ANALYSIS & PDE

Volume 4 No. 1 2011

mathematical sciences publishers

Analysis & PDE

pjm.math.berkeley.edu/apde

See inside back cover or pjm.math.berkeley.edu/apde for submission instructions.

The subscription price for 2011 is US $120/year for the electronic version, and $180/year for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to Mathematical Sciences Publishers, Department of Mathematics, University of California, Berkeley, CA 94720-3840, USA.

Analysis & PDE, at Mathematical Sciences Publishers, Department of Mathematics, University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFLOW™ from Mathematical Sciences Publishers.

PUBLISHED BY
mathematical sciences publishers
http://www.mathscipub.org
A NON-PROFIT CORPORATION
Typeset in LaTeX

ANALYSIS AND PDE
Vol. 4, No. 1, 2011

STRICHARTZ ESTIMATES ON ASYMPTOTICALLY HYPERBOLIC MANIFOLDS

JEAN-MARC BOUCLET

We prove local in time Strichartz estimates without loss for the restriction of the solution of the Schrödinger equation, outside a large compact set, on a class of asymptotically hyperbolic manifolds.

1. The results

Let $(\mathcal{M}, G)$ be a Riemannian manifold of dimension $n \geq 2$ with Riemannian volume density dG and associated Laplace–Beltrami operator Δ_G. The Strichartz estimates for the Schrödinger equation

$$i\partial_t u + \Delta_G u = 0, \qquad u_{|t=0} = u_0, \tag{1-1}$$

are basically estimates of

$$\|u\|_{L^p([0,1],L^q(\mathcal{M},dG))} := \left(\int_0^1 \|u(t,\cdot)\|^p_{L^q(\mathcal{M},dG)} dt \right)^{1/p},$$

in terms of certain L^2 quantities of u_0, when the pair of exponents (p, q) satisfies the *admissibility conditions*

$$\frac{2}{p} + \frac{n}{q} = \frac{n}{2}, \qquad p \geq 2, \quad (p,q) \neq (2,\infty). \tag{1-2}$$

Strichartz estimates play an important role in the proof of local existence results for nonlinear Schrödinger equations (see for instance [Ginibre and Velo 1985; Cazenave 2003; Burq et al. 2004]). We won't consider such applications in this paper and will only focus on the estimates themselves.

We review some classical results. If $\mathcal{M} = \mathbb{R}^n$ with the flat metric, it is well known [Strichartz 1977; Ginibre and Velo 1985; Keel and Tao 1998] that

$$\|u\|_{L^p([0,1],L^q(\mathbb{R}^n))} \lesssim \|u_0\|_{L^2(\mathbb{R}^n)}. \tag{1-3}$$

MSC2000: 35B45, 35S30, 58J40, 58J47.
Keywords: Strichartz estimates, asymptotically hyperbolic, Isozaki–Kiada parametrix, semiclassical functional calculus.

In this model case, the time interval [0, 1] can be replaced by $\mathbb{R}$ and the Strichartz estimates are said to be *global in time*. Furthermore, the conditions (1-2) are seen to be natural by considering the action of the scaling $u(t,x) \mapsto u(t/\lambda^2, x/\lambda)$ on both Schrödinger equation and Strichartz estimates.

(In this paper we will not pursue global in time Strichartz estimates. Although one can expect that they exist, it is not clear how to obtain them by the present method. One may hope to obtain such global in time results at least for initial data spectrally cutoff on the low frequencies by combining the present analysis with the method of [Bouclet and Tzvetkov 2008].)

In more general situations, estimates of the form (1-3) sometimes have to be replaced by

$$\|u\|_{L^p([0,1],L^q(\mathcal{M},dG))} \lesssim \|u_0\|_{H^s(\mathcal{M},dG)}, \quad s \geq 0, \tag{1-4}$$

where

$$\|u_0\|_{H^s(\mathcal{M},dG)} := \|(1-\Delta_G)^{s/2} u_0\|_{L^2(\mathcal{M},dG)},$$

is the natural L^2 Sobolev norm. If $s > 0$, estimates such as (1-4) are called *Strichartz estimates with loss* (of s derivatives). Notice that, under fairly general assumptions on $(\mathcal{M}, G)$, we have the Sobolev embeddings $H^s(\mathcal{M}, dG) \subset L^q(\mathcal{M}, dG)$ for $s > n/2 - n/q$. They show that (1-4) holds automatically if s is large enough and the point of Strichartz estimates with loss (and a fortiori without loss) is to consider smaller s than those given by Sobolev embeddings.

Such inequalities have been proved by Bourgain [1993] for the flat tori $\mathbb{T}^1$ and $\mathbb{T}^2$, for certain values of p, q and *any* $s > 0$ (i.e., with "almost no loss"), and by Burq, Gérard, and Tzvetkov [Burq et al. 2004] for any compact manifold with $s = 1/p$. The techniques of the latter work are actually very robust and can be applied to prove the same results on many noncompact manifolds; the estimates are known to be sharp for $\mathcal{M} = \mathbb{S}^3$ with $p = 2$ and by considering certain subsequences of eigenfunctions of the Laplacian. This counterexample can then be used to construct quasimodes and show that (1-4) cannot hold in general with $s = 0$, even for noncompact manifolds.

A natural question is therefore to find (sufficient) conditions leading to estimates with no loss.

A classical one is the nontrapping condition. We recall that $(\mathcal{M}, G)$ is *nontrapping* if all geodesics escape to infinity (implying that $\mathcal{M}$ is noncompact). It was for instance shown in [Staffilani and Tataru 2002; Robbiano and Zuily 2005; Bouclet and Tzvetkov 2007] that, for nontrapping perturbations of the flat metric on $\mathbb{R}^n$, (1-4) holds with $s = 0$. By a *perturbation* we mean that the departure of G from the flat metric G_{Eucl} is small near infinity and we refer to those papers for more details. In [Hassell et al. 2006], the more general case of nontrapping asymptotically conic manifolds was considered. To emphasize the difference with the asymptotically hyperbolic manifolds studied in this paper, we simply recall that $(\mathcal{M}, G)$ is asymptotically conic if G is close to $dr^2 + r^2 g$, in a neighborhood of infinity diffeomorphic to $(R, +\infty) \times S$, for some fixed metric g on a compact manifold S. The asymptotically Euclidean case corresponds to the case where $S = \mathbb{S}^{n-1}$.

The nontrapping condition, however, has several drawbacks, such as being nongeneric and difficult to check. Moreover, it is not clearly a *necessary* condition to get Strichartz estimates without loss.

In [Bouclet and Tzvetkov 2007], we partially got rid of this condition by considering Strichartz estimates localized near spatial infinity. For long-range perturbations G of the Euclidean metric on $\mathcal{M} = \mathbb{R}^n$ (meaning that $\partial_x^\alpha (G(x) - G_{\text{Eucl}}) = \mathcal{O}(\langle x \rangle^{-\tau - |\alpha|})$ for $\tau > 0$), trapping or not, we proved the existence of

$R > 0$ large enough such that, if $\chi \in C_0^\infty(\mathbb{R}^n)$ satisfies $\chi \equiv 1$ for $|x| \le R$, then

$$\|(1-\chi)u\|_{L^p([0,1];L^q(\mathbb{R}^n,dG))} \lesssim \|u_0\|_{L^2(\mathbb{R}^n,dG)}. \tag{1-5}$$

This shows that the possible loss in Strichartz estimates can only come from a bounded region where the metric is essentially arbitrary (recall that being asymptotically Euclidean is only a condition at infinity). One can loosely interpret this result as follows: as long as the metric is close to a model one for which one has Strichartz estimates without loss, the solution to the Schrödinger equation satisfies Strichartz estimates without loss too.

The first goal of the present paper is to show that the same result holds in (bounded) negative curvature, more precisely for asymptotically hyperbolic (AH) manifolds. We point out, however, that even if our Theorem 1.2 below is formally the same as in the asymptotically Euclidean case [Bouclet and Tzvetkov 2007, Theorem 1], its proof involves new arguments using the negative curvature. One of the messages of this paper is that, by taking advantage of certain curvature effects described at the end of this Section, we prove Strichartz estimates using long time (microlocal) parametrices of the Schrödinger group which are localized in very narrow regions of the phase space, much smaller than those considered in the asymptotically Euclidean situation.

As far as the Schrödinger equation is concerned, Strichartz estimates on negatively curved spaces have been studied in [Banica 2007; Pierfelice 2006; 2008; Anker and Pierfelice 2009] (see [Tataru 2001] for the wave equation). In [Pierfelice 2006], Pierfelice considers perturbations of the Schrödinger equation on the hyperbolic space $\mathbb{H}^n$ by singular time-dependent radial potentials, with radial initial data (and also radial source terms) and derives some weighted Strichartz estimates without loss. The nonradial case for the free Schrödinger equation on $\mathbb{H}^n$ is studied in [Banica 2007] where weighted Strichartz estimates are obtained too. The more general case of certain Lie groups, namely Damek-Ricci spaces, was considered in [Pierfelice 2008] for global in time estimates (see also [Banica et al. 2008] for the two-dimensional case) and further generalized in [Banica and Duyckaerts 2007]. In these last papers, only radial data are considered. This radial assumption was removed in [Anker and Pierfelice 2009]. This last paper also shows, with [Banica et al. 2008], in such geometries, the set of admissible pairs for the Strichartz estimates is contained in a triangle, and thus is much wider than in the (asymptotically) Euclidean case. One expects that such a result remains valid in our context, but this does not clearly follow from the tools presented here and might require refined propagation estimates.

In this article, we give a proof of Strichartz estimates at infinity which is purely (micro)local and so, to a large extent, stable under perturbation. We do not use any Lie group structure or spherical symmetry, nor do we assume any nontrapping condition. We refer to Definition 1.1 below for precise statements and simply quote here that our class of manifolds contains $\mathbb{H}^n$, some of its quotients and perturbations thereof. In particular, we do not assume that the curvature is constant, even near infinity. (Powerful microlocal techniques for AH manifolds have already been developed by Melrose and his school; see [Mazzeo and Melrose 1987] and the references in [Melrose 1995]. These geometric methods, based on compactification and blowup considerations, are perfectly designed for conformally compact manifolds with boundary, but do not clearly apply to the more general manifolds we study here.)

In the next few pages we fix our framework and state our main results precisely, highlighting the key points that allow us to prove them. We conclude the section with an overview of the remainder of the article, on page 7.

Definition 1.1 (AH manifold). $(\mathcal{M}^n, G)$ is *asymptotically hyperbolic* if there exist a compact set $\mathcal{K} \Subset \mathcal{M}$, a real number $R_{\mathcal{K}} > 0$, a compact manifold without boundary S and a function

$$r \in C^\infty(\mathcal{M}, \mathbb{R}) \quad \text{with} \quad r(m) \to +\infty \ \text{ as } \ m \to \infty \tag{1-6}$$

(a coordinate near $\overline{\mathcal{M} \setminus \mathcal{K}}$) such that we have an isometry

$$\Psi : (\mathcal{M} \setminus \mathcal{K}, G) \to \big((R_{\mathcal{K}}, +\infty)_r \times S, \ dr^2 + e^{2r} g(r) \big), \tag{1-7}$$

where $g(r)$ is a family of metrics on S depending smoothly on r such that, for some $\tau > 0$ and some fixed metric g on S, we have

$$\big\| \partial_r^k (g(r) - g) \big\|_{C^\infty(S, T^*S \otimes T^*S)} \lesssim r^{-\tau - k} \quad \text{for } \ r > R_{\mathcal{K}}, \tag{1-8}$$

for all $k \geq 0$ and all seminorms $\| \cdot \|_{C^\infty(S, T^*S \otimes T^*S)}$ in the space of smooth sections of $T^*S \otimes T^*S$.

With no loss of generality, we can assume that the decay rate τ in (1-8) satisfies

$$0 < \tau < 1. \tag{1-9}$$

Therefore, by analogy with the standard terminology in Euclidean scattering, $dr^2 + e^{2r} g(r)$ can be considered as a *long-range* perturbation of the metric $dr^2 + e^{2r} g$. Notice that the conformally compact case quoted above corresponds to the special situation where $g(r)$ is of the form $\tilde{g}(e^{-r})$, for some family of metrics $(\tilde{g}(x))_{0 \leq x \ll 1}$ depending smoothly on $x \in [0, x_0)$ (x_0 small enough) up to $x = 0$. In that case, $g(r)$ is an exponentially small perturbation of $g = \tilde{g}(0)$. The assumption (1-8) is therefore more general.

We next denote by Δ_G the Laplace–Beltrami operator associated to this metric. It is classical that this operator is essentially self-adjoint on $C_0^\infty(\mathcal{M})$ (using for instance the method of [Helffer and Robert 1983]), and therefore generates a unitary group $e^{it\Delta_G}$ on $L^2(\mathcal{M}, dG)$.

Our main result is the following.

Theorem 1.2. *There exists $\chi \in C_0^\infty(\mathcal{M})$, with $\chi \equiv 1$ on a sufficiently large compact set, such that, for all pair (p, q) satisfying (1-2),*

$$\| (1 - \chi) e^{it\Delta_G} u_0 \|_{L^p([0,1]; L^q(\mathcal{M}, dG))} \lesssim \| u_0 \|_{L^2(\mathcal{M}, dG)}, \qquad u_0 \in C_0^\infty(\mathcal{M}). \tag{1-10}$$

This theorem is the AH analogue of Theorem 1 of [Bouclet and Tzvetkov 2007] in the asymptotically Euclidean case.

To be more complete, let us point out that the analysis contained in this paper and a classical argument due to [Staffilani and Tataru 2002] (see also [Bouclet and Tzvetkov 2007, Section 5]), using the local smoothing effect [Doi 1996], would give the following global in space estimates.

Theorem 1.3. *If in addition $(\mathcal{M}, G)$ is nontrapping, then we have global in space Strichartz estimates with no loss: for all pair (p, q) satisfying (1-2),*

$$\| e^{it\Delta_G} u_0 \|_{L^p([0,1]; L^q(\mathcal{M}, dG))} \lesssim \| u_0 \|_{L^2(\mathcal{M}, dG)}, \qquad u_0 \in C_0^\infty(\mathcal{M}).$$

We state this result as a theorem although we won't explicitly prove it. The techniques are fairly well known and don't involve any new argument in the present context. We simply note that resolvent estimates implying the local smoothing effect can be found in [Cardoso and Vodev 2002].

Remark. Theorem 1.2 reduces the proof of potential improvements of Burq–Gérard–Tzvetkov inequalities to local in space estimates of the form

$$\|\chi u\|_{L^p([0,1],L^q(\mathcal{M},dG))} \lesssim \|u_0\|_{H^s(\mathcal{M},dG)},$$

with $0 \le s < 1/p$. It would be interesting to know if such inequalities holds for some trapping AH manifolds.

We now describe, quite informally, the key points of the analysis developed in this paper. Assuming for simplicity that $S = \mathbb{S}^1$ (and thus $n = 2$), we consider the model case where the principal symbol of the Laplacian is

$$p = \rho^2 + e^{-2r}\eta^2.$$

For convenience, we introduce

$$P := -e^{(n-1)r/2}\Delta_G e^{-(n-1)r/2} = -e^{r/2}\Delta_G e^{-r/2},$$

which is self-adjoint with respect to $dr\, d\theta$, instead of $e^{(n-1)r}\, dr\, d\theta = e^r\, dr\, d\theta$ for the Laplacian itself.

Recall first that, by the Keel–Tao TT^* Theorem [1998], proving Strichartz estimates (without loss) mainly reduces to prove certain dispersion estimates. Using the natural semiclassical time scaling $t \mapsto ht$, this basically requires to control the propagator e^{-ithP} for semiclassical times of order h^{-1}. Such a control on the full propagator is out of reach (basically because of trapped trajectories) but, fortunately, studying some of its cutoffs will be sufficient.

After fairly classical reductions, we will work with semiclassical pseudodifferential operators localized where $r \gg 1$ and $p \in I$, I being a (relatively) compact interval of $(0, +\infty)$. We can split the latter region into two areas defined by

$$\Gamma^+ = \left\{ r \gg 1,\ p \in I,\ \rho > -\tfrac{1}{2}p^{1/2} \right\}, \quad \Gamma^- = \left\{ r \gg 1,\ p \in I,\ \rho < \tfrac{1}{2}p^{1/2} \right\},$$

respectively called the outgoing and incoming areas. The main interest of such areas is that one has a very good control on the geodesic flow therein (see Section 3). Basically, geodesics with initial data in outgoing (resp. incoming) areas escape to infinity as $t \to +\infty$ (resp. $t \to -\infty$), which is proved in Proposition 3.3. One thus expects to be able to give long time approximations of the propagator e^{-ithP}, microlocalized in such areas, for large times ($t \ge 0$ in outgoing areas and $t \le 0$ in incoming ones).

In the asymptotically Euclidean case, it turns out that one can give accurate approximations of $e^{-ithP}\chi^\pm$ for times t such that $0 \le \pm t \lesssim h^{-1}$, if $\chi^\pm$ are pseudodifferential cutoffs localized in $\Gamma^\pm$. This is not the AH case: here we are only able to approximate $e^{-ithP}\chi^\pm_{\mathrm{s}}$ for cutoffs $\chi^\pm_{\mathrm{s}}$ localized in much smaller areas, namely

$$\Gamma^+_{\mathrm{s}}(\varepsilon) = \left\{ r \gg 1,\ p \in I,\ \rho > (1-\varepsilon^2)p^{1/2} \right\}, \quad \Gamma^-_{\mathrm{s}}(\varepsilon) = \left\{ r \gg 1,\ p \in I,\ \rho < (\varepsilon^2-1)p^{1/2} \right\},$$

which we call *strongly* outgoing/incoming areas. Here ε will be a fixed small real number. We then obtain approximations of the form

$$e^{-ithP}\chi^\pm_{\mathrm{s}} = J_{S^\pm}(a^\pm)e^{-ithD_r^2}J_{S^\pm}(b^\pm)^* + \mathcal{O}(h^N), \qquad 0 \le \pm t \lesssim h^{-1}. \tag{1-11}$$

Here $e^{-ithD_r^2}$ is the semiclassical group associated to the radial part D_r^2 of P. Here and in the sequel, we shall use the standard notation

$$D_r = i^{-1}\partial_r, \qquad D_\theta = i^{-1}\partial_\theta .$$

The operators $J_{S^\pm}(a^\pm)$ and $J_{S^\pm}(b^\pm)$ are Fourier integral operators with amplitudes $a^\pm, b^\pm$ supported in strongly outgoing $(+)$ / incoming $(-)$ areas and phases essentially of the form

$$S^\pm \approx r\rho + \theta\eta + \frac{e^{-2r}\eta^2}{4\rho},$$

i.e., the sum of the *free phase* $r\rho+\theta\eta$ and of a term whose Hessian is nondegenerate in η, which will be crucial for the final stationary phase argument (the small factor e^{-2r} will be eliminated by a change of variable). The nondegeneracy of the full phase of the parametrix (1-11) in ρ will come of course from $e^{-ithD_r^2}$. This approximation of $S^\pm$ comes basically from (4-34) and (4-35). Although the right-hand side does not depend on $\pm$, it is only defined in the disconnected regions $\{\rho > 0\}$ and $\{\rho < 0\}$.

The approximation (1-11) is the AH Isozaki–Kiada parametrix and a significant part of this paper is devoted to its construction. We mention that it is an adaptation to the AH geometry of an approximation introduced first in [Isozaki and Kitada 1985] to study perturbations of the Euclidean Laplacian by long-range potentials. In the present paper, it will be used very similarly to the usual (semiclassical) Euclidean one as in [Bouclet and Tzvetkov 2007]. Its main interest is to give microlocal approximations of the propagator for times of size h^{-1}. Recall however the big difference with the asymptotically Euclidean case where one is able to consider cutoffs supported in $\Gamma^\pm$ rather than $\Gamma^\pm_{\mathrm{s}}(\varepsilon)$ in the AH case. We therefore have to consider the left parts, namely

$$\Gamma^\pm_{\mathrm{inter}} = \Gamma^\pm \setminus \Gamma^\pm_{\mathrm{s}}(\varepsilon),$$

which we call *intermediate* areas. These areas will only contribute to the dispersion estimates for small times, in view of the following argument. By choosing δ small enough and by splitting the interval $\left(-\frac{1}{2}, 1-\varepsilon^2\right)$ into small intervals of size δ, we can write

$$\Gamma^\pm_{\mathrm{inter}} = \bigcup_{l \lesssim \delta^{-1}} \left\{ r \gg 1,\ \ p \in I,\ \pm\rho p^{-1/2} \in (\sigma_l, \sigma_l+\delta) \right\} = \bigcup_{l \lesssim \delta^{-1}} \Gamma^\pm_{\mathrm{inter}}(l,\varepsilon,\delta).$$

Carefuly consideration of the Hamiltonian flow Φ^t_p of p shows that, for any fixed (small) time t_0, we can choose δ (which depends also on ε) such that

$$\Phi^t_p\left(\Gamma^\pm_{\mathrm{inter}}(l,\varepsilon,\delta)\right) \cap \Gamma^\pm_{\mathrm{inter}}(l,\varepsilon,\delta) = \varnothing \quad \text{for } \pm t \geq t_0. \tag{1-12}$$

By semiclassical propagation, this implies that

$$\chi^\pm_{\mathrm{inter}} e^{-ithP} \chi^{\pm *}_{\mathrm{inter}} = \mathbb{O}(h^\infty) \quad \text{for } \pm t \geq t_0,$$

for pseudodifferential operators $\chi^\pm_{\mathrm{inter}}$ localized in $\Gamma^\pm_{\mathrm{inter}}(l,\varepsilon,\delta)$. Such operators typically appear in the TT^* argument and the estimate above reduces the proof of dispersion estimates to times $|t| \leq t_0$. The latter range of times can then be treated by fairly standard geometric optics approximation.

We interpret (1-12) as a *negative curvature effect* on the geodesic flow, which we can roughly describe as follows, say in the outgoing case. For initial conditions (r,θ,ρ,η) in $\Gamma^+_{\mathrm{inter}}(l,\varepsilon,\delta)$, the bounds

$\frac{1}{2} < \rho \leq (1-\varepsilon^2)p^{1/2}$ yield the lower bound

$$\dot{\rho}^t = 2e^{-2r^t}(\eta^t)^2 \gtrsim \varepsilon^2,$$

over a sufficiently long time, if we set $(r^t, \theta^t, \rho^t, \eta^t) =: \Phi_p^t$. This ensures that $\rho^t/p^{1/2}$ increases fast enough to leave the interval $(\sigma_l, \sigma_l + \delta)$ before $t = t_0$ and give (1-12). In the asymptotically flat case, that is, with r^{-2} instead of e^{-2r}, we have $\dot{\rho}^t = 2(r^t)^{-3}(\eta^t)^2$ and its control from below is not as good, basically because of the "extra" third power of $(r^t)^{-1}$.

Overview of remaining sections. In Section 2, we introduce all the necessary definitions, and some additional results, needed to prove Theorem 1.2. The latter proof is given in Section 2E using microlocal approximations which will be proved in Sections 5, 6 and 7.

In Section 3, we study the properties of the geodesic flow in outgoing/incoming areas required to construct the phases involved in the Isozaki–Kiada parametrix. This parametrix is then constructed in Section 5.

In Section 6 we prove two results: the small semiclassical time approximation of the Schrödinger group by the WKB method and the propagation of the microlocal support (Egorov theorem). These results are essentially well known. We need however to check that all the symbols and phases belong to the natural classes (for AH geometry) of Definition 2.2 below. Furthermore, we use our Egorov theorem to obtain a propagation property in a time scale of size h^{-1}, which is not quite standard.

Finally, in Section 7, we prove dispersion estimates using basically stationary phase estimates in the parametrices obtained in Sections 5 and 6.

Up to the semiclassical functional calculus, which is taken from [Bouclet 2007; Bouclet 2010] and whose results are recalled in Section 2C, this paper is essentially self-contained. This is not only for the reader's convenience, but also because the results of Section 6 do require proofs in the AH setting, although they are in principle well known. The construction of Section 5 is new.

2. The strategy of the proof of Theorem 1.2

2A. *The setup.* Before discussing the proof of Theorem 1.2, we give the form of the Laplacian, volume densities and related objects on AH manifolds.

The isometry (1-7) defines polar coordinates: r is the radial coordinate and S will be called the angular manifold. Coordinates on S will be denoted by $\theta_1, \ldots, \theta_{n-1}$.

A finite atlas on $\mathcal{M}\backslash\mathcal{K}$ is obtained as follows. By (1-7), we have a natural projection $\pi_S : (\mathcal{M}\backslash\mathcal{K}, G) \to S$ defined as the second component of Ψ, that is,

$$\Psi(m) = (r(m), \pi_S(m)) \in (R_{\mathcal{K}}, +\infty) \times S \quad \text{for } m \in \mathcal{M} \setminus \mathcal{K}. \tag{2-1}$$

Choosing a finite cover of the angular manifold by coordinate patches U_ι, say

$$S = \bigcup_{\iota \in \mathcal{I}} U_\iota, \tag{2-2}$$

with corresponding diffeomorphisms

$$\psi_\iota : U_\iota \to \psi_\iota(U_\iota) \subset \mathbb{R}^{n-1}, \tag{2-3}$$

we consider the open sets

$$\mathcal{U}_\iota := \Psi^{-1}\left((R_{\mathcal{K}}, +\infty) \times U_\iota\right) \subset \mathcal{M} \setminus \mathcal{K}$$

and then define diffeomorphisms

$$\Psi_\iota : \mathcal{U}_\iota \to (R_{\mathcal{K}}, +\infty) \times \psi_\iota(U_\iota) \subset \mathbb{R}^n, \tag{2-4}$$

by

$$\Psi_\iota(m) = (r(m), \psi_\iota(\pi_S(m)))\,.$$

The collection $(\mathcal{U}_\iota, \Psi_\iota)_{\iota\in\mathcal{I}}$ is then an atlas on $\mathcal{M} \setminus \mathcal{K}$. If $\theta_1, \ldots, \theta_{n-1}$ are the coordinates in U_ι, that is, $\psi_\iota = (\theta_1, \ldots, \theta_{n-1})$, the coordinates in $\mathcal{U}_\iota$ are then $(r, \theta_1, \ldots, \theta_{n-1})$.

We now give formulas for the Riemannian measure dG and the Laplacian Δ_G on $\mathcal{M} \setminus \mathcal{K}$. In local coordinates $\theta = (\theta_1, \ldots, \theta_{n-1})$ on S, the Riemannian density associated to $g(r)$ reads

$$dg(r) := \det(g(r, \theta))^{1/2} |d\theta_1 \wedge \cdots \wedge d\theta_{n-1}|,$$

where $\det(g(r, \theta)) = \det(g_{jk}(r, \theta))$ if $g(r) = g_{jk}(r, \theta)\, d\theta_j\, d\theta_k$ (using the summation convention). Then, in local coordinates on $\mathcal{M} \setminus \mathcal{K}$, the Riemannian density is

$$dG = e^{(n-1)r} \det(g(r, \theta))^{1/2}\, |dr \wedge d\theta_1 \wedge \cdots \wedge d\theta_{n-1}|. \tag{2-5}$$

Now consider the Laplacian. Slightly abusing the notation, we set

$$c(r, s) = \frac{1}{2} \frac{\partial_r \det(g(r, s))}{\det(g(r, s))} \quad \text{for } r > R_{\mathcal{K}},\ s \in S, \tag{2-6}$$

since, for fixed r, the quotient of $\partial_r \det(g_{jk}(r, \theta))$ by $2\det(g_{jk}(r, \theta))$ is intrinsically defined as a function on S, independently of the choice of the coordinate chart. We then have

$$\Delta_G = \partial_r^2 + e^{-2r} \Delta_{g(r)} + c(r, s)\partial_r + (n-1)\partial_r.$$

It will turn out be convenient to work with the density

$$\widehat{dG} = e^{(1-n)r} dG, \tag{2-7}$$

rather than dG itself. In particular, we will use the following elementary property: for all relatively compact subset $V_\iota' \Subset \psi_\iota(U_\iota)$, all $R > R_{\mathcal{K}}$ and all $1 \le q \le \infty$, we have the equivalence of norms

$$\|u\|_{L^q(\mathcal{M}, \widehat{dG})} \approx \|u \circ \Psi_\iota^{-1}\|_{L^q(\mathbb{R}^n)}, \qquad \operatorname{supp}(u) \subset \Psi_\iota^{-1}\left((R, +\infty) \times V_\iota'\right), \tag{2-8}$$

$L^q(\mathbb{R}^n)$ being the usual Lebesgue space. This is a simple consequence of (1-8) and (2-5) (we consider $R > R_{\mathcal{K}}$ since (1-8) gives an upper bound for $\det g(r, \theta)$ as $r \to R_{\mathcal{K}}$, not a lower bound).

We then have a unitary isomorphism

$$L^2(\mathcal{M}, \widehat{dG}) \ni u \mapsto e^{-(n-1)r/2} u \in L^2(\mathcal{M}, dG), \tag{2-9}$$

and Δ_G is unitarily equivalent to the operator

$$\widehat{\Delta}_G := e^{\gamma_n r} \Delta_G e^{-\gamma_n r}, \quad \gamma_n = \frac{n-1}{2}, \tag{2-10}$$

on $L^2(\mathcal{M}, \widehat{dG})$. This operator reads

$$\widehat{\Delta}_G = \partial_r^2 + e^{-2r}\Delta_{g(r)} + c(r,s)\partial_r - \gamma_n c(r,s) - \gamma_n^2, \tag{2-11}$$

and we will work with

$$P = -\widehat{\Delta}_G - \gamma_n^2. \tag{2-12}$$

If $q_\iota(r, ., \cdot)$ is the principal symbol of $-\Delta_{g(r)}$ in the chart U_ι, namely

$$q_\iota(r,\theta,\xi) = \sum_{1\le k,l\le n-1} g^{kl}(r,\theta)\xi_k\xi_l, \tag{2-13}$$

the principal symbol of P in the chart $\mathcal{U}_\iota$ is then

$$p_\iota = \rho^2 + e^{-2r}q_\iota(r,\theta,\eta), = \rho^2 + q_\iota(r,\theta,e^{-r}\eta). \tag{2-14}$$

The full symbol of P is of the form $p_\iota + p_{\iota,1} + p_{\iota,0}$ with

$$p_{\iota,j} = \sum_{k+|\beta|=j} a_{\iota,k\beta}(r,\theta)\rho^k(e^{-r}\eta)^\beta, \quad j=0,1. \tag{2-15}$$

The terms of degree 1 in η come from the first-order terms of the symbol of $-\Delta_{g(r)}$. In the expression of Δ_G they carry a factor e^{-2r} and therefore, if $j=1$, $k=0$ and $|\beta|=1$ above, we could write $a_{\iota,k\beta}(r,\theta) = e^{-r}b_{\iota,k\beta}(r,\theta)$ for some function $b_{\iota,k\beta}$ bounded as $r\to\infty$. This remark and (1-8) show more precisely that, for all $V \Subset \psi_\iota(U_\iota)$, the coefficients in (2-15) decay as

$$|\partial_r^j\partial_\theta^\alpha a_{\iota,k\beta}(r,\theta)| \le C_{Vj\alpha}\langle r\rangle^{-\tau-1-j}, \quad \theta\in V,\ r\ge R_{\mathcal{K}}+1. \tag{2-16}$$

The decay rate $-\tau-1-j$ will be important to solve transport equations for the Isozaki–Kiada parametrix. This is the main reason of the long-range assumption (1-8).

2B. *Pseudodifferential operators and the spaces $\mathcal{B}_{\mathrm{hyp}}(\Omega)$*. We will consider h-pseudodifferential operators (h-ΨDOs) in a neighborhood of infinity and the calculus will be rather elementary. For instance, we will only consider compositions of operators with symbols supported in the same coordinate patch and no invariance result under diffeomorphism will be necessary.

The first step is to construct a suitable partition of unity near infinity. Using the cover (2-2) and the related diffeomorphisms (2-3), we consider a partition of unity on S of the form

$$\sum_{\iota\in\mathcal{I}} \kappa_\iota\circ\psi_\iota = 1, \quad \text{with } \kappa_\iota\in C_0^\infty(\mathbb{R}^{n-1}),\ \mathrm{supp}(\kappa_\iota)\Subset\psi_\iota(U_\iota), \tag{2-17}$$

and a function $\kappa\in C^\infty(\mathbb{R})$ such that

$$\mathrm{supp}(\kappa)\subset[R_{\mathcal{K}}+1,+\infty), \qquad \kappa\equiv 1 \ \text{ on } [R_{\mathcal{K}}+2,+\infty). \tag{2-18}$$

Then, the functions $(\kappa\otimes\kappa_\iota)\circ\Psi_\iota\in C^\infty(\mathcal{M})$ satisfy

$$\sum_{\iota\in\mathcal{I}}(\kappa\otimes\kappa_\iota)\circ\Psi_\iota(m) = \begin{cases}1 & \text{if } r(m)\ge R_{\mathcal{K}}+2,\\ 0 & \text{if } r(m)\le R_{\mathcal{K}}+1,\end{cases} \tag{2-19}$$

which means that they define a partition of unity near infinity. We could obtain a partition of unity on $\mathcal{M}$ by adding a finite number of compactly supported functions (in coordinate patches) be we won't need it since the whole analysis in this paper will be localized near infinity.

We also consider $\tilde{\kappa} \in C^\infty(\mathbb{R})$ and $\tilde{\kappa}_\iota \in C_0^\infty(\mathbb{R}^{n-1})$, for all $\iota \in \mathcal{I}$, such that

$$\begin{aligned} \tilde{\kappa} &\equiv 1 \quad \text{on } \left(R_{\mathcal{K}} + \tfrac{1}{2}, +\infty\right), & \operatorname{supp}(\tilde{\kappa}) &\subset (R_{\mathcal{K}} + \tfrac{1}{4}, +\infty), \\ \tilde{\kappa}_\iota &\equiv 1 \quad \text{near } \operatorname{supp}(\kappa_\iota), & \operatorname{supp}(\tilde{\kappa}_\iota) &\Subset \psi_\iota(U_\iota). \end{aligned} \tag{2-20}$$

We next choose, for each $\iota \in \mathcal{I}$, two relatively compact open subsets V_ι and V_ι' such that

$$\operatorname{supp}(\kappa_\iota) \Subset V_\iota \Subset V_\iota' \Subset \operatorname{supp}(\tilde{\kappa}_\iota) \quad \text{and} \quad \tilde{\kappa}_\iota \equiv 1 \text{ near } V_\iota'. \tag{2-21}$$

We are now ready to define our ΨDOs. In the following definition, we will say that $a \in C^\infty(\mathbb{R}^{2n})$ is a symbol if either $a \in C_b^\infty(\mathbb{R}^{2n})$ — that is, a is bounded with all derivatives bounded — or

$$a(r, \theta, \rho, \eta) = \sum a_{k\beta}(r, \theta) \rho^k \eta^\beta, \tag{2-22}$$

with $a_{k\beta} \in C_b^\infty(\mathbb{R}^n)$, the sum being finite. We shall give examples below. Notice that throughout this paper, ρ and η will denote respectively the dual variables to r and θ.

Definition 2.1. For $\iota \in \mathcal{I}$, all $h \in (0, 1]$ and all symbol a such that

$$supp(a) \subset [R_{\mathcal{K}} + 1, +\infty) \times V_\iota' \times \mathbb{R}^n, \tag{2-23}$$

we define

$$\widehat{Op}_\iota(a) : C_0^\infty(\mathcal{M}) \to C^\infty(\mathcal{M}),$$

by

$$\left(\widehat{Op}_\iota(a) u\right) \circ \Psi_\iota^{-1}(r, \theta) = a(r, \theta, hD_r, hD_\theta) \left(\tilde{\kappa}(r) \tilde{\kappa}_\iota(\theta) (u \circ \Psi_\iota^{-1})(r, \theta)\right). \tag{2-24}$$

Note the cutoff $\tilde{\kappa} \otimes \tilde{\kappa}_\iota$ in the right-hand side of (2-24). It makes the Schwartz kernel of $\widehat{Op}_\iota(a)$ supported in a closed subset of $\mathcal{M}^2$ strictly contained in the patch $\mathcal{U}_\iota^2$ so that $\widehat{Op}_\iota(a)$ is fully defined by the prescription of $\Psi_{\iota *} \widehat{Op}_\iota(a) \Psi_\iota^*$. For future reference, we recall that the kernel of the latter operator is

$$(2\pi h)^{-n} \iint e^{\frac{i}{h}(r - r')\rho + \frac{i}{h}(\theta - \theta') \cdot \eta} a(r, \theta, \rho, \eta) \, d\rho \, d\eta \, \tilde{\chi}(r') \tilde{\chi}_\iota(\theta'). \tag{2-25}$$

The notation $\widehat{Op}_\iota$ refers to the following relation with the measure $\widehat{dG}$: if $a \in C_b^\infty(\mathbb{R}^{2n})$ satisfies (2-23), then

$$\| \widehat{Op}_\iota(a) \|_{L^2(\mathcal{M}, \widehat{dG}) \to L^2(\mathcal{M}, \widehat{dG})} \lesssim 1, \quad h \in (0, 1]. \tag{2-26}$$

This is a direct consequence of the Calderón–Vaillancourt theorem using (2-8) with $q = 2$, the inclusions in (2-20), and (2-21). In the "gauge" defined by dG, the latter gives

$$\| e^{-\gamma_n r} \widehat{Op}_\iota(a) e^{\gamma_n r} \|_{L^2(\mathcal{M}, dG) \to L^2(\mathcal{M}, dG)} \lesssim 1, \quad h \in (0, 1]. \tag{2-27}$$

Working with the measure $\widehat{dG}$ is to this extent more natural and avoids to deal with exponential weights.

We now describe the typical symbols we shall use in this paper. Using (2-17), (2-18), (2-19) and (2-21), we can write

$$h^2 P = \sum_{\iota \in \mathcal{J}} \widehat{Op}_\iota \left((\kappa \otimes \kappa_\iota) \times (p_\iota + h p_{\iota,1} + h^2 p_{\iota,0}) \right), \quad r > R_{\mathcal{K}} + 2, \tag{2-28}$$

using (2-13), (2-14) and (2-15). One observes that the symbols involved in (2-28) are of the form

$$a_\iota(r, \theta, \rho, \eta) = \tilde{a}_\iota(r, \theta, \rho, e^{-r}\eta), \tag{2-29}$$

with $\tilde{a}_\iota \in S^2(\mathbb{R}^n \times \mathbb{R}^n)$. It will turn out that the functional calculus of $h^2 P$ (or $h^2 \Delta_G$) will involve more generally symbols of this form with $a_\iota \in S^{-\infty}(\mathbb{R}^n \times \mathbb{R}^n)$. For instance, if $f \in C_0^\infty(\mathbb{R})$, the semiclassical principal symbol of $f(h^2 P)$ or $f(-h^2 \Delta_G)$ will be

$$f(\rho^2 + q_\iota(r, \theta, e^{-r}\eta)), \tag{2-30}$$

which, once multiplied by the cutoff $\kappa \otimes \kappa_\iota$, is of the form (2-29) with $\tilde{a}_\iota \in S^{-\infty}(\mathbb{R}^n \times \mathbb{R}^n)$. This type of symbols is the model of functions described in Definition 2.2 below. To state this definition, we introduce the notation

$$D^{j\alpha k\beta}_{\text{hyp}} := e^{r|\beta|} \partial_\eta^\beta \partial_r^j \partial_\theta^\alpha \partial_\rho^k,$$

for all $j, k \in \mathbb{N}_0$ and $\alpha, \beta \in \mathbb{N}_0^{n-1}$.

Definition 2.2. Given an open set $\Omega \subset T^*\mathbb{R}^n_+ = (0, +\infty)_r \times \mathbb{R}^{n-1}_\theta \times \mathbb{R}_\rho \times \mathbb{R}^{n-1}_\theta$, we define

$$\mathcal{B}_{\text{hyp}}(\Omega) = \left\{ a \in C^\infty(\Omega) : D^{j\alpha k\beta}_{\text{hyp}} a \in L^\infty(\Omega) \text{ for all } j, k \in \mathbb{N}_0,\ \alpha, \beta \in \mathbb{N}_0^{n-1} \right\}$$

and

$$\mathcal{S}_{\text{hyp}}(\Omega) = \left\{ a \in C^\infty(\mathbb{R}^{2n}) : \text{supp}(a) \subset \Omega \text{ and } a \in \mathcal{B}_{\text{hyp}}(\Omega) \right\}.$$

A family $(a_\nu)_{\nu \in \Lambda}$ is bounded in $\mathcal{B}_{\text{hyp}}(\Omega)$ if $(D^{j\alpha k\beta}_{\text{hyp}} a_\nu)_{\nu \in \Lambda}$ is bounded in $L^\infty(\Omega)$ for all j, k, α, β.

Note that considering $\Omega \subset T^*\mathbb{R}^n_+$ is not necessary but, since we shall work only in the region where $r \gg 1$, this will be sufficient.

Example 2.3. Consider the following diffeomorphism from $\mathbb{R}^{2n}$ onto itself

$$F_{\text{hyp}} : (r, \theta, \rho, \eta) \mapsto (r, \theta, \rho, e^{-r}\eta). \tag{2-31}$$

If $a_\iota \in S^0(\mathbb{R}^n \times \mathbb{R}^n)$ is supported in $F_{\text{hyp}}(\Omega)$, with $\Omega \subset T^*\mathbb{R}^n_+$, then (2-29) belongs to $\mathcal{S}_{\text{hyp}}(\Omega)$.

Proof. We only need to check that (2-29) belongs to $\mathcal{B}_{\text{hyp}}(\Omega)$. We have

$$\partial_r \left(\tilde{a}_\iota(r, \theta, \rho, e^{-r}\eta) \right) = (\partial_r \tilde{a}_\iota)(r, \theta, \rho, e^{-r}\eta) - e^{-r}\eta \cdot (\partial_\xi \tilde{a}_\iota)(r, \theta, \rho, \xi)_{|\xi = e^{-r}\eta},$$

which is bounded since $\xi \cdot \partial_\xi a_\iota$ is bounded. Similarly

$$e^r \partial_\eta \left(\tilde{a}_\iota(r, \theta, \rho, e^{-r}\eta) \right) = (\partial_\xi \tilde{a}_\iota)(r, \theta, \rho, \xi)_{|\xi = e^{-r}\eta},$$

is bounded too. Derivatives with respect to ρ, θ are harmless and higher-order derivatives in r, η are treated similarly. □

The next lemma gives a characterization of functions in $\mathcal{B}_{\text{hyp}}(\Omega)$.

Lemma 2.4. *Let $\Omega \subset T^*\mathbb{R}^n_+$ be an open subset and assume that*

$$F_{\mathrm{hyp}}(\Omega) \subset \mathbb{R}^n_+ \times B, \quad \text{with } B \text{ bounded}. \tag{2-32}$$

A function $a \in C^\infty(\Omega)$ is of the form

$$a(r,\theta,\rho,\eta) = \tilde{a}(r,\theta,\rho,e^{-r}\eta), \quad \text{with } \tilde{a} \in C_b^\infty(F_{\mathrm{hyp}}(\Omega)), \tag{2-33}$$

if and only if

$$D_{\mathrm{hyp}}^{j\alpha k\beta} a \in L^\infty(\Omega) \quad \text{for all } j,k,\alpha,\beta. \tag{2-34}$$

Here $C_b^\infty(\Omega)$ and $C_b^\infty(F_{\mathrm{hyp}}(\Omega))$ are spaces of smooth functions bounded with all derivatives bounded on Ω and $F_{\mathrm{hyp}}(\Omega)$, respectively.

Proof. That (2-33) implies (2-34) is proved in the same way as Example 2.3: the boundedness of $\xi \cdot \partial_\xi \tilde{a}$ follows from the boundedness of $\xi = e^{-r}\eta$ in $F_{\mathrm{hyp}}(\Omega)$ by (2-32) and the fact that $\tilde{a} \in C_b^\infty(F_{\mathrm{hyp}}(\Omega))$. Conversely, one checks by induction that

$$\tilde{a}(r,\theta,\rho,\xi) := a(r,\theta,\rho,e^r\xi),$$

belongs to $C_b^\infty(F_{\mathrm{hyp}}(\Omega))$, using again the boundedness of ξ on $F_{\mathrm{hyp}}(\Omega)$. ☐

Example 2.5. For all $f \in C_0^\infty(\mathbb{R}^n)$, all $R > R_{\mathcal{K}}$ and all $V \Subset \Psi_\iota(U_\iota)$, (2-30) satisfies the conditions of this lemma with $\Omega = (R,+\infty) \times V \times \mathbb{R}^n$.

Proof. By (1-8), there exists $C > 1$ such that

$$C^{-1}|\xi|^2 \lesssim q_\iota(r,\theta,\xi) \lesssim C|\xi|^2 \quad \text{for } r > R,\ \theta \in V,\ \xi \in \mathbb{R}^{n-1}, \tag{2-35}$$

and, using the notation (2-13),

$$|\partial_r^j \partial_\theta^\alpha g^{kl}(r,\theta)| \le C_{jk} \quad \text{for } r > R,\ \theta \in V. \tag{2-36}$$

Therefore, (2-35) and the compact support of f ensure that $e^{-r}\eta$ and ρ are bounded, hence that (2-32) holds on the support of (2-30). Then, (2-36) implies that $f(\rho^2 + q_\iota(r,\theta,\xi))$ belongs to $C_b^\infty(F_{\mathrm{hyp}}(\Omega))$ (notice that here $F_{\mathrm{hyp}}(\Omega) = (R,+\infty) \times V \times \mathbb{R}^n$). ☐

We conclude this subsection with the following useful remarks. If $a, b \in \mathcal{S}_{\mathrm{hyp}}(\Omega)$ for some Ω (such a, b satisfy (2-23)), we have the composition rule

$$\widehat{Op}_\iota(a)\widehat{Op}_\iota(b) = \widehat{Op}_\iota((a\#b)(h)), \tag{2-37}$$

if $(a\#b)(h)$ denotes the full symbol of $a(r,\theta,hD_r,hD_\theta)b(r,\theta,hD_r,hD_\theta)$. In particular all the terms of the expansion of $(a\#b)(h)$ belong to $\mathcal{S}_{\mathrm{hyp}}(\Omega)$ and are supported in $\mathrm{supp}(a) \cap \mathrm{supp}(b)$. Similarly, for all $N \ge 0$, we have

$$\widehat{Op}_\iota(a)^* = \widehat{Op}_\iota(a_0^* + \cdots + h^N a_N^*) + h^{N+1} R_N(a,h) \tag{2-38}$$

with $a_0^*, \ldots, a_N^* \in \mathcal{S}_{\mathrm{hyp}}(\Omega)$ supported in $\mathrm{supp}(a)$ and $\|R_N(a,h)\|_{L^2(\mathcal{M},\widehat{dG}) \to L^2(\mathcal{M},\widehat{dG})} \lesssim 1$ for $h \in (0,1]$.

2C. *The functional calculus.* In Proposition 2.7 below, we give two pseudodifferential approximations of $f(h^2 P)$ near infinity of $\mathcal{M}$, when $f \in C_0^\infty(\mathbb{R})$. The first approximation, namely (2-43), is given in terms of the "quantization" $\widehat{Op}_\iota$ defined in the previous subsection. This is the one we shall mostly use in this paper. However, at some crucial points, we shall need another approximation, (2-44), which uses *properly supported* ΨDOs.

To define such properly supported operators, we need a function

$$\zeta \in C_0^\infty(\mathbb{R}^n), \qquad \zeta \equiv 1 \text{ near } 0, \qquad \operatorname{supp}(\zeta) \text{ small enough,}$$

which will basically be used as a cutoff near the diagonal. The smallness of the support will be fixed in the following definition.

Definition 2.6. For $\iota \in \mathcal{I}$, all $h \in (0, 1]$ and all symbol a satisfying (2-23), we define

$$Op_{\iota,\mathrm{pr}}(a) : C_0^\infty(\mathcal{M}) \to C^\infty(\mathcal{M}),$$

as the unique operator with kernel supported in $\mathcal{U}_\iota^2$ and such that the kernel of $\Psi_\iota^* \widehat{Op}_\iota(a) \Psi_{\iota *}$ is

$$(2\pi h)^{-n} \iint e^{\frac{i}{h}(r-r')\rho + \frac{i}{h}(\theta-\theta')\cdot\eta} a(r, \theta, \rho, \eta)\, d\rho\, d\eta\, \zeta(r - r', \theta - \theta'). \tag{2-39}$$

The advantage of choosing the support of ζ small enough is that, using (2-23), we can assume that, on the support of (2-39), r' belongs to a neighborhood of $[R_{\mathcal{K}}+1, +\infty)$ and θ' belongs to a neighborhood of V_ι'. For instance, we may assume that $r' \in \tilde{\kappa}^{-1}(1)$ and $\theta' \in \tilde{\kappa}_\iota^{-1}(1)$ so that we can put a factor $\tilde{\kappa}(r')\tilde{\kappa}_\iota(\theta')$ for free to the right-hand side of (2-39). The latter implies, using (2-8), (2-25), and (2-39), the standard off-diagonal fast decay of kernels of ΨDOs and the Calderón–Vaillancourt theorem stating that, for all $a \in C_b^\infty(\mathbb{R}^{2n})$ satisfying (2-23) and all $N \in \mathbb{N}_0$, we have

$$\|\widehat{Op}_\iota(a) - Op_{\iota,\mathrm{pr}}(a)\|_{L^2(\mathcal{M},\widehat{dG}) \to L^2(\mathcal{M},\widehat{dG})} \lesssim h^N, \qquad h \in (0, 1]. \tag{2-40}$$

This shows that, up to remainders of size h^∞, $\widehat{Op}_\iota(a)$ and $Op_{\iota,\mathrm{pr}}(a)$ coincide as bounded operators on $L^2(\mathcal{M}, \widehat{dG})$. Under the same assumptions on a, we also have

$$\|Op_{\iota,\mathrm{pr}}(a)\|_{L^2(\mathcal{M},dG) \to L^2(\mathcal{M},dG)} \lesssim 1, \qquad h \in (0, 1], \tag{2-41}$$

which is a first difference with $\widehat{Op}_\iota(a)$ for which we have only (2-27) in general. The estimate (2-41) is equivalent to the uniform boundedness (with respect to $h \in (0, 1]$) of $e^{\gamma_n r} Op_{\iota,pr}(a) e^{-\gamma_n r}$ on $L^2(\mathcal{M}, \widehat{dG})$. The latter is obtained similarly to (2-26), using the Calderón–Vaillancourt theorem, for we only have to consider the kernel obtained by multiplying (2-39) by $e^{\gamma_n (r-r')}$, which is bounded (as well as its derivatives) on the support of $\zeta(r - r', \theta - \theta')$.

In other words, (2-41) can be interpreted as a boundedness result between (exponentially) weighted L^2 spaces. Similar properties holds for L^q spaces (under suitable assumptions on the symbol a) and they are the main reason for considering properly supported operators. In particular, they lead to following proposition, where we collect the estimates we shall need in this paper. We refer to [Bouclet 2007] for the proof.

Proposition 2.7. *Let $f \in C_0^\infty(\mathbb{R})$ and let $I \Subset (0,+\infty)$ be an open interval containing* $\operatorname{supp}(f)$. *Let $\chi_{\mathcal{K}} \in C_0^\infty(\mathcal{M})$ and $R > R_{\mathcal{K}} + 1$ be such that*

$$\chi_{\mathcal{K}}(m) = 1 \quad \text{if } r(m) \leq R+1.$$

Then, for all $N \geq 0$ and all $\iota \in \mathcal{I}$, we can find symbols

$$a_{\iota,0}(f), \dots, a_{\iota,N}(f) \in \mathcal{S}_{\text{hyp}}\big((R,+\infty) \times V_\iota \times \mathbb{R}^n \cap p_\iota^{-1}(I)\big) \tag{2-42}$$

(*where p_ι is the principal symbol of P in the chart $\mathcal{U}_\iota$*) *such that, if we set*

$$a_\iota^{(N)}(f,h) = a_{\iota,0}(f) + h a_{\iota,1}(f) + \dots + h^N a_{\iota,N}(f),$$

we have

$$(1-\chi_{\mathcal{K}}) f(h^2 P) = \sum_{\iota \in \mathcal{I}} \widehat{Op}_\iota(a_\iota^{(N)}(f,h)) + h^{N+1} \widehat{R}_N(f,h), \tag{2-43}$$

$$(1-\chi_{\mathcal{K}}) f(h^2 P) = \sum_{\iota \in \mathcal{I}} Op_{\iota,\text{pr}}(a_\iota^{(N)}(f,h)) + h^{N+1} R_{N,\text{pr}}(f,h), \tag{2-44}$$

where, for each $q \in [2,\infty]$,

$$\|e^{-\gamma_n r} R_{N,\text{pr}}(f,h)\|_{L^2(\mathcal{M},\widehat{dG}) \to L^q(\mathcal{M},dG)} \lesssim h^{-n(\frac{1}{2}-\frac{1}{q})} \quad \text{for } h \in (0,1], \tag{2-45}$$

and

$$\|\widehat{R}_N(f,h)\|_{L^2(\mathcal{M},\widehat{dG}) \to L^2(\mathcal{M},\widehat{dG})} \lesssim 1 \quad \text{for } h \in (0,1]. \tag{2-46}$$

In addition, for all $\iota \in \mathcal{I}$ and all $q \in [2,\infty]$, we have

$$\|e^{-\gamma_n r} Op_{\iota,\text{pr}}(a_\iota^{(N)}(f,h))\|_{L^2(\mathcal{M},\widehat{dG}) \to L^q(\mathcal{M},dG)} \lesssim h^{-n(\frac{1}{2}-\frac{1}{q})} \quad \text{for } h \in (0,1], \tag{2-47}$$

and, for all $q \in [1,\infty]$ and all $\gamma \in \mathbb{R}$,

$$\|e^{-\gamma r} Op_{\iota,\text{pr}}(a_\iota^{(N)}(f,h)) e^{\gamma r}\|_{L^q(\mathcal{M},\widehat{dG}) \to L^q(\mathcal{M},\widehat{dG})} \lesssim 1 \quad \text{for } h \in (0,1]. \tag{2-48}$$

In this proposition, as well as in further definitions or propositions, the interval I can be considered as a semiclassical energy window, in the sense that the principal symbol of $h^2 P$ will live in I. In the sequel, I will be more explicitly of the form $\left(\frac{1}{4}, 4\right)$ or $\left(\frac{1}{4} - \varepsilon,\ 4 + \varepsilon\right)$; see for instance (2-54).

To make (2-42) more explicit, let us quote for instance that

$$a_{\iota,0}(f)(r,\theta,\rho,\eta) = \kappa(r)\kappa_\iota(\theta) f(\rho^2 + q_\iota(r,\theta,e^{-r}\eta)) \times (1-\chi_{\mathcal{K}})(\Psi_\iota^{-1}(r,\theta)).$$

More generally, (2-42) and Lemma 2.4 show that $a_{\iota,0}(f), \dots, a_{\iota,N}(f)$ are of the form (2-29), with $\tilde{a}_\iota(r,\theta,\rho,\xi)$ compactly supported with respect to (ρ,ξ).

The estimate (2-48) basically means that $Op_{\iota,\text{pr}}(a_\iota^{(N)}(f,h))$ preserves all L^q spaces with any exponential weights. In particular, since $L^q(\mathcal{M},dG) = e^{-\gamma_n r/q} L^q(\mathcal{M},\widehat{dG})$, replacing $\widehat{dG}$ by dG in (2-48) would give a completely equivalent statement. This estimate is the main reason for introducing properly supported operators. Of course, (2-48) holds for other symbols than those involved in the functional calculus of P. We have more generally (see [Bouclet 2007]) for all $\gamma \in \mathbb{R}$,

$$\|e^{-\gamma r} Op_{\iota,\text{pr}}(a_\iota) e^{\gamma r}\|_{L^q(\mathcal{M},\widehat{dG}) \to L^q(\mathcal{M},\widehat{dG})} \lesssim 1 \quad \text{for } h \in (0,1], \tag{2-49}$$

for any $q \in [1, \infty]$ and any

$$a_\iota \in \mathcal{S}_{\text{hyp}}\left((R_{\mathcal{K}}+1, +\infty) \times V'_\iota \times \mathbb{R}^n \cap p_\iota^{-1}(I')\right),$$

provided I' is bounded.

By the unitary equivalence of P and $-\Delta_G - \gamma_n^2$, we would get a very similar pseudodifferential expansion for $f(-h^2\Delta_G)$. (Here we have only described $(1-\chi_{\mathcal{K}})f(h^2P)$ since this will be sufficient for our present purpose, but of course there is a completely analogous result for the compactly supported part $\chi_{\mathcal{K}} f(h^2P)$; see [Bouclet 2007].) Such an approximation of $f(-h^2\Delta_G)$ was used in [Bouclet 2010] to prove the next two propositions.

Proposition 2.8. *Consider a dyadic partition of unit*

$$1 = f_0(\lambda) + \sum_{k\geq 0} f(2^{-k}\lambda),$$

for λ in a neighborhood of $[0, +\infty)$, with

$$f_0 \in C_0^\infty(\mathbb{R}), \qquad f \in C_0^\infty\left([\tfrac{1}{4}, 4]\right). \tag{2-50}$$

Then, for all $\chi \in C_0^\infty(\mathcal{M})$ and all $q \in [2, \infty)$, we have

$$\|(1-\chi)u\|_{L^q(\mathcal{M},dG)} \lesssim \Bigg(\sum_{\substack{h^2=2^{-k}\\ k\geq 0}} \|(1-\chi)f(-h^2\Delta_G)u\|^2_{L^q(\mathcal{M},dG)}\Bigg)^{1/2} + \|u\|_{L^2(\mathcal{M},dG)}.$$

This proposition leads to the following classical reduction.

Proposition 2.9. *Let $\chi \in C_0^\infty(\mathcal{M})$ and (p, q) be an admissible pair. Then (1-10) holds true if and only if there exists C such that*

$$\|(1-\chi)e^{it\Delta_G} f(-h^2\Delta_G)u_0\|_{L^p([0,1];L^q(\mathcal{M},dG))} \leq C\|u_0\|_{L^2(\mathcal{M},dG)}, \tag{2-51}$$

for all $h \in (0, 1]$ and $u_0 \in C_0^\infty(\mathcal{M})$.

This result is essentially well known and proved in [Bouclet 2010] for a class of noncompact manifolds. We simply recall here that the $L^q \to L^q$ boundedness of the spectral cutoffs $f(-h^2\Delta_G)$ is not necessary to prove this result, although the latter slightly simplifies the proof when it is available.

2D. *Outgoing and incoming areas.* Propositions 2.7 and 2.9 lead to a microlocalization of Theorem 1.2: as we shall see more precisely in Section 2E, they allow to reduce the proof of (1-10) to the same estimate in which $1-\chi$ is replaced by h-ΨDOs. But this microlocalization, i.e., the support of the symbols in (2-42), is still too rough to simplify the proof of Theorem 1.2 in a significant way. The purpose of this subsection is to describe convenient regions which will refine this localization.

Definition 2.10. Fix $\iota \in \mathcal{I}$. For $R > R_{\mathcal{K}} + 1$, an open subset $V \Subset V'_\iota$ (see (2-21)), an open interval $I \Subset (0, +\infty)$ and $\sigma \in (-1, 1)$, we define

$$\Gamma_\iota^\pm(R, V, I, \sigma) = \left\{(r, \theta, \rho, \eta) \in \mathbb{R}^{2n} : r > R,\ \theta \in V,\ p_\iota \in I,\ \pm\rho > -\sigma p_\iota^{1/2}\right\},$$

where p_ι is the principal symbol of P in the chart $\mathcal{U}_\iota$ given by (2-14). The open set $\Gamma_\iota^+(R, V, I, \sigma)$ is called an outgoing area, and $\Gamma_\iota^-(R, V, I, \sigma)$ an incoming area.

We note in passing that, except from the localization in θ, these areas are defined using only the variable r, its dual ρ and the principal symbol of P. In particular, up to the choice of the coordinate r, the conditions $r > R$, $p_\iota \in I$ and $\pm\rho > -\sigma p_\iota^{1/2}$ define invariant subsets of $T^*\mathcal{M}$. However the whole analysis in this paper will be localized in charts and we will not use this invariance property.

Let us record some useful properties of outgoing/incoming areas. First, they decrease with respect to V, I, σ and R^{-1}:

$$R_1 \geq R_2, \quad V_1 \subset V_2, \quad I_1 \subset I_2, \quad \sigma_1 \leq \sigma_2 \quad \Longrightarrow \quad \Gamma_\iota^\pm(R_1, V_1, I_1, \sigma_1) \subset \Gamma_\iota^\pm(R_2, V_2, I_2, \sigma_2). \tag{2-52}$$

Second, we have

$$\Gamma_\iota^+\big(R, V, I, \tfrac{1}{2}\big) \cup \Gamma_\iota^-\big(R, V, I, \tfrac{1}{2}\big) = (R, +\infty) \times V \times \mathbb{R}^n \cap p_\iota^{-1}(I). \tag{2-53}$$

Here we have chosen $\sigma = \frac{1}{2}$ but any $\sigma \in (0, 1)$ would work as well.

We will use the following elementary property, proved later as part (ii) of Proposition 4.1.

Proposition 2.11. *Any symbol $a \in \mathcal{S}_{\mathrm{hyp}}\big((R, +\infty) \times V \times \mathbb{R}^n \cap p_\iota^{-1}(I)\big)$ can be written as*

$$a = a^+ + a^-, \qquad \textit{with} \quad a^\pm \in \mathcal{S}_{\mathrm{hyp}}\big(\Gamma_\iota^\pm(R, V, I, \tfrac{1}{2})\big).$$

This splitting into outgoing/incoming areas was sufficient to use the Isozaki–Kiada parametrix in the asymptotically Euclidean case; in the AH case, we will only be able to construct this parametrix in much smaller areas, called *strongly* outgoing/incoming areas, which we now introduce.

We first describe briefly the meaning of such areas, say in the outgoing case. Basically, being in an outgoing area means that ρ is *not too close to* $-p^{1/2}$; the aim of strongly outgoing areas is to guarantee that ρ is *very close to* $p^{1/2}$, which is of course a much stronger restriction. This amounts essentially to chose σ close to -1 in the definition of outgoing areas. We will measure this closeness in term of a small parameter ε. It will actually be convenient to have the other parameters, namely R, V, I, depending also on ε, so we introduce

$$R(\varepsilon) = 1/\varepsilon, \quad V_{\iota,\varepsilon} = \{\theta \in \mathbb{R}^{n-1} : \mathrm{dist}(\theta, V_\iota) < \varepsilon^2\}, \quad I(\varepsilon) = \big(\tfrac{1}{4} - \varepsilon, 4 + \varepsilon\big), \tag{2-54}$$

where we recall that V_ι is defined in (2-21).

Definition 2.12. For all $\varepsilon > 0$ small enough, we set

$$\Gamma_{\iota,\mathrm{s}}^\pm(\varepsilon) := \Gamma_\iota^\pm\big(R(\varepsilon), V_{\iota,\varepsilon}, I(\varepsilon), \varepsilon^2 - 1\big).$$

The open set $\Gamma_{\iota,\mathrm{s}}^+(\varepsilon)$ is called a strongly outgoing area, and $\Gamma_{\iota,\mathrm{s}}^-(\varepsilon)$ a strong incoming area.

The main interest of such areas is to ensure that $e^{-r}|\eta|$ is small if ε is small. Indeed, if $q \in [0, +\infty)$ and $-1 < \sigma < 0$, we have the equivalence

$$\pm\rho > -\sigma(\rho^2 + q)^{1/2} \quad \Longleftrightarrow \quad \pm\rho > 0 \ \text{ and } \ q < \sigma^{-2}(1 - \sigma^2)\rho^2. \tag{2-55}$$

Therefore, there exists C such that, for all ε small enough and $(r, \theta, \rho, \eta) \in \Gamma_{\iota,\mathrm{s}}^\pm(\varepsilon)$,

$$q_\iota(r, \theta, e^{-r}\eta) \leq C\varepsilon^2,$$

which, by (2-35), is equivalent to

$$|e^{-r}\eta| \lesssim \varepsilon. \tag{2-56}$$

Note also that, by (2-52), strongly outgoing/incoming areas decrease with ε.

We now quote a result that motivates, at least partially, the introduction of strongly outgoing/incoming areas. Its proof is given in Section 4A.

Denote by Φ_ι^t the Hamiltonian flow of p_ι. This is of course the geodesic flow written in the chart $\Psi_\iota(\mathcal{U}_\iota) \times \mathbb{R}^n$ of $T^*\mathcal{M}$.

Proposition 2.13. *Fix $\sigma \in (-1, 1)$. There exists $R'_\sigma > 0$ such that for all $R \geq R'_\sigma$ and all $\varepsilon > 0$ small enough, there exists $t_{R,\varepsilon} \geq 0$ such that*

$$\Phi_\iota^t\big(\Gamma_\iota^\pm(R, V_\iota, (\tfrac{1}{4} - \varepsilon, 4 + \varepsilon), \sigma)\big) \subset \Gamma_{\iota,s}^\pm(\varepsilon) \quad \textit{if } \pm t \geq t_{R,\varepsilon}.$$

In particular, for all $\varepsilon > 0$ small enough, there exists $T_\varepsilon > 0$ such that

$$\Phi_\iota^t\big(\Gamma_\iota^\pm(R(\varepsilon), V_\iota, I(\varepsilon), \sigma)\big) \subset \Gamma_{\iota,s}^\pm(\varepsilon) \quad \textit{if } \pm t \geq T_\varepsilon. \tag{2-57}$$

Note that, since p_ι is only defined in the chart $\Psi_\iota(\mathcal{U}_\iota) \times \mathbb{R}^n$, its flow is not complete. We shall however see in Section 3 that, for any initial data $(r, \theta, \rho, \eta) \in \Gamma_\iota^\pm(R(\varepsilon), V_\iota, I, \sigma)$, $\Phi_\iota^t(r, \theta, \rho, \eta)$ is well defined for all $\pm t \geq 0$; that is, $\Phi_\iota^t(r, \theta, \rho, \eta) \in \Psi_\iota(\mathcal{U}_\iota) \times \mathbb{R}^n$ for all $\pm t \geq 0$.

Proposition 2.13 essentially states that the forward flow sends outgoing areas into strongly outgoing areas in finite time, and likewise the backward flow sends incoming areas into strongly incoming ones. This will be interesting for the proof of Proposition 2.24.

The last type of region we need to consider are the *intermediate* areas. They should have two properties: firstly they should essentially cover the complement of strongly outgoing/incoming areas in outgoing/incoming areas and, secondly, be small enough.

To define them we need the following. For all $\varepsilon > 0$ and all $\delta > 0$, we can find $L + 1$ real numbers, $\sigma_0, \dots, \sigma_L$,

$$(\varepsilon/2)^2 - 1 = \sigma_0 < \sigma_1 < \cdots < \sigma_L = \tfrac{1}{2}, \tag{2-58}$$

such that

$$\big((\varepsilon/2)^2 - 1, \tfrac{1}{2}\big) = \bigcup_{l=1}^{L-1} (\sigma_{l-1}, \sigma_{l+1}) \tag{2-59}$$

and

$$|\sigma_{l+1} - \sigma_{l-1}| \leq \delta. \tag{2-60}$$

Note that the intervals overlap in (2-59), since $(\sigma_{l-1}, \sigma_{l+1})$ always contains σ_l.

Definition 2.14. The intermediate outgoing and incoming areas associated to the cover (2-59) are

$$\Gamma_{\iota,\mathrm{inter}}^\pm(\varepsilon, \delta; l) := \big\{(r, \theta, \rho, \eta) \in \mathbb{R}^{2n} : r > R(\varepsilon),\ \theta \in V_\iota,\ p_\iota \in I(\varepsilon),\ \pm\rho/p_\iota^{1/2} \in (-\sigma_{l+1}, -\sigma_{l-1})\big\},$$

for $1 \leq l \leq L - 1$.

Notice that, by definition,

$$\Gamma_{\iota,\mathrm{inter}}^\pm(\varepsilon, \delta; l) \subset \Gamma_\iota^\pm\big(R(\varepsilon), V_\iota, I(\varepsilon), \tfrac{1}{2}\big). \tag{2-61}$$

In the notation, we only specify the parameters which are relevant for our analysis, namely ε, δ, but, of course, intermediate areas depend on the choice of $\sigma_1, \dots, \sigma_L$. Here δ measures the smallness and Proposition 2.16 below will explain how to choose this parameter.

Proposition 2.15. *Fix $\varepsilon > 0$ small enough, $\delta > 0$ and $\sigma_0, \dots, \sigma_L$ satisfying (2-58), (2-59) and (2-60). Then, any symbol*

$$a^{\pm} \in \mathcal{S}_{\mathrm{hyp}}\big(\Gamma_{\iota}^{\pm}(R(\varepsilon), V_{\iota}, I(\varepsilon), \tfrac{1}{2})\big)$$

can be written as

$$a^{\pm} = a_{\mathrm{s}}^{\pm} + a_{1,\mathrm{inter}}^{\pm} + \cdots + a_{L-1,\mathrm{inter}}^{\pm},$$

with $a_{\mathrm{s}}^{\pm} \in \mathcal{S}_{\mathrm{hyp}}(\Gamma_{\iota,\mathrm{s}}^{\pm}(\varepsilon))$ and $a_{l,\mathrm{inter}}^{\pm} \in \mathcal{S}_{\mathrm{hyp}}(\Gamma_{\iota,\mathrm{inter}}^{\pm}(\varepsilon, \delta; l))$.

The proof is given in Section 4A.

We conclude this subsection with the following proposition which will be crucial for the proof of Theorem 1.2 and motivates the introduction of intermediate areas. The proof is given in Section 4A.

Proposition 2.16. *Fix $\underline{t} > 0$. Then, for all $\varepsilon > 0$ small enough, we can find $\delta > 0$ small enough such that, for any choice of $\sigma_0, \dots, \sigma_L$ satisfying (2-58), (2-59) and (2-60), we have, for all $1 \le l \le L-1$,*

$$\Phi_{\iota}^{t}\big(\Gamma_{\iota,\mathrm{inter}}^{\pm}(\varepsilon, \delta; l)\big) \cap \Gamma_{\iota,\mathrm{inter}}^{\pm}(\varepsilon, \delta; l) = \varnothing, \tag{2-62}$$

provided that

$$\pm t \ge \underline{t}.$$

2E. *The main steps of the proof of Theorem 1.2.* We already know from Proposition 2.9 that we only have to find $\chi \in C_0^{\infty}(\mathcal{M})$ such that (2-51) holds, which is equivalent to

$$\|e^{-\gamma_n r}(1-\chi) f(h^2 P) e^{-itP} u_0\|_{L^p([0,1];L^q(\mathcal{M},dG))} \le C \|u_0\|_{L^2(\mathcal{M},\widehat{dG})}, \tag{2-63}$$

using the unitary map (2-9) and (2-11), (2-12).

Before choosing χ, we introduce the following operators. Choose a cutoff $\tilde{f} \in C_0^{\infty}((0,+\infty))$ such that $\tilde{f} f = f$.

Lemma 2.17. *For all $\chi \in C_0^{\infty}(\mathcal{M})$, we can write*

$$(1-\chi)\tilde{f}(h^2 P) = (1-\chi) A_{\mathrm{pr}}(h) + R(h)$$

with $R(h)$ satisfying, for all $q \in [2, \infty]$,

$$\|e^{-\gamma_n r} R(h)\|_{L^2(\mathcal{M},\widehat{dG}) \to L^q(\mathcal{M},dG)} \lesssim 1, \tag{2-64}$$

and $A_{\mathrm{pr}}(h)$ such that, for all $q \in [2, \infty]$,

$$\|e^{-\gamma_n r} A_{\mathrm{pr}}(h)\|_{L^2(\mathcal{M},\widehat{dG}) \to L^q(\mathcal{M},dG)} \lesssim h^{-n\left(\frac{1}{2}-\frac{1}{q}\right)}, \tag{2-65}$$

$$\|e^{-\gamma_n r} A_{\mathrm{pr}}(h) e^{\gamma_n r}\|_{L^{\infty}(\mathcal{M},dG) \to L^{\infty}(\mathcal{M},dG)} \lesssim 1, \tag{2-66}$$

$$\|A_{\mathrm{pr}}(h)^{*} e^{-\gamma_n r}\|_{L^1(\mathcal{M},\widehat{dG}) \to L^2(\mathcal{M},\widehat{dG})} \lesssim h^{-n/2}, \tag{2-67}$$

$$\|e^{\gamma_n r} A_{\mathrm{pr}}(h)^{*} e^{-\gamma_n r}\|_{L^1(\mathcal{M},\widehat{dG}) \to L^1(\mathcal{M},\widehat{dG})} \lesssim 1. \tag{2-68}$$

Proof. This is an immediate consequence of Proposition 2.7. Using (2-44), with N such that $N+1 \ge n/2$, we define $A_{\mathrm{pr}}(h)$ as the sum of the properly supported pseudodifferential operators. We thus have (2-64), (2-65) and (2-66). The estimates (2-67) and (2-68) are obtained by taking the adjoints (with $q = \infty$ in (2-65)) with respect to $\widehat{dG}$. □

Basically, the operators $e^{-\gamma_n r} A_{\rm pr}(h)$ and $A_{\rm pr}(h)^* e^{-\gamma_n r}$ will be used as "ghost cutoffs" to deal with remainder terms of parametrices which will be $\mathcal{O}(h^N)$ in $\mathcal{L}(L^2(\mathcal{M}, \widehat{dG}))$, using the Sobolev embeddings (2-65) and (2-67). They will be "transparent" for the principal terms of the parametrices by (2-66) and (2-68), which uses crucially that they are properly supported.

For ε to be fixed below, we choose $\chi \in C_0^\infty(\mathcal{M})$ such that

$$\chi \equiv 1 \quad \text{for } r(m) \le 3\varepsilon^{-1}.$$

This function will appear in Proposition 2.18 below only trough its support. More precisely, the proposition states that to prove (2-63) for such a χ (with ε small enough), it is sufficient to prove the estimate (2-70) for a class of symbols supported where $r(m) \ge \varepsilon^{-1}$.

Proposition 2.18 (Microlocalization of Strichartz estimates). *To prove (2-63), it is sufficient to show that, for some ε small enough and all*

$$a_\iota \in \mathcal{S}_{\rm hyp}\left((R(\varepsilon), +\infty) \times V_\iota \times \mathbb{R}^n \cap p_\iota^{-1}(I(\varepsilon))\right), \tag{2-69}$$

where we recall that $R(\varepsilon) = \varepsilon^{-1}$ and $I(\varepsilon) = \left(\frac{1}{4} - \varepsilon, 4 + \varepsilon\right)$, we have

$$\|e^{-\gamma_n r} A_{\rm pr}(h) \widehat{Op}_\iota(a_\iota) e^{-itP} u_0\|_{L^p([0,1]; L^q(\mathcal{M}, dG))} \le C \|u_0\|_{L^2(\mathcal{M}, \widehat{dG})}, \tag{2-70}$$

uniformly with respect to $h \in (0, 1]$.

Proof. Choose $\chi_0 \in C_0^\infty(\mathcal{M})$ such that

$$\chi_0 \equiv 1 \quad \text{for } r(m) \le \varepsilon^{-1},$$
$$\chi_0 \equiv 0 \quad \text{for } r(m) \ge 2\varepsilon^{-1}.$$

We then have $(1 - \chi_0) \equiv 1$ near $\text{supp}(1 - \chi)$ so, by the proper support of the kernel of $A_{\rm pr}(h)$, we also have

$$(1 - \chi) A_{\rm pr}(h) = (1 - \chi) A_{\rm pr}(h)(1 - \chi_0),$$

at least for ε small enough. The latter and (2-64) reduces the proof of (2-63) to the study of

$$e^{-\gamma_n r} A_{\rm pr}(h)(1 - \chi_0) f(h^2 P) e^{-itP}.$$

By splitting $(1 - \chi_0) f(h^2 P)$ using (2-43) with $N + 1 \ge n/2$, we obtain the result using (2-46) and (2-65). ☐

We now introduce a second small parameter $\delta > 0$. By Propositions 2.11 and 2.15, for all $\delta > 0$, any a_ι satisfying (2-69) can be written as

$$a_\iota = a_{\rm s}^+ + a_{\rm s}^- + \sum_{l=1}^{L-1} a_{l,\rm inter}^+ + a_{l,\rm inter}^-, \tag{2-71}$$

with

$$a_{\rm s}^\pm \in \mathcal{S}_{\rm hyp}(\Gamma_{\iota,\rm s}^\pm(\varepsilon)), \quad a_{l,\rm inter}^\pm \in \mathcal{S}_{\rm hyp}(\Gamma_{\iota,\rm inter}^\pm(\varepsilon, \delta; l)). \tag{2-72}$$

Proposition 2.19 (Reduction to microlocalized dispersion estimates). *To prove (2-70), it is sufficient to show that, for some ε and δ small enough, we have*

$$\left\| e^{-\gamma_n r} A_{\mathrm{pr}}(h) \widehat{Op}_\iota(a_{\mathrm{s}}^{\pm}) e^{-ithP} \widehat{Op}_\iota(a_{\mathrm{s}}^{\pm})^* A_{\mathrm{pr}}(h)^* e^{-\gamma_n r} \right\|_{L^1(\widehat{dG}) \to L^\infty(dG)} \leq C_\varepsilon |ht|^{-n/2}, \tag{2-73}$$

$$\left\| e^{-\gamma_n r} A_{\mathrm{pr}}(h) \widehat{Op}_\iota(a_{l,\mathrm{inter}}^{\pm}) e^{-ithP} \widehat{Op}_\iota(a_{l,\mathrm{inter}}^{\pm})^* A_{\mathrm{pr}}(h)^* e^{-\gamma_n r} \right\|_{L^1(\widehat{dG}) \to L^\infty(dG)} \leq C_{\varepsilon,\delta} |ht|^{-n/2}, \tag{2-74}$$

for

$$h \in (0, 1] \quad \textit{and} \qquad 0 \leq \pm t \leq 2h^{-1}. \tag{2-75}$$

Recall that the important point in this lemma is (2-75), i.e., that it is sufficient to consider $t \geq 0$ for outgoing localizations, and $t \leq 0$ for incoming ones.

Proof. Define

$$T_{\mathrm{s}}^{\pm}(t, h, \varepsilon) = e^{-\gamma_n r} A_{\mathrm{pr}}(h) \widehat{Op}_\iota(a_{\mathrm{s}}^{\pm}) e^{-itP}, \qquad T_{l,\mathrm{inter}}^{\pm}(t, h, \varepsilon, \delta) = e^{-\gamma_n r} A_{\mathrm{pr}}(h) \widehat{Op}_\iota(a_{l,\mathrm{inter}}) e^{-itP}.$$

By (2-26) and (2-65) (with $q = 2$), we have,

$$\|T_{\mathrm{s}}^{\pm}(t, h, \varepsilon)\|_{L^2(\widehat{dG}) \to L^2(dG)} + \|T_{l,\mathrm{inter}}^{\pm}(t, h, \varepsilon, \delta)\|_{L^2(\widehat{dG}) \to L^2(dG)} \leq C_{\varepsilon,\delta} \quad \text{for } h \in (0, 1],\ t \in \mathbb{R};$$

hence by the Keel–Tao Theorem [1998], the inequality (2-70) would follow from the estimates

$$\|T_{\mathrm{s}}^{\pm}(t, h, \varepsilon) T_{\mathrm{s}}^{\pm}(s, h, \varepsilon)^*\|_{L^1(dG) \to L^\infty(dG)} \leq C_\varepsilon |t - s|^{-n/2}, \tag{2-76}$$

$$\|T_{l,\mathrm{inter}}^{\pm}(t, h, \varepsilon) T_{l,\mathrm{inter}}^{\pm}(s, h, \varepsilon)^*\|_{L^1(dG) \to L^\infty(dG)} \leq C_{\varepsilon,\delta} |t - s|^{-n/2}, \tag{2-77}$$

for $h \in (0, 1]$ and $t, s \in [0, 1]$. Using the time rescaling $t \mapsto ht$, the equality $L^1(dG) = e^{-2\gamma_n r} L^1(\widehat{dG})$, and the fact that the adjoint of (2-9) is given by $e^{\gamma_n r}$, we see that (2-76) and (2-77) are respectively equivalent to (2-73) and (2-74), for $h \in (0, 1]$ and $|t| \leq 2h^{-1}$.

The reduction (2-75) to $\pm t \geq 0$ is obtained similarly to [Bouclet and Tzvetkov 2007, Lemma 4.3]. We only recall here that it is based on the simple observation that the operators $T(t)T(s)^*$ considered above are of the form $Be^{-i(t-s)P}B^*$, so L^∞ bounds on their Schwartz kernel for $\pm(t-s) \geq 0$ give automatically bounds for $\pm(t - s) \leq 0$ by taking the adjoints. □

As we shall see, there are basically two reasons for choosing ε small enough. The next result is the first condition.

Proposition 2.20 (Time h^{-1} Isozaki–Kiada parametrix). *For all $\varepsilon > 0$ small enough and all $a_{\mathrm{s}}^{\pm}$ in $\mathcal{S}_{\mathrm{hyp}}(\Gamma_{\iota,\mathrm{s}}^{\pm}(\varepsilon))$, we can write*

$$e^{-ithP} \widehat{Op}_\iota(a_{\mathrm{s}}^{\pm})^* = E_{\mathrm{IK}}^{\pm}(t, h) + h^n R_{\mathrm{IK}}^{\pm}(t, h),$$

with

$$\|e^{-\gamma_n r} E_{\mathrm{IK}}^{\pm}(t, h) e^{-\gamma_n r}\|_{L^1(\widehat{dG}) \to L^\infty(dG)} \lesssim |ht|^{-n/2}, \tag{2-78}$$

$$\|R_{\mathrm{IK}}^{\pm}(t, h)\|_{L^2(\widehat{dG}) \to L^2(\widehat{dG})} \lesssim 1, \tag{2-79}$$

for

$$h \in (0, 1], \quad 0 \leq \pm t \leq 2h^{-1}.$$

Proof. By (2-38), the result follows from Theorem 5.1 and by a stationary phase argument justified by Propositions 7.2, 7.3, 7.6, Lemma 7.9 and Propositions 7.11, 7.12. □

Proposition 2.20 is mainly an application of the Isozaki–Kiada parametrix. It has the following consequence.

Proposition 2.21 (Time h^{-1} strongly incoming/outgoing dispersion estimates). *For all $\varepsilon > 0$ sufficiently small, (2-73) holds for all h, t satisfying (2-75).*

Proof. We first replace $\widehat{Op}_\iota(a_s^\pm)$ by $Op_{\iota,\mathrm{pr}}(a_s^\pm)$ to the left of e^{-ithP} in (2-73). The remainder term, which is $\mathbb{O}(h^\infty)$ in $\mathcal{L}(L^2(\widehat{dG}))$ by (2-40), produces a term of size $\mathbb{O}(h^\infty)$ in $\mathcal{L}(L^1(\widehat{dG}), L^\infty(dG))$ using (2-65) (with $q = \infty$) and (2-67). We then use Proposition 2.20: the remainder term satisfies

$$\left\| e^{-\gamma_n r} A_{\mathrm{pr}}(h) Op_{\iota,\mathrm{pr}}(a_s^\pm) e^{-ithP} h^n R_{\mathrm{IK}}^\pm(t,h) A_{\mathrm{pr}}(h)^* e^{-\gamma_n r} \right\|_{L^1(\widehat{dG})\to L^\infty(dG)} \lesssim 1 \lesssim |ht|^{-d/2},$$

and the main term $E_{\mathrm{IK}}^\pm(t,h)$ gives the expected contribution via the use of (2-66), (2-68), and (2-49) for $Op_{\iota,\mathrm{pr}}(a_s^\pm)$. □

The second condition on ε will come from Proposition 2.24. It uses Proposition 2.16 which depends on some fixed small time which will be given by the following result.

Proposition 2.22 (Time 1 geometric optics). *There exists $t_{\mathrm{WKB}} > 0$ such that, for all $\varepsilon > 0$ small enough and all symbol $a^\pm \in \mathcal{S}_{\mathrm{hyp}}\left(\Gamma_\iota^\pm(R(\varepsilon), V_\iota, I, \tfrac{1}{2})\right)$, we can write*

$$e^{-ithP}\widehat{Op}_\iota(a^\pm)^* = E_{\mathrm{WKB}}^\pm(t,h) + h^n R_{\mathrm{WKB}}^\pm(t,h),$$

with

$$\begin{gathered} \|e^{-\gamma_n r} E_{\mathrm{WKB}}^\pm(t,h) e^{-\gamma_n r}\|_{L^1(\widehat{dG})\to L^\infty(dG)} \lesssim |ht|^{-n/2}, \\ \|R_{\mathrm{WKB}}^\pm(t,h)\|_{L^2(\widehat{dG})\to L^2(\widehat{dG})} \lesssim 1, \end{gathered} \tag{2-80}$$

for

$$h \in (0,1], \quad 0 \le \pm t \le t_{\mathrm{WKB}}. \tag{2-81}$$

Proof. This follows from the stationary phase theorem, using the parametrix given in Theorem 6.1 and Propositions 7.2, 7.3, 7.6, and 7.8. □

The first consequence of this proposition is the following result on short-time dispersion estimates, whose proof is completely similar to that of Proposition 2.21.

Proposition 2.23 (Time 1 dispersion estimates in intermediate areas). *For all $\varepsilon > 0$, all $\delta > 0$ and all $a_{l,\mathrm{inter}}^\pm$ satisfying (2-72), the estimate (2-74) holds for all h, t satisfying (2-81).*

We can now give the second condition on ε, also giving the choice of δ. The proof is given in on page 65 (Section 6B).

Proposition 2.24 (Negligibility of $1 \lesssim t \lesssim h^{-1}$ dispersion estimates in intermediate areas). *If ε is small enough, we can choose $\delta > 0$ small enough such that, for all $1 \le l \le L-1$, all*

$$b_{l,\mathrm{inter}}^\pm \in \mathcal{S}_{\mathrm{hyp}}\big(\Gamma_{\iota,\mathrm{inter}}^\pm(\varepsilon, \delta; l)\big),$$

and all $N \ge 0$, we have

$$\|\widehat{Op}_\iota(b_{l,\mathrm{inter}}^\pm) e^{-ithP} \widehat{Op}_\iota(b_{l,\mathrm{inter}}^\pm)^*\|_{L^2(\widehat{dG})\to L^2(\widehat{dG})} \le C_{l,N} h^N, \tag{2-82}$$

for

$$h \in (0, 1], \quad t_{\mathrm{WKB}} \leq \pm t \leq 2h^{-1}.$$

This is, at least intuitively, a consequence of Proposition 2.16 with $\underline{t} = t_{\mathrm{WKB}}$ and of the Egorov theorem which shows that $e^{-ithP}\widehat{Op}_\iota(b^{\pm}_{l,\mathrm{inter}})^*$ lives semiclassically in the region $\Phi^t_\iota\big(\mathrm{supp}(b^{\pm}_{l,\mathrm{inter}})\big)$.

We summarize the reasoning above as follows.

Proof of Theorem 1.2. Using Proposition 2.21, we first choose $\varepsilon_0 > 0$ small enough that, for all $\varepsilon \in (0, \varepsilon_0]$, (2-73) holds for $0 \leq \pm t \leq 2h^{-1}$. By possibly decreasing ε_0, we then choose t_{WKB} according to Proposition 2.22, uniformly with respect to $\varepsilon \in (0, \varepsilon_0]$. Next, according to Proposition 2.24, we fix $\varepsilon \in (0, \varepsilon_0]$ and $\delta > 0$ small enough that (2-82) holds for $t_{\mathrm{WKB}} \leq \pm t \leq 2h^{-1}$. Using (2-65), (2-67) and Proposition 2.24 with $N = n$ and $b^{\pm}_{l,\mathrm{inter}} = a^{\pm}_{l,\mathrm{inter}}$ defined by (2-71), we have

$$\big\| e^{-\gamma_n r} A_{\mathrm{pr}}(h)\widehat{Op}_\iota(a^{\pm}_{l,\mathrm{inter}}) e^{-ithP}\widehat{Op}_\iota(a^{\pm}_{l,\mathrm{inter}})^* A_{\mathrm{pr}}(h)^* e^{-\gamma_n r}\big\|_{L^1(\widehat{dG})\to L^\infty(dG)} \leq C_{\varepsilon,\delta} \lesssim |ht|^{-n/2},$$

for $t_{\mathrm{WKB}} \leq \pm t \leq 2h^{-1}$. On the other hand, (2-74) holds for $0 \leq \pm t \leq t_{\mathrm{WKB}}$, using Proposition 2.22. Therefore (2-74) holds for $0 \leq \pm t \leq 2h^{-1}$. By Proposition 2.19, this proves (2-70) for all a_ι satisfying (2-69). By Proposition 2.18, this implies (2-63) which, by Proposition 2.9, implies Theorem 1.2. □

3. Estimates on the geodesic flow near infinity

In this section, we describe some properties of the Hamiltonian flow of functions of the form

$$p(r, \theta, \rho, \eta) = \rho^2 + w(r)q(r, \theta, \eta), \tag{3-1}$$

on $T^*\mathbb{R}^n_+ = \mathbb{R}^+_r \times \mathbb{R}^{n-1}_\theta \times \mathbb{R}_\rho \times \mathbb{R}^{n-1}_\eta$. Here q is an homogeneous polynomial of degree 2 with respect to η and w a positive function. In Section 3B, we will assume that $w(r) = e^{-2r}$ but we start with more general cases in Section 3A.

The motivation for the study of (3-1) comes naturally from the form of the principal symbol p_ι of P given by (2-14).

We emphasize that the symbol p considered in this section is defined on $T^*\mathbb{R}^n_+$ whereas p_ι is only defined on a subset of the form $T^*(R_{\mathscr{K}}, +\infty) \times V_\iota$. The results of Section 3B will nevertheless hold for p_ι as well with no difficulty for we shall have a good localization of the flow in the regions we consider (see Corollary 3.10).

3A. ***A general result.*** Let $w = w(r)$ be a smooth function on $\mathbb{R}^+ = (0, +\infty)$ such that

$$w > 0, \quad w' < 0, \quad \left(\frac{w'}{w}\right)' \geq 0, \tag{3-2}$$

and, for some $0 < \gamma < 1$,

$$\limsup_{r\to+\infty} \int_r^{(1+\gamma)r} \frac{w'}{w} \in [-\infty, 0). \tag{3-3}$$

Note that $\lim_{r\to+\infty} w(r)$ exists, by (3-2), and that (3-3) implies that this limit must be 0. Note also that, for all $R > 0$, we have

$$w(r) \lesssim 1 \quad \text{and} \quad |w'(r)| \lesssim w(r) \quad \text{for } r \in [R, +\infty).$$

These assumptions are satisfied for instance by $w(r) = r^{-2}$ or $w(r) = e^{-2r}$.

We assume that q is an homogeneous polynomial of degree 2 with respect to η of the form

$$q(r, \theta, \eta) = q_0(\theta, \eta) + q_1(r, \theta, \eta) \tag{3-4}$$

with q_0, q_1 homogeneous polynomials of degree 2 with respect to η satisfying, for some $0 < \tau \leq 1$,

$$|\partial_\theta^\alpha \partial_\eta^\beta q_0(\theta, \eta)| \lesssim \langle \eta \rangle^{2-|\beta|}, \tag{3-5}$$

$$|\partial_r^j \partial_\theta^\alpha \partial_\eta^\beta q_1(r, \theta, \eta)| \lesssim \langle r \rangle^{-\tau-j} \langle \eta \rangle^{2-|\beta|}, \tag{3-6}$$

and, for some $C > 0$,

$$C^{-1}|\eta|^2 \leq q(r, \theta, \eta) \leq C|\eta|^2, \tag{3-7}$$

for $(r, \theta, \eta) \in \mathbb{R}^+ \times \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$. The latter implies, by possibly increasing C, that

$$C^{-1}|\eta|^2 \leq q_0(\theta, \eta) \leq C|\eta|^2, \qquad (\theta, \eta) \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}. \tag{3-8}$$

Setting $q' = \partial_r q$ $(= \partial_r q_1)$, we finally assume that,

$$\frac{q'}{q} \times \frac{w}{w'} \to 0 \qquad \text{as} \qquad r \to +\infty, \tag{3-9}$$

uniformly with respect to $\theta \in \mathbb{R}^{n-1}$ and $\eta \in \mathbb{R}^{n-1} \setminus 0$.

The Hamiltonian flow $\Phi^t = (r^t, \theta^t, \rho^t, \eta^t)$, generated by p, is the solution to the system

$$\begin{cases} \dot{r} = 2\rho, \\ \dot{\theta} = w\, \partial q/\partial \eta, \\ \dot{\rho} = -w'q - wq', \\ \dot{\eta} = -w\partial q/\partial \theta, \end{cases} \tag{3-10}$$

with initial condition

$$(r^t, \theta^t, \rho^t, \eta^t)_{|t=0} = (r, \theta, \rho, \eta). \tag{3-11}$$

Our main purpose is to show that, if $\rho > -p^{1/2}$ (with $p = p(r, \theta, \rho, \eta)$) and r is large enough, then Φ^t is defined for all $t \geq 0$ and $r_t \to +\infty$ as $t \to +\infty$ (we will obtain a similar result for $t \leq 0$ provided $\rho < p^{1/2}$). This result relies mainly on the following remark: if $\eta \neq 0$, we can write

$$-w'q - wq' = -\frac{w'}{w}\left(p - \rho^2\right)\left(1 + \frac{w}{w'} \times \frac{q'}{q}\right).$$

Using (3-9) and the negativity of w'/w, this shows that, for all $\varepsilon > 0$, we can find $R > 0$ such that

$$-w'q - wq' \geq -(1-\varepsilon)(p - \rho^2)\frac{w'}{w}, \qquad \text{on } [R, +\infty)_r \times \mathbb{R}^{n-1}_\theta \times \mathbb{R}_\rho \times \mathbb{R}^{n-1}_\eta \tag{3-12}$$

which we shall exploit to prove that $\dot{\rho} \geq 0$.

In the following lemma and in the sequel, we shall use extensively the shorter notation

$$p = p(r, \theta, \rho, \eta).$$

Lemma 3.1. *Denote by* $(-t_-, t_+)$ $(t_\pm \in (0, +\infty])$ *the maximal interval on which the solution of* (3-10), *with initial condition* (3-11), *is defined. Then*

$$t_\pm \geq \frac{r}{2p^{1/2}}.$$

Furthermore, either $r_t \to 0$ *as* $t \to t_+$ *(resp.* $t \to -t_-$*) or* $t_+ = +\infty$ *(resp.* $t_- = +\infty$*).*

Note that, if $p(r, \theta, \rho, \eta) = 0$, i.e., $\rho = 0$ and $\eta = 0$, then it is trivial that $t_\pm = +\infty$.

Proof. We will only consider the case of t_+, the one of t_- being similar. By the conservation of energy we have $|\rho^t| \leq p^{1/2}$ thus, for $t \in [0, t_+)$, $\dot{r}^t$ is bounded,

$$|r^t - r| \leq 2tp^{1/2}, \tag{3-13}$$

and $r^t \geq r - 2tp^{1/2}$. We now argue by contradiction and assume that $t_+ < r/2p^{1/2}$ (in particular, that t_+ is finite). Then $r_+ := r - 2t_+ p^{1/2} > 0$ and $r_t \geq r_+$ for all $t \in [0, t_+)$. Furthermore, by (3-7), we have $|w \partial_\eta q| \leq C(wq + w) \leq C(p + w)$, with w bounded on $[r_+, +\infty)$, hence $\dot{\theta}^t$ is bounded on $[0, t_+)$. One shows similarly that $\dot{\rho}^t$ and $\dot{\eta}^t$ are bounded on $[0, t_+)$, using that $|w'| \lesssim w$ on $[r_+, +\infty)$ for $\dot{\rho}$. This implies that $\lim_{t \to t_+} (r^t, \theta^t, \rho^t, \eta^t)$ exists and belongs to $(0, +\infty) \times \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}^{n-1}$. The solution can therefore be continued beyond t_+, which yields the contradiction.

We now consider the second statement. Assume that $t_+ < +\infty$. We must show that $r^t \to 0$ as $t \to t_+$. Assume that this is wrong. Then there exists $R > 0$ small enough and a sequence $t_k \to t_+$ such that $r^{t_k} \geq R$ for all $k \geq 0$. On the other hand, by energy conservation, we have $|r^t - r^s| \leq 2p^{1/2}|t - s|$ for all $t, s \in [0, t_+)$, hence

$$r^t \geq r^{t_k} - 2p^{1/2}|t - t_k| \geq R/2$$

provided $|t - t_k| \leq R/4p^{1/2}$. Since t_k can be chosen as close to t_+ as we want, there exists $\varepsilon > 0$ small enough such that $r^t \geq R/2$ for $t \in [t_+ - \varepsilon, t_+)$. Then, by the same argument as above, $\lim_{t \to t_+} (r^t, \theta^t, \rho^t, \eta^t)$ exists and belongs to $(0, +\infty) \times \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}^{n-1}$. The solution can be continued beyond t_+; hence $t_+ = +\infty$, which is a contradiction. □

Lemma 3.2. *Let* $0 < \varepsilon < 1$. *For any* $R > 0$ *such that* (3-12) *holds, we have:*

(i) *If* $r^{t_0} \geq R$ *and* $\rho^{t_0} > 0$ *for some* $t_0 \in [0, t_+)$, *then* $t_+ = +\infty$ *and*

$$r^t \geq R, \quad \rho^t \geq \rho^{t_0}, \quad r^t \geq r^{t_0} + 2(t - t_0)\rho^{t_0} \quad \textit{for all } t \geq t_0.$$

(ii) *If* $r^{t_0} \geq R$ *and* $\rho^{t_0} < 0$ *for some* $t_0 \in (-t_-, 0]$, *then* $-t_- = -\infty$ *and*

$$r^t \geq R, \quad \rho^t \leq \rho^{t_0}, \quad r^t \geq r^{t_0} + 2(t - t_0)\rho^{t_0} \quad \textit{for all } t \leq t_0.$$

Proof. As in Lemma 3.1, we only consider the case of t_+. It suffices to show that

$$r^t \geq R \quad \text{for all } t \in [t_0, t_+). \tag{3-14}$$

Indeed, if this is true, Lemma 3.1 shows that $t_+ = +\infty$ and then, by (3-12), we have $\dot{\rho}^t \geq 0$, whence $\rho^t \geq \rho^{t_0}$ and $r^t - r^{t_0} \geq 2\rho^{t_0}(t - t_0)$. Let us prove (3-14). Consider the set

$$I = \left\{ t \in [t_0, t_+) : r^s \geq R \text{ and } \rho^s \geq \rho^{t_0} \text{ for all } s \in [t_0, t] \right\}.$$

It is clearly an interval containing t_0 and we set $T := \sup I$. By continuity, $\rho^t \geq \rho^{t_0}/2 > 0$ for t in a small neighborhood J of t_0. This implies that $\dot{r}^t > 0$ on J, hence that $r^t \geq r^{t_0} \geq R$ on $J \cap [t_0, t_+)$ and thus that $\dot{\rho}^t \geq 0$ on $J \cap [t_0, t_+)$ which in turn shows that $\rho^t \geq \rho^{t_0}$ on $J \cap [t_0, t_+)$. This proves that $T > t_0$. Then, on $[t_0, T)$, we have

$$r^t \geq R, \quad \rho^t \geq \rho^{t_0}. \tag{3-15}$$

Now assume, by contradiction, that $T < t_+$. Then (3-15) holds on $[t_0, T]$ and in particular we have $r^T \geq r^{t_0} + 2(T - t_0)\rho^{t_0} > r^{t_0}$. Thus $r^t \geq R$ in a neighborhood of T and this implies that $\dot{\rho}^t \geq 0$ in this neighborhood. Hence there exists $T' > T$ such that (3-15) holds on $[t_0, T']$ yielding a contradiction. □

To state the next result, we define $l \in (0, +\infty]$ as

$$l = -\limsup_{r \to +\infty} \int_r^{(1+\gamma)r} \frac{w'}{w} \tag{3-16}$$

and we choose an arbitrary $\sigma \in \mathbb{R}$ such that

$$0 < \sigma < \begin{cases} -\dfrac{2}{l} + \left(\dfrac{4}{l^2} + 1\right)^{1/2} & \text{if } l < +\infty, \\ 1 & \text{if } l = +\infty. \end{cases} \tag{3-17}$$

Note that $0 < -\frac{2}{l} + \left(\frac{4}{l^2} + 1\right)^{1/2} < 1$ if l is finite, and that (3-17) is equivalent to

$$(1 - \sigma^2)l/2 > 2\sigma > 0.$$

Proposition 3.3. *For any σ satisfying (3-17), there exists $R_{w,\gamma,\sigma} > 0$ large enough that the following property holds. Let $r > R_{w,\gamma,\sigma}$. Then:*

(i) *If $\rho > -\sigma p^{1/2}$, then $t_+ = +\infty$ and*

$$r^t \geq \max\left((1 - \gamma)r, (1 - \gamma - \sigma\gamma)r + 2\sigma p^{1/2}|t|\right) \tag{3-18}$$

for all $t \geq 0$.

(ii) *If $\rho < \sigma p^{1/2}$, then $-t_- = -\infty$ and (3-18) holds for $t \leq 0$.*

This proposition means that, by choosing initial data with r large enough and $\rho > -\sigma p^{1/2}$ (resp. $\rho < \sigma p^{1/2}$), the forward (resp. backward) trajectory lies in a neighborhood of infinity. In particular, the forward (resp. backward) flow starting at (r, θ, ρ, η), with $\rho > -\sigma p^{1/2}$ (resp. $\rho < \sigma p^{1/2}$) depends only on the values of p on $[(1 - \gamma)r, +\infty) \times \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}^{n-1}$.

Proof. We only consider the case where $\rho > -\sigma p^{1/2}$, the case where $\rho < \sigma p^{1/2}$ being similar. If $l < \infty$, (3-17) allows one to choose $0 < \varepsilon < 1$ such that

$$(1 - \varepsilon)^2(1 - \sigma^2)l/2 \geq 2\sigma. \tag{3-19}$$

If $l = \infty$, we choose an arbitrary $\varepsilon \in (0, 1)$. We next choose R so that (3-12) holds with the above choice of ε. If $\rho \geq \sigma p^{1/2}$ (recall that $p^{1/2} > 0$ since $\rho > -\sigma p^{1/2}$) and $r \geq R$, then Lemma 3.2 shows that the result holds with $R_{w,\gamma,\sigma} = R$. We can therefore assume that $\rho < \sigma p^{1/2}$. Set

$$R_1 = (1 - \gamma)^{-1}R, \quad T = \gamma r/2p^{1/2}. \tag{3-20}$$

By Lemma 3.1, we have $t_+ > T$ and, if $r \geq R_1$,

$$r^t \geq r - 2tp^{1/2} \geq (1-\gamma)r \geq R \quad \text{for } t \in [0, T].$$

Using (3-12), this implies that $\dot{\rho}^t \geq 0$ on $[0, T]$ and hence that $\rho^t \geq -\sigma p^{1/2}$ for all $t \in [0, T]$. We now prove by contradiction that there exists $t \in [0, T]$ such that $\rho^t \geq \sigma p^{1/2}$. If this is wrong, we have $(\rho^t)^2 \leq \sigma^2 p$ on $[0, T]$, thus (3-12) shows that, for all $t \in [0, T]$,

$$\dot{\rho}^t \geq -(1-\varepsilon)(1-\sigma^2)p\frac{w'}{w}(r^t) \geq -(1-\varepsilon)(1-\sigma^2)p\frac{w'}{w}(r + 2tp^{1/2}),$$

using the third estimate of (3-2) and the fact that $r^t \leq r + 2tp^{1/2}$ in the second inequality. By integration over $[0, T]$, we get

$$\rho^T - \rho \geq -(1-\varepsilon)(1-\sigma^2)p^{1/2}\,\frac{1}{2}\int_r^{(1+\gamma)r} \frac{w'}{w}, \tag{3-21}$$

using the second equality in (3-20). Fix R_2 such that, for all $r > R_2$,

$$-\int_r^{(1+\gamma)r} \frac{w'}{w} > \begin{cases} (1-\varepsilon)l & \text{if } l < +\infty, \\ \dfrac{4\sigma}{(1-\varepsilon)(1-\sigma^2)} & \text{if } l = +\infty. \end{cases}$$

With such a choice (and (3-19) if l is finite), we see that, if $r \geq \max(R_1, R_2)$, (3-21) implies that $\rho^T - \rho \geq 2\sigma p^{1/2}$ and hence that $\rho^T \geq \sigma p^{1/2}$ which yields the expected contradiction.

In summary, we have shown that for any $r \geq \max(R_1, R_2)$ and any $\rho > -\sigma p^{1/2}$, there exists $t_0 \in [0, T]$ such that $\rho^{t_0} \geq \sigma p^{1/2} > 0$ and $r^{t_0} \geq R$, hence $t_+ = +\infty$ by Lemma 3.2. Furthermore, $r^t \geq (1-\gamma)r$ on $[0, T]$ and $r^t \geq r^T + 2(t-T)\sigma p^{1/2} \geq (1-(1+\sigma)\gamma)r + 2t\sigma p^{1/2}$ on $[T, +\infty)$. The result follows since

$$\max\big((1-\gamma)r,\ (1-\gamma-\sigma\gamma)r + 2\sigma p^{1/2}t\big) = \begin{cases} (1-\gamma)r & \text{if } t \in [0, T], \\ (1-\gamma-\sigma\gamma)r + 2\sigma p^{1/2}t & \text{if } t > T. \end{cases}$$ □

3B. *The asymptotically hyperbolic case.* We will now prove more precise estimates on the Hamiltonian flow of p when

$$w(r) = e^{-2r}.$$

In that case, the conditions (3-2), (3-3) and (3-9) are fulfilled, with any $0 < \gamma < 1$ in (3-3) and we have $l = +\infty$ in (3-16).

We shall need the following improvement of Proposition 3.3.

Proposition 3.4. *Let $0 < \sigma < 1$. There exist $R_\sigma > 0$ and $C_\sigma > 0$ such that: if $r \geq R_\sigma$ and $\rho > -\sigma p^{1/2}$ (resp. $\rho < \sigma p^{1/2}$), then*

$$r^t \geq r + 2\sigma p^{1/2}|t| - C_\sigma, \qquad \text{for all } t \geq 0 \ (\text{resp. } t \leq 0).$$

The improvement consists in replacing $(1-\gamma-\sigma\gamma)r$ in the estimate (3-18) by $r - C_\sigma$.

Proof. Here again we only consider the case $t \geq 0$. By Proposition 3.3, we may assume that $r^t \geq R$ for all $t \geq 0$, with R large enough so that (3-12) holds with $\varepsilon = \frac{1}{2}$. This implies that

$$\dot{\rho}^t = 2e^{-2r^t}q(r^t, \theta^t, \eta^t) - e^{-2r^t}\partial_r q_1(r^t, \theta^t, \eta^t) \geq e^{-2r^t}q(r^t, \theta^t, \eta^t) = p - (\rho^t)^2. \tag{3-22}$$

If $\rho \geq \sigma p^{1/2}$, then the result follows from Lemma 3.2 (with $C_\sigma = 0$). If $\rho < \sigma p^{1/2}$, we will show that, with $T = 2\sigma p^{-1/2}/(1-\sigma^2)$, there exists $t \in [0, T]$ such that $\rho^t \geq \sigma p^{1/2}$. Assume that this is wrong. Then $(\rho^t)^2 \leq \sigma^2 p$ on $[0, T]$ and by integrating the above estimate on $\dot{\rho}^t$, we get

$$\rho^T - \rho \geq T(1-\sigma^2)p = 2\sigma p^{1/2}.$$

This proves that $\rho^T \geq \sigma p^{1/2}$ which is a contradiction. Therefore, by Lemma 3.2, we see that $r^t - r^T \geq 2\sigma p^{1/2}(t-T)$ for $t \geq T$. On the other hand, we have $r^t \geq r - 2p^{1/2}t$ for $t \in [0, T]$. The latter implies that $r^t \geq r + 2\sigma p^{1/2}t - 2p^{1/2}(1+\sigma)t \geq r + 2\sigma p^{1/2}t - 4\sigma/(1-\sigma)$ for $t \in [0, T]$. This holds in particular for $t = T$ and then for $t \geq T$. Thus the results holds with $C_\sigma = 4\sigma/(1-\sigma)$. ☐

We have so far only studied some localization properties of Φ^t, the Hamiltonian flow of p. We shall now give estimates on derivatives of Φ^t. We start with the following lemma giving some rough estimates. They will serve as a priori estimates for the proof of Proposition 3.8 below.

Lemma 3.5. *For all $0 < \sigma < 1$, there exists $R > 0$ such that, for all $(r, \theta, \rho, \eta) \in T^*\mathbb{R}^n_+$ satisfying*

$$r > R, \quad \pm\rho > -\sigma p^{1/2}, \quad p \in \left(\tfrac{1}{4}, 4\right), \tag{3-23}$$

and all $\pm t \geq 0$, we have

$$\left| e^{r|\beta|} \partial_\eta^\beta \partial_r^j \partial_\theta^\alpha \partial_\rho^k (\Phi^t - \Phi^0)(r, \theta, \rho, \eta) \right| \lesssim \langle t \rangle.$$

Note the $e^{r|\beta|}$ factor in front of the derivatives.

We will need two lemmas. The first one, proved by induction, is a soft version of the classical Faà di Bruno formula.

Lemma 3.6. *Let $\Omega_1 \subset \mathbb{R}^{n_1}$, $\Omega_2 \subset \mathbb{R}^{n_2}$ be open subsets, with $n_1, n_2 \geq 1$. Consider smooth maps $y = (y_1, \dots, y_{n_2}) : \Omega_1 \to \Omega_2$ and $Z : \Omega_1 \times \Omega_2 \to \mathbb{R}^{n_3}$, with $n_3 \geq 1$. Then, for all $|\gamma| \geq 1$,*

$$\partial_x^\gamma \left(Z(x, y(x)) \right) = (\partial_y Z)(x, y(x)) \partial_x^\gamma y(x) + \left(\partial_x^\gamma Z \right)(x, y(x)) + R_\gamma(x)$$

where $R_\gamma(x)$ vanishes identically if $|\gamma| = 1$ and, otherwise, is a linear combination of

$$\left(\partial_x^{\gamma-\gamma'} \partial_y^\nu Z \right)(x, y(x)) \left(\partial_x^{\gamma_1^1} y_1(x) \dots \partial_x^{\gamma_{\nu_1}^1} y_1(x) \right) \dots \left(\partial_x^{\gamma_1^{n_2}} y_{n_2}(x) \dots \partial_x^{\gamma_{\nu_{n_2}}^{n_2}} y_{n_2}(x) \right),$$

with $\gamma, \gamma', \gamma_j^k \in \mathbb{N}_0^{n_1}$, $\nu = (\nu_1, \dots, \nu_{n_2}) \in \mathbb{N}_0^{n_2}$ satisfying $\gamma' \neq 0$, $\nu \neq 0$ and

$$\gamma' \leq \gamma, \qquad 2 \leq |\nu| + |\gamma - \gamma'| \leq |\gamma|, \qquad \gamma_1^1 + \dots + \gamma_{\nu_1}^1 + \dots + \gamma_1^{n_2} + \dots + \gamma_{\nu_{n_2}}^{n_2} = \gamma',$$

and using the convention that $\partial_x^{\gamma_1^k} y_k(x) \dots \partial_x^{\gamma_{\nu_k}^k} y_k(x) \equiv 1$ if $\nu_k = 0$ (if $\nu_k \neq 0$ then $\gamma_1^k, \dots, \gamma_{\nu_k}^k$ are all nonzero).

In the second lemma, we consider the linear differential equation

$$\dot{X} = A(t)X + Y(t), \tag{3-24}$$

where $A(\cdot)$ is a continuous map from $[0, +\infty)$ to the space $\mathcal{M}_{N\times N}(\mathbb{R})$) of $N \times N$ matrices with real entries, for some $N \geq 1$, and $Y(\cdot) \in C([0, +\infty), \mathbb{C}^N)$. We assume that $A(\cdot)$ belongs to a subset $\mathcal{B} \subset C([0, +\infty), \mathcal{M}_{N\times N}(\mathbb{R}))$ for which there exist $\delta_{\mathcal{B}} > 0$ and $C_{\mathcal{B}} > 0$ such that

$$|||A(t)||| \leq C_{\mathcal{B}} e^{-\delta_{\mathcal{B}} t} \quad \text{for all } t \geq 0 \text{ and } A(\cdot) \in \mathcal{B},$$

with $|||\cdot|||$ a matrix norm associated to the norm $\|\cdot\|$ on $\mathbb{C}^N$, i.e., such that $\|MZ\| \leq |||M||| \, \|Z\|$, for all $M \in \mathcal{M}_{N\times N}(\mathbb{R})$ and $Z \in \mathbb{C}^N$.

Lemma 3.7. *There exists $C > 0$ such that, for all $A(\cdot) \in \mathcal{B}$ and all $Y(\cdot)$ satisfying*

$$\int_0^\infty \|Y(t)\| dt < \infty,$$

the solutions $X(\cdot)$ of (3-24) satisfy

$$\|X(t)\| \leq C \left(\|X(0)\| + \int_0^\infty \|Y(s)\| \, ds \right) \quad \textit{for } t \geq 0. \tag{3-25}$$

Proof. First fix $0 < \delta < \delta_{\mathcal{B}}$ and $\varepsilon = \delta_{\mathcal{B}} - \delta$. Choose $T > 0$ such that $C_{\mathcal{B}} e^{-\delta_{\mathcal{B}} t} \leq \varepsilon$ for $t \geq T$. By Gronwall's lemma, we have

$$\|X(t)\| \leq \left(\|X(T)\| + \int_T^\infty \|Y(s)\| \, ds \right) e^{\varepsilon(t-T)} \quad \text{for } t \geq T,$$
$$\|X(t)\| \leq \left(\|X(0)\| + \int_0^T \|Y(s)\| \, ds \right) e^{C_{\mathcal{B}} T} \quad \text{for } t \in [0, T].$$

These two inequalities give, for some C depending only on $C_{\mathcal{B}}$, $\delta_{\mathcal{B}}$, δ and T,

$$\|X(t)\| \leq C \left(\|X(0)\| + \int_0^\infty \|Y(s)\| \, ds \right) e^{\varepsilon t} \quad \text{for } t \geq 0.$$

Used as an a priori estimate in (3-24), this yields

$$\|\dot{X}(t)\| \leq \|Y(t)\| + C C_{\mathcal{B}} e^{-\delta t} \left(\|X(0)\| + \int_0^\infty \|Y(s)\| \, ds \right) \quad \text{for } t \geq 0,$$

which implies (3-25). □

Proof of Lemma 3.5. As before, we only prove the result for $t \geq 0$. For $|\beta| + j + |\alpha| + k = 0$, the result is a consequence of the motion equations (3-10) and energy conservation. Indeed, for $r^t - r$, the estimate follows directly from (3-13). Next, the equation of motion for θ, together with (3-7) and Proposition 3.4, shows that

$$|\dot{\theta}^t| \lesssim e^{-2r^t} |\eta^t| \lesssim e^{-2r^t} \langle \eta^t \rangle^2 \lesssim 1 + p;$$

hence that $|\theta^t - \theta| \lesssim \langle t \rangle$ by integration. One similarly shows that $|\rho^t - \rho| + |\eta^t - \rho| \lesssim \langle t \rangle$. We now consider the derivatives and write, for simplicity, $\partial^\gamma = \partial_\eta^\beta \partial_r^j \partial_\theta^\alpha \partial_\rho^k$. Denoting by H_p is the Hamiltonian vector field of p and applying ∂^γ to (3-10), we obtain

$$e^{r|\beta|} \partial^\gamma \dot{\Phi}^t = (dH_p)(\Phi^t) e^{r|\beta|} \partial^\gamma \Phi^t + R(t),$$

where, by Lemma 3.6, $R(t)$ vanishes if $|\gamma| = 1$ or, if $|\gamma| \geq 2$, is a linear combination of

$$(\partial^\nu H_p)(\Phi^t) e^{r|\beta|} \big(\partial^{\gamma_1^1} r^t \cdots \partial^{\gamma_{\nu_1}^1} r^t \big) \cdots \big(\partial^{\gamma_1^{2n}} \eta_{n-1}^t \cdots \partial^{\gamma_{\nu_{2n}}^{2n}} \eta_{n-1}^t \big). \tag{3-26}$$

Here $\nu = (\nu_1, \dots, \nu_{2n})$ is of length at least 2, all the derivatives of Φ^t involved in $R(t)$ are of strictly smaller order than γ (meaning that $\gamma^i_{l_i} \le \gamma$ and $\gamma^i_{l_i} \ne \gamma$), and

$$2 \le |\nu| \le |\gamma|, \quad \gamma^1_1 + \cdots + \gamma^{2n}_{\nu_{2n}} = \gamma. \tag{3-27}$$

Writing dH_p as a matrix, we have

$$dH_p = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + e^{-2r} \begin{pmatrix} 0 & 0 & 0 & 0 \\ \partial^2_{r\eta} q_1 - 2\partial_\eta q & \partial^2_{\theta\eta} q & 0 & \partial^2_{\eta\eta} q \\ 4\partial_r q_1 - 4q - \partial^2_{rr} q_1 & 2\partial_\theta q - \partial^2_{\theta r} q_1 & 0 & 2\partial_\eta q - \partial^2_{\eta r} q_1 \\ 2\partial_\theta q - \partial^2_{r\theta} q_1 & -\partial^2_{\theta\theta} q & 0 & -\partial^2_{\eta\theta} q \end{pmatrix}.$$

Defining M as the first (constant) matrix of the right-hand side and using Proposition 3.4, we have

$$\left| dH_p(\Phi^t) - M \right| \lesssim e^{-2r^t} \langle \eta^t \rangle^2 \lesssim e^{-2r - 2\sigma t} (\langle \eta \rangle^2 + \langle t \rangle^2) \lesssim e^{-\sigma t},$$

using that $2p^{1/2} \ge 1$ and that $e^{-2r} \langle \eta \rangle^2$ is bounded, by (3-23). We then set

$$A(t) = e^{-tM} \big(dH_p(\Phi^t) - M \big) e^{tM},$$
$$X(t) = e^{-tM} e^{r|\beta|} \partial^\gamma \Phi^t - e^{r|\beta|} \partial^\gamma \Phi^0,$$
$$Y(t) = e^{-tM} R(t) + A(t) e^{r|\beta|} \partial^\gamma \Phi^0,$$

so that

$$\dot{X}(t) = A(t) X(t) + Y(t), \quad X(0) = 0.$$

Noting that $M^2 = 0$, we have

$$\exp(\pm tM) = 1 \pm tM, \quad |\exp(\pm tM)| \lesssim \langle t \rangle; \tag{3-28}$$

thus

$$|A(t)| \lesssim e^{-\sigma t} \langle t \rangle^2 \lesssim e^{-\sigma t/2}. \tag{3-29}$$

To estimate $X(t)$ by Lemma 3.7, we still need to estimate $Y(t)$. We first assume that $\partial^\gamma = \partial^\beta_\eta$ with $|\beta| = 1$. We then have $R(t) = 0$ and

$$A(t) e^{r|\beta|} \partial^\gamma \Phi^0 = e^{-tM} (\partial^\beta_\eta H_p)(\Phi^t) e^r,$$

since $M \partial^\beta_\eta \Phi^0 = 0$. By Proposition 3.4 and (3-23) again, we obtain

$$|(\partial^\beta_\eta H_p)(\Phi^t)| \lesssim e^{-2r - 2\sigma t} \langle \eta^t \rangle \lesssim e^{-r - \sigma t},$$

so that $|Y(t)| \lesssim e^{-\sigma t/2}$. Using (3-29) and Lemma 3.7, we get $|X(t)| \lesssim 1$. Since $M \partial^\beta_\eta \Phi^0 = 0$, we can rewrite $X(t) = e^{-tM} e^r \partial^\beta_\eta (\Phi^t - \Phi^0)$ and, using (3-28), finally get

$$\left| e^r \partial^\beta_\eta (\Phi^t - \Phi^0) \right| \lesssim \langle t \rangle.$$

The other first-order derivatives of $\Phi^t - \Phi^0$ are studied similarly (note that there is no e^r factor then), by showing that $X(t)$ is bounded and using that $X(t) = e^{-tM} \partial^\gamma (\Phi^t - \Phi^0) + (e^{-tM} - 1) \partial^\gamma \Phi^0$ with (3-28) to get

$$|\partial^\gamma (\Phi^t - \Phi^0)| \lesssim \langle t \rangle.$$

For higher-order derivatives, $\partial^\gamma \Phi^0 = 0$ and $\partial^\gamma(\Phi^t - \Phi^0) = \partial^\gamma \Phi^t$. Furthermore, since the derivatives of Φ^t involved in $R(t)$ are of strictly smaller order than γ, we can proceed by induction. By writing x^t for r^t, ρ^t, θ^t and

$$\partial^{\gamma_l^i} = \partial_\eta^{\beta_l^i} \partial_r^{k_l^i} \partial_\theta^{\alpha_l^i} \partial_\rho^{j_l^i}$$

for the derivatives involved in (3-26), with $1 \le i \le 2n$ and $1 \le l \le \nu_i$ (recall that, if $\nu_i = 0$, the corresponding product in (3-26) is 1), the induction hypothesis yields

$$|e^{|\beta_l^i| r} \partial^{\gamma_l^i} x^t| \lesssim \langle t \rangle,$$

since $\partial_\eta^{\beta_l^i} x^t = \partial_\eta^{\beta_l^i}(x^t - x^0)$ if $\beta_l^i \neq 0$. If $n+2 \le i \le 2n$ (and $\nu_i \neq 0$), we also have

$$|e^{|\beta_l^i| r} \partial^{\gamma_l^i} \eta_{i-n-1}^t| \lesssim \langle t \rangle,$$

unless $\partial^{\gamma_l^i} = \partial_\eta^{\beta_l^i}$ with $|\beta_l^i| = 1$, in which case we only have $|\partial^{\gamma_l^i} \eta_{i-n-1}^t| \lesssim \langle t \rangle$. By setting

$$\mathscr{E} = \{n+2 \le i \le 2n : \exists 1 \le l \le \nu_i \text{ such that } \partial^{\gamma_l^i} = \partial_\eta^{\beta_l^i} \text{ with } |\beta_l^i| = 1\},$$

and $N = \#\mathscr{E}$, we thus obtain

$$|(3\text{-}26)| \lesssim e^{Nr} |(\partial^\nu H_p)(\Phi^t)| \langle t \rangle^{|\nu| - N} \prod_{\mathscr{E}} |\partial^{\gamma_l^i} \eta_{i-n-1}^t|.$$

Since the components of H_p are polynomial of degree 2 with respect to the last $n-1$ variables, we only need to consider the case where $N \le 2$, otherwise $\nu_{n+2} + \cdots + \nu_{2n} \ge 3$ and $\partial^\nu H_p \equiv 0$. Furthermore

$$|(\partial^\nu H_p)(\Phi^t)| \lesssim e^{-2r^t} \langle \eta^t \rangle^{2 - \nu_{n+2} - \cdots - \nu_{2n}} \lesssim e^{-2r^t} \langle \eta^t \rangle^{2-N}.$$

For $N \le 2$, we have $\langle \eta^t \rangle^{2-N} \lesssim \langle \eta \rangle^{2-N} + \langle t \rangle^{2-N}$ so, using that $e^{Nr} e^{-2r^t} \lesssim e^{-(2-N)r - 2\sigma t}$, we see that $e^{Nr} e^{-2r^t} \langle \eta^t \rangle^{2-N} \lesssim e^{-\sigma t}$ which finally implies $|(3\text{-}26)| \lesssim \langle t \rangle^{|\nu|} e^{-\sigma t} \lesssim e^{-\sigma t/2}$. Therefore $|Y(t)| \lesssim \langle t \rangle e^{-\sigma t}$ and, by Lemma 3.7, $|X(t)|$ is bounded. The result then follows easily. ☐

The following proposition will be important in Section 4C to construct and estimate phase functions.

Proposition 3.8. *For all* $0 < \sigma < 1$, *there exists* $R > 0$ *such that, for all* $j, k \in \mathbb{N}_0$, $\alpha, \beta \in \mathbb{N}_0^{n-1}$, *with the notation*

$$D_{\text{hyp}}^{j\alpha k\beta} = e^{r|\beta|} \partial_\eta^\beta \partial_r^j \partial_\theta^\alpha \partial_\rho^k,$$

(*introduced before Definition 2.2*) *and* $(l)_+ = \max(0, l)$, *we have*

$$|D_{\text{hyp}}^{j\alpha k\beta}(r^t - r - 2|t|p^{1/2})| \lesssim \left(e^{-r}\langle \eta/p^{1/2}\rangle\right)^{(2-|\beta|)_+} p^{-(k+|\beta|)/2},$$

$$|D_{\text{hyp}}^{j\alpha k\beta}(\theta^t - \theta)| \lesssim e^{-r}\left(e^{-r}\langle \eta/p^{1/2}\rangle\right)^{(1-|\beta|)_+} p^{-(k+|\beta|)/2},$$

$$|D_{\text{hyp}}^{j\alpha k\beta}(\rho^t - \rho)| + |D_{\text{hyp}}^{j\alpha k\beta}(\eta^t - \eta)| \lesssim \left(e^{-r}\langle \eta/p^{1/2}\rangle\right)^{(2-|\beta|)_+} p^{(1-k-|\beta|)/2},$$

and, for all $0 < \varepsilon < 1$,

$$\left|D_{\text{hyp}}^{j\alpha k\beta}(\rho^t \mp p^{1/2})\right| \lesssim \left(e^{-r}\langle \eta/p^{1/2}\rangle\right)^{(2-|\beta|)_+} e^{-4(1-\varepsilon)|t|p^{1/2}} p^{(1-k-|\beta|)/2},$$

uniformly with respect to (r, θ, ρ, η) *and* t *satisfying*

$$r > R, \quad \pm\rho > -\sigma p^{1/2}, \quad \pm t \geq 0. \tag{3-30}$$

Apart from the energy localization and the localization in θ, the conditions (3-30) are the main ones that define outgoing/incoming areas according to Definition 2.10.

Note also that, if (r, θ, ρ, η) are restricted to a subset where p belongs to a compact subset of $(0, +\infty)$, the estimates of Proposition 3.8 read

$$\left|D^{j\alpha k\beta}_{\text{hyp}}(r^t - r - 2|t|p^{1/2})\right| + \left|D^{j\alpha k\beta}_{\text{hyp}}(\rho^t - \rho)\right| + \left|D^{j\alpha k\beta}_{\text{hyp}}(\eta^t - \eta)\right| \lesssim \left(e^{-r}\langle\eta\rangle\right)^{(2-|\beta|)_+}, \tag{3-31}$$

$$\left|D^{j\alpha k\beta}_{\text{hyp}}(\theta^t - \theta)\right| \lesssim e^{-r}\left(e^{-r}\langle\eta\rangle\right)^{(1-|\beta|)_+}, \tag{3-32}$$

$$\left|D^{j\alpha k\beta}_{\text{hyp}}(\rho^t \mp p^{1/2})\right| \lesssim \left(e^{-r}\langle\eta\rangle\right)^{(2-|\beta|)_+} e^{-4(1-\varepsilon)|t|p^{1/2}}. \tag{3-33}$$

Actually the latter estimates are equivalent to Proposition 3.8, in view of the elementary scaling properties

$$(r^t, \theta^t)(r, \theta, \rho, \eta) = (r^{\lambda t}, \theta^{\lambda t})(r, \theta, \rho/\lambda, \eta/\lambda), \tag{3-34}$$

$$(\rho^t, \eta^t)(r, \theta, \rho, \eta) = \lambda(\rho^{\lambda t}, \eta^{\lambda t})(r, \theta, \rho/\lambda, \eta/\lambda), \tag{3-35}$$

valid for $\lambda > 0$. Note that the condition (3-30) is invariant under the scaling $(t, \rho, \eta) \mapsto (\lambda t, \rho/\lambda, \eta/\lambda)$.

To prove Proposition 3.8, we need the following lemma (which will also be useful in proof of Proposition 2.16 in Section 4A).

Lemma 3.9. *For all* $0 < \sigma < 1$, *there exist* $R > 0$ *and* $C > 0$ *such that, for all* (r, θ, ρ, η) *satisfying* (3-23),

$$|\rho^t \mp p^{1/2}| \leq Ce^{-|t|/C} \quad \text{for } \pm t \geq 0. \tag{3-36}$$

In particular, $\rho^t \to \pm p^{1/2}$ *as* $t \to \pm\infty$.

Proof. We consider the case where $t \geq 0$, the case of negative times being similar. Using (3-12), Proposition 3.4 and Lemma 3.5, we can choose R large enough such that $\dot{\rho}^t \geq 0$ and

$$\dot{\rho}^t \lesssim e^{-2r^t}|\eta^t|^2 \lesssim e^{-2r^t}(|\eta| + \langle t\rangle)^2 \lesssim e^{-2r-2\sigma t}(|\eta| + \langle t\rangle)^2 \lesssim e^{-\sigma t}, \tag{3-37}$$

using the fact that $e^{-2r}|\eta|^2 \lesssim p$ in the last estimate. Therefore, ρ^t has a limit as $t \to +\infty$. By the energy conservation and the estimate on $e^{-2r^t}|\eta^t|^2$ given by (3-37), we have $p = (\rho^t)^2 + \mathcal{O}(e^{-\sigma t})$, which shows that $(\rho^t)^2 \to p$. Since ρ^t is nondecreasing and $\rho^0 = \rho > -p^{1/2}$, the limit must be $p^{1/2}$. Then we get (3-36) by integrating the equation of motion for ρ^t between t and $+\infty$, namely

$$p^{1/2} - \rho^t = \int_t^\infty \dot{\rho}^s\, ds = \int_t^\infty e^{-2r^s}\left(2q(r^s, \theta^s, \eta^s) - (\partial_r q_1)(r^s, \theta^s, \eta^s)\right) ds \tag{3-38}$$

where, by Proposition 3.4 and Lemma 3.5, the integrand is $\mathcal{O}(e^{-2r-2\sigma s}(\langle s\rangle + \langle\eta\rangle)^2)$. □

Proof of Proposition 3.8. We only need to prove (3-31), (3-32) and (3-33) with $p \in (\frac{1}{4}, 4)$ and, again, we only consider $t \geq 0$ and $\rho > -\sigma p^{1/2}$. We first assume that $j + |\alpha| + k + |\beta| = 0$. By (3-10), Proposition 3.4 and Lemma 3.5, we have

$$|\dot{\theta}^t| \lesssim e^{-2r-2\sigma t}(|\eta| + \langle t\rangle) \lesssim e^{-2r-\sigma t}\langle\eta\rangle,$$
$$|\dot{\eta}^t| \lesssim e^{-2r-2\sigma t}(|\eta| + \langle t\rangle)^2 \lesssim e^{-2r-\sigma t}\langle\eta\rangle^2;$$

hence $|\eta^t-\eta|\lesssim e^{-2r}\langle\eta\rangle^2$ and $|\theta^t-\theta|\lesssim e^{-2r}\langle\eta\rangle$. In particular, $\eta^t-\eta$ and $\theta^t-\theta$ are bounded. The motion equation for r^t yields

$$r^t-r-2tp^{1/2}=2\int_0^t(\rho^s-p^{1/2})\,ds, \tag{3-39}$$

and, using (3-36), we get $|r^t-r-2tp^{1/2}|\lesssim 1$. The latter estimate and the boundedness $|\eta^t-\eta|$ imply, together with (3-38),

$$|\rho^t-p^{1/2}|\lesssim e^{-2r-4tp^{1/2}}\langle\eta\rangle^2. \tag{3-40}$$

Furthermore, since $|p^{1/2}-\rho|=|\rho^2-p|/|\rho+p^{1/2}|\lesssim e^{-2r}|\eta|^2$, we also have $|\rho^t-\rho|\lesssim e^{-2r}\langle\eta\rangle^2$. Putting (3-40) into (3-39), we obtain $|r^t-r-2tp^{1/2}|\lesssim e^{-2r}\langle\eta\rangle^2$ which completes the proof of (3-31), (3-32) and (3-33) for $j+|\alpha|+k+|\beta|=0$ (note that we can choose $\varepsilon=0$ in this case).

We now prove (3-32) when $j+|\alpha|+k+|\beta|\geq 1$. We first note that, by Lemma 3.5 and the boundedness of $|r^t-r-2tp^{1/2}|$, we have

$$\begin{aligned}\big|D_{\mathrm{hyp}}^{j'\alpha'k'\beta'}(e^{-r^t}\eta^t)\big| &\leq \big|D_{\mathrm{hyp}}^{j'\alpha'k'\beta'}(e^{-r^t}(\eta^t-\eta))\big|+\big|D_{\mathrm{hyp}}^{j'\alpha'k'\beta'}(e^{-r^t}\eta)\big|\\ &\lesssim e^{-2tp^{1/2}}\langle t\rangle^{j'+|\alpha'|+k'+|\beta'|}\big(e^{-r}+(e^{-r}|\eta|)^{(1-|\beta'|)_+}\big)\\ &\lesssim e^{-2tp^{1/2}}\langle t\rangle^{j'+|\alpha'|+k'+|\beta'|}\big(e^{-r}\langle\eta\rangle\big)^{(1-|\beta'|)_+},\end{aligned} \tag{3-41}$$

for all $j'+|\alpha'|+k'+|\beta'|\geq 0$. By writing

$$\theta^t-\theta=\int_0^t e^{-r^s}(\partial_\eta q)(r^s,\theta^s,e^{-r^s}\eta^s)\,ds,$$

and using (3-41), Lemma 3.5 (more precisely, the estimates $\big|D_{\mathrm{hyp}}^{j''\alpha''k''\beta''}r^t\big|+\big|D_{\mathrm{hyp}}^{j''\alpha''k''\beta''}\theta^t\big|\lesssim\langle t\rangle$ if $j''+|\alpha''|+k''+|\beta''|\neq 0$), the Leibniz formula and Lemma 3.6, we obtain (3-32). We obtain similarly (3-33) and then (3-31) (also using that $(e^{-r}\langle\eta\rangle)^2\lesssim e^{-r}\langle\eta\rangle\lesssim 1$). Note that, for $r^t-r-2tp^{1/2}$, (3-31) follows directly from (3-33) and (3-39). □

Corollary 3.10. *Let $V\Subset V'\Subset\mathbb{R}^{n-1}$ be two relatively compact open subsets and let $0<\sigma<1$. There exists $R>0$ and $C>0$ such that the conditions*

$$r>R,\quad \theta\in V,\quad \pm\rho>-\sigma p^{1/2}, \tag{3-42}$$

imply that, for all $\pm t\geq 0$,

$$r^t>r-C\quad\text{and}\quad\theta^t\in V'.$$

In particular, if (3-42) holds, the flow $\Phi^t(r,\theta,\rho,\eta)$ depend only on p on $T^*\big((r-C,+\infty)\times V'\big)$ for $\pm t\geq 0$.

This corollary allows us to localize the estimates of Proposition 3.8 in charts of asymptotically hyperbolic manifolds.

4. The Hamilton–Jacobi and transport equations

In this section, we develop the analytical tools necessary for the Isozaki–Kiada parametrix that will be constructed in Section 5. We mainly construct the phases and amplitudes needed for that parametrix, but also prove certain useful properties of outgoing/incoming areas, including those quoted without proof in Section 2D.

All the statements in this section will hold in a coordinate chart at infinity, associated to a fixed coordinate patch U_ι on the angular manifold. Thus, for notational simplicity, we will drop the corresponding index ι from the notation.

4A. ***Properties of outgoing, incoming and intermediate areas.*** Here we collect some properties of outgoing, incoming and intermediate areas which will be needed for the construction of the Isozaki–Kiada parametrix. We also prove a part of the results quoted without proofs in Section 2D, namely Propositions 2.11, 2.13, 2.15 and 2.16.

In the first proposition below, we use the classes $\mathcal{S}_{\text{hyp}}(\Omega)$ introduced in Definition 2.2.

Proposition 4.1. (i) *Assume that*

$$R_1 > R_2, \qquad V_1 \Subset V_2, \qquad I_1 \Subset I_2, \qquad \sigma_1 < \sigma_2. \tag{4-1}$$

Then we can find $\chi^{\pm}_{1\to 2} \in \mathcal{S}_{\text{hyp}}\big(\Gamma^{\pm}(R_2, V_2, I_2, \sigma_2)\big)$ *such that*

$$\chi^{\pm}_{1\to 2} \equiv 1 \quad \textit{on} \quad \Gamma^{\pm}(R_1, V_1, I_1, \sigma_1).$$

(ii) *Any symbol* $a \in \mathcal{S}_{\text{hyp}}\big((R, +\infty) \times V \times \mathbb{R}^n \cap p^{-1}(I)\big)$ *can be written*

$$a = a^+ + a^-, \quad \textit{with } a^{\pm} \in \mathcal{S}_{\text{hyp}}\big(\Gamma^{\pm}(R, V, I, \tfrac{1}{2})\big).$$

One important point in this proposition is that $\chi^{\pm}_{1\to 2}$ and $a^{\pm}$ can be chosen in $\mathcal{S}_{\text{hyp}}$.

Proof. (i) We may for instance choose

$$\chi^{\pm}_{1\to 2}(r, \theta, \rho, \eta) = \chi_{R_1\to R_2}(r)\chi_{V_1\to V_2}(\theta)\chi_{I_1\to I_2}(p)\chi_{\sigma_1\to\sigma_2}(\pm\rho/p^{1/2}),$$

with $\chi_{R_1\to R_2}, \chi_{\sigma_1\to\sigma_2} \in C^\infty(\mathbb{R})$, $\chi_{V_1\to V_2} \in C_0^\infty(V_2)$ and $\chi_{I_1\to I_2} \in C_0^\infty(I_2)$ such that

$$\operatorname{supp}(\chi_{R_1\to R_2}) \subset (R_2, +\infty), \qquad \operatorname{supp}(\chi_{\sigma_1\to\sigma_2}) \subset (-\sigma_2, +\infty),$$

$$\chi_{R_1\to R_2} \equiv 1 \text{ on } (R_1, +\infty), \quad \chi_{V_1\to V_2} \equiv 1 \text{ on } V_1, \quad \chi_{I_1\to I_2} \equiv 1 \text{ on } I_1, \quad \chi_{\sigma_1\to\sigma_2} \equiv 1 \text{ on } (-\sigma_1, +\infty).$$

Notice that $\rho/p^{1/2}$ is smooth on the support of $\chi_{I_1\to I_2}(p)$. The so defined $\chi^{\pm}_{1\to 2}$ is smooth on $\mathbb{R}^{2n}$, supported in $\Gamma^{\pm}(R_2, V_2, I_2, \sigma_2)$, identically 1 on $\Gamma^{\pm}(R_1, V_1, I_1, \sigma_1)$, and one easily checks that it belongs to $\mathcal{B}_{\text{hyp}}\big(\Gamma^{\pm}(R_2, V_2, I_2, \sigma_2)\big)$, using for instance Lemma 2.4.

(ii) This is very similar to the first case. We may for instance choose

$$a^{\pm}(r, \theta, \rho, \eta) = a(r, \theta, \rho, \eta)\chi^{\pm}_{1/2}(\rho/p^{1/2}),$$

with $\chi^{\pm}_{1/2} \in C^\infty(\mathbb{R})$ such that

$$\chi^{+}_{1/2} + \chi^{-}_{1/2} \equiv 1, \quad \operatorname{supp}(\chi^{+}_{1/2}) \subset \big(-\tfrac{1}{2}, +\infty\big), \quad \operatorname{supp}(\chi^{+}_{1/2}) \subset \big(-\infty, \tfrac{1}{2}\big).$$

Here again $\rho/p^{1/2}$ is smooth on the support of a and $a^{\pm} \in \mathcal{B}_{\text{hyp}}\big(\Gamma^{\pm}(R, V, I, \tfrac{1}{2})\big)$. □

By Proposition 4.1(i), $\Gamma^{\pm}(R_2, V_2, I_2, \sigma_2)$ is a neighborhood of the closure of $\Gamma^{\pm}(R_1, V_1, I_1, \sigma_1)$ under the assumption (4-1). In the following proposition, we make this remark more quantitative.

Proposition 4.2. *Assume* (4-1). *There exists* $\varepsilon > 0$ *such that, for all* $(r', \theta', \rho', \eta') \in \mathbb{R}^{2n}$ *and all* $(r, \theta, \rho, \eta) \in \Gamma^{\pm}(R_1, V_1, I_1, \sigma_1)$,

$$|(r, \theta, \rho, \eta) - (r', \theta', \rho', \eta')| \le \varepsilon \implies (r', \theta', \rho', \eta') \in \Gamma^{\pm}(R_2, V_2, I_2, \sigma_2).$$

Proof. Choose first $\varepsilon_0 > 0$ such that, if $|r - r'| + |\theta - \theta'| \le \varepsilon_0$, $r' > R_2$ and $\theta' \in V_2$. Then, by writing

$$q(r', \theta', e^{-r'}\eta') - q(r', \theta', e^{-r}\eta) = e^{-2r'} q(r', \theta', \eta' - \eta) + (e^{2(r-r')} - 1) q(r', \theta', e^{-r}\eta), \tag{4-2}$$

and using (3-5), (3-6) with the Taylor formula, we get

$$|p(r', \theta', \rho', \eta') - p(r, \theta, \rho, \eta)| \le |\rho^2 - \rho'^2| + C|\eta' - \eta|^2 + C(|r - r'| + |\theta - \theta'|) e^{-2r} |\eta|^2,$$

where $e^{-2r}|\eta|^2$ is bounded, using (3-7). Since ρ is bounded too, we obtain

$$\big|p(r', \theta', \rho', \eta') - p(r, \theta, \rho, \eta)\big| \le C\big|(r, \theta, \rho, \eta) - (r', \theta', \rho', \eta')\big|,$$

provided that $\big|(r, \theta, \rho, \eta) - (r', \theta', \rho', \eta')\big| \le \varepsilon_0$ and therefore,

$$\big|p^{1/2}(r', \theta', \rho', \eta') - p^{1/2}(r, \theta, \rho, \eta)\big| \le C\,\big|(r, \theta, \rho, \eta) - (r', \theta', \rho', \eta')\big|,$$
$$\left|\frac{\rho'}{p^{1/2}(r', \theta', \rho', \eta')} - \frac{\rho}{p^{1/2}(r, \theta, \rho, \eta)}\right| \le C\,\big|(r, \theta, \rho, \eta) - (r', \theta', \rho', \eta')\big|,$$

if $\big|(r, \theta, \rho, \eta) - (r', \theta', \rho', \eta')\big|$ is small enough, using that $I_2 \Subset (0, +\infty)$. The conclusion is then easy. □

Similarly to (2-54), we fix $V_0 \subset \mathbb{R}^{n-1}$ a relatively compact open subset of $\psi_\iota(U_\iota)$ and define

$$R(\varepsilon) = 1/\varepsilon, \qquad V_\varepsilon = \{\theta \in \mathbb{R}^{n-1} : \operatorname{dist}(\theta, V_0) < \varepsilon^2\}. \tag{4-3}$$

In the sequel, we shall need very often the following result on strongly outgoing/incoming areas (see Propositions 4.8, 4.14 and Lemmas 4.11, 4.16). This will for instance be the case when we use Taylor's formula and want to guarantee that the whole segment between two points of a strongly outgoing/incoming area is still contained in such an area.

Proposition 4.3. *For all* $M > 0$, *there exist* $\varepsilon_M > 0$ *and* $C_M > 1$ *such that, for all* $0 < \varepsilon \le \varepsilon_M$, *the following holds*: *if*

$$(r, \theta, \rho, \eta) \in \Gamma_{\mathrm{s}}^{\pm}(\varepsilon), \tag{4-4}$$

and

$$r' - r \ge -M, \quad |\theta' - \theta| < M\varepsilon^2, \quad |\rho' - \rho| < M\varepsilon^2, \quad |\eta' - \eta| < M\varepsilon e^{1/\varepsilon}, \tag{4-5}$$

then, for all $0 \le s \le 1$,

$$(r', \theta', \rho', s\eta') \in \Gamma_{\mathrm{s}}^{\pm}(C_M\varepsilon).$$

In particular, $(r', \theta', \rho', 0) \in \Gamma_{\mathrm{s}}^{\pm}(C_M\varepsilon)$.

Remark. There should not be any confusion between the interpolation parameter $0 \le s \le 1$ and the subscript $_{\mathrm{s}}$, which refers to strongly outgoing/incoming areas (and which are independent of s).

Proof. Using (2-56) and (4-2), we first note the existence of $M' > 0$ such that, for all $0 < \varepsilon < \frac{1}{4}$, if (4-4) and (4-5) hold then

$$\left|p(r', \theta', \rho', s\eta') - p(r, \theta, \rho, \eta)\right| \le M'\varepsilon^2,$$

using in particular that $s\eta' - \eta = s(\eta' - \eta) + (s-1)\eta$. If C_M is large enough and $0 < \varepsilon C_M < \frac{1}{4}$, we obtain

$$0 < \tfrac{1}{4} - C_M\varepsilon < \tfrac{1}{4} - \varepsilon - M'\varepsilon^2 \le p(r', \theta', \rho', s\eta') \le 4 + \varepsilon + M'\varepsilon^2 < 4 + C_M\varepsilon.$$

If $0 < \varepsilon \le \varepsilon_M$ with ε_M small enough, then $p(r', \theta', \rho', s\eta')/p(r, \theta, \rho, \eta) = 1 + \mathcal{O}(\varepsilon^2)$ so that

$$\frac{\pm\rho'}{p(r', \theta', \rho', s\eta')^{1/2}} = \frac{\pm\rho}{p(r, \theta, \rho, \eta)^{1/2}} \frac{p(r, \theta, \rho, \eta)^{1/2}}{p(r', \theta', \rho', s\eta')^{1/2}} \pm \frac{\rho' - \rho}{p(r', \theta', \rho', s\eta')^{1/2}} > 1 - (C_M\varepsilon)^2,$$

by possibly increasing C_M. In addition, $\operatorname{dist}(\theta, V_0) \le |\theta' - \theta| + \operatorname{dist}(\theta, V_0) < (C_M\varepsilon)^2$, by possibly increasing C_M again and decreasing ε_M. Finally, $r' \ge r - M > e^{1/\varepsilon} - M > e^{1/C_M\varepsilon}$, for all $0 < \varepsilon \le \varepsilon_M$ by possibly decreasing ε_M again, so $(r', \theta', \rho', s\eta') \in \Gamma_s^{\pm}(C_M\varepsilon)$. □

We can now prove Proposition 2.13, which states that one can reach a strongly outgoing (incoming) area from an outgoing (incoming) one in finite time, along the geodesic flow.

Proof of Proposition 2.13. We consider only the outgoing case. With no loss of generality, we may assume that $0 < \sigma < 1$. By choosing $R \ge R'_\sigma$ large enough, we can use Proposition 3.4 and Corollary 3.10. By Proposition 3.4, we have $r_t \ge r + ct - C$ for some $C, c > 0$, hence $r_t > R(\varepsilon)$ for all $t \ge t_{R,\varepsilon}$, provided

$$ct_{R,\varepsilon} - C + R > R(\varepsilon). \tag{4-6}$$

By Proposition 3.8, we have $|\theta^t - \theta| \lesssim e^{-r}$ hence $\theta^t \in V_\varepsilon$, for ε small enough and all $t \ge 0$, since $e^{-1/\varepsilon} \ll \varepsilon^2$. Using (3-33) and the energy conservation, we shall have $\rho^t/p^{1/2}(r^t, \theta^t, \rho^t, \eta^t) > 1 - \varepsilon^2$ provided for instance that

$$e^{-p^{1/2}t_{R,\varepsilon}} \le \varepsilon^3, \tag{4-7}$$

with ε small enough. Choosing $t_{R,\varepsilon}$ so that (4-6) and (4-7) hold, we get the result. □

We conclude this part with an explicit construction for cutoffs.

In Section 5, we will need a result similar to Proposition 4.1(i). This is the purpose of the following result.

Proposition 4.4. *We can find* $0 < \nu < 1$ *and a family of cutoffs* $\chi^{\pm}_{\varepsilon^2\to\varepsilon} \in \mathcal{S}_{\text{hyp}}(\Gamma_s^{\pm}(\varepsilon^{1+\nu}))$, *defined for all* ε *small enough, such that*

$$\chi^{\pm}_{\varepsilon^2\to\varepsilon} = 1 \quad \text{on} \quad \Gamma_s^{\pm}(\varepsilon^2) \tag{4-8}$$

and, uniformly on $\mathbb{R}^{2n}$,

$$\left|e^{-2r}|\eta|^j \partial_{r,\theta,\rho,\eta}\chi^{\pm}_{\varepsilon^2\to\varepsilon}\right| + \left|e^{-2r}|\eta|^2 \partial_{\rho,\eta}\partial_{r,\theta}\chi^{\pm}_{\varepsilon^2\to\varepsilon}\right| \lesssim \varepsilon^{1/2}, \qquad j = 1, 2. \tag{4-9}$$

That we can find, for each ε, $\chi^{\pm}_{\varepsilon^2\to\varepsilon} \in \mathcal{S}_{\text{hyp}}(\Gamma_s^{\pm}(\varepsilon^{1+\nu}))$ satisfying (4-8) would follow directly from Proposition 4.1. The important additional point here is the control with respect to ε given by (4-9). Note also that the power $\frac{1}{2}$ is essentially irrelevant: we only mean that the left-hand side of (4-9) is uniformly small as $\varepsilon \to 0$. This rather technical point will only be used in Section 5 to globalize suitably certain phase functions.

Proof. For $0 < \delta < 1$ to be chosen later, we consider the characteristic functions $\bar{\chi}^I_{\varepsilon^{1+\delta}}$ and $\bar{\chi}^V_{\varepsilon^{2+\delta}}$ of $\left(\frac{1}{4}-\varepsilon^{1+\delta}, 4+\varepsilon^{1+\delta}\right)$ and $V + B(0, \varepsilon^{2+\delta})$ respectively. Choose $\zeta^I \in C_0^\infty(\mathbb{R})$, $\zeta^V \in C_0^\infty(\mathbb{R}^{n-1})$ both equal to 1 near 0, such that $\int \zeta^I = \int \zeta^V = 1$ and set

$$\chi^I_{\varepsilon^{1+\delta}}(\lambda) = \int \bar{\chi}^I_{\varepsilon^{1+\delta}}(\mu)\zeta^I\left(\frac{\lambda-\mu}{\varepsilon^{1+2\delta}}\right)\varepsilon^{-1-2\delta}d\mu, \quad \chi^V_{\varepsilon^{2+\delta}}(\theta) = \int \bar{\chi}^V_{\varepsilon^{2+\delta}}(\vartheta)\zeta^V\left(\frac{\theta-\vartheta}{\varepsilon^{2+2\delta}}\right)\varepsilon^{-(n-1)(2+2\delta)}d\vartheta.$$

One then easily checks that, if ε is small enough,

$$\begin{aligned} &\chi^I_{\varepsilon^{1+\delta}} \equiv 1 \ \text{ on } \left(\tfrac{1}{4}-\varepsilon^2, 4+\varepsilon^2\right), && \chi^V_{\varepsilon^{2+\delta}}(\theta) = 1 \ \text{ if } \operatorname{dist}(\theta, V) < \varepsilon^4, \\ &\chi^I_{\varepsilon^{1+\delta}} \equiv 0 \ \text{ outside } \left(\tfrac{1}{4}-\varepsilon^{1+\frac{\delta}{4}}, 4+\varepsilon^{1+\frac{\delta}{4}}\right), && \chi^V_{\varepsilon^{2+\delta}}(\theta) = 0 \ \text{ if } \operatorname{dist}(\theta, V) \geq \varepsilon^{2+\frac{\delta}{2}}. \end{aligned}$$

Choosing $\omega \in C^\infty(\mathbb{R})$ supported in $\left(\frac{1}{4}, \infty\right)$ such that $\omega = 1$ near $\left[\frac{1}{3}, \infty\right)$, we now define

$$\chi^{\pm}_{\varepsilon^2\to\varepsilon}(r,\theta,\rho,\eta) = \omega\left(\frac{r}{R(\varepsilon^{3/2})}\right)\chi^V_{\varepsilon^{2+\delta}}(\theta)\,\chi^I_{\varepsilon^{1+\delta}}(p)\,\omega(\pm\rho)\,\zeta^I\left(e^{-2r}\frac{|\eta|^2}{\varepsilon^{4-\delta}}\right).$$

On the support of $\chi^I_{\varepsilon^{1+\delta}}(p)$, we have $\rho^2 \geq \frac{1}{4} - \mathcal{O}(\varepsilon)$ so the factor $\omega(\pm\rho)$ only determines the sign of ρ. By (2-55) and (2-56), one sees that (4-8) holds with $\nu = \delta/2$, if ε is small enough. Furthermore, $\chi^{\pm}_{\varepsilon^2\to\varepsilon}$ is supported in $\Gamma^{\pm}_{\mathrm{s}}(\varepsilon^{1+\nu})$ and belongs to $\mathcal{B}_{\mathrm{hyp}}(\Gamma^{\pm}_{\mathrm{s}}(\varepsilon^{1+\nu}))$.

We prove (4-9). Since $e^{-2r}|\eta|^2 \lesssim \varepsilon^{4-\delta}$ on the support of $\chi^{\pm}_{\varepsilon^2\to\varepsilon}$, the first-order derivatives satisfy

$$\begin{aligned} &|\partial_r\chi^{\pm}_{\varepsilon^2\to\varepsilon}| \lesssim R(\varepsilon^{3/2})^{-1} + e^{-2r}|\eta|^2(\varepsilon^{-1-2\delta}+\varepsilon^{-4+\delta}) \lesssim 1, \\ &|\partial_\rho\chi^{\pm}_{\varepsilon^2\to\varepsilon}| \lesssim \varepsilon^{-1-2\delta}, \\ &|\partial_\theta\chi^{\pm}_{\varepsilon^2\to\varepsilon}| \lesssim \varepsilon^{-2-2\delta} + \varepsilon^{-1-2\delta}e^{-2r}|\eta|^2 \lesssim \varepsilon^{-2-2\delta}, \\ &|\partial_\eta\chi^{\pm}_{\varepsilon^2\to\varepsilon}| \lesssim e^{-2r}|\eta|(\varepsilon^{-1-2\delta}+\varepsilon^{-4+\delta}) \ll e^{-\varepsilon^{-1/2}}, \end{aligned}$$

using the fact that $e^{-2r}|\eta| \lesssim e^{-r} \leq e^{-\varepsilon^{-1}}$ for the last estimate. Similarly

$$|\partial_\rho\partial_{r,\theta}\chi^{\pm}_{\varepsilon^2\to\varepsilon}| \lesssim \varepsilon^{-2-2\delta}\times\varepsilon^{-1-2\delta} = \varepsilon^{-3-4\delta}, \quad |\partial_\eta\partial_{r,\theta}\chi^{\pm}_{\varepsilon^2\to\varepsilon}| \lesssim e^{-\varepsilon^{-1/2}}.$$

Since $e^{-2r}|\eta|^2e^{-3-4\delta} \lesssim \varepsilon^{1-5\delta}$ and $e^{-2r}|\eta| \ll e^{-\varepsilon^{-1/2}}$, the result follows with $\delta = \frac{1}{10}$ (hence with $\nu = \frac{1}{20}$). □

We finally consider the statements involving intermediate areas.

Proof of Proposition 2.15. By (2-58) and (2-59), we can find $\chi_{-\infty}, \chi_{+\infty} \in C^\infty(\mathbb{R})$ and

$$\chi_l \in C_0^\infty(-\sigma_{l+1}, -\sigma_{l-1}),$$

for $1 \leq l \leq L-1$, such that

$$\operatorname{supp}(\chi_{-\infty}) \subset (-\infty, -\sigma_{L-1}), \quad \operatorname{supp}(\chi_{+\infty}) \in (1-\varepsilon^2, +\infty) \quad \text{and} \quad \chi_{+\infty} + \sum_{l=1}^{L-1}\chi_l + \chi_{-\infty} \equiv 1 \ \text{ on } \mathbb{R}.$$

This simply relies on the overlapping property of the intervals in (2-59). We then obtain the result by considering

$$a_{\mathrm{s}}^{\pm}(r,\theta,\rho,\eta)=a^{\pm}(r,\theta,\rho,\eta)\,\chi_{+\infty}(\pm\rho/p^{1/2}),$$
$$a_{l,\mathrm{inter}}^{\pm}(r,\theta,\rho,\eta)=a^{\pm}(r,\theta,\rho,\eta)\,\chi_{l}(\pm\rho/p^{1/2}),\quad 1\le l\le L-2,$$
$$a_{L-1,\mathrm{inter}}^{\pm}(r,\theta,\rho,\eta)=a^{\pm}(r,\theta,\rho,\eta)\,(\chi_{L-1}+\chi_{-\infty})(\pm\rho/p^{1/2}).$$

since, in the definition of $a_{L-1,\mathrm{inter}}^{\pm}$, the cutoff guarantees that $\pm\rho/p^{1/2}<-\sigma_{L-2}$ and $a^{\pm}$ that $\pm\rho/p^{1/2}>-\frac{1}{2}=-\sigma_L$. □

Proof of Proposition 2.16. We consider the outgoing case, the incoming one being similar. Using Corollary 3.10, we may assume that, if ε is small enough, (3-22) holds for any initial condition such that $r>R(\varepsilon)$, $\theta\in V$ and $\rho\ge-\frac{1}{2}p^{1/2}$. In particular $t\mapsto\rho^t$ is nondecreasing for $t\ge 0$. Assume that $\frac{1}{2}\le\rho/p^{1/2}\le 1-(\varepsilon/2)^2$ and set

$$t_\varepsilon=t_\varepsilon(r,\theta,\rho,\eta):=\sup\left\{t\ge 0:\frac{\rho^s}{p^{1/2}}<\frac{\rho}{p^{1/2}}+\varepsilon^4\ \text{ for all } s\in[0,t]\right\}.$$

Notice that t_ε is finite by Lemma 3.9 and that $\rho^{t_\varepsilon}=\rho+p^{1/2}\varepsilon^4$. If $1-(\varepsilon/2)^2+\varepsilon^4\ge\frac{1}{2}$, we have $|\rho^t/p^{1/2}|\le 1-(\varepsilon/2)^2+\varepsilon^4$ on $[0,t_\varepsilon)$. Thus, if ε is small enough (independent of (r,θ,ρ,η)), we have $(\rho^t)^2/p\le 1-(\varepsilon/2)^2$ for all $t\in[0,t_\varepsilon)$ and then, by (3-22) again, we have $\dot\rho^t\ge(\varepsilon/2)^2p$ on $[0,t_\varepsilon]$, so

$$\rho^{t_\varepsilon}-\rho\ge(\varepsilon/2)^2pt_\varepsilon.$$

This shows that $t_\varepsilon\le\varepsilon^4/(\varepsilon/2)^2p=4\varepsilon^2/p$. Then, for ε small enough such that $4\varepsilon^2/p\le\underline{t}$ for all (r,θ,ρ,η) in

$$\left\{(r,\theta,\rho,\eta)\in\mathbb{R}^{2n}:r>R(\varepsilon),\ \theta\in V,\ p\in I(\varepsilon),\ -\tfrac{1}{2}\le\rho/p^{1/2}\le 1-(\varepsilon/2)^2\right\},\tag{4-10}$$

and with $\delta=\varepsilon^4/2$, we have $\rho^t-\rho\ge 2\delta p^{1/2}$ for all $t\ge\underline{t}$. This implies (2-62) since, for any choice of $\sigma_0,\dots,\sigma_L$ and any l, $\Gamma_{\mathrm{inter}}^{\pm}(\varepsilon,\delta;l)$ is contained in (4-10). □

4B. *Hyperbolic long/short-range symbols.* In this short subsection, we introduce the definitions of short/long-range hyperbolic symbols which will be useful for the resolution of transport equations in Section 4E. We prove in passing Proposition 4.6 below which will be used at several places, in particular in Section 4C.

Definition 4.5. A smooth function $a_\pm$ on $\Gamma_{\mathrm{s}}^{\pm}(\varepsilon)$ is said to be of hyperbolic short range if

$$|\partial_r^j\partial_\theta^\alpha\partial_\rho^k\partial_\eta^\beta a_\pm(r,\theta,\rho,\eta)|\lesssim\langle r-\log\langle\eta\rangle\rangle^{-\tau-1-j},\qquad(r,\theta,\rho,\eta)\in\Gamma_{\mathrm{s}}^{\pm}(\varepsilon),\tag{4-11}$$

and of hyperbolic long range if

$$|\partial_r^j\partial_\theta^\alpha\partial_\rho^k\partial_\eta^\beta a_\pm(r,\theta,\rho,\eta)|\lesssim\langle r-\log\langle\eta\rangle\rangle^{-\tau-j},\qquad(r,\theta,\rho,\eta)\in\Gamma_{\mathrm{s}}^{\pm}(\varepsilon).\tag{4-12}$$

Notice that in this definition, we do not assume that $a\in\mathscr{B}_{\mathrm{hyp}}(\Gamma_{\mathrm{s}}^{\pm}(\varepsilon))$. However, this will be the case in the applications and we now give a simple criterion to check that a symbol $a\in\mathscr{B}_{\mathrm{hyp}}(\Gamma_{\mathrm{s}}^{\pm}(\varepsilon))$ is of hyperbolic short/long range.

For ε small enough, by restricting a to a smaller area $\Gamma_{\mathrm{s}}^{\pm}(\varepsilon/C)$, with $C>1$ large enough (or to $\Gamma_{\mathrm{s}}^{\pm}(\varepsilon^2)$, $\Gamma_{\mathrm{s}}^{\pm}(\varepsilon^3)$ as it will be the case in the applications), using Lemma 2.4 and Proposition 4.3, we have

$$a(r,\theta,\rho,\eta)=a(r,\theta,\rho,0)+\int_0^1(\partial_\xi\tilde a)(r,\theta,\rho,s\xi)_{|\xi=e^{-r}\eta}\,ds\cdot e^{-r}\eta,\tag{4-13}$$

where $\tilde{a}$ belongs to $C_b^\infty(F_{\mathrm{hyp}}(\Gamma_{\mathrm{s}}^\pm(\varepsilon)))$ and $(r,\theta,\rho,s\eta)\in\Gamma_{\mathrm{s}}^\pm(\varepsilon)$ if $(r,\theta,\rho,\eta)\in\Gamma_{\mathrm{s}}^\pm(\varepsilon/C)$. Since

$$|\partial_r^j\partial_\eta^\beta e^{-r}\eta|\lesssim\langle r-\log\langle\eta\rangle\rangle^{-N}\qquad\text{for all } N>0 \text{ and } (r,\theta,\rho,\eta)\in\Gamma_{\mathrm{s}}^\pm(\varepsilon),$$

we obtain that, for $a\in\mathcal{B}_{\mathrm{hyp}}(\Gamma_{\mathrm{s}}^\pm(\varepsilon))$,

$$a \text{ is of hyperbolic short/long range in } \Gamma_{\mathrm{s}}^\pm(\varepsilon/C) \iff a_{|\eta=0} \text{ is of usual short/long range} \tag{4-14}$$

in the sense that

$$\left|\left(\partial_r^j\partial_\theta^\alpha\partial_\rho^k a\right)(r,\theta,\rho,0)\right|\lesssim\langle r\rangle^{-\tau-j}\quad\text{for } (r,\theta,\rho,0)\in\Gamma_{\mathrm{s}}^\pm(\varepsilon),$$

in the long-range case (recall that $0<\tau\le 1$) and

$$\left|\left(\partial_r^j\partial_\theta^\alpha\partial_\rho^k a\right)(r,\theta,\rho,0)\right|\lesssim\langle r\rangle^{-\tau-1-j}\quad\text{for } (r,\theta,\rho,0)\in\Gamma_{\mathrm{s}}^\pm(\varepsilon),$$

in the short-range case.

To calculate $a_{|\eta=0}$ in some applications, we shall use the following elementary result.

Proposition 4.6. *For all $r>0$, all $\theta\in\mathbb{R}^{n-1}$ and all $\pm\rho>0$, we have, for all $\pm t\ge 0$,*

$$(r^t,\theta^t,\rho^t,\eta^t)_{|\eta=0}=(r+2t\rho,\theta,\rho,0), \tag{4-15}$$

$$\partial_\eta(r^t,\theta^t,\rho^t,\eta^t)_{|\eta=0}=\left(0,\int_0^t e^{-2r-4s\rho}\,\mathrm{hess}_\eta[q](r+s\rho,\theta)\,ds,0,\mathrm{Id}\right). \tag{4-16}$$

where $\mathrm{hess}_\eta[q](r,\theta)$ is the Hessian matrix of q with respect to η (which is independent of η).

Proof. One simply checks that the right-hand side of (4-15) is a solution to (3-10) (with $w(r)=e^{-2r}$) for $\pm t\ge 0$. Applying then ∂_η to (3-10), one sees easily as well that the right-hand side of (4-16) is a solution to the corresponding system. □

Remark. If ε is small enough then, on $\Gamma_{\mathrm{s}}^\pm(\varepsilon)$, we have

$$r-\log\langle\eta\rangle\ge 0. \tag{4-17}$$

In particular, in this region, $\langle r-\log\langle\eta\rangle\rangle$ is equivalent to the weight

$$\langle r-\log\langle\eta\rangle\rangle_+:=\max(1,r-\log\langle\eta\rangle)$$

which was introduced by Froese and Hislop [1989]. For the study of global in time estimates, which we hope to consider in a future work, the resolvent estimates proved in [Bouclet 2006] suggest that the hyperbolic short/long-range conditions (4-11)/(4-12) would play the same role as the usual Euclidean short/long-range conditions used in [Bouclet and Tzvetkov 2008].

4C. *The Hamilton–Jacobi equation.* We now use the results of Section 3B to solve the time-independent Hamilton–Jacobi equations giving the phases of the Isozaki–Kiada parametrix.

Lemma 4.7. *There exists $0<\varepsilon_0<1$ such that, for all $0<\varepsilon\le\varepsilon_0$ and all $\pm t\ge 0$, the map*

$$\Psi_t^\pm:(r,\theta,\rho,\eta)\mapsto(r,\theta,\rho^t,\eta^t)$$

is a diffeomorphism from $\Gamma_{\mathrm{s}}^\pm(\varepsilon)$ onto its range and

$$\Gamma_{\mathrm{s}}^\pm(\varepsilon^3)\subset\Psi_t^\pm\left(\Gamma_{\mathrm{s}}^\pm(\varepsilon)\right)\quad\textit{for all } \pm t\ge 0. \tag{4-18}$$

Proof. See Appendix A. □

The power ε^3 in (4-18) is not very important. It is only a rough explicit quantitative bound for the size of a strongly outgoing area contained in $\Psi_t^+(\Gamma_s^+(\varepsilon))$ for all $t \geq 0$ (or a strongly incoming area contained in $\Psi_t^-(\Gamma_s^-(\varepsilon))$ for all $t \leq 0$).

The components of the inverse map $(\Psi_t^{\pm})^{-1}$ are of the form $(r, \theta, \rho_t, \eta_t)$ with

$$\rho_t = \rho_t(r, \theta, \rho, \eta), \qquad \eta_t = \eta_t(r, \theta, \rho, \eta).$$

Here we omit the $\pm$ dependence for notational simplicity. We thus have

$$\rho^t(r, \theta, \rho_t, \eta_t) = \rho, \qquad \eta^t(r, \theta, \rho_t, \eta_t) = \eta, \tag{4-19}$$

at least for all $(r, \theta, \rho, \eta) \in \Gamma_s^{\pm}(\varepsilon_0^3)$ and $\pm t \geq 0$.

Remark. It follows from the proof of Lemma 4.7 and the scaling properties (3-34), (3-35) that $\Psi_t^{\pm}$ is actually a diffeomorphism from the cone generated by $\Gamma_s^{\pm}(\varepsilon_0)$ onto its range, the latter range containing the cone generated by $\Gamma_s^{\pm}(\varepsilon_0^3)$. Therefore (ρ_t, η_t) is actually the restriction to $\Gamma_s^{\pm}(\varepsilon_0^3)$ of a map defined on the cone generated by $\Gamma_s^{\pm}(\varepsilon_0^3)$ and, using (3-35), we have

$$(\rho_t, \eta_t)(r, \theta, \lambda\rho, \lambda\eta) = \lambda(\rho_{\lambda t}, \eta_{\lambda t})(r, \theta, \rho, \eta) \quad \text{if } \pm t \geq 0 \text{ and } (r, \theta, \rho, \eta) \in \Gamma_s^{\pm}(\varepsilon_0^3), \tag{4-20}$$

for all $\lambda > 0$.

Proposition 4.8. *There exists $\varepsilon_1 \leq \varepsilon_0^3$ such that, for all $j, k \in \mathbb{N}_0$, $\alpha, \beta \in \mathbb{N}_0^{n-1}$,*

$$\left|D_{\mathrm{hyp}}^{j\alpha k\beta}(\rho_t - \rho)\right| + \left|D_{\mathrm{hyp}}^{j\alpha k\beta}(\eta_t - \eta)\right| \lesssim 1, \quad (r, \theta, \rho, \eta) \in \Gamma_s^{\pm}(\varepsilon_1), \quad \pm t \geq 0. \tag{4-21}$$

In addition, if $(r, \theta, \rho, 0) \in \Gamma_s^{\pm}(\varepsilon_1)$, we have

$$(\rho_t, \eta_t)_{|\eta=0} = (\rho, 0), \tag{4-22}$$

$$\partial_\eta(\rho_t - \rho, \eta_t - \eta)_{|\eta=0} = (0, 0). \tag{4-23}$$

Proof. By (4-18), any $(r, \theta, \rho, \eta) \in \Gamma_s^{\pm}(\varepsilon_0^3)$ can be written as $\Psi_t^{\pm}(r, \theta, \tilde{\rho}, \tilde{\eta})$ with $(r, \theta, \tilde{\rho}, \tilde{\eta}) \in \Gamma_s^{\pm}(\varepsilon_0)$. Hence

$$\sup_{\Gamma_s^{\pm}(\varepsilon_0^3)} |\rho_t - \rho| + |\eta_t - \eta| \leq \sup_{\Gamma_s^{\pm}(\varepsilon_0)} |\tilde{\rho} - \rho^t(\tilde{r}, \tilde{\theta}, \tilde{\rho}, \tilde{\eta})| + |\tilde{\eta} - \eta^t(\tilde{r}, \tilde{\theta}, \tilde{\rho}, \tilde{\eta})|.$$

By (3-31), the right-hand side is bounded, so we obtain (4-21) for $j + |\alpha| + k + |\beta| = 0$. Then, for ε small enough, using Propositions 3.8 and 4.6, we remark that, for $(r, \theta, \rho, \eta) \in \Gamma_s^{\pm}(\varepsilon)$,

$$\left|\partial_{\rho,\eta}\left(\rho^t - \rho, \eta^t - \eta\right)\right| \leq \int_0^1 \left|\partial_\eta \partial_{\rho,\eta}(\rho^t, \eta^t)(r, \theta, \rho, s\eta)\right| ds |\eta| \lesssim |e^{-r}\eta| \lesssim \varepsilon,$$

since, by Proposition 4.3, $(r, \theta, \rho, s\eta) \in \Gamma_s^{\pm}(\varepsilon_0)$ if $(r, \theta, \rho, \eta) \in \Gamma_s^{\pm}(\varepsilon)$ and ε is small enough. Therefore, if ε is small enough,

$$|\partial_{\rho,\eta}(\rho^t, \eta^t) - \mathrm{Id}_n| \leq \tfrac{1}{2} \quad \text{on} \quad \Gamma_s^{\pm}(\varepsilon), \tag{4-24}$$

for all $\pm t \geq 0$. Here $|\cdot|$ is a matrix norm. We can now prove (4-21) when $j+|\alpha|+k+|\beta| \geq 1$. Assume first that $D_{\text{hyp}}^{j\alpha k\beta} = e^r \partial_\eta^\beta$, with $|\beta| = 1$, and set for simplicity

$$\Xi_t(r,\theta,\rho,\eta) = (\rho_t,\eta_t)(r,\theta,\rho,\eta), \quad \Xi^t(r,\theta,\tilde{\rho},\tilde{\eta}) = (\rho^t,\eta^t)(r,\theta,\tilde{\rho},\tilde{\eta}), \quad \Xi = (\rho,\eta),$$

when $(r,\theta,\rho,\eta) \in \Gamma_{\text{s}}^{\pm}(\varepsilon^3)$, $(r,\theta,\tilde{\rho},\tilde{\eta}) \in \Gamma_{\text{s}}^{\pm}(\varepsilon)$ and $\pm t \geq 0$. Applying $e^r \partial_\eta^\beta$ to (4-19), we get

$$(\partial_{\tilde{\rho},\tilde{\eta}} \Xi^t)(r,\theta,\Xi_t) e^r \partial_\eta^\beta \Xi_t = (0, e^r \partial_\eta^\beta \eta) = e^r \partial_\eta^\beta \Xi,$$

and using that $(\partial_{\tilde{\rho},\tilde{\eta}} \Xi^t) \partial_\eta^\beta \Xi = \partial_{\tilde{\eta}}^\beta \Xi^t$, we obtain

$$(\partial_{\tilde{\rho},\tilde{\eta}} \Xi^t)(r,\theta,\Xi_t) e^r \partial_\eta^\beta (\Xi_t - \Xi) = e^r \left(\partial_{\tilde{\eta}}(\Xi - \Xi^t)\right)_{|(r,\theta,\Xi_t)},$$

where the right-hand side is bounded, by (3-31). Using (4-24), we see that $e^r \partial_\eta^\alpha(\Xi_t - \Xi)$ is bounded on $\Gamma_{\text{s}}^{\pm}(\varepsilon_1)$ for $\pm t \geq 0$, by choosing $\varepsilon_1 \leq \varepsilon_0^3$ and such that (4-24) holds. The other first-order derivatives are treated similarly and are simpler to handle since there is no e^r. When $j+|\alpha|+k+|\beta| \geq 2$, we iterate this process using Lemma 3.6. To complete the proof of the proposition, we finally note that (4-22) and (4-23) are easy consequences of (4-19) and Proposition 4.6. □

By Propositions 4.7 and 4.8, we can define $r_t^s = r_t^s(r,\theta,\rho,\eta)$ and $\theta_t^s = \theta_t^s(r,\theta,\rho,\eta)$ on $\Gamma_{\text{s}}^{\pm}(\varepsilon_1)$ by

$$r_t^s = r^s(r,\theta,\rho_t,\eta_t), \quad \theta_t^s = \theta^s(r,\theta,\rho_t,\eta_t) \quad \text{for } \pm t \geq \pm s \geq 0,$$

where $\pm t \geq \pm s \geq 0$ means more precisely that $t \geq s \geq 0$ if $(r,\theta,\rho,\eta) \in \Gamma_{\text{s}}^{+}(\varepsilon_1)$ and $t \leq s \leq 0$ if $(r,\theta,\rho,\eta) \in \Gamma_{\text{s}}^{-}(\varepsilon_1)$. Here we assume that ε_1 is small enough so that Proposition 3.8 hold for $r > R(\varepsilon_1)$ and $\sigma = \frac{1}{2}$ (for instance), which justifies that r_t^s and θ_t^s are well defined (and that their derivatives can be estimated using Proposition 3.8).

By the classical Hamilton–Jacobi theory, the function $\Sigma_\pm$ defined by

$$\Sigma_\pm(t,r,\theta,\rho,\eta) = r_t^t \rho + \theta_t^t \cdot \eta - t\rho^2 - t e^{-2r_t^t} q(r_t^t,\theta_t^t,\eta) \tag{4-25}$$

solve the following time-dependent eikonal equation, for $(r,\theta,\rho,\eta) \in \Gamma_{\text{s}}^{\pm}(\varepsilon_1)$ and $\pm t \geq 0$,

$$\partial_t \Sigma_\pm = p(r,\theta,\partial_r \Sigma_\pm, \partial_\theta \Sigma_\pm), \quad \Sigma_\pm|_{t=0} = r\rho + \theta \cdot \eta. \tag{4-26}$$

To put it in a more standard way, note that (4-25) is obtained by defining $\Sigma_\pm$ via

$$\Sigma_\pm(t,r,\theta,\rho^t,\eta^t) = r^t \rho^t + \theta^t \cdot \eta^t - t p(r^t,\theta^t,\rho^t,\eta^t). \tag{4-27}$$

(This simple expression uses the fact that p is homogeneous of degree 2 in (ρ,η).) Now assume for a while that

$$S_\pm(r,\theta,\rho,\eta) := r\rho + \theta \cdot \eta + \int_0^{\pm\infty} \partial_t \left(\Sigma_\pm(t,r,\theta,\rho,\eta) - t\rho^2\right) dt \tag{4-28}$$

is well defined on $\Gamma_{\text{s}}^{\pm}(\varepsilon_1)$. Then, at least formally,

$$\partial_{r,\theta} S_\pm(r,\theta,\rho,\eta) = \lim_{t\to\pm\infty} \partial_{r,\theta} \Sigma_\pm(t,r,\theta,\rho,\eta). \tag{4-29}$$

The latter only uses the fact that the term $t\rho^2$ inside the integral is independent of r,θ. If we know in addition that

$$\lim_{t\to\pm\infty} \partial_\rho \Sigma_\pm(t,r,\theta,\rho,\eta) = +\infty, \tag{4-30}$$

then, using the fact that $\Sigma^{\pm}$ are generating functions of Φ^t, that is,

$$\Phi^t(r,\theta,\partial_r\Sigma_\pm,\partial_\theta\Sigma_\pm)=(\partial_\rho\Sigma_\pm,\partial_\eta\Sigma_\pm,\rho,\eta)\quad \text{for } \pm t\geq 0, \tag{4-31}$$

we obtain, on $\Gamma_{\rm s}^{\pm}(\varepsilon_1)$,

$$p(r,\theta,\partial_r S_\pm,\partial_\theta S_\pm)=\lim_{t\to\pm\infty}p(\partial_\rho\Sigma_\pm,\partial_\eta\Sigma_\pm,\rho,\eta)=\rho^2. \tag{4-32}$$

Proposition 4.9. *There exist* $0<\varepsilon_2\leq\varepsilon_1$ *such that we can find* $S_\pm=S_\pm(r,\theta,\rho,\eta)$, *defined on* $\Gamma_{\rm s}^{\pm}(\varepsilon_2)$, *real-valued, satisfying*

$$p(r,\theta,\partial_r S_\pm,\partial_\theta S_\pm)=\rho^2 \quad \textit{on } \Gamma_{\rm s}^{\pm}(\varepsilon_2), \tag{4-33}$$

and such that

$$S_\pm(r,\theta,\rho,\eta)=r\rho+\theta\cdot\eta+\varphi_\pm(r,\theta,\rho,\eta), \tag{4-34}$$

for some $\varphi_\pm\in\mathscr{B}_{\rm hyp}(\Gamma_{\rm s}^{\pm}(\varepsilon_2))$ *satisfying, when* $(r,\theta,\rho,0)\in\Gamma_{\rm s}^{\pm}(\varepsilon_2)$,

$$\varphi_{\pm|\eta=0}=0,\quad e^r\partial_\eta\varphi_{\pm|\eta=0}=0,\quad e^{2r}\,\mathrm{hess}_\eta[\varphi_\pm]_{|\eta=0}=\int_0^{\pm\infty}e^{-4t\rho}\,\mathrm{hess}_\eta[q](r+2t\rho,\theta)dt. \tag{4-35}$$

It is convenient to note that, by possibly decreasing ε_2 and by using Lemma 2.4, (4-13), and the first two equalities in (4-35), we can write

$$\varphi_\pm(r,\theta,\rho,\eta)=\sum_{|\beta|=2}a_\beta^{\pm}(r,\theta,\rho,e^{-r}\eta)e^{-2r}\eta^\beta, \tag{4-36}$$

with $a_\beta^{\pm}\in C_b^\infty(F_{\rm hyp}(\Gamma^{\pm}(\varepsilon_2)))$.

Proof. We consider only the outgoing case. To complete the proof of (4-33), we have to prove the missing details, namely the convergence of the integral in (4-28) (plus its derivability) and the limits (4-29) and (4-30). Defining $(\rho_t^s,\eta_t^s):=(\rho^s,\eta^s)(r,\theta,\rho_t,\eta_t)$, the equations of motion yield

$$r_t^t=r+2\int_0^t\rho_t^s\,ds=r+2t\rho-2\int_0^t\int_s^t e^{-2r_t^u}\big(2q(r_t^u,\theta_t^u,\eta_t^u)-(\partial_r q)(r_t^u,\theta_t^u,\eta_t^u)\big)\,du\,ds. \tag{4-37}$$

By Propositions 3.8 and 4.8, we have the following bounds on $\Gamma_{\rm s}^{+}(\varepsilon_1)$, for $s\geq 0$ and $t\geq 0$,

$$\big|D^{\rm hyp}_{j\alpha k\beta}(r_t^s-r)\big|\lesssim\langle s\rangle,\quad \big|D^{\rm hyp}_{j\alpha k\beta}(\theta_t^s-\theta)\big|\lesssim e^{-r},\quad \big|D^{\rm hyp}_{j\alpha k\beta}(\eta_t^s-\eta)\big|\lesssim 1. \tag{4-38}$$

In addition, using Proposition 3.4 and (4-18), we have, for $s\geq 0$ and $t\geq 0$,

$$r_t^s\geq r+2(1-\varepsilon^6)sp^{1/2}(r,\theta,\rho_t,\eta_t)-C\geq r+s/4-C\quad \text{on } \Gamma_{\rm s}^{+}(\varepsilon^3), \tag{4-39}$$

with ε small enough such that, $p^{1/2}(r,\theta,\rho_t,\eta_t)\geq\frac14$. Using (4-37), (4-38), (4-39), with $\varepsilon_2:=\varepsilon^3\leq\varepsilon_1$ small enough, and Lemma 3.6, we obtain the existence of a bounded family $(a_t)_{t\geq 0}$ in $\mathscr{B}_{\rm hyp}(\Gamma_{\rm s}^{+}(\varepsilon_2))$ such that

$$r_t^t=r+2t\rho+a_t(r,\theta,\rho,\eta)\quad \text{for } t\geq 0. \tag{4-40}$$

One shows similarly that $(\theta_t^t-\theta)\cdot\eta=e^r(\theta_t^t-\theta)\cdot e^{-r}\eta$ is bounded in $\mathscr{B}_{\rm hyp}(\Gamma_{\rm s}^{+}(\varepsilon_2))$ for $t\geq 0$, and hence that

$$\Sigma_+-(r\rho+\theta\cdot\eta+t\rho^2)\quad \text{is bounded in } \mathscr{B}_{\rm hyp}(\Gamma_{\rm s}^{+}(\varepsilon_2))\ \text{ for } t\geq 0, \tag{4-41}$$

which proves (4-30). Then, using (4-26) and (4-31), we note that

$$\partial_t \Sigma_+ - \rho^2 = e^{-2\partial_\rho \Sigma_+} q(\partial_\rho \Sigma_+, \partial_\eta \Sigma_+, \eta). \tag{4-42}$$

Therefore, using (4-39), (4-40), (4-41) and (4-42), we obtain the convergence of the integral in (4-28) and the limit (4-29) as well as the fact that $S_+(r, \theta, \rho, \eta) - r\rho - \theta \cdot \eta$ belongs to $\mathscr{B}_{\mathrm{hyp}}(\Gamma_{\mathrm{s}}^+(\varepsilon_2))$. Finally, the formulas in (4-35) follow directly from (4-42) combined with (4-22) and (4-15). □

Remark 1. By applying ∂_η to (4-41) we see that there exists C such that

$$|\partial_\eta \Sigma_+(t, r, \theta, \rho, \eta) - \theta| \le C e^{-r} \lesssim e^{-R(\varepsilon_2)} \quad \text{for all } (r, \theta, \rho, \eta) \in \Gamma_{\mathrm{s}}^+(\varepsilon_2) \text{ and } t \ge 0.$$

This shows, in the spirit of Corollary 3.10, that the proof above depends only on the definition of $q(r, \theta, \eta)$ for θ in an arbitrarily small neighborhood of $\overline{V}_0$, provided ε_2 is small enough.

Remark 2. Using (3-34), (3-35) and (4-20), one sees that $S_\pm$ is actually well defined on the conical area given by

$$r > R(\varepsilon_2), \quad \theta \in V_{\varepsilon_2}, \quad \pm\rho > (1 - \varepsilon_2^2) p^{1/2},$$

and that

$$\Sigma_\pm(t, r, \theta, \lambda\rho, \lambda\eta) = \lambda \Sigma_\pm(\lambda t, r, \theta, \rho, \eta) \quad \text{if } \lambda > 0.$$

Thus that $S_\pm$ is the restriction to $\Gamma_{\mathrm{s}}^\pm(\varepsilon_2)$ of an homogeneous function of degree 1 with respect to (ρ, η).

We conclude this part with a result useful for considering phases globally defined on $\mathbb{R}^{2n}$ when we shall construct Fourier integral operators.

Proposition 4.10. *For some small enough $\varepsilon_3 > 0$, there exists a family of functions $(S_{\pm,\varepsilon})_{0<\varepsilon\le\varepsilon_3}$, globally defined on $\mathbb{R}^{2n}$, such that*

$$\varphi_{\pm,\varepsilon}(r, \theta, \rho, \eta) := S_{\pm,\varepsilon}(r, \theta, \rho, \eta) - r\rho - \theta \cdot \eta$$

coincides with $\varphi_\pm$ on $\Gamma_{\mathrm{s}}^\pm(\varepsilon)$ and satisfies

$$\operatorname{supp}(\varphi_{\pm,\varepsilon}) \subset \Gamma_{\mathrm{s}}^\pm(\varepsilon^{1/2}), \quad \varphi_{\pm,\varepsilon} \in \mathscr{B}_{\mathrm{hyp}}(\Gamma_{\mathrm{s}}^\pm(\varepsilon^{1/2})), \tag{4-43}$$

$$\left|\partial_{\rho,\eta} \otimes \partial_{r,\theta} \varphi_{\pm,\varepsilon}(r, \theta, \rho, \eta)\right| \le \tfrac{1}{2} \quad \text{for } (r, \theta, \rho, \eta) \in \mathbb{R}^{2n},\ 0 < \varepsilon \le \varepsilon_3, \tag{4-44}$$

with $|\cdot|$ a matrix norm.

In further applications, (4-44) will also be used under the equivalent form

$$\left|\partial_{\rho,\eta} \otimes \partial_{r,\theta} S_{\pm,\varepsilon}(r, \theta, \rho, \eta) - \mathrm{Id}_n\right| \le \tfrac{1}{2} \quad \text{for } (r, \theta, \rho, \eta) \in \mathbb{R}^{2n},\ 0 < \varepsilon \le \varepsilon_3. \tag{4-45}$$

Remark. Although this proposition allows one to assume that they are globally defined, the phases $S_{\pm,\varepsilon}$ solve the Hamilton–Jacobi equations on $\Gamma_{\mathrm{s}}^\pm(\varepsilon_2)$ only.

Proof. We use Proposition 4.4 and consider

$$S_{\pm,\varepsilon}(r, \theta, \rho, \eta) := r\rho + \theta \cdot \eta + \chi_{\varepsilon^{1/2} \to \varepsilon}(r, \theta, \rho, \eta) \varphi_\pm(r, \theta, \rho, \eta), \tag{4-46}$$

with $\varphi_\pm$ defined in Proposition 4.9. We have $S_{\pm,\varepsilon} = S_\pm$ on $\Gamma_{\mathrm{s}}^\pm(\varepsilon)$ and, using(4-9) and (4-36),

$$\left|\partial_{\rho,\eta} \otimes \partial_{r,\theta} S_{\pm,\varepsilon}(r, \theta, \rho, \eta) - \mathrm{Id}_n\right| \lesssim \varepsilon^{1/4} \quad \text{on } \mathbb{R}^{2n}, \tag{4-47}$$

since $e^{-r}|\eta| \lesssim \varepsilon^{1/2}$ on $\Gamma_{\mathrm{s}}^{+}(\varepsilon^{1/2})$. This yields the result if ε is small. □

4D. *Fourier integral operators on $\mathbb{R}^n$.* In this subsection, we derive some basic properties of Fourier integral operators associated to the phases $S_{\pm}$ obtained in Proposition 4.9.

For simplicity, we introduce the shorter notation

$$\mathcal{B}_{\mathrm{s}}^{\pm}(\varepsilon) := \mathcal{B}_{\mathrm{hyp}}(\Gamma_{\mathrm{s}}^{\pm}(\varepsilon)), \quad \mathcal{S}_{\mathrm{s}}^{\pm}(\varepsilon) := \mathcal{S}_{\mathrm{hyp}}^{\pm}(\Gamma_{\mathrm{s}}^{\pm}(\varepsilon)), \tag{4-48}$$

where the classes $\mathcal{B}_{\mathrm{hyp}}$ and $\mathcal{S}_{\mathrm{hyp}}$ were defined in Definition 2.2.

By Propositions 4.9 and 4.10, for all $h \in (0, 1]$, all ε small enough and all $a^{\pm} \in \mathcal{S}_{\mathrm{s}}^{\pm}(\varepsilon)$, we can define the operator

$$J_h^{\pm}(a^{\pm}) : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n), \tag{4-49}$$

as the operator with Schwartz kernel

$$K_h^{\pm}(r, \theta, r', \theta') = (2\pi h)^{-n} \int e^{\frac{i}{h}(S_{\pm}(r,\theta,\rho,\eta) - r'\rho - \theta'\cdot\eta)} a^{\pm}(r, \theta, \rho, \eta)\, d\rho\, d\eta.$$

Since the symbol $a^{\pm}$ is supported in $\Gamma_{\mathrm{s}}^{\pm}(\varepsilon)$, the phase $S_{\pm}$ can be replaced by $S_{\pm,\varepsilon}$ which is globally defined (see Proposition 4.10). Note also that $J_h^{\pm}(a^{\pm})$ maps clearly the Schwartz space into itself since, for fixed h say $h = 1$, it can be considered as the pseudodifferential operator with symbol $e^{i\varphi_{\pm}}a^{\pm} = e^{i\varphi_{\pm,\varepsilon}}a^{\pm}$ which belongs to $C_b^{\infty}(\mathbb{R}^{2n})$.

To obtain the L^2 boundedness of such operators uniformly in $h \in (0, 1]$ as well as the factorization Proposition 4.13 below, which are both consequences of the usual Kuranishi trick, we need a preliminary result.

Consider the maps $(\underline{\rho}_{\pm,\varepsilon}, \underline{\eta}_{\pm,\varepsilon}) : \mathbb{R}^{3n} \to \mathbb{R}^n$ defined by

$$(\underline{\rho}_{\pm,\varepsilon}, \underline{\eta}_{\pm,\varepsilon})(r, \theta, r', \theta', \rho, \eta) := \int_0^1 \partial_{r,\theta} S_{\pm,\varepsilon}(r' + s(r - r'), \theta' + s(\theta - \theta'), \rho, \eta) ds \tag{4-50}$$

so that

$$(r - r')\underline{\rho}_{\pm,\varepsilon} + (\theta - \theta') \cdot \underline{\eta}_{\pm,\varepsilon} = S_{\pm,\varepsilon}(r, \theta, \rho, \eta) - S_{\pm,\varepsilon}(r', \theta', \rho, \eta). \tag{4-51}$$

Lemma 4.11. *For all $(r, \theta, r', \theta') \in \mathbb{R}^{2n}$ and all $0 < \varepsilon \le \varepsilon_3$, the map $(\rho, \eta) \mapsto (\underline{\rho}_{\pm,\varepsilon}, \underline{\eta}_{\pm,\varepsilon})$ is a diffeomorphism from $\mathbb{R}^n$ onto itself. Denoting by $(\overline{\rho}_{\pm,\varepsilon}, \overline{\eta}_{\pm,\varepsilon})$ the corresponding inverse, we have, for all $0 < \varepsilon \le \varepsilon_3$,*

$$\left| \partial_\eta^\beta \partial_r^j \partial_{r'}^{j'} \partial_\theta^\alpha \partial_{\theta'}^{\alpha'} \partial_\rho^k ((\overline{\rho}_{\pm,\varepsilon}, \overline{\eta}_{\pm,\varepsilon}) - (\rho, \eta)) \right| \lesssim 1 \quad \text{on } \mathbb{R}^{3n}. \tag{4-52}$$

Furthermore, there exists $\varepsilon_6 > 0$ such that, for all $0 < \varepsilon \le \varepsilon_6$, we have

$$(r, \theta, \rho, \eta) \in \Gamma_{\mathrm{s}}^{\pm}(\varepsilon) \implies \left(r, \theta, \underline{\rho}_{\pm,\varepsilon}, \underline{\eta}_{\pm,\varepsilon}\right)_{|r=r',\theta=\theta'} \in \Gamma_{\mathrm{s}}^{\pm}(\varepsilon^{1/3}), \tag{4-53}$$

$$\left(r, \theta, \underline{\rho}_{\pm,\varepsilon}, \underline{\eta}_{\pm,\varepsilon}\right)_{|r=r',\theta=\theta'} \in \Gamma_{\mathrm{s}}^{\pm}(\varepsilon^3) \implies (r, \theta, \rho, \eta) \in \Gamma_{\mathrm{s}}^{\pm}(\varepsilon), \tag{4-54}$$

and

$$\left| \partial_\eta^\beta \partial_r^j \partial_{r'}^{j'} \partial_\theta^\alpha \partial_{\theta'}^{\alpha'} \partial_\rho^k ((\overline{\rho}_{\pm,\varepsilon}, \overline{\eta}_{\pm,\varepsilon}) - (\rho, \eta))_{|r=r',\theta=\theta'} \right| \lesssim e^{-|\beta| r} \quad \text{on } \Gamma_{\mathrm{s}}^{+}(\varepsilon^3). \tag{4-55}$$

Proof. The estimate (4-45) implies directly that $(\rho,\eta)\mapsto(\underline{\rho}_{\pm,\varepsilon},\underline{\eta}_{\pm,\varepsilon})$ is a diffeomorphism for all $(r,\theta,r',\theta')\in\mathbb{R}^{2n}$ and $0<\varepsilon\le\varepsilon_3$. Evaluating (4-50) at $(r,\theta,r',\theta',\overline{\rho}_{\pm,\varepsilon},\overline{\eta}_{\pm,\varepsilon})$, namely

$$(\rho,\eta)=(\underline{\rho}_{\pm,\varepsilon},\underline{\eta}_{\pm,\varepsilon})(r,\theta,r',\theta',\overline{\rho}_{\pm,\varepsilon},\overline{\rho}_{\pm,\varepsilon}), \tag{4-56}$$

yields

$$(\rho,\eta)-(\overline{\rho}_{\pm,\varepsilon},\overline{\eta}_{\pm,\varepsilon})=\int_0^1\partial_{r,\theta}\varphi_{\pm,\varepsilon}(r'+s(r-r'),\theta'+s(\theta-\theta'),\overline{\rho}_{\pm,\varepsilon},\overline{\eta}_{\pm,\varepsilon})ds. \tag{4-57}$$

By (4-43) we have $\varphi_{\pm,\varepsilon}\in C_b^\infty(\mathbb{R}^{2n})$, so $(\overline{\rho}_{\pm,\varepsilon},\overline{\eta}_{\pm,\varepsilon})-(\rho,\eta)$ is bounded, for fixed ε. For the derivatives, we apply $\partial_\eta^\beta\partial_r^j\partial_{r'}^{j'}\partial_\theta^\alpha\partial_{\theta'}^{\alpha'}\partial_\rho^k$ to the right-hand side of (4-57) and obtain (4-52) by induction, using Lemma 3.6.

To prove (4-53), we simply notice that $\varphi_{\pm,\varepsilon}$ coincides with $\varphi_\pm$ on $\Gamma_{\rm s}^\pm(\varepsilon^3)$ so that

$$\left|(\rho,\eta)-(\underline{\rho}_{\pm,\varepsilon},\underline{\eta}_{\pm,\varepsilon})_{|r=r',\theta=\theta'}\right|=|\partial_{r,\theta}\varphi_\pm(r,\theta,\rho,\eta)|\lesssim\varepsilon^2,$$

using (2-56) and (4-36). The result follows from Proposition 4.3 and the fact that $\Gamma_{\rm s}^\pm(C\varepsilon)\subset\Gamma_{\rm s}^\pm(\varepsilon^{1/3})$ for ε small enough. To get (4-54), we use directly Proposition A.1 proving that $\Gamma_{\rm s}^\pm(\varepsilon^3)\subset\Psi^t(\Gamma_{\rm s}^\pm(\varepsilon))$ with

$$\Psi^t(r,\theta,\rho,\eta):=\big(r,\theta,\underline{\rho}_{\pm,\varepsilon},\underline{\eta}_{\pm,\varepsilon}\big)_{|r=r',\theta=\theta'}=\big(r,\theta,\partial_rS_\pm(r,\theta,\rho,\eta),\partial_\theta S_\pm(r,\theta,\rho,\eta)\big),$$

which is actually independent of t and ε.

By (4-52), (4-55) holds when $\beta=0$. Consider next the first-order derivatives when $|\beta|=1$ and the other multi-indices are 0. Applying ∂_η^β to (4-56) and evaluating at $r=r'$, $\theta=\theta'$, we get

$$\big(\partial_{\rho,\eta}(\underline{\rho}_{\pm,\varepsilon},\underline{\eta}_{\pm,\varepsilon})\big)\partial_\eta^\beta\big((\overline{\rho}_{\pm,\varepsilon},\overline{\eta}_{\pm,\varepsilon})-(\rho,\eta)\big)=\partial_\eta^\beta\partial_{r,\theta}\varphi_\pm(r,\theta,\overline{\rho}_{\pm,\varepsilon},\overline{\eta}_{\pm,\varepsilon})$$

where we have replaced $\varphi_{\pm,\varepsilon}$ by $\varphi_\pm$ using (4-54). Since $\big(\partial_{\rho,\eta}(\underline{\rho}_{\pm,\varepsilon},\underline{\eta}_{\pm,\varepsilon})\big)^{-1}$ is uniformly bounded and $e^{r\beta}\partial_\eta^\beta\partial_{r,\theta}\varphi_\pm\big(r,\theta,\overline{\rho}_{\pm,\varepsilon},\overline{\eta}_{\pm,\varepsilon}\big)$ is bounded, using (4-54) again, we get the result in this case. Higher-order derivatives are obtained similarly by induction, using Lemma 3.6. □

Proposition 4.12. *For all $0<\varepsilon\le\varepsilon_6$ and all $a^\pm,b^\pm\in\mathcal{S}_{\rm s}^\pm(\varepsilon)$, we have*

$$\Big\|J_h^\pm(a^\pm)J_h^\pm(b^\pm)^*-\sum_{k\le N}h^kc_k^\pm(r,\theta,hD_r,hD_\theta)\Big\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)}\le Ch^{N+1}\quad \textit{for } h\in(0,1], \tag{4-58}$$

where the constant C can be chosen uniformly with respect to $a^\pm$ and $b^\pm$ when they vary in bounded subsets of $\mathcal{S}_{\rm s}^\pm(\varepsilon)$ and where the symbols $c_k^\pm$ are given by

$$c_k^\pm=\sum_{j+|\alpha|=k}\frac{1}{j!\,\alpha!}\partial_{r'}^j\partial_{\theta'}^\alpha D_\rho^jD_\eta^\alpha\big(a(r,\theta,\overline{\rho}_{\pm,\varepsilon},\overline{\eta}_{\pm,\varepsilon})\overline{b(r',\theta',\overline{\rho}_{\pm,\varepsilon},\overline{\eta}_{\pm,\varepsilon})}\,\mathrm{Jac}(\overline{\rho}_{\pm,\varepsilon},\overline{\eta}_{\pm,\varepsilon})\big)_{|r=r',\ \theta=\theta'}, \tag{4-59}$$

with $\mathrm{Jac}(\overline{\rho}_{\pm,\varepsilon},\overline{\eta}_{\pm,\varepsilon})=|\det(\partial_{\rho,\eta}(\overline{\rho}_{\pm,\varepsilon},\overline{\eta}_{\pm,\varepsilon}))|$. *In particular,*

$$c_k^\pm\in\mathcal{S}_{\rm s}^\pm(\varepsilon^{1/3}). \tag{4-60}$$

Proof. The Schwartz kernel of $J_h^{\pm}(a^{\pm}) J_h^{\pm}(b^{\pm})^*$ takes the form

$$(2\pi h)^{-n} \int e^{\frac{i}{h}(S_{\pm,\varepsilon}(r,\theta,\rho,\eta) - S_{\pm,\varepsilon}(r',\theta',\rho,\eta))} a(r,\theta,\rho,\eta)\, \overline{b(r',\theta',\rho,\eta)}\, d\rho\, d\eta$$

and this can be rewritten using the Kuranishi trick, that is, (4-51) and Lemma 4.11, as

$$(2\pi h)^{-n} \int e^{\frac{i}{h}((r-r')\rho + (\theta-\theta')\cdot\eta)} a(r,\theta,\overline{\rho}_{\pm,\varepsilon},\overline{\eta}_{\pm,\varepsilon}) \overline{b(r',\theta',\overline{\rho}_{\pm,\varepsilon},\overline{\eta}_{\pm,\varepsilon})} \operatorname{Jac}(\overline{\rho}_{\pm,\varepsilon},\overline{\eta}_{\pm,\varepsilon})\, d\rho\, d\eta. \tag{4-61}$$

By (4-52), the symbol in (4-61) belongs to $C_b^{\infty}(\mathbb{R}^{3n})$. Therefore, the standard h-pseudodifferential calculus implies that, with c_k defined by (4-59), we obtain the L^2 bound (4-58) by the Calderón–Vaillancourt theorem. In addition, by (4-53) (applied with $(\rho,\eta) = (\overline{\rho}_{\pm,\varepsilon},\overline{\eta}_{\pm,\varepsilon})_{|r=r',\theta=\theta'}$), we have $\operatorname{supp}(c_k^{\pm}) \subset \Gamma_{\mathrm{s}}^{+}(\varepsilon^{1/3})$. One then checks that $c_k^{\pm} \in \mathcal{B}_{\mathrm{s}}^{\pm}(\varepsilon^{1/3})$, using (4-55). □

We note in passing that this proposition shows that, for all $0 < \varepsilon \le \varepsilon_6$ and all $a^{\pm} \in \mathcal{S}_{\mathrm{s}}^{\pm}(\varepsilon)$,

$$\| J_h^{\pm}(a^{\pm}) \|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} \le C \quad \text{for } h \in (0,1]. \tag{4-62}$$

More precisely, the constant C can be chosen independently of $a^{\pm}$ if, for ε fixed, $a^{\pm}$ vary in a bounded subset of $\mathcal{S}_{\mathrm{s}}^{\pm}(\varepsilon)$.

Proposition 4.13. *For all $0 < \varepsilon \le \varepsilon_6$, the following holds: if we are given $a_0^{\pm}, \dots, a_N^{\pm} \in \mathcal{S}_{\mathrm{s}}^{\pm}(\varepsilon)$ with*

$$a_0^{\pm} \gtrsim 1 \quad \text{on } \Gamma_{\mathrm{s}}^{\pm}(\varepsilon^3), \tag{4-63}$$

then, for all $\chi_{\mathrm{s}}^{\pm} \in \mathcal{S}_{\mathrm{s}}^{\pm}(\varepsilon^9)$, we can find $b_0^{\pm}, \dots, b_N^{\pm} \in \mathcal{S}_{\mathrm{s}}^{\pm}(\varepsilon^3)$ such that, if we set

$$a^{\pm}(h) = a_0^{\pm} + \cdots + h^N a_N^{\pm}, \quad b^{\pm}(h) = b_0^{\pm} + \cdots + h^N b_N^{\pm},$$

we have

$$\left\| J_h^{\pm}(a^{\pm}(h)) J_h^{\pm}(b^{\pm}(h))^* - \chi_{\mathrm{s}}^{\pm}(r,\theta,hD_r,hD_\theta) \right\|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} \le C h^{N+1} \quad \textit{for } h \in (0,1].$$

Proof. By Proposition 4.12 and the notation therein, we only need to find $b_0^{\pm}, \dots, b_N^{\pm}$ such that

$$c_0^{\pm} = \chi_{\mathrm{s}}^{\pm}, \qquad c_k^{\pm} = 0 \quad \text{for } k = 1, \dots, N.$$

Using Lemma 4.11 and (4-59), the first equation, $c_0^{\pm} = \chi_{\mathrm{s}}^{\pm}$, is solved explicitly by

$$\overline{b_0^{\pm}(r,\theta,\rho,\eta)} = \left(\chi_{\mathrm{s}}^{\pm}(r,\theta,\underline{\rho}_{\pm,\varepsilon},\underline{\eta}_{\pm,\varepsilon}) \operatorname{Jac}(\underline{\rho}_{\pm,\varepsilon},\underline{\eta}_{\pm,\varepsilon}) \right)_{|r'=r,\ \theta'=\theta} \times \frac{1}{a_0^{\pm}(r,\theta,\rho,\eta)},$$

where $1/a_0^{\pm}$ is well defined since $\chi_{\mathrm{s}}^{\pm}(r,\theta,\underline{\rho}_{\pm,\varepsilon},\underline{\eta}_{\pm,\varepsilon})_{|r'=r,\ \theta'=\theta}$ is supported in $\Gamma_{\mathrm{s}}^{\pm}(\varepsilon^3)$ by (4-54). Thus, $b_0^{\pm}$ is well defined, supported in $\Gamma_{\mathrm{s}}^{\pm}(\varepsilon^3)$ and belongs to $\mathcal{B}_{\mathrm{s}}^{\pm}(\varepsilon^3)$ by (4-50) and Proposition 4.9 (since $(\underline{\rho}_{\pm,\varepsilon},\underline{\eta}_{\pm,\varepsilon})_{|r'=r,\ \theta'=\theta} = \partial_{r,\theta} S_{\pm}$ in $\Gamma_{\mathrm{s}}^{\pm}(\varepsilon^3)$). Furthermore, $b_0^{\pm}(r,\theta,\overline{\rho}_{\pm,\varepsilon},\overline{\eta}_{\pm,\varepsilon})_{|r'=r,\ \theta'=\theta}$ is supported in $\Gamma_{\mathrm{s}}^{\pm}(\varepsilon^9)$. We then find the other symbols by induction for we have a triangular system of equations. More precisely, the k-th equation $c_k \equiv 0$ ($k \ge 1$), reads

$$\left(\overline{b_k^{\pm}(r,\theta,\overline{\rho}_{\pm,\varepsilon},\overline{\eta}_{\pm,\varepsilon})}\, a_0^{\pm}(r,\theta,\overline{\rho}_{\pm,\varepsilon},\overline{\eta}_{\pm,\varepsilon}) \operatorname{Jac}(\overline{\rho}_{\pm,\varepsilon},\overline{\eta}_{\pm,\varepsilon}) \right)_{|r=r',\theta=\theta'} = d_k^{\pm}(r,\theta,\rho,\eta)$$

where $d_k^{\pm}$ is a linear combination of symbols of the form

$$\overline{(\partial^{\gamma} b_{k_2}^{\pm})(r,\theta,\overline{\rho}_{\pm,\varepsilon},\overline{\eta}_{\pm,\varepsilon})_{|r=r',\theta=\theta'}}(\partial^{\gamma'} a_{k_1}^{\pm})(r,\theta,\overline{\rho}_{\pm,\varepsilon},\overline{\eta}_{\pm,\varepsilon})_{|r=r',\theta=\theta'}\delta_{k_1k_2\gamma\gamma'}(r,\theta,\rho,\eta)$$

with $k_2 < k$ and $\delta_{k_1k_2\gamma\gamma'}$ a product of derivatives of order ≥ 1 of $(\overline{\rho}_{\pm,\varepsilon},\overline{\eta}_{\pm,\varepsilon})(r,\theta,r',\eta',\rho,\eta)$ evaluated at $r=r'$, $\theta=\theta'$. By the induction assumption $(\partial^{\gamma} b_{k_2}^{\pm})(r,\theta,\overline{\rho}_{\pm,\varepsilon},\overline{\eta}_{\pm,\varepsilon})_{|r=r',\theta=\theta'}$ is supported in $\Gamma_{\mathrm{s}}^{\pm}(\varepsilon^9)$, so we have

$$(r,\theta,\underline{\rho}_{\pm,\varepsilon},\underline{\eta}_{\pm,\varepsilon})_{|r=r',\theta=\theta'} \in \Gamma_{\mathrm{s}}^{\pm}(\varepsilon^3),$$

using (4-53). Therefore, $\delta_{k_1k_2\gamma\gamma'}(r,\theta,\underline{\rho}_{\pm,\varepsilon},\underline{\eta}_{\pm,\varepsilon})_{|r=r',\theta=\theta'}$ belongs to $\mathscr{B}_{\mathrm{s}}^{\pm}(\varepsilon^3)$ by (4-55) and $b_k^{\pm}$ satisfies the expected properties. □

4E. *The transport equations.* In this subsection, we solve the time-independent transport equations related to the phases constructed in Proposition 4.9. If we define $(v^{\pm},w^{\pm}) = (v^{\pm},w^{\pm})(r,\theta,\rho,\eta)$ by

$$\begin{pmatrix} v^{\pm} \\ w^{\pm} \end{pmatrix} := \begin{pmatrix} (\partial_{\rho} p)(r,\theta,\partial_r S_{\pm},\partial_{\theta} S_{\pm}) \\ (\partial_{\eta} p)(r,\theta,\partial_r S_{\pm},\partial_{\theta} S_{\pm}) \end{pmatrix} = \begin{pmatrix} 2\partial_r S_{\pm} \\ e^{-2r}(\partial_{\eta} q)(r,\theta,\partial_{\theta} S_{\pm}) \end{pmatrix}, \tag{4-64}$$

these transport equations take the form

$$v^{\pm}\partial_r a^{\pm} + w^{\pm}\cdot\partial_{\theta} a^{\pm} + y^{\pm} a^{\pm} = z^{\pm}, \tag{4-65}$$

where $y^{\pm}, z^{\pm}$ are given and $a^{\pm}$ is the unknown function of (r,θ,ρ,η). Such equations arise naturally in the construction of the Isozaki–Kiada parametrix (see Section 5). They can be solved standardly by the method of characteristics and therefore, we start with the study the integral curves of the vector field $(v^{\pm},w^{\pm})$.

Given $(r,\theta,\rho,\eta) \in \Gamma_{\mathrm{s}}^{\pm}(\varepsilon^2)$, with $\varepsilon > 0$ small enough (to be specified below), we denote by

$$r_t^{\pm} = r_t^{\pm}(r,\theta,\rho,\eta), \quad \theta_t^{\pm} = \theta_t^{\pm}(r,\theta,\rho,\eta),$$

the solution to

$$\begin{cases} \dot{r}_t^{\pm} = v^{\pm}(r_t^{\pm},\theta_t^{\pm},\rho,\eta), \\ \dot{\theta}_t^{\pm} = w^{\pm}(r_t^{\pm},\theta_t^{\pm},\rho,\eta), \end{cases} \tag{4-66}$$

with initial data

$$r_0^{\pm}(r,\theta,\rho,\eta) = r, \quad \theta_0^{\pm}(r,\theta,\rho,\eta) = \theta.$$

In this problem, ρ and η are parameters. Equivalently,

$$\phi_t^{\pm} = \phi_t^{\pm}(r,\theta,\rho,\eta) := (r_t^{\pm},\theta_t^{\pm},\rho,\eta) \tag{4-67}$$

is the flow of the autonomous vector field $(v^{\pm},w^{\pm},0,0)$.

Proposition 4.14. *There exists $\varepsilon_4 > 0$ such that for all $(r,\theta,\rho,\eta) \in \Gamma_{\mathrm{s}}^{\pm}(\varepsilon_4^2)$, the solution (r_t^{+},θ_t^{+}) (resp. (r_t^{-},θ_t^{-})) is globally defined on $[0,+\infty)$ (resp. $(-\infty,0]$). There also exists $C > 0$ such that, for all $0 < \varepsilon \leq \varepsilon_4$ and all $(r,\theta,\rho,\eta) \in \Gamma_{\mathrm{s}}^{\pm}(\varepsilon^2)$, we have*

$$(r_t^{\pm},\theta_t^{\pm},\rho,\eta) \in \Gamma_{\mathrm{s}}^{\pm}(\varepsilon) \quad \textit{for } \pm t \geq 0, \tag{4-68}$$

and

$$\left|r_t^{\pm} - r - 2t\rho\right| \leq C\varepsilon^2 \min(1,|t|), \quad \left|\theta_t^{\pm} - \theta\right| \leq Ce^{-r}. \tag{4-69}$$

Furthermore,

$$\left|D_{\mathrm{hyp}}^{j\alpha k\beta}(r_t^\pm - r - 2t\rho)\right| + \left|D_{\mathrm{hyp}}^{j\alpha k\beta}(\theta_t^\pm - \theta)\right| \le C_{j\alpha k\beta}. \tag{4-70}$$

for $(r, \theta, \rho, \eta) \in \Gamma_{\mathrm{s}}^\pm(\varepsilon_4^2)$ *and* $\pm t \ge 0$.

Since $S_{\pm,\varepsilon} = S_\pm$ on $\Gamma_{\mathrm{s}}^\pm(\varepsilon)$, the localization property (4-68) shows that $\phi_t^\pm$ still solves (4-66) on $\Gamma_{\mathrm{s}}^\pm(\varepsilon^2)$ if one replaces $(v^\pm, w^\pm)$ by $(v_\varepsilon^\pm, w_\varepsilon^\pm)$, the latter being obtained by replacing $S_\pm$ by $S_{\pm,\varepsilon}$ in (4-64).

Proof. Here again we only consider the outgoing case. By (4-36), there exists $C_0 \ge 1$ such that, for all $(r, \theta, \rho, \eta) \in \Gamma_{\mathrm{s}}^+(\varepsilon_2)$,

$$|\partial_r S_+ - \rho| \le C_0 e^{-r}|\eta| \quad \text{and} \quad |e^{-2r}(\partial_\eta q)(r, \theta, \partial_\theta S_+)| \le C_0 e^{-2r}|\eta|. \tag{4-71}$$

By (2-56), there exists $C_1 \ge 1$ such that, for all $\varepsilon > 0$ small enough and all $(r, \theta, \rho, \eta) \in \Gamma_{\mathrm{s}}^+(\varepsilon)$, we have

$$e^{-r}|\eta| \le C_1\varepsilon \quad \text{and} \quad e^{-2r}|\eta| \le C_1\varepsilon^2, \tag{4-72}$$

the last inequality following from $e^{-R(\varepsilon)} \le \varepsilon$. If ε small enough, we may also assume that

$$\rho > \tfrac{1}{8} \quad \text{for all } (r, \theta, \rho, \eta) \in \Gamma_{\mathrm{s}}^+(\varepsilon).$$

Now fix $M = 5C_0C_1$, and for $(r, \theta, \rho, \eta) \in \Gamma_{\mathrm{s}}^+(\varepsilon^2)$, consider $\mathcal{T} := \mathcal{T}(r, \theta, \rho, \eta)$ defined by

$$\mathcal{T} = \left\{t \ge 0 : (r_s^+, \theta_s^+) \text{ is defined and } r_s^+ \ge r + s/8,\ |\theta_s^+ - \theta| \le M\varepsilon^2 \text{ for all } s \in [0, t]\right\}.$$

The set $\mathcal{T}$ is clearly an interval containing 0 and, if ε is small enough, Proposition 4.3 shows that $(r_s^+, \theta_s^+, \rho, \eta) \in \Gamma_{\mathrm{s}}^+(\varepsilon)$ for all $s \in \mathcal{T}$. Thus, by (4-71) and (4-72), we have

$$|\dot{r}_s^+ - 2\rho| \le 2C_0C_1\varepsilon \quad \text{and} \quad |\dot{\theta}_s^+| \le C_0C_1\varepsilon^2 \quad \text{for } s \in \mathcal{T},$$

and, by possibly assuming that $C_0C_1\varepsilon < \frac{1}{8}$, we have $\dot{r}_s^+ > 0$ on $\mathcal{T}$. Choosing $C_M \ge 1$ as in Proposition 4.3, we now claim that, if

$$\varepsilon < \varepsilon_2/C_M \quad \text{and} \quad r > R(C_M\varepsilon),$$

then $T := \sup \mathcal{T} = +\infty$. Assume this is wrong. Then T is finite, belongs to $\mathcal{T}$ and, on $[0, T]$, we have

$$r_s^+ \ge r + s/8 \ge r, \quad |\theta_s^+ - \theta| \le C_1\varepsilon^2 < M\varepsilon^2,$$

so, by Proposition 4.3, $(r_s^+, \theta_s^+, \rho, \eta) \in \Gamma_{\mathrm{s}}^+(C_M\varepsilon) \subset \Gamma_{\mathrm{s}}^+(\varepsilon_2)$ and, by (4-71) and (4-72),

$$|r_T^+ - r - 2\rho T| \le C_0 e^{-r}\,|\eta| \int_0^T e^{-s/8}\,ds \le C_0 e^{-r}\,|\eta|\,T \le C_0C_1\varepsilon T, \tag{4-73}$$

$$|\theta_T^+ - \theta| \le C_0 e^{-2r}|\eta| \int_0^T e^{-s/4}\,ds \le 4C_0 e^{-2r}|\eta| < 5C_0C_1\varepsilon^2. \tag{4-74}$$

This implies that $r_T^+ > r + T/8$ and that $|\theta_T^+ - \theta| < M\varepsilon^2$, so the flow can be continued beyond T, yielding a contradiction with the definition of T. The flow is thus well defined for $t \ge 0$. Then (4-69) follows from the first inequalities of (4-73) and (4-74) with an arbitrary $t \ge 0$ instead of T, since $e^{-r}|\eta| \lesssim \varepsilon^2$ for $(r, \theta, \rho, \eta) \in \Gamma_{\mathrm{s}}^+(\varepsilon^2)$. If ε is small enough, Proposition 4.3 shows that (4-68) is a direct consequence of (4-69), using that $e^{-r} \ll \varepsilon^4$.

It remains to prove (4-70) for $j+|\alpha|+k+|\beta|\geq 1$. We consider $\bar{r}_t^+ := r_t^+ - 2t\rho$ and $\bar{\theta}_t^+ := \theta_t^+$, which satisfy

$$\frac{d\bar{r}_t^+}{dt} = \bar{v}(t, \bar{r}_t^+, \bar{\theta}_t^+, \rho, \eta), \qquad \frac{d\bar{\theta}_t^+}{dt} = \overline{w}(t, \bar{r}_t^+, \bar{\theta}_t^+, \rho, \eta), \tag{4-75}$$

with

$$\bar{v}(t, r, \theta, \rho, \eta) = (\partial_r \varphi_+)(r + 2t\rho, \theta, \rho, \eta),$$
$$\overline{w}(t, r, \theta, \rho, \eta) = e^{-2r-4t\rho}\left(\partial_\eta q\right)(r + 2t\rho, \theta, \partial_\theta S_+(r + 2t\rho, \theta, \rho, \eta)).$$

Using (4-36), we have, for all j', α', k', β',

$$\left|D_{\text{hyp}}^{j'\alpha'k'\beta'}(\bar{v}, \overline{w})\right| \lesssim \langle t\rangle^{k'} e^{-4t\rho} \lesssim e^{-2t\rho} \quad \text{for } t\geq 0, \quad \text{on } \Gamma_{\text{s}}^+(\varepsilon_2/C), \tag{4-76}$$

with C such that if $(r, \theta, \rho, \eta)\in\Gamma_{\text{s}}^+(\varepsilon_2/C)$ then $(r+2t\rho, \theta, \rho, \eta)\in\Gamma_{\text{s}}^+(\varepsilon_2/C)$. Note also that if ε is small enough and $(r, \theta, \rho, \eta)\in\Gamma_{\text{s}}^+(\varepsilon^2)$, we have $(\bar{r}_t^+, \bar{\theta}_t^+, \rho, \eta)\in\Gamma_{\text{s}}^+(\varepsilon_2/C)$, using (4-69) and Proposition 4.3. We then obtain (4-70) by induction by applying $D_{\text{hyp}}^{j\alpha k\beta}$ to (4-75). Indeed, using Lemma 3.6 and (4-76), we have

$$\frac{d}{dt} D_{\text{hyp}}^{j\alpha k\beta}\left(\bar{r}_t^+, \bar{\theta}_t^+\right) = \left(\partial_{r,\theta}(\bar{v}, \overline{w})\right) D_{\text{hyp}}^{j\alpha k\beta}(\bar{r}_t^+, \bar{\theta}_t^+) + \mathbb{O}(e^{-2\rho t}),$$

where $\mathbb{O}(e^{-2\rho t}) = 0$ for first-order derivatives and, otherwise, follows from the induction assumption. Since $|\partial_{r,\theta}(\bar{v}, \overline{w})| \lesssim e^{-2\rho t}$, Lemma 3.7 yields the result. □

We now come to the resolution of (4-65) in a way suitable to further purposes.

Proposition 4.15. *There exists $\varepsilon_5 > 0$ such that, for all $0 < \varepsilon \leq \varepsilon_5$ and all $y^\pm \in \mathcal{B}_{\text{hyp}}(\Gamma_{\text{s}}^\pm(\varepsilon))$ of hyperbolic short range in $\Gamma_{\text{s}}^\pm(\varepsilon)$, the function*

$$a_{\text{hom}}^\pm = \exp\int_0^{\pm\infty} y^\pm\circ\phi_s^\pm ds,$$

solves (4-65) on $\Gamma_{\text{s}}^\pm(\varepsilon^2)$ with $z^\pm\equiv 0$, belongs to $\mathcal{B}_{\text{hyp}}(\Gamma_{\text{s}}^\pm(\varepsilon^2))$ and $a_{\text{hom}}^\pm - 1$ is of hyperbolic long range in $\Gamma_{\text{s}}^\pm(\varepsilon^2)$.

In addition, for all $z^\pm\in\mathcal{B}_{\text{hyp}}\left(\Gamma_{\text{s}}^\pm(\varepsilon)\right)$, of hyperbolic short range in $\Gamma_{\text{s}}^\pm(\varepsilon)$, the function

$$a_{\text{inhom}}^\pm = -\int_0^{\pm\infty} z^\pm\circ\phi_s^\pm \exp\left(\int_0^s y^\pm\circ\phi_u^\pm du\right) ds$$

solves (4-65) on $\Gamma_{\text{s}}^\pm(\varepsilon^2)$, belongs to $\mathcal{B}_{\text{hyp}}(\Gamma_{\text{s}}^\pm(\varepsilon^2))$, and is of hyperbolic long range in $\Gamma_{\text{s}}^\pm(\varepsilon^2)$.

Lemma 4.16. *There exists $\varepsilon_5 > 0$ such that, for all j, α, k, β and all $N\geq 0$,*

$$\left|\partial_r^j\partial_\theta^\alpha\partial_\rho^k\partial_\eta^\beta\left(r_t^\pm - r - 2t\rho\right)\right| + \left|\partial_r^j\partial_\theta^\alpha\partial_\rho^k\partial_\eta^\beta(\theta_t^\pm - \theta)\right| \lesssim \langle r - \log\langle\eta\rangle\rangle^{-N},$$

on $\Gamma_{\text{s}}^\pm(\varepsilon_5)$, uniformly with respect to $\pm t\geq 0$.

Proof. By Proposition 4.3, there exists $C > 0$ such that, for all ε small enough and all $s\in[0, 1]$,

$$(r, \theta, \rho, \eta)\in\Gamma_{\text{s}}^\pm(\varepsilon^2) \implies (r, \theta, \rho, s\eta)\in\Gamma_{\text{s}}^\pm(C\varepsilon^2). \tag{4-77}$$

Therefore, if $C\varepsilon^2\leq\varepsilon_4^2$ and if we set $X_t^\pm(r, \theta, \rho, \eta) = (r_t^\pm - r - 2t\rho, \theta_t^\pm - \theta)$, we can write

$$X_t^\pm(r, \theta, \rho, \eta) = X_t^\pm(r, \theta, \rho, 0) + \int_0^1 (e^r\partial_\eta X_t^\pm)(r, \theta, \rho, s\eta)\, ds\cdot e^{-r}\eta,$$

on $\Gamma_{\rm s}^{\pm}(\varepsilon^2)$. The crucial remark is that $X_t^{\pm}(r,\theta,\rho,0)=0$. Indeed, by (4-34) and the first equation in (4-35), we have $\partial_r S_{\pm}\equiv\rho$ and $\partial_\theta S_{\pm}\equiv 0$ at $\eta=0$ (notice that $(r,\theta,\rho,0)\in\Gamma_{\rm s}^{\pm}(\varepsilon_2)$ if $C\varepsilon^2\leq\varepsilon_2$), so the solution to (4-66) is simply $(r+2t\rho,\theta)$ in this case. In addition, by (4-70), $(X_t^{\pm})_{t\geq 0}$ is bounded in $\mathscr{B}_{\rm hyp}(\Gamma_{\rm s}^{\pm}(\varepsilon^2))$. Thus, for all $N\geq 0$,

$$\left|\partial_r^j\partial_\theta^\alpha\partial_\rho^k\partial_\eta^\beta X_t^{\pm}(r,\theta,\rho,\eta)\right|\lesssim e^{-r}\langle\eta\rangle\lesssim\langle r-\log\langle\eta\rangle\rangle^{-N}\quad\text{for }\pm t\geq 0,\ (r,\theta,\rho,\eta)\in\Gamma_{\rm s}^{\pm}(\varepsilon^2),$$

which yields the result. □

Proof of Proposition 4.15. For simplicity we set $\partial^\gamma=\partial_r^j\partial_\theta^\alpha\partial_\rho^k\partial_\eta^\beta$. Then, using Lemma 3.6 with $|\gamma|\geq 1$, $\partial^\gamma\left(y^{\pm}\circ\phi_s^{\pm}\right)$ is the sum of

$$(\partial_r y^{\pm})\circ\phi_s^{\pm}\partial^\gamma r_s^{\pm}+(\partial_\theta y^{\pm})\circ\phi_s^{\pm}\cdot\partial^\gamma\theta_s^{\pm}+\delta_{j0}\delta_{\alpha 0}(\partial_\rho^k\partial_\eta^\beta y^{\pm})\circ\phi_s^{\pm}\tag{4-78}$$

and of a linear combination of

$$(\partial_\rho^{k-k'}\partial_\eta^{\beta-\beta'}\partial_{r,\theta}^{\nu}y^{\pm})\circ\phi_s^{\pm}\big(\partial^{\gamma_1^1}r_s^{\pm}\ldots\partial^{\gamma_{\nu_1}^1}r_s^{\pm}\big)\ldots\big(\partial^{\gamma_1^n}(\theta_s^{\pm})_{n-1}\ldots\partial_x^{\gamma_{\nu_n}^n}(\theta_s^{\pm})_{n-1}\big),\tag{4-79}$$

where $(\theta_s^{\pm})_1,\ldots,(\theta_s^{\pm})_{n-1}$ are the components of $\theta_s^{\pm}$, $(0,0,k',\beta')+\sum\gamma_i^j=\gamma$, using the convention and the notation of Lemma 3.6. By (4-70), we have

$$|(\partial_r y^{\pm})\circ\phi_s^{\pm}\partial^\gamma r_s^{\pm}|\lesssim\langle r_s^{\pm}-\log\langle\eta\rangle\rangle^{-\tau-2}e^{-r|\beta|}\langle s\rangle^{\kappa},$$

where $\kappa=1$ if $k=1$ and $j+|\alpha|+|\beta|=0$, and $\kappa=0$ otherwise. On the other hand, by Lemma 4.16, we have

$$|(\partial_r y^{\pm})\circ\phi_s^{\pm}\partial^\gamma r_s^{\pm}|\lesssim\langle r_s^{\pm}-\log\langle\eta\rangle\rangle^{-\tau-2}\langle r-\log\langle\eta\rangle\rangle^{-\tilde{j}}\langle s\rangle^{\kappa},$$

with the same κ as above and $\tilde{j}=j$ if $j\geq 2$, or $\tilde{j}=0$ for $j\leq 1$. Similarly, we also have

$$|(\partial_\theta y^{\pm})\circ\phi_s^{\pm}\cdot\partial^\gamma\theta_s^{\pm}|\lesssim\langle r_s^{\pm}-\log\langle\eta\rangle\rangle^{-\tau-1}\ \min\big(e^{-|\beta|r},\langle r-\log\langle\eta\rangle\rangle^{-j}\big),$$

while, for the last term of (4-78), we have

$$|\delta_{j0}\delta_{\alpha 0}(\partial_\rho^k\partial_\eta^\beta y^{\pm})\circ\phi_s^{\pm}|\lesssim\min\big(e^{-|\beta|r}e^{-2|\beta||\rho s|},\langle r_s^{\pm}-\log\langle\eta\rangle\rangle^{-\tau-1-j}\big),$$

since $e^{-|\beta|r_s^{\pm}}\lesssim e^{-|\beta|r}e^{-2|\beta||s|}$ for $r_s^{\pm}-r-2\rho s$ is bounded from below and $\rho s\geq 0$. Now, we remark that

$$\left|\big(\partial^{\gamma_1^1}r_s^{\pm}\ldots\partial^{\gamma_{\nu_1}^1}r_s^{\pm}\big)\right|\lesssim\langle s\rangle^{\tilde{\nu}_1}\langle r-\log\langle\eta\rangle\rangle^{-N_0},$$

where $\tilde{\nu}_1$ is the number of $\partial^{\gamma_l^1}=\partial_r^{j_l^1}\partial_\theta^{\alpha_l^1}\partial_\rho^{k_l^1}\partial_\eta^{\beta_l^1}$ for which $j_l^1=0$, $N_0=0$ if $j_l^1\leq 1$ for all l and N_0 is any positive number if $j_l^1\geq 2$ for at least one l. We therefore obtain, if $\beta=\beta'$,

$$|(4\text{-}79)|\lesssim\langle r_s^{\pm}-\log\langle\eta\rangle\rangle^{-\tau-1-\nu_1}\min\big(e^{-r|\beta|}\langle s\rangle^{\nu_1},\langle r-\log\langle\eta\rangle\rangle^{\nu_1-\tilde{\nu}_1-j}\langle s\rangle^{\tilde{\nu}_1}\big),$$

since $\nu_1-\tilde{\nu}_1-j\geq 0$ in the case where no r derivative fall on the components of $\theta_s^{\pm}$ and only r derivatives of order at most 1 fall on $r_s^{\pm}$. If $\beta\neq\beta'$, we have

$$|(4\text{-}79)|\lesssim\min\big(e^{-2|\beta-\beta'||\rho s|}e^{-|\beta|r}\langle s\rangle^{\nu_1},\ \langle r_s^{\pm}-\log\langle\eta\rangle\rangle^{-\tau-1-\nu_1}\langle r-\log\langle\eta\rangle\rangle^{\nu_1-\tilde{\nu}_1-j}\langle s\rangle^{\tilde{\nu}_1}\big).$$

Since $r_s^{\pm}-r-2\rho s$ is bounded from below, $\rho s\geq 0$ (with $|\rho|\gtrsim 1$) and using (4-17), we have

$$\langle r_s^{\pm}-\log\langle\eta\rangle\rangle^{-\tau-1-\nu_1}\lesssim\langle r-\log\langle\eta\rangle+|s|\rangle^{-\tau-1-\nu_1}.$$

All this implies that

$$\left|D_{\text{hyp}}^{j\alpha k\beta}\left(y^{\pm}\circ\phi_s^{\pm}\right)\right|\lesssim\langle s\rangle^{-\tau-1}\quad\text{and}\quad\left|\partial_r^j\partial_\theta^\alpha\partial_\rho^k\partial_\eta^\beta\left(y^{\pm}\circ\phi_s^{\pm}\right)\right|\lesssim\langle r-\log\langle\eta\rangle+|s|\rangle^{-\tau-1}\langle r-\log\langle\eta\rangle\rangle^{-j},$$

and since

$$\int_0^{+\infty}\langle r-\log\langle\eta\rangle+|s|\rangle^{-\tau-1}ds\lesssim\langle r-\log\langle\eta\rangle\rangle^{-\tau}$$

(using (4-17) on strongly outgoing/incoming areas), we see that the function $\int_0^{\pm\infty}y^{\pm}\circ\phi_s^{\pm}\,ds$ belongs to $\mathcal{B}_{\text{hyp}}(\Gamma_{\text{s}}^{\pm}(\varepsilon^2))$ and is of hyperbolic long range. This implies easily that the same holds for $a_{\text{hom}}^{\pm}-1$. One then checks that $a_{\text{hom}}^{\pm}$ solves the homogeneous transport equation by computing $d(a_{\text{hom}}^{\pm}\circ\phi_t^{\pm})/dt$ at $t=0^{\pm}$. One studies similarly the case of $a_{\text{inhom}}^{\pm}$. □

5. An Isozaki–Kiada type parametrix

In this section, we prove an approximation of $e^{-ithP}\widehat{Op}_\iota(\chi_{\text{s}}^{\pm})$ when $\chi_{\text{s}}^{\pm}$ is supported in the strongly outgoing (+) or incoming (−) region $\Gamma_{\iota,\text{s}}^{\pm}(\varepsilon)$ (see Definition 2.12 for these areas and Definition 2.1 for $\widehat{Op}_\iota(\cdot)$). We recall that ι is an arbitrary index corresponding to the chart at infinity we consider and where the symbols are supported (see (2-4) and (2-19)).

Here we will prove an L^2 approximation, valid for times such that $0\leq\pm t\lesssim h^{-1}$. Basically, we will show that, for any N, $e^{-ithP}\widehat{Op}_\iota(\chi_{\text{s}}^{\pm})$ is the sum of a Fourier integral operator and of a term of order h^N in the operator norm of $L^2(\mathcal{M},\widehat{dG})$, uniformly for $0\leq t\lesssim h^{-1}$.

We will therefore essentially prove half of Proposition 2.20, namely the estimate (2-79). The dispersion estimate (2-78), following from a stationary phase argument on the Fourier integral operator, will be proved in Section 7.

In the sequel, we choose an arbitrary $\iota\in\mathcal{I}$ (see (2-2)). Since it will be fixed, we drop it most of the time from the notation (in particular in phases, symbols) and keep it only for the diffeomorphism Ψ_ι, the regions $\Gamma_{\iota,s}^{\pm}(\cdot)$ and (5-3).

In the next result, we use the classes of symbols $\mathcal{S}_{\text{hyp}}(\cdot)$ introduced in Definition 2.2 and the Fourier integral operators (4-49) defined in Section 4D. For these operators, the phases are associated to the Hamiltonian $p=p_\iota$, the principal symbol of P in the ι-th chart (this notation is consistant with (5-3)).

Theorem 5.1. *For all $N\geq0$, there exists $\varepsilon(N)>0$ such that, for all $0<\varepsilon\leq\varepsilon(N)$, the following holds: there exists $a^{\pm}(h)=a_0^{\pm}+\cdots+h^Na_N^{\pm}$ with $a_0^{\pm},\ldots,a_N^{\pm}\in\mathcal{S}_{\text{hyp}}(\Gamma_{\iota,\text{s}}^{\pm}(\varepsilon))$, such that for all*

$$\chi_{\text{s}}^{\pm}\in\mathcal{S}_{\text{hyp}}(\Gamma_{\iota,\text{s}}^{\pm}(\varepsilon^9))\tag{5-1}$$

we can find $b^{\pm}(h)=b_0^{\pm}+\cdots+h^Nb_N^{\pm}$ with

$$b_0^{\pm},\ldots,b_N^{\pm}\in\mathcal{S}_{\text{hyp}}(\Gamma_{\iota,\text{s}}^{\pm}(\varepsilon^3)),\tag{5-2}$$

such that, for all $T>0$, there exists $C>0$ such that

$$\left\|e^{-ithP}\widehat{Op}_\iota(\chi_{\text{s}}^{\pm})-\Psi_\iota^*\left(J_h^{\pm}(a^{\pm}(h))e^{-ithD_r^2}J_h^{\pm}(b^{\pm}(h))^*\right)\left(\Psi_\iota^{-1}\right)^*\right\|_{L^2(\widehat{dG})\to L^2(\widehat{dG})}\leq Ch^{N-1},$$

provided that

$$0\leq\pm t\leq Th^{-1},\qquad h\in(0,1].$$

By the inclusions in (2-20), together with (2-21) and (2-54), the symbols $a^{\pm}(h)$ and $b^{\pm}(h)$ are supported in $(\varepsilon^{-1}, +\infty) \times V_{\iota,\varepsilon} \times \mathbb{R}^n \subset (R_{\mathcal{K}}+1, +\infty) \times V'_\iota \times \mathbb{R}^n$, for ε small. Therefore the Schwartz kernel of the operator $J_h^{\pm}(a^{\pm}(h))e^{-ithD_r^2} J_h^{\pm}(b^{\pm}(h))^*$ is supported in $\big((R_{\mathcal{K}}+1, +\infty) \times V'_\iota\big)^2$ and hence

$$\Psi_\iota^*\big(J_h^{\pm}(a^{\pm}(h))e^{-ithD_r^2} J_h^{\pm}(b^{\pm}(h))^*\big)(\Psi_\iota^{-1})^*$$

is well defined on the whole manifold (by the implicit requirement that its kernel vanishes outside the coordinate patch $\mathcal{U}_\iota \times \mathcal{U}_\iota$ of $\mathcal{M} \times \mathcal{M}$).

We also remark that $\varepsilon(N)$ could certainly be chosen independently of N. However this is useless for the applications we have in mind and we will not consider this refinement.

Before starting the proof, we give some heuristic ideas about our parametrix. It gives a microlocal approximation of e^{ithP} for initial data microlocalized in strongly outgoing/incoming areas. In such areas, $e^{-r}\eta$ is small and r is large, so the geodesic flow is close to the "free" flow of ρ^2 uniformly in the future/past, as a consequence of Proposition 3.8 basically. This closeness at the classical level remains true at the quantum level in the sense that the flow e^{ithP} can be put in the normal form $e^{ithD_r^2}$, i.e., up to the conjugation by time-independent Fourier integral operators. We point out that we state this approximation on a h^{-1} time scale, but it would more generally hold for times of order h^{-N}, for any N. To obtain a semiglobal in time parametrix (one with $t \geq 0$ or $t \leq 0$), we would need to combine our construction with a priori estimates on e^{ithP} of local energy decay type, to control the error terms given by the Duhamel formula.

Let us fix or recall some notation. We set

$$P_\iota = (\Psi_\iota^{-1})^* P (\Psi_\iota)^* = p(r, \theta, D_r, D_\theta) + p_1(r, \theta, D_r, D_\theta) + p_2(r, \theta), \tag{5-3}$$

with p the principal symbol and p_k of degree $2-k$ in (ρ, η) for $k = 1, 2$. For simplicity, we also use the notation (4-48).

Recall finally that, for some fixed $\varepsilon_\iota > 0$ small enough, Proposition 4.10 proves the existence of $S_{\pm}$ solving

$$p(r, \theta, \partial_r S_{\pm}, \partial_\theta S_{\pm}) = \rho^2 \quad \text{for } (r, \theta, \rho, \eta) \in \Gamma^{+}_{\iota,\mathrm{s}}(\varepsilon_\iota). \tag{5-4}$$

Proof of Theorem 5.1. For simplicity we set

$$A_{\pm} = J_h^{\pm}(a^{\pm}(h)), \qquad B_{\pm} = J_h^{\pm}(b^{\pm}(h)).$$

By the Duhamel formula, we have

$$e^{-ithP}\Psi_\iota^* A_{\pm} = \Psi_\iota^* A_{\pm} e^{-ithD_r^2} - \frac{i}{h}\int_0^t e^{-i(t-s)hP}\Psi_\iota^*(h^2 P_\iota A_{\pm} - A_{\pm}h^2 D_r^2)e^{-ishD_r^2}\,ds. \tag{5-5}$$

Multiplying (5-5) by $B_{\pm}^*(\Psi_\iota^{-1})^*$ and defining

$$C_{\pm} := \chi_{\mathrm{s}}^{\pm}(r, \theta, hD_r, hD_\theta)(\tilde{\kappa} \otimes \tilde{\kappa}_\iota) - A_{\pm}B_{\pm}^*, \quad D_{\pm}(s) := (h^2 P_\iota A_{\pm} - A_{\pm}h^2 D_r^2)e^{-ishD_r^2}B_{\pm}^* \tag{5-6}$$

(where $\tilde{\kappa}$ and $\tilde{\kappa}_\iota$ are the cutoffs used in Definition 2.1), we obtain

$$e^{-ithP}\widehat{Op}_\iota(\chi_s^\pm) \\ = \Psi_\iota^* A_\pm e^{-ithD_r} B_\pm^*(\Psi_\iota^{-1})^* + e^{-ithP}\Psi_\iota^* C_\pm(\Psi_\iota^{-1})^* - \frac{i}{h}\int_0^t e^{-i(t-s)hP}\Psi_\iota^* D_\pm(s)(\Psi_\iota^{-1})^* ds. \quad (5\text{-}7)$$

Using (2-8) with $q=2$, the theorem will then be proved if we find $a^\pm(h)$ and $b^\pm(h)$ such that

$$\|C_\pm\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \lesssim h^N \quad \text{and} \quad \|D_\pm(s)\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \lesssim h^{N+1} \quad \text{for } h\in(0,1], \quad (5\text{-}8)$$

uniformly with respect to $0\le \pm s\le Th^{-1}$ for $D_\pm(s)$.

For simplicity we only consider the outgoing case but the incoming one is of course completely similar.

Construction of $a^+(h)$. We first define (v^+, w^+) by (4-64) and also set

$$y^+ := p(r,\theta,\partial_r,\partial_\theta)S_+ + p_1(r,\theta,\partial_r,\partial_\theta)S_+. \quad (5\text{-}9)$$

Lemma 5.2. *There exists $\tilde{\varepsilon}_\iota \le \varepsilon_\iota$ such that y^+ belongs to $\mathscr{B}_{\mathrm{hyp}}(\Gamma_{\iota,s}^+(\tilde{\varepsilon}_\iota))$ and is of hyperbolic short range on $\Gamma_{\iota,s}^+(\tilde{\varepsilon}_\iota)$.*

Proof. This follows from (2-11) and (4-14), since Proposition 4.9 shows that $y^+_{|\eta=0}\equiv 0$. □

Elementary computations show that, for all $a_0^+,\dots,a_N^+\in\mathscr{S}_{\mathrm{hyp}}^+(\varepsilon)$ and $a^+(h)=a_0^+ +\cdots+h^N a_N^+$,

$$h^2 P_\iota J_h^+(a^+(h)) - J_h^+(a^+(h))h^2D_r^2 = \sum_{l=0}^{N+2} h^l J_h^+(d_l^+),$$

where the symbols are given by

$$\begin{aligned} d_l^+ &= \big(p(r,\theta,\partial_r S_+,\partial_\theta S_+)-\rho^2\big)a_l^+ - i\left(v^+\partial_r a_{l-1}^+ + w^+\cdot\partial_\theta a_{l-1}^+ + y^+ a_{l-1}^+\right) + P_\iota a_{l-2}^+ \\ &= -i\left(v^+\partial_r a_{l-1}^+ + w^+\cdot\partial_\theta a_{l-1}^+ + y^+ a_{l-1}^+\right) + P_\iota a_{l-2}^+, \end{aligned} \quad (5\text{-}10)$$

using (5-4) and assuming $\varepsilon\le\varepsilon_\iota$. Here, we have $0\le l\le N+2$ and the convention that $a_{-2}^+ = a_{-1}^+ = a_{N+1}^+ = a_{N+2}^+\equiv 0$. In particular, the first three terms are given by

$$d_0^+ = 0, \quad (5\text{-}11)$$

$$i d_1^+ = v^+\partial_r a_0^+ + w^+\cdot\partial_\theta a_0^+ + y^+ a_0^+, \quad (5\text{-}12)$$

$$i d_2^+ = v^+\partial_r a_1^+ + w^+\cdot\partial_\theta a_1^+ + y^+ a_1^+ + iP_\iota a_0^+. \quad (5\text{-}13)$$

Using Proposition 4.15, Lemma 5.2 and assuming $\hat{\varepsilon}_\iota\le\min(\tilde{\varepsilon}_\iota^2,\varepsilon_5)$ we can define

$$\hat{a}_0^+(r,\theta,\rho,\eta) = \exp\int_0^{+\infty} y^+\circ\phi_s^+(r,\theta,\rho,\eta)\,ds \quad \text{for } (r,\theta,\rho,\eta)\in\Gamma_{\iota,s}^+(\hat{\varepsilon}_\iota),$$

so $\hat{a}_0^+\in\mathscr{B}_{\mathrm{hyp}}(\Gamma_{\iota,s}^+(\hat{\varepsilon}_\iota))$, $\hat{a}_0^+-1$ is of hyperbolic long range in $\Gamma_{\iota,s}^+(\hat{\varepsilon}_\iota)$ and

$$v^+\partial_r\hat{a}_0^+ + w^+\cdot\partial_\theta\hat{a}_0^+ + y^+\hat{a}_0^+\equiv 0 \quad \text{on } \Gamma_{\iota,s}^+(\hat{\varepsilon}_\iota).$$

Since the function $\int_0^\infty y^+\circ\phi_s^+\,ds$ is bounded on $\Gamma_{\iota,s}^+(\hat{\varepsilon}_\iota)$ (see the proof of Proposition 4.15), we also have

$$\hat{a}_0^+(r,\theta,\rho,\eta)\gtrsim 1 \quad \text{for } (r,\theta,\rho,\eta)\in\Gamma_{\iota,s}^+(\hat{\varepsilon}_\iota). \quad (5\text{-}14)$$

Using (2-11) and the fact that $\hat{a}_0^+ - 1$ is of hyperbolic long range, it is easy to check that $P_\iota \hat{a}_0^+$ is of hyperbolic short range in $\Gamma^+_{\iota,s}(\hat{\varepsilon}_\iota^2)$. By Proposition 4.15, we can then define

$$\hat{a}_1^+ = i\int_0^{+\infty} (P_\iota \hat{a}_0^+)\circ\phi_s^+ \exp\left(\int_0^s y^+\circ\phi_u^+\,du\right)ds \quad \text{on } \Gamma^+_{\iota,s}(\hat{\varepsilon}_\iota^2),$$

which belongs to $\mathcal{B}_{\text{hyp}}(\Gamma^+_{\iota,s}(\hat{\varepsilon}_\iota^2))$, is of hyperbolic long range in $\Gamma^+_{\iota,s}(\hat{\varepsilon}_\iota)$ and satisfies

$$v^+\partial_r\hat{a}_1^+ + w^+\cdot\partial_\theta\hat{a}_1^+ + y^+\hat{a}_1^+ \equiv -iP_\iota\hat{a}_0^+ \quad \text{on } \Gamma^+_{\iota,s}(\hat{\varepsilon}_\iota^2).$$

More generally, for $1\le l\le N$, we can define iteratively

$$\hat{a}_l^+ = i\int_0^{+\infty} (P_\iota \hat{a}_{l-1}^+)\circ\phi_s^+ \exp\left(\int_0^s y^+\circ\phi_u^+\,du\right)ds \quad \text{on } \Gamma^+_{\iota,s}(\hat{\varepsilon}_\iota^{2^l}),$$

which belongs to $\mathcal{B}_{\text{hyp}}(\Gamma^+_{\iota,s}(\hat{\varepsilon}_\iota^{2^l}))$, is of hyperbolic long range in $\Gamma^+_{\iota,s}(\hat{\varepsilon}_\iota^{2^l})$ and satisfies

$$v^+\partial_r\hat{a}_l^+ + w^+\cdot\partial_\theta\hat{a}_l^+ + y^+\hat{a}_l^+ \equiv -iP_\iota\hat{a}_{l-1}^+ \quad \text{on } \Gamma^+_{\iota,s}(\hat{\varepsilon}_\iota^{2^l}),$$

using Proposition 4.15 and the fact that $P_\iota\hat{a}_{l-1}^+$ is of hyperbolic short range if $\hat{a}_l^+$ is of hyperbolic long range. Therefore, using Proposition 4.4 with $\varepsilon \le \hat{\varepsilon}_\iota^{2^N}$ and setting

$$a_l^+ = \chi^+_{\varepsilon^2\to\varepsilon}\hat{a}_l^+ \quad \text{for } 0\le l\le N,$$

with the $\hat{a}_l^+$ defined above, we have constructed $a_0^+,\dots,a_N^+ \in \mathcal{S}^+_{\text{hyp}}(\varepsilon)$ with a_0^+ satisfying (4-63), by (5-14). Furthermore,

$$d_l^+ \in \mathcal{S}^+_{\text{hyp}}(\varepsilon) \quad \text{for } \ 0\le l\le N+2 \tag{5-15}$$

and

$$d_l^+ \equiv 0 \ \text{ on } \Gamma^+_{\iota,s}(\varepsilon^2) \ \text{ for } 0\le l\le N. \tag{5-16}$$

Construction of $b^+(h)$. Given $\chi_s^+ \in \mathcal{S}^+_{\text{hyp}}(\varepsilon^9)$, we simply choose the symbols $b_0^+,\dots,b_N^+$ according to Proposition 4.13, with $\varepsilon \le \min(\hat{\varepsilon}_\iota^{2^N}, \varepsilon_6)$.

Justification of the parametrix. Since $\tilde{\kappa}\otimes\tilde{\kappa}_\iota \equiv 1$ near the support of χ_s^+, we have

$$\left\|\chi_s^+(r,\theta,hD_r,hD_\theta) - \chi_s^+(r,\theta,hD_r,hD_\theta)(\tilde{\kappa}\otimes\tilde{\kappa}_\iota)\right\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \lesssim h^M, \qquad h\in(0,1],$$

for all M, using the standard symbolic calculus, the Calderón–Vaillancourt Theorem and the fact that $\mathcal{S}^+_{\text{hyp}}(\varepsilon) \subset C_b^\infty(\mathbb{R}^{2n})$. Using Proposition 4.13, we therefore obtain

$$\|C^+\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \lesssim h^{N+1}, \qquad h\in(0,1].$$

It remains to consider $D_+(s)$, which reads

$$D_+(s) = \sum_{l=0}^{N+2}\sum_{m=0}^{N} h^{l+m} J_h^+(d_l^+)e^{-ishD_r^2}J_h^+(b_m^+)^*.$$

By (4-62) and (5-15), the part of the sum where $l\ge N+1$, has an L^2 operator norm of order h^{N+1}. Once divided by h and integrated over an interval of size at most h^{-1}, the corresponding operator norm is $\mathcal{O}(h^{N-1})$. The control of the other terms of the sum will follow from the next result.

Proposition 5.3. *If ε is small enough, then, for all $0 \le l, m \le N$ and all $M \ge 0$, we have*

$$\big\| J_h^+(d_l^+) e^{-ishD_r^2} J_h^+(b_m^+)^* \big\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \le C_\varepsilon h^M \quad \text{for } h \in (0,1],\ 0 \le s \le Th^{-1}.$$

The proof is based on a fairly elementary nonstationary phase argument. To control the operator norms of the kernels obtained after integrations by parts, we need the following rough lemma.

Lemma 5.4. *For $a \in C_b^\infty(\mathbb{R}^{3n})$ compactly supported with respect to ρ, let us set*

$$[a]_h^+(r,\theta,r',\theta') = (2\pi h)^{-n} \iint e^{\frac{i}{h}(S_{+,\varepsilon}(r,\theta,\rho,\eta) - s\rho^2 - S_{+,\varepsilon}(r',\theta',\rho,\eta))} a(r,\theta,r',\theta',\rho,\eta)\, d\rho\, d\eta,$$

using $S_{+,\varepsilon}$ defined in Proposition 4.10. Denote by $\mathcal{A}_h^+ : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ the operator with Schwartz kernel $[a]_h^+$. Then, there exists $n_0(n) \ge 0$ such that, for all ε small enough,

$$\|\mathcal{A}_h^+\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \le C_\varepsilon h^{-n_0} \langle s \rangle^{n_0} \max_{|\gamma|\le n_0} \sup_{\mathbb{R}^{3n}} \|\partial^\gamma a\|_\infty,$$

for all $h \in (0,1]$, all $s \in \mathbb{R}$ and all $a \in C_b^\infty(\mathbb{R}^{3n})$ satisfying

$$\operatorname{supp}(a) \subset \{|\rho| \le 10\}.$$

Proof. We get this as a simple consequence of the Calderón–Vaillancourt Theorem by interpreting $\mathcal{A}_h^+$ as the pseudodifferential operator with symbol

$$e^{\frac{i}{h}(\varphi_{+,\varepsilon}(r,\theta,\rho,\eta) - s\rho^2 - \varphi_{+,\varepsilon}(r',\theta',\rho,\eta))} a(r,\theta,r',\theta',\rho,\eta),$$

where $\varphi_{+,\varepsilon}$ is defined in Proposition 4.10. □

Proof of Proposition 5.3. We notice first that, by Proposition 4.9 and (4-36),

$$\partial_\rho \big(S_+(r,\theta,\rho,\eta) - s\rho^2 - S_+(r',\theta',\rho,\eta)\big) = r - r' - 2s\rho + \mathcal{O}(\varepsilon^2), \tag{5-17}$$
$$\partial_\eta \big(S_+(r,\theta,\rho,\eta) - s\rho^2 - S_+(r',\theta',\rho,\eta)\big) = \theta - \theta' + \mathcal{O}(e^{-1/\varepsilon}), \tag{5-18}$$

on the support of $d_l^+(r,\theta,\rho,\eta) b_m^+(r',\theta',\rho,\eta)$. On the other hand, by construction, we have

$$d_l^+ = i^{-1}\big(v^+ \partial_r \chi_{\varepsilon^2\to\varepsilon} + w^+ \cdot \partial_\theta \chi_{\varepsilon^2\to\varepsilon}\big) \hat{a}_{l-1}^+ + P_\iota(\chi_{\varepsilon^2\to\varepsilon} \hat{a}_{l-2}^+) - \chi_{\varepsilon^2\to\varepsilon} P_\iota \hat{a}_{l-2}^+$$

(with the convention that $\hat{a}_{-2}^+ = \hat{a}_{-1}^+ \equiv 0$). Using in particular that

$$w^+ = e^{-r}(\partial_\eta q)(r,\theta,e^{-r}\partial_\theta S_+),$$

we conclude that d_l^+ is a sum of terms of the form $c(r,\theta,\rho,\eta)\partial_r^j (e^{-r}\partial_\theta)^\alpha \chi_{\varepsilon^2\to\varepsilon}^+$, with $j + |\alpha| \ge 1$ and $c \in \mathcal{B}_{\mathrm{s}}^+(\varepsilon)$. Using the form of $\chi_{\varepsilon^2\to\varepsilon}^+$ given by Proposition 4.4, we see that, on the support of such terms, at least one of the following properties hold:

$$\varepsilon^{-1} \le r \le \varepsilon^{-2}, \tag{5-19}$$
$$p(r,\theta,\rho,\eta) \le \tfrac{1}{4} - \varepsilon^2 \ \text{ or } \ p(r,\theta,\rho,\eta) \ge 4 + \varepsilon^2, \tag{5-20}$$
$$\varepsilon^{4-2\kappa} \lesssim e^{-2r}|\eta|^2 \lesssim \varepsilon^2, \tag{5-21}$$
$$\operatorname{dist}(\theta, V_\iota) \ge \varepsilon^4, \tag{5-22}$$

for some fixed $0 < \kappa < 1$ in (5-21). For terms such that (5-19) holds on their supports, we have

$$(5\text{-}17) \leq \varepsilon^{-2} - \varepsilon^{-3} - 2s\rho + C \leq -1 - 2s\rho. \tag{5-23}$$

for ε small enough and integrate by parts with respect to ρ. For those satisfying (5-20) on their supports, then we must have

$$\rho^2 - \tfrac{1}{4} \leq -\varepsilon^2 \quad \text{or} \quad \rho^2 - 4 \gtrsim \varepsilon^2,$$

since $e^{-2r}|\eta|^2 \lesssim \varepsilon^2$ in any case, whereas on the support of b_l^+, where $p(r', \theta', \rho, \eta) \in \left(\frac{1}{4} - \varepsilon^3,\, 4 + \varepsilon^3\right)$ and $e^{-2r'}|\eta|^2 \lesssim \varepsilon^6$,

$$\rho^2 - \tfrac{1}{4} \gtrsim -\varepsilon^3 \quad \text{and} \quad \rho^2 - 4 \leq \varepsilon^3,$$

so that the amplitude vanishes identically, again if ε is small enough. For those satisfying (5-21) on their supports, we have $e^r|\eta|^{-1} \lesssim \varepsilon^{\kappa-2}$. Since $e^{-r'}|\eta| \lesssim \varepsilon^3$, we get

$$e^{r-r'} \leq C + (1+\kappa)\ln\varepsilon \ll 0,$$

which implies again that $(5\text{-}17) \leq -1 - 2s\rho$, if ε is small enough. Thus on the supports of terms satisfying either (5-19) or (5-20) or (5-21), we have $|(5\text{-}17)| \gtrsim \langle s\rangle$. By standard integrations by parts, the kernel of corresponding operator can be written, for all M, as in Lemma 5.4 with amplitudes of order $(h/\langle s\rangle)^M$ in $C_b^\infty(\mathbb{R}^{3n})$. Hence, their L^2 operator norms are of order $(h/\langle s\rangle)^{M-n_0}$ with an arbitrary M.

For the remaining terms satisfying (5-22) on their supports, we remark that $|\theta' - \theta| \geq \varepsilon^5$ (otherwise $\operatorname{dist}(\theta, V_\iota) \leq |\theta - \theta'| + \operatorname{dist}(\theta', V_\iota) < \varepsilon^5 + \varepsilon^6 \ll \varepsilon^4$) hence

$$|(5\text{-}18)| \gtrsim \varepsilon^5.$$

Thus, for all $M \geq 0$, the kernel of the corresponding operators can be written as in Lemma 5.4 with amplitudes of order h^M in $C_b^\infty(\mathbb{R}^{3n})$. Since M is arbitrary, their L^2 operator norms are of order h^M if $|s| \lesssim h^{-1}$. □

This completes the proof of Theorem 5.1. □

6. Geometric optics and Egorov's theorem on AH manifolds

As in the previous section, we fix here an arbitrary index ι corresponding a coordinate patch and then drop it from the notation in symbols, phases, intervals, etc.

6A. ***Finite time WKB approximation.*** Next we give a short time parametrix of $e^{-ithP}\widehat{Op}_\iota(\chi^\pm)$ when $\chi^\pm$ is supported in an outgoing $(+)$ or an incoming $(-)$ area. This parametrix is the standard geometric optic (or WKB) approximation which is basically well known. Nevertheless, in the literature, one mostly finds local versions (i.e., with $\chi \in C_0^\infty$) or versions in $\mathbb{R}^n$ for elliptic operators. Here we are neither in a relatively compact set nor in the uniformly elliptic setting, so we recall the construction with some details.

Analogously to Section 5, we prove here an L^2 approximation. The related dispersion estimates leading to (2-80) will be derived in Section 7.

We also emphasize that, although we shall prove this approximation with a specified time orientation ($t \geq 0$ for χ^+ and $t \leq 0$ for χ^-), this result has nothing to do with outgoing/incoming areas; in principle we should be able to state a similar result for any χ supported in $p^{-1}(I)$ and for times $|t| \ll 1$. We restrict the sense of time for only two reasons: firstly, because it is sufficient for our purpose and, secondly, because we can use directly Proposition 3.8 (we should otherwise give a similar result for the geodesic flow for t in an open neighborhood of 0).

Fix

$$I_1 \Subset I_2 \Subset I_3 \Subset (0, +\infty),$$

three relatively compact open subsets of V'_ι (see (2-21)),

$$V_1 \Subset V_2 \Subset V_3 \Subset V'_\iota,$$

and three real numbers

$$-1 < \sigma_1 < \sigma_2 < \sigma_3 < 1.$$

For some R_3 large enough to be fixed below, we also choose arbitrary R_1, R_2 real numbers such that

$$R_1 > R_2 > R_3.$$

Theorem 6.1. *For all R_3 large enough, there exists $t_{\mathrm{WKB}} > 0$ and a function*

$$\Sigma \in C^\infty\big([0, \pm t_{\mathrm{WKB}}] \times \mathbb{R}^{2n}, \mathbb{R}\big)$$

such that, for any

$$\chi^\pm \in \mathcal{S}_{\mathrm{hyp}}\big(\Gamma_\iota^\pm(R_1, V_1, I_1, \sigma_1)\big), \tag{6-1}$$

we can find

$$a_0^\pm(t), \dots, a_N^\pm(t) \in \mathcal{S}_{\mathrm{hyp}}\big(\Gamma_\iota^\pm(R_2, V_2, I_2, \sigma_2)\big),$$

depending smoothly on t for $0 \le \pm t \le t_{\mathrm{WKB}}$, and such that, if we set

$$a_N^\pm(t, h) = a_0^\pm(t) + \dots + h^N a_N^\pm(t),$$

the operator defined on $C_0^\infty(\mathbb{R}^n)$ by the kernel

$$\big[\mathcal{J}_h^\pm\big(t, a_N^\pm(t, h)\big)\big](t, r, \theta, r', \theta') = (2\pi h)^{-n} \iint e^{\frac{i}{h}(\Sigma^\pm(t,r,\theta,\rho,\eta) - r'\rho - \theta'\cdot\eta)} a_N^\pm(t, h, r, \theta, \rho, \eta)\, d\rho\, d\eta,$$

satisfies, with $\mathbf{1}_\iota$ the characteristic function of $(R_3, +\infty) \times V_3$,

$$\big\| e^{-ithP} \widehat{Op}_\iota\big(\chi^\pm\big) - \Psi_\iota^* \mathcal{J}_h^\pm\big(t, a_N^\pm(t, h)\big) \mathbf{1}_\iota \big(\Psi_\iota^{-1}\big)^* \big\|_{L^2(\mathcal{M}, d\widehat{G}) \to L^2(\mathcal{M}, d\widehat{G})} \le C h^{N+1}, \tag{6-2}$$

for

$$0 \le \pm t \le t_{\mathrm{WKB}}, \qquad h \in (0, 1].$$

In addition, the functions $\Sigma^\pm$ are of the form

$$\Sigma^\pm(t, r, \theta, \rho, \eta) = r\rho + \theta\cdot\eta + \big(\Sigma_0^\pm(t, r, \theta, \rho, \eta) - r\rho - \eta\cdot\eta\big)\, \chi_{2\to3}^\pm(r, \theta, \rho, \eta),$$

with $\chi_{2\to3}^\pm \in \mathcal{S}_{\mathrm{hyp}}(\Gamma_\iota^\pm(R_3, V_3, I_3, \sigma_3))$ such that $\chi_{2\to3}^\pm \equiv 1$ on $\Gamma_\iota^\pm(R_2, V_2, I_2, \sigma_2)$, and some bounded family $(\Sigma_0^\pm(t))_{0 \le \pm t \le t_{\mathrm{WKB}}}$ in $\mathcal{B}_{\mathrm{hyp}}(\Gamma_\iota^\pm(R_3, V_3, I_3, \sigma_3))$ satisfying

$$\begin{cases} \partial_t \Sigma_0^\pm + p(r, \theta, \partial_r \Sigma_0^\pm, \partial_\theta \Sigma_0^\pm) = 0, \\ \Sigma_0^\pm(0, r, \theta, \rho, \eta) = r\rho + \theta\cdot\eta, \end{cases} \tag{6-3}$$

and

$$\big| D_{\mathrm{hyp}}^{j\alpha k\beta}\big(\Sigma_0^\pm(t, r, \theta, \rho, \eta) - r\rho - \theta\cdot\eta - tp(r, \theta, \rho, \eta)\big) \big| \le C_{j\alpha k\beta} t^2, \tag{6-4}$$

both for

$$0 \le \pm t \le t_{\mathrm{WKB}} \quad \text{and} \quad (r,\theta,\rho,\eta) \in \Gamma_\iota^{\pm}(R_3, V_3, I_3, \sigma_3).$$

We also have

$$\big(\Sigma^{\pm}(t,r,\theta,\rho,\eta) - r\rho - \theta\cdot\eta\big)_{0\le\pm t\le t_{\mathrm{WKB}}} \quad \text{bounded in} \quad \mathcal{S}_{\mathrm{hyp}}\big(\Gamma_\iota^{\pm}(R_3,V_3,I_3,\sigma_3)\big). \tag{6-5}$$

Finally, for all $0 \le j \le N$,

$$\big(a_j^{\pm}(t)\big)_{0\le\pm t\le t_{\mathrm{WKB}}} \quad \text{is bounded in} \quad \mathcal{S}_{\mathrm{hyp}}\big(\Gamma_\iota^{\pm}(R_2,V_2,I_2,\sigma_2)\big). \tag{6-6}$$

Notice that $V_1 \Subset V_\iota'$, so it makes sense to consider $\widehat{Op}_\iota(\chi^{\pm})$; see (2-23).

In principle it is not necessary to have R_3 large to get such a lemma, but this will be sufficient for our applications. The interest of choosing R_3 large is simply to allow to use directly Proposition 3.8.

Note also that, by (6-6), the kernel of $\mathcal{J}_h^{\pm}\big(t, a_N^{\pm}(t,h)\big)\,\mathbf{1}_\iota$ is supported in $((R_3,+\infty)\times V_3)^2$.

Proof of Theorem 6.1. The proof will occupy the rest of this section.

We need to find $\Sigma_{\pm}$ and $a_N^{\pm}(t,h)$ such that

$$\mathcal{J}_h^{\pm}\big(0, a_N^{\pm}(0,h)\big) = \chi^{\pm}(r,\theta,hD_r,hD_\theta), \tag{6-7}$$

$$(hD_t + h^2 P_\iota)\mathcal{J}_h^{\pm}(t, a_N^{\pm}(t,h)) = h^{N+2} R_N^{\pm}(t,h), \tag{6-8}$$

where $P_\iota = (\Psi_\iota^{-1})^* P \Psi_\iota^*$ and

$$\|R_N^{\pm}(t,h)\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \le C, \qquad h\in(0,1],\ 0\le\pm t\le t_{\mathrm{WKB}}. \tag{6-9}$$

Indeed, if (6-7), (6-8) and (6-9) hold, the equality

$$\Psi_\iota^*\mathcal{J}_h^{\pm}\big(t,a_N^{\pm}(t,h)\big)\,\mathbf{1}_\iota\,\big(\Psi_\iota^{-1}\big)^* - e^{-ithP}\Psi_\iota^*\chi^{\pm}(r,\theta,hD_r,hD_\theta)\mathbf{1}_\iota\,\big(\Psi_\iota^{-1}\big)^* \\ = ih^{N+1}\int_0^t e^{-i(t-s)h\tilde{P}}\Psi_\iota^* R_N(s,h)\mathbf{1}_\iota\,\big(\Psi_\iota^{-1}\big)^*\,ds$$

will yield (6-2) since, for all $M>0$,

$$\|\Psi_\iota^*\chi^{\pm}(r,\theta,hD_r,hD_\theta)\mathbf{1}_\iota\,\big(\Psi_\iota^{-1}\big)^* - \widehat{Op}_\iota(\chi^{\pm})\|_{L^2(\mathcal{M},\widehat{dG})\to L^2(\mathcal{M},\widehat{dG})} \le C_M h^M,$$

by standard off-diagonal decay (see Definition 2.1 for $\widehat{Op}_\iota$), since $\mathbf{1}_\iota \equiv 1$ near $\Pi_{r,\theta}\big(\mathrm{supp}(\chi^{\pm})\big)$.

To get the conditions to be satisfied by $\Sigma^{\pm}$ and $a_0^{\pm},\dots,a_N^{\pm}$ we observe that

$$\big(hD_t + h^2 P_\iota\big)\,\mathcal{J}_h^{\pm}\big(t,a_N^{\pm}(t,h)\big) = \sum_{j=0}^{N+2} h^j \mathcal{J}_h^{\pm}\Big(t, b_j^{\pm}(t)\Big), \tag{6-10}$$

where, if we additionally set $a_{-2}^{\pm} = a_{-1}^{\pm} = a_{N+1}^{\pm} = a_{N+2}^{\pm} \equiv 0$,

$$b_j = (\partial_t\Sigma^{\pm} + p(r,\theta,\partial_r\Sigma^{\pm},\partial_\theta\Sigma^{\pm}))a_j^{\pm} + i^{-1}(\partial_t + \mathcal{T}^{\pm})a_{j-1}^{\pm} + Pa_{j-2}^{\pm}, \tag{6-11}$$

with

$$\mathcal{T}^{\pm} = 2\partial_r\Sigma^{\pm}\partial_r + (\partial_\eta q)(r,\theta,e^{-r}\partial_\theta\Sigma^{\pm})\cdot e^{-r}\partial_\theta + (p+p_1)(r,\theta,\partial_r,\partial_\theta)\Sigma^{\pm}, \tag{6-12}$$

where $q = q_\iota$ is defined in (2-13) and p_1 is the homogeneous part of degree 1 of the full symbol of P_ι. To obtain (6-7), (6-8) and (6-9) it will therefore be sufficient to solve the eikonal equation (6-3), then the transport equations

$$(\partial_t + \mathcal{T}^\pm)a_0^\pm = 0, \qquad a_0^\pm(0,\cdot) = \chi^\pm(.), \tag{6-13}$$

$$(\partial_t + \mathcal{T}^\pm)a_k^\pm = -iP_\iota a_{k-1}^\pm, \qquad a_k^\pm(0,\cdot) = 0, \tag{6-14}$$

for $1 \le k \le N$, and finally to get an L^2 bound for Fourier integral operators of the form $\mathcal{J}_h^\pm(t,a)$ (using the Kuranishi trick).

To solve (6-3), we need the following lemma for which we recall that $(r^t, \theta^t, \rho^t, \eta^t)$ is the Hamiltonian flow of p.

Lemma 6.2. *For all $-1 < \sigma_{\mathrm{eik}} < \sigma'_{\mathrm{eik}} < 1$, all open intervals $I_{\mathrm{eik}} \Subset I'_{\mathrm{eik}} \Subset (0,+\infty)$, all open subsets $V_{\mathrm{eik}} \Subset V'_{\mathrm{eik}} \Subset V'_\iota$ and all $R_{\mathrm{eik}} > R'_{\mathrm{eik}}$ large enough, there exists $t_1 > 0$ small enough that*

$$\Psi_\pm^t : (r,\theta,\rho,\eta) \mapsto (r^t, \theta^t, \rho, \eta)$$

is a diffeomorphism from $\Gamma_\iota^\pm(R'_{\mathrm{eik}}, V'_{\mathrm{eik}}, I'_{\mathrm{eik}}, \sigma'_{\mathrm{eik}})$ onto its range for all $0 \le \pm t < t_1$ and

$$\Gamma_\iota^\pm(R_{\mathrm{eik}}, V_{\mathrm{eik}}, I_{\mathrm{eik}}, \sigma_{\mathrm{eik}}) \subset \Psi_\pm^t\big(\Gamma_\iota^\pm(R'_{\mathrm{eik}}, V'_{\mathrm{eik}}, I'_{\mathrm{eik}}, \sigma'_{\mathrm{eik}})\big) \qquad \text{for all } 0 \le \pm t < t_1.$$

Proof. First choose a $\sigma''_{\mathrm{eik}} \in \mathbb{R}$, and open interval I''_{eik} and open set V''_{eik} such that

$$\sigma'_{\mathrm{eik}} < \sigma''_{\mathrm{eik}} < 1, \qquad I'_{\mathrm{eik}} \Subset I''_{\mathrm{eik}} \Subset (0,+\infty), \qquad V'_{\mathrm{eik}} \Subset V''_{\mathrm{eik}} \Subset V'_\iota.$$

Also choose $R''_{\mathrm{eik}} > 0$ large enough that Proposition 3.8 holds with $\sigma = |\sigma''_{\mathrm{eik}}|$ and $R = R''_{\mathrm{eik}}$. We then choose arbitrary R_{eik} and R'_{eik} such that

$$R_{\mathrm{eik}} > R'_{\mathrm{eik}} > R''_{\mathrm{eik}},$$

and then $\chi^\pm_{\prime\to\prime\prime} \in \mathcal{S}_{\mathrm{hyp}}(\Gamma_\iota^\pm(R''_{\mathrm{eik}}, V''_{\mathrm{eik}}, I''_{\mathrm{eik}}, \sigma''_{\mathrm{eik}}))$ such $\chi^\pm_{\prime\to\prime\prime} \equiv 1$ on $\Gamma_\iota^\pm(R'_{\mathrm{eik}}, V'_{\mathrm{eik}}, I'_{\mathrm{eik}}, \sigma'_{\mathrm{eik}})$. The existence of such a function follows from Proposition 4.1(i). In particular, $\chi^\pm_{\prime\to\prime\prime}$ and $\partial_{r,\theta,\rho,\eta}\chi^\pm_{\prime\to\prime\prime}$ are bounded on $\mathbb{R}^{2n}$. For $\pm t \ge 0$, consider the map

$$\varepsilon_\pm^t : \mathbb{R}^{2n} \ni (r,\theta,\rho,\eta) \mapsto \left(\int_0^t 2\rho^s\,ds, \int_0^t e^{-r^s}(\partial_\eta q)(r^s,\theta^s,e^{-r^s}\eta^s)\,ds\right)\chi^\pm_{\prime\to\prime\prime}(r,\theta,\rho,\eta) \in \mathbb{R}^n, \tag{6-15}$$

so that, by the equations of motion,

$$\Psi_\pm^t = \mathrm{Id}_{\mathbb{R}^{2n}} + (\varepsilon_\pm^t, 0) \quad \text{on } \Gamma_\iota^\pm(R'_{\mathrm{eik}}, V'_{\mathrm{eik}}, I'_{\mathrm{eik}}, \sigma'_{\mathrm{eik}}).$$

By Proposition 3.8 we have $|\partial_{r,\theta,\rho,\eta}\varepsilon_\pm^t| \lesssim |t|$; hence $\mathrm{Id}_{\mathbb{R}^{2n}} + \big(\varepsilon_\pm^t, 0\big)$ is a diffeomorphism from $\mathbb{R}^{2n}$ onto itself, for all $\pm t \ge 0$ small enough. Therefore, it remains to show that, if t is small enough and $(r,\theta,\rho,\eta) \in \Gamma_\iota^\pm(R_{\mathrm{eik}}, V_{\mathrm{eik}}, I_{\mathrm{eik}}, \sigma_{\mathrm{eik}})$ is of the form

$$(r,\theta,\rho,\eta) = (r',\theta',\rho',\eta') + (\varepsilon_\pm^t(r',\theta',\rho',\eta'), 0),$$

then $(r',\theta',\rho',\eta') \in \Gamma_\iota^\pm(R'_{\mathrm{eik}}, V'_{\mathrm{eik}}, I'_{\mathrm{eik}}, \sigma'_{\mathrm{eik}})$. We have trivially $\rho = \rho'$ and $\eta = \eta'$. By Proposition 3.8, $|\varepsilon_\pm^t| \lesssim |t|$ on $\mathbb{R}^{2n}$, so $|r - r'| + |\theta - \theta'| \lesssim |t|$; hence $r' > R'_{\mathrm{eik}}$ and $\theta' \in V'_{\mathrm{eik}}$ if t is small enough. Moreover, by writing

$$q(r,\theta,e^{-r}\eta)-q(r',\theta',e^{-r'}\eta)=q(r,\theta,e^{-r}\eta)-q(r',\theta',e^{-r}\eta)+(1-e^{-2(r'-r)})q(r',\theta',e^{-r}\eta),$$

we see that

$$\left|p(r,\theta,\rho,\eta)-p(r',\theta',\rho,\eta)\right|\lesssim|t|,$$

using the boundedness of $|e^{-r'}\eta|$ and Taylor's formula. Hence

$$p(r',\theta',\rho,\eta)\in I'_{\mathrm{eik}}\quad\text{and}\quad\pm\rho>-\sigma'_{\mathrm{eik}}p(r',\theta',\rho,\eta)^{1/2}$$

if t is small enough, since $p(r,\theta,\rho,\eta)\in I_{\mathrm{eik}}$ and $\pm\rho>-\sigma_{\mathrm{eik}}p(r,\theta,\rho,\eta)^{1/2}$. This completes the proof. □

Now fix $I_{\mathrm{eik}}, I'_{\mathrm{eik}}, V_{\mathrm{eik}}, V'_{\mathrm{eik}}$, and $\sigma_{\mathrm{eik}}, \sigma'_{\mathrm{eik}}$ as in Lemma 6.2, with the additional conditions

$$V_{\mathrm{eik}}=V_3,\qquad I_{\mathrm{eik}}=I_3,\qquad \sigma_{\mathrm{eik}}=\sigma_3.$$

Denote by $\Psi_t^{\pm}$ the inverse of $\Psi_{\pm}^t$ and define $(r_t,\theta_t)=(r_t,\theta_t)(r,\theta,\rho,\eta)$ by

$$\Psi_t^{\pm}(r,\theta,\rho,\eta)=(r_t,\theta_t,\rho,\eta)\in\Gamma_\iota^{\pm}(R'_{\mathrm{eik}},V'_{\mathrm{eik}},I'_{\mathrm{eik}},\sigma'_{\mathrm{eik}}),$$

if $(r,\theta,\rho,\eta)\in\Gamma_\iota^{\pm}(R_{\mathrm{eik}},V_{\mathrm{eik}},I_{\mathrm{eik}},\sigma_{\mathrm{eik}})$ and $0\le\pm t<t_1$. Here t_1, R_{eik} and R'_{eik} are those given by Lemma 6.2.

Proposition 6.3. *For all $R_3>R_{\mathrm{eik}}$, there exists $t_{\mathrm{eik}}>0$ such that*

$$\Sigma_0^{\pm}(t,r,\theta,\rho,\eta):=r_t\rho+\theta_t\cdot\eta+tp(r_t,\theta_t,\rho,\eta),$$

solves (6-3) on $\Gamma_\iota^{\pm}(R_3,V_3,I_3,\sigma_3)$ for $0\le\pm t\le t_{\mathrm{eik}}$, and such that

$$(\Sigma_0^{\pm}(t,r,\theta,\rho,\eta)-r\rho-\theta\cdot\eta)_{0\le\pm t\le t_{\mathrm{eik}}}\quad\textit{is bounded in}\quad\mathscr{B}_{\mathrm{hyp}}\big(\Gamma_\iota^{\pm}(R_3,V_3,I_3,\sigma_3)\big).\tag{6-16}$$

Proof. That $\Sigma_0^{\pm}$ solves the eikonal equation is standard, so we only have to show (6-16). Since

$$\Sigma_0^{\pm}(t,r,\theta,\rho,\eta)=r\rho+\theta\cdot\eta+(r_t-r)\rho+e^r(\theta_t-\theta)\cdot e^{-r}\eta+te^{-2(r_t-r)}q(r_t,\theta_t,e^{-r}\eta),$$

(6-16) would follow from the estimates

$$\left|D_{\mathrm{hyp}}^{j\alpha k\beta}(r_t-r)\right|+\left|D_{\mathrm{hyp}}^{j\alpha k\beta}(e^r(\theta_t-\theta))\right|\le C_{j\alpha k\beta},\tag{6-17}$$

for $0\le\pm t\le\pm t_{\mathrm{eik}}$ and $(r,\theta,\rho,\eta)\in\Gamma_\iota^{\pm}(R_3,V_3,I_3,\sigma_3)$. The equations of motion yield

$$r^t=r+\int_0^t 2\rho^s\,ds,\qquad \theta^t=\theta+\int_0^t e^{-r^s}(\partial_\eta q)(r^s,\theta^s,e^{-r^s}\eta^s)\,ds,\tag{6-18}$$

so, by Proposition 3.8 with R'_{eik} of Lemma 6.2 and by choosing t_{eik} small enough, we see that, for $0\le\pm t\le t_{\mathrm{eik}}$,

$$|\partial_{r,\theta}(r^t,\theta^t)-\mathrm{Id}_n|\le\tfrac12\quad\text{on }\Gamma_\iota^{\pm}(R'_{\mathrm{eik}},V'_{\mathrm{eik}},I'_{\mathrm{eik}},\sigma'_{\mathrm{eik}}),$$

where $|\cdot|$ is a matrix norm. Therefore, by differentiating the identity $(r^t,\theta^t)(r_t,\theta_t,\rho,\eta)=(r,\theta)$ one obtains, similarly to Proposition 4.8,

$$|D_{\mathrm{hyp}}^{j\alpha k\beta}(r_t-r)|+|D_{\mathrm{hyp}}^{j\alpha k\beta}(\theta_t-\theta)|\le C_{j\alpha k\beta},\tag{6-19}$$

for $0 \leq \pm t \leq t_{\text{eik}}$ and $(r, \theta, \rho, \eta) \in \Gamma_\iota^{\pm}(R_3, V_3, I_3, \sigma_3)$. This proves the expected estimates for $r_t - r$. The second equation of (6-18) evaluated at $(r_t, \theta_t, \rho, \eta)$ yields

$$e^r(\theta - \theta_t) = \int_0^t e^{r - r_t^s}(\partial_\eta q)(r_t^s, \theta_t^s, e^{-r_t^s}\eta_t^s)\, ds, \tag{6-20}$$

where $x_t^s = x^s(r_t, \theta_t, \rho, \eta)$ for $x = r, \theta, \eta$.

Combining (6-19) and Proposition 3.8, we have, on $\Gamma_\iota^{\pm}(R_3, V_3, I_3, \sigma_3)$,

$$\left|D_{\text{hyp}}^{j\alpha k\beta}(r_t^s - r)\right| + \left|D_{\text{hyp}}^{j\alpha k\beta}(\theta_t^s - \theta)\right| + \left|D_{\text{hyp}}^{j\alpha k\beta}(\eta_t^s - \eta)\right| \leq C_{j\alpha k\beta} \quad \text{for } 0 \leq \pm t, \pm s \leq t_{\text{eik}},$$

from which the estimate of the second term of (6-17) follows using (6-20). □

We now solve the transport equations. By (6-12), we have to consider the time-dependent vector field $(v_t^{\pm}, w_t^{\pm})$ defined on $\Gamma_\iota^{\pm}(R_3, V_3, I_3, \sigma_3)$, for $0 \leq \pm t \leq t_{\text{eik}}$, by

$$\begin{pmatrix} v_t^{\pm} \\ w_t^{\pm} \end{pmatrix} := \begin{pmatrix} (\partial_\rho p)(r, \theta, \partial_r \Sigma_0^{\pm}, \partial_\theta \Sigma_0^{\pm}) \\ (\partial_\eta p)(r, \theta, \partial_r \Sigma_0^{\pm}, \partial_\theta \Sigma_0^{\pm}) \end{pmatrix} = \begin{pmatrix} 2\partial_r \Sigma_0^{\pm} \\ e^{-2r}(\partial_\eta q)(r, \theta, \partial_\theta \Sigma_0^{\pm}) \end{pmatrix}. \tag{6-21}$$

We then denote by $\phi_{s\to t}^{\pm}$ the flow, from time s to time t, of $(v_t^{\pm}, w_t^{\pm}, 0_{\mathbb{R}^n})$ namely the solution to

$$\partial_t \phi_{s\to t}^{\pm} = (v_t^{\pm}(\phi_{s\to t}^{\pm}), w_t^{\pm}(\phi_{s\to t}^{\pm}), 0), \quad \phi_{s\to s}^{\pm}(r, \theta, \rho, \eta) = (r, \theta, \rho, \eta). \tag{6-22}$$

Lemma 6.4. *For any open interval I_{tr}, any$\sigma_{\text{tr}} \in \mathbb{R}$, and any open subset$V_{\text{tr}} \subset \mathbb{R}^{n-1}$ such that*

$$R_{\text{tr}} > R_3, \quad V_{\text{tr}} \Subset V_3, \quad I_{\text{tr}} \Subset I_3, \quad -1 < \sigma_{\text{tr}} < \sigma_3,$$

there exists $0 < t_2 \leq t_{\text{eik}}$ small enough that

$$\phi_{s\to t}^{\pm} \ \textit{is well defined on}\ \Gamma_\iota^{\pm}(R_{\text{tr}}, V_{\text{tr}}, I_{\text{tr}}, \sigma_{\text{tr}}) \quad \textit{for all}\ 0 \leq \pm s \leq t_2,\ 0 \leq \pm t \leq t_2 \tag{6-23}$$

and

$$\left|D_{j\alpha k\beta}^{\text{hyp}}(\phi_{s\to t}^{\pm} - \text{Id})\right| \lesssim 1 \quad \textit{on}\ \Gamma_\iota^{\pm}(R_{\text{tr}}, V_{\text{tr}}, I_{\text{tr}}, \sigma_{\text{tr}}) \quad \textit{for}\ 0 \leq \pm s, \pm t \leq t_2. \tag{6-24}$$

By (6-23), we mean in particular that

$$\phi_{s\to t}\left(\Gamma_\iota^{\pm}(R_{\text{tr}}, V_{\text{tr}}, I_{\text{tr}}, \sigma_{\text{tr}})\right) \subset \Gamma_\iota^{\pm}(R_3, V_3, I_3, \sigma_3) \quad \text{for } 0 \leq \pm s, \pm t \leq t_2. \tag{6-25}$$

The estimate (6-24) can be restated by saying that the components of $\phi_{s\to t}^{\pm} - \text{Id}$ are bounded families of $\mathscr{B}_{\text{hyp}}\left(\Gamma_\iota^{\pm}(R_{\text{tr}}, V_{\text{tr}}, I_{\text{tr}}, \sigma_{\text{tr}})\right)$ for $0 \leq \pm s, \pm t \leq t_2$.

Proof. For all $\delta > 0$ small enough, we have

$$|r - r'| + |\theta - \theta'| \leq \delta \ \text{and}\ (r, \theta, \rho, \eta) \in \Gamma_\iota^{\pm}(R_{\text{tr}}, V_{\text{tr}}, I_{\text{tr}}, \sigma_{\text{tr}}) \implies (r', \theta', \rho, \eta) \in \Gamma_\iota^{\pm}(R_3, V_3, I_3, \sigma_3) \tag{6-26}$$

by Proposition 4.2. Denoting by $(r_{s\to t}^{\pm}, \theta_{s\to t}^{\pm}, \rho, \eta)$ the components of $\phi_{s\to t}^{\pm}$, they must be solutions of the problem

$$(r_{s\to t}^{\pm}, \theta_{s\to t}^{\pm}) = (r, \theta) + \int_s^t (v_\tau^{\pm}, w_\tau^{\pm})(r_{s\to\tau}^{\pm}, \theta_{s\to\tau}^{\pm}, \rho, \eta)\, d\tau.$$

By (6-16), we have

$$|(v_\tau^{\pm}, w_\tau^{\pm})| + |\partial_{r,\theta}(v_\tau^{\pm}, w_\tau^{\pm})| \leq C, \tag{6-27}$$

on $\Gamma_\iota^{\pm}(R_3, V_3, I_3, \sigma_3)$, for $0 \le \pm\tau \le t_{\text{eik}}$. Therefore, the sequence $u_n^{\pm}(t) = u_n^{\pm}(t, s, r, \theta, \rho, \eta)$ defined by

$$u_0^{\pm}(s) = (r, \theta), \qquad u_{k+1}^{\pm}(t) = (r, \theta) + \int_s^t (v_\tau^{\pm}, w_\tau^{\pm})(u_k^{\pm}(\tau), \rho, \eta)d\tau,$$

is a Cauchy sequence in $C^0([0, \pm t_2], \mathbb{R}^n)$ for all $(r, \theta, \rho, \eta) \in \Gamma_\iota^{\pm}(R_{\text{tr}}, V_{\text{tr}}, I_{\text{tr}}, \sigma_{\text{tr}})$ and $0 \le \pm s \le t_2$, for some t_2 small enough independent of (r, θ, ρ, η). Indeed, using (6-26) and choosing t_2 small enough so that $\sum_{k\ge 0}(Ct_2)^{k+1} \le \delta$, a standard induction using (6-27) shows that

$$|u_{k+1}^{\pm}(t) - u_k^{\pm}(t)| \le (Ct_2)^{k+1},$$

which makes the sequence well defined and convergent. This proves (6-23). We then obtain (6-24) by induction by differentiating the equations in (6-22). This proof is completely similar to that of the estimate (4-70) in Proposition 4.14 (and much simpler since it is local in time) so we omit the details. □

Now denote by $q_t^{\pm} = q_t^{\pm}(r, \theta, \rho, \eta)$ the function defined on $[0, \pm t_{\text{eik}}] \times \Gamma_\iota^{\pm}(R, V, I, \sigma)$ by

$$q_t^{\pm} := (p + p_1)(r, \theta, \partial_r, \partial_\theta)\Sigma_0^{\pm}.$$

This function was involved in (6-12).

Proposition 6.5. *Choose R_{tr}, V_{tr}, I_{tr} and σ_{tr} such that*

$$R_2 > R_{\text{tr}} > R_3, \quad V_2 \Subset V_{\text{tr}} \Subset V_3, \quad I_2 \Subset I_{\text{tr}} \Subset I_3, \quad \sigma_2 < \sigma_{\text{tr}} < \sigma_3.$$

There exists $t_{\text{tr}} > 0$ small enough that, for all $\chi^{\pm}$ satisfying (6-1), the functions

$$a_0^{\pm}, \dots, a_N^{\pm} : [0, \pm t_{\text{tr}}] \times \mathbb{R}^{2n} \to \mathbb{C}$$

vanishing outside $\Gamma_\iota^{\pm}(R_2, V_2, I_2, \sigma_2)$ and defined iteratively on $\Gamma_\iota^{\pm}(R_2, V_2, I_2, \sigma_2)$ by

$$a_0^{\pm}(t) := \chi^{\pm} \circ \phi_{t\to 0}^{\pm} \exp\left(\int_0^t q_s^{\pm} \circ \phi_{t\to s}^{\pm}\right),$$

$$a_k^{\pm}(t) := -\int_0^t i(P_\iota a_{k-1}^{\pm})(s_1, \phi_{t\to s_1}^{\pm}) \exp\left(\int_{s_1}^t q_{s_2}^{\pm} \circ \phi_{t\to s_2}^{\pm} ds_2\right) ds_1 \quad \text{for } 1 \le k \le N$$

are smooth and solve (6-13) and (6-14). Furthermore, for all $0 \le k \le N$,

$$(a_k^{\pm}(t))_{0\le \pm t\le t_{\text{tr}}} \text{ is bounded in } \mathcal{S}_{\text{hyp}}\big(\Gamma_\iota^{\pm}(R_2, V_2, I_2, \sigma_2)\big). \tag{6-28}$$

Proof. Fix R'_{tr}, V'_{tr}, I'_{tr} and σ'_{tr} such that

$$R_2 > R'_{\text{tr}} > R_{\text{tr}}, \quad V_2 \Subset V'_{\text{tr}} \Subset V_{\text{tr}}, \quad I_2 \Subset I'_{\text{tr}} \Subset I_{\text{tr}}, \quad \sigma_2 < \sigma'_{\text{tr}} < \sigma_{\text{tr}}.$$

By choosing $0 < t_{\text{tr}} \le t_2$ small enough, we then have, for all $0 \le \pm s, \pm t \le t_{\text{tr}}$,

$$\phi_{s\to t}^{\pm}\left(\Gamma_\iota^{\pm}(R_1, V_1, I_1, \sigma_1)\right) \subset \Gamma_\iota^{\pm}(R_2, V_2, I_2, \sigma_2), \tag{6-29}$$

$$\phi_{s\to t}^{\pm}\left(\Gamma_\iota^{\pm}(R_2, V_2, I_2, \sigma_2)\right) \subset \Gamma_\iota^{\pm}(R'_{\text{tr}}, V'_{\text{tr}}, I'_{\text{tr}}, \sigma'_{\text{tr}}), \tag{6-30}$$

$$\phi_{s\to t}^{\pm}\left(\Gamma_\iota^{\pm}(R'_{\text{tr}}, V'_{\text{tr}}, I'_{\text{tr}}, \sigma'_{\text{tr}})\right) \subset \Gamma_\iota^{\pm}(R_{\text{tr}}, V_{\text{tr}}, I_{\text{tr}}, \sigma_{\text{tr}}). \tag{6-31}$$

This follows from Proposition 4.2 and the fact that $|\phi^{\pm}_{t\to s} - \mathrm{Id}| \lesssim |t-s|$, which comes from the integration of (6-22) between s and t, using (6-24). By Lemma 6.4, the flow is well defined on $\Gamma^{\pm}_{\iota}(R_{\rm tr}, V_{\rm tr}, I_{\rm tr}, \sigma_{\rm tr})$, therefore the condition (6-31) ensures that we have the pseudo-group property

$$\phi^{\pm}_{t\to u} \circ \phi^{\pm}_{s\to t} = \phi^{\pm}_{s\to u}, \qquad 0 \le \pm s, \pm t, \pm u \le t_{\rm tr}, \tag{6-32}$$

on $\Gamma^{\pm}_{\iota}(R'_{\rm tr}, V'_{\rm tr}, I'_{\rm tr}, \sigma'_{\rm tr})$. In particular, $\phi^{\pm}_{t\to s} \circ \phi^{\pm}_{s\to t} = \mathrm{Id}$ on this set. Therefore, by (6-30), we have

$$\Gamma^{\pm}_{\iota}(R_2, V_2, I_2, \sigma_2) \subset \phi^{\pm}_{t\to s}\left(\Gamma^{\pm}_{\iota}(R'_{\rm tr}, V'_{\rm tr}, I'_{\rm tr}, \sigma'_{\rm tr})\right).$$

This implies that the map

$$(t, r, \theta, \rho, \eta) \mapsto (t, \phi^{\pm}_{s\to t}(r, \theta, \rho, \eta))$$

is a diffeomorphism from $(0, \pm t_{\rm tr}) \times \Gamma^{\pm}_{\iota}(R'_{\rm tr}, V'_{\rm tr}, I'_{\rm tr}, \sigma'_{\rm tr})$ onto its range and that this range contains $(0, \pm t_{\rm tr}) \times \Gamma^{\pm}_{\iota}(R_2, V_2, I_2, \sigma_2)$. Restricted to the latter set, the inverse is given by $(t, \phi^{\pm}_{t\to s})$ which shows that $\phi^{\pm}_{t\to s}$ is smooth with respect to t. Furthermore, by differentiating in t the relation $\phi^{\pm}_{t\to s} \circ \phi^{\pm}_{s\to t} = \mathrm{Id}$, one obtains

$$\partial_t \phi^{\pm}_{t\to s} + (\partial_{r,\theta}\phi^{\pm}_{t\to s}) \cdot (v^{\pm}_t, w^{\pm}_t) = 0, \qquad \text{on } \Gamma^{\pm}_{\iota}(R_2, V_2, I_2, \sigma_2),$$

for $0 < \pm t < t_{\rm tr}$. Using this relation, one easily checks that $a^{\pm}_0$ solves (6-13) on $\Gamma^{\pm}_{\iota}(R_2, V_2, I_2, \sigma_2)$. In addition, if

$$(r, \theta, \rho, \eta) \in \Gamma^{\pm}_{\iota}(R'_{\rm tr}, V'_{\rm tr}, I'_{\rm tr}, \sigma'_{\rm tr}) \setminus \Gamma^{\pm}_{\iota}(R_2, V_2, I_2, \sigma_2),$$

we have $\phi^{\pm}_{t\to 0}(r, \theta, \rho, \eta) \notin \mathrm{supp}(\chi^{\pm})$ otherwise $(r, \theta, \rho, \eta) \in \Gamma^{\pm}_{\iota}(R_2, V_2, I_2, \sigma_2)$ by (6-1), (6-29) and (6-32). This shows that, extended by 0 outside $\Gamma^{\pm}_{\iota}(R_2, V_2, I_2, \sigma_2)$, $a^{\pm}_0$ is smooth. The property (6-28) for $k = 0$ is then a direct consequence of (6-24). We note in passing that we have

$$\mathrm{supp}(a^{\pm}_0(t)) \subset \phi^{\pm}_{0\to t}(\mathrm{supp}(\chi^{\pm})).$$

The proof for the higher-order terms $a^{\pm}_k$, $k \ge 1$, is then obtained similarly by induction using that $\mathrm{supp}(P_\iota a^{\pm}_{k-1}(s_1)) \subset \phi^{\pm}_{0\to s_1}(\mathrm{supp}(\chi^{\pm}))$ for all s_1. $\square$

Proof of Theorem 6.1. There remains to prove (6-4), to globalize $\Sigma^{\pm}_0$, to prove (6-5) and the bound (6-9). By Proposition 4.1, we can choose

$$\chi^{\pm}_{2\to 3} \in \mathscr{S}_{\rm hyp}(\Gamma^{\pm}_{\iota}(R_3, V_3, I_3, \sigma_3)) \ \text{ such that } \ \chi^{\pm}_{2\to 3} \equiv 1 \ \text{ on } \ \Gamma^{\pm}_{\iota}(R_2, V_2, I_2, \sigma_2).$$

We set

$$\Sigma^{\pm}(t, r, \theta, \rho, \eta) = r\rho + \theta\cdot\eta + \chi^{\pm}_{2\to 3}(r, \theta, \rho, \eta) \times \left(\Sigma^{\pm}_0(t, r, \theta, \rho, \eta) - r\rho - \theta\cdot\eta\right).$$

It coincides with $\Sigma^{\pm}_0$ on $[0, \pm t_{\rm eik}] \times \Gamma^{\pm}_{\iota}(R_2, V_2, I_2, \sigma_2)$ so it is a solution to the eikonal equation on $[0, \pm t_{\rm WKB}] \times \Gamma^{\pm}_{\iota}(R_2, V_2, I_2, \sigma_2)$, for any $0 < t_{\rm WKB} \le t_{\rm eik}$. Furthermore, (6-16) implies (6-5) and, by using

$$\Sigma^{\pm}_0(t, r, \theta, \rho, \eta) = r\rho + \theta\cdot\eta + \int_0^t p(r, \theta, \partial_r \Sigma^{\pm}_0(s), \partial_\theta \Sigma^{\pm}_0(s))\, ds, \tag{6-33}$$

we get (6-4) since (6-16) and (6-33) itself show that the components of $(\partial_r \Sigma^{\pm}(s) - \rho, \partial_\theta \Sigma^{\pm}(s) - \eta)$ are $\mathcal{O}(s)$ in $\mathscr{B}_{\rm hyp}(\Gamma^{\pm}_{\iota}(R_3, V_3, I_3, \sigma_3))$.

To prove (6-9), we use the Kuranishi trick which is as follows. By Taylor's formula, we can write

$\Sigma^{\pm}(t, r, \theta, \rho, \eta) - \Sigma^{\pm}(t, r', \theta', \rho, \eta) = (r - r')\tilde{\rho}^{\pm}(t, r, \theta, r', \theta', \rho, \eta) + (\theta - \theta') \cdot \tilde{\eta}^{\pm}(t, r, \theta, r', \theta', \rho, \eta).$

Using again (6-33) and (6-16), we obtain

$$\left|\partial_r^j \partial_\theta^\alpha \partial_{r'}^{j'} \partial_{\theta'}^{\alpha'} \partial_\rho^k \partial_\eta^\beta \left((\tilde{\rho}^{\pm}, \tilde{\eta}^{\pm})(t, r, \theta, r', \theta', \rho, \eta) - (\rho, \eta)\right)\right| \leq C_{j\alpha j'\alpha' k\beta}|t|, \tag{6-34}$$

for $(r, \theta, r', \theta', \rho, \eta \in \mathbb{R}^{3n})$ and $0 \leq \pm t \leq t_{\text{eik}}$. The latter implies that, for all $0 \leq \pm t \leq t_{\text{WKB}}$ small enough and all $(r, \theta, r', \theta') \in \mathbb{R}^{2n}$, the map

$$(\rho, \eta) \mapsto (\tilde{\rho}^{\pm}, \tilde{\eta}^{\pm}),$$

is a diffeomorphism from $\mathbb{R}^n$ onto itself. Furthermore, proceeding similarly to the proof of (4-52) in Lemma 4.11, we see that its inverse $(\tilde{\rho}, \tilde{\eta}) \mapsto (\rho^{\pm}, \eta^{\pm})$ satisfies

$$\left|\partial_r^j \partial_\theta^\alpha \partial_{r'}^{j'} \partial_{\theta'}^{\alpha'} \partial_{\tilde{\rho}}^k \partial_{\tilde{\eta}}^\beta \left((\rho^{\pm}, \eta^{\pm})(t, r, \theta, r', \theta', \tilde{\rho}, \tilde{\eta}) - (\tilde{\rho}, \tilde{\eta})\right)\right| \leq C_{j\alpha j'\alpha' k\beta}, \tag{6-35}$$

on $\mathbb{R}^{3n}$, uniformly with respect to $0 \leq \pm t \leq t_{\text{WKB}}$. Thus, for any bounded family $(a^{\pm}(t))_{0 \leq \pm t \leq t_{\text{WKB}}}$ in $\mathcal{S}_{\text{hyp}}(\Gamma_\iota^{\pm}(R_2, V_2, I_2, \sigma_2))$, the kernel of $\mathcal{J}_h^{\pm}(t, a^{\pm}(t))\mathcal{J}_h^{\pm}(t, a^{\pm}(t))^*$, which reads

$$(2\pi h)^{-n} \int e^{\frac{i}{h}(\Sigma^{\pm}(t,r,\theta,\rho,\eta) - \Sigma^{\pm}(t,r',\theta',\rho,\eta))}\, a^{\pm}(t, r, \theta, \rho, \eta)\, \overline{a^{\pm}(t, r', \theta', \rho, \eta)}\, d\rho\, d\eta, \tag{6-36}$$

can be written as

$$(2\pi h)^{-n} \int e^{\frac{i}{h}((r-r')\tilde{\rho} + (\theta-\theta')\cdot\tilde{\eta})} B(t, r, \theta, r', \theta', \tilde{\rho}, \tilde{\eta})\, d\tilde{\rho}\, d\tilde{\eta}, \tag{6-37}$$

with $B(t, \cdot)$ bounded in $C_b^\infty(\mathbb{R}^{3n})$ as $0 \leq \pm t \leq t_{\text{WKB}}$. By the Calderón–Vaillancourt theorem the operator given by (6-37) is uniformly bounded; hence $\|\mathcal{J}_h^{\pm}(t, a^{\pm}(t))\|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} \leq C$ whenever $0 \leq \pm t \leq t_{\text{WKB}}$ and $h \in (0, 1]$, where C depends only a finite number of seminorms of $a^{\pm}(t)$ in $C_b^\infty(\mathbb{R}^{2n})$. Using (6-10), (6-11) (with $a_k^{\pm}(t)$ solutions to the transport equations) and (6-28), the bound above yields (6-9), which completes the proof of Theorem 6.1. ☐

6B. ***Proof of Proposition 2.24.*** To prove Proposition 2.24, we first need a version of the semiclassical Egorov Theorem in the asymptotically hyperbolic setting. We recall that $\Phi^t = (r^t, \theta^t, \rho^t, \eta^t)$ denotes the Hamiltonian flow of the principal symbol p of P.

Fix an open subset $V \Subset V_\iota'$, an open interval $I \Subset (0, +\infty)$, and $-1 < \sigma < 1$.

Theorem 6.6. *If $R > 0$ is large enough the following holds: for all $T > 0$, all $N \geq 0$ and all*

$$a \in \mathcal{S}_{\text{hyp}}\big(\Gamma_\iota^{\pm}(R, V, I, \sigma)\big), \tag{6-38}$$

we can find

$$a_0(t), \dots, a_N(t) \in \mathcal{S}_{\text{hyp}}\big(\Phi^t(\text{supp}(a))\big) \quad \text{for } 0 \leq \pm t \leq T, \tag{6-39}$$

such that, for all $0 \leq \pm t \leq T$ and all $0 < h \leq 1$,

$$\left\| e^{-ithP}\widehat{Op}_\iota(a)e^{ithP} - \sum_{k=0}^{N} h^k \widehat{Op}_\iota(a_k(t)) \right\|_{L^2(\mathcal{M}, d\widehat{G}) \to L^2(\mathcal{M}, d\widehat{G})} \leq C_{N,T,a} h^{N+1}. \tag{6-40}$$

This theorem is basically well known. Here the main point is to check (6-39), namely that $a_0(t), \dots, a_N(t)$ lie in $\mathcal{B}_{\mathrm{hyp}}\big(\Phi^t(\mathrm{supp}(a))\big)$. Notice that, by Corollary 3.10, $\Phi^t(\mathrm{supp}(a))$ is contained in the same chart as a in which it is therefore sufficient to work.

Using the group property, it is sufficient to prove the result when T is small enough (depending only on V, I, σ). To check this point, we choose open sets V_1, V_2 such that $V \Subset V_1 \Subset V_2 \Subset V_\iota'$. Then, for some $C > 0$ and all R large enough,

$$\Phi^t\big(\Gamma_\iota^\pm(R, V, I, \sigma)\big) \subset \Gamma_\iota^\pm(R - C, V_1, I, \sigma), \qquad \pm t \geq 0,$$
$$\Phi^t\big(\Gamma_\iota^\pm(R - C, V_1, I, \sigma)\big) \subset \Gamma_\iota^\pm(R - 2C, V_2, I, \sigma), \qquad \pm t \geq 0.$$

This follows from Corollary 3.10 and the fact that ρ^t can be assumed to be nondecreasing, using (3-22). Thus, it is sufficient to prove (6-40) for $0 \leq \pm t \leq \varepsilon$ with $\varepsilon > 0$ small enough independent of $a \in \mathcal{S}_{\mathrm{hyp}}\big(\Gamma_\iota^\pm(R - C, V_1, I, \sigma)\big)$. Indeed, if this holds, it holds for a satisfying (6-38) and

$$e^{i\varepsilon hP}\widehat{Op}_\iota(a)e^{-i\varepsilon hP} - \sum_{k=0}^N h^k \widehat{Op}_\iota(a_k(\varepsilon)) + h^{N+1}R_N(h, \varepsilon)$$

with $R_N(h, \varepsilon)$ uniformly bounded on $L^2(\mathcal{M}, \widehat{dG})$ and $a_k(\varepsilon) \in \mathcal{S}_{\mathrm{hyp}}\big(\Gamma_\iota^\pm(R - C, V_1, I, \sigma)\big)$, with $a_k(\varepsilon)$ supported in Φ^ε (supp(a)) more precisely. Conjugating the expression above by $e^{-i\varepsilon hP}$ and then applying the same result with $a_k(\varepsilon)$ instead of a we can write

$$e^{i2\varepsilon hP}\widehat{Op}_\iota(a)e^{-2i\varepsilon hP} - \sum_{k=0}^N h^k \widehat{Op}_\iota(a_k(2\varepsilon)) + h^{N+1}R_N(h, 2\varepsilon),$$

where $a_k(2\varepsilon)$ is supported in $\Phi^{2\varepsilon}(\mathrm{supp}(a))$, which is still contained in $\Gamma_\iota^\pm(R - C, V_1, I, \sigma)$, and thus allows one to iterate the procedure.

The interest of considering small times is justified by the following lemma.

Lemma 6.7. *Fix V_1, I, σ as above. For some $R_1 > 0$ large enough and $\varepsilon > 0$ small enough,*

$$\big|D_{\mathrm{hyp}}^{j\alpha k\beta}\big((\Phi^t)^{-1} - \mathrm{Id}_{2n}\big)\big| \leq C_{j\alpha k\beta} \quad \text{on } \Phi^t\big(\Gamma_\iota^\pm(R_1, V_1, I, \sigma)\big),$$

for all $0 \leq \pm t \leq \varepsilon$.

Proof. Using the identity

$$d(\Phi^t - \mathrm{Id}_{2n}) = \int_0^t dH_p(\Phi^s)d\Phi^s\, ds$$

and Proposition 3.8, we have $|d(\Phi^t - \mathrm{Id}_{2n})| \lesssim |t|$ hence $|(d\Phi^t)^{-1}| \lesssim 1$ on $\Gamma_\iota^\pm(R_1, V_1, I, \sigma)$ if R_1 is large enough and t is small enough. We then obtain the result by applying $D_{\mathrm{hyp}}^{j\alpha k\beta}$ to $\Phi^t \circ (\Phi^t)^{-1}$ and using the Faà di Bruno formula. For instance, if $j = k = |\alpha| = 0$ and $|\beta| = 1$, we have

$$d\Phi^t_{|(\Phi^t)^{-1}} e^r \partial_\eta^\beta((\Phi^t)^{-1} - \mathrm{Id}_{2n}) = (\mathrm{Id}_{2n} - d\Phi^t_{|(\Phi^t)^{-1}})e^r\partial_\eta^\beta \mathrm{Id}_{2n}$$

where, using Proposition 3.8, the right-hand side is bounded for this is simply $e^r\partial_\eta^\beta(\mathrm{Id}_{2n} - \Phi^t)$ evaluated at $(\Phi^t)^{-1}$. Higher-order derivatives are studied similarly by iteration, using Lemma 3.6. ☐

Naturally, $(\Phi^t)^{-1}$ is the reverse Hamiltonian flow, namely flowing $\Phi^t\big(\Gamma_\iota^\pm(R_1, V_1, I, \sigma)\big)$ back to $\Gamma_\iota^\pm(R_1, V_1, I, \sigma)$. More precisely, for $0 \le \pm t \le \varepsilon$,

$$\frac{d}{dt}(\Phi^t)^{-1}(r,\theta,\rho,\eta) = -H_p\big((\Phi^t)^{-1}(r,\theta,\rho,\eta)\big) \quad \text{for } (r,\theta,\rho,\eta) \in \Phi^{\pm\varepsilon}\big(\Gamma_\iota^\pm(R_1, V_1, I, \sigma)\big). \tag{6-41}$$

We prefer to keep the notation $(\Phi^t)^{-1}$ on $\Phi^t\big(\Gamma_\iota^\pm(R_1, V_1, I, \sigma)\big)$ rather than using Φ^{-t}, since we have only studied Φ^t for $t \ge 0$ on outgoing areas and $t \le 0$ on incoming areas.

We have essentially all the tools needed to solve the transport equations considered in the next lemma.

Lemma 6.8. *There exists $C > 0$ such that, for all R large enough, the following holds: for any $a_{\text{ini}} \in \mathcal{S}_{\text{hyp}}\big(\Gamma_\iota^\pm(R, V, I, \sigma)\big)$ and any bounded family $(f(t))_{0\le\pm t\le\varepsilon}$ of $\mathcal{S}_{\text{hyp}}\big(\Gamma_\iota^\pm(R-C, V_1, I, \sigma)\big)$, smooth with respect to t and such that*

$$\operatorname{supp}(f(t)) \subset \Phi^t(\operatorname{supp}(a_{\text{ini}})),$$

the function defined for $0 \le \pm t \le \varepsilon$ by

$$a(t) := \begin{cases} a_{\text{ini}} \circ (\Phi^t)^{-1} + \int_0^t f(s) \circ \Phi^s \circ (\Phi^t)^{-1}\, ds & \textit{on } \Phi^t(\operatorname{supp}(a)), \\ 0 & \textit{outside}, \end{cases}$$

is smooth and satisfies

$$\partial_t a(t) + \{p, a(t)\} = f(t), \quad a(0) = a_{\text{ini}}. \tag{6-42}$$

Furthermore

$$(a(t))_{0\le\pm t\le\varepsilon} \text{ is bounded in } \mathcal{S}_{\text{hyp}}\left(\Gamma_\iota^\pm(R-C, V_1, I, \sigma)\right). \tag{6-43}$$

In (6-43), we consider $\Gamma_\iota^\pm(R-C, V_1, I, \sigma)$ for it is independent of t but, by construction, $a(t)$ is supported in the smaller region $\Phi^t(\operatorname{supp}(a))$.

Proof. To check the smoothness of $a_0(t)$ it suffices to see that $a_{\text{ini}} \circ (\Phi^t)^{-1}$ and $f(s) \circ (\Phi^{t-s})^{-1}$ are defined and smooth in a neighborhood of $\Phi^t(\operatorname{supp}(a))$, while they vanish on the complement of $\Phi^t(\operatorname{supp}(a))$ (relatively to the neighborhood). Indeed $(\Phi^t)^{-1}$ is defined on $\Phi^t(\Gamma_\iota^\pm(R-C, V_1, I, \sigma))$ and if (r,θ,ρ,η) belongs to $\Phi^t(\Gamma_\iota^\pm(R-C, V_1, I, \sigma))$ but doesn't belong to $\Phi^t(\operatorname{supp}(a))$, then $a_{\text{circ}} \circ (\Phi^t)^{-1}(r,\theta,\rho,\eta) = 0$; otherwise, $(\Phi^t)^{-1}(r,\theta,\rho,\eta)$ should belong to $\operatorname{supp}(a)$ and thus (r,θ,ρ,η) should belong to $\Phi^t(\operatorname{supp}(a))$. Similarly,

$$\int_0^t f(s) \circ \Phi^s \circ (\Phi^t)^{-1}(r,\theta,\rho,\eta)\, ds$$

must vanish, otherwise there would be s between 0 and t such that $\Phi^s \circ (\Phi^t)^{-1}(r,\theta,\rho,\eta) \in \Phi^s(\operatorname{supp}(a))$ implying that $(r,\theta,\rho,\eta) \in \Phi^t(\operatorname{supp}(a))$. Then (6-42) follows directly from (6-41) and (6-43) follows from Lemma 6.7. □

Proof of Theorem 6.6. By Lemma 6.8, the solutions of the transport equations (6-42) belong to the set $\mathcal{S}_{\text{hyp}}(\Gamma_\iota^\pm(R-C, V_1, I, \sigma))$. The proof is then standard; see [Robert 1987], for instance. □

Proof of Proposition 2.24. We start by choosing $\varepsilon > 0$ and $\delta > 0$ according to Proposition 2.16 with $\underline{t} = t_{\text{WKB}}$. Then, using (2-26), (2-37), (2-38) and Theorem 6.6, it is straightforward to show that, for all $T \ge t_{\text{WKB}}$ and all $N \ge 0$,

$$\big\| \widehat{Op}_\iota(b^\pm_{l,\text{inter}}) e^{-ithP} \widehat{Op}_\iota(b^\pm_{l,\text{inter}})^* \big\|_{L^2(d\widehat{G}) arrow L^2(d\widehat{G})} \le C_{T,l,N} h^N \quad \text{for } h \in (0,1],\ t_{\text{WKB}} \le \pm t \le T.$$

It is therefore sufficient to show the existence of T large enough such that

$$\big\| \widehat{Op}_\iota(b^{\pm}_{l,\mathrm{inter}}) e^{-ithP} \widehat{Op}_\iota(b^{\pm}_{l,\mathrm{inter}})^* \big\|_{L^2(\widehat{dG}) arrow L^2(\widehat{dG})} \leq C_{l,N} h^N \quad \text{for } h \in (0,1],\ T \leq \pm t \leq 2h^{-1}. \tag{6-44}$$

For simplicity we consider positive times and set $B = \widehat{Op}_\iota(b^{+}_{l,\mathrm{inter}})$. For T to be chosen, we write

$$e^{-ithP} B^* = e^{-i(t-T)hP} B(T)^* e^{-iThP}, \qquad B(T) = e^{-iThP} B e^{iThP}.$$

As above, we may write

$$B(T)^* = \sum_{k \leq N} h^k \widehat{Op}_\iota(b^*_k(T)) + h^{N+1} B_N(h),$$

with $B_N(h)$ uniformly bounded on $L^2(\mathcal{M}, \widehat{dG})$ and

$$b^*_k(T) \in \mathcal{S}_{\mathrm{hyp}}\big(\Phi^T(\mathrm{supp}(b^{+}_{l,\mathrm{inter}}))\big) \subset \mathcal{S}_{\mathrm{hyp}}\big(\Phi^T(\Gamma^{+}_{\iota,\mathrm{inter}}(\varepsilon,\delta;l))\big).$$

By (2-57), for all $\tilde{\varepsilon} > 0$, we can choose $T_{\tilde{\varepsilon}}$ large enough that $\Phi^T\big(\Gamma^{+}_{\iota,\mathrm{inter}}(\varepsilon,\delta;l)\big) \subset \Gamma^{+}_{\iota,s}(\tilde{\varepsilon}^9)$. Thus, if $\tilde{\varepsilon}$ is small enough, Theorem 5.1 allows one to write, for $t \geq T_{\tilde{\varepsilon}}$,

$$e^{-i(t-T_{\tilde{\varepsilon}})hP} \widehat{Op}_\iota(b^*_k(T_{\tilde{\varepsilon}})) = \Psi^*_\iota\big(J^+_h(\tilde{a}_{\tilde{\varepsilon}}(h)) e^{-i(t-T_{\tilde{\varepsilon}})hD_r^2} J^+_h(\tilde{b}_{\tilde{\varepsilon}}(h))^*\big)(\Psi^{-1}_\iota)^* + h^N R_N(t,h),$$

with $R_N(t,h)$ uniformly bounded on $L^2(\mathcal{M}, \widehat{dG})$ for $h \in (0,1]$ and $0 \leq t - T_{\tilde{\varepsilon}} \leq 2h^{-1}$, and

$$\tilde{a}_{\tilde{\varepsilon}}(h) \in \mathcal{S}_{\mathrm{hyp}}\big(\Gamma^{+}_{\iota,s}(\tilde{\varepsilon})\big).$$

We will therefore get (6-44) with $T = T_{\tilde{\varepsilon}}$ if we choose $\tilde{\varepsilon}$ small enough such that, for all N,

$$\big\| b^{+}_{l,\mathrm{inter}}(r,\theta,hD_r,hD_\theta) J^+_h(\tilde{a}_{\tilde{\varepsilon}}(h)) \big\|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} \leq C_N h^N.$$

By the standard composition rule between pseudodifferential and Fourier integral operators (see [Robert 1987]), $b^{+}_{l,\mathrm{inter}}(r,\theta,hD_r,hD_\theta)\, J^+_h(\tilde{a}_{\tilde{\varepsilon}}(h))$ is the sum of an operator with norm of order h^N and of Fourier integral operators with amplitudes vanishing outside the support of

$$b^{+}_{l,\mathrm{inter}}(r,\theta,\partial_r S_+,\partial_\theta S_+)\, \tilde{a}_{\tilde{\varepsilon}}(r,\theta,\rho,\eta,h),$$

where $S_+ = S_+(r,\theta,\rho,\eta)$ is the phase defined in Proposition 4.9. It is therefore sufficient to show that, for $\tilde{\varepsilon}$ small enough, the support of the amplitude above is empty. Indeed, on this support we have

$$\frac{\partial_r S_+}{p(r,\theta,\partial_r S_+,\partial_\theta S_+)^{1/2}} \leq 1 - (\varepsilon/2)^2, \qquad \frac{\rho}{p(r,\theta,\rho,\eta)^{1/2}} > 1 - \tilde{\varepsilon}^2. \tag{6-45}$$

Furthermore, by Proposition 4.9, we also have

$$|\partial_r S_+ - r| + |\partial_\theta S_+ - \eta| \lesssim \tilde{\varepsilon}^2$$

on $\Gamma^{+}_{\iota,s}(\tilde{\varepsilon})$, where $\tilde{a}_{\tilde{\varepsilon}}(h)$ is supported. Since p is bounded from above and from below on $\Gamma^{+}_{\iota,s}(\tilde{\varepsilon})$, we obtain, for all $\tilde{\varepsilon}$ small enough,

$$\frac{\rho}{p(r,\theta,\rho,\eta)^{1/2}} \leq 1 - (\varepsilon/2)^2 + C\tilde{\varepsilon}^2 \leq 1 - (\varepsilon/4)^2,$$

which is clearly incompatible with the second condition of (6-45). □

7. Dispersion estimates

In this section, we prove Propositions 2.20 and 2.22, using respectively the parametrices given in Theorems 5.1 and 6.1. The dispersion estimates will basically follow from the stationary phase theorem, applied to the kernels of these parametrices which are oscillatory integrals. The principle is thus quite simple. One needs however to check some technical points essentially due to the noncompactness of the manifold and, more precisely, to the nonuniform ellipticity of the symbol of the Laplacian.

Here is some heuristic in the case of the Isozaki–Kiada parametrix. We have to consider oscillatory integrals with phases whose model is

$$(r-r')\rho+(\theta-\theta')\cdot\eta-t\rho^2+(e^{-2r}-e^{-2r'})\frac{|\eta|^2}{4\rho},$$

where r, r', θ, θ' are parameters and ρ, η the integration variables. Due to the localization of the amplitudes, we may also assume that (r, θ, ρ, η) and $(r', \theta', \rho, \eta)$ belong to strongly outgoing/incoming area. The critical point satisfies (assuming that it is unique)

$$r-r'-2t\rho-(e^{-2r}-e^{-2r'})\frac{|\eta|^2}{4\rho^2}=0, \tag{7-1}$$

$$\theta-\theta'+(e^{-2r}-e^{-2r'})\frac{\eta}{2\rho}=0, \tag{7-2}$$

where one should also keep in mind that $e^{-r}\eta$ and $e^{-r}\eta'$ are small since the amplitudes are supported in strongly outgoing/incoming areas. In particular, ρ is close to $\pm p^{1/2}$ and thus is far from 0. By (7-1), one obtains at the critical point that, as expected,

$$r\approx r'+2t\rho=r'+2|t\rho|, \tag{7-3}$$

where $t\rho=|t\rho|$ by the sense of time considered in outgoing/incoming areas. This in turn shows that

$$\theta-\theta'\approx e^{-2r'}\left(1-e^{-2t\rho}\right)\frac{\eta}{\rho}.$$

In Proposition 7.2, we check that this intuition is correct, and we improve the localization around critical points in Proposition 7.6. To use the stationary phase theorem, one needs to check the nondegeneracy of the phase. Using the change of variable $\xi=e^{-r}\eta'$, the phase is changed into

$$(r-r')\rho+e^{r'}(\theta-\theta')\cdot\xi-t\rho^2+(e^{-2(r-r')}-1)\frac{|\xi|^2}{4\rho}$$

and its hessian becomes

$$t\left\{\begin{pmatrix}-2 & \\ 0 & \dfrac{e^{-2(r-r')}-1}{2t\rho}\end{pmatrix}-\frac{e^{-2(r-r')}-1}{2t\rho}\begin{pmatrix}0 & \xi/\rho\\ \xi/\rho & 0\end{pmatrix}\right\}. \tag{7-4}$$

Since ξ is small, the second matrix is small compared to the first one. When t is not too large, the entry $(e^{-2(r-r')}-1)/(2t\rho)$ is bounded from above and below (recall (7-3)) and the phase is thus nondegenerate.

This is made more rigorous in Proposition 7.11. When t becomes large the hessian matrix is basically equivalent to

$$\begin{pmatrix} -2t & \\ 0 & -1/(2\rho) \end{pmatrix}$$

which is again nondegenerate but will contribute apparently only through a factor $|t|^{-1/2}$ in the stationary phase theorem. However, recalling the change of variable $e^{-r'}\eta = \xi$ whose Jacobian is $e^{r'(n-1)}$, and using the two factors $e^{-(n-1)r'/2}$, $e^{-(n-1)r/2}$ on both sides of the kernel (written with respect to dG rather than $\widehat{dG}$), we get a factor of the form $e^{(n-1)(r'-r)/2}$ which decays exponentially in t by (7-3) and provides (much more than) the missing $|t|^{-(n-1)/2}$ decay. This is made more rigorous in Proposition 7.12.

The aim of the following subsection is to justify this intuition. In particular, to justify the above approximations (*e.g.* the precise meaning of (7-3) or the smallness of the second matrix in (7-4)) we need to be in an asymptotic regime given by a certain (small) parameter: in the Isozaki–Kiada case, the relevant parameter is ε (the size of the strongly outgoing/incoming areas) and, in the WKB case, it is the range of time.

7A. ***Stationary and nonstationary phase estimates.*** For simplicity, we drop the index ι from the notation, including in outgoing/incoming areas. In both Isozaki–Kiada and WKB parametrices, we have to consider oscillatory integrals of the form

$$(2\pi h)^{-n} \iint e^{\frac{i}{h}\Phi^{\pm}(t,r,\theta,r',\theta',\rho,\eta)} A^{\pm}(t,r,\theta,r',\theta',\rho,\eta)\, d\rho\, d\eta. \tag{7-5}$$

For the Isozaki–Kiada parametrix, the amplitude is independent of t and of the form

$$A^{\pm}_{\mathrm{IK}}(t,r,\theta,r',\theta',\rho,\eta) = a^{\pm}(r,\theta,\rho,\eta)\,\overline{b^{\pm}(r',\theta',\rho,\eta)},$$

with

$$a^{\pm} \in \mathcal{S}_{\mathrm{hyp}}\big(\Gamma^{\pm}_{\mathrm{s}}(\varepsilon)\big) \quad \text{and} \quad b^{\pm} \in \mathcal{S}_{\mathrm{hyp}}\big(\Gamma^{\pm}_{\mathrm{s}}(\varepsilon^3)\big), \tag{7-6}$$

with $\varepsilon > 0$ small to be fixed. The phase reads

$$\Phi^{\pm}_{\mathrm{IK}}(t,r,\theta,r',\theta',\rho,\eta) = S_{\pm,\varepsilon}(r,\theta,\rho,\eta) - t\rho^2 - S_{\pm,\varepsilon}(r',\theta',\rho,\eta),$$

where $S_{\pm,\varepsilon}$ is defined in Proposition 4.10. We recall that it coincides with $S_{\pm}$ on $\Gamma^{\pm}_{\mathrm{s}}(\varepsilon)$ (hence on $\Gamma^{\pm}_{\mathrm{s}}(\varepsilon^3)$ too), where $S_{\pm}$ is given by Proposition 4.9. We can therefore freely replace $S_{\pm,\varepsilon}$ by $S_{\pm}$, or more generally by any other continuation of $S_{\pm}$ outside $\Gamma^{\pm}_{\mathrm{s}}(\varepsilon)$. Here we have $0 \le \pm t \le 2h^{-1}$. The integral (7-5) is well defined for $(r,\theta,r',\theta') \in \mathbb{R}^{2n}$ but, using (7-6), we can assume that

$$r \ge \varepsilon^{-1}, \quad \theta \in V_{\varepsilon}, \quad r' \ge \varepsilon^{-3}, \quad \theta' \in V_{\varepsilon^3}. \tag{7-7}$$

The first goal of this section is to prove that, if ε is small enough, we can use stationary phase estimates.

The second goal is to show a similar result for the WKB parametrix, using t_{WKB} as small parameter (see Theorem 6.1). In this case, we have to consider

$$A^{\pm}_{\mathrm{WKB}}(t,r,\theta,r',\theta',\rho,\eta) = a^{\pm}(t,r,\theta,\rho,\eta),$$

where, for $V_2 \Subset \psi_{\iota}(U_{\iota})$, $I_2 \Subset (0,+\infty)$, $\sigma_2 \in (-1,1)$, some $R_2 > 0$ large enough and some $t_{\mathrm{WKB}} > 0$,

$$(a^{\pm}(t))_{0 \le \pm t \le t_{\mathrm{WKB}}} \text{ is bounded in } \mathcal{S}_{\mathrm{hyp}}\big(\Gamma^{\pm}(R_2, V_2, I_2, \sigma_2)\big). \tag{7-8}$$

In particular, we can assume that

$$r \geq R_2, \quad \theta \in V_2. \tag{7-9}$$

The phase is of the form

$$\Phi^{\pm}_{\mathrm{WKB}}(t, r, \theta, r', \theta', \rho, \eta) = \Sigma^{\pm}(t, r, \theta, \rho, \eta) - r'\rho - \theta' \cdot \eta, \tag{7-10}$$

and we refer to Theorem 6.1 for more details. We only recall here that the phase $\Sigma^{\pm}$ is defined on $[0, \pm t_{\mathrm{WKB}}] \times \mathbb{R}^{2n}$ and solves the eikonal equation (6-3) on $[0, \pm t_{\mathrm{WKB}}] \times \Gamma^{\pm}(R_3, V_3, I_3, \sigma_3)$, with $\Gamma^{\pm}(R_2, V_2, I_2, \sigma_2) \subset \Gamma^{\pm}(R_3, V_3, I_3, \sigma_3)$. Here again, the condition (7-8) implies that we can freely modify $\Sigma^{\pm}$ outside $\Gamma^{\pm}(R_2, V_2, I_2, \sigma_2)$.

Below, we will use the notation $\Phi^{\pm}$ (resp. $A^{\pm}$) either for $\Phi^{\pm}_{\mathrm{IK}}$ or $\Phi^{\pm}_{\mathrm{WKB}}$ (resp. $A^{\pm}_{\mathrm{IK}}$ or $A^{\pm}_{\mathrm{WKB}}$), as long as a single analysis for both cases will be possible. For convenience we also define

$$0 \leq \pm t \leq T(h) := \begin{cases} 2h^{-1} & \text{for Isozaki–Kiada,} \\ t_{\mathrm{WKB}} & \text{for WKB.} \end{cases}$$

In the next lemma, we summarize the basic properties of $A^{\pm}$ and $\Phi^{\pm}$ needed to get a first nonstationary phase result. For simplicity, we set $\partial^{\gamma} = \partial_r^j \partial_\theta^\alpha \partial_{r'}^{j'} \partial_{\theta'}^{\alpha'} \partial_\rho^k \partial_\eta^\beta$.

Lemma 7.1. *In each case, for all* $|\gamma| \geq 0$, *the amplitude satisfies*

$$\left|\partial^{\gamma} A^{\pm}(t, r, \theta, r', \theta', \rho, \eta)\right| \leq C_\gamma \tag{7-11}$$

for all

$$(r, \theta, r', \theta', \rho, \eta) \in \mathbb{R}^{3n}, \quad h \in (0, 1], \quad 0 \leq \pm t \leq T(h), \tag{7-12}$$

and we may assume that the phase satisfies

$$\left|\partial^{\gamma}\big(\Phi^{\pm}(t, r, \theta, r', \theta', \rho, \eta) - (r - r')\rho - (\theta - \theta') \cdot \eta\big)\right| \leq C_\gamma \langle t \rangle, \tag{7-13}$$

under the condition (7-12) too. In particular, for all $|\gamma| \geq 1$,

$$|\partial^{\gamma} \partial_\rho \Phi^{\pm}(t, r, \theta, r', \theta', \rho, \eta)| \leq C_\gamma \langle t \rangle, \tag{7-14}$$

under the condition (7-12).

Proof. If $A^{\pm} = A^{\pm}_{\mathrm{IK}}$, (7-11) follows easily from Definition 2.2, (7-6), (7-8) and the time independence of $A^{\pm}_{\mathrm{IK}}$. If $A^{\pm} = A^{\pm}_{\mathrm{WKB}}$, (7-11) is a direct consequence of (7-8). For the phase, Proposition 4.10 shows that $\Phi^{\pm}_{\mathrm{IK}} - (r - r')\rho - (\theta - \theta') \cdot \eta$ is the sum of a function $f \in C_b^\infty(\mathbb{R}^{3n})$ with $-t\rho^2$; similarly, by Lemma 7.5, $\Phi^{\pm}_{\mathrm{WKB}} - (r - r')\rho - (\theta - \theta') \cdot \eta$ is the sum of some $f \in C_b^\infty(\mathbb{R}^{3n})$ with $-tp(r, \theta, \rho, \eta)$. Since the amplitude is compactly supported with respect to ρ and $p(r, \theta, \rho, \eta)$, we may replace $\Phi^{\pm}_{\mathrm{IK}}$ by $(r - r')\rho - (\theta - \theta') \cdot \eta + f - t\rho^2 \chi_1(\rho)$ and $\Phi^{\pm}_{\mathrm{WKB}}$ by $(r - r')\rho - (\theta - \theta') \cdot \eta + f - tp(r, \theta, \rho, \eta)\chi_1(p(r, \theta, \rho, \eta))$, for some $\chi_1 \in C_0^\infty(\mathbb{R})$. This implies (7-13) and completes the proof. ☐

Now choose $\chi_1 \in C_0^\infty(-1, 1)$, $\chi_2 \in C_0^\infty(\mathbb{R}^{n-1})$, both equal to 1 near 0 and define, for any $c_1, c_2 > 0$,

$$A^{\pm}_{c_1, c_2} = \chi_1\left(\frac{\partial_\rho \Phi^{\pm}}{c_1 \langle t \rangle}\right) \chi_2\left(\frac{\partial_\eta \Phi^{\pm}}{c_2}\right) A^{\pm}.$$

Let $E^{\pm}(t,h)$ be the operator with Schwartz kernel (7-5) and $E^{\pm}_{c_1,c_2}(t,h)$ the operator with Schwartz kernel

$$(2\pi h)^{-n} \iint e^{\frac{i}{h}\Phi^{\pm}(t,r,\theta,r',\theta',\rho,\eta)} A^{\pm}_{c_1,c_2}(t,r,\theta,r',\theta',\rho,\eta)\, d\rho\, d\eta, \tag{7-15}$$

for $h \in (0,1]$ and $0 \le \pm t \le T(h)$.

Proposition 7.2 (Semiclassical finite speed of propagation). *For all $c_1, c_2 > 0$ and all $N \ge 0$, we have*

$$\big\| E^{\pm}(t,h) - E^{\pm}_{c_1,c_2}(t,h) \big\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \le C_{N,A,\Phi,c_1,c_2} h^N \quad \text{for } h \in (0,1],\ 0 \le \pm t \le T(h). \tag{7-16}$$

Moreover, if c_1 is small enough, there exists $C \ge 0$, independent of $\pm t \in [0, T(h)]$ and of $c_2 > 0$, such that

$$r' - r \le C \tag{7-17}$$

on the support of $A^{\pm}_{c_1,c_2}$.

Proof. The kernel of $E^{\pm}(t,h) - E^{\pm}_{c_1,c_2}(t,h)$ is an oscillatory integral similar to (7-15) with amplitude

$$A^{\pm} - A^{\pm}_{c_1,c_2} = \left(1 - \chi_1\left(\frac{\partial_\rho \Phi^{\pm}}{c_1\langle t\rangle}\right)\right)\chi_2\left(\frac{\partial_\eta \Phi^{\pm}}{c_2}\right)A^{\pm} + \left(1 - \chi_2\left(\frac{\partial_\eta \Phi^{\pm}}{c_2}\right)\right)A^{\pm}.$$

On the support of the second term of the right-hand side, we integrate by part M times with

$$\frac{h}{i|\partial_\eta \Phi^{\pm}|^2}\partial_\eta \Phi^{\pm} \cdot \partial_\eta.$$

All derivatives of $\partial_\eta \Phi^{\pm}/|\partial_\eta \Phi^{\pm}|^2$ are bounded since t is bounded in the WKB case and $\partial^\gamma \partial_\eta \Phi^{\pm}_{\mathrm{IK}}$ is independent of t and bounded for $|\gamma| \ge 1$. On the support of the first term, integrate by part M times with

$$\frac{h}{i\partial_\rho \Phi^{\pm}}\partial_\rho.$$

Using (7-14), we have, on the support of the first term, $|\partial^\gamma(1/\partial_\rho \Phi^{\pm})| \lesssim 1$, for all γ. Thus, using also (7-11), we end up in both cases with an integral of the form

$$h^{M-n} \iint e^{\frac{i}{h}\Phi^{\pm}(t,r,\theta,r',\theta',\rho,\xi)} B^{\pm}(t,r,\theta,r',\theta',\rho,\xi)\, d\rho d\xi$$

with $B^{\pm}(t,\cdot)$ bounded in $C_b^\infty(\mathbb{R}^{3n})$, for $0 \le \pm t \le T(h)$. We then interpret this integral as the kernel of a pseudodifferential operator with symbol $h^M \exp\big(i(\Phi^{\pm} - (r-r')\rho - (\theta-\theta')\cdot\eta)/h\big)B^{\pm}$ (in the spirit of Lemma 5.4). By the Calderón–Vaillancourt Theorem and (7-13), its operator norm has order $h^M(\langle t\rangle/h)^{n_0}$, for some universal n_0 depending only on n. Thus we get (7-16) by choosing $M = N + 2n_0$.

To prove the second statement, we consider separately the two cases. For the WKB parametrix, t is bounded. Thus, by (7-13), $\partial_\rho \Phi^{\pm}_{\mathrm{WKB}} - (r - r')$ is bounded and since $|\partial_\rho \Phi^{\pm}_{\mathrm{WKB}}| \lesssim c_1\langle t\rangle$, on the support of $A^{\pm}_{\mathrm{WKB},c_1,c_2}$, $r - r'$ must be bounded too. For the Isozaki–Kiada parametrix, as long as t belongs to a bounded set the same argument holds. We may therefore assume that $\pm t \ge T$ with $T > 0$ a fixed large constant. We then exploit two facts: first, for some $c > 0$, we have $c < \pm\rho < c^{-1}$ and $t\rho \ge 0$ on the support of $A^{\pm}_{\mathrm{IK}}$. Second, $f^{\pm} := \Phi^{\pm}_{\mathrm{IK}} - (r - r')\rho - (\theta - \theta')\cdot\eta + t\rho^2$ is independent of t and bounded, together with all its derivatives on the support of $A^{\pm}_{\mathrm{IK}}$. Then

$$\partial_\rho \Phi^{\pm}_{\mathrm{IK}} = r - r' - 2t\rho + \partial_\rho f^{\pm};$$

hence, on the support of $\chi_1(\partial_\rho \Phi^{\pm}_{\mathrm{IK}}/c_1\langle t\rangle)$, we have

$$r - r' \geq -c_1\langle t\rangle + 2t\rho - \partial_\rho f^{\pm}.$$

If c_1 is small enough and T large enough, we have $2t\rho - c_1\langle t\rangle \geq 0$ for $t \geq T$. This completes the proof. □

Remark. It is clear from the proof that the constant C in (7-17) is uniform with respect to $\varepsilon > 0$ small in the Isozaki–Kiada case (recall that the amplitudes depend respectively on t and ε for the WKB and the IK parametrices).

From now on, we fix $c_1 > 0$ small enough that (7-17) holds.

Proposition 7.3 (Dispersion estimate for times $\leq h$). *For all $c_2 > 0$, and still with $\gamma_n = \dfrac{n-1}{2}$, we have*

$$\|e^{-\gamma_n r} E^{\pm}_{c_1,c_2}(t,h) e^{-\gamma_n r}\|_{L^1(\mathbb{R}^n)\to L^\infty(\mathbb{R}^n)} \leq C_{A,\Phi,c_2}|ht|^{-n/2} \quad \textit{for } 0 < \pm t \leq \min(T(h), h).$$

Note that the condition $\pm t \leq \min(T(h), h)$ is essentially the condition $\pm t \leq h$. We have put it under this form only because of those h such that $h \geq t_{\mathrm{WKB}}$. This will not modify the rest of the analysis. Furthermore, the latter h correspond to bounded frequencies and their contribution to the Strichartz estimates can be treated by Sobolev embeddings.

Proof. In the Isozaki–Kiada case, both $e^{-r'}\eta = \xi$ and $e^{-r}\eta$ are supported in a compact set. In the WKB one, $e^{-r}\eta$ is compactly supported but, using (7-17), this also implies that $e^{-r'}\eta$ is compactly supported. Therefore, in both cases, the change of variable $e^{-r'}\eta = \xi$ shows that the kernel of $E^{\pm}_{c_1,c_2}(t,h)$ is an integral of the form

$$h^{-n} e^{(n-1)r'} \int e^{\frac{i}{h}\Phi^{\pm}(t,r,\theta,r',\theta',\rho,e^{r'}\xi)} B^{\pm}(t,\theta,r',\theta',\rho,\xi)\, d\rho\, d\xi,$$

with $B^{\pm}$ bounded on $[0, \pm T(h)] \times \mathbb{R}^{3n}$ and supported in a region where $|\rho| + |\xi| \lesssim 1$. The kernel of $e^{-\gamma_n r} E^{\pm}_{c_1,c_2}(t) e^{-\gamma_n r}$ is then simply obtained by multiplying the integral above by $e^{-\gamma_n(r+r')}$, so its modulus is controlled by $h^{-n} e^{\gamma_n(r'-r)} \lesssim |ht|^{-n/2}$, by (7-17) and the fact that $0 < \pm t \leq h$. This completes the proof. □

To prove the dispersion estimates for $h \leq \pm t \leq T(h)$ we need to analyze the phases more precisely.

In the following lemma and its proof, we shall use the notation (3-4).

Lemma 7.4. *For all (fixed) $\tilde{\varepsilon} > 0$ small enough, we can find a family of real-valued functions $(\varphi^{\mathrm{st}}_{\pm,\varepsilon})_{0<\varepsilon\ll 1}$ such that*

$$\varphi^{\mathrm{st}}_{\pm,\varepsilon} = \varphi_{\pm} = \varphi_{\pm,\varepsilon} \quad \textit{on } \Gamma^{\pm}_{\mathrm{s}}(\varepsilon), \tag{7-18}$$

$$\varphi^{\mathrm{st}}_{\pm,\varepsilon} \in \mathcal{S}_{\mathrm{hyp}}\left(\Gamma_{\mathrm{s}}(\tilde{\varepsilon})\right), \tag{7-19}$$

and that, if we set

$$R_{\pm,\varepsilon}(r,\theta,\rho,\eta) = \varphi^{\mathrm{st}}_{\pm,\varepsilon}(r,\theta,\rho,\eta) - \frac{q_0(\theta, e^{-r}\eta)}{4\rho}$$

the following holds for $j + |\alpha| \leq 1$:

$$\sup_{\substack{(r,\theta,\eta)\in\mathbb{R}^{2n-1} \\ \pm\rho\in[\frac14,4]}} \left|(e^r\partial_\eta)^\beta \partial_r^j \partial_\theta^\alpha \partial_\rho^k R_{\pm,\varepsilon}(r,\theta,\rho,\eta)\right| \leq \begin{cases} C\varepsilon^{\tau/2} & \textit{if } k + |\beta| \leq 2, \\ C_{\varepsilon j\alpha k\beta} & \textit{if } k + |\beta| \geq 3, \end{cases} \tag{7-20}$$

where τ, the decay rate in (1-8), satisfies (1-9).

Proof. Using (4-35) and Taylor's formula, we can write

$$\varphi_\pm(r,\theta,\rho,\eta)=\int_0^{\pm\infty}e^{-4t\rho}q(r+2t\rho,\theta,e^{-r}\eta)dt+\sum_{|\gamma|=3}a_\gamma(r,\theta,\rho,\eta)(e^{-r}\eta)^\gamma,$$

with $a_\gamma\in\mathcal{B}_{\mathrm{hyp}}(\Gamma_{\mathrm{s}}^\pm(\varepsilon_0))$ for some fixed $\varepsilon_0>0$. Therefore,

$$\varphi_\pm(r,\theta,\rho,\eta)-\frac{q_0(\theta,e^{-r}\eta)}{4\rho}=\int_0^{\pm\infty}e^{-4t\rho}q_1(r+2t\rho,\theta,e^{-r}\eta)dt+\sum_{|\gamma|=3}a_\gamma(r,\theta,\rho,\eta)(e^{-r}\eta)^\gamma,\quad (7\text{-}21)$$

with q_1 satisfying (3-6). Denote by $R(r,\theta,\rho,\eta)$ the right-hand side of (7-21) and choose $\chi_1\in C_0^\infty(\mathbb{R})$ and $\chi_2\in C_0^\infty(\mathbb{R}^{n-1})$ both equal to 1 near 0. For some $\tilde\varepsilon>0$ to be fixed below, choose $\chi_{\tilde\varepsilon}^\pm$ such that

$$\chi_{\tilde\varepsilon}^\pm\in\mathcal{S}_{\mathrm{hyp}}(\Gamma_{\mathrm{s}}^\pm(\tilde\varepsilon)),\qquad \chi_{\tilde\varepsilon}^\pm\equiv1\ \text{ on }\ \Gamma_{\mathrm{s}}^\pm(\tilde\varepsilon^2),$$

using Proposition 4.1. (We don't need Proposition 4.4 here, since ε^0 will be fixed.) We then claim that, if $\tilde\varepsilon$ is small enough (and fixed) and ε with $\tilde\varepsilon'$ is small enough too, the function

$$\varphi_{\pm,\varepsilon}^{\mathrm{st}}(r,\theta,\rho,\eta):=\frac{q_0(\theta,e^{-r}\eta)}{4\rho}+R(r,\theta,\rho,\eta)\chi_{\tilde\varepsilon}^\pm(r,\theta,\rho,\eta)\chi_2(e^{-r}\eta/\varepsilon^{1/2})(1-\chi_1)(\varepsilon^{1/2}r),$$

satisfies (7-18), (7-19) and (7-20). Indeed, by choosing $\tilde\varepsilon$ small enough, we have $\pm\rho\approx1$ on the support of $\chi_{\tilde\varepsilon}^\pm$, so the integral in (7-21) is exponentially convergent. Furthermore, since

$$\left|(e^r\partial_\eta)^\beta\partial_r^j\left((e^{-r}\eta)^\gamma\chi_1(e^{-r}\eta/\varepsilon^{1/2})\right)\right|\le C(\varepsilon^{1/2})^{|\gamma|-|\beta|},$$

for all γ, and using the fact that, if $t\rho\ge0$ and $r\ge0$,

$$\left|(e^r\partial_\eta)^\beta\partial_r^j\partial_\theta^\alpha\partial_\rho^k q_1(r+t\rho,\theta,e^{-r}\eta)\right|\le C|t|^k\langle r\rangle^{-\tau}|e^{-r}\eta|^{2-|\beta|},$$

we get the estimate (7-20). Finally, since $e^r|\eta|\lesssim\varepsilon$ and $r\ge\varepsilon$ on $\Gamma_{\mathrm{s}}^\pm(\varepsilon)$, we have (7-18) for all ε small enough. The property (7-19) is clear thanks to $\chi_{\tilde\varepsilon}^\pm$. □

In the following lemma, we use the notation of Theorem 6.1.

Lemma 7.5. *We can find a family of real-valued functions* $(\Sigma_{\mathrm{st}}^\pm(t))_{0\le\pm t\le t_{\mathrm{WKB}}}$ *such that*

$$\Sigma_{\mathrm{st}}^\pm(t)=\Sigma^\pm(t)\qquad\text{on }\Gamma^\pm(R_2,V_2,I_2,\sigma_2),\quad (7\text{-}22)$$

and, for all k,β,

$$\sup_{\mathbb{R}^{2n}}\left|(e^r\partial_\eta)^\beta\partial_\rho^k\left(\Sigma_{\mathrm{st}}^\pm(t,r,\theta,\rho,\eta)-r\rho-\theta\cdot\eta-tp(r,\theta,\rho,\eta)\right)\right|\le C_{k\beta}t^2.\quad (7\text{-}23)$$

Proof. Using the function $\chi_{2\to3}^\pm$ of Theorem 6.1, the result is straightforward by considering

$$\Sigma_{\mathrm{st}}^\pm(t,r,\theta,\rho,\eta)\\=\chi_{2\to3}^\pm(r,\theta,\rho,\eta)\left(\Sigma_0^\pm(t,r,\theta,\rho,\eta)-r\rho-\theta\cdot\eta-tp(r,\theta,\rho,\eta)\right)+r\rho+\theta\cdot\eta+tp(r,\theta,\rho,\eta),$$

and using (6-4). □

We remark that $\Sigma^\pm$ satisfies (6-5) whereas $\Sigma^\pm_{\mathrm{st}}$ does not. This was the reason for considering $\Sigma^\pm$ first, since the property (6-5) is convenient to prove L^2 bounds for Fourier integral operators.

The estimates (7-20) and (7-23) show that we have good asymptotics for the phases in certain regimes, namely $\varepsilon \to 0$ for the Isozaki–Kiada parametrix and $t \to 0$ for the WKB parametrix. Using Lemmas 7.4 and 7.5, we replace $\varphi_{\pm,\varepsilon}$ by $\varphi^{\mathrm{st}}_{\pm,\varepsilon}$ and $\Sigma^\pm$ by $\Sigma^\pm_{\mathrm{st}}$ in the expression of $\Phi^\pm_{\mathrm{IK}}$ and $\Phi^\pm_{\mathrm{WKB}}$, respectively.

To use a single formalism for both cases, we introduce the parameter

$$\lambda_{\mathrm{st}} := \begin{cases} \varepsilon & \text{for the Isozaki–Kiada parametrix,} \\ t^{\mathrm{st}}_{\mathrm{WKB}} & \text{for the WKB parametrix,} \end{cases}$$

where $t^{\mathrm{st}}_{\mathrm{WKB}} > 0$ will denote the size of the time interval where t will be allowed to live. Using the change of variable $\xi = e^{-r'}\eta$ and factorizing by t in the phase, the integral (7-15) can be written

$$(2\pi h)^{-n} e^{2\gamma_n r'} \int e^{i\frac{t}{h}\tilde{\Phi}^\pm_{\lambda_{\mathrm{st}}}(z,\rho,\xi)} \tilde{A}^\pm_{c_1,c_2\lambda_{\mathrm{st}}}(z,\rho,\xi)\, d\rho\, d\xi,$$

where $h \in (0,1]$,

$$\tilde{\Phi}^\pm_{\lambda_{\mathrm{st}}}(y,\rho,\xi) = \frac{1}{t}\Phi^\pm(t,r,\theta,r',\theta',\rho,e^{r'}\xi), \tag{7-24}$$

$$\tilde{A}^\pm_{c_1,c_2,\lambda_{\mathrm{st}}}(y,\rho,\xi) = A_{c_1,c_2}(t,r,\theta,r',\theta',\rho,e^{r'}\xi), \tag{7-25}$$

and

$$y = (h,t,r,\theta,r',\theta'), \tag{7-26}$$

with r, r' satisfying (7-17) and

$$0 < \pm t \le T(h,\lambda_{\mathrm{st}}) := \begin{cases} 2h^{-1} & \text{for the Isozaki–Kiada parametrix,} \\ t^{\mathrm{st}}_{\mathrm{WKB}} & \text{for the WKB parametrix.} \end{cases}$$

The kernel of $e^{-\gamma_n r} E^\pm_{c_1,c_2}(t,h) e^{-\gamma_n r}$ then becomes

$$(2\pi h)^{-n} e^{\gamma_n(r'-r)} \int e^{i\frac{t}{h}\tilde{\Phi}^\pm_{\lambda_{\mathrm{st}}}(y,\rho,\xi)} \tilde{A}^\pm_{c_1,c_2,\lambda_{\mathrm{st}}}(y,\rho,\xi)\, d\rho\, d\xi.$$

Proposition 7.6 (nonstationary phase). *There exists $C' > 0$ such that the condition*

$$\left|\frac{r-r'}{t}\right| + e^{r'}\left|\frac{\theta-\theta'}{t}\right| \ge C' \tag{7-27}$$

implies that for all $c_2 > 0$, all $N \ge 0$ and all $0 < \lambda_{\mathrm{st}} \ll 1$, we can find $C_{c_2,N,\lambda_{\mathrm{st}}}$ such that, for all

$$h \in (0,1], \quad \pm t \in [h, T(h,\lambda_{\mathrm{st}})], \quad \omega \ge 1, \quad (r,\theta,r',\theta') \in \mathbb{R}^{2n},$$

with r, r' satisfying (7-17), we have

$$\left|(2\pi h)^{-n} e^{\gamma_n(r'-r)} \int e^{i\omega\tilde{\Phi}^\pm_{\lambda_{\mathrm{st}}}(y,\rho,\xi)} \tilde{A}^\pm_{c_1,c_2,\lambda_{\mathrm{st}}}(y,\rho,\xi)\, d\rho\, d\xi\right| \le C_{c_2,N,\lambda_{\mathrm{st}}} h^{-n} \omega^{-N}.$$

Proof. For $t \neq 0$, we define

$$\tilde{\Phi}^{\mathrm{free}}_t := \frac{r-r'}{t}\rho + e^{r'}\frac{\theta-\theta'}{t}\cdot\xi.$$

Then

$$\nabla_{\rho,\xi}\tilde{\Phi}_t^{\text{free}} = \left(\frac{r-r'}{t}, e^{r'}\frac{\theta-\theta'}{t}\right).$$

We then start with the case of $\Phi^{\pm}_{\text{WKB}}$. By Lemma 7.5 and (7-17), $\nabla_{\rho,\xi}(\tilde{\Phi}_{\lambda_{\text{st}}} - \tilde{\Phi}_t^{\text{free}})$ is a function of $(t, r, \theta, r', \rho, \xi)$ which is bounded on the support of the amplitude, as well as all its derivatives in ρ, ξ, uniformly with respect to (t, r, θ, r'). Therefore, if C' is large enough, we have

$$\left|\nabla_{\rho,\xi}\tilde{\Phi}_{\lambda_{\text{st}}}\right| \gtrsim \left|\frac{r-r'}{t}\right| + e^{r'}\left|\frac{\theta-\theta'}{t}\right|, \tag{7-28}$$

and the result follows from standard integrations by parts. Note that, here, we have not used the smallness of λ_{st} (i.e., of t). We shall use it for the case of $\Phi^{\pm}_{\text{IK}}$ which we now consider. Since $\pm\rho \in \left[\frac{1}{4}, 4\right]$ on the support of the amplitude if $\varepsilon = \lambda_{\text{st}}$ is small enough, Lemma 7.4 and Taylor's formula imply that

$$\nabla_{\rho,\xi}(\tilde{\Phi}_{\lambda_{\text{st}}} - \tilde{\Phi}_t^{\text{free}}) = (-2\rho, 0) + \nabla_{\rho,\xi}\frac{q_0(\theta, e^{r'-r}\xi) - q_0(\theta', \xi)}{t\rho} + \varepsilon_\varepsilon(y, \rho, \xi)\left(\frac{r-r'}{t}, \frac{\theta-\theta'}{t}\right),$$

where $\varepsilon_\varepsilon(y, \rho, \xi)$ and all its derivatives in ρ, ξ go to 0 as $\varepsilon \to 0$, uniformly with respect to y (see (7-26)) with r, r' satisfying (7-17) and $(\pm\rho, \xi) \in \left[\frac{1}{4}, 4\right] \times \mathbb{R}^{n-1}$. Furthermore, using (7-17) and the fact that $|\xi| \lesssim \varepsilon^3$ on the support of the amplitude, we have

$$\left|\nabla_{\rho,\xi}\frac{q_0(\theta, e^{r'-r}\xi) - q_0(\theta', \xi)}{t\rho}\right| \lesssim \varepsilon^3\left|\left(\frac{r-r'}{t}, \frac{\theta-\theta'}{t}\right)\right|$$

thus, using that $r' \geq 0$ on the support of the amplitude, we have (7-28) if ε is small enough. In addition, for all $k + |\beta| \geq 2$, we also have

$$\left|\partial_\rho^k \partial_\xi^\beta \tilde{\Phi}_{\lambda_{\text{st}}}\right| \lesssim \left|\left(\frac{r-r'}{t}, \frac{\theta-\theta'}{t}\right)\right|$$

on the support of the amplitude, using (7-17). The result then follows again from integrations by parts. □

We next state a convenient form of the stationary phase theorem with parameters; the demonstration — a simple adaptation of the proof of [Hörmander 1983, Theorem 7.7.5] — is given in Appendix A for completeness.

Proposition 7.7 (Stationary phase theorem). *Let Ω be a set and*

$$f : \mathbb{R}^n \times \Omega \ni (x, y) \mapsto f(x, y) \in \mathbb{R}$$

a function, smooth with respect to x and such that

$$\text{Hess}_x[f](x, y) = S(y) + R(x, y) \quad \textit{for } (x, y) \in \mathbb{R}^n \times \Omega, \tag{7-29}$$

with $S(y)$ a symmetric nonsingular matrix such that

$$|S(y)^{-1}| \lesssim 1 \quad \textit{for } y \in \Omega, \tag{7-30}$$

and $R(x, y)$ a symmetric matrix such that

$$\|S(y)^{-1}R(x, y)\| \leq \tfrac{1}{2} \quad \textit{for } (x, y) \in \mathbb{R}^n \times \Omega, \tag{7-31}$$

where $\|\cdot\|$ *is the Euclidean matrix norm. Then there exists* $N \geq 0$ *such that, for all* $K \Subset \mathbb{R}^n$, *there exists* $C_K > 0$ *satisfying*

$$\left|\int e^{i\omega f(x,y)} u(x)dx\right| \leq C_K \omega^{-n/2} \sup_{|\alpha|\leq N} \|\partial^\alpha u\|_{L^\infty(K)} \sup_{2\leq|\alpha|\leq N} \left(\sup_{x\in K} |\partial^\alpha f(x,y)| + 1\right)^N,$$

for all $y \in \Omega$, *all* $u \in C_0^\infty(K)$ *and all* $\omega \geq 1$.

For the WKB parametrix, we shall use this proposition fairly directly by considering

$$\Omega^{\pm}_{\mathrm{WKB}}\left(t^{\mathrm{st}}_{\mathrm{WKB}}\right) = \left\{(h,t,r,\theta,r',\theta') : h \in (0,1],\ \left|\frac{r-r'}{t}\right| \leq C',\ h \leq \pm t \leq t^{\mathrm{st}}_{\mathrm{WKB}}\right\}.$$

Notice in particular that $r - r'$ is bounded on $\Omega_{\mathrm{WKB}}\left(t^{\mathrm{st}}_{\mathrm{WKB}}\right)$.

Proposition 7.8 (Dispersion estimate for the WKB parametrix). *Fix* $c_2 > 0$. *There exists* $t^{\mathrm{st}}_{\mathrm{WKB}} > 0$ *small enough such that, for all* $y = (h,t,r,\theta,r',\theta') \in \Omega^{\pm}_{\mathrm{WKB}}\left(t^{\mathrm{st}}_{\mathrm{WKB}}\right)$ *and all* $\omega \geq 1$, *we have*

$$\left|(2\pi h)^{-n} e^{\gamma_n(r'-r)} \int e^{i\omega\tilde{\Phi}^{\pm}_{t^{\mathrm{st}}_{\mathrm{WKB}}}(y,\rho,\xi)} \tilde{A}^{\pm}_{c_1,c_2,t^{\mathrm{st}}_{\mathrm{WKB}}}(y,\rho,\xi)\, d\rho\, d\xi\right| \lesssim \omega^{-n/2}.$$

Proof. This is a straightforward application of Proposition 7.7 since, using (7-23), we have

$$\mathrm{Hess}_{\rho,\xi}[\tilde{\Phi}_{t^{\mathrm{st}}_{\mathrm{WKB}}}] = \begin{pmatrix} 2 & 0 \\ 0 & \mathrm{Hess}_\eta(q) \end{pmatrix} + \mathbb{O}(t^{\mathrm{st}}_{\mathrm{WKB}}),$$

where the first matrix of the right-hand side satisfies (7-30) by the uniform ellipticity of q. The conclusion is then clear since all derivatives, in ρ, ξ, of $\tilde{A}^{\pm}_{t^{\mathrm{st}}_{\mathrm{WKB}}}$ are bounded, as well as those of $\tilde{\Phi}_{t^{\mathrm{st}}_{\mathrm{WKB}}}$ of order at least 2, on the support of the amplitude. □

To be in position to use Proposition 7.7 for the Isozaki–Kiada parametrix, we still need two lemmas.

Lemma 7.9 (Sharper localization for IK). *Let* $\chi_0 \in C_0^\infty(\mathbb{R})$ *be equal to* 1 *near* 0 *and set*

$$\chi_\varepsilon(y,\rho) = \chi_0\left(\varepsilon^{-\tau/4}\left(2\rho - \frac{r-r'}{t}\right)\right). \tag{7-32}$$

Then, for all $\varepsilon > 0$ *small enough, all* $N \geq 0$ *and all* $c_2 > 0$, *there exists* $C_{c_2,N,\varepsilon}$ *such that, for all*

$$h \in (0,1], \quad \pm h \leq t \leq 2h^{-1}, \quad \omega \geq 1,$$

and all $(r,\theta,r',\theta') \in \mathbb{R}^{2n}$ *satisfying* (7-7) *and such that*

$$\left|\frac{r-r'}{t}\right| + e^{r'}\left|\frac{\theta-\theta'}{t}\right| \leq C', \tag{7-33}$$

we have

$$\left|(2\pi h)^{-n} e^{\gamma_n(r'-r)} \int e^{i\omega\tilde{\Phi}^{\pm}_\varepsilon(y,\rho,\xi)} (1-\chi_\varepsilon(y,\rho))\tilde{A}^{\pm}_{c_1,c_2,\varepsilon}(y,\rho,\xi)\, d\rho\, d\xi\right| \leq C_{c_2,N,\varepsilon} h^{-n}\omega^{-N}.$$

Proof. By the same analysis as in the proof of Proposition 7.6, using Lemma 7.4 and (7-33), we may write

$$\tilde{\Phi}^{\pm}_\varepsilon(y,\rho,\xi) = \frac{r-r'}{t}\rho - \rho^2 + R^{\pm}_\varepsilon(y,\rho,\xi),$$

where, on the support of the amplitude, we have

$$|\partial_\rho R_\varepsilon^\pm| \lesssim \varepsilon^{\tau/2}, \qquad |\partial_\rho^k \partial_\xi^\beta R_\varepsilon^\pm| \lesssim 1$$

for $k + |\beta| \geq 1$. On the other hand, on the support of $(1 - \chi_\varepsilon(y, \rho))$ we also have, for some $c > 0$,

$$\frac{r - r'}{t} - 2\rho \geq c\varepsilon^{\tau/4} \quad \text{or} \quad \frac{r - r'}{t} - 2\rho \leq -c\varepsilon^{\tau/4}.$$

Therefore, if ε is small enough,

$$|\partial_\rho \tilde{\Phi}_\varepsilon^\pm(y, \rho, \xi)| \gtrsim \varepsilon^{\tau/4},$$

on the support of the amplitude and the result follows from integrations by parts in ρ. □

Basically, the interest of the localization (7-34) is to replace $\frac{1}{4}\rho$ in (7-20) by $2t/(r - r')$ up to a small error. We implement this idea as follows. By Lemma 7.9, we can replace $\tilde{A}^\pm_{c_1,c_2,\varepsilon}(y, \rho, \xi)$ in (7-25) by

$$\chi_\varepsilon(y, \rho) \tilde{A}^\pm_{c_1,c_2,\varepsilon}(y, \rho, \xi). \tag{7-34}$$

If ε is small enough, we have $\pm\rho \in \left[\frac{1}{4}, 4\right]$ on the support of $\tilde{A}^\pm_{c_1,c_2,\varepsilon}$ hence, for some $c > 0$,

$$c|t| \leq r - r' \leq c^{-1}|t|, \tag{7-35}$$

on the support of (7-34), which is stronger than (7-17). Furthermore, the condition (7-33) together with (7-7) implies that we may assume that $|\theta - \theta'| \leq C' e^{-\varepsilon^{-3}} |t|$. From now on we fix

$$c_2 = \varepsilon.$$

Thus, by writing

$$\partial_\eta \Phi^\pm_{\mathrm{IK}} = \theta - \theta' + \partial_\eta \varphi_\pm(r, \theta, \rho, \eta) - \partial_\eta \varphi_\pm(r', \theta', \rho, \eta),$$

with $\varphi_\pm \in \mathcal{B}_{\mathrm{hyp}}(\Gamma^\pm(\varepsilon_2))$, we have$|\partial_\eta \varphi_\pm(r, \theta, \rho, \eta)| \lesssim e^{-r}$ and $|\partial_\eta \varphi_\pm(r', \theta', \rho, \eta)| \lesssim e^{-r'}$ on the support of the amplitude. By (7-7), we have for instance $|\partial_\eta \Phi^\pm_{\mathrm{IK}} - (\theta - \theta')| \leq \varepsilon^2$ if ε is small enough. We may therefore assume that

$$|\theta - \theta'| \leq C'' \varepsilon \frac{|t|}{\langle t \rangle}. \tag{7-36}$$

To be set of parameters for the stationary phase theorem, we will thus choose

$$\Omega^\pm_{\mathrm{IK}}(\varepsilon) = \left\{ (h, t, r, \theta, r', \theta') : h \in (0, 1],\ \pm t \in [h, 2h^{-1}] \text{ and (7-7), (7-33), (7-35), (7-36) hold} \right\}.$$

Before applying Proposition 7.7, we still need to modify the phase $\tilde{\Phi}^\pm_\varepsilon$ outside the support of the new amplitude (7-34).

Lemma 7.10. *We can find $\Psi^\pm_\varepsilon$ smooth and real-valued such that, on the support of (7-34),*

$$\Psi^\pm_\varepsilon(y, \rho, \xi) = \tilde{\Phi}^\pm_\varepsilon(y, \rho, \xi),$$

and

$$\Psi^\pm_\varepsilon(y, \rho, \xi) = \frac{r - r'}{t}\rho + \frac{\theta - \theta'}{t} e^{r'} \xi - \rho^2 - \frac{1 - e^{2(r' - r)}}{2(r - r')} q_0(\theta', \xi) + \psi^\pm_\varepsilon(y, \rho, \xi), \tag{7-37}$$

where, for all $k+|\beta|\le 2$,

$$\sup_{\substack{(\rho,\xi)\in\mathbb{R}^n\\ y\in\Omega^{\pm}_{\mathrm{IK}}(\varepsilon)}} |\langle t\rangle^{|\beta|/2}\partial_\rho^k\partial_\xi^\beta \psi_\varepsilon^{\pm}(y,\rho,\xi)|\to 0 \quad \textit{as } \varepsilon\to 0, \tag{7-38}$$

and for $|k|+|\beta|\ge 3$,

$$\sup_{\substack{(\rho,\xi)\in\mathbb{R}^n\\ y\in\Omega^{\pm}_{\mathrm{IK}}(\varepsilon)}} |\,\partial_\rho^k\partial_\xi^\beta \psi_\varepsilon^{\pm}(y,\rho,\xi)|\le C_{\varepsilon,k,\beta}. \tag{7-39}$$

Proof. We shall basically combine (7-20) with the fact that

$$|2\rho-(r-r')/t|\lesssim \varepsilon^{\tau/4}, \tag{7-40}$$

on the support of (7-34). By Lemma 7.4, the phase reads

$$\frac{r-r'}{t}\rho+\frac{\theta-\theta'}{t}e^{r'}\xi-\rho^2-\frac{q_0(\theta',\xi)-e^{2(r'-r)}q_0(\theta,\xi)}{4\rho t}+\frac{R_{\pm,\varepsilon}(r,\theta,\rho,e^{r'}\xi)-R_{\pm,\varepsilon}(r',\theta',\rho,e^{r'}\xi)}{t}.$$

The last term of this sum satisfies the estimates (7-38) and (7-39): for $0<\pm t\le 1$, it follows from Taylor's formula using (7-33) and Lemma 7.4 with $j+|\alpha|=1$, and for $\pm t\ge 1$ it follows from Lemma 7.4 with $j+|\alpha|=0$. For the term involving q_0 we write

$$\frac{1}{4\rho t}=\frac{1}{2(r-r')}+\left(\frac{1}{4\rho t}-\frac{1}{2(r-r')}\right)\chi_1\left(\frac{2\rho-(r-r')/t}{\varepsilon^{\tau/8}}\right),$$

using (7-40) with ε small enough and $\chi_1\in C_0^\infty(\mathbb{R}^{n-1})$ equal to 1 near 0, and

$$q_0(\theta,e^{r'-r}\xi)=e^{2(r'-r)}q_0(\theta',\xi)+e^{2(r'-r)}(q_0(\theta,\xi)-q_0(\theta',\xi))\chi_2(\xi),$$

with $\chi_2\in C_0^\infty(\mathbb{R}^{n-1})$ equal to 1 near 0. We obtain the estimates (7-38) and (7-39) for

$$\frac{1}{4\rho t}e^{2(r'-r)}(q_0(\theta,\xi)-q_0(\theta',\xi))\chi_2(\xi),$$

using (7-36), and for

$$(1-e^{2(r'-r)})q_0(\theta',\xi)\left(\frac{1}{4\rho t}-\frac{1}{2(r-r')}\right)\chi_1\left(\frac{2\rho-(r-r')/t}{\varepsilon^{\tau/8}}\right)$$

using (7-35). In both cases, we can freely multiply the functions by a compactly support cutoff in ρ using that $\pm\approx 1$ on the support of the amplitude. This completes the proof. □

Proposition 7.11 (bounded times). *There exists* $\varepsilon_{\mathrm{st}}>0$ *such that, for all* $T>0$, *all* $0<\varepsilon\le\varepsilon_{\mathrm{st}}$, *there exists* $C_{\varepsilon,T}$ *such that, for all*

$$h\in(0,1],\qquad h\le\pm t\le T,\qquad (r,\theta,r',\theta')\ \textit{satisfying (7-7), (7-35) and (7-36)}, \tag{7-41}$$

we have

$$\left|(2\pi h)^{-n}e^{\gamma_n(r'-r)}\int e^{i\frac{t}{h}\tilde{\Phi}^{\pm}_\varepsilon(y,\rho,\xi)}\chi_\varepsilon(y,\rho)\tilde{A}_{c_1,\varepsilon,\varepsilon}(y,\rho,\xi)\,d\rho\,d\xi\right|\le C_{\varepsilon,T}|ht|^{-n/2}. \tag{7-42}$$

Proof. By Lemma 7.10, we can replace $\tilde{\Phi}_\varepsilon^\pm$ by $\Psi_\varepsilon^\pm$. We then have

$$\mathrm{Hess}_{\rho,\xi}[\Psi_\varepsilon^\pm] = \begin{pmatrix} 2 & 0 \\ 0 & \dfrac{1-e^{2(r'-r)}}{2(r-r')}\mathrm{Hess}_\eta(q_0) \end{pmatrix} + o(1),$$

where $o(1) \to 0$ as $\varepsilon \to 0$, uniformly with respect to $(\rho, \xi) \in \mathbb{R}^n$ and to the parameters satisfying (7-41). Using the upper bound in (7-35) and the boundedness of t, the positive number

$$\frac{1-e^{2(r'-r)}}{2(r-r')}$$

belongs to a compact subset of $(0, \infty)$, yielding the condition (7-30). We then conclude by applying Proposition 7.7. □

To obtain (7-42), we have used the boundedness of $e^{\gamma_n(r'-r)}$, since $|r-r'|$ was bounded. In principle, the condition (7-35) implies that $e^{\gamma_n(r'-r)}$ decays exponentially in time. We shall exploit the latter below.

Proposition 7.12 (Large times). *There exists $T > 0$ and $\varepsilon'_{\mathrm{st}}$ such that, for all $0 < \varepsilon \le \varepsilon'_{\mathrm{st}}$, there exists C_ε such that, for all*

$$h \in (0, 1], \qquad T \le \pm t \le 2h^{-1}, \qquad (r, \theta, r', \theta') \ \textit{satisfying (7-7), (7-35) and (7-36)}, \tag{7-43}$$

we have

$$\left| (2\pi h)^{-n} e^{\gamma_n(r'-r)} \int e^{i\frac{t}{h}\tilde{\Phi}_\varepsilon^\pm(y,\rho,\xi)} \chi_\varepsilon(y, \rho) \tilde{A}_{c_1,\varepsilon,\varepsilon}(y, \rho, \xi)\, d\rho\, d\xi \right| \le C_\varepsilon |ht|^{-n/2}.$$

Proof. Choose T large enough such that, for $t \ge T$ and r, r' satisfying (7-35), we have $e^{2(r'-r)} \le \frac{1}{2}$. To compensate the factor $1/(r-r')$ in (7-37) (of order $1/|t|$ by (7-35)), we consider the new variable $|t|^{1/2}\zeta = \xi$. By (7-38), if ε is small enough, this new phase satisfies the assumptions of Proposition 7.7. In the corresponding estimate given by Proposition 7.7, derivatives of the new amplitude as well as derivatives of the new phase of order at least 3 will grow at most polynomially with respect to t. This gives a polynomial growth in t of the coefficient in the stationary phase estimate of Proposition 7.7 but such a growth is controlled by the exponential decay of $e^{\gamma_n(r'-r)} \lesssim e^{-c|t|}$. This completes the proof. □

7B. ***Proof of Proposition 2.20.*** By (2-38), up to a remainder of operator norm of size h^n (uniformly in time), we may replace $\widehat{Op}_\iota(a_{\mathrm{s}}^\pm)^*$ by a linear combination of operators of the form $\widehat{Op}_\iota(\tilde{a}_{\mathrm{s}}^\pm)$ with $\mathrm{supp}(\tilde{a}_{\mathrm{s}}^\pm) \subset \mathrm{supp}(a_{\mathrm{s}}^\pm)$. We next apply Theorem 5.1 to order $n+1$ and are left with the study of the Fourier integral operator part. By Proposition 7.2, the amplitude can be modified so that, up to a remainder of operator norm of order h^n uniformly in time, we are left with an operator whose kernel $K^\pm(r, \theta, r', \theta', t, h)$ satisfies

$$|e^{-\gamma_n r} K^\pm(r, \theta, r', \theta', t, h) e^{-\gamma_n r'}| \lesssim |ht|^{-n/2}, \qquad h \in (0, 1], 0 < \pm t \le 2h.$$

Indeed, for $t \le h$, this follows from Proposition 7.3 and for $t \ge h$, from Propositions 7.11 and 7.12 with $\omega = \pm t/h$ and also from Proposition 7.6 and Lemma 7.9 with $N \ge n/2$. □

Proof of Proposition 2.22. It is completely similar to the one of Proposition 2.20 by considering times $0 \le \pm t \le t_{\mathrm{WKB}}^{\mathrm{st}}$ with $t_{\mathrm{WKB}}^{\mathrm{st}}$ small enough to be in position to use both Theorem 6.1 and Proposition 7.8. □

Appendix A. Control on the range of some diffeomorphisms

In this section, we prove a proposition implying Lemma 4.7 and (4-54) in Lemma 4.11. For simplicity, we consider the outgoing case only but the symmetric result holds in the incoming one.

Let us define the following conical subset of $T^*\mathbb{R}^n_+ \setminus 0$,

$$\Gamma^+_{\text{s-con}}(\varepsilon) = \left\{ (r,\theta,\rho,\eta) : r > R(\varepsilon),\ \theta \in V_\varepsilon,\ \rho > (1-\varepsilon^2)(\rho^2 + q(r,\theta,e^{-r}\eta))^{1/2} \right\}, \tag{A-1}$$

which is the cone generated by $\Gamma^+_{\text{s}}(\varepsilon)$.

Proposition A.1. *Assume that, for some $0 < \bar\varepsilon < \frac{1}{4}$, we are given a family of maps $(\Psi^t)_{t\geq 0}$ defined on $\Gamma^+_{s\text{-}con}(\bar\varepsilon)$, of the form*

$$\Psi^t(r,\theta,\rho,\eta) = (r,\theta,\underline{\rho}^t(r,\theta,\rho,\eta),\underline{\eta}^t(r,\theta,\rho,\eta)) \in \mathbb{R}^{2n},$$

satisfying, for all $r > R(\bar\varepsilon)$, $\theta \in V_{\bar\varepsilon}$, $\rho > (1-\bar\varepsilon^2)p^{1/2}$, $t \geq 0$ and $\lambda > 0$,

$$(\underline{\rho}^t,\underline{\eta}^t)(r,\theta,\lambda\rho,\lambda\eta) = \lambda(\underline{\rho}^{\lambda t},\underline{\eta}^{\lambda t})(r,\theta,\rho,\eta), \tag{A-2}$$

$$(\underline{\rho}^t,\underline{\eta}^t)(r,\theta,\rho,0) = (\rho,0), \tag{A-3}$$

and such that

$$(\underline{\rho}^t-\rho)_{t\geq 0} \text{ and (the components of) } (\underline{\eta}^t-\eta)_{t\geq 0} \text{ are bounded in } \mathcal{B}_{\text{hyp}}(\Gamma^+_{\text{s}}(\bar\varepsilon)). \tag{A-4}$$

Then, there exists $0<\tilde\varepsilon\leq\bar\varepsilon$ such that, for all $t\geq 0$ and all $0<\varepsilon\leq\tilde\varepsilon$, Ψ^t is a diffeomorphism from $\Gamma^+_{\text{s}}(\varepsilon)$ onto its range and

$$\Gamma^+_{\text{s}}(\varepsilon^3) \subset \Psi^t\left(\Gamma^+_{\text{s}}(\varepsilon)\right), \qquad t\geq 0,\quad 0<\varepsilon\leq\tilde\varepsilon.$$

Lemma 4.7 is indeed a consequence of Proposition A.1 since Proposition 3.8, (3-35) and (4-15) show that (A-2), (A-3) and (A-4) hold with $(\underline{\rho}^t,\underline{\eta}^t) = (\rho^t,\eta^t)$. Similarly, for Lemma 4.11, we consider

$$(\underline{\rho}^t,\underline{\eta}^t)(r,\theta,\rho,\eta) := (\underline{\rho}_+,\underline{\eta}_+)(r,\theta,r,\theta,\rho,\eta)$$

which is independent of t and satisfies the assumptions (A-2), (A-3), (A-4) by (4-50), Proposition 4.9 and Remark 2 after Proposition 4.9.

To prove the proposition, we need another conical subset of $T^*\mathbb{R}^n_+ \setminus 0$:

$$\tilde\Gamma^+_{\text{s-con}}(\varepsilon) = \left\{ (r,\theta,\rho,\eta) : r > R(\varepsilon),\ \theta \in V_\varepsilon,\ \rho > (1-\varepsilon^2)(\rho^2 + |\eta|^2)^{1/2} \right\}.$$

Using the diffeomorphism F_{hyp} defined by (2-31), we have

$$F^{-1}_{\text{hyp}}\big(\tilde\Gamma^+_{\text{s-con}}(\varepsilon)\big) = \left\{ (r,\theta,\rho,\eta) : r > R(\varepsilon),\ \theta \in V_\varepsilon,\ \rho > (1-\varepsilon^2)(\rho^2 + |e^{-r}\eta|^2)^{1/2} \right\}. \tag{A-5}$$

The latter is of interest in view of the following lemma.

Lemma A.2. *There exists $C>1$ such that, for all $\varepsilon>0$ small enough,*

$$\Gamma^+_{s\text{-}con}(\varepsilon/C) \subset F^{-1}_{\text{hyp}}\big(\tilde\Gamma^+_{s\text{-}con}(\varepsilon)\big) \subset \Gamma^+_{s\text{-}con}(C\varepsilon).$$

Proof. By (3-7), we have, for some $0 < c < 1$,

$$ce^{-2r}|\eta|^2 \le q(r,\theta,e^{-r}\eta) \le c^{-1}|e^{-r}\eta|^2, \qquad r > R(\varepsilon),\ \theta \in V_\varepsilon,\ \eta \in \mathbb{R}^{n-1}.$$

Using (2-55), it suffices to show the existence of $C > 1$ satisfying, for all ε small enough,

$$c^{-1}(1-(\varepsilon/C)^2)^{-2}\left(1-(1-(\varepsilon/C)^2)^2\right) \le (1-\varepsilon^2)^{-2}\left(1-(1-\varepsilon^2)^2\right), \tag{A-6}$$

and

$$(1-\varepsilon^2)^{-2}\left(1-(1-\varepsilon^2)^2\right) \le c(1-(C\varepsilon)^2)^{-2}\left(1-(1-(C\varepsilon)^2)^2\right). \tag{A-7}$$

For $\varepsilon \to 0$, the left-hand side of (A-6) is equivalent to $2c^{-1}(\varepsilon/C)^2$ and the right-hand side to $2\varepsilon^2$. Therefore, (A-6) holds if $c^{-1}/C^2 < 1$ and ε is small enough. We get (A-7) similarly. □

Let us now consider $(1,0) = (1,0,\dots,0) \in \mathbb{R}^n \setminus 0$. For all $0 < \varepsilon < 1$, let us denote by $\mathscr{C}^+(\varepsilon)$ the cone generated by $B((1,0),\varepsilon)$, namely

$$\mathscr{C}^+(\varepsilon) = \{(\lambda\rho,\lambda\eta) \mid \lambda > 0,\ (\rho-1)^2 + |\eta|^2 < \varepsilon^2\}.$$

Since $\rho > 1-\varepsilon > 0$ and $\rho^2/(\rho^2+|\eta|^2) > 1-\varepsilon^2/(1-\varepsilon)^2$ on $B((1,0),\varepsilon)$, it is then not hard to check that, for all ε small enough,

$$\mathscr{C}^+(\varepsilon^2/4) \subset \{\rho > (1-\varepsilon^2)(\rho^2+|\eta|^2)^{1/2}\},$$

and

$$\{\rho > (1-\varepsilon^2)(\rho^2+|\eta|^2)^{1/2}\} \subset \mathscr{C}^+(2\varepsilon),$$

since, if $\rho > (1-\varepsilon^2)(\rho^2+|\eta|^2)^{1/2}$ then $(1,\eta/\rho) \in B((1,0),2\varepsilon)$, using that $1-(1-\varepsilon^2)^2 < 4\varepsilon^2(1-\varepsilon^2)^2$ for ε small enough. In particular, we obtain

$$(R(\varepsilon),+\infty) \times V_\varepsilon \times \mathscr{C}^+(\varepsilon^2/4) \subset \tilde{\Gamma}^+_{\text{s-con}}(\varepsilon) \subset (R(\varepsilon),+\infty) \times V_\varepsilon \times \mathscr{C}^+(2\varepsilon). \tag{A-8}$$

We next recall a standard lemma the simple proof of which we omit.

Lemma A.3. *Let $x_0 \in \mathbb{R}^n$, $\varepsilon > 0$ and $f : B(x_0,\varepsilon) \to \mathbb{R}^n$ such that $f(x_0) = x_0$ and $f - \mathrm{id}$ is $\frac{1}{2}$ Lipschitz (meaning that $|f(x) - x + y - f(y)| \le |x-y|/2$ on $B(x_0,\varepsilon)$. Then f is injective on $B(x_0,\varepsilon)$ and*

$$B(x_0,\varepsilon/2) \subset f(B(x_0,\varepsilon)).$$

Proof of Proposition A.1. Set

$$f_{r,\theta,t}(\rho,\xi) = \left(\underline{\rho}^t(r,\theta,\rho,e^r\xi),\ e^{-r}\underline{\eta}^t(r,\theta,\rho,e^r\xi)\right).$$

By Lemma 2.4 and (A-4), we have, for $k + |\beta| = 2$,

$$|\partial_\rho^k \partial_\eta^\beta f_{r,\theta,t}(\rho,\eta)| \lesssim 1, \qquad t \ge 0,\ (r,\theta,\rho,\xi) \in F_{\text{hyp}}(\Gamma_{\text{s}}^+(\bar{\varepsilon})), \tag{A-9}$$

and, by choosing $\bar{\varepsilon}_1$ small enough, we also have

$$r > R(\bar{\varepsilon}),\quad \theta \in V_{\bar{\varepsilon}},\quad (\rho,\xi) \in B((1,0),\bar{\varepsilon}_1) \quad\Longrightarrow\quad (r,\theta,\rho,\xi) \in F_{\text{hyp}}(\Gamma_{\text{s}}^+(\bar{\varepsilon})).$$

By (A-3) $\partial_{\rho,\xi} f_{r,\theta,t}(\rho,0) = \mathrm{Id}_n$, so (A-9) implies that $f_{r,\theta,t} - \mathrm{Id}_n$ is $\frac{1}{2}$-Lipschitz on $B((1,0),2\varepsilon)$ for all ε small enough, all $t \geq 0$, $r > R(\bar{\varepsilon})$, and $\theta \in V_{\tilde{\varepsilon}}$. Therefore, by Lemma A.3,

$$B((1,0),\varepsilon) \subset f_{t,r,\theta}\left(B((1,0),2\varepsilon)\right), \qquad t \geq 0,\ r > R(\tilde{\varepsilon}),\ \theta \in V_{\tilde{\varepsilon}}.$$

Using (A-2), we can replace the balls in the inclusion above by the cones they generate and, using Lemma A.2 with (A-8), we get

$$\Gamma^+_{\text{s-con}}(\varepsilon/2C) \subset \Psi^t\left(\Gamma^+_{\text{s-con}}(2\sqrt{2}C\varepsilon^{1/2})\right), \qquad t \geq 0, \tag{A-10}$$

for all ε small enough, with the $C > 1$ of Lemma A.2. Since $f_{r,\theta,t} - \mathrm{Id}_n$ is $\frac{1}{2}$-Lipschitz on $B((1,0),2\varepsilon)$ for all $t \geq 0$, (A-2) implies that it is also $\frac{1}{2}$-Lipschitz on the cone generated by $B((1,0),2\varepsilon)$ so $f_{r,\theta,t}$ is injective on this cone. Thus, for all ε small enough and $t \geq 0$, Ψ^t is injective on $\Gamma^+_{\text{s-con}}(\varepsilon)$ and is a diffeomorphism onto its range. By (A-10), we have

$$\Gamma^+_{\text{s}}(\varepsilon^3) \subset \Gamma^+_{\text{s-con}}(\varepsilon^3) \subset \Psi^t\left(\Gamma^+_{\text{s-con}}(\varepsilon)\right),$$

for all $t \geq 0$ and all ε small enough, so the proof will be completed by showing that, for all ε small enough and all $t \geq 0$, the following implication holds:

$$(r,\theta,\rho,\eta) = \Psi^t(r,\theta,\rho_1,\eta_1) \in \Gamma^+_{\text{s}}(\varepsilon^3) \ \text{ with } \ (r,\theta,\rho_1,\eta_1) \in \Gamma^+_{\text{s-con}}(\varepsilon)$$
$$\Longrightarrow \quad p(r,\theta,\rho_1,\eta_1) \in \left(\tfrac{1}{4} - \varepsilon, 4 + \varepsilon\right). \tag{A-11}$$

Assume the first line of (A-11). Using (A-3) at $(\rho_1, 0)$ and the fact that $f_{t,r,\theta} - \mathrm{Id}_n$ is $\frac{1}{2}$-Lipschitz, we have

$$|(\rho, e^{-r}\eta) - (\rho_1, e^{-r}\eta_1)| = |f_{t,r,\theta}(\rho_1, e^{-r}\eta_1) - (\rho_1, e^{-r}\eta_1)| \leq |e^{-r}\eta_1|/2. \tag{A-12}$$

Therefore $|e^{-r}\eta - e^{-r}\eta_1| \leq |e^{-r}\eta_1|/2$ and we get $|\eta_1| \leq 2|\eta|$. Since $e^{-r}|\eta| \lesssim \varepsilon^3$, (A-12) shows that $|\rho - \rho_1| + |e^{-r}(\eta - \eta_1)| \lesssim \varepsilon^3$ hence that

$$|p(r,\theta,\rho_1,\eta_1) - p(r,\theta,\rho,\eta)| \lesssim \varepsilon^3.$$

Since $p(r,\theta,\rho,\eta) \in (\frac{1}{4} - \varepsilon^3, 4 + \varepsilon^3)$, the latter yields (A-11) for ε small enough. □

Proof of Proposition 7.7.
Note first that, for all $y \in \Omega$, the map

$$\mathbb{R}^n \ni x \mapsto \nabla_x f(x,y) \in \mathbb{R}^n$$

is a diffeomorphism since, by considering $F(x,y) := S(y)^{-1}\nabla_x f(x,y)$ and using (7-29), (7-31) and (7-30), $x \mapsto F(x,y) - x$ is $\frac{1}{2}$ Lipschitz. For all $y \in \Omega$, we denote by $x_0 = x_0(y)$ the unique solution to

$$\nabla_x f(x_0, y) = 0.$$

Now consider

$$g(x,y) = f(x,y) - f(x_0,y) - \langle \mathrm{Hess}_x[f](x_0,y)(x - x_0), x - x_0\rangle/2,$$

and, for all $s \in [0,1]$,

$$f_s(x,y) = f(x_0,y) + \langle \mathrm{Hess}_x[f](x_0,y)(x - x_0), x - x_0\rangle/2 + s g(x,y).$$

Notice that $f_1 = f$, that $f_0 - f(x_0, y)$ is quadratic with respect to $x - x_0$ and that

$$\nabla_x f_s(x,y) = \left\{ S(y) + s \int_0^1 R(x_0 + t(x-x_0), y)dt + (1-s)R(x_0,y) \right\} (x - x_0),$$

by Taylor's formula and (7-29). By (7-30), there exists $c > 0$ such that $|S(y)X| \geq 2c|X|$, for all $X \in \mathbb{R}^n$ and all $y \in \Omega$ hence (7-31) implies that

$$|\nabla_x f_s(x,y)| \geq c|x - x_0(y)|, \qquad s \in [0,1], \ (x,y) \in \mathbb{R}^n \times \Omega. \tag{A-13}$$

Lemma A.4. *For all $K \Subset \mathbb{R}^n$ and all integer $k \geq 1$, there exists $C > 0$ and $N > 0$ such that, for all $s \in [0,1]$, all $y \in \Omega$ and all u such that*

$$u \in C_0^{2k-1}(K) \cap C^{2k}(\mathbb{R}^n \setminus \{x_0(y)\}), \tag{A-14}$$

$$\partial_x^\alpha u(x_0(y)) = 0, \qquad |\alpha| < 2k, \tag{A-15}$$

$$\partial_x^\alpha u \in L^\infty(\mathbb{R}^n), \qquad |\alpha| = 2k, \tag{A-16}$$

we have

$$\left| \int e^{i\omega f_s(x,y)} u(x)dx \right| \leq C\omega^{-k} \max_{|\alpha| \leq 2k} \|\partial^\alpha u\|_{L^\infty(K)} \max_{2 \leq |\alpha| \leq 2k} \Big(1 + \sup_{x \in K} |\partial^\alpha f_s|\Big)^N, \qquad \omega \geq 1.$$

Notice that the assumption (A-16) is only a condition near $x_0(y)$. It guarantees the boundedness of $\partial^\alpha u(x)/|x - x_0|^{2k-|\alpha|}$.

Proof. We proceed by induction and consider first $k = 1$. We would like to integrate by part using the operator $|\nabla_x f_s|^{-2} \nabla_x f_s \cdot \nabla_x$ but, since $\nabla_x f_s$ may vanish on the support of u, we consider $L_\delta := (|\nabla_x f_s|^2 + \delta)^{-1} \nabla_x f_s \cdot \nabla_x$ which satisfies

$$i\omega \int e^{i\omega f_s(x,y)} u(x)dx = \lim_{\delta \downarrow 0} \int (L_\delta e^{i\omega f_s(x,y)}) u(x)dx.$$

We then integrate by part at fixed $\delta > 0$, using that

$${}^t L_\delta = -\frac{1}{|\nabla_x f_s|^2 + \delta} \left\{ \nabla_x f_s \cdot \nabla_x + \Delta_x f_s - \frac{2}{|\nabla_x f_s|^2 + \delta} \langle \mathrm{Hess}_x[f_s] \nabla_x f_s, \nabla_x f_s \rangle \right\}.$$

Since $|\Delta_x f_s(x,y)u(x)| \lesssim \max_{|\alpha|=2} \|\Delta_x f_s(\,\cdot\,, y)\|_{L^\infty(K)} \|\partial^\alpha u\|_{L^\infty} |x - x_0(y)|^2$ and using (A-13), by letting $\delta \downarrow 0$ we get

$$\left| i\omega \int e^{i\omega f_s(x,y)} u(x)dx \right| \leq C \max_{|\alpha| \leq 2} \|\partial^\alpha u\|_{L^\infty(K)} \max_{|\alpha|=2} \Big(1 + \sup_{x \in \mathbb{R}^n} |\partial^\alpha f_s|\Big).$$

Here the constant C is independent of y, u, s and ω; it depends only on K and the constant c in (A-13). The result then follows by induction using that

$$|\nabla_x f_s|^{-2} \langle \nabla_x f_s, \partial_x u \rangle, \qquad |\nabla_x f_s|^{-2} (\Delta_x f_s) u, \qquad |\nabla_x f_s|^{-4} \langle \mathrm{Hess}_x[f_s] \nabla_x f_s, \nabla_x f_s \rangle \, u$$

satisfy the assumptions (A-14), (A-15) and (A-16) if u does for $k + 1$. ☐

End of the proof of Proposition 7.7. We next consider $I(s) = \int e^{i\omega f_s(x,y)} u(x)dx$ so that, for all $j \in \mathbb{N}_0$, we have

$$I^{(2j)}(s) = (i\omega)^{2j} \int e^{i\omega f_s(x,y)} g(x,y)^{2j} u(x)dx.$$

Since $\partial_x^\alpha \big(g(x,y)^{2j}\big)_{|x=x_0(y)} = 0$ for all $|\alpha| < 6j$, Lemma A.4 yields, with $k = 3j \geq n/2$,

$$|I^{(2j)}(s)| \leq C\omega^{-n/2} \max_{|\alpha|\leq 6j} \|\partial^\alpha u\|_{L^\infty(K)} \max_{2\leq|\alpha|\leq 6j} \big(1 + \sup_{x\in\mathbb{R}^n} |\partial^\alpha f_s|\big)^N, \qquad s \in [0,1].$$

Since $I(1) = \int e^{i\omega f(x,y)} u(x)dx$, the estimate

$$|I(1) - \sum_{l<2j} I^{(l)}(0)/l!| \leq \sup_{s\in[0,1]} |I^{(2j)}(s)|/(2j)!,$$

reduces the proof to estimating the integrals $I^{(l)}(0)$ whose common phase f_0 is quadratic, up to a constant term and whose amplitude is $u(x)g(x,y)^l$. By Taylor's formula $g(x,y)$ is of order $|x - x_0(y)|^2$ so the derivatives of $u(x)g(x,y)^l$ may be of order $\langle x_0(y)\rangle^{2l}$ on which we have no control. By choosing $\tilde{K}$ a bounded neighborhood of K and applying Lemma A.4 to the subset of Ω on which $x_0(y) \notin \tilde{K}$, we can assume that we consider those y for which $x_0(y) \in \tilde{K}$. We then use the Lemma 7.7.3 of [Hörmander 1983] on oscillatory integrals with quadratic phases, observing that $\|\partial_x^\alpha g(\cdot, y)^l\|_{L^\infty(K_x)}$ is controlled by (products of) of norms $\|\partial_x^\beta f(\cdot, y)\|_{L^\infty(K_x)}$ with $|\beta| \geq 2$, since x is bounded on the support of u and $x_0(y)$ remains bounded. □

Acknowledgments

It is a pleasure to thank Nikolay Tzvetkov for helpful and stimulating discussions about this paper. The author would also like to emphasize the high quality of the referring process and thanks sincerely the anonymous referees for their suggestions and for the very careful reading of the first version of this paper.

References

[Anker and Pierfelice 2009] J.-P. Anker and V. Pierfelice, "Nonlinear Schrödinger equation on real hyperbolic spaces", *Ann. Inst. H. Poincaré Anal. Non Linéaire* **26**:5 (2009), 1853–1869. MR 2010m:35416

[Banica 2007] V. Banica, "The nonlinear Schrödinger equation on hyperbolic space", *Comm. Partial Differential Equations* **32**:10-12 (2007), 1643–1677. MR 2008m:35323

[Banica and Duyckaerts 2007] V. Banica and T. Duyckaerts, "Weighted Strichartz estimates for radial Schrödinger equation on noncompact manifolds", *Dyn. Partial Differ. Equ.* **4**:4 (2007), 335–359. MR 2008j:58036

[Banica et al. 2008] V. Banica, R. Carles, and G. Staffilani, "Scattering theory for radial nonlinear Schrödinger equations on hyperbolic space", *Geom. Funct. Anal.* **18**:2 (2008), 367–399. MR 2010h:35364

[Bouclet 2006] J.-M. Bouclet, "Resolvent estimates for the Laplacian on asymptotically hyperbolic manifolds", *Ann. Henri Poincaré* **7**:3 (2006), 527–561. MR 2007k:58050 Zbl 1155.58012

[Bouclet 2007] J.-M. Bouclet, "Semi-classical functional calculus on manifolds with ends and weighted L^p estimates", 2007. to appear in *Ann. Inst. Fourier*. arXiv 0711.3583

[Bouclet 2010] J.-M. Bouclet, "Littlewood–Paley decompositions on manifolds with ends", *Bull. Soc. Math. France* **138**:1 (2010), 1–37. MR 2638890 Zbl 1198.42013

[Bouclet and Tzvetkov 2007] J.-M. Bouclet and N. Tzvetkov, "Strichartz estimates for long range perturbations", *Amer. J. Math.* **129**:6 (2007), 1565–1609. MR 2009d:58035

[Bouclet and Tzvetkov 2008] J.-M. Bouclet and N. Tzvetkov, "On global Strichartz estimates for non-trapping metrics", *J. Funct. Anal.* **254**:6 (2008), 1661–1682. MR 2009d:35039 Zbl 1168.35005

[Bourgain 1993] J. Bourgain, "Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. I. Schrödinger equations", *Geom. Funct. Anal.* **3**:2 (1993), 107–156. MR 95d:35160a

[Burq et al. 2004] N. Burq, P. Gérard, and N. Tzvetkov, "Strichartz inequalities and the nonlinear Schrödinger equation on compact manifolds", *Amer. J. Math.* **126**:3 (2004), 569–605. MR 2005h:58036

[Cardoso and Vodev 2002] F. Cardoso and G. Vodev, "Uniform estimates of the resolvent of the Laplace–Beltrami operator on infinite volume Riemannian manifolds. II", *Ann. Henri Poincaré* **3**:4 (2002), 673–691. MR 2003j:58054 Zbl 1021.58016

[Cazenave 2003] T. Cazenave, *Semilinear Schrödinger equations*, Courant Lecture Notes in Math. **10**, Courant Institute (NYU), New York, 2003. MR 2004j:35266

[Doi 1996] S.-i. Doi, "Smoothing effects of Schrödinger evolution groups on Riemannian manifolds", *Duke Math. J.* **82**:3 (1996), 679–706. MR 97f:58141

[Froese and Hislop 1989] R. Froese and P. Hislop, "Spectral analysis of second-order elliptic operators on noncompact manifolds", *Duke Math. J.* **58**:1 (1989), 103–129. MR 90k:58234 Zbl 0687.35060

[Ginibre and Velo 1985] J. Ginibre and G. Velo, "The global Cauchy problem for the nonlinear Schrödinger equation revisited", *Ann. Inst. H. Poincaré Anal. Non Linéaire* **2**:4 (1985), 309–327. MR 87b:35150

[Hassell et al. 2006] A. Hassell, T. Tao, and J. Wunsch, "Sharp Strichartz estimates on nontrapping asymptotically conic manifolds", *Amer. J. Math.* **128**:4 (2006), 963–1024. MR 2007d:58053 Zbl 1177.58019

[Helffer and Robert 1983] B. Helffer and D. Robert, "Calcul fonctionnel par la transformation de Mellin et opérateurs admissibles", *J. Funct. Anal.* **53**:3 (1983), 246–268. MR 85i:47052 Zbl 0524.35103

[Hörmander 1983] L. Hörmander, *The analysis of linear partial differential operators, I*, Springer, Berlin, 1983.

[Isozaki and Kitada 1985] H. Isozaki and H. Kitada, "Modified wave operators with time-independent modifiers", *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **32**:1 (1985), 77–104. MR 86j:35125 Zbl 0582.35036

[Keel and Tao 1998] M. Keel and T. Tao, "Endpoint Strichartz estimates", *Amer. J. Math.* **120**:5 (1998), 955–980. MR 2000d:35018 Zbl 0922.35028

[Mazzeo and Melrose 1987] R. R. Mazzeo and R. B. Melrose, "Meromorphic extension of the resolvent on complete spaces with asymptotically constant negative curvature", *J. Funct. Anal.* **75**:2 (1987), 260–310. MR 89c:58133 Zbl 0636.58034

[Melrose 1995] R. B. Melrose, *Geometric scattering theory*, Cambridge University Press, Cambridge, 1995. MR 96k:35129 Zbl 0849.58071

[Pierfelice 2006] V. Pierfelice, "Weighted Strichartz estimates for the radial perturbed Schrödinger equation on the hyperbolic space", *Manuscripta Math.* **120**:4 (2006), 377–389. MR 2008i:35195

[Pierfelice 2008] V. Pierfelice, "Weighted Strichartz estimates for the Schrödinger and wave equations on Damek-Ricci spaces", *Math. Z.* **260**:2 (2008), 377–392. MR 2009d:35272

[Robbiano and Zuily 2005] L. Robbiano and C. Zuily, *Strichartz estimates for Schrödinger equations with variable coefficients*, Mém. Soc. Math. Fr. (N.S.) **101-102**, Soc. Mat. de France, Paris, 2005. MR 2006i:35047

[Robert 1987] D. Robert, *Autour de l'approximation semi-classique*, Progress in Mathematics **68**, Birkhäuser, Boston, 1987. MR 89g:81016 Zbl 0621.35001

[Staffilani and Tataru 2002] G. Staffilani and D. Tataru, "Strichartz estimates for a Schrödinger operator with nonsmooth coefficients", *Comm. Partial Differential Equations* **27**:7-8 (2002), 1337–1372. MR 2003f:35248

[Strichartz 1977] R. S. Strichartz, "Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations", *Duke Math. J.* **44**:3 (1977), 705–714. MR 58 #23577 Zbl 0372.35001

[Tataru 2001] D. Tataru, "Strichartz estimates in the hyperbolic space and global existence for the semilinear wave equation", *Trans. Amer. Math. Soc.* **353**:2 (2001), 795–807. MR 2001k:35218 Zbl 0956.35088

Received 14 Mar 2008. Revised 4 Aug 2009. Accepted 3 Dec 2009.

JEAN-MARC BOUCLET: Jean-Marc.Bouclet@math.univ-lille1.fr
UMR 8524 CNRS, UFR de Mathématiques, Université de Lille 1, 59655 Villeneuve d'Ascq, France

ANALYSIS AND PDE
Vol. 4, No. 1, 2011

msp

ASYMPTOTIC BEHAVIORS OF NONVARIATIONAL ELLIPTIC SYSTEMS

SZU-YU SOPHIE CHEN

We use a method, inspired by Pohozaev's work, to study asymptotic behaviors of nonvariational elliptic systems in dimension $n \geq 3$. As an application, we prove removal of an apparent singularity in a ball and uniqueness of the entire solution. All results apply to changing sign solutions.

In this paper, we study solutions of elliptic systems on $\mathbb{R}^n$, $n \geq 3$.

A classical work by Gidas and Spruck [1981] asserts that any nonnegative solution to $\Delta u + |u|^{\alpha-2}u = 0$ in $\mathbb{R}^n$ with $2 < \alpha < 2n/(n-2)$ (subcritical case) is trivial. For $\alpha = 2n/(n-2)$, Caffarelli, Gidas and Spruck [1989] proved that any nonnegative solution in $\mathbb{R}^n$ is of the form $u = (a + b|x|^2)^{-(n-2)/2}$, where a, b are constants. Such problem for elliptic systems are also studied, for example, in the studies of Lane–Emden type systems; see [Zou 2000; Poláčik etal. 2007; Souplet 2009] and the references therein.

By contrast, the behaviors of changing sign solutions are more delicate. For example, there exists a sequence of changing sign solutions to $\Delta u + |u|^{\alpha-2}u = 0$ in $\mathbb{R}^n$ with $2 < \alpha < 2n/(n-2)$ [Kuzin and Pohozaev 1997]. In this paper, we study under what circumstances a solution to an elliptic system in an exterior domain is asymptotic to $|x|^{-(n-2)}$ at infinity. Such decay is optimal in the sense that infinity is a regular point in the inverted coordinates. It is known [Kuzin and Pohozaev 1997] that there exist solutions to $\Delta u + u^{\alpha-1} = 0$ in $\mathbb{R}^n$ that decay more slowly than $|x|^{-(n-2)}$. Thus, a suitable integrability condition is necessary to exclude such a case.

While the study of changing sign solutions to elliptic systems is interesting by itself, the problem is well motivated by differential geometry. For example, the decay of curvature tensors was studied for Yang–Mills fields [Uhlenbeck 1982], Einstein metrics [Bando et al. 1989] and other generalizations [Tian and Viaclovsky 2005; Chen 2009], just to name a few. A typical system is of the form

$$\Delta(\mathrm{Rm})_{ijkl} = Q_{ijkl}(\mathrm{Rm}, \mathrm{Rm}),$$

where Rm is the Riemannian curvature tensor and Q is a quadratic in Rm. A natural geometric assumption is that $|\,\mathrm{Rm}\,|$ is in $L^{n/2}$. Therefore, $|\,\mathrm{Rm}\,|$ vanishes at infinity and the problem is to find out the decay rate. The study of geometrical systems is more subtle as $(\mathrm{Rm})_{ijkl}$ satisfies an extra relation, the Bianchi identity, and the underlying spaces are not Euclidean.

The technique we use in this paper is based on the method developed in [Chen 2009] on asymptotically flat manifolds, where a special geometric setting is considered. In this paper, we study general nonvariational elliptic systems of the reaction-diffusion type. Our result applies to changing sign solutions and includes the supercritical case (i.e., $\Delta u + Cu^{\alpha-1} = 0$ with $\alpha > 2n/(n-2)$, where C is a constant).

The author was supported by the Miller Institute for Basic Research in Science, and while preparing the manuscript, by NSF grant DMS-0635607.
MSC2000: 35B40, 35J45.
Keywords: elliptic system, decay estimates, Liouville.

Let $V = (V_1, \dots, V_m)$ and $f^i : \mathbb{R}^m \to \mathbb{R}$. Consider the system of equations

$$\sum_{j=1}^{m} A_{ij} \Delta V_j = f^i(V), \tag{1}$$

where A is a constant invertible symmetric matrix and $i = 1, \dots, m$. Note that $f^i(V)$ or V_i may have no sign. We assume the following structure conditions:

(A1) $|f^i(V)| \le C|V|^q$.

(A2) $|\nabla f^i(V)| \le C|V|^{q-1}$.

Let K be a compact subset in $\mathbb{R}^n$.

Theorem 1. *Let $q > (n+2)/n$ and $p = (n/2)(q-1)$. Suppose that f^i satisfies (A1) and (A2). Let $V \in L^p(\mathbb{R}^n \setminus K)$ be a solution to (1) in $\mathbb{R}^n \setminus K$. Then $|V| = O(|x|^{-(n-2)})$ and $|\nabla V| = O(|x|^{-(n-1)})$ at infinity.*

An immediate consequence is a result on singularity removal for affine invariant equations. For scalar equations, the problem was studied in [Gidas and Spruck 1981; Brézis and Lions 1981; Caffarelli et al. 1989].

Let B_1 be the unit ball centered at the origin.

Corollary 2. *Suppose f^i are homogeneous functions of degree $(n+2)/(n-2)$. Let $V \in L^{2n/(n-2)}(B_1)$ be a solution to (1) in $B_1 \setminus \{0\}$. Then V can be extended to a smooth solution to (1) in B_1.*

By performing a linear transformation $W_i = \sum_j A_{ij} V_j$, the system (1) can be reduced to an equation of the diagonal form $\Delta W = \tilde{f}(W)$. The assumptions (A1)–(A2) and other conditions on V or f^i equivalently hold for W and $\tilde{f}$. Therefore, for Theorem 1 and Corollary 2, we may assume without loss of generality the equation is of the diagonal form.

We turn to study the uniqueness of entire solutions for variational systems. Let $P(V)$ be a homogeneous function of degree $q+1$. Suppose that A_{ij} is positive definite and $f^i = \partial P/\partial V^i$ in (1). For scalar equations, there is a large literature on the uniqueness problem; see, for example, [Gidas and Spruck 1981; Bidaut-Véron 1989; Serrin and Zou 2002]; see also [Pucci and Serrin 2007] and the references therein. For systems, when $P(V) \le 0$ and $q > (n+2)/(n-2)$ (supercritical case), the problem was studied by Pucci and Serrin [1986] under some asymptotic assumption of V. Their result also holds for the nonhomogeneous function P (and more general $P(x, V, \nabla V)$) satisfying some inequality.

Theorem 3. *Let $q > (n+2)/n$, $q \ne (n+2)/(n-2)$ and $p = (n/2)(q-1)$. Suppose $P(V)$ is a homogeneous function of degree $q+1$. Suppose that A_{ij} is positive definite and $f^i = \partial P/\partial V^i$ in (1). Let $V \in L^p(\mathbb{R}^n)$ be a solution to (1) in $\mathbb{R}^n$. Then $V \equiv 0$.*

We outline the proofs. To fix notation, we denote by dx the volume element in $\mathbb{R}^n$ and by dS the area element of a hypersurface in $\mathbb{R}^n$. Let $B_r(x)$ and $S_r(x)$ be the ball of radius r and sphere of radius r centered at x, respectively. When x is at the origin, we simply denote by B_r and S_r.

The idea of the proof of Theorem 1 is to compare the size of $\int_{\mathbb{R}^n \setminus B_r} |\nabla V|^2 dx$ (as a function of r) to its derivative $-\int_{S_r} |\nabla V|^2 dS$. Then by the ordinary differential inequality lemma, we get the optimal decay of $|\nabla V|$ and, as a consequence, the decay of $|V|$. In order to relate the two integrands, we use some version of Pohozaev's identity for nonvariational systems. Pohozaev's ingenious idea [1965] is to use a conformal Killing field to prove uniqueness in a star-shaped domain. This idea was generalized nicely by Pucci and Serrin [1986] to general variational systems. Our use of the identity is different from the original one. We apply the identity to an unbounded domain (the complement of a large ball) and use only the size of $|f^i|$. Therefore, our method can be applied to nonvariational systems.

The proof of Theorem 3 is a combination of Theorem 1 and Pohozaev's original idea. Since the solution decays fast enough at infinity, no terms from infinity contribute to the main integrand. We use the identity differently such that we obtain the uniqueness also in the subcritical case, in contrast to the problem in star-shaped regions where one has to restrict to the supercritical case.

Finally, we show that the assumptions in these theorems are sharp.

Example 4. Consider the equation $\Delta u + u^{(n+2)/(n-2)} = 0$ in $\mathbb{R}^n$. By [Caffarelli et al. 1989], nonnegative solutions are of the form $u = (a + b|x|^2)^{-(n-2)/2}$. Therefore, u decays as $|x|^{-(n-2)}$ at infinity. This example shows that in Theorem 3, the assumption $q \neq (n+2)/(n-2)$ is necessary. Consider instead the equation in $B_1 \setminus \{0\}$. There exists a nonnegative radial singular solution with the blow-up rate $|x|^{-(n-2)/2}$ near the origin. Therefore, in Corollary 2, the condition $V \in L^{2n/(n-2)}(B_1)$ is sharp.

Example 5. Consider $\Delta u + u^q = 0$ in $\mathbb{R}^n$. For $q > (n+2)/(n-2)$, there exists a solution asymptotic to $|x|^{-2/(q-1)}$ at infinity [Kuzin and Pohozaev 1997]. Hence, in Theorem 1, the conditions $q = (2p+n)/n$ and $V \in L^p$ are sharp. Moreover, in Theorem 3, the condition $q = (2p+n)/n$ is also sharp.

1. Preliminaries

We collect some standard results in elliptic regularity theory and ordinary differential equations. Lemmas 6–8 follow by an argument similar to [Bando et al. 1989, Section 4].

Let C_s be the Sobolev constant and $\gamma = n/(n-2)$. Suppose that the nonnegative function $u \in C^{0,1}$ satisfies $\Delta u + C_0 u^q \geq 0$ weakly in the sense that

$$\int (-\langle \nabla u, \nabla \phi \rangle + C_0 u^q \phi) dx \geq 0 \quad \text{for all } 0 \leq \phi \in C_o^\infty.$$

Let $\varphi \geq 0$ be a function with compact support and let $s > 1$. Then, by the Cauchy inequality,

$$\begin{aligned} \int \varphi^2 u^{q+s-1}\, dx &\geq C_0^{-1} \int \Big(\frac{4(s-1)}{s^2} |\varphi \nabla u^{s/2}|^2 + \frac{4}{s} \varphi u^{s/2} \langle \nabla \varphi, \nabla u^{s/2} \rangle \Big)\, dx \\ &\geq C_0^{-1} \int \Big(\frac{2}{s^2}(s-1) |\varphi \nabla u^{s/2}|^2 - \frac{2}{(s-1)} |\nabla \varphi|^2 u^s \Big)\, dx. \end{aligned}$$

By the Sobolev inequality, we have

$$\Big(\int (\varphi^2 u^s)^\gamma\, dx \Big)^{1/\gamma} \leq C \int \Big(\frac{s^2 C_0}{2(s-1)} \varphi^2 u^{q+s-1} + \Big(1 + \frac{s^2}{(s-1)^2}\Big) |\nabla \varphi|^2 u^s \Big)\, dx, \tag{2}$$

where $C = C(n, C_s, C_0)$.

In Lemmas 6–8, u is a $C^{0,1}$ function.

Lemma 6. *Let $p > 1$ and $q = (2p+n)/n$. Suppose that the nonnegative function $u \in L^p(B_r)$ satisfies $\Delta u + C_0 u^q \geq 0$ weakly in B_r. Then there exists $\epsilon > 0$ such that if $\int_{B_r} u^p dx < \epsilon$, then*

$$\sup_{B_{r/2}} u \leq C r^{-n/p} \|u\|_{L^p(B_r)}, \qquad \text{where } C = C(n, p, C_s, C_0).$$

Proof. Let $s = p$ in (2). Then

$$\begin{aligned}\Big(\int (\varphi^2 u^p)^\gamma \, dx\Big)^{1/\gamma} &\leq C \int (u^{q-1}(\varphi^2 u^p) + |\nabla\varphi|^2 u^p) \, dx \\ &\leq C\Big(\int_{\{\text{supp } \varphi\}} u^p \, dx\Big)^{2/n} \Big(\int (\varphi^2 u^p)^\gamma \, dx\Big)^{1/\gamma} + C \int |\nabla\varphi|^2 u^p \, dx.\end{aligned}$$

We choose φ to be a cutoff function such that $\varphi = 1$ in $B_{r/2}$ and $\varphi = 0$ outside B_r, with $|\nabla\varphi| \leq C r^{-1}$. We get

$$\Big(\int_{B_{r/2}} u^{p\gamma} dx\Big)^{1/\gamma} \leq \frac{C}{r^2} \int_{B_r} u^p \, dx.$$

Choose a sequence $r_k = (2^{-1} + 2^{-k})r$. Apply (and rescale) the above inequality for B_{r_k} and $B_{r_{k+1}}$ with $p_k = p\gamma^{k-1}$. By Moser iteration, we have $\sup_{B_{r/2}} u \leq C r^{-n/p} \|u\|_{L^p(B_r)}$. □

Lemma 7. *Let $p > n/(n-2)$ and $q = (2p+n)/n$. Suppose that the nonnegative function $u \in L^p(\mathbb{R}^n \setminus B_r)$ satisfies $\Delta u + C_0 u^q \geq 0$ weakly in $\mathbb{R}^n \setminus B_r$. Then there exists $\epsilon > 0$ such that if $\int_{\mathbb{R}^n \setminus B_r} u^p < \epsilon$, then $u = O(|x|^{-\lambda})$ for all $\lambda < n-2$ as $|x| \to \infty$.*

Proof. By Lemma 6, $u = O(|x|^{-n/p})$. Let $s = p((n-2)/n) > 1$ in (2). Then

$$\Big(\int \varphi^{2\gamma} u^p \, dx\Big)^{1/\gamma} \leq C\Big(\int_{\{\text{supp } \varphi\}} u^p \, dx\Big)^{2/n} \Big(\int (\varphi^2 u^{p(n-2/n)})^\gamma \, dx\Big)^{1/\gamma} + C \int |\nabla\varphi|^2 u^{p(n-2)/n} \, dx.$$

φ is chosen to be a cutoff function such that $\varphi = 1$ in $B_{r'} \setminus B_{2r}$ and $\varphi = 0$ outside $B_{2r'} \setminus B_r$ with $|\nabla\varphi| \leq C(1/r + 1/r')$. Let $r' \to \infty$. Then

$$\Big(\int \varphi^{2\gamma} u^p \, dx\Big)^{1/\gamma} \leq C\Big(\int |\nabla\varphi|^n \, dx\Big)^{2/n} \Big(\int_{\{\text{supp } \nabla\varphi\}} u^p \, dx\Big)^{1/\gamma}.$$

And thus,

$$\Big(\int_{\mathbb{R}^n \setminus B_{2r}} u^p \, dx\Big)^{1/\gamma} \leq C\Big(\int_{B_{2r} \setminus B_r} u^p \, dx\Big)^{1/\gamma}.$$

This gives $\int_{\mathbb{R}^n \setminus B_r} u^p = O(r^{-\delta})$ for some small $\delta > 0$. Therefore, by Lemma 6, $u = O(|x|^{-(n/p)-(\delta/p)})$. Let $\lambda_0 = \sup\{\lambda : u = O(|x|^{-\lambda})\}$. By iteration and a contradiction argument, we get that $\lambda_0 = n-2$. □

Suppose that $h \geq 0$ is a C^0 function. The nonnegative function $u \in C^{0,1}$ satisfies $\Delta u + C_0 h u \geq 0$ weakly if

$$\int \big(-\langle \nabla u, \nabla\phi\rangle + C_0 h u \phi\big) \, dx \geq 0 \qquad \text{for all } 0 \leq \phi \in C_o^\infty.$$

Lemma 8. *Let $p > 1$ and $t > n/2$. Suppose that the nonnegative function $h \in L^t(B_r)$ satisfies $\int_{B_r} h^t dx \le C_1/r^{2t-n}$. Suppose also that the nonnegative function $u \in L^p(B_r)$ satisfies $\Delta u + C_0 hu \ge 0$ weakly in B_r. Then $\sup_{B_{r/2}} u \le Cr^{-n/p}\|u\|_{L^p(B_r)}$, where $C = C(n, p, C_s, C_0, C_1)$.*

Proof. The proof is by standard Moser iteration. See Morrey [1966]. □

The following is a basic result in ordinary differential equations [Chen 2009].

Lemma 9. *Suppose that $f(r) \ge 0$ satisfies $f(r) \le -(r/a)f'(r) + C_2 r^{-b}$ for some $a, b > 0$.*

(i) *$a \ne b$. Then there exists a constant C_3 such that*

$$f(r) \le C_3 r^{-a} + \frac{a\,C_2}{a-b} r^{-b}.$$

Therefore, $f(r) = O(r^{-\min\{a,b\}})$ as $r \to \infty$.

(ii) *$a = b$. Then there exists a constant C_3 such that*

$$f(r) \le C_3 r^{-a} + a\,C_2 r^{-a} \ln r.$$

Therefore, $f(r) = O(r^{-a} \ln r)$ as $r \to \infty$.

2. Proof of Theorem 1

As we explained in the introduction, without loss of generality we may assume the equation is of the diagonal form, that is,

$$\Delta V_i = f^i(V). \tag{3}$$

We first derive a version of Pohozaev's identity for nonvariational systems. Let Ω be a domain in $\mathbb{R}^n$ and N be the unit outer normal on $\partial\Omega$. We perform integration by parts repeatedly.

$$\begin{aligned}
&\int_\Omega \sum_{k,l} f^k(V) x_l D_l V_k\, dx \\
&\qquad = \int_\Omega \sum_{j,l} \Delta V_j x_l D_l V_j\, dx \\
&\qquad = \int_\Omega -\sum_{i,j,l} D_i V_j D_i(x_l D_l V_j)\, dx + \int_{\partial\Omega} \sum_{i,j,l} D_i V_j x_l D_l V_j N_i\, dS \\
&\qquad = \int_\Omega \Big(-|\nabla V|^2 - \sum_l D_l(|\nabla V|^2)\frac{x_l}{2}\Big) dx + \int_{\partial\Omega} \sum_{i,j,l} D_i V_j x_l D_l V_j N_i\, dS \\
&\qquad = \Big(\frac{n}{2} - 1\Big) \int_\Omega |\nabla V|^2 dx - \int_{\partial\Omega} \frac{1}{2} \sum_l |\nabla V|^2 x_l N_l\, dS + \int_{\partial\Omega} \sum_{i,j,l} D_i V_j x_l D_l V_j N_i\, dS. \qquad (4)
\end{aligned}$$

It is worth mentioning that $x_l D_l$ is a conformal Killing field in $\mathbb{R}^n$.

We note that $|V|$ and $|\nabla V|$ are $C^{0,1}$ functions. By (3) and (A1)–(A2), we have

$$\Delta |V| \ge -C|V|^q,$$
$$\Delta |\nabla V| \ge -C|V|^{q-1}|\nabla V|,$$

weakly. Since $V \in L^p(\mathbb{R}^n \setminus K)$, there exists a large number R such that $\int_{\mathbb{R}^n \setminus B_R} |V|^p dx < \epsilon$, where ϵ is as in Lemma 6. Applying Lemma 6 to $B_r(x_0)$ where $|x_0| \geq 2r \geq 2R$, we get $|V| = O(|x|^{-n/p})$.

Case 1. If $(n+2)/n < q \leq n/(n-2)$ (or equivalently, $1 < p \leq n/(n-2)$), then $n/p \geq n-2$. By Lemma 6, we have $|V| = O(|x|^{-n/p})$. Let φ be a cutoff function such that $\varphi = 1$ in B_r and $\varphi = 0$ outside B_{2r} with $|\nabla\varphi| \leq Cr^{-1}$. Applying φV_i to (3) and integrating gives

$$\int_{B_r(x_0)} |\nabla V|^2 \, dx \leq C \int_{B_{2r}(x_0)} |V|^{q+1} \, dx + \frac{C}{r^2} \int_{B_{2r}(x_0)} |V|^2 \, dx = O(r^{n-2-(2n/p)}) \leq O(r^{-n+2}),$$

where $|x_0| \geq 2r \gg 1$. By Lemma 8 with $h = |V|^{q-1}$, we obtain $|\nabla V| = O(|x|^{-(n-1)})$ and thus $|V| = O(|x|^{-(n-2)})$.

Case 2. If $n/(n-2) < q$ (or equivalently $p > n/(n-2)$), by Lemma 7, $|V| = O(|x|^{-\lambda})$ for all $\lambda < n-2$. Therefore,

$$\int_{B_r(x_0)} |\nabla V|^2 dx \leq C \int_{B_{2r}(x_0)} |V|^{q+1} dx + \frac{C}{r^2} \int_{B_{2r}(x_0)} |V|^2 dx = O(r^{n-2-2\lambda}),$$

where $|x_0| \geq 2r \gg 1$. Moreover, $|V| \in L^{p'}$ for all $p' > n/(n-2)$. Choose $p' < p$ close to $n/(n-2)$. Hence, $q > (2p'+n)/n$. We can then find $q' > n/2$ such that

$$\int_{B_r(x_0)} (|V|^{q-1})^{q'} dx \leq \frac{C}{r^{2q'-n}}, \qquad \text{where } |x_0| \geq 2r \gg 1.$$

This is possible because λ is close to $n-2$. By Lemma 8, we obtain

$$\sup_{B_{r/2}(x_0)} |\nabla V| \leq \frac{C}{r^{n/2}} \|\nabla V\|_{L^2(B_r(x_0))} = O(r^{-\lambda-1}), \qquad \text{where } |x_0| \geq 2r \gg 1.$$

Let $\Omega = B_R \setminus B_r$ in (4). We have

$$\int_\Omega \sum_{k,l} f^k(V) x_l D_l V_k dx \\ = \left(\frac{n}{2} - 1\right) \int_\Omega |\nabla V|^2 dx - \int_{\partial\Omega} \frac{1}{2} \sum_l |\nabla V|^2 x_l N_l \, dS + \int_{\partial\Omega} \sum_{i,j,l} D_i V_j x_l D_l V_j N_i \, dS. \quad (5)$$

Note that

$$\lim_{R\to\infty} \int_{S_R} R|\nabla V|^2 dS = \lim_{R\to\infty} O(R^{-2\lambda-2+n}) = 0.$$

Let $R \to \infty$ in (5). Then there is no boundary term coming from infinity. We can choose $\Omega = \mathbb{R}^n \setminus B_r$. The boundary terms only occur on S_r. On $\partial\Omega$, $N = -x/r$. Hence,

$$\int_{\mathbb{R}^n \setminus B_r} \sum_{k,l} f^k(V) x_l D_l V_k \, dx = \left(\frac{n}{2} - 1\right) \int_{\mathbb{R}^n \setminus B_r} |\nabla V|^2 dx + \int_{S_r} \frac{r}{2} |\nabla V|^2 dS - r \int_{S_r} |\nabla_N V|^2 dS \\ \geq \left(\frac{n}{2} - 1\right) \int_{\mathbb{R}^n \setminus B_r} |\nabla V|^2 dx - \int_{S_r} \frac{r}{2} |\nabla V|^2 dS.$$

Let $G(r) := \int_{\mathbb{R}^n\setminus B_r} |\nabla V|^2\,dx$. Since $G'(r) = -\int_{S_r} |\nabla V|^2\,dS$, the previous formula becomes

$$G(r) \le -\frac{r}{n-2}G'(r) + \frac{2}{n-2}\int_{\mathbb{R}^n\setminus B_r} \sum_{k,l} f^k(V)x_l D_l V_k\,dx.$$

The key idea is to compare the size of $G(r)$ to that of $G'(r)$. The coefficient in front of $G'(r)$ plays an important role. Here is the only place we use the condition of $|f^i|$. We have

$$\int_{\mathbb{R}^n\setminus B_r} \sum_{k,l} f^k(V)x_l D_l V_k\,dx \le \int_{\mathbb{R}^n\setminus B_r} |V|^q|x||\nabla V|\,dx = O(r^{-\lambda(q+1)+n}).$$

Thus,

$$G(r) \le -\frac{r}{n-2}G'(r) + Cr^{-\lambda(q+1)+n}.$$

Since $q > n/(n-2)$ and λ is close to $n-2$, we have $\lambda(q+1) - n > n-2$. By Lemma 9, this implies $G(r) = O(r^{-(n-2)})$. By the Sobolev inequality, we get

$$\int_{B_{2r}\setminus B_r} |V|^{2n/(n-2)}\,dx = O(r^{-n}).$$

Finally, by Lemma 6 and 8 we obtain $|V| = O(|x|^{-(n-2)})$ and $|\nabla V| = O(|x|^{-(n-1)})$. □

3. Proofs of Corollary 2 and Theorem 3

Proof of Corollary 2. Since the equation is invariant under inversion, we transform the solution to $\mathbb{R}^n \setminus B_1$ and apply Theorem 1.

Let $y = x/|x|^2$. Define $U_i(y) = (1/|y|^{n-2})V_i(y/|y|^2)$. This is called the Kelvin transform with the property that

$$\Delta_y U_i(y) = \frac{1}{|y|^{n+2}}\Delta_x V_i(x).$$

This can also be viewed as the conformal change formula of the conformal Laplacian with zero scalar curvature. Therefore, $U_i(y)$ satisfies

$$\sum_j A_{ij}\Delta_y U_i(y) = \frac{1}{|y|^{n+2}} f^i(|y|^{n-2}U(y)) = f^i(U(y)) \quad \text{in } \mathbb{R}^n \setminus B_1,$$

where we use that f^i is homogeneous of degree $(n+2)/(n-2)$. Moreover,

$$\int_{\mathbb{R}^n\setminus B_1} |U|^{2n/(n-2)}\,dy = \int_{\mathbb{R}^n\setminus B_1} (|V||y|^{-n+2})^{2n/(n-2)}\,dy = \int_{B_1\setminus\{0\}} (|V||x|^{n-2})^{2n/(n-2)}|x|^{-2n}\,dx$$
$$= \int_{B_1\setminus\{0\}} |V|^{2n/(n-2)}\,dx < +\infty.$$

Now we apply Theorem 1 with $p = 2n/(n-2)$ and $q = (n+2)/(n-2)$. We get $|U| = O(|y|^{-(n-2)})$ and $|\nabla U| = O(|y|^{-(n-1)})$. Hence, $|V| = O(1)$ and $|\nabla V| = O(|x|^{-1})$. As a result, $V \in L^\infty(B_1)$ and $\nabla V \in L^p(B_1)$ for all $p < n$.

We show that V is a weak solution to (1) in B_1. Let $\varphi \in H_0^1(B_1, \mathbb{R}^m)$. Let $\eta_k(|x|)$ be a compactly supported function in $B_1 \setminus \{0\}$ such that $\eta_k \to 1$ a.e. in B_1 and $\|\eta_k\|_{L^n(B_1)} \to 0$ as $k \to \infty$. (Such functions were used by Serrin [1964].) Then

$$\int_{B_1} \eta_k \sum_{i,j,l} A_{ij} D_l \varphi_j D_l V_i \, dx = \int_{B_1} -\sum_i f^i(V) \varphi_i \eta_k \, dx - \int_{B_1} \sum_{i,j,l} D_l \eta_k A_{ij} \varphi_j D_l V_i \, dx.$$

The last term can be estimated as follows.

$$\left| \int_{B_1} \sum_{i,j,l} D_l \eta_k A_{ij} \varphi_j D_l V_i \, dx \right| \le C \|\varphi\|_{L^{2n/(n-2)}(B_1)} \|\nabla V\|_{L^2(B_1)} \|\eta_k\|_{L^n(B_1)} \le C \|\eta_k\|_{L^n(B_1)} \to 0$$

as $k \to \infty$. Hence, in the limit

$$\int_{B_1} \sum_{i,j,l} A_{ij} D_l \varphi_j D_l V_i \, dx = \int_{B_1} -\sum_i f^i(V) \varphi_i \, dx.$$

Thus, V is a weak solution in B_1. It follows by elliptic regularity that $V \in C^\infty(B_1)$. □

Proof of Theorem 3. Since A_{ij} is positive definite, there exists an orthogonal matrix M such that

$$M^{-1} A M = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix},$$

where $\lambda_1, \ldots, \lambda_n$ are positive. Let

$$B = M \begin{pmatrix} \sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_n} \end{pmatrix} M^{-1}.$$

By performing a transformation $W_i = \sum_j B_{ij} V_j$, the system can be reduced to $\Delta W_i = \dfrac{\partial \tilde{P}(W)}{\partial W^i}$. Thus, without loss of generality we may assume the equation is of the diagonal form.

Let $\Omega = B_R$ in (4). Therefore, $N = x/R$. We get

$$\int_{B_R} \sum_{k,l} f^k(V) x_l D_l V_k \, dx = \left(\frac{n}{2} - 1\right) \int_{B_R} |\nabla V|^2 dx - \int_{S_R} \frac{R}{2} |\nabla V|^2 dS + R \int_{S_R} |\nabla_N V|^2 dS.$$

Since $f^k = \partial P / \partial V_k$, we have

$$\int_{B_R} -n P(V) \, dx = \left(\frac{n}{2} - 1\right) \int_{B_R} |\nabla V|^2 dx - \int_{S_R} \frac{R}{2} |\nabla V|^2 dS + R \int_{S_R} |\nabla_N V|^2 dS - \int_{S_R} R P(V) \, dS. \quad (6)$$

On the other hand, we also have

$$\int_{B_R} (q+1) P(V) \, dx = \int_{B_R} \sum_k \frac{\partial P}{\partial V_k} V_k \, dx = -\int_{B_R} |\nabla V|^2 dx + \int_{S_R} \sum_j D_N V_j V_j \, dS, \quad (7)$$

where we use the Euler formula for homogeneous functions.

Case 1. $n \geq 4$. By Theorem 1, when $R \to \infty$, (6) becomes

$$\int_{B_R} -nP(V)\,dx = \left(\frac{n}{2}-1\right)\int_{B_R} |\nabla V|^2 dx + O(R^{-(n-2)}) + O(R^{-(q+1)(n-2)+n})$$
$$= \left(\frac{n}{2}-1\right)\int_{B_R} |\nabla V|^2 dx + o(1),$$

where we use conditions on p, q and $n \geq 4$ to get $(q+1)(n-2)-n > 0$. Similarly, (7) gives

$$\int_{B_R} (q+1)P(V)\,dx = -\int_{B_R} |\nabla V|^2 dx + O(R^{-(n-2)}).$$

Combining the above two formulas and noting that $q+1 \neq 2n/(n-2)$, we finally arrive at

$$\int_{B_R} |\nabla V|^2 dx = o(1).$$

We have $|\nabla V| \equiv 0$ and hence $V \equiv 0$.

Case 2. $n = 3$. Note that $\sup |V| \leq (C/|x|^{n/p})\|V\|_{L^p}$. Combining this fact with Theorem 1, we have $|V| = O(|x|^{-\lambda})$, where $\lambda = \max\{1, 3/p\}$. Therefore,

$$\lambda(q+1) - 3 \geq \max\left\{q-2, \frac{3}{p}(q+1)-3\right\} \geq \max\left\{-1+\frac{2p}{3}, -1+\frac{6}{p}\right\} > 0.$$

Then (6) becomes

$$\int_{B_R} -3P(V)\,dx = \left(\frac{3}{2}-1\right)\int_{B_R} |\nabla V|^2 dx + O(R^{-1}) + O(R^{-\lambda(q+1)+3})$$
$$= \left(\frac{3}{2}-1\right)\int_{B_R} |\nabla V|^2 dx + o(1),$$

as in Case 1. The rest of proof is the same as in Case 1. □

Acknowledgments

The author would like to thank Craig Evans for drawing her attention to the reaction-diffusion type elliptic systems. She is grateful for many of his suggestions. The author appreciates James Serrin for his comments and especially suggestions for the references, which help improve the presentation of the work. The author wishes to thank Haïm Brézis for comments and interests. Finally, the author would like to thank the referee and the editor for careful reading and for valuable suggestions.

References

[Bando et al. 1989] S. Bando, A. Kasue, and H. Nakajima, "On a construction of coordinates at infinity on manifolds with fast curvature decay and maximal volume growth", *Invent. Math.* **97**:2 (1989), 313–349. MR 90c:53098 Zbl 0682.53045

[Bidaut-Véron 1989] M.-F. Bidaut-Véron, "Local and global behavior of solutions of quasilinear equations of Emden–Fowler type", *Arch. Rational Mech. Anal.* **107**:4 (1989), 293–324. MR 90f:35066

[Brézis and Lions 1981] H. Brézis and P.-L. Lions, "A note on isolated singularities for linear elliptic equations", pp. 263–266 in *Mathematical analysis and applications, Part A*, edited by L. Nachbin, Adv. in Math. Suppl. Stud. **7**, Academic Press, New York, 1981. MR 83e:35039 Zbl 0468.35036

[Caffarelli et al. 1989] L. A. Caffarelli, B. Gidas, and J. Spruck, "Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth", *Comm. Pure Appl. Math.* **42**:3 (1989), 271–297. MR 90c:35075 Zbl 0702.35085

[Chen 2009] S. S. Chen, "Optimal curvature decays on asymptotically locally euclidean manifolds", preprint, 2009. arXiv 0911.55380

[Gidas and Spruck 1981] B. Gidas and J. Spruck, "Global and local behavior of positive solutions of nonlinear elliptic equations", *Comm. Pure Appl. Math.* **34**:4 (1981), 525–598. MR83f:35045 Zbl0465.35003

[Kuzin and Pohozaev 1997] I. Kuzin and S. Pohozaev, *Entire solutions of semilinear elliptic equations*, Progress in Nonlinear Differential Equations and their Applications **33**, Birkhäuser, Basel, 1997. MR99d:35050 Zbl0882.35003

[Morrey 1966] C.B. Morrey, Jr., *Multiple integrals in the calculus of variations*, Grund. der Math. Wissenschaften **130**, Springer, New York, 1966. MR34 #2380 Zbl0142.38701

[Pohožaev 1965] S.I. Pohožaev, "On the eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$", *Dokl. Akad. Nauk SSSR* **165** (1965), 36–39. In Russian; translated in *Soviet Math. Dokl.* **6** (1965), 1408–1411. MR33 #411

[Poláčik etal. 2007] P. Poláčik, P. Quittner, and P. Souplet, "Singularity and decay estimates in superlinear problems via Liouville-type theorems. I. Elliptic equations and systems", *Duke Math. J.* **139**:3 (2007), 555–579. MR2009b:35131

[Pucci and Serrin 1986] P. Pucci and J. Serrin, "A general variational identity", *Indiana Univ. Math. J.* **35**:3 (1986), 681–703. MR88b:35072 Zbl0625.35027

[Pucci and Serrin 2007] P. Pucci and J. Serrin, *The maximum principle*, Progress in Nonlinear Differential Equations and their Applications **73**, Birkhäuser, Basel, 2007. MR2008m:35001 Zbl1134.35001

[Serrin 1964] J. Serrin, "Local behavior of solutions of quasi-linear equations", *Acta Math.* **111** (1964), 247–302. MR30 #337 Zbl0128.09101

[Serrin and Zou 2002] J. Serrin and H. Zou, "Cauchy–Liouville and universal boundedness theorems for quasilinear elliptic equations and inequalities", *Acta Math.* **189**:1 (2002), 79–142. MR2003j:35107 Zbl1059.35040

[Souplet 2009] P. Souplet, "The proof of the Lane–Emden conjecture in four space dimensions", *Adv. Math.* **221**:5 (2009), 1409–1427. MR2522424 Zbl1171.35035

[Tian and Viaclovsky 2005] G. Tian and J. Viaclovsky, "Bach-flat asymptotically locally Euclidean metrics", *Invent. Math.* **160**:2 (2005), 357–415. MR2006b:53050 Zbl1085.53030

[Uhlenbeck 1982] K.K. Uhlenbeck, "Removable singularities in Yang–Mills fields", *Comm. Math. Phys.* **83**:1 (1982), 11–29. MR83e:53034 Zbl0491.58032

[Zou 2000] H. Zou, "Symmetry of ground states for a semilinear elliptic system", *Trans. Amer. Math. Soc.* **352**:3 (2000), 1217–1245. MR2000j:35075 Zbl0933.35065

Received 24 Nov 2008. Revised 4 Mar 2010. Accepted 15 Apr 2010.

SZU-YU SOPHIE CHEN: sophiechen@math.princeton.edu
Institute for Advanced Study, Einstein Drive, Princeton, NJ 08540, United States
Current address: *Mathematics Department, Fine Hall, Washington Road, Princeton, NJ 08544, United States*

ANALYSIS AND PDE
Vol. 4, No. 1, 2011

GLOBAL REGULARITY FOR THE NAVIER–STOKES EQUATIONS WITH SOME CLASSES OF LARGE INITIAL DATA

MARIUS PAICU AND ZHIFEI ZHANG

Chemin, Gallagher, and Paicu obtained in 2010 a class of large initial data that generate a global smooth solution to the three-dimensional, incompressible Navier–Stokes equation. The data varies slowly in the vertical direction — it is expressed as a function of εx_3 — and it has a norm that blows up as the small parameter goes to zero. This type of initial data can be regarded as an *ill prepared* case, in contrast with the *well prepared* case treated in earlier papers. The data was supposed to evolve in a special domain, namely $\Omega = T_h^2 \times \mathbb{R}_v$. The choice of a periodic domain in the horizontal variable played an important role.

The aim of this article is to study the case where the fluid evolves in the whole space $\mathbb{R}^3$. In this case, we have to overcome the difficulties coming from very low horizontal frequencies. We consider in this paper an intermediate situation between the well prepared case and ill prepared situation (the norms of the horizontal components of initial data are small but the norm of the vertical component blows up as the small parameter goes to zero). The proof uses the analytical-type estimates and the special structure of the nonlinear term of the equation.

1. Introduction

We study in this paper the Navier–Stokes equations with initial data which is slowly varying in the vertical variable. More precisely we consider the system

$$\begin{cases} \partial_t u + u \cdot \nabla u - \Delta u = -\nabla p \quad \text{in } \mathbb{R}_+ \times \Omega, \\ \operatorname{div} u = 0, \\ u|_{t=0} = u_{0,\varepsilon}, \end{cases} \tag{NS}$$

where $\Omega = \mathbb{R}^3$ and $u_{0,\varepsilon}$ is a divergence-free vector field, whose dependence on the vertical variable x_3 will be chosen to be slow, meaning that it depends on εx_3, where ε is a small parameter. The goal is to prove the global existence in time of the solution generated by this type of initial data, with no smallness assumption on its norm.

This type of initial data (slowly varying in the vertical direction) has also been studied by Chemin, Gallagher, and coworkers. The case of *well prepared* initial data, of the form $(\varepsilon u_0^h(x_h, \varepsilon x_3), u_0^3(x_h, \varepsilon x_3))$, was dealt with in [Chemin and Gallagher 2010]; the more difficult case of *ill prepared* initial data, of the form $(u^h(x_h, \varepsilon x_3), \varepsilon^{-1} u^3(x_h, \varepsilon x_3))$, in [Chemin et al. 2011].

Here we consider a class of large initial data lying between those two cases, having the form

$$u_{0,\varepsilon} = (\varepsilon^{\frac{1}{2}} u_0^h(x_h, \varepsilon x_3), \varepsilon^{-\frac{1}{2}} u_0^3(x_h, \varepsilon x_3)).$$

Z. Zhang is supported by NSF of China under Grant 10990013, 11071007, and SRF for ROCS, SEM.
MSC2000: 35B65, 35Q35, 76D99, 76N10.
Keywords: Navier–Stokes equations, global well-posedness, large data.

We recall some classical facts about the Navier–Stokes system, focusing on conditions that imply the global existence of the strong solution.

The first important result about the classical Navier–Stokes system [Leray 1934] asserted that for any initial data of finite energy there exists at least one global in time weak solution that satisfies the energy estimate. This solution is unique in $\mathbb{R}^2$, but it is not known to be unique in $\mathbb{R}^3$. Leray's result uses the structure of the nonlinear terms in order to obtain the energy inequality. The question of regularity of the weak solutions also remains open.

The Fujita–Kato theorem [1964] gives a partial answer for the construction of global unique solution: it allows one to construct a unique local in time solution in the homogeneous Sobolev spaces $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$, or in the Lebesgue space $L^3(\mathbb{R}^3)$ [Kato 1984]. If the initial data is small compared to the viscosity, that is, if $\|u_0\|_{\dot{H}^{\frac{1}{2}}} \le c\nu$, the strong solution exists globally in time. This result was generalized by Cannone, Meyer and Planchon [Cannone et al. 1994] to Besov spaces of negative index of regularity. Those authors proved that, if the initial data belongs to the Besov space $B^{-1+3/p}_{p,\infty}(\mathbb{R}^3)$ and is small in the norm of this Besov space, compared to the viscosity, then the solution is global in time.

Later, Koch and Tataru [2001] obtained a unique global in time solution for the Navier–Stokes equation for small data belonging to a more general space of initial data, namely derivatives of BMO functions.

Concerning the methods for obtaining such results, we recall that the existence of a unique, global in time solution to the Navier–Stokes equation is a standard consequence of the Banach fixed point theorem, as long as the initial data is chosen small enough in some scale-invariant space (with norm invariant under the scaling $\lambda u(\lambda^2 t, \lambda x)$) embedded in $\dot{B}^{-1}_{\infty,\infty}$ (the Besov space), with norm

$$\|f\|_{\dot{B}^{-1}_{\infty,\infty}} \stackrel{\text{def}}{=} \sup_{t>0} t^{\frac{1}{2}} \|e^{t\Delta} f\|_{L^\infty}.$$

See [Cannone et al. 1994; Fujita and Kato 1964; Koch and Tataru 2001; Weissler 1980] for proofs in various scale-invariant function spaces.

These theorems are general results of global existence for small initial data and do not take into account any algebraical properties of the nonlinear terms in the Navier–Stokes equations. Proving such results without any smallness assumption or geometrical invariance hypothesis implying conservation of quantities beyond the scaling is a challenge. Only modest progress has been made in that direction: see [Chemin et al. 2000; 2006; 2011; Chemin and Gallagher 2006; 2009; 2010; Chemin and Zhang 2007; Gallagher et al. 2003; Makhalov and Nikolaenko 2003; Raugel and Sell 1993] and references therein.

Here are some cases where large initial data is known to imply global existence of the solution:

For regular axisymmetric initial data without swirl, the Navier–Stokes system has a unique global in time solution. This result from [Ukhovskii and Iudovich 1968] is based on the conservation of some quantities beyond the scaling regularity level.

The case of large (in some sense) initial data for fluids evolving in thin domains was first considered in [Raugel and Sell 1993]. Roughly speaking, the three-dimensional Navier–Stokes system can be seen as a perturbation of the two-dimensional Navier–Stokes system if the vertical thickness of the domain is small enough. More generally, a solution exists globally in time if the initial data can be split as $v_0 + w_0$, where v_0 is a two-dimensional divergence-free vector field in $L^2(T_h^2)$ and $w_0 \in \dot{H}^{\frac{1}{2}}(T^3)$ satisfies

$$\|w_0\|_{\dot{H}^{\frac{1}{2}}(T^3)} \exp \frac{\|v_0\|^2_{L^2(T_h^2)}}{C\nu^2} \le c\nu.$$

The case of initial data with large vortex in the vertical direction ($\operatorname{rot} u_0^\varepsilon = \operatorname{rot} u_0 + \varepsilon^{-1}(0, 0, 1)$), or equivalently the case of rotating fluids, was studied in [Makhalov and Nikolaenko 2003] for periodic domains and in [Chemin et al. 2000; Chemin et al. 2006] for a rotating fluid in $\mathbb{R}^3$ or in $\mathbb{R}^2 \times (0, 1)$. When the rotation is fast enough the fluid tends to have a two-dimensional behavior far from the boundary of the domain, by the Taylor–Proudman column theorem [Pedlovsky 1979]. For example, when the domain is $\mathbb{R}^3$ the fluctuation of this motion is dispersed to infinity and some Strichatz quantities became small [Chemin et al. 2000], which allow to obtain the global existence of the solution (for ε small enough).

An important issue for the Navier–Stokes equations is to make the best possible use of the algebraical structure of the nonlinear terms. Some results have made crucial use of this structure, and have proved very fruitful.

The case of Navier–Stokes equations with vanishing vertical viscosity was first studied in [Chemin et al. 2000], which contains proofs of local existence for large data in anisotropic Sobolev spaces $H^{0,s}$, with $s > \frac{1}{2}$, and of global existence and uniqueness for small initial data. One of the key observations there is that, even if there is no vertical viscosity and thus no smoothing in the vertical variable, the partial derivative ∂_3 is only applied to the component u_3 in the nonlinear term. The divergence-free condition implies that $\partial_3 u_3$ is regular enough to get good estimates of the nonlinear term.

Global existence of the solution for the anisotropic Navier–Stokes system with high oscillatory initial data was obtained in [Chemin and Zhang 2007].

A different idea, still using the special structure of the Navier–Stokes equation, was used in [Chemin and Gallagher 2006] to construct the first example of periodic initial data which is big in C^{-1}, and strongly oscillating in one direction which generate a global solution. The initial data is given by

$$u_0^N = (N u_h(x_h)\cos(N x_3), -\operatorname{div}_h u_h(x_h)\sin(N x_3)),$$

where $\|u_h\|_{L^2(T_h^2)} \le C(\ln N)^{\frac{1}{9}}$. This result was generalized to $\mathbb{R}^3$ in [Chemin and Gallagher 2009].

The same authors [Chemin and Gallagher 2010] studied the Navier–Stokes equations for initial data that varies slowly in the vertical direction in the well prepared case. The well prepared case means that the norm of the initial data is large but does not blow up when the parameter ε converges to zero. We note that important remarks on the pressure term and the bilinear term were used in this paper in order to obtain the global existence for large data.

The case of slowly varying initial data in the vertical direction (ell prepared initial data) was recently studied in [Chemin et al. 2011]. Here the horizontal components have large norm and the vertical component has a norm that blows up when the parameter goes to zero. After a change of scale, one obtains a Navier–Stokes type system with anisotropic viscosity $-\nu\Delta_\varepsilon u$, where $\Delta_\varepsilon = \Delta_h + \varepsilon^2\partial_3^2$, and anisotropic pressure gradient, namely $-(\nabla_h p, \varepsilon^2\partial_3 p)$. In this equation there is a loss of regularity in the vertical variable in Sobolev estimates.

To overcome this difficulty one needs to work with analytical initial data. The most important tool was developed in [Chemin 2004] and consists in making analytical type estimates, and at the same time to control the size of the analyticity band. This is done by controlling nonlinear quantities that depend on the solution itself. Even in this situation, it is important to take into account very carefully the special structure of the Navier–Stokes equations. In fact, a global in time Cauchy–Kowalewskaya type theorem was obtained in [Chemin et al. 2011]. (Some local in time results for Euler and Prandtl equation with analytic initial data can be found in [Sammartino and Caflisch 1998].)

In [Chemin et al. 2011] the fluid is supposed to evolve in a special domain, $\Omega = T_h^2 \times \mathbb{R}_v$. This choice of domain is justified by the pressure term: the pressure satisfies the elliptic equation $\Delta_\varepsilon p = \partial_i \partial_j (u^i u^j)$, and consequently, $\nabla_h p = (-\Delta_\varepsilon)^{-1} \nabla_h \partial_i \partial_j (u^i u^j)$. Because Δ_ε^{-1} converges to Δ_h^{-1}, it is important to control the low horizontal frequencies, in contrast with the case of the periodic torus in the horizontal variable, where we have only zero horizontal frequency and high horizontal frequencies.

In this paper we investigate the case where the fluid evolves in the full space $\mathbb{R}^3$. In that situation, we are able to solve globally in time the equation (conveniently rescaled in ε) for small analytic-type initial data. To do this, we need to control the low horizontal frequencies very precisely. Note that we can construct functional spaces where the operator $\Delta_h^{-1} \nabla_h (a \nabla_h b)$ is bounded. However, we still need to impose on the initial data more regularity control on low horizontal frequencies; thus we make the assumption $u_0(\,\cdot\,, x_3) \in L^2(\mathbb{R}_h^2) \cap \dot{H}^{-\frac{1}{2}}(\mathbb{R}_h^2)$. In the vertical variable we need to impose analyticity of the data. The method of the proof follows closely the argument of [Chemin et al. 2011], but instead to use pointwise estimates on the Fourier variables, we write an equation with a regularizing term in the vertical variable and we use energy estimates on anisotropic Sobolev spaces of the form $H^{0,s}$ respectively $H^{-\frac{1}{2},s}$.

Our main result in the case of the full space $\mathbb{R}^3$ is the following (for notation see the next section).

Theorem 1.1. *Let $a > 0$ and $\frac{1}{2} > \alpha > 0$. There exist positive constants ε_0 and η such that, for any divergence-free field v_0 satisfying*

$$\|v_0\|_{X^{\frac{1}{2}-\alpha}} \|v_0\|_{X^{\frac{1}{2}+\alpha}} \le \eta, \qquad \|v_0\|_{X^s} \stackrel{\text{def}}{=} \big(\|e^{a\langle D_3\rangle} v_0\|_{H^{0,s}} + a^{-\frac{1}{2}} \|e^{a\langle D_3\rangle} v_0\|_{H^{-\frac{1}{2},s}}\big)$$

and for any $\varepsilon \in (0, \varepsilon_0)$, the Navier–Stokes system (NS) with initial data

$$u_0^\varepsilon = \big(\varepsilon^{\frac{1}{2}} v_0^h(x_h, \varepsilon x_3), \varepsilon^{-\frac{1}{2}} v_0^3(x_h, \varepsilon x_3)\big)$$

has a global smooth solution on $\mathbb{R}^3$.

We emphasize that we obtain the global wellposedness under a smallness condition which is invariant by the scaling of the equation, and this is the main motivation of Theorem 1.1.

As mentioned, to prove the theorem we will first transform the system using the change of scale

$$u^\varepsilon(t, x_h, x_3) = \big(\varepsilon^{\frac{1}{2}} v^h(t, x_h, \varepsilon x_3), \varepsilon^{-\frac{1}{2}} v^3(t, x_h, \varepsilon x_3)\big)$$

into a system of Navier–Stokes type, with a vertical vanishing viscosity, that is the Laplacian operator became $-\nu \Delta_h v - \varepsilon^2 \nu \partial_3 v$ and a changed pressure term which became $-(\nabla_h p, \varepsilon^2 \partial_3 p)$.

Taking advantage of the fact that we're working in $\mathbb{R}^3$, we can also consider a different type of initial data, with larger amplitude but strongly oscillating in the horizontal variables:

$$u_0^\varepsilon = \big(\varepsilon^{-\frac{1}{2}} v_0^h(\varepsilon^{-1} x_h, x_3), \varepsilon^{-\frac{3}{2}} v_0^3(\varepsilon^{-1} x_h, x_3)\big).$$

This type of initial data has $\dot{B}^{-1}_{\infty,\infty}$ norm of the same order as the initial data in the previous theorem. In order to solve the Navier–Stokes equations with this new type of initial data, we make a different change of scale,

$$u^\varepsilon(t, x_h, x_3) = \big(\varepsilon^{-\frac{1}{2}} v^h(\varepsilon^{-2} t, \varepsilon^{-1} x_h, x_3), \varepsilon^{-\frac{3}{2}} v^3(\varepsilon^{-2} t, \varepsilon^{-1} x_h, x_3)\big)$$

and we note that the rescaled system that we obtain is exactly the same as in the proof of Theorem 1.1. Consequently, we obtain:

Theorem 1.2. *Let $a > 0$ and $\frac{1}{2} > \alpha > 0$. There exist positive constants ε_0 and η such that, for any divergence-free field v_0 satisfying*

$$\|v_0\|_{X^{\frac{1}{2}-\alpha}} \|v_0\|_{X^{\frac{1}{2}+\alpha}} \leq \eta$$

and any $\varepsilon \in (0, \varepsilon_0)$, the Navier–Stokes system (NS) with initial data

$$u_0^\varepsilon = \big(\varepsilon^{-\frac{1}{2}} v_0^h(\varepsilon^{-1} x_h, x_3), \varepsilon^{-\frac{3}{2}} v_0^3(\varepsilon^{-1} x_h, x_3)\big)$$

has a global smooth solution on $\mathbb{R}^3$.

2. A simplified model

We start with an equation already studied in [Chemin et al. 2011], where a complete proof of global well-posedness is given. For completeness and because we are going to use the method of energy estimates, we sketch the proof related to the energy estimates. Consider the equation

$$\partial_t u + \gamma u + a(D)(u^2) = 0,$$

where $a(D)$ is a Fourier multiplier of order one. For any function f such that the following definition makes sense, we define $e^{\delta|D|} f = \mathscr{F}^{-1}\big(e^{\delta|\xi|} \hat{f}(\xi)\big)$, where $\hat{f} = \mathscr{F} f$ denotes the Fourier transform and $\mathscr{F}^{-1}$ denotes the inverse of Fourier transform. Then, if the initial data satisfies

$$\|e^{\delta|D|} u_0\|_{H^s} \leq c\gamma \quad \text{with } \delta > 0 \text{ and } s > \tfrac{1}{2} d,$$

we have a global solution in the same space. The idea of the method in [Chemin 2004; Chemin et al. 2011] is the following: we want to control certain analytical quantities on the solution, but we must prevent a decrease in the radius of analyticity of the solution. Introduce $\theta(t)$, representing the loss of analyticity. We set $\Phi(t, \xi) = (\delta - \lambda\theta(t))|\xi|$ and $u_\Phi = \mathscr{F}^{-1}\big(e^{\Phi(t,\xi)} \hat{u}(\xi)\big)$. The function $\theta(t)$ is defined as the unique solution of the ordinary differential equation

$$\dot{\theta}(t) = \|u_\Phi\|_{H^s}, \quad \theta(0) = 0.$$

The computations that follow are performed under the condition $\theta(t) \leq \delta/\lambda$ (which implies $\Phi \geq 0$). The equation satisfied by $\hat{u}_\phi$ is

$$\partial_t \hat{u}_\Phi + \gamma \hat{u}_\Phi + \lambda \dot{\theta}(t)|\xi| \hat{u}_\Phi + a(\xi) e^{\Phi} (\widehat{u^2}) = 0.$$

This contains an extra-regularizing term, since we control a quantity that takes into account the analyticity of the solution. As $\dot{\theta}$ approaches 0, we obtain by an energy estimate in H^s the inequality

$$\frac{1}{2}\frac{d}{dt}\|u_\Phi\|_{H^s}^2 + \gamma\|u_\Phi\|_{H^s}^2 + \lambda\dot{\theta}(t)\||D|^{\frac{1}{2}} u_\Phi\|_{H^s}^2 \leq C|(a(D)(u^2)_\Phi, u_\Phi)_{H^s}|.$$

Following the proof of [Chemin et al. 2011, Lemma 2.1] (which uses the important fact that $e^{\Phi(t,\xi)}$ is a sublinear function, and also the classical Bony decomposition [1981]), we get

$$|a(D)(u^2)_\Phi, u_\Phi)| \leq C\|u_\Phi\|_{H^s} \||D|^{\frac{1}{2}} u_\Phi\|_{H^s}^2.$$

Choosing $\lambda = 4C$ we obtain

$$\dot{\theta}(t) = \|u_\Phi(t)\|_{H^s} \leq 2\|e^{\delta|D|} u_0\|_{H^s} e^{-\gamma t},$$

which, for u_0 small enough, gives

$$\theta(t) \le \gamma^{-1} \|e^{\delta|D|} u_0\|_{H^s} \le \delta\lambda^{-1}.$$

This allows us to obtain the global in time existence of the solution.

3. Structure of the proof

Reduction to a rescaled problem. We seek a solution of the form

$$u_\varepsilon(t,x) \stackrel{\text{def}}{=} \big(\varepsilon^{\frac12} v^h(t,x_h,\varepsilon x_3), \varepsilon^{-\frac12} v^3(t,x_h,\varepsilon x_3)\big).$$

This leads to the rescaled Navier–Stokes system

$$\begin{cases} \partial_t v^h - \Delta_h v^h - \varepsilon^2 \partial_3^2 v^h + \varepsilon^{\frac12} v\cdot\nabla v^h = -\nabla^h q, \\ \partial_t v^3 - \Delta_h v^3 - \varepsilon^2 \partial_3^2 v^3 + \varepsilon^{\frac12} v\cdot\nabla v^3 = -\varepsilon^2 \partial_3 q, \\ \operatorname{div} v = 0, \\ v(0) = v_0(x), \end{cases} \tag{RNS$_\varepsilon$}$$

where $\Delta_h \stackrel{\text{def}}{=} \partial_1^2 + \partial_2^2$ and $\nabla_h \stackrel{\text{def}}{=} (\partial_1, \partial_2)$. As there is no boundary, the rescaled pressure q can be computed with the formula

$$-\Delta_\varepsilon q = \varepsilon^{\frac12} \operatorname{div}_h(v\cdot\nabla v), \qquad \Delta_\varepsilon = \Delta_h + \varepsilon^2\partial_3^2. \tag{3-2}$$

When ε tends to zero, Δ_ε^{-1} looks like Δ_h^{-1}. Thus, for low horizontal frequencies, an expression of $\nabla_h \Delta_h^{-1}$ cannot be estimated in L^2. This is one reason for working in $T_h^2 \times \mathbb{R}_v$ in [Chemin et al. 2011]. To obtain a similar result in $\mathbb{R}^3$, we need to introduce the following anisotropic Sobolev space.

Definition 3.1. Let $s, \sigma \in \mathbb{R}$, $\sigma < 1$. The anisotropic Sobolev space $H^{\sigma,s}$ is defined by

$$H^{\sigma,s} = \{ f \in \mathcal{S}'(\mathbb{R}^3) : \|f\|_{H^{\sigma,s}} < \infty \},$$

where

$$\|f\|_{H^{\sigma,s}}^2 \stackrel{\text{def}}{=} \int_{\mathbb{R}^3} |\xi_h|^{2\sigma} (1+|\xi_3|^2)^s |\hat f(\xi)|^2 d\xi, \qquad \xi = (\xi_h, \xi_3).$$

For any $f, g \in H^{\sigma,s}$, we set

$$(f,g)_{H^{\sigma,s}} \stackrel{\text{def}}{=} \big(|D_h|^\sigma \langle D_3\rangle^s f, |D_h|^\sigma \langle D_3\rangle^s g\big)_{L^2}, \qquad \langle D_3\rangle = (1+|D_3|^2)^{\frac12}.$$

Theorem 3.2. *Let $a > 0$, $\frac12 > \alpha > 0$. There exist two positive constants ε_0 and η such that for any divergence-free fields v_0 satisfying*

$$\|v_0\|_{X^{\frac12-\alpha}} \|v_0\|_{X^{\frac12+\alpha}} \le \eta,$$

and for any $\varepsilon \in (0, \varepsilon_0)$, (RNS$_\varepsilon$) has a global smooth solution on $\mathbb{R}^3$.

Definition of the functional setting. As in [Chemin et al. 2011], the proof relies on exponential decay estimates for the Fourier transform of the solution. Thus, for any locally bounded function Ψ on $\mathbb{R}^+\times\mathbb{R}^3$ and for any function f, continuous in time and compactly supported in Fourier space, we define

$$f_\Psi(t)\stackrel{\text{def}}{=}\mathcal{F}^{-1}\big(e^{\Psi(t,\cdot)}\widehat{f}(t,\cdot)\big).$$

Now we introduce a key quantity we want to control in order to prove the theorem. We define the function $\theta(t)$ by

$$\dot\theta(t)\stackrel{\text{def}}{=}\frac{1}{a}\big(\varepsilon\|v^h_\Phi(t)\|_{H^{\frac12,\frac12-\alpha}}\|v^h_\Phi(t)\|_{H^{\frac12,\frac12+\alpha}}+\|v^3_\Phi(t)\|_{H^{\frac12,\frac12-\alpha}}\|v^3_\Phi(t)\|_{H^{\frac12,\frac12+\alpha}}\big) \\ +\|\nabla_h v_\Phi(t)\|_{H^{0,\frac12-\alpha}}\|\nabla_h v_\Phi(t)\|_{H^{0,\frac12+\alpha}},\quad (3\text{-}3)$$

with $\theta(0)=0$, and we also define

$$\Psi(t)\stackrel{\text{def}}{=}\|v_\Phi(t)\|_{H^{0,\frac12-\alpha}}\|v_\Phi(t)\|_{H^{0,\frac12+\alpha}},$$

where

$$\Phi(t,\xi)\stackrel{\text{def}}{=}a(1-\lambda\theta(t))\langle\xi_3\rangle,\quad (3\text{-}4)$$

for some λ that will be chosen later on. We denoted by $\langle\xi_3\rangle=(1+|\xi_3|^2)^{\frac12}$ which is a sublinear function.

Main steps of the proof.

Proposition 3.3. *Let $s>0$. A constant C exists such that, for any positive λ and for any t satisfying $\theta(t)\le 1/\lambda$, we have*

$$\int_0^t\big(\|\varepsilon^{\frac12}v^h_\Phi(\tau)\|^2_{H^{\frac12,s}}+\|v^3_\Phi(\tau)\|^2_{H^{\frac12,s}}\big)d\tau \\ \le\exp\big(C\theta(t)\big)\Big[\|e^{a\langle D_3\rangle}v_0\|^2_{H^{-\frac12,s}}+Ca\int_0^t\dot\theta(\tau)\|v^h_\Phi(\tau)\|^2_{H^{0,s}}d\tau \\ +C\int_0^t\Psi(\tau)\big(\|\varepsilon^{\frac12}v^h_\Phi(\tau)\|^2_{H^{\frac12,s}}+\|v^3_\Phi(\tau)\|^2_{H^{\frac12,s}}\big)d\tau+\frac{a}{10}\int_0^t\|\nabla_h v^h_\Phi(\tau)\|^2_{H^{0,s}}d\tau\Big].$$

Proposition 3.4. *Let $1>s>0$. Then there exist C and λ_0 such that for any $\lambda\ge\lambda_0$ and for any t satisfying $\theta(t)\le 1/\lambda$, we have*

$$\|v_\Phi(t)\|^2_{H^{0,s}}+\int_0^t\|\nabla_h v_\Phi(\tau)\|^2_{H^{0,s}}d\tau \\ \le\exp\big(C\theta(t)\big)\Big[\|e^{a\langle D_3\rangle}v_0\|^2_{H^{0,s}}+C\int_0^t\Psi(\tau)\|v_\Phi(\tau)\|^2_{H^{1,s}}d\tau+\frac{1}{10}\int_0^t\frac1a\|v^3_\Phi(\tau)\|^2_{H^{\frac12,s}}d\tau\Big].$$

Proposition 3.3 will be proved in Section 4, and Proposition 3.4 in Section 5. For the moment, let us assume that they are true and conclude the proof of Theorem 3.2. As in [Chemin et al. 2011], we use a continuation argument. For any $\lambda\ge\lambda_0$ and η, define

$$\mathcal{T}_\lambda\stackrel{\text{def}}{=}\{T:\theta(T)+\Psi(T)\le 4\eta\}.$$

Similar to the argument in [Chemin et al. 2011], $\mathcal{T}_\lambda$ is of the form $[0, T^*)$ for some positive T^*. Thus, it suffices to prove that $T^* = +\infty$. In order to two propositions, we need to assume that $\theta(T) \le 1/\lambda$, which leads to the condition

$$4\eta \le \frac{1}{\lambda}.$$

We set

$$F_s(t) \stackrel{\text{def}}{=} \|v_\Phi(t)\|^2_{H^{0,s}} + \int_0^t \|\nabla_h v_\Phi(\tau)\|^2_{H^{0,s}} d\tau + \frac{1}{a}\int_0^t \big(\|\varepsilon^{\frac12} v^h_\Phi(\tau)\|^2_{H^{\frac12,s}} + \|v^3_\Phi(\tau)\|^2_{H^{\frac12,s}}\big) d\tau.$$

From Propositions 3.3 and 3.4, it follows that for all $T \in \mathcal{T}_\lambda$,

$$F_s(t) \le \tfrac{10}{9}\exp\big(4C\eta\big)\bigg(\frac{1}{a}\|e^{a\langle D_3\rangle} v_0\|^2_{H^{-\frac12,s}} + \|e^{a\langle D_3\rangle} v_0\|^2_{H^{0,s}} + C\int_0^t \dot\theta(\tau)\|v^h_\Phi(\tau)\|^2_{H^{0,s}} d\tau$$
$$+4C\eta\int_0^t \Big(\frac{1}{a}\|\varepsilon^{\frac12} v^h_\Phi(\tau)\|^2_{H^{\frac12,s}} + \frac{1}{a}\|v^3_\Phi(\tau)\|^2_{H^{\frac12,s}} + \|v_\Phi(\tau)\|^2_{H^{1,s}}\Big) d\tau\bigg).$$

Now we choose η such that

$$\exp(4C\eta) \le \tfrac{9}{8}, \quad 4C\eta\exp(4C\eta) \le \tfrac{1}{4}.$$

With this choice of η, we infer from Gronwall's inequality that

$$F_s(t) < 2\Big(\frac{1}{a}\|e^{a\langle D_3\rangle} v_0\|^2_{H^{-\frac12,s}} + \|e^{a\langle D_3\rangle} v_0\|^2_{H^{0,s}}\Big) \le 2\|v_0\|^2_{X^s}.$$

Taking $s = \frac12 - \alpha$ and $s = \frac12 + \alpha$ respectively, we obtain

$$a^{-1}\theta(t) + \Psi(t) \le 2F^{\frac12}_{\frac12-\alpha}(t)F^{\frac12}_{\frac12+\alpha}(t) < 4\|v_0\|_{X^{\frac12-\alpha}}\|v_0\|_{X^{\frac12+\alpha}} \le 4\eta,$$

which ensures that $T^* = +\infty$, thus concluding the proof of Theorem 3.2. □

4. The action of subadditive phases on products

For any function f, we denote by f^+ the inverse Fourier transform of $|\hat f|$. Let us notice that the map $f \mapsto f^+$ preserves the norm of all $H^{\sigma,s}$ spaces. Throughout this section, Ψ will denote a locally bounded function on $\mathbb{R}^+ \times \mathbb{R}^3$ which satisfies the following inequality

$$\Psi(t,\xi) \le \Psi(t,\xi-\eta) + \Psi(t,\eta). \tag{4-1}$$

Before presenting the product estimates, let us recall the Littlewood–Paley decomposition. Choose two nonnegative even functions $\chi, \varphi \in \mathcal{S}(\mathbb{R})$ supported, respectively, in $\mathcal{B} = \{\xi \in \mathbb{R}, |\xi| \le \frac43\}$ and $\mathcal{C} = \{\xi \in \mathbb{R}, \frac34 \le |\xi| \le \frac83\}$ such that

$$\chi(\xi) + \sum_{j\ge0}\varphi(2^{-j}\xi) = 1 \quad \text{for } \xi \in \mathbb{R},$$
$$\sum_{j\in\mathbb{Z}}\varphi(2^{-j}\xi) = 1 \quad \text{for } \xi \in \mathbb{R}\setminus\{0\}.$$

The frequency localization operators Δ_j^v and S_j^v in the vertical direction are defined by

$$\Delta_j^v f = \begin{cases} \mathcal{F}^{-1}(\varphi(2^{-j}|\xi_3|)\hat{f}) & \text{for } j \geq 0, \\ S_0^v f & \text{for } j = -1, \\ 0 & \text{for } j \leq -2, \end{cases}$$

$$S_j^v f = \mathcal{F}^{-1}\big(\chi(2^{-j}|\xi_3|)\hat{f}\big) = \sum_{j' \leq j-1} \Delta_{j'}^v f.$$

The frequency localization operators $\dot{\Delta}_j^h$ and S_j^h in the horizontal direction are defined by

$$\dot{\Delta}_j^h f = \mathcal{F}^{-1}\big(\varphi(2^{-j}|\xi_h|)\hat{f}\big), \quad S_j^h f = \sum_{j' \leq j-1} \dot{\Delta}_{j'}^h f \qquad \text{for } j \in \mathbb{Z}.$$

It is easy to verify that

$$\|f\|_{H^{\sigma,s}}^2 \approx \sum_{j,k \in \mathbb{Z}} 2^{2js} 2^{2k\sigma} \|\Delta_j^v \dot{\Delta}_k^h f\|_{L^2}^2. \tag{4-2}$$

In the sequel, we will constantly use the Bony's decomposition [1981]:

$$fg = T_f^v g + R_f^v g, \tag{4-3}$$

with

$$T_f^v g = \sum_j S_{j-1}^v f \Delta_j^v g, \quad R_f^v g = \sum_j S_{j+2}^v f \Delta_j^v g.$$

We also use the Bony's decomposition in the horizontal direction:

$$fg = T_f^h g + T_f^h g + R^h(f, g), \tag{4-4}$$

with

$$T_f^h g = \sum_j S_{j-1}^h f \dot{\Delta}_j^h g, \quad R^h(f, g) = \sum_{|j'-j| \leq 1} \dot{\Delta}_j^h f \dot{\Delta}_{j'}^h g.$$

Lemma 4.1 (Bernstein's inequality). *Let $1 \leq p \leq q \leq \infty$. If $f \in L^p(\mathbb{R}^d)$, there exists a constant C independent of f, j such that*

$$\operatorname{supp} \hat{f} \subset \{|\xi| \leq C2^j\} \quad \Longrightarrow \quad \|\partial^\alpha f\|_{L^q} \leq C 2^{j|\alpha| + dj\left(\frac{1}{p} - \frac{1}{q}\right)} \|f\|_{L^p},$$

$$\operatorname{supp} \hat{f} \subset \left\{\frac{1}{C} 2^j \leq |\xi| \leq C2^j\right\} \quad \Longrightarrow \quad \|f\|_{L^p} \leq C 2^{-j|\alpha|} \sup_{|\beta| = |\alpha|} \|\partial^\beta f\|_{L^p}.$$

Lemma 4.2. *Let $s > 0$, $\sigma_1, \sigma_2 < 1$ such that $\sigma_1 + \sigma_2 > 0$ and $\frac{1}{2} > \alpha > 0$. Assume that $a_\Psi \in H^{\sigma_1, \frac{1}{2}+\alpha}$ and $b_\Psi \in H^{\sigma_2, s}$. Then*

$$\|\big[\Delta_j^v \dot{\Delta}_k^h (T_a^v b)\big]_\Psi\|_{L^2} + \|\big[\Delta_j^v \dot{\Delta}_k^h (R_a^v b)\big]_\Psi\|_{L^2} \leq C c_{j,k} 2^{(1-\sigma_1-\sigma_2)k} 2^{-js} \|a_\Psi\|_{H^{\sigma_1, \frac{1}{2}-\alpha}}^{\frac{1}{2}} \|a_\Psi\|_{H^{\sigma_1, \frac{1}{2}+\alpha}}^{\frac{1}{2}} \|b_\Psi\|_{H^{\sigma_2, s}},$$

with the sequence $(c_{j,k})_{j,k \in \mathbb{Z}}$ of positive numbers satisfying

$$\sum_{j,k} c_{j,k}^2 \leq 1.$$

Proof. Let us firstly prove the case when the function Ψ is identically 0. Below we only present the proof of $R^v_a b$, the proof for $T^v_a b$ is very similar. Using Bony's decomposition (4-4) in the horizontal direction, we write

$$\Delta_j \dot\Delta^h_k(R^v_a b) = \sum_{j'} \Delta^v_j \dot\Delta^h_k(S^v_{j'+2}a\Delta^v_{j'}b) = \sum_{j'} \Delta^v_j \dot\Delta^h_k\big(T^h_{S^v_{j'+2}a}\Delta^v_{j'}b + T^h_{\Delta^v_{j'}b}S^v_{j'+2}a + R^h(S^v_{j'+2}a, \Delta^v_{j'}b)\big)$$
$$\stackrel{\text{def}}{=} \mathrm{I}+\mathrm{II}+\mathrm{III}.$$

Considering the support of the Fourier transform of $T^h_{S^v_{j'+2}a}\Delta^v_{j'}b$, we have

$$\mathrm{I} = \sum_{j'\ge j-4}\ \sum_{|k'-k|\le 4} \Delta^v_j \dot\Delta^h_k\big(S^v_{j'+2}S^h_{k'-1}a\Delta^v_{j'}\dot\Delta^h_{k'}b\big).$$

Then, by Lemma 4.1,

$$\begin{aligned}\|\mathrm{I}\|_{L^2} &\le C \sum_{j'\ge j-4}\ \sum_{|k'-k|\le 4} \|S^v_{j'+2}S^h_{k'-1}a\Delta_{j'}\dot\Delta^h_{k'}b\|_{L^2}\\ &\le C \sum_{j'\ge j-4}\ \sum_{|k'-k|\le 4} \|S^v_{j'+2}S^h_{k'-1}a\|_{L^\infty}\|\Delta^v_{j'}\dot\Delta^h_{k'}b\|_{L^2}.\end{aligned}$$

We use Lemma 4.1 again to get

$$\begin{aligned}\|S^v_{j'+2}S^h_{k'-1}a\|_{L^\infty} &\le \sum_{j''\le j'+1}\ \sum_{k''\le k'-2}\|\Delta^v_{j''}\dot\Delta^h_{k''}a\|_{L^\infty} \le C\sum_{j''\le j'+1}\ \sum_{k''\le k'-2} 2^{k''}\|\Delta^v_{j''}\dot\Delta^h_{k''}a\|_{L^\infty_{x_3}L^2_{x_h}}\\ &\le C\sum_{j''\le j'+1}\ \sum_{k''\le k'-2} 2^{j''/2}2^{k''}\|\Delta^v_{j''}\dot\Delta^h_{k''}a\|_{L^2} \le C2^{(1-\sigma_1)k}\|a\|^{\frac12}_{H^{\sigma_1,\frac12-\alpha}}\|a\|^{\frac12}_{H^{\sigma_1,\frac12+\alpha}},\end{aligned}$$

from which it follows that

$$\begin{aligned}\|\mathrm{I}\|_{L^2} &\le C2^{(1-\sigma_1)k}\|a\|^{\frac12}_{H^{\sigma_1,\frac12-\alpha}}\|a\|^{\frac12}_{H^{\sigma_1,\frac12+\alpha}}\sum_{j'\ge j-4}\ \sum_{|k'-k|\le 4}\|\Delta^v_{j'}\dot\Delta^h_{k'}b\|_{L^2}\\ &\le Cc_{j,k}2^{-js}2^{(1-\sigma_1-\sigma_2)k}\|a\|^{\frac12}_{H^{\sigma_1,\frac12-\alpha}}\|a\|^{\frac12}_{H^{\sigma_1,\frac12+\alpha}}\|b\|_{H^{\sigma_2,s}}.\end{aligned}\tag{4-5}$$

Similarly, we have

$$\mathrm{II} = \sum_{j'\ge j-4}\ \sum_{|k'-k|\le 4}\Delta^v_j\dot\Delta^h_k(\Delta^v_{j'}S^h_{k'-1}bS^v_{j'+2}\dot\Delta^h_{k'}a).$$

Then, by Lemma 4.1,

$$\begin{aligned}\|\mathrm{II}\|_{L^2} &\le C\sum_{j'\ge j-4}\sum_{|k'-k|\le 4}\|\Delta^v_{j'}S^h_{k'-1}b\|_{L^2_{x_3}L^\infty_{x_h}}\|S^v_{j'+2}\dot\Delta^h_{k'}a\|_{L^2_{x_h}L^\infty_{x_3}}\\ &\le C2^{-js}2^{(1-\sigma_1-\sigma_2)k}\|a\|^{\frac12}_{H^{\sigma_1,\frac12-\alpha}}\|a\|^{\frac12}_{H^{\sigma_1,\frac12+\alpha}}\|b\|_{H^{\sigma_2,s}}\sum_{j'\ge j-4}\sum_{|k'-k|\le 4}2^{-(j'-j)s}c_{k'}c_{j'}\\ &\le Cc_{j,k}2^{-js}2^{(1-\sigma_1-\sigma_2)k}\|a\|^{\frac12}_{H^{\sigma_1,\frac12-\alpha}}\|a\|^{\frac12}_{H^{\sigma_1,\frac12+\alpha}}\|b\|_{H^{\sigma_2,s}}.\end{aligned}\tag{4-6}$$

We turn to III. We have

$$\mathrm{III} = \sum_{j' \geq j-4} \sum_{\substack{k',k'' \geq k-2 \\ |k'-k''| \leq 1}} \Delta_j^v \dot{\Delta}_k^h (S_{j'+2}^v \dot{\Delta}_{k'}^h a \Delta_{j'}^v \dot{\Delta}_{k''}^h b).$$

So, by Lemma 4.1,

$$\begin{aligned} \|\mathrm{III}\|_{L^2} &\leq C \sum_{j' \geq j-4} \sum_{\substack{k',k'' \geq k-2 \\ |k'-k''| \leq 1}} 2^k \|S_{j'+2}^v \dot{\Delta}_{k'}^h a \Delta_{j'}^v \dot{\Delta}_{k''}^h b\|_{L^2_{x_3} L^1_{x_h}} \\ &\leq C \sum_{j' \geq j-4} \sum_{\substack{k',k'' \geq k-2 \\ |k'-k''| \leq 1}} 2^k \|S_{j'+2}^v \dot{\Delta}_{k'}^h a\|_{L^\infty_{x_3} L^2_{x_h}} \|\Delta_{j'}^v \dot{\Delta}_{k''}^h b\|_{L^2} \\ &\leq C 2^{-js} 2^{(1-\sigma_1-\sigma_2)k} \|a\|^{\frac{1}{2}}_{H^{\sigma_1,\frac{1}{2}-\alpha}} \|a\|^{\frac{1}{2}}_{H^{\sigma_1,\frac{1}{2}+\alpha}} \|b\|_{H^{\sigma_2,s}} \sum_{j' \geq j-4} \sum_{k' \geq k-2} 2^{-(\sigma_1+\sigma_2)(k'-k)} 2^{-(j'-j)s} c_{k'} c_{j'} \\ &\leq C c_{j,k} 2^{-js} 2^{(1-\sigma_1-\sigma_2)k} \|a\|^{\frac{1}{2}}_{H^{\sigma_1,\frac{1}{2}-\alpha}} \|a\|^{\frac{1}{2}}_{H^{\sigma_1,\frac{1}{2}+\alpha}} \|b\|_{H^{\sigma_2,s}}. \end{aligned} \tag{4-7}$$

Summing up (4-5), (4-6), and (4-7), we obtain

$$\|\Delta_j^v \dot{\Delta}_k^h (R_a b)\|_{L^2} \leq C c_{j,k} 2^{-js} 2^{(1-\sigma_1-\sigma_2)k} \|a\|^{\frac{1}{2}}_{H^{\sigma_1,\frac{1}{2}-\alpha}} \|a\|^{\frac{1}{2}}_{H^{\sigma_1,\frac{1}{2}+\alpha}} \|b\|_{H^{\sigma_2,s}}.$$

The lemma is proved in the case when the function Ψ is identically 0. In order to treat the general case, we only need to notice the fact that

$$|\mathcal{F}[\Delta_j \dot{\Delta}_k^h (R_a b)]_\Psi(\xi)| \leq \mathcal{F}[\Delta_j \dot{\Delta}_k^h (R_{a_\Psi^+} b_\Psi^+)](\xi). \qquad \square$$

As a consequence of Lemma 4.2 and (4-2), we have:

Lemma 4.3. *Let* $\frac{1}{2} > \alpha > 0$, $s > 0$, *and* $\sigma_1, \sigma_2 < 1$ *such that* $\sigma_1 + \sigma_2 > 0$. *Let* $\{\tilde{\sigma}_1, \tilde{\sigma}_2\} = \{\sigma_1, \sigma_2\}$. *Then*

$$\|(ab)_\Psi\|_{H^{\sigma_1+\sigma_2-1,s}} \leq C\big(\|a_\Psi\|^{\frac{1}{2}}_{H^{\sigma_1,\frac{1}{2}-\alpha}} \|a_\Psi\|^{\frac{1}{2}}_{H^{\sigma_1,\frac{1}{2}+\alpha}} \|b_\Psi\|_{H^{\sigma_2,s}} + \|a_\Psi\|_{H^{\tilde{\sigma}_1,s}} \|b_\Psi\|^{\frac{1}{2}}_{H^{\tilde{\sigma}_2,\frac{1}{2}-\alpha}} \|b_\Psi\|^{\frac{1}{2}}_{H^{\tilde{\sigma}_2,\frac{1}{2}+\alpha}}\big).$$

5. Classical analytical-type estimates

In this section, we prove Proposition 3.3. In this part, we don't need to use any regularizing effect from the analyticity, but only the fact that the $e^{\Phi(t,\xi_3)}$ is a sublinear function.

Notice that $\partial_t v_\Phi + a\lambda\dot{\theta}(t)\langle D_3\rangle v_\Phi = (\partial_t v)_\Phi$, we find from (RNS$_\varepsilon$) that

$$\begin{cases} \partial_t v_\Phi^h + a\lambda\dot{\theta}(t)\langle D_3\rangle v_\Phi^h - \Delta_h v_\Phi^h - \varepsilon^2\partial_3^2 v_\Phi^h + \varepsilon^{\frac{1}{2}}(v\cdot\nabla v^h)_\Phi = -\nabla_h q_\Phi, \\ \partial_t v_\Phi^3 + a\lambda\dot{\theta}(t)\langle D_3\rangle v_\Phi^3 - \Delta_h v_\Phi^3 - \varepsilon^2\partial_3^2 v_\Phi^3 + \varepsilon^{\frac{1}{2}}(v\cdot\nabla v^3)_\Phi = -\varepsilon^2\partial_3 q_\Phi, \\ \operatorname{div} v_\Phi = 0, \\ v_\Phi(0) = e^{a\langle D_3\rangle} v_0(x). \end{cases} \tag{5-1}$$

Step 1. Estimates on the vertical component $\boldsymbol{v^3_\Phi}$**.** Noting that $\dot\theta(t)\geq 0$, we get from the second equation of (5-1) that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v^3_\Phi(t)\|^2_{H^{-\frac12,s}}+\|\nabla_h v^3_\Phi(t)\|^2_{H^{-\frac12,s}}+\|\varepsilon\partial_3 v^3_\Phi(t)\|^2_{H^{-\frac12,s}}&\\ \leq -\varepsilon^{\frac12}\big((v^h\cdot\nabla_h v^3)_\Phi, v^3_\Phi\big)_{H^{-\frac12,s}}+\varepsilon^{\frac12}\big((v^3\mathrm{div}_h v^h)_\Phi, v^3_\Phi\big)_{H^{-\frac12,s}}-\varepsilon^2\big(\partial_3 q_\Phi, v^3_\Phi\big)_{H^{-\frac12,s}}&\\ \stackrel{\text{def}}{=}\mathrm{I}+\mathrm{II}+\mathrm{III}.&\end{aligned}\tag{5-2}$$

Here we used the fact that $\mathrm{div}\, v=0$, so $v\cdot\nabla v^3=v^h\cdot\nabla_h v^3-v^3\mathrm{div}_h v^h$.

For II, an application of Lemma 4.3 gives

$$\begin{aligned}|\mathrm{II}|&\leq\varepsilon^{\frac12}\|(v^3\mathrm{div}_h v^h)_\Phi\|_{H^{-\frac12,s}}\|v^3_\Phi\|_{H^{-\frac12,s}}\\ &\leq C\varepsilon^{\frac12}\Big(\|v^3_\Phi\|^{\frac12}_{H^{\frac12,\frac12-\alpha}}\|v^3_\Phi\|^{\frac12}_{H^{\frac12,\frac12+\alpha}}\|\nabla_h v^h_\Phi\|_{H^{0,s}}+\|v^3_\Phi\|_{H^{\frac12,s}}\|\nabla_h v^h_\Phi\|^{\frac12}_{H^{0,\frac12-\alpha}}\|\nabla_h v^h_\Phi\|^{\frac12}_{H^{0,\frac12+\alpha}}\Big)\|v^3_\Phi\|_{H^{-\frac12,s}}\\ &\leq C\dot\theta(t)\|v^3_\Phi\|^2_{H^{-\frac12,s}}+\tfrac{1}{100}a\|\nabla_h v^h_\Phi\|^2_{H^{0,s}}+\tfrac{1}{100}\|v^3_\Phi\|^2_{H^{\frac12,s}}.\end{aligned}\tag{5-3}$$

For I, we get by integration by parts that

$$\mathrm{I}=\varepsilon^{\frac12}\big((\mathrm{div}_h v^h v^3)_\Phi, v^3_\Phi\big)_{H^{-\frac12,s}}+\varepsilon^{\frac12}\big((v^h v^3)_\Phi, \nabla_h v^3_\Phi\big)_{H^{-\frac12,s}}\stackrel{\text{def}}{=}\mathrm{I}_1+\mathrm{I}_2.$$

As in (5-3), we have

$$|\mathrm{I}_1|\leq C\dot\theta(t)\|v^3_\Phi\|^2_{H^{-\frac12,s}}+\tfrac{1}{100}a\|\nabla_h v^h_\Phi\|^2_{H^{0,s}}+\tfrac{1}{100}\|v^3_\Phi\|^2_{H^{\frac12,s}},\tag{5-4}$$

and by Lemma 4.3,

$$\begin{aligned}|\mathrm{I}_2|&\leq\varepsilon^{\frac12}\|(v^3v^h)_\Phi\|_{H^{-\frac12,s}}\|\nabla_h v^3_\Phi\|_{H^{-\frac12,s}}\\ &\leq C\varepsilon^{\frac12}\Big(\|v^h_\Phi\|_{H^{\frac12,s}}\|v^3_\Phi\|^{\frac12}_{H^{0,\frac12-\alpha}}\|v^3_\Phi\|^{\frac12}_{H^{0,\frac12+\alpha}}+\|v^h_\Phi\|^{\frac12}_{H^{0,\frac12-\alpha}}\|v^h_\Phi\|^{\frac12}_{H^{0,\frac12+\alpha}}\|v^3_\Phi\|_{H^{\frac12,s}}\Big)\|\nabla_h v^3_\Phi\|_{H^{-\frac12,s}}\\ &\leq C\Psi(t)\big(\varepsilon\|v^h_\Phi\|^2_{H^{\frac12,s}}+\|v^3_\Phi\|^2_{H^{\frac12,s}}\big)+\tfrac{1}{100}\|\nabla_h v^3_\Phi\|^2_{H^{-\frac12,s}}.\end{aligned}\tag{5-5}$$

Now, we turn to the estimates of the pressure. Recall that

$$-\Delta_\varepsilon p=\varepsilon^{\frac12}\big(\partial_i\partial_j(v^iv^j)+\partial_i\partial_3(v^iv^3)-2\partial_3(v^3\mathrm{div}_h v^h)\big).$$

Here and in what follows the indexes i, j run from 1 to 2. Thus, we can write $p=p^1+p^2+p^3$, with

$$p^1=\varepsilon^{\frac12}(-\Delta_\varepsilon)^{-1}\partial_i\partial_j(v^iv^j),\quad p^2=\varepsilon^{\frac12}(-\Delta_\varepsilon)^{-1}\partial_i\partial_3(v^iv^3),\quad p^3=-2\varepsilon^{\frac12}(-\Delta_\varepsilon)^{-1}\partial_3(v^3\mathrm{div}_h v^h).\tag{5-6}$$

Integrating by parts, we get

$$\varepsilon^2(\partial_3 p^1_\Phi, v^3_\Phi)_{H^{-\frac12,s}}=-\varepsilon(p^1_\Phi,\varepsilon\partial_3 v^3_\Phi)_{H^{-\frac12,s}}\leq C\varepsilon^2\|p^1_\Phi\|^2_{H^{-\frac12,s}}+\tfrac{1}{100}\|\varepsilon\partial_3 v^3_\Phi\|^2_{H^{-\frac12,s}},$$

which together with the fact that the operator $\partial_i\partial_j(-\Delta_\varepsilon)^{-1}$ is bounded on $H^{\sigma,s}$ together with Lemma 4.3 implies that

$$\begin{aligned}\varepsilon^2(\partial_3 p^1_\Phi, v^3_\Phi)_{H^{-\frac12,s}}&\leq C\varepsilon^3\|(v^h\otimes v^h)_\Phi\|^2_{H^{-\frac12,s}}+\tfrac{1}{100}\|\varepsilon\partial_3 v^3_\Phi\|^2_{H^{-\frac12,s}}\\ &\leq C\varepsilon^3\|v^h_\Phi\|^2_{H^{\frac12,s}}\|v^h_\Phi\|_{H^{0,\frac12-\alpha}}\|v^h_\Phi\|_{H^{0,\frac12+\alpha}}+\tfrac{1}{100}\|\varepsilon\partial_3 v^3_\Phi\|^2_{H^{-\frac12,s}}\\ &\leq C\varepsilon^2\Psi(t)\varepsilon\|v^h_\Phi\|^2_{H^{\frac12,s}}+\tfrac{1}{100}\|\varepsilon\partial_3 v^3_\Phi\|^2_{H^{-\frac12,s}}.\end{aligned}\tag{5-7}$$

For the term containing p_2, we get by integration by parts that

$$\varepsilon^2(\partial_3 p^2_\Phi, v^3_\Phi)_{H^{-\frac12,s}} = -\varepsilon^{\frac12}\big(\varepsilon^2\partial_3^2(-\Delta_\varepsilon)^{-1}(v^i v^3)_\Phi, \partial_i v^3_\Phi\big)_{H^{-\frac12,s}}.$$

Using the fact that $(\varepsilon\partial_3)^2(-\Delta_\varepsilon)^{-1}$ is bounded on $H^{\sigma,s}$ together with Lemma 4.3, we then have

$$\begin{aligned}\varepsilon^2(\partial_3 p^2_\Phi, v^3_\Phi)_{H^{-\frac12,s}} &\le C\varepsilon^{\frac12}\|(v^3v^h)_\Phi\|_{H^{-\frac12,s}}\|\nabla_h v^3_\Phi\|_{H^{-\frac12,s}}\\ &\le C\Psi(t)\big(\varepsilon\|v^h_\Phi\|^2_{H^{\frac12,s}}+\|v^3_\Phi\|^2_{H^{\frac12,s}}\big)+\tfrac{1}{100}\|\nabla_h v^3_\Phi\|^2_{H^{-\frac12,s}}.\end{aligned}\tag{5-8}$$

For the last term, coming from p_3, we use again the fact that $(\varepsilon\partial_3)^2(-\Delta_\varepsilon)^{-1}$ is bounded on $H^{\sigma,s}$ and obtain

$$\begin{aligned}\varepsilon^2(\partial_3 p^3_\Phi, v^3_\Phi)_{H^{-\frac12,s}} &\le C\varepsilon^{\frac12}\|(v^3\operatorname{div} v^h)_\Phi\|_{H^{-\frac12,s}}\|v^3_\Phi\|_{H^{-\frac12,s}}\\ &\le C\dot\theta(t)\|v^3_\Phi\|^2_{H^{-\frac12,s}}+\tfrac{1}{100}a\|\nabla_h v^h_\Phi\|^2_{H^{0,s}}+\tfrac{1}{100}\|v^3_\Phi\|^2_{H^{\frac12,s}}.\end{aligned}\tag{5-9}$$

Summing up (5-2)–(5-5) and (5-7)–(5-9), we obtain

$$\begin{aligned}&\frac{d}{dt}\|v^3_\Phi(t)\|^2_{H^{-\frac12,s}}+\|v^3_\Phi(t)\|^2_{H^{\frac12,s}}\\ &\qquad\le C\dot\theta(t)\|v^3_\Phi\|^2_{H^{-\frac12,s}}+C\Psi(t)\big(\varepsilon\|v^h_\Phi\|^2_{H^{\frac12,s}}+\|v^3_\Phi\|^2_{H^{\frac12,s}}\big)+\tfrac{1}{20}a\|\nabla_h v^h_\Phi\|^2_{H^{0,s}},\end{aligned}\tag{5-10}$$

where we used the equality $\|\nabla_h v^3_\Phi\|^2_{H^{-\frac12,s}}=\|v^3_\Phi\|^2_{H^{\frac12,s}}$.

Step 2. Estimates on the horizontal component v^h_Φ. From the first equation of (5-1), we infer that

$$\begin{aligned}&\frac12\frac{d}{dt}\|\varepsilon^{\frac12}v^h_\Phi(t)\|^2_{H^{-\frac12,s}}+\|\varepsilon^{\frac12}\nabla_h v^h_\Phi(t)\|^2_{H^{-\frac12,s}}+\varepsilon\|\varepsilon\partial_3 v^h_\Phi(t)\|^2_{H^{-\frac12,s}}\\ &\qquad\le -\varepsilon\big((v\cdot\nabla v^h)_\Phi, \varepsilon^{\frac12}v^h_\Phi\big)_{H^{-\frac12,s}}-\varepsilon\big(\nabla_h q_\Phi, v^h_\Phi\big)_{H^{-\frac12,s}}\\ &\qquad\overset{\text{def}}{=}\mathrm{I}+\mathrm{II}.\end{aligned}\tag{5-11}$$

We rewrite I as

$$\mathrm{I}=-\varepsilon\big((v^h\cdot\nabla_h v^h)_\Phi, \varepsilon^{\frac12}v^h_\Phi\big)_{H^{-\frac12,s}}-\varepsilon\big((v^3\partial_3 v^h)_\Phi, \varepsilon^{\frac12}v^h_\Phi\big)_{H^{-\frac12,s}}\overset{\text{def}}{=}\mathrm{I}_1+\mathrm{I}_2.$$

An application of Lemma 4.3 gives

$$\begin{aligned}|\mathrm{I}_1| &\le \varepsilon\|(v^h\nabla_h v^h)_\Phi\|_{H^{-\frac12,s}}\|\varepsilon^{\frac12}v^h_\Phi\|_{H^{-\frac12,s}}\\ &\le C\varepsilon\Big(\|v^h_\Phi\|^{\frac12}_{H^{\frac12,\frac12-\alpha}}\|v^h_\Phi\|^{\frac12}_{H^{\frac12,\frac12+\alpha}}\|\nabla_h v^h_\Phi\|_{H^{0,s}}+\|v^h_\Phi\|_{H^{\frac12,s}}\|\nabla_h v^h_\Phi\|^{\frac12}_{H^{0,\frac12-\alpha}}\|\nabla_h v^h_\Phi\|^{\frac12}_{H^{0,\frac12+\alpha}}\Big)\|\varepsilon^{\frac12}v^h_\Phi\|_{H^{-\frac12,s}}\\ &\le C\dot\theta(t)\|\varepsilon^{\frac12}v^h_\Phi\|^2_{H^{-\frac12,s}}+\frac{1}{100}a\|\nabla_h v^h_\Phi\|^2_{H^{0,s}}+\frac{1}{100}\|\varepsilon^{\frac12}v^h_\Phi\|^2_{H^{\frac12,s}}.\end{aligned}\tag{5-12}$$

For I_2, we use integration by parts and $\operatorname{div} v=0$ to get

$$\mathrm{I}_2=-\varepsilon\big((\operatorname{div}_h v^h v^h)_\Phi, \varepsilon^{\frac12}v^h_\Phi\big)_{H^{-\frac12,s}}+\big((v^hv^3)_\Phi, \varepsilon^{\frac12}\varepsilon\partial_3 v^h_\Phi\big)_{H^{-\frac12,s}}\overset{\text{def}}{=}\mathrm{I}_{21}+\mathrm{I}_{22}.$$

As in (5-12), we have

$$|I_{21}|\le C\dot\theta(t)\|\varepsilon^{\frac12}v^h_\Phi\|^2_{H^{-\frac12,s}}+\tfrac{1}{100}a\|\nabla_h v^h_\Phi\|^2_{H^{0,s}}+\tfrac{1}{100}\|\varepsilon^{\frac12}v^h_\Phi\|^2_{H^{\frac12,s}},\tag{5-13}$$

and by Lemma 4.3,

$$\begin{aligned}|I_{22}| &\le \|(v^3v^h)_\Phi\|_{H^{-\frac12,s}}\varepsilon^{\frac12}\|\varepsilon\partial_3 v^h_\Phi\|_{H^{-\frac12,s}}\\ &\le C\big(\|v^h_\Phi\|_{H^{0,s}}\|v^3_\Phi\|^{\frac12}_{H^{\frac12,\frac12-\alpha}}\|v^3_\Phi\|^{\frac12}_{H^{\frac12,\frac12+\alpha}}+\|v^h_\Phi\|^{\frac12}_{H^{0,\frac12-\alpha}}\|v^h_\Phi\|^{\frac12}_{H^{0,\frac12+\alpha}}\|v^3_\Phi\|_{H^{\frac12,s}}\big)\varepsilon^{\frac12}\|\varepsilon\partial_3 v^h_\Phi\|_{H^{-\frac12,s}}\\ &\le Ca\dot\theta(t)\|v^h_\Phi\|^2_{H^{0,s}}+C\Psi(t)\|v^3_\Phi\|^2_{H^{\frac12,s}}+\frac{\varepsilon}{100}\|\varepsilon\partial_3 v^h_\Phi\|_{H^{-\frac12,s}}.\end{aligned}\tag{5-14}$$

To deal with the pressure, we write $p=p^1+p^2+p^3$, with p^1, p^2, p^3 defined by (5-6). Using the boundedness of the operator $\partial_i\partial_j(-\Delta_\varepsilon)^{-1}$ on $H^{\sigma,s}$ together with Lemma 4.3, we have

$$\begin{aligned}\varepsilon(\nabla_h p^1_\Phi, v^h_\Phi)_{H^{-\frac12,s}} &= -\varepsilon((-\Delta_\varepsilon)^{-1}\partial_i\partial_j(v^iv^j)_\Phi, \varepsilon^{\frac12}\mathrm{div}_h v^h_\Phi)_{H^{-\frac12,s}}\\ &\le C\varepsilon\|(v^h\otimes v^h)_\Phi\|_{H^{-\frac12,s}}\|\varepsilon^{\frac12}\nabla_h v^h_\Phi\|_{H^{-\frac12,s}}\\ &\le C\varepsilon^{\frac12}\|v^h_\Phi\|^{\frac12}_{H^{0,\frac12-\alpha}}\|v^h_\Phi\|^{\frac12}_{H^{0,\frac12+\alpha}}\|\varepsilon^{\frac12}v^h_\Phi\|_{H^{\frac12,s}}\|\varepsilon^{\frac12}\nabla_h v^h_\Phi\|_{H^{-\frac12,s}}\\ &\le C\Psi(t)\|\varepsilon^{\frac12}v^h_\Phi\|^2_{H^{\frac12,s}}+\tfrac{1}{100}\|\varepsilon^{\frac12}\nabla_h v^h_\Phi\|^2_{H^{-\frac12,s}}.\end{aligned}\tag{5-15}$$

For the term coming from p_2, we integrate by parts to get

$$\varepsilon(\nabla_h p^2_\Phi, v^h_\Phi)_{H^{-\frac12,s}} = -\big(\varepsilon\partial_i\partial_3(-\Delta_\varepsilon)^{-1}(v^iv^3)_\Phi, \varepsilon^{\frac12}\mathrm{div}_h v^h_\Phi\big)_{H^{-\frac12,s}},$$

then note that $\varepsilon\partial_3\partial_i(-\Delta_\varepsilon)^{-1}$ is bounded on $H^{\sigma,s}$. We get, by Lemma 4.3,

$$\begin{aligned}\varepsilon(\nabla_h p^2_\Phi, v^h_\Phi)_{H^{-\frac12,s}} &\le C\|(v^3v^h)_\Phi\|_{H^{-\frac12,s}}\|\varepsilon^{\frac12}\nabla_h v^h_\Phi\|_{H^{-\frac12,s}}\\ &\le Ca\dot\theta(t)\|v^h_\Phi\|^2_{H^{0,s}}+C\Psi(t)\|v^3_\Phi\|^2_{H^{\frac12,s}}+\tfrac{1}{100}\|\varepsilon^{\frac12}\nabla_h v^h_\Phi\|^2_{H^{-\frac12,s}}.\end{aligned}\tag{5-16}$$

Similarly,

$$\begin{aligned}&\varepsilon(\nabla_h p^3_\Phi, v^h_\Phi)_{H^{-\frac12,s}}\\ &\quad\le C\|(v^3\mathrm{div}_h v^h)_\Phi\|_{H^{-\frac12,s}}\|\varepsilon^{\frac12}v^h_\Phi\|_{H^{-\frac12,s}}\\ &\quad\le \big(\|v^3_\Phi\|^{\frac12}_{H^{\frac12,\frac12-\alpha}}\|v^3_\Phi\|^{\frac12}_{H^{\frac12,\frac12+\alpha}}\|\nabla_h v^h_\Phi\|_{H^{0,s}}+\|v^3_\Phi\|_{H^{\frac12,s}}\|\nabla_h v^h_\Phi\|^{\frac12}_{H^{0,\frac12-\alpha}}\|\nabla_h v^h_\Phi\|^{\frac12}_{H^{0,\frac12+\alpha}}\big)\|\varepsilon^{\frac12}v^h_\Phi\|_{H^{-\frac12,s}}\\ &\quad\le C\dot\theta(t)\|\varepsilon^{\frac12}v^h_\Phi\|^2_{H^{-\frac12,s}}+\tfrac{1}{100}a\|\nabla_h v^h_\Phi\|^2_{H^{0,s}}+\tfrac{1}{100}\|v^3_\Phi\|^2_{H^{\frac12,s}}.\end{aligned}\tag{5-17}$$

Summing up (5-11)–(5-17) yields

$$\begin{aligned}&\frac{d}{dt}\|\varepsilon^{\frac12}v^h_\Phi(t)\|^2_{H^{-\frac12,s}}+\|\varepsilon^{\frac12}v^h_\Phi(t)\|^2_{H^{\frac12,s}}\\ &\quad\le C\dot\theta(t)\big(\|\varepsilon^{\frac12}v^h_\Phi\|^2_{H^{-\frac12,s}}+a\|v^h_\Phi\|^2_{H^{0,s}}\big)\\ &\qquad+C\Psi(t)\big(\|v^3_\Phi\|^2_{H^{\frac12,s}}+\|\varepsilon^{\frac12}v^h_\Phi\|^2_{H^{\frac12,s}}\big)+\tfrac{1}{20}a\|\nabla_h v^h_\Phi\|^2_{H^{0,s}}+\tfrac{1}{20}\|v^3_\Phi\|^2_{H^{\frac12,s}}.\end{aligned}\tag{5-18}$$

Now we are in a position to prove Proposition 3.3. Combining the energy estimate (5-10) with (5-18), we obtain

$$\begin{aligned}&\frac{d}{dt}\big(\|\varepsilon^{\frac12}v^h_\Phi(t)\|^2_{H^{-\frac12,s}}+\|v^3_\Phi(t)\|^2_{H^{-\frac12,s}}\big)+\big(\|\varepsilon^{\frac12}v^h_\Phi(t)\|^2_{H^{\frac12,s}}+\|v^3_\Phi(t)\|^2_{H^{\frac12,s}}\big)\\ &\quad\le C\dot\theta(t)\big(\|\varepsilon^{\frac12}v^h_\Phi\|^2_{H^{-\frac12,s}}+\|v^3_\Phi\|^2_{H^{-\frac12,s}}\big)+Ca\dot\theta(t)\|v^h_\Phi\|^2_{H^{0,s}}\\ &\qquad+C\Psi(t)\big(\|\varepsilon^{\frac12}v^h_\Phi\|^2_{H^{-\frac12,s}}+\|v^3_\Phi\|^2_{H^{\frac12,s}}\big)+\tfrac{1}{10}a\|\nabla_h v^h_\Phi\|^2_{H^{0,s}}.\end{aligned}$$

From this and Gronwall's inequality, it follows that

$$\|\varepsilon^{\frac12} v^h_\Phi(t)\|^2_{H^{-\frac12,s}} + \|v^3_\Phi(t)\|^2_{H^{-\frac12,s}} + \int_0^t \big(\|\varepsilon^{\frac12} v^h_\Phi(\tau)\|^2_{H^{\frac12,s}} + \|v^3_\Phi(\tau)\|^2_{H^{\frac12,s}}\big)d\tau$$
$$\le \exp\Big(C\int_0^t \dot\theta(\tau)d\tau\Big)\Big(\|e^{a\langle D_3\rangle}v_0\|^2_{H^{-\frac12,s}} + a\int_0^t \dot\theta(\tau)\|v^h_\Phi(\tau)\|^2_{H^{0,s}}d + C\int_0^t \Psi(\tau)\big(\|\varepsilon^{\frac12} v^h_\Phi(\tau)\|^2_{H^{\frac12,s}}\tau$$
$$+\|v^3_\Phi(\tau)\|^2_{H^{\frac12,s}}\big)d\tau + \tfrac{1}{10}a\int_0^t \|\nabla_h v^h_\Phi(\tau)\|^2_{H^{0,s}}d\tau\Big).$$

This finishes the proof of Proposition 3.3. □

6. Regularizing the effect of analyticity

Let's now prove Proposition 3.4. Here we will encounter two kinds of bad terms, where we lose a vertical derivative. The first one is $(v^3\partial_3 v^h)_\Phi$ and the second term comes from $-\nabla_h p$. In this last term, we really lose a vertical derivative. To compensate this loss, we use the divergence-free condition ($\partial_3 u^3 = -\operatorname{div}_h u^h$) and more important, the fact that the equation contain an extra-regularizing term given by the analyticity of the solution.

Step 1. Estimates on the horizontal component v^h_Φ. Let us recall that v^h_Φ verifies the equations

$$\partial_t v^h_\Phi + a\lambda\dot\theta(t)\langle D_3\rangle v^h_\Phi - \Delta_h v^h_\Phi - \varepsilon^2\partial_3^2 v^h_\Phi + \varepsilon^{\frac12}(v\cdot\nabla v^h)_\Phi = -\nabla_h q_\Phi.$$

Note that $\dot\theta \ge 0$, we perform an energy estimate in $H^{0,s}$ to obtain

$$\frac12\frac{d}{dt}\|v^h_\Phi\|^2_{H^{0,s}} + a\lambda\dot\theta(t)\|v^h_\Phi\|^2_{H^{0,s+\frac12}} + \|\nabla_h v^h_\Phi\|^2_{H^{0,s}} + \|\varepsilon\partial_3 v^h_\Phi\|^2_{H^{0,s}}$$
$$\le \varepsilon^{\frac12}((v^h\otimes v^h)_\Phi, \nabla_h v^h_\Phi)_{H^{0,s}} - \varepsilon^{\frac12}(\partial_3(v^3 v^h)_\Phi, v^h_\Phi)_{H^{0,s}} - (\nabla_h p_\Phi, v^h_\Phi)_{H^{0,s}}$$
$$\stackrel{\text{def}}{=} \mathrm{I} + \mathrm{II} + \mathrm{III}. \tag{6-1}$$

We get by Lemma 4.3 and interpolation that

$$\begin{aligned}|\mathrm{I}| &\le C\varepsilon^{\frac12}\|(v^h\otimes v^h)_\Phi\|_{H^{0,s}}\|\nabla_h v^h_\Phi\|_{H^{0,s}}\\
&\le C\varepsilon^{\frac12}\|v^h_\Phi\|^{\frac12}_{H^{\frac12,\frac12-\alpha}}\|v^h_\Phi\|^{\frac12}_{H^{\frac12,\frac12+\alpha}}\|v^h_\Phi\|_{H^{\frac12,s}}\|\nabla_h v^h_\Phi\|_{H^{0,s}}\\
&\le C\varepsilon\|v^h_\Phi\|_{H^{\frac12,\frac12-\alpha}}\|v^h_\Phi\|_{H^{\frac12,\frac12+\alpha}}\|v^h_\Phi\|^2_{H^{\frac12,s}} + \tfrac{1}{100}\|\nabla_h v^h_\Phi\|_{H^{0,s}}\\
&\le C\varepsilon\|v^h_\Phi\|^{\frac12}_{H^{0,\frac12-\alpha}}\|v^h_\Phi\|^{\frac12}_{H^{0,\frac12+\alpha}}\|\nabla_h v^h_\Phi\|^{\frac12}_{H^{0,\frac12-\alpha}}\|\nabla_h v^h_\Phi\|^{\frac12}_{H^{0,\frac12+\alpha}}\|v^h_\Phi\|_{H^{0,s}}\|v^h_\Phi\|_{H^{1,s}} + \tfrac{1}{100}\|\nabla_h v^h_\Phi\|^2_{H^{0,s}}\\
&\le C\Psi(t)\|v^h_\Phi\|^2_{H^{1,s}} + C\dot\theta(t)\|v^h_\Phi\|^2_{H^{0,s}} + \tfrac{1}{100}\|\nabla_h v^h_\Phi\|^2_{H^{0,s}}.\end{aligned} \tag{6-2}$$

To estimate II, we use Bony's decomposition (4-3) to rewrite it as

$$\mathrm{II} = -\varepsilon^{\frac12}(\partial_3(T^v_{v^h}v_3)_\Phi, v^h_\Phi)_{H^{0,s}} - \varepsilon^{\frac12}(\partial_3(R^v_{v_3}v^h)_\Phi, v^h_\Phi)_{H^{0,s}} \stackrel{\text{def}}{=} \mathrm{II}_1 + \mathrm{II}_2.$$

From the proof of Lemma 4.2, it is easy to find that

$$\begin{aligned}|\mathrm{II}_2| &\le C\||D_3|^{\frac12}(R^v_{v^3}v^h)_\Phi\|_{H^{-\frac12,s}}\||\varepsilon D_3|^{\frac12}|D_h|^{\frac12}v^h_\Phi\|_{H^{0,s}}\\ &\le C\|v^3_\Phi\|^{\frac12}_{H^{\frac12,\frac12-\alpha}}\|v^3_\Phi\|^{\frac12}_{H^{\frac12,\frac12+\alpha}}\|v^h_\Phi\|_{H^{0,s+\frac12}}\|\nabla_\varepsilon v^h\|_{H^{0,s}}\\ &\le Ca\dot\theta(t)\|v^h_\Phi\|^2_{H^{0,s+\frac12}}+\tfrac{1}{100}\|\nabla_\varepsilon v^h\|^2_{H^{0,s}}.\end{aligned}\tag{6-3}$$

Due to $\operatorname{div} v=0$, we rewrite II_1 as

$$\mathrm{II}_1=\varepsilon^{\frac12}((T^v_{v^h}\operatorname{div}_h v^h)_\Phi,v^h_\Phi)_{H^{0,s}}-\varepsilon^{\frac12}((T^v_{\partial_3 v^h}v^3)_\Phi,v^h_\Phi)_{H^{0,s}}\overset{\text{def}}{=}\mathrm{II}_{11}+\mathrm{II}_{12}.$$

Using Lemma 4.2 and interpolation, we have

$$\begin{aligned}|\mathrm{II}_{11}| &\le \varepsilon^{\frac12}\|(T^v_{v^h}\operatorname{div}_h v^h)_\Phi\|_{H^{-\frac12,s}}\|v^h_\Phi\|_{H^{\frac12,s}}\\ &\le C\varepsilon^{\frac12}\|v^h_\Phi\|^{\frac12}_{H^{\frac12,\frac12-\alpha}}\|v^h_\Phi\|^{\frac12}_{H^{\frac12,\frac12+\alpha}}\|\nabla_h v^h_\Phi\|_{H^{0,s}}\|v^h_\Phi\|_{H^{\frac12,s}}\\ &\le C\Psi(t)\|v^h_\Phi\|^2_{H^{1,s}}+C\dot\theta(t)\|v^h_\Phi\|^2_{H^{0,s}}+\tfrac{1}{100}\|\nabla_h v^h_\Phi\|^2_{H^{0,s}}.\end{aligned}\tag{6-4}$$

From the proof of Lemma 4.2, and using the fact that $s<1$, we can conclude that

$$\begin{aligned}|\mathrm{II}_{12}| &\le C\|v^h_\Phi\|_{H^{0,s+\frac12}}\|v^3_\Phi\|_{H^{\frac12,s}}\|\nabla_\varepsilon v^h\|_{H^{0,s}}\\ &\le C\|v^h_\Phi\|^{\frac12}_{H^{0,s+\frac12}}\|v^3_\Phi\|^{\frac12}_{H^{\frac12,\frac12-\alpha}}\|v^3_\Phi\|^{\frac12}_{H^{\frac12,\frac12+\alpha}}\|\nabla_\varepsilon v^h\|_{H^{0,s}}\\ &\le Ca\dot\theta(t)\|v^h_\Phi\|^2_{H^{0,s+\frac12}}+\tfrac{1}{100}\|\nabla_\varepsilon v^h\|^2_{H^{0,s}}.\end{aligned}\tag{6-5}$$

We next turn to the estimate of the pressure. Recall that $p=p^1+p^2+p^3$ with p^1,p^2,p^3 defined by (5-6). Using the boundedness of $(-\Delta_\varepsilon)^{-1}\partial_i\partial_j$ on $H^{\sigma,s}$ together with Lemma 4.3, we get

$$\begin{aligned}(\nabla_h p^1_\Phi,v^h_\Phi)_{H^{0,s}} &= -\varepsilon^{\frac12}((-\Delta_\varepsilon)^{-1}\partial_i\partial_j(v^iv^j)_\Phi,\operatorname{div} v^h_\Phi)_{H^{0,s}}\\ &\le C\varepsilon^{\frac12}\|(v^h\otimes v^h)_\Phi\|_{H^{0,s}}\|\nabla_h v^h_\Phi\|_{H^{0,s}}\\ &\le C\Psi(t)\|v^h_\Phi\|^2_{H^{1,s}}+C\dot\theta(t)\|v^h_\Phi\|^2_{H^{0,s}}+\tfrac{1}{100}\|\nabla_h v^h_\Phi\|^2_{H^{0,s}}.\end{aligned}\tag{6-6}$$

Notice that $\partial_i\partial_j(-\Delta_\varepsilon)^{-1}$ is bounded on $H^{\sigma,s}$. Exactly as in the estimate of II, we obtain

$$(\nabla_h p^2_\Phi,v^h_\Phi)_{H^{0,s}}\le C\Psi(t)\|v^h_\Phi\|^2_{H^{1,s}}+C\dot\theta(t)\|v^h_\Phi\|^2_{H^{0,s}}+Ca\dot\theta(t)\|v^h_\Phi\|^2_{H^{0,s+\frac12}}+\tfrac{1}{100}\|\nabla_\varepsilon v^h\|^2_{H^{0,s}}.\tag{6-7}$$

We write

$$\nabla_h p_3=-2\partial_3|D_3|^{-\frac12}\big(\nabla_h|D_h|^{\frac12}|\varepsilon D_3|^{1/2}(-\Delta_\varepsilon)^{-1}\big)|D_h|^{-\frac12}(v^3\operatorname{div}_h v^h);$$

thus,

$$(\nabla_h p^3_\Phi,v^h_\Phi)_{H^{0,s}}=-2(\big(\nabla_h|D_h|^{\frac12}|\varepsilon D_3|^{\frac12}(-\Delta_\varepsilon)^{-1}\big)|D_h|^{-\frac12}(v^3\operatorname{div}_h v^h),\partial_3\langle D_3\rangle^{-\frac12}v^h_\Phi)_{H^{0,s}}.$$

Note that $\nabla_h|D_h|^{\frac12}|\varepsilon D_3|^{\frac12}(-\Delta_\varepsilon)^{-1}$ is a bounded operator on $H^{\sigma,s}$. Thus we get, by Lemma 4.3,

$$\begin{aligned}&(\nabla_h p^3_\Phi,v^h_\Phi)_{H^{0,s}}\\ &\quad\le C\||D_h|^{-\frac12}(v^3\operatorname{div}_h v^h)\|_{H^{0,s}}\|\partial_3\langle D_3\rangle^{-\frac12}v^h_\Phi\|_{H^{0,s}}\\ &\quad\le C\big(\|v^3_\Phi\|^{\frac12}_{H^{\frac12,\frac12-\alpha}}\|v^3_\Phi\|^{\frac12}_{H^{\frac12,\frac12+\alpha}}\|\nabla_h v^h_\Phi\|_{H^{0,s}}+\|v^3_\Phi\|_{H^{\frac12,s}}\|\nabla_h v^h_\Phi\|^{\frac12}_{H^{0,\frac12-\alpha}}\|\nabla_h v^h_\Phi\|^{\frac12}_{H^{0,\frac12+\alpha}}\big)\|v^h\|_{H^{0,s+\frac12}}\\ &\quad\le Ca\dot\theta(t)\|v^h_\Phi\|^2_{H^{0,s+\frac12}}+\tfrac{1}{100}\|\nabla_h v^h_\Phi\|^2_{H^{0,s}}+\tfrac{1}{100}a^{-1}\|v^3_\Phi\|^2_{H^{\frac12,s}}.\end{aligned}\tag{6-8}$$

Summing up (6-1)–(6-8), we get by taking λ big enough that

$$\frac{d}{dt}\|v_\Phi^h(t)\|_{H^{0,s}}^2+\|\nabla_h v_\Phi^h(t)\|_{H^{0,s}}^2\le C\dot\theta(t)\|v_\Phi^h\|_{H^{0,s}}^2+C\Psi(t)\|v_\Phi^h\|_{H^{1,s}}^2+\tfrac{1}{20}a^{-1}\|v_\Phi^3\|_{H^{\frac12,s}}^2. \tag{6-9}$$

Step 2. Estimates on the vertical component v_Φ^3. Recall that v_Φ^3 satisfies

$$\partial_t v_\Phi^3+\lambda a\dot\theta(t)\langle D_3\rangle v_\Phi^3-\Delta_h v_\Phi^3-\varepsilon^2\partial_3^2 v_\Phi^3+\varepsilon^{\frac12}(v\cdot\nabla v^3)_\Phi=-\varepsilon^2\partial_3 q_\Phi.$$

We perform an energy estimate in $H^{0,s}$ to obtain

$$\begin{aligned}\frac12\frac{d}{dt}\|v_\Phi^3\|_{H^{0,s}}^2+\|\nabla_h v_\Phi^3\|_{H^{0,s}}^2+\|\varepsilon\partial_3 v_\Phi^3\|_{H^{0,s}}^2&\\ \le -\varepsilon^{\frac12}((v^h\cdot\nabla_h v^3)_\Phi, v_\Phi^3)_{H^{0,s}}&+\varepsilon^{\frac12}((v^3\operatorname{div}_h v^h)_\Phi, v_\Phi^3)_{H^{0,s}}-\varepsilon^2(\partial_3 p_\Phi, v_\Phi^3)_{H^{0,s}}\\ \stackrel{\text{def}}{=}\mathrm{I}+\mathrm{II}+\mathrm{III}.&\end{aligned} \tag{6-10}$$

Using Lemma 4.3 and interpolation, we have

$$\begin{aligned}|\mathrm{I}|&\le C\varepsilon^{\frac12}\|(v^h\cdot\nabla_h v^3)_\Phi\|_{H^{-\frac12,s}}\|v_\Phi^3\|_{H^{\frac12,s}}\\ &\le C\varepsilon^{\frac12}\Big(\|v_\Phi^h\|_{H^{\frac12,\frac12-\alpha}}^{\frac12}\|v_\Phi^h\|_{H^{\frac12,\frac12+\alpha}}^{\frac12}\|\nabla_h v_\Phi^3\|_{H^{0,s}}+\|v_\Phi^h\|_{H^{\frac12,s}}\|\nabla_h v_\Phi^3\|_{H^{0,\frac12-\alpha}}^{\frac12}\|\nabla_h v_\Phi^3\|_{H^{0,\frac12+\alpha}}^{\frac12}\Big)\|v_\Phi^3\|_{H^{\frac12,s}}\\ &\le C\Psi(t)\|v_\Phi\|_{H^{1,s}}^2+C\dot\theta(t)\|v_\Phi\|_{H^{0,s}}^2+\tfrac{1}{100}\|\nabla_h v_\Phi\|_{H^{0,s}}^2,\end{aligned} \tag{6-11}$$

and similarly,

$$\begin{aligned}|\mathrm{II}|&\le C\varepsilon^{\frac12}\|(v^3\operatorname{div}_h v^h)_\Phi\|_{H^{-\frac12,s}}\|v_\Phi^3\|_{H^{\frac12,s}}\\ &\le C\Psi(t)\|v_\Phi^3\|_{H^{1,s}}^2+C\dot\theta(t)\|v_\Phi^3\|_{H^{0,s}}^2+\tfrac{1}{100}\|\nabla_h v_\Phi\|_{H^{0,s}}^2.\end{aligned} \tag{6-12}$$

Using the decomposition (5-6), we can similarly obtain

$$|\mathrm{III}|\le C\Psi(t)\|v_\Phi\|_{H^{1,s}}^2+C\dot\theta(t)\|v_\Phi\|_{H^{0,s}}^2+\tfrac{1}{100}\|\nabla_h v_\Phi\|_{H^{0,s}}^2. \tag{6-13}$$

Summing up (6-10)–(6-13), we obtain

$$\frac{d}{dt}\|v_\Phi^3\|_{H^{0,s}}^2+\|\nabla_h v_\Phi^3\|_{H^{0,s}}^2\le C\Psi(t)\|v_\Phi\|_{H^{1,s}}^2+C\dot\theta(t)\|v_\Phi\|_{H^{0,s}}^2+\tfrac{1}{20}\|\nabla_h v_\Phi^h\|_{H^{0,s}}^2. \tag{6-14}$$

Now we combine (6-9) with (6-14) to obtain

$$\frac{d}{dt}\|v_\Phi\|_{H^{0,s}}^2+\|\nabla_h v_\Phi\|_{H^{0,s}}^2\le C\dot\theta(t)\|v_\Phi\|_{H^{0,s}}^2+C\Psi(t)\|v_\Phi\|_{H^{1,s}}^2+\frac{1}{10a}\|v_\Phi^3\|_{H^{\frac12,s}}^2.$$

From this and Gronwall's inequality, we infer that

$$\begin{aligned}\|v_\Phi(t)\|_{H^{0,s}}^2&+\int_0^t\|\nabla_h v_\Phi(\tau)\|_{H^{0,s}}^2\,d\tau\\ &\le\exp\Big(C\int_0^t\dot\theta(\tau)\,d\tau\Big)\Big(\|e^{a\langle D_3\rangle}v_0\|_{H^{0,s}}^2+\int_0^t\Psi(\tau)\|v_\Phi(\tau)\|_{H^{1,s}}^2\,d\tau+\frac{1}{10a}\int_0^t\|v_\Phi^3(\tau)\|_{H^{\frac12,s}}^2\,d\tau\Big),\end{aligned}$$

This finishes the proof of Proposition 3.4. □

Acknowledgments

This work was partly done when Zhifei Zhang was visiting the Department of Mathematics of Paris-Sud University as a postdoctor fellow. Zhang would like to thank the department for its hospitality and support. The authors are indebted to the referee for fruitful remarks.

References

[Bony 1981] J.-M. Bony, "Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires", *Ann. Sci. École Norm. Sup.* (4) **14**:2 (1981), 209–246. MR 84h:35177 Zbl 0495.35024

[Cannone et al. 1994] M. Cannone, Y. Meyer, and F. Planchon, "Solutions auto-similaires des équations de Navier–Stokes", exposé VIII in *Séminaire sur les Équations aux Dérivées Partielles, 1993–1994*, École Polytech., Palaiseau, 1994. MR 95k:35157

[Chemin 2004] J.-Y. Chemin, "Le système de Navier–Stokes incompressible soixante dix ans après Jean Leray", pp. 99–123 in *Actes des Journées Mathématiques à la Mémoire de Jean Leray*, edited by L. Guillopé and R. Didier, Sémin. Congr. **9**, Soc. Math. France, Paris, 2004. MR 2006m:35274 Zbl 1075.35035

[Chemin and Gallagher 2006] J.-Y. Chemin and I. Gallagher, "On the global wellposedness of the 3-D Navier–Stokes equations with large initial data", *Ann. Sci. École Norm. Sup.* (4) **39**:4 (2006), 679–698. MR 2008h:35265 Zbl 1124.35052

[Chemin and Gallagher 2009] J.-Y. Chemin and I. Gallagher, "Wellposedness and stability results for the Navier–Stokes equations in $\mathbf{R}^3$", *Ann. Inst. H. Poincaré Anal. Non Linéaire* **26**:2 (2009), 599–624. MR 2010i:35278 Zbl 1165.35038

[Chemin and Gallagher 2010] J.-Y. Chemin and I. Gallagher, "Large, global solutions to the Navier–Stokes equations, slowly varying in one direction", *Trans. Amer. Math. Soc.* **362**:6 (2010), 2859–2873. MR 2592939 Zbl 1189.35220

[Chemin and Zhang 2007] J.-Y. Chemin and P. Zhang, "On the global wellposedness to the 3-D incompressible anisotropic Navier–Stokes equations", *Comm. Math. Phys.* **272**:2 (2007), 529–566. MR 2008j:35137 Zbl 1132.35068

[Chemin et al. 2000] J.-Y. Chemin, B. Desjardins, I. Gallagher, and E. Grenier, "Fluids with anisotropic viscosity", *M2AN Math. Model. Numer. Anal.* **34**:2 (2000), 315–335. MR 2001m:35261 Zbl 0954.76012

[Chemin et al. 2006] J.-Y. Chemin, B. Desjardins, I. Gallagher, and E. Grenier, *Mathematical geophysics. An introduction to rotating fluids and the Navier–Stokes equations*, Oxford Lecture Series in Mathematics and its Applications **32**, The Clarendon Press Oxford University Press, Oxford, 2006. MR 2007m:76148 Zbl 05029231

[Chemin et al. 2011] J.-Y. Chemin, I. Gallagher, and M. Paicu, "Global regularity for some classes of large solutions to the Navier–Stokes equations", *Ann. Math.* **173**:2 (2011).

[Fujita and Kato 1964] H. Fujita and T. Kato, "On the Navier–Stokes initial value problem. I", *Arch. Rational Mech. Anal.* **16** (1964), 269–315. MR 29 #3774 Zbl 0126.42301

[Gallagher et al. 2003] I. Gallagher, D. Iftimie, and F. Planchon, "Asymptotics and stability for global solutions to the Navier–Stokes equations", *Ann. Inst. Fourier* (*Grenoble*) **53**:5 (2003), 1387–1424. MR 2005b:35219 Zbl 1038.35054

[Kato 1984] T. Kato, "Strong L^p-solutions of the Navier–Stokes equation in $\mathbf{R}^m$, with applications to weak solutions", *Math. Z.* **187**:4 (1984), 471–480. MR 86b:35171 Zbl 0545.35073

[Koch and Tataru 2001] H. Koch and D. Tataru, "Well-posedness for the Navier–Stokes equations", *Adv. Math.* **157**:1 (2001), 22–35. MR 2001m:35257 Zbl 0972.35084

[Leray 1934] J. Leray, "Sur le mouvement d'un liquide visqueux emplissant l'espace", *Acta Math.* **63** (1934), 193–248.

[Makhalov and Nikolaenko 2003] A. S. Makhalov and V. P. Nikolaenko, "Global solvability of three-dimensional Navier–Stokes equations with uniformly high initial vorticity", *Uspekhi Mat. Nauk* **58**:2(350) (2003), 79–110. In Russian: translated in *Russian Math. Surveys*, **58** 2003, 287–318. MR 2004h:35177 Zbl 1059.35099

[Pedlovsky 1979] J. Pedlovsky, *Geophysical fluid dynamics*, Springer, 1979.

[Raugel and Sell 1993] G. Raugel and G. R. Sell, "Navier–Stokes equations on thin 3D domains, I: Global attractors and global regularity of solutions", *J. Amer. Math. Soc.* **6**:3 (1993), 503–568. MR 93j:35134

[Sammartino and Caflisch 1998] M. Sammartino and R. E. Caflisch, "Zero viscosity limit for analytic solutions, of the Navier–Stokes equation on a half-space. I. Existence for Euler and Prandtl equations", *Comm. Math. Phys.* **192**:2 (1998), 433–461. MR 99d:35129a

[Ukhovskii and Iudovich 1968] M. R. Ukhovskii and V. I. Iudovich, "Axially symmetric flows of ideal and viscous fluids filling the whole space", *J. Appl. Math. Mech.* **32** (1968), 52–61. MR 39 #650

[Weissler 1980] F. B. Weissler, "The Navier–Stokes initial value problem in L^p", *Arch. Rational Mech. Anal.* **74**:3 (1980), 219–230. MR 83k:35071 Zbl 0454.35072

Received 27 Mar 2009. Revised 14 May 2010. Accepted 1 Sep 2010.

MARIUS PAICU: Marius.Paicu@math.u-psud.fr
Departement de Mathématique, Université Paris Sud, 91405 Orsay Cedex, France
http://www.math.u-psud.fr/~paicu/

ZHIFEI ZHANG: zfzhang@math.pku.edu.cn
School of Mathematical Sciences, Peking University, 100871 Beijing, China

ANALYSIS AND PDE
Vol. 4, No. 1, 2011

DYNAMICS ON GRASSMANNIANS AND RESOLVENTS OF CONE OPERATORS

JUAN B. GIL, THOMAS KRAINER AND GERARDO A. MENDOZA

The paper proves the existence and elucidates the structure of the asymptotic expansion of the trace of the resolvent of a closed extension of a general elliptic cone operator on a compact manifold with boundary as the spectral parameter tends to infinity. The hypotheses involve only minimal conditions on the symbols of the operator. The results combine previous investigations by the authors on the subject with an analysis of the asymptotics of a family of projections related to the domain. This entails a detailed study of the dynamics of a flow on the Grassmannian of domains.

1. Introduction

In [Gil et al. 2010] we analyzed the behavior of the trace of the resolvent of an elliptic cone operator on a compact manifold as the spectral parameter increases radially assuming, in addition to natural ray conditions on its symbols, that the domain is stationary. We complete this analysis with Theorem 1.4 of the present paper, which describes the behavior of that trace without any restriction on the domain. The main new ingredient is Theorem 4.13 on the asymptotics of a family of projections related to the domain. This involves a fairly detailed analysis of the dynamics of a flow on the Grassmannian of domains.

Let M be a smooth compact n-dimensional manifold with boundary Y. A cone operator on M is an element $A \in x^{-m}\operatorname{Diff}^m_b(M;E)$, $m>0$; here $\operatorname{Diff}^m_b(M;E)$ is the space of b-differential operators of Melrose [1993] acting on sections of a vector bundle $E \to M$ and x is a defining function of Y in M, positive in $\mathring{M}$. Associated with such an operator is a pair of symbols, the c-symbol ${}^c\boldsymbol{\sigma}(A)$ and the wedge symbol $A_\wedge$. The former is a bundle endomorphism closely related to the regular principal symbol of A, indeed ellipticity is defined as the invertibility of ${}^c\boldsymbol{\sigma}(A)$. The wedge symbol is a partial differential operator on N_+Y, the closed inward pointing normal bundle of Y in M, essentially the original operator with coefficients frozen at the boundary. See [Gil et al. 2010, Section 2] for a brief overview and [Gil et al. 2007a, Section 3] for a detailed exposition of basic facts concerning cone operators.

Fix a Hermitian metric on E and a smooth positive b-density $\mathfrak{m}_b$ on M ($x\mathfrak{m}_b$ is a smooth everywhere positive density on M) to define the spaces $x^\gamma L^2_b(M;E)$. Let A be a cone operator. The unbounded operator

$$A : C^\infty_c(\mathring{M};E) \subset x^\gamma L^2_b(M;E) \to x^\gamma L^2_b(M;E) \tag{1.1}$$

admits a variety of closed extensions with domains $\mathcal{D} \subset x^\gamma L^2_b(M;E)$ such that $\mathcal{D}_{\min} \subset \mathcal{D} \subset \mathcal{D}_{\max}$, where $\mathcal{D}_{\min}$ is the domain of the closure of (1.1) and

$$\mathcal{D}_{\max} = \{u \in x^\gamma L^2_b(M;E) : Au \in x^\gamma L^2_b(M;E)\}.$$

Work partially supported by the National Science Foundation, Grants DMS-0901173 and DMS-0901202.
MSC2010: primary 58J35; secondary 37C70, 35P05, 47A10.
Keywords: resolvents, trace asymptotics, manifolds with conical singularities, spectral theory, dynamics on Grassmannians.

When A is c-elliptic, A is Fredholm with any such domain [Lesch 1997, Proposition 1.3.16]. We may assume without loss of generality that $\gamma = -m/2$, since otherwise we may replace A by the operator $x^{-\gamma-m/2} A\, x^{\gamma+m/2} \in x^{-m} \operatorname{Diff}^m_b(M; E)$.

The set of closed extensions is parametrized by the elements of the various Grassmannian manifolds associated with the finite-dimensional space $\mathcal{D}_{\max}/\mathcal{D}_{\min}$, a useful point of view exploited extensively in [Gil et al. 2007a]. Recall that both spaces $\mathcal{D}_{\max}/\mathcal{D}_{\min}$ and $\mathcal{D}_{\wedge,\max}/\mathcal{D}_{\wedge,\min}$ are determined by the set $\{\sigma \in \operatorname{spec}_b(A) : -m/2 < \operatorname{Im}\sigma < m/2\}$, together with certain finite dimensional spaces of functions associated to each element of this set. Also recall that the boundary spectrum of A, denoted by $\operatorname{spec}_b(A)$, is the set of points in $\mathbb{C}$ at which the conormal symbol (indicial family) of A is not invertible. The intersection of this set with horizontal strips in $\mathbb{C}$ is finite.

Associated with N_+Y there are analogous Hilbert spaces $x_\wedge^{-m/2} L^2_b(N_+Y; E_\wedge)$. Here $x_\wedge$ is the function determined by dx on N_+Y, $E_\wedge$ is the pullback of $E|_Y$ to N_+Y, and the density is $x_\wedge^{-1}\mathfrak{m}_Y$ where $\mathfrak{m}_Y$ is the density on Y obtained by contraction of $\mathfrak{m}_b$ with $x\partial_x$. We will drop the subscript $\wedge$ from $x_\wedge$ and $E_\wedge$, and trivialize N_+Y as $Y^\wedge = [0,\infty) \times Y$ using the defining function. The space $x^{-m/2} L^2_b(Y^\wedge; E)$ carries a natural unitary $\mathbb{R}_+$ action $(\varrho, u) \mapsto \kappa_\varrho u$ which after fixing a Hermitian connection on E is given by

$$\kappa_\varrho u(x, y) = \varrho^{m/2} u(\varrho x, y) \quad \text{for } \varrho > 0,\ (x, y) \in Y^\wedge.$$

The minimal and maximal domains, $\mathcal{D}_{\wedge,\min}$ and $\mathcal{D}_{\wedge,\max}$, of $A_\wedge$ are defined in an analogous fashion as those of A, the first of these spaces being the domain of the closure of

$$A_\wedge : C^\infty_c(\mathring{Y}^\wedge; E) \subset x^{-m/2} L^2_b(Y^\wedge; E) \to x^{-m/2} L^2_b(Y^\wedge; E). \tag{1.2}$$

A fundamental property of $A_\wedge$ is its κ-homogeneity, $\kappa_\varrho A_\wedge = \varrho^{-m} A_\wedge \kappa_\varrho$. Thus $\mathcal{D}_{\wedge,\min}$ and $\mathcal{D}_{\wedge,\max}$ are both κ-invariant, hence there is an $\mathbb{R}_+$ action

$$\varrho \mapsto \boldsymbol{\kappa}_\varrho : \mathcal{D}_{\wedge,\max}/\mathcal{D}_{\wedge,\min} \to \mathcal{D}_{\wedge,\max}/\mathcal{D}_{\wedge,\min},$$

which in turn induces for each d'' an action on $\operatorname{Gr}_{d''}(\mathcal{D}_{\wedge,\max}/\mathcal{D}_{\wedge,\min})$, the complex Grassmannian of d''-dimensional subspaces of $\mathcal{D}_{\wedge,\max}/\mathcal{D}_{\wedge,\min}$. Observe that since the quotient is finite dimensional these actions extend holomorphically to $\mathbb{C} \setminus \overline{\mathbb{R}}_-$.

Assuming the c-ellipticity of A, we constructed in [Gil et al. 2007a, Theorem 4.7] and reviewed in [Gil et al. 2010, Section 2] a natural isomorphism

$$\theta : \mathcal{D}_{\max}/\mathcal{D}_{\min} \to \mathcal{D}_{\wedge,\max}/\mathcal{D}_{\wedge,\min},$$

allowing, in particular, passage from a domain $\mathcal{D}$ for A to a domain $\mathcal{D}_\wedge$ for $A_\wedge$ which we shall call the associated domain.

We showed in [Gil et al. 2006] that if

$${}^c\!\boldsymbol{\sigma}(A) - \lambda \text{ is invertible for } \lambda \text{ in a closed sector } \Lambda \subsetneq \mathbb{C} \text{ which is a sector of minimal growth for } A_\wedge \text{ with the associated domain } \mathcal{D}_\wedge \text{ defined via } \mathcal{D}_\wedge/\mathcal{D}_{\wedge,\min} = \theta(\mathcal{D}/\mathcal{D}_{\min}), \tag{1.3}$$

then Λ is also a sector of minimal growth for $A_{\mathcal{D}}$, the operator A with domain $\mathcal{D}$, and for $l \in \mathbb{N}$ sufficiently large, $(A_{\mathcal{D}} - \lambda)^{-l}$ is an analytic family of trace class operators. In [Gil et al. 2010] we gave the asymptotic expansion of $\operatorname{Tr}(A_{\mathcal{D}} - \lambda)^{-l}$ under the condition that $\mathcal{D}$ was stationary. Recall that a subspace $\mathcal{D} \subset \mathcal{D}_{\max}$

with $\mathfrak{D}_{\min} \subset \mathfrak{D}$ is said to be stationary if $\theta(\mathfrak{D}/\mathfrak{D}_{\wedge,\max}) \in \mathrm{Gr}_{d''}(\mathfrak{D}_{\wedge,\max}/\mathfrak{D}_{\wedge,\min})$ is a fixed point of the action $\boldsymbol{\kappa}$. More generally, assuming only (1.3), we now prove:

Theorem 1.4. *Let A be an elliptic cone operator of degree $m > 0$ on M, and let $\mathfrak{D}$ be a domain for A so that (1.3) is satisfied. For any $\varphi \in C^\infty(M; \mathrm{End}(E))$ and $l \in \mathbb{N}$ with $ml > n$,*

$$\mathrm{Tr}(\varphi(A_{\mathfrak{D}} - \lambda)^{-l}) \sim \sum_{j=0}^{\infty} r_j(\lambda^{i\mu_1}, \dots, \lambda^{i\mu_N}, \log\lambda)\lambda^{\nu_j/m} \quad \text{as } |\lambda| \to \infty,$$

where each r_j is a rational function in $N+1$ variables, $N \in \mathbb{N}_0$, with real numbers μ_k, $k = 1, \dots, N$, and $\nu_j > \nu_{j+1} \to -\infty$ as $j \to \infty$. We have $r_j = p_j/q_j$ with p_j, $q_j \in \mathbb{C}[z_1, \dots, z_{N+1}]$ such that $q_j(\lambda^{i\mu_1}, \dots, \lambda^{i\mu_N}, \log\lambda)$ is uniformly bounded away from zero for large λ.

The expansion above is to be understood as the asymptotic expansion of a symbol into its components as discussed in the Appendix. As shown in [Gil et al. 2010],

$$\mathrm{Tr}(\varphi(A_{\mathfrak{D}} - \lambda)^{-l}) \sim \sum_{j=0}^{n-1} \alpha_j \lambda^{(n-lm-j)/m} + \alpha_n \log(\lambda)\lambda^{-l} + s_{\mathfrak{D}}(\lambda),$$

with coefficients $\alpha_j \in \mathbb{C}$ that are independent of the choice of domain $\mathfrak{D}$, and a remainder $s_{\mathfrak{D}}(\lambda)$ of order $\mathcal{O}(|\lambda|^{-l})$. Here we will show that $s_{\mathfrak{D}}(\lambda)$ is in fact a symbol that admits an expansion into components that exhibit in general the structure shown in Theorem 1.4. More precisely, let

$$\mathfrak{M} = \big\{\mathrm{Re}\,\sigma/m : \sigma \in \mathrm{spec}_b(A),\ -m/2 < \mathrm{Im}\,\sigma < m/2\big\}, \tag{1.5}$$

where $\mathrm{spec}_b(A)$ denotes the boundary spectrum of A; see [Melrose 1993]. Set

$\mathfrak{E}$ = additive semigroup generated by

$$\big\{\mathrm{Im}(\sigma - \sigma') : \sigma,\ \sigma' \in \mathrm{spec}_b(A),\ -m/2 < \mathrm{Im}\,\sigma \le \mathrm{Im}\,\sigma' < m/2\big\} \cup (-\mathbb{N}_0), \tag{1.6}$$

which is a discrete subset of $\overline{\mathbb{R}}_-$ without points of accumulation. Then

$$s_{\mathfrak{D}}(\lambda) \sim \sum_{\substack{\nu \in \mathfrak{E} \\ \nu \le -lm}} r_\nu(\lambda^{i\mu_1}, \dots, \lambda^{i\mu_N}, \log\lambda)\lambda^{\nu/m} \ \text{ as } |\lambda| \to \infty, \tag{1.7}$$

where the μ_i are the elements of $\mathfrak{M}$ and the r_ν are rational functions of their arguments as described in the theorem.

An analysis of the arguments of Sections 3 and 4 shows that the structure of the functions r_ν depends strongly on the relation of the domain with the part of the boundary spectrum in the "critical strip" $\{\sigma \in \mathbb{C} : -m/2 < \mathrm{Im}\,\sigma < m/2\}$. This includes what elements of the set $\mathfrak{M}$ actually appear in the r_ν, and whether they are truly rational functions and not just polynomials. We will not follow up on this observation in detail, but only single out here the following two cases because of their special role in the existing literature. When $\mathfrak{D}$ is stationary, the machinery of Sections 3 and 4 is not needed, and we recover the results of [Gil et al. 2010]: the r_ν are just polynomials in $\log\lambda$, and the numbers ν in (1.7) are all integers. If $\mathfrak{D}$ is nonstationary, but the elements of $\mathrm{spec}_b(A)$ in the critical strip are vertically aligned, then again there is no dependence on the elements of $\mathfrak{M}$, but the coefficients are generically rational functions of $\log\lambda$. Note that all second order regular singular operators in the sense of Brüning and Seeley [1987; 1991] (see also [Kirsten et al. 2008a]) have this special property.

By standard arguments, Theorem 1.4 implies corresponding results about the expansion of the heat trace $\mathrm{Tr}\big(\varphi e^{-tA_{\mathcal{D}}}\big)$ as $t \to 0^+$ if $A_{\mathcal{D}}$ is sectorial, and about the structure of the ζ-function if $A_{\mathcal{D}}$ is positive. It has been observed by other authors that the resolvent trace, the heat kernel, and the ζ-function for certain model operators may exhibit so called *unusual* or *exotic* behavior [Falomir et al. 2004; 2003; 2002; Kirsten et al. 2006; 2008a; 2008b; Loya et al. 2007]. This is accounted for in Theorem 1.4 by the fact that the components may have noninteger orders ν_j belonging to the set $\mathfrak{E}$, and that the r_j may be genuine rational functions and not mere polynomials. For example, the former implies that the ζ-function of a positive operator might have poles at unusual locations, and the latter that it might not extend meromorphically to $\mathbb{C}$ at all. Both phenomena have been observed for ζ-functions of model operators.

Earlier investigations on this subject typically relied on separation of variables and special function techniques to carry out the analysis near the boundary. This is one major reason why all previously known results are limited to narrow classes of operators. Here and in [Gil et al. 2010] we develop a new approach which leads to the completely general result Theorem 1.4. This result is new even for Laplacians with respect to warped cone metrics, or, more generally, for c-Laplacians [Gil et al. 2010].

Throughout this paper we assume that the ray conditions (1.3) hold. We will rely heavily on [Gil et al. 2010], where we analyzed $(A_{\mathcal{D}}-\lambda)^{-l}$ with the aid of the formula

$$(A_{\mathcal{D}}-\lambda)^{-l} = \frac{1}{l-1}\partial_\lambda^{l-1}(A_{\mathcal{D}}-\lambda)^{-1},$$

and the representation

$$(A_{\mathcal{D}}-\lambda)^{-1} = B(\lambda) + \big[1 - B(\lambda)(A-\lambda)\big]F_{\mathcal{D}}(\lambda)^{-1}T(\lambda), \tag{1.8}$$

obtained in [Gil et al. 2006]. The analogous formula for $(A_{\wedge,\mathcal{D}_\wedge}-\lambda)^{-1}$ is briefly reviewed in Section 2.

In [Gil et al. 2010] we described in full generality the asymptotic behavior of the operator families $B(\lambda)$, $[1-B(\lambda)(A-\lambda)]$, and $T(\lambda)$, and gave an asymptotic expansion of $F_{\mathcal{D}}(\lambda)^{-1}$ if $\mathcal{D}$ is stationary. Therefore, to complete the picture we only need to show that $F_{\mathcal{D}}(\lambda)^{-1}$ has a full asymptotic expansion and describe its qualitative features for a general domain $\mathcal{D}$.

We end this introduction with an overview of the paper. There is a formula similar to (1.8) concerning the extension of (1.2) with domain $\mathcal{D}_\wedge$. The analysis of $F_{\mathcal{D}}(\lambda)^{-1}$ in the reference just cited was facilitated by the fact that the corresponding operator $F_{\wedge,\mathcal{D}_\wedge}(\lambda)^{-1}$ for $A_{\wedge,\mathcal{D}_\wedge}$ has a simple homogeneity property when $\mathcal{D}$ is stationary. In Section 2 we will establish an explicit connection between the operator $F_{\wedge,\mathcal{D}_\wedge}(\lambda)^{-1}$ and a family of projections for a general domain $\mathcal{D}_\wedge$. This family of projections, previously studied in the context of rays of minimal growth in [Gil et al. 2007a; 2007b], is analyzed further in Sections 3 and 4, and is shown to fully determine the asymptotic structure of $F_{\wedge,\mathcal{D}_\wedge}(\lambda)^{-1}$, summarized in Proposition 2.17. As a consequence, we obtain in Proposition 2.20 a description of the asymptotic structure of $(A_{\wedge,\mathcal{D}_\wedge}-\lambda)^{-1}$.

The family of projections is closely related to the curve through $\mathcal{D}_\wedge/\mathcal{D}_{\wedge,\min}$ determined by the flow defined by $\boldsymbol{\kappa}$ on $\mathrm{Gr}_{d''}(\mathcal{D}_{\wedge,\max}/\mathcal{D}_{\wedge,\min})$. The behavior of an abstract version of $\boldsymbol{\kappa}_\zeta^{-1}(\mathcal{D}_\wedge/\mathcal{D}_{\wedge,\min})$ is analyzed extensively in Section 3. Let $\mathscr{E}$ denote a finite dimensional complex vector space and $\mathfrak{a}:\mathscr{E}\to\mathscr{E}$ an arbitrary linear map. The main technical result of Section 3 is an algorithm (Lemmas 3.5 and 3.11) which is used to obtain a basis of $e^{t\mathfrak{a}}D$ for all sufficiently large t (really, all complex t with $|\mathrm{Im}\, t| \le \theta$ and $\mathrm{Re}\, t$ large); here $D\subset\mathscr{E}$ is a linear subspace. The dependence of the section on t is explicit enough to allow the determination of the nature of the Ω-limit sets of the flow $t\mapsto e^{t\mathfrak{a}}$ on $\mathrm{Gr}_{d''}(\mathscr{E})$ (Proposition 3.3).

The results of Section 3 are used in Section 4 to obtain the asymptotic behavior of the aforementioned family of projections, and consequently of $F_{\wedge,\mathscr{D}_\wedge}(\lambda)^{-1}$ when $\lambda \in \Lambda$ as $|\lambda| \to \infty$, assuming only the ray condition (1.3) for $A_\wedge$ on $\mathscr{D}_\wedge$ (in the equivalent form given by (iii) of Theorem 2.15).

The work comes together in Section 5. There we obtain first the full asymptotics of $F_{\mathscr{D}}(\lambda)^{-1}$ using results from [Gil et al. 2006; 2010] and the asymptotics of $F_{\wedge,\mathscr{D}_\wedge}(\lambda)^{-1}$ obtained earlier. This is then combined with work done in [Gil et al. 2010] on the asymptotics of the rest of the operators in (1.8), giving Theorem 5.6 on the asymptotics of the trace $\operatorname{Tr}(\varphi(A_{\mathscr{D}} - \lambda)^{-l})$. The manipulation of symbols and their asymptotics is carried out within the framework of refined classes of symbols discussed in the Appendix.

2. Resolvent of the model operator

In [Gil et al. 2006; 2007a; 2007b] we studied the existence of sectors of minimal growth and the structure of resolvents for the closed extensions of an elliptic cone operator A and its wedge symbol $A_\wedge$. In particular, in [Gil et al. 2006] we determined that Λ is a sector of minimal growth for $A_{\mathscr{D}}$ if ${}^c\boldsymbol{\sigma}(A) - \lambda$ is invertible for λ in Λ, and if Λ is also a sector of minimal growth for $A_\wedge$ with the associated domain $\mathscr{D}_\wedge$. In this section we will briefly review and refine some of the results concerning the resolvent of $A_{\wedge,\mathscr{D}_\wedge}$ in the closed sector Λ.

The set

$$\text{bg-res}(A_\wedge) = \big\{\lambda \in \mathbb{C} : A_\wedge - \lambda \text{ is injective on } \mathscr{D}_{\wedge,\min} \text{ and surjective on } \mathscr{D}_{\wedge,\max}\big\},$$

introduced in [Gil et al. 2007a], is of interest for a number of reasons, including the property that if $\lambda \in \text{bg-res}(A_\wedge)$ then every closed extension of $A_\wedge - \lambda$ is Fredholm. Using the property

$$\kappa_\varrho A_\wedge = \varrho^{-m} A_\wedge \kappa_\varrho, \tag{2.1}$$

one verifies that bg-res$(A_\wedge)$ is a disjoint union of open sectors in $\mathbb{C}$. Defining $d'' = -\operatorname{ind}(A_{\wedge,\min} - \lambda)$ and $d' = \operatorname{ind}(A_{\wedge\max} - \lambda)$ for λ in one of these sectors, one has that if $(A_{\wedge,\mathscr{D}_\wedge} - \lambda)$ is invertible, then $\dim(\mathscr{D}_\wedge/\mathscr{D}_{\wedge,\min}) = d''$ and $\dim\ker(A_{\wedge,\max} - \lambda) = d'$. The dimension of $\mathscr{D}_{\wedge,\max}/\mathscr{D}_{\wedge,\min}$ is $d' + d''$.

From now on we assume that $\Lambda \neq \mathbb{C}$ is a fixed closed sector such that $\Lambda \setminus 0 \subset \text{bg-res}(A_\wedge)$ and $\operatorname{res} A_{\wedge,\mathscr{D}_\wedge} \cap \Lambda \neq \varnothing$. Without loss of generality we also assume that Λ has nonempty interior. The set $\operatorname{res} A_{\wedge,\mathscr{D}_\wedge} \cap \Lambda$ has discrete complement in Λ and is therefore connected.

Corresponding to (1.8), there is a representation

$$(A_{\wedge,\mathscr{D}_\wedge} - \lambda)^{-1} = B_\wedge(\lambda) + \big[1 - B_\wedge(\lambda)(A_\wedge - \lambda)\big] F_{\wedge,\mathscr{D}_\wedge}(\lambda)^{-1} T_\wedge(\lambda) \quad \text{for } \lambda \in \Lambda \cap \operatorname{res}(A_{\wedge,\mathscr{D}_\wedge}). \tag{2.2}$$

As we shall see in Section 5, if Λ is a sector of minimal growth for $A_{\wedge,\mathscr{D}_\wedge}$, then the asymptotic structure of $F_{\wedge,\mathscr{D}_\wedge}(\lambda)^{-1}$ determines much of the asymptotic structure of the operator $F_{\mathscr{D}}(\lambda)^{-1}$ in (1.8).

If $\mathscr{D}_\wedge$ is κ-invariant, then $F_{\wedge,\mathscr{D}_\wedge}(\lambda)^{-1}$ has the homogeneity property

$$\boldsymbol{\kappa}^{-1}_{|\lambda|^{1/m}} F_{\wedge,\mathscr{D}_\wedge}(\lambda)^{-1} = F_{\wedge,\mathscr{D}_\wedge}(\hat{\lambda})^{-1}, \tag{2.3}$$

and is, in that sense, the principal homogeneous component of $F_{\mathscr{D}}(\lambda)^{-1}$. This facilitates the expansion of $F_{\mathscr{D}}(\lambda)^{-1}$ as shown in [Gil et al. 2010, Proposition 5.17]. However, if $\mathscr{D}_\wedge$ is not κ-invariant, $F_{\wedge,\mathscr{D}_\wedge}(\lambda)^{-1}$ fails to be homogeneous and its asymptotic behavior is more intricate.

The identity (2.2) obtained in [Gil et al. 2006] begins with a choice of a family of operators $K_\wedge(\lambda) : \mathbb{C}^{d''} \to x^{-m/2}L^2_b(Y^\wedge; E)$ which is κ-homogeneous of degree m and such that

$$\begin{pmatrix} A_\wedge - \lambda & K_\wedge(\lambda) \end{pmatrix} : \mathcal{D}_{\wedge,\min} \oplus \mathbb{C}^{d''} \to x^{-m/2}L^2_b(Y^\wedge; E)$$

is invertible for all $\lambda \in \Lambda \setminus 0$. The homogeneity condition on $K_\wedge$ means that

$$K_\wedge(\varrho^m \lambda) = \varrho^m \kappa_\varrho K_\wedge(\lambda) \quad \text{for } \varrho > 0. \tag{2.4}$$

Defining the action of $\mathbb{R}_+$ on $\mathbb{C}^{d''}$ to be the trivial action, this condition on the family $K_\wedge(\lambda)$ becomes the same homogeneity property that the family $A_\wedge - \lambda$ has because of (2.1). Other than this, the choice of $K_\wedge$ is largely at our disposal. That such a family $K_\wedge(\lambda)$ exists is guaranteed by the condition that $\Lambda \setminus 0 \subset \text{bg-res}(A_\wedge)$. We now proceed to make a specific choice of $K_\wedge(\lambda)$.

Let $\lambda_0 \in \mathring{\Lambda}$ be such that $A_{\wedge,\mathcal{D}_\wedge} - \lambda$ is invertible for every $\lambda = e^{i\vartheta}\lambda_0 \in \Lambda$. We fix λ_0 (for convenience on the central axis of the sector) and a cut-off function $\omega \in C_c^\infty([0, 1))$, and define

$$K_\wedge(\lambda) = (A_\wedge - \lambda)\omega(x|\lambda|^{1/m})\kappa_{|\lambda/\lambda_0|^{1/m}} \quad \text{for } \lambda \in \Lambda \setminus 0 \tag{2.5}$$

acting on $\mathcal{D}_\wedge/\mathcal{D}_{\wedge,\min} \cong \mathbb{C}^{d''}$. The factor $\omega(x|\lambda|^{1/m})\kappa_{|\lambda/\lambda_0|^{1/m}}$ in (2.5) is to be understood as the composition

$$\mathcal{D}_\wedge/\mathcal{D}_{\wedge,\min} \xrightarrow{\kappa_{|\lambda/\lambda_0|^{1/m}}} \mathcal{D}_{\wedge,\max}/\mathcal{D}_{\wedge,\min} \cong \mathcal{E}_{\wedge,\max} \subset \mathcal{D}_{\wedge,\max} \xrightarrow{\omega(x|\lambda|^{1/m})} \mathcal{D}_{\wedge,\max},$$

in which the last operator is multiplication by the function $\omega(x|\lambda|^{1/m})$ and we use the canonical identification of $\mathcal{D}_{\wedge,\max}/\mathcal{D}_{\wedge,\min}$ with the orthogonal complement $\mathcal{E}_{\wedge,\max}$ of $\mathcal{D}_{\wedge,\min}$ in $\mathcal{D}_{\wedge,\max}$ using the graph inner product

$$(u, v)_{A_\wedge} = (A_\wedge u, A_\wedge v) + (u, v), \quad u, v \in \mathcal{D}_{\wedge,\max}.$$

By definition, $K_\wedge(\lambda)$ satisfies (2.4) and the family

$$\begin{pmatrix} A_\wedge - \lambda & K_\wedge(\lambda) \end{pmatrix} : \mathcal{D}_{\wedge,\min} \oplus \mathcal{D}_\wedge/\mathcal{D}_{\wedge,\min} \to x^{-m/2}L^2_b(Y^\wedge; E)$$

is invertible for every λ on the arc $\{\lambda \in \Lambda : |\lambda| = |\lambda_0|\}$ through λ_0. Therefore, using κ-homogeneity, it is invertible for every $\lambda \in \Lambda \setminus 0$. If

$$\begin{pmatrix} B_\wedge(\lambda) \\ T_\wedge(\lambda) \end{pmatrix} : x^{-m/2}L^2_b(Y^\wedge; E) \to \mathcal{D}_{\wedge,\min} \oplus \mathcal{D}_\wedge/\mathcal{D}_{\wedge,\min}$$

is the inverse of $\begin{pmatrix} A_{\wedge,\min} - \lambda & K_\wedge(\lambda) \end{pmatrix}$, then $T_\wedge(\lambda)(A_\wedge - \lambda) = 0$ on $\mathcal{D}_{\wedge,\min}$, so it induces a map

$$F_\wedge(\lambda) = [T_\wedge(\lambda)(A_\wedge - \lambda)] : \mathcal{D}_{\wedge,\max}/\mathcal{D}_{\wedge,\min} \to \mathcal{D}_\wedge/\mathcal{D}_{\wedge,\min},$$

whose restriction $F_{\wedge,\mathcal{D}_\wedge}(\lambda) = F_\wedge(\lambda)|_{\mathcal{D}_\wedge/\mathcal{D}_{\wedge,\min}}$ is invertible for $\lambda \in \text{res}(A_{\wedge,\mathcal{D}_\wedge}) \cap \Lambda \setminus 0$ and leads to (2.2). Moreover, since $T_\wedge(\lambda)K_\wedge(\lambda) = 1$, we have

$$\begin{aligned} F_{\wedge,\mathcal{D}_\wedge}(\lambda)^{-1} &= q_\wedge(A_{\wedge,\mathcal{D}_\wedge} - \lambda)^{-1}K_\wedge(\lambda) \\ &= q_\wedge(A_{\wedge,\mathcal{D}_\wedge} - \lambda)^{-1}(A_\wedge - \lambda)\omega(x|\lambda|^{1/m})\kappa_{|\lambda/\lambda_0|^{1/m}}, \end{aligned}$$

where $q_\wedge : \mathcal{D}_{\wedge,\max} \to \mathcal{D}_{\wedge,\max}/\mathcal{D}_{\wedge,\min}$ is the quotient map.

For $\lambda \in \text{bg-res}(A_\wedge)$ let $\mathscr{K}_{\wedge,\lambda} = \ker(A_{\wedge,\max} - \lambda)$. Then, by [Gil et al. 2007a, Lemma 5.7],

$$\lambda \in \operatorname{res}(A_{\wedge,\mathscr{D}_\wedge}) \quad \text{if and only if} \quad \mathscr{D}_{\wedge,\max} = \mathscr{D}_\wedge \oplus \mathscr{K}_{\wedge,\lambda}, \tag{2.6}$$

in which case we let $\pi_{\mathscr{D}_\wedge,\mathscr{K}_{\wedge,\lambda}}$ be the projection on $\mathscr{D}_\wedge$ according to this decomposition. If $B_{\wedge,\max}(\lambda)$ is the right inverse of $A_{\wedge,\max} - \lambda$ with range $\mathscr{K}_{\wedge,\lambda}^\perp$, then

$$(A_{\wedge,\mathscr{D}_\wedge} - \lambda)^{-1} = \pi_{\mathscr{D}_\wedge,\mathscr{K}_{\wedge,\lambda}} B_{\wedge,\max}(\lambda),$$

and $B_{\wedge,\max}(\lambda)(A_{\wedge,\max} - \lambda)$ is the orthogonal projection onto $\mathscr{K}_{\wedge,\lambda}^\perp$. Thus

$$\pi_{\mathscr{D}_\wedge,\mathscr{K}_{\wedge,\lambda}} B_{\wedge,\max}(\lambda)(A_{\wedge,\max} - \lambda) = \pi_{\mathscr{D}_\wedge,\mathscr{K}_{\wedge,\lambda}},$$

and therefore,

$$F_{\wedge,\mathscr{D}_\wedge}(\lambda)^{-1} = q_\wedge\, \pi_{\mathscr{D}_\wedge,\mathscr{K}_{\wedge,\lambda}}\, \omega(x|\lambda|^{1/m}) \boldsymbol{\kappa}_{|\lambda/\lambda_0|^{1/m}}.$$

Let

$$D = \mathscr{D}_\wedge / \mathscr{D}_{\wedge,\min}, \qquad K_{\wedge,\lambda} = (\mathscr{K}_{\wedge,\lambda} + \mathscr{D}_{\wedge,\min})/\mathscr{D}_{\wedge,\min}. \tag{2.7}$$

Again by [Gil et al. 2007a, Lemma 5.7], either of the conditions in (2.6) is equivalent to $D \cap K_{\wedge,\lambda} = 0$, hence to

$$\mathscr{D}_{\wedge,\max}/\mathscr{D}_{\wedge,\min} = D \oplus K_{\wedge,\lambda} \tag{2.8}$$

by dimensional considerations, since $\dim K_{\wedge,\lambda} = \dim \mathscr{K}_{\wedge,\lambda} = d'$. Let then $\pi_{D,K_{\wedge,\lambda}}$ be the projection on D according to the decomposition (2.8). Then $q_\wedge\, \pi_{\mathscr{D}_\wedge,\mathscr{K}_{\wedge,\lambda}} = \pi_{D,K_{\wedge,\lambda}}\, q_\wedge$ and

$$\begin{aligned} F_{\wedge,\mathscr{D}_\wedge}(\lambda)^{-1} &= \pi_{D,K_{\wedge,\lambda}} q_\wedge \omega(x|\lambda|^{1/m}) \boldsymbol{\kappa}_{|\lambda/\lambda_0|^{1/m}} \\ &= \pi_{D,K_{\wedge,\lambda}} \boldsymbol{\kappa}_{|\lambda/\lambda_0|^{1/m}}, \end{aligned} \tag{2.9}$$

since multiplication by $1 - \omega(x|\lambda|^{1/m})$ maps $\mathscr{D}_{\wedge,\max}$ into $\mathscr{D}_{\wedge,\min}$ for every λ.

We will now express $F_{\wedge,D_\wedge}(\lambda)^{-1}$ in terms of projections with $K_{\wedge,\lambda_0}$ in place of $K_{\wedge,\lambda}$. This will of course require replacing D by a family depending on λ.

Fix $\lambda \in \mathring{\Lambda}$, let $S_{\lambda,m}$ be the connected component of $\{\zeta : \zeta^m \lambda \in \mathring{\Lambda}\}$ containing $\mathbb{R}_+$. Since $\Lambda \neq \mathbb{C}$, $S_{\lambda,m}$ omits a ray, and so the map $\mathbb{R}_+ \ni \varrho \mapsto \boldsymbol{\kappa}_\varrho \in \operatorname{Aut}(\mathscr{D}_{\wedge,\max}/\mathscr{D}_{\wedge,\min})$ extends holomorphically to a map

$$S_{\lambda,m} \ni \zeta \mapsto \boldsymbol{\kappa}_\zeta \in \operatorname{Aut}(\mathscr{D}_{\wedge,\max}/\mathscr{D}_{\wedge,\min}).$$

It is an elementary fact that

$$\boldsymbol{\kappa}_\zeta^{-1}(\pi_{D,K_{\wedge,\lambda}})\boldsymbol{\kappa}_\zeta = \pi_{\boldsymbol{\kappa}_\zeta^{-1} D, \boldsymbol{\kappa}_\zeta^{-1} K_{\wedge,\lambda}}.$$

A simple consequence of (2.1) is that $\kappa_\zeta^{-1}\mathscr{K}_{\wedge,\lambda} = \mathscr{K}_{\wedge,\lambda/\zeta^m}$ if $\zeta \in \mathbb{R}_+$, hence also $\boldsymbol{\kappa}_\zeta^{-1} K_{\wedge,\lambda} = K_{\wedge,\lambda/\zeta^m}$ for such ζ since the maps $q_\wedge|_{\mathscr{K}_{\wedge,\lambda}} : \mathscr{K}_{\wedge,\lambda} \to K_{\wedge,\lambda}$ are isomorphisms. Therefore

$$\boldsymbol{\kappa}_\zeta^{-1}(\pi_{D,K_{\wedge,\lambda}})\boldsymbol{\kappa}_\zeta = \pi_{\boldsymbol{\kappa}_\zeta^{-1} D, K_{\wedge,\lambda/\zeta^m}}, \tag{2.10}$$

if $\zeta \in \mathbb{R}_+$. This formula holds also for arbitrary $\zeta \in S_{\lambda,m}$. To see this we make use of the family of isomorphisms $\mathfrak{P}(\lambda') : \mathscr{K}_{\wedge,\lambda_0} \to \mathscr{K}_{\wedge,\lambda'}$ (defined for λ' in the connected component of bg-res$(A_\wedge)$ containing λ_0) constructed in Section 7 of [Gil et al. 2007a]. Its two basic properties are that $\lambda' \mapsto \mathfrak{P}(\lambda')\phi$

is holomorphic for each $\phi \in \mathscr{K}_{\wedge,\lambda_0}$ and that $\kappa_\varrho \mathfrak{P}(\lambda') = \mathfrak{P}(\varrho^m\lambda')$ if $\varrho \in \mathbb{R}_+$. These statements are, respectively, Proposition 7.9 and Lemma 7.11 of [Gil et al. 2007a]. Let

$$f : \mathscr{D}_{\wedge,\max} \to \mathbb{C}$$

be an arbitrary continuous linear map that vanishes on $\mathscr{K}_{\wedge,\lambda}$. For any $\phi \in \mathscr{K}_{\wedge,\lambda_0}$ the function

$$S_{\lambda,m} \ni \zeta \mapsto \langle f, \kappa_\zeta \mathfrak{P}(\lambda/\zeta^m)\phi\rangle \in \mathbb{C}$$

is holomorphic and vanishes on $\mathbb{R}_+$, the latter because $\kappa_\zeta \mathfrak{P}(\lambda/\zeta^m) = \mathfrak{P}(\lambda)$ for such ζ. Therefore $\langle f, \kappa_\zeta \mathfrak{P}(\lambda/\zeta^m)\phi\rangle = 0$ for all $\zeta \in S_{\lambda,m}$. Since f is arbitrary, we must have $\kappa_\zeta \mathfrak{P}(\lambda/\zeta^m)\phi \in \mathscr{K}_{\wedge,\lambda}$. Hence

$$\mathfrak{P}(\lambda/\zeta^m)\phi \in \kappa_\zeta^{-1}\mathscr{K}_{\wedge,\lambda}.$$

Since $\mathfrak{P}(\lambda/\zeta^m) : \mathscr{K}_{\wedge,\lambda_0} \to \mathscr{K}_{\wedge,\lambda/\zeta^m}$ is an isomorphism, we have $\mathscr{K}_{\wedge,\lambda/\zeta^m} = \kappa_\zeta^{-1}\mathscr{K}_{\wedge,\lambda}$ when $\zeta \in S_{\lambda,m}$. This shows that

$$K_{\wedge,\lambda/\zeta^m} = \kappa_\zeta^{-1} K_{\wedge,\lambda},$$

and hence that (2.10) holds for $\zeta \in S_{\lambda,m}$.

The principal branch of the m-th root gives a bijection

$$(\,\cdot\,)^{1/m} : \lambda_0^{-1}\mathring{\Lambda} \to S_{\lambda_0,m}. \tag{2.11}$$

The reader may now verify that for this root, with the notation $\hat{\zeta} = \zeta/|\zeta|$ whenever $\zeta \in \mathbb{C} \setminus 0$, one has

$$\boldsymbol{\kappa}^{-1}_{|\lambda|^{1/m}} F_{\wedge,\mathscr{D}_\wedge}(\lambda)^{-1} = \boldsymbol{\kappa}^{-1}_{|\lambda_0|^{1/m}} \boldsymbol{\kappa}_{(\hat{\lambda}/\hat{\lambda}_0)^{1/m}} \Big(\pi_{\boldsymbol{\kappa}^{-1}_{(\lambda/\lambda_0)^{1/m}} D, K_{\wedge,\lambda_0}}\Big) \boldsymbol{\kappa}^{-1}_{(\hat{\lambda}/\hat{\lambda}_0)^{1/m}} \tag{2.12}$$

when $\lambda \in \mathring{\Lambda} \cap \operatorname{res}(A_{\wedge,\mathscr{D}_\wedge})$. The arguments leading to this formula remain valid if Λ is replaced by a slightly bigger closed sector, so the formula just proved holds in $(\Lambda \setminus 0) \cap \operatorname{res}(A_{\wedge,\mathscr{D}_\wedge})$.

The projection in parentheses in (2.12) is thus a key component of the resolvent of $A_{\wedge,\mathscr{D}_\wedge}$ whose behavior for large $|\lambda|$ will be analyzed in Section 4 under a certain fundamental condition which happens to be equivalent to the condition that Λ is a sector of minimal growth for $A_{\wedge,\mathscr{D}_\wedge}$. We now proceed to discuss this condition.

The condition that the sector Λ with $\Lambda \setminus 0 \subset \operatorname{bg-res}(A_\wedge)$ is a sector of minimal growth for $A_{\wedge,\mathscr{D}_\wedge}$ was shown in [Gil et al. 2007a, Theorem 8.3] to be equivalent to the invertibility of $A_{\wedge,\mathscr{D}_\wedge} - \lambda$ for λ in

$$\Lambda_R = \{\lambda \in \Lambda : |\lambda| \geq R\},$$

together with the uniform boundedness in Λ_R of the projection $\pi_{\kappa^{-1}_{|\lambda|^{1/m}} D, K_{\hat{\lambda}}}$. Further, it was shown in [Gil et al. 2007b] that along a ray containing λ_0, this condition is in turn equivalent to requiring that the curve

$$\varrho \mapsto \boldsymbol{\kappa}_\varrho^{-1} D : [R, \infty) \to \operatorname{Gr}_{d''}(\mathscr{D}_{\wedge,\max}/\mathscr{D}_{\wedge,\min})$$

does not approach the set

$$\mathscr{V}_{K_{\wedge,\lambda_0}} = \big\{D \in \operatorname{Gr}_{d''}(\mathscr{D}_{\wedge,\max}/\mathscr{D}_{\wedge,\min}) : D \cap K_{\wedge,\lambda_0} \neq 0\big\} \tag{2.13}$$

as $\varrho \to \infty$, a condition conveniently phrased in terms of the limiting set

$$\Omega^-(D) = \big\{D' \in \operatorname{Gr}_{d''}(\mathscr{D}_{\wedge,\max}/\mathscr{D}_{\wedge,\min}) : \exists\, \varrho_\nu \to \infty \text{ in } \mathbb{R}_+ \text{ such that } \boldsymbol{\kappa}^{-1}_{\varrho_\nu} D \to D' \text{ as } \nu \to \infty\big\}.$$

A ray $\{r\lambda_0 \in \mathbb{C} : r > 0\}$ contained in bg-res$(A_\wedge)$ is a ray of minimal growth for $A_{\wedge,\mathscr{D}_\wedge}$ if and only if

$$\Omega^-(D) \cap \mathscr{V}_{K_{\wedge,\lambda_0}} = \varnothing.$$

Define

$$\Omega^-_\Lambda(D) = \Big\{D' \in \mathrm{Gr}_{d''}(\mathscr{D}_{\wedge,\max}/\mathscr{D}_{\wedge,\min}) : \\ \exists\, \{\zeta_\nu\}_{\nu=1}^\infty \subset \mathbb{C} \text{ with } \lambda_0\zeta_\nu \in \Lambda \text{ and } |\zeta_\nu| \to \infty \text{ s.t. } \lim_{\nu\to\infty} \boldsymbol{\kappa}^{-1}_{\zeta_\nu^{1/m}} D = D'\Big\}, \tag{2.14}$$

in which we are using the holomorphic extension of $\varrho \mapsto \boldsymbol{\kappa}_\varrho$ to $S_{\lambda_0,m}$ and the m-th root is the principal branch, as specified in (2.11). We can now consolidate all these conditions as follows.

Theorem 2.15. *Let Λ be a closed sector such that $\Lambda \smallsetminus 0 \subset$ bg-res$(A_\wedge)$, and let $\lambda_0 \in \mathring{\Lambda}$. The following statements are equivalent*:

(i) *Λ is a sector of minimal growth for $A_{\wedge,\mathscr{D}_\wedge}$.*

(ii) *There are constants $C,\ R > 0$ such that $\Lambda_R \subset \mathrm{res}(A_{\wedge,\mathscr{D}_\wedge})$ and $\|\pi_{\boldsymbol{\kappa}^{-1}_{\zeta^{1/m}} D, K_{\wedge,\lambda_0}}\|_{\mathscr{L}(\mathscr{D}_{\wedge,\max}/\mathscr{D}_{\wedge,\min})} \le C$ for every ζ such that $\lambda_0\zeta \in \Lambda_R$.*

(iii) $\Omega^-_\Lambda(D) \cap \mathscr{V}_{K_{\wedge,\lambda_0}} = \varnothing$.

Proof. By means of (2.10) we get the identity

$$\pi_{\boldsymbol{\kappa}^{-1}_{\zeta^{1/m}} D, K_{\wedge,\lambda_0}} = \boldsymbol{\kappa}^{-1}_{\hat\zeta^{1/m}} \boldsymbol{\kappa}_{|\lambda_0|^{1/m}} \big(\pi_{\boldsymbol{\kappa}^{-1}_{|\lambda|^{1/m}} D, K_{\wedge,\hat\lambda}}\big) \boldsymbol{\kappa}^{-1}_{|\lambda_0|^{1/m}} \boldsymbol{\kappa}_{\hat\zeta^{1/m}},$$

which is valid for large $\lambda \in \Lambda$, $\zeta = \lambda/\lambda_0$, and $\hat\zeta = \zeta/|\zeta|$. Since $\boldsymbol{\kappa}_{\hat\zeta^{1/m}}$ and $\boldsymbol{\kappa}^{-1}_{\hat\zeta^{1/m}}$ are uniformly bounded, Theorem 8.3 of [Gil et al. 2007a] gives the equivalence of (i) and (ii).

We now prove that (ii) and (iii) are equivalent. Let $\mathscr{E}_{\wedge,\max} = \mathscr{D}_{\wedge,\max}/\mathscr{D}_{\wedge,\min}$ and assume (iii) is satisfied. Since $\Omega^-_\Lambda(D)$ and $\mathscr{V}_{K_{\wedge,\lambda_0}}$ are closed sets in $\mathrm{Gr}_{d''}(\mathscr{E}_{\wedge,\max})$, there is a neighborhood $\mathscr{U}$ of $\mathscr{V}_{K_{\wedge,\lambda_0}}$ and a constant $R > 0$ such that if $|\lambda_0\zeta| > R$ then $\boldsymbol{\kappa}^{-1}_{\zeta^{1/m}} D \notin \mathscr{U}$. Let $\delta : \mathrm{Gr}_{d''}(\mathscr{E}_{\wedge,\max}) \times \mathrm{Gr}_{d'}(\mathscr{E}_{\wedge,\max}) \to \mathbb{R}$ be as in Section 5 of [Gil et al. 2007a]. Since $\mathscr{V}_{K_{\wedge,\lambda_0}}$ is the zero set of the continuous function $\mathscr{V} \mapsto \delta(\mathscr{V}, K_{\wedge,\lambda_0})$, there is a constant $\delta_0 > 0$ such that $\delta(\boldsymbol{\kappa}^{-1}_{\zeta^{1/m}} D, K_{\wedge,\lambda_0}) > \delta_0$ for every ζ such that $\lambda_0\zeta \in \Lambda_R$. Then Lemma 5.12 of the same reference gives (ii).

Conversely, let (ii) be satisfied. Suppose $\Omega^-_\Lambda(D) \cap \mathscr{V}_{K_{\wedge,\lambda_0}} \ne \varnothing$ and let D_0 be an element in the intersection. Thus $D_0 \cap K_{\wedge,\lambda_0} \ne \{0\}$ and there is a sequence $\{\zeta_\nu\}_{\nu=1}^\infty \subset \mathbb{C}$ with $\lambda_0\zeta_\nu \in \Lambda$ such that $|\zeta_\nu| \to \infty$ and

$$D_\nu = \boldsymbol{\kappa}^{-1}_{\zeta_\nu^{1/m}} D \to D_0 \quad \text{ as } \nu \to \infty.$$

If ν is such that $|\lambda_0\zeta_\nu| > R$, then $\lambda_0\zeta_\nu \in \mathrm{res}(A_{\wedge,\mathscr{D}_\wedge})$ and $D \cap K_{\wedge,\lambda_0\zeta_\nu} = \{0\}$, so $D_\nu \cap K_{\wedge,\lambda_0} = \{0\}$. Thus for ν large enough $D_\nu \notin \mathscr{V}_{K_{\wedge,\lambda_0}}$.

Pick $u \in D_0 \cap K_{\wedge,\lambda_0}$ with $\|u\| = 1$. Let π_{D_ν} be the orthogonal projection on D_ν. Since $D_\nu \to D_0$ as $\nu \to \infty$, we have $\pi_{D_\nu} \to \pi_{D_0}$, so $u_\nu = \pi_{D_\nu} u \to \pi_{D_0} u = u$. For ν large, $D_\nu \notin \mathscr{V}_{K_{\wedge,\lambda_0}}$, so $u_\nu - u \ne 0$. Now, since $u_\nu \in D_\nu$, $u \in K_{\wedge,\lambda_0}$, and $u_\nu \to u$,

$$\pi_{D_\nu, K_{\wedge,\lambda_0}}\Big(\frac{u_\nu - u}{\|u_\nu - u\|}\Big) = \frac{u_\nu}{\|u_\nu - u\|} \to \infty \quad \text{ as } \nu \to \infty.$$

But this contradicts (ii). Hence $\Omega^-(D) \cap \mathscr{V}_{K_{\wedge,\lambda_0}} = \varnothing$. ∎

If $\mathscr{D}_\wedge$ is not κ-invariant, the asymptotic analysis of $F_{\wedge,\mathscr{D}_\wedge}(\lambda)^{-1}$ (through the analysis of the projection $\pi_{D,K_{\wedge,\lambda}}$) leads to rational functions of the form

$$r(\lambda^{i\mu_1},\dots,\lambda^{i\mu_N},\log\lambda)=\frac{p(\lambda^{i\mu_1},\dots,\lambda^{i\mu_N},\log\lambda)}{q(\lambda^{i\mu_1},\dots,\lambda^{i\mu_N},\log\lambda)}, \tag{2.16}$$

with $\mu_l\in\mathbb{R}$ for $l=1,\dots,N$, where $q(z_1,\dots,z_{N+1})$ is a polynomial over $\mathbb{C}$ such that

$$\big|q(\lambda^{i\mu_1},\dots,\lambda^{i\mu_N},\log\lambda)\big|>\delta,$$

for some $\delta>0$ and every sufficiently large $\lambda\in\Lambda$, and

$$p(\lambda^{i\mu_1},\dots,\lambda^{i\mu_N},\log\lambda)=\sum_{\alpha,k}a_{\alpha k}(\lambda)\lambda^{i\alpha\mu}\log^k\lambda,$$

with $\mu=(\mu_1,\dots,\mu_N)$, $\alpha\in\mathbb{N}_0^N$, $k\in\mathbb{N}_0$, and coefficients

$$a_{\alpha k}\in C^\infty\big(\Lambda\smallsetminus 0,\mathscr{L}(\mathscr{D}_\wedge/\mathscr{D}_{\wedge,\min},\mathscr{D}_{\wedge,\max}/\mathscr{D}_{\wedge,\min})\big),$$

such that $a_{\alpha k}(\varrho^m\lambda)=\boldsymbol{\kappa}_\varrho a_{\alpha k}(\lambda)$ for every $\varrho>0$.

Proposition 2.17. *If Λ is a sector of minimal growth for $A_{\wedge,\mathscr{D}_\wedge}$, then for $R>0$ large enough, the family $F_{\wedge,\mathscr{D}_\wedge}(\lambda)=F_\wedge(\lambda)|_{\mathscr{D}_\wedge/\mathscr{D}_{\wedge,\min}}$ is invertible for $\lambda\in\Lambda_R$ and $F_{\wedge,\mathscr{D}_\wedge}(\lambda)^{-1}$ has the following properties:*

(i) *$F_{\wedge,\mathscr{D}_\wedge}(\lambda)^{-1}\in C^\infty(\Lambda_R;\mathscr{L}(\mathscr{D}_\wedge/\mathscr{D}_{\wedge,\min},\mathscr{D}_{\wedge,\max}/\mathscr{D}_{\wedge,\min}))$, and for every $\alpha,\beta\in\mathbb{N}_0$ we have*

$$\big\|\boldsymbol{\kappa}^{-1}_{|\lambda|^{1/m}}\partial_\lambda^\alpha\partial_{\bar\lambda}^\beta F_{\wedge,\mathscr{D}_\wedge}(\lambda)^{-1}\big\|=\mathcal{O}(|\lambda|^{\nu/m-\alpha-\beta})\ \text{ as }\ |\lambda|\to\infty, \tag{2.18}$$

with $\nu=0$.

(ii) *For all $j\in\mathbb{N}_0$ there exist rational functions r_j of the form (2.16) and a decreasing sequence of real numbers $0=\nu_0>\nu_1>\cdots\to-\infty$ such that for every $J\in\mathbb{N}$, the difference*

$$F_{\wedge,\mathscr{D}_\wedge}(\lambda)^{-1}-\sum_{j=0}^{J-1}r_j(\lambda^{i\mu_1},\dots,\lambda^{i\mu_N},\log\lambda)\,\lambda^{\nu_j/m} \tag{2.19}$$

satisfies (2.18) with $\nu=\nu_J+\varepsilon$ for any $\varepsilon>0$.

The phases $\mu_1,\dots,\mu_N$ and the exponents ν_j in (2.19) depend on the boundary spectrum of A. In fact, $\mu_1,\dots,\mu_N\in\mathfrak{M}$ and $\nu_j\in\mathfrak{E}$ for all j; see (1.5) and (1.6).

This suggests the introduction of operator-valued symbols with a notion of asymptotic expansion in components that take into account the rational structure above and the κ-homogeneity of the numerators. The idea of course is to have a class of symbols whose structure is preserved under composition, differentiation, and asymptotic summation. In the Appendix we propose such a class, $S^{\nu^+}_{\mathscr{R}}(\Lambda;E,\tilde E)$, a subclass of the operator-valued symbols $S^\infty(\Lambda;E,\tilde E)$ introduced by Schulze, where E and $\tilde E$ are Hilbert spaces with suitable group actions. The space $S^{\nu^+}_{\mathscr{R}}(\Lambda;E,\tilde E)$ is contained in $S^{\nu+\varepsilon}(\Lambda;E,\tilde E)$ for any $\varepsilon>0$.

As reviewed at the beginning of the Appendix, the notion of anisotropic homogeneity in $S^{(\nu)}(\Lambda;E,\tilde E)$ depends on the group actions in E and $\tilde E$. Thus homogeneity is always to be understood with respect to these actions.

In the symbol terminology, we have

$$F_{\wedge,\mathscr{D}_\wedge}(\lambda)^{-1}\in(S^{0^+}_{\mathscr{R}}\cap S^0)(\Lambda_R;\mathscr{D}_\wedge/\mathscr{D}_{\wedge,\min},\mathscr{D}_{\wedge,\max}/\mathscr{D}_{\wedge,\min}),$$

where $\mathscr{D}_\wedge/\mathscr{D}_{\wedge,\min}$ carries the trivial action and $\mathscr{D}_{\wedge,\max}/\mathscr{D}_{\wedge,\min}$ is equipped with $\boldsymbol{\kappa}_\varrho$.

Proof of Proposition 2.17. Since Λ is a sector of minimal growth for $A_{\wedge,\mathscr{D}_\wedge}$, there exists $R>0$ such that $(A_{\wedge,\mathscr{D}_\wedge}-\lambda)$ is invertible for $\lambda\in\Lambda_R$, which by definition is equivalent to the invertibility of $F_{\wedge,\mathscr{D}_\wedge}(\lambda)$. Since the map $\zeta\mapsto\boldsymbol{\kappa}_{\hat\zeta^{1/m}}$ is uniformly bounded (recall that $\hat\zeta=\zeta/|\zeta|$), the relation (2.12) together with Theorem 2.15 give the estimate (2.18) for $\alpha=\beta=0$. If we differentiate with respect to λ (or $\bar\lambda$), then

$$\partial_\lambda F_{\wedge,\mathscr{D}_\wedge}(\lambda)^{-1}=-F_{\wedge,\mathscr{D}_\wedge}(\lambda)^{-1}[\partial_\lambda F_{\wedge,\mathscr{D}_\wedge}(\lambda)]F_{\wedge,\mathscr{D}_\wedge}(\lambda)^{-1}=-F_{\wedge,\mathscr{D}_\wedge}(\lambda)^{-1}[\partial_\lambda F_\wedge(\lambda)]F_{\wedge,\mathscr{D}_\wedge}(\lambda)^{-1}.$$

Now, if we equip $\mathscr{D}_\wedge/\mathscr{D}_{\wedge,\min}$ with the trivial group action and $\mathscr{D}_{\wedge,\max}/\mathscr{D}_{\wedge,\min}$ with $\boldsymbol{\kappa}_\varrho$, then

$$F_\wedge(\lambda):\mathscr{D}_{\wedge,\max}/\mathscr{D}_{\wedge,\min}\to\mathscr{D}_\wedge/\mathscr{D}_{\wedge,\min}$$

is homogeneous of degree zero, hence $\|\partial_\lambda F_\wedge(\lambda)\boldsymbol{\kappa}_{|\lambda|^{1/m}}\|$ is $\mathscr{O}(|\lambda|^{-1})$ as $|\lambda|\to\infty$. Therefore,

$$\big\|\boldsymbol{\kappa}^{-1}_{|\lambda|^{1/m}}\partial_\lambda F_{\wedge,\mathscr{D}_\wedge}(\lambda)^{-1}\big\|=\mathscr{O}(|\lambda|^{-1})\quad\text{ as }|\lambda|\to\infty,$$

since $\boldsymbol{\kappa}^{-1}_{|\lambda|^{1/m}}\partial_\lambda F_{\wedge,\mathscr{D}_\wedge}(\lambda)^{-1}$ can be written as

$$-[\boldsymbol{\kappa}^{-1}_{|\lambda|^{1/m}}F_{\wedge,\mathscr{D}_\wedge}(\lambda)^{-1}][\partial_\lambda F_\wedge(\lambda)\boldsymbol{\kappa}_{|\lambda|^{1/m}}][\boldsymbol{\kappa}^{-1}_{|\lambda|^{1/m}}F_{\wedge,\mathscr{D}_\wedge}(\lambda)^{-1}],$$

and the first and last factors are uniformly bounded by our previous argument. The corresponding estimates for arbitrary derivatives follow by induction.

Next, observe that by (2.12),

$$F_{\wedge,\mathscr{D}_\wedge}(\lambda)^{-1}=\boldsymbol{\kappa}_{\zeta^{1/m}}\big(\pi_{\boldsymbol{\kappa}^{-1}_{\zeta^{1/m}}D,K_{\wedge,\lambda_0}}\big)\boldsymbol{\kappa}^{-1}_{\hat\zeta^{1/m}},$$

with $\zeta=\lambda/\lambda_0$ and $\hat\zeta=\zeta/|\zeta|$. For $\lambda\in\Lambda_R$ let $k(\lambda)=\boldsymbol{\kappa}_{\zeta^{1/m}}$ and $\hat k(\lambda)=\boldsymbol{\kappa}^{-1}_{\hat\zeta^{1/m}}$. Then $k(\lambda)$ is a homogeneous symbol in $S^{(0)}(\Lambda_R;\mathscr{D}_{\wedge,\max}/\mathscr{D}_{\wedge,\min},\mathscr{D}_{\wedge,\max}/\mathscr{D}_{\wedge,\min})$, where the first copy of the quotient is equipped with the trivial action and the target space carries $\boldsymbol{\kappa}_\varrho$. Similarly, $\hat k(\lambda)\in S^{(0)}(\Lambda_R;\mathscr{D}_\wedge/\mathscr{D}_{\wedge,\min},\mathscr{D}_{\wedge,\max}/\mathscr{D}_{\wedge,\min})$ with respect to the trivial action on both spaces.

Finally, the asymptotic expansion claimed in (ii) follows from Theorem 4.13 together with the homogeneity properties of $k(\lambda)$ and $\hat k(\lambda)$. □

As a consequence of Proposition 2.17, and since $B_\wedge(\lambda)$, $1-B_\wedge(\lambda)(A_\wedge-\lambda)$, and $T_\wedge(\lambda)$ in (2.2) are homogeneous of degree $-m$, 0, and $-m$, in their respective classes, we obtain:

Proposition 2.20. *If Λ is a sector of minimal growth for $A_{\wedge,\mathscr{D}_\wedge}$, then for $R>0$ large enough, we have*

$$(A_{\wedge,\mathscr{D}_\wedge}-\lambda)^{-1}\in(S^{(-m)^+}_{\mathfrak{R}}\cap S^{-m})(\Lambda_R;x^{-m/2}L^2_b,\mathscr{D}_{\wedge,\max}),$$

where the spaces are equipped with the standard action κ_ϱ. The components have orders ν^+ with $\nu\in\mathfrak{E}$ and their phases belong to $\mathfrak{M}$; see (1.5) and (1.6).

3. Limiting orbits

We will write $\mathscr{E}$ instead of $\mathscr{D}_{\wedge,\max}/\mathscr{D}_{\wedge,\min}$ and denote by $\mathfrak{a}:\mathscr{E}\to\mathscr{E}$ the infinitesimal generator of the $\mathbb{R}_+$ action $(\varrho,v)\mapsto\boldsymbol{\kappa}_\varrho^{-1}v$ on $\mathscr{E}$, so that $\boldsymbol{\kappa}_\varrho^{-1}D=e^{t\mathfrak{a}}D$ with $t=\log\varrho$. In what follows we allow t to be complex. The spectrum of $\mathfrak{a}$ is related to the boundary spectrum of A by

$$\operatorname{spec}\mathfrak{a}=\big\{-i\sigma-m/2:\sigma\in\operatorname{spec}_b(A),\ -m/2<\operatorname{Im}\sigma<m/2\big\}.\tag{3.1}$$

For each $\lambda \in \operatorname{spec} \mathfrak{a}$ let $\mathscr{E}_\lambda$ be the generalized eigenspace of $\mathfrak{a}$ associated with λ, let $\pi_\lambda : \mathscr{E} \to \mathscr{E}$ be the projection on $\mathscr{E}_\lambda$ according to the decomposition

$$\mathscr{E} = \bigoplus_{\lambda \in \operatorname{spec} \mathfrak{a}} \mathscr{E}_\lambda.$$

Define $N : \mathscr{E} \to \mathscr{E}$ and $N_\lambda : \mathscr{E}_\lambda \to \mathscr{E}_\lambda$ by

$$N = \mathfrak{a} - \sum_{\lambda \in \operatorname{spec} \mathfrak{a}} \lambda \pi_\lambda, \quad N_\lambda = N|_{\mathscr{E}_\lambda},$$

respectively. Thus N is the nilpotent part of $\mathfrak{a}$. Correspondingly, let

$$\mathfrak{a}' : \mathscr{E} \to \mathscr{E}, \quad \mathfrak{a}' = \sum_{\lambda \in \operatorname{spec} \mathfrak{a}} (i \operatorname{Im} \lambda) \pi_\lambda, \tag{3.2}$$

so $\mathfrak{a}'$ is the skew-adjoint component of the semisimple part of $\mathfrak{a}$.

For $\mu \in \operatorname{Re}(\operatorname{spec} \mathfrak{a})$ let

$$\tilde{\mathscr{E}}_\mu = \bigoplus_{\substack{\lambda \in \operatorname{spec} a \\ \operatorname{Re} \lambda = \mu}} \mathscr{E}_\lambda,$$

let $\tilde{\pi}_\mu : \mathscr{E} \to \mathscr{E}$ be the projection on $\tilde{\mathscr{E}}_\mu$ according to the decomposition

$$\mathscr{E} = \bigoplus_{\mu \in \operatorname{Re}(\operatorname{spec} \mathfrak{a})} \tilde{\mathscr{E}}_\mu,$$

and set

$$\tilde{N}_\mu = N|_{\tilde{\mathscr{E}}_\mu} : \tilde{\mathscr{E}}_\mu \to \tilde{\mathscr{E}}_\mu.$$

Fix an auxiliary Hermitian inner product on $\mathscr{E}$ so that $\bigoplus \mathscr{E}_\lambda$ is an orthogonal decomposition of $\mathscr{E}$. Then $\mathfrak{a}'$ is skew-adjoint and $e^{t\mathfrak{a}'}$ is unitary if t is real.

Proposition 3.3. *For every $D \in \operatorname{Gr}_{d''}(\mathscr{E})$ there is $D_\infty \in \operatorname{Gr}_{d''}(\mathscr{E})$ such that*

$$\operatorname{dist}(e^{t\mathfrak{a}} D, e^{t\mathfrak{a}'} D_\infty) \to 0 \quad \text{as } \operatorname{Re} t \to \infty \text{ in } S_\theta = \{t \in \mathbb{C} : |\operatorname{Im} t| \le \theta\} \tag{3.4}$$

for any $\theta > 0$. The set

$$\Omega_\theta^+(D) = \left\{D' \in \operatorname{Gr}_{d''}(\mathscr{E}) : \exists \{t_\nu\} \subset S_\theta : \operatorname{Re} t_\nu \to \infty \text{ and } \lim_{\nu \to \infty} e^{t_\nu \mathfrak{a}} D = D'\right\}$$

is the closure of

$$\{e^{t\mathfrak{a}'} D_\infty : t \in S_\theta\}.$$

We are using Ω^+ for the limit set for consistency with common usage: we are letting $\operatorname{Re} t$ tend to infinity.

If $\mathscr{F}$ is a vector space, we will write $\mathscr{F}[t, t^{-1}]$ for the space of polynomials in t and t^{-1} with coefficients in $\mathscr{F}$ (that is, the $\mathscr{F}$-valued rational functions on $\mathbb{C}$ with a pole only at 0). If $p \in \mathscr{F}[t, t^{-1}]$, let $\mathrm{c}_s(p)$ denote the coefficient of t^s in p, and if $p \neq 0$, let

$$\operatorname{ord}(p) = \max\{s \in \mathbb{Z} : \mathrm{c}_s(p) \neq 0\}.$$

The proof of the proposition hinges on the following lemma, whose proof will be given later.

Lemma 3.5. *Let $D \subset \mathscr{E}$ be an arbitrary nonzero subspace. Define $D^1 = D$ and by induction define*

$$\mu_l = \max\{\mu \in \operatorname{Re}(\operatorname{spec}\mathfrak{a}) : \tilde{\pi}_\mu D^l \neq 0\}, \quad D^{l+1} = \ker \tilde{\pi}_{\mu_l}|_{D^l}, \quad D_{\mu_l} = (D^{l+1})^\perp \cap D^l,$$

starting with $l = 1$. Let L be the smallest l such that $D^{l+1} = 0$. Thus

$$\tilde{\pi}_{\mu_l}|_{D^l} : D_{\mu_l} \to \tilde{\pi}_{\mu_l} D_{\mu_l} \text{ is an isomorphism} \tag{3.6}$$

and $D = \bigoplus_{l=1}^{L} D_{\mu_l}$. Then for each l there are elements

$$\tilde{p}_k^l \in \tilde{\pi}_{\mu_l} D_{\mu_l}[t, 1/t], \quad k = 1, \dots, \dim D_{\mu_l},$$

such that with

$$\tilde{q}_k^l(t) = e^{t\tilde{N}_{\mu_l}} \tilde{p}_k^l(t),$$

we have $\operatorname{ord} \tilde{q}_k^l = 0$ and the elements

$$g_k^l = \mathrm{c}_0(\tilde{q}_k^l),$$

for $k = 1, \dots, \dim D_{\mu_l}$, are independent.

Proof of Proposition 3.3. Suppose $D \subset \mathscr{E}$ is a subspace. With the notation of Lemma 3.5 let

$$D_{\mu_l,\infty} = \operatorname{span}\{g_k^l : k = 1, \dots, \dim D_{\mu_l}\}.$$

Since $e^{t\tilde{N}_{\mu_l}}$ is invertible and $\tilde{q}_k^l(t) = g_k^l + \tilde{h}_k^l(t)$ with $\tilde{h}_k^l(t) = \mathcal{O}(t^{-1})$ for large $\operatorname{Re} t$ ($t \in S_\theta$), the vectors $\tilde{p}_k^l(t)$ form a basis of $\tilde{\pi}_{\mu_l} D_{\mu_l}$ for all sufficiently large t. Using (3.6) we get unique elements

$$p_k^l \in D_{\mu_l}[t, 1/t], \quad \tilde{\pi}_{\mu_l} p_k^l = \tilde{p}_k^l.$$

For each l the $p_k^l(t)$ give a basis of D_{μ_l} if t is large enough, and therefore also the

$$e^{-t\mu_l} p_k^l(t), \quad k = 1, \dots, \dim D_{\mu_l},$$

form a basis of D_{μ_l} for large $\operatorname{Re} t$. Consequently, the vectors

$$e^{t\mathfrak{a}} e^{-t\mu_l} p_k^l(t), \quad k = 1, \dots, \dim D_{\mu_l},\ l = 1, \dots, L,$$

form a basis of $e^{t\mathfrak{a}} D$ for large $\operatorname{Re} t$. We have, with $N_\lambda = N|_{\mathscr{E}_\lambda}$,

$$\begin{aligned}
e^{t\mathfrak{a}} e^{-t\mu_l} p_k^l(t) &= \sum_{\lambda \in \operatorname{spec}\mathfrak{a}} e^{t(\lambda-\mu_l)} e^{tN_\lambda} \pi_\lambda p_k^l(t) \\
&= \sum_{\substack{\lambda \in \operatorname{spec}\mathfrak{a} \\ \operatorname{Re}\lambda = \mu_l}} e^{t(\lambda-\mu_l)} e^{tN_\lambda} \pi_\lambda p_k^l(t) + \sum_{\substack{\lambda \in \operatorname{spec}\mathfrak{a} \\ \operatorname{Re}\lambda < \mu_l}} e^{t(\lambda-\mu_l)} e^{tN_\lambda} \pi_\lambda p_k^l(t) \\
&= e^{t\mathfrak{a}'} e^{t\tilde{N}_{\mu_l}} \tilde{\pi}_{\mu_l} p_k^l(t) + \sum_{\substack{\lambda \in \operatorname{spec}\mathfrak{a} \\ \operatorname{Re}\lambda < \mu_l}} e^{t(\lambda-\mu_l)} e^{tN_\lambda} \pi_\lambda p_k^l(t) \\
&= e^{t\mathfrak{a}'} (g_k^l + \tilde{h}_k^l(t)) + \sum_{\substack{\lambda \in \operatorname{spec}\mathfrak{a} \\ \operatorname{Re}\lambda < \mu_l}} e^{t(\lambda-\mu_l)} e^{tN_\lambda} \pi_\lambda p_k^l(t)
\end{aligned}$$

so $e^{t\mathfrak{a}} e^{-t\mu_l} p_k^l(t) = e^{t\mathfrak{a}'} g_k^l + h_k^l(t)$, where $h_k^l(t) = \mathcal{O}(t^{-1})$ as $\operatorname{Re} t \to \infty$ in S_θ. It follows that (3.4) holds with $D_\infty = \bigoplus_{l=1}^{L} D_{\mu_l,\infty}$. This completes the proof of the first assertion of Proposition 3.3.

Remark 3.7. The formulas for the $v_k^l(t) = e^{t\mathfrak{a}} e^{-t\mu_l} p_k^l(t)$ given in the last displayed line above will eventually give the asymptotics of the projections $\pi_{e^{t\mathfrak{a}}D,K}$ (assuming $\mathscr{V}_K \cap \Omega^+(D) = \varnothing$, see Theorem 2.15). Note that the shift by $m/2$ in (3.1) is irrelevant and that the coefficients of the exponents in the formula for $v_k^l(t)$ belong to

$$\big\{\lambda - \operatorname{Re}\lambda' : \lambda, \lambda' \in \operatorname{spec}\mathfrak{a},\ \operatorname{Re}\lambda \le \operatorname{Re}\lambda'\big\}. \tag{3.8}$$

Because of (3.1), this set is equal to

$$-i\big\{\sigma - i\operatorname{Im}\sigma' : \sigma, \sigma' \in \operatorname{spec}_b(A),\ -m/2 < \operatorname{Im}\sigma \le \operatorname{Im}\sigma' < m/2\big\}. \tag{3.9}$$

If all elements of $\{\sigma \in \operatorname{spec}_b(A) : -m/2 < \operatorname{Im}\sigma < m/2\}$ have the same real part, then all elements of (3.8) have the same imaginary part ν, the operator $\mathfrak{a}'$ is multiplication by $i\nu$, and we can divide each of the $v_k^l(t)$ by $e^{it\nu}$ to obtain a basis of $e^{t\mathfrak{a}}D$ in which the coefficients of the exponents are all real.

To prove the second assertion of the proposition, we note first that (3.4) implies that $\Omega_\theta^+(D)$ is contained in the closure of $\{e^{t\mathfrak{a}'}D_\infty : t \in S_\theta\}$. To prove the opposite inclusion, it is enough to show that

$$e^{t\mathfrak{a}'}D_\infty \in \Omega_\theta^+(D) \tag{3.10}$$

for each $t \in S_\theta$, since $\Omega_\theta^+(D)$ is a closed set. Writing $e^{t\mathfrak{a}'}D_\infty$ as $e^{i\operatorname{Im} t\,\mathfrak{a}'}(e^{\operatorname{Re} t\,\mathfrak{a}'}D_\infty)$ further reduces the problem to the case $\theta = 0$ (that is, t real). While proving (3.10) we will also show that the closure $\mathscr{X}$ of $\{e^{t\mathfrak{a}'}D_\infty : t \in \mathbb{R}\}$ is an embedded torus, equal to $\Omega_0^+(D)$.

Let $\{\lambda_k\}_{k=1}^K$ be an enumeration of the elements of $\operatorname{spec}\mathfrak{a}$. Define $f : \mathbb{R}^K \times \operatorname{Gr}_{d''}(\mathscr{E}) \to \operatorname{Gr}_{d''}(\mathscr{E})$ by

$$f(\tau, D) = e^{\sum i\tau^k \pi_{\lambda_k}} D,$$

$\tau = (\tau^1, \dots, \tau^K)$. This is a smooth map. Since the π_{λ_k} commute with each other, f defines a left action of $\mathbb{R}^K$ on $\operatorname{Gr}_{d''}(\mathscr{E})$. For each $\tau \in \mathbb{R}^K$ define

$$f_\tau : \operatorname{Gr}_{d''}(\mathscr{E}) \to \operatorname{Gr}_{d''}(\mathscr{E}), \quad f_\tau(D) = f(\tau, D),$$

and for each $D \in \operatorname{Gr}_{d''}(\mathscr{E})$ let

$$f^D : \mathbb{R}^K \to \operatorname{Gr}_{d''}(\mathscr{E}), \quad f^D(\tau) = f(\tau, D).$$

The maps f_τ are diffeomorphisms.

We claim that f^{D_∞} factors as the composition of a smooth group homomorphism $\phi : \mathbb{R}^K \to \mathbb{T}^{K'}$ onto a torus and an embedding $h : \mathbb{T}^{K'} \to \operatorname{Gr}_{d''}(\mathscr{E})$,

$$\begin{array}{ccc} \mathbb{R}^K & \xrightarrow{\ \phi\ } & \mathbb{T}^{K'} \\ {\scriptstyle f^{D_\infty}}\big\downarrow & \swarrow{\scriptstyle h} & \\ \operatorname{Gr}_{d''}(\mathscr{E}). & & \end{array}$$

Both ϕ and h depend on D_∞.

To prove the claim we begin by observing that $\{u \in T\mathbb{R}^K : df^{D_\infty}(u) = 0\}$ is translation-invariant. Indeed, let $\tau_0 \in \mathbb{R}^K$, let $v = (v^1, \dots, v^K) \in \mathbb{R}^K$, and let $\gamma : \mathbb{R} \to \mathbb{R}^K$ be the curve $\gamma(t) = tv$. Then

$$f^{D_\infty}(\tau_0 + \gamma(t)) = f_{\tau_0} \circ f^{D_\infty}(\gamma(t)),$$

so

$$df^{D_\infty}\big(\sum v^k\partial_{\tau^k}|_{\tau_0}\big) = df_{\tau_0}\circ df^{D_\infty}\big(\sum v^k\partial_{\tau^k}|_0\big).$$

Since f_{τ_0} is a diffeomorphism,

$$\sum v^k\partial_{\tau^k}|_{\tau_0}\in[\ker df^{D_\infty}: T_{\tau_0}\mathbb{R}^K\to T_{f^{D_\infty}(\tau_0)}\mathrm{Gr}_{d''}(\mathscr{E})] \iff \sum v^k\partial_{\tau^k}|_0\in[\ker df^{D_\infty}: T_0\mathbb{R}^K\to T_{D_\infty}\mathrm{Gr}_{d''}(\mathscr{E})].$$

Thus the kernel of df^{D_∞} is translation-invariant as asserted.

Identify the kernel of $df^{D_\infty}: T_0\mathbb{R}^K\to T_{D_\infty}\mathrm{Gr}_{d''}(\mathscr{E})$ with a subspace $\mathscr{S}$ of $\mathbb{R}^K$ in the standard fashion. Then f^{D_∞} is constant on the translates of $\mathscr{S}$ and if $\mathscr{R}$ is a subspace of $\mathbb{R}^K$ complementary to $\mathscr{S}$, then $f^{D_\infty}|_{\mathscr{R}}$ is an immersion. Renumbering the elements of $\operatorname{spec}\mathfrak{a}'$ (and reordering the components of $\mathbb{R}^K$ accordingly) we may take $\mathscr{R}=\mathbb{R}^{K'}\times 0$.

Since $f^{D_\infty}|_{\mathscr{R}}$ is an immersion, the sets

$$\mathscr{F}_{D'}=\{\tau\in\mathscr{R}: f^{D_\infty}(\tau)=D'\}$$

are discrete for each $D'\in f^{D_\infty}(\mathscr{R})$. Using again the property $f^{D_\infty}(\tau_1+\tau_2)=f_{\tau_1}\circ f^{D_\infty}(\tau_2)$ for arbitrary $\tau_1,\tau_2\in\mathbb{R}^K$, we see that $\mathscr{F}_{D_\infty}$ is an additive subgroup of $\mathscr{R}$ and that f^{D_∞} is constant on the lateral classes of $\mathscr{F}_{D_\infty}$. Therefore $f^{D_\infty}|_{\mathscr{R}}$ factors through a (smooth) homomorphism $\phi:\mathscr{R}\to\mathscr{R}/\mathscr{F}_{D_\infty}$ and a continuous map $\mathscr{R}/\mathscr{F}_{D_\infty}\to\mathrm{Gr}_{d''}(\mathscr{E})$. Since f^{D_∞} is 2π-periodic in all variables, $2\pi\mathbb{Z}^{K'}\subset\mathscr{F}_{D_\infty}$, so $\mathscr{R}/\mathscr{F}_{D_\infty}$ is indeed a torus $\mathbb{T}^{K'}$. Since ϕ is a local diffeomorphism and f^{D_∞} is smooth, h is smooth.

With this, the proof of the second assertion of the proposition goes as follows. Let $L\subset\mathbb{R}^K$ be the subspace generated by $(\operatorname{Im}\lambda_1,\dots,\operatorname{Im}\lambda_K)$. This is a line or the origin. Its image by ϕ is a subgroup H of $\mathbb{T}^{K'}$, so the closure of $\phi(L)$ is a torus $G\subset\mathbb{T}^{K'}$, and $h(\overline{\phi(L)})$ is an embedded torus $\mathscr{X}\subset\mathrm{Gr}_{d''}(\mathscr{E})$. On the other hand, $h\circ\phi(L)=f^{D_\infty}(L)$ is the image of the curve $\gamma: t\to e^{t\mathfrak{a}'}D_\infty$, so the closure of the image of γ is $\mathscr{X}$. Clearly, $\Omega_0^+(D)\subset\mathscr{X}$. The equality of $\Omega_0^+(D)$ and $\mathscr{X}$ is clear if γ is periodic or $L=\{0\}$. So assume that γ is not periodic and $L\neq\{0\}$. Then $H\neq G$ and there is a sequence

$$\{g_\nu\}_{\nu=1}^\infty\subset G\setminus H$$

such that $g_\nu\to e$, the identity element of G. Let v be an element of the Lie algebra of G such that H is the image of $t\mapsto\exp(tv)$. For each ν there is a sequence $\{t_{\nu,\rho}\}_{\rho=1}^\infty$, necessarily unbounded because $g_\nu\notin H$, such that $g_\nu=\lim_{\rho\to\infty}\exp(t_{\nu,\rho}v)$. We may assume that $\{t_{\nu,\rho}\}_{\rho=1}^\infty$ is monotonic, so it diverges to $+\infty$ or to $-\infty$. In the latter case we replace g_ν by its group inverse, so we may assume that $\lim_{\rho\to\infty}t_{\nu,\rho}=\infty$ for all ν. Thus if $g\in H$ is arbitrary, then $h(gg_\nu)\in\Omega_0^+(D)$ and $h(gg_\nu)$ converges to $h(g)$. Since $\Omega_0^+(D)$ is closed, this shows that $h\circ\phi(H)\subset\Omega_0^+(D)$. Consequently, also $\mathscr{X}\subset\Omega_0^+(D)$.

This completes the proof of the second assertion of Proposition 3.3. □

As a consequence of the proof we have that $\Omega_\theta^+(D)$ is a union of embedded tori:

$$\Omega_\theta^+(D)=\bigcup_{s\in[-\theta,\theta]} e^{is\mathfrak{a}'}\overline{\{e^{t\mathfrak{a}'}D_\infty: t\in\mathbb{R}\}}.$$

The proof of Lemma 3.5 will be based on the following lemma. The properties of the elements $\tilde{p}_k^l\in\tilde{\pi}_{\mu_l}D_{\mu_l}[t,1/t]$ whose existence is asserted in Lemma 3.5 pertain only to $\tilde{\mathscr{E}}_{\mu_l}$, $\tilde{N}_{\mu_l}$, and the subspace $\tilde{\pi}_{\mu_l}D_{\mu_l}$ of $\tilde{\mathscr{E}}_{\mu_l}$. For the sake of notational simplicity we let $\mathscr{W}=\tilde{\pi}_{\mu_l}D_{\mu_l}$ and drop the μ_l from the notation. The space $\tilde{\mathscr{E}}$ comes equipped with some Hermitian inner product, and $\tilde{N}$ is nilpotent.

Lemma 3.11. *There is an orthogonal decomposition* $\mathscr{W} = \bigoplus_{j=0}^{J} \bigoplus_{m=0}^{M_j} \mathscr{W}_j^m$ *(with nontrivial summands) and nonzero elements*

$$P_j^m \in \operatorname{Hom}(\mathscr{W}_{j,m}, \mathscr{W}_{j'})[t, t^{-1}], \tag{3.12}$$

where

$$\mathscr{W}_{j,m} = \bigoplus_{m'=m}^{M_j} \mathscr{W}_j^{m'}, \quad \mathscr{W}_j' = \bigoplus_{j'=0}^{j} \mathscr{W}_{j',0},$$

satisfying the following properties:

(1) $P_j^0 = I_{\mathscr{W}_{j,0}}$.

(2) *Let* $Q_j^m(t) = e^{t\tilde{N}} P_j^m(t)$ *and* $n_j^m = \operatorname{ord}(Q_j^m)$. *Then the sequence* $\{n_j^m\}_{m=0}^{M_j}$ *is strictly decreasing and consists of nonnegative numbers.*

(3) *Let*

$$G_j^m = \mathrm{c}_{n_j^m}(Q_j^m), \quad \mathscr{V}_j^m = G_j^m(\mathscr{W}_j^m). \tag{3.13}$$

Then

$$\mathscr{W}_{j,m+1} = (G_j^m)^{-1}\Big(\bigoplus_{j'=0}^{j-1} \bigoplus_{m'=0}^{M_{j'}} \mathscr{V}_{j'}^{m'} + \bigoplus_{m'=0}^{m-1} \mathscr{V}_j^{m'}\Big). \tag{3.14}$$

(4) *There are unique maps* $F_{j',j}^{m',m+1} : \mathscr{W}_{j,m+1} \to \mathscr{W}_{j'}^{m'}$ *such that*

$$G_j^m + \sum_{j'=0}^{j-1} \sum_{m'=0}^{M_{j'}} G_{j'}^{m'} F_{j',j}^{m',m+1} + \sum_{m'=0}^{m-1} G_j^{m'} F_{j,j}^{m',m+1} = 0 \tag{3.15}$$

holds on $\mathscr{W}_{j,m+1}$, *and*

$$P_j^{m+1} = P_j^m + \sum_{j'=0}^{j-1} \sum_{m'=0}^{M_{j'}} t^{n_j^m - n_{j'}^{m'}} P_{j'}^{m'} F_{j',j}^{m',m+1} + \sum_{m'=0}^{m-1} t^{n_j^m - n_j^{m'}} P_j^{m'} F_{j,j}^{m',m+1}. \tag{3.16}$$

The lemma is a definition by induction if we adopt the convention that spaces with negative indices and summations where the upper index is less than the lower index are the zero space. In the inductive process that will constitute the proof of the lemma we will first define $\mathscr{W}_{j,m+1} \subset \mathscr{W}_{j,m}$ using (3.14) starting with suitably defined spaces $\mathscr{W}_{j,0}$ and then define

$$\mathscr{W}_j^m = \mathscr{W}_{j,m} \cap \mathscr{W}_{j,m+1}^{\perp}.$$

Note that the right hand side of (3.14) depends only on $\mathscr{W}_{j,m}$, P_j^m (through G_j^m) and the spaces $\mathscr{V}_{j'}^{m'}$ with $j' < j$ and m' arbitrary, or $j' = j$ and $m' < m$. The relation (3.15) follows from (3.14) and induction, and then (3.16) (where P_j^m actually means its restriction to $\mathscr{W}_{j,m+1}$) is a definition by induction; it clearly gives that the $P_j^m(t)$ have values in $\mathscr{W}_j'$ as required in (3.12).

We will illustrate the lemma and its proof with an example and then give a proof.

Example 3.17. Suppose $\tilde{\mathscr{E}}$ is spanned by elements $e_{j,k}$ ($j = 0, 1$ and $k = 1, \dots, K_j$) and that the Hermitian inner product is defined so that these vectors are orthonormal. Define the linear operator $\tilde{N} : \tilde{\mathscr{E}} \to \tilde{\mathscr{E}}$ so that $\tilde{N} e_{j,1} = 0$ and $\tilde{N} e_{j,k} = e_{j,k-1}$ for $1 < k \le K_j$. Thus $\tilde{N}^k e_{j,k} = 0$ and $\tilde{N}^k e_{j,k+1} = e_{j,1} \neq 0$. Pick integers $0 \le s_0 < s_1 < \min\{K_0, K_1\}$, and let

$$\mathscr{W} = \operatorname{span}\{e_{0,s_0+1}, e_{1,s_1+1}, e_{0,s_1+1} + e_{1,s_1}\}.$$

If $w \in \mathscr{W}$ and $w \neq 0$, then $e^{t\tilde{N}} w$ is a polynomial of degree exactly s_0 or s_1. Let $\mathscr{W}_{0,0} = \mathscr{W} \cap \ker \tilde{N}^{s_0+1}$, that is,

$$\mathscr{W}_{0,0} = \operatorname{span}\{e_{0,s_0+1}\}.$$

Then $e^{t\tilde{N}} w$ is polynomial of degree s_0 if $w \in \mathscr{W}_{0,0}$. Let $\mathscr{W}_{1,0} = \mathscr{W} \cap \ker \tilde{N}^{s_1+1} \cap \mathscr{W}_{0,0}^{\perp}$. Thus

$$\mathscr{W}_{1,0} = \operatorname{span}\{e_{1,s_1+1}, e_{0,s_1+1} + e_{1,s_1}\},$$

and $e^{t\tilde{N}} w$ is polynomial of degree exactly s_1 if $w \in \mathscr{W}_{1,0}$ and $w \neq 0$. With these spaces we have

$$\mathscr{W} = \mathscr{W}_{0,0} \oplus \mathscr{W}_{1,1}$$

as an orthogonal sum. By (1) of Lemma 3.11, $P_0^0 = I_{\mathscr{W}_{0,0}}$. So $e^{t\tilde{N}} P_0^0$ is the restriction of

$$e^{t\tilde{N}} = \sum_{k=0}^{s_0} \frac{t^k}{k!} \tilde{N}^k$$

to $\mathscr{W}_{0,0}$, $n_0^0 = s_0$, and G_0^0 is $(1/s_0!)\tilde{N}^{s_0}$ restricted to $\mathscr{W}_{0,0}$. Thus $\mathscr{V}_0^0 = \operatorname{span}\{e_{0,1}\}$. The space $\mathscr{W}_{0,1}$, defined using (3.14), is the zero space by the convention on sums where the upper index is less than the lower index. Thus $M_0 = 0$. We next analyze what the lemma says when $j = 1$. As when $j = 0$, $P_1^0 = I_{\mathscr{W}_{1,0}}$, so $e^{t\tilde{N}} P_1^0$ is the restriction of

$$e^{t\tilde{N}} = \sum_{k=0}^{s_1} \frac{t^k}{k!} \tilde{N}^k$$

to $\mathscr{W}_{1,0}$. Hence $n_1^0 = s_1$, and $G_1^0 = (1/s_1!)\tilde{N}^{s_1}|_{\mathscr{W}_{1,0}}$. The preimage of $\mathscr{V}_0^0$ by G_1^0 is

$$\mathscr{W}_{1,1} = \operatorname{span}\{e_{0,s_1+1} + e_{1,s_1}\},$$

and so $\mathscr{W}_1^0 = \operatorname{span}\{e_{1,s_1+1}\}$ and $\mathscr{V}_1^0 = \operatorname{span}\{e_{1,1}\}$. With $w = e_{0,s_1+1} + e_{1,s_1}$ we have

$$G_1^0 w = \frac{1}{s_1!} e_{0,1} = G_0^0 \frac{s_0!}{s_1!} e_{0,s_0+1},$$

so with $F_{0,1}^{0,1} : \mathscr{W}_{1,1} \to \mathscr{W}_0^0$ defined by

$$F_{0,1}^{0,1} w = -\frac{s_0!}{s_1!} e_{0,s_0+1}$$

we have $G_1^0 + G_0^0 F_{0,1}^{0,1} = 0$. Formula (3.16) reads

$$P_1^1(t) = I_{\mathscr{W}_{1,1}} + t^{s_1 - s_0} F_{0,1}^{0,1}$$

in this instance, and

$$e^{t\tilde{N}} P_1^1(t) w = \sum_{k=0}^{s_1} \frac{t^k}{k!} \tilde{N}^k w - \frac{s_0!\, t^{s_1-s_0}}{s_1!} \sum_{k=0}^{s_0} \frac{t^k}{k!} \tilde{N}^k e_{0,s_0+1}.$$

In the first sum the highest order term is $t^{s_1}/s_1!\,e_{0,1}$, while in the second it is $t^{s_0}/s_0!\,e_{0,1}$. Taking into account the coefficient of the second sum we see that $e^{t\tilde N}P_1^1(t)w$ has order $< s_1$. A more detailed calculation gives that the order is s_1-1, and that the leading coefficient is given by the map

$$w \mapsto \Big(\frac{1}{(s_1-1)!} - \frac{s_0!}{s_1!(s_0-1)!}\Big)e_{0,2} + e_{1,1};$$

its image spans $\mathscr{V}_1^1$. Note that $\mathscr{V}_0^0+\mathscr{V}_1^0+\mathscr{V}_1^1$ is a direct sum and is invariant under $\tilde N$.

Proof of Lemma 3.11. We note first that the properties of the objects in the lemma are such that

$$D_{\mu,\infty} = \sum_{j=0}^{J}\sum_{m=0}^{M_j}\mathscr{V}_j^m \tag{3.18}$$

is a direct sum. Indeed, suppose we have $w_j^m \in \mathscr{W}_j^m$, $j=0,\dots,J$, $m=0,\dots,M_j$ such that

$$\sum_{j=0}^{J}\sum_{m=0}^{M_j} G_j^m w_j^m = 0.$$

If some w_j^m is nonzero, let

$$j_0 = \max\{j : \exists m \text{ such that } w_j^m \neq 0\}, \quad m_0 = \max\{m : w_{j_0}^m \neq 0\},$$

so that $w_{j_0}^{m_0} \neq 0$. Thus

$$G_{j_0}^{m_0}w_{j_0}^{m_0} = -\sum_{j=0}^{j_0-1}\sum_{m=0}^{M_j} G_j^m w_j^m - \sum_{m=0}^{m_0-1} G_{j_0}^m w_{j_0}^m \in \sum_{j=0}^{j_0-1}\sum_{m=0}^{M_j}\mathscr{V}_j^m - \sum_{m=0}^{m_0-1}\mathscr{V}_{j_0}^m,$$

therefore $w_{j_0}^{m_0} \in \mathscr{W}_{j_0,m_0+1}$ by (3.14). But also $w_{j_0}^{m_0} \in \mathscr{W}_{j_0}^{m_0}$, a space which by definition is orthogonal to $\mathscr{W}_{j_0,m_0+1}$. Consequently $w_{j_0}^{m_0}=0$, a contradiction. It follows that (3.18) is a direct sum as claimed, and in particular that the maps

$$G_j^m|_{\mathscr{W}_j^m} : \mathscr{W}_j^m \to \mathscr{V}_j^m$$

are isomorphisms.

Note that $e^{t\tilde N}w$ is a nonzero polynomial whenever $w \in \mathscr{W}\setminus 0$ and let

$$\{s_j\}_{j=0}^J = \{\deg e^{t\tilde N}w : w \in \mathscr{W},\ w \neq 0\}$$

be an enumeration of the degrees of these polynomials, in increasing order. Let $\mathscr{W}_{-1,0}=\{0\}\subset\mathscr{W}$ and inductively define

$$\mathscr{W}_{j,0} = \mathscr{W}\cap\ker\tilde N^{s_j+1}\cap\mathscr{W}_{j-1,0}^{\perp}, \quad j=0,\dots,J.$$

Thus $\mathscr{W}_{j,0}\subset\mathscr{W}$ and $\mathscr{W} = \bigoplus_{j=0}^{J}\mathscr{W}_{j,0}$ is an orthogonal decomposition of $\mathscr{W}$; moreover,

$$\tilde N^{s_j}|_{\mathscr{W}_{j,0}} : \mathscr{W}_{j,0} \to \tilde{\mathscr{E}}$$

is injective for $j=0,\dots,J$, and if $w\in\mathscr{W}_{j,0}\setminus 0$ then $e^{t\tilde N}w$ is a polynomial of degree exactly s_j. The spaces $\mathscr{W}_j^m$ will be defined so that $\bigoplus_m \mathscr{W}_j^m = \mathscr{W}_{j,0}$.

Let $P_0^0(t) = I_{\mathcal{W}_{0,0}}$, let $Q_0^0(t) = e^{t\tilde{N}} P_0^0(t)$. Then $\operatorname{ord}(Q_0^0) = s_0$ and $G_0^0 = 1/s_0!\, \tilde{N}^{s_0}|_{\mathcal{W}_{0,0}}$. By (3.14), $\mathcal{W}_{0,1}$ is the preimage of the zero vector space. Since $\tilde{N}^{s_0}$ is injective on $\mathcal{W}_{0,0}$, $\mathcal{W}_{0,1} = 0$, $\mathcal{W}_0^0 = \mathcal{W}_{0,0}$ and $M_0 = 0$. Let $\mathcal{V}_0^0 = G_0^0(\mathcal{W}_0^0)$. This proves the lemma if $J = 0$.

We continue the proof using induction on J. Suppose that $J \geq 1$ and that the lemma has been proved for $\mathcal{W}' = \bigoplus_{j=0}^{J-1} \mathcal{W}_{j,0}$, so we have all objects described in the statement of the lemma, for $\mathcal{W}'$. The corresponding objects for $\mathcal{W}_{J,0}$ are then defined by induction in the second index, as follows.

First, let $P_J^0(t) = I_{\mathcal{W}_{J,0}}$, $Q_J^0 = e^{t\tilde{N}} P_J^0$ (a polynomial in t of degree $n_J^0 = s_J$) and $G_J^0 = \mathrm{c}_{s_J}(Q_J^0)$.

Next, suppose we have found

$$\mathcal{W}_{J,0} \supset \cdots \supset \mathcal{W}_{J,M-1} \quad \text{and} \quad P_j^m \in L(\mathcal{W}_{j,m}, \mathcal{W})[t, t^{-1}]$$

so that the properties described in the lemma are satisfied for $j < J$ and all m, or $j = J$ and $m \leq M-1$. As discussed, it follows that

$$\sum_{m=0}^{M-2} \mathcal{V}_J^m + \sum_{j=0}^{J-1} \sum_{m=0}^{M_j} \mathcal{V}_j^m$$

is a direct sum and that the maps

$$G_j^m|_{\mathcal{W}_j^m} : \mathcal{W}_j^m \to \mathcal{V}_j^m \tag{3.19}$$

defined so far are isomorphisms. Suppose further that the $n_J^m = \operatorname{ord}(Q_J^m)$, $m = 0, \ldots, M-1$, are non-negative and strictly decrease as m increases. In agreement with (3.14), let

$$\mathcal{W}_{J,M} = (G_J^{M-1})^{-1}\Bigl(\sum_{m=0}^{M-2} \mathcal{V}_J^m + \sum_{j=0}^{J-1} \sum_{m=0}^{M_j} \mathcal{V}_j^m\Bigr),$$

a subspace of the domain $\mathcal{W}_{J,M-1}$ of G_J^{M-1}. Define $\mathcal{W}_J^{M-1} = \mathcal{W}_{J,M-1} \cap \mathcal{W}_{J,M}^{\perp}$. If $w \in \mathcal{W}_{J,M}$, then

$$G_J^{M-1} w = \sum_{m=0}^{M-2} v_J^m + \sum_{j=0}^{J-1} \sum_{m=0}^{M_j} v_j^m$$

uniquely with $v_j^m \in \mathcal{V}_j^m$. Since the maps (3.19) are isomorphisms, there are unique maps

$$F_{j,J}^{m,M} : \mathcal{W}_{J,M} \to \mathcal{W}_j^m,$$

$j = 0, \ldots, J-1$ and $m = 0, \ldots, M_j$, or $j = J$ and $m = 0, \ldots, M-2$ such that

$$G_J^{M-1} + \sum_{m=0}^{M-2} G_J^m F_{J,J}^{m,M} + \sum_{j=0}^{J-1} \sum_{m=0}^{M_j} G_j^m F_{j,J}^{m,M} = 0$$

on $\mathcal{W}_{J,M}$, that is, (3.15) holds. Define

$$P_J^M = P_J^{M-1} + \sum_{m=0}^{M-2} t^{n_J^{M-1} - n_J^m} P_J^m F_{J,J}^{m,M} + \sum_{j=0}^{J-1} \sum_{m=0}^{M_j} t^{n_J^{M-1} - n_j^m} P_j^m F_{j,J}^{m,M}$$

so (3.16) holds. Let $Q_J^M = e^{t\tilde{N}} P_J^M$. Because of (3.13), each term on the right in

$$Q_J^M = Q_J^{M-1} + \sum_{m=0}^{M-2} t^{n_J^{M-1} - n_J^m} Q_J^m F_{J,J}^{m,M} + \sum_{j=0}^{J-1} \sum_{m=0}^{M_j} t^{n_J^{M-1} - n_j^m} Q_j^m F_{j,J}^{m,M}.$$

has order n_J^{M-1}, so $\mathrm{c}_n(Q_J^M)=0$ if $n\geq n_J^{M-1}$. If $Q_J^M\neq 0$, let $n_J^M=\mathrm{ord}(Q_J^M)$. *A fortiori* $n_J^M<n_J^{M-1}$.

We now show that if $Q_J^M=0$, then $\mathcal{W}_{J,M}=0$, so $M_J=M-1$ and the inductive construction stops.

Let $F_{j,j}^{m,m+1}:\mathcal{W}_{j,m+1}\to\mathcal{W}_{j,m}$ be the inclusion map. Note that the combination of indices just used does not appear in (3.15): these maps are not defined in the statement of the lemma. With this notation

$$P_J^m=\sum_{m'=0}^{m-1} t^{n_J^{m-1}-n_J^{m'}} P_J^{m'} F_{J,J}^{m',m}+\tilde{H}_J^m \tag{3.20}$$

for $m=1,\dots,M$ and some $\tilde{H}_J^m\in L(\mathcal{W}_{J,m},\mathcal{W}')[t,t^{-1}]$. Let $\mathcal{P}_m$ be the set of finite strictly increasing sequences $\boldsymbol{\nu}=(\nu_0,\nu_1,\dots,\nu_k)$ of elements of $\{0,\dots,m\}$ with $\nu_0=0$ and $\nu_k=m$. For $\nu=(\nu_0,\dots,\nu_k)\in\mathcal{P}_m$ $(m\geq 1)$ define

$$\begin{aligned}
F_J^{\boldsymbol{\nu}}&=F_{J,J}^{\nu_0,\nu_1}\circ\cdots\circ F_{J,J}^{\nu_{m-1},\nu_m},\\
n_J^{\boldsymbol{\nu}}&=(n_J^{\nu_1-1}-n_J^{\nu_0})+(n_J^{\nu_2-1}-n_J^{\nu_1})+\cdots+(n_J^{\nu_k-1}-n_J^{\nu_{k-1}}).
\end{aligned}$$

Since the $n_J^{m'}$ strictly decrease as m' increases, the numbers $n_J^{\boldsymbol{\nu}}$ are strictly negative except when $\boldsymbol{\nu}$ is the maximal sequence $\boldsymbol{\nu}_{\max}$ in $\{0,\dots,m\}$, in which case $n_J^{\boldsymbol{\nu}_{\max}}=0$ and $F^{\boldsymbol{\nu}_{\max}}$ is the inclusion of $\mathcal{W}_{J,m}$ in $\mathcal{W}_{J,0}$. It is not hard to prove by induction on m, using (3.20), that

$$P_J^m=P_J^0\sum_{\boldsymbol{\nu}\in\mathcal{P}_m} t^{n_J^{\boldsymbol{\nu}}} F_J^{\boldsymbol{\nu}}+H_J^m \tag{3.21}$$

for all $m\geq 1$ where $H_J^m\in L(\mathcal{W}_{J,m},\mathcal{W}')[t,t^{-1}]$. If $Q_J^M=0$, then $P_J^M=0$, so, since $\tilde{N}^{s_J}H_J^M=0$,

$$\tilde{N}^{s_J}P_J^M=\sum_{\boldsymbol{\nu}\in\mathcal{P}_M} t^{n_J^{\boldsymbol{\nu}}}\tilde{N}^{s_J}F_J^{\boldsymbol{\nu}}=0.$$

In particular, $\tilde{N}^{s_J}F_J^{\boldsymbol{\nu}_{\max}}=\mathrm{c}_0(\tilde{N}^{s_J}P_J^M)=0$. Since $\tilde{N}^{s_J}$ is injective on $\mathcal{W}_{J,0}$, we conclude that the inclusion of $\mathcal{W}_{J,M}$ in $\mathcal{W}_{J,0}$ is zero. This means that $\mathcal{W}_{J,M}=0$, so the inductive construction stops with $M_J=M-1$.

We will now show that there is a finite M such that $Q_J^M=0$. The inductive construction gives, as long as $Q_J^m\neq 0$, the numbers $n_J^m=\mathrm{ord}(Q_J^m)$ which form a strictly decreasing sequence in m, with $n_J^0=s_J$. Suppose $n_J^{M-1}\geq 0$, $Q_J^M\neq 0$, and $n_J^M<0$. In particular, the coefficient of t^0 in Q_J^M vanishes. Using (3.21) with $m=M$ we have

$$e^{t\tilde{N}}P_J^M=\sum_{\boldsymbol{\nu}\in\mathcal{P}_M}\sum_{s=0}^{s_J}\frac{t^{s+n_J^{\boldsymbol{\nu}}}}{s!}\tilde{N}^sF_J^{\boldsymbol{\nu}}+e^{t\tilde{N}}H_J^M.$$

The coefficient of t^0 is

$$\mathrm{c}_0(\mathrm{e}^{t\tilde{N}}P_J^M)=\sum_{\boldsymbol{\nu}\in\mathcal{P}_M}\frac{1}{(-n_J^{\boldsymbol{\nu}})!}\tilde{N}^{-n_J^{\boldsymbol{\nu}}}F_J^{\boldsymbol{\nu}}+\mathrm{c}_0(e^{t\tilde{N}}H_J^M);$$

recall that $n_J^{\boldsymbol{\nu}}\leq 0$. Since H_J^M maps into $\mathcal{W}'$, we have $\tilde{N}^{s_J}\,\mathrm{c}_0(H_J^M)=0$, and since $\tilde{N}^s|_{\mathcal{W}_{J,0}}=0$ if $s>s_J$, $\tilde{N}^{s_J}\tilde{N}^{-n_J^{\boldsymbol{\nu}}}=0$ if $n_J^{\boldsymbol{\nu}}\neq 0$. Thus

$$\tilde{N}^{s_J}\,\mathrm{c}_0(\mathrm{e}^{t\tilde{N}}P_J^M)=\tilde{N}^{s_J}F_J^{\boldsymbol{\nu}_{\max}},$$

where $\boldsymbol{\nu}_{\max}=(0,1,\dots,M)$. Since $c_0(e^{t\tilde{N}}P_J^M)=0$ by hypothesis, since $F_J^{\boldsymbol{\nu}_{\max}}$ is the inclusion of $\mathcal{W}_{J,M}$ in $\mathcal{W}_{J,0}$, and since $\tilde{N}^{s_J}$ is injective on $\mathcal{W}_{J,0}$, $\mathcal{W}_{J,M}=0$. $\square$

Proof of Lemma 3.5. Apply Lemma 3.11 to each of the spaces $\mathscr{W}_{\mu_l} = \tilde{\pi}_{\mu_l} D_{\mu_l}$. The corresponding objects are labeled adjoining l as a subindex. Get in particular, decompositions

$$\tilde{\pi}_{\mu_l} D_{\mu_l} = \bigoplus_{j=0}^{J_l} \bigoplus_{m=0}^{M_{j,l}} \mathscr{W}^m_{j,l} \subset \tilde{\mathscr{E}}_{\mu_l}$$

for each l, and operators $G^m_{j,l} : \mathscr{W}^m_{j,l} \to \mathscr{V}^m_{j,l} \subset \tilde{\mathscr{E}}_{\mu_l}$ such that

$$\bigoplus_{j=0}^{J_l} \bigoplus_{m=0}^{M_{j,l}} G^m_{j,l}|_{\mathscr{W}^m_{j,l}} : \bigoplus_{j=0}^{J_l} \bigoplus_{m=0}^{M_{j,l}} \mathscr{W}^m_{j,l} \to D_{\mu_l,\infty} = \bigoplus_{j=0}^{J_l} \bigoplus_{m=0}^{M_{j,l}} \mathscr{V}^m_{j,l}$$

is an isomorphism. Let $d^m_{j,l} = \dim \mathscr{W}^m_{j,l}$ and pick a basis

$$w^m_{j,l.k}, \quad 1 \le k \le d^m_{j,l}$$

of $\mathscr{W}^m_{j,l}$, $j = 0, \dots, J_l$, $m = 0, \dots, M_{j,l}$. Then $\tilde{p}^m_{j,l,k}(t) = t^{-n^m_{j,l}} P^m_{j,l}(t) w^m_{j,l,k} \in \mathscr{W}_{\mu_l}$. These elements

$$\tilde{p}^m_{j,l,k} \in \mathscr{W}_{\mu_l}[t, t^{-1}], \quad \text{for } j = 0, \dots, J_l,\ m = 0, \dots, M_{j,l},\ l = 1, \dots d^m_{j,l},$$

are the ones Lemma 3.5 claims exist. Indeed, since $Q^m_{j,l}(t) = e^{t\tilde{N}_{\mu_l}} P^m_{j,l}(t)$,

$$\lim_{\substack{\operatorname{Re} t \to \infty \\ t \in S_\theta}} e^{t\tilde{N}_{\mu_l}} t^{-n^m_{j,l}} P^m_{j,l}(t) w^m_{j,l,k} = G^m_{j,l} w^m_{j,l,l}.$$

Since the $G^m_{j,l} w^m_{j,l,k}$ form a basis of $D_{\mu,\infty}$, the $t^{-n^m_{j,l}} P^m_{j,l}(t) w^m_{j,l,k}$, form a basis of $\mathscr{W}_{\mu_l}$ for all $t \in S_\theta$ with large enough real part. □

4. Asymptotics of the projection

With the setup and (slightly changed) notation leading to and in the proof of Proposition 3.3, given a subspace $D \subset \mathscr{E}$ and the linear map $\mathfrak{a} : \mathscr{E} \to \mathscr{E}$ we have, for fixed $\theta \ge 0$ and $t \in S_\theta = \{t \in \mathbb{C} : |\operatorname{Im} t| \le \theta\}$,

$$e^{t\mathfrak{a}} D = \operatorname{span}\{v_k(t)\}, \quad \operatorname{Re} t \gg 0$$

with

$$v_k(t) = e^{t\mathfrak{a}'} g_k(t) + \sum_{\substack{\lambda \in \operatorname{spec} \mathfrak{a} \\ \operatorname{Re} \lambda < \mu_k}} e^{t(\lambda - \mu_k)} \hat{p}_{k,\lambda}(t). \tag{4.1}$$

The $g_k(t)$ are polynomials in $1/t$ with values in $\tilde{\mathscr{E}}_{\mu_k}$, the collection of vectors

$$g_{\infty,k} = \lim_{t \to \infty} g_k(t)$$

is a basis of D_∞, the μ_k form a finite sequence, possibly with repetitions, of elements in $\{\operatorname{Re} \lambda : \lambda \in \operatorname{spec} \mathfrak{a}\}$, and we have

$$\hat{p}_{k,\lambda}(t) = e^{tN_\lambda} \pi_\lambda p_k(t),$$

where the $p(t)$ are polynomials in t and $1/t$ with values in $\mathscr{E}$. The additive semigroup $\mathfrak{S}_{\mathfrak{a}} \subset \mathbb{C}$ (possibly without identity) generated by the set (3.8) is a subset of $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \le 0\}$ and has the property that $\{\vartheta \in \mathfrak{S}_{\mathfrak{a}} : \operatorname{Re}\vartheta > \mu\}$ is finite for every $\mu \in \mathbb{R}$.

Proposition 4.2. *Let $K \in \operatorname{Gr}_{d'}(\mathscr{E})$ be complementary to D, and suppose that*

$$\mathscr{V}_K \cap \Omega^+_\theta(D) = \varnothing. \tag{4.3}$$

There are polynomials $p_\vartheta(z^1, \dots, z^N, t)$ with values in $\operatorname{End}(\mathscr{E})$ *and $\mathbb{C}$-valued polynomials*

$$q_\vartheta(z^1, \dots, z^N, t)$$

such that

$$\exists\, C,\ R_0 > 0 \text{ such that } \left|q_\vartheta(e^{it \operatorname{Im}\lambda_1}, \dots, e^{it \operatorname{Im}\lambda_N}, t)\right| > C \text{ if } t \in S_\theta,\ \operatorname{Re} t > R_0 \tag{4.4}$$

and such that

$$\pi_{e^{t\mathfrak{a}}D,K} = \sum_{\vartheta \in \mathfrak{S}_{\mathfrak{a}}} \frac{e^{t\vartheta} p_\vartheta(e^{it \operatorname{Im}\lambda_1}, \dots, e^{it \operatorname{Im}\lambda_N}, t)}{q_\vartheta(e^{it \operatorname{Im}\lambda_1}, \dots, e^{it \operatorname{Im}\lambda_N}, t)}, \quad t \in S_\theta,\ \operatorname{Re} t > R_0,$$

with uniform convergence in norm in the indicated subset of S_θ.

Proof. Let $K \subset \mathscr{E}$ be complementary to D as indicated in the statement of the proposition, let $\boldsymbol{u} = [u_1, \dots, u_{d'}]$ be an ordered basis of K. Write $\boldsymbol{g}$ for an ordering of the basis $\{g_{\infty,k}\}$ of D_∞. With the $v_k(t)$ ordered as the $g_{\infty,k}$ to form $\boldsymbol{v}(t)$, we have

$$\begin{bmatrix}\boldsymbol{v}(t) & \boldsymbol{u}\end{bmatrix} = \begin{bmatrix}\boldsymbol{g} & \boldsymbol{u}\end{bmatrix} \cdot \begin{bmatrix}\alpha(t) & 0 \\ \beta(t) & I\end{bmatrix}, \tag{4.5}$$

where

$$\alpha(t) = \sum_k \sum_{\substack{\lambda \in \operatorname{spec} a \\ \operatorname{Re}\lambda \le \mu_k}} e^{t(\lambda - \mu_k)} \alpha_{k,\lambda}(t), \quad \beta(t) = \sum_k \sum_{\substack{\lambda \in \operatorname{spec} a \\ \operatorname{Re}\lambda \le \mu_k}} e^{t(\lambda - \mu_k)} \beta_{k,\lambda}(t). \tag{4.6}$$

The entries of the matrices $\alpha_{k,\lambda}(t)$ and $\beta_{k,\lambda}(t)$ are both polynomials in t and $1/t$, but only in $1/t$ if $\operatorname{Re}\lambda = \mu_k$. Define

$$\alpha^{(0)}(t) = \sum_k \sum_{\substack{\lambda \in \operatorname{spec} a \\ \operatorname{Re}\lambda = \mu_k}} e^{t(\lambda - \mu_k)} \alpha_{k,\lambda}(t), \quad \tilde\alpha(t) = \sum_k \sum_{\substack{\lambda \in \operatorname{spec} a \\ \operatorname{Re}\lambda < \mu_k}} e^{t(\lambda - \mu_k)} \alpha_{k,\lambda}(t), \tag{4.7}$$

and likewise $\beta^{(0)}(t)$ and $\tilde\beta(t)$. Note that $\tilde\alpha(t)$ and $\tilde\beta(t)$ decrease exponentially as $\operatorname{Re} t \to \infty$ with $|\operatorname{Im} t|$ bounded.

The hypothesis (4.3) implies that $\begin{bmatrix}\alpha(t) & 0 \\ \beta(t) & I\end{bmatrix}$ is invertible for every sufficiently large $\operatorname{Re} t$, so $\alpha(t)$ is invertible for such t. In fact,

$$\text{there are } C,\ R_0 > 0 \text{ such that } \left|\det(\alpha(t))\right| > C \text{ if } t \in S_\theta,\ \operatorname{Re} t > R_0. \tag{4.8}$$

For suppose this is not the case. Then there is a sequence $\{t_\nu\}$ in S_θ with $\operatorname{Re} t_\nu \to \infty$ as $\nu \to \infty$ such that $\det \alpha(t_\nu) \to 0$. Since both $\alpha(t_\nu)$ and $\beta(t_\nu)$ are bounded, we may assume, passing to a subsequence, that they converge. It follows that $e^{t_\nu a} D$ converges, by definition, to an element $D' \in \Omega^+_\theta(D)$. Also the matrix in (4.5) converges. The vanishing of the determinant of the limiting matrix implies that $K \cap D' \neq \{0\}$, contradicting (4.3). Thus (4.8) holds.

If $\phi \in \mathscr{E}$ then of course

$$\phi = \begin{bmatrix} \boldsymbol{g} & \boldsymbol{u} \end{bmatrix} \cdot \begin{bmatrix} \varphi^1 \\ \varphi^2 \end{bmatrix},$$

where the φ^i are columns of scalars. Substituting

$$\begin{bmatrix} \boldsymbol{g} & \boldsymbol{u} \end{bmatrix} = \begin{bmatrix} \boldsymbol{v}(t) & \boldsymbol{u} \end{bmatrix} \cdot \begin{bmatrix} \alpha(t)^{-1} & 0 \\ -\beta(t)\alpha(t)^{-1} & I \end{bmatrix}$$

gives

$$\phi = \begin{bmatrix} \boldsymbol{v}(t) & \boldsymbol{u} \end{bmatrix} \cdot \begin{bmatrix} \alpha(t)^{-1} & 0 \\ -\beta(t)\alpha(t)^{-1} & I \end{bmatrix} \begin{bmatrix} \varphi^1 \\ \varphi^2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{v}(t) & \boldsymbol{u} \end{bmatrix} \cdot \begin{bmatrix} \alpha(t)^{-1}\varphi^1 \\ -\beta(t)\alpha(t)^{-1}\varphi^1 + \varphi^2 \end{bmatrix};$$

hence

$$\phi = \boldsymbol{v}(t) \cdot \alpha(t)^{-1}\varphi^1 + \boldsymbol{u} \cdot (-\beta(t)\alpha(t)^{-1}\varphi^1 + \varphi^2).$$

This is the decomposition of ϕ according to $\mathscr{E} = e^{t\mathfrak{a}}D \oplus K$; therefore

$$\pi_{e^{t\mathfrak{a}}D,K}\phi = \boldsymbol{v}(t) \cdot \alpha(t)^{-1}\varphi^1.$$

Replacing $\boldsymbol{v}(t) = \boldsymbol{g} \cdot \alpha(t) + \boldsymbol{u} \cdot \beta(t)$ we obtain

$$\pi_{e^{t\mathfrak{a}}D,K}\phi = (\boldsymbol{g} \cdot \alpha(t) + \boldsymbol{u} \cdot \beta(t))\alpha(t)^{-1}\varphi^1 = (\boldsymbol{g} + \boldsymbol{u} \cdot \beta(t)\alpha(t)^{-1})\varphi^1. \tag{4.9}$$

The matrix $\alpha^{(0)}(t)$ is invertible because of (4.8) and the decomposition $\alpha(t) = \alpha^{(0)}(t) + \tilde{\alpha}(t)$, so

$$\begin{aligned} \beta(t)\alpha(t)^{-1} &= \beta(t)\alpha^{(0)}(t)^{-1}(I + \tilde{\alpha}(t)\alpha^{(0)}(t)^{-1})^{-1} \\ &= \beta(t)\alpha^{(0)}(t)^{-1}\sum_{l=0}^{\infty}(-1)^l[\tilde{\alpha}(t)\alpha^{(0)}(t)^{-1}]^l. \end{aligned} \tag{4.10}$$

The series converges absolutely and uniformly in $\{t \in S_\theta : \operatorname{Re} t > R_0\}$ for some real $R_0 \in \mathbb{R}$. The entries of $\alpha^{(0)}(t)$ are expressions

$$\sum_{\lambda \in \operatorname{spec} \mathfrak{a}} e^{it \operatorname{Im} \lambda} \sum_{\nu=0}^{N} c_{\lambda,\nu} t^{-\nu};$$

hence

$$\det \alpha^{(0)}(t) = q(e^{it \operatorname{Im} \lambda_1}, \dots, e^{it \operatorname{Im} \lambda_N}, 1/t),$$

for some polynomial $q(z^1, \dots, z^N, 1/t)$. Note that because of (4.8),

$$\text{there are } C,\ R_0 > 0 \text{ such that } \left|\det(\alpha^0(t))\right| > C \text{ if } t \in S_\theta,\ \operatorname{Re} t > R_0. \tag{4.11}$$

Since $\alpha^{(0)}(t)^{-1} = (\det \alpha^{(0)}(t))^{-1}\Delta(t)^\dagger$ where $\Delta(t)^\dagger$ is the matrix of cofactors of $\alpha^{(0)}(t)$, (4.10) and (4.6) give

$$\beta(t)\alpha(t)^{-1} = \sum_{\vartheta \in \mathfrak{S}_{\mathfrak{a}}} r_\vartheta(t)e^{t\vartheta} \tag{4.12}$$

where $\mathfrak{S}_{\mathfrak{a}}$ was defined before the statement of Proposition 4.2 as the additive semigroup generated by $\{\lambda - \operatorname{Re}\lambda' : \lambda, \lambda' \in \operatorname{spec} \mathfrak{a},\ \operatorname{Re}\lambda \leq \operatorname{Re}\lambda'\}$ and $r_\vartheta(t)$ is a matrix whose entries are of the form

$$\frac{p_\vartheta(e^{it \operatorname{Im} \lambda_1}, \dots, e^{it \operatorname{Im} \lambda_N}, t, 1/t)}{q(e^{it \operatorname{Im} \lambda_1}, \dots, e^{it \operatorname{Im} \lambda_N}, 1/t)^{n_\vartheta}},$$

for some polynomial $p_\vartheta(z^1,\dots,z^N,t,1/t)$ and nonnegative integers n_ϑ. Multiplying the numerator and denominator by the same nonnegative (integral) power of t we replace the dependence on $1/t$ by polynomial dependence in $e^{it\operatorname{Im}\lambda_1},\dots,e^{it\operatorname{Im}\lambda_N}$, t only. This gives the structure of the "coefficients" of the $e^{t\vartheta}$ stated in the proposition for the expansion of $\pi_{e^{t\mathfrak{a}}D,K}$. □

The terms in (4.12) with $\operatorname{Re}\vartheta=0$ come from $\beta^{(0)}(t)\alpha^{(0)}(t)^{-1}$. So the principal part of $\pi_{e^{t\mathfrak{a}}D,K}$ is

$$\boldsymbol{\sigma}(\pi_{e^{t\mathfrak{a}}D,K})\phi=(\boldsymbol{g}+\boldsymbol{u}\cdot\beta^{(0)}(t)\alpha^{(0)}(t)^{-1})\varphi^1$$

This principal part is not itself a projection, but

$$\|\boldsymbol{\sigma}(\pi_{e^{t\mathfrak{a}}D,K})-\pi_{e^{t\mathfrak{a}'}D_\infty,K}\|\to 0\quad\text{ as }\operatorname{Re}t\to\infty,\ t\in S_\theta.$$

We now restate Proposition 4.2 as an asymptotics for the family (2.10) using the notation $\boldsymbol{\kappa}$ for the action on $\mathscr{E}$ and express the asymptotics of $\pi_{\boldsymbol{\kappa}^{-1}_{\zeta^{1/m}}D,K}$ in terms of the boundary spectrum of A exploiting (3.1). Condition (4.14) below corresponds to our geometric condition in part (iii) of Theorem 2.15 expressing the fact that Λ is a sector of minimal growth for $A_{\wedge,\mathscr{D}_\wedge}$. The Ω-limit set is the one defined in (2.14). Recall that by $\zeta^{1/m}$ we mean the root defined by the principal branch of the logarithm on $\mathbb{C}\setminus\overline{\mathbb{R}}_-$. We let $\lambda_0\neq 0$ be an element in the central axis of Λ and define $\tilde\Lambda=\{\zeta:\zeta\lambda_0\in\Lambda\}$; this is a closed sector not containing the negative real axis.

Let $\mathfrak{S}\subset\mathbb{C}$ be the additive semigroup generated by

$$\{\sigma-i\operatorname{Im}\sigma':\sigma,\sigma'\in\operatorname{spec}_b(A),\ -m/2<\operatorname{Im}\sigma\le\operatorname{Im}\sigma'<m/2\}.$$

Thus $-i\mathfrak{S}=\mathfrak{S}_{\mathfrak{a}}$. Let $\sigma_1,\dots,\sigma_N$ be an enumeration of the elements of

$$\Sigma=\operatorname{spec}_b(A)\cap\{-m/2<\operatorname{Im}\sigma<m/2\}.$$

Theorem 4.13. *Let $K\in\operatorname{Gr}_{d'}(\mathscr{E})$ be complementary to D, suppose that*

$$\mathscr{V}_K\cap\Omega^-_\Lambda(D)=\varnothing. \tag{4.14}$$

Then there are polynomials $p_\vartheta(z^1,\dots,z^N,t)$ with values in $\operatorname{End}(\mathscr{E})$ and $\mathbb{C}$-valued polynomials

$$q_\vartheta(z^1,\dots,z^N,t)$$

such that

$$\exists\, C,\ R_0>0\ \text{such that}\ \left|q_\vartheta(\zeta^{i\operatorname{Re}\sigma_1/m},\dots,\zeta^{i\operatorname{Re}\sigma_N/m},t)\right|>C\ \text{if}\ \zeta\in\tilde\Lambda,\ |\zeta|>R_0, \tag{4.15}$$

and such that

$$\pi_{\boldsymbol{\kappa}^{-1}_{\zeta^{1/m}}D,K}=\sum_{\vartheta\in\mathfrak{S}}\frac{\zeta^{-i\vartheta/m}p_\vartheta(\zeta^{i\operatorname{Re}\sigma_1/m},\dots,\zeta^{i\operatorname{Re}\sigma_N/m},m^{-1}\log\zeta)}{q_\vartheta(\zeta^{i\operatorname{Re}\sigma_1/m},\dots,\zeta^{i\operatorname{Re}\sigma_N/m},m^{-1}\log\zeta)},\qquad\zeta\in\tilde\Lambda,\ |\zeta|>R_0,$$

with uniform convergence in norm in the indicated subset of $\tilde\Lambda$.

The elements $\vartheta\in\mathfrak{S}$ are of course finite sums $\vartheta=\sum n_{jk}(\sigma_j-i\operatorname{Im}\sigma_k)$ for some nonnegative integers n_{jk}, with $\sigma_j,\sigma_k\in\Sigma$ and $\operatorname{Im}\sigma_j\le\operatorname{Im}\sigma_k$. Separating real and imaginary parts we may write $\zeta^{-i\vartheta/m}$ as a product of factors

$$\frac{\zeta^{n_{jk}(\operatorname{Im}\sigma_j-\operatorname{Im}\sigma_k)/m}}{\zeta^{in_{jk}\operatorname{Re}\sigma_k/m}}.$$

We thus see that we may also organize the series expansion of $\pi_{\kappa^{-1}_{\zeta^{1/m}}D,K}$ in the theorem as

$$\pi_{\kappa^{-1}_{\zeta^{1/m}}D,K}=\sum_{\vartheta\in\mathfrak{S}_{\mathbb{R}}}\frac{\zeta^{-i\vartheta/m}\,\tilde{p}_\vartheta(\zeta^{i\operatorname{Re}\sigma_1/m},\dots,\zeta^{i\operatorname{Re}\sigma_N/m},m^{-1}\log\zeta)}{\tilde{q}_\vartheta(\zeta^{i\operatorname{Re}\sigma_1/m},\dots,\zeta^{i\operatorname{Re}\sigma_N/m},m^{-1}\log\zeta)},$$

where $\mathfrak{S}_{\mathbb{R}}\subset\mathbb{R}$ is the additive semigroup generated by

$$\big\{\operatorname{Im}\sigma-\operatorname{Im}\sigma':\sigma,\sigma'\in\Sigma,\ \operatorname{Im}\sigma'\le\operatorname{Im}\sigma'\big\}$$

and $\tilde{p}_\vartheta$, $\tilde{q}_\vartheta$ are still polynomials.

Remark 4.16. If Σ lies on a line $\operatorname{Re}\sigma=c_0$, then $-i\mathfrak{S}\subset\overline{\mathbb{R}}_--ic_0$. Also in this case, the coefficients of the exponents in (4.1) can be assumed to have vanishing imaginary part (see Remark 3.7). Assuming this, the coefficients of the exponents in (4.7) are real, in particular $\det\alpha^{(0)}(t)$ is just a polynomial in $1/t$, the coefficients r_ϑ in the expansion (4.12) can be written as rational functions of t only. Consequently, in the expansion of the projection in Theorem 4.13, the powers $-i\vartheta$ are real ≤ 0 and the coefficients can be written as rational functions of $\log\zeta$.

5. Asymptotic structure of the resolvent

For the analysis of $(A_{\mathcal{D}}-\lambda)^{-l}$ for $l\in\mathbb{N}$ sufficiently large we make use of the representation (1.8) of the resolvent as

$$(A_{\mathcal{D}}-\lambda)^{-1}=B(\lambda)+G_{\mathcal{D}}(\lambda),\tag{5.1}$$

where $B(\lambda)$ is a parametrix of $(A_{\min}-\lambda)$ and

$$G_{\mathcal{D}}(\lambda)=[1-B(\lambda)(A-\lambda)]F_{\mathcal{D}}(\lambda)^{-1}T(\lambda).\tag{5.2}$$

The starting point of our analysis is

$$(A_{\mathcal{D}}-\lambda)^{-l}=\frac{1}{(l-1)!}\partial_\lambda^{l-1}(A_{\mathcal{D}}-\lambda)^{-1}\qquad\text{for any } l\in\mathbb{N}.$$

We are thus led to further analyze the asymptotic structure of the pieces involved in the representation of the resolvent. In [Gil et al. 2010] we described in full generality the behavior of

$$B(\lambda),\qquad 1-B(\lambda)(A-\lambda),\qquad T(\lambda),$$

and we analyzed $F_{\mathcal{D}}(\lambda)^{-1}$ in the special case that $\mathcal{D}$ is stationary. In the case of a general domain $\mathcal{D}$, we now obtain as a consequence of Theorem 4.13 the following result.

Proposition 5.3. *For $R>0$ large enough we have*

$$F_{\mathcal{D}}(\lambda)^{-1}\in(S^{0+}_{\mathfrak{R}}\cap S^0)(\Lambda_R;\mathcal{D}_\wedge/\mathcal{D}_{\wedge,\min},\mathcal{D}_{\max}/\mathcal{D}_{\min}).$$

The components of $F_{\mathcal{D}}(\lambda)^{-1}$ have orders ν^+ with $\nu\in\mathfrak{E}$, the semigroup defined in (1.6), and their phases belong to the set $\mathfrak{M}$ defined in (1.5).

Here $S^0(\Lambda_R;\mathscr{D}_\wedge/\mathscr{D}_{\wedge,\min},\mathscr{D}_{\max}/\mathscr{D}_{\min})$ denotes the standard space of (anisotropic) operator-valued symbols of order zero on Λ_R (see the Appendix), where $\mathscr{D}_\wedge/\mathscr{D}_{\wedge,\min}$ carries the trivial group action, and $\mathscr{D}_{\max}/\mathscr{D}_{\min}$ is equipped with the group action $\tilde{\kappa}_\varrho=\theta^{-1}\boldsymbol{\kappa}_\rho\theta$. The symbol class

$$S^{0^+}_{\mathcal{R}}(\Lambda_R;\mathscr{D}_\wedge/\mathscr{D}_{\wedge,\min},\mathscr{D}_{\max}/\mathscr{D}_{\min})$$

is discussed in the Appendix (see Definition A.7). Recall that $\Lambda_R=\{\lambda\in\Lambda:|\lambda|\geq R\}$.

Proof of Proposition 5.3. We follow the line of reasoning of [Gil et al. 2010, Propositions 5.10 and 5.17]. The crucial point is that we now know from Theorem 4.13 and Proposition 2.17 that $F_{\wedge,\mathscr{D}_\wedge}(\lambda)^{-1}$ belongs to the symbol class

$$(S^{0^+}_{\mathcal{R}}\cap S^0)(\Lambda_R;\mathscr{D}_\wedge/\mathscr{D}_{\wedge,\min},\mathscr{D}_{\wedge,\max}/\mathscr{D}_{\wedge,\min}),$$

where the actions on $\mathscr{D}_\wedge/\mathscr{D}_{\wedge,\min}$ and $\mathscr{D}_{\wedge,\max}/\mathscr{D}_{\wedge,\min}$ are, respectively, the trivial action as above and $\boldsymbol{\kappa}_\varrho$. The components of $F_{\wedge,\mathscr{D}_\wedge}(\lambda)^{-1}$ have orders ν^+ with $\nu\in\mathfrak{E}$, and their phases belong to the set $\mathfrak{M}$. Consequently, $\Phi_0(\lambda)=\theta^{-1}F_{\wedge,\mathscr{D}_\wedge}(\lambda)^{-1}$ belongs to

$$(S^{0^+}_{\mathcal{R}}\cap S^0)(\Lambda_R;\mathscr{D}_\wedge/\mathscr{D}_{\wedge,\min},\mathscr{D}_{\max}/\mathscr{D}_{\min}),$$

and we have the same statement about the orders and phases of its components.

Phrased in the terminology of the present paper, we proved (see [Gil et al. 2010, Proposition 5.10]) that the operator family

$$F(\lambda)=[T(\lambda)(A-\lambda)]:\mathscr{D}_{\max}/\mathscr{D}_{\min}\to\mathscr{D}_\wedge/\mathscr{D}_{\wedge,\min}$$

belongs to the symbol class

$$(S^{0^+}_{\mathcal{R}}\cap S^0)(\Lambda_R;\mathscr{D}_{\max}/\mathscr{D}_{\min},\mathscr{D}_\wedge/\mathscr{D}_{\wedge,\min}),$$

and that

$$F(\lambda)\Phi_0(\lambda)-1=R(\lambda)\in S^{-1+\varepsilon}(\Lambda_R;\mathscr{D}_\wedge/\mathscr{D}_{\wedge,\min},\mathscr{D}_\wedge/\mathscr{D}_{\wedge,\min})$$

for any $\varepsilon>0$. More precisely, $F(\lambda)$ is an anisotropic log-polyhomogeneous operator-valued symbol. We thus can infer further that in fact

$$R(\lambda)\in S^{(-1)^+}_{\mathcal{R}}(\Lambda_R;\mathscr{D}_\wedge/\mathscr{D}_{\wedge,\min},\mathscr{D}_\wedge/\mathscr{D}_{\wedge,\min}),$$

and that the components of $R(\lambda)$ have orders ν^+ with $\nu\in\mathfrak{E}$, $\nu\leq-1$, and phases belonging to the set $\mathfrak{M}$. The usual Neumann series argument then yields the existence of a symbol

$$R_1(\lambda)\in S^{(-1)^+}_{\mathcal{R}}(\Lambda_R;\mathscr{D}_\wedge/\mathscr{D}_{\wedge,\min},\mathscr{D}_\wedge/\mathscr{D}_{\wedge,\min})$$

such that $F(\lambda)\Phi_0(\lambda)(1+R_1(\lambda))=1$ for $\lambda\in\Lambda_R$. Consequently $F_{\mathscr{D}}(\lambda)^{-1}=\Phi_0(\lambda)(1+R_1(\lambda))$ lies in

$$(S^{0^+}_{\mathcal{R}}\cap S^0)(\Lambda_R;\mathscr{D}_\wedge/\mathscr{D}_{\wedge,\min},\mathscr{D}_{\max}/\mathscr{D}_{\min}),$$

and its components have the structure that was claimed. □

With Proposition 5.3 and our results in [Gil et al. 2010, Section 5] at our disposal, we now obtain a general theorem about the asymptotics of the finite rank contribution $G_{\mathscr{D}}(\lambda)$ in the representation (5.1) of the resolvent. Before stating it we recall and rephrase the relevant results from that paper about the other pieces involved in (5.2) using the terminology of the present paper.

Concerning $T(\lambda)$ we have [Gil et al. 2010, Proposition 5.5]:

(i) For any cut-off function $\omega \in C_c^\infty([0,1))$ the function $T(\lambda)(1-\omega)$ is rapidly decreasing on Λ taking values in $\mathscr{L}(x^{-m/2}H_b^s, \mathscr{D}_\wedge/\mathscr{D}_{\wedge,\min})$, and

$$t(\lambda) = T(\lambda)\omega \in S^{-m}(\Lambda; \mathscr{K}^{s,-m/2}, \mathscr{D}_\wedge/\mathscr{D}_{\wedge,\min}).$$

Here $\mathscr{K}^{s,-m/2}$ is equipped with the (normalized) dilation group action κ_ϱ, and we give $\mathscr{D}_\wedge/\mathscr{D}_{\wedge,\min}$ again the trivial action.

(ii) The family $t(\lambda)$ admits a full asymptotic expansion into anisotropic homogeneous components. In particular, we have

$$t(\lambda) \in S_{\mathfrak{R}}^{(-m)^+}(\Lambda; \mathscr{K}^{s,-m/2}, \mathscr{D}_\wedge/\mathscr{D}_{\wedge,\min}).$$

The spaces $\mathscr{K}^{s,-m/2}$ are weighted cone Sobolev spaces on $Y^\wedge$. We discussed them in [Gil et al. 2006, Section 2] and reviewed the definition in [Gil et al. 2010, Section 4] (see also [Schulze 1991], where different weight functions as $x \to \infty$ are considered). Note that $\mathscr{K}^{0,-m/2} = x^{-m/2}L_b^2(Y^\wedge; E)$.

Concerning $1 - B(\lambda)(A-\lambda)$ Proposition 5.20 of [Gil et al. 2010] gives, for any $\varphi \in C^\infty(M; \mathrm{End}(E))$:

(iii) The operator function $P(\lambda) = \varphi[1 - B(\lambda)(A-\lambda)]$ is a smooth function

$$\Lambda_R \to \mathscr{L}(\mathscr{D}_{\max}/\mathscr{D}_{\min}, x^{-m/2}H_b^s),$$

which is defined for $R > 0$ large enough. Let $\omega \in C_c^\infty([0,1))$ be an arbitrary cut-off function. Then $(1-\omega)P(\lambda)$ is rapidly decreasing on Λ_R, and

$$p(\lambda) = \omega P(\lambda) \in S^0(\Lambda_R; \mathscr{D}_{\max}/\mathscr{D}_{\min}, \mathscr{K}^{s,-m/2});$$

here $\mathscr{K}^{s,-m/2}$ is equipped with the (normalized) dilation group action κ_ϱ, and the quotient $\mathscr{D}_{\max}/\mathscr{D}_{\min}$ is equipped with the group action $\tilde{\kappa}_\varrho$.

(iv) $p(\lambda)$ is an anisotropic log-polyhomogeneous operator-valued symbol on Λ_R. In particular,

$$p(\lambda) = \omega P(\lambda) \in S_{\mathfrak{R}}^{0^+}(\Lambda_R; \mathscr{D}_{\max}/\mathscr{D}_{\min}, \mathscr{K}^{s,-m/2}).$$

With $\mathfrak{M}$ as in (1.5) and $\mathfrak{E}$ as in (1.6) we have:

Theorem 5.4. *Let $\varphi \in C^\infty(M; \mathrm{End}(E))$, and let $\omega, \tilde{\omega} \in C_c^\infty([0,1))$ be arbitrary cut-off functions. For $R > 0$ large enough the operator family $G_{\mathscr{D}}(\lambda)$ is defined on Λ_R, and*

$$(1-\omega)\varphi G_{\mathscr{D}}(\lambda),\ \varphi G_{\mathscr{D}}(\lambda)(1-\omega) \in \mathscr{S}\big(\Lambda_R, l^1(x^{-m/2}H_b^s, x^{-m/2}H_b^t)\big).$$

Moreover,

$$\omega\varphi G_{\mathscr{D}}(\lambda)\tilde{\omega} \in (S_{\mathfrak{R}}^{(-m)^+} \cap S^{-m})(\Lambda_R; \mathscr{K}^{s,-m/2}, \mathscr{K}^{t,-m/2}),$$

where the spaces $\mathscr{K}^{s,-m/2}$ and $\mathscr{K}^{t,-m/2}$ are equipped with the group action κ_ϱ. In fact, $\omega\varphi G_{\mathscr{D}}(\lambda)\tilde{\omega}$ takes values in the trace class operators, and all statements about symbol estimates and asymptotic expansions hold in trace class norms. The components have orders ν^+ with $\nu \in \mathfrak{E}$, $\nu \le -m$, and their phases belong to $\mathfrak{M}$.

Corollary 5.5. *For $R > 0$ sufficiently large and $\varphi \in C^\infty(M; \mathrm{End}(E))$, the operator family $\varphi G_D(\lambda)$ is a smooth family of trace class operators in $x^{-m/2}L^2_b$ for $\lambda \in \Lambda_R$, and $\mathrm{Tr}(\varphi G_D(\lambda)) \in (S^{(-m)^+}_{\mathfrak{R}} \cap S^{-m})(\Lambda_R)$. The components have orders ν^+ with $\nu \in \mathfrak{E}$, $\nu \le -m$, and their phases belong to the set $\mathfrak{M}$.*

Theorem 5.4 and Corollary 5.5 follow at once from the previous results about the pieces involved in the representation (5.2) for $G_{\mathfrak{D}}(\lambda)$ and the properties of the operator-valued symbol class discussed in the Appendix. In the statement of Corollary 5.5 the scalar symbol spaces are also anisotropic with anisotropy m. In particular, this means that $\mathrm{Tr}(\varphi G_D(\lambda)) = O(|\lambda|^{-1})$ as $|\lambda| \to \infty$.

We are now in the position to prove the trace expansion claimed in Theorem 1.4. To this end, we need the following result [Gil et al. 2010, Theorem 4.4]:

(v) Let $\varphi \in C^\infty(M; \mathrm{End}(E))$. If $ml > n$, then $\varphi \partial_\lambda^{l-1} B(\lambda)$ is a smooth family of trace class operators in $x^{-m/2}L^2_b$, and the trace $\mathrm{Tr}(\varphi \partial_\lambda^{l-1} B(\lambda))$ is a log-polyhomogeneous symbol on Λ. For large λ we have
$$\mathrm{Tr}(\varphi \partial_\lambda^{l-1} B(\lambda)) \sim \sum_{j=0}^{n-1} \alpha_j \lambda^{(n-lm-j)/m} + \alpha_n \log(\lambda)\lambda^{-l} + r(\lambda),$$
where
$$r(\lambda) \in (S^{(-lm)^+}_{\mathfrak{R}} \cap S^{-lm})(\Lambda).$$

Now, combining (v) with Corollary 5.5, we finally obtain:

Theorem 5.6. *Let $\Lambda \subset \mathbb{C}$ be a closed sector. Assume that $A \in x^{-m}\,\mathrm{Diff}^m_b(M; E)$, $m > 0$, with domain $\mathfrak{D} \subset x^{-m/2}L^2_b$ satisfies the ray conditions (1.3). Then Λ is a sector of minimal growth for $A_{\mathfrak{D}}$, and for $ml > n$, $(A_{\mathfrak{D}} - \lambda)^{-l}$ is an analytic family of trace class operators on Λ_R for some $R > 0$. Moreover, for $\varphi \in C^\infty(M; \mathrm{End}(E))$,*
$$\mathrm{Tr}(\varphi(A_{\mathfrak{D}} - \lambda)^{-l}) \in (S^{(n-lm)^+}_{\mathfrak{R},\mathrm{hol}} \cap S^{n-lm})(\Lambda_R).$$
The components have orders ν^+ with $\nu \in \mathfrak{E}$, $\nu \le n - lm$, where $\mathfrak{E}$ is the semigroup defined in (1.6), and their phases belong to the set $\mathfrak{M}$ defined in (1.5).

More precisely, we have the expansion
$$\mathrm{Tr}(\varphi(A_{\mathfrak{D}} - \lambda)^{-l}) \sim \sum_{j=0}^{n-1} \alpha_j \lambda^{(n-lm-j)/m} + \alpha_n \log(\lambda)\lambda^{-l} + s_{\mathfrak{D}}(\lambda),$$
with constants $\alpha_j \in \mathbb{C}$ independent of the choice of domain $\mathfrak{D}$, and a domain dependent remainder $s_{\mathfrak{D}}(\lambda) \in (S^{(-lm)^+}_{\mathfrak{R},\mathrm{hol}} \cap S^{-lm})(\Lambda_R)$.

If all elements of the set $\{\sigma \in \mathrm{spec}_b(A) : -m/2 < \mathrm{Im}\,\sigma < m/2\}$ are vertically aligned, then the coefficients r_ν in the expansion (1.7) of $s_{\mathfrak{D}}(\lambda)$ are rational functions of $\log\lambda$ only. This is because, in this case, the series representation of the projection in Theorem 4.13 contains only real powers of ζ and rational functions of $\log\zeta$; see Remark 4.16. This simplifies the structure of $F_{\wedge,\mathfrak{D}_\wedge}(\lambda)^{-1}$ according to Section 2, and consequently the structure of $F_{\mathfrak{D}}(\lambda)^{-1}$ (see the proof of Proposition 5.3). As recalled in this section, the terms coming from $B(\lambda)$ and the other pieces in the representation (5.2) of $G_{\mathfrak{D}}(\lambda)$ do not generate phases.

If $\mathfrak{D}$ is stationary, then the expansion (1.7) of $s_{\mathfrak{D}}(\lambda)$ is even simpler: the r_ν are just polynomials in $\log\lambda$, and the numbers ν are all integers. To see this recall that if $\mathfrak{D}_\wedge$ is κ-invariant, then $F_{\wedge,\mathfrak{D}_\wedge}(\lambda)^{-1}$ is homogeneous, see (2.3), so it belongs to the class

$$S^{(0)}(\Lambda_R;\mathscr{D}_\wedge/\mathscr{D}_{\wedge,\min},\mathscr{D}_{\wedge,\max}/\mathscr{D}_{\wedge,\min})\subset(S^{0^+}_{\mathfrak{R}}\cap S^0)(\Lambda_R;\mathscr{D}_\wedge/\mathscr{D}_{\wedge,\min},\mathscr{D}_{\wedge,\max}/\mathscr{D}_{\wedge,\min}).$$

Consequently, by the proof of Proposition 5.3, $F_{\mathscr{D}}(\lambda)^{-1}$ is log-polyhomogeneous. This property propagates throughout the rest of the results in this section and gives the structure of $s_{\mathscr{D}}(\lambda)$ just asserted.

Appendix: A class of symbols

Let $\Lambda\subset\mathbb{C}$ be a closed sector. Let E and $\tilde{E}$ be Hilbert spaces equipped with strongly continuous group actions κ_ϱ and $\tilde{\kappa}_\varrho$, $\varrho>0$, respectively. Recall that the space $S^\nu(\Lambda;E,\tilde{E})$ of anisotropic operator-valued symbols on the sector Λ of order $\nu\in\mathbb{R}$ is defined as the space of all $a\in C^\infty(\Lambda,\mathscr{L}(E,\tilde{E}))$ such that for all $\alpha,\beta\in\mathbb{N}_0$

$$\big\|\tilde{\kappa}^{-1}_{|\lambda|^{1/m}}\partial^\alpha_\lambda\partial^\beta_{\bar{\lambda}}a(\lambda)\kappa_{|\lambda|^{1/m}}\big\|_{\mathscr{L}(E,\tilde{E})}=\mathcal{O}(|\lambda|^{\nu/m-\alpha-\beta})\quad\text{ as }|\lambda|\to\infty\text{ in }\Lambda. \tag{A.1}$$

By $S^{(\nu)}(\Lambda;E,\tilde{E})$ we denote the space of anisotropic homogeneous functions of degree $\nu\in\mathbb{R}$, that is, all $a\in C^\infty(\Lambda\setminus\{0\},\mathscr{L}(E,\tilde{E}))$ such that

$$a(\varrho^m\lambda)=\varrho^\nu\tilde{\kappa}_\varrho a(\lambda)\kappa_\varrho^{-1}\quad\text{ for }\varrho>0\text{ and }\lambda\in\Lambda\setminus\{0\}. \tag{A.2}$$

Clearly $\chi(\lambda)S^{(\nu)}(\Lambda;E,\tilde{E})\subset S^\nu(\Lambda;E,\tilde{E})$ with the obvious meaning of notation, where $\chi\in C^\infty(\mathbb{R}^2)$ is any excision function of the origin. When $E=\tilde{E}=\mathbb{C}$ equipped with the trivial group action the spaces are dropped from the notation.

Such symbol classes were introduced by Schulze in his theory of pseudodifferential operators on manifolds with singularities, see [Schulze 1991]. In particular, classical symbols, that is, symbols that admit asymptotic expansions into homogeneous components, play an important role and were used in [Gil et al. 2006] for the construction of a parameter-dependent parametrix of $A_{\min}-\lambda$. As illustrated in the present paper, for a general domain $\mathscr{D}$, the structure of $(A_{\mathscr{D}}-\lambda)^{-1}$ is rather involved, and classical symbols do not suffice to describe it. We are therefore led to introduce a new class of (anisotropic) operator-valued symbols that admit expansions of a more general kind. As it turns out, this class occurs naturally and is well adapted to describe the structure of resolvents in the general case.

Remark A.3. The operator-valued symbol classes $S^\nu(\Lambda;E,\tilde{E})$ and $S^{(\nu)}(\Lambda;E,\tilde{E})$, as well as the spaces $S^{(\nu^+)}_{\mathfrak{R}}(\Lambda;E,\tilde{E})$ and $S^{\nu^+}_{\mathfrak{R}}(\Lambda;E,\tilde{E})$ defined in this Appendix, all depend on the choice of the group actions on E and $\tilde{E}$. They also depend on the anisotropy parameter m that appears in (A.1) and (A.2). However, in order to avoid an overload of notation, we will not emphasize this dependence. In this paper, the anisotropy m is always the order of the cone operator A under study, and the group actions are explicitly defined when necessary.

Recall that $V[z_1,\dots,z_M]$ denotes the space of polynomials in the variables z_j, $j=1\dots,M$, with coefficients in V for any vector space V. We shall make use of this in particular for $V=\mathbb{C}$ and $V=S^{(0)}(\Lambda;E,\tilde{E})$. In what follows, all holomorphic powers and logarithms on $\mathring{\Lambda}$ are defined using a holomorphic branch of the logarithm with cut $\Gamma\not\subset\Lambda$.

Definition A.4. Let $\nu\in\mathbb{R}$. We define $S^{(\nu^+)}_{\mathfrak{R}}(\Lambda;E,\tilde{E})$ as the space of all $\mathscr{L}(E,\tilde{E})$-valued functions $s(\lambda)$ of the following form:

There exist polynomials $p \in S^{(0)}(\Lambda; E, \tilde{E})[z_1, \dots, z_{N+1}]$ and $q \in \mathbb{C}[z_1, \dots, z_{N+1}]$ in $N+1$ variables, $N = N(s) \in \mathbb{N}_0$, and real numbers $\mu_k = \mu_k(s)$, $k = 1, \dots, N$, such that the following holds:

(a) $|q(\lambda^{i\mu_1}, \dots, \lambda^{i\mu_N}, \log \lambda)| \geq c > 0$ for $\lambda \in \Lambda$ with $|\lambda|$ sufficiently large;

(b) $s(\lambda) = r(\lambda)\lambda^{\nu/m}$, where

$$r(\lambda) = \frac{p(\lambda^{i\mu_1}, \dots, \lambda^{i\mu_N}, \log \lambda)}{q(\lambda^{i\mu_1}, \dots, \lambda^{i\mu_N}, \log \lambda)}. \tag{A.5}$$

To clarify the notation, we note that

$$p(\lambda^{i\mu_1}, \dots, \lambda^{i\mu_N}, \log \lambda) = \sum_{|\alpha|+k \leq M} a_{\alpha,k}(\lambda)\lambda^{i\mu_1\alpha_1} \cdots \lambda^{i\mu_N\alpha_N} \log^k \lambda$$

as a function $\Lambda \setminus \{0\} \to \mathscr{L}(E, \tilde{E})$ with certain $a_{\alpha,k}(\lambda) \in S^{(0)}(\Lambda; E, \tilde{E})$. We call the μ_k the phases and ν^+ the order of $s(\lambda)$.

Every $s(\lambda) \in S_{\mathscr{R}}^{(\nu^+)}(\Lambda; E, \tilde{E})$ is an operator function defined everywhere on Λ except at $\lambda = 0$ and the zero set of $q(\lambda^{i\mu_1}, \dots, \lambda^{i\mu_N}, \log \lambda)$. The latter is a discrete subset of $\Lambda \setminus \{0\}$, and it is finite outside any neighborhood of zero in view of (a).

Proposition A.6. (1) *$S_{\mathscr{R}}^{(\nu^+)}(\Lambda; E, \tilde{E})$ is a vector space.*

(2) *Let $\hat{E}$ be a third Hilbert space with group action $\hat{\kappa}_\varrho$, $\varrho > 0$. Composition of operator functions induces a map*

$$S_{\mathscr{R}}^{(\nu_1^+)}(\Lambda; \tilde{E}, \hat{E}) \times S_{\mathscr{R}}^{(\nu_2^+)}(\Lambda; E, \tilde{E}) \to S_{\mathscr{R}}^{((\nu_1+\nu_2)^+)}(\Lambda; E, \hat{E}).$$

(3) *For $\alpha, \beta \in \mathbb{N}_0$ we have*

$$\partial_\lambda^\alpha \partial_{\bar{\lambda}}^\beta : S_{\mathscr{R}}^{(\nu^+)}(\Lambda; E, \tilde{E}) \to S_{\mathscr{R}}^{((\nu-m\alpha-m\beta)^+)}(\Lambda; E, \tilde{E}).$$

(4) *Let $s(\lambda) \in S_{\mathscr{R}}^{(\nu^+)}(\Lambda; E, \tilde{E})$. Then*

$$\chi(\lambda)s(\lambda) \in S^{\nu+\varepsilon}(\Lambda; E, \tilde{E}),$$

for any $\varepsilon > 0$ and any excision function $\chi \in C^\infty(\mathbb{R}^2)$ of the set where $s(\lambda)$ is undefined.

(5) *Let $s(\lambda) \in S_{\mathscr{R}}^{(\nu^+)}(\Lambda; E, \tilde{E})$ and assume that*

$$\left\| \tilde{\kappa}_{|\lambda|^{1/m}}^{-1} s(\lambda) \kappa_{|\lambda|^{1/m}} \right\|_{\mathscr{L}(E,\tilde{E})} = \mathcal{O}(|\lambda|^{\nu/m-\varepsilon})$$

as $|\lambda| \to \infty$ for some $\varepsilon > 0$. Then $s(\lambda) \equiv 0$ on Λ.

In particular, $S_{\mathscr{R}}^{(\nu_1^+)}(\Lambda; E, \tilde{E}) \cap S_{\mathscr{R}}^{(\nu_2^+)}(\Lambda; E, \tilde{E}) = \{0\}$ whenever $\nu_1 \neq \nu_2$.

Proof. (1) and (2) are obvious. For (3) note that

$$\partial_\lambda^\alpha \partial_{\bar{\lambda}}^\beta : S^{(\nu_0)}(\Lambda; E, \tilde{E}) \to S^{(\nu_0-m\alpha-m\beta)}(\Lambda; E, \tilde{E}),$$

for any ν_0. Consequently, $\partial_\lambda^\alpha \partial_{\bar{\lambda}}^\beta$ acts in the spaces

$$S^{(\nu_0)}(\Lambda; E, \tilde{E})[\lambda^{i\mu_1}, \dots, \lambda^{i\mu_N}, \log \lambda] \to S^{(\nu_0-m\alpha-m\beta)}(\Lambda; E, \tilde{E})[\lambda^{i\mu_1}, \dots, \lambda^{i\mu_N}, \log \lambda],$$
$$\mathbb{C}[\lambda^{i\mu_1}, \dots, \lambda^{i\mu_N}, \log \lambda] \to S^{(-m\alpha-m\beta)}(\Lambda)[\lambda^{i\mu_1}, \dots, \lambda^{i\mu_N}, \log \lambda],$$

with the obvious meaning of notation (the latter is a special case of the former in view of $\mathbb{C} \subset S^{(0)}(\Lambda)$). Statement (3) is an immediate consequence of these observations.

Statement (4) follows at once in view of property (a) in Definition A.4 (and using (3) to estimate higher derivatives). Note also that, for large λ, the numerator in (A.5) can be regarded as a polynomial in $\log\lambda$ of operator-valued symbols of order zero.

In the proof of (5) we may without loss of generality assume that $\nu = 0$, so $s(\lambda)$ is of the form (A.5). Since $|q(\lambda^{i\mu_1}, \dots, \lambda^{i\mu_N}, \log\lambda)| = \mathcal{O}(\log^M|\lambda|)$ as $|\lambda| \to \infty$ we see that it is sufficient to consider the case $q \equiv 1$, so $s(\lambda) = p(\lambda^{i\mu_1}, \dots, \lambda^{i\mu_N}, \log\lambda)$. For this case we will prove that if

$$\|\tilde{\kappa}^{-1}_{|\lambda|^{1/m}} s(\lambda) \kappa_{|\lambda|^{1/m}}\|_{\mathscr{L}(E,\tilde{E})} \to 0$$

as $|\lambda| \to \infty$, then $s(\lambda) \equiv 0$ on Λ. For this proof we can without loss of generality further assume that $s(\lambda)$ contains no logarithmic terms, so we have $s(\lambda) = p(\lambda^{i\mu_1}, \dots, \lambda^{i\mu_N})$. Moreover, we can assume that the numbers $\mu_1, \dots, \mu_N \in \mathbb{R}$ are independent over the rationals, for if this is not the case we can choose rationally independent numbers $\tilde{\mu}_1, \dots, \tilde{\mu}_K \in \mathbb{R}$ such that $\mu_j = \sum_{k=1}^K z_{jk}\tilde{\mu}_k$ with coefficients $z_{jk} \in \mathbb{Z}$, and so

$$\lambda^{i\mu_j} = \prod_{k=1}^K (\lambda^{i\tilde{\mu}_k})^{z_{jk}}$$

for every $j = 1, \dots, N$. Consequently, there are numbers $N_j \in \mathbb{N}$, $j = 1, \dots, K$, and a polynomial $\tilde{p} \in S^{(0)}(\Lambda; E, \tilde{E})[z_1, \dots, z_K]$ such that

$$\lambda^{i\tilde{\mu}_1 N_1} \cdots \lambda^{i\tilde{\mu}_K N_K} p(\lambda^{i\mu_1}, \dots, \lambda^{i\mu_N}) = \tilde{p}(\lambda^{i\tilde{\mu}_1}, \dots, \lambda^{i\tilde{\mu}_K}),$$

and both assertion and assumption are valid for p if and only if they hold for $\tilde{p}$. So we can indeed assume that the numbers μ_j, $j = 1, \dots, N$, are independent over the rationals.

Now let $\lambda_0 \in \Lambda$ be arbitrary with $|\lambda_0| = 1$, and consider the function $f : (0, \infty) \to \mathscr{L}(E, \tilde{E})$ defined by

$$f(\varrho) = \tilde{\kappa}^{-1}_{\varrho} p(\varrho^{im\mu_1}\lambda_0^{i\mu_1}, \dots, \varrho^{im\mu_N}\lambda_0^{i\mu_N})\kappa_{\varrho}.$$

This function is of the form

$$f(\varrho) = \sum_{|\alpha|\le M} a_\alpha (\varrho^{i\mu_1})^{\alpha_1} \cdots (\varrho^{i\mu_N})^{\alpha_N}$$

for certain $a_\alpha \in \mathscr{L}(E, \tilde{E})$, and by assumption $\|f(\varrho)\|_{\mathscr{L}(E,\tilde{E})} \to 0$ as $\varrho \to \infty$. Let $p_0(z) = \sum_{|\alpha|\le M} a_\alpha z^\alpha$ for $z = (z_1, \dots, z_N) \in \mathbb{C}^N$, and consider the curve

$$\varrho \mapsto (\varrho^{i\mu_1}, \dots, \varrho^{i\mu_N}) \in \mathbb{S}^1 \times \dots \times \mathbb{S}^1$$

on the N-torus. The image of this curve for $\varrho > \varrho_0$ is a dense subset of the N-torus, where $\varrho_0 > 0$ can be chosen arbitrarily, because the μ_j are independent over the rationals. The function f is merely the operator polynomial $p_0(z)$ restricted to that curve. Since $f(\varrho) \to 0$ as $\varrho \to \infty$, this implies that for any $\varepsilon > 0$ we have $\|p_0(z)\| < \varepsilon$ for all z in a dense subset of the N-torus. This shows that $p_0(z)$ is the zero polynomial, and so the function $f(\varrho) = 0$ for all $\varrho > 0$.

Consequently, the function $p(\lambda^{i\mu_1}, \dots, \lambda^{i\mu_N})$ vanishes along the ray through λ_0, and because λ_0 was arbitrary the proof is complete. □

Definition A.7. For $\nu \in \mathbb{R}$ define $S_{\mathfrak{R}}^{\nu^+}(\Lambda; E, \tilde{E})$ as the space of all operator-valued symbols $a(\lambda)$ that admit an asymptotic expansion

$$a(\lambda) \sim \sum_{j=0}^{\infty} \chi_j(\lambda) s_j(\lambda), \tag{A.8}$$

where $s_j(\lambda) \in S_{\mathfrak{R}}^{(\nu_j^+)}(\Lambda; E, \tilde{E})$, $\nu = \nu_0 > \nu_1 > \cdots$ and $\nu_j \to -\infty$ as $j \to \infty$, and $\chi_j(\lambda)$ is a suitable excision function of the set where $s_j(\lambda)$ is undefined.

We call $s_j(\lambda)$ the component of order ν_j^+ of $a(\lambda)$. The components are uniquely determined by the symbol $a(\lambda)$ (see Proposition A.9).

Familiar symbol classes like classical (polyhomogeneous) symbols, symbols that admit asymptotic expansions into homogeneous components of complex degrees, or log-polyhomogeneous symbols are all particular cases of the class defined in Definition A.7. In particular, the denominators q in (A.5) are equal to one in all those cases.

Of particular interest in the context of this paper are symbols $a(\lambda)$ with the property that all components $s_j(\lambda)$ have orders ν_j^+ with $\nu_j \in \mathfrak{C}$, the semigroup defined in (1.6), and phases in the set $\mathfrak{M}$ defined in (1.5).

Proposition A.9. (1) *$S_{\mathfrak{R}}^{\nu^+}(\Lambda; E, \tilde{E})$ is a vector space. For any $\varepsilon > 0$ we have the inclusion*

$$S_{\mathfrak{R}}^{\nu^+}(\Lambda; E, \tilde{E}) \subset S^{\nu+\varepsilon}(\Lambda; E, \tilde{E}).$$

(2) *Let $a(\lambda) \in S_{\mathfrak{R}}^{\nu^+}(\Lambda; E, \tilde{E})$. The components $s_j(\lambda)$ in (A.8) are uniquely determined by $a(\lambda)$.*

(3) *Let $\hat{E}$ be a third Hilbert space with group action $\hat{\kappa}_\varrho$, $\varrho > 0$. Composition of operator functions induces a map*

$$S_{\mathfrak{R}}^{\nu_1^+}(\Lambda; \tilde{E}, \hat{E}) \times S_{\mathfrak{R}}^{\nu_2^+}(\Lambda; E, \tilde{E}) \to S_{\mathfrak{R}}^{(\nu_1+\nu_2)^+}(\Lambda; E, \hat{E}).$$

The components of the composition of two symbols are obtained by formally multiplying the asymptotic expansions (A.8) of the factors.

(4) *For $\alpha, \beta \in \mathbb{N}_0$ we have*

$$\partial_\lambda^\alpha \partial_{\bar{\lambda}}^\beta : S_{\mathfrak{R}}^{\nu^+}(\Lambda; E, \tilde{E}) \to S_{\mathfrak{R}}^{(\nu-m\alpha-m\beta)^+}(\Lambda; E, \tilde{E}).$$

If $s_j(\lambda)$ are the components of $a(\lambda) \in S_{\mathfrak{R}}^{\nu^+}(\Lambda; E, \tilde{E})$, the components of $\partial_\lambda^\alpha \partial_{\bar{\lambda}}^\beta a(\lambda)$ are $\partial_\lambda^\alpha \partial_{\bar{\lambda}}^\beta s_j(\lambda)$.

(5) *Let $a_j(\lambda) \in S_{\mathfrak{R}}^{\nu_j^+}(\Lambda; E, \tilde{E})$, where $\nu_j \to -\infty$ as $j \to \infty$, and let $\bar{\nu} = \max \nu_j$. Let $a(\lambda)$ be an operator-valued symbol such that $a(\lambda) \sim \sum_{j=0}^{\infty} a_j(\lambda)$.*

Then $a(\lambda) \in S_{\mathfrak{R}}^{\bar{\nu}^+}(\Lambda; E, \tilde{E})$, and the component of $a(\lambda)$ of order M^+ is obtained by adding the components of that order of the $a_j(\lambda)$. This is a finite sum for each $M \le \bar{\nu}$ and will yield a nontrivial result for at most countably many values of M that form a sequence tending to $-\infty$.

Proof. Everything follows from Proposition A.6 and standard arguments. Because of its importance we will, however, prove (2):

To this end, assume that $0 \sim \sum_{j=0}^{\infty} \chi_j(\lambda) s_j(\lambda)$ with $s_j(\lambda) \in S_{\mathfrak{R}}^{(\nu_j^+)}(\Lambda; E, \tilde{E})$, $\nu_j > \nu_{j+1} \to -\infty$ as $j \to \infty$. We need to prove that all $s_j(\lambda)$ are zero. Because

$$\chi_0(\lambda) s_0(\lambda) \sim -\sum_{j=1}^{\infty} \chi_j(\lambda) s_j(\lambda),$$

we see that $\chi_0(\lambda)s_0(\lambda) \in S^{\nu_1+\varepsilon}(\Lambda; E, \tilde{E})$ for every $\varepsilon > 0$. Choose $\varepsilon > 0$ such that $\nu_1 + \varepsilon < \nu_0$. Then

$$\big\|\tilde{\kappa}^{-1}_{|\lambda|^{1/m}}\chi_0(\lambda)s_0(\lambda)\kappa_{|\lambda|^{1/m}}\big\|_{\mathscr{L}(E,\tilde{E})} = \mathcal{O}(|\lambda|^{(\nu_1+\varepsilon)/m})$$

as $|\lambda| \to \infty$, and by Proposition A.6(5) we obtain that $s_0(\lambda) \equiv 0$ on Λ. Consequently all $s_j(\lambda)$ are zero by induction, and (2) is proved. □

By $S^{\nu^+}_{\mathfrak{R},\mathrm{hol}}(\Lambda; E, \tilde{E})$ we denote the class of symbols $a(\lambda) \in S^{\nu^+}_{\mathfrak{R}}(\Lambda; E, \tilde{E})$ that are holomorphic in $\mathring{\Lambda}$. Let $s_j(\lambda)$ be the components of $a(\lambda) \in S^{\nu^+}_{\mathfrak{R},\mathrm{hol}}(\Lambda; E, \tilde{E})$. By Proposition A.9, $\partial_{\bar{\lambda}}s_j(\lambda)$ are the components of $\partial_{\bar{\lambda}}a(\lambda) \equiv 0$, and consequently all components $s_j(\lambda)$ are holomorphic.

In the case of holomorphic scalar symbols (or, more generally, holomorphic operator-valued symbols with trivial group actions), we can improve the description of the components as follows.

Proposition A.10. *Let $a(\lambda) \in S^{\nu^+}_{\mathfrak{R},\mathrm{hol}}(\Lambda)$, $a(\lambda) \sim \sum_{j=0}^{\infty}\chi_j(\lambda)s_j(\lambda)$ with components $s_j(\lambda)$ of order ν_j^+.*

For every $j \in \mathbb{N}_0$ there exist polynomials $p_j, q_j \in \mathbb{C}[z_1, \dots, z_{N_j+1}]$ in $N_j + 1$ variables with constant coefficients, $N_j \in \mathbb{N}_0$, and real numbers μ_{jk}, $k = 1, \dots, N_j$, such that the following holds:

(a) $|q_j(\lambda^{i\mu_{j1}}, \dots, \lambda^{i\mu_{jN_j}}, \log\lambda)| \geq c_j > 0$ *for $\lambda \in \Lambda$ with $|\lambda|$ sufficiently large;*
(b) $s_j(\lambda) = r_j(\lambda^{i\mu_{j1}}, \dots, \lambda^{i\mu_{jN_j}}, \log\lambda)\lambda^{\nu_j/m}$, *where $r_j = p_j/q_j$.*

Proof. We already know that the components $s_j(\lambda)$ are holomorphic. We just need to show that in this case the numerator polynomials p in Definition A.4 can be chosen to have constant coefficients rather than homogeneous coefficient functions. This, however, follows from Lemma A.11 below. □

Lemma A.11. *Let $f_1(\lambda), \dots, f_M(\lambda)$ be holomorphic functions on $\Lambda \setminus \{0\}$, and let p be an element of $S^{(0)}(\Lambda)[z_1, \dots, z_M]$. Assume that the function $p(f_1(\lambda), \dots, f_M(\lambda))$ is holomorphic on $\mathring{\Lambda}$, except possibly on a discrete set.*

Then there is a polynomial $p_0 \in \mathbb{C}[z_1, \dots, z_M]$ with constant coefficients such that

$$p(f_1(\lambda), \dots, f_M(\lambda)) = p_0(f_1(\lambda), \dots, f_M(\lambda))$$

as functions on $\Lambda \setminus \{0\}$.

Proof. Since all singularities are removable, we know that $p(f_1(\lambda), \dots, f_M(\lambda))$ is holomorphic everywhere on $\mathring{\Lambda}$. We have

$$p(f_1(\lambda), \dots, f_M(\lambda)) = \sum_{|\alpha|\leq D} a_\alpha(\lambda/|\lambda|)f_1(\lambda)^{\alpha_1}\cdots f_M(\lambda)^{\alpha_M}.$$

Let $\lambda_0 \in \mathring{\Lambda}$. Define

$$p_0(z_1, \dots, z_M) = \sum_{|\alpha|\leq D} a_\alpha(\lambda_0/|\lambda_0|)z_1^{\alpha_1}\cdots z_M^{\alpha_M}.$$

Then clearly

$$p(f_1(\lambda), \dots, f_M(\lambda)) = p_0(f_1(\lambda), \dots, f_M(\lambda))$$

on the ray through λ_0. By uniqueness of analytic continuation this equality necessarily holds everywhere on $\mathring{\Lambda}$, and by continuity then also on $\Lambda \setminus \{0\}$. □

References

[Brüning and Seeley 1987] J. Brüning and R. Seeley, "The resolvent expansion for second order regular singular operators", *J. Funct. Anal.* **73**:2 (1987), 369–429. MR 88g:35151

[Brüning and Seeley 1991] J. Brüning and R. Seeley, "The expansion of the resolvent near a singular stratum of conical type", *J. Funct. Anal.* **95**:2 (1991), 255–290. MR 93g:58146

[Falomir et al. 2002] H. Falomir, P. A. G. Pisani, and A. Wipf, "Pole structure of the Hamiltonian ζ-function for a singular potential", *J. Phys. A* **35**:26 (2002), 5427–5444. MR 2003g:81054 Zbl 1066.81008

[Falomir et al. 2003] H. Falomir, M. A. Muschietti, P. A. G. Pisani, and R. Seeley, "Unusual poles of the ζ-functions for some regular singular differential operators", *J. Phys. A* **36**:39 (2003), 9991–10010. MR 2004k:58049 Zbl 1043.58017

[Falomir et al. 2004] H. Falomir, M. A. Muschietti, and P. A. G. Pisani, "On the resolvent and spectral functions of a second order differential operator with a regular singularity", *J. Math. Phys.* **45**:12 (2004), 4560–4577. MR 2005m:81093 Zbl 1064.81038

[Gil et al. 2006] J. B. Gil, T. Krainer, and G. A. Mendoza, "Resolvents of elliptic cone operators", *J. Funct. Anal.* **241**:1 (2006), 1–55. MR 2007j:58032 Zbl 1109.58023

[Gil et al. 2007a] J. B. Gil, T. Krainer, and G. A. Mendoza, "Geometry and spectra of closed extensions of elliptic cone operators", *Canad. J. Math.* **59**:4 (2007), 742–794. MR 2008f:58030 Zbl 1126.58014

[Gil et al. 2007b] J. B. Gil, T. Krainer, and G. A. Mendoza, "On rays of minimal growth for elliptic cone operators", pp. 33–50 in *Modern trends in pseudo-differential operators*, edited by J. Toft et al., Oper. Theory Adv. Appl. **172**, Birkhäuser, Basel, 2007. MR 2008c:58022 Zbl 1126.58015

[Gil et al. 2010] J. B. Gil, T. Krainer, and G. A. Mendoza, "Trace expansions for elliptic cone operators with stationary domains", *Trans. Amer. Math. Soc.* **362**:12 (2010), 6495–6522. MR 2678984 Zbl 05833229

[Kirsten et al. 2006] K. Kirsten, P. Loya, and J. Park, "The very unusual properties of the resolvent, heat kernel, and zeta function for the operator $-d^2/dr^2 - 1/(4r^2)$", *J. Math. Phys.* **47**:4 (2006), 043506, 27. MR 2007c:58050 Zbl 1111.58025

[Kirsten et al. 2008a] K. Kirsten, P. Loya, and J. Park, "Exotic expansions and pathological properties of ζ-functions on conic manifolds", *J. Geom. Anal.* **18**:3 (2008), 835–888. MR 2009j:58051 Zbl 1154.58016

[Kirsten et al. 2008b] K. Kirsten, P. Loya, and J. Park, "Functional determinants for general self-adjoint extensions of Laplace-type operators resulting from the generalized cone", *Manuscripta Math.* **125**:1 (2008), 95–126. MR 2009h:58073 Zbl 1142.58017

[Lesch 1997] M. Lesch, *Operators of Fuchs type, conical singularities, and asymptotic methods*, Teubner-Texte zur Math. **136**, B. G. Teubner, Stuttgart, 1997.

[Loya et al. 2007] P. Loya, P. McDonald, and J. Park, "Zeta regularized determinants for conic manifolds", *J. Funct. Anal.* **242**:1 (2007), 195–229. MR 2007g:58036 Zbl 1146.58024

[Melrose 1993] R. B. Melrose, *The Atiyah–Patodi–Singer index theorem*, Research Notes in Mathematics **4**, A K Peters, Wellesley, MA, 1993. MR 96g:58180 Zbl 0796.58050

[Schulze 1991] B.-W. Schulze, *Pseudo-differential operators on manifolds with singularities*, Studies in Mathematics and its Applications **24**, North-Holland, Amsterdam, 1991. MR 93b:47109 Zbl 0747.58003

Received 30 Jun 2009. Revised 10 Mar 2010. Accepted 11 Jun 2010.

JUAN B. GIL: jgil@psu.edu
Department of Mathematics, Penn State Altoona, 3000 Ivyside Park, Altoona, PA 16601, United States

THOMAS KRAINER: krainer@psu.edu
Department of Mathematics, Penn State Altoona, 3000 Ivyside Park, Altoona, PA 16601, United States

GERARDO A. MENDOZA: gmendoza@temple.edu
Department of Mathematics, Temple University, Philadelphia, PA 19122, United States

ANALYSIS AND PDE
Vol. 4, No. 1, 2011

THREE-TERM COMMUTATOR ESTIMATES AND THE REGULARITY OF $\frac{1}{2}$-HARMONIC MAPS INTO SPHERES

FRANCESCA DA LIO AND TRISTAN RIVIÈRE

We prove the regularity of weak $\frac{1}{2}$-harmonic maps from the real line into a sphere. A key step is the formulation of the $\frac{1}{2}$-harmonic map equation in the form of a nonlocal linear Schrödinger type equation with *three-term commutators* on the right-hand side. We then establish a sharp estimate for these three-term commutators.

1. Introduction

Starting in the early 1950s, the analysis of critical points of conformal invariant lagrangians has attracted much interest, due to their importance in physics and geometry. (See the introduction of [Rivière 2008] for an overview.) We recall some classical examples of such operators and their associated variational problems:

The most elementary example of a two-dimensional conformal invariant lagrangian is the Dirichlet energy

$$E(u)=\int_D |\nabla u(x,y)|^2\,dx\,dy, \tag{1}$$

where $D\subseteq R^2$ is an open set and ∇u is the gradient of $u : D\to\mathbb{R}$. We recall that a map $\phi:\mathbb{C}\to\mathbb{C}$ is conformal if it satisfies

$$\left|\frac{\partial\phi}{\partial x}\right|=\left|\frac{\partial\phi}{\partial y}\right|,\quad \left\langle\frac{\partial\phi}{\partial x},\frac{\partial\phi}{\partial y}\right\rangle=0,\quad \det\nabla\phi\geq 0,\quad \nabla\phi\neq 0, \tag{2}$$

where $\langle\cdot,\cdot\rangle$ denotes the standard Euclidean inner product in $\mathbb{R}^n$.

MSC2000: 58E20, 35B65, 35J20, 35J60, 35S99.
Keywords: harmonic map, nonlinear elliptic PDE, regularity of solutions, commutator estimates.

For every $u \in W^{1,2}(D,\mathbb{R})$ and every conformal map ϕ with $\deg\phi = 1$, we have

$$E(u) = E(u\circ\phi) = \int_{\phi^{-1}(D)} \big|(\nabla\circ\phi)u(x,y)\big|^2\,dx\,dy.$$

The critical points of this functional are the harmonic functions satisfying

$$\Delta u = 0 \quad \text{in } D. \tag{3}$$

We can extend E to maps taking values in $\mathbb{R}^m$ by setting

$$E(u) = \int_D \big|\nabla u(x,y)\big|^2\,dx\,dy = \int_D \sum_{i=1}^m \big|\nabla u_i(x,y)\big|^2\,dx\,dy, \tag{4}$$

where the u_i are the components of u. The lagrangian (4) is still conformally invariant and each component of its critical points satisfies (3).

We can define the lagrangian (4) also on the set of maps taking values in a compact submanifold $\mathcal{N} \subseteq \mathbb{R}^m$ without boundary. We have

$$-\Delta u \perp T_u\mathcal{N},$$

where $T_u\mathcal{N}$ is the tangent plane to $\mathcal{N}$ at the point $u \in \mathcal{N}$; equivalently, we can write

$$-\Delta u = A(u)(\nabla u,\nabla u) := A(u)(\partial_x u,\partial_x u) + A(u)(\partial_y u,\partial_y u), \tag{5}$$

where $A(u)$ is the second fundamental form at a point $u\in\mathcal{N}$; see [Hélein 2002], for instance. Equation (5) is called the *harmonic map equation* into $\mathcal{N}$.

When $\mathcal{N}$ is an oriented hypersurface of $\mathbb{R}^m$ the harmonic map equation reads as

$$-\Delta u = n\langle\nabla n,\nabla u\rangle, \tag{6}$$

where n denotes the composition of u with the unit normal vector field ν to $\mathcal{N}$.

All these examples belong to the class of conformal invariant coercive lagrangians whose corresponding Euler–Lagrange equation is of the form

$$-\Delta u = f(u,\nabla u), \tag{7}$$

where $f : \mathbb{R}^2 \times (\mathbb{R}^m \otimes \mathbb{R}^2) \to \mathbb{R}^m$ is a continuous function satisfying

$$C^{-1}|p|^2 \le |f(\xi,p)| \le C|p|^2 \quad \text{for all } \xi, p,$$

for some positive constant C. One of the main issues concerning equations of the form (7) is the regularity of solutions $u \in W^{1,2}(D,\mathcal{N})$. We observe that (7) is critical in dimension $n = 2$ for the $W^{1,2}$-norm. Indeed, if we plug into the nonlinearity $f(u,\nabla u)$ the information that $u \in W^{1,2}(D,\mathcal{N})$, we obtain $\Delta u \in L^1(D)$, so ∇u belongs to $L^{2,\infty}_{\text{loc}}(D)$, the weak L^2 space [Stein 1970], which has the same homogeneity of L^2. Hence we are back in some sense to the initial situation. This shows that the equation is critical.

In general, $W^{1,2}$ solutions to (7) are not smooth in dimensions greater than 2; for a counterexample, see [Rivière 2007]. For an exposition of regularity and compactness results for such equations, we refer the reader to [Giaquinta 1983].

We next recall the approach introduced by F. Hélein [2002] to prove the regularity of harmonic maps from a domain D of $\mathbb{R}^2$ into the unit sphere S^{m-1} of $\mathbb{R}^m$. In this case the Euler–Lagrange equation is

$$-\Delta u = u|\nabla u|^2. \tag{8}$$

Shatah [1988] observed in that $u \in W^{1,2}(D, S^{m-1})$ is a solution of (8) if and only if the conservation law

$$\operatorname{div}(u_i \nabla u_j - u_j \nabla u_i) = 0 \quad \text{for all } i, j \in \{1, \dots, m\} \tag{9}$$

holds. Using (9) and the fact that $\sum_{j=1}^m u_j \nabla u_j = 0$ when $|u| \equiv 1$, Hélein rewrote (8) in the form

$$-\Delta u = \nabla^{\perp} B \cdot \nabla u, \tag{10}$$

where $\nabla^{\perp} B = (\nabla^{\perp} B_{ij})$ with $\nabla^{\perp} B_{ij} = u_i \nabla u_j - u_j \nabla u_i$ (for every vector field $v : \mathbb{R}^2 \to \mathbb{R}^n$, $\nabla^{\perp} v$ denotes the $\pi/2$ rotation of the gradient ∇v, namely $\nabla^{\perp} v = (-\partial_y v, \partial_x v)$).

The right-hand side of (10) can be written as a sum of Jacobians:

$$\nabla^{\perp} B_{ij} \nabla u_j = \partial_x u_j \partial_y B_{ij} - \partial_y u_j \partial_x B_{ij}.$$

This particular structure allows us to apply to (8) the following result:

Theorem 1.1 [Wente 1969]. *Let D be a smooth bounded domain of $\mathbb{R}^2$. Let a and b be measurable functions in D whose gradients are in $L^2(D)$. Then there exists a unique solution $\varphi \in W^{1,2}(D)$ to*

$$\begin{cases} -\Delta \varphi = \dfrac{\partial a}{\partial x}\dfrac{\partial b}{\partial y} - \dfrac{\partial a}{\partial y}\dfrac{\partial b}{\partial x} & \text{in } D, \\ \varphi = 0 & \text{on } \partial D. \end{cases} \tag{11}$$

There exists a constant $C > 0$ independent of a and b such that

$$\|\varphi\|_\infty + \|\nabla \varphi\|_{L^2} \le C \|\nabla a\|_{L^2} \|\nabla b\|_{L^2}.$$

In particular φ is continuous in D.

Theorem 1.1 applied to (10) leads, via a standard localization argument in elliptic PDEs, to the estimate

$$\|\nabla u\|_{L^2(B_r(x_0))} \le C\, \|\nabla B\|_{L^2(B_r(x_0))}\, \|\nabla u\|_{L^2(B_r(x_0))} + C r\, \|\nabla u\|_{L^2(\partial B_r(x_0))} \tag{12}$$

for every $x_0 \in D$ and $r > 0$ such that $B_r(x_0) \subset D$. Assume we are considering radii $r < r_0$ such that $\max_{x_0 \in D} C\, \|\nabla B\|_{L^2(B_r(x_0))} < \frac{1}{2}$. Then (12) implies a Morrey estimate

$$\sup_{x_0, r>0} r^{-\beta} \int_{B_r(x_0)} |\nabla u|^2 \, dx < \infty \tag{13}$$

for some $\beta > 0$, which itself implies the Hölder continuity of u by a standard embedding result [Giaquinta 1983]. Finally a bootstrap argument implies that u is in fact C^∞, and even analytic: see [Hildebrandt and Widman 1975; Morrey 1966].

In the present work we are interested in one-dimensional quadratic lagrangians invariant under the trace of conformal maps that keep invariant the half-space $\mathbb{R}^2_+$: the Möbius group.

A typical example, which we will call the *L-energy* (L for "line"), is the lagrangian

$$L(u) = \int_{\mathbb{R}} \left|\Delta^{1/4} u(x)\right|^2 dx, \tag{14}$$

where u is a map from $\mathbb{R}$ into a k-dimensional submanifold $\mathcal{N}$ of $\mathbb{R}^m$ which is at least C^2, compact and without boundary. In fact $L(u)$ coincides with $\|u\|^2_{\dot{H}^{1/2}(\mathbb{R})}$ (for the definition of the seminorm $\|\cdot\|_{\dot{H}^{1/2}(\mathbb{R})}$ see Section 2). A more tractable way to look at this norm is given by the identity

$$\int_{\mathbb{R}} |\Delta^{1/4} u(x)|^2 \, dx = \inf\left\{\int_{\mathbb{R}^2_+} |\nabla \tilde{u}|^2 \, dx : \tilde{u} \in W^{1,2}(\mathbb{R}^2, \mathbb{R}^m) \text{ with trace } \tilde{u} = u\right\}.$$

The Lagrangian L extends to map u in the function space

$$\dot{H}^{1/2}(\mathbb{R}, \mathcal{N}) = \{u \in \dot{H}^{1/2}(\mathbb{R}, \mathbb{R}^m) : u(x) \in \mathcal{N} \text{ a.e.}\}.$$

The operator $\Delta^{1/4}$ on $\mathbb{R}$ is defined by means of the Fourier transform (denoted by $\hat{\ }$) as

$$\widehat{\Delta^{1/4} u} = |\xi|^{1/2} \hat{u}.$$

Denote by $\pi_{\mathcal{N}}$ the orthogonal projection onto $\mathcal{N}$, which happens to be a C^l map in a sufficiently small neighborhood of $\mathcal{N}$ if $\mathcal{N}$ is assumed to be C^{l+1}. We now introduce the notion of $\frac{1}{2}$-harmonic map into a manifold.

Definition 1.2. A map $u \in \dot{H}^{1/2}(\mathbb{R}, \mathcal{N})$ is called a *weak $\frac{1}{2}$-harmonic* map into $\mathcal{N}$ if

$$\frac{d}{dt} L(\pi_{\mathcal{N}}(u + t\phi))_{|t=0} = 0 \quad \text{for any } \phi \in \dot{H}^{1/2}(\mathbb{R}, \mathbb{R}^m) \cap L^\infty(\mathbb{R}, \mathbb{R}^m).$$

In short, a weak $\frac{1}{2}$-harmonic map is a *critical point of L in $\dot{H}^{1/2}(\mathbb{R}, \mathcal{N})$ for perturbations in the target.*

We encounter $\frac{1}{2}$-harmonic maps into the circle S^1, for instance, in the asymptotic of equations in phase-field theory for fractional reaction-diffusion such as

$$\epsilon^2 \, \Delta^{1/2} u + u(1 - |u|^2) = 0$$

where u is a complex-valued wavefunction.

In this paper we consider the case $\mathcal{N} = S^{m-1}$. We first write (deferring the proof till Theorem 5.2) the Euler–Lagrange equation associated to L in $\dot{H}^{1/2}(\mathbb{R}, S^{m-1})$:

Proposition 1.3. *Let T be the operator defined by*

$$T(Q, u) := \Delta^{1/4}(Q \Delta^{1/4} u) - Q \Delta^{1/2} u + \Delta^{1/4} u \Delta^{1/4} Q, \tag{15}$$

for $Q \in \dot{H}^{1/2}(\mathbb{R}^n, \mathcal{M}_{l \times m}(\mathbb{R}))$ $l \geq 1$ and $u \in \dot{H}^{1/2}(\mathbb{R}^n, \mathbb{R}^m)$. (Here n and l are natural numbers and $\mathcal{M}_{l \times m}(\mathbb{R})$ denotes the space of $l \times m$ real matrices.)

A map u in $\dot{H}^{1/2}(\mathbb{R}, S^{m-1})$ is a weak $\frac{1}{2}$-harmonic map if and only if it satisfies the Euler–Lagrange equation

$$\Delta^{1/4}(u \wedge \Delta^{1/4} u) = T(u\wedge, u). \tag{16}$$

The Euler–Lagrange equation (16) will often be completed by the following "structure equation", which is a consequence of the fact that $u \in S^{m-1}$ almost everywhere:

Proposition 1.4. *Let S be the operator given by*

$$S(Q,u) := \Delta^{1/4}(Q\Delta^{1/4}u) - \mathcal{R}(Q\nabla u) + \mathcal{R}(\Delta^{1/4}Q\mathcal{R}\Delta^{1/4}u) \tag{17}$$

for $Q \in \dot{H}^{1/2}(\mathbb{R}^n, \mathcal{M}_{l\times m}(\mathbb{R}))$ and $u \in \dot{H}^{1/2}(\mathbb{R}^n, \mathbb{R}^m)$, where n and l are natural numbers and $\mathcal{R}$ is the Fourier multiplier of symbol $m(\xi) = i\xi/|\xi|$.

All maps in $\dot{H}^{1/2}(\mathbb{R}, S^{m-1})$ satisfy

$$\Delta^{1/4}(u\cdot\Delta^{1/4}u) = S(u\cdot, u) - \mathcal{R}(\Delta^{1/4}u\cdot\mathcal{R}\Delta^{1/4}u). \tag{18}$$

We will first show that $\dot{H}^{1/2}$ solutions to the $\frac{1}{2}$-harmonic map equation (16) are Hölder continuous. This regularity result will be a direct consequence of a Morrey-type estimate we will establish:

$$\sup_{\substack{x_0\in\mathbb{R}\\ r>0}} r^{-\beta}\int_{B_r(x_0)} \left|\Delta^{1/4}u\right|^2 dx < \infty. \tag{19}$$

For this purpose, in the spirit of what we have just presented regarding Hélein's proof of the regularity of harmonic maps from a two-dimensional domain into a round sphere, we will take advantage of a "regularity gain" in the right-hand sides of (16) and (18), where the different terms $T(u\wedge, u)$, $S(u\cdot, u)$ and $\mathcal{R}(\Delta^{1/4}u\cdot\mathcal{R}\Delta^{1/4}u)$ play more or less the role played by $\nabla^{\perp}B\cdot\nabla u$ in (10). More precisely, we will establish, for every $u \in \dot{H}^{1/2}(\mathbb{R}, \mathbb{R}^m)$ and $Q \in H^{1/2}(\mathbb{R}, \mathcal{M}_{l\times m}(\mathbb{R}))$, the estimates

$$\|T(Q,u)\|_{\dot{H}^{-1/2}(\mathbb{R})} \le C\,\|Q\|_{\dot{H}^{1/2}(\mathbb{R})}\,\|u\|_{\dot{H}^{1/2}(\mathbb{R})}, \tag{20}$$

$$\|S(Q,u)\|_{\dot{H}^{-1/2}(\mathbb{R})} \le C\,\|Q\|_{\dot{H}^{1/2}(\mathbb{R})}\,\|u\|_{\dot{H}^{1/2}(\mathbb{R})}, \tag{21}$$

$$\left\|\mathcal{R}(\Delta^{1/4}u\cdot\mathcal{R}\Delta^{1/4}u)\right\|_{\dot{H}^{-1/2}(\mathbb{R})} \le C\,\|u\|^2_{\dot{H}^{1/2}(\mathbb{R})}. \tag{22}$$

The phrase "regularity gain" is illustrated by the fact that, for such u and Q, the individual terms in T and S (such as $\Delta^{1/4}(Q\Delta^{1/4}u)$ or $Q\Delta^{1/2}u$) are not in $\dot{H}^{-1/2}$, but the special linear combinations of them constituting T and S do lie in $\dot{H}^{-1/2}$. In a similar way, *in two dimensions*, $J(a,b) := \frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial a}{\partial y}\frac{\partial b}{\partial x}$ satisfies

$$\|J(a,b)\|_{\dot{H}^{-1}} \le C\,\|a\|_{\dot{H}^1}\,\|b\|_{\dot{H}^1} \tag{23}$$

as a direct consequence of Wente's result (Theorem 1.1), whereas the individual terms $\frac{\partial a}{\partial x}\frac{\partial b}{\partial y}$ and $\frac{\partial a}{\partial y}\frac{\partial b}{\partial x}$ are not in $\dot{H}^{-1}$.

The estimates (20) and (21) are in fact consequences of the *three-term commutator estimates* in the next two theorems, which are valid in arbitrary dimension and which are two of the main results of this paper. We recall that *BMO* denotes the space of *bounded mean oscillations* functions of John and Nirenberg (see for instance [Grafakos 2009])

$$\|u\|_{BMO(\mathbb{R}^n)} = \sup_{\substack{x_0\in\mathbb{R}^n\\ r>0}} \frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} \left|u(x) - \frac{1}{|B_r(x_0)|}\int u(y)\,dy\right| dx.$$

Theorem 1.5. *For $n \in \mathbb{N}^*$, $u \in BMO(\mathbb{R}^n)$, and $Q \in \dot{H}^{1/2}(\mathbb{R}^n, \mathcal{M}_{l\times m}(\mathbb{R}))$, set*

$$T(Q,u) := \Delta^{1/4}(Q\Delta^{1/4}u) - Q\Delta^{1/2}u + \Delta^{1/4}u\Delta^{1/4}Q,$$

Then $T(Q,u) \in H^{-1/2}(\mathbb{R}^n)$ and there exists $C > 0$, depending only on n, such that

$$\|T(Q,u)\|_{H^{-1/2}(\mathbb{R}^n)} \le C\ \|Q\|_{\dot{H}^{1/2}(\mathbb{R}^n)}\|u\|_{BMO(\mathbb{R}^n)}\,. \tag{24}$$

Theorem 1.6. *For $n \in \mathbb{N}^*$, $u \in BMO(\mathbb{R}^n)$, and $Q \in \dot{H}^{1/2}(\mathbb{R}^n, \mathcal{M}_{l\times m}(\mathbb{R}))$, set*

$$S(Q,u) := \Delta^{1/4}[Q\Delta^{1/4}u] - \mathcal{R}(Q\nabla u) + \mathcal{R}(\Delta^{1/4}Q\mathcal{R}\Delta^{1/4}u),$$

where $\mathcal{R}$ is the Fourier multiplier of symbol $m(\xi) = i\xi/|\xi|$. Then $S(Q,u) \in H^{-1/2}(\mathbb{R}^n)$ and there exists C depending only on n such that

$$\|S(Q,u)\|_{H^{-1/2}(\mathbb{R}^n)} \le C\ \|Q\|_{\dot{H}^{1/2}(\mathbb{R}^n)}\|u\|_{BMO(\mathbb{R}^n)}\,. \tag{25}$$

The estimates (20) and (21) follow from Theorems 1.5 and 1.6 as a consequence of the embedding $\dot{H}^{1/2}(\mathbb{R}) \hookrightarrow BMO(\mathbb{R})$.

The parallel between the structures T and S for $H^{1/2}$ in one hand and the Jacobian structure J for H^1 in the other can be pushed further as follows. As a consequence of a result of R. Coifman, P. L. Lions, Y. Meyer and S. Semmes [Coifman et al. 1993], the Wente estimate (23) can be deduced from a more general one. Set, for any $i, j \in \{1,\dots,n\}$ and $a, b \in \dot{H}^1(\mathbb{R}^n)$,

$$J_{ij}(a,b) := \frac{\partial a}{\partial x_i}\frac{\partial b}{\partial x_j} - \frac{\partial a}{\partial x_j}\frac{\partial b}{\partial x_i},$$

and form the matrix $J(a,b) := (J_{ij}(a,b))_{ij=1,\dots,n}$. The main result in [Coifman et al. 1993] implies

$$\|J(a,b)\|_{\dot{H}^{-1}(\mathbb{R}^n)} \le C\ \|a\|_{\dot{H}^1(\mathbb{R}^n)}\ \|b\|_{BMO(\mathbb{R}^n)}\,, \tag{26}$$

which is reminiscent of (24) and (25). Recall also that (26) is a consequence of a commutator estimate by Coifman, R. Rochberg and G. Weiss [Coifman et al. 1976].

Theorems 1.5 and 1.6 will follow respectively Theorems 1.7 and (27) below, which are their "dual versions". Recall first that $\mathcal{H}^1(\mathbb{R}^n)$ denotes the Hardy space of L^1 functions f on $\mathbb{R}^n$satisfying

$$\int_{\mathbb{R}^n}\sup_{t\in\mathbb{R}}|\phi_t * f|(x)\,dx < \infty,$$

where $\phi_t(x) := t^{-n}\ \phi(t^{-1}x)$ and where ϕ is some function in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ satisfying $\int_{\mathbb{R}^n}\phi(x)\,dx = 1$. Recall the famous result by Fefferman saying that the dual space to $\mathcal{H}^1$ is BMO.

Theorem 1.7. *For $u, Q \in \dot{H}^{1/2}(\mathbb{R}^n)$, set*

$$R(Q,u) = \Delta^{1/4}(Q\Delta^{1/4}u) - \Delta^{1/2}(Qu) + \Delta^{1/4}((\Delta^{1/4}Q)u).$$

Then $R(Q,u) \in \mathcal{H}^1(\mathbb{R}^n)$ and

$$\|R(Q,u)\|_{\mathcal{H}^1(\mathbb{R}^n)} \le C\|Q\|_{\dot{H}^{1/2}(\mathbb{R}^n)}\|u\|_{\dot{H}^{1/2}(\mathbb{R}^n)}. \tag{27}$$

Theorem 1.8. *For $u, Q \in \dot{H}^{1/2}(\mathbb{R}^n)$ and $u \in BMO(\mathbb{R}^n)$, set*

$$\tilde{S}(Q,u) = \Delta^{1/4}(Q\Delta^{1/4}u) - \nabla(Q\mathcal{R}u) + \mathcal{R}\Delta^{1/4}(\Delta^{1/4}Q\mathcal{R}u),$$

where $\mathcal{R}$ is the Fourier multiplier of symbol $m(\xi) = i\xi/|\xi|$. Then $\tilde{S}(Q,u) \in \mathcal{H}^1$ and

$$\|\tilde{S}(Q,u)\|_{\mathcal{H}^1} \leq C\|Q\|_{\dot{H}^{1/2}(\mathbb{R}^n)}\|u\|_{\dot{H}^{1/2}(\mathbb{R}^n)}. \tag{28}$$

We say a few words on the proof of the estimates (27) and (28). The compensations of the three different terms in $R(Q,u)$ will be clear from the Littlewood–Paley decomposition of the different products that we present in Section 3. As usual, we denote by $\Pi_1(f,g)$ the high-low contribution (respectively from f and g), by $\Pi_2(f,g)$ the low-high contribution, and by $\Pi_3(f,g)$ the high-high contribution. We also use the notation $\Pi_k(\Delta^\alpha(fg))$, for $k = 1,2,3$ and $\alpha = \frac{1}{4}, \frac{1}{2}$, as an alternative for $\Delta^\alpha(\Pi_k(f,g))$.

We will use the following decompositions for the operators $\Pi_k(R(Q,u))$:

$$\begin{aligned}
\Pi_1(R(Q,u)) &= \underbrace{\Pi_1(\Delta^{1/4}(Q\Delta^{1/4}u))} + \underbrace{\Pi_1(-\Delta^{1/2}(Qu) + \Delta^{1/4}((\Delta^{1/4}Q)u))},\\
\Pi_2(R(Q,u)) &= \underbrace{\Pi_2(\Delta^{1/4}(Q\Delta^{1/4}u) - \Delta^{1/2}(Qu))} + \underbrace{\Pi_2(\Delta^{1/4}((\Delta^{1/4}Q)u))},\\
\Pi_3(R(Q,u)) &= \underbrace{\Pi_3(\Delta^{1/4}(Q\Delta^{1/4}u))} - \underbrace{\Pi_3(\Delta^{1/2}(Qu))} + \underbrace{\Pi_3(\Delta^{1/4}((\Delta^{1/4}Q)u))}.
\end{aligned}$$

Finally, injecting the Morrey estimate (19) in equations (16) and (18), a classical elliptic-type bootstrap argument leads to the following result (see [Lio and Riviere 2010] for details).

Theorem 1.9. *Any weak $\frac{1}{2}$-harmonic map in $\dot{H}^{1/2}(\mathbb{R}, S^{m-1})$ belongs to $H^s_{\mathrm{loc}}(\mathbb{R}, S^{m-1})$ for every $s \in \mathbb{R}$, and is therefore C^∞.*

The paper is organized as follows. After a section with preliminary definitions and notation, we prove in Section 3 we prove the three-term commutator estimates (Theorems 1.5 and 1.6).

In Section 4 we prove some L-energy decrease control estimates on dyadic annuli for general solutions to certain linear nonlocal systems of equations, which include (16) and (18).

In Section 5 we derive the Euler–Lagrange equation (16) associated to the lagrangian (14); this is Proposition 1.3. We then prove Proposition 1.4. We finally use the results of the previous section to deduce the Morrey-type estimate (19) for $\frac{1}{2}$-harmonic maps into a sphere.

In the Appendix we study geometric localization properties of the $\dot{H}^{1/2}$-norm on the real line for $\dot{H}^{1/2}$-functions in general and we prove some preliminary results.

2. Definitions and notation

For $n \geq 1$, let $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ denote respectively the spaces of Schwartz functions and tempered distributions. Given a function v we will denote either by $\hat{v}$ or by $\mathcal{F}[v]$ the Fourier Transform of v:

$$\hat{v}(\xi) = \mathcal{F}[v](\xi) = \int_{\mathbb{R}^n} v(x)e^{-i\langle\xi,x\rangle}\,dx.$$

Throughout the paper we use the convention that x, y denote space variables and ξ, ζ phase variables.

We recall the definition of fractional Sobolev spaces. For some of the material on the next page, see [Tartar 2007], for instance.

Definition 2.1. For s real,

$$H^s(\mathbb{R}^n) = \begin{cases} \{v \in L^2(\mathbb{R}^n) : |\xi|^s \mathcal{F}[v] \in L^2(\mathbb{R}^n)\} & \text{if } s \geq 0, \\ \{v \in \mathcal{S}'(\mathbb{R}^n) : (1+|\xi|^2)^{s/2}\mathcal{F}[v] \in L^2(\mathbb{R}^n)\} & \text{if } s < 0. \end{cases}$$

It is known that $H^{-s}(\mathbb{R}^n)$ is the dual of $H^s(\mathbb{R}^n)$.

For $0 < s < 1$, we mention an alternative characterization of $H^s(\mathbb{R}^n)$, which does not use the Fourier transform.

Lemma 2.2. *For* $0 < s < 1$, *the condition* $u \in H^s(\mathbb{R}^n)$ *is equivalent to* $u \in L^2(\mathbb{R}^n)$ *and*

$$\left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x)-u(y))^2}{|x-y|^{n+2s}}\, dx\, dy \right)^{1/2} < \infty.$$

For $s > 0$ we set

$$\|u\|_{H^s(\mathbb{R}^n)} = \|u\|_{L^2(\mathbb{R}^n)} + \||\xi|^s \mathcal{F}[v]\|_{L^2(\mathbb{R}^n)} \quad \text{and} \quad \|u\|_{\dot{H}^s(\mathbb{R}^n)} = \||\xi|^s \mathcal{F}[v]\|_{L^2(\mathbb{R}^n)}.$$

For an open set $\Omega \subset \mathbb{R}^n$, $H^s(\Omega)$ is the space of the restrictions of functions from $H^s(\mathbb{R}^n)$, and

$$\|u\|_{\dot{H}^s(\Omega)} = \inf \left\{ \|U\|_{\dot{H}^s(\mathbb{R}^n)} : U = u \text{ on } \Omega \right\}.$$

If $0 < s < 1$, then $f \in H^s(\Omega)$ if and only if $f \in L^2(\Omega)$ and

$$\left(\int_{\Omega} \int_{\Omega} \frac{(u(x)-u(y))^2}{|x-y|^{n+2s}}\, dx\, dy \right)^{1/2} < \infty.$$

Moreover,

$$\|u\|_{\dot{H}^s(\Omega)} \simeq \left(\int_{\Omega} \int_{\Omega} \frac{(u(x)-u(y))^2}{|x-y|^{n+2s}}\, dx\, dy \right)^{1/2} < \infty.$$

Finally, for a submanifold $\mathcal{N}$ of $\mathbb{R}^m$, we can define

$$H^s(\mathbb{R}, \mathcal{N}) = \{u \in H^s(\mathbb{R}, \mathbb{R}^m) : u(x) \in \mathcal{N} \text{ a.e.}\}.$$

We introduce the so-called Littlewood–Paley or dyadic decomposition of unity. Let $\phi(\xi)$ be a radial Schwartz function supported on $\{\xi : |\xi| \leq 2\}$ and equal to 1 on $\{\xi : |\xi| \leq 1\}$. Let $\psi(\xi)$ be the function $\psi(\xi) := \phi(\xi) - \phi(2\xi)$; thus ψ is a bump function supported on the annulus $\{\xi : \frac{1}{2} \leq |\xi| \leq 2\}$.

We put $\psi_0 = \phi$, $\psi_j(\xi) = \psi(2^{-j}\xi)$ for $j \neq 0$. The functions ψ_j, for $j \in \mathbb{Z}$, are supported on $\{\xi : 2^{j-1} \leq |\xi| \leq 2^{j+1}\}$. Moreover $\sum_{j\in\mathbb{Z}} \psi_j(x) = 1$.

We then set $\phi_j(\xi) := \sum_{k=-\infty}^{j} \psi_k(\xi)$. The function ϕ_j is supported on $\{\xi,\ |\xi| \leq 2^{j+1}\}$.

We recall the definition of the homogeneous Besov spaces $\dot{B}^s_{p,q}(\mathbb{R}^n)$ and homogeneous Triebel–Lizorkin spaces $\dot{F}^s_{pq}(\mathbb{R}^n)$ in terms of the dyadic decomposition.

Definition 2.3. Let $s \in \mathbb{R}$, $0 < p, q \leq \infty$. For $f \in \mathcal{S}'(\mathbb{R}^n)$, set

$$\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)} = \begin{cases} \left(\sum_{j=-\infty}^{\infty} 2^{jsq} \|\mathcal{F}^{-1}[\psi_j \mathcal{F}[f]]\|^q_{L^p(\mathbb{R}^n)} \right)^{1/q} & \text{if } q < \infty, \\ \sup_{j\in\mathbb{Z}} 2^{js} \|\mathcal{F}^{-1}[\psi_j \mathcal{F}[f]]\|_{L^p(\mathbb{R}^n)} & \text{if } q = \infty. \end{cases} \tag{29}$$

The *homogeneous Besov space with indices* s, p, q, denoted by $\dot{B}^s_{p,q}(\mathbb{R}^n)$, is the space of all tempered distributions f for which $\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)}$ is finite.

Let $s \in \mathbb{R}$, $0 < p, q < \infty$. Again for $f \in \mathcal{S}'(\mathbb{R}^n)$, set

$$\|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} = \Big\| \Big(\sum_{j=-\infty}^{\infty} 2^{jsq} \big|\mathcal{F}^{-1}[\psi_j \mathcal{F}[f]]\big|^q \Big)^{1/q} \Big\|_{L^p}.$$

The *homogeneous Triebel–Lizorkin space with indices* s, p, q, denoted by $\dot{F}^s_{p,q}(\mathbb{R}^n)$, is the space of all tempered distributions f for which $\|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}$ is finite.

It is known that $\dot{H}^s(\mathbb{R}^n) = \dot{B}^s_{2,2}(\mathbb{R}^n) = \dot{F}^s_{2,2}(\mathbb{R}^n)$.

Finally we denote by $\mathcal{H}^1(\mathbb{R}^n)$ the homogeneous Hardy space in $\mathbb{R}^n$. It is known that $\mathcal{H}^1(\mathbb{R}^n) \simeq F^0_{2,1}$; thus we have

$$\|f\|_{\mathcal{H}^1(\mathbb{R}^n)} \simeq \int_{\mathbb{R}} \Big(\sum_j |\mathcal{F}^{-1}[\psi_j \mathcal{F}[f]]|^2 \Big)^{1/2} dx.$$

We recall that in dimension $n = 1$, the space $\dot{H}^{1/2}(\mathbb{R})$ is continuously embedded in the Besov space $\dot{B}^0_{\infty,\infty}(\mathbb{R})$. More precisely we have

$$\dot{H}^{1/2}(\mathbb{R}) \hookrightarrow BMO(\mathbb{R}) \hookrightarrow \dot{B}^0_{\infty,\infty}(\mathbb{R}); \tag{30}$$

see, for instance, [Runst and Sickel 1996, p. 31] or [Triebel 1983, p. 129].

The s-fractional Laplacian of a function $u : \mathbb{R}^n \to \mathbb{R}$ is defined as a pseudodifferential operator of symbol $|\xi|^{2s}$:

$$\widehat{\Delta^s u}(\xi) = |\xi|^{2s} \hat{u}(\xi). \tag{31}$$

It can also be defined as

$$\Delta^s u(x) = p.v. \int_{\mathbb{R}} \frac{u(y) - y(x)}{|x-y|^{n+2s}}\, dy,$$

where $p.v.$ denotes the Cauchy principal value.

In the case $s = \frac{1}{2}$, we can write $\Delta^{1/2} u = -\mathcal{R}(\nabla u)$ where $\mathcal{R}$ is Fourier multiplier of symbol $\frac{i}{|\xi|} \sum_{k=1}^n \xi_k$:

$$\widehat{\mathcal{R}X}(\xi) = \frac{1}{|\xi|} \sum_{k=1}^n i \xi_k \hat{X}_k(\xi)$$

for every $X : \mathbb{R}^n \to \mathbb{R}^n$; thus $\mathcal{R} = \Delta^{-1/2} \mathrm{div}$.

We denote by $B_r(\bar{x})$ the ball of radius r and center $\bar{x}$. If $\bar{x} = 0$ we simply write B_r. If $x, y \in \mathbb{R}^n$, $x \cdot y$ denote the scalar product between x, y.

For every function $f : \mathbb{R}^n \to \mathbb{R}$ we denote by $M(f)$ the maximal function of f, namely

$$M(f) = \sup_{\substack{r>0 \\ x \in \mathbb{R}^n}} |B(x,r)|^{-1} \int_{B(x,r)} |f(y)|\, dy. \tag{32}$$

3. Three-term commutator estimates: proof of Theorems 1.5 and 1.6

We consider the dyadic decomposition introduced in Section 2. For every $j \in \mathbb{Z}$ and $f \in \mathcal{S}'(\mathbb{R}^n)$ we define the Littlewood–Paley projection operators P_j and $P_{\le j}$ by

$$\widehat{P_j f} = \psi_j \hat{f}, \qquad \widehat{P_{\le j} f} = \phi_j \hat{f}.$$

Informally, P_j is a frequency projection to the annulus $\{2^{j-1} \le |\xi| \le 2^j\}$, while $P_{\le j}$ is a frequency projection to the ball $\{|\xi| \le 2^j\}$. We will set $f_j = P_j f$ and $f^j = P_{\le j} f$.

We observe that $f^j = \sum_{k=-\infty}^{j} f_k$ and $f = \sum_{k=-\infty}^{+\infty} f_k$, where the convergence is in $\mathcal{S}'(\mathbb{R}^n)$.

Given $f, g \in \mathcal{S}'(\mathbb{R})$ we can split the product fg as

$$fg = \Pi_1(f,g) + \Pi_2(f,g) + \Pi_3(f,g), \tag{33}$$

where

$$\Pi_1(f,g) = \sum_{-\infty}^{+\infty} f_j g^{j-4} = \sum_{-\infty}^{+\infty} f_j \sum_{-\infty}^{j-4} g_k,\ \Pi_2(f,g) = \sum_{-\infty}^{+\infty} g_j f^{j-4} = \sum_{-\infty}^{+\infty} f_j \sum_{j+4}^{+\infty} g_k,\ \Pi_3(f,g) = \sum_{-\infty}^{+\infty} f_j \sum_{j-4}^{j+4} g_k.$$

This is an example of decomposition into *paraproducts* (see [Grafakos 2009], for example). Informally, the first paraproduct Π_1 is an operator that allows high frequencies of f ($\sim 2^j$) multiplied by low frequencies of g ($\ll 2^j$) to produce high frequencies in the output; Π_2 multiplies low frequencies of f with high frequencies of g to produce high frequencies in the output; and Π_3 multiplies high frequencies of f with high frequencies of g to produce comparable or lower frequencies in the output.

For every j, we have

$$\operatorname{supp} \mathcal{F}[f^{j-4} g_j] \subset \{2^{j-2} \le |\xi| \le 2^{j+2}\} \quad \text{and} \quad \operatorname{supp} \mathcal{F}\Big[\sum_{k=j-3}^{j+3} f_j g_k\Big] \subset \{|\xi| \le 2^{j+5}\}.$$

Lemma 3.1. *For every $f \in \mathcal{S}'$ we have $\sup_{j \in Z} |f^j| \le M(f)$.*

Proof. We have

$$\begin{aligned} f^j &= \mathcal{F}^{-1}[\phi_j] \star f = 2^j \int_R \mathcal{F}^{-1}[\phi](2^j(x-y)) f(y)\, dy = \int_{\mathbb{R}} \mathcal{F}^{-1}[\phi](z) f(x - 2^{-j} z)\, dz \\ &= \sum_{k=-\infty}^{+\infty} \int_{B_{2^k} \setminus B_{2^{k-1}}} \mathcal{F}^{-1}[\phi](z) f(x - 2^{-j} z)\, dz \\ &\le \sum_{k=-\infty}^{+\infty} \max_{B_{2^k} \setminus B_{2^{k-1}}} |\mathcal{F}^{-1}[\phi](z)| \int_{B_{2^k} \setminus B_{2^{k-1}}} |f(x - 2^{-j} z)|\, dz \\ &\le \sum_{k=-\infty}^{+\infty} \max_{B_{2^k} \setminus B_{2^{k-1}}} 2^k |\mathcal{F}^{-1}[\phi](z)| 2^{j-k} \int_{B(x,2^{k-j}) \setminus B(x,2^{k-1-j})} |f(z)|\, dz \\ &\le M(f) \sum_{k=-\infty}^{+\infty} \max_{B_{2^k} \setminus B_{2^{k-1}}} 2^k |\mathcal{F}^{-1}[\phi](z)| \le C M(f). \end{aligned}$$

In the last inequality we use the fact that $\mathcal{F}^{-1}[\phi]$ is in $\mathcal{S}(\mathbb{R}^n)$, and thus

$$\sum_{k=-\infty}^{+\infty} \max_{B_{2^k}\setminus B_{2^{k-1}}} 2^k |\mathcal{F}^{-1}[\phi](z)| \le 2\int_{\mathbb{R}} |\mathcal{F}^{-1}[\phi](z)|\,d\xi < \infty. \qquad \square$$

Proof of Theorem 1.7. We need to estimate $\Pi_1(R(Q,u))$, $\Pi_2(R(Q,u))$ and $\Pi_3(R(Q,u))$. Consistently with our earlier convention, we write, for example, $\Pi_1(\Delta^{1/4}(Q\Delta^{1/4}u))$ to mean

$$\Delta^{1/4}(\Pi_1(Q,\Delta^{1/4}u)) = \sum_{j=-\infty}^{\infty} \Delta^{1/4}(Q_j(\Delta^{1/4}u^{j-4})).$$

- Estimate of $\left\|\Pi_1(\Delta^{1/4}(Q\Delta^{1/4}u))\right\|_{\mathcal{H}^1}$. This expression equals

$$\begin{aligned}\int_{\mathbb{R}^n}\Bigl(\sum_{j=-\infty}^{\infty} 2^j Q_j^2(\Delta^{1/4}u^{j-4})^2\Bigr)^{1/2} dx &\le \int_{\mathbb{R}^n} \sup_j |\Delta^{1/4}u^{j-4}|\bigl(\textstyle\sum_j 2^j Q_j^2\bigr)^{1/2} dx \\ &\le \Bigl(\int_{\mathbb{R}^n}(M(\Delta^{1/4}u))^2\,dx\Bigr)^{1/2}\Bigl(\int_R \textstyle\sum_j 2^j Q_j^2\,dx\Bigr)^{1/2} \\ &\le C\|u\|_{\dot H^{1/2}}\|Q\|_{\dot H^{1/2}}. \end{aligned} \tag{34}$$

- Estimate of $\Pi_1\bigl(\Delta^{1/4}(\Delta^{1/4}Qu)-\Delta^{1/2}(Qu)\bigr)$. We show that this term lies in $\dot B^0_{1,1}$ ($\mathcal{H}^1 \hookrightarrow \dot B^0_{1,1}$). To this purpose we use the "commutator structure" of the term above:

$$\begin{aligned}&\bigl\|\Pi_1\bigl(\Delta^{1/4}(\Delta^{1/4}Q)u-\Delta^{1/2}(Qu)\bigr)\bigr\|_{\dot B^0_{1,1}} \\ &\quad= \sup_{\|h\|_{\dot B^0_{\infty,\infty}}\le 1}\int_{\mathbb{R}^n}\sum_j\sum_{|t-j|\le 3}\bigl(\Delta^{1/4}(u^{j-4}\Delta^{1/4}Q_j)-\Delta^{1/2}(u^{j-4}Q_j)\bigr)h_t\,dx \\ &\quad= \sup_{\|h\|_{\dot B^0_{\infty,\infty}}\le 1}\int_{\mathbb{R}^n}\sum_j\sum_{|t-j|\le 3}\mathcal{F}[u^{j-4}]\,\mathcal{F}[\Delta^{1/4}Q_j\Delta^{1/4}h_t - Q_j\Delta^{1/2}h_t]\,d\xi \\ &\quad= \sup_{\|h\|_{\dot B^0_{\infty,\infty}}\le 1}\int_{\mathbb{R}^n}\sum_j\sum_{|t-j|\le 3}\mathcal{F}[u^{j-4}](\xi) \\ &\qquad\qquad \times\Bigl(\int_{\mathbb{R}^n}\mathcal{F}[Q_j](\zeta)\mathcal{F}[\Delta^{1/4}h_t](\xi-\zeta)(|\zeta|^{1/2}-|\xi-\zeta|^{1/2})\,d\zeta\Bigr)\,d\xi. \end{aligned}\tag{35}$$

Note that in (35) we have $|\xi|\le 2^{j-3}$ and $2^{j-2}\le|\zeta|\le 2^{j+2}$. Thus $|\xi/\zeta|\le\frac{1}{2}$, allowing us to write

$$\bigl||\zeta|^{\frac12}-|\xi-\zeta|^{\frac12}\bigr| = |\zeta|^{\frac12}\Bigl(1-\Bigl|1-\frac{\xi}{\zeta}\Bigr|^{\frac12}\Bigr) = |\zeta|^{\frac12}\frac{\xi}{\zeta}\Bigl(1+\Bigl|1-\frac{\xi}{\zeta}\Bigr|^{\frac12}\Bigr)^{-1} = |\zeta|^{\frac12}\sum_{l=0}^{\infty}\frac{c_l}{l!}\Bigl(\frac{\xi}{\zeta}\Bigr)^{l+1}$$

for appropriate coefficients c_l. Thus the expression on the last two lines of (35) equals

$$\sup_{\|h\|_{\dot B^0_{\infty,\infty}}\le 1}\int_{\mathbb{R}^n}\sum_j\sum_{|t-j|\le 3}\mathcal{F}[u^{j-4}](\xi)\Bigl(\int_{\mathbb{R}^n}|\zeta|^{1/2}\mathcal{F}[Q_j](\zeta)\mathcal{F}[\Delta^{1/4}h_t](\xi-\zeta)\sum_{l=0}^{\infty}\frac{c_l}{l!}\Bigl(\frac{\xi}{\zeta}\Bigr)^{l+1}d\zeta\Bigr)\,d\xi. \tag{36}$$

Next, for $k \in \mathbb{Z}$ and $g \in \mathcal{S}'$, we set

$$S_k g = \mathcal{F}^{-1}[\xi^{-(k+1)}|\xi|^{1/2}\mathcal{F}g].$$

We note that if $h \in \dot{B}^s_{\infty,\infty}$ then $S_k h \in \dot{B}^{s+1/2+k}_{\infty,\infty}$ and if $h \in \dot{H}^s$ then $S_k h \in \dot{H}^{s+1/2+k}$.

Finally, if $Q \in \dot{H}^{1/2}$ then $\nabla^{k+1}(Q) \in \dot{H}^{-k-1/2}$.

It follows that (36) is bounded above by

$$\begin{aligned} C \sup_{\|h\|_{\dot{B}^0_{\infty,\infty}} \le 1} & \sum_{l=0}^{\infty} \frac{c_l}{l!} \int_{\mathbb{R}^n} \sum_j \sum_{|t-j|\le 3} (i)^{-(l+1)} \mathcal{F}[\nabla^{l+1} u^{j-4}]\, \mathcal{F}[S_l Q_j \Delta^{1/4} h_t)](\xi)\, d\xi \\ &\le C \sup_{\|h\|_{\dot{B}^0_{\infty,\infty}} \le 1} \|h\|_{\dot{B}^0_{\infty,\infty}} \sum_{l=0}^{\infty} \frac{c_l}{l!} \int_{\mathbb{R}^n} \sum_j 2^{j/2} |\nabla^{l+1} u^{j-4}|\,|S_l Q_j|\, dx \\ &\le C \sum_{l=0}^{\infty} \frac{c_l}{l!} \int_{\mathbb{R}^n} \sum_j |2^{-(l+1/2)j} \nabla^{l+1} u^{j-4}|\,|2^{(l+1)j} S_l Q_j|\, dx \\ &\le C \sum_{l=0}^{\infty} \frac{c_l}{l!} \Big(\int_{\mathbb{R}^n} \sum_j 2^{-2(l+1/2)j} |\nabla^{l+1} u^{j-4}|^2\, dx \Big)^{1/2} \Big(\int_{\mathbb{R}^n} \sum_j 2^{2(l+1)j} |S_l Q_j|^2\, dx \Big)^{1/2}. \end{aligned}$$

By Plancherel's theorem, this equals

$$\begin{aligned} C \sum_{l=0}^{\infty} \frac{c_l}{l!} & \Big(\int_{\mathbb{R}^n} \sum_j 2^{-2(l+1/2)j} |\xi|^{2l} |\mathcal{F}[\nabla u^{j-4}]|^2 d\xi \Big)^{1/2} \Big(\int_{\mathbb{R}^n} \sum_j 2^{2(l+1)j} |\xi|^{-2(l+1/2)} |\mathcal{F}[Q_j]|^2 d\xi \Big)^{1/2} \\ &\le C \sum_{l=0}^{\infty} \frac{c_l}{l!} 2^{-3l} \Big(\int_{\mathbb{R}^n} \sum_j 2^{-j} |\mathcal{F}[\nabla u^{j-4}]|^2 d\xi \Big)^{1/2} \Big(\int_{\mathbb{R}^n} \sum_j 2^{j} |\mathcal{F}[Q_j]|^2 d\xi \Big)^{1/2} \\ &\le C \sum_{l=0}^{\infty} \frac{c_l}{l!} 2^{-3l} \|Q\|_{\dot{H}^{1/2}} \|u\|_{\dot{H}^{1/2}}, \end{aligned}$$

where we have used the fact that for every vector field X we have

$$\int_{\mathbb{R}^n} \sum_{j=-\infty}^{+\infty} 2^{-j} (X^{j-4})^2\, dx = \int_{\mathbb{R}^n} \sum_{k,l} X_k X_l \sum_{\substack{j-4\ge k \\ j-4\ge l}} 2^{-j}\, dx \lesssim \int_{\mathbb{R}^n} \sum_{j=-\infty}^{+\infty} 2^{-j} (X_j)^2\, dx. \tag{37}$$

- Estimate of $\left\| \Pi_2(\Delta^{1/4}(\Delta^{1/4} Q u)) \right\|_{\mathcal{H}^1}$: as in (34).
- Estimate of $\left\| \Pi_2\big(\Delta^{1/4}(Q\Delta^{1/4} u) - \Delta^{1/2}(Qu)\big) \right\|_{\dot{B}^0_{1,1}}$: analogous to (35).
- Estimate of $\left\| \Pi_3(\Delta^{1/2}(Qu)) \right\|_{\mathcal{H}^1}$. We show that this lies in the smaller space $\dot{B}^0_{1,1}$ (we always have $\dot{B}^0_{1,1} \hookrightarrow \mathcal{H}^1$). We first observe that if $h \in \dot{B}^0_{\infty,\infty}$ then $\Delta^{1/2} h \in \dot{B}^{-1}_{\infty,\infty}$ and

$$\Delta^{1/2} h^{j+6} = \sum_{k=-\infty}^{j+6} \Delta^{1/2} h_k \le \sup_{k\in\mathbb{N}} |2^{-k} \Delta^{1/2} h_k| \sum_{k=-\infty}^{j+6} 2^k \le C\, 2^j \|h\|_{\dot{B}^0_{\infty,\infty}}. \tag{38}$$

Thus

$$\begin{aligned}\left\|\Pi_3(\Delta^{1/2}(Qu))\right\|_{\dot B^0_{1,1}} &= \sup_{\|h\|_{\dot B^0_{\infty,\infty}}\le 1}\int_{\mathbb{R}^n}\sum_j\sum_{|k-j|\le 3}\Delta^{1/2}(Q_ju_k)h\\ &= \sup_{\|h\|_{\dot B^0_{\infty,\infty}}\le 1}\int_{\mathbb{R}^n}\sum_j\sum_{|k-j|\le 3}\Delta^{1/2}(Q_ju_k)[h^{j+6}]\,dx\\ &= \sup_{\|h\|_{\dot B^0_{\infty,\infty}}\le 1}\int_{\mathbb{R}^n}\sum_j\sum_{|k-j|\le 3}(Q_ju_k)[\Delta^{1/2}h^{j+6}]\,dx\\ &\le C\sup_{\|h\|_{\dot B^0_{\infty,\infty}}\le 1}\|h\|_{\dot B^0_{\infty,\infty}}\int_{\mathbb{R}^n}\sum_j\sum_{|k-j|\le 3}2^j|Q_ju_k|\,dx\\ &\le C\Big(\int_{\mathbb{R}^n}\sum_j 2^jQ_j^2dx\Big)^{1/2}\Big(\int_{\mathbb{R}^n}\sum_j 2^ju_j^2dx\Big)^{1/2}\le C\|Q\|_{\dot H^{1/2}}\|u\|_{\dot H^{1/2}}.\end{aligned}\tag{39}$$

• Estimate of $\Pi_3(\Delta^{1/4}(Q\Delta^{1/4}u))$. To show that this is in $\dot B^0_{1,1}$, we observe that if $h\in\dot B^0_{\infty,\infty}$ then $\Delta^{1/4}h\in B^{-1/2}_{\infty,\infty}$, and by arguing as in (38) we get

$$\|\Delta^{1/4}h_j\|_{L^\infty}\le 2^{j/2}\|h\|_{\dot B^0_{\infty,\infty}}.$$

Thus

$$\begin{aligned}\left\|\Pi_3(\Delta^{1/4}(Q,\Delta^{1/4}u))\right\|_{\dot B^0_{1,1}} &= \sup_{\|h\|_{\dot B^0_{\infty,\infty}}\le 1}\int_{\mathbb{R}^n}\sum_j\sum_{|k-j|\le 3}\Delta^{1/4}(Q_j\Delta^{1/4}u_k)h\\ &= \sup_{\|h\|_{\dot B^0_{\infty,\infty}}\le 1}\int_{\mathbb{R}^n}\sum_j\sum_{|k-j|\le 3}(Q_j\Delta^{1/4}u_k)[\Delta^{1/4}h^{j+6}]\,dx\\ &\le C\sup_{\|h\|_{\dot B^0_{\infty,\infty}}\le 1}\|h\|_{\dot B^0_{\infty,\infty}}\int_{\mathbb{R}^n}\sum_j\sum_{|k-j|\le 3}2^{j/2}|Q_j\Delta^{1/4}u_k|\,dx\\ &\le C\Big(\int_{\mathbb{R}^n}\sum_j 2^jQ_j^2\,dx\Big)^{1/2}\Big(\int_{\mathbb{R}^n}\sum_j(\Delta^{1/4}u_j)^2\,dx\Big)^{1/2}\\ &\le C\|Q\|_{\dot H^{1/2}}\|u\|_{\dot H^{1/2}}.\end{aligned}\tag{40}$$

• Estimate of $\Pi_3(\Delta^{1/4}(\Delta^{1/4}Qu))$: analogous to (40). □

Proof of Theorem 1.5. We use Theorem 1.7 and the duality between BMO and $\mathscr{H}^1$. For all $h,Q\in\dot H^{1/2}$ and $u\in BMO$ we have

$$\begin{aligned}\int_{\mathbb{R}^n}\big(\Delta^{1/4}(Q\Delta^{1/4}u)-Q\Delta^{1/2}u+\Delta^{1/4}Q\Delta^{1/4}u\big)h\,dx &= \int_{\mathbb{R}^n}\big(\Delta^{1/4}(Q\Delta^{1/4}h)-\Delta^{1/2}(Qh)+\Delta^{1/4}(h\Delta^{1/4}Q)\big)u\,dx\\ &\le C\|u\|_{BMO}\|R(Q,h)\|_{\mathscr{H}^1};\end{aligned}$$

by Theorem 1.7, this is at most

$$C\|u\|_{BMO}\|Q\|_{\dot H^{1/2}}\|h\|_{\dot H^{1/2}}.$$

Hence

$$\|T(Q,u)\|_{\dot H^{-1/2}}=\sup_{\|h\|_{\dot H^{1/2}}\le 1}\int_{\mathbb{R}^n}T(Q,u)h\,dx\le C\|u\|_{BMO}\|Q\|_{\dot H^{1/2}}.$$ □

Proof of Theorem 1.8. We observe that $\mathcal{R}$ is a Fourier multiplier of order zero; thus $\mathcal{R} : H^{-1/2} \to H^{-1/2}$, $\mathcal{R} : \mathcal{H}^1 \to \mathcal{H}^1$, and $\mathcal{R} : \dot{B}^0_{1,1} \to \dot{B}^0_{1,1}$. See [Taylor 1991] and [Sickel and Triebel 1995].

The estimates are very similar to the ones in Theorem 1.7; thus we will write down only one:

- Estimate of $\Pi_1\big(\mathcal{R}\Delta^{1/4}(\Delta^{1/4}Q\mathcal{R}u) - \nabla(Q\mathcal{R}u)\big)$. We observe that $\nabla u = \Delta^{1/4}\mathcal{R}\Delta^{1/4}u$. Hence

$$
\begin{aligned}
&\big\|\Pi_1\big(\mathcal{R}\Delta^{1/4}(\Delta^{1/4}Q\mathcal{R}u) - \nabla(Q\mathcal{R}u)\big)\big\|_{\dot{B}^0_{1,1}} \\
&\quad = \sup_{\|h\|_{\dot{B}^0_{\infty,\infty}}\le 1} \int_{\mathbb{R}^n} \sum_j \sum_{|t-j|\le 3} \big(\mathcal{R}\Delta^{1/4}(\Delta^{1/4}Q_j\mathcal{R}u^{j-4}) - \nabla(Q_j\mathcal{R}u^{j-4})\big)h_t\,dx \\
&\quad \simeq \sup_{\|h\|_{\dot{B}^0_{\infty,\infty}}\le 1} \int_{\mathbb{R}^n} \sum_j \sum_{|t-j|\le 3} \mathcal{R}u^{j-4}\big(\mathcal{R}\Delta^{1/4}h_t\Delta^{1/4}Q_j - \nabla h_t Q_j\big)\,dx \\
&\quad \simeq \sup_{\|h\|_{\dot{B}^0_{\infty,\infty}}\le 1} \int_{\mathbb{R}^n} \sum_j \sum_{|t-j|\le 3} \mathcal{F}[\mathcal{R}u^{j-4}](\xi) \\
&\qquad\qquad \times \bigg(\int_{\mathbb{R}^n} \mathcal{F}[Q_j](\zeta)\mathcal{F}[\mathcal{R}\Delta^{1/4}h_t](\xi-\zeta)\big(|\zeta|^{1/2} - |\xi-\zeta|^{1/2}\big)\,d\zeta\bigg)\,d\xi. \qquad (41)
\end{aligned}
$$

Now we can proceed exactly as in (35) and get

$$
\sup_{\|h\|_{\dot{B}^0_{\infty,\infty}}\le 1} \int_{\mathbb{R}^n} \sum_j \sum_{|t-j|\le 3} \big(\mathcal{R}\Delta^{1/4}(\Delta^{1/4}Q_j\mathcal{R}u^{j-4}) - \nabla(Q_j\mathcal{R}u^{j-4})\big)\,h_t\,dx \le C\|Q\|_{\dot{H}^{1/2}}\|u\|_{\dot{H}^{1/2}}. \qquad \square
$$

Proof of Theorem 1.6. This follows from Theorem 1.8 and the duality between $\mathcal{H}^1$ and BMO. $\square$

Lemma 3.2. *Let $u \in \dot{H}^{1/2}(\mathbb{R}^n)$, then $\mathcal{R}(\Delta^{1/4}u \cdot \mathcal{R}\Delta^{1/4}u) \in \mathcal{H}^1$, and*

$$
\|\mathcal{R}(\Delta^{1/4}u \cdot \mathcal{R}\Delta^{1/4}u)\|_{\mathcal{H}^1} \le C\|u\|^2_{\dot{H}^{1/2}}.
$$

Proof. Since $\mathcal{R} : \mathcal{H}^1 \to \mathcal{H}^1$, it is enough to verify that $\Delta^{1/4}u \cdot \mathcal{R}\Delta^{1/4}u \in \mathcal{H}^1$.

- Estimate of $\Pi_1(\Delta^{1/4}u, \mathcal{R}\Delta^{1/4}u)$:

$$
\begin{aligned}
\big\|\Pi_1(\Delta^{1/4}u, \mathcal{R}\Delta^{1/4}u)\big\|_{\mathcal{H}^1} &= \int_{\mathbb{R}^n} \bigg(\sum_{j=-\infty}^{+\infty} [\Delta^{1/4}u_j(\mathcal{R}\Delta^{1/4}u)^{j-4})^2\bigg)^{1/2} dx \\
&\le \int_{\mathbb{R}^n} \sup_j \big|(\mathcal{R}\Delta^{1/4}u)^{j-4}\big| \bigg(\sum_{j=0}^{+\infty}[\Delta^{1/4}u_j]^2\bigg)^{1/2} dx \\
&\le \bigg(\int_{\mathbb{R}^n} |M(\mathcal{R}\Delta^{1/4}u)|^2\,dx\bigg)^{1/2} \bigg(\int_{\mathbb{R}^n} \sum_{j=-\infty}^{+\infty} [\Delta^{1/4}u_j]^2\,dx\bigg)^{1/2} \\
&\le C\|u\|^2_{\dot{H}^{1/2}}. \qquad (42)
\end{aligned}
$$

The estimate of the $\mathcal{H}^1$ norm of $\Pi_2(\Delta^{1/4}u \cdot \mathcal{R}\Delta^{1/4}u)$ is similar to (42).

• Estimate of $\Pi_3(\Delta^{1/4}u\cdot\mathcal{R}\Delta^{1/4}u)$:

$$
\begin{aligned}
&\left\|\Pi_1(\Delta^{1/4}u,\mathcal{R}\Delta^{1/4}u)\right\|_{\mathcal{H}^1}\\
&\quad=\sup_{\|h\|_{\dot B^0_{\infty,\infty}}\le 1}\int_{\mathbb{R}^n}\sum_j\sum_{|k-j|\le 3}\Delta^{1/4}u_j\mathcal{R}(\Delta^{1/4}u_k)\Big(h^{j-6}+\sum_{t=j-5}^{j+6}h_t\Big)\,dx\\
&\quad=\sup_{\|h\|_{\dot B^0_{\infty,\infty}}\le 1}\int_{\mathbb{R}^n}\sum_j\sum_{|k-j|\le 3}\big(\Delta^{1/4}u_j\mathcal{R}(\Delta^{1/4}u_k)-u_j\nabla u_k+\tfrac{1}{2}\nabla(u_ju_k)\big)\Big(h^{j-6}+\sum_{t=j-5}^{j+6}h_t\Big)\,dx. \qquad (43)
\end{aligned}
$$

We only estimate the terms with h^{j-6}, the estimates with h_t being similar. We have

$$
\begin{aligned}
&\sup_{\|h\|_{\dot B^0_{\infty,\infty}}\le 1}\int_{\mathbb{R}^n}\sum_j\sum_{|k-j|\le 3}(\Delta^{1/4}u_j\mathcal{R}(\Delta^{1/4}u_k)-u_j\nabla u_k)h^{j-6}\,dx\\
&=\sup_{\|h\|_{\dot B^0_{\infty,\infty}}\le 1}\int_{\mathbb{R}^n}\sum_j\sum_{|k-j|\le 3}\mathcal{F}[h^{j-6}](x)\left(\int_{\mathbb{R}^n}\mathcal{F}[u_j]\,\mathcal{F}[\mathcal{R}\Delta^{1/4}u_k][|y|^{1/2}-|x-y|^{1/2}]\,dy\right)dx.
\end{aligned}
$$

By arguing as in (35), we can show that this is bounded above by $C\|u\|^2_{\dot H^{1/2}}$. Finally we also have

$$
\begin{aligned}
&\sup_{\|h\|_{\dot B^0_{\infty,\infty}}\le 1}\int_{\mathbb{R}^n}\sum_j\sum_{|k-j|\le 3}\tfrac{1}{2}\nabla(u_ju_k)h^{j-6}\,dx\\
&=\sup_{\|h\|_{\dot B^0_{\infty,\infty}}\le 1}\int_{\mathbb{R}^n}\sum_j\sum_{|k-j|\le 3}\tfrac{1}{2}(u_ju_k)\nabla h^{j-6}\,dx\\
&\le C\sup_{\|h\|_{\dot B^0_{\infty,\infty}}\le 1}\|h\|_{\dot B^0_{\infty,\infty}}\int_{\mathbb{R}^n}\sum_j\sum_{|k-j|\le 3}2^ju_ju_k\,dx\le C\left(\int_{\mathbb{R}^n}\sum_j 2^ju_j^2\,dx\right)^{1/2}=C\|u\|^2_{\dot H^{1/2}}.
\end{aligned}
$$

□

Theorem 1.8 and Lemma 3.2 imply:

Corollary 3.3. *Let $n\in\dot H^{1/2}(\mathbb{R}^n,S^{m-1})$. Then $\Delta^{1/4}[n\cdot\Delta^{1/4}n]\in\mathcal{H}^1(\mathbb{R}^n)$.*

Proof. Since $n\cdot\nabla n=0$ (see proof of Proposition 1.4), we can write

$$
\begin{aligned}
\Delta^{1/4}[n\cdot\Delta^{1/4}n]&=\Delta^{1/4}[n\cdot\Delta^{1/4}n]-\mathcal{R}[n\cdot\nabla n]+\mathcal{R}[\Delta^{1/4}n\cdot\mathcal{R}\Delta^{1/4}n]-\mathcal{R}[\Delta^{1/4}n\cdot\mathcal{R}\Delta^{1/4}n]\\
&=S(n\cdot,n)-\mathcal{R}[\Delta^{1/4}n\cdot\mathcal{R}\Delta^{1/4}n]. \qquad (44)
\end{aligned}
$$

The estimate in the corollary's conclusion is a consequence of Theorem 1.8 and Lemma 3.2, which imply respectively that $S(n\cdot,n)\in\mathcal{H}^1$ and $\mathcal{R}(\Delta^{1/4}n\cdot\mathcal{R}\Delta^{1/4}n)\in\mathcal{H}^1$. □

4. *L*-energy decrease controls

We now provide (in Propositions 4.1 and 4.2) localization estimates of solutions to the equations

$$\Delta^{1/4}(M\Delta^{1/4}u)=T(Q,u) \qquad (45)$$

and

$$\Delta^{1/4}(p\cdot\Delta^{1/4}u)=S(q\cdot,u)-\mathcal{R}(\Delta^{1/4}u\cdot\mathcal{R}\Delta^{1/4}u), \qquad (46)$$

where $Q \in \dot{H}^{1/2}(\mathbb{R}, \mathcal{M}_{l\times m}(\mathbb{R}))$, $M \in \dot{H}^{1/2}(\mathbb{R}, \mathcal{M}_{l\times m}(\mathbb{R}))$, $l \geq 1$ and $p, q \in \dot{H}^{1/2}(\mathbb{R}, \mathbb{R}^m)$.

Such estimates will be crucial to obtaining Morrey-type estimates for half-harmonic maps into the sphere (see Section 5). As observed in Section 1, half-harmonic maps into the sphere satisfy both equations (16) and (18), which are (45) and (46) with (M, Q) and (p, q) replaced by $(u\wedge, u\wedge)$ and (u, u), respectively. Roughly speaking, we show that the L^2 norm of $M\Delta^{1/4}u$ in a sufficiently small ball (u being a solution of either (45) or (46)), is controlled by the L^2 norm of the same function in annuli outside the ball multiplied by a "crushing" factor.

To this end we consider a dyadic decomposition of unity (Section 2). For convenience set

$$A_h = B_{2^{h+1}} \setminus B_{2^{h-1}}, \quad A'_h = B_{2^h} \setminus B_{2^{h-1}},$$

for $h \in \mathbb{Z}$. Choose a dyadic decomposition $\varphi_j \in C_0^\infty(\mathbb{R})$, so

$$\operatorname{supp}(\varphi_j) \subset A_j \quad \text{and} \quad \sum_{-\infty}^{+\infty} \varphi_j = 1. \tag{47}$$

Also define, for $h \in \mathbb{Z}$,

$$\chi_h := \sum_{-\infty}^{h-1} \varphi_j, \quad \bar{u}_h = |B_{2^k}|^{-1} \int_{B_{2^k}} u(x)\,dx, \quad \bar{u}^h = |A_h|^{-1} \int_{A_h} u(x)\,dx, \quad \bar{u}'^h = |A'_h|^{-1} \int_{A'_h} u(x)\,dx.$$

Proposition 4.1. *Let $Q \in \dot{H}^{1/2}(\mathbb{R}, \mathcal{M}_{l\times m}(\mathbb{R}))$, $M \in \dot{H}^{1/2}(\mathbb{R}, \mathcal{M}_{l\times m}(\mathbb{R}))$, $l \geq 1$, and let $u \in \dot{H}^{1/2}(\mathbb{R}, \mathbb{R}^m)$ be a solution of (45). Then for $k < 0$ with $|k|$ large enough we have*

$$\|M\Delta^{1/4}u\|^2_{L^2(B_{2^k})} - \tfrac{1}{4}\|\Delta^{1/4}u\|^2_{L^2(B_{2^k})} \leq C\left(\sum_{h=k}^{\infty} 2^{(k-h)/2}\|M\Delta^{1/4}u\|^2_{L^2(A_h)} + \sum_{h=k}^{\infty} 2^{(k-h)/2}\|\Delta^{1/4}u\|^2_{L^2(A_h)}\right). \tag{48}$$

Proposition 4.2. *Let $p, q \in \dot{H}^{1/2}(\mathbb{R}, \mathbb{R}^m)$ and let $u \in \dot{H}^{1/2}(\mathbb{R}, \mathbb{R}^m)$ be a solution of (46). Then for $k < 0$ with $|k|$ large enough we have*

$$\|p\cdot\Delta^{1/4}u\|^2_{L^2(B_{2^k})} - \tfrac{1}{4}\|\Delta^{1/4}u\|^2_{L^2(B_{2^k})} \leq C\left(\sum_{h=k}^{\infty} 2^{(k-h)/2}\|p\cdot\Delta^{1/4}u\|^2_{L^2(A_h)} + \sum_{h=k}^{\infty} 2^{(k-h)/2}\|\Delta^{1/4}u\|^2_{L^2(A_h)}\right). \tag{49}$$

For the proof, we need some estimates.

Lemma 4.3. *Let $u \in \dot{H}^{1/2}(\mathbb{R})$. Then, for all $k \in \mathbb{Z}$,*

$$\sum_{h=k}^{+\infty} 2^{k-h}\|\varphi_h(u-\bar{u}_k)\|_{\dot{H}^{1/2}(\mathbb{R})} \leq C\left(\sum_{s\leq k} 2^{s-k}\|u\|_{\dot{H}^{1/2}(A_s)} + \sum_{s\geq k} 2^{k-s}\|u\|_{\dot{H}^{1/2}(A_s)}\right). \tag{50}$$

Proof of Lemma 4.3. We first have

$$\|\varphi_h(u-\bar{u}_k)\|_{\dot{H}^{1/2}(\mathbb{R})} \leq \|\varphi_h(u-\bar{u}^h)\|_{\dot{H}^{1/2}(\mathbb{R})} + \|\varphi_h\|_{\dot{H}^{1/2}(\mathbb{R})}|\bar{u}_k - \bar{u}^h|. \tag{51}$$

We estimate separately the two terms on the right-hand side of (51). We have

$$\begin{aligned}\|\varphi_h(u-\bar u^h)\|^2_{\dot H^{1/2}(\mathbb{R})} &= \int_{A_h}\int_{A_h}\frac{\left|\varphi_h(u-\bar u^h)(x)-\varphi_h(u-\bar u^h)(y)\right|^2}{|x-y|^2}\,dx\,dy\\ &\le 2\biggl(\int_{A_h}\int_{A_h}\frac{|u(x)-u(y)|^2}{|x-y|^2}\,dx\,dy+\|\nabla\varphi_h\|^2_\infty\int_{A_h}\int_{A_h}|u-\bar u^h|^2\,dx\,dy\biggr)\\ &\le C\biggl(\|u\|^2_{\dot H^{1/2}(A_h)}+2^{-h}\int_{A_h}|u-\bar u^h|^2\,dx\biggr)\le C\|u\|^2_{\dot H^{1/2}(A_h)},\end{aligned}\tag{52}$$

where we used the fact that $\|\nabla\varphi_h\|_\infty\le C\,2^{-h}$ and the embedding $\dot H^{1/2}(\mathbb{R})\hookrightarrow BMO(\mathbb{R})$.

Now we estimate $|\bar u_k-\bar u^h|$. We can write $\bar u_k=\sum_{l=-\infty}^{k-1}2^{l-k}\bar u'^l$. Moreover,

$$\begin{aligned}|\bar u_k-\bar u^h| &\le |\bar u^h-\bar u'^h|+|\bar u_k-\bar u'^h|\\ &\le C|A_h|^{-1}\int_{A_h}|u-\bar u^h|\,dx+\sum_{l=-\infty}^{k-1}2^{l-k}\sum_{s=l}^{h-1}|\bar u'^{s+1}-\bar u'^s|\\ &\le C|A_h|^{-1}\int_{A_h}|u-\bar u^h|\,dx+\sum_{l=-\infty}^{k-1}2^{l-k}\sum_{s=l}^{h-1}|A_{s+1}|^{-1}\int_{A_{s+1}}|u-\bar u^{s+1}|\,dx\\ &\le C\biggl(\|u\|_{\dot H^{1/2}(A_h)}+\sum_{l=-\infty}^{k-1}2^{l-k}\sum_{s=l}^{h-1}\|u\|_{\dot H^{1/2}(A_{s+1})}\biggr).\end{aligned}\tag{53}$$

Combining (52) and (53) we get

$$\begin{aligned}\|\varphi_h(u-\bar u^h)\|_{\dot H^{1/2}(\mathbb{R})} &\le \|\varphi_h(u-\bar u^h)\|_{\dot H^{1/2}(\mathbb{R})}+\|\varphi_h\|_{\dot H^{1/2}(\mathbb{R})}|\bar u_k-\bar u^h|\\ &\le C\biggl(\|u\|_{\dot H^{1/2}(A_h)}+\sum_{l=-\infty}^{k-1}2^{l-k}\sum_{s=l}^{h-1}\|u\|_{\dot H^{1/2}(A_{s+1})}\biggr).\end{aligned}\tag{54}$$

Multiplying both sides of (54) by 2^{k-h} and summing up from $h=k$ to $+\infty$ we get

$$\begin{aligned}&\sum_{h=k}^{+\infty}2^{k-h}\biggl(\sum_{l=-\infty}^{k-1}2^{l-k}\sum_{s=l+1}^{h}\|u\|_{\dot H^{1/2}(A_s)}\biggr)\\ &\qquad\le C\sum_{s\le k}\|u\|_{\dot H^{1/2}(A_s)}\biggl(\sum_{h\ge k}\sum_{l\le s}2^{l-h}\biggr)+\sum_{s\ge k}\|u\|_{\dot H^{1/2}(A_s)}\biggl(\sum_{h\ge s}\sum_{l\le k}2^{l-h}\biggr)\\ &\qquad\le C\sum_{s\le k}2^{s-k}\|u\|_{\dot H^{1/2}(A_s)}+\sum_{s\ge k}2^{k-s}\|u\|_{\dot H^{1/2}(A_s)}.\end{aligned}$$

□

Now we recall the value of the Fourier transform of some functions that will be used in the sequel. We have

$$\mathcal{F}[|x|^{-1/2}](\xi)=|\xi|^{-1/2}.\tag{55}$$

The Fourier transforms of $|x|$, $x|x|^{-1/2}$, and $|x|^{1/2}$ are the tempered distributions defined, for every $\varphi\in\mathcal{S}(\mathbb{R})$, as follows (with $\mathbb{1}_X$ the characteristic function of $\mathbb{1}$):

$$\langle \mathscr{F}[|x|], \varphi \rangle = \langle \mathscr{F}[x/|x|] \star \mathscr{F}[x], \varphi \rangle = \langle p.v.(x^{-1}) \star (\delta)_0'(x), \varphi \rangle = p.v. \int_{\mathbb{R}} \frac{\varphi(x) - \varphi(0) - \mathbb{1}_{B_1(0)}\phi'(0)x}{x^2} dx,$$

$$\langle \mathscr{F}[x\,|x|^{-1/2}], \varphi \rangle = \langle \mathscr{F}[x] \star \mathscr{F}[|x|^{-1/2}], \varphi \rangle = \langle (\delta)_0'(x) \star |x|^{-1/2}, \varphi \rangle = p.v. \int_{\mathbb{R}} (\varphi(x) - \varphi(0)) \frac{x}{|x|} \frac{1}{|x|^{3/2}} dx,$$

$$\langle \mathscr{F}[|x|^{1/2}], \varphi \rangle = p.v. \int_{\mathbb{R}} \frac{\varphi(x) - \varphi(0)}{|x|^{3/2}} dx.$$

Next we introduce the operators

$$F(Q,a) = \Delta^{1/4}(Qa) - Q\Delta^{1/4}a + \Delta^{1/4}Qa, \tag{56}$$

$$G(Q,a) = \mathscr{R}\Delta^{1/4}(Qa) - Q\Delta^{1/4}\mathscr{R}a + \Delta^{1/4}Q\mathscr{R}a. \tag{57}$$

We observe that $T(Q,u) = F(Q, \Delta^{1/4}u)$ and $S(Q,u) = \mathscr{R}G(Q, \Delta^{1/4}u)$.

We now state turn to lemmas where we consider M, u as in Proposition 4.1 or p, u as in Proposition 4.2, and estimate the $\dot{H}^{1/2}$ norm of $w = \Delta^{-1/4}(M\Delta^{1/4}u)$ or $w = \Delta^{-1/4}(p \cdot \Delta^{1/4}u)$ in B_{2^k} in terms of the $\dot{H}^{1/2}$ norm of w in annuli outside the ball and the L^2 norm of $\Delta^{1/4}u$ in annuli inside and outside the ball B_{2^k}. The key point is that each term is multiplied by a crushing factor.

Lemma 4.4. *Assume the hypotheses of Proposition 4.1. There exist $C > 0$ and $\bar{n} > 0$, independent of u and M, such that for all $\eta \in (0, \frac{1}{4})$, all $k < k_0$ (where $k_0 \in \mathbb{Z}$ depends on η and $\|Q\|_{\dot{H}^{1/2}(\mathbb{R})}$), and all $n \geq \bar{n}$, we have*

$$\begin{aligned} &\|\chi_{k-4}(w - \bar{w}_{k-4})\|_{\dot{H}^{1/2}(\mathbb{R})} \\ &\qquad \leq \eta\|\chi_{k-4}\Delta^{1/4}u\|_{L^2} + C\left(\sum_{h=k}^{\infty} 2^{(k-h)/2}\|\Delta^{1/4}u\|_{L^2(A_h)} + \sum_{h=k-n}^{+\infty} 2^{k-h}\|w\|_{\dot{H}^{1/2}(A_h)}\right), \end{aligned} \tag{58}$$

where $w = \Delta^{-1/4}(M\Delta^{1/4}u)$ and we recall that $\chi_{k-4} \equiv 1$ on $B_{2^{k-5}}$ and $\chi_{k-4} \equiv 0$ on $B^c_{2^{k-4}}$.

Lemma 4.5. *Assume the hypotheses of Proposition 4.2. There exist $C > 0$ and $\bar{n} > 0$, independent of u and M, such that for all $(0, \frac{1}{4})$, all $k < k_0$ (where $k_0 \in \mathbb{Z}$ depends on η and the $\dot{H}^{1/2}$ norms of Q and u in $\mathbb{R}$), and all $n \geq \bar{n}$, we have*

$$\begin{aligned} &\|\chi_{k-4}(w - \bar{w}_{k-4})\|_{\dot{H}^{1/2}(\mathbb{R})} \\ &\qquad \leq \eta\|\chi_{k-4}\Delta^{1/4}u\|_{L^2(\mathbb{R})} + C\left(\sum_{h=k}^{\infty} 2^{(k-h)/2}\|\Delta^{1/4}u\|_{L^2(A_h)} + \sum_{h=k-n}^{k-3} 2^{h-k}\|w\|_{\dot{H}^{1/2}(A_h)}\right), \end{aligned} \tag{59}$$

where $w = \Delta^{-1/4}(p \cdot \Delta^{1/4}u)$.

Proof of Lemma 4.4. Fix $\eta \in (0, \frac{1}{4})$. We first consider $k < 0$ large enough in absolute value so that $\|\chi_k(Q - \bar{Q}_k)\|_{\dot{H}^{1/2}(\mathbb{R})} \leq \varepsilon$, where $\varepsilon \in (0,1)$ will be determined later. We write

$$F(Q, \Delta^{1/4}u) = F(Q_1, \Delta^{1/4}u) + F(Q_2, \Delta^{1/4}u),$$

where

$$Q_1 = \chi_k(Q - \bar{Q}_k) \quad \text{and} \quad Q_2 = (1 - \chi_k)(Q - \bar{Q}_k).$$

By construction, we have

$$\operatorname{supp} Q_2 \subseteq B^c_{2^{k-1}} \quad \text{and} \quad \|Q_2\|_{\dot H^{1/2}(\mathbb{R})} \le \|Q\|_{\dot H^{1/2}(\mathbb{R})}.$$

For brevity, set

$$W := \chi_{k-4}(w - \bar w_{k-4}).$$

We rewrite (45) as

$$\Delta^{1/2}(W) = -\Delta^{1/2}\Big(\sum_{h=k-4}^{+\infty} \varphi_h (w - \bar w_{k-4})\Big) + F(Q_1, \Delta^{1/4}u) + F(Q_2, \Delta^{1/4}u). \tag{60}$$

We take the scalar product of both sides with W and integrate over $\mathbb{R}$. From Corollary A.8 it follows that

$$\lim_{N\to+\infty} \int_{\mathbb{R}} \Delta^{1/2}\Big(\sum_{h=N}^{+\infty} \varphi_h (w - \bar w_{k-4})\Big)\cdot W\,dx = \lim_{N\to+\infty} \int_{\mathbb{R}} \Delta^{1/2}\big((1-\chi_{N-1})(w - \bar w_{k-4})\big)\cdot W\,dx = 0.$$

This allows us to interchange the infinite sum with the integral and the operator $\Delta^{1/2}$ in the expression

$$\int_{\mathbb{R}} \Delta^{1/2}\Big(\sum_{h=k-4}^{+\infty} \varphi_h (w - \bar w_{k-4})\Big)\cdot W\,dx = \sum_{h=k-4}^{+\infty} \int_{\mathbb{R}} \Delta^{1/2}\big(\varphi_h (w - \bar w_{k-4})\big)\cdot W\,dx.$$

Thus we get from (60) the equality

$$\begin{aligned}&\int_{\mathbb{R}} \big|\Delta^{1/4}(W)\big|^2\,dx \\ &= -\sum_{h=k-4}^{+\infty} \int_{\mathbb{R}} \Delta^{1/2}(\varphi_h(w-\bar w_{k-4}))\cdot W\,dx + \int_{\mathbb{R}} F(Q_1, \Delta^{1/4}u)\cdot W\,dx + \int_{\mathbb{R}} F(Q_2, \Delta^{1/4}u)\cdot W\,dx.\end{aligned} \tag{61}$$

Step 1: estimate of the sum. We split the sum in (61) into two parts: $k-4 \le h \le k-3$ and $h \ge k-2$.

Step 1a. We have

$$-\sum_{h=k-4}^{k-3} \int_{\mathbb{R}} \Delta^{1/2}(\varphi_h(w-\bar w_{k-4}))\cdot W\,dx \le \|W\|_{\dot H^{1/2}(\mathbb{R})}\Big(\sum_{h=k-4}^{k-3} \big\|\varphi_h(w-\bar w_{k-4})\big\|_{\dot H^{1/2}(\mathbb{R})}\Big).$$

By Lemma 4.3, the right-hand side is bounded above by

$$\begin{aligned}&\|W\|_{\dot H^{1/2}(\mathbb{R})}\Big(\sum_{h=k-4}^{k-3}\Big(\|w\|_{\dot H^{1/2}(A_h)} + \sum_{l=-\infty}^{k-5} 2^{l-(k-4)} \sum_{s=l+1}^{h} \|w\|_{\dot H^{1/2}(A_s)}\Big)\Big) \\ &\qquad \le C\|W\|_{\dot H^{1/2}(\mathbb{R})}\Big(\sum_{h=-\infty}^{k-3} 2^{h-k}\|w\|_{\dot H^{1/2}(A_h)}\Big).\end{aligned} \tag{62}$$

From the localization theorem A.1 it follows that

$$\sum_{h=-\infty}^{k-6} \|w\|^2_{\dot H^{1/2}(A_h)} \le \tilde C\,\|W\|^2_{\dot H^{1/2}(\mathbb{R})},$$

where $\tilde{C} > 0$ is independent of k and w. Thus we can find $n_1 \geq 6$ such that

$$C \sum_{h=-\infty}^{k-n} 2^{h-k} \|w\|_{\dot{H}^{1/2}(A_h)} \leq \tfrac{1}{8} \|W\|_{\dot{H}^{1/2}(\mathbb{R})} \quad \text{for all } n \geq n_1,$$

with the same constant C appearing on the last line of (62). Then for $n \geq n_1$ we have

$$\sum_{h=k-4}^{k-3} \int_{\mathbb{R}} \Delta^{1/2}\big(\varphi_h(w - \bar{w}_{k-4})\big) \cdot W\, dx \\ \leq \tfrac{1}{8} \|W\|^2_{\dot{H}^{1/2}(\mathbb{R})} + C \|W\|_{\dot{H}^{1/2}(\mathbb{R})} \bigg(\sum_{h=k-n}^{k-3} 2^{h-k} \|w\|_{\dot{H}^{1/2}(A_h)} \bigg). \tag{63}$$

Step 1b. To estimate the part of the sum in (61) with $h \geq k-2$, we use the fact that the supports of φ_h and of χ_{k-4} are disjoint; in particular $0 \notin \operatorname{supp}\big(\varphi_h(w - \bar{w}_{k-4}) \star W\big)$. We have

$$\begin{aligned} \sum_{h=k-2}^{+\infty} \int_{\mathbb{R}} \Delta^{1/2}\big(\varphi_h(w - \bar{w}_{k-4})\big) \cdot W\, dx \\ &= \sum_{h=k-2}^{+\infty} \int_{\mathbb{R}} \mathcal{F}^{-1}(|\xi|)(x) \big(\varphi_h(w - \bar{w}_{k-4})\big) \star W\, dx \\ &\leq \sum_{h=k-2}^{+\infty} \big\|\mathcal{F}^{-1}(|\xi|)\big\|_{L^\infty(B_{2^{h+2}} \setminus B_{2^{h-2}})} \big\|\varphi_h(w - \bar{w}_{k-4})\big\|_{L^1} \|W\|_{L^1} \\ &\leq C \sum_{h=k-2}^{+\infty} 2^{-2h} 2^{h/2} \big\|\varphi_h(w - \bar{w}_{k-4})\big\|_{L^2(\mathbb{R})} 2^{k/2} \|W\|_{L^2(\mathbb{R})}. \end{aligned} \tag{64}$$

By Theorem A.5 and Lemma 4.3 the sum on this last line is bounded above by

$$\begin{aligned} \sum_{h=k-2}^{+\infty} 2^{k-4-h} \big\|\varphi_h(w - \bar{w}_{k-4})\big\|_{\dot{H}^{1/2}(\mathbb{R})} \|W\|_{\dot{H}^{1/2}(\mathbb{R})} \\ &\leq \sum_{h=k-2}^{+\infty} 2^{k-4-h} \bigg(\|w\|_{\dot{H}^{1/2}(A_h)} + \sum_{l=-\infty}^{k-5} 2^{l-(k-4)} \sum_{s=l+1}^{h} \|w\|_{\dot{H}^{1/2}(A_s)} \bigg) \|W\|_{\dot{H}^{1/2}(\mathbb{R})} \\ &\leq \bigg(\sum_{h=k-4}^{+\infty} 2^{k-h-4} \|w\|_{\dot{H}^{1/2}(A_h)} + \sum_{s \leq k-4} \|w\|_{\dot{H}^{1/2}(A_s)} \bigg(\sum_{h \geq k-4} \sum_{l \leq s-1} 2^{l-h} \bigg) \\ &\qquad + \sum_{s \geq k-4} \|w\|_{\dot{H}^{1/2}(A_s)} \bigg(\sum_{h \geq s-1} \sum_{l \leq k-4} 2^{l-h} \bigg) \bigg) \|W\|_{\dot{H}^{1/2}(\mathbb{R})} \\ &\leq \bigg(\sum_{h=k-4}^{+\infty} 2^{k-4-h} \|w\|_{\dot{H}^{1/2}(A_h)} + \sum_{h=-\infty}^{k-5} 2^{h-(k-4)} \|w\|_{\dot{H}^{1/2}(A_h)} \bigg) \|W\|_{\dot{H}^{1/2}(\mathbb{R})}. \end{aligned} \tag{65}$$

Finally, set $n > \bar{n} = \max(n_1, n_2)$, where $n_2 \geq 6$ is such that

$$C \sum_{h=-\infty}^{k-n} 2^{h-(k-4))} \|w\|_{\dot{H}^{1/2}(A_h)} \leq \tfrac{1}{8} \|W\|_{\dot{H}^{1/2}(\mathbb{R})} \quad \text{for } n \geq n_2.$$

We conclude from (63)–(65) that

$$\sum_{h=k-4}^{+\infty} \int_{\mathbb{R}} \Delta^{1/2}(\varphi_h(w-\bar{w}_{k-4}))\cdot W \le \tfrac{1}{4}\|W\|^2_{\dot{H}^{1/2}(\mathbb{R})} + C\|W\|_{\dot{H}^{1/2}(\mathbb{R})} \sum_{h=k-n}^{+\infty} 2^{k-h}\|w\|_{\dot{H}^{1/2}(A_h)}. \quad (66)$$

Step 2: estimate of $\int_{\mathbb{R}} F(Q_1,\Delta^{1/4}u)\cdot W\,dx$, the second term on the right-hand side of (61). We write

$$F(Q_1,\Delta^{1/4}u) = F(Q_1,\chi_{k-4}\Delta^{1/4}u) + \sum_{h=k-4}^{k+1} F(Q_1,\varphi_h\Delta^{1/4}u) + \sum_{h=k+2}^{+\infty} F(Q_1,\varphi_h\Delta^{1/4}u). \quad (67)$$

By Theorem 1.7, the integral involving the first term on the right can be estimated as follows:

$$\begin{aligned}\int_{\mathbb{R}} F(Q_1,\chi_{k-4}\Delta^{1/4}u)\cdot W\,dx &\le C\|Q_1\|_{\dot{H}^{1/2}(\mathbb{R})}\|\chi_{k-4}\Delta^{1/4}u\|_{L^2}\|W\|_{\dot{H}^{1/2}(\mathbb{R})}\\ &\le C\varepsilon\|\chi_{k-4}\Delta^{1/4}u\|_{L^2}\|W\|_{\dot{H}^{1/2}(\mathbb{R})}\\ &\le \tfrac{1}{16}\|\chi_{k-4}\Delta^{1/4}u\|_{L^2}\|W\|_{\dot{H}^{1/2}(\mathbb{R})},\end{aligned} \quad (68)$$

where in the last inequality we have made use of the choice of $\varepsilon > 0$ (see beginning of proof on page 166).

We also use Theorem 1.7 for the integral involving the second term on the right-hand side of (67):

$$\sum_{h=k-4}^{k+1} \int_{\mathbb{R}} F(Q_1,\varphi_h\Delta^{1/4}u)\cdot W\,dx \le C\sum_{h=k-4}^{k+1} \|Q_1\|_{\dot{H}^{1/2}(\mathbb{R})}\|\varphi_h\Delta^{1/4}u\|_{L^2(\mathbb{R})}\|W\|_{\dot{H}^{1/2}(\mathbb{R})}. \quad (69)$$

Next we want to deal with the term in (67) involving the infinite sum. Again by Corollary A.8 we can exchange the summation with the integral and write

$$\int_{\mathbb{R}}\Big(\sum_{h=k+2}^{+\infty} F(Q_1,\varphi_h\Delta^{1/4}u)\Big)\cdot W\,dx = \sum_{h=k+2}^{+\infty}\int_{\mathbb{R}} F(Q_1,\varphi_h\Delta^{1/4}u)\cdot W\,dx.$$

If $h \ge k+2$, we have $F(Q_1,\varphi_h\Delta^{1/4}u)\cdot W = Q_1\Delta^{1/4}(\varphi_h\Delta^{1/4}u)\cdot W$, since the supports of Q_1 and φ_h are disjoint, as are the supports of χ_{k-4} and φ_h. Hence we can write

$$\begin{aligned}\sum_{h=k+2}^{+\infty}\int_{\mathbb{R}} F(Q_1,\varphi_h\Delta^{1/4}u)\cdot W\,dx &= \sum_{h=k+2}^{+\infty}\int_{\mathbb{R}} Q_1\Delta^{1/4}(\varphi_h\Delta^{1/4}u)\cdot W\,dx\\ &= \sum_{h=k+2}^{+\infty}\int_{\mathbb{R}} \mathcal{F}^{-1}(|\xi|^{1/2})(x)\big((Q_1\varphi_h\Delta^{1/4}u)\star W\big)\\ &= \sum_{h=k+2}^{+\infty} \|\mathcal{F}^{-1}(|\xi|^{1/2})\|_{L^\infty(B_{2^{h+2}}\setminus B_{2^{h-2}})}\|Q_1\varphi_h\Delta^{1/4}u\|_{L^1}\|W\|_{L^1}\\ &\le C\sum_{h=k+2}^{+\infty} 2^{-3h/2}\|Q_1\varphi_h\Delta^{1/4}u\|_{L^1}\|W\|_{L^1}.\end{aligned} \quad (70)$$

By Theorem A.5 we finally get

$$\sum_{h=k+2}^{+\infty}\int_{\mathbb{R}} F(Q_1,\varphi_h\Delta^{1/4}u)\cdot W\,dx \le C\sum_{h=k+2}^{+\infty} 2^{k-h}\|Q_1\|_{\dot H^{1/2}(\mathbb{R})}\|\,\|\varphi_h\Delta^{1/4}u\|_{L^2(\mathbb{R})}\|W\|_{\dot H^{1/2}(\mathbb{R})}$$
$$\le C\sum_{h=k+2}^{+\infty} 2^{k-h}\|\varphi_h\Delta^{1/4}u\|_{L^2}\|W\|_{\dot H^{1/2}(\mathbb{R})}.$$

Step 3: estimate of $\int_{\mathbb{R}} F(Q_2,\Delta^{1/4}u)\cdot W\,dx$, the last term in (61). As in Step 2, we write

$$F(Q_2,\Delta^{1/4}u)=F(Q_2,\chi_{k-4}\Delta^{1/4}u)+\sum_{h=k-4}^{k+1}F(Q_2,\varphi_h\Delta^{1/4}u)+\sum_{h=k+2}^{+\infty}F(Q_2,\varphi_h\Delta^{1/4}u). \tag{71}$$

For the first term, since the support of Q_2 is included in $B^c_{2^{k-1}}$, we have

$$F(Q_2,\chi_{k-4}\Delta^{1/4}u)\cdot W=\Delta^{1/4}\big(Q_2(\chi_{k-4}\Delta^{1/4}u)\big)\cdot W.$$

Observe that $Q_2=\sum_{h=k-1}^{+\infty}\varphi_h(Q_2-(\bar Q_2)_{k-1})$, $((\bar Q_2)_{k-1}=0)$ and by using Corollary A.8 we get

$$\int_{\mathbb{R}} F(Q_2,\chi_{k-4}\Delta^{1/4}u)\cdot W\,dx$$
$$=\sum_{h=k-1}^{+\infty}\int_{\mathbb{R}}\Delta^{1/4}\big((\varphi_h(Q_2-(\bar Q_2)_{k-1}))(\chi_{k-4}\Delta^{1/4}u)\big)\cdot W$$
$$\le C\sum_{h=k-1}^{+\infty}\int_{\mathbb{R}}\mathcal{F}^{-1}(|\xi|^{1/2})\big(((\chi_{k-4}\Delta^{1/4}u)\varphi_h(Q_2-(\bar Q_2)_{k-1})\big)\star W\big)$$
$$\le C\|W\|_{L^1}\sum_{h=k-1}^{+\infty}\|\mathcal{F}^{-1}(|\xi|^{1/2})\|_{L^\infty(B_{2^{h+2}}\setminus B_{2^{h-2}})}\|(\chi_{k-4}\Delta^{1/4}u)\varphi_h(Q_2-(\bar Q_2)_{k-1})\|_{L^1}$$
$$\le C\|\chi_{k-4}\Delta^{1/4}u\|_{L^2}\|W\|_{\dot H^{1/2}(\mathbb{R})}\sum_{h=k-1}^{+\infty}2^{-h/2}2^{k/2}\|\varphi_h(Q_2-(\bar Q_2)_{k-1})\|_{\dot H^{1/2}(\mathbb{R})}.$$

From Lemma 4.3, possibly by choosing a smaller k, we get

$$C\sum_{h=k-1}^{+\infty}2^{(k-h)/2}\|\varphi_h(Q_2-(\bar Q_2)_{k-1})\|_{\dot H^{1/2}(\mathbb{R})}\le \tfrac14\eta<\tfrac1{16}.$$

Therefore

$$\int_{\mathbb{R}} F(Q_2,\chi_{k-4}\Delta^{1/4}u)\cdot W\,dx\le\tfrac14\eta\|\chi_{k-4}\Delta^{1/4}u\|_{L^2}\|W\|_{\dot H^{1/2}(\mathbb{R})}.$$

Now turning to the second term in (71), we bound the corresponding integral using Theorem 1.7:

$$\sum_{h=k-4}^{k+1}\int_{\mathbb{R}} F(Q_2,\varphi_h\Delta^{1/4}u)\cdot W\,dx\le C\sum_{h=k-4}^{k+1}\|Q\|_{\dot H^{1/2}(\mathbb{R})}\|\varphi_h\Delta^{1/4}u\|_{L^2}\|W(w-\bar w_{k-4})\|_{\dot H^{1/2}(\mathbb{R})}. \tag{72}$$

Finally we consider the last term in (71). By Corollary A.8 we can write

$$\int_{\mathbb{R}}\Big(\sum_{h=k+2}^{+\infty} F(Q_2,\varphi_h\Delta^{1/4}u)\Big)\cdot W\,dx=\sum_{h=k+2}^{+\infty}\int_{\mathbb{R}} F(Q_2,\varphi_h\Delta^{1/4}u)\cdot W\,dx.$$

Next, since the support of Q_2 is included in $B^c_{2^{k-1}}$, we have for $h\ge k+2$ the equality

$$\begin{aligned}F(Q_2,\varphi_h\Delta^{1/4}u)\cdot W&=\big(\Delta^{1/4}(Q_2\,\varphi_h\Delta^{1/4}u)-Q_2\Delta^{1/4}(\varphi_h\Delta^{1/4}u)+\Delta^{1/4}Q_2\varphi_h\Delta^{1/4}u\big)\cdot W\\&=\Delta^{1/4}(Q_2\varphi_h\Delta^{1/4}u)\cdot W.\end{aligned}$$

Therefore

$$\begin{aligned}&\sum_{h=k+2}^{+\infty}\int_{\mathbb{R}} F(Q_2,\varphi_h\Delta^{1/4}u)\cdot W\,dx \qquad (73)\\
&\quad=\sum_{h=k+2}^{+\infty}\int_{\mathbb{R}}\Delta^{1/4}(Q_2\varphi_h\Delta^{1/4}u)\cdot W\,dx=\sum_{h=k+2}^{+\infty}\int_{\mathbb{R}}\mathcal{F}[\Delta^{1/4}(Q_2\varphi_h\Delta^{1/4}u)]\,\mathcal{F}[W]\,d\xi\\
&\quad=\sum_{h=k+2}^{+\infty}\int_{\mathbb{R}}|\xi|^{1/2}\mathcal{F}[(Q_2\varphi_h\Delta^{1/4}u)]\,\mathcal{F}[W]\,d\xi\\
&\quad=\sum_{h=k+2}^{+\infty}\int_{\mathbb{R}}\mathcal{F}^{-1}(|\xi|^{1/2})\left(\big(\varphi_h\Delta^{1/4}u(Q_2-(\bar Q_2)_{k-1})\big)\star W\right)\,dx\\
&\quad\le\sum_{h=k+2}^{+\infty}\|\mathcal{F}^{-1}[|\xi^{1/2}]\|_{L^\infty(B_{2^{h+2}}\setminus B_{2^{h-2}})}\,\big\|\big(\varphi_h\Delta^{1/4}u(Q_2-(\bar Q_2)_{k-1})\big)\star W\big\|_{L^1(\mathbb{R})}.\end{aligned}$$

Now choose $\psi_h\in C_0^\infty(\mathbb{R})$ such that $\psi_h\equiv 1$ in $B_{2^{h+1}}\setminus B_{2^{h-1}}$ and $\operatorname{supp}\psi\subset B_{2^{h+2}}\setminus B_{2^{h-2}}$. Thus

$$\begin{aligned}(73)&\le C\sum_{h=k+2}^{+\infty}2^{-3h/2}\,\|\psi_h(Q_2-(\bar Q_2)_{k-1})\|_{L^2(\mathbb{R})}\,\|\varphi_h\Delta^{1/4}u\|_{L^2(\mathbb{R})}\,\|W\|_{L^1(\mathbb{R})}\\
&\le C\sum_{h=k+2}^{+\infty}2^{k-h}\|\psi_h(Q_2-(\bar Q_2)_{k-1})\|_{\dot H^{1/2}(\mathbb{R})}\,\|\varphi_h\Delta^{1/4}u\|_{L^2(\mathbb{R})}\,\|W\|_{\dot H^{1/2}(\mathbb{R})} \qquad (74)\\
&\le C\Big(\sum_{h=k+2}^{+\infty}2^{k-h}\|\psi_h(Q_2-(\bar Q_2)_{k-1})\|^2_{\dot H^{1/2}(\mathbb{R})}\Big)^{1/2}\Big(\sum_{h=k+2}^{+\infty}2^{k-h}\|\varphi_h\Delta^{1/4}u\|^2_{L^2}\Big)^{1/2}\|W\|_{\dot H^{1/2}(\mathbb{R})}.\end{aligned}$$

where we have applied Theorem A.5 and Cauchy–Schwartz.

From Lemma 4.3 (with φ replaced by ψ) and Theorem A.1 we deduce that

$$\Big(\sum_{h=k+2}^{+\infty}2^{k-h}\|\psi_h(Q_2-(\bar Q_2)_{k-1})\|^2_{\dot H^{1/2}(\mathbb{R})}\Big)^{1/2}\le C\|Q\|_{\dot H^{1/2}(\mathbb{R})}.$$

Thus

$$\sum_{h=k+2}^{+\infty} \int_{\mathbb{R}} F(Q_2, \varphi_h \Delta^{1/4} u) \cdot W \, dx \le C \|W\|_{\dot H^{1/2}(\mathbb{R})} \Big(\sum_{h=k+1}^{+\infty} 2^{k-h} \|\varphi_h \Delta^{1/4} u\|^2_{L^2(\mathbb{R})} \Big)^{1/2}$$
$$\le C \|W\|_{\dot H^{1/2}(\mathbb{R})} \Big(\sum_{h=k+1}^{+\infty} 2^{(k-h)/2} \|\varphi_h \Delta^{1/4} u\|_{L^2(\mathbb{R})} \Big). \quad (75)$$

By combining (68), (69), (70), (72) and (75) we obtain (for some constant C depending on Q)

$$\int_{\mathbb{R}} F(Q, \Delta^{1/4} u) \cdot W \, dx$$
$$\le \tfrac{1}{2} \eta \|\chi_{k-4} \Delta^{1/4} u\|_{L^2} \|W\|_{\dot H^{1/2}(\mathbb{R})} + C \sum_{h=k-4}^{+\infty} 2^{(k-h)/2} \|\Delta^{1/4} u\|_{L^2(A_h)} \|W\|_{\dot H^{1/2}(\mathbb{R})}. \quad (76)$$

Finally for all $n \ge \bar n$ we have

$$\|W\|_{\dot H^{1/2}(\mathbb{R})}$$
$$\le \eta \|\chi_{k-4} \Delta^{1/4} u\|_{\dot H^{1/2}(\mathbb{R})} + C \Big(\sum_{h=k-n}^{+\infty} 2^{k-h} \|w\|_{\dot H^{1/2}(A_s)} + \sum_{h=k-4}^{+\infty} 2^{(k-h)/2} \|\Delta^{1/4} u\|_{L^2(A_h)} \Big), \quad (77)$$

concluding the proof of Lemma 4.4. □

Proof of Lemma 4.5. The proof is similar to the preceding one, so we just sketch it. As before, we fix $\eta \in \big(0, \frac{1}{4}\big)$. We consider $k < 0$ such that

$$\|\chi_k (q - \bar q_k)\|_{\dot H^{1/2}(\mathbb{R})} \le \varepsilon \quad \text{and} \quad \|\chi_k \Delta^{1/4} u\|_{L^2(\mathbb{R})} \le \varepsilon,$$

with $\varepsilon > 0$ to be determined later.

We observe that (46) is equivalent to

$$\mathcal{R} \Delta^{1/4} (p \cdot \Delta^{1/4} u) = G(q \,\cdot, \Delta^{1/4} u) - \Delta^{1/4} u \cdot (\mathcal{R} \Delta^{1/4} u). \quad (78)$$

We write

$$G(q \,\cdot, \Delta^{1/4} u) = G(q_1 \,\cdot, \Delta^{1/4} u) + G(q_2 \,\cdot, \Delta^{1/4} u),$$

where

$$q_1 = \chi_k (q - \bar q_k) \quad \text{and} \quad q_2 = (1 - \chi_k)(q - \bar q_k).$$

We observe that $\operatorname{supp} q_2 \subseteq B^c_{2^{k-1}}$ and $\|q_1\|_{\dot H^{1/2}(\mathbb{R})} \le \varepsilon$. We also set

$$u_1 = \chi_k \Delta^{1/4} u, \quad u_2 = (1 - \chi_k) \Delta^{1/4} u, \quad w = \Delta^{-1/4} (p \cdot \Delta^{1/4} u), \quad W = \chi_{k-4} (w - \bar w_{k-4}).$$

We rewrite (78) as

$$\mathcal{R} \Delta^{1/2} (W) = -\mathcal{R} \Delta^{1/2} \Big(\sum_{h=k-4}^{+\infty} \varphi_h (w - \bar w_{k-4}) \Big)$$
$$+ G(q_1 \,\cdot, \Delta^{1/4} u) + G(q_2 \,\cdot, \Delta^{1/4} u) + u_1 \cdot (\mathcal{R} \Delta^{1/4} u) + u_2 \cdot (\mathcal{R} \Delta^{1/4} u). \quad (79)$$

We multiply (79) by W and integrate over $\mathbb{R}$. By using again Corollary A.8 we get

$$\begin{aligned}\int_{\mathbb{R}} |\Delta^{1/4}(W)|^2\,dx \\ = -\sum_{h=k-4}^{+\infty}\int_{\mathbb{R}} \mathcal{R}\Delta^{1/2}(\varphi_h(w-\bar{w}_{k-4}))(W)\,dx + \int_{\mathbb{R}} G(q_1\cdot,\Delta^{1/4}u)(W)\,dx + \int_{\mathbb{R}} G(q_2\cdot,\Delta^{1/4}u)(W)\,dx \\ + \int_{\mathbb{R}} u_1\cdot(\mathcal{R}\Delta^{1/4}u)(W)\,dx + \int_{\mathbb{R}} u_2\cdot(\mathcal{R}\Delta^{1/4}u)(W)\,dx.\end{aligned} \tag{80}$$

The last term vanishes, since u_2 and χ_{k-4} have disjoint supports. Estimating $\int_{\mathbb{R}} G(Q_1,\Delta^{1/4}u)(W)\,dx$ and $\int_{\mathbb{R}} G(Q_2,\Delta^{1/4}u)(W)\,dx$ is analogous to what we did for the terms $\int_{\mathbb{R}} F(Q_1,\Delta^{1/4}u)(W)\,dx$ and $\int_{\mathbb{R}} F(Q_2,\Delta^{1/4}u)(W)\,dx$ of (61). We therefore concentrate on the other two terms in the right-hand side of (80).

To estimate the sum term, we split it into two parts: one sum for $k-4\le h\le k-3$ and one for $h\ge k-2$. For the first part we write

$$\begin{aligned}-\sum_{h=k-4}^{k-3}\int_{\mathbb{R}} \mathcal{R}\Delta^{1/2}(\varphi_h(w-\bar{w}_{k-4}))(W)\,dx \\ \le \sum_{h=k-4}^{k-3} \big\|\Delta^{1/2}(\varphi_h(w-\bar{w}_{k-4}))\big\|_{\dot{H}^{-1/2}(\mathbb{R})}\,\|(W)\|_{\dot{H}^{1/2}(\mathbb{R})} \\ \le C\sum_{h=k-4}^{k-3}\Big(\|w\|_{\dot{H}^{1/2}(A_h)} + \sum_{l=-\infty}^{k-5} 2^{l-(k-4)}\sum_{s=l+1}^{h}\|w\|_{\dot{H}^{1/2}(A_s)}\Big)\|W\|_{\dot{H}^{1/2}(\mathbb{R})},\end{aligned} \tag{81}$$

where the second inequality follows from Lemma 4.3. Let $n_1\ge 6$ be such that

$$C\sum_{h=-\infty}^{k-n_1} 2^{h-k}\|w\|_{\dot{H}^{1/2}(A_h)} \le \tfrac{1}{8}\|W\|_{\dot{H}^{1/2}(\mathbb{R})}.$$

If $n\ge n_1$ we have

$$(81) \le \tfrac{1}{8}\|W\|^2_{\dot{H}^{1/2}(\mathbb{R})} + C\|W\|_{\dot{H}^{1/2}(\mathbb{R})}\Big(\sum_{h=k-n}^{k-3} 2^{h-k}\|w\|_{\dot{H}^{1/2}(A_h)}\Big). \tag{82}$$

For the second part of the sum ($h\ge k-2$) we use the fact that $\operatorname{supp}\big(\varphi_h(w-\bar{w}_{k-4})\star W\big)$ is contained in $B_{2^{h+2}}\setminus B_{2^{h-2}}$; in particular, it does not contain 0.

$$\begin{aligned}\sum_{h=k-2}^{+\infty}\int_{\mathbb{R}} \mathcal{R}\Delta^{1/2}(\varphi_h(w-\bar{w}_{k-4}))(W)\,dx &= \sum_{h=k-2}^{+\infty}\int_{\mathbb{R}} \xi\mathcal{F}[\varphi_h(w-\bar{w}_{k-4})](\xi)\mathcal{F}[W](\xi)\,d\xi \\ &= \sum_{h=k-2}^{+\infty}\int_{\mathbb{R}} \mathcal{F}^{-1}(\xi)(x)\big(\varphi_h(w-\bar{w}_{k-4})\star W\big)\,dx \\ &= \sum_{h=k-2}^{+\infty}\int_{\mathbb{R}} \delta_0'(x)\,(\varphi_h(w-\bar{w}_{k-4})\star W)\,(x)\,dx = 0.\end{aligned}$$

Step 2: estimate of $\int_{\mathbb{R}} u_1 \cdot (\mathcal{R}\Delta^{1/4}u)(W)\,dx$. We have

$$\int_{\mathbb{R}} u_1 \cdot (\mathcal{R}\Delta^{1/4}u)(W)\,dx = \int_{\mathbb{R}} u_1 \cdot (\mathcal{R}u_1)(W)\,dx + \sum_{h=k}^{+\infty} \int_{\mathbb{R}} u_1 \cdot (\mathcal{R}\varphi_h \Delta^{1/4}u)(W)\,dx. \tag{83}$$

By applying Lemma 3.2 and using the embedding of $\mathcal{H}^1(\mathbb{R})$ into $\dot{H}^{-1/2}(\mathbb{R})$ we get

$$\begin{aligned}\int_{\mathbb{R}} u_1 \cdot (\mathcal{R}u_1)(W)\,dx &\le C\|u_1 \cdot (\mathcal{R}u_1)\|_{\mathcal{H}^1}\|(W)\|_{\dot{H}^{1/2}(\mathbb{R})} \le C\|u_1\|_{L^2}^2\|(W)\|_{\dot{H}^{1/2}(\mathbb{R})}\\ &\le C\varepsilon\|\chi_k\Delta^{1/4}u\|_{L^2}\|(W)\|_{\dot{H}^{1/2}(\mathbb{R})} \le \frac{1}{4}\varepsilon\|\chi_k\Delta^{1/4}u\|_{L^2}\|(W)\|_{\dot{H}^{1/2}(\mathbb{R})}.\end{aligned}$$

By choosing $\varepsilon > 0$ smaller if needed, we may suppose that $C\varepsilon <$.

Now we observe that for $h \ge k$ the supports of φ_h and χ_{k-4} are disjoint. Thus

$$\begin{aligned}\sum_{h=k}^{+\infty} \int_{\mathbb{R}} u_1 \cdot (\mathcal{R}\varphi_h \Delta^{1/4}u)(W)\,dx &= \sum_{h=k}^{+\infty} \int_{\mathbb{R}} \mathcal{F}^{-1}\Big[\frac{\xi}{|\xi|}\Big](x)\,\big((\varphi_h\Delta^{1/4}u) \star (u_1 W)\big)\,dx\\ &\le C\sum_{h=k}^{+\infty} \||x|^{-1}\|_{L^\infty(B_{2^{h+2}}\setminus B_{2^{h-2}})}\|(\varphi_h\Delta^{1/4}u)\star(u_1W)\|_{L^1}\\ &\le C\sum_{h=k}^{+\infty} 2^{-h}2^{h/2}2^{k/2}\|\varphi_h\Delta^{1/4}u\|_{L^2(\mathbb{R})}\|u_1\|_{L^2(\mathbb{R})}\|(W)\|_{\dot{H}^{1/2}(\mathbb{R})}\\ &\le C\varepsilon\sum_{h=k}^{+\infty} 2^{(k-2)/2}\|\varphi_h\Delta^{1/4}u\|_{L^2(\mathbb{R})}\|(W)\|_{\dot{H}^{1/2}(\mathbb{R})}\\ &\le \frac{\eta}{4}\sum_{h=k}^{+\infty} 2^{(k-2)/2}\|\varphi_h\Delta^{1/4}u\|_{L^2(\mathbb{R})}\|(W)\|_{\dot{H}^{1/2}(\mathbb{R})}.\end{aligned}$$

□

Proof of Proposition 4.1. From Lemma 4.4, there exist $C > 0$ and $\bar{n} > 0$ such that for all $n > \bar{n}$, $0 < \eta < \frac{1}{4}$, $k < k_0$ (k_0 depending on η and the $\dot{H}^{1/2}$ norm of Q), every solution to (45) satisfies (77) and thus also

$$\|W\|_{\dot{H}^{1/2}(\mathbb{R})}^2 \le \eta^2\|\chi_{k-4}\Delta^{1/4}u\|_{L^2}^2 + C\,2^{n/2}\sum_{h=k-n}^{+\infty} 2^{k-h}\|w\|_{\dot{H}^{1/2}(A_h)}^2 + C\sum_{h=k-4}^{+\infty} 2^{(k-h)/2}\|\Delta^{1/4}u\|_{L^2(\mathbb{R})}^2. \tag{84}$$

Now we can fix $n \ge \bar{n}$ and we can replace in the second term of (84) $C\,2^{n/2}$ by C.

From Lemma A.3 it follows that there are $C_1, C_2 > 0$ and $m_1 > 0$ (independent of n and k) such that if $m \ge m_1$ we have

$$\|W\|_{\dot{H}^{1/2}(\mathbb{R})}^2 \ge C_1\int_{B_{2^{k-n-m}}} |M\Delta^{1/4}u|^2\,dx - C_2\sum_{h=k-n-m}^{+\infty} 2^{k-h}\int_{B_{2^h}\setminus B_{2^{h-1}}} |M\Delta^{1/4}u|^2\,dx. \tag{85}$$

Finally from Lemma A.4 it follows that there is $C > 0$ such that for all $\gamma \in (0,1)$ there exists $m_2 > 0$ such that if $m \ge m_2$ we have

$$\begin{aligned}\sum_{h=k-n}^{+\infty} 2^{k-h}\|w\|^2_{\dot H^{1/2}(A_h)} \\ &= \sum_{h=k-n}^{+\infty} 2^{k-h}\|\Delta^{-1/4}(M\Delta^{1/4}u)\|^2_{\dot H^{1/2}(A_h)} \\ &\le \gamma\int_{|\xi|\le 2^{k-n-m}} |M\Delta^{1/4}u|^2\,dx + \sum_{h=k-n-m}^{+\infty} 2^{(k-h)/2}\int_{2^h\le|\xi|\le 2^{h+1}} |M\Delta^{1/4}u|^2\,dx.\end{aligned} \tag{86}$$

By combining (84), (85) and (86) we get

$$\begin{aligned}C_1\|M\Delta^{1/4}u\|^2_{L^2(B_{2^{k-n-m}})} \le C\sum_{h=k-n-m}^{\infty} 2^{(k-h)/2}\|M\Delta^{1/4}u\|^2_{L^2(A_h)} + C_2\sum_{h=k-n-m}^{+\infty} 2^{(k-h)/2}\|\Delta^{1/4}u\|_{L^2(A_h)} \\ +\eta^2\|\chi_{k-4}\Delta^{1/4}u\|^2_{L^2(\mathbb{R})} + C\gamma\|M\Delta^{1/4}u\|^2_{L^2(B_{2^{k-n-m}})}.\end{aligned} \tag{87}$$

Now choose $\gamma,\eta>0$ so that $C_1^{-1}C\gamma<\frac{1}{4}$ and $C_1^{-1}\eta^2<\frac{1}{4}$. With these choices we get for some constant $C>0$

$$\begin{aligned}\|M\Delta^{1/4}u\|^2_{L^2(B_{2^{k-n-m}})} - \tfrac{1}{4}\|\Delta^{1/4}u\|^2_{L^2(B_{2^{k-n-m}})} \\ \le C\Bigg(\sum_{h=k-n-m}^{\infty} 2^{(k-h)/2}\|M\Delta^{1/4}u\|^2_{L^2(A_h)} + \sum_{h=k-n-m}^{+\infty} 2^{(k-h)/2}\|\Delta^{1/4}u\|_{L^2(A_h)}\Bigg).\end{aligned} \tag{88}$$

We observe that in the final estimate (88) the index m can be fixed as well. Thus by replacing in (88) $k-n-m$ by k we get (48) and we conclude the proof. □

The proof of Proposition 4.2 is analogous and we omit it.

5. Morrey estimates and Hölder continuity of $\frac{1}{2}$-harmonic maps into the sphere

We consider the $(m-1)$-dimensional sphere $S^{m-1}\subset\mathbb{R}^m$. Let $\Pi_{S^{m-1}}$ be the orthogonal projection on S^{m-1}. We also consider the *Dirichlet energy* defined by

$$L(u) = \int_{\mathbb{R}} |\Delta^{1/4}u(x)|^2\,dx \quad \text{for } u:\mathbb{R}\to S^{m-1}. \tag{89}$$

Definition 5.1. We say that $u\in H^{1/2}(\mathbb{R},S^{m-1})$ is a weak $\frac{1}{2}$-harmonic map if

$$\frac{d}{dt}L(\Pi_{S^{m-1}}(u+t\phi))_{|t=0} = 0 \tag{90}$$

for every map $\phi\in H^{1/2}(\mathbb{R},\mathbb{R}^m)\cap L^\infty(\mathbb{R},\mathbb{R}^m)$. In other words, weak $\frac{1}{2}$-harmonic maps are the critical points of the functional (89) with respect to perturbations of the form $\Pi_{S^{m-1}}(u+t\phi)$.

We denote by $\bigwedge(\mathbb{R}^m)$ the exterior algebra (or Grassmann algebra) of $\mathbb{R}^m$. If $(e_i)_{i=1,\dots,m}$ is the canonical orthonormal basis of $\mathbb{R}^m$, every element $v\in\bigwedge_p(\mathbb{R}^m)$ can be written as $v=\sum_I v_I e_I$, where $I=\{i_1,\dots,i_p\}$ with $1\le i_1\le\cdots\le i_p\le m$, $v_I:=v_{i_1,\dots,i_p}$, and $e_I:=e_{i_1}\wedge\cdots\wedge e_{i_p}$.

By $\llcorner$ we denote the interior multiplication $\llcorner : \bigwedge_p(\mathbb{R}^m)\times\bigwedge_q(\mathbb{R}^m)\to\bigwedge_{q-p}(\mathbb{R}^m)$ defined as follows: Let $e_I = e_{i_1}\wedge\cdots\wedge e_{i_p}$, $e_J = e_{j_1}\wedge\cdots\wedge e_{j_q}$, with $q\ge p$. Then $e_I\llcorner e_J = 0$ if $I\not\subset J$; otherwise $e_I\llcorner e_J = (-1)^M e_K$, where e_K is a $(q-p)$-vector and M is the number of pairs $(i,j)\in I\times J$ with $j>i$.

By the symbol $\bullet$ we denote the first order contraction between multivectors. We recall that it satisfies $\alpha\bullet\beta = \alpha\llcorner\beta$ if β is a 1-vector and $\alpha\bullet(\beta\wedge\gamma) = (\alpha\bullet\beta)\wedge\gamma + (-1)^{pq}(\alpha\bullet\gamma)\wedge\beta$, if β and γ are respectively a p-vector and a q-vector.

Finally by the symbol $*$ we denote the Hodge star operator, $* : \bigwedge_p(\mathbb{R}^m)\to\bigwedge_{m-p}(\mathbb{R}^m)$, defined by $*\beta = (e_1\wedge\cdots\wedge e_n)\bullet\beta$.

Next we write the Euler equation associated to the functional (89).

Theorem 5.2. *All weak $\frac{1}{2}$-harmonic maps $u\in H^{1/2}(\mathbb{R},S^{m-1})$ satisfy in a weak sense the equation*

$$\int_{\mathbb{R}}(\Delta^{1/2}u)\cdot v\,dx = 0, \tag{91}$$

for every $v\in H^{1/2}(\mathbb{R},\mathbb{R}^m)\cap L^\infty(\mathbb{R},\mathbb{R}^m)$ such that $v\in T_{u(x)}S^{m-1}$ almost everywhere, or equivalently the equation

$$\Delta^{1/2}u\wedge u = 0 \quad in\ \mathcal{D}', \tag{92}$$

or yet

$$\Delta^{1/4}(u\wedge\Delta^{1/4}u) = T(Q,u) \quad in\ \mathcal{D}', \tag{93}$$

with $Q = u\wedge$.

Proof. The proof of (91) is analogous that of Lemma 1.4.10 in [Hélein 2002]. For v as in the statement, we have

$$\Pi_{S^{m-1}}(u+tv) = u + tw_t,$$

where

$$w_t = \int_0^1 \frac{\partial\Pi_{S^{m-1}}}{\partial y_j}(u+tsv)v^j\,ds.$$

Hence

$$L(\Pi_{S^{m-1}}(u+tv)) = \int_{\mathbb{R}}|\Delta^{1/4}u|^2\,dx + 2t\int_{\mathbb{R}}\Delta^{1/2}u\cdot w_t\,dx + o(t),$$

as $t\to 0$. Thus (90) is equivalent to

$$\lim_{t\to 0}\int_{\mathbb{R}}\Delta^{1/2}u\cdot w_t\,dx = 0.$$

Since $\Pi_{S^{m-1}}$ is smooth it follows that $w_t\to w_0 = d\Pi_{S^{m-1}}(u)(v)$ in $H^{1/2}(\mathbb{R},\mathbb{R}^m)\cap L^\infty(\mathbb{R},\mathbb{R}^m)$ and therefore

$$\int_{\mathbb{R}}\Delta^{1/4}u\,d\Pi_{S^{m-1}}(u)(v)\,dx = 0.$$

Since $v\in T_{u(x)}S^{m-1}$ a.e., we have $d\Pi_{S^{m-1}}(u)(v) = v$ a.e. and (91) follows.

To prove (92), we take $\varphi\in C_0^\infty(\mathbb{R},\bigwedge_{m-2}(\mathbb{R}^m))$. Then

$$\int_{\mathbb{R}}\varphi\wedge u\wedge\Delta^{1/2}u\,dx = \left(\int_{\mathbb{R}}*(\varphi\wedge u)\cdot\Delta^{1/2}u\,dx\right)e_1\wedge\cdots\wedge e_m. \tag{94}$$

We claim that

$$v = *(\varphi \wedge u) \in \dot{H}^{1/2}(\mathbb{R}, \mathbb{R}^m) \quad \text{and} \quad v(x) \in T_{u(x)} S^{m-1} \text{ a.e.}$$

That $v \in H^{1/2}(\mathbb{R}, \mathbb{R}^m) \cap L^\infty(\mathbb{R}, \mathbb{R}^m)$ follows form the fact that its components are the product of two functions in $\dot{H}^{1/2}(\mathbb{R}, \mathbb{R}^m) \cap L^\infty(\mathbb{R}, \mathbb{R}^m)$, which is an algebra. Moreover,

$$v \cdot u = *(u \wedge \varphi) \cdot u = *(u \wedge \varphi \wedge u) = 0. \tag{95}$$

It follows from (91) and (94) that

$$\int_{\mathbb{R}} \varphi \wedge u \wedge \Delta^{1/2} u \, dx = 0.$$

This shows that $\Delta^{1/2} u \wedge u = 0$ in $\mathcal{D}'$, concluding the proof of (92).

To prove (93) it is enough to observe that $\Delta^{1/2} u \wedge u = 0$ and $\Delta^{1/4} u \wedge \Delta^{1/4} u = 0$. □

Next we show that any map $u \in \dot{H}^{1/2}(\mathbb{R}, \mathbb{R}^m)$ such that $|u| = 1$ a.e. satisfies the structural equation (18).

Proof of Proposition 1.4. We observe that if $u \in \dot{H}^{1/2}(\mathbb{R}, S^{m-1})$ then Leibniz's rule holds. Thus

$$\nabla |u|^2 = 2u \cdot \nabla u \text{ in } \quad \mathcal{D}'. \tag{96}$$

Indeed, the equality (96) holds trivially if $u \in C_0^\infty(\mathbb{R}, \mathbb{R}^{m-1})$. Let $u \in \dot{H}^{1/2}(\mathbb{R}, S^{m-1})$ and let u_j be a sequence in $C_0^\infty(\mathbb{R}, \mathbb{R}^m)$ converging to u in $\dot{H}^{1/2}(\mathbb{R}, \mathbb{R}^m)$ as $j \to +\infty$. Then $\nabla u_j \to \nabla u$ as $j \to +\infty$ in $\dot{H}^{-1/2}(\mathbb{R}, \mathbb{R}^{m-1})$. Thus $u_j \cdot \nabla u_j \to u \cdot \nabla u$ in $\mathcal{D}'$ and (96) follows.

If $u \in \dot{H}^{1/2}(\mathbb{R}, S^{m-1})$, then $\nabla |u|^2 = 0$ and thus $u \cdot \nabla u = 0$ in $\mathcal{D}'$ as well. Thus u satisfies (18) and this conclude the proof. □

By combining Theorem 5.2, Proposition 1.4 and the results of the previous section we get the Hölder regularity of weak $\frac{1}{2}$-harmonic maps.

Theorem 5.3. *Let $u \in \dot{H}^{1/2}(\mathbb{R}, S^{m-1})$ be a harmonic map. Then $u \in C^{0,\alpha}_{\text{loc}}(\mathbb{R}, S^{m-1})$.*

Proof. From Theorem 5.2 it follows that u satisfies (93). Moreover, since $|u| = 1$, Proposition 1.4 implies that u satisfies (18) as well. Propositions 4.1 and 4.2 yield for $k < 0$, with $|k|$ large enough,

$$\|u \wedge \Delta^{1/4} u\|^2_{L^2(B_{2^k})} \le C \sum_{h=k}^{\infty} 2^{(k-h)/2} \|\Delta^{1/4} u\|^2_{L^2(A_h)} + \tfrac{1}{4} \|\Delta^{1/4} u\|^2_{L^2(B_{2^k})}, \tag{97}$$

$$\|u \cdot \Delta^{1/4} u\|^2_{L^2(B_{2^k})} \le C \sum_{h=k}^{\infty} 2^{(k-h)/2} \|\Delta^{1/4} u\|^2_{L^2(A_h)} + \tfrac{1}{4} \|\Delta^{1/4} u\|^2_{L^2(B_{2^k})}. \tag{98}$$

Since

$$\|\Delta^{1/4} u\|^2_{L^2(B_{2^k})} = \|u \cdot \Delta^{1/4} u\|^2_{L^2(B_{2^k})} + \|u \wedge \Delta^{1/4} u\|^2_{L^2(B_{2^k})},$$

we get

$$\|\Delta^{1/4} u\|^2_{L^2(B_{2^k})} \le C \sum_{h=k}^{\infty} 2^{(k-h)/2} \|\Delta^{1/4} u\|^2_{L^2(A_h)}. \tag{99}$$

Now observe that for some $C>0$ (independent of k) we have

$$C^{-1}\sum_{h=-\infty}^{k-1}\|\Delta^{1/4}u\|^2_{L^2(A_h)}\le\|\Delta^{1/4}u\|^2_{L^2(B_{2^k})}\le C\sum_{h=-\infty}^{k}\|\Delta^{1/4}u\|^2_{L^2(A_h)}.$$

From this and (98) it follows that

$$\sum_{h=-\infty}^{k-1}\|\Delta^{1/4}u\|^2_{L^2(A_h)}\le C\sum_{h=k}^{\infty}2^{(k-h)/2}\|\Delta^{1/4}u\|^2_{L^2(A_h)}.$$

By applying Proposition A.9 and using again (99) we get for $r>0$ small enough and some $\beta\in(0,1)$

$$\int_{B_r}|\Delta^{1/4}u|^2\,dx\le Cr^\beta. \tag{100}$$

Condition (100) yields that u belongs to the Morrey–Campanato space $\mathscr{L}^{2,-\beta}$ (see [Adams 1975], page 79), and thus $u\in C^{0,\beta/2}(\mathbb{R})$ (see [Adams 1975; Giaquinta 1983], for instance). □

Appendix

We prove here some results used in the previous sections. The first is that the $\dot H^{1.2}([a,b])$ norm, where $-\infty\le a<b\le+\infty$, can be localized in space. This result, besides being of independent interest, is used in Section 4 for localization estimates. For simplicity we will suppose that $[a,b]=[-1,1]$.

Theorem A.1 (Localization of $H^{1/2}((-1,1))$ norm). *Let $u\in H^{1/2}((-1,1))$. For some $C>0$ we have*

$$\|u\|^2_{\dot H^{1/2}((-1,1))}\simeq\sum_{j=-\infty}^{0}\|u\|^2_{\dot H^{1/2}(A_j)},$$

where $A_j=B_{2^{j+1}}\setminus B_{2^{j-1}}$.

Proof. For every $i\in\mathbb{Z}$, we set $A'_i=B_{2^i}\setminus B_{2^{i-1}}$ and $\bar u'_i=|A'_i|^{-1}\int_{A'_i}u(x)\,dx$. We have

$$\begin{aligned}\|u\|^2_{\dot H^{1/2}((-1,1))}&\simeq\int_{[-1,1]}\int_{[-1,1]}\frac{|u(x)-u(y)|^2}{|x-y|^2}\,dx\,dy\\&=\sum_{i,j=-\infty}^{0}\int_{A'_i}\int_{A'_j}\frac{|u(x)-u(y)|^2}{|x-y|^2}\,dx\,dy\\&=\sum_{i=-\infty}^{0}\int_{A'_i}\int_{A'_i}\frac{|u(x)-u(y)|^2}{|x-y|^2}\,dx\,dy+2\sum_{j=-\infty}^{0}\sum_{i>j+1}\int_{A'_i}\int_{A'_j}\frac{|u(x)-u(y)|^2}{|x-y|^2}\,dx\,dy\\&\qquad+2\sum_{j=-\infty}^{0}\int_{A'_j}\int_{A'_{j+1}}\frac{|u(x)-u(y)|^2}{|x-y|^2}\,dx\,dy.\end{aligned} \tag{101}$$

We first observe that

$$\sum_{i,j=-\infty}^{0}\int_{A'_i}\int_{A'_j}\frac{|u(x)-u(y)|^2}{|x-y|^2}\,dx\,dy\le\sum_{i,j=-\infty}^{0}\int_{A_i}\int_{A_j}\frac{|u(x)-u(y)|^2}{|x-y|^2}\,dx\,dy \tag{102}$$

and

$$\sum_{j=-\infty}^{0} \int_{A'_j} \int_{A'_{j+1}} \frac{|u(x)-u(y)|^2}{|x-y|^2}\, dx\, dy \le \sum_{j=-\infty}^{0} \int_{A_j} \int_{A_j} \frac{|u(x)-u(y)|^2}{|x-y|^2}\, dx\, dy. \tag{103}$$

It remains to estimate the double sum in (101). We have

$$\begin{aligned}
&\sum_{j=-\infty}^{0} \sum_{i>j+1} \int_{A'_i}\int_{A'_j} \frac{|u(x)-u(y)|^2}{|x-y|^2}\, dx\, dy \\
&\quad \le C \sum_{j=-\infty}^{0} \sum_{i\ge j+2} 2^{-2i} \int_{A'_i}\int_{A'_j} |u(x)-u(y)|^2\, dx\, dy \\
&\quad \le C\Big(\sum_{j=-\infty}^{0} \sum_{i\ge j+2} 2^{-2i} \int_{A'_i}\int_{A'_j} |\bar{u}'_i - \bar{u}'_j|^2\, dx\, dy + \sum_{j=-\infty}^{0} \sum_{i\ge j+2} 2^{-2i} \int_{A'_i}\int_{A'_j} |u(x)-\bar{u}'_i|^2\, dx\, dy \\
&\qquad\qquad + \sum_{j=-\infty}^{0} \sum_{i\ge j+2} 2^{-2i} \int_{A'_i}\int_{A'_j} |u(y)-\bar{u}'_j|^2\, dx\, dy\Big) \\
&\quad \le C\Big(\sum_{j=-\infty}^{0} \sum_{i\ge j+2} 2^{-2i} 2^{i+j} |\bar{u}'_i - \bar{u}'_j|^2 + \sum_{j=-\infty}^{0} \sum_{i\ge j+2} 2^{-2i} 2^{j} \int_{A'_i} |u(x)-\bar{u}'_i|^2 dx \\
&\qquad\qquad + \sum_{j=-\infty}^{0} \sum_{i\ge j+2} 2^{-2i} 2^{i} \int_{A'_j} |u(y)-\bar{u}'_j|^2 dy\Big).
\end{aligned}$$

Denote by W, W_x, W_y the three double sums in the last parentheses. We have

$$\begin{aligned}
W_x &= \sum_{i=-\infty}^{0} \sum_{j\le i-2} 2^{-2i} 2^{j} \int_{A'_i} |u(x)-\bar{u}'_i|^2 dx = \sum_{i=-\infty}^{0} 2^{-2i} \int_{A'_i} |u(x)-\bar{u}'_i|^2\, dx \Big(\sum_{j\le i-2} 2^{j}\Big) \\
&\le C \sum_{i=-\infty}^{0} |A'_i|^{-1} \int_{A'_i} |u(x)-\bar{u}'_i|^2 dx \le C \sum_{i=-\infty}^{0} \int_{A'_i}\int_{A'_i} \frac{|u(x)-u(y)|^2}{|x-y|^2}\, dx\, dy,
\end{aligned} \tag{104}$$

where in the last inequality we used the fact that, for every i,

$$\begin{aligned}
|A'_i|^{-1} \int_{A'_i} |u(x)-\bar{u}'_i|^2 dx &\le |A'_i|^{-1} \int_{A'_i} \Big| u(x) - |A'_i|^{-1} \int_{A'_i} u(y)\, dy \Big|^2 dx \\
&\le |A'_i|^{-2} \int_{A'_i}\int_{A'_i} |u(x)-u(y)|^2\, dx\, dy \le C \int_{A'_i}\int_{A'_i} \frac{|u(x)-u(y)|^2}{|x-y|^2}\, dx\, dy.
\end{aligned}$$

A similar calculation yields

$$W_y \le C \sum_{j=-\infty}^{0} \int_{A'_j}\int_{A'_j} \frac{|u(x)-u(y)|^2}{|x-y|^2}\, dx\, dy. \tag{105}$$

Finally, to estimate $W=\sum_{j=-\infty}^{0}\sum_{i\ge j+2}2^{-2i}2^{i+j}|\bar u_i'-\bar u_j'|^2$, we first observe that

$$|\bar u_i'-\bar u_j'|^2\le (i-j)\sum_{j}^{i-1}|\bar u_{l+1}'-\bar u_l'|^2 \quad\text{and}\quad |\bar u_{l+1}-\bar u_l|^2\le |A_l|^{-1}\int_{A_l}|u-\bar u_l|^2\,dx,$$

where $\bar u_l=|A_l|^{-1}\int_{A_l}u(x)\,dx$. Setting $a_l=|A_l|^{-1}\int_{A_l}|u-\bar u_l|^2\,dx$, we then have

$$W\le\sum_{j=-\infty}^{0}\sum_{i\ge j+2}(i-j)2^{j-i}\sum_{j}^{i-1}a_l\le\sum_{l=-\infty}^{0}a_l\sum_{j=-\infty}^{l}\sum_{i-j\ge l+1-j}(i-j)2^{j-i}.$$

We observe that

$$\begin{aligned}\sum_{j=-\infty}^{l}\sum_{i-j\ge l+1-j}(i-j)2^{j-i}&\le\sum_{j=-\infty}^{l}\int_{l+1-j}^{+\infty}2^{-x}x\,dx=\sum_{j=-\infty}^{l}2^{-(l+1-j)}(l+2-j)\\&\le\int_{1}^{+\infty}2^{-t}(t+1)\,dx\le C,\end{aligned}\tag{106}$$

for some constant C independent of l. It follows that

$$W\le C\sum_{l=-\infty}^{0}a_l\le C\int_{A_l}\int_{A_l}\frac{|u(x)-u(y)|^2}{|x-y|^2}\,dx\,dy.\tag{107}$$

By combining (102), (103), (104), (105), and (107) we finally obtain

$$\|u\|^2_{\dot H^{1/2}((-1,1))}\lesssim\sum_{l=-\infty}^{0}\|u\|^2_{\dot H^{1/2}(A_l)}.$$

Next we show that

$$\sum_{l=-\infty}^{0}\|u\|^2_{\dot H^{1/2}(A_l)}\lesssim\|u\|^2_{\dot H^{1/2}((-1,1))}.\tag{108}$$

For every l we have $A_l=C_l\cup D_l$, where $C_l=B_{2^{l+1}}\setminus B_{2^l}$ and $D_l=B_{2^l}\setminus B_{2^{l-1}}$. Thus

$$\begin{aligned}&\|u\|^2_{\dot H^{1/2}(A_l)}\\&=\int_{C_l}\int_{C_l}\frac{|u(x)-u(y)|^2}{|x-y|^2}\,dx\,dy+\int_{D_{l,h}}\int_{D_l}\frac{|u(x)-u(y)|^2}{|x-y|^2}\,dx\,dy+2\int_{D_{l,h}}\int_{C_l}\frac{|u(x)-u(y)|^2}{|x-y|^2}\,dx\,dy.\end{aligned}$$

Since $\bigcup_l(C_l\times C_l)$, $\bigcup_l(D_l\times C_l)$, and $\bigcup_l(D_l\times C_l)$ are disjoint unions contained in $[0,1]\times[0,1]$, we have

$$\sum_l\int_{C_l}\int_{C_l}\frac{|u(x)-u(y)|^2}{|x-y|^2}\,dx\,dy\le\int_{[-1,1]}\int_{[-1,1]}\frac{|u(x)-u(y)|^2}{|x-y|^2}\,dx\,dy,$$

$$\sum_l \int_{D_{l,h}} \int_{C_l} \frac{|u(x)-u(y)|^2}{|x-y|^2}\,dx\,dy \le \int_{[-1,1]} \int_{[-1,1]} \frac{|u(x)-u(y)|^2}{|x-y|^2}\,dx\,dy,$$
$$\sum_l \int_{D_{l,h}} \int_{D_l} \frac{|u(x)-u(y)|^2}{|x-y|^2}\,dx\,dy \le \int_{[-1,1]} \int_{[-1,1]} \frac{|u(x)-u(y)|^2}{|x-y|^2}\,dx\,dy.$$

It follows that

$$\sum_{l=-\infty}^{0} \|u\|^2_{\dot H^{1/2}(A_l)} \le \bar C \int_{[-1,1]} \int_{[-1,1]} \frac{|u(x)-u(y)|^2}{|x-y|^2}\,dx\,dy = \bar C \|u\|^2_{\dot H^{1/2}((-1,1))}. \qquad \square$$

Remark A.2. By analogous computations one can show that for all $r > 0$ we have

$$\|u\|^2_{\dot H^{1/2}(\mathbb{R})} \simeq \sum_{j=-\infty}^{+\infty} \|u\|^2_{\dot H^{1/2}(A^r_j)},$$

where $A^r_j = B_{2^{j+1}r} \setminus B_{2^{j-1}r}$, where the equivalence constants do not depend on r.

Next we compare the $\dot H^{1/2}$ norm of $\Delta^{-1/4}(M\Delta^{1/4}u)$ with the L^2 norm of $M\Delta^{1/4}u$, where $u \in \dot H^{1/2}(\mathbb{R})$ and $M \in \dot H^{1/2}(\mathbb{R}, \mathcal{M}_{t\times m}(\mathbb{R}))$, for $t \ge 1$.

In the sequel, for $\rho \ge \sigma \ge 0$, we denote by $\mathbb{1}_{|x|\le\rho}$, $\mathbb{1}_{\rho\le|x|}$, and $\mathbb{1}_{\rho\le|x|\le\sigma}$ the characteristic functions of the sets of points $x \in \mathbb{R}$ satisfying the respective inequalities.

Lemma A.3. *Let $M \in \dot H^{1/2}(\mathbb{R}, \mathcal{M}_{t\times m}(\mathbb{R}))$, with $m \ge 1$ and $t \ge 1$, and let and $u \in \dot H^{1/2}(\mathbb{R})$. There exist $C_1 > 0$, $C_2 > 0$ and $n_0 \in \mathbb{N}$, independent of u and M, such that, for any $r \in (0,1)$, $n > n_0$ and any $x_0 \in \mathbb{R}$, we have*

$$\|\Delta^{-1/4}(M\Delta^{1/4}u)\|^2_{\dot H^{1/2}(B_r(x_0))} \ge C_1 \int_{B_{r/2^n}(x_0)} |M\Delta^{1/4}u|^2\,dx - C_2 \sum_{h=-n}^{+\infty} 2^{-h} \int_{B_{2^h r}(x_0)\setminus B_{2^{h-1}r}(x_0)} |M\Delta^{1/4}u|^2\,dx.$$

Proof. For notational simplicity we take $x_0 = 0$, but the estimates made will be independent of x_0. We write

$$\Delta^{-1/4}(M\Delta^{1/4}u) = \Delta^{-1/4}(\mathbb{1}_{|x|\le r/2^n} M\Delta^{1/4}u) + \Delta^{-1/4}\big((1-\mathbb{1}_{|x|\le r/2^n})M\Delta^{1/4}u\big),$$

where $n > 0$ is large enough; the threshold will be determined later in the proof. We have

$$\begin{aligned}
&\big\|\Delta^{-1/4}(M\Delta^{1/4}u)\big\|_{\dot H^{1/2}(B_r)} \\
&\ge \big\|\Delta^{-1/4}(\mathbb{1}_{r/2^n} M\Delta^{1/4}u)\big\|_{\dot H^{1/2}(B_r)} - \big\|\Delta^{-1/4}((1-\mathbb{1}_{|x|\le r/2^n})M\Delta^{1/4}u)\big\|_{\dot H^{1/2}(B_r)} \\
&\ge \big\|\Delta^{-1/4}(\mathbb{1}_{r/2^n} M\Delta^{1/4}u)\big\|_{\dot H^{1/2}(B_r)} - \big\|\Delta^{-1/4}(\mathbb{1}_{r/2^n\le|x|\le 4r} M\Delta^{1/4}u)\big\|_{\dot H^{1/2}(B_r)} \\
&\qquad - \big\|\Delta^{-1/4}(\mathbb{1}_{|x|\ge 4r} M\Delta^{1/4}u)\big\|_{\dot H^{1/2}(B_r)} \\
&\ge \big\|\Delta^{-1/4}(\mathbb{1}_{|x|\le r/2^n} M\Delta^{1/4}u)\big\|_{\dot H^{1/2}(B_r)} - \big\|\Delta^{-1/4}(\mathbb{1}_{r/2^n\le|x|\le 4r} M\Delta^{1/4}u)\big\|_{\dot H^{1/2}(\mathbb{R})} \\
&\qquad - \big\|\Delta^{-1/4}(\mathbb{1}_{|x|\ge 4r} M\Delta^{1/4}u)\big\|_{\dot H^{1/2}(B_r)}.
\end{aligned} \tag{109}$$

We estimate the last three terms in (109).

• Estimate of $\left\|\Delta^{-1/4}(\mathbb{1}_{r/2^n\le|x|\le 4r} M\Delta^{1/4}u)\right\|_{\dot H^{1/2}(\mathbb{R})}$. This expression is equal to

$$\int_{r/2^n\le|x|\le 4r}|M\Delta^{1/4}u|^2\,dx=\sum_{h=-n}^{1}\int_{2^hr\le|x|\le 2^{h+1}r}|M\Delta^{1/4}u|^2\,dx. \tag{110}$$

• Estimate of $\left\|\Delta^{-1/4}(\mathbb{1}_{|x|\ge 4r} M\Delta^{1/4}u)\right\|_{\dot H^{1/2}(B_r)}$. Setting $g:=\mathbb{1}_{|x|\ge 4r}M\Delta^{1/4}u$, we have

$$\begin{aligned}
\|\Delta^{-1/4}g\|^2_{\dot H^{1/2}(B_r)} &= \int_{B_r}\int_{B_r}\frac{\left|(|x|^{-2}\star g)(t)-(|x|^{-2}\star g)(s)\right|^2}{|t-s|^2}\,dt\,ds\\
&=\int_{B_r}\int_{B_r}\frac{1}{|t-s|^2}\left(\int_{|x|\ge 4r}g(x)\left(|t-x|^{-1/2}-|s-x|^{-1/2}\right)dx\right)^2dt\,ds\\
\text{(mean-value thm.)}&\le C\int_{B_r}\int_{B_r}\left(\int_{|x|\ge 4r}|g(x)|\max\left(|t-x|^{-3/2},|s-x|^{-3/2}\right)dx\right)^2dt\,ds\\
&\le C\int_{B_r}\int_{B_r}\left(\sum_{h=4}^{+\infty}\int_{2^hr\le|x|\le 2^{h+1}r}|g(x)|\max\left(|t-x|^{-3/2},|s-x|^{-3/2}\right)dx\right)^2dt\,ds\\
&\le C\int_{B_r}\int_{B_r}\left(\sum_{h=4}^{+\infty}\int_{2^hr\le|x|\le 2^{h+1}r}|g(x)|\,2^{-3h/2}r^{-3/2}\,dx\right)^2dt\,ds\\
\text{(Hölder inequality)}&\le C\int_{B_r}\int_{B_r}\left(\sum_{h=4}^{+\infty}2^{-h}r^{-1}\left(\int_{2^hr\le|x|\le 2^{h+1}r}|g(x)|^2\,dx\right)^{-1/2}\right)^2dt\,ds\\
\text{(Cauchy–Schwarz)}&\le C\left(\sum_{h=4}^{+\infty}2^{-h}\right)\left(\sum_{h=4}^{+\infty}2^{-h}\int_{2^hr\le|x|\le 2^{h+1}r}|M\Delta^{1/4}u|^2\,dx\right)\\
&\le C\left(\sum_{h=4}^{+\infty}2^{-h}\int_{2^hr\le|x|\le 2^{h+1}r}|M\Delta^{1/4}u|^2\,dx\right).
\end{aligned}\tag{111}$$

• Estimate of $\|\Delta^{-1/4}(\mathbb{1}_{|x|\le r/2^n}M\Delta^{1/4}u)\|_{\dot H^{1/2}(B_r)}$. We set

$$A_h^r:=\{x:2^{h-1}r\le|x|\le 2^{h+1}r\}.$$

By the localization theorem A.1 there exists a constant $\tilde C>0$ (independent of r) such that

$$\begin{aligned}
&\|\Delta^{-1/4}(\mathbb{1}_{|x|\le r/2^n}M\Delta^{1/4}u)\|^2_{\dot H^{1/2}(\mathbb{R})}\\
&\quad\le \tilde C\sum_{h=-\infty}^{+\infty}\|\Delta^{-1/4}(\mathbb{1}_{|x|\le r/2^n}M\Delta^{1/4}u)\|^2_{\dot H^{1/2}(A_h^r)}\\
&\quad\le \tilde C\,\|\Delta^{-1/4}(\mathbb{1}_{|x|\le r/2^n}M\Delta^{1/4}u)\|^2_{\dot H^{1/2}(B_r)}+\tilde C\sum_{h=0}^{+\infty}\|\Delta^{-1/4}(\mathbb{1}_{|x|\le r/2^n}M\Delta^{1/4}u)\|^2_{\dot H^{1/2}(A_h^r)}.
\end{aligned}\tag{112}$$

• Estimate of $\sum_{h=0}^{+\infty} \|\Delta^{-1/4}(\mathbb{1}_{|x|\le r/2^n} M\Delta^{1/4}u)\|^2_{\dot H^{1/2}(A_h^r)}$. Setting

$$f(x) := \mathbb{1}_{|x|\le r/2^n} M\Delta^{1/4}u$$

and working as in the first three lines of (111), we can write for this sum the upper bound

$$\begin{aligned} C\sum_{h=0}^{+\infty}\int_{A_h^r}\int_{A_h^r}\Big(\int_{|x|\le r/2^n} |f(x)|\max\big(|t-x|^{-3/2},|s-x|^{-3/2}\big)\,dx\Big)^2 dt\ ds \\ \le C\sum_{h=0}^{+\infty}\int_{A_h^r}\int_{A_h^r}\max(|t|^{-3},|s|^{-3})\frac{r}{2^n}\Big(\int_{|x|\le r/2^n}|f(x)|^2\,dx\Big)\,dt\,ds \\ = \frac{C}{2^n}\sum_{h=0}^{+\infty}2^{-h}\Big(\int_{|x|\le r/2^n}|f(x)|^2\,dx\Big) \le \frac{C}{2^n}\int_{|x|\le r/2^n}|M\Delta^{1/4}u|^2\,dx. \end{aligned} \tag{113}$$

If n is large enough that $C\,\tilde C/2^n < \frac{1}{2}$, we get, combining (109), (110), (111), (112) and (113), for some C_1, C_2 positive,

$$\|\Delta^{-1/4}(M\Delta^{1/4}u)\|^2_{\dot H^{1/2}(B_r)} \ge C_1\int_{B_{r/2^n}}|M\Delta^{1/4}u|^2\,dx - C_2\sum_{h=-n}^{+\infty}2^{-h}\int_{B_{2^{h+1}r}\setminus B_{2^h r}}|M\Delta^{1/4}u|^2\,dx,$$

which ends the proof of the lemma. □

We now compare the $\dot H^{1/2}$ norm of $\Delta^{-1/4}(M\Delta^{1/4}u)$ in the annuli $A_h = B_{2^{h+1}}(x_0)\setminus B_{2^{h-1}}(x_0)$ with the L^2 norm in the same annuli of $M\Delta^{1/4}u$. This result, like the previous one, was used in the proof of Proposition 4.1.

Lemma A.4. *Let $M \in \dot H^{1/2}(\mathbb{R}, \mathcal{M}_{t\times m}t \ge 1(\mathbb{R}))$, $m \ge 1$, $t \ge 1$, and $u \in \dot H^{1/2}(\mathbb{R})$. There exists $C > 0$ such that for every $\gamma \in (0,1)$, for all $n \ge n_0 \in \mathbb{N}$ (n_0 dependent on γ and independent of u and M), for every $k \in \mathbb{Z}$, and any $x_0 \in \mathbb{R}$, we have*

$$\begin{aligned} \sum_{h=k}^{+\infty}2^{k-h}\|\Delta^{-1/4}(M\Delta^{1/4}u)\|^2_{\dot H^{1/2}(B_{2^{h+1}}(x_0)\setminus B_{2^{h-1}}(x_0))} \\ \le \gamma\int_{B_{2^{k-n}}(x_0)}|M\Delta^{1/4}u|^2dx + \sum_{h=k-n}^{+\infty}2^{(k-h)/2}\int_{B_{2^{h+1}}(x_0)\setminus B_{2^{h-1}}(x_0)}|M\Delta^{1/4}u|^2dx. \end{aligned}$$

Proof. Again we take $x_0 = 0$, but the estimates will be independent of x_0. Given $h \in \mathbb{Z}$ and $l \ge 3$ we set $A_h = B_{2^{h+1}}\setminus B_{2^{h-1}}$ and $D_{l,h} = B_{2^{h+l}}\setminus B_{2^{h-l}}$.

Fix $\gamma \in (0,1)$. We have, for $w = \Delta^{-1/4}(M\Delta^{1/4}u)$ and for any $l \ge 3$ (to be chosen later),

$$\begin{aligned} \|w\|^2_{\dot H^{1/2}(A_h)} = \int_{A_h}\int_{A_h}\frac{|w(x)-w(y)|^2}{|x-y|^2}\,dx\,dy \\ \le 2\,\|\Delta^{-1/4}\mathbb{1}_{D_{l,h}}M\Delta^{1/4}u\|^2_{\dot H^{1/2}(A_h)} + 2\,\|\Delta^{-1/4}(1-\mathbb{1}_{D_{l,h}})M\Delta^{1/4}u\|^2_{\dot H^{1/2}(A_h)}. \end{aligned} \tag{114}$$

The first of these two terms is bounded above by

$$\|\Delta^{-1/4}\mathbb{1}_{D_{l,h}} M\Delta^{1/4}u\|^2_{\dot H^{1/2}(\mathbb{R})} = \int_{D_{l,h}} |M\Delta^{1/4}u|^2\,dx = \sum_{s=h-l}^{h+l-1}\int_{B_{2^{s+1}}\setminus B_{2^s}} |M\Delta^{1/4}u|^2\,dx. \tag{115}$$

Multiplying by 2^{k-h} and summing up from $h=k$ to $+\infty$ we get

$$\sum_{h=k}^{+\infty} 2^{k-h}\|\Delta^{-1/4}\mathbb{1}_{D_{l,h}} M\Delta^{1/4}u\|^2_{\dot H^{1/2}(A_h)} \le C\,2^{-l}\sum_{h=k-l}^{+\infty}\int_{B_{2^{h+1}}\setminus B_{2^{h-1}}} |M\Delta^{1/4}u|^2\,dx. \tag{116}$$

To estimate the remaining term on the right-hand side of (114), set $g=(1-\mathbb{1}_{D_{l,h}})M\Delta^{1/4}u$ and write, as in the first two lines of (111),

$$\begin{aligned}
&\|\Delta^{-1/4}g\|^2_{\dot H^{1/2}(A_h)}\\
&= \int_{A_h}\int_{A_h}\frac{1}{|t-s|^2}\left(\int_{|x|<2^{h-l}\text{ or }|x|>2^{l+h}} g(x)\left(|t-x|^{-1/2}-|s-x|^{-1/2}\right)dx\right)^2 dt\,ds\\
&\le 2\int_{A_h}\int_{A_h}\frac{1}{|t-s|^2}\left(\int_{|x|>2^{l+h}}(\text{same})\right)^2 dt\,ds + 2\int_{A_h}\int_{A_h}\frac{1}{|t-s|^2}\left(\int_{|x|<2^{h-l}}(\text{same})\right)^2 dt\,ds.
\end{aligned}\tag{117}$$

For the first of these last two terms we can write, following the same steps as in (111) and using the fact that, since $l\ge 3$, we have $|x-t|,|x-s|\ge 2^{q-1}$ for every $s,t\in A_h$ and $2^q\le |x|\le 2^{q+1}$:

$$\begin{aligned}
&\int_{A_h}\int_{A_h}\frac{1}{|t-s|^2}\left(\int_{|x|>2^{l+h}} g(x)\left(|t-x|^{-1/2}-|s-x|^{-1/2}\right)dx\right)^2 dt\,ds\\
&\qquad\le C\,2^{h-l}\left(\sum_{q=h+l}^{\infty}2^{-q}\int_{2^q\le|x|\le 2^{q+1}}|g(x)|^2\,dx\right).
\end{aligned}\tag{118}$$

Multiplying the right-hand side by 2^{k-h}, where $k\in\mathbb{Z}$, taking the sum from $h=k$ to $+\infty$, interchanging the summations, and using the fact that $g(x)=M\Delta^{1/4}u(x)$ when $2^q\le|x|\le 2^{q+1}$, we get the value

$$\begin{aligned}
&C\,2^{-l}\sum_{q=k+l}^{+\infty}2^{k-q}(q-l-k)\left(\int_{2^q\le|x|\le 2^{q+1}}|M\Delta^{1/4}u|^2\,dx\right)\\
&\qquad\le C\,2^{-l}\sum_{q=k+l}^{+\infty}2^{(k-q)/2}\left(\int_{2^q\le|x|\le 2^{q+1}}|M\Delta^{1/4}u|^2\,dx\right),
\end{aligned}\tag{119}$$

which is therefore an upper bound for the contribution to $\sum_{h=k}^{+\infty}2^{k-h}\|w\|^2_{\dot H^{1/2}(A_h)}$ of the term in (117) containing the integral over $|x|>2^{l+h}$.

We still have to estimate the contribution of the term containing the integral over $|x|<2^{h-l}$. We can assume that $h\ge k$. Again following the same reasoning as in (111) and the using the inequalities $|x-s|,|x-t|\ge 2^{h-2}$ applicable to this case, we write

$$\int_{A_h}\int_{A_h} \frac{1}{|t-s|^2}\left(\int_{|x|<2^{h-l}} g(x)\left(|t-x|^{-1/2}-|s-x|^{-1/2}\right)dx\right)^2 dt\,ds$$
$$\leq C\int_{A_h}\int_{A_h} 2^{-3h}2^{h-l}\left(\int_{|x|<2^{h-l}} |g(x)|^2\,dx\right)dt\,ds = C\,2^{-l}\int_{|x|<2^{h-l}} |M\Delta^{1/4}u|^2\,dx$$
$$= C\,2^{-l}\left(\int_{|x|<2^{k-l}} |M\Delta^{1/4}u|^2\,dx + \sum_{q=k-l}^{h-l-1}\int_{2^q\leq|x|<2^{q+1}} |M\Delta^{1/4}u|^2\,dx\right). \tag{120}$$

Multiply the right-hand side of (120) by 2^{k-h}, take the sum from $h=k$ to $+\infty$, interchange the double summation, evaluate the geometric series, and rename q to h as the index of the remaining summation, to obtain the upper bound

$$C\,2^{-l+1}\int_{|x|<2^{k-l}} |M\Delta^{1/4}u|^2\,dx + C\,2^{-2l}\sum_{h=k-l}^{+\infty}\int_{2^h\leq|x|<2^{h+1}} 2^{k-h}|M\Delta^{1/4}u|^2\,dx \tag{121}$$

for the contribution to $\sum_{h=k}^{+\infty} 2^{k-h}\|w\|^2_{\dot H^{1/2}(A_h)}$ of the term under consideration (second term on the last line of (117)).

Now choose l so that $C\,2^{-l}<\gamma<1$, and set $n_0=l$. Then, for all $n\geq n_0$,

$$\sum_{h=k}^{+\infty} 2^{k-h}\left(C\,2^{-l}\int_{|x|<2^{k-l}} |M\Delta^{1/4}u|^2\,dx + C\,2^{-2l}\sum_{s=k-l}^{h-l}\int_{2^s\leq|x|\leq 2^{s+1}} |M\Delta^{1/4}u|^2\,dx\right)$$
$$\leq \gamma\int_{|x|<2^{k-n}} |M\Delta^{1/4}u|^2\,dx + \sum_{h=k-n}^{+\infty}\int_{2^h\leq|x|\leq 2^{h+1}} 2^{k-h}|M\Delta^{1/4}u|^2\,dx.$$

By combining (114), (116), (119) and (121), for $n\geq n_0$ we finally get

$$\sum_{h=k}^{+\infty} 2^{k-h}\|\Delta^{-1/4}(M\Delta^{1/4}u)\|^2_{\dot H^{1/2}(A_h)}$$
$$\leq \gamma\int_{|x|<2^{k-n}} |M\Delta^{1/4}u|^2\,dx + \sum_{h=k-n}^{+\infty}\int_{2^{h-1}\leq|x|\leq 2^{h+1}} 2^{(k-h)/2}|M\Delta^{1/4}u|^2\,dx. \qquad \square$$

Next we show a sort of Poincaré inequality for functions in $\dot H^{1/2}(\mathbb{R})$ having compact support. Recall that, for Ω an open subset of $\mathbb{R}$, the extension by 0 of a function in $H_0^{1/2}(\Omega)=\overline{C_0^\infty(\Omega)}^{H^{1/2}}$ is, generally speaking, not in $H^{1/2}(\mathbb{R})$. This is why Lions and Magenes [1972] introduced the set $H_{00}^{1/2}(\Omega)$ for which the Poincaré inequality holds.

Theorem A.5. *Let $v\in\dot H^{1/2}(\mathbb{R})$ be such that* $\operatorname{supp} v\subset(-1,1)$. *Then $v\in L^2([-1,1])$ and*

$$\int_{[-1,1]} |v(x)|^2\,dx \leq C\|v\|^2_{\dot H^{1/2}}((-2,2)).$$

Proof.

$$\int_{[-1,1]} |v(x)|^2\,dx \le C\int_{1\le|y|\le 2}\int_{|x|\le 1}\frac{|v(x)|^2}{|x-y|^2}\,dx\,dy \le C\int_{1\le|y|\le 2}\int_{|x|\le 1}\frac{|v(x)-v(y)|^2}{|x-y|^2}\,dx\,dy$$
$$\le C\int_{|y|\le 2}\int_{|x|\le 2}\frac{|v(x)-v(y)|^2}{|x-y|^2}\,dx\,dy = C\|v\|^2_{\dot H^{1/2}}([-2,2]). \qquad \square$$

From Theorem A.5 it follows that

$$\|v\|_{L^2((-r,r))} \le C r^{1/2}\|v\|_{\dot H^{1/2}(\mathbb{R})}.$$

The next three results justify the interchanging of infinite sums, pseudodifferential operators, and integrals that we performed several times to obtain the localization estimates in Section 4.

In Lemma A.6 (resp. A.7) we consider a function $g \in \dot H^{1/2}(\mathbb{R})\cap L^\infty(\mathbb{R})$ (resp. $f \in \dot H^{1/2}(\mathbb{R})\cap L^\infty(\mathbb{R})$) whose support is contained in B_{2^k} (resp. $B^c_{2^N}$). We estimate the L^2-norm of $\Delta^{1/4}g$ (resp. $\Delta^{1/4}f$) in annuli $A_h = B_{2^h}\setminus B_{2^{h-1}}$ with $h \gg k$ (resp. $h \ll N$).

Lemma A.6. *Let $g \in \dot H^{1/2}(\mathbb{R})\cap L^\infty(\mathbb{R})$ be such that* supp $g \subset B_{2^k}(\mathbb{R})$. *Then for all $h > k+3$ we have*

$$\|\Delta^{1/4}g\|_{L^2(A_h)} \le C\,2^{k-h}, \tag{122}$$

where $A_h = B_{2^h}\setminus B_{2^{h-1}}$ and C depends on $\|g\|_{\dot H^{1/2}(\mathbb{R})}$, $\|g\|_{L^\infty(\mathbb{R})}$.

Proof. We fix $h > k+3$ and let $x \in A_h$. We set $\bar g_k = |B_{2^k}|^{-1}\int_{B_{2^k}} g(x)\,dx$. We have

$$\Delta^{1/4}g(x) = \lim_{\varepsilon\to 0}\int_{|x-y|\ge\varepsilon}\frac{g(y)-g(x)}{|x-y|^{3/2}}\,dy = \lim_{\varepsilon\to 0}\int_{\substack{|x-y|\ge\varepsilon\\ y\in B_{2^k}}}\frac{g(y)-g(x)}{|x-y|^{3/2}}\,dy$$
$$\le C\,2^{-3h/2}2^k|B_{2^k}|^{-1}\int_{B_{2^k}}|g(y)-\bar g_k|\,dy + 2^{-3h/2}\int_{B_{2^k}}|g(x)-\bar g_k|\,dy$$
$$\le C\,2^{-3h/2}2^k(\|g\|_{\dot H^{1/2}(\mathbb{R})}+\|g\|_{L^\infty(\mathbb{R})}).$$

In the last inequality we used the fact that $\dot H^{1/2}(\mathbb{R}) \hookrightarrow BMO(\mathbb{R})$. It follows that

$$\int_{A_h}|\Delta^{1/4}g(x)|^2\,dx \le C\,2^{2k-2h}\big(\|g\|^2_{L^\infty(\mathbb{R})}+\|g\|^2_{\dot H^{1/2}(\mathbb{R})}\big).$$

Thus (122) holds. $\square$

Lemma A.7. *Let $f \in \dot H^{1/2}(\mathbb{R})\cap L^\infty(\mathbb{R})$ be such that* supp $f \subset B^c_{2^N}(\mathbb{R})$. *For all $h < N-3$, we have*

$$\|\Delta^{1/4}f\|_{L^2(A_h)}dx \le C\,2^{(h-N)/2}, \tag{123}$$

where C depends on $\|f\|_{\dot H^{1/2}(\mathbb{R})}$ and $\|f\|_{L^\infty}$.

Proof. Fix $h < N-3$ and $x \in A_h$. We have

$$\begin{aligned}\Delta^{1/4} f(x) &= \lim_{\varepsilon\to 0}\int_{|x-y|\geq\varepsilon}\frac{f(y)-f(x)}{|x-y|^{3/2}}\,dy\\ &= \lim_{\substack{\varepsilon\to 0\\ \varepsilon<2^{N-1}}}\left(\int_{2^{N-1}\geq|x-y|\geq\varepsilon}\frac{f(y)-f(x)}{|x-y|^{3/2}}\,dy+\int_{|x-y|\geq 2^{N-1}}\frac{f(y)-f(x)}{|x-y|^{3/2}}\,dy\right).\end{aligned}\tag{124}$$

We observe that if $|x-y| < 2^{N-2}$ and $x \in A_h$ then $|y| < 2^{N-1}$ and thus $f(y) = f(x) = 0$. Hence

$$\begin{aligned}(124) &= \int_{2^{N-2}\leq|x-y|\leq 2^N}\frac{f(y)-f(x)}{|x-y|^{3/2}}\,dy+\int_{2^N\leq|x-y|}\frac{f(y)-f(x)}{|x-y|^{3/2}}\,dy\\ &\leq C[2^{-3/2N}2^N(\|f\|_{\dot H^{1/2}(\mathbb{R})}+\|f\|_{L^\infty(\mathbb{R})})+2^{-N/2}\|f\|_{L^\infty(\mathbb{R})}]\\ &\leq C\,2^{-N/2}(\|f\|_{\dot H^{1/2}(\mathbb{R})}+\|f\|_{L^\infty(\mathbb{R})}).\end{aligned}\tag{125}$$

From (125) it follows that

$$\int_{A_h}|\Delta^{1/4} f(x)|^2\,dx \leq C\,2^{-N+h}\big(\|f\|^2_{\dot H^{1/2}(\mathbb{R})}+|f\|^2_{L^\infty(\mathbb{R})}\big)$$

and thus (123) holds. □

Corollary A.8. *Let $g \in H^{1/2}(\mathbb{R})\cap L^\infty(\mathbb{R})$ with* $\operatorname{supp} g \in B_{2^k}$, *for some $k \in \mathbb{Z}$ and for every $N > 0$ let f_N be a sequence in $H^{1/2}(\mathbb{R})\cap L^\infty(\mathbb{R})$ such that $\|f_N\|_{\dot H^{1/2}(\mathbb{R})}+\|f_N\|_{L^\infty(\mathbb{R})} \leq C$ (C independent of N) and* $\operatorname{supp} f_N \subset B^c_{2^N}$ *Then*

$$\lim_{N\to+\infty}\int_{\mathbb{R}}\Delta^{1/4} f_N(x)\Delta^{1/4} g(x)\,dx = 0.\tag{126}$$

Proof. We split the integral in (126) as follows:

$$\begin{aligned}&\int_{\mathbb{R}}\Delta^{1/4} f_N(x)\Delta^{1/4} g(x)\,dx\\ &\quad=\sum_{h=-\infty}^{k+2}\int_{A_h}\Delta^{1/4} f_N(x)\Delta^{1/4} g(x)\,dx+\sum_{h=k+3}^{N-2}\int_{A_h}\Delta^{1/4} f_N(x)\Delta^{1/4} g(x)\,dx\\ &\qquad+\sum_{h=N-1}^{+\infty}\int_{A_h}\Delta^{1/4} f_N(x)\Delta^{1/4} g(x)\,dx.\end{aligned}\tag{127}$$

We estimate the three summations in (127). We take $N \gg k$.

By applying Lemma A.7 we have

$$\sum_{h=-\infty}^{k+2} \int_{A_h} \Delta^{1/4} f_N(x) \Delta^{1/4} g(x)\, dx \le \sum_{h=-\infty}^{k+2} \Big(\int_{A_h} |\Delta^{1/4} f_N(x)|^2\, dx\Big)^{1/2} \Big(\int_{A_h} |\Delta^{1/4} g(x)|^2\, dx\Big)^{1/2}$$
$$\le C\|g\|_{\dot H^{1/2}}(\|f_N\|_{\dot H^{1/2}(\mathbb{R})} + \|f_N\|_{L^\infty(\mathbb{R})}) \sum_{h=-\infty}^{k+2} 2^{(h-N)/2}$$
$$\le C\, 2^{(k-N)/2}. \tag{128}$$

By Lemma A.6 we have

$$\sum_{h=N-1}^{+\infty} \int_{A_h} \Delta^{1/4} f(x) \Delta^{1/4} g(x)\, dx \le C\|f_N\|_{\dot H^{1/2}(\mathbb{R})}(\|g\|_{\dot H^{1/2}(\mathbb{R})} + |g\|_{L^\infty(\mathbb{R})}) \sum_{h=N-1}^{+\infty} 2^{k-h}$$
$$\le C\, 2^{k-N}. \tag{129}$$

Finally, by applying Lemmas A.6 and A.7 we get

$$\sum_{h=k+3}^{N-2} \int_{A_h} \Delta^{1/4} f_N(x) \Delta^{1/4} g(x)\, dx \le C\, 2^{k-N/2} \sum_{h=k+3}^{+\infty} 2^{-h/2} \le C\, 2^{(k-N)/2}. \tag{130}$$

By combining (127), (128) and (129) we get (126) and we can conclude. □

We conclude with the following technical result, used in the proof of Theorem 5.3.

Proposition A.9. *Let $(a_k)_k$ be a sequence of positive real numbers satisfying $\sum_{k=-\infty}^{+\infty} a_k^2 < \infty$ and*

$$\sum_{-\infty}^{n} a_k^2 \le C \sum_{k=n+1}^{+\infty} 2^{(n+1-k)/2} a_k^2 \quad \textit{for every } n \le 0. \tag{131}$$

There are $0 < \beta < 1$, $C > 0$ and $\bar n < 0$ such that for $n \le \bar n$ we have

$$\sum_{-\infty}^{n} a_k^2 \le C(2^n)^\beta.$$

Proof. For $n < 0$, we set $A_n = \sum_{-\infty}^{n} a_k^2$. We have $a_k^2 = A_k - A_{k-1}$ and thus

$$A_n \le C \sum_{k=n+1}^{+\infty} 2^{(n+1-k)/2}(A_k - A_{k-1}) \le C(1 - 1/\sqrt{2}) \sum_{k=n+1}^{+\infty} 2^{(n+1-k)/2} A_k - C A_n.$$

Therefore

$$A_n \le \tau \sum_{n+1}^{+\infty} 2^{(n+1-k)/2} A_k, \tag{132}$$

with

$$\tau = \frac{C}{C+1}\Big(1 - \frac{1}{\sqrt{2}}\Big) < 1 - \frac{1}{\sqrt{2}}.$$

The relation (132) implies the estimate

$$A_n \le \tau A_{n+1} + \tau \sum_{n+2}^{+\infty} 2^{(n+1-k)/2} A_k. \tag{133}$$

Now we apply induction on A_{n+1} in (133) and we get

$$\begin{aligned}
(133) &\le \tau^2 \Big(\sum_{n+2}^{+\infty} 2^{(n+2-k)/2} A_k \Big) + \frac{\tau}{\sqrt{2}} \Big(\sum_{n+2}^{+\infty} 2^{(n+2-k)/2} A_k \Big) \\
&= \tau(\tau + 1/\sqrt{2}) \Big(\sum_{n+2}^{+\infty} 2^{(n+2-k)/2} A_k \Big) \\
&= \tau(\tau + 1/\sqrt{2}) \Big(A_{n+2} + 1/\sqrt{2} \sum_{n+3}^{+\infty} 2^{(n+3-k)/2} A_k \Big) \\
&\le \tau(\tau + 1/\sqrt{2})^2 \sum_{n+3}^{+\infty} 2^{(n+3-k)/2} A_k \text{ (by applying induction on } A_{n+2}) \\
&\le \cdots \le \tau(\tau + 1/\sqrt{2})^{-n} \sum_{k=0}^{+\infty} 2^{-k} A_k \\
&\le \tau(\tau + 1/\sqrt{2})^{-n} \Big(\sum_{k=0}^{\infty} 2^{-k} \Big) \Big(\sum_{k=-\infty}^{+\infty} a_k^2 \Big) \\
&\le 2\tau(\tau + 1/\sqrt{2})^{-n} \sum_{k=-\infty}^{+\infty} a_k^2 \\
&\le C\gamma^{-n},
\end{aligned}$$

with $\gamma = \tau(\tau + 1/\sqrt{2})^{-n}$. Therefore for some $\beta \in (0, 1)$ and for all $n < 0$ we have $A_n \le C(2^n)^\beta$. □

Acknowledgement

The authors thank the referee for useful comments and remarks.

References

[Adams 1975] D. R. Adams, "A note on Riesz potentials", *Duke Mathematical J.* **42**:4 (1975), 765–778. MR 56 #16361 Zbl 0336.46038

[Coifman et al. 1976] R. R. Coifman, R. Rochberg, and G. Weiss, "Factorization theorems for Hardy spaces in several variables", *Ann. of Math.* (2) **103**:3 (1976), 611–635. MR 54 #843 Zbl 0326.32011

[Coifman et al. 1993] R. Coifman, P.-L. Lions, Y. Meyer, and S. Semmes, "Compensated compactness and Hardy spaces", *J. Math. Pures Appl.* (9) **72**:3 (1993), 247–286. MR 95d:46033 Zbl 0864.42009

[Giaquinta 1983] M. Giaquinta, *Multiple integrals in the calculus of variations and nonlinear elliptic systems*, Annals of Math. Studies **105**, Princeton Univ. Press, 1983. MR 86b:49003 Zbl 0516.49003

[Grafakos 2009] L. Grafakos, *Modern Fourier analysis*, 2nd ed., Graduate Texts in Mathematics **250**, Springer, New York, 2009. MR 2463316 Zbl 1158.42001

[Hélein 2002] F. Hélein, *Harmonic maps, conservation laws and moving frames*, 2nd ed., Cambridge Tracts in Mathematics **150**, Cambridge University Press, Cambridge, 2002. MR 2003g:58024 Zbl 1010.58010

[Hildebrandt and Widman 1975] S. Hildebrandt and K.-O. Widman, "Some regularity results for quasilinear elliptic systems of second order", *Math. Z.* **142** (1975), 67–86. MR 51 #13446 Zbl 0317.35040

[Lio and Riviere 2010] F. D. Lio and T. Riviere, "Analysis of solutions to critical non-local Schrödinger equation and applications", 2010.

[Lions and Magenes 1972] J.-L. Lions and E. Magenes, *Non-homogeneous boundary value problems and applications, I*, Grundlehren der Math. Wiss. **181**, Springer, New York, 1972. MR 50 #2670 Zbl 0223.35039

[Morrey 1966] C. B. Morrey, *Multiple integrals in the calculus of variations*, Grundlehren der Math. Wiss. **130**, Springer, New York, 1966. MR 34 #2380 Zbl 0142.38701

[Rivière 2007] T. Rivière, "Conservation laws for conformally invariant variational problems", *Invent. Math.* **168**:1 (2007), 1–22. MR 2008d:58010 Zbl 1128.58010

[Rivière 2008] T. Rivière, "The role of integrability by compensation in conformal geometric analysis", 2008, Available at http://www.math.ethz.ch/~riviere/papers/integrabcomp.pdf. To appear in *Analytic aspects of problems from riemannian geometry*, Soc. Math. de France.

[Runst and Sickel 1996] T. Runst and W. Sickel, *Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations*, Series in Nonlinear Analysis and Applications **3**, de Gruyter., Berlin, 1996. MR 98a:47071 Zbl 0873.35001

[Shatah 1988] J. Shatah, "Weak solutions and development of singularities of the SU(2) σ-model", *Comm. Pure Appl. Math.* **41**:4 (1988), 459–469. MR 89f:58044 Zbl 0686.35081

[Sickel and Triebel 1995] W. Sickel and H. Triebel, "Hölder inequalities and sharp embeddings in function spaces of B^s_{pq} and F^s_{pq} type", *Z. Anal. Anwendungen* **14**:1 (1995), 105–140. MR 96h:46042

[Stein 1970] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Math. Series **30**, Princeton Univ. Press, 1970. MR 44 #7280 Zbl 0207.13501

[Tartar 2007] L. Tartar, *An introduction to Sobolev spaces and interpolation spaces*, Lecture Notes of the Unione Matematica Italiana **3**, Springer, Berlin, and UMI, Bologna, 2007. MR 2008g:46055 Zbl 1126.46001

[Taylor 1991] M. E. Taylor, *Pseudodifferential operators and nonlinear PDE*, Progress in Mathematics **100**, Birkhäuser, Boston, 1991. MR 92j:35193 Zbl 0746.35062

[Triebel 1983] H. Triebel, *Theory of function spaces*, Monographs in Mathematics **78**, Birkhäuser, Basel, 1983. MR 86j:46026 Zbl 0546.46027

[Wente 1969] H. C. Wente, "An existence theorem for surfaces of constant mean curvature", *J. Math. Anal. Appl.* **26** (1969), 318–344. MR 39 #4788 Zbl 0181.11501

Received 23 Jul 2009. Accepted 10 Mar 2010.

FRANCESCA DA LIO: dalio@math.unipd.it
Dipartimento di Matematica Pura ed Applicata, Università degli Studi di Padova, Via Trieste 63, 35121 Padova, Italy

and

Department of Mathematics, ETH Zürich, Rämistrasse 101, 8092 Zürich, Switzerland

TRISTAN RIVIÈRE: riviere@math.ethz.ch
Department of Mathematics, ETH Zürich, Rämistrasse 101, 8092 Zürich, Switzerland
http://www.math.ethz.ch/~riviere/

Guidelines for Authors

Authors may submit manuscripts in PDF format on-line at the Submission page at pjm.math.berkeley.edu/apde.

Originality. Submission of a manuscript acknowledges that the manuscript is original and and is not, in whole or in part, published or under consideration for publication elsewhere. It is understood also that the manuscript will not be submitted elsewhere while under consideration for publication in this journal.

Language. Articles in APDE are usually in English, but articles written in other languages are welcome.

Required items. A brief abstract of about 150 words or less must be included. It should be self-contained and not make any reference to the bibliography. If the article is not in English, two versions of the abstract must be included, one in the language of the article and one in English. Also required are keywords and subject classifications for the article, and, for each author, postal address, affiliation (if appropriate), and email address.

Format. Authors are encouraged to use LaTeX but submissions in other varieties of TeX, and exceptionally in other formats, are acceptable. Initial uploads should be in PDF format; after the refereeing process we will ask you to submit all source material.

References. Bibliographical references should be complete, including article titles and page ranges. All references in the bibliography should be cited in the text. The use of BibTeX is preferred but not required. Tags will be converted to the house format, however, for submission you may use the format of your choice. Links will be provided to all literature with known web locations and authors are encouraged to provide their own links in addition to those supplied in the editorial process.

Figures. Figures must be of publication quality. After acceptance, you will need to submit the original source files in vector graphics format for all diagrams in your manuscript: vector EPS or vector PDF files are the most useful.

Most drawing and graphing packages (Mathematica, Adobe Illustrator, Corel Draw, MATLAB, etc.) allow the user to save files in one of these formats. Make sure that what you are saving is vector graphics and not a bitmap. If you need help, please write to graphics@mathscipub.org with details about how your graphics were generated.

White space. Forced line breaks or page breaks should not be inserted in the document. There is no point in your trying to optimize line and page breaks in the original manuscript. The manuscript will be reformatted to use the journal's preferred fonts and layout.

Proofs. Page proofs will be made available to authors (or to the designated corresponding author) at a Web site in PDF format. Failure to acknowledge the receipt of proofs or to return corrections within the requested deadline may cause publication to be postponed.

ANALYSIS & PDE

Volume 4 No. 1 2011