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THE CORONA THEOREM FOR THE DRURY–ARVESON HARDY SPACE AND OTHER HOLOMORPHIC BESOV–SOBOLEV SPACES ON THE UNIT BALL IN $\mathbb{C}^n$

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We prove that the multiplier algebra of the Drury–Arveson Hardy space H_n^2 on the unit ball in $\mathbb{C}^n$ has no corona in its maximal ideal space, thus generalizing the corona theorem of L. Carleson to higher dimensions. This result is obtained as a corollary of the Toeplitz corona theorem and a new Banach space result: the Besov–Sobolev space B_p^σ has the "baby corona property" for all $\sigma \geq 0$ and $1 < p < \infty$. In addition we obtain infinite generator and semi-infinite matrix versions of these theorems.

1. Introduction

Lennart Carleson [1962] demonstrated the absence of a corona in the maximal ideal space of $H^\infty(\mathbb{D})$ by showing that if $\{g_j\}_{j=1}^N$ is a finite set of functions in $H^\infty(\mathbb{D})$ satisfying

$$\sum_{j=1}^N |g_j(z)| \geq c > 0, \quad z \in \mathbb{D}, \tag{1-1}$$

then there are functions $\{f_j\}_{j=1}^N$ in $H^\infty(\mathbb{D})$ with

$$\sum_{j=1}^N f_j(z) g_j(z) = 1, \quad z \in \mathbb{D}, \tag{1-2}$$

Fuhrmann [1968] extended Carleson's corona theorem to the finite matrix case. Rosenblum [1980] and Tolokonnikov [1980] proved the corona theorem for infinitely many generators $N = \infty$. This was further generalized to the one-sided infinite matrix setting by Vasyunin (see [Tolokonnikov 1981]). Finally Treil

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[1988] showed that the generalizations stop there by producing a counterexample to the two-sided infinite matrix case.

Hörmander noted a connection between the corona problem and the Koszul complex, and in the late 1970s Tom Wolff gave a simplified proof using the theory of the $\bar{\partial}$ equation and Green's theorem (see [Garnett 1981]). This proof has since served as a model for proving corona type theorems for other Banach algebras.

While there is a large literature on corona theorems in one complex dimension (see [Nikolski 2002], for example), progress in higher dimensions has been limited. Indeed, apart from the simple cases in which the maximal ideal space of the algebra can be identified with a compact subset of $\mathbb{C}^n$, no corona theorem has been proved until now in higher dimensions. Instead, partial results have been obtained, such as the beautiful Toeplitz corona theorem for Hilbert function spaces with a complete Nevanlinna–Pick kernel, the H^p corona theorem on the ball and polydisk, and results restricting N to 2 generators in (1-1) (the case $N = 1$ is trivial). In particular, N. Varopoulos [1977] published a lengthy classic paper in an unsuccessful attempt to prove the corona theorem for the multiplier algebra $H^\infty(\mathbb{B}_n)$ of the classical Hardy space $H^2(\mathbb{B}_n)$ of holomorphic functions on the ball with square integrable boundary values. His BMO estimates for solutions with $N = 2$ generators remain largely unimproved to this day (though see [Costea et al. 2010] for the extension to an infinite number of generators). A related result for $N = 2$ and $H^p(\mathbb{B}_n)$ with $1 < p < \infty$ was studied in [Amar 1991]. The case $N = 2$ is easier compared to $N > 2$ because of certain algebraic simplifications that arise. We will discuss these partial results in more detail below.

In many ways H_n^2, and not the more familiar space $H^2(\mathbb{B}_n)$, is the natural generalization to higher dimensions of the classical Hardy space on the disk. For example, H_n^2 is universal among Hilbert function spaces with the complete Pick property, and its multiplier algebra $M_{H_n^2}$ is the correct home for the multivariate von Neumann inequality (see [Agler and McCarthy 2002], for instance). See [Arveson 1998] for more on the space H_n^2, including the model theory of n-contractions. Because of the connections the space H_n^2 has with operator theory, there is current interest in understanding the related function theory of this space.

Our main result is that the corona theorem, namely the absence of a corona in the maximal ideal space, holds for the multiplier algebra $M_{H_n^2}$ of the Hilbert space H_n^2, the celebrated Drury–Arveson Hardy space on the ball in n dimensions. This result provides yet more evidence that the space H_n^2 is the "correct" generalization to several variables.

Theorem 1. *If $\{g_j\}_{j=1}^N$ is a finite set of functions in $M_{H_n^2}$ satisfying*

$$1 \geq \sum_{j=1}^{N} |g_j(z)|^2 \geq \delta^2 > 0 \quad \text{for all } z \in \mathbb{B}_n,$$

then there are functions $\{f_j\}_{j=1}^N$ in $M_{H_n^2}$ satisfying

(i) $\sum_{j=1}^N f_j(z) g_j(z) = 1, \quad z \in \mathbb{B}_n$;

(ii) $\|f_j\|_{M_{H_n^2}} \leq C_{n,\delta,N}$ *for all* $j = 1, \dots, N$.

There is a close relationship between the corona problem as stated and a related "baby" corona problem. In the case when $p = 2$, thanks to the Toeplitz corona Theorem, as explained in the next section, this connection in fact becomes an equivalence in certain situations and an application of the Toeplitz corona Theorem then will give the statement in Theorem 1. Because of this close connection between the corona problem and the "baby" corona problem, in this paper we will actually prove that the Besov–Sobolev space B_p^σ has the "baby corona property" for all $\sigma \geq 0$ and $1 < p < \infty$. The precise statements for the "baby corona property" are given below in Theorem 2. In addition, when formulated appropriately, we will obtain infinite generator and semi-infinite matrix versions of these results, see Corollary 4.

More generally, Theorem 1 holds for the multiplier algebras $M_{B_2^\sigma(\mathbb{B}_n)}$ of the Besov–Sobolev spaces $B_2^\sigma(\mathbb{B}_n)$, $0 \leq \sigma \leq \frac{1}{2}$, on the unit ball $\mathbb{B}_n$ in $\mathbb{C}^n$. These function spaces will be defined later, but the space $B_2^\sigma(\mathbb{B}_n)$ consists roughly of those holomorphic functions f whose derivatives of order $\frac{n}{2} - \sigma$ lie in the classical Hardy space $H^2(\mathbb{B}_n) = B_2^{n/2}(\mathbb{B}_n)$. In particular $H_n^2 = B_2^{1/2}(\mathbb{B}_n)$. Again, we will study these more general corona problems by studying the easier "baby" corona problem.

1.1. *The Toeplitz corona problem in $\mathbb{C}^n$.* In this section we connect the corona problem to the "baby" corona problem, and then formulate the analogous "baby" corona problems in the Besov–Sobolev spaces B_p^σ when $1 < p < \infty$ and $0 \leq \sigma < \infty$.

Let X be a Hilbert space of holomorphic functions in an open set Ω in $\mathbb{C}^n$ that is a reproducing kernel Hilbert space with a *complete irreducible Nevanlinna–Pick kernel* (see [Agler and McCarthy 2002] for the definition). The following *Toeplitz corona theorem* is due to Ball, Trent and Vinnikov [Ball et al. 2001] (see also [Ambrozie and Timotin 2002; Agler and McCarthy 2002, Theorem 8.57]).

For $f = (f_\alpha)_{\alpha=1}^N \in \bigoplus^N X$ and $h \in X$, define

$$\mathbb{M}_f h = (f_\alpha h)_{\alpha=1}^N \quad \text{and} \quad \|f\|_{\text{Mult}(X,\oplus^N X)} = \|\mathbb{M}_f\|_{X \to \oplus^N X} = \sup_{\|h\|_X \leq 1} \|\mathbb{M}_f h\|_{\oplus^N X}.$$

Note that

$$\max_{1 \leq \alpha \leq N} \|\mathcal{M}_{f_\alpha}\|_{M_X} \leq \|f\|_{\text{Mult}(X,\oplus^N X)} \leq \sqrt{\textstyle\sum_{\alpha=1}^N \|\mathcal{M}_{f_\alpha}\|_{M_X}^2}.$$

Toeplitz corona theorem. *Let X be a Hilbert function space in an open set Ω in $\mathbb{C}^n$ with an irreducible complete Nevanlinna–Pick kernel. Let $\delta > 0$ and $N \in \mathbb{N}$. For $g_1, \dots, g_N \in M_X$, there is equivalence between*:

- (*"baby corona property"*) *For every $h \in X$, there are $f_1, \dots, f_N \in X$ such that*

$$\|f_1\|_X^2 + \cdots + \|f_N\|_X^2 \leq \frac{1}{\delta}\|h\|_X^2, \qquad g_1(z) f_1(z) + \cdots + g_N(z) f_N(z) = h(z) \quad \textit{for } z \in \Omega. \tag{1-3}$$

- (*"multiplier corona property"*) *There are $\varphi_1, \dots, \varphi_N \in M_X$ such that*

$$\|\varphi\|_{\text{Mult}(X,\oplus^N X)} \leq 1, \qquad g_1(z)\varphi_1(z) + \cdots + g_N(z)\varphi_N(z) = \sqrt{\delta} \quad \textit{for } z \in \Omega. \tag{1-4}$$

The *baby corona theorem* is said to hold for X if, whenever $g_1, \dots, g_N \in M_X$ satisfy

$$|g_1(z)|^2 + \cdots + |g_N(z)|^2 \geq c > 0 \quad \text{for } z \in \Omega, \tag{1-5}$$

then $g_1, \dots, g_N$ satisfy the baby corona property (1-3). The Toeplitz corona theorem thus provides a useful tool for reducing the multiplier corona property (1-4) to the more tractable, but still very difficult, baby corona property (1-3) for multiplier algebras $M_{B_p^\sigma(\mathbb{B}_n)}$ of certain of the Besov–Sobolev spaces $B_p^\sigma(\mathbb{B}_n)$ when $p = 2$: see below. The case of $M_{B_p^\sigma(\mathbb{B}_n)}$ when $p \neq 2$ must be handled by more classical methods and remains largely unsolved.

Remark. A standard abstract argument applies to show that the absence of a corona for the multiplier algebra M_X, i.e., the density of the linear span of point evaluations in the maximal ideal space of M_X, is equivalent to the following assertion: for each finite set $\{g_j\}_{j=1}^N \subset M_X$ such that (1-5) holds for some $c > 0$, there are $\{\varphi_j\}_{j=1}^N \subset M_X$ and $\delta > 0$ such that condition (1-4) holds. See for example Lemma 9.2.6 in [Nikolski 2002] or the proof of Criterion 3.5 on page 39 of [Sawyer 2009].

1.2. *The baby corona theorem.*

Notation. For sequences $f(z) = (f_i(z))_{i=1}^\infty \in \ell^2$ we will write

$$|f(z)| = \sqrt{\textstyle\sum_{i=1}^\infty |f_i(z)|^2}.$$

When considering sequences of vectors such as $\nabla^m f(z) = (\nabla^m f_i(z))_{i=1}^\infty$, the same notation $|\nabla^m f(z)| = \sqrt{\sum_{i=1}^\infty |\nabla^m f_i(z)|^2}$ will be used with $|\nabla^m f_i(z)|$ denoting the Euclidean length of the vector $\nabla^m f_i(z)$. Thus the symbol $|\cdot|$ is used in at least three different ways: to denote the absolute value of a complex number, the length of a finite vector in $\mathbb{C}^N$ and the norm of a sequence in ℓ^2. Later it will also be used to denote the Hilbert–Schmidt norm of a tensor, namely the square root of the sum of the squares of the coefficients in the standard basis. In all cases the meaning should be clear from the context.

Recall that $B_p^\sigma(\mathbb{B}_n; \ell^2)$ consists of all $f = (f_i)_{i=1}^\infty \in H(\mathbb{B}_n; \ell^2)$ such that

$$\|f\|_{B_p^\sigma(\mathbb{B}_n;\ell^2)} \equiv \sum_{k=0}^{m-1} |\nabla^k f(0)| + \left(\int_{\mathbb{B}_n} \left| (1-|z|^2)^{m+\sigma} \nabla^m f(z) \right|^p d\lambda_n(z) \right)^{1/p} < \infty, \tag{1-6}$$

for some $m > \frac{n}{p} - \sigma$. By a result in [Beatrous 1986] the right side is finite for some $m > \frac{n}{p} - \sigma$ if and only if it is finite for all $m > \frac{n}{p} - \sigma$. As usual we will write $B_p^\sigma(\mathbb{B}_n)$ for the scalar-valued space.

We now state our baby corona theorem for the ℓ^2-valued Banach spaces $B_p^\sigma(\mathbb{B}_n; \ell^2)$, $\sigma \geq 0$, $1 < p < \infty$. Observe that for $\sigma < 0$, $M_{B_p^\sigma(\mathbb{B}_n)} = B_p^\sigma(\mathbb{B}_n)$ is a subalgebra of $C(\overline{\mathbb{B}}_n)$ and so has no corona. The $N = 2$ generator case of Theorem 2 when $\sigma \in \left[0, \frac{1}{p}\right) \cup \left(\frac{n}{p}, \infty\right)$ and $1 < p < \infty$ is due to Ortega and Fàbrega [2000], who also obtained the $N = 2$ generator case when $\sigma = \frac{n}{p}$ and $1 < p \leq 2$. See Theorem A in that reference. Ortega and Fàbrega [2006] prove analogous results with scalar-valued Hardy–Sobolev spaces in place of the Besov–Sobolev spaces.

Let

$$\|\mathbb{M}_g\|_{B_p^\sigma(\mathbb{B}_n) \to B_p^\sigma(\mathbb{B}_n;\ell^2)}$$

denote the norm of the multiplication operator $\mathbb{M}_g$ from $B_p^\sigma(\mathbb{B}_n)$ to the ℓ^2-valued Besov–Sobolev space $B_p^\sigma(\mathbb{B}_n; \ell^2)$.

Theorem 2. *Let $\delta > 0$, $\sigma \geq 0$ and $1 < p < \infty$. There is a constant $C_{n,\sigma,p,\delta}$ such that, given a sequence $g = (g_i)_{i=1}^{\infty} \in M_{B_p^\sigma(\mathbb{B}_n) \to B_p^\sigma(\mathbb{B}_n;\ell^2)}$ satisfying*

$$\|\mathbb{M}_g\|_{B_p^\sigma(\mathbb{B}_n) \to B_p^\sigma(\mathbb{B}_n;\ell^2)} \leq 1, \qquad \sum_{j=1}^{\infty} |g_j(z)|^2 \geq \delta^2 > 0 \quad \textit{for } z \in \mathbb{B}_n,$$

there is for each $h \in B_p^\sigma(\mathbb{B}_n)$ a vector-valued function $f \in B_p^\sigma(\mathbb{B}_n;\ell^2)$ satisfying

$$\|f\|_{B_p^\sigma(\mathbb{B}_n;\ell^2)} \leq C_{n,\sigma,p,\delta} \|h\|_{B_p^\sigma(\mathbb{B}_n)}, \qquad \sum_{j=1}^{\infty} g_j(z) f_j(z) = h(z) \quad \textit{for } z \in \mathbb{B}_n. \tag{1-7}$$

Corollary 3. *Let $0 \leq \sigma \leq \frac{1}{2}$. Then the Banach algebra $M_{B_2^\sigma(\mathbb{B}_n)}$ has no corona; that is, the analogue of Theorem 1 holds. As particular cases we obtain that the multiplier algebra of the Drury–Arveson space $H_n^2 = B_2^{1/2}(\mathbb{B}_n)$ has no corona (Theorem 1) and that the multiplier algebra of the n-dimensional Dirichlet space $\mathfrak{D}(\mathbb{B}_n) = B_2^0(\mathbb{B}_n)$ has no corona.*

The corollary follows immediately from the finite generator case $p = 2$ of Theorem 2 and the Toeplitz corona theorem (and the remark on page 502) since the spaces $B_2^\sigma(\mathbb{B}_n)$ have an irreducible complete Nevanlinna–Pick kernel when $0 \leq \sigma \leq \frac{1}{2}$; see for example [Arcozzi et al. 2008].

Note that in dimension $n = 1$ and $\sigma = \frac{1}{2}$, Corollary 3 gives a new proof of Carleson's classical corona theorem, similar to that in [Andersson and Carlsson 2001]. Of course it is the Toeplitz corona theorem that yields the difficult L^∞ estimate there. Additionally, when $n = 1$ and $\sigma = 0$, we have that the multiplier algebra of the Dirichlet space has no corona, recovering a result from [Tolokonnikov 1991]. See also [Xiao 1998] for the case of $n = 1$ and $0 \leq \sigma < \frac{1}{2}$.

We also have a semi-infinite matricial corona theorem.

Corollary 4. *Let $0 \leq \sigma \leq \frac{1}{2}$. Let $\mathcal{H}_1$ be a finite m-dimensional Hilbert space and let $\mathcal{H}_2$ be an infinite-dimensional separable Hilbert space. Suppose that $F \in \mathcal{M}_{B_2^\sigma(\mathbb{B}_n)(\mathcal{H}_1 \to \mathcal{H}_2)}$ satisfies*

$$\delta^2 I_m \leq F^*(z) F(z) \leq I_m.$$

Then there is $G \in \mathcal{M}_{B_2^\sigma(\mathbb{B}_n)(\mathcal{H}_2 \to \mathcal{H}_1)}$ such that

$$G(z) F(z) = I_m, \qquad \|G\|_{\mathcal{M}_{B_2^\sigma(\mathbb{B}_n)(\mathcal{H}_2 \to \mathcal{H}_1)}} \leq C_{\sigma,n,\delta,m}.$$

This corollary follows immediately from the case $p = 2$ of Theorem 2 and the Toeplitz corona theorem together with Theorem (MCT) in [Trent and Zhang 2006]. We follow the notation in that reference. We already commented above on the special case of this corollary for the Hardy space $B_2^{1/2}(\mathbb{B}_1) = H^2(\mathbb{D})$ on the disk. The case $m = 1$ of this corollary for the classical Dirichlet space $B_2^0(\mathbb{B}_1) = \mathfrak{D}(\mathbb{D})$ on the disk is due to Trent [2004a]. Our method yields information about the dependence of the constants on the parameters δ, σ, p and n in Theorem 2. However, this information is not sharp and more precise information would be desirable.

Remark. It is an open question [Trent 2004a] for the Dirichlet space $B_2^0(\mathbb{B}_1)$ in one dimension whether or not in Theorem 2 the boundedness condition on the column operator, $\|\mathbb{M}_g\|_{B_2^0(\mathbb{B}_1) \to B_2^0(\mathbb{B}_1;\ell^2)} \leq 1$, can be replaced by a similar (but weaker — see Lemma 1 in [Trent 2004a]) boundedness condition for the row operator, $\|\mathcal{M}_g\|_{B_2^0(\mathbb{B}_1;\ell^2)\to B_2^0(\mathbb{B}_1)} \le 1$. The question also appears to be open for the Besov–Sobolev spaces $B_2^\sigma(\mathbb{B}_n)$, with $0 \le \sigma < \frac{n}{2}$. (The two operators are not dual to one another for these spaces.)

Prior results. The baby corona problem for $H^2(\mathbb{B}_n)$ was first formulated and proved via L^2 methods by Mats Andersson [1994b]. It is noteworthy that the approach used in that work allowed for one to obtain estimates independent of the number of generators N. The case of two generators in $H^p(\mathbb{B}_n)$, $1 < p < \infty$, was handled by Éric Amar [1991]. His proof could be extended to handle more generators but doing so will result in a constant that depends upon the number of generators N. Andersson and Carlsson [2000] solved the baby corona problem for $H^2(\mathbb{B}_n)$ and obtained the analogous (baby) H^p corona theorem on the ball $\mathbb{B}_n$ for $1 < p < \infty$ and with constants independent of the number of generators and sharp information in terms of the estimates in terms of δ and the dimension n. The interested reader can also see [Andersson 1994a; Andersson and Carlsson 2001; 1994; Krantz and Li 1995], where the problem is studied.

Partial results on the corona problem restricted to $N = 2$ generators and BMO in place of L^∞ estimates have been obtained for $H^\infty(\mathbb{B}_n)$ (the multiplier algebra of $H^2(\mathbb{B}_n) = B_2^{n/2}(\mathbb{B}_n)$) by Varopoulos [1977]. Note that the techniques used in this paper also yield BMO estimates for the $H^\infty(\mathbb{B}_n)$ corona problem, which appear in [Costea et al. 2010]. This classical corona problem remains open (Problem 19.3.7 in [Rudin 1980]), along with the corona problems for the multiplier algebras of $B_2^\sigma(\mathbb{B}_n)$, $\frac{1}{2} < \sigma < \frac{n}{2}$.

More recently, J. M. Ortega and J. Fàbrega [2000] obtained partial results with $N = 2$ generators in (1-3) for the algebras $M_{B_2^\sigma(\mathbb{B}_n)}$ when $0 \le \sigma < \frac{1}{2}$, i.e., from the Dirichlet space $B_2^0(\mathbb{B}_n)$ up to but not including the Drury–Arveson Hardy space $H_n^2 = B_2^{1/2}(\mathbb{B}_n)$. To handle $N = 2$ generators they exploit the fact that a 2×2 antisymmetric matrix consists of just one entry up to sign, so that as a consequence the form Ω_1^2 in the Koszul complex below is $\bar\partial$-closed. Ortega and Fàbrega's paper has proved to be of enormous influence in our work, as the basic groundwork and approach we use are set out there.

In [Treil and Wick 2005] the H^p corona theorem on the polydisk $\mathbb{D}^n$ is obtained (see also [Lin 1994; Trent 2004b]). The Hardy space $H^2(\mathbb{D}^n)$ on the polydisk fails to have the complete Nevanlinna–Pick property, and consequently the Toeplitz corona theorem only holds in a more complicated sense that a family of kernels must be checked for positivity instead of just one (see [Amar 2003; Trent and Wick 2009]). As a result the corona theorem for the algebra $H^\infty(\mathbb{D}^n)$ on the polydisk remains open for $n \ge 2$. Finally, even the baby corona problems, apart from that for H^p, remain open on the polydisk.

1.3. ***Plan of the paper.*** We will prove Theorem 2 using the Koszul complex and a factorization of Andersson and Carlsson, an explicit calculation of Charpentier's solution operators, and generalizations of the integration by parts formulas of Ortega and Fàbrega, together with new estimates for boundedness of operators on certain real-variable analogues of the holomorphic Besov–Sobolev spaces.

More precisely, to treat $N > 2$ generators in (1-7), it is just as easy to treat the case $N = \infty$, and this has the advantage of not requiring bookkeeping of constants depending on N. We will

(1) use the Koszul complex for infinitely many generators,

(2) invert higher order forms in the $\bar\partial$ equation, and

(3) devise new estimates for the Charpentier solution operators for these equations, including
   (a) the use of sharp estimates — (5-7), (5-8), and (5-9) — on Euclidean expressions $\left|(\overline{w-z})\,\dfrac{\partial}{\partial \overline{w}} f\right|$ in terms of the invariant derivative $|\widetilde{\nabla} f|$,
   (b) the use of the exterior calculus together with the explicit form of Charpentier's solution kernels in Theorems 8 and 10 to handle "rogue" Euclidean factors $\overline{w_j - z_j}$ (see Section 7), and
   (c) the application of generalized operator estimates of Schur type in Lemma 24 to obtain appropriate boundedness of solution operators.

Remark. We emphasize that the crucial new ingredient in our approach, as compared to previous work, is the use of the Besov–Sobolev norms

$$\|f\|_{B^\sigma_{p,m}(\mathbb{B}_n;\ell^2)} \approx \sum_{j=0}^{m-1} |\nabla^j f(0)| + \left(\int_{\mathbb{B}_n} \left|(1-|z|^2)^\sigma D^m f(z)\right|^p \, d\lambda_n(z)\right)^{1/p}$$

given by Arcozzi, Rochberg and Sawyer [Arcozzi et al. 2006] in terms of the *almost invariant holomorphic derivative*

$$D_a f(z) = -f'(z)\big((1-|a|^2)P_a + (1-|a|^2)^{1/2} Q_a\big),$$

given in (5-1) below. This derivative neatly separates the normal and tangential components of the Euclidean derivative, and permits a key exchange between Charpentier's solution kernel in (2-6),

$$\mathscr{C}^{0,q}_n(w,z) \sim \frac{(1-w\bar{z})^{n-1-q}(1-|w|^2)^q}{\triangle(w,z)^n}(\bar{z}_k - \overline{w}_k).$$

and appropriate derivatives D_a of the forms F in the Koszul complex. The point is that the Euclidean portion $\bar{z}_k - \overline{w}_k$ of the kernel $\mathscr{C}^{0,q}_n(w,z)$ is generally *not* dominated by the corresponding invariant portion (see (2-1))

$$\sqrt{\triangle(w,z)} = \left|P_w(z-w) + \sqrt{1-|w|^2}\,Q_w(z-w)\right|$$

appearing in the denominator. However, this complication is offset by the fact that the almost invariant derivative $D_a f(z)$ is correspondingly *larger* than the Euclidean derivative $(1-|a|^2)f'(z)$, and this is exploited in the following exchange formula (5-7):

$$\left|(\overline{z-w})^\alpha \frac{\partial^m}{\partial \overline{w}^\alpha} F(w)\right| \le C\left(\frac{\sqrt{\triangle(w,z)}}{1-|w|^2}\right)^m \left|\overline{D}^m F(w)\right|,$$

which permits control of the solution by the $B^\sigma_{p,m}$ norm using the *larger* derivative $\overline{D}^m F$. It is likely that the Charpentier kernel can be replaced in these arguments by more general kernels with appropriate estimates, and this would be key to extending our baby corona theorem to strictly pseudoconvex domains Ω. This extension will be pursued in subsequent work.

In addition to these novel elements in the proof, we make crucial use of the beautiful integration by parts formula of [Ortega and Fàbrega 2000], and in order to obtain ℓ^2-valued results, we use the clever factorization of the Koszul complex in [Andersson and Carlsson 2000] but adapted to ℓ^2.

Here is a brief outline of the approach of the proof.

We are given an infinite vector of multipliers $g=(g_i)_{i=1}^{\infty}\in M_{B_p^\sigma(\mathbb{B}_n)\to B_p^\sigma(\mathbb{B}_n;\ell^2)}$ that satisfy

$$\|\mathbb{M}_g\|_{B_p^\sigma(\mathbb{B}_n)\to B_p^\sigma(\mathbb{B}_n;\ell^2)}\leq 1 \quad\text{and}\quad \inf_{\mathbb{B}_n}|g|\geq\delta>0,$$

and an element $h\in B_p^\sigma(\mathbb{B}_n)$. We wish to find $f=(f_i)_{i=1}^{\infty}\in B_p^\sigma(\mathbb{B}_n;\ell^2)$ such that

$$(1)\ \mathcal{M}_g f=g\cdot f=h,\qquad (2)\ \bar\partial f=0,\qquad (3)\ \|f\|_{B_p^\sigma(\mathbb{B}_n;\ell^2)}\leq C_{n,\sigma,p,\delta}\|h\|_{B_p^\sigma(\mathbb{B}_n)}.$$

An obvious first attempt at a solution is $f=\dfrac{\bar g}{|g|^2}h$, which clearly satisfies (1), can be shown to satisfy (3), but fails to satisfy (2) in general.

To rectify this we use the Koszul complex in Section 4, which employs *any* solution to the $\bar\partial$ problem on forms of bidegree $(0,q)$, $1\leq q\leq n$, to produce a correction term $\Lambda_g\Gamma_0^2$ so that

$$f=\frac{\bar g}{|g|^2}h-\Lambda_g\Gamma_0^2$$

now satisfies (1) and (2); but (3) is now in doubt without specifying the exact nature of the correction term $\Lambda_g\Gamma_0^2$.

In Section 2 we explicitly calculate Charpentier's solution operators to the $\bar\partial$ equation for use in solving the $\bar\partial$ problems arising in the Koszul complex. These solution operators are remarkably simple in form and moreover are superbly adapted for obtaining estimates in real-variable analogues of the Besov–Sobolev spaces in the ball. In particular, the kernels $K(w,z)$ of these solution operators involve expressions like

$$\frac{(1-w\bar z)^{n-1-q}(1-|w|^2)^q(\overline{w-z})}{\triangle(w,z)^n},\tag{1-8}$$

where

$$\sqrt{\triangle(w,z)}=\left|P_z(w-z)+\sqrt{1-|z|^2}\,Q_z(w-z)\right|$$

is the length of the vector $w-z$ shortened by multiplying by $\sqrt{1-|z|^2}$ its projection $Q_z(w-z)$ onto the orthogonal complement of the complex line through z. Also useful is the identity

$$\sqrt{\triangle(w,z)}=|1-w\bar z|\,|\varphi_z(w)|,$$

where φ_z is the involutive automorphism of the ball that interchanges z and 0; in particular this shows that $d(w,z)=\sqrt{\triangle(w,z)}$ is a quasimetric on the ball.

In Section 5.1 we introduce real-variable analogues $\Lambda_{p,m}^\sigma(\mathbb{B}_n)$ of the Besov–Sobolev spaces $B_p^\sigma(\mathbb{B}_n)$ along with ℓ^2-valued variants, that are based on the geometry inherent in the complex structure of the ball and reflected in the solution kernels in (1-8). In particular these norms involve modifications D of the invariant derivative $\widetilde\nabla$ in the ball:

$$Df(w)=(1-|w|^2)P_w\nabla f+\sqrt{1-|w|^2}\,Q_w\nabla f.$$

Three crucial inequalities are then developed to facilitate the boundedness of the Charpentier solution operators, most notably

$$\left|(\overline{z-w})^{\alpha}\frac{\partial^m}{\partial\overline{w}^{\alpha}}F(w)\right| \leq C\,\triangle(w,z)^{m/2}\left|(1-|w|^2)^{-m}\,\overline{D}^m F(w)\right|, \tag{1-9}$$

for $F\in H^\infty(\mathbb{B}_n;\ell^2)$, which controls the product of Euclidean lengths with Euclidean derivatives on the left, in terms of the product of the smaller length $\sqrt{\triangle(w,z)}$ and the larger derivative $(1-|w|^2)^{-1}\,\overline{D}$ on the right. We caution the reader that our definition of $\overline{D}^m$ is *not* simply the composition of m copies of $\overline{D}$; see Definition 18 below.

In Section 3 we recall the clever integration by parts formulas of Ortega and Fàbrega involving the left side of (1-9), and extend them to the Charpentier solution operators for higher degree forms. If we differentiate (1-8), the power of $\triangle(w,z)$ in the denominator can increase and the integration by parts in Lemma 14 below will temper this singularity on the diagonal. On the other hand the radial integration by parts in Corollary 16 below will temper singularities on the boundary of the ball.

In Section 6 we use Schur's test to establish the boundedness of positive operators with kernels of the form

$$\frac{(1-|z|^2)^a(1-|w|^2)^b\sqrt{\triangle(w,z)}^{\,c}}{|1-\overline{w}z|^{a+b+c+n+1}}.$$

The case $c=0$ is standard (see [Zhu 2005], for example) and the extension to the general case follows from an automorphic change of variables. These results are surprisingly effective in dealing with the ameliorated solution operators of Charpentier.

Finally in Section 7 we put these pieces together to prove Theorem 2.

An Electronic Supplement collects many of the technical modifications of existing proofs in the literature mentioned below that would otherwise interrupt the main flow of this paper.

2. Charpentier's solution kernels for $(0,q)$-forms on the ball

Charpentier proved the following formula for $(0,q)$-forms:

Theorem 5 [Charpentier 1980, Theorem I.1, page 127]. *For $q\geq 0$ and all forms $f(\xi)\in C^1(\overline{\mathbb{B}}_n)$ of degree $(0,q+1)$, we have, for $z\in\mathbb{B}_n$,*

$$f(z)=C_q\int_{\mathbb{B}_n}\overline{\partial}f(\xi)\wedge\mathscr{C}_n^{0,q+1}(\xi,z)+c_q\overline{\partial}_z\left(\int_{\mathbb{B}_n}f(\xi)\wedge\mathscr{C}_n^{0,q}(\xi,z)\right).$$

Here $\mathscr{C}_n^{0,q}(\xi,z)$ is a $(n,n-q-1)$-form in ξ on the ball and a $(0,q)$-form in z on the ball that is defined in Definition 7 below. Using Theorem 5, we can solve $\overline{\partial}_z u=f$ for a $\overline{\partial}$-closed $(0,q+1)$-form f as follows. Set

$$u(z)\equiv c_q\int_{\mathbb{B}_n}f(\xi)\wedge\mathscr{C}_n^{0,q}(\xi,z).$$

Taking $\overline{\partial}_z$ of this we see from Theorem 5 and $\overline{\partial}f=0$ that

$$\overline{\partial}_z u=c_q\overline{\partial}_z\left(\int_{\mathbb{B}_n}f(\xi)\wedge\mathscr{C}_n^{0,q}(\xi,z)\right)=f(z).$$

It is essential for our proof to explicitly compute the kernels $\mathscr{C}_n^{0,q}$ when $0 \le q \le n-1$. The case $q=0$ is given in [Charpentier 1980] and we briefly recall the setup. Denote by $\triangle : \mathbb{C}^n \times \mathbb{C}^n \to [0,\infty)$ the map

$$\triangle(w,z) \equiv |1-w\bar{z}|^2 - (1-|w|^2)(1-|z|^2).$$

It is convenient to record the many faces of $\triangle(w,z)$:

$$\begin{aligned}
\triangle(w,z) &= |1-w\bar{z}|^2 - (1-|w|^2)(1-|z|^2)\\
&= (1-|z|^2)|w-z|^2 + |\bar{z}(w-z)|^2\\
&= (1-|w|^2)|w-z|^2 + |\overline{w}(w-z)|^2\\
&= |1-w\bar{z}|^2|\varphi_w(z)|^2\\
&= |1-w\bar{z}|^2|\varphi_z(w)|^2\\
&= \left|P_w(z-w) + \sqrt{1-|w|^2}\,Q_w(z-w)\right|^2\\
&= \left|P_z(z-w) + \sqrt{1-|z|^2}\,Q_z(z-w)\right|^2.
\end{aligned} \tag{2-1}$$

To compute the kernels $\mathscr{C}_n^{0,q}$ we start with the closed Cauchy–Leray form (see [Rudin 1980, 16.4.5], for example)

$$\mu(\xi,w,z) \equiv \frac{1}{(\xi(w-z))^n}\sum_{i=1}^n (-1)^{i-1}\xi_i \left[\textstyle\bigwedge_{j\neq i} d\xi_j\right] \textstyle\bigwedge_{i=1}^n d(w_i-z_i).$$

One then lifts the form μ via a section s to give a closed form on $\mathbb{C}^n \times \mathbb{C}^n$. Namely, for $s : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$ one defines

$$s^*\mu(w,z) \equiv \frac{1}{\big(s(w,z)(w-z)\big)^n}\sum_{i=1}^n (-1)^{i-1} s_i(w,z) \left[\textstyle\bigwedge_{j\neq i} ds_j\right] \textstyle\bigwedge_{i=1}^n d(w_i-z_i).$$

Now we fix s to be the following section used by Charpentier:

$$s(w,z) \equiv \overline{w}(1-w\bar{z}) - \bar{z}(1-|w|^2). \tag{2-2}$$

Simple computations [Ortega and Fàbrega 2000] demonstrate that

$$s(w,z)(w-z) = \triangle(w,z). \tag{2-3}$$

Definition 6. We define the Cauchy kernel on $\mathbb{B}_n \times \mathbb{B}_n$ to be

$$\mathscr{C}_n(w,z) \equiv s^*\mu(w,z) \tag{2-4}$$

for the section s given in (2-2).

Definition 7. For $0 \le p \le n$ and $0 \le q \le n-1$ we let $\mathscr{C}_n^{p,q}$ be the component of $\mathscr{C}_n(w,z)$ that has bidegree (p,q) in z and bidegree $(n-p, n-q-1)$ in w.

Thus if η is a $(p,q+1)$-form in w, then $\mathscr{C}_n^{p,q} \wedge \eta$ is a (p,q)-form in z and a multiple of the volume form in w. We now prepare to give explicit formulas for Charpentier's solution kernels $\mathscr{C}_n^{0,q}(w,z)$. First we introduce some notation.

Notation. Let $\omega_n(z) = \bigwedge_{j=1}^n dz_j$. For n a positive integer and $0 \le q \le n-1$ let P_n^q denote the collection of all permutations ν on $\{1, \dots, n\}$ that map to $\{i_\nu, J_\nu, L_\nu\}$ where J_ν is an increasing multi-index with $\operatorname{card}(J_\nu) = n-q-1$ and $\operatorname{card}(L_\nu) = q$. Let $\epsilon_\nu \equiv \operatorname{sgn}(\nu) \in \{-1, 1\}$ denote the signature of the permutation ν.

Note that the number of increasing multi-indices of length $n-q-1$ is $\frac{n!}{(q+1)!(n-q-1)!}$, while the number of increasing multi-indices of length q are $\frac{n!}{q!(n-q)!}$. Since we are only allowed certain combinations of J_ν and L_ν (they must have disjoint intersection and they must be increasing multi-indices), it is straightforward to see that the total number of permutations in P_n^q that we are considering is $\frac{n!}{(n-q-1)!q!}$.

From [Øvrelid 1971] we obtain that Charpentier's kernel takes the (abstract) form

$$\mathscr{C}_n^{0,q}(w, z) = \frac{1}{\triangle(w, z)^n} \sum_{\nu \in P_n^q} \operatorname{sgn}(\nu) s_{i_\nu} \bigwedge_{j \in J_\nu} \bar\partial_w s_j \bigwedge_{l \in L_\nu} \bar\partial_z s_l \wedge \omega_n(w).$$

Fundamental for us will be the explicit formula for Charpentier's kernel given in the next theorem. It is convenient to isolate the following factor common to all summands in the formula:

$$\Phi_n^q(w, z) \equiv \frac{(1 - w\bar z)^{n-1-q}(1 - |w|^2)^q}{\triangle(w, z)^n}, \quad 0 \le q \le n-1. \tag{2-5}$$

Theorem 8. *Let n be a positive integer and suppose that $0 \le q \le n-1$. Then*

$$\mathscr{C}_n^{0,q}(w, z) = \sum_{\nu \in P_n^q} (-1)^q \Phi_n^q(w, z) \operatorname{sgn}(\nu)(\overline{w}_{i_\nu} - \bar z_{i_\nu}) \bigwedge_{j \in J_\nu} d\overline{w}_j \bigwedge_{l \in L_\nu} d\bar z_l \wedge \omega_n(w). \tag{2-6}$$

Remark. We can rewrite the formula for $\mathscr{C}_n^{0,q}(w, z)$ in (2-6) as

$$\mathscr{C}_n^{0,q}(w, z) = \Phi_n^q(w, z) \sum_{|J|=q} \sum_{k \notin J} (-1)^{\mu(k,J)} (\bar z_k - \overline{w}_k) d\bar z^J \wedge d\overline{w}^{(J \cup \{k\})^c} \wedge \omega_n(w), \tag{2-7}$$

where $J \cup \{k\}$ denotes the increasing multi-index obtained by rearranging the integers $\{k, j_1, \dots j_q\}$ as

$$J \cup \{k\} = \{j_1, \dots j_{\mu(k,J)-1}, k, j_{\mu(k,J)}, \dots j_q\}.$$

Thus k occupies the $\mu(k, J)$-th position in $J \cup \{k\}$. The notation $(J \cup \{k\})^c$ refers to the increasing multi-index obtained by rearranging the integers in $\{1, 2, \dots, n\} \setminus (J \cup \{k\})$. To see (2-7), we note that in (2-6) the permutation ν takes the n-tuple $(1, 2, \dots n)$ to (i_ν, J_ν, L_ν). In (2-7) the n-tuple $\big(k, (J \cup \{k\})^c, J\big)$ corresponds to (i_ν, J_ν, L_ν), and so $\operatorname{sgn}(\nu)$ becomes in (2-7) the signature of the permutation that takes $(1, 2, \dots, n)$ to $(k, (J \cup \{k\})^c, J)$. This in turn equals $(-1)^{\mu(k,J)}$ with $\mu(k, J)$ as above.

We observe at this point that the functional coefficient in the summands in (2-6) looks like

$$(-1)^q \Phi_n^q(w, z)(\overline{w}_{i_\nu} - \bar z_{i_\nu}) = (-1)^q \frac{(1 - w\bar z)^{n-q-1}(1 - |w|^2)^q}{\triangle(w, z)^n} (\overline{w}_{i_\nu} - \bar z_{i_\nu}),$$

which behaves like a fractional integral operator of order 1 in the Bergman metric on the diagonal relative to invariant measure.

Finally, we will adopt the usual convention of writing

$$\mathscr{C}_n^{0,q} f(z) = \int_{\mathbb{B}_n} f(w) \wedge \mathscr{C}_n^{0,q}(w, z),$$

when we wish to view $\mathscr{C}_n^{0,q}$ as an operator taking $(0,q+1)$-forms f in w to $(0,q)$-forms $\mathscr{C}_n^{0,q} f$ in z. The proof of Theorem 8 is carried out in the Electronic Supplement. Here we present a relatively short and elegant proof pointed out to us by a referee. It is helpful to make the following elementary observation.

Remark. If a form λ has odd degree, then any power λ^ℓ with $\ell \geq 2$ necessarily vanishes, by the alternating property. On the other hand, if one of the forms λ_1, λ_2 has even degree, then the binomial theorem holds for the sum:

$$(\lambda_1+\lambda_2)^\ell = \sum_{j=0}^{\ell} \binom{\ell}{j} (\lambda_1)^j (\lambda_2)^{\ell-j}.$$

Note that the wedge power $\bigwedge_{i=1}^{\ell} \Big(\sum_{k_i=1}^{\infty} \frac{\bar\partial \bar g_{k_i}}{|g|^2} e_{k_i} \Big)$ above doesn't vanish since the form $\frac{\bar\partial \bar g_{k_i}}{|g|^2} e_{k_i}$ has degree 2.

Proof of Theorem 8. Consider the section $s(w,z)$ in (2-2) and the associated $(1,0)$-form $s \cdot dw = \sum_{j=1}^n s_j \, dw_j$ and the $(1,1)$-form $\bar\partial(s \cdot dw)$. We claim that

$$\triangle(w,z)^{-n} (s \cdot dw) \wedge \big(\bar\partial(s \cdot dw)\big)^{n-1} \tag{2-8}$$

is the term $K(w,z)$ of total bidegree $(n,n-1)$ in the Cauchy kernel $\mathscr{C}_n(w,z) = s^* \mu(w,z)$. To see this we first recall formula (2.2) in [Øvrelid 1971], which reads

$$K(w,z) = c_n \triangle(w,z)^{-n} \sum_{i=1}^{n} (-1)^{i-1} s_i \Big(\bigwedge_{j \neq i} (\bar\partial s_j) \Big) \wedge \omega(w),$$

where $\omega(w) = dw_1 \wedge \cdots \wedge dw_n$ and the product over $j \neq i$ is taken with increasing j. Now we note that each term in the expansion of the product in (2-8) must contain a permutation of the product $dw_1 \wedge \cdots \wedge dw_n$. Thus by factoring out the term $\omega(w)$ we compute

$$\begin{aligned}\triangle(w,z)^{-n} (s \cdot dw) \wedge \big(\bar\partial(s \cdot dw)\big)^{n-1} &= \triangle(w,z)^{-n} \Big(\sum_{i=1}^n s_i dw_i \Big) \wedge \Big(\sum_{j=1}^n (\bar\partial s_j) \wedge dw_j \Big)^{n-1} \\ &= \triangle(w,z)^{-n} \sum_{i=1}^{n} (-1)^{i-1} s_i \Big(\bigwedge_{j \neq i} (\bar\partial s_j) \Big) \wedge \omega(w) = K(w,z),\end{aligned}$$

where the factor $(-1)^{i-1}$ arises since the terms $(\bar\partial s_j) \wedge dw_j$ of total degree 2 commute, while the term dw_i anticommutes, with terms of degree 1.

Now we analyze (2-8) with the aid of the forms

$$\beta = \bar\partial\partial|w|^2 = \bar\partial \sum_{i=1}^n (dw_i)\overline{w}_i = d\overline{w} \cdot dw = \sum_{i=1}^n d\overline{w}_i \wedge dw_i, \quad \mu = dw \cdot d\bar z = \sum_{i=1}^n dw_i \wedge d\bar z_i,$$

where δ is the interior product, given by

$$\delta\alpha = \alpha \lrcorner (w \cdot dw) = \alpha \lrcorner \Big(\sum_{k=1}^n w_k dw_k \Big).$$

We have both

$$\delta\beta = (d\overline{w}\cdot dw) \lrcorner (w\cdot dw) = \left(\sum_{i=1}^n d\overline{w}_i \wedge dw_i\right) \lrcorner \left(\sum_{j=1}^n w_j\, dw_j\right) = -\sum_{i=1}^n w_i\, d\overline{w}_i = -w\cdot d\overline{w}$$

and

$$\delta\mu = (dw\cdot d\bar{z}) \lrcorner (w\cdot dw) = \left(\sum_{i=1}^n dw_i \wedge d\bar{z}_i\right) \lrcorner \left(\sum_{j=1}^n w_j\, dw_j\right) = \sum_{i=1}^n w_i\, d\bar{z}_i = w\cdot d\bar{z}.$$

Now we compute, using

$$s(w,z)\cdot dw \equiv (1-w\bar{z})(\overline{w}\cdot dw) - (1-|w|^2)(\bar{z}\cdot dw), \tag{2-9}$$

that

$$\begin{aligned}\lambda \equiv \overline{\partial}(s\cdot dw) &= -(w\cdot d\bar{z})\wedge(\overline{w}\cdot dw) + (1-|w|^2)(dw\cdot d\bar{z}) + (1-w\bar{z})(d\overline{w}\cdot dw) + (w\cdot d\overline{w})\wedge(\bar{z}\cdot dw)\\ &= (\overline{w}\cdot dw)\wedge\delta\mu + (\bar{z}\cdot dw)\wedge\delta\beta + (1-|w|^2)\mu + (1-w\bar{z})\beta.\end{aligned}$$

Consider the form $(s\cdot dw)\wedge\lambda^{n-1}$. Since λ has degree two, the remark on page 510 shows that the power λ^{n-1} can be expanded by the binomial theorem. Let $\lambda = A + B$, where

$$A = (\overline{w}\cdot dw)\wedge\delta\mu + (\bar{z}\cdot dw)\wedge\delta\beta, \quad B = (1-|w|^2)\mu + (1-w\bar{z})\beta.$$

We claim the formula

$$(s\cdot dw)\wedge\lambda^{n-1} = (s\cdot dw)\wedge B^{n-1} + (n-1)(s\cdot dw)\wedge A\wedge B^{n-2}. \tag{2-10}$$

To see this we expand the left-hand side using the binomial theorem to get

$$(s\cdot dw)\wedge(A+B)^{n-1} = (s\cdot dw)\wedge\bigl(A^{n-1} + (n-1)A^{n-2}\wedge B + \cdots + (n-1)A\wedge B^{n-2} + B^{n-1}\bigr).$$

We want this to equal

$$(s\cdot dw)\wedge B^{n-1} + (n-1)(s\cdot dw)\wedge A\wedge B^{n-2},$$

which will be the case if

$$(s\cdot dw)\wedge A^k = 0 \quad \text{for all } k\geq 2,$$

which in turn follows from $(s\cdot dw)\wedge A^2 = 0$. However, using

$$\delta\beta = -w\cdot d\overline{w} \text{ and } \delta\mu = w\cdot d\bar{z}, \tag{2-11}$$

we obtain

$$A = (\overline{w}\cdot dw)\wedge(w\cdot d\bar{z}) - (\bar{z}\cdot dw)\wedge(w\cdot d\overline{w}). \tag{2-12}$$

Hence we can write

$$A^2 = A_1 + A_2 + A_3 + A_4$$

with

$$\begin{aligned} A_1 &= (\overline{w}\cdot dw)\wedge(w\cdot d\bar{z})\wedge(\overline{w}\cdot dw)\wedge(w\cdot d\bar{z}), & A_2 &= -(\overline{w}\cdot dw)\wedge(w\cdot d\bar{z})\wedge(\bar{z}\cdot dw)\wedge(w\cdot d\overline{w}),\\ A_3 &= -(\bar{z}\cdot dw)\wedge(w\cdot d\overline{w})\wedge(\overline{w}\cdot dw)\wedge(w\cdot d\bar{z}), & A_4 &= (\bar{z}\cdot dw)\wedge(w\cdot d\overline{w})\wedge(\bar{z}\cdot dw)\wedge(w\cdot d\overline{w}).\end{aligned}$$

Now

$$A_1 = (\overline{w}\cdot dw)\wedge w\cdot d\bar{z}\wedge(\overline{w}\cdot dw)\wedge(w\cdot d\bar{z}) = -\big((\overline{w}\cdot dw)\wedge\overline{w}\cdot dw\big)\wedge\big((w\cdot d\bar{z})\wedge(w\cdot d\bar{z})\big) = 0,$$

and similarly $A_4 = 0$. We also compute that

$$A_2 = -(\overline{w}\cdot dw)\wedge w\cdot d\bar{z}\wedge(\bar{z}\cdot dw)\wedge(w\cdot d\overline{w}) = -(\bar{z}\cdot dw)\wedge w\cdot d\overline{w}\wedge(\overline{w}\cdot dw)\wedge(w\cdot d\bar{z}) = A_3,$$

so that

$$A^2 = -2A_2.$$

Now we note, using (2-9), that

$$(s\cdot dw)\wedge A_2 = (1-w\bar{z})\big((\overline{w}\cdot dw)\wedge A_2\big) - (1-|w|^2)\big((\bar{z}\cdot dw)\wedge A_2\big)$$

vanishes, since $(\overline{w}\cdot dw)\wedge A_2$ contains two factors $\overline{w}\cdot dw$, and since $(\bar{z}\cdot dw)\wedge A_2$ contains two factors $\bar{z}\cdot dw$. Thus we have proved that

$$(s\cdot dw)\wedge A^2 = -(s\cdot dw)\wedge 2A_2 = 0.$$

This completes the proof of (2-10).

Now we continue by using (2-9), (2-11) and (2-12) to obtain

$$\begin{aligned}(s\cdot dw)\wedge A &= (1-w\bar{z})\big((\overline{w}\cdot dw)\wedge A\big) - (1-|w|^2)\big((\bar{z}\cdot dw)\wedge A\big)\\ &= -(1-w\bar{z})\big((\overline{w}\cdot dw)\wedge(\bar{z}\cdot dw)\wedge(w\cdot d\overline{w})\big) - (1-|w|^2)\big((\bar{z}\cdot dw)\wedge(\overline{w}\cdot dw)\wedge(w\cdot d\bar{z})\big)\\ &= -(\bar{z}\cdot dw)\wedge(\overline{w}\cdot dw)\wedge\big((1-|w|^2)\,\delta\mu + (1-w\bar{z})\,\delta\beta\big).\end{aligned}$$

We can now simplify the second term on the right side of (2-10) to obtain

$$\begin{aligned}(s\cdot dw)\wedge\lambda^{n-1} &= (s\cdot dw)\wedge B^{n-1} + (n-1)(s\cdot dw)\wedge A\wedge B^{n-2}\\ &= (s\cdot dw)\wedge B^{n-1} - (\bar{z}\cdot dw)\wedge(\overline{w}\cdot dw)\wedge\big((1-|w|^2)\,\delta\mu + (1-w\bar{z})\,\delta\beta\big)\wedge(n-1)B^{n-2}\\ &= (s\cdot dw)\wedge B^{n-1} - (\bar{z}\cdot dw)\wedge(\overline{w}\cdot dw)\wedge(\delta B^{n-1}).\end{aligned} \tag{2-13}$$

Now we note that

$$(\bar{z}\cdot dw)\wedge(\overline{w}\cdot dw)\wedge(\delta B^{n-1}) + \delta\big((\bar{z}\cdot dw)\wedge(\overline{w}\cdot dw)\big)\wedge B^{n-1} = \delta\big((\bar{z}\cdot dw)\wedge(\overline{w}\cdot dw)\wedge B^{n-1}\big) = 0,$$

since the left side has full degree in dw and the form $(\bar{z}\cdot dw)\wedge(\overline{w}\cdot dw)$ has even degree. As a consequence we obtain the formula

$$(s\cdot dw)\wedge\lambda^{n-1} = \big[(s\cdot dw) + \delta\big((\bar{z}\cdot dw)\wedge(\overline{w}\cdot dw)\big)\big]\wedge B^{n-1}.$$

Now the simple computation

$$\begin{aligned}&(s\cdot dw) + \delta\big((\bar{z}\cdot dw)\wedge(\overline{w}\cdot dw)\big)\\ &\qquad = (1-w\bar{z})(\overline{w}\cdot dw) - (1-|w|^2)(\bar{z}\cdot dw) + (\bar{z}\cdot w)(\overline{w}\cdot dw) - (\bar{z}\cdot dw)(\overline{w}\cdot w)\\ &\qquad = \overline{w}\cdot dw - \bar{z}\cdot dw\end{aligned}$$

shows that

$$(s\cdot dw)\wedge\lambda^{n-1}=(\overline{w}-\bar{z})\cdot dw\wedge B^{n-1}. \tag{2-14}$$

Now the product rule in the remark on page 510 gives

$$B^{n-1}=\big((1-|w|^2)\mu+(1-w\bar{z})\beta\big)^{n-1}=\sum_{q=0}^{n-1}\binom{n-1}{q}\big((1-|w|^2)\mu\big)^q\wedge\big((1-w\bar{z})\beta\big)^{n-1-q},$$

and so taking the terms of bidegree $(0,q)$ in z in the formula (2-14) we obtain

$$\mathscr{C}_n^{0,q}=\binom{n-1}{q}\frac{(1-|w|^2)^q(1-w\bar{z})^{n-1-q}}{\triangle(w-z)^n}(\overline{w}-\bar{z})\cdot dw\wedge\mu^q\wedge\beta^{n-1-q}. \tag{2-15}$$

Finally we note that this coincides with our formula

$$\mathscr{C}_n^{0,q}(w,z)=\Phi_n^q(w,z)\sum_{|J|=q}\sum_{k\notin J}(-1)^{\mu(k,J)}(\bar{z}_k-\overline{w}_k)d\bar{z}^J\wedge d\overline{w}^{(J\cup\{k\})^c}\wedge\omega_n(w). \tag{2-16}$$

This can be seen by writing

$$(\overline{w}-\bar{z})\cdot dw=\sum_{k=1}^n(\overline{w}_k-\bar{z}_k)dw_k \text{ and } \mu^q=(dw\cdot d\bar{z})^q=\sum_J(-1)^\nu dw^J\wedge d\bar{z}^J,$$

with $|J|=q$, and then noting that in order to have a nonzero term in (2-15), we must have $k\notin J$ and the summand from $\beta^{n-1-q}=(dw\cdot d\overline{w})^{n-1-q}$ must be

$$(-1)^\omega dw^{(J\cup\{k\})^c}\wedge d\overline{w}^{(J\cup\{k\})^c}.$$

One then checks that the powers of -1 work out correctly. □

Remark. One might wonder if the special form of the right hand side of the recursion formula (2-10) can be put to good use in estimating the Besov–Sobolev norms of solutions to the $\bar\partial$-equation. This formula neatly exhibits a factoring of the solution operator that may be helpful, but we are unable to take advantage of this at this point, and must revert instead to the use of the explicit Charpentier formula (2-6) together with the exchange formula (5-7).

2.1. ***Ameliorated kernels.*** We now wish to define right inverses with improved behavior at the boundary. We consider the case when the right side f of the $\bar\partial$ equation is a $(p,q+1)$-form in $\mathbb{B}_n$.

As usual for a positive integer $s>n$ we will "project" the formula $\bar\partial\mathscr{C}_s^{p,q}f=f$ in $\mathbb{B}_s$ for a $\bar\partial$-closed form f in $\mathbb{B}_s$ to a formula $\bar\partial\mathscr{C}_{n,s}^{p,q}f=f$ in $\mathbb{B}_n$ for a $\bar\partial$-closed form f in $\mathbb{B}_n$. To accomplish this we define *ameliorated* operators $\mathscr{C}_{n,s}^{p,q}$ by

$$\mathscr{C}_{n,s}^{p,q}=\mathsf{R}_n\mathscr{C}_s^{p,q}\mathsf{E}_s,$$

where, for $n<s$, E_s (R_n) is the extension (restriction) operator that takes forms $\Omega=\sum\eta_{I,J}dw^I\wedge d\overline{w}^J$ in $\mathbb{B}_n$ ($\mathbb{B}_s$) and extends (restricts) them to $\mathbb{B}_s$ ($\mathbb{B}_n$) by

$$\mathsf{E}_s\Big(\sum\eta_{I,J}dw^I\wedge d\overline{w}^J\Big)\equiv\sum(\eta_{I,J}\circ R)\,dw^I\wedge d\overline{w}^J,$$

$$\mathsf{R}_n\Big(\sum\eta_{I,J}dw^I\wedge d\overline{w}^J\Big)\equiv\sum_{I,J\subset\{1,2,\dots,n\}}(\eta_{I,J}\circ E)\,dw^I\wedge d\overline{w}^J.$$

Here R is the natural orthogonal projection from $\mathbb{C}^s$ to $\mathbb{C}^n$ and E is the natural embedding of $\mathbb{C}^n$ into $\mathbb{C}^s$. In other words, we extend a form by taking the coefficients to be constant in the extra variables, and we restrict a form by discarding all wedge products of differentials involving the extra variables and restricting the coefficients accordingly.

For $s > n$ we observe that the operator $\mathscr{C}_{n,s}^{p,q}$ has integral kernel

$$\mathscr{C}_{n,s}^{p,q}(w,z) \equiv \int_{\sqrt{1-|w|^2}\mathbb{B}_{s-n}} \mathscr{C}_s^{p,q}\big((w,w'),(z,0)\big)\,dV(w'), \quad z,w \in \mathbb{B}_n, \tag{2-17}$$

where $\mathbb{B}_{s-n}$ denotes the unit ball in $\mathbb{C}^{s-n}$ with respect to the orthogonal decomposition $\mathbb{C}^s = \mathbb{C}^n \oplus \mathbb{C}^{s-n}$, and dV denotes Lebesgue measure. If $f(w)$ is a $\bar\partial$-closed form on $\mathbb{B}_n$ then $f(w,w') = f(w)$ is a $\bar\partial$-closed form on $\mathbb{B}_s$ and we have for $z \in \mathbb{B}_n$,

$$\begin{aligned} f(z) = f(z,0) &= \bar\partial \int_{\mathbb{B}_s} \mathscr{C}_s^{p,q}\big((w,w'),(z,0)\big) f(w)\,dV(w)\,dV(w') \\ &= \bar\partial \int_{\mathbb{B}_n} \bigg(\int_{\sqrt{1-|w|^2}\mathbb{B}_{s-n}} \mathscr{C}_s^{p,q}\big((w,w'),(z,0)\big)\,dV(w') \bigg) f(w)\,dV(w) = \bar\partial \int_{\mathbb{B}_n} \mathscr{C}_{n,s}^{p,q}(w,z) f(w)\,dV(w). \end{aligned}$$

We have proved the following:

Theorem 9. *For all $s > n$ and $\bar\partial$-closed forms f in $\mathbb{B}_n$, we have*

$$\bar\partial\, \mathscr{C}_{n,s}^{p,q} f = f \quad \text{in } \mathbb{B}_n.$$

We will use only the case $p = 0$ of this theorem and from now on we restrict our attention to this case. The operators $\mathscr{C}_{n,s}^{0,0}$ have been computed in [Ortega and Fàbrega 2000] and are given by

$$\mathscr{C}_{n,s}^{0,0} f(z) = \int_{\mathbb{B}_n} \sum_{j=0}^{n-1} c_{n,j,s} \frac{(1-|w|^2)^{s-n+j}(1-|z|^2)^j}{(1-\overline{w}z)^{s-n+j}(1-w\bar z)^j} \mathscr{C}_n^{0,0}(w,z) \wedge f(w), \tag{2-18}$$

where

$$\mathscr{C}_n^{0,0}(w,z) = c_0 \frac{(1-w\bar z)^{n-1}}{\big(|1-w\bar z|^2 - (1-|w|^2)(1-|z|^2)\big)^n} \sum_{j=1}^n (-1)^{j-1}(\overline{w}_j - \bar z_j) \bigwedge_{k\neq j} d\overline{w}_k \bigwedge_{\ell=1}^n dw_\ell.$$

A similar result holds for the operators $\mathscr{C}_{n,s}^{0,q}$. Define

$$\begin{aligned} \Phi_{n,s}^q(w,z) &= \Phi_n^q(w,z) \left(\frac{1-|w|^2}{1-\overline{w}z} \right)^{s-n} \sum_{j=0}^{n-q-1} c_{j,n,s} \left(\frac{(1-|w|^2)(1-|z|^2)}{|1-w\bar z|^2} \right)^j \\ &= \frac{(1-w\bar z)^{n-1-q}(1-|w|^2)^q}{\triangle(w,z)^n} \left(\frac{1-|w|^2}{1-\overline{w}z} \right)^{s-n} \sum_{j=0}^{n-q-1} c_{j,n,s} \left(\frac{(1-|w|^2)(1-|z|^2)}{|1-w\bar z|^2} \right)^j \\ &= \sum_{j=0}^{n-q-1} c_{j,n,s} \frac{(1-w\bar z)^{n-1-q-j}(1-|w|^2)^{s-n+q+j}(1-|z|^2)^j}{(1-\overline{w}z)^{s-n+j}\triangle(w,z)^n}. \end{aligned}$$

Note that the numerator and denominator are *balanced* in the sense that the sum of the exponents in the denominator minus the corresponding sum in the numerator (counting $\triangle(w,z)$ double) equals $s+n+j-(s+j-1) = n+1$, the exponent of the invariant measure of the ball $\mathbb{B}_n$.

Theorem 10. *Suppose that* $s > n$ *and* $0 \le q \le n-1$. *Then*

$$\begin{aligned}\mathscr{C}^{0,q}_{n,s}(w,z) &= \mathscr{C}^{0,q}_n(w,z)\left(\frac{1-|w|^2}{1-\overline{w}z}\right)^{s-n}\sum_{j=0}^{n-q-1} c_{j,n,s}\left(\frac{(1-|w|^2)(1-|z|^2)}{|1-w\bar{z}|^2}\right)^j\\ &= \Phi^q_{n,s}(w,z)\sum_{|J|=q}\sum_{k\notin J}(-1)^{\mu(k,J)}(\bar{z}_k-\overline{w}_k)\,d\bar{z}^J\wedge d\overline{w}^{(J\cup\{k\})^c}\wedge\omega_n(w).\end{aligned}$$

Proof. For $s > n$ recall that the kernels of the ameliorated operators $\mathscr{C}^{0,q}_{n,s}$ are given in (2-17). For ease of notation, we will set $k = s-n$, so we have $\mathbb{C}^s = \mathbb{C}^n \oplus \mathbb{C}^k$. Suppose that $0 \le q \le n-1$. Recall from (2-6) that

$$\begin{aligned}\mathscr{C}^{0,q}_s(w,z) &= (-1)^q\frac{(1-w\bar{z})^{s-q-1}(1-|w|^2)^q}{\triangle(w,z)^s}\sum_{\nu\in P^q_s}\operatorname{sgn}(\nu)(\overline{w}_{i_\nu}-\bar{z}_{i_\nu})\bigwedge_{j\in J_\nu}d\overline{w}_j\bigwedge_{l\in L_\nu}d\bar{z}_l\wedge\omega_s(w)\\ &= \sum_{\nu\in P^q_s}F^q_{s,i_\nu}(w,z)\bigwedge_{j\in J_\nu}d\overline{w}_j\bigwedge_{l\in L_\nu}d\bar{z}_l\wedge\omega_s(w),\end{aligned}$$

where

$$F^q_{s,i_\nu}(w,z) = \Phi^q_s(w,z)(\overline{w}_{i_\nu}-\bar{z}_{i_\nu}) = \frac{(1-w\bar{z})^{s-q-1}(1-|w|^2)^q}{\triangle(w,z)^s}(\overline{w}_{i_\nu}-\bar{z}_{i_\nu}).$$

To compute the ameliorations of these kernels, we need only focus on the functional coefficient $F^q_{s,i_\nu}(w,z)$ of the kernel. It is easy to see that the ameliorated kernel can only give a contribution in the variables when $1 \le i_\nu \le n$, since when $n+1 \le i_\nu \le s$ the functional kernel becomes radial in certain variables and thus reduces to zero upon integration.

Then for any $1 \le i \le n$ the corresponding functional coefficient $F^q_{s,i}(w,z)$ has amelioration $F^q_{n,s,i}(w,z)$ given by

$$\begin{aligned}F^q_{n,s,i}(w,z) &= \int_{\sqrt{1-|w|^2}\mathbb{B}_{s-n}} F^q_{s,i}((w,w'),(z,0))\,dV(w')\\ &= \int_{\sqrt{1-|w|^2}\mathbb{B}_k}\frac{(1-w\bar{z})^{s-q-1}(1-|w|^2-|w'|^2)^q(\bar{z}_i-\overline{w}_i)}{\triangle\big((w,w'),(z,0)\big)^s}\,dV(w')\\ &= (\bar{z}_i-\overline{w}_i)(1-w\bar{z})^{s-q-1}\int_{\sqrt{1-|w|^2}\mathbb{B}_k}\frac{(1-|w|^2-|w'|^2)^q}{\triangle\big((w,w'),(z,0)\big)^s}\,dV(w').\end{aligned}$$

Theorem 10 is a thus a consequence of the following elementary formula, which will find application in the next section as well:

$$\begin{aligned}(1-w\bar{z})^{s-q-1}&\int_{\sqrt{1-|w|^2}\mathbb{B}_{s-n}}\frac{(1-|w|^2-|w'|^2)^q}{\triangle\big((w,w'),(z,0)\big)^s}\,dV(w')\\ &= \frac{\pi^{s-n}}{(s-n)!}\Phi^q_n(w,z)\left(\frac{1-|w|^2}{1-\overline{w}z}\right)^{s-n}\sum_{j=0}^{n-q-1}c_{j,n,s}\left(\frac{(1-|w|^2)(1-|z|^2)}{|1-w\bar{z}|^2}\right)^j. \qquad (2\text{-}19)\end{aligned}$$

□

3. Integration by parts

We begin with an integration by parts formula involving a covariant derivative in [Ortega and Fàbrega 2000, Lemma 2.1, page 57] that reduces the singularity of the solution kernel on the diagonal at the expense of differentiating the form. However, in order to prepare for a generalization to higher order forms, we replace the covariant derivative with the notion of $\overline{\mathfrak{X}}_{z,w}$-derivative defined in (3-2) below.

Recall Charpentier's explicit solution $\mathscr{C}_n^{0,0}\eta$ to the $\bar\partial$ equation $\bar\partial\,\mathscr{C}_n^{0,0}\eta=\eta$ in the ball $\mathbb{B}_n$ when η is a $\bar\partial$-closed $(0,1)$-form with coefficients in $C(\overline{\mathbb{B}}_n)$: the kernel is given by

$$\mathscr{C}_n^{0,0}(w,z)=c_0\frac{(1-w\bar z)^{n-1}}{\triangle(w,z)^n}\sum_{j=1}^n(-1)^{j-1}(\overline{w}_j-\bar z_j)\bigwedge_{k\neq j}d\overline{w}_k\bigwedge_{\ell=1}^n dw_\ell,$$

for $(w,z)\in\mathbb{B}_n\times\mathbb{B}_n$, where

$$\triangle(w,z)=|1-w\bar z|^2-(1-|w|^2)(1-|z|^2).$$

Define the Cauchy operator $\mathscr{S}_n$ on $\partial\mathbb{B}_n\times\mathbb{B}_n$ with kernel

$$\mathscr{S}_n(\zeta,z)=c_1\frac{1}{(1-\bar\zeta z)^n}d\sigma(\zeta),\qquad(\zeta,z)\in\partial\mathbb{B}_n\times\mathbb{B}_n.$$

Let $\eta=\sum_{j=1}^n\eta_j\,d\overline{w}_j$ be a $(0,1)$-form with smooth coefficients. Define a vector field acting in the variable $w=(w_1,\dots,w_n)$ and parametrized by $z=(z_1,\dots,z_n)$ by

$$\overline{\mathfrak{X}}=\overline{\mathfrak{X}}_{z,w}=\sum_{j=1}^n(\overline{w}_j-\bar z_j)\frac{\partial}{\partial\overline{w}_j}. \tag{3-1}$$

It will usually be understood from the context what the acting variable w and the parameter variable z are in $\overline{\mathfrak{X}}_{z,w}$ and we will then omit the subscripts and simply write $\overline{\mathfrak{X}}$ for $\overline{\mathfrak{X}}_{z,w}$.

Definition 11. For $m\geq 0$, define the m-th order derivative $\overline{\mathfrak{X}}^m\eta$ of a $(0,1)$-form $\eta=\sum_{k=1}^n\eta_k(w)\,d\overline{w}_k$ to be the $(0,1)$-form obtained by componentwise differentiation holding monomials in $\overline{w}-\bar z$ fixed:

$$\overline{\mathfrak{X}}^m\eta(w)=\sum_{k=1}^n(\overline{\mathfrak{X}}^m\eta_k)(w)d\overline{w}_k=\sum_{k=1}^n\biggl(\sum_{|\alpha|=m}^n\overline{(w-z)^\alpha}\frac{\partial^m\eta_k}{\partial\overline{w}^\alpha}(w)\biggr)d\overline{w}_k. \tag{3-2}$$

Lemma 12 (compare [Ortega and Fàbrega 2000, Lemma 2.1]). *For all $m\geq 0$ and smooth $(0,1)$-forms $\eta=\sum_{k=1}^n\eta_k(w)\,d\overline{w}_k$, we have*

$$\begin{aligned}\mathscr{C}_n^{0,0}\eta(z)&\equiv\int_{\mathbb{B}_n}\mathscr{C}_n^{0,0}(w,z)\wedge\eta(w)\\&=\sum_{j=0}^{m-1}c_j\int_{\partial\mathbb{B}_n}\mathscr{S}_n(w,z)(\overline{\mathfrak{X}}^j\eta)[\overline{\mathfrak{X}}](w)\,d\sigma(w)+c_m\int_{\mathbb{B}_n}\mathscr{C}_n^{0,0}(w,z)\wedge\overline{\mathfrak{X}}^m\eta(w).\end{aligned} \tag{3-3}$$

Here the $(0,1)$-form $\overline{\mathfrak{X}}^j\eta$ acts on the vector field $\overline{\mathfrak{X}}$ in the usual way:

$$(\overline{\mathfrak{X}}^j\eta)\big[\overline{\mathfrak{X}}\big]=\Big(\sum_{k=1}^n\overline{\mathfrak{X}}^j\eta_k(w)\,d\overline{w}_k\Big)\Big(\sum_{i=1}^n(\overline{w}_i-\bar{z}_i)\frac{\partial}{\partial\overline{w}_i}\Big)=\sum_{k=1}^n(\overline{w}_k-\bar{z}_k)\overline{\mathfrak{X}}^j\eta_k(w).$$

We can also rewrite the final integral in (3-3) as

$$\int_{\mathbb{B}_n}\mathscr{C}_n^{0,0}(w,z)\wedge\overline{\mathfrak{X}}^m\eta(w)=\int_{\mathbb{B}_n}\Phi_n^0(w,z)(\overline{\mathfrak{X}}^m\eta)\big[\overline{\mathfrak{X}}\big](w)\,dV(w).$$

Lemma 12 is proved by following verbatim the proof of Lemma 2.1 of [Ortega and Fàbrega 2000].

We now extend Lemma 12 to $(0,q+1)$-forms. Let

$$\eta=\sum_{|I|=q+1}\eta_I(w)\,d\overline{w}^I$$

be a $(0,q+1)$-form with smooth coefficients. Given a $(0,q+1)$-form $\eta=\sum_{|I|=q+1}\eta_I\,d\overline{w}^I$ and an increasing sequence J of length $|J|=q$, we define the interior product $\eta\lrcorner\,d\overline{w}^J$ of η and $d\overline{w}^J$ by

$$\eta\lrcorner\,d\overline{w}^J=\sum_{|I|=q+1}\eta_I\,d\overline{w}^I\lrcorner\,d\overline{w}^J=\sum_{k\notin J}(-1)^{\mu(k,J)}\eta_{J\cup\{k\}}\,d\overline{w}_k,\tag{3-4}$$

since $d\overline{w}^I\lrcorner\,d\overline{w}^J=(-1)^{\mu(k,J)}d\overline{w}_k$ if $k\in I\setminus J$ is the $\mu(k,J)$-th index in I, and 0 otherwise. Recall the vector field $\overline{\mathfrak{X}}$ defined in (3-1). The key connection between it and $\eta\lrcorner\,d\overline{w}^J$ is

$$\begin{aligned}(\eta\lrcorner\,d\overline{w}^J)(\overline{\mathfrak{X}})&=\Big(\sum_{k=1}^n(-1)^{\mu(k,J)}\eta_{J\cup\{k\}}d\overline{w}_k\Big)\Big(\sum_{j=1}^n(\overline{w}_j-\bar{z}_j)\frac{\partial}{\partial\overline{w}_j}\Big)\\&=\sum_{k=1}^n(\overline{w}_k-\bar{z}_k)(-1)^{\mu(k,J)}\eta_{J\cup\{k\}}.\end{aligned}\tag{3-5}$$

We now define an m-th order derivative $\overline{\mathfrak{D}}^m\eta$ of a $(0,q+1)$-form η using the interior product. In the case $q=0$ we will have $\overline{\mathfrak{D}}^m\eta=(\overline{\mathfrak{X}}^m\eta)\big[\overline{\mathfrak{X}}\big]$ for a $(0,1)$-form η.

Remark. We are motivated by the fact that the Charpentier kernel $\mathscr{C}_n^{0,q}(w,z)$ takes $(0,q+1)$-forms in w to $(0,q)$-forms in z. Thus in order to express the solution operator $\mathscr{C}_n^{0,q}$ in terms of a volume integral rather than the integration of a form in w and z, our definition of $\overline{\mathfrak{D}}^m\eta$, even when $m=0$, must include an appropriate exchange of w-differentials for z-differentials.

Definition 13. Let $m\geq 0$. For a $(0,q+1)$-form $\eta=\sum_{|I|=q+1}\eta_I\,d\overline{w}^I$ in the variable w, define the $(0,q)$-form $\overline{\mathfrak{D}}^m\eta$ in the variable z by

$$\overline{\mathfrak{D}}^m\eta(w)=\sum_{|J|=q}\overline{\mathfrak{X}}^m(\eta\lrcorner\,d\overline{w}^J)\big[\overline{\mathfrak{X}}\big](w)d\bar{z}^J.$$

Again it is usually understood what the acting and parameter variables are in $\overline{\mathfrak{D}}^m$, but we will write $\overline{\mathfrak{D}}^m_{z,w}\eta(w)$ when this may not be the case. Note that for a $(0,q+1)$-form $\eta=\sum_{|I|=q+1}\eta_I\,d\overline{w}^I$, we have

$$\eta=\sum_{|J|=q}(\eta\lrcorner\,d\overline{w}^J)\wedge d\overline{w}^J,$$

and using (3-2) the preceding definition yields

$$\begin{aligned}\overline{\mathfrak{D}}^m \eta(w) &= \sum_{|J|=q} \overline{\mathfrak{X}}^m (\eta \lrcorner d\overline{w}^J)\big[\overline{\mathfrak{X}}\big](w)\, d\bar{z}^J \\ &= \sum_{|J|=q} \sum_{k=1}^{n} (\overline{w}_k - \bar{z}_k)(-1)^{\mu(k,J)} (\overline{\mathfrak{X}}^m \eta_{J\cup\{k\}})(w) d\bar{z}^J \\ &= \sum_{|J|=q} \sum_{k=1}^{n} (\overline{w}_k - \bar{z}_k)(-1)^{\mu(k,J)} \bigg(\sum_{|\alpha|=m} (\overline{w-z})^\alpha \frac{\partial^m \eta_{J\cup\{k\}}}{\partial \overline{w}^\alpha}(w) \bigg) d\bar{z}^J . \end{aligned} \tag{3-6}$$

Thus the effect of $\overline{\mathfrak{D}}^m$ on a basis element $\eta_I \, d\overline{w}^I$ is to replace a differential $d\overline{w}_k$ from $d\overline{w}^I$ $(I = J \cup \{k\})$ with the factor $(-1)^{\mu(k,J)}(\overline{w}_k - \bar{z}_k)$ (and this is accomplished by acting a $(0,1)$-form on $\overline{\mathfrak{X}}$), replace the remaining differential $d\overline{w}^J$ with $d\bar{z}^J$, and then to apply the differential operator $\overline{\mathfrak{X}}^m$ to the coefficient η_I. We will refer to the factor $(\overline{w}_k - \bar{z}_k)$ introduced above as a *rogue factor* since it is not associated with a derivative $\partial/\partial\overline{w}_k$ in the way that $(\overline{w-z})^\alpha$ is associated with $\partial^m/\partial\overline{w}^\alpha$. The point of this distinction will be explained in Section 7 on estimates for solution operators.

The following lemma expresses $\mathscr{C}_n^{0,q}\eta(z)$ in terms of integrals involving $\overline{\mathfrak{D}}^j \eta$ for $0 \le j \le m$. Note that the overall effect is to reduce the singularity of the kernel on the diagonal by m factors of $\sqrt{\triangle(w,z)}$, at the cost of increasing by m the number of derivatives hitting the form η. Recall from (2-5) that

$$\Phi_n^\ell(w,z) \equiv \frac{(1-w\bar{z})^{n-1-\ell}(1-|w|^2)^\ell}{\triangle(w,z)^n}.$$

We define the operator Φ_n^ℓ on forms η by

$$\Phi_n^\ell \eta(z) = \int_{\mathbb{B}_n} \Phi_n^\ell(w,z)\eta(w)\, dV(w).$$

Lemma 14. *Let $q \ge 0$. For all $m \ge 0$ we have*

$$\mathscr{C}_n^{0,q}\eta(z) = \sum_{k=0}^{m-1} c_k \mathscr{S}_n(\overline{\mathfrak{D}}^j \eta)(z) + \sum_{\ell=0}^{q} c_\ell \Phi_n^\ell(\overline{\mathfrak{D}}^m \eta)(z). \tag{3-7}$$

The proof is simply a reprise of that of Lemma 12 (see the proof of Lemma 2.1 of [Ortega and Fàbrega 2000]) complicated by the algebra that reduces matters to $(0,1)$-forms.

3.1. ***The radial derivative.*** Recall the radial derivative $R = \sum_{j=1}^n w_j \dfrac{\partial}{\partial w_j}$. The following lemma is essentially Lemma 2.2 on page 58 of [Ortega and Fàbrega 2000].

Lemma 15. *Let $b > -1$. For $\Psi \in C(\overline{\mathbb{B}}_n) \cap C^\infty(\mathbb{B}_n)$ we have*

$$\int_{\mathbb{B}_n} (1-|w|^2)^b \Psi(w)\, dV(w) = \int_{\mathbb{B}_n} (1-|w|^2)^{b+1} \Big(\frac{n+b+1}{b+1} I + \frac{1}{b+1} R \Big) \Psi(w)\, dV(w).$$

Remark. Typically this lemma is applied with

$$\Psi(w) = \frac{1}{(1-\overline{w}z)^s} \psi(w,z),$$

where z is a parameter in the ball $\mathbb{B}_n$ and

$$R\Psi(w) = \frac{1}{(1-\overline{w}z)^s} R\psi(w,z)$$

since $\frac{1}{(1-\overline{w}z)^s}$ is antiholomorphic in w.

We will also need to iterate Lemma 15, and for this purpose it is convenient to introduce for $m \geq 1$ the notation

$$R_b = R_{b,n} = \frac{n+b+1}{b+1} I + \frac{1}{b+1} R, \quad R_b^m = R_{b+m-1} R_{b+m-2} \dots R_b = \prod_{k=1}^{m} R_{b+m-k}.$$

Corollary 16. *Let $b > -1$. For $\Psi \in C(\overline{\mathbb{B}}_n) \cap C^\infty(\mathbb{B}_n)$ we have*

$$\int_{\mathbb{B}_n} (1-|w|^2)^b \Psi(w)\, dV(w) = \int_{\mathbb{B}_n} (1-|w|^2)^{b+m} R_b^m \Psi(w)\, dV(w).$$

Remark. The important point in Corollary 16 is that combinations of radial derivatives R and the identity I are played off against powers of $1-|w|^2$. It will sometimes be convenient to write this identity as

$$\int_{\mathbb{B}_n} F(w)\, dV(w) = \int_{\mathbb{B}_n} \mathcal{R}_b^m F(w)\, dV(w),$$

where

$$\mathcal{R}_b^m \equiv (1-|w|^2)^{b+m} R_b^m (1-|w|^2)^{-b}. \tag{3-8}$$

In this form the identity is valid for F such that $\Psi(w) = (1-|w|^2)^{-b} F(w)$ lies in $C(\overline{\mathbb{B}}_n) \cap C^\infty(\mathbb{B}_n)$.

3.2. ***Integration by parts in ameliorated kernels.*** We must now extend Lemma 14 and Corollary 16 to the ameliorated kernels $\mathcal{C}_{n,s}^{0,q}$ given by

$$\mathcal{C}_{n,s}^{0,q} = \mathsf{R}_n \mathcal{C}_s^{0,q} \mathsf{E}_s.$$

Since Corollary 16 already applies to very general functions $\Psi(w)$, we need only consider an extension of Lemma 14. The procedure for doing this is to apply Lemma 14 to $\mathcal{C}_s^{0,q}$ in s dimensions, and then integrate out the additional variables using (2-19).

Lemma 17. *Suppose that $s > n$ and $0 \leq q \leq n-1$. For all $m \geq 0$ and smooth $(0, q+1)$-forms η in $\overline{\mathbb{B}}_n$ we have the formula*

$$\mathcal{C}_{n,s}^{0,q} \eta(z) = \sum_{k=0}^{m-1} c'_{k,n,s} \mathcal{S}_{n,s}(\overline{\mathcal{D}}^k \eta)[\overline{\mathcal{X}}](z) + \sum_{\ell=0}^{q} c_{\ell,n,s} \Phi_{n,s}^\ell (\overline{\mathcal{D}}^m \eta)(z),$$

where the ameliorated operators $\mathcal{S}_{n,s}$ and $\Phi_{n,s}^\ell$ have kernels given by

$$\mathcal{S}_{n,s}(w,z) = c_{n,s} \frac{(1-|w|^2)^{s-n-1}}{(1-\overline{w}z)^s} = c_{n,s} \left(\frac{1-|w|^2}{1-\overline{w}z}\right)^{s-n-1} \frac{1}{(1-\overline{w}z)^{n+1}},$$

$$\Phi_{n,s}^\ell(w,z) = \Phi_n^\ell(w,z) \left(\frac{1-|w|^2}{1-\overline{w}z}\right)^{s-n} \sum_{j=0}^{n-\ell-1} c_{j,n,s} \left(\frac{(1-|w|^2)(1-|z|^2)}{|1-w\bar{z}|^2}\right)^j.$$

Proof. Recall that for a smooth $(0,q+1)$-form $\eta(w)=\sum_{|I|=q+1}\eta_I\,d\overline{w}^I$ in $\overline{\mathbb{B}}_n$, the $(0,q)$-form $\overline{\mathfrak{D}}^m\mathsf{E}_s\eta$ is given by

$$\begin{aligned}\overline{\mathfrak{D}}^m\mathsf{E}_s\eta(w)&=\sum_{|J|=q}\overline{\mathfrak{D}}^m(\eta\lrcorner\,d\overline{w}^J)\,d\bar{z}^J=\sum_{|J|=q}\overline{\mathfrak{D}}^m\biggl(\sum_{k\notin J}(-1)^{\mu(k,J)}\eta_{J\cup\{k\}}(w)\,d\overline{w}_k\biggr)\,d\bar{z}^J\\&=\sum_{|J|=q}\overline{\mathfrak{D}}^m\biggl(\sum_{k\notin J}(-1)^{\mu(k,J)}\eta_{J\cup\{k\}}(w)\,d\overline{w}_k\biggr)\,d\bar{z}^J\\&=\sum_{|J|=q}\sum_{k\notin J}(-1)^{\mu(k,J)}\biggl(\sum_{|\alpha|=m}(\overline{w_k-z_k})\overline{(w-z)^\alpha}\,\frac{\partial^m}{\partial\overline{w}^\alpha}\eta_{J\cup\{k\}}(w)\biggr),\end{aligned}$$

where $J\cup\{k\}$ is a multi-index with entries in $\mathfrak{I}_n\equiv\{1,2,\dots,n\}$ since the coefficient η_I vanishes if I is not contained in $\mathfrak{I}_n$. Moreover, the multi-index α lies in $(\mathfrak{I}_n)^m$ since the coefficients η_I are constant in the variable $w'=(w_{n+1},\dots,w_s)$. Thus

$$\overline{\mathfrak{D}}^m_{(z,0),(w,w')}\mathsf{E}_s\eta=\overline{\mathfrak{D}}^m_{z,w}\eta=\overline{\mathfrak{D}}^m\eta,$$

and we compute

$$\begin{aligned}&\mathsf{R}_n\Phi_s^\ell\bigl(\overline{\mathfrak{D}}^m_{(z,0),(w,w')}\mathsf{E}_s\eta\bigr)(z)\\&\qquad=\Phi_s^\ell(\overline{\mathfrak{D}}^m\eta)((z,0))\\&\qquad=\sum_{|J|=q}\sum_{k\in\mathfrak{I}_n\setminus J}(-1)^{\mu(k,J)}\sum_{|\alpha|=m}\Phi_s^\ell\biggl((\overline{w_k-z_k})\overline{(w-z)^\alpha}\,\frac{\partial^m}{\partial\overline{w}^\alpha}\eta_{J\cup\{k\}}((w,w'))\biggr)((z,0)),\end{aligned}$$

where $J\cup\{k\}\subset\mathfrak{I}_n$ and $\alpha\in(\mathfrak{I}_n)^m$ and

$$\begin{aligned}&\Phi_s^\ell\biggl((\overline{w_k-z_k})\overline{(w-z)^\alpha}\,\frac{\partial^m}{\partial\overline{w}^\alpha}\eta_{J\cup\{k\}}(w)\biggr)((z,0))\\&\quad=\int_{\mathbb{B}_s}\frac{(1-w\bar{z})^{s-1-\ell}(1-|w|^2-|w'|^2)^\ell}{\triangle((w,w'),(z,0))^s}\,(\overline{w_k-z_k})\overline{(w-z)^\alpha}\,\frac{\partial^m}{\partial\overline{w}^\alpha}\eta_{J\cup\{k\}}(w)\,dV((w,w'))\\&\quad=\int_{\mathbb{B}_n}\biggl\{(1-w\bar{z})^{s-\ell-1}\int_{\mathbb{B}_{s-n}}\frac{(1-|w|^2-|w'|^2)^\ell}{\triangle((w,w'),(z,0))^s}\,dV(w')\biggr\}(\overline{w_k-z_k})\overline{(w-z)^\alpha}\,\frac{\partial^m}{\partial\overline{w}^\alpha}\eta_{J\cup\{k\}}(w)\,dV(w).\end{aligned}$$

By (2-19) the term in braces on the previous line equals

$$\frac{\pi^{s-n}}{(s-n)!}\Phi_n^\ell(w,z)\biggl(\frac{1-|w|^2}{1-\overline{w}z}\biggr)^{s-n}\sum_{j=0}^{n-\ell-1}c_{j,n,s}\biggl(\frac{(1-|w|^2)(1-|z|^2)}{|1-w\bar{z}|^2}\biggr)^j,$$

and now performing the sum $\sum_{|J|=q}\sum_{k\in\mathfrak{I}_n\setminus J}(-1)^{\mu(k,J)}\sum_{|\alpha|=m}$ yields

$$\mathsf{R}_n\Phi_s^\ell\Bigl(\overline{\mathfrak{D}}^m_{(z,0)}\mathsf{E}_s\eta\Bigr)(z)=\Phi_s^\ell(\overline{\mathfrak{D}}^m_z\,\eta)((z,0))=\Phi_{n,s}^\ell(\overline{\mathfrak{D}}^m_z\,\eta)(z).\tag{3-9}$$

An even easier calculation using formula (1) in 1.4.4 on page 14 of [Rudin 1980] shows that

$$\mathsf{R}_n\mathcal{S}_s(\mathsf{E}_s\overline{\mathfrak{D}}^k_z\eta)((z,0))=\mathcal{S}_s(\overline{\mathfrak{D}}^k_z\eta)((z,0))=\mathcal{S}_{n,s}(\overline{\mathfrak{D}}^k_z\eta)(z),\tag{3-10}$$

and now the conclusion of Lemma 17 follows from (3-9), (3-10), the definition $\mathscr{C}_{n,s}^{0,q} = \mathsf{R}_n \mathscr{C}_s^{0,q} \mathsf{E}_s$, and Lemma 14. $\square$

4. The Koszul complex

Here we briefly review the algebra behind the Koszul complex as presented for example in [Lin 1994] in the finite-dimensional setting. A more detailed treatment in that setting can be found in Section 5.5.3 of [Sawyer 2009]. Fix h holomorphic as in (1-7). Now if $g = (g_j)_{j=1}^{\infty}$ satisfies $|g|^2 = \sum_{j=1}^{\infty} |g_j|^2 \geq \delta^2 > 0$, define

$$\Omega_0^1 = \frac{\bar{g}}{|g|^2} = \left(\frac{\bar{g}_j}{|g|^2}\right)_{j=1}^{\infty} = \left(\Omega_0^1(j)\right)_{j=1}^{\infty},$$

which we view as a 1-tensor (in $\ell^2 = \mathbb{C}^{\infty}$) of $(0,0)$-forms with components $\Omega_0^1(j) = \bar{g}_j/|g|^2$. Then $f = \Omega_0^1 h$ satisfies $\mathcal{M}_g f = f \cdot g = h$, but in general fails to be holomorphic. The Koszul complex provides a scheme which we now recall for solving a sequence of $\bar{\partial}$ equations that result in a correction term $\Lambda_g \Gamma_0^2$ that, when subtracted from f above, yields a holomorphic solution to the equality in (1-7). See below.

The 1-tensor of $(0,1)$-forms

$$\bar{\partial}\Omega_0 = \left(\bar{\partial}\frac{\bar{g}_j}{|g|^2}\right)_{j=1}^{\infty} = \left(\bar{\partial}\Omega_0^1(j)\right)_{j=1}^{\infty}$$

is given by

$$\bar{\partial}\Omega_0^1(j) = \bar{\partial}\frac{\bar{g}_j}{|g|^2} = \frac{|g|^2\bar{\partial}\bar{g}_j - \bar{g}_j\bar{\partial}|g|^2}{|g|^4} = \frac{1}{|g|^4}\sum_{k=1}^{\infty} g_k \overline{(g_k \partial g_j - g_j \partial g_k)},$$

and can be written as

$$\bar{\partial}\Omega_0^1 = \Lambda_g \Omega_1^2 \equiv \left(\sum_{k=1}^{\infty} \Omega_1^2(j,k) g_k\right)_{j=1}^{\infty},$$

where the antisymmetric 2-tensor Ω_1^2 of $(0,1)$-forms is given by

$$\Omega_1^2 = \left[\Omega_1^2(j,k)\right]_{j,k=1}^{\infty} = \left[\frac{\overline{g_k \partial g_j - g_j \partial g_k}}{|g|^4}\right]_{j,k=1}^{\infty}.$$

and $\Lambda_g \Omega_1^2$ denotes its contraction by the vector g in the final variable.

We can repeat this process and by induction we have

$$\bar{\partial}\Omega_q^{q+1} = \Lambda_g \Omega_{q+1}^{q+2}, \quad 0 \leq q \leq n, \tag{4-1}$$

where Ω_q^{q+1} is an alternating $(q+1)$-tensor of $(0,q)$-forms. Recall that h is holomorphic. When $q = n$ we have that $\Omega_n^{n+1} h$ is $\bar{\partial}$-closed and this allows us to solve a chain of $\bar{\partial}$ equations

$$\bar{\partial}\Gamma_{q-2}^{q} = \Omega_{q-1}^{q} h - \Lambda_g \Gamma_{q-1}^{q+1},$$

for alternating q-tensors Γ^q_{q-2} of $(0, q-2)$-forms, using the ameliorated Charpentier solution operators $\mathscr{C}^{0,q}_{n,s}$ defined in (2-17). (Note that our notation suppresses the dependence of Γ on h.) With the convention that $\Gamma^{n+2}_n \equiv 0$ we have

$$\bar\partial\big(\Omega^{q+1}_q h - \Lambda_g \Gamma^{q+2}_q\big) = 0, \quad 0 \le q \le n, \tag{4-2}$$

and

$$\bar\partial\Gamma^{q+1}_{q-1} = \Omega^{q+1}_q h - \Lambda_g \Gamma^{q+2}_q, \quad 1 \le q \le n.$$

Now

$$f \equiv \Omega^1_0 h - \Lambda_g \Gamma^2_0$$

is holomorphic by (4-2) with $q = 0$, and since Γ^2_0 is antisymmetric, we compute that $\Lambda_g \Gamma^2_0 \cdot g = \Gamma^2_0(g, g) = 0$ and

$$\mathcal{M}_g f = f \cdot g = \Omega^1_0 h \cdot g - \Lambda_g \Gamma^2_0 \cdot g = h - 0 = h.$$

Thus $f = (f_i)^\infty_{i=1}$ is a vector of holomorphic functions satisfying the equality in (1-7). The inequality in (1-7) is the subject of the remaining sections of the paper.

4.1. ***Wedge products and factorization of the Koszul complex.*** Here we record the remarkable factorization of the Koszul complex in [Andersson and Carlsson 2000]. To describe the factorization we introduce an exterior algebra structure on $\ell^2 = \mathbb{C}^\infty$. Let $\{e_1, e_2, \dots\}$ be the usual basis in $\mathbb{C}^\infty$, and for an increasing multiindex $I = (i_1, \dots, i_\ell)$ of integers in $\mathbb{N}$, define

$$e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_\ell},$$

where we use $\wedge$ to denote the wedge product in the exterior algebra $\Lambda^*(\mathbb{C}^\infty)$ of $\mathbb{C}^\infty$, as well as for the wedge product on forms in $\mathbb{C}^n$. Note that $\{e_I : |I| = r\}$ is a basis for the alternating r-tensors on $\mathbb{C}^\infty$.

If $f = \sum_{|I|=r} f_I e_I$ is an alternating r-tensor on $\mathbb{C}^\infty$ with values that are $(0, k)$-forms in $\mathbb{C}^n$, which may be viewed as a member of the exterior algebra of $\mathbb{C}^\infty \otimes \mathbb{C}^n$, and if $g = \sum_{|J|=s} g_J e_J$ is an alternating s-tensor on $\mathbb{C}^\infty$ with values that are $(0, \ell)$-forms in $\mathbb{C}^n$, then as in [Andersson and Carlsson 2000] we define the wedge product $f \wedge g$ in the exterior algebra of $\mathbb{C}^\infty \otimes \mathbb{C}^n$ to be the alternating $(r+s)$-tensor on $\mathbb{C}^\infty$ with values that are $(0, k+\ell)$-forms in $\mathbb{C}^n$ given by

$$\begin{aligned} f \wedge g &= \Big(\sum_{|I|=r} f_I e_I\Big) \wedge \Big(\sum_{|J|=s} g_J e_J\Big) = \sum_{\substack{|I|=r \\ |J|=s}} (f_I \wedge g_J)(e_I \wedge e_J) \\ &= \sum_{|K|=r+s} \Big(\pm \sum_{I+J=K} f_I \wedge g_J\Big) e_K. \end{aligned} \tag{4-3}$$

Note that we simply write the exterior product of an element from $\Lambda^*(\mathbb{C}^\infty)$ with an element from $\Lambda^*(\mathbb{C}^n)$ as juxtaposition, without an explicit wedge symbol. This should cause no confusion since the basis we use in $\Lambda^*(\mathbb{C}^\infty)$ is $\{e_i\}^\infty_{i=1}$, while the basis we use in $\Lambda^*(\mathbb{C}^n)$ is $\{dz_j, d\hat{z}_j\}^n_{j=1}$, quite different in both appearance and interpretation.

In terms of this notation we then have the following factorization in Theorem 3.1 of [Andersson and Carlsson 2000]:

$$\Omega_0^1 \wedge \bigwedge_{i=1}^{\ell} \widetilde{\Omega}_0^1 = \left(\sum_{k_0=1}^{\infty} \frac{\overline{g}_{k_0}}{|g|^2} e_{k_0} \right) \wedge \bigwedge_{i=1}^{\ell} \left(\sum_{k_i=1}^{\infty} \frac{\overline{\partial} \overline{g}_{k_i}}{|g|^2} e_{k_i} \right) = -\frac{1}{\ell+1} \Omega_{\ell}^{\ell+1}, \tag{4-4}$$

where

$$\Omega_0^1 = \left(\frac{\overline{g}_i}{|g|^2} \right)_{i=1}^{\infty} \quad \text{and} \quad \widetilde{\Omega}_0^1 = \left(\frac{\overline{\partial} \overline{g}_i}{|g|^2} \right)_{i=1}^{\infty}.$$

The factorization in [Andersson and Carlsson 2000] is proved in the finite-dimensional case, but this extends to the infinite-dimensional case by continuity. Since the ℓ^2 norm is quasimultiplicative on wedge products by Lemma 5.1 in that reference we have

$$\left| \Omega_{\ell}^{\ell+1} \right|^2 \leq C_{\ell} \left| \Omega_0^1 \right|^2 \left| \widetilde{\Omega}_0^1 \right|^{2\ell}, \quad 0 \leq \ell \leq n, \tag{4-5}$$

where the constant C_{ℓ} depends only on the number of factors ℓ in the wedge product, and *not* on the underlying dimension of the vector space (which is infinite for $\ell^2 = \mathbb{C}^{\infty}$).

It will be useful in the next section to consider also tensor products

$$\widetilde{\Omega}_0^1 \otimes \widetilde{\Omega}_0^1 = \left(\sum_{i=1}^{\infty} \frac{\overline{\partial} \overline{g}_i}{|g|^2} e_i \right) \otimes \left(\sum_{j=1}^{\infty} \frac{\overline{\partial} \overline{g}_j}{|g|^2} e_j \right) = \sum_{i,j=1}^{\infty} \frac{\overline{\partial} \overline{g}_i \otimes \overline{\partial} \overline{g}_j}{|g|^4} e_i \otimes e_j, \tag{4-6}$$

and more generally $\mathcal{X}^{\alpha} \widetilde{\Omega}_0^1 \otimes \mathcal{X}^{\beta} \widetilde{\Omega}_0^1$, where $\mathcal{X}^m$ denotes the vector derivative defined in Definition 19 below. We will use the fact that the ℓ^2-norm is *multiplicative* on tensor products.

5. An almost invariant holomorphic derivative

We continue to consider ℓ^2-valued spaces. We refer the reader to [Arcozzi et al. 2006] for the definition of the Bergman tree $\mathcal{T}_n$ and the corresponding pairwise disjoint decomposition of the ball $\mathbb{B}_n$:

$$\mathbb{B}_n = \dot{\bigcup_{\alpha \in \mathcal{T}_n}} K_{\alpha},$$

where the sets K_{α} are comparable to balls of radius one in the Bergman metric β on the ball $\mathbb{B}_n$:

$$\beta(z, w) = \frac{1}{2} \ln \frac{1 + |\varphi_z(w)|}{1 - |\varphi_z(w)|}$$

(see Proposition 1.21 in [Zhu 2005]). This decomposition gives an analogue in $\mathbb{B}_n$ of the standard decomposition of the upper half-plane $\mathbb{C}_+$ into dyadic squares whose distance from the boundary $\partial\mathbb{C}_+$ equals their side length. We also recall from [Arcozzi et al. 2006] the differential operator D_a which on the Bergman kube K_{α}, and provided $a \in K_{\alpha}$, is close to the invariant gradient $\widetilde{\nabla}$, and which has the additional property that $D_a^m f(z)$ is holomorphic for $m \geq 1$ and $z \in K_{\alpha}$ when f is holomorphic. For our purposes the powers $D_a^m f$, $m \geq 1$, are easier to work with than the corresponding powers $\widetilde{\nabla}^m f$, which fail to be holomorphic. It is shown in the same paper that D_a^m can be used to define an equivalent norm on the Besov space $B_p(\mathbb{B}_n) = B_p^0(\mathbb{B}_n)$, and it is a routine matter to extend this result to the Besov–Sobolev space $B_p^\sigma(\mathbb{B}_n)$ when $\sigma \geq 0$ and $m > 2(\frac{n}{p} - \sigma)$. The further extension to ℓ^2 -valued functions is also routine.

We define

$$\nabla_z = \left(\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}\right) \quad \text{and} \quad \overline{\nabla}_z = \left(\frac{\partial}{\partial \bar{z}_1}, \dots, \frac{\partial}{\partial \bar{z}_n}\right),$$

so that the usual Euclidean gradient is given by the pair $(\nabla_z, \overline{\nabla}_z)$. Fix $\alpha \in \mathcal{T}_n$ and let $a = c_\alpha$. Recall that the gradient with invariant length given by

$$\widetilde{\nabla} f(a) = (f \circ \varphi_a)'(0) = f'(a)\varphi_a'(0) = -f'(a)\big((1-|a|^2)P_a + (1-|a|^2)^{1/2}Q_a\big)$$

fails to be holomorphic in a. To rectify this, we define, as in [Arcozzi et al. 2006],

$$D_a f(z) = f'(z)\varphi_a'(0) = -f'(z)\big((1-|a|^2)P_a + (1-|a|^2)^{1/2}Q_a\big), \tag{5-1}$$

for $z \in \mathbb{B}_n$.

In order to deal with functions f on $\mathbb{B}_n$ that are not necessarily holomorphic, we use a notion of higher-order derivative D^m introduced in [Arcozzi et al. 2006], based on iterating D_a rather than $\widetilde{\nabla}$.

Definition 18. For $m \in \mathbb{N}$ and $f \in C^\infty(\mathbb{B}_n; \ell^2)$ smooth in $\mathbb{B}_n$ we define $\Theta^m f(a, z) = D_a^m f(z)$ for $a, z \in \mathbb{B}_n$, and then set

$$D^m f(z) = \Theta^m f(z, z) = D_z^m f(z), \quad z \in \mathbb{B}_n.$$

Note that in this definition, we iterate the operator D_z holding z fixed, and then evaluate the result at the same z. We obtain that for $f \in H(\mathbb{B}_n; \ell^2)$ (see [Arcozzi et al. 2006] and [Beatrous 1986]),

$$\|f\|_{B^\sigma_{p,m}(\mathbb{B}_n;\ell^2)} \approx \sum_{j=0}^{m-1} |\nabla^j f(0)| + \left(\int_{\mathbb{B}_n} \left|(1-|z|^2)^\sigma D^m f(z)\right|^p d\lambda_n(z)\right)^{1/p}.$$

We remind the reader that $|D_a^m f(z)| = \sqrt{\sum_{i=1}^\infty |D_a^m f_i(z)|^2}$ if $f = (f_i)_{i=1}^\infty$.

We will also need to know that the pointwise multipliers in $M_{B_p^\sigma(\mathbb{B}_n) \to B_p^\sigma(\mathbb{B}_n;\ell^2)}$ are bounded. Indeed, standard arguments show that

$$M_{B_p^\sigma(\mathbb{B}_n) \to B_p^\sigma(\mathbb{B}_n;\ell^2)} \subset H^\infty(\mathbb{B}_n; \ell^2) \cap B_p^\sigma(\mathbb{B}_n; \ell^2). \tag{5-2}$$

5.1. ***Real variable analogues of Besov–Sobolev spaces.*** In order to handle the operators arising from integration by parts formulas below, we will need yet more general equivalent norms on $B^\sigma_{p,m}(\mathbb{B}_n; \ell^2)$.

Definition 19. We denote by $\mathcal{X}^m$ the vector of all differential operators of the form $X_1 X_2 \dots X_m$ where each X_i is either $1 - |z|^2$ times the identity operator I, the operator $\overline{D}$, or the operator $(1-|z|^2)R$. Just as in Definition 18, we calculate the products $X_1 X_2 \dots X_m$ by composing $\overline{D}_a$ and $(1-|a|^2)R$ and then setting $a = z$ at the end. Note that $\overline{D}_a$ and $(1-|a|^2)R$ commute since the first is an antiholomorphic derivative and the coefficient z in $R = z \cdot \nabla$ is holomorphic. Similarly we denote by $\mathcal{Y}^m$ the corresponding products of $(1-|z|^2)I$, D (instead of $\overline{D}$) and $(1-|z|^2)R$.

In the iterated derivative $\mathcal{X}^m$ we are differentiating only with the *antiholomorphic* derivative $\overline{D}$ or the *holomorphic* derivative R. When f is holomorphic, we thus have $\mathcal{X}^m f \sim \{(1-|z|^2)^m R^k f\}_{k=0}^m$. The reason we allow $1-|z|^2$ times the identity I to occur in $\mathcal{X}^m$ is that this produces a norm (as opposed to just a seminorm) without including the term $\sum_{k=0}^{m-1} |\nabla^k f(0)|$. We define the norm $\|\cdot\|_{B^\sigma_{p,m}(\mathbb{B}_n;\ell^2)}$ for *smooth* f on the ball $\mathbb{B}_n$ by

$$\|f\|_{B^\sigma_{p,m}(\mathbb{B}_n;\ell^2)} \equiv \left(\sum_{k=0}^m \int_{\mathbb{B}_n} \left|(1-|z|^2)^{m+\sigma} R^k f(z)\right|^p d\lambda_n(z) \right)^{1/p},$$

and note that provided $m+\sigma > \frac{n}{p}$, this gives an equivalent norm for the Besov–Sobolev space $B^\sigma_p(\mathbb{B}_n;\ell^2)$ of holomorphic functions on $\mathbb{B}_n$ (see [Beatrous 1986], for instance). These considerations motivate the following two definitions of a *real-variable* analogue of the norm $\|\cdot\|_{B^\sigma_{p,m}(\mathbb{B}_n;\ell^2)}$.

Definition 20. We define the *norms* $\|\cdot\|_{\Lambda^\sigma_{p,m}(\mathbb{B}_n;\ell^2)}$ and $\|\cdot\|_{\Phi^\sigma_{p,m}(\mathbb{B}_n;\ell^2)}$ for $f = (f_i)_{i=1}^\infty$ smooth on the ball $\mathbb{B}_n$ by

$$\begin{aligned} \|f\|_{\Lambda^\sigma_{p,m}(\mathbb{B}_n;\ell^2)} &\equiv \left(\int_{\mathbb{B}_n} \left|(1-|z|^2)^\sigma \mathcal{X}^m f(z)\right|^p d\lambda_n(z) \right)^{1/p}, \\ \|f\|_{\Phi^\sigma_{p,m}(\mathbb{B}_n;\ell^2)} &\equiv \left(\int_{\mathbb{B}_n} \left|(1-|z|^2)^\sigma \mathcal{Y}^m f(z)\right|^p d\lambda_n(z) \right)^{1/p}. \end{aligned} \tag{5-3}$$

It is *not* true that either of the norms $\|\cdot\|_{\Lambda^\sigma_{p,m}(\mathbb{B}_n;\ell^2)}$ or $\|\cdot\|_{\Phi^\sigma_{p,m}(\mathbb{B}_n;\ell^2)}$ are independent of m for large m when acting on smooth functions. However, these norms are equivalent when restricted to holomorphic vector functions (see [Arcozzi et al. 2006] and [Beatrous 1986]):

Lemma 21. *Let $1 < p < \infty$, $\sigma \ge 0$ and $m > 2(\frac{n}{p} - \sigma)$. If f is a holomorphic vector function, then*

$$\|f\|_{B^\sigma_{p,m}(\mathbb{B}_n;\ell^2)} \approx \|f\|_{\Lambda^\sigma_{p,m}(\mathbb{B}_n;\ell^2)} \approx \|f\|_{\Phi^\sigma_{p,m}(\mathbb{B}_n;\ell^2)}. \tag{5-4}$$

The norms $\|\cdot\|_{\Lambda^\sigma_{p,m}(\mathbb{B}_n;\ell^2)}$ arise in the integration by parts in iterated Charpentier kernels in Section 7, while the norms $\|\cdot\|_{\Phi^\sigma_{p,m}(\mathbb{B}_n;\ell^2)}$ are useful for estimating the holomorphic function g in the Koszul complex. For this latter purpose we will use the following multilinear inequality whose scalar version is, after translating notation, Theorem 3.5 in [Ortega and Fàbrega 2000].

Proposition 22. *Suppose that $1 < p < \infty$, $0 \le \sigma < \infty$, $M \ge 1$, $m > 2(\frac{n}{p} - \sigma)$ and $\alpha = (\alpha_0, \dots, \alpha_M) \in \mathbb{Z}_+^{M+1}$ with $|\alpha| = m$. For $g \in M_{B^\sigma_p(\mathbb{B}_n) \to B^\sigma_p(\mathbb{B}_n;\ell^2)}$ and $h \in B^\sigma_p(\mathbb{B}_n)$ we have*

$$\int_{\mathbb{B}_n} (1-|z|^2)^{p\sigma} \left|(\mathcal{Y}^{\alpha_1} g)(z)\right|^p \dots \left|\left(\mathcal{Y}^{\alpha_M} g\right)(z)\right|^p \left|(\mathcal{Y}^{\alpha_0} h)(z)\right|^p d\lambda_n(z) \\ \le C_{n,M,\sigma,p} \|\mathbb{M}_g\|^{Mp}_{B^\sigma_p(\mathbb{B}_n) \to B^\sigma_p(\mathbb{B}_n;\ell^2)} \|h\|^p_{B^\sigma_p(\mathbb{B}_n)}.$$

Remark. The inequalities for $M = 1$ in Proposition 22 actually characterize multipliers g in the sense that a function $g \in B^\sigma_p(\mathbb{B}_n;\ell^2) \cap H^\infty(\mathbb{B}_n;\ell^2)$ is in $M_{B^\sigma_p(\mathbb{B}_n) \to B^\sigma_p(\mathbb{B}_n;\ell^2)}$ *if and only if* the inequalities with $M = 1$ in Proposition 22 hold. This follows from noting that each term in the Leibniz expansion of $\mathcal{Y}^m(gh)$ occurs on the left side of the display above with $M = 1$.

Proposition 22 is proved by adapting the proof of Theorem 3.5 in [Ortega and Fàbrega 2000] to ℓ^2-valued functions. This argument uses the complex interpolation theorem of Beatrous [1986] and Ligocka [1987], which extends to Hilbert space valued functions with the same proof. In order to apply this extension we will need the following operator norm inequality.

If $\varphi \in M_{B_p^\sigma(\mathbb{B}_n)\to B_p^\sigma(\mathbb{B}_n;\ell^2)}$ and $f = \sum_{|I|=\kappa} f_I e_I \in B_p^\sigma(\mathbb{B}_n;\otimes^{\kappa-1}\ell^2)$, we define

$$\mathbb{M}_\varphi f = \varphi \otimes f = \varphi \otimes \Bigg(\sum_{|I|=\kappa-1} f_I e_I\Bigg) = \sum_{|I|=\kappa-1} (\varphi f_I)\otimes e_I,$$

where $I=(i_1,\dots,i_{\kappa-1})\in\mathbb{N}^{\kappa-1}$ and $e_I = e_{i_1}\otimes\cdots\otimes e_{i_{\kappa-1}}$.

Lemma 23. *Suppose that $\sigma\ge 0$, $1<p<\infty$ and $\kappa\ge 1$. There is a constant $C_{n,\sigma,p,\kappa}$ such that*

$$\|\mathbb{M}_g\|_{B_p^\sigma(\mathbb{B}_n;\otimes^{\kappa-1}\ell^2)\to B_p^\sigma(\mathbb{B}_n;\otimes^\kappa\ell^2)} \le C_{n,\sigma,p,\kappa}\,\|\mathbb{M}_g\|_{B_p^\sigma(\mathbb{B}_n;\ell^2)\to B_p^\sigma(\mathbb{B}_n;\ell^2)}. \tag{5-5}$$

In the case $p=2$ we have equality:

$$\|\mathbb{M}_\varphi\|_{B_2^\sigma(\mathbb{B}_n;\otimes^{\kappa-1}\ell^2)\to B_2^\sigma(\mathbb{B}_n;\otimes^\kappa\ell^2)} = \|\mathbb{M}_\varphi\|_{B_2^\sigma(\mathbb{B}_n)\to B_2^\sigma(\mathbb{B}_n;\ell^2)}. \tag{5-6}$$

The proof of Lemma 23 uses the well-known technique of extending bounded linear operators on L^p to ℓ^2-valued L^p with the same norm (see, for instance, page 451 in [Stein 1993]). It turns out that in order to prove (5-5) for $p\ne 2$ we will need the case $M=1$ of Proposition 22. Fortunately, the case $M=1$ does not require inequality (5-5), thus avoiding circularity. The proofs of Proposition 22 and Lemma 23 reduce to modifying existing arguments in the literature and the details can be found in the Electronic Supplement.

Three crucial inequalities. In order to establish appropriate inequalities for the Charpentier solution operators, we will need to control terms of the form

$$(\overline{w-z})^\alpha \frac{\partial^m}{\partial\overline{w}^\alpha}F(w),\quad \overline{D}^m_{(z)}\triangle(w,z),\quad \overline{D}^m_{(z)}\big((1-w\bar z)^k\big)\quad\text{and}\quad R^m_{(z)}\big((1-\overline{w}z)^k\big)$$

inside the integral for T as given in the integration by parts formula in Lemma 14. Here we are using the subscript (z) in parentheses to indicate the variable being differentiated. This is to avoid confusion with the notation D_a introduced in (5-1). For $z,w\in\mathbb{B}_n$ and $m\in\mathbb{N}$, we have the crucial estimates

$$\left|(\overline{w-z})^\alpha \frac{\partial^m}{\partial\overline{w}^\alpha}F(w)\right| \le C\left(\frac{\sqrt{\triangle(w,z)}}{1-|w|^2}\right)^m |\overline{D}^m F(w)|,\quad F\in H(\mathbb{B}_n;\ell^2),\ m=|\alpha|, \tag{5-7}$$

$$\begin{cases} \big|D_{(z)}\triangle(w,z)\big| \le C\big((1-|z|^2)\triangle(w,z)^{1/2}+\triangle(w,z)\big),\\ \big|(1-|z|^2)R_{(z)}\triangle(w,z)\big| \le C(1-|z|^2)\sqrt{\triangle(w,z)}, \end{cases} \tag{5-8}$$

$$\begin{cases} \big|D^m_{(z)}\big((1-\overline{w}z)^k\big)\big| \le C|1-\overline{w}z|^k\left(\dfrac{1-|z|^2}{|1-\overline{w}z|}\right)^{m/2},\\ \big|(1-|z|^2)^m R^m_{(z)}\big((1-\overline{w}z)^k\big)\big| \le C|1-\overline{w}z|^k\left(\dfrac{1-|z|^2}{|1-\overline{w}z|}\right)^m. \end{cases} \tag{5-9}$$

Proof of (5-7). We view D_a as a differentiation operator in the variable w, so that

$$D_a = -\nabla_w\big((1-|a|^2)P_a + \sqrt{1-|a|^2}\,Q_a\big).$$

A basic calculation is then:

$$\begin{aligned}(1-\bar{a}z)\varphi_a(z)\cdot(D_a)^t &= \big(P_a(z-a)+\sqrt{1-|a|^2}\,Q_a(z-a)\big)\big((1-|a|^2)P_a\nabla_w+\sqrt{1-|a|^2}\,Q_a\nabla_w\big)\\ &= P_a(z-a)(1-|a|^2)P_a\nabla_w+\sqrt{1-|a|^2}\,Q_a(z-a)\sqrt{1-|a|^2}\,Q_a\nabla_w\\ &= (1-|a|^2)(z-a)\cdot\nabla_w.\end{aligned}$$

From this we conclude the inequality

$$\left|(z_i-a_i)\frac{\partial}{\partial w_i}F(w)\right| \le \big|(z-a)\cdot\nabla F(w)\big| \le \left|\frac{1-\bar{a}z}{1-|a|^2}\varphi_a(z)\right|\,|D_aF(w)| = \frac{\sqrt{\triangle(a,z)}}{1-|a|^2}\,|D_aF(w)|,$$

as well as its conjugate

$$\left|(\overline{z_i-a_i})\frac{\partial}{\partial\overline{w}_i}F(w)\right| \le \frac{\sqrt{\triangle(a,z)}}{1-|a|^2}|\overline{D}_aF(w)|.$$

Moreover, we can iterate this inequality to obtain

$$\left|(\overline{z-a})^\alpha\frac{\partial^m}{\partial\overline{w}^\alpha}F(w)\right| \le C\left(\frac{\sqrt{\triangle(a,z)}}{1-|a|^2}\right)^m\big|(\overline{D}_a)^mF(w)\big|,$$

for a multi-index of length m. With $a=w$ this becomes the first estimate (5-7). □

Proof of (5-8). Recall from (5-1) that

$$D_af(z) = -\big((1-|a|^2)P_a\nabla f(z)+(1-|a|^2)^{1/2}Q_a\nabla f(z)\big).$$

We let $a=z$. By the unitary invariance of

$$\triangle(w,z) = |1-\overline{w}z|^2-(1-|z|^2)(1-|w|^2),$$

we may assume that $z=(|z|,0,\dots,0)$. Then we have

$$\begin{aligned}\frac{\partial}{\partial z_j}\triangle(w,z) &= \frac{\partial}{\partial z_j}\big((1-\overline{w}z)(1-\bar{z}w)-(1-\bar{z}z)(1-|w|^2)\big)\\ &= -\overline{w}_j(1-\bar{z}w)+\bar{z}_j(1-|w|^2) = (\bar{z}_j-\overline{w}_j)+\overline{w}_j(\bar{z}w)-\bar{z}_j|w|^2\\ &= (\bar{z}_j-\overline{w}_j)(1-|z|^2)+\bar{z}_j|z|^2-\overline{w}_j|z|^2+\overline{w}_j(\bar{z}w)-\bar{z}_j|w|^2\\ &= (\bar{z}_j-\overline{w}_j)(1-|z|^2)+\bar{z}_j\big(|z|^2-|w|^2\big)+\overline{w}_j\big(\bar{z}(w-z)\big).\end{aligned}$$

Now $Q_z\nabla f=(0,\partial f/\partial z_2,\dots,\partial f/\partial z_n)$, and thus a typical term in $Q_z\nabla\triangle$ is $\dfrac{\partial}{\partial z_j}\triangle(w,z)$ with $j\ge 2$. From $z=(|z|,0,\dots,0)$ and $j\ge 2$ we have $z_j=0$ and so

$$\frac{\partial}{\partial z_j}\triangle(w,z) = (\bar{z}_j-\overline{w}_j)(1-|z|^2)-(\bar{z}_j-\overline{w}_j)\bar{z}(w-z),\qquad j\ge 2.$$

Now (2-1) implies

$$\triangle(w,z) = (1-|z|^2)|w-z|^2+|\bar{z}(w-z)|^2, \tag{5-10}$$

which together with the above shows that

$$\begin{aligned}\sqrt{1-|z|^2}\,\big|Q_z\nabla\triangle(w,z)\big| &\le C|z-w|(1-|z|^2)^{3/2}+C\sqrt{1-|z|^2}\,|z-w|\,|\bar{z}(w-z)|\\ &\le C(1-|z|^2)\triangle(w,z)^{1/2}+C\triangle(w,z).\end{aligned}\tag{5-11}$$

As for $P_z\nabla D=(\partial f/\partial z_1,0,\dots,0)$ we use (5-10) to obtain

$$\begin{aligned}\big|P_z\nabla\triangle(w,z)\big| &= \Big|(\bar{z}_1-\overline{w}_1)(1-|z|^2)+\bar{z}_1\left(|z|^2-|w|^2\right)+\overline{w}_1\bar{z}(w-z)\Big|\\ &\le |z-w|(1-|z|^2)+\big||z|^2-|w|^2\big|+|\bar{z}(w-z)|\le C\sqrt{\triangle(w,z)}+2\big||z|-|w|\big|.\end{aligned}$$

However,

$$\begin{aligned}\triangle(w,z) &\ge (1-|w|\,|z|)^2-(1-|z|^2)(1-|w|^2)\\ &= 1-2|w|\,|z|+|w|^2|z|^2-\big(1-|z|^2-|w|^2+|z|^2|w|^2\big)\\ &= |z|^2+|w|^2-2|w|\,|z|=\big(|z|-|w|\big)^2,\end{aligned}$$

and so altogether we have the estimate

$$|P_z\nabla\triangle(w,z)|\le C\sqrt{\triangle(w,z)}.\tag{5-12}$$

Combining (5-11) and (5-12) with the definition (5-1) completes the proof of the first line in (5-8). The second line in (5-8) follows from (5-12) since $R_{(z)}=P_z\nabla$. □

Proof of (5-9). We compute

$$\begin{aligned}D_{(z)}(1-\overline{w}z)^k &= k(1-\overline{w}z)^{k-1}D_{(z)}(1-\overline{w}z)\\ &= k(1-\overline{w}z)^{k-1}\big((1-|z|^2)P_z\nabla+\sqrt{1-|z|^2}\,Q_z\nabla\big)(1-\overline{w}z)\\ &= -k(1-\overline{w}z)^{k-1}\big((1-|z|^2)P_z\overline{w}+\sqrt{1-|z|^2}\,Q_z\overline{w}\big),\\ R_{(z)}(1-\overline{w}z)^k &= k(1-\overline{w}z)^{k-1}(-\overline{w}z).\end{aligned}$$

Since $|w|^2+|a|^2\le 2$ we have

$$|Q_z\overline{w}|^2=|Q_z(\overline{w}-\bar{z})|^2\le|\overline{w}-\bar{z}|^2=|w|^2+|z|^2-2\,\mathrm{Re}(w\bar{z})\le 2\,\mathrm{Re}(1-w\bar{z})\le 2|1-w\bar{z}|,$$

which yields

$$\big|D_{(z)}\big((1-\overline{w}z)^k\big)\big|\le C|1-\overline{w}z|^k\,\frac{(1-|z|^2)+\sqrt{(1-|z|^2)|1-\overline{w}z|}}{|1-\overline{w}z|}\le C|1-\overline{w}z|^k\sqrt{\frac{1-|z|^2}{|1-\overline{w}z|}}.$$

Iteration then yields (5-9). □

6. Schur's test

Here we characterize boundedness of the positive operators that arise as majorants of the solution operators below. The case $c=0$ of the following lemma is Theorem 2.10 in [Zhu 2005].

Lemma 24. *Let $a, b, c, t \in \mathbb{R}$. Then the operator*

$$T_{a,b,c} f(z) = \int_{\mathbb{B}_n} \frac{(1-|z|^2)^a (1-|w|^2)^b \left(\sqrt{\triangle(w,z)}\right)^c}{|1-w\bar{z}|^{n+1+a+b+c}} f(w)\, dV(w)$$

is bounded on $L^p\left(\mathbb{B}_n; (1-|w|^2)^t\, dV(w)\right)$ if and only if $c > -2n$ and

$$-pa < t+1 < p(b+1). \tag{6-1}$$

The proof of Lemma 24 is a straightforward application of the argument in Theorem 2.10 of [Zhu 2005] together with an automorphic change of variable. Details can be found in the Electronic Supplement.

Remark. We will also use the trivial consequence of Lemma 24 that the operator

$$T_{a,b,c,d} f(z) = \int_{\mathbb{B}_n} \frac{(1-|z|^2)^a (1-|w|^2)^b \sqrt{\triangle(w,z)}^{\,c}}{|1-w\bar{z}|^{n+1+a+b+c+d}} f(w)\, dV(w)$$

is bounded on $L^p(\mathbb{B}_n; (1-|w|^2)^t\, dV(w))$ if $c > -2n$, $d \le 0$ and (6-1) holds. This is simply because $|1-w\bar{z}| \le 2$.

7. Operator estimates

We must show that $f = \Omega_0^1 h - \Lambda_g \Gamma_0^2 \in B_p^\sigma(\mathbb{B}_n; \ell^2)$, where Γ_0^2 is an antisymmetric 2-tensor of $(0,0)$-forms that solves

$$\bar{\partial}\Gamma_0^2 = \Omega_1^2 h - \Lambda_g \Gamma_{,1}^{\,3},$$

and inductively where Γ_q^{q+2} is an alternating $(q+2)$-tensor of $(0,q)$-forms that solves

$$\bar{\partial}\Gamma_q^{q+2} = \Omega_{q+1}^{q+2} h - \Lambda_g \Gamma_{q+1}^{q+3},$$

up to $q = n-1$ (since $\Gamma_n^{n+2} = 0$ and the $(0,n)$-form Ω_n^{n+1} is $\bar{\partial}$-closed). Using the Charpentier solution operators $\mathscr{C}_{n,s}^{0,q}$ on $(0,q+1)$-forms we can write

$$f = \mathscr{F}^0 + \mathscr{F}^1 + \cdots + \mathscr{F}^n,$$

with

$$\begin{aligned}
\mathscr{F}^0 &= \Omega_0^1 h - \Lambda_g \Gamma_0^2, \\
\mathscr{F}^1 &= \Omega_0^1 h - \Lambda_g \mathscr{C}_{n,s_1}^{0,0}\left(\Omega_1^2 h - \Lambda_g \Gamma_1^3\right), \\
\mathscr{F}^2 &= \Omega_0^1 h - \Lambda_g \mathscr{C}_{n,s_1}^{0,0}\left(\Omega_1^2 h - \Lambda_g \mathscr{C}_{n,s_2}^{0,1}(\Omega_2^3 h - \Lambda_g \Gamma_2^4)\right), \\
&\ \ \vdots \\
\mathscr{F}^n &= \Omega_0^1 h - \Lambda_g \mathscr{C}_{n,s_1}^{0,0} \Omega_1^2 h + \Lambda_g \mathscr{C}_{n,s_1}^{0,0} \Lambda_g \mathscr{C}_{n,s_2}^{0,1} \Omega_2^3 h - \Lambda_g \mathscr{C}_{n,s_1}^{0,0} \Lambda_g \mathscr{C}_{n,s_2}^{0,1} \Lambda_g \mathscr{C}_{n,s_3}^{0,2} \Omega_3^4 h - \cdots \\
&\qquad + (-1)^n \Lambda_g \mathscr{C}_{n,s_1}^{0,0} \ldots \Lambda_g \mathscr{C}_{n,s_n}^{0,n-1} \Omega_n^{n+1} h.
\end{aligned}$$

The goal is to establish

$$\|f\|_{B_p^\sigma(\mathbb{B}_n;\ell^2)} \le C_{n,\sigma,p,\delta}(g) \|h\|_{B_p^\sigma(\mathbb{B}_n)},$$

which we accomplish by showing that

$$\|\mathcal{F}^{\mu}\|_{B^{\sigma}_{p,m_1}(\mathbb{B}_n;\ell^2)} \le C_{n,\sigma,p,\delta}(g)\|h\|_{\Lambda^{\sigma}_{p,m_\mu}(\mathbb{B}_n)}, \quad 0\le\mu\le n, \tag{7-1}$$

for a choice of integers m_μ satisfying

$$\frac{n}{p}-\sigma<m_1<m_2<\cdots<m_\ell<\cdots<m_n.$$

Recall that we defined both of the norms $\|F\|_{B^{\sigma}_{p,m_\mu}(\mathbb{B}_n;\ell^2)}$ and $\|F\|_{\Lambda^{\sigma}_{p,m_\mu}(\mathbb{B}_n;\ell^2)}$ for smooth vector functions F in the ball $\mathbb{B}_n$.

Note on constants. We often indicate via subscripts, such as n,σ,p,δ, the important parameters on which a given constant C depends, especially when the constant appears in a basic inequality. However, at times in mid-argument, we will often revert to suppressing some or all of the subscripts in the interests of readability.

The norms $\|\cdot\|_{\Lambda^{\sigma}_{p,m}(\mathbb{B}_n;\ell^2)}$ in (5-3) above will now be used to estimate the composition of Charpentier solution operators in each function

$$\mathcal{F}^{\mu}=\Lambda_g\mathcal{C}^{0,0}_{n,s_1}\dots\Lambda_g\mathcal{C}^{0,\mu-1}_{n,s_\mu}\Omega^{\mu+1}_{\mu}h$$

as follows. More precisely we will use the specialized variants of the seminorms given by

$$\|F\|^p_{\Lambda^{\sigma}_{p,m',m''}(\mathbb{B}_n;\ell^2)}\equiv\int_{\mathbb{B}_n}\big|(1-|z|^2)^{\sigma}\big((1-|z|^2)^{m'}R^{m'}\big)\overline{D}^{m''}F(z)\big|^p\,d\lambda_n(z),$$

where we take m'' derivatives in $\overline{D}$ followed by m' derivatives in the invariant radial operator $(1-|z|^2)R$. Recall from Definition 19 that $\mathcal{X}^m$ denotes the vector of all differential operators of the form $X_1X_2\dots X_m$ where each X_i is either I, $\overline{D}$, or $(1-|z|^2)R$, and where by definition $1-|z|^2$ is held constant in composing operators. It will also be convenient at times to use the notation

$$\mathcal{R}^m\equiv(1-|z|^2)^m(R^k)^m_{k=0}, \tag{7-2}$$

which should cause no confusion with the related operators $\mathcal{R}^m_b$ introduced in (3-8). Note that $\mathcal{R}^m$ is simply $\mathcal{X}^m$ when none of the operators $\overline{D}$ appear. We will make extensive use the multilinear estimate in Proposition 22.

Let us fix our attention on the function $\mathcal{F}^{\mu}=\mathcal{F}^{\mu}_0$ and write

$$\begin{aligned}\mathcal{F}^{\mu}_0&=\Lambda_g\mathcal{C}^{0,0}_{n,s_1}\big(\Lambda_g\mathcal{C}^{0,1}_{n,s_2}\dots\Lambda_g\mathcal{C}^{0,\mu-1}_{n,s_\mu}\Omega^{\mu+1}_{\mu}h\big)=\Lambda_g\mathcal{C}^{0,0}_{n,s_1}(\mathcal{F}^{\mu}_1),\\ \mathcal{F}^{\mu}_1&=\Lambda_g\mathcal{C}^{0,1}_{n,s_2}\big(\Lambda_g\mathcal{C}^{0,2}_{n,s_3}\dots\Lambda_g\mathcal{C}^{0,\mu-1}_{n,s_\mu}\Omega^{\mu+1}_{\mu}h\big)=\Lambda_g\mathcal{C}^{0,1}_{n,s_2}(\mathcal{F}^{\mu}_2),\\ \mathcal{F}^{\mu}_q&=\Lambda_g\mathcal{C}^{0,q}_{n,s_{q+1}}(\mathcal{F}^{\mu}_{q+1}),\end{aligned}$$

and so on, where $\mathcal{F}^{\mu}_q$ is a $(0,q)$-form. We now perform the integration by parts in Lemma 17 in each iterated Charpentier operator $\mathcal{F}^{\mu}_q=\Lambda_g\mathcal{C}^{0,q}_{n,s_{q+1}}(\mathcal{F}^{\mu}_{q+1})$ to obtain

$$\begin{aligned}\mathscr{F}_q^{\mu} &= \Lambda_g \mathscr{C}_{n,s_{q+1}}^{0,q} \mathscr{F}_{q+1}^{\mu} \\ &= \sum_{j=0}^{m'_{q+1}-1} c'_{j,n,s_{q+1}} \Lambda_g \mathscr{S}_{n,s_{q+1}} (\overline{\mathscr{D}}^j \mathscr{F}_{q+1}^{\mu})(z) + \sum_{\ell=0}^{\mu} c_{\ell,n,s_{q+1}} \Lambda_g \Phi_{n,s_{q+1}}^{\ell} \big(\overline{\mathscr{D}}^{m'_{q+1}} \mathscr{F}_{q+1}^{\mu}\big)(z).\end{aligned} \tag{7-3}$$

Now we compose these formulas for $\mathscr{F}_k^{\mu}$ to obtain an expression for $\mathscr{F}^{\mu}$ that is a complicated sum of compositions of the individual operators in (7-3) above. For now we will concentrate on the main terms $\Lambda_g \Phi_{n,s_{k+1}}^{\mu} \Big(\overline{\mathscr{D}}^{m'_{k+1}} \mathscr{F}_{k+1}^{\mu}\Big)$ that arise in the second sum above when $\ell = \mu$. We will see that the same considerations apply to any of the other terms in (7-3). Recall from Lemma 17 that the "boundary" operators $\mathscr{S}_{n,s_{q+1}}$ are projections of operators on $\partial \mathbb{B}_{s_q}$ to the ball $\mathbb{B}_n$ and have (balanced) kernels even simpler than those of the operators $\Phi_{n,s_{q+1}}^{\ell}$. The composition of these main terms is

$$\begin{aligned}\big(\Lambda_g \Phi_{n,s_1}^{\mu} \overline{\mathscr{D}}^{m'_1}\big) \mathscr{F}_1^{\mu} &= \big(\Lambda_g \Phi_{n,s_1}^{\mu} \overline{\mathscr{D}}^{m'_1}\big)\big(\Lambda_g \Phi_{n,s_2}^{\mu} \overline{\mathscr{D}}^{m'_2}\big) \mathscr{F}_2^{\mu} \\ &= \big(\Lambda_g \Phi_{n,s_1}^{\mu} \overline{\mathscr{D}}^{m'_1}\big)\big(\Lambda_g \Phi_{n,s_2}^{\mu} \overline{\mathscr{D}}^{m'_2}\big) \dots \big(\Lambda_g \Phi_{n,s_{\mu}}^{\mu} \overline{\mathscr{D}}^{m'_{\mu}}\big) \Omega_{\mu}^{\mu+1} h.\end{aligned} \tag{7-4}$$

At this point we would like to take absolute values inside all of these integrals and use the crucial inequalities (5-7)–(5-9) to obtain a composition of positive operators of the type considered in Lemma 24. However, there is a difficulty in using inequality (5-7) to estimate the derivative $\overline{\mathscr{D}}^m$ on $(0, q+1)$-forms η given by (3-6):

$$\overline{\mathscr{D}}^m \eta(z) = \sum_{|J|=q} \sum_{k \notin J} \sum_{|\alpha|=m} (-1)^{\mu(k,J)} (\overline{w_k - z_k}) \overline{(w-z)^{\alpha}} \frac{\partial^m}{\partial \overline{w}^{\alpha}} \eta_{J \cup \{k\}}(w).$$

The problem is that the factor $\overline{w_k - z_k}$ has no derivative $\partial/\partial \overline{w}_k$ naturally associated with it, as do the other factors in $\overline{(w-z)^{\alpha}}$. We refer to the factor $\overline{w_k - z_k}$ as a *rogue factor*, as it requires special treatment in order to apply (5-7). Note that we cannot simply estimate $\overline{w_k - z_k}$ by $|w-z|$ because this is much larger in general than the estimate $\sqrt{\triangle(w,z)}$ obtained in (5-7) (where the difference in size between $|w-z|$ and $\sqrt{\triangle(w,z)}$ is compensated by the difference in size between $\partial/\partial \overline{w}_k$ and $\overline{D}$).

We now describe how to circumvent this difficulty in the composition of operators in (7-4). Let us write each $\overline{\mathscr{D}}^{m'_{q+1}} \mathscr{F}_{q+1}^{\mu}$ as

$$\sum_{|J|=q} \sum_{k \notin J} \sum_{|\alpha|=m'_{q+1}} (-1)^{\mu(k,J)} (\overline{w_k - z_k}) \overline{(w-z)^{\alpha}} \frac{\partial^m}{\partial \overline{w}^{\alpha}} (\mathscr{F}_{q+1}^{\mu})_{J \cup \{k\}}(w),$$

where $(\mathscr{F}_{q+1}^{\mu})_{J \cup \{k\}}$ is the coefficient of the form $\mathscr{F}_{q+1}^{\mu}$ with differential $d\overline{w}^{J \cup \{k\}}$. We now replace each of these sums with just one of the summands, say

$$(\overline{w_k - z_k}) \overline{(w-z)^{\alpha}} \frac{\partial^m}{\partial \overline{w}^{\alpha}} (\mathscr{F}_{q+1}^{\mu})_{J \cup \{k\}}(w). \tag{7-5}$$

Here the factor $\overline{w_k - z_k}$ is a rogue factor, not associated with a corresponding derivative $\partial/\partial \overline{w}_k$. We will refer to k as the *rogue index* associated with the rogue factor when it is not convenient to explicitly display the variables.

The key fact in treating the rogue factor $\overline{w_k - z_k}$ is that its presence in (7-5) means that the coefficient $(\mathcal{F}^{\mu}_{q+1})_I$ of the form $\mathcal{F}^{\mu}_{q+1}$ that multiplies it *must* have k in the multi-index I. Since

$$\mathcal{F}^{\mu}_{q+1} = \Lambda_g \mathscr{C}^{0,q+1}_{n,s_{q+2}}(\mathcal{F}^{\mu}_{q+2}),$$

the form of the ameliorated Charpentier kernel $\mathscr{C}^{0,q+1}_{n,s_{q+2}}$ in Theorem 10 shows that the coefficients of $\mathscr{C}^{0,q+1}_{n,s_{q+2}}(w,z)$ that multiply the rogue factor *must* have the differential $d\bar{z}_k$ in them. In turn, this means that the differential $d\overline{w}_k$ must be *missing* in the coefficient of $\mathscr{C}^{0,q+1}_{n,s_{q+2}}(w,z)$, and hence finally that the coefficients $(\mathcal{F}^{\mu}_{q+2})_H$ with multi-index H that survive the wedge products in the integration *must* have $k \in H$. This observation can be repeated, and we now derive an important consequence.

Returning to (7-4), each summand in $\overline{\mathscr{D}}^{m'_{q+1}} \mathcal{F}^{\mu}_{q+1}$ has a rogue factor with associated rogue index k_{q+1}. Thus the function in (7-4) is a sum of terms of the form

$$\begin{aligned}\big(\Lambda_g \Phi^{\mu}_{n,s_1}(\overline{w_{k_1} - z_{k_1}})\,\overline{\mathscr{X}}^{m'_1}\big) \circ \big(\Lambda_g \Phi^{\mu}_{n,s_2}(\overline{w_{k_2} - z_{k_2}})\,\overline{\mathscr{X}}^{m'_2}\big)_{I_1} \circ \cdots \circ \big(\Lambda_g \Phi^{\nu}_{n,s_\nu}(\overline{w_{k_\nu} - z_{k_\nu}})\,\overline{\mathscr{X}}^{m'_\nu}\big)_{I_{\nu-1}} \\ \circ \cdots \circ \big(\Lambda_g \Phi^{\mu-1}_{n,s_\mu}(\overline{w_{k_\mu} - z_{k_\mu}})\,\overline{\mathscr{X}}^{m'_\mu}\big)_{I_{\mu-1}} \circ \big(\Omega^{\mu+1}_{\mu} h\big)_{I_\mu},\end{aligned}$$

where the subscript I_ν on the form $\Lambda_g \Phi^{\nu}_{n,s_\nu}(\overline{w_{k_\nu} - z_{k_\nu}})\,\overline{\mathscr{X}}^{m'_\nu}$ indicates that we are composing with the component of $\Lambda_g \Phi^{\nu}_{n,s_\nu}(\overline{w_{k_\nu} - z_{k_\nu}})\,\overline{\mathscr{X}}^{m'_\nu}$ corresponding to the multi-index $I_{\nu-1}$, i.e., the component with the differential $d\bar{z}^{I_{\nu-1}}$. The notation will become exceedingly unwieldy if we attempt to identify the different variables associated with each of the iterated integrals, so we refrain from this in general. The considerations of the previous paragraph now show that we must have $\{k_1\} = I_1$, $\{k_2\} \cup I_1 = I_2$, and more generally

$$\{k_\nu\} \cup I_{\nu-1} = I_\nu, \quad 1 < \nu \le \mu.$$

In particular we see that the associated rogue indices $k_1, k_2, \dots k_\mu$ are all distinct and that as sets

$$\{k_1, k_2, \dots, k_\mu\} = I_\mu.$$

Denoting by ζ the variable in the final form $\Omega^{\mu+1}_{\mu} h$, we can thus write each rogue factor $\overline{w_{k_\nu} - z_{k_\nu}}$ as

$$\overline{w_{k_\nu} - z_{k_\nu}} = (\overline{w_{k_\nu} - \zeta_{k_\nu}})(\overline{z_{k_\nu} - \zeta_{k_\nu}}),$$

and since $k_\nu \in I_\mu$, there is a factor of the form $(\partial/\partial\bar{\zeta}_{k_\nu})(\partial^{|\beta|} g_i / \partial\bar{\zeta}^{\beta})$ in each summand of the component $(\Omega^{\mu+1}_{\mu} h)_{I_\mu}$ of $\Omega^{\mu+1}_{\mu} h$. So we are able to associate the rogue factor $\overline{w_{k_\nu} - z_{k_\nu}}$ with derivatives of g as follows:

$$\left((\overline{w_{k_\nu} - z_{k_\nu}})\frac{\partial}{\partial\bar{\zeta}_{k_\nu}}\right)\frac{\partial^{|\beta|} g_i}{\partial\bar{\zeta}^{\beta}} - \left((\overline{z_{k_\nu} - \zeta_{k_\nu}})\frac{\partial}{\partial\bar{\zeta}_{k_\nu}}\right)\frac{\partial^{|\gamma|} g_j}{\partial\bar{\zeta}^{\gamma}}. \tag{7-6}$$

Thus it is indeed possible to

(1) apply the radial integration by parts in Corollary 16,

(2) then take absolute values and ℓ^2-norms inside all the integrals,

(3) and then apply the crucial inequalities (5-7)–(5-9).

One of the difficulties remaining after this is that we are now left with additional factors of the form

$$\frac{\sqrt{\triangle(w,\zeta)}}{1-|w|^2} \quad \text{and} \quad \frac{\sqrt{\triangle(z,\zeta)}}{1-|z|^2},$$

resulting from an application of (5-7) to the derivatives in (7-6). These factors are still rogue in the sense that the variable pairs occurring in them, namely (w,ζ) and (z,ζ), do not consist of consecutive variables in the iterated integrals of (7-4). This is rectified by using the fact that

$$d(w,z) = \sqrt{\triangle(w,z)}$$

is a quasimetric, which in turn follows from the identity

$$\sqrt{\triangle(w,z)} = |1-w\bar{z}|\,|\varphi_z(w)| = \delta(w,z)^2\rho(w,z),$$

where $\rho(w,z) = |\varphi_z(w)|$ is the invariant pseudohyperbolic metric on the ball (Corollary 1.22 in [Zhu 2005]) and where $\delta(w,z) = |1-w\bar{z}|^{1/2}$ satisfies the triangle inequality on the ball (Proposition 5.1.2 in [Rudin 1980]). Using the quasisubadditivity of $d(w,z)$ we can, with some care, redistribute appropriate factors back to the iterated integrals where they can be favorably estimated using Lemma 24. It is simplest to illustrate this procedure in specific cases, so we defer further discussion of this point until we treat in detail the cases $\mu = 0, 1, 2$ below. We again emphasize that all these observations regarding rogue factors in (7-4) apply equally well to the rogue factors in the other terms $\Phi^{\ell}_{n,s_{q+1}}(\overline{\mathcal{D}}^{m'_q}\mathcal{F}^{\mu}_{q+1})(z)$ in (7-3), as well as to the boundary terms $\mathcal{S}_{n,s_{q+1}}(\overline{\mathcal{D}}^{j}\mathcal{F}^{\mu}_{q+1})(z)$ in (7-3).

The other difficulty remaining is that in order to obtain a favorable estimate using Lemma 24 for the iterated integrals resulting from the bullet items above, it is necessary to generate additional powers of $1-|z|^2$ (we are using z as a generic variable in the iterated integrals here). This is accomplished by applying the radial integrations by parts in Corollary 16 to the *previous* iterated integral. Of course such a possibility is impossible for the first of the iterated integrals, but there we are only applying the radial derivative R thanks to the fact that our candidate f from the Koszul complex is holomorphic. As a result, we see from (5-8) that $(1-|z|^2)R$, unlike D, generates positive powers of $1-|z|^2$ even when acting on $\triangle(w,z)$. This procedure is also best illustrated in specific cases and will be treated in the next subsection.

So ignoring these technical issues for the moment, the integrals that result from taking absolute values and ℓ^2-norms inside (7-4) are now estimated using Lemma 24 and the remark following it. Note that we only use *scalar-valued* Schur estimates since all the integrals to which that lemma and remark are applied have positive integrands. Here is the rough idea. Suppose that $\{T_1, T_2, \dots, T_\mu\}$ is a collection of Charpentier solution operators and that for a sequence of large integers

$$\{m'_1, m''_1, m'_2, , m''_2 \dots, m'_{\mu+1}, m''_{\mu+1}\},$$

we have the inequalities

$$\|T_j F\|_{\Lambda^{\sigma}_{p,m'_j,m''_j}(\mathbb{B}_n;\ell^2)} \le C_j \|F\|_{\Lambda^{\sigma}_{p,m'_{j+1},m''_{j+1}}(\mathbb{B}_n;\ell^2)}, \quad 1 \le j \le \ell+1, \tag{7-7}$$

for the class of smooth functions F that arise as TG for some Charpentier solution operator T and some smooth G. Then we can estimate $\|T_1 \circ T_2 \circ \cdots \circ T_\mu \Omega\|_{B^\sigma_{p,m}(\mathbb{B}_n;\ell^2)}$ by

$$\begin{aligned} \|T_1 \circ T_2 \circ \cdots \circ T_\ell \Omega\|_{\Lambda^\sigma_{p,m'_1,m''_1}(\mathbb{B}_n;\ell^2)} &\le C_1 \|T_2 \circ \cdots \circ T_\ell \Omega\|_{\Lambda^\sigma_{p,m'_2,m''_2}(\mathbb{B}_n;\ell^2)} \\ &\le C_1 C_2 \|T_3 \circ \cdots \circ T_\ell \Omega\|_{\Lambda^\sigma_{p,m'_3,m''_3}(\mathbb{B}_n;\ell^2)} \\ &\le C_1 C_2 \ldots C_\ell \|\Omega\|_{\Lambda^\sigma_{p,m'_{\ell+1},m''_{\ell+1}}(\mathbb{B}_n;\ell^2)}. \end{aligned}$$

Finally we will show that if Ω is one of the forms Ω^{q+1}_q in the Koszul complex, then

$$\|\Omega\|_{\Lambda^\sigma_{p,m'_{\ell+1},m''_{\ell+1}}(\mathbb{B}_n;\ell^2)} \le \|\Omega\|_{\Lambda^\sigma_{p,m'_{\ell+1}+m''_{\ell+1}}(\mathbb{B}_n;\ell^2)} \le C_{n,\sigma,p,\delta}(g)\|h\|_{B^\sigma_{p,m}(\mathbb{B}_n)},$$

and so altogether this proves that

$$\|f\|_{B^\sigma_p(\mathbb{B}_n;\ell^2)} \le C_{n,\sigma,p,\delta}(g)\|h\|_{B^\sigma_{p,m}(\mathbb{B}_n)}.$$

We now make some brief comments on how to obtain the inequalities in (7-7). Complete details will be given in the cases $\mu = 0, 1, 2$ below, and the general case $0 \le \mu \le n$ is no different from these three cases. We note that from (2-6) the kernel of $\mathscr{C}^{0,q}_n$ typically looks like a sum of terms

$$\frac{(1-w\bar{z})^{n-1-q}(1-|w|^2)^q}{\triangle(w,z)^n}(\bar{z}_j - \overline{w}_j) \tag{7-8}$$

times a wedge product of differentials in which the differential $d\overline{w}_j$ is missing. We again emphasize that the rogue factor $\bar{z}_j - \overline{w}_j$ cannot simply be estimated by $|\bar{z}_j - \overline{w}_j|$, as the formula (2-1) shows that

$$\sqrt{\triangle(w,z)} = \left|P_z(z-w) + \sqrt{1-|z|^2}\,Q_z(z-w)\right|$$

can be much smaller than $|z-w|$. As we mentioned above, it is possible to exploit the fact that any surviving term in the form $\Omega^{\mu+1}_\mu$ must then involve the derivative $\partial/\partial\overline{w}_j$ hitting a component of g. This permits us to absorb part of the complex tangential component of $z-w$ into the almost invariant derivative D which is larger than the usual gradient in the complex tangential directions. This results in a good estimate for the rogue factor $(\bar{z}_j - \overline{w}_j)$ in (7-8) based on the smaller quantity $\sqrt{\triangle(w,z)}$. We have already integrated by parts to write (7-8) as (recall that the factors $\bar{z}_j - \overline{w}_j$ are already incorporated into $\overline{\mathscr{D}}^m_z \eta(w)$)

$$\int_{\mathbb{B}_n} \frac{(1-w\bar{z})^{n-1-q}(1-|w|^2)^q}{\triangle(w,z)^n}\overline{\mathscr{D}}^m \eta(w)\, dV(w),$$

plus boundary terms which we ignore for the moment. Then we use the three crucial inequalities (5-7), (5-8), and (5-9) to help show that the resulting iterated kernels can be factored (after accounting for all rogue factors $\bar{z}_j - \overline{w}_j$) into operators that satisfy the hypotheses of Lemma 24 or the subsequent remark.

Definition 25. The expression $\widehat{\Omega}^{\ell+1}_\ell$ denotes the form $\Omega^{\ell+1}_\ell$ but with every occurrence of the derivative $\partial/\partial\overline{w}_j$ replaced by the derivative $\overline{D}_j$.

We can rewrite (5-7) in the form

$$\left|(\bar{z}_j - \overline{w}_j)\overline{\mathscr{D}}^m_{z,w}\Omega^{\ell+1}_\ell(w)\right| \le \left(\frac{\sqrt{\triangle(w,z)}}{1-|w|^2}\right)^{m+1} \left|\overline{D}^m \widehat{\Omega}^{\ell+1}_\ell(w)\right|,$$

Recall that each summand of $\Omega_\ell^{\ell+1}$ includes a product of exactly ℓ distinct derivatives $\partial/\partial\overline{w}_j$ applied to components of g. Thus the entries of $\overline{D}^m\widehat{\Omega}_\ell^{\ell+1}(w)$ consist of $m+\ell$ derivatives distributed among components of g. Using the factorization of $\Omega_\ell^{\ell+1}$ in (4-4), we obtain the corresponding factorization for $\widehat{\Omega}_\ell^{\ell+1}$:

$$\Omega_0^1 \wedge \bigwedge_{i=1}^{\ell} \widehat{\Omega}_0^1 = -\frac{1}{\ell+1}\widehat{\Omega}_\ell^{\ell+1}, \tag{7-9}$$

where

$$\Omega_0^1 = \left(\frac{\overline{g}_i}{|g|^2}\right)_{i=1}^{\infty} \quad \text{and} \quad \widehat{\Omega}_0^1 = \left(\frac{\overline{D}\overline{g}_i}{|g|^2}\right)_{i=1}^{\infty}.$$

It is important for this purpose of using Lemma 24 and the subsequent remark to first apply the integration by parts Lemma 14 to temper the singularity due to negative powers of $\triangle(w,z)$, and to use the integration by parts Corollary 16 to infuse enough powers of $1-|w|^2$ for use in the subsequent iterated integral.

Finally it follows from Proposition 22 together with the factorization (4-4) that

$$\left\|(1-|z|^2)^\sigma \mathscr{X}^m \widehat{\Omega}_\mu^{\mu+1} h(z)\right\|_{L^p(\lambda_n;\ell^2)} \le C\,\|\mathbb{M}_g\|^{m+\mu}_{B_p^\sigma(\mathbb{B}_n)\to B_p^\sigma(\mathbb{B}_n;\ell^2)}\|h\|_{B_p^\sigma(\mathbb{B}_n)}. \tag{7-10}$$

We defer the proof of (7-10) until page 538 when further calculations are available.

Remark. At this point we observe from (7-1) that the exponent $m+\mu$ in (7-10) is at most m_n+n, and thus we may take $\kappa = m_n+n$. We leave it to the interested reader to estimate the size of m_n.

Taking into account all of the above, the conclusion is that with $\kappa = m_n+n$,

$$\|f\|_{B_p^\sigma(\mathbb{B}_n;\ell^2)} \le C_{n,\sigma,p,\delta}\,\|\mathbb{M}_g\|^{\kappa}_{B_p^\sigma(\mathbb{B}_n)\to B_p^\sigma(\mathbb{B}_n;\ell^2)}\|h\|_{B_p^\sigma(\mathbb{B}_n)}.$$

As the arguments described above are rather complicated we illustrate them by considering the three cases $\mu = 0,1,2$ in complete detail in the next subsection before proceeding to the general case.

7.1. ***Estimates in special cases.*** Here we prove the estimates (7-1) for $\mu=0,1,2$. Recall that

$$\mathscr{F}^0 = \Omega_0^1 h, \quad \mathscr{F}^1 = \Lambda_g \mathscr{C}^{0,0}_{n,s_1}\Omega_1^2 h, \quad \mathscr{F}^2 = \Lambda_g \mathscr{C}^{0,0}_{n,s_1}\Lambda_g \mathscr{C}^{0,1}_{n,s_2}\Omega_2^3 h.$$

To obtain the estimate for $\mathscr{F}^0$ we use the multilinear inequality in Proposition 22.

In estimating $\mathscr{F}^1$ we confront for the first time a rogue factor $\bar{z}_{k-w_k}$ that we must associate with a derivative $\partial/\partial\overline{w}_k$ occurring in each surviving summand of the k-th component of the form Ω_1^2. After applying the integration by parts formula in 17 as in [Ortega and Fàbrega 2000], we use the crucial inequalities (5-7)–(5-9) and the Schur-type operator estimates in Lemma 24 with $c=0$ to obtain the desired estimates. Finally we must also deal with the boundary terms in the integration by parts formula for ameliorated Charpentier kernels in Lemma 17. This requires using the radial derivative integration by parts formula in Corollary 16 as in [Ortega and Fàbrega 2000], and also requires dealing with the corresponding rogue factors.

The final trick in the proof arises in estimating $\mathscr{F}^2$. This time there are two iterated integrals each with a rogue factor. The problematic rogue factor $\overline{z_k-\zeta_k}$ occurs in the *first* of the iterated integrals since there is *no* derivative $\partial/\partial\overline{\zeta}_k$ hitting the second iterated integral with which to associate the rogue factor $\overline{z_k-\zeta_k}$. Instead we decompose the factor as $(\overline{w_k-z_k})-(\overline{\zeta_k-w_k})$ and associate each of these summands with a derivative $\partial/\partial\overline{w}_k$ already occurring in Ω_2^3. Then we can apply the crucial inequality (5-7) and use the fact that $\sqrt{\triangle(w,z)}$ is a quasimetric to redistribute the estimates appropriately. As a result of this redistribution we are forced to use Lemma 24 with $c=\pm 1$ this time as well as $c=0$. In applying the Schur-type estimates in Lemma 24 to the *second* iterated integral, we require a sufficiently large power of $1-|w|^2$ to be carried over from the first iterated integral. To ensure this we again use the radial derivative integration by parts formula in Corollary 16.

The estimate (7-1) for general μ involves no new ideas. There are now μ rogue terms and we need to apply Lemma 24 with $c=0,\pm 1,\dots,\pm(\mu-1)$. With this noted the arguments needed are those used above in the cases $\mu=0,1,2$.

The estimate for $\mathcal{F}^0$. We begin with the estimate

$$\|\mathcal{F}^0\|_{B^{\sigma}_{p,m}(\mathbb{B}_n;\ell^2)}=\|\Omega_0^1 h\|_{B^{\sigma}_{p,m}(\mathbb{B}_n;\ell^2)}\le C_{n,\sigma,p,\delta}\,\|\mathbb{M}_g\|^m_{B^{\sigma}_p(\mathbb{B}_n)\to B^{\sigma}_p(\mathbb{B}_n;\ell^2)}\,\|h\|_{B^{\sigma}_{p,m}(\mathbb{B}_n)},$$

for $m+\sigma>\frac{n}{p}$. However, for later use we prove instead the more general estimate with $\mathcal{X}$ in place of R, except that m must then be chosen twice as large:

$$\int_{\mathbb{B}_n}\left|(1-|z|^2)^{\sigma}\mathcal{X}^m(\Omega_0^1 h)(z)\right|^p d\lambda_n(z)\le C_{n,\sigma,p,\delta}\,\|\mathbb{M}_g\|^{mp}_{B^{\sigma}_p(\mathbb{B}_n)\to B^{\sigma}_p(\mathbb{B}_n;\ell^2)}\,\|h\|^p_{B^{\sigma}_p(\mathbb{B}_n)},\tag{7-11}$$

for $m>2(\frac{n}{p}-\sigma)$. Recall that $\mathcal{X}^m$ is the differential operator of order m given in Definition 19 that is adapted to the complex geometry of the unit ball $\mathbb{B}_n$. It will be in estimating iterated Charpentier integrals below that the derivatives R^m and $\overline{\mathcal{D}}^m$ will arise from integration by parts in the previous iterated integral, and this will require estimates using $\mathcal{X}^m$.

By Leibniz's rule for $\mathcal{X}^m$ we have

$$\mathcal{X}^m(\Omega_0^1 h)=\sum_{k=0}^{m}c_k(\mathcal{X}^k\Omega_0^1)(\mathcal{X}^{m-k}h)$$

and

$$\mathcal{X}^k(\Omega_0^1)=\mathcal{X}^k\left(\frac{\overline{g}}{|g|^2}\right)=\sum_{\ell=0}^{k}c_\ell(\mathcal{X}^{k-\ell}\overline{g})(\mathcal{X}^{\ell}|g|^{-2}).\tag{7-12}$$

It suffices to prove

$$\int_{\mathbb{B}_n}\left|(1-|z|^2)^{\sigma}\left(\sum_{k=0}^{m}\sum_{\ell=0}^{k}c_k c_\ell(\mathcal{X}^{k-\ell}\overline{g})(\mathcal{X}^{\ell}|g|^{-2})(\mathcal{X}^{m-k}h)\right)\right|^p d\lambda_n \le C_{n,\sigma,p,\delta}\,\|\mathbb{M}_g\|^{mp}_{B^{\sigma}_p(\mathbb{B}_n)\to B^{\sigma}_p(\mathbb{B}_n;\ell^2)}\,\|h\|^p_{B^{\sigma}_p(\mathbb{B}_n)},$$

and hence

$$\int_{\mathbb{B}_n}(1-|z|^2)^{p\sigma}|\mathcal{X}^{k-\ell}\overline{g}|^p\left|\mathcal{X}|g|^{-2}\right|^p|\mathcal{X}^{m-k}h|^p d\lambda_n \le C_{n,\sigma,p,\delta}\,\|\mathbb{M}_g\|^{mp}_{B^{\sigma}_p(\mathbb{B}_n)\to B^{\sigma}_p(\mathbb{B}_n;\ell^2)}\,\|h\|^p_{B^{\sigma}_p(\mathbb{B}_n)},\tag{7-13}$$

for each fixed $0\le\ell\le k\le m$.

Now we can profitably estimate both $|\mathscr{X}^{m-k}h|$ and $|\mathscr{X}^{k-\ell}\overline{g}|$ as they are, but we must be more careful with $\bigl|\mathscr{X}|g|^{-2}\bigr|$. In the case $\ell=1$, we assume for convenience that $\mathscr{X}$ annihilates g_i (if not it will annihilate $\overline{g}_i$ unless $\mathscr{X}=I$, and the estimates are similar) and obtain

$$\bigl|\mathscr{X}|g|^{-2}\bigr|^2=\left|-|g|^{-4}\sum_{i=1}^{\infty}g_i\mathscr{X}\overline{g}_i\right|^2\le|g|^{-8}\left(\sum_{i=1}^{\infty}|g_i|^2\right)\left(\sum_{i=1}^{\infty}|\mathscr{X}\overline{g}_i|^2\right)\le|g|^{-6}\sum_{i=1}^{\infty}|\mathscr{X}\overline{g}_i|^2.$$

Similarly, when $\ell=2$,

$$\begin{aligned}\bigl|\mathscr{X}^2|g|^{-2}\bigr|^2&=\left|-|g|^{-4}\sum_{i=1}^{\infty}g_i\mathscr{X}^2\overline{g}_i+2|g|^{-6}\sum_{i\neq j}(g_i\mathscr{X}\overline{g}_i)(g_j\mathscr{X}\overline{g}_j)\right|^2\\&\le 2|g|^{-6}\sum_{i=1}^{\infty}|\mathscr{X}^2\overline{g}_i|^2+4|g|^{-8}\left(\sum_{i=1}^{\infty}|\mathscr{X}\overline{g}_i|^2\right)^2,\end{aligned}$$

and the general case is

$$\begin{aligned}&\bigl|\mathscr{X}^\ell|g|^{-2}\bigr|^2\\&\le C_\ell|g|^{-6}\sum_{i=1}^{\infty}|\mathscr{X}^\ell\overline{g}_i|^2+C_{\ell-1}|g|^{-8}\left(\sum_{i=1}^{\infty}|\mathscr{X}^{\ell-1}\overline{g}_i|^2\right)\left(\sum_{i=1}^{\infty}|\mathscr{X}\overline{g}_i|^2\right)+\cdots+C_0|g|^{-4-2\ell}\left(\sum_{i=1}^{\infty}|\mathscr{X}\overline{g}_i|^2\right)^\ell\\&=\sum_{\substack{1\le\alpha_1\le\alpha_2\le\cdots\le\alpha_M\\ \alpha_1+\alpha_2+\cdots+\alpha_M=\ell}}c_\alpha|g|^{-4-2\ell}\prod_{m=1}^{M}\left(\sum_{i=1}^{\infty}|\mathscr{X}^{\alpha_m}\overline{g}_i|^2\right).\end{aligned}\tag{7-14}$$

We can ignore the powers of $|g|$ since $|g|$ is bounded above and below by (5-2) and the hypotheses of Theorem 2. Fixing α we see that the left side of (7-13) is thus at most

$$C_{n,\sigma,p,\delta}\int_{\mathbb{B}_n}(1-|z|^2)^{p\sigma}\,|\mathscr{X}^{k-\ell}\overline{g}|^p\,|\mathscr{Y}^{m-k}h|^p\left(\prod_{j=1}^{M}|\mathscr{X}^{\alpha_j}\overline{g}|^p\right)d\lambda_n.$$

Since

$$|\mathscr{X}^{k-\ell}\overline{g}|^2=\sum_{i=1}^{\infty}|\mathscr{X}^{k-\ell}\overline{g}_i|^2$$

and $k-\ell$ could vanish (unlike the exponents α_ℓ, which are positive), we see that altogether after renumbering, it suffices to prove

$$\int_{\mathbb{B}_n}(1-|z|^2)^{p\sigma}\,|\mathscr{Y}^{\alpha_1}h|^p\,|\mathscr{Y}^{\alpha_2}g|^p\ldots|\mathscr{Y}^{\alpha_M}g|^p\,d\lambda_n\le C_{n,\sigma,p,\delta}\,\|\mathbb{M}_g\|^{Mp}_{B_p^\sigma(\mathbb{B}_n)\to B_p^\sigma(\mathbb{B}_n;\ell^2)}\,\|h\|^p_{B_p^\sigma(\mathbb{B}_n)}\tag{7-15}$$

for each fixed $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_M)$ where $M\ge 2$, $|\alpha|=m$ and at most one of $\alpha_2,\ldots,\alpha_M$ is zero. We have used here that $|\overline{D}\overline{g}|=|Dg|$. Now Proposition 22 yields (7-15) for each $0\le k\le m$ and $|\alpha|=m-k$. Summing these estimates completes the proof of (7-11).

We can now prove the more general inequality (7-10). Indeed, using the factorization (4-4) of $\widehat{\Omega}_{\mu}^{\mu+1}$ together with the Leibniz formula gives

$$\begin{aligned}\mathscr{X}^m\big(\widehat{\Omega}_{\mu}^{\mu+1}h\big) = \mathscr{X}^m\big(\Omega_0^1\wedge(\widehat{\Omega}_0^1)^{\mu}h\big) &= \sum_{\substack{\alpha\in\mathbb{Z}_+^{\mu+2}\\|\alpha|=m}}(\mathscr{X}^{\alpha_0}\Omega_0^1)\wedge\bigwedge_{j=1}^{\mu}\big(\mathscr{X}^{\alpha_j}\widehat{\Omega}_0^1\big)(\mathscr{X}^{\alpha_{\mu+1}}h)\\ &= \sum_{\substack{\alpha\in\mathbb{Z}_+^{\mu+2}\\|\alpha|=m}}\bigg((\mathscr{X}^{\alpha_0}\Omega_0^1)\wedge\bigwedge_{j=1}^{\mu}\big(\mathscr{X}^{\alpha_j+1}\Omega_0^1\big)\bigg)(\mathscr{X}^{\alpha_{\mu+1}}h),\end{aligned}$$

where we have used that $\widehat{\Omega}_0^1$ already has an $\mathscr{X}$ derivative in each summand, and so $\mathscr{X}^{\alpha_j}\widehat{\Omega}_0^1$ can be written as $\mathscr{X}^{\alpha_j+1}\Omega_0^1$. Now use (7-12) and (7-14) to see that $\big|\mathscr{X}^m(\widehat{\Omega}_{\mu}^{\mu+1}h)\big|$ is controlled by a tensor product of at most $m+\mu$ factors, and then apply Proposition 22 as above to complete the proof of (7-10).

The estimate for $\mathscr{F}^1$. The estimate in (7-1) with $\mu=1$ will follow from (7-10) and the estimate

$$\big\|(1-|z|^2)^{\sigma}\mathscr{Y}^{m_1}\big(\Lambda_g\mathscr{C}_{n,s}^{0,0}\Omega_1^2h\big)\big\|_{L^p(\lambda_n)}^p \le C\int_{\mathbb{B}_n}\big|(1-|z|^2)^{\sigma}\mathscr{X}^{m_2}(\widehat{\Omega}_1^2h)(z)\big|^p\,d\lambda_n(z), \tag{7-16}$$

where, as in Definition 25, we define $\widehat{\Omega}_1^2$ to be Ω_1^2 with ∂ replaced by D throughout:

$$\widehat{\Omega}_1^2 = \sum_{j,k=1}^{N}\overline{\frac{g_k Dg_j - g_j Dg_k}{|g|^4}}\,e_j\wedge e_k,$$

and where $Dh=\sum_{k=1}^n(D_kh)\,dz_k$ and D_k is the k-th component of D. We are using here the following observation regarding the interior product $\Omega_1^2h \lrcorner\, d\overline{w}_k$:

$$\text{For each summand of } \Omega_1^2h\lrcorner\, d\overline{w}_k, \text{ there is a unique } 1\le i\le N \text{ such that } \partial g_i/\partial\overline{w}_k \text{ occurs as a factor in the summand.} \tag{7-17}$$

We rewrite (7-16) as

$$\big\|(1-|z|^2)^{\sigma}\mathscr{R}^{m_1''}D^{m_1'}\big(\Lambda_g\mathscr{C}_{n,s}^{0,0}\Omega_1^2h\big)\big\|_{L^p(\lambda_n)}^p \le C\int_{\mathbb{B}_n}\big|(1-|z|^2)^{\sigma}\mathscr{R}^{m_2''}\overline{D}^{m_2'}\big(\widehat{\Omega}_1^2h\big)(z)\big|^p\,d\lambda_n(z), \tag{7-18}$$

where $\mathscr{R}^m=(1-|z|^2)^m(R^k)_{k=0}^m$ as in (7-2). As mentioned above, we only need to prove the case $m_1''=0$ since (7-1) only requires that we estimate $\|\mathscr{F}^1\|_{B_{p,m}^{\sigma}(\mathbb{B}_n)}$. However, when considering the estimate for $\mathscr{F}^2$ in (7-1) we will no longer have the luxury of using the norm $\|\cdot\|_{B_{p,m}^{\sigma}(\mathbb{B}_n)}$ in the second iterated integral occurring there, and so we will consider the more general case now in preparation for what comes later. As we will see however, it is necessary to choose m_1' sufficiently large in order to obtain (7-18). It is useful to recall that the operator $(1-|z|^2)R$ is "smaller" than $\overline{D}$ in the sense that

$$\begin{aligned}\overline{D} &= (1-|z|^2)P_z\overline{\nabla}+\sqrt{1-|z|^2}\,Q_z\overline{\nabla},\\ (1-|z|^2)R &= (1-|z|^2)P_z\nabla.\end{aligned}$$

To prove (7-18) we will ignore the contraction Λ_g since if derivatives hit g in the contraction, the estimates are similar if not easier. Note also that $|\Lambda_g F| \le |g|\,|F|$ for the contraction $\Lambda_g F$ of any tensor F.

We will also initially suppose that $m_1'' = 0$ and later take m_1'' sufficiently large. Now we apply Lemma 17 to $\mathscr{C}_{n,s}^{0,0}\Omega_1^2 h$ and obtain

$$\begin{aligned}\mathscr{C}_{n,s}^{0,0}\Omega_1^2 h(z) &= c_0\mathscr{C}_{n,s}^{0,0}\big(\overline{\mathscr{D}}^{m_2'}\Omega_1^2 h\big)(z) + \text{boundary terms}\\ &= \int_{\mathbb{B}_n}\Phi_{n,s}^0(w,z)\overline{\mathscr{D}}^{m_2'}(\Omega_1^2 h)\,dV(w) + \text{boundary terms}.\end{aligned}\tag{7-19}$$

A typical term above looks like

$$\int_{\mathbb{B}_n}\left(\frac{1-|w|^2}{1-\overline{w}z}\right)^{s-n}\frac{(1-w\bar{z})^{n-1}}{\triangle(w,z)^n}\overline{\mathscr{D}}^{m_2'}(\Omega_1^2 h)\,dV(w)\tag{7-20}$$

where we are discarding the sum of (balanced) factors

$$\left(\frac{(1-|w|^2)(1-|z|^2)}{|1-w\bar{z}|^2}\right)^j$$

for $1 \le j \le n-1$ in Lemma 17, which turn out to only help with the estimates. This can be seen from (5-9) and its trivial counterpart

$$\left|D_{(z)}^m(1-|z|^2)^k\right| + \left|(1-|z|^2)^m R_{(z)}^m(1-|z|^2)^k\right| \le C(1-|z|^2)^k.$$

Recall from the general discussion above that in the integral (7-20) there are rogue factors $\overline{w_k - z_k}$ in $\overline{\mathscr{D}}^{m_2'}(\Omega_1^2 h)(w)$ that must be associated with a $\partial/\partial\overline{w}_k$ derivative that hits some factor of each summand in the k-th component $\Omega_1^2 \lrcorner\, d\overline{w}_k$ of $\Omega_1^2 \approx \overline{g_i\partial g_j - g_j\partial g_i}$. Thus we can apply (5-7) to the components of $\Omega_1^2 h(z)$ to obtain

$$\begin{aligned}\left|\overline{\mathscr{D}}^{m_2'}\Omega_1^2 h(z)\right| &\approx \left|\sum_{k=1}^n\sum_{|\alpha|=m_2'}(\overline{w_k - z_k})\overline{(w-z)^\alpha}\,\frac{\partial^{m_2'}}{\partial\overline{w}^\alpha}\big(\Omega_1^2 h \lrcorner\, d\overline{w}_k\big)\right|\\ &\le C\left(\frac{\sqrt{\triangle(w,z)}}{1-|w|^2}\right)^{m_2'+1}\left|\overline{D}^{m_2'}(\widehat{\Omega}_1^2 h)(w)\right|.\end{aligned}\tag{7-21}$$

Thus we get

$$\begin{aligned}&(1-|z|^2)^\sigma\left|D^{m_1'}\mathscr{C}_{n,s}^{0,0}\Omega_1^2 h(z)\right|\\ &\quad\le \int_{\mathbb{B}_n}(1-|z|^2)^\sigma\left|D_{(z)}^{m_1'}\left(\frac{(1-|w|^2)^{s-n}(1-w\bar{z})^{n-1}}{(1-\overline{w}z)^{s-n}\triangle(w,z)^n}\right)\right|\left(\frac{\sqrt{\triangle(w,z)}}{1-|w|^2}\right)^{m_2'+1}\left|\overline{D}^{m_2'}(\widehat{\Omega}_1^2 h)(w)\right|dV(w)\\ &\quad\equiv S_{m_1',m_2'}^s f(z),\end{aligned}\tag{7-22}$$

where

$$f(w) = (1-|w|^2)^\sigma\left|\overline{D}^{m_2'}\big(\widehat{\Omega}_1^2 h\big)(w)\right|.\tag{7-23}$$

Now we iterate the estimate (5-8),

$$\left|D_{(z)}\triangle(w,z)\right| \le C(1-|z|^2)\triangle(w,z)^{1/2} + \triangle(w,z),$$

to obtain

$$\left|D_{(z)}^{m_1'}\left(\frac{(1-|w|^2)^{s-n}(1-w\bar{z})^{n-1}}{(1-\overline{w}z)^{s-n}\triangle(w,z)^n}\right)\right| \\ \leq \frac{(1-|z|^2)^{m_1'}(1-|w|^2)^{s-n}\triangle(w,z)^{m_1'/2}}{|1-w\bar{z}|^{s-2n+1}\triangle(w,z)^{n+m_1'}}+\cdots+\frac{(1-|w|^2)^{s-n}}{|1-w\bar{z}|^{s-2n+1}\triangle(w,z)^n}+OK, \tag{7-24}$$

where the terms in OK are obtained when some of the derivatives D hit the factor $(1-\overline{w}z)^{s-n}$ in the denominator or factors $D\triangle(w,z)$ already in the numerator. Leaving the OK terms for later, we combine all the estimates above to get that if we plug the first term on the right in (7-24) into the left side of (7-18), then the result is dominated by

$$\int_{\mathbb{B}_n}\frac{(1-|z|^2)^{m_1'+\sigma}(1-|w|^2)^{s-n-m_2'-1-\sigma}\triangle(w,z)^{m_1'+m_2'+1/2}}{|1-w\bar{z}|^{s-2n+1}\triangle(w,z)^{n+m_1'}}f(w)\,dV(w) \\ =\int_{\mathbb{B}_n}\frac{(1-|z|^2)^{m_1'+\sigma}(1-|w|^2)^{s-n-1-m_2'-\sigma}}{|1-w\bar{z}|^{s-2n+1}}\sqrt{\triangle(w,z)}^{\,m_2'-m_1'-2n+1}f(w)\,dV(w).$$

Now for convenience choose $m_2'=m_1'+2n-1$ so that the factor of $\sqrt{\triangle(w,z)}$ disappears. We then get

$$(1-|z|^2)^{\sigma}\left|D^{m_1'}\mathscr{C}_{n,s}^{0,0}\Omega_1^2h(z)\right|\leq\int_{\mathbb{B}_n}\frac{(1-|z|^2)^{m_1'+\sigma}(1-|w|^2)^{s-3n-m_1'-\sigma}}{|1-w\bar{z}|^{s-2n+1}}f(w)\,dV(w). \tag{7-25}$$

Lemma 24 shows that the operator

$$T_{a,b,0}f(z)=\int_{\mathbb{B}_n}\frac{(1-|z|^2)^a(1-|w|^2)^b}{|1-w\bar{z}|^{n+1+a+b}}f(w)\,dV(w)$$

is bounded on $L^p\big(\mathbb{B}_n;(1-|w|^2)^t\,dV(w)\big)$ if and only if

$$-pa<t+1<p(b+1).$$

We apply this lemma with $t=-n-1$, $a=m_1'+\sigma$ and $b=s-3n-m_1'-\sigma$. Note that the sums of the exponents in the numerator and denominator of (7-25) are equal if we write the integral in terms of invariant measure $d\lambda_n(w)=(1-|w|^2)^{-n-1}\,dV(w)$. We conclude that $S_{m_1',m_2'}^s$ is bounded on $L^p(d\lambda_n)$ provided T is, and that this latter happens if and only if

$$-p(m_1'+\sigma)<-n<p\left(s-3n+1-m_1'-\sigma\right).$$

This requires $m_1'+\sigma>\frac{n}{p}$ and $s>3n-1+m_1'+\sigma-\frac{n}{p}$.

Remark. Suppose instead that we choose m_2' to be a positive integer satisfying $c=m_2'-m_1'-2n+1>-2n$. Then we would be dealing with the operator $T_{a,b,c}$, where $a=m_1'+\sigma$ and

$$b=s-n-1-m_2'-\sigma=s-3n-c-m_1'-\sigma.$$

By Lemma 24, $T_{a,b,c}$ is bounded on $L^p(d\lambda_n)$ if and only if

$$-p(m_1'+\sigma)<-n<p(s-3n+1-c-m_1'-\sigma),$$

i.e., $m_1' + \sigma > \frac{n}{p}$ and $s > c + 3n - 1 + m_1' + \sigma - \frac{n}{p}$. Thus we can use *any* value of $c > -2n$ provided we choose $m_2' \geq m_1'$ and s large enough.

Now we turn to the second displayed term on the right side of (7-24), which leads to the operator $T_{a,b,0}$ with $a = \sigma,\ b = s - 3n - \sigma$. This time we will *not* in general have the required boundedness condition $\sigma > \frac{n}{p}$. It is for this reason that we must return to (7-18) and insist that m_1'' be chosen sufficiently large that $m_1'' + \sigma > \frac{n}{p}$. For convenience we let $m_1' = 0$ for now. Indeed, it follows from the second line in the crucial inequality (5-8) that the second displayed term on the right side of (7-24) is

$$\frac{(1-|z|^2)^{m_1''}(1-|w|^2)^{s-n}\triangle(w,z)^{m_1''/2}}{|1-w\bar{z}|^{s-2n+1}\triangle(w,z)^{n+m_1''}} + \text{better terms.}$$

Using this expression and choosing $m_2' = m_1'' + 2n - 1$ so that the term $\sqrt{\triangle(w,z)}$ disappears from the ensuing integral, we obtain the following analogue of (7-25):

$$(1-|z|^2)^{\sigma}(1-|z|^2)^{m_1''}\left|\mathcal{R}^{m_1''}\mathcal{C}_{n,s}^{0,0}\Omega_1^2 h(z)\right| \leq \int_{\mathbb{B}_n} \frac{(1-|z|^2)^{m_1''+\sigma}(1-|w|^2)^{s-3n-m_1''-\sigma}}{|1-w\bar{z}|^{s-2n+1}} f(w)\,dV(w).$$

The corresponding operator $T_{a,b,0}$ has $a = m_1'' + \sigma$ and $b = s - 3n - m_1'' - \sigma$ and is bounded on $L^p(\lambda_n)$ when $-p(m_1'' + \sigma) < -n < p(s - 3n + 1 - m_1'' - \sigma)$. Thus there is no unnecessary restriction on σ if m_1'' and s are chosen appropriately large. Note that the only difference between this operator $T_{a,b,0}$ and the previous one is that m_1' has been replaced by m_1''.

The arguments above are easily modified to handle the general case of (7-18) provided $m_1'' + \sigma > \frac{n}{p}$ and s is chosen sufficiently large.

Now we return to consider the OK terms in (7-24). For this we use the inequality (5-9):

$$\left|D_{(z)}^m\{(1-\overline{w}z)^k\}\right| \leq C\,|1-\overline{w}z|^k\left(\frac{1-|z|^2}{|1-\overline{w}z|}\right)^{m/2}.$$

We ignore the derivative $(1-|z|^2)R$, since the second line in (5-9) shows that it satisfies a better estimate. We also write m_1 and m_2 in place of m_1' and m_2' now. As a result, one of the extremal OK terms in (7-24) is

$$\frac{(1-|z|^2)^{m_1/2}(1-|w|^2)^{s-n}}{|1-w\bar{z}|^{s-2n+1+(m_1/2)}\triangle(w,z)^n},$$

which when combined with the other estimates leads to the integral operator

$$\int_{\mathbb{B}_n} \frac{(1-|z|^2)^{m_1/2+\sigma}(1-|w|^2)^{s-n-1-m_2-\sigma}}{|1-w\bar{z}|^{s-2n+1+(m_1/2)}}\sqrt{\triangle(w,z)}^{\,m_2-2n-1} f(w)\,dV(w).$$

This is $T_{a,b,c}$ with $a = \frac{m_1}{2} + \sigma,\ b = s - n - 1 - m_2 - \sigma$, and $c = m_2 - 2n - 1$. This is bounded on $L^p(\lambda_n)$ provided $m_2 \geq 2$ and

$$-p\left(\frac{m_1}{2} + \sigma\right) < -n < p(s - n - m_2 - \sigma),$$

i.e., $\frac{m_1}{2} + \sigma > \frac{n}{p}$ and $s > n + m_2 + \sigma - \frac{n}{p}$. The intermediate OK terms are handled similarly. Note that the crux of the matter is that all of the positive operators have the form $T_{a,b,c}$, and moreover, if s and the $m's$ are chosen appropriately large, then $T_{a,b,c}$ is bounded on $L^p(\lambda_n)$.

Boundary terms for $\mathscr{F}^1$. Now we turn to estimating the boundary terms in (7-19). A typical term is

$$\mathscr{S}_{n,s}\big(\overline{\mathscr{D}}^k(\Omega_1^2 h)\big)\big[\overline{\mathscr{Z}}\big](z)=\int_{\mathbb{B}_n}\frac{(1-|w|^2)^{s-n-1}}{(1-\overline{w}z)^s}\overline{\mathscr{D}}^k(\Omega_1^2 h)\big[\overline{\mathscr{Z}}\big](w)\,dV(w), \tag{7-26}$$

with $0\le k\le m-1$ upon appealing to Lemma 17.

We now apply the operator $(1-|z|^2)^{m_1+\sigma}R^{m_1}$ to the integral on the right side of (7-26); using the inequalities (5-7)–(5-9) we obtain that the absolute value of the result is dominated by

$$\begin{aligned}\int_{\mathbb{B}_n}&\frac{(1-|z|^2)^{m_1+\sigma}(1-|w|^2)^{s-n-1}}{|1-\overline{w}z|^{s+m_1}}\left(\frac{\sqrt{\triangle(w,z)}}{1-|w|^2}\right)^{k+1}\big|\overline{D}^k(\widehat{\Omega}_1^2 h)\big|\,dV(w)\\&=\int_{\mathbb{B}_n}\frac{(1-|z|^2)^{m_1+\sigma}(1-|w|^2)^{s-n-2-k-\sigma}\sqrt{\triangle(w,z)}^{\,k+1}}{|1-\overline{w}z|^{s+m_1}}\big|(1-|w|^2)^{\sigma}\,\overline{D}^k\big(\widehat{\Omega}_1^2 h\big)(w)\big|\,dV(w).\end{aligned}$$

The operator in question here is $T_{a,b,c}$ with $a=m_1+\sigma$, $b=s-n-2-k-\sigma$, and $c=k+1$, since

$$a+b+c+n+1=s+m_1.$$

Lemma 24 applies to prove the desired boundedness on $L^p(\lambda_n)$ provided $m_1+\sigma>\frac{n}{p}$.

However, if k fails to satisfy $k+1>2\big(\frac{n}{p}-\sigma\big)$, then the derivative $D^{k+1}\Omega$ cannot be used to control the norm $\|\Omega\|_{B_p^\sigma(\mathbb{B}_n)}$. To compensate for a small k, we must then apply Corollary 16 to the right side of (7-26) (which for fixed z is in $C(\overline{\mathbb{B}}_n)\cap C^\infty(\mathbb{B}_n)$) before differentiating and taking absolute values inside the integral. This then leads to operators of the form

$$(1-|z|^2)^{m_1+\sigma}R^{m_1}\left(\int_{\mathbb{B}_n}\frac{(1-|w|^2)^{s-n-1}}{(1-\overline{w}z)^s}(1-|w|^2)^m R^m\big[\overline{\mathscr{D}}^k(\Omega_1^2 h)(w)\big]\,dV(w)\right),$$

which are dominated by

$$\int_{\mathbb{B}_n}\frac{(1-|z|^2)^{m_1+\sigma}(1-|w|^2)^{s-n-1}}{|1-\overline{w}z|^{s+m_1}}\left(\frac{\sqrt{\triangle(w,z)}}{1-|w|^2}\right)^{k+1}\big|\mathscr{R}^m\,\overline{D}^k(\widehat{\Omega}_1^2 h)(w)\big|\,dV(w),$$

which is

$$\int_{\mathbb{B}_n}\frac{(1-|z|^2)^{m_1+\sigma}(1-|w|^2)^{s-n-2-k-\sigma}\sqrt{\triangle(w,z)}^{\,k+1}}{|1-\overline{w}z|^{s+m_1}}\big|(1-|w|^2)^{\sigma}\mathscr{R}^m\,\overline{D}^k(\widehat{\Omega}_1^2 h)(w)\big|\,dV(w).$$

This latter operator is $T_{a,b,c}H(z)$, with

$$a=m_1+\sigma,\quad b=s-n-2-k-\sigma,\quad c=k+1,\quad\text{and}\quad H(w)=\big|(1-|w|^2)^{\sigma}R_{b'}^m\overline{D}^k(\widehat{\Omega}_1^2 h)(w)\big|.$$

Note that for $m>2\big(\frac{n}{p}-\sigma\big)$ we do indeed now have $\|H\|_{L^p(\lambda_n)}\approx\|\widehat{\Omega}_1^2 h\|_{B_p^\sigma(\mathbb{B}_n)}$. The operator here is the same as that above and so Lemma 24 applies to prove the desired boundedness on $L^p(\lambda_n)$.

The estimate for $\mathscr{F}^2$. Our next task is to obtain the estimate (7-1) for $\mu=2$. For this we will show that

$$\begin{aligned}\int_{\mathbb{B}_n}&\big|(1-|z|^2)^{m_1+\sigma}R^{m_1}\Lambda_g\mathscr{C}_{n,s_1}^{0,0}\Lambda_g\mathscr{C}_{n,s_2}^{0,1}\Omega_2^3\big|^p\,d\lambda_n(z)\\&\le C\int_{\mathbb{B}_n}\big|(1-|z|^2)^{\sigma}(1-|z|^2)^{m_3''}R^{m_3''}\overline{D}^{m_3'}(\widehat{\Omega}_2^3 h)(z)\big|^p\,d\lambda_n(z).\end{aligned}\tag{7-27}$$

Unlike the previous argument, this time we will have to deal with a rogue term $\bar{z}_2 - \bar{\xi}_2$ where there is no derivative $\partial/\partial\bar{\xi}_2$ to associate to it. Again we ignore the contractions Λ_g. Then we use Lemma 17 to perform integration by parts m_2' times in the first iterated integral and m_3' times in the second iterated integral. We also use Corollary 16 to perform integration by parts in the *radial* derivative m_2'' times in the *first* iterated integral (for fixed z, we have $\mathscr{C}^{0,1}_{n,s_2}\Omega^3_2 \in C(\overline{\mathbb{B}}_n) \cap C^\infty(\mathbb{B}_n)$ by standard estimates [Charpentier 1980]), so that the additional factor $(1-|\xi|^2)^{m_2''}$ can be used crucially in the second iterated integral, and also m_3'' times in the *second* iterated integral for use in acting on Ω^3_2.

Recall from Lemma 17 that

$$\begin{aligned}\mathscr{C}^{0,q}_{n,s}\eta(z) &= \text{boundary terms (depending on } m)\\ &+\sum_{\ell=0}^{q}\int_{\mathbb{B}_n}\frac{(1-w\bar{z})^{n-1-\ell}(1-|w|^2)^\ell}{\triangle(w,z)^n}\left(\frac{1-|w|^2}{1-\overline{w}z}\right)^{s-n}\left(\sum_{j=0}^{n-\ell-1}c_{j,\ell,n,s}\left(\frac{(1-|w|^2)(1-|z|^2)}{|1-w\bar{z}|^2}\right)^j\right)\overline{\mathscr{D}}^m\eta(z).\end{aligned}$$

Recall also that $\overline{\mathscr{D}}^m$ already has the rogue terms built in, as can be seen from (3-6). Now we use the right side above with $q=\ell=j=0$ to substitute for $\mathscr{C}^{0,0}_{n,s_1}$, and the right side above with $q=\ell=1$ and $j=0$ to substitute for $\mathscr{C}^{0,1}_{n,s_2}$. Then a typical part of the resulting kernel of the operator $\mathscr{C}^{0,0}_{n,s_1}\mathscr{C}^{0,1}_{n,s_2}\Omega^3_2(z)$ is

$$\begin{aligned}&\int_{\mathbb{B}_n}\frac{(1-\xi\bar{z})^{n-1}}{\triangle(\xi,z)^n}\left(\frac{1-|\xi|^2}{1-\bar{\xi}z}\right)^{s_1-n}(\bar{z}_2-\bar{\xi}_2)(1-|\xi|^2)^{m_2'}R^{m_2'}\overline{\mathscr{D}}^{m_2''} \qquad (7\text{-}28)\\ &\times\int_{\mathbb{B}_n}\frac{(1-w\bar{\xi})^{n-2}(1-|w|^2)}{\triangle(w,\xi)^n}\left(\frac{1-|w|^2}{1-\overline{w}\xi}\right)^{k_2-n}(\overline{w}_1-\bar{\xi}_1)(1-|w|^2)^{m_3'}R^{m_3'}\overline{\mathscr{D}}^{m_3''}(\Omega^3_2h)(w)\,dV(w)\,dV(\xi),\end{aligned}$$

where we have arbitrarily chosen $\bar{z}_2-\bar{\xi}_2$ and $\overline{w}_1-\bar{\xi}_1$ as the rogue factors.

Remark. It is important to note that the differential operators $\overline{\mathscr{D}}^{m_2}_\xi$ are conjugate in the variable z and hence *vanish* on the kernels of the boundary terms $\mathscr{S}_{n,s}(\overline{\mathscr{D}}^k\Omega^3_2h)(z)$ in the integration by parts formula (3-7) associated to the Charpentier solution operator $\mathscr{C}^{0,1}_{n,s_2}$, since these kernels are *holomorphic*. As a result the operator $\overline{\mathscr{D}}^{m_2'}$ hits only the factor $\overline{\mathscr{D}}^k\Omega^3_2h$ and a typical term is

$$(\overline{z_i-\zeta_i})\,\frac{\partial}{\partial\bar{z}_i}\big((\overline{w_i-z_i})\,\Omega^3_2h\big) = -(\overline{z_i-\zeta_i})\,\Omega^3_2h,$$

where the derivative $\partial/\partial\overline{w}_i$ must occur in each surviving term in Ω^3_2h, and this term which is then handled like the rogue terms.

Now we recall the factorization (4-4) with $\ell=2$,

$$\Omega^3_2 = -4\Omega^1_0\wedge\widetilde{\Omega}^1_0\wedge\widetilde{\Omega}^1_0,$$

and that $\Omega^3_2(w)$ must have both derivatives $\partial g/\partial\overline{w}_1$ and $\partial g/\partial\overline{w}_2$ occurring in it, one surviving in each of the factors $\widetilde{\Omega}^1_0$, along with other harmless powers of g that we ignore. Thus we may replace $\widetilde{\Omega}^1_0\wedge\widetilde{\Omega}^1_0$ with $\partial/\partial\overline{w}_2\Omega^1_0\wedge\partial/\partial\overline{w}_1\Omega^1_0$. If we use

$$\bar{z}_2-\bar{\xi}_2 = (\bar{z}_2-\overline{w}_2)-(\bar{\xi}_2-\overline{w}_2),$$

we can write the iterated integral above as

$$\int_{\mathbb{B}_n} \frac{(1-\xi\bar{z})^{n-1}}{\triangle(\xi,z)^n}\left(\frac{1-|\xi|^2}{1-\bar{\xi}z}\right)^{s_1-n} \\ \times \int_{\mathbb{B}_n} (1-|\xi|^2)^{m_2''} R^{m_2''}\overline{\mathfrak{D}}^{m_2'}\left(\frac{(1-w\bar{\xi})^{n-2}(1-|w|^2)}{\triangle(w,\xi)^n}\left(\frac{1-|w|^2}{1-\overline{w}\xi}\right)^{s_2-n}\right) \\ \times\left((1-|w|^2)^{m_3''}R^{m_3''}(\bar{\xi}_2-\overline{w}_2)\frac{\partial}{\partial\overline{w}_2}\overline{\mathfrak{D}}^{m_3'-\ell}\Omega_0^1\right)\wedge\left((1-|w|^2)^{m_3''}R^{m_3''}(\bar{\xi}_1-\overline{w}_1)\frac{\partial}{\partial\overline{w}_1}\overline{\mathfrak{D}}^{\ell}\Omega_0^1\right) \\ \times dV(w)\,dV(\xi)$$

minus the same expression but with the rogue factor $\bar{\xi}_2-\overline{w}_2$ on the third line replaced by the rogue factor $\bar{z}_2-\overline{w}_2$. We have temporarily ignored the wedge products with terms that do not include derivatives of g, as these terms are bounded and so harmless.

Now we apply $(1-|z|^2)^{\sigma}(1-|z|^2)^{m_1''}R^{m_1''}D^{m_1'}$ to these operators. Using the crucial inequalities (5-7)–(5-9), together with the factorization (7-9) with $\ell=2$,

$$\widehat{\Omega}_2^3=-4\Omega_0^1\wedge\widehat{\Omega}_0^1\wedge\widehat{\Omega}_0^1,$$

the result of this application on the first integral is then dominated by

$$\int_{\mathbb{B}_n}\frac{(1-|z|^2)^{\sigma}|1-\xi\bar{z}|^{n-1}}{\triangle(\xi,z)^{m_1'+m_1''+n}}\left[(1-|z|^2)\sqrt{\triangle(\xi,z)}\right]^{m_1''}\left(\left[(1-|z|^2)\sqrt{\triangle(\xi,z)}\right]^{m_1'}+\triangle(\xi,z)^{m_1'}\right)\left|\frac{1-|\xi|^2}{1-\xi\bar{z}}\right|^{s_1-n} \\ \times\left(\int_{\mathbb{B}_n}\frac{(1-|\xi|^2)^{m_2''}|1-w\bar{\xi}|^{n-2}(1-|w|^2)}{\triangle(w,\xi)^{m_2'+m_2''+n}}\left(\frac{\sqrt{\triangle(\xi,z)}}{1-|\xi|^2}\right)^{m_2'}\left[(1-|\xi|^2)\sqrt{\triangle(w,\xi)}\right]^{m_2''}\right. \\ \times\left(\left[(1-|\xi|^2)\sqrt{\triangle(w,\xi)}\right]^{m_2'}+\triangle(w,\xi)^{m_2'}\right)\left|\frac{1-|w|^2}{1-w\bar{\xi}}\right|^{s_2-n}\left(\frac{\sqrt{\triangle(w,\xi)}}{1-|w|^2}\right)^{m_3'}\left(\frac{\sqrt{\triangle(w,\xi)}}{1-|w|^2}\right)^2 \\ \left.\times\left|(1-|w|^2)^{m_3''}R^{m_3''}\overline{D}^{m_3'}(\widehat{\Omega}_2^3h)(w)\right|dV(w)\right)dV(\xi), \quad (7\text{-}29)$$

and the result of this application on the second integral is dominated by exactly the same expression but with one of the two factors $\sqrt{\triangle(w,\xi)}/(1-|w|^2)$ that occur at the end of the third line in (7-29) replaced by the factor $\sqrt{\triangle(w,z)}/(1-|w|^2)$. The ignored wedge products have now been reinstated in $\widehat{\Omega}_2^3$.

Now for the iterated integral in (7-29), we can separate it into the composition of two operators of the form treated previously. One factor is the operator

$$\int_{\mathbb{B}_n}\frac{(1-|z|^2)^{\sigma}|1-\xi\bar{z}|^{n-1}}{\triangle(\xi,z)^{m_1'+m_1''+n}}\left[(1-|z|^2)\sqrt{\triangle(\xi,z)}\right]^{m_1''}\left(\left[(1-|z|^2)\sqrt{\triangle(\xi,z)}\right]^{m_1'}+\triangle(\xi,z)^{m_1'}\right) \\ \times\left(\frac{\sqrt{\triangle(\xi,z)}}{1-|\xi|^2}\right)^{m_2'}\left|\frac{1-|\xi|^2}{1-\xi\bar{z}}\right|^{s_1-n}(1-|\xi|^2)^{-\sigma}F(\xi)\,dV(\xi), \quad (7\text{-}30)$$

and the other factor is the operator $F(\xi)$ given by

$$\int_{\mathbb{B}_n} \frac{(1-|\xi|^2)^\sigma |1-w\bar{\xi}|^{n-2}(1-|w|^2)}{\triangle(w,\xi)^{m_2'+m_2''+n}} \big[(1-|\xi|^2)\sqrt{\triangle(w,\xi)}\big]^{m_2''} \Big(\big[(1-|\xi|^2)\sqrt{\triangle(w,\xi)}\big]^{m_2'} + \triangle(w,\xi)^{m_2'}\Big) \\ \times \left(\frac{\sqrt{\triangle(w,\xi)}}{1-|w|^2}\right)^{m_3'+2} \left|\frac{1-|w|^2}{1-w\bar{\xi}}\right|^{s_2-n} (1-|w|^2)^{-\sigma} f(w)\, dV(w), \quad (7\text{-}31)$$

where $f(w) = (1-|w|^2)^\sigma \big|(1-|w|^2)^{m_3''} R^{m_3''} \bar{D}^{m_3'} (\widehat{\Omega}_2^3 h)(w)\big|$. We now show how Lemma 24 applies to obtain the appropriate boundedness.

We will in fact compare the corresponding kernels to that in (7-25). When we consider the summand $\triangle(\xi,z)^{m_1'}$ at the end of the first line of (7-30), the first operator has kernel

$$\frac{(1-|z|^2)^{\sigma+m_1''}(1-|\xi|^2)^{s_1-n-m_2'-\sigma}}{|1-\xi\bar{z}|^{s_1-2n+1}\triangle(\xi,z)^{m_1'+m_1''+n-(m_1''+2m_1'+m_2')/2}} = \frac{(1-|z|^2)^{\sigma+m_1''}(1-|\xi|^2)^{s_1-3n-m_1''-\sigma}}{|1-\xi\bar{z}|^{s_1-2n+1}}, \quad (7\text{-}32)$$

if we choose $m_2' = m_1''+2n$ so that the factor $\triangle(\xi,z)$ disappears. This is exactly the same as the kernel of the operator in (7-25) in the previous alternative argument but with m_1'' in place of m_1' there. When we consider instead the summand $\big[(1-|z|^2)\sqrt{\triangle(\xi,z)}\big]^{m_1'}$ on the first line of (7-30), we obtain the kernel in (7-32) but with $m_1''+m_1'$ in place of m_1''.

When we consider the summand $\triangle(w,\xi)^{m_2'}$ at the end of the second line of (7-31), the second operator has kernel

$$\frac{(1-|\xi|^2)^{m_2''+\sigma}(1-|w|^2)^{1+s_2-n-m_3'-2-\sigma}}{|1-w\bar{\xi}|^{s_2-2n+2}\triangle(w,\xi)^{m_2'+m_2''+n-(m_2''+2m_2'+m_3'+2)/2}} \\ = \frac{(1-|\xi|^2)^{m_2''+\sigma}(1-|w|^2)^{s_2-3n+1-m_2''-\sigma}}{|1-w\bar{\xi}|^{s_2-2n+2}}. \quad (7\text{-}33)$$

if we choose $m_3' = m_2''+2n-2$, and this is also bounded on $L^p(d\lambda_n)$ for m_2'' and s_2 sufficiently large.

Remark. It is here in choosing m_2'' large that we are using the full force of Corollary 16 to perform integration by parts in the *radial* derivative m_2'' times in the first iterated integral.

When we consider instead the summand $\big[(1-|z|^2)\sqrt{\triangle(\xi,z)}\big]^{m_2'}$ on the first line of (7-31), we obtain the kernel in (7-33) but with $m_2''+m_2'$ in place of m_2''.

To handle the case of (7-29) in which the factor $\sqrt{\triangle(w,z)}/(1-|w|^2)$ replaces one of the factors $\sqrt{\triangle(w,\xi)}/(1-|w|^2)$, we must first deal with the rogue factor $\sqrt{\triangle(w,z)}$ whose variable pair (w,z) doesn't match that of either of the denominators $\triangle(\xi,z)$ or $\triangle\,(w,\xi)$. For this we use the fact that

$$\sqrt{\triangle(w,z)} = |1-w\bar{z}|\,|\varphi_z(w)| = \delta(w,z)^2\rho(w,z),$$

where $\rho(w,z) = |\varphi_z(w)|$ is the invariant pseudohyperbolic metric on the ball (Corollary 1.22 in [Zhu 2005]) and where $\delta(w,z) = |1-w\bar{z}|^{1/2}$ satisfies the triangle inequality on the ball (Proposition 5.1.2 in [Rudin 1980]). Thus we have

$$\rho(w,z) \le \rho(\xi,z)+\rho(w,\xi), \quad \delta(w,z) \le \delta(\xi,z)+\delta(w,\xi),$$

and so also

$$\begin{aligned}\sqrt{\triangle(w,z)} &\le 2\big[\delta(\xi,z)^2+\delta(w,\xi)^2\big]\big(|\varphi_z(\xi)|+|\varphi_\xi(w)|\big)\\ &=2\left(1+\frac{|1-w\bar{\xi}|}{|1-\xi\bar{z}|}\right)\sqrt{\triangle(\xi,z)}+2\left(1+\frac{|1-\xi\bar{z}|}{|1-w\bar{\xi}|}\right)\sqrt{\triangle(w,\xi)}.\end{aligned}$$

Thus we can write

$$\begin{aligned}\frac{\sqrt{\triangle(w,z)}}{1-|w|^2} \lesssim \frac{1-|\xi|^2}{1-|w|^2}\frac{\sqrt{\triangle(\xi,z)}}{1-|\xi|^2}+\frac{|1-w\bar{\xi}|}{1-|w|^2}\frac{1-|\xi|^2}{|1-\xi\bar{z}|}\frac{\sqrt{\triangle(\xi,z)}}{1-|\xi|^2}\\ +\frac{\sqrt{\triangle(w,\xi)}}{1-|w|^2}+\frac{|1-\xi\bar{z}|}{1-|\xi|^2}\frac{1-|\xi|^2}{|1-w\bar{\xi}|}\frac{\sqrt{\triangle(w,\xi)}}{1-|w|^2}.\end{aligned}\tag{7-34}$$

All of the terms on the right side of (7-34) are of an appropriate form to distribute throughout the iterated integral, and again Lemma 24 applies to obtain the appropriate boundedness.

For example, the final two terms on the right side of (7-34) that involve $\sqrt{\triangle(w,\xi)}/(1-|w|^2)$ are handled in the same way as the operator in (7-29) by taking $m_3'=m_2''+2n-2$ and $m_2'=m_1''+2n$, and taking s_1 and s_2 large as required by the extra factors

$$\frac{|1-\xi\bar{z}|}{1-|\xi|^2}\frac{1-|\xi|^2}{|1-w\bar{\xi}|}.$$

With these choices the first two terms on the right side of (7-34) that involve $\sqrt{\triangle(\xi,z)}/(1-|\xi|^2)$ are then handled using Lemma 24 with $c=\pm 1$ as follows.

If we substitute the first term

$$\frac{1-|\xi|^2}{1-|w|^2}\frac{\sqrt{\triangle(\xi,z)}}{1-|\xi|^2}$$

on the right in (7-34) for the factor $\sqrt{\triangle(w,z)}/(1-|w|^2)$, we get a composition of two operators as in (7-30) and (7-31) but with the kernel in (7-30) multiplied by $\sqrt{\triangle(\xi,z)}/(1-|\xi|^2)$ and the kernel in (7-31) multiplied by $(1-|\xi|^2)/(1-|w|^2)$ and divided by $\sqrt{\triangle(w,\xi)}/(1-|w|^2)$. If we consider the summand $\triangle(\xi,z)^{m_1'}$ at the end of the first line of (7-30), and with the choice $m_2'=m_1''+2n$ already made, the first operator then has kernel

$$\frac{\sqrt{\triangle(\xi,z)}}{1-|\xi|^2}\frac{(1-|z|^2)^{\sigma+m_1''}(1-|\xi|^2)^{s_1-3n-m_1''-\sigma}}{|1-\xi\bar{z}|^{s_1-2n+1}}=\frac{(1-|z|^2)^{m_1''+\sigma}(1-|\xi|^2)^{s_1-m_1''-3n-1-\sigma}\sqrt{\triangle(\xi,z)}}{|1-\xi\bar{z}|^{s_1-2n+1}},$$

and hence is of the form $T_{a,b,c}$ with

$$a=m_1''+\sigma,\quad b=s_1-3n-1-m_1''-\sigma,\quad c=1,$$

since $a+b+c+n+1=s_1-n-1$. Now we apply Lemma 24 to conclude that this operator is bounded on $L^p(\lambda_n)$ if and only if

$$-p(m_1''+\sigma)<-n<p(s_1-3n-m_1''-\sigma),$$

i.e., $m_1''+\sigma>\frac{n}{p}$ and $s_1>m_1''+\sigma+3n-\frac{n}{p}$.

Next we consider the summand $\triangle(w,\xi)^{m_2'}$ at the end of the first line of (7-31). With the choice $m_3' = m_2'' + 2n - 2$ already made, the second operator has kernel

$$\frac{1-|\xi|^2}{1-|w|^2}\left(\frac{\sqrt{\triangle(w,\xi)}}{1-|w|^2}\right)^{-1}\frac{(1-|\xi|^2)^{m_2''+\sigma}(1-|w|^2)^{s_2-3n+1-m_2''-\sigma}}{|1-w\bar{\xi}|^{s_2-2n+2}}$$
$$= \frac{(1-|\xi|^2)^{m_2''+\sigma+1}(1-|w|^2)^{s_2-3n+1-m_2''-\sigma}\sqrt{\triangle(w,\xi)}^{\,-1}}{|1-w\bar{\xi}|^{s_2-2n+2}},$$

and hence is of the form $T_{a,b,c}$ with

$$a = m_2'' + \sigma + 1, \quad b = s_2 - 3n + 1 - m_2'' - \sigma, \quad c = -1.$$

This operator is bounded on $L^p(\lambda_n)$ if and only if

$$-p(m_2'' + \sigma + 1) < -n < p(s_2 - 3n + 2 - m_2'' - \sigma),$$

i.e., $m_2'' + \sigma > \frac{n}{p} - 1$ and $s_2 > m_2'' + \sigma + 3n - 2 - \frac{n}{p}$.

If we now substitute the second term

$$\frac{|1-w\bar{\xi}|}{1-|w|^2}\,\frac{1-|\xi|^2}{|1-\xi\bar{z}|}\,\frac{\sqrt{\triangle(\xi,z)}}{1-|\xi|^2}$$

on the right in (7-34) for the factor $\sqrt{\triangle(w,z)}/(1-|w|^2)$ we similarly get a composition of two operators that are each bounded on $L^p(\lambda_n)$ for m_i and s_i chosen large enough.

Boundary terms for $\mathscr{F}^2$. Now we must address in $\mathscr{F}^2$ the boundary terms that arise in the integration by parts formula (3-7). Suppose the first operator $\mathscr{C}^{0,0}_{n,s_1}$ is replaced by a boundary term, but not the second. We proceed by applying Corollary 16 to the boundary term. Since the differential operator $(1-|z|^2)^{m_1+\sigma}R^{m_1}$ hits only the kernel of the boundary term, we can apply the remark following Lemma 24 to the first iterated integral and the lemma itself to the second iterated integral in the manner indicated in the above arguments. If the second operator $\mathscr{C}^{0,1}_{n,s_2}$ is replaced by a boundary term, then as mentioned in the remark on page 543, the operators $\overline{D}^{m_2}$ hit only the factors $\overline{\mathscr{D}}^{m_3}$, and this produces rogue terms that are handled as above. If the first operator $\mathscr{C}^{0,0}_{n,s_1}$ was also replaced by a boundary term, then in addition we would have radial derivatives R^m hitting the second boundary term. Since radial derivatives are holomorphic, they hit only the holomorphic kernel and not the antiholomorphic factors in $\overline{\mathscr{D}}^{m_3}$, and so these terms can also be handled as above.

7.2. ***The estimates for general $\mathscr{F}^\mu$.*** In view of inequality (7-10), it suffices to establish the inequality

$$\begin{aligned}\|\mathscr{F}^\mu\|^p_{B_p^\sigma(\mathbb{B}_n)} &= \int_{\mathbb{B}_n}\left|(1-|z|^2)^{m_1+\sigma}R^{m_1}\Lambda_g\mathscr{C}^{0,0}_{n,s_1}\dots\Lambda_g\mathscr{C}^{0,\mu-1}_{n,s_\mu}\Omega^{\mu+1}_\mu h\right|^p d\lambda_n(z)\\ &\le C_{\sigma,n,p,\delta}\int_{\mathbb{B}_n}\left|(1-|z|^2)^\sigma\mathscr{X}^{m_\mu}(\widehat{\Omega}^{\mu+1}_\mu h)(z)\right|^p d\lambda_n(z).\end{aligned} \tag{7-35}$$

Recall that the absolute value $|F|$ of an element F in the exterior algebra is the square root of the sum of the squares of the coefficients of F in the standard basis.

The case $\mu > 2$ involves no new ideas, and is merely complicated by straightforward algebra. The reason is that the solution operator $\Lambda_g \mathscr{C}^{0,0}_{n,s_1} \ldots \Lambda_g \mathscr{C}^{0,\mu-1}_{n,s_\mu}$ acts *separately* in each entry of the form $\Omega^{\mu+1}_\mu h$, an element of the exterior algebra of $\mathbb{C}^\infty \otimes \mathbb{C}^n$ which we view as an alternating ℓ^2-tensor of $(0,\mu)$ forms in $\mathbb{C}^n$. These operators decompose as a sum of simpler operators with the basic property that their kernels are *identical*, except that the rogue factors in each kernel differ according to the entry. Nevertheless, there are always exactly μ distinct rogue factors in each kernel and after splitting, the μ rogue factors can be associated in one-to-one fashion with each of the derivatives $\partial/\partial \overline{w}_j$ in the corresponding entry of

$$\Omega^{\mu+1}_\mu h = -(\mu+1)\left(\sum_{k_0=1}^{\infty} \frac{\overline{g}_{k_0}}{|g|^2} e_{k_0}\right) \wedge \bigwedge_{i=1}^{\mu} \left(\sum_{k_i=1}^{\infty} \frac{\overline{\partial}\,\overline{g}_{k_i}}{|g|^2} e_{k_i}\right) h.$$

After applying the crucial inequalities, this effectively results in replacing each derivative $\partial/\partial \overline{w}_j$ by the derivative $\overline{D}_j$, and consequently we can write the resulting form as $\widehat{\Omega}^{\mu+1}_\mu h$.

This completes our proof of Theorem 2.

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