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We use a method, inspired by Pohozaev's work, to study asymptotic behaviors of nonvariational elliptic systems in dimension $n \geq 3$. As an application, we prove removal of an apparent singularity in a ball and uniqueness of the entire solution. All results apply to changing sign solutions.

In this paper, we study solutions of elliptic systems on $\mathbb{R}^n$, $n \geq 3$.

A classical work by Gidas and Spruck [1981] asserts that any nonnegative solution to $\Delta u + |u|^{\alpha-2}u = 0$ in $\mathbb{R}^n$ with $2 < \alpha < 2n/(n-2)$ (subcritical case) is trivial. For $\alpha = 2n/(n-2)$, Caffarelli, Gidas and Spruck [1989] proved that any nonnegative solution in $\mathbb{R}^n$ is of the form $u = (a + b|x|^2)^{-(n-2)/2}$, where a, b are constants. Such problem for elliptic systems are also studied, for example, in the studies of Lane–Emden type systems; see [Zou 2000; Poláčik etal. 2007; Souplet 2009] and the references therein.

By contrast, the behaviors of changing sign solutions are more delicate. For example, there exists a sequence of changing sign solutions to $\Delta u + |u|^{\alpha-2}u = 0$ in $\mathbb{R}^n$ with $2 < \alpha < 2n/(n-2)$ [Kuzin and Pohozaev 1997]. In this paper, we study under what circumstances a solution to an elliptic system in an exterior domain is asymptotic to $|x|^{-(n-2)}$ at infinity. Such decay is optimal in the sense that infinity is a regular point in the inverted coordinates. It is known [Kuzin and Pohozaev 1997] that there exist solutions to $\Delta u + u^{\alpha-1} = 0$ in $\mathbb{R}^n$ that decay more slowly than $|x|^{-(n-2)}$. Thus, a suitable integrability condition is necessary to exclude such a case.

While the study of changing sign solutions to elliptic systems is interesting by itself, the problem is well motivated by differential geometry. For example, the decay of curvature tensors was studied for Yang–Mills fields [Uhlenbeck 1982], Einstein metrics [Bando et al. 1989] and other generalizations [Tian and Viaclovsky 2005; Chen 2009], just to name a few. A typical system is of the form

$$\Delta(\mathrm{Rm})_{ijkl} = Q_{ijkl}(\mathrm{Rm}, \mathrm{Rm}),$$

where Rm is the Riemannian curvature tensor and Q is a quadratic in Rm. A natural geometric assumption is that $|\,\mathrm{Rm}\,|$ is in $L^{n/2}$. Therefore, $|\,\mathrm{Rm}\,|$ vanishes at infinity and the problem is to find out the decay rate. The study of geometrical systems is more subtle as $(\mathrm{Rm})_{ijkl}$ satisfies an extra relation, the Bianchi identity, and the underlying spaces are not Euclidean.

The technique we use in this paper is based on the method developed in [Chen 2009] on asymptotically flat manifolds, where a special geometric setting is considered. In this paper, we study general nonvariational elliptic systems of the reaction-diffusion type. Our result applies to changing sign solutions and includes the supercritical case (i.e., $\Delta u + Cu^{\alpha-1} = 0$ with $\alpha > 2n/(n-2)$, where C is a constant).

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Let $V = (V_1, \dots, V_m)$ and $f^i : \mathbb{R}^m \to \mathbb{R}$. Consider the system of equations

$$\sum_{j=1}^{m} A_{ij} \Delta V_j = f^i(V), \tag{1}$$

where A is a constant invertible symmetric matrix and $i = 1, \dots, m$. Note that $f^i(V)$ or V_i may have no sign. We assume the following structure conditions:

(A1) $|f^i(V)| \le C|V|^q$.

(A2) $|\nabla f^i(V)| \le C|V|^{q-1}$.

Let K be a compact subset in $\mathbb{R}^n$.

Theorem 1. *Let $q > (n+2)/n$ and $p = (n/2)(q-1)$. Suppose that f^i satisfies* (A1) *and* (A2). *Let $V \in L^p(\mathbb{R}^n \setminus K)$ be a solution to* (1) *in $\mathbb{R}^n \setminus K$. Then $|V| = O(|x|^{-(n-2)})$ and $|\nabla V| = O(|x|^{-(n-1)})$ at infinity.*

An immediate consequence is a result on singularity removal for affine invariant equations. For scalar equations, the problem was studied in [Gidas and Spruck 1981; Brézis and Lions 1981; Caffarelli et al. 1989].

Let B_1 be the unit ball centered at the origin.

Corollary 2. *Suppose f^i are homogeneous functions of degree $(n+2)/(n-2)$. Let $V \in L^{2n/(n-2)}(B_1)$ be a solution to* (1) *in $B_1 \setminus \{0\}$. Then V can be extended to a smooth solution to* (1) *in B_1.*

By performing a linear transformation $W_i = \sum_j A_{ij} V_j$, the system (1) can be reduced to an equation of the diagonal form $\Delta W = \tilde{f}(W)$. The assumptions (A1)–(A2) and other conditions on V or f^i equivalently hold for W and $\tilde{f}$. Therefore, for Theorem 1 and Corollary 2, we may assume without loss of generality the equation is of the diagonal form.

We turn to study the uniqueness of entire solutions for variational systems. Let $P(V)$ be a homogeneous function of degree $q+1$. Suppose that A_{ij} is positive definite and $f^i = \partial P/\partial V^i$ in (1). For scalar equations, there is a large literature on the uniqueness problem; see, for example, [Gidas and Spruck 1981; Bidaut-Véron 1989; Serrin and Zou 2002]; see also [Pucci and Serrin 2007] and the references therein. For systems, when $P(V) \le 0$ and $q > (n+2)/(n-2)$ (supercritical case), the problem was studied by Pucci and Serrin [1986] under some asymptotic assumption of V. Their result also holds for the nonhomogeneous function P (and more general $P(x, V, \nabla V)$) satisfying some inequality.

Theorem 3. *Let $q > (n+2)/n$, $q \ne (n+2)/(n-2)$ and $p = (n/2)(q-1)$. Suppose $P(V)$ is a homogeneous function of degree $q+1$. Suppose that A_{ij} is positive definite and $f^i = \partial P/\partial V^i$ in* (1). *Let $V \in L^p(\mathbb{R}^n)$ be a solution to* (1) *in $\mathbb{R}^n$. Then $V \equiv 0$.*

We outline the proofs. To fix notation, we denote by dx the volume element in $\mathbb{R}^n$ and by dS the area element of a hypersurface in $\mathbb{R}^n$. Let $B_r(x)$ and $S_r(x)$ be the ball of radius r and sphere of radius r centered at x, respectively. When x is at the origin, we simply denote by B_r and S_r.

The idea of the proof of Theorem 1 is to compare the size of $\int_{\mathbb{R}^n \setminus B_r} |\nabla V|^2 dx$ (as a function of r) to its derivative $-\int_{S_r} |\nabla V|^2 dS$. Then by the ordinary differential inequality lemma, we get the optimal decay of $|\nabla V|$ and, as a consequence, the decay of $|V|$. In order to relate the two integrands, we use some version of Pohozaev's identity for nonvariational systems. Pohozaev's ingenious idea [1965] is to use a conformal Killing field to prove uniqueness in a star-shaped domain. This idea was generalized nicely by Pucci and Serrin [1986] to general variational systems. Our use of the identity is different from the original one. We apply the identity to an unbounded domain (the complement of a large ball) and use only the size of $|f^i|$. Therefore, our method can be applied to nonvariational systems.

The proof of Theorem 3 is a combination of Theorem 1 and Pohozaev's original idea. Since the solution decays fast enough at infinity, no terms from infinity contribute to the main integrand. We use the identity differently such that we obtain the uniqueness also in the subcritical case, in contrast to the problem in star-shaped regions where one has to restrict to the supercritical case.

Finally, we show that the assumptions in these theorems are sharp.

Example 4. Consider the equation $\Delta u + u^{(n+2)/(n-2)} = 0$ in $\mathbb{R}^n$. By [Caffarelli et al. 1989], nonnegative solutions are of the form $u = (a + b|x|^2)^{-(n-2)/2}$. Therefore, u decays as $|x|^{-(n-2)}$ at infinity. This example shows that in Theorem 3, the assumption $q \neq (n+2)/(n-2)$ is necessary. Consider instead the equation in $B_1 \setminus \{0\}$. There exists a nonnegative radial singular solution with the blow-up rate $|x|^{-(n-2)/2}$ near the origin. Therefore, in Corollary 2, the condition $V \in L^{2n/(n-2)}(B_1)$ is sharp.

Example 5. Consider $\Delta u + u^q = 0$ in $\mathbb{R}^n$. For $q > (n+2)/(n-2)$, there exists a solution asymptotic to $|x|^{-2/(q-1)}$ at infinity [Kuzin and Pohozaev 1997]. Hence, in Theorem 1, the conditions $q = (2p+n)/n$ and $V \in L^p$ are sharp. Moreover, in Theorem 3, the condition $q = (2p+n)/n$ is also sharp.

1. Preliminaries

We collect some standard results in elliptic regularity theory and ordinary differential equations. Lemmas 6–8 follow by an argument similar to [Bando et al. 1989, Section 4].

Let C_s be the Sobolev constant and $\gamma = n/(n-2)$. Suppose that the nonnegative function $u \in C^{0,1}$ satisfies $\Delta u + C_0 u^q \geq 0$ weakly in the sense that

$$\int (-\langle \nabla u, \nabla \phi \rangle + C_0 u^q \phi) dx \geq 0 \quad \text{for all } 0 \leq \phi \in C_o^\infty.$$

Let $\varphi \geq 0$ be a function with compact support and let $s > 1$. Then, by the Cauchy inequality,

$$\begin{aligned}\int \varphi^2 u^{q+s-1}\, dx &\geq C_0^{-1} \int \Big(\frac{4(s-1)}{s^2} |\varphi \nabla u^{s/2}|^2 + \frac{4}{s} \varphi u^{s/2} \langle \nabla \varphi, \nabla u^{s/2} \rangle \Big)\, dx \\ &\geq C_0^{-1} \int \Big(\frac{2}{s^2}(s-1) |\varphi \nabla u^{s/2}|^2 - \frac{2}{(s-1)} |\nabla \varphi|^2 u^s \Big)\, dx.\end{aligned}$$

By the Sobolev inequality, we have

$$\Big(\int (\varphi^2 u^s)^\gamma\, dx \Big)^{1/\gamma} \leq C \int \Big(\frac{s^2 C_0}{2(s-1)} \varphi^2 u^{q+s-1} + \Big(1 + \frac{s^2}{(s-1)^2}\Big) |\nabla \varphi|^2 u^s \Big)\, dx, \tag{2}$$

where $C = C(n, C_s, C_0)$.

In Lemmas 6–8, u is a $C^{0,1}$ function.

Lemma 6. *Let $p > 1$ and $q = (2p+n)/n$. Suppose that the nonnegative function $u \in L^p(B_r)$ satisfies $\Delta u + C_0 u^q \geq 0$ weakly in B_r. Then there exists $\epsilon > 0$ such that if $\int_{B_r} u^p dx < \epsilon$, then*

$$\sup_{B_{r/2}} u \leq C r^{-n/p} \|u\|_{L^p(B_r)}, \quad \text{where } C = C(n, p, C_s, C_0).$$

Proof. Let $s = p$ in (2). Then

$$\begin{aligned}\left(\int (\varphi^2 u^p)^\gamma \, dx\right)^{1/\gamma} &\leq C \int (u^{q-1}(\varphi^2 u^p) + |\nabla\varphi|^2 u^p) \, dx \\ &\leq C\left(\int_{\{\text{supp } \varphi\}} u^p \, dx\right)^{2/n} \left(\int (\varphi^2 u^p)^\gamma \, dx\right)^{1/\gamma} + C \int |\nabla\varphi|^2 u^p \, dx.\end{aligned}$$

We choose φ to be a cutoff function such that $\varphi = 1$ in $B_{r/2}$ and $\varphi = 0$ outside B_r, with $|\nabla\varphi| \leq C r^{-1}$. We get

$$\left(\int_{B_{r/2}} u^{p\gamma} dx\right)^{1/\gamma} \leq \frac{C}{r^2} \int_{B_r} u^p \, dx.$$

Choose a sequence $r_k = (2^{-1} + 2^{-k})r$. Apply (and rescale) the above inequality for B_{r_k} and $B_{r_{k+1}}$ with $p_k = p\gamma^{k-1}$. By Moser iteration, we have $\sup_{B_{r/2}} u \leq C r^{-n/p} \|u\|_{L^p(B_r)}$. □

Lemma 7. *Let $p > n/(n-2)$ and $q = (2p+n)/n$. Suppose that the nonnegative function $u \in L^p(\mathbb{R}^n \setminus B_r)$ satisfies $\Delta u + C_0 u^q \geq 0$ weakly in $\mathbb{R}^n \setminus B_r$. Then there exists $\epsilon > 0$ such that if $\int_{\mathbb{R}^n \setminus B_r} u^p < \epsilon$, then $u = O(|x|^{-\lambda})$ for all $\lambda < n-2$ as $|x| \to \infty$.*

Proof. By Lemma 6, $u = O(|x|^{-n/p})$. Let $s = p((n-2)/n) > 1$ in (2). Then

$$\left(\int \varphi^{2\gamma} u^p \, dx\right)^{1/\gamma} \leq C\left(\int_{\{\text{supp } \varphi\}} u^p \, dx\right)^{2/n} \left(\int (\varphi^2 u^{p(n-2/n)})^\gamma \, dx\right)^{1/\gamma} + C \int |\nabla\varphi|^2 u^{p(n-2)/n} \, dx.$$

φ is chosen to be a cutoff function such that $\varphi = 1$ in $B_{r'} \setminus B_{2r}$ and $\varphi = 0$ outside $B_{2r'} \setminus B_r$ with $|\nabla\varphi| \leq C(1/r + 1/r')$. Let $r' \to \infty$. Then

$$\left(\int \varphi^{2\gamma} u^p \, dx\right)^{1/\gamma} \leq C\left(\int |\nabla\varphi|^n \, dx\right)^{2/n} \left(\int_{\{\text{supp } \nabla\varphi\}} u^p \, dx\right)^{1/\gamma}.$$

And thus,

$$\left(\int_{\mathbb{R}^n \setminus B_{2r}} u^p \, dx\right)^{1/\gamma} \leq C\left(\int_{B_{2r} \setminus B_r} u^p \, dx\right)^{1/\gamma}.$$

This gives $\int_{\mathbb{R}^n \setminus B_r} u^p = O(r^{-\delta})$ for some small $\delta > 0$. Therefore, by Lemma 6, $u = O(|x|^{-(n/p)-(\delta/p)})$. Let $\lambda_0 = \sup\{\lambda : u = O(|x|^{-\lambda})\}$. By iteration and a contradiction argument, we get that $\lambda_0 = n-2$. □

Suppose that $h \geq 0$ is a C^0 function. The nonnegative function $u \in C^{0,1}$ satisfies $\Delta u + C_0 h u \geq 0$ weakly if

$$\int \bigl(-\langle \nabla u, \nabla\phi\rangle + C_0 h u \phi\bigr) \, dx \geq 0 \quad \text{for all } 0 \leq \phi \in C_o^\infty.$$

Lemma 8. *Let $p > 1$ and $t > n/2$. Suppose that the nonnegative function $h \in L^t(B_r)$ satisfies $\int_{B_r} h^t\,dx \le C_1/r^{2t-n}$. Suppose also that the nonnegative function $u \in L^p(B_r)$ satisfies $\Delta u + C_0 hu \ge 0$ weakly in B_r. Then $\sup_{B_{r/2}} u \le Cr^{-n/p}\|u\|_{L^p(B_r)}$, where $C = C(n, p, C_s, C_0, C_1)$.*

Proof. The proof is by standard Moser iteration. See Morrey [1966]. □

The following is a basic result in ordinary differential equations [Chen 2009].

Lemma 9. *Suppose that $f(r) \ge 0$ satisfies $f(r) \le -(r/a)f'(r) + C_2 r^{-b}$ for some $a, b > 0$.*

(i) *$a \ne b$. Then there exists a constant C_3 such that*
$$f(r) \le C_3 r^{-a} + \frac{a\,C_2}{a-b} r^{-b}.$$
Therefore, $f(r) = O(r^{-\min\{a,b\}})$ as $r \to \infty$.

(ii) *$a = b$. Then there exists a constant C_3 such that*
$$f(r) \le C_3 r^{-a} + a\,C_2 r^{-a}\ln r.$$
Therefore, $f(r) = O(r^{-a}\ln r)$ as $r \to \infty$.

2. Proof of Theorem 1

As we explained in the introduction, without loss of generality we may assume the equation is of the diagonal form, that is,
$$\Delta V_i = f^i(V). \tag{3}$$
We first derive a version of Pohozaev's identity for nonvariational systems. Let Ω be a domain in $\mathbb{R}^n$ and N be the unit outer normal on $\partial\Omega$. We perform integration by parts repeatedly.
$$\begin{aligned}
&\int_\Omega \sum_{k,l} f^k(V) x_l D_l V_k\,dx \\
&\quad= \int_\Omega \sum_{j,l} \Delta V_j x_l D_l V_j\,dx \\
&\quad= \int_\Omega -\sum_{i,j,l} D_i V_j D_i(x_l D_l V_j)\,dx + \int_{\partial\Omega} \sum_{i,j,l} D_i V_j x_l D_l V_j N_i\,dS \\
&\quad= \int_\Omega \Big(-|\nabla V|^2 - \sum_l D_l(|\nabla V|^2)\frac{x_l}{2}\Big)dx + \int_{\partial\Omega} \sum_{i,j,l} D_i V_j x_l D_l V_j N_i\,dS \\
&\quad= \Big(\frac{n}{2}-1\Big)\int_\Omega |\nabla V|^2\,dx - \int_{\partial\Omega} \frac{1}{2}\sum_l |\nabla V|^2 x_l N_l\,dS + \int_{\partial\Omega} \sum_{i,j,l} D_i V_j x_l D_l V_j N_i\,dS. \qquad (4)
\end{aligned}$$
It is worth mentioning that $x_l D_l$ is a conformal Killing field in $\mathbb{R}^n$.

We note that $|V|$ and $|\nabla V|$ are $C^{0,1}$ functions. By (3) and (A1)–(A2), we have
$$\Delta|V| \ge -C|V|^q,$$
$$\Delta|\nabla V| \ge -C|V|^{q-1}|\nabla V|,$$

weakly. Since $V \in L^p(\mathbb{R}^n \setminus K)$, there exists a large number R such that $\int_{\mathbb{R}^n \setminus B_R} |V|^p dx < \epsilon$, where ϵ is as in Lemma 6. Applying Lemma 6 to $B_r(x_0)$ where $|x_0| \geq 2r \geq 2R$, we get $|V| = O(|x|^{-n/p})$.

Case 1. If $(n+2)/n < q \leq n/(n-2)$ (or equivalently, $1 < p \leq n/(n-2)$), then $n/p \geq n-2$. By Lemma 6, we have $|V| = O(|x|^{-n/p})$. Let φ be a cutoff function such that $\varphi = 1$ in B_r and $\varphi = 0$ outside B_{2r} with $|\nabla \varphi| \leq Cr^{-1}$. Applying φV_i to (3) and integrating gives

$$\int_{B_r(x_0)} |\nabla V|^2\, dx \leq C \int_{B_{2r}(x_0)} |V|^{q+1}\, dx + \frac{C}{r^2} \int_{B_{2r}(x_0)} |V|^2\, dx = O(r^{n-2-(2n/p)}) \leq O(r^{-n+2}),$$

where $|x_0| \geq 2r \gg 1$. By Lemma 8 with $h = |V|^{q-1}$, we obtain $|\nabla V| = O(|x|^{-(n-1)})$ and thus $|V| = O(|x|^{-(n-2)})$.

Case 2. If $n/(n-2) < q$ (or equivalently $p > n/(n-2)$), by Lemma 7, $|V| = O(|x|^{-\lambda})$ for all $\lambda < n-2$. Therefore,

$$\int_{B_r(x_0)} |\nabla V|^2 dx \leq C \int_{B_{2r}(x_0)} |V|^{q+1} dx + \frac{C}{r^2} \int_{B_{2r}(x_0)} |V|^2 dx = O(r^{n-2-2\lambda}),$$

where $|x_0| \geq 2r \gg 1$. Moreover, $|V| \in L^{p'}$ for all $p' > n/(n-2)$. Choose $p' < p$ close to $n/(n-2)$. Hence, $q > (2p'+n)/n$. We can then find $q' > n/2$ such that

$$\int_{B_r(x_0)} (|V|^{q-1})^{q'} dx \leq \frac{C}{r^{2q'-n}}, \qquad \text{where } |x_0| \geq 2r \gg 1.$$

This is possible because λ is close to $n-2$. By Lemma 8, we obtain

$$\sup_{B_{r/2(x_0)}} |\nabla V| \leq \frac{C}{r^{n/2}} \|\nabla V\|_{L^2(B_r(x_0))} = O(r^{-\lambda-1}), \qquad \text{where } |x_0| \geq 2r \gg 1.$$

Let $\Omega = B_R \setminus B_r$ in (4). We have

$$\int_\Omega \sum_{k,l} f^k(V) x_l D_l V_k dx = \left(\frac{n}{2} - 1\right) \int_\Omega |\nabla V|^2 dx - \int_{\partial\Omega} \frac{1}{2} \sum_l |\nabla V|^2 x_l N_l\, dS + \int_{\partial\Omega} \sum_{i,j,l} D_i V_j x_l D_l V_j N_i\, dS. \tag{5}$$

Note that

$$\lim_{R\to\infty} \int_{S_R} R|\nabla V|^2 dS = \lim_{R\to\infty} O(R^{-2\lambda-2+n}) = 0.$$

Let $R \to \infty$ in (5). Then there is no boundary term coming from infinity. We can choose $\Omega = \mathbb{R}^n \setminus B_r$. The boundary terms only occur on S_r. On $\partial\Omega$, $N = -x/r$. Hence,

$$\begin{aligned}\int_{\mathbb{R}^n\setminus B_r} \sum_{k,l} f^k(V) x_l D_l V_k\, dx &= \left(\frac{n}{2} - 1\right) \int_{\mathbb{R}^n\setminus B_r} |\nabla V|^2 dx + \int_{S_r} \frac{r}{2} |\nabla V|^2 dS - r \int_{S_r} |\nabla_N V|^2 dS \\ &\geq \left(\frac{n}{2} - 1\right) \int_{\mathbb{R}^n\setminus B_r} |\nabla V|^2 dx - \int_{S_r} \frac{r}{2} |\nabla V|^2 dS.\end{aligned}$$

Let $G(r) := \int_{\mathbb{R}^n\setminus B_r} |\nabla V|^2\,dx$. Since $G'(r) = -\int_{S_r} |\nabla V|^2\,dS$, the previous formula becomes

$$G(r) \le -\frac{r}{n-2}G'(r) + \frac{2}{n-2}\int_{\mathbb{R}^n\setminus B_r} \sum_{k,l} f^k(V)x_l D_l V_k\,dx.$$

The key idea is to compare the size of $G(r)$ to that of $G'(r)$. The coefficient in front of $G'(r)$ plays an important role. Here is the only place we use the condition of $|f^i|$. We have

$$\int_{\mathbb{R}^n\setminus B_r} \sum_{k,l} f^k(V)x_l D_l V_k\,dx \le \int_{\mathbb{R}^n\setminus B_r} |V|^q|x||\nabla V|\,dx = O(r^{-\lambda(q+1)+n}).$$

Thus,

$$G(r) \le -\frac{r}{n-2}G'(r) + Cr^{-\lambda(q+1)+n}.$$

Since $q > n/(n-2)$ and λ is close to $n-2$, we have $\lambda(q+1) - n > n-2$. By Lemma 9, this implies $G(r) = O(r^{-(n-2)})$. By the Sobolev inequality, we get

$$\int_{B_{2r}\setminus B_r} |V|^{2n/(n-2)}\,dx = O(r^{-n}).$$

Finally, by Lemma 6 and 8 we obtain $|V| = O(|x|^{-(n-2)})$ and $|\nabla V| = O(|x|^{-(n-1)})$. □

3. Proofs of Corollary 2 and Theorem 3

Proof of Corollary 2. Since the equation is invariant under inversion, we transform the solution to $\mathbb{R}^n \setminus B_1$ and apply Theorem 1.

Let $y = x/|x|^2$. Define $U_i(y) = (1/|y|^{n-2})V_i(y/|y|^2)$. This is called the Kelvin transform with the property that

$$\Delta_y U_i(y) = \frac{1}{|y|^{n+2}}\Delta_x V_i(x).$$

This can also be viewed as the conformal change formula of the conformal Laplacian with zero scalar curvature. Therefore, $U_i(y)$ satisfies

$$\sum_j A_{ij}\Delta_y U_i(y) = \frac{1}{|y|^{n+2}} f^i(|y|^{n-2}U(y)) = f^i(U(y)) \quad \text{in } \mathbb{R}^n \setminus B_1,$$

where we use that f^i is homogeneous of degree $(n+2)/(n-2)$. Moreover,

$$\begin{aligned}\int_{\mathbb{R}^n\setminus B_1} |U|^{2n/(n-2)}\,dy &= \int_{\mathbb{R}^n\setminus B_1} (|V||y|^{-n+2})^{2n/(n-2)}\,dy = \int_{B_1\setminus\{0\}} (|V||x|^{n-2})^{2n/(n-2)}|x|^{-2n}\,dx \\ &= \int_{B_1\setminus\{0\}} |V|^{2n/(n-2)}\,dx < +\infty.\end{aligned}$$

Now we apply Theorem 1 with $p = 2n/(n-2)$ and $q = (n+2)/(n-2)$. We get $|U| = O(|y|^{-(n-2)})$ and $|\nabla U| = O(|y|^{-(n-1)})$. Hence, $|V| = O(1)$ and $|\nabla V| = O(|x|^{-1})$. As a result, $V \in L^\infty(B_1)$ and $\nabla V \in L^p(B_1)$ for all $p < n$.

We show that V is a weak solution to (1) in B_1. Let $\varphi \in H_0^1(B_1, \mathbb{R}^m)$. Let $\eta_k(|x|)$ be a compactly supported function in $B_1 \setminus \{0\}$ such that $\eta_k \to 1$ a.e. in B_1 and $\|\eta_k\|_{L^n(B_1)} \to 0$ as $k \to \infty$. (Such functions were used by Serrin [1964].) Then

$$\int_{B_1} \eta_k \sum_{i,j,l} A_{ij} D_l \varphi_j D_l V_i \, dx = \int_{B_1} -\sum_i f^i(V) \varphi_i \eta_k \, dx - \int_{B_1} \sum_{i,j,l} D_l \eta_k A_{ij} \varphi_j D_l V_i \, dx.$$

The last term can be estimated as follows.

$$\left| \int_{B_1} \sum_{i,j,l} D_l \eta_k A_{ij} \varphi_j D_l V_i \, dx \right| \leq C \|\varphi\|_{L^{2n/(n-2)}(B_1)} \|\nabla V\|_{L^2(B_1)} \|\eta_k\|_{L^n(B_1)} \leq C \|\eta_k\|_{L^n(B_1)} \to 0$$

as $k \to \infty$. Hence, in the limit

$$\int_{B_1} \sum_{i,j,l} A_{ij} D_l \varphi_j D_l V_i \, dx = \int_{B_1} -\sum_i f^i(V) \varphi_i \, dx.$$

Thus, V is a weak solution in B_1. It follows by elliptic regularity that $V \in C^\infty(B_1)$. □

Proof of Theorem 3. Since A_{ij} is positive definite, there exists an orthogonal matrix M such that

$$M^{-1} A M = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix},$$

where $\lambda_1, \dots, \lambda_n$ are positive. Let

$$B = M \begin{pmatrix} \sqrt{\lambda}_1 & & \\ & \ddots & \\ & & \sqrt{\lambda}_n \end{pmatrix} M^{-1}.$$

By performing a transformation $W_i = \sum_j B_{ij} V_j$, the system can be reduced to $\Delta W_i = \dfrac{\partial \tilde{P}(W)}{\partial W^i}$. Thus, without loss of generality we may assume the equation is of the diagonal form.

Let $\Omega = B_R$ in (4). Therefore, $N = x/R$. We get

$$\int_{B_R} \sum_{k,l} f^k(V) x_l D_l V_k \, dx = \left(\frac{n}{2} - 1\right) \int_{B_R} |\nabla V|^2 dx - \int_{S_R} \frac{R}{2} |\nabla V|^2 dS + R \int_{S_R} |\nabla_N V|^2 dS.$$

Since $f^k = \partial P / \partial V_k$, we have

$$\int_{B_R} -n P(V) \, dx = \left(\frac{n}{2} - 1\right) \int_{B_R} |\nabla V|^2 dx - \int_{S_R} \frac{R}{2} |\nabla V|^2 dS + R \int_{S_R} |\nabla_N V|^2 dS - \int_{S_R} R P(V) \, dS. \quad (6)$$

On the other hand, we also have

$$\int_{B_R} (q+1) P(V) \, dx = \int_{B_R} \sum_k \frac{\partial P}{\partial V_k} V_k \, dx = -\int_{B_R} |\nabla V|^2 dx + \int_{S_R} \sum_j D_N V_j V_j \, dS, \quad (7)$$

where we use the Euler formula for homogeneous functions.

Case 1. $n \geq 4$. By Theorem 1, when $R \to \infty$, (6) becomes

$$\begin{aligned}\int_{B_R} -nP(V)\,dx &= \Big(\frac{n}{2}-1\Big)\int_{B_R} |\nabla V|^2\,dx + O(R^{-(n-2)}) + O(R^{-(q+1)(n-2)+n}) \\ &= \Big(\frac{n}{2}-1\Big)\int_{B_R} |\nabla V|^2\,dx + o(1),\end{aligned}$$

where we use conditions on p, q and $n \geq 4$ to get $(q+1)(n-2)-n > 0$. Similarly, (7) gives

$$\int_{B_R} (q+1)P(V)dx = -\int_{B_R} |\nabla V|^2 dx + O(R^{-(n-2)}).$$

Combining the above two formulas and noting that $q+1 \neq 2n/(n-2)$, we finally arrive at

$$\int_{B_R} |\nabla V|^2 dx = o(1).$$

We have $|\nabla V| \equiv 0$ and hence $V \equiv 0$.

Case 2. $n = 3$. Note that $\sup|V| \leq (C/|x|^{n/p})\|V\|_{L^p}$. Combining this fact with Theorem 1, we have $|V| = O(|x|^{-\lambda})$, where $\lambda = \max\{1, 3/p\}$. Therefore,

$$\lambda(q+1) - 3 \geq \max\Big\{q-2,\ \frac{3}{p}(q+1)-3\Big\} \geq \max\Big\{-1+\frac{2p}{3},\ -1+\frac{6}{p}\Big\} > 0.$$

Then (6) becomes

$$\begin{aligned}\int_{B_R} -3P(V)\,dx &= \Big(\frac{3}{2}-1\Big)\int_{B_R} |\nabla V|^2\,dx + O(R^{-1}) + O(R^{-\lambda(q+1)+3}) \\ &= \Big(\frac{3}{2}-1\Big)\int_{B_R} |\nabla V|^2\,dx + o(1),\end{aligned}$$

as in Case 1. The rest of proof is the same as in Case 1. □

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