ANALYSIS & PDE

Volume 4 No. 1 2011

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mathematical sciences publishers

ANALYSIS AND PDE
Vol. 4, No. 1, 2011

THREE-TERM COMMUTATOR ESTIMATES AND THE REGULARITY OF $\frac{1}{2}$-HARMONIC MAPS INTO SPHERES

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We prove the regularity of weak $\frac{1}{2}$-harmonic maps from the real line into a sphere. A key step is the formulation of the $\frac{1}{2}$-harmonic map equation in the form of a nonlocal linear Schrödinger type equation with *three-term commutators* on the right-hand side. We then establish a sharp estimate for these three-term commutators.

1. Introduction

Starting in the early 1950s, the analysis of critical points of conformal invariant lagrangians has attracted much interest, due to their importance in physics and geometry. (See the introduction of [Rivière 2008] for an overview.) We recall some classical examples of such operators and their associated variational problems:

The most elementary example of a two-dimensional conformal invariant lagrangian is the Dirichlet energy

$$E(u) = \int_D |\nabla u(x, y)|^2 \, dx \, dy, \tag{1}$$

where $D \subseteq R^2$ is an open set and ∇u is the gradient of $u : D \to \mathbb{R}$. We recall that a map $\phi : \mathbb{C} \to \mathbb{C}$ is conformal if it satisfies

$$\left|\frac{\partial \phi}{\partial x}\right| = \left|\frac{\partial \phi}{\partial y}\right|, \quad \left\langle \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y} \right\rangle = 0, \quad \det \nabla \phi \geq 0, \quad \nabla \phi \neq 0, \tag{2}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard Euclidean inner product in $\mathbb{R}^n$.

MSC2000: 58E20, 35B65, 35J20, 35J60, 35S99.
Keywords: harmonic map, nonlinear elliptic PDE, regularity of solutions, commutator estimates.

For every $u \in W^{1,2}(D, \mathbb{R})$ and every conformal map ϕ with $\deg \phi = 1$, we have

$$E(u) = E(u \circ \phi) = \int_{\phi^{-1}(D)} \left|(\nabla \circ \phi)u(x,y)\right|^2 dx\, dy.$$

The critical points of this functional are the harmonic functions satisfying

$$\Delta u = 0 \quad \text{in } D. \tag{3}$$

We can extend E to maps taking values in $\mathbb{R}^m$ by setting

$$E(u) = \int_D \left|\nabla u(x,y)\right|^2 dx\, dy = \int_D \sum_{i=1}^m \left|\nabla u_i(x,y)\right|^2 dx\, dy, \tag{4}$$

where the u_i are the components of u. The lagrangian (4) is still conformally invariant and each component of its critical points satisfies (3).

We can define the lagrangian (4) also on the set of maps taking values in a compact submanifold $\mathcal{N} \subseteq \mathbb{R}^m$ without boundary. We have

$$-\Delta u \perp T_u\mathcal{N},$$

where $T_u\mathcal{N}$ is the tangent plane to $\mathcal{N}$ at the point $u \in \mathcal{N}$; equivalently, we can write

$$-\Delta u = A(u)(\nabla u, \nabla u) := A(u)(\partial_x u, \partial_x u) + A(u)(\partial_y u, \partial_y u), \tag{5}$$

where $A(u)$ is the second fundamental form at a point $u \in \mathcal{N}$; see [Hélein 2002], for instance. Equation (5) is called the *harmonic map equation* into $\mathcal{N}$.

When $\mathcal{N}$ is an oriented hypersurface of $\mathbb{R}^m$ the harmonic map equation reads as

$$-\Delta u = n\langle \nabla n, \nabla u\rangle, \tag{6}$$

where n denotes the composition of u with the unit normal vector field ν to $\mathcal{N}$.

All these examples belong to the class of conformal invariant coercive lagrangians whose corresponding Euler–Lagrange equation is of the form

$$-\Delta u = f(u, \nabla u), \tag{7}$$

where $f : \mathbb{R}^2 \times (\mathbb{R}^m \otimes \mathbb{R}^2) \to \mathbb{R}^m$ is a continuous function satisfying

$$C^{-1}|p|^2 \le |f(\xi, p)| \le C|p|^2 \quad \text{for all } \xi, p,$$

for some positive constant C. One of the main issues concerning equations of the form (7) is the regularity of solutions $u \in W^{1,2}(D, \mathcal{N})$. We observe that (7) is critical in dimension $n = 2$ for the $W^{1,2}$-norm. Indeed, if we plug into the nonlinearity $f(u, \nabla u)$ the information that $u \in W^{1,2}(D, \mathcal{N})$, we obtain $\Delta u \in L^1(D)$, so ∇u belongs to $L^{2,\infty}_{\text{loc}}(D)$, the weak L^2 space [Stein 1970], which has the same homogeneity of L^2. Hence we are back in some sense to the initial situation. This shows that the equation is critical.

In general, $W^{1,2}$ solutions to (7) are not smooth in dimensions greater than 2; for a counterexample, see [Rivière 2007]. For an exposition of regularity and compactness results for such equations, we refer the reader to [Giaquinta 1983].

We next recall the approach introduced by F. Hélein [2002] to prove the regularity of harmonic maps from a domain D of $\mathbb{R}^2$ into the unit sphere S^{m-1} of $\mathbb{R}^m$. In this case the Euler–Lagrange equation is

$$-\Delta u = u|\nabla u|^2. \tag{8}$$

Shatah [1988] observed in that $u \in W^{1,2}(D, S^{m-1})$ is a solution of (8) if and only if the conservation law

$$\operatorname{div}(u_i \nabla u_j - u_j \nabla u_i) = 0 \quad \text{for all } i, j \in \{1, \dots, m\} \tag{9}$$

holds. Using (9) and the fact that $\sum_{j=1}^m u_j \nabla u_j = 0$ when $|u| \equiv 1$, Hélein rewrote (8) in the form

$$-\Delta u = \nabla^\perp B \cdot \nabla u, \tag{10}$$

where $\nabla^\perp B = (\nabla^\perp B_{ij})$ with $\nabla^\perp B_{ij} = u_i \nabla u_j - u_j \nabla u_i$ (for every vector field $v : \mathbb{R}^2 \to \mathbb{R}^n$, $\nabla^\perp v$ denotes the $\pi/2$ rotation of the gradient ∇v, namely $\nabla^\perp v = (-\partial_y v, \partial_x v)$).

The right-hand side of (10) can be written as a sum of Jacobians:

$$\nabla^\perp B_{ij} \nabla u_j = \partial_x u_j \partial_y B_{ij} - \partial_y u_j \partial_x B_{ij}.$$

This particular structure allows us to apply to (8) the following result:

Theorem 1.1 [Wente 1969]. *Let D be a smooth bounded domain of $\mathbb{R}^2$. Let a and b be measurable functions in D whose gradients are in $L^2(D)$. Then there exists a unique solution $\varphi \in W^{1,2}(D)$ to*

$$\begin{cases} -\Delta \varphi = \dfrac{\partial a}{\partial x}\dfrac{\partial b}{\partial y} - \dfrac{\partial a}{\partial y}\dfrac{\partial b}{\partial x} & \text{in } D, \\ \varphi = 0 & \text{on } \partial D. \end{cases} \tag{11}$$

There exists a constant $C > 0$ independent of a and b such that

$$\|\varphi\|_\infty + \|\nabla \varphi\|_{L^2} \le C \|\nabla a\|_{L^2} \|\nabla b\|_{L^2}.$$

In particular φ is continuous in D.

Theorem 1.1 applied to (10) leads, via a standard localization argument in elliptic PDEs, to the estimate

$$\|\nabla u\|_{L^2(B_r(x_0))} \le C\, \|\nabla B\|_{L^2(B_r(x_0))}\, \|\nabla u\|_{L^2(B_r(x_0))} + C r\, \|\nabla u\|_{L^2(\partial B_r(x_0))} \tag{12}$$

for every $x_0 \in D$ and $r > 0$ such that $B_r(x_0) \subset D$. Assume we are considering radii $r < r_0$ such that $\max_{x_0 \in D} C\, \|\nabla B\|_{L^2(B_r(x_0))} < \frac{1}{2}$. Then (12) implies a Morrey estimate

$$\sup_{x_0, r > 0} r^{-\beta} \int_{B_r(x_0)} |\nabla u|^2 \, dx < \infty \tag{13}$$

for some $\beta > 0$, which itself implies the Hölder continuity of u by a standard embedding result [Giaquinta 1983]. Finally a bootstrap argument implies that u is in fact C^∞, and even analytic: see [Hildebrandt and Widman 1975; Morrey 1966].

In the present work we are interested in one-dimensional quadratic lagrangians invariant under the trace of conformal maps that keep invariant the half-space $\mathbb{R}^2_+$: the Möbius group.

A typical example, which we will call the *L-energy* (L for "line"), is the lagrangian

$$L(u) = \int_{\mathbb{R}} \left|\Delta^{1/4} u(x)\right|^2 dx, \tag{14}$$

where u is a map from $\mathbb{R}$ into a k-dimensional submanifold $\mathcal{N}$ of $\mathbb{R}^m$ which is at least C^2, compact and without boundary. In fact $L(u)$ coincides with $\|u\|^2_{\dot{H}^{1/2}(\mathbb{R})}$ (for the definition of the seminorm $\|\cdot\|_{\dot{H}^{1/2}(\mathbb{R})}$ see Section 2). A more tractable way to look at this norm is given by the identity

$$\int_{\mathbb{R}} |\Delta^{1/4} u(x)|^2\, dx = \inf\left\{\int_{\mathbb{R}^2_+} |\nabla \tilde{u}|^2\, dx : \tilde{u} \in W^{1,2}(\mathbb{R}^2, \mathbb{R}^m) \text{ with trace } \tilde{u} = u\right\}.$$

The Lagrangian L extends to map u in the function space

$$\dot{H}^{1/2}(\mathbb{R}, \mathcal{N}) = \{u \in \dot{H}^{1/2}(\mathbb{R}, \mathbb{R}^m) : u(x) \in \mathcal{N} \text{ a.e.}\}.$$

The operator $\Delta^{1/4}$ on $\mathbb{R}$ is defined by means of the Fourier transform (denoted by $\hat{\ }$) as

$$\widehat{\Delta^{1/4} u} = |\xi|^{1/2} \hat{u}.$$

Denote by $\pi_{\mathcal{N}}$ the orthogonal projection onto $\mathcal{N}$, which happens to be a C^l map in a sufficiently small neighborhood of $\mathcal{N}$ if $\mathcal{N}$ is assumed to be C^{l+1}. We now introduce the notion of $\frac{1}{2}$-harmonic map into a manifold.

Definition 1.2. A map $u \in \dot{H}^{1/2}(\mathbb{R}, \mathcal{N})$ is called a *weak $\frac{1}{2}$-harmonic* map into $\mathcal{N}$ if

$$\frac{d}{dt} L(\pi_{\mathcal{N}}(u + t\phi))_{|t=0} = 0 \quad \text{for any } \phi \in \dot{H}^{1/2}(\mathbb{R}, \mathbb{R}^m) \cap L^\infty(\mathbb{R}, \mathbb{R}^m).$$

In short, a weak $\frac{1}{2}$-harmonic map is a *critical point of L in $\dot{H}^{1/2}(\mathbb{R}, \mathcal{N})$ for perturbations in the target*.

We encounter $\frac{1}{2}$-harmonic maps into the circle S^1, for instance, in the asymptotic of equations in phase-field theory for fractional reaction-diffusion such as

$$\epsilon^2\, \Delta^{1/2} u + u(1 - |u|^2) = 0$$

where u is a complex-valued wavefunction.

In this paper we consider the case $\mathcal{N} = S^{m-1}$. We first write (deferring the proof till Theorem 5.2) the Euler–Lagrange equation associated to L in $\dot{H}^{1/2}(\mathbb{R}, S^{m-1})$:

Proposition 1.3. *Let T be the operator defined by*

$$T(Q, u) := \Delta^{1/4}(Q \Delta^{1/4} u) - Q \Delta^{1/2} u + \Delta^{1/4} u \Delta^{1/4} Q, \tag{15}$$

for $Q \in \dot{H}^{1/2}(\mathbb{R}^n, \mathcal{M}_{l\times m}(\mathbb{R}))$ $l \geq 1$ and $u \in \dot{H}^{1/2}(\mathbb{R}^n, \mathbb{R}^m)$. (Here n and l are natural numbers and $\mathcal{M}_{l\times m}(\mathbb{R})$ denotes the space of $l \times m$ real matrices.)

A map u in $\dot{H}^{1/2}(\mathbb{R}, S^{m-1})$ is a weak $\frac{1}{2}$-harmonic map if and only if it satisfies the Euler–Lagrange equation

$$\Delta^{1/4}(u \wedge \Delta^{1/4} u) = T(u\wedge, u). \tag{16}$$

The Euler–Lagrange equation (16) will often be completed by the following "structure equation", which is a consequence of the fact that $u \in S^{m-1}$ almost everywhere:

Proposition 1.4. *Let S be the operator given by*

$$S(Q,u) := \Delta^{1/4}(Q\Delta^{1/4}u) - \mathcal{R}(Q\nabla u) + \mathcal{R}(\Delta^{1/4}Q\mathcal{R}\Delta^{1/4}u) \tag{17}$$

for $Q \in \dot{H}^{1/2}(\mathbb{R}^n, \mathcal{M}_{l\times m}(\mathbb{R}))$ and $u \in \dot{H}^{1/2}(\mathbb{R}^n, \mathbb{R}^m)$, where n and l are natural numbers and $\mathcal{R}$ is the Fourier multiplier of symbol $m(\xi) = i\xi/|\xi|$.

All maps in $\dot{H}^{1/2}(\mathbb{R}, S^{m-1})$ satisfy

$$\Delta^{1/4}(u\cdot\Delta^{1/4}u) = S(u\cdot, u) - \mathcal{R}(\Delta^{1/4}u\cdot\mathcal{R}\Delta^{1/4}u). \tag{18}$$

We will first show that $\dot{H}^{1/2}$ solutions to the $\frac{1}{2}$-harmonic map equation (16) are Hölder continuous. This regularity result will be a direct consequence of a Morrey-type estimate we will establish:

$$\sup_{\substack{x_0\in\mathbb{R}\\ r>0}} r^{-\beta}\int_{B_r(x_0)} \left|\Delta^{1/4}u\right|^2 dx < \infty. \tag{19}$$

For this purpose, in the spirit of what we have just presented regarding Hélein's proof of the regularity of harmonic maps from a two-dimensional domain into a round sphere, we will take advantage of a "regularity gain" in the right-hand sides of (16) and (18), where the different terms $T(u\wedge, u)$, $S(u\cdot, u)$ and $\mathcal{R}(\Delta^{1/4}u\cdot\mathcal{R}\Delta^{1/4}u)$ play more or less the role played by $\nabla^{\perp}B\cdot\nabla u$ in (10). More precisely, we will establish, for every $u \in \dot{H}^{1/2}(\mathbb{R}, \mathbb{R}^m)$ and $Q \in H^{1/2}(\mathbb{R}, \mathcal{M}_{l\times m}(\mathbb{R}))$, the estimates

$$\|T(Q,u)\|_{\dot{H}^{-1/2}(\mathbb{R})} \le C\,\|Q\|_{\dot{H}^{1/2}(\mathbb{R})}\,\|u\|_{\dot{H}^{1/2}(\mathbb{R})}, \tag{20}$$

$$\|S(Q,u)\|_{\dot{H}^{-1/2}(\mathbb{R})} \le C\,\|Q\|_{\dot{H}^{1/2}(\mathbb{R})}\,\|u\|_{\dot{H}^{1/2}(\mathbb{R})}, \tag{21}$$

$$\left\|\mathcal{R}(\Delta^{1/4}u\cdot\mathcal{R}\Delta^{1/4}u)\right\|_{\dot{H}^{-1/2}(\mathbb{R})} \le C\,\|u\|^2_{\dot{H}^{1/2}(\mathbb{R})}. \tag{22}$$

The phrase "regularity gain" is illustrated by the fact that, for such u and Q, the individual terms in T and S (such as $\Delta^{1/4}(Q\Delta^{1/4}u)$ or $Q\Delta^{1/2}u$) are not in $\dot{H}^{-1/2}$, but the special linear combinations of them constituting T and S do lie in $\dot{H}^{-1/2}$. In a similar way, *in two dimensions*, $J(a,b) := \frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial a}{\partial y}\frac{\partial b}{\partial x}$ satisfies

$$\|J(a,b)\|_{\dot{H}^{-1}} \le C\,\|a\|_{\dot{H}^1}\,\|b\|_{\dot{H}^1} \tag{23}$$

as a direct consequence of Wente's result (Theorem 1.1), whereas the individual terms $\frac{\partial a}{\partial x}\frac{\partial b}{\partial y}$ and $\frac{\partial a}{\partial y}\frac{\partial b}{\partial x}$ are not in $\dot{H}^{-1}$.

The estimates (20) and (21) are in fact consequences of the *three-term commutator estimates* in the next two theorems, which are valid in arbitrary dimension and which are two of the main results of this paper. We recall that *BMO* denotes the space of *bounded mean oscillations* functions of John and Nirenberg (see for instance [Grafakos 2009])

$$\|u\|_{BMO(\mathbb{R}^n)} = \sup_{\substack{x_0\in\mathbb{R}^n\\ r>0}} \frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} \left|u(x) - \frac{1}{|B_r(x_0)|}\int u(y)\,dy\right| dx.$$

Theorem 1.5. *For $n \in \mathbb{N}^*$, $u \in BMO(\mathbb{R}^n)$, and $Q \in \dot{H}^{1/2}(\mathbb{R}^n, \mathcal{M}_{l\times m}(\mathbb{R}))$, set*

$$T(Q,u) := \Delta^{1/4}(Q\Delta^{1/4}u) - Q\Delta^{1/2}u + \Delta^{1/4}u\Delta^{1/4}Q,$$

Then $T(Q,u) \in H^{-1/2}(\mathbb{R}^n)$ and there exists $C > 0$, depending only on n, such that

$$\|T(Q,u)\|_{H^{-1/2}(\mathbb{R}^n)} \leq C\, \|Q\|_{\dot{H}^{1/2}(\mathbb{R}^n)} \|u\|_{BMO(\mathbb{R}^n)}\,. \tag{24}$$

Theorem 1.6. *For $n \in \mathbb{N}^*$, $u \in BMO(\mathbb{R}^n)$, and $Q \in \dot{H}^{1/2}(\mathbb{R}^n, \mathcal{M}_{l\times m}(\mathbb{R}))$, set*

$$S(Q,u) := \Delta^{1/4}[Q\Delta^{1/4}u] - \mathcal{R}(Q\nabla u) + \mathcal{R}(\Delta^{1/4}Q\mathcal{R}\Delta^{1/4}u),$$

where $\mathcal{R}$ is the Fourier multiplier of symbol $m(\xi) = i\xi/|\xi|$. Then $S(Q,u) \in H^{-1/2}(\mathbb{R}^n)$ and there exists C depending only on n such that

$$\|S(Q,u)\|_{H^{-1/2}(\mathbb{R}^n)} \leq C\, \|Q\|_{\dot{H}^{1/2}(\mathbb{R}^n)} \|u\|_{BMO(\mathbb{R}^n)}\,. \tag{25}$$

The estimates (20) and (21) follow from Theorems 1.5 and 1.6 as a consequence of the embedding $\dot{H}^{1/2}(\mathbb{R}) \hookrightarrow BMO(\mathbb{R})$.

The parallel between the structures T and S for $H^{1/2}$ in one hand and the Jacobian structure J for H^1 in the other can be pushed further as follows. As a consequence of a result of R. Coifman, P. L. Lions, Y. Meyer and S. Semmes [Coifman et al. 1993], the Wente estimate (23) can be deduced from a more general one. Set, for any $i, j \in \{1, \ldots, n\}$ and $a, b \in \dot{H}^1(\mathbb{R}^n)$,

$$J_{ij}(a,b) := \frac{\partial a}{\partial x_i}\frac{\partial b}{\partial x_j} - \frac{\partial a}{\partial x_j}\frac{\partial b}{\partial x_i},$$

and form the matrix $J(a,b) := (J_{ij}(a,b))_{ij=1,\ldots,n}$. The main result in [Coifman et al. 1993] implies

$$\|J(a,b)\|_{\dot{H}^{-1}(\mathbb{R}^n)} \leq C\, \|a\|_{\dot{H}^1(\mathbb{R}^n)}\, \|b\|_{BMO(\mathbb{R}^n)}\,, \tag{26}$$

which is reminiscent of (24) and (25). Recall also that (26) is a consequence of a commutator estimate by Coifman, R. Rochberg and G. Weiss [Coifman et al. 1976].

Theorems 1.5 and 1.6 will follow respectively Theorems 1.7 and (27) below, which are their ”dual versions”. Recall first that $\mathcal{H}^1(\mathbb{R}^n)$ denotes the Hardy space of L^1 functions f on $\mathbb{R}^n$satisfying

$$\int_{\mathbb{R}^n} \sup_{t\in\mathbb{R}} |\phi_t * f|(x)\, dx < \infty,$$

where $\phi_t(x) := t^{-n}\, \phi(t^{-1}x)$ and where ϕ is some function in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ satisfying $\int_{\mathbb{R}^n} \phi(x)\, dx = 1$. Recall the famous result by Fefferman saying that the dual space to $\mathcal{H}^1$ is BMO.

Theorem 1.7. *For $u, Q \in \dot{H}^{1/2}(\mathbb{R}^n)$, set*

$$R(Q,u) = \Delta^{1/4}(Q\Delta^{1/4}u) - \Delta^{1/2}(Qu) + \Delta^{1/4}((\Delta^{1/4}Q)u).$$

Then $R(Q,u) \in \mathcal{H}^1(\mathbb{R}^n)$ and

$$\|R(Q,u)\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq C\|Q\|_{\dot{H}^{1/2}(\mathbb{R}^n)}\|u\|_{\dot{H}^{1/2}(\mathbb{R}^n)}. \tag{27}$$

Theorem 1.8. *For $u, Q \in \dot{H}^{1/2}(\mathbb{R}^n)$ and $u \in BMO(\mathbb{R}^n)$, set*

$$\tilde{S}(Q,u) = \Delta^{1/4}(Q\Delta^{1/4}u) - \nabla(Q\mathcal{R}u) + \mathcal{R}\Delta^{1/4}(\Delta^{1/4}Q\mathcal{R}u),$$

where $\mathcal{R}$ is the Fourier multiplier of symbol $m(\xi) = i\xi/|\xi|$. Then $\tilde{S}(Q,u) \in \mathcal{H}^1$ and

$$\|\tilde{S}(Q,u)\|_{\mathcal{H}^1} \leq C\|Q\|_{\dot{H}^{1/2}(\mathbb{R}^n)}\|u\|_{\dot{H}^{1/2}(\mathbb{R}^n)}. \tag{28}$$

We say a few words on the proof of the estimates (27) and (28). The compensations of the three different terms in $R(Q,u)$ will be clear from the Littlewood–Paley decomposition of the different products that we present in Section 3. As usual, we denote by $\Pi_1(f,g)$ the high-low contribution (respectively from f and g), by $\Pi_2(f,g)$ the low-high contribution, and by $\Pi_3(f,g)$ the high-high contribution. We also use the notation $\Pi_k(\Delta^\alpha(fg))$, for $k = 1,2,3$ and $\alpha = \frac{1}{4}, \frac{1}{2}$, as an alternative for $\Delta^\alpha(\Pi_k(f,g))$.

We will use the following decompositions for the operators $\Pi_k(R(Q,u))$:

$$\Pi_1(R(Q,u)) = \underbrace{\Pi_1(\Delta^{1/4}(Q\Delta^{1/4}u))} + \underbrace{\Pi_1(-\Delta^{1/2}(Qu) + \Delta^{1/4}((\Delta^{1/4}Q)u))},$$

$$\Pi_2(R(Q,u)) = \underbrace{\Pi_2(\Delta^{1/4}(Q\Delta^{1/4}u) - \Delta^{1/2}(Qu))} + \underbrace{\Pi_2(\Delta^{1/4}((\Delta^{1/4}Q)u))},$$

$$\Pi_3(R(Q,u)) = \underbrace{\Pi_3(\Delta^{1/4}(Q\Delta^{1/4}u))} - \underbrace{\Pi_3(\Delta^{1/2}(Qu))} + \underbrace{\Pi_3(\Delta^{1/4}((\Delta^{1/4}Q)u))}.$$

Finally, injecting the Morrey estimate (19) in equations (16) and (18), a classical elliptic-type bootstrap argument leads to the following result (see [Lio and Riviere 2010] for details).

Theorem 1.9. *Any weak $\frac{1}{2}$-harmonic map in $\dot{H}^{1/2}(\mathbb{R}, S^{m-1})$ belongs to $H^s_{\mathrm{loc}}(\mathbb{R}, S^{m-1})$ for every $s \in \mathbb{R}$, and is therefore C^∞.*

The paper is organized as follows. After a section with preliminary definitions and notation, we prove in Section 3 we prove the three-term commutator estimates (Theorems 1.5 and 1.6).

In Section 4 we prove some L-energy decrease control estimates on dyadic annuli for general solutions to certain linear nonlocal systems of equations, which include (16) and (18).

In Section 5 we derive the Euler–Lagrange equation (16) associated to the lagrangian (14); this is Proposition 1.3. We then prove Proposition 1.4. We finally use the results of the previous section to deduce the Morrey-type estimate (19) for $\frac{1}{2}$-harmonic maps into a sphere.

In the Appendix we study geometric localization properties of the $\dot{H}^{1/2}$-norm on the real line for $\dot{H}^{1/2}$-functions in general and we prove some preliminary results.

2. Definitions and notation

For $n \geq 1$, let $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ denote respectively the spaces of Schwartz functions and tempered distributions. Given a function v we will denote either by $\hat{v}$ or by $\mathcal{F}[v]$ the Fourier Transform of v:

$$\hat{v}(\xi) = \mathcal{F}[v](\xi) = \int_{\mathbb{R}^n} v(x)e^{-i\langle\xi,x\rangle}\,dx.$$

Throughout the paper we use the convention that x, y denote space variables and ξ, ζ phase variables.

We recall the definition of fractional Sobolev spaces. For some of the material on the next page, see [Tartar 2007], for instance.

Definition 2.1. For s real,

$$H^s(\mathbb{R}^n) = \begin{cases} \{v \in L^2(\mathbb{R}^n) : |\xi|^s \mathscr{F}[v] \in L^2(\mathbb{R}^n)\} & \text{if } s \geq 0, \\ \{v \in \mathscr{S}'(\mathbb{R}^n) : (1+|\xi|^2)^{s/2}\mathscr{F}[v] \in L^2(\mathbb{R}^n)\} & \text{if } s < 0. \end{cases}$$

It is known that $H^{-s}(\mathbb{R}^n)$ is the dual of $H^s(\mathbb{R}^n)$.

For $0 < s < 1$, we mention an alternative characterization of $H^s(\mathbb{R}^n)$, which does not use the Fourier transform.

Lemma 2.2. *For $0 < s < 1$, the condition $u \in H^s(\mathbb{R}^n)$ is equivalent to $u \in L^2(\mathbb{R}^n)$ and*

$$\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(u(x)-u(y))^2}{|x-y|^{n+2s}}\,dx\,dy\right)^{1/2} < \infty.$$

For $s > 0$ we set

$$\|u\|_{H^s(\mathbb{R}^n)} = \|u\|_{L^2(\mathbb{R}^n)} + \||\xi|^s \mathscr{F}[v]\|_{L^2(\mathbb{R}^n)} \quad \text{and} \quad \|u\|_{\dot{H}^s(\mathbb{R}^n)} = \||\xi|^s \mathscr{F}[v]\|_{L^2(\mathbb{R}^n)}.$$

For an open set $\Omega \subset \mathbb{R}^n$, $H^s(\Omega)$ is the space of the restrictions of functions from $H^s(\mathbb{R}^n)$, and

$$\|u\|_{\dot{H}^s(\Omega)} = \inf\left\{\|U\|_{\dot{H}^s(\mathbb{R}^n)} : U = u \text{ on } \Omega\right\}.$$

If $0 < s < 1$, then $f \in H^s(\Omega)$ if and only if $f \in L^2(\Omega)$ and

$$\left(\int_{\Omega}\int_{\Omega} \frac{(u(x)-u(y))^2}{|x-y|^{n+2s}}\,dx\,dy\right)^{1/2} < \infty.$$

Moreover,

$$\|u\|_{\dot{H}^s(\Omega)} \simeq \left(\int_{\Omega}\int_{\Omega} \frac{(u(x)-u(y))^2}{|x-y|^{n+2s}}\,dx\,dy\right)^{1/2} < \infty.$$

Finally, for a submanifold $\mathcal{N}$ of $\mathbb{R}^m$, we can define

$$H^s(\mathbb{R}, \mathcal{N}) = \{u \in H^s(\mathbb{R}, \mathbb{R}^m) : u(x) \in \mathcal{N} \text{ a.e.}\}.$$

We introduce the so-called Littlewood–Paley or dyadic decomposition of unity. Let $\phi(\xi)$ be a radial Schwartz function supported on $\{\xi : |\xi| \leq 2\}$ and equal to 1 on $\{\xi : |\xi| \leq 1\}$. Let $\psi(\xi)$ be the function $\psi(\xi) := \phi(\xi) - \phi(2\xi)$; thus ψ is a bump function supported on the annulus $\{\xi : \frac{1}{2} \leq |\xi| \leq 2\}$.

We put $\psi_0 = \phi$, $\psi_j(\xi) = \psi(2^{-j}\xi)$ for $j \neq 0$. The functions ψ_j, for $j \in \mathbb{Z}$, are supported on $\{\xi : 2^{j-1} \leq |\xi| \leq 2^{j+1}\}$. Moreover $\sum_{j\in\mathbb{Z}} \psi_j(x) = 1$.

We then set $\phi_j(\xi) := \sum_{k=-\infty}^{j} \psi_k(\xi)$. The function ϕ_j is supported on $\{\xi,\ |\xi| \leq 2^{j+1}\}$.

We recall the definition of the homogeneous Besov spaces $\dot{B}^s_{p,q}(\mathbb{R}^n)$ and homogeneous Triebel–Lizorkin spaces $\dot{F}^s_{pq}(\mathbb{R}^n)$ in terms of the dyadic decomposition.

Definition 2.3. Let $s \in \mathbb{R}$, $0 < p, q \leq \infty$. For $f \in \mathscr{S}'(\mathbb{R}^n)$, set

$$\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)} = \begin{cases} \left(\sum_{j=-\infty}^{\infty} 2^{jsq} \|\mathscr{F}^{-1}[\psi_j \mathscr{F}[f]]\|^q_{L^p(\mathbb{R}^n)}\right)^{1/q} & \text{if } q < \infty, \\ \sup_{j\in\mathbb{Z}} 2^{js} \|\mathscr{F}^{-1}[\psi_j \mathscr{F}[f]]\|_{L^p(\mathbb{R}^n)} & \text{if } q = \infty. \end{cases} \tag{29}$$

The *homogeneous Besov space with indices* s, p, q, denoted by $\dot{B}^s_{p,q}(\mathbb{R}^n)$, is the space of all tempered distributions f for which $\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)}$ is finite.

Let $s \in \mathbb{R}$, $0 < p, q < \infty$. Again for $f \in \mathcal{S}'(\mathbb{R}^n)$, set

$$\|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} = \Big\| \Big(\sum_{j=-\infty}^{\infty} 2^{jsq} \big|\mathcal{F}^{-1}[\psi_j \mathcal{F}[f]]\big|^q \Big)^{1/q} \Big\|_{L^p}.$$

The *homogeneous Triebel–Lizorkin space with indices* s, p, q, denoted by $\dot{F}^s_{p,q}(\mathbb{R}^n)$, is the space of all tempered distributions f for which $\|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}$ is finite.

It is known that $\dot{H}^s(\mathbb{R}^n) = \dot{B}^s_{2,2}(\mathbb{R}^n) = \dot{F}^s_{2,2}(\mathbb{R}^n)$.

Finally we denote by $\mathcal{H}^1(\mathbb{R}^n)$ the homogeneous Hardy space in $\mathbb{R}^n$. It is known that $\mathcal{H}^1(\mathbb{R}^n) \simeq F^0_{2,1}$; thus we have

$$\|f\|_{\mathcal{H}^1(\mathbb{R}^n)} \simeq \int_{\mathbb{R}} \Big(\sum_j |\mathcal{F}^{-1}[\psi_j \mathcal{F}[f]]|^2 \Big)^{1/2} dx.$$

We recall that in dimension $n = 1$, the space $\dot{H}^{1/2}(\mathbb{R})$ is continuously embedded in the Besov space $\dot{B}^0_{\infty,\infty}(\mathbb{R})$. More precisely we have

$$\dot{H}^{1/2}(\mathbb{R}) \hookrightarrow BMO(\mathbb{R}) \hookrightarrow \dot{B}^0_{\infty,\infty}(\mathbb{R}); \tag{30}$$

see, for instance, [Runst and Sickel 1996, p. 31] or [Triebel 1983, p. 129].

The s-fractional Laplacian of a function $u : \mathbb{R}^n \to \mathbb{R}$ is defined as a pseudodifferential operator of symbol $|\xi|^{2s}$:

$$\widehat{\Delta^s u}(\xi) = |\xi|^{2s} \hat{u}(\xi). \tag{31}$$

It can also be defined as

$$\Delta^s u(x) = p.v. \int_{\mathbb{R}} \frac{u(y) - y(x)}{|x - y|^{n+2s}}\, dy,$$

where $p.v.$ denotes the Cauchy principal value.

In the case $s = \frac{1}{2}$, we can write $\Delta^{1/2} u = -\mathcal{R}(\nabla u)$ where $\mathcal{R}$ is Fourier multiplier of symbol $\dfrac{i}{|\xi|} \sum_{k=1}^{n} \xi_k$:

$$\widehat{\mathcal{R}X}(\xi) = \frac{1}{|\xi|} \sum_{k=1}^{n} i \xi_k \hat{X}_k(\xi)$$

for every $X : \mathbb{R}^n \to \mathbb{R}^n$; thus $\mathcal{R} = \Delta^{-1/2}\mathrm{div}$.

We denote by $B_r(\bar{x})$ the ball of radius r and center $\bar{x}$. If $\bar{x} = 0$ we simply write B_r. If $x, y \in \mathbb{R}^n$, $x \cdot y$ denote the scalar product between x, y.

For every function $f : \mathbb{R}^n \to \mathbb{R}$ we denote by $M(f)$ the maximal function of f, namely

$$M(f) = \sup_{\substack{r>0\\ x\in\mathbb{R}^n}} |B(x,r)|^{-1} \int_{B(x,r)} |f(y)|\, dy. \tag{32}$$

3. Three-term commutator estimates: proof of Theorems 1.5 and 1.6

We consider the dyadic decomposition introduced in Section 2. For every $j \in \mathbb{Z}$ and $f \in \mathcal{S}'(\mathbb{R}^n)$ we define the Littlewood–Paley projection operators P_j and $P_{\le j}$ by

$$\widehat{P_j f} = \psi_j \hat{f}, \qquad \widehat{P_{\le j} f} = \phi_j \hat{f}.$$

Informally, P_j is a frequency projection to the annulus $\{2^{j-1} \le |\xi| \le 2^j\}$, while $P_{\le j}$ is a frequency projection to the ball $\{|\xi| \le 2^j\}$. We will set $f_j = P_j f$ and $f^j = P_{\le j} f$.

We observe that $f^j = \sum_{k=-\infty}^{j} f_k$ and $f = \sum_{k=-\infty}^{+\infty} f_k$, where the convergence is in $\mathcal{S}'(\mathbb{R}^n)$.

Given $f, g \in \mathcal{S}'(\mathbb{R})$ we can split the product fg as

$$fg = \Pi_1(f,g) + \Pi_2(f,g) + \Pi_3(f,g), \tag{33}$$

where

$$\Pi_1(f,g) = \sum_{-\infty}^{+\infty} f_j g^{j-4} = \sum_{-\infty}^{+\infty} f_j \sum_{-\infty}^{j-4} g_k, \ \Pi_2(f,g) = \sum_{-\infty}^{+\infty} g_j f^{j-4} = \sum_{-\infty}^{+\infty} f_j \sum_{j+4}^{+\infty} g_k, \ \Pi_3(f,g) = \sum_{-\infty}^{+\infty} f_j \sum_{j-4}^{j+4} g_k.$$

This is an example of decomposition into *paraproducts* (see [Grafakos 2009], for example). Informally, the first paraproduct Π_1 is an operator that allows high frequencies of f ($\sim 2^j$) multiplied by low frequencies of g ($\ll 2^j$) to produce high frequencies in the output; Π_2 multiplies low frequencies of f with high frequencies of g to produce high frequencies in the output; and Π_3 multiplies high frequencies of f with high frequencies of g to produce comparable or lower frequencies in the output.

For every j, we have

$$\operatorname{supp} \mathcal{F}[f^{j-4} g_j] \subset \{2^{j-2} \le |\xi| \le 2^{j+2}\} \quad \text{and} \quad \operatorname{supp} \mathcal{F}\Big[\sum_{k=j-3}^{j+3} f_j g_k\Big] \subset \{|\xi| \le 2^{j+5}\}.$$

Lemma 3.1. *For every $f \in \mathcal{S}'$ we have $\sup_{j \in Z} |f^j| \le M(f)$.*

Proof. We have

$$\begin{aligned}
f^j &= \mathcal{F}^{-1}[\phi_j] \star f = 2^j \int_R \mathcal{F}^{-1}[\phi](2^j(x-y)) f(y)\, dy = \int_{\mathbb{R}} \mathcal{F}^{-1}[\phi](z) f(x - 2^{-j} z)\, dz \\
&= \sum_{k=-\infty}^{+\infty} \int_{B_{2^k} \setminus B_{2^{k-1}}} \mathcal{F}^{-1}[\phi](z) f(x - 2^{-j} z)\, dz \\
&\le \sum_{k=-\infty}^{+\infty} \max_{B_{2^k} \setminus B_{2^{k-1}}} |\mathcal{F}^{-1}[\phi](z)| \int_{B_{2^k} \setminus B_{2^{k-1}}} |f(x - 2^{-j} z)|\, dz \\
&\le \sum_{k=-\infty}^{+\infty} \max_{B_{2^k} \setminus B_{2^{k-1}}} 2^k |\mathcal{F}^{-1}[\phi](z)| 2^{j-k} \int_{B(x, 2^{k-j}) \setminus B(x, 2^{k-1-j})} |f(z)|\, dz \\
&\le M(f) \sum_{k=-\infty}^{+\infty} \max_{B_{2^k} \setminus B_{2^{k-1}}} 2^k |\mathcal{F}^{-1}[\phi](z)| \le C M(f).
\end{aligned}$$

In the last inequality we use the fact that $\mathcal{F}^{-1}[\phi]$ is in $\mathcal{S}(\mathbb{R}^n)$, and thus

$$\sum_{k=-\infty}^{+\infty} \max_{B_{2^k}\setminus B_{2^{k-1}}} 2^k |\mathcal{F}^{-1}[\phi](z)| \le 2\int_{\mathbb{R}} |\mathcal{F}^{-1}[\phi](z)|\,d\xi < \infty. \qquad \square$$

Proof of Theorem 1.7. We need to estimate $\Pi_1(R(Q,u))$, $\Pi_2(R(Q,u))$ and $\Pi_3(R(Q,u))$. Consistently with our earlier convention, we write, for example, $\Pi_1(\Delta^{1/4}(Q\Delta^{1/4}u))$ to mean

$$\Delta^{1/4}(\Pi_1(Q,\Delta^{1/4}u)) = \sum_{j=-\infty}^{\infty} \Delta^{1/4}(Q_j(\Delta^{1/4}u^{j-4})).$$

• Estimate of $\left\|\Pi_1(\Delta^{1/4}(Q\Delta^{1/4}u))\right\|_{\mathcal{H}^1}$. This expression equals

$$\begin{aligned}\int_{\mathbb{R}^n}\Big(\sum_{j=-\infty}^{\infty} 2^j Q_j^2(\Delta^{1/4}u^{j-4})^2\Big)^{1/2} dx &\le \int_{\mathbb{R}^n} \sup_j |\Delta^{1/4}u^{j-4}|\big(\textstyle\sum_j 2^j Q_j^2\big)^{1/2} dx \\ &\le \Big(\int_{\mathbb{R}^n} (M(\Delta^{1/4}u))^2\,dx\Big)^{1/2}\Big(\int_R \textstyle\sum_j 2^j Q_j^2\,dx\Big)^{1/2} \\ &\le C\|u\|_{\dot H^{1/2}}\|Q\|_{\dot H^{1/2}}. \end{aligned} \tag{34}$$

• Estimate of $\Pi_1\big(\Delta^{1/4}(\Delta^{1/4}Qu)-\Delta^{1/2}(Qu)\big)$. We show that this term lies in $\dot B^0_{1,1}$ ($\mathcal{H}^1 \hookrightarrow \dot B^0_{1,1}$). To this purpose we use the "commutator structure" of the term above:

$$\begin{aligned}&\big\|\Pi_1\big(\Delta^{1/4}(\Delta^{1/4}Q)u-\Delta^{1/2}(Qu)\big)\big\|_{\dot B^0_{1,1}} \\ &\quad= \sup_{\|h\|_{\dot B^0_{\infty,\infty}}\le 1}\int_{\mathbb{R}^n}\sum_j\sum_{|t-j|\le 3}\big(\Delta^{1/4}(u^{j-4}\Delta^{1/4}Q_j)-\Delta^{1/2}(u^{j-4}Q_j)\big)h_t\,dx \\ &\quad= \sup_{\|h\|_{\dot B^0_{\infty,\infty}}\le 1}\int_{\mathbb{R}^n}\sum_j\sum_{|t-j|\le 3}\mathcal{F}[u^{j-4}]\,\mathcal{F}[\Delta^{1/4}Q_j\Delta^{1/4}h_t - Q_j\Delta^{1/2}h_t]\,d\xi \\ &\quad= \sup_{\|h\|_{\dot B^0_{\infty,\infty}}\le 1}\int_{\mathbb{R}^n}\sum_j\sum_{|t-j|\le 3}\mathcal{F}[u^{j-4}](\xi) \\ &\qquad\qquad \times\Big(\int_{\mathbb{R}^n}\mathcal{F}[Q_j](\zeta)\mathcal{F}[\Delta^{1/4}h_t](\xi-\zeta)(|\zeta|^{1/2}-|\xi-\zeta|^{1/2})\,d\zeta\Big)\,d\xi. \end{aligned} \tag{35}$$

Note that in (35) we have $|\xi|\le 2^{j-3}$ and $2^{j-2}\le|\zeta|\le 2^{j+2}$. Thus $|\xi/\zeta|\le\frac{1}{2}$, allowing us to write

$$\big||\zeta|^{\frac12}-|\xi-\zeta|^{\frac12}\big| = |\zeta|^{\frac12}\Big(1-\Big|1-\frac{\xi}{\zeta}\Big|^{\frac12}\Big) = |\zeta|^{\frac12}\frac{\xi}{\zeta}\Big(1+\Big|1-\frac{\xi}{\zeta}\Big|^{\frac12}\Big)^{-1} = |\zeta|^{\frac12}\sum_{l=0}^{\infty}\frac{c_l}{l!}\Big(\frac{\xi}{\zeta}\Big)^{l+1}$$

for appropriate coefficients c_l. Thus the expression on the last two lines of (35) equals

$$\sup_{\|h\|_{\dot B^0_{\infty,\infty}}\le 1}\int_{\mathbb{R}^n}\sum_j\sum_{|t-j|\le 3}\mathcal{F}[u^{j-4}](\xi)\Big(\int_{\mathbb{R}^n}|\zeta|^{1/2}\mathcal{F}[Q_j](\zeta)\mathcal{F}[\Delta^{1/4}h_t](\xi-\zeta)\sum_{l=0}^{\infty}\frac{c_l}{l!}\Big(\frac{\xi}{\zeta}\Big)^{l+1}d\zeta\Big)\,d\xi. \tag{36}$$

Next, for $k \in \mathbb{Z}$ and $g \in \mathcal{S}'$, we set

$$S_k g = \mathcal{F}^{-1}[\xi^{-(k+1)}|\xi|^{1/2}\mathcal{F}g].$$

We note that if $h \in \dot{B}^s_{\infty,\infty}$ then $S_k h \in \dot{B}^{s+1/2+k}_{\infty,\infty}$ and if $h \in \dot{H}^s$ then $S_k h \in \dot{H}^{s+1/2+k}$.

Finally, if $Q \in \dot{H}^{1/2}$ then $\nabla^{k+1}(Q) \in \dot{H}^{-k-1/2}$.

It follows that (36) is bounded above by

$$\begin{aligned}
C \sup_{\|h\|_{\dot{B}^0_{\infty,\infty}}\le 1} &\sum_{l=0}^{\infty} \frac{c_l}{l!} \int_{\mathbb{R}^n} \sum_j \sum_{|t-j|\le 3} (i)^{-(l+1)} \mathcal{F}[\nabla^{l+1} u^{j-4}]\, \mathcal{F}[S_l Q_j \Delta^{1/4} h_t)](\xi)\, d\xi \\
&\le C \sup_{\|h\|_{\dot{B}^0_{\infty,\infty}}\le 1} \|h\|_{\dot{B}^0_{\infty,\infty}} \sum_{l=0}^{\infty} \frac{c_l}{l!} \int_{\mathbb{R}^n} \sum_j 2^{j/2} |\nabla^{l+1} u^{j-4}|\,|S_l Q_j|\, dx \\
&\le C \sum_{l=0}^{\infty} \frac{c_l}{l!} \int_{\mathbb{R}^n} \sum_j |2^{-(l+1/2)j} \nabla^{l+1} u^{j-4}|\,|2^{(l+1)j} S_l Q_j|\, dx \\
&\le C \sum_{l=0}^{\infty} \frac{c_l}{l!} \Big(\int_{\mathbb{R}^n} \sum_j 2^{-2(l+1/2)j} |\nabla^{l+1} u^{j-4}|^2\, dx \Big)^{1/2} \Big(\int_{\mathbb{R}^n} \sum_j 2^{2(l+1)j} |S_l Q_j|^2\, dx \Big)^{1/2}.
\end{aligned}$$

By Plancherel's theorem, this equals

$$\begin{aligned}
C \sum_{l=0}^{\infty} \frac{c_l}{l!} &\Big(\int_{\mathbb{R}^n} \sum_j 2^{-2(l+1/2)j} |\xi|^{2l} |\mathcal{F}[\nabla u^{j-4}]|^2 d\xi \Big)^{1/2} \Big(\int_{\mathbb{R}^n} \sum_j 2^{2(l+1)j} |\xi|^{-2(l+1/2)} |\mathcal{F}[Q_j]|^2 d\xi \Big)^{1/2} \\
&\le C \sum_{l=0}^{\infty} \frac{c_l}{l!} 2^{-3l} \Big(\int_{\mathbb{R}^n} \sum_j 2^{-j} |\mathcal{F}[\nabla u^{j-4}]|^2 d\xi \Big)^{1/2} \Big(\int_{\mathbb{R}^n} \sum_j 2^{j} |\mathcal{F}[Q_j]|^2 d\xi \Big)^{1/2} \\
&\le C \sum_{l=0}^{\infty} \frac{c_l}{l!} 2^{-3l} \|Q\|_{\dot{H}^{1/2}} \|u\|_{\dot{H}^{1/2}},
\end{aligned}$$

where we have used the fact that for every vector field X we have

$$\int_{\mathbb{R}^n} \sum_{j=-\infty}^{+\infty} 2^{-j} (X^{j-4})^2\, dx = \int_{\mathbb{R}^n} \sum_{k,l} X_k X_l \sum_{\substack{j-4\ge k\\ j-4\ge l}} 2^{-j}\, dx \lesssim \int_{\mathbb{R}^n} \sum_{j=-\infty}^{+\infty} 2^{-j} (X_j)^2\, dx. \tag{37}$$

- Estimate of $\big\|\Pi_2(\Delta^{1/4}(\Delta^{1/4} Qu))\big\|_{\mathcal{H}^1}$: as in (34).
- Estimate of $\big\|\Pi_2\big(\Delta^{1/4}(Q\Delta^{1/4}u) - \Delta^{1/2}(Qu)\big)\big\|_{\dot{B}^0_{1,1}}$: analogous to (35).
- Estimate of $\big\|\Pi_3(\Delta^{1/2}(Qu))\big\|_{\mathcal{H}^1}$. We show that this lies in the smaller space $\dot{B}^0_{1,1}$ (we always have $\dot{B}^0_{1,1} \hookrightarrow \mathcal{H}^1$). We first observe that if $h \in \dot{B}^0_{\infty,\infty}$ then $\Delta^{1/2} h \in \dot{B}^{-1}_{\infty,\infty}$ and

$$\Delta^{1/2} h^{j+6} = \sum_{k=-\infty}^{j+6} \Delta^{1/2} h_k \le \sup_{k\in\mathbb{N}} |2^{-k} \Delta^{1/2} h_k| \sum_{k=-\infty}^{j+6} 2^k \le C\, 2^j \|h\|_{\dot{B}^0_{\infty,\infty}}. \tag{38}$$

Thus

$$\begin{aligned}
\left\|\Pi_3(\Delta^{1/2}(Qu))\right\|_{\dot{B}^0_{1,1}} &= \sup_{\|h\|_{\dot{B}^0_{\infty,\infty}}\le 1} \int_{\mathbb{R}^n} \sum_j \sum_{|k-j|\le 3} \Delta^{1/2}(Q_j u_k) h \\
&= \sup_{\|h\|_{\dot{B}^0_{\infty,\infty}}\le 1} \int_{\mathbb{R}^n} \sum_j \sum_{|k-j|\le 3} \Delta^{1/2}(Q_j u_k)[h^{j+6}]\, dx \\
&= \sup_{\|h\|_{\dot{B}^0_{\infty,\infty}}\le 1} \int_{\mathbb{R}^n} \sum_j \sum_{|k-j|\le 3} (Q_j u_k)[\Delta^{1/2} h^{j+6}]\, dx \\
&\le C \sup_{\|h\|_{\dot{B}^0_{\infty,\infty}}\le 1} \|h\|_{\dot{B}^0_{\infty,\infty}} \int_{\mathbb{R}^n} \sum_j \sum_{|k-j|\le 3} 2^j |Q_j u_k|\, dx \\
&\le C \left(\int_{\mathbb{R}^n} \sum_j 2^j Q_j^2 \, dx \right)^{1/2} \left(\int_{\mathbb{R}^n} \sum_j 2^j u_j^2 \, dx \right)^{1/2} \le C \|Q\|_{\dot{H}^{1/2}} \|u\|_{\dot{H}^{1/2}}. \qquad (39)
\end{aligned}$$

• Estimate of $\Pi_3(\Delta^{1/4}(Q\Delta^{1/4}u))$. To show that this is in $\dot{B}^0_{1,1}$, we observe that if $h \in \dot{B}^0_{\infty,\infty}$ then $\Delta^{1/4}h \in B^{-1/2}_{\infty,\infty}$, and by arguing as in (38) we get

$$\|\Delta^{1/4} h_j\|_{L^\infty} \le 2^{j/2} \|h\|_{\dot{B}^0_{\infty,\infty}}.$$

Thus

$$\begin{aligned}
\left\|\Pi_3(\Delta^{1/4}(Q, \Delta^{1/4}u))\right\|_{\dot{B}^0_{1,1}} &= \sup_{\|h\|_{\dot{B}^0_{\infty,\infty}}\le 1} \int_{\mathbb{R}^n} \sum_j \sum_{|k-j|\le 3} \Delta^{1/4}(Q_j \Delta^{1/4} u_k) h \\
&= \sup_{\|h\|_{\dot{B}^0_{\infty,\infty}}\le 1} \int_{\mathbb{R}^n} \sum_j \sum_{|k-j|\le 3} (Q_j \Delta^{1/4} u_k)[\Delta^{1/4} h^{j+6}]\, dx \\
&\le C \sup_{\|h\|_{\dot{B}^0_{\infty,\infty}}\le 1} \|h\|_{\dot{B}^0_{\infty,\infty}} \int_{\mathbb{R}^n} \sum_j \sum_{|k-j|\le 3} 2^{j/2} |Q_j \Delta^{1/4} u_k|\, dx \\
&\le C \left(\int_{\mathbb{R}^n} \sum_j 2^j Q_j^2 \, dx \right)^{1/2} \left(\int_{\mathbb{R}^n} \sum_j (\Delta^{1/4} u_j)^2 \, dx \right)^{1/2} \\
&\le C \|Q\|_{\dot{H}^{1/2}} \|u\|_{\dot{H}^{1/2}}. \qquad (40)
\end{aligned}$$

• Estimate of $\Pi_3(\Delta^{1/4}(\Delta^{1/4}Qu))$: analogous to (40). □

Proof of Theorem 1.5. We use Theorem 1.7 and the duality between BMO and $\mathcal{H}^1$. For all $h, Q \in \dot{H}^{1/2}$ and $u \in BMO$ we have

$$\begin{aligned}
\int_{\mathbb{R}^n} \big(\Delta^{1/4}(Q\Delta^{1/4}u) - Q\Delta^{1/2}u + \Delta^{1/4}Q\Delta^{1/4}u\big) h\, dx &= \int_{\mathbb{R}^n} \big(\Delta^{1/4}(Q\Delta^{1/4}h) - \Delta^{1/2}(Qh) + \Delta^{1/4}(h\Delta^{1/4}Q)\big) u\, dx \\
&\le C \|u\|_{BMO} \|R(Q,h)\|_{\mathcal{H}^1};
\end{aligned}$$

by Theorem 1.7, this is at most

$$C\|u\|_{BMO} \|Q\|_{\dot{H}^{1/2}} \|h\|_{\dot{H}^{1/2}}.$$

Hence

$$\|T(Q,u)\|_{\dot{H}^{-1/2}} = \sup_{\|h\|_{\dot{H}^{1/2}}\le 1} \int_{\mathbb{R}^n} T(Q,u) h\, dx \le C \|u\|_{BMO} \|Q\|_{\dot{H}^{1/2}}. \qquad □$$

Proof of Theorem 1.8. We observe that $\mathcal{R}$ is a Fourier multiplier of order zero; thus $\mathcal{R}: H^{-1/2} \to H^{-1/2}$, $\mathcal{R}: \mathcal{H}^1 \to \mathcal{H}^1$, and $\mathcal{R}: \dot{B}^0_{1,1} \to \dot{B}^0_{1,1}$. See [Taylor 1991] and [Sickel and Triebel 1995].

The estimates are very similar to the ones in Theorem 1.7; thus we will write down only one:

- Estimate of $\Pi_1\big(\mathcal{R}\Delta^{1/4}(\Delta^{1/4}Q\mathcal{R}u) - \nabla(Q\mathcal{R}u)\big)$. We observe that $\nabla u = \Delta^{1/4}\mathcal{R}\Delta^{1/4}u$. Hence

$$
\begin{aligned}
&\big\|\Pi_1\big(\mathcal{R}\Delta^{1/4}(\Delta^{1/4}Q\mathcal{R}u) - \nabla(Q\mathcal{R}u)\big)\big\|_{\dot{B}^0_{1,1}} \\
&\qquad = \sup_{\|h\|_{\dot{B}^0_{\infty,\infty}}\le 1} \int_{\mathbb{R}^n} \sum_j \sum_{|t-j|\le 3} \big(\mathcal{R}\Delta^{1/4}(\Delta^{1/4}Q_j\mathcal{R}u^{j-4}) - \nabla(Q_j\mathcal{R}u^{j-4})\big)h_t\,dx \\
&\qquad \simeq \sup_{\|h\|_{\dot{B}^0_{\infty,\infty}}\le 1} \int_{\mathbb{R}^n} \sum_j \sum_{|t-j|\le 3} \mathcal{R}u^{j-4}\big(\mathcal{R}\Delta^{1/4}h_t\,\Delta^{1/4}Q_j - \nabla h_t\,Q_j\big)\,dx \\
&\qquad \simeq \sup_{\|h\|_{\dot{B}^0_{\infty,\infty}}\le 1} \int_{\mathbb{R}^n} \sum_j \sum_{|t-j|\le 3} \mathcal{F}[\mathcal{R}u^{j-4}](\xi) \\
&\qquad\qquad \times \bigg(\int_{\mathbb{R}^n} \mathcal{F}[Q_j](\zeta)\mathcal{F}[\mathcal{R}\Delta^{1/4}h_t](\xi-\zeta)\big(|\zeta|^{1/2} - |\xi-\zeta|^{1/2}\big)\,d\zeta\bigg)\,d\xi. \qquad (41)
\end{aligned}
$$

Now we can proceed exactly as in (35) and get

$$
\sup_{\|h\|_{\dot{B}^0_{\infty,\infty}}\le 1} \int_{\mathbb{R}^n} \sum_j \sum_{|t-j|\le 3} \big(\mathcal{R}\Delta^{1/4}(\Delta^{1/4}Q_j\mathcal{R}u^{j-4}) - \nabla(Q_j\mathcal{R}u^{j-4})\big)\,h_t\,dx \le C\|Q\|_{\dot{H}^{1/2}}\|u\|_{\dot{H}^{1/2}}. \qquad \square
$$

Proof of Theorem 1.6. This follows from Theorem 1.8 and the duality between $\mathcal{H}^1$ and BMO. $\square$

Lemma 3.2. *Let $u \in \dot{H}^{1/2}(\mathbb{R}^n)$, then $\mathcal{R}(\Delta^{1/4}u \cdot \mathcal{R}\Delta^{1/4}u) \in \mathcal{H}^1$, and*

$$
\|\mathcal{R}(\Delta^{1/4}u \cdot \mathcal{R}\Delta^{1/4}u)\|_{\mathcal{H}^1} \le C\|u\|^2_{\dot{H}^{1/2}}.
$$

Proof. Since $\mathcal{R}: \mathcal{H}^1 \to \mathcal{H}^1$, it is enough to verify that $\Delta^{1/4}u \cdot \mathcal{R}\Delta^{1/4}u \in \mathcal{H}^1$.

- Estimate of $\Pi_1(\Delta^{1/4}u, \mathcal{R}\Delta^{1/4}u)$:

$$
\begin{aligned}
\big\|\Pi_1(\Delta^{1/4}u, \mathcal{R}\Delta^{1/4}u)\big\|_{\mathcal{H}^1} &= \int_{\mathbb{R}^n} \bigg(\sum_{j=-\infty}^{+\infty} \big[\Delta^{1/4}u_j(\mathcal{R}\Delta^{1/4}u)^{j-4}\big)^2\bigg)^{1/2} dx \\
&\le \int_{\mathbb{R}^n} \sup_j \big|(\mathcal{R}\Delta^{1/4}u)^{j-4}\big| \bigg(\sum_{j=0}^{+\infty}[\Delta^{1/4}u_j]^2\bigg)^{1/2} dx \\
&\le \bigg(\int_{\mathbb{R}^n} |M(\mathcal{R}\Delta^{1/4}u)|^2\,dx\bigg)^{1/2} \bigg(\int_{\mathbb{R}^n} \sum_{j=-\infty}^{+\infty} [\Delta^{1/4}u_j]^2\,dx\bigg)^{1/2} \\
&\le C\|u\|^2_{\dot{H}^{1/2}}. \qquad (42)
\end{aligned}
$$

The estimate of the $\mathcal{H}^1$ norm of $\Pi_2(\Delta^{1/4}u \cdot \mathcal{R}\Delta^{1/4}u)$ is similar to (42).

• Estimate of $\Pi_3(\Delta^{1/4}u\cdot\mathcal{R}\Delta^{1/4}u)$:

$$
\begin{aligned}
&\left\|\Pi_1(\Delta^{1/4}u,\mathcal{R}\Delta^{1/4}u)\right\|_{\mathcal{H}^1}\\
&=\sup_{\|h\|_{\dot B^0_{\infty,\infty}}\le 1}\int_{\mathbb{R}^n}\sum_j\sum_{|k-j|\le 3}\Delta^{1/4}u_j\mathcal{R}(\Delta^{1/4}u_k)\Big(h^{j-6}+\sum_{t=j-5}^{j+6}h_t\Big)\,dx\\
&=\sup_{\|h\|_{\dot B^0_{\infty,\infty}}\le 1}\int_{\mathbb{R}^n}\sum_j\sum_{|k-j|\le 3}\big(\Delta^{1/4}u_j\mathcal{R}(\Delta^{1/4}u_k)-u_j\nabla u_k+\tfrac12\nabla(u_ju_k)\big)\Big(h^{j-6}+\sum_{t=j-5}^{j+6}h_t\Big)\,dx. \qquad (43)
\end{aligned}
$$

We only estimate the terms with h^{j-6}, the estimates with h_t being similar. We have

$$
\begin{aligned}
&\sup\nolimits_{\|h\|_{\dot B^0_{\infty,\infty}}\le 1}\int_{\mathbb{R}^n}\sum_j\sum_{|k-j|\le 3}(\Delta^{1/4}u_j\mathcal{R}(\Delta^{1/4}u_k)-u_j\nabla u_k)h^{j-6}\,dx\\
&=\sup\nolimits_{\|h\|_{\dot B^0_{\infty,\infty}}\le 1}\int_{\mathbb{R}^n}\sum_j\sum_{|k-j|\le 3}\mathcal{F}[h^{j-6}](x)\left(\int_{\mathbb{R}^n}\mathcal{F}[u_j]\,\mathcal{F}[\mathcal{R}\Delta^{1/4}u_k][|y|^{1/2}-|x-y|^{1/2}]\,dy\right)dx.
\end{aligned}
$$

By arguing as in (35), we can show that this is bounded above by $C\|u\|^2_{\dot H^{1/2}}$. Finally we also have

$$
\begin{aligned}
&\sup\nolimits_{\|h\|_{\dot B^0_{\infty,\infty}}\le 1}\int_{\mathbb{R}^n}\sum_j\sum_{|k-j|\le 3}\tfrac12\nabla(u_ju_k)h^{j-6}\,dx\\
&=\sup\nolimits_{\|h\|_{\dot B^0_{\infty,\infty}}\le 1}\int_{\mathbb{R}^n}\sum_j\sum_{|k-j|\le 3}\tfrac12(u_ju_k)\nabla h^{j-6}\,dx\\
&\le C\sup\nolimits_{\|h\|_{\dot B^0_{\infty,\infty}}\le 1}\|h\|_{\dot B^0_{\infty,\infty}}\int_{\mathbb{R}^n}\sum_j\sum_{|k-j|\le 3}2^ju_ju_k\,dx\le C\left(\int_{\mathbb{R}^n}\sum_j 2^ju_j^2\,dx\right)^{1/2}=C\|u\|^2_{\dot H^{1/2}}.
\end{aligned}
$$

□

Theorem 1.8 and Lemma 3.2 imply:

Corollary 3.3. *Let $n\in\dot H^{1/2}(\mathbb{R}^n,S^{m-1})$. Then $\Delta^{1/4}[n\cdot\Delta^{1/4}n]\in\mathcal{H}^1(\mathbb{R}^n)$.*

Proof. Since $n\cdot\nabla n=0$ (see proof of Proposition 1.4), we can write

$$
\begin{aligned}
\Delta^{1/4}[n\cdot\Delta^{1/4}n]&=\Delta^{1/4}[n\cdot\Delta^{1/4}n]-\mathcal{R}[n\cdot\nabla n]+\mathcal{R}[\Delta^{1/4}n\cdot\mathcal{R}\Delta^{1/4}n]-\mathcal{R}[\Delta^{1/4}n\cdot\mathcal{R}\Delta^{1/4}n]\\
&=S(n\cdot,n)-\mathcal{R}[\Delta^{1/4}n\cdot\mathcal{R}\Delta^{1/4}n]. \qquad (44)
\end{aligned}
$$

The estimate in the corollary's conclusion is a consequence of Theorem 1.8 and Lemma 3.2, which imply respectively that $S(n\cdot,n)\in\mathcal{H}^1$ and $\mathcal{R}(\Delta^{1/4}n\cdot\mathcal{R}\Delta^{1/4}n)\in\mathcal{H}^1$. □

4. L-energy decrease controls

We now provide (in Propositions 4.1 and 4.2) localization estimates of solutions to the equations

$$\Delta^{1/4}(M\Delta^{1/4}u)=T(Q,u) \qquad (45)$$

and

$$\Delta^{1/4}(p\cdot\Delta^{1/4}u)=S(q\cdot,u)-\mathcal{R}(\Delta^{1/4}u\cdot\mathcal{R}\Delta^{1/4}u), \qquad (46)$$

where $Q \in \dot{H}^{1/2}(\mathbb{R}, \mathcal{M}_{l\times m}(\mathbb{R}))$, $M \in \dot{H}^{1/2}(\mathbb{R}, \mathcal{M}_{l\times m}(\mathbb{R}))$, $l \geq 1$ and $p, q \in \dot{H}^{1/2}(\mathbb{R}, \mathbb{R}^m)$.

Such estimates will be crucial to obtaining Morrey-type estimates for half-harmonic maps into the sphere (see Section 5). As observed in Section 1, half-harmonic maps into the sphere satisfy both equations (16) and (18), which are (45) and (46) with (M, Q) and (p, q) replaced by $(u\wedge, u\wedge)$ and (u, u), respectively. Roughly speaking, we show that the L^2 norm of $M\Delta^{1/4}u$ in a sufficiently small ball (u being a solution of either (45) or (46)), is controlled by the L^2 norm of the same function in annuli outside the ball multiplied by a "crushing" factor.

To this end we consider a dyadic decomposition of unity (Section 2). For convenience set

$$A_h = B_{2^{h+1}} \setminus B_{2^{h-1}}, \quad A'_h = B_{2^h} \setminus B_{2^{h-1}},$$

for $h \in \mathbb{Z}$. Choose a dyadic decomposition $\varphi_j \in C_0^\infty(\mathbb{R})$, so

$$\operatorname{supp}(\varphi_j) \subset A_j \quad \text{and} \quad \sum_{-\infty}^{+\infty} \varphi_j = 1. \tag{47}$$

Also define, for $h \in \mathbb{Z}$,

$$\chi_h := \sum_{-\infty}^{h-1} \varphi_j, \quad \bar{u}_h = |B_{2^k}|^{-1} \int_{B_{2^k}} u(x)\,dx, \quad \bar{u}^h = |A_h|^{-1} \int_{A_h} u(x)\,dx, \quad \bar{u}'^h = |A'_h|^{-1} \int_{A'_h} u(x)\,dx.$$

Proposition 4.1. *Let $Q \in \dot{H}^{1/2}(\mathbb{R}, \mathcal{M}_{l\times m}(\mathbb{R}))$, $M \in \dot{H}^{1/2}(\mathbb{R}, \mathcal{M}_{l\times m}(\mathbb{R}))$, $l \geq 1$, and let $u \in \dot{H}^{1/2}(\mathbb{R}, \mathbb{R}^m)$ be a solution of (45). Then for $k < 0$ with $|k|$ large enough we have*

$$\begin{aligned} &\|M\Delta^{1/4}u\|^2_{L^2(B_{2^k})} - \tfrac{1}{4}\|\Delta^{1/4}u\|^2_{L^2(B_{2^k})} \\ &\qquad \leq C\bigg(\sum_{h=k}^{\infty} 2^{(k-h)/2}\|M\Delta^{1/4}u\|^2_{L^2(A_h)} + \sum_{h=k}^{\infty} 2^{(k-h)/2}\|\Delta^{1/4}u\|^2_{L^2(A_h)}\bigg). \end{aligned} \tag{48}$$

Proposition 4.2. *Let $p, q \in \dot{H}^{1/2}(\mathbb{R}, \mathbb{R}^m)$ and let $u \in \dot{H}^{1/2}(\mathbb{R}, \mathbb{R}^m)$ be a solution of (46). Then for $k < 0$ with $|k|$ large enough we have*

$$\begin{aligned} &\|p \cdot \Delta^{1/4}u\|^2_{L^2(B_{2^k})} - \tfrac{1}{4}\|\Delta^{1/4}u\|^2_{L^2(B_{2^k})} \\ &\qquad \leq C\bigg(\sum_{h=k}^{\infty} 2^{(k-h)/2}\|p \cdot \Delta^{1/4}u\|^2_{L^2(A_h)} + \sum_{h=k}^{\infty} 2^{(k-h)/2}\|\Delta^{1/4}u\|^2_{L^2(A_h)}\bigg). \end{aligned} \tag{49}$$

For the proof, we need some estimates.

Lemma 4.3. *Let $u \in \dot{H}^{1/2}(\mathbb{R})$. Then, for all $k \in \mathbb{Z}$,*

$$\sum_{h=k}^{+\infty} 2^{k-h}\|\varphi_h(u - \bar{u}_k)\|_{\dot{H}^{1/2}(\mathbb{R})} \leq C\bigg(\sum_{s\leq k} 2^{s-k}\|u\|_{\dot{H}^{1/2}(A_s)} + \sum_{s\geq k} 2^{k-s}\|u\|_{\dot{H}^{1/2}(A_s)}\bigg). \tag{50}$$

Proof of Lemma 4.3. We first have

$$\|\varphi_h(u - \bar{u}_k)\|_{\dot{H}^{1/2}(\mathbb{R})} \leq \|\varphi_h(u - \bar{u}^h)\|_{\dot{H}^{1/2}(\mathbb{R})} + \|\varphi_h\|_{\dot{H}^{1/2}(\mathbb{R})}|\bar{u}_k - \bar{u}^h|. \tag{51}$$

We estimate separately the two terms on the right-hand side of (51). We have

$$\begin{aligned}
\|\varphi_h(u-\bar{u}^h)\|^2_{\dot H^{1/2}(\mathbb{R})} &= \int_{A_h}\int_{A_h} \frac{\left|\varphi_h(u-\bar{u}^h)(x)-\varphi_h(u-\bar{u}^h)(y)\right|^2}{|x-y|^2}\,dx\,dy \\
&\le 2\left(\int_{A_h}\int_{A_h}\frac{|u(x)-u(y)|^2}{|x-y|^2}\,dx\,dy + \|\nabla\varphi_h\|^2_\infty \int_{A_h}\int_{A_h}|u-\bar{u}^h|^2\,dx\,dy\right) \\
&\le C\left(\|u\|^2_{\dot H^{1/2}(A_h)} + 2^{-h}\int_{A_h}|u-\bar{u}^h|^2\,dx\right) \le C\|u\|^2_{\dot H^{1/2}(A_h)},
\end{aligned} \tag{52}$$

where we used the fact that $\|\nabla\varphi_h\|_\infty \le C\,2^{-h}$ and the embedding $\dot H^{1/2}(\mathbb{R}) \hookrightarrow BMO(\mathbb{R})$.

Now we estimate $|\bar{u}_k-\bar{u}^h|$. We can write $\bar{u}_k=\sum_{l=-\infty}^{k-1} 2^{l-k}\bar{u}'^l$. Moreover,

$$\begin{aligned}
|\bar{u}_k-\bar{u}^h| &\le |\bar{u}^h-\bar{u}'^h| + |\bar{u}_k-\bar{u}'^h| \\
&\le C|A_h|^{-1}\int_{A_h}|u-\bar{u}^h|\,dx + \sum_{l=-\infty}^{k-1} 2^{l-k}\sum_{s=l}^{h-1}|\bar{u}'^{s+1}-\bar{u}'^s| \\
&\le C|A_h|^{-1}\int_{A_h}|u-\bar{u}^h|\,dx + \sum_{l=-\infty}^{k-1} 2^{l-k}\sum_{s=l}^{h-1}|A_{s+1}|^{-1}\int_{A_{s+1}}|u-\bar{u}^{s+1}|\,dx \\
&\le C\left(\|u\|_{\dot H^{1/2}(A_h)} + \sum_{l=-\infty}^{k-1} 2^{l-k}\sum_{s=l}^{h-1}\|u\|_{\dot H^{1/2}(A_{s+1})}\right).
\end{aligned} \tag{53}$$

Combining (52) and (53) we get

$$\begin{aligned}
\|\varphi_h(u-\bar{u}^h)\|_{\dot H^{1/2}(\mathbb{R})} &\le \|\varphi_h(u-\bar{u}^h)\|_{\dot H^{1/2}(\mathbb{R})} + \|\varphi_h\|_{\dot H^{1/2}(\mathbb{R})}|\bar{u}_k-\bar{u}^h| \\
&\le C\left(\|u\|_{\dot H^{1/2}(A_h)} + \sum_{l=-\infty}^{k-1} 2^{l-k}\sum_{s=l}^{h-1}\|u\|_{\dot H^{1/2}(A_{s+1})}\right).
\end{aligned} \tag{54}$$

Multiplying both sides of (54) by 2^{k-h} and summing up from $h=k$ to $+\infty$ we get

$$\begin{aligned}
&\sum_{h=k}^{+\infty} 2^{k-h}\left(\sum_{l=-\infty}^{k-1} 2^{l-k}\sum_{s=l+1}^{h}\|u\|_{\dot H^{1/2}(A_s)}\right) \\
&\qquad \le C\sum_{s\le k}\|u\|_{\dot H^{1/2}(A_s)}\left(\sum_{h\ge k}\sum_{l\le s}2^{l-h}\right) + \sum_{s\ge k}\|u\|_{\dot H^{1/2}(A_s)}\left(\sum_{h\ge s}\sum_{l\le k}2^{l-h}\right) \\
&\qquad \le C\sum_{s\le k}2^{s-k}\|u\|_{\dot H^{1/2}(A_s)} + \sum_{s\ge k}2^{k-s}\|u\|_{\dot H^{1/2}(A_s)}. \qquad \square
\end{aligned}$$

Now we recall the value of the Fourier transform of some functions that will be used in the sequel. We have

$$\mathcal{F}[|x|^{-1/2}](\xi) = |\xi|^{-1/2}. \tag{55}$$

The Fourier transforms of $|x|$, $x|x|^{-1/2}$, and $|x|^{1/2}$ are the tempered distributions defined, for every $\varphi\in\mathcal{S}(\mathbb{R})$, as follows (with $\mathbb{1}_X$ the characteristic function of $\mathbb{1}$):

$$\langle \mathcal{F}[|x|], \varphi\rangle = \langle \mathcal{F}[x/|x|] \star \mathcal{F}[x], \varphi\rangle = \langle p.v.(x^{-1}) \star (\delta)'_0(x), \varphi\rangle = p.v. \int_{\mathbb{R}} \frac{\varphi(x) - \varphi(0) - \mathbb{1}_{B_1(0)}\phi'(0)x}{x^2}\, dx,$$

$$\langle \mathcal{F}[x\,|x|^{-1/2}], \varphi\rangle = \langle \mathcal{F}[x] \star \mathcal{F}[|x|^{-1/2}], \varphi\rangle = \langle (\delta)'_0(x) \star |x|^{-1/2}, \varphi\rangle = p.v. \int_{\mathbb{R}} \big(\varphi(x) - \varphi(0)\big) \frac{x}{|x|} \frac{1}{|x|^{3/2}}\, dx,$$

$$\langle \mathcal{F}[|x|^{1/2}], \varphi\rangle = p.v. \int_{\mathbb{R}} \frac{\varphi(x) - \varphi(0)}{|x|^{3/2}}\, dx.$$

Next we introduce the operators

$$F(Q, a) = \Delta^{1/4}(Qa) - Q\Delta^{1/4}a + \Delta^{1/4}Qa, \tag{56}$$

$$G(Q, a) = \mathcal{R}\Delta^{1/4}(Qa) - Q\Delta^{1/4}\mathcal{R}a + \Delta^{1/4}Q\mathcal{R}a. \tag{57}$$

We observe that $T(Q, u) = F(Q, \Delta^{1/4}u)$ and $S(Q, u) = \mathcal{R}G(Q, \Delta^{1/4}u)$.

We now state turn to lemmas where we consider M, u as in Proposition 4.1 or p, u as in Proposition 4.2, and estimate the $\dot H^{1/2}$ norm of $w = \Delta^{-1/4}(M\Delta^{1/4}u)$ or $w = \Delta^{-1/4}(p \cdot \Delta^{1/4}u)$ in B_{2^k} in terms of the $\dot H^{1/2}$ norm of w in annuli outside the ball and the L^2 norm of $\Delta^{1/4}u$ in annuli inside and outside the ball B_{2^k}. The key point is that each term is multiplied by a crushing factor.

Lemma 4.4. *Assume the hypotheses of Proposition 4.1. There exist $C > 0$ and $\bar n > 0$, independent of u and M, such that for all $\eta \in \left(0, \frac14\right)$, all $k < k_0$ (where $k_0 \in \mathbb{Z}$ depends on η and $\|Q\|_{\dot H^{1/2}(\mathbb{R})}$), and all $n \ge \bar n$, we have*

$$\|\chi_{k-4}(w - \bar w_{k-4})\|_{\dot H^{1/2}(\mathbb{R})} \le \eta\|\chi_{k-4}\Delta^{1/4}u\|_{L^2} + C\left(\sum_{h=k}^{\infty} 2^{(k-h)/2}\|\Delta^{1/4}u\|_{L^2(A_h)} + \sum_{h=k-n}^{+\infty} 2^{k-h}\|w\|_{\dot H^{1/2}(A_h)}\right), \tag{58}$$

where $w = \Delta^{-1/4}(M\Delta^{1/4}u)$ and we recall that $\chi_{k-4} \equiv 1$ on $B_{2^{k-5}}$ and $\chi_{k-4} \equiv 0$ on $B^c_{2^{k-4}}$.

Lemma 4.5. *Assume the hypotheses of Proposition 4.2. There exist $C > 0$ and $\bar n > 0$, independent of u and M, such that for all $\left(0, \frac14\right)$, all $k < k_0$ (where $k_0 \in \mathbb{Z}$ depends on η and the $\dot H^{1/2}$ norms of Q and u in $\mathbb{R}$), and all $n \ge \bar n$, we have*

$$\|\chi_{k-4}(w - \bar w_{k-4})\|_{\dot H^{1/2}(\mathbb{R})} \le \eta\|\chi_{k-4}\Delta^{1/4}u\|_{L^2(\mathbb{R})} + C\left(\sum_{h=k}^{\infty} 2^{(k-h)/2}\|\Delta^{1/4}u\|_{L^2(A_h)} + \sum_{h=k-n}^{k-3} 2^{h-k}\|w\|_{\dot H^{1/2}(A_h)}\right), \tag{59}$$

where $w = \Delta^{-1/4}(p \cdot \Delta^{1/4}u)$.

Proof of Lemma 4.4. Fix $\eta \in \left(0, \frac14\right)$. We first consider $k < 0$ large enough in absolute value so that $\|\chi_k(Q - \bar Q_k)\|_{\dot H^{1/2}(\mathbb{R})} \le \varepsilon$, where $\varepsilon \in (0, 1)$ will be determined later. We write

$$F(Q, \Delta^{1/4}u) = F(Q_1, \Delta^{1/4}u) + F(Q_2, \Delta^{1/4}u),$$

where

$$Q_1 = \chi_k(Q - \bar Q_k) \quad \text{and} \quad Q_2 = (1 - \chi_k)(Q - \bar Q_k).$$

By construction, we have

$$\operatorname{supp} Q_2 \subseteq B^c_{2^{k-1}} \quad \text{and} \quad \|Q_2\|_{\dot H^{1/2}(\mathbb{R})} \le \|Q\|_{\dot H^{1/2}(\mathbb{R})}.$$

For brevity, set

$$W := \chi_{k-4}(w - \bar w_{k-4}).$$

We rewrite (45) as

$$\Delta^{1/2}(W) = -\Delta^{1/2}\Big(\sum_{h=k-4}^{+\infty} \varphi_h(w - \bar w_{k-4})\Big) + F(Q_1, \Delta^{1/4}u) + F(Q_2, \Delta^{1/4}u). \tag{60}$$

We take the scalar product of both sides with W and integrate over $\mathbb{R}$. From Corollary A.8 it follows that

$$\lim_{N\to+\infty} \int_{\mathbb{R}} \Delta^{1/2}\Big(\sum_{h=N}^{+\infty} \varphi_h(w - \bar w_{k-4})\Big)\cdot W\,dx = \lim_{N\to+\infty} \int_{\mathbb{R}} \Delta^{1/2}\big((1-\chi_{N-1})(w - \bar w_{k-4})\big)\cdot W\,dx = 0.$$

This allows us to interchange the infinite sum with the integral and the operator $\Delta^{1/2}$ in the expression

$$\int_{\mathbb{R}} \Delta^{1/2}\Big(\sum_{h=k-4}^{+\infty} \varphi_h(w - \bar w_{k-4})\Big)\cdot W\,dx = \sum_{h=k-4}^{+\infty} \int_{\mathbb{R}} \Delta^{1/2}\big(\varphi_h(w - \bar w_{k-4})\big)\cdot W\,dx.$$

Thus we get from (60) the equality

$$\begin{aligned}&\int_{\mathbb{R}} \big|\Delta^{1/4}(W)\big|^2\,dx \\ &= -\sum_{h=k-4}^{+\infty} \int_{\mathbb{R}} \Delta^{1/2}(\varphi_h(w-\bar w_{k-4}))\cdot W\,dx + \int_{\mathbb{R}} F(Q_1, \Delta^{1/4}u)\cdot W\,dx + \int_{\mathbb{R}} F(Q_2, \Delta^{1/4}u)\cdot W\,dx.\end{aligned} \tag{61}$$

Step 1: estimate of the sum. We split the sum in (61) into two parts: $k-4 \le h \le k-3$ and $h \ge k-2$.

Step 1a. We have

$$-\sum_{h=k-4}^{k-3} \int_{\mathbb{R}} \Delta^{1/2}(\varphi_h(w-\bar w_{k-4}))\cdot W\,dx \le \|W\|_{\dot H^{1/2}(\mathbb{R})}\Big(\sum_{h=k-4}^{k-3} \big\|\varphi_h(w-\bar w_{k-4})\big\|_{\dot H^{1/2}(\mathbb{R})}\Big).$$

By Lemma 4.3, the right-hand side is bounded above by

$$\begin{aligned}&\|W\|_{\dot H^{1/2}(\mathbb{R})}\Big(\sum_{h=k-4}^{k-3}\Big(\|w\|_{\dot H^{1/2}(A_h)} + \sum_{l=-\infty}^{k-5} 2^{l-(k-4)} \sum_{s=l+1}^{h} \|w\|_{\dot H^{1/2}(A_s)}\Big)\Big) \\ &\qquad\qquad \le C\|W\|_{\dot H^{1/2}(\mathbb{R})}\Big(\sum_{h=-\infty}^{k-3} 2^{h-k}\|w\|_{\dot H^{1/2}(A_h)}\Big).\end{aligned} \tag{62}$$

From the localization theorem A.1 it follows that

$$\sum_{h=-\infty}^{k-6} \|w\|^2_{\dot H^{1/2}(A_h)} \le \tilde C\,\|W\|^2_{\dot H^{1/2}(\mathbb{R})},$$

where $\tilde{C} > 0$ is independent of k and w. Thus we can find $n_1 \geq 6$ such that

$$C \sum_{h=-\infty}^{k-n} 2^{h-k} \|w\|_{\dot{H}^{1/2}(A_h)} \leq \tfrac{1}{8} \|W\|_{\dot{H}^{1/2}(\mathbb{R})} \quad \text{for all } n \geq n_1,$$

with the same constant C appearing on the last line of (62). Then for $n \geq n_1$ we have

$$\sum_{h=k-4}^{k-3} \int_{\mathbb{R}} \Delta^{1/2}\big(\varphi_h(w - \bar{w}_{k-4})\big) \cdot W \, dx \\ \leq \tfrac{1}{8} \|W\|^2_{\dot{H}^{1/2}(\mathbb{R})} + C \|W\|_{\dot{H}^{1/2}(\mathbb{R})} \Bigg(\sum_{h=k-n}^{k-3} 2^{h-k} \|w\|_{\dot{H}^{1/2}(A_h)} \Bigg). \tag{63}$$

Step 1b. To estimate the part of the sum in (61) with $h \geq k-2$, we use the fact that the supports of φ_h and of χ_{k-4} are disjoint; in particular $0 \notin \operatorname{supp}\big(\varphi_h(w - \bar{w}_{k-4}) \star W\big)$. We have

$$\sum_{h=k-2}^{+\infty} \int_{\mathbb{R}} \Delta^{1/2}\big(\varphi_h(w - \bar{w}_{k-4})\big) \cdot W \, dx \\ = \sum_{h=k-2}^{+\infty} \int_{\mathbb{R}} \mathcal{F}^{-1}(|\xi|)(x) \big(\varphi_h(w - \bar{w}_{k-4})\big) \star W \, dx \\ \leq \sum_{h=k-2}^{+\infty} \big\|\mathcal{F}^{-1}(|\xi|)\big\|_{L^\infty(B_{2^{h+2}} \setminus B_{2^{h-2}})} \big\|\varphi_h(w - \bar{w}_{k-4})\big\|_{L^1} \|W\|_{L^1} \\ \leq C \sum_{h=k-2}^{+\infty} 2^{-2h} 2^{h/2} \big\|\varphi_h(w - \bar{w}_{k-4})\big\|_{L^2(\mathbb{R})} 2^{k/2} \|W\|_{L^2(\mathbb{R})}. \tag{64}$$

By Theorem A.5 and Lemma 4.3 the sum on this last line is bounded above by

$$\sum_{h=k-2}^{+\infty} 2^{k-4-h} \big\|\varphi_h(w - \bar{w}_{k-4})\big\|_{\dot{H}^{1/2}(\mathbb{R})} \|W\|_{\dot{H}^{1/2}(\mathbb{R})} \\ \leq \sum_{h=k-2}^{+\infty} 2^{k-4-h} \Big(\|w\|_{\dot{H}^{1/2}(A_h)} + \sum_{l=-\infty}^{k-5} 2^{l-(k-4)} \sum_{s=l+1}^{h} \|w\|_{\dot{H}^{1/2}(A_s)} \Big) \|W\|_{\dot{H}^{1/2}(\mathbb{R})} \\ \leq \Bigg(\sum_{h=k-4}^{+\infty} 2^{k-h-4} \|w\|_{\dot{H}^{1/2}(A_h)} + \sum_{s \leq k-4} \|w\|_{\dot{H}^{1/2}(A_s)} \Big(\sum_{h \geq k-4} \sum_{l \leq s-1} 2^{l-h} \Big) \\ + \sum_{s \geq k-4} \|w\|_{\dot{H}^{1/2}(A_s)} \Big(\sum_{h \geq s-1} \sum_{l \leq k-4} 2^{l-h} \Big) \Bigg) \|W\|_{\dot{H}^{1/2}(\mathbb{R})} \\ \leq \Bigg(\sum_{h=k-4}^{+\infty} 2^{k-4-h} \|w\|_{\dot{H}^{1/2}(A_h)} + \sum_{h=-\infty}^{k-5} 2^{h-(k-4)} \|w\|_{\dot{H}^{1/2}(A_h)} \Bigg) \|W\|_{\dot{H}^{1/2}(\mathbb{R})}. \tag{65}$$

Finally, set $n > \bar{n} = \max(n_1, n_2)$, where $n_2 \geq 6$ is such that

$$C \sum_{h=-\infty}^{k-n} 2^{h-(k-4))} \|w\|_{\dot{H}^{1/2}(A_h)} \leq \tfrac{1}{8} \|W\|_{\dot{H}^{1/2}(\mathbb{R})} \quad \text{for } n \geq n_2.$$

We conclude from (63)–(65) that

$$\sum_{h=k-4}^{+\infty} \int_{\mathbb{R}} \Delta^{1/2}(\varphi_h(w-\bar{w}_{k-4}))\cdot W \le \tfrac{1}{4}\|W\|^2_{\dot{H}^{1/2}(\mathbb{R})} + C\|W\|_{\dot{H}^{1/2}(\mathbb{R})} \sum_{h=k-n}^{+\infty} 2^{k-h}\|w\|_{\dot{H}^{1/2}(A_h)}. \tag{66}$$

Step 2: estimate of $\int_{\mathbb{R}} F(Q_1, \Delta^{1/4}u)\cdot W\,dx$, the second term on the right-hand side of (61). We write

$$F(Q_1, \Delta^{1/4}u) = F(Q_1, \chi_{k-4}\Delta^{1/4}u) + \sum_{h=k-4}^{k+1} F(Q_1, \varphi_h\Delta^{1/4}u) + \sum_{h=k+2}^{+\infty} F(Q_1, \varphi_h\Delta^{1/4}u). \tag{67}$$

By Theorem 1.7, the integral involving the first term on the right can be estimated as follows:

$$\begin{aligned}\int_{\mathbb{R}} F(Q_1, \chi_{k-4}\Delta^{1/4}u)\cdot W\,dx &\le C\|Q_1\|_{\dot{H}^{1/2}(\mathbb{R})}\|\chi_{k-4}\Delta^{1/4}u\|_{L^2}\|W\|_{\dot{H}^{1/2}(\mathbb{R})}\\ &\le C\varepsilon\|\chi_{k-4}\Delta^{1/4}u\|_{L^2}\|W\|_{\dot{H}^{1/2}(\mathbb{R})}\\ &\le \tfrac{1}{16}\|\chi_{k-4}\Delta^{1/4}u\|_{L^2}\|W\|_{\dot{H}^{1/2}(\mathbb{R})},\end{aligned} \tag{68}$$

where in the last inequality we have made use of the choice of $\varepsilon > 0$ (see beginning of proof on page 166).

We also use Theorem 1.7 for the integral involving the second term on the right-hand side of (67):

$$\sum_{h=k-4}^{k+1} \int_{\mathbb{R}} F(Q_1, \varphi_h\Delta^{1/4}u)\cdot W\,dx \le C \sum_{h=k-4}^{k+1} \|Q_1\|_{\dot{H}^{1/2}(\mathbb{R})}\|\varphi_h\Delta^{1/4}u\|_{L^2(\mathbb{R})}\|W\|_{\dot{H}^{1/2}(\mathbb{R})}. \tag{69}$$

Next we want to deal with the term in (67) involving the infinite sum. Again by Corollary A.8 we can exchange the summation with the integral and write

$$\int_{\mathbb{R}} \Big(\sum_{h=k+2}^{+\infty} F(Q_1, \varphi_h\Delta^{1/4}u)\Big)\cdot W\,dx = \sum_{h=k+2}^{+\infty} \int_{\mathbb{R}} F(Q_1, \varphi_h\Delta^{1/4}u)\cdot W\,dx.$$

If $h \ge k+2$, we have $F(Q_1, \varphi_h\Delta^{1/4}u)\cdot W = Q_1\Delta^{1/4}(\varphi_h\Delta^{1/4}u)\cdot W$, since the supports of Q_1 and φ_h are disjoint, as are the supports of χ_{k-4} and φ_h. Hence we can write

$$\begin{aligned}\sum_{h=k+2}^{+\infty} \int_{\mathbb{R}} F(Q_1, \varphi_h\Delta^{1/4}u)\cdot W\,dx &= \sum_{h=k+2}^{+\infty} \int_{\mathbb{R}} Q_1\Delta^{1/4}(\varphi_h\Delta^{1/4}u)\cdot W\,dx\\ &= \sum_{h=k+2}^{+\infty} \int_{\mathbb{R}} \mathcal{F}^{-1}(|\xi|^{1/2})(x)\big((Q_1\varphi_h\Delta^{1/4}u)\star W\big)\\ &= \sum_{h=k+2}^{+\infty} \|\mathcal{F}^{-1}(|\xi|^{1/2})\|_{L^\infty(B_{2^{h+2}}\setminus B_{2^{h-2}})}\|Q_1\varphi_h\Delta^{1/4}u\|_{L^1}\|W\|_{L^1}\\ &\le C\sum_{h=k+2}^{+\infty} 2^{-3h/2}\|Q_1\varphi_h\Delta^{1/4}u\|_{L^1}\|W\|_{L^1}.\end{aligned} \tag{70}$$

By Theorem A.5 we finally get

$$\begin{aligned}\sum_{h=k+2}^{+\infty}\int_{\mathbb{R}} F(Q_1,\varphi_h\Delta^{1/4}u)\cdot W\,dx &\le C\sum_{h=k+2}^{+\infty} 2^{k-h}\|Q_1\|_{\dot H^{1/2}(\mathbb{R})}\|\,\|\varphi_h\Delta^{1/4}u\|_{L^2(\mathbb{R})}\|W\|_{\dot H^{1/2}(\mathbb{R})}\\ &\le C\sum_{h=k+2}^{+\infty} 2^{k-h}\|\varphi_h\Delta^{1/4}u\|_{L^2}\|W\|_{\dot H^{1/2}(\mathbb{R})}.\end{aligned}$$

Step 3: estimate of $\int_{\mathbb{R}} F(Q_2,\Delta^{1/4}u)\cdot W\,dx$, the last term in (61). As in Step 2, we write

$$F(Q_2,\Delta^{1/4}u)=F(Q_2,\chi_{k-4}\Delta^{1/4}u)+\sum_{h=k-4}^{k+1}F(Q_2,\varphi_h\Delta^{1/4}u)+\sum_{h=k+2}^{+\infty}F(Q_2,\varphi_h\Delta^{1/4}u). \tag{71}$$

For the first term, since the support of Q_2 is included in $B^c_{2^{k-1}}$, we have

$$F(Q_2,\chi_{k-4}\Delta^{1/4}u)\cdot W=\Delta^{1/4}\big(Q_2(\chi_{k-4}\Delta^{1/4}u)\big)\cdot W.$$

Observe that $Q_2=\sum_{h=k-1}^{+\infty}\varphi_h(Q_2-(\bar Q_2)_{k-1})$, $((\bar Q_2)_{k-1}=0)$ and by using Corollary A.8 we get

$$\begin{aligned}&\int_{\mathbb{R}} F(Q_2,\chi_{k-4}\Delta^{1/4}u)\cdot W\,dx\\ &\quad=\sum_{h=k-1}^{+\infty}\int_{\mathbb{R}}\Delta^{1/4}\big((\varphi_h(Q_2-(\bar Q_2)_{k-1}))(\chi_{k-4}\Delta^{1/4}u)\big)\cdot W\\ &\quad\le C\sum_{h=k-1}^{+\infty}\int_{\mathbb{R}}\mathcal{F}^{-1}(|\xi|^{1/2})\big(((\chi_{k-4}\Delta^{1/4}u)\varphi_h(Q_2-(\bar Q_2)_{k-1})\big)\star W\big)\\ &\quad\le C\|W\|_{L^1}\sum_{h=k-1}^{+\infty}\|\mathcal{F}^{-1}(|\xi|^{1/2})\|_{L^\infty(B_{2^{h+2}}\setminus B_{2^{h-2}})}\|(\chi_{k-4}\Delta^{1/4}u)\varphi_h(Q_2-(\bar Q_2)_{k-1})\|_{L^1}\\ &\quad\le C\|\chi_{k-4}\Delta^{1/4}u\|_{L^2}\|W\|_{\dot H^{1/2}(\mathbb{R})}\sum_{h=k-1}^{+\infty}2^{-h/2}2^{k/2}\|\varphi_h(Q_2-(\bar Q_2)_{k-1})\|_{\dot H^{1/2}(\mathbb{R})}.\end{aligned}$$

From Lemma 4.3, possibly by choosing a smaller k, we get

$$C\sum_{h=k-1}^{+\infty}2^{(k-h)/2}\|\varphi_h(Q_2-(\bar Q_2)_{k-1})\|_{\dot H^{1/2}(\mathbb{R})}\le \tfrac14\eta<\tfrac1{16}.$$

Therefore

$$\int_{\mathbb{R}} F(Q_2,\chi_{k-4}\Delta^{1/4}u)\cdot W\,dx\le \tfrac14\eta\|\chi_{k-4}\Delta^{1/4}u\|_{L^2}\|W\|_{\dot H^{1/2}(\mathbb{R})}.$$

Now turning to the second term in (71), we bound the corresponding integral using Theorem 1.7:

$$\sum_{h=k-4}^{k+1}\int_{\mathbb{R}} F(Q_2,\varphi_h\Delta^{1/4}u)\cdot W\,dx\le C\sum_{h=k-4}^{k+1}\|Q\|_{\dot H^{1/2}(\mathbb{R})}\,\|\varphi_h\Delta^{1/4}u\|_{L^2}\,\|W(w-\bar w_{k-4})\|_{\dot H^{1/2}(\mathbb{R})}. \tag{72}$$

Finally we consider the last term in (71). By Corollary A.8 we can write

$$\int_{\mathbb{R}}\Big(\sum_{h=k+2}^{+\infty} F(Q_2,\varphi_h\Delta^{1/4}u)\Big)\cdot W\,dx=\sum_{h=k+2}^{+\infty}\int_{\mathbb{R}} F(Q_2,\varphi_h\Delta^{1/4}u)\cdot W\,dx.$$

Next, since the support of Q_2 is included in $B^c_{2^{k-1}}$, we have for $h\geq k+2$ the equality

$$\begin{aligned}F(Q_2,\varphi_h\Delta^{1/4}u)\cdot W&=\big(\Delta^{1/4}(Q_2\,\varphi_h\Delta^{1/4}u)-Q_2\Delta^{1/4}(\varphi_h\Delta^{1/4}u)+\Delta^{1/4}Q_2\varphi_h\Delta^{1/4}u\big)\cdot W\\&=\Delta^{1/4}(Q_2\varphi_h\Delta^{1/4}u)\cdot W.\end{aligned}$$

Therefore

$$\begin{aligned}&\sum_{h=k+2}^{+\infty}\int_{\mathbb{R}} F(Q_2,\varphi_h\Delta^{1/4}u)\cdot W\,dx \qquad (73)\\&=\sum_{h=k+2}^{+\infty}\int_{\mathbb{R}}\Delta^{1/4}(Q_2\varphi_h\Delta^{1/4}u)\cdot W\,dx=\sum_{h=k+2}^{+\infty}\int_{\mathbb{R}}\mathcal{F}[\Delta^{1/4}(Q_2\varphi_h\Delta^{1/4}u)]\,\mathcal{F}[W]\,d\xi\\&=\sum_{h=k+2}^{+\infty}\int_{\mathbb{R}}|\xi|^{1/2}\mathcal{F}[(Q_2\varphi_h\Delta^{1/4}u)]\,\mathcal{F}[W]\,d\xi\\&=\sum_{h=k+2}^{+\infty}\int_{\mathbb{R}}\mathcal{F}^{-1}(|\xi|^{1/2})\,\big(\big(\varphi_h\Delta^{1/4}u(Q_2-(\bar{Q}_2)_{k-1})\big)\star W\big)\;dx\\&\leq\sum_{h=k+2}^{+\infty}\|\mathcal{F}^{-1}[|\xi^{1/2}]\|_{L^\infty(B_{2^{h+2}}\setminus B_{2^{h-2}})}\,\big\|\big(\varphi_h\Delta^{1/4}u(Q_2-(\bar{Q}_2)_{k-1})\big)\star W\big\|_{L^1(\mathbb{R})}.\end{aligned}$$

Now choose $\psi_h\in C^\infty_0(\mathbb{R})$ such that $\psi_h\equiv 1$ in $B_{2^{h+1}}\setminus B_{2^{h-1}}$ and $\operatorname{supp}\psi\subset B_{2^{h+2}}\setminus B_{2^{h-2}}$. Thus

$$\begin{aligned}(73)&\leq C\sum_{h=k+2}^{+\infty}2^{-3h/2}\,\|\psi_h(Q_2-(\bar{Q}_2)_{k-1})\|_{L^2(\mathbb{R})}\,\|\varphi_h\Delta^{1/4}u\|_{L^2(\mathbb{R})}\,\|W\|_{L^1(\mathbb{R})}\\&\leq C\sum_{h=k+2}^{+\infty}2^{k-h}\|\psi_h(Q_2-(\bar{Q}_2)_{k-1})\|_{\dot{H}^{1/2}(\mathbb{R})}\,\|\varphi_h\Delta^{1/4}u\|_{L^2(\mathbb{R})}\,\|W\|_{\dot{H}^{1/2}(\mathbb{R})} \qquad (74)\\&\leq C\Big(\sum_{h=k+2}^{+\infty}2^{k-h}\|\psi_h(Q_2-(\bar{Q}_2)_{k-1})\|^2_{\dot{H}^{1/2}(\mathbb{R})}\Big)^{1/2}\Big(\sum_{h=k+2}^{+\infty}2^{k-h}\|\varphi_h\Delta^{1/4}u\|^2_{L^2}\Big)^{1/2}\|W\|_{\dot{H}^{1/2}(\mathbb{R})}.\end{aligned}$$

where we have applied Theorem A.5 and Cauchy–Schwartz.

From Lemma 4.3 (with φ replaced by ψ) and Theorem A.1 we deduce that

$$\Big(\sum_{h=k+2}^{+\infty}2^{k-h}\|\psi_h(Q_2-(\bar{Q}_2)_{k-1})\|^2_{\dot{H}^{1/2}(\mathbb{R})}\Big)^{1/2}\leq C\|Q\|_{\dot{H}^{1/2}(\mathbb{R})}.$$

Thus

$$\sum_{h=k+2}^{+\infty}\int_{\mathbb{R}} F(Q_2,\varphi_h\Delta^{1/4}u)\cdot W\,dx \le C\|W\|_{\dot H^{1/2}(\mathbb{R})}\Big(\sum_{h=k+1}^{+\infty} 2^{k-h}\|\varphi_h\Delta^{1/4}u\|^2_{L^2(\mathbb{R})}\Big)^{1/2}$$
$$\le C\|W\|_{\dot H^{1/2}(\mathbb{R})}\Big(\sum_{h=k+1}^{+\infty} 2^{(k-h)/2}\|\varphi_h\Delta^{1/4}u\|_{L^2(\mathbb{R})}\Big). \tag{75}$$

By combining (68), (69), (70), (72) and (75) we obtain (for some constant C depending on Q)

$$\int_{\mathbb{R}} F(Q,\Delta^{1/4}u)\cdot W\,dx$$
$$\le \tfrac{1}{2}\eta\|\chi_{k-4}\Delta^{1/4}u\|_{L^2}\|W\|_{\dot H^{1/2}(\mathbb{R})} + C\sum_{h=k-4}^{+\infty} 2^{(k-h)/2}\|\Delta^{1/4}u\|_{L^2(A_h)}\|W\|_{\dot H^{1/2}(\mathbb{R})}. \tag{76}$$

Finally for all $n\ge\bar n$ we have

$$\|W\|_{\dot H^{1/2}(\mathbb{R})}$$
$$\le \eta\|\chi_{k-4}\Delta^{1/4}u\|_{\dot H^{1/2}(\mathbb{R})} + C\Big(\sum_{h=k-n}^{+\infty} 2^{k-h}\|w\|_{\dot H^{1/2}(A_s)} + \sum_{h=k-4}^{+\infty} 2^{(k-h)/2}\|\Delta^{1/4}u\|_{L^2(A_h)}\Big), \tag{77}$$

concluding the proof of Lemma 4.4. □

Proof of Lemma 4.5. The proof is similar to the preceding one, so we just sketch it. As before, we fix $\eta\in\big(0,\frac14\big)$. We consider $k<0$ such that

$$\|\chi_k(q-\bar q_k)\|_{\dot H^{1/2}(\mathbb{R})}\le\varepsilon \quad\text{and}\quad \|\chi_k\Delta^{1/4}u\|_{L^2(\mathbb{R})}\le\varepsilon,$$

with $\varepsilon>0$ to be determined later.

We observe that (46) is equivalent to

$$\mathcal{R}\Delta^{1/4}(p\cdot\Delta^{1/4}u) = G(q\,\cdot,\Delta^{1/4}u) - \Delta^{1/4}u\cdot(\mathcal{R}\Delta^{1/4}u). \tag{78}$$

We write

$$G(q\,\cdot,\Delta^{1/4}u) = G(q_1\,\cdot,\Delta^{1/4}u) + G(q_2\,\cdot,\Delta^{1/4}u),$$

where

$$q_1=\chi_k(q-\bar q_k) \quad\text{and}\quad q_2=(1-\chi_k)(q-\bar q_k).$$

We observe that $\operatorname{supp} q_2\subseteq B^c_{2^{k-1}}$ and $\|q_1\|_{\dot H^{1/2}(\mathbb{R})}\le\varepsilon$. We also set

$$u_1=\chi_k\Delta^{1/4}u,\quad u_2=(1-\chi_k)\Delta^{1/4}u,\quad w=\Delta^{-1/4}(p\cdot\Delta^{1/4}u),\quad W=\chi_{k-4}(w-\bar w_{k-4}).$$

We rewrite (78) as

$$\mathcal{R}\Delta^{1/2}(W) = -\mathcal{R}\Delta^{1/2}\Big(\sum_{h=k-4}^{+\infty}\varphi_h(w-\bar w_{k-4})\Big)$$
$$+G(q_1\,\cdot,\Delta^{1/4}u)+G(q_2\,\cdot,\Delta^{1/4}u)+u_1\cdot(\mathcal{R}\Delta^{1/4}u)+u_2\cdot(\mathcal{R}\Delta^{1/4}u). \tag{79}$$

We multiply (79) by W and integrate over $\mathbb{R}$. By using again Corollary A.8 we get

$$\int_{\mathbb{R}} |\Delta^{1/4}(W)|^2\, dx \tag{80}$$
$$= -\sum_{h=k-4}^{+\infty} \int_{\mathbb{R}} \mathcal{R}\Delta^{1/2}(\varphi_h(w-\bar{w}_{k-4}))(W)\, dx + \int_{\mathbb{R}} G(q_1\cdot, \Delta^{1/4}u)(W)\, dx + \int_{\mathbb{R}} G(q_2\cdot, \Delta^{1/4}u)(W)\, dx$$
$$+ \int_{\mathbb{R}} u_1\cdot(\mathcal{R}\Delta^{1/4}u)(W)\, dx + \int_{\mathbb{R}} u_2\cdot(\mathcal{R}\Delta^{1/4}u)(W)\, dx.$$

The last term vanishes, since u_2 and χ_{k-4} have disjoint supports. Estimating $\int_{\mathbb{R}} G(Q_1, \Delta^{1/4}u)(W)\, dx$ and $\int_{\mathbb{R}} G(Q_2, \Delta^{1/4}u)(W)\, dx$ is analogous to what we did for the terms $\int_{\mathbb{R}} F(Q_1, \Delta^{1/4}u)(W)\, dx$ and $\int_{\mathbb{R}} F(Q_2, \Delta^{1/4}u)(W)\, dx$ of (61). We therefore concentrate on the other two terms in the right-hand side of (80).

To estimate the sum term, we split it into two parts: one sum for $k-4 \le h \le k-3$ and one for $h \ge k-2$. For the first part we write

$$-\sum_{h=k-4}^{k-3} \int_{\mathbb{R}} \mathcal{R}\Delta^{1/2}(\varphi_h(w-\bar{w}_{k-4}))(W)\, dx$$
$$\le \sum_{h=k-4}^{k-3} \left\|\Delta^{1/2}(\varphi_h(w-\bar{w}_{k-4}))\right\|_{\dot{H}^{-1/2}(\mathbb{R})} \|(W)\|_{\dot{H}^{1/2}(\mathbb{R})}$$
$$\le C\sum_{h=k-4}^{k-3} \left(\|w\|_{\dot{H}^{1/2}(A_h)} + \sum_{l=-\infty}^{k-5} 2^{l-(k-4)} \sum_{s=l+1}^{h} \|w\|_{\dot{H}^{1/2}(A_s)}\right) \|W\|_{\dot{H}^{1/2}(\mathbb{R})}, \tag{81}$$

where the second inequality follows from Lemma 4.3. Let $n_1 \ge 6$ be such that

$$C\sum_{h=-\infty}^{k-n_1} 2^{h-k} \|w\|_{\dot{H}^{1/2}(A_h)} \le \tfrac{1}{8}\|W\|_{\dot{H}^{1/2}(\mathbb{R})}.$$

If $n \ge n_1$ we have

$$(81) \le \tfrac{1}{8}\|W\|^2_{\dot{H}^{1/2}(\mathbb{R})} + C\|W\|_{\dot{H}^{1/2}(\mathbb{R})} \left(\sum_{h=k-n}^{k-3} 2^{h-k}\|w\|_{\dot{H}^{1/2}(A_h)}\right). \tag{82}$$

For the second part of the sum ($h \ge k-2$) we use the fact that $\operatorname{supp}\big(\varphi_h(w-\bar{w}_{k-4}) \star W\big)$ is contained in $B_{2^{h+2}} \setminus B_{2^{h-2}}$; in particular, it does not contain 0.

$$\sum_{h=k-2}^{+\infty} \int_{\mathbb{R}} \mathcal{R}\Delta^{1/2}(\varphi_h(w-\bar{w}_{k-4}))(W)\, dx = \sum_{h=k-2}^{+\infty} \int_{\mathbb{R}} \xi \mathcal{F}[\varphi_h(w-\bar{w}_{k-4})](\xi)\mathcal{F}[W](\xi)\, d\xi$$
$$= \sum_{h=k-2}^{+\infty} \int_{\mathbb{R}} \mathcal{F}^{-1}(\xi)(x)\big(\varphi_h(w-\bar{w}_{k-4}) \star W\big)\, dx$$
$$= \sum_{h=k-2}^{+\infty} \int_{\mathbb{R}} \delta_0'(x)\, (\varphi_h(w-\bar{w}_{k-4}) \star W)\, (x)\, dx = 0.$$

Step 2: estimate of $\int_{\mathbb{R}} u_1 \cdot (\mathcal{R}\Delta^{1/4}u)(W)\,dx$. We have

$$\int_{\mathbb{R}} u_1 \cdot (\mathcal{R}\Delta^{1/4}u)(W)\,dx = \int_{\mathbb{R}} u_1 \cdot (\mathcal{R}u_1)(W)\,dx + \sum_{h=k}^{+\infty} \int_{\mathbb{R}} u_1 \cdot (\mathcal{R}\varphi_h \Delta^{1/4}u)(W)\,dx. \tag{83}$$

By applying Lemma 3.2 and using the embedding of $\mathcal{H}^1(\mathbb{R})$ into $\dot{H}^{-1/2}(\mathbb{R})$ we get

$$\begin{aligned}\int_{\mathbb{R}} u_1 \cdot (\mathcal{R}u_1)(W)\,dx &\le C\|u_1 \cdot (\mathcal{R}u_1)\|_{\mathcal{H}^1} \|(W)\|_{\dot{H}^{1/2}(\mathbb{R})} \le C\|u_1\|_{L^2}^2 \|(W)\|_{\dot{H}^{1/2}(\mathbb{R})}\\ &\le C\varepsilon\|\chi_k \Delta^{1/4}u\|_{L^2}\|(W)\|_{\dot{H}^{1/2}(\mathbb{R})} \le \frac{1}{4}\varepsilon\|\chi_k \Delta^{1/4}u\|_{L^2}\|(W)\|_{\dot{H}^{1/2}(\mathbb{R})}.\end{aligned}$$

By choosing $\varepsilon > 0$ smaller if needed, we may suppose that $C\varepsilon <$.

Now we observe that for $h \ge k$ the supports of φ_h and χ_{k-4} are disjoint. Thus

$$\begin{aligned}\sum_{h=k}^{+\infty} \int_{\mathbb{R}} u_1 \cdot (\mathcal{R}\varphi_h \Delta^{1/4}u)(W)\,dx &= \sum_{h=k}^{+\infty} \int_{\mathbb{R}} \mathcal{F}^{-1}\Big[\frac{\xi}{|\xi|}\Big](x)\,\big((\varphi_h\Delta^{1/4}u) \star (u_1 W)\big)\,dx\\ &\le C \sum_{h=k}^{+\infty} \||x|^{-1}\|_{L^\infty(B_{2^{h+2}}\setminus B_{2^{h-2}})} \|(\varphi_h \Delta^{1/4}u) \star (u_1 W)\|_{L^1}\\ &\le C \sum_{h=k}^{+\infty} 2^{-h}2^{h/2}2^{k/2}\|\varphi_h\Delta^{1/4}u\|_{L^2(\mathbb{R})}\|u_1\|_{L^2(\mathbb{R})}\|(W)\|_{\dot{H}^{1/2}(\mathbb{R})}\\ &\le C\varepsilon \sum_{h=k}^{+\infty} 2^{(k-2)/2}\|\varphi_h\Delta^{1/4}u\|_{L^2(\mathbb{R})}\|(W)\|_{\dot{H}^{1/2}(\mathbb{R})}\\ &\le \frac{\eta}{4} \sum_{h=k}^{+\infty} 2^{(k-2)/2}\|\varphi_h\Delta^{1/4}u\|_{L^2(\mathbb{R})}\|(W)\|_{\dot{H}^{1/2}(\mathbb{R})}.\end{aligned}$$

□

Proof of Proposition 4.1. From Lemma 4.4, there exist $C > 0$ and $\bar{n} > 0$ such that for all $n > \bar{n}$, $0 < \eta < \frac{1}{4}$, $k < k_0$ (k_0 depending on η and the $\dot{H}^{1/2}$ norm of Q), every solution to (45) satisfies (77) and thus also

$$\begin{aligned}&\|W\|_{\dot{H}^{1/2}(\mathbb{R})}^2\\ &\quad\le \eta^2\|\chi_{k-4}\Delta^{1/4}u\|_{L^2}^2 + C\,2^{n/2}\sum_{h=k-n}^{+\infty} 2^{k-h}\|w\|_{\dot{H}^{1/2}(A_h)}^2 + C\sum_{h=k-4}^{+\infty} 2^{(k-h)/2}\|\Delta^{1/4}u\|_{L^2(\mathbb{R})}^2.\end{aligned} \tag{84}$$

Now we can fix $n \ge \bar{n}$ and we can replace in the second term of (84) $C\,2^{n/2}$ by C.

From Lemma A.3 it follows that there are $C_1, C_2 > 0$ and $m_1 > 0$ (independent of n and k) such that if $m \ge m_1$ we have

$$\|W\|_{\dot{H}^{1/2}(\mathbb{R})}^2 \ge C_1 \int_{B_{2^{k-n-m}}} |M\Delta^{1/4}u|^2\,dx - C_2 \sum_{h=k-n-m}^{+\infty} 2^{k-h} \int_{B_{2^h}\setminus B_{2^{h-1}}} |M\Delta^{1/4}u|^2\,dx. \tag{85}$$

Finally from Lemma A.4 it follows that there is $C > 0$ such that for all $\gamma \in (0,1)$ there exists $m_2 > 0$ such that if $m \ge m_2$ we have

$$
\begin{aligned}
\sum_{h=k-n}^{+\infty} & 2^{k-h}\|w\|^2_{\dot H^{1/2}(A_h)} \\
&= \sum_{h=k-n}^{+\infty} 2^{k-h}\|\Delta^{-1/4}(M\Delta^{1/4}u)\|^2_{\dot H^{1/2}(A_h)} \\
&\le \gamma\int_{|\xi|\le 2^{k-n-m}} |M\Delta^{1/4}u|^2\,dx + \sum_{h=k-n-m}^{+\infty} 2^{(k-h)/2}\int_{2^h\le|\xi|\le 2^{h+1}} |M\Delta^{1/4}u|^2\,dx.
\end{aligned}
\tag{86}
$$

By combining (84), (85) and (86) we get

$$
\begin{aligned}
C_1\|M\Delta^{1/4}u\|^2_{L^2(B_{2^{k-n-m}})} \le C\sum_{h=k-n-m}^{\infty} 2^{(k-h)/2}\|M\Delta^{1/4}u\|^2_{L^2(A_h)} + C_2\sum_{h=k-n-m}^{+\infty} 2^{(k-h)/2}\|\Delta^{1/4}u\|_{L^2(A_h)} \\
+\eta^2\|\chi_{k-4}\Delta^{1/4}u\|^2_{L^2(\mathbb{R})} + C\gamma\|M\Delta^{1/4}u\|^2_{L^2(B_{2^{k-n-m}})}.
\end{aligned}
\tag{87}
$$

Now choose $\gamma,\eta>0$ so that $C_1^{-1}C\gamma<\frac{1}{4}$ and $C_1^{-1}\eta^2<\frac{1}{4}$. With these choices we get for some constant $C>0$

$$
\begin{aligned}
\|M\Delta^{1/4}u\|^2_{L^2(B_{2^{k-n-m}})} &- \tfrac{1}{4}\|\Delta^{1/4}u\|^2_{L^2(B_{2^{k-n-m}})} \\
&\le C\left(\sum_{h=k-n-m}^{\infty} 2^{(k-h)/2}\|M\Delta^{1/4}u\|^2_{L^2(A_h)} + \sum_{h=k-n-m}^{+\infty} 2^{(k-h)/2}\|\Delta^{1/4}u\|_{L^2(A_h)}\right).
\end{aligned}
\tag{88}
$$

We observe that in the final estimate (88) the index m can be fixed as well. Thus by replacing in (88) $k-n-m$ by k we get (48) and we conclude the proof. □

The proof of Proposition 4.2 is analogous and we omit it.

5. Morrey estimates and Hölder continuity of $\frac{1}{2}$-harmonic maps into the sphere

We consider the $(m-1)$-dimensional sphere $S^{m-1}\subset\mathbb{R}^m$. Let $\Pi_{S^{m-1}}$ be the orthogonal projection on S^{m-1}. We also consider the *Dirichlet energy* defined by

$$
L(u)=\int_{\mathbb{R}} |\Delta^{1/4}u(x)|^2\,dx \quad \text{for } u:\mathbb{R}\to S^{m-1}. \tag{89}
$$

Definition 5.1. We say that $u\in H^{1/2}(\mathbb{R},S^{m-1})$ is a weak $\frac{1}{2}$-harmonic map if

$$
\frac{d}{dt}L(\Pi_{S^{m-1}}(u+t\phi))_{|t=0}=0 \tag{90}
$$

for every map $\phi\in H^{1/2}(\mathbb{R},\mathbb{R}^m)\cap L^\infty(\mathbb{R},\mathbb{R}^m)$. In other words, weak $\frac{1}{2}$-harmonic maps are the critical points of the functional (89) with respect to perturbations of the form $\Pi_{S^{m-1}}(u+t\phi)$.

We denote by $\bigwedge(\mathbb{R}^m)$ the exterior algebra (or Grassmann algebra) of $\mathbb{R}^m$. If $(e_i)_{i=1,\dots,m}$ is the canonical orthonormal basis of $\mathbb{R}^m$, every element $v\in\bigwedge_p(\mathbb{R}^m)$ can be written as $v=\sum_I v_I e_I$, where $I=\{i_1,\dots,i_p\}$ with $1\le i_1\le\cdots\le i_p\le m$, $v_I:=v_{i_1,\dots,i_p}$, and $e_I:=e_{i_1}\wedge\cdots\wedge e_{i_p}$.

By $\llcorner$ we denote the interior multiplication $\llcorner : \bigwedge_p(\mathbb{R}^m) \times \bigwedge_q(\mathbb{R}^m) \to \bigwedge_{q-p}(\mathbb{R}^m)$ defined as follows: Let $e_I = e_{i_1} \wedge \cdots \wedge e_{i_p}$, $e_J = e_{j_1} \wedge \cdots \wedge e_{j_q}$, with $q \geq p$. Then $e_I \llcorner e_J = 0$ if $I \not\subset J$; otherwise $e_I \llcorner e_J = (-1)^M e_K$, where e_K is a $(q-p)$-vector and M is the number of pairs $(i, j) \in I \times J$ with $j > i$.

By the symbol $\bullet$ we denote the first order contraction between multivectors. We recall that it satisfies $\alpha \bullet \beta = \alpha \llcorner \beta$ if β is a 1-vector and $\alpha \bullet (\beta \wedge \gamma) = (\alpha \bullet \beta) \wedge \gamma + (-1)^{pq}(\alpha \bullet \gamma) \wedge \beta$, if β and γ are respectively a p-vector and a q-vector.

Finally by the symbol $*$ we denote the Hodge star operator, $* : \bigwedge_p(\mathbb{R}^m) \to \bigwedge_{m-p}(\mathbb{R}^m)$, defined by $*\beta = (e_1 \wedge \cdots \wedge e_n) \bullet \beta$.

Next we write the Euler equation associated to the functional (89).

Theorem 5.2. *All weak $\frac{1}{2}$-harmonic maps $u \in H^{1/2}(\mathbb{R}, S^{m-1})$ satisfy in a weak sense the equation*

$$\int_{\mathbb{R}} (\Delta^{1/2} u) \cdot v \, dx = 0, \tag{91}$$

for every $v \in H^{1/2}(\mathbb{R}, \mathbb{R}^m) \cap L^\infty(\mathbb{R}, \mathbb{R}^m)$ such that $v \in T_{u(x)} S^{m-1}$ almost everywhere, or equivalently the equation

$$\Delta^{1/2} u \wedge u = 0 \quad \text{in } \mathcal{D}', \tag{92}$$

or yet

$$\Delta^{1/4}(u \wedge \Delta^{1/4} u) = T(Q, u) \quad \text{in } \mathcal{D}', \tag{93}$$

with $Q = u \wedge$.

Proof. The proof of (91) is analogous that of Lemma 1.4.10 in [Hélein 2002]. For v as in the statement, we have

$$\Pi_{S^{m-1}}(u + tv) = u + t w_t,$$

where

$$w_t = \int_0^1 \frac{\partial \Pi_{S^{m-1}}}{\partial y_j}(u + tsv) v^j \, ds.$$

Hence

$$L(\Pi_{S^{m-1}}(u + tv)) = \int_{\mathbb{R}} |\Delta^{1/4} u|^2 \, dx + 2t \int_{\mathbb{R}} \Delta^{1/2} u \cdot w_t \, dx + o(t),$$

as $t \to 0$. Thus (90) is equivalent to

$$\lim_{t \to 0} \int_{\mathbb{R}} \Delta^{1/2} u \cdot w_t \, dx = 0.$$

Since $\Pi_{S^{m-1}}$ is smooth it follows that $w_t \to w_0 = d\Pi_{S^{m-1}}(u)(v)$ in $H^{1/2}(\mathbb{R}, \mathbb{R}^m) \cap L^\infty(\mathbb{R}, \mathbb{R}^m)$ and therefore

$$\int_{\mathbb{R}} \Delta^{1/4} u \, d\Pi_{S^{m-1}}(u)(v) \, dx = 0.$$

Since $v \in T_{u(x)} S^{m-1}$ a.e., we have $d\Pi_{S^{m-1}}(u)(v) = v$ a.e. and (91) follows.

To prove (92), we take $\varphi \in C_0^\infty(\mathbb{R}, \bigwedge_{m-2}(\mathbb{R}^m))$. Then

$$\int_{\mathbb{R}} \varphi \wedge u \wedge \Delta^{1/2} u \, dx = \left(\int_{\mathbb{R}} *(\varphi \wedge u) \cdot \Delta^{1/2} u \, dx \right) e_1 \wedge \cdots \wedge e_m. \tag{94}$$

We claim that

$$v = *(\varphi \wedge u) \in \dot{H}^{1/2}(\mathbb{R}, \mathbb{R}^m) \quad \text{and} \quad v(x) \in T_{u(x)} S^{m-1} \text{ a.e.}$$

That $v \in H^{1/2}(\mathbb{R}, \mathbb{R}^m) \cap L^\infty(\mathbb{R}, \mathbb{R}^m)$ follows form the fact that its components are the product of two functions in $\dot{H}^{1/2}(\mathbb{R}, \mathbb{R}^m) \cap L^\infty(\mathbb{R}, \mathbb{R}^m)$, which is an algebra. Moreover,

$$v \cdot u = *(u \wedge \varphi) \cdot u = *(u \wedge \varphi \wedge u) = 0. \tag{95}$$

It follows from (91) and (94) that

$$\int_{\mathbb{R}} \varphi \wedge u \wedge \Delta^{1/2} u \, dx = 0.$$

This shows that $\Delta^{1/2} u \wedge u = 0$ in $\mathcal{D}'$, concluding the proof of (92).

To prove (93) it is enough to observe that $\Delta^{1/2} u \wedge u = 0$ and $\Delta^{1/4} u \wedge \Delta^{1/4} u = 0$. □

Next we show that any map $u \in \dot{H}^{1/2}(\mathbb{R}, \mathbb{R}^m)$ such that $|u| = 1$ a.e. satisfies the structural equation (18).

Proof of Proposition 1.4. We observe that if $u \in \dot{H}^{1/2}(\mathbb{R}, S^{m-1})$ then Leibniz's rule holds. Thus

$$\nabla |u|^2 = 2u \cdot \nabla u \text{ in } \quad \mathcal{D}'. \tag{96}$$

Indeed, the equality (96) holds trivially if $u \in C_0^\infty(\mathbb{R}, \mathbb{R}^{m-1})$. Let $u \in \dot{H}^{1/2}(\mathbb{R}, S^{m-1})$ and let u_j be a sequence in $C_0^\infty(\mathbb{R}, \mathbb{R}^m)$ converging to u in $\dot{H}^{1/2}(\mathbb{R}, \mathbb{R}^m)$ as $j \to +\infty$. Then $\nabla u_j \to \nabla u$ as $j \to +\infty$ in $\dot{H}^{-1/2}(\mathbb{R}, \mathbb{R}^{m-1})$. Thus $u_j \cdot \nabla u_j \to u \cdot \nabla u$ in $\mathcal{D}'$ and (96) follows.

If $u \in \dot{H}^{1/2}(\mathbb{R}, S^{m-1})$, then $\nabla |u|^2 = 0$ and thus $u \cdot \nabla u = 0$ in $\mathcal{D}'$ as well. Thus u satisfies (18) and this conclude the proof. □

By combining Theorem 5.2, Proposition 1.4 and the results of the previous section we get the Hölder regularity of weak $\frac{1}{2}$-harmonic maps.

Theorem 5.3. *Let $u \in \dot{H}^{1/2}(\mathbb{R}, S^{m-1})$ be a harmonic map. Then $u \in C^{0,\alpha}_{\text{loc}}(\mathbb{R}, S^{m-1})$.*

Proof. From Theorem 5.2 it follows that u satisfies (93). Moreover, since $|u| = 1$, Proposition 1.4 implies that u satisfies (18) as well. Propositions 4.1 and 4.2 yield for $k < 0$, with $|k|$ large enough,

$$\|u \wedge \Delta^{1/4} u\|^2_{L^2(B_{2^k})} \le C \sum_{h=k}^{\infty} 2^{(k-h)/2} \|\Delta^{1/4} u\|^2_{L^2(A_h)} + \tfrac{1}{4} \|\Delta^{1/4} u\|^2_{L^2(B_{2^k})}, \tag{97}$$

$$\|u \cdot \Delta^{1/4} u\|^2_{L^2(B_{2^k})} \le C \sum_{h=k}^{\infty} 2^{(k-h)/2} \|\Delta^{1/4} u\|^2_{L^2(A_h)} + \tfrac{1}{4} \|\Delta^{1/4} u\|^2_{L^2(B_{2^k})}. \tag{98}$$

Since

$$\|\Delta^{1/4} u\|^2_{L^2(B_{2^k})} = \|u \cdot \Delta^{1/4} u\|^2_{L^2(B_{2^k})} + \|u \wedge \Delta^{1/4} u\|^2_{L^2(B_{2^k})},$$

we get

$$\|\Delta^{1/4} u\|^2_{L^2(B_{2^k})} \le C \sum_{h=k}^{\infty} 2^{(k-h)/2} \|\Delta^{1/4} u\|^2_{L^2(A_h)}. \tag{99}$$

Now observe that for some $C > 0$ (independent of k) we have

$$C^{-1}\sum_{h=-\infty}^{k-1}\|\Delta^{1/4}u\|^2_{L^2(A_h)} \le \|\Delta^{1/4}u\|^2_{L^2(B_{2^k})} \le C\sum_{h=-\infty}^{k}\|\Delta^{1/4}u\|^2_{L^2(A_h)}.$$

From this and (98) it follows that

$$\sum_{h=-\infty}^{k-1}\|\Delta^{1/4}u\|^2_{L^2(A_h)} \le C\sum_{h=k}^{\infty}2^{(k-h)/2}\|\Delta^{1/4}u\|^2_{L^2(A_h)}.$$

By applying Proposition A.9 and using again (99) we get for $r > 0$ small enough and some $\beta \in (0, 1)$

$$\int_{B_r}|\Delta^{1/4}u|^2\,dx \le Cr^\beta. \tag{100}$$

Condition (100) yields that u belongs to the Morrey–Campanato space $\mathscr{L}^{2,-\beta}$ (see [Adams 1975], page 79), and thus $u \in C^{0,\beta/2}(\mathbb{R})$ (see [Adams 1975; Giaquinta 1983], for instance). □

Appendix

We prove here some results used in the previous sections. The first is that the $\dot H^{1.2}([a,b])$ norm, where $-\infty \le a < b \le +\infty$, can be localized in space. This result, besides being of independent interest, is used in Section 4 for localization estimates. For simplicity we will suppose that $[a,b]=[-1,1]$.

Theorem A.1 (Localization of $H^{1/2}((-1,1))$ norm). *Let $u \in H^{1/2}((-1,1))$. For some $C > 0$ we have*

$$\|u\|^2_{\dot H^{1/2}((-1,1))} \simeq \sum_{j=-\infty}^{0}\|u\|^2_{\dot H^{1/2}(A_j)},$$

where $A_j = B_{2^{j+1}} \setminus B_{2^{j-1}}$.

Proof. For every $i \in \mathbb{Z}$, we set $A'_i = B_{2^i} \setminus B_{2^{i-1}}$ and $\bar u'_i = |A'_i|^{-1}\int_{A'_i}u(x)\,dx$. We have

$$\begin{aligned}
\|u\|^2_{\dot H^{1/2}((-1,1))} &\simeq \int_{[-1,1]}\int_{[-1,1]}\frac{|u(x)-u(y)|^2}{|x-y|^2}\,dx\,dy\\
&= \sum_{i,j=-\infty}^{0}\int_{A'_i}\int_{A'_j}\frac{|u(x)-u(y)|^2}{|x-y|^2}\,dx\,dy\\
&= \sum_{i=-\infty}^{0}\int_{A'_i}\int_{A'_i}\frac{|u(x)-u(y)|^2}{|x-y|^2}\,dx\,dy + 2\sum_{j=-\infty}^{0}\sum_{i>j+1}\int_{A'_i}\int_{A'_j}\frac{|u(x)-u(y)|^2}{|x-y|^2}\,dx\,dy\\
&\qquad + 2\sum_{j=-\infty}^{0}\int_{A'_j}\int_{A'_{j+1}}\frac{|u(x)-u(y)|^2}{|x-y|^2}\,dx\,dy.
\end{aligned} \tag{101}$$

We first observe that

$$\sum_{i,j=-\infty}^{0}\int_{A'_i}\int_{A'_j}\frac{|u(x)-u(y)|^2}{|x-y|^2}\,dx\,dy \le \sum_{i,j=-\infty}^{0}\int_{A_i}\int_{A_j}\frac{|u(x)-u(y)|^2}{|x-y|^2}\,dx\,dy \tag{102}$$

and

$$\sum_{j=-\infty}^{0} \int_{A'_j} \int_{A'_{j+1}} \frac{|u(x)-u(y)|^2}{|x-y|^2}\, dx\, dy \leq \sum_{j=-\infty}^{0} \int_{A_j} \int_{A_j} \frac{|u(x)-u(y)|^2}{|x-y|^2}\, dx\, dy. \tag{103}$$

It remains to estimate the double sum in (101). We have

$$\begin{aligned}
&\sum_{j=-\infty}^{0} \sum_{i>j+1} \int_{A'_i}\int_{A'_j} \frac{|u(x)-u(y)|^2}{|x-y|^2}\, dx\, dy \\
&\quad \leq C \sum_{j=-\infty}^{0} \sum_{i\geq j+2} 2^{-2i} \int_{A'_i}\int_{A'_j} |u(x)-u(y)|^2\, dx\, dy \\
&\quad \leq C\Big(\sum_{j=-\infty}^{0} \sum_{i\geq j+2} 2^{-2i} \int_{A'_i}\int_{A'_j} |\bar{u}'_i - \bar{u}'_j|^2\, dx\, dy + \sum_{j=-\infty}^{0} \sum_{i\geq j+2} 2^{-2i} \int_{A'_i}\int_{A'_j} |u(x)-\bar{u}'_i|^2\, dx\, dy \\
&\qquad\qquad + \sum_{j=-\infty}^{0} \sum_{i\geq j+2} 2^{-2i} \int_{A'_i}\int_{A'_j} |u(y)-\bar{u}'_j|^2\, dx\, dy\Big) \\
&\quad \leq C\Big(\sum_{j=-\infty}^{0} \sum_{i\geq j+2} 2^{-2i} 2^{i+j} |\bar{u}'_i - \bar{u}'_j|^2 + \sum_{j=-\infty}^{0} \sum_{i\geq j+2} 2^{-2i} 2^{j} \int_{A'_i} |u(x)-\bar{u}'_i|^2 dx \\
&\qquad\qquad + \sum_{j=-\infty}^{0} \sum_{i\geq j+2} 2^{-2i} 2^{i} \int_{A'_j} |u(y)-\bar{u}'_j|^2 dy\Big).
\end{aligned}$$

Denote by W, W_x, W_y the three double sums in the last parentheses. We have

$$\begin{aligned}
W_x &= \sum_{i=-\infty}^{0} \sum_{j\leq i-2} 2^{-2i} 2^{j} \int_{A'_i} |u(x)-\bar{u}'_i|^2 dx = \sum_{i=-\infty}^{0} 2^{-2i} \int_{A'_i} |u(x)-\bar{u}'_i|^2\, dx \Big(\sum_{j\leq i-2} 2^{j}\Big) \\
&\leq C \sum_{i=-\infty}^{0} |A'_i|^{-1} \int_{A'_i} |u(x)-\bar{u}'_i|^2 dx \leq C \sum_{i=-\infty}^{0} \int_{A'_i} \int_{A'_i} \frac{|u(x)-u(y)|^2}{|x-y|^2}\, dx\, dy,
\end{aligned} \tag{104}$$

where in the last inequality we used the fact that, for every i,

$$\begin{aligned}
|A'_i|^{-1} \int_{A'_i} |u(x)-\bar{u}'_i|^2 dx &\leq |A'_i|^{-1} \int_{A'_i} \left| u(x) - |A'_i|^{-1} \int_{A'_i} u(y)\, dy \right|^2 dx \\
&\leq |A'_i|^{-2} \int_{A'_i} \int_{A'_i} |u(x)-u(y)|^2\, dx\, dy \leq C \int_{A'_i} \int_{A'_i} \frac{|u(x)-u(y)|^2}{|x-y|^2}\, dx\, dy.
\end{aligned}$$

A similar calculation yields

$$W_y \leq C \sum_{j=-\infty}^{0} \int_{A'_j} \int_{A'_j} \frac{|u(x)-u(y)|^2}{|x-y|^2}\, dx\, dy. \tag{105}$$

Finally, to estimate $W = \sum_{j=-\infty}^{0} \sum_{i \geq j+2} 2^{-2i} 2^{i+j} |\bar{u}_i' - \bar{u}_j'|^2$, we first observe that

$$|\bar{u}_i' - \bar{u}_j'|^2 \leq (i-j) \sum_{j}^{i-1} |\bar{u}_{l+1}' - \bar{u}_l'|^2 \quad \text{and} \quad |\bar{u}_{l+1} - \bar{u}_l|^2 \leq |A_l|^{-1} \int_{A_l} |u - \bar{u}_l|^2 \, dx,$$

where $\bar{u}_l = |A_l|^{-1} \int_{A_l} u(x)\, dx$. Setting $a_l = |A_l|^{-1} \int_{A_l} |u - \bar{u}_l|^2 \, dx$, we then have

$$W \leq \sum_{j=-\infty}^{0} \sum_{i \geq j+2} (i-j) 2^{j-i} \sum_{j}^{i-1} a_l \leq \sum_{l=-\infty}^{0} a_l \sum_{j=-\infty}^{l} \sum_{i-j \geq l+1-j} (i-j) 2^{j-i}.$$

We observe that

$$\begin{aligned} \sum_{j=-\infty}^{l} \sum_{i-j \geq l+1-j} (i-j) 2^{j-i} \leq \sum_{j=-\infty}^{l} \int_{l+1-j}^{+\infty} 2^{-x} x \, dx &= \sum_{j=-\infty}^{l} 2^{-(l+1-j)} (l+2-j) \\ &\leq \int_{1}^{+\infty} 2^{-t} (t+1) \, dx \leq C, \end{aligned} \tag{106}$$

for some constant C independent of l. It follows that

$$W \leq C \sum_{l=-\infty}^{0} a_l \leq C \int_{A_l} \int_{A_l} \frac{|u(x) - u(y)|^2}{|x-y|^2} \, dx \, dy. \tag{107}$$

By combining (102), (103), (104), (105), and (107) we finally obtain

$$\|u\|^2_{\dot{H}^{1/2}((-1,1))} \lesssim \sum_{l=-\infty}^{0} \|u\|^2_{\dot{H}^{1/2}(A_l)}.$$

Next we show that

$$\sum_{l=-\infty}^{0} \|u\|^2_{\dot{H}^{1/2}(A_l)} \lesssim \|u\|^2_{\dot{H}^{1/2}((-1,1))}. \tag{108}$$

For every l we have $A_l = C_l \cup D_l$, where $C_l = B_{2^{l+1}} \setminus B_{2^l}$ and $D_l = B_{2^l} \setminus B_{2^{l-1}}$. Thus

$$\begin{aligned} &\|u\|^2_{\dot{H}^{1/2}(A_l)} \\ &= \int_{C_l} \int_{C_l} \frac{|u(x) - u(y)|^2}{|x-y|^2} \, dx \, dy + \int_{D_{l,h}} \int_{D_l} \frac{|u(x) - u(y)|^2}{|x-y|^2} \, dx \, dy + 2 \int_{D_{l,h}} \int_{C_l} \frac{|u(x) - u(y)|^2}{|x-y|^2} \, dx \, dy. \end{aligned}$$

Since $\bigcup_l (C_l \times C_l)$, $\bigcup_l (D_l \times C_l)$, and $\bigcup_l (D_l \times C_l)$ are disjoint unions contained in $[0,1] \times [0,1]$, we have

$$\sum_l \int_{C_l} \int_{C_l} \frac{|u(x) - u(y)|^2}{|x-y|^2} \, dx \, dy \leq \int_{[-1,1]} \int_{[-1,1]} \frac{|u(x) - u(y)|^2}{|x-y|^2} \, dx \, dy,$$

$$\sum_l \int_{D_{l,h}} \int_{C_l} \frac{|u(x)-u(y)|^2}{|x-y|^2}\, dx\, dy \le \int_{[-1,1]} \int_{[-1,1]} \frac{|u(x)-u(y)|^2}{|x-y|^2}\, dx\, dy,$$
$$\sum_l \int_{D_{l,h}} \int_{D_l} \frac{|u(x)-u(y)|^2}{|x-y|^2}\, dx\, dy \le \int_{[-1,1]} \int_{[-1,1]} \frac{|u(x)-u(y)|^2}{|x-y|^2}\, dx\, dy.$$

It follows that

$$\sum_{l=-\infty}^{0} \|u\|^2_{\dot H^{1/2}(A_l)} \le \bar C \int_{[-1,1]} \int_{[-1,1]} \frac{|u(x)-u(y)|^2}{|x-y|^2}\, dx\, dy = \bar C \|u\|^2_{\dot H^{1/2}((-1,1))}. \qquad \square$$

Remark A.2. By analogous computations one can show that for all $r > 0$ we have

$$\|u\|^2_{\dot H^{1/2}(\mathbb{R})} \simeq \sum_{j=-\infty}^{+\infty} \|u\|^2_{\dot H^{1/2}(A_j^r)},$$

where $A_j^r = B_{2^{j+1}r} \setminus B_{2^{j-1}r}$, where the equivalence constants do not depend on r.

Next we compare the $\dot H^{1/2}$ norm of $\Delta^{-1/4}(M\Delta^{1/4}u)$ with the L^2 norm of $M\Delta^{1/4}u$, where $u \in \dot H^{1/2}(\mathbb{R})$ and $M \in \dot H^{1/2}(\mathbb{R}, \mathcal{M}_{t\times m}(\mathbb{R}))$, for $t \ge 1$.

In the sequel, for $\rho \ge \sigma \ge 0$, we denote by $\mathbb{1}_{|x|\le\rho}$, $\mathbb{1}_{\rho\le|x|}$, and $\mathbb{1}_{\rho\le|x|\le\sigma}$ the characteristic functions of the sets of points $x \in \mathbb{R}$ satisfying the respective inequalities.

Lemma A.3. *Let $M \in \dot H^{1/2}(\mathbb{R}, \mathcal{M}_{t\times m}(\mathbb{R}))$, with $m \ge 1$ and $t \ge 1$, and let and $u \in \dot H^{1/2}(\mathbb{R})$. There exist $C_1 > 0$, $C_2 > 0$ and $n_0 \in \mathbb{N}$, independent of u and M, such that, for any $r \in (0,1)$, $n > n_0$ and any $x_0 \in \mathbb{R}$, we have*

$$\|\Delta^{-1/4}(M\Delta^{1/4}u)\|^2_{\dot H^{1/2}(B_r(x_0))} \ge C_1 \int_{B_{r/2^n}(x_0)} |M\Delta^{1/4}u|^2\, dx - C_2 \sum_{h=-n}^{+\infty} 2^{-h} \int_{B_{2^h r}(x_0)\setminus B_{2^{h-1}r}(x_0)} |M\Delta^{1/4}u|^2\, dx.$$

Proof. For notational simplicity we take $x_0 = 0$, but the estimates made will be independent of x_0. We write

$$\Delta^{-1/4}(M\Delta^{1/4}u) = \Delta^{-1/4}(\mathbb{1}_{|x|\le r/2^n} M\Delta^{1/4}u) + \Delta^{-1/4}\big((1-\mathbb{1}_{|x|\le r/2^n})M\Delta^{1/4}u\big),$$

where $n > 0$ is large enough; the threshold will be determined later in the proof. We have

$$\begin{aligned}
&\big\|\Delta^{-1/4}(M\Delta^{1/4}u)\big\|_{\dot H^{1/2}(B_r)} \\
&\quad\ge \big\|\Delta^{-1/4}(\mathbb{1}_{r/2^n} M\Delta^{1/4}u)\big\|_{\dot H^{1/2}(B_r)} - \big\|\Delta^{-1/4}((1-\mathbb{1}_{|x|\le r/2^n})M\Delta^{1/4}u)\big\|_{\dot H^{1/2}(B_r)} \\
&\quad\ge \big\|\Delta^{-1/4}(\mathbb{1}_{r/2^n} M\Delta^{1/4}u)\big\|_{\dot H^{1/2}(B_r)} - \big\|\Delta^{-1/4}(\mathbb{1}_{r/2^n\le|x|\le 4r} M\Delta^{1/4}u)\big\|_{\dot H^{1/2}(B_r)} \\
&\qquad\qquad - \big\|\Delta^{-1/4}(\mathbb{1}_{|x|\ge 4r} M\Delta^{1/4}u)\big\|_{\dot H^{1/2}(B_r)} \\
&\quad\ge \big\|\Delta^{-1/4}(\mathbb{1}_{|x|\le r/2^n} M\Delta^{1/4}u)\big\|_{\dot H^{1/2}(B_r)} - \big\|\Delta^{-1/4}(\mathbb{1}_{r/2^n\le|x|\le 4r} M\Delta^{1/4}u)\big\|_{\dot H^{1/2}(\mathbb{R})} \\
&\qquad\qquad - \big\|\Delta^{-1/4}(\mathbb{1}_{|x|\ge 4r} M\Delta^{1/4}u)\big\|_{\dot H^{1/2}(B_r)}. \qquad (109)
\end{aligned}$$

We estimate the last three terms in (109).

- Estimate of $\left\|\Delta^{-1/4}(\mathbb{1}_{r/2^n\le|x|\le 4r}M\Delta^{1/4}u)\right\|_{\dot H^{1/2}(\mathbb{R})}$. This expression is equal to

$$\int_{r/2^n\le|x|\le 4r}|M\Delta^{1/4}u|^2\,dx=\sum_{h=-n}^{1}\int_{2^hr\le|x|\le 2^{h+1}r}|M\Delta^{1/4}u|^2\,dx. \tag{110}$$

- Estimate of $\left\|\Delta^{-1/4}(\mathbb{1}_{|x|\ge 4r}M\Delta^{1/4}u)\right\|_{\dot H^{1/2}(B_r)}$. Setting $g:=\mathbb{1}_{|x|\ge 4r}M\Delta^{1/4}u$, we have

$$\begin{aligned}
\|\Delta^{-1/4}g\|^2_{\dot H^{1/2}(B_r)}&=\int_{B_r}\int_{B_r}\frac{\left|(|x|^{-2}\star g)(t)-(|x|^{-2}\star g)(s)\right|^2}{|t-s|^2}dt\,ds\\
&=\int_{B_r}\int_{B_r}\frac{1}{|t-s|^2}\left(\int_{|x|\ge 4r}g(x)\left(|t-x|^{-1/2}-|s-x|^{-1/2}\right)dx\right)^2dt\,ds\\
\text{(mean-value thm.)}&\le C\int_{B_r}\int_{B_r}\left(\int_{|x|\ge 4r}|g(x)|\max\left(|t-x|^{-3/2},|s-x|^{-3/2}\right)dx\right)^2dt\,ds\\
&\le C\int_{B_r}\int_{B_r}\left(\sum_{h=4}^{+\infty}\int_{2^hr\le|x|\le 2^{h+1}r}|g(x)|\max\left(|t-x|^{-3/2},|s-x|^{-3/2}\right)dx\right)^2dt\,ds\\
&\le C\int_{B_r}\int_{B_r}\left(\sum_{h=4}^{+\infty}\int_{2^hr\le|x|\le 2^{h+1}r}|g(x)|\,2^{-3h/2}r^{-3/2}\,dx\right)^2dt\,ds\\
\text{(Hölder inequality)}&\le C\int_{B_r}\int_{B_r}\left(\sum_{h=4}^{+\infty}2^{-h}r^{-1}\left(\int_{2^hr\le|x|\le 2^{h+1}r}|g(x)|^2\,dx\right)^{-1/2}\right)^2dt\,ds\\
\text{(Cauchy–Schwarz)}&\le C\left(\sum_{h=4}^{+\infty}2^{-h}\right)\left(\sum_{h=4}^{+\infty}2^{-h}\int_{2^hr\le|x|\le 2^{h+1}r}|M\Delta^{1/4}u|^2\,dx\right)\\
&\le C\left(\sum_{h=4}^{+\infty}2^{-h}\int_{2^hr\le|x|\le 2^{h+1}r}|M\Delta^{1/4}u|^2\,dx\right).
\end{aligned}\tag{111}$$

- Estimate of $\|\Delta^{-1/4}(\mathbb{1}_{|x|\le r/2^n}M\Delta^{1/4}u)\|_{\dot H^{1/2}(B_r)}$. We set

$$A_h^r:=\{x:2^{h-1}r\le|x|\le 2^{h+1}r\}.$$

By the localization theorem A.1 there exists a constant $\tilde C>0$ (independent of r) such that

$$\begin{aligned}
&\|\Delta^{-1/4}(\mathbb{1}_{|x|\le r/2^n}M\Delta^{1/4}u)\|^2_{\dot H^{1/2}(\mathbb{R})}\\
&\quad\le\tilde C\sum_{h=-\infty}^{+\infty}\|\Delta^{-1/4}(\mathbb{1}_{|x|\le r/2^n}M\Delta^{1/4}u)\|^2_{\dot H^{1/2}(A_h^r)}\\
&\quad\le\tilde C\,\|\Delta^{-1/4}(\mathbb{1}_{|x|\le r/2^n}M\Delta^{1/4}u)\|^2_{\dot H^{1/2}(B_r)}+\tilde C\sum_{h=0}^{+\infty}\|\Delta^{-1/4}(\mathbb{1}_{|x|\le r/2^n}M\Delta^{1/4}u)\|^2_{\dot H^{1/2}(A_h^r)}.
\end{aligned}\tag{112}$$

• Estimate of $\sum_{h=0}^{+\infty}\|\Delta^{-1/4}(\mathbb{1}_{|x|\le r/2^n}M\Delta^{1/4}u)\|^2_{\dot H^{1/2}(A_h^r)}$. Setting

$$f(x):=\mathbb{1}_{|x|\le r/2^n}M\Delta^{1/4}u$$

and working as in the first three lines of (111), we can write for this sum the upper bound

$$\begin{aligned}C\sum_{h=0}^{+\infty}\int_{A_h^r}\int_{A_h^r}&\Big(\int_{|x|\le r/2^n}|f(x)|\max\big(|t-x|^{-3/2},|s-x|^{-3/2}\big)\,dx\Big)^2dt\;ds\\ &\le C\sum_{h=0}^{+\infty}\int_{A_h^r}\int_{A_h^r}\max(|t|^{-3},|s|^{-3})\frac{r}{2^n}\Big(\int_{|x|\le r/2^n}|f(x)|^2\,dx\Big)\,dt\,ds\\ &=\frac{C}{2^n}\sum_{h=0}^{+\infty}2^{-h}\Big(\int_{|x|\le r/2^n}|f(x)|^2\,dx\Big)\le\frac{C}{2^n}\int_{|x|\le r/2^n}|M\Delta^{1/4}u|^2\,dx.\end{aligned}\tag{113}$$

If n is large enough that $C\,\tilde C/2^n<\frac{1}{2}$, we get, combining (109), (110), (111), (112) and (113), for some C_1, C_2 positive,

$$\|\Delta^{-1/4}(M\Delta^{1/4}u)\|^2_{\dot H^{1/2}(B_r)}\ge C_1\int_{B_{r/2^n}}|M\Delta^{1/4}u|^2\,dx-C_2\sum_{h=-n}^{+\infty}2^{-h}\int_{B_{2^{h+1}r}\setminus B_{2^h r}}|M\Delta^{1/4}u|^2\,dx,$$

which ends the proof of the lemma. □

We now compare the $\dot H^{1/2}$ norm of $\Delta^{-1/4}(M\Delta^{1/4}u)$ in the annuli $A_h=B_{2^{h+1}}(x_0)\setminus B_{2^{h-1}}(x_0)$ with the L^2 norm in the same annuli of $M\Delta^{1/4}u$. This result, like the previous one, was used in the proof of Proposition 4.1.

Lemma A.4. *Let $M\in\dot H^{1/2}(\mathbb{R},\mathcal{M}_{t\times m}t\ge 1(\mathbb{R}))$, $m\ge 1, t\ge 1$, and $u\in\dot H^{1/2}(\mathbb{R})$. There exists $C>0$ such that for every $\gamma\in(0,1)$, for all $n\ge n_0\in\mathbb{N}$ (n_0 dependent on γ and independent of u and M), for every $k\in\mathbb{Z}$, and any $x_0\in\mathbb{R}$, we have*

$$\begin{aligned}\sum_{h=k}^{+\infty}&2^{k-h}\|\Delta^{-1/4}(M\Delta^{1/4}u)\|^2_{\dot H^{1/2}(B_{2^{h+1}}(x_0)\setminus B_{2^{h-1}}(x_0))}\\ &\le\gamma\int_{B_{2^{k-n}}(x_0)}|M\Delta^{1/4}u|^2dx+\sum_{h=k-n}^{+\infty}2^{(k-h)/2}\int_{B_{2^{h+1}}(x_0)\setminus B_{2^{h-1}}(x_0)}|M\Delta^{1/4}u|^2dx.\end{aligned}$$

Proof. Again we take $x_0=0$, but the estimates will be independent of x_0. Given $h\in\mathbb{Z}$ and $l\ge 3$ we set $A_h=B_{2^{h+1}}\setminus B_{2^{h-1}}$ and $D_{l,h}=B_{2^{h+l}}\setminus B_{2^{h-l}}$.

Fix $\gamma\in(0,1)$. We have, for $w=\Delta^{-1/4}(M\Delta^{1/4}u)$ and for any $l\ge 3$ (to be chosen later),

$$\begin{aligned}\|w\|^2_{\dot H^{1/2}(A_h)}&=\int_{A_h}\int_{A_h}\frac{|w(x)-w(y)|^2}{|x-y|^2}\,dx\,dy\\ &\le 2\,\|\Delta^{-1/4}\mathbb{1}_{D_{l,h}}M\Delta^{1/4}u\|^2_{\dot H^{1/2}(A_h)}+2\,\|\Delta^{-1/4}(1-\mathbb{1}_{D_{l,h}})M\Delta^{1/4}u\|^2_{\dot H^{1/2}(A_h)}.\end{aligned}\tag{114}$$

The first of these two terms is bounded above by

$$\|\Delta^{-1/4}\mathbb{1}_{D_{l,h}} M\Delta^{1/4}u\|^2_{\dot H^{1/2}(\mathbb{R})} = \int_{D_{l,h}} |M\Delta^{1/4}u|^2\,dx = \sum_{s=h-l}^{h+l-1}\int_{B_{2^{s+1}}\setminus B_{2^s}} |M\Delta^{1/4}u|^2\,dx. \tag{115}$$

Multiplying by 2^{k-h} and summing up from $h=k$ to $+\infty$ we get

$$\sum_{h=k}^{+\infty} 2^{k-h}\|\Delta^{-1/4}\mathbb{1}_{D_{l,h}} M\Delta^{1/4}u\|^2_{\dot H^{1/2}(A_h)} \le C\,2^{-l}\sum_{h=k-l}^{+\infty}\int_{B_{2^{h+1}}\setminus B_{2^{h-1}}} |M\Delta^{1/4}u|^2\,dx. \tag{116}$$

To estimate the remaining term on the right-hand side of (114), set $g=(1-\mathbb{1}_{D_{l,h}})M\Delta^{1/4}u$ and write, as in the first two lines of (111),

$$\begin{aligned}
&\|\Delta^{-1/4}g\|^2_{\dot H^{1/2}(A_h)}\\
&= \int_{A_h}\int_{A_h}\frac{1}{|t-s|^2}\left(\int_{|x|<2^{h-l}\text{ or }|x|>2^{l+h}} g(x)\,\big(|t-x|^{-1/2}-|s-x|^{-1/2}\big)\,dx\right)^2 dt\,ds\\
&\le 2\int_{A_h}\int_{A_h}\frac{1}{|t-s|^2}\left(\int_{|x|>2^{l+h}}(\text{same})\right)^2 dt\,ds + 2\int_{A_h}\int_{A_h}\frac{1}{|t-s|^2}\left(\int_{|x|<2^{h-l}}(\text{same})\right)^2 dt\,ds.
\end{aligned} \tag{117}$$

For the first of these last two terms we can write, following the same steps as in (111) and using the fact that, since $l\ge 3$, we have $|x-t|,|x-s|\ge 2^{q-1}$ for every $s,t\in A_h$ and $2^q\le |x|\le 2^{q+1}$:

$$\begin{aligned}
&\int_{A_h}\int_{A_h}\frac{1}{|t-s|^2}\left(\int_{|x|>2^{l+h}} g(x)\,\big(|t-x|^{-1/2}-|s-x|^{-1/2}\big)\,dx\right)^2 dt\,ds\\
&\qquad\qquad \le C\,2^{h-l}\left(\sum_{q=h+l}^{\infty} 2^{-q}\int_{2^q\le|x|\le 2^{q+1}} |g(x)|^2\,dx\right).
\end{aligned} \tag{118}$$

Multiplying the right-hand side by 2^{k-h}, where $k\in\mathbb{Z}$, taking the sum from $h=k$ to $+\infty$, interchanging the summations, and using the fact that $g(x)=M\Delta^{1/4}u(x)$ when $2^q\le|x|\le 2^{q+1}$, we get the value

$$\begin{aligned}
&C\,2^{-l}\sum_{q=k+l}^{+\infty} 2^{k-q}(q-l-k)\left(\int_{2^q\le|x|\le 2^{q+1}}|M\Delta^{1/4}u|^2\,dx\right)\\
&\qquad\qquad \le C\,2^{-l}\sum_{q=k+l}^{+\infty} 2^{(k-q)/2}\left(\int_{2^q\le|x|\le 2^{q+1}}|M\Delta^{1/4}u|^2\,dx\right),
\end{aligned} \tag{119}$$

which is therefore an upper bound for the contribution to $\sum_{h=k}^{+\infty} 2^{k-h}\|w\|^2_{\dot H^{1/2}(A_h)}$ of the term in (117) containing the integral over $|x|>2^{l+h}$.

We still have to estimate the contribution of the term containing the integral over $|x|<2^{h-l}$. We can assume that $h\ge k$. Again following the same reasoning as in (111) and the using the inequalities $|x-s|,|x-t|\ge 2^{h-2}$ applicable to this case, we write

$$\begin{aligned}\int_{A_h}\int_{A_h}\frac{1}{|t-s|^2}\Big(\int_{|x|<2^{h-l}} g(x)\,\big(|t-x|^{-1/2}-|s-x|^{-1/2}\big)\,dx\Big)^2 dt\,ds\\ \le C\int_{A_h}\int_{A_h} 2^{-3h}2^{h-l}\Big(\int_{|x|<2^{h-l}}|g(x)|^2\,dx\Big)\,dt\,ds = C\,2^{-l}\int_{|x|<2^{h-l}}|M\Delta^{1/4}u|^2\,dx\\ = C\,2^{-l}\Big(\int_{|x|<2^{k-l}}|M\Delta^{1/4}u|^2\,dx+\sum_{q=k-l}^{h-l-1}\int_{2^q\le|x|<2^{q+1}}|M\Delta^{1/4}u|^2\,dx\Big).\end{aligned}\tag{120}$$

Multiply the right-hand side of (120) by 2^{k-h}, take the sum from $h=k$ to $+\infty$, interchange the double summation, evaluate the geometric series, and rename q to h as the index of the remaining summation, to obtain the upper bound

$$C\,2^{-l+1}\int_{|x|<2^{k-l}}|M\Delta^{1/4}u|^2\,dx+C\,2^{-2l}\sum_{h=k-l}^{+\infty}\int_{2^h\le|x|<2^{h+1}}2^{k-h}|M\Delta^{1/4}u|^2\,dx\tag{121}$$

for the contribution to $\sum_{h=k}^{+\infty}2^{k-h}\|w\|^2_{\dot H^{1/2}(A_h)}$ of the term under consideration (second term on the last line of (117)).

Now choose l so that $C\,2^{-l}<\gamma<1$, and set $n_0=l$. Then, for all $n\ge n_0$,

$$\begin{aligned}\sum_{h=k}^{+\infty}2^{k-h}\Big(C\,2^{-l}\int_{|x|<2^{k-l}}|M\Delta^{1/4}u|^2\,dx+C\,2^{-2l}\sum_{s=k-l}^{h-l}\int_{2^s\le|x|\le 2^{s+1}}|M\Delta^{1/4}u|^2\,dx\Big)\\ \le\gamma\int_{|x|<2^{k-n}}|M\Delta^{1/4}u|^2\,dx+\sum_{h=k-n}^{+\infty}\int_{2^h\le|x|\le 2^{h+1}}2^{k-h}|M\Delta^{1/4}u|^2\,dx.\end{aligned}$$

By combining (114), (116), (119) and (121), for $n\ge n_0$ we finally get

$$\begin{aligned}\sum_{h=k}^{+\infty}2^{k-h}\|\Delta^{-1/4}(M\Delta^{1/4}u)\|^2_{\dot H^{1/2}(A_h)}\\ \le\gamma\int_{|x|<2^{k-n}}|M\Delta^{1/4}u|^2\,dx+\sum_{h=k-n}^{+\infty}\int_{2^{h-1}\le|x|\le 2^{h+1}}2^{(k-h)/2}|M\Delta^{1/4}u|^2\,dx.\end{aligned}\qquad\square$$

Next we show a sort of Poincaré inequality for functions in $\dot H^{1/2}(\mathbb{R})$ having compact support. Recall that, for Ω an open subset of $\mathbb{R}$, the extension by 0 of a function in $H_0^{1/2}(\Omega)=\overline{C_0^\infty(\Omega)}^{H^{1/2}}$ is, generally speaking, not in $H^{1/2}(\mathbb{R})$. This is why Lions and Magenes [1972] introduced the set $H_{00}^{1/2}(\Omega)$ for which the Poincaré inequality holds.

Theorem A.5. *Let $v\in\dot H^{1/2}(\mathbb{R})$ be such that* $\operatorname{supp} v\subset(-1,1)$. *Then $v\in L^2([-1,1])$ and*

$$\int_{[-1,1]}|v(x)|^2\,dx\le C\|v\|^2_{\dot H^{1/2}((-2,2))}.$$

Proof.

$$\int_{[-1,1]} |v(x)|^2\,dx \le C\int_{1\le|y|\le 2}\int_{|x|\le 1}\frac{|v(x)|^2}{|x-y|^2}\,dx\,dy \le C\int_{1\le|y|\le 2}\int_{|x|\le 1}\frac{|v(x)-v(y)|^2}{|x-y|^2}\,dx\,dy$$
$$\le C\int_{|y|\le 2}\int_{|x|\le 2}\frac{|v(x)-v(y)|^2}{|x-y|^2}\,dx\,dy = C\|v\|^2_{\dot H^{1/2}}([-2,2]). \qquad \square$$

From Theorem A.5 it follows that

$$\|v\|_{L^2((-r,r))} \le C r^{1/2}\|v\|_{\dot H^{1/2}(\mathbb{R})}.$$

The next three results justify the interchanging of infinite sums, pseudodifferential operators, and integrals that we performed several times to obtain the localization estimates in Section 4.

In Lemma A.6 (resp. A.7) we consider a function $g\in \dot H^{1/2}(\mathbb{R})\cap L^\infty(\mathbb{R})$ (resp. $f\in \dot H^{1/2}(\mathbb{R})\cap L^\infty(\mathbb{R})$) whose support is contained in B_{2^k} (resp. $B^c_{2^N}$). We estimate the L^2-norm of $\Delta^{1/4}g$ (resp. $\Delta^{1/4}f$) in annuli $A_h = B_{2^h}\setminus B_{2^{h-1}}$ with $h\gg k$ (resp. $h\ll N$).

Lemma A.6. *Let $g\in \dot H^{1/2}(\mathbb{R})\cap L^\infty(\mathbb{R})$ be such that* $\operatorname{supp} g\subset B_{2^k}(\mathbb{R})$. *Then for all $h>k+3$ we have*

$$\|\Delta^{1/4}g\|_{L^2(A_h)} \le C\,2^{k-h}, \tag{122}$$

where $A_h = B_{2^h}\setminus B_{2^{h-1}}$ and C depends on $\|g\|_{\dot H^{1/2}(\mathbb{R})}$, $\|g\|_{L^\infty(\mathbb{R})}$.

Proof. We fix $h>k+3$ and let $x\in A_h$. We set $\bar g_k = |B_{2^k}|^{-1}\int_{B_{2^k}} g(x)\,dx$. We have

$$\Delta^{1/4}g(x) = \lim_{\varepsilon\to 0}\int_{|x-y|\ge\varepsilon}\frac{g(y)-g(x)}{|x-y|^{3/2}}\,dy = \lim_{\varepsilon\to 0}\int_{\substack{|x-y|\ge\varepsilon\\ y\in B_{2^k}}}\frac{g(y)-g(x)}{|x-y|^{3/2}}\,dy$$
$$\le C\,2^{-3h/2}2^k|B_{2^k}|^{-1}\int_{B_{2^k}}|g(y)-\bar g_k|\,dy + 2^{-3h/2}\int_{B_{2^k}}|g(x)-\bar g_k|\,dy$$
$$\le C\,2^{-3h/2}2^k(\|g\|_{\dot H^{1/2}(\mathbb{R})}+\|g\|_{L^\infty(\mathbb{R})}).$$

In the last inequality we used the fact that $\dot H^{1/2}(\mathbb{R})\hookrightarrow BMO(\mathbb{R})$. It follows that

$$\int_{A_h}|\Delta^{1/4}g(x)|^2\,dx \le C\,2^{2k-2h}\big(\|g\|^2_{L^\infty(\mathbb{R})}+\|g\|^2_{\dot H^{1/2}(\mathbb{R})}\big).$$

Thus (122) holds. $\square$

Lemma A.7. *Let $f\in \dot H^{1/2}(\mathbb{R})\cap L^\infty(\mathbb{R})$ be such that* $\operatorname{supp} f\subset B^c_{2^N}(\mathbb{R})$. *For all $h<N-3$, we have*

$$\|\Delta^{1/4}f\|_{L^2(A_h)}dx \le C\,2^{(h-N)/2}, \tag{123}$$

where C depends on $\|f\|_{\dot H^{1/2}(\mathbb{R})}$ and $\|f\|_{L^\infty}$.

Proof. Fix $h < N-3$ and $x \in A_h$. We have

$$
\begin{aligned}
\Delta^{1/4} f(x) &= \lim_{\varepsilon\to 0} \int_{|x-y|\ge\varepsilon} \frac{f(y)-f(x)}{|x-y|^{3/2}}\,dy \\
&= \lim_{\substack{\varepsilon\to 0\\ \varepsilon<2^{N-1}}} \left(\int_{2^{N-1}\ge|x-y|\ge\varepsilon} \frac{f(y)-f(x)}{|x-y|^{3/2}}\,dy + \int_{|x-y|\ge 2^{N-1}} \frac{f(y)-f(x)}{|x-y|^{3/2}}\,dy \right). \qquad (124)
\end{aligned}
$$

We observe that if $|x-y| < 2^{N-2}$ and $x \in A_h$ then $|y| < 2^{N-1}$ and thus $f(y) = f(x) = 0$. Hence

$$
\begin{aligned}
(124) &= \int_{2^{N-2}\le|x-y|\le 2^N} \frac{f(y)-f(x)}{|x-y|^{3/2}}\,dy + \int_{2^N\le|x-y|} \frac{f(y)-f(x)}{|x-y|^{3/2}}\,dy \\
&\le C[2^{-3/2N}2^N(\|f\|_{\dot H^{1/2}(\mathbb{R})} + \|f\|_{L^\infty(\mathbb{R})}) + 2^{-N/2}\|f\|_{L^\infty(\mathbb{R})}] \\
&\le C\,2^{-N/2}(\|f\|_{\dot H^{1/2}(\mathbb{R})} + \|f\|_{L^\infty(\mathbb{R})}). \qquad (125)
\end{aligned}
$$

From (125) it follows that

$$
\int_{A_h} |\Delta^{1/4} f(x)|^2\,dx \le C\,2^{-N+h}\big(\|f\|^2_{\dot H^{1/2}(\mathbb{R})} + |f\|^2_{L^\infty(\mathbb{R})}\big)
$$

and thus (123) holds. □

Corollary A.8. *Let $g \in H^{1/2}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ with* supp $g \in B_{2^k}$*, for some $k \in \mathbb{Z}$ and for every $N > 0$ let f_N be a sequence in $H^{1/2}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ such that $\|f_N\|_{\dot H^{1/2}(\mathbb{R})} + \|f_N\|_{L^\infty(\mathbb{R})} \le C$ (C independent of N) and* supp $f_N \subset B^c_{2^N}$ *Then*

$$
\lim_{N\to+\infty} \int_{\mathbb{R}} \Delta^{1/4} f_N(x)\Delta^{1/4} g(x)\,dx = 0. \qquad (126)
$$

Proof. We split the integral in (126) as follows:

$$
\begin{aligned}
&\int_{\mathbb{R}} \Delta^{1/4} f_N(x)\Delta^{1/4} g(x)\,dx \\
&\quad= \sum_{h=-\infty}^{k+2} \int_{A_h} \Delta^{1/4} f_N(x)\Delta^{1/4} g(x)\,dx + \sum_{h=k+3}^{N-2} \int_{A_h} \Delta^{1/4} f_N(x)\Delta^{1/4} g(x)\,dx \\
&\qquad + \sum_{h=N-1}^{+\infty} \int_{A_h} \Delta^{1/4} f_N(x)\Delta^{1/4} g(x)\,dx. \qquad (127)
\end{aligned}
$$

We estimate the three summations in (127). We take $N \gg k$.

By applying Lemma A.7 we have

$$\sum_{h=-\infty}^{k+2}\int_{A_h}\Delta^{1/4}f_N(x)\Delta^{1/4}g(x)\,dx \le \sum_{h=-\infty}^{k+2}\Big(\int_{A_h}|\Delta^{1/4}f_N(x)|^2\,dx\Big)^{1/2}\Big(\int_{A_h}|\Delta^{1/4}g(x)|^2\,dx\Big)^{1/2}$$
$$\le C\|g\|_{\dot H^{1/2}}(\|f_N\|_{\dot H^{1/2}(\mathbb{R})}+\|f_N\|_{L^\infty(\mathbb{R})})\sum_{h=-\infty}^{k+2}2^{(h-N)/2}$$
$$\le C\,2^{(k-N)/2}. \tag{128}$$

By Lemma A.6 we have

$$\sum_{h=N-1}^{+\infty}\int_{A_h}\Delta^{1/4}f(x)\Delta^{1/4}g(x)\,dx \le C\|f_N\|_{\dot H^{1/2}(\mathbb{R})}(\|g\|_{\dot H^{1/2}(\mathbb{R})}+|g\|_{L^\infty(\mathbb{R})})\sum_{h=N-1}^{+\infty}2^{k-h}$$
$$\le C\,2^{k-N}. \tag{129}$$

Finally, by applying Lemmas A.6 and A.7 we get

$$\sum_{h=k+3}^{N-2}\int_{A_h}\Delta^{1/4}f_N(x)\Delta^{1/4}g(x)\,dx \le C\,2^{k-N/2}\sum_{h=k+3}^{+\infty}2^{-h/2}\le C\,2^{(k-N)/2}. \tag{130}$$

By combining (127), (128) and (129) we get (126) and we can conclude. □

We conclude with the following technical result, used in the proof of Theorem 5.3.

Proposition A.9. *Let $(a_k)_k$ be a sequence of positive real numbers satisfying $\sum_{k=-\infty}^{+\infty}a_k^2<\infty$ and*

$$\sum_{-\infty}^{n}a_k^2\le C\sum_{k=n+1}^{+\infty}2^{(n+1-k)/2}a_k^2 \quad \textit{for every } n\le 0. \tag{131}$$

There are $0<\beta<1$, $C>0$ and $\bar n<0$ such that for $n\le\bar n$ we have

$$\sum_{-\infty}^{n}a_k^2\le C(2^n)^\beta.$$

Proof. For $n<0$, we set $A_n=\sum_{-\infty}^{n}a_k^2$. We have $a_k^2=A_k-A_{k-1}$ and thus

$$A_n\le C\sum_{k=n+1}^{+\infty}2^{(n+1-k)/2}(A_k-A_{k-1})\le C(1-1/\sqrt{2})\sum_{k=n+1}^{+\infty}2^{(n+1-k)/2}A_k-CA_n.$$

Therefore

$$A_n\le\tau\sum_{n+1}^{+\infty}2^{(n+1-k)/2}A_k, \tag{132}$$

with

$$\tau=\frac{C}{C+1}\Big(1-\frac{1}{\sqrt{2}}\Big)<1-\frac{1}{\sqrt{2}}.$$

The relation (132) implies the estimate

$$A_n \le \tau A_{n+1} + \tau \sum_{n+2}^{+\infty} 2^{(n+1-k)/2} A_k. \tag{133}$$

Now we apply induction on A_{n+1} in (133) and we get

$$\begin{aligned}
(133) &\le \tau^2 \Big(\sum_{n+2}^{+\infty} 2^{(n+2-k)/2} A_k \Big) + \frac{\tau}{\sqrt{2}} \Big(\sum_{n+2}^{+\infty} 2^{(n+2-k)/2} A_k \Big) \\
&= \tau(\tau + 1/\sqrt{2}) \Big(\sum_{n+2}^{+\infty} 2^{(n+2-k)/2} A_k \Big) \\
&= \tau(\tau + 1/\sqrt{2}) \Big(A_{n+2} + 1/\sqrt{2} \sum_{n+3}^{+\infty} 2^{(n+3-k)/2} A_k \Big) \\
&\le \tau(\tau + 1/\sqrt{2})^2 \sum_{n+3}^{+\infty} 2^{(n+3-k)/2} A_k \text{ (by applying induction on } A_{n+2}) \\
&\le \cdots \le \tau(\tau + 1/\sqrt{2})^{-n} \sum_{k=0}^{+\infty} 2^{-k} A_k \\
&\le \tau(\tau + 1/\sqrt{2})^{-n} \Big(\sum_{k=0}^{\infty} 2^{-k} \Big) \Big(\sum_{k=-\infty}^{+\infty} a_k^2 \Big) \\
&\le 2\tau(\tau + 1/\sqrt{2})^{-n} \sum_{k=-\infty}^{+\infty} a_k^2 \\
&\le C\gamma^{-n},
\end{aligned}$$

with $\gamma = \tau(\tau + 1/\sqrt{2})^{-n}$. Therefore for some $\beta \in (0, 1)$ and for all $n < 0$ we have $A_n \le C(2^n)^\beta$. □

Acknowledgement

The authors thank the referee for useful comments and remarks.

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Received 23 Jul 2009. Accepted 10 Mar 2010.

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Analysis & PDE, at Mathematical Sciences Publishers, Department of Mathematics, University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFLOW™ from Mathematical Sciences Publishers.

PUBLISHED BY

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Typeset in LaTeX

ANALYSIS & PDE

Volume 4 No. 1 2011