0(x). \tag{2-16}$$

From (2-15) and (2-16) it follows that

$$h\|g_h\|_{L^2(S)} \simeq \|f_h\|_{L^2(S)} \simeq \left\|\Psi(-h^2\Delta_g)v_0\right\|_{L^2(S)}, \tag{2-17}$$

where by $\alpha \simeq \beta$ we mean that there is $C > 0$ such that $C^{-1}\alpha < \beta < C\alpha$.

Indeed, to prove (2-17) notice that w defined by (2-13) satisfies

$$(hD_t - h\sqrt{-\Delta_g})w = 0$$

and (since Δ_g and D_t commute) we have

$$f_h = w|_{t=0} = \left[(\sqrt{-\Delta_g})^{-1}D_t w\right]\Big|_{t=0} = (\sqrt{-\Delta_g})^{-1}(D_t w|_{t=0}) = (\sqrt{-\Delta_g})^{-1}g_h.$$

Due to the spectral localization and since $g_h = -(1/2h)\Psi(-h^2\Delta_g)v_0$ we deduce (2-17).

By the L^2 continuity of the (classical) Fourier integral operator K introduced in Proposition 2.3 we deduce

$$\left\|K(f_h, g_h)\right\|_{L^2(S)} \le C\left(\|f_h\|_{L^2(S)} + h\|g_h\|_{L^2(S)}\right) \simeq \left\|\Psi(-h^2\Delta_g)v_0\right\|_{L^2(S)}. \tag{2-18}$$

The solution $v(x,t)$ of (1-4) can be written as

$$v(x,t) = \frac{1}{2\pi h}\int_0^\infty e^{-it\sigma^2/h} 2\sigma\hat{v}(x, -\frac{\sigma^2}{h})\,d\sigma = \frac{1}{2\pi h}\int_0^\infty e^{-i\frac{t\sigma^2}{h}} 2\sigma \int_{s\in\mathbb{R}} e^{-i\frac{s\sigma}{h}} w(x,s)\,ds\,d\sigma. \tag{2-19}$$

The next step is to use (2-7) to obtain a representation of $v(x,t)$ near the glancing region: notice that the glancing part of the stationary wave $\hat{w}(x, \sigma/h)$ is given by

$$1_{\{\sigma^2 + r(x',0,\eta')\in[-c,c]\}}\hat{w}\left(x, \frac{\sigma}{h}\right) = 1_{\{\sigma^2 + r(x',0,\eta')\in[-c,c]\}}\hat{v}\left(x, -\frac{\sigma^2}{h}\right) = 1_{\{-\tau + r(x',0,\eta')\in[-c,c]\}}\hat{v}\left(x, \frac{\tau}{h}\right), \tag{2-20}$$

with $\tau = -\sigma^2$ and where $c > 0$ is sufficiently small. The equality in (2-20) follows from (2-13) and from the fact that $\hat{v}$ is essentially supported for the second variable in the interval $[-\beta_0, -\alpha_0]$. Consequently we can apply Equation (2-7) and determine a representation for v near the glancing region (for the Schrödinger equation) as

$$\begin{aligned} v(x,t) = \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n} e^{(i/h)(\theta(x,\xi) - t\xi_1^2)} 2\xi_1 \Bigg[a(x,\xi/h)\left(\mathrm{Ai}(\zeta(x,\xi/h)) - A_+(\zeta(x,\xi/h))\frac{\mathrm{Ai}(\zeta_0(\xi/h))}{A_+(\zeta_0(\xi/h))}\right) \\ + b(x,\xi/h)\left(\mathrm{Ai}'(\zeta(x,\xi/h)) - A'_+(\zeta(x,\xi/h))\frac{\mathrm{Ai}(\zeta_0(\xi/h))}{A_+(\zeta_0(\xi/h))}\right)\Bigg] \widehat{K(f_h, g_h)}\left(\frac{\xi}{h}\right)d\xi, \end{aligned} \tag{2-21}$$

where a, b and K are those defined in Proposition 2.3 and f_h, g_h are given by (2-15) and (2-16). The initial data f_h, g_h are both supported, like v_0, away from ∂S, so their $\dot{H}^\sigma(S)$ norms for $\alpha < n/2$ will be comparable to the norms of the nonhomogeneous Sobolev space $H^\sigma(\mathbb{R}^n)$. For this reason we shall henceforth work with the latter norms on the data f_h, g_h.

Remark 2.6. It is enough to prove semiclassical Strichartz estimates only for the "outgoing" piece corresponding to the oscillatory term $A_+(\zeta)\,\mathrm{Ai}(\zeta_0)/A_+(\zeta_0)$, since the direct term, corresponding to $\mathrm{Ai}(\zeta)$, has already been dealt with (see Remark 2.2).

We deduce from (2-18) and (2-21) that, to finish the proof of Theorem 1.3, we need only show that the operator A_h defined, for f supported away from ∂S and spectrally localized at the frequency $1/h$ (that is, such that $f=\Psi(-h^2\Delta_g)f$), by

$$A_h f(x,t)=\frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n} 2\xi_1\Big(a(x,\xi/h)A_+\big(\zeta(x,\xi/h)\big)+b(x,\xi/h)A'_+\big(\zeta(x,\xi/h)\big)\Big) \times e^{(i/h)(\theta(x,\xi)-t\xi_1^2)}\frac{\mathrm{Ai}(\zeta_0(\xi/h))}{A_+(\zeta_0(\xi/h))}\hat{f}\Big(\frac{\xi}{h}\Big)\,d\xi, \tag{2-22}$$

satisfies

$$\|A_h f\|_{L^q((0,T],L^r(\mathbb{R}^n))}\le Ch^{-1/q}\|f\|_{L^2(\mathbb{R}^n)}. \tag{2-23}$$

Remark 2.7. We introduce a cutoff function $\chi_1\in C_0^\infty(\mathbb{R}^n)$ equal to 1 on the support of f and to 0 near ∂S. Since χ_1 is supported away from the boundary it follows from [Burq et al. 2004b, Proposition 2.1] (which applies here in its adjoint form) that $\Psi(-h^2\Delta_g)\chi_1 f$ is a pseudodifferential operator and can be written in local coordinates as

$$\Psi(-h^2\Delta_g)\chi_1 f=d(x,hD_x)\chi_2 f+O_{L^2(S)}(h^\infty), \tag{2-24}$$

where $\chi_2\in C_0^\infty(\mathbb{R}^n)$ is equal to 1 on the support of χ_1 and where $d(x,D_x)$ is defined for x in the suitable coordinate patch using the usual pseudodifferential quantization rule,

$$d(x,D_x)f(x)=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\xi}d(x,\xi)\hat{f}(\xi)\,d\xi,\quad d\in C_0^\infty,$$

with symbol d compactly supported for $|\xi|_g^2:=\langle\xi,\xi\rangle_g\in[\alpha_0,\beta_0]$, which follows by the condition of the support of Ψ. Since the principal part of the Laplace operator Δ_g is uniformly elliptic, we can introduce a smooth radial function $\psi\in C_0^\infty\big([\frac{1}{\delta}\alpha_0^{1/2},\delta\beta_0^{1/2}]\big)$ for some $\delta\ge 1$ such that $\psi(|\xi|)d=d$ everywhere. In what follows we shall prove (2-23) where, instead of f we shall write $\psi(|\xi|)f$, keeping in mind that f is supported away from the boundary and localized at spatial frequency $1/h$.

The proof of Theorem 1.3 will be completed once we prove (2-23). To do that, we split the operator A_h into two parts, namely a main term and a diffractive term. To this end, let $\chi(s)$ be a smooth function satisfying

$$\operatorname{supp}\chi\subset(-\infty,-1],\qquad \operatorname{supp}(1-\chi)\subset[-2,\infty).$$

We write this operator as a sum $A_h=M_h+D_h$, by decomposing

$$A_+\big(\zeta(x,\xi)\big)=(\chi A_+)\big(\zeta(x,\xi)\big)+\big((1-\chi)A_+\big)\big(\zeta(x,\xi)\big),$$

and by letting the "main term" be defined for f, as in Remark 2.7, by

$$M_h f(x,t)=\frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n} 2\xi_1\Big(a(x,\xi/h)(\chi A_+)\big(\zeta(x,\xi/h)\big)+b(x,\xi/h)(\chi A'_+)\big(\zeta(x,\xi/h)\big)\Big) \times e^{(i/h)(\theta(x,\xi)-t\xi_1^2)}\frac{\mathrm{Ai}(\zeta_0(\xi/h))}{A_+(\zeta_0(\xi/h))}\psi(|\xi|)\hat{f}\Big(\frac{\xi}{h}\Big)\,d\xi. \tag{2-25}$$

The diffractive term is then defined for f as before by

$$D_h f(x,t)=\frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n}2\xi_1\Big(a(x,\xi/h)\big((1-\chi)A_+\big)\big(\zeta(x,\xi/h)\big)+b(x,\xi/h)\big((1-\chi)A'_+\big)\big(\zeta(x,\xi/h)\big)\Big)\\ \times e^{(i/h)(\theta(x,\xi)-t\xi_1^2)}\frac{\mathrm{Ai}(\zeta_0(\xi/h))}{A_+(\zeta_0(\xi/h))}\psi(|\xi|)\hat{f}\Big(\frac{\xi}{h}\Big)\,d\xi. \tag{2-26}$$

We analyze these operators separately following the ideas of [Smith and Sogge 1995].

The main term M_h. To estimate the main term M_h we first use the fact that

$$\Big|\frac{\mathrm{Ai}(s)}{A_+(s)}\Big|\le 2,\quad \text{for } s\in\mathbb{R}. \tag{2-27}$$

Consequently, since the term $\mathrm{Ai}(\zeta_0)/A_+(\zeta_0)$ acts like a multiplier, as does ξ_1, which by virtue of (2-1) is localized in the interval $[\alpha_0,\beta_0]$, the estimates for M_h will follow from showing that the operator

$$f\to\frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n}\Big(a(x,\xi/h)(\chi A_+)\big(\zeta(x,\xi/h)\big)+b(x,\xi/h)(\chi A'_+)\big(\zeta(x,\xi/h)\big)\Big)\\ \times e^{(i/h)(\theta(x,\xi)-t\xi_1^2)}\psi(|\xi|)\hat{f}\Big(\frac{\xi}{h}\Big)\,d\xi \tag{2-28}$$

satisfies the same bounds as in (2-23) for f spectrally localized at frequency $1/h$. Following [Zworski 1990, Lemma 4.1], we write χA_+ and $(\chi A_+)'$ in terms of their Fourier transform to express the phase function of this operator

$$\phi(t,x,\xi)=-t\xi_1^2+\theta(x,\xi)-\tfrac{2}{3}(-\zeta)^{3/2}(x,\xi), \tag{2-29}$$

which satisfies the eikonal equation (2-9). Let its symbol be $c_m(x,\xi/h)$, with $c_m(x,\xi)\in\mathcal{S}^0_{2/3,1/3}(\mathbb{R}^n\times\mathbb{R}^n)$ and we also denote the operator defined in (2-28) by W_h^m, thus

$$W_h^m f(x,t)=\frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n}e^{(i/h)\phi(t,x,\xi)}c_m(x,\xi/h)\psi(|\xi|)\hat{f}\Big(\frac{\xi}{h}\Big)\,d\xi.$$

Proposition 2.8. *Let (q,r) be an admissible pair with $q>2$, let $T>0$ be sufficiently small and for $f=d(x,D_x)\chi_2 f+O_{L^2(\Omega)}(h^\infty)$ as in Remark 2.7 let*

$$W_h f(x,t):=W_h^m f(x,t)=\frac{1}{(2\pi h)^n}\int e^{(i/h)\phi(t,x,\xi)}c_m(x,\xi/h)\psi(|\xi|)\hat{f}\Big(\frac{\xi}{h}\Big)\,d\xi.$$

Then the following estimates hold:

$$\|W_h f\|_{L^q((0,T],L^r(\mathbb{R}^n))}\le Ch^{-1/q}\|f\|_{L^2(\mathbb{R}^n)}. \tag{2-30}$$

The proof occupies the rest of this section. The first step is a TT* argument. Explicitly,

$$\widehat{W_h^*(F)}\Big(\frac{\xi}{h}\Big)=\int e^{-(i/h)\phi(s,y,\xi)}F(y,s)\overline{c_m(y,\xi/h)}\,dy\,ds,$$

and if we set

$$(T_h F)(x,t) = (W_h W_h^* F)(x,t) = \frac{1}{(2\pi h)^n} \int e^{(i/h)(\phi(t,x,\xi)-\phi(s,y,\xi))} c_m(x,\xi/h)\overline{c_m(y,\xi/h)}\psi^2(|\xi|)F(y,s)\,d\xi\,ds\,dy, \quad (2\text{-}31)$$

then inequality (2-30) is equivalent to

$$\|T_h F\|_{L^q((0,T],L^r(\mathbb{R}^n))} \le C h^{-2/q} \|F\|_{L^{q'}((0,T],L^{r'}(\mathbb{R}^n))}, \quad (2\text{-}32)$$

where q' and r' satisfy $1/q + 1/q' = 1$ and $1/r + 1/r' = 1$. To see, for instance, that (2-32) implies (2-30), notice that the dual version of (2-30) is

$$\|W_h^* F\|_{L^2(\mathbb{R}^n)} \le C h^{-1/q} \|F\|_{L^{q'}((0,T],L^{r'}(\mathbb{R}^n))},$$

and we have

$$\|W_h^* F\|^2_{L^2(\mathbb{R}^n)} = \int W_h W_h^* F \bar{F}\,dt\,dx \le \|T_h F\|_{L^q((0,T],L^r(\mathbb{R}^n))} \|F\|_{L^{q'}((0,T],L^{r'}(\mathbb{R}^n))}. \quad (2\text{-}33)$$

Therefore we only need to prove (2-32). Since the symbols are of type $(\frac{2}{3}, \frac{1}{3})$ and not of type $(1,0)$, before starting the proof of (2-32) for the operator T_h we need to make a further decomposition: Let $\rho \in C_0^\infty(\mathbb{R})$ satisfy $\rho(s) = 1$ near 0 and $\rho(s) = 0$ if $|s| \ge 1$. Let

$$T_h F = T_h^f F + T_h^s F,$$

where

$$T_h^s F(x,t) = \int K_h^s(t,x,s,y) F(y,s)\,ds\,dy \quad (2\text{-}34)$$

and

$$K_h^s(t,x,s,y) = \frac{1}{(2\pi h)^n} \int e^{(i/h)(\phi(t,x,\xi)-\phi(s,y,\xi))} \big(1 - \rho(h^{-1/3}|t-s|)\big) \times c_m(x,\xi/h)\overline{c_m(y,\xi/h)}\psi^2(|\xi|)\,d\xi, \quad (2\text{-}35)$$

while

$$T_h^f F(x,t) = \int K_h^f(t,x,s,y) F(y,s)\,ds\,dy, \quad (2\text{-}36)$$

and

$$K_h^f(t,x,s,y) = \frac{1}{(2\pi h)^n} \int e^{(i/h)(\phi(t,x,\xi)-\phi(s,y,\xi))} \rho\big(h^{-1/3}|t-s|\big) \times c_m(x,\xi/h)\overline{c_m(y,\xi/h)}\psi^2(|\xi|)\,d\xi. \quad (2\text{-}37)$$

Remark 2.9. The two pieces will be handled differently. The kernel of T_h^f is supported in a suitable small set and it will be estimated by "freezing" the coefficients. To estimate T_h^s we shall use the stationary phase method for type $(1,0)$ symbols. For type $(\frac{2}{3}, \frac{1}{3})$ symbols, these stationary phase arguments break down if $|t-s|$ is smaller than $h^{1/3}$, which motivates the decomposition. We use here the same arguments found in [Smith and Sogge 1995].

- The "stationary phase admissible" term T_h^s:

Proposition 2.10. *There is a constant $1 < C_0 < \infty$ such that the kernel K_h^s of T_h^s satisfies*

$$|K_h^s(t,x,s,y)| \le C_N h^N \quad \text{for all } N \quad \text{if } \frac{|t-s|}{|x-y|} \notin [C_0^{-1}, C_0]. \tag{2-38}$$

Moreover, there is a function $\xi_c(t,x,s,y)$ which is smooth in the variables (t,s), uniformly over (x,y), so that if $C_0^{-1} \le |t-s|/|x-y| \le C_0$, then

$$|K_h^s(t,x,s,y)| \lesssim h^{-n}\Big(1+\frac{|t-s|}{h}\Big)^{-n/2} \quad \text{for } |t-s| \ge h^{1/3}. \tag{2-39}$$

Proof. We shall use the stationary phase lemma to evaluate the kernel K_h^s of T_h^s. The critical points occur when $|t-s| \simeq |x-y|$. For some constant C_0 and for $|\xi| \in \operatorname{supp}\psi$, ξ_1 in a small neighborhood of 1, we have

$$|\nabla_\xi(\phi(t,x,\xi)-\phi(s,y,\xi))| \simeq |t-s|+|x-y| \ge h^{1/3} \quad \text{if } \frac{|t-s|}{|x-y|} \notin [C_0^{-1}, C_0].$$

Since $c \in S^0_{2/3,1/3}$, an integration by parts leads to (2-38). If $|t-s| \simeq |x-y|$ we introduce a cutoff function $\kappa\big(|x-y|/|t-s|\big)$, with $\kappa \in C_0^\infty(\mathbb{R}\setminus\{0\})$. The phase function can be written as

$$\phi(t,x,\xi)-\phi(s,y,\xi) = (t-s)\Theta(t,x,s,y,\xi) \quad \text{for} \quad |t-s| \simeq |x-y| \ge h^{1/3}.$$

We want to apply the stationary phase method with parameter $|t-s|/h \ge h^{-2/3} \gg 1$ to estimate K_h^s. For x, y, t, s fixed we must show that the critical points of Θ are nondegenerate.

Lemma 2.11. *If T is sufficiently small, the phase function $\Theta(t,x,s,y,\xi)$ admits a unique, nondegenerate critical point ξ_c. Moreover, for $0 \le t, s \le T$, the function $\xi_c(t,x,s,y)$ solving $\nabla_\xi\Theta(t,x,s,y,\xi_c)=0$ is smooth in t and s, with uniform bounds on derivatives as x and y vary, and we have*

$$|\partial_{t,s}^\alpha \partial_{x,y}^\gamma \xi_c(t,x,s,y)| \le C_{\alpha,\gamma} h^{-|\alpha|/3} \quad \text{if } |x-y| \ge h^{1/3}. \tag{2-40}$$

Proof. The phase $\Theta(t,x,s,y,\xi)$ has the form

$$\begin{aligned}\Theta(t,x,s,y,\xi) &= \xi_1^2 + \frac{1}{t-s}\big(\phi(0,x,\xi)-\phi(0,y,\xi)\big)\\ &= \xi_1^2 + \frac{1}{t-s}\sum_{j=1}^n (x_j-y_j)\partial_{x_j}\phi(0,z_{x,y},\xi),\end{aligned} \tag{2-41}$$

for some $z_{x,y}$ close to x, y (if T is sufficiently small then $|t-s| \simeq |x-y|$ is small), and using the eikonal equations (2-9) we can write

$$\Theta(t,x,s,y,\xi) = \langle\nabla_x\phi, \nabla_x\phi\rangle_g(0,z_{x,y},\xi) - \frac{1}{t-s}\sum_{j=1}^n (x_j-y_j)\partial_{x_j}\phi(0,z_{x,y},\xi).$$

Write $\langle\nabla_x\phi,\nabla_x\phi\rangle_g = \sum_{j,k} g^{j,k}\partial_{x_j}\phi\partial_{x_k}\phi$. We compute $\nabla_\xi\Theta$ explicitly: for each $l \in \{1,\dots,n\}$ we have

$$\partial_{\xi_l}\Theta(t,x,s,y,\xi) = \sum_{j=1}^n \partial^2_{\xi_l,x_j}\phi(0,z_{x,y},\xi)\bigg(2\sum_{k=1}^n g^{j,k}(z_{x,y})\partial_{x_k}\phi(0,z_{x,y},\xi) - \frac{x_j-y_j}{t-s}\bigg). \tag{2-42}$$

Thus

$$\nabla_\xi \Theta(t,x,s,y,\xi) = \nabla^2_{\xi,x}\phi(0,z_{x,y},\xi) \begin{pmatrix} 2\sum_k g^{1,k}(z_{x,y})\partial_{x_k}\phi(0,z_{x,y},\xi) - \frac{x_1-y_1}{(t-s)} \\ \vdots \\ 2\sum_k g^{n,k}(z_{x,y})\partial_{x_k}\phi(0,z_{x,y},\xi) - \frac{x_n-y_n}{(t-s)} \end{pmatrix}, \tag{2-43}$$

where $\nabla^2_{\xi,x}\phi = (\partial^2_{\xi_l,x_j}\phi)_{l,j\in\{1,\dots,n\}}$ is the matrix $n\times n$ whose elements are the second derivatives of ϕ with respect to ξ and x. We need the following lemma:

Lemma 2.12 [Smith and Sogge 1994, Lemma 3.9]. *For ξ in a conic neighborhood of the ξ_1 axis the mapping*

$$x \to \nabla_\xi\big(\theta(x,\xi) - \tfrac{2}{3}(-\zeta)^{3/2}(x,\xi)\big)$$

is a diffeomorphism on the complement of the hypersurface $\zeta = 0$, with uniform bounds on the Jacobian of the inverse mapping.

Corollary 2.13. *If T is small enough and $|x-y| \simeq |t-s| \le 2T$ then*

$$\det(\nabla^2_{\xi,x}\phi)(0,z_{x,y},\xi) \neq 0. \tag{2-44}$$

We now complete the proof of Lemma 2.11. A critical point for Θ satisfies $\nabla_\xi\Theta(t,x,s,y,\xi)=0$ and from (2-43) and (2-44) this translates into

$$\big((g^{j,k}(z_{x,y}))_{j,k}\big)(\nabla_x\phi)^t(0,z_{x,y},\xi) = \frac{x-y}{t-s}. \tag{2-45}$$

Since $(g^{j,k})_{j,k}$ is invertible and using again (2-44) we can apply the implicit function's theorem to obtain (for T small enough) a critical point $\xi_c = \xi_c(t,x,s,y)$ for Θ. To show that ξ_c is nondegenerate we compute

$$\begin{aligned}\partial_{\xi_q}\partial_{\xi_l}\Theta(t,x,s,y,\xi) = \sum_{j=1}^n \partial^3_{\xi_q,\xi_l,x_j}\phi(0,z_{x,y},\xi)\bigg(2\sum_{k=1}^n g^{j,k}(z_{x,y})\partial_{x_k}\phi(0,z_{x,y},\xi) - \frac{(x_j-y_j)}{(t-s)}\bigg)\\ + 2\sum_{j=1}^n \partial^2_{\xi_l,x_j}\phi(0,z_{x,y},\xi)\bigg(\sum_{k=1}^n g^{j,k}(z_{x,y})\partial^2_{\xi_q,x_k}\phi(0,z_{x,y},\xi)\bigg).\end{aligned} \tag{2-46}$$

Consequently at the critical point $\xi=\xi_c$ the hessian matrix $\nabla^2_{\xi,\xi}\Theta$ is given by

$$\nabla^2_{\xi,\xi}\Theta(t,x,s,y,\xi_c) = 2(\nabla^2_{\xi,x}\phi)(g^{ij}(z_{x,y}))_{i,j}(\nabla^2_{\xi,x}\phi)\,\big|_{(0,z_{x,y},\xi_c)},$$

and therefore for T small enough, the critical point ξ_c is nondegenerate by (2-44). □

On the support of κ it follows that the kernel K^s_h has the form

$$K^s_h(t,x,s,y) = \frac{1}{(2\pi h)^n}\int e^{(i/h)|t-s|\Theta(t,x,s,y,\xi)}\psi^2(|\xi|)\big(1-\rho(h^{-1/3}|t-s|)\big)\times c_m(x,\xi/h)\overline{c_m(y,\xi/h)}\,d\xi, \tag{2-47}$$

where, if $\omega = |t-s|/h$ and $\xi_1 \simeq 1$, the symbol satisfies

$$\left|\partial_{t,s}^{\alpha}\partial_{\omega}^{k}\sigma_h\big(t,x,s,y,\omega\xi/|t-s|\big)\right| \le C_{\alpha,k}h^{-|\alpha|/3}\big(|t-s|^{3/2}/h\big)^{-2k/3},$$

where we have set

$$\sigma_h\big(t,x,s,y,\omega\xi/|t-s|\big) = \big(1-\rho(h^{-1/3}|t-s|)\big)c_m\big(x,\omega\xi/|t-s|\big)\overline{c_m\big(y,\omega\xi/|t-s|\big)}.$$

Indeed, since $c_m \in S^0_{2/3,1/3}$, for $\alpha = 0$ one has

$$|\partial_\omega^k\sigma_h| \le |\xi||t-s|^{-k}\left|(\partial_\xi^k c_m)\big(t,x,\omega\xi/|t-s|\big)\right| \le C_{0,k}|t-s|^{-k}(\omega/|t-s|)^{-2k/3} = C_{0,k}|t-s|^{-k}h^{2k/3}.$$

We conclude using the next lemma with $\omega = |t-s|/h$ and $\delta = |t-s|^{3/2} \ge h^{1/2} \gg h$.

Lemma 2.14. *Suppose that $\Theta(z,\xi) \in C^\infty(\mathbb{R}^{2(n+1)}\times\mathbb{R}^n)$ is real, $\nabla_\xi\Theta(z,\xi_c(z)) = 0$, $\nabla_\xi\Theta(z,\xi) \ne 0$ if $\xi \ne \xi_c(z)$, and*

$$|\det \nabla^2_{\xi\xi}\Theta| \ge c_0 > 0 \quad \textit{if } |\xi| \le 1.$$

Suppose also that

$$|\partial_z^\alpha\partial_\xi^\beta\Theta(z,\xi)| \le C_{\alpha,\beta}h^{-|\alpha|/3} \quad \textit{for all } \alpha,\beta.$$

In addition, suppose that the symbol $\sigma_h(z,\xi,\omega)$ vanishes when $|\xi| \ge 1$ and satisfies

$$|\partial_z^\alpha\partial_\xi^\gamma\partial_\omega^k\sigma_h(z,\xi,\omega)| \le C_{k,\alpha,\gamma}h^{-(|\alpha|+|\gamma|)/3}(\delta/h)^{-2k/3} \quad \textit{for all } k,\alpha,\gamma,$$

where on the support of σ_h we have $\omega \ge h^{-2/3}$ and $\delta > 0$. Then we can write

$$\int_{\mathbb{R}^n} e^{i\omega\Theta(z,\xi)}\sigma_h(z,\xi,\omega)\,d\xi = \omega^{-n/2}e^{i\omega\Theta(z,\xi_c(z))}b_h(z,\omega),$$

where b_h satisfies

$$|\partial_\omega^k\partial_z^\alpha b_h(z,\omega)| \le C_{k,\alpha}h^{-|\alpha|/3}(\delta/h)^{-2k/3}$$

and where each of the constants depend only on c_0 and the size of finitely many of the constants $C_{\alpha,\beta}$ and $C_{k,\alpha,\gamma}$ above. In particular, the constants are uniform in δ if $1 \ge \delta \ge h$.

This lemma, used in [Smith and Sogge 1995, Lemma 2.6] and also in Grieser's thesis [1992], follows easily from the proof of the standard stationary phase lemma [Sogge 1993, page 45]. Its application concludes the proof of Proposition 2.10. □

For each t, s, let $T_h^s(t,s)$ be the "frozen" operator defined by

$$T_h^s(t,s)g(x) = \int K_h^s(t,x,s,y)g(y)\,dy.$$

From Proposition 2.10 we deduce

$$\|T_h^s(t,s)g\|_{L^\infty(\mathbb{R}^n)} \le C\max\big(h^{-n},(h|t-s|)^{-n/2}\big)\|g\|_{L^1(\mathbb{R}^n)}. \tag{2-48}$$

Lemma 2.15. *If T is small enough then for t, s fixed the frozen operators $T_h^s(t,s)$, $T_h^f(t,s)$ are bounded on $L^2(\mathbb{R}^n)$; that is, for all $g \in L^2(\mathbb{R}^n)$ we have*

$$\|T_h^s(t,s)g\|_{L^2(\mathbb{R}^n)} \le C\|g\|_{L^2(\mathbb{R}^n)}. \tag{2-49}$$

Proof. If $f \in L^2(\mathbb{R}^n)$ then

$$\|W_h f(\cdot\,,t)\|^2_{L^2(\mathbb{R}^n)} = \frac{1}{(2\pi h)^{2n}} \int_{\xi,\eta}\int_x e^{(i/h)(\phi(t,x,\xi)-\phi(t,x,\eta))} c_m(x,\xi/h)\overline{c_m(x,\eta/h)} \times \psi(|\xi|)\psi(|\eta|)\hat{f}\Big(\frac{\xi}{h}\Big)\hat{\bar{f}}\Big(\frac{\eta}{h}\Big)\,dx\,d\xi\,d\eta. \tag{2-50}$$

From Lemma 2.12 it follows that the mapping

$$\chi := \left(x \to -t(\xi_1+\eta_1,0,\dots,0) + \int_0^1 \nabla_\xi \phi(0,x,(1-w)\xi + w\eta)\,dw\right)$$

is a diffeomorphism away from the hypersurface $\zeta = 0$ with uniform bounds on the Jacobian of χ^{-1}. This change of variables reduces the problem to the L^2-continuity of semiclassical pseudodifferential operators with symbols of type $(\frac{2}{3},\frac{1}{3})$. □

Interpolation between (2-48) and (2-49) with weights $1-2/r$ and $2/r$ respectively yields

$$\|T_h^s(t,s)g\|_{L^r(\mathbb{R}^n)} \le Ch^{-n(1-2/r)}\left(1+\frac{|t-s|}{h}\right)^{-n(1/2-1/r)} \|g\|_{L^{r'}(\mathbb{R}^n)} \tag{2-51}$$

and hence

$$\|T_h^s F\|_{L^q(0,T],L^r(\mathbb{R}^n)} \le Ch^{-n/2(1-2/r)} \left\| \int_{1\ll\frac{|t-s|}{h}}^T |t-s|^{-n/2(1-2/r)} \|F(.,s)\|_{L^{r'}(\mathbb{R}^n)}\,ds \right\|_{L^{q'}((0,T])}.$$

Since $n\big(\frac{1}{2}-\frac{1}{r}\big) = \frac{2}{q} < 1$ the application $|t|^{-2/q} : L^{q'} \to L^q$ is bounded and by Hardy–Littlewood–Sobolev inequality we deduce

$$\|T_h^s F\|_{L^q((0,T],L^r(\mathbb{R}^n))} \le Ch^{-2/q}\|F\|_{L^{q'}((0,T],L^{r'}(\mathbb{R}^n))}. \tag{2-52}$$

- The "frozen" term T_h^f:

To estimate T_h^f it suffices to obtain bounds for its kernel K_h^f with both the variables (t,x) and (s,y) restricted to lie in a cube of $\mathbb{R}^{n+1}$ of side length comparable to $h^{1/3}$. Let us decompose S_T into disjoint cubes $Q = Q_x \times Q_t$ of side length $h^{1/3}$. We then have

$$\|T_h^f F\|^q_{L^q([0,T],L^r(\mathbb{R}^n))} = \int_0^T \Bigg(\sum_{Q=Q_x\times Q_t} \|\chi_Q T_h^f F\|^r_{L^r(Q_x)}\Bigg)^{q/r} dt = \sum_Q \|\chi_Q T_h^f F\|^q_{L^q([0,T],L^r(\mathbb{R}^n))},$$

where by χ_Q we denoted the characteristic function of the cube Q. In fact, by the definition, the integral kernel $K_h^f(t,x,s,y)$ of T_h^f vanishes if $|t-s| \ge h^{1/3}$. If $|t-s| \le h^{1/3}$ and $|x-y| \ge C_0 h^{1/3}$, then the phase

$$\phi(t,x,\xi) - \phi(s,y,\xi)$$

has no critical points with respect to ξ_1 (on the support of ψ), so that

$$|K_h^f(t,x,s,y)| \le C_N h^N \quad \text{for all } N \qquad \text{if } |x-y| \ge C_0 h^{1/3}.$$

It therefore suffices to estimate $\|\chi_Q T_h^f \chi_{Q^*} F\|_{L^q([0,T],L^r(\mathbb{R}^n))}$, where Q^* is the dilate of Q by some fixed factor independent of h. Since $q > 2 > q'$, $r \ge 2 \ge r'$, where q', r' are such that $1/q + 1/q' = 1$,

$1/r + 1/r' = 1$, we shall obtain

$$\sum_Q \|\chi_Q T_h^f \chi_{Q^*} F\|^q_{L^q([0,T],L^r(\mathbb{R}^n))} \le C_1 \sum_Q \|\chi_{Q^*} F\|^q_{L^{q'}([0,T],L^{r'}(\mathbb{R}^n))} \le C_2 \|F\|^q_{L^{q'}([0,T],L^{r'}(\mathbb{R}^n))}. \tag{2-53}$$

To prove (2-53) we shall use the following proposition:

Proposition 2.16. *Let $b(\xi) \in L^\infty(\mathbb{R}^n)$ be elliptic near $\xi_1 \simeq 1$, $b_h(\xi) := b(\xi/h)$, then for $h \ll |t-s| \le h^{1/3}$, $h \ll |x-y| \le h^{1/3}$ the operator defined by*

$$B_h f(x,t) = \frac{1}{(2\pi h)^n} \int e^{(i/h)\phi(t,x,\xi)} \psi(|\xi|) b_h(\xi) \hat{f}\Big(\frac{\xi}{h}\Big)\, d\xi \tag{2-54}$$

satisfies

$$\|B_h f\|_{L^q((0,T],L^r(\mathbb{R}^n))} \le C h^{-1/q} \|f\|_{L^2(\mathbb{R}^n)}. \tag{2-55}$$

Proof. We use again the TT* argument. Since $b(\xi)$ acts as an L^2 multiplier we can apply the stationary phase theorem in the integral

$$\int e^{(i/h)(\phi(t,x,\xi)-\phi(s,y,\xi))} \psi(|\xi|)\, d\xi$$

to obtain

$$\|B_h B_h^* F\|_{L^q((0,T],L^r((\mathbb{R}^n)))} \lesssim h^{-2/q} \|F\|_{L^{q'}((0,T],L^{r'}(\mathbb{R}^n))}.$$

Notice that we haven't used the special properties of the phase function at $t = 0$. □

Let now Q be a fixed cube in $\mathbb{R}^{n+1}$ of side length $h^{1/3}$. Let

$$b_h(t,x,s,y,\xi) = \rho(h^{-1/3}|t-s|) c_m(x,\xi/h) \overline{c_m(y,\xi/h)},$$

and write

$$b_h(t,x,s,y,\xi) = b_h(0,0,s,y,\xi) + \int_0^t \partial_t b_h(r,0,s,y,\xi)\, dr + \dots + \int_0^t \cdots \int_0^{x_n} \partial_t \cdots \partial_{x_n} b_h(r,z_1,\dots,z_n,s,y,\xi)\, dr\, dz. \tag{2-56}$$

If the symbol c is independent of t and x, the estimates (2-30) follow from Proposition 2.16. We use this, for instance, to deduce

$$\|\chi_Q T_h^f \chi_{Q*} F\|_{L^q((0,T],L^r(\mathbb{R}^n))} \le C h^{-n/2(1/2-1/r)} \tag{2-57}$$
$$\times \Bigg(\bigg\| \iint e^{(i/h)(x\xi-\phi(s,y,\xi))} \psi(|\xi|) b_h(0,0,s,y,\xi) F(y,s)\, d\xi\, ds\, dy \bigg\|_{L^2(\mathbb{R}^n)}$$
$$+ \dots + \int_0^{h^{1/3}} \int_0^{h^{1/3}} \bigg\| \iint e^{(i/h)(x\xi-\phi(s,y,\xi))} \partial_t \dots \partial_{x_n} \psi(|\xi|) b_h(r,z,s,y,\xi) F(y,s)\, d\xi\, ds\, dy \bigg\|_{L^2(\mathbb{R}^n)} dr\, dz \Bigg).$$

Each derivative of $b_h(t,x,s,y,\xi)$ loses a factor of $h^{-1/3}$, but this is compensated by the integral over (r,z), so that it suffices to establish uniform estimates for fixed (r,z). By duality, we have to establish the estimate

$$\bigg\| \iint e^{(i/h)\phi(s,y,\xi)} \psi(|\xi|) b_h(0,0,s,y,\xi) \hat{f}\Big(\frac{\xi}{h}\Big)\, d\xi \bigg\|_{L^q((0,T],L^r(\mathbb{R}^n))} \le C \|f\|_{L^2(\mathbb{R}^n)},$$

which follows by using the same argument of freezing the variables (s, y) together with Proposition 2.16.

The diffractive term D_h**.** To estimate the diffractive term we shall proceed again as in [Smith and Sogge 1995, Section 2].

Lemma 2.17. *For $x_n \geq 0$ and for ξ in a small conic neighborhood of the positive ξ_1 axis, the symbol q of S_h can be written in the form*

$$\begin{aligned} q(x,\xi) &:= \big(a(x,\xi)((1-\chi)A_+)(\zeta(x,\xi)) + b(x,\xi)((1-\chi)A_+)'(\zeta(x,\xi))\big)\frac{\mathrm{Ai}(\zeta_0(\xi))}{A_+(\zeta_0(\xi))} \\ &= p(x,\xi,\zeta(x,\xi)), \end{aligned}$$

where, for some $c > 0$

$$\big|\partial_\xi^\alpha \partial_\zeta^j \partial_{x'}^\beta \partial_{x_n}^k p\big(x,\xi,\zeta(x,\xi)\big)\big| \leq C_{\alpha,j,\beta,k}\xi_1^{1/6-|\alpha|+2k/3} e^{-cx_n^{3/2}\xi_1 - |\zeta|^{3/2}/2}.$$

Proof. Since

$$\big|\partial_\zeta^k((1-\chi)A_+)(\zeta)\big| \leq C_{k,\epsilon} e^{(2/3+\epsilon)|\zeta|^{3/2}} \quad \text{for all } \epsilon > 0$$

and a and b belong to $S^{1/6}_{1,0}$, the result will follow by showing that $\dfrac{\mathrm{Ai}}{A_+}(\zeta_0(\xi)) = \tilde p(x,\xi',\zeta(x,\xi))$ in the region $\zeta(x,\xi) \geq -2$, where, if $\xi' = (\xi_1,\dots,\xi_{n-1})$,

$$\big|\partial_{\xi'}^\alpha \partial_\zeta^j \partial_{x'}^\beta \partial_{x_n}^k \tilde p(x,\xi',\zeta)\big| \leq C_{\alpha,j,\beta,k,\epsilon}\xi_1^{-|\alpha|+2k/3} e^{-cx_n^{3/2}\xi_1-(4/3-\epsilon)|\zeta|^{3/2}}. \tag{2-58}$$

At $x_n = 0$, one has $\zeta = \zeta_0$, $\partial_{x_n}\zeta < 0$. It follows that for some $c > 0$

$$\zeta_0(x,\xi) \geq \zeta(x,\xi) + cx_n\xi_1^{2/3}.$$

By the asymptotic behavior of the Airy function we have, in the region $\zeta(x,\xi) \geq -2$

$$\Big|\Big(\frac{\mathrm{Ai}}{A_+}\Big)^{(k)}(\zeta_0)\Big| \leq C_{k,\epsilon} e^{-cx_n^{3/2}\xi_1-(4/3-\epsilon)|\zeta(x,\xi)|^{3/2}}. \tag{2-59}$$

We introduce a new variable $\tau(x,\xi) = \xi_1^{1/3}\zeta(x,\xi)$. At $x_n = 0$ one has $\tau = -\xi_n$, so that we can write $\xi_n = \sigma(x,\xi',\tau)$, where σ is homogeneous of degree 1 in (ξ',τ). We set

$$\tilde p(x,\xi',\zeta) = \frac{\mathrm{Ai}}{A_+}(-\xi_1^{-1/3}\sigma(x,\xi',\xi^{1/3}\zeta)).$$

The estimates (2-58) will follow by showing that

$$\Big|\partial_{\xi'}^\alpha \partial_\tau^j \partial_{x'}^\beta \partial_{x_n}^k \frac{\mathrm{Ai}}{A_+}(-\xi_1^{-1/3}\sigma(x,\xi',\tau))\Big| \leq C_{\alpha,j,\beta,k,\epsilon}\xi_1^{-|\alpha|-j+2k/3} e^{-cx_n^{3/2}\xi_1-(4/3-\epsilon)|\tau|^{3/2}\xi_1^{-1/2}}. \tag{2-60}$$

For $k = 0$, the estimates (2-60) follow from (2-59), together with the fact that

$$\Big|\partial_{\xi'}^\alpha \partial_\tau^j \partial_{x'}^\beta \frac{\mathrm{Ai}}{A_+}(-\xi_1^{-1/3}\sigma(x,\xi',\tau))\Big| \leq C_{\alpha,\beta,j}\big(x_n\xi_1^{2/3} + \xi_1^{-1/3}|\tau|\big)\xi_1^{-|\alpha|-j},$$

which, in turn, holds by homogeneity, together with the fact that $\sigma(x,\xi',\tau) = 0$ if $x_n = \tau = 0$. If $k > 0$, the estimate (2-60) follows by observing that the effect of differentiating in x_n is similar to multiplying by a symbol of order $2/3$. This concludes the proof of Lemma 2.17. ☐

Lemma 2.18. *The Schwartz kernel of the diffractive term D_h can be written in the form*

$$\int e^{i(\theta(x,\xi)-ht\xi_1^2)}\psi(h|\xi|)q(x,\xi)\,d\xi = \int e^{i(\theta(x,\xi)-ht\xi_1^2+\sigma\xi_1^{-2/3}\zeta(x,\xi)+\sigma^3/3\xi_1^2-\langle y,\xi\rangle)}\psi(h|\xi|)c_d(x,\xi,\sigma)\,d\sigma\,d\xi, \quad (2\text{-}61)$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product and where

$$|\partial_\xi^\alpha\partial_\sigma^j\partial_{x'}^\beta\partial_{x_n}^k c_d(x,\xi,\sigma)| \le C_{\alpha,j,\beta,k,N}\xi_1^{-1/2-|\alpha|-2j/3+2k/3}e^{-cx_n^{3/2}\xi_1}(1+\xi_1^{-4/3}\sigma^2)^{-N/2} \quad \textit{for all } N.$$

Proof. The symbol c_d of the Schwartz kernel of D_h can be expressed as a product of two symbols

$$c_d(x,\xi,\sigma)=c_1(x,\xi,\sigma\xi_1^{-2/3})c_2\big(x,\xi,\zeta(x,\xi)\big),$$

where

$$c_1(x,\xi,\sigma\xi_1^{-2/3})=\xi_1^{-2/3}\Psi_+(\xi_1^{-2/3}\sigma)\big(a(x,\xi)+\sigma\xi_1^{-2/3}b(x,\xi)\big)\in S^{-1/2}_{2/3,1/3}(\mathbb{R}^n_x,\mathbb{R}^{n+1}_{\xi,\sigma})$$

comes from the Fourier transform of A_+ (here Ψ_+ is a symbol of order 0) and where c_2 satisfies for all $N\ge 0$ (for $\sigma^2\xi_1^{-4/3}+\zeta(x,\xi)=0$)

$$\big|\partial_{\xi'}^\alpha\partial_\sigma^j\partial_{x'}^\beta\partial_{x_n}^k c_2\big(x,\xi',-(\sigma^2\xi_1^{-4/3})\big)\big| \le C_{\alpha,j,\beta,k,N}\xi_1^{-2j/3}|\sigma\xi_1^{-2/3}|^j\xi_1^{-|\alpha|+2k/3}e^{-cx_n^{3/2}\xi_1}(1+\xi_1^{-4/3}\sigma^2)^{-N/2}, \quad (2\text{-}62)$$

which follows from (2-58). We use the exponential factor $e^{-cx_n^{3/2}\xi_1}$ to deduce from (2-62) that

$$|x_n^j\partial_{x_n}^k c_2(x,\xi',-(\sigma^2\xi_1^{-4/3}))| \le C_{j,k,N}(x_n\xi_1^{2/3})^j e^{-c(x_n\xi_1^{2/3})^{3/2}}\xi_1^{2/3(k-j)}(1+\xi_1^{-4/3}\sigma^2)^{-N/2} \quad \text{for all } N. \ \square$$

From now on we proceed as for the main term and we reduce the problem to considering the operator

$$W_h^d f(x,t)=\frac{1}{(2\pi h)^n}\int e^{(i/h)\tilde\phi(t,x,\xi,\sigma)}c_d(x,\xi/h,\sigma)\psi(|\xi|)\hat f\Big(\frac{\xi}{h}\Big)\,d\xi,$$

where $x_n^j\partial_{x_n}^k c_d\in S^{2(k-j)/3}_{2/3,1/3}(\mathbb{R}^{n-1}_{x'}\times\mathbb{R}^n_\xi)$ uniformly over x_n and where we have set

$$\tilde\phi(t,x,\xi,\sigma):=-t\xi_1^2+\theta(x,\xi)+\sigma\xi_1^{1/3}\zeta(x,\xi)+\tfrac13\xi_1\sigma^3, \quad (2\text{-}63)$$

obtained after the changes of variables $\sigma\to\sigma\xi_1$, $\xi\to\xi/h$ in (2-61). Using the freezing arguments behind the proof of the estimates for T_h^f and Minkowski inequality we have

$$\begin{aligned}\|W_h^d f\|_{L^q((0,T],L^r(\mathbb{R}^n))} &\le \Big\|\frac{1}{(2\pi h)^n}\int e^{(i/h)\tilde\phi(t,x,\xi,\sigma)}c_d(x',0,\xi/h,\sigma)\psi(|\xi|)\hat f\Big(\frac{\xi}{h}\Big)\,d\sigma\,d\xi\Big\|_{L^q((0,T],L^r(\mathbb{R}^n))}\\ &+h^{-2/3}\int_0^{h^{2/3}}\Big\|\frac{1}{(2\pi h)^n}\int e^{(i/h)\tilde\phi(t,x,\xi,\sigma)}h^{2/3}\partial_{x_n}c_d(x',r,\xi/h,\sigma)\psi(|\xi|)\hat f\Big(\frac{\xi}{h}\Big)\,d\sigma\,d\xi\Big\|_{L^q((0,T],L^r(\mathbb{R}^{n-1}))}dr\\ &+h^{2/3}\int_{r>h^{2/3}}\frac{dr}{r^2}\Big\|\frac{1}{(2\pi h)^n}\int e^{\frac{i}{h}\tilde\phi(t,x,\xi,\sigma)}h^{-2/3}r^2\partial_{x_n}c_d(x',r,\xi/h,\sigma)\psi(|\xi|)\hat f\Big(\frac{\xi}{h}\Big)\,d\sigma\,d\xi\Big\|_{L^q((0,T],L^r(\mathbb{R}^{n-1}))}.\end{aligned}$$

Since $c_d(x',0,\xi,\sigma)$ and $h^{2/3}(1+h^{-4/3}r^2)\partial_{x_n}c_d(x',r,\xi,\sigma)$ are symbols of order 0 and type $(\frac23,\frac13)$ with uniform estimates over r, the estimates for the diffractive term also follow from Proposition 2.8. Indeed, the term on the second line loses a factor $h^{-2/3}$, but this is compensated by the integral over $r \le h^{2/3}$. The term on the last line can be bounded by above by

$$h^{2/3}\int_{r>h^{2/3}} \frac{dr}{r^2} \left\| \frac{1}{(2\pi h)^n} \int e^{(i/h)\tilde{\phi}(t,x,\xi,\sigma)} (h^{-2/3} r^2 \partial_{x_n} c_d(x', r, \xi/h, \sigma)) \psi(|\xi|) \hat{f}\Big(\frac{\xi}{h}\Big)\, d\sigma\, d\xi \right\|_{L^q((0,T],L^r(\mathbb{R}^n))}$$
$$\le \left\| \frac{1}{(2\pi h)^n} \int e^{(i/h)\tilde{\phi}(t,x,\xi,\sigma)} (h^{-2/3} r^2 \partial_{x_n} c_d(x', r, \xi/h)) \psi(|\xi|) \hat{f}\Big(\frac{\xi}{h}\Big)\, d\sigma\, d\xi \right\|_{L^q((0,T],L^r(\mathbb{R}^n))}.$$

We conclude by using the same arguments as in the proof of Proposition 2.8, where now W_h is replaced by operators with symbols $c_d(x', 0, \xi, \sigma)$, However, for this term we can't directly apply Lemma 2.11, since the expansion of the Airy function giving the phase function (2-29) is available only for $\zeta(x, \xi/h) \le -1$. Writing the phase function of (2-61) in the form $\tilde{\phi}(t, x, \xi, \sigma) - \langle y, \xi\rangle$, we notice that at $t = 0$ this phase is homogeneous of degree 1 in ξ and the proof of the nondegeneracy of the critical points in the TT* argument of Lemma 2.11 reduces to checking that the Jacobian J of the mapping

$$(\xi, \sigma) \to (\nabla_x(\theta(x, \xi) + \sigma\zeta(x, \xi)), \zeta(x, \xi) + \sigma^2) \tag{2-64}$$

does not vanish at the critical point of the phase of (2-61). Hence we will obtain a phase function $\breve{\phi}(t, x, \xi)$ which will satisfy $\nabla^2_{x,\xi}\breve{\phi}(0, x, \xi) \neq 0$ and this will hold also for small $|t| \le T$ and we can use the same argument as in Lemma 2.11. To prove that the Jacobian of the application (2-64) doesn't vanish we use [Smith and Sogge 1994, Lemma A.2]. Precisely, at this (critical) point $\sigma = \zeta(x, \xi) = 0$, $y = 0$, and $\nabla_{x'}\zeta(x, \xi) = 0$. Since $\partial_{x_n}\zeta(x, \xi) \neq 0$ and $\partial_{\xi_n}\zeta(x, \xi) \neq 0$ at this point, the result follows by the nonvanishing of $|\nabla_{x'}\nabla_{\xi'}\theta(x, \xi)|$. In fact we have

$$\det \begin{pmatrix} \nabla_{x'}\nabla_{\xi'}\theta & \nabla_{\xi'}\partial_{x_n}\theta & \nabla_{\xi'}\zeta \\ \partial_{\xi_n}\nabla_{x'}\theta & \partial_{\xi_n}\partial_{x_n}\theta & \partial_{\xi_n}\zeta \\ \nabla_{x'}\zeta & \partial_{x_n}\zeta & 2\sigma \end{pmatrix}\Bigg|_{\sigma^2=-\zeta=0} \neq 0.$$

3. Strichartz estimates for the classical Schrödinger equation outside a strictly convex obstacle in $\mathbb{R}^n$

In this section we prove Theorem 1.7 under Assumption 1.6. We shall work with the Laplace operator with constant coefficients $\Delta_D = \sum_{j=1}^n \partial_j^2$ acting on $L^2(\Omega)$ to avoid technicalities, where Ω is the exterior in $\mathbb{R}^n$ of a strictly convex domain Θ.

In the proof of Theorem 1.7 we distinguish two main steps. We start by performing a time rescaling which transforms the Equation (1-8) into a semiclassical problem. Due to the finite speed of propagation (proved by Lebeau [1992]), we can use the (local) semiclassical result of Theorem 1.3 together with the smoothing effect (following Staffilani and Tataru [2002] and Burq [2002]) to obtain classical Strichartz estimates near the boundary. Outside a fixed neighborhood of $\partial\Omega$ we use a method suggested by Staffilani and Tataru [2002] which considers the Schrödinger flow as a solution of a problem in the whole space $\mathbb{R}^n$, for which the Strichartz estimates are known.

We start by proving that using Theorem 1.3 on a compact manifold with strictly concave boundary we can deduce sharp Strichartz estimates for the semiclassical Schrödinger flow on Ω. More precisely, we prove the following result, and then show how it can be used to prove Theorem 1.7.

Proposition 3.1. *Given (q, r) satisfying the scaling condition* (1-3) *with $q > 2$ there exists a constant $C > 0$ such that the* (*classical*) *Schrödinger flow on $\Omega\times\mathbb{R}$ with Dirichlet boundary condition and spectrally localized initial data $\Psi(-h^2\Delta_D)u_0$, where $\Psi \in C_0^\infty(\mathbb{R}\setminus 0)$, satisfies*

$$\left\| e^{it\Delta_D}\Psi(-h^2\Delta_D)u_0\right\|_{L^q(\mathbb{R})L^r(\Omega)} \le C\left\|\Psi(-h^2\Delta_D)u_0\right\|_{L^2(\Omega)}. \tag{3-1}$$

Proof. We use a method similar to the one given in our recent paper [Ivanovici and Planchon 2009] in collaboration with F. Planchon. Let $\tilde{\Psi} \in C_0^\infty(\mathbb{R}\setminus\{0\})$ be such that $\tilde{\Psi} = 1$ on the support of Ψ, hence

$$\tilde{\Psi}(-h^2\Delta_D)\Psi(-h^2\Delta_D) = \Psi(-h^2\Delta_D).$$

Following [Burq 2002; Ivanovici and Planchon 2009], we split $e^{it\Delta_D}\Psi(-h^2\Delta_D)u_0(x)$ as a sum of two terms,

$$\tilde{\Psi}(-h^2\Delta_D)\chi\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0 + \tilde{\Psi}(-h^2\Delta_D)(1-\chi)\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0,$$

where $\chi \in C_0^\infty(\mathbb{R}^n)$ equals 1 in a neighborhood of $\partial\Omega$.

- Study of $\tilde{\Psi}(-h^2\Delta_D)(1-\chi)\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0$:
  Set $w_h(x, t) = (1-\chi)\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0(x)$. Then w_h satisfies

$$\begin{cases} i\partial_t w_h + \Delta_D w_h = -[\Delta_D, \chi]\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0, \\ w_h|_{t=0} = (1-\chi)\Psi(-h^2\Delta_D)u_0. \end{cases} \tag{3-2}$$

Since χ is equal to 1 near the boundary $\partial\Omega$, the solution to (3-2) also solves a problem in the whole space $\mathbb{R}^n$. Consequently, the Duhamel formula gives

$$w_h(t, x) = e^{it\Delta}(1-\chi)\Psi(-h^2\Delta_D)u_0 - \int_0^t e^{i(t-s)\Delta}[\Delta_D, \chi]\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0(s)\,ds, \tag{3-3}$$

where Δ denotes the free Laplacian on $\mathbb{R}^n$ and therefore the contribution of $e^{it\Delta}(1-\chi)\Psi(-h^2\Delta_D)u_0$ satisfies the usual Strichartz estimates. For the second term on the right in (3-3) we use the next lemma:

Lemma 3.2 [Christ and Kiselev 2001]. *Consider a bounded operator*

$$T : L^{q'}(\mathbb{R}, B_1) \to L^q(\mathbb{R}, B_2)$$

given by a locally integrable kernel $K(t, s)$ with values in bounded operators from B_1 to B_2, where B_1 and B_2 are Banach spaces. Suppose that $q' < q$. Then the operator

$$\tilde{T}f(t) = \int_{s<t} K(t, s)f(s)\,ds$$

is bounded from $L^{q'}(\mathbb{R}, B_1)$ to $L^q(\mathbb{R}, B_2)$ and

$$\|\tilde{T}\|_{L^{q'}(\mathbb{R},B_1)\to L^q(\mathbb{R},B_2)} \le C(1-2^{-(1/q-1/q')})^{-1}\|T\|_{L^{q'}(\mathbb{R},B_1)\to L^q(\mathbb{R},B_2)}.$$

Since $q > 2$, this lemma allows us to replace the study of the second term in the right-hand side of (3-3) by that of

$$\int_0^\infty e^{i(t-s)\Delta}[\Delta_D, \chi]\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0(s)(s)\,ds =: U_0U_0^* f(x, t),$$

where $U_0 = e^{it\Delta}$ is bounded from $L^2(\mathbb{R}^n)$ to $L^q(\mathbb{R}, L^r(\mathbb{R}^n))$ and U_0^* is bounded from $L^2(\mathbb{R}, H^{-1/2}_{\text{comp}})$ to $L^2(\mathbb{R}^n)$ and where we set $f := [\Delta_D, \chi]\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0$ which belongs to $L^2 H^{-1/2}_{\text{comp}}(\Omega)$ by Burq et al. [2004a, Proposition 2.7]. The estimates for w_h follow as in [Burq et al. 2004a] and we find

$$\|w_h\|_{L^q(\mathbb{R}, L^r(\Omega))} \le C\|(1-\chi)\Psi(-h^2\Delta_D)u_0\|_{L^2(\mathbb{R}^n)} + \|[\Delta_D, \chi]\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0\|_{L^2(\mathbb{R}, H^{-1/2}_{\text{comp}}(\Omega))}. \tag{3-4}$$

The last term in (3-4) can be estimated using [Burq et al. 2004a, Proposition 2.7] by

$$C\|\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0\|_{L^2(\mathbb{R}, H^{1/2}_{\text{comp}}(\Omega))} \le C\|\Psi(-h^2\Delta_D)u_0\|_{L^2(\Omega)}. \tag{3-5}$$

Finally, we conclude this part using [Ivanovici and Planchon 2008, Theorem 1.1] which gives

$$\|\Psi(-h^2\Delta_D)w_h\|_{L^r(\Omega)} \le \|w_h\|_{L^r(\Omega)}. \tag{3-6}$$

- Study of $\tilde{\Psi}(-h^2\Delta_D)\chi\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0$:
  Let $\varphi \in C_0^\infty((-1, 2))$ equal to 1 on [0, 1]. For $l \in \mathbb{Z}$ set

$$v_{h,l} = \varphi(t/h - l)\chi\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0, \tag{3-7}$$

$$V_{h,l} = \Big(\varphi(t/h - l)[\Delta_D, \chi] + i\frac{\varphi'(t/h - l)}{h}\chi\Big)\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0. \tag{3-8}$$

The quantity in (3-7) is a solution to

$$\begin{cases} i\partial_t v_{h,l} + \Delta_D v_{h,l} = V_{h,l}, \\ v_{h,l}|_{t<hl-h} = 0, \quad v_{h,l}|_{t>hl+2h} = 0. \end{cases} \tag{3-9}$$

Let $Q \subset \mathbb{R}^n$ be an open cube sufficiently large such that $\partial\Omega$ is contained in the interior of Q. We denote by S the punctured torus obtained from removing the obstacle Θ (recall that $\Omega = \mathbb{R}^n \setminus \Theta$) in the compact manifold obtained from Q with periodic boundary conditions on ∂Q. Notice that S, when defined in this way, coincides with the Sinai billiard. Let $\Delta_S := \sum_{j=1}^n \partial_j^2$ denote the Laplace operator on the compact domain S.

On S, we may define a spectral localization operator using eigenvalues λ_k and eigenvectors e_k of Δ_S: if $f = \sum_k c_k e_k$, then

$$\Psi(-h^2\Delta_S)f = \sum_k \Psi(-h^2\lambda_k^2)c_k e_k. \tag{3-10}$$

Remark 3.3. In a neighborhood of the boundary, the domains of Δ_S and Δ_D coincide, thus if $\tilde{\chi} \in C_0^\infty(\mathbb{R}^n)$ is supported near $\partial\Omega$ then $\Delta_S\tilde{\chi} = \Delta_D\tilde{\chi}$.

Now let $\tilde{\chi} \in C_0^\infty(\mathbb{R}^n)$ be equal to 1 on the support of χ and be supported in a neighborhood of $\partial\Omega$ such that, on its support, the operator $-\Delta_D$ coincides with $-\Delta_S$. From their respective definitions, we know that $v_{h,l} = \tilde{\chi}v_{h,l}$ and $V_{h,l} = \tilde{\chi}V_{h,l}$; consequently $v_{h,l}$ will also solve, on the compact domain S, the equation

$$\begin{cases} i\partial_t v_{h,l} + \Delta_S v_{h,l} = V_{h,l}, \\ v_{h,l}|_{t<h(l-1/2)\pi} = 0, \quad v_{h,l}|_{t>h(l+1)\pi} = 0. \end{cases} \tag{3-11}$$

Writing the Duhamel formula for the last equation in (3-11) on S, applying $\tilde\Psi(-h^2\Delta_D)$, and using that $\tilde\chi v_{h,l}=v_{h,l}$, $\tilde\chi V_{h,l}=V_{h,l}$ and writing

$$\tilde\Psi(-h^2\Delta_D)\tilde\chi=\chi_1\tilde\Psi(-h^2\Delta_S)\tilde\chi+(1-\chi_1)\tilde\Psi(-h^2\Delta_D)\tilde\chi+\chi_1\big(\tilde\Psi(-h^2\Delta_D)-\tilde\Psi(-h^2\Delta_S)\big)\tilde\chi \tag{3-12}$$

for some $\chi_1\in C_0^\infty(\mathbb{R}^n)$ equal to 1 on the support of $\tilde\chi$, we obtain

$$\begin{aligned}\tilde\Psi(-h^2\Delta_D)v_{h,l}(x,t)=\chi_1\int_{hl-l}^{t}&e^{i(t-s)\Delta_S}\tilde\Psi(-h^2\Delta_S)V_{h,l}(x,s)\,ds\\&+(1-\chi_1)\int_{hl-l}^{t}\tilde\Psi(-h^2\Delta_D)e^{i(t-s)\Delta_S}V_{h,l}(x,s)\,ds\\&\qquad+\chi_1\big(\tilde\Psi(-h^2\Delta_D)-\tilde\Psi(-h^2\Delta_S)\big)v_{h,l}.\end{aligned} \tag{3-13}$$

Denote by $v_{h,l,m}$ the first term of (3-13), by $v_{h,l,f}$ the second one, and by $v_{h,l,s}$ the last one. We deal with them separately. To estimate the $L_t^qL^r(\Omega)$ norm of $v_{h,l,f}$ we notice that it is supported away from the boundary and therefore the estimates will follow as in the previous part of this section. Indeed, notice that since $v_{h,l}$ also solves (3-7) on Ω, we can use the Duhamel formula on Ω so that in the integral we can define $v_{h,l,f}$ to have Δ_D instead of Δ_S. We then estimate the $L_t^qL^r(\Omega)$ norm of $v_{h,l,f}$ by applying the Minkowski inequality and using the sharp Strichartz estimates for $(1-\chi_1)\tilde\Psi(-h^2\Delta_D)e^{i(t-s)\Delta_D}V_{h,l}$ deduced in the first part of the proof of Proposition 3.1 and obtain, denoting $I_l^h=[hl-h,hl+2h]$,

$$\|v_{h,l,f}\|_{L^q(I_l^h,L^r(\Omega))}\le C\int_{I_l^h}\|V_{h,l}(x,s)\|_{L^2(\Omega)}\,ds. \tag{3-14}$$

For the last term $v_{h,l,s}$ we use the following lemma, which will be proved in Appendix B:

Lemma 3.4. *Let $\chi_1\in C_0^\infty(\mathbb{R}^n)$ be equal to 1 on a fixed neighborhood of the support of $\tilde\chi$. Then we have*

$$\|v_{h,l,s}\|_{L^q(I_l^h,L^r(\Omega))}\le C_Nh^N\|V_{h,l}(x,s)\|_{L^2(I_l^h,H_0^{n(\frac12-\frac1r)-\frac12}(\Omega))}\quad \textit{for all } N\in\mathbb{N}. \tag{3-15}$$

To estimate the main contribution $v_{h,l,m}$ we use the Minkowski inequality, which yields

$$\|v_{h,l,m}\|_{L^q(I_l^h,L^r(\Omega))}=\|v_{h,l,m}\|_{L^q(I_l^h,L^r(S))}\le C\int_{I_l^h}\big\|e^{i(t-s)\Delta_S}\tilde\Psi(-h^2\Delta_S)V_{h,l}(x,s)\big\|_{L^q(I_l^h,L^r(S))}\,ds. \tag{3-16}$$

Applying Theorem 1.3 for the linear semiclassical Schrödinger flow on S, the term to integrate in (3-16) is bounded by $C\|\tilde\Psi(-h^2\Delta_S)V_{h,l}(x,s)\|_{L^2(S)}$. Using [Ivanovici and Planchon 2008, Theorem 1.1] and the fact that $\tilde\chi V_{h,l}=V_{h,l}$ (so that taking the norm over Ω or S makes no difference) we obtain

$$\|v_{h,l,m}\|_{L^q(I_l^h,L^r(\Omega))}\le C\int_{I_l^h}\|V_{h,l}(x,s)\|_{L^2(\Omega)}\,ds. \tag{3-17}$$

After applying the Cauchy–Schwartz inequality in Equations (3-14) and (3-17) it remains to estimate the $L^2(I_l^h,H^\sigma(\Omega))$ norm of $V_{h,l}$, where $\sigma\in\big\{0,n\big(\frac12-\frac1r\big)-\frac12\big\}$. We do this using the precise form (3-8) and obtain

$$\begin{aligned}\|V_{h,l}\|_{L^2(I_l^h,H^\sigma(\Omega))}\le{}&C\big\|\varphi(t/h-l)[\Delta_D,\chi]\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0\big\|_{L^2(I_l^h,H^\sigma(\Omega))}\\&+Ch^{-1}\big\|\varphi'(t/h-l)\chi\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0\big\|_{L^2(I_l^h,H^\sigma(\Omega))}.\end{aligned} \tag{3-18}$$

Since the operator $[\Delta_D, \chi]\Psi(-h^2\Delta_D)$ is bounded from $H^{\sigma+1}$ to H^{σ}, we deduce from (3-13), (3-14), (3-18), (3-19), and Lemma 3.4 the following bound (the last two lines differing only in the superscript of H_0):

$$\begin{aligned}\|\tilde{\Psi}(-h^2\Delta_D)v_{h,l}\|_{L^q(I_l^h,L^r(\Omega))} \le{}& Ch^{1/2}\big\|\tilde{\varphi}(t/h-l)\tilde{\chi}\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0\big\|_{L^2(I_l^h,H_0^1(\Omega))}\\ &+Ch^{-1/2}\big\|\tilde{\varphi}(t/h-l)\tilde{\chi}\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0\big\|_{L^2(I_l^h,L^2(\Omega))}\\ &+C_Nh^{N+1/2}\big\|\tilde{\varphi}(t/h-l)\tilde{\chi}\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0\big\|_{L^2(I_l^h,H_0^{n(\frac12-\frac1r)+\frac12}(\Omega))}\\ &+C_Nh^{N-1/2}\big\|\tilde{\varphi}(t/h-l)\tilde{\chi}\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0\big\|_{L^2(I_l^h,H_0^{n(\frac12-\frac1r)-\frac12}(\Omega))},\end{aligned}\tag{3-19}$$

where $\tilde{\varphi}\in C_0^\infty(\mathbb{R})$ is chosen equal to 1 on the support of φ. Since $q\ge 2$ we estimate

$$\begin{aligned}&\big\|\tilde{\Psi}(-h^2\Delta_D)\chi\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0\big\|^q_{L^q(\mathbb{R},L^r(\Omega))}\\ &\quad\le C\sum_{l=-\infty}^{\infty}\|\tilde{\Psi}(-h^2\Delta_D)v_{h,l}\|^q_{L^q(I_l^h,L^r(\Omega))}\\ &\quad\le Ch^{q/2}\Big(\sum_{l=-\infty}^{\infty}\big\|\tilde{\varphi}(t/h-l)\tilde{\chi}\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0\big\|^2_{L^2(I_l^h,H_0^1(\Omega))}\Big)^{q/2}\\ &\qquad\quad+Ch^{-q/2}\Big(\sum_{l=-\infty}^{\infty}\big\|\tilde{\varphi}(t/h-l)\tilde{\chi}\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0\big\|^2_{L^2(I_l^h,L^2(\Omega))}\Big)^{q/2}\\ &\qquad\quad+C_Nh^{q(N+1/2)}\Big(\sum_{l=-\infty}^{\infty}\big\|\tilde{\varphi}(t/h-l)\tilde{\chi}\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0\big\|^2_{L^2(I_l^h,H_0^{n(\frac12-\frac1r)+\frac12}(\Omega))}\Big)^{q/2}\\ &\qquad\quad+C_Nh^{q(N-1/2)}\Big(\sum_{l=-\infty}^{\infty}\big\|\tilde{\varphi}(t/h-l)\tilde{\chi}\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0\big\|^2_{L^2(I_l^h,H_0^{n(\frac12-\frac1r)-\frac12}(\Omega))}\Big)^{q/2}.\end{aligned}\tag{3-20}$$

The almost-orthogonality of the supports of $\tilde{\varphi}(\cdot-l)$ in time allows us to estimate the term on the third line of (3-20) by

$$Ch^{q/2}\big\|\tilde{\chi}\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0\big\|^q_{L^2(\mathbb{R},H_0^1(\Omega))},\tag{3-21}$$

the one on the fourth line by

$$Ch^{-q/2}\big\|\tilde{\chi}\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0\big\|^q_{L^2(\mathbb{R},L^2(\Omega))},\tag{3-22}$$

the term on the fifth line by

$$C_Nh^{q(N+1/2)}\big\|\tilde{\chi}\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0\big\|^q_{L^2(\mathbb{R},H_0^{n(\frac12-\frac1r)+\frac12}(\Omega))},\tag{3-23}$$

and the one on the last line of (3-20) by

$$C_Nh^{q(N-1/2)}\big\|\tilde{\chi}\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0\big\|^q_{L^2(\mathbb{R},H_0^{n(\frac12-\frac1r)-\frac12}(\Omega))}.\tag{3-24}$$

We need the following smoothing effect on a nontrapping domain:

Proposition 3.5 [Burq et al. 2004a, Proposition 2.7]. *Assume that $\Omega = \mathbb{R}^n \setminus \mathbb{O}$, where $\mathbb{O} \neq \varnothing$ is a compact nontrapping obstacle. For every $\tilde{\chi} \in C_0^\infty(\mathbb{R}^n)$, $n \geq 2$, $\sigma \in [-1/2, 1]$, one has*

$$\|\tilde{\chi}\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0\|_{L^2(\mathbb{R}, H_0^{\sigma+1/2}(\Omega))} \leq C\|\Psi(-h^2\Delta_D)u_0\|_{H^\sigma(\Omega)}. \tag{3-25}$$

Remark 3.6. This is proved in [Burq et al. 2004a] for $\sigma \in [0, 1]$, but for spectrally localized data the result also follows using the estimates (2.15) of [Burq et al. 2004a, Proposition 2.7].

We apply Proposition 3.5 with $\sigma = \frac{1}{2}$ in (3-21), with $\sigma = -\frac{1}{2}$ in (3-22) and with $\sigma = n\bigl(\frac{1}{2} - \frac{1}{r}\bigr) = \frac{2}{q} \in [0, 1]$ in (3-23). In (3-24) we use that $n\bigl(\frac{1}{2} - \frac{1}{r}\bigr) - \frac{1}{2} \leq \frac{1}{2}$ to estimate the $L^2\bigl(\mathbb{R}, H^{n(\frac{1}{2}-\frac{1}{r})-\frac{1}{2}}(\Omega)\bigr)$ norm by the $L^2(\mathbb{R}, H^{1/2}(\Omega))$ norm and use Proposition 3.5 with $\sigma = 0$. This yields

$$\|\tilde{\Psi}(-h^2\Delta_D)\chi\Psi(-h^2\Delta_D)e^{it\Delta_D}u_0\|_{L^q(\mathbb{R}, L^r(\Omega))} \leq C\|\Psi(-h^2\Delta_D)u_0\|_{L^2(\Omega)}. \tag{3-26}$$

Here we used the spectral localization Ψ to estimate $\|\Psi(-h^2\Delta_D)u_0\|_{H^\sigma(\Omega)}$ by $h^{-\sigma}\|\Psi(-h^2\Delta_D)u_0\|_{L^2(\Omega)}$. This achieves the proof of Proposition 3.1. □

In the rest of this section we show how Proposition 3.1 implies Theorem 1.7.

Lemma 3.7 [Ivanovici and Planchon 2008, Theorem 1.1]. *Let $\Psi_0 \in C_0^\infty(\mathbb{R})$, $\Psi \in C_0^\infty((1/2, 2))$ satisfy*

$$\Psi_0(\lambda) + \sum_{j\geq 1}\Psi(2^{-2j}\lambda) = 1, \quad \textit{for all } \lambda \in \mathbb{R}.$$

Then for all $r \in [2, \infty)$ we have

$$\|f\|_{L^r(\Omega)} \leq C_r\Biggl(\|\Psi_0(-\Delta_D)f\|_{L^r(\Omega)} + \Biggl(\sum_{j=1}^{\infty}\|\Psi(-2^{-2j}\Delta_D)f\|_{L^r(\Omega)}^2\Biggr)^{1/2}\Biggr). \tag{3-27}$$

Applying Lemma 3.7 to $f = e^{it\Delta_D}u_0$ and taking the L^q norm in time yields

$$\|e^{it\Delta_D}u_0\|_{L^q(\mathbb{R}, L^r(\Omega))} \leq \Biggl\|\|e^{it\Delta_D}\Psi_0(-\Delta_D)u_0\|_{L^r(\Omega)} + \Biggl(\sum_{j\geq 1}\|e^{it\Delta_D}\Psi(-2^{-2j}\Delta_D)u_0\|_{L^r(\Omega)}^2\Biggr)^{1/2}\Biggr\|_{L^q(\mathbb{R})}$$

which, by the Minkowski inequality, leads to $\|e^{it\Delta_D}u_0\|_{L^q(\mathbb{R}, L^r(\Omega))} \leq C\|u_0\|_{L^2(\Omega)}$. The proof of Theorem 1.7 is complete.

4. Applications

In this section we sketch the proofs of Theorem 1.8 and Theorem 1.9.

We start with Theorem 1.8. From Theorem 1.7 we have an estimate of the linear flow of the Schrödinger equation

$$\|e^{-it\Delta_D}u_0\|_{L^5(\mathbb{R}, L^{30/11}(\Omega))} \leq C\|u_0\|_{L^2(\Omega)}. \tag{4-1}$$

One may shift the regularity by 1 and obtain

$$\|e^{-it\Delta_D}u_0\|_{L^5(\mathbb{R}, W^{1,30/11}(\Omega))} \leq C\|u_0\|_{H_0^1(\Omega)}. \tag{4-2}$$

Hence for small $T > 0$ the left-hand side of (4-1) and (4-2) will be small; for such T let $X_T := L^5\bigl((0, T], W^{1,30/11}(\Omega)\bigr)$. One may then set up the usual fixed point argument in X_T, as if $u \in X_T$ then $u^5 \in L^1\bigl([0, T], H^1(\Omega)\bigr)$.

Let us proceed with Theorem 1.9. From [Planchon and Vega 2009], one has a time-global control on the solution u, at the level of $\dot{H}^{\frac{1}{4}}$ regularity:

$$u \in L^4\big((0,+\infty), L^4(\Omega)\big).$$

By interpolation with either mass or energy conservation, combined with the local existence theory, one may bootstrap this time-global control into

$$u \in L^{p-1}\big((0,+\infty), L^\infty(\Omega)\big),$$

from which scattering in $H_0^1(\Omega)$ follows immediately.

Appendices

A. ***Finite speed of propagation for the semiclassical equation.*** In this appendix we recall some properties of the semiclassical Schrödinger flow. For further discussion and proofs, see [Lebeau 1992].

Let S be a compact manifold with smooth boundary ∂S.

Definition A.1. We say that a symbol $q(y,\eta) \in S^m_{\rho,\delta}$ is of type (ρ,δ) and of order m if, for any α and β, there exists $C_{\alpha,\beta} > 0$ such that

$$|\partial_y^\beta \partial_\eta^\alpha q(y,\eta)| \le C_{\alpha,\beta}(1+|\eta|)^{m-\rho|\alpha|+\delta|\beta|}.$$

For $q \in S^m_{1,0}$ we let $Op_h(q) = Q(y,hD,h)$ be the h-pseudodifferential operator defined by

$$Op_h(q)f(y) = \frac{1}{(2\pi h)^n}\int e^{(i/h)(y-\tilde{y})\eta} q(y,\eta,h) f(\tilde{y})\, d\tilde{y}.$$

We set $y = (x,t) \in S \times \mathbb{R}$ and denote $\eta = (\xi,\tau)$ the dual variable of y. Near a point $x_0 \in \partial S$ we can choose a system of local coordinates such that S is given by $S = \big\{x = (x', x_n) : x_n > 0\big\}$. We define the tangential operators

$$Op_{h,\mathrm{tang}}(q) f(y) = \frac{1}{(2\pi h)^{n-1}} \int e^{(i/h)(y'-\tilde{y}')\eta'} q(y,\eta',h) f(\tilde{x}', x_n, \tilde{t})\, d\tilde{y}'\, d\eta',$$

where $y = (x', x_n, t)$, $y' = (x', t)$, $\tilde{y}' = (\tilde{x}', \tilde{t})$, $\eta = (\xi', \xi_n, \tau)$, $\eta' = (\xi', \tau)$, and where the symbol $q(y,\eta',h)$ lies in $S^m_{1,0,\mathrm{tang}}$; in other words, for any α and β, there exists $C_{\alpha,\beta} > 0$ such that

$$|\partial_y^\alpha \partial_{\eta'}^\beta q(y,\eta',h)| \le C_{\alpha,\beta}(1+|\eta'|)^{m-|\beta|}.$$

Let g be a Riemannian metric on S such that ∂S is strictly concave and (S,g) satisfies Assumption 1.1. Let $v_0 \in L^2(S)$ be compactly supported outside a small neighborhood of the boundary, take $\Psi \in C_0^\infty((\alpha_0,\beta_0))$, and let $v(x,t) = e^{iht\Delta_g}\Psi(-h^2\Delta_g)v_0$ denote the linear semiclassical Schrödinger flow with initial data at time $t=0$ equal to $\Psi(-h^2\Delta_g)v_0$ and such that $\|\Psi(-h^2\Delta_g)v_0\|_{L^2(S)} \lesssim 1$.

Let $\pi : T^*(\bar{S}\times\mathbb{R}) \to T^*(\partial S \times \mathbb{R}) \cup T^*(S\times\mathbb{R})$ be the canonical projection, defined by

$$\pi|_{T^*(S\times\mathbb{R})} = \mathrm{Id}, \qquad \pi(y,\eta) = (y, \eta|_{T^*(\partial S\times\mathbb{R})}) \quad \text{for } (y,\eta) \in T^*(\bar{S}\times\mathbb{R})|_{\partial S\times\mathbb{R}}.$$

Writing $y = (x,t)$ and $\eta = (\xi,\tau)$, we introduce the characteristic set

$$\Sigma_b := \pi\big\{(y,\eta) : \eta = (\xi,\tau),\ \tau + |\xi|_g^2 = 0,\ -\beta_0 \le \tau \le -\alpha_0\big\},$$

where $|\xi|_g^2 = \langle \xi, \xi \rangle_g =: \xi_n^2 + r(x, \xi')$ denotes the inner product given by the metric g and where, due to the strict concavity of the boundary we have $\partial_{x_n} r(x', 0, \eta') < 0$.

Definition A.2. We say that a point $\rho_0 = (y_0, \eta_0) \in T_b^*(\partial S \times \mathbb{R}) := T^*(\partial S \times \mathbb{R}) \cup T^*(S \times \mathbb{R})$ *does not* belong to the b-wave front set $WF_b(v)$ of v if there exists a h-pseudodifferential operator of symbol $q(y, \eta, h)$ [or $q(y, \eta', h)$ if $\rho_0 \in T^*(\partial S \times \mathbb{R})$] with compact support in (y, η), elliptic at ρ_0, and a smooth function $\phi \in C_0^\infty$ equal to 1 near y_0, such that for every $\sigma \geq 0$ and $N \geq 0$ there exists $C_N > 0$ such that

$$\| Op_h(q) \phi v \|_{H^\sigma(S \times \mathbb{R})} \leq C_N h^N.$$

We then write $\rho_0 \notin WF_b(v)$.

Proposition A.3 (elliptic regularity [Lebeau 1992, Theorem 3.1]). *Let $q(y, \eta)$ a symbol such that $q = 0$ on a neighborhood of Σ_b. Then for every $\sigma \geq 0$ and $N \geq 0$ there exists $C_N > 0$ such that*

$$\| Op_h(q) v \|_{H^\sigma(S)} \leq C_N h^N.$$

This is proved in [Lebeau 1992] for eigenfunctions of the Laplace operator, but the same arguments apply in this setting. From Proposition A.3 and [Lebeau 1992, Sections 2, 3] we have:

Corollary A.4. *There exists a constant $D > 0$ such that*

$$WF_b(v) \subset \Sigma_b \cap \bigl\{ -\tau \in [\alpha_0, \beta_0], |\xi|_g \leq D \bigr\}.$$

Corollary A.5 [Lebeau 1992, Chapter 3]. *Let $\varphi \in C_0^\infty(\mathbb{R})$ be equal to 1 near the interval $[-\beta_0, -\alpha_0]$. Then for any bounded interval I and any $N \geq 1$ there exists $C_N > 0$ such that*

$$\bigl| (1 - \varphi)(h D_t) v \bigr| \leq C_N h^N \quad \textit{for all } t \in I. \tag{A-1}$$

Corollary A.6 (elliptic regularity at "∞"). *Let $\vartheta \in C_0^\infty(\mathbb{R}^n)$ be equal to 1 on $\{|\xi|_g \leq D\}$. Then, for all $N \geq 1$, there exists $C_N > 0$ such that*

$$\bigl| (1 - \vartheta)(h D_x) v \bigr| \leq C_N h^N. \tag{A-2}$$

Proposition A.7. *Let $y_0 \notin \mathrm{pr}_y(WF_b(v))$, where by pr_y we mean the projection on the variable $y = (x, t)$. Then there exists $\phi \in C_0^\infty$ with $\phi = 1$ near y_0 and such that for every $\sigma \geq 0$ and $N \geq 0$, there exists $C_N > 0$ such that*

$$\| \phi v \|_{H^\sigma(S)} \leq C_N h^N.$$

Proof. Let φ, ϑ be as defined in Corollaries A.5 and A.6. Using Proposition A.3 again, we get

$$v(x, t) = \varphi(h D_t) \vartheta(h D_x) v + O(h^\infty). \tag{A-3}$$

Now let $y_0 = (x_0, t_0) \notin \mathrm{pr}_y(WF_b(v))$. It follows that for every $\eta \neq 0$, $(y_0, \eta) \notin WF_b(v)$ and in particular for every $\eta_0 \in \operatorname{supp} \vartheta \times \operatorname{supp} \varphi$ there exists a symbols $q_0(y, \eta, h)$ with compact support in (y, η) near (y_0, η_0) and elliptic at (y_0, η_0), and there exists $\phi_0 \in C_0^\infty$ equal to 1 in a neighborhood U_0 of y_0 such that for every $\sigma \geq 0$ and every $N \geq 0$, there exists $C_N > 0$ such that

$$\| Op_h(q_0) \phi v \|_{H^\sigma(S)} \leq C_N h^N.$$

After shrinking U_0 if necessary, suppose that q_0 is elliptic on $U_0 \times W_0$, where W_0 is an open neighborhood of η_0. Then it follows that on U_0, for every $\sigma \geq 0$ and $N \geq 0$, there exists $C_N > 0$ such that

$$\|\phi v\|_{H^\sigma(U_0)} \leq C_N h^N.$$

Since the set $\operatorname{supp} \vartheta \times \operatorname{supp} \varphi$ is compact there exist η^α, $\alpha \in \{1, \dots, N\}$ for some fixed $N \geq 1$ and for each η^α there exist symbols q_α elliptic on some neighborhoods $U_\alpha \times W_\alpha$ of (y_0, η^α) and smooth functions $\phi_\alpha \in C_0^\infty$ equal to 1 on the neighborhoods U_α of y_0, such that $\operatorname{supp} \vartheta \times \operatorname{supp} \varphi \subset \bigcup_{j=1}^N W_\alpha$. Suppose that $\phi \in C_0^\infty$ is equal to 1 in an open neighborhood of y_0 strictly included in the intersection $\bigcap_{\alpha=1}^N U_\alpha$ (which has nonempty interior) and supported in the compact set $\bigcap_{\alpha=1}^N \operatorname{supp} \phi_\alpha$. Considering a partition of unity associated to $(U_\alpha \times W_\alpha)_\alpha$ and using (A-3) we deduce that ϕ satisfies Proposition A.7. □

Proposition A.8 [Burq 1993, Lemma B.7]. *Let $v(x,t) = e^{ith\Delta_g}\Psi(-h^2\Delta_g)v_0$ as before, $v_0 \in L^2(S)$ and let Q be a h-pseudodifferential operator of order* 0, *$t_0 > 0$ and $\tilde{\psi} \in C_0^\infty((-2t_0, -t_0))$. Let w denote the solution to*

$$\begin{cases} (ih\partial_t + h^2\Delta_g)w = ih\tilde{\psi}(t)Q(v) & \text{on } S \times \mathbb{R}, \\ w|_{\partial S} = 0, \quad w|_{t<-2t_0} = 0. \end{cases} \tag{A-4}$$

If $\rho_0 \in WF_b(w)$ then the broken bicharacteristic starting from ρ_0 has a nonempty intersection with $WF_b(v) \cap \{t \in \operatorname{supp} \tilde{\psi}\}$.

B. ***Proof of Lemma 3.4.*** In this section (M, Δ_M) denotes either (S, Δ_S) or (Ω, Δ_D), respectively. This notation will be used to refer both domains at the same time. Let $\tilde{\chi} \in C_0^\infty(\mathbb{R}^n)$ be such that $\Delta_D \tilde{\chi} = \Delta_S \tilde{\chi}$.

Let $\varphi_0 \in C^\infty(\mathbb{R})$ be supported in the interval $[-4, 4]$ and $\varphi \in C^\infty(\mathbb{R})$ be supported in $[-4, -1] \cup [1, 4]$ such that for all $\xi \in \mathbb{R}$

$$\varphi_0(\xi) + \sum_{k \geq 1} \varphi(2^{-k}\xi) = 1.$$

If $\hat{\Psi}$ denotes the Fourier transform of Ψ, we write it using the preceding sum as

$$\hat{\Psi}(\xi) = \hat{\Psi}(\xi)\Big(\varphi_0(\xi) + \sum_{k \geq 1} \varphi(2^{-k}\xi)\Big)$$

and denote by $\phi_k \in \mathcal{S}(\mathbb{R})$ the functions such that $\hat{\phi}_0(\xi) = \hat{\Psi}(\xi)\varphi_0(\xi)$, $\hat{\phi}_k(\xi) = \hat{\Psi}(\xi)\varphi(2^{-k}\xi)$. We denote by $\mathcal{S}(\mathbb{R})$ the Schwartz space of rapidly decreasing functions. Hence we have

$$\Psi(\lambda) = \sum_{k \in \mathbb{N}} \phi_k(\lambda), \quad \text{where} \quad \|\hat{\phi}_k\|_{L^\infty} = \|\hat{\Psi}(\xi)\varphi(2^{-k}\xi)\|_{L^\infty} \leq C_N 2^{-kN} \quad \text{for all } N \in \mathbb{N}. \tag{B-5}$$

For $k \in \mathbb{N}$, write

$$\phi_k(h\sqrt{-\Delta_M})\tilde{\chi} v_{h,l} = \frac{1}{2\pi} \int_{\operatorname{supp} \hat{\phi}_k} e^{i\xi h\sqrt{-\Delta_M}} \tilde{\chi} v_{h,l} \hat{\phi}_k(\xi)\, d\xi. \tag{B-6}$$

On the support of $\hat{\phi}_k(\xi)$, $|\xi| \simeq 2^k$ and for $k \leq \frac{1}{2}\log_2(1/h)$, for example, we see, by the finite speed of propagation of the wave operator, that on a time interval of size $2^k h \leq h^{1/2}$ we remain in a fixed neighborhood of the boundary of Ω where Δ_D coincides with Δ_S, therefore we can introduce χ_1 equal to 1 on a fixed neighborhood of the support of $\tilde{\chi}$ (independent of k, h) such that, for every $k \le \frac{1}{2}\log_2(1/h)$,

$$\chi_1\phi_k(h\sqrt{-\Delta_S})\tilde{\chi}v_{h,l} = \chi_1\phi_k(h\sqrt{-\Delta_\Omega})\tilde{\chi}v_{h,l}. \tag{B-7}$$

Since $v_{h,l,s} = \chi_1\big(\tilde{\Psi}(-h^2\Delta_D) - \tilde{\Psi}(-h^2\Delta_S)\big)v_{h,l}$ and $v_{h,l} = \tilde{\chi}v_{h,l}$, we obtain, using (B-7)

$$v_{h,l,s} = \chi_1\Bigg(\sum_{k\ge\frac{1}{4}\log_2(1/h)} \big(\phi_k(h\sqrt{-\Delta_\Omega}) - \phi_k(h\sqrt{-\Delta_S})\big)\Bigg)\tilde{\chi}v_{h,l}. \tag{B-8}$$

To estimate the $L^q(I_l^h, L^r(\Omega))$ norm of $v_{h,l,s}$ it will be enough to estimate separately the norms of $\chi_1\phi_k(h\sqrt{-\Delta_M})\tilde{\chi}v_{h,l}$ for $k \ge \frac{1}{4}\log_2(1/h)$ where $(M, \Delta_M) \in \big\{(\Omega, \Delta_D), (S, \Delta_S)\big\}$. Using the Cauchy–Schwartz inequality and the Sobolev embeddings gives

$$\begin{aligned}
\|\chi_1\phi_k(h\sqrt{-\Delta_M})\tilde{\chi}v_{h,l}\|_{L^q(I_l^h,L^r(\Omega))} &\le Ch^{1/q}\|\chi_1\phi_k(h\sqrt{-\Delta_M})\tilde{\chi}v_{h,l}\|_{L^\infty(I_l^h,L^r(\Omega))} \\
&\le Ch^{1/q}\|\chi_1\phi_k(h\sqrt{-\Delta_M})\tilde{\chi}v_{h,l}\|_{L^\infty(I_l^h,H^{n(\frac{1}{2}-\frac{1}{r})}(\Omega))} \\
&\le C_N h^{1/q}2^{-kN}\|\tilde{\chi}v_{h,l}\|_{L^\infty(I_l^h,H^{n(\frac{1}{2}-\frac{1}{r})}(\Omega))} \quad \text{for all } N \in \mathbb{N},
\end{aligned} \tag{B-9}$$

where in the last line we used (B-5). We estimate the last term in (B-9) writing the Duhamel formula for $v_{h,l}$ only on Ω using the Equation (3-7), since in this case the smoothing effect yields (see [Staffilani and Tataru 2002], [Burq et al. 2004a], or the dual estimates of (3-25) in Proposition 3.5)

$$\|\tilde{\chi}v_{h,l}\|_{L^\infty(I_l^h,H^{n(\frac{1}{2}-\frac{1}{r})}(\Omega))} \le C\|V_{h,l}\|_{L^2(I_l^h,H^{n(\frac{1}{2}-\frac{1}{r})-\frac{1}{2}}(\Omega))}. \tag{B-10}$$

Since we consider here only large values $k \ge \frac{1}{4}\log_2(1/h)$, each 2^{-k} is bounded by $h^{1/4}$, therefore, after summing over k we obtain

$$\|v_{h,l,s}\|_{L^q(I_l^h,L^r(\Omega))} \le C_N h^{1/q+N/4}\|V_{h,l}\|_{L^2(I_l^h,H^{n(\frac{1}{2}-\frac{1}{r})-\frac{1}{2}}(\Omega))} \quad \text{for all } N \in \mathbb{N}. \tag{B-11}$$

Acknowledgments

The author thanks Nicolas Burq for many helpful and stimulating discussions and Fabrice Planchon for having suggested to her the applications and for his careful reading of the paper. She also thanks Michael Taylor for having sent her the manuscript "Boundary problems for the wave equations with grazing and gliding rays" and the referees for helpful comments and suggestions which greatly improved the presentation.

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Received 16 Jan 2009. Revised 13 Aug 2009. Accepted 12 Sep 2009.

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ANALYSIS AND PDE
Vol. 3, No. 3, 2010

BERGMAN METRICS AND GEODESICS IN THE SPACE OF KÄHLER METRICS ON TORIC VARIETIES

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A guiding principle in Kähler geometry is that the infinite-dimensional symmetric space $\mathcal{H}$ of Kähler metrics in a fixed Kähler class on a polarized projective Kähler manifold M should be well approximated by finite-dimensional submanifolds $\mathcal{B}_k \subset \mathcal{H}$ of Bergman metrics of height k (Yau, Tian, Donaldson). The Bergman metric spaces are symmetric spaces of type $G_{\mathbb{C}}/G$ where $G = U(d_k + 1)$ for certain d_k. This article establishes some basic estimates for Bergman approximations for geometric families of toric Kähler manifolds.

The approximation results are applied to the endpoint problem for geodesics of $\mathcal{H}$, which are solutions of a homogeneous complex Monge–Ampère equation in $A \times X$, where $A \subset \mathbb{C}$ is an annulus. Donaldson, Arezzo and Tian, and Phong and Sturm raised the question whether $\mathcal{H}$-geodesics with fixed endpoints can be approximated by geodesics of $\mathcal{B}_k$. Phong and Sturm proved weak C^0-convergence of Bergman to Monge–Ampère geodesics on a general Kähler manifold. Our approximation results show that one has $C^2(A \times X)$ convergence in the case of toric Kähler metrics, extending our earlier result on $\mathbb{CP}^1$.

1. Introduction

This is the first in a series of articles on the Riemannian geometry of the space

$$\mathcal{H} = \{\varphi \in C^\infty(M) : \omega_\varphi = \omega_0 + dd^c\varphi > 0\} \tag{1}$$

of Kähler metrics in the class $[\omega_0]$ of a polarized projective Kähler manifold (M, ω_0, L), equipped with

MSC2000: 14M25, 35P20, 35S30, 53C22, 53C55.
Keywords: Kähler metrics, Bergman kernels, toric varieties, Monge–Ampère.
Research partially supported by National Science Foundation grants DMS-0604805 and DMS-0904252.

the Riemannian metric $g_{\mathcal{H}}$ of Mabuchi [1987], Semmes [1992], and Donaldson [1999]:

$$\|\psi\|^2_{g_{\mathcal{H}},\varphi} = \int_M |\psi|^2 \, \frac{\omega_\varphi^m}{m!}, \quad \text{where } \varphi \in \mathcal{H} \text{ and } \psi \in T_\varphi\mathcal{H} \simeq C^\infty(M). \tag{2}$$

Here, $L \to M$ is an ample line bundle with $c_1(L) = [\omega_0]$. Formally, $(\mathcal{H}, g_{\mathcal{H}})$ is an infinite-dimensional nonpositively curved symmetric space of the type $G_{\mathbb{C}}/G$, where $G = \mathrm{SDiff}_{\omega_0}(M)$ is the group of Hamiltonian symplectic diffeomorphisms of (M, ω_0). This statement is only formal since G does not possess a complexification and $\mathcal{H}$ is an incomplete, infinite-dimensional space. An attractive approach to the infinite-dimensional geometry is to approximate it by a sequence of finite-dimensional submanifolds $\mathcal{B}_k \subset \mathcal{H}$ of so-called Bergman (or Fubini–Study) metrics. The space $\mathcal{B}_k$ of Bergman metrics may be identified with the finite-dimensional symmetric space $\mathrm{GL}(d_k+1, \mathbb{C})/U(d_k+1)$, where d_k is a certain dimension. Thus, $\mathcal{B}_k$ is equipped with a finite-dimensional symmetric space metric $g_{\mathcal{B}_k}$, which is not the same as the submanifold Riemannian metric induced on it by $g_{\mathcal{H}}$. The purpose of the series is to show that much of the symmetric space geometry of $(\mathcal{B}_k, g_{\mathcal{B}_k})$ tends to the infinite-dimensional symmetric space geometry of $(\mathcal{H}, g_{\mathcal{H}})$ as $k \to \infty$.

To put the problem and results in perspective, we recall that at the level of individual metrics $\omega \in \mathcal{H}$, there exists a well-developed approximation theory: Given ω, one can define a canonical sequence of Bergman metrics $\omega_k \in \mathcal{B}_k$ which approximates ω in the C^∞ topology (see (9)), in much the same way that smooth functions can be approximated by Bernstein polynomials [Yau 1992; Tian 1990]; see also [Catlin 1999; Zelditch 1998; 2009]. The approximation theory is based on microlocal analysis in the complex domain, specifically Bergman kernel asymptotics on and off the diagonal [Boutet de Monvel and Sjöstrand 1976; Catlin 1999; Zelditch 1998; Donaldson 2001; Phong and Sturm 2009]. The same methods are used in [Rubinstein and Zelditch $\geq$ 2010a] to prove that the geometry of $(\mathcal{B}_k, g_{\mathcal{B}_k})$ tends to the geometry of $(\mathcal{H}, g_{\mathcal{H}})$ at the infinitesimal level: for example, that the Riemann metric, connection and curvature tensor of $\mathcal{B}_k$ tend to the Riemann metric, connection and curvature of $\mathcal{H}$. But our principal aim in this series is to extend the approximation from pointwise or infinitesimal objects to more global aspects of the geometry, such as $\mathcal{B}_k$-geodesics or harmonic maps to $(\mathcal{B}_k, g_{\mathcal{B}_k})$. These more global approximation problems are much more difficult than the infinitesimal ones. The obstacles are analogous to those involved in complexifying $\mathrm{SDiff}_{\omega_0}(M)$. We will explain this comparison in more detail in Section 1F at the end of this introduction.

This article is concerned with the approximation of $g_{\mathcal{H}}$-geodesic segments ω_t in $\mathcal{H}$ with fixed endpoints by $g_{\mathcal{B}_k}$-geodesic segments in $\mathcal{B}_k$. As recalled in Section 1A, the geodesic equation for the Kähler potentials φ_t of ω_t is a complex homogeneous Monge–Ampère equation. Little is known about the solutions of the Dirichlet problem at present beyond the regularity result that $\varphi_t \in C^{1,\alpha}([0, T] \times M)$ for all $\alpha < 1$ if the endpoint metrics are smooth (see [Chen 2000; Chen and Tian 2008] for results and background). It is therefore natural to study the approximation of Monge–Ampère $g_{\mathcal{H}}$-geodesics φ_t by the much simpler $g_{\mathcal{B}_k}$-geodesics $\varphi_k(t, z)$, which are defined by one-parameter subgroups of $\mathrm{GL}(d_k+1, \mathbb{C})$ (see (24)). The problem of approximating $\mathcal{H}$-geodesic segments between two smooth endpoints by $\mathcal{B}_k$-geodesic segments was raised by Donaldson [2001], Arezzo and Tian [2003] and Phong and Sturm [2006] and was studied in depth by Phong and Sturm [2006; 2007]. Phong and Sturm [2006] proved that $\varphi_k(t, z) \to \varphi_t$ in a weak C^0 sense on $[0, 1] \times M$ (see (13)); a C^0 result with a remainder estimate was later proved by Berndtsson [2009] for a somewhat different approximation.

In this article, we study the $g_{\mathcal{B}_k}$-approximation of $g_{\mathcal{H}}$-geodesics in the case of a polarized projective toric Kähler manifold. Our main result is that a $g_{\mathcal{H}}$ geodesic segment of toric Kähler metrics with fixed endpoints is approximated in C^2 by a sequence $\varphi_k(t, z)$ of toric $g_{\mathcal{B}_k}$-geodesic segments. More precisely, for any $T \in \mathbb{R}_+$, we have $\varphi_k(t, z) \to \varphi_t(z)$ in $C^2([0, T] \times M)$, generalizing the results of Song and Zelditch [2007a] in the case of $\mathbb{CP}^1$. It is natural to study convergence of two (space-time) derivatives since the Kähler metric $\omega_\varphi = \omega_0 + dd^c\varphi$ involves two derivatives. In the course of the proof, we introduce methods which have many other applications to global approximation problems on toric Kähler manifolds, and which should also have applications to nontoric Kähler manifolds.

Here, as in [Song and Zelditch 2007b; Rubinstein and Zelditch 2010; $\geq$ 2010b], we restrict to the toric setting because, at this stage, it is possible to obtain much stronger results than for general Kähler manifolds and because it is one of the few settings where we can see clearly what is involved in the classical limit as $k \to \infty$. The simplifying feature of toric Kähler manifolds is that they are completely integrable on both the classical and quantum level. In Riemannian terms, the submanifolds of toric metrics of $\mathcal{H}$ and $\mathcal{B}_k$ form totally geodesic flats. Hence in the toric case, the geodesic equation along the flat is linearized by the Legendre transform, with the consequence that there exists an explicit formula for the Monge–Ampère geodesic φ_t between two smooth toric endpoint metrics. In particular, the explicit formula shows that geodesics between smooth toric endpoints are smooth. We use this explicit solution throughout the article, starting from (29). Thus, in the toric case we only need to prove C^2-convergence of the Bergman approximation. An analogous result on a general Kähler manifold would require an improvement on the known regularity results on Monge–Ampère geodesics in addition to a convergence result. We refer to [Chen and Tian 2008] for the state of the art on the regularity theory.

1A. ***Background.*** To state our results, we need some notation and background. Let $L \to M^m$ be an ample holomorphic line bundle over a compact complex manifold of dimension m. Let $\omega_0 \in H^{(1,1)}(M, \mathbb{Z})$ denote an integral Kähler form. Fixing a reference hermitian metric h_0 on L, we may write other hermitian metrics on L as

$$h_\varphi = e^{-\varphi} h_0,$$

and then the space of hermitian metrics h on L with curvature $(1, 1)$-forms ω_h in the class of ω_0 may (by the $\partial\bar{\partial}$ lemma) be identified with the space $\mathcal{H}$ of relative Kähler potentials (1). We may then identify the tangent space $T_\varphi\mathcal{H}$ at $\varphi \in \mathcal{H}$ with $C^\infty(M)$. Following [Mabuchi 1987; Semmes 1992; Donaldson 2001], we define the Riemannian metric (2) on $\mathcal{H}$. With this Riemannian metric, $\mathcal{H}$ is formally an infinite-dimensional nonpositively curved symmetric space.

The space $\mathcal{B}_k$ of Bergman (or Fubini–Study) metrics of height k is defined as follows: Let $H^0(M, L^k)$ denote the space of holomorphic sections of the k-th power $L^k \to M$ of L and let $d_k + 1 = \dim H^0(M, L^k)$. We let $\mathcal{B}H^0(M, L^k)$ denote the manifold of all bases $\underline{s} = \{s_0, \ldots, s_{d_k}\}$ of $H^0(M, L^k)$. Given a basis, we define the Kodaira embedding

$$\iota_{\underline{s}} : M \to \mathbb{CP}^{d_k}, \quad z \mapsto [s_0(z), \ldots, s_{d_k}(z)]. \tag{3}$$

We then define a Bergman metric (or equivalently, Fubini–Study) metric of height k to be a metric of the form

$$h_{\underline{s}} := (\iota_{\underline{s}}^* h_{\mathrm{FS}})^{1/k} = \frac{h_0}{\left(\sum_{j=0}^{d_k} |s_j(z)|^2_{h_0^k}\right)^{1/k}}, \tag{4}$$

where h_{FS} is the Fubini–Study hermitian metric on $\mathcal{O}(1) \to \mathbb{CP}^{d_k}$. We then define

$$\mathcal{B}_k = \{h_{\underline{s}} : \underline{s} \in \mathcal{B}H^0(M, L^k)\}. \tag{5}$$

Here, h_0^k is the k-th tensor power of the Hermitian metric h_0 on $\mathcal{O}(1)$ and $h_{\underline{s}}$ is independent of the choice of h_0. We use the same notation for the associated space of potentials φ such that $h_{\underline{s}} = e^{-\varphi} h_0$ and for the associated Kähler metrics ω_φ. We observe that with a choice of basis of $H^0(M, L^k)$ we may identify $\mathcal{B}_k$ with the symmetric space $\mathrm{GL}(d_k+1, \mathbb{C})/U(d_k+1)$ since $\mathrm{GL}(d_k+1, \mathbb{C})$ acts transitively on the set of bases, while $\iota_{\underline{s}}^* h_{\mathrm{FS}}$ is unchanged if we replace the basis $\underline{s}$ by a unitary change of basis.

Several further identifications are important. The first is that $\mathcal{B}_k$ may be identified with the space $\mathcal{I}_k$ of hermitian inner products on $H^0(M, L^k)$, the correspondence being that a basis is identified with an inner product for which the basis is hermitian orthonormal. As in [Donaldson 2001; 2005], we define maps

$$\mathrm{Hilb}_k : \mathcal{H} \to \mathcal{I}_k,$$

by the rule that a hermitian metric $h \in \mathcal{H}$ induces the inner products on $H^0(M, L^k)$,

$$\|s\|^2_{\mathrm{Hilb}_k(h)} = R \int_M |s(z)|^2_{h^k} dV_h, \quad \text{where } dV_h = \frac{\omega_h^m}{m!} \text{ and } R = \frac{d_k+1}{\mathrm{Vol}(M, dV_h)}. \tag{6}$$

Further, we define the identifications

$$\mathrm{FS}_k : \mathcal{I}_k \simeq \mathcal{B}_k$$

as follows: an inner product $G = \langle\ ,\ \rangle$ on $H^0(M, L^k)$ determines a G-orthonormal basis $\underline{s} = \underline{s}_G$ of $H^0(M, L^k)$ and an associated Kodaira embedding (3) and Bergman metric (4). Thus,

$$\mathrm{FS}_k(G) = h_{\underline{s}_G}. \tag{7}$$

The right side is independent of the choice of h_0 and the choice of orthonormal basis. As observed in [Donaldson 2001; Phong and Sturm 2006], $\mathrm{FS}_k(G)$ is characterized by the fact that for any G-orthonormal basis $\{s_j\}$ of $H^0(M, L^k)$, we have

$$\sum_{j=0}^{d_k} |s_j(z)|^2_{\mathrm{FS}_k(G)} \equiv 1 \quad \text{for all } z \in M. \tag{8}$$

Metrics in $\mathcal{B}_k$ are defined by an algebro-geometric construction. By analogy with the approximation of real numbers by rational numbers, we say that $h \in \mathcal{H}$ (or its curvature form ω_h) has *height* k if $h \in \mathcal{B}_k$. A basic fact is that the union

$$\mathcal{B} = \bigcup_{k=1}^{\infty} \mathcal{B}_k$$

of Bergman metrics is dense in the C^∞-topology in the space $\mathcal{H}$ [Tian 1990; Zelditch 1998]. Indeed,

$$\frac{\mathrm{FS}_k \circ \mathrm{Hilb}_k(h)}{h} = 1 + O(k^{-2}), \tag{9}$$

where the remainder is estimated in $C^r(M)$ for any $r > 0$; the left side moreover has a complete asymptotic expansion. See [Donaldson 2002; Phong and Sturm 2006] for precise statements.

Now that we have defined the spaces $\mathcal{H}$ and $\mathcal{B}_k$, we can compare Monge–Ampère geodesics and Bergman geodesics. Geodesics of $\mathcal{H}$ satisfy the Euler–Lagrange equations for the energy functional determined by (2); see (68). By [Mabuchi 1987; Semmes 1992; Donaldson 1999], the geodesics of $\mathcal{H}$ in this metric are the paths $h_t = e^{-\varphi_t} h_0$ which satisfy the equation

$$\ddot{\varphi} - \tfrac{1}{2} |\nabla \dot{\varphi}|^2_{\omega_\varphi} = 0, \tag{10}$$

which may be interpreted as a homogeneous complex Monge–Ampère equation on $A \times M$, where A is an annulus [Semmes 1992; Donaldson 1999].

Geodesics in $\mathcal{B}_k$ with respect to the symmetric space metric are given by orbits of certain one-parameter subgroups $\sigma_k^t = e^{tA_k}$ of $\mathrm{GL}(d_k+1, \mathbb{C})$. In the identification of $\mathcal{B}_k$ with the symmetric space $\mathcal{I}_k \simeq \mathrm{GL}(d_k+1, \mathbb{C})/U(d_k+1)$ of inner products, the one-parameter subgroup $e^{tA_k} \in \mathrm{GL}(d_k+1)$ changes an orthonormal basis $\underline{\hat{s}}^{(0)}$ for the initial inner product G_0 to an orthonormal basis $e^{tA_k} \cdot \underline{\hat{s}}^{(0)}$ for G_t, where G_t is a geodesic of $\mathcal{I}_k$. Geometrically, a Bergman geodesic may be visualized as the path of metrics on M obtained by holomorphically embedding M using a basis of $H^0(M, L^k)$ and then moving the embedding under the one-parameter subgroup e^{tA_k} of motions of $\mathbb{CP}^{d_k}$. The difficulty is to interpret this simple extrinsic motion in intrinsic terms on M.

In this article, we only study the endpoint problem for the geodesic equation. We are given $h_0, h_1 \in \mathcal{H}$ and let $h(t)$ denote the Monge–Ampère geodesic between them. We then consider the geodesic $G_k(t)$ of $\mathcal{I}_k$ between $G_k(0) = \mathrm{Hilb}_k(h_0)$ and $G_k(1) = \mathrm{Hilb}_k(h_1)$ or equivalently between $\mathrm{FS}_k \circ \mathrm{Hilb}_k(h_0)$ and $\mathrm{FS}_k \circ \mathrm{Hilb}_k(h_1)$. Without loss of generality, we may assume that the change of orthonormal basis (or change of inner product) matrix $\sigma_k = e^{A_k}$ between $\mathrm{Hilb}_k(h_0)$, $\mathrm{Hilb}_k(h_1)$ is diagonal with entries $e^{\lambda_0}, \dots, e^{\lambda_{d_k}}$ for some $\lambda_j \in \mathbb{R}$. Let $\underline{\hat{s}}^{(t)} = e^{tA_k} \cdot \underline{\hat{s}}^{(0)}$, where e^{tA_k} is diagonal with entries $e^{\lambda_j t}$. Define

$$h_k(t) := \mathrm{FS}_k \circ G_k(t) = h_{\underline{\hat{s}}^{(t)}} =: h_0 e^{-\varphi_k(t)}. \tag{11}$$

It follows immediately from (8) that

$$\varphi_k(t; z) = \frac{1}{k} \log \sum_{j=0}^{d_k} e^{2\lambda_j t} |\hat{s}_j^{(0)}|^2_{h_0^k}. \tag{12}$$

We emphasize that $\varphi_k(t; z)$ is the intrinsic $\mathcal{B}_k$ geodesic between the endpoints $\mathrm{FS}_k \circ \mathrm{Hilb}_k(h_0)$ and $\mathrm{FS}_k \circ \mathrm{Hilb}_k(h_1)$. It is of course quite distinct from the Hilb_k-image of the Monge–Ampère geodesic; the latter is not intrinsic to $\mathcal{B}_k$ and one cannot gain any information on the $\mathcal{H}$-geodesic by studying it.

We summarize the notation for hermitian metrics and geodesics of metrics:

- For any metric h on L, h^k denotes the induced metric on L^k, and for any metric H on L^k, $H^{1/k}$ is the induced metric on L;
- Given $h_0 \in \mathcal{H}$, $h_t = e^{-\varphi_t} h_0$ is the Monge–Ampère geodesic;
- $h_k = \mathrm{FS} \circ \mathrm{Hilb}_k(h) \in \mathcal{B}_k$ is the natural approximating Bergman metric to h, and $h_k(t) = e^{-\varphi_k(t)} h_0$ is the Bergman geodesic (11).

The main result of [Phong and Sturm 2006] is that the Monge–Ampère geodesic φ_t is approximated by the one-parameter subgroup Bergman geodesic $\varphi_k(t, z)$ in the following weak C^0 sense:

$$\varphi_t(z) = \lim_{\ell\to\infty}\Big[\sup_{k\geq\ell}\varphi_k(t,z)\Big]^* \quad \text{uniformly as } \ell\to\infty, \tag{13}$$

where u^* is the upper envelope of u, that is, $u^*(\zeta_0) = \lim_{\epsilon\to 0}\sup_{|\zeta-\zeta_0|<\epsilon} u(\zeta)$. In particular, without taking the upper envelope, $\sup_{k\geq\ell}\varphi_k(t,z)\to\varphi(t,z)$ almost everywhere as $\ell\to\infty$. See also [Berndtsson 2009] for the subsequent proof of an analogous result for the adjoint bundle $L^k\otimes K$ (where K is the canonical bundle) with an error estimate $\|\varphi_k(t)-\varphi(t)\|_{C^0} = O(\log k\,/\,k)$.

1B. ***Statement of results.*** Our purpose is to show that the degree of convergence of $h_k(t)\to h_t$ or equivalently of $\varphi_k(t,z)\to\varphi_t(z)$ is much stronger that C^0 for toric hermitian metrics on the invariant line bundle $L\to M$ over a smooth toric Kähler manifold. We recall that a toric variety M of dimension m carries the holomorphic action of a complex torus $(\mathbb{C}^*)^m$ with an open dense orbit. The associated real torus $\mathbf{T}^m = (S^1)^m$ acts on M in a Hamiltonian fashion with respect to any invariant Kähler metric ω, that is, it possesses a moment map

$$\mu : M\to P$$

with image a convex lattice polytope. Here, and henceforth, P denotes the closed polytope; its interior is denoted P^o (see Section 2 for background). Objects associated to M are called toric if they are invariant or equivariant with respect to the torus action (real or complex, depending on the context). We define the space of toric hermitian metrics by

$$\mathcal{H}_{\mathbf{T}^m} = \{\varphi\in\mathcal{H} : (e^{i\theta})^*\varphi = \varphi \text{ for all } e^{i\theta}\in\mathbf{T}^m\}. \tag{14}$$

Here, we assume the reference metric h_0 is $\mathbf{T}^m$-invariant. We note that since $\mathbf{T}^m$ has a moment map, it automatically lifts to L and hence it makes sense to say that $h_0 : L\to\mathbb{C}$ is invariant under it. With a slight abuse of notation carried over from [Donaldson 2001], we also let φ denote the full Kähler potential on the open orbit, that is, $\omega_\varphi = dd^c\varphi$ on the open orbit. It is clearly $\mathbf{T}^m$-invariant.

Our main result is:

Theorem 1.1. *Let $L\to M$ be a very ample toric line bundle over a smooth compact toric variety M. Let $\mathcal{H}_T$ denote the space of toric hermitian metrics on L. Let $h_0, h_1\in\mathcal{H}_T$ and let h_t be the Monge–Ampère geodesic between them. Let $h_k(t)$ be the Bergman geodesic between* $\mathrm{Hilb}_k(h_0)$ *and* $\mathrm{Hilb}_k(h_1)$ *in $\mathcal{B}_k$. Let $h_k(t) = e^{-\varphi_k(t,z)}h_0$ and let $h_t = e^{-\varphi_t(z)}h_0$. Then*

$$\lim_{k\to\infty}\varphi_k(t,z) = \varphi_t(z)$$

in $C^2([0,1]\times M)$. In fact, there exists C independent of k such that

$$\|\varphi_k-\varphi\|_{C^2([0,1]\times M)} \leq Ck^{-1/3+\epsilon} \quad \textit{for all } \epsilon > 0.$$

Our methods show moreover that away from the divisor at infinity $\mathcal{D}$ (see Section 2), the function $\varphi_k(t,z)$ has an asymptotic expansion in powers of k^{-1}, and converges in C^∞ to φ_t. But the asymptotics become complicated near $\mathcal{D}$, and require a “multiscale” analysis involving distance to boundary facets. It is therefore not clear whether φ_k has an asymptotic expansion in k^{-1} globally on M. At least, no such asymptotics follow from the known Bergman kernel asymptotics, on or off the diagonal. The analysis of these regimes for general toric varieties seems to be fundamental in “quantum mechanical approximations” on toric varieties.

As mentioned above, the Monge–Ampère equation can be linearized in the toric case and solved explicitly (17); we give a simple new proof in Section 2. The geodesic arcs are easily seen to be C^∞ when the endpoints are C^∞. Hence the C^2-convergence result does not improve the known regularity results on Monge–Ampère geodesics of toric metrics, but pertains only to the degree of convergence of Bergman to Monge–Ampère geodesics in a setting where the latter are known to be smooth; it is possible that the methods can be developed to give regularity results, but this is a distant prospect (see the remarks at the end of this introduction).

1C. ***Outline of the proof.*** We now outline the proof of Theorem 1.1. We start with the fact that the Legendre transform of the Kähler potential linearizes the Monge–Ampère equation (see Section 2G and [Abreu 2003; Guan 1999; Donaldson 2002]). The Legendre transform $\mathscr{L}\varphi$ of the open-orbit Kähler potential φ, a convex function on $\mathbb{R}^m$ in logarithmic coordinates, is the so-called dual symplectic potential

$$u_\varphi(x) = \mathscr{L}\varphi(x), \tag{15}$$

a convex function on the convex polytope P. Under this Legendre transform, the complex Monge–Ampère equation on $\mathscr{H}_{\mathbf{T}^m}$ linearizes to the equation $\ddot{u} = 0$ and is thus solved by

$$u_t = u_{\varphi_0} + t(u_{\varphi_1} - u_{\varphi_0}). \tag{16}$$

Hence the solution φ_t of the geodesic equation on $\mathscr{H}$ is solved in the toric setting by

$$\varphi_t = \mathscr{L}^{-1} u_t. \tag{17}$$

Our goal is to show that $\varphi_k(t; z) \to \mathscr{L}^{-1} u_t$ as in (16) in a strong sense.

The second simplifying feature of the toric setting occurs on the quantum level. The Bergman geodesic is obtained by applying the FS_k map to the one-parameter subgroup e^{tA_k}. In general, it is difficult to understand what kind of asymptotic behavior is possessed by the operators e^{tA_k}. But on a toric variety, there exists a natural basis of the space of holomorphic sections $H^0(M, L^k)$ furnished by monomial sections z^α which are orthogonal with respect to all torus-invariant inner products, and with respect to which all change of basis operators e^{tA_k} are diagonal; we refer to Section 2 or to [Shiffman et al. 2004] for background. Hence, we only need to analyze the eigenvalues of e^{A_k}. The exponents α of the monomials are lattice points $\alpha \in kP$ in the k-th dilate of the polytope P corresponding to M. The eigenvalues in the toric case are given by

$$\lambda_\alpha := \tfrac{1}{2} \log \frac{\mathscr{Q}_{h_0^k}(\alpha)}{\mathscr{Q}_{h_1^k}(\alpha)}, \tag{18}$$

where $\mathscr{Q}_{h_0^k}(\alpha)$ is a *norming constant* for a toric inner product. By a norming constant for a toric hermitian inner product G on $H^0(M, L^k)$ we mean the associated L^2 norm-squares of the monomials

$$\mathscr{Q}_G(\alpha) = \|s_\alpha\|_G^2. \tag{19}$$

In particular, if $h \in \mathscr{H}_{\mathbf{T}^m}$, the norming constants for $\mathrm{Hilb}_k(h)$ are given by

$$\mathscr{Q}_{h^k}(\alpha) = \|s_\alpha\|_{h^k}^2 := \int_{M_P} |s_\alpha(z)|_{h^k}^2 \, dV_h. \tag{20}$$

Thus, an orthonormal basis of $H^0(M, L^k)$ with respect to $\mathrm{Hilb}_k(h)$ for $h \in \mathcal{H}_T$ is given by

$$\left\{ \frac{s_\alpha}{\sqrt{\mathcal{Q}_{h^k}(\alpha)}} : \alpha \in kP \cap \mathbb{Z}^m \right\}.$$

An equivalent, and in a sense dual (see Section 3), formulation is in terms of the functions

$$\mathcal{P}_{h^k}(\alpha, z) := \frac{|s_\alpha(z)|^2_{h^k}}{\mathcal{Q}_{h^k}(\alpha)}, \tag{21}$$

and their special values

$$\mathcal{P}_{h^k}(\alpha) := \mathcal{P}_{h^k}\Big(\alpha, \mu_h^{-1}\Big(\frac{\alpha}{k}\Big)\Big) = \frac{\Big|s_\alpha\Big(\mu_h^{-1}\Big(\frac{\alpha}{k}\Big)\Big)\Big|^2_{h^k}}{\mathcal{Q}_{h^k}(\alpha)}. \tag{22}$$

Given toric hermitian metrics $h_0, h_1 \in \mathcal{H}_{\mathbf{T}^m}$, the change of basis matrix $e^{A_k} = \sigma_{h_0,h_1,k}$ from the monomial orthonormal basis for $\mathrm{Hilb}_k(h_0)$ to that for $\mathrm{Hilb}_k(h_1)$ is diagonal, and the eigenvalues are given by

$$\mathrm{Sp}(e^{A_k} e^{A_k^*}) := \left\{ e^{2\lambda_\alpha(k)} = \frac{\mathcal{Q}_{h_0^k}(\alpha)}{\mathcal{Q}_{h_1^k}(\alpha)} : \alpha \in kP \right\}. \tag{23}$$

Hence, for a $\mathcal{B}_k$-geodesic, (12) becomes

$$\varphi_k(t, z) = \frac{1}{k} \log Z_k(t, z), \tag{24}$$

where

$$Z_k(t, z) = \sum_{\alpha \in kP \cap \mathbb{Z}^m} \left(\frac{\mathcal{Q}_{h_0^k}(\alpha)}{\mathcal{Q}_{h_1^k}(\alpha)} \right)^t \frac{|s_\alpha(z)|^2_{h_0^k}}{\mathcal{Q}_{h_0^k}(\alpha)}. \tag{25}$$

It is interesting to observe that the relative Kähler potential (24) is the logarithm of an exponential sum, hence has the form of a free energy of a statistical mechanical problem with states parametrized by $\alpha \in kP$ and with Boltzmann weights

$$\left(\frac{\mathcal{Q}_{h_0^k}(\alpha)}{\mathcal{Q}_{h_1^k}(\alpha)} \right)^t.$$

Thus, our goal is to prove that

$$\frac{1}{k} \log \sum_{\alpha \in kP \cap \mathbb{Z}^m} \left(\frac{\mathcal{Q}_{h_0^k}(\alpha)}{\mathcal{Q}_{h_1^k}(\alpha)} \right)^t \frac{|s_\alpha(z)|^2_{h_0^k}}{\mathcal{Q}_{h_0^k}(\alpha)} \to \varphi_t(z) \quad \text{in } C^2([0, 1] \times M). \tag{26}$$

1D. *Heuristic proof.* We next sketch a heuristic proof which makes the pointwise convergence obvious. The first step is to obtain good asymptotics of the norming constants (20). As in [Song and Zelditch 2007a], they may be expressed in terms of the symplectic potential by

$$\mathcal{Q}_{h^k}(\alpha) = \int_P e^{-k(u_\varphi(x) + \langle (\alpha/k) - x, \nabla u_\varphi(x) \rangle)} \, dx \tag{27}$$

As k tends to ∞ the integral is dominated by the unique point $x = \alpha/k$, where the "phase function" is maximized. The Hessian is always nondegenerate and by complex stationary phase we obtain the asymptotics

$$\mathcal{Q}_{h^k}(\alpha_k) \sim k^{-m/2} e^{2ku_\varphi(\alpha)}.$$

The complex stationary phase (or steepest descent) method does not apply near the boundary ∂P, causing serious complications, but in this heuristic sketch we ignore this aspect.

If we then replace each term in Z_k by its asymptotics, we obtain

$$\varphi_k(t, e^{\rho/2}) \sim \frac{1}{k} \log \sum_{\alpha \in P \cap (1/k)\mathbb{Z}^m} e^{2k(u_0(\alpha)+t(u_1(\alpha)-u_0(\alpha))+\langle\rho,\alpha\rangle)}. \tag{28}$$

The exponent $(u_0(\alpha)+t(u_1(\alpha)-u_0(\alpha))+\langle\rho,\alpha\rangle)$ is convex and therefore has a unique minimum point. This suggests applying a discrete analogue of complex stationary phase to the sum (28), a Dedekind–Riemann sum which is asymptotic to the integral

$$\int_P e^{2k(u_0(\alpha)+t(u_1(\alpha)-u_0(\alpha))+\langle\rho,\alpha\rangle)}\, d\alpha.$$

Taking $\frac{1}{k}$ times the log of the integral and applying complex stationary phase gives the asymptote

$$\max_{\alpha \in P} \bigl\{u_0(\alpha) + t(u_1(\alpha) - u_0(\alpha)) + \langle\rho, \alpha\rangle\bigr\}.$$

But this is the Legendre transform of the ray of symplectic potentials

$$u_{\varphi_0}(\alpha) + t(u_{\varphi_1}(\alpha) - u_{\varphi_0}(\alpha)),$$

and thus is the Monge–Ampère geodesic.

This is the core idea of the proof. We now give the rigorous version.

1E. *Outline of the rigorous proof.* The main difficulty in the proof of Theorem 1.1 is that the norms have very different asymptotic regimes according to the position of the normalized lattice point α/k relative to the boundary ∂P of the polytope. Even in the simplest case of $\mathbb{CP}^m$, the different positions correspond to the regimes of the central limit theorem, large deviations theorems and Poisson law of rare events for multinomial coefficients. In determining the asymptotics of (24), we face the difficulty that these Boltzmann weights might be exponentially growing or decaying in k as $k \to \infty$.

To simplify the comparison between the Bergman and Monge–Ampère geodesics, we take advantage of the explicit solution (17) of geodesic equation to rewrite $Z_k(t, z)$ in the form

$$e^{-k\varphi_t(z)} Z_k(t, z) = \sum_{\alpha \in kP \cap \mathbb{Z}^m} \mathcal{R}_k(t, \alpha) \frac{|s_\alpha(z)|^2_{h_t^k}}{\mathcal{Q}_{h_t^k}(\alpha)} = \sum_{\alpha \in kP \cap \mathbb{Z}^m} \mathcal{R}_k(t, \alpha) \mathcal{P}_{h_t^k}(\alpha, z), \tag{29}$$

where as usual $h_t = e^{-\varphi_t} h_0$ (with φ_t as in (17)), and where

$$\mathcal{R}_k(t, \alpha) := \frac{\mathcal{Q}_{h_t^k}(\alpha)}{(\mathcal{Q}_{h_0^k}(\alpha))^{1-t} (\mathcal{Q}_{h_1^k}(\alpha))^t}. \tag{30}$$

One of the key ideas is that $\mathcal{R}_k(t,\alpha)$ has at least one order a semiclassical symbol in k, that is, it has at least to some extent an asymptotic expansion in powers of k. Once this is established, it is possible to prove that

$$\frac{1}{k}\log\sum_{\alpha\in kP\cap\mathbb{Z}^m}\mathcal{R}_k(t,\alpha)\mathcal{P}_{h_t^k}(\alpha,z)\to 0 \tag{31}$$

in the C^2-topology on $[0,1]\times M$.

The proof of Theorem 1.1 consists of four main ingredients:

- the Localization Lemma 1.2, which states that the sum over α localizes to a ball of radius $O(k^{-1/2+\delta})$ around the point $\mu_{h_t}(z)$. Here and hereafter, δ can be taken to be any sufficiently small positive constant;
- Bergman/Szegő asymptotics (see Section 4B), which allow one to make comparisons between the sum in Z_k and sums with known asymptotics;
- the Regularity Lemma 1.3, which states that the summands $\mathcal{R}_k(t,\alpha)$ one is averaging have sufficiently smooth asymptotics as $k\to\infty$, allowing one to Taylor expand to order at least one around the point $\mu_{h_t}(z)$;
- joint asymptotics of the Fourier coefficients (21) and particularly their special values $\mathcal{P}_{h^k}(\alpha)$ in the parameters k and distance to ∂P (see Proposition 6.1). We use a complex stationary phase method in the "interior region" far from ∂P and local Bargmann–Fock models near ∂P.

The localization lemma is needed not just for $\mathcal{R}_k(t,\alpha)$ but also for summands which arise from differentiation with respect to (t,z).

Lemma 1.2 (localization of sums). *Let $B_k(t,\alpha):\mathbb{Z}^m\cap kP\to\mathbb{C}$ be a family of lattice point functions satisfying $|B_k(t,\alpha)|\le C_0k^M$ for some $C_0, M\ge 0$. Then, there exists $C>0$ so that for any $\delta>0$,*

$$\sum_{\alpha\in kP\cap\mathbb{Z}^m}B_k(t,\alpha)\frac{|s_\alpha(z)|^2_{h_t^k}}{\mathcal{Q}_{h_t^k}(\alpha)}=\sum_{\alpha:\left|\frac{\alpha}{k}-\mu_{h_t}(z)\right|\le k^{-1/2+\delta}}B_k(t,\alpha)\frac{|s_\alpha(z)|^2_{h_t^k}}{\mathcal{Q}_{h_t^k}(\alpha)}\;+\;O_\delta(k^{-C}).$$

The proof is based on integration by parts. One could localize to the smaller scale

$$\left|\frac{\alpha}{k}-\mu_{h_t}(z)\right|\le C\frac{\log k}{\sqrt{k}},$$

but then the argument only brings errors of the order $(\log k)^{-M}$ for all M and that complicates later applications.

The regularity lemma concerns the behavior of the *Fourier multiplier* $R_k(t,\alpha)$ (30). The sum (25) formally resembles the Berezin covariant symbol of a Toeplitz Fourier multiplier, that is, the restriction to the diagonal of the Schwartz kernel of the operator; we refer to [Shiffman et al. 2003; Zelditch 2009] for discussion of such Toeplitz Fourier multipliers operators on toric varieties and their Berezin symbols. However, the resemblance is a priori just formal — it is not obvious that $R_k(t,\alpha)$ has asymptotics in k. As mentioned above, the nature of the asymptotics is most difficult near ∂P; it is not obvious that smooth convergence holds along $\mathcal{D}$, the divisor at infinity.

Definition. We define the metric volume ratio to be the function on $[0,1]\times P$ defined by

$$\mathcal{R}_\infty(t,x) := \Big(\frac{\det \nabla^2 u_t(x)}{(\det \nabla^2 u_0(x))^{1-t}(\det \nabla^2 u_1(x))^t}\Big)^{1/2}.$$

The purpose of introducing $R_k(t,\alpha)$ is explained by the following result.

Lemma 1.3 (regularity). *The volume ratio $\mathcal{R}_\infty(t,x)\in C^\infty([0,1]\times P)$. Further, for $0\le j\le 2$,*

$$\Big(\frac{\partial}{\partial t}\Big)^j \mathcal{R}_k(t,\alpha) = \Big(\frac{\partial}{\partial t}\Big)^j \mathcal{R}_\infty\Big(t,\frac{\alpha}{k}\Big) + O(k^{-1/3}),$$

where the O symbol is uniform in (t,α).

This lemma is the subtlest part of the analysis. If the $\mathcal{R}_k$ function were replaced by a fixed function $f(x)$ evaluated at α/k then the convergence problem reduces to generalizations of convergence of Bernstein polynomial approximations to smooth functions [Zelditch 2009], and only requires now standard Bergman kernel asymptotics. However, the actual $R_k(t,\alpha)$ do not a priori have this form, and much more is required for their analysis than asymptotics (on and off diagonal) of Bergman kernels. The analysis uses a mixture of complex stationary phase arguments in directions where α/k is not too close to ∂P, while for directions close to ∂P we use an approximation by the "linear" Bargmann–Fock model (see Section 2F and Section 6D).

The somewhat unexpected $k^{-1/3}$ remainder estimate has its origin in this mixture of complex stationary phase and Bargmann–Fock asymptotics. Both methods are valid for k satisfying

$$\frac{C\log k}{k} \le \delta_k \le C'\frac{1}{\sqrt{k}\log k}.$$

In this region, the stationary phase remainder is of order $(k\delta_k)^{-1}$ while the Bargmann–Fock remainder is of order $k\delta_k^2$; the two remainders agree when $\delta_k = k^{-2/3}$, and then the remainder is $O(k^{-1/3})$. For smaller δ_k the Bargmann–Fock approximation is more accurate and for larger δ_k the stationary phase approximation is more accurate. This matter is discussed in detail in Section 6D.

The rest of the proof of the C^2-convergence may be roughly outlined as follows: We calculate two logarithmic derivatives of $e^{-k\varphi_t(z)}Z_k(t,z)$ of (29) with respect to (t,ρ). Using the Localization Lemma 1.2 we can drop the terms in the resulting sums corresponding to α for which $\big|\frac{\alpha}{k}-\mu_{h_t}(z)\big| > k^{-1/2+\delta}$. In the remaining terms we use the Regularity Lemma 1.3 to approximate the summands by their Taylor expansions to order one around $\mu_{h_t}(z)$. This reduces the expressions to derivatives of the diagonal Szegő kernel

$$\Pi_{h_t^k}(z,z) = \sum_{\alpha\in kP\cap\mathbb{Z}^m} \frac{|s_\alpha(z)|^2_{h_t^k}}{\mathcal{Q}_{h_t^k}(\alpha)} \tag{32}$$

for the metric h_t^k on $H^0(M,L^k)$ induced by Monge–Ampère geodesic h_t. Here, we use the smoothness of h_t. The known asymptotic expansion of this kernel (Section 4B) implies the C^2-convergence of $e^{k\varphi_t(z)}Z_k(t,z)$.

As indicated in this sketch, the key problem is to analyze the joint asymptotics of norming constants $\mathcal{Q}_h^k(\alpha)$ and the dual constants $\mathcal{P}_{h^k}(\alpha)$ (22) in (k,α). Norming constants are a complete set of invariants of toric Kähler metrics. Initial results (but not joint asymptotics in the boundary regime) were obtained in [Shiffman et al. 2004]; norms are also an important component of Donaldson's numerical analysis of canonical metrics on toric varieties [Donaldson 2005]. Song and Zelditch [2007a] studied the joint asymptotics of $\mathcal{Q}_h^k(\alpha)$ up to the boundary of the polytope [0, 1] associated to $\mathbb{CP}^1$. In this article, we emphasize the dual constants (22).

1F. ***Bergman approximation and complexification.*** Having described our methods and results, we return to the discussion of their relation to Kähler quantization and to the obstacles in complexifying $\mathrm{Diff}_{\omega_0}(M)$. Further discussion is given in [Rubinstein and Zelditch $\geq$ 2010b].

We may distinguish two intuitive ideas as to the nature of Monge–Ampère geodesics. The first heuristic idea, due to Semmes [1992] and Donaldson [2001], is to view HCMA (homogeneous complex Monge–Ampère) geodesics as one-parameter subgroups of $G_{\mathbb{C}}$, where $G = \mathrm{SDiff}_{\omega_0}(M)$. One-parameter subgroups of $\mathrm{SDiff}_{\omega_0}(M)$ are defined by Hamiltonian flows of initial Hamiltonians $\dot{\varphi}_0$ with respect to ω_0. A complexified one-parameter subgroup is the analytic continuation in time of such a Hamiltonian flow [Semmes 1992; Donaldson 2001]. This idea is heuristic inasmuch as Hamiltonian flows need not possess analytic continuations in time; moreover, no genuine complexification of $\mathrm{SDiff}_{\omega_0}(M)$ exists.

The second intuitive idea, backed up by [Phong and Sturm 2006] and this article, is to view HCMA geodesics as classical limits of $\mathcal{B}_k$ geodesics. The latter have a very simple extrinsic interpretation as one-parameter motions $e^{tA_k}\iota_{\underline{s}}(M)$ of a holomorphic embedding $\iota_{\underline{s}} : M \to \mathbb{CP}^{d_k}$. But the passage to the classical limit is quite nonstandard from the point of view of Kähler quantization. The problem is that the approximating one parameter subgroups e^{tA_k} of operators on $H^0(M, L^k)$, which change an orthonormal basis for an initial inner product to a path of orthonormal bases for the geodesic of inner products, are not a priori complex Fourier integral operators or any known kind of quantization of classical dynamics.

The heuristic view taken in this article and series is that e^{tA_k} should be approximately the analytic continuation of the Kähler quantization of a classical Hamiltonian flow. To explain this, let us recall the basic ideas of Kähler quantization.

Traditionally, Kähler quantization refers to the quantization of a polarized Kähler manifold (M, ω, L) by Hilbert spaces $H^0(M, L^k)$ of holomorphic sections of high powers of a holomorphic line bundle $L \to M$ with Chern class $c_1(L) = [\omega]$. The Kähler form determines a hermitian metric h such that $\mathrm{Ric}(h) = \omega$. The hermitian metric induces inner products $\mathrm{Hilb}_k(h)$ on $H^0(M, L^k)$. In this quantization theory, functions H on M are quantized as hermitian (Toeplitz) operators $\hat{H} := \Pi_{h^k} H \Pi_{h^k}$ on $H^0(M, L^k)$, and canonical transformations of (M, ω) are quantized as unitary operators on $H^0(M, L^k)$. Quantum dynamics is given by unitary groups $e^{itk\hat{H}}$ (see [Berman et al. 2008; Boutet de Monvel and Sjöstrand 1976; Zelditch 1998] for references).

In the case of Bergman geodesics with fixed endpoints, H should be $\dot{\varphi}_0$, the initial tangent vector to the HCMA geodesic with the fixed endpoints. The quantization of the Hamiltonian flow of $\dot{\varphi}$ should then be $e^{itk\hat{H}}$ and its analytic continuation should be $e^{tk\hat{H}}$. The change of basis operator e^{tA_k} should then be approximately the same as $e^{tk\hat{H}}$. But proving this and taking the classical limit is necessarily nonstandard when the classical analytic continuation of the Hamiltonian flow of $\dot{\varphi}$ does not exist. Moreover, we only know that $\dot{\varphi} \in C^{0,1}$.

This picture of the Bergman approximation to HCMA geodesics is validated in this article in the case of the Dirichlet problem on projective toric Kähler manifolds. Y. Rubinstein and the second author have also verified for the initial value problem on toric Kähler manifolds [Rubinstein and Zelditch $\geq$ 2010b]. The same authors are currently investigating it for general Kähler metrics on Riemann surfaces.

1G. ***Final remarks and further results and problems.*** An obvious question within the toric setting is whether $\varphi_k(t) \to \varphi_t$ in a stronger topology than C^2 on a toric variety. It seems possible that the methods of this paper could be extended to C^k-convergence. The methods of this paper easily imply C^k-convergence for all k away from ∂P or equivalently the divisor at infinity, but the degree of convergence along this set has yet to be investigated. As mentioned above, we do not see why φ_k should have an asymptotic expansion in k, but this aspect may deserve further exploration. We also mention that our methods can be extended to prove C^2-convergence of Berndtsson's approximations in [Berndtsson 2009].

In subsequent articles on the toric case, we build on the methods introduced here to prove convergence theorems. In [Song and Zelditch 2007b], we develop the methods of this article to prove that the geodesic rays constructed in [Phong and Sturm 2007] from test configurations are $C^{1,1}$ and no better on a toric variety. Test configuration geodesic rays are solutions of a kind of initial value problem; we refer to [Phong and Sturm 2007; Song and Zelditch 2007b] for the definitions and results. For test configuration geodesics, the analogue of $\mathcal{R}_k$ is not even smooth in t. The smooth initial value problem is studied in [Rubinstein and Zelditch $\geq$ 2010b]. In a different direction, a C^2 convergence result has been proved for completely general harmonic maps of Riemannian manifolds with boundary into toric varieties [Rubinstein 2008; Rubinstein and Zelditch 2010]. This includes the Wess–Zumino–Witten model, where the manifold is a Riemann surface with boundary.

We believe that the techniques of this paper extend to other settings with a high degree of symmetry of the kinds discussed in [Donaldson 2008]. Recently, Feng [2009] adapted our methods to "toric" metrics on abelian varieties, that is, metrics which are invariant under a real Lagrangian torus action. Associated to the torus action is a torus-valued moment map. Abelian varieties are simpler than toric varieties in that the image of the moment map is the full torus; that is, there is no boundary to the image. Consequently, Feng [2009] is able to improve Lemma 1.3 and then Theorem 1.1 to give C^∞ convergence and complete asymptotics expansions. The general Kähler case involves significant further obstacles. A basic problem in generalizing the results is to construct a useful localized basis of sections on a general (M, ω). In the toric case, we use the basis of $\mathbf{T}^m$-invariant states $\hat{s}_\alpha = z^\alpha$, which "localize" on the so-called "Bohr–Sommerfeld tori", that is, the inverse images $\mu^{-1}(\frac{\alpha}{k})$ of lattice points under the moment map μ. Such Bohr–Sommerfeld states also exist on any Riemann surface; in subsequent work, we hope to relate them to the convergence problem for HCMA geodesics on Riemann surfaces.

We briefly speculate on the higher-dimensional general Kähler case. There are a number of plausible substitutes for the Bohr–Sommerfeld basis on a general Kähler manifold. A rather traditional one is to study the asymptotics of e^{A_k} on a basis of *coherent states* $\Phi^w_{h^k}$. Here,

$$\Phi^w_{h^k}(z) = \frac{\Pi_{h^k}(z, w)}{\sqrt{\Pi_{h^k}(w, w)}}$$

are L^2 normalized Szegő kernels pinned down in the second argument. Intuitively, $\Phi^w_{h^k}$ is like a gaussian bump centered at w with shape determined by the metric h. It is thus more localized than the monomials z^α, which are only gaussian transverse to the tori. Under the change of basis operators e^{tA_k}, both the center and shape should change. Like the monomials z^α, coherent states have some degree of orthogonality. There are in addition other well localized bases depending on the Kähler metric which may be used in the analysis.

Our main result (Theorem 1.1) may be viewed heuristically as showing that as $k \to \infty$ the change of basis operators e^{tA_k} tend to a path f_t of diffeomorphisms changing the initial Kähler metric ω_0 into the metric ω_t along the Monge–Ampère geodesic. This suggests that

$$e^{tA_k}\Phi^w_{h^k} \sim \Phi^{f_t(w)}_{h_t^k},$$

where h_t is the Monge–Ampère geodesic and f_t is the Moser path of diffeomorphisms such that $f_t^*\omega_0 = \omega_t$. We leave the exact degree of asymptotic similarity vague at this time since even the regularity of the Moser path is currently an open problem.

2. Background on toric varieties

In this section, we review the necessary background on toric Kähler manifolds. In addition to standard material on Kähler and symplectic potentials, moment maps and polytopes, we also present some rather nonstandard material on almost analytic extensions of Kähler potentials and moment maps that are needed later on. We also give a simple proof that the Legendre transform from Kähler potentials to symplectic potentials linearizes the Monge–Ampère equation.

Let M be a complex manifold. We use the standard notation

$$\partial = \frac{\partial}{\partial z} = \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big), \quad \bar\partial = \frac{\partial}{\partial \bar z} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big), \quad d = \partial + \bar\partial, \quad d^c := \frac{i}{4\pi}(\bar\partial - \partial), \quad dd^c = \frac{i}{2\pi}\partial\bar\partial.$$

The last three are real operators.

Let $L \to M$ be a holomorphic line bundle. The Chern form of a hermitian metric h on L is defined by

$$c_1(h) = \omega_h := -\frac{\sqrt{-1}}{2\pi}\partial\bar\partial \log \|e_L\|_h^2, \tag{33}$$

where e_L denotes a local holomorphic frame (= nonvanishing section) of L over an open set $U \subset M$, and $\|e_L\|_h = h(e_L, e_L)^{1/2}$ denotes the h-norm of e_L. We say that (L, h) is positive if the (real) 2-form ω_h is a positive $(1, 1)$ form, that is, defines a Kähler metric. We write $\|e_L(z)\|_h^2 = e^{-\varphi}$ or locally $h = e^{-\varphi}$, and then refer to φ as the Kähler potential of ω_h in U. In this notation,

$$\omega_h = \frac{\sqrt{-1}}{2\pi}\partial\bar\partial\varphi = dd^c\varphi. \tag{34}$$

If we fix a hermitian metric h_0 and let $h = e^{-\varphi}h_0$, and put $\omega_0 = \omega_{h_0}$, then

$$\omega_h = \omega_0 + dd^c\varphi. \tag{35}$$

The metric h induces hermitian metrics h^k on $L^k = L \otimes \cdots \otimes L$ given by $\|s^{\otimes k}\|_{h_N} = \|s\|_h^k$.

We now specialize to toric Kähler manifolds; for background, we refer to [Abreu 2003; Donaldson 2002; Guan 1999; Shiffman et al. 2004]. A toric Kähler manifold is a Kähler manifold (M, J, ω) on which the complex torus $(\mathbb{C}^*)^m$ acts holomorphically with an open orbit M^o. Choosing a basepoint m_0 on the open orbit identifies $M^o \equiv (\mathbb{C}^*)^m$ and give the point $z = e^{\rho/2+i\varphi}m_0$ the holomorphic coordinates

$$z = e^{\rho/2+i\varphi} \in (\mathbb{C}^*)^m, \quad \rho, \varphi \in \mathbb{R}^m. \tag{36}$$

The real torus $\mathbf{T}^m \subset (\mathbb{C}^*)^m$ acts in a Hamiltonian fashion with respect to ω. Its moment map

$$\mu = \mu_\omega : M \to P \subset \mathbf{t}^* \simeq \mathbb{R}^m$$

(where $\mathbf{t}$ is the Lie algebra of $\mathbf{T}^m$) with respect to ω defines a singular torus fibration over a convex lattice polytope P; as in the introduction, P is understood to be the closed polytope. We recall that the moment map of a Hamiltonian torus action with respect to a symplectic form ω is the map $\mu_\omega : M \to \mathbf{t}^*$ defined by $d\langle \mu_\omega(z), \xi\rangle = \iota_{\xi^\#}\omega$, where $\xi^\#$ is the vector field on M induced by the vector $\xi \in \mathbf{t}$. Over the open orbit one thus has a symplectic identification

$$\mu : M^o \simeq P^o \times \mathbf{T}^m.$$

We let x denote the Euclidean coordinates on P. The components $(I_1, \dots, I_m)$ of the moment map are called action variables for the torus action. The symplectically dual variables on $\mathbf{T}^m$ are called the angle variables. Given a basis of $\mathbf{t}$ or equivalently of the action variables, we denote by $\{\partial/\partial\theta_j\}$ the corresponding generators (Hamiltonian vector fields) of the $\mathbf{T}^m$ action. Under the complex structure J, we also obtain generators $\partial/\partial\rho_j$ of the $\mathbb{R}^m_+$ action.

The action variables are globally defined smooth functions but fail to be coordinates at points where the generators of the $\mathbf{T}^m$ action vanish. We denote the set of such points by $\mathcal{D}$ and refer to it as the divisor at infinity. If $p \in \mathcal{D}$ and $\mathbf{T}^m_p$ denotes the isotropy group of p, then the generating vector fields of $\mathbf{T}^m_p$ become linearly dependent at P. Since we are proving C^2 estimates, we need to replace them near points of $\mathcal{D}$ by vector fields with norms bounded below. We discuss good choices of coordinates near points of $\mathcal{D}$ below.

We assume M is smooth and that P is a Delzant polytope. It is defined by a set of linear inequalities

$$\ell_r(x) := \langle x, v_r\rangle - \lambda_r \geq 0, \quad r = 1, \dots, d,$$

where v_r is a primitive element of the lattice and inward-pointing normal to the r-th $(m-1)$-dimensional facet $F_r = \{\ell_r = 0\}$ of P. We recall that a facet is a highest-dimensional face of a polytope. The inverse image $\mu^{-1}(\partial P)$ of the boundary of P is the divisor at infinity $\mathcal{D} \subset M$. For $x \in \partial P$ we denote by

$$\mathcal{F}(x) = \{r : \ell_r(x) = 0\}$$

the set of facets containing x. To measure when $x \in P$ is near the boundary we further define

$$\mathcal{F}_\epsilon(x) = \{r : |\ell_r(x)| < \epsilon\}. \tag{37}$$

The simplest toric varieties are linear Kähler manifolds (V, ω) carrying a linear holomorphic torus action. They provide local models near a corner of P or equivalently near a fixed point of the $\mathbf{T}^m$ action. As discussed in [Guillemin and Sternberg 1982; Lerman and Tolman 1997], a linear symplectic torus action is determined by a choice of m elements β_j of the weight lattice of the Lie algebra of the torus. The vector space then decomposes $(V, \omega) = \bigoplus (V_i, \omega_i)$ of orthogonal symplectic subspaces so that the moment map has the form

$$\mu_{BF}(v_1, \dots, v_m) = \sum |v_j|^2 \beta_j. \tag{38}$$

The image of the moment map is the orthant $\mathbb{R}^m_+$. This provides a useful local model at corners. We refer to these as Bargmann–Fock models; they play a fundamental role in this article (see Section 2F).

2A. ***Slice-orbit coordinates.*** We will also need local models at points near codimension r faces, and therefore supplement the coordinates (36) on the open orbit with holomorphic coordinates valid in neighborhoods of points of $\mathcal{D}$. An atlas of coordinate charts for M generalizing the usual affine charts of $\mathbb{CP}^m$ is given in [Shiffman et al. 2004, Section 3.2], and we briefly recall the definitions. For each vertex $v_0 \in P$, we define the chart U_{v_0} by

$$U_{v_0} := \{z \in M_P : \chi_{v_0}(z) \neq 0\}, \tag{39}$$

where

$$\chi_\alpha(z) = z^\alpha = z_1^{\alpha_1} \cdots z_m^{\alpha_m}.$$

Throughout the article we use standard multiindex notation, and put $|\alpha| = \alpha_1 + \cdots + \alpha_m$. Since P is Delzant, we can choose lattice points $\alpha^1, \ldots, \alpha^m$ in P such that each α^j is in an edge incident to the vertex v_0, and the vectors $v^j := \alpha^j - v_0$ form a basis of $\mathbb{Z}^m$. We define

$$\eta : (\mathbb{C}^*)^m \to (\mathbb{C}^*)^m, \quad \eta(z) = \eta_j(z) := (z^{v^1}, \ldots, z^{v^m}). \tag{40}$$

The map η is a $\mathbf{T}^m$-equivariant biholomorphism with inverse

$$z : (\mathbb{C}^*)^m \to (\mathbb{C}^*)^m, \quad z(\eta) = (\eta^{\Gamma e^1}, \ldots, \eta^{\Gamma e^m}), \tag{41}$$

where e^j is the standard basis for $\mathbb{C}^m$, and Γ is an $m \times m$-matrix with $\det \Gamma = \pm 1$ and integer coefficients defined by

$$\Gamma v^j = e^j, \quad v^j = \alpha^j - v_0. \tag{42}$$

The corner of P at v_0 is transformed to the standard corner of the orthant $\mathbb{R}^m_+$ by the affine linear transformation

$$\tilde{\Gamma} : \mathbb{R}^m \ni u \to \Gamma u - \Gamma v_0 \in \mathbb{R}^m, \tag{43}$$

which preserves $\mathbb{Z}^m$, carries P to a polytope $Q_{v_0} \subset \{x \in \mathbb{R}^m : x_j \geq 0\}$ and carries the facets F_j incident at v_0 to the coordinate hyperplanes $= \{x \in Q_{v_0}; x_j = 0\}$. The map η extends to a homeomorphism

$$\eta : U_{v_0} \to \mathbb{C}^m, \quad \eta(z_0) = 0, \tag{44}$$

where z_0 is the fixed point corresponding to v_0. By this homeomorphism, the set $\mu_P^{-1}(\bar{F}_j)$ corresponds to the set $\{\eta \in \mathbb{C}^m : \eta_j = 0\}$. If $\bar{F}$ be a closed face with $\dim F = m - r$ which contains v_0, then there are facets $F_{i_1}, \ldots, F_{i_r}$ incident at v_0 such that $\bar{F} = \bar{F}_{i_1} \cap \cdots \cap \bar{F}_{i_r}$. The subvariety $\mu_P^{-1}(\bar{F})$ corresponding $\bar{F}$ is expressed by

$$\mu_P^{-1}(\bar{F}) \cap U_{v_0} = \{\eta \in \mathbb{C}^m : \eta_{i_j} = 0,\ j = 1, \ldots, r\}. \tag{45}$$

When working near a point of $\mu_P^{-1}(\bar{F})$, we simplify notation by writing

$$\eta = (\eta', \eta'') \in \mathbb{C}^m = \mathbb{C}^r \times \mathbb{C}^{m-r}, \tag{46}$$

where $\eta' = (\eta_{i_j})$ as in (45) and where η'' are the remaining η_j's, so that $(0, \eta'')$ is a local coordinate of the submanifold $\mu_P^{-1}(\bar{F})$. When the point $(0, \eta'')$ lies in the open orbit of $\mu_P^{-1}(\bar{F})$, we often write $\eta'' = e^{i\theta'' + \rho''/2}$. In practice, we simplify notation by tacitly treating the corner at v_0 as if it were the standard corner of $\mathbb{R}^m_+$, omit mention of Γ and always use (z', z'') instead of η. It is straightforward to rewrite all the expressions we use in terms of the more careful coordinate charts just mentioned.

These coordinates may be described more geometrically as *slice-orbit* coordinates. Set $P_0 \in \mu_P^{-1}(\bar{F})$ and let $(\mathbb{C}^*)^m_{P_0}$ denote its stabilizer (isotropy subgroup). Then there always exists a local slice at P_0, that is, a local analytic subspace $S \subset M$ containing P_0, invariant under $(\mathbb{C}^*)^m_{P_0}$, and such that the natural $(\mathbb{C}^*)^m$-equivariant map of the normal bundle of the orbit $(\mathbb{C}^*)^m \cdot P_0$, namely

$$[\zeta, P] \in (\mathbb{C}^*)^m \times_{(\mathbb{C}^*)^m_z} S \mapsto \zeta P \in M, \tag{47}$$

is a biholomorphism onto $(\mathbb{C}^*)^m \cdot S$. The terminology is taken from [Sjamaar 1995, Theorem 1.23]. The slice S can be taken to be the image of a ball in the hermitian normal space $T_{P_0}((\mathbb{C}^*)^m P_0)^\perp$ to the orbit under any local holomorphic embedding $w : T_{P_0}((\mathbb{C}^*)^m P_0)^\perp \to M$ with $w(P_0) = P_0$ and $dw_{P_0} = \mathrm{Id}$. The affine coordinates η'' above define the slice $S = \eta^{-1}\{(z', z''(P_0)) : z' \in (\mathbb{C}^*)^r\}$. The local orbit-slice coordinates are then defined by

$$P = (z', e^{i\theta''+\rho''/2}) \Longleftrightarrow \eta(P) = e^{i\theta''+\rho''/2}(z', 0), \tag{48}$$

where $(z', 0) \in S$ is the point on the slice with affine holomorphic coordinates $z' = (\eta')$.

As will be seen below, toric functions are smooth functions of the variables e^{ρ_j} away from $\mathcal{D}$, and of the variables $|z_j|^2$ at points near $\mathcal{D}$. We introduce the following polar coordinates centered at a point $P \in \mathcal{D}$:

$$r_j := |z_j| = e^{\rho_j/2}. \tag{49}$$

They are polar coordinates along the slice. The gradient vector field of r_j is denoted $\partial/\partial r_j$. As with polar vector fields, it is not well-defined at $r_j = 0$. But to prove C^ℓ estimates of functions which are smooth functions of r_j^2 it is sufficient to prove C^ℓ estimates with respect to the vector fields $\partial/\partial r_j$ or $\partial/\partial(r_j^2)$.

2B. ***Kähler potential in the open orbit and symplectic potential.*** Now consider the Kähler metrics ω in $\mathcal{H}$ (see (1)). We recall that on any simply connected open set, a Kähler metric may be locally expressed as $\omega = 2i\partial\bar{\partial}\varphi$, where φ is a locally defined function which is unique up to the addition $\varphi \mapsto \varphi + f(z) + \overline{f(z)}$ of the real part of a holomorphic or antiholomorphic function f. Here, $a \in \mathbb{R}$ is a real constant which depends on the choice of coordinates. Thus, a Kähler metric $\omega \in \mathcal{H}$ has a Kähler potential φ over the open orbit $M^o \subset M$. In fact, there is a canonical choice of the open-orbit Kähler potential once one fixes the image P of the moment map:

$$\varphi(z) = \log \sum_{\alpha \in P} |z^\alpha|^2 = \log \sum_{\alpha \in P} e^{\langle \alpha, \rho \rangle}. \tag{50}$$

Invariance under the real torus action implies that φ only depends on the ρ-variables, so that we may write it in the form

$$\varphi(z) = \varphi(\rho) = F(e^\rho). \tag{51}$$

The notation $\varphi(z) = \varphi(\rho)$ is an abuse of notation, but is rather standard since [Donaldson 2002]. For instance, the Fubini–Study Kähler potential is $\varphi(z) = \log(1 + |z|^2) = \log(1 + e^\rho) = F(e^\rho)$. Note that the Kähler potential $\log(1 + |z|^2)$ extends to $\mathbb{C}^m$ from the open orbit $(\mathbb{C}^*)^m$, although the coordinates (ρ, θ) are only valid on the open orbit. This is a typical situation.

On the open orbit, we then have

$$\omega_\varphi = \frac{i}{2} \sum_{j,k} \frac{\partial^2 \varphi(\rho)}{\partial \rho_k \partial \rho_j} \frac{dz_j}{z_j} \wedge \frac{d\bar{z}_k}{\bar{z}_k}. \tag{52}$$

Positivity of ω_φ implies that $\varphi(\rho) = F(e^\rho)$ is a strictly convex function of $\rho \in \mathbb{R}^n$. The moment map with respect to ω_φ is given on the open orbit by

$$\mu_{\omega_\varphi}(z_1, \dots, z_m) = \nabla_\rho \varphi(\rho) = \nabla_\rho F(e^{\rho_1}, \dots, e^{\rho_m}), \quad (z = e^{\rho/2+i\theta}). \tag{53}$$

Here, and henceforth, we subscript moments maps either by the hermitian metric h or by a local Kähler potential φ. The formula (53) follows from the fact that the generators $\partial/\partial\theta_j$ of the $\mathbf{T}^m$ actions are Hamiltonian vector fields with respect to ω_φ with Hamiltonians $\partial\varphi(\rho)/\partial\rho_j$, since

$$\iota_{\partial/\partial\theta_j} \omega_\varphi = d \frac{\partial \varphi}{\partial \rho_j}. \tag{54}$$

The moment map is a homeomorphism from $\rho \in \mathbb{R}^m$ to the interior P^o of P and extends as a smooth map from $M \to \overline{P}$ with critical points on the divisor at infinity $\mathcal{D}$. Hence, the Hamiltonians (54) extend to $\mathcal{D}$.

Note that the local Kähler potential on the open orbit is not the same as the global smooth *relative Kähler potential* in (1) with respect to a background Kähler metric ω_0. That is, given a reference metric ω_0 with Kähler potential φ_0, it follows by the $\partial\bar{\partial}$ lemma that $\omega = \omega_0 + dd^c\varphi$ with $\varphi \in C^\infty(M)$. As discussed in [Donaldson 2002, Proposition 3.1.7], the Kähler potential φ on the open orbit defines a singular potential on M which satisfies $dd^c\varphi = \omega + H$ where H is a fixed current supported on $\mathcal{D}$. We generally denote Kähler potentials by φ and in each context explain which type we mean.

By (52), a $\mathbf{T}^m$-invariant Kähler potential defines a real convex function on $\rho \in \mathbb{R}^m$. Its Legendre dual is the *symplectic potential* u_φ: for $x \in P$ there is a unique ρ such that $\mu_\varphi(e^{\rho/2}) = \nabla_\rho \varphi = x$. Then the Legendre transform is defined to be the convex function

$$u_\varphi(x) = \langle x, \rho_x \rangle - \varphi(\rho_x), \quad e^{\rho_x/2} = \mu_\varphi^{-1}(x) \Longleftrightarrow \rho_x = 2 \log \mu_\varphi^{-1}(x) \tag{55}$$

on P. The gradient $\nabla_x u_\varphi$ is an inverse to μ_{ω_φ} on $M_\mathbb{R}$ on the open orbit, or equivalently on P, in the sense that $\nabla u_\varphi(\mu_{\omega_\varphi}(z)) = z$ as long as $\mu_{\omega_\varphi}(z) \notin \partial P$.

The symplectic potential has canonical logarithmic singularities on ∂P. According to [Abreu 2003, Proposition 2.8] or [Donaldson 2002, Proposition 3.1.7], there is a one-to-one correspondence between $\mathbf{T}^m_\mathbb{R}$-invariant Kähler potentials ψ on M_P and symplectic potentials u in the class S of continuous convex functions on $\overline{P}$ such that $u - u_0$ is smooth on $\overline{P}$ where

$$u_0(x) = \sum_k \ell_k(x) \log \ell_k(x). \tag{56}$$

Thus, $u_\varphi(x) = u_0(x) + f_\varphi(x)$ where $f_\varphi \in C^\infty(\overline{P})$. We note that u_0 and u_φ are convex, that $u_0 = 0$ on ∂P and hence $u_\varphi = f_\varphi$ on ∂P. By convexity, $\max_P u_0 = 0$.

We denote by $G_\varphi = \nabla_x^2 u_\varphi$ the Hessian of the symplectic potential. It has simple poles on ∂P. It follows that $\nabla_\rho^2 \varphi$ has a kernel along $\mathcal{D}$. The kernel of $G_\varphi^{-1}(x)$ on $T_x \partial P$ is the linear span of the normals

μ_r for $r \in \mathcal{F}(x)$. We also denote by $H_\varphi(\rho) = \nabla^2_\rho \varphi(e^\rho)$ the Hessian of the Kähler potential on the open orbit in ρ coordinates. By Legendre duality,

$$H_\varphi(\rho) = G_\varphi^{-1}(x), \quad \mu(e^\rho) = x. \tag{57}$$

This relation may be extended to $\mathcal{D} \to \partial P$. The kernel of the left side is the Lie algebra of the isotropy group G_p of any point $p \in \mu^{-1}(x)$. The volume density has the form

$$\det(G_\varphi^{-1}) = \delta_\varphi(x) \cdot \prod_{r=1}^{d} \ell_r(x), \tag{58}$$

for some positive smooth function δ_φ [Abreu 2003]. We note that $\log \prod_{r=1}^{d} \ell_r(x)$ is known in convex optimization as the logarithmic barrier function of P.

2C. ***Kähler potential near $\mathcal{D}$.*** We also need smooth local Kähler potentials in neighborhoods of points $z_0 \in \mathcal{D}$. We note that the open orbit Kähler potential (50) is well-defined near $z = 0$. Local expressions for the Kähler potential at other points of $\mathcal{D}$ essentially amount to making an affine transformation of P to transform a given corner of P to 0, and in these coordinates the local Kähler potential near any point of $\mathcal{D}$ can be expressed in the form (50). For instance, on $\mathbb{CP}^1$, a Kähler potential valid at $z = \infty$ is given in the coordinates $w = 1/z$ by $\log(1 + |w|^2)$. It differs on the open orbit from the canonical Kähler potential $\log(1 + |z|^2)$ by the term $\log |z|^2$ whose $i\partial\bar\partial$ is a delta function at $z = 0$, supported on $\mathcal{D}$ away from the point $w = 0$ that one is studying. In [Song 2005] the reader can find further explicit examples of toric Kähler potentials in affine coordinate charts. Hence, in what follows, we will always use (50) as the local expression of the Kähler potential, without explicitly writing in the affine change of variables.

We will however need to be explicit about the use of slice-orbit coordinates z'_j, ρ''_j (see (48)) in the local expressions of the Kähler potential. The coordinates near z_0 depend on $\mathcal{F}_\epsilon(z_0)$ from (37). For each $z_0 \in \mathcal{D}$ corresponding to a codimension r face of P, after an affine transformation changing the face to $x' = 0$, we may write the Kähler potential as the canonical one in slice-orbit coordinates, $F(|z'|^2, e^{\rho''})$ Section 2A (48). Since $0 \in P$, F is smooth up to the boundary face $z' = 0$. The fact that F is smooth up to the boundary also follows from the general fact that a smooth $\mathbf{T}^m$-invariant function $g \in C^\infty_{\mathbf{T}^m}(M)$ may be expressed in the form $g(z) = \hat F_g(\mu_\varphi(z))$ where as $\hat F_g \in C^\infty(\mathbb{R}^m)$. This is known as the divisibility property of $\mathbf{T}^m$-invariant smooth functions [Lerman and Tolman 1997]. It implies that F is a smooth function of the polar coordinates r_j^2 near points of $\mathcal{D}$ in the sense of (49).

2D. ***Almost analytic extensions.*** In analyzing the Bergman/Szegő kernel and the functions (21), we make use of the *almost analytic extension* $\varphi(z, w)$ to $M \times M$ of a Kähler potential for a Kähler ω; for background on almost analytic extensions; see [Boutet de Monvel and Sjöstrand 1976; Melin and Sjöstrand 1975]. It is defined near the totally real antidiagonal $(z, \bar z) \in M \times M$ by

$$\varphi_{\mathbb{C}}(x + h, x + k) \sim \sum_{\alpha,\beta} \frac{\partial^{\alpha+\beta}\varphi}{\partial z^\alpha \partial \bar z^\beta}(x) \frac{h^\alpha}{\alpha!} \frac{k^\beta}{\beta!}. \tag{59}$$

When φ is real analytic on M, the almost analytic extension $\varphi(z, w)$ is holomorphic in z and antiholomorphic in w and is the unique such function for which $\varphi(z) = \varphi(z, z)$. In the general C^∞ case, the almost analytic extension is a smooth function with the right side of (59) as its C^∞ Taylor expansion along the antidiagonal, for which $\bar{\partial}\varphi(z, w) = 0$ to infinite order on the antidiagonal. It is only defined in a small neighborhood $(M \times M)_\delta = \{(z, w) : d(z, w) < \delta\}$ of the antidiagonal in $M \times M$, where $d(z, w)$ refers to the distance between z and w with respect to the Kähler metric ω. It is well defined up to a smooth function vanishing to infinite order on the diagonal; the latter is negligible for our purposes (cf. Proposition 1.1 of [Boutet de Monvel and Sjöstrand 1976].)

The analytic continuation $\varphi(z, w)$ of the Kähler potential was used by Calabi [1953] in the analytic case to define a Kähler distance function, known as the Calabi diastasis function:

$$D(z, w) := \varphi(z, w) + \varphi(w, z) - (\varphi(z) + \varphi(w)). \tag{60}$$

Calabi showed that

$$D(z, w) = d(z, w)^2 + O(d(z, w)^4), \;\; dd^c_w D(z, w)|_{z=w} = \omega. \tag{61}$$

One has the same notion in the almost analytic sense.

The gradient of the almost analytic extension of the Kähler potential in the toric case defines the almost analytic extension $\mu_{\mathbb{C}}(z, w)$ of the moment map. We are mainly interested in the case where $w = e^{i\theta} z$ lies on the $\mathbf{T}^m$-orbit of z, and by (53) we have,

$$i\,\mu_{\mathbb{C}}(z, e^{i\theta} z) = \nabla_\theta \varphi_{\mathbb{C}}(z, e^{i\theta} z) = \nabla_\theta F_{\mathbb{C}}(e^{i\theta}|z|^2), \tag{62}$$

where F is defined in (51). We sometimes drop the subscript in $F_{\mathbb{C}}$ and $\mu_{\mathbb{C}}$ since there is only one interpretation of their extension; but we emphasize that $\varphi(z, e^{i\theta} z) = F_{\mathbb{C}}(e^{i\theta}|z|^2)$ is very different from $\varphi(e^{i\theta} z) = F(|e^{i\theta} z|^2) = F(|z|^2)$. For example, the moment map of the Bargmann–Fock model $(\mathbb{C}^m, |z|^2)$ is $\mu(z) = (|z_1|^2, \dots, |z_m|^2)$, whose analytic extension is $(z_1\bar{w}_1, \dots, z_m\bar{w}_m)$. Similarly that of the Fubini–Study metric on $\mathbb{CP}^m$ is (in multiindex notation)

$$\mu_{\mathrm{FS},\mathbb{C}}(z, w) = \frac{z \cdot \bar{w}}{1 + z \cdot \bar{w}}.$$

In Section 2F we further illustrate the notation in the basic examples of Bargmann–Fock and Fubini–Study models. We also observe that (62) continues to hold for the Kähler potential $F(|z'|^2, e^{\rho''})$ in slice-orbit coordinates. That is, we have

$$i\,\mu(z', e^{\rho''/2}) = \nabla_{\theta',\theta''} F_{\mathbb{C}}(e^{i\theta'}|z'|^2, e^{i\theta'' + \rho''})|_{(\theta',\theta'')=(0,0)}. \tag{63}$$

The complexified moment map is a map

$$\mu_{\mathbb{C}} \to (M \times M)_\delta \to \mathbb{C}^m. \tag{64}$$

The invariance of μ under the torus action implies that $\mu_{\mathbb{C}}(e^{i\theta} z, e^{i\theta} w) = \mu_{\mathbb{C}}(z, w)$. The following proposition will clarify the discussion of critical point sets later on (see, for example, Lemma 5.2).

Proposition 2.1. *For δ sufficiently small so that $\mu_{\mathbb{C}}(z, w)$ is well-defined, we have*

(1) $\operatorname{Im} \mu_{\mathbb{C}}(z, e^{i\theta} z) = \frac{1}{2} \nabla_\theta D(z, e^{i\theta} z)$.

(2) $\mu_{\mathbb{C}}(z, e^{i\theta} z) = \mu_{\mathbb{C}}(z, z)$ *with* $(z, e^{i\theta} z) \in (M \times M)_\delta$ *if and only if* $e^{i\theta} z = z$.

Proof. The proof of the identity (1) is immediate from the definitions; we only note that the diastasis function is a kind of real part, and that the imaginary part originates in the factor of i in (62). One can check the factors of i in the Bargmann–Fock model, where $\mu_{\mathbb{C}}(z, e^{i\theta}z) = e^{i\theta}|z|^2$ while $D(z, e^{i\theta}z) = 2(\cos\theta - 1)|z|^2 + 2i(\sin\theta)|z|^2$ (in vector notation).

By (61), $D(z, w)$ has a strict global minimum at $w = z$ which is nondegenerate. It is therefore isolated for each z. Since its Hessian at $w = z$ is the identify with respect to ω, the isolating neighborhood has a uniform size as z varies. Thus, there exists a $\delta > 0$ so that $\mu_{\mathbb{C}}(z, w) = \mu_{\mathbb{C}}(z, z)$ in $(M \times M)_\delta$ if and only if $z = w$. This is true both in the real analytic case and the almost-analytic case. □

2E. ***Hilbert spaces of holomorphic sections.*** On the "quantum level", a toric Kähler variety (M, ω) induces the sequence of spaces $H^0(M, L^k)$ of holomorphic sections of powers of the holomorphic toric line bundle L with $c_1(L) = \frac{1}{2\pi}[\omega]$. The $(\mathbb{C}^*)^m$ action lifts to $H^0(M, L^k)$ as a holomorphic representation which is unitary on $\mathbf{T}^m$. Corresponding to the lattice points $\alpha \in kP$, there is a natural basis $\{s_\alpha\}$ (denoted χ_α^P in [Shiffman et al. 2004]) of $H^0(M, L^k)$ given by joint eigenfunctions of the $(\mathbb{C}^*)^m$ action. It is well-known that the joint eigenvalues are precisely the lattice points $\mathbb{Z}^m \cap kP$ in the k-th dilate of P. On the open orbit $s_\alpha(z) = \chi_\alpha(z)e^k$ where e is a frame and where as above $\chi_\alpha(z) = z^\alpha = z_1^{\alpha_1} \cdots z_m^{\alpha_m}$. Hence, the s_α are referred to as monomials. For further background, we refer to [Shiffman et al. 2004]. A hermitian metric h on L induces the Hilbert space inner products (6) on $H^0(M, L^k)$.

As is evident from (21), we will need formulae for the monomials which are valid near $\mathcal{D}$. By (40) and (42), we have

$$\chi_{\alpha^j}(z) = \eta_j(z)\chi_{v^0}(z), \quad z \in (\mathbb{C}^*)^m, \tag{65}$$

and by (43) we then have

$$|\chi_\alpha(z)|^2 = |\eta^{\tilde{\Gamma}(\alpha)}|^2. \tag{66}$$

As mentioned above, for simplicity of notation we suppress the transformation $\tilde{\Gamma}$ and coordinates η, and we will use the orbit-slice coordinates of (48). Thus, we denote the monomials corresponding to lattice points α near a face F by $(z')^{\alpha'} e^{\langle (i\theta'' + \rho''/2), \alpha'' \rangle}$, where $\tilde{\Gamma}(\alpha) = (\alpha', \alpha'')$ with α'' in the coordinate hyperplane corresponding under $\tilde{\Gamma}$ to F and with α' in the normal space.

2F. ***Examples: Bargmann–Fock and Fubini–Study models.*** As mentioned above the Bargmann–Fock model is the linear model. It plays a fundamental role in this article because it provides an approximation for objects on any toric variety on balls of radius $\log k/\sqrt{k}$ and also near $\mathcal{D}$. Although it and the Fubini–Study model are elementary examples, we go over them because the notation is used frequently later on.

The Bargmann–Fock models on $\mathbb{C}^m$ correspond to choices of a positive definite hermitian matrix H on $\mathbb{C}^m$. A toric Bargmann–Fock model is one in which H commutes with the standard $\mathbf{T}^m$ action, that is, is a diagonal matrix. We denote its diagonal elements by $H_{j\bar{j}}$. The Kähler metric on $\mathbb{C}^m$ is thus $i\partial\bar{\partial}\varphi_{BF,H}(z)$ where the global Kähler potential is

$$\varphi_{BF,H}(z) = \sum_{j=1}^m H_{j\bar{j}}|z_j|^2 = F(|z_1|^2, \dots, |z_m|^2), \text{ with } F(y_1, \dots, y_m) = \sum_j H_{j\bar{j}} y_j.$$

For simplicity we often only consider the case $H = I$. Putting $|z_j|^2 = e^{\rho_j}$ and using (53), it follows that $\mu_{BF,H}(z_1, \dots, z_m) = (H_{1\bar{1}}|z_1|^2, \dots, H_{m\bar{m}}|z_m|^2) : \mathbb{C}^m \to \mathbb{R}_+^m$ as in (38). The symplectic potential

Legendre dual to $\varphi_{BF,H}$ is given by

$$u_{BF,H}(x) = -\varphi_{BF,H}(\mu_{BF}^{-1}(x)) + 2\langle \log \mu_{BF,H}^{-1}(x), x\rangle = -\sum_j x_j + \sum_{j=1}^m x_j \log \frac{x_j}{H_{j\bar{j}}}. \tag{67}$$

In this case, $G_{BF,H}$ is the diagonal matrix with entries $\dfrac{1}{x_j H_{j\bar{j}}}$, so

$$\det G_{BF,H} = \frac{1}{\det H} \prod_j \frac{1}{x_j}.$$

The off-diagonal analytic extension of the Kähler potential in the sense of (59) is then

$$\varphi_{BF,H}(z, \bar{w}) = \sum_{j=1}^m H_{j\bar{j}} z_j \bar{w}_j = F(z_1\bar{w}_1, \dots, z_m\bar{w}_m),$$

and in particular,

$$\varphi_{BF,H}(z, e^{i\theta} z) = \sum_{j=1}^m H_{j\bar{j}} e^{i\theta_j} |z_j|^2 = F(e^{i\theta_1}|z_1|^2, \dots, e^{i\theta_m}|z_m|^2).$$

Henceforth we often write the right side in the multiindex notation $F_{\mathbb{C}}(e^{i\theta}|z|^2)$. We observe, as claimed in (62), that $\nabla_\theta F_{BF,\mathbb{C}}(e^{i\theta}|z|^2)|_{\theta=0} = i\mu_{BF}(z)$.

Quantization of the Bargmann–Fock model with $H = I$ produces the Bargmann–Fock (Hilbert) space

$$\mathcal{H}^2(\mathbb{C}^m, (2\pi)^{-m} k^m e^{-k|z|^2} dz \wedge d\bar{z})$$

of entire functions which are L^2 relative to the displayed weight. It is infinite-dimensional and a basis is given by the monomials z^α where $\alpha \in \mathbb{R}_+^m \cap \mathbb{Z}^m$. In Section 3A we compute their L^2 norms. For $H \neq I$ one uses the volume form $e^{-k\langle Hz,z\rangle}(i\partial\bar{\partial}\langle Hz, z\rangle)^m/m! = e^{-k\langle Hz,z\rangle} \det H \, dz \wedge d\bar{z}$.

Toric Fubini–Study metrics provide compact models which are similar to Bargmann–Fock models. In a local analysis we always use the latter. A Fubini–Study metric on $\mathbb{CP}^m$ is determined by a positive hermitian form H on $\mathbb{C}^{m+1}$ and a toric Fubini–Study metric is a diagonal one $\sum_{j=0}^m H_{j\bar{j}}|Z_j|^2$. In the affine chart $Z_0 \neq 0$, for example, a local Fubini–Study Kähler potential is

$$\varphi_{\mathrm{FS},H}(z_1, \dots, z_m) = \log\Big(1 + \sum_j h_{j\bar{j}}|z_j|^2\Big),$$

where $h_{j\bar{j}} = H_{j\bar{j}}/H_{0\bar{0}}$. This is a valid Kähler potential near $z = 0$ but of course has logarithmic singularities on the hyperplane at infinity. The almost analytic extension of the Fubini–Study Kähler potential is given in the affine chart by $\log(1 + \sum_j h_{j\bar{j}} z_j \bar{w}_j)$. Thus (62) asserts that

$$i\, \frac{\sum_j h_{j\bar{j}}|z_j|^2}{1 + \sum_j h_{j\bar{j}}|z_j|^2} = \nabla_\theta \log\Big(1 + \sum_j h_{j\bar{j}} e^{i\theta_j}|z_j|^2\Big)\Big|_{\theta=0}.$$

Quantization produces the Hilbert spaces $H^0(\mathbb{CP}^m, \mathbb{O}(k))$, where $\mathbb{O}(k) \to \mathbb{CP}^m$ is the kth power of the hyperplane section bundle. Sections lift to homogeneous holomorphic polynomials on $\mathbb{C}^{m+1}$, and correspond to lattice points in $k\Sigma$ where Σ is the unit simplex in $\mathbb{R}^m$.

2G. ***Linearization of the Monge–Ampère equation.*** It is known that the Legendre transform linearizes the Monge–Ampère geodesic equation. Since it is important for this article, we present a simple proof that does not seem to exist in the literature.

Proposition 2.2. *Let M_P^c be a toric variety. Then under the Legendre transform $\varphi \mapsto u_\varphi$ the complex Monge–Ampère equation on $\mathcal{H}_{\mathbf{T}^m}$ linearizes to the equation $u'' = 0$. Hence the Legendre transform of a geodesic φ_t has the form $u_t = u_0 + t(u_1 - u_0)$.*

Proof. It suffices to show that the energy functional

$$E = \int_0^1 \int_M \dot{\varphi}_t^2 d\mu_{\varphi_t} dt \tag{68}$$

is Euclidean on paths of symplectic potentials. For each t let us push forward the integral $\int_M \dot{\varphi}_t^2 d\mu_\varphi$ under the moment map μ_{φ_t}. The integrand is by assumption invariant under the real torus action, so the push forward is a diffeomorphism on the real points. The volume measure $d\mu_{\varphi_t}$ pushes forward to dx. The function $\partial_t \varphi_t(\rho)$ pushes forward to the function $\psi_t(x) = \dot{\varphi}_t(\rho_{x,t})$ where $\mu_{\varphi_t}(\rho_{x,t}) = x$. By (55), the symplectic potential at time t is

$$u_t(x) = \langle x, \rho_{x,t}\rangle - \varphi_t(\rho_{x,t}).$$

We note that

$$\dot{u}_t = \langle x, \partial_t \rho_{x,t}\rangle - \dot{\varphi}_t(\rho_{x,t}) - \langle \nabla_\rho \varphi_t(\rho_{x,t}), \partial_t \rho_{x,t}\rangle. \tag{69}$$

The outer terms cancel, and thus, our integral is just

$$\int_0^1 \int_P |\dot{u}_t|^2 dx\, dt.$$

Clearly the Euler–Lagrange equations are linear. □

3. The functions $\mathcal{P}_{h^k}$ and $\mathcal{Q}_{h^k}$

We now introduce the key players in the analysis, the norming constants $\mathcal{Q}_{h^k}(\alpha)$ (20) and the dual constants $\mathcal{P}_{h^k}(\alpha)$ of (22). The duality is given in the following:

Proposition 3.1.

$$\mathcal{Q}_{h_k}(\alpha) = \frac{e^{ku_\varphi(\alpha/k)}}{\mathcal{P}_{h^k}(\alpha)}. \tag{70}$$

Proof. By (55), it follows that

$$\left\| s_\alpha\left(\mu_h^{-1}\left(\frac{\alpha}{k}\right)\right)\right\|_{h^k}^2 = \left|\chi_\alpha\left(\mu_h^{-1}\left(\frac{\alpha}{k}\right)\right)\right|^2 e^{-k\varphi_h(\mu_h^{-1}(\alpha/k))} = e^{ku_{\varphi_h}(\alpha/k)}. \qquad \square$$

Corollary 3.2.

$$\mathcal{R}_k(t,\alpha) = \frac{(\mathcal{P}_{h_0^k}(\alpha))^{1-t}(\mathcal{P}_{h_1^k}(\alpha))^t}{\mathcal{P}_{h_t^k}(\alpha)}.$$

Proof. We need to show that

$$\frac{\mathcal{Q}_{h_t^k}(\alpha)}{(\mathcal{Q}_{h_0^k}(\alpha))^{1-t}(\mathcal{Q}_{h_1^k}(\alpha))^t} = \frac{(\mathcal{P}_{h_0^k}(\alpha))^{1-t}(\mathcal{P}_{h_1^k}(\alpha))^t}{\mathcal{P}_{h_t^k}(\alpha)}. \tag{71}$$

By Proposition 3.1, the left side of (71) equals

$$\frac{\big|\chi_\alpha\big(\mu_{h_t}^{-1}\big(\frac{\alpha}{k}\big)\big)\big|^2 e^{-k\varphi_t(\mu_{h_t}^{-1}(\alpha/k))}}{\mathcal{P}_{h_t^k}(\alpha)}\left(\frac{\mathcal{P}_{h_0^k}(\alpha)}{\big|\chi_\alpha\big(\mu_0^{-1}\big(\frac{\alpha}{k}\big)\big)\big|^2 e^{-k\varphi_0(\mu_0^{-1}(\alpha/k))}}\right)^{1-t}\left(\frac{\mathcal{P}_{h_1^k}(\alpha)}{\big|\chi_\alpha\big(\mu_1^{-1}\big(\frac{\alpha}{k}\big)\big)\big|^2 e^{-k\varphi_1(\mu_1^{-1}(\alpha/k))}}\right)^{t}.$$

By the equality in the proof of Proposition 3.1, the left side of (71) equals

$$e^{k(u_t(\alpha/k)+(1-t)u_0(\alpha/k)+tu_1(\alpha/k))}\times\frac{(\mathcal{P}_{h_0^k}(\alpha))^{1-t}(\mathcal{P}_{h_1^k}(\alpha))^t}{\mathcal{P}_{h_t^k}(\alpha)}.$$

But $u_t(x)+(1-t)u_0(x)+tu_1(x)=0$ on a toric variety, and this gives the stated equality. □

Further, we relate the full $\mathcal{P}_{h^k}(\alpha,z)$ to the Szegő kernel. The Szegő (or Bergman) kernels of a positive hermitian line bundle $(L,h)\to(M,\omega)$ over a Kähler manifold are the kernels of the orthogonal projections $\Pi_{h^k}:L^2(M,L^k)\to H^0(M,L^k)$ onto the spaces of holomorphic sections with respect to the inner product $\mathrm{Hilb}_k(h)$ (6). Thus, we have

$$\Pi_{h^k}s(z)=\int_M\Pi_{h^k}(z,w)\cdot s(w)\frac{\omega_h^m}{m!},\tag{72}$$

where the $\cdot$ denotes the h-hermitian inner product at w. Let e_L be a local holomorphic frame for $L\to M$ over an open set $U\subset M$ of full measure, and let $\{s_j^k=f_je_L^{\otimes k}:j=1,\dots,d_k\}$ be an orthonormal basis for $H^0(M,L^k)$ with $d_k=\dim H^0(M,L^k)$. Then the Szegő kernel can be written in the form

$$\Pi_{h^k}(z,w):=F_{h^k}(z,w)\,e_L^{\otimes k}(z)\otimes\overline{e_L^{\otimes k}(w)},\tag{73}$$

where

$$F_{h^k}(z,w)=\sum_{j=1}^{d_k}f_j(z)\overline{f_j(w)}.\tag{74}$$

Since the Szegő kernel is a section of the bundle $(L^k)\otimes(L^k)^*\to M\times M$, it often simplifies the analysis to lift it to a scalar kernel $\hat{\Pi}_{h^k}(x,y)$ on the associated unit circle bundle $X\to M$ of (L,h). Here, $X=\partial D_h^*$ is the boundary of the unit disc bundle with respect to h^{-1} in the dual line bundle L^*. We use local product coordinates $x=(z,t)\in M\times S^1$ on X, where $x=e^{it}\|e_L(z)\|_he_L^*(z)\in X$. To avoid confusing the S^1 action on X with the $\mathbf{T}^m$ action on M we use e^{it} for the former and $e^{i\theta}$ (multiindex notation) for the latter. We note that the $\mathbf{T}^m$ action lifts to X and combines with the S^1 action to produce a $(S^1)^{m+1}$ action. We refer to [Zelditch 1998; 2009; Shiffman and Zelditch 2002] for background and for more on lifting the Szegő kernel of a toric variety.

The equivariant lift of a section $s=fe_L^{\otimes k}\in H^0(M,L^k)$ is given explicitly by

$$\hat{s}(z,t)=e^{ikt}\|e_L^{\otimes k}\|_{h^k}f(z)=e^{k[-(1/2)\varphi(z)+it]}f(z)\,.\tag{75}$$

The Szegő kernel thus lifts to $X\times X$ as the scalar kernel

$$\hat{\Pi}_k(z,t;w,t')=e^{k[-(1/2)\varphi(z)-(1/2)\varphi(w)+i(t-t')]}F_k(z,w)\,.\tag{76}$$

Since it is S^1- equivariant we often put $t=t'=0$.

Proposition 3.3. $$\mathcal{P}_{h^k}(\alpha, z) = (2\pi)^{-m} \int_{\mathbf{T}^m} \hat{\Pi}_{h^k}(e^{i\theta}z, 0; z, 0)e^{-i\langle\alpha,\theta\rangle}d\theta.$$

Proof. We recall that $\chi_\alpha(z) = z^\alpha$ is the local representative of s_α in the open orbit with respect to an invariant frame. Since $\{\chi_\alpha/\sqrt{\mathcal{Q}_{h^k}(\alpha)}\}$ is the local expression of an orthonormal basis, we have

$$F_{h^k}(z, w) = \sum_{\alpha\in kP\cap\mathbb{Z}^m} \frac{\chi_\alpha(z)\overline{\chi_\alpha(w)}}{\mathcal{Q}_{h^k}(\alpha)};$$

hence

$$\hat{\Pi}_{h^k}(z, 0; w, 0) = \sum_{\alpha\in kP\cap\mathbb{Z}^m} \frac{\chi_\alpha(z)\overline{\chi_\alpha(w)}e^{-k(\varphi(z)+\varphi(w))/2}}{\mathcal{Q}_{h^k}(\alpha)}.$$

It follows that

$$\Pi_{h^k}(e^{i\theta}z, 0; z, 0) = \sum_{\alpha\in kP\cap\mathbb{Z}^m} \frac{|\chi_\alpha(z)|^2e^{-k\varphi(z)}e^{i\langle\alpha,\theta\rangle}}{\mathcal{Q}_{h^k}(\alpha)}.$$

Integrating against $e^{-i\langle\alpha,\theta\rangle}$ sifts out the α term. □

Corollary 3.4. *We have*

$$\mathcal{P}_{h^k}(\alpha) = (2\pi)^{-m} \int_{\mathbf{T}^m} \hat{\Pi}_{h^k}\left(e^{i\theta}\mu_h^{-1}\left(\frac{\alpha}{k}\right), 0; \mu_h^{-1}\left(\frac{\alpha}{k}\right), 0\right)e^{-i\langle\alpha,\theta\rangle}d\theta. \tag{77}$$

3A. ***Bargmann–Fock model.*** As discussed in Section 2F, the Hilbert space in this model has the orthogonal basis z^α with $\alpha \in \mathbb{R}^m_+ \cap \mathbb{Z}^m$. The Bargmann–Fock norming constants when $H = I$ are given by

$$\mathcal{Q}_{h^k_{BF}}(\alpha) = k^{-|\alpha|-m}\alpha!, \quad \text{where } \alpha! := \alpha_1!\ldots\alpha_m!.$$

It follows that an orthonormal basis of holomorphic monomials is given by $\{k^{(|\alpha|+m/2)}z^\alpha/\sqrt{\alpha!}\}$.

We therefore have

$$\frac{|s_\alpha(z)|^2_{h^k_{BF}}}{\mathcal{Q}_{h^k_{BF}}(\alpha)} = k^{|\alpha|+m}\frac{|z^\alpha|^2}{\alpha!}e^{-k|z|^2}, \tag{78}$$

and in particular,

$$\mathcal{P}_{h^k_{BF}}(\alpha) = k^m e^{-|\alpha|}\frac{\alpha^\alpha}{\alpha!}, \tag{79}$$

where $\alpha^\alpha = 1$ when $\alpha = 0$. Here, we use that $u_{BF}\left(\frac{\alpha}{k}\right) = \frac{\alpha}{k}\log\frac{\alpha}{k} - \frac{\alpha}{k}$, so that

$$e^{ku_{BF}(\alpha/k)} = e^{-|\alpha|}\frac{k^{-|\alpha|}}{\alpha^\alpha}$$

and $\mathcal{Q}_{h^k_{BF}}(\alpha) = k^{-m-|\alpha|}\alpha!$. We observe that $\mathcal{P}_{h^k_{BF}}(\alpha)$ depends on k only through the factor k^m.

Precisely the same formula holds if we replace I by a positive diagonal H with elements $H_{j\bar{j}}$. By a change of variables we obtain $\mathcal{Q}_{h^k_{BF,H}}(\alpha) = \prod_{j=1}^m H_{j\bar{j}}^{-\alpha_j}\mathcal{Q}_{h^k_{BF}}(\alpha)$, and also by (67) we have $u_{BF,H}(x) = u_{BF}(x) + \sum_j x_j \log H_{j\bar{j}}$. Hence, by Proposition 3.1,

$$\mathcal{P}_{h^k_{BF,H}}(\alpha) = \mathcal{P}_{h^k_{BF}}(\alpha)\prod_{j=1}^m H_{j\bar{j}}^{-\alpha_j}e^{\sum_j \alpha_j \log H_{j\bar{j}}} = \mathcal{P}_{h^k_{BF}}(\alpha).$$

3A1. $\mathbb{CP}^m$. In the Fubini–Study model, a basis of $H^0(\mathbb{CP}^m, \mathbb{O}(k))$ is given by monomials with $\alpha \in k\Sigma$ (see Section 2F), and the norming constants are given by

$$\mathcal{Q}_{h^k_{\mathrm{FS}}}(\alpha) = \binom{k}{\alpha} := \binom{k}{\alpha_1, \dots, \alpha_m}^{-1}. \tag{80}$$

Recall that multinomial coefficients are defined for $\alpha_1 + \cdots + \alpha_m \leq k$ by

$$\binom{k}{\alpha_1, \dots, \alpha_m} = \frac{k!}{\alpha_1! \cdots \alpha_m!(k - |\alpha|)!},$$

where, as above, $|\alpha| = \alpha_1 + \cdots + \alpha_m$.

We further have

$$|s_\alpha(z)|^2_{h^k_{\mathrm{FS}}} = |z^\alpha|^2 e^{-k \log(1+|z|^2)}$$

and therefore,

$$\mathcal{P}_{h^k_{\mathrm{FS}}}(\alpha, z) = \binom{k}{\alpha_1, \dots, \alpha_m} |z^\alpha|^2 e^{-k \log(1+|z|^2)},$$

and since

$$e^{-ku_{\mathrm{FS}}(\alpha/k)} = \left| s_\alpha\left(\mu_{\mathrm{FS}}^{-1}\left(\frac{\alpha}{k}\right)\right)\right|^2_{h^k_{\mathrm{FS}}} = \left(\frac{\alpha}{k}\right)^\alpha \left(1 - \frac{|\alpha|}{k}\right)^{k-|\alpha|},$$

we have

$$\mathcal{P}_{h^k_{\mathrm{FS}}}(\alpha) = \frac{k!}{\alpha_1! \cdots \alpha_m!(k - |\alpha|)!} \left(\frac{\alpha}{k}\right)^\alpha \left(1 - \frac{|\alpha|}{k}\right)^{k-|\alpha|}.$$

4. The Szegő kernel of a toric variety

We will use Proposition 3.3 to reduce the joint asymptotics of $\mathcal{P}_{h^k} \alpha, z)$ in (k, α) to asymptotics of the Bergman–Szegő kernel off the diagonal. We now review some general facts about diagonal and off-diagonal expansions of these kernels, for which complete details can be found in [Shiffman and Zelditch 2002], and we also consider some special properties of toric Bergman–Szegő kernels which are very convenient for calculations; to some extent they derive from [Shiffman et al. 2004], but the latter only considered Szegő kernels for powers of Bergman metrics.

The Szegő kernels $\hat{\Pi}_{h^k}(x, y)$ are the Fourier coefficients of the total Szegő projector $\hat{\Pi}_h(x, y) : \mathcal{L}^2(X) \to \mathcal{H}^2(X)$, where $\mathcal{H}^2(X)$ is the Hardy space of boundary values of holomorphic functions on D^* (the kernel of $\bar{\partial}_b$ in $L^2(X)$). Thus,

$$\hat{\Pi}_{h^k}(x, y) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ikt} \hat{\Pi}_h(e^{it} x, y)\, dt.$$

The properties we need of $\hat{\Pi}_{h^k}(x, y)$ are based on the Boutet de Monvel–Sjöstrand construction [1976] of an oscillatory integral parametrix for the Szegő kernel:

$$\begin{gathered} \hat{\Pi}(x, y) = S(x, y) + E(x, y)\,, \\ \text{with } S(x, y) = \int_0^\infty e^{i\lambda\psi(x,y)} s(x, y, \lambda) d\lambda\,, \qquad E(x, y) \in \mathscr{C}^\infty(X \times X)\,. \end{gathered} \tag{81}$$

The phase function ψ is of positive type and is given in the local coordinates above by

$$\psi(z,t;w,t') = \frac{1}{i}\bigl(1 - e^{\varphi(z,w)-(\varphi(z)+\varphi(w))/2}e^{i(t-t')}\bigr). \tag{82}$$

Here, $\varphi(z,w)$ is the almost analytic extension of the local Kähler potential with respect to the frame, that is, $h = e^{-\varphi(z)}$; see (59) for the notion of almost analytic extension. The amplitude $s(z,t;w,t',\lambda)$ is a semiclassical amplitude as in [Boutet de Monvel and Sjöstrand 1976, Theorem 1.5], that is, it admits an expansion $s \sim \sum_{j=0}^{\infty} \lambda^{m-j} s_j(x,y) \in S^m(X \times X \times \mathbb{R}^+)$.

The phase $\psi(z,t;w,t')$ is the generating function for the graph of the identity map along the symplectic cone $\Sigma \subset T^*X$ defined by $\Sigma = \{(x, r\alpha_x) : r > 0\}$, where α_x is the Chern connection one form. Hence the singularity of $\hat{\Pi}(x,y)$ only occurs on the diagonal and the symbol s is understood to be supported in a small neighborhood $(M \times M)_\delta$ of the antidiagonal. It will be useful to make the cutoff explicit by introducing a smooth cutoff function $\chi(d(z,w))$, where χ is a smooth even function on $\mathbb{R}$ and $d(z,w)$ denotes the distance between z, w in the base Kähler metric.

As above, we denote the k-th Fourier coefficient of these operators relative to the S^1 action by $\hat{\Pi}_{h^k} = S_{h^k} + E_{h^k}$. Since E is smooth, we have $E_{h^k}(x,y) = O(k^{-\infty})$, where $O(k^{-\infty})$ denotes a quantity which is uniformly $O(k^{-n})$ on $X \times X$ for all positive n. Hence $E_{h^k}(z,w)$ is negligible for all the calculations and estimates of this article, and further it is only necessary to use a finite number of terms of the symbol s. For simplicity of notation, we will use the entire symbol.

It follows that (with $x = (z,t)$, $y = (w,0)$ and with $\chi(d(z,w))$ as above),

$$\begin{aligned}\hat{\Pi}_{h^k}(x,y) &= S_{h^k}(x,y) + O(k^{-\infty}) \\ &= k \int_0^\infty \int_0^{2\pi} e^{ik(-t+\lambda\psi(z,t;w,0))} \chi(d(z,w)) s(z,t;w,0,k\lambda)\, dt\, d\lambda + O(k^{-\infty}).\end{aligned} \tag{83}$$

The integral is a damped complex oscillatory integral since (61) implies that

$$\operatorname{Im} \psi(x,y) \ge C d(x,y)^2, \quad (x, y \in X), \tag{84}$$

for (x,y) sufficiently close to the diagonal, as one sees by Taylor expanding the phase around the diagonal; see [Boutet de Monvel and Sjöstrand 1976, Corollary 1.3]. It follows from (83) and from (84) that the Szegő kernel $\Pi_{h^k}(z,w)$ on M is "gaussian" in small balls $d(z,w) \le \log k/\sqrt{k}$, that is,

$$|\hat{\Pi}_{h^k}(z,\varphi;w,\varphi')| \le Ck^m e^{-kd(z,w)^2} + O(k^{-\infty}), \qquad \text{when } d(z,w) \le \frac{\log k}{\sqrt{k}}, \tag{85}$$

and on the complement $d(z,w) \ge \log k/\sqrt{k}$ it is rapidly decaying. This rapid decay can be improved to long range (subgaussian) exponential decay off the diagonal given by the global Agmon estimates,

$$|\hat{\Pi}_{h^k}(z,\varphi;w,\varphi')| \le Ck^m e^{-\sqrt{k}d(z,w)}. \tag{86}$$

We refer to [Christ 2003; Lindholm 2001] for background and references.

It is helpful to eliminate the integrals in (83) by complex stationary phase. Expressed in a local frame and local coordinates on M, the result is this:

Proposition 4.1. *Let (L,h) be a C^∞ positive hermitian line bundle, and let $h = e^{-\varphi}$ in a local frame. Then in this frame, there exists a semiclassical amplitude $A_k(z,w) \sim k^m a_0(z,w) + k^{m-1}a_1(z,w) + \cdots$*

in the parameter k^{-1} such that

$$\hat{\Pi}_{h^k}(z, 0; w, 0) = e^{k(\varphi(z,w)-(\varphi(z)+\varphi(w))/2)}\chi_k(d(z, w))\, A_k(z, w) + O(k^{-\infty}),$$

where, as above, $\chi_k(d(z, w)) = \chi\Big(\frac{k^{1/2}}{\log k} d(z, w)\Big)$ is a cutoff to a $\frac{\log k}{\sqrt{k}}$-neighborhood of the diagonal.

Proof. This follows from the scaling asymptotics of [Shiffman and Zelditch 2002] or from [Berman et al. 2008, Theorem 3.1]. We refer there for a detailed proof of the scaling asymptotics and only sketch a somewhat intuitive proof.

The integral (83) is a complex oscillatory integral with a positive complex phase. With no loss of generality we may set $\varphi' = 0$. Taking the λ-derivative gives one critical point equation

$$1 - e^{\varphi(z,w)-(\varphi(z)+\varphi(w))/2} e^{it} = 0 \tag{87}$$

and the critical point equation in t implies that $\lambda = 1$. The λ-critical point equation can only be satisfied for complex t with imaginary part equal to the negative of the Calabi diastasis function (60), that is,

$$\operatorname{Im} t = D(z, w),$$

and with real part equal to $-\operatorname{Im}\varphi(z, w)$. To obtain asymptotics, we therefore have to deform the integral over S^1 to the circle $|\zeta| = e^{-D(z,w)}$. Since $d(z, w) \le C(\log k/\sqrt{k})$ by assumption, the deformed contour is a slightly rescaled circle by the amount $(\log k/\sqrt{k})$; in the complete proofs, the contour is held fixed and the integrand is rescaled as in [Shiffman and Zelditch 2002]. The contour deformation is possible modulo an error $O(k^{-M})$ of arbitrarily rapid polynomial decay because the integrand may be replaced by the parametrix (up to any order in λ) which has a holomorphic dependence on the $\mathbb{C}^*$ action on L^*, hence in $e^{i\theta}$ to a neighborhood of S^1 in $\mathbb{C}$. This is immediately visible in the phase and with more work is visible in the amplitude (this is the only incompleteness in the proof; the statement can be derived from [Shiffman and Zelditch 2002; Christ 2003]). We need to use a cutoff to a neighborhood of the diagonal of $M \times M$, but it may be chosen to be independent of θ.

By deforming the circle of integration from the unit circle to $|\zeta| = e^{D(z,w)}$ and then changing variables $t \mapsto t + iD(z, w)$ to bring it back to the unit circle, we obtain

$$\hat{\Pi}_{h^k}(x, y) \sim k\int_0^\infty\int_0^{2\pi} e^{ik(-t-iD(z,w)-\lambda\psi(z,t+iD(z,w);w,0))} s(z, t+iD(z, w); w, 0, k\lambda)\,dt\,d\lambda \mod k^{-\infty}.$$

The new critical point equations state that $\lambda = 1$ and that $e^{i\operatorname{Im}\varphi(z,w)}e^{it} = 1$. The calculation shows that $\psi = 0$ on the critical set so the phase factor on the critical set equals $e^{\varphi(z,w)-(1/2)(\varphi(z)+\varphi(w))}$. The Hessian of the phase on the critical set is $\left(\begin{smallmatrix} 0 & 1 \\ 1 & i \end{smallmatrix}\right)$, as in the diagonal case, and the rest of the calculation proceeds as in [Zelditch 1998]. (As mentioned above, a complete proof is contained in [Shiffman and Zelditch 2002]). □

4A. *Toric Bergman–Szegő kernels.* In the toric case, we may simplify the expression for the Szegő kernels in Proposition 4.1 using the almost analytic extension (see Equation (59)) of the Kähler potential $\varphi(z, w)$ to $M \times M$, which has the form

$$F_{\mathbb{C}}(z \cdot \bar{w}) = \text{ the almost analytic extension of } F(|z|^2) \text{ to } M \times M. \tag{88}$$

The almost analytic extension will be illustrated in some analytic examples below, where it coincides with the analytic continuation.

Thus, we have:

Proposition 4.2. *For any hermitian toric positive line bundle over a toric variety, the Szegő kernel for the metrics h_φ^k have the asymptotic expansions in a local frame on M,*

$$\Pi_{h^k}(z, w) \sim e^{k(F_{\mathbb{C}}(z\cdot\bar{w})-(F(|z|^2)+F(|w|^2))/2}A_k(z, w) \mod k^{-\infty},$$

where

$$A_k(z, w) \sim k^m\Big(a_0(z, w)+\frac{a_1(z, w)}{k}+\cdots\Big)$$

is a semiclassical symbol of order m.

As an example, the Bargmann–Fock(–Heisenberg) Szegő kernel with $k=1$ and $H=I$ is given (up to a constant C_m depending only on the dimension) by

$$\hat{\Pi}_{h_{BF}}(z, \theta, w, \varphi)=e^{z\cdot\bar{w}-(|z|^2+|w|^2)/2}e^{i(\theta-\varphi)}=\sum_{\alpha\in\mathbb{Z}^n}\frac{z^\alpha\overline{w^\alpha}}{\alpha!}e^{-(|z|^2+|w|^2)/2}e^{i(\theta-\varphi)}.$$

The higher Szegő kernels are Heisenberg dilates of this kernel:

$$\hat{\Pi}_{h^k_{BF}}(x, y)=\frac{1}{\pi^m}k^m e^{ik(t-s)}e^{k(\zeta\cdot\bar{\eta}-(1/2)|\zeta|^2-(1/2)|\eta|^2)}, \tag{89}$$

where $x=(\zeta, t)$, $y=(\eta, s)$. In this case, the almost analytic extension is analytic and $F_{BF,\mathbb{C}}(z, w)=z\cdot\bar{w}$.

A second example is the Fubini–Study Szegő kernel on $\mathbb{O}(k)$, which lifts to $S^{2m-1}\times S^{2m-1}$ as

$$\hat{\Pi}_{h^k_{\mathrm{FS}}}(x, y)=\sum_J\frac{(k+m)!}{\pi^m j_0!\cdots j_m!}x^J\bar{y}^J=\frac{(k+m)!}{\pi^m k!}\langle x, y\rangle^k. \tag{90}$$

Recalling that

$$x=e^{i\theta}\frac{e(z)}{\|e(z)\|}$$

in a local frame e over an affine chart, the Szegő kernel has the local form on $\mathbb{C}^m\times\mathbb{C}^m$ of

$$\hat{\Pi}_{h^k_{\mathrm{FS}}}(z, 0; w, 0)=\frac{(k+m)!}{\pi^m k!}\exp\biggl(k\log\frac{(1+z\cdot\bar{w})}{(\sqrt{1+|z|^2}\sqrt{1+|w|^2})}\biggr). \tag{91}$$

Thus, $F_{\mathrm{FS},\mathbb{C}}(z, w)=\log(1+z\cdot\bar{w})$.

4B. ***Asymptotics of derivatives of toric Bergman–Szegő kernels.*** One of the key ingredients in of Theorem 1.1 is the asymptotics of derivatives of the contracted Bergman–Szegő kernel

$$\Pi_{h_t^k}(z, z)=F_{h_t^k}(z, z)\|e_L^k(z)\|^2_{h^k}=\hat{\Pi}_{h^k}(z, 0; z, 0) \tag{92}$$

in (t, z). (The notation is slightly ambiguous since in (73) it is used for the uncontracted kernel, but it is standard and we hope no confusion will arise since one is scalar-valued and the other is not.) These derivatives allow us to make simple comparisons to derivatives of $\varphi_k(t, z)$. Since we are ultimately interested in C^k norms we need asymptotics of derivatives with respect to nonvanishing vector fields.

We can use the vector fields $(\partial/\partial\rho_j)$ away from $\mathfrak{D}$ and the vector fields $(\partial/\partial r_j)$ near $\mathfrak{D}$. The calculations are very similar, but we carry them both out in some detail here. Later we will tend to suppress the calculations with $(\partial/\partial r_j)$ to avoid duplication; the reader can check in this section that the calculations and estimates are valid.

Only the leading coefficient and the order of asymptotics are relevant. The undifferentiated diagonal asymptotics are of the following form: for any $h \in P(M,\omega)$ we have

$$\Pi_{h^k}(z,z) = \sum_{i=0}^{d_k} \|s_i(z)\|^2_{h_k} = a_0 k^m + a_1(z)k^{m-1} + a_2(z)k^{m-2} + \cdots, \tag{93}$$

where a_0 is constant and as above $d_k + 1 = \dim H^0(M, L^k)$.

We first consider derivatives with respect to ρ. Calculating ρ derivatives of $\Pi_{h^k}(e^{\rho/2}, e^{\rho/2})$ is equivalent to calculating θ-derivatives of $\Pi_{h_t^k}(e^{i\theta}z, z)$. Using (62) we have

$$\Pi_{h_t^k}(e^{i\theta}z,z) = \sum_{\alpha\in kP\cap\mathbb{Z}^m} \frac{e^{i\langle\alpha,\theta\rangle}|z^\alpha|^2 e^{-kF_t(e^{i\theta}|z|^2)}}{\mathcal{Q}_{h_t^k}(\alpha)}.$$

The results are globally valid but are not useful near $\mathfrak{D}$ since on each stratum some of the vector fields generating the $(\mathbb{C}^*)^m$ action vanish.

Below, we use the tensor product notation $\Big(\frac{\alpha}{k} - \mu_{h_t}(e^{\rho/2})\Big)^{\otimes 2}_{ij}$ for $\Big(\frac{\alpha_i}{k} - \mu_{h_t}(e^{\rho/2})_i\Big)\Big(\frac{\alpha_j}{k} - \mu_{h_t}(e^{\rho/2})_j\Big)$.

Proposition 4.3. *For $i, j = 1, \dots, m$ we have*:

(1) $\displaystyle k^{-m} \sum_{\alpha\in kP\cap\mathbb{Z}^m} \Big(\frac{\alpha}{k} - \mu_{h_t}(e^{\rho/2})\Big) \frac{e^{\langle\alpha,\rho\rangle - k\varphi_t(e^{\rho/2})}}{\mathcal{Q}_{h_t^k}(\alpha)} = O(k^{-2});$

(2) $\displaystyle \frac{1}{\Pi_{h_t^k}(z,z)} \Big(- \sum_{\alpha\in kP\cap\mathbb{Z}^m} \Big(\frac{\partial}{\partial t}\log \mathcal{Q}_{h_t^k}(\alpha)\Big) \frac{e^{\langle\alpha,\rho\rangle - k\varphi_t(e^{\rho/2})}}{\mathcal{Q}_{h_t^k}(\alpha)}\Big) - k\frac{\partial}{\partial t}\varphi_t = O(k^{-1});$

(3) $\displaystyle \frac{1}{\Pi_{h_t^k}(z,z)} \Big(k^2 \sum_{\alpha\in kP\cap\mathbb{Z}^m} \Big(\frac{\alpha}{k} - \mu_{h_t}(e^{\rho/2})\Big)^{\otimes 2}_{ij} \frac{e^{\langle\alpha,\rho\rangle - k\varphi_t(e^{\rho/2})}}{\mathcal{Q}_{h_t^k}(\alpha)}\Big) - k\frac{\partial^2\varphi_t}{\partial\rho_i\partial\rho_j} = O(k^{-1});$

(4) $\displaystyle \frac{1}{\Pi_{h_t^k}(z,z)} \Big(k \sum_{\alpha\in kP\cap\mathbb{Z}^m} \Big(\frac{\alpha}{k} - \mu_{h_t}(e^{\rho/2})\Big)_i \Big(\frac{\partial}{\partial t}\log \mathcal{Q}_{h_t^k}(\alpha)\Big) \frac{e^{\langle\alpha,\rho\rangle - k\varphi_t(e^{\rho/2})}}{\mathcal{Q}_{h_t^k}(\alpha)}\Big) - k\frac{\partial^2\varphi_t}{\partial\rho_i\partial t} = O(k^{-1}).$

Proof. To prove (1), we differentiate and use (53)–(62) and (93) to obtain

$$O(k^{m-1}) = \nabla_\rho \Pi_{h_t^k}(e^{\rho/2}, e^{\rho/2}) = k \sum_{\alpha\in kP\cap\mathbb{Z}^m} \Big(\frac{\alpha}{k} - \mu_{h_t}(e^{\rho/2})\Big) \frac{e^{\langle\alpha,\rho\rangle - k\varphi_t(e^{\rho/2})}}{\mathcal{Q}_{h_t^k}(\alpha)}.$$

To prove (2) we differentiate

$$\log \Pi_{h_t^k}(e^{\rho/2}, e^{\rho/2}) = \log \sum_{\alpha\in kP\cap\mathbb{Z}^m} \frac{e^{\langle\alpha,\rho\rangle - k\varphi_t(e^{\rho/2})}}{\mathcal{Q}_{h_t^k}(\alpha)}$$

with respect t to produce the left side. Since the leading coefficient of (93) is independent of t, the t derivative has the order of magnitude of the right side of (2).

To prove (3), we take a second derivative of (1) in ρ (or θ) to get

$$\nabla_\rho^2 \Pi_{h_t^k}(e^{\rho/2}, e^{\rho/2}) = -k\nabla\mu_{h_t}(e^{\rho/2})\Pi_{h_t^k}(e^{\rho/2}, e^{\rho/2}) + k^2 \sum_{\alpha\in kP\cap\mathbb{Z}^m} \Big(\frac{\alpha}{k} - \mu_{h_t}(e^{\rho/2})\Big)^{\otimes 2} \frac{e^{\langle\alpha,\rho\rangle - k\varphi_t(e^{\rho/2})}}{\mathcal{Q}_{h_t^k}(\alpha)}.$$

Then (3) follows from (93) and the fact that $\nabla\mu_{h_t}(e^{\rho/2}) = \nabla^2\varphi$. Similar calculations show (4). □

In our applications, we actually need asymptotics of logarithmic derivatives. They follow in a straightforward way from Proposition 4.3, using that $\Pi_{h^k}(z,z) \sim k^m$. We record the results for future reference.

Proposition 4.4. *We have*

$$\frac{1}{k}\nabla_\rho \log \sum_{\alpha\in kP\cap\mathbb{Z}^m} \frac{|S_\alpha(z)|^2_{h_t^k}}{\mathcal{Q}_{h_t^k}(\alpha)} = \frac{\sum_\alpha \Big(\frac{\alpha}{k} - \mu_{h_t}(z)\Big) \frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}}{\sum_\alpha \frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}} = O\Big(\frac{1}{k^2}\Big),$$

$$\frac{1}{k}\frac{\partial}{\partial t} \log \sum_{\alpha\in kP\cap\mathbb{Z}^m} \frac{|S_\alpha(z)|^2_{h_t^k}}{\mathcal{Q}_{h_t^k}(\alpha)} = \frac{\sum_\alpha \partial_t \log\Big(\frac{1}{\mathcal{Q}_{h_t^k}(\alpha)}\Big) \frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}}{\sum_\alpha \frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}} - \frac{\partial\varphi_t}{\partial t} = O\Big(\frac{1}{k^2}\Big).$$

Proposition 4.5. *We have*

(1) $$\frac{1}{k}\nabla_\rho^2 \log \sum_{\alpha\in kP\cap\mathbb{Z}^m} \frac{|S_\alpha(z)|^2_{h_t^k}}{\mathcal{Q}_{h_t^k}(\alpha)} = \frac{1}{k}\sum_{\alpha,\beta} (\alpha-\beta)^{\otimes 2} \frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)} \frac{e^{\langle\beta,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\beta)} \Big(\sum_\alpha \frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\Big)^{-2} - \frac{\partial^2\varphi_t}{\partial\rho_i\partial\rho_j} = O\Big(\frac{1}{k^2}\Big),$$

(2) $$\frac{1}{k}\frac{\partial}{\partial t}\nabla_\rho \log \sum_{\alpha\in kP\cap\mathbb{Z}^m} \frac{|S_\alpha(z)|^2_{h_t^k}}{\mathcal{Q}_{h_t^k}(\alpha)} = \frac{1}{k} \frac{\sum_{\alpha,\beta} (\alpha-\beta)\ \partial_t \log\Big(\frac{\mathcal{Q}_{h_t^k}(\beta)}{\mathcal{Q}_{h_t^k}(\alpha)}\Big) \frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)} \frac{e^{\langle\beta,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\beta)}}{\Big(\sum_\alpha \frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\Big)^2} - \frac{\partial^2\varphi_t}{\partial\rho_i\partial t} = O\Big(\frac{1}{k^2}\Big),$$

(3) $$\frac{1}{k}\frac{\partial^2}{\partial t^2} \log \sum_{\alpha\in kP\cap\mathbb{Z}^m} \frac{|S_\alpha(z)|^2_{h_t^k}}{\mathcal{Q}_{h_t^k}(\alpha)}$$
$$= \frac{1}{k} \frac{\sum_{\alpha,\beta} \Big(\partial_t^2 \log \frac{1}{\mathcal{Q}_{h_t^k}(\alpha)} + \Big(\partial_t \log\frac{1}{\mathcal{Q}_{h_t^k}}\Big)\Big(\partial_t \log \frac{\mathcal{Q}_{h_t^k}(\beta)}{\mathcal{Q}_{h_t^k}(\alpha)}\Big) \frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\Big) \frac{e^{\langle\beta,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\beta)}}{\Big(\sum_\alpha \frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\Big)^2} - \frac{\partial^2\varphi_t}{\partial t^2} = O\Big(\frac{1}{k^2}\Big).$$

Finally, we consider the analogous derivatives with respect to the radial coordinates r_j near $\mathcal{D}$. We assume z is close to the component of $\mathcal{D}$ given in local slice orbit coordinates by $z' = 0$ and let $r' = (r_j)_{j=1}^p$ denote polar coordinates in this slice as discussed in Section 2. The Szegő kernel then has the form

$$\Pi_{h_t^k}(z,z) = \sum_{\alpha\in kP\cap\mathbb{Z}^m} \frac{\prod_{j=1}^p r_j^{2\alpha_j} e^{\langle\rho'',\alpha''\rangle} e^{-kF_t(r_1^2,\dots,r_p^2,e^{\rho_{p+1}},\dots,e^{\rho_m})}}{\mathcal{Q}_{h_t^k}(\alpha)}. \tag{94}$$

The coefficients of the expansion (93) are smooth functions of r_j^2 and the expansion may be differentiated any number of times.

The behavior of $\Pi_{h_t^k}(z, z)$ for $z \in \mathcal{D}$ has the new aspect that many of the terms vanish. The extreme case is where z is a fixed point. We choose the slice coordinates so that it has coordinates $z = 0$. We observe that only the term with $\alpha = 0$ in (94) is nonzero, and the α-th term vanishes to order $|\alpha|$.

Since

$$\frac{\partial}{\partial r_j} = \frac{2}{r_j} \frac{\partial}{\partial \rho_j}$$

where both are defined, the calculations above are only modified by the presence of new factors of $\frac{2}{r_j}$ in each space derivative. Since we are applying the derivative to functions of r_j^2, it is clear that the apparent poles will be canceled. Indeed, the r_j derivative removes any lattice point α with vanishing α_j component. Comparing these derivatives with derivatives of (94) gives the following:

Proposition 4.6. *For $n = 1, \dots, p$, we have*

$$\frac{1}{k} \frac{\partial}{\partial r_n} \log \Pi_{h_t^k}(z, z) = \frac{\displaystyle\sum_{\alpha:\alpha_n \neq 0} \frac{2\Big(\frac{\alpha_n}{k} - \mu_{tn}(z)\Big)}{r_n} \frac{\prod_{j=1}^p r_j^{2\alpha_j} e^{\langle \rho'', \alpha'' \rangle} e^{-kF_t(r_1^2, \dots, r_p^2, e^{\rho_{p+1}}, \dots, e^{\rho_m})}}{\mathcal{Q}_{h_t^k}(\alpha)}}{\displaystyle\sum_{\alpha} \frac{\prod_{j=1}^p r_j^{2\alpha_j} e^{\langle \rho'', \alpha'' \rangle} e^{-kF_t(r_1^2, \dots, r_p^2, e^{\rho_{p+1}}, \dots, e^{\rho_m})}}{\mathcal{Q}_{h_t^k}(\alpha)}} = O\Big(\frac{1}{k^2}\Big).$$

In effect, the exponent α is taken to $\alpha - (0, \dots, 1_n, \dots)$ in the sum or removed if $\alpha_n = 0$, where $(0, \dots, 1_n, \dots)$ is the lattice point with only a 1 in the n-th coordinate. There are similar formulae for the second derivatives

$$\frac{\partial^2}{\partial r_n \partial r_i}, \quad \frac{\partial^2}{\partial r_n \partial t}, \quad \frac{\partial^2}{\partial r_n \partial \rho_i}.$$

The only important point to check is that the modification changing α to $\alpha - (0, \dots, 1_n, \dots)$ does not affect the proofs in Sections 7 and 8.

5. Localization of sums: proof of the Localization Lemma 1.2

The following proposition immediately implies Lemma 1.2:

Proposition 5.1. *Given (t, z), and for any $\delta, C > 0$, there exists $C' > 0$ such that*

$$\frac{|s_\alpha(z)|^2_{h_t^k}}{\mathcal{Q}_{h_t^k}(\alpha)} = \mathcal{P}_{h_t^k}(\alpha, z) = O(k^{-C}) \quad \text{if } \Big|\frac{\alpha}{k} - \mu_{h_t}(z)\Big| \geq C' k^{-1/2+\delta}.$$

Proof. The proof is based on integration by parts. All of the essential issues occur in the Bargmann–Fock model, so we first illustrate with that case.

5A. ***The Bargmann–Fock case.*** To analyze the decay of $\mathcal{P}_{h^k_{BF}}(\alpha, z)$ as a function of lattice points α, it seems simplest to use the following integral formula (suppressing the factor k^m and normalizing the volume of $\mathbf{T}^m$ to equal one):

$$k^{|\alpha|} \frac{|z^\alpha|^2}{\alpha!} e^{-k|z|^2} = (2\pi)^{-m} \int_{T^m} e^{-k\left((1-e^{i\theta})|z|^2 - i\langle \alpha/k, \theta \rangle\right)} d\theta = e^{-k|z|^2} (2\pi)^{-m} \int_{T^m} e^{k\left(e^{i\theta}|z|^2 - i\langle \alpha/k, \theta \rangle\right)} d\theta. \quad (95)$$

Here we denote $e^{i\theta}|z|^2$ by $\langle e^{i\theta}z, z\rangle$ for simplicity.

The rightmost expression in (95) is $e^{-k|z|^2}$ times a complex oscillatory integral with phase

$$\Phi_{z,\alpha/k}(\theta) = (e^{i\theta} - 1)|z|^2 - i\Big\langle \frac{\alpha}{k}, \theta\Big\rangle.$$

We observe that (consistent with Proposition 2.1),

$$\nabla_\theta \Phi_{z,\alpha/k}(\theta) = i\Big(e^{i\theta}|z|^2 - \frac{\alpha}{k}\Big) = 0 \iff e^{i\theta}|z|^2 = |z|^2 = \frac{\alpha}{k}.$$

Further, we claim that

$$|\nabla_\theta \Phi_{z,(\alpha/k)}(\theta)| \geq \Big||z|^2 - \frac{\alpha}{k}\Big|. \tag{96}$$

Indeed, the function

$$f_{z,\alpha}(\theta) := \Big|e^{i\theta}|z|^2 - \frac{\alpha}{k}\Big|^2 = \sum_{j=1}^m \Big(\cos\theta_j\ |z_j|^2 - \frac{\alpha_j}{k}\Big)^2 + (\sin\theta_j\ |z_j|^2)^2$$

on $\mathbf{T}^m$ has a strict global minimum at $\theta = 0$ as long as $|z_j|^2 \neq 0$ and $\alpha_j/k \neq 0$ for all j. We note that this discussion of global minima is possible only because the Kähler potential admits a global analytic continuation in (z, w); in general, one can only analyze critical points near the diagonal.

We integrate by parts with the operator

$$\mathscr{L} = \frac{1}{k}\,\frac{1}{|\nabla_\theta \Phi_{z,\alpha/k}|^2}\overline{\nabla_\theta \Phi_{z,\alpha/k}} \cdot \nabla_\theta\,; \tag{97}$$

that is, we apply its transpose

$$\mathscr{L}^t = -\frac{1}{k}\,\frac{1}{|\nabla_\theta \Phi_{z,\alpha/k}|^2}\nabla_\theta \Phi_{z,\alpha/k} \cdot \nabla_\theta - \frac{1}{k}\nabla_\theta \cdot \frac{1}{|\nabla_\theta \Phi_{z,\alpha/k}|^2}\nabla_\theta \Phi_{z,\alpha/k} \tag{98}$$

to the amplitude. The second (divergence) term is -1 times

$$\frac{1}{k}\,\frac{\nabla \cdot \nabla \Phi_{z,\alpha/k}}{|\nabla \Phi_{z,\alpha/k}|^2} + \frac{1}{k}\,\frac{\langle \nabla^2 \Phi_{z,\alpha/k} \cdot \nabla \Phi_{z,\alpha/k}, \nabla \Phi_{z,\alpha/k}\rangle}{|\nabla \Phi_{z,\alpha/k}|^4}. \tag{99}$$

We will need to take into account the k-dependence of the coefficients, and therefore introduce some standard spaces of semiclassical symbols. We denote by $S^n_\delta(\mathbf{T}^m)$ the class of smooth functions $a_k(\theta)$ on $\mathbf{T}^m \times \mathbb{N}$ satisfying

$$\sup_{e^{i\theta} \in \mathbf{T}^m} \big|D^\gamma_\theta a_k(\theta)\big| \leq C k^{n+|\gamma|\delta}. \tag{100}$$

Here we use multiindex notation $D^\gamma_\theta = \prod_{j=1}^m (-i\partial/\partial\theta_j)^{\gamma_j}$. Thus, each D_{θ_j} derivative gives rise to an extra order of k^δ in estimates of a_k. We note that products of symbols satisfy

$$S^{n_1}_\delta \times S^{n_2}_\delta \subset S^{n_1+n_2}_\delta.$$

We now claim that, with δ the same as in the statement of the proposition,

$$\frac{\nabla_\theta \Phi_{z,\alpha/k}}{|\nabla_\theta \Phi_{z,\alpha/k}|^2} \in S^{1/2-\delta}_{1/2-\delta} \tag{101}$$

while the quantity in (99) — note in particular the prefactor $1/k$ — lies in $S^{-2\delta}_{1/2-\delta}$.

To prove the claim, we first observe that the sup norm estimates are correct by (96) and from the fact that $\nabla\Phi_{z,\alpha/k}/|\nabla\Phi_{z,\alpha/k}|$ is a unit vector. We further consider derivatives of (101) and (96). Each θ derivative essentially introduces one more factor of $k|\nabla_\theta\Phi_{z,\alpha/k}|$ and hence raises the order by $k^{1/2-\delta}$. This continues to be true for iterated derivatives, proving the claim.

Now we observe that

$$\mathscr{L}^t : S^n_{1/2-\delta} \to S^{n-2\delta}_{1/2-\delta}. \tag{102}$$

Indeed, the first term of $\mathscr{L}^t$ is the composition of (i) ∇_θ, which raises the order by $\frac{1}{2}-\delta$, (ii) multiplication by an element of $S^{1/2-\delta}_{1/2-\delta}$, which again raises the order by $\frac{1}{2}-\delta$, and (iii) $1/k$, which lowers the order by 1. The second term is $1/k$ times an element of $S^{1-2\delta}_{1/2-\delta}$ and thus also lowers the order by 2δ.

It follows that each partial integration by $\mathscr{L}$ introduces decay of $k^{-2\delta}$. Hence, for any $M>0$,

$$(95) = e^{-k|z|^2}(2\pi)^{-m}\int_{T^m} e^{k\left(e^{i\theta}|z|^2 - i\langle\alpha/k,\theta\rangle\right)}((\mathscr{L}^t)^M 1)d\theta = O(k^{-2\delta})^M e^{-k|z|^2}\int_{T^m} e^{k\operatorname{Re}(e^{i\theta}|z|^2)}d\theta = O(k^{-2\delta M})$$

in this region.

5B. ***General case.*** We now generalize this argument from the model case to the general one. With no loss of generality we may choose coordinates so that z lies in a fixed compact subset of $\mathbb{C}^m$, where the open orbit is identified with $(\mathbb{C}^*)^m$. In the open orbit we continue to write $|z|^2 = e^\rho$. The first step is to obtain a useful oscillatory integral formula for $\mathscr{P}_{h^k}(\alpha, z)$. By Propositions 3.3 and 4.2, we have

$$\mathscr{P}_{h^k}(\alpha, z) = (2\pi)^{-m}\int_{T^m} e^{k(F_{\mathbb{C}}(e^{i\theta}|z|^2)-F(|z|^2))}\,\chi(d(z,e^{i\theta}z))\,A_k(z,e^{i\theta}z,0)e^{i\langle\alpha,\theta\rangle}d\theta + O(k^{-\infty}). \tag{103}$$

The phase is given by

$$\Phi_{z,\alpha/k}(\theta) = F_{\mathbb{C}}(e^{i\theta}|z|^2) - F(|z|^2) - i\left\langle\frac{\alpha}{k},\theta\right\rangle, \tag{104}$$

where as above, $F_{\mathbb{C}}(e^{i\theta}|z|^2)$ is the almost analytic continuation of the Kähler potential $F(|z|^2)$ to $M\times M$. By (84) and (61), it satisfies

$$\operatorname{Re}(F_{\mathbb{C}}(e^{i\theta}|z|^2) - F(|z|^2)) \le -Cd(z,e^{i\theta}z)^2 \quad \text{for some } C>0. \tag{105}$$

Hence, the integrand (103) is rapidly decaying on the set of θ where $d(z,e^{i\theta}z)^2 \ge C(\log k)/k$ (see also (86)), and we may replace $\chi(d(z,e^{i\theta}z))$ by $\chi(k^{1/2-\delta'}d(z,e^{i\theta}z)) \in S^0_{1/2-\delta'}$, since the contribution from $1-\chi(k^{1/2-\delta'}d(z,e^{i\theta}z))$ is rapidly decaying. Here, δ' is an arbitrarily small constant and we may choose it so that $\delta' < \delta$ in the proposition. (We did not use such cutoffs in the Bargmann–Fock case since the real analytic potential had a global analytic extension with obvious properties, but as in Section 2D, it is necessary for almost analytic extensions).

The set $d(z,e^{i\theta}z) \le C(k^{\delta'}/\sqrt{k})$ depends strongly on the position of z relative to $\mathscr{D}$, or equivalently on the position of $\mu_h(z)$ relative to ∂P. For instance, if z is a fixed point then $d(z,e^{i\theta}z) = 0$ for all θ. However, we will not need to analyze these sets until the next section.

We now generalize the integration by parts argument. Our goal is to prove that $\mathscr{P}_{h_t^k}(\alpha,z) = O(k^{-C})$ if $|\alpha/k - \mu_{h_t}(z)| \ge Ck^{-1/2+\delta}$. Now, the gradient in θ of the phase of (103) is given by

$$\nabla_\theta\Phi_{z,\alpha/k}(\theta) = \nabla_\theta F_{\mathbb{C}}(e^{i\theta}|z|^2) - i\frac{\alpha}{k} = i\left(\mu_{\mathbb{C}}(z,e^{i\theta}z) - \frac{\alpha}{k}\right), \tag{106}$$

where $\mu_{\mathbb{C}}(z, e^{i\theta}z)$ is the almost analytic extension of the moment map (see Section 2D). The following lemma is obvious, but we display it to highlight the relations between the small parameters δ of the proposition and δ' in our choice of cutoffs.

Lemma 5.2. *If* $\left|\frac{\alpha}{k} - \mu_{h_t}(z)\right| \geq Ck^{-1/2+\delta}$ *and if* $d(z, e^{i\theta}z) \leq Ck^{-1/2+\delta'}$ *with* $\delta' < \delta$, *then*

$$\left|\left(\mu_{h_t}(z, e^{i\theta}z) - \frac{\alpha}{k}\right)\right| \geq C'k^{-1/2+\delta}.$$

Proof. By Proposition 2.1,

$$\begin{aligned}\left|\left(\mu_{h_t}(z, e^{i\theta}z) - \frac{\alpha}{k}\right)\right|^2 &= \left|\left(\operatorname{Re}\mu_{h_t}(z, e^{i\theta}z) - \frac{\alpha}{k}\right)\right|^2 + \left|\tfrac{1}{2}\nabla_\theta D(z, e^{i\theta}z)\right|^2 \\ &\geq \left|\left(\mu_{h_t}(z) - \frac{\alpha}{k}\right)\right|^2 + O(d(e^{i\theta}z, z)).\end{aligned}$$ □

It follows that, under the assumption $\left|\frac{\alpha}{k} - \mu_{h_t}(z)\right| \geq Ck^{-1/2+\delta}$ of the proposition, we may integrate by parts with the operator

$$\mathscr{L} = \frac{1}{k}|\nabla_\theta \Phi_{z,\alpha/k}|^{-2}\nabla_\theta \Phi_{z,\alpha/k} \cdot \nabla_\theta \tag{107}$$

The transpose $\mathscr{L}^t$ has the same form (98) as for the Bargmann–Fock example, the only significant change being that it is now applied to a nonconstant amplitude A_k and to the cutoff

$$\chi(k^{1/2-\delta'}d(z, e^{i\theta}z)) \in S^0_{1/2-\delta'} \tag{108}$$

as well as to its own coefficients. Differentiations of A_k preserve the orders of terms; the only significant change in the symbol analysis in the Bargmann–Fock case is that differentiations of $\chi(k^{1/2-\delta'}d(z, e^{i\theta}z))$ bring only improvements of order $k^{-\delta'}$ rather than $k^{-\delta}$. However, the order still decreases by at least $2\delta'$ on each partial integration, and therefore repeated integration by parts again gives the estimate

$$|\mathcal{P}_{h^k}(\alpha, z)| = O\left((k^{-\delta'})^M \int_{\mathbf{T}^m} e^{k(\operatorname{Re} F(e^{i\theta}|z|^2) - F(|z|^2)}d\theta\right) = O((k^{-\delta'})^M).$$ □

Remark. It is natural to use integration by parts in this estimate since the decay in $\mu_{h_t}(z) - \alpha/k$ must use the imaginary part of the phase and is not a matter of being far from the center of the gaussian.

5C. ***Further details on the phase.*** For future reference (see Lemma 6.2), we Taylor expand the phase (104) in the θ variable to obtain

$$\Phi_{z,\alpha/k}(\theta) = i\langle \mu(z) - \alpha/k, \theta\rangle + \langle H_{\alpha/k}\theta, \theta\rangle + R_3\left(k, e^{i\theta}\mu^{-1}\left(\frac{\alpha}{k}\right)\right), \tag{109}$$

where $R_3 = O(|\theta|^3)$. Here, $H_{\alpha/k} = \nabla^2 F(\mu^{-1}(\alpha/k))$ denotes the Hessian of φ at α/k (see (57) in Section 2B). Indeed, we have

$$\begin{aligned} F_{\mathbb{C}}(e^{i\theta}|z|^2) - F(|z|^2) &= \int_0^1 \frac{d}{dt} F_{\mathbb{C}}(e^{it\theta}|z|^2)\,dt = \int_0^1 \langle \nabla_\theta F(e^{it\theta}|z|^2), i\theta\rangle dt \\ &= \langle \nabla_\rho F(e^\rho)), (i\theta)\rangle + \int_0^1 (t-1)\nabla^2_\rho (F(e^{it\theta+\rho}))(i\theta)^2/2dt \\ &= i\langle \mu(z), \theta\rangle + \nabla^2_\rho (F(e^\rho))(i\theta)^2 + R_3\Big(k, e^{i\theta}\mu^{-1}\Big(\frac{\alpha}{k}\Big), \theta\Big) \\ &= i\langle \mu(z), \theta\rangle + \langle H_z\theta, \theta\rangle + R_3\Big(k, \theta, \mu^{-1}\Big(\frac{\alpha}{k}\Big)\Big), \end{aligned} \tag{110}$$

in the notation (57), where $H_z = \nabla^2_\rho F(|z|^2)$ and where

$$R_3(k, \theta, \rho) := \int_0^1 (t-1)^2 \langle \nabla^3_\rho (F(e^{it\theta+\rho})), (i\theta)^3/3!\rangle\, dt. \tag{111}$$

6. Proof of the Regularity Lemma 1.3 and joint asymptotics of $\mathcal{P}_{h^k}(\alpha)$

The first statement that $\mathcal{R}_\infty(t, x)$ is C^∞ up to the boundary follows from (58),

$$\mathcal{R}_\infty(t, x) = \left(\frac{\delta_{\varphi_t}(x)\cdot \prod_{r=1}^d \ell_r(x)}{\big(\delta_{\varphi_0}(x)\cdot \prod_{r=1}^d \ell_r(x)\big)^{1-t}\big(\delta_{\varphi_1}(x)\cdot \prod_{r=1}^d \ell_r(x)\big)^t}\right)^{1/2} = \Big(\frac{\delta_{\varphi_t}(x)}{\delta_{\varphi_0}(x)^{1-t}\delta_{\varphi_1}(x)^t}\Big)^{1/2}, \tag{112}$$

where the functions δ_φ are positive, bounded below by strictly positive constants, and C^∞ up to ∂P.

We now consider the asymptotics of $\mathcal{R}_k(t, \alpha)$. We determine the asymptotics of the ratio by first determining the asymptotics of the factors of the ratio. We could use either the expression (30) in terms of norming constants $\mathcal{Q}^k_h(\alpha)$ for the dual expression in terms of $\mathcal{P}_{h^k}(\alpha)$ in Corollary 3.2. Each approach has its advantages and each seems of interest in the geometry of toric varieties, but for the sake of simplicity we only consider $\mathcal{P}_{h^k}(\alpha)$ here. In [Song and Zelditch 2007a] we take the opposite approach of focusing on the norming constants. The advantage of using $\mathcal{P}_{h^k}(\alpha)$ is that it may be represented by a smooth complex oscillatory integral up to the boundary, while $\mathcal{Q}^k_h(\alpha)$ are singular oscillatory integrals over P. A disadvantage of $\mathcal{P}_{h^k}(\alpha)$ is that it does not extend to a smooth function on $\overline{P}$ and has singularities on ∂P.

The asymptotics of $\mathcal{P}_{h^k}(\alpha)$ are straightforward applications of steepest descent in compact subsets of $M\backslash\mathcal{D}$ but become nonuniform at $\mathcal{D}$. To gain insight into the general problem we again consider first the Bargmann–Fock model, where by (79) we have

$$\mathcal{P}_{h^k_{BF}}(\alpha) = k^m e^{-|\alpha|}\frac{\alpha^\alpha}{\alpha!} = (2\pi)^{-m} k^m \int_{\mathbf{T}^m} e^{k\langle e^{i\theta}-1-i\theta, \alpha/k\rangle} d\theta. \tag{113}$$

As observed before, the factors of k cancel so "asymptotics" means asymptotics as $\alpha \to \infty$. This indicates that we do not have asymptotics when α ranges over a bounded set, or equivalently when α/k is (C/k)-close to a corner. On the other hand, steepest descent asymptotics applies in a coordinate α_j as long as $\alpha_j \to \infty$. Our aim in general is to obtain steepest descent asymptotics of $\mathcal{P}_{h^k}(\alpha)$ in directions far from facets and Bargmann–Fock asymptotics in directions near a facet.

6A. ***Asymptotics of*** $\mathcal{P}_{h^k}(\boldsymbol{\alpha})$**.** The analysis of $\mathcal{P}_{h^k}(\alpha)$ is closely related to the analysis of $\mathcal{P}_{h^k}(\alpha, z)$ in Section 5B, and in a sense is a continuation of it. But the arguments are now more than integrations-by-parts. We obtain the asymptotics of $\mathcal{P}_{h^k}(\alpha)$ from the integral representation analogous to (103) (see also Proposition 4.2 and Corollary 3.4). Modulo rapidly decaying functions in k, we have (in the notation of Proposition 4.2):

$$\mathcal{P}_{h^k}(\alpha) \sim (2\pi)^{-m} \int_{\mathbf{T}^m} e^{-k(F_{\mathbb{C}}(e^{i\theta}\mu_h^{-1}(\alpha/k)) - F(\mu_h^{-1}(\alpha/k)))} A_k(e^{i\theta}\mu_h^{-1}(\alpha/k), \mu_h^{-1}(\alpha/k), 0, k) e^{i\langle\alpha,\theta\rangle} d\theta. \tag{114}$$

This largely reduces the asymptotic calculation of $\mathcal{P}_{h^k}(\alpha)$ to facts about the off-diagonal asymptotics of the Szegő kernel (compare Proposition 4.2).

The integral (114) is the oscillatory integral (103) but with $z = \mu^{-1}\big(\frac{\alpha}{k}\big)$. Hence, as in (104), its phase is

$$\Phi_{\alpha/k}(\theta) = F_{\mathbb{C}}\Big(e^{i\theta}\mu^{-1}\Big(\frac{\alpha}{k}\Big)\Big) - F\Big(\mu^{-1}\Big(\frac{\alpha}{k}\Big)\Big) - i\Big\langle\frac{\alpha}{k}, \theta\Big\rangle. \tag{115}$$

As in (84) and (105) (but with i included in as part of the phase),

$$\operatorname{Re}\Phi_{\alpha/k}(\theta) \le -Cd\Big(\mu^{-1}\Big(\frac{\alpha}{k}\Big), e^{i\theta}\mu^{-1}\Big(\frac{\alpha}{k}\Big)\Big)^2 \quad \text{for some } C > 0. \tag{116}$$

Specializing (106) to our $z = \mu^{-1}\big(\frac{\alpha}{k}\big)$, we get

$$\nabla_\theta\Phi_{\alpha/k}(\theta) = \nabla_\theta F_{\mathbb{C}}\Big(e^{i\theta}\mu^{-1}\Big(\frac{\alpha}{k}\Big)\Big) - i\frac{\alpha}{k} = i\Big(\mu_{\mathbb{C}}\Big(\mu^{-1}\Big(\frac{\alpha}{k}\Big), e^{i\theta}\mu^{-1}\Big(\frac{\alpha}{k}\Big)\Big) - \frac{\alpha}{k}\Big). \tag{117}$$

By Proposition 2.1, the complex phase has a critical point at values of θ such that $d(z, e^{i\theta}z) \le \delta$, and $e^{i\theta}\mu^{-1}(\alpha/k) = \mu^{-1}(\alpha/k)$. For $\alpha/k \notin \partial P$, the only critical point is therefore $\theta = 0$. The phase then equals zero, and hence at the critical point the real part of the phase is at its maximum of zero.

For $\alpha/k \notin \partial P$, the critical point $\theta = 0$ is nondegenerate. Specializing (109) to $z = \mu^{-1}(\alpha/k)$, we have

$$\begin{aligned} F_{\mathbb{C}}\Big(e^{i\theta}\mu_h^{-1}\Big(\frac{\alpha}{k}\Big)\Big) - F\Big(\mu_h^{-1}\Big(\frac{\alpha}{k}\Big)\Big) &= \int_0^1 \frac{d}{dt} F_{\mathbb{C}}\Big(e^{it\theta}\mu_h^{-1}\Big(\frac{\alpha}{k}\Big)\Big)\,dt \\ &= i\Big\langle\frac{\alpha}{k}, \theta\Big\rangle + i\langle H_{\alpha/k}\theta, \theta\rangle + R_3\Big(k, \theta, \mu^{-1}\Big(\frac{\alpha}{k}\Big)\Big), \end{aligned} \tag{118}$$

where R_3 is defined in (111). Hence,

$$\Phi_{\alpha/k}(\theta) = \langle H_{\alpha/k}\theta, \theta\rangle + R_3\Big(\theta, k, \mu^{-1}\Big(\frac{\alpha}{k}\Big)\Big), \tag{119}$$

and finally

$$\mathcal{P}_{h^k}(\alpha) \sim (2\pi)^{-m} \int_{\mathbf{T}^m} e^{-k\langle H_{\alpha/k}\theta,\theta\rangle} e^{kR_3(\theta,k,\mu^{-1}(\alpha/k))} A_k\Big(\mu_h^{-1}\Big(\frac{\alpha}{k}\Big), e^{i\theta}\mu_h^{-1}\Big(\frac{\alpha}{k}\Big), 0, k\Big)\,d\theta. \tag{120}$$

Nondegeneracy of the phase is the statement that $H_{\alpha/k}$ is a nondegenerate symmetric matrix, and this follows from strict convexity of the Kähler potential or symplectic potential, see (57). But as discussed in Section 2B, $H_{\alpha/k}$ has a kernel when $\alpha/k \in \partial P$. Hence the stationary phase expansion is nonuniform for $\alpha/k \in P$ and is not possible when $\alpha/k \in \partial P$. This explains why we need to break up the analysis into several cases, and why we cannot rely on the complex stationary phase method for all of them.

Specializing (85) and (86), we have

$$\left|\Pi_{h^k}\left(e^{i\theta}\mu_h^{-1}\left(\frac{\alpha}{k}\right),\mu_h^{-1}\left(\frac{\alpha}{k}\right)\right)\right| \le Ck^m e^{-Ckd(\alpha/k),e^{i\theta}\alpha/k))^2} + O(e^{-C\sqrt{k}d(z,e^{i\theta}z)}). \tag{121}$$

Hence, the integrand of (114) is negligible off the set of θ where $d(\mu^{-1}(\frac{\alpha}{k}), e^{i\theta}\mu^{-1}(\frac{\alpha}{k})) \le C(\log k)/\sqrt{k}$. We now observe that for $d(z, e^{i\theta}z) \le Ck^{\delta}/\sqrt{k}$,

$$d(e^{i\theta}z, z)^2 \sim \sum_j (1-\cos\theta_j)\ell_j(\mu(z)), \tag{122}$$

where we sum over j such that $|\ell_j(\mu(z))| \ll 1$ (we will make this precise in the next definition). In particular,

$$d\left(e^{i\theta}\mu_h^{-1}\left(\frac{\alpha}{k}\right),\mu_h^{-1}\left(\frac{\alpha}{k}\right)\right)^2 \sim \sum_j (1-\cos\theta_j)\ell_j\left(\frac{\alpha}{k}\right). \tag{123}$$

Indeed, both in small balls in the interior and near the boundary, the calculation is universal and hence is accurately reflected in the Bargmann–Fock model with all $H_j = 1$, where the distance squared equals

$$\sum_{j=1}^{m} |e^{i\theta_j}z_j - z_j|^2 = 2\sum_{j=1}^{m} |z_j|^2(1-\cos\theta_j) = 2\sum_{j=1}^{m} \ell_j(\mu(z))(1-\cos\theta_j). \tag{124}$$

This motivates the following terminology:

Definition. Let $0 < \delta_k \ll 1$.

- $x \in P$ is δ_k-*close to* the facet $F_j = \{\ell_j = 0\}$ if $\ell_j(x) \le \delta_k$.
- $x \in P$ is δ_k-*far from* the facet $F_j = \{\ell_j = 0\}$ if $\ell_j(x) \ge \delta_k$.
- x is a δ_k-*interior point* if it is δ_k-far from all facets.

There are m possible cases according to the number of facets to which x is δ_k-close. Of course, x can be δ_k-close to at most m facets, in which case it is δ_k-close to the corner defined by the intersection of these facets. We thus define

$$\mathcal{F}_{\delta_k}(x) = \{r : |\ell_r(x)| < \delta_k\}. \tag{125}$$

We also let

$$\delta_k^{\#}(x) = \#\mathcal{F}_{\delta_k}(x) \tag{126}$$

denote the number of δ_k-close facets to x. Dual to the sets $\mathcal{F}_{\delta_k}$ above are the sets

$$\mathcal{F}_{F_{i_1},\dots,F_{i_r}} = \bigl\{x : \mathcal{F}_{\delta_k}(x) = \{i_1,\dots,i_r\}\bigr\}. \tag{127}$$

The asymptotics of $\mathcal{P}_{h^k}(\alpha)$ depend to the leading order on the determinant of the inverse of the Hessian of the phase of (114) (see also (103)) at $\theta = 0$. This Hessian is the same as the Hessian of the Kähler potential discussed in Section 2B, and we recall that its inverse is the Hessian G of the symplectic potential. Hence, the asymptotics are in terms of the determinant of G, which has first order poles on

∂P. This indicates that the asymptotics are not uniform up to ∂P. We saw this as well in the explicit example of the Bargmann–Fock case. We define

$$\mathcal{G}_{\varphi,\delta_k}(x) = \Big(\delta_\varphi(x) \cdot \prod_{j \notin \mathcal{F}_{\delta_k}(x)} \ell_j(x)\Big)^{-1}, \tag{128}$$

where the functions δ_φ are defined in Section 2B. When x is δ_k-far from all facets, then $\mathcal{G}_\varphi(x) = \det G_\varphi$; compare (58). We also define $\mathcal{P}_{h^k_{BF}}(k\ell_j(x))$ to be the unique real analytic extension of (79) to all $x \in [0, \infty)$. We then consider Bargmann–Fock type functions of type (79) adapted to the corners of our polytope P:

$$\mathcal{P}_{P,k,\delta_k}(x) = \prod_{j \in \mathcal{F}_{\delta_k}(x)} \mathcal{P}_{h^k_{BF}}\Big(k\ell_j\Big(\frac{\alpha}{k}\Big)\Big) \tag{129}$$

and

$$\tilde{\mathcal{P}}_{P,k}(x) = \prod_{j \in \mathcal{F}_{\delta_k}(x)} k^{-1}(2\pi\ell_j(x))^{1/2}\mathcal{P}_{h^k_{BF}}(k\ell_j(x)). \tag{130}$$

When we straighten out the corners by affine maps to be standard octants and separate variables $x = (x', x'')$ into directions near and far from ∂P, then $\mathcal{P}_{P,k,\delta_k}(x)$ is by definition a function of the near variables x' and $\mathcal{G}_{\varphi,\delta_k}(x)$ is by definition a function of the far variables x''.

The main result of this section is this:

Proposition 6.1. $$\mathcal{P}_{h^k}(\alpha) = C_m k^{m/2}\sqrt{\det G_\varphi\Big(\frac{\alpha}{k}\Big)}\,\tilde{\mathcal{P}}_{P,k}\Big(\frac{\alpha}{k}\Big)\Big(1 + R_k\Big(\frac{\alpha}{k}, h\Big)\Big), \tag{131}$$
where $R_k = O(k^{-1/3})$ and C_m is a positive constant depending only on m. The expansion is uniform in the metric h and may be differentiated in the metric parameter h twice with a remainder of the same order.

Equivalently, with $\delta^\#_k$ defined in (126) *and by letting $\delta_k = k^{-2/3}$,*

$$\mathcal{P}_{h^k}(\alpha) = C_m k^{(m-\delta^\#_k(\alpha/k))/2}\sqrt{\mathcal{G}_{\varphi,\delta_k}\Big(\frac{\alpha}{k}\Big)}\mathcal{P}_{P,k,\delta_k}\Big(\frac{\alpha}{k}\Big)\Big(1 + R_k\Big(\frac{\alpha}{k}, h\Big)\Big), \tag{132}$$

where again $R_k = O(k^{-1/3})$.

The factor $k^{(m-\delta^\#_k(\alpha/k))/2}$ is due to the fact that we apply complex stationary phase in $m - \delta^\#_k\Big(\dfrac{\alpha}{k}\Big)$ variables to a complex oscillatory integral with symbol of order $k^{(m-\delta^\#_k(\alpha/k))}$.

As a check, let us consider the m-dimensional Bargmann–Fock case where $\delta^\#_k(\alpha/k) = r$, and with no loss of generality we will assume that the first r facets are the close ones. The factor k^m in the symbol of the Szegő kernel is then split into k^r (absorbed in $\mathcal{P}_{P,k,\delta_k}$) and k^{m-r} in the far factor. As discussed in Section 3A, the far factor should have the form

$$k^{m-r}\prod_{j=r+1}^{m} e^{-\alpha_j}\frac{\alpha_j^{\alpha_j}}{\alpha_j!} \sim k^{m-r}\prod_{j=r+1}^{m}\alpha_j^{-1/2}.$$

The asymptotic factor in Proposition 6.1,

$$k^{(m-\delta^\#_k(\alpha/k))/2}\Big(\prod_{j=r+1}^{m}\frac{k}{\alpha_j}\Big)^{1/2},$$

matches this expression. Here, and throughout the proof, we always straighten out the corner to a standard octant when doing calculations in coordinates.

Secondly, as a check on the remainder, we note that it arises from two sources. As will be seen in the proof, in far directions the stationary phase remainder has the form

$$O\Big(\frac{1}{kd((\alpha/k),\partial P)}\Big),$$

while in the near directions it has the form $O(k(d(\alpha/k,\partial P))^2)$. When $d(\alpha/k,\partial P)\sim k^{-2/3}$ the remainders match.

We break up the proof into cases according to the distance of α/k to the various facets as $k\to\infty$. Since we are studying joint asymptotics in (α,k), α may change with k.

6B. *Interior asymptotics.*

α/k is δ-far from all facets. We first consider the case where α/k is δ-far from all facets as an introduction to the problems we face. In this case, we obtain asymptotics of the integral (114) by a complex stationary phase argument. But it is not quite standard even in this interior case. In the next section, we go on to consider the same expansion when δ depends on k.

Lemma 6.2. *Assume that there exists $\delta>0$ such that $\ell_j(\alpha/k)\geq\delta$ for all j, that is, that α/k is δ-far from all facets. Then there exist bounded smooth functions $A_{-j}(x)$ on $\overline{P}$ such that*

$$\mathcal{P}_{h^k}(\alpha)\sim C_m k^{m/2}\sqrt{\det G_\varphi\Big(\frac{\alpha}{k}\Big)}\Big(1+\frac{A_{-1}(\alpha/k)}{k}+\frac{A_{-2}(\alpha/k)}{k^2}+\cdots+O_\delta(k^{-M})\Big).$$

Here, $G_\varphi=\nabla^2u$ (see Section 2B) *and $G_\varphi(\alpha/k)$ is its value at α/k; its norm is $O(\delta^{-1})$ and its determinant is $O(\delta^{-m})$.*

Before going into the proof, we note that the only assumption on the limit points of α/k is that they are δ-far from facets. The lattice points α are implicitly allowed to vary with k. Asymptotics of the left side clearly depend on the asymptotics of the points α/k, and the lemma states how they do so.

Proof. We now apply the complex stationary phase method, or more precisely its proof. The usual complex stationary phase theorem applies to exponents $k\Phi(\theta)$, where $\Phi(\theta)$ is a positive phase function with a nondegenerate critical point at $\theta=0$. In our case, the phase is also k-dependent since it depends on α/k and the asymptotics of (120) therefore depend on the asymptotics of α/k in the domain $d(\alpha/k,\partial P)\geq\delta$. Our stated asymptotics also depend on the behavior of α/k in the same way.

Although the exact statement of complex stationary phase [Hörmander 1990, Theorem 7.7.5] does not apply, the proof applies without difficulty in this region. Namely, we introduce a cutoff $\chi_\delta(\theta)=\chi(\delta^{-1}\theta)\in C^\infty(\mathbf{T}^m)$ which is equal to 1 in a δ-neighborhood of $\theta=0$ and which vanishes outside a 2δ-neighborhood of $\theta=0$. We decompose the integral into its χ_δ and $1-\chi_\delta$ parts. A standard integration by parts argument, essentially the same as in the Localization Lemma 1.2 shows that the $1-\chi_\delta$ term is $=O(\delta^{-M}k^M)$ for all $M>0$. In the χ_δ part the integral may be viewed as an integral over $\mathbb{R}^m$ and we may apply the Plancherel theorem as in the standard stationary phase argument to obtain

$$\mathcal{P}_{h^k}(\alpha) \sim \frac{C_m}{\sqrt{\det(kH_{\alpha/k})}} \int_{\mathbb{R}^m} e^{-\langle (kH_{\alpha/k})^{-1}\xi,\xi\rangle} \mathcal{F}_{\theta\to\xi}\Big(e^{kR_3(\theta,k,\mu^{-1}(\alpha/k))} A_k\Big(\mu_h^{-1}\Big(\frac{\alpha}{k}\Big), e^{i\theta}\mu_h^{-1}\Big(\frac{\alpha}{k}\Big), 0\Big)\Big)(\xi)\,d\xi, \tag{133}$$

where $\mathcal{F}_{\theta\to\xi}$ is the Fourier transform.

The stationary phase expansion [Hörmander 1990, Theorem 7.7.5] is asymptotic to

$$\Big(\frac{2\pi}{k}\Big)^{m/2} \frac{e^{(i\pi/4)\,\mathrm{sgn} H_{\alpha/4}}}{\sqrt{|\det H_{\alpha/k}|}} \sum_j^{\infty} k^{-j}\mathcal{P}_{\alpha/k,j} A_k\Big(\mu_h^{-1}\Big(\frac{\alpha}{k}\Big), e^{i\theta}\mu_h^{-1}\Big(\frac{\alpha}{k}\Big), 0\Big)\Big|_{\theta=0}, \tag{134}$$

where

$$\mathcal{P}_{\alpha/k,j} A_k(0) = \sum_{\nu-\mu=j}\sum_{2\nu\geq 3\mu} \frac{i^{-j}2^{-\nu}}{\mu!\,\nu!} \langle H_{\alpha/k}^{-1} D_\theta, D_\theta\rangle^\nu (A_k R_3^\mu)|_{\theta=0}. \tag{135}$$

The only change in the standard argument is that we have a family of quadratic forms $H_{\frac{\alpha}{k}}$ depending on parameters (α, k) rather than a fixed one. But the standard proof is valid for this modification. As in the standard proof, we expand the exponential in (133) and evaluate the terms and the remainder of the exponential factor just as in [Hörmander 1990, Theorem 7.7.5], to obtain (134), which becomes

$$\begin{aligned}\Big(\det\big(k^{-1}G_\varphi\big(\tfrac{\alpha}{k}\big)\big)\Big)^{1/2} \sum_{j=0}^{M} k^{-j}\big(\big\langle G_\varphi\big(\tfrac{\alpha}{k}\big)D_\theta, D_\theta\big\rangle\big)^j \chi_\delta e^{kR_3(k,\theta,\mu^{-1}(\alpha/k))} A_k\big(\mu_h^{-1}\big(\tfrac{\alpha}{k}\big), e^{i\theta}\mu_h^{-1}\big(\tfrac{\alpha}{k}\big), 0, k\big)\Big|_{\theta=0} \\ +O\Big(k^{-M}\sup_{\theta\in \mathrm{Supp}\,\chi_\delta}\Big|\langle G_\varphi(\tfrac{\alpha}{k})D_\theta, D_\theta\rangle^M \chi_\delta e^{kR_3(k,\theta,\mu^{-1}(\alpha/k))} A_k\big(\mu_h^{-1}(\tfrac{\alpha}{k}), e^{i\theta}\mu_h^{-1}(\tfrac{\alpha}{k}), 0, k\big)\Big|\Big).\end{aligned} \tag{136}$$

Here, G_x is the Hessian of the symplectic potential, that is, the inverse of $H_{\mu^{-1}(x)}$. (See Section 2B.) We recall that G_x has poles x_j^{-1} of order one when $x \in \partial P$. When $d(\alpha/k, \partial P) \geq \delta$, its norm is therefore $O(\delta^{-1})$ and its determinant is $O(\delta^{-m})$. Since R_3 vanishes to order 3 at the critical point, the terms of the expansion can be arranged into terms of descending order as in the standard proof. If we recall that the leading term of S is k^m, we obtain the statement of Proposition 6.1 in the δ-interior case. □

α/k is δ_k-far from facets with $k\delta_k \to \infty$. We continue to study the complex oscillatory integral (114) but now allow α/k to become δ_k-close to some facet, and obtain a stationary phase expansion (with very possibly a slow decrease in the steps) under the condition that $k\delta_k \to \infty$. This should be feasible since the phase $k\Phi_{\alpha/k}$ is still rapidly oscillating in this region, albeit at different rates in different directions according to the proximity of α/k to a particular facet. The principal complication is as follows:

- The Hessian $G_\varphi(\alpha/k)$ now has components which blow up like δ_k^{-1} near the close facets. In the stationary phase expansion, we get factors of

  $$k^{-j}\Big\langle G_\varphi\Big(\frac{\alpha}{k}\Big)D_\theta, D_\theta\Big\rangle^j A_k\Big(\mu_h^{-1}\Big(\frac{\alpha}{k}\Big), e^{i\theta}\mu_h^{-1}\Big(\frac{\alpha}{k}\Big), 0, k\Big) R_3\Big(k, \theta, \mu_h^{-1}\Big(\frac{\alpha}{k}\Big)\Big)^\mu$$

  both in the expansion and remainder. We must verify that these terms still are of descending order.

As a guide, we note that by (95), the Bargmann–Fock phase with $\mu_h(z) = \alpha/k$ is given by

$$\Phi_{BF,\alpha/k}(\theta) = \Big\langle \frac{\alpha}{k}, e^{i\theta} - i\theta\Big\rangle = \Big\langle \cos\theta + i(\sin\theta - \theta), \frac{\alpha}{k}\Big\rangle,$$

while the amplitude is constant. In this case, the phase factors into single-variable factors and one can employ the complex stationary phase method separately to each. In the general case, we will roughly split the variables θ into two groups (θ', θ''), depending on α/k, so that the θ' variables are paired with the small components of α/k while the θ'' variables are paired with its large components. The complex stationary phase method applies equally to either $d\theta'$ or $d\theta''$ integral, but the orders of the terms are determined by the proximity of α/k to the facets.

Lemma 6.3. *Let $\{\delta_k\}$ be a sequence such that $k\delta_k \to \infty$. Assume that $\ell_j(\alpha/k) \geq \delta_k$ for all j, that is, that α/k is δ_k far from all facets. Then in the notation of Lemma 6.2, we have*

$$\mathcal{P}_{h^k}(\alpha) \sim C_m k^{m/2} \sqrt{\det G_\varphi\Big(\frac{\alpha}{k}\Big)} \Big(1 + \frac{A_{-1}(\alpha/k)}{k} + \frac{A_{-2}(\alpha/k)}{k^2} + \cdots + \frac{A_{-M}(\alpha/k)}{k^2} + O(k\delta_k)^{-M}\Big),$$

where now

$$A_{-j}\Big(\frac{\alpha}{k}\Big) \leq D\delta_k^{-1} = Cd\Big(\frac{\alpha}{k}, \partial P\Big)^{-j}.$$

Remark. One may regard this as an expansion in the semiclassical parameter $(k\delta_k)^{-1} = (kd(\alpha/k, \partial P))^{-1}$.

Proof. We need to prove that the expansion (136) may be rearranged into terms of decreasing order and that the remainder can be made to have an arbitrarily small order k^{-M} by taking sufficiently many terms.

To analyze the expansion (136), we begin with a decomposition of the inverse Hessian $G_{\alpha/k}$, that is, the Hessian of the symplectic potential, which has the form $u_0 + g$, where $g \in C^\infty(\overline{P})$ and where u_0 is the canonical symplectic potential (56). We continue to fix a small $\delta > 0$ as in the previous section, and consider the facets to which α/k is δ-close. We use the affine transformation to map these δ-close facets to the hyperplanes $x'_j = 0$. In these coordinates, we may write the symplectic potential as

$$u_\varphi(x) = \sum_{j \in \mathcal{F}_{\delta_k}} x'_j \log x'_j + g(x), \tag{137}$$

where the Hessian of g is bounded with bounded derivatives near α/k. The Hessian $G_{\alpha/k}$ then decomposes into the sum

$$G_\varphi(x) = \sum_{j \in \mathcal{F}_{\delta_k}(\alpha/k)} \frac{1}{x'_j} \delta_{jj} + \nabla^2 g := G^s_\varphi(x) + \nabla^2 g, \tag{138}$$

where $\nabla^2 g$ is smooth up to the boundary in a neighborhood of $\mathcal{F}_{\delta_k}(\alpha/k)$. The notation $G_\varphi(x)^s$ refers to the "singular part" of G_x. The choice of δ is not important; we are allowing α/k to become δ_k close to some facets, and for any choice of δ, the sum will include such facets.

The decomposition (138) of the inverse Hessian induces a block decomposition of the Hessian operator $\langle G_{\alpha/k} D_\theta, D_\theta\rangle$. The change of variables to x above induces an affine change of the θ variables, as follows. We are using the coordinates (x', x'') on P with x' denoting the linear coordinates in the directions of the normals to the facets $\mathcal{F}_{\delta_k}(\alpha/k)$. The normals corresponding to $\mathcal{F}_{\delta_k}(\alpha/k)$ generate the isotropy algebra of the subtorus $(\mathbf{T}^m)'$ fixing the near facets. We have $\mathbf{T}^m = (\mathbf{T}^m)' \times (\mathbf{T}^m)''$, and denote the corresponding coordinates by (θ', θ'').

The Hessian operator in these coordinates has the form

$$\Big\langle G_\varphi\Big(\frac{\alpha}{k}\Big) D_\theta, D_\theta\Big\rangle = \sum_{j \in \mathcal{F}_{\delta_k}(\alpha/k)} \frac{k}{\alpha'_j} D^2_{\theta'_j\theta'_j} + \Big\langle G_\varphi\Big(\frac{\alpha}{k}\Big)'' D_\theta, D_\theta\Big\rangle, \tag{139}$$

where the second term has bounded coefficients. Evidently, the change to the interior stationary phase expansion is entirely due to the singular part of the Hessian operator

$$\Big\langle G_\varphi^s\Big(\frac{\alpha}{k}\Big)D_\theta, D_\theta\Big\rangle := \sum_{j\in\mathcal{F}_{\delta_k}(\alpha/k)} \frac{k}{\alpha'_j}\, D^2_{\theta'_j\theta'_j}. \tag{140}$$

We now consider the order of magnitude of the terms in the j-th term (135), which has the form

$$k^{-\nu}\Big\langle G_\varphi\Big(\frac{\alpha}{k}\Big)D_\theta, D_\theta\Big\rangle^\nu A_k\Big(\mu_h^{-1}\Big(\frac{\alpha}{k}\Big), e^{i\theta}\mu_h^{-1}\Big(\frac{\alpha}{k}\Big), 0, k\Big)\, R_3\Big(k,\theta,\mu_h^{-1}\Big(\frac{\alpha}{k}\Big)\Big)^\mu\Big|_{\theta=0}, \tag{141}$$

with $\nu-\mu=j$ and $2\nu\geq 3\mu$. The latter constraint is evident from the fact that R_3 vanishes to order 3.

Using (139), $\langle G_\varphi(\alpha/k)D_\theta, D_\theta\rangle^\nu$ becomes a sum of terms of which the most singular is

$$\Big\langle G_\varphi^s\Big(\frac{\alpha}{k}\Big)D_\theta, D_\theta\Big\rangle^\nu := \Big(\sum_{j\in\mathcal{F}_{\delta_k}(\alpha/k)} \frac{k}{\alpha'_j} D^2_{\theta'_j\theta'_j}\Big)^\nu.$$

We will only discuss the terms generated by this operator; the discussion is similar but simpler for the other terms. In the extreme case of $\langle G''_{\alpha/k}D_\theta, D_\theta\rangle^\nu$, the discussion is essentially the same as in the previous section; in particular, (141) has order k^{-j}.

The problem with each application of $\langle G_\varphi^s(\alpha/k)D_\theta, D_\theta\rangle$ is that it raises the order by the maximum of k/α'_j, which may be as large as $k\delta_k$. Although we have an overall k^{-j} and constraints $\nu-\mu=j$, $2\nu\geq 3\mu$, it is not hard to check that these are not sufficient to produce negative exponents of k.

The key fact which saves the situation is that the phase $\Phi_{\alpha/k}$ and amplitude S depend on θ as functions of $e^{i\theta}|\mu^{-1}(\alpha/k)|^2$. Although R_3 has a more complicated θ-dependence, its third and higher derivatives are the same as those of $\Phi_{\alpha/k}$, and it is obvious that only these contribute to (141). Hence derivatives in θ bring in factors of $|\mu^{-1}(\alpha/k)|^2$ by the chain rule. Due to the behavior of the moment map near a facet, these chain rule factors cancel a square root of the blowing up factor in $G_\varphi(\alpha/k)$. This turns out to be sufficient for a descending series due to the power k^{-j} and constraint $2\nu\geq 3\mu$.

Before giving all the details, let us consider what should be the "worst" terms of (141), that is, the ones with the least decay in k. Each factor of R_3 comes with a factor of k, so one would expect terms with large μ to be worst. The worst term will be one with a maximum μ and where a maximum number of applications on operator $\langle G^s_{\alpha/k}D_\theta, D_\theta\rangle^\nu$ is applied to the chain-rule factors $e^{i\theta}\big(|\mu^{-1}(\alpha/k)|^2\big)_j$ (the j-th component of this vector), obtained from an application of some $D_{\theta'_j}$ to S or to R_3. If instead we differentiate S or R_3 again, we pull out another chain rule factor, which cancels more of the bad coefficient k/α'_j.

We now give the rigorous argument. The terms of (141) have the form

$$k^{-\nu+\mu}G_\varphi\Big(\frac{\alpha}{k}\Big)^{i_1j_1}\cdots G_\varphi\Big(\frac{\alpha}{k}\Big)^{i_\nu j_\nu} D^{\beta_1}R_3\cdots D^{\beta_\mu}R_3 D^{\beta_{\mu+1}}S, \tag{142}$$

where $|\beta|=2\nu$ and where D^{β_q} denote universal constant multiples of the multinomial differential operators $\partial^{\beta_q}/(\partial\theta^{n_1}\cdots\partial\theta^{n_{\beta_q}})$, where the union of the indices agrees with $\{i_1, j_1, \dots, i_\nu, j_\nu\}$. We need each $|\beta_q|\geq 3$ for $q\leq\mu$ to remove the zero of R_3. If we only consider the most singular term, then we need

$i_q = j_q \in \mathcal{F}_{\delta_k}(\alpha/k)$. In this case our term becomes

$$k^{-\nu+\mu}\Bigg(\prod_{\substack{1\le j\le\nu\\ q_j\in\mathcal{F}_\delta(\alpha/k)}}\frac{k}{\alpha'_{q_j}}\Bigg)D^{\beta_1}R_3\cdots D^{\beta_\mu}R_3D^{\beta_{\mu+1}}S, \tag{143}$$

For each factor k/α'_{q_j}, there exist two factors of the associated differential operator $\partial/\partial\theta_{q_j}$. When one is applied to either R_3 or S it pulls out a chain rule factor $e^{i\theta_{q_j}}|\mu^{-1}(\alpha/k))_j|^2$. If the second derivative is applied to this factor, it will not introduce any new factors of $|\mu^{-1}(\alpha/k))_j|^2$. We now estimate (143) by

$$|(143)| \le k^{-\nu+\mu}\Bigg(\prod_{\substack{1\le j\le\nu\\ q_j\in\mathcal{F}_\delta(\alpha/k)}}\frac{k}{\alpha'_{q_j}}\Bigg)\prod_{j=1}^{\mu}\Big|\mu^{-1}\Big(\frac{\alpha}{k}\Big)_{q_j}\Big|^2. \tag{144}$$

Now $\mu^{-1}(x) = \nabla u_\varphi(x)$ in ρ coordinates. So the square of the q_j-th component of $\mu^{-1}(\alpha/k)$ equals $\log(\alpha_{q_j}/k)$ plus a bounded remainder in ρ coordinates; here as above we are using the x_j coordinates adapted to α/k. It follows that in the z coordinates adapted to the facets of $\mathcal{D}$ corresponding to the hyperplanes $x'_j=0$, with $|z_j|^2=e^{\rho_j}$, we have $|\mu^{-1}(\alpha/k)_{q_j}|^2\le C\,\alpha_j/k$. The constant C comes from the smooth part of the symplectic potential and has a uniform bound. As a check, we note that for the approximating Bargmann–Fock model we have $|z_j|^2=\alpha_j/k$. It follows from (144) and $k/\alpha'_j\le Cd(\alpha/k,\partial P)^{-1}$ that

$$\big|(143)\big| \le Ck^{-\nu+\mu}\Bigg(\prod_{\substack{1\le j\le\nu\\ q_j\in\mathcal{F}_\delta(\alpha/k)}}\frac{k}{\alpha'_{q_j}}\Bigg)\prod_{j=1}^{\mu}\frac{k}{\alpha'_{q_j}} \le Ck^{-\nu+\mu}d\Big(\frac{\alpha}{k},\partial P\Big)^{-\nu+\mu} = C\Big(kd\Big(\frac{\alpha}{k},\partial P\Big)\Big)^{-j}. \tag{145}$$

Effectively, the "semiclassical parameter" has changed from k^{-1} to $k^{-1}d(\alpha/k,\partial P)^{-1}$, a natural parameter in boundary problems. As long as $kd(\alpha/k,\partial P)\to\infty$ at some fixed rate, we obtain a descending expansion. □

6C. *Boundary zones: corner zone.* Having dealt with the case where $|\alpha_j/k|\ge\delta_k$, we now turn to the complementary cases where $d(\mu(z),\partial P)\le\delta_k$, that is, at least for one j, $|\alpha_j/k|\le\delta_k$ or equivalently, α/k is δ_k-close to at least one facet. The choice of the scale δ_k is so that it is small enough to justify the Bargmann–Fock approximation in the "near" variables.

In this section, we consider the extreme corner case where $\mu(z)$ lies in a δ_k-corner, that is, where there exists a vertex $v\in\partial P$ so that $d(\mu(z),v)\le\delta_k$. Putting $v=0$, the assumption becomes that $|\mu(z)|\le C\delta_k$. Our main object is to determine the scale δ_k so that the Bargmann–Fock approximation is valid. That is, for $z=\mu^{-1}(\alpha/k)$ we should have, in the multiindex notation of Section 2F (see (113)),

$$\mathcal{P}_{h^k}(\alpha) \sim \mathcal{P}_{h^k_{BF}}(\alpha) = k^m(2\pi)^{-m}\int_{\mathbf{T}^m}\exp\Big(-k\sum_{j=1}^m H_{j\bar j}\Big\langle e^{i\theta_j}-1+i\theta_j,\frac{\alpha_j}{k}\Big\rangle\Big)\,d\theta. \tag{146}$$

Lemma 6.4. *If $\mu(z)$ lies in a δ_k-corner, then*

$$\mathcal{P}_{h^k}(\alpha) = C_m\mathcal{P}_{h^k_{BF}}(\alpha)(1+O(\delta_k)+O(k\delta_k^2)) = C_m\mathcal{P}_{h^k_{BF}}(\alpha)(1+O(k\delta_k^2)).$$

Proof. We may assume that $v = 0$ and that the corner is a standard octant. The phase is

$$k\Big(F_{\mathbb{C}}(|z|^2 e^{i\theta}) - F(|z|^2) - \Big\langle \frac{\alpha}{k}, \theta \Big\rangle\Big). \tag{147}$$

We Taylor expand $F(w)$ at $w = 0$:

$$F_{\mathbb{C}}(e^{i\theta}|z|^2) = F(0) + F'(0)e^{i\theta}|z|^2 + O(|z|^4),$$

so that

$$F_{\mathbb{C}}(|z|^2 e^{i\theta}) - F(|z|^2) = F'(0)|z|^2(e^{i\theta} - 1)) + O(|z|^4).$$

Since $|z|^2 = O(\delta_k)$, we see that k times the quartic remainder is $O(k\delta_k^2) = o(1)$ as long as $\delta_k = o\big(1/\sqrt{k}\big)$. Hence this part of the exponential is a symbol of order zero and may be absorbed into the amplitude. Further we note that $F'(0)|z|^2 = \mu(z) + O(|z|^4)$ and therefore we have

$$k\left(F_{\mathbb{C}}(|z|^2 e^{i\theta}) - F(|z|^2) - i\Big\langle \frac{\alpha}{k}, \theta \Big\rangle\right) = k\mu(z)\big((1-\cos\theta) + i(\sin\theta - \theta)\big) + O(|z|^4)).$$

It follows that when $\mu(z) = \alpha/k = O(\delta_k)$, the phase equals

$$\alpha((1-\cos\theta) + i(\sin\theta - \theta)) + O(k\delta_k^2).$$

Absorbing the $e^{O(k\delta_k^2)} = 1 + O(k\delta_k^2)$ term into the amplitude produces an oscillatory integral with the same phase function as for the Bargmann–Fock kernel.

Now let us consider the amplitude of the integral. We continue with the notation of Proposition 4.2. The amplitude has a semiclassical expansion $A_k(z, w) \sim k^m a_0(z, w) + k^{m-1} a_1(z, w) + \cdots$. Further, the $\mathbf{T}^m$-invariance implies that $A_k(e^{i\theta} z, e^{i\theta} w) = A_k(z, w)$. The leading order amplitude equals 1 when $z = w$ and thus

$$a_0(z, e^{i\theta} w) = 1 + Ce^{i\theta}|z|^2 + O(|z|^4),$$

hence the full symbol satisfies

$$A_k(z, e^{i\theta} z) = k^m(1 + Ce^{i\theta}|z|^2 + \cdots) + O(\delta_k^2).$$

When $\mu(z) = \alpha/k = O(\delta_k)$ we thus have

$$A_k(z, e^{i\theta} z) = k^m\left(1 + Ce^{i\theta}\frac{\alpha}{k} + O(\delta_k^2)\right).$$

Therefore, $\mathcal{P}_{h^k}(\alpha) = \mathcal{P}_{h^k_{BF}}(\alpha)\big(1 + O(\delta_k) + O(k\delta_k^2)\big)$ in the corner region. □

6D. ***Boundary zones: mixed boundary zone.*** Now consider the general case where $d(\mu(z), \partial P) \le \delta_k$, but $\mu(z)$ is not necessarily in a corner. Thus, at least one component $\alpha_j/k = O(\delta_k)$ but not all components need to satisfy this condition. We refer to this case as *mixed* since some components are small and some are not.

The basic idea to handle this case is to split the components into near and far parts, to use Taylor expansions and Bargmann–Fock approximations in the near components, and to use complex stationary phase in the far components. By Section 6B, complex stationary phase works for any sequence δ_k satisfying $k\delta_k \to \infty$, and by Section 6C the Taylor–Bargmann–Fock approximation works whenever $\delta_k = o\big(1/\sqrt{k}\big)$, so we have some flexibility in choosing δ_k.

Remark. In fact, we see that both the complex stationary phase and the Bargmann–Fock approximations are valid for k satisfying (for instance)

$$\frac{C\log k}{k} \leq \delta_k \leq C' \frac{1}{\sqrt{k}\log k},$$

although the remainder estimates will not be equally sharp by both methods. In fact, the stationary phase remainder is of order $(k\delta_k)^{-1}$ while the Bargmann–Fock remainder is of order $k\delta_k^2$; the two remainders agree when $\delta_k = k^{-2/3}$ and for small δ_k the Bargmann–Fock remainder is smaller.

We first choose linear coordinates so that $\mu(z) = \alpha/k$ is δ_k' close to the first r facets and δ_k' far from the $p := m - r$ remaining facets, and by an affine map we position the first r facets as the first r coordinate hyperplanes at $x = 0$, and the remaining facets as the remaining coordinate hyperplanes. We use coordinates (x', x'') relative to this splitting. We also write the z variables as (z', z'') in the corresponding slice-orbit coordinates and (θ', θ'') as the associated coordinates on $\mathbf{T}^m$.

We now introduce two small scales, a smaller one δ_k' to define the nearest facets, and a larger one δ_k''. The Bargmann–Fock approximation will be used in the x' variables which are δ_k' close to a facet. It is sometimes advantageous to use the Bargmann–Fock approximation in the x'' variables which are δ_k'' small, but the complex phase method is also applicable. In the following, we continue to use the notation above Proposition 6.1.

Lemma 6.5. *Assume $\mu(z)$ lies in the mixed boundary zone $\{|x'| \leq \delta_k', |x''| \leq \delta_k''\}$. If*

$$\eta_k = k^{-1}(\delta_k'')^{-1} + k(\delta_k')^2 + k(\delta_k')^2\delta_k'' + \delta_k' \to 0,$$

then $\mathcal{P}_{h^k}(\alpha)$ has an asymptotic expansion

$$\mathcal{P}_{h^k}(\alpha) = C_m k^{m-p/2}\sqrt{\mathcal{G}_{\varphi,\delta_k}\left(\frac{\alpha}{k}\right)}\mathcal{P}_{P,k,\delta_k'}(\alpha)(1+O(\eta_k)).$$

Our strategy for obtaining asymptotics of $\mathcal{P}_{h^k}(\alpha)$ in this case is as follows:

- We employ steepest descent in the p directions which are δ_k''-far from all facets, that is, in the x'' variables. This removes the x'' variables and produces an expansion analogous to that of Lemma 6.2.
- In the remaining x' variables, we Taylor expand the phase and amplitude in the directions δ_k-close to ∂P as in Section 6C.
- We thus obtain universal asymptotics to leading order depending only on the number of facets to which α/k is δ_k-close.

Proof. We are still working on the oscillatory integral with phase (114), but we now treat it as an iterated complex oscillatory integral in the variables (θ', θ'') defined above. We first consider the $d\theta''$ integral,

$$I_k\left(\theta', \frac{\alpha}{k}\right) := (2\pi)^{-p}\int_{\mathbf{T}^p} e^{k(F_{\mathbb{C}}(e^{i\theta}\mu_h^{-1}(\alpha/k)) - F(\mu_h^{-1}(\alpha/k)))} A_k\left(e^{i\theta}\mu_h^{-1}\left(\frac{\alpha}{k}\right), \mu_h^{-1}\left(\frac{\alpha}{k}\right), 0, k\right) e^{-i\langle\alpha,\theta\rangle} d\theta'', \quad (148)$$

where p is the number of θ'' variables. We also let $r = m - p$ be the number of θ' variables. We now verify that we may apply the complex stationary phase method to the $d\theta''$ integral for fixed θ'. Throughout this section, we put $z = \mu^{-1}(\alpha/k)$ and often write $|z'|^2$, $|z''|^2$ for the modulus square of the associated complex coordinate components of this point in the open orbit.

First we simplify the complex phase. As in Section 6C, we Taylor expand $F_{\mathbb{C}}(e^{i\theta'}|z'|^2, e^{i\theta''}|z''|^2)$ in the z' variable (and only in the z' variable) to obtain

$$F_{\mathbb{C}}(e^{i\theta'}|z'|^2, e^{i\theta''}|z''|^2) = F_{\mathbb{C}}(0, e^{i\theta''}|z''|^2) + F_1'(0, e^{i\theta''}|z''|^2)e^{i\theta'}|z'|^2 + O(|z'|^4),$$

where F_1 is the z'-derivative of F. The phase is then

$$\begin{aligned} k\Big(F_{\mathbb{C}}(e^{i\theta'}|z'|^2, e^{i\theta''}|z''|^2) - F(|z'|^2, |z''|^2) - i\Big\langle\frac{\alpha'}{k},\theta'\Big\rangle - i\Big\langle\frac{\alpha''}{k},\theta''\Big\rangle\Big) \\ = k\big(F_{\mathbb{C}}(0, e^{i\theta''}|z''|^2) - F(0,|z''|^2)\big) + k\big(F_1'(0, e^{i\theta''}|z''|^2)e^{i\theta'}|z'|^2 - F_1'(0,|z''|^2)|z'|^2\big) \\ -k\Big(i\Big\langle\frac{\alpha'}{k},\theta'\Big\rangle + i\Big\langle\frac{\alpha''}{k},\theta''\Big\rangle\Big) + O(k|z'|^4). \end{aligned} \tag{149}$$

We now absorb the exponentials of the terms $kO(|z'|^4)$, $ki\langle\alpha'/k,\theta'\rangle$ of the phase (149) into the amplitude, that is, we take the new amplitude A_k'' to be the old one A_k multiplied by this factor. The term $kO(|z'|^4)$ is $o(1)$, while $ki\langle\alpha'/k,\theta'\rangle$ is constant in θ'', so their exponentials are symbols in θ'' and may be absorbed into the amplitude. Moreover, the term $-F_1'(0,|z''|^2)|z'|^2$ is independent of θ'' so its exponential may also be absorbed into the amplitude.

The phase function for the $d\theta''$ integral thus simplifies to

$$k\big(F_{\mathbb{C}}(0, e^{i\theta''}|z''|^2) - F(0,|z''|^2)\big) + k\big(F_1'(0, e^{i\theta''}|z''|^2)e^{i\theta'}|z'|^2\big) - ki\Big\langle\frac{\alpha''}{k},\theta''\Big\rangle. \tag{150}$$

Due to the presence of $|z'|^2$, the terms $k(F_1'(0, e^{i\theta''}|z''|^2)e^{i\theta'}|z'|^2 - F_1'(0,|z_2|^2)|z'|^2)$ are $O(k\delta')$, hence of much lower order than the remaining terms. To simplify the phase further, we now argue that their exponentials can also be absorbed into the amplitude, albeit as exponentially growing rather than polynomially growing factors in k. Since $F_1'(0,|z_2|^2)|z'|^2$ is independent of θ'', it can be factored out of the θ'' integral, so the key factor is

$$E_k(\theta'') := e^{k(F_1'(0,e^{i\theta''}|z''|^2)e^{i\theta'}|z'|^2)}, \tag{151}$$

where in the notation for E_k we omit its dependence on the parameters $|z''|^2, |z'|^2, \theta'$. Thus we would like to show that complex stationary phase method applies to the complex oscillatory integral with phase

$$\Phi''(\theta'') := F_{\mathbb{C}}(0, e^{i\theta''}|z''|^2) - F(0,|z''|^2) - i\Big\langle\frac{\alpha''}{k},\theta''\Big\rangle \tag{152}$$

and with the amplitude $A_k''(\theta'')$ given by the original amplitude A_k multiplied by

$$\exp k\Big(F_1'(0, e^{i\theta''}|z''|^2)e^{i\theta'}|z'|^2 - F_1'(0,|z''|^2)|z'|^2 + i\Big\langle\frac{\alpha''}{k},\theta''\Big\rangle + O(|z'|^4)\Big).$$

The amplitude is of exponential growth but its growth is of strictly lower exponential growth than the phase factor.

The next (not very important) observation is that by (116), the real part of complex phase damps the integral so that the integrand is negligible on the complement of the set

$$|\theta''| \le C\frac{\delta'}{d''(\mu(z),\partial P)} \tag{153}$$

modulo rapidly decaying errors. This follows by splitting up the sum in (122)–(123) into the close facets to z and the far facets. The integrand is negligible unless $|\operatorname{Re}\Phi| \leq C(\log k)/k$; hence it is negligible unless

$$\begin{aligned} d(e^{i\theta}z, z)^2 &\sim \sum_{j\in\mathcal{F}_{\delta_k}(\mu(z))} (1-\cos\theta_j'')\ell_j''(\mu(z)) + O(|z'|^2) \\ &\sim \sum_{j\in\mathcal{F}_{\delta_k}(\mu(z))} (\theta_j'')^2\ell_j''(\mu(z)) + O(\delta_k') \\ &\leq C\frac{\log k}{k} \iff \theta_j^2 \leq \frac{O(\delta_k') + O\left(\frac{\log k}{k}\right)}{d''(\mu(z),\partial P)}, \quad \text{for all } j\in\mathcal{F}_{\delta_k}(\mu(z)). \end{aligned} \tag{154}$$

Under the assumption that $d''(\mu(z),\partial P) \geq \delta_k''$, the integrand is rapidly decaying unless $\theta_j^2 \leq C\delta_k'/\delta_k''$. We could introduce a cutoff of the form

$$\chi\left(\sqrt{\frac{\delta_k''}{\delta_k'}}\theta\right),$$

but for our purposes, it suffices to use a smooth cutoff $\chi_\delta(\theta'')$ around $\theta''=0$ with a fixed small δ so that we may use local θ'' coordinates. We then break up the integral using $1=\chi_\delta+(1-\chi_\delta)$. The $(1-\chi_\delta)$ term is rapidly decaying and may be neglected.

We observe that $\nabla_{\theta''}F_{\mathbb{C}}(0, e^{i\theta''}|z''|^2) = i\mu_{\mathbb{C}}''(|z''|, e^{i\theta''}|z''|)$ is the complexified moment map for the subtoric variety $z'=0$, and we can use Proposition 2.1 to see that its only critical point in the domain of integration is at $\theta''=0$. We denote the Hessian of the phase (152) at $\theta''=0$ by

$$H_{|z''|^2}'' = \nabla_{\theta''}^2\Phi''(\theta'')|_{\theta''=0} = \nabla_{\theta''}^2 F_{\mathbb{C}}(0, e^{i\theta''}|z''|^2)|_{\theta''=0}, \tag{155}$$

and observe that it equals $iD\mu_{\mathbb{C}}''(|z''|, e^{i\theta''}|z''|)$, the derivative of the moment map from the subtoric variety to its polytope. By the same calculation that led to (138), the $\theta''-\theta''$ block of the inverse Hessian operator has the form

$$G_\varphi''(x'') = \sum_{j=1}^{p}\frac{1}{x_j''}\,\delta_{jj} + \nabla^2 g := (G_\varphi'')^s(x'') + \nabla^2 g, \tag{156}$$

where $|x''| \geq \delta_k''$.

We now must verify that the complex stationary phase expansion

$$(\det k^{-1}G_\varphi''(|z''|^2))^{1/2}\sum_{j=1}^{M} k^{-j}(\langle G_\varphi''(|z''|^2)D_{\theta''}, D_{\theta''}\rangle)^j\chi_\delta A_k''(\theta'')|_{\theta''=0} \tag{157}$$

is a descending expansion in well-defined steps and that the remainder

$$k^{-M}\sup_{\theta''\in\operatorname{Supp}\chi_\delta}\left|\langle G_\varphi''(|z''|^2)D_{\theta''}, D_{\theta''}\rangle^M A_k''(\theta'')^M\,\chi_\delta A_k''(\theta'')\right| \tag{158}$$

is of arbitrarily small order as M increases.

We first note that the Hessian operator $k^{-1}\langle G''_\varphi(|z''|^2)D_{\theta''}, D_{\theta''}\rangle$ brings in a net order of $k^{-1}(\delta''_k)^{-1}$, since the coefficients $1/x''$ in the singular part are bounded by $(\delta''_k)^{-1}$. The maximal order terms arise from applying the Hessian operator to the factor E_k. Each derivative can bring down a factor of

$$kF'_1(0, e^{i\theta''}|z''|^2)e^{i\theta'}|z'|^2) = O(k\delta'_k\delta''_k).$$

Since there are two θ'' derivatives for each $k^{-1}(\delta''_k)^{-1}$ the maximum order in k from a single factor of $k^{-1}\langle G''_\varphi(|z''|^2)D_{\theta''}, D_{\theta''}\rangle$ applied to A''_k is of order

$$\eta_k = k^{-1}(\delta''_k)^{-1}((k\delta'_k)^2(\delta''_k)^2 + k\delta'_k\delta''_k) = k(\delta'_k)^2\delta''_k + \delta'_k.$$

In particular this is the order of magnitude of the subdominant term. Therefore, to obtain a descending expansion in steps of at least $k^{-\epsilon_0}$, we obtain the following necessary and sufficient condition on (δ'_k, δ''_k):

$$\eta_k \le Ck^{-\epsilon_0}. \tag{159}$$

Under this condition, the series and remainder will go down in steps of $k^{-\epsilon_0}$.

With these choices of (δ'_k, δ''_k), the complex stationary phase expansion gives an asymptotic expansion in powers of $k^{-\epsilon_0}$. Recalling that the unique critical point occurs at $\theta'' = 0$, the remaining $d\theta'$ integral is given by the dimensional constant $C_m(2\pi)^{-r}$ times

$$\mathcal{P}_{h^k}(\alpha) \sim \big(\det(k^{-1}G''_\varphi(|z''|^2)\big)^{1/2} \int_{\mathbf{T}^r} e^{ik\langle\alpha'/k,\theta'\rangle} \sum_{j=1}^{M} k^{-j} \big(\langle G''_\varphi(|z''|^2)D_{\theta''}, D_{\theta''}\rangle\big)^j \chi_\delta A''_k(\theta', 0)\, d\theta', \tag{160}$$

plus the integral of the remainder (158), which is uniform in θ' and integrates to a remainder of the same order. Here we wrote the amplitude as $A''_k(\theta', \theta'')$ and set $\theta'' = 0$ after the differentiations.

The differentiations leave the factor E_k (151) while bringing down polynomials in the derivatives of its phase. The same is true of the factor $e^{kO(\|z'\|^4)}$ that we absorbed into the amplitude. We now collect these factors and note that the exponent is simply the original phase (149) evaluated at $\theta'' = 0$:

$$\Phi'(\theta'; |z'|^2, |z''|^2) := F_{\mathbb{C}}(e^{i\theta'}|z'|^2, |z''|^2) - F(|z'|^2, |z''|^2) - i\Big\langle \frac{\alpha'}{k}, \theta'\Big\rangle. \tag{161}$$

We also collect the derivatives of this phase and the other factors of A_k and find that

$$\sum_{j=1}^{M} k^{-j} \big(\langle G''_\varphi(|z''|^2)D_{\theta''}, D_{\theta''}\rangle\big)^j \chi_\delta A''_k(\theta', 0) = e^{k\Phi'(\theta';|z'|^2,|z''|^2)}\tilde{A}_k(\theta'), \tag{162}$$

where $\tilde{A}_k(\theta')$ is a classical symbol in k whose order is the order m of the original symbol A_k. The integral (160) then takes the form

$$\mathcal{P}_{h^k}(\alpha) \sim C_m \big(\det(k^{-1}G''_\varphi(|z''|^2))\big)^{1/2} \int_{\mathbf{T}^r} e^{k\Phi'(\theta';|z'|^2,|z''|^2)}\tilde{A}_k(\theta')\, d\theta'. \tag{163}$$

This is a corner type integral as studied in Section 6C, with $|z''|^2$ as an additional parameter. The asymptotics of (163) are given by Lemma 6.4. It is only necessary to keep track of the powers of $|z'|^2$, $|z''|^2$ and of the parameter $k^{-1}(\delta''_k)^{-1}(k\delta'_k)^2$ in the analysis of $\tilde{A}_k$.

To do so, we first observe that

$$\nabla_{\theta'} F_{\mathbb{C}}(e^{i\theta'}|z'|^2, |z''|^2) = i\mu'_{\mathbb{C}}((z', z''), (e^{i\theta'}z', z'')), \tag{164}$$

that is, it is the $'$ component of the complexified moment map. By definition of (z', z'') it equals α'/k when $\theta' = 0$. It follows that $F_1'(0, |z''|^2)|z'|^2 = \alpha'/k$, and the almost analytic extension satisfies

$$F_1'(0, |z''|^2)e^{i\theta'}|z'|^2 = e^{i\theta'}\frac{\alpha'}{k}, \tag{165}$$

where (as previously) the multiplication is componentwise. If we then Taylor expand the phase, we obtain

$$\Phi'(\theta'; |z'|^2, |z''|^2) = F_1'(0, |z''|^2)|z'|^2(1 - e^{i\theta'}) + O(|z'|^4) = \frac{\alpha'}{k}(1 - e^{i\theta'}) + O(|z'|^4). \tag{166}$$

If we absorb the $e^{kO(|z|^4)}$ factor into the amplitude, the integral has now been converted to the form (146) with a more complicated amplitude.

We next observe that

$$\tilde{A}_k = k^m(1 + O(|z'|^2)). \tag{167}$$

Hence, the assumption $|z'|^2 = O(\delta_k')$ implies that to leading order

$$\begin{aligned} \mathcal{P}_{h^k}(\alpha) &\sim \sqrt{\det k^{-1}\, G_\varphi''(|z''|^2)}\, k^m \int_{T^r} e^{-k((e^{i\theta'}-1-i\theta))\alpha'/k} d\theta'(1 + O(\delta_k')) \\ &= k^{m-p/2}\sqrt{\det G_\varphi''(|z''|^2)}\, \mathcal{P}_{h^k_{BF}}(\alpha')(1 + O(\delta_k')). \end{aligned} \tag{168}$$

This completes the proof of the lemma. □

6E. *Completion of proof of Proposition 6.1.*

6E1. *Asymptotic expansion for $\mathcal{P}_{h^k}(\alpha)$.* The error terms for the asymptotics of $\mathcal{P}_{h^k}(\alpha)$ are $k^{-1}(\delta_k'')^{-1}$ in the corner zone, $k(\delta_k')^2$ in the interior zone and $\eta_k = k^{-1}(\delta_k'')^{-1} + k(\delta_k')^2 + k(\delta_k')^2\delta_k'' + \delta_k'$ in the mixed zone. To minimize these terms, we let

$$k^{-1}(\delta_k'')^{-1} = k(\delta_k')^2 \quad \text{and} \quad 0 < \delta_k' \le \delta_k''.$$

By elementary calculation, the optimal choice for δ_k' and δ_k'' is given by

$$\delta_k' = \delta_k'' = k^{-2/3} \quad \text{and} \quad k^{-1}(\delta_k'')^{-1} = k(\delta_k')^2$$

and

$$k^{-1}(\delta_k'')^{-1} = k(\delta_k')^2 = k^{-1/3}, \quad \eta_k \sim O(k^{-1/3}).$$

We let $\delta_k = k^{-2/3}$ and break up the estimate into four cases.

(1) $|x'|, |x''| \le \delta_k$: this is the corner case handled in Lemma 6.4 if $k(\delta_k)^2 \to 0$.

$$\mathcal{P}_{h^k}(\alpha) = C_m \mathcal{P}_{h^k_{BF}}(\alpha)(1 + O(k^{-1/3})).$$

(2) $|x'|, |x''| \geq \delta_k$. By Lemma 6.3, the stationary phase is valid and

$$\mathcal{P}_{h^k}(\alpha) \sim C_m k^{m/2} \sqrt{\det G_\varphi\left(\frac{\alpha}{k}\right)}\ \left(1 + O(k^{-1/3})\right).$$

(3) $|x'| \leq \delta_k$ and $|x''| \geq \delta_k$. By Lemma 6.5,

$$\mathcal{P}_{h^k}(\alpha) = C_m k^{m-(p/2)} \sqrt{\det G''_\varphi\left(\frac{\alpha}{k}\right)}\, \mathcal{P}_{P,k,\delta'_k}(\alpha')(1 + O(k^{-1/3})).$$

(4) $|x''| \leq \delta_k$ and $|x'| \geq \delta_k$. This case is the same as case (3) by switching x' and x''.

Combining the formulas above, the asymptotics for $\mathcal{P}_{h^k}(\alpha)$ is given by (132)

$$\mathcal{P}_{h^k}(\alpha) \;=\; C_m k^{1/2\,(m-\delta_k^\#(\alpha/k))} \sqrt{\mathcal{G}_{\varphi,\delta_k}\left(\frac{\alpha}{k}\right)}\, \mathcal{P}_{P,k,\delta_k}\left(\frac{\alpha}{k}\right)\left(1 + R_k\left(\frac{\alpha}{k}, h\right)\right),$$

where $R_k(\alpha/k, h) = O(k^{-1/3})$.

On the other hand, (131) is derived by the calculation

$$\begin{aligned} k^{1/2(m-\delta_k^\#)} & \sqrt{\mathcal{G}_{\varphi,\delta_k}\left(\frac{\alpha}{k}\right)}\mathcal{P}_{P,k,\delta_k}\left(\frac{\alpha}{k}\right) \\ &= k^{(m/2)} \sqrt{\det G_\varphi\left(\frac{\alpha}{k}\right)}\tilde{\mathcal{P}}_{P,k} \prod_{j \notin \mathcal{F}_{\delta_k}(x)} \left(2\pi k\ell_j\left(\frac{\alpha}{k}\right)\right)^{-1/2} e^{|k\ell_j(\alpha/k)|} \frac{k\ell_j\left(\frac{\alpha}{k}\right)}{k\ell_j\left(\frac{\alpha}{k}\right)^{k\ell_j(\alpha/k)}} \\ &= k^{m/2}\sqrt{\det G_\varphi\left(\frac{\alpha}{k}\right)}\tilde{\mathcal{P}}_{P,k}(1 + O(k^{-1/3})), \end{aligned}$$

where the last equality follows from the Stirling approximation.

6E2. *Derivatives with respect to metric parameters.* Now suppose that $h = h_t$ is a smooth one-parameter family of metrics. We would like to obtain asymptotics $(\partial/\partial t)^j \mathcal{P}_{h_t^k}(\alpha)$ for $j = 1, 2$.

Proposition 6.6. *For $j = 1, 2$, there exist amplitudes S_j of order zero such that*

$$\left(\frac{\partial}{\partial t}\right)^j \mathcal{P}_{h_t^k}(\alpha) = C_m k^{(m-\delta_k^\#(\alpha/k))/2} \sqrt{\mathcal{G}_{\varphi_t,\delta_k}\left(\frac{\alpha}{k}\right)}\, \mathcal{P}_{P,k,\delta_k}\left(\frac{\alpha}{k}\right)\left(S_j(t,\alpha,k) + R_k\left(\frac{\alpha}{k}, h_t\right)\right),$$

where $R_k = O(k^{-1/3})$. The expansion is uniform in h and may be differentiated in h twice with a remainder of the same order.

Proof. Such time derivatives may also be represented in the form (114)

$$\left(\frac{\partial}{\partial t}\right)^j \mathcal{P}_{h_t^k}(\alpha) = (2\pi)^{-m} \int_{\mathbf{T}^m} e^{-k(F_t(e^{i\theta}\mu_{h_t}^{-1}(\alpha/k)) - F_t(\mu_{h_t}^{-1}(\alpha/k)))} A_{k,j}(k,t,\alpha,\theta) e^{i\langle\alpha,\theta\rangle} d\theta, \tag{169}$$

with a new amplitude $A_{k,j}$ that is obtained by a combination of differentiations of the original amplitude in t and of multiplications by t derivatives of the phase. It is easy to see that t derivatives of the amplitude do not change the estimates above since they do not change the order in growth in k of the amplitude. However, t derivatives of the phase bring down factors $k(\partial/\partial t)^j (F_t(e^{i\theta}\mu_{h_t}^{-1}(\alpha/k)) - F_t(\mu_{h_t}^{-1}(\alpha/k))$. The second derivative can bring down two factors with $j = 1$ or one factor with $j = 2$. We now verify that, despite the extra factor of k, the new oscillatory integral still satisfies the same estimates as before.

The key point is that, by the calculation (118), the phase $F_t(e^{i\theta}\mu_{h_t}^{-1}(\alpha/k)) - F_t(\mu_{h_t}^{-1}(\alpha/k)) - i\langle\alpha/k,\theta\rangle$ for any metric h vanishes to order 2 at the critical point $\theta = 0$; the first derivative vanishes because $\nabla_\theta F_t(e^{i\theta}z)|_{\theta=0} = i\mu_{h_t}(z)$. Hence, the t derivative of the h_t-dependent Taylor expansion (118) for a one-parameter family h_t of metrics also vanishes to order 2, that is,

$$\left(\frac{\partial}{\partial t}\right)^j \left(F_t\left(e^{i\theta}\mu_{h_t}^{-1}\left(\frac{\alpha}{k}\right)\right) - F_t\left(\mu_{h_t}^{-1}\left(\frac{\alpha}{k}\right)\right)\right) = O(|\theta|^2). \tag{170}$$

Thus, for each new power of k one obtains by differentiating the phase factor in t one obtains a factor which vanishes to order 2 at $\theta = 0$. As a check, we note that in the Bargmann–Fock model, the phase has the form $\sum_j (e^{i\theta_j} - 1 - i\theta_j)\alpha_j/k$.

We start with the first derivative, repeating the asymptotic analysis but with the new amplitude S_1. In the "interior region" the stationary phase calculation in Lemma 6.2 proceeds as before, but the leading term (now of one higher order than before) vanishes since it contains the value of (170) at the critical point as a factor. Therefore the asymptotics start at the same order as before but with the value of the second θ-derivative of the amplitude at $\theta = 0$.

In the corner and mixed boundary zones we obtain an integral of the same type as the ones studied in Lemma 6.4 and Lemma 6.5, respectively, but again with an amplitude of one higher order given by the t-derivative of the phase. The only change in the calculation is in the Taylor expansion of the amplitude in (167) in the z' variable, which now has the form

$$\tilde{A}_{k,1} = k\left(\frac{\partial}{\partial t}\right)\left(F_t\left(e^{i\theta}\mu_{h_t}^{-1}\left(\frac{\alpha}{k}\right)\right) - F_t\left(\mu_{h_t}^{-1}\left(\frac{\alpha}{k}\right)\right)\right) + O(|z'|^2), \tag{171}$$

so that the final integral now has the form

$$(2\pi)^{-m}k^m \int_{T^r} e^{-k(e^{i\theta'}-1-i\theta')\alpha'/k}\left(k\left(\frac{\partial}{\partial t}\right)\left(F_t\left(e^{i\theta}\mu_{h_t}^{-1}\left(\frac{\alpha}{k}\right)\right) - F_t\left(\mu_{h_t}^{-1}\left(\frac{\alpha}{k}\right)\right)\right)\right)_{\theta''=0} d\theta'.$$

As noted in (170)

$$\begin{aligned}
k\left(\frac{\partial}{\partial t}\right)\left(F_t\left(e^{i\theta}\mu_{h_t}^{-1}\left(\frac{\alpha}{k}\right)\right) - F_t\left(\mu_{h_t}^{-1}\left(\frac{\alpha}{k}\right)\right)\right) &= k\left(\frac{\partial}{\partial t}\right)\left(F_t\left(e^{i\theta}\mu_{h_t}^{-1}\left(\frac{\alpha}{k}\right)\right) - F_t\left(\mu_{h_t}^{-1}\left(\frac{\alpha}{k}\right)\right) - i\left\langle\frac{\alpha}{k},\theta\right\rangle\right)\\
&= k\frac{\partial}{\partial t}\int_0^1 (1-s)\frac{\partial^2}{\partial s^2}\left(F_t\left(e^{is\theta}\mu_{h_t}^{-1}\left(\frac{\alpha}{k}\right)\right)\right)ds\\
&= O\left(k|\theta|^2\frac{\alpha}{k}\right).
\end{aligned}$$

Since the stationary phase method applies as long as $|\alpha| \to \infty$ we may assume that $|\alpha| \le C$ and we see that the factor is then bounded. Here, we have suppressed the subscript $\mathbb{C}$ for the almost-analytic extension to simplify the writing.

As an independent check, we use integration by parts in θ'. We use a cutoff function χ supported near $\theta' = 0$ to decompose the integral into a term supported near $\theta' = 0$ and one supported away from $\theta' = 0$.

We use the integration by parts operator

$$\mathscr{L} = \frac{1}{\big((e^{i\theta'}-1)\alpha'\big)^2}\big(e^{i\theta'}-1\big)\alpha'\cdot\nabla_\theta,$$

where we note that the factors of k cancel. The operator is well defined for $\theta' \neq 0$ and repeated partial integration gives decay in α' in case $|\alpha'| \to \infty$. On the support of χ the denominator is not well defined but the vanishing of the phase to order two shows that $\mathscr{L}^t(S_1)$ is bounded.

Now we consider second time derivatives. The second $\partial/\partial t$ could be applied to the phase factor $e^{k\Phi_t}$ again or it could be applied again to (171), and then we have

$$\tilde{A}_{k,2} = k\Big(\frac{\partial}{\partial t}\Big)\bigg(F_t\Big(e^{i\theta}\mu_{h_t}^{-1}\Big(\frac{\alpha}{k}\Big)\Big) - F_t\Big(\mu_{h_t}^{-1}\Big(\frac{\alpha}{k}\Big)\Big)\bigg)^2 \\ + k\Big(\frac{\partial^2}{\partial t^2}\Big)\bigg(F_t\Big(e^{i\theta}\mu_{h_t}^{-1}\Big(\frac{\alpha}{k}\Big)\Big) - F_t\Big(\mu_{h_t}^{-1}\Big(\frac{\alpha}{k}\Big)\Big)\bigg)^2 + O(|z'|^2). \quad (172)$$

The first term contains the factor k^2 and after cancellation it induces a term of order $|\alpha'|^2$. In addition this term vanishes to order four at $\theta = 0$. Hence the stationary phase calculation in the case of the first derivative equally shows that the first two terms vanish and thus the factors of k^2 are canceled. In the regime where stationary phase is not applicable, $|\alpha'|^2$ may be assumed bounded, and additionally one can integrate by parts twice. Thus again this term is bounded. □

6F. ***Completion of the proof of Lemma 1.3.*** So far we have only considered the asymptotics of $\mathcal{P}_{h^k}(t, z)$. We now take the ratios to complete the proof of Lemma 1.3.

Lemma 6.7. *With δ_φ defined by* (58), *we have*

$$\mathcal{R}_k(t,\alpha) = \bigg(\frac{\det\nabla^2 u_t(\frac{\alpha}{k})}{(\det\nabla^2 u_0(\frac{\alpha}{k}))^{1-t}(\det\nabla^2 u_1(\frac{\alpha}{k}))^t}\bigg)^{1/2}(1+O(k^{-1/3})).$$

The asymptotic may be differentiated twice with the same order of remainder.

Proof. Combining Corollary 3.2 and Proposition 6.1, we have

$$\mathcal{R}_k(t,\alpha) = \frac{\sqrt{\det G_{\varphi_t}(\frac{\alpha}{k})}\,\tilde{\mathcal{P}}_{P,k}(\frac{\alpha}{k})}{\Big(\sqrt{\det G_{\varphi_0}(\frac{\alpha}{k})}\,\tilde{\mathcal{P}}_{P,k}(\frac{\alpha}{k})\Big)^{1-t}\Big(\sqrt{\det G_{\varphi_1}(\frac{\alpha}{k})}\,\tilde{\mathcal{P}}_{P,k}(\frac{\alpha}{k})\Big)^{t}}(1+O(k^{-1/3})). \quad (173)$$

We observe that the factors of $\tilde{\mathcal{P}}_{P,k}$ cancel out, leaving

$$\mathcal{R}_k(t,\alpha) = \frac{\sqrt{\det G_{\varphi_t}(\frac{\alpha}{k})}}{\Big(\sqrt{\det G_{\varphi_0}(\frac{\alpha}{k})}\Big)^{1-t}\Big(\sqrt{\det G_{\varphi_1}(\frac{\alpha}{k})}\Big)^{t}}(1+O(k^{-1/3})). \quad (174)$$

By Proposition 6.6, the asymptotic in (173) may be differentiated twice with the same order of remainder, completing the proof. □

Remark. By (58), we also have

$$\mathcal{R}_k(t,\alpha) = \left(\frac{\delta_{\varphi_0}^{1-t}\delta_{\varphi_1}^{t}}{\delta_{\varphi_t}}\right)^{-1/2}(1+O(k^{-1/3})).$$

Indeed, the factors of $\ell_j(\alpha/k)$ are independent of the metrics and cancel out. Also $\left(\delta_{\varphi_0}^{1-t}\delta_{\varphi_1}^{t}/\delta_{\varphi_t}\right)^{-1/2}$ is smooth on P.

The following simpler estimate on logarithmic derivatives is sufficient for much of the proof of the main results:

Lemma 6.8. *Both $\partial_t \log \mathcal{R}_k(t,\alpha)$ and $\partial_t^2 \log \mathcal{R}_k(t,\alpha)$ are uniformly bounded.*

Proof. We first note that

$$\partial_t \log \mathcal{R}_k(t,\alpha) = \log \mathcal{P}_{h_1^k}(\alpha) - \log \mathcal{P}_{h_0^k}(\alpha) - \partial_t \log \mathcal{P}_{h_t^k}(\alpha). \tag{175}$$

By Proposition 6.1,

$$\log \mathcal{P}_{h^k}(\alpha) = \tfrac{1}{2}\log\det\left(k^{-1}G_\varphi\left(\frac{\alpha}{k}\right)\right) + \log\tilde{\mathcal{P}}_{P,k}\left(\frac{\alpha}{k}\right) + \log C_m + O(k^{-1/3}). \tag{176}$$

As in Lemma 6.7, the Bargmann–Fock terms cancel between the h_0 and h_1 terms, while the metric factors simplify asymptotically to $\frac{1}{2}\log\left(\delta_{\varphi_1}\delta_{\varphi_0}\right)$, and this is clearly bounded. To complete the proof that $\partial_t \log \mathcal{R}_k(t,\alpha)$ is uniformly bounded, we need the final ratio to be bounded. By Proposition 6.6, we see that in the "interior" region both numerator and denominator have asymptotics which differ only in the value of a zeroth order amplitude at $\theta = 0$ and that it equals 1 in the case of the denominator. Hence, the ratio is bounded in the interior. Towards the boundary, the denominator is comparable with the Bargmann–Fock model and is bounded below by one. The numerator is also bounded by Proposition 6.6, and therefore the ratio is everywhere bounded.

Now we consider the case of $\partial_t^2 \log \mathcal{R}_k(t,\alpha)$, which simplifies to

$$\partial_t^2 \log \mathcal{R}_k(t,\alpha) = -\frac{\partial_t^2 \mathcal{P}_{h_t^k}(\alpha)}{\mathcal{P}_{h_t^k}(\alpha)} + \left(\frac{\partial_t \mathcal{P}_{h_t^k}(\alpha)}{\mathcal{P}_{h_t^k}(\alpha)}\right)^2. \tag{177}$$

As we have just argued, the second factor is bounded. The same argument applies to the first term by Proposition 6.6. ☐

7. C^0 and C^1-convergence

We begin with the rather simple proof of C^0-convergence with remainder bounds.

7A. C^0-convergence.

Proposition 7.1. $\dfrac{1}{k}\log Z_k(t,z)e^{-k\varphi_t(z)} = O\left(\dfrac{\log k}{k}\right)$ *uniformly for* $(t,z) \in [0,1]\times M$.

We will derive the proposition from the following result, which in turn is an immediate consequence of Lemma 6.7:

Lemma 7.2 (upper/lower bound lemma). *There exist $C, c > 0$ such that*

$$c \le \mathcal{R}_k(t,\alpha) \le C.$$

Proof of Proposition 7.1. By the upper/lower bound lemma, there exist constants $c, C > 0$ such that

$$c\Pi_{h_t^k}(z,z) \le \sum_{\alpha\in kP\cap\mathbb{Z}^m} \mathcal{R}_k(t,\alpha)\frac{|S_\alpha(z)|^2_{h_t^k}}{\mathcal{Q}_{h_t^k}(\alpha)} \le C\Pi_{h_t^k}(z,z). \tag{178}$$

Hence,

$$\frac{1}{k}\log\Pi_{h_t^k}(z,z) \le \frac{1}{k}\log\sum_{\alpha\in kP\cap\mathbb{Z}^m} \mathcal{R}_k(t,\alpha)\frac{|S_\alpha(z)|^2_{h_t^k}}{\mathcal{Q}_{h_t^k}(\alpha)} \le \frac{1}{k}\log\Pi_{h_t^k}(z,z) + O\Big(\frac{1}{k}\Big) = O\Big(\frac{\log k}{k}\Big), \tag{179}$$

where the last estimate follows from (93). □

7B. C^1-convergence. We now discuss first derivatives in (t, z). In the z variable the vector fields $\partial/\partial\rho_j$ vanish on $\mathcal{D}$, so we can only use them to estimate C^1 norms in directions δ_k far from the boundary. In directions close to the boundary we may choose coordinates so that derivatives in z' near $z' = 0$ define the C^1 norm.

The estimates in the ρ and z' derivatives are similar. We carry out the calculations in detail in the ρ variables and then indicate how to carry out the analogous estimates in the z variable.

We also consider the t derivative. The key distinction between t and z derivatives is the following:

- z or ρ derivatives bring down derivatives of the phase, which have the form $k(\mu_{h_t}(z) - \alpha/k)$. The factor of k raises the order of asymptotics while the factor $(\mu_{h_t}(z)-\alpha/k)$ lowers it by the Localization Lemma.
- t derivatives do not apply to the phase and only differentiate $\mathcal{R}_k(t,\alpha)$ and $\mathcal{Q}_{h_t^k}(\alpha)$.

Proposition 7.3. *Uniformly for $(t, z) \in [0, 1] \times M$, we have*:

(1) $\displaystyle\frac{1}{k}\left|\frac{\partial}{\partial\rho_i}\log\sum_{\alpha\in kP\cap\mathbb{Z}^m} \mathcal{R}_k(t,\alpha)\frac{|S_\alpha(z)|^2_{h_t^k}}{\mathcal{Q}_{h_t^k}(\alpha)}\right| = O(k^{-1/2+\delta}).$

(2) *The same estimate is valid for the derivative $\partial/\partial r_n$ in directions near $\mathcal{D}$, as in Proposition 4.6.*

(3) $\displaystyle\frac{1}{k}\left|\frac{\partial}{\partial t}\log\sum_{\alpha\in kP\cap\mathbb{Z}^m} \mathcal{R}_k(t,\alpha)\frac{|S_\alpha(z)|^2_{h_t^k}}{\mathcal{Q}_{h_t^k}(\alpha)}\right| = O(k^{-1/3}).$

Proof. For (1), we write $\displaystyle\frac{1}{k}\left|\nabla_\rho\log\sum_{\alpha\in kP\cap\mathbb{Z}^m} \mathcal{R}_k(t,\alpha)\frac{|S_\alpha(z)|^2_{h_t^k}}{\mathcal{Q}_{h_t^k}(\alpha)}\right| = \left|\frac{\sum_{\alpha\in kP\cap\mathbb{Z}^m}\Big(\frac{\alpha}{k}-\mu_{h_t}(z)\Big)\mathcal{R}_k(t,\alpha)\frac{|S_\alpha|^2_{h_t^k}}{\mathcal{Q}_{h_t^k}(\alpha)}}{\sum_{\alpha\in kP\cap\mathbb{Z}^m}\mathcal{R}_k(t,\alpha)\frac{|S_\alpha|^2_{h_t^k}}{\mathcal{Q}_{h_t^k}(\alpha)}}\right|.$

The right-hand side can be rewritten as

$$\left|\frac{\displaystyle\sum_{\substack{\alpha\in kP\cap\mathbb{Z}^m\\ |\alpha/k-\mu_{h_t}(z)|\le k^{-1/2+\delta}}}\Big(\frac{\alpha}{k}-\mu_{h_t}(z)\Big)\mathcal{R}_k(t,\alpha)\frac{|S_\alpha|^2_{h_t^k}}{\mathcal{Q}_{h_t^k}(\alpha)}}{\displaystyle\sum_{\alpha\in kP\cap\mathbb{Z}^m}\mathcal{R}_k(t,\alpha)\frac{|S_\alpha|^2_{h_t^k}}{\mathcal{Q}_{h_t^k}(\alpha)}}\right| + O(k^{-M}),$$

which in turn is bounded above by

$$Ck^{-1/2+\delta}\left|\frac{\sum_{\substack{\alpha\in kP\cap\mathbb{Z}^m\\ |\alpha/k-\mu_{h_t}(z)|\le k^{-1/2+\delta}}} |S_\alpha|^2_{h_t^k}/\mathcal{Q}_{h_t^k}(\alpha)}{\sum_{\alpha\in kP\cap\mathbb{Z}^m} |S_\alpha|^2_{h_t^k}/\mathcal{Q}_{h_t^k}(\alpha)}\right| + O(k^{-M}) \le\ Ck^{-1/2+\delta},$$

proving (1). Here we have applied the Localization Lemma 1.2 and Lemma 7.2 to $\mathcal{R}_k$.

Regarding the derivatives $\partial/\partial r_n$ in (2), the only change to the argument is in summing only α with $\alpha_n \neq 0$ and then changing α to $\alpha - (0,\dots,1_n,\dots,0)$ as explained in Proposition 4.6. Clearly the localization and the estimates only change by $1/k$.

We now consider the ∂_t derivative. By Proposition 4.4, we have

$$\begin{aligned}
&\frac{1}{k}\frac{\partial}{\partial t}\log\sum_{\alpha\in kP\cap\mathbb{Z}^m}\mathcal{R}_k(t,\alpha)\frac{|S_\alpha(z)|^2_{h_t^k}}{\mathcal{Q}_{h_t^k}(\alpha)}\\
&\qquad=\frac{1}{k}\,\frac{\sum_\alpha \mathcal{R}_k(t,\alpha)\partial_t\log\Big(\frac{\mathcal{R}_k(t,\alpha)}{\mathcal{Q}_{h_t^k}(\alpha)}\Big)\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_t^k(\alpha)}}{\sum_\alpha \mathcal{R}_k(t,\alpha)\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}} - \frac{\partial}{\partial t}\varphi_t\\
&\qquad=\frac{1}{k}\,\frac{\sum_\alpha \mathcal{R}_k(t,\alpha)\,\partial_t\log\Big(\frac{\mathcal{R}_k(t,\alpha)}{\mathcal{Q}_{h_t^k}(\alpha)}\Big)\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}}{\sum_\alpha \mathcal{R}_k(t,\alpha)\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}} - \frac{1}{k}\,\frac{\sum_\alpha \partial_t\log\Big(\frac{1}{\mathcal{Q}_{h_t^k}(\alpha)}\Big)\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}}{\sum_\alpha \frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}} + O(k^{-1})\\
&\qquad=\frac{1}{k}\,\frac{\sum_\alpha \mathcal{R}_k(t,\alpha)\,\partial_t\log\mathcal{R}_k(t,\alpha)\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}}{\sum_\alpha \mathcal{R}_k(t,\alpha)\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}}\\
&\qquad\quad+\frac{1}{k}\left(\frac{\sum_\alpha \partial_t\log\mathcal{Q}_{h_t^k}(\alpha)\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}}{\sum_\alpha \frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}} - \frac{\sum_\alpha \mathcal{R}_k(t,\alpha)\,\partial_t\log\mathcal{Q}_{h_t^k}(\alpha)\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}}{\sum_\alpha \mathcal{R}_k(t,\alpha)\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}}\right) + O(k^{-1}).
\end{aligned}$$

Notice that $\mathcal{Q}_{h_t^k} = \mathcal{R}_k(t,\alpha)(\mathcal{Q}_{h_0^k}(\alpha))^{1-t}(\mathcal{Q}_{h_1^k}(\alpha))^t$, so

$$\partial_t\log\mathcal{Q}_k(t,\alpha) = \partial_t\log\mathcal{R}_k(t,\alpha) + \log\frac{\mathcal{Q}_{h_1^k}(t,\alpha)}{\mathcal{Q}_{h_0^k}(t,\alpha)}. \tag{180}$$

It follows easily from the fact proved in Lemma 1.3 (or more precisely the simpler Lemma 6.8) that $\mathcal{R}_k(t,\alpha) = O(1)$ and $\partial_t\log\mathcal{R}_k(t,\alpha) = O(1)$. Also the rightmost term in (180) is $O(k)$ uniformly in α. Replacing $\mathcal{R}_k$ by $\mathcal{R}_\infty$ plus an error of order $k^{-1/3}$, we obtain, as needed,

$$\frac{1}{k}\frac{\partial}{\partial t}\log\sum_{\alpha\in kP\cap\mathbb{Z}^m}\mathcal{R}_k(t,\alpha)\frac{|S_\alpha(z)|^2_{h_t^k}}{\mathcal{Q}_{h_t^k}(\alpha)} = O(k^{-1/3}). \qquad \square$$

8. C^2-convergence

We now consider second derivatives in ρ, t. Again we must separately consider derivatives in the interior and near the boundary. The following proposition completes the proof of Theorem 1.1.

Proposition 8.1. *Uniformly for $(t, z) \in [0, 1] \times M$, we have, for any $\delta > 0$,*

(1) $$\frac{1}{k}\left|\frac{\partial^2}{\partial\rho_i\partial\rho_j}\log\sum_{\alpha\in kP\cap\mathbb{Z}^m}\mathcal{R}_k(t,\alpha)\frac{|S_\alpha(z)|^2_{h_t^k}}{\mathcal{Q}_{h_t^k}(\alpha)}\right| = O(k^{-(1/3)+2\delta});$$

(2) $$\frac{1}{k}\left|\frac{\partial^2}{\partial t\partial\rho_j}\log\sum_{\alpha\in kP\cap\mathbb{Z}^m}\mathcal{R}_k(t,\alpha)\frac{|S_\alpha(z)|^2_{h_t^k}}{\mathcal{Q}_{h_t^k}(\alpha)}\right| = O(k^{-(1/3)+2\delta});$$

(3) $$\frac{1}{k}\left|\frac{\partial^2}{\partial t^2}\log\sum_{\alpha\in kP\cap\mathbb{Z}^m}\mathcal{R}_k(t,\alpha)\frac{|S_\alpha(z)|^2_{h_t^k}}{\mathcal{Q}_{h_t^k}(\alpha)}\right| = O(k^{-(1/3)+2\delta}).$$

(4) *The same estimates are valid for the derivative $\partial/\partial r_n$ in directions near $\mathcal{D}$ as in Proposition 4.6.*

We break up the proof into the four cases. To simplify the exposition, we introduce some new notation for localizing sums over lattice points. By the Localization Lemma 1.2, sums over lattice points can be localized to a ball of radius $O(k^{-1/2+\delta})$ around $\mu_{h_t}(z)$. We emphasize that although there are three metrics at play, it is the metric h_t along the Monge–Ampère geodesic that is used to localize the sum. We introduce a notation for localized sums over pairs of lattice points: let

$$\widetilde{\sum_{\alpha,\beta}} F(\alpha,\beta) := \sum_{\substack{|\alpha/k-\mu_{h_t}(z)|\le k^{-1/2+\delta}\\ |\beta/k-\mu_{h_t}(z)|\le k^{-1/2+\delta}}} F(\alpha,\beta). \tag{181}$$

Notation. Throughout the calculations in Sections 6.5 and 8B, $(\alpha-\beta)$ stands for $(\alpha-\beta)_i$ and $(\alpha-\beta)^2$ stands for $(\alpha-\beta)_i(\alpha-\beta)_j$.

8A. ***Second space derivatives in the interior.*** In this section we prove case (1). We have

$$\begin{aligned}
&\frac{1}{k}\left|\frac{\partial^2}{\partial\rho_i\partial\rho_j}\log\sum_{\alpha\in kP\cap\mathbb{Z}^m}\mathcal{R}_k(t,\alpha)\frac{|S_\alpha(z)|^2_{h_t^k}}{\mathcal{Q}_{h_t^k}(\alpha)}\right| \\
&\quad= \frac{1}{k}\left|\frac{\frac{1}{2}\sum_{\alpha,\beta}(\alpha-\beta)^2\mathcal{R}_k(t,\alpha)\mathcal{R}_k(t,\beta)\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\frac{e^{\langle\beta,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\beta)}}{\left(\sum_\alpha\mathcal{R}_k(t,\alpha)\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\right)^2} - k\frac{\partial^2}{\partial\rho_i\partial\rho_j}\varphi_t\right| \\
&\quad\equiv \frac{1}{k}\left|\frac{\frac{1}{2}\sum_{\alpha,\beta}(\alpha-\beta)^2\mathcal{R}_k(t,\alpha)\mathcal{R}_k(t,\beta)\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\frac{e^{\langle\beta,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\beta)}}{\left(\sum_\alpha\mathcal{R}_k(t,\alpha)\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\right)^2} - \frac{\frac{1}{2}\sum_{\alpha,\beta}(\alpha-\beta)^2\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\frac{e^{\langle\beta,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\beta)}}{\left(\sum_\alpha\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\right)^2}\right|,
\end{aligned} \tag{182}$$

modulo $O(1/k)$ by Proposition 4.3. We also completed the square and used that the sum over α is a probability measure to replace $\alpha^2-\alpha\beta$ by $\frac{1}{2}(\alpha-\beta)^2$ using the symmetry in α and β in the sum. We also use Proposition 4.5 to write $\partial^2\varphi_t/\partial\rho_i\partial\rho_j$ as a sum over lattice points.

By the Localization Lemma 1.2, each sum over lattice points can be localized to a ball of radius $O(k^{-1/2+\delta})$ around $\mu_{h_t}(z)$. Then, by Lemma 1.3 each occurrence of $\mathcal{R}_k(t,\alpha)$ or $\mathcal{R}_k(t,\beta)$ may be replaced by $\mathcal{R}_\infty(t,\alpha/k)$ plus an error of order $k^{-1/3}$. Since

$$\frac{1}{k}(\alpha-\beta)^2 = O(k^{2\delta}),$$

the total error is of order $k^{2\delta-1/3}$. Since δ is arbitrarily small, this term is decaying. Further, after replacing $\mathcal{R}_k(t,\beta)$ by $\mathcal{R}_\infty(t,\alpha/k)$ we may then replace α/k and β/k by $\mu_{h_t}(z)$ at the expense of another error of order $k^{-1/2+\delta}$. By modifying (182) accordingly, we have

$$\frac{1}{k}\left|\frac{\partial^2}{\partial\rho_i\partial\rho_j}\log\sum_{\alpha\in kP\cap\mathbb{Z}^m}\mathcal{R}_k(t,\alpha)\frac{|S_\alpha(z)|^2_{h_t^k}}{\mathcal{Q}_{h_t^k}(\alpha)}\right| + O(k^{-(1/3)+2\delta})$$
$$\equiv \frac{1}{k}\left|\frac{\frac{1}{2}\widetilde{\sum_{\alpha,\beta}}(\alpha-\beta)^2\,\mathcal{R}_\infty(t,\mu_{h_t}(e^{\rho/2}))^2\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\frac{e^{\langle\beta,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\beta)}}{\left(\sum_\alpha\mathcal{R}_\infty(t,\mu_{h_t}(e^{\rho/2}))\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\right)^2} - \frac{\frac{1}{2}\widetilde{\sum_{\alpha,\beta}}(\alpha-\beta)^2\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\frac{e^{\langle\beta,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\beta)}}{\left(\sum_\alpha\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\right)^2}\right| \equiv 0, \tag{183}$$

where $\equiv$ means that the lines agree modulo errors of order $O(k^{-(1/3)+2\delta})$. In the last estimate, we use that $\mathcal{R}_\infty(t,\mu_{h_t}(e^{\rho/2}))^2$ cancels out in the first term. This completes the proof in the spatial interior case.

The modifications when z is close to ∂P are just as in the case of the first derivatives.

8B. ***Mixed space-time derivatives.*** The mixed space-time derivative is given by

$$\frac{1}{k}\left|\frac{\partial^2}{\partial\rho_i\partial t}\log\sum_{\alpha\in kP\cap\mathbb{Z}^m}\mathcal{R}_k(t,\alpha)\frac{|S_\alpha(z)|^2_{h_t^k}}{\mathcal{Q}_{h_t^k}(\alpha)}\right|$$
$$= \frac{1}{k}\left|\frac{1}{2}\,\frac{\sum_{\alpha,\beta}(\alpha-\beta)\mathcal{R}_k(t,\beta)\mathcal{R}_k(t,\alpha)\,\partial_t\log\left(\frac{\mathcal{R}_k(t,\alpha)\mathcal{Q}_{h_t^k}(\beta)}{\mathcal{R}_k(t,\beta)\mathcal{Q}_{h_t^k}(\alpha)}\right)\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\frac{e^{\langle\beta,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\beta)}}{\left(\sum_\alpha\mathcal{R}_k(t,\alpha)\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\right)^2} - k\frac{\partial^2}{\partial\rho_i\partial t}\varphi_t\right|.$$

It suffices to prove that

$$\frac{1}{k}\left|\frac{\sum_{\alpha,\beta}(\alpha-\beta)\,\partial_t\log(\mathcal{R}_k(t,\alpha))\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\frac{e^{\langle\beta,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\beta)}}{\left(\sum_\alpha\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\right)^2}\right| = O(k^{-1/2+\delta})$$

and

$$\frac{1}{k}\left|\frac{1}{2}\frac{\displaystyle\sum_{\alpha,\beta}(\alpha-\beta)\mathcal{R}_k(t,\beta)\mathcal{R}_k(t,\alpha)\partial_t\log\Big(\frac{\mathcal{Q}_{h_t^k}(\beta)}{\mathcal{Q}_{h_t^k}(\alpha)}\Big)\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\frac{e^{\langle\beta,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\beta)}}{\Big(\displaystyle\sum_{\alpha}\mathcal{R}_k(t,\alpha)\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\Big)^2}-k\,\frac{\partial^2}{\partial\rho_i\partial t}\varphi_t\right|=O(k^{-(1/3)+2\delta}).$$

The first estimate follows by the Localization Lemma 1.2 and from Lemma 6.8, i.e., $\partial_t\log\mathcal{R}_k(t,\alpha)=O(1)$. The second estimate is very similar to that in Section 8A, specifically in (183), so we do not write it out in full. In outline, we first apply the Localization Lemma and replace each $\mathcal{R}_k(t,\alpha)$ by $\mathcal{R}_\infty(\mu_{h_t}(z))$ with $z=e^{\rho/2}$. The errors in making these replacements are of order $k^{-1/3+\delta}$ because

$$\partial_t\log\frac{\mathcal{Q}_{h_t^k}(\beta)}{\mathcal{Q}_{h_t^k}(\alpha)}=O(k\,|u_t(\alpha)-u_t(\beta)|)=O(k^{1/2+\delta}),$$

and because $\alpha-\beta=O(k^{1/2+\delta})$ in the localized sum. We then express $\partial^2\varphi_t/\partial\rho_i\partial t$ in terms of the Szegő kernel, that is, as a sum over lattice points, using Proposition 4.5, and cancel the $\partial^2\varphi_t/\partial\rho_i\partial t$ term. The sum of the remainders is then of order $k^{-1/3+\delta}$, completing the proof in this mixed case.

8C. ***Second time derivatives.*** The proof in this case follows the same pattern, although the estimates are somewhat more involved. The main steps are to localize the sums over lattice points, to replace each $\mathcal{R}_k$ by $\mathcal{R}_\infty$, then to cancel out $\mathcal{R}_\infty$ after all replacements, and to see that the resulting lattice point sum cancels $\partial^2\varphi_t/\partial\rho_i\partial t$. The complications are only due to the number of estimates that are required to justify the replacements.

The second time derivative equals

$$\begin{aligned}\frac{1}{k}\frac{\partial^2}{\partial t^2}&\log\sum_{\alpha\in kP\cap\mathbb{Z}^m}\mathcal{R}_k(t,\alpha)\frac{|S_\alpha(z)|^2_{h_t^k}}{\mathcal{Q}_{h_t^k}(\alpha)}\\ &=\frac{1}{k}\frac{\displaystyle\sum_{\alpha,\beta}\mathcal{R}_k(t,\beta)\mathcal{R}_k(t,\alpha)\Big(\partial_t\log\Big(\frac{\mathcal{R}_k(t,\alpha)}{\mathcal{Q}_{h_t^k}(\alpha)}\Big(\frac{\mathcal{R}_k(t,\beta)}{\mathcal{Q}_{h_t^k}(\beta)}\Big)^{-1}\Big)\Big)^2\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\frac{e^{\langle\beta,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\beta)}}{\Big(\displaystyle\sum_{\alpha}\mathcal{R}_k(t,\alpha)\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\Big)^2}\\ &\qquad+\frac{1}{k}\left(\frac{\displaystyle\sum_{\alpha,\beta}\mathcal{R}_k(t,\beta)\mathcal{R}_k(t,\alpha)\partial_t^2\log\Big(\frac{\mathcal{R}_k(t,\alpha)}{\mathcal{Q}_{h_t^k}(\alpha)}\Big)\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\frac{e^{\langle\beta,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\beta)}}{\Big(\displaystyle\sum_{\alpha}\mathcal{R}_k(t,\alpha)\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\Big)^2}-k\frac{\partial^2}{\partial t^2}\varphi_t\right).\end{aligned}\tag{184}$$

On the middle line of (184), the square term in the numerator is a simplification of

$$\Big(\partial_t\log\Big(\frac{\mathcal{R}_k(t,\alpha)}{\mathcal{Q}_{h_t^k}(\alpha)}\Big(\frac{\mathcal{R}_k(t,\beta)}{\mathcal{Q}_{h_t^k}(\beta)}\Big)^{-1}\Big)\Big)\Big(\partial_t\log\frac{\mathcal{R}_k(t,\alpha)}{\mathcal{Q}_{h_t^k}(\alpha)}\Big),$$

using the fact that the expression is antisymmetric in (α,β) and that we are summing over α,β — similar to what we did in (182).

To simplify the notation, we introduce the abbreviations $\mathcal{R}(\alpha)=\mathcal{R}_k(t,\alpha)$, $\mathcal{T}(\alpha)=\dfrac{1}{\mathcal{Q}_{h_t^k}(\alpha)}$, $f'=\dfrac{\partial f}{\partial t}$, and we write (184) $=N/D$, where the numerator has the schematic form

$$N=\sum_{\alpha,\beta}\biggl(\Bigl(\frac{\mathcal{R}'}{\mathcal{R}}(\alpha)+\frac{\mathcal{T}'}{\mathcal{T}}(\alpha)\Bigr)'+\frac{1}{2}\Bigl(\frac{\mathcal{R}'}{\mathcal{R}}(\alpha)+\frac{\mathcal{T}'}{\mathcal{T}}(\alpha)-\Bigl(\frac{\mathcal{R}'}{\mathcal{R}}(\beta)+\frac{\mathcal{T}'}{\mathcal{T}}(\beta)\Bigr)\Bigr)^2\biggr)\mathcal{R}(\alpha)\mathcal{T}(\alpha)\mathcal{R}(\beta)\mathcal{T}(\beta)e^{\langle\alpha,\rho\rangle}e^{\langle\beta,\rho\rangle}, \tag{185}$$

and where the denominator is $D=\bigl(\sum_\alpha\mathcal{R}(\alpha)\mathcal{T}(\alpha)\bigr)^2$. We omit the factors $\dfrac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\dfrac{e^{\langle\beta,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\beta)}$ from the notation since they are always present.

We now compare N and D to the corresponding expressions in the second time derivative of the Szegő kernel in Proposition 4.5. In the latter case, $\mathcal{R}\equiv 1$ so any terms with t-derivatives of $\mathcal{R}$ above do not occur in the third comparison expression of Proposition 4.5. Terms with no t derivatives of $\mathcal{R}$ will be precisely as in the comparison except that $\mathcal{R}$ is replaced by 1. So we consider a subsum of N:

$$N_1=\sum_{\alpha,\beta}\biggl(\Bigl(\frac{\mathcal{T}'}{\mathcal{T}}(\alpha)\Bigr)'+\frac{1}{2}\Bigl(\frac{\mathcal{T}'}{\mathcal{T}}(\alpha)-\frac{\mathcal{T}'}{\mathcal{T}}(\beta)\Bigr)^2\biggr)\mathcal{R}(\alpha)\mathcal{T}(\alpha)\mathcal{R}(\beta)\mathcal{T}(\beta). \tag{186}$$

If we now replace all occurrences of $\mathcal{R}_k(t,\alpha)$ by $\mathcal{R}_\infty(\mu_{h_t}(z))$ in both numerator and denominator we get the Szegő kernel expression (the third comparison expression of Proposition 4.5) of order $1/k^2$. (This is verified in more detail at the end of the proof.) So we are left with estimating two remainder terms: first, the difference $N_1-\tilde{N}_1$, where $\tilde{N}_1$ is a sum of terms in which we replace at least one $\mathcal{R}(\alpha)$ by $\mathcal{R}_\infty(\mu_{h_t}(z))$ (or with β). Second, we must estimate $N-N_1$.

We first consider $N_1-\tilde{N}_1$. It arises by substituting at least one $\mathcal{R}(\alpha)-\mathcal{R}_\infty(\mu_{h_t}(z))=O(k^{-1/3})$ for one of the $\mathcal{R}(\alpha)$'s in N_1. We apply the Localization Lemma 1.2 to replace N_1 (and D) by sums over $\alpha/k,\beta/k\in B(\mu_{h_t}(z),k^{-1/2+\delta})$. We thus need to estimate the following expression, when at least one $\mathcal{R}(\alpha)$ is replaced by $\mathcal{R}(\alpha)-\mathcal{R}_\infty(\mu_{h_t}(z))$:

$$\frac{1}{k}\,\frac{\displaystyle\widetilde{\sum_{\alpha,\beta}}\mathcal{R}_k(t,\beta)\mathcal{R}_k(t,\alpha)\biggl(\partial_t\log\frac{\mathcal{Q}_{h_t^k}(\beta)}{\mathcal{Q}_{h_t^k}(\alpha)}\biggr)\bigl(-\partial_t\log\mathcal{Q}_{h_t^k}(\alpha)\bigr)\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\frac{e^{\langle\beta,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\beta)}}{\biggl(\displaystyle\sum_{\alpha\in B(\mu_{h_t}(z),k^{-1/2+\delta})}\mathcal{R}_k(t,\alpha)\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\biggr)^2}$$
$$+\frac{1}{k}\left(\frac{\displaystyle\widetilde{\sum_{\alpha,\beta}}\mathcal{R}_k(t,\beta)\mathcal{R}_k(t,\alpha)\partial_t^2\log\Bigl(\frac{1}{\mathcal{Q}_{h_t^k}(\alpha)}\Bigr)\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\frac{e^{\langle\beta,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\beta)}}{\biggl(\displaystyle\sum_{\alpha\in B(\mu_{h_t}(z),k^{-1/2+\delta})}\mathcal{R}_k(t,\alpha)\frac{e^{\langle\alpha,\rho\rangle}}{(\mathcal{Q}_{h_t^k}(\alpha))}\biggr)^2}-k\frac{\partial^2}{\partial t^2}\varphi_t\right).$$

Due to the factor $1/k$ outside the sum, it suffices to prove that

$$\biggl(\Bigl(\frac{\mathcal{T}'}{\mathcal{T}}(\alpha)\Bigr)'+\frac{1}{2}\Bigl(\frac{\mathcal{T}'}{\mathcal{T}}(\alpha)-\frac{\mathcal{T}'}{\mathcal{T}}(\beta)\Bigr)^2\biggr)=O(k^{1+2\delta}).$$

By Proposition 3.1, we have

$$\frac{\mathcal{T}'}{\mathcal{T}} = -\frac{\mathcal{P}'}{\mathcal{P}} + ku_t'\Big(\frac{\alpha}{k}\Big).$$

Since $u_t = (1-t)u_0 + tu_1$, we have

$$\frac{\mathcal{T}'}{\mathcal{T}}(\alpha) = -\frac{\mathcal{P}'}{\mathcal{P}} + k(u_1-u_0)\Big(\frac{\alpha}{k}\Big) = -\frac{\mathcal{P}'}{\mathcal{P}} + k(f_1-f_0)\Big(\frac{\alpha}{k}\Big),$$

where we recall from Section 2B that $u_\varphi = u_0 + f_\varphi$ with f_φ smooth up to the boundary of P. It follows that

$$\frac{\mathcal{T}'}{\mathcal{T}}(\alpha) - \frac{\mathcal{T}'}{\mathcal{T}}(\beta) = -\frac{\mathcal{P}'}{\mathcal{P}}(\alpha) + \frac{\mathcal{P}'}{\mathcal{P}}(\beta) + k(f_1-f_0)\Big(\frac{\alpha}{k}\Big) - k(f_1-f_0)\Big(\frac{\beta}{k}\Big), \tag{187}$$

$$\Big(\frac{\mathcal{T}'}{\mathcal{T}}(\alpha)\Big)' = -\Big(\frac{\mathcal{P}'}{\mathcal{P}}\Big)' = O(1), \tag{188}$$

with

$$k(f_1-f_0)\Big(\frac{\alpha}{k}\Big) - k(f_1-f_0)\Big(\frac{\beta}{k}\Big) = kO\Big(\Big|\frac{\alpha}{k} - \frac{\beta}{k}\Big|\Big) = O(k^{1/2+\delta}).$$

Further, by Lemma 6.7 (using Proposition 6.6), the factors of

$$\frac{(\frac{\partial}{\partial t})\mathcal{P}_{h_t^k}(\alpha)}{\mathcal{P}_{h_t^k}(\alpha)} = \frac{\big(S_1(t,\alpha,k) + R_k(\frac{\alpha}{k},h)\big)}{S_0(t,\alpha,k)} = O(1),$$

and similarly $(\mathcal{P}'/\mathcal{P})' = O(1)$. Since (187) is squared, it has terms as large as $O(k^{1+2\delta})$. Taking into account the overall factor of $\frac{1}{k}$ and the presence of at least one factor of size $k^{-1/3}$ coming from the replacement of at least one $\mathcal{R}_k(t,\alpha)$ by $\mathcal{R}_\infty(\mu_{h_t}(z))$, we see that $N_1 - \tilde{N}_1$ has order $k^{-(1/3)+2\delta}$ and again this decays for sufficiently small δ.

Now we estimate $N - N_1$, which consists of terms with at least one t-derivative of $\mathcal{R}$. By Lemma 6.7, the terms with no t derivatives on $\mathcal{T}$ give the terms

$$\frac{1}{k}\,\frac{\widetilde{\sum}_{\alpha,\beta}\mathcal{R}_k(t,\beta)\mathcal{R}_k(t,\alpha)\Big(\partial_t \log\dfrac{\mathcal{R}_k(t,\alpha)}{\mathcal{R}_k(t,\beta)}\Big)(\partial_t\log\mathcal{R}_k(t,\alpha))\dfrac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\dfrac{e^{\langle\beta,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\beta)}}{\Big(\sum_\alpha \mathcal{R}_k(t,\alpha)\dfrac{e^{\langle\alpha,\rho\rangle}}{(\mathcal{Q}_{h_t^k}(\alpha))}\Big)^2}$$
$$+\frac{1}{k}\,\frac{\widetilde{\sum}_{\alpha,\beta}\mathcal{R}_k(t,\beta)\mathcal{R}_k(t,\alpha)\partial_t^2\log(\mathcal{R}_k(t,\alpha))\dfrac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\dfrac{e^{\langle\beta,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\beta)}}{\Big(\sum_\alpha \mathcal{R}_k(t,\alpha)\dfrac{e^{\langle\alpha,\rho\rangle}}{(\mathcal{Q}_{h_t^k}(\alpha))}\Big)^2} = O(k^{-1}),$$

by Lemma 1.3.

This leaves us with the terms

$$\Big(\frac{\mathcal{R}'}{\mathcal{R}}(\alpha) - \frac{\mathcal{R}'}{\mathcal{R}}(\beta)\Big)\Big(\frac{\mathcal{T}'}{\mathcal{T}}(\alpha) - \frac{\mathcal{T}'}{\mathcal{T}}(\beta)\Big).$$

Again by Lemma 6.8, the first term is $O(1)$ while the second factor is (187) and has size $kk^{-1/2+\delta}$. Here, we again use Propositions 3.1 and 6.6. Due to the overall factor of $1/k$ this term has size $k^{-1/2+\delta}$.

Therefore, as stated above, up to errors of order $k^{-(1/3)+\delta}$, (184) is simplified to $-\partial^2\varphi_t/\partial t^2$ plus

$$\frac{1}{k}\left(\frac{\displaystyle\tilde{\sum_{\alpha,\beta}}\mathcal{R}_\infty(\mu_{h_t}(e^{\rho/2}))\mathcal{R}_\infty(\mu_{h_t}(e^{\rho/2}))\Big(\partial_t^2\log\frac{1}{\mathcal{Q}_{h_t^k}(\alpha)}+\Big(\partial_t\log\frac{1}{\mathcal{Q}_{h_t^k}}\Big)\Big)\Big(\partial_t\log\frac{\mathcal{Q}_{h_t^k}(\beta)}{\mathcal{Q}_{h_t^k}(\alpha)}\Big)\frac{e^{\langle\alpha,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\alpha)}\frac{e^{\langle\beta,\rho\rangle}}{\mathcal{Q}_{h_t^k}(\beta)}}{\Big(\displaystyle\sum_{\alpha\in B(\mu_{h_t}(z),k^{-1/2+\delta})}\mathcal{R}_\infty(\mu_{h_t}(e^{\rho/2}))\frac{e^{\langle\alpha,\rho\rangle}}{(\mathcal{Q}_{h_t^k}(\alpha))}\Big)^2}\right).$$

As before, we cancel the factors of $\mathcal{R}_\infty(\mu_{h_t}(e^{\rho/2}))$. The resulting difference then cancels to order $k^{-1/2+\delta}$ by Proposition 4.5(3).

This completes the proof of the second time derivative estimate, and hence of the main theorem.

Acknowledgments

The authors thank D. H. Phong and J. Sturm for their support of this project. We also thank them and Y. A. Rubinstein for many detailed corrections. The second author's collaboration with Rubinstein, subsequent to the initial version of this article, has led to a deepened understanding of the global approximation problem, which is reflected in the revised version of the introduction.

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Received 2 Mar 2009. Revised 14 Nov 2009. Accepted 14 Dec 2009.

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Volume 3 No. 3 2010