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LIFSHITZ TAILS FOR GENERALIZED ALLOY-TYPE RANDOM SCHRÖDINGER OPERATORS

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We study Lifshitz tails for random Schrödinger operators where the random potential is alloy-type in the sense that the single site potentials are independent, identically distributed, but they may have various function forms. We suppose the single site potentials are distributed in a finite set of functions, and we show that under suitable symmetry conditions, they have a Lifshitz tail at the bottom of the spectrum except for special cases. When the single site potential is symmetric with respect to all the axes, we give a necessary and sufficient condition for the existence of Lifshitz tails. As an application, we show that certain random displacement models have a Lifshitz singularity at the bottom of the spectrum, and also complete our previous study (2009) of continuous Anderson type models.

1. Introduction

Consider the continuous alloy-type (or Anderson) random Schrödinger operator

$$H_\omega = -\Delta + V_0 + V_\omega, \qquad \text{where } V_\omega(x) = \sum_{\gamma\in\mathbb{Z}^d} \omega_\gamma V(x-\gamma) \tag{1-1}$$

on $\mathbb{R}^d$, $d \geq 1$, where

- V_0 is a periodic potential;
- V is a compactly supported single site potential;
- $(\omega_\gamma)_{\gamma\in\mathbb{Z}^d}$ are independent identically distributed random coupling constants.

Let Σ be the almost sure spectrum of H_ω and $E_- = \inf \Sigma$. When V has a fixed sign, it is well known that, if a =ess-inf(ω_0) and b =ess-sup(ω_0), then $E_- = \inf(\sigma(-\Delta + V_{\bar{b}}))$ if $V \leq 0$ and $E_- = \inf(\sigma(-\Delta + V_{\bar{a}}))$ if $V \geq 0$. Here, $\bar{x}$ is the constant vector $\bar{x} = (x)_{\gamma\in\mathbb{Z}^d}$.

For E a real energy, the *integrated density of states* is defined by

$$N(E) = \lim_{L\to+\infty} \frac{\#\{\text{eigenvalues of } H^N_{\omega,L} \ \leq E\}}{L^d}, \tag{1-2}$$

where

$$H^N_{\omega,L} = -\Delta + V_0 + V_\omega \qquad \text{on } L^2(C_L(0)), \tag{1-3}$$

with Neumann boundary conditions, where $C_L(0)$ is defined by (1-4). It is well-known that $N(E)$ exists and is non-random, i.e., $N(E)$ is independent of ω, almost surely; it has been the object of a lot of studies.

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In particular, it is well known that the integrated density of states of the Hamiltonian admits a Lifshitz tail near E_-, i.e.,

$$\lim_{E\to E_-^+} \frac{\log|\log N(E)|}{\log(E-E_-)} < 0.$$

Actually, the limit can often be computed and in many cases is equal to $-d/2$; we refer to [Carmona and Lacroix 1990; Kirsch 1989; 1985; Pastur and Figotin 1992; Stollmann 2001; Veselić 2004; 2008] for extensive reviews and more precise statements.

In the present paper, we mainly consider a generalized Bernoulli alloy-type model that we define below: we allow the single site potential to have various function forms (with a discrete distribution). We give a necessary and sufficient condition to have Lifshitz tail under a symmetry assumption on the single site potentials. The results we obtain are then applied to the random displacement models studied recently by Baker, Loss and Stolz [2008; 2009], and also to complete the study of the occurrence of Lifshitz tails for alloy-type models initiated in [Klopp and Nakamura 2009].

1.1. ***The model.*** We now describe our model. We let $d \geq 1$ and we study operators on $\mathcal{H} = L^2(\mathbb{R}^d)$. By

$$C_\ell(x) = \{y \in \mathbb{R}^d \mid 0 \leq y_j - x_j \leq \ell,\ j = 1, \dots, d\}, \tag{1-4}$$

we denote the cube with edge $\ell > 0$ and x as the lower right corner. Let $V_0 \in C^0(\mathbb{R}^d)$ be a background potential, periodic with respect to $\mathbb{Z}^d$.

Let $v_k \in C_c^0(C_1(0))$, $k = 1, \dots, M$, be single site potentials where $M \in \mathbb{N}$. We consider the random Schrödinger operator:

$$H_\omega = -\Delta + V_0 + V_\omega \qquad \text{on } \mathcal{H} = L^2(\mathbb{R}^d),$$

where

$$V_\omega(x) = \sum_{\gamma\in\mathbb{Z}^d} v_{\omega(\gamma)}(x-\gamma)$$

is the random potential and $\{\omega(\gamma) \mid \gamma \in \mathbb{Z}^d\}$ are independent, identically distributed random variables with values in $\{1, \dots, M\}$.

To fix ideas, let us assume

$$\inf \sigma(H_\omega) = 0, \quad \text{a.s. } \omega, \tag{1-5}$$

which can always be achieved by shifting V_0 by a constant.

We set

$$H_k^N = -\Delta + V_0 + v_k \quad \text{on } L^2(C_1(0)),$$

with Neumann boundary conditions on the boundary $\partial C_1(0)$.

Assumption A. (1) V_0 is symmetric about the plane $\{x \mid x_d = 1/2\}$. (2) There exists $m \in \{1, \dots, M\}$ such that

$$\inf \sigma(H_k^N) = 0 \qquad \text{for } k = 1, \dots, m,$$
$$\inf \sigma(H_k^N) > 0 \qquad \text{for } k > m.$$

(3) For $k = 1, \dots, m$, $v_k(x)$ is symmetric about $\{x_d = 1/2\}$.

Remark 1.1. Note that in this assumption, we only require symmetry with respect to a single coordinate hyperplane that we chose to be the d-th one.

If one assumes that V_0 and the $(v_k)_{1\le k\le M}$ are reflection symmetric with respect to all the coordinate planes [Baker et al. 2008; 2009; Klopp and Nakamura 2009], the standard characterization of the almost sure spectrum [Pastur and Figotin 1992; Kirsch 1989] and lower bounding H_ω by the direct sum of its Neumann restrictions to the cubes $(C_1(\gamma))_{\gamma\in\mathbb{Z}^d}$ show that, as a consequence of (1-5), one obtains

- for all $k\in\{1,\dots,M\}$, $\inf\sigma(H_k^N)\ge 0$;
- there exists $k\in\{1,\dots,M\}$ such that $\inf\sigma(H_k^N)=0$.

1.2. *The results.* We study the Lifshitz singularity for the integrated density of states (IDS) at the zero energy. Recall that the IDS is defined by (1-2).

We first consider a relatively easy case:

Theorem 1.2. *Suppose Assumption A holds with $m<M$. Then*

$$\limsup_{E\to+0}\frac{\log|\log N(E)|}{\log E}\le -\frac{1}{2}. \tag{1-6}$$

We expect that (1-6) holds with $-d/2$ on the right-hand side, which is known to be optimal; see [Klopp and Nakamura 2009, Theorem 0.2 and Section 2.2], for example.

If $m=M$, then we need further classification of the potential functions. We denote the standard basis of $\mathbb{R}^d$ by

$$\mathbf{e}_j=(\delta_{ji})_{i=1}^d\in\mathbb{R}^d,\quad j=1,\dots,d,$$

and we define an operator $H^N_{k\ell(j)}$ on $L^2(U_j)$ as

$$U_j=C_1(0)\cup C_1(\mathbf{e}_j),\quad j=1,\dots,d. \tag{1-7}$$

We set

$$H^N_{k\ell(j)}=\begin{cases}-\Delta+V_0(x)+v_k(x) & \text{on } C_1(0),\\ -\Delta+V_0(x)+v_\ell(x-\mathbf{e}_j) & \text{on } C_1(\mathbf{e}_j),\end{cases} \tag{1-8}$$

with Neumann boundary conditions on ∂U_j, where $k,\ell\in\{1,\dots,m\}$ and $j\in\{1,\dots,d\}$. We define

$$v_j\sim_j v_\ell \quad\overset{\text{def}}{\iff}\quad \inf\sigma(H^N_{k\ell(j)})=0. \tag{1-9}$$

Namely, $v_k\sim_j v_\ell$ implies that the coupling of two local Hamiltonians H_k^N and H_ℓ^N does not increase the ground state energy. We note that $v_k\not\sim_j v_\ell$ generically for $k\ne\ell$.

Theorem 1.3. *Suppose Assumption A holds with $m=M$, and that $v_k\not\sim_d v_\ell$ for some $k\ne\ell$. Then* (1-6) *holds*, *i.e.*, *H_ω has Lifshitz singularities at the zero energy.*

To obtain a more precise result on the existence and the absence of Lifshitz singularities, we make a stronger symmetry assumption on the potentials.

Assumption B. In addition to satisfying Assumption A, V_0 and v_k are symmetric about $\{x\mid x_j=1/2\}$ for all $j=1,\dots,d$, and $k=1,\dots,m=M$.

Theorem 1.4. *Suppose Assumption B holds.*

(i) *If* $v_k \not\sim_j v_\ell$ *for some* j *and* $k \neq \ell$, *then* (1-6) *holds.*

(ii) *If* $v_k \sim_j v_\ell$ *for all* j *and* k, ℓ, *then the van Hove property holds*, *namely*, *there exists* $C > 0$ *such that*

$$\frac{1}{C} E^{d/2} \leq N(E) \leq C E^{d/2}. \tag{1-10}$$

In (1-10), the asymptotic is new only for E small; for E large, it is a consequence of Weyl's law. The example in Section 3 of [Klopp and Nakamura 2009] is a special case of Theorem 1.4(ii).

In a previous paper [Klopp and Nakamura 2009], we used the concavity of the ground state energy with respect to the random parameters, and also used an operator theoretical trick to reduce the problem to the monotonous perturbation case. These methods are not available under the assumptions of the present paper. Instead, we employ a quadratic inequality similar to the Poincaré inequality, and take advantage of the positivity of certain Dirichlet-to-Neumann operators to obtain a lower bound of the ground state energy for Schrödinger operators on a strip. This estimate is quasi one-dimensional, and this is why we obtain Lifshitz tail estimate with the exponent corresponding to the one-dimensional case. We do believe that this method can be refined to obtain the optimal exponent, though we have not been successful so far.

This paper is organized as follows. We discuss the eigenvalue estimate on a strip in Section 2 and prove our main theorems in Section 3. We discuss an application to random displacement models in Section 4, and an application to the model studied in [Klopp and Nakamura 2009] in Section 5.

Throughout this paper, we use the following notations: $\mathbb{P}(\,\cdot\,)$ denotes the probability measure for the random potential, and $\mathbb{E}(\,\cdot\,)$ denotes the expectation; $\mathfrak{D}(A)$ denotes the definition domain of an operator A; $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the inner product of L^2-spaces; $\partial\Omega$ denotes the boundary of a domain Ω; and $\#\Lambda$ denotes the cardinality of a set Λ.

2. Lower bounds on the ground state energy

Throughout this section, we suppose $v_1, \dots v_m$ satisfy Assumption A. Let $a > 0$,

$$\Omega_0 = [0,1]^{d-1} \times [-a, 0] \subset \mathbb{R}^d,$$

and let $W_0 \in C^0(\Omega_0)$ be a real-valued function on Ω_0. We set

$$P_0^N = -\Delta + W_0 \quad \text{on } L^2(\Omega_0)$$

with Neumann boundary conditions. Let $L \in \mathbb{N}$,

$$\Omega_1 = [0,1]^{d-1} \times [0, L]$$

and let $W_1 \in C^0(\Omega_1)$ such that

$$W_1 = V_0 + v_{k(\ell)}(x - \ell \mathbf{e}_d) \quad \text{if } x \in C_1(\ell \mathbf{e}_d),\ \ell = 0, \dots, L-1,$$

where $\{k(\ell)\}_{\ell=0}^{L-1}$ is a sequence with values in $\{1, \dots, m\}$. We then set

$$\Omega = \Omega_0 \cup \Omega_1, \qquad W(x) = \begin{cases} W_0(x) & \text{if } x \in \Omega_0, \\ W_1(x) & \text{if } x \in \Omega_1, \end{cases}$$

and set

$$P^N = -\Delta + W \quad \text{on } L^2(\Omega),$$

with Neumann boundary conditions. The main result of this section is this:

Theorem 2.1. *Suppose* $\inf \sigma(P_0^N) > 0$, *and suppose* $v_{k(\ell)} \sim_d v_{k(\ell')}$ *for* $\ell, \ell' \in \{0, \ldots, L-1\}$. *Then there exists* $C > 0$ *such that* C *is independent of* L *and of the sequence* $\{k(\ell)\}$, *and such that*

$$\inf \sigma(P^N) \geq \frac{1}{CL^2}.$$

In the following, we suppose $v_k \sim_d v_\ell$ for all k, ℓ for simplicity (and without loss of generality). We prove Theorem 2.1 by a series of lemmas. First, we show a variant of the classical Poincaré inequality. Let Γ be the trace operator from $H^1(\Omega_1)$ to $L^2(S)$ with $S = [0,1]^{d-1} \times \{0\}$, i.e.,

$$\Gamma\varphi(x') = \varphi(x', 0) \quad \text{for } x' \in [0,1]^{d-1},\ \varphi \in C^0(\Omega_1),$$

and Γ extends to a bounded operator from $H^1(\Omega_1)$ to $L^2(S)$.

Lemma 2.2. *Let* $\varphi \in H^1(\Omega_1)$. *Then*

$$\frac{2}{L}\|\Gamma\varphi\|^2_{L^2(S)} + \|\nabla\varphi\|^2_{L^2(\Omega_1)} \geq \frac{1}{L^2}\|\varphi\|^2_{L^2(\Omega_1)}.$$

Proof. It suffices to show the estimate for $\varphi \in C^1(\Omega_1)$. Since

$$\varphi(x', t) = \varphi(x', 0) + \int_0^t \partial_{x_d}\varphi(x', s)\,ds, \qquad x' \in [0,1]^{d-1}, t \in [0, L],$$

we have

$$|\varphi(x', t)| \leq |\varphi(x', 0)| + \int_0^t |\partial_{x_d}\varphi(x', s)|\,ds \leq |\varphi(x', 0)| + \sqrt{t}\Big(\int_0^L |\nabla\varphi(x', s)|^2 ds\Big)^{1/2}$$

by the Cauchy-Schwarz inequality. This implies

$$\begin{aligned}
\|\varphi\|^2_{L^2(\Omega_1)} &\leq \iint_0^L \bigg\{|\varphi(x', 0)| \ + \sqrt{t}\Big(\int_0^L |\nabla\varphi(x', s)|^2 ds\Big)^{1/2}\bigg\}^2 dt\,dx' \\
&\leq 2\iint_0^L |\varphi(x', 0)|^2 ds\,dx' + 2\int_0^L t\,dt \times \|\nabla\varphi\|^2_{L^2(\Omega_1)} \\
&= 2L\|\Gamma\varphi\|^2_{L^2(S)} + L^2\|\nabla\varphi\|^2_{L^2(\Omega_1)}
\end{aligned}$$

and the claim follows. □

For $k \in \{1, \ldots, M\}$, we set

$$q_k(\varphi, \psi) = \int_{C_1(0)} (\nabla\varphi \cdot \nabla\overline{\psi} + v_k\varphi\overline{\psi})\,dx, \qquad \varphi, \psi \in H^1(C_1(0)),$$

which is the quadratic form corresponding to H_k^N. Let Ψ_k be the positive ground state for H_k^N, which is unique up to a constant. Since $\inf \sigma(H_k^N) = 0$, we expect φ/Ψ_k is close to a constant if $q_k(\varphi, \varphi)$ is close to 0, and this observation is justified by the following lemma.

Lemma 2.3. *There exists $c_1 > 0$ such that*

$$\|\nabla(\varphi/\Psi_k)\|^2_{L^2(C_1(0))} \le c_1 q_k(\varphi,\varphi), \quad \varphi \in H^1(C_1(0)), k = 1, \dots, m.$$

Proof. This is a consequence of the so-called *ground state transform*. It suffices to show the inequality when $\varphi \in C^1(C_1(0))$. We set $f = \varphi/\Psi_k$. Then we have

$$\begin{aligned} q_k(\varphi,\varphi) &= \langle \nabla(f\Psi_k), \nabla(f\Psi_k)\rangle + \langle v_k f\Psi_k, f\Psi_k\rangle \\ &= \|\Psi_k(\nabla f)\|^2 + \langle \Psi_k\nabla f, f\nabla\Psi_k\rangle + \langle f\nabla\Psi_k, \Psi_k\nabla f\rangle \\ &\qquad\qquad + \langle f\nabla\Psi_k, f\nabla\Psi_k\rangle + \langle v_k f\Psi_k, f\Psi_k\rangle \\ &= \|\Psi_k(\nabla f)\|^2 + \langle \nabla(|f|^2\Psi_k), \nabla\Psi_k\rangle + \langle v_k|f|^2\Psi_k, \Psi_k\rangle \\ &= \|\Psi_k(\nabla f)\|^2 + q_k(|f|^2\Psi_k, \Psi_k). \end{aligned}$$

Since $q_k(|f|^2\Psi_k, \Psi_k) = \langle (H_k^N)^{1/2}|f|^2\Psi_k, (H_k^N)^{1/2}\Psi_k\rangle = 0$, we have

$$q_k(\varphi,\varphi) = \|\Psi_k(\nabla f)\|^2 \ge (\inf|\Psi_k|)^2\|\nabla f\|^2,$$

and we may choose $c_1 = (\min_k \inf|\Psi_k|)^{-2}$. □

Lemma 2.4. *Suppose $v_k \sim_d v_\ell$. Then there exists $\mu_1, \mu_2 > 0$ such that*

$$\mu_1\Psi_k(x',0) = \mu_2\Psi_\ell(x',0), \qquad \text{for } x' \in [0,1]^{d-1}.$$

Proof. Consider $H^N_{k\ell(d)}$ in U_d (see (1-7) and (1-8) in Section 1), and let $\Phi \in L^2(U_d)$ be the positive ground state of $H^N_{k\ell(j)}$. We set

$$\varphi_1 = \Phi\lceil_{C_1(0)}, \quad \varphi_2(\cdot) = \Phi(\cdot + \mathbf{e}_d)\lceil_{C_1(0)}.$$

Then φ_1, φ_2 are positive and $q_k(\varphi_1,\varphi_1) = q_\ell(\varphi_2,\varphi_2) = 0$. By the variational principle and the uniqueness of the ground states, we learn

$$\varphi_1 = \mu_1\Psi_k, \quad \varphi_2 = \mu_2\Psi_\ell$$

with some $\mu_1, \mu_2 > 0$. By Assumption A, Ψ_k and Ψ_ℓ are symmetric about $\{x_d = 1/2\}$, and hence

$$\mu_1\Psi_k(x',0) = \mu_1\Psi_k(x',1) = \varphi_1(x',1) = \varphi_2(x',0) = \mu_2\Psi_\ell(x',0)$$

for $x' \in [0,1]^{d-1}$, where we have used the continuity of Φ on $\{x_d = 1\}$. □

Now, let Ω_1 and W_1 be as in the beginning of Section 2, and define

$$P_1^N = -\triangle + W_1 \quad \text{on } L^2(\Omega_1)$$

with Neumann boundary conditions. We set

$$Q_1(\varphi,\psi) = \int_{\Omega_1} (\nabla\varphi\cdot\nabla\overline{\psi} + W_1\varphi\overline{\psi})\,dx = \langle (P_1^N)^{1/2}\varphi, (P_1^N)^{1/2}\psi\rangle$$

for $\varphi, \psi \in H^1(\Omega_1) = \mathcal{D}((P_1^N)^{1/2})$.

Lemma 2.5. *There exists $c_2 > 0$ such that c_2 is independent of L and of the sequence $\{k(\ell)\}$, and*

$$\frac{1}{L}\|\Gamma\varphi\|^2_{L^2(S)} + Q_1(\varphi,\varphi) \geq \frac{1}{c_2 L^2}\|\varphi\|^2_{L^2(\Omega_1)}$$

for $\varphi \in H^1(\Omega_1)$.

Proof. By Lemma 2.4, there exist $\mu_1, \dots, \mu_m > 0$ such that

$$\mu_1\Psi_1(x',0) = \mu_2\Psi_2(x',0) = \cdots = \mu_m\Psi_m(x',0).$$

We set

$$\Psi(x) = \mu_{k(\ell)}\Psi_{k(\ell)}(x - \ell\mathbf{e}_d) \quad \text{if } \ell \leq x_d \leq \ell+1,$$

and then $\Psi \in H^1(\Omega_1)$ by the above observation. Moreover, Ψ is the ground state for P_1^N, unique up to a constant. We apply Lemma 2.2 to φ/Ψ, and we have

$$\begin{aligned}
\frac{1}{L^2}\|\varphi\|^2_{L^2(\Omega_1)} &\leq \frac{1}{L^2}(\sup\Psi)^2\|\varphi/\Psi\|^2_{L^2(\Omega_1)} \\
&\leq \frac{(\sup\Psi)^2}{L}\|\Gamma(\varphi/\Psi)\|^2_{L^2(S)} + (\sup\Psi)^2\|\nabla(\varphi/\Psi)\|^2_{L^2(\Omega_1)} \\
&\leq \left(\frac{\sup\Psi}{\inf\Psi}\right)^2\frac{1}{L}\|\Gamma\varphi\|^2_{L^2(S)} + c_1(\sup\Psi)^2 Q_1(\varphi,\varphi),
\end{aligned}$$

where we have used Lemma 2.3 in the last inequality. The claim follows immediately. □

We next consider $P_0 = -\Delta + W_0$ on $L^2(\Omega_0)$ and its Dirichlet-to-Neumann operator. As in Theorem 2.1, we suppose

$$\alpha = \inf\sigma(P_0^N) > 0.$$

We set

$$P_0' = -\Delta + W_0 \quad \text{on } L^2(\Omega_0) \text{ with } \mathfrak{D}((P_0')^{1/2}) = \{\varphi \in H^1(\Omega_0) \mid \Gamma\varphi = 0\},$$

where Γ is the trace operator from $H^1(\Omega_1)$ to $L^2(S)$. Then P_0' defines a self-adjoint operator, and each $\varphi \in \mathfrak{D}(P_0')$ satisfies Dirichlet boundary conditions on S and Neumann boundary conditions on $\partial\Omega_0 \setminus S$. Let $\lambda < \alpha$. By a standard argument of the theory of elliptic boundary value problems (see [Folland 1995], for instance), for any $g \in H^{3/2}(S)$, there exists a unique $\psi \in H^2(\Omega_0)$ such that

$$(-\Delta + W_0 - \lambda)\psi = 0, \qquad \Gamma\psi = g \tag{2-1}$$

and that satisfies Neumann boundary conditions on $\partial\Omega_0 \setminus S$. Then the map

$$T(\lambda) \; : \; g \mapsto \Gamma(\partial_\nu\psi) \in H^{1/2}(S)$$

defines a bounded linear map from $H^{3/2}(S)$ to $H^{1/2}(S)$, where $\partial_\nu = \partial/\partial x_d$ is the outer normal derivative on S. We consider $T(\lambda)$ as an operator on $L^2(S)$, and it is called the *Dirichlet-to-Neumann operator*.

Lemma 2.6. *$T(\lambda)$ is a symmetric operator. If $\lambda_0 < \alpha$, then $T(\lambda) \geq \varepsilon$ for $0 \leq \lambda \leq \lambda_0$ with some $\varepsilon > 0$.*

Proof. Let $\varphi, \psi \in H^2(\Omega_0)$ such that $\Gamma\varphi = f$, $\Gamma\psi = g$, and

$$(-\Delta + W_0 - \lambda)\varphi = (-\Delta + W_0 - \lambda)\psi = 0,$$

with Neumann boundary conditions on $\partial\Omega_0 \setminus S$. By Green's formula we have

$$\begin{aligned} 0 &= \langle(-\triangle + W_0 - \lambda)\varphi, \psi\rangle - \langle\varphi, (-\triangle + W_0 - \lambda)\psi\rangle \\ &= -\int_S \partial_\nu\varphi \cdot \overline{\psi} + \int_S \varphi \cdot \partial_\nu \overline{\psi} = -\langle T(\lambda) f, g\rangle_{L^2(S)} + \langle f, T(\lambda) g\rangle_{L^2(S)}, \end{aligned}$$

and hence $T(\lambda)$ is symmetric. Similarly, we have

$$\begin{aligned} 0 = \langle(-\triangle + W_0 - \lambda)\varphi, \varphi\rangle &= -\int_S \partial_\nu\varphi \cdot \overline{\varphi} + \int_{\Omega_0} |\nabla\varphi|^2 + \int_{\Omega_0} (W_0 - \lambda)|\varphi|^2 \\ &= -\langle T(\lambda) f, f\rangle + Q_0(\varphi, \varphi) - \lambda\|\varphi\|^2, \end{aligned}$$

where $Q_0(\varphi, \varphi) = \int_{\Omega_0} \big(|\nabla\varphi|^2 + W_0|\varphi|^2\big) dx$. Hence, we learn that

$$\langle T(\lambda) f, f\rangle = Q_2(\varphi, \varphi) - \lambda\|\varphi\|^2 \geq Q_0(\varphi, \varphi) - \lambda_0\|\varphi\|^2.$$

The form in the right-hand side is equivalent to $\|\varphi\|^2_{H^1(\Omega_0)}$ since $\lambda_0 < \alpha$. Hence, it is bounded from below by $\varepsilon\|f\|^2_{L^2(S)}$ with some $\varepsilon > 0$ by virtue of the boundedness of the trace operator from $H^1(\Omega_0)$ to $L^2(S)$. □

We note that $T(\lambda)$ extends to a self-adjoint operator on $L^2(S)$ by the Friedrichs extension, though we do not use the fact in this paper.

Proof of Theorem 2.1. Let φ be the ground state of P^N on Ω with the ground state energy $\lambda \geq 0$. If $\lambda \geq \lambda_0 > 0$ with some fixed λ_0 (independently of L), then the statement is obvious, and hence we may assume $0 \leq \lambda \leq \lambda_0 < \alpha = \inf \sigma(P_0^N)$ without loss of generality.

Let $f = \Gamma\varphi \in H^{3/2}(S)$. Since φ satisfies Neumann boundary conditions on $\partial\Omega_0 \setminus S$, we learn $\partial_\nu\varphi\lceil_S = T(\lambda)\varphi$. On the other hand, by Green's formula, we have

$$\begin{aligned} \int_{\Omega_1} P^N\varphi \cdot \overline{\varphi} &= \int_S \partial_n\varphi \cdot \overline{\varphi} + \int_{\Omega_1} |\nabla\varphi|^2 + W_1|\varphi|^2 \\ &= \langle T(\lambda) f, f\rangle_{L^2(S)} + Q_1(\varphi, \varphi) \\ &\geq \varepsilon\|f\|^2_{L^2(S)} + Q_1(\varphi, \varphi) \end{aligned}$$

by Lemma 2.6. Now, we apply Lemma 2.5 to learn that the right-hand side is bounded from below by $(1/c_2L^2)\|\varphi\|^2_{L^2(\Omega_1)}$. Since $P^N\varphi = \lambda\varphi$ and $\|\varphi\|_{L^2(\Omega_1)} \neq 0$, this implies $\lambda \geq 1/c_2L^2$ for large enough L. □

3. Proof of the main theorems

We now discuss the proofs of Theorems 1.2 and 1.3, and we prove Theorem 1.4 at the end of the section. We thus suppose Assumption A with either $m < M$ or that there exists k, k' such that $v_k \not\sim_d v_{k'}$.

For notational simplicity, we assume the reflections of v_k at $\{x_d = 1/2\}$ are included in the possible set of potentials $\{v_k\}$. This does not change the conditions on $\{v_1, \dots, v_m\}$, but we might need to add the reflections of $\{v_{m+1}, \dots, v_M\}$. This does not affect the following arguments.

We write

$$\Lambda = \{p \in \mathbb{Z}^{d-1} \mid 0 \leq p_j \leq L - 1,\ j = 1, \dots, d - 1\}$$

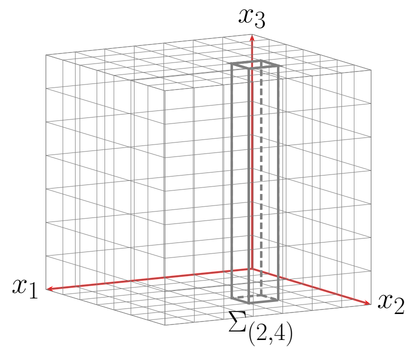

Figure 1. Chopping the cube into strips.

and, for $p \in \Lambda$, we set

$$\Sigma_p = \bigcup_{k=0}^{L-1} C_1((p,k))$$

so that $C_L(0)$ is decomposed (see Figure 1) as

$$C_L(0) = \bigcup_{p \in \Lambda} \Sigma_p$$

which is a disjoint union except for the boundaries of the strips.

For a given V_ω and $p \in \Lambda$, we consider the restriction of H_ω to Σ_p, i.e.,

$$\tilde{H}_p^N = \Delta + V_0 + \sum_{\ell=0}^{L-1} v_{\omega((p,\ell))}(x-(p,\ell)) \quad \text{on } L^2(\Sigma_p)$$

with Neumann boundary conditions on $\partial\Sigma_p$. By the standard Neumann bracketing, we learn

$$H_{\omega,L}^N \geq \bigoplus_{p \in \Lambda} \tilde{H}_p^N \quad \text{on } L^2(C_L(0)) \cong \bigoplus_{p \in \Lambda} L^2(\Sigma_p),$$

and hence, in particular,

$$\inf \sigma(H_{\omega,L}^N) \geq \min_{p \in \Lambda} \inf \sigma(\tilde{H}_p^N). \tag{3-1}$$

Under our assumptions, one of the following holds for each $p \in \Lambda$:

$(a)_p$: $\omega((p,\ell)) > m$ for some ℓ, or $v_{\omega((p,\ell))} \nsim_d v_{\omega((p,\ell'))}$ for some $\ell, \ell' \in \{0, \dots, L-1\}$.

$(b)_p$: For all $\ell, \ell' \in \{0, \dots, L-1\}$, $\omega((p,\ell)) \leq m$ and $v_{\omega((p,\ell))} \sim_d v_{\omega((p,\ell'))}$.

We note that the probability of Condition $(b)_p$ to occur is less than μ^{-L} with some $\mu < 1$ independent of L. Since $\{\omega(\gamma)\}$ are independent, we have

$$\mathbb{P}\big((b)_p \text{ holds for some } p \in \Lambda\big) \leq L^d \mu^{-L}, \tag{3-2}$$

which is small if L is large. For the moment, then, we suppose Condition $(a)_p$ holds for all $p \in \Lambda$.

We denote by $V^p(x)$ the potential function of $\tilde{H}_p^N$ on Σ_p. Let

$$\hat{\Sigma}_p = (p + [0,1]^{d-1}) \times (\mathbb{R}/(2L\mathbb{Z}))$$

and set $\hat{V}^p(x) = V^p(x', |x_d|)$ for $x = (x', x_d) \in (p + [0,1]^{d-1}) \times [-L, L) \cong \hat{\Sigma}_p$, i.e., $\hat{V}^p$ is the extension of $\tilde{V}^p$ by the reflection at $\{x_d = 0\}$. We note $\hat{V}^p$ is continuous on $\hat{\Sigma}_p$. We now consider

$$\hat{H}_p^N = \triangle + \hat{V}^p \quad \text{on } L^2(\hat{\Sigma}_p)$$

with Neumann boundary conditions. It is easy to see

$$\inf \sigma(\tilde{H}_p^N) \geq \inf \sigma(\hat{H}_p^N). \tag{3-3}$$

In fact, if Φ is the ground state of $\tilde{H}_p^N$, then we extend Φ by reflection to obtain $\hat{\Phi} \in H^1(\hat{\Sigma}_p)$ and we have

$$\frac{\langle \hat{H}_p^N \hat{\Phi}, \hat{\Phi}\rangle}{\|\hat{\Phi}\|^2} = \frac{\langle \tilde{H}_p^N \Phi, \Phi\rangle}{\|\Phi\|^2} = \inf \sigma(\tilde{H}_p^N)$$

and the claim (3-3) follows by the variational principle.

Since we assume Condition $(a)_p$, Σ_p can be decomposed to subsegments $\Sigma_p = \bigcup_{j=1}^K \Xi_j$ such that each Ξ_j satisfies the following conditions: We write

$$\Xi_j = \bigcup_{\ell=0}^{\nu} C_1(p, \kappa + \ell), \quad \kappa \in \mathbb{Z},\ 0 \leq \nu < L$$

and

$$\hat{V}^p(x) = v_{\beta(\ell)}(x - (p, \ell)) \quad \text{for } x \in C_1(p, \kappa + \ell),\ \ell \in \{0, \dots, \nu\},$$

with $\beta(\ell) \in \{1. \dots, M\}$. Then either one of the following holds:

(i) $\beta(0) \in \{m+1, \dots, M\}$; $\beta(\ell) \in \{1, \dots, m\}$ for $\ell \geq 1$; and $v_{\beta(\ell)} \sim_d v_{\beta(\ell')}$ for $\ell, \ell' \in \{1, \dots, \nu\}$.

(ii) $\beta(\ell) \in \{1, \dots, m\}$ for all ℓ; $v_{\beta(0)} \not\sim_d v_{\beta(1)}$; and $v_{\beta(\ell)} \sim_d v_{\beta(\ell')}$ for $\ell, \ell' \in \{2, \dots, \nu\}$.

The proof of this claim is an easy, though somewhat lengthy, combinatorial exercise. We omit the details.

We again decompose $\hat{H}_p^N$. We denote the restriction of $\hat{H}_p^N$ to Ξ_j by P_j on $L^2(\Xi_j)$ with Neumann boundary conditions. Then again by Neumann bracketing, we learn that

$$\hat{H}_p^N \geq \bigoplus_{j=1}^{\kappa} P_j \quad \text{on } L^2(\hat{\Sigma}_p) \cong \bigoplus_{j=1}^{\kappa} L^2(\Xi_j),$$

and in particular,

$$\inf \sigma(\hat{H}_p^N) \geq \min_j \inf \sigma(P_j). \tag{3-4}$$

Now if (i) holds for Ξ_j, then we set $a = 1$ and use Theorem 2.1 for P_j. Since $\inf \sigma(H_{\beta(0)}^N) > 0$ by Assumption A and $\nu \leq L$, we learn that

$$\inf \sigma(P_j) \geq \frac{1}{C(\nu-1)^2} \geq \frac{1}{C(L-1)^2}.$$

If (ii) holds for Ξ_j, then we set $a = 2$ and use Theorem 2.1 for P_j. Since $v_{\beta(0)} \nsim_d v_{\beta(1)}$, we have $\inf \sigma(H^N_{\beta(0)\beta(1)(d)}) > 0$. Thus we have

$$\inf \sigma(P_j) \geq \frac{1}{C(\nu-2)^2} \geq \frac{1}{C(L-2)^2}.$$

Combining these with (3-1), (3-3) and (3-4), we conclude that

$$\inf \sigma(H^N_{\omega,L}) \geq \frac{c_3}{L^2} \tag{3-5}$$

with some $c_3 > 0$, provided Condition $(a)_p$ holds for all $p \in \Lambda$.

Proof of Theorems 1.2 and 1.3. For $E > 0$, we set

$$\sqrt{\frac{c_3}{E}} < L \leq \sqrt{\frac{c_3}{E}} + 1,$$

so that, by virtue of (3-5),

$$\inf \sigma(H^N_{\omega,L}) > E$$

provided Condition $(a)_p$ holds for all $p \in \Lambda$. As noted in (3-2), the probability of the events that Condition $(b)_p$ holds for some $p \in \Lambda$ is bounded by

$$\mathbb{P}\big((b)_p \text{ holds for some } p \in \Lambda\big) \leq L^d \mu^{-L} \leq c_4 E^{-d/2} e^{-c_5 E^{-1/2}}$$

with some $c_4, c_5 > 0$. On the other hand, since the potential $V_0 + V_\omega$ is uniformly bounded, we have

$$\#\{\text{eigenvalues of } H^N_{\omega,L} \leq \alpha\} \leq c_6 L^d$$

for any ω with some $c_6 > 0$. Thus we have

$$\begin{aligned} L^{-d}\mathbb{E}\big(\#\{\text{e.v. of } H^N_{\omega,L} \leq E\}\big) &\leq L^{-d}(c_6 L^d)\mathbb{P}\big((b)_p \text{ holds for some } p \in \Lambda\big) \\ &\leq c_4 c_6 E^{-d/2} e^{-c_5 E^{-1/2}} \leq c_7 e^{-(c_5-\varepsilon)E^{-1/2}} \end{aligned}$$

for $0 < \varepsilon < c_5$ with some $c_7 > 0$. By the Neumann bracketing again, we have

$$N(E) \leq L^{-d}\mathbb{E}\big(\#\{\text{e.v. of } H^N_{\omega,L} \leq E\}\big) \leq c_7 e^{-(c_5-\varepsilon)E^{-1/2}}$$

and Theorems 1.2 and 1.3 follow immediately from this estimate. □

In fact, we have proved that

$$\liminf_{E\to+0} \frac{|\log N(E)|}{E^{-1/2}} > 0,$$

and this statement is slightly stronger than (1-6).

Proof of Theorem 1.4. Statement (i) is an immediate consequence of Assumption B and Theorem 1.3. We just replace the x_d-axis by the x_j-axis where $v_k \nsim_j v_\ell$ for some k, ℓ.

For (ii), we use the ground state transform as in the proof of Lemmas 2.3–2.5. Under our conditions, there exist $\mu_1, \dots, \mu_m > 0$ such that

$$\mu_1 \Psi_1(x) = \mu_2 \Psi_2(x) = \cdots = \mu_m \Psi_m(x) \quad \text{for } x \in \partial C_1(0).$$

For given $H^N_{\omega,L}$, we set

$$\Phi(x) = \mu_k \Psi_k(x) \quad \text{if } x \in C_1(\gamma) \text{ with } \omega(\gamma) = k.$$

Then it is easy to see that Φ is the positive ground state of $H^N_{\omega,L}$ with the energy 0. Let $Q(\cdot\,,\cdot)$ be the quadratic form corresponding to $H^N_{\omega,L}$. For $\varphi \in H^1(C_L(0))$, we set $f = \varphi/\Phi$. As in the proof of Lemma 2.3, we have

$$Q(\varphi, \varphi) = \|\Phi(\nabla f)\|^2$$

and hence

$$(\inf \Phi)^2 \|\nabla f\|^2 \le Q(\varphi, \varphi) \le (\sup \Phi)^2 \|\nabla f\|^2.$$

This implies

$$K^{-2} \frac{\|\nabla f\|^2}{\|f\|^2} \le \frac{Q(\varphi, \varphi)}{\|\varphi\|^2} \le K^2 \frac{\|\nabla f\|^2}{\|f\|^2},$$

where $K = \max_k \sup(\mu_k \Psi_k)/\min_k \inf(\mu_k \Psi_k)$. By the min-max principle, we learn that

$$K^{-2}\,\#\{\text{e.v. of } (-\triangle)^N_L \le E\} \le \#\{\text{e.v. of } H^N_{\omega,L} \le E\} \le K^2\,\#\{\text{e.v. of } (-\triangle)^N_L \le E\},$$

where $(-\triangle)^N_L$ is the Laplacian on $C_L(0)$ with Neumann boundary conditions. Taking the limit $L \to +\infty$, we have

$$K^{-2} c_d E^{d/2} \le N(E) \le K^2 c_d E^{d/2}, \tag{3-6}$$

where c_d is the volume of the unit ball in $\mathbb{R}^d$. This completes the proof of Theorem 1.4. □

4. Application to random displacement models

We now consider a model recently studied by Baker, Loss and Stolz [2008; 2009]. Combining their results with Theorem 1.2, we show that this model exhibits Lifshitz singularities at the ground state energy.

We consider a random Schrödinger operator of the form:

$$H_\omega = -\triangle + V_\omega \quad \text{on } L^2(\mathbb{R}^d),$$

where

$$V_\omega(x) = \sum_{\gamma \in \mathbb{Z}^d} q(x - \gamma - \omega(\gamma))$$

with independent, identically distributed random variables $\{\omega(\gamma) \mid \gamma \in \mathbb{Z}^d\}$ taking values in $C_1(0)$.

Assumption C. (1) There exists $\delta \in (0, 1/2)$ such that $\omega(\gamma)$ takes values in a finite set

$$\Theta \subset \{x \in \mathbb{R}^d \mid \delta \le x_j \le 1 - \delta, \quad \text{for all } j \in \{1, \dots, d\}\}.$$

Moreover

$$\Theta \supset \Delta = \{x \in \mathbb{R}^d \mid x_j = \delta \text{ or } 1 - \delta, \quad \text{for all } j \in \{1, \dots, d\}\}$$

and $\mathbb{P}(\omega(\gamma) = x) > 0$ for $x \in \Delta$.

(2) $q \in C_0(\mathbb{R}^d)$ and it is supported in $\{x \mid |x_j| \le \delta,\ j \in \{1, \dots, d\}\}$. Moreover, q is symmetric about $\{x \mid x_j = 0\}$, $j = 1, \dots, d$.

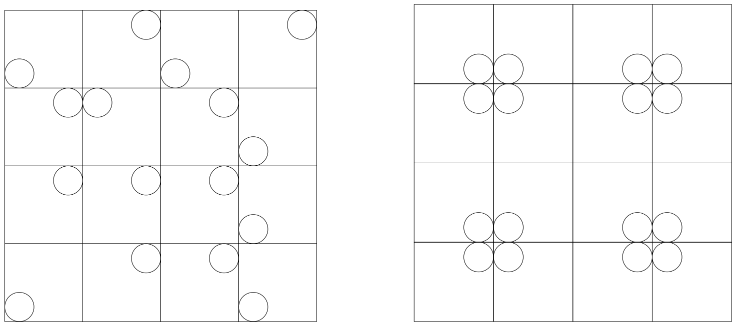

Figure 2. An example in two dimensions, showing a typical random configuration (left) and the minimizing configuration (right).

(3) Let $H_q^N = -\Delta + q$ on $L^2(\{|x| \le 1\})$ with Neumann boundary conditions, and let ϕ be the ground state. Then ϕ is not a constant outside Supp q. Note that this is relevant only if the ground state energy is 0.

Let $H_{1,\beta}^N = -\Delta + q(x-\beta)$ on $L^2(C_1(0))$ with Neumann boundary conditions, where $\beta \in \Theta$. Baker, Loss and Stolz [2008] showed that $\inf \sigma(H_{1,\beta}^N)$ takes its minimum (with respect to β) if and only if $\beta \in \Delta$. In particular, they showed that for $H_{\omega,2\ell}^N$ the Neumann restriction of H_ω to $C_{2\ell}(0)$ the minimal value of the ground state energy was obtained for clustered configuration (see Figure 2).

We cannot directly apply our result to this model, since $q(x-\beta)$ is not symmetric for $\beta \in \Delta$. However, they also showed that if we consider the operator H_ω restricted to $C_2(0)$ and if $d \ge 2$, then the minimum is attained by 2^d symmetric configurations, which are equivalent to each other by translations (see [Baker et al. 2009] and Figure 3). Thus, we can apply our results by considering H_ω as a $2\mathbb{Z}^d$-ergodic random Schrödinger operators, i.e., by considering $C_2(0)$ as the unit cell. Then this model satisfies Assumption A with $M = (\#\Theta)^{2^d}$ and $m = 2^d$.

Theorem 4.1. *Let $d \ge 2$, and suppose Assumption C for some $\delta \in (0, 1/2)$. Then* (1-6) *holds at the bottom of the spectrum of H_ω, a.s.*

We note that if $d = 1$, this result does not hold, and the IDS may have logarithmic singularity at the bottom of the spectrum [Baker et al. 2009]. In view of our results, such singularities can occur for the lack of symmetry of the minimizing configurations.

5. The alloy-type model studied in [Klopp and Nakamura 2009]

In a previous paper on Lifshitz tails for sign indefinite alloy-type random Schrödinger operators [Klopp and Nakamura 2009], we studied the model (1-1) for a single site potential V satisfying the reflection symmetry Assumption B.

We now recall some of the results of that work. Let the support of the random variables $(\omega_\gamma)_\gamma$ be contained in $[a, b]$ and assume both a and b belong to the essential support of the random variables.

Now consider the operator $H_\lambda^N = -\Delta + \lambda V$ with Neumann boundary conditions on the cube $C_1(0) = [0, 1]^d$. Its spectrum is discrete, and we let $E_-(\lambda)$ be its ground state energy. It is a simple eigenvalue

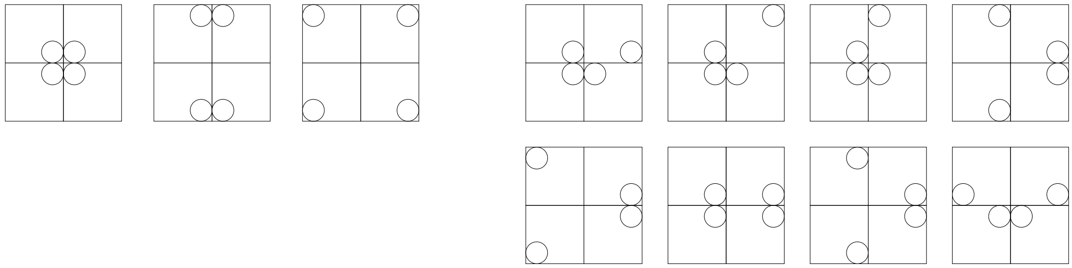

Figure 3. Left: the minimal 2×2 configurations in two dimensions. Right: other 2×2 configurations in two dimensions.

and $\lambda \mapsto E_-(\lambda)$ is a real analytic concave function defined on $\mathbb{R}$. Let E_- be the infimum of the almost sure spectrum of H_ω then

Proposition 5.1 [Klopp and Nakamura 2009]. *Under Assumption B*,

$$E_- = \inf(E_-(a), E_-(b)).$$

As for Lifshitz tails, we proved

Theorem 5.2 [Klopp and Nakamura 2009]. *Suppose that Assumption B is satisfied*, *and that*

$$E_-(a) \neq E_-(b). \tag{5-1}$$

Then

$$\limsup_{E \to E_-^+} \frac{\log|\log N(E)|}{\log(E - E_-)} \leq -\frac{d}{2} - \alpha_+,$$

where we have set $c = a$ if $E_-(a) < E_-(b)$ and $c = b$ if $E_-(a) > E_-(b)$, and

$$\alpha_+ = -\frac{1}{2} \liminf_{\varepsilon \to 0} \frac{\log\bigl|\log \mathbb{P}(\{|c - \omega_0| \leq \varepsilon\})\bigr|}{\log \varepsilon} \geq 0.$$

The technique developed in [Klopp and Nakamura 2009] did not allow us to treat the case $E_-(a) = E_-(b)$. Clearly, if the random variables $(\omega_\gamma)_\gamma$ are non trivial and Bernoulli distributed, i.e., if

$$\mathbb{P}(\omega_0 = a) + \mathbb{P}(\omega_0 = b) = 1 \quad \text{and} \quad \mathbb{P}(\omega_0 = a) > 0, \quad \mathbb{P}(\omega_0 = b) > 0,$$

Theorem 1.4 tells us that the Lifshitz tails hold if and only if $aV \not\sim_j bV$ for some $j \in \{1, \ldots, d\}$ (see (1-9)). So we are just left with the case when the random variables $(\omega_\gamma)_\gamma$ are not Bernoulli distributed.

We prove

Theorem 5.3. *Suppose Assumption B is satisfied and that*

$$E_-(a) = E_-(b). \tag{5-2}$$

Assume moreover that the independent, *identically distributed random variables $(\omega_\gamma)_\gamma$ are not Bernoulli distributed*, *that is*, $\mathbb{P}(\omega_0 = a) + \mathbb{P}(\omega_0 = b) < 1$. *Then*

$$\limsup_{E \to E_-^+} \frac{\log|\log N(E)|}{\log(E - E_-)} \leq -\frac{1}{2}. \tag{5-3}$$

So we show that Lifshitz tails also hold in this case. As already noted we believe that (5-3) is not optimal and that $-1/2$ should be replaced by $-d/2$. Moreover, depending on the tail of the distributions of the random variables $(\omega_\gamma)_\gamma$ near a and b, the lim sup in (5-3) should be a limit, the inequality should become an equality, the exponent $-1/2$ should be replaced by $-d/2$ plus a possibly vanishing constant (see of [Klopp and Nakamura 2009, Section 0] for the case $E_-(a) \neq E_-(b)$).

Combining Theorems 5.2 and 5.3 with the Wegner estimates obtained in [Klopp 1995; Hislop and Klopp 2002] and the multiscale analysis as developed in [Germinet and Klein 2001], we learn:

Theorem 5.4. *Assume that Assumption B holds. and that the common distribution of the random variables admits an absolutely continuous density. Then the bottom edge of the spectrum of H_ω exhibits complete localization in the sense of* [Germinet and Klein 2001].

This result improves upon Theorem 0.3 of [Klopp and Nakamura 2009].

5.1. ***The proof of Theorem 5.3.*** Recall that $H^N_{\omega,L}$ is defined in (1-3). It is well known that, at E, a continuity point of $N(E)$, the sequence

$$N^N_L(E) = \mathbb{E}\left(\frac{\#\{\text{eigenvalues of } H^N_{\omega,L} \ \leq E\}}{L^d}\right)$$

is decreasing and converges to $N(E)$ [Pastur and Figotin 1992; Kirsch 1989]. As

$$N^N_L(E) \leq C\,\mathbb{P}(\{\inf\sigma(H^N_{\omega,L}) \leq E\}), \tag{5-4}$$

it is sufficient to prove an upper bound for $\mathbb{P}(\{\inf\sigma(H^N_{\omega,L}) \leq E\})$ for a well chosen value of L.

Define $E_{-,L}(\omega) = \inf\sigma(H^N_{\omega,L})$. It only depends on $(\omega_\gamma)_{\gamma\in Z_L}$, where

$$Z_L = \{\gamma \in \mathbb{Z}^d \mid 0 \leq \gamma_j < L,\ j = 1, \dots, d\}.$$

Lemma 5.5. *The function $\omega \mapsto E_{-,L}(\omega)$ is real analytic and strictly concave on $[a,b]^{Z_L}$.*

Proof. Though this is certainly a well known result, for the sake of completeness, we give the proof. The ground state being simple, $\omega \mapsto E_{-,L}(\omega)$ is real analytic in ω.

As H_ω depends affinely on ω, by the variational characterization of the ground state energy, $E_{-,L}(\omega)$ is the infimum of a family of affine functions of ω. So it is concave.

The strict concavity is obtained using perturbation theory. Let $\varphi_L(\omega)$ be the unique normalized positive ground state associated to $E_{-,L}(\omega)$ and $H^N_{\omega,L}$. The ground state energy being simple, this ground state is a real analytic function of ω; differentiating once the eigenvalue equation and the normalization condition of the ground state, as the ground state is normalized and real, one obtains

$$(H^N_{\omega,L} - E_{-,L}(\omega))\partial_{\omega_\gamma}\varphi_L(\omega) = \left(\partial_{\omega_\gamma}E_{-,L}(\omega) - V(\,\cdot - \gamma)\right)\varphi_L(\omega), \tag{5-5}$$

$$\langle \partial_{\omega_\gamma}\varphi_L(\omega), \varphi_L(\omega)\rangle = 0. \tag{5-6}$$

A second differentiation yields

$$\begin{aligned}(H^N_{\omega,L} - E_{-,L}(\omega))\partial^2_{\omega_\gamma\omega_\beta}\varphi_L(\omega) = \partial^2_{\omega_\gamma\omega_\beta}E_{-,L}(\omega)\varphi_L(\omega) + \left(\partial_{\omega_\gamma}E_{-,L}(\omega) - V(\,\cdot - \gamma)\right)\partial_{\omega_\beta}\varphi_L(\omega)\\ + \left(\partial_{\omega_\beta}E_{-,L}(\omega) - V(\,\cdot - \beta)\right)\partial_{\omega_\gamma}\varphi_L(\omega).\end{aligned}$$

Hence, using (5-5) and (5-6), we compute

$$\begin{aligned}\partial^2_{\omega_\gamma\omega_\beta}E_{-,L}(\omega) &= -\langle V(\cdot-\gamma)\partial_{\omega_\beta}\varphi_L(\omega),\varphi_L(\omega)\rangle - \langle V(\cdot-\beta)\partial_{\omega_\gamma}\varphi_L(\omega),\varphi_L(\omega)\rangle\\ &= -2\mathrm{Re}(\langle (H^N_{\omega,L}-E_{-,L}(\omega))^{-1}\psi_\beta,\psi_\gamma\rangle),\end{aligned}$$

where

- $\psi_\gamma = \Pi V(\cdot-\gamma)\varphi_L(\omega)$;
- Π is the orthogonal projector on the orthogonal to $\varphi_L(\omega)$.

Hence, for $(a_\gamma)_\gamma$ complex numbers,

$$\sum_{\gamma,\beta}\partial^2_{\omega_\gamma\omega_\beta}E_{-,L}(\omega)a_\gamma\overline{a_\beta} = -2\mathrm{Re}(\langle (H^N_{\omega,L}-E_{-,L}(\omega))^{-1}\Pi u_a,\Pi u_a\rangle)$$

where

$$u_a = \Big(\sum_\gamma a_\gamma V(\cdot-\gamma)\Big)\varphi_L(\omega).$$

Note that, as V is not trivial, the assumption $E_-(a)=E_-(b)$ implies that V changes sign, that is, there exists $x_+\neq x_-$ such that $V(x_-)\cdot V(x_+)<0$. Now, the vector Πu_a vanishes if and only if u_a is colinear to $\varphi_L(\omega)$ which cannot happen as V is not constant and $\varphi_L(\omega)$ does not vanish on open sets by the unique continuation principle. On the other hand, $E_{-,L}(\omega)$ being a simple eigenvalue associated to $\varphi_L(\omega)$, $\Pi(H^N_{\omega,L}-E_{-,L}(\omega))^{-1}\Pi\geq c\,\Pi$ for some $c>0$. So the Hessian of $\omega\mapsto E_{-,L}(\omega)$ is positive definite. This completes the proof of Lemma 5.5. □

We now turn to the proof of Theorem 5.3. As the random variables are not Bernoulli distributed, that is, $\mathbb{P}(\omega_0=a)+\mathbb{P}(\omega_0=b)<1$, we can fix $\varepsilon>0$ sufficiently small such that

$$\mathbb{P}(\omega_0\in[a,\ a+\varepsilon))+\mathbb{P}(\omega_0\in(b-\varepsilon,\ b])<1.$$

By strict concavity of $E_-(\lambda)$, one has $E_-(a)<E_-(a+\varepsilon)$ and $E_-(b)<E_-(b-\varepsilon)$.

In Section 2, we proved:

Lemma 5.6. *Assume $E_-(a)=E_-(b)$. There exists $C>0$ with the following property: For any $L\geq 0$, if $\omega\in\{a,\ b,\ a+\varepsilon,\ b-\varepsilon\}^{Z_L}$ is such that*

$$\forall p\in\Lambda\ \ \exists\ell\in\{0,\dots,L-1\}\ \text{ such that }\ \omega_{(p,\ell)}\in\{a+\varepsilon,\ b-\varepsilon\}, \tag{P}$$

then

$$E_{-,L}(\omega)\geq E_-(a)+\frac{1}{CL^2}. \tag{5-7}$$

To complete the proof of Theorem 5.3, we first extend Lemma 5.6 using the concavity of the ground state energy:

Lemma 5.7. *Assume $E_-(a)=E_-(b)$. There exists $C>0$ satisfying the following property: For all $L\geq 0$, if $\omega\in\Omega_L$ is such that*

$$\forall p\in\Lambda\ \ \exists\ell\in\{0,\dots,L-1\}\ \text{ such that }\ \omega_{(p,\ell)}\in[a+\varepsilon,\ b-\varepsilon], \tag{P$'$}$$

then (5-7) *holds.* (The constant C is the same as in Lemma 5.6.)

We postpone the proof of this result to complete that of Theorem 5.3. Pick $E > E_-(a) = E_-(b)$. We use (5-4) and pick $L = c(E - E_-(a))^{1/2}$. Pick $c > 0$ sufficiently small that $Cc^2 < 1$. Then Lemma 5.6 tells us that, if $\omega \in [a, b]^{Z_L}$ satisfies (P′), then $E_-(\omega) > E$. So, the set $\Omega_L(E) := \{\omega \in \Omega_L \mid E_-(\omega) > E\}$ satisfies

$$\Omega_L \setminus \Omega_L(E) \subset \big\{\omega \in \Omega_L \mid \exists p \in \Lambda \text{ such that } \omega_{(p,\ell)} \in [a,\ a+\varepsilon) \cup (b-\varepsilon,\ b] \text{ for all } \ell\big\}.$$

Hence,

$$\begin{aligned}\mathbb{P}(\Omega_L \setminus \Omega_L(E)) &\le \sum_{p\in\Lambda} \mathbb{P}(\{\omega_{(p,\ell)} \in [a,\ a+\varepsilon) \cup (b-\varepsilon,\ b] \text{ for all } \ell\}) \\ &= L^{d-1}(\mathbb{P}(\omega_0 \in [a,\ a+\varepsilon)) + \mathbb{P}(\omega_0 \in (b-\varepsilon,\ b]))^L.\end{aligned}$$

This yields the announced exponential decay and completes the proof of Theorem 5.3. □

Proof of Lemma 5.7. We will proceed in two steps. First, we prove that, if ω satisfies (P′) and all its coordinates that are not in $[a+\varepsilon,\ b-\varepsilon]$ are either equal to a or to b, then (5-7) holds (with the same constant as in Lemma 5.6). This comes from the concavity of the ground state and the fact that any such point is a convex combination of points satisfying (P). Indeed, take such a point ω and let $\Gamma(\omega)$ be the set of coordinates such that $\omega_\gamma \in [a+\varepsilon,\ b-\varepsilon]$. Define $K(\omega) = \{a+\varepsilon,\ b-\varepsilon\}^{\Gamma(\omega)}$. Then there exists a convex combination $(\mu_\eta)_{\eta\in K(\omega)}$ such that

$$(\omega_\gamma)_{\gamma\in\Gamma(\omega)} = \sum_{\eta\in K(\omega)} \mu_\eta \eta, \qquad \sum_{\eta\in K(\omega)} \mu_\eta = 1, \quad \mu_\eta \ge 0.$$

Hence,

$$\omega = \sum_{\eta\in K(\omega)} \mu_\eta \tilde{\eta} \text{ where } (\tilde{\eta})_\gamma = \begin{cases} \eta_\gamma & \text{if} \gamma \in \Gamma(\omega), \\ \omega_\gamma & \text{if } \gamma \notin \Gamma(\omega). \end{cases}$$

That ω satisfies (5-7) then follows from the concavity of $\omega \mapsto E_{-,L}(\omega)$, that is Lemma 5.5, and from Lemma 5.6.

To complete the proof of Lemma 5.7, it suffices to show that a point ω satisfying (P′) can be written a convex combination of points of the type defined above. This is done as above. Indeed, pick ω satisfying (P′). Define $L(\omega) = \{a, b\}^{(Z_L\setminus\Gamma(\omega))}$. Then there exists a convex combination $(\mu_\eta)_{\eta\in L(\omega)}$ such that

$$(\omega_\gamma)_{\gamma\in(Z_L\setminus\Gamma(\omega))} = \sum_{\eta\in L(\omega)} \mu_\eta \eta, \quad \sum_{\eta\in L(\omega)} \mu_\eta = 1, \quad \mu_\eta \ge 0.$$

Hence,

$$\omega = \sum_{\eta\in L(\omega)} \mu_\eta \tilde{\eta} \text{ where } (\tilde{\eta})_\gamma = \begin{cases} \eta_\gamma & \text{if } \gamma \notin \Gamma(\omega), \\ \omega_\gamma & \text{if } \gamma \in \Gamma(\omega). \end{cases}$$

That ω satisfies (5-7) then follows from the concavity of $\omega \mapsto E_{-,L}(\omega)$ and from the first step. This completes the proof of Lemma 5.7. □

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