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SOBOLEV SPACE ESTIMATES FOR A CLASS OF BILINEAR PSEUDODIFFERENTIAL OPERATORS LACKING SYMBOLIC CALCULUS

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The reappearance of what is sometimes called exotic behavior for linear and multilinear pseudodifferential operators is investigated. The phenomenon is shown to be present in a recently introduced class of bilinear pseudodifferential operators which can be seen as more general variable coefficient counterparts of the bilinear Hilbert transform and other singular bilinear multipliers operators. We prove that such operators are unbounded on products of Lebesgue spaces but bounded on spaces of smooth functions (this is the exotic behavior referred to). In addition, by introducing a new way to approximate the product of two functions, estimates on a new paramultiplication are obtained.

1. Introduction

An anomalous yet recurrent phenomenon. This article is a continuation of recent work devoted to the development of a theory of bilinear and multilinear pseudodifferential operators which are the x-dependent counterparts of the singular multipliers modeled by the bilinear Hilbert transform. In particular we will further study the class of bilinear pseudodifferential operators $BS^0_{1,1;\pi/4}$ and show that it has a sometimes called *exotic* or *forbidden* behavior regarding boundedness on function spaces.

By a bilinear pseudodifferential operator we mean an operator, defined a priori on test functions, of the form

$$T_\sigma(f,g)(x)=\int_{\mathbb{R}^{2n}}\sigma(x,\xi,\eta)\hat{f}(\xi)\hat{g}(\eta)e^{ix\cdot(\xi+\eta)}\,d\xi\,d\eta.$$

Two main types of x-dependent classes of symbols have been studied in the literature. One is the Coifman–Meyer type $BS^m_{\rho,\delta}(\mathbb{R}^n)$, $0\le\delta\le\rho\le 1$, $m\in\mathbb{R}$, of symbols satisfying estimates of the form

$$|\partial_x^\alpha\partial_\xi^\beta\partial_\eta^\gamma\sigma(x,\xi,\eta)|\le C_{\alpha\beta\gamma}(1+|\xi|+|\eta|)^{m+\delta|\alpha|-\rho(|\beta|+|\gamma|)}, \tag{1-1}$$

for all multi-indices α,β,γ.

The other type corresponds to classes denoted by $BS^m_{\rho,\delta;\theta}(\mathbb{R}^n)$, $0\le\delta\le\rho\le 1$, $m\in\mathbb{R}$, $-\pi/2<\theta\le\pi/2$, and consisting of symbols satisfying

$$|\partial_x^\alpha\partial_\xi^\beta\partial_\eta^\gamma\sigma(x,\xi,\eta)|\le C_{\alpha\beta\gamma;\theta}(1+|\eta-\tan(\theta)\xi|)^{m+\delta|\alpha|-\rho(|\beta|+|\gamma|)} \tag{1-2}$$

(where for $\theta=\pi/2$ the estimates are interpreted to decay in terms of $1+|\xi|$ only). Both types can be

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seen as bilinear analogs of the classical Hörmander classes $S^m_{\rho,\delta}(\mathbb{R}^n)$ of linear pseudodifferential operators

$$T_\tau(f)(x) = \int_{\mathbb{R}^n} \tau(x,\xi)\widehat{f}(\xi)e^{ix\cdot\xi}\,d\xi,$$

with symbols satisfying

$$|\partial_x^\alpha \partial_\xi^\beta \tau(x,\xi)| \le C_{\alpha\beta}(1+|\xi|)^{m+\delta|\alpha|-\rho|\beta|}. \tag{1-3}$$

As the name indicates, the first type of bilinear classes was introduced by Coifman and Meyer [1975; 1978a; 1978b] at least in the case $m=0$, $\rho=1$ and $\delta=0$. It is now well understood that the operators in $BS^0_{1,0}$ are examples of certain singular integrals and fit within the general multilinear Calderón–Zygmund theory developed in [Grafakos and Torres 2002]; see also [Christ and Journé 1987; Kenig and Stein 1999]. For other values of the parameters, the classes $BS^m_{\rho,\delta}$ were studied in [Bényi 2003; Bényi and Torres 2003; 2004; Bényi et al. 2006; 2010].

The general classes $BS^m_{\rho,\delta;\theta}$ with x-dependent symbols were first introduced in [Bényi et al. 2006]. A connection to the bilinear Hilbert transform and the work of Lacey and Thiele [1997; 1999] is given by the study in the x-independent case of singular multipliers in one dimension satisfying

$$|\partial_\xi^\beta \partial_\eta^\gamma \sigma(\xi,\eta)| \le C_{\beta\gamma}|\eta-\tan(\theta)\xi|^{-|\beta|-|\gamma|}.$$

This type of multipliers was investigated in [Gilbert and Nahmod 2000; 2001; 2002; Muscalu et al. 2002]. We also recall that if for τ in $S^0_{1,0}(\mathbb{R})$ we define

$$\sigma(x,\xi,\eta) = \tau(x,\xi-\eta), \tag{1-4}$$

then σ is in $BS^0_{1,0;\pi/4}$. These operators have a certain modulation invariance:

$$T_\sigma(e^{iw\cdot}f, e^{iw\cdot}g)(x) = e^{i2wx}T_\sigma(f,g)(x)$$

for all $w\in\mathbb{R}$. Such a T_σ fits then within the more general framework of modulation invariant bilinear singular integrals of [Bényi et al. 2009]. Boundedness properties for symbols in the classes $BS^0_{1,0;\theta}(\mathbb{R})$, not necessarily of the form (1-4), were obtained in [Bernicot 2008; 2010]. See [Torres 2009] for further motivation and references.

In this article we want to discuss the reappearance of the exotic phenomenon for the parameters $m=0$ and $\rho=\delta=1$. Namely, the unboundedness on L^p spaces of operators in $BS^0_{1,0;\theta}$, but their boundedness on spaces of smooth functions.

In the linear case this phenomenon for $S^0_{1,1}$ is by now well understood through works such as [Stein 1993; Meyer 1981a; Runst 1985; Bourdaud 1988; Hörmander 1988; Torres 1990]. It is intimately related to the lack of calculus for the adjoints of operators in such class and, ultimately, this behavior has been interpreted through the $T(1)$-Theorem of David and Journé [1984]. The class $S^0_{1,1}$ is the largest class of linear pseudodifferential operators with Calderón–Zygmund kernels but their exotic behavior on L^p spaces is given by the fact that for T in the class $S^0_{1,1}$, the distribution $T^*(1)$ is in general not in *BMO* (though $T(1)$ is). Here T^* is the formal transpose of T. Moreover, the boundedness of an operator T in $S^0_{1,1}$ on several other spaces of function is related to the action (properly defined) of T^* on polynomials; see [Torres 1991] and the relation to the work of Hörmander [1989] found in [Torres 1990]. By comparison, the smaller classes $S^0_{1,\delta}$ with $\delta < 1$ are closed by transposition and hence the operators in such classes do satisfy the hypotheses of the $T(1)$-Theorem and are bounded on L^p for $1 < p < \infty$.

Likewise, in the bilinear case, the class $BS^0_{1,1}$ is the largest class of pseudodifferential operators with bilinear Calderón–Zygmund kernels. But again, T^{*1} and T^{*2}, the two formal transposes of an operator T in $BS^0_{1,1}$, may fail to satisfy the hypotheses of the $T(1)$-Theorem for bilinear Calderón–Zygmund operators in [Grafakos and Torres 2002]. A symbolic calculus for the transposes hold in the smaller classes $BS^0_{1,\delta}$ with $\delta < 1$ [Bényi and Torres 2003; Bényi et al. 2010], rendering the boundedness of operators in $BS^0_{1,\delta}$. Though unbounded on product of L^p spaces, the class $BS^0_{1,1}$ is still bounded on product of Sobolev spaces [Bényi and Torres 2003]. For the Coifman–Meyer symbols there is then a complete analogy with the linear situation.

For the newer more singular classes $BS^0_{1,0;\theta}$ a symbolic calculus for the transposes was shown to exist in [Bényi et al. 2006] and extended in [Bernicot 2010]. Hence, the boundedness on product of L^p spaces of operators in such classes and of the form (1-4) can be easily obtained from the new $T(1)$-Theorem for modulation invariant singular integrals in [Bényi et al. 2009]. The class $BS^0_{1,0;\theta}$ also produced bounded operators on Sobolev spaces of positive smoothness as shown in [Bernicot 2008]. All these developments motivate us to look for exotic behavior in the larger classes $BS^0_{1,1;\theta}$.

New results. In this article, we show with an example that there exit modulation invariant operators in the class $BS^0_{1,1;\theta}$ that fail to be bounded on a product of L^p spaces (Proposition 2.1). This immediately implies that an arbitrary operator T in $BS^0_{1,1;\theta}$ may not have both $T^{*1}(1,1)$ and $T^{*2}(1,1)$ in BMO, as defined in [Bényi et al. 2009]. It follows also that a symbolic calculus for the transposes in those classes is not possible. Nevertheless, as the reader may expect after the above introduction, we shall show that the classes are bounded on product of Sobolev spaces. For simplicity in the presentation we will only consider the case $BS^0_{1,1;\pi/4}$. The corresponding results for other values of θ in $(-\pi/2, \pi/2) \setminus \{-\pi/4\}$ (avoiding the degenerate directions) can be obtained in similar way.

In the case of modulation invariant operators, we obtained boundedness on product of Sobolev spaces with positive smoothness (Theorem 3.1). Surprisingly if we do not assume modulation invariance we can only obtain the corresponding result on Sobolev spaces of smoothness bigger than $\frac{1}{2}$ (Theorem 3.3). We do not know if the result is sharp, but a better result does not seem attainable with our techniques. Table 1 summarizes the known results and the new ones and puts in evidence the parallel situation in several classes of pseudodifferential operators.

As a byproduct of our results, we also improve on some known estimates on paramultiplication by introducing a new way to approximate the pointwise product of two functions with errors better localized in the frequency plane (see Section 4 for precise statements).

Further definitions and notation. We recall the maximal Hardy–Littlewood operator M defined for a function $f \in L^1_{\mathrm{loc}}(\mathbb{R})$ by

$$M(f)(x) = \sup_{\substack{B \text{ ball} \\ B \ni x}} \frac{1}{|B|} \int_B |f(y)|\, dy.$$

Class/symbol estimates	Lebesgue spaces	Sobolev spaces
(linear) $S^0_{1,0}$ $\lvert\partial_x^\beta\partial_\xi^\alpha\sigma(x,\xi)\rvert \le C_{\alpha\beta}(1+\lvert\xi\rvert)^{-\lvert\alpha\rvert}$	$L^p \to L^p$ $1<p<\infty$	$W^{s,p}\to W^{s,p}$ $1<p<\infty,\ s>0$
(linear) $S^0_{1,1}$ $\lvert\partial_x^\beta\partial_\xi^\alpha\sigma(x,\xi)\rvert \le C_{\alpha\beta}(1+\lvert\xi\rvert)^{\lvert\beta\rvert-\lvert\alpha\rvert}$	unbounded	$W^{s,p}\to W^{s,p}$ $1<p<\infty,\ s>0$
(bilinear) $BS^0_{1,0}$ $\lvert\partial_x^\beta\partial_{\xi,\eta}^\alpha\sigma(x,\xi,\eta)\rvert \le C_{\alpha\beta}(1+\lvert\xi\rvert+\lvert\eta\rvert)^{-\lvert\alpha\rvert}$	$L^p\times L^q\to L^t$ $1<p,q<\infty$ $1/p+1/q=1/t$	$W^{s,p}\times W^{s,q}\to W^{s,t}$ $1<p,q,t<\infty,\ s>0$ $1/p+1/q=1/t$
(bilinear) $BS^0_{1,1}$ $\lvert\partial_x^\beta\partial_{\xi,\eta}^\alpha\sigma(x,\xi,\eta)\rvert \le C_{\alpha\beta}(1+\lvert\xi\rvert+\lvert\eta\rvert)^{\lvert\beta\rvert-\lvert\alpha\rvert}$	unbounded	$W^{s,p}\times W^{s,q}\to W^{s,t}$ $1<p,q,t<\infty,\ s>0$ $1/p+1/q=1/t$
(bilinear) $BS^0_{1,0;\pi/4}$ $\lvert\partial_x^\beta\partial_{\xi,\eta}^\alpha\sigma(x,\xi,\eta)\rvert \le C_{\alpha\beta}(1+\lvert\xi-\eta\rvert)^{-\lvert\alpha\rvert}$	$L^p\times L^q\to L^t$ $1<p,q<\infty$ $1/p+1/q=1/t<\frac{3}{2}$	$W^{s,p}\times W^{s,q}\to W^{s,t}$ $1<p,q,t<\infty,\ s>0$ $1/p+1/q=1/t$
(bilinear) $BS^0_{1,1;\pi/4}$ $\lvert\partial_x^\beta\partial_{\xi,\eta}^\alpha\sigma(x,\xi-\eta)\rvert \le C_{\alpha\beta}(1+\lvert\xi-\eta\rvert)^{\lvert\beta\rvert-\lvert\alpha\rvert}$	unbounded	$W^{s,p}\times W^{s,q}\to W^{s,t}$ $1<p,q,t<\infty,\ s>0$ $1/p+1/q=1/t$
(bilinear) $BS^0_{1,1;\pi/4}$ $\lvert\partial_x^\beta\partial_{\xi,\eta}^\alpha\sigma(x,\xi,\eta)\rvert \le C_{\alpha\beta}(1+\lvert\xi-\eta\rvert)^{\lvert\beta\rvert-\lvert\alpha\rvert}$	unbounded	$W^{s,p}\times W^{s,q}\to W^{s,t}$ $1<p,q,t<\infty,\ s>\frac{1}{2}$ $1/p+1/q=1/t$

Table 1. Summary of the boundedness properties of pseudodifferential operators on Lebesgue and Sobolev spaces.

We write $M^2 = M\circ M$ for the composition of the maximal operator with itself.

For a function f in the Schwartz space $\mathcal{S}$ of smooth and rapidly decreasing functions, we will define the Fourier transform by

$$\widehat{f}(\xi)=\int_{\mathbb{R}} f(x)e^{-ix\cdot\xi}\,dx.$$

With this definition, the inverse Fourier transform is given by $f^\vee(\xi)=(2\pi)^{-1}\widehat{f}(-\xi)$. Both the Fourier transform and its inverse can be extended as usual to the dual space of tempered distributions $\mathcal{S}'$.

For a bounded symbol σ, the bilinear operator

$$T_\sigma(f,g)(x)=\int e^{ix(\xi+\eta)}\widehat{f}(\xi)\widehat{g}(\eta)\sigma(x,\xi,\eta)\,d\xi\,d\eta$$

is well defined and gives a bounded function for each pair of functions f, g in $\mathcal{S}$. Moreover, for σ in $BS^0_{1,1;\pi/4}$, the operator T_σ clearly maps $\mathcal{S}\times\mathcal{S}$ into $\mathcal{S}'$ continuously. This justifies many limiting arguments and computations that we will perform without further comment.

The formal transposes, T^{*1} and T^{*2}, of an operator $T:\mathcal{S}\times\mathcal{S}\to\mathcal{S}'$ are defined by

$$\langle T^{*1}(h,g),f\rangle=\langle T(f,g),h\rangle=\langle T^{*2}(f,h),g\rangle,$$

where $\langle\cdot,\cdot\rangle$ is the usual pairing between distributions and test functions.

We will use the notation $\Psi_{2^{-k}}$ for the L^1-normalized function $2^k\Psi(2^k\cdot)$ and consider the Littlewood–Paley characterization of Sobolev spaces $W^{s,p}$, $1<p<\infty$, $s\geq 0$. That is, for a function Ψ in $\mathcal{S}$ with spectrum contained in $\{\xi:\ 2^{-1}\leq|\xi|\leq 2\}$ and another function Φ also in $\mathcal{S}$ and with spectrum included in $\{|\xi|\leq 1\}$, and such that

$$\widehat{\Phi}(\xi)+\sum_{k\geq 0}\widehat{\Psi}(2^{-k}\xi)=1 \tag{1-5}$$

for all ξ, we have

$$\|f\|_{W^{s,p}}\approx\|\Phi*f\|_{L^p}+\biggl\|\biggl(\sum_{k\geq 0}2^{2ks}\,|\Psi_{2^{-k}}*f|^2\biggr)^{1/2}\biggr\|_{L^p}. \tag{1-6}$$

Here $\|\cdot\|_{L^p}$ denotes the usual norm of the Lebesgue space $L^p(\mathbb{R})$. For $s=0$, the norm $\|\cdot\|_{W^{0,p}}$ is equivalent to $\|\cdot\|_{L^p}$. Also, by *BMO* we mean as usual the classical John–Nirenberg space of functions of bounded mean oscillation.

By homogeneity considerations, we will investigate boundedness properties of the form

$$T:W^{s,p}\times W^{s,q}\to W^{s,t}, \tag{1-7}$$

where the exponents satisfy $1\leq p,q,t\leq\infty$ and the Hölder relation

$$\frac{1}{p}+\frac{1}{q}=\frac{1}{t}. \tag{1-8}$$

2. Unboundedness on Lebesgue spaces

We first show that for $s=0$ the bound (1-7) may fail for $BS^0_{1,1;\pi/4}(\mathbb{R})$.

Proposition 2.1. *There exists a symbol $\tau\in S^0_{1,1}$ such that the operator T_σ with symbol $\sigma(x,\xi,\eta)=\tau(x,\xi-\eta)$ is in $BS^0_{1,1;\pi/4}$ and is not bounded from $L^p\times L^q$ into L^t for any exponents p,q,t satisfying* (1-8).

Proof. As in [Bényi and Torres 2003], we adapt to the bilinear situation a by now classical counterexample in the linear setting; see [Bourdaud 1988]. Let ψ be a function in $\mathcal{S}$ satisfying $\widehat{\psi}\geq 0$, $\widehat{\psi}(\xi)\neq 0$ only for $\frac{5}{7}<|\xi|<\frac{5}{3}$, and $\widehat{\psi}(\xi)=1$ for $\frac{5}{6}\leq|\xi|<\frac{4}{3}$. Consider the symbol

$$\tau(x,\xi)=\sum_{j\geq 4}e^{-i2^jx}\widehat{\psi}(2^{-j}\xi),$$

which is easily seen to be in $S^0_{1,1}$. Select another function ψ_1 in $\mathcal{S}$ satisfying supp $(\widehat{\psi}_1)\subset[0,\frac{1}{3}]$ and define

$$f=\sum_{j=4}^{m}a_je^{i2^jx}\psi_1(x),$$

for arbitrarily coefficients a_j. For $\sigma(x,\xi,\eta)=\tau(x,\eta-\xi)$, we have

$$T_\sigma(f,\psi_1)(x)=\sum_{j,k\geq 4} a_k e^{-i2^jx}\int_{\mathbb{R}^2} e^{ix(\xi+\eta)}\widehat{\psi}(2^{-j}(\eta-\xi))\widehat{\psi}_1(\xi-2^k)\widehat{\psi}_1(\eta)\,d\xi\,d\eta. \tag{2-1}$$

For each k, the integration at most takes place where $0\leq\eta\leq\frac{1}{3}$ and $2^k\leq\xi\leq 2^k+\frac{1}{3}$, which implies

$$-2^k-\tfrac{1}{3}\leq\eta-\xi\leq\tfrac{1}{3}-2^k,$$

and then for each j,

$$-2^{k-j}-\tfrac{1}{3}2^{-j}\leq 2^{-j}(\eta-\xi)\leq\tfrac{1}{3}2^{-j}-2^{k-j}. \tag{2-2}$$

Note that since $j,k\geq 4$, if $k>j$ we have

$$\tfrac{1}{3}2^{-j}-2^{k-j}<-\tfrac{5}{3},$$

while if $k<j$

$$-2^{k-j}-\tfrac{1}{3}2^{-j}>-\tfrac{5}{7}.$$

It follows from (2-2) that the only nonzero term in (2-1) is the one with $j=k$ and also

$$\widehat{\psi}(2^{-j}(\eta-\xi))=1$$

where the integrand is not zero. We obtain

$$T_\sigma(f,\psi_1)(x)=\sum_{j=4}^m a_j e^{-i2^jx}e^{i2^jx}\psi_1^2(x)=\Bigl(\sum_{j=4}^m a_j\Bigr)\psi_1^2(x).$$

If we assume that the operator T_σ is bounded from $L^p\times L^q$ into L^t, we could conclude then that

$$\Bigl|\sum_{j=4}^m a_j\Bigr|\lesssim\|f\|_{L^p}\lesssim\Bigl(\sum_{j=4}^m |a_j|^2\Bigr)^{1/2}, \tag{2-3}$$

where the last inequality follows from the Littlewood–Paley square function characterization of the L^p norm of f and the constants involved depend on ψ_1 but are independent of m. Since the a_j are arbitrary (2-3) is not possible. □

3. Sobolev space estimates

We will show that the class $BS^0_{1,1;\pi/4}$ produces bounded operators on product of Sobolev spaces. The situations in the modulation invariant and the general case are slightly different.

The modulation invariant case. We first consider the case of bilinear operators obtained from linear ones as in the previous section. That is, the symbol σ takes the form

$$\sigma(x,\xi,\eta)=\tau(x,\xi-\eta),$$

where τ belongs to the linear class $S^0_{1,1}$.

Theorem 3.1. *Let τ be a linear symbol in $S^0_{1,1}$ and consider the bilinear operator T_σ, where $\sigma(x,\xi,\eta) = \tau(x,\xi-\eta)$. If $s > 0$ and $1 < p,q,t < \infty$ satisfy the Hölder relation (1-8), then T_σ is bounded from $W^{s,p} \times W^{s,q}$ into $W^{s,t}$.*

Proof. We begin by recalling the Coifman–Meyer reduction for symbols in $S^0_{1,1}$, which is by now a standard technique. (For details see [Coifman and Meyer 1978b, Chapter II, Section 9] for example.) The symbol τ can be decomposed into an absolutely convergent sum of *reduced symbols* of the form

$$\tau(x,\xi) = \sum_{j=0}^{\infty} m_j(2^j x)\hat{\psi}(2^{-j}\xi),$$

where ψ is a smooth function whose Fourier transform is supported on $\{\xi :\ 2^{-1} \le |\xi| \le 2\}$ and $\{m_j\}_{j\ge 0}$ is a uniformly bounded collection of $C^r(\mathbb{R})$ functions where r can be taken arbitrarily large. Due to this reduction, we need only to study a symbol of the form

$$\sigma(x,\xi,\eta) = \sum_{j\ge 0} m_j(2^j x)\hat{\psi}(2^{-j}(\xi-\eta)) := \sum_{j\ge 0}\sigma_j(x,\xi,\eta).$$

We use the notation of [Bourdaud 1988]. We expand m_j into an inhomogeneous Littlewood–Paley decomposition using (1-5) so that

$$m_j = \sum_{k\ge 0} m_{j,k} \tag{3-1}$$

with the spectrum of $m_{j,k}$ contained in the dyadic annulus $\{\xi :\ 2^{k-1} \le |\xi| \le 2^{k+1}\}$ for $k \ge 1$, and in the ball $\{\xi,\ |\xi| \le 2\}$ for $k = 0$. Then we define for $h \ge j$ the function $n_{j,h}(x) := m_{j,h-j}(2^j x)$. Due to the regularity of the function m_j, we have the following properties for $h \ge j+1$:

$$\operatorname{supp} \hat{n}_{j,h} \subset \{\xi :\ 2^{h-1} \le |\xi| \le 2^{h+1}\} \tag{3-2}$$

and

$$\|n_{j,h}\|_{L^\infty} \le C_r 2^{(j-h)r}, \tag{3-3}$$

where, we mention again, the number r can be chosen as large as we want. For $h = j$ we have

$$\operatorname{supp} \hat{n}_{j,j} \subset \{\xi :\ |\xi| \le 2^{j+1}\} \tag{3-4}$$

and

$$\|n_{j,j}\|_{L^\infty} \le C_r. \tag{3-5}$$

Note also that

$$m_j(2^j x) = m_{j,k}(2^j x) + \sum_{h\ge j+1} m_{j,h-j}(2^j x) = n_{j,j}(x) + \sum_{h\ge j+1} n_{j,h}(x). \tag{3-6}$$

Writing T_j for the bilinear operator with symbol $\hat{\psi}(2^{-j}(\xi-\eta))$, we get

$$T_\sigma(f,g)(x) = \sum_{j\ge 0} m_j(2^j x) T_j(f,g)(x).$$

To study the norm of $T_\sigma(f,g)$ in the Sobolev space $W^{s,t}$, and with the functions Ψ and Φ as in (1-6), we need to estimate terms of the form $\Phi * T_\sigma(f,g)$ and, say for $k-2\geq 0$,

$$\Psi_{2^{-k}} * T_\sigma(f,g) := \sum_{j\geq 0} \Psi_{2^{-k}} * \big(m_j(2^j\cdot)T_j(f,g)\big) = I_k(f,g) + II_k(f,g),$$

where

$$I_k(f,g) := \sum_{j=0}^{k-2} \Psi_{2^{-k}} * \big(m_j(2^j\cdot)T_j(f,g)\big),$$

$$II_k(f,g) := \sum_{j\geq k-2} \Psi_{2^{-k}} * \big(m_j(2^j\cdot)T_j(f,g)\big).$$

We treat only I_k and II_k. The estimate for the other terms can be achieved with the same arguments (they are actually easier). For notational convenience, we identify $\Psi_{2^{-k}}$ with the convolution operator it defines (and similarly with other functions).

Estimate for I. We further decompose $m_j(2^j\cdot)$ and $T_j(f,g)$. Using (3-1), (3-6), and (1-5) we have

$$m_j(2^jx) = \Phi_{2^{-k}}(m_j(2^j\cdot))(x) + \sum_{l\geq k} n_{j,l}(x).$$

We also decompose $T_j(f,g)(x)$ as $\Phi_{2^{-k}}\big(T_j(f,g)\big)(x) + \sum_{p\geq k} \Psi_{2^{-p}}\big(T_j(f,g)\big)(x)$. Then

$$I_k(f,g) = \sum_{j=0}^{k-2} \Psi_{2^{-k}}\big(\Phi_{2^{-k}}(m_j(2^j\cdot))\Phi_{2^{-k}}(T_j(f,g))\big) + \sum_{j=0}^{k-2}\sum_{l\geq k} \Psi_{2^{-k}}\big(n_{j,l}\Phi_{2^{-k}}(T_j(f,g))\big) \tag{3-7}$$
$$+ \sum_{j=0}^{k-2}\sum_{p\geq k} \Psi_{2^{-k}}\big(\Phi_{2^{-k}}(m_j(2^j\cdot))\Psi_{2^{-p}}(T_j(f,g))\big) + \sum_{j=0}^{k-2}\sum_{l,p\geq k} \Psi_{2^{-k}}\big(n_{j,l}\Psi_{2^{-p}}(T_j(f,g))\big).$$

Using the notation $\tilde{\phi}$ for a generic smooth function with bounded spectrum and $\tilde{\psi}$ for a generic smooth function with a spectrum contained in an annulus around 0, we claim that we can write I_k as a sum of terms of three different form:

$$I_k(f,g) = \sum_{0\leq j\leq k-2} \Psi_{2^{-k}}(T_{\sigma_j}(f,g)) \approx (1)_k + (2)_k + (3)_k,$$

where

$$(1)_k := \sum_{j\leq k-2} \Psi_{2^{-k}}\big(n_{j,k}\tilde{\phi}_{2^{-k}}(T_j(f,g))\big),$$

$$(2)_k := \sum_{j\leq k-2} \Psi_{2^{-k}}\big(\tilde{\phi}_{2^{-k}}(m_j(2^j\cdot))\tilde{\psi}_{2^{-k}}(T_j(f,g))\big),$$

$$(3)_k := \sum_{l\geq k}\sum_{j\leq k-2} \Psi_{2^{-k}}\big(n_{j,l}\tilde{\psi}_{2^{-l}}(T_j(f,g))\big).$$

Let us explain this reduction. The first sum in (3-7) can be written as a finite linear combination of terms taking the form $(1)_k$ and $(2)_k$. Indeed, consider one of the general terms

$$\Psi_{2^{-k}}\big(\Phi_{2^{-k}}(m_j(2^j\,\cdot))\Phi_{2^{-k}}(T_j(f,g))\big).$$

Denote by ξ the frequency variable of $m_j(2^j\,\cdot)$ and by η that of $T_j(f,g)$. We have a nonvanishing contribution if

$$|\eta|\le 2^k,\qquad |\xi|\le 2^k\quad\text{ and }\quad |\eta+\xi|\simeq 2^k,$$

where we have used that the spectrum of the product is included in Minkowski sum of spectra. Consequently, this is possible only if $|\xi|\simeq 2^k$, which corresponds to $(1)_k$ (recall that $n_{j,l}$ has spectrum in $\{|\xi|\approx 2^l\}$), or $|\eta|\simeq 2^k$, which corresponds to $(2)_k$.

Concerning the second sum in (3-7), it can also be reduced to the sum for $l\approx k$ (as the other terms vanish) and it is a finite sum of terms like $(1)_k$. Similar reasoning for the third term in (3-7) gives that it is controlled by $(2)_k$. Finally, the general term in the fourth sum in (3-7) is nonzero if

$$2^p\pm 2^l\approx 2^k.$$

But, since the inner double sum has $l,p\ge k$, the general term is nonzero only for $l\approx p$. We see then that the double sum (over l and p) reduces to one sum over only one parameter. It follows that the fourth sum in (3-7) is similar to $(3)_k$.

We now study each of the model sums $(1)_k$, $(2)_k$, $(3)_k$.

The sum with $(1)_k$. We use the estimate (3-3) for $n_{j,k}$ with $r>s$ and Young's inequality to obtain

$$\begin{aligned}\|2^{ks}(1)_k\|_{l^2(k\in\mathbb{N})} &\lesssim \Big\|\sum_{j+2\le k}2^{(j-k)r}2^{ks}M\big(\tilde{\phi}_{2^{-k}}(T_j(f,g))\big)\Big\|_{l^2(k\in\mathbb{N})}\\ &\lesssim \Big\|\sum_{j+2\le k}2^{js}2^{(j-k)(r-s)}M\big(\tilde{\phi}_{2^{-k}}(T_j(f,g))\big)\Big\|_{l^2(k\in\mathbb{N})}\lesssim \|2^{js}M^2(T_j(f,g))\|_{l^2(j\in\mathbb{N})}.\end{aligned}$$

Therefore,

$$\big\|\,\|2^{ks}(1)_k\|_{l^2(k\in\mathbb{N})}\big\|_{L^t}\lesssim\big\|\,\|2^{js}M^2(T_j(f,g))\|_{l^2(j\in\mathbb{N})}\big\|_{L^t}\tag{3-8}$$

and from the Fefferman–Stein vector-valued inequality [1971] for the maximal operator M, we deduce that

$$\big\|\,\|2^{ks}(1)_k\|_{l^2(k\in\mathbb{N})}\big\|_{L^t}\lesssim\big\|\,\|2^{js}T_j(f,g)\|_{l^2(j\in\mathbb{N})}\big\|_{L^t}.$$

We can use now a linearization argument. By writing $r_j(\omega)$ for Rademacher functions ($\omega\in[0,1]$), we know that (see, e.g., Appendix C in [Grafakos 2004]):

$$\big\|\,\|2^{ks}(1)_k\|_{l^2(k\in\mathbb{N})}\big\|_{L^t}\lesssim\Big\|\Big\|\sum_j 2^{js}r_j(\omega)T_j(f,g)\Big\|_{L^t(\omega\in[0,1])}\Big\|_{L^t}.$$

By Fubini's Theorem, we have

$$\Big\|\,\|2^{ks}(1)_k\|_{l^2(k\in\mathbb{N})}\Big\|_{L^t}\lesssim\Big\|\Big\|\sum_j 2^{js}r_j(\omega)T_j(f,g)\Big\|_{L^t}\Big\|_{L^t(\omega\in[0,1])}.$$

Now for each $\omega \in [0,1]$, the operator $(f,g) \to \sum_j 2^{js} r_j(\omega) T_j(f,g)$ is the bilinear operator associated to the symbol

$$\sum_j 2^{js} r_j(\omega) \widehat{\Psi}(2^{-j}(\xi-\eta)) \in BS^s_{1,0;\pi/4}.$$

It follows from [Bényi et al. 2006] and [Bernicot 2010] (since the symbol is x-independent) that these bilinear operators are bounded from $W^{s,p} \times W^{s,q}$ into L^t (uniformly on $\omega \in [0,1]$) and the proof in this case is complete.

The sum with $(2)_k$. This term is the most difficult to estimate. Using again the boundedness of the functions m_j in $C^r \hookrightarrow L^\infty$, we can estimate

$$\|2^{ks}(2)_k\|_{l^2(k\in\mathbb{N})} \lesssim \left\| \sum_{j+2\le k} 2^{ks} M\left(\widetilde{\psi}_{2^{-k}}\left[T_j(f,g)\right]\right)(x) \right\|_{l^2(k\in\mathbb{N})}. \tag{3-9}$$

We observe that

$$\begin{aligned}\widetilde{\psi}_{2^{-k}}\left[T_j(f,g)\right](x) &= \int \widetilde{\psi}_{2^{-k}}(x-z) \int \widehat{\Psi}(2^{-j}(\xi-\eta)) \widehat{f}(\xi)\widehat{g}(\eta) e^{iz(\xi+\eta)}\, d\xi\, d\eta\, dz \\ &= \int \widehat{\widetilde{\psi}}(2^{-k}(\xi+\eta)) \widehat{\Psi}(2^{-j}(\xi-\eta)) \widehat{f}(\xi)\widehat{g}(\eta) e^{ix(\xi+\eta)}\, d\xi\, d\eta.\end{aligned}$$

We must have $|\xi+\eta| \approx 2^k$ and $|\xi-\eta| \approx 2^j$. But we only have terms with $2^j < 2^k/4$, so we deduce that $|\xi| \approx |\eta| \approx 2^k$. It follows that we can further localize in the frequency plane adding a new function $\overline{\psi}$ (whose spectrum is contained in an annulus) such that

$$\widetilde{\psi}_{2^{-k}}(T_j(f,g))(x) = \widetilde{\psi}_{2^{-k}}\left(T_j(\overline{\psi}_{2^{-k}} f, \overline{\psi}_{2^{-k}} g)\right)(x)$$

Going back to (3-9) we obtain by the Cauchy–Schwartz inequality (there are k terms in the inner sum)

$$\|2^{ks}(2)_k\|_{l^2(k\in\mathbb{N})} \lesssim \left\| 2^{ks} k^{1/2} \left\| M\left(\widetilde{\psi}_{2^{-k}}\left(T_j(\overline{\psi}_{2^{-k}}(f), \overline{\psi}_{2^{-k}}(g))\right)\right)\right\|_{l^2(j\in\mathbb{N})} \right\|_{l^2(k\in\mathbb{N})}.$$

We then obtain similarly as in the previous case

$$\begin{aligned}\big\| \|2^{ks}(2)_k\|_{l^2(k\in\mathbb{N})} \big\|_{L^t} &\lesssim \left\| \left\| 2^{ks} k^{1/2} \left\| M^2\left(T_j(\overline{\psi}_{2^{-k}}(f), \overline{\psi}_{2^{-k}}(g))\right) \right\|_{l^2(j\in\mathbb{N})} \right\|_{l^2(k\in\mathbb{N})} \right\|_{L^t} \\ &\lesssim \left\| \left\| 2^{ks} k^{1/2} \left\| T_j(\overline{\psi}_{2^{-k}}(f), \overline{\psi}_{2^{-k}}(g)) \right\|_{l^2(j\in\mathbb{N})} \right\|_{l^2(k\in\mathbb{N})} \right\|_{L^t}.\end{aligned}$$

We linearize in j as before and use the fact that $k^{1/2} \lesssim 2^{ks}$ (as $s>0$) to obtain

$$\begin{aligned}\big\| \|2^{ks}(2)_k\|_{l^2(k)} \big\|_{L^t} &\lesssim \left\| \left\| \left\| \sum_j r_j(\omega) T_j\left(2^{ks}\overline{\psi}_{2^{-k}}(f), 2^{ks}\overline{\psi}_{2^{-k}}(g)\right) \right\|_{L^1(\omega\in[0,1])} \right\|_{l^2(k\in\mathbb{N})} \right\|_{L^t} \\ &\lesssim \left\| \left\| \left\| \sum_j r_j(\omega) T_j\left(2^{ks}\overline{\psi}_{2^{-k}}(f), 2^{ks}\overline{\psi}_{2^{-k}}(g)\right) \right\|_{l^2(k\in\mathbb{N})} \right\|_{L^t} \right\|_{L^1(\omega\in[0,1])}.\end{aligned}$$

For each $\omega \in [0, 1]$, we can invoke a vector-valued result for bilinear operators of [Grafakos and Martell 2004]. More precisely, as explained when we dealt with $(1)_k$, for each $\omega \in [0, 1]$ the bilinear operator $(f, g) \to \sum_j r_j(\omega) T_j(2^{ks}\overline{\psi}_{2^{-k}}(f), 2^{ks}\overline{\psi}_{2^{-k}}(g))$ is bounded from $L^p \times L^q$ to L^t (since it is associated to a symbol independent of x). Then, Theorem 9.1 in [Grafakos and Martell 2004] implies that the operator admits an l^2-valued bilinear extension, which yields

$$\Big\| \|2^{ks}(2)_k\|_{l^2(k\in\mathbb{N})} \Big\|_{L^t} \lesssim \Bigg\| \bigg\| \Big\| 2^{ks}\overline{\psi}_{2^{-k}}(f) \Big\|_{l^2(k\in\mathbb{N})} \bigg\|_{L^p} \bigg\| \Big\| 2^{ks}\overline{\psi}_{2^{-k}}(g) \Big\|_{l^2(k\in\mathbb{N})} \bigg\|_{L^q} \Bigg\|_{L^1(\omega)},$$

with estimates uniformly in $\omega \in [0, 1]$. This concludes the proof of the case $(2)_k$.

The sum with $(3)_k$. The analysis in this case is entirely analogous as the case $(1)_k$ and so we leave the details to the reader.

Estimate for II. In this case, we decompose the term $II_k(f, g)$ with quantities appearing as a linear combination of terms of the form

$$(1)_k = \sum_{j\geq k-2} \Psi_{2^{-k}}\big(n_{j,j}\widetilde{\phi}_{2^{-j}}(T_j(f, g))\big) \quad \text{or} \quad (2)_k = \sum_{j\geq k-2} \sum_{l\geq j} \Psi_{2^{-j}}\big(n_{j,l}(x)\widetilde{\psi}_{2^{-l}}(T_j(f, g))\big).$$

Indeed with a similar reasoning as before and since $j \geq k-2$, the general quantity in II_k has a nonvanishing contribution only if the frequency variables of $m_j(2^j\,\cdot)$ or $T_j(f, g)$ are contained in $\{|\xi| \lesssim 2^j\}$ (which corresponds to $(1)_k$) or if the two frequency variables are contained in $\{|\xi| \simeq 2^l\}$ for some $l \geq j$ (which corresponds to $(2)_k$).

The study of $(2)_k$ is similar to the one of $(1)_k$ with the help of fast decays in l (see (3-3)), so we only write the proof for $(1)_k$. By the estimates on $n_{j,j}$, we have

$$\big\| \|2^{ks}(1)_k\|_{l^2(k\in\mathbb{N})} \big\|_{L^t} \lesssim \Bigg\| \bigg\| \sum_{j\geq k-2} 2^{(k-j)s} 2^{js} M^2(T_j(f, g)) \bigg\|_{l^2(k\in\mathbb{N})} \Bigg\|_{L^t}.$$

Using $s > 0$ and Young's inequality for the l^2-norm on k, we get the bound

$$\big\| \|2^{js} M^2(T_j(f, g))\|_{l^2(j\in\mathbb{N})} \big\|_{L^t}.$$

We have already studied such quantities in the first case — see (3-8) — and proved the appropriate bounds. □

Remark 3.2. Since $\sigma(x, \xi, \eta) = \tau(x, \xi - \eta)$ is bounded, the function $T_\sigma(1, 1)$ (rigorously defined in [Bényi et al. 2009]) is given by

$$T_\sigma(1, 1) = \sigma(\,\cdot\,, 0, 0) \in L^\infty \subset BMO.$$

If the transposes of T_σ are also given by symbols in the classes $BS^0_{1,1;\theta}$ or even by some bounded functions, then we can use the bilinear $T(1)$-Theorem of [Bényi et al. 2009] (since T_σ is modulation invariant) to conclude that T is bounded on the product of Lebesgue spaces. The counterexample of the previous section shows that this is not always the case, so the classes $BS^0_{1,1;\theta}$ cannot be closed by transposition. As mentioned in the introduction the smaller classes $BS^0_{1,0;\theta}$ are.

The general case. In this subsection, we consider general symbols in the class $BS^0_{1,1;\pi/4}$. We obtain a slightly less general result than the one in the previous case.

Theorem 3.3. *If $\sigma \in BS_{1,1;\pi/4}$ and $s > \frac{1}{2}$, then the bilinear operator T_σ is bounded from $W^{s,p} \times W^{s,q}$ into $W^{s,t}$ for all exponents $1 < p, q, t < \infty$ satisfying the Hölder condition (1-8).*

Proof. We want to adapt the proof of Theorem 3.1. We briefly indicate the extra difficulties faced.

Reduction to elementary symbols. We first reduce the problem to the study of elementary symbols taking the following form

$$\sigma(x,\xi,\eta) = \sum_{\substack{j\geq 0\\ l\in\mathbb{Z}}} m_{j,l}(2^j x)\widehat{\Psi}\big(2^{-j}(\xi-\eta)\big)\widehat{\Psi}\big(l+2^{-j}(\xi+\eta)\big). \tag{3-10}$$

Let us give a sketch of such a reduction. Multiplying the symbol σ by

$$\widehat{\Psi}\big(2^{-j}(\xi-\eta)\big)\widehat{\Psi}\big(l+2^{-j}(\xi+\eta)\big),$$

we localize it in frequency to the domain

$$\big\{(\xi,\eta) : |\xi-\eta| \simeq 2^j \text{ and } |\xi+\eta+l\,2^j| \simeq 2^j,\big\}$$

which can be compared to a ball of radius 2^j. This compactly supported symbols $\sigma_{j,l}$ satisfy

$$|\partial_x^\alpha \partial_{\xi,\eta}^\beta \sigma_{j,l}(x,\xi,\eta)| \leq C_{\alpha\beta} 2^{j(\alpha-\beta)}.$$

As usually, we decompose this symbol into a Fourier series, obtaining

$$\sigma_{j,l}(x,\xi,\eta) = \sum_{a,b\in\mathbb{Z}^2} \gamma_{a,b}(x) e^{i(a\xi+b\eta)}\widehat{\Psi}\big(2^{-j}(\xi-\eta)\big)\widehat{\Psi}\big(l+2^{-j}(\xi+\eta)\big).$$

The modulation term $e^{i(a\xi+b\eta)}$ does not play a role, as it corresponds to translation in physical space (which does not modify the Lebesgue norms), it remains for us to check that the coefficients $\gamma_{a,b}$ are fast decreasing in (a,b) and satisfies the desired smoothness in x. To do so, we remark that, for $\alpha \in \mathbb{N}$, integration by parts yields

$$\begin{aligned}
\big|\partial_x^\alpha \gamma_{a,b}(2^{-j}x)\big| &\lesssim 2^{-j\alpha-2j}\left|\iint e^{-i(a\xi+b\eta)}\partial_x^\alpha \sigma_{j,l}(2^{-j}x,\xi,\eta)\,d\xi\,d\eta\right| \\
&\lesssim 2^{-j\alpha-2j}\big(1+|a|+|b|\big)^{-M}\left|\iint e^{-i(a\xi+b\eta)}\big(1+\partial_\xi^M+\partial_\eta^M\big)\partial_x^\alpha\sigma_{j,l}(2^{-j}x,\xi,\eta)\,d\xi\,d\eta\right| \\
&\lesssim \big(1+|a|+|b|\big)^{-M},
\end{aligned}$$

where M is an integer that can be chosen as large as we wish. So we conclude that the functions $\gamma_{a,b}(2^{-j}\,\cdot)$ are uniformly bounded in C^r (for r arbitrarily large) with fast decays in (a,b). This operation (expansion in Fourier series) allows us to reduce the study of σ to reduced symbols taking the form (3-10).

Study of elementary symbols. We adapt the proof of Theorem 3.1 and use the same notation. We have to study the sum

$$\sum_{\substack{j\geq 0\\ l\in\mathbb{Z}}} m_{j,l}(2^j x) T_{j,l}(f,g), \tag{3-11}$$

where $T_{j,l}$ is the bilinear operator associated to the x-independent symbol

$$\widehat{\Psi}(2^{-j}(\xi-\eta))\widehat{\Psi}(l+2^{-j}(\xi+\eta)).$$

We can proceed as in the modulation invariant case and consider the different cases, eventually arriving to the point where we need to linearize with respect to the parameter j. But now, we also have to linearize according to the new parameter l. When we estimate the square function of $T_{j,l}$, we have to study $\Psi_{2^{-k}}(T_{j,l}(f,g))$ and we are interested only in the indices j,l satisfying $|\xi+\eta|\approx 2^k$ with $|\xi-\eta|\approx 2^j$ and $|\xi+\eta+l\,2^j|\approx 2^j$. However, due to the use of the Cauchy–Schwartz inequality in l, we will have an extra term bounded by $2^{(k-j)/2}$, which corresponds to the square root of the number of indices l satisfying all these conditions. For the study of $(1)_k$ and $(3)_k$ there is no problem, since r can be chosen satisfying $r>s+\frac{1}{2}$. However, for the study of $(2)_k$ we will need $2^{k(s+1/2)}k^{1/2}\leq 2^{ks}2^{ks}$ and so we need to assume that $s>\frac{1}{2}$. □

Remark 3.4. It is interesting to note that without the modulation invariance, an extra exponent $\frac{1}{2}$ appears. We do not know if our result is optimal or not. Moreover, unlike the modulation invariance case, we also do not know whether a general operator T_σ with symbol $\sigma\in BS_{1,1;\pi/4}$, and whose two adjoints satisfy similar assumptions, is bounded on product of Lebesgue spaces. To address this question, it would be interesting to obtained (if possible) a $T(1)$-Theorem as in [Bényi et al. 2009] but without assuming modulation invariance.

4. An improvement on paramultiplication

In this section, we will use x-independent symbols in $BS_{1,1;\pi/4}$ (and also in the smaller class $BS_{1,0;\pi/4}$) to describe a new paramultiplication operation. We will obtain an improvement over the classical paramultiplication first studied in [Bony 1981] in the L^2 setting and extended in [Meyer 1981a; 1981b] to L^p norms. The classical paraproducts and their properties hold for multidimensional variables, however our improvement works (at least at this moment) only in the one dimensional case.

We start with the classical definition.

Definition 4.1. Let f and b be two smooth functions and let Φ and Ψ be as in (1-5) and (1-6). We assume that for all $\eta\in\operatorname{supp}\widehat{\Phi}$ and $\xi\in\operatorname{supp}\widehat{\Psi}$ we have

$$|\eta|\leq \tfrac{1}{2}|\xi|.$$

Then paramultiplication by b is defined by

$$\Pi_b(f):=\sum_{k\in\mathbb{Z}}\Phi_{2^k}(f)\Psi_{2^k}(b).$$

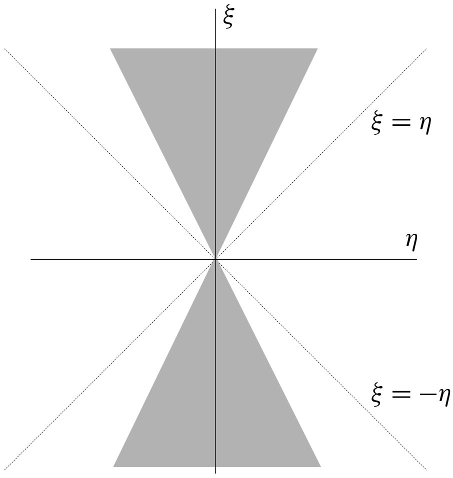

Figure 1. Support of the bilinear symbol associated to the paraproduct Π.

The operator $(b, f) \to \Pi_b(f)$ can essentially be thought as a bilinear multiplier whose symbol is a smooth decomposition of the characteristic function of the cone in Figure 1.

The following two propositions are well-known properties for paraproducts (see [Bony 1981, Theorems 2.1 and 2.5], for example, for the original results involving L^2-Sobolev spaces and [Meyer 1981a; 1981b] for extensions to other Sobolev spaces):

Proposition 4.2. *For all $s > 0$ and $p \in (1, \infty)$ the linear operator Π_b is bounded on the Sobolev space $W^{s,p}$, satisfies*

$$\|\Pi_b\|_{W^{s,p} \to W^{s,p}} \lesssim \|b\|_{L^\infty},$$

and the operation can be extended to an L^∞ function b.

The paramultiplication approximates pointwise multiplication in the following sense.

Proposition 4.3. *Let $1 < t < \infty$ and $s > 1/t$. For $f \in W^{s,t}$ and $g \in W^{s,t}$, we have*

$$\left\| fg - \Pi_f(g) - \Pi_g(f) \right\|_{W^{2s-1/t,t}} \lesssim \|f\|_{W^{s,t}} \, \|g\|_{W^{s,t}}.$$

The exponent of regularity $2s - \frac{1}{t}$ is bigger than s for $ts > 1$. This gain is very important. The result is essentially due to the fact that, in frequency space, the error term has only a contribution from f and g when

$$\{|\xi| \approx |\eta|\},$$

i.e., in a cone along the two main diagonals.

Using the new bilinear operators (whose singularities are localized on a line in the frequency plane), we can define a new paramultiplication operation $\widetilde{\Pi}$ such that the error term will be concentrated in the frequency plane exactly in a strip (of fixed width) around the two diagonals. In this way, we will be able to get a better gain for the exponent of regularity.

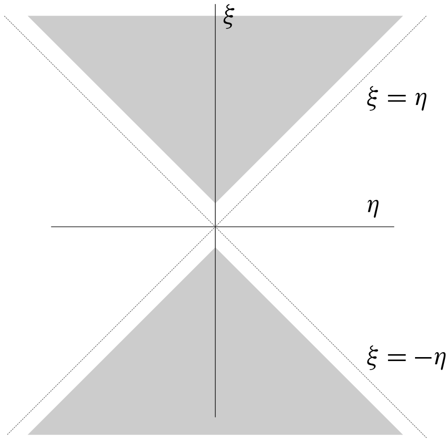

Figure 2. Support of the bilinear symbol associated to the new paraproduct $\widetilde{\Pi}$.

Definition 4.4. Let Θ be a smooth function on $\mathbb{R}$ whose Fourier transform $\widehat{\Theta}$ satisfies

$$\omega \geq 2 \implies \widehat{\Theta}(\omega) = 1 \quad \text{and} \quad -\infty < \omega \leq 1 \implies \widehat{\Theta}(\omega) = 0.$$

Then we define, for $b, f \in \mathcal{S}(\mathbb{R})$, the *improved paramultiplication* by b (written $\widetilde{\Pi}_b(f)$) by

$$\widetilde{\Pi}_b(f)(x) = \int_{\mathbb{R}^2} e^{ix(\xi+\eta)} \hat{b}(\xi) \hat{f}(\eta) \big(\widehat{\Theta}(\xi-\eta)\widehat{\Theta}(\xi+\eta) + \widehat{\Theta}(\eta-\xi)\widehat{\Theta}(-\xi-\eta)\big)\, d\xi\, d\eta. \tag{4-1}$$

The new bilinear multiplier $(b, f) \to \widetilde{\Pi}_b(f)$ is associated to a bilinear symbol, corresponding to a smooth version of the characteristic function of the region in Figure 2. We remark that this new region approximates the domain $\{(\xi, \eta), |\xi| \geq |\eta|\}$ better than the region in Figure 1.

This new operation satisfies a similar property to the one in Proposition 4.2.

Proposition 4.5. *Let $s \geq 0$ and let $1 < p, q, t < \infty$ be exponents satisfying (1-8). For every $\epsilon > 0$ and $b \in W^{\epsilon,p}(\mathbb{R})$, the improved paramultiplication by b is well defined and produce a bounded operation from $W^{s,q}$ to $W^{s,t}$. In fact, there exists a constant $C = C(s, \epsilon, p, q, t)$ such that for all functions $f \in W^{s,q}$,*

$$\left\| \widetilde{\Pi}_b(f) \right\|_{W^{s,t}} \leq C \|b\|_{W^{\epsilon,p}} \|f\|_{W^{s,q}}.$$

Moreover if $s = 0$, the exponent $\epsilon = 0$ is allowed.

Proof. The new paramultiplication is given by two terms, which can be studied by identical arguments. We only deal with the first term but for simplicity in the notation we still write

$$\widetilde{\Pi}_f(b)(x) = \int_{\mathbb{R}^2} e^{ix(\xi+\eta)} \hat{b}(\xi) \hat{f}(\eta) \widehat{\Theta}(\xi-\eta)\widehat{\Theta}(\xi+\eta)\, d\xi\, d\eta.$$

We note that this function $\widetilde{\Pi}_b(f)$ corresponds to the operator $T_\sigma(b, f)$ associated to the bilinear symbol

$$\sigma(\xi, \eta) = \widehat{\Theta}(\xi-\eta)\widehat{\Theta}(\xi+\eta).$$

We need to show that T_σ is continuous from $W^{\epsilon,p} \times W^{s,q}$ to $W^{s,r}$.

The case $s=0$. We compute the Fourier transform of $T_\sigma(b,f)$,

$$\widehat{T_\sigma(b,f)}(\omega)=\int_{\xi+\eta=\omega}\hat{b}(\xi)\hat{f}(\eta)\widehat{\Theta}(\xi-\eta)\widehat{\Theta}(\eta+\xi)\,d\xi\,d\eta$$
$$=\widehat{\Theta}(\omega)\int_{\xi+\eta=\omega}\hat{b}(\xi)\hat{f}(\eta)\widehat{\Theta}(\xi-\eta)\,d\xi\,d\eta=\widehat{\Theta}(\omega)\widehat{T_\tau(b,f)}(\omega),$$

where τ is given by $\tau(\xi,\eta)=\widehat{\Theta}(\xi-\eta)$. So in fact we can write $T_\sigma(b,f)$ as the convolution product between Θ and $T_\tau(b,f)$. Since the function Θ in Definition 4.4 is smooth, the convolution operation by Θ is bounded on L^t. We obtain also

$$\|T_\sigma(b,f)\|_{L^t}\lesssim\|T_\tau(b,f)\|_{L^t}\,.$$

Now the bilinear operator T_τ is associated to the symbol τ which satisfies the Hörmander multiplier conditions related to the frequency line $\{\xi=\eta\}$. That is,

$$\left|\partial_\xi^\alpha\partial_\eta^\beta\tau(\xi,\eta)\right|\lesssim|\xi-\eta|^{-\alpha-\beta}$$

for all α and β. It follows from [Gilbert and Nahmod 2000] that this bilinear operator maps $L^p\times L^q$ to L^t and we obtain the desired result

$$\|T_\sigma(b,f)\|_{L^t}\lesssim\|b\|_{L^p}\|f\|_{L^q}.$$

Note that for the case $s=0$ no regularity on b is really needed.

The case $s>0$. Let Φ and Ψ be as in (1-5) and (1-6). We study first $\Phi*T_\sigma(f,g)$. We have

$$\widehat{\Phi*T_\sigma(b,f)}(\omega)=\widehat{\Phi}(\omega)\,\widehat{\Theta}(\omega)\,\widehat{T_\tau(b,f)}(\omega).$$

The spectral condition over Φ and Θ imply that $\omega\approx1$. So for ξ and η (the frequency variables of b and f) satisfying $\xi-\eta\geq1$ and $\xi+\eta=\omega\approx1$, we deduce that either η is bounded or $-\xi\approx\eta\gg1$. Therefore, we can find a smooth function ζ and an other one $\tilde{\psi}$ (whose spectrum is contained in an annulus around 0) such that

$$\Phi*T_\sigma(b,f)=\Phi*T_\sigma(b,\zeta*f)+\sum_{l\geq0}\Phi*T_\sigma\big(\tilde{\psi}_{2^{-l}}*b,\tilde{\psi}_{2^{-l}}*f\big).$$

Using $0<\epsilon$, we get by the Cauchy–Schwartz inequality

$$\left|\Phi*T_\sigma(b,f)\right|\leq\left|\Phi*T_\sigma(b,\zeta*f)\right|+\bigg(\sum_{l\geq0}2^{2\epsilon l}\left|M\big(T_\sigma(\tilde{\psi}_{2^{-l}}*b,\tilde{\psi}_{2^{-l}}*f)\big)\right|^2\bigg)^{1/2}.\tag{4-2}$$

By the same reasoning for an integer $k\geq1$, if ξ and η satisfy $\eta\geq\xi+1$ and $1<\xi+\eta=\omega\approx2^k$, we deduce that either $\eta\approx2^k$ or $-\xi\approx\eta\gg2^k$. So we can find a smooth function $\tilde{\psi}$ (for convenience we keep the same notation), whose spectrum is contained in an annulus around 0 such that for all integer k large enough

$$\Psi_{2^{-k}}*T_\sigma(b,f)=\Psi_{2^{-k}}*T_\sigma\big(b,\tilde{\psi}_{2^{-k}}*f\big)+\sum_{l\geq k}\Psi_{2^{-k}}*T_\sigma\big(\tilde{\psi}_{2^{-l}}*b,\tilde{\psi}_{2^{-l}}*f\big).$$

Using the same ϵ, we get by the Minkowski and Cauchy–Schwartz inequalities

$$\begin{aligned}
&\Big(\sum_k 2^{2ks}\,|\Psi_{2^{-k}} * T_\sigma(b,f)|^2\Big)^{1/2} \\
&\lesssim \Big(\sum_k 2^{2ks} M\big(T_\sigma(b,\tilde{\psi}_{2^{-k}} * f)\big)^2\Big)^{1/2} + \sum_{l\ge 0}\Big(\sum_{k\le l} 2^{2ks}\Big|\Psi_{2^{-k}} * T_\sigma(\tilde{\psi}_{2^{-l}} * b, \tilde{\psi}_{2^{-l}} * f)\Big|^2\Big)^{1/2} \\
&\lesssim \Big(\sum_k 2^{2ks} M\big(T_\sigma(b,\tilde{\psi}_{2^{-k}} * f)\big)^2\Big)^{1/2} + \sum_{l\ge 0} 2^{ls} M\big(T_\sigma(\tilde{\psi}_{2^{-l}} * b, \tilde{\psi}_{2^{-l}} * f)\big) \\
&\lesssim \Big(\sum_k 2^{2ks} M\big(T_\sigma(b,\tilde{\psi}_{2^{-k}} * f)\big)^2\Big)^{1/2} + \Big(\sum_{l\ge 0} 2^{2l(s+\epsilon)}\Big|M\big(T_\sigma(\tilde{\psi}_{2^{-l}} * b, \tilde{\psi}_{2^{-l}} * f)\big)\Big|^2\Big)^{1/2}. \qquad (4\text{-}3)
\end{aligned}$$

From (4-2) and (4-3), using the $L^q - L^t$ boundedness of $T_\sigma(b,\cdot)$ (proved in the first case), the vector-valued Fefferman–Stein inequality, and its bilinear version [Grafakos and Martell 2004, Theorem 9.1], we obtain the desired result:

$$\begin{aligned}
\|T_\sigma(b,f)\|_{W^{s,t}} &\lesssim \Big\| \big|\Phi * T_\sigma(b,f)\big| + \Big(\sum_{k\ge 0} 2^{2sk}\big|\Psi_{2^{-k}} * T_\sigma(b,f)\big|^2\Big)^{1/2}\Big\|_{L^t} \\
&\lesssim \|b\|_{L^p}\Big\| |\zeta * f| + \Big(\sum_{k\ge 0} 2^{2sk}\big|\tilde{\psi}_{2^{-k}} * f\big|^2\Big)^{1/2}\Big\|_{L^q} \\
&\qquad + \Big\|\Big(\sum_{l\ge 0} 2^{2l\epsilon}\big|\tilde{\psi}_{2^{-l}} * b\big|^2\Big)^{1/2}\Big\|_{L^p}\Big\|\Big(\sum_{k\ge 0} 2^{2sk}\big|\tilde{\psi}_{2^{-k}} * f\big|^2\Big)^{1/2}\Big\|_{L^q} \\
&\lesssim \|b\|_{W^{\epsilon,p}}\|f\|_{W^{s,q}}.
\end{aligned}$$

□

Remark 4.6. We note that our new bilinear operation needs an extra regularity assumption $b \in W^{\epsilon,p}$ to keep the regularity of the function f (the case $s > 0$). This is due to the fact that the high frequencies of b play a role in the high frequency of $\tilde{\Pi}_b(f)$ (which is natural) but in the low frequencies of $\tilde{\Pi}_b(f)$ too. This last phenomenom does not appear in the classical paramultiplication operation. This point can be observed in the Figures 1 and 2. Let ω be the frequency variable of the paraproduct. For small ω, say $\omega \simeq 2$, the contributions of b and f correspond to the intersection of the cone in Figures 1 and 2 and the line $\{\omega = \xi + \eta\}$. In the first case (Figure 1) this intersection is bounded set, whereas in the second case (Figure 2) it is not bounded and contains also high frequencies of b.

We now obtain an improvement on Proposition 4.3.

Proposition 4.7. *Let $t \in (1,\infty)$ and $s \ge 1/t$. If $f \in W^{s,t}$ and $g \in W^{s,t}$, then*

$$\big\| fg - \tilde{\Pi}_f(g) - \tilde{\Pi}_g(f)\big\|_{W^{2s,t}} \lesssim \|f\|_{W^{s,t}}\|g\|_{W^{s,t}}.$$

Remark 4.8. As already mentioned, in the classical paramultiplication calculus, the regularity result is true for $s \ge 1/t$ and the gain is only $s - 1/t$.

Proof. Let us denote by D the difference operator

$$D(f,g) := fg - \widetilde{\Pi}_f(g) - \widetilde{\Pi}_g(f).$$

It corresponds to the bilinear operator associated to the symbol τ given by

$$\tau(\xi,\eta) := 1 - \widehat{\Theta}(\eta-\xi)\widehat{\Theta}(\eta+\xi) - \widehat{\Theta}(-\eta+\xi)\widehat{\Theta}(-\eta-\xi) - \widehat{\Theta}(\xi-\xi)\widehat{\Theta}(\eta+\xi) - \widehat{\Theta}(\eta-\xi)\widehat{\Theta}(-\eta-\xi).$$

This symbol is supported in the complement of the cone drawn in Figure 2 and the one symmetric to it. Consequently, it is supported in two strips (around the two diagonals)

$$\operatorname{supp}(\tau) \subset \big\{(\xi,\eta) : |\xi-\eta| \le 3\big\} \cup \big\{(\xi,\eta) : |\xi+\eta| \le 3\big\}.$$

We can then reproduce a similar reasonning as used for Proposition 4.5. The symbol τ can be decomposed in two parts τ_1, τ_2; the first one supported in $\big\{(\xi,\eta) : |\xi+\eta| \le 3\big\}$ and the second one supported in $\big\{(\xi,\eta) : |\xi-\eta| \le 3\big\}$.

The bilinear multiplier associated to τ_1 has only low frequencies, hence

$$\|T_{\tau_1}(f,g)\|_{W^{2s,t}} \lesssim \|T_{\tau_1}(f,g)\|_{L^t}.$$

Using Proposition 4.5 with exponents $t, p, q \in (1,\infty)$ satisfying (1-8), it follows that

$$\|T_{\tau_1}(f,g)\|_{W^{2s,t}} \lesssim \|f\|_{L^p}\|g\|_{L^q} \lesssim \|f\|_{W^{s,t}}\|g\|_{W^{s,t}},$$

where we have used the Sobolev embeding $W^{s,t} \subset L^p$ since $s \ge \frac{1}{t} > \frac{1}{t} - \frac{1}{p}$ (and similarly with q).

Concerning the second part τ_2, it is easy to check that, on its support, $1+|\xi+\eta|$, $1+|\xi|$ and $1+|\eta|$ are comparable and in addition

$$\max\big\{1+|\xi+\eta|, 1+|\xi|,\ 1+|\eta|\big\} - \min\big\{1+|\xi+\eta|, 1+|\xi|,\ 1+|\eta|\big\} \lesssim 1. \tag{4-4}$$

We claim that T_{τ_2} is bounded from $L^t \times L^t$ into L^t. Indeed, the symbol τ_2 is supported around the diagonal $\xi = \eta$ and it takes the form

$$\tau_2(\xi,\eta) = m(\xi-\eta),$$

for a smooth function m supported on $[-3,3]$. It follows that

$$T_{\tau_2}(f,g)(x) = \int \widehat{m}(y) f(x-y) g(x+y)\, dy. \tag{4-5}$$

Since $m \in \mathcal{S}(\mathbb{R})$ we have, in particular, that $\widehat{m} \in L^1 \cap L^\infty$, and using Minkowski's inequality we easily deduce that T_{τ_2} is bounded from $L^\infty \times L^\infty$ to L^∞ and from $L^1 \times L^1$ to L^1. By (complex) bilinear interpolation, we conclude that T is bounded from $L^t \times L^t$ to L^t, for $1 < t < \infty$.

It remains to estimate T_{τ_2} in the Sobolev space. We let the reader verify that, as in similar previously done computations (and using (4-4)), T_{τ_2} can be decomposed as

$$T_{\tau_2}(f,g) = \sum_{k \ge 0} \Psi_{2^{-k}} T_{\tau_2}(\Psi^1_{2^{-k}} f, \Psi^2_{2^{-k}} g), \tag{4-6}$$

for some smooth frequency truncations Ψ, Ψ^1, Ψ^2. It follows that

$$\begin{aligned}
&\left\| \left(\sum_{k\geq 0} 2^{k4s} \left|\Psi_{2^{-k}} T_{\tau_2}(\Psi^1_{2^{-k}} f, \Psi^2_{2^{-k}} g)\right|^2 \right)^{1/2} \right\|_{L^t} \\
&\qquad \lesssim \left\| \left(\sum_{k\geq 0} 2^{k4s} \left|T_{\tau_2}(\Psi^1_{2^{-k}} f, \Psi^2_{2^{-k}} g)\right|^2 \right)^{1/2} \right\|_{L^t} \\
&\qquad \lesssim \left\| \left(\sum_{k\geq 0} 2^{ks} |\Psi^1_{2^{-k}} f|^2 \right)^{1/2} \right\|_{L^t} \left\| \left(\sum_{k\geq 0} 2^{k2s} |\Psi^2_{2^{-k}} g|^2 \right)^{1/2} \right\|_{L^t} \\
&\qquad \lesssim \|f\|_{W^{s,t}} \|g\|_{W^{s,t}},
\end{aligned}$$

where we have used the L^t boundedness of the operator T_{τ_2} and its l^2-vector-valued extension (given again by Theorem 9.1 of [Grafakos and Martell 2004]). □

Remark 4.9. The previous proof relies on the boundedness from $L^t \times L^t$ to L^t of T_{τ_2}. This property does not hold in the classical paraproduct situation.

We have given a proof by interpolation, where the specific form of τ_2 plays an important role. We would like to describe now a direct proof of the boundedness for the simpler case $t = 2$. The arguments are based on the geometric fact that the symbol τ_2 is supported on a strip around the diagonal with bounded width.

We can use in the L^2 case a partition of frequencies given by Δ_k a smooth truncation on the interval $[k-4, k+4]$:

$$\widehat{\Delta_k(f)}(\xi) = \chi(\xi - k)\widehat{f}(\xi),$$

where χ is a smooth function, supported on $[-4, 4]$ and equal to 1 on $[-3, 3]$. Then, by Plancherel's equality, we have

$$\|T_{\tau_2}(f, g)\|_{L^2} \lesssim \left(\sum_{k\in\mathbb{Z}} \|\Delta_k(T_{\tau_2}(f, g)\|^2_{L^2} \right)^{1/2}.$$

By (4-4), it follows that with other similar truncation operators Δ^1 and Δ^2,

$$\begin{aligned}
\|T_{\tau_2}(f, g)\|_{L^2} &\lesssim \left(\sum_{k\in\mathbb{Z}} \left\|\Delta_k(T_{\tau_2}(\Delta^1_k(f), \Delta^2_k(g)))\right\|^2_{L^2} \right)^{1/2} \\
&\lesssim \left(\sum_{k\in\mathbb{Z}} \left\| \mathbf{1}_{|\xi-k|\leq 4} \int \left| \widehat{\Delta^1_k(f)}(\eta)\widehat{\Delta^2_k(g)}(\xi-\eta)\right| \, d\eta \right\|^2_{L^2} \right)^{1/2} \\
&\lesssim \left(\sum_{k\in\mathbb{Z}} \left\|\widehat{\Delta^1_k(f)}\right\|^2_{L^2} \left\|\widehat{\Delta^2_k(g)}\right\|^2_{L^2} \right)^{1/2},
\end{aligned}$$

where we have used that each interval $[k-4, k+4]$ has bounded length. Since the collection of intervals $([k-4, k+4])_{k\in\mathbb{Z}}$ is a bounded covering, we can conclude the boundedness of T_{τ_2} from $L^2 \times L^2$ into L^2. (Note that the same argument does not apply in L^p.)

Remark 4.10. Our new definition of paramultiplication is based on bilinear operators associated to x-independent symbols of the class $BS_{1,0;\pi/4}$. We could use the Sobolev boundedness (proved in the first sections of the current paper) in order to define other kind of paramultiplications with an x-dependent symbol but we will not carry here such analysis any further.

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