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WKB p-forms are constructed as approximate solutions to boundary value problems associated with semiclassical Witten Laplacians. Naturally distorted Neumann or Dirichlet boundary conditions are considered.

1. Introduction

Motivation. In order to compute accurately the small eigenvalues, that is, of order $\mathcal{O}(e^{-C/h})$ with $C > 0$, of a self-adjoint Witten Laplacian acting on 0-forms,

$$\Delta^{(0)}_{f,h} = -h^2\Delta + |\nabla f(x)|^2 - h\Delta f(x)\,,$$

as the small parameter $h > 0$ tends to 0, we need Wentzel–Kramers–Brillouin (WKB) approximations of the 1-eigenforms associated with the small eigenvalues of $\Delta^{(1)}_{f,h}$, the self-adjoint Witten Laplacian acting on 1-forms. The function f is assumed to be a Morse function on some bounded domain $\overline{\Omega}$ with or without boundary.

In [Helffer et al. 2004], which improves the previous works [Bovier et al. 2004; 2005] done in a probabilistic point of view, the authors compute accurately the small eigenvalues of $\Delta^{(0)}_{f,h}$ in the case of a manifold without boundary. In this case, the WKB approximations of 1-eigenforms are the one provided in the work by Helffer and Sjöstrand [1985], where the analysis is done for general p-forms.

In the case without boundary, it is moreover well known, since the article by Witten [1982], that the dimension of the spectral subspace associated with the small eigenvalues (i.e., smaller than h) of $\Delta^{(p)}_{f,h}$, the self-adjoint Witten Laplacian acting on p-forms, is $m_p(f)$, the number of critical points of f with index p. Furthermore, the corresponding eigenvectors are concentrated around these critical points (see also [Helffer and Sjöstrand 1985; Helffer et al. 2004; Helffer 1988]).

According to [Chang and Liu 1995; Helffer and Nier 2006; Koldan et al. 2009; Le Peutrec 2008], in the case of a compact manifold with boundary, these last statements require the introduction of *generalized* critical points of f with index p (see Definition 2.6). For a self-adjoint Witten Laplacian $\Delta^{(p)}_{f,h}$ with Neumann or Dirichlet type boundary conditions, $\Delta^{(p)}_{f,h}$ admits $m_p(f)$ eigenvalues, where $m_p(f)$ is the number of *generalized* critical points of f with index p. Moreover, the corresponding p-eigenforms are concentrated around these *generalized* critical points, which can belong to the boundary. The proper definition of *generalized* critical point of f relies on the additional assumption that f has no critical point on the boundary $\partial\Omega$ and that $f|_{\partial\Omega}$ is also a Morse function (see Assumption 2.5). This definition is

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different for Neumann or Dirichlet type boundary conditions, but, in both cases, the interior *generalized* critical points of f with index p are the usual critical points with index p (see again Definition 2.6).

Hence, in the case of a manifold with boundary, some WKB approximations of 1-eigenforms have to be constructed near some *generalized* critical points which lie on the boundary. This was done in [Helffer and Nier 2006] for Dirichlet type boundary conditions. Nevertheless, the construction there relies on some specific trick which cannot be extended to the construction of local WKB 1-forms in the Neumann case. In order to treat this last case (see [Le Peutrec 2008]), a finer treatment of the three geometries involved in the boundary problem (boundary, metric, Morse function) is carried out.

It happens that the Neumann case for 1-forms contains all the technical obstructions for a general WKB ansatz for p-eigenforms. Moreover, this construction can be extended to the Dirichlet case, for general p-forms, using "dual" computations.

Therefore we show in this paper how to construct local WKB p-forms localized near the boundary in both Neumann and Dirichlet cases. However, only the construction of local WKB p-forms is considered here and the comparison with the corresponding p-eigenforms has only be treated in the case $p = 1$, in [Helffer and Nier 2006; Le Peutrec 2008].

Main results. Before enunciating our results, let us introduce some notation used in their statements. We refer in particular the reader to Definition 2.3 and connected material behind.

The operators $\boldsymbol{n}$ and $\boldsymbol{t}$ denote the normal and tangential components, and j^* the canonical pull-back associated with the embedding $j : \partial\Omega \to \overline{\Omega}$. They are defined in the next section.

The function Φ is the degenerate Agmon distance to the generalized critical point U associated with the function f. This is the only nonnegative solution to $|\nabla\Phi|^2 = |\nabla f|^2$ around U (Sections 4A and 4D).

Recall also that for a p-form b_h, the notation $b_h = \mathbb{O}(h^\infty)$ means that, for each N in $\mathbb{N}$, we have $b_h = \mathbb{O}(h^N)$ in the sense that $\|b_h\| \le C_N h^N$ for some $C_N > 0$. Here $\|\cdot\|$ is the L^2-norm over the p-forms inherited from the Riemannian structure.

Lastly, for $A \in \mathcal{L}(T_x^*\overline{\Omega})$, $x \in \overline{\Omega}$ ($T_x^*\overline{\Omega}$ denoting the cotangential space at x), and a p-form $\omega_1 \wedge \cdots \wedge \omega_p$, $A^{(p)}(\omega_1 \wedge \cdots \wedge \omega_p)(x)$ denotes the following p-form (see also Definition B.1):

$$(A\omega_1 \wedge \cdots \wedge \omega_p) + \cdots + (\omega_1 \wedge \cdots \wedge A\omega_p).$$

Theorem 1.1 (Neumann case). *Let U be a generalized critical point of f with index p on the boundary, for Neumann type boundary conditions. There exists locally, in a neighborhood of U, a $\mathscr{C}^\infty$ solution u_p^{WKB} to*

$$\Delta_{f,h}^{(p)} u_p^{\mathrm{WKB}} = e^{-\Phi/h}\mathbb{O}(h^\infty), \tag{1-1}$$

$$\boldsymbol{n} u_p^{\mathrm{WKB}} = 0 \quad \text{on } \partial\Omega, \tag{1-2}$$

$$\boldsymbol{n} d_{f,h} u_p^{\mathrm{WKB}} = 0 \quad \text{on } \partial\Omega, \tag{1-3}$$

where u_p^{WKB} has the form

$$u_p^{\mathrm{WKB}} = a_h e^{-\Phi/h},$$

with $a_h \sim \sum_k a^k h^k$, $a^0(U) = \boldsymbol{t} a^0(U) \neq 0$, and

$$a^0(U) \in \mathrm{Ker}\big(2(\mathrm{Hess}(f|_{\partial\Omega})(U) j^*)^{(p)} - \mathrm{Tr}(\mathrm{Hess}(f|_{\partial\Omega} - \Phi|_{\partial\Omega})(U))\big).$$

When restricted to tangential p-forms, $2(\mathrm{Hess}(f|_{\partial\Omega})(U)j^*)^{(p)} - \mathrm{Tr}(\mathrm{Hess}(f|_{\partial\Omega} - \Phi|_{\partial\Omega})(U))$ *has a one-dimensional kernel. The tangential form a^0 is then unique up to multiplication by a constant.*

Theorem 1.2 (Dirichlet case). *Let U be a generalized critical point of f with index p on the boundary, for Dirichlet type boundary conditions. There exists locally, in a neighborhood of U, a $\mathscr{C}^\infty$ solution u_p^{WKB} to*

$$\Delta_{f,h}^{(p)} u_p^{\mathrm{WKB}} = e^{-\Phi/h}\mathbb{O}(h^\infty) \tag{1-4}$$

$$\boldsymbol{t} u_p^{\mathrm{WKB}} = 0 \text{ on } \partial\Omega \tag{1-5}$$

$$\boldsymbol{t} d_{f,h}^* u_p^{\mathrm{WKB}} = 0 \text{ on } \partial\Omega\,, \tag{1-6}$$

where u_p^{WKB} has the form

$$u_p^{\mathrm{WKB}} = a_h e^{-\Phi/h},$$

with $a_h \sim \sum_k a^k h^k$, $a^0(U) = \boldsymbol{n} a^0(U) \neq 0$, and

$$a^0(U) \in \mathrm{Ker}\big(2(\mathrm{Hess}(f|_{\partial\Omega})(U)j^*)^{(p)} - \mathrm{Tr}(\mathrm{Hess}(f|_{\partial\Omega} - \Phi|_{\partial\Omega})(U))\big).$$

When restricted to normal p-forms, $2(\mathrm{Hess}(f|_{\partial\Omega})(U)j^*)^{(p)} - \mathrm{Tr}(\mathrm{Hess}(f|_{\partial\Omega} - \Phi|_{\partial\Omega})(U))$ *has a one-dimensional kernel. The normal form a^0 is then unique up to multiplication by a constant.*

Remark 1.3. When $B \in \mathscr{L}(T_x^*\partial\Omega)$, $x \in \partial\Omega$, note that $Bj^* = Bj^*\boldsymbol{t} \in \mathscr{L}(T_x^*\overline{\Omega}; T_x^*\partial\Omega) \subset \mathscr{L}(T_x^*\overline{\Omega})$ and $(Bj^*)^{(p)} \neq (B)^{(p)}j^*$. For example, if $\vec{n}$ is the outgoing normal at the boundary and $\vec{n}^*$ its dual for the Riemannian scalar product, then for $\omega \wedge \vec{n}^*$ with $\omega = \boldsymbol{t}\omega$,

$$(Bj^*)^{(p)}(\omega \wedge \vec{n}^*) = ((Bj^*)^{(p-1)}\omega) \wedge \vec{n}^* \quad (= (B^{(p-1)}(j^*\omega)) \wedge \vec{n}^*\ ).$$

To prove these results and make some explicit computations, we are going to work in local coordinates. To carry out properly the analysis, we need to choose suitably these local coordinates with respect to the geometry of the problem. Some "adapted coordinates" will then be defined in Section 3A. They will be more finely specified in Sections 4A and 4D; see (4-6) and (4-30). The last statements of Theorems 1.1 and 1.2 simply specify the polarization of $a^0(U)$ which is imposed, while solving degenerate transport equations (see Sections 4C and 4F). Again, this is more explicit later, choosing the suitable coordinate system. In particular, with the coordinate formulation, the fact that $a^0(U)$ lies in a given one-dimensional space appears clearly in (4-25) after Proposition 4.1 for the Neumann case and in (4-49) after Proposition 4.4 for the Dirichlet case. These theorems are respectively proved in Sections 4C and 4F.

When the metric is Euclidean, $g = \sum_{i=1}^n (dx^i)^2$, the manifold Ω is locally $\mathbb{R}^n_- = \mathbb{R}^{n-1} \times (-\infty, 0)$, the boundary $\partial\Omega$ is locally $\partial\Omega = \{x^n = 0\}$, and the function f is of the form

$$f(x) = -x^n - \tfrac{1}{2}|\lambda_1|(x^1)^2 - \cdots - \tfrac{1}{2}|\lambda_p|(x^p)^2 + \tfrac{1}{2}|\lambda_{p+1}|(x^{p+1})^2 + \tfrac{1}{2}|\lambda_{n-1}|(x^{n-1})^2$$

in the Neumann case, or

$$f(x) = +x^n - \tfrac{1}{2}|\lambda_1|(x^1)^2 - \cdots - \tfrac{1}{2}|\lambda_p|(x^p)^2 + \tfrac{1}{2}|\lambda_{p+1}|(x^{p+1})^2 + \tfrac{1}{2}|\lambda_{n-1}|(x^{n-1})^2$$

in the Dirichlet case, the "adapted coordinates" are simply $(x^1, \ldots, x^n)$. The general case is more involved because the three geometries of the boundary, of the metric (curvature), and of the level sets of the function f do not match.

Our goal consists in reducing the analysis to a problem on the boundary, hence to a problem in a manifold without boundary. Once this is done, we will be able to apply the results of [Helffer and Sjöstrand 1985], obtained in the case of a manifold without boundary, to this reduced problem.

2. Generalities about Witten Laplacians

On both manifolds with or without boundary. Let $\overline{\Omega}$ be a $\mathscr{C}^\infty$ connected compact oriented Riemannian manifold with dimension $n \in \mathbb{N}^*$. We will denote by g_0 the given Riemannian metric on $\overline{\Omega}$; Ω and $\partial\Omega$ will denote respectively its interior and its boundary.

The cotangent and tangent bundles on Ω are denoted by $T^*\Omega$ and $T\Omega$, respectively, and the corresponding exterior fiber bundles by $\Lambda T^*\Omega = \bigoplus_{p=0}^n \Lambda^p T^*\Omega$ and $\Lambda T\Omega = \bigoplus_{p=0}^n \Lambda^p T\Omega$. The fiber bundles $\Lambda T\partial\Omega = \bigoplus_{p=0}^{n-1} \Lambda^p T\partial\Omega$ and $\Lambda T^*\partial\Omega = \bigoplus_{p=0}^{n-1} \Lambda^p T^*\partial\Omega$ are defined similarly. The space of $\mathscr{C}^\infty$, $\mathscr{C}_0^\infty$, L^2, H^s, etc. sections in any of these fiber bundles, E, on $O = \Omega$ or $O = \partial\Omega$, will be denoted respectively by $\mathscr{C}^\infty(O; E)$, $\mathscr{C}_0^\infty(O; E)$, $L^2(O; E)$, $H^s(O; E)$, etc.

When no confusion is possible we will simply use the short notation $\Lambda^p\mathscr{C}^\infty$, $\Lambda^p\mathscr{C}_0^\infty$, $\Lambda^p L^2$ and $\Lambda^p H^s$ for $E = \Lambda^p T^*\Omega$ or $E = \Lambda^p T^*\partial\Omega$.

Note that the L^2 spaces are those associated with the unit volume form for the Riemannian structure on Ω or $\partial\Omega$ (Ω and $\partial\Omega$ are oriented).

The notation $\mathscr{C}^\infty(\overline{\Omega}; E)$ is used for the set of $\mathscr{C}^\infty$ sections up to the boundary.

Let d be the exterior differential on $\mathscr{C}_0^\infty(\Omega; \Lambda T^*\Omega)$,

$$d^{(p)} : \mathscr{C}_0^\infty(\Omega; \Lambda^p T^*\Omega) \to \mathscr{C}_0^\infty(\Omega; \Lambda^{p+1} T^*\Omega),$$

and d^* its formal adjoint with respect to the L^2-scalar product inherited from the Riemannian structure,

$$d^{(p),*} : \mathscr{C}_0^\infty(\Omega; \Lambda^{p+1} T^*\Omega) \to \mathscr{C}_0^\infty(\Omega; \Lambda^p T^*\Omega).$$

Remark 2.1. Note that d and d^* are both well defined on $\mathscr{C}^\infty(\overline{\Omega}; \Lambda T^*\Omega)$.

For a function $f \in \mathscr{C}^\infty(\overline{\Omega}; \mathbb{R})$ and $h > 0$, we introduce distorted operators defined on $\mathscr{C}^\infty(\overline{\Omega}; \Lambda T^*\Omega)$:

$$d_{f,h} = e^{-f(x)/h}\,(hd)\,e^{f(x)/h} \qquad \text{and} \qquad d^*_{f,h} = e^{f(x)/h}\,(hd^*)\,e^{-f(x)/h}.$$

The Witten Laplacian is the differential operator defined on $\mathscr{C}^\infty(\overline{\Omega}; \Lambda T^*\Omega)$ by

$$\Delta_{f,h} = d^*_{f,h} d_{f,h} + d_{f,h} d^*_{f,h} = (d_{f,h} + d^*_{f,h})^2. \tag{2-1}$$

The last equality follows from the property $dd = d^*d^* = 0$ which implies

$$d_{f,h} d_{f,h} = d^*_{f,h} d^*_{f,h} = 0. \tag{2-2}$$

This means, by restriction to the p-forms in $\mathscr{C}^\infty(\overline{\Omega}; \Lambda^p T^*\Omega)$, that

$$\Delta^{(p)}_{f,h} = d^{(p),*}_{f,h} d^{(p)}_{f,h} + d^{(p-1)}_{f,h} d^{(p-1),*}_{f,h} .$$

We next give some uselful relations involving exterior and interior products (denoted respectively by $\wedge$ and $\boldsymbol{i}$), gradients (denoted by ∇) and Lie derivatives (denoted by $\mathscr{L}$):

$$(df\wedge)^* = \boldsymbol{i}_{\nabla f} \quad (\text{in } L^2(\overline{\Omega};\Lambda^p T^*\Omega)), \tag{2-3}$$
$$d_{f,h} = hd + df\wedge, \tag{2-4}$$
$$d^*_{f,h} = hd^* + \boldsymbol{i}_{\nabla f}, \tag{2-5}$$
$$d\circ \boldsymbol{i}_X + \boldsymbol{i}_X\circ d = \mathscr{L}_X, \tag{2-6}$$
$$\Delta_{f,h} = h^2(d+d^*)^2 + |\nabla f|^2 + h(\mathscr{L}_{\nabla f} + \mathscr{L}^*_{\nabla f}), \tag{2-7}$$

where X denotes a vector field on Ω or $\overline{\Omega}$.

Remark 2.2. The operators introduced depend on the Riemannian metric g_0 but we omit this dependence for conciseness.

On manifolds with boundary.

Definition 2.3. We denote by $\vec{n}_\sigma$ the outgoing normal at $\sigma\in\partial\Omega$ and by $\vec{n}^*_\sigma$ the 1-form dual to $\vec{n}_\sigma$ for the Riemannian scalar product.

For any $\omega\in\mathscr{C}^\infty(\overline{\Omega};\Lambda^pT^*\Omega)$, the form $\boldsymbol{t}\omega$ is the element of $\mathscr{C}^\infty(\partial\Omega;\Lambda^pT^*\Omega)$ defined by

$$(\boldsymbol{t}\omega)_\sigma(X_1,\dots,X_p) = \omega_\sigma(X_1^T,\dots,X_p^T) \quad \text{for all } \sigma\in\partial\Omega,$$

with the decomposition into the tangential and normal components to $\partial\Omega$ at σ; i.e., $X_i = X_i^T \oplus x_i^\perp \vec{n}_\sigma$. Moreover,

$$(\boldsymbol{t}\omega)_\sigma = \boldsymbol{i}_{\vec{n}_\sigma}(\vec{n}^*_\sigma\wedge\omega_\sigma)\,.$$

The projected form $\boldsymbol{t}\omega$, which depends on the choice of $\vec{n}_\sigma$ (hence on g_0), can be compared with the canonical pull-back $j^*\omega$ associated with the embedding $j:\partial\Omega\to\overline{\Omega}$. Actually, the exact relationship is $j^*\omega = j^*(\boldsymbol{t}\omega)$.

The normal part of ω on $\partial\Omega$ is defined by

$$\boldsymbol{n}\omega = \omega|_{\partial\Omega} - \boldsymbol{t}\omega \quad \in \mathscr{C}^\infty(\partial\Omega;\Lambda^pT^*\Omega).$$

In the sequel, the form $\omega\in\mathscr{C}^\infty(\overline{\Omega};\Lambda^pT^*\Omega)$ will be said tangential or normal if $\omega=\boldsymbol{t}\omega$ or $\omega=\boldsymbol{n}\omega$, respectively, at any point of the boundary.

Definition 2.4. We denote by $\dfrac{\partial f}{\partial n}(\sigma)$ or $\partial_n f(\sigma)$ the normal derivative of f at σ:

$$\frac{\partial f}{\partial n}(\sigma) = \partial_n f(\sigma) := \langle \nabla f(\sigma)\,|\,\vec{n}_\sigma\rangle.$$

Assumption 2.5. The functions $f\in\mathscr{C}^\infty(\overline{\Omega},\mathbb{R})$ and $f|_{\partial\Omega}\in\mathscr{C}^\infty(\partial\Omega,\mathbb{R})$ are Morse functions. Moreover, the function f has no critical point on $\partial\Omega$.

The Neumann realization of the Witten Laplacian, denoted by $\Delta^N_{f,h}$, is the self-adjoint realization of $\Delta_{f,h}$ whose domain is

$$D(\Delta^N_{f,h}) = \{\omega\in\Lambda H^2(\Omega) : \boldsymbol{n}\omega = 0,\ \boldsymbol{n}d_{f,h}\omega = 0\}.$$

An analogous statement holds for the Dirichlet realization $\Delta^D_{f,h}$, the domain now being

$$D(\Delta^D_{f,h}) = \{\omega\in\Lambda H^2(\Omega) : \boldsymbol{t}\omega = 0,\ \boldsymbol{t}d^*_{f,h}\omega = 0\}.$$

See [Chang and Liu 1995; Helffer and Nier 2006; Le Peutrec 2008] for these results.

Definition 2.6. A point $U \in \overline{\Omega}$ is called a generalized critical point of f with index p if either $U \in \Omega$ and U is a critical point of f with index p, or $U \in \partial\Omega$ and

- in the Neumann case, U is a critical point with index p of $f|_{\partial\Omega}$ such that $\partial_n f(U) < 0$;
- in the Dirichlet case, U is a critical point with index $p-1$ of $f|_{\partial\Omega}$ such that $\partial_n f(U) > 0$.

Remark 2.7. This convention implies that the index p of a generalized critical point U on the boundary satisfies $p \in \{0, \dots, n-1\}$ in the Neumann case and $p \in \{1, \dots, n\}$ in the Dirichlet case.

We end this section by giving the statement extending to the case of a manifold with boundary the analysis done by Witten [1982]; see [Chang and Liu 1995; Helffer and Nier 2006; Le Peutrec 2008].

Theorem 2.8. *Under Assumption 2.5, there exists $h_0 > 0$ such that $\Delta^N_{f,h}$ and $\Delta^D_{f,h}$ have, for $h \in (0, h_0]$, the following property: For any $p \in \{0, \dots, n\}$, the spectral subspaces*

$$\operatorname{Ran} 1_{[0,h^{3/2})}(\Delta^{N,(p)}_{f,h}) \quad or \quad \operatorname{Ran} 1_{[0,h^{3/2})}(\Delta^{D,(p)}_{f,h})$$

have rank $m_p(f)$, the number of generalized critical points of f with index p in the respective cases (Neumann or Dirichlet).

The proofs in [Helffer and Nier 2006; Le Peutrec 2008] in fact show that the corresponding eigenvectors are concentrated around these critical points.

3. Preliminaries, coordinate systems

Since more than two geometries overlap around a generalized critical point of f with index p on the boundary and since systems of PDE are considered, the choice of the proper coordinate systems is a crucial point for making the analysis possible.

3A. *Existence of an adapted local coordinate system.*

Definition 3.1. Let σ be a point on the boundary $\partial\Omega$. An adapted local coordinate system around σ is a local coordinate system $(x^1, \dots, x^n) = (x', x^n)$ centered at σ satisfying the following properties:

(i) $dx^1, \dots, dx^n$ is an orthonormal positively oriented basis of $T^*_\sigma(\overline{\Omega})$, the cotangent space at σ.

(ii) The boundary $\partial\Omega$ corresponds locally to $x^n = 0$ and the interior Ω to $x^n < 0$.

(iii) $(\partial/\partial x^n)|_{\partial\Omega} = \vec{n}$, the outgoing normal at the boundary. Moreover, $(\partial/\partial x^n)$ is unitary and normal to $\{x^n = \text{Constant}\}$.

Such a coordinate system is more specific than the one provided by the collar theorem in [Schwarz 1995; Duff 1952; Duff and Spencer 1952]. Moreover, owing to the analysis done in [Petersen 1998, 117–122], it can be proven that such a system always exists. This is the aim of the next result.

Proposition 3.2. *A local coordinate system satisfying Definition 3.1 always exists.*

Proof. As in [Petersen 1998, 119–120], we look at

$$T\partial\Omega^\perp = \{v \in T_\sigma\overline{\Omega} : \sigma \in \partial\Omega,\, v \in (T_\sigma\partial\Omega)^\perp \subset T_\sigma\overline{\Omega}\},$$

where $(T_\sigma\partial\Omega)^\perp$ is the orthogonal complement of $T\partial\Omega$ in $T_\sigma\overline{\Omega}$ (so $T_\sigma\overline{\Omega} = T_\sigma\partial\Omega \oplus^\perp (T_\sigma\partial\Omega)^\perp$ for each $\sigma \in \partial\Omega$). Then, the map $\exp^\perp$ introduced in [Petersen 1998] is a diffeomorphism from an open neighborhood of the zero section in $T\partial\Omega^\perp$ onto its image in $\overline{\Omega}$. It means, choosing a point σ near the boundary $\partial\Omega$, that there exists an unique geodesic ν joining σ to a point σ_b on the boundary which satisfies $\dot\nu(\sigma_b) \in T\partial\Omega^\perp$. It is equivalent to say that there exists an unique geodesic ν joining σ to σ_b with $\dot\nu(\sigma_b) = \vec{n}_{\sigma_b}$.

Now let $-x^n$ be the geodesic distance to $\partial\Omega$ and take x' such that $x'|_{\partial\Omega}$ is a coordinate system on the boundary and x' is constant along the geodesics parametrized by x^n. The second point of the definition is then satisfied and $\partial/\partial x^n$ is unitary. Moreover, the choice of $x'|_{\partial\Omega}$ is arbitrary and we can choose it centered at U such that $dx^1, \dots, dx^n$ is an orthonormal basis of $T^*_U(\overline{\Omega})$ positively oriented. Then the first point of the definition is also satisfied.

We now verify that the third point of the definition is fulfilled. Write

$$\begin{aligned}\frac{\partial}{\partial x^n}\Big\langle \frac{\partial}{\partial x^n}\,\Big|\,\frac{\partial}{\partial x^i}\Big\rangle_\sigma &= \Big\langle \nabla_{\partial/\partial x^n}\frac{\partial}{\partial x^n}\,\Big|\,\frac{\partial}{\partial x^i}\Big\rangle_\sigma + \Big\langle \frac{\partial}{\partial x^n}\,\Big|\,\nabla_{\partial/\partial x^n}\frac{\partial}{\partial x^i}\Big\rangle_\sigma = 0 + \Big\langle \frac{\partial}{\partial x^n}\,\Big|\,\nabla_{\partial/\partial x^n}\frac{\partial}{\partial x^i}\Big\rangle_\sigma \\ &= \Big\langle \frac{\partial}{\partial x^n}\,\Big|\,\nabla_{\partial/\partial x^i}\frac{\partial}{\partial x^n}\Big\rangle_\sigma = \frac{1}{2}\frac{\partial}{\partial x^i}\Big\langle \frac{\partial}{\partial x^n}\,\Big|\,\frac{\partial}{\partial x^n}\Big\rangle_\sigma = 0,\end{aligned}$$

where we used the fact that ∇ is the Levi-Civita connection and $\nabla_{\partial/\partial x^n}\partial/\partial x^n = 0$ since x^n is a geodesic curve. Hence,

$$\Big\langle \frac{\partial}{\partial x^n}\,\Big|\,\frac{\partial}{\partial x^i}\Big\rangle_\sigma = \Big\langle \frac{\partial}{\partial x^n}\,\Big|\,\frac{\partial}{\partial x^i}\Big\rangle_{\sigma_b} = \Big\langle \vec{n}_{\sigma_b}\,\Big|\,\frac{\partial}{\partial x^i}\Big\rangle_{\sigma_b} = 0,$$

which gives the third point of the definition. □

Remark 3.3. In an adapted local coordinate system (x', x^n) around σ, remark that the metric g_0 can be written as

$$g_0(x) = (dx^n)^2 + \sum_{1\le i,j<n} g_{ij}(x)\,dx^i\,dx^j.$$

Moreover, it can be convenient to work with matrices and we write $G_0(x) = (g_{ij}(x))_{ij}$, $G_0^{-1}(x) = (g^{ij}(x))_{ij}$. Remember that $g_{ij} = \langle(\partial/\partial x^i) \mid (\partial/\partial x^j)\rangle$, $g^{ij} = \langle dx^i \mid dx^j\rangle$, and $dx^i(\partial/\partial x^j) = \delta_{ij}$.

Hence, in the (x', x^n) coordinate system, $G_0^{\pm1}(x)$ has the form

$$G_0^{\pm1}(x) = \begin{pmatrix} & & & 0 \\ & G_0^{\pm1'}(x) & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}, \quad \text{with } G_0^{\pm1}(0) = \mathrm{Id}_n .$$

3B. *Separating the x^n-variable.*

Lemma 3.4. (1) *Let f_1 belong to $\mathscr{C}^\infty(\overline{\Omega}, \mathbb{R})$ and let $U \in \partial\Omega$ be a critical point of $f_1|_{\partial\Omega}$ such that*

$$\frac{\partial f_1}{\partial n}(U) \neq 0. \tag{3-1}$$

Assume also that $\alpha \in \mathscr{C}^\infty(\partial\Omega, \mathbb{R})$ is a local solution to $|\nabla_T\alpha|^2 = |\nabla_T f_1|^2$ around U.

Then there exists a neighborhood $\mathcal{V}$ *of* U *in* $\overline{\Omega}$ *such that the eikonal equation*

$$|\nabla\Phi_\pm|^2 = |\nabla f_1|^2 \tag{3-2}$$

with boundary conditions

$$\Phi_\pm|_{\partial\Omega\cap\mathcal{V}} = \alpha, \quad \partial_n\Phi_\pm|_{\partial\Omega\cap\mathcal{V}} = \pm\frac{\partial f_1}{\partial n}\Big|_{\partial\Omega\cap\mathcal{V}}$$

admits a unique local smooth real-valued solution. (On the boundary, (3-2) is to be interpreted as saying that $|\nabla\Phi_\pm|^2 = |\partial_n\Phi_\pm|^2 + |\nabla_T\Phi_\pm|^2$; see details in the proof.)

(2) *There exist local coordinates* $(\bar{x}^1, \dots, \bar{x}^n) = (\bar{x}', \bar{x}^n)$ *in a neighborhood of* U *in* $\overline{\Omega}$ *with*

$$(\bar{x}', \bar{x}^n)(U) = 0$$

where the function $\Phi_\pm$ *and the metric* g_0 *have the form*

$$\Phi_\pm = \mp\bar{x}^n + \alpha(\bar{x}') \quad \textit{and} \quad g_0 = g_{nn}(\bar{x})\,(d\bar{x}^n)^2 + \sum_{i,j=1}^{n-1} g_{ij}(\bar{x})\,d\bar{x}^i\,d\bar{x}^j.$$

Moreover, the boundary $\partial\Omega$ *is locally defined by* $\{\bar{x}^n = 0\}$ *and* Ω *corresponds to*

$$\Big\{\operatorname{sgn}\Big(\frac{\partial f_1}{\partial n}(U)\Big)\,\bar{x}^n > 0\Big\}. \tag{3-3}$$

Proof. (1) Take an adapted local coordinate system (x', x^n) around U in order to write (3-2) as

$$|\partial_{x^n}\Phi_\pm|^2 + |\nabla_T\Phi_\pm|^2 = |\partial_{x^n} f_1|^2 + |\nabla_T f_1|^2$$

(see Appendix A for the exact meaning of ∇_T in the interior).

In particular, we obtain on the boundary

$$|\partial_n\Phi_\pm|^2 + |\nabla_T\Phi_\pm|^2 = |\partial_n f_1|^2 + |\nabla_T\alpha|^2.$$

The first point is then a direct consequence of the Hamilton–Jacobi theorem, due to the condition

$$\frac{\partial f_1}{\partial n}(U) \neq 0.$$

(2) As in [Helffer and Sjöstrand 1985], set

$$f_+ = \Phi_+ - \Phi_- \quad \text{and} \quad f_- = \Phi_+ + \Phi_-,$$

and note the relations

$$\Phi_- = -\tfrac{1}{2}f_+ + \tfrac{1}{2}f_-, \quad \Phi_+ = \tfrac{1}{2}f_+ + \tfrac{1}{2}f_-, \tag{3-4}$$

$$\nabla f_+ \cdot \nabla f_- = 0, \tag{3-5}$$

$$f_+|_{\partial\Omega\cap\mathcal{V}} = 0, \quad f_-|_{\partial\Omega\cap\mathcal{V}} = 2\alpha, \tag{3-6}$$

$$\frac{\partial f_+}{\partial n}\Big|_{\partial\Omega\cap\mathcal{V}} = 2\frac{\partial f_1}{\partial n}\Big|_{\partial\Omega\cap\mathcal{V}} \neq 0, \quad \frac{\partial f_-}{\partial n}\Big|_{\partial\Omega\cap\mathcal{V}} = 0. \tag{3-7}$$

Let $(\bar{x}^1,\dots,\bar{x}^{n-1}) = \bar{x}'$ denote a set of coordinates on $\partial\Omega$ in a neighborhood of U (then contained in $\mathcal{V}$) and such that $\bar{x}^j(U)=0$. We extend them to a neighborhood of U in $\overline{\Omega}$ as constant along the integral curve of the vector field ∇f_+. Then we take $\bar{x}^n = -\frac{1}{2} f_+(x)$ for the last coordinate.

In these coordinates, the functions $\Phi_\pm$ and the metric g_0 have the forms announced in the lemma.

Further, by (3-6), (3-7), and (3-1), the boundary $\partial\Omega$ is locally defined by $\{\bar{x}^n=0\}$ and Ω corresponds to the set in (3-3). □

In the sequel, we will apply part (1) of this lemma in the Neumann and Dirichlet cases in order to specify the Agmon distance, associated with the function f, to a generalized critical point U with index p on the boundary.

Then, using part (2) of the lemma and Proposition 3.2.11 of [Le Peutrec 2008] (in the Neumann case) or Proposition 3.3.9 of [Helffer and Nier 2006] (in the Dirichlet case), we view $\Delta^{(p),N}_{f,h}$ and $\Delta^{(p),D}_{f,h}$ locally in $\mathcal{V}$ around $U\in\partial\Omega$ as the restrictions to $\mathcal{V}$ of $\mathcal{A}^{(p)}_N$ and $\mathcal{A}^{(p)}_D$, the latter being the self-adjoint Witten Laplacian operators on $\mathbb{R}^n_- = \mathbb{R}^{n-1}\times(-\infty,0)$ (possibly after choosing $-\bar{x}^n$ instead of $\bar{x}^n$) whose domains are

$$D(\mathcal{A}_N) = \{\omega\in\Lambda H^2(\mathbb{R}^n_-) : \boldsymbol{n}\omega = \boldsymbol{n} d_{f,h}\omega = 0\},\quad D(\mathcal{A}_D) = \{\omega\in\Lambda H^2(\mathbb{R}^n_-) : \boldsymbol{t}\omega = \boldsymbol{t} d^*_{f,h}\omega = 0\}$$

(see also [Koldan et al. 2009]), and which satisfy

$$\dim\operatorname{Ker}\mathcal{A}^{(p)}_N = 1,\quad \sigma(\mathcal{A}^{(p)}_N)\setminus\{0\}\subset[Ch^{6/5},+\infty), \tag{3-8}$$

$$\dim\operatorname{Ker}\mathcal{A}^{(p)}_D = 1,\quad \sigma(\mathcal{A}^{(p)}_D)\setminus\{0\}\subset[Ch^{6/5},+\infty). \tag{3-9}$$

4. WKB construction near the boundary for $\Delta^{(p)}_{f,h}$, with p in $\{0,\dots,n\}$

4A. ***Local WKB construction in the Neumann case.*** Let U be a generalized critical point of f with index p in the Neumann case, that is, a critical point with index $p\in\{0,\dots,n-1\}$ of $f|_{\partial\Omega}$ satisfying $\frac{\partial f}{\partial n}(U)<0$, and take an adapted local coordinate system (x',x^n) around U.

Let Φ and φ be respectively the Agmon distance to U associated with the function f and its restriction to the boundary. The Agmon distance associated with f, that is, with the metric $|\nabla f(x)|^2\,dx^2$, is denoted by d_{Ag}: $\Phi(x)=d_{\mathrm{Ag}}(x,U)$. Recall that, locally,

$$|\nabla f|^2 = |\nabla\Phi|^2$$

and that Φ is smooth near U; see [Helffer and Sjöstrand 1984]. Moreover, φ is nothing but the Agmon distance to U on the boundary and satisfies locally, on the boundary,

$$|\nabla_T f|^2 = |\nabla\varphi|^2.$$

We now use Lemma 3.4(1) with $f_1 = f$ and $\alpha=\varphi$. The function Φ_+ of the lemma is consequently Φ and we have locally

$$|\partial_n\Phi|^2 + |\nabla_T\Phi|^2 = |\nabla\Phi|^2 = |\nabla f|^2, \tag{4-1}$$

$$\Phi|_{\partial\Omega} = \varphi, \tag{4-2}$$

$$\partial_n\Phi|_{\partial\Omega} = \frac{\partial f}{\partial n}\Big|_{\partial\Omega}. \tag{4-3}$$

Moreover,
$$\partial^2_{x^n x^n}(f-\Phi)(0) = \partial^2_{nn}(f-\Phi)(0) = 0. \tag{4-4}$$
Indeed, we can write in the coordinates (x', x^n), for the metric g_0,
$$|\partial_{x^n}\Phi|^2 + |\nabla_T \Phi|^2_{g_0} = |\partial_{x^n} f|^2 + |\nabla_T f|^2_{g_0},$$
where $|\nabla_T\Phi|^2_{g_0} = \mathbb{O}(|x|^2)$ and $|\nabla_T f|^2_{g_0} = \mathbb{O}(|x|^2)$ because 0 is a critical point of $f|_{\partial\Omega}$ in the coordinates (x', x^n) (see for example Appendix A). Then apply ∂_{x^n} to the last equation:
$$\partial_{x^n}|\partial_{x^n}\Phi|^2 + \mathbb{O}(|x|) = \partial_{x^n}|\partial_{x^n} f|^2 + \mathbb{O}(|x|),$$
that is, using (4-3),
$$2\partial^2_{x^n x^n}(f-\Phi)\partial_{x^n} f = \mathbb{O}(|x|),$$
which yields the result. According to [Helffer and Sjöstrand 1985, 279–280], there exist local coordinates $(\bar{x}', \bar{x}^n)$ centered at U, where $\bar{x}' = (\bar{x}^1, \dots, \bar{x}^{n-1})$ are Morse coordinates for $f|_{\partial\Omega}$ around U, such that $d\bar{x}^1, \dots, d\bar{x}^{n-1}, dx^n$ is orthonormal at U, and
$$\begin{aligned} f(\bar{x}', 0) &= \tfrac{1}{2}\lambda_1(\bar{x}^1)^2 + \cdots + \tfrac{1}{2}\lambda_{n-1}(\bar{x}^{n-1})^2 + f(U), \\ \varphi(\bar{x}') &= \tfrac{1}{2}|\lambda_1|(\bar{x}^1)^2 + \cdots + \tfrac{1}{2}|\lambda_{n-1}|(\bar{x}^{n-1})^2, \end{aligned} \tag{4-5}$$
with $\lambda_i < 0$ for $i \in \{1, \dots, p\}$ and $\lambda_i > 0$ for $i \in \{p+1, \dots, n-1\}$. Furthermore, the coordinates (x', x^n) can be chosen such that $dx^1, \dots, dx^{n-1}$ and $d\bar{x}^1, \dots, d\bar{x}^{n-1}$ coincide at U, and even such that $x'|_{\partial\Omega} = \bar{x}'|_{\partial\Omega}$ since $x'|_{\partial\Omega}$ can be chosen freely.

Specification of the coordinate system for Theorem 1.1. In the rest of the paper we are going to work in an adapted local coordinate system $x = (x', x^n)$ around U such that
$$dx^i = d\bar{x}^i \text{ at } U \quad \text{for all } i \in \{1, \dots, n-1\}. \tag{4-6}$$

4B. ***First boundary conditions in the Neumann case.*** We first write out the function $a_h(x) = a(x, h)$ in our coordinate system:
$$a(x,h) = \sum_{I \in \mathcal{I}} a_I(x,h)\, dx^I = \sum_{I' \in \mathcal{I}'} a_{I'}(x,h)\, dx^{I'} + \sum_{I_n \in \mathcal{I}_n} a_{I_n}(x,h)\, dx^{I_n}, \tag{4-7}$$
where
$$\begin{aligned} \mathcal{I} &:= \left\{(i_1, \dots, i_p) \in \{1, \dots, n\}^p,\ i_1 < \cdots < i_p\right\}, \\ \mathcal{I}' &:= \left\{(i_1, \dots, i_p) \in \{1, \dots, n\}^p,\ i_1 < \cdots < i_p < n\right\}, \\ \mathcal{I}_n &:= \left\{(i_1, \dots, i_p) \in \{1, \dots, n\}^p,\ i_1 < \cdots < i_p = n\right\}, \end{aligned}$$
and $dx^{(i_1, \dots, i_p)} = dx^{i_1} \wedge \cdots \wedge dx^{i_p}$. We will use in the sequel the Einstein summation convention to write (4-7) without the summation symbols:
$$a(x,h) = a_I(x,h)\, dx^I = a_{I'}(x,h)\, dx^{I'} + a_{I_n}(x,h)\, dx^{I_n}.$$

The first boundary condition (1-2) simply says that
$$a_{I_n}((x', 0), h) \sim \sum_k a^k_{I_n}(x', 0) h^k \equiv 0 \quad \text{for all } I_n \in \mathcal{I}_n, \tag{4-8}$$

which is equivalent to

$$a^k_{I_n}(x',0) \equiv 0 \quad \text{for all } k \in \mathbb{N} \text{ and } I_n \in \mathcal{I}_n. \tag{4-9}$$

The rest of this subsection specifies some consequence of these conditions. These consequences will be used in the next subsection to prove Theorem 1.1.

Proposition 4.1. *In the notation of Appendices A and B, the following relations are satisfied for every tangential p-form $b(x) = b_I(x)\,dx^I$, that is, every p-form $b(x)$ satisfying $b_{I_n}(x',0) \equiv 0$ for all $I_n \in \mathcal{I}_n$:*

$$\begin{aligned}
\boldsymbol{t}((2\mathcal{L}_{\nabla\Phi} + \mathcal{R}_1)b) &= (2\mathcal{L}_{\nabla\tilde{\Phi}} \otimes \mathrm{Id} + \mathcal{R}^T_{\mathrm{Neu}})b_{I'}dx^{I'} + 2\frac{\partial\Phi}{\partial x^n}\boldsymbol{i}_{\partial/\partial x^n}db,\\
\boldsymbol{n}((2\mathcal{L}_{\nabla\Phi} + \mathcal{R}_1)b) &= 2\Big(\frac{\partial b_{I_n}}{\partial x^n}\frac{\partial\Phi}{\partial x^n} + \ell_{I_n}(x',0)\Big)\,dx^{I_n},
\end{aligned}$$

where the ℓ_{I_n} are $\mathscr{C}^\infty(\partial\Omega)$-linear combinations of the $b_{I'}$ (for I' in $\mathcal{I}'$) that do not depend on the b_{I_n} (for I_n in $\mathcal{I}_n$) and $\mathcal{R}^T_{\mathrm{Neu}}$ is an order-zero differential operator given on the boundary, in the coordinates (x', x^n), by the matrix

$$\mathcal{R}^T_{\mathrm{Neu}}(x',0) = \begin{pmatrix} & & 0\\ & \mathcal{R}^{T'}_{\mathrm{Neu}}(x') & \vdots\\ & & 0\\ 0 & \cdots & 0 \;\; \beta(x') \end{pmatrix}^{(p)} - \gamma(x')\,\mathrm{Id},$$

where

$$\beta(0) = 0, \quad \gamma(0) = \mathrm{Tr}\left(\mathrm{Hess}(f|_{\partial\Omega} - \varphi)(0)\right), \quad \textit{and} \quad \mathcal{R}^{T'}_{\mathrm{Neu}}(0) = 2\,\mathrm{Hess}(f|_{\partial\Omega})(0).$$

In particular, this is true for a^k for k in $\mathbb{N}$ when (4-9) is fulfilled.

The following elementary result is important to notice here and also while verifying the final compatibility conditions (see pages 242–244).

Lemma 4.2. *Let $b(x)$ be a* tangential *p-form. The p-form*

$$\boldsymbol{i}_{\vec{n}}(db)$$

is then tangential and the equivalence

$$\boldsymbol{i}_{\vec{n}}(db) = 0 \iff \boldsymbol{n}\,db = 0$$

is locally valid on the boundary $\partial\Omega$. In particular, this is true for a^k for k in $\mathbb{N}$ when (4-9) is fulfilled.

Proof. On the boundary $\partial\Omega$, we have, in the coordinate system (x', x^n),

$$\boldsymbol{i}_{\partial/\partial x^n}(db) = \boldsymbol{i}_{\partial/\partial x^n}\boldsymbol{n}\,db + \boldsymbol{i}_{\partial/\partial x^n}\boldsymbol{t}\,db = \boldsymbol{i}_{\partial/\partial x^n}\boldsymbol{n}\,db + 0 = \boldsymbol{i}_{\partial/\partial x^n}(db)_{I_n}\,dx^{I_n} = (-1)^p(db)_{I_n}\,dx^{I_n\setminus\{n\}},$$

which leads to the result. $\square$

Lemma 4.3. *For every tangential p-form $b(x)$, we have*

$$\begin{aligned}
\boldsymbol{t}((\mathcal{L}_{\nabla\Phi} - \mathcal{L}_{\nabla\tilde{\Phi}})b) = \boldsymbol{t}((\mathcal{L}_{\nabla_T\Phi} - \mathcal{L}_{\nabla\tilde{\Phi}})b) &= \frac{\partial\Phi}{\partial x^n}\boldsymbol{i}_{\partial/\partial x^n}db,\\
\boldsymbol{n}((\mathcal{L}_{\nabla\Phi} - \mathcal{L}_{\nabla\tilde{\Phi}})b) &= \Big(\frac{\partial b_{I_n}}{\partial x^n}\frac{\partial\Phi}{\partial x^n} + \tilde{\ell}_{I_n}(x',0)\Big)\,dx^{I_n},
\end{aligned}$$

where the $\tilde{\ell}_{I_n}$ are $\mathscr{C}^\infty(\partial\Omega)$-linear combinations of the $b_{I'}$ (for I' in $\mathscr{I}'$) that do not depend on the b_{I_n} (for I_n in $\mathscr{I}_n$).

Proof. On the boundary $\partial\Omega$, we have the decomposition

$$(\mathscr{L}_{\nabla\Phi} - \mathscr{L}_{\nabla\tilde{\Phi}})b = \mathscr{L}_{(\partial\Phi/\partial x^n)(\partial/\partial x^n)}b + (\mathscr{L}_{\nabla_T\Phi} - \mathscr{L}_{\nabla\tilde{\Phi}})b. \tag{4-10}$$

Thanks to Cartan's formula (2-6), we can rewrite (4-10) as

$$(\mathscr{L}_{\nabla\Phi}-\mathscr{L}_{\nabla\tilde{\Phi}})b=\boldsymbol{i}_{(\partial\Phi/\partial x^n)(\partial/\partial x^n)}db+d(\boldsymbol{i}_{(\partial\Phi/\partial x^n)(\partial/\partial x^n)}b)+\boldsymbol{i}_{(\nabla_T\Phi-\nabla\tilde{\Phi})}db+d(\boldsymbol{i}_{(\nabla_T\Phi-\nabla\tilde{\Phi})}b). \tag{4-11}$$

Using Lemma 4.2, the first term on the right side of (4-11) is tangential:

$$\boldsymbol{i}_{(\partial\Phi/\partial x^n)(\partial/\partial x^n)}\, db = \frac{\partial\Phi}{\partial x^n}\boldsymbol{i}_{\partial/\partial x^n}\, db.$$

Moreover, since $\nabla_T\Phi = \nabla\tilde{\Phi}$ on the boundary (see Appendix A), the term $\boldsymbol{i}_{(\nabla_T\Phi-\nabla\tilde{\Phi})}db$ of the right side equals 0 on $\partial\Omega$. Hence, we can write on $\partial\Omega$

$$(\mathscr{L}_{\nabla\Phi} - \mathscr{L}_{\nabla\tilde{\Phi}})b = \boldsymbol{i}_{(\partial\Phi/\partial x^n)(\partial/\partial x^n)}db + d(\boldsymbol{i}_{(\partial\Phi/\partial x^n)(\partial/\partial x^n)}b) + d(\boldsymbol{i}_{(\nabla_T\Phi-\nabla\tilde{\Phi})}b). \tag{4-12}$$

Let us study in a first time the term $d(\boldsymbol{i}_{(\partial\Phi/\partial x^n)(\partial/\partial x^n)}b)$. Writing

$$b = b_I dx^I = b_{I'}dx^{I'} + b_{I_n}dx^{I_n},$$

we deduce (in $\overline{\Omega}$) that

$$\boldsymbol{i}_{(\partial\Phi/\partial x^n)(\partial/\partial x^n)}b = b_{I_n}\boldsymbol{i}_{(\partial\Phi/\partial x^n)(\partial/\partial x^n)}dx^{I_n} = (-1)^{p-1}b_{I_n}\frac{\partial\Phi}{\partial x^n}dx^{I_n\setminus\{n\}},$$

and, applying d to this last relation, we obtain on $\partial\Omega$ (remembering that $b_{I_n} = 0$ on $\partial\Omega$)

$$\begin{aligned} d(\boldsymbol{i}_{(\partial\Phi/\partial x^n)(\partial/\partial x^n)}b) &= (-1)^{p-1}\sum_{i=1}^{n}\frac{\partial}{\partial x^i}\Big(b_{I_n}\frac{\partial\Phi}{\partial x^n}\Big)\, dx^i \wedge dx^{I_n\setminus\{n\}} \\ &= (-1)^{p-1}\frac{\partial b_{I_n}}{\partial x^n}\frac{\partial\Phi}{\partial x^n}\, dx^n \wedge dx^{I_n\setminus\{n\}} + 0 = \frac{\partial b_{I_n}}{\partial x^n}\frac{\partial\Phi}{\partial x^n}\, dx^{I_n}. \end{aligned} \tag{4-13}$$

Now recall that $\mathscr{I} \ni I = (i_1, \dots, i_p)$ with $1 \le i_1 \le \cdots \le i_p \le n$, and denote by $\mathrm{ind}(i_k)$ the integer k. Looking at the third term of the right side of (4-12), we write

$$\begin{aligned} \boldsymbol{i}_{(\nabla_T\Phi-\nabla\tilde{\Phi})}b_I\, dx^I &= b_I\, dx^I(\nabla_T\Phi - \nabla\tilde{\Phi}) = b_I\sum_{j\in I}(-1)^{\mathrm{ind}(j)+1}(\nabla_T\Phi - \nabla\tilde{\Phi})_j\, dx^{I\setminus\{j\}} \\ &= b_I\sum_{j\in I}(-1)^{\mathrm{ind}(j)+1}\alpha_j\, dx^{I\setminus\{j\}}, \end{aligned}$$

where, due to (A-2) and (A-3), for all j in $\{1, \dots, n\}$,

$$\alpha_j = (\nabla_T\Phi - \nabla\tilde{\Phi})_j = \sum_{i=1}^{n} g^{ij}\Big(\frac{\partial\Phi}{\partial x^i}(x) - \frac{\partial\Phi}{\partial x^i}(x', 0)\Big).$$

Moreover, due to the block diagonal form of G_0^{-1}, for all j in $\{1,\dots,n\}$, α_j satisfies, again by (A-2) and (A-3),

$$\alpha_n(x)\equiv 0 \qquad\text{and}\qquad \alpha_j(x',0)\equiv 0 \quad\text{for all } j\in\{1,\dots,n-1\},$$

Hence, we obtain on $\partial\Omega$

$$\begin{aligned}
d(\boldsymbol{i}_{(\nabla_T\Phi-\nabla\tilde{\Phi})}b_I\,dx^I)(x',0) &= \sum_{l=1}^{n}\sum_{j\in I}(-1)^{\mathrm{ind}(j)+1}\frac{\partial}{\partial x^l}(b_I\alpha_j)(x',0)\,dx^l\wedge dx^{I\setminus\{j\}}\\
&= 0+\sum_{j\in I}(-1)^{\mathrm{ind}(j)+1}\frac{\partial}{\partial x^n}(b_I\alpha_j)(x',0)\,dx^n\wedge dx^{I\setminus\{j\}}\\
&= \sum_{j\in I'}(-1)^{\mathrm{ind}(j)+1}\frac{\partial}{\partial x^n}(b_{I'}\alpha_j)(x',0)\,dx^n\wedge dx^{I'\setminus\{j\}}\\
&\qquad\qquad+\sum_{j\in I_n\setminus\{n\}}(-1)^{\mathrm{ind}(j)+1}\frac{\partial}{\partial x^n}(b_{I_n}\alpha_j)(x',0)\,dx^n\wedge dx^{I_n\setminus\{j\}}\\
&= \sum_{j\in I'}(-1)^{\mathrm{ind}(j)+1}\frac{\partial}{\partial x^n}(b_{I'}\alpha_j)(x',0)\,dx^n\wedge dx^{I'\setminus\{j\}},
\end{aligned}$$

where we used $\alpha_j(x',0)\equiv 0$ at the second line and $\alpha_n(x)\equiv 0$ at the second to last line. Using again $\alpha_j(x',0)\equiv 0$ allows us to write on $\partial\Omega$

$$\begin{aligned}
d(\boldsymbol{i}_{(\nabla_T\Phi-\nabla\tilde{\Phi})}b_I\,dx^I)(x',0) &= b_{I'}\sum_{j\in I'}(-1)^{\mathrm{ind}(j)+1}\frac{\partial\alpha_j}{\partial x^n}(x',0)\,dx^n\wedge dx^{I'\setminus\{j\}}\\
&= b_{I'}\sum_{j\in I'}(-1)^{\mathrm{ind}(j)+p}\frac{\partial\alpha_j}{\partial x^n}(x',0)\,dx^{I'\setminus\{j\}}\wedge dx^n\\
&=: \tilde{\ell}_{I_n}(x',0)\,dx^{I_n},
\end{aligned}\tag{4-14}$$

where the $\tilde{\ell}_{I_n}$ are $\mathscr{C}^\infty(\partial\Omega)$-linear combinations of the $b_{I'}$ (for I' in $\mathscr{I}'$) that do not depend on the b_{I_n} (for I_n in $\mathscr{I}_n$). Combining (4-12), (4-13), and (4-14) leads to the result announced in Lemma 4.3. ☐

Proof of Proposition 4.1. From Section B2 we have

$$\mathscr{L}_{\nabla\Phi}-\mathscr{L}^*_{\nabla\Phi}+\mathscr{L}_{\nabla f}+\mathscr{L}^*_{\nabla f}=2\mathscr{L}_{\nabla\Phi}+\mathscr{R}_1,$$

where $\mathscr{R}_1$ is an order-zero differential operator. Writing $\mathscr{R}_1=\mathscr{R}_1^T+\mathscr{R}_1^N$, we deduce from (B-1), since $b_I\,dx^I=b_{I'}dx^{I'}$ on the boundary, that

$$\begin{aligned}
\boldsymbol{t}(\mathscr{R}_1(b_I\,dx^I)) &= b_{I'}(x',0)\mathscr{R}_1^T(dx^{I'}),\\
\boldsymbol{n}(\mathscr{R}_1(b_I\,dx^I)) &= b_{I'}(x',0)\mathscr{R}_1^N(dx^{I'})=\tilde{\ell}'_{I_n}(x',0)\,dx^{I_n},
\end{aligned}$$

where the $\tilde{\ell}'_{I_n}$ are $\mathscr{C}^\infty(\partial\Omega)$-linear combinations of the $b_{I'}$ (for I' in $\mathscr{I}'$) that do not depend on the b_{I_n} (for I_n in $\mathscr{I}_n$).

Moreover, from (4-1)–(4-4), $f-\Phi$ satisfies the assumptions of Corollary B.5; thus $\mathscr{R}_1^T$ is given on the boundary, in the coordinates (x',x^n), by

$$\mathscr{R}_1^T(x',0) = \begin{pmatrix} & & & 0 \\ & \mathscr{R}_1^{T'}(x') & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & \beta(x') \end{pmatrix}^{(p)} - \gamma(x')\,\mathrm{Id},$$

where β and γ are $\mathscr{C}^\infty$ functions that satisfy

$$\beta(0)=0, \quad \gamma(0)=\mathrm{Tr}(\mathrm{Hess}(f|_{\partial\Omega}-\varphi)(0)), \quad \text{and} \quad \mathscr{R}_1^{T'}(0) = 2\,\mathrm{Hess}(f|_{\partial\Omega}-\varphi)(0).$$

Having in mind Lemma 4.3, we now look at the term $2\mathscr{L}_{\nabla\tilde{\Phi}}+\mathscr{R}_1$. From Proposition B.3, we write

$$2\mathscr{L}_{\nabla\tilde{\Phi}} = 2\mathscr{L}_{\nabla\tilde{\Phi}} \otimes \mathrm{Id} + \mathscr{R}_3,$$

where $\mathscr{R}_3 = \mathscr{R}_3^T + \mathscr{R}_3^N$ is an order-zero differential operator such that, since $\tilde{\Phi}$ satisfies the assumptions of Corollary B.5,

$$\begin{aligned} \boldsymbol{t}(\mathscr{R}_3(b_I\,dx^I)) &= b_{I'}(x',0)\mathscr{R}_3^T(dx^{I'}), \\ \boldsymbol{n}(\mathscr{R}_3(b_I\,dx^I)) &= b_{I'}(x',0)\mathscr{R}_3^N(dx^{I'}) = \tilde{\ell}''_{I_n}(x',0)\,dx^{I_n}, \end{aligned}$$

where the $\tilde{\ell}''_{I_n}$ are $\mathscr{C}^\infty(\partial\Omega)$-linear combinations of the $b_{I'}$ (for I' in $\mathscr{I}'$) that do not depend on the b_{I_n}, and $\mathscr{R}_3^T$ is given on the boundary, in the coordinates (x',x^n), by

$$\mathscr{R}_3^T(x',0) = \begin{pmatrix} & & & 0 \\ & \mathscr{R}_3^{T'}(x') & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix}^{(p)},$$

with

$$\mathscr{R}_3^{T'}(0) = 2\,\mathrm{Hess}(\tilde{\Phi}|_{\partial\Omega})(0) = 2\,\mathrm{Hess}(\varphi)(0).$$

Note that, according to Remark B.4, the (n,n)-entry of the matrix is indeed 0 since $\partial^2\tilde{\Phi}/(\partial x^n)^2 \equiv 0$.

Set $\mathscr{R}_{\mathrm{Neu}} = \mathscr{R}_1 + \mathscr{R}_3$ and $\tilde{\ell}^{(3)}_{I_n} = \tilde{\ell}'_{I_n} + \tilde{\ell}''_{I_n}$ for I_n in $\mathscr{I}_n$. Then $\mathscr{R}_{\mathrm{Neu}}$ is an order-zero differential operator satisfying

$$2\mathscr{L}_{\nabla\tilde{\Phi}} + \mathscr{R}_1 = 2\mathscr{L}_{\nabla\tilde{\Phi}} \otimes \mathrm{Id} + \mathscr{R}_{\mathrm{Neu}} \tag{4-15}$$

and

$$\begin{aligned} \boldsymbol{t}(\mathscr{R}_{\mathrm{Neu}}(b_I\,dx^I)) &= b_{I'}(x',0)\mathscr{R}_{\mathrm{Neu}}^T(dx^{I'}), \\ \boldsymbol{n}(\mathscr{R}_{\mathrm{Neu}}(b_I\,dx^I)) &= b_{I'}(x',0)\mathscr{R}_{\mathrm{Neu}}^N(dx^{I'}) = \tilde{\ell}^{(3)}_{I_n}(x',0)\,dx^{I_n}, \end{aligned} \tag{4-16}$$

where the $\tilde{\ell}^{(3)}_{I_n}$ are $\mathscr{C}^\infty(\partial\Omega)$-linear combinations of the $b_{I'}$ (for I' in $\mathscr{I}'$) that do not depend on the b_{I_n} (for I_n in $\mathscr{I}_n$). Moreover, $\mathscr{R}_{\mathrm{Neu}}^T$ is given on the boundary, in the coordinates (x',x^n), by

$$\mathscr{R}_{\mathrm{Neu}}^T(x',0) = \begin{pmatrix} & & & 0 \\ & \mathscr{R}_1^{T'}(x',0)+\mathscr{R}_3^{T'}(x',0) & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & \beta(x') \end{pmatrix}^{(p)} - \gamma(x')\,\mathrm{Id},$$

where

$$\beta(0)=0, \quad \gamma(0)=\operatorname{Tr}\left(\operatorname{Hess}(f|_{\partial\Omega}-\varphi)(0)\right), \quad \text{and} \quad \mathcal{R}_1^{T'}(0)+\mathcal{R}_3^{T'}(0)=2\operatorname{Hess}(f|_{\partial\Omega})(0).$$

Now look at the term $2\mathcal{L}_{\nabla\tilde{\Phi}}\otimes\mathrm{Id}$. By Cartan's formula (2-6), we have

$$(2\mathcal{L}_{\nabla\tilde{\Phi}}\otimes\mathrm{Id})b = db_{I'}(\nabla\tilde{\Phi})\,dx^{I'}+db_{I_n}(\nabla\tilde{\Phi})\,dx^{I_n},$$

and, using the boundary condition satisfied by the b_{I_n} (for I_n in $\mathcal{I}_n$) and the fact that $\nabla\tilde{\Phi}$ is a tangential vector field, we obtain

$$(2\mathcal{L}_{\nabla\tilde{\Phi}}\otimes\mathrm{Id})b=\sum_{i=1}^{n-1}\frac{\partial b_{I'}}{\partial x^i}(\nabla\tilde{\Phi})_i\,dx^{I'}=(2\mathcal{L}_{\nabla\tilde{\Phi}}\otimes\mathrm{Id})b_{I'}\,dx^{I'}. \tag{4-17}$$

Set $\ell_{I_n}=\tilde{\ell}_{I_n}+\frac{1}{2}\tilde{\ell}_{I_n}^{(3)}$ for I_n in $\mathcal{I}_n$. Writing

$$(2\mathcal{L}_{\nabla\Phi}+\mathcal{R}_1)b=2(\mathcal{L}_{\nabla\Phi}-\mathcal{L}_{\nabla\tilde{\Phi}})b+(2\mathcal{L}_{\nabla\tilde{\Phi}}+\mathcal{R}_1)b$$

and using (4-15)–(4-17), we obtain Proposition 4.1 after the application of Lemma 4.3. □

4C. *Proof of Theorem 1.1.* We shall first consider a WKB-approximation for

$$(\Delta_{f,h}^{(p)}-E(h))u_p^{\mathrm{WKB}}=e^{-\Phi/h}\mathcal{O}(h^\infty) \tag{4-18}$$

with $E(h)=O(h^2)$ and the boundary conditions (1-2) and (1-3) and then check $E(h)=O(h^\infty)$.

Writing

$$d_{f,h}(e^{-\Phi/h}a^k)=e^{-\Phi/h}\big(h\,da^k+d(f-\Phi)\wedge a^k\big)\quad\text{for all } k\in\mathbb{N},$$

where, due to (1-2) and (4-3), a^k and $d(f-\Phi)$ are tangential forms, the second boundary condition (1-3) corresponds to

$$\boldsymbol{n}(da^k)=0\quad\text{for all } k\in\mathbb{N}. \tag{4-19}$$

We now recall a relation that will be very useful; see [Helffer and Sjöstrand 1985] for a complete proof:

$$\begin{aligned} e^{\Phi/h}\Delta_{f,h}e^{-\Phi/h}&=h^2(d+d^*)^2+|\nabla f|^2-|\nabla\Phi|^2+h(\mathcal{L}_{\nabla\Phi}-\mathcal{L}_{\nabla\Phi}^*+\mathcal{L}_{\nabla f}+\mathcal{L}_{\nabla f}^*)\\ &=h^2(d+d^*)^2+h(\mathcal{L}_{\nabla\Phi}-\mathcal{L}_{\nabla\Phi}^*+\mathcal{L}_{\nabla f}+\mathcal{L}_{\nabla f}^*). \end{aligned} \tag{4-20}$$

We then write, in the notation of Section B2,

$$\mathcal{L}_{\nabla\Phi}-\mathcal{L}_{\nabla\Phi}^*+\mathcal{L}_{\nabla f}+\mathcal{L}_{\nabla f}^*=2\mathcal{L}_{\nabla\Phi}+\mathcal{R}_1=2\mathcal{L}_{\nabla\Phi}\otimes\mathrm{Id}+\mathcal{R},$$

where $\mathcal{R}$ and $\mathcal{R}_1$ are order-zero differential operators defined in Section B2.

By looking for $E(h)\sim\sum_{k=1}^{\infty}h^{k+1}E_k$, the interior equation (4-18) reads

$$e^{\Phi/h}(\Delta_{f,h}-E(h))e^{-\Phi/h}=h^2\big((d+d^*)^2-h^{-2}E(h)\big)+h\,(2\mathcal{L}_{\nabla\Phi}\otimes\mathrm{Id}+\mathcal{R}).$$

We now verify that it is possible to construct a solution u_p^{WKB} to (4-18) in Ω which can be extended to $\overline{\Omega}$ and satisfying the boundary conditions (1-2) and (1-3). The construction of an interior WKB solution in Ω is standard as an inductive Cauchy problem, once the a^k are known on $\partial\Omega$; see [Dimassi and Sjöstrand 1999; Helffer 1988]. Actually the noncharacteristic Cauchy problems

$$(2\mathscr{L}_{\nabla\Phi}\otimes\mathrm{Id}+\mathscr{R})a^k=-(d+d^*)^2a^{k-1}+\sum_{\ell=1}^{k}E_\ell a^{k-\ell}\quad\text{in }\overline{\Omega}\tag{4-21}$$

are solved by induction, with the convention $a_{-1}=0$.

Hence the problem is reduced to the solving of the system made of the boundary conditions (4-9), (4-19) and of the compatibility equation on the boundary (see Section B2 for the meaning of the notation):

$$(2\mathscr{L}_{\nabla\Phi}+\mathscr{R}_1)\,a^k=-(d+d^*)^2a^{k-1}+\sum_{\ell=1}^{k}E_\ell a^{k-\ell}\quad\text{on }\partial\Omega.\tag{4-22}$$

Owing to Proposition 4.1 (with the notation of Section 4B) and to (4-3), the system (4-22), (4-9), (4-19) is equivalent to the following differential system on $\partial\Omega$:

$$\begin{aligned}-\boldsymbol{t}(d+d^*)^2a^{k-1}+\sum_{\ell=1}^{k}E_\ell a^{k-\ell}&=(2\mathscr{L}_{\nabla\tilde{\Phi}}\otimes\mathrm{Id}+\mathscr{R}^T_{\mathrm{Neu}})a^k_{I'}\,dx^{I'}+2\frac{\partial f}{\partial x^n}\boldsymbol{i}_{\partial/\partial x^n}da^k,\\ -\boldsymbol{n}(d+d^*)^2a^{k-1}-2\ell_{I_n}(x',0)\,dx^{I_n}&=2\frac{\partial f}{\partial n}\frac{\partial a^k_{I_n}}{\partial x^n}\,dx^{I_n},\\ a^k_{I_n}|_{\partial\Omega}\equiv 0\quad\text{and}\quad \boldsymbol{n}(da^k)&=0\quad\text{for all }k\in\mathbb{N},\end{aligned}$$

where the ℓ_{I_n} are $\mathscr{C}^\infty(\partial\Omega)$-linear combinations of the $a^k_{I'}$ (for I' in $\mathscr{I}'$) that do not depend on the $a^k_{I_n}$ (for I_n in $\mathscr{I}_n$). Note also, owing to Lemma 4.2, that the first line of this system simply reads

$$-\boldsymbol{t}(d+d^*)^2a^{k-1}+\sum_{\ell=1}^{k}E_\ell a^{k-\ell}=(2\mathscr{L}_{\nabla\tilde{\Phi}}\otimes\mathrm{Id}+\mathscr{R}^T_{\mathrm{Neu}})a^k_{I'}\,dx^{I'}.\tag{4-23}$$

Moreover, since $dx^i=d\bar{x}^i$ for $i\in\{1,\dots,n-1\}$ at the point U, it follows from Corollary B.5, (4-5), and the results in [Helffer and Sjöstrand 1985, 271–275] that $\mathscr{R}^T_{\mathrm{Neu}}(0)$ restricted to tangential forms is symmetric with the one-dimensional kernel $\mathbb{R}dx^1\wedge\cdots\wedge dx^p$.

Since $a^k_{I'}dx^{I'}$ is tangential and $2\mathscr{L}_{\nabla\tilde{\Phi}}\otimes\mathrm{Id}$ only differentiates the $a^k_{I'}$ tangentially, because

$$(2\mathscr{L}_{\nabla\tilde{\Phi}}\otimes\mathrm{Id})a^k_{I'}\,dx^{I'}=\sum_{i=1}^{n-1}\frac{\partial a^k_{I'}}{\partial x^i}(\nabla\tilde{\Phi})_i\,dx^{I'},$$

it turns out that (4-23) can be rewritten as a tangential system that can be solved according to the analysis of the boundaryless case done in [Helffer and Sjöstrand 1985]. Here are the details: thanks to Lemma 4.2, the complete system (4-21), (4-22), (4-9) and (4-19) becomes equivalent to the system

$$\left.\begin{aligned}(2\mathscr{L}_{\nabla\tilde{\Phi}}\otimes\mathrm{Id}+\mathscr{R}^T_{\mathrm{Neu}})a^k_{I'}dx^{I'}&=-\boldsymbol{t}(d+d^*)^2a^{k-1}+\sum_{\ell=1}^{k-1}E_\ell a^{k-\ell}+E_ka^0 &&\text{on }\partial\Omega,\\ (2\mathscr{L}_{\nabla\Phi}\otimes\mathrm{Id}+\mathscr{R})a^k&=-(d+d^*)^2a^{k-1}+\sum_{\ell=1}^{k}E_\ell a^{k-\ell} &&\text{on }\overline{\Omega},\\ a_{I_n}|_{\partial\Omega}&\equiv 0 &&\text{for }I_n\in\mathscr{I}_n.\end{aligned}\right\}\tag{$\mathscr{S}_{\mathrm{Neu}}$}$$

The first line is a degenerate matrix transport equation, which can be solved following [Helffer and Sjöstrand 1985, page 275] and [Helffer 1988, pages 13–14]: for $k=0$, the homogeneous boundary equation

$$(2\mathscr{L}_{\nabla\tilde{\Phi}}\otimes \mathrm{Id}+\mathscr{R}^{T}_{\mathrm{Neu}})a^0_{I'}dx^{I'}=0$$

admits some solution if and only if

$$a^0_{I'}(0)\,dx^{I'}\in \mathrm{Ker}(\mathscr{R}^{T}_{\mathrm{Neu}}(0)), \tag{4-25}$$

and the solution is unique once $a^0_{I'}(0)\,dx^{I'}$ has been chosen. This shows the uniqueness of a^0 up to multiplication by a constant. Note also that the formulation of Theorem 1.1 is a coordinate-free rewriting of this condition for $a^0(U)$. Indeed, it has already been mentioned that, when restricted to tangential p-forms, the kernel of $\mathscr{R}^{T}_{\mathrm{Neu}}(0)$ is one-dimensional. Recall moreover that, *in our coordinate system* (see Proposition 4.1), at $U\cong 0$,

$$\begin{aligned}\mathscr{R}^{T}_{\mathrm{Neu}}(U)&=2\begin{pmatrix} & & 0\\ \mathrm{Hess}(f|_{\partial\Omega})(U) & & \vdots\\ 0 & \cdots & 0\end{pmatrix}^{(p)}-\mathrm{Tr}\,(\mathrm{Hess}(f|_{\partial\Omega}-\Phi|_{\partial\Omega})(U))\\ &=2A^{(p)}-\mathrm{Tr}\,(\mathrm{Hess}(f|_{\partial\Omega}-\Phi|_{\partial\Omega})(U)),\end{aligned}$$

and that, for a tangential p-form $dx^{I'}=dx^{i_1}\wedge\cdots\wedge dx^{i_p}$,

$$A^{(p)}dx^{I'}=(A\,dx^{i_1})\wedge dx^{i_2}\wedge\cdots\wedge dx^{i_p}+\cdots+dx^{i_1}\wedge\cdots\wedge dx^{i_{p-1}}\wedge(A\,dx^{i_p}),$$

where, for $\ell\in\{1,\dots,p\}$,

$$A\,dx^{i_\ell}=(\mathrm{Hess}(f|_{\partial\Omega})(U))\,dx^{i_\ell}+0.dx^n=(\mathrm{Hess}(f|_{\partial\Omega})(U)\,j^*)\,dx^{i_\ell}.$$

Lastly, note that in the previous equation, we wrote $(\mathrm{Hess}(f|_{\partial\Omega})(U))\,dx^{i_\ell}$ with a slight abuse of notation, since $\mathrm{Hess}(f|_{\partial\Omega})(U)\in\mathscr{L}(T^*_U\partial\Omega)$ and $dx^{i_\ell}(U)\in T^*_U\overline{\Omega}$. Indeed, the proper notation would be $(\mathrm{Hess}(f|_{\partial\Omega})(U)\,j^*)\,dx^{i_\ell}$, where

$$(\mathrm{Hess}(f|_{\partial\Omega})(U)\,j^*)\,dx^{i_\ell}\in\mathscr{L}(T^*_U\overline{\Omega};T^*_U\partial\Omega)\subset\mathscr{L}(T^*_U\overline{\Omega};T^*_U\overline{\Omega}).$$

Now take $a^0(0)=dx^1\wedge\cdots\wedge dx^p\in\mathrm{Ker}(\mathscr{R}^{T}_{\mathrm{Neu}}(0))$. For $k=1$, we have to solve the boundary equation

$$(2\mathscr{L}_{\nabla\tilde{\Phi}}\otimes \mathrm{Id}+\mathscr{R}^{T}_{\mathrm{Neu}})a^1_{I'}dx^{I'}=-\boldsymbol{t}(d+d^*)^2a^0+E_1a^0.$$

Choose then E_1 such that

$$-\boldsymbol{t}(d+d^*)^2a^0(0)+E_1a^0(0)\in\mathrm{Ran}(\mathscr{R}^{T}_{\mathrm{Neu}}(0))=\big(\mathrm{Ker}(\mathscr{R}^{T}_{\mathrm{Neu}}(0))\big)^{\perp},$$

where the last equality follows from both the symmetry of $\mathscr{R}^{T}_{\mathrm{Neu}}(0)$ and $G_0(0)=\mathrm{Id}_n$. This is equivalent to choosing E_1 such that

$$E_1=\frac{\langle\boldsymbol{t}(d+d^*)^2a^0(0)\,|\,a^0(0)\rangle_{g_0(0)}}{\|a^0(0)\|^2_{g_0(0)}},$$

and this is indeed possible since $\operatorname{Ker}(\mathcal{R}^T_{\mathrm{Neu}}(0)) = \mathbb{R}a^0(0) \neq \{0\}$. Next take $a^1_{I'}(0)\,dx^{I'}$ such that

$$\mathcal{R}^T_{\mathrm{Neu}}(0)(a^1_{I'}(0)\,dx^{I'}) = -\boldsymbol{t}(d+d^*)^2 a^0(0) + E_1 a^0(0).$$

Then, for each $k > 2$, choose E_k such that the compatibility condition

$$-\boldsymbol{t}(d+d^*)^2 a^{k-1}(0) + \sum_{\ell=1}^{k-1} E_\ell a^{k-\ell}(0) + E_k a^0(0) \in \big(\operatorname{Ker}(\mathcal{R}^T_{\mathrm{Neu}}(0))\big)^{\perp}$$

is satisfied, or, more precisely, such that

$$E_k = \frac{\langle \boldsymbol{t}(d+d^*)^2 a^{k-1}(0) - \sum_{\ell=1}^{k-1} E_\ell a^{k-\ell}(0) \mid a^0(0)\rangle_{g_0(0)}}{\|a^0(0)\|^2_{g_0(0)}},$$

and take $a^{k-1}_{I'}(0)\,dx^{I'}$ in

$$(\mathcal{R}^T_{\mathrm{Neu}}(0))^{-1}\Big(-\boldsymbol{t}(d+d^*)^2 a^{k-1}(0) + \sum_{\ell=1}^{k-1} E_\ell a^{k-\ell}(0) + E_k a^0(0)\Big).$$

Thus, at every step $k \in \mathbb{N}$, the first and third lines of the system $(\mathcal{S}_{\mathrm{Neu}})$ fully determine the Cauchy data $a^k(x', 0)$ and the number E_k. The first line fully determines the restrictions of the $a_{I'}$ to $\partial\Omega$. The second line solves the interior problem with these Cauchy data and contains, with the two other lines, thanks to Lemma 4.2, the second trace condition (4-19).

We now check that $E(h) = \mathcal{O}(h^\infty)$. We prove this by comparing with the half-space problem, for which we know by (3-8) that the first eigenvalue is 0 with multiplicity one and that the second one is larger than $Ch^{6/5}$. Take a cut-off function $\chi \in \mathcal{C}^\infty_0(\overline{\Omega})$ satisfying $\chi = 1$ in a neighborhood of U and $\partial\chi/\partial n|_{\partial\Omega} = 0$. Then set

$$u^K_p = \chi e^{-\Phi/h} \sum_{k=0}^{K} a^k h^k = \chi e^{-\Phi/h} A^K_h.$$

From $\partial\chi/\partial n|_{\partial\Omega} \equiv 0$ and

$$d_{f,h}(\chi A^K_h) = (hd + df\wedge)\chi A^K_h = h\,d\chi \wedge A^K_h + \chi d_{f,h} A^K_h,$$

the form $u^K_p \in \Lambda^1 H^2(\mathbb{R}^n_-)$ belongs to the domain of $\mathcal{A}^{(p)}_N$ and the approximations u^K_p and $E^K(h) = \sum_{k=1}^K E_k h^{k+1}$ satisfy

$$\begin{aligned}
(\mathcal{A}^{(p)}_N - E^K(h))u^K_p &= h^{K+2}\rho^K e^{-\Phi/h} - h^2[\Delta,\chi]u^K_p = \mathcal{O}(h^{K+2}) && \text{in } \overline{\mathbb{R}^n_-},\\
\boldsymbol{n}u^K_p &= 0 && \text{on } \mathbb{R}^{n-1}\times\{0\},\\
\boldsymbol{n}d_{f,h}u^K_p &= 0 && \text{on } \mathbb{R}^{n-1}\times\{0\},
\end{aligned}$$

for some $\mathcal{C}^\infty$ 1-form ρ^K defined in a neighborhood of U and independent of h. From a direct Laplace method we obtain

$$\|u^K_p\| \sim ch^{(n+1)/4},$$

and the spectral theorem then implies that there exists an eigenvalue $\lambda(h)$ of $\mathcal{A}_N^{(p)}$ such that

$$|E^K(h)-\lambda(h)| = \mathcal{O}(h^{K+2-(n+1)/4}).$$

Choosing the integer K large enough, we deduce from the inclusion

$$\sigma(\mathcal{A}_N^{(p)}) \setminus \{0\} \subset [Ch^{6/5}, +\infty)$$

combined with the estimate $E^K(h) = \mathcal{O}(h^2)$ that $\lambda(h)=0$. The number K being arbitrary, the construction of the previous quasimode is then possible only if $E_k = 0$ for all $k \in \mathbb{N}^*$. □

4D. *Local WKB construction in the Dirichlet case.* Let U be a generalized critical point of f with index p in the Dirichlet case, i.e., a critical point of index $p-1$, with $p \in \{1,\dots,n\}$, of $f|_{\partial\Omega}$ satisfying $(\partial f/\partial n)(U) > 0$, and again take an adapted local coordinate system (x', x^n) around U, as in Section 4A.

Let φ be the Agmon distance to U on the boundary and use Lemma 3.4(1) with $f_1 = f$ and $\alpha = \varphi$. Denoting by Φ the function Φ_- of the lemma, Φ is then the Agmon distance to U and we have locally

$$|\partial_n \Phi|^2 + |\nabla_T \Phi|^2 = |\nabla\Phi|^2 = |\nabla f|^2, \tag{4-26}$$

$$\Phi|_{\partial\Omega} = \varphi, \tag{4-27}$$

$$\partial_n \Phi|_{\partial\Omega} = -\frac{\partial f}{\partial n}\Big|_{\partial\Omega}. \tag{4-28}$$

Moreover, the following relation is satisfied (see the proof of (4-4) and replace $\partial_n\Phi|_{\partial\Omega} = \partial_n f|_{\partial\Omega}$ by $\partial_n\Phi|_{\partial\Omega} = -\partial_n f|_{\partial\Omega}$):

$$\partial^2_{x^n x^n}(f+\Phi)(0) = \partial^2_{nn}(f+\Phi)(0) = 0. \tag{4-29}$$

As in Section 4A, there exist other local coordinates $(\bar{x}', \bar{x}^n)$ centered at U, with $\bar{x}' = (\bar{x}^1, \dots, \bar{x}^{n-1})$ and $d\bar{x}^1, \dots, d\bar{x}^{n-1}, dx^n$ orthonormal at U, such that (4-5) is satisfied with $\lambda_i < 0$ for $i \in \{1,\dots,p-1\}$ and $\lambda_i > 0$ for $i \in \{p,\dots,n-1\}$. Furthermore, the coordinates (x', x^n) can be chosen in such a way that $dx^1,\dots,dx^{n-1}$ and $d\bar{x}^1,\dots,d\bar{x}^{n-1}$ coincide at U and even such that $x'|_{\partial\Omega} = \bar{x}'|_{\partial\Omega}$.

Specification of the coordinate system for Theorem 1.2. In the rest of this section, we are again going to work in an adapted local coordinate system $x = (x', x^n)$ around U such that

$$dx^i = d\bar{x}^i \quad \text{at } U \qquad \text{for all } i \in \{1,\dots,n-1\}. \tag{4-30}$$

The proof is quite close to the one for the Neumann case, but here it turns out to be more natural to make "dual computations". In particular, we will work with d^* where we worked with d in the Neumann case. This leads to somewhat more complicated computations.

4E. *First boundary conditions in the Dirichlet case.* Writing

$$a_h(x) = a(x,h) = a_I(x,h)\,dx^I = a_{I'}(x,h)\,dx^{I'} + a_{I_n}(x,h)\,dx^{I_n},$$

the first boundary condition (1-5) is equivalent to

$$a^k_{I'}(x',0) \equiv 0 \quad \text{for all } k \in \mathbb{N} \text{ and } I' \in \mathcal{I}'. \tag{4-31}$$

The rest of this subsection specifies some consequences of these conditions, in the same spirit as those specified in the Section 4B concerning the Neumann case.

4E1. *About $\mathscr{L}+\mathscr{L}^*$.* The relation

$$\mathscr{L}_{\nabla\Phi}-\mathscr{L}^*_{\nabla\Phi}+\mathscr{L}_{\nabla f}+\mathscr{L}^*_{\nabla f}=-2\mathscr{L}^*_{\nabla\Phi}+\mathscr{L}_{\nabla(f+\Phi)}+\mathscr{L}^*_{\nabla(f+\Phi)}$$

is obviously satisfied, and using again Proposition B.3, we can write

$$\mathscr{L}^*_{\nabla(f+\Phi)}+\mathscr{L}_{\nabla(f+\Phi)}=\mathscr{R}_4,$$

where $\mathscr{R}_4$ is an order-zero differential operator.

Writing $\mathscr{R}_4=\mathscr{R}_4^T+\mathscr{R}_4^N$, we deduce from (B-2), since $a_I^k\,dx^I=a_{I_n}^k\,dx^{I_n}$ on the boundary, that

$$\begin{aligned}\boldsymbol{t}(\mathscr{R}_4(a_I^k\,dx^I))&=a_{I_n}^k(x',0)\mathscr{R}_4^N(dx^{I_n})=\tilde{\ell}'_{I'}(x',0)\,dx^{I'},\\ \boldsymbol{n}(\mathscr{R}_4(a_I^k\,dx^I))&=a_{I_n}^k(x',0)\mathscr{R}_4^T(dx^{I_n}),\end{aligned}$$

where the $\tilde{\ell}'_{I'}$ are $\mathscr{C}^\infty(\partial\Omega)$-linear combinations of the $a_{I_n}^k$ (for I_n in $\mathscr{I}_n$) that do not depend on the $a_{I'}^k$ (for I' in $\mathscr{I}'$).

Moreover, by (4-26)–(4-29), here $f+\Phi$ satisfies the assumptions of Corollary B.5; thus $\mathscr{R}_4^T$ is given on the boundary, in the coordinates (x',x^n), by

$$\mathscr{R}_4^T(x',0)=\begin{pmatrix} & & 0\\ & \mathscr{R}_4^{T'}(x') & \vdots\\ & & 0\\ 0 & \cdots\quad 0 & \delta(x')\end{pmatrix}^{(p)}-\kappa(x')\,\mathrm{Id},$$

where δ, κ are $\mathscr{C}^\infty$ functions which satisfy

$$\delta(0)=0,\quad \kappa(0)=\mathrm{Tr}(\mathrm{Hess}(f|_{\partial\Omega}+\varphi)(0)),\quad \mathscr{R}_4^{T'}(0)=2\,\mathrm{Hess}(f|_{\partial\Omega}+\varphi)(0).$$

4E2. *Expression of the codifferential d^*.* As already mentioned, to make a study similar to the one done in Section 4B for the Neumann case, we need to work with d^*, so we must have a handy expression for this operator.

For a differential form ω we set, in the coordinate system (x',x^n),

$$\nabla_i=\nabla_{x^i},\quad \boldsymbol{a}_i^*\omega=dx^i\wedge\omega,\quad \boldsymbol{a}_i\omega=\boldsymbol{i}_{\nabla x^i}\omega.$$

Then d and d^* have the following form (see [Cycon et al. 1987, pages 238–247]):

$$d=\sum_{i=1}^n\boldsymbol{a}_i^*\nabla_i=-\sum_{i=1}^n(\nabla_i)^*\boldsymbol{a}_i^*,\tag{4-32}$$

$$d^*=-\sum_{i=1}^n\boldsymbol{a}_i\nabla_i.\tag{4-33}$$

Recall also the characteristic relations

$$\boldsymbol{a}_i^*\boldsymbol{a}_j^*+\boldsymbol{a}_j^*\boldsymbol{a}_i^*=0,\quad \boldsymbol{a}_i\boldsymbol{a}_j+\boldsymbol{a}_j\boldsymbol{a}_i=0,\quad \boldsymbol{a}_i^*\boldsymbol{a}_j+\boldsymbol{a}_j\boldsymbol{a}_i^*=g^{ij},\quad \text{for all } i,j\in\{1,\dots,n\}.$$

Denoting by ∂_i the operator defined by components with differentiation in a fixed coordinate system,

$$\partial_i(\omega_I\,dx^I) = \frac{\partial \omega_I}{\partial x^i}\,dx^I,$$

∇_i becomes (see again [Cycon et al. 1987, pages 238–247])

$$\nabla_i = \partial_i - \sum_{j,l,m} \Gamma^j_{il} g_{jm} \boldsymbol{a}^*_l \boldsymbol{a}_m, \tag{4-34}$$

where the Γ^j_{il} are the Christoffel symbols. Then d^* becomes

$$\begin{aligned} d^* &= -\sum_i \boldsymbol{a}_i \partial_i + \sum_{i,j,l,m} \Gamma^j_{il} g_{jm} \boldsymbol{a}_i \boldsymbol{a}^*_l \boldsymbol{a}_m \\ &= -\sum_i \boldsymbol{a}_i \partial_i + \sum_{i,j,l,m} \Gamma^j_{il} g_{jm} \left(\boldsymbol{a}_i \boldsymbol{a}^*_l + \boldsymbol{a}^*_l \boldsymbol{a}_i\right) \boldsymbol{a}_m - \sum_{i,j,l,m} \Gamma^j_{il} g_{jm} \boldsymbol{a}^*_l \boldsymbol{a}_i \boldsymbol{a}_m \\ &= -\sum_i \boldsymbol{a}_i \partial_i + \sum_{i,j,l,m} \Gamma^j_{il} g_{jm} g^{il} \boldsymbol{a}_m - \sum_{i,j,l,m} \Gamma^j_{il} g_{jm} \boldsymbol{a}^*_l \boldsymbol{a}_i \boldsymbol{a}_m. \end{aligned} \tag{4-35}$$

4E3. *Results.*

Proposition 4.4. *In the notation of Appendix A and Section 4E1, the following relations are satisfied for every normal p-form $b(x) = b_I(x)\,dx^I$ (that is, every p-form $b(x)$ satisfying $b_{I'}(x',0) \equiv 0$ for all $I' \in \mathcal{I}'$):*

$$\boldsymbol{t}((-2\mathcal{L}^*_{\nabla\Phi} + \mathcal{R}_4)b) = 2\left(\frac{\partial b_{I'}}{\partial x^n}\frac{\partial \Phi}{\partial x^n} + \ell_{I'}(x',0)\right) dx^{I'},$$

$$\boldsymbol{n}((-2\mathcal{L}^*_{\nabla\Phi} + \mathcal{R}_4)b) = (2\mathcal{L}_{\nabla\tilde{\Phi}} \otimes \mathrm{Id} + \mathcal{R}^T_{\mathrm{Dir}})b_{I_n}\,dx^{I_n} - 2\frac{\partial \Phi}{\partial x^n}\,dx^n \wedge d^*b,$$

where the $\ell_{I'}$ are $\mathscr{C}^\infty(\partial\Omega)$-linear combinations of the b_{I_n} (for I_n in $\mathcal{I}_n$) that do not depend on the $b_{I'}$ (for I' in $\mathcal{I}'$) and $\mathcal{R}^T_{\mathrm{Dir}}$ is an order-zero differential operator given in the coordinates (x', x^n), on the boundary by the following matrix, by

$$\mathcal{R}^T_{\mathrm{Dir}}(x',0) = \begin{pmatrix} & & & 0 \\ & \mathcal{R}^{T'}_{\mathrm{Dir}}(x') & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & \delta(x') \end{pmatrix}^{(p)} - \kappa_2(x')\,\mathrm{Id},$$

where

$$\delta(0) = 0, \qquad \kappa_2(0) = \mathrm{Tr}(\mathrm{Hess}(f|_{\partial\Omega} - \varphi)(0)), \qquad \mathcal{R}^{T'}_{\mathrm{Dir}}(0) = 2\,\mathrm{Hess}(f|_{\partial\Omega})(0).$$

In particular, this is true for a^k for k in $\mathbb{N}$, when (4-31) is fulfilled.

Lemma 4.5. *Let $b(x)$ be a* normal *p-form. The p-form $\vec{n}^* \wedge d^*b$ is then normal and the following equivalence is locally valid on the boundary $\partial\Omega$:*

$$\vec{n}^* \wedge d^*b = 0 \iff \boldsymbol{t}\,d^*b = 0.$$

In particular, this is true for a^k for k in $\mathbb{N}$, when (4-31) is fulfilled.

Proof. On the boundary $\partial\Omega$, we can write, in the coordinate system (x', x^n),

$$\begin{aligned} dx^n \wedge d^*b = dx^n \wedge \boldsymbol{n} d^*b + dx^n \wedge \boldsymbol{t} d^*b = 0 + dx^n \wedge \boldsymbol{t} d^*b = dx^n \wedge (d^*b)_{I'} dx^{I'} \\ = (-1)^{p-1} (d^*b)_{I'} dx^{I'} \wedge dx^n. \end{aligned}$$

Since $dx^n = \vec{n}^*$, this leads to the result. □

Lemma 4.6. *For every tangential p-form $b(x)$,*

$$\begin{aligned} \boldsymbol{n}((\mathscr{L}^*_{\nabla\Phi} - \mathscr{L}^*_{\nabla\tilde{\Phi}})b) &= \frac{\partial\Phi}{\partial x^n}\, dx^n \wedge d^*b, \\ \boldsymbol{t}((\mathscr{L}^*_{\nabla\Phi} - \mathscr{L}^*_{\nabla\tilde{\Phi}})b) &= \Big(-\frac{\partial b_{I'}}{\partial x^n}\frac{\partial\Phi}{\partial x^n} + \tilde{\ell}_{I'}(x', 0)\Big)\, dx^{I'}, \end{aligned}$$

where the $\tilde{\ell}_{I'}$ are $\mathscr{C}^\infty(\partial\Omega)$-linear combinations of the b_{I_n} (for I_n in $\mathscr{I}_n$) that do not depend on the $b_{I'}$ (for I' in $\mathscr{I}_n$).

Proof. Owing to (2-3) and to Cartan's formula (2-6), we write, in the coordinates (x', x^n),

$$\begin{aligned} &(\mathscr{L}^*_{\nabla\Phi} - \mathscr{L}^*_{\nabla\tilde{\Phi}})b \\ &\quad= d^*(d\Phi\wedge b) + d\Phi\wedge d^*b + d^*(d\tilde{\Phi}\wedge b) + d\tilde{\Phi}\wedge d^*b \\ &\quad= d^*\Big(\frac{\partial\Phi}{\partial x^n}\, dx^n\wedge b\Big) + \frac{\partial\Phi}{\partial x^n}\, dx^n\wedge d^*b + d^*((d_T\Phi - d\tilde{\Phi})\wedge b) + (d_T\Phi - d\tilde{\Phi})\wedge d^*b, \end{aligned} \tag{4-36}$$

where the function $\tilde{\Phi}$ is defined in Appendix A.

The second summand on the last line of (4-36) is normal by Lemma 4.5. Moreover, since $d_T\Phi = d\tilde{\Phi}$ on the boundary, the last summand also equals 0 on $\partial\Omega$. Hence, on $\partial\Omega$,

$$(\mathscr{L}^*_{\nabla\Phi} - \mathscr{L}^*_{\nabla\tilde{\Phi}})b = \frac{\partial\Phi}{\partial x^n} dx^n \wedge d^*b + d^*\Big(\frac{\partial\Phi}{\partial x^n} dx^n \wedge b\Big) + d^*((d_T\Phi - d\tilde{\Phi})\wedge b). \tag{4-37}$$

We study first the second summand on the right-hand side. Writing

$$b = b_I dx^I = b_{I'} dx^{I'} + b_{I_n} dx^{I_n},$$

we deduce that, in $\overline{\Omega}$,

$$\frac{\partial\Phi}{\partial x^n}\, dx^n \wedge b = \frac{\partial\Phi}{\partial x^n} b_{I'} dx^n \wedge dx^{I'}.$$

Applying d^* to this last relation (see (4-35)) and recalling that $b_{I'} = 0$ on $\partial\Omega$, we obtain on $\partial\Omega$

$$\begin{aligned} d^*\Big(\frac{\partial\Phi}{\partial x^n}\, dx^n\wedge b\Big) &= -\sum_i \boldsymbol{a}_i \partial_i \Big(\frac{\partial\Phi}{\partial x^n} b_{I'}\, dx^n \wedge\, dx^{I'}\Big) + \sum_{i,j,l,m} \Gamma^j_{il} g_{jm} g^{il} \boldsymbol{a}_m \Big(\frac{\partial\Phi}{\partial x^n} b_{I'}\, dx^n \wedge\, dx^{I'}\Big) \\ &\qquad - \sum_{i,j,l,m} \Gamma^j_{il} g_{jm} \boldsymbol{a}^*_l \boldsymbol{a}_i \boldsymbol{a}_m \Big(\frac{\partial\Phi}{\partial x^n} b_{I'}\, dx^n \wedge\, dx^{I'}\Big) \\ &= -\sum_i \boldsymbol{a}_i \partial_i \Big(\frac{\partial\Phi}{\partial x^n} b_{I'}\, dx^n \wedge\, dx^{I'}\Big) + 0 \\ &= -\boldsymbol{i}_{\nabla x^n} \frac{\partial\Phi}{\partial x^n}\frac{\partial b_{I'}}{\partial x^n}\, dx^n \wedge\, dx^{I'} = -\frac{\partial\Phi}{\partial x^n}\frac{\partial b_{I'}}{\partial x^n}\, dx^{I'}. \end{aligned} \tag{4-38}$$

We used on the last line the fact that G_0^{-1} is block diagonal with $g^{nn} \equiv 1$.

Now look at the third term of the right-hand side of (4-37) and write, in view of (A-5),

$$(d_T\Phi - d\tilde{\Phi}) \wedge b_I\, dx^I = \sum_{i=1}^{n-1} \left(\frac{\partial \Phi}{\partial x^i}(x) - \frac{\partial \Phi}{\partial x^i}(x', 0) \right) b_I\, dx^i \wedge dx^I =: \sum_{i=1}^{n-1} \alpha_i b_I\, dx^i \wedge dx^I,$$

where, $\alpha_i = \dfrac{\partial \Phi}{\partial x^i}(x) - \dfrac{\partial \Phi}{\partial x^i}(x', 0)$ for i in $\{1, \dots, n-1\}$. Hence we have

$$\alpha_j(x', 0) \equiv 0 \quad \text{for all } j \in \{1, \dots, n-1\}.$$

Taking (4-35) again into account, we therefore obtain, on $\partial\Omega$,

$$\begin{aligned}
&d^*((d_T\Phi - d\tilde{\Phi}) \wedge b_I\, dx^I)(x', 0) \\
&\qquad = -\sum_i \boldsymbol{a}_i \partial_i \sum_{j=1}^{n-1} \alpha_j b_I\, dx^j \wedge dx^I \\
&\qquad\qquad + \Big(\sum_{i,j,l,m} \Gamma^j_{il} g_{jm} g^{il} \boldsymbol{a}_m - \sum_{i,j,l,m} \Gamma^j_{il} g_{jm} \boldsymbol{a}_l^* \boldsymbol{a}_i \boldsymbol{a}_m \Big) \sum_{j=1}^{n-1} \alpha_j b_I\, dx^j \wedge dx^I \\
&\qquad = -\sum_i \boldsymbol{a}_i \partial_i \sum_{j=1}^{n-1} \alpha_j b_I\, dx^j \wedge dx^I = -\boldsymbol{a}_n \sum_{j=1}^{n-1} \frac{\partial}{\partial x^n}(\alpha_j b_I)\, dx^j \wedge dx^I,
\end{aligned}$$

where we used $\alpha_j(x', 0) \equiv 0$ twice on the last line. Now, since $g^{ni} = g^{in} = 0$ for i in $\{1, \dots, n-1\}$, we can write, for all $I' \in \mathcal{I}'$,

$$\boldsymbol{a}_n dx^{I'} = \boldsymbol{i}_{\nabla x^n} dx^{I'} = 0.$$

This implies

$$\begin{aligned}
d^*((d_T\Phi - d\tilde{\Phi}) \wedge b_I\, dx^I)(x', 0) &= -\boldsymbol{a}_n \sum_{j=1}^{n-1} \frac{\partial}{\partial x^n}(\alpha_j b_I)\, dx^j \wedge dx^I = -\boldsymbol{a}_n \sum_{j=1}^{n-1} \frac{\partial}{\partial x^n}(\alpha_j b_{I_n})\, dx^j \wedge dx^{I_n} \\
&= (-1)^{p+1} \sum_{j=1}^{n-1} \frac{\partial}{\partial x^n}(\alpha_j b_{I_n})\, dx^j \wedge dx^{I_n \setminus \{n\}} \\
&= (-1)^{p+1} \sum_{j=1}^{n-1} b_{I_n} \frac{\partial \alpha_j}{\partial x^n}(x', 0)\, dx^j \wedge dx^{I_n \setminus \{n\}} \\
&=: \tilde{\ell}_{I'}(x', 0)\, dx^{I'},
\end{aligned} \tag{4-39}$$

where the $\tilde{\ell}_{I'}$ are $\mathscr{C}^\infty(\partial\Omega)$-linear combinations of the b_{I_n} (for I_n in $\mathcal{I}_n$) that do not depend on the $b_{I'}$ (for I' in $\mathcal{I}'$).

Combining (4-37), (4-38), and (4-39) leads to the result announced in Lemma 4.6. □

Proof of Proposition 4.4. Having in mind Lemma 4.6, we now look at the term $-2\mathscr{L}^*_{\nabla\tilde{\Phi}} + \mathscr{R}_4$. Again by Proposition B.3, we can write

$$-2\mathscr{L}^*_{\nabla\tilde{\Phi}} = 2\mathscr{L}_{\nabla\tilde{\Phi}} + \mathscr{R}_5 = 2\mathscr{L}_{\nabla\tilde{\Phi}} \otimes \mathrm{Id} + \mathscr{R}_5 + \mathscr{R}_6,$$

where $\mathcal{R}_5 = \mathcal{R}_5^T + \mathcal{R}_5^N$ and $\mathcal{R}_6 = \mathcal{R}_6^T + \mathcal{R}_6^N$ are order-zero differential operators satisfying, for $i \in \{5, 6\}$ (since $b_I\, dx^I = b_{I_n}\, dx^{I_n}$ on the boundary),

$$\begin{aligned} \boldsymbol{t}(\mathcal{R}_i(b_I\, dx^I)) &= b_{I_n}(x',0)\mathcal{R}_i^N(dx^{I_n}) = \tilde{\ell}^{i\prime}_{I'}(x',0)\, dx^{I'}, \\ \boldsymbol{n}(\mathcal{R}_i(b_I\, dx^I)) &= b_{I_n}(x',0)\mathcal{R}_i^T(dx^{I_n}). \end{aligned}$$

Here the $\tilde{\ell}^{i\prime}_{I'}(x',0)$ are $\mathscr{C}^\infty(\partial\Omega)$-linear combinations of the b_{I_n} (for I_n in $\mathcal{I}_n$) that do not depend on the $b_{I'}$ (for I' in $\mathcal{I}'$). Moreover, since $\tilde{\Phi}$ satisfies the assumptions of Corollary B.5, $\mathcal{R}_5^T$ and $\mathcal{R}_6^T$ are given on the boundary, in the coordinates (x', x^n), by

$$\mathcal{R}_5^T = \begin{pmatrix} & & & 0 \\ & \mathcal{R}_5^{T'} & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix}^{(p)} - \zeta(x')\,\mathrm{Id} \quad \text{and} \quad \mathcal{R}_6^T = \begin{pmatrix} & & & 0 \\ & \mathcal{R}_6^{T'} & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix}^{(p)},$$

where

$$\zeta(0) = -2\,\mathrm{Tr}(\mathrm{Hess}(\tilde{\Phi}|_{\partial\Omega})(0)) = -2\,\mathrm{Tr}(\mathrm{Hess}(\varphi)(0)), \quad \mathcal{R}_5^{T'}(0) = -4\,\mathrm{Hess}(\varphi)(0), \quad \mathcal{R}_6^{T'}(0) = 2\,\mathrm{Hess}(\varphi)(0).$$

Set $\mathcal{R}_{\mathrm{Dir}} = \mathcal{R}_4 + \mathcal{R}_5 + \mathcal{R}_6$ and $\tilde{\ell}^{(3)}_{I'} = \tilde{\ell}'_{I'} + \tilde{\ell}^{5\prime}_{I'} + \tilde{\ell}^{6\prime}_{I'}$ for I' in $\mathcal{I}'$. Then $\mathcal{R}_{\mathrm{Dir}}$ is an order-zero differential operator satisfying

$$-2\mathcal{L}_{\nabla\tilde{\Phi}} + \mathcal{R}_4 = 2\mathcal{L}_{\nabla\tilde{\Phi}} \otimes \mathrm{Id} + \mathcal{R}_{\mathrm{Dir}} \tag{4-40}$$

and

$$\begin{aligned} \boldsymbol{t}(\mathcal{R}_{\mathrm{Dir}}(b_I\, dx^I)) &= b_{I_n}(x',0)\mathcal{R}_{\mathrm{Dir}}^N(dx^{I_n}) = \tilde{\ell}^{(3)}_{I'}(x',0)\, dx^{I'}, \\ \boldsymbol{n}(\mathcal{R}_{\mathrm{Dir}}(b_I\, dx^I)) &= b_{I_n}(x',0)\mathcal{R}_{\mathrm{Dir}}^T(dx^{I_n}), \end{aligned} \tag{4-41}$$

where the $\tilde{\ell}^{(3)}_{I'}$ are $\mathscr{C}^\infty(\partial\Omega)$-linear combinations of the b_{I_n} (for I_n in $\mathcal{I}_n$) that do not depend on the $b_{I'}$ (for I' in $\mathcal{I}'$). Moreover, $\mathcal{R}_{\mathrm{Dir}}^T$ is given on the boundary, in the coordinates (x', x^n), by

$$\mathcal{R}_{\mathrm{Dir}}^T(x',0) = \begin{pmatrix} & & & 0 \\ & \mathcal{R}_{\mathrm{Dir}}^{T'}(x',0) & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & \delta(x') \end{pmatrix}^{(p)} - \kappa_2(x')\,\mathrm{Id},$$

where

$$\begin{aligned} &\delta(0) = 0, \\ &\kappa_2(0) = \kappa(0) + \zeta(0) = \mathrm{Tr}\,(\mathrm{Hess}(f|_{\partial\Omega} + \varphi)(0)) - 2\,\mathrm{Tr}\,(\mathrm{Hess}(\varphi)(0)) = \mathrm{Tr}\,(\mathrm{Hess}(f|_{\partial\Omega} - \varphi)(0)), \\ &\mathcal{R}_{\mathrm{Dir}}^{T'}(0) = \mathcal{R}_4^{T'}(0) + \mathcal{R}_5^{T'}(0) + \mathcal{R}_6^{T'}(0) = 2\,\mathrm{Hess}(f|_{\partial\Omega} + \varphi)(0) - 2\,\mathrm{Hess}(\varphi)(0) = 2\,\mathrm{Hess}(f|_{\partial\Omega})(0). \end{aligned}$$

We now look at the term $2\mathcal{L}_{\nabla\tilde{\Phi}} \otimes \mathrm{Id}$. By Cartan's formula (2-6),

$$(2\mathcal{L}_{\nabla\tilde{\Phi}} \otimes \mathrm{Id})b = db_{I'}(\nabla\tilde{\Phi})\, dx^{I'} + db_{I_n}(\nabla\tilde{\Phi})\, dx^{I_n},$$

and, using the boundary conditions satisfied by the b_I (for I in $\mathcal{I}$) and the fact that $\nabla\tilde{\Phi}$ is a tangential vector field, we obtain

$$(2\mathcal{L}_{\nabla\tilde{\Phi}} \otimes \mathrm{Id})b = \sum_{i=1}^{n-1} \frac{\partial b_{I_n}}{\partial x^i}(\nabla\tilde{\Phi})_i \, dx^{I_n} = 2\mathcal{L}_{\nabla\tilde{\Phi}} \otimes \mathrm{Id}\, b_{I_n}\, dx^{I_n}. \tag{4-42}$$

Set $\ell_{I'} = -\tilde{\ell}_{I'} + \frac{1}{2}\tilde{\ell}^{(3)}_{I'}$ and write

$$(-2\mathcal{L}^*_{\nabla\Phi} + \mathcal{R}_3)b = -2(\mathcal{L}^*_{\nabla\Phi} - \mathcal{L}^*_{\nabla\tilde{\Phi}})b + (-2\mathcal{L}^*_{\nabla\tilde{\Phi}} + \mathcal{R}_3)b.$$

Using (4-40)–(4-42), Proposition 4.4 is then a direct consequence of Lemma 4.6. □

4F. *Proof of Theorem 1.2.* Although the calculations are different, the scheme of the proof is the same as for Theorem 1.1. Consider first a WKB-approximation for

$$(\Delta^{(p)}_{f,h} - E(h))u^{\mathrm{WKB}}_p = e^{-\Phi/h}\mathcal{O}(h^\infty), \tag{4-43}$$

with $E(h) = O(h^2)$ and the boundary conditions (1-5) and (1-6).

From

$$d^*_{f,h}(e^{-\Phi/h}a^k) = e^{-\Phi/h}\big(hd^*a^k + \boldsymbol{i}_{\nabla(f+\Phi)}a^k\big) \quad \text{for all } k \in \mathbb{N},$$

where, due to (1-5) and (4-28), a^k is a normal form and $\nabla(f+\Phi)$ is a tangential vector field, the second boundary condition (1-6) corresponds to

$$\boldsymbol{t}(d^*a^k) = 0 \quad \text{for all } k \in \mathbb{N}, \tag{4-44}$$

We now recall that, in the notation of Section B2 and Section 4E1,

$$e^{\Phi/h}\Delta_{f,h}e^{-\Phi/h} = h^2(d+d^*)^2 + h(2\mathcal{L}_{\nabla\Phi} \otimes \mathrm{Id} + \mathcal{R}) = h^2(d+d^*)^2 + h(-2\mathcal{L}^*_{\nabla\Phi} + \mathcal{R}_4).$$

By looking for $E(h) \sim \sum_{k=1}^{\infty} h^{k+1}E_k$, the interior equation (4-43) reads, as in Section 4C,

$$e^{\Phi/h}(\Delta_{f,h} - E(h))e^{-\Phi/h} = h^2\big((d+d^*)^2 - h^{-2}E(h)\big) + h(2\mathcal{L}_{\nabla\Phi} \otimes \mathrm{Id} + \mathcal{R}).$$

Hence, as in Section 4C, the construction of an interior WKB solution in Ω is standard as an inductive Cauchy problem, once the a^k are known on $\partial\Omega$, since the noncharacteristic Cauchy problems

$$\big(2\mathcal{L}_{\nabla\Phi} \otimes \mathrm{Id} + \mathcal{R}\big)a^k = -(d+d^*)^2a^{k-1} + \sum_{\ell=1}^{k} E_\ell a^{k-\ell} \quad \text{in } \overline{\Omega} \tag{4-45}$$

are solved by induction with the convention $a_{-1} = 0$.

The problem is then reduced to solving the system made of the boundary conditions (4-31) and (4-44) and of the compatibility equation

$$\big(-2\mathcal{L}^*_{\nabla\Phi} + \mathcal{R}_4\big)\, a^k = -(d+d^*)^2a^{k-1} + \sum_{\ell=1}^{k} E_\ell a^{k-\ell} \quad \text{on } \partial\Omega \tag{4-46}$$

(see Section 4E1 for the notation).

Owing to Proposition 4.4 (with the notation of Section 4E3) and to (4-28), the system (4-46), (4-31), (4-44) is equivalent to the following differential system on $\partial\Omega$:

$$
\begin{aligned}
-\boldsymbol{n}(d+d^*)^2a^{k-1}+\sum_{\ell=1}^{k}E_\ell a^{k-\ell} &= (2\mathscr{L}_{\nabla\tilde\Phi}\otimes\mathrm{Id}+\mathscr{R}^T_{\mathrm{Dir}})a^k_{I_n}\,dx^{I_n}+2\frac{\partial f}{\partial x^n}\,dx^n\wedge d^*a^k,\\
-\boldsymbol{t}(d+d^*)^2a^{k-1}-2\ell_{I'}(x',0)\,dx^{I'} &= -2\frac{\partial f}{\partial n}\frac{\partial a^k_{I'}}{\partial x^n}\,dx^{I'},\\
a^k_{I'}|_{\partial\Omega}\equiv 0 \quad\text{and}\quad \boldsymbol{t}(d^*a^k)=0 &\quad \text{for all } k\in\mathbb{N},
\end{aligned}
$$

where the $\ell_{I'}$ are $\mathscr{C}^\infty(\partial\Omega)$-linear combinations of the $a^k_{I_n}$ (for I_n in $\mathscr{I}_n$) which do not depend on the $a^k_{I'}$ (for I' in $\mathscr{I}'$). Note also, according to Lemma 4.5, that the first line of the last system reads

$$
-\boldsymbol{n}(d+d^*)^2a^{k-1}+\sum_{\ell=1}^{k}E_\ell a^{k-\ell}=(2\mathscr{L}_{\nabla\tilde\Phi}\otimes\mathrm{Id}+\mathscr{R}^T_{\mathrm{Dir}})a^k_{I_n}\,dx^{I_n}. \tag{4-47}
$$

Moreover, since $dx^i=d\bar x^i$ for $i\in\{1,\dots,n-1\}$ at the point U, it follows from Corollary B.5, (4-5), and the results in [Helffer and Sjöstrand 1985, pages 271–275] that $\mathscr{R}^T_{\mathrm{Dir}}(0)$ restricted to normal forms is symmetric with the one-dimensional kernel $\mathbb{R}dx^1\wedge\cdots\wedge dx^{p-1}\wedge dx^n$.

Since $a^k_{I_n}\,dx^{I_n}$ is normal and $2\mathscr{L}_{\nabla\tilde\Phi}\otimes\mathrm{Id}$ only differentiates the $a^k_{I_n}$ tangentially, because

$$
(2\mathscr{L}_{\nabla\tilde\Phi}\otimes\mathrm{Id})a^k_{I_n}\,dx^{I_n}=\sum_{i=1}^{n-1}\frac{\partial a^k_{I_n}}{\partial x^i}(\nabla\tilde\Phi)_i\,dx^{I_n},
$$

it turns out (4-47) can be rewritten as a tangential system that can be solved according to the analysis of the boundaryless case done in [Helffer and Sjöstrand 1985]. Here are the details: thanks to Lemma 4.5, the complete system becomes equivalent to

$$
\left.
\begin{aligned}
(2\mathscr{L}_{\nabla\tilde\Phi}\otimes\mathrm{Id}+\mathscr{R}^T_{\mathrm{Dir}})a^k_{I_n}\,dx^{I_n} &= -\boldsymbol{n}(d+d^*)^2a^{k-1}+\sum_{\ell=1}^{k-1}E_\ell a^{k-\ell}+E_ka^0 \text{ on } \partial\Omega,\\
(2\mathscr{L}_{\nabla\Phi}\otimes\mathrm{Id}+\mathscr{R})a^k &= -(d+d^*)^2a^{k-1}+\sum_{\ell=1}^{k}E_\ell a^{k-\ell} \qquad \text{on } \overline\Omega,\\
a_{I'}|_{\partial\Omega}\equiv 0 &\qquad\qquad \text{for all } I'\in\mathscr{I}'.
\end{aligned}
\right\}\tag{$\mathscr{S}_{\mathrm{Dir}}$}
$$

The first line is again a homogeneous degenerate matrix transport equation which can be solved following [Helffer and Sjöstrand 1985; Helffer 1988]: for $k=0$, take

$$
a^0(0)=dx^1\wedge\cdots\wedge dx^{p-1}\wedge dx^n\in\mathrm{Ker}(\mathscr{R}^T_{\mathrm{Dir}}(0)). \tag{4-49}
$$

The formulation of Theorem 1.2 is just a coordinate-free rewriting of this condition for $a^0(U)$. Recall that, *in our coordinate system* (see Proposition 4.4), at $U\cong 0$,

$$
\mathscr{R}^T_{\mathrm{Dir}}(U)=2\begin{pmatrix} & & 0\\ & \mathrm{Hess}(f|_{\partial\Omega})(U) & \vdots\\ 0 & \cdots & 0\end{pmatrix}^{(p)}-\mathrm{Tr}(\mathrm{Hess}(f|_{\partial\Omega}-\Phi|_{\partial\Omega})(U))
=2A^{(p)}-\mathrm{Tr}(\mathrm{Hess}(f|_{\partial\Omega}-\Phi|_{\partial\Omega})(U)),
$$

and that, for a normal p-form $dx^{I_n} = dx^{i_1} \wedge \cdots \wedge dx^{i_{p-1}} \wedge dx^n$,

$$A^{(p)} dx^{I_n} = (A\, dx^{i_1}) \wedge dx^{i_2} \cdots \wedge dx^n + \cdots + dx^{i_1} \wedge \cdots \wedge (A\, dx^{i_{p-1}}) \wedge dx^n + 0,$$

where, for $\ell \in \{1, \ldots, p-1\}$,

$$A\, dx^{i_\ell} = (\mathrm{Hess}(f|_{\partial\Omega})(U))\ dx^{i_\ell} + 0.dx^n = \big(\mathrm{Hess}(f|_{\partial\Omega})(U)\, j^*\big)\ dx^{i_\ell}.$$

Finally, as in the analogous part of the proof in the Neumann case, the writing $(\mathrm{Hess}(f|_{\partial\Omega})(U))\, dx^{i_\ell}$ is a slight abuse of notation, the proper one being $(\mathrm{Hess}(f|_{\partial\Omega})(U)\, j^*)\, dx^{i_\ell}$.

Then, for $k > 0$, choose E_k such that the compatibility condition

$$-\boldsymbol{n}(d + d^*)^2 a^{k-1}(0) + \sum_{\ell=1}^{k-1} E_\ell a^{k-\ell}(0) + E_k a^0(0) \in \big(\mathrm{Ker}(\mathcal{R}^T_{\mathrm{Dir}}(0))\big)^\perp$$

is satisfied and take $a^k_{I_n}(0)\, dx^{I_n}$ in

$$(\mathcal{R}^T_{\mathrm{Dir}}(0))^{-1}\Big(-\boldsymbol{n}(d + d^*)^2 a^{k-1}(0) + \sum_{\ell=1}^{k-1} E_\ell a^{k-\ell}(0) + E_k a^0(0)\Big).$$

Thus, at every step $k \in \mathbb{N}$, the first and the third line of $(\mathcal{S}_{\mathrm{Dir}})$ fully determine the Cauchy data $a^k(x', 0)$ and the number E_k. The second line solves the interior problem with these Cauchy data and contains, with the two other lines, thanks to Lemma 4.5, the second trace condition (4-44). Checking $E(h) = O(h^\infty)$ is then identical to the end of the proof of Theorem 1.1 done in Section 4C after choosing a cut-off function χ which satisfies $\nabla\chi = \nabla_T \chi$ on the boundary $\partial\Omega$. □

Appendices: Computations in adapted local coordinate systems

In the two appendices below we work in an adapted local coordinate system (x', x^n) around $U \in \partial\Omega$ so as to be able to apply the results both to the Neumann and Dirichlet cases.

Appendix A. A modified Agmon distance

Define $\tilde{\Phi}$ around U in the coordinates (x', x^n) by

$$\tilde{\Phi}(x', x^n) = \Phi(x', 0) \quad \text{for all } x = (x', x^n), \tag{A-1}$$

and note the following relation satisfied for all x around U, in the coordinates (x', x^n), due to the form of $G_0^{\pm 1}$ (see Remark 3.3):

$$d\tilde{\Phi}(x) = d_T \tilde{\Phi}(x) + \frac{\partial \tilde{\Phi}}{\partial x^n}(x) dx^n = d_T \tilde{\Phi}(x),$$

$$\nabla \tilde{\Phi}(x) = \nabla_T \tilde{\Phi}(x) + \frac{\partial \tilde{\Phi}}{\partial x^n}(x) \frac{\partial}{\partial x^n} = \nabla_T \tilde{\Phi}(x).$$

For a vector (or a vector field) $X = \sum_{i=1}^n X_i \frac{\partial}{\partial x^i}$, with the identification $X = \begin{pmatrix} X_1 \\ \vdots \\ X_N \end{pmatrix}$, the tangential and normal parts of X are defined as

$$X_T = \begin{pmatrix} X_1 \\ \vdots \\ X_{n-1} \\ 0 \end{pmatrix}, \qquad X_N = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ X_n \end{pmatrix}.$$

Similarly, for a (n,n)-matrix $A(x) = (a_{ij}(x))_{i,j}$, define $A_T(x)$ and $A_N(x)$ by

$$A_T = \begin{pmatrix} & & & 0 \\ & A' & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & a_{nn} \end{pmatrix}, \qquad A_N = \begin{pmatrix} & & & a_{1n} \\ & [0] & & \vdots \\ & & & a_{n-1n} \\ a_{n1} & \cdots & a_{nn-1} & 0 \end{pmatrix}.$$

Recall moreover that, for a vector (or a vector field) X and a $\mathscr{C}^\infty$ function ψ, the identification $\langle \nabla\psi \mid X\rangle_{g_0} = d\psi(X)$ leads to

$$\nabla\psi = G_0^{-1} \begin{pmatrix} \partial\psi/\partial x^1 \\ \vdots \\ \partial\psi/\partial x^n \end{pmatrix}.$$

Hence, due to the form of G_0^{-1} (see Remark 3.3), the following relations are satisfied:

$$(\nabla\psi)_T = \nabla_T\psi = G_0^{-1} \begin{pmatrix} \partial\psi/\partial x^1 \\ \vdots \\ \partial\psi/\partial x^{n-1} \\ 0 \end{pmatrix}, \qquad (\nabla\psi)_N = \frac{\partial\psi}{\partial x^n}\frac{\partial}{\partial x^n} = G_0^{-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \partial\psi/\partial x^n \end{pmatrix}.$$

In the Neumann case, we will compare $\mathscr{L}_{\nabla\Phi}$ and $\mathscr{L}_{\nabla\tilde{\Phi}}$ and the following relations can sometimes be convenient:

$$\nabla\Phi - \nabla\tilde{\Phi} = G_0^{-1} \begin{pmatrix} \frac{\partial\Phi}{\partial x^1}(x) - \frac{\partial\Phi}{\partial x^1}(x',0) \\ \vdots \\ \frac{\partial\Phi}{\partial x^{n-1}}(x) - \frac{\partial\Phi}{\partial x^{n-1}}(x',0) \\ \frac{\partial\Phi}{\partial x^n}(x) \end{pmatrix}, \tag{A-2}$$

$$\nabla_T\Phi - \nabla\tilde{\Phi} = G_0^{-1} \begin{pmatrix} \frac{\partial\Phi}{\partial x^1}(x) - \frac{\partial\Phi}{\partial x^1}(x',0) \\ \vdots \\ \frac{\partial\Phi}{\partial x^{n-1}}(x) - \frac{\partial\Phi}{\partial x^{n-1}}(x',0) \\ 0 \end{pmatrix}. \tag{A-3}$$

We will compare $\mathscr{L}^*_{\nabla\Phi}$ and $\mathscr{L}^*_{\nabla\tilde{\Phi}}$ in the Dirichlet case and the following relations are also convenient:

$$d\Phi - d\tilde{\Phi} = \sum_{i=1}^{n-1}\Big(\frac{\partial\Phi}{\partial x^i}(x) - \frac{\partial\Phi}{\partial x^i}(x',0)\Big)\,dx^i + \frac{\partial\Phi}{\partial x^n}(x)\,dx^n, \tag{A-4}$$

$$d_T\Phi - d\tilde{\Phi} = \sum_{i=1}^{n-1}\Big(\frac{\partial\Phi}{\partial x^i}(x) - \frac{\partial\Phi}{\partial x^i}(x',0)\Big)\,dx^i. \tag{A-5}$$

Appendix B. About $\mathscr{L}+\mathscr{L}^*$

B1. ***For a general $\mathscr{C}^\infty$ function h.*** Here we give similar results to those found in [Helffer and Sjöstrand 1985, Appendix A].

Let h be a $\mathscr{C}^\infty$ function from $\overline{\Omega}$ to $\mathbb{R}$ and write

$$\nabla h = \sum_{i=1}^{n}(\nabla h)_i\frac{\partial}{\partial x^i}.$$

Following [Helffer and Sjöstrand 1985], we make the following algebraic definition:

Definition B.1. For a Euclidean space $(E,\langle\cdot\,|\,\cdot\rangle)$ and $A\in\mathscr{L}(E)$, $A^{(p)}$ and $\Gamma^{(p)}(A)$ denote respectively the linear application $A^{(p)}\in\mathscr{L}(\Lambda^pE)$ and the application $\Gamma^{(p)}(A) = A\otimes\cdots\otimes A$:

$$\begin{aligned}A^{(p)}(\omega_1\wedge\cdots\wedge\omega_p) &= (A\omega_1\wedge\cdots\wedge\omega_p)+\cdots+(\omega_1\wedge\cdots\wedge A\omega_p),\\ \Gamma^{(p)}(A)(\omega_1\wedge\cdots\wedge\omega_p) &= (A\omega_1)\wedge\cdots\wedge(A\omega_p),\end{aligned}$$

with the obvious convention $A^{(0)}=0$ and $\Gamma^{(0)}(A)=1$.

Remark B.2. Under the canonical identification $\Lambda^1E=E$, note that $A^{(1)}=A$. Moreover, if A^* denotes the adjoint of A according to the scalar product on E, the adjoint of $A^{(p)}$ is simply $(A^{(p)})^*=(A^*)^{(p)}=:A^{(p),*}$. Recall that Λ^pE is a Euclidean space with the scalar product $\langle\cdot\,|\,\cdot\rangle_p$:

$$\langle\omega_1\wedge\cdots\wedge\omega_p\,|\,\mu_1\wedge\cdots\wedge\mu_p\rangle_p = \det\big(\langle\omega_i\,|\,\mu_j\rangle\big)_{i,j}\,.$$

We also remark that for a p-form $a_I^k\,dx^I = a_{I'}^k\,dx^{I'}+a_{I_n}^k\,dx^{I_n}$ we have, in the notation of Appendix A,

$$A^{(p)} = A_T^{(p)}+A_N^{(p)}$$

and

$$\begin{aligned}\boldsymbol{t}(A^{(p)}(a_I^k\,dx^I)) &= a_{I'}^k(x',0)A_T^{(p)}(dx^{I'})+a_{I_n}^k(x',0)A_N^{(p)}(dx^{I_n}),\\ \boldsymbol{n}(A^{(p)}(a_I^k\,dx^I)) &= a_{I_n}^k(x',0)A_T^{(p)}(dx^{I_n})+a_{I'}^k(x',0)A_N^{(p)}(dx^{I'}).\end{aligned}$$

For any order-zero differential operator $\mathscr{A}=A^{(p)}+\psi\,\mathrm{Id}$, where ψ is a $\mathscr{C}^\infty$ function, we define the order-zero differential operators

$$\mathscr{A}^T = A_T^{(p)}+\psi\,\mathrm{Id}\quad\text{and}\quad\mathscr{A}^N = A_N^{(p)}.$$

(If $\psi\equiv 0$ then $\mathscr{A}^T$ coincides with $A^{(p)}_T$ and $\mathscr{A}^N$ with $A^{(p)}_N$.) Our aim is to work with tangential forms in the Neumann case (i.e., $a^k_I dx^I = a^k_{I'} dx^{I'}$ on $\partial\Omega$) and with normal forms in the Dirichlet case (i.e., $a^k_I dx^I = a^k_{I_n} dx^{I_n}$ on $\partial\Omega$). Hence, for any tangential form in the Neumann case we write

$$\begin{aligned} \boldsymbol{t}(\mathscr{A}(a^k_I dx^I)) &= a^k_{I'}(x',0)A^{(p)}_T(dx^{I'}) + \psi(x',0)a^k_{I'}(x',0)\,dx^{I'} = \boldsymbol{t}(\mathscr{A}^T(a^k_I dx^I)),\\ \boldsymbol{n}(\mathscr{A}(a^k_I dx^I)) &= a^k_{I'}(x',0)A^{(p)}_N(dx^{I'}) = \boldsymbol{n}(\mathscr{A}^N(a^k_I dx^I)), \end{aligned} \tag{B-1}$$

and for any normal form in the Dirichlet case, we write

$$\begin{aligned} \boldsymbol{t}(\mathscr{A}(a^k_I dx^I)) &= a^k_{I_n}(x',0)A^{(p)}_N(dx^{I'}) = \boldsymbol{t}(\mathscr{A}^N(a^k_I dx^I)),\\ \boldsymbol{n}(\mathscr{A}(a^k_I dx^I)) &= a^k_{I_n}(x',0)A^{(p)}_T(dx^{I_n}) + \psi(x',0)a^k_{I_n}(x',0)\,dx^{I_n} = \boldsymbol{n}(\mathscr{A}^T(a^k_I dx^I)). \end{aligned} \tag{B-2}$$

Proposition B.3. *In the coordinates (x',x^n), we have $\mathscr{L}_{\nabla h} = \mathscr{L}_{\nabla h}\otimes \mathrm{Id} + \mathscr{R}_h$ and*

$$\mathscr{L}_{\nabla h} + \mathscr{L}^*_{\nabla h} = \mathscr{R}_h + \mathscr{R}^*_h - \left(\sum_{i=1}^n \left(\frac{\partial(\nabla h)_i}{\partial x^i} + \frac{1}{2}(\nabla h)_i \frac{\partial \ln\det G_0}{\partial x^i}\right)\right)\mathrm{Id} - \sum_{i=1}^n (\nabla h)_i \left(G_0 \frac{\partial (G_0^{-1})}{\partial x^i}\right)^{(p)},$$

where $(\mathscr{L}_{\nabla h}\otimes\mathrm{Id})a^k_I dx^I = (\mathscr{L}_{\nabla h}(a^k_I))\,dx^I$, $\mathscr{R}_h$ is the order-zero differential operator given by the matrix

$$\mathscr{R}_h(x) = \left(\frac{\partial(\nabla h)_j}{\partial x^i}\right)^{(p)}_{i,j} =: A^{(p)}_h,$$

and $\left(\frac{\partial(\nabla h)_j}{\partial x^i}\right)_{i,j}$ and $G_0\frac{\partial(G_0^{-1})}{\partial x^i}$ are viewed as endomorphisms of $T^*_x\overline{\Omega}$. Further, the matrix of $\mathscr{R}^*_h$ is

$$\mathscr{R}^*_h := A^{(p),*}_h = (G_0\,{}^t\!A_h G_0^{-1})^{(p)}.$$

Remark B.4. According to the computations in Appendix A, $(\nabla h)_n = \partial h/\partial x^n$. Moreover, due to the form of $G_0^{\pm 1}$, note that

$$\mathscr{R}_h + \mathscr{R}^*_h - \sum_{i=1}^n (\nabla h)_i \left(G_0 \frac{\partial [G_0^{-1}]}{\partial x^i}\right)^{(p)}$$

is given by the matrix

$$\begin{pmatrix} A'_h + G'_0\,{}^t\!A'_h G_0^{-1'} - \sum_{i=1}^n (\nabla h)_i G'_0 \dfrac{\partial[G_0^{-1'}]}{\partial x^i} & \left(\dfrac{\partial^2 h}{\partial x^n \partial x^i}\right)_{i,1} + G'_0\left(\dfrac{\partial(\nabla h)_i}{\partial x^n}\right)_{i,1} \\ \left(\dfrac{\partial(\nabla h)_j}{\partial x^n}\right)_{1,j} + \left(\dfrac{\partial^2 h}{\partial x^n\partial x^j}\right)_{1,j} G_0^{-1'} & \dfrac{\partial^2 h}{(\partial x^n)^2} \end{pmatrix}^{(p)}.$$

Corollary B.5. *In the coordinates (x',x^n), assume that the function h admits a critical point at 0, that $\partial h/\partial x^n \equiv 0$ on the boundary $\partial\Omega$, and that $((\partial^2 h)/(\partial x^n)^2)(0) = 0$. Then the following relations are true:*

$$\mathscr{R}_h(0) = \mathscr{R}^*_h(0) = \begin{pmatrix} & & 0 \\ & \mathrm{Hess}(h|_{\partial\Omega})(0) & \vdots \\ 0 & \cdots & 0 \end{pmatrix}^{(p)}$$

and

$$(\mathscr{L}_{\nabla h}+\mathscr{L}^*_{\nabla h})(0)=2\mathscr{R}_h(0)-\mathrm{Tr}(\mathrm{Hess}(h|_{\partial\Omega})(0))\,\mathrm{Id}\,.$$

Proof. Since (x',x^n) are adapted local coordinates around $U\cong 0$ and 0 is a critical point of h, note first that, for all i in $\{1,\dots,n\}$,

$$(\nabla h)_i=\sum_{j=1}^n g^{ij}\frac{\partial h}{\partial x^j}=\frac{\partial h}{\partial x^i}+\mathbb{O}(|x|^2).$$

This implies

$$\mathscr{R}_h(x)=\Big(\frac{\partial(\nabla h)_j}{\partial x^i}\Big)^{(p)}_{i,j}=(\mathrm{Hess}(h))^{(p)}+\mathbb{O}(|x|).$$

At 0, in particular, since $\partial h/\partial x^n\equiv 0$ on the boundary and $((\partial^2 h)/(\partial x^n)^2)(0)=0$, we have

$$\mathscr{R}_h(0)=\begin{pmatrix} & & 0\\ & \mathrm{Hess}(h|_{\partial\Omega})(0) & \vdots\\ 0 & \cdots & 0\end{pmatrix}^{(p)}.$$

Moreover, we deduce from $G_0^{\pm1}(0)=\mathrm{Id}_n$ and the symmetry of $\mathrm{Hess}(h|_{\partial\Omega})(0)$,

$$\mathscr{R}^*_h(0)=\mathscr{R}_h(0).$$

At last, we obtain from $\dfrac{\partial^2 h}{(\partial x^n)^2}(0)=0$ that

$$-\Big(\sum_{i=1}^n\frac{\partial(\nabla h)_i}{\partial x^i}\Big)\,\mathrm{Id}=-\,\mathrm{Tr}\,(\mathrm{Hess}(h|_{\partial\Omega})(0))\quad\text{at }0,$$

which leads to the end of the proof, using that, for all i in $\{1,\dots,n\}$,

$$(\nabla h)_i(0)=\frac{\partial h}{\partial x^i}(0)=0.$$ ☐

Proof of Proposition B.3. The first equality is proved in [Helffer and Sjöstrand 1985, pages 334–336]. There is also a proof of the second equality in the same paper, but we need to be more precise here. From the first equality, let us deduce

$$\mathscr{L}^*_{\nabla h}=(\mathscr{L}_{\nabla h}\otimes\mathrm{Id})^*+\mathscr{R}^*_h.$$

Remarking that the scalar product of two p-forms ω and η is given by

$$\langle\omega\mid\eta\rangle_{g_0}=\langle\omega\mid\Gamma^{(p)}(G_0^{-1})\eta\rangle_{g_e},$$

where g_e is the Euclidean metric $\sum_{i=1}^n(dx^i)^2$, we obtain

$$\mathscr{R}^*_h=\Gamma^{(p)}(G_0)({}^t\!A_h)^{(p)}\Gamma^{(p)}(G_0^{-1})=(G_0{}^t\!A_hG_0^{-1})^{(p)}.$$

Now look at the term $(\mathscr{L}_{\nabla h}\otimes\mathrm{Id})^*$. Take first two p-forms $\alpha\omega$ and $\beta\eta$ where α, β are $\mathscr{C}^\infty_0(\Omega,\mathbb{R})$ functions, and ω, η are two p-forms dx^I and dx^J. Denoting by $V_{g_0}(dx)$ the normalized volume form,

$V_{g_0}(dx)$ satisfies

$$V_{g_0}(dx) = (\det G_0(x))^{1/2} dx^1 \wedge \cdots \wedge dx^n =: \nu(x)\, dx^1 \wedge \cdots \wedge dx^n.$$

Hence we deduce

$$\langle \alpha\omega \mid (\mathscr{L}_{\nabla h} \otimes \mathrm{Id})^* \beta\eta \rangle_{g_0} = \langle \mathscr{L}_{\nabla h}(\alpha)\omega \mid \eta \rangle_{g_0} = \int (\mathscr{L}_{\nabla h}(\alpha))\beta \langle \omega \mid \eta \rangle_{g_0(x)} (\det G_0(x))^{1/2} dx.$$

Using Cartan's formula (2-6), $\mathscr{L}_{\nabla h}(\alpha) = d\alpha(\nabla h) = \sum_{i=1}^n \frac{\partial \alpha}{\partial x^i} (\nabla h)_i$ and we obtain

$$\begin{aligned}\int (\mathscr{L}_{\nabla h}(\alpha))\beta \langle \omega \mid \eta \rangle_{g_0(x)} (\det G_0(x))^{1/2} dx &= \int \Big(\sum_{i=1}^n \frac{\partial \alpha}{\partial x^i} (\nabla h)_i \beta \Big) \langle \omega \mid \eta \rangle_{g_0(x)} \nu \, dx \\ &= -\int \alpha \sum_{i=1}^n \frac{\partial}{\partial x^i} \big((\nabla h)_i \beta \langle \omega \mid \eta \rangle_{g_0(x)} \nu \big)\, dx.\end{aligned}$$

Now write

$$\begin{aligned}&\int \alpha \sum_{i=1}^n \frac{\partial}{\partial x^i} \big((\nabla h)_i \beta \langle \omega \mid \eta \rangle_{g_0(x)} \nu \big)\, dx \\ &\quad = -\int \alpha \sum_{i=1}^n \Big(\frac{\partial (\nabla h)_i}{\partial x^i} \beta \langle \omega \mid \eta \rangle_{g_0(x)} \nu \Big)\, dx - \int \alpha \sum_{i=1}^n \Big((\nabla h)_i \frac{\partial \beta}{\partial x^i} \langle \omega \mid \eta \rangle_{g_0(x)} \nu \Big)\, dx \\ &\qquad\qquad - \int \alpha \sum_{i=1}^n \Big((\nabla h)_i \beta \frac{\partial}{\partial x^i} (\langle \omega \mid \eta \rangle_{g_0(x)}) \nu \Big)\, dx - \int \alpha \sum_{i=1}^n \Big((\nabla h)_i \beta \langle \omega \mid \eta \rangle_{g_0(x)} \frac{\partial \nu}{\partial x^i} \Big)\, dx \\ &\quad = -\int \alpha \sum_{i=1}^n \Big(\frac{\partial (\nabla h)_i}{\partial x^i} \beta \langle \omega \mid \eta \rangle_{g_0(x)} \nu \Big)\, dx - \int \alpha (\mathscr{L}_{\nabla h}(\beta)) \langle \omega \mid \eta \rangle_{g_0(x)} \nu \, dx \\ &\qquad\qquad - \int \alpha \sum_{i=1}^n \Big((\nabla h)_i \beta \frac{\partial}{\partial x^i} (\langle \omega \mid \eta \rangle_{g_0(x)}) \nu \Big)\, dx - \int \alpha \sum_{i=1}^n \Big((\nabla h)_i \beta \langle \omega \mid \eta \rangle_{g_0(x)} \frac{\partial \nu}{\partial x^i} \Big)\, dx.\end{aligned}$$

Noting that, for all i in $\{1, \dots, n\}$,

$$\begin{aligned}\frac{\partial}{\partial x^i} \Gamma^{(p)}(G_0^{-1}) &= \Big(\frac{\partial G_0^{-1}}{\partial x^i} \otimes G_0^{-1} \otimes \cdots \otimes G_0^{-1} \Big) + \cdots + \Big(G_0^{-1} \otimes \cdots \otimes G_0^{-1} \otimes \frac{\partial G_0^{-1}}{\partial x^i} \Big) \\ &= \Gamma^{(p)}(G_0^{-1}) \Big(G_0 \frac{\partial [G_0^{-1}]}{\partial x^i} \Big)^{(p)},\end{aligned}$$

we deduce that, for all i in $\{1, \dots, n\}$,

$$\frac{\partial}{\partial x^i} \langle \omega \mid \eta \rangle_{g_0(x)} = \Big\langle \omega \,\Big|\, \Big(G_0 \frac{\partial [G_0^{-1}]}{\partial x^i} \Big)^{(p)} \eta \Big\rangle_{g_0(x)}.$$

Consequently,

$$(\mathscr{L}_{\nabla h} \otimes \mathrm{Id})^* = -\mathscr{L}_{\nabla h} \otimes \mathrm{Id} - \Big(\sum_{i=1}^n \Big(\frac{\partial (\nabla h)_i}{\partial x^i} + \frac{(\nabla h)_i}{\nu} \frac{\partial \nu}{\partial x^i} \Big) \Big) \mathrm{Id} - \sum_{i=1}^n (\nabla h)_i \Big(G_0 \frac{\partial [G_0^{-1}]}{\partial x^i} \Big)^{(p)},$$

which leads to the second result of Proposition B.3. □

B2. ***Application to*** $\mathscr{L}_{\nabla\Phi} - \mathscr{L}^*_{\nabla\Phi} + \mathscr{L}_{\nabla f} + \mathscr{L}^*_{\nabla f}$**.** Let us first write

$$\mathscr{L}_{\nabla\Phi} - \mathscr{L}^*_{\nabla\Phi} + \mathscr{L}_{\nabla f} + \mathscr{L}^*_{\nabla f} = 2\mathscr{L}_{\nabla\Phi} + \mathscr{L}_{\nabla(f-\Phi)} + \mathscr{L}^*_{\nabla(f-\Phi)}.$$

By Proposition B.3, we deduce

$$\mathscr{L}^*_{\nabla(f-\Phi)} + \mathscr{L}_{\nabla(f-\Phi)} = \mathscr{R}_1,$$

where $\mathscr{R}_1$ is an order-zero differential operator.

Next, using the first equality of Proposition B.3, we get

$$2\mathscr{L}_{\nabla\Phi} = 2\mathscr{L}_{\nabla\Phi} \otimes \mathrm{Id} + \mathscr{R}_2,$$

where $\mathscr{R}_2$ is an order-zero differential operator too.

Consequently, setting $\mathscr{R} = \mathscr{R}_1 + \mathscr{R}_2$, we obtain

$$\mathscr{L}_{\nabla\Phi} - \mathscr{L}^*_{\nabla\Phi} + \mathscr{L}_{\nabla f} + \mathscr{L}^*_{\nabla f} = 2\mathscr{L}_{\nabla\Phi} \otimes \mathrm{Id} + \mathscr{R},$$

where $\mathscr{R}$ is an order-zero differential operator.

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