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MEAN CURVATURE MOTION OF GRAPHS WITH CONSTANT CONTACT ANGLE AT A FREE BOUNDARY

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We consider the motion by mean curvature of an n-dimensional graph over a time-dependent domain in $\mathbb{R}^n$ intersecting $\mathbb{R}^n$ at a constant angle. In the general case, we prove local existence for the corresponding quasilinear parabolic equation with a free boundary and derive a continuation criterion based on the second fundamental form. If the initial graph is concave, we show this is preserved and that the solution exists only for finite time. This corresponds to a symmetric version of mean curvature motion of a network of hypersurfaces with triple junctions with constant contact angle at the junctions.

1. Time-dependent graphs with a contact angle condition

We consider a moving hypersurface Σ_t in $\mathbb{R}^{n+1}$ with normal velocity equal to its mean curvature. We assume Σ_t to be a graph over a time-dependent open set $D(t) \subset \mathbb{R}^n$, not necessarily bounded or connected. The (properly embedded) intersection $(n-1)$-submanifold

$$\Gamma(t) = \Sigma_t \cap \mathbb{R}^n = \partial D(t)$$

is a *moving boundary*. Along $\Gamma(t)$ we impose a constant-angle condition

$$\langle N, e_{n+1} \rangle_{|\Gamma(t)} = \beta,$$

where $0 < \beta < 1$ is a constant and N is the upward unit normal of Σ_t. *Mean curvature motion* (mcm) is defined by the law

$$V_N = H,$$

where $V_N = \langle V, N \rangle$, with $V = \partial_t F$ the velocity vector in a given parametrization $F(t)$ of Σ_t (V depends on the parametrization, while V_N does not). A particular parametrization yields *mean curvature flow*:

$$\partial_t F = HN.$$

For graphs, it is natural to consider *graph mean curvature motion*: If $\Sigma_t =$ graph $w(t)$ for a function $w(t) : D(t) \to \mathbb{R}$, imposing $\langle \partial_t F, N \rangle = H$ with $F(y, t) = [y, w(y, t)]$ for $y \in D(t)$, we find

$$w_t = \sqrt{1 + |Dw|^2}\, H$$

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(and the velocity is vertical, $\partial_t F = w_t e_{n+1}$). With the contact angle condition, we obtain a free boundary problem for a quasilinear PDE

$$\begin{cases} w_t = g^{ij}(Dw)w_{ij} & \text{in} \quad D(t), \\ w = 0, \quad \beta\sqrt{1+|Dw|^2} = 1 & \text{on} \quad \partial D(t), \end{cases}$$

where $g^{ij}(Dw) = \delta^{ij} - w_i w_j/(1+|Dw|^2)$ is the inverse metric matrix.

Remark. It is easy to see that the constant-angle boundary condition is incompatible with mean curvature flow parametrized over a fixed domain D_0: on ∂D_0 we would have $\langle F, e_{n+1}\rangle = 0$, leading to $\langle \partial_t F, e_{n+1}\rangle = 0$, which is incompatible with $\partial_t F = HN$ and $\langle N, e_{n+1}\rangle = \beta$. If we parametrize over a time-dependent domain, mean curvature flow leads to a normal velocity for the moving boundary that is difficult to control; hence we chose to analyze the geometry of the motion in terms of the graph mcm parametrization.

To establish short-time existence (in parabolic Hölder spaces) we will work with a third parametrization of the motion, defined over a fixed domain:

$$F(t) : D_0 \to \mathbb{R}^{n+1}, \quad F(x,t) = [\varphi(x,t), u(x,t)] \in \mathbb{R}^n \times \mathbb{R},$$

where $\varphi(t) : D_0 \to D(t)$ is a diffeomorphism and F is a solution of the parabolic system

$$F_t = g^{ij}(DF)F_{ij},$$

where $g_{ij} = \langle F_i, F_j\rangle$ is the induced metric on Σ_t and g^{ij} is the inverse metric matrix.

In the first part of the paper (Sections 3 to 8) we prove the following short-time existence theorem (on $Q := D_0 \times [0, T]$), where by *boundary-orthogonal* we mean that certain orthogonality conditions at the boundary, specified in Section 3, are satisfied.

Theorem 1.1. *Let $\Sigma_0 \subset \mathbb{R}^{n+1}$ be a $C^{3+\bar{\alpha}}$ graph over $D_0 \subset \mathbb{R}^n$ satisfying the contact and angle conditions at ∂D_0. There exist $T > 0$ depending only on Σ_0, a parametrization $F_0 = [\varphi_0, u_0] \in C^{2+\alpha}(D_0)$ of Σ_0 (where $\alpha = \bar{\alpha}^2$ and φ_0 is a boundary-orthogonal diffeomorphism of D_0), and a unique solution $F \in C^{2+\alpha,1+\alpha/2}(Q^T; \mathbb{R}^{n+1})$ of the system*

$$\begin{cases} \partial_t F - g^{ij}(DF)\partial_i\partial_j F = 0, & F = [\varphi, u] \in \mathbb{R}^n \times \mathbb{R}, \\ u_{|\partial D_0} = 0, & N^{n+1}(DF)_{|\partial D_0} = \beta, \end{cases}$$

with initial data F_0, where $\varphi(t) : D_0 \to D(t) \subset \mathbb{R}^n$ is a boundary-orthogonal diffeomorphism as well.

The system and boundary conditions are discussed in more detail in Section 3. Sections 4, 5, and 6 deal with compatibility at $t = 0$, linearization and the verification that the boundary conditions satisfy complementarity. In particular, adjusting the initial diffeomorphism φ_0 to ensure compatibility (Section 4) leads to the loss of differentiability seen in Theorem 1.1. The required estimates in Hölder spaces for the linearized system are described in Section 7 and the proof is concluded (by a fixed-point argument) in Section 8. While the general scheme is standard, details are included since we are dealing with a free boundary problem with somewhat nonstandard boundary conditions. Free boundary-type problems for mean curvature motion of graphs have apparently not been considered previously.

We describe the evolution equations in the rotationally symmetric case in Section 9 (including a stationary example for the exterior problem) and the extension to the case of a graph motion Σ_t intersecting fixed support hypersurfaces orthogonally in Section 10.

The original motivation for this work was to establish (by classical parabolic PDE methods) existence-uniqueness for mean curvature motion of networks of surfaces meeting along triple junctions with constant-angle conditions. One can use a motion Σ_t of graphs with constant contact angle to produce examples of *triple junction motion*: three hypersurfaces moving by mean curvature meeting along an $(n-1)$-dimensional submanifold $\Sigma(t)$ so that the three normals make constant angles (say, 120 degrees) along $\Gamma(t)$. We simply reflect on $\mathbb{R}^n$, so the hypersurfaces are Σ_t, $\bar{S}igma_t$, and $\mathbb{R}^n - \bar{D}(t)$. If $\Sigma_t = \text{graph } w(t)$ with $w > 0$, the system is *embedded* in $\mathbb{R}^{n+1}$. This is mean curvature motion of a "symmetric triple junction of graphs".

Short-time existence holds for general triple junctions of graphs moving by mean curvature with constant 120-degree angles at the junction, provided a compatibility condition holds along the junction (see Section 16). Since the free-boundary problem is easier to understand in the symmetric case, we decided to do this first. In addition, in the present case it is possible to go further towards a geometric global existence result. In the second part of the paper (Sections 11–15), motivated by recent work on lens-type curve networks [Schnürer et al. 2007], we consider continuation criteria and the preservation of concavity. Since we chose to develop these results for graph motion with a free boundary, although the general lines of proof (via maximum principles) have precedents, the details of the arguments are new. For example, Section 12 contains an extension of the maximum principle for symmetric tensors with Neumann-type boundary conditions given in [Stahl 1996], which in our setting allows one to show preservation of weak concavity in general. Section 14 includes a continuation criterion for the flow. The results obtained in Sections 11–15 are summarized in the following theorem, where h denotes the second fundamental form, pulled back to a symmetric 2-tensor on $D(t)$.

Theorem 1.2. *If Σ_0 is weakly concave ($h \le 0$ at $t = 0$), this property is preserved by the evolution. Let $T_{\max}$ be the maximal existence time for the evolution. If the mean curvature of Σ_0 is strictly negative ($\sup_{\Sigma_0} H = H_0 < 0$), then $T_{\max}$ is finite. Assuming $T_{\max} < \infty$, we have*

$$\limsup_{t \to T_{\max}} [\sup_{\Gamma_t} (|h|_g + |\nabla^{\tan} h^{\tan}|_g)] = \infty$$

(if $n = 2$, in the concave case). If there is no gradient blowup at $T_{\max}$, the hypersurface contracts to a compact convex subset of $\mathbb{R}^n$ as $t \to T_{\max}$.

Remark. We have not yet proved that the diameter tends to zero as $t \to T_{\max}$, though this seems likely based on the experience with curves [Schnürer et al. 2007], in the absence of gradient blowup. It is an interesting question (even in the concave case, for $n = 2$) whether gradient blowup can really occur, that is, whether $\sup_{\Gamma_t} |\nabla^{\tan} h^{\tan}|_g$ can diverge as $t \to T_{\max}$, while $|h|_g$ remains bounded on Γ_t).

2. Normal velocity of the moving boundary

The evolution is naturally supplied with initial data Σ_0, a graph meeting $\mathbb{R}^{n+1}$ at the prescribed angle. Since we are interested in classical solutions in the parabolic Hölder space $C^{2+\alpha,1+\alpha/2}$, we expect an additional compatibility condition at $t = 0$. We discuss this first for graph mcm $w(y, t)$.

Denote by $\Gamma(t)$ a global parametrization of $\partial D(t)$ (with domain in a fixed manifold, and space variables left implicit). Differentiating in t the *contact condition* $w(\Gamma(t), t) = 0$, we find

$$w_t + \langle Dw, \dot{\Gamma}(t) \rangle = 0.$$

Denote by n_t the unit normal vector field to $\Gamma(t)$, chosen so that the directional derivative $d_n w > 0$. The contact condition also implies the gradient of w is purely normal:

$$Dw_{|\partial D(t)} = (d_{n_t} w) n_t.$$

Combining this with the angle condition, and bearing in mind that $d_{n_t} w_{|\Gamma(t)} > 0$, we find

$$d_{n_t} w = \frac{\beta_0}{\beta} \text{ on } \partial D(t), \quad \beta_0 := \sqrt{1-\beta^2}.$$

(In fact, this is a more convenient form of the angle boundary condition for w, since it is linear.) Thus, on $\partial D(t)$,

$$\frac{1}{\beta} H = \sqrt{1+(d_{n_t} w)^2} H = \frac{\partial w}{\partial t} = -\langle \dot{\Gamma}(t), n_t \rangle d_{n_t} w = -\dot{\Gamma}_n(t) \frac{\beta_0}{\beta},$$

and we find the normal velocity of the moving boundary, independent of the parametrization of Γ_t:

$$\dot{\Gamma}_n = -\frac{1}{\beta_0} H_{|\Gamma(t)}.$$

In particular, this must hold at $t = 0$. Note that we don't get a compatibility condition in the usual sense (of a constraint on the 2-jet of the initial data), but instead an equation of motion for the moving boundary. Later, in the fixed-domain formulation, we will have to deal with a real compatibility condition.

Remark. For more general (nonsymmetric, nonflat) triple junctions with 120-degree angles, the condition

$$H^1 + H^2 = H^3 \quad \text{on } \Gamma(t)$$

must hold at the junction (for graphs, oriented by the upward normal); this gives a geometric constraint on the initial data, for classical evolution in $C^{2+\alpha,1+\alpha/2}$. This automatically holds in the symmetric case ($w^2 = -w^1$, $w^3 \equiv 0$), since $H^3 = 0$ and $H^I = \mathrm{tr}_{g^I} d^2 w^I$ for $I = 1, 2$.

3. Choice of gauge

It is traditional in moving boundary problems to parametrize the time-dependent domain $D(t)$ of the unknown $w(y,t)$ by a time-dependent diffeomorphism:

$$y = \varphi(x,t), \quad \varphi(t) : D_0 \to D(t),$$

and then derive the equation satisfied by the coordinate-changed function from the equation for w; see, for example, [Baconneau and Lunardi 2004; Solonnikov 2003]. Motivated by work on curve networks [Mantegazza et al. 2004], we will, instead, consider a general parametrization

$$F : D_0 \times [0,T] \to \mathbb{R}^{n+1}, \quad F(x,t) = [\varphi(x,t), u(x,t)] \in \mathbb{R}^n \times \mathbb{R},$$

and derive an equation for F directly from the definition of mean curvature motion,

$$\langle \partial_t F, N \rangle = H.$$

We'll still assume $\varphi(t) : D_0 \to D(t)$ is a diffeomorphism.

The first and second fundamental forms are given by

$$g_{ij} = \langle F_i, F_j \rangle, \quad A(F_i, F_j) = \langle F_{ij}, N \rangle,$$

where we have set $DF = F_i e_i$ and $D^2F(e_i, e_j) = F_{ij}$, with (e_i) the standard basis of $\mathbb{R}^{n+1}$. The mean curvature is the trace of A in the induced metric:

$$H = \langle g^{ij}(DF)F_{ij}, N \rangle.$$

The equation for F is

$$\langle \partial_t F - g^{ij}(DF)F_{ij}, N \rangle = 0.$$

There is a natural gauge choice yielding a quasilinear parabolic system

$$\partial_t F - g^{ij}(DF)F_{ij} = 0.$$

We will sometimes refer to this as the *split gauge*, since in terms of the components $F = [\varphi, u]$ we have the essentially decoupled system

$$\begin{cases} \partial_t u - g^{ij}(D\varphi, Du)u_{ij} = 0, \\ \partial_t \varphi - g^{ij}(D\varphi, Du)\varphi_{ij} = 0. \end{cases} .$$

The splitting is useful in stating the boundary conditions

$$\begin{cases} u_{|\partial D_0} = 0 & \text{(contact condition)}, \\ N^{n+1}(D\varphi, Du)_{|\partial D_0} = \beta & \text{(angle condition)}. \end{cases}$$

We immediately see a problem: we have two scalar boundary conditions for $n+1$ unknowns, and no moving boundary to help! Our solution to this is to introduce $n-1$ additional orthogonality conditions at the boundary for the parametrization $\varphi(t)$. We impose

$$\langle D_\tau \varphi, D_n \varphi \rangle_{|\partial D_0} = 0 \quad \text{(orthogonality condition)},$$

for any $\tau \in T\partial D_0$, where n denotes the inward unit normal to D_0. (We fix a tubular neighborhood $\mathcal{N}$ of ∂D_0 and extend n to $\mathcal{N}$ so that $d_n n = 0$ in $\mathcal{N}$.)

Geometrically, the orthogonality boundary condition has a precedent in a method often adopted when dealing with the evolution of hypersurfaces in $\mathbb{R}^{n+1}$ intersecting a fixed n-dimensional support surface orthogonally (see [Struwe 1988], for example), where one replaces vanishing inner product of the unit normals — a single scalar condition — by a stronger Neumann-type condition for the parametrization corresponding to $n-1$ scalar conditions. (More details are given in Section 10.)

The system must also be supplied with initial data. We assume given an initial hypersurface Σ_0, the graph of a $C^{3+\bar{\alpha}}$ function $\tilde{u}_0(x)$ defined in the $C^{3+\bar{\alpha}}$ domain $D_0 \subset \mathbb{R}^n$. (The reason for this choice of differentiability class will be seen later.) It would seem natural to set $\varphi_0 = \mathrm{Id}_{D_0}$, but this causes problems related to compatibility; see Section 4. We do require the 1-jet of φ_0 at the boundary to be that of the identity:

$$\varphi_{0|\partial D_0} = \mathrm{Id}, \quad D\varphi_{0|\partial D} = \mathbb{I}.$$

(In particular, the orthogonality condition holds at $t = 0$.)

We need a more explicit expression for the unit normal, and for that we use the "vector product"

$$\begin{aligned}\tilde{N}(D\varphi, Du) &:= (-1)^n \det\begin{bmatrix} e_1 & \cdots & e_{n+1} \\ DF^1 & \cdots & DF^{n+1}\end{bmatrix} = (-1)^n \det\begin{bmatrix} e_1 & \dots & e_n & e_{n+1} \\ D\varphi^1 & \dots & D\varphi^n & Du\end{bmatrix} \\ &:= [J(D\varphi, Du), J_\varphi] \in \mathbb{R}^n \times \mathbb{R},\end{aligned}$$

where $DF^i \in \mathbb{R}^n$ for $i = 1, \dots n+1$, $J_\varphi > 0$ is the Jacobian of φ and $(-1)^n$ is introduced to make sure the last component is positive. $J(D\varphi, Du)$ is an $\mathbb{R}^n$-valued multilinear form, linear in the components u_i of Du, and of weight $n-1$ in the components of $D\varphi$. It is easy to check that $J(\mathbb{I}, Du) = -Du$. The unit normal is

$$N(D\varphi, Du) = \tilde{N}(D\varphi, Du)/\big(|J(D\varphi, Du)|^2 + (J_\varphi)^2\big)^{1/2}.$$

Thus the angle condition may be stated in the form

$$\beta\big[|J(Du, D\varphi)|^2 + (J_\varphi)^2\big]^{1/2}_{|\partial D_0} = J_{\varphi\,|\partial D_0},$$

and we lose nothing by squaring it:

$$B(D\varphi, Du) := \beta^2 |J(Du, D\varphi)|^2 - \beta_0^2 (J_\varphi)^2_{|\partial D_0} = 0.$$

4. Compatibility and the choice of φ_0

Assume $D\varphi_{0|\partial D_0} = \mathbb{I}$. Differentiating in t the contact condition $u_{|\partial D_0} = 0$ and evaluating at $t = 0$, we find

$$0 = g^{ij}(\mathbb{I}, Du_0) u_{0ij} \equiv g_0^{ij} u_{0ij} \text{ on } \partial D_0.$$

To interpret this condition, consider the mean curvature at $t = 0$, on ∂D_0:

$$H_0 = \frac{1}{v_0}\big[\langle J(\mathbb{I}, Du_0), g_0^{ij}\varphi_{0ij}\rangle + J_{\varphi_0} g_0^{ij} u_{0ij}\big],$$

where

$$v_0 = \big[|J(\mathbb{I}, Du_0)|^2 + J^2_{\varphi_0}\big]^{1/2}_{|\partial D_0} = (|Du_0|^2 + 1)^{1/2}_{|\partial D_0} = \frac{1}{\beta},$$

using the equality

$$J(\mathbb{I}, Du_0) = -Du_0 = -(D_n u_0) n = -\frac{\beta_0}{\beta} n$$

on ∂D_0. (Recall that $\beta_0 := \sqrt{1-\beta^2}$.) Thus the compatibility condition is equivalent to

$$H_{0|\partial D_0} = -\beta_0 g_0^{ij} \langle \varphi_{0ij}, n\rangle_{|\partial D_0}.$$

This implies we can't choose $\varphi_0 \equiv \mathrm{Id}$ (on all of D_0), unless $H_{0|\partial D_0} \equiv 0$, a constraint not present in the geometric problem (as seen above).[1] Instead, regarding H_0 as given (by Σ_0), and using

$$g_0^{ij} = \delta_{ij} - \frac{u_{0i} u_{0j}}{v_0^2} = \delta_{ij} - \beta_0^2 n^i n^j,$$

[1] The compatibility condition $H_{0|\partial D_0} = 0$ does occur for graph mcm with Dirichlet boundary conditions in a mean-convex domain [Huisken 1989].

we find the compatibility constraint

$$\big\langle(\delta_{ij}-\beta_0^2 n^i n^j)\varphi_{0ij}, n\big\rangle = -\frac{1}{\beta_0} H_0 \text{ on } \partial D_0.$$

Given the zero- and first-order constraints on φ_0, this can also be written as

$$n^i n^j \langle \varphi_{0ij}, n\rangle = -\frac{1}{\beta^2\beta_0} H_0 \text{ on } \partial D_0.$$

The next lemma, whose proof is given in Appendix A, shows that this can be solved.

Lemma 4.1. *Let $D_0 \subset \mathbb{R}^n$ be a uniformly $C^{3+\alpha}$ domain (possibly unbounded), $h \in C^\alpha(\partial D_0)$* $(0<\alpha<1)$.

(i) *One can find a diffeomorphism $\varphi \in \mathrm{Diff}^{2+\alpha}(D_0)$ satisfying on ∂D_0*

$$\varphi = \mathrm{Id}, \quad d\varphi = \mathbb{I}, \quad n\cdot d^2\varphi(n,n) = h.$$

(ii) *More generally, given a nonvanishing vector field*

$$e \in C^{1+\alpha}(\partial D_0; \mathbb{R}^n)$$

with $\langle e, u\rangle \neq 0$ on ∂D_0, one can find $\varphi \in \mathrm{Diff}^{2+\alpha}(D_0)$ satisfying on ∂D_0

$$\varphi = \mathrm{Id}, \quad d_n\varphi = e, \quad n\cdot d^2\varphi(n,n) = h.$$

If ∂D_0 has two components, we may even require φ to satisfy the conditions in parts (i) *and* (ii) *at the two components with different functions h.* (This will be needed in Section 10).

As usual, a domain is *uniformly* $C^{3+\alpha}$ if at each boundary point there are local charts to the upper half-space (of class $C^{3+\alpha}$), defined on balls of uniform radius, and with uniform bounds on the $C^{3+\alpha}$ norms of the charts and their inverses.

Remark 4.2. In particular, φ satisfies the orthogonality conditions at ∂D_0.

Remark 4.3. It is at this step in the proof that we have a drop in regularity: for $C^{2+\alpha}$ local solutions, we require $C^{3+\alpha}$ initial data. While this is not unexpected in free-boundary problems (see, for example, [Baconneau and Lunardi 2004]), I don't know a counterexample to the lemma if D_0 is assumed to be a $C^{2+\alpha}$ domain.

Remark 4.4. In our application of the lemma, we in fact have $h \in C^{1+\alpha}(\partial D_0)$, but this does not imply higher regularity for φ.

5. Linearization

The evolution equation and boundary conditions in split gauge are

$$\begin{cases} F_t - g^{ij}(DF)F_{ij} &= 0,\\ u_{|\partial D_0} &= 0,\\ B(D\varphi, Du)_{|\partial D_0} &= 0,\\ \mathbb{O}(D\varphi)_{|\partial D_0} &= 0,\end{cases}$$

where

$$\mathbb{O}(D\varphi) := \langle D^T\varphi, D_n\varphi\rangle.$$

Here $D^T\varphi = D\varphi - (d_n\varphi)\langle\,\cdot\,, n\rangle$ is an $\mathbb{R}^n$-valued $(n-1)$-form on ∂D_0. We'll prove short-time existence for this system (with initial data u_0, φ_0) in $C^{2+\alpha,1+\alpha/2}$ by the usual fixed-point argument based on linear parabolic theory. Given $\bar{F} = [\bar{\varphi}, \bar{u}]$ in a suitable ball in this Hölder space with center $F_0 = [\varphi_0, u_0]$, it suffices to consider the pseudolinearization of the system:

$$F_t - g^{ij}(DF_0)F_{ij} = [g^{ij}(D\bar{F}) - g^{ij}(DF_0)]\bar{F}_{ij} =: \mathscr{F}(\bar{F}, F_0) =: \bar{\mathscr{F}}. \qquad \text{(LPDE)}$$

A fixed point of the map $\bar{F} \mapsto F$ corresponds to a solution of the quasilinear equation.

For the nonlinear boundary conditions, we need the honest linearization at F_0. For the angle condition, a computation using the boundary constraints on u_0 and φ_0 yields

$$\tfrac{1}{2}\mathscr{L}_0 B[D\varphi, Du] = \beta\beta_0 d_n u - \beta_0^2\langle d_n\varphi, n\rangle.$$

The corresponding linear boundary condition will be

$$\beta\beta_0 d_n u - \beta_0^2\langle d_n\varphi, n\rangle = \mathscr{B}(D\bar{F}, DF_0) := \bar{\mathscr{B}},$$

where

$$2\mathscr{B}(DF^1, DF^2) := B(D\varphi_1, Du_1) - B(D\varphi_2, Du_2) - \mathscr{L}_0 B\big(D(\varphi_1 - \varphi_2), D(u_1 - u_2)\big),$$

and we used

$$-\tfrac{1}{2}\mathscr{L}_0[D\varphi_0, Du_0]_{|\partial D_0} = \beta\beta_0 d_n u_0 - \beta_0^2\langle d_n\varphi_0, n\rangle_{|\partial D_0} = 0.$$

Also, $B(D\varphi_0, Du_0)_{|\partial D_0} = 0$, so at a fixed point $B(D\varphi, Du)_{|\partial D_0} = 0$.

Linearizing the orthogonality boundary condition, we find that $\mathscr{L}_0\mathbb{O}[D\varphi]$ is the $(n-1)$-form on ∂D_0 given by

$$\mathscr{L}_0\mathbb{O}[D\varphi](v) = (\partial_j\varphi^i + \partial_i\varphi^j)n^j(\delta_{ik} - n^k n^i)v^k$$

(summing over repeated indices). The corresponding linear boundary condition is

$$\langle d_n\varphi, \mathrm{proj}^T(\,\cdot\,)\rangle + \langle D^T\varphi, n\rangle = -\Omega(D\bar{\varphi}, D\varphi_0) =: \bar{\Omega},$$

where proj^T denotes orthogonal projection $\mathbb{R}^n \to T\partial D_0$, and

$$\Omega(D\varphi_1, D\varphi_2) := \mathbb{O}(D\varphi_1) - \mathbb{O}(D\varphi_2) - \mathscr{L}_0\mathbb{O}[D\varphi_1 - D\varphi_2],$$

and we used

$$\mathscr{L}_0\mathbb{O}[D\varphi_0]_{|\partial D_0} = \langle (d_n\varphi_0)^T, \cdot\,\rangle + \langle D^T\varphi_0, n\rangle_{|\partial D_0} = 0.$$

6. Complementarity

We wish to apply linear existence theory to the system

$$F_t - g^{ij}(DF_0)F_{ij} = \bar{\mathscr{F}},$$

with boundary conditions at ∂D_0

$$\begin{cases} u = 0 \\ \beta\beta_0 d_n u + \beta_0^2\langle d_n\varphi, n\rangle = \bar{\mathscr{B}}, \\ \langle d_n\varphi, \mathrm{proj}^T(\,\cdot\,)\rangle + \langle D^T\varphi, n\rangle = -\bar{\Omega} \end{cases} \qquad \text{(LBC)}$$

and initial conditions

$$u_{t=0} = u_0, \qquad \varphi_{t=0} = \varphi_0.$$

It is easy to see that the initial data satisfy the linearized boundary conditions, and above we constructed φ_0 so as to guarantee $g^{ij}(Du_0, D\varphi_0)u_{0ij\,|\partial D_0} = 0$. (There is no first-order compatibility condition for φ_0.) Thus the linear system satisfies the required compatibility at $t = 0$.

Since the linearized boundary conditions are slightly nonstandard, we must verify they satisfy the Lopatinski–Shapiro complementarity conditions. We *fix* $x_0 \in \partial D_0$ and introduce adapted coordinates (ρ, σ) in a neighborhood $\mathcal{N}_0 \subset \mathcal{N}$ of x_0 in D_0:

$$x \in \mathcal{N}_0 \implies x = \Gamma_0(\sigma) + \rho n(\sigma), \quad \sigma = (\sigma_a) \in \mathcal{U},$$

where $\mathcal{U} \subset \mathbb{R}^{n-1}$ is open and $\Gamma_0 : \mathcal{U} \to \mathbb{R}^n$ is a local chart for ∂D_0 at x_0. This defines a basis of tangential vector fields in $\Gamma_0(\mathcal{U})$, and we may assume that at x_0, we have $\langle \tau_a, \tau_b \rangle = \delta_{ab}$ and $\nabla_{\tau_a} \tau_b(x_0) = 0$ (for the induced connection on $T\partial D_0$). Let U and ψ be defined in $(-\rho_1, 0) \times \mathcal{U} \times [0, T]$ by

$$U(\rho, \sigma, t) = u(\Gamma_0(\sigma) + \rho n(\sigma), t), \qquad \psi(\rho, \sigma, t) = \varphi(\Gamma_0(\sigma) + \rho n(\sigma), t).$$

In these coordinates, the induced metric is written in block form as

$$[g(DF_0)] = \begin{bmatrix} |\psi_\rho|^2 + (U_\rho)^2 & \langle \psi_\rho, \psi_a \rangle + U_\rho U_a \\ \langle \psi_\rho, \psi_a \rangle + U_\rho U_a & \langle \psi_a, \psi_b \rangle + U_a U_b \end{bmatrix}_{|t=0} = \begin{bmatrix} 1/\beta^2 & 0 \\ 0 & \mathbb{I}_{n-1} \end{bmatrix}$$

at $t = 0$ and x_0.

We have

$$U_{\rho\rho} = D^2u(n, n),$$

since $d_n n = 0$, and

$$U_{ab} = D^2u(\tau_a, \tau_b) + Du \cdot \nabla_{\tau_a} \tau_b = D^2u(\tau_a, \tau_b) \quad \text{at } x_0.$$

We don't need $U_{\rho a}$, since $g_{\rho a} = 0$ at x_0.

Thus

$$\mathrm{tr}_{g_0} D^2u(x_0) = \beta^2 D^2u(n, n) + \sum_a D^2u(\tau_a, \tau_a) = \beta^2 U_{\rho\rho} + \sum_a U_{aa} := \beta^2 U_{\rho\rho} + \Delta_\sigma U,$$

and, likewise,

$$\mathrm{tr}_{g_0} D^2\varphi(x_0) = \beta^2 \psi_{\rho\rho} + \Delta_\sigma \psi.$$

For the linearized orthogonality operator, note that, at x_0,

$$\mathcal{L}_0 \mathbb{O}[D\psi] = (\langle \psi_\rho, \tau_a \rangle + \langle \psi_a, n \rangle)\tau_a.$$

Putting everything together, the linear system to consider at x_0 is

$$\begin{cases} U_t - \beta^2 U_{\rho\rho} - \Delta_\sigma U = 0, \\ \psi_t - \beta^2 \psi_{\rho\rho} - \Delta_\sigma \psi = 0, \end{cases}$$

with boundary conditions

$$\begin{cases} U|_{\rho=0} = 0, \\ \beta_0\langle \psi_\rho, n\rangle + \beta U_\rho|_{\rho=0} = b(\sigma, t), \\ \langle \psi_\rho, \tau_a\rangle + \langle \psi_a, n\rangle|_{\rho=0} = \omega_a(\sigma, t), \quad a = 1, \dots n-1. \end{cases}$$

Now take the Fourier transform in $\sigma \in \mathbb{R}^{n-1}$ (corresponding to $\xi \in \mathbb{R}^{n-1}$), Laplace transform in t (corresponding to $p \in \mathbb{C}$) to obtain

$$\hat{U}(\rho, \xi, p) \in \mathbb{C},\ \hat{\psi}(\rho, \xi, p) \in \mathbb{C}^n;\ \xi \in \mathbb{R}^{n-1}, \quad p \in \mathbb{C}, \quad \rho < 0.$$

In transformed variables, we obtain the following system of linear ODE in $\rho < 0$, for fixed (ξ, p):

$$\begin{cases} \beta^2 \hat{U}_{\rho\rho} - (p + |\xi|^2)\hat{U} = 0, \\ \beta^2 \hat{\psi}_{\rho\rho} - (p + |\xi|^2)\hat{\psi} = 0 \end{cases}$$

Writing the solution in the form

$$\begin{bmatrix} \hat{U}(\rho) \\ \hat{\psi}(\rho) \end{bmatrix} = e^{i\rho\gamma} \begin{bmatrix} \hat{U}(0) \\ \hat{\psi}(0) \end{bmatrix},$$

we find the characteristic equation $\beta^2\gamma^2 + p + |\xi|^2 = 0$, and choose the root γ so that $i\gamma = (1/\beta)\sqrt{\Delta}$ (where $\Delta = p + |\xi|^2$ and we take the branch of the square root defined by $\operatorname{Re}\sqrt{\Delta} > 0$). Here $(p, \xi) \in \mathcal{A}$, where

$$\mathcal{A} = \left\{(p, \xi) \in \mathbb{C} \times \mathbb{R}^{n-1} : |p| + |\xi| > 0,\ \operatorname{Re} p > -|\xi|^2\right\}.$$

Thus the solutions decay as $\rho \to -\infty$.

Let $\mathcal{W}^+$ be the space of such decaying solutions; it has complex dimension $n-1$. The relevant boundary operator on $\mathcal{W}^+$ is

$$\mathbb{B}\begin{bmatrix} \hat{U} \\ \hat{\psi} \end{bmatrix} = \begin{bmatrix} \hat{U} \\ \beta_0\langle \hat{\psi}_\rho, n\rangle + \beta \hat{U}_\rho \\ \langle \hat{\psi}_\rho, \tau_a\rangle + i\xi_a\langle \hat{\Psi}, n\rangle \end{bmatrix}_{|\rho=0} = \begin{bmatrix} \hat{U}(0) \\ \beta_0(i\gamma)\langle \hat{\psi}(0), n\rangle + i\beta\gamma\,\hat{U}(0) \\ (i\gamma)\langle \hat{\psi}(0), \tau_a\rangle + i\xi_a\langle \hat{\psi}(0), n\rangle \end{bmatrix}$$

(a vector in $\mathbb{C} \times \mathbb{C} \times \mathbb{C}^{n-1}$).

The *complementarity condition* (see [Eidelman and Zhitarashu 1998], for example) is the statement that $\mathbb{B}$ is a linear isomorphism from $\mathcal{W}^+$ to $\mathbb{C}^{n+1}$. With respect to the basis $\left\{\hat{U}(0), \langle \hat{\psi}(0), n\rangle, \langle \hat{\psi}(0), \tau_a\rangle\right\}$ of $\mathcal{W}^+$, the matrix of $\mathbb{B}$ is (in block form)

$$[\mathbb{B}] = \begin{bmatrix} 1 & 0 & [0]_{1\times(n-1)} \\ -\sqrt{\Delta} & -(\beta_0/\beta)\sqrt{\Delta} & [0]_{1\times(n-1)} \\ [0]_{(n-1)\times 1} & [i\xi_a]_{(n-1)\times 1} & -(\sqrt{\Delta}/\beta)\mathbb{I}_{n-1} \end{bmatrix}.$$

This is triangular with nonzero diagonal entries for every $(p, \xi) \in \mathcal{A}$. Hence $\mathbb{B}$ is an isomorphism.

7. Estimates in Hölder spaces

For the fixed-point argument based on the linear system, we need estimates for $\|\mathcal{F}\|_\alpha$, $\|\mathcal{B}\|_{1+\alpha}$, $\|\boldsymbol{\Omega}\|_{1+\alpha}$ of two types, namely mapping and contraction estimates.

More precisely, for $T > 0$, $R > 0$ and $Q^T = D_0 \times [0, T]$ consider the open ball

$$B_R^T = \{F \in C^{2+\alpha,1+\alpha/2}(Q^T, \mathbb{R}^{n+1}) : \|F - F_0\|_{2+\alpha} < R,\ F|_{t=0} = F_0\}.$$

($F_0 = [\varphi_0, u_0]$ is defined from the initial surface Σ_0, via Lemma 4.1.) Solving the linear system with right-hand side defined by $\bar{F} \in B_R^T$ defines a map $\mathbb{F} : \bar{F} \mapsto F$, and we need to verify that, for suitable choices of T and R, $\mathbb{F}$ maps into B_R^T and is a contraction.

The argument that follows is standard, and the experienced reader may want to skip to the statement of local existence in Theorem 8.1. On the other hand, the result is not covered by any general theorem proved in detail in a reference known to the author, and some readers may find it useful to have all the details included. Another reason is that, although the "right-hand sides" are clearly quadratic, without explicit expressions one might run into trouble with compositions — which cause problems in Hölder spaces — or when appealing to Taylor-remainder arguments if the domain is not convex.

The estimates required to document that $\mathbb{F}$ maps into B_R^T are of the form

$$\|\mathcal{F}(\bar{F}, F_0)\|_\alpha + \|\mathcal{B}(D\bar{F}, DF_0)\|_{1+\alpha} + \|\Omega(D\bar{\varphi}, D\varphi_0)\|_{1+\alpha} \to 0 \quad \text{as } T \to 0_+,$$

and the contraction estimates are of the form

$$\|\mathcal{F}(F^1, F^0) - \mathcal{F}(F^2, F^0)\|_\alpha + \|\mathcal{B}(DF^1, DF^2)\|_{1+\alpha} + \|\Omega(D\varphi^1, D\varphi^2)\|_{1+\alpha} \le \mu(T)\|F^1 - F^2\|_{2+\alpha},$$

where $\mu(T) \to 0$ as $T \to 0_+$.

Notation. The $(\alpha, \alpha/2)$ norms are taken on Q^T, the $(1+\alpha, (1+\alpha)/2)$ norms on $\partial D_0 \times [0, T])$. Double bars without an index refer to the $(2+\alpha, 1+\alpha/2)$ norm, single bars to supremum norms over Q^T, and parabolic norms are indexed by their spatial regularity (α for $(\alpha, \alpha/2)$, etc.) In general, we use brackets for Hölder-type difference quotients.

We deal with the estimates for the forcing term $\mathcal{F}$ first. Consider the map

$$\mathcal{G} : \mathrm{Imm}(\mathbb{R}^n, \mathbb{R}^{n+1}) \to \mathrm{GL}_n$$

that associates to the linear immersion A the inverse matrix of $(\langle A_i, A_j\rangle)_{i=1}^n$, inner products of the rows of A. $\mathcal{G}$ is smooth, in particular locally Lipschitz in the space $\mathcal{W}$ of linear immersions. Hence, if F^1, F^2 are maps $Q^T \to \mathbb{R}^{n+1}$ such that $DF^i \in C^{\alpha,\alpha/2}(Q^T)$ and $DF^i(z) \in K$ for all $z \in Q^T$, where $K \subset \mathcal{W}$ is a fixed compact set, we have the bound

$$\|\mathcal{G}(DF^1) - \mathcal{G}(DF^2)\|_\alpha \le c_K \|D(F^1 - F^2)\|_\alpha.$$

In fact our maps F^i are in $C^{2+\alpha,1+\alpha/2}$, so $DF^i \in C^{1+\alpha,(1+\alpha)/2}$. From this higher regularity we obtain the decay as $T \to 0_+$. Assuming $F^1|_{t=0} = F^2|_{t=0}$, we have

$$|D(F^1 - F^2)| \le [D(F^1 - F^2)]_t^{(1+\alpha)/2} T^{(1+\alpha)/2}.$$

Now recall the elementary fact that if $D \subset \mathbb{R}^n$ is a uniformly C^1 domain (not necessarily convex or bounded) and $f \in C^1(D)$ with $\alpha \in (0, 1)$, we have for the α-Hölder difference quotient $|f|^\alpha$ the estimate $[f]^\alpha \le C_D \|f\|_{C^1}$. (Here "uniformly C^1" means that D can be covered by countably many balls of a fixed radius, which are domains of C^1 manifold-with-boundary local charts for D, with uniform C^1 bounds for the charts and their inverses. The constant C_D depends on those bounds.) Applying this to DF, where $F = F^1 - F^2$ vanishes identically at $t = 0$, and assuming $T < 1$, we obtain

$$[DF]_x^a \le c\big(|DF| + |D^2F|\big) \le c\big([DF]_t^{(1+\alpha)/2} T^{(1+\alpha)/2} + [D^2F]_t^{\alpha/2} T^{\alpha/2}\big) \le c\|F\| T^{\alpha/2},$$

(where c depends on D_0) and similarly for the Hölder difference quotient in t:

$$[DF]_t^{\alpha/2} \le [DF]_t^{1+\alpha/2} T^{1/2} \le \|F\| T^{1/2},$$

so we have

$$\|D(F^1 - F^2)\|_\alpha \le c\|F^1 - F^2\| T^{\alpha/2}.$$

We conclude, under the assumption $F^1 = F^2$ at $t = 0$

$$\|\mathcal{G}(DF^1) - \mathcal{G}(DF^2)\|_\alpha \le c_K \|F^1 - F^2\| T^{\alpha/2}.$$

In particular, applying this to $\bar{F}$ and F_0, we find

$$\big\|\big(\mathcal{G}(D\bar{F}) - \mathcal{G}(DF_0)\big) D^2\bar{F}\big\|_\alpha \le c_K \|\bar{F} - F_0\| T^{\alpha/2} \|\bar{F}\|,$$

and for F^1 and F^2 coinciding at $t = 0$

$$\big\|\big(\mathcal{G}(DF^1) - \mathcal{G}(DF^2)\big) D^2F^1\big\|_\alpha \le c_K \|F^1 - F^2\| T^{\alpha/2} \|F^1\|,$$

as well as

$$\big\|\big(\mathcal{G}(DF^2) - \mathcal{G}(DF_0)\big)(D^2F^1 - D^2F^2)\big\|_\alpha \le c_K \|F^2 - F_0\| T^{\alpha/2} \|F^1 - F^2\|,$$

so we have the mapping and contraction estimates for $\mathcal{F}(\bar{F}, F_0)$ and $\mathcal{F}(F^1, F_0) - \mathcal{F}(F^2, F_0)$.

Lemma 7.1. *Assume $\bar{F}$, F_0, F^1, F^2 are in $C^{2+\alpha,1+\alpha/2}(Q^T; \mathbb{R}^{n+1})$ and have the same initial values, and that $D\bar{F}$, DF_0, DF^1, DF^2 all take values in the compact subset K of* $\mathrm{Imm}(\mathbb{R}^n, \mathbb{R}^{n+1})$. *Then*

$$\|\mathcal{F}(\bar{F}, F_0)\|_\alpha \le c_K \|\bar{F} - F_0\| \|\bar{F}\| T^{\alpha/2},$$
$$\|\mathcal{F}(F^1, F_0) - \mathcal{F}(F^2, F_0)\|_\alpha \le c_K \big(\|F^1\| + \|F^2 - F_0\|\big) T^{\alpha/2} \|F^1 - F^2\|.$$

In particular, if $\bar{F} \in B_R^T$,

$$\|\mathcal{F}(\bar{F}, F_0)\|_\alpha \le c_0 R T^{\alpha/2}.$$

If $\bar{F}^1, \bar{F}^2 \in B_R^T$, we have

$$\|\mathcal{F}(\bar{F}^1, F_0) - \mathcal{F}(\bar{F}^2, F_0)\|_\alpha \le c_0 T^{\alpha/2} \|\bar{F}^1 - \bar{F}^2\|.$$

(*The constant c_0 depends only on the data at $t = 0$, and we assume $T < 1$, $R < 1$.*)

Turning to the orthogonality boundary condition, first observe that

$$\begin{aligned}
\Omega(D\varphi^1, D\varphi^2)& \\
&= \langle D^T\varphi^1, d_n\varphi^1\rangle - \langle D^T\varphi^2, d_n\varphi^2\rangle - \mathcal{L}_0\mathbb{O}[D\varphi^1 - D\varphi^2]\\
&= \big\langle D^T(\varphi^1 - \varphi^2), d_n\varphi^1\big\rangle + \big\langle D^T\varphi^2, d_n(\varphi^1 - \varphi^2)\big\rangle - \big\langle d_n(\varphi^1 - \varphi^2), D^T\varphi_0\big\rangle - \big\langle D^T(\varphi^1 - \varphi^2), d_n\varphi_0\big\rangle\\
&= \big\langle D^T\varphi^1 - D^T\varphi^2, d_n\varphi^1 - d_n\varphi_0\big\rangle + \big\langle d_n\varphi^1 - d_n\varphi^2, D^T\varphi^2 - D^T\varphi_0\big\rangle,
\end{aligned}$$

which has quadratic structure. Using a local frame $(\tau_a)_{a=1}^{n-1}$ for $T\partial D_0$, we find the components

$$\Omega_a(D\varphi^1, D\varphi^2) = \big[\partial_i(\varphi^1-\varphi^2)\partial_j(\varphi^1-\varphi_0) + \partial_j(\varphi^1-\varphi^2)\partial_i(\varphi^2-\varphi_0)\big] n^j \tau_a^i.$$

The summation convention is $i, j = 1, \dots, n$, so Ω_a is a sum of terms of the form

$$b(x)D(\varphi^1-\varphi^2)D(\varphi^3-\varphi^4),$$

where $b(x) = n^j\tau_a^i$ and the φ^I coincide at $t = 0$. It is then not hard to show that

$$\big\|b(x)D(\varphi^1-\varphi^2)D(\varphi^3-\varphi^4)\big\|_{1+\alpha} \le c\|b\|_{1+\alpha}\|\varphi^1-\varphi^2\|\|\varphi^3-\varphi^4\|T^\alpha,$$

with c depending on the C^1 norms of local charts for D_0. To bound the norm $\|n\otimes\tau_a\|_{1+\alpha}$, note that $|n||\tau_a| \le 1$, $|D(n\otimes\tau_a)| \le |Dn| + |D\tau_a|$, and $[D(n\otimes\tau_a)]_x^\alpha \le [Dn]_x^\alpha + [D\tau_a]_x^\alpha$. Since $n = -(\beta/\beta_0)Du_0$ on ∂D_0 and ∂D_0 is a level set of u_0, we clearly have

$$\|Dn\|_\alpha + \|D\tau_a\|_\alpha \le c\|D^2u_0\|_\alpha \le c\|u_0\|.$$

We summarize the conclusion in the following lemma:

Lemma 7.2. *Assume $\bar\varphi, \varphi_0, \varphi^1, \varphi^2 \in C^{2+\alpha,1+\alpha/2}(Q^T;\mathbb{R}^n)$ have the same initial values. Then*

$$\|\Omega(D\bar\varphi, D\varphi_0)\|_{1+\alpha} \le c_0\|u_0\|\|\bar\varphi-\varphi_0\|^2T^\alpha$$

and

$$\|\Omega(D\varphi^1, D\varphi^2)\|_{1+\alpha} \le c_0\|u_0\|\big(\|\varphi^1-\varphi_0\| + \|\varphi^2-\varphi_0\|\big)T^\alpha\|\varphi^1-\varphi^2\|$$

with c_0 depending only on the data at $t = 0$. In particular, if $\bar F = [\bar\varphi, \bar u] \in B_R^T$, we have

$$\|\Omega(D\bar\varphi, D\varphi_0)\|_{1+\alpha} \le c_0R^2T^\alpha,$$

and for $\bar F^I = [\bar\varphi^I, \bar u^I] \in B_R^T$, $I = 1, 2$, we have

$$\|\Omega(D\bar\varphi^1, D\bar\varphi^2)\|_{1+\alpha} \le c_0RT^\alpha\|\bar\varphi^1-\bar\varphi^2\|.$$

To explain the estimates for the angle condition, we write the normal vector as a multilinear form on DF^i

$$\tilde N(DF) = J_n(DF) := (-1)^n\sum_{i=1}^{n+1}(-1)^{i-1}(DF^1\wedge\dots\widehat{DF^i}\wedge\dots DF^{n+1})e_i \in \mathbb{R}^{n+1}$$

(DF^i omitted in the i-th term of the sum), where $DF^i \in \mathbb{R}^n$ for $i = 1, \dots, n+1$ and we identify the n-multivector in $\mathbb{R}^n$ with a scalar, using the standard volume form. The angle condition has the form

$$\beta^2|\tilde N|^2 - \langle\tilde N, e_{n+1}\rangle^2 = 0 \text{ on } \partial D_0,$$

and we set

$$B(DF) := \beta^2|J_n(DF)|^2 - \langle J_n(DF), e_{n+1}\rangle^2,$$

with linearization at $DF_0 = [\mathbb{I}_n|Du_0]$

$$\mathscr{L}_0B[DF] = 2\beta^2\big\langle J_n(DF_0), DJ_n(DF_0)[DF]\big\rangle - 2\big\langle J_n(DF_0), e_{n+1}\big\rangle\big\langle DJ_n(DF_0)[DF], e_{n+1}\big\rangle.$$

Under the assumption $F^1 = F^2$ at $t = 0$, we need an estimate in $C^{1+\alpha,(1+\alpha)/2}$ for

$$\begin{aligned}
&\mathcal{B}(DF^1, DF^2)\\
&\quad := B(DF^1) - B(DF^2) - \mathcal{L}_0 B[DF^1 - DF^2]\\
&\quad = \beta^2\big(|J_n(DF^1)|^2 - |J_n(DF^2)|^2 - 2\big\langle J_n(DF_0), DJ_n(DF_0)[DF^1 - DF^2]\big\rangle\big)\\
&\qquad - \big(\langle J_n(DF^1), e_{n+1}\rangle^2 - \langle J_n(DF^2), e_{n+1}\rangle^2 - 2\langle J_n(DF_0), e_{n+1}\rangle\big\langle DJ_n(DF_0)[DF^1 - DF^2], e_{n+1}\big\rangle\big).
\end{aligned}$$

It will suffice to estimate the expression in the first parenthesis; the second is analogous.

We need the following algebraic observation: if $T_0 = [\mathbb{I}_n | Du_0]$ and T are $n \times (n+1)$ matrices, the expression

$$|J_n(T_0 + T)|^2 - |J_n(T_0)|^2 - 2\big\langle J_n(T_0), DJ_n(T_0)[T]\big\rangle$$

is a linear combination (with constant coefficients) of terms of the form

$$u_{0i}\, p_{(2)}(T), \qquad u_{0i} u_{0j}\, p_{(2)}(T), \qquad p_{(2)}(T),$$

where the $p_{(2)}(T)$ are polynomials in the entries of T (with constant coefficients) with terms of degree $2 \le deg \le 2n$.

Thus $\mathcal{B}(DF^1, DF^2)$ is a linear combination (with constant coefficients) of terms

$$u_{0i}\, p_{(2)}(DF^1 - DF^2), \qquad u_{0i} u_{0j}\, p_{(2)}(DF^1 - DF^2), \qquad p_{(2)}(DF^1 - DF^2),$$

with the $p_{(2)}$ as described, and hence it is a linear combination of terms of the form

$$u_{0i}(F_k^{1j} - F_k^{2j})^d, \quad u_{0i} u_{0l}(F_k^{1j} - F_k^{2j})^d, \quad (F_k^{1j} - F_k^{2j})^d$$

(where $2 \le d \le 2n,\ 1 \le j \le n+1,\ 1 \le i, l, k \le n$), which we write symbolically as

$$\mathcal{B}(DF^1, DF^2) \sim \sum_{2 \le d \le 2n} b(x)(DF^1 - DF^2)^d,$$

where $b(x)$ is constant or $u_{0i}(x)$ or $u_{0i}(x)u_{0j}(x)$. For the degree d terms $G^{(d)} \sim b(x)(DF^1 - DF^2)^d$, it is not hard to show the bound

$$\|G^{(d)}\|_{1+\alpha} \le c\|b\|_{1+\alpha}\|F^1 - F^2\|^d T^\alpha, \quad 2 \le d \le 2n.$$

We conclude:

Lemma 7.3. *Assume $\bar{F}, F_0, F^1, F^2$ are in $C^{2+\alpha,1+\alpha/2}(Q^T; \mathbb{R}^{n+1})$ and have the same initial values. Then*

$$\|\mathcal{B}(D\bar{F}, DF_0)\|_{1+\alpha} \le c(1 + \|u_0\|^2)(1 + \|\bar{F} - F_0\|^{2n-2})T^\alpha \|\bar{F} - F_0\|^2.$$

$$\|\mathcal{B}(DF^1, DF^2)\|_{1+\alpha} \le c(1 + \|u_0\|^2)(1 + \|F^1 - F^2\|^{2n-2})T^\alpha \|F^1 - F^2\|^2$$

with c depending only on F_0. In particular, if $\bar{F} \in B_R^T$ then

$$\|\mathcal{B}(D\bar{F}, DF_0)\|_{1+\alpha} \le c_0 R^2 T^\alpha,$$

and if $\bar{F}^1, \bar{F}^2 \in B_R^T$ then

$$\|\mathcal{B}(D\bar{F}^1, D\bar{F}^2)\|_{1+\alpha} \le c_0 T^\alpha \|\bar{F}^1 - \bar{F}^2\|,$$

with c_0 depending only on F_0.

8. Local existence

Given a $C^{3+\bar{\alpha}}$ graph Σ_0 over a uniformly $C^{3+\bar{\alpha}}$ domain $D_0 \subset \mathbb{R}^n$ (for arbitrary $\bar{\alpha} \in (0,1)$) satisfying the contact and angle conditions, let $\varphi_0 \in \mathrm{Diff}^{2+\bar{\alpha}}$ be the diffeomorphism given by Lemma 4.1 (with the 1-jet of the identity at ∂D_0 and 2-jet determined by the mean curvature of Σ_0 at ∂D_0). Then find $u_0 \in C^{2+\alpha}(D_0)$ so that $F_0 = [\varphi_0, u_0] \in C^{2+\alpha}(D_0; \mathbb{R}^{n+1})$ parametrizes Σ_0 over D_0 ($\alpha = \bar{\alpha}^2 < \bar{\alpha}$).

(Precisely, if $[z, \tilde{u_0}(z)]$ parametrizes Σ_0 as a graph, and φ_0 is given by Lemma 4.1, let $u_0 = \tilde{u_0} \circ \varphi_0$; so $u_0 \in C^{2+\alpha}$.)

We obtained in Section 7 all the estimates needed for a fixed-point argument in the set

$$B_R^T = \{F \in C^{2+\alpha, 1+\alpha/2}(Q^T, \mathbb{R}^{n+1}) : \|F - F_0\| < R, F|_{t=0} = F_0\}.$$

Choose $R < 1$ and $T_0 < 1$ small enough (depending only on F_0) so that, for $F \in B_R^{T_0}$, $F(t) = [\varphi(t), u(t)]$ defines an embedding of D_0, with $\varphi(t)$ a diffeomorphism onto its image $D(t)$. Let $K \subset \mathrm{Imm}(\mathbb{R}^n, \mathbb{R}^{n+1})$ be a compact set containing $DF(z)$ for all $F \in B_R$ and $z \in Q^{T_0}$. Now consider $T < T_0$.

Given $\bar{F} \in B_R^T$, solve the linear system (LPDE)/(LBC) with initial data F_0 to get $F \in C^{2+\alpha, 1+\alpha/2}(Q^T)$. (This is possible since the complementarity and compatibility conditions hold for the linear system.) This defines a map $\mathbb{F} : \bar{F} \mapsto F$.

From linear parabolic theory (see [Eidelman and Zhitarashu 1998, theorem VI.21], for example), we have

$$\|F - F_0\| \le M\big(\|\mathcal{F}(\bar{F}, F_0)\|_\alpha + \|\mathcal{B}(D\bar{F}, DF_0)\|_{1+\alpha} + \|\Omega(D\bar{\varphi}, D\varphi_0)\|_{1+\alpha}\big),$$

where $M > 0$ depends on the $C^{\alpha,\alpha/2}$ norm of the coefficients of the linear system, that is, ultimately on $\|F_0\|$.

From Lemmas 7.1–7.3, it follows that

$$\|F - F_0\| \le Mc_0(RT^{\alpha/2} + R^2 T^\alpha) < R$$

provided T is chosen small enough (depending only on F_0.) Thus $\mathbb{F}$ maps B_R^T to itself.

Similarly, if $\mathbb{F}(\bar{F}^i) = F^i$ for $i = 1, 2$, standard estimates for the linear system solved by $F^1 - F^2$ give

$$\|F^1 - F^2\| \le M\big(\|\mathcal{F}(\bar{F}^1, \bar{F}^2)\|_\alpha + \|\mathcal{B}(D\bar{F}^1, D\bar{F}^2)\|_{1+\alpha} + \|\Omega(D\bar{\varphi}^1, D\bar{\varphi}^2)\|_{1+\alpha}\big)$$

Again the estimates in Lemmas 7.1–7.3 imply

$$\|F^1 - F^2\| \le Mc_0(T^{\alpha/2} + T^\alpha)\|\bar{F}^1 - \bar{F}^2\| < \tfrac{1}{2}\|\bar{F}^1 - \bar{F}^2\|,$$

assuming T is small enough (depending only on F_0). This concludes the argument for local existence.

Theorem 8.1. *Let $\Sigma_0 \subset \mathbb{R}^{n+1}$ be a $C^{3+\bar{\alpha}}$ graph over $D_0 \subset \mathbb{R}^n$ satisfying the contact and angle conditions at ∂D_0. With $\alpha = \bar{\alpha}^2$, there exists a parametrization $F_0 = [\varphi_0, u_0] \in C^{2+\alpha}(D_0)$ of Σ_0, a number $T > 0$ depending only on F_0 and a unique solution $F \in C^{2+\alpha, 1+\alpha/2}(Q^T; \mathbb{R}^{n+1})$ of the system*

$$\begin{cases} \partial_t F - g^{ij}(DF)\partial_i \partial_j F = 0, \quad F = [\varphi, u] \\ u_{|\partial D_0} = 0, \quad N^{n+1}(D\varphi, Du)_{|\partial D_0} = \beta, \quad \langle D^T \varphi, d_n \varphi \rangle_{|\partial D_0} = 0 \end{cases}$$

with initial data F_0. For each $t \in [0, T]$, $F(t)$ is a $C^{2+\alpha}$ embedding parametrizing a surface Σ_t which satisfies the contact and angle conditions and moves by mean curvature. In addition, $F(t)$ satisfies the orthogonality condition at ∂D_0.

The hypersurfaces Σ_t are graphs. For each $t \in [0, T]$, $\varphi(t) : D_0 \to D(t)$ is a diffeomorphism and $\Sigma_t = \mathrm{graph}(w(t))$ for $w(t) : D(t) \to \mathbb{R}$ given by $w(t) = u(t) \circ \varphi^{-1}(t)$. (Since $w(t)$ lies in $C^{2+\alpha^2}(D(t))$, it is less regular than $u(t)$ or $\varphi(t)$.) $D(t)$ is a uniformly $C^{2+\alpha}$ domain.

Remark. This theorem does not address the geometric uniqueness of the motion, given Σ_0. It only asserts uniqueness for solutions of the parametrized flow (including the orthogonality boundary condition) in the given regularity class.

9. Rotational symmetry

In this section we record the equations for two rotationally symmetric instances of the problem:

(i) D_0 and $D(t)$ are disks, and $u > 0$ (*lens case*).

(ii) D_0 and $D(t)$ are complements of disks in $\mathbb{R}^n$ (*exterior case*). For simplicity we restrict to $n = 2$.

Let $F(r) = [\varphi(r), u(r)]$ parametrize a hypersurface Σ, where $\varphi(r) = \phi(r)e_r$ is a diffeomorphism onto its image. Here e_r and e_θ are orthonormal vectors, outward normal and counterclockwise tangent, respectively, to the circles $r = \text{const}$. The unit upward normal vector and mean curvature are

$$N = \frac{[-u_r e_r, \phi_r]}{\sqrt{u_r^2 + \phi_r^2}} \quad \text{and} \quad H = \frac{1}{(\phi_r^2 + u_r^2)^{3/2}} \big(\phi_r \mathcal{M}(\phi_r, u_r)[D^2 u] - \langle u_r e_r, \vec{\mathcal{M}}(\phi_r, u_r[D^2\varphi])\rangle\big),$$

where

$$\mathcal{M}(\phi_r, u_r)[D^2 u] = u_{rr} + (\phi_r^2 + u_r^2)\frac{u_r \phi_r}{\phi^2}, \quad \vec{\mathcal{M}}(\phi_r, u_r)[D^2\varphi] = \Big[\phi_{rr} + (\phi_r^2 + u_r^2)\Big(\frac{r\phi_r}{\phi^2} - \frac{1}{\phi}\Big)\Big]e_r.$$

Simplifying we get

$$H = \frac{1}{(\phi_r^2 + u_r^2)^{3/2}}\Big[\phi_r u_{rr} - u_r \phi_{rr} + (\phi_r^2 + u_r^2)\frac{u_r}{\phi}\Big].$$

Now consider the time-dependent case $F(r, t) = [\phi(r, t)e_r, u(r, t)]$. From the expressions above, one finds easily that the equation $\langle \partial_t F, N\rangle = H$ takes the form

$$\phi_r\Big(u_t - \frac{1}{\phi_r^2 + u_r^2}\mathcal{M}(\phi_r, u_r)[D^2 u]\Big) = u_r\Big\langle e_r, \varphi_t - \frac{1}{\phi_r^2 + u_r^2}\vec{\mathcal{M}}(\phi_r, u_r)[D^2\varphi]\Big\rangle.$$

In split gauge, we consider the system

$$\begin{cases} u_t - \dfrac{1}{\phi_r^2 + u_r^2}\mathcal{M}(\phi_r, u_r)[D^2 u] = 0, \\ \varphi_t - \dfrac{1}{\phi_r^2 + u_r^2}\vec{\mathcal{M}}(\phi_r, u_r)[D^2\varphi] = 0. \end{cases}$$

Note that $\phi(r, t) = r$ solves the ϕ equation, and that in this case the u equation becomes

$$w_t - \frac{w_{rr}}{1 + w_r^2} - \frac{w_r}{r} = 0.$$

This can be compared with the equation for curve networks,

$$w_t - \frac{w_{xx}}{1+w_x^2} = 0.$$

The boundary conditions are easily stated (we assume D_0 is the unit disk or its complement). The "contact condition" at $r = 1$ is $u = 0$. For the "angle condition" at $r = 1$, we find

$$u_r^2 = \frac{\beta_0^2}{\beta^2}\phi_r^2, \quad \beta_0 := \sqrt{1-\beta^2}.$$

Assuming $\phi_r > 0$ at $r = 1$, this resolves as

$$\begin{aligned} \beta u_r + \beta_0 \phi_r &= 0 \quad \text{at } r = 1 \quad \text{(lens case)}, \\ \beta u_r - \beta_0 \phi_r &= 0 \quad \text{at } r = 1 \quad \text{(exterior case)}. \end{aligned}$$

(For lenses, one also has at $r = 0$: $u_r = 0$ and $\phi_r = 1$.) Thus in both cases one can work with *linear* Dirichlet/Neumann-type boundary conditions.

One reason to consider the exterior case is that, unlike the lens case, it admits stationary solutions. Geometrically one just has to consider one-half of a catenoid truncated at an appropriate height. For example, for 120-degree junctions the equation for stationary solutions

$$\begin{cases} \dfrac{u_{rr}}{1+u_r^2} + \dfrac{u_r}{r} = 0 & \text{in}\{r > 1\}, \\ u_{r\,|r=1} = \sqrt{3}, & u_{|r=1} = 0. \end{cases}$$

admits the explicit solution

$$u(r) = \frac{\sqrt{3}}{2}\bigl(\ln(2r + \sqrt{4r^2-3}) - \ln 3\bigr), \quad r > \sqrt{3}/2.$$

Problem. It would be interesting to consider the nonlinear dynamical stability of this solution (even linear stability is yet to be considered). One may even work with bounded domains by introducing a fixed boundary at some $R > 1$ intersecting the surface orthogonally (see Section 10).

10. Fixed supporting hypersurfaces

Extending the local existence theorem to the case of hypersurfaces intersecting a fixed hypersurface $\mathcal{S}$ orthogonally presents no essential difficulty. The case of vertical support surface leads directly to graph evolution with a standard Neumann condition on a fixed boundary; we consider the complementary case where $\mathcal{S}$ is a graph. Let $\mathcal{S} \subset \mathbb{R}^{n+1}$ be a C^4 embedded hypersurface (not necessarily connected), the graph over $\mathcal{D} \subset \mathbb{R}^n$ of $B \in C^4(\mathcal{D})$, oriented by the upward unit normal

$$\nu(y) := \frac{1}{v_B}\tilde{\nu}(y), \qquad \tilde{\nu}(y) := [-DB(y), 1] \in \mathbb{R}^n \times \mathbb{R}, \qquad v_B := \sqrt{1+|DB(y)|^2}.$$

We assume ν to be nowhere vertical in $\mathcal{D}$ ($DB \neq 0$). To state the problem in the graph parametrization, we consider a time-dependent domain $D(t) \subset \mathbb{R}^n$ with a boundary consisting of two components $\partial_1 D(t)$

and $\partial_2 D(t)$, both moving. The hypersurface Σ_t is the graph of $w(\,\cdot\,,t)$ over $D(t)$ solving the parabolic equation

$$w_t - g^{ij}(Dw)w_{ij} = 0 \quad \text{ in } E := \bigcup_{t\in[0,T]} D(t)\times\{t\} \in \mathbb{R}^{n+1}\times[0,T]$$

with boundary conditions

$$w(\,\cdot\,,t)_{|\partial_1 D(t)} = 0, \qquad \sqrt{1+|Dw|^2}_{|\partial_1 D(t)} = 1/\beta$$

(as before), and on $\partial_2 D(t)$

$$w = B, \qquad \nabla w\cdot\nabla B = -1.$$

(The first-order condition on $\partial_2 D(t)$ is equivalent to $\langle \nu, N\rangle = 0$.)

Differentiating in t the boundary condition $w = B$ leads easily to an equation for the normal velocity of the interface $\Gamma(t) = \partial_2 D(t)$:

$$\dot{\Gamma}_n = \frac{vH}{B_n - w_n}.$$

Note that w_n at $\partial_2 D(t)$ can be computed from B_n, since

$$-1 = \nabla w\cdot\nabla B = w_n B_n + |\nabla^T B|^2;$$

in particular neither B_n nor w_n can vanish (so both have constant sign on connected components of $\partial_2 D$), and one easily computes: $w_n - B_n = -v_B^2/B_n$.

Let $\Lambda = \Sigma\cap\mathcal{S}$ be the intersection $(n-1)$-manifold, the graph of w (or B) over $\partial_2 D$. Given the graph parametrizations of Σ and $\mathcal{S}$, say

$$G(y) = [y, w(y)], \qquad \mathbb{B}(y) = [y, B(y)], \qquad y\in\partial_2 D,$$

and $\tau\in T\partial_2 D$, we have the tangent vectors

$$G_n := [n, w_n]\in T\Sigma, \qquad G_B := [\nabla B, -1] = -v_B\nu \in T\Sigma, \qquad G_\tau := [\tau, \nabla w\cdot\tau]\in T\Lambda,$$

and the second fundamental forms of Σ and $\mathcal{S}$ (for $e\in\mathbb{R}^n$ arbitrary):

$$A(dGe, dGe) = \frac{1}{v}d^2 w(e,e), \qquad \mathcal{A}(d\mathbb{B}e, d\mathbb{B}e) = \frac{1}{v_B}d^2 B(e,e).$$

From the equality $\langle\nu, N\rangle = 0$ at $\partial_2 D$, it follows easily that (compare [Stahl 1996])

$$A(G_\tau, \nu) = -\mathcal{A}(G_\tau, N), \quad \tau\in T\partial D.$$

For the remainder of this section, we concentrate on the boundary conditions at $\partial_2 D_0$. To establish short-time existence, we consider as before the parametrized flow

$$F_t - \mathrm{tr}_g d^2 F = 0, \qquad g = g(dF), \qquad F = [\varphi, u].$$

The contact and angle boundary conditions are

$$u_{|\partial_2 D_0} = B\circ\varphi_{|\partial_2 D_0}, \qquad \langle N, \nu\circ\varphi\rangle_{|\partial_2 D_0} = 0.$$

Again we have two scalar boundary conditions for $n+1$ components. Here the solution is easier than at the junction. With the notation $F_n = dFn = [\varphi_n, u_n]$, we replace the angle condition by the "vector Neumann condition"

$$F_n \perp T\mathcal{S} \quad \text{or} \quad F_n = -\alpha v_B \nu \text{ on } \partial_2 D_0,$$

where $\alpha : \partial_2 D_0 \to \mathbb{R}$, or equivalently (since this leads to $\alpha = -u_n$)

$$\varphi_n = -u_n(\nabla B \circ \varphi) \text{ on } \partial_2 D_0.$$

Clearly the Neumann condition implies the angle condition $\langle N, \nu \circ \varphi \rangle = 0$, but not conversely. This linear Neumann-type condition can easily be incorporated into the fixed-point existence scheme described earlier.

There is one issue to consider: the zero- and first-order compatibility conditions must hold at $\partial_2 D_0$ at $t = 0$. The initial hypersurface Σ_0 uniquely determines w_0 and $D_0 \subset \mathbb{R}^n$ (satisfying $w_0 = B$ and $\nabla w_0 \cdot \nabla B = -1$ on $\partial_2 D_0$), and then once $\varphi_0 \in \mathrm{Diff}(D_0)$ is fixed, $u_0 = w_0 \circ \varphi_0$ is also determined. We may assume

$$\varphi_0 = id, \qquad \varphi_{0n} = \nabla B \text{ on } \partial_2 D_0,$$

so

$$u_{0n} = \nabla w_0 \cdot \varphi_{0n} = \nabla w_0 \cdot \nabla B = -1 \text{ on } \partial_2 D_0,$$

and then the Neumann condition $F_{0n\,|\partial_2 D_0} = -v_B \nu$ holds at $t = 0$, on $\partial_2 D_0$.

The first-order compatibility condition is

$$\mathrm{tr}_g d^2 u_0 = u_t = \nabla B \cdot \varphi_t = \nabla B \cdot \mathrm{tr}_g d^2 \varphi_0 \text{ on } \partial D_0,$$

or equivalently

$$\mathrm{tr}_g \langle \nu, d^2 F_0 \rangle = 0 \text{ on } \partial D_0.$$

(This is not a mean curvature condition; the mean curvature of Σ_0 is $H = \mathrm{tr}_g \langle N, d^2 F_0 \rangle$.)

From now on we omit the subscript 0 but continue to discuss compatibility at $t = 0$. First observe that the Neumann condition leads to a splitting of the induced metric. Given $\tau \in T\partial_2 D_0$, let $F_\tau = dF\tau \in T\Lambda$. Then (recalling $u_n = -1$ on $\partial_2 D_0$)

$$\langle F_\tau, F_n \rangle = \big\langle [\tau, dB\tau], [\varphi_n, u_n] \big\rangle = \nabla B \cdot \tau - \nabla B \cdot \tau = 0.$$

Thus we have

$$\mathrm{tr}_g \langle \nu, d^2 F \rangle = g^{ab} \langle \nu, d^2 F(\tau_a, \tau_b) \rangle + g^{nn} \langle \nu, d^2 F(F_n, F_n) \rangle,$$

for a local basis $\{T_a = dF\tau_a\}_{a=1}^{n-1}$ of $T\Lambda$ with $g_{ab} = \langle T_a, T_b \rangle$ and $g_{nn} = |F_n|^2 = v_B^2$.

Differentiating in n the condition $u_n = \nabla w \cdot \varphi_n$ (assuming, as usual, that n is extended to a tubular neighborhood $\mathcal{N}$ of $\partial_2 D_0$ as a self-parallel vector field) we find

$$u_{nn} = d^2 w(n, \nabla B) + \nabla w \cdot d^2 \varphi(n, n).$$

(This is legitimate, since $u = w \circ \varphi$ throughout $\mathcal{N}$.) This is used to compute

$$\begin{aligned}\langle \nu, d^2F(n,n)\rangle &= \frac{1}{v_B}[u_{nn} - \nabla B \cdot d^2\varphi(n,n)]\\ &= \frac{1}{v_B}\big[d^2w(n,\nabla B) + (\nabla w - \nabla B)\cdot d^2\varphi(n,n)\big]\\ &= -vA(G_n,\nu) + \frac{1}{v_B}(w_n - B_n)n\cdot d^2\varphi(n,n).\end{aligned}$$

Bearing in mind the expression for $w_n - B_n$ found earlier, the compatibility condition may be stated in the form

$$\frac{v_B}{B_n} n\cdot d^2\varphi(n,n) = -vA(G_n,\nu) + g^{ab}\langle d^2F(\tau_a,\tau_b),\nu\rangle.$$

We are now in the same situation as in Section 4. Given the 1-jet of φ_0 on $\partial_2 D_0$, we extend φ_0 to a tubular neighborhood $\mathcal{N}$ of $\partial_2 D_0$ (and then to all of D_0) so that $n\cdot d^2\varphi(n,n)$ has the value on $\partial_2 D_0$ dictated by the compatibility condition, using Lemma 4.1(ii). We just need to verify that the right-hand side of the expression above depends only on Σ_0, $\mathcal{S}$ and the 1-jet of φ_0 over $\partial_2 D_0$. Clearly only the term $g^{ab}\langle\nu, d^2F(\tau_a,\tau_b)\rangle$ is potentially an issue.

Fix $p\in\partial_2 D_0$ and let $\{\tau_a\}$ be an orthonormal frame for $T\partial_2 D_0$ near p, parallel at p for the connection induced on $\partial_2 D_0$ from $\mathbb{R}^n$. If $\mathcal{K}$ denotes the second fundamental form of $\partial_2 D_0$ in $\mathbb{R}^n$, we have

$$\tau_a(\tau_b) = \mathcal{K}(\tau_a,\tau_b)n \quad \text{at } p;$$

on the left-hand-side, τ_b is regarded as a vector-valued function in $\mathbb{R}^n$. Still computing at p, this implies

$$\begin{aligned}d^2F(\tau_a,\tau_b) = \tau_a(dF\tau_b) - dF(\tau_a(\tau_b)) &= \tau_a(d\mathbb{B}\tau_b) - \mathcal{K}(\tau_a,\tau_b)F_n\\ &= d^2\mathbb{B}(\tau_a,\tau_b) + \mathcal{K}(\tau_a,\tau_b)\mathbb{B}_n - \mathcal{K}(\tau_a,\tau_n)F_n,\end{aligned}$$

where $F_n = -v\nu$ and $\mathbb{B}_n = d\mathbb{B}n \in T\mathcal{S}$. Hence

$$\langle\nu, d^2F(\tau_a,\tau_b)\rangle = \langle\nu, d^2\mathbb{B}(\tau_a,\tau_b)\rangle + v\mathcal{K}(\tau_a,\tau_b) = \mathcal{A}(T_a,T_b) + v\mathcal{K}(\tau_a,\tau_b).$$

This clearly depends only on $\mathcal{S}$ and on Σ_0. We summarize the discussion in a lemma.

Lemma 10.1. *Let $\Sigma_0 = \mathrm{graph}(w_0)$ be a C^3 graph over $D_0\subset\mathbb{R}^n$ (a uniformly C^3 domain) intersecting a fixed hypersurface $\mathcal{S} = \mathrm{graph}(B)$ over ∂D_0. Consider the parametrized mean curvature motion with Neumann boundary condition*

$$\begin{gathered}F\in C^{2,1}(D_0\times[0,T])\to\mathbb{R}^{n+1}, \qquad F = [\varphi,u],\\ F_t - \mathrm{tr}_g d^2F = 0, \qquad g = g(dF), \qquad u\circ\varphi = B \quad \textit{and} \quad F_n\perp T\mathcal{S} \textit{ on } \partial D_0.\end{gathered}$$

Then $\varphi_0\in\mathrm{Diff}(D_0)$ can be chosen so that (with $u_0 = w_0\circ\varphi_0$) the initial data $F_0 = [\varphi_0,u_0]$ satisfies the zero- and first-order compatibility conditions at $t=0$ and ∂D_0:

$$\varphi_{0n} = -u_{0n}(\nabla B\circ\varphi_0), \qquad \langle\nu\circ\varphi_0, \mathrm{tr}_{g_0}d^2F_0\rangle = 0.$$

Remark. Differentiating $dw\tau_a = dB\tau_a$ along τ_b, we find

$$d^2w(\tau_a,\tau_b) - d^2B(\tau_a,\tau_b) = (w_n - B_n)\mathcal{K}(\tau_a,\tau_b)$$

(reminding us that, although $w \equiv B$ on ∂D_0, the tangential components of their Hessians do not coincide.) From this follows the expression for $\mathscr{K}$ in terms of A and $\mathscr{A}$:

$$\mathscr{K}(\tau_a, \tau_b) = \frac{1}{w_n - B_n}\big[v A(T_a, T_b) - v_B \mathscr{A}(T_a, T_b)\big].$$

It is also easy to express the corresponding traces in terms of the mean curvatures H^Λ and $\mathscr{H}^\Lambda$ of Λ in Σ and $\mathscr{S}$:

$$H^\Lambda = \frac{v}{v_B} g^{ab} A(T_a, T_b), \qquad \mathscr{H}^\Lambda = \frac{v_B}{v} g^{ab} \mathscr{A}(T_a, T_b).$$

11. Boundary conditions for the second fundamental form

To understand the long-term behavior of a graph (Σ_t) in $\mathbb{R}^{n+1}$ moving by mean curvature and intersecting $\mathbb{R}^n$ at a constant angle, we need to consider the evolution of its second fundamental form. Working in the graph parametrization the boundary conditions are easy to state and linear:

$$w_{|\partial D(t)} = 0, \qquad d_n w_{|\partial D(t)} = \frac{\beta_0}{\beta},$$

where $n = n_t$ is the inner unit normal to $\partial D(t)$. It is possible to reparametrize the Σ_t over a different time-dependent domain $\mathscr{D}(t)$, obtaining mean curvature flow

$$\mathscr{F}_t : \mathscr{D}(t) \to \mathbb{R}^{n+1}, \qquad \partial_t \mathscr{F} = HN,$$

with boundary conditions

$$\mathscr{F}^{n+1}_{|\partial\mathscr{D}(t)} = 0, \qquad N^{n+1}_{|\partial\mathscr{D}(t)} = \beta.$$

For this parametrization the evolution equation for the second fundamental form (and its covariant derivatives of arbitrary order) is well-understood [Huisken 1984]. The disadvantage is that the unit normal $N_{|\mathscr{F}_t}$ depends nonlinearly on the components of $\mathscr{F}$, and as a result the boundary conditions for the second fundamental form (which are needed for global estimates over spacetime domains) do not admit simple expressions. Therefore we choose to work with graph flow at the cost of having to derive and understand a new set of evolution equations. The equations for h and the mean curvature H are derived in Appendix B. In this section we derive boundary conditions. The development is similar that in [Stahl 1996] for MCF of hypersurfaces intersecting a fixed boundary orthogonally.

It is easy to see that h splits on $\partial D(t)$: if $\tau \in T\partial D(t)$ is a tangential vector field, and $n = n_t$ is the inner unit normal

$$h(n, \tau) = \frac{1}{v} d^2 w(n, \tau) = \frac{1}{v}(\tau(w_n) - Dw \cdot \bar{\nabla}_\tau n) = 0 \text{ on } \partial D(t),$$

since $w_n \equiv \beta_0/\beta$ on the boundary and $\bar{\nabla}_\tau n \in T\partial D(t)$ ($\bar{\nabla}$ is the euclidean connection). In particular, it follows that $h(Dw, \tau) = 0$ on $\partial D(t)$.

Remark. Already this simple fact cannot be shown for $a(\nu, \tau)$, the second fundamental form in the MCF parametrization, regarded as a quadratic form on $\mathscr{D}(t)$.

Boundary condition for H. In Section 2 we derived the equation for the normal velocity of the moving boundary $\Gamma_t = \partial D(t)$:

$$\dot{\Gamma}_n = -\frac{v}{w_n} H = -\frac{1}{\beta_0} H \text{ at } \partial D(t).$$

Since $\langle N, e_{n+1}\rangle(\Gamma(t), t) \equiv \beta$ on $\partial D(t)$ we have

$$\langle \partial_t N, e_{n+1}\rangle = -\langle \partial_k N, e_{n+1}\rangle \dot{\Gamma}^k,$$

where $\partial_k N = -g^{ij} h_{ik} G_j$ with e_{n+1} component

$$\langle \partial_k N, e_{n+1}\rangle = -g^{ij} w_j h_{ik} = -\frac{1}{v^2} h(Dw, \partial_k) = -\frac{1}{v^2} w_n h(n, \partial_k).$$

Hence we find, on $\partial D(t)$,

$$\langle \partial_t N, e_{n+1}\rangle = \frac{w_n}{v^2} h(n, \dot{\Gamma}) = \frac{w_n}{v^2} \dot{\Gamma}_n h(n, n) = -\beta H h_{nn}. \tag{11-1}$$

(We set $h_{nn} := h(n, n)$). Denote by ∇^Σ the gradient of Σ_t, in the induced metric ($\nabla^\Sigma f = g^{ij} f_i G_j$). Using $\partial_t N = -\nabla^\Sigma H - H v^{-1} \nabla^\Sigma v$, combined with the expressions (valid on $\partial D(t)$)

$$\langle \nabla^\Sigma H, e_{n+1}\rangle = g^{ij} H_i \langle G_j, e_{n+1}\rangle = g^{ij} H_i w_j = \frac{1}{v^2} w_i H_i = \frac{w_n}{v^2} H_n = \beta\beta_0 H_n,$$

$$\langle \nabla^\Sigma v, e_{n+1}\rangle = \frac{v_n w_n}{v^2} = \frac{w_n^2}{v^2} h_{nn} = \beta_0^2 h_{nn},$$

we find on $\partial D(t)$

$$\langle \partial_t N, e_{n+1}\rangle = -\beta\beta_0 (H_n + \beta_0 H h_{nn}). \tag{11-2}$$

Comparing expressions for $\langle \partial_t N, e_{n+1}\rangle$ in (11-1) and (11-2) yields a Neumann-type condition for H. We state this as a lemma (including the evolution equation derived in Appendix B). Here $L = L_g$ denotes the operator $L[f] = \partial_t f - \mathrm{tr}_g D^2 f$ and $\omega = Dw/v$, a vector field in $D(t)$.

Lemma 11.1. *For the surfaces Σ_t evolving by graph mean curvature motion with constant contact angle, the mean curvature satisfies*

$$\begin{cases} L[H] = |h|_g^2 H + H h^2(\omega, \omega) - H^2 h(\omega, \omega) & \text{on } D(t), \\ d_n H = (\beta^2/\beta_0) H h_{nn} & \text{on } \partial D(t). \end{cases}$$

Boundary conditions for h_{ij}. Fix $p \in \partial D(t)$ and let (τ_a) be an orthonormal frame for $T_p \partial D(t)$ (in the induced metric) satisfying $\nabla^\Gamma_{\tau_a} \tau_b(p) = 0$, where ∇^Γ is the connection induced on Γ_t by the euclidean connection d, or, equivalently, by ∇, the Levi-Civita connection of the metric g in $D(t)$. We extend the τ_a to a tubular neighborhood of Γ_t so that $\bar{\nabla}_n \tau_a = 0$. Differentiating $h(n, \tau_b) = 0$ along τ_a, we find

$$(\nabla_{\tau_a} h)(n, \tau_b) = -h(\nabla_{\tau_a} n, \tau_b) - h(n, \nabla_{\tau_a} \tau_b). \tag{11-3}$$

The second fundamental form $\mathcal{H}(\tau, \tau')$ of Γ_t in $(D(t), \text{eucl})$ (equivalently, in $(D(t), g)$) is defined by

$$d_{\tau_a} \tau_b = \nabla^\Gamma_{\tau_a} \tau_b + \mathcal{H}(\tau_a, \tau_b) n \quad \text{on } \partial D(t).$$

To relate $\mathcal{H}$ to $h_{|\partial D(t)}$, note that since $w = 0$ on $\partial D(t)$ we have

$$h(\tau_a, \tau_b) = \frac{1}{v} d^2 w(\tau_a, \tau_b) = \frac{1}{v}(\tau_a(\tau_b w) - Dw \cdot d_{\tau_a} \tau_b) = -d_{\tau_a} \tau_b \cdot \frac{Dw}{v} = -\beta_0 \mathcal{H}(\tau_a, \tau_b).$$

(So we see that Γ_t convex with respect to n corresponds to Σ_t *concave* over $D(t)$, as expected.) In (B-1) in the appendix we observe that $\nabla_{\partial_i}\partial_j = (h_{ij}/v)Dw$. Then

$$\begin{aligned}\nabla_{\tau_a}\tau_b &= \tau_a^i\big((\tau_b^j)_i\partial_j + \tau_b^j\nabla_{\partial_i}\partial_j\big) = d_{\tau_a}\tau_b + \frac{1}{v}\tau_a^i\tau_b^j h_{ij}Dw\\ &= \nabla^\Gamma_{\tau_a}\tau_b + \mathcal{K}(\tau_a,\tau_b)n + \frac{w_n}{v}h(\tau_a,\tau_b)n = \Big(-\frac{1}{\beta_0}+\beta_0\Big)h(\tau_a,\tau_b)n = -\frac{\beta^2}{\beta_0}h(\tau_a,\tau_b)n\end{aligned}$$

at p, given our assumption $\nabla^\Gamma_{\tau_a}\tau_b(p) = 0$. We use this immediately to compute, at p,

$$\nabla_{\tau_a}n = \langle\nabla_{\tau_a}n, \tau_b\rangle_g\tau_b = -\langle n, \nabla_{\tau_a}\tau_b\rangle_g\tau_b = \frac{\beta^2}{\beta_0}|n|_g^2 h(\tau_a,\tau_b)\tau_b = \frac{1}{\beta_0}h(\tau_a,\tau_b)\tau_b,$$

since $|n|_g^2 = g_{ij}n^in^j = 1 + w_n^2 = \beta^{-2}$ at p. Using these expressions for $\nabla_{\tau_a}n$ and $\nabla_{\tau_a}\tau_b$ in (11-3) and recalling the Codazzi equations, we obtain

$$(\nabla_n h)(\tau_a,\tau_b) = (\nabla_{\tau_a}h)(n,\tau_b) = -\frac{1}{\beta_0}\sum_c h(\tau_a,\tau_c)h(\tau_c,\tau_b) + \frac{\beta^2}{\beta_0}h(\tau_a,\tau_b)h_{nn}.$$

This can also be written in the form

$$\beta_0(\nabla_n h)(\tau,\tau') = -(h^{\tan})^2(\tau,\tau') + \beta^2 h_{nn}h(\tau,\tau'). \tag{11-4}$$

It turns out the expression for the n-directional derivative of $h(\tau,\tau')$ is exactly the same (at $\partial D(t)$):

$$\beta_0 d_n(h(\tau,\tau')) = -(h^{\tan})^2(\tau,\tau') + \beta^2 h_{nn}h(\tau,\tau'). \tag{11-5}$$

The reason is that $\nabla_n\tau_a = 0$ at the boundary, also for the g-connection

$$\nabla_n\tau_a = d_n(\tau_a) + n^i\tau_a^j\nabla_{\partial_i}\partial_j = 0 + \frac{1}{v}h(n,\tau_a)Dw = 0,$$

so in fact

$$(\nabla_n h)(\tau_a,\tau_b) = n(h(\tau_a,\tau_b)) = d_n(h(\tau_a,\tau_b)).$$

As done in [Stahl 1996], we combine this with the result for H_n to compute $(\nabla_n h)(n,n)$. From

$$H_n = \nabla_n(\mathrm{tr}_g h) = \mathrm{tr}_g(\nabla_n h) = \beta^2(\nabla_n h)(n,n) + \sum_a(\nabla_n h)(\tau_a,\tau_a).$$

Here we used $|n|_g^2 = \beta^{-2}$ on $\partial D(t)$, which also implies $H = \beta^2 h_{nn} + \sum_a h(\tau_a,\tau_a)$. Using also $|h^{\tan}|^2 = \sum(h^{\tan})^2(\tau_a,\tau_a)$, we find for $(\nabla_n h)(n,n)$

$$\beta^2(\nabla_n h)(n,n) = \frac{\beta^2}{\beta_0}Hh_{nn} + \frac{1}{\beta_0}|h^{\tan}|^2 - \frac{\beta^2}{\beta_0}(H-\beta^2 h_{nn})h_{nn} = \frac{1}{\beta_0}(|h^{\tan}|^2 + \beta^4 h_{nn}^2) = \frac{1}{\beta_0}|h|_g^2,$$

since $g^{nn} = \beta^2$ at $\partial D(t)$. Equivalently,

$$\beta_0(\nabla_n h)(n,n) = \frac{1}{\beta^2}|h|_g^2 \text{ on } \partial D(t).$$

It is easy to obtain the corresponding expression for the euclidean connection. Noting that

$$\nabla_n n = d_n n + n^i n^j\frac{1}{v}h_{ij}Dw = \beta_0 h_{nn}n \quad \text{at } \partial D(t),$$

we find

$$(d_n h)(n,n) = n(h_{nn}) = (\nabla_n h)(n,n) + 2h(\nabla_n n, n) = (\nabla_n h)(n,n) + 2\beta_0 h_{nn}^2,$$

so that

$$\beta_0 d_n(h(n,n)) = \frac{1}{\beta^2}|h|_g^2 + 2\beta_0^2 h_{nn}^2 \text{ on } \partial D(t).$$

We record these results as a lemma, including also the evolution equations derived in Appendix B.

Lemma 11.2. *Under graph mean curvature motion with constant contact angle, the second fundamental form satisfies the following tensorial evolution equations, where C_{ij} and $\overline{C}_{ij}$ are symmetric 2-tensors cubic in h; see* (B-2) *and* (B-3). *Recall that $\omega = Dw/v$ and d_ω denotes directional derivative.*

(i) *For the operator $L = L_g$,*

$$L[h_{ij}] = -2[h_i^k d_\omega(h_{jk}) + h_j^k d_\omega(h_{ik})] + \overline{C}_{ij} \quad \text{on } D(t),$$

with boundary conditions on $\partial D(t)$ given by

$$\begin{cases} h(n,\tau) = 0, \\ \beta_0 d_n(h(\tau,\tau')) = -(h^{\tan})^2(\tau,\tau') + \beta^2 h_{nn} h(\tau,\tau'), \\ \beta_0 d_n(h(n,n)) = |h|_g^2/\beta^2 + 2\beta_0^2 h_{nn}^2. \end{cases}$$

(ii) *For the operator $\partial_t - \Delta_g$, where Δ_g is the Laplace–Beltrami operator of g*

$$(\partial_t - \Delta_g)[h]_{ij} = H(\nabla_\omega h)_{ij} + H_i h(\omega, \partial_j) + H_j h(\omega, \partial_i) + C_{ij} \quad \text{on } D(t),$$

with boundary conditions on $\partial D(t)$ given by

$$\begin{cases} h(n,\tau) = 0, \\ \beta_0 (\nabla_n h)(\tau,\tau') = -(h^{\tan})^2(\tau,\tau') + \beta^2 h_{nn} h(\tau,\tau'), \\ \beta_0 (\nabla_n h)(n,n) = |h|_g^2/\beta^2. \end{cases}$$

It is also useful to compute the boundary condition for $|h|_g^2$. Using Lemma 11.2(ii), we have at $\partial D(t)$

$$\begin{aligned}
(\beta_0/2) d_n |h|_g^2 &= \beta_0 \langle \nabla_n h, h \rangle_g \\
&= \beta_0 \beta^4 (\nabla_n h)(n,n) h_{nn} + \beta_0 \sum_{b,c} (\nabla_n h)(\tau_a, \tau_b) h(\tau_a, \tau_b) \\
&= \beta^2 |h|_g^2 h_{nn} + \sum_{a,b} [-(h^{\tan})^2(\tau_a,\tau_b) + \beta^2 h_{nn} h(\tau_a,\tau_b)] h(\tau_a,\tau_b) \\
&= \beta^2 \big(|h|_g^2 + |h^{\tan}|_g^2\big) h_{nn} - \operatorname{tr}_g (h^{\tan})^3
\end{aligned}$$

Since $\operatorname{tr}_g h^3 = \beta^6 (h_{nn})^3 + \operatorname{tr}_g (h^{\tan})^3$ on $\partial D(t)$, we may state this in a slightly different form. Including also the evolution equation for $|h|_g^2$ (see Appendix B), we have the following lemma:

Lemma 11.3. *Under graph mean curvature flow, the function $|h|_g^2$ satisfies the evolution equation and Neumann boundary condition*

$$\begin{cases} (\partial_t - \Delta_g)|h|_g^2 = -2|\nabla h|_g^2 + H d_\omega |h|_g^2 + 2|h|_g^4 - 4H h^3(\omega,\omega) - 2H|h|_g^2 h(\omega,\omega), \\ (\beta_0/2) d_n |h|_g^2 = 2\beta^2 |h|_g^2 h_{nn} - \operatorname{tr}_g(h^3) \quad \text{on } \partial D(t). \end{cases}$$

12. A maximum principle for symmetric 2-tensors

By the local existence theorem, for suitable initial data we have a mean curvature motion $F = [\varphi, u] \in C^{2+\alpha,1+\alpha/2}(Q_0, \mathbb{R}^{n+1})$, where $Q_0 = D_0 \times [0, T]$ and, for each $t \in [0, T]$, $\varphi_t : D_0 \to D(t)$ is a $C^{2+\alpha}$ diffeomorphism. In particular, with $\delta = \alpha^2$, $w_t = u_t \circ \varphi_t^{-1} : D(t) \to \mathbb{R}$ defines a graph mcm $w \in C^{2+\delta,1+\delta/2}(E; \mathbb{R})$ in an open spacetime domain

$$E = \bigcup_{t\in(0,T)} D(t) \times \{t\} \subset \mathbb{R}^n \times \mathbb{R}.$$

We have a $C^{2+\alpha,1+\alpha/2}$ diffeomorphism

$$\Phi : \bar{Q}_0 \to \bar{E}, \qquad \Phi(x, t) = (\varphi_t(x), t),$$

which, for any $t_0 > 0$, restricts to a diffeomorphism $Q_{t_0} \to E_{t_0}$, where

$$Q_{t_0} = D_0 \times (t_0, T), \qquad E_{t_0} = \bigcup_{t\in(t_0,T)} D(t) \times \{t\}.$$

The parabolic boundary of E is the disjoint union of base and lateral boundary:

$$\partial_p E = (\bar{D}_0 \times \{0\}) \sqcup \partial_l E, \qquad \partial_l E = \bigcup_{t\in(0,T)} \partial D(t) \times \{t\}.$$

(The notions of parabolic boundary, base and lateral boundary have general definitions for arbitrary bounded spacetime domains [Lieberman 1996], but using Φ it is easy to see that they are given by the above sets.) In particular, note that Φ defines a diffeomorphism

$$Q_{t_0} \cup \partial_l Q_{t_0} \to E_{t_0} \cup \partial_l E_{t_0},$$

for each $t_0 > 0$. This diffeomorphism is $C^{k+\alpha,(k+\alpha)/2}$ up to the lateral boundary, if D_0 is a $C^{k+\alpha}$ domain and $F \in C^{k+\alpha,(k+\alpha)/2)}(Q_0)$.

Denote by L the operator $L = \partial_t - g^{ij}(Dw)\partial_i\partial_j$, so $Lw = 0$ in E and $w = 0$ on $\partial_l E$. The following height bound is immediate.

Lemma 12.1. *Assume $0 < w_0 < M$ in D_0. Then $0 \le w \le M$ in $\bar{E}$ (and vanishes only on $\partial_l E$).*

Proof. This follows from the weak maximum principle for the operator L, since $0 \le w \le M$ holds on the parabolic boundary $\partial_p E$. □

It is well-known that the function $v = \sqrt{1 + |Dw|^2}$ solves the evolution equation (assuming $Dw \in C^{2,1}(\bar{E})$ — see [Guan 1996], for example)

$$L[v] + \frac{2}{v} g^{ij} v_i v_j = -v|h|_g^2, \text{ or } L[v] = -\frac{2}{v}|Dv|_g^2 - v|h|_g^2.$$

From the maximum principle, we have the following global bound on v (equivalently, on $|Dw|$):

Lemma 12.2. *Assume w is a solution with $Dw \in C^{2,1}(\bar{E})$. Then, on $\bar{E}$,*

$$v(z) \le \max\{\sup_{D(t_0)} v(x, t_0), 1/\beta\}.$$

Proof. By the weak maximum principle, $\max_{\bar{E}} v = \max_{\partial_p E} v$. Note that $v_{|S} \equiv 1/\beta$. □

It follows from this lemma that $g_{ij}(t)$ is uniformly equivalent to the euclidean metric in $D(t)$: If $v \le \bar{v}$ in $\bar{E}$, and X is a vector field in $D(t)$, then

$$|X|_e^2 \le |X|_g^2 = g_{ij} X^i X^j = |X|_e^2 + (X \cdot Dw)^2 \le |X|_e^2 (1 + |Dw|^2) \le \bar{v}^2 |X|_e^2.$$

Also, if $\omega := v^{-1} Dw$ then

$$|\omega|_e^2 = \frac{|Dw|_e^2}{v^2} = 1 - \frac{1}{v^2} \le 1 - \frac{1}{\bar{v}^2}.$$

The main result in this section is a maximum principle for symmetric 2-tensors satisfying a parabolic equation on a spacetime domain such as E (image of a cylinder under a diffeomorphism of the special type Φ).

We recall the boundary point lemma for scalar equations, which holds for open spacetime domains $\Omega \subset \mathbb{R}^n \times \mathbb{R}_+$ satisfying an *interior ball condition*:

For each $P = (p, \bar{t}) \in \partial_l \Omega$ there is a ball B (in the euclidean metric in $\mathbb{R}^{n+1}$) which is tangent to $\partial_l \Omega$ only at P and satisfies:

(i) *The line segment from P to the center of the ball is not parallel to the t axis.*

(ii) $B \cap \{t \le \bar{t}\} \subset \Omega \cap \{t \le \bar{t}\}$.

For the domain of interest the interior ball condition follows from the fact that $\partial_l E = \Phi(\partial D_0 \times (0, T))$, with $\Phi \in C^{2,1}(D_0 \times (0, T))$ of the special form above.

Lemma 12.3 [Protter and Weinberger 1984, Theorem 6, page 174]. *Let $\Omega \subset \mathbb{R}^n \times \mathbb{R}_+$ be a connected open set satisfying the interior ball condition. Assume $f \in C^{2,1}(\Omega)$ satisfies the uniformly parabolic inequality*

$$\partial_t f - \mathrm{tr}_g d^2 f - d_X f \le 0.$$

Here $g = g_t$ is a Riemannian metric in each section $\Omega(t)$, and X_t is a bounded vector field in $\Omega(t)$. Denote by $n = n_t$ the inner unit normal of $\Omega(t)$. Assume the supremum M of f in $\Omega_{\bar{t}} := \Omega \cap \{t \le \bar{t}\}$ is attained at the point $P \in \partial\Omega(\bar{t})$, and that $f < M$ for $t < \bar{t}$. Then $d_n f(P) < 0$.

We now state the hypotheses of our tensorial maximum principle.

$E \subset \mathbb{R}^n \times [0, T]$ is the image of a cylinder $D_0 \times (0, T)$ under a $C^{3,2}$ diffeomorphism Φ of the form $\Phi(x, t) = (\varphi_t(x), t)$ with $\varphi_t : D_0 \to D(t)$ a C^3 diffeomorphism up to the boundary for each $t \in [0, T]$ (here $D(t)$ is the t-level set of E); $\bar{D}_0 \subset \mathbb{R}^n$ is assumed to be the image of the closed unit ball under a C^3 diffeomorphism. In particular, the lateral boundary $\partial_l E$ is of class $C^{3,2}$. On $\partial_l E$ we have the *inner* unit normal $n = n_t \in \mathbb{R}^n$. Extend n_t to a vector field in all of $\bar{D}(t)$ so that it is in $C^{2,1}(\bar{E}, \mathbb{R}^n)$, arbitrarily except for the requirements that $|n| \le 1$ pointwise and $d_n n = 0$ in a tubular neighborhood of $\partial D(t)$ (equivalently, $n^i \partial_i n^j = 0$ for each j). Fix $R > 0$ so that $D(t) \subset B_R(0)$ for each $t \in [0, T]$.

The assumptions on the coefficients are given next:

- $g = g_t$ is a t-dependent Riemannian metric in $\bar{D}(t)$, uniformly equivalent to the euclidean metric for $t \in [0, T]$;
- $X = X_t$ is a bounded t-dependent vector field in $\bar{D}(t)$;
- $q = q(z, m)$ assigns to each $z \in \bar{E}$ and each m in $\mathbb{S}$ (the space of quadratic forms in $\mathbb{R}^n$) a quadratic form $q \in \mathbb{S}$. q is assumed to be $C^{2,1}$ in z, locally Lipschitz in m (uniformly in $z \in \bar{E}$);
- $b = b(z, m) \in \mathbb{S}$ is defined for $z \in \partial_l E$, with the same regularity assumptions as q.

We state the next theorem in terms of the Laplace–Beltrami heat operator $\partial_t - \Delta_g$ and the g-Riemannian connection ∇, but the result also holds for L and the "euclidean connection" d.

Theorem 12.4. *Assume $m \in C^{2,1}(\bar{E}; \mathbb{S})$ satisfies in E the tensorial differential inequality*

$$\partial_t m_{ij} - (\Delta_g m)_{ij} \le (\nabla_X m)_{ij} + q_{ij}(\,\cdot\,, m(\,\cdot\,)),$$

and on $\partial_l E$ the boundary condition

$$(\nabla_n m)_{ij}(z) \ge b_{ij}(z, m(z)).$$

Suppose the functions q and b satisfy the following null eigenvector conditions: for any $\hat{m} \in \mathbb{S}$ and any null eigenvector $V \in \mathbb{R}^n$ of $\hat{m}$ (meaning that $\hat{m}_{ij} V^j = 0$ for all i), we have $q_{ij}(z, \hat{m}) V^i V^j \le 0$ for all $z \in \bar{E}$ and $b_{ij}(z, \hat{m}) V^i V^j \ge 0$ for all $z \in \partial_l E$. Then weak concavity of m at $t = 0$ is preserved:

$$m \le 0 \ \textit{in}\ D(0) \quad \Longrightarrow \quad m \le 0 \ \textit{in}\ \bar{E}.$$

Proof. The assumptions imply that there is a $K > 0$ (depending only on E and on the functions X, g, n, q, and b) satisfying

$$|n|_{C^{2,1}(\bar{E})} \le K, \quad |X(z)|_{\mathrm{eucl}} \le K, \quad |g(z)| + |g^{-1}(z)| \le K, \quad z \in \bar{E},$$

and if $m, \hat{m} \in C^{2,1}(\bar{E}, \mathbb{S})$ satisfy (for some $\mu : \bar{E} \to \mathbb{R}_+$)

$$-\mu(z) g \le m(z) - \hat{m}(z) \le \mu(z) g,$$

(where the inequality of quadratic forms has the usual meaning) then also

$$\begin{aligned} q(z, m(z)) &\le q(z, \hat{m}(z)) + K\mu(z) g, \quad z \in \bar{E}, \\ b(z, m(z)) &\ge b(z, \hat{m}(z)) - K\mu(z) g, \quad z \in \partial_l E. \end{aligned}$$

Now, for $z \in \bar{E}$, $z = (x, t)$ define

$$\varphi(z) := -2Kn(z) \cdot x := 2Ks(z),$$

where we use the euclidean inner product and, on $\partial_l E$, s is the "support function" of $\partial D(t)$ (positive if $D(t)$ is convex and contains the origin). It is clear that we can find $M = M(R, K) > 0$ depending only on K, R and $|n|_{C^{2,1}}$ so that

$$|\varphi|_{C^{2,1}} \le M, \qquad |d\varphi|_g^2 + |\Delta_g \varphi| \le M, \qquad |X \cdot d\varphi| \le M.$$

We assume also $M \ge K$. Now, given m as in the statement of the theorem and given constants $\epsilon > 0$, $\gamma > 0$, and $\delta > 0$, define for $E^\delta := E \cap \{t < \delta\}$

$$\hat{m}(z) := m(z) - (\epsilon t + \gamma e^{\varphi(z)}) g, \quad z \in \bar{E}^\delta.$$

Clearly $\hat{m} \in C^{2,1}(\bar{E}^\delta; \mathbb{S})$. We now derive the constraints on δ, ϵ, and γ. It will turn out that δ must be taken small enough (depending only on K, R), $\epsilon > 0$ is arbitrary, and γ is ϵ times a constant depending only on K, R.

The following inequalities are easily derived:

$$
\begin{gathered}
q(z, m(z)) \leq q(z, \hat{m}(z)) + K(\epsilon t + \gamma e^{\varphi(z)})g, \\
\nabla_X m = \nabla_X \hat{m} + \gamma (e^{\varphi} d_X \varphi) g \leq \nabla_X \hat{m} + (\gamma e^{\varphi} M) g, \\
\partial_t \hat{m} = \partial_t m - \epsilon g - (\gamma e^{\varphi} \partial_t \varphi) g \leq \partial_t m + (\gamma e^{\varphi} M) g - \epsilon g, \\
\Delta_g \hat{m} = \Delta_g d^2 \hat{m} - \gamma e^{\varphi} (|d\varphi|_g^2 + \Delta_g \varphi) g \geq \Delta_g m - (\gamma e^{\varphi} M) g, \\
b(z, m(z)) \geq b(z, \hat{m}(z)) - K(\epsilon t + \gamma e^{\varphi}) g.
\end{gathered}
$$

We use this to compute

$$
\begin{aligned}
\partial_t \hat{m} - \Delta_g \hat{m} &\leq \partial_t m - \Delta_g m + (2\gamma e^{\varphi} M) g - \epsilon g \\
&\leq q(z, m(z)) + \nabla_X m + (2\gamma e^{\varphi} M) g - \epsilon g \\
&\leq q(z, \hat{m}(z)) + \nabla_X \hat{m} + K(\epsilon t + \gamma e^{\varphi}) g + (3M\gamma e^{\varphi}) g - \epsilon g \\
&\leq q(z, \hat{m}(z)) + \nabla_X \hat{m} + M\epsilon t g + 4M\gamma e^{\varphi} g - \epsilon g,
\end{aligned}
$$

using $K \leq M$ in the last step. We conclude the inequality

$$
\partial_t \hat{m} - \Delta_g \hat{m} \leq q(z, \hat{m}(z)) + \nabla_X \hat{m} - (\epsilon/2) g \tag{12-1}
$$

will hold in E^{δ}, provided the constants are selected so that, for $z \in E^{\delta}$

$$
4M\gamma e^{\varphi(z)} + M\epsilon t \leq \epsilon/2. \tag{12-2}
$$

Turning to boundary points $z = (x, t) \in \partial_l E$, note that $d_n \varphi = -2K$, so that

$$
\begin{aligned}
\nabla_n \hat{m}(z) &= \nabla_n m(z) - (\gamma e^{\varphi(z)} d_n \varphi(z)) g \\
&\geq b(z, m(z)) - (\gamma e^{\varphi(z)} d_n \varphi(z)) g \\
&\geq b(z, \hat{m}(z)) - K(\epsilon t + \gamma e^{\varphi(z)}) g - (\gamma e^{\varphi(z)} d_n \varphi(z)) g \\
&\geq b(z, \hat{m}(z)) + K(\gamma e^{\varphi(z)} - \epsilon t) g,
\end{aligned}
$$

implying the inequality

$$
\nabla_n \hat{m}(z) \geq b(z, \hat{m}), \quad z \in \partial_l E^{\delta} \tag{12-3}
$$

will hold provided the constants are so chosen that, on $\partial_l E^{\delta}$

$$
\epsilon t \leq \gamma e^{\varphi(z)}. \tag{12-4}
$$

Bearing in mind that $e^{-2KR} \leq e^{\varphi(z)} \leq e^{2KR}$ on E, it is not hard to arrange for (12-2) and (12-4) to hold, or equivalently, for

$$
\epsilon t \leq \gamma e^{\varphi(z)}, \qquad 10M\gamma e^{\varphi(z)} \leq \epsilon.
$$

Given $\epsilon > 0$, define γ so that $10M\gamma e^{2KR} = \epsilon$. Then the second inequality holds, and so will the first, provided that

$$
\epsilon t \leq \gamma e^{-2KR} = (\epsilon/10M) e^{-4KR},
$$

which is true for any $\epsilon > 0$, if δ is defined by $\delta := e^{-4KR}/10M$ (recall $t \in [0, \delta]$).

Note that, since $m \le 0$ at $t = 0$, it follows that $\hat{m}$ is negative definite at $t = 0$, and hence also for small time, and we *claim* that this persists throughout $\bar{E}^\delta$ so that (letting $\epsilon \to 0$) $m \le 0$ in $\bar{E}^\delta$. Restarting the argument at $t = \delta$, we see that this is enough to prove the theorem.

To prove the claim, suppose for a contradiction that $\hat{m}$ acquires a null eigenvector $0 \ne V \in \mathbb{R}^n$ at a point $z_1 = (x_1, t_1) \in \bar{E}^\delta$ with $t_1 \in (0, \delta]$ the first time this happens.

Let $\hat{f}(z) := \hat{m}_{ij} V^i V^j$ for $z \in E^\delta$ (that is, we "extend" V to E^δ as a constant vector). It follows from (12-1) that $\hat{f}$ satisfies in E^δ

$$\partial_t \hat{f} \le (\Delta_g \hat{m})_{ij} V^i V^j + (\nabla_X \hat{m})_{ij} V^i V^j + q_{ij}(\,\cdot\,, \hat{m}) V^i V^j - \tfrac{1}{2}\epsilon |V|_g^2.$$

A short, standard Riemannian calculation using the fact that V is a null eigenvector for $\hat{m}$ shows that

$$d_X \hat{f} = (\nabla_X \hat{m})_{ij} V^i V^j, \quad \Delta_g \hat{f} = (\Delta_g \hat{m})_{ij} V^i V^j.$$

Using the null eigenvector condition for q, we find that $\hat{f}$ satisfies in E^δ the strict inequality

$$\partial_t \hat{f} < \mathrm{tr}_g d^2 \hat{f} + d_X \hat{f}.$$

This shows x_1 cannot be an interior point of $D(t_1)$, for then (as a first-time interior maximum point for $\hat{f}$) we would have $\Delta_g \hat{f}(z_1) \le 0$ and $d\hat{f}(z_1) = 0$, contradicting $\partial_t \hat{f}(z_1) \ge 0$. Thus $x_1 \in \partial D(t_1)$. Since $\hat{f}$ satisfies the differential inequality just stated and $z_1 = (x_1, t_1)$ is a first-time boundary maximum in $\bar{E}^\delta$, the parabolic Hopf lemma (Lemma 12.3) implies $d_n \hat{f}(z_1) < 0$. On the other hand, as seen in (12-3),

$$d_n \hat{f} = (\nabla_n \hat{m})_{ij} V^i V^j \ge b_{ij}(z_1, \hat{m}(z_1)) V^i V^j \ge 0,$$

from the null eigenvector condition on the boundary. This contradiction concludes the proof. □

Corollary 12.5. *Suppose $m \in C^{2,1}(\bar{E}, \mathbb{S})$ satisfies the same differential inequality with the same hypotheses on the coefficients as in Theorem 12.4 (including the null eigenvector condition for q), and the boundary conditions*

$$\begin{cases} m(z)(n, \tau) = 0, & \forall z = (x, t) \in \partial_l E,\ \tau \in T_x \partial D(t) \\ (\nabla_n m)(n, n) \ge b_{nn}(z, m(z)) & \\ (\nabla_n m)(\tau, \tau) \ge b^{\tan}(z, m(z))(\tau, \tau), & \tau \in T_x \partial D(t), \end{cases}$$

for functions $b_{nn}(z, \hat{m})$ from $\partial_l E \times \mathbb{S}$ to $\mathbb{R}$ and $b^{\tan}$ assigning to $(z, \hat{m}) \in \partial_l E \times \mathbb{S}$, $z = (x, t)$, a quadratic form in $T_x \partial D(t)$. Suppose $b_{nn} \ge 0$ in $E \times \mathbb{S}$ and $b^{\tan}$ satisfies, for each $\hat{m} \in \mathbb{S}$,

$$\hat{m}_{ij} \tau^i = 0 \text{ for some } \tau \in T_x \partial D(t) \implies b^{\tan}(z, \hat{m})(\tau, \tau) \ge 0.$$

Then, as in the theorem, weak concavity is preserved:

$$m \le 0 \text{ at } t = 0 \implies m \le 0 \text{ in } \bar{E}.$$

Proof. As for the theorem, with the following change in the last part of the proof: If $0 \ne V \in \mathbb{R}^n$ is a null eigenvector of $\hat{m}$ (defined as in the proof of the theorem) at a boundary point $z_1 = (x_1, t_1) \in \partial_l E$, write

$$V = V^n n + V^T, \quad V^T \in T_{x_1} \partial D(t_1).$$

Assume first that $V^n \neq 0$. Then, noting that $\hat{m}$ splits at the boundary if m does, we see that n is a null eigenvector of $\hat{m}$ at z_1, so we define $\hat{f}(z) = \hat{m}_{ij}(z)n^i(z_1)n^j(z_1)$ and repeat the argument. At z_1, $(\nabla_n \hat{m})(n, n) \geq b_{nn}(z_1, \hat{m}(z_1)) \geq 0$ leads to a contradiction with the parabolic Hopf lemma, as before.

If $V^n = 0$, then $V^T \in T_{x_1}\partial D(t_1)$ must be a null eigenvector of $\hat{m}$ at the boundary point z_1, and then we run the argument with $\hat{f}(z) = \hat{m}(z)(V^T, V^T)$, leading to a contradiction, as before. □

Corollary 12.6. *Let $w \in C^{4,2}(E)$ define a mcm of graphs with constant-angle boundary conditions, where E is as in the statement of Theorem 12.4. Then weak concavity is preserved*:

$$h \leq 0 \ \text{ at } t = 0 \quad \Longrightarrow \quad h \leq 0 \text{ in } \bar{E}.$$

Proof. From Lemma 11.2, h satisfies $(\partial_t - \Delta_g)h_{ij} = H\nabla_\omega h_{ij} + q(z, h)_{ij}$ and

$$q(z, h)_{ij} = H_i h(\omega, \partial_j) + H_j h(\omega, \partial_i) + |h|_g^2 h_{ij} + Hh(\partial_i, \omega)h(\partial_j, \omega) - Hh(\omega, \omega)h_{ij},$$

where H_i, H_j, H and ω^i are regarded as fixed functions of $z \in E$. Clearly q satisfies the null eigenvector condition, since $q_{ij}V^iV^j = 0$ when $h_{ij}V^j = 0$ for all i. In addition, expressions obtained for $d_n h$ in Lemma 11.2 show that the boundary conditions in Corollary 12.5 are satisfied with

$$b_{nn}(z, \hat{m}) \equiv 0, \qquad b^{\tan}(z, \hat{m}) = -((\hat{m})^{\tan})^2 + \beta^2 \hat{m}_{nn}\hat{m}^{\tan}.$$

Hence the claim follows from Corollary 12.5. □

For less regular solutions, we may apply the theorem to a domain $E_{t_0} = E \cap \{t > t_0\}$ for arbitrarily small $\delta > 0$. Thus, assuming $h < 0$ at $t = 0$ (strictly negative definite), we conclude from Corollary 12.6 that $h \leq 0$ for all t.

Remark. It seems plausible that a slightly different version of the result in this section could be used to strengthen the conclusions in [Stahl 1996].

Finite existence time. It is not difficult to derive that the flow is defined only for finite time in the concave case.

Lemma 12.7. *Let $w \in C^{4,2}(E)$, $E \subset \mathbb{R}^n \times [0, T)$, define a graph mcm Σ_t with constant-angle boundary conditions on a moving boundary. Assume that Σ_0 (and hence Σ_t, for all t) is weakly concave. Assume that $H_{|t=0} \leq H_0 < 0$, where H_0 is a negative constant, and that $T = \sup\{t \in [0, T) : D(t) \neq \varnothing\}$. Then $T \leq t_* = 1/(2H_0^2 c_n)$, where $c_n > 0$ depends only on n and an upper bound for v in E.*

The proof is based on the evolution equation and boundary condition for H (see Appendix B; we have $\omega = Dw/v$):

$$L[H] = |h|_g^2 H + Hh^2(\omega, \omega) - H^2 h(\omega, \omega), \quad H_n = (\beta^2/\beta_0)Hh_{nn}.$$

Since $h^2(\omega, \omega) \geq 0$, $|h|_g^2 \geq (1/n)H^2$ and (given that $h \leq 0$) $h(\omega, \omega) \geq |Dw|^2 H$, we have

$$L[H] \leq \frac{1}{n}H^3 + |Dw|^2 H^3 \leq c_n H^3,$$

where c_n depends on n and on $\sup_E |v|$, already known to be finite. Let $\phi(t)$ solve the ODE $\dot{\phi} = c_n\phi^3$, $\phi(0) = H_0$, so

$$\phi(t) = H_0[1 - 2c_n H_0^2 t]^{-1/2}, \quad 0 \leq t < t_* := \frac{1}{2H_0^2 c_n}.$$

Then, with $\psi := \frac{1}{n}(H^2 + H\phi + \phi^2) > 0$ and setting $\chi = H - \phi$, we have $L[\chi] \le \psi\chi$ in E and

$$\chi_n = \frac{\beta^2}{\beta_0}(\chi + \phi)h_{nn} \ge \frac{\beta^2}{\beta_0}\chi \quad \text{on } \partial_l E,$$

since $\phi < 0$ and $h_{nn} \le 0$. Given that $\chi \le 0$ at $t = 0$, it follows from the maximum principle that $\chi \le 0$, or $H \le \phi$ in $[0, \min\{T, t_*\})$. This shows $t_* < T$ is impossible, since $\phi \to -\infty$ as $t \to t_*$.

Remark 12.8. It would be natural to try to show that a negative upper bound H_0 on the mean curvature (at $t = 0$) is preserved, at least under the assumption of concavity. Unfortunately, the evolution equation for H (under graph mcm) does not lend itself to a maximum principle argument. Letting $u := H - H_0$, we have

$$L[u] = |h|_g^2 u + uh^2(\omega, \omega) - u(H + H_0)h(\omega, \omega) + H_0 Q \quad \text{in } E,$$

with

$$Q := |h|_g^2 + h^2(\omega, \omega) - H_0 h(\omega, \omega). \tag{12-5}$$

At a point where $u = 0$, we would need to show $L[u] \le 0$. But it is not true that $Q \ge 0$ at such a point. (Note that $u_n \ge 0$ does hold at boundary points.)

The exception is if $n = 2$ (under an additional condition). Let $\hat{\omega} = \omega/|\omega|_g$, $\tilde{\omega} = \omega^\perp/|\omega|_g$. It is easy to check that $\mathcal{B} = \{\hat{\omega}, \tilde{\omega}\}$ is a g-orthonormal frame at each point where $\omega \ne 0$. Then with

$$a := h(\hat{\omega}, \hat{\omega}), \qquad b := h(\hat{\omega}, \tilde{\omega}), \qquad c := h(\tilde{\omega}, \tilde{\omega}),$$

we have

$$h^2(\hat{\omega}, \hat{\omega}) - Hh(\hat{\omega}, \hat{\omega}) = a^2 + b^2 - (a + c)a = b^2 - ac = -\Delta,$$

where Δ, the determinant of the matrix of h in $\mathcal{B}$, is nonnegative if $h \le 0$. In particular,

$$h^2(\omega, \omega) - Hh(\omega, \omega) = -|\omega|_g^2 \Delta \le 0$$

in the concave case. Now consider the expression (12-5) for Q, at a point where $u = 0$, or $H = H_0$. Since $|\omega|_g^2 = |Dw|^2$ we can write

$$\begin{aligned} Q &= |h|_g^2 + h^2(\omega, \omega) - Hh(\omega, \omega) \\ &= a^2 + 2b^2 + c^2 + |Dw|^2(b^2 - ac) \\ &= b^2(2 + |Dw|^2) + a^2 - |Dw|^2 ac + c^2, \end{aligned}$$

so $Q \ge 0$ provided $|Dw|^2 \le 2$. This last condition is equivalent to $\upsilon \le \sqrt{3}$, and hence (Lemma 12.2) is preserved by the evolution if it holds at $t = 0$. Thus:

Proposition 12.9. *Assume $n = 2$, $h \le 0$, and $\upsilon \le \sqrt{3}$ on Σ_0 (in particular, $\beta \ge 1/\sqrt{3}$). Then $H \le H_0 < 0$ at $t = 0$ implies $H \le H_0$ for all $t \in [0, T_{\max})$.*

13. Global bounds from boundary bounds for $\nabla^n h$

In this section we begin to develop a continuation criterion for solutions of graph mean curvature motion with constant contact angle based on the second fundamental form. Our first observation is that the supremum of $|h|_g$ on the moving boundary controls its value in the interior. Recall we already have a bound on $\sup_E \upsilon$ (Lemma 12.2) and it is a well known-fact for mean curvature flow of graphs that this implies interior bounds for the second fundamental form and its covariant derivatives [Ecker and Huisken 1991; Ecker 2004]. In the next lemma we describe a global bound for mean curvature motion of graphs with moving boundaries.

Lemma 13.1. *Let $w : E \to \mathbb{R}$ be a (sufficiently regular) solution of graph mcm in a spacetime domain $E \subset \mathbb{R}^n \times [0, T]$, where $T < \infty$. Assume the first derivative bound $v(x, t) \le \bar{v}$ holds globally in $\bar{E}$. Then if the bound $|h|_g \le h_0$ holds on the parabolic boundary $\partial_p E$, we also have the global bound*

$$|h|_g \le a_0 \quad \text{in } \bar{E}$$

for a constant a_0 depending only on $n, \bar{v}, h_0, T$ and the initial data of w.

Proof. The proof is simpler under the assumption that h is negative definite, that is, the concave case. (As shown in the previous section, this condition is preserved if it holds at $t = 0$.) We give the details in this case only.

The norms of tensors in $D(t)$ will always be taken with respect to the induced metric g, so we write $|h|$ for $|h|_g$, $|\nabla h|$ for $|\nabla h|_g$, and $|Df|^2 = g^{ij} f_i f_j$ for a function f.

Recall the evolution equations $L[v] = -v|h|^2 - 2|Dv|^2/v$ (so $L[v^2] = -2v^2|h|^2 - 6|Dv|^2$) and

$$L[|h|^2] = -2|\nabla h|^2 + 2|h|^4 - 4Hh^3(\omega, \omega) - 2H|h|^2 h(\omega, \omega).$$

In the concave case $H \le 0$ and h^3 is negative definite, so we get

$$L[|h|^2] \le -2|\nabla h|^2 + 2|h|^4.$$

The idea then is to apply the maximum principle to $f = |h|^2 v^2$. In the evolution equation for f,

$$L[f] = v^2 L[|h|^2] + |h|^2 L[v^2] - 2\langle D|h|^2, Dv^2\rangle_g,$$

the terms $\pm 2v^2|h|^4$ cancel exactly, and we have the inequality

$$L[f] \le -2v^2|\nabla h|^2 - 6|h|^2|Dv|^2 - 2\langle D|h|^2, Dv^2\rangle_g.$$

The term with the inner product can be estimated in two ways:

$$|\langle D|h|^2, Dv^2\rangle_g| \le \big|D|h|^2\big|\,|Dv^2| \le 4\big|h|v|\nabla h\big|\,|Dv| \le 2v^2|\nabla h|^2 + 2|h|^2|Dv|^2$$

and

$$\langle D|h|^2, Dv^2\rangle_g = \frac{1}{v^2}\langle D(|h|^2 v^2), Dv^2\rangle_g - \frac{|h|^2}{v^2}|Dv^2|^2 = \frac{1}{v^2}\langle Df, Dv^2\rangle_g - 4|h|^2|Dv|^2.$$

Using the second expression, we have

$$L[f] \le -2v^2|\nabla h|^2 - 6|h|^2|Dv|^2 - \frac{1}{v^2}\langle Df, Dv^2\rangle_g + 4|h|^2|Dv|^2 - \langle D|h|^2, Dv^2\rangle_g,$$

and then estimating the remaining inner product term from the first expression

$$L[f] \le -2v^2|\nabla h|^2 - 6|h|^2|Dv|^2 - \frac{1}{v^2}\langle Df, Dv^2\rangle_g + 4|h|^2|Dv|^2 + 2v^2|\nabla h|^2 + 2|h|^2|Dv|^2,$$

yielding after cancellation

$$L[f] \le -\frac{1}{v^2}\langle Df, Dv^2\rangle_g.$$

Applying the (weak) maximum principle to f, we conclude

$$\max_{\bar{E}} |h|^2 \le \max_{\bar{E}} f \le \max_{\partial_p E} f \le \bar{v}^2 \max_{\partial_p E} |h|^2,$$

which implies the result (for the concave case) with an explicit constant $a_0 = \bar{v} h_0$.

In the general case, we have

$$L[|h|^2] \le -2|\nabla h|^2 + c_n |h|^4.$$

Then the proof follows the same lines as [Ecker 2004, Proposition 3.21]. We apply the maximum principle to $f = |h|^2(\eta \circ v^2)$, for a carefully chosen function $\eta(s)$. □

Evolution of $|\nabla h|^2$. In the calculation that follows, we adopt the usual convention that in symbols such as $\nabla^2 h * (\nabla h)^{(2)} * h^{(3)}$ and $(\nabla^j h)^{(p)} = \nabla^j h * \cdots * \nabla^j h$ (p times), $*$ denotes some unspecified g-contraction of the tensors in question.

For the time derivative, we have

$$\begin{aligned}\partial_t |\nabla h|^2 &= 2\langle \partial_t(\nabla h), \nabla h\rangle + \partial_t(g^{ij} g^{pq} g^{rs})(\nabla_i h)_{pr} (\nabla_j h)_{qs} \\ &= 2\langle \partial_t(\nabla h), \nabla h\rangle + 3(\partial_t g^{ij})\langle \nabla_i h, \nabla_j h\rangle,\end{aligned}$$

using the Codazzi identity.

For the Hessian (using $\nabla_k \partial_l = h_{kl}\omega$, derived as (B-1) in Appendix B), we get

$$\begin{aligned}\nabla^2_{k,l} |\nabla h|^2 &= 2\langle \nabla_l(\nabla_k \nabla h), \nabla h\rangle + 2\langle \nabla_k \nabla h, \nabla_l \nabla h\rangle - h_{kl} d_\omega |\nabla h|^2 \\ &= 2\langle \nabla^2_{k,l}(\nabla h), \nabla h\rangle + 2\langle h_{kl} \nabla_\omega \nabla h, \nabla h\rangle + 2\langle \nabla_k \nabla h, \nabla_l \nabla h\rangle - h_{kl} d_\omega |\nabla h|^2 \\ &= 2\langle \nabla^2_{k,l}(\nabla h), \nabla h\rangle + 2\langle \nabla_k \nabla h, \nabla_l \nabla h\rangle,\end{aligned}$$

after cancellation. Taking traces we find

$$(\partial_t - \Delta)|\nabla h|^2 = -2|\nabla^2 h|^2 + 2\langle (\partial_t - \Delta)(\nabla h), \nabla h\rangle + 3(\partial_t g^{ij})\langle \nabla_i h, \nabla_j h\rangle.$$

Commutation of covariant derivatives introduces the Riemann curvature tensor, and the time derivative of the connection is also needed:

$$(\partial_t - \Delta)(\nabla h) = \nabla[(\partial_t - \Delta)h] + (\nabla \mathrm{Rm}) * h + \mathrm{Rm} * (\nabla h) + (\partial_t \Gamma) * h,$$

where (see appendix)

$$\partial_t h = \nabla d H + H \nabla_\omega h + T + h^{(3)}, \quad T_{ij} = H_i h(\omega, \partial_j) + H_j h(\omega, \partial_i),$$

which combined with $\Gamma = h\omega$ and $\partial_t \omega = \nabla H + h^{(2)}$ is easily seen to imply

$$\partial_t \Gamma = (\nabla d H)\omega + \nabla h * h + h^{(3)} \sim \nabla^2 h + \nabla h * h + h^{(3)}.$$

From the Gauss equation, $\mathrm{Rm} \sim h * h$. Thus

$$\langle (\partial_t - \Delta)(\nabla h), \nabla h\rangle \sim \langle \nabla[(\partial_t - \Delta)h], \nabla h\rangle + \nabla^2 h * \nabla h * h + (\nabla h)^{(2)} * h^{(2)} + \nabla h * h^{(4)}.$$

On the other hand, from the evolution equation for h (Appendix B) we have

$$\langle \nabla[(\partial_t - \Delta)h], \nabla h\rangle = \langle \nabla(H\nabla_\omega h + T + h^{(3)}), \nabla h\rangle = \langle \nabla(H \nabla_\omega h), \nabla h\rangle + \langle \nabla T, \nabla h\rangle + (\nabla h)^{(2)} * h^{(2)}.$$

Computing the terms on the right, we find

$$\langle\nabla(H\nabla_\omega h),\nabla h\rangle=\langle\nabla_\omega h,\nabla_{\nabla H}h\rangle+H\langle\nabla(\nabla_\omega h),\nabla h\rangle=\langle\nabla_\omega h,\nabla_{\nabla H}h\rangle+\nabla^2 h*\nabla h*h,$$

and using the Codazzi identity

$$\langle\nabla T,\nabla h\rangle=2\langle\nabla_\omega h,\nabla_{\nabla H}h\rangle+\nabla^2h*\nabla h*h+(\nabla h)^{(2)}*h^{(2)}.$$

Putting together these results, we have

$$\langle(\partial_t-\Delta)(\nabla h),\nabla h\rangle=3\langle\nabla_\omega h,\nabla_{\nabla H}h\rangle+\nabla^2h*\nabla h*h+(\nabla h)^{(2)}*h^{(2)}+\nabla h*h^{(4)}.$$

On the other hand, using the expression for $\partial_t g^{ij}$ given in the appendix we find

$$3\partial_t g^{ij}\langle\nabla_i h,\nabla_j h\rangle=-6\langle\nabla_\omega h,\nabla_{\nabla H}h\rangle+(\nabla h)^{(2)}*h^{(2)}.$$

So we have cancellation, and obtain the evolution equation

$$(\partial_t-\Delta)|\nabla h|^2=-2|\nabla^2h|^2+\nabla^2h*\nabla h*h+(\nabla h)^{(2)}*h^{(2)}+\nabla h*h^{(4)}.$$

Remark. Without the cancellation, the right-hand side would involve terms of type $(\nabla h)^{(3)}$, which would be a problem for the argument that follows.

Given this calculation, the following lemma has a very simple proof.

Lemma 13.2. *For a solution $w\in C^{5,3}(E)$, assume we have a uniform bound for h: $|h|\le a_0$ in E. Then there are constants $\alpha>0$, $C>0$ depending only on the dimension and a_0, so that the function*

$$f(x,t)=\alpha|\nabla h|^2+|h|^2$$

is a subsolution in E, that is, $(\partial_t-\Delta)f\le C$.

Proof. The calculation above implies that

$$(\partial_t-\Delta)|\nabla h|^2\le-2|\nabla^2h|^2+c_n\big(a_0|\nabla^2h|\,|\nabla h|+a_0^2|\nabla h|^2+a_0^4|\nabla h|\big),$$

while the evolution equation for $|h|^2$ implies that

$$(\partial_t-\Delta h)|h|^2\le-2|\nabla h|^2+c_n(a_0^2|\nabla h|+a_0^4).$$

Clearly we may choose α small enough to satisfy the claim. □

Our next goal is to extend this argument to higher covariant derivatives of h. It turns out this does not involve a cancellation similar to the one noted above. The terms appearing in each expression below all have the same weight, the weight of a term $T=(\nabla^{j_1}h)^{(p_1)}*\cdots*(\nabla^{j_r}h)^{(p_r)}$ being the positive integer

$$w[T]=\sum_{i=1}^r p_i(j_i+1)$$

— in particular, $w[\nabla^j h]=j+1$ and $w[(\nabla^j h)^{(p)}]=p(j+1)$ for $j\ge0$, and $p\ge1$. We introduce a convenient notation for the "error terms". For integers $w_0\ge1$ and $n\ge0$, the notation $\tilde E^{w_0,n}$ is used for a generic term of weight w_0 and involving covariant derivatives of h of order at most n; i.e.,

$$T=\tilde E^{w_0,n}\quad\text{means}\quad w[T]=w_0,\quad j_i\le n.$$

The symbol $E^{w_0,n}$ denotes such a term satisfying the additional restrictions

$$p_i = \begin{cases} 1 & \text{if } j_i = n, \\ 1 \text{ or } 2 & \text{if } j_i = n-1 \text{ and } n \geq 1. \end{cases}$$

Sometimes the same notation is used for the real vector space spanned by terms of the given type. For example, above we showed that

$$(\partial_t - \Delta)|h|^2 = -2|\nabla h|^2 + E^{4,1} \quad \text{and} \quad (\partial_t - \Delta)|\nabla h|^2 = -2|\nabla^2 h|^2 + E^{6,2}. \tag{13-1}$$

These symbols have some useful properties. For example, one sees by induction that

$$\nabla(E^{n+3,n+1}) \subset E^{n+4,n+2}, \quad n \geq 0,$$

using the easily checked fact that

$$E^{n+3,n+1} = (\nabla^{n+1} h) * h + (\nabla^n h) * [\nabla h + h^{(2)}] + \tilde{E}^{n+3,n-1}, \quad n > 1. \tag{13-2}$$

The property (13-1) generalizes to higher n:

Lemma 13.3. $(\partial_t - \Delta)|\nabla^n h|^2 = -2|\nabla^{n+1} h|^2 + E^{2n+4,n+1}$ *for* $n \geq 0$.

Proof (*for* $n \geq 2$). With the natural multiindex notation,

$$\partial_t |\nabla h|^2 = 2\langle \partial_t(\nabla^n h), \nabla^n h\rangle + \partial_t(g^{IJ} g^{pr} g^{qs})(\nabla^n_I h)_{pq}(\nabla^n_J h)_{rs}, \quad |I| = |J| = n.$$

Using the Codazzi identity and the curvature tensor repeatedly, we obtain

$$\begin{aligned} \partial_t(g^{IJ} g^{pr} g^{qs})(\nabla^n_I h)_{pq}(\nabla^n_J h)_{rs} = (n+2)(\partial_t g^{ij})\langle \nabla_i \nabla^{n-1} h, \nabla_j \nabla^{n-1} h\rangle \\ + (\partial_t g^{ij})\mathrm{Rm}[\nabla^{n-2} h]_i * \mathrm{Rm}[\nabla^{n-2} h]_j + (\partial_t g^{ij})(\nabla^n h)_i * \mathrm{Rm}[\nabla^{n-2} h]_j. \end{aligned}$$

Since $\partial_t g^{ij} = \nabla h + h^{(2)}$ (see (B-4) in Appendix B) and $\mathrm{Rm} = h * h$, this reduces to

$$(\nabla^n h)^{(2)} * (\nabla h + h^{(2)}) + (\nabla^{n-2} h)^{(2)} * (\nabla h + h^{(2)}) * h^{(4)} + (\nabla^n h) * (\nabla^{n-2} h) * (\nabla h + h^{(2)}) * h^{(2)},$$

which is in $E^{2n+4,n+1}$.

Turning to space derivatives, we have (as for $n = 1$)

$$\Delta|\nabla^n h|^2 = 2\langle \Delta(\nabla^n h), \nabla^n h\rangle + 2|\nabla^{n+1} h|^2,$$

and therefore

$$(\partial_t - \Delta)|\nabla^n h|^2 = -2|\nabla^{n+1} h|^2 + 2\langle (\partial_t - \Delta)(\nabla^n h), \nabla^n h\rangle + E^{2n+4,n+1}.$$

The conclusion of the lemma is now an immediate consequence of the next claim, and of the expression (13-2) for a general term in $E^{n+3,n+1}$. □

Claim. $(\partial_t - \Delta)[\nabla^n h] \in E^{n+3,n+1}$ for $n \geq 0$.

Proof. We work by induction on n, the cases $n = 0, 1$ having already been checked:

$$(\partial_t - \Delta)h = H\nabla_\omega h + T + h^{(3)} \in E^{3,1}, \quad (\partial_t - \Delta)(\nabla h) = \nabla[(\partial_t - \Delta)h] + E^{4,2} \in E^{4,2}.$$

For the induction step, it is enough to show that

$$(\partial_t - \Delta)[\nabla^{n+1} h] = \nabla\big[(\partial_t - \Delta)(\nabla^n h)\big] + E^{n+4,n+2},$$

since $\nabla E^{n+3,n+1} \subset E^{n+4,n+2}$.

For the time derivative part, we have, for any multiindex iI of length $n+1$ (with $n = |I|$),

$$\begin{aligned}\partial_t[\nabla^{n+1} h]_{iI} &= \partial_t\big[\partial_i(\nabla^n h[\partial_I]) - (\nabla^n h)(\nabla_i \partial_I)\big] = \partial_i(\partial_t(\nabla^n h[\partial_I])) - \partial_t(\nabla^n h[\nabla_i \partial_I]) \\ &= \nabla_i(\partial_t(\nabla^n h))[\partial_I] + \partial_t(\nabla^n h)(\nabla_i \partial_I) - \partial_t(\nabla^n h)(\nabla_i \partial_I) - \nabla^n h[\partial_t(\nabla_i \partial_I)].\end{aligned}$$

For a multiindex $I = i_1 \dots i_n$ of length n denote by I^k_p the multiindex of length n obtained from I by setting its k-th entry i_k equal to p. It is then clear that

$$\partial_t(\nabla_i \partial_I) = \sum_{k=1}^{n} \sum_p (\partial_t \Gamma^p_{i i_k}) \partial_{I^k_p}.$$

In symbolic notation, the preceding calculation is summarized as

$$\partial_t[\nabla^{n+1} h] = \nabla(\partial_t \nabla^n h) + (\nabla^n h) * (\partial_t \Gamma).$$

Since $\partial_t \Gamma \in E^{3,2}$, this says

$$\partial_t[\nabla^{n+1} h] = \nabla(\partial_t \nabla^n h) + E^{n+4,n+2}.$$

Covariant derivatives in space may be dealt with in the usual way. Again for a multiindex iI of length $n+1$, we have for first-order derivatives

$$\begin{aligned}\nabla_k(\nabla^{n+1}_{iI} h) &= \nabla_k(\nabla_i(\nabla^n_I h)) - \nabla_k(\nabla^n h(\nabla_i \partial_I)) \\ &= \nabla_i(\nabla_k(\nabla^n_I h)) - \nabla_k(\nabla^n h(\nabla_i \partial_I)) + \mathrm{Rm}_{ik}[\nabla^n_I h],\end{aligned}$$

and for second-order covariant derivatives

$$\begin{aligned}\nabla_l(\nabla_k(\nabla^{n+1}_{iI} h)) &= \nabla_l(\nabla_i(\nabla_k(\nabla^n_I h))) + \nabla_l(\mathrm{Rm}_{ik}[\nabla^n_I h]) - \nabla_l(\nabla_k(\nabla^n h(\nabla_i \partial_I))) \\ &= \nabla_i(\nabla_l(\nabla_k(\nabla^n_I h))) + \mathrm{Rm}_{il}[\nabla_k(\nabla^n_I h)] + \nabla(\mathrm{Rm} * \nabla^n h) + \nabla^2(\nabla^n h * h) \\ &= \nabla_i(\nabla^2_{l,k}(\nabla^n_I h)) + \nabla_i(\nabla_{\nabla_l \partial_k} \nabla^n_I h) + \mathrm{Rm} * \nabla^{n+1} h + \nabla(\mathrm{Rm} * \nabla^n h) + \nabla^2(\nabla^n h * h),\end{aligned}$$

$$\begin{aligned}\nabla^2_{l,k}(\nabla^{n+1}_{iI} h) &= \nabla_i(\nabla^2_{l,k}(\nabla^n_I h)) - \nabla_{\nabla_l \partial_k}(\nabla^{n+1}_{iI} h) + \nabla(\Gamma * \nabla^{n+1} h) \\ &\qquad + \mathrm{Rm} * \nabla^{n+1} h + \nabla(\mathrm{Rm} * \nabla^n h) + \nabla^2(\nabla^n h * h) \\ &= \nabla_i(\nabla^2_{l,k}(\nabla^n_I h)) + \Gamma * \nabla^{n+2} h + \nabla(\Gamma * \nabla^{n+1} h) + \mathrm{Rm} * \nabla^{n+1} h + \nabla(\mathrm{Rm} * \nabla^n h) + \nabla^2(\nabla^n h * h).\end{aligned}$$

Taking traces with g^{kl} and using the expressions $\mathrm{Rm} = h^{(2)}$, $\nabla \mathrm{Rm} = \nabla h * h$, $\Gamma = h\omega$, $\nabla\Gamma = \nabla h + h^{(2)}$, it follows easily that

$$\Delta(\nabla^{n+1} h) = \nabla(\Delta(\nabla^n h)) + E^{n+4,n+2},$$

and therefore

$$(\partial_t - \Delta)[\nabla^{n+1} h] = \nabla[(\partial_t - \Delta)(\nabla^n h)] + E^{n+4,n+2},$$

proving the claim and the lemma. □

The analog of Lemma 13.2 for higher covariant derivatives of h follows easily from these remarks.

Lemma 13.4. *For a solution $w \in C^{n+5,[(n+5)/2]+1}(E)$ assume we have a uniform bound for h and its first n covariant derivatives $|\nabla^j h| \le a_j$ in E, $j = 0, \dots, n$. Then there are constants $\alpha > 0, C > 0$ depending only on the dimension and the a_j, so that the function*

$$f_{n+1}(x,t) = \alpha|\nabla^{n+1}h|^2 + |\nabla^n h|^2$$

is a subsolution in E, that is, $(\partial_t - \Delta) f_{n+1} \le C$.

Proof. In the proof we denote by C_n a generic positive constant depending only on dimension and the a_j, $j = 0, \dots, n$. We have

$$(\partial_t - \Delta)|\nabla^n h|^2 = -2|\nabla^{n+1}h|^2 + E^{2n+4,n+1}, \quad (\partial_t - \Delta)|\nabla^{n+1}h|^2 = -2|\nabla^{n+1}h|^2 + E^{2n+6,n+2},$$

where

$$\begin{aligned} E^{2n+4,n+1} &= \nabla^{n+1}h * \nabla^n h * h + (\nabla^n h)^{(2)} * \tilde{E}^{2,1} + (\nabla^n h) * \tilde{E}^{n+3,n-1} + \tilde{E}^{2n+4,n-1}, \\ E^{2n+6,n+2} &= \nabla^{n+2}h * \nabla^{n+1}h * h + (\nabla^{n+1}h)^{(2)} * \tilde{E}^{2,1} + (\nabla^{n+1}h) * \tilde{E}^{n+4,n} + \tilde{E}^{2n+6,n}. \end{aligned}$$

This implies

$$\begin{aligned} (\partial_t - \Delta)|\nabla^n h|^2 &\le -2|\nabla^{n+1}h|^2 + C_n|\nabla^{n+1}h| + C_n, \\ (\partial_t - \Delta)|\nabla^{n+1}h|^2 &\le -2|\nabla^{n+2}h|^2 + C_n|\nabla^{n+2}h||\nabla^{n+1}h| + C_n(|\nabla^{n+1}h|^2 + |\nabla^{n+1}h| + 1). \end{aligned}$$

It is easy to see from these inequalities that α can be chosen sufficiently small so that the conclusion of the lemma will hold. □

14. Hölder gradient estimate for the second fundamental form

Notation. In this section, parabolic Hölder spaces are denoted by a single superscript; i.e., $C^{2+\alpha,(1+\alpha)/2}$ becomes $C^{2+\alpha}$, etc. Capital X, Y, etc., denote general points in the spacetime domain E. This follows the notation used in [Lieberman 1996].

A continuation criterion for the solution $w(y,t)$ in E^T in terms of a bound on the norm $|h|_g$ of the second fundamental form would follow from an a priori $C^{3+\delta}(E^T)$ bound on a solution, assuming $|h|_g \le a_0$ in $\bar{E}^T$; equivalently, from a global a priori Hölder gradient bound $|\nabla h|_\delta \le M$ in $\bar{E}^T$ (for suitably controlled M). In this section we show how such a bound follows from the a priori estimates of linear parabolic theory applied to the evolution equations for v, H, and the Weingarten operator, under an additional hypothesis.

Assuming $w \in C^{2+\delta}(E^T)$ is a solution, satisfying in addition $|h|_g \le a_0$ in E^T, we already observed the maximum principle implies bounds

$$0 \le w \le w_0, \qquad 1 \le v \le \bar{v} \text{ in } \bar{E}^T,$$

depending only on the initial data and β (we assume $w \ge 0$, at $t = 0$, vanishing only on ∂D_0.) In particular, g is uniformly equivalent to the euclidean metric on E^T. In this section, bounds depending on a_0, $\bar{v}$, and the initial data will be denoted generically by a constant $M > 0$ (dependence on β will not be recorded explicitly). The bound on h implies a uniform C^2 bound for the spacetime domain E^T, which we can express in terms of a diffeomorphism $\Phi : D_0 \times [0, T] \to E^T$ by

$$|\Phi|_{C^2} \leq M.$$

We will also need to assume a uniform gradient bound on the boundary for the second fundamental form:

$$|(\nabla_\tau h)(\tau, \tau)| \leq a_1 \quad \text{for all } \tau \in T\partial D(t) \text{ with } |\tau| = 1.$$

Estimates depending a_0, a_1, $\bar{v}$ and the initial data will be given in terms of constants denoted generically by M_1.

In fact E^T is a bounded domain in $\mathbb{R}^n \times [0, T]$ of class $C^{2+\delta}$ with bounds controlled by M_1. (This statement includes some regularity in t, so it is not immediate from the uniform bound assumed for $\nabla^{\tan} h^{\tan}$ on $\partial_l E$). To see this, consider the equation satisfied by $w_k = \partial_k w$, written in "divergence form" with Dirichlet boundary conditions

$$\begin{cases} \partial_t w_k - \partial_i (g^{ij} \partial_j w_k) = g^k := (\partial_k g^{ij}) w_{ij} - (\partial_i g^{ij}) \partial_j w_k, \\ w_{k|\partial_l E} := \varphi^k = (\beta_0/\beta) n^k, \quad w_{k|t=0} = \partial_k w_0. \end{cases}$$

Assuming $\partial_k w \in C^{1+\delta}(E)$, the following estimate holds [Lieberman 1996, Theorem 4.27]:

$$|w_k|_{1+\delta} \leq C\big(\sup_E |w_k| + \|g^k\|_{1,n+1+\delta} + |\varphi^k|_{1+\delta;\partial_l E} + |\partial_k w_0|_{1+\delta; D_0}\big).$$

Here $\|g^k\|_{1,n+1+\delta}$ is the norm in the spacetime Morrey space $L^{1,n+1+\delta}(E)$

$$\|g^k\|_{1,n+1+\delta} = \sup_{\substack{Y \in E, \\ r < \operatorname{diam} E}} \left(r^{-(n+1+\delta)} \int_{E[Y,r]} |g^n| \, dX \right).$$

In the present case this can easily be estimated, since

$$|\partial_k g^{ij}| = |h^i_k \omega^j + h^j_k \omega^i| \leq M, \qquad |\partial_j w_k| \leq \bar{v} a_0 \leq M \implies |g^k| \leq M,$$

and $|E[Y, r]| \leq C r^{n+2}$, while $\delta \in (0, 1)$. Thus $\|g^k\|_{1,n+1+\delta} \leq M$.

Since $|\nabla_\tau(\nabla_\tau n)| \leq c\big(|(\nabla_\tau h)(\tau, \tau)| + |h|\big) \leq M_1$, it follows that n is C^2 in space variables on $\partial_l E$. On the other hand, $Dw = \omega/\beta$ on $\partial D(t)$, and ω is a solution of $\partial_t \omega^k = \operatorname{tr}_g D^2 \omega^k + |h|^2_g \omega^k$, hence n is also C^1 in time on $\partial_l E$. We conclude $|\varphi^k|_{1+\delta;\partial_l E} \leq (\beta_0/\beta)|n|_{1+\delta;\partial_l E} \leq M_1$.

Therefore we have $|Dw|_{1+\delta} \leq M_1$, and $|w|_{2+\delta} \leq M_1$ (note that C depends on $|g^{ij}|_{C^\delta}$ and other constants also controlled by M.) In particular, E^T is a $C^{2+\delta}$ domain with chart constants controlled by M_1. (In fact, in a neighborhood of any point $P \in \partial_l E$ with $\partial_{y_2} w \neq 0$, a boundary chart Ψ is given by $\Psi(y_1, y_2, t) = (y_1, w(y, t), t)$.)

The first-order term in the evolution equation for h (or for the Weingarten operator) involves DH; hence the next step is to obtain a global gradient bound $|DH|_{1+\alpha} \leq M_1$ in $\bar{E}^T$. The mean curvature satisfies the "divergence form" equation with Neumann boundary conditions

$$\begin{cases} \partial_t H - \partial_j (g^{ij}(X) H_i) + \partial_j (g^{ij})(X) \partial_i H - c(X) H = 0, \\ d_n H = (\beta^2/\beta_0) H h_{nn} := \psi \text{ on } \partial D(t), \quad H_{|t=0} = H_0, \end{cases}$$

where

$$c := |h|_g^2 - h^2(\omega, \omega) + Hh(\omega, \omega).$$

Then with the regularity conditions for the domain and the coefficients

$$\partial_l E \in C^{1+\delta}, \qquad n \in C^{\delta}(\partial_l E), \qquad \partial_j(g^{ij}) \in L^{1,n+1+\delta}(E), \qquad c \in L^{1,n+1+\delta}(E),$$

and assuming $H \in C^{1+\delta}(E)$, or $w \in C^{3+\delta}(E)$, we have the bound

$$|H|_{1+\delta;\bar{E}} \le C\big(\sup_E |H| + |\psi|_{\delta;\partial_l E} + |H_0|_{1+\delta;D_0}\big).$$

As noted earlier

$$\|\partial_j g^{ij}\|_{1,n+1+\delta} + \|c\|_{1,n+1+\delta} \le M,$$

hence C is controlled by M. In addition, $|w|_{2+\delta} \le M_1$ implies $|h|_\delta \le M_1$, and hence $|\psi|_{\delta;\partial_l E} \le M_1$. We conclude $|H|_{1+\delta} \le M_1$, and state it as a lemma.

Lemma 14.1. *Let $w \in C^{3+\delta}(E^T)$ be a classical solution of graph mean curvature motion with contact and constant-angle boundary conditions. Assume that $|h|_g \le a_0$ on $\partial_l E$ and that $|(\nabla_\tau h)(\tau, \tau)| \le a_1$ on $\partial_l E$. Then we have a global gradient bound for H:*

$$\sup_{\bar{E}^T} |DH|_\delta \le M_1,$$

for a constant M_1 depending on $\delta, \bar{v}, a_0, a_1$ and the initial data w_0.

Corollary 14.2. *Under the same hypotheses as Lemma 14.1, we have a global gradient bound*

$$\sup_{\bar{E}} |\nabla h|_g \le M_1,$$

for a positive constant M_1 depending on $\delta, \bar{v}, a_0, a_1$ and the initial data w_0.

Proof. The bound on the components $(\nabla_n h)(\tau, \tau)$ and $(\nabla_n h)(n, n)$ on the lateral boundary $\partial_l E$ follows immediately from the expressions in Section 11. The bound on $(\nabla_\tau h)(\tau, \tau)$ over $\partial_l E$ is hypothesized, and then the bound on the remaining component $(\nabla_\tau h)(n, n)$ follows from the global gradient bound $|DH| \le M$ implied by Lemma 14.1. Thus $|\nabla h| \le M_1$ on $\partial_l E$, and then the global bound follows from Lemma 13.2 and the maximum principle. □

To improve the conclusion of Corollary 14.2 to a Hölder gradient bound, it is natural to consider the evolution equation for h with the Neumann-type boundary conditions derived in Section 11. One is then faced with the problem that those boundary conditions do not control components such as $(\nabla_\tau h)(\tau, \tau)$ on $\partial_t E$. So as a preliminary step we consider the evolution equation for v, which has the advantage that the boundary values are constant. Written in linear form, we have

$$\begin{cases} \partial_t v - g^{ij}(X) v_{ij} + b^i(X)\partial_i v + c(X) v = 0, \\ v_{|\partial_t E} = 1/\beta, \quad v_{|t=0} = v_0, \end{cases}$$

where

$$g^{ij}(X) = \delta_{ij} - \frac{w_i w_j}{1 + |Dw|^2}(X), \qquad b^i(X) = \frac{2g^{ij} w_k w_{kj}}{1 + |Dw|^2}(X), \qquad c(X) = |h|_g^2(X).$$

We clearly have $g^{ij} \in C^\delta$ (since $Dw \in C^\delta$), as well as $b^i, c \in C^\delta$ (since $h \in C^\delta$), and $\partial_l E \in C^{2+\delta}$ with bounds controlled by M_1 in all cases as observed earlier. Therefore assuming $v \in C^{2+\delta}$ (equivalently, $w \in C^{3+\delta}$) we have the bound

$$|v|_{2+\delta;\bar{E}} \le C\Big(\sup_E v + \frac{1}{\beta}\Big),$$

with C controlled by M_1. Thus $|D^2 v|_\delta \le M_1$. Recalling $v^{-1}\partial_i v = h(\partial_i, \omega)$, this implies that

$$|(\nabla_\tau h)(n,n)|_{\delta;\partial_l E} = |(\nabla_n h)(\tau,n)|_{\delta;\partial_l E} \le M_1 \quad \text{for all } \tau \in T\partial D(t) \text{ and } |\tau| = 1.$$

Since $H = \beta^2 h_{nn} + h(\tau,\tau)$ on $\partial_l E$, it follows from Lemma 14.1 that we also have $|(\nabla_\tau h)(\tau,\tau)|_{\delta;\partial_l E} \le M_1$. For the remaining components of ∇h, this bound follows directly from the boundary conditions

$$|(\nabla_n h)(\tau,\tau)|_{\delta;\partial_l E} + |(\nabla_n h)(n,n)|_{\delta;\partial_l E} \le M_1.$$

Now consider the evolution of the components of the Weingarten operator, written in divergence form with Neumann boundary conditions

$$\begin{cases} \partial_t h^k_j - \partial_i(g^{il}\partial_l h^k_j) = f^k_j & \text{in } E^T, \quad f^k_j := H_j h^k_l \omega^l - H_l h^l_j \omega^k + h^{(3)k}_j - (\partial_i g^{il})(\partial_l h^k_j), \\ d_n(h^k_j) = \varphi^k_j & \text{on } \partial_l E, \quad h^k_{j\,|t=0} = h^k_{j0}. \end{cases}$$

The same theorem quoted above gives the estimate (assuming $h^k_j \in C^{1+\delta}$ or $w \in C^{3+\delta}$)

$$|h^k_j|_{1+\delta;\bar{E}} \le C\big(\sup_E |h^k_j| + \|f^k_j\|_{1,n+1+\delta} + |\varphi^k_j|_{\delta;\partial_l E} + |h^k_{0j}|_{1+\delta;D_0}\big).$$

Note that

$$d_n(h^k_j) = g^{ik}(\nabla_n h)_{ij} = \beta^2 (\nabla_n h)(n,\partial_j) n^k + (\nabla_n h)(\tau,\partial_j)\tau^k \text{ on } \partial_l E.$$

From this and the above discussion it follows that $|\varphi^k_j|_{\delta;\partial_l E} \le M_1$. The bound $\|f^k_j\|_{1,n+1+\delta} \le M_1$ follows from Lemma 14.1 and Corollary 14.2. We conclude $|h^k_j|_{1+\delta;\bar{E}} \le M_1$. The $1+\delta$ estimate for h^k_j clearly implies the following lemma:

Lemma 14.3. *Let $w \in C^{3+\delta}(E^T)$ be a classical solution of graph mean curvature motion with contact and constant-angle boundary conditions. Assume that $|h|_g \le a_0$ on $\partial_l E$ and that $|(\nabla_\tau h)(\tau,\tau)|_{\partial_l E} \le a_1$. Then we have a global Hölder gradient bound for h:*

$$|\nabla h|_{\delta;\bar{E}^T} \le M_1,$$

for a constant M_1 depending on $\delta, \bar{v}, a_0, a_1$ and the initial data w_0.

Remark. This is clearly equivalent to a global a priori $C^{3+\delta}$ bound for w on $\bar{E}^T$, $|w|_{3+\delta} \le M_1$.

Lemma 14.3 is the main step in the derivation of a "continuation criterion" for this flow.

Proposition 14.4. *Assume the maximal existence time $T_{\max}$ is finite. Then (for $n = 2$, in the concave case)*

$$\limsup_{t\to T_{\max}} \sup_{\partial D(t)} (|h|_g + |(\nabla_\tau h)(\tau,\tau)|) = \infty.$$

Proof. For $w_0 \in C^{3+\bar{\alpha}}(D_0)$ satisfying the contact angle condition (with $\bar{\alpha} \in (0,1)$ arbitrary) and $\alpha = \bar{\alpha}^2$, Theorem 8.1 yields a unique solution $F = [u, \varphi]$ of mcm with contact angle/orthogonality boundary conditions in a maximal time interval $[0, T_{\max}]$ with $F \in C^{2+\alpha}(Q_0^{T_{\max}})$, $Q_0^{T_{\max}} = Q \times [0, T_{\max})$; this is also the unique solution in $F \in C^{2+\delta^2}(Q_0^{T_{\max}})$, where $\delta = \alpha^2$. Then $w = u \circ \varphi^{-1} \in C^{2+\delta}(E^{T_{\max}})$ is a solution of graph mcm, which for any $t_0 > 0$ is in $C^{3+\delta}(E_{t_0}^{T_{\max}})$. By contradiction, assume $|h|_g + |(\nabla_\tau h)(\tau, \tau)|$ is bounded in $E_{t_0}^T$ for any $T < T_{\max}$ (with bound independent of T). Then Lemma 14.3 applies, giving an a priori bound $|\nabla h|_{\delta; \bar{E}_{t_0}^T} \le M_1$, for T arbitrarily close to $T_{\max}$. In particular, $|w(\,\cdot\,, T)|_{C^{3+\delta}(D(T))} \le M_1$, and for T close enough to $T_{\max}$ we can use Theorem 8.1 again, with initial data $w(\,\cdot\,, T)$, to find a solution $F' = [u', \varphi'] \in C^{2+\delta^2}(Q_0^{T'})$ (where $T' > T_{\max}$), extending F. This contradicts the maximality of $T_{\max}$. □

15. Behavior at the extinction time

In this section we consider the behavior of Σ_t as t approaches the maximal existence time T, in the concave case. We assume $H \le H_0 < 0$ at $t = 0$, so T is finite. Let $K_t \subset \mathbb{R}^{n+1}$ be the compact convex set bounded by Σ_t. Since $H \le 0$, $\{K_t\}$ is a decreasing family, and the intersection

$$K_T = \bigcap_{0 \le t < T} K_t \subset \mathbb{R}^{n+1}$$

is compact, convex and nonempty. It turns out that K_T has zero $(n+1)$-volume. In this section we use the support function to show this when $n = 2$ (following the argument in [Stahl 1996]), under the assumption that there is no gradient blowup.

Assume the origin $0 \in \mathbb{R}^n$ is a point of K_T. The support function of K_t (with respect to this origin) is the function $p(\,\cdot\,, t)$ on $D(t)$ given by

$$p(y, t) = \big\langle G(y, t), N(y, t)\big\rangle, \quad G(y, t) = [y, w(y, t)].$$

Since K_t is convex, $p > 0$ in $D(t)$; the evolution equations and boundary conditions for p are easily computed. From $L[G] = 0$ and $L[N] = |h|_g^2 N$, we have

$$L[p] = \langle L[G], N\rangle + \langle G, L[N]\rangle - 2g^{kl}\langle \partial_k G, \partial_l N\rangle = |h|_g^2 p + 2H,$$

and, since $\langle d_n G, N\rangle = 0$,

$$p_{n|\partial D(t)} = \langle G, d_n N\rangle = -A(G^T, N),$$

where, with $y^T := y - (y \cdot n)n \in T_y \partial D(t)$, the tangential component $G^T := G - \langle G, N\rangle N$ is easily seen to be, at $\partial D(t)$,

$$G^T = \frac{1}{v^2}[w_n^2 y^T + y, 0].$$

Since $h(y^T, n) = 0$ at $\partial D(t)$, this implies $A(G^T, n) = \beta^2 (y \cdot n) h(n, n)$. Note that $p(y) = -\beta_0 (y \cdot n)$ on $\partial D(t)$, so we have

$$p_{n|\partial D(t)} = \frac{\beta^2}{\beta_0} p h_{nn},$$

which is reminiscent of the boundary condition for H. We also have the upper bound

$$p \le \|G\| \le \max_{D(0)} \|G_0\| := p_0,$$

since the K_t are nested.

Proposition 15.1. *Let $n = 2$. Assume that*

$$\limsup_{t\to T} \sup_{y\in\partial D(t)} |h|_g = \infty$$

at the maximal existence time T. Then

$$\liminf_{t\to T} \inf_{y\in D(t)} p(y,t) = 0.$$

Proof. Reasoning by contradiction, assume $p > 2\delta > 0$ for $t \in [0, T)$. We claim that this implies an upper bound for $|H|$ (and hence for $|h|$, since $|h|^2 \le nH^2$) contradicting the fact that $\limsup_{t\to T} \sup_{\Gamma_t} |h| = \infty$.

To prove the claim, consider the function

$$f(y,t) := \frac{|H|}{p-\delta} = -\frac{H}{p-\delta}.$$

Using the evolution equations and boundary conditions for H and p we find (with $\hat\omega := \omega/|\omega|_g$, see Remark 15.2 below)

$$L[f] = f(-\delta|h|_g^2 + 2pf) + |\omega|_g^2\big(h^2(\hat\omega,\hat\omega) - Hh(\hat\omega,\hat\omega)\big) - \frac{2}{p-\delta} g^{kl}\partial_k f \partial_l p$$

and

$$f_{n|\partial D(t)} = -\delta\frac{\beta^2}{\beta_0} h_{nn}\frac{|h|^2}{(p-\delta)^2} \ge 0.$$

Since $|h|_g^2 \ge \frac{1}{n}H^2 = \frac{1}{n}f^2(p-\delta)^2$ we get

$$L[f] \le f\Big(-\frac{(p-\delta)^2\delta}{n}f^2 + 2pf\Big) + |\omega|_g^2\big(h^2(\hat\omega,\hat\omega) - Hh(\hat\omega,\hat\omega)\big) - \frac{2}{p-\delta}\langle\nabla f, \nabla p\rangle_g.$$

Now recall from Remark 12.8 that if $n = 2$

$$h^2(\hat\omega,\hat\omega) - Hh(\hat\omega,\hat\omega) = -\Delta \le 0,$$

so

$$L[f] \le f\Big(-\frac{(p-\delta)^2\delta}{n}f^2 + 2pf\Big) - \frac{2}{p-\delta}\langle\nabla f, \nabla p\rangle_g.$$

Let $\delta > 0$ be so small that $\sup_{D(0)} f_{|t=0} < 2np_0/\delta^3$. We claim this persists for all $t \in [0, T)$. If not, assume $f(y_0, t_0) = 2np_0/\delta^3$ with $t_0 > 0$ smallest possible and let y_0 be a local maximum of $f(\cdot, t_0)$. Since $f_n \ge 0$ at $\partial D(t_0)$, the boundary point lemma implies that $z_0 = (y_0, t_0)$ can't be a boundary point of E. Thus $y_0 \in \partial D(t_0)$ is an interior point, so $L[f]_{|z_0} \ge 0$ and $\nabla f(z_0) = 0$: hence

$$\frac{(p-\delta)^2\delta}{n} f(z_0) \le 2p(z_0), \quad \text{or} \quad f(z_0) \le \frac{2np(z_0)}{\delta(p-\delta)^2} \le \frac{2np_0}{\delta(p-\delta)^2},$$

which is not possible since $p-\delta > \delta$. Thus $f(y,t) < 4p_0/\delta^3$ in E, which implies the bound $|H| \le 4p_0^2/\delta^3$ for $t \in [0, T)$, contradicting the maximality of T. □

Remark 15.2. It is easy to verify that the vector fields

$$\omega = \frac{1}{v}[w_1, w_2], \quad \tilde\omega = v\omega^\perp = [-w_2, w_1]$$

in $D(t) \subset \mathbb{R}^2$ satisfy

$$\langle \omega, \tilde{\omega} \rangle_g = 0, \qquad |\omega|_g^2 = |\tilde{\omega}|_g^2 = |Dw|_e^2 := w_1^2 + w_2^2.$$

Thus we may think of $\{\omega, \tilde{\omega}\}$ as a "conformal pseudoframe" (ω and $\tilde{\omega}$ vanish when $Dw = 0$), defined on all of $D(t)$. Moreover, at the boundary $\partial D(t)$,

$$\omega = \beta_0 n, \quad \tilde{\omega} = \frac{\beta_0}{\beta} n^{\perp} = \frac{\beta_0}{\beta} \tau,$$

where $\{\tau, n\}$ is an euclidean-orthonormal frame along Γ_t. Thus ω and $\tilde{\omega}$ supply canonical extensions of n, τ to the interior of $D(t)$ as uniformly bounded vector fields.

It follows from the proposition that K_T cannot contain a half-ball of positive radius centered at a point of $\mathbb{R}^2$; in particular, $\mathrm{vol}_3(K_T) = 0$. Based on the experience with curve networks [Schnürer et al. 2007], one is led to expect that K_T is a point ($\mathrm{diam} K_T = 0$), at least under the same assumption as the proposition (no gradient blowup). We have not been able to show this yet; existence of self-similar solutions and comparison arguments appropriate to the free-boundary setting appear to be needed for the usual approach to work.

16. Final comments

Local existence. We state here a local existence theorem for configurations of graphs over domains with moving boundaries. In this setting, a *triple junction configuration* consists of three embedded hypersurfaces Σ^1, Σ^2, Σ^3 in $\mathbb{R}^{n+1}$, graphs of functions w^I defined over time-dependent domains $D^1(t)$, $D^2(t) \subset \mathbb{R}^n$ (D^1 covered by one graph, D^2 by two graphs), satisfying the following conditions:

(1) The Σ^I intersect along an $(n-1)$-dimensional graph $\Lambda(t)$ (the "junction"), along which the upward unit normals satisfy the relation: $N_1 + N_2 = N_3$.

(2) If a fixed support hypersurface $S \subset \mathbb{R}^{n+1}$ is given (also a graph, not necessarily connected), the Σ^I intersect S orthogonally.

Topologically, in the case of bounded domains one has the following examples:

(i) Lens type: two disks or two annuli covering $D^2(t)$ and one annulus covering $D^1(t)$.

(ii) Exterior type: two annuli covering $D^2(t)$ and one disk covering $D^1(t)$.

The boundary component of the annuli disjoint from the junction intersects the support hypersurface S orthogonally for each t.

Let Σ_0^I ($I = 1, 2, 3$) be graphs of $C^{3+\alpha}$ functions over $C^{3+\alpha}$ domains D_0^1, $D_0^2 \subset \mathbb{R}^n$, defining a triple junction configuration and satisfying the compatibility condition for the mean curvatures on the common boundary Γ_0 of D_0^1 and D_0^2

$$H^1 + H^2 = H^3.$$

Then there exists $T > 0$ depending only on the initial data, and functions $w^I \in C^{2+\alpha, 1+\alpha/2}(Q^I)$, $Q^I \subset \mathbb{R}^n \times [0, T)$, so that the graphs of $w^I(., t) : D^I(t) \to \mathbb{R}$ define a triple junction configuration for each $t \in [0, T)$ moving by mean curvature.

The proof will be given elsewhere.

Uniqueness. An interesting issue we have not addressed here is whether one has breakdown of uniqueness for initial data of lower regularity, or if the "orthogonality condition" at the junction is removed. For curve networks, nonuniqueness has been considered in [Mazzeo and Sáez 2007]; but neither a drop in regularity (from initial data to solution in Hölder spaces) nor the orthogonality condition play a role in the case of curves.

Appendix A. Proof of Lemma 4.1

Throughout the proof n denotes the inner unit normal at ∂D, extended to a tubular neighborhood $\mathcal{N}$ in $\mathbb{R}^n$ so that $D_n n = 0$. Since D is uniformly $C^{3+\alpha}$, it follows that $n \in C^{2+\alpha}(\partial D)$ with uniform bounds. Denote by ρ the oriented distance to the ∂D (so $D\rho = n$ in $\mathcal{N}$). Let $\zeta \in C^3(\bar{D})$ be a cutoff function with $\zeta \equiv 1$ in $\mathcal{N}_1 \subset \mathcal{N}$, $\zeta \equiv 0$ in $D \setminus \mathcal{N}$.

We find φ of the form

$$\varphi(x) = x + \zeta(x) f(x) n(x)$$

with $f \in C^{2+\alpha}(\mathcal{N})$. The 1-jet conditions on φ at ∂D translate to these conditions on f:

$$f_{|\partial D} = 0, \qquad Df_{|\partial D} = 0, \qquad D^2 f(n,n)_{|\partial D} = \Delta f_{|\partial D} = h.$$

Lemma A.1. *Let D be a uniformly $C^{3+\alpha}$ domain with boundary distance function $\rho > 0$. Let $h \in C^\alpha(\partial D)$ be bounded. There exists an extension $g \in C^\infty(D) \cap C(\bar{D})$ such that $g_{|\partial D} = h$, $\sup_{\bar{D}} |g| \le \sup_{\partial D} |h|$ and $\rho^2 g \in C^{2+\alpha}(\bar{D})$.*

Given this lemma, all we have to do is set $f = \frac{1}{2}\rho^2 g$, which clearly satisfies all the requirements (in particular, $\Delta f = h$ at ∂D.)

To verify that φ is a diffeomorphism, it suffices to check that $|\zeta f n|_{C^1}$ (in $\mathcal{N} \subset \{\rho < \rho_0\}$) is small if ρ_0 is small. This is easily seen: $|\zeta f n|_{C^0} \le \frac{1}{2}\rho_0^2 |g|_{C^0}$; from $|D\zeta| \le c\rho_0^{-1}$ it follows that $|f D\zeta| \le c\rho_0 |g|_{C^0}$; and $|Df| \le \frac{1}{2}\rho_0^\alpha \|g\|_{C^{2+\alpha}(\bar{D})}$ on $\mathcal{N}$, since $Df \in C^{1+\alpha}(\bar{D})$ and $Df_{|\partial D} = 0$. Finally, with $\mathcal{A}$ the second fundamental form of ∂D,

$$|Dn| \le |\mathcal{A}|_{C^0} \implies |f Dn| \le \tfrac{1}{2}\rho_0^2 |g|_{C^0} |\mathcal{A}|_{C^0}.$$

A word about Lemma A.1. (This is probably in the literature, but I don't know a reference.) If D is the upper half-space, we solve $\Delta g = 0$ in D with boundary values h. Then the estimate

$$[D^2(\rho^2 P * h)]^\alpha(\bar{D}) \le c|h|_{C^\alpha(\partial D)}$$

follows by direct computation with the Poisson kernel P; for the rest of the norm, use interpolation. Then transfer the estimate to a general domain using "adapted local charts" in which ρ in D corresponds to the vertical coordinate in the upper half-space. (It is easy to see that at each boundary point there is a $C^{2+\alpha}$ adapted chart with uniform bounds.)

Appendix B. Evolution equations for the second fundamental form

We consider mean curvature motion of graphs:

$$G(y,t) = [y, w(y,t)], \quad y \in D(t) \subset \mathbb{R}^n, \qquad w_t = g^{ij} w_{ij} = v H, \qquad v = \sqrt{1 + |Dw|^2}.$$

In this appendix we include evolution equations for geometric quantities, in terms of the operators

$$\partial_t - \Delta_g, \quad L = \partial_t - \mathrm{tr}_g d^2.$$

It is often convenient to use the vector field in $D(t)$

$$\omega := \frac{1}{v} Dw.$$

Since $-\omega$ is the $\mathbb{R}^n$ component of the unit normal N and $L[N] = |h|_g^2 N$, we have

$$L[\omega^i] = |h|_g^2 \omega^i, \quad |h|_g^2 := g^{ik} g^{jl} h_{ij} h_{kl}.$$

Here $h = (h_{ij})$ is the pullback to $D(t)$ of the second fundamental form A:

$$h(\partial_i, \partial_j) = h_{ij} = A(G_i, G_j) = \frac{1}{v} w_{ij}.$$

First, denoting by ∇ the pullback to $D(t)$ of the induced connection ∇^Σ (that is, $G_*(\nabla_X Y) = \nabla^\Sigma_{G_* X} G_* Y$ for any vector fields X, Y in $D(t)$), and using the definition

$$\nabla^\Sigma_{G_i} G_j = G_{ij} - \langle G_{ij}, N\rangle N = [0, w_{ij}] - \frac{1}{v^2} w_{ij}[-Dw, 1] = \frac{w_{ij}}{v^2}[Dw, |Dw|^2] = \frac{w_{ij}}{v^2} G_* Dw,$$

we conclude that

$$\nabla_{\partial_i} \partial_j = \frac{1}{v} h_{ij} Dw = h_{ij} \omega. \tag{B-1}$$

From this one derives easily a useful expression relating the Laplace–Beltrami operator and the operator $\mathrm{tr}_g d^2$ acting on functions

$$\Delta_g f = \mathrm{tr}_g d^2 f - \frac{H}{v} w_m f_m = \mathrm{tr}_g d^2 f - H d_\omega f.$$

We also have, for the covariant derivatives of h with respect to the euclidean connection and to $\nabla = \nabla^g$:

$$\partial_m(h_{ij}) = \nabla_m h_{ij} + [h_{jm} h_{ik} + h_{im} h_{jk}]\omega^k.$$

(Here ∇h is the symmetric $(3, 0)$-tensor with components: $\nabla_m h_{ij} = (\nabla_{\partial_m} h)(\partial_i, \partial_j)$.)

Iterating this and taking g-traces yields, using the Codazzi identity and the easily verified relation $\partial_i \omega^k = h_i^k := g^{jk} h_{ij}$,

$$\begin{aligned}
\mathrm{tr}_g d^2(h_{ij}) &= g^{mk} \partial_m(\partial_k(h_{ij})) \\
&= g^{mk} (\nabla^2_{\partial_m, \partial_k} h)(\partial_i, \partial_j) + H \nabla_\omega h_{ij} + 2[h_i^k \nabla_k h_{jp} + h_j^k \nabla_k h_{ip}]\omega^p + [H_i h_{jp} + H_j h_{ip}]\omega^p \\
&\qquad + 2\big[h_{ip}(h^2)_{jq} + (h^2)_{ip} h_{jq} + H h_{ip} h_{jq}\big]\omega^p \omega^q + 2(h^3)_{ij} + 2(h^2)_{ij} h(\omega, \omega).
\end{aligned}$$

Here the powers h^2 and h^3 of h are the symmetric 2-tensors defined used the metric:

$$(h^2)_{ij} := g^{kp} h_{ik} h_{pj} = h_i^k h_{pj}, \quad (h^3)_{ij} := g^{kp} g^{lq} h_{ik} h_{pl} h_{qj}.$$

Note also that

$$[h_i^k \nabla_k h_{jp} + h_j^k \nabla_k h_{ip}]\omega^p = \nabla_\omega (h^2)_{ij},$$

using the Codazzi identity.

Evolution equations for h. Starting from $G_t = vHe_{n+1} = H\big(N + v^{-1}[Dw, |Dw|^2]\big) = HN + HG_*\omega$ and $N_t = -\nabla^\Sigma H - Hv^{-1}\nabla^\Sigma v$ (where $\nabla^\Sigma f = g^{ij} f_j G_i$ and $\nabla f = g^{ij} f_j \partial_i$) we have

$$\partial_t(h_{ij}) = \langle (HN)_{ij}, N\rangle - \langle G_{ij}, \nabla^\Sigma H\rangle - \frac{H}{v}\langle G_{ij}, \nabla^\Sigma v\rangle + \langle (HG_*\omega)_{ij}, N\rangle.$$

Using the easily derived facts

$$\langle N_{ij}, N\rangle = -h^2(\partial_i, \partial_j), \quad H_{ij} - \langle G_{ij}, \nabla^\Sigma H\rangle = (\nabla dH)(\partial_i, \partial_j), \quad \frac{1}{v}\langle G_{ij}, \nabla^\Sigma v\rangle = h(\omega, \omega)h_{ij},$$

we obtain

$$\partial_t(h_{ij}) = (\nabla dH)(\partial_i, \partial_j) - Hh^2(\partial_i, \partial_j) - Hh(\omega, \omega)h_{ij} + \langle (HG_*\omega)_{ij}, N\rangle,$$

where

$$\langle (HG_*\omega)_{ij}, N\rangle = H_i\langle (G_*\omega)_j, N\rangle + H_j\langle (G_*\omega)_i, N\rangle + H\langle (G_*\omega)_{ij}, N\rangle.$$

To identify the terms, computation shows that

$$\langle (G_*\omega)_i, N\rangle = h(\omega, \partial_i),$$

and hence, using also

$$\nabla^\Sigma_{G_i}(G_*\omega) = G_*(\nabla_{\partial_i}\omega), \qquad \nabla_{\partial_i}\omega = (h_i^p + \omega^q h_{iq}\omega^p)\partial_p = \sum_p h_{ip}\partial_p,$$

we obtain (using $\omega^k\partial_j(h_{ik}) = \nabla_\omega h_{ij} + 2h(\partial_i, \omega)h(\partial_j, \omega)$) that

$$\begin{aligned}\langle (G_*\omega)_{ij}, N\rangle &= \partial_j(\omega^k h_{ik}) - \langle \nabla^\Sigma_{G_i}(G_*\omega), \partial_j N\rangle = h_j^k h_{ik} + \omega^k\partial_j(h_{ik}) + h(\partial_j, \nabla_{\partial_i}\omega)\\ &= (\nabla_\omega h)_{ij} + (h^2)_{ij} + 2h(\omega, \partial_i)h(\omega, \partial_j) + \sum_p h_{ip}h_{jp}\\ &= (\nabla_\omega h)_{ij} + 2(h^2)_{ij} + 3h(\omega, \partial_i)h(\omega, \partial_j),\end{aligned}$$

since $\sum_p h_{ip}h_{jp} = (h^2)_{ij} + h(\omega, \partial_i)h(\omega, \partial_j)$. Combining all the terms yields

$$\begin{aligned}\partial_t(h_{ij}) = (\nabla dH)(\partial_i, \partial_j) + H\nabla_\omega h_{ij} + H_i h(\omega, \partial_j) + H_j h(\omega, \partial_i)\\ + H(h^2)_{ij} + 3Hh(\omega, \partial_i)h(\omega, \partial_j) - Hh(\omega, \omega)h_{ij}.\end{aligned}$$

From this expression and Simons' identity (in tensorial form)

$$\nabla dH = \Delta_g h + |h|_g^2 h - Hh^2,$$

we obtain easily a tensorial "heat equation" for h:

$$[(\partial_t - \Delta_g)h]_{ij} = H\nabla_\omega h_{ij} + H_i h(\omega, \partial_j) + H_j h(\omega, \partial_i) + C_{ij},$$

with

$$C_{ij} := |h|_g^2 h_{ij} + 3Hh(\partial_i, \omega)h(\partial_j, \omega) - Hh(\omega, \omega)h_{ij}. \tag{B-2}$$

Using the earlier computation relating $\Delta_g h$ (the tensorial Laplacian of h) and $\mathrm{tr}_g d^2 h$, we obtain from this the evolution equation in terms of L:

$$L[h_{ij}] = -2[h_i^k\nabla_\omega h_{jk} + h_j^k\nabla_\omega h_{ik}] + \tilde{C}_{ij},$$

where

$$\tilde{C}_{ij} := C_{ij} - 2\big[h(\partial_i,\omega)h^2(\partial_j,\omega) + h^2(\partial_i,\omega)h(\partial_j,\omega)\big] - 2(h^3)_{ij} - 2(h^2)_{ij}h(\omega,\omega) - 2Hh(\partial_i,\omega)h(\partial_j,\omega).$$

We may also write this purely in terms of the euclidean connection d:

$$L[h_{ij}] = -2[h_i^k d_\omega h_{jk} + h_j^k d_\omega h_{ik}] + \bar{C}_{ij},$$

where

$$\bar{C}_{ij} = C_{ij} + 2\big[h(\partial_i,\omega)h^2(\partial_j,\omega) + h^2(\partial_i,\omega)h(\partial_j,\omega)\big] - 2(h^3)_{ij} - 2(h^2)_{ij}h(\omega,\omega) - 2Hh(\partial_i,\omega)h(\partial_j,\omega). \tag{B-3}$$

Time derivatives and evolution equations for ω and g. The time derivative of ω is simply minus the time derivative of the $\mathbb{R}^n$ component of N. In addition, one computes easily that $(\nabla v)/v = S(\omega)$, where

$$S(X) := S(X^i\partial_i) = h_j^i X^j \partial_i$$

is the Weingarten operator. Hence

$$\partial_t\omega = \nabla H + \frac{H}{v}\nabla v = \nabla H + HS(\omega). \tag{B-4}$$

For the metric and “inverse metric” tensors it follows from $\partial_t g_{ij} = (w_i w_j)_t$ and $w_{it} = (vH)_i$ that

$$\partial_t g_{ij} = v^2(H_i\omega^j + H_j\omega^i) + v^2 H\big(h(\omega,\partial_i)\omega^j + h(\omega,\partial_j)\omega^i\big).$$

Then, using $\partial_t g^{ij} = -g^{ik}\partial_t g_{kl} g^{lj}$, we have

$$\partial_t g^{ij} = -\big[(\nabla H)^i\omega^j + (\nabla H^j)\omega^i\big] - H\big[S(\omega)^i\omega^j + S(\omega)^j\omega^i\big].$$

Since we know the evolution equation of ω, it is easy to obtain that of g^{ij}:

$$L[g^{ij}] = -L[\omega^i\omega^j] = -L[\omega^i]\omega^j + 2g^{kl}(\partial_k\omega^i)(\partial_l\omega^j) - \omega^i L[\omega^j].$$

Using $\partial_k\omega^i = h_k^i$, we find

$$L[g^{ij}] = -2|h|_g^2\omega^i\omega^j + 2(h^2)^{ij}.$$

It is also easy to see that $\partial_k g^{ij} = -(h_k^i\omega^j + h_k^j\omega^i)$.

Evolution of the mean curvature. To compute the evolution equation for $H = g^{ij}h_{ij}$, we just need to remember that g^{ij} is time-dependent:

$$(\partial_t - \Delta_g)H = (\partial_t g^{ij})(h_{ij}) + \mathrm{tr}_g[(\partial_t - \Delta_g)h] = -2h(\nabla H,\omega) - 2Hh^2(\omega,\omega) + \mathrm{tr}_g[(\partial_t - \Delta_g)h].$$

The result is

$$(\partial_t - \Delta_g)H = Hd_\omega H + |h|_g^2 H + Hh^2(\omega,\omega) - H^2 h(\omega,\omega).$$

Since

$$L[f] = (\partial_t - \Delta_g)f - Hd_\omega f$$

(for any f), we see that the equation in terms of L has no first-order terms:

$$L[H] = |h|_g^2 H + Hh^2(\omega,\omega) - H^2 h(\omega,\omega).$$

One can also derive $L[H]$ from the expression $L[g^{ij}h_{ij}] = L[g^{ij}]h_{ij} + g^{ij}L[h_{ij}] - 2g^{kl}(\partial_k g^{ij})(\partial_l h_{ij})$.

Evolution of the Weingarten operator. The tensorial Laplacian of S is the $(1,1)$ tensor $\Delta_g S$ with components $\Delta_g h^k_j$. We have

$$\Delta_g h^k_j = g^{ik}\Delta_g h_{ij} \quad \text{or} \quad \langle(\Delta_g S)X, Y\rangle_g = (\Delta_g h)(X, Y).$$

The evolution equation is easily obtained:

$$\begin{aligned}(\partial_t - \Delta_g)h^k_j &= (\partial_t g^{ik})h_{ij} + g^{ik}(\partial_t - \Delta_g)h_{ij}\\ &= H\nabla_\omega h^k_j + H_j h^k_l \omega^l - H_l h^l_j \omega^k + |h|^2_g h^k_j + 2HS(\omega)^k h(\omega, \partial_j) - Hh(\omega,\omega)h^k_j - Hh(S(\omega), \partial_j)\omega^k.\end{aligned}$$

Remark. Since the components of ∇S are given by

$$(\nabla_\omega S)(\partial_j) = (\nabla_\omega h^k_j)\partial_k, \quad \nabla_\omega h^k_j = d_\omega(h^k_j) + h^2(\omega, \partial_j)\omega^k - h(\omega, \partial_j)S(\omega)^k,$$

we see that upon setting $j = k$ and adding over k we recover the evolution equation for H.

The evolution equation for h^k_j in terms of L follows from the calculation

$$\begin{aligned}L[h^k_j] &= L[g^{ik}]h_{ij} + g^{ik}L[h_{ij}] - 2g^{mn}(\partial_m g^{ik})(\partial_n h_{ij})\\ &= -2(\nabla_\omega h^k_m)h^m_j + (\partial_j |h|^2_g)\omega^k\\ &\qquad + |h|^2_g h^k_j - Hh(\omega,\omega)h^k_j + HS(\omega)^k h(\partial_j, \omega) + 2h^3(\partial_j, \omega)\omega^k - 2(h^2)^k_p \omega^p h(\partial_j, \omega).\end{aligned}$$

Setting $j = k$ and adding over k, we recover the earlier expression for $L[H]$.

Evolution of $|h|^2_g$. That g^{ij} is time-dependent introduces an additional term in the usual expression

$$(\partial_t - \Delta_g)|h|^2_g = -2|\nabla h|^2_g + 2\langle h, (\partial_t - \Delta_g)h\rangle_g + 2(\partial_t g^{ij})(h^2)_{ij}.$$

Using the expressions given earlier, one easily finds

$$\begin{aligned}(\partial_t - \Delta_g)|h|^2_g &= -2|\nabla h|^2_g + Hd_\omega |h|^2_g + 2|h|^4_g - 4Hh^3(\omega,\omega) - 2H|h|^2_g h(\omega,\omega),\\ L[|h|^2_g] &= -2|\nabla h|^2_g + 2|h|^4_g - 4Hh^3(\omega,\omega) - 2H|h|^2_g h(\omega,\omega).\end{aligned}$$

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LIFSHITZ TAILS FOR GENERALIZED ALLOY-TYPE RANDOM SCHRÖDINGER OPERATORS

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We study Lifshitz tails for random Schrödinger operators where the random potential is alloy-type in the sense that the single site potentials are independent, identically distributed, but they may have various function forms. We suppose the single site potentials are distributed in a finite set of functions, and we show that under suitable symmetry conditions, they have a Lifshitz tail at the bottom of the spectrum except for special cases. When the single site potential is symmetric with respect to all the axes, we give a necessary and sufficient condition for the existence of Lifshitz tails. As an application, we show that certain random displacement models have a Lifshitz singularity at the bottom of the spectrum, and also complete our previous study (2009) of continuous Anderson type models.

1. Introduction

Consider the continuous alloy-type (or Anderson) random Schrödinger operator

$$H_\omega = -\Delta + V_0 + V_\omega, \qquad \text{where } V_\omega(x) = \sum_{\gamma\in\mathbb{Z}^d} \omega_\gamma V(x-\gamma) \tag{1-1}$$

on $\mathbb{R}^d$, $d \geq 1$, where

- V_0 is a periodic potential;
- V is a compactly supported single site potential;
- $(\omega_\gamma)_{\gamma\in\mathbb{Z}^d}$ are independent identically distributed random coupling constants.

Let Σ be the almost sure spectrum of H_ω and $E_- = \inf\Sigma$. When V has a fixed sign, it is well known that, if a =ess-inf(ω_0) and b =ess-sup(ω_0), then $E_- = \inf(\sigma(-\Delta + V_{\bar b}))$ if $V \leq 0$ and $E_- = \inf(\sigma(-\Delta + V_{\bar a}))$ if $V \geq 0$. Here, $\bar x$ is the constant vector $\bar x = (x)_{\gamma\in\mathbb{Z}^d}$.

For E a real energy, the *integrated density of states* is defined by

$$N(E) = \lim_{L\to+\infty} \frac{\#\{\text{eigenvalues of } H^N_{\omega,L} \ \leq E\}}{L^d}, \tag{1-2}$$

where

$$H^N_{\omega,L} = -\Delta + V_0 + V_\omega \qquad \text{on } L^2(C_L(0)), \tag{1-3}$$

with Neumann boundary conditions, where $C_L(0)$ is defined by (1-4). It is well-known that $N(E)$ exists and is non-random, i.e., $N(E)$ is independent of ω, almost surely; it has been the object of a lot of studies.

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In particular, it is well known that the integrated density of states of the Hamiltonian admits a Lifshitz tail near E_-, i.e.,

$$\lim_{E\to E_-^+} \frac{\log|\log N(E)|}{\log(E-E_-)} < 0.$$

Actually, the limit can often be computed and in many cases is equal to $-d/2$; we refer to [Carmona and Lacroix 1990; Kirsch 1989; 1985; Pastur and Figotin 1992; Stollmann 2001; Veselić 2004; 2008] for extensive reviews and more precise statements.

In the present paper, we mainly consider a generalized Bernoulli alloy-type model that we define below: we allow the single site potential to have various function forms (with a discrete distribution). We give a necessary and sufficient condition to have Lifshitz tail under a symmetry assumption on the single site potentials. The results we obtain are then applied to the random displacement models studied recently by Baker, Loss and Stolz [2008; 2009], and also to complete the study of the occurrence of Lifshitz tails for alloy-type models initiated in [Klopp and Nakamura 2009].

1.1. ***The model.*** We now describe our model. We let $d \geq 1$ and we study operators on $\mathcal{H} = L^2(\mathbb{R}^d)$. By

$$C_\ell(x) = \{y \in \mathbb{R}^d \mid 0 \leq y_j - x_j \leq \ell,\ j = 1, \dots, d\}, \tag{1-4}$$

we denote the cube with edge $\ell > 0$ and x as the lower right corner. Let $V_0 \in C^0(\mathbb{R}^d)$ be a background potential, periodic with respect to $\mathbb{Z}^d$.

Let $v_k \in C_c^0(C_1(0))$, $k = 1, \dots, M$, be single site potentials where $M \in \mathbb{N}$. We consider the random Schrödinger operator:

$$H_\omega = -\Delta + V_0 + V_\omega \qquad \text{on } \mathcal{H} = L^2(\mathbb{R}^d),$$

where

$$V_\omega(x) = \sum_{\gamma\in\mathbb{Z}^d} v_{\omega(\gamma)}(x-\gamma)$$

is the random potential and $\{\omega(\gamma) \mid \gamma \in \mathbb{Z}^d\}$ are independent, identically distributed random variables with values in $\{1, \dots, M\}$.

To fix ideas, let us assume

$$\inf \sigma(H_\omega) = 0, \quad \text{a.s. } \omega, \tag{1-5}$$

which can always be achieved by shifting V_0 by a constant.

We set

$$H_k^N = -\Delta + V_0 + v_k \quad \text{on } L^2(C_1(0)),$$

with Neumann boundary conditions on the boundary $\partial C_1(0)$.

Assumption A. (1) V_0 is symmetric about the plane $\{x \mid x_d = 1/2\}$. (2) There exists $m \in \{1, \dots, M\}$ such that

$$\inf \sigma(H_k^N) = 0 \qquad \text{for } k = 1, \dots, m,$$
$$\inf \sigma(H_k^N) > 0 \qquad \text{for } k > m.$$

(3) For $k = 1, \dots, m$, $v_k(x)$ is symmetric about $\{x_d = 1/2\}$.

Remark 1.1. Note that in this assumption, we only require symmetry with respect to a single coordinate hyperplane that we chose to be the d-th one.

If one assumes that V_0 and the $(v_k)_{1\le k\le M}$ are reflection symmetric with respect to all the coordinate planes [Baker et al. 2008; 2009; Klopp and Nakamura 2009], the standard characterization of the almost sure spectrum [Pastur and Figotin 1992; Kirsch 1989] and lower bounding H_ω by the direct sum of its Neumann restrictions to the cubes $(C_1(\gamma))_{\gamma\in\mathbb{Z}^d}$ show that, as a consequence of (1-5), one obtains

- for all $k\in\{1,\dots,M\}$, $\inf\sigma(H_k^N)\ge 0$;
- there exists $k\in\{1,\dots,M\}$ such that $\inf\sigma(H_k^N)=0$.

1.2. *The results.* We study the Lifshitz singularity for the integrated density of states (IDS) at the zero energy. Recall that the IDS is defined by (1-2).

We first consider a relatively easy case:

Theorem 1.2. *Suppose Assumption A holds with $m<M$. Then*

$$\limsup_{E\to+0}\frac{\log|\log N(E)|}{\log E}\le -\frac{1}{2}. \tag{1-6}$$

We expect that (1-6) holds with $-d/2$ on the right-hand side, which is known to be optimal; see [Klopp and Nakamura 2009, Theorem 0.2 and Section 2.2], for example.

If $m=M$, then we need further classification of the potential functions. We denote the standard basis of $\mathbb{R}^d$ by

$$\mathbf{e}_j=(\delta_{ji})_{i=1}^d\in\mathbb{R}^d,\quad j=1,\dots,d,$$

and we define an operator $H_{k\ell(j)}^N$ on $L^2(U_j)$ as

$$U_j=C_1(0)\cup C_1(\mathbf{e}_j),\quad j=1,\dots,d. \tag{1-7}$$

We set

$$H_{k\ell(j)}^N=\begin{cases}-\Delta+V_0(x)+v_k(x) & \text{on } C_1(0),\\ -\Delta+V_0(x)+v_\ell(x-\mathbf{e}_j) & \text{on } C_1(\mathbf{e}_j),\end{cases} \tag{1-8}$$

with Neumann boundary conditions on ∂U_j, where $k,\ell\in\{1,\dots,m\}$ and $j\in\{1,\dots,d\}$. We define

$$v_j\sim_j v_\ell \quad\overset{\text{def}}{\Longleftrightarrow}\quad \inf\sigma(H_{k\ell(j)}^N)=0. \tag{1-9}$$

Namely, $v_k\sim_j v_\ell$ implies that the coupling of two local Hamiltonians H_k^N and H_ℓ^N does not increase the ground state energy. We note that $v_k\not\sim_j v_\ell$ generically for $k\ne\ell$.

Theorem 1.3. *Suppose Assumption A holds with $m=M$, and that $v_k\not\sim_d v_\ell$ for some $k\ne\ell$. Then* (1-6) *holds*, *i.e.*, *H_ω has Lifshitz singularities at the zero energy.*

To obtain a more precise result on the existence and the absence of Lifshitz singularities, we make a stronger symmetry assumption on the potentials.

Assumption B. In addition to satisfying Assumption A, V_0 and v_k are symmetric about $\{x\mid x_j=1/2\}$ for all $j=1,\dots,d$, and $k=1,\dots,m=M$.

Theorem 1.4. *Suppose Assumption B holds.*

(i) *If* $v_k \not\sim_j v_\ell$ *for some* j *and* $k \neq \ell$, *then* (1-6) *holds.*

(ii) *If* $v_k \sim_j v_\ell$ *for all* j *and* k, ℓ, *then the van Hove property holds, namely, there exists* $C > 0$ *such that*

$$\frac{1}{C} E^{d/2} \leq N(E) \leq C E^{d/2}. \tag{1-10}$$

In (1-10), the asymptotic is new only for E small; for E large, it is a consequence of Weyl's law. The example in Section 3 of [Klopp and Nakamura 2009] is a special case of Theorem 1.4(ii).

In a previous paper [Klopp and Nakamura 2009], we used the concavity of the ground state energy with respect to the random parameters, and also used an operator theoretical trick to reduce the problem to the monotonous perturbation case. These methods are not available under the assumptions of the present paper. Instead, we employ a quadratic inequality similar to the Poincaré inequality, and take advantage of the positivity of certain Dirichlet-to-Neumann operators to obtain a lower bound of the ground state energy for Schrödinger operators on a strip. This estimate is quasi one-dimensional, and this is why we obtain Lifshitz tail estimate with the exponent corresponding to the one-dimensional case. We do believe that this method can be refined to obtain the optimal exponent, though we have not been successful so far.

This paper is organized as follows. We discuss the eigenvalue estimate on a strip in Section 2 and prove our main theorems in Section 3. We discuss an application to random displacement models in Section 4, and an application to the model studied in [Klopp and Nakamura 2009] in Section 5.

Throughout this paper, we use the following notations: $\mathbb{P}(\,\cdot\,)$ denotes the probability measure for the random potential, and $\mathbb{E}(\,\cdot\,)$ denotes the expectation; $\mathfrak{D}(A)$ denotes the definition domain of an operator A; $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the inner product of L^2-spaces; $\partial\Omega$ denotes the boundary of a domain Ω; and $\#\Lambda$ denotes the cardinality of a set Λ.

2. Lower bounds on the ground state energy

Throughout this section, we suppose $v_1, \dots v_m$ satisfy Assumption A. Let $a > 0$,

$$\Omega_0 = [0,1]^{d-1} \times [-a, 0] \subset \mathbb{R}^d,$$

and let $W_0 \in C^0(\Omega_0)$ be a real-valued function on Ω_0. We set

$$P_0^N = -\triangle + W_0 \quad \text{on } L^2(\Omega_0)$$

with Neumann boundary conditions. Let $L \in \mathbb{N}$,

$$\Omega_1 = [0,1]^{d-1} \times [0, L]$$

and let $W_1 \in C^0(\Omega_1)$ such that

$$W_1 = V_0 + v_{k(\ell)}(x - \ell \mathbf{e}_d) \quad \text{if } x \in C_1(\ell \mathbf{e}_d),\ \ell = 0, \dots, L-1,$$

where $\{k(\ell)\}_{\ell=0}^{L-1}$ is a sequence with values in $\{1, \dots, m\}$. We then set

$$\Omega = \Omega_0 \cup \Omega_1, \qquad W(x) = \begin{cases} W_0(x) & \text{if } x \in \Omega_0, \\ W_1(x) & \text{if } x \in \Omega_1, \end{cases}$$

and set

$$P^N = -\triangle + W \quad \text{on } L^2(\Omega),$$

with Neumann boundary conditions. The main result of this section is this:

Theorem 2.1. *Suppose* $\inf \sigma(P_0^N) > 0$, *and suppose* $v_{k(\ell)} \sim_d v_{k(\ell')}$ *for* $\ell, \ell' \in \{0, \dots, L-1\}$. *Then there exists* $C > 0$ *such that* C *is independent of* L *and of the sequence* $\{k(\ell)\}$, *and such that*

$$\inf \sigma(P^N) \geq \frac{1}{CL^2}.$$

In the following, we suppose $v_k \sim_d v_\ell$ for all k, ℓ for simplicity (and without loss of generality). We prove Theorem 2.1 by a series of lemmas. First, we show a variant of the classical Poincaré inequality. Let Γ be the trace operator from $H^1(\Omega_1)$ to $L^2(S)$ with $S = [0, 1]^{d-1} \times \{0\}$, i.e.,

$$\Gamma\varphi(x') = \varphi(x', 0) \quad \text{for } x' \in [0, 1]^{d-1},\ \varphi \in C^0(\Omega_1),$$

and Γ extends to a bounded operator from $H^1(\Omega_1)$ to $L^2(S)$.

Lemma 2.2. *Let* $\varphi \in H^1(\Omega_1)$. *Then*

$$\frac{2}{L}\|\Gamma\varphi\|^2_{L^2(S)} + \|\nabla\varphi\|^2_{L^2(\Omega_1)} \geq \frac{1}{L^2}\|\varphi\|^2_{L^2(\Omega_1)}.$$

Proof. It suffices to show the estimate for $\varphi \in C^1(\Omega_1)$. Since

$$\varphi(x', t) = \varphi(x', 0) + \int_0^t \partial_{x_d}\varphi(x', s)\,ds, \qquad x' \in [0, 1]^{d-1}, t \in [0, L],$$

we have

$$|\varphi(x', t)| \leq |\varphi(x', 0)| + \int_0^t |\partial_{x_d}\varphi(x', s)|\,ds \leq |\varphi(x', 0)| + \sqrt{t}\Big(\int_0^L |\nabla\varphi(x', s)|^2 ds\Big)^{1/2}$$

by the Cauchy-Schwarz inequality. This implies

$$\begin{aligned}
\|\varphi\|^2_{L^2(\Omega_1)} &\leq \int\!\!\int_0^L \bigg\{|\varphi(x', 0)| \ + \sqrt{t}\Big(\int_0^L |\nabla\varphi(x', s)|^2 ds\Big)^{1/2}\bigg\}^2 dt dx' \\
&\leq 2\int\!\!\int_0^L |\varphi(x', 0)|^2 ds dx' + 2\int_0^L t dt \times \|\nabla\varphi\|^2_{L^2(\Omega_1)} \\
&= 2L\|\Gamma\varphi\|^2_{L^2(S)} + L^2\|\nabla\varphi\|^2_{L^2(\Omega_1)}
\end{aligned}$$

and the claim follows. □

For $k \in \{1, \dots, M\}$, we set

$$q_k(\varphi, \psi) = \int_{C_1(0)} (\nabla\varphi \cdot \nabla\overline{\psi} + v_k\varphi\overline{\psi})\,dx, \qquad \varphi, \psi \in H^1(C_1(0)),$$

which is the quadratic form corresponding to H_k^N. Let Ψ_k be the positive ground state for H_k^N, which is unique up to a constant. Since $\inf \sigma(H_k^N) = 0$, we expect φ/Ψ_k is close to a constant if $q_k(\varphi, \varphi)$ is close to 0, and this observation is justified by the following lemma.

Lemma 2.3. *There exists* $c_1 > 0$ *such that*

$$\|\nabla(\varphi/\Psi_k)\|^2_{L^2(C_1(0))} \le c_1 q_k(\varphi,\varphi), \quad \varphi \in H^1(C_1(0)), k = 1, \dots, m.$$

Proof. This is a consequence of the so-called *ground state transform*. It suffices to show the inequality when $\varphi \in C^1(C_1(0))$. We set $f = \varphi/\Psi_k$. Then we have

$$\begin{aligned} q_k(\varphi,\varphi) &= \langle \nabla(f\Psi_k), \nabla(f\Psi_k)\rangle + \langle v_k f\Psi_k, f\Psi_k\rangle \\ &= \|\Psi_k(\nabla f)\|^2 + \langle \Psi_k\nabla f, f\nabla\Psi_k\rangle + \langle f\nabla\Psi_k, \Psi_k\nabla f\rangle \\ &\qquad\qquad + \langle f\nabla\Psi_k, f\nabla\Psi_k\rangle + \langle v_k f\Psi_k, f\Psi_k\rangle \\ &= \|\Psi_k(\nabla f)\|^2 + \langle \nabla(|f|^2\Psi_k), \nabla\Psi_k\rangle + \langle v_k|f|^2\Psi_k, \Psi_k\rangle \\ &= \|\Psi_k(\nabla f)\|^2 + q_k(|f|^2\Psi_k, \Psi_k). \end{aligned}$$

Since $q_k(|f|^2\Psi_k, \Psi_k) = \langle (H_k^N)^{1/2}|f|^2\Psi_k, (H_k^N)^{1/2}\Psi_k\rangle = 0$, we have

$$q_k(\varphi,\varphi) = \|\Psi_k(\nabla f)\|^2 \ge (\inf|\Psi_k|)^2\|\nabla f\|^2,$$

and we may choose $c_1 = (\min_k \inf|\Psi_k|)^{-2}$. □

Lemma 2.4. *Suppose* $v_k \sim_d v_\ell$. *Then there exists* $\mu_1, \mu_2 > 0$ *such that*

$$\mu_1\Psi_k(x',0) = \mu_2\Psi_\ell(x',0), \qquad \textit{for } x' \in [0,1]^{d-1}.$$

Proof. Consider $H^N_{k\ell(d)}$ in U_d (see (1-7) and (1-8) in Section 1), and let $\Phi \in L^2(U_d)$ be the positive ground state of $H^N_{k\ell(j)}$. We set

$$\varphi_1 = \Phi\lceil_{C_1(0)}, \quad \varphi_2(\,\cdot\,) = \Phi(\,\cdot + \mathbf{e}_d)\lceil_{C_1(0)}.$$

Then φ_1, φ_2 are positive and $q_k(\varphi_1,\varphi_1) = q_\ell(\varphi_2,\varphi_2) = 0$. By the variational principle and the uniqueness of the ground states, we learn

$$\varphi_1 = \mu_1\Psi_k, \quad \varphi_2 = \mu_2\Psi_\ell$$

with some $\mu_1, \mu_2 > 0$. By Assumption A, Ψ_k and Ψ_ℓ are symmetric about $\{x_d = 1/2\}$, and hence

$$\mu_1\Psi_k(x',0) = \mu_1\Psi_k(x',1) = \varphi_1(x',1) = \varphi_2(x',0) = \mu_2\Psi_\ell(x',0)$$

for $x' \in [0,1]^{d-1}$, where we have used the continuity of Φ on $\{x_d = 1\}$. □

Now, let Ω_1 and W_1 be as in the beginning of Section 2, and define

$$P_1^N = -\triangle + W_1 \quad \text{on } L^2(\Omega_1)$$

with Neumann boundary conditions. We set

$$Q_1(\varphi,\psi) = \int_{\Omega_1} (\nabla\varphi\cdot\nabla\overline{\psi} + W_1\varphi\overline{\psi})\,dx = \langle (P_1^N)^{1/2}\varphi, (P_1^N)^{1/2}\psi\rangle$$

for $\varphi, \psi \in H^1(\Omega_1) = \mathfrak{D}((P_1^N)^{1/2})$.

Lemma 2.5. *There exists $c_2 > 0$ such that c_2 is independent of L and of the sequence $\{k(\ell)\}$, and*

$$\frac{1}{L}\|\Gamma\varphi\|^2_{L^2(S)} + Q_1(\varphi,\varphi) \ge \frac{1}{c_2 L^2}\|\varphi\|^2_{L^2(\Omega_1)}$$

for $\varphi \in H^1(\Omega_1)$.

Proof. By Lemma 2.4, there exist $\mu_1, \dots, \mu_m > 0$ such that

$$\mu_1\Psi_1(x',0) = \mu_2\Psi_2(x',0) = \cdots = \mu_m\Psi_m(x',0).$$

We set

$$\Psi(x) = \mu_{k(\ell)}\Psi_{k(\ell)}(x-\ell\mathbf{e}_d) \quad \text{if } \ell \le x_d \le \ell+1,$$

and then $\Psi \in H^1(\Omega_1)$ by the above observation. Moreover, Ψ is the ground state for P_1^N, unique up to a constant. We apply Lemma 2.2 to φ/Ψ, and we have

$$\begin{aligned}\frac{1}{L^2}\|\varphi\|^2_{L^2(\Omega_1)} &\le \frac{1}{L^2}(\sup\Psi)^2\|\varphi/\Psi\|^2_{L^2(\Omega_1)}\\ &\le \frac{(\sup\Psi)^2}{L}\|\Gamma(\varphi/\Psi)\|^2_{L^2(S)} + (\sup\Psi)^2\|\nabla(\varphi/\Psi)\|^2_{L^2(\Omega_1)}\\ &\le \Big(\frac{\sup\Psi}{\inf\Psi}\Big)^2\frac{1}{L}\|\Gamma\varphi\|^2_{L^2(S)} + c_1(\sup\Psi)^2 Q_1(\varphi,\varphi),\end{aligned}$$

where we have used Lemma 2.3 in the last inequality. The claim follows immediately. □

We next consider $P_0 = -\triangle + W_0$ on $L^2(\Omega_0)$ and its Dirichlet-to-Neumann operator. As in Theorem 2.1, we suppose

$$\alpha = \inf\sigma(P_0^N) > 0.$$

We set

$$P_0' = -\triangle + W_0 \quad \text{on } L^2(\Omega_0) \text{ with } \mathfrak{D}((P_0')^{1/2}) = \{\varphi \in H^1(\Omega_0) \mid \Gamma\varphi = 0\},$$

where Γ is the trace operator from $H^1(\Omega_1)$ to $L^2(S)$. Then P_0' defines a self-adjoint operator, and each $\varphi \in \mathfrak{D}(P_0')$ satisfies Dirichlet boundary conditions on S and Neumann boundary conditions on $\partial\Omega_0 \setminus S$. Let $\lambda < \alpha$. By a standard argument of the theory of elliptic boundary value problems (see [Folland 1995], for instance), for any $g \in H^{3/2}(S)$, there exists a unique $\psi \in H^2(\Omega_0)$ such that

$$(-\triangle + W_0 - \lambda)\psi = 0, \qquad \Gamma\psi = g \tag{2-1}$$

and that satisfies Neumann boundary conditions on $\partial\Omega_0 \setminus S$. Then the map

$$T(\lambda)\ :\ g \mapsto \Gamma(\partial_\nu\psi) \in H^{1/2}(S)$$

defines a bounded linear map from $H^{3/2}(S)$ to $H^{1/2}(S)$, where $\partial_\nu = \partial/\partial x_d$ is the outer normal derivative on S. We consider $T(\lambda)$ as an operator on $L^2(S)$, and it is called the *Dirichlet-to-Neumann operator*.

Lemma 2.6. *$T(\lambda)$ is a symmetric operator. If $\lambda_0 < \alpha$, then $T(\lambda) \ge \varepsilon$ for $0 \le \lambda \le \lambda_0$ with some $\varepsilon > 0$.*

Proof. Let $\varphi, \psi \in H^2(\Omega_0)$ such that $\Gamma\varphi = f$, $\Gamma\psi = g$, and

$$(-\triangle + W_0 - \lambda)\varphi = (-\triangle + W_0 - \lambda)\psi = 0,$$

with Neumann boundary conditions on $\partial\Omega_0 \setminus S$. By Green's formula we have

$$\begin{aligned}0 &= \langle(-\Delta + W_0 - \lambda)\varphi, \psi\rangle - \langle\varphi, (-\Delta + W_0 - \lambda)\psi\rangle \\ &= -\int_S \partial_\nu\varphi \cdot \overline{\psi} + \int_S \varphi \cdot \partial_\nu\overline{\psi} = -\langle T(\lambda)f, g\rangle_{L^2(S)} + \langle f, T(\lambda)g\rangle_{L^2(S)},\end{aligned}$$

and hence $T(\lambda)$ is symmetric. Similarly, we have

$$\begin{aligned}0 = \langle(-\Delta + W_0 - \lambda)\varphi, \varphi\rangle &= -\int_S \partial_\nu\varphi \cdot \overline{\varphi} + \int_{\Omega_0} |\nabla\varphi|^2 + \int_{\Omega_0} (W_0 - \lambda)|\varphi|^2 \\ &= -\langle T(\lambda)f, f\rangle + Q_0(\varphi, \varphi) - \lambda\|\varphi\|^2,\end{aligned}$$

where $Q_0(\varphi, \varphi) = \int_{\Omega_0}\big(|\nabla\varphi|^2 + W_0|\varphi|^2\big)dx$. Hence, we learn that

$$\langle T(\lambda)f, f\rangle = Q_2(\varphi, \varphi) - \lambda\|\varphi\|^2 \geq Q_0(\varphi, \varphi) - \lambda_0\|\varphi\|^2.$$

The form in the right-hand side is equivalent to $\|\varphi\|^2_{H^1(\Omega_0)}$ since $\lambda_0 < \alpha$. Hence, it is bounded from below by $\varepsilon\|f\|^2_{L^2(S)}$ with some $\varepsilon > 0$ by virtue of the boundedness of the trace operator from $H^1(\Omega_0)$ to $L^2(S)$. □

We note that $T(\lambda)$ extends to a self-adjoint operator on $L^2(S)$ by the Friedrichs extension, though we do not use the fact in this paper.

Proof of Theorem 2.1. Let φ be the ground state of P^N on Ω with the ground state energy $\lambda \geq 0$. If $\lambda \geq \lambda_0 > 0$ with some fixed λ_0 (independently of L), then the statement is obvious, and hence we may assume $0 \leq \lambda \leq \lambda_0 < \alpha = \inf\sigma(P_0^N)$ without loss of generality.

Let $f = \Gamma\varphi \in H^{3/2}(S)$. Since φ satisfies Neumann boundary conditions on $\partial\Omega_0 \setminus S$, we learn $\partial_\nu\varphi\restriction_S = T(\lambda)\varphi$. On the other hand, by Green's formula, we have

$$\begin{aligned}\int_{\Omega_1} P^N\varphi \cdot \overline{\varphi} &= \int_S \partial_n\varphi \cdot \overline{\varphi} + \int_{\Omega_1} |\nabla\varphi|^2 + W_1|\varphi|^2 \\ &= \langle T(\lambda)f, f\rangle_{L^2(S)} + Q_1(\varphi, \varphi) \\ &\geq \varepsilon\|f\|^2_{L^2(S)} + Q_1(\varphi, \varphi)\end{aligned}$$

by Lemma 2.6. Now, we apply Lemma 2.5 to learn that the right-hand side is bounded from below by $(1/c_2L^2)\|\varphi\|^2_{L^2(\Omega_1)}$. Since $P^N\varphi = \lambda\varphi$ and $\|\varphi\|_{L^2(\Omega_1)} \neq 0$, this implies $\lambda \geq 1/c_2L^2$ for large enough L. □

3. Proof of the main theorems

We now discuss the proofs of Theorems 1.2 and 1.3, and we prove Theorem 1.4 at the end of the section. We thus suppose Assumption A with either $m < M$ or that there exists k, k' such that $v_k \not\sim_d v_{k'}$.

For notational simplicity, we assume the reflections of v_k at $\{x_d = 1/2\}$ are included in the possible set of potentials $\{v_k\}$. This does not change the conditions on $\{v_1, \ldots, v_m\}$, but we might need to add the reflections of $\{v_{m+1}, \ldots, v_M\}$. This does not affect the following arguments.

We write

$$\Lambda = \{p \in \mathbb{Z}^{d-1} \mid 0 \leq p_j \leq L - 1, j = 1, \ldots, d - 1\}$$

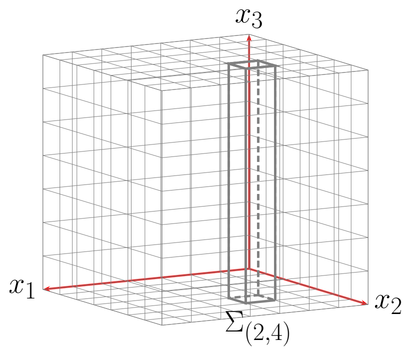

Figure 1. Chopping the cube into strips.

and, for $p \in \Lambda$, we set

$$\Sigma_p = \bigcup_{k=0}^{L-1} C_1((p,k))$$

so that $C_L(0)$ is decomposed (see Figure 1) as

$$C_L(0) = \bigcup_{p \in \Lambda} \Sigma_p$$

which is a disjoint union except for the boundaries of the strips.

For a given V_ω and $p \in \Lambda$, we consider the restriction of H_ω to Σ_p, i.e.,

$$\tilde{H}_p^N = \Delta + V_0 + \sum_{\ell=0}^{L-1} v_{\omega((p,\ell))}(x-(p,\ell)) \quad \text{on } L^2(\Sigma_p)$$

with Neumann boundary conditions on $\partial\Sigma_p$. By the standard Neumann bracketing, we learn

$$H_{\omega,L}^N \geq \bigoplus_{p \in \Lambda} \tilde{H}_p^N \quad \text{on } L^2(C_L(0)) \cong \bigoplus_{p \in \Lambda} L^2(\Sigma_p),$$

and hence, in particular,

$$\inf \sigma(H_{\omega,L}^N) \geq \min_{p \in \Lambda} \inf \sigma(\tilde{H}_p^N). \tag{3-1}$$

Under our assumptions, one of the following holds for each $p \in \Lambda$:

$(a)_p$: $\omega((p,\ell)) > m$ for some ℓ, or $v_{\omega((p,\ell))} \not\sim_d v_{\omega((p,\ell'))}$ for some $\ell, \ell' \in \{0, \dots, L-1\}$.

$(b)_p$: For all $\ell, \ell' \in \{0, \dots, L-1\}$, $\omega((p,\ell)) \leq m$ and $v_{\omega((p,\ell))} \sim_d v_{\omega((p,\ell'))}$.

We note that the probability of Condition $(b)_p$ to occur is less than μ^{-L} with some $\mu < 1$ independent of L. Since $\{\omega(\gamma)\}$ are independent, we have

$$\mathbb{P}\big((b)_p \text{ holds for some } p \in \Lambda\big) \leq L^d \mu^{-L}, \tag{3-2}$$

which is small if L is large. For the moment, then, we suppose Condition $(a)_p$ holds for all $p \in \Lambda$.

We denote by $V^p(x)$ the potential function of $\tilde{H}_p^N$ on Σ_p. Let

$$\hat{\Sigma}_p = (p + [0,1]^{d-1}) \times (\mathbb{R}/(2L\mathbb{Z}))$$

and set $\hat{V}^p(x) = V^p(x', |x_d|)$ for $x = (x', x_d) \in (p + [0,1]^{d-1}) \times [-L, L) \cong \hat{\Sigma}_p$, i.e., $\hat{V}^p$ is the extension of $\tilde{V}^p$ by the reflection at $\{x_d = 0\}$. We note $\hat{V}^p$ is continuous on $\hat{\Sigma}_p$. We now consider

$$\hat{H}_p^N = \triangle + \hat{V}^p \quad \text{on } L^2(\hat{\Sigma}_p)$$

with Neumann boundary conditions. It is easy to see

$$\inf \sigma(\tilde{H}_p^N) \geq \inf \sigma(\hat{H}_p^N). \tag{3-3}$$

In fact, if Φ is the ground state of $\tilde{H}_p^N$, then we extend Φ by reflection to obtain $\hat{\Phi} \in H^1(\hat{\Sigma}_p)$ and we have

$$\frac{\langle \hat{H}_p^N \hat{\Phi}, \hat{\Phi}\rangle}{\|\hat{\Phi}\|^2} = \frac{\langle \tilde{H}_p^N \Phi, \Phi\rangle}{\|\Phi\|^2} = \inf \sigma(\tilde{H}_p^N)$$

and the claim (3-3) follows by the variational principle.

Since we assume Condition $(a)_p$, Σ_p can be decomposed to subsegments $\Sigma_p = \bigcup_{j=1}^K \Xi_j$ such that each Ξ_j satisfies the following conditions: We write

$$\Xi_j = \bigcup_{\ell=0}^{\nu} C_1(p, \kappa + \ell), \quad \kappa \in \mathbb{Z},\ 0 \leq \nu < L$$

and

$$\hat{V}^p(x) = v_{\beta(\ell)}(x - (p, \ell)) \quad \text{for } x \in C_1(p, \kappa + \ell),\ \ell \in \{0, \dots, \nu\},$$

with $\beta(\ell) \in \{1 \dots, M\}$. Then either one of the following holds:

(i) $\beta(0) \in \{m+1, \dots, M\}$; $\beta(\ell) \in \{1, \dots, m\}$ for $\ell \geq 1$; and $v_{\beta(\ell)} \sim_d v_{\beta(\ell')}$ for $\ell, \ell' \in \{1, \dots, \nu\}$.

(ii) $\beta(\ell) \in \{1, \dots, m\}$ for all ℓ; $v_{\beta(0)} \not\sim_d v_{\beta(1)}$; and $v_{\beta(\ell)} \sim_d v_{\beta(\ell')}$ for $\ell, \ell' \in \{2, \dots, \nu\}$.

The proof of this claim is an easy, though somewhat lengthy, combinatorial exercise. We omit the details.

We again decompose $\hat{H}_p^N$. We denote the restriction of $\hat{H}_p^N$ to Ξ_j by P_j on $L^2(\Xi_j)$ with Neumann boundary conditions. Then again by Neumann bracketing, we learn that

$$\hat{H}_p^N \geq \bigoplus_{j=1}^{\kappa} P_j \quad \text{on } L^2(\hat{\Sigma}_p) \cong \bigoplus_{j=1}^{\kappa} L^2(\Xi_j),$$

and in particular,

$$\inf \sigma(\hat{H}_p^N) \geq \min_j \inf \sigma(P_j). \tag{3-4}$$

Now if (i) holds for Ξ_j, then we set $a = 1$ and use Theorem 2.1 for P_j. Since $\inf \sigma(H_{\beta(0)}^N) > 0$ by Assumption A and $\nu \leq L$, we learn that

$$\inf \sigma(P_j) \geq \frac{1}{C(\nu-1)^2} \geq \frac{1}{C(L-1)^2}.$$

If (ii) holds for Ξ_j, then we set $a = 2$ and use Theorem 2.1 for P_j. Since $v_{\beta(0)} \nsim_d v_{\beta(1)}$, we have $\inf \sigma(H^N_{\beta(0)\beta(1)(d)}) > 0$. Thus we have

$$\inf \sigma(P_j) \geq \frac{1}{C(\nu-2)^2} \geq \frac{1}{C(L-2)^2}.$$

Combining these with (3-1), (3-3) and (3-4), we conclude that

$$\inf \sigma(H^N_{\omega,L}) \geq \frac{c_3}{L^2} \tag{3-5}$$

with some $c_3 > 0$, provided Condition $(a)_p$ holds for all $p \in \Lambda$.

Proof of Theorems 1.2 and 1.3. For $E > 0$, we set

$$\sqrt{\frac{c_3}{E}} < L \leq \sqrt{\frac{c_3}{E}} + 1,$$

so that, by virtue of (3-5),

$$\inf \sigma(H^N_{\omega,L}) > E$$

provided Condition $(a)_p$ holds for all $p \in \Lambda$. As noted in (3-2), the probability of the events that Condition $(b)_p$ holds for some $p \in \Lambda$ is bounded by

$$\mathbb{P}\big((b)_p \text{ holds for some } p \in \Lambda\big) \leq L^d \mu^{-L} \leq c_4 E^{-d/2} e^{-c_5 E^{-1/2}}$$

with some $c_4, c_5 > 0$. On the other hand, since the potential $V_0 + V_\omega$ is uniformly bounded, we have

$$\#\{\text{eigenvalues of } H^N_{\omega,L} \leq \alpha\} \leq c_6 L^d$$

for any ω with some $c_6 > 0$. Thus we have

$$\begin{aligned} L^{-d}\mathbb{E}\big(\#\{\text{e.v. of } H^N_{\omega,L} \leq E\}\big) &\leq L^{-d}(c_6 L^d)\mathbb{P}\big((b)_p \text{ holds for some } p \in \Lambda\big) \\ &\leq c_4 c_6 E^{-d/2} e^{-c_5 E^{-1/2}} \leq c_7 e^{-(c_5-\varepsilon)E^{-1/2}} \end{aligned}$$

for $0 < \varepsilon < c_5$ with some $c_7 > 0$. By the Neumann bracketing again, we have

$$N(E) \leq L^{-d}\mathbb{E}\big(\#\{\text{e.v. of } H^N_{\omega,L} \leq E\}\big) \leq c_7 e^{-(c_5-\varepsilon)E^{-1/2}}$$

and Theorems 1.2 and 1.3 follow immediately from this estimate. □

In fact, we have proved that

$$\liminf_{E \to +0} \frac{|\log N(E)|}{E^{-1/2}} > 0,$$

and this statement is slightly stronger than (1-6).

Proof of Theorem 1.4. Statement (i) is an immediate consequence of Assumption B and Theorem 1.3. We just replace the x_d-axis by the x_j-axis where $v_k \nsim_j v_\ell$ for some k, ℓ.

For (ii), we use the ground state transform as in the proof of Lemmas 2.3–2.5. Under our conditions, there exist $\mu_1, \ldots, \mu_m > 0$ such that

$$\mu_1 \Psi_1(x) = \mu_2 \Psi_2(x) = \cdots = \mu_m \Psi_m(x) \quad \text{for } x \in \partial C_1(0).$$

For given $H^N_{\omega,L}$, we set

$$\Phi(x) = \mu_k \Psi_k(x) \quad \text{if } x \in C_1(\gamma) \text{ with } \omega(\gamma) = k.$$

Then it is easy to see that Φ is the positive ground state of $H^N_{\omega,L}$ with the energy 0. Let $Q(\,\cdot\,,\cdot\,)$ be the quadratic form corresponding to $H^N_{\omega,L}$. For $\varphi \in H^1(C_L(0))$, we set $f = \varphi/\Phi$. As in the proof of Lemma 2.3, we have

$$Q(\varphi, \varphi) = \|\Phi(\nabla f)\|^2$$

and hence

$$(\inf \Phi)^2 \|\nabla f\|^2 \le Q(\varphi, \varphi) \le (\sup \Phi)^2 \|\nabla f\|^2.$$

This implies

$$K^{-2} \frac{\|\nabla f\|^2}{\|f\|^2} \le \frac{Q(\varphi, \varphi)}{\|\varphi\|^2} \le K^2 \frac{\|\nabla f\|^2}{\|f\|^2},$$

where $K = \max_k \sup(\mu_k \Psi_k) / \min_k \inf(\mu_k \Psi_k)$. By the min-max principle, we learn that

$$K^{-2}\,\#\{\text{e.v. of } (-\triangle)^N_L \le E\} \le \#\{\text{e.v. of } H^N_{\omega,L} \le E\} \le K^2\,\#\{\text{e.v. of } (-\triangle)^N_L \le E\},$$

where $(-\triangle)^N_L$ is the Laplacian on $C_L(0)$ with Neumann boundary conditions. Taking the limit $L \to +\infty$, we have

$$K^{-2} c_d E^{d/2} \le N(E) \le K^2 c_d E^{d/2}, \tag{3-6}$$

where c_d is the volume of the unit ball in $\mathbb{R}^d$. This completes the proof of Theorem 1.4. □

4. Application to random displacement models

We now consider a model recently studied by Baker, Loss and Stolz [2008; 2009]. Combining their results with Theorem 1.2, we show that this model exhibits Lifshitz singularities at the ground state energy.

We consider a random Schrödinger operator of the form:

$$H_\omega = -\triangle + V_\omega \quad \text{on } L^2(\mathbb{R}^d),$$

where

$$V_\omega(x) = \sum_{\gamma \in \mathbb{Z}^d} q(x - \gamma - \omega(\gamma))$$

with independent, identically distributed random variables $\{\omega(\gamma) \mid \gamma \in \mathbb{Z}^d\}$ taking values in $C_1(0)$.

Assumption C. (1) There exists $\delta \in (0, 1/2)$ such that $\omega(\gamma)$ takes values in a finite set

$$\Theta \subset \{x \in \mathbb{R}^d \mid \delta \le x_j \le 1 - \delta, \quad \text{for all } j \in \{1, \dots, d\}\}.$$

Moreover

$$\Theta \supset \Delta = \{x \in \mathbb{R}^d \mid x_j = \delta \text{ or } 1 - \delta, \quad \text{for all } j \in \{1, \dots, d\}\}$$

and $\mathbb{P}(\omega(\gamma) = x) > 0$ for $x \in \Delta$.

(2) $q \in C_0(\mathbb{R}^d)$ and it is supported in $\{x \mid |x_j| \le \delta,\ j \in \{1, \dots, d\}\}$. Moreover, q is symmetric about $\{x \mid x_j = 0\}$, $j = 1, \dots, d$.

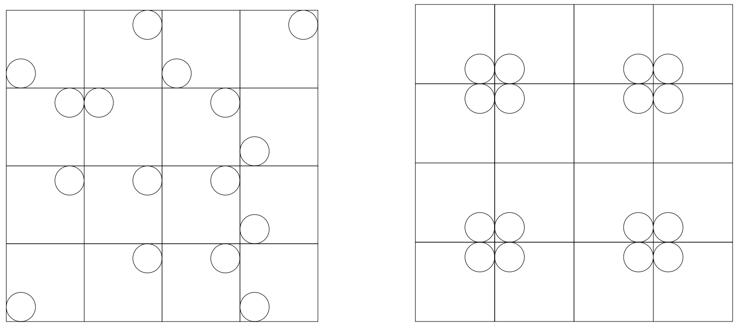

Figure 2. An example in two dimensions, showing a typical random configuration (left) and the minimizing configuration (right).

(3) Let $H_q^N = -\Delta + q$ on $L^2(\{|x| \le 1\})$ with Neumann boundary conditions, and let ϕ be the ground state. Then ϕ is not a constant outside Supp q. Note that this is relevant only if the ground state energy is 0.

Let $H_{1,\beta}^N = -\Delta + q(x-\beta)$ on $L^2(C_1(0))$ with Neumann boundary conditions, where $\beta \in \Theta$. Baker, Loss and Stolz [2008] showed that $\inf \sigma(H_{1,\beta}^N)$ takes its minimum (with respect to β) if and only if $\beta \in \Delta$. In particular, they showed that for $H_{\omega,2\ell}^N$ the Neumann restriction of H_ω to $C_{2\ell}(0)$ the minimal value of the ground state energy was obtained for clustered configuration (see Figure 2).

We cannot directly apply our result to this model, since $q(x-\beta)$ is not symmetric for $\beta \in \Delta$. However, they also showed that if we consider the operator H_ω restricted to $C_2(0)$ and if $d \ge 2$, then the minimum is attained by 2^d symmetric configurations, which are equivalent to each other by translations (see [Baker et al. 2009] and Figure 3). Thus, we can apply our results by considering H_ω as a $2\mathbb{Z}^d$-ergodic random Schrödinger operators, i.e., by considering $C_2(0)$ as the unit cell. Then this model satisfies Assumption A with $M = (\#\Theta)^{2^d}$ and $m = 2^d$.

Theorem 4.1. *Let $d \ge 2$, and suppose Assumption C for some $\delta \in (0, 1/2)$. Then* (1-6) *holds at the bottom of the spectrum of H_ω, a.s.*

We note that if $d = 1$, this result does not hold, and the IDS may have logarithmic singularity at the bottom of the spectrum [Baker et al. 2009]. In view of our results, such singularities can occur for the lack of symmetry of the minimizing configurations.

5. The alloy-type model studied in [Klopp and Nakamura 2009]

In a previous paper on Lifshitz tails for sign indefinite alloy-type random Schrödinger operators [Klopp and Nakamura 2009], we studied the model (1-1) for a single site potential V satisfying the reflection symmetry Assumption B.

We now recall some of the results of that work. Let the support of the random variables $(\omega_\gamma)_\gamma$ be contained in $[a, b]$ and assume both a and b belong to the essential support of the random variables.

Now consider the operator $H_\lambda^N = -\Delta + \lambda V$ with Neumann boundary conditions on the cube $C_1(0) = [0, 1]^d$. Its spectrum is discrete, and we let $E_-(\lambda)$ be its ground state energy. It is a simple eigenvalue

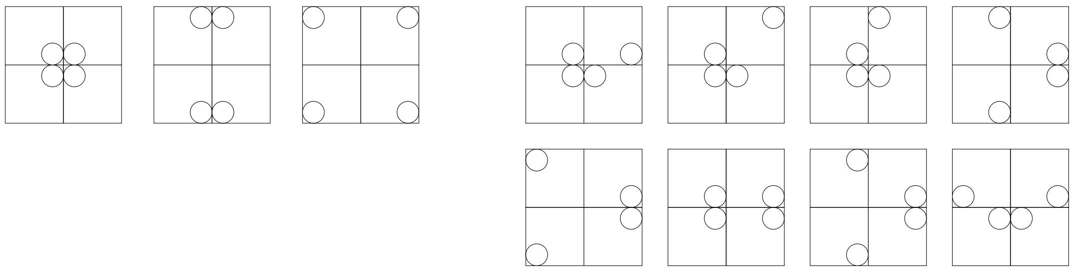

Figure 3. Left: the minimal 2×2 configurations in two dimensions. Right: other 2×2 configurations in two dimensions.

and $\lambda \mapsto E_-(\lambda)$ is a real analytic concave function defined on $\mathbb{R}$. Let E_- be the infimum of the almost sure spectrum of H_ω then

Proposition 5.1 [Klopp and Nakamura 2009]. *Under Assumption B*,

$$E_- = \inf(E_-(a), E_-(b)).$$

As for Lifshitz tails, we proved

Theorem 5.2 [Klopp and Nakamura 2009]. *Suppose that Assumption B is satisfied*, *and that*

$$E_-(a) \neq E_-(b). \tag{5-1}$$

Then

$$\limsup_{E \to E_-^+} \frac{\log|\log N(E)|}{\log(E - E_-)} \leq -\frac{d}{2} - \alpha_+,$$

where we have set $c = a$ if $E_-(a) < E_-(b)$ and $c = b$ if $E_-(a) > E_-(b)$, and

$$\alpha_+ = -\frac{1}{2} \liminf_{\varepsilon \to 0} \frac{\log\big|\log \mathbb{P}(\{|c - \omega_0| \leq \varepsilon\})\big|}{\log \varepsilon} \geq 0.$$

The technique developed in [Klopp and Nakamura 2009] did not allow us to treat the case $E_-(a) = E_-(b)$. Clearly, if the random variables $(\omega_\gamma)_\gamma$ are non trivial and Bernoulli distributed, i.e., if

$$\mathbb{P}(\omega_0 = a) + \mathbb{P}(\omega_0 = b) = 1 \quad \text{and} \quad \mathbb{P}(\omega_0 = a) > 0, \quad \mathbb{P}(\omega_0 = b) > 0,$$

Theorem 1.4 tells us that the Lifshitz tails hold if and only if $aV \not\sim_j bV$ for some $j \in \{1, \ldots, d\}$ (see (1-9)). So we are just left with the case when the random variables $(\omega_\gamma)_\gamma$ are not Bernoulli distributed.

We prove

Theorem 5.3. *Suppose Assumption B is satisfied and that*

$$E_-(a) = E_-(b). \tag{5-2}$$

Assume moreover that the independent, *identically distributed random variables $(\omega_\gamma)_\gamma$ are not Bernoulli distributed*, *that is*, $\mathbb{P}(\omega_0 = a) + \mathbb{P}(\omega_0 = b) < 1$. *Then*

$$\limsup_{E \to E_-^+} \frac{\log|\log N(E)|}{\log(E - E_-)} \leq -\frac{1}{2}. \tag{5-3}$$

So we show that Lifshitz tails also hold in this case. As already noted we believe that (5-3) is not optimal and that $-1/2$ should be replaced by $-d/2$. Moreover, depending on the tail of the distributions of the random variables $(\omega_\gamma)_\gamma$ near a and b, the lim sup in (5-3) should be a limit, the inequality should become an equality, the exponent $-1/2$ should be replaced by $-d/2$ plus a possibly vanishing constant (see of [Klopp and Nakamura 2009, Section 0] for the case $E_-(a) \neq E_-(b)$).

Combining Theorems 5.2 and 5.3 with the Wegner estimates obtained in [Klopp 1995; Hislop and Klopp 2002] and the multiscale analysis as developed in [Germinet and Klein 2001], we learn:

Theorem 5.4. *Assume that Assumption B holds. and that the common distribution of the random variables admits an absolutely continuous density. Then the bottom edge of the spectrum of H_ω exhibits complete localization in the sense of* [Germinet and Klein 2001].

This result improves upon Theorem 0.3 of [Klopp and Nakamura 2009].

5.1. ***The proof of Theorem 5.3.*** Recall that $H^N_{\omega,L}$ is defined in (1-3). It is well known that, at E, a continuity point of $N(E)$, the sequence

$$N^N_L(E) = \mathbb{E}\left(\frac{\#\{\text{eigenvalues of } H^N_{\omega,L} \le E\}}{L^d}\right)$$

is decreasing and converges to $N(E)$ [Pastur and Figotin 1992; Kirsch 1989]. As

$$N^N_L(E) \le C\,\mathbb{P}(\{\inf \sigma(H^N_{\omega,L}) \le E\}), \tag{5-4}$$

it is sufficient to prove an upper bound for $\mathbb{P}(\{\inf \sigma(H^N_{\omega,L}) \le E\})$ for a well chosen value of L.

Define $E_{-,L}(\omega) = \inf \sigma(H^N_{\omega,L})$. It only depends on $(\omega_\gamma)_{\gamma \in Z_L}$, where

$$Z_L = \{\gamma \in \mathbb{Z}^d \mid 0 \le \gamma_j < L,\ j = 1, \dots, d\}.$$

Lemma 5.5. *The function $\omega \mapsto E_{-,L}(\omega)$ is real analytic and strictly concave on $[a,b]^{Z_L}$.*

Proof. Though this is certainly a well known result, for the sake of completeness, we give the proof. The ground state being simple, $\omega \mapsto E_{-,L}(\omega)$ is real analytic in ω.

As H_ω depends affinely on ω, by the variational characterization of the ground state energy, $E_{-,L}(\omega)$ is the infimum of a family of affine functions of ω. So it is concave.

The strict concavity is obtained using perturbation theory. Let $\varphi_L(\omega)$ be the unique normalized positive ground state associated to $E_{-,L}(\omega)$ and $H^N_{\omega,L}$. The ground state energy being simple, this ground state is a real analytic function of ω; differentiating once the eigenvalue equation and the normalization condition of the ground state, as the ground state is normalized and real, one obtains

$$(H^N_{\omega,L} - E_{-,L}(\omega))\partial_{\omega_\gamma}\varphi_L(\omega) = \big(\partial_{\omega_\gamma} E_{-,L}(\omega) - V(\cdot - \gamma)\big)\,\varphi_L(\omega), \tag{5-5}$$

$$\langle \partial_{\omega_\gamma}\varphi_L(\omega), \varphi_L(\omega)\rangle = 0. \tag{5-6}$$

A second differentiation yields

$$\begin{aligned}(H^N_{\omega,L} - E_{-,L}(\omega))\partial^2_{\omega_\gamma\omega_\beta}\varphi_L(\omega) = \partial^2_{\omega_\gamma\omega_\beta} E_{-,L}(\omega)\varphi_L(\omega) &+ \big(\partial_{\omega_\gamma} E_{-,L}(\omega) - V(\cdot - \gamma)\big)\,\partial_{\omega_\beta}\varphi_L(\omega)\\ &+ \big(\partial_{\omega_\beta} E_{-,L}(\omega) - V(\cdot - \beta)\big)\,\partial_{\omega_\gamma}\varphi_L(\omega).\end{aligned}$$

Hence, using (5-5) and (5-6), we compute

$$\begin{aligned}\partial^2_{\omega_\gamma\omega_\beta} E_{-,L}(\omega) &= -\langle V(\cdot-\gamma)\partial_{\omega_\beta}\varphi_L(\omega), \varphi_L(\omega)\rangle - \langle V(\cdot-\beta)\partial_{\omega_\gamma}\varphi_L(\omega), \varphi_L(\omega)\rangle \\ &= -2\mathrm{Re}(\langle (H^N_{\omega,L} - E_{-,L}(\omega))^{-1}\psi_\beta, \psi_\gamma\rangle),\end{aligned}$$

where

- $\psi_\gamma = \Pi V(\cdot - \gamma)\varphi_L(\omega)$;
- Π is the orthogonal projector on the orthogonal to $\varphi_L(\omega)$.

Hence, for $(a_\gamma)_\gamma$ complex numbers,

$$\sum_{\gamma,\beta} \partial^2_{\omega_\gamma\omega_\beta} E_{-,L}(\omega) a_\gamma \overline{a_\beta} = -2\mathrm{Re}(\langle (H^N_{\omega,L} - E_{-,L}(\omega))^{-1}\Pi u_a, \Pi u_a\rangle)$$

where

$$u_a = \Bigl(\sum_\gamma a_\gamma V(\cdot-\gamma)\Bigr)\varphi_L(\omega).$$

Note that, as V is not trivial, the assumption $E_-(a) = E_-(b)$ implies that V changes sign, that is, there exists $x_+ \neq x_-$ such that $V(x_-)\cdot V(x_+) < 0$. Now, the vector Πu_a vanishes if and only if u_a is colinear to $\varphi_L(\omega)$ which cannot happen as V is not constant and $\varphi_L(\omega)$ does not vanish on open sets by the unique continuation principle. On the other hand, $E_{-,L}(\omega)$ being a simple eigenvalue associated to $\varphi_L(\omega)$, $\Pi(H^N_{\omega,L} - E_{-,L}(\omega))^{-1}\Pi \geq c\,\Pi$ for some $c > 0$. So the Hessian of $\omega \mapsto E_{-,L}(\omega)$ is positive definite. This completes the proof of Lemma 5.5. □

We now turn to the proof of Theorem 5.3. As the random variables are not Bernoulli distributed, that is, $\mathbb{P}(\omega_0 = a) + \mathbb{P}(\omega_0 = b) < 1$, we can fix $\varepsilon > 0$ sufficiently small such that

$$\mathbb{P}(\omega_0 \in [a,\ a+\varepsilon)) + \mathbb{P}(\omega_0 \in (b-\varepsilon,\ b]) < 1.$$

By strict concavity of $E_-(\lambda)$, one has $E_-(a) < E_-(a+\varepsilon)$ and $E_-(b) < E_-(b-\varepsilon)$.

In Section 2, we proved:

Lemma 5.6. *Assume $E_-(a) = E_-(b)$. There exists $C > 0$ with the following property: For any $L \geq 0$, if $\omega \in \{a,\ b,\ a+\varepsilon,\ b-\varepsilon\}^{Z_L}$ is such that*

$$\forall p \in \Lambda\ \ \exists \ell \in \{0, \dots, L-1\}\ \textit{ such that }\ \omega_{(p,\ell)} \in \{a+\varepsilon,\ b-\varepsilon\}, \tag{P}$$

then

$$E_{-,L}(\omega) \geq E_-(a) + \frac{1}{CL^2}. \tag{5-7}$$

To complete the proof of Theorem 5.3, we first extend Lemma 5.6 using the concavity of the ground state energy:

Lemma 5.7. *Assume $E_-(a) = E_-(b)$. There exists $C > 0$ satisfying the following property: For all $L \geq 0$, if $\omega \in \Omega_L$ is such that*

$$\forall p \in \Lambda\ \ \exists \ell \in \{0, \dots, L-1\}\ \textit{ such that }\ \omega_{(p,\ell)} \in [a+\varepsilon,\ b-\varepsilon], \tag{P$'$}$$

then (5-7) *holds.* (The constant C is the same as in Lemma 5.6.)

We postpone the proof of this result to complete that of Theorem 5.3. Pick $E > E_-(a) = E_-(b)$. We use (5-4) and pick $L = c(E - E_-(a))^{1/2}$. Pick $c > 0$ sufficiently small that $Cc^2 < 1$. Then Lemma 5.6 tells us that, if $\omega \in [a,b]^{Z_L}$ satisfies (P′), then $E_-(\omega) > E$. So, the set $\Omega_L(E) := \{\omega \in \Omega_L \mid E_-(\omega) > E\}$ satisfies

$$\Omega_L \setminus \Omega_L(E) \subset \Big\{\omega \in \Omega_L \mid \exists p \in \Lambda \text{ such that } \omega_{(p,\ell)} \in [a,\ a+\varepsilon) \cup (b-\varepsilon,\ b] \text{ for all } \ell\Big\}.$$

Hence,

$$\begin{aligned}\mathbb{P}(\Omega_L \setminus \Omega_L(E)) &\le \sum_{p\in\Lambda} \mathbb{P}(\{\omega_{(p,\ell)} \in [a,\ a+\varepsilon) \cup (b-\varepsilon,\ b] \text{ for all } \ell\}) \\ &= L^{d-1}(\mathbb{P}(\omega_0 \in [a,\ a+\varepsilon)) + \mathbb{P}(\omega_0 \in (b-\varepsilon,\ b]))^L.\end{aligned}$$

This yields the announced exponential decay and completes the proof of Theorem 5.3. □

Proof of Lemma 5.7. We will proceed in two steps. First, we prove that, if ω satisfies (P′) and all its coordinates that are not in $[a+\varepsilon,\ b-\varepsilon]$ are either equal to a or to b, then (5-7) holds (with the same constant as in Lemma 5.6). This comes from the concavity of the ground state and the fact that any such point is a convex combination of points satisfying (P). Indeed, take such a point ω and let $\Gamma(\omega)$ be the set of coordinates such that $\omega_\gamma \in [a+\varepsilon,\ b-\varepsilon]$. Define $K(\omega) = \{a+\varepsilon,\ b-\varepsilon\}^{\Gamma(\omega)}$. Then there exists a convex combination $(\mu_\eta)_{\eta\in K(\omega)}$ such that

$$(\omega_\gamma)_{\gamma\in\Gamma(\omega)} = \sum_{\eta\in K(\omega)} \mu_\eta \eta, \qquad \sum_{\eta\in K(\omega)} \mu_\eta = 1, \quad \mu_\eta \ge 0.$$

Hence,

$$\omega = \sum_{\eta\in K(\omega)} \mu_\eta \tilde\eta \text{ where } (\tilde\eta)_\gamma = \begin{cases} \eta_\gamma & \text{if} \gamma \in \Gamma(\omega), \\ \omega_\gamma & \text{if } \gamma \notin \Gamma(\omega). \end{cases}$$

That ω satisfies (5-7) then follows from the concavity of $\omega \mapsto E_{-,L}(\omega)$, that is Lemma 5.5, and from Lemma 5.6.

To complete the proof of Lemma 5.7, it suffices to show that a point ω satisfying (P′) can be written a convex combination of points of the type defined above. This is done as above. Indeed, pick ω satisfying (P′). Define $L(\omega) = \{a,b\}^{(Z_L\setminus\Gamma(\omega))}$. Then there exists a convex combination $(\mu_\eta)_{\eta\in L(\omega)}$ such that

$$(\omega_\gamma)_{\gamma\in(Z_L\setminus\Gamma(\omega))} = \sum_{\eta\in L(\omega)} \mu_\eta \eta, \quad \sum_{\eta\in L(\omega)} \mu_\eta = 1, \quad \mu_\eta \ge 0.$$

Hence,

$$\omega = \sum_{\eta\in L(\omega)} \mu_\eta \tilde\eta \text{ where } (\tilde\eta)_\gamma = \begin{cases} \eta_\gamma & \text{if } \gamma \notin \Gamma(\omega), \\ \omega_\gamma & \text{if } \gamma \in \Gamma(\omega). \end{cases}$$

That ω satisfies (5-7) then follows from the concavity of $\omega \mapsto E_{-,L}(\omega)$ and from the first step. This completes the proof of Lemma 5.7. □

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ANALYTIC CONTINUATION OF THE RESOLVENT OF THE LAPLACIAN AND THE DYNAMICAL ZETA FUNCTION

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Let $s_0 < 0$ be the abscissa of absolute convergence of the dynamical zeta function $Z(s)$ for several disjoint strictly convex compact obstacles $K_i \subset \mathbb{R}^N$, $i = 1, \dots, \kappa_0$, $\kappa_0 \geq 3$, and let

$$R_\chi(z) = \chi(-\Delta_D - z^2)^{-1}\chi, \quad \chi \in C_0^\infty(\mathbb{R}^N),$$

be the cutoff resolvent of the Dirichlet Laplacian $-\Delta_D$ in the closure of $\mathbb{R}^N \setminus \bigcup_{i=1}^{\kappa_0} K_i$. We prove that there exists $\sigma_1 < s_0$ such that the cutoff resolvent $R_\chi(z)$ has an analytic continuation for

$$\operatorname{Im} z < -\sigma_1, \quad |\operatorname{Re} z| \geq J_1 > 0.$$

1. Introduction

Let K be a subset of $\mathbb{R}^N$ $(N \geq 2)$ of the form $K = K_1 \cup K_2 \cup \dots \cup K_{\kappa_0}$, where the K_i are compact strictly convex disjoint domains in $\mathbb{R}^N$ with C^∞ boundaries $\Gamma_i = \partial K_i$ and $\kappa_0 \geq 3$. Set $\Omega = \overline{\mathbb{R}^N \setminus K}$ and $\Gamma = \partial K$. We assume that K satisfies the following (no-eclipse) condition:

$$\begin{gathered}\text{for every pair } K_i, K_j \text{ of different connected components of } K\text{, the convex hull}\\ \text{of } K_i \cup K_j \text{ has no common points with any other connected component of } K.\end{gathered} \tag{H}$$

With this condition, the *billiard flow* ϕ_t defined on the *cosphere bundle* $S^*(\Omega)$ in the standard way is called an open billiard flow. It has singularities, however its restriction to the *nonwandering set* Λ has only simple discontinuities at reflection points. Moreover, Λ is compact, ϕ_t is hyperbolic and transitive on Λ, and it follows from [Stoyanov 1999] that ϕ_t is non-lattice; therefore, by a result of Bowen [1973], it is topologically weak-mixing on Λ.

Given a periodic reflecting ray $\gamma \subset \Omega$ with m_γ reflections, denote by d_γ the period (return time) of γ, by T_γ the primitive period (length) of γ and by P_γ the linear Poincaré map associated to γ. Denote by Π the set of all periodic rays in Ω and let $\lambda_{i,\gamma}$, for $i = 1, \dots, N-1$, denote the eigenvalues of P_γ with $|\lambda_{i,\gamma}| > 1$ [Petkov and Stoyanov 1992].

Let $\mathcal{P}$ be the set of primitive periodic rays. Set

$$\delta_\gamma = -\frac{1}{2}\log(\lambda_{1,\gamma}\dots\lambda_{N-1,\gamma}) \quad \text{for } \gamma \in \mathcal{P}, \qquad r_\gamma = \begin{cases} 0 & \text{if } m_\gamma \text{ is even,} \\ 1 & \text{if } m_\gamma \text{ is odd,} \end{cases}$$

and consider the *dynamical zeta function*

$$Z(s) = \sum_{m=1}^{\infty} \frac{1}{m} \sum_{\gamma \in \mathcal{P}} (-1)^{m r_\gamma} e^{m(-sT_\gamma + \delta_\gamma)}.$$

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Keywords: open billiard, periodic rays, zeta function.

It is easy to show that there exists $s_0 \in \mathbb{R}$ such that for $\operatorname{Re} s > s_0$ the series $Z(s)$ is absolutely convergent and s_0 is minimal with this property. The number s_0 is called *abscissa of absolute convergence*. On the other hand, using symbolic dynamics and the results of [Parry and Pollicott 1990], it follows that $Z(s)$ is meromorphic for $\operatorname{Re} s > s_0 - a,\ a > 0$ [Ikawa 1990] and $Z(s)$ is analytic for $\operatorname{Re} s \geq s_0$. According to [Stoyanov 2001] (for $N = 2$) and [Stoyanov 2007] (for $N \geq 3$ under some additional conditions), there exists $0 < \varepsilon < a$ so that the dynamical zeta function $Z(s)$ admits an analytic continuation for $\operatorname{Re} s \geq s_0 - \varepsilon$.

The *cutoff resolvent*, defined by

$$R_\chi(z) = \chi(-\Delta_K - z^2)^{-1}\chi : L^2(\Omega) \to L^2(\Omega)$$

for $\operatorname{Im} z < 0$, where $\chi \in C_0^\infty(\mathbb{R}^N)$, $\chi = 1$ on K, and Δ_K is the Dirichlet Laplacian in Ω, has a meromorphic continuation in $\mathbb{C}$ for N odd with poles z_j such that $\operatorname{Im} z_j > 0$ and in $\mathbb{C} \setminus \{\boldsymbol{i}\,\mathbb{R}^+\}$ for N even. The analytic properties and the estimates of $R_\chi(z)$ play a crucial role in many problems related to the local energy decay, distribution of the resonances etc. In the physical literature and in works concerning numerical calculation of resonances [Cvitanović and Eckhardt 1989; Wirzba 1999; Lin 2002; Lin and Zworski 2002; Lin et al. 2003] the following conjecture is often made.

Conjecture 1.1. *The poles μ_j (with $\operatorname{Re} \mu_j < 0$) of $Z(s)$ and the poles z_j of $R_\chi(z)$ are related by $\boldsymbol{i} z_j = \mu_j$.*

At least one would expect that the poles z_j of $R_\chi(z)$ lie in sufficiently small neighborhoods of $-\boldsymbol{i}\mu_j$. Presumably for this reason the numbers $-\boldsymbol{i}\mu_j$ are called *pseudopoles* of $R_\chi(z)$.

The case of several disjoint disks has been treated in many works (see [Wirzba 1999] for a comprehensive list of references), and a certain method for numerical computation of the resonances has been used. Although it is not rigorously known whether the numerically found resonances approximate the (true) resonances in the exterior of the discs, and whether the dynamical zeta function has an analytic continuation to the left of the line of absolute convergence, this way of computation is widely accepted in the physical literature.

In the case of two strictly convex disjoint domains it was proved [Ikawa 1982; Gérard 1988] that the poles of $R_\chi(\lambda)$ are contained in small neighborhoods of the pseudopoles

$$m\frac{\pi}{d} + \boldsymbol{i}\alpha_k, \quad m \in \mathbb{Z},\ k \in \mathbb{N}.$$

Here $d > 0$ is the distance between the obstacles and $\alpha_k > 0$ are determined by the eigenvalues λ_j of the Poincaré map related to the unique primitive periodic ray.

It is known that the conjecture above is true for convex cocompact hyperbolic manifolds $X = \boldsymbol{\Gamma} \setminus \mathbb{H}^{n+1}$, where $\boldsymbol{\Gamma}$ is a discrete group of isometries with only hyperbolic elements admitting a finite fundamental domain (then X is a manifold of constant negative curvature). More precisely, the zeros of the corresponding Selberg's zeta function coincide with the poles (resonances) of the Laplacian Δ_g on X [Patterson and Perry 2001].

The case of several convex obstacles is generally much more complicated. However the case $s_0 > 0$ is easier, since we know that for $-s_0 \leq \operatorname{Im} z \leq 0$ the cutoff resolvent $R_\chi(z)$ is analytic [Ikawa 2000].

In the following we assume that $s_0 < 0$.

The first problem is to examine the link between the analyticity of $Z(s)$ for $\operatorname{Re} s > s_0$ and the behavior of $R_\chi(z)$ for $0 \leq \operatorname{Im} z < -s_0$. (The parameters z and s are connected by the equality $s = \boldsymbol{i} z$).

Theorem 1.2 [Ikawa 1988]. *Assume $s_0 < 0$ and $N = 3$. Then for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ so that the cutoff resolvent $R_\chi(z)$ is analytic for* $\operatorname{Im} z < -(s_0 + \varepsilon)$, $|\operatorname{Re} z| \geq C_\varepsilon$.

A similar result for a control problem has been established by Burq [1993]. The proofs in [Ikawa 1988; Burq 1993] are based on the construction of an asymptotic solution $U_M(x, s; k)$ with boundary data $m(x; k) = e^{\boldsymbol{i} k\psi(x)} h(x)$, $k \in \mathbb{R}$, $k \geq 1$, where ψ is a phase function and $h \in C^\infty(\Gamma)$ has a small support. More precisely, $U_M(\,\cdot\,, s; k)$ is $C^\infty(\Omega)$-valued function for $\operatorname{Re} s > s_0$, and we have

$$(\Delta_x - s^2) U_M(\,\cdot\,, s; k) = 0 \qquad \text{for } x \in \mathring{\Omega} \qquad \text{if } \operatorname{Re} s > s_0, \tag{1-1}$$

$$U_M(\,\cdot\,, s; k) \in L^2(\mathring{\Omega}) \qquad \text{if } \operatorname{Re} s > 0, \tag{1-2}$$

$$U_M(x, s; k) = m(x; k) + r_M(x, s; k) \qquad \text{on } \Gamma \qquad \text{if } \operatorname{Re} s > s_0, \tag{1-3}$$

where, for $r_M(x, s; k)$ and $\operatorname{Re} s > s_0 + d > s_0$, $|s + \boldsymbol{i} k| \leq c$, we have the estimates

$$\|r_M(\,\cdot\,, s; k)\|_{L^\infty(\Gamma)} \leq C_{d,\psi,h} k^{-M}. \tag{1-4}$$

To obtain the leading term of $U_M(x, s; k)$ it is necessary to justify the convergence of series of the form

$$\sum_{n=0}^{\infty} \sum_{\substack{|\boldsymbol{j}|=n+3 \\ j_{n+2}=l}} e^{-s\varphi_{\boldsymbol{j}}(x)} a_{\boldsymbol{j}}(x, s; k), \tag{1-5}$$

where $\boldsymbol{j} = (j_0, \ldots, j_{n+2})$ is a configuration (word) of length $|\boldsymbol{j}| = n + 3$, the $\varphi_{\boldsymbol{j}}(x)$ are phase functions and the amplitudes $a_{\boldsymbol{j}}(x, s; k)$ depend on the complex parameter $s \in \mathbb{C}$ and a real parameter $k \geq 1$ (see Sections 3 and 5 for the notation and more details). These parameters are not connected but to have (1-4) we must take $|s + \boldsymbol{i} k| \leq c$. The main difficulty is to establish the summability of the series above and to obtain suitable C^p estimates of their traces on Γ for $\operatorname{Re} s > s_0$. The absolute convergence of $Z(s)$ makes it possible to study the absolute convergence of these series and to get estimates which lead to the properties in (1-1)–(1-4). This might seem a bit surprising since the dynamical zeta function $Z(s)$ is determined by the periods of periodic rays and the corresponding Poincaré maps, and formally from $Z(s)$ one gets almost no information about the dynamics of the rays in a whole neighborhood of the nonwandering set. As it turns out, the absolute convergence of $Z(s)$ is a strong condition which enables us to justify the absolute convergence of (1-5).

The existence of a domain $\{z \in \mathbb{C} : \operatorname{Re} z \in [E - \delta, E + \delta],\ 0 \leq \operatorname{Im} z \leq h_\delta\}$ free of resonances was proved in [Nonnenmacher and Zworski 2009] for the operator $-h^2\Delta + V(x)$, $V(x) \in C_0^\infty(\mathbb{R}^n)$, assuming that the trapping set of the Hamiltonian flow Φ^t of $|\xi|^2 + V(x)$ has a hyperbolic dynamics similar to that of the billiard flow in the exterior of K. The existence of a resonance-free domain in that work is established under the hypothesis $\Pr(1/2) < 0$, where $\Pr(s)$ is the topological pressure associated with the (negative infinitesimal) unstable Jacobian of the flow Φ^t. In our situation this condition is equivalent to $\Pr(g) < 0$, where $\Pr(g)$ is the pressure of the function g associated with the symbolic dynamics related to the flow (see Section 3 for the definition of g and its pressure). It is shown in Section 3 below that $C_1 \Pr(g) \leq s_0 \leq C_2 \Pr(g)$ for some constants $C_1 > 0$, $C_2 > 0$, so $\Pr(g) < 0$ if and only if $s_0 < 0$. It should be mentioned that the techniques and tools in [Nonnenmacher and Zworski 2009] are different from those in [Ikawa 1988; Burq 1993] and the present work.

In the case $\operatorname{Re} s < s_0$, it is an interesting problem to examine the link between the analytic continuation of $R_\chi(z)$ for $\operatorname{Im} z \geq -s_0$ and that of the dynamical zeta function $Z(s)$. Several years ago, Ikawa [1994] announced a result concerning a *local* analytic continuation of $R_\chi(z)$ in a neighborhood of a point z_0 in the region

$$\mathcal{D}_{\alpha,\varepsilon} = \{z \in \mathbb{C} : \operatorname{Im} z \leq -s_0 + |\operatorname{Re} z|^{-\alpha},\ |\operatorname{Re} z| \geq C_\varepsilon\}, \quad 0 < \alpha < 1,$$

assuming the following conditions:

(i) $Z(s)$ is analytic in a neighborhood of $\boldsymbol{i} z_0$ and

$$|Z(\boldsymbol{i} z_0)| \leq |z_0|^{1-\varepsilon}, \quad 0 < \varepsilon < 1; \tag{1-6}$$

(ii) if $w(\eta) > 0$ is an eigenfunction of the Ruelle operator $L_{-s_0\tilde{f}+\tilde{g}}$ with eigenvalue 1, then the constants

$$M = \max_{\xi,\eta \in \Sigma_A^+} \frac{w(\xi)}{w(\eta)}, \quad m = \min_{\xi \in \Sigma_A^+} e^{-s_0\tilde{f}(\xi)+\tilde{g}(\xi)}$$

satisfy the inequality $(M/m)\sqrt{\theta} < 1$ with a global constant $0 < \theta < 1$ depending on the expanding properties of the billiard flow [Ikawa 1988; 1990]. We refer to Section 3 for the notation Σ_A^+, $\tilde{f}$, $\tilde{g}$.

Also in [Ikawa 1994] it was announced that (ii) holds in the case of three balls centered at the vertices of an equilateral triangle, provided the radii of the balls are sufficiently small. In general condition (ii) is rather restrictive. On the other hand, it is difficult to check condition (i) if we have no precise information about the spectral properties of $\tilde{L}_s = L_{-s\tilde{f}+\tilde{g}}$ for $\operatorname{Re} s$ close to s_0. In [Ikawa 1994] there are no comments on when (i) holds or whether this happens at all. As we show in Section 5, the estimate (1-6) for $z \in D_{\alpha,\varepsilon}$ is related to the behavior of the iterations of the Ruelle operator $\tilde{L}_s$ introduced in Section 3. It does not look like the tools required to do this were available back in 1994. To our knowledge a proof of the result announced by Ikawa has not been published anywhere.

Starting with [Dolgopyat 1998], there has been considerable progress in the analysis of the spectral properties of the Ruelle transfer operators $\tilde{L}_s$ related to hyperbolic systems. The so-called Dolgopyat type estimates for the norms of the iterations $\tilde{L}_s^n$ [Dolgopyat 1998; Stoyanov 2001; 2007] imply an estimate for the zeta function $Z(s)$ in a strip $s_0 - \varepsilon \leq \operatorname{Re} s \leq s_0$, $\varepsilon > 0$ (see Section 3 and Appendix C for details). Note also that the information given by the estimates of the iterations and the behavior of the spectrum of $\tilde{L}_s$ is richer than that related to the zeta function $Z(s)$.

Assuming certain regularity of the family of local unstable manifolds $W_\varepsilon^u(x)$ of the billiard flow over the nonwandering set Λ (see Appendix C) and that the Dolgopyat type estimates (3-3) hold for the related operator $\tilde{L}_s$ for some class of functions, in this paper we prove the following main result:

Theorem 1.3. *Let* $s_0 < 0$. *Suppose that the estimates* (3-3) *for the operator* $\tilde{L}_s$ *hold and that the map* $\Lambda \ni x \mapsto W_\varepsilon^u(x)$ *is Lipschitz. Then there exist* $\sigma_1 < s_0$ *and* $J_1 > 0$ *such that the cutoff resolvent* $R_\chi(z)$ *is analytic in*

$$\mathcal{S} = \{z \in \mathbb{C} : \operatorname{Im} z < -\sigma_1,\ |\operatorname{Re} z| \geq J_1\}.$$

Moreover, there exists an integer $m \in \mathbb{N}$ *such that*

$$\|R_\chi(z)\|_{L^2(\mathring{\Omega}) \to L^2(\mathring{\Omega})} \leq C(1+|z|)^m, \quad z \in \mathcal{S}. \tag{1-7}$$

The geometric assumptions in this theorem are always satisfied for $N = 2$. In particular, the Dolgopyat type estimates (3-3) stated in Section 3 below always hold when $N = 2$ [Stoyanov 2001]. For $N \geq 3$ it follows from some general results in [Stoyanov 2007] that (3-3) hold under certain assumptions about the flow on Λ. These assumptions are listed in detail at the beginning of Appendix C. It seems likely that most of these assumptions are either always satisfied or not really necessary in proving the estimates (3-3) for open billiard flows. In fact, it was shown very recently in [Stoyanov 2009] that one of the conditions[1] imposed in [Stoyanov 2007] (and in [Petkov and Stoyanov 2009] as well) is always satisfied for pinched open billiard flows. Apart from that in [Stoyanov 2009] a class of examples with $N \geq 3$ is described for which the results in this paper can be applied.

Our argument in Sections 7–8 shows that the integer m in (1-7) depends on σ_1 and N, however we have not tried to get precise information about m. It seems that to obtain an optimal growth in (1-7) is a difficult problem.

We stress that the Dolgopyat type estimates only apply to a special class of functions on Λ, namely to Lipschitz functions on Λ that are constant on any local stable manifold $W^s_{\mathrm{loc}}(x)$ of the billiard flow ϕ_t (see Section 3 below for details). The estimates for the iterations of the Ruelle operator were originally obtained for the Ruelle operator $\mathscr{L}_s$ related to a coding given by a Markov family of rectangles (see [Petkov and Stoyanov 2009; Stoyanov 2007] and Appendix C for the notation). For the proof of Theorem 1.3 we need Dolgopyat type estimates for the iterations of the Ruelle operator $\tilde{L}_s$ related to the symbolic coding using the connected components of K. The link between the operators $\tilde{L}_s$ and $\mathscr{L}_s$ and the estimates leading to (3-3) are given in [Petkov and Stoyanov 2009, Section 3]; see also Proposition C.5.

We mention that our result implies the existence of an analytic continuation of $R_\chi(z)$ in a strip $0 \leq \operatorname{Im} z \leq -\sigma_1$, $|\operatorname{Re} z| > J_1$, without any restrictions on the eigenfunction $w(\eta)$ and the behavior of $Z(s)$ for $\sigma_1 \leq \operatorname{Re} s \leq s_0$. The estimate (1-7) enables us to obtain a scattering expansion with an exponential decay rate of the remainder for the solutions of the Dirichlet problem

$$\begin{cases} (\partial_t^2 - \Delta)u(t,x) = 0, \quad x \in \mathring{\Omega}, \, u|_{\mathbb{R}\times\Gamma} = 0, \\ u|_{t=0} = f \in C_0^\infty(\mathring{\Omega}), \, \partial_t u|_{t=0} = g \in C_0^\infty(\mathring{\Omega}). \end{cases} \tag{1-8}$$

Set $\mathscr{H} = \dot{H}(\mathring{\Omega}) \oplus L^2(\mathring{\Omega})$ and $\mathscr{D}^j = H^j(\mathring{\Omega}) \oplus H^{j-1}(\mathring{\Omega})$ for $j \geq 2$, where the space $\dot{H}(\mathring{\Omega})$ is the closure of $C_0^\infty(\mathring{\Omega})$ with respect to the norm

$$\|v\|_{\dot{H}(\mathring{\Omega})} = \left(\int_\Omega |\nabla v(x)|^2 dx \right)^{1/2}.$$

Corollary 1.4. *Let N be odd and let $\chi \in C_0^\infty(\mathbb{R}^n)$ be equal to 1 in a neighborhood of K. Let $u(t,x)$ be the solution of* (1-8) *with initial data $(\chi f, \chi g)$. Then under the assumptions of Theorem 1.3 there exists $L \in \mathbb{N}$ such that for every $\varepsilon > 0$ and for $t > 0$ sufficiently large we have*

$$\chi u(t,x) = \sum_{\operatorname{Im} z_l \leq -\sigma_1} \sum_{j=1}^{m(z_l)} w_{z_l,j}(x) e^{itz_l} t^{j-1} + E(t)(f,g),$$

where

$$\|E(t)(f,g)\|_{\mathscr{H}} \leq C_\varepsilon e^{(\sigma_1+\varepsilon)t} \|(f,g)\|_{\mathscr{D}^L}.$$

[1]This is the nondegeneracy of the symplectic form over the nonwandering set Λ; see condition (ND) in Appendix C.

Here $\sigma_1 < s_0$ is as in Theorem 1.3, the z_l are the resonances with $\operatorname{Im} z_l \le -\sigma_1$, *$m_l(z_l)$ is the multiplicity of z_l, and $w_{z_l,j}$ is related to the cutoff resonances states corresponding to z_l.*

A similar result was established by Ikawa [1988] with σ_1 replaced by $s_0 < 0$. Recently, a local decay result for the solutions of the wave equation related to hyperbolic convex cocompact manifolds $\mathbf{\Gamma} \setminus \mathbb{H}^{n+1}$ was proved by C. Guillarmou and F. Naud [2009]. They obtain an exponentially decreasing remainder related to the abscissa δ of absolute convergence of the Poincaré series

$$P_s(m, m') = \sum_{\gamma \in \mathbf{\Gamma}} e^{-s d_h(m, \gamma m')}, \quad m, m' \in \mathbb{H}^{n+1},$$

d_h being the hyperbolic distance. To improve this result, one would have to establish a polynomial growth of the corresponding cutoff resolvent for $\delta - \varepsilon \le \operatorname{Re} s \le \delta$, $|\operatorname{Im} s| \ge C_\varepsilon$ and small $\varepsilon > 0$, and an analog of Corollary 1.4 can be conjectured for convex cocompact manifolds (for which Dolgopyat type estimates are known). For other results concerning scattering expansions for trapping obstacles the reader could consult [Tang and Zworski 2000] and the references given there.

The proof of Theorem 1.3 is long and technical. The reason for this is that we are trying to exploit some quite weak information coming from the Dolgopyat type estimates for some *restrictive* class of functions defined on a symbolic model to build approximations of the resolvent of a boundary value problem based on infinite series which are not absolutely convergent. This reflects the geometric situation and we have to deal with infinite series related to reflections of trapping rays. In this direction it appears the present work is the first one where infinite series of this kind are used for a WKB construction.

Below we discuss the main steps in the proof of Theorem 1.3.

As in [Ikawa 1988; 1994], the idea is to construct an approximative solution $U_M(x, s; k)$ for

$$\sigma_1 \le \operatorname{Re} s \le s_0, \quad |\operatorname{Im} s| \ge J_1,\ k \ge 1,$$

so that $U_M(x, s; k)$ satisfies the conditions (1-1)–(1-3). For our analysis in Section 8 we need to study the Dirichlet problem for $(\Delta_x - s^2)$ with initial data

$$m(x; k) = G(x) e^{\boldsymbol{i} k \langle x, \eta \rangle}\big|_{x \in \Gamma_j} = G(x) e^{\boldsymbol{i} k \varphi(x)}\big|_{x \in \Gamma_j}$$

coming from a representation by using the Fourier transform. On the other hand, it is convenient to pass to data $m(x, s; k) = e^{-s\varphi(x)} b_1(x, s; k)$ with $b_1(x, s; k) = e^{(s + \boldsymbol{i} k)\varphi(x)} G(x)$ and to work with two parameters $s \in \mathbb{C}$ and $k \ge 1$. After the preparation in Sections 3–5, we construct in Section 6 the first approximation $V^{(0)}(x, s; k)$. The first step in the construction of $V^{(0)}(x, s; k)$ is the analysis of the series

$$w_{0,j}(x, s; k) = \sum_{n=-2}^{\infty} \sum_{|\boldsymbol{j}| = n+3,\, j_{n+2} = j} e^{-s\varphi_{\boldsymbol{j}}(x)} a_{\boldsymbol{j}}(x, s; k) = \sum_{n=-2}^{\infty} U_{n+2, j}(x, s; k),\ x \in \Gamma_j,$$

where $\boldsymbol{j} = (j_0, \dots, j_n, j_{n+1}, j_{n+2})$ are configurations of length $|\boldsymbol{j}| = n + 3$, $\varphi_{\boldsymbol{j}}(x)$ are phase functions and $a_{\boldsymbol{j}}(x, s; k)$ are amplitudes determined by a recurrent procedure starting with $m(x, s; k)$. This series corresponds to the sum of the leading terms of the asymptotic solutions constructed after an infinite number of reflections. The analysis of $w_{0,j}(x, s; k)$ is given in Sections 3–5. The main goal there is to justify the existence of $w_{0,j}(x, s; k)$ and to obtain an analytic continuation of $w_{0,j}(x, s; k)$ from $\operatorname{Re} s > s_0$ to a strip $\sigma_0 \le \operatorname{Re} s \le s_0$ with $\sigma_0 < s_0$. To do this, as in the analysis of Dirichlet series with complex parameter, the strategy is to establish suitable estimates for $U_{n+2,j}(x,s;k)$ and to apply a *summation by packages*. The structure of $U_{n+2,j}$ is rather complicated since the phases $\varphi_{\boldsymbol{j}}(x)$ and the amplitudes $a_{\boldsymbol{j}}(x,s;k)$ are related to the dynamics of the reflecting rays having $|\boldsymbol{j}|$ reflections and issued from the convex front $\{(x,\nabla\varphi(x)) : x \in \operatorname{supp} h\}$. It seems unlikely that an explicit relationship exists between $U_{n+2,j}(x,s;k)$ and the iterations $L^n_{-s\tilde{f}+\tilde{g}}$ of the Ruelle operator $L_{-s\tilde{f}+\tilde{g}}$; see Sections 3 and 5). Consequently, one would not expect a particular relationship between $\sum_{n=-2}^{\infty} U_{n+2,j}(x,s;k)$ and the zeta function $Z(s)$. Thus, it appears the situation considered here is rather different from the case of convex cocompact surfaces where it is known that the singularities of the Selberg zeta function coincide with the singularities of the corresponding Poincaré series which in turn is related to the resolvent of the Laplacian [Patterson and Perry 2001].

It was observed by Ikawa [1994] that $U_{n+2,j}(x,s;k)$ can be compared with $L^n_{-s\tilde{f}+\tilde{g}}\mathcal{M}_{n,s}(x)\mathcal{G}_s\tilde{v}_s(\xi)$, where $\mathcal{M}_{n,s}(x)$ and $\mathcal{G}_s$ are suitable operators defined by means of billiard trajectories issued from appropriate unstable or stable manifolds, while $\tilde{v}_s(\xi)$ is a function related to the boundary data $m(x,s;k) = e^{-s\varphi(x)}h$. The precise definitions with some small but essential differences[2] are given in Section 3.

The crucial step in this direction is Theorem 3.2, which provides an estimate of the form

$$\|L^n_{-s\tilde{f}+\tilde{g}}\mathcal{M}_{n,s}(x)\mathcal{G}_s\tilde{v}_s(\xi) - U_{n+2,l}(x,s;k)\|_{C^p(\Gamma)} \le C_p(s,\varphi,h)(\theta+ca)^n \quad \text{for all } p \in \mathbb{N} \text{ and } n \in \mathbb{N},$$

where $a = s_0 - \operatorname{Re} s$ and $c > 0$, $0 < \theta < 1$, $C_p > 0$ are global constants. The assumption concerning the Dolgopyat type estimates (3-3) of $\tilde{L}_s$ is not required for the proof of Theorem 3.2. A statement similar to part (a) of Theorem 3.2 (corresponding to $p = 0$) was announced by Ikawa [1994], however as far as we know no proof has ever been published. The proof of Theorem 3.2 is long and technical, however we consider it in detail since it is of fundamental importance for the considerations later on. It is essential to notice that the link between $U_{n+2,j}$ and the iterations of the Ruelle operator $L_{-s\tilde{f}+\tilde{g}}$ is crucial and allows us to find suitable estimates and deduce the convergence of $w_{0,j}(x,s;k)$. This could be considered as a mathematical interpretation of the interaction between the terms with complex phases in $U_{n+2,j}$. The proof of Theorem 3.2 in the case $p = 0$ is given in Section 3, while Section 4 deals with $p \ge 1$.

In Section 5 we obtain estimates for $w_{0,j}(x,s;k)$ applying Theorem 3.2. The convergence of this series is reduced to that of the series $\sum_{n=0}^{\infty} L^n_{-s\tilde{f}+\tilde{g}}\mathcal{M}_{n,s}(x)\mathcal{G}_s\tilde{v}_s(\xi)$. Here the Dolgopyat type estimates (3-3) for the iterations $L^n_{-s\tilde{f}+\tilde{g}}$ play a crucial role and we can justify the analyticity of $w_{0,j}(x,s;k)$ for $\operatorname{Re} s \ge \sigma_0$ with $\sigma_0 < s_0$. The estimates of $w_{0,j}(x,s;k)$ for $\sigma_0 \le \operatorname{Re} s \le s_0$ are different from those in the domain of absolute convergence $\operatorname{Re} s > s_0$.

In Section 6 we construct outgoing parametrices P_h, P_g, P_e respectively for the hyperbolic, glancing and elliptic sets of $T^*(\Gamma_j)$ related to a fixed strictly convex obstacle K_j. We set $\mathcal{S}_j(s) = P_h + P_g + P_e$ and define the first approximation

$$V^{(0)}(x,s;k) = \sum_{j=1}^{\kappa_0}\Big(\mathcal{S}_j(s)w_{0,j}\Big)(x,s;k),\ x \in \Omega,$$

which is an analytic function for $s \in \mathcal{D}_0 = \{s \in \mathbb{C} : \sigma_0 \le \operatorname{Re} s \le 1,\ |\operatorname{Im} s| \ge J \ge 2\}$. Here the estimates for $U_{n+2,j}(x,s;k)$ obtained in Section 5 are crucial for the convergence of the series $\mathcal{S}_j(s)w_{0,j}$. Next,

[2] In fact, it is difficult to see how the original definitions of the operators $\mathcal{M}_{n,s}$ and $\mathcal{G}_s$ in [Ikawa 1994] would work without the changes we have made in Section 3 below.

we need to examine the leading terms of the traces of $V^{(0)}$ on $\Gamma_l, l \neq j$, and for this purpose we use a microlocal analysis based on the frequency set introduced in [Guillemin and Sternberg 1977] and [Gérard 1988] as well as a global construction of asymptotic solution with oscillatory boundary data $e^{-\boldsymbol{i} s\varphi_j(x)} b(x, s; k)$ with frequency set in the hyperbolic domain given by Ikawa [1988]. Thus, we show that $V^{(0)}(x, s; k)$ satisfies the conditions

$$\begin{cases} (\Delta_x - s^2)V^{(0)}(x, s; k) = 0 & \text{for } x \in \mathring{\Omega}, s \in \mathscr{D}_0, \\ V^{(0)}(x, s; k) \in L^2(\mathring{\Omega}) & \text{for } \operatorname{Re} s > 0, \\ V^{(0)}(x, s; k) = m(x, s; k) + s^{-1} R_1(x, s; k) \quad \text{on } \Gamma & \text{for } s \in \mathscr{D}_0, \end{cases}$$

with estimates

$$\|R_1(x, s; k)\|_{C^p(\Gamma)} \le C_p \langle s + \boldsymbol{i} k\rangle^{p+2} \, |s|^{p+(N+3)/2+\beta_0}, \quad 0 < \beta_0 < 1, \quad \text{for all } p \in \mathbb{N},$$

where $\langle z\rangle = 1 + |z|$. The main point here is that $R_1(x, s; k)$ is analytic for $s \in \mathscr{D}_0$. Finite higher order approximations $V^{(j)}(x, s; k)$, $j = 0, \dots, M-1$, are examined in Section 7, and we show that

$$\sum_{j=0}^{M-1} V^{(j)}(x, s; k) = m(x, s; k) + s^{-M} \mathscr{Q}_M(x, s; k), \quad x \in \Gamma, \ s \in \mathscr{D}_0,$$

with estimates

$$\|\mathscr{Q}_M(x, s; k)\|_{C^0(\Gamma)} \le C_M |s|^{N(M)} \langle s + \boldsymbol{i} k\rangle^{L(M)}, \quad s \in \mathscr{D}_0,$$

where $N(M) > M$ depends on M and $L(M) \to \infty$ as $M \to \infty$ and $Q_M(x, s; k)$ is analytic for $s \in \mathscr{D}_0$. The situation here is quite different from the absolutely convergent case treated in [Ikawa 1988; Burq 1993], where we have $N(M) = 0$ for $\operatorname{Re} s > s_0 + d > s_0$. We need a finite number $M - 1 > (N-3)/2$ of higher order approximations, so we fix M and, applying a version of the three lines theorem, we choose $\sigma_1 < s_0$ close to s_0 so that for

$$s \in \{s \in \mathbb{C} : \sigma_1 \le \operatorname{Re} s \le s_0 + c, \ |\operatorname{Im} s| \ge J, \ |s + \boldsymbol{i} k| \le |\sigma_0| + c\}, \quad s_0 + c \ge 1$$

we get an estimate

$$\|\mathscr{Q}_M(x, s; k)\|_{C^0(\Gamma)} \le B_M k^{\alpha},$$

with $0 < \alpha < M - (N-1)/2$. The final step of our argument is in Section 8, where we solve an integral equation on the boundary Γ. To do this, we invert in $L^2(\Gamma)$ an operator $I + Q(s; k)$ and we apply the last estimate to show that $Q(s; k)$ has a small $L^2(\Gamma)$ norm for $k \ge k_1$.

Depending on how much details the reader is prepared to see in trying to understand the proof of our main result, we would suggest three different ways to proceed. The shortest one is to start by reading Section 2 and only the beginning of Section 3 concerning the definitions of $u_j(x, s)$ and the statement of Theorem 3.2, however omitting the proof of this theorem in Sections 3–4. Then one should read the definition of $w_{0,j}(x, s)$ in Section 5, and skipping the proof of the estimates (5-8) of $w_{0,j}$ in Section 5, one could go directly to the constructions in Section 6, followed by Sections 7 and 8. The arguments in Sections 6–8 use only the estimates (5-8) and some geometrical facts from Section 2 and Appendix B, so the reader should be able to understand the proof of Theorem 1.3 in Section 8 modulo the omitted technical details.

The second way to proceed is to read Section 2 and then to follow the dynamical proofs in Section 3, assuming the estimate (3-3). One could then proceed as above up to Section 8. In this way at a first reading Section 4 could be skipped, if the reader is not interested in the details of the estimates of the derivatives of $U_{n+2,j}$. Finally, a complete reading would start with Section 2 and then Appendices A and C, to understand the estimates (3-3) and the restrictions on the class of functions for which we have Dolgopyat type estimates based on [Stoyanov 2007] and [Petkov and Stoyanov 2009]. Then one can proceed as in the second way.

2. Preliminaries

This section contains some basic facts about the dynamics of the billiard flow in the exterior Ω of K. Our main reference is [Ikawa 1988], whose notation we follow for the most part; see also [Burq 1993] and [Petkov and Stoyanov 1992].

Throughout the paper we use the symbols c and C to denote positive *global constants* depending only on K. These constants might be different in different expressions. Notation of the form C_p, c_p will be used to denote global constants that depend on K and possibly on the number p. We assume throughout that K is as in Section 1.

Denote by A the $\kappa_0 \times \kappa_0$ matrix with entries $A(i, j) = 1$ if $i \neq j$ and $A(i,i) = 0$ for all i, and set

$$\begin{aligned}
\Sigma_A &= \{(\dots, \eta_{-m}, \dots, \eta_{-1}, \eta_0, \eta_1, \dots, \eta_m, \dots) : 1 \le \eta_j \le \kappa_0,\ \eta_j \in \mathbb{N},\ \eta_j \neq \eta_{j+1} \text{ for all } j \in \mathbb{Z}\},\\
\Sigma_A^+ &= \{(\eta_0, \eta_1, \dots, \eta_m, \dots) : 1 \le \eta_j \le \kappa_0,\ \eta_j \in \mathbb{N},\ \eta_j \neq \eta_{j+1} \text{ for all } j \ge 0\},\\
\Sigma_A^- &= \{(\dots, \eta_{-m}, \dots, \eta_{-1}, \eta_0) : 1 \le \eta_j \le \kappa_0,\ \eta_j \in \mathbb{N},\ \eta_{j-1} \neq \eta_j \text{ for all } j \le 0\}.
\end{aligned}$$

Let

$$\mathrm{pr}_1 : S^*(\Omega) = \Omega \times \mathbb{S}^{N-1} \to \Omega \quad \text{and} \quad \mathrm{pr}_2 : S^*(\Omega) \to \mathbb{S}^{N-1}$$

be the natural projections. Introduce the shift operator

$$\sigma : \Sigma_A \to \Sigma_A \quad (\text{or } \sigma : \Sigma_A^+ \to \Sigma_A^+)$$

by $(\sigma(\xi))_i = \xi_{i+1}$ for $i \in \mathbb{Z}$ and $\xi \in \Sigma_A$ (or for $i \in \mathbb{N}$ and $\xi \in \Sigma_A^+$).

Fix a large ball B_0 containing K in its interior. For any $x \in \Gamma = \partial K$ we will denote by $\nu(x)$ the outward unit normal to Γ at x.

For any $\delta > 0$ and $V \subset \Omega$ denote by $S_\delta^*(V)$ the set of those $(x, u) \in S^*(\Omega)$ such that $x \in V$ and there exist $y \in \Gamma$ and $t \ge 0$ with $y + tu = x$, $y + su \in \mathbb{R}^N \setminus K$ for all $s \in (0, t)$ and $\langle u, \nu(y) \rangle \ge \delta$.

Condition (H) implies:

Lemma 2.1 [Ikawa 1988, Lemma 3.1]. *There exist constants $\delta_0 > 0$ and $d_0 > 0$ such that for all $i, j = 1, \dots, \kappa_0$, if a ray issued from $x \in \Gamma_i$ with direction u hits Γ_j at a point $y \in \Gamma_j$ such that $\langle u, \nu(y) \rangle \ge -\delta_0$, then the forward ray issued from (y, v) with $v = u - 2\langle u, \nu(y)\rangle \nu(y)$ does not meet a d_0 neighborhood of $\bigcup_{l \neq j} K_l$.*

That is, there exists a constant $\delta' > 0$ such that if for some $(y, v) \in S^*(\Omega)$ with $y \in \Gamma$, both its forward and backward billiard trajectories have common points with Γ, then $\delta' \le \langle v, \nu(y) \rangle$.

Let $z_0 = (x_0, u_0) \in S^*(\Omega)$. Denote by $X_1(z_0), X_2(z_0), \dots, X_m(z_0), \dots$ the successive *reflection points* (if any) of the *forward trajectory* $\gamma_+(z_0) = \{\mathrm{pr}_1(\phi_t(z_0)) : 0 \le t\}$. If $\gamma_+(z_0)$ is bounded (that is, if it has infinitely many reflection points), we will say that it *has a forward itinerary* $\eta = (\eta_1, \eta_2, \dots)$ (or that it follows the *configuration* η) if $X_j(z_0) \in \partial K_{\eta_j}$ for all $j \geq 1$. Similarly, we denote by $\gamma_-(z_0)$ the *backward trajectory* determined by z_0 and by $\dots, X_{-m}(z_0), \dots, X_{-1}(z_0), X_0(z_0)$ its backward reflection points, if any. For any $j \in \mathbb{Z}$ for which $X_j(z_0)$ exists, denote by $\Xi_j(z_0)$ the *direction* of $\gamma(z_0) = \gamma_-(z_0) \cup \gamma_+(z_0)$ at $X_j(z_0) = \mathrm{pr}_1(\phi_{t_j}(z_0))$; that is,

$$\Xi_j(z_0) = \lim_{t \searrow t_j} \mathrm{pr}_2(\phi_t(z_0)).$$

Thus, $\phi_{t_j}(z_0) = (X_j(z_0), \Xi_j(z_0))$. A finite string $\boldsymbol{j} = (j_0, j_1, j_2, \dots, j_m)$ of numbers $j_i = 1, 2, \dots, \kappa_0$ will be called an *admissible configuration* (of *length* $|\boldsymbol{j}| = m+1$) if $j_i \neq j_{i+1}$ for all $i = 0, 1, \dots, m-1$. We will say that a billiard trajectory γ with successive reflection points $x_0, x_1, \dots, x_m$ *follows the configuration* $\boldsymbol{j}$ if $x_i \in \Gamma_{j_i}$ for all $i = 0, 1, \dots, m$.

A *phase function* on an open set $\mathcal{U}$ in $\mathbb{R}^N$ is a smooth (C^∞) function $\varphi : \mathcal{U} \to \mathbb{R}$ such that $\|\nabla\varphi\| = 1$ everywhere in $\mathcal{U}$. For $x \in \mathcal{U}$ the level surface

$$\mathcal{C}_\varphi(x) = \{y \in \mathcal{U} : \varphi(y) = \varphi(x)\}$$

has a unit normal field $\pm\nabla\varphi(y)$.

Remark 2.2. In this section and the next two, the C^∞ smoothness assumption can be replaced by C^k for any $k \geq 1$.

Definition 2.3. A phase function φ defined on $\mathcal{U}$ is said to *satisfy condition* ($\mathcal{P}$) on $\mathcal{V}$ if

(i) the normal curvatures of $\mathcal{C}_\varphi$ with respect to the normal field $-\nabla\varphi$ are nonnegative at every point of $\mathcal{C}_\varphi$, and

(ii) $\mathcal{U}^+(\varphi) = \{y + t\nabla\varphi(y) : t \geq 0,\ y \in \mathcal{U} \cap \mathcal{V}\} \supset \bigcup_{i \neq j} K_i$.

A natural extension of φ on $\mathcal{U}^+(\varphi)$ is obtained by setting $\varphi(y + t\nabla\varphi(y)) = \varphi(y) + t$ for $t \geq 0$ and $y \in \mathcal{U} \cap \mathcal{V}$.

Given a phase function φ satisfying condition ($\mathcal{P}$) on Γ_j and $i \neq j$, denote by $\mathcal{U}_i(\varphi)$ *the set of all points x of the form* $x = X_1(y, \nabla\varphi(y)) + t\, \Xi_1(y, \nabla\varphi(y))$, where $y \in \mathcal{U} \cap \Gamma_j$ and $t \geq 0$ are such that $X_1(y, \nabla\varphi(y)) \in \Gamma_{i,(j)}$, where

$$\Gamma_{i,(j)} = \left\{ x \in \Gamma_i : \left\langle \nu(x), \frac{y - x}{\|y - x\|} \right\rangle \geq \delta_0 \text{ for all } y \in \Gamma_j \right\}.$$

Then, setting $\varphi_i(x) = \varphi(X_1(y, \nabla\varphi(y))) + t$, one gets a phase function φ_i satisfying condition ($\mathcal{P}$) on Γ_i [Ikawa 1988]. The operator sending φ to φ_i is denoted by Φ^i_j, that is, $\Phi^i_j(\varphi) = \varphi_i$.

Given an admissible configuration $\boldsymbol{j} = (j_0, j_1, \dots, j_m)$ and a phase function φ satisfying condition ($\mathcal{P}$) on Γ_{j_0}, define

$$\varphi_{\boldsymbol{j}} = \Phi^{j_m}_{j_{m-1}} \circ \Phi^{j_{m-1}}_{j_{m-2}} \circ \dots \Phi^{j_2}_{j_1} \circ \Phi^{j_1}_{j_0}(\varphi).$$

Notice that for any z in the domain $\mathcal{U}_{\boldsymbol{j}}(\varphi)$ of $\varphi_{\boldsymbol{j}}$ there exists $(x, u) \in S^*(\Gamma_{j_0})$ such that $x \in \mathcal{U}$ and $\gamma_+(x, u)$ *follows the configuration* $\boldsymbol{j}$, that is, it has at least m reflection points and $X_i(x, u) \in \Gamma_{j_i}$ for all $i = 1, \dots, m$, and $z = X_m(x, u) + t\, \Xi_m(x, u)$ for some $t \geq 0$. Set

$$X^{-l}(z, \varphi_{\boldsymbol{j}}) = X_{m-l}(x, u), \quad 0 \leq l \leq m.$$

Several well-known facts about the dynamics of the billiard in Ω, phase functions and related objects will be frequently used throughout the paper and for convenience of the reader we state them here.

The following is a consequence of the hyperbolicity of the billiard flow in the exterior of K and can be derived from the works of Sinai on general dispersing billiards [Sinai 1970; Sinai 1979] and from Ikawa's papers on open billiards, such as [Ikawa 1988]; see also [Burq 1993]. In this particular form it can be found in [Sjöstrand 1990]; see also [Petkov and Stoyanov 1992, Chapter 10].

Proposition 2.4. *There exist global constants $C > 0$ and $\alpha \in (0, 1)$ such that for any admissible configuration $\boldsymbol{j} = (j_0, j_1, \dots, j_m)$ and any two billiard trajectories in Ω with successive reflection points $x_0, x_1, \dots, x_m$ and $y_0, y_1, \dots, y_m$, both following the configuration $\boldsymbol{j}$, we have*

$$\|x_i - y_i\| \le C\,(\alpha^i + \alpha^{m-i}), \quad 0 \le i \le m.$$

C and α can be chosen so that if there exists a phase function φ satisfying condition $(\mathcal{P})$ on some open set $\mathcal{U}$ containing x_0 and y_0 and such that

$$\nabla\varphi(x_0) = \frac{x_1 - x_0}{\|x_1 - x_0\|} \quad \text{and} \quad \nabla\varphi(y_0) = \frac{y_1 - y_0}{\|y_1 - y_0\|},$$

then $\|x_i - y_i\| \le C\,\alpha^{m-i}$ for $0 \le i \le m$.

Next, given a vector $a = (a_1, \dots, a_N) \in \mathbb{R}^N$, set

$$D_a = a_1 \frac{\partial}{\partial x_1} + \cdots + a_N \frac{\partial}{\partial x_N},$$

and for any C^1 vector field $f : U \to \mathbb{R}^N$ ($U \subset \mathbb{R}^N$) and any $V \subset U$ set $\|f\|_0(V) = \sup_{x\in V} \|f(x)\|$ and $\|f\|_0 = \|f\|_0(U)$. Assuming f has continuous derivatives of all orders up to $p \ge 1$, set

$$\|f\|_p(x) = \max_{a^{(1)},\dots,a^{(p)}\in\mathbb{S}^{N-1}} \|(D_{a^{(1)}} \dots D_{a^{(p)}} f)(x)\|, \quad \|f\|_p(V) = \sup_{x\in V} \|f\|_p(x), \quad \|f\|_p = \|f\|_p(U),$$
$$\|f\|_{(p)}(x) = \max_{0\le j\le p} \|f\|_j(x), \quad \|f\|_{(p)}(V) = \sup_{x\in V} \|f\|_{(p)}(x), \quad \|f\|_{(p)} = \|f\|_{(p)}(U).$$

Similarly, for $x \in \Gamma$ and $V \subset \Gamma$ set

$$\|f\|_{\Gamma,p}(x) = \max_{a^{(1)},\dots,a^{(p)}\in S_x\Gamma} \|(D_{a^{(1)}} \dots D_{a^{(p)}} f)(x)\|, \quad \|f\|_{\Gamma,p}(V) = \sup_{x\in V} \|f\|_{\Gamma,p}(x), \quad \|f\|_{\Gamma,p} = \|f\|_{\Gamma,p}(U),$$

where $S_x\Gamma$ is the unit sphere in the tangent plane $T_x\Gamma$ to Γ at x. Finally, set

$$\|f\|_{\Gamma,(p)}(x) = \max_{0\le j\le p} \|f\|_{\Gamma,j}(x), \quad \|f\|_{\Gamma,(p)}(V) = \sup_{x\in V} \|f\|_{(p)}(x), \quad \|f\|_{\Gamma,(p)} = \|f\|_{\Gamma,(p)}(U).$$

Remark 2.5. It follows easily from the definitions that for any $\delta > 0$ and any integer $p \ge 1$ there exists a constant $A_p = A_p(\delta, K) > 0$ such that if ψ is a phase function which is at least C^{p+1}-smooth on some subset V of Ω and $x \in V \cap \Gamma$ with $(x, \nabla\psi(x)) \in S^*_\delta(V)$, then $\|\nabla\psi\|_p(x) \le A_p\,\|\nabla\psi\|_{\Gamma,p}(x)$.

The following comprises Proposition 5.4 in [Ikawa 1982], Propositions 3.11 and 3.12 in [Ikawa 1988] and Lemma 4.1 in [Ikawa 1987]; see also the proof of the estimate (3.64) in [Burq 1993].

Proposition 2.6. *For every integer $p \geq 1$ there exist global constants $C_p > 0$ and $\alpha \in (0, 1)$ such that for any admissible configuration $\boldsymbol{j} = (j_0, j_1, \dots, j_m)$ and any phase functions φ and ψ satisfying condition $(\mathcal{P})$ on Γ_{j_0} on some open set $\mathcal{U}$, we have*

$$\|\nabla\varphi_{\boldsymbol{j}}\|_p(x) \leq C_p \,\|\nabla\varphi\|_{(p)}(\mathcal{U} \cap B_0) \quad \text{for any } x \in \mathcal{U}_{\boldsymbol{j}}(\varphi) \cap B_0, \tag{2-1}$$

and

$$\|\nabla\varphi_{\boldsymbol{j}} - \nabla\psi_{\boldsymbol{j}}\|_p(x) \leq C_p\, \alpha^m \,\|\nabla\varphi - \nabla\psi\|_p(\mathcal{U} \cap B_0), \tag{2-2}$$

$$\left\|X^{-l}(\,\cdot\,, \nabla\varphi_{\boldsymbol{j}}) - X^{-l}(\,\cdot\,, \nabla\psi_{\boldsymbol{j}})\right\|_{\Gamma,p}(x) \leq C_p\, \alpha^{m-l} \,\|\nabla\varphi - \nabla\psi\|_{(p)}(\mathcal{U} \cap B_0) \tag{2-3}$$

for any $x \in \mathcal{U}_{\boldsymbol{j}}(\varphi) \cap \mathcal{U}_{\boldsymbol{j}}(\psi) \cap B_0$ and $0 \leq l < m$. Finally, we can choose $C_p > 0$ so that

$$\|X^{-l}(\,\cdot\,, \nabla\varphi_{\boldsymbol{j}})\|_{\Gamma,p}(x) \leq C_p\, \alpha^l \quad \text{for all } x \in \mathcal{U}_{\boldsymbol{j}}(\varphi) \cap B_0 \text{ and } 0 \leq l < m. \tag{2-4}$$

Given x in the domain $\mathcal{U}$ of a phase function φ, introduce

$$\Lambda_\varphi(x) = \left(\frac{G_\varphi(x)}{G_\varphi(X^{-1}(x, \nabla\varphi))}\right)^{1/(N-1)},$$

where $G_\varphi(y)$ is the Gaussian curvature of $C_\varphi(y)$ at y. It follows from [Ikawa 1988] (or [Burq 1993]) that there exist global constants $0 < \alpha_1 < \alpha < 1$ such that

$$0 < \alpha_1 \leq \Lambda_\varphi(y) \leq \alpha < 1 \tag{2-5}$$

for any phase function φ and any $y \in \mathcal{U}(\varphi)$.

Now for any $\boldsymbol{j} = (j_0 = 1, j_1, \dots, j_m)$ and any $x \in \mathcal{U}_{\boldsymbol{j}}(\varphi)$, slightly changing a definition from [Ikawa 1988], set

$$(A_{\boldsymbol{j}}(\varphi)\, h)(x) = \Lambda_{\varphi,\boldsymbol{j}}(x)\, h(X^{-m}(x, \nabla\varphi_{\boldsymbol{j}})),$$

where

$$\Lambda_{\varphi,\boldsymbol{j}}(x) = \Lambda_{\varphi_{(j_1,\dots,j_m)}}(x)\, \Lambda_{\varphi_{(j_1,\dots,j_{m-1})}}(X^{-1}(x, \nabla\varphi_{\boldsymbol{j}}))\, \dots\, \Lambda_\varphi(X^{-m}(x, \nabla\varphi_{\boldsymbol{j}})) \in (0, 1).$$

The following facts can be derived from [Ikawa 1982; 1988]; see also [Burq 1993, Proposition 5.1].

Proposition 2.7. *For every integer $p \geq 1$ there exists a global constant $C_p > 0$ such that for any admissible configuration $\boldsymbol{j} = (j_0, j_1, \dots, j_m)$ and any phase function φ satisfying condition $(\mathcal{P})$ on Γ_{j_0} on some open set $\mathcal{U}$, we have $\|\Lambda_{\varphi,\boldsymbol{j}}\|_p(x) \leq C_p \,\|\nabla\varphi\|_{(p)}(\mathcal{U} \cap B_0)$ for $x \in \mathcal{U}_{\boldsymbol{j}}(\varphi) \cap B_0$.*

3. Ruelle operator and asymptotic solutions

Given $\xi \in \Sigma_A$, let $\dots, P_{-2}(\xi), P_{-1}(\xi), P_0(\xi), P_1(\xi), P_2(\xi), \dots$ be the successive reflection points of the unique billiard trajectory in the exterior of K such that $P_j(\xi) \in K_{\xi_j}$ for all $j \in \mathbb{Z}$. Set

$$f(\xi) = \|P_0(\xi) - P_1(\xi)\|.$$

Following [Ikawa 1988] (see also Appendix A), one constructs a sequence $\{\varphi_{\xi,j}\}_{j=-\infty}^{\infty}$ of phase functions such that for each j, $\varphi_{\xi,j}$ is defined and smooth in a neighborhood $U_{\xi,j}$ of the segment $[P_j(\xi), P_{j+1}(\xi)]$ in Ω and:

(i) $\|\nabla\varphi_{\xi,j}\| = 1$ on $U_{\xi,j}$ and $\nabla\varphi_{\xi,j}$ satisfies part (i) of condition $(\mathcal{P})$ on $U_{\xi,j}$;

(ii) $\nabla\varphi_{\xi,j}(P_j(\xi)) = \dfrac{P_{j+1}(\xi) - P_j(\xi)}{\|P_{j+1}(\xi) - P_j(\xi)\|}$;

(iii) $\varphi_{\xi,j} = \varphi_{\xi,j+1}$ on $\Gamma_{\xi_{j+1}} \cap U_{\xi,j} \cap U_{\xi,j+1}$;

(iv) for each $x \in U_{\xi,j}$ the surface $C_{\xi,j}(x) = \{y \in U_{\xi,j} : \varphi_{\xi,j}(y) = \varphi_{\xi,j}(x)\}$ is strictly convex with respect to its normal field $\nabla\varphi_{\xi,j}$.

More precisely, one can proceed as follows. Given $\xi \in \Sigma_A$, let $\xi^- = (\dots, \xi_{-2}, \xi_{-1}, \xi_0)$ and let ψ_{ξ^-} be the phase function with $\psi_{\xi^-}(P_0) = 0$ and $\nabla\psi_{\xi^-}(P_0) = (P_1 - P_0)/\|P_1 - P_0\|$ constructed in Proposition A.1(a). Set $\varphi_{\xi,0} = \psi_{\xi^-}$ and $\varphi_{\xi,j} = (\psi_{\xi^-})_{(\xi_0,\xi_1,\dots,\xi_j)}$ for any $j > 0$. For $j < 0$, setting $\xi^{(j)} = (\dots, \xi_{j-2}, \xi_{j-1}, \xi_j)$ and using again Proposition A.1, we get a phase function $\psi_{\xi^{(j)}}$ with $\psi_{\xi^{(j)}}(P_j) = 0$ and $\nabla\psi_{\xi^{(j)}}(P_j) = (P_{j+1} - P_j)/\|P_{j+1} - P_j\|$. By the uniqueness of the phase functions ψ_η (see Proposition A.1(c)), it follows that there exists a constant c_j such that $\psi_{\xi^-} = (\psi_{\xi^{(j)}} + c_j)_{(\xi_j,\xi_{j+1},\dots,\xi_0)}$ (locally near the segment $[P_0, P_1]$). Setting $\varphi_{\xi,j} = \psi_{\xi^{(j)}} + c_j$, one obtains a phase function defined on some naturally determined[3] open set $\mathcal{U}_{\xi^-,j}$ such that

$$(\varphi_{\xi,j})_{(\xi_j,\xi_{j+1},\dots,\xi_{-1},\xi_0)} = \psi_{\xi^-}, \quad j < 0. \tag{3-1}$$

This completes the construction of the phase functions $\varphi_{\xi,j}$.

It follows from Proposition 2.6 that for any $p \geq 1$ there exists a global constant $C_p > 0$ such that

$$\|\nabla\varphi_{\xi,j}\|_{(p)} \leq C_p \quad \text{for all } \xi \in \Sigma_A \text{ and } j \in \mathbb{Z}. \tag{3-2}$$

Remark 3.1. The construction above can be carried out for $j < 0$ for any $\xi \in \Sigma_A^-$ and any billiard trajectory γ in Ω with reflection points $\dots, P_{-2}(\xi), P_{-1}(\xi), P_0(\xi)$ such that $P_j(\xi) \in K_{\xi_j}$ for all $j \leq 0$. Then one defines a phase function ψ_{ξ^-} with $\psi_{\xi^-}(P_0) = 0$ as above, and using (3-1) one gets a sequence $\{\varphi_{\xi,j}\}_{j\leq 0}$ of phase functions such that for each $j < 0$, $\varphi_{\xi,j}$ is defined and smooth in a neighborhood $U_{\xi,j}$ of the segment $[P_j(\xi), P_{j+1}(\xi)]$ in Ω and satisfies conditions (i)–(iv). Moreover (3-2) holds for any $p \geq 1$ and any $j \leq 0$.

For any $y \in U_{\xi,j}$ denote by $G_{\xi,j}(y)$ the *Gauss curvature* of $C_{\xi,j}(x)$ at y. Now define $g : \Sigma_A \to \mathbb{R}$ by

$$g(\xi) = \frac{1}{N-1} \log \frac{G_{\xi,1}(P_1(\xi))}{G_{\xi,0}(P_0(\xi))}.$$

Clearly, $g(\xi) = \log \Lambda_{\varphi_{\xi,1}}(P_1(\xi))$, where Λ_φ is the function introduced in Section 2.

Given a function $F : \Sigma_A \to \mathbb{C}$ and an integer $n \geq 0$, set

$$\operatorname{var}_n F = \sup\{|F(\xi) - F(\eta)| : \xi_i = \eta_i \quad \text{for } |i| < n\},$$

and for $0 < \theta < 1$ we define $\|F\|_\theta = \sup_n (\operatorname{var}_n F)/\theta^n$, $|||F|||_\theta = \|F\|_\infty + \|F\|_\theta$ and introduce the space $\mathcal{F}_\theta(\Sigma_A) = \{F : |||F|||_\theta < \infty\}$. Clearly $\mathcal{F}_\theta(\Sigma_A)$ is the space of all Lipschitz functions with respect to the metric d_θ on Σ_A defined by $d_\theta(\xi, \xi) = 0$ and $d_\theta(\xi, \eta) = \theta^n$, where $n \geq 0$ is the least integer with $\xi_i = \eta_i$ for $|i| < n$.

[3] See the proof of Proposition A.1(a).

It follows from Proposition 2.4 that $f, g \in \mathscr{F}_\alpha(\Sigma_A)$. By Sinai's Lemma [Parry and Pollicott 1990], there exist $\tilde{f}, \tilde{g} \in \mathscr{F}_{\sqrt{\alpha}}(\Sigma_A)$ depending on future coordinates only and $\chi_1, \chi_2 \in \mathscr{F}_{\sqrt{\alpha}}(\Sigma_A)$ such that

$$f(\xi) = \tilde{f}(\xi) + \chi_1(\xi) - \chi_1(\sigma\xi), \quad g(\xi) = \tilde{g}(\xi) + \chi_2(\xi) - \chi_2(\sigma\xi), \quad \xi \in \Sigma_A.$$

As in the proof of Sinai's Lemma, for any $k = 1, \dots, \kappa_0$ choose and fix an arbitrary sequence

$$\eta^{(k)} = (\dots, \eta^{(k)}_{-m}, \dots, \eta^{(k)}_{-1}, \eta^{(k)}_0) \in \Sigma_A^- \quad \text{with } \eta^{(k)}_0 \neq k.$$

Then for any $\xi \in \Sigma_A$ (or $\xi \in \Sigma_A^+$) set

$$e(\xi) = (\dots, \eta^{(\xi_0)}_{-m}, \dots, \eta^{(\xi_0)}_{-1}, \eta^{(\xi_0)}_0 = \xi_0, \xi_1, \dots, \xi_m, \dots) \in \Sigma_A.$$

Then we have

$$\chi_1(\xi) = \sum_{n=0}^{\infty} \big(f(\sigma^n(\xi)) - f(\sigma^n e(\xi))\big),$$

and the function χ_2 is defined similarly, replacing f by g.

Setting $\chi(\xi, s) = -s\chi_1(\xi) + \chi_2(\xi)$, for the function $R(\xi, s) = -s\, f(\xi) + g(\xi) + \boldsymbol{i}\pi$ we have $R(\xi, s) = \tilde{R}(\xi, s) + \chi(\xi, s) - \chi(\sigma\xi, s)$ for $\xi \in \Sigma_A$, $s \in \mathbb{C}$, where $\tilde{R}(\xi, s) = -s\, \tilde{f}(\xi) + \tilde{g}(\xi) + \boldsymbol{i}\pi$ depends on future coordinates of ξ only (so it can be regarded as a function on $\Sigma_A^+ \times \mathbb{C}$). Below we need the *Ruelle transfer operator* $L_s : C(\Sigma_A^+) \to C(\Sigma_A^+)$ defined by

$$L_s u(\xi) = \sum_{\sigma\eta = \xi} e^{\tilde{R}(\eta, s)}\, u(\eta)$$

for any continuous (complex-valued) function u on Σ_A^+ and any $\xi \in \Sigma_A^+$. Notice that

$$L_s^n u(\xi) = (-1)^n \sum_{\sigma\eta = \xi} e^{-s\tilde{f}(\eta) + \tilde{g}(\eta)} u(\eta) = (-1)^n\, L^n_{-s\tilde{f} + \tilde{g}} u(\xi), \quad n \geq 0,$$

hence $\|L_s^n\|_\infty = \big\| L^n_{-s\tilde{f}+\tilde{g}} \big\|_\infty$. Set $\tilde{L}_s = L_{-s\tilde{f}+\tilde{g}}$.

Define the map $\Phi : \Sigma_A \to \Lambda_{\partial K} = \Lambda \cap S^*_{\partial K}(\Omega)$ by

$$\Phi(\xi) = \left(P_0(\xi), \frac{P_1(\xi) - P_0(\xi)}{\|P_1(\xi) - P_0(\xi)\|} \right).$$

Then Φ is a bijection such that $\Phi \circ \sigma = B \circ \Phi$, where $B : \Lambda_{\partial K} \to \Lambda_{\partial K}$ is the *billiard ball map*. It is well-known — and relatively easy to see — that there exist global constants $0 < \alpha' < \alpha < 1$, $C > 0$ and $c > 0$ (α is actually the constant from Proposition 2.4) such that

$$c\, d_{\alpha'}(\xi, \theta) \leq \operatorname{dist}(\Phi(\xi), \Phi(\eta)) \leq C\, d_\alpha(\xi, \eta), \quad \xi, \eta \in \Sigma_A,$$

where dist is the Euclidean distance in $S^*(\Omega) \subset \mathbb{R}^N \times \mathbb{S}^{N-1}$. Thus, if $h : \Lambda_{\partial K} \to \mathbb{C}$ is Lipschitz, then $h \circ \Phi \in \mathscr{F}_\alpha(\Sigma_A)$, and if $v \in \mathscr{F}_{\alpha'}(\Sigma_A)$, then $v \circ \Phi^{-1}$ is a Lipschitz function on $\Lambda_{\partial K}$.

Let $\pi : \Sigma_A \to \Sigma_A^+$ be the natural projection. For any function $v : \Sigma_A^+ \to \mathbb{C}$ the function $v \circ \pi : \Sigma_A \to \mathbb{C}$ depends on future coordinates only, so $(v \circ \pi) \circ \Phi^{-1} : \Lambda_{\partial K} \to \mathbb{C}$ is constant on local stable manifolds. Conversely, if $h : \Lambda_{\partial K} \to \mathbb{C}$ is constant on local stable manifolds, then $v = h \circ \Phi : \Sigma_A \to \mathbb{C}$ depends on future coordinates only, so it can be regarded as a function on Σ_A^+. For any $(p, u) \in S^*(\Omega)$ sufficiently close to Λ, let $\omega(p,u) \in S^*_{\partial K}(\Omega)$ be the backward shift of (p,u) along the flow to the first point at the boundary. That is, $\omega(p,u) = (q,u) \in S^*_{\partial K}(\Omega)$, where $p = q + tu$ and $(p,u) = \phi_t(q,u)$ for some $t \geq 0$ and $\langle u, \nu(q)\rangle > 0$. Thus, $\omega : V_0 \to S^*_{\partial K}(\Omega)$ is a smooth map defined on an open subset V_0 of $S^*(\Omega)$ containing Λ.

Denote by $C^{\mathrm{Lip}}_u(\Lambda_{\partial K})$ the space of Lipschitz functions $h : \Lambda_{\partial K} \to \mathbb{C}$ such that $h \circ \omega$ is constant on any local stable manifold $W^s_{\mathrm{loc}}(x)$ of the flow ϕ_t contained in the interior of $V_0 \setminus S^*_{\partial K}(\Omega)$. For such h let $\mathrm{Lip}(h)$ denote the Lipschitz constant of h, and for $t \in \mathbb{R}$, $|t| \geq 1$, define

$$\|h\|_{\mathrm{Lip},t} = \|h\|_0 + \frac{\mathrm{Lip}(h)}{|t|}, \quad \|h\|_0 = \sup_{x \in \Lambda_{\partial K}} |h(x)|.$$

To estimate the norm of $\tilde{L}^n_s$, we will apply Dolgopyat type estimates [Dolgopyat 1998] established in the case of open billiard flows in [Stoyanov 2001] for $N = 2$ and in [Stoyanov 2007] for $N \geq 3$ under certain assumptions (see Appendix C). It follows from these results that there exist constants $\sigma_0 < s_0$, $t_0 > 1$ and $0 < \rho < 1$ such that for $s = \tau + \boldsymbol{i}\, t$ with $\tau \geq \sigma_0$, $|t| \geq t_0$ and $n = p[\log|t|] + l$, $p \in \mathbb{N}$, $0 \leq l \leq [\log|t|] - 1$, and for any function $v \in C(\Sigma^+_A)$ of the form $v = h \circ \Phi$ with $h \in C^{\mathrm{Lip}}_u(\Lambda_{\partial K})$, we have

$$\|\tilde{L}^n_s v\|_\infty \leq C\rho^{p[\log|t|]} e^{l\,\mathrm{Pr}(-\tau \tilde{f} + \tilde{g})} \|h\|_{\mathrm{Lip},t}. \tag{3-3}$$

Here $\mathrm{Pr}\,(F)$ denotes the *topological pressure* of F, defined by

$$\mathrm{Pr}(F) = \sup_{\mu \in \mathcal{M}_\sigma} \Big(h_\mu(\sigma) + \int_{\Sigma^+_A} F\, d\mu \Big),$$

where $\mathcal{M}_\sigma$ is the set of probability measures on Σ^+_A invariant with respect to σ and $h_\mu(\sigma)$ is the measure-theoretic entropy of σ with respect to μ.

The abscissa of absolute convergence s_0 introduced in Section 1 is determined by the equality

$$\mathrm{Pr}(-s_0 f + g) = 0.$$

Thus,

$$h_\nu(\sigma) - s_0 \int f\, d\nu + \int g\, d\nu \leq 0 \quad \text{for all } \nu \in \mathcal{M}_\sigma\,.$$

Let ν_g be the equilibrium state of g such that $\mathrm{Pr}(g) = h_{\nu_g}(\sigma) + \int g\, d\nu_g$. Then $\mathrm{Pr}(g) \leq s_0 \int f\, d\nu_g$. Next, let $\nu_0 \in \mathcal{M}_\sigma$ be the equilibrium state of $-s_0 f + g$ with

$$h_{\nu_0}(\sigma) - s_0 \int f\, d\nu_0 + \int g\, d\nu_0 = 0\,.$$

This yields $s_0 \int f\, d\nu_0 = h_{\nu_0}(\sigma) + \int g\, d\nu_0 \leq \mathrm{Pr}(g)$. Consequently,

$$\frac{\mathrm{Pr}(g)}{\int f\, d\nu_g} \leq s_0 \leq \frac{\mathrm{Pr}(g)}{\int f\, d\nu_0},$$

and we deduce that $s_0 < 0$ if only if $\mathrm{Pr}(g) < 0$.

We will deal with oscillatory data on Γ_1 (which can be replaced by any Γ_j) of the form

$$u_1(x,s) = e^{-s\,\varphi(x)}\, h(x), \quad x \in \Gamma_1,\ s \in \mathbb{C},\ \sigma_0 \leq \mathrm{Re}\, s \leq 1.$$

Here φ is a C^∞ phase function defined on some open subset $\mathcal{U} = \mathcal{U}(\varphi)$ and satisfying condition ($\mathcal{P}$) on Γ_1 (see Section 2) and h is a $C^\infty(\Gamma)$ function with small support on Γ_1. In fact, using a C^∞ extension, we may assume that h is a C^∞ function on $\mathbb{R}^N$, so in particular h is C^∞ on $\mathcal{U}$, as well. For every configuration $\boldsymbol{j} = (j_0, j_1, \dots, j_m)$, $j_0 = 1$, $|\boldsymbol{j}| = m + 1$, we can construct a function $u_{\boldsymbol{j}}(x, s)$ following a recurrent procedure [Ikawa 1994]. We construct a sequence of phase functions $\varphi_{\boldsymbol{j}}(x)$ and amplitudes $a_{\boldsymbol{j}}(x)$ and define

$$u_{\boldsymbol{j}}(x, s) = (-1)^{|\boldsymbol{j}|-1} e^{-s\varphi_{\boldsymbol{j}}(x)} a_{\boldsymbol{j}}(x).$$

For the configurations $\boldsymbol{j}$ and $\boldsymbol{j}' = (j_0, j_1, \dots, j_m, j_{m+1})$, we have

$$\begin{aligned} u_{j_0}(x, s) &= u_1(x, s) && \text{on } \Gamma_1, \\ u_{\boldsymbol{j}}(x, s) + u_{\boldsymbol{j}'}(x, s) &= 0 && \text{on } \Gamma_{j_{m+1}}. \end{aligned}$$

The phase functions $\varphi_{\boldsymbol{j}}$ and their domains $\mathcal{U}_{\boldsymbol{j}}(\varphi)$ are determined following the procedure in Section 2. In particular, each $\varphi_{\boldsymbol{j}}$ satisfies condition ($\mathcal{P}$) on Γ_{j_m}, so it follows from item (ii) of that condition that $\Gamma_i \subset \mathcal{U}_{\boldsymbol{j}}(\varphi)$ for every $i = 1, \dots, \kappa_0$, $i \neq j_m$. The amplitudes $a_{\boldsymbol{j}}(x)$ are determined on $\mathcal{U}_{\boldsymbol{j}}(\varphi)$ as the solutions of the transport equations

$$2\langle \nabla\varphi_{\boldsymbol{j}}, \nabla a_{\boldsymbol{j}} \rangle + (\Delta\varphi_{\boldsymbol{j}}) a_{\boldsymbol{j}} = 0.$$

More precisely, using the notation of Section 2 (see also [Ikawa 1988, Section 4] and [Ikawa 1994, Section 4.1]), we will assume that $a_{\boldsymbol{j}}(x)$ has the form

$$a_{\boldsymbol{j}}(x) = (A_{\boldsymbol{j}}(\varphi)h)(x), \quad x \in \mathcal{U}_{\boldsymbol{j}}(\varphi). \tag{3-4}$$

Next, let $\mu = (\mu_0 = 1, \mu_1, \dots) \in \Sigma_A^+$. It follows from [Ikawa 1988] that there exists a unique point $y(\mu) \in \Gamma_1$ such that the ray $\gamma(y, \varphi)$ issued from a point $y(\mu)$ in direction $\nabla\varphi(y(\mu))$ follows the configuration μ. Let $Q_0(\mu) = y(\mu)$, $Q_1(\mu), \dots,$ be the consecutive reflection points of this ray. Define

$$f_i^+(\mu) = \| Q_i(\mu) - Q_{i+1}(\mu) \|, \quad g_i^+(\mu) = \frac{1}{N-1} \log \frac{G^{\varphi}_{\mu,i}(Q_{i+1}(\mu))}{G^{\varphi}_{\mu,i}(Q_i(\mu))} < 0,$$

where $G^{\varphi}_{\mu,i}(y)$ denotes the Gaussian curvature of the surface

$$C^{\varphi}_{\mu,i}(x) = \left\{ z \in \mathcal{U}_{(\mu_0,\mu_1,\dots,\mu_i)}(\varphi) : \varphi_{(\mu_0,\mu_1,\dots,\mu_i)}(z) = \varphi_{(\mu_0,\mu_1,\dots,\mu_i)}(x) \right\}$$

at y. As for $g(\xi)$, the function $g_i^+(\mu)$ can be expressed by means of the function Λ_φ introduced in Section 2, namely $g_i^+(\mu) = \log \Lambda_{\varphi_{(\mu_0,\mu_1,\dots,\mu_i)}}(Q_{i+1}(\mu))$.

Using the points $Q_j(\mu)$ constructed above, define $\tilde{v} \in \mathcal{F}_\theta(\Sigma_A^+)$ by

$$\tilde{v}_s(\xi) = e^{-s\,\varphi(Q_0(\xi))}\, h(Q_0(\xi))$$

if $\xi_0 = 1$ and $\tilde{v}_s(\xi) = 0$ otherwise. Here the function h comes from the boundary data $u_1(x, s)$.

Next, for $s \in \mathbb{C}$ and $\xi \in \Sigma_A^+$ with $\xi_0 = 1$, following [Ikawa 1994], set

$$\phi^+(\xi, s) = \sum_{n=0}^{\infty} \left(-s\,[f(\sigma^n e(\xi)) - f_n^+(\xi)] + [g(\sigma^n e(\xi)) - g_n^+(\xi)] \right). \tag{3-5}$$

Formally, define $\phi^+(\xi, s) = 0$ when $\xi_0 \neq 1$, thus obtaining a function $\phi^+ : \Sigma_A^+ \times \mathbb{C} \to \mathbb{C}$.

Now for any $s \in \mathbb{C}$ define the operator $\mathcal{G}_s : C(\Sigma_A^+) \to C(\Sigma_A^+)$ by

$$(\mathcal{G}_s v)(\xi) = \sum_{\sigma\eta=\xi} e^{-\phi^+(\eta,s)-s\tilde{f}(\eta)+\tilde{g}(\eta)}\, v(\eta), \quad v \in C(\Sigma_A^+),\ \xi \in \Sigma_A^+.$$

(Although similar, this is different from the corresponding definition in [Ikawa 1994].)

Fix an arbitrary $l = 1, \dots, \kappa_0$ *and an arbitrary point* $x_0 \in \Gamma_l$. Define the function $\phi^-(x_0; \cdot, \cdot) : \Sigma_A \times \mathbb{C} \to \mathbb{C}$ (depending on l as well) as follows. First, set $\phi^-(x_0; \eta, s) = 0$ if $\eta_0 \neq l$. Next, assume that $\eta \in \Sigma_A$ satisfies $\eta_0 = l$. There exists a unique billiard trajectory in Ω with successive reflection points $\tilde{P}_i(x_0; \eta) \in \partial K_{\eta_i}$ $(-\infty < i \leq 0)$ such that $x_0 = \tilde{P}_{-1}(x_0; \eta) + t\nabla\psi_{\eta^-}(\tilde{P}_{-1}(x_0; \eta))$ for some $t > 0$. (See the beginning of this section and Appendix A for the definition of ψ_{η^-}.) Notice that in general the segment $[\tilde{P}_{-1}(x_0; \eta), x_0]$ may intersect the interior of K_l. Denote $\tilde{P}_0(x_0; \eta) = x_0$, and for any $i < 0$ set

$$f_i^-(x_0; \eta) = \|\tilde{P}_{i+1}(x_0; \eta) - \tilde{P}_i(x_0; \eta)\|, \quad g_i^-(x_0; \eta) = \frac{1}{N-1} \log \frac{G_{\eta,i}(\tilde{P}_{i+1}(x_0; \eta))}{G_{\eta,i}(\tilde{P}_i(x_0; \eta))}.$$

Then define

$$\phi^-(x_0; \eta, s) = -s \sum_{i=-1}^{-\infty} \big(f(\sigma^i(\eta)) - f_i^-(x_0; \eta)\big) + \sum_{i=-1}^{-\infty} \big(g(\sigma^i(\eta)) - g_i^-(x_0; \eta)\big).$$

We will show later that this series is absolutely convergent.

Next, define the operator $\mathcal{M}_{n,s}(x_0) : C(\Sigma_A^+) \to C(\Sigma_A^+)$ (depending also on l) by

$$(\mathcal{M}_{n,s}(x_0)v)\,(\xi) = \sum_{\sigma\eta=\xi} e^{-\phi^-(x_0;\sigma^{n+1}e(\eta),s)-\chi(\sigma^{n+1}e(\eta),s)-s\tilde{f}(\eta)+\tilde{g}(\eta)}\, v(\eta)$$

for any $v \in C(\Sigma_A^+)$, any $x_0 \in \Gamma$ and any $\xi \in \Sigma_A^+$.

Let $s_0 \in \mathbb{R}$ be the abscissa of absolute convergence of the dynamical zeta function (pages 427–428) determined by $\Pr(-s_0\tilde{f} + \tilde{g}) = 0$.

The first part of the following theorem is similar to (4-10) in [Ikawa 1994]:

Theorem 3.2. *There exist global constants* $c > 0$, $a > 0$, $\theta \in (0, 1)$ *and* $C_p > 0$ *for every integer* $p \geq 0$ *such that for any choice of* $l = 1, \dots, \kappa_0$ *and* $x_0 \in \Gamma_l$ *the following hold*:

(a) *For all integers* $n \geq 1$, *all* $\xi \in \Sigma_A^+$ *with* $\xi_0 = l$ *and all* $s \in \mathbb{C}$ *with* $\mathrm{Re}\, s \geq s_0 - a$ *we have*

$$\Big|(L_s^n \mathcal{M}_{n,s}(x_0)\mathcal{G}_s\tilde{v}_s)(\xi) - \sum_{\substack{|j|=n+3\\ j_{n+2}=l}} u_j(x_0, s)\Big| \\ \leq C_0\,(\theta + c\,a)^n\, e^{C_0[\mathrm{Re}(s)\,(1+\|\varphi\|_{\Gamma,0})+\|\nabla\varphi\|_{\Gamma,(1)}]}\big((|s| + \|\nabla\varphi\|_{\Gamma,(1)})\,\|h\|_{\Gamma,0} + \|h\|_{\Gamma,(1)}\big). \quad (3\text{-}6)$$

(b) *For all* $n \geq 1$, *all* $\xi \in \Sigma_A^+$ *with* $\xi_0 = l$ *and all* $s \in \mathbb{C}$ *with* $\mathrm{Re}\, s \geq s_0 - a$ *we have*

$$\Big\|\big(L_s^n \mathcal{M}_{n,s}(\cdot)\mathcal{G}_s\tilde{v}_s\big)\,(\xi) - \sum_{\substack{|j|=n+3\\ j_{n+2}=l}} u_j(\cdot, s)\Big\|_{\Gamma,p} \\ \leq C_p\,(\theta + c\,a)^n\, e^{C_p[|\mathrm{Re}\, s|\,(1+\|\varphi\|_{\Gamma,0})+\|\nabla\varphi\|_{\Gamma,(1)}]} \sum_{i=0}^{p} (|s|\|\nabla\varphi\|_{\Gamma,i} + \|\nabla\varphi\|_{\Gamma,i+1})^{i+1}\|h\|_{\Gamma,p-i}\,. \quad (3\text{-}7)$$

In this section we deal with part (a). The proof of part (b) is given in Section 4 below.

Proof of Theorem 3.2(a). Fix l, $x_0 \in \Gamma_l$ and $\xi \in \Sigma_A^+$ with $\xi_0 = l$. Then for any $s \in \mathbb{C}$ and $n \geq 1$, using [Ikawa 1994, Section 4.1], setting $\boldsymbol{j} = (1, j_1, j_2, \dots, j_{n+1}, l)$, we get

$$u_{(1,j_1,j_2,\dots,j_{n+1},l)}(x_0, s) = (-1)^{n+2}\, e^{-s\,[\varphi(Q_0(\boldsymbol{j}))+f_0^+(x_0;\boldsymbol{j})+\cdots+f_{n+1}^+(x_0;\boldsymbol{j})]}\, a_{\boldsymbol{j}}(x_0), \tag{3-8}$$

where $f_i^+(x_0; \boldsymbol{j}) = \| Q_i(x_0; \boldsymbol{j}) - Q_{i+1}(x_0; \boldsymbol{j}) \|$ $(i = 0, 1, \dots, n+1)$, $Q_i(x_0; \boldsymbol{j})$ being the reflection points of the billiard trajectory issued from a point $y \in \Gamma_1$ in direction $\nabla\varphi(y)$ which follows the configuration $\boldsymbol{j}$ for its first $n+1$ reflections and is such that $Q_{n+2}(x_0; \boldsymbol{j}) = x_0$. Notice that the segment $[Q_{n+1}(x_0; \boldsymbol{j}), x_0]$ may intersect the interior of K_l.[4] Then there is exactly one such trajectory. Given a function

$$F(\xi) : \Sigma_A^+ \to \mathbb{C},$$

introduce the notation

$$F_n(\xi) = F(\xi) + F(\sigma(\xi)) + \cdots + F(\sigma^{n-1}(\xi)).$$

We have

$$\begin{aligned}
(L_s^n \mathcal{M}_{n,s}(x_0)\mathcal{G}_s\tilde{v}_s)(\xi) &= (-1)^n \sum_{\sigma^n\eta=\xi} e^{-s\tilde{f}_n(\eta)+\tilde{g}_n(\eta)}\, (\mathcal{M}_{n,s}(x_0)\mathcal{G}_s\tilde{v}_s)(\eta)\\
&= (-1)^n \sum_{\sigma^n\eta=\xi} e^{-s\tilde{f}_n(\eta)+\tilde{g}_n(\eta)} \sum_{\sigma\zeta=\eta} e^{-\phi^-(x_0;\sigma^{n+1}e(\zeta),s)-\chi(\sigma^{n+1}e(\zeta),s)-s\tilde{f}(\zeta)+\tilde{g}(\zeta)}\\
&\qquad\qquad \times \sum_{\sigma\mu=\zeta} e^{-\phi^+(\mu,s)+\chi(e(\mu),s)-s\tilde{f}(\mu)+\tilde{g}(\mu)}\, \tilde{v}_s(\mu)\\
&= (-1)^n \sum_{\substack{\sigma^{n+2}\mu=\xi\\ \mu_0=1}} e^{-s\tilde{f}_{n+2}(\mu)+\tilde{g}_{n+2}(\mu)}\, W^{(n+2)}(x_0; \mu, s),
\end{aligned}\tag{3-9}$$

where the function

$$W^{(n+2)}(x_0; \cdot, \cdot) = W_{1,l}^{(n+2)}(x_0; \cdot, \cdot) : \Sigma_A^+ \times \mathbb{C} \to \mathbb{C}$$

is defined by $W^{(n+2)}(x_0; \mu, s) = 0$ when $\mu_0 \neq 1$ or $\mu_{n+2} \neq l$ and otherwise (i.e., when $\mu_0 = 1$ and $\mu_{n+2} = l$) by

$$W^{(n+2)}(x_0; \mu, s) = e^{z(x_0;\mu,s)}\, e^{-s\,\varphi(Q_0(\mu))}\, h(Q_0(\mu)), \tag{3-10}$$

where we have set

$$z(x_0; \mu, s) = -\phi^-(x_0; \sigma^{n+1}e(\sigma\mu), s) - \chi(\sigma^{n+1}e(\sigma\mu), s) - \phi^+(\mu, s) + \chi(e(\mu), s). \tag{3-11}$$

Clearly, in (3-9) the summation is over sequences

$$\mu = (1, j_1, j_2, \dots, j_{n+1}, l, \xi_1, \xi_2, \dots) = (\boldsymbol{j}, \xi), \tag{3-12}$$

with $\mu_{n+2} = l$, where $\boldsymbol{j} = (1, j_1, j_2, \dots, j_{n+1}, l)$. It follows from (3-9) that

$$[L_s^n \mathcal{M}_{n,s}(x_0)\mathcal{G}_s\tilde{v}_s](\xi) = (-1)^n \big[L_{-s\tilde{f}+\tilde{g}}^{n+2}(W^{(n+2)}(x_0; \cdot, s))\big](\xi). \tag{3-13}$$

[4]In fact one can define the functions $f_i^+(x_0; \boldsymbol{j})$ $(i = 0, 1, \dots, n+1)$ and therefore $u_{\boldsymbol{j}}(x_0, s)$ for any $x_0 \in \mathcal{U}_{\boldsymbol{j}}(\varphi)$ in a similar way. Just consider the (unique) billiard trajectory issued from a point $y = Q_0(x_0; \boldsymbol{j}) \in \Gamma_1$ in direction $\nabla\varphi(y)$ following the configuration $\boldsymbol{j}$ for its first $n+1$ reflections and such that if v is the reflected direction of the trajectory at $Q_{n+1}(x_0; \boldsymbol{j})$, then $x_0 = Q_{n+1}(x_0, \boldsymbol{j}) + t\,v$ for some $t \geq 0$.

It follows from Propositions 2.4 and 2.6 that there exist global constants $C > 0$ and $\alpha \in (0, 1)$ such that

$$|f(\sigma^n e(\xi)) - f_n^+(\xi)| \le C\,\alpha^n, \quad |g(\sigma^n e(\xi)) - g_n^+(\xi)| \le C\,\|\nabla\varphi\|_{\Gamma,(1)}\,\alpha^n, \tag{3-14}$$

for all $\xi \in \Sigma_A$ and all integers $n \ge 1$, so by (3-5),

$$\phi^+(\mu, s) = (|s| + \|\nabla\varphi\|_{\Gamma,(1)})\,O(\alpha^n) + \sum_{i=0}^{n+1}\big(-s\,[f(\sigma^i e(\mu)) - f_i^+(\mu)] + [g(\sigma^i e(\mu)) - g_i^+(\mu)]\big).$$

Thus, using the definitions of $\tilde f$, $\tilde g$ and χ and the fact that $\chi(\sigma^{n+2}e(\mu), s) = \chi(\sigma^{n+1}e(\sigma\mu), s) + |s|\,O(\alpha^n)$, we get

$$\begin{aligned}
-s[f_0^+(\mu) + f_1^+(\mu) + \cdots + f_{n+1}^+(\mu)] + [g_0^+(\mu) + g_1^+(\mu) + \cdots + g_{n+1}^+(\mu)]&\\
= (s + \|\nabla\varphi\|_{\Gamma,(1)})\,O(\alpha^n) - \phi^+(\mu, s) - s[f(e(\mu)) + f(\sigma e(\mu)) + \cdots + f(\sigma^{n+1}e(\mu)]&\\
+[g(e(\mu)) + g(\sigma e(\mu)) + \cdots + g(\sigma^{n+1}e(\mu)]&\\
= (|s| + \|\nabla\varphi\|_{\Gamma,(1)})\,O(\alpha^n) - \phi^+(\mu, s) - s\tilde f_{n+2}(\mu) + \tilde g_{n+2}(\mu) + \chi(e(\mu), s) - \chi(\sigma^{n+1}\,e(\sigma\mu), s).&
\end{aligned}$$

Now, *fix for a moment $n \ge 1$ and μ as in* (3-12), and set $\eta = \sigma^{n+1}e(\sigma(\mu))$. Then we have

$$\eta = \sigma^{n+1}e(\sigma(\mu)) = (\ldots, *, *, \mu_1, \mu_2, \ldots, \mu_{n+1}; \mu_{n+2} = l, \mu_{n+3}, \ldots), \tag{3-15}$$

and as for ϕ^+ one gets

$$\phi^-(x_0; \eta, s) = (|s| + \|\nabla\varphi\|_{\Gamma,(1)})\,O(\alpha^n) - s\sum_{i=-1}^{-n-1}[f(\sigma^i\eta) - f_i^-(x_0; \eta)] + \sum_{i=-1}^{-n-1}[g(\sigma^i\eta) - g_i^-(x_0; \eta)].$$

From these estimates and (3-11) one derives

$$\begin{aligned}
z(x_0; \mu, s) &= s\tilde f_{n+2}(\mu) - \tilde g_{n+2}(\mu) - \phi^-(x_0; \eta, s) - s\sum_{i=0}^{n+1} f_i^+(\mu) + \sum_{i=0}^{n+1} g_i^+(\mu) + (|s| + \|\nabla\varphi\|_{\Gamma,(1)})\,O(\alpha^n)\\
&= s\tilde f_{n+2}(\mu) - \tilde g_{n+2}(\mu) - sc(x_0; \mu) + d(x_0; \mu) + (|s| + \|\nabla\varphi\|_{\Gamma,(1)})\,O(\alpha^n),
\end{aligned} \tag{3-16}$$

where

$$c(x_0; \mu) = -\sum_{i=0}^{n+1}[f(\sigma^i\eta) - f_i^-(x_0; \eta)] + \sum_{i=0}^{n+1} f_i^+(\mu), \quad d(x_0; \mu) = -\sum_{i=-1}^{-n-1}[g(\sigma^i\eta) - g_i^-(x_0; \eta)] + \sum_{i=0}^{n+1} g_i^+(\mu).$$

We will show that

$$\Big|c(x_0; \mu) - \sum_{i=0}^{n+1} f_i^+(x_0; \boldsymbol{j})\Big| \le C\,\alpha^n \tag{3-17}$$

and

$$\big|e^{d(x_0;\mu)}\,h(Q_0(\mu)) - (A_{\boldsymbol{j}}(\varphi)h)(x_0)\big| \le C\,(\|\nabla\varphi\|_{\Gamma,(1)}\,\|h\|_{\Gamma,0} + \|h\|_{\Gamma,(1)})\,\theta^n, \tag{3-18}$$

for some global constant $C > 0$, where

$$\theta = \sqrt{\alpha} \in (0, 1).$$

There exists a unique ray $\gamma(y, \varphi)$ issued from a point $y = y_n(x_0; \mu) \in \Gamma_1$ in direction $\nabla\varphi(y)$, following the configuration μ for its first $n+1$ reflections and such that if $\tilde{Q}_i(x_0; \mu)$ $(1 \leq i \leq n+1)$ are its first $n+1$ reflection points and v is the reflected direction of the trajectory at $Q_{n+1}(x_0; \boldsymbol{j})$, then

$$x_0 = Q_{n+1}(x_0, \boldsymbol{j}) + t\, v$$

for some $t \geq 0$. Set $\tilde{Q}_{n+2}(x_0; \mu) = x_0$. Notice that as before the segment $[\tilde{Q}_{n+1}(x_0; \mu), x_0]$ may intersect the interior of K_l (or be tangent to Γ_l at x_0).

Before we continue, let us make a few simple (but essential) remarks concerning the sequences of points

$$Q_0(\mu) \in \Gamma_1 = \Gamma_{\mu_0},\ Q_1(\mu) \in \Gamma_{\mu_1}, \dots,\ Q_{n+1}(\mu) \in \Gamma_{\mu_{n+1}},\ Q_{n+2}(\mu) \in \Gamma_{\mu_{n+2}} = \Gamma_l, \dots, \tag{3-19}$$

$$\tilde{Q}_0(x_0; \mu) \in \Gamma_1 = \Gamma_{\mu_0},\ \tilde{Q}_1(x_0; \mu) \in \Gamma_{\mu_1}, \dots,\ \tilde{Q}_{n+1}(x_0; \mu) \in \Gamma_{\mu_{n+1}},\ \tilde{Q}_{n+2}(x_0; \mu) \in \Gamma_l, \tag{3-20}$$

$$\dots,\ P_{\eta_{-n-1}}(\eta) \in \Gamma_{\eta_{-n-1}} = \Gamma_{\mu_1}, \dots,\ P_{-1}(\eta) \in \Gamma_{\eta_{-1}} = \Gamma_{\mu_{n+1}},\ P_0(\eta) \in \Gamma_{\eta_0} = \Gamma_{\mu_{n+2}} = \Gamma_l, \dots, \tag{3-21}$$

$$\dots,\ \tilde{P}_{\eta_{-n-1}}(x_0; \eta) \in \Gamma_{\eta_{-n-1}} = \Gamma_{\mu_1}, \dots,\ \tilde{P}_{-1}(x_0; \mu) \in \Gamma_{\eta_{-1}} = \Gamma_{\mu_{n+1}},\ \tilde{P}_0(x_0; \eta) \in \Gamma_{\eta_0} = \Gamma_{\mu_{n+2}} = \Gamma_l. \tag{3-22}$$

It is clear that the sequences (3-19) and (3-20) "start" from the same convex level surface $\varphi = c$, therefore by Proposition 2.4 there exist constants $C > 0$ and $\alpha \in (0, 1)$ such that

$$\| Q_i(\mu) - \tilde{Q}_i(x_0; \mu) \| \leq C\, \alpha^{n+2-i}, \quad 0 \leq i \leq n+2. \tag{3-23}$$

(Notice that $\tilde{Q}_{n+2}(x_0; \mu) = x_0 \in \Gamma_l$, so $\| Q_{n+2}(\mu) - \tilde{Q}_{n+2}(x_0; \mu) \| \leq \mathrm{diam}(K) \leq C$.) Similarly, the right ends of sequences (3-21) and (3-22) determine points on the same unstable manifold of the billiard flow ϕ_t, so by Proposition 2.4 these sequences "converge backwards", that is,

$$\| P_i(\eta) - \tilde{P}_i(x_0; \eta) \| \leq C\, \alpha^{|i|}, \quad i \leq 0. \tag{3-24}$$

On the other hand, the sequences (3-19) and (3-21) continue indefinitely to the right following the same patterns. Thus, these sequences converge forwards; more precisely, using Proposition 2.4 again, we have

$$\| Q_i(\mu) - P_{i-n-2}(\eta) \| \leq C\, \alpha^i, \quad 1 \leq i. \tag{3-25}$$

Similarly, the sequences (3-20) and (3-22) converge forwards to $\tilde{Q}_{n+2}(x_0; \mu) = \tilde{P}_0(x_0; \eta) = x_0$:

$$\| \tilde{Q}_i(x_0; \mu) - \tilde{P}_{i-n-2}(x_0; \eta) \| \leq C\, \alpha^i, \quad 1 \leq i \leq n+2. \tag{3-26}$$

It now follows from (3-2) and (3-24) that

$$|g(\sigma^i(\eta)) - g_i^-(x_0; \eta)| = \left| \frac{1}{N-1} \log \frac{G_{\eta,i}(P_{i+1}(\eta))}{G_{\eta,i}(P_i(\eta))} - \frac{1}{N-1} \log \frac{G_{\eta,i}(\tilde{P}_{i+1}(x_0; \eta))}{G_{\eta,i}(\tilde{P}_i(x_0; \eta))} \right| \leq C\, \alpha^{|i|} \tag{3-27}$$

for all $i \leq 0$. In particular, the second series in (3-5) is absolutely convergent, and by (3-27) and Proposition 2.7, $|d(x_0; \mu)| \leq C$ for some global constant $C > 0$.

Next, setting

$$\tilde{a}_i(x_0; \mu) = \frac{1}{N-1} \log \frac{G^{\varphi}_{\mu,i}(\tilde{Q}_{i+1}(x_0; \mu))}{G^{\varphi}_{\mu,i}(\tilde{Q}_i(x_0; \mu))} \tag{3-28}$$

and using (3-23) and Proposition 2.6, one gets

$$
\begin{aligned}
|\tilde{a}_i(x_0;\mu) - g_i^+(\mu)| &= \frac{1}{N-1}\left|\log\frac{G^{\varphi}_{\mu,i}(\tilde{Q}_{i+1}(x_0;\mu))}{G^{\varphi}_{\mu,i}(\tilde{Q}_i(x_0;\mu))} - \log\frac{G^{\varphi}_{\mu,i}(Q_{i+1}(\mu))}{G^{\varphi}_{\mu,i}(Q_i(\mu))}\right| \\
&\le C\,\|\nabla\varphi\|_{\Gamma,(1)}\big(\|\tilde{Q}_i(x_0;\mu) - Q_i(\mu)\| + \|\tilde{Q}_{i+1}(x_0;\mu) - Q_{i+1}(\mu)\|\big) \\
&\le C\,\|\nabla\varphi\|_{\Gamma,(1)}\,\alpha^{n+2-i}, \qquad (3\text{-}29)
\end{aligned}
$$

for all $i = 0, 1, \dots, n+2$.

Next, notice that by construction $\varphi_{\eta,i} = (\varphi_{\eta,-n-2})_{(\mu_1,\dots,\mu_{n+2+i})} + c$ for $-n-1 \le i \le -1$. Thus, by (2-2), (3-2) and (3-25), for all $-n-1 \le i \le -1$ we have

$$
\begin{aligned}
|g^+_{n+2+i}(\mu) - g(\sigma^i\eta)| &= \frac{1}{N-1}\left|\log\frac{G^{\varphi}_{\mu,n+2+i}(Q_{n+2+i+1}(\mu))}{G^{\varphi}_{\mu,n+2+i}(Q_{n+2+i}(\mu))} - \log\frac{G_{\eta,i}(P_{i+1}(\eta))}{G_{\eta,i}(P_i(\eta))}\right| \\
&\le C\,\big(\|\nabla\varphi_{(\mu_1,\dots,\mu_{n+2+i})} - \nabla(\varphi_{\eta,-n-2})_{(\mu_1,\dots,\mu_{n+2+i})}\|_{\Gamma,(1)} \\
&\qquad + \|Q_{n+2+i+1}(\mu) - P_{i+1}(\eta)\| + \|Q_{n+2+i}(\mu) - P_i(\eta)\|\big) \\
&\le C\,\|\nabla\varphi - \nabla(\varphi_{\eta,-n-2})\|_{\Gamma,(1)}\,\alpha^{n+2+i} + C\,\alpha^{n+2+i} \le C\,\|\nabla\varphi\|_{\Gamma,(1)}\,\alpha^{n+2+i}. \qquad (3\text{-}30)
\end{aligned}
$$

In a similar way (3-26) implies

$$
|\tilde{a}_{n+2+i}(x_0;\mu) - g_i^-(x_0;\eta)| \le C\,\|\nabla\varphi\|_{\Gamma,(1)}\,\alpha^{n+2+i}, \quad -n-1 \le i \le -1. \qquad (3\text{-}31)
$$

To prove (3-18), notice that $(A_{\boldsymbol{j}}(\varphi)h)(x_0) = \Lambda_{\varphi,\boldsymbol{j}}(x_0)\,h(\tilde{Q}_0(x_0;\mu))$. The definition of $\Lambda_{\varphi,\boldsymbol{j}}$ and $\tilde{Q}_{n+2}(x_0;\mu) = x_0$ gives

$$
\log\Lambda_{\varphi,\boldsymbol{j}}(x_0) = \log\Lambda_{\varphi,\boldsymbol{j}}(\tilde{Q}_{n+2}(x_0;\mu)) = \sum_{i=0}^{n+1}\tilde{a}_i(x_0;\mu). \qquad (3\text{-}32)
$$

Next, assume for simplicity that n is odd (the other case is similar), and set $m = (n+1)/2$. Using (3-27)–(3-31), we get

$$
\begin{aligned}
\log\Lambda_{\varphi,\boldsymbol{j}}(x_0) - d(x_0;\mu) &= \sum_{i=0}^{n+1}\tilde{a}_i(x_0;\mu) + \sum_{i=-1}^{-n-1}[g(\sigma^i\eta) - g_i^-(x_0;\eta)] - \sum_{i=0}^{n+1} g_i^+(\mu) \\
&= \sum_{i=-m-1}^{-n-1}[g(\sigma^i\eta) - g_i^-(x_0;\eta)] + \sum_{i=0}^{m}\big[\tilde{a}_i(x_0;\mu) - g_i^+(\mu)\big] \\
&\qquad + \sum_{i=m+1}^{n+1}\big[\tilde{a}_i(x_0;\mu) - g^-_{i-n-2}(x_0;\eta)\big] + \sum_{i=-1}^{-m}[g(\sigma^i\eta) - g^+_{n+2+i}(\mu)] \\
&= O(\alpha^m)\,\|\nabla\varphi\|_{\Gamma,(1)} = O(\theta^n)\,\|\nabla\varphi\|_{\Gamma,(1)}. \qquad (3\text{-}33)
\end{aligned}
$$

Since, by (3-23),

$$
|h(\tilde{Q}_0(x_0;\mu)) - h(Q_0(\mu))| = \|h\|_{\Gamma,1}\,O(\alpha^n), \qquad (3\text{-}34)
$$

this gives

$$
\begin{aligned}
&|e^{d(x_0;\mu)}\,h(Q_0(\mu)) - (A_{\boldsymbol{j}}(\varphi)h)(x_0)| \\
&\qquad \le |e^{d(x_0;\mu)} - e^{\log \Lambda_{\varphi,\boldsymbol{j}}(x_0)}|\,\|h(Q_0(\mu))\| + \Lambda_{\varphi,\boldsymbol{j}}(x_0)\,\|h(Q_0(\mu)) - h(\tilde{Q}_0(x_0;\mu)\| \\
&\qquad \le e^{\max\{d(x_0;\mu),\log \Lambda_{\varphi,j}(x_0)\}}\,|d(x_0;\mu) - \log \Lambda_{\varphi,\boldsymbol{j}}(x_0)|\,\|h\|_{\Gamma,0} + \|h\|_{\Gamma,(1)}\,O(\alpha^n) \\
&\qquad \le C\,(\|\nabla\varphi\|_{\Gamma,(1)}\,\|h\|_{\Gamma,0} + \|h\|_{\Gamma,(1)})\,\theta^n,
\end{aligned}
$$

which proves (3-18).

Similarly to (3-27) one gets $|f(\sigma^i(\eta)) - f_i^-(x_0;\eta)| \le C\,\alpha^{|i|}$, and also

$$
|f_i^+(\mu) - f_i^+(x_0;\boldsymbol{j})| = |\|Q_i(\mu) - Q_{i+1}(\mu)\| - \|Q_i(x_0;\boldsymbol{j}) - Q_{i+1}(x_0;\boldsymbol{j})\|| \le C\,\alpha^{n+2-i}.
$$

Combining these two estimates yields (3-17).

Next, using the notation from the beginning of this proof, notice that for any μ as in (3-12) we have $Q_i(x_0;\boldsymbol{j}) = \tilde{Q}_i(x_0;\mu)$ for all $i = 0, 1\ldots, n+2$, and therefore $f_i^+(x_0;\boldsymbol{j}) = \|\tilde{Q}_i(x_0;\mu) - \tilde{Q}_{i+1}(x_0;\mu)\|$ for all $i = 0, 1, \ldots, n+1$. (This has been used already in the proof of (3-17).)

Define the function

$$
\tilde{W}^{(n+2)}(x_0;\cdot,\cdot) = \tilde{W}^{(n+2)}_{1,l}(x_0;\cdot,\cdot) : \Sigma_A^+ \times \mathbb{C} \to \mathbb{C}
$$

by $\tilde{W}^{(n+2)}(x_0;\mu,s) = 0$ when $\mu_0 \ne 1$ or $\mu_{n+2} \ne l$ and

$$
\begin{aligned}
\tilde{W}^{(n+2)}(x_0;\mu,s) = e^{s\tilde{f}_{n+2}(\mu) - \tilde{g}_{n+2}(\mu) - s\,\varphi(\tilde{Q}_0(x_0;\mu)) - s\,\sum_{i=0}^{n+1}\|\tilde{Q}_i(x_0;\mu) - \tilde{Q}_{i+1}(x_0;\mu)\|}& \\
\times\,\Lambda_{\varphi,\boldsymbol{j}}(x_0)\,h(\tilde{Q}_0(x_0;\mu)),& \quad (3\text{-}35)
\end{aligned}
$$

whenever $\mu_0 = 1$ and $\mu_{n+2} = l$, where $\boldsymbol{j} = \boldsymbol{j}^{(n+2)}(\mu)$ is defined by (3-12).

Using (3-8), we can now write

$$
\begin{aligned}
&\sum_{\substack{|\boldsymbol{j}|=n+3\\ j_0=1\\ j_{n+2}=l}} u_{\boldsymbol{j}}(x_0, -\boldsymbol{i}\,s) \\
&\qquad = (-1)^n \sum_{\substack{\sigma^{n+2}\mu=\xi\\ \mu_0=1}} e^{-s\,\varphi(\tilde{Q}_0(x_0;\mu)) - s\sum_{i=0}^{n+1}\|\tilde{Q}_i(x_0;\mu) - \tilde{Q}_{i+1}(x_0;\mu)\|}\,\Lambda_{\varphi,\boldsymbol{j}}(x_0)\,h(\tilde{Q}_0(x_0;\mu)) \\
&\qquad = (-1)^n \sum_{\sigma^{n+2}\mu=\xi} e^{-s\tilde{f}_{n+2}(\mu) + \tilde{g}_{n+2}(\mu)}\,\tilde{W}^{(n+2)}(x_0;\mu,s) = (-1)^n \left[L^{n+2}_{-s\tilde{f}+\tilde{g}}(\tilde{W}^{(n+2)}(x_0;\cdot,s))\right](\xi).
\end{aligned}
$$

This and (3-13) imply

$$
\begin{aligned}
&\left|(L_s^n\mathcal{M}_{n,s}(x_0)\mathcal{G}_s\tilde{v}_s)(\xi) - \sum_{\substack{|\boldsymbol{j}|=n+3\\ j_{n+2}=l}} u_{\boldsymbol{j}}(x_0,s)\right| \\
&\qquad\qquad = \left|L^{n+2}_{-s\tilde{f}+\tilde{g}}\left[(W^{(n+2)}(x_0;\cdot,s) - \tilde{W}^{(n+2)}(x_0;\cdot,s))\right](\xi)\right|. \quad (3\text{-}36)
\end{aligned}
$$

Standard estimates for Ruelle transfer operators yield that there exists a global constant $C > 0$ such that

$$
\|L^p_{-s\tilde{f}+\tilde{g}}H\|_\infty \le C\,e^{C|\mathrm{Re}\,s|}\,e^{p\,\mathrm{Pr}(-\mathrm{Re}(s)\tilde{f}+\tilde{g})}\,\|H\|_\infty, \quad p \ge 0, \quad s \in \mathbb{C}, \qquad (3\text{-}37)
$$

for any continuous function $H : \Sigma_A^+ \to \mathbb{C}$.

Remark 3.3. The estimate above can be derived, for example, from [Stoyanov 2005]; see the proof of Theorem 2.2, Case 1 there, which uses arguments from [Bowen 1975] (see also the proof of [Parry and Pollicott 1990, Theorem 2.2]). More precisely, since $f, g \in \mathscr{F}_\alpha(\Sigma_A)$, where $\alpha > 0$ is as in Proposition 2.4, we have $\tilde{f}, \tilde{g} \in \mathscr{F}_\theta(\Sigma_A^+)$, where $\theta = \sqrt{\alpha} \in (0,1)$. Setting $u = -\mathrm{Re}(s)\,\tilde{f} + \tilde{g}$, $v = -\mathrm{Im}(s)\,\tilde{f}$, $\lambda = e^{\mathrm{Pr}(-\mathrm{Re}(s)\,\tilde{f}+\tilde{g})}$, we have $-s\tilde{f} + \tilde{g} = u + \boldsymbol{i}\,v$, and $\lambda > 0$ is the maximal eigenvalue of the operator L_u on $\mathscr{F}_\theta(\Sigma_A^+)$. Let $h \in \mathscr{F}_\theta(\Sigma_A^+)$ be a positive corresponding eigenfunction, that is, $L_u h = \lambda\, h$. It is then easy to check (see, [Stoyanov 2005, (2.2)], for example) that

$$\|L^p_{-s\tilde{f}+\tilde{g}} H\|_\infty \le \frac{\|h\|_\infty}{\min h}\, \lambda^p\, \|H\|_\infty$$

for any $p \ge 0$ and any continuous functions H on Σ_A^+. To estimate $\dfrac{\|h\|_\infty}{\min h}$ one can use [Stoyanov 2005, (3.6)], for example, from which it follows that

$$\frac{\|h\|_\infty}{\min h} \le K = e^{2\theta\, b/(1-\theta)}\, \lambda^M\, e^{M\,\|u\|_\infty},$$

where $M \ge 1$ is a constant (one can take $M = 2$ in the situation considered here) and $b = \max\{1, \|u\|_\theta\}$. Clearly, $\|u\|_\theta \le |\mathrm{Re}\, s|\, \|\tilde{f}\|_\theta + \|\tilde{g}\|_\theta \le C\,(|\mathrm{Re}\, s| + 1)$ and similarly, $\|u\|_\infty \le C\,(|\mathrm{Re}\, s| + 1)$, so (3-37) follows.

To use (3-37), we need to estimate $\sup_{\xi \in \Sigma_A^+} \left| \big(W^{(n+2)}(x_0; \cdot, s) - \tilde{W}^{(n+2)}(x_0; \cdot, s) \big)(\xi) \right|$.

Fix for a moment $s \in \mathbb{C}$. According to the definitions of $W^{(n+2)}$ and $\tilde{W}^{(n+2)}$, it is enough to consider $\mu \in \Sigma_A^+$ with $\mu_0 = 1$ and $\mu_{n+2} = l$. For such μ, using (3-10), (3-16), (3-32), (3-33) and (3-35), we have

$$\begin{aligned} |W^{(n+2)}(x_0; \mu, s) - \tilde{W}^{(n+2)}(x_0; \mu, s)| &= \left| e^{s\,\tilde{f}_{n+2}(\mu) - \tilde{g}_{n+2}(\mu) - s\,\varphi(\tilde{Q}_0(x_0;\mu)) - s\sum_{i=0}^{n+1} f_i^+(x_0;\boldsymbol{j}) + \sum_{i=0}^{n+1} \tilde{a}_i(x_0;\mu)} \right| \\ \times \left| e^{(s+\|\nabla\varphi\|_{\Gamma,(1)})O(\theta^n) - s[c(x_0;\mu) - \sum_{i=0}^{n+1} f_i^+(x_0;\boldsymbol{j})] - s[\varphi(Q_0(\mu)) - \varphi(\tilde{Q}_0(x_0;\mu)]}\, h(Q_0(\mu)) - h(\tilde{Q}_0(x_0;\mu)) \right|&. \end{aligned} \tag{3-38}$$

To estimate (3-38), first notice that by (3-15) and Proposition 2.4,

$$|f(\sigma^i \mu) - f(\sigma^{i-(n+2)}\eta)| \le C\,\alpha^i, \quad 0 \le i \le n+2.$$

Using this, (3-24), (3-26) and Proposition 2.4 again, one gets

$$\begin{aligned} \left| \tilde{f}_{n+2}(\mu) - \sum_{i=0}^{n+1} f_i^+(x_0;\boldsymbol{j}) \right| &\le C + \left| f_{n+2}(\mu) - \sum_{i=0}^{n+1} f_i^+(x_0;\boldsymbol{j}) \right| \\ &\le C + \sum_{i=0}^{n+1} |f(\sigma^i\mu) - f_i^+(x_0;\boldsymbol{j})| \le C \end{aligned} \tag{3-39}$$

for some global constant $C > 0$. Similarly, it follows from (3-15), (3-29) and (3-30) that

$$\left| \tilde{g}_{n+2}(\mu) - \sum_{i=0}^{n+1} \tilde{a}_i(x_0;\mu) \right| \le C\, \|\varphi\|_{\Gamma,(1)}. \tag{3-40}$$

Next, notice that

$$|e^{(s+\|\nabla\varphi\|_{\Gamma,(1)})O(\theta^n)} - 1| \le C\, e^{C(|\mathrm{Re}\, s| + \|\nabla\varphi\|_{\Gamma,(1)})}\, (|s| + \|\nabla\varphi\|_{\Gamma,(1)})\theta^n.$$

Using this together with (3-17), (3-18), (3-39) and (3-40) in (3-38) we obtain

$$\begin{aligned}
&|W^{(n+2)}(x_0;\mu,s) - \tilde{W}^{(n+2)}(x_0;\mu,s)| \\
&\qquad \le C\, e^{C[|\mathrm{Re}\, s|\,(1+\|\varphi\|_{\Gamma,0})+\|\nabla\varphi\|_{\Gamma,(1)}]} \left| e^{(s+\|\nabla\varphi\|_{\Gamma,(1)})O(\theta^n)}\, h(Q_0(\mu)) - h(\tilde{Q}_0(x_0;\mu)) \right| \\
&\qquad \le C\, e^{C[|\mathrm{Re}\, s|\,(1+\|\varphi\|_{\Gamma,0})+\|\nabla\varphi\|_{\Gamma,(1)}]} \left| e^{(s+\|\nabla\varphi\|_{\Gamma,(1)})O(\theta^n)} - 1 \right| |h(Q_0(\mu))| \\
&\qquad\qquad\qquad + C\, e^{C[|\mathrm{Re}\, s|\,(1+\|\varphi\|_{\Gamma,0})+\|\nabla\varphi\|_{\Gamma,(1)}]} \left| h(Q_0(\mu)) - h(\tilde{Q}_0(x_0;\mu)) \right| \\
&\qquad \le C\, e^{C[|\mathrm{Re}\, s|\,(1+\|\varphi\|_{\Gamma,0})+\|\nabla\varphi\|_{\Gamma,(1)}]} \big((|s| + \|\nabla\varphi\|_{\Gamma,(1)})\, \|h\|_{\Gamma,0} + \|h\|_{\Gamma,(1)} \big)\, \theta^n.
\end{aligned}$$

Thus, choosing the global constant $C > 0$ sufficiently large, combining the above with (3-37) gives

$$\begin{aligned}
&\left| L^{n+2}_{-s\tilde{f}+\tilde{g}} \big[(W^{(n+2)}(x_0;\cdot,s) - \tilde{W}^{(n+2)}(x_0;\cdot,s)) \big](\xi) \right| \\
&\qquad \le C\, e^{C[|\mathrm{Re}\, s|\,(1+\|\varphi\|_{\Gamma,0})+\|\nabla\varphi\|_{\Gamma,(1)}]} \big((|s| + \|\nabla\varphi\|_{\Gamma,(1)})\, \|h\|_{\Gamma,0} + \|h\|_{\Gamma,(1)} \big) (e^{\Pr(-\mathrm{Re}(s)\,\tilde{f}+\tilde{g})}\,\theta)^{n+2}. \qquad (3\text{-}41)
\end{aligned}$$

Next we have (see [Parry and Pollicott 1990, Chapter 4], for example)

$$\frac{d}{ds} \Pr(-s\tilde{f}+\tilde{g}) \Big|_{s=s_0} = -\int_{\Sigma_A^+} \tilde{f}\, d\nu = -\int_{\Sigma_A^+} f\, d\nu = -c_0 < 0,$$

where ν is the equilibrium state of $(-s_0\tilde{f}+\tilde{g})$. Recall that $\Pr(-s_0\tilde{f}+\tilde{g}) = 0$, so $e^{\Pr(-\mathrm{Re}(s)\,\tilde{f}+\tilde{g})} < 1$ for $\mathrm{Re}\, s > s_0$. Now assume $s_0 - a \le \mathrm{Re}\, s$ with some small constant $a > 0$. Then

$$e^{\Pr(-\mathrm{Re}\, s\,\tilde{f}+\tilde{g})} = 1 + c_0(s_0 - \mathrm{Re}\, s) + O((\mathrm{Re}\, s - s_0)^2) \le 1 + c_1 a,$$

for some constant $c_1 > 0$. Thus,

$$e^{\Pr(-\mathrm{Re}\, s\,\tilde{f}+\tilde{g})}\theta \le \theta + c\,a,$$

for some global constant $c = c_1\theta > 0$. Combining this with (3-41) completes the proof of (3-6). □

4. Estimates for the derivatives

In this section we prove Theorem 3.2(b). Throughout we assume that $p \ge 1$.

For any $x \in \Gamma_l$ close to x_0 and any $\eta \in \Sigma_A$ with $\eta_0 = l$ define the points $\tilde{P}_j(x;\eta)$ and the functions $f_i^-(x;\eta)$, $g_i^-(x;\eta)$, $\phi^-(x;\eta,s)$, etc., as in the beginning of Section 3 replacing the point x_0 by x. We will assume that the segment $[\tilde{P}_{-1}(x_0;\eta), x_0]$ has no common points with the interior of K_l and x is close enough to x_0 so that the same holds with x_0 replaced by x.

By Proposition A.1 there exists a unique phase function ψ_η (also depending on x_0) defined in a neighborhood U of x_0 in Γ_l, such that $\psi_\eta(x_0) = 0$ and the backward trajectory $\gamma_-(x, \nabla\psi_\eta(x))$ of any point $x \in U$ with $\psi_\eta(x) = 0$ has an itinerary $(\dots, \eta_{-l}, \dots, \eta_{-1}, \eta_0)$, that is

$$\nabla\psi_\eta(x) = \frac{\tilde{P}_0(x;\eta) - \tilde{P}_{-1}(x;\eta)}{\|\tilde{P}_0(x;\eta) - \tilde{P}_{-1}(x;\eta)\|},$$

for any $x \in \mathscr{C}_{\psi_\eta} \cap U$. (Notice that in general ψ_η is different from the functions $\varphi_{\eta,j}$ defined in the beginning of Section 3.) For any $i < 0$, denoting $J = (\eta_i, \eta_{i+1}, \dots, \eta_{-1}, \eta_0)$, we can write $\psi_\eta = (\psi_{\eta,i})_J$ for some phase function $\psi_{\eta,i}$ (defined on some naturally defined open subset $V_{\eta,i}$ of $\mathbb{R}^N$) satisfying Ikawa's condition $(\mathscr{P})$ on Γ_{η_i}. We then have $\tilde{P}_i(x;\eta) = X^{-i}(x, \nabla(\psi_{\eta,i})_J)$. As in the discussion leading up to

(3-2), one derives the existence of a global constant $C_p > 0$ such that $\|\psi_{\eta,i}\|_{(p)}(V_{\eta,i} \cap B_0) \le C_p$ for all η and $i < 0$. Using (2-4) in Proposition 2.6 with $\varphi = \psi_{\eta,m}$ for some $m \ge i$ and replacing C_p with a larger global constant if necessary, we get

$$\|\tilde{P}_i(\,\cdot\,;\eta)\|_{\Gamma,p}(x) \le C_p\,\alpha^{|i|}, \quad i < 0. \tag{4-1}$$

Similarly, for any $\mu \in \Sigma_A^+$ with $\mu_0 = 0$ and $\mu_{n+2} = k$ we have

$$\|\tilde{Q}_i(\,\cdot\,;\eta)\|_{\Gamma,p}(x) \le C_p\,\alpha^{n+2-i}, \qquad 0 \le i \le n+2, \tag{4-2}$$

$$\|\tilde{Q}_i(\,\cdot\,;\mu) - \tilde{P}_{i-n-2}(\,\cdot\,;\eta)\|_{\Gamma,p}(x) \le C_p\,\alpha^{i}, \qquad 0 \le i \le n+2. \tag{4-3}$$

Next, recall the function Λ_φ from the beginning of this section. By Proposition 2.6,

$$\|\nabla\varphi_J\|_{\Gamma,p} \le C_p\,\|\nabla\varphi\|_{\Gamma,(p)}, \tag{4-4}$$

for any finite admissible configuration J.

Since for any $i < 0$ we have $g_i^-(x;\eta) = \log\Lambda_{\psi_{\eta,i}}(\tilde{P}_{i+1}(x;\eta))$, it follows from (4-1)–(4-3) and from Proposition 2.7 that for any $p \ge 1$ there exists a global constant $C_p > 0$ such that

$$\|g_i^-(\,\cdot\,;\eta)\|_{\Gamma,p}(x) \le C_p\,\alpha^{|i|}, \quad i < 0. \tag{4-5}$$

Similarly, according to (3-28) and Proposition 2.6,

$$\|\tilde{a}_i(\,\cdot\,;\mu)\|_p(x) \le C_p\,\|\nabla\varphi\|_{\Gamma,(p)}\,\alpha^{n+2-i}, \quad 0 \le i \le n+2, \tag{4-6}$$

and as in the proof of (3-31) one gets,

$$\|\tilde{a}_i(\,\cdot\,;\mu) - g^-_{i-n-2}(\,\cdot\,;\eta)\|_p(x) \le C_p\,\|\nabla\varphi\|_{\Gamma,(p+1)}\,\alpha^{i}, \quad 0 \le i \le n+2. \tag{4-7}$$

Next, given x as above, μ and n with $\mu_{n+2} = l$, define $W^{(n+2)}(x;\mu,s)$ by (3-10), η by (3-15) and $\tilde{W}^{(n+2)}(x;\mu,s)$ by (3-35) replacing x_0 by x. We will estimate the derivatives of

$$W^{(n+2)}(x;\mu,s) - \tilde{W}^{(n+2)}(x;\mu,s)$$

with respect to x.

First look at the first derivatives $D_v[W^{(n+2)}(\,\cdot\,;\mu,s) - \tilde{W}^{(n+2)}(\,\cdot\,;\mu,s)](x)$, where $v \in S_x\Gamma$. Writing $\phi^-(x;\eta,s) = -s\,\phi_1^-(x;\eta) + \phi_2^-(x;\eta)$, where

$$\phi_1^-(x;\eta) = \sum_{i=-1}^{-\infty}\big(f(\sigma^i(\eta)) - f_i^-(x;\eta)\big), \quad \phi_2^-(x,\eta) = \sum_{i=-1}^{-\infty}\big(g(\sigma^i(\eta)) - g_i^-(x;\eta)\big),$$

we see that for any $x, x' \in \Gamma_l$ close to x_0 we have

$$\phi_1^-(x;\eta) - \phi_1^-(x';\eta) = -\psi_\eta(x) + \psi_\eta(x'),$$

so $D_v(\phi_1^-(\,\cdot\,;\eta))(x) = D_v(\psi_\eta(x))$. Therefore, by (3-11),

$$D_v z(\,\cdot\,;\mu,s)(x) = -s\,D_v\psi_\eta(x) + \sum_{i=-1}^{-\infty} D_v(g_i^-(\,\cdot\,;\eta))(x). \tag{4-8}$$

Next, using the notation $\boldsymbol{j} = (\mu_0, \mu_1, \mu_2, \dots, \mu_{n+2})$ and

$$\tilde{z}(x;\, \mu, s) = s\, \tilde{f}_{n+2}(\mu) - \tilde{g}_{n+2}(\mu) - s\, (\varphi_{\mu_0})_{\boldsymbol{j}}(x),$$

it follows from (3-38) that

$$\begin{aligned} W^{(n+2)}(\,\cdot\,;\, \mu, s) - \tilde{W}^{(n+2)}(\,\cdot\,;\, \mu, s)(x) \\ &= e^{z(x;\mu,s) - s\,\varphi(Q_0(\mu))}\, h(Q_0(\mu)) - e^{\tilde{z}(x;\mu,s)}\, \Lambda_{\varphi,\boldsymbol{j}}(\tilde{Q}_{n+2}(x;\, \mu))\, h(\tilde{Q}_0(x;\, \mu)) \\ &= I(x) + II(x), \end{aligned} \tag{4-9}$$

where

$$I(x) = \big(e^{z(x;\mu,s) - s\,\varphi(Q_0(\mu))} - e^{\tilde{z}(x;\mu,s) + \log \Lambda_{\varphi,\boldsymbol{j}}(\tilde{Q}_{n+2}(x;\mu))}\big)\, h(Q_0(\mu)),$$
$$II(x) = e^{\tilde{z}(x;\mu,s)}\, \Lambda_{\varphi,\boldsymbol{j}}(\tilde{Q}_{n+2}(x;\, \mu))\big(h(Q_0(\mu)) - h(\tilde{Q}_0(x;\, \mu))\big).$$

Let $\mathbb{O}$ be a small compact connected neighborhood of x in Γ. Fix temporarily μ, s, n and η with (3-15), and set

$$A(y) = z(y;\, \mu, s) - s\,\varphi(Q_0(\mu)), \quad B(y) = \tilde{z}(x;\, \mu, s) + \log \Lambda_{\varphi,\boldsymbol{j}}(\tilde{Q}_{n+2}(x;\, \mu)), \quad y \in \mathbb{O}.$$

To estimate $I(x)$ we first write $\|A\|_0(\mathbb{O}) = O(|s| + |s|\, \|\varphi\|_{\Gamma,0} + \|\nabla\varphi\|_{\Gamma,(1)})$, using the estimates in Section 3, and also

$$|e^A|_{\Gamma,0}(\mathbb{O}) \le C\, e^{C[|\mathrm{Re}\, s|\,(1 + \|\varphi\|_{\Gamma,0}) + \|\nabla\varphi\|_{\Gamma,(1)}]}. \tag{4-10}$$

It follows from (4-6) and (3-40) that $|\tilde{g}_{n+2}(\mu)| \le C\, \|\nabla\varphi\|_{\Gamma,(1)}$. Combining this with the definition of $\tilde{z}(x;\, \mu, s)$ and (3-39) implies

$$\|\tilde{z}(\,\cdot\,;\, \mu, s)\|_0(\mathbb{O}) = O(|s| + |s|\, \|\varphi\|_{\Gamma,0} + \|\nabla\varphi\|_{\Gamma,(1)}), \quad \|B\|_0(\mathbb{O}) = O(|s| + |s|\, \|\varphi\|_{\Gamma,0} + \|\nabla\varphi\|_{\Gamma,(1)}).$$

Next, we will estimate the derivatives of A and B. For any $q \ge 1$ and any $y \in \mathbb{O}$, using (4-8), (2-1) and (4-5), we get

$$\begin{aligned} \|A\|_{\Gamma,q}(y) &= \|s\, \phi_1^-(\,\cdot\,;\, \eta) - \phi_2^-(\,\cdot\,;\, \eta)\|_{\Gamma,q}(y) \\ &\le |s|\, \|\nabla\psi_\eta\|_{\Gamma,q}(y) + \sum_{i=-1}^{-\infty} \|g_i^-(\,\cdot\,;\, \eta)\|_{\Gamma,q}(y) \le |s|\, C_q + C_q \sum_{i=-1}^{-\infty} \alpha^{|i|} \le C_q\,(|s| + 1). \end{aligned} \tag{4-11}$$

Thus, for any $q \ge 0$,

$$\|e^A\|_{\Gamma,q}(\mathbb{O}) \le C_q \|e^A\|_{\Gamma,0}(\mathbb{O})\, (\max\nolimits_{1\le i\le q} \|A\|_{\Gamma,i}(\mathbb{O}))^q \le C_q\, e^{C\,[|\mathrm{Re}\, s|\,(1+|\varphi|_{\Gamma,0}) + \|\nabla\varphi\|_{\Gamma,(1)}]}\, (|s| + 1)^q.$$

Similarly, (4-4) gives

$$\|\tilde{z}(\,\cdot\,;\, \mu, s)\|_{\Gamma,q}(y) = \|s\, (\varphi_{\mu_0})_{\boldsymbol{j}}\|_{\Gamma,q}(y) \le C_q\, |s|\, \|\nabla\varphi\|_{\Gamma,(q)},$$

while (3-31) and (4-6) imply

$$\|\log \Lambda_{\varphi,\boldsymbol{j}}(\,\cdot\,)\|_{\Gamma,q}(y) \le \sum_{i=0}^{n+1} \|\tilde{a}_i(\,\cdot\,;\, \mu)\|_{\Gamma,q}(y) \le C_q\, \|\nabla\varphi\|_{\Gamma,(q)} \quad \text{for any } q \ge 0,$$

so

$$\|B\|_{\Gamma,q}(y) \le C_q\, (|s| + 1)\, \|\nabla\varphi\|_{\Gamma,(q)}, \quad y \in \mathbb{O}.$$

The next step is to estimate the derivatives of $A - B$. By Proposition 2.6 and (2-1) we have

$$\|\nabla \psi_\eta - \nabla(\varphi_{\mu_0})_J\|_{\Gamma,q}(\mathcal{O}) \le C_q\, \alpha^n\, \|\nabla \psi_\eta - \nabla \varphi_{\mu_0}\|_{\Gamma,(q)} \le C_q\, \alpha^n\, \|\nabla \varphi\|_{\Gamma,(q)}.$$

Again set $m = (n+1)/2$, assuming for simplicity that n is odd, and write $\theta = \sqrt{\alpha} \in (0, 1)$. As in the proof of (3-18), for any $y \in \mathcal{O}$ and any $q \ge 1$, using (4-5), (4-6) and (4-7), we have

$$\begin{aligned}
\|A - B\|_{\Gamma,q}(y) &\le \Big\| -s\, \psi_\eta + \sum_{i=-1}^{-\infty} g_i^-(\,\cdot\,;\eta) + s\,(\varphi_{\mu_0})_J - \sum_{i=0}^{n+1} \tilde{a}_i(\,\cdot\,;\mu) \Big\|_{\Gamma,q}(y) \\
&\le |s|\, \|\psi_\eta - (\varphi_{\mu_0})_J\|_{\Gamma,q}(y) + \sum_{i=-m-1}^{-\infty} \|g_i^-(\,\cdot\,;\eta)\|_{\Gamma,q}(y) \\
&\qquad\qquad + \sum_{i=0}^{m} \|\tilde{a}_i(\,\cdot\,;\mu)\|_{\Gamma,q}(y) + \sum_{i=m+1}^{n+1} \|\tilde{a}_i(\,\cdot\,;\mu) - g_{i-n-2}^-(\,\cdot\,;\eta)\|_{\Gamma,q}(y) \\
&\le C_q (|s|\, \|\nabla \varphi\|_{\Gamma,(q)} + \|\nabla \varphi\|_{\Gamma,(q+1)})\, \theta^n.
\end{aligned}$$

From Section 3, a similar estimate holds for $q = 0$. Consequently,

$$\begin{aligned}
\|e^{B-A}\|_{\Gamma,q}(\mathcal{O}) &\le C_q \|e^{B-A}\|_0(\mathcal{O}) \left(\max\nolimits_{1\le i\le q} \|B - A\|_{\Gamma,i}(\mathcal{O})\right)^q \\
&\le C_q\, e^{C\,(|\mathrm{Re}\, s| + \|\nabla\varphi\|_{\Gamma,(1)})}\, (|s|\, \|\nabla \varphi\|_{\Gamma,(q)} + \|\nabla \varphi\|_{\Gamma,(q+1)})^q \theta^{nq}.
\end{aligned}$$

Finally, as in the estimate just after (3-40), it follows that

$$\|e^{B-A} - 1\|_0(\mathcal{O}) \le C\, e^{C\,(|\mathrm{Re}\, s| + \|\nabla\varphi\|_{\Gamma,(1)})}\, (|s| + \|\nabla \varphi\|_{\Gamma,(1)})\, \theta^n.$$

This, together with (4-10) and (4-11), implies that for any $q \ge 1$,

$$\|(I)\|_{\Gamma,q}(\mathcal{O}) \le C_q\, \|h\|_0(\Gamma)\, e^{C\,[|\mathrm{Re}\, s|\, (1+\|\varphi\|_{\Gamma,0}) + \|\nabla\varphi\|_{\Gamma,(1)}]}\, (|s|\, \|\nabla \varphi\|_{\Gamma,(q)} + \|\nabla \varphi\|_{\Gamma,(q+1)})^q\, \theta^n.$$

Using similar estimates, for any $q \ge 1$ one gets

$$|II|_{\Gamma,q}(\mathcal{O}) \le C_q\, \alpha^n\, e^{C\,[|\mathrm{Re}\, s|\, (1+\|\varphi\|_{\Gamma,0}) + \|\nabla\varphi\|_{\Gamma,(1)}]} \sum_{r=0}^{q-1} (|s| + 1)^{r+1}\, (\|\nabla \varphi\|_{\Gamma,(r)})^{r+1}\, \|h\|_{\Gamma,q-r}(\mathcal{O}).$$

It now follows from (4-9) and the estimates for I and II found above that for any $p \ge 1$ we have

$$\begin{aligned}
&\|W^{(n+2)}(\,\cdot\,;\mu, s) - \tilde{W}^{(n+2)}(\,\cdot\,;\mu, s)\|_{\Gamma,(p)}(\mathcal{O}) \\
&\qquad \le C_p\, \theta^n\, e^{C\,[|\mathrm{Re}\, s|\, (1+\|\varphi\|_{\Gamma,0}) + \|\nabla\varphi\|_{\Gamma,(1)}]} \times \sum_{r=0}^{q} \big(|s|\, \|\nabla \varphi\|_{\Gamma,(r)} + \|\nabla \varphi\|_{\Gamma,(r+1)}\big)^{r+1}\, \|h\|_{\Gamma,q-r}(\mathcal{O}).
\end{aligned}$$

Combining this with (3-6), (3-36) and the argument from the end of Section 3 completes the proof of Theorem 3.2. □

5. Estimates for $w_{0,j}(x, s)$

Our purpose in this section is to prove that the series

$$w_{0,j}(x, s) = \sum_{n=n_j}^{\infty} \sum_{\substack{|\boldsymbol{j}|=n+3 \\ j_{n+2}=j}} u_{\boldsymbol{j}}(x, s),\ x \in \Gamma_j$$

is convergent and that $w_{0,j}(x,s)$ is an analytic function for $s \in \mathcal{D}_1$ with values in $C^\infty(\Gamma_j)$. Since we deal with initial data $m(x,s) = u_1(x,s)$ on Γ_1 we set $n_1 = -2$ and $n_j = -1$, $j = 2, \dots, \kappa_0$. Theorem 3.2 clearly reduces the problem to the convergence of the series

$$\sum_{n=0}^{\infty} (L_s^n \mathcal{M}_{n,s}(x) \mathcal{G}_s \tilde{v}_s)(\xi),\ x \in \Gamma_j.$$

Throughout this and the following sections we will use the notation

$$E_p(s,\varphi,h) = \begin{cases} e^{C_p[|\operatorname{Re} s|\,(1+\|\varphi\|_{\Gamma,0})+\|\nabla\varphi\|_{\Gamma,(1)}]} \sum_{j=0}^{p} (|s|\,\|\nabla\varphi\|_{\Gamma,j} + \|\nabla\varphi\|_{\Gamma,j+1})^{j+1} \|h\|_{\Gamma,p-j} & \text{if } p \geq 1, \\ C_0 e^{C_p[|\operatorname{Re} s|\,(1+\|\varphi\|_{\Gamma,0})+\|\nabla\varphi\|_{\Gamma,(1)}]} [(|s| + \|\nabla\varphi\|_{\Gamma,(1)})\,\|h\|_{\Gamma,0} + \|h\|_{\Gamma,(1)}] & \text{if } p = 0, \end{cases}$$

where as before by C_p we denote positive global constants depending on p which may change from line to line.

First we will establish for $\sigma_0 \leq \operatorname{Re} s \leq 1$ the inequality

$$\|L_s^n \mathcal{M}_{n,s}(\,\cdot\,) - L_s^{n-1} \mathcal{M}_{n-1,s}(\,\cdot\,) L_s\|_{\Gamma,p} \leq C_p E_p(s,\varphi,h)\theta^n, \tag{5-1}$$

where $L_s = -L_{-s\tilde{f}+\tilde{g}}$ and $\sigma_0 < s_0$. The precise choice of σ_0 depends on the estimates (3-3) and will be discussed below. For this purpose we write

$$(L_s^n \mathcal{M}_{n,s} - L_s^{n-1} \mathcal{M}_{n-1,s} L_s) w(\xi) = -L_s^{n+1} [Y^{(n)}(x;s,\mu) - \tilde{Y}^{(n)}(x;s,\mu)](\xi),$$

where

$$Y^{(n)}(x;s,\mu) = \exp\bigl(-\phi^-(x;\sigma^{n+1}e(\mu),s) - \chi(\sigma^{n+1}e(\mu),s)\bigr) w(\mu),$$
$$\tilde{Y}^{(n)}(x;s,\mu) = \exp\bigl(-\phi^-(x;\sigma^{n}e(\sigma\mu),s) - \chi(\sigma^{n}e(\sigma\mu),s)\bigr) w(\mu).$$

The inequality (5-1) follows from the estimates

$$\bigl\|\phi^-(x;\sigma^{n+1}e(\xi),s) - \phi^-(x;\sigma^n e(\sigma(\xi)),s)\bigr\|_{\Gamma,p} \leq C_p E_p(s,\varphi,h)\theta^n, \tag{5-2}$$

$$|\chi(\sigma^{n+1}e(\xi),s) - \chi(\sigma^n e(\sigma(\xi)),s)| \leq C(1+|s|)\theta^n, \tag{5-3}$$

and the form of the operators $\mathcal{M}_{n,s}(x)$. The estimate (5-3) is a consequence of the choice of χ_1, χ_2 and the fact that $f, g \in \mathcal{F}_\theta(\Sigma_A)$. To prove (5-2), notice that

$$\left| \sum_{i=-1}^{-\infty} \bigl(f(\sigma^{n+1+i}e(\xi)) - f\bigl(\sigma^{n+i}e(\sigma(\xi))\bigr)\bigr) \right| \leq C\theta^n,$$

and similar estimates hold for g. The terms involving f and g are independent of x and they are not important for the estimates of the derivatives. To deal with the terms depending on x, recall that

$$\phi^-(x;\eta) = -s\phi_1^-(x;\eta) + \phi_2^-(x;\eta),$$

with $D_\upsilon(\phi_1^-(\,\cdot\,;\eta)(x) = D_\upsilon(\psi_\eta(x))$. Here and below we use the notation of the previous section. On the other hand,

$$\|\nabla\psi_{\sigma^{n+1}e(\mu)}(x) - \nabla\psi_{\sigma^n e(\sigma(\mu))}(x)\|_{\Gamma,p} \leq C_p \alpha^n. \tag{5-4}$$

In fact, the backward trajectories $\gamma_-(x, \nabla\psi_{\sigma^{n+1}e(\mu)}(x))$ and $\gamma_-(x, \nabla\psi_{\sigma^n e(\sigma(\mu))}(x))$ follow an itinerary $(\mu_{n+1}, \mu_n, \dots, \mu_1)$ and we can apply Proposition 2.6. Now we repeat the argument used in the previous section for the estimate of $\|A - B\|_{\Gamma,p}$. Set $m = (n+1)/2$ and assume for simplicity that n is odd. For fixed n we set $\eta = \sigma^{n+1}e(\mu)$, $\tilde{\eta} = \sigma^n e(\sigma(\mu))$. The estimate of

$$\|\phi_1^-(x;\eta) - \phi_1^-(x;\tilde{\eta})\|_{\Gamma,p}$$

follows from (5-4). Next we write

$$\begin{aligned}\sum_{i=-1}^{-\infty} &\big(g_i^-(x;\eta) - g_i^-(x;\tilde{\eta})\big)\\ &= \sum_{i=-m-1}^{-\infty}\big(g_i^-(x;\eta) - g_i^-(x;\tilde{\eta})\big) + \sum_{i=m+1}^{n+1}\big(g_{i-n-2}^-(x;\eta) - \tilde{a}_i(x;\mu)\big) - \sum_{i=m+1}^{n+1}\big(g_{i-n-2}^-(x,\tilde{\eta}) - \tilde{a}_i(x;\mu)\big).\end{aligned}$$

The $\|\cdot\|_{\Gamma,p}$ norms of the sums from $i = m+1$ to $n+1$ can be estimated as in Section 4 by using (4-7), since

$$\begin{aligned}\eta &= \sigma^{n+1}e(\mu) = (\dots,*,*,\mu_0,\mu_1,\dots,\mu_{n+1} = l,\mu_{n+2},\dots),\\ \tilde{\eta} &= \sigma^n e(\sigma(\mu)) = (\dots,*,*,\mu_1,\dots,\mu_{n+1} = l,\mu_{n+2},\dots),\end{aligned}$$

and

$$\begin{aligned}\sum_{i=m+1}^{n+1}\big\|g_{i-n-2}^-(x;\eta) - \tilde{a}_i(x;\mu)\big\|_{\Gamma,p} &\le \sum_{i=m+1}^{n+1}\alpha^i,\\ \sum_{i=m+1}^{n+1}\big\|g_{i-n-2}^-(x;\tilde{\eta}) - \tilde{a}_i(x;\mu)\big\|_{\Gamma,p} &\le \sum_{i=m+1}^{n+1}\alpha^i.\end{aligned}$$

To estimate the sums from $i = -m-1$ to $-\infty$, we apply (4-5) and this completes the proof of (5-1).

From the representation

$$L_s^n\mathcal{M}_{n,s} = \sum_{k=1}^{n}(L_s^k\mathcal{M}_{k,s} - L_s^{k-1}\mathcal{M}_{k-1,s}L_s)L_s^{n-k} + \mathcal{M}_{0,s}L_s^n,$$

we get

$$\sum_{n=1}^{\infty} L_s^n\mathcal{M}_{n,s}w = \sum_{n=1}^{\infty}\Big(\sum_{k=1}^{n}(L_s^k\mathcal{M}_{k,s} - L_s^{k-1}\mathcal{M}_{k-1,s}L_s)L_s^{n-k}w + \mathcal{M}_{0,s}L_s^n w\Big).$$

Since $s_0 \in \mathbb{R}$ is the abscissa of absolute convergence, for $\operatorname{Re} s > s_0$ we have $\Pr(-\operatorname{Re}(s)\tilde{f} + \tilde{g}) < 0$ and $\|L_s^n\|_\infty \le 1$ for all n. Consequently, the double sum in the right hand side is absolutely convergent for $\operatorname{Re} s > s_0$ and we can change the order of summation. Applying Fubini's theorem, we are going to examine

$$\sum_{n=0}^{\infty} L_s^n\mathcal{M}_{n,s}\mathcal{G}_s\tilde{v}_s = (\mathcal{M}_{0,s} + \mathcal{Q}_s)\sum_{n=0}^{\infty} L_s^n\mathcal{G}_s\tilde{v}_s, \tag{5-5}$$

where

$$\mathcal{Q}_s = \sum_{k=1}^{\infty}(L_s^k\mathcal{M}_{k,s} - L_s^{k-1}\mathcal{M}_{k-1,s}L_s).$$

According to (5-1), the series defining $\mathcal{Q}_s$ is absolutely convergent for $\sigma_0 \le \operatorname{Re} s \le 1$ and

$$\|\mathcal{Q}_s\|_{\Gamma,p} \le C_p E_p(s,\varphi,h).$$

Consequently, the problem of the analytic continuation of the left hand side of (5-5) for $\operatorname{Re} s < s_0$ is reduced to that of the series $\sum_{n=0}^{\infty} L_s^n w_s$, with $w_s = \mathcal{G}_s\tilde{v}_s$.

The analysis of $\sum_{n=0}^{\infty} L_s^n w_s$ is based on Dolgopyat type estimates (3-3); we must show that, with Φ and $C_u^{\mathrm{Lip}}(\Lambda_{\partial K})$ as in Appendix C, we have $w_s = h_s \circ \Phi$ for some $h_s \in C_u^{\mathrm{Lip}}(\Lambda_{\partial K})$. This assertion is proved in the same appendix, where we show that for $|\mathrm{Re}\, s| \le a$ we have $\|h_s\|_{\mathrm{Lip},t} \le C_0$ with C_0 independent of s. Thus for $s = \tau + \boldsymbol{i}\, t$, $\sigma_0 \le \tau \le 1$, $|t| \ge t_0 > 1$, we get

$$\begin{aligned}\sum_{n=0}^{\infty} \|\tilde{L}_s^n w_s\|_\infty &\le \sum_{p=0}^{\infty} \sum_{l=0}^{[\log|t|]-1} C\rho^{p[\log|t|]} e^{l\,\mathrm{Pr}(-\tau\tilde{f}+\tilde{g})} \|h_s\|_{\mathrm{Lip},t} \\ &\le \frac{CC_0}{1-\rho^{[\log|t|]}} \sum_{l=0}^{[\log|t|]-1} e^{l\,\mathrm{Pr}(-\tau\tilde{f}+\tilde{g})} \le C_1 \max\{\log|t|, |t|^{\mathrm{Pr}(-\tau\tilde{f}+\tilde{g})}\}.\end{aligned}$$

On the other hand, for σ_0 sufficiently close to s_0 we have

$$\mathrm{Pr}(-\sigma_0\tilde{f}+\tilde{g}) = \tilde{\beta}_0 < 1.$$

Combining this with the estimate for $\mathcal{Q}_s$, we conclude that for $\sigma_0 \le \mathrm{Re}\, s$ and $|t| \ge t_0 > 1$ we have

$$\Big\| \sum_{n=0}^{\infty} L_s^n \mathcal{M}_{n,s} \mathcal{G}_s \tilde{v}_s \Big\|_{\Gamma,0} \le C_2 |t|^{1+\tilde{\beta}_0}.$$

The analysis in [Ikawa 1982, Section 5] implies that the series defining $w_{0,j}(x,s)$ is absolutely convergent for $x \in \Gamma_j$, $\mathrm{Re}\, s \ge s_0 + d > s_0$ and we have

$$\|w_{0,j}(x,s)\|_{\Gamma_j,0} \le C_{j,d}, \quad \mathrm{Re}\, s \ge s_0 + d. \tag{5-6}$$

On the other hand, the analytic continuation of the series $\sum_{n=0}^{\infty} L_s^n \mathcal{M}_{n,s} \mathcal{G}_s \tilde{v}_s$ established above, together with an application of Theorem 3.2(a) with a sufficiently small $\varepsilon = s_0 - \mathrm{Re}\, s > 0$, guarantee an analytic continuation of $w_{0,j}(x,s)$ for $x \in \Gamma_j$, $\mathrm{Re}\, s \ge \sigma_0$, $|\mathrm{Im}\, s| \ge t_0$ with $\sigma_0 = s_0 - \varepsilon$. Applying Theorem 3.2(a) once more for $s = \sigma_0 + \boldsymbol{i}\, t$, we get the estimate

$$\|w_{0,j}(x, \sigma_0 + \boldsymbol{i}\, t)\|_{\Gamma_j,0} \le D_j |t|^{1+\tilde{\beta}_0}.$$

The same argument works for all $l = 1, \ldots, \kappa_0$ and we get the same estimate for

$$w_{0,l}(x,s) = \sum_{n=n_l}^{\infty} \sum_{\substack{|\boldsymbol{j}|=n+3 \\ j_0=1 \\ j_{n+2}=l}} u_{\boldsymbol{j}}(x,s),\ x \in \Gamma_l.$$

Clearly, we can choose $0 < \tilde{\beta}_0 < 1$ independent of $l = 1, \ldots, \kappa_0$.

Now we will obtain $C^p(\Gamma_j)$ estimates for $w_{0,j}(x,s)$. To examine the regularity of the functions $w_{0,j}(x,s)$ on Γ_j, set

$$U_{n+2,j}(x,s) = \sum_{\substack{|\boldsymbol{j}|=n+3 \\ j_{n+2}=j}} u_{\boldsymbol{j}}(x,s).$$

We start with an estimate of the $C^p(\Gamma_j)$ norms of $U_{n+2,j}(x,s)\big|_{\Gamma_j}$. To this end, applying Theorem 3.2(b) with $p \ge 1$, we must estimate the norms $\|L^s \mathcal{M}_{n,s}(\,\cdot\,) w_s\|_{\Gamma_j,p}$, where $w_s = \mathcal{G}_s \tilde{v}_s$ and L_s^n are independent of $x \in \Gamma$. We write

$$
\begin{aligned}
&L_s^n \mathcal{M}_{n,s} w_s \\
&\quad = \mathcal{M}_{0,s} L_s^n w_s + \sum_{k=1}^{m} (L_s^k \mathcal{M}_{k,s} - L_s^{k-1} \mathcal{M}_{k-1,s} L_s) L_s^{n-k} w_s + \sum_{k=m+1}^{n} (L_s^k \mathcal{M}_{k,s} - L_s^{k-1} \mathcal{M}_{k-1,s} L_s) L_s^{n-k} w_s \\
&\quad =: B_0 + B_1 + B_2,
\end{aligned}
$$

where $m = [n/2]$. We apply the estimate (3-3) combined with $\|h_s\|_{\mathrm{Lip},t} \le C_0$, $t = \operatorname{Im} s$, and we obtain

$$\|L_s^n w_s\|_0 \le C\rho^n e^{\log|t|[\Pr(-s\tilde{f}+\tilde{g})-\log\rho]} \le C\rho^n |t|^{\beta_0} \quad \text{for all } n \in \mathbb{N},$$

with $0 < \rho < 1$ and $\beta_0 = \Pr(-\sigma_0 \tilde{f} + \tilde{g}) - \log\rho > 0$. Increasing ρ, we can arrange $\beta_0 < 1$ but this is not important for our argument (see also Remark C.4).

For the term B_0 we get

$$\|B_0\|_{\Gamma_j,p} \le C_p |\operatorname{Im} s|^{\beta_0} E_p(s, \varphi, h) \rho^n.$$

In the same way for the term B_1 we have

$$\|B_1\|_{\Gamma_j,p} \le C'_p |\operatorname{Im} s|^{\beta_0} E_p(s, \varphi, h) \sum_{k=1}^{m} \theta^k \rho^m \le C''_p |\operatorname{Im} s|^{\beta_0} E_p(s, \varphi, h) (\sqrt{\rho})^n.$$

Finally, for B_2 we obtain

$$\|B_2\|_{\Gamma_j,p} \le D_p |\operatorname{Im} s|^{\beta_0} E_p(s, \varphi, h) \sum_{k=m+1}^{n} \theta^k \le D'_p |\operatorname{Im} s|^{\beta_0} E_p(s, \varphi, h) \theta^{m+1}.$$

So, replacing θ by another global constant $0 < \tilde{\theta} < 1$ with $\tilde{\theta} \ge \max\{\sqrt{\rho}, \sqrt{\theta}\}$, we arrange an estimate

$$\|L_s^n \mathcal{M}_{n,s} w_s\|_{\Gamma_j,p} \le B_p |\operatorname{Im} s|^{\beta_0} E_p(s, \varphi, h) \tilde{\theta}^n.$$

Thus, with global constants C_p, D_p we deduce

$$\|U_{n+2,j}(x,s)\|_{\Gamma_j,p} \le C_p |\operatorname{Im} s|^{\beta_0} E_p(s, \varphi, h)(\theta^n + \tilde{\theta}^n) \le D_p |\operatorname{Im} s|^{\beta_0} E_p(s, \varphi, h) \tilde{\theta}^n \quad \text{for all } n \in \mathbb{N}. \tag{5-7}$$

Consequently, the series $w_{0,j}(x,s)$ is convergent in the $C^p(\Gamma_j)$ norm and for $\sigma_0 \le \tau \le s_0 + 1$ we have

$$\|w_{0,j}(x, \tau + \boldsymbol{i}t)\|_{\Gamma_j,p} \le B_p |t|^{\beta_0} E_p(s, \varphi, h), \quad p \ge 1, \tag{5-8}$$

where the constants B_p are independent of j. Summing over $l = 1, \dots, \kappa_0$, we obtain the same estimate for $\|w_0(x, \tau + \boldsymbol{i}t)\|_{\Gamma,p}$ and for $\operatorname{Re} s \ge \sigma_0$ the trace $w_0(x,s)$ is an analytic function in s with values in $C^\infty(\Gamma)$.

Observe that by contracting the domain $\sigma_0 \le \operatorname{Re} s \le s_0 + 1$ we may obtain better bounds for the $C^p(\Gamma)$ norms. For example, we treat below the case $p = 0$ and the same argument works for $p \ge 1$. In the domain $\sigma_0 \le \operatorname{Re} s \le s_0 + d$, $d > 0$, $\operatorname{Im} s \ge t_0$, we apply the Phragmen–Lindelöf theorem [Titchmarsh 1968, 5.65]. Notice that when we decrease $d > 0$ the constant $C_{j,d}$ in (5-6) change but we always have the bound (5-6). Consequently, for $\sigma_0 \le \tau \le s_0 + d$ we deduce

$$\|w_{0,j}(x, \tau + \boldsymbol{i}t)\|_{\Gamma_j,0} \le B|t|^{\kappa(\tau)}, \quad t \ge 2,$$

where $\kappa(x)$ is a linear function such that

$$\kappa(\sigma_0) = 1 + \tilde{\beta}_0, \quad \kappa(s_0 + d) = 0.$$

It is clear that if $d > 0$ is small enough, there exist σ_0' with $\sigma_0 < \sigma_0' < s_0$ and $0 < \beta < 1$ so that for $\tau \geq \sigma_0'$ we have

$$\|w_{0,j}(x, \tau + \boldsymbol{i}\, t)\|_{\Gamma_j,0} \leq A_j\, |t|^{\beta}, \quad t \geq t_0,$$

and similarly we treat the case $t \leq -t_0$. Finally, for $\tau \geq \sigma_0'$, $|t| \geq t_0$ we have

$$\|w_{0,j}(x, \tau + \boldsymbol{i}\, t)\|_{\Gamma_j,0} \leq A_j |t|^{\beta}. \tag{5-9}$$

Here the constants A_j depend on the norms of $\nabla\varphi$ and h.

Remark 5.1. In the following we will not use the estimate (5-9); however a similar argument based on the Phragmen–Lindelöf theorem will be crucial in Section 7, where we need to control the behavior of the remainder $\mathcal{Q}_M(x, s; k)$ and its bounds when $|\operatorname{Im} s| \to \infty$. On the other hand, (5-9) is related to the assumption (1-6) of Ikawa mentioned in the Introduction. The estimate (1-6) can be established choosing $\sigma_0' < s_0$ close to s_0 and applying (3-3). This is not necessary for our exposition and we leave the details to the reader.

6. The leading term $V^{(0)}(x, s; k)$

Our purpose here is to apply the construction in Section 3 with boundary data

$$m(x, s; k) = e^{\boldsymbol{i}\, k\psi(x)} b(x, s; k), \quad x \in \Gamma_j,$$

where $k \geq 1$ and $s \in \mathcal{D}_0 = \{s \in \mathbb{C} : \sigma_0 \leq \operatorname{Re} s \leq 1,\ |\operatorname{Im} s| \geq J > 0\}$, with some constant J to be chosen below. We suppose that there exists a phase function $\varphi(x)$ satisfying condition $(\mathcal{P})$ in Γ_j such that $\varphi(x)|_{\Gamma_j} = \psi(x)$ for $x \in \operatorname{supp}_x b(x, s; k)$. The amplitude $b(x, s; k)$ is analytic with respect to $s \in \mathcal{D}_0$ and $\bigcup_{s,k} \operatorname{supp}_x b \subset \Gamma_j$,

$$\|b(x, s; k)\|_{\Gamma_j,p} \leq C_p \quad \text{for all } k \geq 1,\ s \in \mathcal{D}_0,\ p \in \mathbb{N}.$$

In the following we will use the notation $\langle z\rangle = (1 + |z|)$. For our construction it is convenient to write the oscillatory data $m(x, s; k)$ with phase $e^{-s\psi(x)}$ and we set

$$m(x, s; k) = e^{-s\psi(x)} e^{(s+\boldsymbol{i}\, k)\psi(x)} b(x, s; k) = e^{-s\psi(x)} b_1(x, s; k).$$

Then

$$\|b_1(x, s; k)\|_{\Gamma_j,p} \leq C_p' \langle s + \boldsymbol{i}\, k\rangle^p \quad \text{for all } p \in \mathbb{N}.$$

Thus our data depends on *two parameters* $s \in \mathcal{D}_0$ and $k \geq 1$. The complex parameter s will be related to the convergence of the series $w_{0,j}(x, s; k)$ constructed in Section 5 starting with initial data $m(x, s; k)$, while the real parameter k is connected with the oscillatory data $G(x)e^{\boldsymbol{i}\, k\langle x,\eta\rangle}|_{y\in\Gamma_j}$, $|\eta| \leq 1 - \delta_1/2 < 1$, coming from a Fourier transform (see Section 8). Note that up to the end of Section 7 the parameters s and k will not be related and the estimates obtained depend on expressions of the form $\langle s + \boldsymbol{i}\, k\rangle^M$. After the application of Phragmen–Lindelöf argument at the end of Section 7, we take $|s+\boldsymbol{i}\, k| \leq \text{Const}$ in order to get bounds by powers of k. We consider amplitudes $b(x, s; k)$ depending on s and k to cover higher order approximations in Section 7. Starting with boundary data $e^{-s\psi}b_1$ and following the procedure in Sections 3–5, we can justify the convergence of the series $w_{0,j}(x,s;k)$ which are analytic for $s \in \mathcal{D}_0$.

Now we will discuss the domain where the parameter s is running. For $\operatorname{Im} z < 0$ we define the resolvent $(-\Delta_K - z^2)^{-1}$ of the Dirichlet Laplacian $-\Delta_K$ related to K by the spectral calculus and we get

$$\|(-\Delta_K - z^2)^{-1}\|_{L^2(\Omega)\to L^2(\Omega)} \le \frac{C}{|z||\operatorname{Im} z|}, \quad \operatorname{Im} z < 0.$$

The cutoff resolvent $\psi(-\Delta_K - z^2)^{-1}\psi$, $\psi \in C_0^\infty(\Omega)$, has a meromorphic continuation in $\mathbb{C}$ for N odd and in $\mathbb{C} \setminus \boldsymbol{i}\mathbb{R}^+$ for N even. This resolvent is called *outgoing*. Setting $z = -\boldsymbol{i}s$, we obtain an outgoing resolvent $(\Delta_K - s^2)^{-1}$ which is a bounded operator in $L^2(\Omega)$ for $\operatorname{Re} s > 0$ and the analytic singularities of $\psi(\Delta_K - s^2)^{-1}\psi$ are included in $\operatorname{Re} s < 0$. Set $\boldsymbol{\Omega}_j = \mathbb{R}^N \setminus K_j$ and suppose that $K \subset \{x \in \mathbb{R}^N : |x| < \rho_0\}$. Since the real parameter $k \ge 1$ is positive, we assume in this and in the following sections that $\operatorname{Im} s < 0$. To treat the case $\operatorname{Im} s > 0$, we must take $k \le -1$ and repeat the argument. For our analysis it is more convenient to consider the outgoing resolvent $\mathcal{R}(s)$ acting on functions $f \in H^2(\Gamma)$ defined for s outside the set of resonances (and also for $s \notin \boldsymbol{i}\mathbb{R}^+$ for N even). More precisely, given $f \in H^2(\Gamma)$ we define $\mathcal{R}(s)f = v(x,s)$, where $v(x,s)$ is the unique outgoing solution of the problem

$$\begin{cases} (\Delta - s^2)v = 0, & x \in \mathring{\boldsymbol{\Omega}}, \\ v|_\Gamma = f. \end{cases}$$

Here outgoing means that

$$v(r\theta) = r^{-(N-1)/2} e^{-sr}(w(\theta) + o(1)) \quad \text{and} \quad \partial_r v + sv = o(1)v \quad \text{as} \quad r \to +\infty,$$

uniformly with respect to $\theta \in S^{N-1}$, with some $w \in C^\infty(\mathbb{S}^{N-1})$. This condition is equivalent to

$$v|_{|x|\ge\rho_1} = (S_0(s)u)\big|_{|x|\ge\rho_1}, \tag{6-1}$$

for some $\rho_1 \gg \rho_0$ and a compactly supported (in a compact set independent of s) function u, where

$$S_0(s) = (\Delta - s^2)^{-1} : L^2_{\text{comp}}(\mathbb{R}^N) \to H^2_{\text{loc}}(\mathbb{R}^N)$$

is the outgoing resolvent of the Laplacian in $\mathbb{R}^N$. If we replace K above by the strictly convex obstacle K_j, we can choose $J \ge 2$ so that the outgoing resolvents

$$\mathcal{R}_j(s) : H^{p+2}(\Gamma_j) \to H^{p+1}(\boldsymbol{\Omega}_j \cap \{|x| \le R\}), \quad p \in \mathbb{N}$$

are analytic [Vainberg 1989; Gérard 1988] for

$$s \in \mathcal{D}_0 = \big\{s \in \mathbb{C} : \sigma_0 \le \operatorname{Re} s \le 1, |\operatorname{Im} s| \ge J\big\},$$

and $w_j = \mathcal{R}_j(s)f$ is outgoing solution of the problem

$$\begin{cases} (\Delta - s^2)w_j = 0, & x \in \boldsymbol{\Omega}_j, \\ w_j\big|_{\Gamma_j} = f. \end{cases}$$

Moreover, for $s \in \mathcal{D}_0$ and $R \ge \rho_0 + 1$ we have the estimate

$$\|\mathcal{R}_j(s)f\|_{H^{p+1}(\boldsymbol{\Omega}_j\cap\{|x|\le R\})} \le C_{R,p}\langle s\rangle^{p+2}\|f\|_{H^{p+2}(\Gamma_j)}, \quad j = 1, \dots, \kappa_0, \tag{6-2}$$

with some constant $C_{R,p} > 0$. This estimate was established for $p = 0$ in [Gérard 1988, Proposition A.II.2]. For completeness we give the argument for $p \geq 1$. Let $\chi \in C_0^\infty(\mathbb{R}^N)$ be a cutoff function such that $\chi(x) = 1$ for $|x| \leq R$ and $\chi(x) = 0$ for $|x| \geq R+1$. Set $w_j = \mathcal{R}_j(s)f$ and observe that

$$\Delta(\chi w_j) = 2\langle \nabla\chi, \nabla w_j\rangle + s^2\chi w_j + \Delta(\chi)w_j = F_j.$$

The function χw_j is a solution of the Dirichlet problem in $\omega_R = (|x| \leq R+1) \cap \Omega_j$ and the standard estimates for boundary problems imply

$$\|\chi w_j\|_{H^2(\omega_R)} \leq C_{R,2}\big(\|F_j\|_{L^2(\omega_R)} + \|f\|_{H^{3/2}(\Gamma_j)}\big).$$

To estimate $\|\chi w_j\|_{L^2(\omega_R)}$, write $w_j = e(f) - (\Delta_{K_j} - s^2)^{-1}(\Delta - s^2)e(f)$, where $e(f)$ is extension operator from $H^2(\Gamma_j)$ to $H^{5/2}_{\mathrm{comp}}(\omega_{R-1})$. This implies $\|\chi w_j\|_{L^2(\omega_R)} \leq B_R\langle s\rangle\|f\|_{H^2(\Gamma_j)}$, since for strictly convex obstacles we have (see for instance [Vainberg 1989, Chapter X])

$$\|\chi(\Delta_{K_j} - s^2)^{-1}\chi\|_{L^2\to L^2} \leq C\langle s\rangle^{-1}.$$

In the same way one estimates $\|\Delta(\chi)w_j\|_{L^2(\omega_R)}$ by using another cutoff, and applying (6-2) for $p = 0$ we obtain this estimate for $p = 1$. The general case can be considered by using an inductive argument. More precise estimates than (6-2) can be obtained following a construction of outgoing parametrix for the Dirichlet problem outside K_j [Gérard 1988, Appendix II].

Finally, notice that for v with $\operatorname{supp} v \subset \{|x| \leq R\}$ we have from [Vainberg 1989] the estimates

$$\|S_0(s)v\|_{H^{p+1}(|x|\leq R)} \leq C_{R,p}\|v\|_{H^p(|x|\leq R)}, \quad p \in \mathbb{N},\ s \in \mathcal{D}_0. \tag{6-3}$$

For our construction we need to introduce some pseudodifferential operators depending on the parameter $s \in \mathcal{D}_0$. For this purpose we will use the notation and the results in [Gérard 1988, A.I and A.II] (see also [Stefanov and Vodev 1995, Appendix]). Given a set $X \subset \mathbb{R}^{N-1}$, we denote by $\tilde{C}^\infty(X)$ the space of the functions $u(x,s)$, $s \in \mathcal{D}_0$, such that $u(\cdot,s) \in C^\infty(X)$ and $p(u(\cdot,s)) = \mathcal{O}(\langle s\rangle^{-\infty})$ for all seminorms p in $C^\infty(X)$. In a similar way we define distributions $\tilde{D}'(X)$. Next, given two open sets $X \subset \mathbb{R}^{N-1}$, $Y \subset \mathbb{R}^{N-1}$, consider the spaces of symbols $a(x,y,\eta,s) \in S^{m,l}_{\rho,\delta}(X \times Y)$ such that for every compact $U \subset X \times Y$, all multiindices α, β, γ and $s \in \mathcal{D}_0$ we have

$$\sup_{(x,y)\in U}\big|\partial_x^\alpha\partial_y^\beta\partial_\eta^\gamma a(x,y,\eta,s)\big| \leq C_{\alpha,\beta,\gamma,U}|s|^{l+\rho|\gamma|+\delta|\alpha+\beta|}(1+|\eta|)^{m-|\gamma|}.$$

Consider the pseudodifferential operator $\mathrm{Op}(a) \in L^{m,l}_{\rho,\delta}(X)$ defined by

$$(\mathrm{Op}(a)u)(x,s) = \Big(\frac{s}{2\pi}\Big)^{N-1}\int e^{-s\langle x-y,\eta\rangle}a(x,y,\eta,s)u(y,s)\,dy\,d\eta,$$

where the support of $a(x,y,\eta,s) \in S^{m,l}_{\rho,\delta}(X \times Y)$ with respect to (y,η) is uniformly bounded for $s \in \mathcal{D}_0$ and $a(x,y,\eta,s)$ is analytic for $s \in \mathcal{D}_0$. The operator $\mathrm{Op}(a)$ maps $\tilde{C}_0^\infty(Y)$ into $\tilde{C}^\infty(X)$. Below we will take $Y = \Gamma_j$ and the symbols $a(x,y,\eta,s)$ will have compact supports with respect to (y,η). Moreover, we will work with symbols in $S^{m,l}_{0,0}$. We say that $\mathrm{Op}(a)$ is properly supported if the kernel $K(x,y,s)$ of $\mathrm{Op}(a)$ is properly supported uniformly with respect to s. Recall that $K(x,y,s)$ is properly supported if both projections from the support of $K(x,y,s)$ to X and Y are proper maps (see [Hörmander 1985a, Definition 18.1.21]). We refer to [Gérard 1988, A.I] for the properties of pseudodifferential operators depending on s. Notice that a properly supported pseudodifferential operator $\mathrm{Op}(a)$ can be defined also by a symbol $a(x, \eta, s)$. A properly supported pseudodifferential operator $\mathrm{Op}(a)$ is called *elliptic* at $(x_0, \eta_0) \in T^*(X)$ if $a(x, \eta, s)$ satisfies the estimate

$$|a(x, \eta, s)| \geq C\langle s\rangle^p, \quad p \geq 0,\ (x, \eta) \in \mathcal{V},\ s \in \mathcal{D}_0,$$

$\mathcal{V}$ being a neighborhood of (x_0, η_0) independent of s.

Next, consider Fourier integral operators with real phase function $\varphi(x, \eta)$ and complex parameter $s \in \mathcal{D}_0$ having the form

$$I(u)(x, s) = \Big(\frac{s}{2\pi}\Big)^{N-1} \int e^{-s(\varphi(x,\eta)-\langle y,\eta\rangle)} a(x, y, \eta, s) u(y, s)\, dy\, d\eta,$$

where as above the support of $a(x, y, \eta, s) \in S^{m,l}_{\rho,\delta}(X \times Y)$ with respect to (y, η) is uniformly bounded for $s \in \mathcal{D}_0$ and $a(x, y, \eta, s)$ is analytic for $s \in \mathcal{D}_0$. For example, the local parametrix constructed in the hyperbolic region defined below is a Fourier integral operator in this form.

To examine the asymptotic behavior with respect to the parameter s we will use the frequency set $\widetilde{WF}(u)$ introduced in [Gérard 1988]; see also [Guillemin and Sternberg 1977; Stefanov and Vodev 1995]. (The notation $\widetilde{WF}(u)$ is used to avoid the confusion with the wave front set $WF(u)$ of a distribution). We recall the definition of $\widetilde{WF}(u)$ only for the so-called *finite points* $(x, \eta) \in T^*(X)$, since this is sufficient for our argument. Let $u(x, s) \in \tilde{\mathcal{D}}'(X)$ be a distribution depending on the parameter s so that for every compact $X' \subset X$ there exists M such that $u(x, s)|_{X'} \in H^{-M}(X')$ and $\|u(\cdot\,, s)\big|_{X'}\|_{H^{-M}} \leq C_M\langle s\rangle^{-M}$. We say that $(x_0, \eta_0) \in T^*(X)$ is not in $\widetilde{WF}(u)$ if there exists $\mathrm{Op}(a) \in L^{0,0}_{\rho,\delta}(X)$, $\rho + \delta < 1$, properly supported and elliptic at (x_0, η_0) such that for every compact $U \subset X$ we have

$$\big\|(\mathrm{Op}(a)u)(x, s)\big\|_{C^j(U)} \leq C_{U,M,j}\langle s\rangle^{-M} \quad \text{for all } j \in \mathbb{N},\ M \in \mathbb{N},\ s \in \mathcal{D}_0.$$

If $\mathcal{U}$ is a neighborhood of K and if the distribution kernel $Q(x, y, s)$ of an operator

$$\mathfrak{Q}(s) : C^\infty(\Gamma) \to C^\infty(\mathcal{U} \setminus K)$$

belongs to $\tilde{C}^\infty(\mathcal{U} \setminus K \times \Gamma)$, we will say briefly that $\mathfrak{Q}(s)u$ is a *negligible term*. The terms having behavior $\mathcal{O}(\langle s\rangle^{-M})$ with large M will also be called negligible. It is important to note that a series of negligible terms in general is not negligible, and one needs to have uniform estimates with respect to s of the terms of the series to conclude that such a series is negligible.

6.1. *Construction of the operators P_h, P_g, P_e.* In the analysis below we fix $j \in \{1, \dots, \kappa_0\}$. Consider the *hyperbolic*, *glancing* and *elliptic sets* on $T^*(\Gamma_j)$ defined respectively by

$$\mathcal{H} = \{(y, \eta) \in T^*(\Gamma_j) : |\eta| < 1\}, \quad \mathcal{G} = \{(y, \eta) \in T^*(\Gamma_j) : |\eta| = 1\}, \quad \mathcal{E} = \{(y, \eta) \in T^*(\Gamma_j) : |\eta| > 1\},$$

where (y, η) are local coordinates in $T^*(\Gamma_j)$. Let $\chi_0 \in C_0^\infty(T^*(\Gamma_j))$ be a function such that $0 \leq \chi_0 \leq 1$ and $\chi_0(y, \eta) = 0$ in a small neighborhood G_0 of $\mathcal{G} \cup \mathcal{E}$, while $\chi_0(y, \eta) = 1$ for

$$(y, \theta) \in G_1,\ G_1 \subset T^*(\Gamma_j) \setminus G_0 \subset \mathcal{H}.$$

Choosing a finite covering of Γ_j, we may suppose that in local coordinates (y, η) we have $\chi_0(y, \eta) = 1$ for $y \in \Gamma_j$, $|\eta| \leq 1 - \delta_1$, where $\sqrt{1 - \delta_0^2} < 1 - \delta_1 < 1$ and $\delta_0 \in (0, 1)$ is a global constant chosen as in

Lemma 2.1. Thus if a ray $\gamma_{\rm in}$ issued from $\bigcup_{l\neq j} K_l$ meets Γ_j at $y \in \Gamma_j$ with direction $\xi \in \mathbb{S}^{N-1}$ so that $\chi_0(y, \xi|_{T_y(\Gamma_j)}) \neq 1$, then the reflected or diffractive outgoing ray $\gamma_{\rm out}$ issued from $(y, \xi - 2\langle \xi, \nu(y)\rangle \nu(y))$ does not meet a neighborhood of $\bigcup_{\nu\neq j} K_\nu$ depending only on δ_0.

Consider a finite partition of unity of the set $\operatorname{supp}(\chi_0) \subset \mathcal{H}$ and, as in [Gérard 1988], a finite partition of unity of psedodifferential operators to localize the construction. Let $(y_0, \eta_0) \in \operatorname{supp}(\chi_0) \subset \mathcal{H}$ and let $\chi(y, \eta) \in C_0^\infty(T^*(\Gamma_j))$, $0 \leq \chi(y, \eta) \leq 1$, be a function such that $\chi = 1$ in a neighborhood of (y_0, η_0). Let $\tilde{\mathcal{U}}_j$ be a small neighborhood of K_j and let $\mathcal{U}_j = \tilde{\mathcal{U}}_j \setminus K_j$. Let $\Gamma_\chi \subset \Gamma_j$ be the projection of supp $\chi(x, \eta)$ on Γ_j.

We will omit again the dependence on k in the notation if the context is clear. Given boundary data $u(y, s)$, in the hyperbolic region we construct an outgoing parametrix $H_{h,\chi} : \tilde{C}^\infty(\Gamma_\chi) \to \tilde{C}^\infty(\mathcal{U}_j)$ of the form

$$(H_{h,\chi}u)(x, s) = \Big(\frac{s}{2\pi}\Big)^{N-1} \int e^{-s(\psi(x,\eta) - \langle y, \eta\rangle)} \sum_{\nu=0}^{M} a_\nu(x, y, \eta) s^{-\nu} u(y, s)\, dy\, d\eta.$$

We have

$$\begin{cases} (\Delta_x - s^2)(H_{h,\chi}u)(x, s) = s^{-M} A_M(s)u, & x \in \mathcal{U}_j, \\ (H_{h,\chi}u)(x, s)\big|_{\Gamma_j} = \operatorname{Op}(\chi)u, \end{cases}$$

where

$$A_M(s)u = \Big(\frac{s}{2\pi}\Big)^{N-1} \int e^{-s(\psi(x,\eta) - \langle y, \eta\rangle)} (\Delta_x - s^2)\big(a_M(x, y, \eta)\big) u(y, s)\, dy\, d\eta.$$

The construction of $H_{h,\chi}$ is given in [Gérard 1988, A.II.2]. Here the phase $\psi(x, \eta)$ satisfies the equation

$$|\nabla_x \psi|^2 = 1, \quad \psi|_{\Gamma_j} = \langle x, \eta\rangle, \quad (x, \eta) \text{ close to } (y_0, \eta_0).$$

The amplitudes $a_\nu(x, y, \eta)$ are determined from the transport equations with initial data

$$a_0|_{x\in\Gamma_j} = \chi(y, \eta), a_\nu|_{x\in\Gamma_j} = 0, \quad \nu \geq 1.$$

Notice that a_ν depend only on $\chi(y, \eta)$ and the integration in $H_{h,\chi}u$ is over a compact domain with respect to y and η, so for $s \in \mathcal{D}_0$ the integral is well defined. Applying a finite partition of unity, we construct an outgoing parametrix $H_h : \tilde{C}^\infty(\Gamma_j) \to \tilde{C}^\infty(\mathcal{U}_j)$ such that

$$\begin{cases} (\Delta_x - s^2)(H_h u)(x, s) = s^{-M} B_M(s)u, & x \in \mathcal{U}_j, \\ (H_h u)(x, s)\big|_{\Gamma_j} = \operatorname{Op}(\chi_0)u, \end{cases}$$

where the operator $B_M(s)$ is analytic with respect to s and satisfies the estimates

$$\|B_M(s)u\|_{H^p(\mathcal{U}_j)} \leq C_p |s|^{p+2} \|u\|_{0,\Gamma_j} \quad \text{for all } p \in \mathbb{N},$$

with some global constants. Let $\Psi(x) \in C_0^\infty(\mathcal{U}_j)$ be a cutoff function such that $\Psi(x) = 1$ in a small neighborhood of K_j. Then we obtain

$$(\Delta_x - s^2)[\Psi H_h u] = s^{-M} \Psi B_M(s)u + [\Delta, \Psi] H_h u, \quad x \in \mathcal{U}_j,$$

and we define the outgoing parametrix

$$(P_h u)(x, s) = \Psi H_h u - S_0(s)(s^{-M} \Psi B_M(s)u + [\Delta, \Psi] H_h u), \quad x \in \Omega_j.$$

Thus we get

$$\begin{cases}(\Delta_x - s^2)(P_h u)(x,s) = 0 & \text{for } x \in \Omega_j,\ s \in \mathscr{D}_0,\\ (P_h u)(\,\cdot\,,s) \in L^2(\Omega_j) & \text{if } \operatorname{Re} s > 0,\\ (P_h u)(x,s)\big|_{\Gamma_j} = \operatorname{Op}(\chi_0)u + \mathscr{Q}_h(s)u, & \end{cases}$$

where for large M we obtain a negligible operator $\mathscr{Q}_h(s)$ coming from the trace of the action of $S_0(s)$. Here we use the fact that the frequency set of $S_0(s)w$ is given by the outgoing rays issued from $\widetilde{WF}(w)$ and the outgoing rays issued from $[\Delta,\Psi]H_h u$ do not meet Γ_j. Notice that the operator P_h depends analytically on s.

Let $\chi_1(x,\eta) + \chi_2(x,\eta) = 1 - \chi_0(x,\eta)$, where, for $\varepsilon_0 > 0$ small enough, $\chi_1(x,\eta) \in C_0^\infty(T^*(\Gamma_j))$ is a function with support in

$$\{(x,\eta) : 1-\delta_1 \le 1-2\varepsilon_0 \le |\eta| \le 1+2\varepsilon_0\},$$

while $\chi_2(x,\eta) \in C^\infty(T^*(\Gamma_j))$ has support in

$$\{(x,\eta) : |\eta| \ge 1+\varepsilon_0\}.$$

In the glancing region following the construction in [Gérard 1988, A.II.3] and in [Stefanov and Vodev 1995, A.3]), we construct an outgoing parametrix H_g such that

$$\begin{cases}(\Delta_x - s^2)(H_g u) = s^{-M} B_g(s)u & \text{for } x \in \mathscr{U}_j,\\ (H_g u)(\,\cdot\,,s) \in L^2(\Omega_j) & \text{if } \operatorname{Re} s > 0,\\ H_g u|_{\Gamma_j} = \operatorname{Op}(\chi_1)u + s^{-M} B'_g(s)u, & \end{cases}$$

where $B_g(s)$ and $B'_g(s)$ are Fourier–Airy operators with complex parameter. The only difference with the construction in [Gérard 1988] is that we have $s^{-M}B_g(s)$ and $s^{-M}B'_g(s)$ instead of operators with kernel in $\tilde{C}^\infty(\mathscr{U}_j \times \Gamma_j)$ and $\tilde{C}^\infty(\Gamma_j \times \Gamma_j)$, respectively. For this purpose, as in the hyperbolic case, we use a finite sum of amplitudes instead of an asymptotic infinite sum of symbols. The advantage is that our parametrix H_g, as well as $B_g(s)$ and $B'_g(s)$, depend analytically on s. Now define

$$(P_g u)(x,s) = \Psi H_g u - S_0(s)\big(s^{-M}\Psi B_g(s)u + [\Delta,\Psi]H_g u\big), \quad x \in \Omega_j.$$

In the elliptic region the construction of a parametrix in [Gérard 1988, A.II.4] is given by a Fourier integral operator with big parameter λ and complex phase function. When λ is complex, there are some difficulties to justify this construction [Stefanov and Vodev 1995, A.4]. For this reason in the elliptic region we introduce $P_e u = \mathscr{R}_j(s)(\operatorname{Op}(\chi_2)u)$ keeping the analytic dependence on s. Thus, setting $\mathscr{S}_j(s) = P_h + P_g + P_e$, we have

$$\begin{cases}(\Delta_x - s^2)(\mathscr{S}_j(s)u)(x,s) = 0 & \text{for } x \in \Omega_j,\ s \in \mathscr{D}_0,\\ (\mathscr{S}_j(s)u)(\,\cdot\,,s) \in L^2(\Omega_j), & \text{if } \operatorname{Re} s > 0,\\ (\mathscr{S}_j(s)u)(x,s)\big|_{\Gamma_j} = u + \mathscr{Q}_j(s)u, & \end{cases}$$

where for large M the operator $\mathscr{Q}_j(s)$ is negligible.

Our strategy is to apply the construction above to the function

$$w_{0,j}(x,s) = \sum_{n=n_j}^{\infty} U_{n+2,j}(x,s)\big|_{\Gamma_j},$$

where

$$U_{n+2,j}(x,s) = \sum_{\substack{|\boldsymbol{j}|=n+3\\ j_{n+2}=j}} u_{\boldsymbol{j}}(x,s),$$

the $u_{\boldsymbol{j}}(x,s)$ being defined in Section 3 starting with initial data $e^{-s\varphi}b_1(x,s;\cdot)$. Recall that in the previous section we obtained estimates for the $C^p(\Gamma_j)$ norms of $U_{n+2,j}(x,s)$ for $s \in \mathcal{D}_0$. Thus applying P_h, P_g and P_e to $w_{0,j}(x,s)$ we obtain convergent series. Consequently, the function $(\mathcal{S}_j(s)w_{0,j})(x,s)$ is analytic for $s \in \mathcal{D}_0$ with values in $C^\infty(\overline{\Omega_j})$ and here we use the fact that $w_{0,j}(x,s) \in C^\infty(\Gamma_j)$. It is convenient to introduce the following.

Definition 6.1. Let $\omega \subset \mathbb{R}^N$ be an open set and let $\mathcal{D}$ be a domain in $\mathbb{C}$. We say that the function $U(x,s;k)$ *satisfies condition* (S) in $(\omega,\mathcal{D})$ if

(i) for $k \geq 1$, $U(\cdot,s;k)$ is a $C^\infty(\overline{\omega})$-valued analytic function in $\mathcal{D}$,

(ii) $U(\cdot,s;k) \in L^2(\omega)$ for $\operatorname{Re} s > 0$, and

(iii) $(\Delta_x - s^2)U(x,s;k) = 0$ in ω for every $s \in \mathcal{D}$.

It is clear that $(\mathcal{S}_j(s)(s)w_{0,j})(x,s)$ satisfies condition (S) in $(\Omega_j,\mathcal{D}_0)$. Taking the sum over $j = 1,\dots,\kappa_0$, we conclude that the function

$$V^{(0)}(x,s) = \sum_{j=1}^{\kappa_0} (\mathcal{S}_j(s)w_{0,j})(x,s)$$

satisfies condition (S) in $(\mathring{\Omega},\mathcal{D}_0)$.

6.2. ***Traces of*** $\mathcal{S}_j(s)w_{0,j}$ ***on*** Γ_l. The analysis of the traces $(\mathcal{S}_j(s)w_{0,j})(x,s)\big|_{\Gamma_l}$, $l \neq j$, is more difficult. The main contributions come from $(P_h w_{0,j})\big|_{\Gamma_l}$, where $l \neq j$. Our goal is to find the leading term of $P_h(U_{n+2,j}(x,s)\big|_{\Gamma_j})\big|_{\Gamma_l}$, $l \neq j$. Let $\boldsymbol{j}$ be a configuration such that $|\boldsymbol{j}| = n+3$, $j_{n+2} = j$ and let $e^{-s\varphi_{\boldsymbol{j}}(x)}a_{\boldsymbol{j}}(x,s)$ be a term in $U_{n+2,j}(x,s)$. For $x \in \Gamma_j$ consider

$$\begin{aligned}\operatorname{Op}(\chi_0)(e^{-s\varphi_{\boldsymbol{j}}(x)}a_{\boldsymbol{j}}(x,s)|_{\Gamma_j}) &= \int e^{-s(\langle x-y,\eta\rangle+\varphi_{\boldsymbol{j}}(y))}\chi_0(y,\eta)a_{\boldsymbol{j}}(y,s)\,dy\,d\eta\\ &= \sum_{\mu=1}^{T}\int e^{-s(\langle x-y,\eta\rangle+\varphi_{\boldsymbol{j}}(y))}\chi_0(y,\eta)a_{\boldsymbol{j}}(y,s)\beta_\mu(y,\eta)\,dy\,d\eta = \sum_{\mu=1}^{T} I_\mu(x,s),\end{aligned}$$

where the $\beta_\mu \in C_0^\infty(T^*(\Gamma_j))$ are cutoff functions such that $\sum_{\mu=1}^{T}\beta_\mu(y,\eta) = 1$ for $(y,\eta) \in \operatorname{supp}\chi_0(y,\eta)$.

For $I_\mu(x,s)$ we will apply the stationary phase argument with big complex parameter $s \in \mathcal{D}_0$; see, for instance, [Gérard 1988, Lemma 2.3]. The critical points of $I_\mu(x,s)$ satisfy the equations $x = y$, $\eta = \nabla_y\varphi(y)$, and the matrix

$$G_{\boldsymbol{j}}(y) = \begin{pmatrix}\varphi_{\boldsymbol{j},y,y} & -I\\ -I & 0\end{pmatrix}$$

is invertible with

$$(G_{\boldsymbol{j}}(y))^{-1} = \begin{pmatrix}0 & -I\\ -I & -\varphi_{\boldsymbol{j},y,y}\end{pmatrix}.$$

An application of the stationary phase argument yields

$$\begin{aligned}\operatorname{Op}(\chi_0)(e^{-s\varphi_{\boldsymbol{j}}(x)}a_{\boldsymbol{j}}(x,s)|_{\Gamma_j})\\ =e^{-s\varphi_{\boldsymbol{j}}(x)}\Big[\chi_0(x,\nabla_y\varphi_{\boldsymbol{j}}(x))a_{\boldsymbol{j}}(x,s)+\sum_{q=1}^{M-1}L_{q,\boldsymbol{j}}(y,D_y,D_\eta)(\chi_0 a_{\boldsymbol{j}})(x,\nabla_y\varphi_{\boldsymbol{j}}(x))s^{-q}+A_{M,\boldsymbol{j}}(x,s)s^{-M}\Big],\\ x\in\Gamma_j.\quad (6\text{-}4)\end{aligned}$$

Here $L_{q,\boldsymbol{j}}(y,D_y,D_\eta)$ are operators of order $2q$ and the form of $(G_{\boldsymbol{j}}(y))^{-1}$ shows that all terms in $L_{q,\boldsymbol{j}}$ contain derivatives with respect to one of the variables η_i, $i=1,\dots,N-1$. Thus, the terms in (6-4) with coefficients s^{-q}, for $1\le q\le M-1$, vanish if $|\nabla_y\varphi_{\boldsymbol{j}}(x)|\le 1-\delta_1$.

For $s\in\mathcal{D}_0$ we have

$$P_h\Big[\sum_{\substack{j=1\\ j\neq l}}^{\kappa_0}U_{n+2,j}\big|_{\Gamma_j}\Big]=\mathcal{R}_j(s)\Big[\big(\operatorname{Op}(\chi_0)+\mathcal{Q}_h(s)\big)\Big(\sum_{\substack{j=1\\ j\neq l}}^{\kappa_0}U_{n+2,j}\big|_{\Gamma_j}\Big)\Big],$$

and for large M, the operator $\mathcal{Q}_{h,j,l}u=(\mathcal{R}_j(s)\mathcal{Q}_h(s)u)\big|_{\Gamma_l}$, $j\neq l$, is negligible.

The leading contribution in the traces on Γ_l comes from the trace of the terms

$$\mathcal{R}_j(s)\big(e^{-s\varphi_j(x)}\chi_0(x,\nabla_y\varphi_{\boldsymbol{j}}(x))a_{\boldsymbol{j}}(x,s)\big|_{\Gamma_j}\big),$$

that is from the action of $\mathcal{R}_j(s)$ on the leading term in (6-4). To examine this contribution we construct, as [Ikawa 1988, Section 4], an asymptotic outgoing *global* solution

$$v_{\boldsymbol{j},M}(x,s)=e^{-s\psi_{\boldsymbol{j}}(x)}\sum_{\mu=1}^{M}c_{\boldsymbol{j},\mu}(x,s)s^{-\mu}$$

of the problem

$$\begin{cases}(\Delta_x-s^2)v_{\boldsymbol{j},M}(x,s)=s^{-M}r_{\boldsymbol{j},M}(x,s) & \text{for } x\in\Omega_j,\\ v_{\boldsymbol{j},M}(x,s)\big|_{\Gamma_j}=e^{-s\varphi_j(x)}\chi_0(x,\nabla_y\varphi_{\boldsymbol{j}}(x))a_{\boldsymbol{j}}(x,s)\big|_{\Gamma_j}.\end{cases}$$

We have $\psi_{\boldsymbol{j}}(x)=\varphi_{\boldsymbol{j}}(x)$ on Γ_j and the phase $\psi_{\boldsymbol{j}}(x)$ is defined following the procedure in Section 2. Moreover, $\psi_{\boldsymbol{j}}(x)$ satisfies condition $(\mathcal{P})$ on Γ_j. Next, the amplitudes $c_{\boldsymbol{j},\mu}(x,s)$ are determined globally by the transport equations. It is easy to see that

$$c_{\boldsymbol{j},0}(x,s)\big|_{\Gamma_l}=-a_{(\boldsymbol{j},l)}(x,s)\big|_{\Gamma_l},\quad l\neq j,$$

where $(\boldsymbol{j},l)$ is the configuration $(j_0,j_1,\dots,j_{n+2}=j,l)$. This follows from the definition of $a_{(\boldsymbol{j},l)}(x,s)$ in Section 3 and from the transport equation for the leading term $c_{\boldsymbol{j},0}$ [Ikawa 1988, Section 4] combined with the fact that if $c_{\boldsymbol{j},0}(x,s)|_{\Gamma_l}\neq 0$, then x must lie on a ray issued from $(y,\nabla_y\varphi_{\boldsymbol{j}}(y))$ with $\chi_0(y,\nabla_y\varphi_{\boldsymbol{j}}(y))=1$. The minus appears since for the configurations $(\boldsymbol{j},l)$ we have to include the factor $(-1)^{n+4}$. Next, choose a function $\Phi\in C_0^\infty(|x|\le\rho_0+1)$ equal to 1 in a neighborhood of K and introduce

$$V_{\boldsymbol{j},M}(x,s)=\Phi v_{\boldsymbol{j},M}(x,s)-S_0(s)\big(s^{-M}r_{\boldsymbol{j},M}(x,s)+[\Delta,\Phi]v_{\boldsymbol{j},M}(x,s)\big).$$

We have $(\Delta_x-s^2)V_{\boldsymbol{j},M}(x,s)=0$ in Ω_j and for M large the traces

$$V_{\boldsymbol{j},M}(x,s)\big|_{\Gamma_l}-\mathcal{R}_j(s)\big(e^{-s\varphi_j(x)}\chi_0(x,\nabla_y\varphi_{\boldsymbol{j}}(x))a_{\boldsymbol{j}}(x,s)\big|_{\Gamma_j}\big)\big|_{\Gamma_l},\quad l=1,\dots,\kappa_0$$

are negligible terms coming from the action of $S_0(s)$. We obtain this first for the trace on Γ_j and then use the estimates for the resolvent $\mathcal{R}_j(s)$. On the other hand, for large M we get $V_{\boldsymbol{j},M}(x,s)\big|_{\Gamma_l} = v_{\boldsymbol{j},M}(x,s)\big|_{\Gamma_l}$ modulo negligible terms related to the action of $S_0(s)$. Thus the leading term of the trace on Γ_l is $e^{-s\varphi_{\boldsymbol{j}}(x)} c_{\boldsymbol{j},0}(x,s)\big|_{\Gamma_l}$.

Next, consider $e^{-s\varphi_{\boldsymbol{j}}(x)} b_{\boldsymbol{j}}(x,s)|_{\Gamma_j}$ with $b_{\boldsymbol{j}}(x,s)|_{\Gamma_j} = 0$ for $|\nabla_y \varphi_{\boldsymbol{j}}(x)| \le 1-\delta_1$. Moreover, assume that if $b_{\boldsymbol{j}}(x,s) \neq 0$ for $x \in \Gamma_j$, then x is lying on a segment issued from some obstacle K_l, with $l \neq j$. From (6-4) we see that the terms with coefficients s^{-q}, $1 \le q \le M-1$, have these properties. According to [Gérard 1988, Theorem A.II.12], the frequency set of $\mathcal{R}_j(s)(e^{-s\varphi_{\boldsymbol{j}}(x)} b_{\boldsymbol{j}}(x,s)|_{\Gamma_j})$ is included in the set determined by the outgoing rays issued from $\widetilde{WF}(e^{-s\varphi_{\boldsymbol{j}}(x)} b_{\boldsymbol{j}}(x,s)|_{\Gamma_j})$. According to Lemma 2.1, our choice of δ_1 shows that these rays do not meet a neighborhood of $\bigcup_{l\neq j} K_l$. Consequently, the traces of $\mathcal{R}_j(s)(e^{-s\varphi_{\boldsymbol{j}}(x)} b_{\boldsymbol{j}}(x,s)|_{\Gamma_j})$ on Γ_l, $l \neq j$, are negligible. It is clear also that all terms with factors s^{-q} will produce traces with this factor.

For fixed n and fixed j, with $l \neq j$, we take the finite sum over the configurations $|\boldsymbol{j}| = n+3$ of all terms having coefficient s^{-q}, $1 \le q \le M$, in the trace $\mathcal{R}_j(s)(\mathrm{Op}(\chi_0)U_{n+2,j}|_{\Gamma_j})\big|_{\Gamma_l}$ and we denote this sum by $s^{-1}R_{h,n,j,l}(x,s)$. Since we cannot estimate directly the series with the contributions s^{-q}, we are going to include in $s^{-1}R_{h,n,j,l}(x,s)$ all terms mentioned above as negligible and appearing with coefficients s^{-q}, $1 \le q \le M$.

Thus for fixed n, summing over $j = 1, \dots, \kappa_0$ with $j \neq l$ and $\boldsymbol{j}$, we obtain all configurations $\boldsymbol{j}$ with $|\boldsymbol{j}| = n+4$, $j_{n+3} = l$ and we conclude that

$$\Big(P_h \sum_{\substack{j=1\\ j\neq l}}^{\kappa_0} U_{n+2,j}\big|_{\Gamma_j}\Big)\Big|_{\Gamma_l} = -\sum_{\substack{|\boldsymbol{j}|=n+4\\ j_{n+3}=l}} e^{-s\varphi_{\boldsymbol{j}}(x)} a_{\boldsymbol{j}}(x,s)\big|_{\Gamma_l} + s^{-1}R_{h,n,j,l}(x,s) + \mathcal{Q}_{h,j,l}\Big(\sum_{\substack{j=1\\ j\neq l}}^{\kappa_0} U_{n+2,j}\big|_{\Gamma_j}\Big). \tag{6-5}$$

To treat $(P_g w_{0,j})|_{\Gamma_l}$, $l \neq j$, we apply the same argument. According to the results in [Gérard 1988, Appendix II], the frequency set of $\mathcal{R}_j(s)\big(\mathrm{Op}(\chi_1)U_{n+2,j}(x,s)\big|_{\Gamma_j}\big)$ is related to the outgoing rays issued from the frequency set of

$$\mathrm{Op}(\chi_1)\Big(\sum_{\substack{|\boldsymbol{j}|=n+3\\ j_{n+2}=j}} e^{-s\varphi_{\boldsymbol{j}}(x)} a_{\boldsymbol{j}}(y,s)\big|_{\Gamma_j}\Big).$$

For every $\boldsymbol{j}$ the frequency set of $\mathrm{Op}(\chi_1)\big(e^{-s\varphi_{\boldsymbol{j}}(y)} a_{\boldsymbol{j}}(y,\cdot)\big|_{\Gamma_j}\big)$ is given by $(y, \nabla_y \varphi_{\boldsymbol{j}}(y))$ such that

$$y \in \operatorname{supp} a_{\boldsymbol{j}}(y,\cdot)\big|_{\Gamma_j}, \quad |\nabla_y \varphi_{\boldsymbol{j}}(y)| \ge 1-\delta_1.$$

If $y \in \Gamma_j$ has this property and $a_{\boldsymbol{j}}(y,\cdot)|_{\Gamma_j} \neq 0$ for some configuration $\boldsymbol{j}$, then y is lying on a segment issued from some Γ_μ, $\mu \neq j$. Our choice of δ_1 guarantees that the outgoing rays mentioned above pass outside a neighborhood of $\bigcup_{l\neq j} K_j$. Thus, we deduce

$$\Big(P_g \sum_{\substack{j=1\\ j\neq l}}^{\kappa_0} U_{n+2,j}\big|_{\Gamma_j}\Big)\Big|_{\Gamma_l} = s^{-M} R_{g,n,j,l}(x,s). \tag{6-6}$$

Here the series $\sum_{n=0}^{\infty} R_{g,n,j,l}$ is convergent but we cannot show that $s^{-M}\sum_{n=0}^{\infty} R_{g,n,j,l}$ is negligible. In fact, the results of Theorem 3.2 cannot be applied to this series and for this reason we take $M = 1$

in (6-6) and consider $R_{g,n,j,l}$ together with the terms $R_{h,n,j,l}$. A similar analysis can be applied to $\mathcal{R}_j(s)(\mathrm{Op}(\chi_2)U_{n+2,j}|_{\Gamma_j})\big|_{\Gamma_l}$ since there are no outgoing rays issued from the elliptic region, and we get

$$\big(\mathcal{R}_j(s)(\mathrm{Op}(\chi_2)U_{n+2,j})\big|_{\Gamma_j}\big)|_{\Gamma_l} = \mathcal{Q}_{e,j,l}(U_{n+2,j}\big|_{\Gamma_j}),$$

where the operator $\mathcal{Q}_{e,j,l}$ has kernel in $\tilde{C}^\infty(\Gamma_l \times \Gamma_j)$.

Summing over n and $j = 1, \dots, \kappa_0$, we conclude that for $x \in \Gamma$ we have

$$V^{(0)}(x, s; k) = m(x, s; k) + s^{-1}R_1(x, s; k) + s^{-M}\mathcal{Q}_{M,0}(x, s; k), \tag{6-7}$$

where the notation makes explicit the dependence on k. The cancellation of the leading terms follows from the equality

$$\big(a_{(j,l)}(x, s) + a_j(x, s)\big)\big|_{x\in\Gamma_l} = 0, \quad l \neq j,$$

and the representation (6-5). The negligible terms coming from the action of $\mathcal{Q}_{h,j,l}, \mathcal{Q}_{e,j.l},\ j, l = 1, \dots, \kappa_0$ to $w_{0,j}$ are included in $s^{-M}\mathcal{Q}_{M,0}(x, s; k)$, while $R_1(x, s; k)$ is the sum over n, j and l of the contributions $R_{h,n,j,l}(x, s; k)$ and $R_{g,n,j,l}(x, s; k)$ coming from (6-6), with $M = 1$. Applying the estimates for $U_{n+2,j}|_{\Gamma_j}$ and the analyticity of P_h, P_g and P_e, we deduce that $Q_{M,0}(x, s; k)$ and $V^{(0)}(x, s; k)|_\Gamma$ are analytic for $s \in \mathcal{D}_0$. Thus we conclude that $R_1(x, s; k)$ is analytic for $s \in \mathcal{D}_0$. We can prove directly that $R_1(x, s; k)$ is analytic examining the series

$$\sum_{n=n_j}^{\infty} P_{h,n,j,l}(x, s; k), \qquad \sum_{n=n_j}^{\infty} P_{g,n,j,l}(x, s; k).$$

In fact, it suffices to obtain estimates $|P_{h,n,j,l}| \leq B_{h,j,l}\tilde{\theta}^n$ for all $n \in \mathbb{N}$, and we treat this question in the next subsection. Thus the analyticity of $R_1(x, s; k)$ is not related to the analyticity of $V^{(0)}$ and Q_M and we may work with a parametrix P_e which is not analytic in s (see [Stefanov and Vodev 1995, A.4] and Section 8). This could simplify a little bit our argument, but we arrange $V^{(0)}$ to be analytic in order to have similarity with the construction in [Ikawa 1988]. On the other hand, to obtain estimates for the outgoing resolvent better than (6-2) we must use an approximation by a parametrix.

6.3. *Estimates of $R_1(x, s; k)$.* To estimate $R_1(x, s; k)$ we need to estimate $R_{h,n,j,l}$ and $R_{g,n,j,l}$. To deal with $R_{h,n,j,l}$, we use the equality (6-5). Notice that the trace

$$\Big(P_h \sum_{\substack{j=1\\ j\neq l}}^{\kappa_0} U_{n+2,j}\big|_{\Gamma_j}\Big)\big|_{\Gamma_l}$$

is given by the trace on Γ_l of

$$S_0(s)\Big(\big(s^{-M}B_M(s) + [\Delta, \Psi]H_h\big)\sum_{\substack{j=1\\ j\neq l}}^{\kappa_0} U_{n+2,j}\big|_{\Gamma_j}\Big).$$

The term involving s^{-M} is easy to handle, and we treat the term with $[\Delta, \Psi]$. Applying the estimates (5-7) with $p = 0$ and applying the L^2 estimates for the action of the Fourier integral operator H_h, we get

$$\Big\|[\Delta, \Psi]H_h\Big(\sum_{\substack{j=1\\ j\neq l}}^{\kappa_0} U_{n+2,j}\Big)\big|_{\Gamma_j}\Big\|_0 \leq C_{j,l}|s|^{2+\beta_0}\langle s + \boldsymbol{i}\,k\rangle\tilde{\theta}^n,$$

where β_0 and $0 < \tilde{\theta} < 1$ were introduced in Section 5 and $\langle s + \boldsymbol{i}\, k\rangle$ comes from (5-7). Next for $g \in C^0(\mathbb{R}^N)$ with compact support we write $S_0(s)g = E_s * g$, where $E_s(x)$ is the kernel of $S_0(s)$. This kernel has the form

$$E_s(x) = \frac{\boldsymbol{i}}{4}\Big(\frac{s}{2\pi|x|}\Big)^{\gamma} H_{\gamma}^{(1)}(s|x|), \quad \gamma = \frac{N-2}{2},$$

where $H_{\gamma}^{(1)}(z)$ is the Hankel function of first type. Since $\Gamma_l \cap \operatorname{supp}\Psi = \varnothing$, we can estimate the C^p norms of $(S_0(s)[\Delta, \Psi]w)\big|_{\Gamma_l}$ exploiting the estimates for the derivatives of $H_{\gamma}^{(1)}(z)$. Thus, setting $\beta_N = (N-3)/2 + \beta_0$, we deduce

$$\Big\| S_0(s)[\Delta, \Psi] H_h \sum_{\substack{j=1\\ j\neq l}}^{\kappa_0} U_{n+2,j}\big|_{\Gamma_j} \Big\|_{\Gamma_l, p} \le B_{j,l,p} \langle s + \boldsymbol{i}\, k\rangle |s|^{2+p+\beta_N} \tilde{\theta}^n. \tag{6-8}$$

Next, for the sum

$$\sum_{\substack{|\boldsymbol{j}|=n+4\\ j_{n+3}=l}} e^{-s\varphi_{\boldsymbol{j}}(x)} a_{\boldsymbol{j}}(x, s)\big|_{\Gamma_l}$$

in (6-5) we apply Theorem 3.2(b). Consequently, summing over n, we obtain estimates for

$$s^{-1} \sum_{n=n_j}^{\infty} P_{h,n,j,l},$$

with the same order as in (6-8).

The analysis of $R_{g,n,j,l}$ is very similar. To estimate

$$[\Delta, \Psi] H_g \Big(\sum_{\substack{j=1\\ j\neq l}}^{\kappa_0} U_{n+2,j}\big|_{\Gamma_j} \Big),$$

we observe that outside a small neighborhood of K_j the parametrix H_g in the glancing domain can be written as a Fourier integral operator with real phase and we may estimate $\big(S_0(s)[\Delta, \Psi] H_g w\big)\big|_{\Gamma_l}$ as in the hyperbolic case discussed above. For the remainder $\mathcal{Q}_{0,M}(x, s; k)$ we have

$$\|\mathcal{Q}_{M,0}(x, s; k)\|_{\Gamma, p} \le D_p \langle s + \boldsymbol{i}\, k\rangle^{p+2} |s|^{p+2+\beta_0}, \quad p \in \mathbb{N}, \tag{6-9}$$

where $\langle s + \boldsymbol{i}\, k\rangle^{p+2}$ comes form the estimates of the amplitude $b_1(x, s; k)$. Finally, we get the following crude estimates

$$\|R_1(x, s; k)\|_{\Gamma, p} \le C_p \langle s + \boldsymbol{i}\, k\rangle^{p+2} |s|^{p+3+\beta_N}, \quad s \in \mathcal{D}_0,\ p \in \mathbb{N} \tag{6-10}$$

and the term $s^{-1}\|R_1(x, s; k)\|_{\Gamma,0}$ has no order $\mathcal{O}(|s|^{-1})$ for all $s \in \mathcal{D}_0$.

It is important to note that in the domain of absolute convergence $\operatorname{Re} s > s_0 + d > s_0$ we have better estimates for $R_1(x, s; k)$. First, in this domain, for all γ and $|x| \le R$ the series

$$D_x^{\gamma}\Big(\sum_{n=1}^{\infty} \sum_{|\boldsymbol{j}|=n} e^{-s\varphi_{\boldsymbol{j}}(x)} a_{\boldsymbol{j}}(x, s) \Big) \tag{6-11}$$

are absolutely convergent [Ikawa 1988]. Next Proposition 2.6 shows that the phases $\varphi_{\boldsymbol{j}}(x)$ and their derivatives are uniformly bounded with respect to $\boldsymbol{j}$ and by recurrence we obtain the absolute convergence of the series

$$\sum_{n=1}^{\infty} \sum_{|\boldsymbol{j}|=n} e^{-s\varphi_{\boldsymbol{j}}(x)} L_{q,\boldsymbol{j}}(x, D_x) a_{\boldsymbol{j}}(x, s),$$

$L_{q,\boldsymbol{j}}(x, D_x)$ being partial differential operators of order q independent of $\boldsymbol{j}$ and n with coefficients uniformly bounded with respect to $\boldsymbol{j}$. Now in the equality (6-4) we can sum over the configurations $\boldsymbol{j}$ and after the action of $\mathcal{R}_j(s)$ the sum of all terms with coefficients s^{-q}, $1 \le q \le M-1$, and the remainder yield contributions which can be included in $\mathcal{Q}_{M,0}$. To deal with the traces of

$$\sum_{n=0}^{\infty} \sum_{\substack{|\boldsymbol{j}|=n+3 \\ j_{n+2}=j}} \mathcal{R}_j(s)\big(\chi_0(x, \nabla_y \varphi_{\boldsymbol{j}}(x)) a_{\boldsymbol{j}}(x, s) e^{-s\varphi_{\boldsymbol{j}}(x)}\big|_{\Gamma_j}\big),$$

we can exploit the estimates in [Ikawa 1988, Sections 4 and 5] for the amplitudes $c_{\boldsymbol{j},\mu}(x, s)$ of the asymptotic solutions $v_{\boldsymbol{j},M}(x, s)$. In the same way, we can estimate and sum the negligible contributions $s^{-M} R_{g,n,j,l}$ coming from the glancing region and show that they yield a negligible term. Thus, for $\operatorname{Re} s > s_0 + d > s_0$ we deduce

$$\|R_1(x, s; k)\|_{\Gamma,p} \le C_{p,d} \langle s + \boldsymbol{i} k\rangle^{p+2} |s|^p, \quad p \in \mathbb{N}, \tag{6-12}$$

while for $|s + \boldsymbol{i} k| \le a + 1$ we obtain

$$\|R_1(x, s; k)\|_{\Gamma,p} \le C'_{p,d} k^p, \quad p \in \mathbb{N}. \tag{6-13}$$

7. Higher order terms of the asymptotic solution

Our purpose is to improve (6-7) by higher order approximations $V^{(j)}(x, s; k)$, $j = 1, \dots, M-1$, where M is an integer such that $M > (N-1)/2$. In particular, for $N = 2$ we can take $M = 1$ and the construction in Section 6 is sufficient. Recall that the term $R_1(x, s; k)$ in the previous section has the form

$$\sum_{n=n_j}^{\infty} \sum_{j,l=1}^{\kappa_0} (R_{h,n,j,l}(x, s; k) + R_{g,n,j,l}(x, s; k)),$$

with $n_1 = -2$ and $n_j = -1$ for $j \neq 1$. Fix j and l and set

$$e^{-s\varphi_n(x)} m_{1,n}^{(j,l)}(x, s; k) = R_{h,n,j,l}(x, s; k) + R_{g,n,j,l}(x, s; k), \quad x \in \Gamma_l,$$

where $\varphi_n(x)$ is one of the phases $\varphi_{\boldsymbol{j}}(x)$ in $U_{n+2,j}(x, s; k)$. The choice of φ_n is not important and we omit in the notation the dependence on (j, l). The analysis in the previous section shows that we have the estimates

$$\big\|m_{1,n}^{(j,l)}(x, s; k)\big\|_{\Gamma_l,p} \le D_p \langle s + \boldsymbol{i} k\rangle^{p+2} |s|^{p+3+\beta_N} \tilde{\theta}^n, \quad \text{for all } n \in \mathbb{N}, \tag{7-1}$$

where $0 < \tilde{\theta} < 1$ is the same as in Section 5. Here and below we denote by $F^{(j,l)}$ some terms depending on the traces on K_j and K_l, $j, l = 1, \dots, \kappa_0$, while $\boldsymbol{j}$, $\boldsymbol{j}'$ denote configurations. Now for fixed n we apply the construction of Sections 3 and 6 to the oscillatory data $e^{-s\varphi_n(x)} m_{1,n}^{(j,l)}(x,s;k)$ and we obtain a series $\sum_{m=-1}^{\infty} U_{1,n,m}^{(j,l)}(x,s;k)$ with

$$U_{1,n,m}^{(j,l)}(x,s;k) = \sum_{\substack{|\boldsymbol{j}'|=m+3\\ j'_{m+2}=l}} (-1)^{m+2} e^{-s\varphi_{1,n,\boldsymbol{j}'}(x)} a_{1,n,\boldsymbol{j}'}^{(j,l)}(x,s;k),$$

where the phase functions $\varphi_{1,n,\boldsymbol{j}'}(x)$ depend on the configurations $\boldsymbol{j}'$. Taking the summation over n, we are going to study the double series

$$w_{1,j,l}(x,s;k) = \sum_{n=n_j}^{\infty} \sum_{m=-1}^{\infty} U_{1,n,m}^{(j,l)}(x,s;k)\big|_{\Gamma_l}, \quad x \in \Gamma_l. \tag{7-2}$$

We repeat the argument of Section 5 for $\sigma_0 \le \operatorname{Re} s \le 1$ and applying (7-1) and Theorem 3.2(b), we get the estimates

$$\big\| U_{1,n,m}^{(j,l)}(x,s;k) \big\|_{\Gamma_l,p} \le D'_p \langle s + \boldsymbol{i}k\rangle^{p+3} |s|^{p+4+\beta_N+\beta_0} \tilde{\theta}^{n+m} \quad \text{for all } n \in \mathbb{N} \text{ and } m \in \mathbb{N}, \tag{7-3}$$

with constants D'_p independent of $n, m \in \mathbb{N}$. Thus, the double series defining $w_{1,j,l}(x,s;k)$ is convergent. Applying $\mathcal{G}_l(s)$ to $w_{1,j,l}(x,s;k)$ and exploiting (7-3), we justify the convergence of the corresponding series and for $s \in \mathcal{D}_0$ we obtain analytic terms. The function

$$V^{(1)}(x,s;k) = -s^{-1} \sum_{j,l=1}^{\kappa_0} \mathcal{G}_l(s)(w_{1,j,l}(x,s;k))$$

satisfies condition (S) in $(\mathring{\Omega}, \mathcal{D}_0)$ and for $s \in \mathcal{D}_0$ and $x \in \Gamma$ we get

$$V^{(0)}(x,s;k) + V^{(1)}(x,s;k) = m(x,s;k) + s^{-2} R_2(x,s;k) + s^{-M} \mathcal{Q}_{M,1}(x,s;k). \tag{7-4}$$

Here $R_2(x,s;k)$ and $\mathcal{Q}_{M,1}(x,s;k)$ are analytic for $s \in \mathcal{D}_0$, $\mathcal{Q}_{M,1}$ satisfies the same estimates as in (6-9), while for $R_2(x,s;k)$ we have

$$\|R_2(x,s;k)\|_{\Gamma,p} \le C_p \langle s + \boldsymbol{i}k\rangle^{p+3} |s|^{p+6+2\beta_N} \quad \text{for all } p \in \mathbb{N}. \tag{7-5}$$

For $\operatorname{Re} s > s_0 + d > s_0$ we obtain again better estimates, since we can choose $\varphi_n(x) = \varphi_{\boldsymbol{j}}(x)$ and

$$m_{1,n}^{(j,l)}(x,s;k) = c_{\boldsymbol{j},1}(x,s;k)\big|_{\Gamma_l},$$

where $c_{\boldsymbol{j},1}(x,s;k)$ is the coefficient in front of s^{-1} in the asymptotic solution $v_{\boldsymbol{j},M}(x,s;k)$ introduced in Section 6. Exploiting the convergence of the series (6-11), we deduce that in this domain the growth in the right hand side of (7-5) is $\langle s + \boldsymbol{i}k\rangle^{p+3}|s|^p$.

Repeating this procedure, we construct $V^{(j)}(x,s;k)$, $0 \le j \le M-1$, which are analytic functions for $s \in \mathcal{D}_0$ with values in $C^\infty(\Omega)$. They satisfy condition (S) in $(\mathring{\Omega}, \mathcal{D}_0)$ and we have

$$\sum_{j=0}^{M-1} V^{(j)}(x,s;k) = m(x,s;k) + s^{-M} \mathcal{Q}_M(x,s;k), \quad x \in \Gamma, \tag{7-6}$$

with polynomial estimates

$$\|\mathcal{Q}_M(x,s;k)\|_{\Gamma,0} \le C_M \langle s + \boldsymbol{i}k\rangle^{L(M)} |s|^{N(M)}, \quad s \in \mathcal{D}_0. \tag{7-7}$$

Here $\mathcal{Q}_M(x, s; k)$ is analytic for $s \in \mathcal{D}_0$ and C_M depend on the norms of the derivatives of $\psi(x)$ and $b(x, s; k)$ involved in the boundary data $m(x, s; k)$ introduced in the beginning of Section 6. Thus, we establish *crude* estimates with orders $N(M)$, $L(M)$ depending on M and it seems quite difficult to obtain more precise estimates for $s \in \mathcal{D}_0$. Of course, we have $N(M) > M$, however we will apply the estimates above for fixed M and the precise value of $N(M)$ is not important for our argument. For $\operatorname{Re} s \geq s_0 + d > s_0$, $\operatorname{Im} s \leq -J$ the absolutely convergence of (6-11) implies

$$\|\mathcal{Q}_M(x, s; k)\|_{\Gamma,0} \leq C_{M,d} \langle s + \boldsymbol{i} k\rangle^{L(M)}. \tag{7-8}$$

The constant $C_{M,d}$ depends on d but $L(M)$ is independent of d. Now we fix an integer $M \in \mathbb{N}$ so that $M > (N-1)/2$, $N(M)$ and $L(M)$ are fixed. Next, we fix $d > 0$ small enough so that

$$d \frac{N(M)}{s_0 + d - \sigma_0} < M - \frac{N-1}{2}.$$

In the domain $\{s \in \mathbb{C} : \sigma_0 \leq \operatorname{Re} s \leq s_0 + d < 0, \operatorname{Im} s \leq -J\}$ consider the function

$$F(x, s; k) = \frac{\mathcal{Q}_M(x, s; k)}{(s + \boldsymbol{i} k)^{L(M)}},$$

which is analytic with respect to s. The estimates (7-7) and (7-8) combined with the Phragmen–Lindelöf theorem [Titchmarsh 1968] show that for $s \in \{s \in \mathbb{C} : \operatorname{Re} s = t,\ \sigma_0 \leq t \leq s_0 + d,\ \operatorname{Im} s \leq -J\}$, we have

$$\|F(x, s; k)\|_{\Gamma,0} \leq A_M |s|^{\kappa(t)},$$

$\kappa(t)$ being the linear function such that $\kappa(\sigma_0) = N(M)$, $\kappa(s_0 + d) = 0$. We can choose $\sigma_1 < s_0$ so that $0 \leq \kappa(t) \leq \alpha$ for $\sigma_1 \leq t \leq s_0 + d$ with some $0 < \alpha < M - (N-1)/2$. Thus, for $\sigma_1 \leq \operatorname{Re} s \leq s_0 + d$, $\operatorname{Im} s \leq -J$, $|s + \boldsymbol{i} k| \leq |\sigma_0| + 1$ we get

$$\|\mathcal{Q}_M(x, s; k)\|_{\Gamma,0} \leq A_M |s + \boldsymbol{i} k|^{L(M)} |s|^{\alpha} \leq B_M k^{\alpha}, \quad k \geq 1. \tag{7-9}$$

Moreover, the constant B_M depends on the derivatives of $\nabla\psi$ and $b(x, s; k)$ involved in the boundary data $m(x, s; k)$ as well as on some global constants depending only on K. The restriction $\sigma_1 \leq \operatorname{Re} s \leq s_0 + d$ with $s_0 + d < 0$ was used only to guarantee that the factor $(s + \boldsymbol{i} k)^{L(M)} \neq 0$ in this domain. For $\operatorname{Re} s > s_0 + d$ we can apply the estimate (7-8) to obtain (7-9) with another constant A_M and $\alpha = 0$. Consequently, for some fixed c such that $s_0 + c \geq 1$ the estimates (7-9) hold for

$$s \in \mathcal{D}_1 = \{s \in \mathbb{C} : \sigma_1 \leq \operatorname{Re} s \leq s_0 + c,\ \operatorname{Im} s \leq -J,\ |s + \boldsymbol{i} k| \leq |\sigma_0| + c\}.$$

8. Integral equation on the boundary

In this section we define for $s \in \mathcal{D}_1$ an operator $R(s, k) : L^2(\Gamma) \to C^\infty(\mathring{\Omega})$, where $k > J + |\sigma_0| + c$ will be taken sufficiently large and $\mathcal{D}_1$ is the domain introduced in the previous section. The operator $R(s, k)$ satisfies

$$\begin{cases} (\Delta_x - s^2) R(s, k) f = 0 & \text{for } x \in \mathring{\Omega}, \\ R(s, k) f \in L^2(\Omega) & \text{if } \operatorname{Re} s > 0, \\ R(s, k) f\big|_{\Gamma} = f, & \end{cases} \tag{8-1}$$

and to arrange the boundary condition we will solve an integral equation on Γ. After the construction of a solution $\sum_{j=0}^{M-1} V^{(j)}(x,s;k)$ with the properties in Section 7, it was mentioned in [Ikawa 1988, Proposition 2.4] that the existence of $R(s,k)$ can be obtained by the argument in [Ikawa 1987]. On the other hand, [Ikawa 1987] deals with the case of two strictly convex obstacles and in that case the geometry of the trapping rays is rather different from that in [Ikawa 1988] and our paper. For the sake of completeness we will discuss briefly how we can construct $R(s,k)$ by using the construction in Sections 6 and 7 in the hyperbolic region and those in [Ikawa 1982; 1988; Stefanov and Vodev 1995] in the glancing and elliptic regions.

Fix $M > (N-1)/2$ and $0 < \alpha < M - (N-1)/2$ as in the previous section and $j \in \{1, \dots, \kappa_0\}$. Let $Y \subset \Gamma_j$ and let $F \in L^2(\Gamma_j)$ with $\operatorname{supp} F \subset Y$. As in Section 6, choose local coordinates (y, η) in $T^*(Y)$ with $y = (y_1, \dots, y_{N-1}) \in W \subset \mathbb{R}^{N-1}$, and write

$$F(y) = (2\pi)^{-N+1} \int e^{i\langle y,\eta\rangle} \hat{F}(\eta)\, d\eta = \Big(\frac{k}{2\pi}\Big)^{N-1} G(y) \int e^{ik\langle y,\eta\rangle} \hat{F}(k\eta)\, d\eta,$$

where $G(y) \in C_0^\infty(\mathbb{R}^{N-1})$, $G(y) = 1$ on $\operatorname{supp} F(y)$ and

$$\hat{F}(\eta) = \int e^{-i\langle y,\eta\rangle} F(y)\, dy.$$

Consider a partition of unity $\chi_0(\eta) + \chi_1(\eta) + \chi_2(\eta) = 1$ with C^∞ functions $\chi_i(\eta)$ between 0 and 1 and such that

$$\begin{aligned}
&\operatorname{supp} \chi_0(\eta) \subset \{\eta : |\eta| \le 1 - \delta_1/2\},\\
&\operatorname{supp} \chi_1(\eta) \subset \{\eta : 1 - \tfrac{2}{3}\delta_1 \le |\eta| \le 1 + \tfrac{2}{3}\delta_1\},\\
&\operatorname{supp} \chi_2(\eta) \subset \{\eta : |\eta| \ge 1 + \delta_1/2\},
\end{aligned}$$

$0 < \delta_1 < 1$ being the constant in Section 6. Set

$$F_i(y) = \Big(\frac{k}{2\pi}\Big)^{N-1} G(y) \int e^{ik\langle y,\eta\rangle} \chi_i(\eta) \hat{F}(k\eta) d\eta,\ i = 0, 1, 2.$$

To treat F_0 we will apply the results of Sections 3–7. Consider the function

$$\psi(y;\eta) = \langle y, \eta\rangle, \quad y \in W,\ |\eta| < 1 - \delta_1/2.$$

We can construct a phase function $\varphi = \varphi(x;\eta)$ defined in $\mathcal{V}_j$ such that

(i) $\varphi\big|_{\operatorname{supp} G} = \psi(y;\eta)$, $y \in W$,

(ii) $(\partial\varphi/\partial\nu)(x;\eta)\big|_{\mathcal{V}_j \cap \Gamma_j} \ge \delta_2 > 0$, $y \in W$,

(iii) the phase $\varphi(x;\eta)$ satisfies condition $(\mathcal{P})$ on Γ_j.

The local existence of $\varphi(x;\eta)$ satisfying the conditions (i)–(ii) has been discussed in [Ikawa 1987; 1988]. To arrange (iii), we use a suitable continuation and we treat this problem in Appendix B below. Starting with the oscillatory data $m_0(y;\eta) = (2\pi)^{-N+1} G(y) e^{ik\langle y,\eta\rangle}$, $|\eta| \le 1 - \delta_1/2$ and applying the argument of Sections 6 and 7, we construct an approximative solution $V_0(x,s;k,\eta)$ which satisfies condition (S) in $(\mathring{\Omega}, \mathcal{D}_1)$ and such that

$$V_0(y,s;k,\eta) = m_0(y;\eta) + s^{-M} \mathcal{Q}_M(y,s;k,\eta), \quad x \in \Gamma.$$

Moreover, for $\mathcal{Q}_M(x, s; k, \eta)$ we have the estimate (7-9) and it is clear that the constants B_M and α in (7-9) can be chosen uniformly with respect to η, $|\eta| \le 1 - \delta_1/2$. Define the operator

$$U_0(s; k)F = \int V_0(x, s; k, \eta)\chi_0(\eta)\hat{F}(k\eta)k^{N-1}d\eta$$

with values in $C^\infty(\Omega)$ so that $U_0(s; k)F$ satisfies condition (S) in $(\mathring{\Omega}, \mathcal{D}_1)$ and

$$U_0(s; k)F|_\Gamma = F_0 + s^{-M}\int \mathcal{Q}_M(x, s; k, \eta)\chi_0(\eta)\hat{F}(k\eta)k^{N-1}d\eta = F_0 + L_0(s; k)F.$$

Therefore

$$\begin{aligned}\|L_0(s; k)F\|^2_{L^2(\Gamma)} &\le C_0\Big(\int_{|\eta|\le 1-\delta_1/2} k^{-M+(N-1)/2+\alpha}|\hat{F}(k\eta)|k^{(N-1)/2}d\eta\Big)^2\\ &\le C_0k^{-2M+N-1+2\alpha}\int_{|\eta|\le 1-\delta_1/2} d\eta \int_{\mathbb{R}^{N-1}} |\hat{F}(k\eta)|^2k^{N-1}d\eta \le C_1k^{-2M+N-1+2\alpha}\|F\|^2_{L^2(\Gamma)},\end{aligned}$$

with a constant $C_1 > 0$ depending only on K. Moreover, for $s \in \mathcal{D}_1$ we obtain the estimate

$$\|U_0(s; k)F\|_{L^2(\Omega\cap\{|x|\le R\})} \le C_{0,R}k^{m_0}\|F\|_{L^2}. \tag{8-2}$$

To prove this, it is sufficient to show that

$$\|V_0(x, s; k, \eta)\|_{L^2(\Omega\cap\{|x|\le R\})} \le C'_{0,R}k^{p_0}, \quad s \in \mathcal{D}_1, \tag{8-3}$$

uniformly with respect to $|\eta| \le 1 - \delta_1/2$. On the other hand,

$$V_0(x, s; k, \eta) = V^{(0)}(x, s; k, \eta) - \sum_{m=1}^{M-1} V^{(m)}(x, s; k, \eta)s^{-m},$$

$$V^{(m)}(x, s; k, \eta) = \sum_{j_1, j_2, \dots, j_m=1}^{\kappa_0} \mathcal{S}_{j_m}(s)w_{j_1,j_2,\dots,j_m}(x, s; k, \eta).$$

Here the $w_{j_1,j_2,\dots,j_m}(x, s; k, \eta)$, $x \in \Gamma_l$, are infinite series and the estimates of $\|V^{(m)}\|_{L^2(\Omega\cap\{|x|\le R\})}$ follow from the estimates for the operators H_h, H_g, $S_0(s)$, P_e and the estimates for $\|w_{j_1,j_2,\dots,j_m}\|_{H^2(\Gamma_m)}$. According to the recurrence procedure in Section 7, we deduce that

$$\|w_{j_1,j_2,\dots,j_m}\|_{H^2(\Gamma_m)} \le D_l|s|^{q(m)}, \quad s \in \mathcal{D}_1,\ m = 0, \dots M - 1,$$

for some integers $q(m)$, and we get (8-3) with $p_0 = \sup_m q(m)$.

To deal with $F_1(y)$, introduce $\xi(y, \eta) \in \mathbb{S}^{N-1}$ such that

$$\xi(y, \eta) - \langle\nu(y), \xi(y, \eta)\rangle = \eta,\ (y, \eta) \in \Xi = \operatorname{supp} G \times \big\{\eta : -\tfrac{2}{3}\delta_1 \le |\eta| - 1 \le \tfrac{2}{3}\delta_1\big\},$$

and consider

$$\zeta(y, \eta) = \xi(y, \eta) - 2\langle\nu(y), \xi(y, \eta)\rangle\nu(y) \in \mathbb{S}^{N-1}.$$

Our choice of δ_1 in Section 6 and Lemma 2.1 show that at least one of the rays $\{y + t\xi(y, \eta) : t \ge 0\}$, $\{y + t\zeta(y, \eta) : t \le 0\}$ does not meet a d_0-neighborhood of $\bigcup_{l\ne j} K_l$. For every fixed $(y_0, \eta_0) \in \Xi$ we have the property above for at least one of the rays related to $\xi(y_0, \eta_0)$ and $\zeta(y_0, \eta_0)$ and the same is true for (y, η) sufficiently close to (y_0, η_0). Consider a microlocal partition of unity

$$\sum_{\mu=1}^{M_1} \psi_\mu(y)\Xi_\mu(\eta) = 1$$

on Ξ so that $\operatorname{supp} \Xi_\mu \subset \{\eta : -\delta_1 \leq |\eta| - 1 \leq \delta_1\}$, while for $(y, \eta) \in \operatorname{supp} \psi_\mu \Xi_\mu$, we have the property of the rays mentioned above. We fix μ and assume first that the outgoing rays $\{y + t\xi(y, \eta) : t \geq 0\}$, $(y, \eta) \in \operatorname{supp} \psi_\mu \Xi_\mu$ do not meet a neighborhood of $\bigcup_{l \neq j} K_l$. Consider boundary data

$$\tilde{m}_\mu(y; k, \eta) = (2\pi)^{-N+1} G(y)\psi_\mu(y)e^{ik\langle y,\eta\rangle}, \quad \eta \in \operatorname{supp} \Xi_\mu.$$

Following [Ikawa 1988, Proposition 4.7] (see also [Ikawa 1982, Proposition 7.5]), for every $M \geq 1$ there exists a function $Z_{\mu,M}(x, s; k, \eta)$ which satisfies condition (S) in $(\Omega_j, \mathcal{D}_1)$ as well as the conditions

$$\|Z_{\mu,M}(\,\cdot\,, s; k, \eta)\|_{C^p(\Omega_j \cap \{|x| \leq R\})} \leq C_{R,p} k^p \quad \text{for all } p \in \mathbb{N} \tag{8-4}$$

and

$$Z_{\mu,M}(y, s; k, \eta) = \tilde{m}_\mu(y; k, \eta) + r^{-M} D_{\mu,M}(y, s; k, \eta), \quad y \in \Gamma,$$

with $\|D_{\mu,M}(\,\cdot\,, s; k, \eta)\|_{\Gamma,p} \leq C_p k^p$ for all $p \in \mathbb{N}$. The constants in these estimates are uniform with respect to η and μ and they depend only on the geometry of K.

The construction of Z_μ in [Ikawa 1982] is long and technical. We sketch below the main points. The starting point is to introduce oscillatory boundary data

$$(2\pi)^{-N+1} G(y)\psi_\mu(y)h(t)e^{ik(\langle y,\eta\rangle - t)}, \quad \eta \in \operatorname{supp} \Xi_\mu,$$

depending on y and t with $h \in C_0^\infty(\mathbb{R}^+)$, $\operatorname{supp} h \subset (T, T+1)$, $T > 1$ and to construct an asymptotic solution $w_\mu(x, t; k, \eta)$ of the wave equation $(\partial_t^2 - \Delta_x)u = 0$ for $t \geq 0$ with

$$\operatorname{supp} w_\mu(x, t; \cdot, \cdot) \subset \{(x, t) : t \geq 0\}$$

and *big parameter* k. We omit in the notation here and below the dependence on M. In the glancing region we have two phase functions $\varphi_\pm = \theta(y, \eta) \pm \frac{2}{3}\rho^{3/2}(y, \eta)$ [Ikawa 1982; Gérard 1988; Stefanov and Vodev 1995] and $\varphi_\pm$ are constructed so that their traces on $\operatorname{supp} G \cap \Gamma_j$ coincide with $\langle y, \eta\rangle$. The outgoing rays are propagating with directions $\nabla\varphi_+$, while the incoming rays are propagating with directions $\nabla\varphi_-$. The proofs in [Ikawa 1982; 1988] work assuming N odd and one considers the Laplace transform

$$\hat{w}_\mu(x, s; k, \eta) = \int_{-\infty}^{\infty} e^{-st} w_\mu(x, t; k, \eta)\,dt, \quad s \in \mathcal{D}_1.$$

The assumption that N is odd is used only by applying the strong Huygens principle to guarantee that for every fixed $x \in \Omega_j$ the support of w_μ with respect to t is compact, hence the integral is convergent. For N even we apply the finite speed of propagations and the fact that the supports of the solutions of the transport equations are propagating along the rays $\{y + t\nabla\varphi_+(y, \eta) : t \geq 0\}$ to show that for $|x| \leq \rho_0$ the solution $w_\mu(x, t; k, \eta)$ vanishes for t large. This justifies the existence of $\hat{w}_\mu(x, s; k, \eta)$ for $|x| \leq \rho_0$.

Next, using the notation of Section 6, consider

$$Z_\mu(x,s;k,\eta) = \frac{1}{\hat{h}(s+\boldsymbol{i}k)}\big(\Phi\hat{w}_\mu - S_0(s)(\Phi(\Delta_x - s^2)\hat{w}_\mu + [\Delta,\Phi]\hat{w}_\mu)\big), \tag{8-5}$$

where h is chosen so that $\hat{h}(s+\boldsymbol{i}k) \neq 0$ for $|s+\boldsymbol{i}k| \leq |\sigma_0| + c$. Now let μ be such that the rays $\{y + t\zeta(y,\eta) : t \leq 0\}, (y,\eta) \in \operatorname{supp}\psi_\mu \Xi_\mu$, do not meet a neighborhood of $\bigcup_{l\neq j} K_l$. In this case we repeat the procedure in [Ikawa 1982, Section 7] and [Ikawa 1988, Section 4] to construct an asymptotic solution $w_\mu(x,t;k,\eta)$ of the wave equation for $t \leq 0$ with $\operatorname{supp} w_\mu(x,t;\cdot,\cdot) \subset \{(x,t); t \leq 0\}$ starting with oscillatory boundary data

$$(2\pi)^{-N+1} G(y)\psi_\mu(y)h(-t)e^{-\boldsymbol{i}k(-\langle y,\eta\rangle - t)}, \quad \eta \in \operatorname{supp}\Xi_\mu.$$

We express $\langle y,\eta\rangle$ by the trace of the phase function $\varphi_-\big|_{\Gamma_j}$ related to the incoming directions and we consider for $|x| \leq \rho_0$ the Laplace transform

$$\hat{w}_\mu(x,s;k,\eta) = \int_{-\infty}^{\infty} e^{st} w_\mu(x,t;k,\eta)\,dt, \quad s \in \mathfrak{D}_1.$$

Next, we define $Z_\mu(x,s;k,\eta)$ by (8-5) using the outgoing parametrix $S_0(s)$ and deduce the estimates (8-4). Finally, we introduce

$$U_1(s;k)F = \sum_{\mu=1}^{M_1} \int Z_\mu(x,s;k,\eta)\Xi_\mu(\eta)\chi_1(\eta)\hat{F}(k\eta)k^{N-1}d\eta,$$

and conclude that $U_1(s;k)F$ is analytic for $s \in \mathfrak{D}_1$ and satisfies

$$\begin{cases} (\Delta_x - s^2)U_1(s;k)F = 0, & x \in \Omega_j, \\ U_1(s;k)F\big|_\Gamma = F_1 + L_1(s;k)F. \end{cases}$$

As above, exploiting the estimates (8-4), we obtain

$$\|L_1(s;k)F\|_{L^2(\Gamma)} \leq C_M k^{-M}\|F\|_{L^2(\Gamma)}, s \in \mathfrak{D}_1$$

and

$$\|U_1(s;k)F\|_{L^2(\mathring{\Omega}\cap\{|x|\leq R\})} \leq C_{1,R}k^{(N-1)/2}\|F\|_{L^2}. \tag{8-6}$$

Now we pass to the analysis of the term F_2 in the elliptic region. Let $\tilde{\mathcal{U}}_j$ be a small neighborhood of K_j and let $\mathcal{U}_j = \tilde{\mathcal{U}}_j \setminus K_j$. Following [Stefanov and Vodev 1995, A.4], we construct a parametrix $H_e : \tilde{C}^\infty(\operatorname{supp} G) \to \tilde{C}^\infty(\mathcal{U}_j)$ as a Fourier integral operator with complex phase function $\tilde{\varphi}(x,\eta)$ and big parameter k having the form

$$(H_e u)(x,s) = \Big(\frac{s}{2\pi}\Big)^{N-1} \int e^{\boldsymbol{i}k(\tilde{\varphi}(x,\eta) - \langle y,\eta\rangle)}\tilde{a}(x,\eta,k)u(y)\,dy\,d\eta,$$

so that

$$\begin{cases} (\Delta_s - s^2)H_e u = K_e u, & x \in \mathcal{U}_j, \\ H_e u\big|_{\Gamma_j} = \operatorname{Op}(G\chi_2)u, \end{cases}$$

where

$$\operatorname{Op}(G\chi_2)u = \Big(\frac{k}{2\pi}\Big)^{N-1}\int e^{\boldsymbol{i}k\langle x-y,\eta\rangle}G(x)\chi_2(\eta)u(y)\,dy\,d\eta.$$

The last operator is defined for $u \in C^\infty(\Gamma_j)$ but it can be prolonged to $F \in L^2(\Gamma_j)$ since the symbol $\chi_2(\eta)$ lies in $S^{0,0}_{0,0}$ [Gérard 1988, Proposition A.I.6].

Assume that locally the boundary Γ_j is given by the equation $x_N = 0$ and let locally $\mathcal{U}_j \subset \{x_N \geq 0\}$. To satisfy the equation $(\Delta_x - s^2)H_e u = 0$ modulo negligible terms, we must choose $\tilde{\varphi}$ so that

$$|\nabla\tilde{\varphi}|^2 = -\Big(\frac{s}{k}\Big)^2 = \gamma^2, \quad \tilde{\varphi}\big|_{\Gamma_j} = \langle x, \eta\rangle. \tag{8-7}$$

For $|s + \boldsymbol{i}k| \leq |\sigma_0| + c$ we see that $\gamma = 1 + \mathbb{O}(k^{-1})$ is a complex parameter close to 1 and we may repeat the argument in [Stefanov and Vodev 1995, A.4] and [Gérard 1988, A.II.4] to construct $\tilde{\varphi}$ with the properties

$$\operatorname{Im}\tilde{\varphi}(x,\eta) \geq c_0 x_N(1+|\eta|), \quad c_0 > 0, \qquad \text{and} \qquad |\operatorname{Re}\tilde{\varphi}(x,\eta)| \leq c_0'(1+|\eta|).$$

The phase $\tilde{\varphi}$ satisfies the eikonal equation modulo $\mathbb{O}(x_N^\infty)$, the amplitudes satisfy the corresponding transport equations modulo $\mathbb{O}(x_N^\infty)$ and $\tilde{a}(x,\eta,k) \in S^{0,0}_{0,0}$. Notice that the sign of $\operatorname{Im}\tilde{\varphi}(x,\eta)$ is related to the choice $k > 0$. We have

$$\operatorname{Re}\big(ik(\tilde{\varphi}(x,\eta) - \langle y,\eta\rangle)\big) = -k\operatorname{Im}\tilde{\varphi}(x,\eta) \leq -c_0 k x_N(1+|\eta|),$$

and the integral $H_e F$ is convergent for $x_N > 0$ and $F \in L^2(Y)$. Moreover, we have

$$\sup_{x_N \geq 0} x_N^m e^{-c_0 x_N(1+|\eta|)} \leq c_m(1+|\eta|)^{-m}k^{-m} \qquad \text{for all } m \in \mathbb{N},$$

and this implies that the kernel of K_e is in $\tilde{C}^\infty(\mathcal{U}_j \times \operatorname{supp} G)$ and we obtain $K_e = \mathbb{O}(|k|^{-\infty})$ uniformly with respect to $x_N \in [0, \varepsilon]$.

Next, let $\Psi(x) \in C_0^\infty(\mathcal{U}_j)$ be a cutoff function such that $\Psi(x) = 1$ in a small neighborhood of K_j. Define

$$U_2(s;k)F = \big[\Psi H_e - S_0(s)(\Psi K_e + [\Delta, \Psi]H_e)\big]F.$$

Then $U_2(s;k)F$ satisfies

$$\begin{cases}(\Delta_x - s^2)U_2(s;k)F = 0, x \in \mathring{\Omega}, \quad s \in \mathcal{D}_1,\\ U_2(s;k)F\big|_\Gamma = F_2 + L_2(s;k)F,\end{cases}$$

but $U_2(s;k)F$ is not analytic with respect to s which will not be important for the proof of Theorem 1.3 below. On the other hand, the trace on Γ of $S_0(s)[\Delta, \Psi]H_e F$ is negligible and the same is true for the trace of $S_0(s)\Psi K_e F$. Thus, $\|L_2(s;k)F\|_{L^2(\Gamma)} \leq C_M k^{-M}\|F\|_{L^2(\Gamma)}$ for all $M \in \mathbb{N}$. Moreover, we have the estimate

$$\|U_2(s;k)F\|_{L^2(\Omega_j \cap \{|x| \leq R\})} \leq C_{2,R}\|F\|_{L^2(\Gamma)}, \tag{8-8}$$

which is a consequence of L^2 estimates of $\Psi H_e F$ and $[\Delta, \Psi]H_e F$. In fact, the estimate of $\|[\Delta, \Psi]H_e F\|_{L^2}$ is easy since $\Psi = 1$ in a neighborhood of Ω_j and the kernel of $[\Delta, \Psi]H_e$ lies in $\tilde{C}^\infty(\mathcal{U}_j \times \operatorname{supp} G)$. To estimate $\|\Psi H_e F\|_{L^2}$, observe that for small $x_N \geq 0$, H_e is a Fourier integral operator with nondegenerate phase function of positive type $\phi(x,y,\eta) = \tilde{\varphi}(x,\eta) - \langle y,\eta\rangle$ [Hörmander 1985b, Definition 25.4.3]. Thus, we can estimate

$$\|(H_e F)(x_N, \cdot, s; k)\|_{L^2(\mathcal{U}_j \cap \{x_N = z\})} \leq B\|F\|_{L^2(\Gamma)}$$

uniformly with respect to $z \in [0, \varepsilon]$ [Hörmander 1985b, Theorem 25.5.6] and this leads to (8-8). Finally, introduce

$$L_Y(s;k)F = U_0(s;k)F + U_1(s;k)F + U_2(s;k)F,$$

and conclude that $L_Y(s;k)F|_\Gamma = F + \sum_{i=0}^{2} L_i(s;k)F = F + Q_Y(s;k)F$, with

$$\|Q_Y(s;k)F\|_{L^2(\Gamma)} \le B_Y k^{-M+(N-1)/2+\alpha} \|F\|_{L^2(\Omega)}.$$

By using a partition of unity on Γ, we define an operator

$$L(s;k) : L^2(\Gamma) \ni f \to L(s;k)f \in C^\infty(\mathring{\Omega})$$

and deduce that $L(s;k)f$ satisfies

$$\begin{cases} (\Delta_x - s^2)L(s,k)f = 0 & \text{for } x \in \mathring{\Omega}, \\ L(s,k)f \in L^2(\Omega) & \text{if } \operatorname{Re} s > 0, \\ L(s,k)F\big|_\Gamma = f + Q(s;k)f, & \end{cases}$$

with

$$\|Q(s;k)f\|_{L^2(\Gamma)} \le Bk^{-M+(N-1)/2+\alpha} \|f\|_{L^2(\Gamma)}.$$

Choosing k_1 sufficiently large, the operator $I + Q(s;k) : L^2(\Gamma) \to L^2(\Gamma)$ is invertible for $s \in \mathscr{D}_1$ and $k \ge k_1$. We define

$$R(s,k)f = L(s;k)(I + Q(s;k))^{-1} f : L^2(\Gamma) \to C^\infty(\mathring{\Omega}),$$

and it is clear that $R(s,k)f$ for $s \in \mathscr{D}_1$ satisfies (8-1).

Proof of Theorem 1.3. Given $g \in L^2(\mathring{\Omega})$ and $\chi \in C_0^\infty(\mathring{\Omega})$ with $\operatorname{supp} \chi \subset \{|x| \le \rho\}$, $\rho \ge \rho_0$, by (6-3) we obtain $S_0(s)(\chi g) \in H^1(|x| \le \rho)$ and this yields $[S_0(s)(\chi g)]\big|_\Gamma \in H^{1/2}(\Gamma)$. Setting $s = \boldsymbol{i}z$, consider for $\operatorname{Im} z < 0$,

$$v = S_0(\boldsymbol{i}z)(\chi g) - R(\boldsymbol{i}z;k)\big([S_0(\boldsymbol{i}z)(\chi g)]\big|_\Gamma\big). \tag{8-9}$$

Then for the cutoff resolvent $R_\chi(z)$ introduced in Section 1 we get

$$R_\chi(z)(\chi g) = \chi v, \quad \operatorname{Im} z < 0.$$

The operators $\chi S_0(\boldsymbol{i}z)\chi$ and $R_\chi(z)$ admit respectively analytic and meromorphic continuation from $\operatorname{Im} z < 0$ to $\{z \in \mathbb{C} : \operatorname{Im} z \le -\sigma_1, \operatorname{Re} z < -J_1\}$, where $-J_1 = \min\{-J, |\sigma_0| + c - k_1\}$. Thus,

$$\chi R(\boldsymbol{i}z;k)\big([S_0(\boldsymbol{i}z)(\chi g)]\big|_\Gamma\big)$$

is also meromorphic in this domain and to show that it is analytic for $\boldsymbol{i}z \in \mathscr{D}_1$ it suffices to prove that this operator is bounded. For $\boldsymbol{i}z \in \mathscr{D}_1$ this follows from the estimates (8-2), (8-6), (8-8) above and we obtain a polynomial bound for

$$\|\chi R(\boldsymbol{i}z;k)\|_{L^2(\Gamma) \to L^2(\mathring{\Omega})}.$$

Consequently, $R_\chi(z)$ admits an analytic continuation and we get (1-7) for $\operatorname{Re} z < -J_1 < 0$. Next to cover the case $\operatorname{Re} z > J_1 > 0$, we can use the fact that the poles of $R_\chi(z)$ are symmetric with respect to $\boldsymbol{i}\mathbb{R}^+$ or repeat the argument with $k \ll 0$. $\square$

To obtain Corollary 1.4 we establish the estimate

$$\|R_\chi(z)\|_{H^L(\mathring{\Omega})\to L^2(\mathring{\Omega})} \le C(1+|z|)^{m-L}, \quad z \in \mathcal{S},$$

where $m \in \mathbb{N}$ is the integer in (1-7) and $L \in \mathbb{N}$, $L > m$. The proof goes repeating that in the nontrapping case [Tang and Zworski 2000, Theorem 1] and we omit the details. □

Appendix A : Stable and instable manifolds for open billiards

Let $z_0 = (x_0, u_0) \in S^*(\Omega)$. For convenience we will assume that $x_0 \notin K$. Assume that the *backward trajectory* $\gamma_-(z_0)$ determined by z_0 is bounded, and let $\eta \in \Sigma_A^-$ be its itinerary.

Given $x \in \mathbb{R}^N$ and $\varepsilon > 0$, by $B(x, \varepsilon)$ we denote the *open ball* with center x and radius ε in $\mathbb{R}^N$.

In this section we use some tools from [Ikawa 1988] to construct the *local unstable manifold*[5] $W^u_{\text{loc}}(z_0)$ of z_0 in $S^*(\Omega)$ and show that it is Lipschitz in z_0 (and η). In a similar way one deals with local stable manifolds.

Notice that if the boundary Γ of K is only C^k ($k \ge 2$) the C^∞ smoothness below should be replaced by C^k.

Proposition A.1. *There exists a constant $\varepsilon_0 > 0$ such that for any $z_0 = (x_0, u_0) \in S^*_{\delta_0}(\Omega \cap B_0)$ whose backward trajectory $\gamma_-(z_0)$ has an infinite number of reflection points $X_j = X_j(z_0)$ ($j \le 0$) and $\eta \in \Sigma_A^-$ is its itinerary, the following hold*:

(a) *There exists a smooth (C^∞) phase function $\psi = \psi_\eta$ satisfying part* (i) *of condition* ($\mathcal{P}$) *on $\mathcal{U} = B(x_0, \varepsilon_0) \cap \Omega$ such that $\psi(x_0) = 0$, $u_0 = \nabla\psi(x_0)$, and such that for any $x \in C_\psi(x_0) \cap \mathcal{U}^+(\psi)$ the billiard trajectory $\gamma_-(x, \nabla\psi(x))$ has an itinerary η and therefore $d(\phi_t(x, \nabla\psi(x)), \phi_t(z_0)) \to 0$ as $t \to -\infty$. That is,*

$$W^u_{\text{loc}}(z_0) = \{(x, \nabla\psi(x)) : x \in C_\psi(x_0) \cap \mathcal{U}^+(\psi)\}$$

is the local unstable manifold of z_0. Moreover, for any $p \ge 1$ there exists a global constant $C_p > 0$ (independent of z_0 and η) such that

$$\|\nabla\psi_\eta\|_{(p)}(\mathcal{U}) \le C_p. \tag{A-1}$$

(b) *If $(y, v) \in S^*(\Omega \cap B_0)$ is such that $y \in C_\psi(x_0)$ and $\gamma_-(y, v)$ has the same itinerary η, then $v = \nabla\psi(y)$, that is, $(y, v) \in W^u_{\text{loc}}(z_0)$.*

(c) *There exist a constant $\alpha \in (0, 1)$ depending only on the obstacle K and for every $p \ge 1$ a constant $C_p > 0$ such that for any integer $r \ge 1$ and any $\zeta, \eta \in \Sigma_A^-$ with $\zeta_j = \eta_j$ for $-r \le j \le 0$, we have $\|\nabla\psi_\eta - \nabla\psi_\zeta\|_p(V) \le C_p\,\alpha^r$, where $V = \mathcal{U}(\psi_\eta) \cap \mathcal{U}(\psi_\zeta)$.*

Proof. (a) Take $\varepsilon_0 > 0$ so small that whenever $(x, u) \in S^*_{\delta_0/2}(\Omega \cap B_0)$ and $(y, v) \in S^*(\Omega)$ is such that $\|x - y\| < \varepsilon_0$ and $\|u - v\| < \varepsilon_0$ we have $(y, v) \in S^*_{\delta_0}(\Omega)$. Then define $\mathcal{U} = B(x_0, \varepsilon_0) \cap \Omega$ as in the statement. Next, set

$$d_{-m} = \|X_{-m+1} - X_{-m}\| \quad \text{and} \quad u_{-m} = \frac{X_{-m+1} - X_{-m}}{\|X_{-m+1} - X_{-m}\|} \in \mathbb{S}^{n-1}, \quad m \ge 1.$$

[5] Notice that $W^u_{\text{loc}}(z_0)$ and $W^u_\varepsilon(z_0)$ (see Appendix C) coincide in a neighborhood of z_0.

Given any integer $m \geq 1$, consider the linear phase function $\psi^{(m)} = \psi^{(m,\eta)}$ in Ω such that $\nabla\psi^{(m)} \equiv u_{-m}$ and $\psi^{(m)}(X_{-m}) = -(d_{-m} + d_{-m+1} + \cdots + d_{-1})$. Then define

$$\psi_m^{(m)} = \psi_m^{(m,\eta)} = \Phi_{\eta_{-1}}^{\eta_0} \circ \Phi_{\eta_{-2}}^{\eta_{-1}} \circ \cdots \circ \Phi_{\eta_{-m+1}}^{\eta_{-m+2}} \circ \Phi_{\eta_{-m}}^{\eta_{-m+1}} (\psi^{(m)}).$$

Clearly $\psi_m^{(m)}$ is a smooth phase function defined everywhere on $\mathcal{U}$ (in fact, on a much larger subset of Ω) with $\psi_m^{(m)}(X_0) = 0$. Moreover, it follows from Proposition 2.6 that

$$\|\nabla\psi_m^{(m)} - \nabla\psi_{m+1}^{(m+1)}\|_p(\mathcal{U}) \leq C_p\,\alpha^m, \quad m \geq 1, \tag{A-2}$$

for some global constant $C_p > 0$ depending only on K and p. Here we use the fact that

$$\|\nabla\psi^{(m)} - \nabla\psi^{(m+1)}\|_{(p)} \leq C,$$

due to the special choice of the phase functions $\psi^{(m)}$ and $\psi^{(m+1)}$. Since

$$\psi_m^{(m)}(X_0) = \psi_{m+1}^{(m+1)}(X_0) = 0,$$

it now follows that there exists a constant $C_p > 0$ such that

$$\|\psi_m^{(m)}(x) - \psi_{m+1}^{(m+1)}(x)\| \leq C_p\,\alpha^m \qquad \text{for } x \in \mathcal{U} \cap B_0.$$

This implies that for every $x \in \mathcal{U}$ there exists $\psi(x) = \lim_{m\to\infty} \psi_m^{(m)}(x)$. Now (A-2) shows that ψ is C^∞-smooth in $\mathcal{U}$ and

$$\|\nabla\psi_m^{(m)} - \nabla\psi\|_p(\mathcal{U}) \leq C_p\,\alpha^m, \quad m \geq 1. \tag{A-3}$$

In particular, $\|\nabla\psi\| \equiv 1$ in $\mathcal{U}$. Extending ψ in a trivial way along straight line rays, we get a phase function ψ satisfying part (i) of condition $(\mathcal{P})$ in $\mathcal{U}$.

We now show that $W = \{(x, \nabla\psi(x)) : x \in C_\psi(x_0) \cap \mathcal{U}^+(\psi)\}$ is the local unstable manifold of z_0. Given $x \in C_\psi(x_0) \cap \mathcal{U}^+(\psi)$ sufficiently close to x_0 and an arbitrary integer $r \geq 0$, consider the points $X^{-r}(x, \psi_m^{(m)}) \in \partial K_{\eta_{-r}}$ for $m \geq r$. By Proposition 2.4, there exist global constants $C > 0$ and $\alpha \in (0, 1)$ such that $\|X^{-r}(x, \psi_m^{(m)}) - X^{-r}(x, \psi_{m'}^{(m')})\| \leq C\,\alpha^{m-r}$ for $m' \geq m > r$. Thus, there exists $X^{-r} = \lim_{m\to\infty} X^{-r}(x, \psi_m^{(m)}) \in \partial K_{\eta_{-r}}$ and

$$\|X^{-r}(x, \psi_m^{(m)}) - X^{-r}\| \leq C\,\alpha^{m-r}, \quad m > r. \tag{A-4}$$

It is now easy to see that $\{X^{-j}\}_{j=0}^\infty$ are the successive reflection points of a billiard trajectory in Ω and this is the trajectory $\gamma_-(x, \nabla\psi)$. The backward itinerary of the latter is obviously η. Moreover, (A-3) implies $d(\phi_t(x, \nabla\psi(x)), \phi_t(z_0)) \to 0$ as $t \to -\infty$, so $(x, \nabla\psi(x)) \in W_{\mathrm{loc}}^u(z_0)$.

Finally, by (2-1),

$$\|\psi_m^{(m)}\|_{(p)}(\mathcal{U}) \leq C_p\,\|\psi^{(m)}\|_{(p)} \leq C_p,$$

and combining this with (A-3) gives (A-1).

(b) Let $(y, v) \in S^*(\Omega)$ be such that $y \in C_\psi(x_0)$ and $\gamma_-(y, v)$ has the same itinerary η. Define the phase functions $\varphi_m^{(m)}$ and $\varphi^{(m)}$ as in part (a) replacing the point $z_0 = (x_0, u_0)$ by $z = (y, v)$, and let $\varphi(x) = \lim_{m\to\infty} \varphi_m^{(m)}(x)$. Then by part (a), we have

$$W_{\mathrm{loc}}^u(z) = \{(x, \nabla\psi(x)) : x \in C_\varphi(y) \cap \mathcal{U}^+(\phi)\}.$$

On the other hand, it follows from Proposition 2.6 that there exist constants $C > 0$ and $\alpha \in (0, 1)$ such that $\|\nabla \psi_m^{(m)} - \nabla \varphi_m^{(m)}\| \le C\,\alpha^m$ for all $m \ge 0$, which implies $\varphi = \psi$. Thus, $v = \nabla\varphi(y) = \nabla\psi(y) \in W^u_{\mathrm{loc}}(z_0)$.

(c) Choose the constants $\alpha \in (0, 1)$ and $C_p > 0$ $(p = 1, \dots, k)$ as in part (a). Let $\zeta, \eta \in \Sigma_A^-$ be such that $\zeta_j = \eta_j$ for all $-r \le j \le 0$ for some $r \ge 1$. Construct the phase functions $\psi_m^{(m,\eta)}$ and $\psi_m^{(m,\zeta)}$ $(m \ge 1)$ as in part (a); then

$$\psi_\eta = \lim_{m\to\infty} \psi_m^{(m,\eta)}, \qquad \psi_\zeta = \lim_{m\to\infty} \psi_m^{(m,\zeta)}.$$

It follows from Proposition 2.6 that $\|\nabla\psi^{(r,\eta)} - \nabla\psi^{(r,\zeta)}\| \le C_p\,\alpha^r$. Combining this with (A-3) with $m = r$ for η and then with η replaced by ζ, one gets

$$\|\nabla\psi_\eta - \nabla\psi_\zeta\| \le \|\nabla\psi_\eta - \nabla\psi^{(r,\eta)}\| + \|\nabla\psi^{(r,\eta)} - \nabla\psi^{(r,\zeta)}\| + \|\nabla\psi^{(r,\zeta)} - \nabla\psi_\zeta\| \le C_p\,\alpha^r.$$

This proves the assertion. □

Appendix B: Construction of a phase function satisfying condition $(\mathcal{P})$

Consider a local representation $x_N = h(y)$ of the boundary Γ_j with $y = (y_1, \dots, y_{N-1}) \in W \subset \mathbb{R}^{N-1}$. We wish to construct a phase function $\varphi(x;\eta)$ such that

$$\varphi(y, h(y); \eta) = \langle y, \eta\rangle,\ (y, h(y)) \in U,\ \eta = (\eta_1, \dots, \eta_{N-1}),$$

U being a small neighborhood of a fixed point $x_0 \in \Gamma_j$ so that $\varphi(x;\eta)$ satisfies conditions (i)–(iii) of Section 8. Assume that $|\eta| \le 1 - \mu$, where $0 < \mu < 1$. It is convenient to consider a little more general problem with boundary data given by a smooth function $\chi(y)$ such that $|\nabla_y \chi(y)| \le 1 - \mu$ for $y \in W$. We will construct a phase function $\varphi(x)$ such that

$$\varphi(y, h(y)) = \chi(y), \qquad y \in W, \tag{B-1}$$

omitting the dependence on η in the notation. From the boundary condition (B-1) we determine the derivatives of φ on the boundary Γ_j. Set

$$\varphi_y = (\varphi_{y_1}, \dots, \varphi_{y_{N-1}}), \qquad h_y = (h_{y_1}, \dots, h_{y_{N-1}}), \qquad \chi_y = (\chi_{y_1}, \dots, \chi_{y_{N-1}}).$$

We have $\varphi_y + \varphi_{x_N} h_y = \chi_y$, so setting $\varphi_{x_N} = \sqrt{1 - |\varphi_y|^2}$ and solving the system

$$\varphi_y + \sqrt{1 - |\varphi_y|^2}\, h_y = \chi_y,$$

we get

$$(1 - |\varphi_y|^2)|h_y|^2 = |\chi_y|^2 + |\varphi_y|^2 - 2\langle \chi_y, \varphi_y\rangle.$$

On the other hand,

$$2\langle \chi_y, \varphi_y\rangle + 2\sqrt{1 - |\varphi_y|^2}\,\langle h_y, \chi_y\rangle = 2|\chi_y|^2,$$

which gives

$$(1 + |h_y|^2)(1 - |\varphi_y|^2) - 2\langle h_y, \chi_y\rangle\sqrt{1 - |\varphi_y|^2} + |\chi_y|^2 - 1 = 0.$$

Consequently, for $\varphi_{x_N} = \sqrt{1 - |\varphi_y|^2}$ we obtain

$$\varphi_{x_N}(y, h(y)) = \frac{1}{1 + |h_y|^2}\Big(\langle h_y, \chi_y\rangle + \sqrt{\langle h_y, \chi_y\rangle^2 + (1 - |\chi_y|^2)(1 + |h_y|^2)}\Big).$$

Now it is easy to see that we have the condition

$$\langle \nabla\varphi(x), \nu(x)\rangle \geq \delta_0 > 0, \quad x = (y, h(y)) \in U. \tag{B-2}$$

In fact in local coordinates $x = (y, h(y))$ the outward normal to Γ_j is given by

$$\nu(x) = \frac{1}{\sqrt{1+|h_y|^2}}(-h_y, 1),$$

and we deduce

$$\langle \nabla\varphi(x), \nu(x)\rangle = \frac{1}{\sqrt{1+|h_y|^2}}[(1+|h_y|^2)\varphi_{x_N} - \langle h_y, \chi_y\rangle] \geq \sqrt{1-|\chi_y|^2} \geq \sqrt{2\mu-\mu^2} > 0.$$

By using (B-2) and a standard argument, we can solve locally the eikonal equation $|\nabla\varphi(x)| = 1$ with initial data

$$\begin{aligned}&\varphi(y, h(y)) = \chi(y),\\ &\nabla_x\varphi(y, h(y)) = \big(\varphi_y(y, h(y)), \varphi_{x_N}(y, h(y))\big), \quad (y, h(y)) \in U.\end{aligned}$$

This argument works for local boundary condition $\chi(y) = \langle y, \eta\rangle$, $|\eta| \leq 1-\delta_1/2$, and we obtain a phase function $\varphi(x;\eta)$, $x = (y, h(y))$, $y \in W$. As in [Ikawa 1988; Burq 1993], we show that the principal curvatures of the wave front

$$\mathcal{G}_\varphi(z) = \{y \in \mathbb{R}^N : \varphi(y;\eta) = \varphi(z;\eta)\}$$

are strictly positive for every $z = (y, h(y)) \in U$.

In order to satisfy condition $(\mathcal{P})$ on Γ_j, we will construct a suitable continuation of $\varphi(x;\eta)$. For this purpose fix a point $x_0 = (y_0, h(y_0)) \in U$. Without loss of generality, we can assume that $\varphi(x_0;\eta) = 0$. Consider a sphere S_0 passing through x_0 with center O in the interior of K_j so that the unit outward normal ν_0 of S_0 at x_0 coincides with $\nabla\varphi(x_0;\eta)$.

Choosing local coordinates $(\theta, z(\theta))$, $\theta \in W \subset \mathbb{R}^{N-1}$ on S_0, let $\Xi_0 = \{(\theta, z(\theta)) : |\theta-\theta_0| \leq 2\varepsilon\} \subset S_0$ be a small neighborhood of $x_0 = (\theta_0, z(\theta_0))$. Consider the trace $\Phi(\theta) = \varphi(\theta, z(\theta))$ of φ on Ξ_0. (We omit again the dependence on η in the notation.) Since $\Phi(\theta_0) = 0$ and $\nabla_\theta\Phi(\theta_0) = 0$, we have

$$|\Phi(\theta)| \leq C_0\varepsilon^2, \quad |\nabla_\theta\Phi(\theta)| \leq C_1\varepsilon, \quad \theta \in \Xi_0.$$

Choose a smooth cutoff function $\alpha(\theta)$, $0 \leq \alpha(\theta) \leq 1$, such that $\alpha(\theta) = 1$ for $|\theta-\theta_0| \leq \varepsilon/2$, $\alpha(\theta) = 0$ for $|\theta-\theta_0| \geq \varepsilon$ with $|\nabla_\theta\alpha| \leq C_2\varepsilon^{-1}$. Set $\chi(\theta) = \alpha(\theta)\Phi(\theta)$. Then for small $\varepsilon > 0$ we have

$$|\nabla_\theta\chi(\theta)| \leq (C_0C_2 + C_1)\varepsilon < 1-\mu < 1.$$

By the procedure above we construct a phase function $\Psi(x)$ so that $\Psi(\theta, z(\theta)) = \chi(\theta)$, $|\theta-\theta_0| \leq 2\varepsilon$. For $\Xi' = \{(\theta, z(\theta)) : \varepsilon \leq |\theta-\theta_0| \leq 2\varepsilon\} \subset \Xi_0$, it is easy to see that $\nabla_x\Psi\big|_{\Xi'}$ coincides with the unit normal ν_0 to S_0. Thus if $x = z + t\nu_0(z)$, $t \geq 0$ with $z \in \Xi'$, we have $\Psi(x) = t$ and for such x the phase $\Psi(x)$ coincides with the phase function $\tilde{\Psi}(x)$ defined *globally* in a neighborhood of S_0 and having boundary data $\tilde{\Psi}(x) = 0$ for all $x \in S_0$. Consequently, we may consider $\tilde{\Psi}(x)$ as a continuation of $\Psi(x)$, so $\Psi(x)$ is defined globally outside a small neighborhood of the center O of S_0 lying in the interior of K_j. It is clear that Ψ satisfies condition $(\mathcal{P})$ on S_0. On the other hand, for $\Xi_1 = \{(\theta, z(\theta)) : |\theta-\theta_0| \leq \varepsilon/2\}$ we have $\Psi\big|_{\Xi_1} = \varphi\big|_{\Xi_1}$ and locally in a neighborhood of x_0 the phases $\Psi(x)$ and $\varphi(x)$ coincide. Thus, we can consider $\Psi(x)$ as a continuation of $\varphi(x)$.

Appendix C: Dolgopyat type estimates for open billiards

Here we first state the assumptions about the billiard flow and the nonwandering set Λ under which the results in [Stoyanov 2007] imply the Dolgopyat type estimates (3-3). Following [Petkov and Stoyanov 2009], we then explain how to apply these in the situation described in Section 6. Full details of the arguments can be found in [Petkov and Stoyanov 2009].

For $x \in \Lambda$ and a sufficiently small $\varepsilon > 0$ let

$$W^s_\varepsilon(x) = \{y \in S^*(\Omega) : d(\phi_t(x), \phi_t(y)) \le \varepsilon \quad \text{for all } t \ge 0,\ d(\phi_t(x), \phi_t(y)) \to 0 \text{ as } t \to \infty\},$$
$$W^u_\varepsilon(x) = \{y \in S^*(\Omega) : d(\phi_t(x), \phi_t(y)) \le \varepsilon \quad \text{for all } t \le 0,\ d(\phi_t(x), \phi_t(y)) \to 0 \text{ as } t \to -\infty\}\}$$

be the (strong) *stable* and *unstable manifolds* of size ε. Then $E^u(x) = T_x W^u_\varepsilon(x)$ and $E^s(x) = T_x W^s_\varepsilon(x)$.

The following *pinching condition*[6] is one of the assumptions mentioned above:

There exist constants $C > 0$ and $0 < \alpha \le \beta$ such that for every $x \in \Lambda$ we have

$$C^{-1}\, e^{\alpha_x t}\, \|u\| \le \|d\phi_t(x) \cdot u\| \le C\, e^{\beta_x t}\, \|u\|, \quad u \in E^u(x),\ t > 0, \tag{P}$$

for some constants $\alpha_x, \beta_x > 0$ depending on x but independent of u and t with

$$\alpha \le \alpha_x \le \beta_x \le \beta \quad \text{and} \quad 2\alpha_x - \beta_x \ge \alpha \quad \text{for all } x \in \Lambda.$$

When $N = 2$ this condition is always satisfied. For $N \ge 3$, some general conditions on K that imply (P) are given in [Stoyanov 2009]. According to general regularity results, (P) implies that $W^u_\varepsilon(x)$ and $W^s_\varepsilon(x)$ are Lipschitz in $x \in \Lambda$. In fact, it follows from [Hasselblatt 1994; 1997] that, assuming (P), the map $\Lambda \ni x \mapsto E^u(x)$ is $C^{1+\varepsilon}$ with $\varepsilon = 2\inf_{x\in\Lambda}(\alpha_x/\beta_x) - 1 > 0$, in the sense that this map has a linearization at any $x \in \Lambda$ that depends (uniformly Hölder) continuously on x. The same applies to the map $\Lambda \ni x \mapsto E^s(x)$.

Next, we need some definitions from [Stoyanov 2007]. Given $z \in \Lambda$, let

$$\exp^u_z : E^u(z) \to W^u_{\varepsilon_0}(z) \quad \text{and} \quad \exp^s_z : E^s(z) \to W^s_{\varepsilon_0}(z)$$

be the corresponding *exponential maps*. A vector $b \in E^u(z) \setminus \{0\}$ will be called *tangent to* Λ at z if there exist infinite sequences $\{v^{(m)}\} \subset E^u(z)$ and $\{t_m\} \subset \mathbb{R} \setminus \{0\}$ such that $\exp^u_z(t_m v^{(m)}) \in \Lambda \cap W^u_\varepsilon(z)$ for all m, $v^{(m)} \to b$ and $t_m \to 0$ as $m \to \infty$. It is easy to see that a vector $b \in E^u(z) \setminus \{0\}$ is tangent to Λ at z if there exists a C^1 curve $z(t)$ $(0 \le t \le a)$ in $W^u_\varepsilon(z)$ for some $a > 0$ with $z(0) = z$ and $\dot z(0) = b$, and $z(t) \in \Lambda$ for arbitrarily small $t > 0$. In a similar way one defines tangent vectors to Λ in $E^s(z)$.

Denote by $d\alpha$ the standard symplectic form on $T^*(\mathbb{R}^N) = \mathbb{R}^N \times \mathbb{R}^N$. The following condition says that $d\alpha$ is in some sense nondegenerate on the "tangent space" of Λ near some its points:

There exist $z_0 \in \Lambda$, $\varepsilon > 0$ and $\mu_0 > 0$ such that, for any $\hat z \in \Lambda \cap W^u_\varepsilon(z_0)$ and any unit vector $b \in E^u(\hat z)$ tangent to Λ at $\hat z$, there exist $\tilde z \in \Lambda \cap W^u_\varepsilon(z_0)$ arbitrarily close to $\hat z$ and a unit vector $a \in E^s(\tilde z)$ tangent to Λ at $\tilde z$ with $|d\alpha(a, b)| \ge \mu_0$. (ND)

Remark C.2. Clearly this is always true for $N = 2$. It was shown very recently in [Stoyanov 2009] that for $N \ge 3$ this conditions is always satisfied for open billiard flows satisfying the pinching condition (P).

[6] It appears that in the proof of the estimates (3-3), in the case of open billiard flows (and some geodesic flows), one should be able to replace condition (P) by just assuming Lipschitzness of the stable and unstable laminations. This will be the subject of future work.

It follows from the hyperbolicity of Λ that if $\varepsilon > 0$ is sufficiently small, there exists $\delta > 0$ such that if $x, y \in \Lambda$ and $d(x, y) < \delta$, then $W^s_\varepsilon(x)$ and $\phi_{[-\varepsilon,\varepsilon]}(W^u_\varepsilon(y))$ intersect at exactly one point $[x, y] \in \Lambda$ [Katok and Hasselblatt 1995]. That is, there exists a unique $t \in [-\varepsilon, \varepsilon]$ such that $\phi_t([x, y]) \in W^u_\varepsilon(y)$. Setting $\Delta(x, y) = t$, defines the so called *temporal distance function*. Given $E \subset \Lambda$, we will denote by $\mathrm{Int}_\Lambda(E)$ and $\partial_\Lambda E$ the interior and the boundary of the subset E of Λ in the topology of Λ, and by $\mathrm{diam}(E)$ the diameter of E. Following [Dolgopyat 1998], a subset R of Λ will be called a *rectangle* if it has the form $R = [U, S] = \{[x, y] : x \in U, y \in S\}$, where U and S are subsets of $W^u_\varepsilon(z) \cap \Lambda$ and $W^s_\varepsilon(z) \cap \Lambda$, respectively, for some $z \in \Lambda$ that coincide with the closures of their interiors in $W^u_\varepsilon(z) \cap \Lambda$ and $W^s_\varepsilon(z) \cap \Lambda$.

Let $\mathcal{R} = \{R_i\}_{i=1}^k$ be a Markov family of rectangles $R_i = [U_i, S_i]$ for Λ (for the definition, see [Bowen 1973], [Dolgopyat 1998] or [Stoyanov 2007] for instance). Set $R = \bigcup_{i=1}^k R_i$, denote by $\mathcal{P} : R \to R$ the corresponding Poincaré map, and by τ the first return time associated with $\mathcal{R}$. Then $\mathcal{P}(x) = \phi_{\tau(x)}(x) \in R$ for any $x \in R$. Notice that τ is constant on each stable fiber of each R_i. We will assume that the *size* $\chi = \max_i \mathrm{diam}(R_i)$ of the Markov family $\mathcal{R} = \{R_i\}_{i=1}^k$ is sufficiently small so that each rectangle R_i is between two boundary components Γ_{p_i} and Γ_{q_i} of K, that is for any $x \in R_i$, the first backward reflection point of the billiard trajectory γ determined by x belongs to Γ_{p_i}, while the first forward reflection point of γ belongs to Γ_{q_i}.

Moreover, using the fact that the intersection of Λ with each cross-section to the flow ϕ_t is a Cantor set, we may assume that the Markov family $\mathcal{R}$ is chosen in such a way that

(i) for any $i = 1, \ldots, k$ we have $\partial_\Lambda U_i = \varnothing$.

Finally, partitioning each R_i into finitely many smaller rectangles if necessary and removing some unnecessary rectangles from the family formed in this way, we may assume that

(ii) for every $x \in R$ the billiard trajectory of x from x to $\mathcal{P}(x)$ makes exactly one reflection.

From now on we will assume that $\mathcal{R} = \{R_i\}_{i=1}^k$ is a fixed Markov family for ϕ_t of size $\chi < \varepsilon_0/2$ satisfying conditions (i) and (ii). Set

$$U = \bigcup_{i=1}^k U_i.$$

The map $\tilde{\sigma} : U \to U$ is given by $\tilde{\sigma} = \pi^{(U)} \circ \mathcal{P}$, where $\pi^{(U)} : R \to U$ is the *projection* along stable leaves.

Let $\mathcal{A} = (\mathcal{A}_{ij})_{i,j=1}^k$ be the matrix given by $\mathcal{A}_{ij} = 1$ if $\mathcal{P}(R_i) \cap R_j \neq \varnothing$ and $\mathcal{A}_{ij} = 0$ otherwise. Consider the symbol space

$$\Sigma_{\mathcal{A}} = \{(i_j)_{j=-\infty}^{\infty} : 1 \le i_j \le k,\ \mathcal{A}_{i_j i_{j+1}} = 1 \text{ for all } j\},$$

with the product topology and the *shift map* $\sigma : \Sigma_{\mathcal{A}} \to \Sigma_{\mathcal{A}}$ given by $\sigma((i_j)) = ((i'_j))$, where $i'_j = i_{j+1}$ for all j. As in [Bowen 1973] one defines a natural map $\Psi : \Sigma_{\mathcal{A}} \to R$. Namely, given any $(i_j)_{j=-\infty}^{\infty} \in \Sigma_{\mathcal{A}}$ there is exactly one point $x \in R_{i_0}$ such that $\mathcal{P}^j(x) \in R_{i_j}$ for all integers j. We then set $\Psi((i_j)) = x$. One checks that $\Psi \circ \sigma = \mathcal{P} \circ \Psi$ on R. It follows from the condition (i) above that *the map* Ψ *is a bijection*.

In a similar way one deals with the one-sided subshift

$$\Sigma^+_{\mathcal{A}} = \{(i_j)_{j=0}^{\infty} : 1 \le i_j \le k,\ \mathcal{A}_{i_j i_{j+1}} = 1 \text{ for all } j \ge 0\},$$

where the *shift map* $\sigma : \Sigma^+_{\mathcal{A}} \to \Sigma^+_{\mathcal{A}}$ is defined in the same way. There exists a unique map $\psi : \Sigma^+_{\mathcal{A}} \to U$ such that $\psi \circ \pi = \pi^{(U)} \circ \Psi$, where $\pi : \Sigma_{\mathcal{A}} \to \Sigma^+_{\mathcal{A}}$ is the natural projection.

Notice that the *roof function* $r : \Sigma_{\mathscr{A}} \to [0, \infty)$ defined by $r(\xi) = \tau(\Psi(\xi))$ depends only on the forward coordinates of $\xi \in \Sigma_{\mathscr{A}}$. Indeed, if $\xi_+ = \eta_+$, where $\xi_+ = (\xi_j)_{j=0}^{\infty}$, then for $x = \Psi(\xi)$ and $y = \Psi(\eta)$ we have $x, y \in R_i$ for $i = \xi_0 = \eta_0$ and $\mathscr{P}^j(x)$ and $\mathscr{P}^j(y)$ belong to the same R_{i_j} for all $j \geq 0$. This implies that x and y belong to the same local stable fibre in R_i and by condition (ii), it follows that $\tau(x) = \tau(y)$. Thus, $r(\xi) = r(\eta)$. So, we can define a *roof function* $r : \Sigma_{\mathscr{A}}^+ \to [0, \infty)$ such that $r \circ \pi = \tau \circ \Psi$.

Let $B(\Sigma_{\mathscr{A}}^+)$ be the space of bounded functions $g : \Sigma_{\mathscr{A}}^+ \to \mathbb{C}$ with its standard sup norm $\| \cdot \|_0$. Given a function $g \in B(\Sigma_{\mathscr{A}}^+)$, the *Ruelle transfer operator* $\mathscr{L}_g : B(\Sigma_{\mathscr{A}}^+) \to B(\Sigma_{\mathscr{A}}^+)$ is defined by $(\mathscr{L}_g h)(\eta) = \sum_{\sigma(\eta)=\xi} e^{g(\eta)} h(\eta)$. Denote by $C^{\mathrm{Lip}}(U)$ the space of Lipschitz functions $h : U \to \mathbb{C}$, and for $h \in C^{\mathrm{Lip}}(U)$ let $\mathrm{Lip}(h)$ denote the Lipschitz constant of h. For $t \in \mathbb{R}$, $|t| \geq 1$, define

$$\|h\|_{\mathrm{Lip},t} = \|h\|_0 + \frac{\mathrm{Lip}(h)}{|t|}, \quad \|h\|_0 = \sup_{x \in U} |h(x)|.$$

Given a real-valued function g on $\Sigma_{\mathscr{A}}^+$ with $g \circ \psi^{-1} \in C^{\mathrm{Lip}}(U)$, there is a unique $s(g) \in \mathbb{R}$ such that

$$\mathrm{Pr}(-s(g)\, r + g) = 0.$$

If $G : \Lambda \to \mathbb{C}$ is a continuous function such that $(g \circ \psi^{-1} \circ \pi^{(U)})(x) = \int_0^{\tau(x)} G(\phi_t(x))\, dt$, with $x \in \mathbb{R}$, then $s(g) = \mathrm{Pr}_{\phi_t}(G)$, the topological pressure of G with respect to the flow ϕ_t on Λ [Parry and Pollicott 1990, Chapter 6].

The following is an immediate consequence of the main result in [Stoyanov 2007], taking into account the particular considerations for open billiard flows in [Stoyanov 2009].

Theorem C.3. *Assume the billiard flow ϕ_t over Λ satisfies conditions* (P) *and* (ND). *Let $g : \Sigma_{\mathscr{A}}^+ \to \mathbb{R}$ be such that $g \circ \psi^{-1} \in C^{\mathrm{Lip}}(U)$. Then there exist constants $a > 0$, $t_0 \geq 1$, $\sigma(g) < s(g)$, $C > 0$ and $0 < \rho < 1$ such that, for any $s = \tau + \boldsymbol{i}\, t$ with $\tau \geq \sigma(g)$, $|\tau| \leq a$ and $|t| \geq t_0$, any integer $n \geq 1$ and any function $\upsilon : \Sigma_{\mathscr{A}}^+ \to \mathbb{C}$ with $\upsilon \circ \psi^{-1} \in C^{\mathrm{Lip}}(U)$, writing $n = p[\log |t|] + l$, $p \in \mathbb{N}$, $0 \leq l \leq [\log |t|] - 1$, we have*

$$\|(\mathscr{L}_{-sr+g}^n\, \upsilon) \circ \psi^{-1}\|_{\mathrm{Lip},t} \leq C \rho^{p[\log |t|]} e^{l \mathrm{Pr}(-\tau r + g)} \|\upsilon \circ \psi^{-1}\|_{\mathrm{Lip},t}. \tag{C-1}$$

Remark C.4. Another way to state the estimate above is the following [Dolgopyat 1998; Stoyanov 2007]: For every $g : \Sigma_{\mathscr{A}}^+ \to \mathbb{R}$ with $g \circ \psi^{-1} \in C^{\mathrm{Lip}}(U)$ and every $\varepsilon > 0$ there exist constants $0 < \rho < 1$, $a_0 > 0$ and $C > 0$ such that for any integer $m > 0$, any $s = \tau + \boldsymbol{i}\, t \in \mathbb{C}$ with $|\tau| \leq a_0$, $|t| \geq 1/a_0$ and any function $\upsilon : \Sigma_{\mathscr{A}}^+ \to \mathbb{C}$ with $\upsilon \circ \psi^{-1} \in C^{\mathrm{Lip}}(U)$ we have:

$$\|(\mathscr{L}_{-sr+g}^m\, \upsilon) \circ \psi^{-1}\|_{\mathrm{Lip},t} \leq C \rho^m |t|^{\varepsilon} \|\upsilon \circ \psi^{-1}\|_{\mathrm{Lip},t}.$$

In the remaining part of this section, following [Petkov and Stoyanov 2009], we show how to apply the Dolgopyat type estimates (C-1) to obtain the estimates of $\|L_s^n \mathscr{G}_s \tilde{\upsilon}_s\|_{\Gamma,0}$ required in Section 5. The problem is that the operator L_s acts on $C(\Sigma_A^+)$, that is, it is related to the coding of billiard trajectories by means of the components of K, while the Dolgopyat type estimates apply to Ruelle transfer operators $\mathscr{L}_{-sr+g}$ defined by means of Markov families and acting on functions υ such that $\upsilon \circ \psi^{-1}$ is Lipschitz with respect to the standard metric in the phase space. Here we describe how the two types of Ruelle transfer operators relate, and show that the function $(\mathscr{G}_s \tilde{\upsilon}_s) \circ \psi^{-1}$ is Lipschitz. This makes it possible to apply (C-1).

Apart from the coding described above, we can also use the coding of the flow over Λ by using the boundary components of K described in Section 3. We will use the notation from there, notably $f(\xi)$, $g(\xi)$, $\eta^{(k)}$ for any $k = 1, \dots, \kappa_0$, $e(\xi)$, $\chi_f = \chi_1$, $\chi_g = \chi_2$, $\tilde{f}(\xi)$ and $\tilde{g}(\xi)$. Define the map $\Phi : \Sigma_A \to \Lambda_{\partial K} = \Lambda \cap S^*_\Lambda(\Omega)$ by

$$\Phi(\xi) = \left(P_0(\xi), \frac{P_1(\xi) - P_0(\xi)}{\| P_1(\xi) - P_0(\xi) \|} \right).$$

Then Φ is a bijection such that $\Phi \circ \sigma = B \circ \Phi$, where $B : \Lambda_{\partial K} \to \Lambda_{\partial K}$ is the billiard ball map. As before, given any function $G \in B(\Sigma_A^+)$, the *Ruelle transfer operator* $L_G : B(\Sigma_A^+) \to B(\Sigma_A^+)$ is defined by $(L_G H)(\xi) = \sum_{\sigma(\eta)=\xi} e^{G(\eta)} H(\eta)$.

Let $\omega : V_0 \to S^*_{\partial K}(\Omega)$ be the backward shift along the flow defined in Section 3 on some neighborhood V_0 of Λ in $S^*(\Omega)$. Consider the bijection $\mathcal{S} = \Phi^{-1} \circ \omega \circ \Psi : \Sigma_{\mathcal{A}} \to \Sigma_A$. Its restriction to $\Sigma_{\mathcal{A}}^+$ defines a bijection $\mathcal{S} : \Sigma_{\mathcal{A}}^+ \to \Sigma_A^+$. Moreover $\mathcal{S} \circ \sigma = \sigma \circ \mathcal{S}$. Define the function $g' : \Sigma_{\mathcal{A}} \to \mathbb{R}$ by $g'(\underline{i}) = g(\mathcal{S}(\underline{i}))$.

Next, for any $i = 1, \dots, k$, choose

$$\underline{\hat{j}}^{(i)} = (\dots, j^{(i)}_{-m}, \dots, j^{(i)}_{-1}) \quad \text{such that } (\underline{\hat{j}}^{(i)}, i) \in \Sigma^-_{\mathcal{A}}.$$

It is convenient to make this choice in such a way that $\underline{\hat{j}}^{(i)}$ *corresponds to the local unstable manifold* $U_i \subset \Lambda \cap W^u_\varepsilon(z_i)$, that is, the backward itinerary of every $z \in U_i$ coincides with $\underline{\hat{j}}^{(i)}$. Now for any $\underline{i} = (i_0, i_1, \dots) \in \Sigma_{\mathcal{A}}^+$ (or $\underline{i} \in \Sigma_{\mathcal{A}}$) set

$$\hat{e}(\underline{i}) = (\underline{\hat{j}}^{(i_0)}; i_0, i_1, \dots) \in \Sigma_{\mathcal{A}}.$$

According to the choice of $\underline{\hat{j}}^{(i_0)}$, we then have $\Psi(\hat{e}(\underline{i})) = \psi(\underline{i}) \in U_{i_0}$. (Notice that without this special choice we would only have that $\Psi(\hat{e}(\underline{i}))$ and $\psi(\underline{i}) \in U_{i_0}$ lie on the same stable leaf in R_{i_0}.) Next, define

$$\hat{\chi}_g(\underline{i}) = \sum_{n=0}^{\infty} \big(g'(\sigma^n(\underline{i})) - g'(\sigma^n \, \hat{e}(\underline{i})) \big) \quad \text{for } \underline{i} \in \Sigma_{\mathcal{A}}.$$

As before, the function $\hat{g} : \Sigma_{\mathcal{A}} \to \mathbb{R}$ given by $\hat{g}(\underline{i}) = g'(\underline{i}) - \hat{\chi}_g(\underline{i}) + \hat{\chi}_g(\sigma \, \underline{i})$ depends on future coordinates only, so it can be regarded as a function on $\Sigma_{\mathcal{A}}^+$.

We will now describe a natural relationship between the operators

$$\mathcal{L}_V : B(\Sigma_{\mathcal{A}}^+) \to B(\Sigma_{\mathcal{A}}^+) \quad \text{and} \quad L_v : B(\Sigma_A^+) \to B(\Sigma_A^+),$$

with v appropriately defined by means of V.

First define $\Gamma : B(\Sigma_A) \to B(\Sigma_{\mathcal{A}})$ by $\Gamma(v) = v \circ \Phi^{-1} \circ \omega \circ \Psi = v \circ \mathcal{S}$. Since by property (ii) of the Markov family, $\omega : R \to \Lambda_{\partial K}$ is a bijectiion, it follows that Γ is a bijection and $\Gamma^{-1}(V) = V \circ \Psi^{-1} \circ \omega^{-1} \circ \Phi$. Moreover, Γ induces a bijection $\Gamma : B(\Sigma_A^+) \to B(\Sigma_{\mathcal{A}}^+)$. Indeed, assume that $v \in B(\Sigma_A)$ depends on future coordinates only. Then $v \circ \Phi^{-1}$ is constant on local stable manifolds in $S^*_\Lambda(\Omega)$. Hence $v \circ \Phi^{-1} \circ \omega$ is constant on local stable manifolds on R, and therefore $\Gamma(v) = v \circ \Phi^{-1} \circ \omega \circ \Psi$ depends on future coordinates only.

Next, let $v, w \in B(\Sigma_A^+)$ and let $V = \Gamma(v)$, $W = \Gamma(w)$. Given $\underline{i}, \underline{j} \in \Sigma_{\mathcal{A}}^+$ with $\sigma(\underline{j}) = \underline{i}$, setting $\xi = \mathcal{S}(\underline{i})$ and $\eta = \mathcal{S}(\underline{j})$, we have $\sigma(\eta) = \xi$. Thus,

$$\mathcal{L}_W V(\underline{i}) = \sum_{\sigma(\underline{j})=\underline{i}} e^{W(\underline{j})} V(\underline{j}) = \sum_{\sigma(\underline{j})=\underline{i}} e^{w(\mathcal{S}(\underline{j}))} v(\mathcal{S}(\underline{j})) = L_w v(\xi) \quad \text{for all } \underline{i} \in \Sigma_{\mathcal{A}}^+.$$

This shows that $(L_w v) \circ \mathcal{S} = \mathcal{L}_{\Gamma(w)} \Gamma(v)$.

The equality

$$\Pr(-\tau r + \hat{g}) = \Pr(-\tau \tilde{f} + \tilde{g}) \tag{C-2}$$

and the following proposition are established in [Petkov and Stoyanov 2009, Section 3].

Proposition C.5. *Assume that the map $\Lambda \ni x \mapsto W^u_\varepsilon(x)$ is Lispchitz. Then there exist Lipschitz functions $\delta_1, \delta_2 : U \to \mathbb{R}$ such that setting $\hat{\delta}_s(\underline{i}) = e^{s\,\delta_1(\psi(\underline{i}))+\delta_2(\psi(\underline{i}))}$, we have*

$$\big(L^n_{-s\,\tilde{f}+\tilde{g}} u\big)(\mathcal{S}(\underline{i})) = \frac{1}{\hat{\delta}_s(\underline{i})} \cdot \mathcal{L}^n_{-s\,r+\hat{g}}\big(\hat{\delta}_s \cdot (u \circ \mathcal{S})\big)(\underline{i}), \quad \underline{i} \in \Sigma^+_{\mathcal{A}},\ s \in \mathbb{C}, \tag{C-3}$$

for any $u \in C(\Sigma^+_A)$ and any integer $n \geq 1$.

Combining (C-1)–(C-3), we deduce:

Theorem C.6 [Petkov and Stoyanov 2009]. *Assume the billiard flow ϕ_t over Λ satisfies conditions* (P) *and* (ND). *There exist constants $a > 0$, $\sigma_0 < s_0$, $t_0 \geq 1$, $C' > 0$ and $0 < \rho < 1$ so that for any $s = \tau + \boldsymbol{i}\,t \in \mathbb{C}$ with $\tau \geq \sigma_0$, $|\tau| \leq a$, $|t| \geq t_0$, any integer $n \geq 1$ and any function $u : \Sigma^+_A \to \mathbb{R}$ with $u \circ \mathcal{S} \circ \psi^{-1} \in C^{\mathrm{Lip}}(U)$, writing $n = p[\log|t|] + l$, $p \in \mathbb{N}$, $0 \leq l \leq [\log|t|] - 1$, we have*

$$\big\| \big(L^n_{-s\tilde{f}+\tilde{g}}\, u\big) \circ \mathcal{S} \circ \psi^{-1} \big\|_{\mathrm{Lip},t} \leq C' \rho^{p[\log|t|]} e^{l P(-\tau \tilde{f}+\tilde{g})} \|u \circ \mathcal{S} \circ \psi^{-1}\|_{\mathrm{Lip},t}. \tag{C-4}$$

The estimate (3-3) is a consequence of (C-4) and it could hold even if the condition (P) is not fulfilled (see Remark C.2 for condition (ND)).

Next, for the needs of Section 5, we have to estimate $\|L^n_{-s\tilde{f}+\tilde{g}} \mathcal{G}_s \tilde{v}_s\|_{\Gamma,0}$, where the operator $\mathcal{G}_s$ is defined in Section 3. For any integer $n \geq 0$ we have

$$\begin{aligned} L^n_{-s\tilde{f}+\tilde{g}} \mathcal{G}_s v(\xi) &= \sum_{\sigma^n \eta = \xi} \sum_{\sigma\zeta = \eta} e^{-s\,\tilde{f}_n(\eta)+\tilde{g}_n(\eta)} e^{-\phi^+(\zeta,s) - s\tilde{f}(\zeta)+\tilde{g}(\zeta)} v(\zeta) \\ &= \sum_{\sigma^{n+1}\zeta = \xi} e^{-s\,\tilde{f}_{n+1}(\zeta)+\tilde{g}_{n+1}(\zeta)} e^{-\phi^+(\zeta,s)} v(\zeta) = L^{n+1}_{-s\tilde{f}+\tilde{g}} (e^{-\phi^+(\,\cdot\,,s)} v)\,(\xi). \end{aligned}$$

Thus, it is enough to estimate

$$\big\| L^{n+1}_{-s\tilde{f}+\tilde{g}} (e^{-\phi^+(\,\cdot\,,s)}\,\tilde{v}_s) \big\|_{\Gamma,0}.$$

As in Sections 3–5, we will consider these operators over Γ_1.

Given $s \in \mathbb{C}$, consider the functions $w_s : U_1 \to \mathbb{R}$ and $\hat{w}_s : \Sigma^+_{\mathcal{A}} \to \mathbb{R}$ defined by

$$w_s(x) = w_s(\psi(\underline{i})) = \hat{w}_s(\underline{i}) = e^{-\phi^+(\xi,s)} \tilde{v}_s(\xi), \qquad \text{for } x = \psi(\underline{i}) \in U_1,\ \underline{i} \in \Sigma^+_{\mathcal{A}},\ \xi = \mathcal{S}(\underline{i}).$$

In order to use the Dolgopyat type estimate (3-3), we have to show that w_s is Lispchitz on U_1. We will deal in details with

$$w^{(1)}_s(x) = e^{s\,\sum_{n=0}^\infty [f(\sigma^n e(\xi)) - f^+_n(\xi)] - s\,\varphi(Q_0(\xi))} h(Q_0(\xi));$$

in a similar way one can deal with $w^{(2)}_s(x) = e^{-\sum_{n=0}^\infty [g(\sigma^n e(\xi)) - g^+_n(\xi)]}$. It follows from the definitions of $\phi^+(\xi, s)$ and $\tilde{v}_s$ in Section 3 that $w_s(x) = w^{(1)}_s(x)\, w^{(2)}_s(x)$.

Fix an arbitrary point $y_1 \in \Lambda$ such that $\eta^{(1)} \in \Sigma^-_A$ corresponds to the local unstable manifold $W^u_{\mathrm{loc}}(y_1)$, i.e. the backward itinerary of every $z \in W^u_{\mathrm{loc}}(y_1) \cap V_0$ coincides with $\eta^{(1)}$. It follows from the Lipschitzness of the stable and unstable laminations that the map $\mathcal{H}_1 : U_1 \to W^u_{\mathrm{loc}}(y_1)$ defined by $\mathcal{H}_1(x) =$

$\phi_{\Delta(x,y_1)}([x, y_1])$ is Lipschitz. Here Δ is the temporal distance function defined in the beginning of this section.

Next, consider the N-dimensional submanifold $X = \{(q, q + t\nabla\varphi(q) : q \in \Gamma_1, 0 < t\}$ of $S^*(\mathbb{R}^N)$ and the (stable) holonomy map $\mathcal{H} : W^u_{\text{loc}}(y_1) \cap \Lambda \to X$ defined by $\mathcal{H}(y) = W^s_{\text{loc}}(y) \cap X$. Since φ satisfies Ikawa's condition $(\mathcal{P})$, it is easy to see that $W^s_{\text{loc}}(y)$ is transversal to X, so $\mathcal{H}(y) = W^s_{\text{loc}}(y) \cap X$ is well-defined for $y \in W^u_{\text{loc}}(y_1) \cap \Lambda$. Moreover, it follows from our assumptions that the stable (and unstable) holonomy maps for the billiard flow ϕ_t are Lispchitz. In particular, $\mathcal{H}$ is Lipschitz.

We can now write down $w^{(1)}_s(x)$ using the maps $\mathcal{H}$ and $\mathcal{H}_1$ as follows. Given $x \in U_1$, we have $x = \psi(\underline{i})$ for some $\underline{i} \in \Sigma^+_{\mathcal{A}}$, with $i_0 = 1$. Setting $\xi = \mathcal{S}(\underline{i})$, we then have $\xi_0 = 1$. For any integer $m > 1$ consider

$$B_m = \sum_{n=0}^{m-1} [f(\sigma^n e(\xi)) - f^+_n(\xi)] - \varphi(Q_0(\xi)).$$

Setting

$$y = \mathcal{H}_1(x) \in W^u_{\text{loc}}(y_1), \quad z = \mathcal{H}(y),$$

we have that $z \in W^s_{\text{loc}}(y)$, and moreover $\omega(z) = (Q_0(\xi), \nabla\varphi(Q_0(\xi)))$. Thus,

$$Q_0(\xi) = \text{pr}_1(\omega(z)) = \text{pr}_1\big(\omega(\mathcal{H}(\mathcal{H}_1(x)))\big)$$

is Lipschitz in $x \in U_1$. Next, set $\varepsilon(u) = \|\text{pr}_1(u) - \text{pr}_1(\omega(u))\|$; then $u = \phi_{\varepsilon(u)}(\omega(y))$ and $\varepsilon(u)$ is a smooth function on an open subset of $S^*(\Omega)$ (where ω is defined and takes values in $S^*_{\Gamma_1}(\Omega)$). For B_m we have

$$B_m = O(\theta^m) + \varepsilon(y) - \varepsilon(z) - \varphi(\omega(z)) = O(\theta^m) + \varepsilon(y) - \varphi(z),$$

and letting $m \to \infty$ we get

$$w^{(1)}_s(x) = e^{s[\varepsilon(y) - \varphi(z)]}\, h(\omega(z)) = e^{s[\varepsilon(\mathcal{H}_1(x)) - \varphi(\mathcal{H}(\mathcal{H}_1(x)))]}\, h\big(\omega(\mathcal{H}(\mathcal{H}_1(x)))\big),$$

so $w^{(1)}_s(x)$ is Lipschitz in $x \in U_1$. Moreover, for $x \in U_1$ and bounded $\text{Re}\, s$ we obtain an uniform bound for the Lipschitz norm of $w^{(1)}_s(x)$. The same argument works for $w^{(2)}_s(x)$.

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