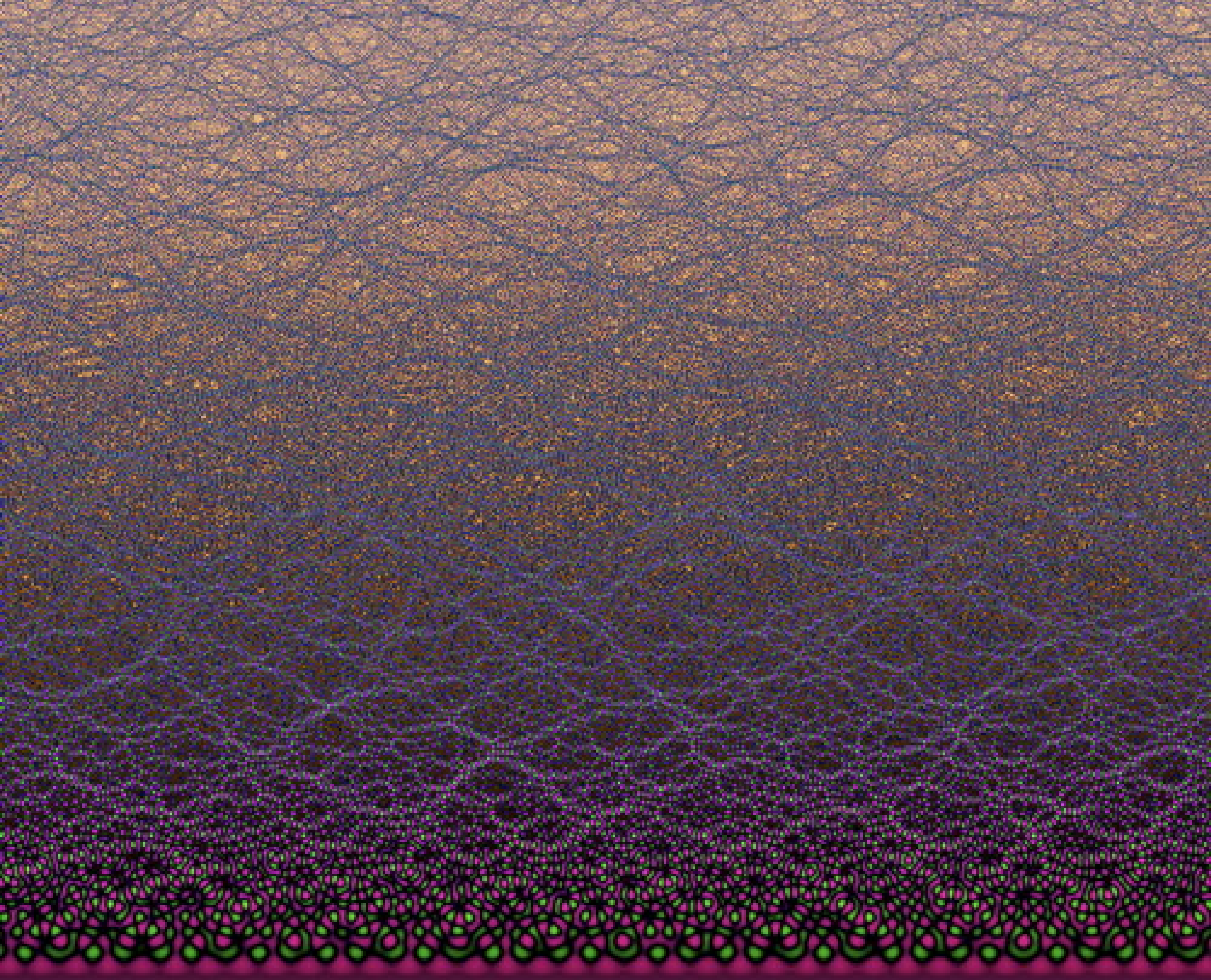
Analysis & PDE

Volume 4 No. 4 2011



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Analysis & PDE, at Mathematical Sciences Publishers, Department of Mathematics, University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFLOW™ from Mathematical Sciences Publishers.

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Typeset in LaTeX

ANALYSIS AND PDE
Vol. 4, No. 4, 2011

msp

THE CORONA THEOREM FOR THE DRURY–ARVESON HARDY SPACE AND OTHER HOLOMORPHIC BESOV–SOBOLEV SPACES ON THE UNIT BALL IN $\mathbb{C}^n$

ŞERBAN COSTEA, ERIC T. SAWYER AND BRETT D. WICK

We prove that the multiplier algebra of the Drury–Arveson Hardy space H_n^2 on the unit ball in $\mathbb{C}^n$ has no corona in its maximal ideal space, thus generalizing the corona theorem of L. Carleson to higher dimensions. This result is obtained as a corollary of the Toeplitz corona theorem and a new Banach space result: the Besov–Sobolev space B_p^σ has the "baby corona property" for all $\sigma \geq 0$ and $1 < p < \infty$. In addition we obtain infinite generator and semi-infinite matrix versions of these theorems.

1. Introduction

Lennart Carleson [1962] demonstrated the absence of a corona in the maximal ideal space of $H^\infty(\mathbb{D})$ by showing that if $\{g_j\}_{j=1}^N$ is a finite set of functions in $H^\infty(\mathbb{D})$ satisfying

$$\sum_{j=1}^N |g_j(z)| \geq c > 0, \quad z \in \mathbb{D}, \tag{1-1}$$

then there are functions $\{f_j\}_{j=1}^N$ in $H^\infty(\mathbb{D})$ with

$$\sum_{j=1}^N f_j(z) g_j(z) = 1, \quad z \in \mathbb{D}, \tag{1-2}$$

Fuhrmann [1968] extended Carleson's corona theorem to the finite matrix case. Rosenblum [1980] and Tolokonnikov [1980] proved the corona theorem for infinitely many generators $N = \infty$. This was further generalized to the one-sided infinite matrix setting by Vasyunin (see [Tolokonnikov 1981]). Finally Treil

Sawyer's research was supported in part by a grant from the National Science and Engineering Research Council of Canada. Wick's research was supported in part by National Science Foundation DMS Grant # 0752703.
MSC2000: 30H05, 32A37.
Keywords: Besov–Sobolev Spaces, corona Theorem, several complex variables, Toeplitz corona theorem.

[1988] showed that the generalizations stop there by producing a counterexample to the two-sided infinite matrix case.

Hörmander noted a connection between the corona problem and the Koszul complex, and in the late 1970s Tom Wolff gave a simplified proof using the theory of the $\bar{\partial}$ equation and Green's theorem (see [Garnett 1981]). This proof has since served as a model for proving corona type theorems for other Banach algebras.

While there is a large literature on corona theorems in one complex dimension (see [Nikolski 2002], for example), progress in higher dimensions has been limited. Indeed, apart from the simple cases in which the maximal ideal space of the algebra can be identified with a compact subset of $\mathbb{C}^n$, no corona theorem has been proved until now in higher dimensions. Instead, partial results have been obtained, such as the beautiful Toeplitz corona theorem for Hilbert function spaces with a complete Nevanlinna–Pick kernel, the H^p corona theorem on the ball and polydisk, and results restricting N to 2 generators in (1-1) (the case $N=1$ is trivial). In particular, N. Varopoulos [1977] published a lengthy classic paper in an unsuccessful attempt to prove the corona theorem for the multiplier algebra $H^\infty(\mathbb{B}_n)$ of the classical Hardy space $H^2(\mathbb{B}_n)$ of holomorphic functions on the ball with square integrable boundary values. His BMO estimates for solutions with $N=2$ generators remain largely unimproved to this day (though see [Costea et al. 2010] for the extension to an infinite number of generators). A related result for $N=2$ and $H^p(\mathbb{B}_n)$ with $1<p<\infty$ was studied in [Amar 1991]. The case $N=2$ is easier compared to $N>2$ because of certain algebraic simplifications that arise. We will discuss these partial results in more detail below.

In many ways H_n^2, and not the more familiar space $H^2(\mathbb{B}_n)$, is the natural generalization to higher dimensions of the classical Hardy space on the disk. For example, H_n^2 is universal among Hilbert function spaces with the complete Pick property, and its multiplier algebra $M_{H_n^2}$ is the correct home for the multivariate von Neumann inequality (see [Agler and McCarthy 2002], for instance). See [Arveson 1998] for more on the space H_n^2, including the model theory of n-contractions. Because of the connections the space H_n^2 has with operator theory, there is current interest in understanding the related function theory of this space.

Our main result is that the corona theorem, namely the absence of a corona in the maximal ideal space, holds for the multiplier algebra $M_{H_n^2}$ of the Hilbert space H_n^2, the celebrated Drury–Arveson Hardy space on the ball in n dimensions. This result provides yet more evidence that the space H_n^2 is the "correct" generalization to several variables.

Theorem 1. *If $\{g_j\}_{j=1}^N$ is a finite set of functions in $M_{H_n^2}$ satisfying*

$$1 \geq \sum_{j=1}^{N} |g_j(z)|^2 \geq \delta^2 > 0 \quad \text{for all } z \in \mathbb{B}_n,$$

then there are functions $\{f_j\}_{j=1}^N$ in $M_{H_n^2}$ satisfying

(i) $\sum_{j=1}^N f_j(z) g_j(z) = 1, \quad z \in \mathbb{B}_n$;

(ii) $\|f_j\|_{M_{H_n^2}} \leq C_{n,\delta,N}$ *for all* $j=1,\dots,N$.

There is a close relationship between the corona problem as stated and a related "baby" corona problem. In the case when $p = 2$, thanks to the Toeplitz corona Theorem, as explained in the next section, this connection in fact becomes an equivalence in certain situations and an application of the Toeplitz corona Theorem then will give the statement in Theorem 1. Because of this close connection between the corona problem and the "baby" corona problem, in this paper we will actually prove that the Besov–Sobolev space B_p^σ has the "baby corona property" for all $\sigma \geq 0$ and $1 < p < \infty$. The precise statements for the "baby corona property" are given below in Theorem 2. In addition, when formulated appropriately, we will obtain infinite generator and semi-infinite matrix versions of these results, see Corollary 4.

More generally, Theorem 1 holds for the multiplier algebras $M_{B_2^\sigma(\mathbb{B}_n)}$ of the Besov–Sobolev spaces $B_2^\sigma(\mathbb{B}_n)$, $0 \leq \sigma \leq \frac{1}{2}$, on the unit ball $\mathbb{B}_n$ in $\mathbb{C}^n$. These function spaces will be defined later, but the space $B_2^\sigma(\mathbb{B}_n)$ consists roughly of those holomorphic functions f whose derivatives of order $\frac{n}{2} - \sigma$ lie in the classical Hardy space $H^2(\mathbb{B}_n) = B_2^{n/2}(\mathbb{B}_n)$. In particular $H_n^2 = B_2^{1/2}(\mathbb{B}_n)$. Again, we will study these more general corona problems by studying the easier "baby" corona problem.

1.1. ***The Toeplitz corona problem in $\mathbb{C}^n$.*** In this section we connect the corona problem to the "baby" corona problem, and then formulate the analogous "baby" corona problems in the Besov–Sobolev spaces B_p^σ when $1 < p < \infty$ and $0 \leq \sigma < \infty$.

Let X be a Hilbert space of holomorphic functions in an open set Ω in $\mathbb{C}^n$ that is a reproducing kernel Hilbert space with a *complete irreducible Nevanlinna–Pick kernel* (see [Agler and McCarthy 2002] for the definition). The following *Toeplitz corona theorem* is due to Ball, Trent and Vinnikov [Ball et al. 2001] (see also [Ambrozie and Timotin 2002; Agler and McCarthy 2002, Theorem 8.57]).

For $f = (f_\alpha)_{\alpha=1}^N \in \bigoplus^N X$ and $h \in X$, define

$$\mathbb{M}_f h = (f_\alpha h)_{\alpha=1}^N \quad \text{and} \quad \|f\|_{\mathrm{Mult}(X,\oplus^N X)} = \|\mathbb{M}_f\|_{X \to \oplus^N X} = \sup_{\|h\|_X \leq 1} \|\mathbb{M}_f h\|_{\oplus^N X}.$$

Note that

$$\max_{1 \leq \alpha \leq N} \|\mathcal{M}_{f_\alpha}\|_{M_X} \leq \|f\|_{\mathrm{Mult}(X,\oplus^N X)} \leq \sqrt{\textstyle\sum_{\alpha=1}^N \|\mathcal{M}_{f_\alpha}\|_{M_X}^2}.$$

Toeplitz corona theorem. *Let X be a Hilbert function space in an open set Ω in $\mathbb{C}^n$ with an irreducible complete Nevanlinna–Pick kernel. Let $\delta > 0$ and $N \in \mathbb{N}$. For $g_1, \ldots, g_N \in M_X$, there is equivalence between*:

- *("baby corona property") For every $h \in X$, there are $f_1, \ldots, f_N \in X$ such that*

$$\|f_1\|_X^2 + \cdots + \|f_N\|_X^2 \leq \frac{1}{\delta}\|h\|_X^2, \qquad g_1(z)f_1(z) + \cdots + g_N(z)f_N(z) = h(z) \quad \text{for } z \in \Omega. \tag{1-3}$$

- *("multiplier corona property") There are $\varphi_1, \ldots, \varphi_N \in M_X$ such that*

$$\|\varphi\|_{\mathrm{Mult}(X,\oplus^N X)} \leq 1, \qquad g_1(z)\varphi_1(z) + \cdots + g_N(z)\varphi_N(z) = \sqrt{\delta} \quad \text{for } z \in \Omega. \tag{1-4}$$

The *baby corona theorem* is said to hold for X if, whenever $g_1, \ldots, g_N \in M_X$ satisfy

$$|g_1(z)|^2 + \cdots + |g_N(z)|^2 \geq c > 0 \quad \text{for } z \in \Omega, \tag{1-5}$$

then $g_1, \dots, g_N$ satisfy the baby corona property (1-3). The Toeplitz corona theorem thus provides a useful tool for reducing the multiplier corona property (1-4) to the more tractable, but still very difficult, baby corona property (1-3) for multiplier algebras $M_{B_p^\sigma(\mathbb{B}_n)}$ of certain of the Besov–Sobolev spaces $B_p^\sigma(\mathbb{B}_n)$ when $p = 2$: see below. The case of $M_{B_p^\sigma(\mathbb{B}_n)}$ when $p \neq 2$ must be handled by more classical methods and remains largely unsolved.

Remark. A standard abstract argument applies to show that the absence of a corona for the multiplier algebra M_X, i.e., the density of the linear span of point evaluations in the maximal ideal space of M_X, is equivalent to the following assertion: for each finite set $\{g_j\}_{j=1}^N \subset M_X$ such that (1-5) holds for some $c > 0$, there are $\{\varphi_j\}_{j=1}^N \subset M_X$ and $\delta > 0$ such that condition (1-4) holds. See for example Lemma 9.2.6 in [Nikolski 2002] or the proof of Criterion 3.5 on page 39 of [Sawyer 2009].

1.2. *The baby corona theorem.*

Notation. For sequences $f(z) = (f_i(z))_{i=1}^\infty \in \ell^2$ we will write

$$|f(z)| = \sqrt{\textstyle\sum_{i=1}^\infty |f_i(z)|^2}.$$

When considering sequences of vectors such as $\nabla^m f(z) = (\nabla^m f_i(z))_{i=1}^\infty$, the same notation $|\nabla^m f(z)| = \sqrt{\sum_{i=1}^\infty |\nabla^m f_i(z)|^2}$ will be used with $|\nabla^m f_i(z)|$ denoting the Euclidean length of the vector $\nabla^m f_i(z)$. Thus the symbol $|\cdot|$ is used in at least three different ways: to denote the absolute value of a complex number, the length of a finite vector in $\mathbb{C}^N$ and the norm of a sequence in ℓ^2. Later it will also be used to denote the Hilbert–Schmidt norm of a tensor, namely the square root of the sum of the squares of the coefficients in the standard basis. In all cases the meaning should be clear from the context.

Recall that $B_p^\sigma(\mathbb{B}_n; \ell^2)$ consists of all $f = (f_i)_{i=1}^\infty \in H(\mathbb{B}_n; \ell^2)$ such that

$$\|f\|_{B_p^\sigma(\mathbb{B}_n;\ell^2)} \equiv \sum_{k=0}^{m-1} |\nabla^k f(0)| + \left(\int_{\mathbb{B}_n} \left| (1-|z|^2)^{m+\sigma} \nabla^m f(z) \right|^p d\lambda_n(z) \right)^{1/p} < \infty, \tag{1-6}$$

for some $m > \frac{n}{p} - \sigma$. By a result in [Beatrous 1986] the right side is finite for some $m > \frac{n}{p} - \sigma$ if and only if it is finite for all $m > \frac{n}{p} - \sigma$. As usual we will write $B_p^\sigma(\mathbb{B}_n)$ for the scalar-valued space.

We now state our baby corona theorem for the ℓ^2-valued Banach spaces $B_p^\sigma(\mathbb{B}_n; \ell^2)$, $\sigma \geq 0$, $1 < p < \infty$. Observe that for $\sigma < 0$, $M_{B_p^\sigma(\mathbb{B}_n)} = B_p^\sigma(\mathbb{B}_n)$ is a subalgebra of $C(\overline{\mathbb{B}}_n)$ and so has no corona. The $N = 2$ generator case of Theorem 2 when $\sigma \in \left[0, \frac{1}{p}\right) \cup \left(\frac{n}{p}, \infty\right)$ and $1 < p < \infty$ is due to Ortega and Fàbrega [2000], who also obtained the $N = 2$ generator case when $\sigma = \frac{n}{p}$ and $1 < p \leq 2$. See Theorem A in that reference. Ortega and Fàbrega [2006] prove analogous results with scalar-valued Hardy–Sobolev spaces in place of the Besov–Sobolev spaces.

Let

$$\|\mathbb{M}_g\|_{B_p^\sigma(\mathbb{B}_n) \to B_p^\sigma(\mathbb{B}_n;\ell^2)}$$

denote the norm of the multiplication operator $\mathbb{M}_g$ from $B_p^\sigma(\mathbb{B}_n)$ to the ℓ^2-valued Besov–Sobolev space $B_p^\sigma(\mathbb{B}_n; \ell^2)$.

Theorem 2. *Let $\delta > 0$, $\sigma \geq 0$ and $1 < p < \infty$. There is a constant $C_{n,\sigma,p,\delta}$ such that, given a sequence $g = (g_i)_{i=1}^{\infty} \in M_{B_p^\sigma(\mathbb{B}_n) \to B_p^\sigma(\mathbb{B}_n;\ell^2)}$ satisfying*

$$\|\mathbb{M}_g\|_{B_p^\sigma(\mathbb{B}_n) \to B_p^\sigma(\mathbb{B}_n;\ell^2)} \leq 1, \qquad \sum_{j=1}^{\infty} |g_j(z)|^2 \geq \delta^2 > 0 \quad \text{for } z \in \mathbb{B}_n,$$

there is for each $h \in B_p^\sigma(\mathbb{B}_n)$ a vector-valued function $f \in B_p^\sigma(\mathbb{B}_n; \ell^2)$ satisfying

$$\|f\|_{B_p^\sigma(\mathbb{B}_n;\ell^2)} \leq C_{n,\sigma,p,\delta} \|h\|_{B_p^\sigma(\mathbb{B}_n)}, \qquad \sum_{j=1}^{\infty} g_j(z) f_j(z) = h(z) \quad \text{for } z \in \mathbb{B}_n. \tag{1-7}$$

Corollary 3. *Let $0 \leq \sigma \leq \frac{1}{2}$. Then the Banach algebra $M_{B_2^\sigma(\mathbb{B}_n)}$ has no corona; that is, the analogue of Theorem 1 holds. As particular cases we obtain that the multiplier algebra of the Drury–Arveson space $H_n^2 = B_2^{1/2}(\mathbb{B}_n)$ has no corona (Theorem 1) and that the multiplier algebra of the n-dimensional Dirichlet space $\mathfrak{D}(\mathbb{B}_n) = B_2^0(\mathbb{B}_n)$ has no corona.*

The corollary follows immediately from the finite generator case $p = 2$ of Theorem 2 and the Toeplitz corona theorem (and the remark on page 502) since the spaces $B_2^\sigma(\mathbb{B}_n)$ have an irreducible complete Nevanlinna–Pick kernel when $0 \leq \sigma \leq \frac{1}{2}$; see for example [Arcozzi et al. 2008].

Note that in dimension $n = 1$ and $\sigma = \frac{1}{2}$, Corollary 3 gives a new proof of Carleson's classical corona theorem, similar to that in [Andersson and Carlsson 2001]. Of course it is the Toeplitz corona theorem that yields the difficult L^∞ estimate there. Additionally, when $n = 1$ and $\sigma = 0$, we have that the multiplier algebra of the Dirichlet space has no corona, recovering a result from [Tolokonnikov 1991]. See also [Xiao 1998] for the case of $n = 1$ and $0 \leq \sigma < \frac{1}{2}$.

We also have a semi-infinite matricial corona theorem.

Corollary 4. *Let $0 \leq \sigma \leq \frac{1}{2}$. Let $\mathcal{H}_1$ be a finite m-dimensional Hilbert space and let $\mathcal{H}_2$ be an infinite-dimensional separable Hilbert space. Suppose that $F \in \mathcal{M}_{B_2^\sigma(\mathbb{B}_n)(\mathcal{H}_1 \to \mathcal{H}_2)}$ satisfies*

$$\delta^2 I_m \leq F^*(z) F(z) \leq I_m.$$

Then there is $G \in \mathcal{M}_{B_2^\sigma(\mathbb{B}_n)(\mathcal{H}_2 \to \mathcal{H}_1)}$ such that

$$G(z) F(z) = I_m, \qquad \|G\|_{\mathcal{M}_{B_2^\sigma(\mathbb{B}_n)(\mathcal{H}_2 \to \mathcal{H}_1)}} \leq C_{\sigma,n,\delta,m}.$$

This corollary follows immediately from the case $p = 2$ of Theorem 2 and the Toeplitz corona theorem together with Theorem (MCT) in [Trent and Zhang 2006]. We follow the notation in that reference. We already commented above on the special case of this corollary for the Hardy space $B_2^{1/2}(\mathbb{B}_1) = H^2(\mathbb{D})$ on the disk. The case $m = 1$ of this corollary for the classical Dirichlet space $B_2^0(\mathbb{B}_1) = \mathfrak{D}(\mathbb{D})$ on the disk is due to Trent [2004a]. Our method yields information about the dependence of the constants on the parameters δ, σ, p and n in Theorem 2. However, this information is not sharp and more precise information would be desirable.

Remark. It is an open question [Trent 2004a] for the Dirichlet space $B_2^0(\mathbb{B}_1)$ in one dimension whether or not in Theorem 2 the boundedness condition on the column operator, $\|\mathbb{M}_g\|_{B_2^0(\mathbb{B}_1) \to B_2^0(\mathbb{B}_1;\ell^2)} \leq 1$, can be replaced by a similar (but weaker — see Lemma 1 in [Trent 2004a]) boundedness condition for the row operator, $\|\mathcal{M}_g\|_{B_2^0(\mathbb{B}_1;\ell^2)\to B_2^0(\mathbb{B}_1)} \le 1$. The question also appears to be open for the Besov–Sobolev spaces $B_2^\sigma(\mathbb{B}_n)$, with $0 \le \sigma < \frac{n}{2}$. (The two operators are not dual to one another for these spaces.)

Prior results. The baby corona problem for $H^2(\mathbb{B}_n)$ was first formulated and proved via L^2 methods by Mats Andersson [1994b]. It is noteworthy that the approach used in that work allowed for one to obtain estimates independent of the number of generators N. The case of two generators in $H^p(\mathbb{B}_n)$, $1 < p < \infty$, was handled by Éric Amar [1991]. His proof could be extended to handle more generators but doing so will result in a constant that depends upon the number of generators N. Andersson and Carlsson [2000] solved the baby corona problem for $H^2(\mathbb{B}_n)$ and obtained the analogous (baby) H^p corona theorem on the ball $\mathbb{B}_n$ for $1 < p < \infty$ and with constants independent of the number of generators and sharp information in terms of the estimates in terms of δ and the dimension n. The interested reader can also see [Andersson 1994a; Andersson and Carlsson 2001; 1994; Krantz and Li 1995], where the problem is studied.

Partial results on the corona problem restricted to $N = 2$ generators and BMO in place of L^∞ estimates have been obtained for $H^\infty(\mathbb{B}_n)$ (the multiplier algebra of $H^2(\mathbb{B}_n) = B_2^{n/2}(\mathbb{B}_n)$) by Varopoulos [1977]. Note that the techniques used in this paper also yield BMO estimates for the $H^\infty(\mathbb{B}_n)$ corona problem, which appear in [Costea et al. 2010]. This classical corona problem remains open (Problem 19.3.7 in [Rudin 1980]), along with the corona problems for the multiplier algebras of $B_2^\sigma(\mathbb{B}_n)$, $\frac{1}{2} < \sigma < \frac{n}{2}$.

More recently, J. M. Ortega and J. Fàbrega [2000] obtained partial results with $N = 2$ generators in (1-3) for the algebras $M_{B_2^\sigma(\mathbb{B}_n)}$ when $0 \le \sigma < \frac{1}{2}$, i.e., from the Dirichlet space $B_2^0(\mathbb{B}_n)$ up to but not including the Drury–Arveson Hardy space $H_n^2 = B_2^{1/2}(\mathbb{B}_n)$. To handle $N = 2$ generators they exploit the fact that a 2×2 antisymmetric matrix consists of just one entry up to sign, so that as a consequence the form Ω_1^2 in the Koszul complex below is $\bar\partial$-closed. Ortega and Fàbrega's paper has proved to be of enormous influence in our work, as the basic groundwork and approach we use are set out there.

In [Treil and Wick 2005] the H^p corona theorem on the polydisk $\mathbb{D}^n$ is obtained (see also [Lin 1994; Trent 2004b]). The Hardy space $H^2(\mathbb{D}^n)$ on the polydisk fails to have the complete Nevanlinna–Pick property, and consequently the Toeplitz corona theorem only holds in a more complicated sense that a family of kernels must be checked for positivity instead of just one (see [Amar 2003; Trent and Wick 2009]). As a result the corona theorem for the algebra $H^\infty(\mathbb{D}^n)$ on the polydisk remains open for $n \ge 2$. Finally, even the baby corona problems, apart from that for H^p, remain open on the polydisk.

1.3. ***Plan of the paper.*** We will prove Theorem 2 using the Koszul complex and a factorization of Andersson and Carlsson, an explicit calculation of Charpentier's solution operators, and generalizations of the integration by parts formulas of Ortega and Fàbrega, together with new estimates for boundedness of operators on certain real-variable analogues of the holomorphic Besov–Sobolev spaces.

More precisely, to treat $N > 2$ generators in (1-7), it is just as easy to treat the case $N = \infty$, and this has the advantage of not requiring bookkeeping of constants depending on N. We will

(1) use the Koszul complex for infinitely many generators,

(2) invert higher order forms in the $\bar\partial$ equation, and

(3) devise new estimates for the Charpentier solution operators for these equations, including
  (a) the use of sharp estimates — (5-7), (5-8), and (5-9) — on Euclidean expressions $\left|(\overline{w-z})\,\frac{\partial}{\partial\overline{w}}f\right|$ in terms of the invariant derivative $|\widetilde{\nabla}f|$,
  (b) the use of the exterior calculus together with the explicit form of Charpentier's solution kernels in Theorems 8 and 10 to handle "rogue" Euclidean factors $\overline{w_j-z_j}$ (see Section 7), and
  (c) the application of generalized operator estimates of Schur type in Lemma 24 to obtain appropriate boundedness of solution operators.

Remark. We emphasize that the crucial new ingredient in our approach, as compared to previous work, is the use of the Besov–Sobolev norms

$$\|f\|_{B^\sigma_{p,m}(\mathbb{B}_n;\ell^2)} \approx \sum_{j=0}^{m-1} |\nabla^j f(0)| + \left(\int_{\mathbb{B}_n} \left|(1-|z|^2)^\sigma D^m f(z)\right|^p d\lambda_n(z)\right)^{1/p}$$

given by Arcozzi, Rochberg and Sawyer [Arcozzi et al. 2006] in terms of the *almost invariant holomorphic derivative*

$$D_a f(z) = -f'(z)\big((1-|a|^2)P_a + (1-|a|^2)^{1/2}Q_a\big),$$

given in (5-1) below. This derivative neatly separates the normal and tangential components of the Euclidean derivative, and permits a key exchange between Charpentier's solution kernel in (2-6),

$$\mathscr{C}_n^{0,q}(w,z) \sim \frac{(1-w\bar{z})^{n-1-q}(1-|w|^2)^q}{\triangle(w,z)^n}(\bar{z}_k-\overline{w}_k).$$

and appropriate derivatives D_a of the forms F in the Koszul complex. The point is that the Euclidean portion $\bar{z}_k-\overline{w}_k$ of the kernel $\mathscr{C}_n^{0,q}(w,z)$ is generally *not* dominated by the corresponding invariant portion (see (2-1))

$$\sqrt{\triangle(w,z)} = \left|P_w(z-w) + \sqrt{1-|w|^2}\,Q_w(z-w)\right|$$

appearing in the denominator. However, this complication is offset by the fact that the almost invariant derivative $D_a f(z)$ is correspondingly *larger* than the Euclidean derivative $(1-|a|^2)f'(z)$, and this is exploited in the following exchange formula (5-7):

$$\left|(\overline{z-w})^\alpha \frac{\partial^m}{\partial\overline{w}^\alpha}F(w)\right| \le C\left(\frac{\sqrt{\triangle(w,z)}}{1-|w|^2}\right)^m \left|\overline{D}^m F(w)\right|,$$

which permits control of the solution by the $B^\sigma_{p,m}$ norm using the *larger* derivative $\overline{D}^m F$. It is likely that the Charpentier kernel can be replaced in these arguments by more general kernels with appropriate estimates, and this would be key to extending our baby corona theorem to strictly pseudoconvex domains Ω. This extension will be pursued in subsequent work.

In addition to these novel elements in the proof, we make crucial use of the beautiful integration by parts formula of [Ortega and Fàbrega 2000], and in order to obtain ℓ^2-valued results, we use the clever factorization of the Koszul complex in [Andersson and Carlsson 2000] but adapted to ℓ^2.

Here is a brief outline of the approach of the proof.

We are given an infinite vector of multipliers $g = (g_i)_{i=1}^{\infty} \in M_{B_p^\sigma(\mathbb{B}_n) \to B_p^\sigma(\mathbb{B}_n;\ell^2)}$ that satisfy

$$\|\mathbb{M}_g\|_{B_p^\sigma(\mathbb{B}_n) \to B_p^\sigma(\mathbb{B}_n;\ell^2)} \le 1 \quad \text{and} \quad \inf_{\mathbb{B}_n} |g| \ge \delta > 0,$$

and an element $h \in B_p^\sigma(\mathbb{B}_n)$. We wish to find $f = (f_i)_{i=1}^{\infty} \in B_p^\sigma(\mathbb{B}_n;\ell^2)$ such that

$$(1)\ \ \mathcal{M}_g f = g \cdot f = h, \qquad (2)\ \ \bar{\partial} f = 0, \qquad (3)\ \ \|f\|_{B_p^\sigma(\mathbb{B}_n;\ell^2)} \le C_{n,\sigma,p,\delta} \|h\|_{B_p^\sigma(\mathbb{B}_n)}.$$

An obvious first attempt at a solution is $f = \dfrac{\bar{g}}{|g|^2} h$, which clearly satisfies (1), can be shown to satisfy (3), but fails to satisfy (2) in general.

To rectify this we use the Koszul complex in Section 4, which employs *any* solution to the $\bar{\partial}$ problem on forms of bidegree $(0,q)$, $1 \le q \le n$, to produce a correction term $\Lambda_g \Gamma_0^2$ so that

$$f = \frac{\bar{g}}{|g|^2} h - \Lambda_g \Gamma_0^2$$

now satisfies (1) and (2); but (3) is now in doubt without specifying the exact nature of the correction term $\Lambda_g \Gamma_0^2$.

In Section 2 we explicitly calculate Charpentier's solution operators to the $\bar{\partial}$ equation for use in solving the $\bar{\partial}$ problems arising in the Koszul complex. These solution operators are remarkably simple in form and moreover are superbly adapted for obtaining estimates in real-variable analogues of the Besov–Sobolev spaces in the ball. In particular, the kernels $K(w,z)$ of these solution operators involve expressions like

$$\frac{(1 - w\bar{z})^{n-1-q} (1 - |w|^2)^q (\overline{w - z})}{\triangle(w,z)^n}, \tag{1-8}$$

where

$$\sqrt{\triangle(w,z)} = \left| P_z(w - z) + \sqrt{1 - |z|^2}\, Q_z(w - z) \right|$$

is the length of the vector $w - z$ shortened by multiplying by $\sqrt{1 - |z|^2}$ its projection $Q_z(w - z)$ onto the orthogonal complement of the complex line through z. Also useful is the identity

$$\sqrt{\triangle(w,z)} = |1 - w\bar{z}|\, |\varphi_z(w)|,$$

where φ_z is the involutive automorphism of the ball that interchanges z and 0; in particular this shows that $d(w,z) = \sqrt{\triangle(w,z)}$ is a quasimetric on the ball.

In Section 5.1 we introduce real-variable analogues $\Lambda_{p,m}^\sigma(\mathbb{B}_n)$ of the Besov–Sobolev spaces $B_p^\sigma(\mathbb{B}_n)$ along with ℓ^2-valued variants, that are based on the geometry inherent in the complex structure of the ball and reflected in the solution kernels in (1-8). In particular these norms involve modifications D of the invariant derivative $\widetilde{\nabla}$ in the ball:

$$Df(w) = (1 - |w|^2) P_w \nabla f + \sqrt{1 - |w|^2}\, Q_w \nabla f.$$

Three crucial inequalities are then developed to facilitate the boundedness of the Charpentier solution operators, most notably

$$\left|(\overline{z-w})^{\alpha}\frac{\partial^m}{\partial\overline{w}^{\alpha}}F(w)\right| \le C\,\triangle(w,z)^{m/2}\left|(1-|w|^2)^{-m}\,\overline{D}^m F(w)\right|, \tag{1-9}$$

for $F \in H^\infty(\mathbb{B}_n;\ell^2)$, which controls the product of Euclidean lengths with Euclidean derivatives on the left, in terms of the product of the smaller length $\sqrt{\triangle(w,z)}$ and the larger derivative $(1-|w|^2)^{-1}\,\overline{D}$ on the right. We caution the reader that our definition of $\overline{D}^m$ is *not* simply the composition of m copies of $\overline{D}$; see Definition 18 below.

In Section 3 we recall the clever integration by parts formulas of Ortega and Fàbrega involving the left side of (1-9), and extend them to the Charpentier solution operators for higher degree forms. If we differentiate (1-8), the power of $\triangle(w,z)$ in the denominator can increase and the integration by parts in Lemma 14 below will temper this singularity on the diagonal. On the other hand the radial integration by parts in Corollary 16 below will temper singularities on the boundary of the ball.

In Section 6 we use Schur's test to establish the boundedness of positive operators with kernels of the form

$$\frac{(1-|z|^2)^a(1-|w|^2)^b\sqrt{\triangle(w,z)}^{\,c}}{|1-\overline{w}z|^{a+b+c+n+1}}.$$

The case $c=0$ is standard (see [Zhu 2005], for example) and the extension to the general case follows from an automorphic change of variables. These results are surprisingly effective in dealing with the ameliorated solution operators of Charpentier.

Finally in Section 7 we put these pieces together to prove Theorem 2.

An Electronic Supplement collects many of the technical modifications of existing proofs in the literature mentioned below that would otherwise interrupt the main flow of this paper.

2. Charpentier's solution kernels for $(0,q)$-forms on the ball

Charpentier proved the following formula for $(0,q)$-forms:

Theorem 5 [Charpentier 1980, Theorem I.1, page 127]. *For $q \ge 0$ and all forms $f(\xi) \in C^1(\overline{\mathbb{B}}_n)$ of degree $(0,q+1)$, we have, for $z \in \mathbb{B}_n$,*

$$f(z) = C_q\int_{\mathbb{B}_n}\overline{\partial} f(\xi)\wedge\mathscr{C}_n^{0,q+1}(\xi,z) + c_q\overline{\partial}_z\left(\int_{\mathbb{B}_n} f(\xi)\wedge\mathscr{C}_n^{0,q}(\xi,z)\right).$$

Here $\mathscr{C}_n^{0,q}(\xi,z)$ is a $(n,n-q-1)$-form in ξ on the ball and a $(0,q)$-form in z on the ball that is defined in Definition 7 below. Using Theorem 5, we can solve $\overline{\partial}_z u = f$ for a $\overline{\partial}$-closed $(0,q+1)$-form f as follows. Set

$$u(z) \equiv c_q\int_{\mathbb{B}_n} f(\xi)\wedge\mathscr{C}_n^{0,q}(\xi,z).$$

Taking $\overline{\partial}_z$ of this we see from Theorem 5 and $\overline{\partial} f = 0$ that

$$\overline{\partial}_z u = c_q\overline{\partial}_z\left(\int_{\mathbb{B}_n} f(\xi)\wedge\mathscr{C}_n^{0,q}(\xi,z)\right) = f(z).$$

It is essential for our proof to explicitly compute the kernels $\mathscr{C}_n^{0,q}$ when $0 \le q \le n-1$. The case $q=0$ is given in [Charpentier 1980] and we briefly recall the setup. Denote by $\triangle : \mathbb{C}^n \times \mathbb{C}^n \to [0,\infty)$ the map

$$\triangle(w,z) \equiv |1-w\bar{z}|^2 - (1-|w|^2)(1-|z|^2).$$

It is convenient to record the many faces of $\triangle(w,z)$:

$$\begin{aligned}
\triangle(w,z) &= |1-w\bar{z}|^2 - (1-|w|^2)(1-|z|^2) \\
&= (1-|z|^2)|w-z|^2 + |\bar{z}(w-z)|^2 \\
&= (1-|w|^2)|w-z|^2 + |\overline{w}(w-z)|^2 \\
&= |1-w\bar{z}|^2 |\varphi_w(z)|^2 \\
&= |1-w\bar{z}|^2 |\varphi_z(w)|^2 \\
&= \left| P_w(z-w) + \sqrt{1-|w|^2}\, Q_w(z-w) \right|^2 \\
&= \left| P_z(z-w) + \sqrt{1-|z|^2}\, Q_z(z-w) \right|^2 .
\end{aligned} \tag{2-1}$$

To compute the kernels $\mathscr{C}_n^{0,q}$ we start with the closed Cauchy–Leray form (see [Rudin 1980, 16.4.5], for example)

$$\mu(\xi,w,z) \equiv \frac{1}{(\xi(w-z))^n} \sum_{i=1}^n (-1)^{i-1} \xi_i \left[\bigwedge\nolimits_{j\neq i} d\xi_j\right] \bigwedge\nolimits_{i=1}^n d(w_i - z_i).$$

One then lifts the form μ via a section s to give a closed form on $\mathbb{C}^n \times \mathbb{C}^n$. Namely, for $s : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$ one defines

$$s^*\mu(w,z) \equiv \frac{1}{\big(s(w,z)(w-z)\big)^n} \sum_{i=1}^n (-1)^{i-1} s_i(w,z) \left[\bigwedge\nolimits_{j\neq i} ds_j\right] \bigwedge\nolimits_{i=1}^n d(w_i - z_i).$$

Now we fix s to be the following section used by Charpentier:

$$s(w,z) \equiv \overline{w}(1-w\bar{z}) - \bar{z}(1-|w|^2). \tag{2-2}$$

Simple computations [Ortega and Fàbrega 2000] demonstrate that

$$s(w,z)(w-z) = \triangle(w,z). \tag{2-3}$$

Definition 6. We define the Cauchy kernel on $\mathbb{B}_n \times \mathbb{B}_n$ to be

$$\mathscr{C}_n(w,z) \equiv s^*\mu(w,z) \tag{2-4}$$

for the section s given in (2-2).

Definition 7. For $0 \le p \le n$ and $0 \le q \le n-1$ we let $\mathscr{C}_n^{p,q}$ be the component of $\mathscr{C}_n(w,z)$ that has bidegree (p,q) in z and bidegree $(n-p, n-q-1)$ in w.

Thus if η is a $(p,q+1)$-form in w, then $\mathscr{C}_n^{p,q} \wedge \eta$ is a (p,q)-form in z and a multiple of the volume form in w. We now prepare to give explicit formulas for Charpentier's solution kernels $\mathscr{C}_n^{0,q}(w,z)$. First we introduce some notation.

Notation. Let $\omega_n(z) = \bigwedge_{j=1}^n dz_j$. For n a positive integer and $0 \le q \le n-1$ let P_n^q denote the collection of all permutations ν on $\{1, \dots, n\}$ that map to $\{i_\nu, J_\nu, L_\nu\}$ where J_ν is an increasing multi-index with $\mathrm{card}(J_\nu) = n-q-1$ and $\mathrm{card}(L_\nu) = q$. Let $\epsilon_\nu \equiv \mathrm{sgn}(\nu) \in \{-1, 1\}$ denote the signature of the permutation ν.

Note that the number of increasing multi-indices of length $n-q-1$ is $\frac{n!}{(q+1)!(n-q-1)!}$, while the number of increasing multi-indices of length q are $\frac{n!}{q!(n-q)!}$. Since we are only allowed certain combinations of J_ν and L_ν (they must have disjoint intersection and they must be increasing multi-indices), it is straightforward to see that the total number of permutations in P_n^q that we are considering is $\frac{n!}{(n-q-1)!q!}$.

From [Øvrelid 1971] we obtain that Charpentier's kernel takes the (abstract) form

$$\mathscr{C}_n^{0,q}(w,z) = \frac{1}{\triangle(w,z)^n} \sum_{\nu \in P_n^q} \mathrm{sgn}(\nu) s_{i_\nu} \bigwedge_{j \in J_\nu} \bar\partial_w s_j \bigwedge_{l \in L_\nu} \bar\partial_z s_l \wedge \omega_n(w).$$

Fundamental for us will be the explicit formula for Charpentier's kernel given in the next theorem. It is convenient to isolate the following factor common to all summands in the formula:

$$\Phi_n^q(w,z) \equiv \frac{(1-w\bar z)^{n-1-q}(1-|w|^2)^q}{\triangle(w,z)^n}, \quad 0 \le q \le n-1. \tag{2-5}$$

Theorem 8. *Let n be a positive integer and suppose that $0 \le q \le n-1$. Then*

$$\mathscr{C}_n^{0,q}(w,z) = \sum_{\nu \in P_n^q} (-1)^q \Phi_n^q(w,z)\, \mathrm{sgn}(\nu)(\bar w_{i_\nu} - \bar z_{i_\nu}) \bigwedge_{j \in J_\nu} d\bar w_j \bigwedge_{l \in L_\nu} d\bar z_l \wedge \omega_n(w). \tag{2-6}$$

Remark. We can rewrite the formula for $\mathscr{C}_n^{0,q}(w,z)$ in (2-6) as

$$\mathscr{C}_n^{0,q}(w,z) = \Phi_n^q(w,z) \sum_{|J|=q} \sum_{k \notin J} (-1)^{\mu(k,J)} (\bar z_k - \bar w_k) d\bar z^J \wedge d\bar w^{(J\cup\{k\})^c} \wedge \omega_n(w), \tag{2-7}$$

where $J \cup \{k\}$ denotes the increasing multi-index obtained by rearranging the integers $\{k, j_1, \dots j_q\}$ as

$$J \cup \{k\} = \{j_1, \dots j_{\mu(k,J)-1}, k, j_{\mu(k,J)}, \dots j_q\}.$$

Thus k occupies the $\mu(k,J)$-th position in $J \cup \{k\}$. The notation $(J \cup \{k\})^c$ refers to the increasing multi-index obtained by rearranging the integers in $\{1, 2, \dots, n\} \setminus (J \cup \{k\})$. To see (2-7), we note that in (2-6) the permutation ν takes the n-tuple $(1, 2, \dots n)$ to (i_ν, J_ν, L_ν). In (2-7) the n-tuple $\big(k, (J\cup\{k\})^c, J\big)$ corresponds to (i_ν, J_ν, L_ν), and so $\mathrm{sgn}(\nu)$ becomes in (2-7) the signature of the permutation that takes $(1, 2, \dots, n)$ to $(k, (J \cup \{k\})^c, J)$. This in turn equals $(-1)^{\mu(k,J)}$ with $\mu(k,J)$ as above.

We observe at this point that the functional coefficient in the summands in (2-6) looks like

$$(-1)^q \Phi_n^q(w,z)(\bar w_{i_\nu} - \bar z_{i_\nu}) = (-1)^q \frac{(1-w\bar z)^{n-q-1}(1-|w|^2)^q}{\triangle(w,z)^n} (\bar w_{i_\nu} - \bar z_{i_\nu}),$$

which behaves like a fractional integral operator of order 1 in the Bergman metric on the diagonal relative to invariant measure.

Finally, we will adopt the usual convention of writing

$$\mathscr{C}_n^{0,q} f(z) = \int_{\mathbb{B}_n} f(w) \wedge \mathscr{C}_n^{0,q}(w,z),$$

when we wish to view $\mathscr{C}_n^{0,q}$ as an operator taking $(0,q+1)$-forms f in w to $(0,q)$-forms $\mathscr{C}_n^{0,q} f$ in z. The proof of Theorem 8 is carried out in the Electronic Supplement. Here we present a relatively short and elegant proof pointed out to us by a referee. It is helpful to make the following elementary observation.

Remark. If a form λ has odd degree, then any power λ^ℓ with $\ell \geq 2$ necessarily vanishes, by the alternating property. On the other hand, if one of the forms λ_1, λ_2 has even degree, then the binomial theorem holds for the sum:

$$(\lambda_1+\lambda_2)^\ell = \sum_{j=0}^{\ell} \binom{\ell}{j} (\lambda_1)^j (\lambda_2)^{\ell-j}.$$

Note that the wedge power $\bigwedge_{i=1}^{\ell} \Big(\sum_{k_i=1}^{\infty} \frac{\bar\partial \bar g_{k_i}}{|g|^2} e_{k_i} \Big)$ above doesn't vanish since the form $\frac{\bar\partial \bar g_{k_i}}{|g|^2} e_{k_i}$ has degree 2.

Proof of Theorem 8. Consider the section $s(w,z)$ in (2-2) and the associated $(1,0)$-form $s \cdot dw = \sum_{j=1}^n s_j \, dw_j$ and the $(1,1)$-form $\bar\partial(s\cdot dw)$. We claim that

$$\triangle(w,z)^{-n}(s\cdot dw) \wedge \big(\bar\partial(s\cdot dw)\big)^{n-1} \tag{2-8}$$

is the term $K(w,z)$ of total bidegree $(n,n-1)$ in the Cauchy kernel $\mathscr{C}_n(w,z) = s^*\mu(w,z)$. To see this we first recall formula (2.2) in [Øvrelid 1971], which reads

$$K(w,z) = c_n \triangle(w,z)^{-n} \sum_{i=1}^{n} (-1)^{i-1} s_i \Big(\bigwedge_{j\neq i} (\bar\partial s_j) \Big) \wedge \omega(w),$$

where $\omega(w) = dw_1 \wedge \cdots \wedge dw_n$ and the product over $j \neq i$ is taken with increasing j. Now we note that each term in the expansion of the product in (2-8) must contain a permutation of the product $dw_1 \wedge \cdots \wedge dw_n$. Thus by factoring out the term $\omega(w)$ we compute

$$\begin{aligned}\triangle(w,z)^{-n}(s\cdot dw) \wedge \big(\bar\partial(s\cdot dw)\big)^{n-1} &= \triangle(w,z)^{-n} \Big(\sum_{i=1}^n s_i \, dw_i \Big) \wedge \Big(\sum_{j=1}^n (\bar\partial s_j) \wedge dw_j \Big)^{n-1} \\ &= \triangle(w,z)^{-n} \sum_{i=1}^n (-1)^{i-1} s_i \Big(\bigwedge_{j\neq i} (\bar\partial s_j) \Big) \wedge \omega(w) = K(w,z),\end{aligned}$$

where the factor $(-1)^{i-1}$ arises since the terms $(\bar\partial s_j) \wedge dw_j$ of total degree 2 commute, while the term dw_i anticommutes, with terms of degree 1.

Now we analyze (2-8) with the aid of the forms

$$\beta = \bar\partial\partial |w|^2 = \bar\partial \sum_{i=1}^n (dw_i)\bar w_i = d\bar w \cdot dw = \sum_{i=1}^n d\bar w_i \wedge dw_i, \quad \mu = dw \cdot d\bar z = \sum_{i=1}^n dw_i \wedge d\bar z_i,$$

where δ is the interior product, given by

$$\delta\alpha = \alpha \lrcorner (w\cdot dw) = \alpha \lrcorner \Big(\sum_{k=1}^n w_k \, dw_k \Big).$$

We have both

$$\delta\beta = (d\overline{w}\cdot dw) \lrcorner (w\cdot dw) = \left(\sum_{i=1}^n d\overline{w}_i \wedge dw_i\right) \lrcorner \left(\sum_{j=1}^n w_j\, dw_j\right) = -\sum_{i=1}^n w_i\, d\overline{w}_i = -w\cdot d\overline{w}$$

and

$$\delta\mu = (dw\cdot d\bar{z}) \lrcorner (w\cdot dw) = \left(\sum_{i=1}^n dw_i \wedge d\bar{z}_i\right) \lrcorner \left(\sum_{j=1}^n w_j\, dw_j\right) = \sum_{i=1}^n w_i\, d\bar{z}_i = w\cdot d\bar{z}.$$

Now we compute, using

$$s(w,z)\cdot dw \equiv (1-w\bar{z})(\overline{w}\cdot dw) - (1-|w|^2)(\bar{z}\cdot dw), \tag{2-9}$$

that

$$\begin{aligned}\lambda \equiv \bar{\partial}(s\cdot dw) &= -(w\cdot d\bar{z})\wedge(\overline{w}\cdot dw) + (1-|w|^2)(dw\cdot d\bar{z}) + (1-w\bar{z})(d\overline{w}\cdot dw) + (w\cdot d\overline{w})\wedge(\bar{z}\cdot dw)\\ &= (\overline{w}\cdot dw)\wedge\delta\mu + (\bar{z}\cdot dw)\wedge\delta\beta + (1-|w|^2)\mu + (1-w\bar{z})\beta.\end{aligned}$$

Consider the form $(s\cdot dw)\wedge\lambda^{n-1}$. Since λ has degree two, the remark on page 510 shows that the power λ^{n-1} can be expanded by the binomial theorem. Let $\lambda = A+B$, where

$$A = (\overline{w}\cdot dw)\wedge\delta\mu + (\bar{z}\cdot dw)\wedge\delta\beta, \quad B = (1-|w|^2)\mu + (1-w\bar{z})\beta.$$

We claim the formula

$$(s\cdot dw)\wedge\lambda^{n-1} = (s\cdot dw)\wedge B^{n-1} + (n-1)(s\cdot dw)\wedge A\wedge B^{n-2}. \tag{2-10}$$

To see this we expand the left-hand side using the binomial theorem to get

$$(s\cdot dw)\wedge(A+B)^{n-1} = (s\cdot dw)\wedge\left(A^{n-1} + (n-1)A^{n-2}\wedge B + \cdots + (n-1)A\wedge B^{n-2} + B^{n-1}\right).$$

We want this to equal

$$(s\cdot dw)\wedge B^{n-1} + (n-1)(s\cdot dw)\wedge A\wedge B^{n-2},$$

which will be the case if

$$(s\cdot dw)\wedge A^k = 0 \quad \text{for all } k\ge 2,$$

which in turn follows from $(s\cdot dw)\wedge A^2 = 0$. However, using

$$\delta\beta = -w\cdot d\overline{w} \text{ and } \delta\mu = w\cdot d\bar{z}, \tag{2-11}$$

we obtain

$$A = (\overline{w}\cdot dw)\wedge(w\cdot d\bar{z}) - (\bar{z}\cdot dw)\wedge(w\cdot d\overline{w}). \tag{2-12}$$

Hence we can write

$$A^2 = A_1 + A_2 + A_3 + A_4$$

with

$$\begin{aligned} A_1 &= \ (\overline{w}\cdot dw)\wedge(w\cdot d\bar{z})\wedge(\overline{w}\cdot dw)\wedge(w\cdot d\bar{z}), &\quad A_2 &= -(\overline{w}\cdot dw)\wedge(w\cdot d\bar{z})\wedge(\bar{z}\cdot dw)\wedge(w\cdot d\overline{w}),\\ A_3 &= -(\bar{z}\cdot dw)\wedge(w\cdot d\overline{w})\wedge(\overline{w}\cdot dw)\wedge(w\cdot d\bar{z}), &\quad A_4 &= \ (\bar{z}\cdot dw)\wedge(w\cdot d\overline{w})\wedge(\bar{z}\cdot dw)\wedge(w\cdot d\overline{w}).\end{aligned}$$

Now

$$A_1 = (\overline{w} \cdot dw) \wedge w \cdot d\bar{z} \wedge (\overline{w} \cdot dw) \wedge (w \cdot d\bar{z}) = -\big((\overline{w} \cdot dw) \wedge \overline{w} \cdot dw\big) \wedge \big((w \cdot d\bar{z}) \wedge (w \cdot d\bar{z})\big) = 0,$$

and similarly $A_4 = 0$. We also compute that

$$A_2 = -(\overline{w} \cdot dw) \wedge w \cdot d\bar{z} \wedge (\bar{z} \cdot dw) \wedge (w \cdot d\overline{w}) = -(\bar{z} \cdot dw) \wedge w \cdot d\overline{w} \wedge (\overline{w} \cdot dw) \wedge (w \cdot d\bar{z}) = A_3,$$

so that

$$A^2 = -2A_2.$$

Now we note, using (2-9), that

$$(s \cdot dw) \wedge A_2 = (1 - w\bar{z})\big((\overline{w} \cdot dw) \wedge A_2\big) - (1 - |w|^2)\big((\bar{z} \cdot dw) \wedge A_2\big)$$

vanishes, since $(\overline{w} \cdot dw) \wedge A_2$ contains two factors $\overline{w} \cdot dw$, and since $(\bar{z} \cdot dw) \wedge A_2$ contains two factors $\bar{z} \cdot dw$. Thus we have proved that

$$(s \cdot dw) \wedge A^2 = -(s \cdot dw) \wedge 2A_2 = 0.$$

This completes the proof of (2-10).

Now we continue by using (2-9), (2-11) and (2-12) to obtain

$$\begin{aligned}(s \cdot dw) \wedge A &= (1 - w\bar{z})\big((\overline{w} \cdot dw) \wedge A\big) - (1 - |w|^2)\big((\bar{z} \cdot dw) \wedge A\big)\\ &= -(1 - w\bar{z})\big((\overline{w} \cdot dw) \wedge (\bar{z} \cdot dw) \wedge (w \cdot d\overline{w})\big) - (1 - |w|^2)\big((\bar{z} \cdot dw) \wedge (\overline{w} \cdot dw) \wedge (w \cdot d\bar{z})\big)\\ &= -(\bar{z} \cdot dw) \wedge (\overline{w} \cdot dw) \wedge \big((1 - |w|^2)\,\delta\mu + (1 - w\bar{z})\,\delta\beta\big).\end{aligned}$$

We can now simplify the second term on the right side of (2-10) to obtain

$$\begin{aligned}(s \cdot dw) \wedge \lambda^{n-1} &= (s \cdot dw) \wedge B^{n-1} + (n-1)(s \cdot dw) \wedge A \wedge B^{n-2}\\ &= (s \cdot dw) \wedge B^{n-1} - (\bar{z} \cdot dw) \wedge (\overline{w} \cdot dw) \wedge \big((1 - |w|^2)\,\delta\mu + (1 - w\bar{z})\,\delta\beta\big) \wedge (n-1)\,B^{n-2}\\ &= (s \cdot dw) \wedge B^{n-1} - (\bar{z} \cdot dw) \wedge (\overline{w} \cdot dw) \wedge (\delta B^{n-1}).\end{aligned} \tag{2-13}$$

Now we note that

$$(\bar{z} \cdot dw) \wedge (\overline{w} \cdot dw) \wedge (\delta B^{n-1}) + \delta\big((\bar{z} \cdot dw) \wedge (\overline{w} \cdot dw)\big) \wedge B^{n-1} = \delta\big((\bar{z} \cdot dw) \wedge (\overline{w} \cdot dw) \wedge B^{n-1}\big) = 0,$$

since the left side has full degree in dw and the form $(\bar{z} \cdot dw) \wedge (\overline{w} \cdot dw)$ has even degree. As a consequence we obtain the formula

$$(s \cdot dw) \wedge \lambda^{n-1} = \big[(s \cdot dw) + \delta\big((\bar{z} \cdot dw) \wedge (\overline{w} \cdot dw)\big)\big] \wedge B^{n-1}.$$

Now the simple computation

$$\begin{aligned}&(s \cdot dw) + \delta\big((\bar{z} \cdot dw) \wedge (\overline{w} \cdot dw)\big)\\ &\qquad = (1 - w\bar{z})(\overline{w} \cdot dw) - (1 - |w|^2)(\bar{z} \cdot dw) + (\bar{z} \cdot w)(\overline{w} \cdot dw) - (\bar{z} \cdot dw)(\overline{w} \cdot w)\\ &\qquad = \overline{w} \cdot dw - \bar{z} \cdot dw\end{aligned}$$

shows that

$$(s \cdot dw) \wedge \lambda^{n-1} = (\overline{w} - \bar{z}) \cdot dw \wedge B^{n-1}. \tag{2-14}$$

Now the product rule in the remark on page 510 gives

$$B^{n-1} = \big((1-|w|^2)\mu + (1-w\bar{z})\beta\big)^{n-1} = \sum_{q=0}^{n-1} \binom{n-1}{q} \big((1-|w|^2)\mu\big)^q \wedge \big((1-w\bar{z})\beta\big)^{n-1-q},$$

and so taking the terms of bidegree $(0,q)$ in z in the formula (2-14) we obtain

$$\mathscr{C}_n^{0,q} = \binom{n-1}{q} \frac{(1-|w|^2)^q (1-w\bar{z})^{n-1-q}}{\triangle(w-z)^n} (\overline{w} - \bar{z}) \cdot dw \wedge \mu^q \wedge \beta^{n-1-q}. \tag{2-15}$$

Finally we note that this coincides with our formula

$$\mathscr{C}_n^{0,q}(w,z) = \Phi_n^q(w,z) \sum_{|J|=q} \sum_{k \notin J} (-1)^{\mu(k,J)} (\bar{z}_k - \overline{w}_k) d\bar{z}^J \wedge d\overline{w}^{(J \cup \{k\})^c} \wedge \omega_n(w). \tag{2-16}$$

This can be seen by writing

$$(\overline{w} - \bar{z}) \cdot dw = \sum_{k=1}^{n} (\overline{w}_k - \bar{z}_k) dw_k \text{ and } \mu^q = (dw \cdot d\bar{z})^q = \sum_J (-1)^\nu dw^J \wedge d\bar{z}^J,$$

with $|J| = q$, and then noting that in order to have a nonzero term in (2-15), we must have $k \notin J$ and the summand from $\beta^{n-1-q} = (dw \cdot d\overline{w})^{n-1-q}$ must be

$$(-1)^\omega dw^{(J \cup \{k\})^c} \wedge d\overline{w}^{(J \cup \{k\})^c}.$$

One then checks that the powers of -1 work out correctly. □

Remark. One might wonder if the special form of the right hand side of the recursion formula (2-10) can be put to good use in estimating the Besov–Sobolev norms of solutions to the $\bar{\partial}$-equation. This formula neatly exhibits a factoring of the solution operator that may be helpful, but we are unable to take advantage of this at this point, and must revert instead to the use of the explicit Charpentier formula (2-6) together with the exchange formula (5-7).

2.1. ***Ameliorated kernels.*** We now wish to define right inverses with improved behavior at the boundary. We consider the case when the right side f of the $\bar{\partial}$ equation is a $(p, q+1)$-form in $\mathbb{B}_n$.

As usual for a positive integer $s > n$ we will "project" the formula $\bar{\partial}\mathscr{C}_s^{p,q} f = f$ in $\mathbb{B}_s$ for a $\bar{\partial}$-closed form f in $\mathbb{B}_s$ to a formula $\bar{\partial}\mathscr{C}_{n,s}^{p,q} f = f$ in $\mathbb{B}_n$ for a $\bar{\partial}$-closed form f in $\mathbb{B}_n$. To accomplish this we define *ameliorated* operators $\mathscr{C}_{n,s}^{p,q}$ by

$$\mathscr{C}_{n,s}^{p,q} = \mathsf{R}_n \mathscr{C}_s^{p,q} \mathsf{E}_s,$$

where, for $n < s$, E_s (R_n) is the extension (restriction) operator that takes forms $\Omega = \sum \eta_{I,J} dw^I \wedge d\overline{w}^J$ in $\mathbb{B}_n$ ($\mathbb{B}_s$) and extends (restricts) them to $\mathbb{B}_s$ ($\mathbb{B}_n$) by

$$\mathsf{E}_s\Big(\sum \eta_{I,J} dw^I \wedge d\overline{w}^J\Big) \equiv \sum (\eta_{I,J} \circ R)\, dw^I \wedge d\overline{w}^J,$$

$$\mathsf{R}_n\Big(\sum \eta_{I,J} dw^I \wedge d\overline{w}^J\Big) \equiv \sum_{I,J \subset \{1,2,\ldots,n\}} (\eta_{I,J} \circ E)\, dw^I \wedge d\overline{w}^J.$$

Here R is the natural orthogonal projection from $\mathbb{C}^s$ to $\mathbb{C}^n$ and E is the natural embedding of $\mathbb{C}^n$ into $\mathbb{C}^s$. In other words, we extend a form by taking the coefficients to be constant in the extra variables, and we restrict a form by discarding all wedge products of differentials involving the extra variables and restricting the coefficients accordingly.

For $s > n$ we observe that the operator $\mathscr{C}_{n,s}^{p,q}$ has integral kernel

$$\mathscr{C}_{n,s}^{p,q}(w,z) \equiv \int_{\sqrt{1-|w|^2}\mathbb{B}_{s-n}} \mathscr{C}_s^{p,q}\big((w,w'),(z,0)\big)\,dV(w'), \qquad z,w\in\mathbb{B}_n, \tag{2-17}$$

where $\mathbb{B}_{s-n}$ denotes the unit ball in $\mathbb{C}^{s-n}$ with respect to the orthogonal decomposition $\mathbb{C}^s = \mathbb{C}^n \oplus \mathbb{C}^{s-n}$, and dV denotes Lebesgue measure. If $f(w)$ is a $\bar\partial$-closed form on $\mathbb{B}_n$ then $f(w,w') = f(w)$ is a $\bar\partial$-closed form on $\mathbb{B}_s$ and we have for $z\in\mathbb{B}_n$,

$$\begin{aligned} f(z) = f(z,0) &= \bar\partial \int_{\mathbb{B}_s} \mathscr{C}_s^{p,q}\big((w,w'),(z,0)\big) f(w)\,dV(w)\,dV(w') \\ &= \bar\partial \int_{\mathbb{B}_n}\bigg(\int_{\sqrt{1-|w|^2}\mathbb{B}_{s-n}} \mathscr{C}_s^{p,q}\big((w,w'),(z,0)\big)\,dV(w')\bigg) f(w)\,dV(w) = \bar\partial\int_{\mathbb{B}_n}\mathscr{C}_{n,s}^{p,q}(w,z) f(w)\,dV(w). \end{aligned}$$

We have proved the following:

Theorem 9. *For all $s > n$ and $\bar\partial$-closed forms f in $\mathbb{B}_n$, we have*

$$\bar\partial\,\mathscr{C}_{n,s}^{p,q} f = f \quad \text{in } \mathbb{B}_n.$$

We will use only the case $p = 0$ of this theorem and from now on we restrict our attention to this case. The operators $\mathscr{C}_{n,s}^{0,0}$ have been computed in [Ortega and Fàbrega 2000] and are given by

$$\mathscr{C}_{n,s}^{0,0} f(z) = \int_{\mathbb{B}_n} \sum_{j=0}^{n-1} c_{n,j,s} \frac{(1-|w|^2)^{s-n+j}(1-|z|^2)^j}{(1-\overline{w}z)^{s-n+j}(1-w\bar z)^j}\mathscr{C}_n^{0,0}(w,z)\wedge f(w), \tag{2-18}$$

where

$$\mathscr{C}_n^{0,0}(w,z) = c_0\frac{(1-w\bar z)^{n-1}}{\big(|1-w\bar z|^2-(1-|w|^2)(1-|z|^2)\big)^n}\sum_{j=1}^n(-1)^{j-1}(\overline{w}_j-\bar z_j)\bigwedge_{k\neq j} d\overline{w}_k \bigwedge_{\ell=1}^n dw_\ell.$$

A similar result holds for the operators $\mathscr{C}_{n,s}^{0,q}$. Define

$$\begin{aligned} \Phi_{n,s}^q(w,z) &= \Phi_n^q(w,z)\left(\frac{1-|w|^2}{1-\overline{w}z}\right)^{s-n}\sum_{j=0}^{n-q-1} c_{j,n,s}\left(\frac{(1-|w|^2)(1-|z|^2)}{|1-w\bar z|^2}\right)^j \\ &= \frac{(1-w\bar z)^{n-1-q}(1-|w|^2)^q}{\triangle(w,z)^n}\left(\frac{1-|w|^2}{1-\overline{w}z}\right)^{s-n}\sum_{j=0}^{n-q-1} c_{j,n,s}\left(\frac{(1-|w|^2)(1-|z|^2)}{|1-w\bar z|^2}\right)^j \\ &= \sum_{j=0}^{n-q-1} c_{j,n,s}\frac{(1-w\bar z)^{n-1-q-j}(1-|w|^2)^{s-n+q+j}(1-|z|^2)^j}{(1-\overline{w}z)^{s-n+j}\,\triangle(w,z)^n}. \end{aligned}$$

Note that the numerator and denominator are *balanced* in the sense that the sum of the exponents in the denominator minus the corresponding sum in the numerator (counting $\triangle(w,z)$ double) equals $s+n+j-(s+j-1) = n+1$, the exponent of the invariant measure of the ball $\mathbb{B}_n$.

Theorem 10. *Suppose that $s > n$ and $0 \le q \le n-1$. Then*

$$\begin{aligned}\mathscr{C}^{0,q}_{n,s}(w,z) &= \mathscr{C}^{0,q}_{n}(w,z)\left(\frac{1-|w|^2}{1-\overline{w}z}\right)^{s-n}\sum_{j=0}^{n-q-1} c_{j,n,s}\left(\frac{(1-|w|^2)(1-|z|^2)}{|1-w\bar{z}|^2}\right)^j \\ &= \Phi^q_{n,s}(w,z)\sum_{|J|=q}\sum_{k\notin J}(-1)^{\mu(k,J)}(\bar{z}_k-\overline{w}_k)\,d\bar{z}^J\wedge d\overline{w}^{(J\cup\{k\})^c}\wedge\omega_n(w).\end{aligned}$$

Proof. For $s > n$ recall that the kernels of the ameliorated operators $\mathscr{C}^{0,q}_{n,s}$ are given in (2-17). For ease of notation, we will set $k = s-n$, so we have $\mathbb{C}^s = \mathbb{C}^n \oplus \mathbb{C}^k$. Suppose that $0 \le q \le n-1$. Recall from (2-6) that

$$\begin{aligned}\mathscr{C}^{0,q}_{s}(w,z) &= (-1)^q\frac{(1-w\bar{z})^{s-q-1}(1-|w|^2)^q}{\triangle(w,z)^s}\sum_{\nu\in P^q_s}\operatorname{sgn}(\nu)(\overline{w}_{i_\nu}-\bar{z}_{i_\nu})\bigwedge_{j\in J_\nu}d\overline{w}_j\bigwedge_{l\in L_\nu}d\bar{z}_l\wedge\omega_s(w) \\ &= \sum_{\nu\in P^q_s}F^q_{s,i_\nu}(w,z)\bigwedge_{j\in J_\nu}d\overline{w}_j\bigwedge_{l\in L_\nu}d\bar{z}_l\wedge\omega_s(w),\end{aligned}$$

where

$$F^q_{s,i_\nu}(w,z) = \Phi^q_s(w,z)(\overline{w}_{i_\nu}-\bar{z}_{i_\nu}) = \frac{(1-w\bar{z})^{s-q-1}(1-|w|^2)^q}{\triangle(w,z)^s}(\overline{w}_{i_\nu}-\bar{z}_{i_\nu}).$$

To compute the ameliorations of these kernels, we need only focus on the functional coefficient $F^q_{s,i_\nu}(w,z)$ of the kernel. It is easy to see that the ameliorated kernel can only give a contribution in the variables when $1 \le i_\nu \le n$, since when $n+1 \le i_\nu \le s$ the functional kernel becomes radial in certain variables and thus reduces to zero upon integration.

Then for any $1 \le i \le n$ the corresponding functional coefficient $F^q_{s,i}(w,z)$ has amelioration $F^q_{n,s,i}(w,z)$ given by

$$\begin{aligned}F^q_{n,s,i}(w,z) &= \int_{\sqrt{1-|w|^2}\mathbb{B}_{s-n}}F^q_{s,i}((w,w'),(z,0))\,dV(w') \\ &= \int_{\sqrt{1-|w|^2}\mathbb{B}_k}\frac{(1-w\bar{z})^{s-q-1}(1-|w|^2-|w'|^2)^q(\bar{z}_i-\overline{w}_i)}{\triangle\big((w,w'),(z,0)\big)^s}\,dV(w') \\ &= (\bar{z}_i-\overline{w}_i)(1-w\bar{z})^{s-q-1}\int_{\sqrt{1-|w|^2}\mathbb{B}_k}\frac{(1-|w|^2-|w'|^2)^q}{\triangle\big((w,w'),(z,0)\big)^s}\,dV(w').\end{aligned}$$

Theorem 10 is a thus a consequence of the following elementary formula, which will find application in the next section as well:

$$\begin{aligned}&(1-w\bar{z})^{s-q-1}\int_{\sqrt{1-|w|^2}\mathbb{B}_{s-n}}\frac{(1-|w|^2-|w'|^2)^q}{\triangle\big((w,w'),(z,0)\big)^s}\,dV(w') \\ &\qquad = \frac{\pi^{s-n}}{(s-n)!}\Phi^q_n(w,z)\left(\frac{1-|w|^2}{1-\overline{w}z}\right)^{s-n}\sum_{j=0}^{n-q-1}c_{j,n,s}\left(\frac{(1-|w|^2)(1-|z|^2)}{|1-w\bar{z}|^2}\right)^j. \qquad (2\text{-}19)\end{aligned}$$

□

3. Integration by parts

We begin with an integration by parts formula involving a covariant derivative in [Ortega and Fàbrega 2000, Lemma 2.1, page 57] that reduces the singularity of the solution kernel on the diagonal at the expense of differentiating the form. However, in order to prepare for a generalization to higher order forms, we replace the covariant derivative with the notion of $\overline{\mathfrak{X}}_{z,w}$-derivative defined in (3-2) below.

Recall Charpentier's explicit solution $\mathscr{C}_n^{0,0}\eta$ to the $\bar\partial$ equation $\bar\partial\mathscr{C}_n^{0,0}\eta=\eta$ in the ball $\mathbb{B}_n$ when η is a $\bar\partial$-closed $(0,1)$-form with coefficients in $C(\overline{\mathbb{B}}_n)$: the kernel is given by

$$\mathscr{C}_n^{0,0}(w,z)=c_0\frac{(1-w\bar z)^{n-1}}{\triangle(w,z)^n}\sum_{j=1}^n(-1)^{j-1}(\overline{w}_j-\bar z_j)\bigwedge_{k\neq j}d\overline{w}_k\bigwedge_{\ell=1}^n dw_\ell,$$

for $(w,z)\in\mathbb{B}_n\times\mathbb{B}_n$, where

$$\triangle(w,z)=|1-w\bar z|^2-(1-|w|^2)(1-|z|^2).$$

Define the Cauchy operator $\mathscr{S}_n$ on $\partial\mathbb{B}_n\times\mathbb{B}_n$ with kernel

$$\mathscr{S}_n(\zeta,z)=c_1\frac{1}{(1-\bar\zeta z)^n}d\sigma(\zeta),\qquad(\zeta,z)\in\partial\mathbb{B}_n\times\mathbb{B}_n.$$

Let $\eta=\sum_{j=1}^n\eta_j\,d\overline{w}_j$ be a $(0,1)$-form with smooth coefficients. Define a vector field acting in the variable $w=(w_1,\dots,w_n)$ and parametrized by $z=(z_1,\dots,z_n)$ by

$$\overline{\mathfrak{X}}=\overline{\mathfrak{X}}_{z,w}=\sum_{j=1}^n(\overline{w}_j-\bar z_j)\frac{\partial}{\partial\overline{w}_j}. \tag{3-1}$$

It will usually be understood from the context what the acting variable w and the parameter variable z are in $\overline{\mathfrak{X}}_{z,w}$ and we will then omit the subscripts and simply write $\overline{\mathfrak{X}}$ for $\overline{\mathfrak{X}}_{z,w}$.

Definition 11. For $m\geq 0$, define the m-th order derivative $\overline{\mathfrak{X}}^m\eta$ of a $(0,1)$-form $\eta=\sum_{k=1}^n\eta_k(w)\,d\overline{w}_k$ to be the $(0,1)$-form obtained by componentwise differentiation holding monomials in $\overline{w}-\bar z$ fixed:

$$\overline{\mathfrak{X}}^m\eta(w)=\sum_{k=1}^n(\overline{\mathfrak{X}}^m\eta_k)(w)d\overline{w}_k=\sum_{k=1}^n\left(\sum_{|\alpha|=m}\overline{(w-z)^\alpha}\frac{\partial^m\eta_k}{\partial\overline{w}^\alpha}(w)\right)d\overline{w}_k. \tag{3-2}$$

Lemma 12 (compare [Ortega and Fàbrega 2000, Lemma 2.1]). *For all $m\geq 0$ and smooth $(0,1)$-forms $\eta=\sum_{k=1}^n\eta_k(w)\,d\overline{w}_k$, we have*

$$\begin{aligned}\mathscr{C}_n^{0,0}\eta(z)&\equiv\int_{\mathbb{B}_n}\mathscr{C}_n^{0,0}(w,z)\wedge\eta(w)\\&=\sum_{j=0}^{m-1}c_j\int_{\partial\mathbb{B}_n}\mathscr{S}_n(w,z)(\overline{\mathfrak{X}}^j\eta)[\overline{\mathfrak{X}}](w)\,d\sigma(w)+c_m\int_{\mathbb{B}_n}\mathscr{C}_n^{0,0}(w,z)\wedge\overline{\mathfrak{X}}^m\eta(w).\end{aligned} \tag{3-3}$$

Here the $(0,1)$-form $\overline{\mathfrak{X}}^j\eta$ acts on the vector field $\overline{\mathfrak{X}}$ in the usual way:

$$(\overline{\mathfrak{X}}^j\eta)[\overline{\mathfrak{X}}]=\left(\sum_{k=1}^n \overline{\mathfrak{X}}^j\eta_k(w)\,d\overline{w}_k\right)\left(\sum_{i=1}^n(\overline{w}_i-\bar z_i)\frac{\partial}{\partial\overline{w}_i}\right)=\sum_{k=1}^n(\overline{w}_k-\bar z_k)\overline{\mathfrak{X}}^j\eta_k(w).$$

We can also rewrite the final integral in (3-3) as

$$\int_{\mathbb{B}_n}\mathscr{C}_n^{0,0}(w,z)\wedge\overline{\mathfrak{X}}^m\eta(w)=\int_{\mathbb{B}_n}\Phi_n^0(w,z)(\overline{\mathfrak{X}}^m\eta)[\overline{\mathfrak{X}}](w)\,dV(w).$$

Lemma 12 is proved by following verbatim the proof of Lemma 2.1 of [Ortega and Fàbrega 2000].

We now extend Lemma 12 to $(0,q+1)$-forms. Let

$$\eta=\sum_{|I|=q+1}\eta_I(w)\,d\overline{w}^I$$

be a $(0,q+1)$-form with smooth coefficients. Given a $(0,q+1)$-form $\eta=\sum_{|I|=q+1}\eta_I\,d\overline{w}^I$ and an increasing sequence J of length $|J|=q$, we define the interior product $\eta\lrcorner\,d\overline{w}^J$ of η and $d\overline{w}^J$ by

$$\eta\lrcorner\,d\overline{w}^J=\sum_{|I|=q+1}\eta_I\,d\overline{w}^I\lrcorner\,d\overline{w}^J=\sum_{k\notin J}(-1)^{\mu(k,J)}\eta_{J\cup\{k\}}\,d\overline{w}_k,\tag{3-4}$$

since $d\overline{w}^I\lrcorner\,d\overline{w}^J=(-1)^{\mu(k,J)}d\overline{w}_k$ if $k\in I\setminus J$ is the $\mu(k,J)$-th index in I, and 0 otherwise. Recall the vector field $\overline{\mathfrak{X}}$ defined in (3-1). The key connection between it and $\eta\lrcorner\,d\overline{w}^J$ is

$$\begin{aligned}(\eta\lrcorner\,d\overline{w}^J)(\overline{\mathfrak{X}})&=\left(\sum_{k=1}^n(-1)^{\mu(k,J)}\eta_{J\cup\{k\}}d\overline{w}_k\right)\left(\sum_{j=1}^n(\overline{w}_j-\bar z_j)\frac{\partial}{\partial\overline{w}_j}\right)\\&=\sum_{k=1}^n(\overline{w}_k-\bar z_k)(-1)^{\mu(k,J)}\eta_{J\cup\{k\}}.\end{aligned}\tag{3-5}$$

We now define an m-th order derivative $\overline{\mathfrak{D}}^m\eta$ of a $(0,q+1)$-form η using the interior product. In the case $q=0$ we will have $\overline{\mathfrak{D}}^m\eta=(\overline{\mathfrak{X}}^m\eta)[\overline{\mathfrak{X}}]$ for a $(0,1)$-form η.

Remark. We are motivated by the fact that the Charpentier kernel $\mathscr{C}_n^{0,q}(w,z)$ takes $(0,q+1)$-forms in w to $(0,q)$-forms in z. Thus in order to express the solution operator $\mathscr{C}_n^{0,q}$ in terms of a volume integral rather than the integration of a form in w and z, our definition of $\overline{\mathfrak{D}}^m\eta$, even when $m=0$, must include an appropriate exchange of w-differentials for z-differentials.

Definition 13. Let $m\geq 0$. For a $(0,q+1)$-form $\eta=\sum_{|I|=q+1}\eta_I\,d\overline{w}^I$ in the variable w, define the $(0,q)$-form $\overline{\mathfrak{D}}^m\eta$ in the variable z by

$$\overline{\mathfrak{D}}^m\eta(w)=\sum_{|J|=q}\overline{\mathfrak{X}}^m(\eta\lrcorner\,d\overline{w}^J)[\overline{\mathfrak{X}}](w)d\bar z^J.$$

Again it is usually understood what the acting and parameter variables are in $\overline{\mathfrak{D}}^m$, but we will write $\overline{\mathfrak{D}}^m_{z,w}\eta(w)$ when this may not be the case. Note that for a $(0,q+1)$-form $\eta=\sum_{|I|=q+1}\eta_I\,d\overline{w}^I$, we have

$$\eta=\sum_{|J|=q}(\eta\lrcorner\,d\overline{w}^J)\wedge d\overline{w}^J,$$

and using (3-2) the preceding definition yields

$$\begin{aligned}\overline{\mathfrak{D}}^m \eta(w) &= \sum_{|J|=q} \overline{\mathfrak{X}}^m (\eta \lrcorner d\overline{w}^J)[\overline{\mathfrak{X}}](w)\, d\bar{z}^J \\ &= \sum_{|J|=q} \sum_{k=1}^{n} (\overline{w}_k - \bar{z}_k)(-1)^{\mu(k,J)} (\overline{\mathfrak{X}}^m \eta_{J\cup\{k\}})(w) d\bar{z}^J \\ &= \sum_{|J|=q} \sum_{k=1}^{n} (\overline{w}_k - \bar{z}_k)(-1)^{\mu(k,J)} \Bigg(\sum_{|\alpha|=m} (\overline{w-z})^\alpha \frac{\partial^m \eta_{J\cup\{k\}}}{\partial \overline{w}^\alpha}(w) \Bigg) d\bar{z}^J . \end{aligned} \tag{3-6}$$

Thus the effect of $\overline{\mathfrak{D}}^m$ on a basis element $\eta_I d\overline{w}^I$ is to replace a differential $d\overline{w}_k$ from $d\overline{w}^I$ $(I = J \cup \{k\})$ with the factor $(-1)^{\mu(k,J)}(\overline{w}_k - \bar{z}_k)$ (and this is accomplished by acting a $(0,1)$-form on $\overline{\mathfrak{X}}$), replace the remaining differential $d\overline{w}^J$ with $d\bar{z}^J$, and then to apply the differential operator $\overline{\mathfrak{X}}^m$ to the coefficient η_I. We will refer to the factor $(\overline{w}_k - \bar{z}_k)$ introduced above as a *rogue factor* since it is not associated with a derivative $\partial/\partial\overline{w}_k$ in the way that $(\overline{w-z})^\alpha$ is associated with $\partial^m/\partial\overline{w}^\alpha$. The point of this distinction will be explained in Section 7 on estimates for solution operators.

The following lemma expresses $\mathscr{C}_n^{0,q}\eta(z)$ in terms of integrals involving $\overline{\mathfrak{D}}^j\eta$ for $0 \le j \le m$. Note that the overall effect is to reduce the singularity of the kernel on the diagonal by m factors of $\sqrt{\triangle(w,z)}$, at the cost of increasing by m the number of derivatives hitting the form η. Recall from (2-5) that

$$\Phi_n^\ell(w,z) \equiv \frac{(1-w\bar{z})^{n-1-\ell}(1-|w|^2)^\ell}{\triangle(w,z)^n}.$$

We define the operator Φ_n^ℓ on forms η by

$$\Phi_n^\ell \eta(z) = \int_{\mathbb{B}_n} \Phi_n^\ell(w,z)\eta(w)\, dV(w).$$

Lemma 14. *Let $q \ge 0$. For all $m \ge 0$ we have*

$$\mathscr{C}_n^{0,q}\eta(z) = \sum_{k=0}^{m-1} c_k \mathscr{S}_n(\overline{\mathfrak{D}}^j \eta)(z) + \sum_{\ell=0}^{q} c_\ell \Phi_n^\ell(\overline{\mathfrak{D}}^m \eta)(z). \tag{3-7}$$

The proof is simply a reprise of that of Lemma 12 (see the proof of Lemma 2.1 of [Ortega and Fàbrega 2000]) complicated by the algebra that reduces matters to $(0,1)$-forms.

3.1. ***The radial derivative.*** Recall the radial derivative $R = \sum_{j=1}^n w_j \dfrac{\partial}{\partial w_j}$. The following lemma is essentially Lemma 2.2 on page 58 of [Ortega and Fàbrega 2000].

Lemma 15. *Let $b > -1$. For $\Psi \in C(\overline{\mathbb{B}}_n) \cap C^\infty(\mathbb{B}_n)$ we have*

$$\int_{\mathbb{B}_n} (1-|w|^2)^b \Psi(w)\, dV(w) = \int_{\mathbb{B}_n} (1-|w|^2)^{b+1} \Big(\frac{n+b+1}{b+1} I + \frac{1}{b+1} R \Big) \Psi(w)\, dV(w).$$

Remark. Typically this lemma is applied with

$$\Psi(w) = \frac{1}{(1-\overline{w}z)^s} \psi(w,z),$$

where z is a parameter in the ball $\mathbb{B}_n$ and

$$R\Psi(w) = \frac{1}{(1-\overline{w}z)^s} R\psi(w,z)$$

since $\frac{1}{(1-\overline{w}z)^s}$ is antiholomorphic in w.

We will also need to iterate Lemma 15, and for this purpose it is convenient to introduce for $m \geq 1$ the notation

$$R_b = R_{b,n} = \frac{n+b+1}{b+1} I + \frac{1}{b+1} R, \quad R_b^m = R_{b+m-1} R_{b+m-2} \dots R_b = \prod_{k=1}^{m} R_{b+m-k}.$$

Corollary 16. *Let $b > -1$. For $\Psi \in C(\overline{\mathbb{B}}_n) \cap C^\infty(\mathbb{B}_n)$ we have*

$$\int_{\mathbb{B}_n} (1-|w|^2)^b \Psi(w)\, dV(w) = \int_{\mathbb{B}_n} (1-|w|^2)^{b+m} R_b^m \Psi(w)\, dV(w).$$

Remark. The important point in Corollary 16 is that combinations of radial derivatives R and the identity I are played off against powers of $1-|w|^2$. It will sometimes be convenient to write this identity as

$$\int_{\mathbb{B}_n} F(w)\, dV(w) = \int_{\mathbb{B}_n} \mathcal{R}_b^m F(w)\, dV(w),$$

where

$$\mathcal{R}_b^m \equiv (1-|w|^2)^{b+m} R_b^m (1-|w|^2)^{-b}. \tag{3-8}$$

In this form the identity is valid for F such that $\Psi(w) = (1-|w|^2)^{-b} F(w)$ lies in $C(\overline{\mathbb{B}}_n) \cap C^\infty(\mathbb{B}_n)$.

3.2. ***Integration by parts in ameliorated kernels.*** We must now extend Lemma 14 and Corollary 16 to the ameliorated kernels $\mathcal{C}_{n,s}^{0,q}$ given by

$$\mathcal{C}_{n,s}^{0,q} = \mathsf{R}_n \mathcal{C}_s^{0,q} \mathsf{E}_s.$$

Since Corollary 16 already applies to very general functions $\Psi(w)$, we need only consider an extension of Lemma 14. The procedure for doing this is to apply Lemma 14 to $\mathcal{C}_s^{0,q}$ in s dimensions, and then integrate out the additional variables using (2-19).

Lemma 17. *Suppose that $s > n$ and $0 \leq q \leq n-1$. For all $m \geq 0$ and smooth $(0,q+1)$-forms η in $\overline{\mathbb{B}}_n$ we have the formula*

$$\mathcal{C}_{n,s}^{0,q} \eta(z) = \sum_{k=0}^{m-1} c'_{k,n,s} \mathcal{S}_{n,s}(\overline{\mathcal{D}}^k \eta)[\overline{\mathcal{X}}](z) + \sum_{\ell=0}^{q} c_{\ell,n,s} \Phi_{n,s}^\ell (\overline{\mathcal{D}}^m \eta)(z),$$

where the ameliorated operators $\mathcal{S}_{n,s}$ and $\Phi_{n,s}^\ell$ have kernels given by

$$\mathcal{S}_{n,s}(w,z) = c_{n,s} \frac{(1-|w|^2)^{s-n-1}}{(1-\overline{w}z)^s} = c_{n,s} \left(\frac{1-|w|^2}{1-\overline{w}z} \right)^{s-n-1} \frac{1}{(1-\overline{w}z)^{n+1}},$$

$$\Phi_{n,s}^\ell(w,z) = \Phi_n^\ell(w,z) \left(\frac{1-|w|^2}{1-\overline{w}z} \right)^{s-n} \sum_{j=0}^{n-\ell-1} c_{j,n,s} \left(\frac{(1-|w|^2)(1-|z|^2)}{|1-w\overline{z}|^2} \right)^j.$$

Proof. Recall that for a smooth $(0,q+1)$-form $\eta(w)=\sum_{|I|=q+1}\eta_I\,d\overline{w}^I$ in $\mathbb{B}_n$, the $(0,q)$-form $\overline{\mathfrak{D}}^m\mathsf{E}_s\eta$ is given by

$$\begin{aligned}\overline{\mathfrak{D}}^m\mathsf{E}_s\eta(w)&=\sum_{|J|=q}\overline{\mathfrak{D}}^m(\eta\lrcorner\,d\overline{w}^J)\,d\overline{z}^J=\sum_{|J|=q}\overline{\mathfrak{D}}^m\Bigl(\sum_{k\notin J}(-1)^{\mu(k,J)}\eta_{J\cup\{k\}}(w)\,d\overline{w}_k\Bigr)\,d\overline{z}^J\\&=\sum_{|J|=q}\overline{\mathfrak{D}}^m\Bigl(\sum_{k\notin J}(-1)^{\mu(k,J)}\eta_{J\cup\{k\}}(w)\,d\overline{w}_k\Bigr)\,d\overline{z}^J\\&=\sum_{|J|=q}\sum_{k\notin J}(-1)^{\mu(k,J)}\Bigl(\sum_{|\alpha|=m}(\overline{w_k-z_k})\overline{(w-z)^\alpha}\,\frac{\partial^m}{\partial\overline{w}^\alpha}\eta_{J\cup\{k\}}(w)\Bigr),\end{aligned}$$

where $J\cup\{k\}$ is a multi-index with entries in $\mathfrak{I}_n\equiv\{1,2,\dots,n\}$ since the coefficient η_I vanishes if I is not contained in $\mathfrak{I}_n$. Moreover, the multi-index α lies in $(\mathfrak{I}_n)^m$ since the coefficients η_I are constant in the variable $w'=(w_{n+1},\dots,w_s)$. Thus

$$\overline{\mathfrak{D}}^m_{(z,0),(w,w')}\mathsf{E}_s\eta=\overline{\mathfrak{D}}^m_{z,w}\eta=\overline{\mathfrak{D}}^m\eta,$$

and we compute

$$\begin{aligned}&\mathsf{R}_n\Phi_s^\ell\bigl(\overline{\mathfrak{D}}^m_{(z,0),(w,w')}\mathsf{E}_s\eta\bigr)(z)\\&\qquad=\Phi_s^\ell(\overline{\mathfrak{D}}^m\eta)((z,0))\\&\qquad=\sum_{|J|=q}\sum_{k\in\mathfrak{I}_n\setminus J}(-1)^{\mu(k,J)}\sum_{|\alpha|=m}\Phi_s^\ell\Bigl((\overline{w_k-z_k})\overline{(w-z)^\alpha}\,\frac{\partial^m}{\partial\overline{w}^\alpha}\eta_{J\cup\{k\}}((w,w'))\Bigr)((z,0)),\end{aligned}$$

where $J\cup\{k\}\subset\mathfrak{I}_n$ and $\alpha\in(\mathfrak{I}_n)^m$ and

$$\begin{aligned}&\Phi_s^\ell\Bigl((\overline{w_k-z_k})\overline{(w-z)^\alpha}\,\frac{\partial^m}{\partial\overline{w}^\alpha}\eta_{J\cup\{k\}}(w)\Bigr)((z,0))\\&\quad=\int_{\mathbb{B}_s}\frac{(1-w\overline{z})^{s-1-\ell}(1-|w|^2-|w'|^2)^\ell}{\triangle((w,w'),(z,0))^s}\,(\overline{w_k-z_k})\overline{(w-z)^\alpha}\,\frac{\partial^m}{\partial\overline{w}^\alpha}\eta_{J\cup\{k\}}(w)\,dV((w,w'))\\&\quad=\int_{\mathbb{B}_n}\Bigl\{(1-w\overline{z})^{s-\ell-1}\int_{\mathbb{B}_{s-n}}\frac{(1-|w|^2-|w'|^2)^\ell}{\triangle((w,w'),(z,0))^s}\,dV(w')\Bigr\}(\overline{w_k-z_k})\overline{(w-z)^\alpha}\,\frac{\partial^m}{\partial\overline{w}^\alpha}\eta_{J\cup\{k\}}(w)\,dV(w).\end{aligned}$$

By (2-19) the term in braces on the previous line equals

$$\frac{\pi^{s-n}}{(s-n)!}\Phi_n^\ell(w,z)\Bigl(\frac{1-|w|^2}{1-\overline{w}z}\Bigr)^{s-n}\sum_{j=0}^{n-\ell-1}c_{j,n,s}\Bigl(\frac{(1-|w|^2)(1-|z|^2)}{|1-w\overline{z}|^2}\Bigr)^j,$$

and now performing the sum $\sum_{|J|=q}\sum_{k\in\mathfrak{I}_n\setminus J}(-1)^{\mu(k,J)}\sum_{|\alpha|=m}$ yields

$$\mathsf{R}_n\Phi_s^\ell\Bigl(\overline{\mathfrak{D}}^m_{(z,0)}\mathsf{E}_s\eta\Bigr)(z)=\Phi_s^\ell(\overline{\mathfrak{D}}^m_z\,\eta)((z,0))=\Phi_{n,s}^\ell(\overline{\mathfrak{D}}^m_z\,\eta)(z).\tag{3-9}$$

An even easier calculation using formula (1) in 1.4.4 on page 14 of [Rudin 1980] shows that

$$\mathsf{R}_n\mathcal{S}_s(\mathsf{E}_s\overline{\mathfrak{D}}^k_z\eta)((z,0))=\mathcal{S}_s(\overline{\mathfrak{D}}^k_z\eta)((z,0))=\mathcal{S}_{n,s}(\overline{\mathfrak{D}}^k_z\eta)(z),\tag{3-10}$$

and now the conclusion of Lemma 17 follows from (3-9), (3-10), the definition $\mathscr{C}^{0,q}_{n,s} = \mathsf{R}_n \mathscr{C}^{0,q}_s \mathsf{E}_s$, and Lemma 14. □

4. The Koszul complex

Here we briefly review the algebra behind the Koszul complex as presented for example in [Lin 1994] in the finite-dimensional setting. A more detailed treatment in that setting can be found in Section 5.5.3 of [Sawyer 2009]. Fix h holomorphic as in (1-7). Now if $g = (g_j)_{j=1}^{\infty}$ satisfies $|g|^2 = \sum_{j=1}^{\infty} |g_j|^2 \geq \delta^2 > 0$, define

$$\Omega_0^1 = \frac{\bar{g}}{|g|^2} = \left(\frac{\bar{g}_j}{|g|^2}\right)_{j=1}^{\infty} = \left(\Omega_0^1(j)\right)_{j=1}^{\infty},$$

which we view as a 1-tensor (in $\ell^2 = \mathbb{C}^{\infty}$) of $(0,0)$-forms with components $\Omega_0^1(j) = \bar{g}_j/|g|^2$. Then $f = \Omega_0^1 h$ satisfies $\mathcal{M}_g f = f \cdot g = h$, but in general fails to be holomorphic. The Koszul complex provides a scheme which we now recall for solving a sequence of $\bar{\partial}$ equations that result in a correction term $\Lambda_g \Gamma_0^2$ that, when subtracted from f above, yields a holomorphic solution to the equality in (1-7). See below.

The 1-tensor of $(0,1)$-forms

$$\bar{\partial}\Omega_0 = \left(\bar{\partial}\frac{\bar{g}_j}{|g|^2}\right)_{j=1}^{\infty} = \left(\bar{\partial}\Omega_0^1(j)\right)_{j=1}^{\infty}$$

is given by

$$\bar{\partial}\Omega_0^1(j) = \bar{\partial}\frac{\bar{g}_j}{|g|^2} = \frac{|g|^2\bar{\partial}\bar{g}_j - \bar{g}_j\bar{\partial}|g|^2}{|g|^4} = \frac{1}{|g|^4}\sum_{k=1}^{\infty} g_k \overline{(g_k \partial g_j - g_j \partial g_k)},$$

and can be written as

$$\bar{\partial}\Omega_0^1 = \Lambda_g \Omega_1^2 \equiv \left(\sum_{k=1}^{\infty} \Omega_1^2(j,k) g_k\right)_{j=1}^{\infty},$$

where the antisymmetric 2-tensor Ω_1^2 of $(0,1)$-forms is given by

$$\Omega_1^2 = \left[\Omega_1^2(j,k)\right]_{j,k=1}^{\infty} = \left[\frac{\overline{g_k \partial g_j - g_j \partial g_k}}{|g|^4}\right]_{j,k=1}^{\infty}.$$

and $\Lambda_g \Omega_1^2$ denotes its contraction by the vector g in the final variable.

We can repeat this process and by induction we have

$$\bar{\partial}\Omega_q^{q+1} = \Lambda_g \Omega_{q+1}^{q+2}, \quad 0 \leq q \leq n, \tag{4-1}$$

where Ω_q^{q+1} is an alternating $(q+1)$-tensor of $(0,q)$-forms. Recall that h is holomorphic. When $q = n$ we have that $\Omega_n^{n+1} h$ is $\bar{\partial}$-closed and this allows us to solve a chain of $\bar{\partial}$ equations

$$\bar{\partial}\Gamma_{q-2}^{q} = \Omega_{q-1}^{q} h - \Lambda_g \Gamma_{q-1}^{q+1},$$

for alternating q-tensors Γ_{q-2}^{q} of $(0,q-2)$-forms, using the ameliorated Charpentier solution operators $\mathscr{C}_{n,s}^{0,q}$ defined in (2-17). (Note that our notation suppresses the dependence of Γ on h.) With the convention that $\Gamma_n^{n+2} \equiv 0$ we have

$$\bar{\partial}\big(\Omega_q^{q+1}h - \Lambda_g \Gamma_q^{q+2}\big) = 0, \quad 0 \le q \le n, \tag{4-2}$$

and

$$\bar{\partial}\Gamma_{q-1}^{q+1} = \Omega_q^{q+1}h - \Lambda_g \Gamma_q^{q+2}, \quad 1 \le q \le n.$$

Now

$$f \equiv \Omega_0^1 h - \Lambda_g \Gamma_0^2$$

is holomorphic by (4-2) with $q = 0$, and since Γ_0^2 is antisymmetric, we compute that $\Lambda_g \Gamma_0^2 \cdot g = \Gamma_0^2(g,g) = 0$ and

$$\mathcal{M}_g f = f \cdot g = \Omega_0^1 h \cdot g - \Lambda_g \Gamma_0^2 \cdot g = h - 0 = h.$$

Thus $f = (f_i)_{i=1}^{\infty}$ is a vector of holomorphic functions satisfying the equality in (1-7). The inequality in (1-7) is the subject of the remaining sections of the paper.

4.1. ***Wedge products and factorization of the Koszul complex.*** Here we record the remarkable factorization of the Koszul complex in [Andersson and Carlsson 2000]. To describe the factorization we introduce an exterior algebra structure on $\ell^2 = \mathbb{C}^\infty$. Let $\{e_1, e_2, \dots\}$ be the usual basis in $\mathbb{C}^\infty$, and for an increasing multiindex $I = (i_1, \dots, i_\ell)$ of integers in $\mathbb{N}$, define

$$e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_\ell},$$

where we use $\wedge$ to denote the wedge product in the exterior algebra $\Lambda^*(\mathbb{C}^\infty)$ of $\mathbb{C}^\infty$, as well as for the wedge product on forms in $\mathbb{C}^n$. Note that $\{e_I : |I| = r\}$ is a basis for the alternating r-tensors on $\mathbb{C}^\infty$.

If $f = \sum_{|I|=r} f_I e_I$ is an alternating r-tensor on $\mathbb{C}^\infty$ with values that are $(0,k)$-forms in $\mathbb{C}^n$, which may be viewed as a member of the exterior algebra of $\mathbb{C}^\infty \otimes \mathbb{C}^n$, and if $g = \sum_{|J|=s} g_J e_J$ is an alternating s-tensor on $\mathbb{C}^\infty$ with values that are $(0,\ell)$-forms in $\mathbb{C}^n$, then as in [Andersson and Carlsson 2000] we define the wedge product $f \wedge g$ in the exterior algebra of $\mathbb{C}^\infty \otimes \mathbb{C}^n$ to be the alternating $(r+s)$-tensor on $\mathbb{C}^\infty$ with values that are $(0,k+\ell)$-forms in $\mathbb{C}^n$ given by

$$\begin{aligned} f \wedge g &= \Big(\sum_{|I|=r} f_I e_I\Big) \wedge \Big(\sum_{|J|=s} g_J e_J\Big) = \sum_{\substack{|I|=r \\ |J|=s}} (f_I \wedge g_J)(e_I \wedge e_J) \\ &= \sum_{|K|=r+s} \Big(\pm \sum_{I+J=K} f_I \wedge g_J\Big) e_K. \end{aligned} \tag{4-3}$$

Note that we simply write the exterior product of an element from $\Lambda^*(\mathbb{C}^\infty)$ with an element from $\Lambda^*(\mathbb{C}^n)$ as juxtaposition, without an explicit wedge symbol. This should cause no confusion since the basis we use in $\Lambda^*(\mathbb{C}^\infty)$ is $\{e_i\}_{i=1}^{\infty}$, while the basis we use in $\Lambda^*(\mathbb{C}^n)$ is $\{dz_j, d\bar{z}_j\}_{j=1}^{n}$, quite different in both appearance and interpretation.

In terms of this notation we then have the following factorization in Theorem 3.1 of [Andersson and Carlsson 2000]:

$$\Omega_0^1 \wedge \bigwedge_{i=1}^{\ell} \widetilde{\Omega}_0^1 = \left(\sum_{k_0=1}^{\infty} \frac{\overline{g}_{k_0}}{|g|^2} e_{k_0} \right) \wedge \bigwedge_{i=1}^{\ell} \left(\sum_{k_i=1}^{\infty} \frac{\overline{\partial} \overline{g}_{k_i}}{|g|^2} e_{k_i} \right) = -\frac{1}{\ell+1} \Omega_\ell^{\ell+1}, \tag{4-4}$$

where

$$\Omega_0^1 = \left(\frac{\overline{g}_i}{|g|^2} \right)_{i=1}^{\infty} \quad \text{and} \quad \widetilde{\Omega}_0^1 = \left(\frac{\overline{\partial} \overline{g}_i}{|g|^2} \right)_{i=1}^{\infty}.$$

The factorization in [Andersson and Carlsson 2000] is proved in the finite-dimensional case, but this extends to the infinite-dimensional case by continuity. Since the ℓ^2 norm is quasimultiplicative on wedge products by Lemma 5.1 in that reference we have

$$\left| \Omega_\ell^{\ell+1} \right|^2 \le C_\ell \left| \Omega_0^1 \right|^2 \left| \widetilde{\Omega}_0^1 \right|^{2\ell}, \quad 0 \le \ell \le n, \tag{4-5}$$

where the constant C_ℓ depends only on the number of factors ℓ in the wedge product, and *not* on the underlying dimension of the vector space (which is infinite for $\ell^2 = \mathbb{C}^\infty$).

It will be useful in the next section to consider also tensor products

$$\widetilde{\Omega}_0^1 \otimes \widetilde{\Omega}_0^1 = \left(\sum_{i=1}^{\infty} \frac{\overline{\partial} \overline{g}_i}{|g|^2} e_i \right) \otimes \left(\sum_{j=1}^{\infty} \frac{\overline{\partial} \overline{g}_j}{|g|^2} e_j \right) = \sum_{i,j=1}^{\infty} \frac{\overline{\partial} \overline{g}_i \otimes \overline{\partial} \overline{g}_j}{|g|^4} e_i \otimes e_j, \tag{4-6}$$

and more generally $\mathscr{X}^\alpha \widetilde{\Omega}_0^1 \otimes \mathscr{X}^\beta \widetilde{\Omega}_0^1$, where $\mathscr{X}^m$ denotes the vector derivative defined in Definition 19 below. We will use the fact that the ℓ^2-norm is *multiplicative* on tensor products.

5. An almost invariant holomorphic derivative

We continue to consider ℓ^2-valued spaces. We refer the reader to [Arcozzi et al. 2006] for the definition of the Bergman tree $\mathscr{T}_n$ and the corresponding pairwise disjoint decomposition of the ball $\mathbb{B}_n$:

$$\mathbb{B}_n = \dot{\bigcup_{\alpha \in \mathscr{T}_n}} K_\alpha,$$

where the sets K_α are comparable to balls of radius one in the Bergman metric β on the ball $\mathbb{B}_n$:

$$\beta(z, w) = \frac{1}{2} \ln \frac{1 + |\varphi_z(w)|}{1 - |\varphi_z(w)|}$$

(see Proposition 1.21 in [Zhu 2005]). This decomposition gives an analogue in $\mathbb{B}_n$ of the standard decomposition of the upper half-plane $\mathbb{C}_+$ into dyadic squares whose distance from the boundary $\partial\mathbb{C}_+$ equals their side length. We also recall from [Arcozzi et al. 2006] the differential operator D_a which on the Bergman kube K_α, and provided $a \in K_\alpha$, is close to the invariant gradient $\widetilde{\nabla}$, and which has the additional property that $D_a^m f(z)$ is holomorphic for $m \ge 1$ and $z \in K_\alpha$ when f is holomorphic. For our purposes the powers $D_a^m f$, $m \ge 1$, are easier to work with than the corresponding powers $\widetilde{\nabla}^m f$, which fail to be holomorphic. It is shown in the same paper that D_a^m can be used to define an equivalent norm on the Besov space $B_p(\mathbb{B}_n) = B_p^0(\mathbb{B}_n)$, and it is a routine matter to extend this result to the Besov–Sobolev space $B_p^\sigma(\mathbb{B}_n)$ when $\sigma \geq 0$ and $m > 2(\frac{n}{p} - \sigma)$. The further extension to ℓ^2 -valued functions is also routine.

We define

$$\nabla_z = \left(\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}\right) \quad \text{and} \quad \overline{\nabla}_z = \left(\frac{\partial}{\partial \bar{z}_1}, \dots, \frac{\partial}{\partial \bar{z}_n}\right),$$

so that the usual Euclidean gradient is given by the pair $(\nabla_z, \overline{\nabla}_z)$. Fix $\alpha \in \mathcal{T}_n$ and let $a = c_\alpha$. Recall that the gradient with invariant length given by

$$\widetilde{\nabla} f(a) = (f \circ \varphi_a)'(0) = f'(a)\varphi_a'(0) = -f'(a)\big((1-|a|^2)P_a + (1-|a|^2)^{1/2} Q_a\big)$$

fails to be holomorphic in a. To rectify this, we define, as in [Arcozzi et al. 2006],

$$D_a f(z) = f'(z)\varphi_a'(0) = -f'(z)\big((1-|a|^2)P_a + (1-|a|^2)^{1/2} Q_a\big), \tag{5-1}$$

for $z \in \mathbb{B}_n$.

In order to deal with functions f on $\mathbb{B}_n$ that are not necessarily holomorphic, we use a notion of higher-order derivative D^m introduced in [Arcozzi et al. 2006], based on iterating D_a rather than $\widetilde{\nabla}$.

Definition 18. For $m \in \mathbb{N}$ and $f \in C^\infty(\mathbb{B}_n; \ell^2)$ smooth in $\mathbb{B}_n$ we define $\Theta^m f(a,z) = D_a^m f(z)$ for $a, z \in \mathbb{B}_n$, and then set

$$D^m f(z) = \Theta^m f(z,z) = D_z^m f(z), \quad z \in \mathbb{B}_n.$$

Note that in this definition, we iterate the operator D_z holding z fixed, and then evaluate the result at the same z. We obtain that for $f \in H(\mathbb{B}_n; \ell^2)$ (see [Arcozzi et al. 2006] and [Beatrous 1986]),

$$\|f\|_{B_{p,m}^\sigma(\mathbb{B}_n;\ell^2)} \approx \sum_{j=0}^{m-1} |\nabla^j f(0)| + \left(\int_{\mathbb{B}_n} \left|(1-|z|^2)^\sigma D^m f(z)\right|^p d\lambda_n(z)\right)^{1/p}.$$

We remind the reader that $|D_a^m f(z)| = \sqrt{\sum_{i=1}^\infty |D_a^m f_i(z)|^2}$ if $f = (f_i)_{i=1}^\infty$.

We will also need to know that the pointwise multipliers in $M_{B_p^\sigma(\mathbb{B}_n) \to B_p^\sigma(\mathbb{B}_n;\ell^2)}$ are bounded. Indeed, standard arguments show that

$$M_{B_p^\sigma(\mathbb{B}_n) \to B_p^\sigma(\mathbb{B}_n;\ell^2)} \subset H^\infty(\mathbb{B}_n; \ell^2) \cap B_p^\sigma(\mathbb{B}_n; \ell^2). \tag{5-2}$$

5.1. ***Real variable analogues of Besov–Sobolev spaces.*** In order to handle the operators arising from integration by parts formulas below, we will need yet more general equivalent norms on $B_{p,m}^\sigma(\mathbb{B}_n; \ell^2)$.

Definition 19. We denote by $\mathcal{X}^m$ the vector of all differential operators of the form $X_1 X_2 \dots X_m$ where each X_i is either $1-|z|^2$ times the identity operator I, the operator $\overline{D}$, or the operator $(1-|z|^2)R$. Just as in Definition 18, we calculate the products $X_1 X_2 \dots X_m$ by composing $\overline{D}_a$ and $(1-|a|^2)R$ and then setting $a = z$ at the end. Note that $\overline{D}_a$ and $(1-|a|^2)R$ commute since the first is an antiholomorphic derivative and the coefficient z in $R = z \cdot \nabla$ is holomorphic. Similarly we denote by $\mathcal{Y}^m$ the corresponding products of $(1-|z|^2)I$, D (instead of $\overline{D}$) and $(1-|z|^2)R$.

In the iterated derivative $\mathscr{X}^m$ we are differentiating only with the *antiholomorphic* derivative $\overline{D}$ or the *holomorphic* derivative R. When f is holomorphic, we thus have $\mathscr{X}^m f \sim \{(1-|z|^2)^m R^k f\}_{k=0}^m$. The reason we allow $1-|z|^2$ times the identity I to occur in $\mathscr{X}^m$ is that this produces a norm (as opposed to just a seminorm) without including the term $\sum_{k=0}^{m-1} |\nabla^k f(0)|$. We define the norm $\|\cdot\|_{B^\sigma_{p,m}(\mathbb{B}_n;\ell^2)}$ for *smooth* f on the ball $\mathbb{B}_n$ by

$$\|f\|_{B^\sigma_{p,m}(\mathbb{B}_n;\ell^2)} \equiv \left(\sum_{k=0}^m \int_{\mathbb{B}_n} \left|(1-|z|^2)^{m+\sigma} R^k f(z)\right|^p d\lambda_n(z)\right)^{1/p},$$

and note that provided $m+\sigma > \frac{n}{p}$, this gives an equivalent norm for the Besov–Sobolev space $B^\sigma_p(\mathbb{B}_n;\ell^2)$ of holomorphic functions on $\mathbb{B}_n$ (see [Beatrous 1986], for instance). These considerations motivate the following two definitions of a *real-variable* analogue of the norm $\|\cdot\|_{B^\sigma_{p,m}(\mathbb{B}_n;\ell^2)}$.

Definition 20. We define the *norms* $\|\cdot\|_{\Lambda^\sigma_{p,m}(\mathbb{B}_n;\ell^2)}$ and $\|\cdot\|_{\Phi^\sigma_{p,m}(\mathbb{B}_n;\ell^2)}$ for $f=(f_i)_{i=1}^\infty$ smooth on the ball $\mathbb{B}_n$ by

$$\begin{aligned}\|f\|_{\Lambda^\sigma_{p,m}(\mathbb{B}_n;\ell^2)} &\equiv \left(\int_{\mathbb{B}_n} \left|(1-|z|^2)^\sigma \mathscr{X}^m f(z)\right|^p d\lambda_n(z)\right)^{1/p},\\ \|f\|_{\Phi^\sigma_{p,m}(\mathbb{B}_n;\ell^2)} &\equiv \left(\int_{\mathbb{B}_n} \left|(1-|z|^2)^\sigma \mathscr{Y}^m f(z)\right|^p d\lambda_n(z)\right)^{1/p}.\end{aligned} \tag{5-3}$$

It is *not* true that either of the norms $\|\cdot\|_{\Lambda^\sigma_{p,m}(\mathbb{B}_n;\ell^2)}$ or $\|\cdot\|_{\Phi^\sigma_{p,m}(\mathbb{B}_n;\ell^2)}$ are independent of m for large m when acting on smooth functions. However, these norms are equivalent when restricted to holomorphic vector functions (see [Arcozzi et al. 2006] and [Beatrous 1986]):

Lemma 21. *Let $1<p<\infty$, $\sigma \ge 0$ and $m > 2(\frac{n}{p}-\sigma)$. If f is a holomorphic vector function, then*

$$\|f\|_{B^\sigma_{p,m}(\mathbb{B}_n;\ell^2)} \approx \|f\|_{\Lambda^\sigma_{p,m}(\mathbb{B}_n;\ell^2)} \approx \|f\|_{\Phi^\sigma_{p,m}(\mathbb{B}_n;\ell^2)}. \tag{5-4}$$

The norms $\|\cdot\|_{\Lambda^\sigma_{p,m}(\mathbb{B}_n;\ell^2)}$ arise in the integration by parts in iterated Charpentier kernels in Section 7, while the norms $\|\cdot\|_{\Phi^\sigma_{p,m}(\mathbb{B}_n;\ell^2)}$ are useful for estimating the holomorphic function g in the Koszul complex. For this latter purpose we will use the following multilinear inequality whose scalar version is, after translating notation, Theorem 3.5 in [Ortega and Fàbrega 2000].

Proposition 22. *Suppose that $1<p<\infty$, $0\le\sigma<\infty$, $M\ge 1$, $m>2(\frac{n}{p}-\sigma)$ and $\alpha=(\alpha_0,\ldots,\alpha_M)\in \mathbb{Z}_+^{M+1}$ with $|\alpha|=m$. For $g\in M_{B^\sigma_p(\mathbb{B}_n)\to B^\sigma_p(\mathbb{B}_n;\ell^2)}$ and $h\in B^\sigma_p(\mathbb{B}_n)$ we have*

$$\int_{\mathbb{B}_n} (1-|z|^2)^{p\sigma} \left|(\mathscr{Y}^{\alpha_1} g)(z)\right|^p \cdots \left|\left(\mathscr{Y}^{\alpha_M} g\right)(z)\right|^p \left|(\mathscr{Y}^{\alpha_0} h)(z)\right|^p d\lambda_n(z)$$
$$\le C_{n,M,\sigma,p} \|\mathbb{M}_g\|^{Mp}_{B^\sigma_p(\mathbb{B}_n)\to B^\sigma_p(\mathbb{B}_n;\ell^2)} \|h\|^p_{B^\sigma_p(\mathbb{B}_n)}.$$

Remark. The inequalities for $M=1$ in Proposition 22 actually characterize multipliers g in the sense that a function $g\in B^\sigma_p(\mathbb{B}_n;\ell^2)\cap H^\infty(\mathbb{B}_n;\ell^2)$ is in $M_{B^\sigma_p(\mathbb{B}_n)\to B^\sigma_p(\mathbb{B}_n;\ell^2)}$ *if and only if* the inequalities with $M=1$ in Proposition 22 hold. This follows from noting that each term in the Leibniz expansion of $\mathscr{Y}^m(gh)$ occurs on the left side of the display above with $M=1$.

Proposition 22 is proved by adapting the proof of Theorem 3.5 in [Ortega and Fàbrega 2000] to ℓ^2-valued functions. This argument uses the complex interpolation theorem of Beatrous [1986] and Ligocka [1987], which extends to Hilbert space valued functions with the same proof. In order to apply this extension we will need the following operator norm inequality.

If $\varphi \in M_{B_p^\sigma(\mathbb{B}_n)\to B_p^\sigma(\mathbb{B}_n;\ell^2)}$ and $f = \sum_{|I|=\kappa} f_I e_I \in B_p^\sigma(\mathbb{B}_n; \otimes^{\kappa-1}\ell^2)$, we define

$$\mathbb{M}_\varphi f = \varphi \otimes f = \varphi \otimes \Bigg(\sum_{|I|=\kappa-1} f_I e_I \Bigg) = \sum_{|I|=\kappa-1} (\varphi f_I) \otimes e_I,$$

where $I = (i_1, \dots, i_{\kappa-1}) \in \mathbb{N}^{\kappa-1}$ and $e_I = e_{i_1} \otimes \cdots \otimes e_{i_{\kappa-1}}$.

Lemma 23. *Suppose that $\sigma \geq 0$, $1 < p < \infty$ and $\kappa \geq 1$. There is a constant $C_{n,\sigma,p,\kappa}$ such that*

$$\|\mathbb{M}_g\|_{B_p^\sigma(\mathbb{B}_n;\otimes^{\kappa-1}\ell^2)\to B_p^\sigma(\mathbb{B}_n;\otimes^{\kappa}\ell^2)} \leq C_{n,\sigma,p,\kappa} \|\mathbb{M}_g\|_{B_p^\sigma(\mathbb{B}_n;\ell^2)\to B_p^\sigma(\mathbb{B}_n;\ell^2)}. \tag{5-5}$$

In the case $p = 2$ we have equality:

$$\|\mathbb{M}_\varphi\|_{B_2^\sigma(\mathbb{B}_n;\otimes^{\kappa-1}\ell^2)\to B_2^\sigma(\mathbb{B}_n;\otimes^{\kappa}\ell^2)} = \|\mathbb{M}_\varphi\|_{B_2^\sigma(\mathbb{B}_n)\to B_2^\sigma(\mathbb{B}_n;\ell^2)}. \tag{5-6}$$

The proof of Lemma 23 uses the well-known technique of extending bounded linear operators on L^p to ℓ^2-valued L^p with the same norm (see, for instance, page 451 in [Stein 1993]). It turns out that in order to prove (5-5) for $p \neq 2$ we will need the case $M = 1$ of Proposition 22. Fortunately, the case $M = 1$ does not require inequality (5-5), thus avoiding circularity. The proofs of Proposition 22 and Lemma 23 reduce to modifying existing arguments in the literature and the details can be found in the Electronic Supplement.

Three crucial inequalities. In order to establish appropriate inequalities for the Charpentier solution operators, we will need to control terms of the form

$$(\overline{w-z})^\alpha \frac{\partial^m}{\partial \overline{w}^\alpha} F(w), \quad \overline{D}^m_{(z)} \triangle(w,z), \quad \overline{D}^m_{(z)}\big((1-w\bar{z})^k\big) \quad \text{and} \quad R^m_{(z)}\big((1-\overline{w}z)^k\big)$$

inside the integral for T as given in the integration by parts formula in Lemma 14. Here we are using the subscript (z) in parentheses to indicate the variable being differentiated. This is to avoid confusion with the notation D_a introduced in (5-1). For $z, w \in \mathbb{B}_n$ and $m \in \mathbb{N}$, we have the crucial estimates

$$\left| (\overline{w-z})^\alpha \frac{\partial^m}{\partial \overline{w}^\alpha} F(w) \right| \leq C \left(\frac{\sqrt{\triangle(w,z)}}{1-|w|^2} \right)^m |\overline{D}^m F(w)|, \quad F \in H(\mathbb{B}_n;\ell^2),\ m = |\alpha|, \tag{5-7}$$

$$\begin{cases} \big|D_{(z)}\triangle(w,z)\big| \leq C\big((1-|z|^2)\triangle(w,z)^{1/2} + \triangle(w,z)\big), \\ \big|(1-|z|^2)R_{(z)}\triangle(w,z)\big| \leq C(1-|z|^2)\sqrt{\triangle(w,z)}, \end{cases} \tag{5-8}$$

$$\begin{cases} \big|D^m_{(z)}\big((1-\overline{w}z)^k\big)\big| \leq C|1-\overline{w}z|^k \left(\dfrac{1-|z|^2}{|1-\overline{w}z|} \right)^{m/2}, \\ \big|(1-|z|^2)^m R^m_{(z)}\big((1-\overline{w}z)^k\big)\big| \leq C|1-\overline{w}z|^k \left(\dfrac{1-|z|^2}{|1-\overline{w}z|} \right)^{m}. \end{cases} \tag{5-9}$$

Proof of (5-7). We view D_a as a differentiation operator in the variable w, so that

$$D_a = -\nabla_w\big((1-|a|^2)P_a + \sqrt{1-|a|^2}\,Q_a\big).$$

A basic calculation is then:

$$\begin{aligned}(1-\bar{a}z)\varphi_a(z)\cdot(D_a)^t &= \big(P_a(z-a)+\sqrt{1-|a|^2}\,Q_a(z-a)\big)\big((1-|a|^2)P_a\nabla_w + \sqrt{1-|a|^2}\,Q_a\nabla_w\big)\\ &= P_a(z-a)(1-|a|^2)P_a\nabla_w + \sqrt{1-|a|^2}\,Q_a(z-a)\sqrt{1-|a|^2}\,Q_a\nabla_w\\ &= (1-|a|^2)(z-a)\cdot\nabla_w.\end{aligned}$$

From this we conclude the inequality

$$\left|(z_i-a_i)\frac{\partial}{\partial w_i}F(w)\right| \le \big|(z-a)\cdot\nabla F(w)\big| \le \left|\frac{1-\bar{a}z}{1-|a|^2}\varphi_a(z)\right|\,|D_aF(w)| = \frac{\sqrt{\triangle(a,z)}}{1-|a|^2}\,|D_aF(w)|,$$

as well as its conjugate

$$\left|(\overline{z_i-a_i})\frac{\partial}{\partial \overline{w}_i}F(w)\right| \le \frac{\sqrt{\triangle(a,z)}}{1-|a|^2}|\overline{D}_aF(w)|.$$

Moreover, we can iterate this inequality to obtain

$$\left|(\overline{z-a})^\alpha\frac{\partial^m}{\partial \overline{w}^\alpha}F(w)\right| \le C\left(\frac{\sqrt{\triangle(a,z)}}{1-|a|^2}\right)^m\big|(\overline{D}_a)^mF(w)\big|,$$

for a multi-index of length m. With $a=w$ this becomes the first estimate (5-7). $\square$

Proof of (5-8). Recall from (5-1) that

$$D_af(z) = -\big((1-|a|^2)P_a\nabla f(z) + (1-|a|^2)^{1/2}Q_a\nabla f(z)\big).$$

We let $a=z$. By the unitary invariance of

$$\triangle(w,z) = |1-\overline{w}z|^2-(1-|z|^2)(1-|w|^2),$$

we may assume that $z=(|z|,0,\dots,0)$. Then we have

$$\begin{aligned}\frac{\partial}{\partial z_j}\triangle(w,z) &= \frac{\partial}{\partial z_j}\big((1-\overline{w}z)(1-\bar{z}w)-(1-\bar{z}z)(1-|w|^2)\big)\\ &= -\overline{w}_j(1-\bar{z}w)+\bar{z}_j(1-|w|^2) = (\bar{z}_j-\overline{w}_j)+\overline{w}_j(\bar{z}w)-\bar{z}_j|w|^2\\ &= (\bar{z}_j-\overline{w}_j)(1-|z|^2)+\bar{z}_j|z|^2-\overline{w}_j|z|^2+\overline{w}_j(\bar{z}w)-\bar{z}_j|w|^2\\ &= (\bar{z}_j-\overline{w}_j)(1-|z|^2)+\bar{z}_j\big(|z|^2-|w|^2\big)+\overline{w}_j\big(\bar{z}(w-z)\big).\end{aligned}$$

Now $Q_z\nabla f = (0,\partial f/\partial z_2,\dots,\partial f/\partial z_n)$, and thus a typical term in $Q_z\nabla\triangle$ is $\dfrac{\partial}{\partial z_j}\triangle(w,z)$ with $j\ge 2$. From $z=(|z|,0,\dots,0)$ and $j\ge 2$ we have $z_j=0$ and so

$$\frac{\partial}{\partial z_j}\triangle(w,z) = (\bar{z}_j-\overline{w}_j)(1-|z|^2)-(\bar{z}_j-\overline{w}_j)\bar{z}(w-z), \qquad j\ge 2.$$

Now (2-1) implies

$$\triangle(w,z) = (1-|z|^2)|w-z|^2+|\bar{z}(w-z)|^2, \tag{5-10}$$

which together with the above shows that

$$\begin{aligned}\sqrt{1-|z|^2}\,\left|Q_z\nabla\triangle(w,z)\right| &\leq C|z-w|(1-|z|^2)^{3/2}+C\sqrt{1-|z|^2}\,|z-w|\,|\bar{z}(w-z)|\\ &\leq C(1-|z|^2)\triangle(w,z)^{1/2}+C\triangle(w,z).\end{aligned} \tag{5-11}$$

As for $P_z\nabla D=(\partial f/\partial z_1,0,\dots,0)$ we use (5-10) to obtain

$$\begin{aligned}\left|P_z\nabla\triangle(w,z)\right| &= \left|(\bar{z}_1-\bar{w}_1)(1-|z|^2)+\bar{z}_1\left(|z|^2-|w|^2\right)+\bar{w}_1\bar{z}(w-z)\right|\\ &\leq |z-w|(1-|z|^2)+\left||z|^2-|w|^2\right|+|\bar{z}(w-z)|\leq C\sqrt{\triangle(w,z)}+2\big||z|-|w|\big|.\end{aligned}$$

However,

$$\begin{aligned}\triangle(w,z) &\geq (1-|w|\,|z|)^2-(1-|z|^2)(1-|w|^2)\\ &= 1-2|w|\,|z|+|w|^2|z|^2-\left(1-|z|^2-|w|^2+|z|^2|w|^2\right)\\ &= |z|^2+|w|^2-2|w|\,|z|=\left(|z|-|w|\right)^2,\end{aligned}$$

and so altogether we have the estimate

$$|P_z\nabla\triangle(w,z)|\leq C\sqrt{\triangle(w,z)}. \tag{5-12}$$

Combining (5-11) and (5-12) with the definition (5-1) completes the proof of the first line in (5-8). The second line in (5-8) follows from (5-12) since $R_{(z)}=P_z\nabla$. □

Proof of (5-9). We compute

$$\begin{aligned}D_{(z)}(1-\bar{w}z)^k &= k(1-\bar{w}z)^{k-1}D_{(z)}(1-\bar{w}z)\\ &= k(1-\bar{w}z)^{k-1}\big((1-|z|^2)P_z\nabla+\sqrt{1-|z|^2}\,Q_z\nabla\big)(1-\bar{w}z)\\ &= -k(1-\bar{w}z)^{k-1}\big((1-|z|^2)P_z\bar{w}+\sqrt{1-|z|^2}\,Q_z\bar{w}\big),\\ R_{(z)}(1-\bar{w}z)^k &= k(1-\bar{w}z)^{k-1}(-\bar{w}z).\end{aligned}$$

Since $|w|^2+|a|^2\leq 2$ we have

$$|Q_z\bar{w}|^2=|Q_z(\bar{w}-\bar{z})|^2\leq|\bar{w}-\bar{z}|^2=|w|^2+|z|^2-2\operatorname{Re}(w\bar{z})\leq 2\operatorname{Re}(1-w\bar{z})\leq 2|1-w\bar{z}|,$$

which yields

$$\left|D_{(z)}\big((1-\bar{w}z)^k\big)\right|\leq C|1-\bar{w}z|^k\,\frac{(1-|z|^2)+\sqrt{(1-|z|^2)|1-\bar{w}z|}}{|1-\bar{w}z|}\leq C|1-\bar{w}z|^k\sqrt{\frac{1-|z|^2}{|1-\bar{w}z|}}.$$

Iteration then yields (5-9). □

6. Schur's test

Here we characterize boundedness of the positive operators that arise as majorants of the solution operators below. The case $c=0$ of the following lemma is Theorem 2.10 in [Zhu 2005].

Lemma 24. *Let $a, b, c, t \in \mathbb{R}$. Then the operator*

$$T_{a,b,c} f(z) = \int_{\mathbb{B}_n} \frac{(1-|z|^2)^a (1-|w|^2)^b \left(\sqrt{\triangle(w,z)}\right)^c}{|1-w\bar{z}|^{n+1+a+b+c}} f(w)\, dV(w)$$

is bounded on $L^p\big(\mathbb{B}_n; (1-|w|^2)^t\, dV(w)\big)$ if and only if $c > -2n$ and

$$-pa < t+1 < p(b+1). \tag{6-1}$$

The proof of Lemma 24 is a straightforward application of the argument in Theorem 2.10 of [Zhu 2005] together with an automorphic change of variable. Details can be found in the Electronic Supplement.

Remark. We will also use the trivial consequence of Lemma 24 that the operator

$$T_{a,b,c,d} f(z) = \int_{\mathbb{B}_n} \frac{(1-|z|^2)^a (1-|w|^2)^b \sqrt{\triangle(w,z)}^{\,c}}{|1-w\bar{z}|^{n+1+a+b+c+d}} f(w)\, dV(w)$$

is bounded on $L^p(\mathbb{B}_n; (1-|w|^2)^t\, dV(w))$ if $c > -2n$, $d \le 0$ and (6-1) holds. This is simply because $|1-w\bar{z}| \le 2$.

7. Operator estimates

We must show that $f = \Omega_0^1 h - \Lambda_g \Gamma_0^2 \in B_p^\sigma(\mathbb{B}_n; \ell^2)$, where Γ_0^2 is an antisymmetric 2-tensor of $(0,0)$-forms that solves

$$\bar{\partial}\Gamma_0^2 = \Omega_1^2 h - \Lambda_g \Gamma_{,1}^3,$$

and inductively where Γ_q^{q+2} is an alternating $(q+2)$-tensor of $(0,q)$-forms that solves

$$\bar{\partial}\Gamma_q^{q+2} = \Omega_{q+1}^{q+2} h - \Lambda_g \Gamma_{q+1}^{q+3},$$

up to $q = n-1$ (since $\Gamma_n^{n+2} = 0$ and the $(0,n)$-form Ω_n^{n+1} is $\bar{\partial}$-closed). Using the Charpentier solution operators $\mathscr{C}_{n,s}^{0,q}$ on $(0, q+1)$-forms we can write

$$f = \mathscr{F}^0 + \mathscr{F}^1 + \cdots + \mathscr{F}^n,$$

with

$$\begin{aligned}
\mathscr{F}^0 &= \Omega_0^1 h - \Lambda_g \Gamma_0^2,\\
\mathscr{F}^1 &= \Omega_0^1 h - \Lambda_g \mathscr{C}_{n,s_1}^{0,0}\big(\Omega_1^2 h - \Lambda_g \Gamma_1^3\big),\\
\mathscr{F}^2 &= \Omega_0^1 h - \Lambda_g \mathscr{C}_{n,s_1}^{0,0}\big(\Omega_1^2 h - \Lambda_g \mathscr{C}_{n,s_2}^{0,1}(\Omega_2^3 h - \Lambda_g \Gamma_2^4)\big),\\
&\ \vdots\\
\mathscr{F}^n &= \Omega_0^1 h - \Lambda_g \mathscr{C}_{n,s_1}^{0,0} \Omega_1^2 h + \Lambda_g \mathscr{C}_{n,s_1}^{0,0} \Lambda_g \mathscr{C}_{n,s_2}^{0,1} \Omega_2^3 h - \Lambda_g \mathscr{C}_{n,s_1}^{0,0} \Lambda_g \mathscr{C}_{n,s_2}^{0,1} \Lambda_g \mathscr{C}_{n,s_3}^{0,2} \Omega_3^4 h - \cdots\\
&\qquad + (-1)^n \Lambda_g \mathscr{C}_{n,s_1}^{0,0} \ldots \Lambda_g \mathscr{C}_{n,s_n}^{0,n-1} \Omega_n^{n+1} h.
\end{aligned}$$

The goal is to establish

$$\|f\|_{B_p^\sigma(\mathbb{B}_n;\ell^2)} \le C_{n,\sigma,p,\delta}(g) \|h\|_{B_p^\sigma(\mathbb{B}_n)},$$

which we accomplish by showing that

$$\|\mathscr{F}^{\mu}\|_{B^{\sigma}_{p,m_1}(\mathbb{B}_n;\ell^2)} \leq C_{n,\sigma,p,\delta}(g)\|h\|_{\Lambda^{\sigma}_{p,m_\mu}(\mathbb{B}_n)}, \quad 0 \leq \mu \leq n, \tag{7-1}$$

for a choice of integers m_μ satisfying

$$\frac{n}{p} - \sigma < m_1 < m_2 < \cdots < m_\ell < \cdots < m_n.$$

Recall that we defined both of the norms $\|F\|_{B^{\sigma}_{p,m_\mu}(\mathbb{B}_n;\ell^2)}$ and $\|F\|_{\Lambda^{\sigma}_{p,m_\mu}(\mathbb{B}_n;\ell^2)}$ for smooth vector functions F in the ball $\mathbb{B}_n$.

Note on constants. We often indicate via subscripts, such as n, σ, p, δ, the important parameters on which a given constant C depends, especially when the constant appears in a basic inequality. However, at times in mid-argument, we will often revert to suppressing some or all of the subscripts in the interests of readability.

The norms $\|\cdot\|_{\Lambda^{\sigma}_{p,m}(\mathbb{B}_n;\ell^2)}$ in (5-3) above will now be used to estimate the composition of Charpentier solution operators in each function

$$\mathscr{F}^{\mu} = \Lambda_g \mathscr{C}^{0,0}_{n,s_1} \ldots \Lambda_g \mathscr{C}^{0,\mu-1}_{n,s_\mu} \Omega^{\mu+1}_{\mu} h$$

as follows. More precisely we will use the specialized variants of the seminorms given by

$$\|F\|^p_{\Lambda^{\sigma}_{p,m',m''}(\mathbb{B}_n;\ell^2)} \equiv \int_{\mathbb{B}_n} \left|(1-|z|^2)^{\sigma}\left((1-|z|^2)^{m'} R^{m'}\right) \overline{D}^{m''} F(z)\right|^p d\lambda_n(z),$$

where we take m'' derivatives in $\overline{D}$ followed by m' derivatives in the invariant radial operator $(1-|z|^2)R$. Recall from Definition 19 that $\mathscr{X}^m$ denotes the vector of all differential operators of the form $X_1 X_2 \ldots X_m$ where each X_i is either I, $\overline{D}$, or $(1-|z|^2)R$, and where by definition $1-|z|^2$ is held constant in composing operators. It will also be convenient at times to use the notation

$$\mathscr{R}^m \equiv (1-|z|^2)^m (R^k)^m_{k=0}, \tag{7-2}$$

which should cause no confusion with the related operators $\mathscr{R}^m_b$ introduced in (3-8). Note that $\mathscr{R}^m$ is simply $\mathscr{X}^m$ when none of the operators $\overline{D}$ appear. We will make extensive use the multilinear estimate in Proposition 22.

Let us fix our attention on the function $\mathscr{F}^{\mu} = \mathscr{F}^{\mu}_0$ and write

$$\begin{aligned}
\mathscr{F}^{\mu}_0 &= \Lambda_g \mathscr{C}^{0,0}_{n,s_1}\left(\Lambda_g \mathscr{C}^{0,1}_{n,s_2} \ldots \Lambda_g \mathscr{C}^{0,\mu-1}_{n,s_\mu} \Omega^{\mu+1}_{\mu} h\right) = \Lambda_g \mathscr{C}^{0,0}_{n,s_1}(\mathscr{F}^{\mu}_1),\\
\mathscr{F}^{\mu}_1 &= \Lambda_g \mathscr{C}^{0,1}_{n,s_2}\left(\Lambda_g \mathscr{C}^{0,2}_{n,s_3} \ldots \Lambda_g \mathscr{C}^{0,\mu-1}_{n,s_\mu} \Omega^{\mu+1}_{\mu} h\right) = \Lambda_g \mathscr{C}^{0,1}_{n,s_2}(\mathscr{F}^{\mu}_2),\\
\mathscr{F}^{\mu}_q &= \Lambda_g \mathscr{C}^{0,q}_{n,s_{q+1}}(\mathscr{F}^{\mu}_{q+1}),
\end{aligned}$$

and so on, where $\mathscr{F}^{\mu}_q$ is a $(0,q)$-form. We now perform the integration by parts in Lemma 17 in each iterated Charpentier operator $\mathscr{F}^{\mu}_q = \Lambda_g \mathscr{C}^{0,q}_{n,s_{q+1}}(\mathscr{F}^{\mu}_{q+1})$ to obtain

$$\begin{aligned}\mathscr{F}_q^\mu &= \Lambda_g \mathscr{C}_{n,s_{q+1}}^{0,q} \mathscr{F}_{q+1}^\mu \qquad (7\text{-}3)\\ &= \sum_{j=0}^{m'_{q+1}-1} c'_{j,n,s_{q+1}} \Lambda_g \mathscr{S}_{n,s_{q+1}} (\overline{\mathscr{D}}^j \mathscr{F}_{q+1}^\mu)(z) + \sum_{\ell=0}^{\mu} c_{\ell,n,s_{q+1}} \Lambda_g \Phi_{n,s_{q+1}}^\ell (\overline{\mathscr{D}}^{m'_{q+1}} \mathscr{F}_{q+1}^\mu)(z).\end{aligned}$$

Now we compose these formulas for $\mathscr{F}_k^\mu$ to obtain an expression for $\mathscr{F}^\mu$ that is a complicated sum of compositions of the individual operators in (7-3) above. For now we will concentrate on the main terms $\Lambda_g \Phi_{n,s_{k+1}}^\mu \left(\overline{\mathscr{D}}^{m'_{k+1}} \mathscr{F}_{k+1}^\mu\right)$ that arise in the second sum above when $\ell = \mu$. We will see that the same considerations apply to any of the other terms in (7-3). Recall from Lemma 17 that the "boundary" operators $\mathscr{S}_{n,s_{q+1}}$ are projections of operators on $\partial\mathbb{B}_{s_q}$ to the ball $\mathbb{B}_n$ and have (balanced) kernels even simpler than those of the operators $\Phi_{n,s_{q+1}}^\ell$. The composition of these main terms is

$$\begin{aligned}\left(\Lambda_g \Phi_{n,s_1}^\mu \overline{\mathscr{D}}^{m'_1}\right)\mathscr{F}_1^\mu &= \left(\Lambda_g \Phi_{n,s_1}^\mu \overline{\mathscr{D}}^{m'_1}\right)\left(\Lambda_g \Phi_{n,s_2}^\mu \overline{\mathscr{D}}^{m'_2}\right)\mathscr{F}_2^\mu\\ &= \left(\Lambda_g \Phi_{n,s_1}^\mu \overline{\mathscr{D}}^{m'_1}\right)\left(\Lambda_g \Phi_{n,s_2}^\mu \overline{\mathscr{D}}^{m'_2}\right)\dots\left(\Lambda_g \Phi_{n,s_\mu}^\mu \overline{\mathscr{D}}^{m'_\mu}\right)\Omega_\mu^{\mu+1} h. \qquad (7\text{-}4)\end{aligned}$$

At this point we would like to take absolute values inside all of these integrals and use the crucial inequalities (5-7)–(5-9) to obtain a composition of positive operators of the type considered in Lemma 24. However, there is a difficulty in using inequality (5-7) to estimate the derivative $\overline{\mathscr{D}}^m$ on $(0,q+1)$-forms η given by (3-6):

$$\overline{\mathscr{D}}^m \eta(z) = \sum_{|J|=q} \sum_{k\notin J} \sum_{|\alpha|=m} (-1)^{\mu(k,J)} (\overline{w_k - z_k}) \overline{(w-z)^\alpha} \frac{\partial^m}{\partial \overline{w}^\alpha} \eta_{J\cup\{k\}}(w).$$

The problem is that the factor $\overline{w_k - z_k}$ has no derivative $\partial/\partial\overline{w}_k$ naturally associated with it, as do the other factors in $\overline{(w-z)^\alpha}$. We refer to the factor $\overline{w_k - z_k}$ as a *rogue factor*, as it requires special treatment in order to apply (5-7). Note that we cannot simply estimate $\overline{w_k - z_k}$ by $|w-z|$ because this is much larger in general than the estimate $\sqrt{\triangle(w,z)}$ obtained in (5-7) (where the difference in size between $|w-z|$ and $\sqrt{\triangle(w,z)}$ is compensated by the difference in size between $\partial/\partial\overline{w}_k$ and $\overline{D}$).

We now describe how to circumvent this difficulty in the composition of operators in (7-4). Let us write each $\overline{\mathscr{D}}^{m'_{q+1}} \mathscr{F}_{q+1}^\mu$ as

$$\sum_{|J|=q} \sum_{k\notin J} \sum_{|\alpha|=m'_{q+1}} (-1)^{\mu(k,J)} (\overline{w_k - z_k}) \overline{(w-z)^\alpha} \frac{\partial^m}{\partial \overline{w}^\alpha} (\mathscr{F}_{q+1}^\mu)_{J\cup\{k\}}(w),$$

where $(\mathscr{F}_{q+1}^\mu)_{J\cup\{k\}}$ is the coefficient of the form $\mathscr{F}_{q+1}^\mu$ with differential $d\overline{w}^{J\cup\{k\}}$. We now replace each of these sums with just one of the summands, say

$$(\overline{w_k - z_k}) \overline{(w-z)^\alpha} \frac{\partial^m}{\partial \overline{w}^\alpha} (\mathscr{F}_{q+1}^\mu)_{J\cup\{k\}}(w). \qquad (7\text{-}5)$$

Here the factor $\overline{w_k - z_k}$ is a rogue factor, not associated with a corresponding derivative $\partial/\partial\overline{w}_k$. We will refer to k as the *rogue index* associated with the rogue factor when it is not convenient to explicitly display the variables.

The key fact in treating the rogue factor $\overline{w_k - z_k}$ is that its presence in (7-5) means that the coefficient $(\mathscr{F}^{\mu}_{q+1})_I$ of the form $\mathscr{F}^{\mu}_{q+1}$ that multiplies it *must* have k in the multi-index I. Since

$$\mathscr{F}^{\mu}_{q+1} = \Lambda_g \mathscr{C}^{0,q+1}_{n,s_{q+2}}(\mathscr{F}^{\mu}_{q+2}),$$

the form of the ameliorated Charpentier kernel $\mathscr{C}^{0,q+1}_{n,s_{q+2}}$ in Theorem 10 shows that the coefficients of $\mathscr{C}^{0,q+1}_{n,s_{q+2}}(w,z)$ that multiply the rogue factor *must* have the differential $d\bar{z}_k$ in them. In turn, this means that the differential $d\overline{w}_k$ must be *missing* in the coefficient of $\mathscr{C}^{0,q+1}_{n,s_{q+2}}(w,z)$, and hence finally that the coefficients $(\mathscr{F}^{\mu}_{q+2})_H$ with multi-index H that survive the wedge products in the integration *must* have $k \in H$. This observation can be repeated, and we now derive an important consequence.

Returning to (7-4), each summand in $\overline{\mathfrak{D}}^{m'_{q+1}} \mathscr{F}^{\mu}_{q+1}$ has a rogue factor with associated rogue index k_{q+1}. Thus the function in (7-4) is a sum of terms of the form

$$\begin{aligned}\big(\Lambda_g \Phi^{\mu}_{n,s_1}(\overline{w_{k_1}-z_{k_1}})\,\overline{\mathfrak{X}}^{m'_1}\big) \circ \big(\Lambda_g \Phi^{\mu}_{n,s_2}(\overline{w_{k_2}-z_{k_2}})\,\overline{\mathfrak{X}}^{m'_2}\big)_{I_1} \circ \cdots \circ \big(\Lambda_g \Phi^{\nu}_{n,s_\nu}(\overline{w_{k_\nu}-z_{k_\nu}})\,\overline{\mathfrak{X}}^{m'_\nu}\big)_{I_{\nu-1}} \\ \circ \cdots \circ \big(\Lambda_g \Phi^{\mu-1}_{n,s_\mu}(\overline{w_{k_\mu}-z_{k_\mu}})\,\overline{\mathfrak{X}}^{m'_\mu}\big)_{I_{\mu-1}} \circ \big(\Omega^{\mu+1}_{\mu} h\big)_{I_\mu},\end{aligned}$$

where the subscript I_ν on the form $\Lambda_g \Phi^{\nu}_{n,s_\nu}(\overline{w_{k_\nu}-z_{k_\nu}})\,\overline{\mathfrak{X}}^{m'_\nu}$ indicates that we are composing with the component of $\Lambda_g \Phi^{\nu}_{n,s_\nu}(\overline{w_{k_\nu}-z_{k_\nu}})\,\overline{\mathfrak{X}}^{m'_\nu}$ corresponding to the multi-index $I_{\nu-1}$, i.e., the component with the differential $d\bar{z}^{I_{\nu-1}}$. The notation will become exceedingly unwieldy if we attempt to identify the different variables associated with each of the iterated integrals, so we refrain from this in general. The considerations of the previous paragraph now show that we must have $\{k_1\} = I_1$, $\{k_2\} \cup I_1 = I_2$, and more generally

$$\{k_\nu\} \cup I_{\nu-1} = I_\nu, \quad 1 < \nu \le \mu.$$

In particular we see that the associated rogue indices $k_1, k_2, \ldots k_\mu$ are all distinct and that as sets

$$\{k_1, k_2, \ldots, k_\mu\} = I_\mu.$$

Denoting by ζ the variable in the final form $\Omega^{\mu+1}_{\mu} h$, we can thus write each rogue factor $\overline{w_{k_\nu}-z_{k_\nu}}$ as

$$\overline{w_{k_\nu}-z_{k_\nu}} = (\overline{w_{k_\nu}-\zeta_{k_\nu}})(\overline{z_{k_\nu}-\zeta_{k_\nu}}),$$

and since $k_\nu \in I_\mu$, there is a factor of the form $(\partial/\partial\bar{\zeta}_{k_\nu})(\partial^{|\beta|} g_i/\partial\bar{\zeta}^{\beta})$ in each summand of the component $(\Omega^{\mu+1}_{\mu} h)_{I_\mu}$ of $\Omega^{\mu+1}_{\mu} h$. So we are able to associate the rogue factor $\overline{w_{k_\nu}-z_{k_\nu}}$ with derivatives of g as follows:

$$\left((\overline{w_{k_\nu}-z_{k_\nu}})\frac{\partial}{\partial\bar{\zeta}_{k_\nu}}\right)\frac{\partial^{|\beta|} g_i}{\partial\bar{\zeta}^{\beta}} - \left((\overline{z_{k_\nu}-\zeta_{k_\nu}})\frac{\partial}{\partial\bar{\zeta}_{k_\nu}}\right)\frac{\partial^{|\gamma|} g_j}{\partial\bar{\zeta}^{\gamma}}. \tag{7-6}$$

Thus it is indeed possible to

(1) apply the radial integration by parts in Corollary 16,

(2) then take absolute values and ℓ^2-norms inside all the integrals,

(3) and then apply the crucial inequalities (5-7)–(5-9).

One of the difficulties remaining after this is that we are now left with additional factors of the form

$$\frac{\sqrt{\triangle(w,\zeta)}}{1-|w|^2} \quad \text{and} \quad \frac{\sqrt{\triangle(z,\zeta)}}{1-|z|^2},$$

resulting from an application of (5-7) to the derivatives in (7-6). These factors are still rogue in the sense that the variable pairs occurring in them, namely (w,ζ) and (z,ζ), do not consist of consecutive variables in the iterated integrals of (7-4). This is rectified by using the fact that

$$d(w,z) = \sqrt{\triangle(w,z)}$$

is a quasimetric, which in turn follows from the identity

$$\sqrt{\triangle(w,z)} = |1-w\bar{z}|\,|\varphi_z(w)| = \delta(w,z)^2\rho(w,z),$$

where $\rho(w,z) = |\varphi_z(w)|$ is the invariant pseudohyperbolic metric on the ball (Corollary 1.22 in [Zhu 2005]) and where $\delta(w,z) = |1-w\bar{z}|^{1/2}$ satisfies the triangle inequality on the ball (Proposition 5.1.2 in [Rudin 1980]). Using the quasisubadditivity of $d(w,z)$ we can, with some care, redistribute appropriate factors back to the iterated integrals where they can be favorably estimated using Lemma 24. It is simplest to illustrate this procedure in specific cases, so we defer further discussion of this point until we treat in detail the cases $\mu = 0, 1, 2$ below. We again emphasize that all these observations regarding rogue factors in (7-4) apply equally well to the rogue factors in the other terms $\Phi^{\ell}_{n,s_{q+1}}(\overline{\mathcal{D}}^{m'_q}\mathcal{F}^{\mu}_{q+1})(z)$ in (7-3), as well as to the boundary terms $\mathcal{S}_{n,s_{q+1}}(\overline{\mathcal{D}}^{j}\mathcal{F}^{\mu}_{q+1})(z)$ in (7-3).

The other difficulty remaining is that in order to obtain a favorable estimate using Lemma 24 for the iterated integrals resulting from the bullet items above, it is necessary to generate additional powers of $1-|z|^2$ (we are using z as a generic variable in the iterated integrals here). This is accomplished by applying the radial integrations by parts in Corollary 16 to the *previous* iterated integral. Of course such a possibility is impossible for the first of the iterated integrals, but there we are only applying the radial derivative R thanks to the fact that our candidate f from the Koszul complex is holomorphic. As a result, we see from (5-8) that $(1-|z|^2)R$, unlike D, generates positive powers of $1-|z|^2$ even when acting on $\triangle(w,z)$. This procedure is also best illustrated in specific cases and will be treated in the next subsection.

So ignoring these technical issues for the moment, the integrals that result from taking absolute values and ℓ^2-norms inside (7-4) are now estimated using Lemma 24 and the remark following it. Note that we only use *scalar-valued* Schur estimates since all the integrals to which that lemma and remark are applied have positive integrands. Here is the rough idea. Suppose that $\{T_1, T_2, \dots, T_\mu\}$ is a collection of Charpentier solution operators and that for a sequence of large integers

$$\{m'_1, m''_1, m'_2, , m''_2 \dots, m'_{\mu+1}, m''_{\mu+1}\},$$

we have the inequalities

$$\|T_j F\|_{\Lambda^{\sigma}_{p,m'_j,m''_j}(\mathbb{B}_n;\ell^2)} \le C_j \|F\|_{\Lambda^{\sigma}_{p,m'_{j+1},m''_{j+1}}(\mathbb{B}_n;\ell^2)}, \quad 1 \le j \le \ell+1, \tag{7-7}$$

for the class of smooth functions F that arise as TG for some Charpentier solution operator T and some smooth G. Then we can estimate $\|T_1 \circ T_2 \circ \cdots \circ T_\mu \Omega\|_{B^\sigma_{p,m}(\mathbb{B}_n;\ell^2)}$ by

$$\begin{aligned}\|T_1 \circ T_2 \circ \cdots \circ T_\ell \Omega\|_{\Lambda^\sigma_{p,m'_1,m''_1}(\mathbb{B}_n;\ell^2)} &\le C_1 \|T_2 \circ \cdots \circ T_\ell \Omega\|_{\Lambda^\sigma_{p,m'_2,m''_2}(\mathbb{B}_n;\ell^2)} \\ &\le C_1 C_2 \, \|T_3 \circ \cdots \circ T_\ell \Omega\|_{\Lambda^\sigma_{p,m'_3,m''_3}(\mathbb{B}_n;\ell^2)} \\ &\le C_1 C_2 \ldots C_\ell \, \|\Omega\|_{\Lambda^\sigma_{p,m'_{\ell+1},m''_{\ell+1}}(\mathbb{B}_n;\ell^2)}.\end{aligned}$$

Finally we will show that if Ω is one of the forms Ω^{q+1}_q in the Koszul complex, then

$$\|\Omega\|_{\Lambda^\sigma_{p,m'_{\ell+1},m''_{\ell+1}}(\mathbb{B}_n;\ell^2)} \le \|\Omega\|_{\Lambda^\sigma_{p,m'_{\ell+1}+m''_{\ell+1}}(\mathbb{B}_n;\ell^2)} \le C_{n,\sigma,p,\delta}(g)\|h\|_{B^\sigma_{p,m}(\mathbb{B}_n)},$$

and so altogether this proves that

$$\|f\|_{B^\sigma_p(\mathbb{B}_n;\ell^2)} \le C_{n,\sigma,p,\delta}(g)\|h\|_{B^\sigma_{p,m}(\mathbb{B}_n)}.$$

We now make some brief comments on how to obtain the inequalities in (7-7). Complete details will be given in the cases $\mu = 0, 1, 2$ below, and the general case $0 \le \mu \le n$ is no different from these three cases. We note that from (2-6) the kernel of $\mathscr{C}^{0,q}_n$ typically looks like a sum of terms

$$\frac{(1 - w\bar{z})^{n-1-q}(1 - |w|^2)^q}{\triangle(w, z)^n}(\bar{z}_j - \overline{w}_j) \tag{7-8}$$

times a wedge product of differentials in which the differential $d\overline{w}_j$ is missing. We again emphasize that the rogue factor $\bar{z}_j - \overline{w}_j$ cannot simply be estimated by $|\bar{z}_j - \overline{w}_j|$, as the formula (2-1) shows that

$$\sqrt{\triangle(w, z)} = \left|P_z(z - w) + \sqrt{1 - |z|^2}\, Q_z(z - w)\right|$$

can be much smaller than $|z - w|$. As we mentioned above, it is possible to exploit the fact that any surviving term in the form $\Omega^{\mu+1}_\mu$ must then involve the derivative $\partial/\partial\overline{w}_j$ hitting a component of g. This permits us to absorb part of the complex tangential component of $z-w$ into the almost invariant derivative D which is larger than the usual gradient in the complex tangential directions. This results in a good estimate for the rogue factor $(\bar{z}_j - \overline{w}_j)$ in (7-8) based on the smaller quantity $\sqrt{\triangle(w, z)}$. We have already integrated by parts to write (7-8) as (recall that the factors $\bar{z}_j - \overline{w}_j$ are already incorporated into $\overline{\mathfrak{D}}^m_z \eta(w)$)

$$\int_{\mathbb{B}_n} \frac{(1 - w\bar{z})^{n-1-q}(1 - |w|^2)^q}{\triangle(w, z)^n} \overline{\mathfrak{D}}^m \eta(w) \, dV(w),$$

plus boundary terms which we ignore for the moment. Then we use the three crucial inequalities (5-7), (5-8), and (5-9) to help show that the resulting iterated kernels can be factored (after accounting for all rogue factors $\bar{z}_j - \overline{w}_j$) into operators that satisfy the hypotheses of Lemma 24 or the subsequent remark.

Definition 25. The expression $\widehat{\Omega}^{\ell+1}_\ell$ denotes the form $\Omega^{\ell+1}_\ell$ but with every occurrence of the derivative $\partial/\partial\overline{w}_j$ replaced by the derivative $\overline{D}_j$.

We can rewrite (5-7) in the form

$$\left|(\bar{z}_j - \overline{w}_j)\overline{\mathfrak{D}}^m_{z,w}\Omega^{\ell+1}_\ell(w)\right| \le \left(\frac{\sqrt{\triangle(w, z)}}{1 - |w|^2}\right)^{m+1} \left|\overline{D}^m \widehat{\Omega}^{\ell+1}_\ell(w)\right|,$$

Recall that each summand of $\Omega_\ell^{\ell+1}$ includes a product of exactly ℓ distinct derivatives $\partial/\partial\overline{w}_j$ applied to components of g. Thus the entries of $\overline{D}^m\widehat{\Omega}_\ell^{\ell+1}(w)$ consist of $m+\ell$ derivatives distributed among components of g. Using the factorization of $\Omega_\ell^{\ell+1}$ in (4-4), we obtain the corresponding factorization for $\widehat{\Omega}_\ell^{\ell+1}$:

$$\Omega_0^1 \wedge \bigwedge_{i=1}^{\ell} \widehat{\Omega}_0^1 = -\frac{1}{\ell+1}\widehat{\Omega}_\ell^{\ell+1}, \tag{7-9}$$

where

$$\Omega_0^1 = \left(\frac{\bar{g}_i}{|g|^2}\right)_{i=1}^{\infty} \quad \text{and} \quad \widehat{\Omega}_0^1 = \left(\frac{\overline{D}\bar{g}_i}{|g|^2}\right)_{i=1}^{\infty}.$$

It is important for this purpose of using Lemma 24 and the subsequent remark to first apply the integration by parts Lemma 14 to temper the singularity due to negative powers of $\triangle(w,z)$, and to use the integration by parts Corollary 16 to infuse enough powers of $1-|w|^2$ for use in the subsequent iterated integral.

Finally it follows from Proposition 22 together with the factorization (4-4) that

$$\left\|(1-|z|^2)^\sigma \mathscr{X}^m \widehat{\Omega}_\mu^{\mu+1} h(z)\right\|_{L^p(\lambda_n;\ell^2)} \le C\,\|\mathbb{M}_g\|^{m+\mu}_{B_p^\sigma(\mathbb{B}_n)\to B_p^\sigma(\mathbb{B}_n;\ell^2)}\|h\|_{B_p^\sigma(\mathbb{B}_n)}. \tag{7-10}$$

We defer the proof of (7-10) until page 538 when further calculations are available.

Remark. At this point we observe from (7-1) that the exponent $m+\mu$ in (7-10) is at most m_n+n, and thus we may take $\kappa = m_n+n$. We leave it to the interested reader to estimate the size of m_n.

Taking into account all of the above, the conclusion is that with $\kappa = m_n+n$,

$$\|f\|_{B_p^\sigma(\mathbb{B}_n;\ell^2)} \le C_{n,\sigma,p,\delta}\,\|\mathbb{M}_g\|^{\kappa}_{B_p^\sigma(\mathbb{B}_n)\to B_p^\sigma(\mathbb{B}_n;\ell^2)}\|h\|_{B_p^\sigma(\mathbb{B}_n)}.$$

As the arguments described above are rather complicated we illustrate them by considering the three cases $\mu = 0,1,2$ in complete detail in the next subsection before proceeding to the general case.

7.1. *Estimates in special cases.* Here we prove the estimates (7-1) for $\mu = 0,1,2$. Recall that

$$\mathscr{F}^0 = \Omega_0^1 h, \quad \mathscr{F}^1 = \Lambda_g \mathscr{C}_{n,s_1}^{0,0}\Omega_1^2 h, \quad \mathscr{F}^2 = \Lambda_g \mathscr{C}_{n,s_1}^{0,0}\Lambda_g \mathscr{C}_{n,s_2}^{0,1}\Omega_2^3 h.$$

To obtain the estimate for $\mathscr{F}^0$ we use the multilinear inequality in Proposition 22.

In estimating $\mathscr{F}^1$ we confront for the first time a rogue factor $\bar{z}_{k-w_k}$ that we must associate with a derivative $\partial/\partial\overline{w}_k$ occurring in each surviving summand of the k-th component of the form Ω_1^2. After applying the integration by parts formula in 17 as in [Ortega and Fàbrega 2000], we use the crucial inequalities (5-7)–(5-9) and the Schur-type operator estimates in Lemma 24 with $c = 0$ to obtain the desired estimates. Finally we must also deal with the boundary terms in the integration by parts formula for ameliorated Charpentier kernels in Lemma 17. This requires using the radial derivative integration by parts formula in Corollary 16 as in [Ortega and Fàbrega 2000], and also requires dealing with the corresponding rogue factors.

The final trick in the proof arises in estimating $\mathscr{F}^2$. This time there are two iterated integrals each with a rogue factor. The problematic rogue factor $\overline{z_k-\zeta_k}$ occurs in the *first* of the iterated integrals since there is *no* derivative $\partial/\partial\overline{\zeta}_k$ hitting the second iterated integral with which to associate the rogue factor $\overline{z_k - \zeta_k}$. Instead we decompose the factor as $(\overline{w_k - z_k}) - (\overline{\zeta_k - w_k})$ and associate each of these summands with a derivative $\partial/\partial\overline{w}_k$ already occurring in Ω_2^3. Then we can apply the crucial inequality (5-7) and use the fact that $\sqrt{\triangle(w,z)}$ is a quasimetric to redistribute the estimates appropriately. As a result of this redistribution we are forced to use Lemma 24 with $c = \pm 1$ this time as well as $c = 0$. In applying the Schur-type estimates in Lemma 24 to the *second* iterated integral, we require a sufficiently large power of $1 - |w|^2$ to be carried over from the first iterated integral. To ensure this we again use the radial derivative integration by parts formula in Corollary 16.

The estimate (7-1) for general μ involves no new ideas. There are now μ rogue terms and we need to apply Lemma 24 with $c = 0, \pm 1, \dots, \pm(\mu - 1)$. With this noted the arguments needed are those used above in the cases $\mu = 0, 1, 2$.

The estimate for $\mathscr{F}^0$. We begin with the estimate

$$\|\mathscr{F}^0\|_{B^\sigma_{p,m}(\mathbb{B}_n;\ell^2)} = \|\Omega_0^1 h\|_{B^\sigma_{p,m}(\mathbb{B}_n;\ell^2)} \le C_{n,\sigma,p,\delta} \|\mathbb{M}_g\|^m_{B^\sigma_p(\mathbb{B}_n)\to B^\sigma_p(\mathbb{B}_n;\ell^2)} \|h\|_{B^\sigma_{p,m}(\mathbb{B}_n)},$$

for $m + \sigma > \frac{n}{p}$. However, for later use we prove instead the more general estimate with $\mathscr{X}$ in place of R, except that m must then be chosen twice as large:

$$\int_{\mathbb{B}_n} \left|(1 - |z|^2)^\sigma \mathscr{X}^m (\Omega_0^1 h)(z)\right|^p d\lambda_n(z) \le C_{n,\sigma,p,\delta} \|\mathbb{M}_g\|^{mp}_{B^\sigma_p(\mathbb{B}_n)\to B^\sigma_p(\mathbb{B}_n;\ell^2)} \|h\|^p_{B^\sigma_p(\mathbb{B}_n)}, \tag{7-11}$$

for $m > 2(\frac{n}{p} - \sigma)$. Recall that $\mathscr{X}^m$ is the differential operator of order m given in Definition 19 that is adapted to the complex geometry of the unit ball $\mathbb{B}_n$. It will be in estimating iterated Charpentier integrals below that the derivatives R^m and $\overline{\mathscr{D}}^m$ will arise from integration by parts in the previous iterated integral, and this will require estimates using $\mathscr{X}^m$.

By Leibniz's rule for $\mathscr{X}^m$ we have

$$\mathscr{X}^m(\Omega_0^1 h) = \sum_{k=0}^m c_k (\mathscr{X}^k \Omega_0^1)(\mathscr{X}^{m-k} h)$$

and

$$\mathscr{X}^k(\Omega_0^1) = \mathscr{X}^k\left(\frac{\overline{g}}{|g|^2}\right) = \sum_{\ell=0}^k c_\ell (\mathscr{X}^{k-\ell}\overline{g})(\mathscr{X}^\ell |g|^{-2}). \tag{7-12}$$

It suffices to prove

$$\int_{\mathbb{B}_n} \left|(1 - |z|^2)^\sigma \left(\sum_{k=0}^m \sum_{\ell=0}^k c_k c_\ell (\mathscr{X}^{k-\ell}\overline{g})(\mathscr{X}^\ell |g|^{-2})(\mathscr{X}^{m-k} h)\right)\right|^p d\lambda_n \le C_{n,\sigma,p,\delta} \|\mathbb{M}_g\|^{mp}_{B^\sigma_p(\mathbb{B}_n)\to B^\sigma_p(\mathbb{B}_n;\ell^2)} \|h\|^p_{B^\sigma_p(\mathbb{B}_n)},$$

and hence

$$\int_{\mathbb{B}_n} (1 - |z|^2)^{p\sigma} |\mathscr{X}^{k-\ell}\overline{g}|^p \, |\mathscr{X}|g|^{-2}|^p \, |\mathscr{X}^{m-k} h|^p d\lambda_n \le C_{n,\sigma,p,\delta} \|\mathbb{M}_g\|^{mp}_{B^\sigma_p(\mathbb{B}_n)\to B^\sigma_p(\mathbb{B}_n;\ell^2)} \|h\|^p_{B^\sigma_p(\mathbb{B}_n)}, \tag{7-13}$$

for each fixed $0 \le \ell \le k \le m$.

Now we can profitably estimate both $|\mathscr{X}^{m-k}h|$ and $|\mathscr{X}^{k-\ell}\overline{g}|$ as they are, but we must be more careful with $\left|\mathscr{X}|g|^{-2}\right|$. In the case $\ell=1$, we assume for convenience that $\mathscr{X}$ annihilates g_i (if not it will annihilate $\overline{g}_i$ unless $\mathscr{X}=I$, and the estimates are similar) and obtain

$$\left|\mathscr{X}|g|^{-2}\right|^2=\left|-|g|^{-4}\sum_{i=1}^{\infty}g_i\mathscr{X}\overline{g}_i\right|^2\leq|g|^{-8}\left(\sum_{i=1}^{\infty}|g_i|^2\right)\left(\sum_{i=1}^{\infty}|\mathscr{X}\overline{g}_i|^2\right)\leq|g|^{-6}\sum_{i=1}^{\infty}|\mathscr{X}\overline{g}_i|^2.$$

Similarly, when $\ell=2$,

$$\begin{aligned}\left|\mathscr{X}^2|g|^{-2}\right|^2&=\left|-|g|^{-4}\sum_{i=1}^{\infty}g_i\mathscr{X}^2\overline{g}_i+2|g|^{-6}\sum_{i\neq j}(g_i\mathscr{X}\overline{g}_i)(g_j\mathscr{X}\overline{g}_j)\right|^2\\&\leq 2|g|^{-6}\sum_{i=1}^{\infty}|\mathscr{X}^2\overline{g}_i|^2+4|g|^{-8}\left(\sum_{i=1}^{\infty}|\mathscr{X}\overline{g}_i|^2\right)^2,\end{aligned}$$

and the general case is

$$\begin{aligned}&\left|\mathscr{X}^{\ell}|g|^{-2}\right|^2\\&\leq C_\ell|g|^{-6}\sum_{i=1}^{\infty}|\mathscr{X}^{\ell}\overline{g}_i|^2+C_{\ell-1}|g|^{-8}\left(\sum_{i=1}^{\infty}|\mathscr{X}^{\ell-1}\overline{g}_i|^2\right)\left(\sum_{i=1}^{\infty}|\mathscr{X}\overline{g}_i|^2\right)+\cdots+C_0|g|^{-4-2\ell}\left(\sum_{i=1}^{\infty}|\mathscr{X}\overline{g}_i|^2\right)^{\ell}\\&=\sum_{\substack{1\leq\alpha_1\leq\alpha_2\leq\cdots\leq\alpha_M\\ \alpha_1+\alpha_2+\cdots+\alpha_M=\ell}}c_\alpha|g|^{-4-2\ell}\prod_{m=1}^{M}\left(\sum_{i=1}^{\infty}|\mathscr{X}^{\alpha_m}\overline{g}_i|^2\right).\end{aligned}\tag{7-14}$$

We can ignore the powers of $|g|$ since $|g|$ is bounded above and below by (5-2) and the hypotheses of Theorem 2. Fixing α we see that the left side of (7-13) is thus at most

$$C_{n,\sigma,p,\delta}\int_{\mathbb{B}_n}(1-|z|^2)^{p\sigma}\,|\mathscr{X}^{k-\ell}\overline{g}|^p\,|\mathscr{Y}^{m-k}h|^p\left(\prod_{j=1}^{M}|\mathscr{X}^{\alpha_j}\overline{g}|^p\right)d\lambda_n.$$

Since

$$|\mathscr{X}^{k-\ell}\overline{g}|^2=\sum_{i=1}^{\infty}|\mathscr{X}^{k-\ell}\overline{g}_i|^2$$

and $k-\ell$ could vanish (unlike the exponents α_ℓ, which are positive), we see that altogether after renumbering, it suffices to prove

$$\int_{\mathbb{B}_n}(1-|z|^2)^{p\sigma}\,|\mathscr{Y}^{\alpha_1}h|^p\,|\mathscr{Y}^{\alpha_2}g|^p\ldots|\mathscr{Y}^{\alpha_M}g|^p\,d\lambda_n\leq C_{n,\sigma,p,\delta}\,\|\mathbb{M}_g\|^{Mp}_{B_p^\sigma(\mathbb{B}_n)\to B_p^\sigma(\mathbb{B}_n;\ell^2)}\|h\|^p_{B_p^\sigma(\mathbb{B}_n)}\tag{7-15}$$

for each fixed $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_M)$ where $M\geq 2$, $|\alpha|=m$ and at most one of $\alpha_2,\ldots,\alpha_M$ is zero. We have used here that $|\overline{D}\overline{g}|=|Dg|$. Now Proposition 22 yields (7-15) for each $0\leq k\leq m$ and $|\alpha|=m-k$. Summing these estimates completes the proof of (7-11).

We can now prove the more general inequality (7-10). Indeed, using the factorization (4-4) of $\widehat{\Omega}_\mu^{\mu+1}$ together with the Leibniz formula gives

$$\begin{aligned}\mathscr{X}^m\big(\widehat{\Omega}_\mu^{\mu+1}h\big) = \mathscr{X}^m\big(\Omega_0^1 \wedge (\widehat{\Omega}_0^1)^\mu h\big) &= \sum_{\substack{\alpha\in\mathbb{Z}_+^{\mu+2}\\|\alpha|=m}} (\mathscr{X}^{\alpha_0}\Omega_0^1)\wedge \bigwedge_{j=1}^{\mu}(\mathscr{X}^{\alpha_j}\widehat{\Omega}_0^1)(\mathscr{X}^{\alpha_{\mu+1}}h)\\ &= \sum_{\substack{\alpha\in\mathbb{Z}_+^{\mu+2}\\|\alpha|=m}} \bigg((\mathscr{X}^{\alpha_0}\Omega_0^1)\wedge \bigwedge_{j=1}^{\mu}(\mathscr{X}^{\alpha_j+1}\Omega_0^1)\bigg)(\mathscr{X}^{\alpha_{\mu+1}}h),\end{aligned}$$

where we have used that $\widehat{\Omega}_0^1$ already has an $\mathscr{X}$ derivative in each summand, and so $\mathscr{X}^{\alpha_j}\widehat{\Omega}_0^1$ can be written as $\mathscr{X}^{\alpha_j+1}\Omega_0^1$. Now use (7-12) and (7-14) to see that $\big|\mathscr{X}^m(\widehat{\Omega}_\mu^{\mu+1}h)\big|$ is controlled by a tensor product of at most $m+\mu$ factors, and then apply Proposition 22 as above to complete the proof of (7-10).

The estimate for $\mathscr{F}^1$. The estimate in (7-1) with $\mu=1$ will follow from (7-10) and the estimate

$$\big\|(1-|z|^2)^\sigma \mathscr{Y}^{m_1}\big(\Lambda_g \mathscr{C}_{n,s}^{0,0}\Omega_1^2 h\big)\big\|_{L^p(\lambda_n)}^p \le C\int_{\mathbb{B}_n}\big|(1-|z|^2)^\sigma \mathscr{X}^{m_2}(\widehat{\Omega}_1^2 h)(z)\big|^p\,d\lambda_n(z), \tag{7-16}$$

where, as in Definition 25, we define $\widehat{\Omega}_1^2$ to be Ω_1^2 with ∂ replaced by D throughout:

$$\widehat{\Omega}_1^2 = \sum_{j,k=1}^{N}\frac{\overline{g_k Dg_j - g_j Dg_k}}{|g|^4}\,e_j\wedge e_k,$$

and where $Dh=\sum_{k=1}^n (D_k h)\,dz_k$ and D_k is the k-th component of D. We are using here the following observation regarding the interior product $\Omega_1^2 h \lrcorner\, d\overline{w}_k$:

$$\begin{gathered}\text{For each summand of } \Omega_1^2 h \lrcorner\, d\overline{w}_k \text{, there is a unique } 1\le i\le N\\ \text{such that } \partial g_i/\partial\overline{w}_k \text{ occurs as a factor in the summand.}\end{gathered} \tag{7-17}$$

We rewrite (7-16) as

$$\big\|(1-|z|^2)^\sigma \mathscr{R}^{m_1''}D^{m_1'}\big(\Lambda_g\mathscr{C}_{n,s}^{0,0}\Omega_1^2h\big)\big\|_{L^p(\lambda_n)}^p \le C\int_{\mathbb{B}_n}\big|(1-|z|^2)^\sigma\mathscr{R}^{m_2''}\overline{D}^{m_2'}\big(\widehat{\Omega}_1^2h\big)(z)\big|^p\,d\lambda_n(z), \tag{7-18}$$

where $\mathscr{R}^m=(1-|z|^2)^m(R^k)_{k=0}^m$ as in (7-2). As mentioned above, we only need to prove the case $m_1''=0$ since (7-1) only requires that we estimate $\|\mathscr{F}^1\|_{B_{p,m}^\sigma(\mathbb{B}_n)}$. However, when considering the estimate for $\mathscr{F}^2$ in (7-1) we will no longer have the luxury of using the norm $\|\cdot\|_{B_{p,m}^\sigma(\mathbb{B}_n)}$ in the second iterated integral occurring there, and so we will consider the more general case now in preparation for what comes later. As we will see however, it is necessary to choose m_1' sufficiently large in order to obtain (7-18). It is useful to recall that the operator $(1-|z|^2)R$ is "smaller" than $\overline{D}$ in the sense that

$$\begin{aligned}\overline{D} &= (1-|z|^2)P_z\overline{\nabla}+\sqrt{1-|z|^2}\,Q_z\overline{\nabla},\\ (1-|z|^2)R &= (1-|z|^2)P_z\nabla.\end{aligned}$$

To prove (7-18) we will ignore the contraction Λ_g since if derivatives hit g in the contraction, the estimates are similar if not easier. Note also that $|\Lambda_g F| \le |g|\,|F|$ for the contraction $\Lambda_g F$ of any tensor F.

We will also initially suppose that $m_1'' = 0$ and later take m_1'' sufficiently large. Now we apply Lemma 17 to $\mathscr{C}^{0,0}_{n,s}\Omega^2_1 h$ and obtain

$$\begin{aligned}\mathscr{C}^{0,0}_{n,s}\Omega^2_1 h(z) &= c_0\mathscr{C}^{0,0}_{n,s}\big(\overline{\mathscr{D}}^{m_2'}\Omega^2_1 h\big)(z) + \text{boundary terms}\\ &= \int_{\mathbb{B}_n}\Phi^0_{n,s}(w,z)\overline{\mathscr{D}}^{m_2'}(\Omega^2_1 h)\,dV(w) + \text{boundary terms}.\end{aligned} \tag{7-19}$$

A typical term above looks like

$$\int_{\mathbb{B}_n}\left(\frac{1-|w|^2}{1-\overline{w}z}\right)^{s-n}\frac{(1-w\overline{z})^{n-1}}{\triangle(w,z)^n}\overline{\mathscr{D}}^{m_2'}(\Omega^2_1 h)\,dV(w) \tag{7-20}$$

where we are discarding the sum of (balanced) factors

$$\left(\frac{(1-|w|^2)(1-|z|^2)}{|1-w\overline{z}|^2}\right)^j$$

for $1\le j\le n-1$ in Lemma 17, which turn out to only help with the estimates. This can be seen from (5-9) and its trivial counterpart

$$\left|D^m_{(z)}(1-|z|^2)^k\right| + \left|(1-|z|^2)^m R^m_{(z)}(1-|z|^2)^k\right| \le C(1-|z|^2)^k.$$

Recall from the general discussion above that in the integral (7-20) there are rogue factors $\overline{w_k - z_k}$ in $\overline{\mathscr{D}}^{m_2'}(\Omega^2_1 h)(w)$ that must be associated with a $\partial/\partial\overline{w}_k$ derivative that hits some factor of each summand in the k-th component $\Omega^2_1 \lrcorner\, d\overline{w}_k$ of $\Omega^2_1 \approx \overline{g_i\partial g_j - g_j\partial g_i}$. Thus we can apply (5-7) to the components of $\Omega^2_1 h(z)$ to obtain

$$\begin{aligned}\left|\overline{\mathscr{D}}^{m_2'}\Omega^2_1 h(z)\right| &\approx \left|\sum_{k=1}^n\sum_{|\alpha|=m_2'}(\overline{w_k - z_k})\overline{(w-z)^\alpha}\,\frac{\partial^{m_2'}}{\partial\overline{w}^\alpha}\big(\Omega^2_1 h \lrcorner\, d\overline{w}_k\big)\right|\\ &\le C\left(\frac{\sqrt{\triangle(w,z)}}{1-|w|^2}\right)^{m_2'+1}\left|\overline{D}^{m_2'}(\widehat{\Omega}^2_1 h)(w)\right|.\end{aligned} \tag{7-21}$$

Thus we get

$$\begin{aligned}&(1-|z|^2)^\sigma\left|D^{m_1'}\mathscr{C}^{0,0}_{n,s}\Omega^2_1 h(z)\right|\\ &\quad\le \int_{\mathbb{B}_n}(1-|z|^2)^\sigma\left|D^{m_1'}_{(z)}\left(\frac{(1-|w|^2)^{s-n}(1-w\overline{z})^{n-1}}{(1-\overline{w}z)^{s-n}\triangle(w,z)^n}\right)\right|\left(\frac{\sqrt{\triangle(w,z)}}{1-|w|^2}\right)^{m_2'+1}\left|\overline{D}^{m_2'}(\widehat{\Omega}^2_1 h)(w)\right|dV(w)\\ &\quad\equiv S^s_{m_1',m_2'}f(z),\end{aligned} \tag{7-22}$$

where

$$f(w) = (1-|w|^2)^\sigma\left|\overline{D}^{m_2'}\big(\widehat{\Omega}^2_1 h\big)(w)\right|. \tag{7-23}$$

Now we iterate the estimate (5-8),

$$\left|D_{(z)}\triangle(w,z)\right| \le C(1-|z|^2)\triangle(w,z)^{1/2} + \triangle(w,z),$$

to obtain

$$\left| D^{m_1'}_{(z)}\left(\frac{(1-|w|^2)^{s-n}(1-w\bar{z})^{n-1}}{(1-\overline{w}z)^{s-n}\triangle(w,z)^n} \right) \right| \leq \frac{(1-|z|^2)^{m_1'}(1-|w|^2)^{s-n}\triangle(w,z)^{m_1'/2}}{|1-w\bar{z}|^{s-2n+1}\triangle(w,z)^{n+m_1'}} + \cdots + \frac{(1-|w|^2)^{s-n}}{|1-w\bar{z}|^{s-2n+1}\triangle(w,z)^n} + OK, \quad (7\text{-}24)$$

where the terms in OK are obtained when some of the derivatives D hit the factor $(1-\overline{w}z)^{s-n}$ in the denominator or factors $D\triangle(w,z)$ already in the numerator. Leaving the OK terms for later, we combine all the estimates above to get that if we plug the first term on the right in (7-24) into the left side of (7-18), then the result is dominated by

$$\int_{\mathbb{B}_n} \frac{(1-|z|^2)^{m_1'+\sigma}(1-|w|^2)^{s-n-m_2'-1-\sigma}\triangle(w,z)^{m_1'+m_2'+1/2}}{|1-w\bar{z}|^{s-2n+1}\triangle(w,z)^{n+m_1'}} f(w)\,dV(w) = \int_{\mathbb{B}_n} \frac{(1-|z|^2)^{m_1'+\sigma}(1-|w|^2)^{s-n-1-m_2'-\sigma}}{|1-w\bar{z}|^{s-2n+1}} \sqrt{\triangle(w,z)}^{\,m_2'-m_1'-2n+1} f(w)\,dV(w).$$

Now for convenience choose $m_2' = m_1' + 2n - 1$ so that the factor of $\sqrt{\triangle(w,z)}$ disappears. We then get

$$(1-|z|^2)^{\sigma}\left| D^{m_1'}\mathscr{C}^{0,0}_{n,s}\Omega_1^2 h(z)\right| \leq \int_{\mathbb{B}_n} \frac{(1-|z|^2)^{m_1'+\sigma}(1-|w|^2)^{s-3n-m_1'-\sigma}}{|1-w\bar{z}|^{s-2n+1}} f(w)\,dV(w). \qquad (7\text{-}25)$$

Lemma 24 shows that the operator

$$T_{a,b,0}f(z) = \int_{\mathbb{B}_n} \frac{(1-|z|^2)^a(1-|w|^2)^b}{|1-w\bar{z}|^{n+1+a+b}} f(w)\,dV(w)$$

is bounded on $L^p\big(\mathbb{B}_n; (1-|w|^2)^t\,dV(w)\big)$ if and only if

$$-pa < t+1 < p(b+1).$$

We apply this lemma with $t=-n-1$, $a = m_1'+\sigma$ and $b = s-3n-m_1'-\sigma$. Note that the sums of the exponents in the numerator and denominator of (7-25) are equal if we write the integral in terms of invariant measure $d\lambda_n(w) = (1-|w|^2)^{-n-1}\,dV(w)$. We conclude that $S^s_{m_1',m_2'}$ is bounded on $L^p(d\lambda_n)$ provided T is, and that this latter happens if and only if

$$-p(m_1'+\sigma) < -n < p\left(s-3n+1-m_1'-\sigma\right).$$

This requires $m_1'+\sigma > \frac{n}{p}$ and $s > 3n-1+m_1'+\sigma-\frac{n}{p}$.

Remark. Suppose instead that we choose m_2' to be a positive integer satisfying $c = m_2'-m_1'-2n+1 > -2n$. Then we would be dealing with the operator $T_{a,b,c}$, where $a = m_1'+\sigma$ and

$$b = s-n-1-m_2'-\sigma = s-3n-c-m_1'-\sigma.$$

By Lemma 24, $T_{a,b,c}$ is bounded on $L^p(d\lambda_n)$ if and only if

$$-p(m_1'+\sigma) < -n < p(s-3n+1-c-m_1'-\sigma),$$

i.e., $m_1' + \sigma > \frac{n}{p}$ and $s > c + 3n - 1 + m_1' + \sigma - \frac{n}{p}$. Thus we can use *any* value of $c > -2n$ provided we choose $m_2' \geq m_1'$ and s large enough.

Now we turn to the second displayed term on the right side of (7-24), which leads to the operator $T_{a,b,0}$ with $a = \sigma,\ b = s - 3n - \sigma$. This time we will *not* in general have the required boundedness condition $\sigma > \frac{n}{p}$. It is for this reason that we must return to (7-18) and insist that m_1'' be chosen sufficiently large that $m_1'' + \sigma > \frac{n}{p}$. For convenience we let $m_1' = 0$ for now. Indeed, it follows from the second line in the crucial inequality (5-8) that the second displayed term on the right side of (7-24) is

$$\frac{(1-|z|^2)^{m_1''}(1-|w|^2)^{s-n}\triangle(w,z)^{m_1''/2}}{|1-w\bar{z}|^{s-2n+1}\triangle(w,z)^{n+m_1''}} + \text{better terms.}$$

Using this expression and choosing $m_2' = m_1'' + 2n - 1$ so that the term $\sqrt{\triangle(w,z)}$ disappears from the ensuing integral, we obtain the following analogue of (7-25):

$$(1-|z|^2)^{\sigma}(1-|z|^2)^{m_1''}\left|\mathcal{R}^{m_1''}\mathscr{C}_{n,s}^{0,0}\Omega_1^2 h(z)\right| \leq \int_{\mathbb{B}_n} \frac{(1-|z|^2)^{m_1''+\sigma}(1-|w|^2)^{s-3n-m_1''-\sigma}}{|1-w\bar{z}|^{s-2n+1}} f(w)\,dV(w).$$

The corresponding operator $T_{a,b,0}$ has $a = m_1'' + \sigma$ and $b = s - 3n - m_1'' - \sigma$ and is bounded on $L^p(\lambda_n)$ when $-p(m_1''+\sigma) < -n < p(s-3n+1-m_1''-\sigma)$. Thus there is no unnecessary restriction on σ if m_1'' and s are chosen appropriately large. Note that the only difference between this operator $T_{a,b,0}$ and the previous one is that m_1' has been replaced by m_1''.

The arguments above are easily modified to handle the general case of (7-18) provided $m_1'' + \sigma > \frac{n}{p}$ and s is chosen sufficiently large.

Now we return to consider the OK terms in (7-24). For this we use the inequality (5-9):

$$\left|D_{(z)}^{m}\{(1-\overline{w}z)^k\}\right| \leq C\,|1-\overline{w}z|^k\left(\frac{1-|z|^2}{|1-\overline{w}z|}\right)^{m/2}.$$

We ignore the derivative $(1-|z|^2)R$, since the second line in (5-9) shows that it satisfies a better estimate. We also write m_1 and m_2 in place of m_1' and m_2' now. As a result, one of the extremal OK terms in (7-24) is

$$\frac{(1-|z|^2)^{m_1/2}(1-|w|^2)^{s-n}}{|1-w\bar{z}|^{s-2n+1+(m_1/2)}\triangle(w,z)^n},$$

which when combined with the other estimates leads to the integral operator

$$\int_{\mathbb{B}_n} \frac{(1-|z|^2)^{m_1/2+\sigma}(1-|w|^2)^{s-n-1-m_2-\sigma}}{|1-w\bar{z}|^{s-2n+1+(m_1/2)}}\sqrt{\triangle(w,z)}^{\,m_2-2n-1} f(w)\,dV(w).$$

This is $T_{a,b,c}$ with $a = \frac{m_1}{2} + \sigma,\ b = s - n - 1 - m_2 - \sigma$, and $c = m_2 - 2n - 1$. This is bounded on $L^p(\lambda_n)$ provided $m_2 \geq 2$ and

$$-p\left(\frac{m_1}{2} + \sigma\right) < -n < p(s - n - m_2 - \sigma),$$

i.e., $\frac{m_1}{2} + \sigma > \frac{n}{p}$ and $s > n + m_2 + \sigma - \frac{n}{p}$. The intermediate OK terms are handled similarly. Note that the crux of the matter is that all of the positive operators have the form $T_{a,b,c}$, and moreover, if s and the $m's$ are chosen appropriately large, then $T_{a,b,c}$ is bounded on $L^p(\lambda_n)$.

Boundary terms for $\mathscr{F}^1$. Now we turn to estimating the boundary terms in (7-19). A typical term is

$$\mathscr{S}_{n,s}\big(\overline{\mathscr{D}}^k(\Omega_1^2 h)\big)\big[\overline{\mathscr{X}}\big](z)=\int_{\mathbb{B}_n}\frac{(1-|w|^2)^{s-n-1}}{(1-\overline{w}z)^s}\overline{\mathscr{D}}^k(\Omega_1^2 h)\big[\overline{\mathscr{X}}\big](w)\,dV(w), \tag{7-26}$$

with $0\le k\le m-1$ upon appealing to Lemma 17.

We now apply the operator $(1-|z|^2)^{m_1+\sigma}R^{m_1}$ to the integral on the right side of (7-26); using the inequalities (5-7)–(5-9) we obtain that the absolute value of the result is dominated by

$$\begin{aligned}\int_{\mathbb{B}_n}&\frac{(1-|z|^2)^{m_1+\sigma}(1-|w|^2)^{s-n-1}}{|1-\overline{w}z|^{s+m_1}}\left(\frac{\sqrt{\triangle(w,z)}}{1-|w|^2}\right)^{k+1}\big|\overline{D}^k(\widehat{\Omega}_1^2 h)\big|\,dV(w)\\&=\int_{\mathbb{B}_n}\frac{(1-|z|^2)^{m_1+\sigma}(1-|w|^2)^{s-n-2-k-\sigma}\sqrt{\triangle(w,z)}^{\,k+1}}{|1-\overline{w}z|^{s+m_1}}\big|(1-|w|^2)^{\sigma}\,\overline{D}^k\big(\widehat{\Omega}_1^2 h\big)(w)\big|\,dV(w).\end{aligned}$$

The operator in question here is $T_{a,b,c}$ with $a=m_1+\sigma$, $b=s-n-2-k-\sigma$, and $c=k+1$, since

$$a+b+c+n+1=s+m_1.$$

Lemma 24 applies to prove the desired boundedness on $L^p(\lambda_n)$ provided $m_1+\sigma>\frac{n}{p}$.

However, if k fails to satisfy $k+1>2\big(\frac{n}{p}-\sigma\big)$, then the derivative $D^{k+1}\Omega$ cannot be used to control the norm $\|\Omega\|_{B_p^\sigma(\mathbb{B}_n)}$. To compensate for a small k, we must then apply Corollary 16 to the right side of (7-26) (which for fixed z is in $C(\overline{\mathbb{B}}_n)\cap C^\infty(\mathbb{B}_n)$) before differentiating and taking absolute values inside the integral. This then leads to operators of the form

$$(1-|z|^2)^{m_1+\sigma}R^{m_1}\left(\int_{\mathbb{B}_n}\frac{(1-|w|^2)^{s-n-1}}{(1-\overline{w}z)^s}(1-|w|^2)^m R^m\big[\overline{\mathscr{D}}^k(\Omega_1^2 h)(w)\big]\,dV(w)\right),$$

which are dominated by

$$\int_{\mathbb{B}_n}\frac{(1-|z|^2)^{m_1+\sigma}(1-|w|^2)^{s-n-1}}{|1-\overline{w}z|^{s+m_1}}\left(\frac{\sqrt{\triangle(w,z)}}{1-|w|^2}\right)^{k+1}\big|\mathfrak{R}^m\,\overline{D}^k(\widehat{\Omega}_1^2 h)(w)\big|\,dV(w),$$

which is

$$\int_{\mathbb{B}_n}\frac{(1-|z|^2)^{m_1+\sigma}(1-|w|^2)^{s-n-2-k-\sigma}\sqrt{\triangle(w,z)}^{\,k+1}}{|1-\overline{w}z|^{s+m_1}}\big|(1-|w|^2)^{\sigma}\mathfrak{R}^m\,\overline{D}^k(\widehat{\Omega}_1^2 h)(w)\big|\,dV(w).$$

This latter operator is $T_{a,b,c}H(z)$, with

$$a=m_1+\sigma,\quad b=s-n-2-k-\sigma,\quad c=k+1,\quad\text{and}\quad H(w)=\big|(1-|w|^2)^{\sigma}R_{b'}^m\overline{D}^k(\widehat{\Omega}_1^2 h)(w)\big|.$$

Note that for $m>2\big(\frac{n}{p}-\sigma\big)$ we do indeed now have $\|H\|_{L^p(\lambda_n)}\approx\|\widehat{\Omega}_1^2 h\|_{B_p^\sigma(\mathbb{B}_n)}$. The operator here is the same as that above and so Lemma 24 applies to prove the desired boundedness on $L^p(\lambda_n)$.

The estimate for $\mathscr{F}^2$. Our next task is to obtain the estimate (7-1) for $\mu=2$. For this we will show that

$$\begin{aligned}\int_{\mathbb{B}_n}&\big|(1-|z|^2)^{m_1+\sigma}R^{m_1}\Lambda_g\mathscr{C}_{n,s_1}^{0,0}\Lambda_g\mathscr{C}_{n,s_2}^{0,1}\Omega_2^3\big|^p\,d\lambda_n(z)\\&\le C\int_{\mathbb{B}_n}\big|(1-|z|^2)^{\sigma}(1-|z|^2)^{m_3''}R^{m_3''}\overline{D}^{m_3'}(\widehat{\Omega}_2^3 h)(z)\big|^p\,d\lambda_n(z).\end{aligned}\tag{7-27}$$

Unlike the previous argument, this time we will have to deal with a rogue term $\bar{z}_2-\bar{\xi}_2$ where there is no derivative $\partial/\partial\bar{\xi}_2$ to associate to it. Again we ignore the contractions Λ_g. Then we use Lemma 17 to perform integration by parts m_2' times in the first iterated integral and m_3' times in the second iterated integral. We also use Corollary 16 to perform integration by parts in the *radial* derivative m_2'' times in the *first* iterated integral (for fixed z, we have $\mathscr{C}^{0,1}_{n,s_2}\Omega^3_2 \in C(\overline{\mathbb{B}}_n)\cap C^\infty(\mathbb{B}_n)$ by standard estimates [Charpentier 1980]), so that the additional factor $(1-|\xi|^2)^{m_2''}$ can be used crucially in the second iterated integral, and also m_3'' times in the *second* iterated integral for use in acting on Ω^3_2.

Recall from Lemma 17 that

$$\mathscr{C}^{0,q}_{n,s}\eta(z) = \text{boundary terms (depending on } m)$$
$$+\sum_{\ell=0}^{q}\int_{\mathbb{B}_n}\frac{(1-w\bar{z})^{n-1-\ell}(1-|w|^2)^\ell}{\triangle(w,z)^n}\left(\frac{1-|w|^2}{1-\overline{w}z}\right)^{s-n}\left(\sum_{j=0}^{n-\ell-1}c_{j,\ell,n,s}\left(\frac{(1-|w|^2)(1-|z|^2)}{|1-w\bar{z}|^2}\right)^j\right)\overline{\mathfrak{D}}^m\eta(z).$$

Recall also that $\overline{\mathfrak{D}}^m$ already has the rogue terms built in, as can be seen from (3-6). Now we use the right side above with $q=\ell=j=0$ to substitute for $\mathscr{C}^{0,0}_{n,s_1}$, and the right side above with $q=\ell=1$ and $j=0$ to substitute for $\mathscr{C}^{0,1}_{n,s_2}$. Then a typical part of the resulting kernel of the operator $\mathscr{C}^{0,0}_{n,s_1}\mathscr{C}^{0,1}_{n,s_2}\Omega^3_2(z)$ is

$$\int_{\mathbb{B}_n}\frac{(1-\xi\bar{z})^{n-1}}{\triangle(\xi,z)^n}\left(\frac{1-|\xi|^2}{1-\bar{\xi}z}\right)^{s_1-n}(\bar{z}_2-\bar{\xi}_2)(1-|\xi|^2)^{m_2'}R^{m_2'}\overline{\mathfrak{D}}^{m_2''} \tag{7-28}$$
$$\times\int_{\mathbb{B}_n}\frac{(1-w\bar{\xi})^{n-2}(1-|w|^2)}{\triangle(w,\xi)^n}\left(\frac{1-|w|^2}{1-\overline{w}\xi}\right)^{k_2-n}(\overline{w}_1-\bar{\xi}_1)(1-|w|^2)^{m_3'}R^{m_3'}\overline{\mathfrak{D}}^{m_3''}(\Omega^3_2h)(w)\,dV(w)\,dV(\xi),$$

where we have arbitrarily chosen $\bar{z}_2-\bar{\xi}_2$ and $\overline{w}_1-\bar{\xi}_1$ as the rogue factors.

Remark. It is important to note that the differential operators $\overline{\mathfrak{D}}^{m_2}_\zeta$ are conjugate in the variable z and hence *vanish* on the kernels of the boundary terms $\mathscr{S}_{n,s}(\overline{\mathfrak{D}}^k\Omega^3_2h)(z)$ in the integration by parts formula (3-7) associated to the Charpentier solution operator $\mathscr{C}^{0,1}_{n,s_2}$, since these kernels are *holomorphic*. As a result the operator $\overline{\mathfrak{D}}^{m_2'}$ hits only the factor $\overline{\mathfrak{D}}^k\Omega^3_2h$ and a typical term is

$$(\overline{z_i-\zeta_i})\,\frac{\partial}{\partial\bar{z}_i}\big((\overline{w_i-z_i})\,\Omega^3_2h\big) = -(\overline{z_i-\zeta_i})\,\Omega^3_2h,$$

where the derivative $\partial/\partial\overline{w}_i$ must occur in each surviving term in Ω^3_2h, and this term which is then handled like the rogue terms.

Now we recall the factorization (4-4) with $\ell=2$,

$$\Omega^3_2 = -4\Omega^1_0\wedge\widetilde{\Omega}^1_0\wedge\widetilde{\Omega}^1_0,$$

and that $\Omega^3_2(w)$ must have both derivatives $\partial g/\partial\overline{w}_1$ and $\partial g/\partial\overline{w}_2$ occurring in it, one surviving in each of the factors $\widetilde{\Omega}^1_0$, along with other harmless powers of g that we ignore. Thus we may replace $\widetilde{\Omega}^1_0\wedge\widetilde{\Omega}^1_0$ with $\partial/\partial\overline{w}_2\Omega^1_0\wedge\partial/\partial\overline{w}_1\Omega^1_0$. If we use

$$\bar{z}_2-\bar{\xi}_2 = (\bar{z}_2-\overline{w}_2)-(\bar{\xi}_2-\overline{w}_2),$$

we can write the iterated integral above as

$$\int_{\mathbb{B}_n} \frac{(1-\xi\bar{z})^{n-1}}{\triangle(\xi,z)^n}\left(\frac{1-|\xi|^2}{1-\bar{\xi}z}\right)^{s_1-n}$$
$$\times \int_{\mathbb{B}_n} (1-|\xi|^2)^{m_2''} R^{m_2''}\overline{\mathfrak{D}}^{m_2'}\left(\frac{(1-w\bar{\xi})^{n-2}(1-|w|^2)}{\triangle(w,\xi)^n}\left(\frac{1-|w|^2}{1-\overline{w}\xi}\right)^{s_2-n}\right)$$
$$\times\left((1-|w|^2)^{m_3''}R^{m_3''}(\bar{\xi}_2-\overline{w}_2)\frac{\partial}{\partial\overline{w}_2}\overline{\mathfrak{D}}^{m_3'-\ell}\Omega_0^1\right)\wedge\left((1-|w|^2)^{m_3''}R^{m_3''}(\bar{\xi}_1-\overline{w}_1)\frac{\partial}{\partial\overline{w}_1}\overline{\mathfrak{D}}^{\ell}\Omega_0^1\right)$$
$$\times\, dV(w)\, dV(\xi)$$

minus the same expression but with the rogue factor $\bar{\xi}_2-\overline{w}_2$ on the third line replaced by the rogue factor $\bar{z}_2-\overline{w}_2$. We have temporarily ignored the wedge products with terms that do not include derivatives of g, as these terms are bounded and so harmless.

Now we apply $(1-|z|^2)^{\sigma}(1-|z|^2)^{m_1''}R^{m_1''}D^{m_1'}$ to these operators. Using the crucial inequalities (5-7)–(5-9), together with the factorization (7-9) with $\ell=2$,

$$\widehat{\Omega}_2^3=-4\Omega_0^1\wedge\widehat{\Omega}_0^1\wedge\widehat{\Omega}_0^1,$$

the result of this application on the first integral is then dominated by

$$\int_{\mathbb{B}_n}\frac{(1-|z|^2)^{\sigma}|1-\xi\bar{z}|^{n-1}}{\triangle(\xi,z)^{m_1'+m_1''+n}}\left[(1-|z|^2)\sqrt{\triangle(\xi,z)}\right]^{m_1''}\left(\left[(1-|z|^2)\sqrt{\triangle(\xi,z)}\right]^{m_1'}+\triangle(\xi,z)^{m_1'}\right)\left|\frac{1-|\xi|^2}{1-\xi\bar{z}}\right|^{s_1-n}$$
$$\times\left(\int_{\mathbb{B}_n}\frac{(1-|\xi|^2)^{m_2''}|1-w\bar{\xi}|^{n-2}(1-|w|^2)}{\triangle(w,\xi)^{m_2'+m_2''+n}}\left(\frac{\sqrt{\triangle(\xi,z)}}{1-|\xi|^2}\right)^{m_2'}\left[(1-|\xi|^2)\sqrt{\triangle(w,\xi)}\right]^{m_2''}\right.$$
$$\times\left(\left[(1-|\xi|^2)\sqrt{\triangle(w,\xi)}\right]^{m_2'}+\triangle(w,\xi)^{m_2'}\right)\left|\frac{1-|w|^2}{1-w\bar{\xi}}\right|^{s_2-n}\left(\frac{\sqrt{\triangle(w,\xi)}}{1-|w|^2}\right)^{m_3'}\left(\frac{\sqrt{\triangle(w,\xi)}}{1-|w|^2}\right)^2$$
$$\left.\times\left|(1-|w|^2)^{m_3''}R^{m_3''}\overline{D}^{m_3'}(\widehat{\Omega}_2^3h)(w)\right|dV(w)\right)dV(\xi), \quad (7\text{-}29)$$

and the result of this application on the second integral is dominated by exactly the same expression but with one of the two factors $\sqrt{\triangle(w,\xi)}/(1-|w|^2)$ that occur at the end of the third line in (7-29) replaced by the factor $\sqrt{\triangle(w,z)}/(1-|w|^2)$. The ignored wedge products have now been reinstated in $\widehat{\Omega}_2^3$.

Now for the iterated integral in (7-29), we can separate it into the composition of two operators of the form treated previously. One factor is the operator

$$\int_{\mathbb{B}_n}\frac{(1-|z|^2)^{\sigma}|1-\xi\bar{z}|^{n-1}}{\triangle(\xi,z)^{m_1'+m_1''+n}}\left[(1-|z|^2)\sqrt{\triangle(\xi,z)}\right]^{m_1''}\left(\left[(1-|z|^2)\sqrt{\triangle(\xi,z)}\right]^{m_1'}+\triangle(\xi,z)^{m_1'}\right)$$
$$\times\left(\frac{\sqrt{\triangle(\xi,z)}}{1-|\xi|^2}\right)^{m_2'}\left|\frac{1-|\xi|^2}{1-\xi\bar{z}}\right|^{s_1-n}(1-|\xi|^2)^{-\sigma}F(\xi)\,dV(\xi), \quad (7\text{-}30)$$

and the other factor is the operator $F(\xi)$ given by

$$\int_{\mathbb{B}_n} \frac{(1-|\xi|^2)^\sigma |1-w\bar{\xi}|^{n-2}(1-|w|^2)}{\triangle(w,\xi)^{m_2'+m_2''+n}} \left[(1-|\xi|^2)\sqrt{\triangle(w,\xi)}\right]^{m_2''} \left(\left[(1-|\xi|^2)\sqrt{\triangle(w,\xi)}\right]^{m_2'} + \triangle(w,\xi)^{m_2'}\right) \times \left(\frac{\sqrt{\triangle(w,\xi)}}{1-|w|^2}\right)^{m_3'+2} \left|\frac{1-|w|^2}{1-w\bar{\xi}}\right|^{s_2-n} (1-|w|^2)^{-\sigma} f(w)\, dV(w), \quad (7\text{-}31)$$

where $f(w) = (1-|w|^2)^\sigma \left|(1-|w|^2)^{m_3''} R^{m_3''} \bar{D}^{m_3'} (\widehat{\Omega}_2^3 h)(w)\right|$. We now show how Lemma 24 applies to obtain the appropriate boundedness.

We will in fact compare the corresponding kernels to that in (7-25). When we consider the summand $\triangle(\xi,z)^{m_1'}$ at the end of the first line of (7-30), the first operator has kernel

$$\frac{(1-|z|^2)^{\sigma+m_1''}(1-|\xi|^2)^{s_1-n-m_2'-\sigma}}{|1-\xi\bar{z}|^{s_1-2n+1}\triangle(\xi,z)^{m_1'+m_1''+n-(m_1''+2m_1'+m_2')/2}} = \frac{(1-|z|^2)^{\sigma+m_1''}(1-|\xi|^2)^{s_1-3n-m_1''-\sigma}}{|1-\xi\bar{z}|^{s_1-2n+1}}, \quad (7\text{-}32)$$

if we choose $m_2' = m_1'' + 2n$ so that the factor $\triangle(\xi,z)$ disappears. This is exactly the same as the kernel of the operator in (7-25) in the previous alternative argument but with m_1'' in place of m_1' there. When we consider instead the summand $\left[(1-|z|^2)\sqrt{\triangle(\xi,z)}\right]^{m_1'}$ on the first line of (7-30), we obtain the kernel in (7-32) but with $m_1''+m_1'$ in place of m_1''.

When we consider the summand $\triangle(w,\xi)^{m_2'}$ at the end of the second line of (7-31), the second operator has kernel

$$\frac{(1-|\xi|^2)^{m_2''+\sigma}(1-|w|^2)^{1+s_2-n-m_3'-2-\sigma}}{|1-w\bar{\xi}|^{s_2-2n+2}\triangle(w,\xi)^{m_2'+m_2''+n-(m_2''+2m_2'+m_3'+2)/2}} = \frac{(1-|\xi|^2)^{m_2''+\sigma}(1-|w|^2)^{s_2-3n+1-m_2''-\sigma}}{|1-w\bar{\xi}|^{s_2-2n+2}}. \quad (7\text{-}33)$$

if we choose $m_3' = m_2'' + 2n - 2$, and this is also bounded on $L^p(d\lambda_n)$ for m_2'' and s_2 sufficiently large.

Remark. It is here in choosing m_2'' large that we are using the full force of Corollary 16 to perform integration by parts in the *radial* derivative m_2'' times in the first iterated integral.

When we consider instead the summand $\left[(1-|z|^2)\sqrt{\triangle(\xi,z)}\right]^{m_2'}$ on the first line of (7-31), we obtain the kernel in (7-33) but with $m_2''+m_2'$ in place of m_2''.

To handle the case of (7-29) in which the factor $\sqrt{\triangle(w,z)}/(1-|w|^2)$ replaces one of the factors $\sqrt{\triangle(w,\xi)}/(1-|w|^2)$, we must first deal with the rogue factor $\sqrt{\triangle(w,z)}$ whose variable pair (w,z) doesn't match that of either of the denominators $\triangle(\xi,z)$ or $\triangle\,(w,\xi)$. For this we use the fact that

$$\sqrt{\triangle(w,z)} = |1-w\bar{z}|\,|\varphi_z(w)| = \delta(w,z)^2\rho(w,z),$$

where $\rho(w,z) = |\varphi_z(w)|$ is the invariant pseudohyperbolic metric on the ball (Corollary 1.22 in [Zhu 2005]) and where $\delta(w,z) = |1-w\bar{z}|^{1/2}$ satisfies the triangle inequality on the ball (Proposition 5.1.2 in [Rudin 1980]). Thus we have

$$\rho(w,z) \le \rho(\xi,z) + \rho(w,\xi), \quad \delta(w,z) \le \delta(\xi,z) + \delta(w,\xi),$$

and so also

$$\begin{aligned}\sqrt{\triangle(w,z)} &\le 2\big[\delta(\xi,z)^2+\delta(w,\xi)^2\big]\big(|\varphi_z(\xi)|+|\varphi_\xi(w)|\big)\\ &= 2\left(1+\frac{|1-w\bar{\xi}|}{|1-\xi\bar{z}|}\right)\sqrt{\triangle(\xi,z)}+2\left(1+\frac{|1-\xi\bar{z}|}{|1-w\bar{\xi}|}\right)\sqrt{\triangle(w,\xi)}.\end{aligned}$$

Thus we can write

$$\begin{aligned}\frac{\sqrt{\triangle(w,z)}}{1-|w|^2} \lesssim \frac{1-|\xi|^2}{1-|w|^2}\,\frac{\sqrt{\triangle(\xi,z)}}{1-|\xi|^2}+\frac{|1-w\bar{\xi}|}{1-|w|^2}\,\frac{1-|\xi|^2}{|1-\xi\bar{z}|}\,\frac{\sqrt{\triangle(\xi,z)}}{1-|\xi|^2}\\ +\frac{\sqrt{\triangle(w,\xi)}}{1-|w|^2}+\frac{|1-\xi\bar{z}|}{1-|\xi|^2}\,\frac{1-|\xi|^2}{|1-w\bar{\xi}|}\,\frac{\sqrt{\triangle(w,\xi)}}{1-|w|^2}.\end{aligned} \tag{7-34}$$

All of the terms on the right side of (7-34) are of an appropriate form to distribute throughout the iterated integral, and again Lemma 24 applies to obtain the appropriate boundedness.

For example, the final two terms on the right side of (7-34) that involve $\sqrt{\triangle(w,\xi)}/(1-|w|^2)$ are handled in the same way as the operator in (7-29) by taking $m_3'=m_2''+2n-2$ and $m_2'=m_1''+2n$, and taking s_1 and s_2 large as required by the extra factors

$$\frac{|1-\xi\bar{z}|}{1-|\xi|^2}\,\frac{1-|\xi|^2}{|1-w\bar{\xi}|}.$$

With these choices the first two terms on the right side of (7-34) that involve $\sqrt{\triangle(\xi,z)}/(1-|\xi|^2)$ are then handled using Lemma 24 with $c=\pm1$ as follows.

If we substitute the first term

$$\frac{1-|\xi|^2}{1-|w|^2}\,\frac{\sqrt{\triangle(\xi,z)}}{1-|\xi|^2}$$

on the right in (7-34) for the factor $\sqrt{\triangle(w,z)}/(1-|w|^2)$, we get a composition of two operators as in (7-30) and (7-31) but with the kernel in (7-30) multiplied by $\sqrt{\triangle(\xi,z)}/(1-|\xi|^2)$ and the kernel in (7-31) multiplied by $(1-|\xi|^2)/(1-|w|^2)$ and divided by $\sqrt{\triangle(w,\xi)}/(1-|w|^2)$. If we consider the summand $\triangle(\xi,z)^{m_1'}$ at the end of the first line of (7-30), and with the choice $m_2'=m_1''+2n$ already made, the first operator then has kernel

$$\frac{\sqrt{\triangle(\xi,z)}}{1-|\xi|^2}\,\frac{(1-|z|^2)^{\sigma+m_1''}(1-|\xi|^2)^{s_1-3n-m_1''-\sigma}}{|1-\xi\bar{z}|^{s_1-2n+1}}=\frac{(1-|z|^2)^{m_1''+\sigma}(1-|\xi|^2)^{s_1-m_1''-3n-1-\sigma}\sqrt{\triangle(\xi,z)}}{|1-\xi\bar{z}|^{s_1-2n+1}},$$

and hence is of the form $T_{a,b,c}$ with

$$a=m_1''+\sigma,\quad b=s_1-3n-1-m_1''-\sigma,\quad c=1,$$

since $a+b+c+n+1=s_1-n-1$. Now we apply Lemma 24 to conclude that this operator is bounded on $L^p(\lambda_n)$ if and only if

$$-p(m_1''+\sigma)<-n<p(s_1-3n-m_1''-\sigma),$$

i.e., $m_1''+\sigma>\frac{n}{p}$ and $s_1>m_1''+\sigma+3n-\frac{n}{p}$.

Next we consider the summand $\triangle(w,\xi)^{m_2'}$ at the end of the first line of (7-31). With the choice $m_3' = m_2'' + 2n - 2$ already made, the second operator has kernel

$$\frac{1-|\xi|^2}{1-|w|^2}\left(\frac{\sqrt{\triangle(w,\xi)}}{1-|w|^2}\right)^{-1}\frac{(1-|\xi|^2)^{m_2''+\sigma}(1-|w|^2)^{s_2-3n+1-m_2''-\sigma}}{|1-w\bar{\xi}|^{s_2-2n+2}} \\ = \frac{(1-|\xi|^2)^{m_2''+\sigma+1}(1-|w|^2)^{s_2-3n+1-m_2''-\sigma}\sqrt{\triangle(w,\xi)}^{-1}}{|1-w\bar{\xi}|^{s_2-2n+2}},$$

and hence is of the form $T_{a,b,c}$ with

$$a = m_2''+\sigma+1, \quad b = s_2-3n+1-m_2''-\sigma, \quad c=-1.$$

This operator is bounded on $L^p(\lambda_n)$ if and only if

$$-p(m_2''+\sigma+1) < -n < p(s_2-3n+2-m_2''-\sigma),$$

i.e., $m_2''+\sigma > \frac{n}{p}-1$ and $s_2 > m_2''+\sigma+3n-2-\frac{n}{p}$.

If we now substitute the second term

$$\frac{|1-w\bar{\xi}|}{1-|w|^2}\,\frac{1-|\xi|^2}{|1-\xi\bar{z}|}\,\frac{\sqrt{\triangle(\xi,z)}}{1-|\xi|^2}$$

on the right in (7-34) for the factor $\sqrt{\triangle(w,z)}/(1-|w|^2)$ we similarly get a composition of two operators that are each bounded on $L^p(\lambda_n)$ for m_i and s_i chosen large enough.

Boundary terms for $\mathscr{F}^2$. Now we must address in $\mathscr{F}^2$ the boundary terms that arise in the integration by parts formula (3-7). Suppose the first operator $\mathscr{C}^{0,0}_{n,s_1}$ is replaced by a boundary term, but not the second. We proceed by applying Corollary 16 to the boundary term. Since the differential operator $(1-|z|^2)^{m_1+\sigma}R^{m_1}$ hits only the kernel of the boundary term, we can apply the remark following Lemma 24 to the first iterated integral and the lemma itself to the second iterated integral in the manner indicated in the above arguments. If the second operator $\mathscr{C}^{0,1}_{n,s_2}$ is replaced by a boundary term, then as mentioned in the remark on page 543, the operators $\bar{D}^{m_2}$ hit only the factors $\bar{\mathscr{D}}^{m_3}$, and this produces rogue terms that are handled as above. If the first operator $\mathscr{C}^{0,0}_{n,s_1}$ was also replaced by a boundary term, then in addition we would have radial derivatives R^m hitting the second boundary term. Since radial derivatives are holomorphic, they hit only the holomorphic kernel and not the antiholomorphic factors in $\bar{\mathscr{D}}^{m_3}$, and so these terms can also be handled as above.

7.2. *The estimates for general* $\mathscr{F}^\mu$. In view of inequality (7-10), it suffices to establish the inequality

$$\|\mathscr{F}^\mu\|^p_{B^\sigma_p(\mathbb{B}_n)} = \int_{\mathbb{B}_n}\left|(1-|z|^2)^{m_1+\sigma}R^{m_1}\Lambda_g\mathscr{C}^{0,0}_{n,s_1}\ldots\Lambda_g\mathscr{C}^{0,\mu-1}_{n,s_\mu}\Omega^{\mu+1}_\mu h\right|^p d\lambda_n(z) \\ \le C_{\sigma,n,p,\delta}\int_{\mathbb{B}_n}\left|(1-|z|^2)^\sigma\mathscr{X}^{m_\mu}(\widehat{\Omega}^{\mu+1}_\mu h)(z)\right|^p d\lambda_n(z). \tag{7-35}$$

Recall that the absolute value $|F|$ of an element F in the exterior algebra is the square root of the sum of the squares of the coefficients of F in the standard basis.

The case $\mu > 2$ involves no new ideas, and is merely complicated by straightforward algebra. The reason is that the solution operator $\Lambda_g \mathscr{C}_{n,s_1}^{0,0} \ldots \Lambda_g \mathscr{C}_{n,s_\mu}^{0,\mu-1}$ acts *separately* in each entry of the form $\Omega_\mu^{\mu+1} h$, an element of the exterior algebra of $\mathbb{C}^\infty \otimes \mathbb{C}^n$ which we view as an alternating ℓ^2-tensor of $(0,\mu)$ forms in $\mathbb{C}^n$. These operators decompose as a sum of simpler operators with the basic property that their kernels are *identical*, except that the rogue factors in each kernel differ according to the entry. Nevertheless, there are always exactly μ distinct rogue factors in each kernel and after splitting, the μ rogue factors can be associated in one-to-one fashion with each of the derivatives $\partial/\partial \overline{w}_j$ in the corresponding entry of

$$\Omega_\mu^{\mu+1} h = -(\mu+1)\left(\sum_{k_0=1}^{\infty} \frac{\overline{g}_{k_0}}{|g|^2} e_{k_0}\right) \wedge \bigwedge_{i=1}^{\mu} \left(\sum_{k_i=1}^{\infty} \frac{\overline{\partial} \overline{g}_{k_i}}{|g|^2} e_{k_i}\right) h.$$

After applying the crucial inequalities, this effectively results in replacing each derivative $\partial/\partial \overline{w}_j$ by the derivative $\overline{D}_j$, and consequently we can write the resulting form as $\widehat{\Omega}_\mu^{\mu+1} h$.

This completes our proof of Theorem 2.

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Received 10 Mar 2010. Revised 25 May 2010. Accepted 23 Jun 2010.

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ANALYSIS AND PDE
Vol. 4, No. 4, 2011

SOBOLEV SPACE ESTIMATES FOR A CLASS OF BILINEAR PSEUDODIFFERENTIAL OPERATORS LACKING SYMBOLIC CALCULUS

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The reappearance of what is sometimes called exotic behavior for linear and multilinear pseudodifferential operators is investigated. The phenomenon is shown to be present in a recently introduced class of bilinear pseudodifferential operators which can be seen as more general variable coefficient counterparts of the bilinear Hilbert transform and other singular bilinear multipliers operators. We prove that such operators are unbounded on products of Lebesgue spaces but bounded on spaces of smooth functions (this is the exotic behavior referred to). In addition, by introducing a new way to approximate the product of two functions, estimates on a new paramultiplication are obtained.

1. Introduction

An anomalous yet recurrent phenomenon. This article is a continuation of recent work devoted to the development of a theory of bilinear and multilinear pseudodifferential operators which are the x-dependent counterparts of the singular multipliers modeled by the bilinear Hilbert transform. In particular we will further study the class of bilinear pseudodifferential operators $BS^0_{1,1;\pi/4}$ and show that it has a sometimes called *exotic* or *forbidden* behavior regarding boundedness on function spaces.

By a bilinear pseudodifferential operator we mean an operator, defined a priori on test functions, of the form

$$T_\sigma(f,g)(x) = \int_{\mathbb{R}^{2n}} \sigma(x,\xi,\eta)\hat{f}(\xi)\hat{g}(\eta)e^{ix\cdot(\xi+\eta)}\,d\xi\,d\eta.$$

Two main types of x-dependent classes of symbols have been studied in the literature. One is the Coifman–Meyer type $BS^m_{\rho,\delta}(\mathbb{R}^n)$, $0\le\delta\le\rho\le 1$, $m\in\mathbb{R}$, of symbols satisfying estimates of the form

$$|\partial_x^\alpha\partial_\xi^\beta\partial_\eta^\gamma\sigma(x,\xi,\eta)| \le C_{\alpha\beta\gamma}(1+|\xi|+|\eta|)^{m+\delta|\alpha|-\rho(|\beta|+|\gamma|)}, \tag{1-1}$$

for all multi-indices α,β,γ.

The other type corresponds to classes denoted by $BS^m_{\rho,\delta;\theta}(\mathbb{R}^n)$, $0\le\delta\le\rho\le 1$, $m\in\mathbb{R}$, $-\pi/2<\theta\le\pi/2$, and consisting of symbols satisfying

$$|\partial_x^\alpha\partial_\xi^\beta\partial_\eta^\gamma\sigma(x,\xi,\eta)| \le C_{\alpha\beta\gamma;\theta}(1+|\eta-\tan(\theta)\xi|)^{m+\delta|\alpha|-\rho(|\beta|+|\gamma|)} \tag{1-2}$$

(where for $\theta=\pi/2$ the estimates are interpreted to decay in terms of $1+|\xi|$ only). Both types can be

Torres' research was supported in part by the National Science Foundation under grant DMS 0800492.
MSC2000: primary 47G30; secondary 42B15, 42C10, 35S99.
Keywords: bilinear pseudodifferential operators, exotic class, transposes, asymptotic expansion, elementary symbols, Littlewood–Paley theory, Sobolev space estimates, T(1)-Theorem.

seen as bilinear analogs of the classical Hörmander classes $S^m_{\rho,\delta}(\mathbb{R}^n)$ of linear pseudodifferential operators

$$T_\tau(f)(x) = \int_{\mathbb{R}^n} \tau(x,\xi)\hat{f}(\xi)e^{ix\cdot\xi}\,d\xi,$$

with symbols satisfying

$$|\partial_x^\alpha \partial_\xi^\beta \tau(x,\xi)| \le C_{\alpha\beta}(1+|\xi|)^{m+\delta|\alpha|-\rho|\beta|}. \tag{1-3}$$

As the name indicates, the first type of bilinear classes was introduced by Coifman and Meyer [1975; 1978a; 1978b] at least in the case $m = 0$, $\rho = 1$ and $\delta = 0$. It is now well understood that the operators in $BS^0_{1,0}$ are examples of certain singular integrals and fit within the general multilinear Calderón–Zygmund theory developed in [Grafakos and Torres 2002]; see also [Christ and Journé 1987; Kenig and Stein 1999]. For other values of the parameters, the classes $BS^m_{\rho,\delta}$ were studied in [Bényi 2003; Bényi and Torres 2003; 2004; Bényi et al. 2006; 2010].

The general classes $BS^m_{\rho,\delta;\theta}$ with x-dependent symbols were first introduced in [Bényi et al. 2006]. A connection to the bilinear Hilbert transform and the work of Lacey and Thiele [1997; 1999] is given by the study in the x-independent case of singular multipliers in one dimension satisfying

$$|\partial_\xi^\beta \partial_\eta^\gamma \sigma(\xi,\eta)| \le C_{\beta\gamma}|\eta - \tan(\theta)\xi|^{-|\beta|-|\gamma|}.$$

This type of multipliers was investigated in [Gilbert and Nahmod 2000; 2001; 2002; Muscalu et al. 2002]. We also recall that if for τ in $S^0_{1,0}(\mathbb{R})$ we define

$$\sigma(x,\xi,\eta) = \tau(x,\xi-\eta), \tag{1-4}$$

then σ is in $BS^0_{1,0;\pi/4}$. These operators have a certain modulation invariance:

$$T_\sigma(e^{iw\cdot}f, e^{iw\cdot}g)(x) = e^{i2wx}T_\sigma(f,g)(x)$$

for all $w \in \mathbb{R}$. Such a T_σ fits then within the more general framework of modulation invariant bilinear singular integrals of [Bényi et al. 2009]. Boundedness properties for symbols in the classes $BS^0_{1,0;\theta}(\mathbb{R})$, not necessarily of the form (1-4), were obtained in [Bernicot 2008; 2010]. See [Torres 2009] for further motivation and references.

In this article we want to discuss the reappearance of the exotic phenomenon for the parameters $m = 0$ and $\rho = \delta = 1$. Namely, the unboundedness on L^p spaces of operators in $BS^0_{1,0;\theta}$, but their boundedness on spaces of smooth functions.

In the linear case this phenomenon for $S^0_{1,1}$ is by now well understood through works such as [Stein 1993; Meyer 1981a; Runst 1985; Bourdaud 1988; Hörmander 1988; Torres 1990]. It is intimately related to the lack of calculus for the adjoints of operators in such class and, ultimately, this behavior has been interpreted through the $T(1)$-Theorem of David and Journé [1984]. The class $S^0_{1,1}$ is the largest class of linear pseudodifferential operators with Calderón–Zygmund kernels but their exotic behavior on L^p spaces is given by the fact that for T in the class $S^0_{1,1}$, the distribution $T^*(1)$ is in general not in *BMO* (though $T(1)$ is). Here T^* is the formal transpose of T. Moreover, the boundedness of an operator T in $S^0_{1,1}$ on several other spaces of function is related to the action (properly defined) of T^* on polynomials; see [Torres 1991] and the relation to the work of Hörmander [1989] found in [Torres 1990]. By comparison, the smaller classes $S^0_{1,\delta}$ with $\delta < 1$ are closed by transposition and hence the operators in such classes do satisfy the hypotheses of the $T(1)$-Theorem and are bounded on L^p for $1 < p < \infty$.

Likewise, in the bilinear case, the class $BS^0_{1,1}$ is the largest class of pseudodifferential operators with bilinear Calderón–Zygmund kernels. But again, T^{*1} and T^{*2}, the two formal transposes of an operator T in $BS^0_{1,1}$, may fail to satisfy the hypotheses of the $T(1)$-Theorem for bilinear Calderón–Zygmund operators in [Grafakos and Torres 2002]. A symbolic calculus for the transposes hold in the smaller classes $BS^0_{1,\delta}$ with $\delta < 1$ [Bényi and Torres 2003; Bényi et al. 2010], rendering the boundedness of operators in $BS^0_{1,\delta}$. Though unbounded on product of L^p spaces, the class $BS^0_{1,1}$ is still bounded on product of Sobolev spaces [Bényi and Torres 2003]. For the Coifman–Meyer symbols there is then a complete analogy with the linear situation.

For the newer more singular classes $BS^0_{1,0;\theta}$ a symbolic calculus for the transposes was shown to exist in [Bényi et al. 2006] and extended in [Bernicot 2010]. Hence, the boundedness on product of L^p spaces of operators in such classes and of the form (1-4) can be easily obtained from the new $T(1)$-Theorem for modulation invariant singular integrals in [Bényi et al. 2009]. The class $BS^0_{1,0;\theta}$ also produced bounded operators on Sobolev spaces of positive smoothness as shown in [Bernicot 2008]. All these developments motivate us to look for exotic behavior in the larger classes $BS^0_{1,1;\theta}$.

New results. In this article, we show with an example that there exit modulation invariant operators in the class $BS^0_{1,1;\theta}$ that fail to be bounded on a product of L^p spaces (Proposition 2.1). This immediately implies that an arbitrary operator T in $BS^0_{1,1;\theta}$ may not have both $T^{*1}(1,1)$ and $T^{*2}(1,1)$ in *BMO*, as defined in [Bényi et al. 2009]. It follows also that a symbolic calculus for the transposes in those classes is not possible. Nevertheless, as the reader may expect after the above introduction, we shall show that the classes are bounded on product of Sobolev spaces. For simplicity in the presentation we will only consider the case $BS^0_{1,1;\pi/4}$. The corresponding results for other values of θ in $(-\pi/2, \pi/2) \setminus \{-\pi/4\}$ (avoiding the degenerate directions) can be obtained in similar way.

In the case of modulation invariant operators, we obtained boundedness on product of Sobolev spaces with positive smoothness (Theorem 3.1). Surprisingly if we do not assume modulation invariance we can only obtain the corresponding result on Sobolev spaces of smoothness bigger than $\frac{1}{2}$ (Theorem 3.3). We do not know if the result is sharp, but a better result does not seem attainable with our techniques. Table 1 summarizes the known results and the new ones and puts in evidence the parallel situation in several classes of pseudodifferential operators.

As a byproduct of our results, we also improve on some known estimates on paramultiplication by introducing a new way to approximate the pointwise product of two functions with errors better localized in the frequency plane (see Section 4 for precise statements).

Further definitions and notation. We recall the maximal Hardy–Littlewood operator M defined for a function $f \in L^1_{\mathrm{loc}}(\mathbb{R})$ by

$$M(f)(x) = \sup_{\substack{B \text{ ball} \\ B \ni x}} \frac{1}{|B|} \int_B |f(y)|\, dy.$$

Class/symbol estimates	Lebesgue spaces	Sobolev spaces
(linear) $S^0_{1,0}$ $\lvert\partial_x^\beta\partial_\xi^\alpha\sigma(x,\xi)\rvert \le C_{\alpha\beta}(1+\lvert\xi\rvert)^{-\lvert\alpha\rvert}$	$L^p \to L^p$ $1<p<\infty$	$W^{s,p} \to W^{s,p}$ $1<p<\infty,\ s>0$
(linear) $S^0_{1,1}$ $\lvert\partial_x^\beta\partial_\xi^\alpha\sigma(x,\xi)\rvert \le C_{\alpha\beta}(1+\lvert\xi\rvert)^{\lvert\beta\rvert-\lvert\alpha\rvert}$	unbounded	$W^{s,p} \to W^{s,p}$ $1<p<\infty,\ s>0$
(bilinear) $BS^0_{1,0}$ $\lvert\partial_x^\beta\partial_{\xi,\eta}^\alpha\sigma(x,\xi,\eta)\rvert \le C_{\alpha\beta}(1+\lvert\xi\rvert+\lvert\eta\rvert)^{-\lvert\alpha\rvert}$	$L^p\times L^q \to L^t$ $1<p,q<\infty$ $1/p+1/q=1/t$	$W^{s,p}\times W^{s,q} \to W^{s,t}$ $1<p,q,t<\infty,\ s>0$ $1/p+1/q=1/t$
(bilinear) $BS^0_{1,1}$ $\lvert\partial_x^\beta\partial_{\xi,\eta}^\alpha\sigma(x,\xi,\eta)\rvert \le C_{\alpha\beta}(1+\lvert\xi\rvert+\lvert\eta\rvert)^{\lvert\beta\rvert-\lvert\alpha\rvert}$	unbounded	$W^{s,p}\times W^{s,q} \to W^{s,t}$ $1<p,q,t<\infty,\ s>0$ $1/p+1/q=1/t$
(bilinear) $BS^0_{1,0;\pi/4}$ $\lvert\partial_x^\beta\partial_{\xi,\eta}^\alpha\sigma(x,\xi,\eta)\rvert \le C_{\alpha\beta}(1+\lvert\xi-\eta\rvert)^{-\lvert\alpha\rvert}$	$L^p\times L^q \to L^t$ $1<p,q<\infty$ $1/p+1/q=1/t<\frac{3}{2}$	$W^{s,p}\times W^{s,q} \to W^{s,t}$ $1<p,q,t<\infty,\ s>0$ $1/p+1/q=1/t$
(bilinear) $BS^0_{1,1;\pi/4}$ $\lvert\partial_x^\beta\partial_{\xi,\eta}^\alpha\sigma(x,\xi-\eta)\rvert \le C_{\alpha\beta}(1+\lvert\xi-\eta\rvert)^{\lvert\beta\rvert-\lvert\alpha\rvert}$	unbounded	$W^{s,p}\times W^{s,q} \to W^{s,t}$ $1<p,q,t<\infty,\ s>0$ $1/p+1/q=1/t$
(bilinear) $BS^0_{1,1;\pi/4}$ $\lvert\partial_x^\beta\partial_{\xi,\eta}^\alpha\sigma(x,\xi,\eta)\rvert \le C_{\alpha\beta}(1+\lvert\xi-\eta\rvert)^{\lvert\beta\rvert-\lvert\alpha\rvert}$	unbounded	$W^{s,p}\times W^{s,q} \to W^{s,t}$ $1<p,q,t<\infty,\ s>\frac{1}{2}$ $1/p+1/q=1/t$

Table 1. Summary of the boundedness properties of pseudodifferential operators on Lebesgue and Sobolev spaces.

We write $M^2 = M\circ M$ for the composition of the maximal operator with itself.

For a function f in the Schwartz space $\mathcal{S}$ of smooth and rapidly decreasing functions, we will define the Fourier transform by

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-ix\cdot\xi}\,dx.$$

With this definition, the inverse Fourier transform is given by $f^\vee(\xi) = (2\pi)^{-1}\widehat{f}(-\xi)$. Both the Fourier transform and its inverse can be extended as usual to the dual space of tempered distributions $\mathcal{S}'$.

For a bounded symbol σ, the bilinear operator

$$T_\sigma(f,g)(x) = \int e^{ix(\xi+\eta)}\widehat{f}(\xi)\widehat{g}(\eta)\sigma(x,\xi,\eta)\,d\xi\,d\eta$$

is well defined and gives a bounded function for each pair of functions f, g in $\mathcal{S}$. Moreover, for σ in $BS^0_{1,1;\pi/4}$, the operator T_σ clearly maps $\mathcal{S}\times\mathcal{S}$ into $\mathcal{S}'$ continuously. This justifies many limiting arguments and computations that we will perform without further comment.

The formal transposes, T^{*1} and T^{*2}, of an operator $T : \mathcal{S} \times \mathcal{S} \to \mathcal{S}'$ are defined by

$$\langle T^{*1}(h, g), f \rangle = \langle T(f, g), h \rangle = \langle T^{*2}(f, h), g \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the usual pairing between distributions and test functions.

We will use the notation $\Psi_{2^{-k}}$ for the L^1-normalized function $2^k \Psi(2^k \cdot)$ and consider the Littlewood–Paley characterization of Sobolev spaces $W^{s,p}$, $1 < p < \infty$, $s \geq 0$. That is, for a function Ψ in $\mathcal{S}$ with spectrum contained in $\{\xi : 2^{-1} \leq |\xi| \leq 2\}$ and another function Φ also in $\mathcal{S}$ and with spectrum included in $\{|\xi| \leq 1\}$, and such that

$$\widehat{\Phi}(\xi) + \sum_{k \geq 0} \widehat{\Psi}(2^{-k}\xi) = 1 \tag{1-5}$$

for all ξ, we have

$$\|f\|_{W^{s,p}} \approx \|\Phi * f\|_{L^p} + \left\| \left(\sum_{k \geq 0} 2^{2ks} |\Psi_{2^{-k}} * f|^2 \right)^{1/2} \right\|_{L^p}. \tag{1-6}$$

Here $\|\cdot\|_{L^p}$ denotes the usual norm of the Lebesgue space $L^p(\mathbb{R})$. For $s = 0$, the norm $\|\cdot\|_{W^{0,p}}$ is equivalent to $\|\cdot\|_{L^p}$. Also, by *BMO* we mean as usual the classical John–Nirenberg space of functions of bounded mean oscillation.

By homogeneity considerations, we will investigate boundedness properties of the form

$$T : W^{s,p} \times W^{s,q} \to W^{s,t}, \tag{1-7}$$

where the exponents satisfy $1 \leq p, q, t \leq \infty$ and the Hölder relation

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{t}. \tag{1-8}$$

2. Unboundedness on Lebesgue spaces

We first show that for $s = 0$ the bound (1-7) may fail for $BS^0_{1,1;\pi/4}(\mathbb{R})$.

Proposition 2.1. *There exists a symbol $\tau \in S^0_{1,1}$ such that the operator T_σ with symbol $\sigma(x, \xi, \eta) = \tau(x, \xi - \eta)$ is in $BS^0_{1,1;\pi/4}$ and is not bounded from $L^p \times L^q$ into L^t for any exponents p, q, t satisfying* (1-8).

Proof. As in [Bényi and Torres 2003], we adapt to the bilinear situation a by now classical counterexample in the linear setting; see [Bourdaud 1988]. Let ψ be a function in $\mathcal{S}$ satisfying $\widehat{\psi} \geq 0$, $\widehat{\psi}(\xi) \neq 0$ only for $\frac{5}{7} < |\xi| < \frac{5}{3}$, and $\widehat{\psi}(\xi) = 1$ for $\frac{5}{6} \leq |\xi| < \frac{4}{3}$. Consider the symbol

$$\tau(x, \xi) = \sum_{j \geq 4} e^{-i 2^j x} \widehat{\psi}(2^{-j}\xi),$$

which is easily seen to be in $S^0_{1,1}$. Select another function ψ_1 in $\mathcal{S}$ satisfying supp $(\widehat{\psi}_1) \subset [0, \frac{1}{3}]$ and define

$$f = \sum_{j=4}^{m} a_j e^{i 2^j x} \psi_1(x),$$

for arbitrarily coefficients a_j. For $\sigma(x,\xi,\eta)=\tau(x,\eta-\xi)$, we have

$$T_\sigma(f,\psi_1)(x)=\sum_{j,k\geq 4} a_k e^{-i2^j x}\int_{\mathbb{R}^2} e^{ix(\xi+\eta)}\widehat{\psi}(2^{-j}(\eta-\xi))\widehat{\psi}_1(\xi-2^k)\widehat{\psi}_1(\eta)\,d\xi\,d\eta. \tag{2-1}$$

For each k, the integration at most takes place where $0\leq\eta\leq\frac{1}{3}$ and $2^k\leq\xi\leq 2^k+\frac{1}{3}$, which implies

$$-2^k-\tfrac{1}{3}\leq\eta-\xi\leq\tfrac{1}{3}-2^k,$$

and then for each j,

$$-2^{k-j}-\tfrac{1}{3}2^{-j}\leq 2^{-j}(\eta-\xi)\leq\tfrac{1}{3}2^{-j}-2^{k-j}. \tag{2-2}$$

Note that since $j,k\geq 4$, if $k>j$ we have

$$\tfrac{1}{3}2^{-j}-2^{k-j}<-\tfrac{5}{3},$$

while if $k<j$

$$-2^{k-j}-\tfrac{1}{3}2^{-j}>-\tfrac{5}{7}.$$

It follows from (2-2) that the only nonzero term in (2-1) is the one with $j=k$ and also

$$\widehat{\psi}(2^{-j}(\eta-\xi))=1$$

where the integrand is not zero. We obtain

$$T_\sigma(f,\psi_1)(x)=\sum_{j=4}^m a_j e^{-i2^j x}e^{i2^j x}\psi_1^2(x)=\left(\sum_{j=4}^m a_j\right)\psi_1^2(x).$$

If we assume that the operator T_σ is bounded from $L^p\times L^q$ into L^t, we could conclude then that

$$\left|\sum_{j=4}^m a_j\right|\lesssim\|f\|_{L^p}\lesssim\left(\sum_{j=4}^m|a_j|^2\right)^{1/2}, \tag{2-3}$$

where the last inequality follows from the Littlewood–Paley square function characterization of the L^p norm of f and the constants involved depend on ψ_1 but are independent of m. Since the a_j are arbitrary (2-3) is not possible. □

3. Sobolev space estimates

We will show that the class $BS^0_{1,1;\pi/4}$ produces bounded operators on product of Sobolev spaces. The situations in the modulation invariant and the general case are slightly different.

The modulation invariant case. We first consider the case of bilinear operators obtained from linear ones as in the previous section. That is, the symbol σ takes the form

$$\sigma(x,\xi,\eta)=\tau(x,\xi-\eta),$$

where τ belongs to the linear class $S^0_{1,1}$.

Theorem 3.1. *Let τ be a linear symbol in $S^0_{1,1}$ and consider the bilinear operator T_σ, where $\sigma(x,\xi,\eta) = \tau(x,\xi-\eta)$. If $s > 0$ and $1 < p,q,t < \infty$ satisfy the Hölder relation (1-8), then T_σ is bounded from $W^{s,p} \times W^{s,q}$ into $W^{s,t}$.*

Proof. We begin by recalling the Coifman–Meyer reduction for symbols in $S^0_{1,1}$, which is by now a standard technique. (For details see [Coifman and Meyer 1978b, Chapter II, Section 9] for example.) The symbol τ can be decomposed into an absolutely convergent sum of *reduced symbols* of the form

$$\tau(x,\xi) = \sum_{j=0}^{\infty} m_j(2^j x)\widehat{\psi}(2^{-j}\xi),$$

where ψ is a smooth function whose Fourier transform is supported on $\{\xi :\ 2^{-1} \le |\xi| \le 2\}$ and $\{m_j\}_{j\ge 0}$ is a uniformly bounded collection of $C^r(\mathbb{R})$ functions where r can be taken arbitrarily large. Due to this reduction, we need only to study a symbol of the form

$$\sigma(x,\xi,\eta) = \sum_{j\ge 0} m_j(2^j x)\widehat{\psi}(2^{-j}(\xi-\eta)) := \sum_{j\ge 0} \sigma_j(x,\xi,\eta).$$

We use the notation of [Bourdaud 1988]. We expand m_j into an inhomogeneous Littlewood–Paley decomposition using (1-5) so that

$$m_j = \sum_{k\ge 0} m_{j,k} \tag{3-1}$$

with the spectrum of $m_{j,k}$ contained in the dyadic annulus $\{\xi :\ 2^{k-1} \le |\xi| \le 2^{k+1}\}$ for $k \ge 1$, and in the ball $\{\xi,\ |\xi| \le 2\}$ for $k = 0$. Then we define for $h \ge j$ the function $n_{j,h}(x) := m_{j,h-j}(2^j x)$. Due to the regularity of the function m_j, we have the following properties for $h \ge j+1$:

$$\operatorname{supp} \widehat{n}_{j,h} \subset \{\xi :\ 2^{h-1} \le |\xi| \le 2^{h+1}\} \tag{3-2}$$

and

$$\|n_{j,h}\|_{L^\infty} \le C_r 2^{(j-h)r}, \tag{3-3}$$

where, we mention again, the number r can be chosen as large as we want. For $h = j$ we have

$$\operatorname{supp} \widehat{n}_{j,j} \subset \{\xi :\ |\xi| \le 2^{j+1}\} \tag{3-4}$$

and

$$\|n_{j,j}\|_{L^\infty} \le C_r. \tag{3-5}$$

Note also that

$$m_j(2^j x) = m_{j,k}(2^j x) + \sum_{h\ge j+1} m_{j,h-j}(2^j x) = n_{j,j}(x) + \sum_{h\ge j+1} n_{j,h}(x). \tag{3-6}$$

Writing T_j for the bilinear operator with symbol $\widehat{\psi}(2^{-j}(\xi-\eta))$, we get

$$T_\sigma(f,g)(x) = \sum_{j\ge 0} m_j(2^j x) T_j(f,g)(x).$$

To study the norm of $T_\sigma(f,g)$ in the Sobolev space $W^{s,t}$, and with the functions Ψ and Φ as in (1-6), we need to estimate terms of the form $\Phi * T_\sigma(f,g)$ and, say for $k-2\geq 0$,

$$\Psi_{2^{-k}} * T_\sigma(f,g) := \sum_{j\geq 0} \Psi_{2^{-k}} * \big(m_j(2^j\,\cdot)T_j(f,g)\big) = I_k(f,g) + II_k(f,g),$$

where

$$I_k(f,g) := \sum_{j=0}^{k-2} \Psi_{2^{-k}} * \big(m_j(2^j\,\cdot)T_j(f,g)\big),$$
$$II_k(f,g) := \sum_{j\geq k-2} \Psi_{2^{-k}} * \big(m_j(2^j\,\cdot)T_j(f,g)\big).$$

We treat only I_k and II_k. The estimate for the other terms can be achieved with the same arguments (they are actually easier). For notational convenience, we identify $\Psi_{2^{-k}}$ with the convolution operator it defines (and similarly with other functions).

Estimate for I. We further decompose $m_j(2^j\,\cdot)$ and $T_j(f,g)$. Using (3-1), (3-6), and (1-5) we have

$$m_j(2^j x) = \Phi_{2^{-k}}(m_j(2^j\,\cdot))(x) + \sum_{l\geq k} n_{j,l}(x).$$

We also decompose $T_j(f,g)(x)$ as $\Phi_{2^{-k}}\big(T_j(f,g)\big)(x) + \sum_{p\geq k} \Psi_{2^{-p}}\big(T_j(f,g)\big)(x)$. Then

$$I_k(f,g) = \sum_{j=0}^{k-2} \Psi_{2^{-k}}\big(\Phi_{2^{-k}}(m_j(2^j\,\cdot))\Phi_{2^{-k}}(T_j(f,g))\big) + \sum_{j=0}^{k-2}\sum_{l\geq k} \Psi_{2^{-k}}\big(n_{j,l}\Phi_{2^{-k}}(T_j(f,g))\big) \tag{3-7}$$
$$+ \sum_{j=0}^{k-2}\sum_{p\geq k} \Psi_{2^{-k}}\big(\Phi_{2^{-k}}(m_j(2^j\,\cdot))\Psi_{2^{-p}}(T_j(f,g))\big) + \sum_{j=0}^{k-2}\sum_{l,p\geq k} \Psi_{2^{-k}}\big(n_{j,l}\Psi_{2^{-p}}(T_j(f,g))\big).$$

Using the notation $\widetilde{\phi}$ for a generic smooth function with bounded spectrum and $\widetilde{\psi}$ for a generic smooth function with a spectrum contained in an annulus around 0, we claim that we can write I_k as a sum of terms of three different form:

$$I_k(f,g) = \sum_{0\leq j\leq k-2} \Psi_{2^{-k}}(T_{\sigma_j}(f,g)) \approx (1)_k + (2)_k + (3)_k,$$

where

$$(1)_k := \sum_{j\leq k-2} \Psi_{2^{-k}}\big(n_{j,k}\widetilde{\phi}_{2^{-k}}(T_j(f,g))\big),$$
$$(2)_k := \sum_{j\leq k-2} \Psi_{2^{-k}}\big(\widetilde{\phi}_{2^{-k}}(m_j(2^j\,\cdot))\widetilde{\psi}_{2^{-k}}(T_j(f,g))\big),$$
$$(3)_k := \sum_{l\geq k}\sum_{j\leq k-2} \Psi_{2^{-k}}\big(n_{j,l}\widetilde{\psi}_{2^{-l}}(T_j(f,g))\big).$$

Let us explain this reduction. The first sum in (3-7) can be written as a finite linear combination of terms taking the form $(1)_k$ and $(2)_k$. Indeed, consider one of the general terms

$$\Psi_{2^{-k}}\big(\Phi_{2^{-k}}(m_j(2^j\,\cdot))\Phi_{2^{-k}}(T_j(f,g))\big).$$

Denote by ξ the frequency variable of $m_j(2^j\,\cdot)$ and by η that of $T_j(f,g)$. We have a nonvanishing contribution if

$$|\eta|\le 2^k,\qquad |\xi|\le 2^k\quad\text{and}\quad |\eta+\xi|\simeq 2^k,$$

where we have used that the spectrum of the product is included in Minkowski sum of spectra. Consequently, this is possible only if $|\xi|\simeq 2^k$, which corresponds to $(1)_k$ (recall that $n_{j,l}$ has spectrum in $\{|\xi|\approx 2^l\}$), or $|\eta|\simeq 2^k$, which corresponds to $(2)_k$.

Concerning the second sum in (3-7), it can also be reduced to the sum for $l\approx k$ (as the other terms vanish) and it is a finite sum of terms like $(1)_k$. Similar reasoning for the third term in (3-7) gives that it is controlled by $(2)_k$. Finally, the general term in the fourth sum in (3-7) is nonzero if

$$2^p\pm 2^l\approx 2^k.$$

But, since the inner double sum has $l,p\ge k$, the general term is nonzero only for $l\approx p$. We see then that the double sum (over l and p) reduces to one sum over only one parameter. It follows that the fourth sum in (3-7) is similar to $(3)_k$.

We now study each of the model sums $(1)_k$, $(2)_k$, $(3)_k$.

The sum with $(1)_k$. We use the estimate (3-3) for $n_{j,k}$ with $r>s$ and Young's inequality to obtain

$$\begin{aligned}\|2^{ks}(1)_k\|_{l^2(k\in\mathbb{N})}&\lesssim\Big\|\sum_{j+2\le k}2^{(j-k)r}2^{ks}M\big(\tilde{\phi}_{2^{-k}}(T_j(f,g))\big)\Big\|_{l^2(k\in\mathbb{N})}\\&\lesssim\Big\|\sum_{j+2\le k}2^{js}2^{(j-k)(r-s)}M\big(\tilde{\phi}_{2^{-k}}(T_j(f,g))\big)\Big\|_{l^2(k\in\mathbb{N})}\lesssim\|2^{js}M^2(T_j(f,g))\|_{l^2(j\in\mathbb{N})}.\end{aligned}$$

Therefore,

$$\big\|\|2^{ks}(1)_k\|_{l^2(k\in\mathbb{N})}\big\|_{L^t}\lesssim\big\|\|2^{js}M^2(T_j(f,g))\|_{l^2(j\in\mathbb{N})}\big\|_{L^t}\tag{3-8}$$

and from the Fefferman–Stein vector-valued inequality [1971] for the maximal operator M, we deduce that

$$\big\|\|2^{ks}(1)_k\|_{l^2(k\in\mathbb{N})}\big\|_{L^t}\lesssim\big\|\|2^{js}T_j(f,g)\|_{l^2(j\in\mathbb{N})}\big\|_{L^t}.$$

We can use now a linearization argument. By writing $r_j(\omega)$ for Rademacher functions ($\omega\in[0,1]$), we know that (see, e.g., Appendix C in [Grafakos 2004]):

$$\big\|\|2^{ks}(1)_k\|_{l^2(k\in\mathbb{N})}\big\|_{L^t}\lesssim\Big\|\Big\|\sum_j 2^{js}r_j(\omega)T_j(f,g)\Big\|_{L^t(\omega\in[0,1])}\Big\|_{L^t}.$$

By Fubini's Theorem, we have

$$\Big\|\|2^{ks}(1)_k\|_{l^2(k\in\mathbb{N})}\Big\|_{L^t}\lesssim\Big\|\Big\|\sum_j 2^{js}r_j(\omega)T_j(f,g)\Big\|_{L^t}\Big\|_{L^t(\omega\in[0,1])}.$$

Now for each $\omega \in [0,1]$, the operator $(f,g) \to \sum_j 2^{js} r_j(\omega) T_j(f,g)$ is the bilinear operator associated to the symbol

$$\sum_j 2^{js} r_j(\omega) \widehat{\Psi}(2^{-j}(\xi-\eta)) \in BS^s_{1,0;\pi/4}.$$

It follows from [Bényi et al. 2006] and [Bernicot 2010] (since the symbol is x-independent) that these bilinear operators are bounded from $W^{s,p} \times W^{s,q}$ into L^t (uniformly on $\omega \in [0,1]$) and the proof in this case is complete.

The sum with $(2)_k$. This term is the most difficult to estimate. Using again the boundedness of the functions m_j in $C^r \hookrightarrow L^\infty$, we can estimate

$$\|2^{ks}(2)_k\|_{l^2(k\in\mathbb{N})} \lesssim \Bigg\| \sum_{j+2\le k} 2^{ks} M\left(\widetilde{\psi}_{2^{-k}}\left[T_j(f,g)\right]\right)(x)\Bigg\|_{l^2(k\in\mathbb{N})}. \tag{3-9}$$

We observe that

$$\begin{aligned}\widetilde{\psi}_{2^{-k}}\left[T_j(f,g)\right](x) &= \int \widetilde{\psi}_{2^{-k}}(x-z)\int \widehat{\Psi}(2^{-j}(\xi-\eta))\widehat{f}(\xi)\widehat{g}(\eta)e^{iz(\xi+\eta)}\,d\xi\,d\eta\,dz\\ &= \int \widehat{\widetilde{\psi}}(2^{-k}(\xi+\eta))\widehat{\Psi}(2^{-j}(\xi-\eta))\widehat{f}(\xi)\widehat{g}(\eta)e^{ix(\xi+\eta)}\,d\xi\,d\eta.\end{aligned}$$

We must have $|\xi+\eta| \approx 2^k$ and $|\xi-\eta| \approx 2^j$. But we only have terms with $2^j < 2^k/4$, so we deduce that $|\xi| \approx |\eta| \approx 2^k$. It follows that we can further localize in the frequency plane adding a new function $\overline{\psi}$ (whose spectrum is contained in an annulus) such that

$$\widetilde{\psi}_{2^{-k}}(T_j(f,g))(x) = \widetilde{\psi}_{2^{-k}}\big(T_j(\overline{\psi}_{2^{-k}}f, \overline{\psi}_{2^{-k}}g)\big)(x)$$

Going back to (3-9) we obtain by the Cauchy–Schwartz inequality (there are k terms in the inner sum)

$$\|2^{ks}(2)_k\|_{l^2(k\in\mathbb{N})} \lesssim \Big\|2^{ks}k^{1/2}\big\|M\big(\widetilde{\psi}_{2^{-k}}\big(T_j(\overline{\psi}_{2^{-k}}(f),\overline{\psi}_{2^{-k}}(g))\big)\big)\big\|_{l^2(j\in\mathbb{N})}\Big\|_{l^2(k\in\mathbb{N})}.$$

We then obtain similarly as in the previous case

$$\begin{aligned}\big\|\|2^{ks}(2)_k\|_{l^2(k\in\mathbb{N})}\big\|_{L^t} &\lesssim \Bigg\|\Big\|2^{ks}k^{1/2}\Big\|M^2\big(T_j(\overline{\psi}_{2^{-k}}(f),\overline{\psi}_{2^{-k}}(g))\big)\Big\|_{l^2(j\in\mathbb{N})}\Big\|_{l^2(k\in\mathbb{N})}\Bigg\|_{L^t}\\ &\lesssim \Bigg\|\Big\|2^{ks}k^{1/2}\Big\|T_j(\overline{\psi}_{2^{-k}}(f),\overline{\psi}_{2^{-k}}(g))\Big\|_{l^2(j\in\mathbb{N})}\Big\|_{l^2(k\in\mathbb{N})}\Bigg\|_{L^t}.\end{aligned}$$

We linearize in j as before and use the fact that $k^{1/2} \lesssim 2^{ks}$ (as $s>0$) to obtain

$$\begin{aligned}\big\|\|2^{ks}(2)_k\|_{l^2(k)}\big\|_{L^t} &\lesssim \Bigg\|\Big\|\Big\|\sum_j r_j(\omega)T_j\big(2^{ks}\overline{\psi}_{2^{-k}}(f),2^{ks}\overline{\psi}_{2^{-k}}(g)\big)\Big\|_{L^1(\omega\in[0,1])}\Big\|_{l^2(k\in\mathbb{N})}\Bigg\|_{L^t}\\ &\lesssim \Bigg\|\Big\|\Big\|\sum_j r_j(\omega)T_j\big(2^{ks}\overline{\psi}_{2^{-k}}(f),2^{ks}\overline{\psi}_{2^{-k}}(g)\big)\Big\|_{l^2(k\in\mathbb{N})}\Big\|_{L^t}\Bigg\|_{L^1(\omega\in[0,1])}.\end{aligned}$$

For each $\omega \in [0,1]$, we can invoke a vector-valued result for bilinear operators of [Grafakos and Martell 2004]. More precisely, as explained when we dealt with $(1)_k$, for each $\omega \in [0,1]$ the bilinear operator $(f,g) \to \sum_j r_j(\omega) T_j(2^{ks}\overline{\psi}_{2^{-k}}(f), 2^{ks}\overline{\psi}_{2^{-k}}(g))$ is bounded from $L^p \times L^q$ to L^t (since it is associated to a symbol independent of x). Then, Theorem 9.1 in [Grafakos and Martell 2004] implies that the operator admits an l^2-valued bilinear extension, which yields

$$\left\| \|2^{ks}(2)_k\|_{l^2(k\in\mathbb{N})} \right\|_{L^t} \lesssim \left\| \left\| \left\| 2^{ks}\overline{\psi}_{2^{-k}}(f) \right\|_{l^2(k\in\mathbb{N})} \right\|_{L^p} \left\| \left\| 2^{ks}\overline{\psi}_{2^{-k}}(g) \right\|_{l^2(k\in\mathbb{N})} \right\|_{L^q} \right\|_{L^1(\omega)},$$

with estimates uniformly in $\omega \in [0,1]$. This concludes the proof of the case $(2)_k$.

The sum with $(3)_k$. The analysis in this case is entirely analogous as the case $(1)_k$ and so we leave the details to the reader.

Estimate for II. In this case, we decompose the term $II_k(f,g)$ with quantities appearing as a linear combination of terms of the form

$$(1)_k = \sum_{j\geq k-2} \Psi_{2^{-k}}\big(n_{j,j}\widetilde{\phi}_{2^{-j}}(T_j(f,g))\big) \quad \text{or} \quad (2)_k = \sum_{j\geq k-2}\sum_{l\geq j} \Psi_{2^{-j}}\big(n_{j,l}(x)\widetilde{\psi}_{2^{-l}}(T_j(f,g))\big).$$

Indeed with a similar reasoning as before and since $j \geq k-2$, the general quantity in II_k has a nonvanishing contribution only if the frequency variables of $m_j(2^j\,\cdot)$ or $T_j(f,g)$ are contained in $\{|\xi| \lesssim 2^j\}$ (which corresponds to $(1)_k$) or if the two frequency variables are contained in $\{|\xi| \simeq 2^l\}$ for some $l \geq j$ (which corresponds to $(2)_k$).

The study of $(2)_k$ is similar to the one of $(1)_k$ with the help of fast decays in l (see (3-3)), so we only write the proof for $(1)_k$. By the estimates on $n_{j,j}$, we have

$$\left\| \|2^{ks}(1)_k\|_{l^2(k\in\mathbb{N})} \right\|_{L^t} \lesssim \left\| \left\| \sum_{j\geq k-2} 2^{(k-j)s} 2^{js} M^2(T_j(f,g)) \right\|_{l^2(k\in\mathbb{N})} \right\|_{L^t}.$$

Using $s > 0$ and Young's inequality for the l^2-norm on k, we get the bound

$$\left\| \|2^{js}M^2(T_j(f,g))\|_{l^2(j\in\mathbb{N})} \right\|_{L^t}.$$

We have already studied such quantities in the first case — see (3-8) — and proved the appropriate bounds. □

Remark 3.2. Since $\sigma(x,\xi,\eta) = \tau(x,\xi-\eta)$ is bounded, the function $T_\sigma(1,1)$ (rigorously defined in [Bényi et al. 2009]) is given by

$$T_\sigma(1,1) = \sigma(\cdot,0,0) \in L^\infty \subset BMO.$$

If the transposes of T_σ are also given by symbols in the classes $BS^0_{1,1;\theta}$ or even by some bounded functions, then we can use the bilinear $T(1)$-Theorem of [Bényi et al. 2009] (since T_σ is modulation invariant) to conclude that T is bounded on the product of Lebesgue spaces. The counterexample of the previous section shows that this is not always the case, so the classes $BS^0_{1,1;\theta}$ cannot be closed by transposition. As mentioned in the introduction the smaller classes $BS^0_{1,0;\theta}$ are.

The general case. In this subsection, we consider general symbols in the class $BS^0_{1,1;\pi/4}$. We obtain a slightly less general result than the one in the previous case.

Theorem 3.3. *If $\sigma \in BS_{1,1;\pi/4}$ and $s > \frac{1}{2}$, then the bilinear operator T_σ is bounded from $W^{s,p} \times W^{s,q}$ into $W^{s,t}$ for all exponents $1 < p, q, t < \infty$ satisfying the Hölder condition (1-8).*

Proof. We want to adapt the proof of Theorem 3.1. We briefly indicate the extra difficulties faced.

Reduction to elementary symbols. We first reduce the problem to the study of elementary symbols taking the following form

$$\sigma(x,\xi,\eta) = \sum_{\substack{j\geq 0\\ l\in\mathbb{Z}}} m_{j,l}(2^j x)\widehat{\Psi}\big(2^{-j}(\xi-\eta)\big)\widehat{\Psi}\big(l+2^{-j}(\xi+\eta)\big). \tag{3-10}$$

Let us give a sketch of such a reduction. Multiplying the symbol σ by

$$\widehat{\Psi}\big(2^{-j}(\xi-\eta)\big)\widehat{\Psi}\big(l+2^{-j}(\xi+\eta)\big),$$

we localize it in frequency to the domain

$$\big\{(\xi,\eta) : |\xi-\eta| \simeq 2^j \text{ and } |\xi+\eta+l\,2^j| \simeq 2^j,\big\}$$

which can be compared to a ball of radius 2^j. This compactly supported symbols $\sigma_{j,l}$ satisfy

$$|\partial_x^\alpha \partial_{\xi,\eta}^\beta \sigma_{j,l}(x,\xi,\eta)| \leq C_{\alpha\beta} 2^{j(\alpha-\beta)}.$$

As usually, we decompose this symbol into a Fourier series, obtaining

$$\sigma_{j,l}(x,\xi,\eta) = \sum_{a,b\in\mathbb{Z}^2} \gamma_{a,b}(x)e^{i(a\xi+b\eta)}\widehat{\Psi}\big(2^{-j}(\xi-\eta)\big)\widehat{\Psi}\big(l+2^{-j}(\xi+\eta)\big).$$

The modulation term $e^{i(a\xi+b\eta)}$ does not play a role, as it corresponds to translation in physical space (which does not modify the Lebesgue norms), it remains for us to check that the coefficients $\gamma_{a,b}$ are fast decreasing in (a,b) and satisfies the desired smoothness in x. To do so, we remark that, for $\alpha \in \mathbb{N}$, integration by parts yields

$$\begin{aligned}
\big|\partial_x^\alpha \gamma_{a,b}(2^{-j}x)\big| &\lesssim 2^{-j\alpha-2j}\left|\iint e^{-i(a\xi+b\eta)}\partial_x^\alpha \sigma_{j,l}(2^{-j}x,\xi,\eta)\,d\xi\,d\eta\right| \\
&\lesssim 2^{-j\alpha-2j}\big(1+|a|+|b|\big)^{-M}\left|\iint e^{-i(a\xi+b\eta)}\big(1+\partial_\xi^M+\partial_\eta^M\big)\partial_x^\alpha \sigma_{j,l}(2^{-j}x,\xi,\eta)\,d\xi\,d\eta\right| \\
&\lesssim \big(1+|a|+|b|\big)^{-M},
\end{aligned}$$

where M is an integer that can be chosen as large as we wish. So we conclude that the functions $\gamma_{a,b}(2^{-j}\,\cdot)$ are uniformly bounded in C^r (for r arbitrarily large) with fast decays in (a,b). This operation (expansion in Fourier series) allows us to reduce the study of σ to reduced symbols taking the form (3-10).

Study of elementary symbols. We adapt the proof of Theorem 3.1 and use the same notation. We have to study the sum

$$\sum_{\substack{j\geq 0\\ l\in\mathbb{Z}}} m_{j,l}(2^j x)T_{j,l}(f,g), \tag{3-11}$$

where $T_{j,l}$ is the bilinear operator associated to the x-independent symbol

$$\widehat{\Psi}(2^{-j}(\xi-\eta))\widehat{\Psi}(l+2^{-j}(\xi+\eta)).$$

We can proceed as in the modulation invariant case and consider the different cases, eventually arriving to the point where we need to linearize with respect to the parameter j. But now, we also have to linearize according to the new parameter l. When we estimate the square function of $T_{j,l}$, we have to study $\Psi_{2^{-k}}(T_{j,l}(f,g))$ and we are interested only in the indices j,l satisfying $|\xi+\eta|\approx 2^k$ with $|\xi-\eta|\approx 2^j$ and $|\xi+\eta+l\,2^j|\approx 2^j$. However, due to the use of the Cauchy–Schwartz inequality in l, we will have an extra term bounded by $2^{(k-j)/2}$, which corresponds to the square root of the number of indices l satisfying all these conditions. For the study of $(1)_k$ and $(3)_k$ there is no problem, since r can be chosen satisfying $r>s+\frac{1}{2}$. However, for the study of $(2)_k$ we will need $2^{k(s+1/2)}k^{1/2}\leq 2^{ks}2^{ks}$ and so we need to assume that $s>\frac{1}{2}$. □

Remark 3.4. It is interesting to note that without the modulation invariance, an extra exponent $\frac{1}{2}$ appears. We do not know if our result is optimal or not. Moreover, unlike the modulation invariance case, we also do not know whether a general operator T_σ with symbol $\sigma\in BS_{1,1;\pi/4}$, and whose two adjoints satisfy similar assumptions, is bounded on product of Lebesgue spaces. To address this question, it would be interesting to obtained (if possible) a $T(1)$-Theorem as in [Bényi et al. 2009] but without assuming modulation invariance.

4. An improvement on paramultiplication

In this section, we will use x-independent symbols in $BS_{1,1;\pi/4}$ (and also in the smaller class $BS_{1,0;\pi/4}$) to describe a new paramultiplication operation. We will obtain an improvement over the classical paramultiplication first studied in [Bony 1981] in the L^2 setting and extended in [Meyer 1981a; 1981b] to L^p norms. The classical paraproducts and their properties hold for multidimensional variables, however our improvement works (at least at this moment) only in the one dimensional case.

We start with the classical definition.

Definition 4.1. Let f and b be two smooth functions and let Φ and Ψ be as in (1-5) and (1-6). We assume that for all $\eta\in\operatorname{supp}\widehat{\Phi}$ and $\xi\in\operatorname{supp}\widehat{\Psi}$ we have

$$|\eta|\leq \tfrac{1}{2}|\xi|.$$

Then paramultiplication by b is defined by

$$\Pi_b(f):=\sum_{k\in\mathbb{Z}}\Phi_{2^k}(f)\Psi_{2^k}(b).$$

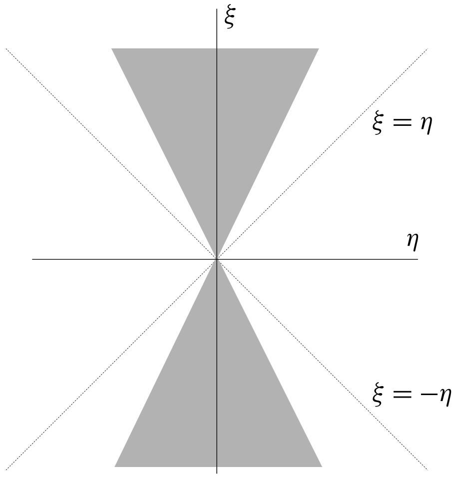

Figure 1. Support of the bilinear symbol associated to the paraproduct Π.

The operator $(b, f) \to \Pi_b(f)$ can essentially be thought as a bilinear multiplier whose symbol is a smooth decomposition of the characteristic function of the cone in Figure 1.

The following two propositions are well-known properties for paraproducts (see [Bony 1981, Theorems 2.1 and 2.5], for example, for the original results involving L^2-Sobolev spaces and [Meyer 1981a; 1981b] for extensions to other Sobolev spaces):

Proposition 4.2. *For all $s > 0$ and $p \in (1, \infty)$ the linear operator Π_b is bounded on the Sobolev space $W^{s,p}$, satisfies*

$$\|\Pi_b\|_{W^{s,p} \to W^{s,p}} \lesssim \|b\|_{L^\infty},$$

and the operation can be extended to an L^∞ function b.

The paramultiplication approximates pointwise multiplication in the following sense.

Proposition 4.3. *Let $1 < t < \infty$ and $s > 1/t$. For $f \in W^{s,t}$ and $g \in W^{s,t}$, we have*

$$\left\| fg - \Pi_f(g) - \Pi_g(f) \right\|_{W^{2s-1/t,t}} \lesssim \|f\|_{W^{s,t}} \|g\|_{W^{s,t}}.$$

The exponent of regularity $2s - \frac{1}{t}$ is bigger than s for $ts > 1$. This gain is very important. The result is essentially due to the fact that, in frequency space, the error term has only a contribution from f and g when

$$\{|\xi| \approx |\eta|\},$$

i.e., in a cone along the two main diagonals.

Using the new bilinear operators (whose singularities are localized on a line in the frequency plane), we can define a new paramultiplication operation $\widetilde{\Pi}$ such that the error term will be concentrated in the frequency plane exactly in a strip (of fixed width) around the two diagonals. In this way, we will be able to get a better gain for the exponent of regularity.

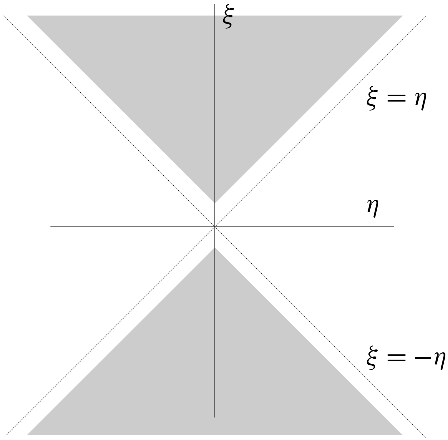

Figure 2. Support of the bilinear symbol associated to the new paraproduct $\widetilde{\Pi}$.

Definition 4.4. Let Θ be a smooth function on $\mathbb{R}$ whose Fourier transform $\widehat{\Theta}$ satisfies

$$\omega \geq 2 \implies \widehat{\Theta}(\omega)=1 \quad \text{and} \quad -\infty<\omega \leq 1 \implies \widehat{\Theta}(\omega)=0.$$

Then we define, for $b, f \in \mathcal{S}(\mathbb{R})$, the *improved paramultiplication* by b (written $\widetilde{\Pi}_b(f)$) by

$$\widetilde{\Pi}_b(f)(x)=\int_{\mathbb{R}^2} e^{ix(\xi+\eta)}\hat{b}(\xi)\hat{f}(\eta)\big(\widehat{\Theta}(\xi-\eta)\widehat{\Theta}(\xi+\eta)+\widehat{\Theta}(\eta-\xi)\widehat{\Theta}(-\xi-\eta)\big)\,d\xi\,d\eta. \tag{4-1}$$

The new bilinear multiplier $(b, f) \to \widetilde{\Pi}_b(f)$ is associated to a bilinear symbol, corresponding to a smooth version of the characteristic function of the region in Figure 2. We remark that this new region approximates the domain $\{(\xi,\eta), |\xi| \geq |\eta|\}$ better than the region in Figure 1.

This new operation satisfies a similar property to the one in Proposition 4.2.

Proposition 4.5. *Let $s \geq 0$ and let $1<p,q,t<\infty$ be exponents satisfying (1-8). For every $\epsilon>0$ and $b \in W^{\epsilon,p}(\mathbb{R})$, the improved paramultiplication by b is well defined and produce a bounded operation from $W^{s,q}$ to $W^{s,t}$. In fact, there exists a constant $C=C(s,\epsilon,p,q,t)$ such that for all functions $f \in W^{s,q}$,*

$$\left\|\widetilde{\Pi}_b(f)\right\|_{W^{s,t}} \leq C\|b\|_{W^{\epsilon,p}}\|f\|_{W^{s,q}}.$$

Moreover if $s=0$, the exponent $\epsilon=0$ is allowed.

Proof. The new paramultiplication is given by two terms, which can be studied by identical arguments. We only deal with the first term but for simplicity in the notation we still write

$$\widetilde{\Pi}_f(b)(x)=\int_{\mathbb{R}^2} e^{ix(\xi+\eta)}\hat{b}(\xi)\hat{f}(\eta)\widehat{\Theta}(\xi-\eta)\widehat{\Theta}(\xi+\eta)\,d\xi\,d\eta.$$

We note that this function $\widetilde{\Pi}_b(f)$ corresponds to the operator $T_\sigma(b, f)$ associated to the bilinear symbol

$$\sigma(\xi,\eta)=\widehat{\Theta}(\xi-\eta)\widehat{\Theta}(\xi+\eta).$$

We need to show that T_σ is continuous from $W^{\epsilon,p} \times W^{s,q}$ to $W^{s,r}$.

The case $s = 0$. We compute the Fourier transform of $T_\sigma(b, f)$,

$$\begin{aligned}\widehat{T_\sigma(b, f)}(\omega) &= \int_{\xi+\eta=\omega} \widehat{b}(\xi)\widehat{f}(\eta)\widehat{\Theta}(\xi-\eta)\widehat{\Theta}(\eta+\xi)\,d\xi\,d\eta \\ &= \widehat{\Theta}(\omega)\int_{\xi+\eta=\omega} \widehat{b}(\xi)\widehat{f}(\eta)\widehat{\Theta}(\xi-\eta)\,d\xi\,d\eta = \widehat{\Theta}(\omega)\widehat{T_\tau(b, f)}(\omega),\end{aligned}$$

where τ is given by $\tau(\xi, \eta) = \widehat{\Theta}(\xi - \eta)$. So in fact we can write $T_\sigma(b, f)$ as the convolution product between Θ and $T_\tau(b, f)$. Since the function Θ in Definition 4.4 is smooth, the convolution operation by Θ is bounded on L^t. We obtain also

$$\|T_\sigma(b, f)\|_{L^t} \lesssim \|T_\tau(b, f)\|_{L^t}\,.$$

Now the bilinear operator T_τ is associated to the symbol τ which satisfies the Hörmander multiplier conditions related to the frequency line $\{\xi = \eta\}$. That is,

$$\left|\partial_\xi^\alpha \partial_\eta^\beta \tau(\xi, \eta)\right| \lesssim |\xi-\eta|^{-\alpha-\beta}$$

for all α and β. It follows from [Gilbert and Nahmod 2000] that this bilinear operator maps $L^p \times L^q$ to L^t and we obtain the desired result

$$\|T_\sigma(b, f)\|_{L^t} \lesssim \|b\|_{L^p}\|f\|_{L^q}.$$

Note that for the case $s = 0$ no regularity on b is really needed.

The case $s > 0$. Let Φ and Ψ be as in (1-5) and (1-6). We study first $\Phi * T_\sigma(f, g)$. We have

$$\widehat{\Phi * T_\sigma(b, f)}(\omega) = \widehat{\Phi}(\omega)\,\widehat{\Theta}(\omega)\,\widehat{T_\tau(b, f)}(\omega).$$

The spectral condition over Φ and Θ imply that $\omega \approx 1$. So for ξ and η (the frequency variables of b and f) satisfying $\xi - \eta \geq 1$ and $\xi + \eta = \omega \approx 1$, we deduce that either η is bounded or $-\xi \approx \eta \gg 1$. Therefore, we can find a smooth function ζ and an other one $\widetilde{\psi}$ (whose spectrum is contained in an annulus around 0) such that

$$\Phi * T_\sigma(b, f) = \Phi * T_\sigma(b, \zeta * f) + \sum_{l\geq 0} \Phi * T_\sigma\big(\widetilde{\psi}_{2^{-l}} * b, \widetilde{\psi}_{2^{-l}} * f\big).$$

Using $0 < \epsilon$, we get by the Cauchy–Schwartz inequality

$$\left|\Phi * T_\sigma(b, f)\right| \leq \left|\Phi * T_\sigma(b, \zeta * f)\right| + \left(\sum_{l\geq 0} 2^{2\epsilon l}\left|M\big(T_\sigma(\widetilde{\psi}_{2^{-l}} * b, \widetilde{\psi}_{2^{-l}} * f)\big)\right|^2\right)^{1/2}. \tag{4-2}$$

By the same reasoning for an integer $k \geq 1$, if ξ and η satisfy $\eta \geq \xi + 1$ and $1 < \xi + \eta = \omega \approx 2^k$, we deduce that either $\eta \approx 2^k$ or $-\xi \approx \eta \gg 2^k$. So we can find a smooth function $\widetilde{\psi}$ (for convenience we keep the same notation), whose spectrum is contained in an annulus around 0 such that for all integer k large enough

$$\Psi_{2^{-k}} * T_\sigma(b, f) = \Psi_{2^{-k}} * T_\sigma\big(b, \widetilde{\psi}_{2^{-k}} * f\big) + \sum_{l\geq k} \Psi_{2^{-k}} * T_\sigma\big(\widetilde{\psi}_{2^{-l}} * b, \widetilde{\psi}_{2^{-l}} * f\big).$$

Using the same ϵ, we get by the Minkowski and Cauchy–Schwartz inequalities

$$\begin{aligned}
&\Big(\sum_k 2^{2ks}\,|\Psi_{2^{-k}} * T_\sigma(b,f)|^2\Big)^{1/2} \\
&\lesssim \Big(\sum_k 2^{2ks} M\big(T_\sigma(b,\widetilde{\psi}_{2^{-k}} * f)\big)^2\Big)^{1/2} + \sum_{l\geq 0}\Big(\sum_{k\leq l} 2^{2ks}\,\Big|\Psi_{2^{-k}} * T_\sigma(\widetilde{\psi}_{2^{-l}} * b,\widetilde{\psi}_{2^{-l}} * f)\Big|^2\Big)^{1/2} \\
&\lesssim \Big(\sum_k 2^{2ks} M\big(T_\sigma(b,\widetilde{\psi}_{2^{-k}} * f)\big)^2\Big)^{1/2} + \sum_{l\geq 0} 2^{ls} M\big(T_\sigma(\widetilde{\psi}_{2^{-l}} * b,\widetilde{\psi}_{2^{-l}} * f)\big) \\
&\lesssim \Big(\sum_k 2^{2ks} M\big(T_\sigma(b,\widetilde{\psi}_{2^{-k}} * f)\big)^2\Big)^{1/2} + \Big(\sum_{l\geq 0} 2^{2l(s+\epsilon)}\,\Big|M\big(T_\sigma(\widetilde{\psi}_{2^{-l}} * b,\widetilde{\psi}_{2^{-l}} * f)\big)\Big|^2\Big)^{1/2}. \qquad (4\text{-}3)
\end{aligned}$$

From (4-2) and (4-3), using the $L^q - L^t$ boundedness of $T_\sigma(b,\cdot)$ (proved in the first case), the vector-valued Fefferman–Stein inequality, and its bilinear version [Grafakos and Martell 2004, Theorem 9.1], we obtain the desired result:

$$\begin{aligned}
\|T_\sigma(b,f)\|_{W^{s,t}} &\lesssim \Big\|\,\big|\Phi * T_\sigma(b,f)\big| + \Big(\sum_{k\geq 0} 2^{2sk}\big|\Psi_{2^{-k}} * T_\sigma(b,f)\big|^2\Big)^{1/2}\Big\|_{L^t} \\
&\lesssim \|b\|_{L^p}\Big\|\,|\zeta * f| + \Big(\sum_{k\geq 0} 2^{2sk}\big|\widetilde{\psi}_{2^{-k}} * f\big|^2\Big)^{1/2}\Big\|_{L^q} \\
&\qquad\qquad + \Big\|\Big(\sum_{l\geq 0} 2^{2l\epsilon}\big|\widetilde{\psi}_{2^{-l}} * b\big|^2\Big)^{1/2}\Big\|_{L^p}\Big\|\Big(\sum_{k\geq 0} 2^{2sk}\big|\widetilde{\psi}_{2^{-k}} * f\big|^2\Big)^{1/2}\Big\|_{L^q} \\
&\lesssim \|b\|_{W^{\epsilon,p}}\|f\|_{W^{s,q}}.
\end{aligned}$$
□

Remark 4.6. We note that our new bilinear operation needs an extra regularity assumption $b \in W^{\epsilon,p}$ to keep the regularity of the function f (the case $s > 0$). This is due to the fact that the high frequencies of b play a role in the high frequency of $\widetilde{\Pi}_b(f)$ (which is natural) but in the low frequencies of $\widetilde{\Pi}_b(f)$ too. This last phenomenom does not appear in the classical paramultiplication operation. This point can be observed in the Figures 1 and 2. Let ω be the frequency variable of the paraproduct. For small ω, say $\omega \simeq 2$, the contributions of b and f correspond to the intersection of the cone in Figures 1 and 2 and the line $\{\omega = \xi + \eta\}$. In the first case (Figure 1) this intersection is bounded set, whereas in the second case (Figure 2) it is not bounded and contains also high frequencies of b.

We now obtain an improvement on Proposition 4.3.

Proposition 4.7. *Let $t \in (1,\infty)$ and $s \geq 1/t$. If $f \in W^{s,t}$ and $g \in W^{s,t}$, then*

$$\big\|fg - \widetilde{\Pi}_f(g) - \widetilde{\Pi}_g(f)\big\|_{W^{2s,t}} \lesssim \|f\|_{W^{s,t}}\|g\|_{W^{s,t}}.$$

Remark 4.8. As already mentioned, in the classical paramultiplication calculus, the regularity result is true for $s \geq 1/t$ and the gain is only $s - 1/t$.

Proof. Let us denote by D the difference operator

$$D(f,g) := fg - \widetilde{\Pi}_f(g) - \widetilde{\Pi}_g(f).$$

It corresponds to the bilinear operator associated to the symbol τ given by

$$\tau(\xi,\eta) := 1 - \widehat{\Theta}(\eta-\xi)\widehat{\Theta}(\eta+\xi) - \widehat{\Theta}(-\eta+\xi)\widehat{\Theta}(-\eta-\xi) - \widehat{\Theta}(\xi-\xi)\widehat{\Theta}(\eta+\xi) - \widehat{\Theta}(\eta-\xi)\widehat{\Theta}(-\eta-\xi).$$

This symbol is supported in the complement of the cone drawn in Figure 2 and the one symmetric to it. Consequently, it is supported in two strips (around the two diagonals)

$$\operatorname{supp}(\tau) \subset \big\{(\xi,\eta) : |\xi-\eta| \le 3\big\} \cup \big\{(\xi,\eta) : |\xi+\eta| \le 3\big\}.$$

We can then reproduce a similar reasonning as used for Proposition 4.5. The symbol τ can be decomposed in two parts τ_1,τ_2; the first one supported in $\big\{(\xi,\eta) : |\xi+\eta| \le 3\big\}$ and the second one supported in $\big\{(\xi,\eta) : |\xi-\eta| \le 3\big\}$.

The bilinear multiplier associated to τ_1 has only low frequencies, hence

$$\|T_{\tau_1}(f,g)\|_{W^{2s,t}} \lesssim \|T_{\tau_1}(f,g)\|_{L^t}.$$

Using Proposition 4.5 with exponents $t,p,q \in (1,\infty)$ satisfying (1-8), it follows that

$$\|T_{\tau_1}(f,g)\|_{W^{2s,t}} \lesssim \|f\|_{L^p}\|g\|_{L^q} \lesssim \|f\|_{W^{s,t}}\|g\|_{W^{s,t}},$$

where we have used the Sobolev embeding $W^{s,t} \subset L^p$ since $s \ge \frac{1}{t} > \frac{1}{t} - \frac{1}{p}$ (and similarly with q).

Concerning the second part τ_2, it is easy to check that, on its support, $1+|\xi+\eta|$, $1+|\xi|$ and $1+|\eta|$ are comparable and in addition

$$\max\big\{1+|\xi+\eta|, 1+|\xi|,\ 1+|\eta|\big\} - \min\big\{1+|\xi+\eta|, 1+|\xi|,\ 1+|\eta|\big\} \lesssim 1. \tag{4-4}$$

We claim that T_{τ_2} is bounded from $L^t \times L^t$ into L^t. Indeed, the symbol τ_2 is supported around the diagonal $\xi=\eta$ and it takes the form

$$\tau_2(\xi,\eta) = m(\xi-\eta),$$

for a smooth function m supported on $[-3,3]$. It follows that

$$T_{\tau_2}(f,g)(x) = \int \widehat{m}(y) f(x-y) g(x+y)\,dy. \tag{4-5}$$

Since $m \in \mathcal{S}(\mathbb{R})$ we have, in particular, that $\widehat{m} \in L^1 \cap L^\infty$, and using Minkowski's inequality we easily deduce that T_{τ_2} is bounded from $L^\infty \times L^\infty$ to L^∞ and from $L^1 \times L^1$ to L^1. By (complex) bilinear interpolation, we conclude that T is bounded from $L^t \times L^t$ to L^t, for $1<t<\infty$.

It remains to estimate T_{τ_2} in the Sobolev space. We let the reader verify that, as in similar previously done computations (and using (4-4)), T_{τ_2} can be decomposed as

$$T_{\tau_2}(f,g) = \sum_{k\ge 0} \Psi_{2^{-k}} T_{\tau_2}(\Psi^1_{2^{-k}} f, \Psi^2_{2^{-k}} g), \tag{4-6}$$

for some smooth frequency truncations Ψ, Ψ^1, Ψ^2. It follows that

$$\begin{aligned}
&\Big\| \Big(\sum_{k\geq 0} 2^{k4s} \big| \Psi_{2^{-k}} T_{\tau_2}(\Psi^1_{2^{-k}} f, \Psi^2_{2^{-k}} g) \big|^2 \Big)^{1/2} \Big\|_{L^t} \\
&\qquad \lesssim \Big\| \Big(\sum_{k\geq 0} 2^{k4s} \big| T_{\tau_2}(\Psi^1_{2^{-k}} f, \Psi^2_{2^{-k}} g) \big|^2 \Big)^{1/2} \Big\|_{L^t} \\
&\qquad \lesssim \Big\| \Big(\sum_{k\geq 0} 2^{ks} |\Psi^1_{2^{-k}} f|^2 \Big)^{1/2} \Big\|_{L^t} \Big\| \Big(\sum_{k\geq 0} 2^{k2s} |\Psi^2_{2^{-k}} g|^2 \Big)^{1/2} \Big\|_{L^t} \\
&\qquad \lesssim \|f\|_{W^{s,t}} \|g\|_{W^{s,t}},
\end{aligned}$$

where we have used the L^t boundedness of the operator T_{τ_2} and its l^2-vector-valued extension (given again by Theorem 9.1 of [Grafakos and Martell 2004]). □

Remark 4.9. The previous proof relies on the boundedness from $L^t \times L^t$ to L^t of T_{τ_2}. This property does not hold in the classical paraproduct situation.

We have given a proof by interpolation, where the specific form of τ_2 plays an important role. We would like to describe now a direct proof of the boundedness for the simpler case $t = 2$. The arguments are based on the geometric fact that the symbol τ_2 is supported on a strip around the diagonal with bounded width.

We can use in the L^2 case a partition of frequencies given by Δ_k a smooth truncation on the interval $[k-4, k+4]$:

$$\widehat{\Delta_k(f)}(\xi) = \chi(\xi - k) \widehat{f}(\xi),$$

where χ is a smooth function, supported on $[-4, 4]$ and equal to 1 on $[-3, 3]$. Then, by Plancherel's equality, we have

$$\|T_{\tau_2}(f, g)\|_{L^2} \lesssim \Big(\sum_{k\in\mathbb{Z}} \|\Delta_k(T_{\tau_2}(f, g)\|^2_{L^2} \Big)^{1/2}.$$

By (4-4), it follows that with other similar truncation operators Δ^1 and Δ^2,

$$\begin{aligned}
\|T_{\tau_2}(f, g)\|_{L^2} &\lesssim \Big(\sum_{k\in\mathbb{Z}} \big\| \Delta_k(T_{\tau_2}(\Delta^1_k(f), \Delta^2_k(g))) \big\|^2_{L^2} \Big)^{1/2} \\
&\lesssim \Big(\sum_{k\in\mathbb{Z}} \Big\| \mathbf{1}_{|\xi-k|\leq 4} \int \Big| \widehat{\Delta^1_k(f)}(\eta) \widehat{\Delta^2_k(g)}(\xi - \eta) \Big| \, d\eta \Big\|^2_{L^2} \Big)^{1/2} \\
&\lesssim \Big(\sum_{k\in\mathbb{Z}} \Big\| \widehat{\Delta^1_k(f)} \Big\|^2_{L^2} \Big\| \widehat{\Delta^2_k(g)} \Big\|^2_{L^2} \Big)^{1/2},
\end{aligned}$$

where we have used that each interval $[k-4, k+4]$ has bounded length. Since the collection of intervals $([k-4, k+4])_{k\in\mathbb{Z}}$ is a bounded covering, we can conclude the boundedness of T_{τ_2} from $L^2 \times L^2$ into L^2. (Note that the same argument does not apply in L^p.)

Remark 4.10. Our new definition of paramultiplication is based on bilinear operators associated to x-independent symbols of the class $BS_{1,0;\pi/4}$. We could use the Sobolev boundedness (proved in the first sections of the current paper) in order to define other kind of paramultiplications with an x-dependent symbol but we will not carry here such analysis any further.

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Received 23 Apr 2010. Revised 2 Sep 2010. Accepted 14 Oct 2010.

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ANALYSIS AND PDE
Vol. 4, No. 4, 2011

SOLITON DYNAMICS FOR GENERALIZED KdV EQUATIONS IN A SLOWLY VARYING MEDIUM

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We consider the problem of existence and global behavior of solitons for generalized Korteweg–de Vries equations (gKdV) with a slowly varying (in space) perturbation. We prove that such slowly varying media induce on the soliton dynamics large dispersive effects at large times. We also prove that, unlike the unperturbed case, there is no pure-soliton solution to the perturbed gKdV.

1. Introduction and main results

In this work we consider the following *generalized Korteweg–de Vries equation* (gKdV) on the real line:

$$u_t + (u_{xx} + f(x,u))_x = 0 \quad \text{in } \mathbb{R}_t \times \mathbb{R}_x. \tag{1-1}$$

Here $u = u(t,x)$ is a real-valued function and $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is a nonlinear function. This represents a generalization of the *Korteweg-de Vries* equation (KdV), which is the case $f(x,s) \equiv s^2$:

$$u_t + (u_{xx} + u^2)_x = 0 \quad \text{in } \mathbb{R}_t \times \mathbb{R}_x. \tag{1-2}$$

Another physically important case is the cubic one, $f(x,s) \equiv s^3$, when (1-1) is often called the (*focusing*) *modified* KdV equation (mKdV), while the case of an arbitrary integer power is what mathematicians generally refer to as the gKdV:

$$u_t + (u_{xx} + u^m)_x = 0 \quad \text{in } \mathbb{R}_t \times \mathbb{R}_x; \quad m \geq 2 \text{ integer}. \tag{1-3}$$

The original KdV equation arises in physics as a model of propagation of dispersive long waves, as

Research supported in part by a CONICYT-Chile and an *Allocation de Recherche* grants.
MSC2000: primary 35Q51, 35Q53; secondary 37K10, 37K40.
Keywords: generalized KdV equations, soliton dynamics, slowly varying medium.

pointed out by J. S. Russel in 1834 [Miura 1976]. The exact formulation of the KdV equation comes from [Korteweg and de Vries 1895]. The equation was rediscovered decades later in a numerical study [Zabusky and Kruskal 1965], after which a great amount of literature — physical, numerical and mathematical — has emerged on the subject; see for example [Bona et al. 1980; Kalisch and Bona 2000; Shih 1980; Mizumachi 2003; Miura 1976].

This continuing, focused research on the KdV (and gKdV) equation can be in part explained by some striking algebraic properties. One of the first properties is the existence of localized, exponentially decaying, stable smooth solutions called *solitons*. For (1-3), solitons are solutions of the form

$$u(t,x) := Q_c(x-x_0-ct), \quad Q_c(s) := c^{\frac{1}{m-1}} Q(c^{1/2}s), \tag{1-4}$$

where x_0 and $c > 0$ are real numbers and Q is an explicit Schwartz function satisfying the second-order nonlinear differential equation $Q'' - Q + Q^m = 0$:

$$Q(x) = \left(\frac{m+1}{2\cosh^2\left(\frac{m-1}{2}x\right)} \right)^{\frac{1}{m-1}}. \tag{1-5}$$

This solution represents a "solitary wave" defined for all time moving to the right without *any change* in shape, velocity, or amplitude.

In addition, Equation (1-3) remains invariant under space and time *translations*. From Noether's theorem, these symmetries are related to *conserved quantities*, invariant under the gKdV flow, usually called *mass* and *energy*:

$$M[u](t) := \int_{\mathbb{R}} u^2(t,x)\,dx = M[u](0) \qquad \text{(mass)}, \tag{1-6}$$

$$E[u](t) := \frac{1}{2}\int_{\mathbb{R}} u_x^2(t,x)\,dx - \frac{1}{m+1}\int_{\mathbb{R}} u^{m+1}(t,x)\,dx = E[u](0) \qquad \text{(energy)}. \tag{1-7}$$

We now review some facts about the gKdV equation (1-3), with $m \geq 2$ an integer. The Cauchy problem for Equation (1-1) (that is, the problem with initial condition $u = u_0$ at $t = 0$) is *locally well-posed* for $u_0 \in H^1(\mathbb{R})$ [Kenig et al. 1993]. In the case $m < 5$, any $H^1(\mathbb{R})$ solution is global in time, thanks to the conservation equation (1-6), (1-7) and the Gagliardo–Nirenberg inequality

$$\int_{\mathbb{R}} u^{p+1} \leq K(p) \Big(\int_{\mathbb{R}} u^2 \Big)^{\frac{p+3}{4}} \Big(\int_{\mathbb{R}} u_x^2 \Big)^{\frac{p-1}{4}}. \tag{1-8}$$

For $m = 5$, solitons are known to be *unstable* and the Cauchy problem for the corresponding gKdV equation has finite-time blow-up solutions; see [Merle 2001; Martel and Merle 2002b; 2002a] and references therein. It is believed that for $m > 5$ the situation is the same. Consequently, in this work, *we will discard high-order nonlinearities*, at leading order.

In addition, there exists another conservation law, valid only for $L^1(\mathbb{R})$ solutions:

$$\int_{\mathbb{R}} u(t,x)\,dx = \text{constant}. \tag{1-9}$$

The problem to be considered in this paper possesses a long and extensive physical literature. We now briefly describe the main results concerning the propagation of solitons in a slowly varying medium.

Statement of the problem; historical review. The dynamical problem of soliton interaction with a slowly varying medium is by now a classical problem in nonlinear wave propagation. By the soliton-medium interaction we mean, loosely speaking, the following problem: In (1-1), consider a nonlinear function $f = f(t, x, s)$, slowly varying in space and time, possibly of small amplitude, satisfying

$$f(t, x, s) \sim s^m \quad \text{as } x \to \pm\infty, \quad \text{for all time}$$

(that is, (1-1) behaves like a gKdV equation at spatial infinity). Consider a soliton solution of the corresponding variable-coefficient equation (1-1) with this nonlinearity, at some early time. We expect that this solution does interact with the medium in space and time, here represented by the nonlinearity $f(t, x, s)$. In a slowly varying medium this interaction, small locally in time, may be important in the long-time behavior of the solution. The resulting solution after the interaction is precisely the object of study. In particular, one considers whether changes in size, position, or shape may be present after some large time, or even the creation or destruction of solitons.

We review some relevant works in this direction. The early works of Fermi, Pasta and Ulam [Fermi et al. 1955] and of Zabusky and Kruskal [1965] established complete integrability for KdV and other equations, leading to a new branch of research devoted to the study of the dynamics of KdV solitons in a slowly varying (*in time*) medium. (See [Miura 1976] for a review.) In [Kaup and Newell 1978; Karpman and Maslov 1977] the focus is on perturbations of integrable equations, and in particular the perturbed (in time τ) gKdV equation

$$u_\tau + (\beta(\varepsilon\tau)u_{xx} + \alpha(\varepsilon\tau)u^m)_x = 0, \quad m = 2, 3; \quad \alpha, \beta > 0. \tag{1-10}$$

This last equation models, for example, the propagation of a wave governed by the KdV equation along a *canal of varying depth*, among many other physical situations [Karpman and Maslov 1977; Asano 1974].

Note that this equation leaves invariant (1-6) and (1-9), but the corresponding energy for this equation is not conserved anymore. After the transformation

$$t := \int_0^\tau \beta(\varepsilon s)\, ds, \quad \tilde{u}(t, x) := \Big(\frac{\alpha}{\beta}\Big)^{\frac{1}{m-1}}(\varepsilon\tau)u(\tau, x),$$

the preceding equation becomes

$$\tilde{u}_t + (\tilde{u}_{xx} + \tilde{u}^m)_x = \varepsilon\gamma(\varepsilon t)\tilde{u}, \quad \text{where} \quad \varepsilon\gamma(\varepsilon t) := \frac{1}{m-1}\partial_t\Big(\log\Big(\frac{\alpha}{\beta}\Big)\,\varepsilon\tau(t)\Big). \tag{1-11}$$

The authors performed a perturbative analysis using inverse scattering theory to describe the dynamics of a soliton (for the integrable equation) in this variable regime. Of interest is the existence of a dispersive shelf-like tail behind the soliton, a phenomenon related to the lack of energy conservation (1-7) for the equation (1-11).

The problem was subsequently addressed in several other works and for different integrable models; see, for example, [Ko and Kuehl 1978; Fernandez et al. 1979; Grimshaw 1979a; Grimshaw 1979b].

Moreover, using inverse-scattering techniques, the production of a *second* (and small) solitary wave was pointed out in [Wright 1980] — see also [Grimshaw and Pudjaprasetya 2004] — but a satisfactory analytical proof of this phenomenon is still out of reach. See [Newell 1985, pp. 87–97] for a more detailed account.

Another important motivation comes from Lochak's interesting observation that, based in heuristic conservation laws, well-modulated solitons of (1-11) are good candidates for *adiabatically stable* objects for this infinite-dimensional dynamical system. See [Lochak 1984; Lochak and Meunier 1988] for details.

In this paper we address the problem of soliton dynamics in the case of an inhomogeneous medium, slowly varying in space but constant in time. This model, from the mathematical point of view, introduces several difficulties, as we will see below; but at the same time it reproduces the creation of a shelf-like tail behind the soliton, as computationally attested by physicists. Our main result is that, as a consequence of this tail, there is no pure soliton solution (unlike gKdV) for this regime. This result illustrates the *lack of pure solutions* of nontrivial perturbations of gKdV equations.

Setting and hypotheses. We come back to the general equation (1-1), and consider a small parameter $\varepsilon > 0$. Following (1-10), we will assume throughout that the nonlinearity f is a slowly varying x-dependent function of the power cases, independent of time, plus a (possibly zero) linear term:

$$\begin{cases} f(x,s) := -\lambda s + a_\varepsilon(x) s^m, \quad \lambda \geq 0,\ m = 2, 3 \text{ and } 4. \\ a_\varepsilon(x) := a(\varepsilon x) \in C^3(\mathbb{R}). \end{cases} \tag{1-12}$$

We will suppose the parameter λ fixed and independent of ε. Concerning the function a we will assume that there exist constants $K, \gamma > 0$ such that

$$\begin{cases} 1 < a(r) < 2, \quad a'(r) > 0 & \text{for } r \in \mathbb{R}, \\ 0 < a(r) - 1 \leq K e^{\gamma r} & \text{for } r \leq 0, \\ 0 < 2 - a(r) \leq K e^{-\gamma r} & \text{for } r \geq 0. \end{cases} \tag{1-13}$$

Thus

$$\lim_{r \to -\infty} a(r) = 1 \qquad \text{and} \qquad \lim_{r \to +\infty} a(r) = 2;$$

however, the special choices (1 and 2) of these limits are irrelevant for the results of this paper. The only necessary conditions are that

$$0 < a_{-\infty} := \lim_{r \to -\infty} a(r) \ < \lim_{r \to +\infty} a(r) =: a_\infty < +\infty.$$

Finally, to deal with a special stability property of the mass in Theorems 3.1 and 6.1 (see also (6-22)), we will need an additional, but not very restrictive, hypothesis: there exists $K > 0$ such that for $m = 2, 3, 4$,

$$|(a^{1/m})^{(3)}(s)| \leq K (a^{1/m})'(s) \qquad \text{for all } s \in \mathbb{R}. \tag{1-14}$$

This condition is often satisfied (provided a' is not be a compactly supported function).

Recapitulating, given $0 \le \lambda < 1$, $\varepsilon_0 > 0$, and a function a satisfying (1-13) and (1-14), we will consider the following equation, for which we use the abbreviation "aKdV" (after the potential a):

$$\begin{cases} u_t + (u_{xx} - \lambda u + a_\varepsilon(x)u^m)_x = 0 & \text{in } \mathbb{R}_t \times \mathbb{R}_x, \\ a_\varepsilon(x) = a(\varepsilon x), \quad \text{with } m = 2,3,4 \text{ and } 0 < \varepsilon \le \varepsilon_0. \end{cases} \tag{1-15}$$

The main issue that we will study in this paper is the interaction problem between a soliton and a slowly varying medium, here represented by the *potential* a_ε. In other words, we intend to study for (1-15) whether it is possible to generalize the well-known soliton-like solution Q of gKdV. It is well-known that in the case $f(t,x,s) = f(s)$, and under reasonable assumptions (see for example [Berestycki and Lions 1983]), there exist soliton-like solutions, constructed via *ground states* of the corresponding elliptic equation for a *bound state*. However, in this paper our objective will be the study of soliton solutions for a variable-coefficient equation, where there is no obvious ground state.

As a heuristic introduction to the results to be proved, consider that (1-15) has the form of a gKdV equation at infinity:

$$\begin{cases} u_t + (u_{xx} - \lambda u + 1u^m)_x = 0 & \text{as } x \to -\infty, \\ u_t + (u_{xx} - \lambda u + 2u^m)_x = 0 & \text{as } x \to +\infty. \end{cases} \tag{1-16}$$

In particular, if Q is the soliton (1-5) of the standard gKdV equation, one should be able of to construct a soliton-like solution $u(t)$ of (1-15) such that

$$u(t) \sim Q(\cdot - (1-\lambda)t) \quad \text{as } t \to -\infty,$$

in some sense to be defined. Indeed, $Q(\cdot - (1-\lambda)t)$ is an actual solution for the first equation in (1-16) on the whole real line, moving toward the left if $\lambda > 1$, toward the right if $\lambda < 1$, and stationary if $\lambda = 1$.

On the other hand, after passing the interaction region, by stability properties, this solution should behave, for small ε, like

$$2^{-\frac{1}{m-1}} Q_{c_\infty}\big(x - (c_\infty - \lambda)t - \rho(t)\big) + \text{smaller-order terms in } \varepsilon \quad \text{as } t \to +\infty, \tag{1-17}$$

where c_∞ is a unknown positive number (a limiting scaling parameter) and $\rho(t)$ is small compared with $(c_\infty - \lambda)t$. In fact, note that if $v = v(t)$ is a solution of (1-3) then $u(t) := 2^{-\frac{1}{m-1}} v(t)$ is a solution of

$$u_t + (u_{xx} - \lambda u + 2u^m)_x = 0 \quad \text{in } \mathbb{R}_t \times \mathbb{R}_x. \tag{1-18}$$

In conclusion, this heuristic suggests that even if the potential varies slowly, the soliton will experience nontrivial transformations on its scaling and shape, of the same order as that of the amplitude variation of the potential a.

Before we state our results, some important facts are in order. First, Equation (1-15) is unfortunately no longer invariant under scaling and space translations. Moreover, a nonzero solution of (1-15) might *lose or gain* some mass, depending on the sign of u, in the sense that, in the case of rapidly decaying functions, the quantity

$$M[u](t) = \frac{1}{2} \int_{\mathbb{R}} u^2(t,x)\,dx \tag{1-19}$$

satisfies the identity

$$\partial_t M[u](t) = -\frac{\varepsilon}{m+1}\int_{\mathbb{R}} a'(\varepsilon x)u^{m+1}. \tag{1-20}$$

Another key observation is the following: in the cubic case $m = 3$, with our choice of a_ε, the mass is always nonincreasing.

On the other hand, when $\lambda \geq 0$, the novel energy

$$E_a[u](t) := \frac{1}{2}\int_{\mathbb{R}} u_x^2(t,x)\,dx + \frac{\lambda}{2}\int_{\mathbb{R}} u^2(t,x)\,dx - \frac{1}{m+1}\int_{\mathbb{R}} a_\varepsilon(x)u^{m+1}(t,x)\,dx \tag{1-21}$$

remains conserved for all time. Moreover, a simple energy balance at $\pm\infty$ allows one to determine heuristically the limiting scaling in (1-17) in certain cases. For example, if $\lambda = 0$, and we suppose that the *lower-order terms* are of zero mass at infinity, we have from (1-17)

$$E_{a\equiv 1}[u](-\infty) = E[Q] \sim 2^{-\frac{2}{m-1}} c_\infty^{\frac{2}{m-1}+\frac{1}{2}} E[Q] = E_{a\equiv 2}[u](+\infty), \quad E[Q] \neq 0$$

(see Section A.6 in the Appendix). This implies that $c_\infty \sim 2^{\frac{4}{m+3}} > 1$. These informal arguments suggest the following definition.

Definition 1.0 (Pure generalized soliton solution for aKdV). Let $0 \leq \lambda < 1$ be a fixed number. We will say that (1-15) admits a *pure* generalized soliton-like solution (of scaling 1) if there exist a C^1 real valued function $\rho = \rho(t)$ defined for all large times and a global in time $H^1(\mathbb{R})$ solution $u(t)$ of (1-15) such that

$$\lim_{t\to-\infty} \|u(t) - Q(\cdot - (1-\lambda)t)\|_{H^1(\mathbb{R})} = \lim_{t\to+\infty} \left\|u(t) - 2^{-\frac{1}{m-1}} Q_{c_\infty}(\cdot - \rho(t))\right\|_{H^1(\mathbb{R})} = 0,$$

with $\lim_{t\to+\infty}\rho(t) = +\infty$, and where $c_\infty = c_\infty(\lambda)$ is the scaling suggested by the energy conservation law (1-21).

Remark. Note that the existence of a translation parameter $\rho(t)$ is a necessary condition: it is even present in the orbital stability of small perturbations of solitons for gKdV. See [Benjamin 1972; Bona et al. 1987; Cazenave and Lions 1982], for example. We have not included the case $\rho(t) \to -\infty$ as $t \to +\infty$, corresponding to a reflected soliton, but we hope to consider this case elsewhere.

Previous analytic results on soliton dynamics in a slowly varying medium. The problem of describing analytically the soliton dynamics of different integrable models in a slowly varying medium has received some increasing attention during the last years. Concerning the KdV equation, our belief is that the first result in this direction was given in [Dejak and Jonsson 2006; Dejak and Sigal 2006]. These works considered the long time dynamics of solitary waves (solitons) over slowly varying perturbations of KdV and modified KdV equations

$$u_t + \left(u_{xx} - b(t,x)u + u^m\right)_x = 0 \quad \text{on } \mathbb{R}_t \times \mathbb{R}_x, \quad m = 2, 3, \tag{1-22}$$

and where b is assumed having small size and small variation, in the sense that for ε small,

$$|\partial_t^n \partial_x^p b| \leq \varepsilon^{n+p+1} \quad \text{for } 0 \leq n + p \leq 2.$$

(Actually their conclusions hold in more generality, but for our purposes we state the closest version to our approach; see [Dejak and Jonsson 2006] for the detailed version.) With these hypotheses the authors showed that if $m = 2$ and the initial condition u_0 satisfies the *orbital stability* condition

$$\inf_{\substack{0<c_0<c<c_1\\ a\in\mathbb{R}}} \|u_0 - Q_c(\cdot - a)\|_{H^1(\mathbb{R})} \le \varepsilon^{2s}, \quad s < \tfrac{1}{2},\ c_0, c_1 \text{ given},$$

then for any for time $t \le K\varepsilon^{-s}$ the solution can be decomposed as

$$u(t,x) = Q_{c(t)}(x - \rho(t)) + w(t,x),$$

where $\|w(t)\|_{H^1(\mathbb{R})} \le K\varepsilon^s$ and $\rho(t), c(t)$ satisfies the following differential system

$$\rho'(t) = c(t) - b(t, a(t)) + O(\varepsilon^{2s}), \quad c'(t) = O(\varepsilon^{2s});$$

during the interval of time considered. In the cubic case ($m = 3$) the results are slightly better; see [Dejak and Jonsson 2006].

Our model can be written as a generalized, time-independent Dejak–Jonsson–Sigal equation of the type (1-22), after writing $v(t,x) := \tilde{a}(\varepsilon x)u(t,x)$, with $\tilde{a}(\varepsilon x) := a^{\frac{1}{m-1}}(\varepsilon x)$. From these considerations we expect to recover and to improve the results obtained by those authors.

Holmer [$\ge$ 2011] has announced some improvements on the Dejak–Sigal results, by assuming b of amplitude $O_{L^\infty}(1)$. He proves that

$$\sup_{t \lesssim \delta\varepsilon^{-1}|\log \varepsilon|} \|w(t)\|_{H^1(\mathbb{R})} \lesssim \varepsilon^{1/2-\delta}, \quad \text{for some } \delta > 0.$$

In this paper we have preferred to avoid the inclusion of a time-dependent potential, and to treat the infinite time prescribed and pure data, instead of the standard Cauchy problem. This choice will have positive consequences for our main results, Theorems 1.1 and 1.2, where we will describe with accuracy the dynamical problem, including its asymptotics as $t \to +\infty$.

The soliton-potential interaction can be considered also in the case of the nonlinear Schrödinger equation

$$iu_t + u_{xx} - V(\varepsilon x)u + |u|^2 u = 0 \quad \text{on } \mathbb{R}_t \times \mathbb{R}_x; \tag{1-23}$$

see [Muñoz $\ge$ 2011b], for example. Results similar to the ones just mentioned were obtained in [Holmer and Zworski 2008; Holmer et al. 2007a; 2007b; Jonsson et al. 2006; Fröhlich et al. 2004]. Finally we point out the recent [Perelman 2009], concerning the critical quintic NLS equation.

Main results. Let

$$T_\varepsilon := \frac{1}{1-\lambda}\varepsilon^{-1-\frac{1}{100}} > 0. \tag{1-24}$$

This parameter can be understood as the *interaction time* between the soliton and the potential. In other words, at time $t = -T_\varepsilon$ the soliton should remain almost unperturbed, and at time $t = T_\varepsilon$ the soliton should have completely crossed the influence region of the potential. Note that the asymptotic $\lambda \sim 1$ is a degenerate case and it will not be considered in this work.

In Theorems 1.1, 1.2, and 1.3 we will show that, under suitable assumptions, a pure soliton-like solution as in Definition 1.0 does not exist, in the sense that the lower order terms appearing after the interaction always have positive mass. This phenomenon will be a consequence of the dispersion produced during the crossing of the soliton with the main core of the potential a_ε.

We will from now on assume the validity of assumptions (1-12), (1-13), and (1-14). Our first result describes the dynamics of the *pure* soliton-like solution for the aKdV equation (1-15).

Theorem 1.1 (Dynamics of interaction of solitons for gKdV equations in a variable medium). *Let $m = 2, 3, 4$, and let $0 \le \lambda \le \lambda_0 := \frac{5-m}{m+3}$ be a fixed number. There exists a small constant $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ the following statements hold.*

(1) Existence of a soliton-like solution. *There exists a solution $u \in C(\mathbb{R}, H^1(\mathbb{R}))$ of (1-15), global in time, such that*

$$\lim_{t\to-\infty} \big\|u(t) - Q(\cdot - (1-\lambda)t)\big\|_{H^1(\mathbb{R})} = 0, \tag{1-25}$$

with conserved energy $E_a[u](t) = (\lambda-\lambda_0)M[Q] \le 0$. This solution is unique if $m = 3$, or if $m = 2, 4$ and $\lambda > 0$.

(2) Soliton-potential interaction. *There exist $K > 0$ and numbers $c_\infty(\lambda) \ge 1$, $\rho_\varepsilon, \tilde{T}_\varepsilon \in \mathbb{R}$ such that the solution $u(t)$ above satisfies*

$$\big\|u(\tilde{T}_\varepsilon) - 2^{-1/(m-1)} Q_{c_\infty}(x - \rho_\varepsilon)\big\|_{H^1(\mathbb{R})} \le K\varepsilon^{1/2}. \tag{1-26}$$

Moreover,

$$c_\infty(\lambda = 0) = 2^{\frac{4}{m+3}} \quad \textit{and} \quad c_\infty(\lambda = \lambda_0) = 1. \tag{1-27}$$

Finally we have the bounds

$$|T_\varepsilon - \tilde{T}_\varepsilon| \le \frac{T_\varepsilon}{100} \quad \textit{and} \quad (1-\lambda)T_\varepsilon \le \rho_\varepsilon \le (2c_\infty(\lambda) - \lambda - 1)T_\varepsilon. \tag{1-28}$$

Note that $\lambda_0 = \lambda_0(m)$ is always less than 1 for $m = 2, 3, 4$, while $\lambda_0 = 0$ for $m = 5$ (the L^2-critical case). Also, for $\lambda = \lambda_0$ we have $E_a[u](t) = (\lambda-\lambda_0)M[Q] = 0$; and if $\lambda < \lambda_0$ we have $E_a[u](t) < 0$ for all $t \in \mathbb{R}$. For the consequences of this property and a detailed study of $c_\infty(\lambda)$, see Lemma 4.4.

Remark. The proof of Theorem 1.1 is based on the construction of an approximate solution of (1-15) in the interaction region, satisfying certain symmetries. However, at some point we formally obtain an *infinite mass term* (see [Martel and Merle 2011; 2010] for a similar problem). It turns out that to obtain a localized solution we need to break the symmetry of this solution (see Proposition 4.7 for the details). This lack of symmetry leads to the error $\varepsilon^{1/2}$ in the theorem. At this price we describe completely the interaction, a completely new result.

The next step is understanding the long time behavior of our generalized soliton solution.

Theorem 1.2 (Long time behavior). *Suppose, in addition to the assumptions of Theorem 1.1, that $0 < \lambda \le \lambda_0$ for the cases $m = 2, 4$, and $0 \le \lambda \le \lambda_0$ if $m = 3$. Let $0 < \beta < \frac{1}{2}(c_\infty(\lambda) - \lambda)$. There exists a* *constant* $\varepsilon_0 > 0$ *such that, for all* $0 < \varepsilon \leq \varepsilon_0$, *there exist* $K, c^+ > 0$ *and a* C^1*-function* $\rho_2(t)$ *defined in* $[T_\varepsilon, +\infty)$ *such that the function*

$$w^+(t,\cdot) := u(t,\cdot) - 2^{\frac{-1}{m-1}} Q_{c^+}(\cdot - \rho_2(t))$$

has the following properties:

(1) Stability. *For any* $t \geq T_\varepsilon$,

$$\|w^+(t)\|_{H^1(\mathbb{R})} + |c^+ - c_\infty(\lambda)| + |\rho_2'(t) - (c_\infty(\lambda) - \lambda)| \leq K\varepsilon^{1/2}. \tag{1-29}$$

(2) Asymptotic stability.

$$\lim_{t\to+\infty} \|w^+(t)\|_{H^1(x>\beta t)} = 0. \tag{1-30}$$

(3) Bounds on the parameters. *Define* $\theta := \frac{1}{m-1} - \frac{1}{4} > 0$. *The limit*

$$\lim_{t\to+\infty} E_a[w^+](t) =: E^+ \tag{1-31}$$

exists and satisfies the identity

$$E^+ = \frac{(c^+)^{2\theta}}{2^{2/(m-1)}}(\lambda_0 c^+ - \lambda)M[Q] + (\lambda - \lambda_0)M[Q], \tag{1-32}$$

and for all $m = 2, 3, 4$ *and* $0 < \lambda \leq \lambda_0$ *there exists* $K(\lambda) > 0$ *such that*

$$\frac{1}{K} \limsup_{t\to+\infty} \|w^+(t)\|^2_{H^1(\mathbb{R})} \leq E^+ \leq K\varepsilon. \tag{1-33}$$

In the case $m = 3$, *we have* $\frac{3}{2}E^+ = (\frac{c^+}{c_\infty})^{3/2} - 1$ *if* $\lambda = 0$, *and*

$$\frac{1}{K} \limsup_{t\to+\infty} \|w^+(t)\|^2_{H^1(\mathbb{R})} \leq \left(\frac{c^+}{c_\infty}\right)^{2\theta} - 1 \leq K\varepsilon \quad \text{if } \lambda > 0. \tag{1-34}$$

Remarks. (a) The stability and asymptotic stability of solitary waves for generalized KdV equations have been widely studied since the 1980s. The main ideas in our proof of (1-29) and (1-30) appear in the literature; see [Benjamin 1972; Cazenave and Lions 1982; Bona et al. 1987; Martel et al. 2002; Pego and Weinstein 1994], for example.

(b) The sign of $a'(\cdot)$ is a sufficient condition to ensure stability, but it is conceivable that it can be replaced by a weaker one, say $a'(s) > 0$ for all $s > s_0$.

Changes for decreasing potentials. Suppose the potential $a(\cdot)$ satisfies instead $a'(s) < 0$ and

$$1 = \lim_{s\to-\infty} a(s) > a(s) > \lim_{t\to+\infty} a(s) = \tfrac{1}{2}.$$

Statement (1) of Theorem 1.1 remains true, except that we do not know whether the solution is unique. Part (2) holds true with the coefficient $2^{\frac{1}{m-1}}$ in front of Q_{c_∞}, $\frac{\lambda}{\lambda_0} < c_\infty(\lambda) < 1$, and $c_\infty = 2^{-p}$ for $\lambda = 0$ (see Lemma 4.4 for this). Bounds similar to (1-28) hold true, with the obvious changes. By contrast, we have no analog for Theorem 1.2: long-time stability for decreasing potentials remains an open question.

A fundamental question arises from Theorems 1.1 and 1.2: Is the solution a pure soliton (Definition 1.0) for the aKdV equation with $a_\varepsilon \equiv 2$? This question is equivalent to deciding whether

$$\limsup_{t\to+\infty} \|w^+(t)\|_{H^1(\mathbb{R})} = 0.$$

Our last result shows that this behavior cannot happen.

Theorem 1.3 (Nonexistence of pure soliton-like solutions for aKdV). *With the assumptions and notation of Theorems 1.1 and 1.2, suppose in addition that* $m = 2, 3, 4$ *with* $0 < \lambda \le \lambda_0$. *There exists* $\varepsilon_0 > 0$ *such that, for all* $0 < \varepsilon < \varepsilon_0$,

$$\limsup_{t\to+\infty} \|w^+(t)\|_{H^1(\mathbb{R})} > 0. \tag{1-35}$$

Remark. In addition to the classical problem of extending the results to more general potentials $a(\cdot)$, several related questions arise naturally, which we are as yet unable to solve:

(1) Is every solution of (1-15) with $H^1(\mathbb{R})$ data globally bounded in time? In Proposition 2.2 we prove that every solution is globally well defined for all positive times, and uniformly bounded if $\lambda > 0$ or $m = 3$. However, for the cases $m = 2, 4$ and $\lambda = 0$ we only have been able to find an exponential upper bound on the H^1-norm of the solution. Is every solution described in Theorem 1.1 globally bounded?

(2) In the cases $m = 2, 4$ and $\lambda = 0$, is the solution constructed in Theorem 1.1 unique? Is it stable for large times? (Compare Theorem 6.1.)

(3) What is the behavior of the solution for a coefficient $\lambda_0 < \lambda < 1$? We believe in this situation the soliton still survives, but is reflected by the potential, propagating to the left for large t. (See [Muñoz $\ge$ 2011a].)

(4) [Added in proof] We have recently proved a quantitative lower bound on the defect [Muñoz 2011].

(5) Is there scattering modulo the soliton solution, at infinity?

The case of a time-dependent potential. As might be expected, our results are also valid, with easier proofs, for the time-dependent gKdV equation

$$u_t + (u_{xx} - \lambda u + a(\varepsilon t)u^m)_x = 0 \quad \text{in } \mathbb{R}_t \times \mathbb{R}_x, \tag{1-36}$$

where a satisfies (1-13)–(1-14), with the time variable in place of r. This equation is invariant under scaling and space translations. In addition, the L^1 integral and the mass $M[u]$ remain constants and the energy

$$\tilde{E}[u](t) := \frac{1}{2}\int_{\mathbb{R}} u_x^2 + \frac{\lambda}{2}\int_{\mathbb{R}} u^2 - \frac{a(\varepsilon t)}{m+1}\int_{\mathbb{R}} u^{m+1}$$

satisfies

$$\partial_t \tilde{E}[u](t) = -\frac{\varepsilon a'(\varepsilon t)}{m+1}\int_{\mathbb{R}} u^{m+1}.$$

Theorems 1.1 and 1.2 hold with $c_\infty(\lambda = 0) = 2^{4/(5-m)}$ (because of mass conservation), for any $\lambda \ge 0$ and $m = 2, 3, 4$ (follow Lemma 4.4 to see this). We leave the details to the reader.

Sketch of proofs. Our arguments combine techniques adapted from [Martel 2005; Martel et al. 2010; Martel and Merle 2008; 2011; 2007; 2010] with some new computations. We separate the analysis into three time intervals: $t \ll -\varepsilon^{-1}$, $|t| \leq \varepsilon$, and $\varepsilon^{-1} \ll t$. On each interval the solution possesses a specific behavior:

$t \ll -\varepsilon^{-1}$**:** In this interval of time we prove that $u(t)$ remains very close to a soliton solution, with no change in the scaling and shift parameters (Theorem 3.1). This is possible for very large negative times, where the soliton is still far from the interacting region $|t| \leq \varepsilon^{-1}$.

$|t| \leq \varepsilon^{-1}$**:** Here the soliton-potential interaction leads the dynamics of $u(t)$. The novelty here is the construction of an *approximate solution* of (1-15) with high order of accuracy such that: (a) at time $t \sim -\varepsilon^{-1}$ this solution is close to the soliton solution and therefore to $u(t)$; (b) it describes the soliton-potential interaction inside this interval, in particular we show the existence of a dispersive tail behind the soliton; and (c) it is close to $u(t)$ in the whole interval $[-\varepsilon^{-1}, \varepsilon^{-1}]$, uniformly on time, apart from a modulation on a translation parameter (Theorem 4.1).

$t \gg \varepsilon^{-1}$**:** Here some *stability* properties (Theorem 6.1) are used to establish the convergence of the solution $u(t)$ to a soliton-like solution with modified parameters.

Additionally, by using a contradiction argument, it will be possible to show that the residue of the interaction at time $t \sim \varepsilon^{-1}$ is still present at infinity. This gives the conclusion of the main Theorems 1.1 and 1.3. Indeed, recall the L^1 conserved quantity from (1-9). This expression is in general useless when the equation is considered on the whole real line $\mathbb{R}$, but it has some striking applications in the blow-up theory (see [Merle 2001]). In our case, it will be useful in discarding the existence of a pure soliton-like solution.

Accordingly, the paper is organized as follows. In Section 2 we introduce some basic tools to study the interaction and asymptotic problems. Section 3 is devoted to the construction of the soliton like solution for large negative time. Sections 4 and 5 deal with the proof of Theorem 1.1. In Section 6 we prove the asymptotic behavior as $t \to +\infty$, namely Theorem 1.2. Finally we prove Theorem 1.3 (Section 7).

Remark. We believe that the main results of this paper are also valid for general subcritical nonlinearities, with stable solitons. In this case the scaling property of the soliton is no longer valid, so in order to construct an approximate solution one should modify the main argument of the proof.

2. Preliminaries

Throughout this paper, C, K, and $\gamma > 0$ will denote constants independent of ε, possibly changing from one line to another.

To treat the case $\lambda > 0$ we need to extend the energy (1-7) by adding a mass term. We therefore introduce a new energy function $E_1[u]$, the particular case of (1-21) when $a \equiv 1$.

The Cauchy problem. First we develop a suitable local well-posedness theory for the Cauchy problem associated to (1-15).

Let $u_0 \in H^s(\mathbb{R})$, $s \geq 1$, $\lambda \geq 0$. We consider the initial value problem

$$\begin{cases} u_t + (u_{xx} - \lambda u + a_\varepsilon(x) u^m)_x = 0 & \text{in } \mathbb{R}_t \times \mathbb{R}_x \\ u(t=0) = u_0, \end{cases} \tag{2-1}$$

where $m = 2, 3, 4$. The analogous problem for the standard gKdV equation (1-3) has been extensively studied. For dealing with (2-1), we will follow closely the contraction method developed in [Kenig et al. 1993]. The following result is proved with standard techniques based on the Picard iteration procedure and the tools developed in this last reference:

Proposition 2.1 (Local well-posedness in $H^s(\mathbb{R})$). (See also [Kenig et al. 1993]). *Suppose $u_0 \in H^s(\mathbb{R})$, $s \geq 1$. Then (2-1) has a unique (in a certain sense) solution $u \in C(I, H^s(\mathbb{R}))$ defined in a maximal interval of existence $I \ni 0$. Moreover*:

(1) Blow-up alternative. *If* $\sup I < +\infty$, *then*

$$\lim_{t \uparrow \sup I} \|u(t)\|_{H^s(\mathbb{R})} = +\infty. \tag{2-2}$$

The same conclusion holds if $\inf I > -\infty$.

(2) Energy conservation. *For any $t \in I$ the energy $E_a[u](t)$ from (1-21) remains constant.*

(3) Mass variation. *For all $t \in I$ the mass $M[u](t)$ defined in (1-19) satisfies (1-20).*

(4) *Suppose $u_0 \in L^1(\mathbb{R}) \cap H^1(\mathbb{R})$. Then (1-9) is well defined and remains constant for all $t \in I$.* □

Once local-in-time existence is established, the next step is to ask for the possibility of a *global well-posedness theorem*. In many cases the proof reduces to the *use of conservation laws* to obtain bounds on the norm of the solution for every time. In the case of gKdV equations ($m \leq 4$) this was proved in [Kenig et al. 1993] using mass and energy conservation; however, in our case (1-20) is not enough to control the L^2 norm of the solution. As stated in the Introduction, global existence for cubic case $m = 3$ follows from the mass decreasing property. However, to deal with the remaining cases, we will modify our arguments by introducing a perturbed mass, almost decreasing in time, in order to prove global existence. Indeed, define for each $t \in I$ and $m = 2, 3, 4$ the quantity

$$\hat{M}[u](t) := \frac{1}{2} \int_{\mathbb{R}} a_\varepsilon^{1/m}(x) u^2(t, x)\, dx. \tag{2-3}$$

It is clear that $\hat{M}[u](t)$ is well defined, for any time $t \in I$ and solution $u \in H^1(\mathbb{R})$ of (2-1). For all $t \in I$ we have the equivalence property

$$M[u](t) \leq \hat{M}[u](t) \leq 2^{1/m} M[u](t). \tag{2-4}$$

This modified mass enjoys a striking property:

Proposition 2.2 (Global existence in $H^1(\mathbb{R})$). *Consider the solution $u(t)$ of the Cauchy problem (2-1) with $u(0) = u_0 \in H^1(\mathbb{R})$ and maximal interval of existence I. Then $u(t)$ is continuously well defined in $H^1(\mathbb{R})$ for any $t \geq 0$. More precisely*:

(1) Cubic case. *Suppose* $m = 3$, $\lambda \geq 0$. *Then* $I = (\tilde{t}_0, +\infty)$ *for some* $-\infty \leq \tilde{t}_0 < 0$ *and there exists* $K = K(\|u_0\|_{H^1(\mathbb{R})}) > 0$ *such that*

$$\sup_{t\geq 0} \|u(t)\|_{H^1(\mathbb{R})} \leq K. \tag{2-5}$$

(2) Almost monotonicity of the modified mass $\hat{M}$ and global existence. *For any* $m = 2, 3, 4$ *and for all* $t \in I$ *we have*

$$\partial_t \hat{M}[u](t) = -\frac{3}{2}\varepsilon \int_{\mathbb{R}} (a^{1/m})'(\varepsilon x) u_x^2 - \frac{\varepsilon}{2} \int_{\mathbb{R}} [\lambda (a^{1/m})' - \varepsilon^2 (a^{1/m})^{(3)}](\varepsilon x) u^2. \tag{2-6}$$

In particular, (a) I *is again of the form* $(\tilde{t}_0, +\infty)$; (b) *item if* $\lambda > 0$ *there exists* $\varepsilon_0 > 0$ *small such that* (2-5) *holds*; *and* (c) *if* $\lambda = 0$ *and* $m = 2, 4$, *there exists* $K = K(\|u_0\|_{H^1(\mathbb{R})})$ *such that we have for all* $t \geq 0$ *the exponential bound*

$$\|u(t)\|_{H^1(\mathbb{R})} \leq K e^{K\varepsilon^3 t}. \tag{2-7}$$

Proof of Proposition 2.2. First we consider the cubic case, $m = 3$. From (1-20) we have

$$M[u](t) \leq M[u](0) \quad \text{for any } t \in I,\ t \geq 0.$$

This bound implies global existence for positive times. Indeed, the bound rules out a L^2 blow-up in (positive) *finite and infinite time*, namely (2-2). In order to control the $H^1(\mathbb{R})$ norm, we use energy conservation, the Gagliardo–Nirenberg inequality (1-8), and the preceding bound on the mass. Indeed, for any $t \in I$, $t \geq 0$, and redefining the constant K if necessary, we have

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}} u_x^2 &= E_a[u](0) - \frac{1}{2}\lambda \int_{\mathbb{R}} u^2 + \frac{1}{m+1}\int_{\mathbb{R}} a_\varepsilon u^{m+1} \\ &\leq E_a[u](0) + \lambda M[u](0) + K\|u(t)\|_{L^2(\mathbb{R})}^{(m+3)/2} \|u_x(t)\|_{L^2(\mathbb{R})}^{(m-1)/2}.\end{aligned}$$

Noticing that $\frac{1}{4}(m-1) < 1$ for $m = 2, 3, 4$, we have

$$\int_{\mathbb{R}} u_x^2 \leq K(\lambda, \|u_0\|_{H^1(\mathbb{R})});$$

for a large constant K. This implies the $H^1(\mathbb{R})$ global existence for all positive times and the uniform bound in time (2-5).

To prove (2-6), we proceed by formally taking the time derivative. Every step can be rigorously justified by introducing mollifiers. From (1-15) we have

$$\begin{aligned}\partial_t \hat{M}[u](t) &= \int_{\mathbb{R}} a_\varepsilon^{1/m} u u_t = \int_{\mathbb{R}} (a_\varepsilon^{1/m} u)_x (u_{xx} - \lambda u + a_\varepsilon u^m) \\ &= \varepsilon \int_{\mathbb{R}} \big((a^{1/m})'(\varepsilon x) u u_{xx} - \tfrac{1}{2}(a^{1/m})'(\varepsilon x) u_x^2\big) - \frac{\lambda}{2}\varepsilon \int_{\mathbb{R}} (a^{1/m})'(\varepsilon x) u^2 \\ &\qquad + \varepsilon \int_{\mathbb{R}} a_\varepsilon (a^{1/m})'(\varepsilon x) u^{m+1} - \frac{\varepsilon}{m+1}\int_{\mathbb{R}} (a^{1/m+1})'(\varepsilon x) u^{m+1} \\ &= -\frac{1}{2}\varepsilon \int_{\mathbb{R}} [\lambda (a^{1/m})'(\varepsilon x) - \varepsilon^2 (a^{1/m})^{(3)}(\varepsilon x)] u^2 - \frac{3}{2}\varepsilon \int_{\mathbb{R}} (a^{1/m})'(\varepsilon x) u_x^2.\end{aligned}$$

This proves (2-6). Now, in order to establish global $H^1(\mathbb{R})$ existence for positive times, we first control the L^2 norm using $\hat{M}[u](t)$. Let us consider the case $\lambda > 0$. In this case, taking ε_0 small enough, and thanks to (1-14), we have

$$\partial_t \hat{M}[u](t) \leq 0,$$

and thus $\hat{M}[u](t) \leq \hat{M}[u](0)$ for all $t \in I$, $t \geq 0$. The rest of the proof is identical to the cubic case.

Now we consider the last case, namely $m = 2, 4$ and $\lambda = 0$. Here the argument above is not valid anymore; we have only the existence of $K > 0$ independent of ε such that

$$\partial_t \hat{M}[u](t) \leq K\varepsilon^3 \hat{M}[u](t).$$

This implies that, for any $t \in I$ with $t \geq 0$,

$$M[u](t) \leq \hat{M}[u](t) \leq K\hat{M}[u](0)e^{K\varepsilon^3 t}.$$

This bound rules out a L^2 blow-up in *finite time* for positive times. To control the $H^1(\mathbb{R})$ norm, we use the same argument from the preceding case. Indeed, for any $t \in I$, redefining the constant K if necessary, we have

$$\int_{\mathbb{R}} u_x^2 \leq Ke^{K\varepsilon^3 t}.$$

This implies the global $H^1(\mathbb{R})$ existence for positive times. □

Remark (Mass monotonicity). Consider a solution $u(t) \in H^1(\mathbb{R})$ of (1-15) and define the modified mass

$$\tilde{M}[u](t) := \begin{cases} M[u](t) & \text{if } m = 3, \\ \hat{M}[u](t) & \text{if } m = 2, 4 \text{ and } \lambda > 0. \end{cases} \tag{2-8}$$

Proposition 2.2 implies that there exists $\varepsilon_0 > 0$ such that $\tilde{M}[u](t) - \tilde{M}[u](t_0) \leq 0$, for all $0 < \varepsilon \leq \varepsilon_0$ and all $t \in \mathbb{R}$ with $t \geq t_0$.

Spectral properties of the linearized gKdV operator. We next consider some important properties of the linearized operator associated to (1-15). Fix $c > 0$ and $m = 2, 3, 4$, and let

$$\mathcal{L}w(y) := -w_{yy} + cw - mQ_c^{m-1}(y)w, \quad \text{where} \quad Q_c(y) := c^{\frac{1}{m-1}} Q(\sqrt{c}\, y). \tag{2-9}$$

Here $w = w(y)$. We also denote $\mathcal{L}_0 := \mathcal{L}_{c=1}$.

Lemma 2.3 (Spectral properties of $\mathcal{L}$). (See [Martel and Merle 2009].) *The operator $\mathcal{L}$ defined on $L^2(\mathbb{R})$ by (2-9) has domain $H^2(\mathbb{R})$, is self-adjoint, and satisfies the following properties*:

(1) First eigenvalue. *There exists a unique $\lambda_m > 0$ such that $\mathcal{L}Q_c^{\frac{m+1}{2}} = -\lambda_m Q_c^{\frac{m+1}{2}}$.*

(2) *The kernel of $\mathcal{L}$ is spanned by Q_c', and we have $\mathcal{L}(\Lambda Q_c) = -Q_c$, where*

$$\Lambda Q_c := \partial_{c'} Q_{c'}\big|_{c'=c} = \frac{1}{c}\Big(\frac{1}{m-1} Q_c + \frac{1}{2} x Q_c'\Big). \tag{2-10}$$

The continuous spectrum of $\mathcal{L}$ is given by $\sigma_{\mathrm{cont}}(\mathcal{L}) = [c, +\infty)$.

(3) Inverse. *For all $h \in L^2(\mathbb{R})$ such that $\int_{\mathbb{R}} hQ_c' = 0$, there exists a unique $\hat{h} \in H^2(\mathbb{R})$ such that $\int_{\mathbb{R}} \hat{h}Q_c' = 0$ and $\mathcal{L}\hat{h} = h$. Moreover, if h is even (resp. odd), then $\hat{h}$ is even (resp. odd).*

(4) Regularity in the Schwartz space $\mathcal{S}$. *For $h \in H^2(\mathbb{R})$, $\mathcal{L}h \in \mathcal{S}$ implies $h \in \mathcal{S}$.*

(5) Coercivity.

(a) *There exist $K, \sigma_c > 0$ such, that for all $w \in H^1(\mathbb{R})$,*

$$\mathcal{B}[w,w] := \int_{\mathbb{R}} (w_x^2 + cw^2 - mQ_c^{m-1}w^2) \geq \sigma_c \int_{\mathbb{R}} w^2 - K\left|\int_{\mathbb{R}} wQ_c\right|^2 - K\left|\int_{\mathbb{R}} wQ_c'\right|^2.$$

In particular, if $\int_{\mathbb{R}} wQ_c = \int_{\mathbb{R}} wQ_c' = 0$, then the functional $\mathcal{B}[w,w]$ is positive definite in $H^1(\mathbb{R})$.

(b) *The same conclusion holds if $\int_{\mathbb{R}} wQ_c = \int_{\mathbb{R}} wxQ_c = 0$,.*

Now we introduce some notation, taken from [Martel and Merle 2007]. We denote by $\mathcal{Y}$ the set of C^∞ functions f such that for all $j \in \mathbb{N}$ there exist $K_j, r_j > 0$ such that for all $x \in \mathbb{R}$ we have

$$|f^{(j)}(x)| \leq K_j(1+|x|)^{r_j} e^{-|x|}.$$

Now we recall a function used to describe the effect of *dispersion* on the solution. Set $\varphi(x) := -\dfrac{Q'(x)}{Q(x)}$; then φ is an odd function and has the following properties (see [Martel and Merle 2009]):

(1) $\lim_{x\to-\infty} \varphi(x) = -1$ and $\lim_{x\to+\infty} \varphi(x) = 1$.

(2) For all $x \in \mathbb{R}$, we have $|\varphi'(x)| + |\varphi''(x)| + |\varphi^{(3)}(x)| \leq Ce^{-|x|}$.

(3) $\varphi' \in \mathcal{Y}$ and $1-\varphi^2 \in \mathcal{Y}$.

For $c > 0$, we then set

$$\varphi_c(x) := -\frac{Q_c'}{Q_c} = \sqrt{c}\,\varphi(\sqrt{c}x). \tag{2-11}$$

Remark. The same function φ has been used in [Martel and Merle 2007] to describe the main-order nonlinearity effect on the phase of two colliding solitons for the quartic KdV equation. Here φ will describe the dispersive tail behind the soliton produced by the interaction with the potential a_ε. For details, see Lemma 4.3.

We conclude the section with a result taken from [Martel and Merle 2007].

Lemma 2.4 (Nontrivial kernel). *There exists a unique even solution of the problem*

$$\mathcal{L}_0 V_0 = mQ^{m-1}, \quad V_0 \in \mathcal{Y}.$$

This solution satisfies

$$(\mathcal{L}_0(1+V_0))' = 0$$

and it is given, in the notation of Lemma 2.3, by

$$V_0(y) = \begin{cases} -\frac{1}{2}\Lambda Q(y) & \text{for } m=2, \\ -Q^2(y) & \text{for } m=3, \\ \frac{1}{3}[Q'(y)\int_0^y Q^2 - 2Q^3(y)] & \text{for } m=4. \end{cases}$$

3. Construction of a soliton-like solution

Following the plan on page 583, we deal first with large negative times, by finding a *pure* soliton-like solution of (1-15) that agrees as $t \to -\infty$, to exponential order in time, with $Q(\cdot - (1-\lambda)t)$, where Q is a soliton for the gKdV equation. Specifically:

Theorem 3.1 (Existence and uniqueness of a pure soliton-like solution). *Suppose* $0 \le \lambda < 1$. *There exists* $\varepsilon_0 > 0$ *small enough such that the following holds for any* $0 < \varepsilon < \varepsilon_0$.

(1) Existence. *There exists a solution* $u \in C(\mathbb{R}, H^1(\mathbb{R}))$ *of* (1-15) *such that*

$$\lim_{t \to -\infty} \left\| u(t) - Q(\cdot - (1-\lambda)t) \right\|_{H^1(\mathbb{R})} = 0, \tag{3-1}$$

and energy $E_a[u](t) = (\lambda - \lambda_0) M[Q]$. *Moreover, there exist constants* $K, \gamma > 0$ *such that for all time* $t \le -\frac{1}{2} T_\varepsilon$ *and* $s \ge 1$,

$$\left\| u(t) - Q(\cdot - (1-\lambda)t) \right\|_{H^s(\mathbb{R})} \le K \varepsilon^{-1} e^{\varepsilon \gamma t}. \tag{3-2}$$

In particular,

$$\left\| u(-T_\varepsilon) - Q(\cdot + (1-\lambda)T_\varepsilon) \right\|_{H^1(\mathbb{R})} \le K \varepsilon^{-1} e^{-\gamma \varepsilon^{-\frac{1}{100}}} \le K \varepsilon^{10}. \tag{3-3}$$

(2) Uniqueness. *This solution is unique if* $m = 3$, *or if* $m = 2, 4$ *and* $0 < \lambda < 1$.

The proof is outlined in Section A.1, and is closely modeled on [Martel 2005], where the existence of a unique N-soliton solution for gKdV equations was established. Other proofs exist, but Martel's method has the advantage of giving an explicit uniform bound in time, (3-2). (This bound is a consequence of compactness properties.) Basically, the result follows from the fact that inside the region $x \le -\frac{1}{2} T_\varepsilon$ the potential a_ε equals 1 to exponential order (see (1-13)). In other words, the aKdV equation (1-15) behaves asymptotically as a gKdV equation, for which soliton solutions exist globally.

Remarks. (a) The energy identity in part (1) of the theorem follows from (3-1), the identities in Section A.6, and the energy conservation law in Proposition 2.1.

(b) The uniqueness of $u(t)$ in the general case is an interesting open question.

(c) For the solution $u(t)$ given by Theorem 3.1, it follows easily from the negativity of the energy E_a and the Gagliardo–Nirenberg inequality (1-8)

$$\frac{1}{K} \|u(t)\|_{H^1(\mathbb{R})} \le \|u(t)\|_{L^2(\mathbb{R})} \le K \|u(t)\|_{H^1(\mathbb{R})} \quad \text{for all } t \in \mathbb{R}, \tag{3-4}$$

for some constant $K > 0$. Moreover, if $m = 3$ or $m = 2, 4$ and $\lambda > 0$, we have

$$\sup_{t \in \mathbb{R}} \|u(t)\|_{H^1(\mathbb{R})} \le K \left\| u\left(-\tfrac{1}{2} T_\varepsilon\right) \right\|_{H^1(\mathbb{R})}. \tag{3-5}$$

This last estimate shows that, to understand the limiting behavior at large times of $u(t)$, it is enough to consider the L^2-norm.

4. Description of the soliton-potential interaction

Once we have proven the existence (and uniqueness) of a pure soliton-like solution for early times, the next step in the study of the soliton-potential interaction. This nonlinear interaction regime is essentially limited to the region $[-T_\varepsilon, T_\varepsilon]$, since $a_\varepsilon(-T_\varepsilon) \sim 1$ and $a_\varepsilon(T_\varepsilon) \sim 2$, by (1-12) and (1-13).

Here we have a stability result saying that the soliton survives the interaction, that the perturbation induced by the potential a_ε is significant, of order one, and that it affects the scaling and shift parameters (the scaling being predicted by conservation of energy). The soliton exits the interaction region as a first-order solution of the aKdV equation (1-15) with $a_\varepsilon \equiv 2$, plus a dispersive term of order $\varepsilon^{1/2}$ in $H^1(\mathbb{R})$.

Recall that we defined $\lambda_0 := \frac{5-m}{m+3}$ in Theorem 1.1. This parameter plays a crucial role in determining the asymptotic behavior.

Theorem 4.1 (Dynamics of the soliton in the interaction region). *Suppose $0 \leq \lambda \leq \lambda_0$. There exist constants $\varepsilon_0 > 0$ and $c_\infty(\lambda) > 1$ such that the following holds for any $0 < \varepsilon < \varepsilon_0$. Let $u = u(t)$ be a globally defined H^1 solution of (1-15) such that*

$$\left\| u(-T_\varepsilon) - Q(\cdot + (1-\lambda)T_\varepsilon) \right\|_{H^1(\mathbb{R})} \leq K\varepsilon^{1/2}. \tag{4-1}$$

Then there exist $K_0 = K_0(K) > 0$ and $\rho(T_\varepsilon), \rho_1(T_\varepsilon) \in \mathbb{R}$ such that

$$\left\| u(T_\varepsilon + \rho_1(T_\varepsilon)) - 2^{-1/(m-1)} Q_{c_\infty}(\cdot - \rho(T_\varepsilon)) \right\|_{H^1(\mathbb{R})} \leq K_0\varepsilon^{1/2}. \tag{4-2}$$

In addition, $c_\infty(\lambda = 0) = 2^p$, with $p = \frac{4}{m+3}$, and $c_\infty(\lambda = \lambda_0) = 1$. Finally, for ε_0 sufficiently small, we have

$$|\rho_1(T_\varepsilon)| \leq \frac{T_\varepsilon}{100} \quad \text{and} \quad (1-\lambda)T_\varepsilon \leq \rho(T_\varepsilon) \leq (2c_\infty(\lambda) - \lambda - 1)T_\varepsilon. \tag{4-3}$$

Remarks. (a) Though Theorem 3.1 ensures exponential decay for the error term at time $t = -T_\varepsilon$ — see (3-3) and (4-1) — we are unable to get a better estimate on the solution at time $t = T_\varepsilon$. This is due to the emergence of dispersive terms of order $\varepsilon^{1/2}$, hard to describe using soliton-based functions. This new phenomenon is similar to a recent description obtained by Martel and Merle [2011; 2010] for the collision of two solitons of similar sizes for the BBM and KdV equations.

(b) We do not know whether Theorem 4.1 is still valid in the range $\lambda > \lambda_0$. Computations suggest that in this regime the soliton might be *reflected* after the interaction. We hope to consider this regime in a forthcoming publication. (See [Muñoz $\geq$ 2011a].)

As mentioned in the Introduction, to prove this theorem we first construct an approximate solution of (1-15) that describes the first-order interaction between the soliton and the potential on the interval of time $[-T_\varepsilon, T_\varepsilon]$. This requires several steps and will occupy us for the rest of this section, culminating in Proposition 4.7. Then, in Section 5, we will prove that the actual solution describing the interaction of the soliton and the potential a_ε is sufficiently close to our approximate solution.

Our first step is the introduction of suitable notation.

Decomposition of the approximate solution. We look for $\tilde{u}(t,x)$, the approximate solution for (1-1), as a suitable modulation of the soliton $Q(x-(1-\lambda)t)$, which solves the KdV equation

$$u_t + (u_{xx} - \lambda u + u^m)_x = 0. \tag{4-4}$$

Let $c = c(\varepsilon t) \geq 1$ be a bounded function to be chosen later and set

$$y := x - \rho(t), \qquad R(t,x) := \frac{Q_{c(\varepsilon t)}(y)}{\tilde{a}(\varepsilon\rho(t))}, \tag{4-5}$$

where

$$\tilde{a}(s) := a^{\frac{1}{m-1}}(s), \qquad \rho(t) := -(1-\lambda)T_\varepsilon + \int_{-T_\varepsilon}^{t} (c(\varepsilon s) - \lambda)\, ds. \tag{4-6}$$

The parameter $\tilde{a}$ is intended to describe the shape variation of the soliton along the interaction.

The form of $\tilde{u}(t,x)$ will be the sum of the soliton plus a correction term:

$$\tilde{u}(t,x) := R(t,x) + w(t,x), \tag{4-7}$$

$$w(t,x) := \varepsilon A_c(\varepsilon t; y), \tag{4-8}$$

where $A_c := A_{c(\varepsilon t)}(\varepsilon t; y) = c^{\frac{1}{m-1}} A(\varepsilon t; \sqrt{c}\, y)$ and A is a function to be determined.

We want to estimate the error produced by inserting $\tilde{u}$ as defined in (4-8) into (1-1). For this, let

$$S[\tilde{u}](t,x) := \tilde{u}_t + (\tilde{u}_{xx} - \lambda\tilde{u} + a_\varepsilon \tilde{u}^m)_x. \tag{4-9}$$

Proposition 4.2 (First decomposition of $S[\tilde{u}]$). *Let $\mathcal{L}$ be the linear operator defined in (2-9). The following nonlinear decomposition of the error term $S[\tilde{u}]$ is valid for every $t \in [-T_\varepsilon, T_\varepsilon]$:*

$$S[\tilde{u}](t,x) = \varepsilon\big(F_1 - (\mathcal{L}A_c)_y\big)(\varepsilon t; y) + \varepsilon^2\big((A_c)_t + c'(\varepsilon t)\Lambda A_c\big)(\varepsilon t; y) + \varepsilon^2 \mathscr{E}(t,x),$$

where $\Lambda A_c(y) := \frac{1}{c}\big(\frac{1}{m-1} A_c(y) + \frac{1}{2} y (A_c)_y(y)\big)$ (compare Lemma 2.3),

$$F_1(\varepsilon t; y) := \frac{c'(\varepsilon t)}{\tilde{a}(\varepsilon\rho(t))}\Lambda Q_c(y) + \frac{a'(\varepsilon\rho(t))}{\tilde{a}^m(\varepsilon\rho(t))}\Big(-\frac{1}{m-1}(c(\varepsilon t)-\lambda)Q_c(y) + (yQ_c^m(y))_y\Big) \tag{4-10}$$

and $\mathscr{E}(t,x)$ is a bounded function in $[-T_\varepsilon, T_\varepsilon] \times \mathbb{R}$.

We prove this result in Section A.2.

Next, if we want to improve the approximation $\tilde{u}$, the unknown function A_c must be such that

$$(\mathcal{L}A_c)_y(\varepsilon t; y) = F_1(\varepsilon t; y) \quad \text{ for all } y \in \mathbb{R}. \tag{Ω}$$

Then the error term will be reduced to the second-order quantity

$$S[\tilde{u}] = \varepsilon^2\big[(A_c)_t + c'(\varepsilon t)\Lambda A_c\big](\varepsilon t; y) + \varepsilon^2 \mathscr{E}(t,x).$$

We prove the solvability of (Ω), which is of independent interest, in the next few pages, concluding with Lemma 4.5. However, we will see that it is not always possible to find a solution of finite mass. In fact, we will look for solutions such that time and space variables are separated:

$$A_c(t,y) = b(\varepsilon t)\varphi_c(y) + d(\varepsilon t) + h(\varepsilon t)\hat{A}_c(y); \tag{4-11}$$

where $b(s)$, $d(s)$ and $h(s)$ are exponentially decreasing in s, φ_c is the bounded function defined in (2-11) and $\hat{A}_c \in \mathcal{Y}$ (recall that $\lim_{\pm\infty} \varphi_c = \pm\sqrt{c}$, and that $c \geq 1$).

This choice is motivated by the fact that a function A_c as in (4-11) satisfies this **I**mportant **P**roperty:

Property IP. Any spatial derivative of $A_c(\varepsilon t, \cdot)$ is a localized $\mathcal{Y}$-function, and there exist $K, \gamma > 0$ such that $\|A_c(\varepsilon t, \cdot)\|_{L^\infty(\mathbb{R})} \leq K e^{-\gamma\varepsilon|t|}$ for all $t \in \mathbb{R}$.

Solution of a time-independent model problem. As a stepping stone to the solution of (Ω), we address the following existence problem. Given a bounded, even function $F = F(y)$, we seek a *bounded* solution $A = A(y)$ to the model problem

$$(\mathcal{L}_0 A)' = F, \tag{4-12}$$

where $\mathcal{L}_0 := -\partial^2_{yy} + 1 - mQ^{m-1}(y)$ as in (2-9). In the spirit of [Martel and Merle 2007, Proposition 2.1] and [Muñoz 2010, Proposition 3.2], we have:

Lemma 4.3 (Existence theory for (4-12)). *Let $F \in \mathcal{Y}$ be even and satisfy the orthogonality condition*

$$\int_{\mathbb{R}} FQ = 0. \tag{4-13}$$

Let $\beta = \frac{1}{2}\int_{\mathbb{R}} F$. For any $\delta \in \mathbb{R}$, the problem (4-12) has a bounded solution A of the form

$$A(y) = \beta\varphi(y) + \delta + A_1(y), \qquad \text{with } A_1(y) \in \mathcal{Y}. \tag{4-14}$$

This solution is unique in $L^2(\mathbb{R})$ up to the addition of a constant times Q'.

Proof. Write $A := \beta\varphi + \delta(1 + V_0) + A_1$, where $\beta, \delta \in \mathbb{R}$ and $A_1 \in \mathcal{Y}$ are to be determined. Inserting this decomposition in (4-12), we have $(\mathcal{L}_0 A_1)' = F - \beta(\mathcal{L}_0\varphi)'$, namely

$$\mathcal{L}_0 A_1 = H - \beta\mathcal{L}_0\varphi + \gamma, \quad H(y) := \int_{-\infty}^{y} F(s)\,ds, \tag{4-15}$$

and where $\gamma := \mathcal{L}_0 A_1(0) - \int_{-\infty}^{0} H(s)\,ds$. Without loss of generality we can suppose the constant term γ equals $-\beta$, because from Lemma 2.4 we have $\mathcal{L}_0(1 + V_0) = 1$, so any constant term can be associated to the free parameter δ.

Now, from Lemma 2.3 the problem (4-12) is solvable if and only if

$$\int_{\mathbb{R}} (H - \beta(\mathcal{L}_0\varphi + 1))Q' = \int_{\mathbb{R}} HQ' = \int_{\mathbb{R}} FQ = 0,$$

which is (4-13) (recall that $\mathcal{L}_0 Q' = 0$). Thus there exists a solution A_1 of (4-15) satisfying $\int_{\mathbb{R}} A_1 Q' = 0$. Moreover, since

$$\lim_{y\to-\infty} (H - \beta(\mathcal{L}_0\varphi + 1))(y) = 0 \quad \text{and} \quad \lim_{y\to+\infty} (H - \beta(\mathcal{L}_0\varphi + 1))(y) = \int_{\mathbb{R}} F - 2\beta,$$

we get $A_1 \in \mathcal{Y}$ provided $\beta = \frac{1}{2}\int_{\mathbb{R}} F$, by Lemma 2.3. This finishes the proof. $\square$

Existence of dynamical parameters. Now we show the existence of a dynamical system involving the evolution of first-order scaling and translation parameters on the main interaction region. This system is related to the orthogonality condition $\int_{\mathbb{R}} F_1 Q_c = 0$; see proof of Lemma 4.5 below.

Lemma 4.4 (Existence of dynamical parameters). *Let $m = 2, 3, 4$ and let λ_0, p, $a(\cdot)$ be as in Theorem 4.1 and (1-13). The system of of differential equations*

$$\begin{cases} c'(\varepsilon t) = p\, c(\varepsilon t)\Big(c(\varepsilon t) - \dfrac{\lambda}{\lambda_0}\Big)\dfrac{a'}{a}(\varepsilon\rho(t)), & c(-\varepsilon T_\varepsilon) = 1, \\ \rho'(t) = c(\varepsilon t) - \lambda, & \rho(-T_\varepsilon) = -(1-\lambda)T_\varepsilon \end{cases} \tag{4-16}$$

has a unique solution (ρ, c) with c bounded, positive, monotone, defined for all $t \geq -T_\varepsilon$, and having the same regularity as $a(\varepsilon\,\cdot)$. In addition:

(1) *If $\lambda = \lambda_0$, one has $c \equiv 1$.*

(2) *If $0 \leq \lambda < \lambda_0$ then for all $t \geq -T_\varepsilon$ one has $c(\varepsilon t) > 1$ and $\lim_{t\to+\infty} c(\varepsilon t) = c_\infty + O(\varepsilon^{10})$, where $c_\infty = c_\infty(\lambda) > 1$ is the unique solution of the algebraic equation*

$$c_\infty^{\lambda_0}\Big(c_\infty - \frac{\lambda}{\lambda_0}\Big)^{1-\lambda_0} = 2^p\Big(1 - \frac{\lambda}{\lambda_0}\Big)^{1-\lambda_0}, \quad c_\infty > 1. \tag{4-17}$$

Moreover, $\lambda \in [0, \lambda_0] \mapsto c_\infty(\lambda) \geq 1$ is a smooth decreasing function and $c_\infty(\lambda = 0) = 2^p$.

In the case $\lambda = 0$, there exists a simple implicit expression for $c(\varepsilon t)$:

$$\rho'(t) = c(\varepsilon t) = \frac{a^p(\varepsilon\rho(t))}{a^p(-\varepsilon T_\varepsilon)}.$$

Using the strict monotonicity of a, from this identity we can find explicitly $c(\varepsilon t)$.

Remark. The critical value λ_0 can be seen as the exact value of λ such that the solution $u(t)$ constructed in Theorem 3.1 has zero energy. Indeed, Theorem 3.1 gives $E_a[u] = (\lambda - \lambda_0)M[Q]$. Hence $E_a[u]$ is zero, positive or negative depending on whether $\lambda = \lambda_0$, $\lambda > \lambda_0$, or $\lambda < \lambda_0$. Because of this the study of the soliton dynamics for $\lambda > \lambda_0$ is an open question.

Proof of Lemma 4.4. The local existence of a solution (c, ρ) of (4-16) is a direct consequence of the Cauchy–Lipschitz–Picard theorem.

Now we use (4-16) to prove a priori estimates on the solution c. Note that

$$\frac{c(\varepsilon t) - \lambda}{c(\varepsilon t)\Big(c(\varepsilon t) - \dfrac{\lambda}{\lambda_0}\Big)} c'(\varepsilon t) = p(c(\varepsilon t) - \lambda)\frac{a'}{a}(\varepsilon\rho) = p\rho'(t)\frac{a'}{a}(\varepsilon\rho).$$

In particular,

$$(1-\lambda_0)\partial_t \log\Big(c(\varepsilon t) - \frac{\lambda}{\lambda_0}\Big) + \lambda_0 \partial_t \log c(\varepsilon t) = p\,\partial_t \log a(\varepsilon\rho).$$

Integrating on $[-T_\varepsilon, t]$ and using $c(-\varepsilon T_\varepsilon) = 1$, we obtain

$$c^{\lambda_0}(\varepsilon t)\Big(c(\varepsilon t) - \frac{\lambda}{\lambda_0}\Big)^{1-\lambda_0} = (1 - \frac{\lambda}{\lambda_0})^{1-\lambda_0} \frac{a^p(\varepsilon\rho(t))}{a^p(-\varepsilon(1-\lambda)T_\varepsilon)}. \tag{4-18}$$

Since $1 \le a \le 2$, the function c is bounded and ρ is bounded on compact sets; this yields global existence. One proves in particular that $c' > 0$ and

$$c^{\lambda_0}(\varepsilon t) < a^p(\varepsilon\rho), \quad \text{and thus} \quad 1 \le c(\varepsilon t) \le 2^{\frac{4}{5-m}}. \tag{4-19}$$

The limit $\lim_{t\to+\infty} c(\varepsilon t)$ exists and is of the form $c_\infty + O(\varepsilon^{10})$, where c_∞ is a solution of (4-17), after passing to the limit in (4-18). To prove the uniqueness of the solution of (4-17), consider for $\mu \ge 1$ the smooth function

$$g(\mu;\lambda) := \mu^{\lambda_0}\Big(\mu - \frac{\lambda}{\lambda_0}\Big)^{1-\lambda_0} - 2^p\Big(1 - \frac{\lambda}{\lambda_0}\Big)^{1-\lambda_0}.$$

Note that in the case $\lambda < \lambda_0$ we have $g(1;\lambda) < 0$ and

$$\partial_\mu g(\mu;\lambda) = \mu^{\lambda_0-1}\Big(\mu - \frac{\lambda}{\lambda_0}\Big)^{-\lambda_0}(\mu-\lambda) \ge \Big(1 - \frac{\lambda}{\lambda_0}\Big)^{-\lambda_0} > 0.$$

This implies that there exists a unique $c_\infty(\lambda) > 1$ such that $g(c_\infty(\lambda);\lambda) = 0$. This proves uniqueness. The smoothness of the application $\lambda \in [0,\lambda_0] \mapsto c_\infty(\lambda)$ is an easy consequence of the implicit function theorem.

Finally we prove that $\lambda \mapsto c_\infty(\lambda)$ is a decreasing map. To do this, we differentiate (4-17), obtaining

$$\begin{aligned}\frac{c_\infty(\lambda)^{\lambda_0-1}(c_\infty(\lambda)-\lambda)}{\Big(c_\infty(\lambda) - \frac{\lambda}{\lambda_0}\Big)^{\lambda_0}} c_\infty'(\lambda) &= \Big(\frac{1}{\lambda_0} - 1\Big)\left(\frac{c_\infty^{\lambda_0}(\lambda)}{\Big(c_\infty(\lambda) - \frac{\lambda}{\lambda_0}\Big)^{\lambda_0}} - \frac{2^p}{\Big(1 - \frac{\lambda}{\lambda_0}\Big)^{\lambda_0}}\right) \\ &\le \Big(\frac{1}{\lambda_0} - 1\Big)\Big(1 - \frac{\lambda}{\lambda_0}\Big)^{-\lambda_0}(1-2^p) < 0.\end{aligned}$$

□

We can now state the promised result on the solvability of (Ω):

Lemma 4.5. *Suppose $0 \le \lambda \le \lambda_0$ and $c(\varepsilon t)$ is given by (4-16). There exists a solution $A_c = A_c(\varepsilon t; y)$ of*

$$(\mathcal{L}A_c)'(\varepsilon t; y) = F_1(\varepsilon t; y), \tag{4-20}$$

satisfying Property IP and the following conditions:

(1) *For every $t \in [-T_\varepsilon, T_\varepsilon]$,*

$$\begin{cases} A_c(\varepsilon t;\cdot) \in L^\infty(\mathbb{R}), & A_c(\varepsilon t; y) = b(\varepsilon t)(\varphi_c(y) - c^{1/2}) + h(\varepsilon t)\hat{A}_c(y), \\ \hat{A}_c \in \mathcal{Y}, & |b(\varepsilon t)| + |h(\varepsilon t)| \le K e^{-\gamma\varepsilon|t|}. \end{cases} \tag{4-21}$$

(2) $\lim_{y\to+\infty} A_c(y) = 0$.

Remark. The function A_c models, to first order in ε, the shelf-like tail behind the soliton, a dispersive effect of the soliton-potential interaction.

Proof. Step 1: *Reduction to a time-independent problem.* We suppose c given as in Lemma 4.4. Note that F_1 in (4-10) can be written as

$$F_1(\varepsilon t; y) = \frac{a'}{\tilde{a}^m}\big[pc(c-\frac{\lambda}{\lambda_0})\Lambda Q_c - \frac{1}{m-1}(c-\lambda)Q_c + (yQ_c^m)'\big](y).$$

Consider the functions

$$\tilde{F}_1(y) := p\Lambda Q - \frac{1}{m-1}Q + (yQ^m)' \quad \text{and} \quad \hat{F}_1(y) := \frac{1}{m-1}Q - \frac{p}{\lambda_0}\Lambda Q = \frac{1}{m-1}Q - \frac{4}{5-m}\Lambda Q.$$

We claim that if $c(\varepsilon t)$ satisfies (4-16) then every term in F_1 has the correct scaling. More precisely:

Claim 4.6. *Suppose $\tilde{A}(y)$, $\hat{A}(y)$ solve the stationary problems*

$$(\mathcal{L}_0\tilde{A})' = \tilde{F}_1, \quad (\mathcal{L}_0\hat{A})' = \hat{F}_1. \tag{4-22}$$

Then, for all $t \in \mathbb{R}$,

$$A_c(\varepsilon t; y) := \frac{a'(\varepsilon\rho)}{\tilde{a}^m(\varepsilon\rho)} c^{\frac{1}{m-1}-\frac{1}{2}}(\varepsilon t)\big(\tilde{A} + \lambda c^{-1}(\varepsilon t)\hat{A}\big)(c^{1/2}(\varepsilon t)y)$$

is a solution of (4-20).

Indeed, we have

$$\begin{aligned}
&(\mathcal{L}A_c)' \\
&= \frac{a'(\varepsilon\rho)}{\tilde{a}^m(\varepsilon\rho)} c^{\frac{1}{m-1}+1}\big(-\tilde{A}'' + \tilde{A} - mQ^{m-1}\tilde{A}\big)'(c^{1/2}y) + \lambda\frac{a'(\varepsilon\rho)}{\tilde{a}^m(\varepsilon\rho)} c^{\frac{1}{m-1}}\big(-\hat{A}'' + \hat{A} - mQ^{m-1}\hat{A}\big)'(c^{1/2}y) \\
&= \frac{a'(\varepsilon\rho)}{\tilde{a}^m(\varepsilon\rho)} c^{\frac{1}{m-1}+1}\tilde{F}_1(c^{1/2}y) + \lambda\frac{a'(\varepsilon\rho)}{\tilde{a}^m(\varepsilon\rho)} c^{\frac{1}{m-1}}\hat{F}_1(c^{1/2}y) \\
&= \frac{a'(\varepsilon\rho)}{\tilde{a}^m(\varepsilon\rho)}\Big(pc^2\Lambda Q_c - \frac{1}{m-1}cQ_c + (yQ_c^m)'\Big) + \lambda\frac{a'(\varepsilon\rho)}{\tilde{a}^m(\varepsilon\rho)}\Big(\frac{1}{m-1}Q_c - \frac{p}{\lambda_0}c\Lambda Q_c\Big) = F_1(\varepsilon t; y).
\end{aligned}$$

This proves the claim, which in turn reduces the problem to the time-independent case.

Step 2: *There exists solutions $\tilde{A}$, $\hat{A}$ of (4-22) satisfying (4-14).* By Lemma 4.3, this follows once we verify the orthogonality conditions

$$\int_{\mathbb{R}} \tilde{F}_1 Q = \int_{\mathbb{R}} \hat{F}_1 Q = 0.$$

To this end, we use the identities in Section A.6:

$$\begin{aligned}
\int_{\mathbb{R}} \tilde{F}_1 Q &= p\int_{\mathbb{R}} \Lambda Q Q - \frac{1}{m-1}\int_{\mathbb{R}} Q^2 + \int_{\mathbb{R}} Q(yQ^m)_y \\
&= p\int_{\mathbb{R}} \Lambda Q Q - \frac{1}{m-1}\int_{\mathbb{R}} Q^2 + \frac{1}{m+1}\int_{\mathbb{R}} Q^{m+1} = \frac{5-m}{4(m-1)}\Big(p - \frac{4}{m+3}\Big)\int_{\mathbb{R}} Q^2 = 0.
\end{aligned}$$

Similarly,

$$\int_{\mathbb{R}} \hat{F}_1 Q = -\frac{4}{5-m}\int_{\mathbb{R}} \Lambda Q Q + \frac{1}{m-1}\int_{\mathbb{R}} Q^2 = -\frac{4}{5-m} \times \frac{5-m}{4(m-1)}\int_{\mathbb{R}} Q^2 + \frac{1}{m-1}\int_{\mathbb{R}} Q^2 = 0.$$

Thus, by virtue of Lemma 4.3, there exist solutions $\tilde{A}$, $\hat{A}$ of (4-22) of the form

$$\begin{cases} \tilde{A}(y) = \tilde{\beta}\varphi(y) + \tilde{\delta} + \tilde{A}_1(y), & \tilde{A}_1 \in \mathcal{Y}, \\ \hat{A}(y) = \hat{\beta}\varphi(y) + \hat{\delta} + \hat{A}_1(y), & \hat{A}_1 \in \mathcal{Y}, \end{cases}$$

where $\tilde{\beta}, \hat{\beta}, \tilde{\delta}, \hat{\delta} \in \mathbb{R}$. Moreover, $\tilde{\beta}, \hat{\beta}$ are given by

$$\tilde{\beta} := \frac{1}{2}\int_{\mathbb{R}} \tilde{F}_1 = \frac{1}{2}\int_{\mathbb{R}}\Big(p\Lambda Q - \frac{1}{m-1}Q\Big) = \frac{1}{2}\Big(p\big(\frac{1}{m-1} - \frac{1}{2}\big) - \frac{1}{m-1}\Big)\int_{\mathbb{R}} Q = -\frac{3}{2(m+3)}\int_{\mathbb{R}} Q < 0,$$

for each $m = 2, 3, 4$. On the other hand,

$$\hat{\beta} := \frac{1}{2}\int_{\mathbb{R}} \hat{F}_1 = \frac{1}{2}\int_{R}\Big(\frac{1}{m-1}Q - \frac{4}{5-m}\Lambda Q\Big) = \frac{1}{2}\Big(\frac{1}{m-1} - \frac{4}{5-m} \times \frac{3-m}{2(m-1)}\Big)\int_{\mathbb{R}} Q = \frac{1}{2(5-m)}\int_{\mathbb{R}} Q > 0,$$

for each $m = 2, 3, 4$.

Step 3: *Conclusion.* Finally, to get $\lim_{y\to+\infty} \tilde{A}(y) = \lim_{y\to+\infty} \hat{A}(y) = 0$ we choose $\tilde{\delta} = -\tilde{\beta}$ and $\hat{\delta} = -\hat{\beta}$. This proves the last part of the lemma. With this choice we have

$$\tilde{A}(y) = \tilde{\beta}(\varphi(y) - 1) + \tilde{A}_1(y), \quad \hat{A}(y) = \hat{\beta}(\varphi(y) - 1) + \hat{A}_1(y), \quad \tilde{A}_1, \hat{A}_1 \in \mathcal{Y}.$$

Using Claim 4.6, an actual solution $A_c(\varepsilon t; y)$ of (4-20) is obtained by considering

$$\begin{aligned} A_c(\varepsilon t; y) &:= \frac{a'(\varepsilon\rho)}{\tilde{a}^m(\varepsilon\rho)} c^{\frac{1}{m-1}}(\varepsilon t)\big[\tilde{A} + \lambda c^{-1}(\varepsilon t)\hat{A}\big](c^{1/2}y) \\ &=: b(\varepsilon t)(\varphi_c(y) - c^{1/2}) + h(\varepsilon t)\hat{A}_c(y), \quad \hat{A}_c \in \mathcal{Y}, \end{aligned}$$

where

$$b(\varepsilon t) := \frac{a'(\varepsilon\rho)c^{\frac{1}{m-1}-1}}{\tilde{a}^m(\varepsilon\rho)}(\tilde{\beta} + \lambda c^{-1}(\varepsilon t)\hat{\beta}), \quad h(\varepsilon t) := \frac{a'(\varepsilon\rho)}{\tilde{a}^m(\varepsilon\rho)}.$$

This finishes the proof of Lemma 4.5. □

Remark. We emphasize that A_c lies in $L^2(\mathbb{R})$ *in all cases*, even if it is exponentially decreasing in time. This nonsummable solution must be modified in order to obtain a finite-mass solution.

Before continuing with the construction of the approximate solution, we need some crucial estimates on the parameter $c(\varepsilon t)$.

Remark (Bounds for $c(\varepsilon t)$). From the bound on $c(\varepsilon t)$ in (4-18) we conclude that, for all $t \in [-T_\varepsilon, T_\varepsilon]$,

$$1 \le c(\varepsilon t) \le 2^{\frac{4}{5-m}}.$$

***Correction to the solution of problem* (Ω).** Consider the cutoff function $\eta \in C^\infty(\mathbb{R})$ satisfying the following properties:

$$\begin{cases} 0 \le \eta(s) \le 1, \quad 0 \le \eta'(s) \le 1 & \text{for any } s \in \mathbb{R}; \\ \eta(s) \equiv 0 & \text{for } s \le -1, \\ \eta(s) \equiv 1 & \text{for } s \ge 1. \end{cases} \tag{4-23}$$

Define

$$\eta_\varepsilon(y) := \eta(\varepsilon y + 2), \tag{4-24}$$

and for the solution $A_c = A_c(\varepsilon t; y)$ of (4-20) constructed in Lemma 4.5, set

$$A_\#(\varepsilon t; y) := \eta_\varepsilon(y) A_c(\varepsilon t; y). \tag{4-25}$$

Now redefine

$$\tilde{u} := R + w = R + \varepsilon A_\#. \tag{4-26}$$

where R is the modulated soliton from (4-5).

The following proposition, which deals with the error associated to this cut-off function and the new approximate solution $\tilde{u}$, is the principal result of this section.

Proposition 4.7 (Construction of an approximate solution for (1-15)). *There exist constants $\varepsilon_0, K > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ the following holds.*

(1) *For the localized function $A_\#$ of (4-25), we have:*

  (a) New behavior. *For all $t \in [-T_\varepsilon, T_\varepsilon]$,*

$$\begin{cases} A_\#(\varepsilon t, y) = 0 & \text{for all } y \le -\frac{3}{\varepsilon}, \\ A_\#(\varepsilon t, y) = A_c(\varepsilon t, y) & \text{for all } y \ge -\frac{1}{\varepsilon}. \end{cases} \tag{4-27}$$

  (b) Integrable solution. *For all $t \in [-T_\varepsilon, T_\varepsilon]$, $A_\#(\varepsilon t, \cdot) \in H^1(\mathbb{R})$ with*

$$\|\varepsilon A_\#(\varepsilon t, \cdot)\|_{H^1(\mathbb{R})} \le K \varepsilon^{\frac{1}{2}} e^{-\gamma\varepsilon|t|}. \tag{4-28}$$

(2) *The error associated to the new function $\tilde{u}$ satisfies*

$$\|S[\tilde{u}](t)\|_{H^2(\mathbb{R})} \le K \varepsilon^{\frac{3}{2}} e^{-\gamma\varepsilon|t|}, \tag{4-29}$$

and the following integral estimate holds:

$$\int_{\mathbb{R}} \|S[\tilde{u}](t)\|_{H^2(\mathbb{R})} dt \le K \varepsilon^{1/2}.$$

Proof. The proof of (4-27) is direct from the definition. To prove (4-28) it is enough to recall that

$$\|\eta_c'\|_{L^2(\mathbb{R})} \le K \varepsilon^{-1/2}.$$

For the proof of (4-29), see Section A.3. □

Recomposition of the solution. We now present some important estimates concerning our approximate solution, showing that $\tilde{u}$ at time $\pm T_\varepsilon$ behaves as a modulated soliton with the scaling given by rough computations at infinity. We start out with some model H^1-estimates.

Lemma 4.8 (First estimates on $\tilde{u}$).

(1) Decay away from zero. *If $f = f(y) \in \mathcal{Y}$, there exist constants $K, \gamma > 0$ such that, for all $t \in [-T_\varepsilon, T_\varepsilon]$,*

$$\left\| a'(\varepsilon x) f(y) \right\|_{H^1(\mathbb{R})} \le K e^{-\gamma\varepsilon|t|}. \tag{4-30}$$

(2) Almost-soliton solution. *The following estimates hold for all* $t \in [-T_\varepsilon, T_\varepsilon]$.

$$\|\tilde u_t + (c-\lambda)\tilde u_x\|_{H^1(\mathbb{R})} \le K\varepsilon e^{-\gamma\varepsilon|t|}, \quad \|\tilde u_t + (c-\lambda)\tilde u_x\|_{L^\infty(\mathbb{R})} \le K\varepsilon e^{-\gamma\varepsilon|t|}. \tag{4-31}$$

$$\tilde u_{xx} - \lambda\tilde u + a_\varepsilon \tilde u^m = (c-\lambda)\tilde u + O_{L^2(\mathbb{R})}(\varepsilon e^{-\gamma\varepsilon|t|}). \tag{4-32}$$

$$\|(\tilde u_{xx} - c\tilde u + a_\varepsilon\tilde u^m)_x\|_{H^1(\mathbb{R})} \le K\varepsilon e^{-\gamma\varepsilon|t|} + K\varepsilon^2. \tag{4-33}$$

Proof. The proof of (4-30) is a direct consequence of (1-13) and the fact that $\rho'(t) = c(\varepsilon t) - \lambda \ge 1-\lambda$, for all $t \in \mathbb{R}$.

Now let us prove (4-31). From (4-26) we obtain

$$\tilde u_t + (c-\lambda)\tilde u_x = \varepsilon\frac{c'}{\tilde a}\Lambda Q_c - \varepsilon\frac{\tilde a'}{\tilde a^2}(c-\lambda)Q_c + \varepsilon\big((A_\#)_t + c(A_\#)_x\big) = \varepsilon\big((A_\#)_t + c(A_\#)_x\big) + O_{H^1(\mathbb{R})}(\varepsilon e^{-\gamma\varepsilon|t|}).$$

Now, from (A-34) in the Appendix, we know that

$$\varepsilon\big((A_\#)_t + c(A_\#)_x\big) = \varepsilon^2(c-\lambda)\eta_c' A_c + O_{H^1(\mathbb{R})}(\varepsilon^{\frac{3}{2}} e^{-\gamma\varepsilon|t|}) = O_{H^1(\mathbb{R})}(\varepsilon^{\frac{3}{2}} e^{-\gamma\varepsilon|t|}).$$

This completes the proof of the H^1-estimate. The L^∞-estimate then follows from the continuous Sobolev embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(R)$.

Concerning (4-32), note that from (4-28) we have

$$\begin{aligned}\tilde u_{xx} - \lambda\tilde u + a_\varepsilon\tilde u^m &= (c-\lambda)\tilde u + \varepsilon\big((A_\#)_{xx} + m a_\varepsilon R^{m-1} A_\#\big) + O_{L^2(\mathbb{R})}(\varepsilon e^{-\gamma\varepsilon|t|}) + O(\varepsilon^2|A_\#|^2)\\ &= (c-\lambda)\tilde u + O_{L^2(\mathbb{R})}(\varepsilon e^{-\gamma\varepsilon|t|}).\end{aligned}$$

For (4-33), note that $\big(\tilde u_{xx} - c\tilde u + a_\varepsilon\tilde u^m\big)_x = S[\tilde u] - ((c-\lambda)\tilde u_x + \tilde u_t)$. The conclusion now follows from (4-29) and (4-31). ☐

The next result describes the behavior of the almost solution $\tilde u$ at the endpoints $t = -T_\varepsilon, T_\varepsilon$.

Proposition 4.9 (Behavior at $t = \pm T_\varepsilon$). *There exist constants* $K, \varepsilon_0 > 0$ *such that for every* $0 < \varepsilon < \varepsilon_0$ *the approximate solution* $\tilde u$ *constructed in Proposition 4.7 has these properties*:

(1) Closeness to Q at time $t = -T_\varepsilon$.

$$\|\tilde u(-T_\varepsilon) - Q(\cdot + (1-\lambda)T_\varepsilon)\|_{H^1(\mathbb{R})} \le K\varepsilon^{10}. \tag{4-34}$$

(2) Closeness to $2^{-1/(m-1)}Q_{c_\infty}$ at time $t = T_\varepsilon$. *Let* $c_\infty(\lambda) > 1$ *be as defined in Lemma 4.4. Then*

$$\|\tilde u(T_\varepsilon) - 2^{-1/(m-1)}Q_{c_\infty}(\cdot - \rho(T_\varepsilon))\|_{H^1(\mathbb{R})} \le K\varepsilon^{10}. \tag{4-35}$$

Proof. By definition,

$$\tilde u(-T_\varepsilon) - Q(\cdot - \rho(-T_\varepsilon)) = R(-T_\varepsilon) - Q(\cdot + (1-\lambda)T_\varepsilon) + w(-T_\varepsilon).$$

From Proposition 4.7 we have

$$\|w(\pm T_\varepsilon)\|_{H^1(\mathbb{R})} = \|\varepsilon A_\#(\pm T_\varepsilon)\|_{H^1(\mathbb{R})} \le K\varepsilon^{1/2} e^{-\gamma\varepsilon^{-\frac{1}{100}}} \le K\varepsilon^{10},$$

for ε small enough. On the other hand, from $\rho(-T_\varepsilon) = -(1-\lambda)T_\varepsilon$ and using the monotonicity of a, we have

$$1 \le c(-\varepsilon T_\varepsilon) \le a^{\frac{4}{5-m}}(\varepsilon\rho(-T_\varepsilon)) \le 1+\varepsilon^{10}.$$

In conclusion we have

$$\|R(-T_\varepsilon) - Q(\cdot + (1-\lambda)T_\varepsilon)\|_{H^1(\mathbb{R})} \le K\varepsilon^{10},$$

as desired. The estimate (4-35) is totally analogous, and we skip the details. □

To summarize this section: we have constructed and approximate solution $\tilde{u}$ describing the soliton-potential interaction in principle. In the next section we will show that the *solution* u constructed in Theorem 3.1 actually behaves like $\tilde{u}$ inside the interaction box $[-T_\varepsilon, T_\varepsilon]$.

5. First stability results

Our next goal is to prove that the approximate solution $\tilde{u}$ describes the actual dynamics of interaction in the interval $[-T_\varepsilon, T_\varepsilon]$. This is the principal result of this section:

Proposition 5.1 (Exact solution close to the approximate solution $\tilde{u}$). *Let $\kappa > \frac{1}{100}$. There exists $\varepsilon_0 > 0$ such that the following holds for any $0 < \varepsilon < \varepsilon_0$. Suppose that*

$$\|S[\tilde{u}](t)\|_{H^2(\mathbb{R})} \le K\varepsilon^{1+\kappa}e^{-\gamma\varepsilon|t|}, \quad \int_{\mathbb{R}} \|S[\tilde{u}](t)\|_{H^2(\mathbb{R})}\,dt \le K\varepsilon^{\kappa}, \tag{5-1}$$

and

$$\|u(-T_\varepsilon) - \tilde{u}(-T_\varepsilon)\|_{H^1(\mathbb{R})} \le K\varepsilon^{\kappa}, \tag{5-2}$$

where $u = u(t)$ is an $H^1(\mathbb{R})$ solution of (1-15) in a vicinity of $t = -T_\varepsilon$. Then $u(t)$ is defined for any $t \in [-T_\varepsilon, T_\varepsilon]$ and there exist $K_0 = K_0(\kappa, K)$ and a C^1-function $\rho_1 : [-T_\varepsilon, T_\varepsilon] \to \mathbb{R}$ such that, for all $t \in [-T_\varepsilon, T_\varepsilon]$,

$$\|u(t+\rho_1(t)) - \tilde{u}(t)\|_{H^1(\mathbb{R})} \le K_0\varepsilon^{\kappa}, \quad |\rho_1'(t)| \le K_0\varepsilon^{\kappa}.$$

Remark. Note that u has to be modulated in order to get the correct result. However, in this case we have not modulated on the scaling and spatial translation parameters because (1-15) is not invariant under these transformations. Nevertheless, we still have another degeneracy, due to time translations, which fortunately allows control of the dynamics of the solution u for every $t \in [-T_\varepsilon, T_\varepsilon]$. In this sense, the *new time* $s(t) := t + \rho_1(t)$ can be interpreted as a *retarded* or *advanced* time of the actual solution with respect to the approximate solution. Moreover, for ε small enough, we have

$$s'(t) = 1 + \rho'(t) > \frac{99}{100} > 0,$$

for all $t \in [-T_\varepsilon, T_\varepsilon]$. This means that we can invert $s(t)$ on $s([-T_\varepsilon, T_\varepsilon]) \subseteq \frac{99}{100}[-T_\varepsilon, T_\varepsilon]$.

From the proof we do not know the sign of $\rho_1'(t)$, so in particular we do not know if the solution u is retarded or in advance with respect to the approximate solution $\tilde{u}$.

Proof of Proposition 5.1. Let $K^* > 1$ be a constant to be fixed later. Recall from Proposition 2.2 that $u(t)$ is globally well defined in $H^1(\mathbb{R})$. Since $\|u(-T_\varepsilon) - \tilde{u}(-T_\varepsilon)\|_{H^1(\mathbb{R})} \le K\varepsilon^\kappa$, by continuity in time in $H^1(\mathbb{R})$, we can define $-T_\varepsilon < T^* \le T_\varepsilon$ by

$$T^* := \sup \left\{ \begin{array}{r} T \in [-T_\varepsilon, T_\varepsilon] : \text{for all } t \in [-T_\varepsilon, T],\ \text{there exists } r(t) \in \mathbb{R} \\ \text{with } \|u(t + r(t)) - \tilde{u}(t)\|_{H^1(\mathbb{R})} \le K^* \varepsilon^\kappa \end{array} \right\}.$$

The goal is to prove that $T^* = T_\varepsilon$ for K^* large enough. To achieve this, we assume otherwise and reach a contradiction with the definition of T^* via some independent estimates for $\|u(t + r(t)) - \tilde{u}(t)\|_{H^1(\mathbb{R})}$ on $[-T_\varepsilon, T^*]$, for a special modulation parameter $r(t)$.

Modulation. By using the implicit function theorem we will construct a modulation parameter and to estimate its variation in time:

Lemma 5.2 (Modulation in time). *Assume $0 < \varepsilon < \varepsilon_0(K^*)$ small enough. There exists a unique C^1 function $\rho_1(t)$ such that the function*

$$z(t) = u(t + \rho_1(t)) - \tilde{u}(t) \tag{5-3}$$

satisfies, for all $t \in [-T_\varepsilon, T^*]$,

$$\int_{\mathbb{R}} z(t, x) Q_c'(y)\, dx = 0. \tag{5-4}$$

For all $t \in [-T_\varepsilon, T^*]$, we have

$$|\rho_1(-T_\varepsilon)| + \|z(-T_\varepsilon)\|_{H^1(\mathbb{R})} \le K\varepsilon^\kappa, \quad \|z(t)\|_{H^1(\mathbb{R})} \le 2K^*\varepsilon^\kappa. \tag{5-5}$$

In addition, $z(t)$ satisfies the equation

$$z_t + (1 + \rho_1')\big(z_{xx} - \lambda z + a_\varepsilon[(\tilde{u} + z)^m - \tilde{u}^m]\big)_x - \rho_1' \tilde{u}_t + (1 + \rho_1') S[\tilde{u}] = 0. \tag{5-6}$$

Finally, there exist $K, \gamma > 0$ independent of K^* such that for every $t \in [-T_\varepsilon, T^*]$

$$|\rho_1'(t)| \le \frac{K}{c(\varepsilon t) - \lambda}\big(\|z\|_{L^2(\mathbb{R})} + \varepsilon e^{-\gamma\varepsilon|t|}\|z(t)\|_{L^2(\mathbb{R})} + \|z(t)\|^2_{L^2(\mathbb{R})} + \|S[\tilde{u}]\|_{L^2(\mathbb{R})}\big). \tag{5-7}$$

Proof. The proof of (5-4) and (5-5) is a by now well-known consequence of the implicit function theorem; see, e.g., [Martel and Merle 2007]. The proof of (5-6) follows after a simple calculation using (1-15).

To prove (5-7), we take the time derivative of (5-4) and substitute replace z_t from (5-6) to obtain

$$\begin{aligned} 0 = (1 + \rho_1') \int_{\mathbb{R}} \{z_{xx} - cz + a_\varepsilon[(\tilde{u} + z)^m - \tilde{u}^m]\} Q_c'' \\ + \rho_1' \int_{\mathbb{R}} (\tilde{u}_t - (c - \lambda) z_x) Q_c' - (1 + \rho_1') \int_{\mathbb{R}} S[\tilde{u}] Q_c' + \varepsilon c'(\varepsilon t) \int_{\mathbb{R}} z \Lambda Q_c'. \end{aligned}$$

Now note that

$$\rho_1' \int_{\mathbb{R}} (\tilde{u}_t - (c - \lambda) z_x) Q_c' = -\frac{\rho_1'}{\tilde{a}} \bigg((c - \lambda) \int_{\mathbb{R}} Q_c'^2 + O(\varepsilon + \|z(t)\|_{L^2(\mathbb{R})}) \bigg).$$

On the other hand,

$$\int_{\mathbb{R}} \big(z_{xx} - cz + a_\varepsilon[(\tilde{u} + z)^m - \tilde{u}^m]\big) Q_c'' = -\int_{\mathbb{R}} z \mathcal{L} Q_c'' + O(\varepsilon e^{-\gamma\varepsilon|t|}\|z(t)\|_{L^2(\mathbb{R})}) + O(\|z(t)\|^2_{L^2(\mathbb{R})}).$$

Collecting these estimates and using the fact that $\|z(t)\|_{H^1(\mathbb{R})}$ is small, we get desired result. $\square$

Control in the Q_c direction. We recall from (1-7) that the energy of the function $u(t+\rho_1(t))$ is conserved; moreover, $E_a[u(t+\rho_1(t))] = E_a[u](t)$ for any $t \in [-T_\varepsilon, T^*]$. In what follows, we will made use of this identity to estimate z against the degenerate direction Q_c. First we prove that the approximate solution $\tilde{u}$ has almost conserved energy.

Lemma 5.3 (Almost conservation of energy). *For the approximate solution $\tilde{u}$ from Proposition 4.7,*

$$\partial_t E_a[\tilde{u}](t) = -\int_{\mathbb{R}} (\tilde{u}_{xx} - \lambda\tilde{u} + a_\varepsilon \tilde{u}^m) S[\tilde{u}]. \tag{5-8}$$

In particular, there exists $K > 0$ independent of K^* such that

$$\big|E_a[\tilde{u}](t) - E_a[\tilde{u}](-T_\varepsilon)\big| \le K\varepsilon^\kappa. \tag{5-9}$$

Proof. From (4-9) we have

$$\begin{aligned}\int_{\mathbb{R}} S[\tilde{u}](\tilde{u}_{xx} - \lambda\tilde{u} + a_\varepsilon\tilde{u}^m) &= \int_{\mathbb{R}} \tilde{u}_t(\tilde{u}_{xx} - \lambda\tilde{u} + a_\varepsilon\tilde{u}^m) \\ &= -\partial_t \frac{1}{2}\int_{\mathbb{R}} \tilde{u}_x^2 - \partial_t \frac{\lambda}{2}\int_{\mathbb{R}} \tilde{u}^2 + \frac{1}{m+1}\partial_t \int_{\mathbb{R}} a_\varepsilon \tilde{u}^{m+1} = -\partial_t E_a[\tilde{u}](t),\end{aligned}$$

which shows (5-8). For (5-9), we have, from the Cauchy–Schwarz inequality,

$$\big|\partial_t E_a[\tilde{u}](t)\big| \le K\big\|S[\tilde{u}](t)\big\|_{L^2(\mathbb{R})},$$

for some constant $K > 0$. After integrating and considering (5-1), we get the result. $\square$

Lemma 5.4 (Control in the Q_c direction). *There exist $K, \gamma > 0$ independent of K^* and such that, for $0 < \varepsilon < \varepsilon_0$ small enough,*

$$\left|\int_{\mathbb{R}} Q_c(y) z\right| \le \frac{K}{c(\varepsilon t) - \lambda}\big(\varepsilon^\kappa + \varepsilon^{1/2} e^{-\varepsilon\gamma|t|}\|z(t)\|_{L^2(\mathbb{R})} + \|z(t)\|_{H^1(\mathbb{R})}^2\big).$$

Proof. We expand the expression of the conserved energy $E_a[u(t+\rho_1)]$ and make use of the identity $u(t+\rho_1) = \tilde{u}(t) + z(t)$ to obtain

$$\begin{aligned}E_a[\tilde{u}+z](t) = E_a[\tilde{u}](t) - \int_{\mathbb{R}} z(\tilde{u}_{xx} - \lambda\tilde{u} + a_\varepsilon\tilde{u}^m) + \frac{1}{2}\int_{\mathbb{R}} z_x^2 + \frac{\lambda}{2}\int_{\mathbb{R}} z^2 \\ - \frac{1}{m+1}\int_{\mathbb{R}} a_\varepsilon\big((\tilde{u}+z)^{m+1} - \tilde{u}^{m+1} - (m+1)\tilde{u}^m z\big).\end{aligned}$$

Note that

$$\int_{\mathbb{R}} z(\tilde{u}_{xx} - \lambda\tilde{u} + a_\varepsilon\tilde{u}^m)(t) = \int_{\mathbb{R}} z(\tilde{u}_{xx} - \lambda\tilde{u} + a_\varepsilon\tilde{u}^m)(-T_\varepsilon) + \big(E_a[\tilde{u}](t) - E_a[\tilde{u}](-T_\varepsilon)\big) + O\big(\|z(t)\|_{H^1(\mathbb{R})}^2\big).$$

We now use (4-32):

$$\int_{\mathbb{R}} z(\tilde{u}_{xx} - \lambda u + a_\varepsilon\tilde{u}^m) = (c-\lambda)\int_{\mathbb{R}} \tilde{u}z + O\big(\varepsilon e^{-\gamma\varepsilon|t|}\|z(t)\|_{L^2(\mathbb{R})}\big).$$

The conclusion follows from this identity and (5-9). $\square$

Energy functional for z. Consider the functional

$$\mathcal{F}(t) := \frac{1}{2}\int_{\mathbb{R}} (z_x^2 + c(\varepsilon t)z^2) - \frac{1}{m+1}\int_{\mathbb{R}} a_\varepsilon\big((\tilde{u}+z)^{m+1} - \tilde{u}^{m+1} - (m+1)\tilde{u}^m z\big). \tag{5-10}$$

Lemma 5.5 (Modified coercivity for $\mathcal{F}$). *There exist $K, \nu_0 > 0$, independent of K^* and ε, such that, for every $t \in [-T_\varepsilon, T_\varepsilon]$,*

$$\mathcal{F}(t) \geq \nu_0 \|z(t)\|_{H^1(\mathbb{R})}^2 - \left|\int_{\mathbb{R}} Q_c(y) z\right|^2 - K(\varepsilon e^{-\gamma\varepsilon|t|} + \varepsilon^2)\|z(t)\|_{L^2(\mathbb{R})}^2 - K\|z(t)\|_{L^2(\mathbb{R})}^3.$$

Proof. We write $\mathcal{F}(t)$ as the sum of

$$\frac{1}{2}\int_{\mathbb{R}} (z_x^2 + cz^2 - m a_\varepsilon \tilde{u}^{m-1} z^2) \tag{5-11}$$

and

$$-\frac{1}{m+1}\int_{\mathbb{R}} a_\varepsilon\big((\tilde{u}+z)^{m+1} - \tilde{u}^{m+1} - (m+1)\tilde{u}^m z - \frac{1}{2}m(m+1)\tilde{u}^{m-1}z^2\big). \tag{5-12}$$

In the case $m = 2$ the term (5-12) is identically zero, and for $m = 3, 4$ we have $|(5\text{-}12)| \leq K\|z(t)\|_{L^2(\mathbb{R})}^3$. The other summand is

$$(5\text{-}11) = \frac{1}{2}\int_{\mathbb{R}} (z_x^2 + c(\varepsilon t)z^2 - mQ_c^{m-1}z^2) - \varepsilon\frac{ma'(\varepsilon\rho)}{2a(\varepsilon\rho)}\int_{\mathbb{R}} yQ_c^{m-1}z^2 + O\big(\varepsilon^2\|z(t)\|_{L^2(\mathbb{R})}^2\big).$$

It is clear that

$$\left|\varepsilon\frac{ma'(\varepsilon\rho)}{2a(\varepsilon\rho)}\int_{\mathbb{R}} yQ_c^{m-1}z^2\right| \leq K\varepsilon e^{-\gamma\varepsilon|t|}\|z(t)\|_{L^2(\mathbb{R})}^2.$$

Finally, by Lemma 2.3, there exist constants $K, \nu_0 > 0$ such that, for all $t \in [-T_\varepsilon, T^*]$,

$$\frac{1}{2}\int_{\mathbb{R}} (z_x^2 + c(\varepsilon t)z^2 - mQ_c^{m-1}z^2) \geq \nu_0\|z(t)\|_{H^1(\mathbb{R})}^2 - K\left|\int_{\mathbb{R}} Q_c z\right|^2. \qquad \square$$

Now we use a coercivity argument to obtain independent estimates for $\mathcal{F}(T^*)$.

Lemma 5.6 (Estimates on $\mathcal{F}(T^*)$). *The following properties hold for any $t \in [-T_\varepsilon, T^*]$.*

(1) First time derivative.

$$\mathcal{F}'(t) = -\int_{\mathbb{R}} z_t\big(z_{xx} - cz + a_\varepsilon((\tilde{u}+z)^m - \tilde{u}^m)\big) + \tfrac{1}{2}\varepsilon c'(\varepsilon t)\int_{\mathbb{R}} z^2 - \int_{\mathbb{R}} a_\varepsilon \tilde{u}_t\big((\tilde{u}+z)^m - \tilde{u}^m - m\tilde{u}^{m-1}z\big).$$

(2) Integration in time. *There exist constants $K, \gamma > 0$ such that*

$$\mathcal{F}(t) - \mathcal{F}(-T_\varepsilon) \leq K(K^*)^4\varepsilon^{4\kappa - \frac{1}{100}} + K(K^*)^3\varepsilon^{3\kappa - \frac{1}{100}} + KK^*\varepsilon^{2\kappa} + K\int_{-T_\varepsilon}^{t} \varepsilon e^{-\varepsilon\gamma|t|}\|z(t)\|_{H^1(\mathbb{R})}^2 dt. \tag{5-13}$$

Proof. Statement (1) amounts to a simple computation. Let us consider (5-13). Substituting (5-6) into the equality in (1) we can write $\mathcal{F}'(t)$ as a sum of the four terms

$$(c(\varepsilon t)-\lambda)(1+\rho_1')\int_{\mathbb{R}} a_\varepsilon\big((\tilde u+z)^m-\tilde u^m\big)z_x, \tag{5-14}$$

$$-\rho_1'\int_{\mathbb{R}} \tilde u_t\big(z_{xx}-cz+a_\varepsilon((\tilde u+z)^m-\tilde u^m)\big), \tag{5-15}$$

$$(1+\rho_1')\int_{\mathbb{R}} S[\tilde u]\big(z_{xx}-cz+a_\varepsilon((\tilde u+z)^m-\tilde u^m)\big), \tag{5-16}$$

$$\tfrac12\varepsilon c'(\varepsilon t)\int_{\mathbb{R}} z^2-\int_{\mathbb{R}} a_\varepsilon\tilde u_t\big((\tilde u+z)^m-\tilde u^m-m\tilde u^{m-1}z\big). \tag{5-17}$$

We consider first the case $m=2$. After some simplifications, we get for (5-14) the value

$$(c-\lambda)(1+\rho_1')\int_{\mathbb{R}} a_\varepsilon\big(2\tilde u z+z^2\big)z_x=-(c-\lambda)(1+\rho_1')\int_{\mathbb{R}}\Big(a_\varepsilon\tilde u_x z^2+\varepsilon a'(\varepsilon x)\tilde u z^2+\frac13\varepsilon a'(\varepsilon x)z^3\Big).$$

Hence

$$\left|(5\text{-}14)+(c-\lambda)(1+\rho_1')\int_{\mathbb{R}} a_\varepsilon\tilde u_x z^2\right|\le K\varepsilon e^{-\gamma\varepsilon|t|}\|z(t)\|^2_{L^2(\mathbb{R})}+K\varepsilon\|z(t)\|^3_{H^1(\mathbb{R})}.$$

On the other hand,

$$(5\text{-}15)=-\rho_1'\int_{\mathbb{R}}(\tilde u_t+(c-\lambda)\tilde u_x)\big(z_{xx}-cz+a_\varepsilon(2\tilde u z+z^2)\big)+(c-\lambda)\rho_1'\int_{\mathbb{R}} a_\varepsilon\tilde u_x z^2$$
$$+(c-\lambda)\rho_1'\int_{\mathbb{R}} z\big(\tilde u_{xx}-c\tilde u+a_\varepsilon\tilde u^2\big)_x-(c-\lambda)\rho_1'\varepsilon\int_{\mathbb{R}} a'(\varepsilon x)\tilde u^2 z.$$

Using the estimates (4-30), (4-33) and (4-31) we then obtain

$$\left|(5\text{-}15)-(c-\lambda)\rho_1'\int_{\mathbb{R}} a_\varepsilon\tilde u_x z^2\right|\le K\varepsilon|\rho_1'|e^{-\gamma\varepsilon|t|}\|z(t)\|_{H^1(\mathbb{R})}.$$

We also have

$$(5\text{-}16)=(1+\rho_1')\int_{\mathbb{R}} z\big(S[\tilde u]_{xx}-cS[\tilde u]+2a_\varepsilon\tilde u S[\tilde u]+a_\varepsilon zS[\tilde u]\big);$$

thus using (5-7)

$$|(5\text{-}16)|\le K\|z(t)\|_{L^2(\mathbb{R})}\|S[\tilde u](t)\|_{H^2(\mathbb{R})}.$$

Finally,

$$(5\text{-}17)=\tfrac12\varepsilon c'(\varepsilon t)\int_{\mathbb{R}} z^2-\int_{\mathbb{R}} a_\varepsilon(\tilde u_t+(c-\lambda)\tilde u_x)z^2+(c-\lambda)\int_{\mathbb{R}} a_\varepsilon\tilde u_x z^2.$$

We get then from (4-31)

$$\left|(5\text{-}17)-(c-\lambda)\int_{\mathbb{R}} a_\varepsilon\tilde u_x z^2\right|\le K\varepsilon e^{-\gamma\varepsilon|t|}\|z(t)\|^2_{L^2(\mathbb{R})}.$$

Collecting these estimates and (5-7), we obtain, after an integration,

$$|\mathcal{F}(t)-\mathcal{F}(-T_\varepsilon)|\le K(K^*)^3\varepsilon^{3\kappa-\frac{1}{100}}+KK^*\varepsilon^{2\kappa}+K\int_{-T_c}^{t}\varepsilon e^{-\gamma\varepsilon|s|}\|z(s)\|^2_{L^2(\mathbb{R})}\,ds.$$

The cases $m=3,4$ are similar, but more involved. From (5-14)–(5-17), and after some integration by parts, we obtain for $\mathcal{F}'(t)$ the expression

$$(c-\lambda)(1+\rho_1')\Big(\int_{\mathbb{R}} a_\varepsilon\Big((\tilde u+z)^m-\tilde u^m-m\tilde u^{m-1}z-\frac{m}{2}(m-1)\tilde u^{m-2}z^2\Big)z_x \tag{5-18}$$

$$-\frac{m}{2}\varepsilon\int_{\mathbb{R}} a'(\varepsilon x)\tilde u^{m-1}z^2-\frac{\varepsilon}{6}m(m-1)\int_{\mathbb{R}} a'(\varepsilon x)\tilde u^{m-2}z^3-\frac{m}{2}\int_{\mathbb{R}} a_\varepsilon(\tilde u^{m-1})_x z^2-\frac{m}{6}(m-1)\int_{\mathbb{R}} a_\varepsilon(\tilde u^{m-2})_x z^3\Big) \tag{5-19}$$

$$-\rho_1'\int_{\mathbb{R}}(\tilde u_t+(c-\lambda)\tilde u_x)\big(z_{xx}-cz+a_\varepsilon((\tilde u+z)^m-\tilde u^m)\big)+(c-\lambda)\rho_1'\Big(\int_{\mathbb{R}} z(\tilde u_{xx}-c\tilde u+a_\varepsilon\tilde u^m)_x-\varepsilon\int_{\mathbb{R}} a'(\varepsilon x)\tilde u^m z\Big)$$

$$+(c-\lambda)(1+\rho_1')\int_{\mathbb{R}}\tilde u_x a_\varepsilon\Big((\tilde u+z)^m-\tilde u^m-m\tilde u^{m-1}z-\frac{m}{2}(m-1)\tilde u^{m-2}z^2-\frac{m}{6}(m-1)(m-2)\tilde u^{m-3}z^3\Big) \tag{5-20}$$

$$+\frac{m}{2}(c-\lambda)\rho_1'\Big(\int_{\mathbb{R}} a_\varepsilon(\tilde u^{m-1})_x z^2+\frac{m-1}{3}\int_{\mathbb{R}} a_\varepsilon(\tilde u^{m-2})_x z^3\Big) \tag{5-21}$$

$$\begin{aligned}&+(1+\rho_1')\int_{\mathbb{R}} z\big(S[\tilde u]_{xx}-cS[\tilde u]+ma_\varepsilon\tilde u^{m-1}S[\tilde u]\big)+(1+\rho_1')\int_{\mathbb{R}} a_\varepsilon\big((\tilde u+z)^m-\tilde u^m-m\tilde u^{m-1}z\big)S[\tilde u]\\&+\frac{\varepsilon}{2}c'\int_{\mathbb{R}} z^2-\int_{\mathbb{R}} a_\varepsilon(\tilde u_t+(c-\lambda)\tilde u_x)\big((\tilde u+z)^m-\tilde u^m-m\tilde u^{m-1}z\big)\\&\qquad+\frac{m}{2}(c-\lambda)\Big(\int_{\mathbb{R}} a_\varepsilon(\tilde u^{m-1})_x z^2+\frac{m-1}{3}\int_{\mathbb{R}} a_\varepsilon(\tilde u^{m-2})_x z^3\Big).\end{aligned} \tag{5-22}$$

Note that the last two terms in (5-19) disappear, as do (5-21) and (5-22). With (5-18) and (5-20), we need a little more care. For $m=3$,

$$|(5\text{-}18)+(5\text{-}20)|=\Big|\frac14\varepsilon(c-\lambda)(1+\rho_1')\int_{\mathbb{R}} a'(\varepsilon x)z^4\Big|\le\varepsilon\|z(t)\|^4_{L^2(\mathbb{R})}.$$

In the case $m=4$,

$$\begin{aligned}(5\text{-}18)+(5\text{-}20)&=(c-\lambda)(1+\rho_1')\int_{\mathbb{R}} a_\varepsilon[z_x(4\tilde u z^3+z^4)+\tilde u_x z^4]\\&=-\varepsilon(c-\lambda)(1+\rho_1')\int_{\mathbb{R}} a'(\varepsilon x)(\tilde u z^4+z^5).\end{aligned}$$

Consequently we have

$$|(5\text{-}18)+(5\text{-}20)|\le K\varepsilon e^{-\gamma\varepsilon|t|}\|z(t)\|^4_{L^2(\mathbb{R})}+K\varepsilon\|z(t)\|^5_{L^2(\mathbb{R})}.$$

Finally, using (4-30), (4-33), (4-31) we obtain

$$\begin{aligned}\mathcal{F}'(t)\le K\varepsilon\|z(t)\|^4_{H^1(\mathbb{R})}+K\varepsilon e^{-\gamma\varepsilon|t|}\|z(t)\|^2_{L^2(\mathbb{R})}+K\varepsilon\|z(t)\|^3_{H^1(\mathbb{R})}&\\+K|\rho_1'(t)|\varepsilon e^{-\gamma\varepsilon|t|}\|z(t)\|_{H^1(\mathbb{R})}+K\|S[\tilde u](t)\|_{H^2(\mathbb{R})}\|z(t)\|_{L^2(\mathbb{R})}&.\end{aligned}$$

Integrating and using (5-7), we obtain

$$\mathcal{F}(t)-\mathcal{F}(-T_\varepsilon)\le K(K^*)^4\varepsilon^{4\kappa-\frac{1}{100}}+K(K^*)^3\varepsilon^{3\kappa-\frac{1}{100}}+KK^*\varepsilon^{2\kappa}+K\int_{-T_\varepsilon}^t\varepsilon e^{-\gamma\varepsilon|s|}\|z(s)\|^2_{H^1(\mathbb{R})}\,ds,$$

completing the proof. □

We are finally in a position to show that $T^* < T_\varepsilon$ leads to a contradiction.

End of proof of Proposition 5.1. From Lemma 5.2, $\mathcal{F}(-T_\varepsilon) \le Kc^{2\kappa}$, and from Lemmas 5.5, 5.4 and (5-13) we get

$$\begin{aligned}
\|z(t)\|^2_{L^2(\mathbb{R})} \le K\left|\int_{\mathbb{R}} zQ_c(y)\right|^2 &+ K\varepsilon^{2\kappa} + K(K^*)^4\varepsilon^{4\kappa-\frac{1}{100}} \\
&+ K(K^*)^3\varepsilon^{3\kappa-\frac{1}{100}} + KK^*\varepsilon^{2\kappa} + K\int_{-T_\varepsilon}^t \varepsilon e^{-\gamma\varepsilon|t|}\|z(t)\|^2_{L^2(\mathbb{R})}dt \\
\le K\Big|\varepsilon^\kappa + K^*\varepsilon^{\frac12+\kappa}e^{-\gamma\varepsilon|t|} &+ (K^*)^2\varepsilon^{2\kappa} + \|S[\tilde u]\|_{L^2(\mathbb{R})}\Big|^2 + K\varepsilon^{2\kappa} \\
+ K(K^*)^4\varepsilon^{4\kappa-\frac{1}{100}} &+ K(K^*)^3\varepsilon^{3\kappa-\frac{1}{100}} + KK^*\varepsilon^{2\kappa} + K\int_{-T_\varepsilon}^t \varepsilon e^{-\gamma\varepsilon|s|}\|z(s)\|^2_{L^2(\mathbb{R})}ds \\
\le K\varepsilon^{2\kappa} + K(K^*)^4\varepsilon^{4\kappa-\frac{1}{100}} &+ K(K^*)^3\varepsilon^{3\kappa-\frac{1}{100}} + KK^*\varepsilon^{2\kappa} + K\int_{-T_\varepsilon}^t \varepsilon e^{-\gamma\varepsilon|s|}\|z(s)\|^2_{L^2(\mathbb{R})}ds.
\end{aligned}$$

Using Gronwall's inequality (see [Verhulst 2006], for instance) we conclude that, for some large constant $K > 0$ independent of K^* and ε,

$$\|z(t)\|^2_{H^1(\mathbb{R})} \le K\varepsilon^{2\kappa} + K(K^*)^4\varepsilon^{4\kappa-\frac{1}{100}} + K(K^*)^3\varepsilon^{3\kappa-\frac{1}{100}} + KK^*\varepsilon^{2\kappa}.$$

Taking ε small and K^* large enough, we then obtain that for all $t \in [-T_\varepsilon, T^*]$,

$$\|z(t)\|^2_{H^1(\mathbb{R})} \le \tfrac12 (K^*)^2\varepsilon^{2\kappa}.$$

This contradicts the definition of T^* and concludes the proof of Proposition 5.1. □

Proof of Theorem 4.1. We are now able to prove Theorem 4.1, showing that the solution $u(t)$ given by Theorem 3.1 is close to the approximate solution $\tilde u(t)$ constructed in Proposition 4.7 at time $t = -T_\varepsilon$.

Behavior at $t = -T_\varepsilon$. From (3-3), Proposition 4.9, and more specifically (4-34) we have

$$\|u(-T_\varepsilon) - \tilde u(-T_\varepsilon)\|_{H^1(\mathbb{R})} \le K\varepsilon^{10}.$$

Behavior at $t = T_\varepsilon$. Thanks to the above estimate and (4-29) we can invoke Proposition 5.1 with $\kappa := \frac12$ to obtain the existence of $K_0, \varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$

$$\|u(T_\varepsilon + \rho_1(T_\varepsilon)) - \tilde u(T_\varepsilon)\|_{H^1(\mathbb{R})} \le K_0\varepsilon^{1/2}, \quad |\rho_1(T_\varepsilon)| \le K_0\varepsilon^{-\frac12-\frac{1}{100}} \le \frac{T_\varepsilon}{100}.$$

Therefore from (4-35) and the triangle inequality,

$$\|u(T_\varepsilon + \rho_1(T_\varepsilon)) - 2^{-1/(m-1)}Q_{c_\infty}(\cdot - \rho(T_\varepsilon))\|_{H^1(\mathbb{R})} \le K_0\varepsilon^{1/2}.$$

(see also (4-5).) Finally note that $(1-\lambda)T_\varepsilon \le \rho(T_\varepsilon) \le (2c_\infty(\lambda) - \lambda - 1)T_\varepsilon$. This finishes the proof.

6. Asymptotics for large times

Recall that for large times ($t \ge T_\varepsilon$) the soliton-like solution is expected to be far away from the region where a_ε varies. Roughly speaking, the solution's stability and asymptotic stability properties will follow from the fact that in this region (1-13) the equation behaves like the gKdV equation

$$u_t + (u_{xx} - \lambda u + 2u^m)_x = 0 \quad \text{in } \{t \geq T_\varepsilon\} \times \mathbb{R}_x.$$

The purpose of this section is to lay out this argument in a rigorous way. We start by restating the asymptotic behavior, already described in Theorem 1.2. Recall the parameters λ_0 and $c_\infty(\lambda)$ from Theorems 1.1 and 4.1.

Theorem 6.1 (Stability and asymptotic stability in H^1). *Suppose $m = 2, 4$ with $0 < \lambda \leq \lambda_0$, or $m = 3$ with $0 \leq \lambda \leq \lambda_0$. Let $0 < \beta < \frac{1}{2}(c_\infty(\lambda) - \lambda)$. There exists $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$ the following statements hold. Suppose that for some time $t_1 \geq \frac{1}{2} T_\varepsilon$ with $t_1 \leq X_0 \leq 2t_1$ we have*

$$\left\| u(t_1) - 2^{-1/(m-1)} Q_{c_\infty}(x - X_0) \right\|_{H^1(\mathbb{R})} \leq \varepsilon^{1/2}, \tag{6-1}$$

where $u(t)$ is an H^1-solution of (1-15). Then $u(t)$ is defined for every $t \geq t_1$ and there exists $K, c^+ > 0$ and a C^1-function $\rho_2(t)$ defined in $[t_1, +\infty)$ with these properties:

(1) Stability.

$$\sup_{t \geq t_1} \left\| u(t) - 2^{-1/(m-1)} Q_{c_\infty}(\cdot - \rho_2(t)) \right\|_{H^1(\mathbb{R})} \leq K \varepsilon^{1/2}, \tag{6-2}$$

where

$$|\rho_2(t_1) - X_0| \leq K\varepsilon^{1/2} \quad \textit{and} \quad |\rho_2'(t) - c_\infty(\lambda) + \lambda| \leq K\varepsilon^{1/2} \quad \textit{for all } t \geq t_1.$$

(2) Asymptotic stability.

$$\lim_{t \to +\infty} \left\| u(t) - 2^{-1/(m-1)} Q_{c^+}(\cdot - \rho_2(t)) \right\|_{H^1(x > \beta t)} = 0. \tag{6-3}$$

In addition,

$$\lim_{t \to +\infty} \rho_2'(t) = c^+ - \lambda, \qquad |c^+ - c_\infty| \leq K\varepsilon^{1/2}. \tag{6-4}$$

Remarks. (a) We do not know if the stability results are valid in the cases $m = 2, 4$ and $\lambda = 0$. Clearly, the stability property as stated above is false if $\limsup_{t \to +\infty} \|u(t)\|_{L^2(\mathbb{R})} = +\infty$.

(b) For $0 < \lambda < \lambda_0$ the asymptotic stability property (6-3) holds for any $\beta > -\lambda$, provided ε_0 is small enough. We make use of this property in [Muñoz ≥ 2011a], but we do not pursue it here.

Proof of Theorem 6.1(1): stability. The proof of stability is standard and similar to that of Proposition 5.1. For this reason we omit many details, inviting the reader to consult [Benjamin 1972; Martel et al. 2002], where the proof originates. Even then the argument will occupy us until page 613.

Assume that, for some fixed $K > 0$,

$$\left\| u(t_1) - 2^{-1/(m-1)} Q_{c_\infty}(\cdot - X_0) \right\|_{H^1(\mathbb{R})} \leq K\varepsilon^{1/2}. \tag{6-5}$$

From the local and global Cauchy theory exposed in Proposition 2.1 and Theorems 3.1 and 4.1, we know that the solution u is well defined for all $t \geq t_1$.

To simplify the calculations, note that from (1-18) the function $v := 2^{1/(m-1)}u$ solves

$$v_t + (v_{xx} - \lambda v + \tfrac{1}{2}a_\varepsilon v^m)_x = 0 \quad \text{on } \mathbb{R}_t \times \mathbb{R}_x.$$

Then (6-5) becomes

$$\|v(t_1) - Q_{c_\infty}(\cdot - X_0)\|_{H^1(\mathbb{R})} \le \tilde{K}\varepsilon^{1/2}. \tag{6-6}$$

With a slight abuse of notation we will rename $v := u$ and $\tilde{K} := K$, and we will assume the validity of (6-6) for u. The parameters X_0 and c_∞ remain unchanged.

Let $D_0 > 2K$ be a large number to be chosen later, and set

$$T^* := \sup \left\{ \begin{array}{r} t \ge t_1 : \forall\, t' \in [t_1, t) \text{ there is a smooth } \tilde{\rho}_2(t') \in \mathbb{R} \text{ with } |\tilde{\rho}_2'(t') - c_\infty + \lambda| \le \frac{1}{100}, \\ |\tilde{\rho}_2(t_1) - X_0| \le \frac{1}{100}, \text{ and } \|u(t') - Q_{c_\infty}(\cdot - \tilde{\rho}_2(t'))\|_{H^1(\mathbb{R})} \le D_0\varepsilon^{1/2} \end{array} \right\}. \tag{6-7}$$

Observe that $T^* > t_1$ is well defined since $D_0 > 2K$ and because of (6-5) and the continuity of $t \mapsto u(t)$ in $H^1(\mathbb{R})$. The goal is to prove $T^* = +\infty$, and thus (6-2).

Therefore, for the sake of contradiction, *suppose* $T^* < +\infty$. Using modulation theory around the soliton, we will decompose the solution on $[t_1, T^*]$, and so find a special $\rho_2(t)$ satisfying the hypotheses in (6-7), but with

$$\sup_{t \in [t_1, T^*]} \big\| u(t) - Q_{c_\infty}(\cdot - \rho_2(t)) \big\|_{H^1(\mathbb{R})} \le \tfrac{1}{2} D_0 \varepsilon^{1/2}, \tag{6-8}$$

in contradiction with the definition of T^*.

Lemma 6.2 (Modulated decomposition). *For $\varepsilon > 0$ small enough, independent of T^*, there exist C^1 functions ρ_2, c_2, defined on $[t_1, T^*]$, with $c_2(t) > 0$, such that the function $z(t)$ given by*

$$z(t,x) := u(t,x) - R(t,x), \tag{6-9}$$

where $R(t,x) := Q_{c_2(t)}(x - \rho_2(t))$, satisfies the following conditions for all $t \in [t_1, T^*]$:

$$\int_{\mathbb{R}} R(t,x) z(t,x)\, dx = \int_{\mathbb{R}} (x - \rho_2(t)) R(t,x) z(t,x)\, dx = 0 \quad \text{(orthogonality)}, \tag{6-10}$$

$$\|z(t)\|_{H^1(\mathbb{R})} + |c_2(t) - c_\infty| \le K D_0 \varepsilon^{1/2}, \tag{6-11}$$

$$\|z(t_1)\|_{H^1(\mathbb{R})} + |\rho_2(t_1) - X_0| + |c_2(t_1) - c_\infty| \le K\varepsilon^{1/2}, \tag{6-12}$$

where K does not depend on D_0. In addition, $z(t)$ now satisfies the modified gKdV equation

$$z_t + \big(z_{xx} - \lambda z + \tfrac{1}{2}a_\varepsilon((R+z)^m - R^m) + (\tfrac{1}{2}a_\varepsilon(x) - 1)Q_{c_2}^m\big)_x$$
$$+ c_2'(t)\Lambda Q_{c_2} + (c_2 - \lambda - \rho_2')(t) Q_{c_2}' = 0. \tag{6-13}$$

Furthermore, for some constant $\gamma > 0$ independent of ε, we have the improved estimates

$$|\rho_2'(t) + \lambda - c_2(t)|$$
$$\le K(m-3)\Big(\int_{\mathbb{R}} e^{-\gamma|x-\rho_2(t)|} z^2(t,x)\, dx\Big)^{\frac{1}{2}} + K\int_{\mathbb{R}} e^{-\gamma|x-\rho_2(t)|} z^2(t,x)\, dx + K e^{-\gamma\varepsilon t} \tag{6-14}$$

and

$$\frac{|c_2'(t)|}{c_2(t)} \leq K\int_{\mathbb{R}} e^{-\gamma|x-\rho_2(t)|}z^2(t,x)\,dx + Ke^{-\gamma\varepsilon t}\|z(t)\|_{H^1(\mathbb{R})} + K\varepsilon e^{-\varepsilon\gamma t}. \tag{6-15}$$

Remark. From (6-11) and taking ε small enough we have an improved the bound on $\rho_2(t)$. Indeed, for all $t\in[t_1,T^*]$,

$$|\rho_2'(t)-c_\infty+\lambda|+|\rho_2(t_1)-X_0|\leq 2D_0\varepsilon^{1/2}.$$

Thus, to reach a contradiction, we only need to show (6-8).

Proof of Lemma 6.2. As in Lemmas A.1.4 and 5.2, the proof of (6-9)–(6-12) is based in an application of the implicit function theorem, and is very similar to the proof of [Martel and Merle 2008, Lemma A.1].

Equation (6-13) also follows from a simple computation, completely similar to (A-11) and (5-6).

Now we claim that from the definition of T^* we can obtain an extra estimate on the parameter $\rho_2(t)$:

$$\rho_2(t)\geq \tfrac{1}{10}(c_\infty(\lambda)-\lambda)t_1 \quad \text{for any } t\geq t_1. \tag{6-16}$$

Indeed, from (6-7) and after integration between t_1 and $t\in[t_1,T^*]$ we have the bound

$$\big|\rho_2(t)-\rho_2(t_1)-(c_\infty-\lambda)(t-t_1)\big|\leq \tfrac{1}{100}(t-t_1),\qquad \big|\rho_2(t_1)-X_0\big|\leq\tfrac{1}{100}.$$

Thus we have

$$\big|\rho_2(t)-(c_\infty-\lambda)t\big|\leq\tfrac{1}{100}(t-t_1+1)+\big|(c_\infty-\lambda)t_1-X_0\big|.$$

In particular, for any $t\in[t_1,T^*]$ (recall that $\rho_2(t_1)\sim X_0>0$)

$$\rho_2(t)\geq(c_\infty-\lambda)t-\tfrac{1}{100}(t-t_1+1)\geq\tfrac{1}{10}c_\infty t.$$

This implies that the soliton is far away from the potential interaction region.

Now we prove (6-14) and (6-15). Set $y:=x-\rho_2(t)$. Taking the time derivative in the first orthogonality condition in (6-10) and using (6-13) we obtain

$$\begin{aligned}0=-c_2'(t)\int_{\mathbb{R}}\Lambda Q_{c_2}(Q_{c_2}-z)+(c_2-\lambda-\rho_2')(t)\int_{\mathbb{R}}Q_{c_2}'z-\frac12\int_{\mathbb{R}}Q_{c_2}^m\big((a_\varepsilon-2)z\big)_x\\ -\frac{\varepsilon}{2(m+1)}\int_{\mathbb{R}}a'(\varepsilon x)Q_{c_2}^{m+1}(y)+\frac12\int_{\mathbb{R}}Q_{c_2}'a_\varepsilon\big((R+z)^m-R^m-mR^{m-1}z\big).\end{aligned}$$

By scaling arguments,

$$\int_{\mathbb{R}}\Lambda Q_{c_2}Q_{c_2}=\theta c_2^{2\theta-1}(t)\int_{\mathbb{R}}Q^2. \tag{6-17}$$

Then, by redefining γ if necessary,

$$\left|\varepsilon\int_{\mathbb{R}}a'(\varepsilon x)Q_{c_2}^{m+1}(y)\right|\leq K\varepsilon e^{-\gamma\varepsilon c_2(t)\rho_2(t)}\leq K\varepsilon e^{-\gamma\varepsilon t}.$$

Similarly, from (6-16) and following (A-13) we have

$$\left|\int_{\mathbb{R}}Q_{c_2}^m\big((a_\varepsilon-2)z\big)_x\right|\leq K\|z(t)\|_{H^1(\mathbb{R})}e^{-\gamma\varepsilon t}.$$

Finally, note that for $\gamma > 0$ independent of ε,

$$\left|\int_{\mathbb{R}} Q'_{c_2} a_\varepsilon\big((R+z)^m - R^m - mR^{m-1}z\big)\right| \le K\int_{\mathbb{R}} e^{-\gamma|y|}z^2.$$

Collecting these estimates together, we have

$$\frac{|c_2'(t)|}{c_2(t)} \le K\int_{\mathbb{R}} e^{-\gamma|y|}z^2 + K\big|c_2(t)-\lambda-\rho_2'(t)\big|\bigg(\int_{\mathbb{R}} e^{-\gamma|y|}z^2\bigg)^{\frac{1}{2}} + Ke^{-\gamma\varepsilon t}\|z(t)\|_{H^1(\mathbb{R})} + K\varepsilon e^{-\gamma\varepsilon t}. \tag{6-18}$$

On the other hand, by using the second orthogonality condition in (6-10), we have

$$\begin{aligned}0 = (c_2-\lambda-\rho_2')(t)\int_{\mathbb{R}} z(yR)_x + c_2'(t)\int_{\mathbb{R}} y\Lambda Q_{c_2} z + \frac{1}{2}(c_2-\lambda-\rho_2')(t)\int_{\mathbb{R}} Q_{c_2}^2\\ + \int_{\mathbb{R}}(yR)_x\Big(\tfrac{1}{2}a_\varepsilon\big((R+z)^m - R^m - mR^{m-1}z\big) + \big(\tfrac{1}{2}a_\varepsilon(x)-1\big)Q_{c_2}^m\Big)\\ + \int_{\mathbb{R}}(yR)_x(z_{xx}-c_2 z + mR^{m-1}z) + \frac{m}{2}\int_{\mathbb{R}}(yR)_x(a_\varepsilon-2)R^{m-1}z.\end{aligned}$$

Note that by integration by parts,

$$\int_{\mathbb{R}}(yR)_x(z_{xx}-c_2z+mR^{m-1}z) = \int_{\mathbb{R}} z(2R+(m-3)R^m) = (m-3)\int_{\mathbb{R}} zR^m.$$

Using the same arguments as in the precedent computations, we have

$$\big|(c_2-\lambda-\rho_2')(t)\big| \le K(m-3)\bigg(1+\frac{|c_2'(t)|}{c_2(t)}\bigg)\bigg(\int_{\mathbb{R}} z^2e^{-\gamma|y|}\bigg)^{\frac{1}{2}} + K\int_{\mathbb{R}} z^2e^{-\gamma|y|} + \bigg|\int_{\mathbb{R}} Q_{c_2}^m(y)(a_\varepsilon-2).\bigg|$$

From (6-16) and following (A-13) we have

$$\left|\int_{\mathbb{R}} Q_{c_2}^m(y)(a_\varepsilon-2)\right| \le Ke^{-\gamma\varepsilon t}.$$

Putting together (6-18) and the last estimates, we finally obtain the bounds in (6-11), and further we obtain (6-14) and (6-15), as desired. □

Almost conserved quantities and monotonicity. We continue with a proof completely analogous to that of Proposition A.1.1. Recall from (2-8) the definition of the modified mass $\tilde{M}$.

Lemma 6.3 (Almost conservation of modified mass and energy). *Consider $\tilde{M} = \tilde{M}[R]$ and $E_a = E_a[R]$, the modified mass and energy of the soliton R of (6-9). For all $t \in [t_1, T^*]$ we have*

$$\tilde{M}[R](t) = \tfrac{1}{2}c_2^{2\theta}(t)\int_{\mathbb{R}} Q^2 + O(e^{-\varepsilon\gamma t}), \tag{6-19}$$

$$E_a[R](t) = \tfrac{1}{2}c_2^{2\theta}(t)(\lambda-\lambda_0 c_2(t))\int_{\mathbb{R}} Q^2 + O(e^{-\varepsilon\gamma t}). \tag{6-20}$$

Furthermore, we have the bound

$$\big|E_a[R](t_1)-E_a[R](t)+(c_2(t_1)-\lambda)\big(\tilde{M}[R](t_1)-\tilde{M}[R](t)\big)\big| \le K\left|\Big(\frac{c_2(t)}{c_2(t_1)}\Big)^{2\theta}-1\right|^2 + Ke^{-\varepsilon\gamma t_1}. \tag{6-21}$$

Proof. We start by showing (6-19). We consider the case $m = 2, 4$, the case $m = 3$ being easier. Note that from (2-8) that

$$\tilde{M}[R](t) = \hat{M}[R](t) = \frac{1}{2}\int_{\mathbb{R}} \Big(\frac{a_\varepsilon}{2}\Big)^{1/m} R^2 = \frac{1}{2} c_2^{2\theta}(t) \int_{\mathbb{R}} Q^2 + \frac{1}{2}\int_{\mathbb{R}} \Big(\Big(\frac{a_\varepsilon(x)}{2}\Big)^{1/m} - 1\Big) R^2.$$

From (6-16)-(6-17) and following the calculations in (A-13),

$$\Big|\int_{\mathbb{R}} (a_\varepsilon^{1/m}(x) - 2^{1/m}) R^2\Big| \le K e^{-\gamma\varepsilon t},$$

for some constants $K, \gamma > 0$. Now we consider (6-20). Here we have

$$\begin{aligned} E_a[R](t) &= \frac{1}{2}\int_{\mathbb{R}} R_x^2 + \frac{\lambda}{2}\int_{\mathbb{R}} R^2 - \frac{1}{2(m+1)}\int_{\mathbb{R}} a_\varepsilon R^{m+1} \\ &= c_2^{2\theta}(t)\Big[c_2(t)\Big(\frac{1}{2}\int_{\mathbb{R}} Q'^2 - \frac{1}{m+1}\int_{\mathbb{R}} Q^{m+1}\Big) + \frac{\lambda}{2}\int_{\mathbb{R}} Q^2\Big] + \frac{1}{m+1}\int_{\mathbb{R}} \big(1 - \tfrac{1}{2}a_\varepsilon\big) R^{m+1}. \end{aligned}$$

Similarly to a recent computation, we have

$$\Big|\int_{\mathbb{R}} (2 - a_\varepsilon(x)) R^{m+1}\Big| \le K e^{-\gamma\varepsilon t},$$

for some constants $K, \gamma > 0$. On the other hand, from Section A.6 we have

$$\frac{1}{2}\int_{\mathbb{R}} Q'^2 - \frac{1}{m+1}\int_{\mathbb{R}} Q^{m+1} = -\frac{\lambda_0}{2}\int_{\mathbb{R}} Q^2 \quad \text{and} \quad \lambda_0 = \frac{5-m}{m+3},$$

and thus

$$E_a[R](t) = \tfrac{1}{2} c_2^{2\theta}(t)\big(\lambda - \lambda_0 c_2(t)\big)\int_{\mathbb{R}} Q^2 + O(e^{-\gamma\varepsilon t}).$$

Combining both identities we have

$$E_a[R](t) + \big(c_2(t_1) - \lambda\big)\hat{M}[R](t) = c_2^{2\theta}(t)\big(c_2(t_1) - \lambda_0 c_2(t)\big) M[Q] + O(e^{-\varepsilon\gamma t}).$$

In particular,

$$\begin{aligned} &E_a[R](t_1) - E_a[R](t) + (c_2(t_1) - \lambda)(\hat{M}[R](t_1) - \hat{M}[R](t)) \\ &\qquad = \lambda_0 M[Q]\Big(c_2^{2\theta+1}(t) - c_2^{2\theta+1}(t_1) - \frac{c_2(t_1)}{\lambda_0}\big(c_2^{2\theta}(t) - c_2^{2\theta}(t_1)\big)\Big) + O(e^{-\varepsilon\gamma t_1}). \end{aligned}$$

To obtain the last estimate (6-21) we perform a Taylor development up to the second order (around $y = y_0$) of the function $g(y) := y^{\frac{2\theta+1}{2\theta}}$; and where $y := c_2^{2\theta}(t)$ and $y_0 := c_2^{2\theta}(t_1)$. Note that $\frac{2\theta+1}{2\theta} = \frac{1}{\lambda_0}$ and $y_0^{1/2\theta} = c_2(t_1)$. The conclusion follows at once. □

To establish some stability properties for the function $u(t)$ we recall the mass $\tilde{M}[u]$ introduced in (2-8). We have that for $m = 3$ and $0 \le \lambda \le \lambda_0$; and for $m = 2, 4$ and $0 < \lambda \le \lambda_0$,

$$\tilde{M}[u](t) - \tilde{M}[u](t_1) \le 0. \tag{6-22}$$

for any $t \in [t_1, T^*]$. This result is a consequence of the remark on page 586.

Now our objective is to estimate the quadratic term involved in (6-21). Following [Martel et al. 2002], we should use a “mass conservation” identity. However, since the mass is not conserved, estimate (6-22) is not enough to obtain a satisfactory estimate. Instead we will introduce a virial-type identity in the next lemma.

Let $\phi \in C(\mathbb{R})$ be an *even* function satisfying

$$\begin{cases} \phi' \leq 0 & \text{on } [0,+\infty), \\ \phi(x) = 1 & \text{on } [0,1], \\ \phi(x) = e^{-x} & \text{on } [2,+\infty), \\ e^{-x} \leq \phi(x) \leq 3e^{-x} & \text{on } [0,+\infty). \end{cases} \tag{6-23}$$

Now, set $\psi(x) := \int_0^x \phi$. It is clear that ψ an odd function. Moreover, for $|x| \geq 2$,

$$\psi(+\infty) - \psi(|x|) = e^{-|x|}. \tag{6-24}$$

Finally, for $A > 0$, set

$$\psi_A(x) := A\big(\psi(+\infty) + \psi(x/A)\big) > 0, \quad e^{-|x|/A} \leq \psi_A'(x) \leq 3e^{-|x|/A}. \tag{6-25}$$

Note that $\lim_{x\to-\infty} \psi(x) = 0$.

Lemma 6.4 (Virial-type estimate). *There exist $K, A_0, \delta_0 > 0$ such that for all $t \in [t_1, T^*]$ and for some $\gamma = \gamma(c_\infty, A_0) > 0$,*

$$\partial_t \int_{\mathbb{R}} z^2(t,x)\psi_{A_0}(x-\rho_2(t)) \leq -\delta_0 \int_{\mathbb{R}} (z_x^2 + z^2)(t,x) e^{-\frac{1}{A_0}|x-\rho_2(t)|} + KA_0\|z(t)\|_{H^1(\mathbb{R})} e^{-\gamma\varepsilon t}. \tag{6-26}$$

For the proof, see Section A.4.

We can improve the estimate (6-21):

Corollary 6.5 (Quadratic control of the variation of $c_2(t)$).

$$\begin{aligned} &\big|E_a[R](t_1) - E_a[R](t) + (c_2(t_1)-\lambda)\big(\tilde M[R](t_1) - \tilde M[R](t)\big)\big| \\ &\qquad\qquad \leq K\|z(t)\|_{H^1(\mathbb{R})}^4 + K\|z(t_1)\|_{H^1(\mathbb{R})}^4 + Ke^{-\varepsilon\gamma t_1}. \end{aligned} \tag{6-27}$$

Proof. From (6-15) and taking A_0 large enough (but fixed and independent of ε) in Lemma 6.4, we have after an integration of (6-26) that

$$|c_2(t) - c_2(t_1)| \leq KA_0\|z(t)\|_{L^2(\mathbb{R})}^2 + KA_0\|z(t_1)\|_{L^2(\mathbb{R})}^2 + KA_0 D_0 \varepsilon^{-1/2} e^{-\gamma\varepsilon t_1}.$$

Substituting this in (6-21) and taking γ even smaller, we get the conclusion. □

Energy estimates. Let us now introduce the second-order functional

$$\mathcal{F}_2(t) := \frac{1}{2}\int_{\mathbb{R}} \Big(z_x^2 + \big(\lambda + (c_2(t_1)-\lambda)(\tfrac{a_\varepsilon}{2})^{1/m}\big)z^2\Big) - \frac{1}{2(m+1)}\int_{\mathbb{R}} a_\varepsilon\big((R+z)^{m+1} - R^{m+1} - (m+1)R^m z\big).$$

This functional, related to the Weinstein functional, have the following properties.

Lemma 6.6 (Energy expansion). *Consider the energy $E_a[u]$ and the mass $\tilde M[u]$ defined in (1-21) and (2-8). For all $t \in [t_1, T^*]$,*

$$E_a[u](t) + (c_2(t_1)-\lambda)\tilde M[u](t) = E_a[R] + (c_2(t_1)-\lambda)\tilde M[R] + \mathcal{F}_2(t) + O(e^{-\gamma\varepsilon t}\|z(t)\|_{H^1(\mathbb{R})}).$$

Proof. Using the orthogonality condition (6-10), we have

$$E_a[u](t)=E_a[R]-\int_{\mathbb{R}} z(a_\varepsilon-2)R^m+\frac{1}{2}\int_{\mathbb{R}} z_x^2+\frac{\lambda}{2}\int_{\mathbb{R}} z^2-\frac{1}{m+1}\int_{\mathbb{R}} a_\varepsilon\big((R+z)^{m+1}-R^{m+1}-(m+1)R^m z\big).$$

Moreover, following (A-13), we easily get

$$\left|\int_{\mathbb{R}} z(a_\varepsilon-2)R^m\right| \le Ke^{-\gamma\varepsilon t}\|z(t)\|_{H^1(\mathbb{R})}.$$

Similarly,

$$\hat{M}[u](t)=\hat{M}[R]+\hat{M}[z]+\int_{\mathbb{R}}\big((\tfrac{a_\varepsilon}{2})^{1/m}-1\big)Rz=\hat{M}[R]+\hat{M}[z]+O(e^{-\varepsilon\gamma t}\|z(t)\|_{H^1(\mathbb{R})}).$$

Combining these estimates, we have

$$\begin{aligned}E_a[u](t)+(c_2(t_1)-\lambda)\tilde{M}[u](t)=E_a[R]+(c_2(t_1)-\lambda)\tilde{M}[R]+\frac{1}{2}\int_{\mathbb{R}}\Big(z_x^2+\big((c_2(t_1)-\lambda)(\tfrac{a_\varepsilon}{2})^{1/m}+\lambda\big)z^2\Big)\\ -\frac{1}{2(m+1)}\int_{\mathbb{R}} a_\varepsilon\big((R+z)^{m+1}-R^{m+1}-(m+1)R^m z\big)+O(e^{-\gamma\varepsilon t}\|z(t)\|_{H^1(\mathbb{R})}).\end{aligned}$$

This concludes the proof. □

Lemma 6.7 (Modified coercivity for $\mathcal{F}_2$). *There exists $\varepsilon_0>0$ such that for all $0<\varepsilon<\varepsilon_0$ the following hold. There exist $K,\tilde{\lambda}_0>0$, independent of K^* such that for every $t\in[t_1,T^*]$*

$$\mathcal{F}_2(t)\ge\tilde{\lambda}_0\|z(t)\|^2_{H^1(\mathbb{R})}-K\varepsilon e^{-\gamma\varepsilon t}\|z(t)\|^2_{L^2(\mathbb{R})}+O(\|z(t)\|^3_{L^2(\mathbb{R})}). \tag{6-28}$$

Proof. Note that

$$\begin{aligned}\mathcal{F}_2(t)=\frac{1}{2}\int_{\mathbb{R}}\Big(z_x^2+\big((c_2(t_1)-\lambda)(\tfrac{a_\varepsilon}{2})^{1/m}+\lambda\big)z^2\Big)\\ -\frac{m}{2}\int_{\mathbb{R}} Q_{c_2}^{m-1}z^2+O(\|z(t)\|^3_{H^1(\mathbb{R})})+O(e^{-\gamma\varepsilon t}\|z(t)\|^2_{H^1(\mathbb{R})}).\end{aligned}$$

Now take $R_0>0$ independent of ε, to be fixed later. Consider the function

$$\phi_{R_0}(t,x):=\phi\Big(\frac{x-\rho_2(t)}{R_0}\Big),$$

where ϕ is defined in (6-23). We split the analysis according to the decomposition $1=\phi_{R_0}+(1-\phi_{R_0})$. Inside the region $|x-\rho_2(t)|\le R_0$, we have, a consequence of (1-13),

$$2-a_\varepsilon(x)\le Ke^{-\gamma\varepsilon|x|}\le Ke^{\gamma\varepsilon R_0}e^{-\gamma\varepsilon\rho_2(t)}.$$

Outside this region, we have $\phi_{R_0}\ge e^{-R_0}$. Thus

$$\int_{\mathbb{R}}\phi_{R_0}\big((c_2(t_1)-\lambda)(\frac{a_\varepsilon}{2})^{1/m}+\lambda\big)z^2\ge\big(c_2(t_1)-Ke^{\gamma\varepsilon R_0}e^{-\gamma\varepsilon\rho_2(t)}\big)\int_{\mathbb{R}}\phi_{R_0}z^2,$$

for fixed $K,\gamma>0$.

On the other hand, $|(1-\phi_{R_0})Q_{c_2}| \le Ke^{-\gamma R_0}$, and thus

$$\int_{\mathbb{R}}(1-\phi_{R_0})\big((c_2(t_1)-\lambda)(\tfrac{a_\varepsilon}{2})^{1/m}+\lambda\big)z^2-\frac{m}{2}\int_{\mathbb{R}}(1-\phi_{R_0})Q_{c_2}^{m-1}z^2$$
$$\ge \big((c_2(t_1)-\lambda)(\tfrac{1}{2})^{1/m}+\lambda-Ke^{-\gamma R_0}\big)\int_{\mathbb{R}}(1-\phi_{R_0})z^2, \quad (6\text{-}29)$$

for fixed $K,\gamma>0$. Taking $R_0=R_0(m,\lambda)$ large enough, we have

$$(6\text{-}29) \ge \frac{1}{2^{1/m}}c_2(t_1)\int_{\mathbb{R}}(1-\phi_{R_0})z^2.$$

Therefore,

$$\mathcal{F}_2(t) \ge \frac{1}{2}\int_{\mathbb{R}}\phi_{R_0}\big(z_x^2+c_2(t_1)z^2-mQ_{c_2}^{m-1}z^2\big)+\frac{1}{2}\int_{\mathbb{R}}(1-\phi_{R_0})\big(z_x^2+(\tfrac{1}{2})^{1/m}c_2(t_1)z^2\big)$$
$$-Ke^{\gamma\varepsilon R_0}e^{-\gamma\varepsilon\rho_2(t)}\int_{\mathbb{R}}\phi_{R_0}z^2+O\big(\|z(t)\|^3_{H^1(\mathbb{R})}\big)+O\big(e^{-\gamma\varepsilon t}\|z(t)\|^2_{H^1(\mathbb{R})}\big).$$

Taking R_0 even larger if necessary (but independent of ε), and using a localization argument as in [Martel and Merle 2002b], we conclude that there exists $\tilde{\lambda}_0>0$ such that

$$\mathcal{F}_2(t) \ge \tilde{\lambda}_0\int_{\mathbb{R}}(z_x^2+z^2)-Ke^{\gamma\varepsilon R_0}e^{-\gamma\varepsilon\rho_2(t)}\int_{\mathbb{R}}\phi_{R_0}z^2+O\big(\|z(t)\|^3_{H^1(\mathbb{R})}\big)+O\big(e^{-\gamma\varepsilon t}\|z(t)\|^2_{H^1(\mathbb{R})}\big).$$

Finally, taking ε_0 smaller if necessary, we have

$$\mathcal{F}_2(t) \ge \tilde{\lambda}_0\int_{\mathbb{R}}(z_x^2+z^2)+O(\|z(t)\|^3_{H^1(\mathbb{R})})+O(e^{-\gamma\varepsilon t}\|z(t)\|^2_{H^1(\mathbb{R})}),$$

for a new constant $\tilde{\lambda}_0>0$. □

Conclusion of the proof of Theorem 6.1(1). Now we prove that our assumption $T^*<+\infty$ must lead to a contradiction. Indeed, from Lemmas 6.6 and 6.7, we have for all $t\in[t_1,T^*]$ and for some constant $K>0$,

$$\|z(t)\|^2_{H^1(\mathbb{R})} \le K\mathcal{F}_2(t_1)+E_a[u](t)-E_a[u](t_1)+(c_2(t_1)-\lambda)[\tilde{M}[u](t)-\tilde{M}[u](t_1)]$$
$$+E_a[R](t_1)-E_a[R](t)+(c_2(t_1)-\lambda)[\tilde{M}[R](t_1)-\tilde{M}[R](t)]$$
$$+K\varepsilon\sup_{t\in[t_1,T^*]}e^{-\gamma\varepsilon t}\|z(t)\|_{L^2(\mathbb{R})}+K\sup_{t\in[t_1,T^*]}\|z(t)\|^3_{L^2(\mathbb{R})}.$$

From Lemmas 6.2 and 6.3, Corollary 6.5 and the conservation we have

$$\|z(t)\|^2_{H^1(\mathbb{R})} \le K\varepsilon+(c_2(t_1)-\lambda)\big(\tilde{M}[u](t)-\tilde{M}[u](t_1)\big)$$
$$+K\sup_{t\in[t_1,T^*]}\|z(t)\|^4_{H^1(\mathbb{R})}+Ke^{-\varepsilon\gamma t_1}(1+D_0\varepsilon^{1/2})+KD_0^3\varepsilon^{3/2}.$$

Finally, from (6-22) we have $\tilde{M}[u](t)-\tilde{M}[u](t_1)\le 0$. Collecting the preceding estimates we have for $\varepsilon>0$ small and $D_0=D_0(K)$ large enough

$$\|z(t)\|^2_{H^1(\mathbb{R})} \le \tfrac{1}{4}D_0^2\varepsilon,$$

which contradicts the definition of T^*. The conclusion is that

$$\sup_{t\geq t_1} \left\| u(t) - 2^{-1/(m-1)} Q_{c_2(t)}(\cdot - \rho_2(t)) \right\|_{H^1(\mathbb{R})} \leq K\varepsilon^{1/2}.$$

Using (6-11), we finally get (6-2). □

Proof of Theorem 6.1(2): asymptotic stability. Now we prove (6-3) in Theorem 6.1, following closely [Martel and Merle 2005; 2008].

We continue with the notation introduced in the proof of the stability property (6-2). We have to show the existence of $K, c^+ > 0$ such that

$$\lim_{t\to+\infty} \|u(t) - Q_{c^+}(\cdot - \rho_2(t))\|_{H^1(x>\frac{1}{10}c_\infty t)} = 0 \quad \text{and} \quad |c_\infty - c^+| \leq K\varepsilon^{1/2}.$$

From the stability result above stated it is easy to check that the decomposition proved in Lemma 6.2 and all its conclusions hold *for any time* $t \geq t_1$.

Monotony for mass and energy. The next step is to prove some monotonicity formulae for local mass and energy. Let $K_0 > 0$ and set

$$\phi(x) := \frac{2}{\pi}\arctan(e^{x/K_0}). \tag{6-30}$$

It is clear that $\lim_{x\to+\infty}\phi(x) = 1$ and $\lim_{x\to-\infty}\phi(x) = 0$. In addition, $\phi(-x) = 1-\phi(x)$ for all $x \in \mathbb{R}$, and

$$0 < \phi'(x) = \frac{2}{\pi K_0}\frac{e^{x/K}}{1+e^{2x/K_0}}; \qquad \phi^{(3)}(x) \leq \frac{1}{K_0^2}\phi'(x).$$

Moreover, we have $1-\phi(x) \leq Ke^{-x/K_0}$ as $x \to +\infty$, and $\phi(x) \leq Ke^{x/K_0}$ as $x \to -\infty$.

Let $\sigma, x_0 > 0$. We define, for $t, t_0 \geq t_1$, and $\tilde{y}(x_0) := x - (\rho_2(t_0) + \sigma(t-t_0) + x_0)$,

$$I_{x_0,t_0}(t) := \int_{\mathbb{R}} u^2(t,x)\phi(\tilde{y}(x_0))\,dx, \quad \tilde{I}_{x_0,t_0}(t) := \int_{\mathbb{R}} u^2(t,x)\phi(\tilde{y}(-x_0))\,dx, \tag{6-31}$$

and

$$J_{x_0,t_0} := \int_{\mathbb{R}} \Big(u_x^2 + u^2 - \frac{2a_\varepsilon}{m+1}u^{m+1}\Big)(t,x)\phi(\tilde{y}(x_0))\,dx.$$

Lemma 6.8 (Monotony formulae). *Suppose* $0 < \sigma < \frac{1}{2}(c_\infty(\lambda) - \lambda)$ *and* $K_0 > \sqrt{\frac{2}{\sigma}}$. *There exists* $K, \varepsilon_0 > 0$ *small enough such that for all* $0 < \varepsilon < \varepsilon_0$ *and for all* $t, t_0 \geq t_1$ *with* $t_0 \geq t$ *we have*

$$I_{x_0,t_0}(t_0) - I_{x_0,t_0}(t) \leq K\big(e^{-x_0/K_0} + \varepsilon^{-1}e^{-\gamma\varepsilon T_\varepsilon}e^{-\varepsilon\gamma x_0/K_0}\big). \tag{6-32}$$

On the other hand, if $t \geq t_0$ *and* $\rho_2(t_0) \geq t_1 + x_0$,

$$\tilde{I}_{x_0,t_0}(t) - \tilde{I}_{x_0,t_0}(t_0) \leq K\big(e^{-x_0/K_0} + \varepsilon^{-1}e^{-\varepsilon\gamma\rho_2(t_0)}e^{\varepsilon\gamma x_0/K_0}\big), \tag{6-33}$$

and finally if $t_0 \geq t$,

$$J_{x_0,t_0}(t_0) - J_{x_0,t_0}(t) \leq K\big(e^{-x_0/K_0} + \varepsilon^{-1}e^{-\gamma\varepsilon T_\varepsilon}e^{-\varepsilon\gamma x_0/K_0}\big). \tag{6-34}$$

The proof is given in Section A.5.

Conclusion of the proof of Theorem 6.1(2). Consider $0 < \varepsilon < \varepsilon_0$ and $u(t)$ satisfying (6-1). From Lemma 6.2, we can decompose $u(t)$ for all $t \geq t_1$ such that $u(t,x) = 2^{-1/(m-1)} Q_{c_2(t)}(x - \rho_2(t)) + z(t,x)$, where z satisfies (6-10), (6-11), (6-12), (6-14) and (6-15). An application of Lemma 6.4 followed by integration in time shows that there exists $K = K(D_0) > 0$ such that

$$\int_{t_1}^{+\infty}\int_{\mathbb{R}} (z_x^2 + z)(t,x) e^{-\frac{1}{A_0}|x-\rho_2(t)|} \leq K(D_0)\varepsilon. \tag{6-35}$$

Now we claim that

$$c^+ := \lim_{t\to+\infty} c_2(t) < +\infty \quad \text{and} \quad |c^+ - c_\infty| \leq K\varepsilon^{1/2}. \tag{6-36}$$

In fact, note that from (6-35) there exists a sequence $t_n \uparrow +\infty$, $t_n \in [n, n+1)$ such that

$$\lim_{n\to+\infty} \int_{\mathbb{R}} (z_x^2 + z)(t_n, x) e^{-\frac{1}{A_0}|x-\rho_2(t_n)|} = 0. \tag{6-37}$$

From (6-37), (6-14), and (6-15), and taking $A_0 > 0$ so large that $1/A_0 < \gamma$, we get

$$|c_2'(t)| \leq K \int_{\mathbb{R}} z^2(t,x) e^{-\frac{1}{A_0}|x-\rho_2(t)|} + K e^{-\gamma\varepsilon t}.$$

This, combined with (6-35) and (6-12), allows us to conclude (6-36). This proves the first part of (6-4).

The next step is to prove that

$$\limsup_{t\to+\infty} \int_{\mathbb{R}} (z_x^2 + z^2)(t, x+\rho_2(t))\phi(x - x_0) \leq K e^{-x_0/2K_0} + K\varepsilon^{-1} e^{-\varepsilon\gamma T_\varepsilon} e^{-\varepsilon\gamma x_0/K_0}.$$

This follows from the decay properties of R and the estimate

$$\limsup_{t\to+\infty} \int_{\mathbb{R}} (u_x^2 + u^2)(t, x+\rho_2(t))\phi(x - x_0) \leq K e^{-x_0/2K_0} + K\varepsilon^{-1} e^{-\varepsilon\gamma T_\varepsilon} e^{-\varepsilon\gamma x_0/K_0}, \tag{6-38}$$

which we prove now. We start from (6-34): we have for $t_0 \geq t_1$,

$$J_{x_0,t_0}(t_0) \leq J_{x_0,t_0}(t_1) + K e^{-x_0/K_0} + K\varepsilon^{-1} e^{-\varepsilon\gamma T_\varepsilon} e^{-\varepsilon\gamma x_0/K_0}.$$

From the equivalence between the energy and H^1-norm (we are in a subcritical case), we have

$$\begin{aligned}\int_{\mathbb{R}} (u_x^2 + u^2)(t_0, x+\rho_2(t_0))\phi(x - x_0) &\leq K \int_{\mathbb{R}} (u_x^2 + u^2)(t_1, x+\rho_2(t_1))\phi(x - y_0)\\ &\quad + K e^{-x_0/2K} + K\varepsilon^{-1} e^{-\varepsilon\gamma T_\varepsilon} e^{-\varepsilon\gamma x_0/K_0},\end{aligned}$$

where $y_0 := \rho_2(t_0) - \rho_2(t_1) + \sigma(t_1 - t_0) + x_0$. Now we send $t_0 \to +\infty$ noticing that $y_0 \to +\infty$. This gives (6-38), as desired.

We next prove that

$$\lim_{n\to+\infty} \int_{\mathbb{R}} (z_x^2 + z^2)(t_n, x)\phi(x - \rho_2(t_n) + x_0)\, dx = 0. \tag{6-39}$$

where $(t_n)_{n\in\mathbb{N}}$ is the sequence from (6-39). Indeed, for any $x_1 > 0$,

$$\begin{aligned}\int_{\mathbb{R}} &(z_x^2+z^2)(t_n, x+\rho_2(t_n))\phi(x+x_0)\\ &\le K(e^{\frac{x_0}{A_0}}+e^{\frac{x_1}{A_0}})\int_{\mathbb{R}}(z_x^2+z^2)(t_n, x+\rho_2(t_n))e^{-\frac{|x|}{A_0}}+K\int_{\mathbb{R}}(z_x^2+z^2)(t_n, x+\rho_2(t_n))\phi(x-x_1).\end{aligned}$$

Using (6-39) we are able to take the limit $n\to+\infty$ in this inequality, with x_0, x_1 fixed. Taking the limit $x_1\to+\infty$ yields the conclusion.

We finally prove that the above result holds for any sequence $t_n\to+\infty$. Let $\beta < c_\infty(\lambda)-\lambda$ to be fixed. We want to prove that for ε small enough,

$$\lim_{t\to+\infty}\int_{\mathbb{R}}(z_x^2+z^2)(t,x)\phi(x-\beta t)\,dx = 0.$$

First, we claim that for any $t_2, t_3 > t_1$ with $t_2 < t_3$ and $\rho_2(t_2) > x_0+t_1$, we have

$$\int_{\mathbb{R}} u^2(t_3,x)\phi(x-y_3)\,dx \le \int_{\mathbb{R}} u^2(t_2,x)\phi(x-y_2)\,dx + Ke^{-x_0/K_0}+K\varepsilon^{-1}e^{-\gamma\varepsilon\rho_2(t_2)}e^{\gamma\varepsilon x_0/K_0}, \tag{6-40}$$

where $y_3 := \rho_2(t_2)+\frac{1}{2}\beta(t_3-t_2)-x_0$ and $y_2 := \rho_2(t_2)-x_0$. In fact, the left-hand side of this inequality corresponds to $\tilde{I}_{x_0,t_2}(t_3)$ and the right one is $\tilde{I}_{x_0,t_2}(t_2)$, with $\sigma := \frac{1}{2}\beta$ (see (6-31) for the definitions). Thus (6-40) a consequence of Lemma 6.8, more specifically of (6-33).

Now the rest of the proof is similar to [Martel and Merle 2005]. Since $\int_{\mathbb{R}} z(t, x+\rho_2(t))R(x) = 0$, we have

$$\left|\int_{\mathbb{R}} z(t,x+\rho_2(t))R(x)\phi(x+x_0)\right| \le K\varepsilon^{1/2}e^{-x_0/2K_0}.$$

Second, we use the decomposition $u(t,x) = 2^{-1/(m-1)}Q_{c_2(t)}(x-\rho_2(t))+z(t,x)$ in (6-40) to get

$$\begin{aligned}\int_{\mathbb{R}} &z^2(t_3,x)\phi(x-y_3)\,dx\\ &\le \int_{\mathbb{R}} z^2(t_2,x)\phi(x-y_2)\,dx + Ke^{-x_0/2K_0}+K\varepsilon^{-1}e^{-\gamma\varepsilon\rho_2(t_2)}e^{\gamma\varepsilon x_0/K_0}+K|c_2(t_2)-c_2(t_3)|.\end{aligned} \tag{6-41}$$

Third, consider $t > t_1$ large, and define $t' \in (t_1, t)$ such that $\beta t := \rho_2(t') + \frac{\beta}{2}(t-t')-x_0$. Note that $t'\to+\infty$ as $t\to+\infty$. Since $t_n\in[n, n+1)$ there exists $n = n(t)$ such that $0 < t-t_n \le 2$, and then

$$\beta t := \rho_2(t_n)+\tfrac{1}{2}\beta(t-t_n)-\tilde{x}_0, \qquad \text{with } |\tilde{x}_0-x_0|\le 10.$$

Now we apply (6-41) between $t_3 = t$ and $t_2 = t_n$. We get

$$\begin{aligned}\int_{\mathbb{R}} &z^2(t,x)\phi(x-\beta t)\,dx\\ &\le \int_{\mathbb{R}} z^2(t_n,x)\phi(x-\rho_2(t_n)+\tilde{x}_0)\,dx + Ke^{-x_0/2K_0}+K\varepsilon^{-1}e^{-\gamma\varepsilon\rho_2(t_n)}e^{\gamma\varepsilon x_0/K_0}+K|c_2(t)-c_2(t_n)|.\end{aligned}$$

Since $n(t)\to+\infty$ as $t\to+\infty$, by (6-39) and (6-36) we obtain

$$\limsup_{t\to+\infty}\int_{\mathbb{R}} z^2(t,x)\phi(x-\beta t)\,dx \le Ke^{-x_0/2K_0},$$

and since x_0 is arbitrary (because of $\lim_{t\to+\infty}\rho_2(t_n)=+\infty$), we get the desired result. The same result is still valid for z_x. We have

$$\limsup_{t\to+\infty}\int_{\mathbb{R}} z_x^2(t,x)\phi(x-\beta t)\,dx \le Ke^{-x_0/2K_0}.$$

Finally, let

$$\begin{aligned} w^+(t,x) &:= u(t,x) - 2^{-1/(m-1)}Q_{c^+}(x-\rho_2(t)) \\ &= z(t,x) + 2^{-1/(m-1)}\big(Q_{c_2(t)}(x-\rho_2(t)) - Q_{c^+}(x-\rho_2(t))\big). \end{aligned}$$

From this and (6-36) we obtain (6-3). □

7. Proof of the main theorems

Proof of Theorem 1.1. We will combine Theorems 3.1 and 4.1 to obtain the global solution $u(t)$ with the required properties. This method was employed earlier in [Martel and Merle 2007; Martel et al. 2010; Muñoz 2010].

By Theorem 3.1 there exists a solution u of (1-15) satisfying $u \in C(\mathbb{R}, H^1(\mathbb{R}))$ and (3-1). This solution also satisfies, from (3-3),

$$\big\|u(-T_\varepsilon) - Q(\cdot + (1-\lambda)T_\varepsilon)\big\|_{H^1(\mathbb{R})} \le K\varepsilon^{10},$$

for ε small enough. In addition, u is unique if $m=2,4$ and $\lambda>0$, or if $m=3$ and $\lambda\ge 0$. This proves Theorem 1.1(1).

To obtain Theorem 1.1(2) we invoke (4-2) and (4-3) in Theorem 4.1. We define $\tilde{T}_\varepsilon := T_\varepsilon + \rho_1(T_\varepsilon)$, and $\rho_\varepsilon := \rho(T_\varepsilon)$. Now (1-27) and (1-28) are straightforward.

Proof of Theorem 1.2. Suppose $m=2,3,4$ with $\lambda>0$ for $m=2,4$. Define $t_1 := T_\varepsilon + \rho_1(T_\varepsilon)$ and $X_0 := \rho(T_\varepsilon)$. Then, from the preceding estimates and Theorem 6.1 we have *stability* and *asymptotic stability* at infinity. In other words, there exist a constant $c^+>0$ and a C^1 function $\rho_2(t)\in\mathbb{R}$ such that

$$w^+(t) := u(t) - 2^{-1/(m-1)}Q_{c^+}(\cdot - \rho_2(t))$$

satisfies (6-2) and (6-3). This proves (1-29) and (1-30).

We finally prove (1-32) and (1-33). From the energy conservation, we have for all $t\ge t_1$,

$$E_a[u](-\infty) = E_a\big[2^{-1/(m-1)}Q_{c^+}(\cdot - \rho_2(t)) + w^+(t)\big]$$

In particular, from (6-3) and Section A.6 we have as $t\to+\infty$

$$(\lambda-\lambda_0)M[Q] = \frac{(c^+)^{2\theta}}{2^{2/(m-1)}}(\lambda - \lambda_0 c^+)M[Q] + E^+. \tag{7-1}$$

From this identity $E^+ := \lim_{t\to+\infty} E_a[w^+](t)$ is well defined. This proves (1-32). To deal with (1-33), note that from the stability result (6-2) and the Morrey embedding we have, for any $\lambda > 0$,

$$E[w^+](t) = \frac{1}{2}\int_{\mathbb{R}} (w_x^+)^2(t) + \frac{\lambda}{2}\int_{\mathbb{R}} (w^+)^2(t) - \frac{1}{m+1}\int_{\mathbb{R}} a_\varepsilon (w^+)^{m+1}(t)$$
$$\geq \frac{1}{2}\int_{\mathbb{R}} (w_x^+)^2(t) + \frac{\lambda}{2}\int_{\mathbb{R}} (w^+)^2(t) - K\varepsilon^{(m-1)/2}\int_{\mathbb{R}} a_\varepsilon (w^+)^2(t) \geq \mu \|w^+(t)\|^2_{H^1(\mathbb{R})}$$

for some $\mu = \mu(\lambda) > 0$. Passing to the limit we obtain (1-33).

Now we prove the bound (1-34). First, the treat the cubic case with $\lambda = 0$. Here, from (7-1) we have

$$E^+ = \lambda_0\left(\frac{(c^+)^{3/2}}{2^{2/(m-1)}} - 1\right) M[Q].$$

Since in this case we have $2^{2/(m-1)} = 2 = c_\infty^{3/2}$, $M[Q] = 2$ and $\lambda_0 = \frac{1}{3}$, we obtain $\frac{3}{2}E^+ = \left(\frac{c^+}{c_\infty}\right)^{3/2} - 1$.

Now we deal with the case $\lambda > 0$. After some algebraic manipulations, the equation for c_∞ in (4-17) becomes

$$\frac{c_\infty^{2\theta}}{2^{2/(m-1)}}(\lambda_0 c_\infty - \lambda) M[Q] = (\lambda_0 - \lambda) M[Q].$$

On the other hand, from (7-1) and (1-33) we have

$$\mu \limsup_{t\to+\infty} \|w^+(t)\|^2_{H^1(\mathbb{R})} \leq \frac{(c^+)^{2\theta}}{2^{2/(m-1)}}(\lambda_0 c^+ - \lambda) M[Q] - (\lambda_0 - \lambda) M[Q].$$

Putting together both estimates, we get

$$\tilde{\mu} \limsup_{t\to+\infty} \|w^+(t)\|^2_{H^1(\mathbb{R})} \leq (c^+)^{2\theta+1} - c_\infty^{2\theta+1} - \frac{\lambda}{\lambda_0}\big((c^+)^{2\theta} - c_\infty^{2\theta}\big),$$

for some $\tilde{\mu} > 0$. Arguing as in Lemma 6.3 we have

$$\tilde{\mu} \limsup_{t\to+\infty} \|w^+(t)\|^2_{H^1(\mathbb{R})} \leq \frac{1}{\lambda_0}(c_\infty - \lambda)\big((c^+)^{2\theta} - c_\infty^{2\theta}\big) + O\big(\big|(c^+)^{2\theta} - c_\infty^{2\theta}\big|^2\big).$$

From this inequality and the bound $|c^+ - c_\infty| \leq K\varepsilon$ we get

$$\left(\frac{c^+}{c_\infty}\right)^{2\theta} - 1 \geq \tilde{\mu} \limsup_{t\to+\infty} \|w^+(t)\|^2_{H^1(\mathbb{R})},$$

as desired.

Proof of Theorem 1.3. In this section we prove (1-35), which implies there is no pure soliton at infinity. This will require several additional arguments, including the fundamental Lemma 7.5 and a monotonicity formula that implies that any such soliton would have polynomial decay and be L^1-integrable, in contradiction with the change of scaling.

Suppose, for a contradiction, that (1-35) is false. Then

$$\lim_{t\to+\infty} \|w^+(t)\|_{H^1(\mathbb{R})} = 0.$$

This, together with subcriticality, implies that $E^+ = 0$. Therefore, by using (7-1), and after some basic algebraic manipulations we see that c^+ must satisfy the algebraic equation

$$(c^+)^{\lambda_0}\Big(c^+ - \frac{\lambda}{\lambda_0}\Big)^{1-\lambda_0} = 2^p\Big(1 - \frac{\lambda}{\lambda_0}\Big)^{1-\lambda_0}$$

(compare with (4-17)). This relation and the uniqueness of c_∞ give

$$c^+ = c_\infty(\lambda). \tag{7-2}$$

In other words, the soliton solution is *pure* (cf. Definition 1.0).

Now consider the decomposition result for $u(t)$ from Lemma 6.2. We claim that $z(t)$ also vanishes at infinity. Indeed, from Lemma 6.2, the fact that

$$u(t) = R(t) + z(t) = w^+(t) + 2^{-1/(m-1)} Q_{c_\infty}(\cdot - \rho_2(t)) \quad \text{for } t \ge t_1,$$

and the estimates (6-11) and (6-36), we have

$$\lim_{t\to+\infty} \|z(t)\|_{H^1(\mathbb{R})} = 0, \tag{7-3}$$

$\lim_{t\to+\infty} u(t, \cdot + \rho_2(t)) = 2^{-1/(m-1)} Q_{c_\infty}$ in $H^1(\mathbb{R})$, and $\lim_{t\to+\infty} \rho_2'(t) - (c_\infty(\lambda) - \lambda) = 0$.

Lemma 7.1 (Monotony of mass backwards in time). *Suppose $u(t)$ solution of (1-15) constructed in Theorem 3.1, satisfying (6-2) and (6-3). Define*

$$\mathcal{M}[u](t) := \int_{\mathbb{R}} \frac{u^2(t,x)}{a_\varepsilon(x)}\,dx. \tag{7-4}$$

Then, under the additional hypothesis $\lambda > 0$ for $m = 2, 3, 4$, we have, for all $t, t' \ge t_1$ with $t' \ge t$,

$$\mathcal{M}[u](t) - \mathcal{M}[u](t') \le K e^{-\varepsilon\gamma t}. \tag{7-5}$$

Proof. A simple computation tell us that the time derivative of $\mathcal{M}[u](t)$ is given by

$$\partial_t \int_{\mathbb{R}} \frac{u^2}{a_\varepsilon} = 2\varepsilon \int_{\mathbb{R}} u_x^2 \frac{a_\varepsilon'}{a_\varepsilon^2} + \varepsilon \int_{\mathbb{R}} u^2 \Big(\lambda \frac{a_\varepsilon'}{a_\varepsilon^2} - \varepsilon^2 \Big(\frac{a_\varepsilon'}{a_\varepsilon^2}\Big)''\Big) - 2\varepsilon \int_{\mathbb{R}} \frac{a_\varepsilon'}{a_\varepsilon} u^{m+1}.$$

Replacing u by $R + z$ (see Lemma 6.2) and using assumption (1-14) and estimates similar to (A-13), plus the smallness of $\|z(t)\|_{H^1(\mathbb{R})}$, we get

$$\partial_t \mathcal{M}[u](t) \ge -K\varepsilon e^{-\varepsilon\gamma t},$$

for some $K, \gamma > 0$. The result follows after integration. □

Remark. The estimate (7-5) is valid under the additional assumption $0 < \lambda \le \lambda_0$. This extra hypothesis unfortunately does not hold in the case $m = 3$, $\lambda = 0$.

Lemma 7.1 allows us to prove a version of Theorem 3.1 for positive times.

Proposition 7.2 (Backward uniqueness). *Suppose $m = 2, 3, 4$. Let $\beta \in \mathbb{R}$ and $0 < \lambda \le \lambda_0$. There exist constants $K, \gamma, \varepsilon_0 > 0$ and a unique solution $v = v_\beta \in C([\frac{1}{2}T_\varepsilon, +\infty), H^1(\mathbb{R}))$ of (1-15) such that*

$$\lim_{t\to+\infty} \left\| v(t) - 2^{-1/(m-1)} Q_{c_\infty}(\cdot - (c_\infty(\lambda) - \lambda)t - \beta) \right\|_{H^1(\mathbb{R})} = 0. \tag{7-6}$$

Furthermore, for all $t \ge \frac{1}{2}T_\varepsilon$ and $s \ge 1$ the function $v(t)$ satisfies

$$\left\| v(t) - 2^{-1/(m-1)} Q_{c_\infty}(\cdot - (c_\infty(\lambda) - \lambda)t - \beta) \right\|_{H^s(\mathbb{R})} \le K\varepsilon^{-1} e^{-\varepsilon\gamma t}. \tag{7-7}$$

Finally, suppose that there exists $\tilde{v}(t) \in H^1(\mathbb{R})$ solution of (1-15) such that

$$\lim_{t\to+\infty} \left\| \tilde{v}(t) - 2^{-1/(m-1)} Q_{c_\infty}(\cdot - \rho_2(t)) \right\|_{H^1(\mathbb{R})} = 0. \tag{7-8}$$

Then $\tilde{v} \equiv v_\beta$ for some $\beta \in \mathbb{R}$.

Proof. Given $\beta \in \mathbb{R}$, the proof of existence and uniqueness of the solution v_β satisfying (7-6) and (7-7) is identical to the proof of Theorem 3.1 in Section 3 and Section A.1. First we construct a sequence of functions v_n as in (A-1) for times $t \sim T_n$. Next, we prove a decomposition lemma as in Lemma A.1.4. This yields a version of (7-5) for $\mathcal{M}[v_n](t)$. The main difference is given in the estimates (A-14) and (A-15), where now we introduce the modified mass $\mathcal{M}[v_n](t)$ defined in (7-4). The energy functional in (A-18) is now given by $E_a[v_n](t) + (c_\infty(\lambda) - \lambda)\mathcal{M}[v_n](t)$. The rest of the proof, including the uniqueness, adapts *mutatis mutandis*.

Now consider a solution $\tilde{v}$ of (1-15) satisfying (7-8). Using monotonicity arguments, similar to the proof of Proposition A.1.7, we show the existence of $\beta \in \mathbb{R}$ such that

$$\left\| \tilde{v}(t) - 2^{-1/(m-1)} Q_{c_\infty}(\cdot - (c_\infty(\lambda) - \lambda)t - \beta) \right\|_{H^1(\mathbb{R})} \le K\varepsilon^{-1} e^{-\varepsilon\gamma t},$$

for some $K, \gamma > 0$. This implies that there exists $\beta \in \mathbb{R}$ such that $\tilde{v}$ satisfies (7-6). The conclusion follows from the uniqueness of $v(t)$. $\square$

As a consequence of this result together with (7-3), the solution $u(t)$ constructed in Theorem 3.1 satisfies the following exponential decay at infinity: there exist $K, \gamma > 0$ and $\beta \in \mathbb{R}$ such that, for all $t \ge t_1$, if $\tilde{\rho}_2(t) := (c_\infty(\lambda) - \lambda)t + \beta$, then

$$\tilde{z}(t) := u(t) - 2^{-1/(m-1)} Q_{c_\infty}(\cdot - \tilde{\rho}_2(t)), \quad \text{satisfies} \quad \|\tilde{z}(t)\|_{H^2(\mathbb{R})} \le K\varepsilon^{-1} e^{-\varepsilon\gamma t}. \tag{7-9}$$

Now we prove that this strong H^1-convergence gives rise to strange localization properties.

Lemma 7.3 (L^2-exponential decay on the left). *There exist $K, \tilde{x}_0 > 0$ large enough such that for all $t \ge T_0$ and for all $x_0 \ge \tilde{x}_0$*

$$\|u(t, \cdot + \tilde{\rho}_2(t))\|^2_{L^2(x \le -x_0)} \le K e^{-x_0/K}. \tag{7-10}$$

Proof. Suppose $x_0 > 0$, $t, t_0 \ge t_1$ and $\sigma > 0$ from (6-3). Consider the modified mass

$$\tilde{I}_{t_0,x_0}(t) := \frac{1}{2} \int_{\mathbb{R}} \frac{u^2(t,x)}{a_\varepsilon(x)} (1 - \phi(y))\, dx,$$

with $y := x - (\tilde\rho_2(t_0) + \sigma(t - t_0) - x_0)$ and ϕ defined in (6-30). For this quantity we claim that for $x_0 > \tilde x_0$ and for all $t \geq t_0$,

$$\tilde I_{t_0,x_0}(t_0) - \tilde I_{t_0,x_0}(t) \leq K e^{-x_0/K}\big(1 + e^{-\frac{1}{2}\sigma(t-t_0)/K}\big). \tag{7-11}$$

Let us assume this result for a moment. After taking the limit $t \to +\infty$ and using (6-3), we have $\lim_{t\to+\infty} \tilde I_{t_0,x_0}(t) = 0$ and thus

$$\tilde I_{t_0,x_0}(t_0) \leq K e^{-x_0/K}.$$

Now (7-10) follows from the fact that $t_0 \geq t_1$ is arbitrary.

Finally, let us prove (7-11). A direct calculation tell us that

$$\begin{aligned}\frac12 \partial_t \int_{\mathbb R} \frac{(1-\phi(y))}{a_\varepsilon} u^2 = {} & \frac32 \int_{\mathbb R} \frac{\phi'}{a_\varepsilon} u_x^2 + \frac32 \varepsilon \int_{\mathbb R} \frac{a_\varepsilon'}{a_\varepsilon^2}(1-\phi) u_x^2 - \frac{m}{m+1} \int_{\mathbb R} \phi' u^{m+1} \\ & + \frac12 \int_{\mathbb R} u^2 \Big((\sigma+\lambda)\frac{\phi'}{a_\varepsilon} - \frac{\phi^{(3)}}{a_\varepsilon} + 3\varepsilon\phi'' \frac{a_\varepsilon'}{a_\varepsilon^2} + 3\varepsilon^2 \phi' \Big(\frac{a_\varepsilon'}{a_\varepsilon^2}\Big)' \Big) \\ & + \frac{\varepsilon}{2} \int_{\mathbb R} u^2 \Big(\lambda \frac{a_\varepsilon'}{a_\varepsilon^2} - \varepsilon^2 \Big(\frac{a_\varepsilon'}{a_\varepsilon^2}\Big)'' \Big)(1-\phi) - \varepsilon \int_{\mathbb R} \frac{a_\varepsilon'}{a_\varepsilon} u^{m+1}(1-\phi).\end{aligned}$$

Using (7-9), we have

$$\Big| \int_{\mathbb R} \phi' u^{m+1} \Big| \leq K \varepsilon^{(m-1)/2} \int_{\mathbb R} \phi' \tilde z^2 + K e^{-\frac12 \sigma(t-t_0)} e^{-x_0/K},$$

$$\Big| \int_{\mathbb R} \frac{a_\varepsilon'}{a_\varepsilon} u^{m+1}(1-\phi) \Big| \leq K e^{-\frac12 \sigma(t-t_0)} e^{-x_0/K} + K\varepsilon^{(m-1)/2} \int_{\mathbb R} \frac{a_\varepsilon'}{a_\varepsilon} \tilde z^2 (1-\phi).$$

After this, it is easy to conclude that

$$\frac12 \partial_t \int_{\mathbb R} \frac{(1-\phi(y))}{a_\varepsilon} u^2 \geq -K e^{-\frac12\sigma(t-t_0)} e^{-x_0/K}.$$

The conclusion follows after integration in time. □

The proof of decay on the right-hand side of the soliton requires more care, and is valid under the assumption $\limsup_{t\to+\infty} \|w^+(t)\|_{H^1(\mathbb R)} = 0$ and $\lambda > 0$. We do not expect to have exponential decay in a general situation, but for our purposes we only need a polynomial decay. The following result is due to Y. Martel.

Lemma 7.4 (L^2-polynomial decay on the right the soliton solution). *There exist $K, \tilde x_0 > 0$ large enough but independent of ε, such that for all $t \geq T_0$ and for all $x_0 \geq \tilde x_0$*

$$\int_{\mathbb R} (x - x_0)_+^2 \tilde z^2(t, x + \tilde\rho_2(t))\, dx \leq K,$$

where $x_+ := \max\{x, 0\}$.

Proof. Take $x_0 > 0$, $t_0, t \geq t_1$ and define

$$\hat I_{t_0,x_0}(t) := \int_{\mathbb R} \tilde z^2(t,x)\phi(\tilde y)\, dx; \quad \tilde y := x - (\tilde\rho_2(t_0) + \tilde\sigma(t - t_0) + x_0),$$

and

$$\hat{J}_{t_0,x_0}(t) := \int_{\mathbb{R}} \tilde{z}_x^2(t,x)\phi(\tilde{y})\,dx.$$

Here ϕ is the cut-off function defined in (6-30), and $\tilde{\sigma}$ is a fixed constant satisfying $\tilde{\sigma} > 2(c_\infty(\lambda)-\lambda)$. We claim that there exists $K>0$ such that (for simplicity we omit the dependence if no confusion is present)

$$|\partial_t \hat{I}_{t_0,x_0}(t)| \le K\int_{\mathbb{R}} (\tilde{z}_x^2+\tilde{z}^2)[\phi' + \varepsilon a'(\varepsilon x)\phi]dx + K\|\tilde{z}(t)\|_{H^1(\mathbb{R})} e^{-\varepsilon(t-t_0)/K} e^{-\varepsilon x_0/K}, \tag{7-12}$$

and

$$|\partial_t \hat{J}_{t_0,x_0}(t)| \le K\int_{\mathbb{R}} (\tilde{z}_{xx}^2+\tilde{z}_x^2+\tilde{z}^2)[\phi' + \varepsilon a'(\varepsilon x)\phi]dx + K\|\tilde{z}(t)\|_{H^2(\mathbb{R})} e^{-\varepsilon(t-t_0)/K} e^{-\varepsilon x_0/K}. \tag{7-13}$$

Indeed, these estimates are proved in the same way as in Lemma 6.4 and Section A.4. For the sake of brevity we skip the details.

From Proposition 7.2 and the exponential decay of z we have that both right-hand sides in (7-12)-(7-13) are integrable between t_0 and $+\infty$. We get

$$\hat{I}_{t_0,x_0}(t_0) \le K\int_{t_0}^{+\infty}\int_{\mathbb{R}} (\tilde{z}_x^2+\tilde{z}^2)[\phi' + \varepsilon a'(\varepsilon x)\phi]dx\,dt + K\varepsilon^{-1}\sup_{t\ge t_0}\|\tilde{z}(t)\|_{H^1(\mathbb{R})} e^{-\varepsilon x_0/K}. \tag{7-14}$$

In the same line, we have

$$\hat{J}_{t_0,x_0}(t_0) \le K\int_{t_0}^{+\infty}\int_{\mathbb{R}} (\tilde{z}_{xx}^2+\tilde{z}_x^2+\tilde{z}^2)[\phi' + \varepsilon a'(\varepsilon x)\phi]dx\,dt + K\varepsilon^{-1}\sup_{t\ge t_0}\|\tilde{z}(t)\|_{H^2(\mathbb{R})} e^{-\varepsilon x_0/K}. \tag{7-15}$$

Note that both quantities above are *integrable* with respect to x_0.

Set $\xi_0(\tilde{y}) := \phi(\tilde{y})$ and $\xi_j(\tilde{y}) := \int_{-\infty}^{\tilde{y}} \xi_{j-1}(s)\,ds$, for $j=1,2$. Recall that the ξ_j are positive and increasing functions on $\mathbb{R}$, with $\xi_j(\tilde{y})\to 0$ as $\tilde{y}\to-\infty$, and $\xi_j(\tilde{y})-\tilde{y}^j\to 0$ as $\tilde{y}\to+\infty$. Integrating (7-14) from x_0 to $+\infty$ and using Fubini's theorem we obtain

$$\int_{\mathbb{R}} \xi_1(\tilde{y}(t_0))\tilde{z}^2(t_0) \le K\int_{t_0}^{+\infty}\int_{\mathbb{R}} (\tilde{z}_x^2+\tilde{z}^2)[\xi_0 + \varepsilon a'(\varepsilon x)\xi_1] + K\varepsilon^{-2}\sup_{t\ge t_0}\|\tilde{z}(t)\|_{H^1(\mathbb{R})} e^{-\varepsilon x_0/K}. \tag{7-16}$$

Similarly, from (7-15),

$$\int_{\mathbb{R}} \xi_1(\tilde{y}(t_0))\tilde{z}_x^2(t_0,x)\,dx \le K\varepsilon^{-3}e^{-2\varepsilon\gamma t_0} + K\varepsilon^{-2}\sup_{t\ge t_0}\|\tilde{z}(t)\|_{H^2(\mathbb{R})} e^{-\varepsilon x_0/K}. \tag{7-17}$$

In conclusion, thanks to the exponential decay of $\tilde{z}$ and (7-16)–(7-17), we have

$$\int_{t_0}^{+\infty}\int_{\mathbb{R}} \xi_1(x-\tilde{\rho}_2(t)-x_0)(\tilde{z}_x^2+\tilde{z}^2)(t,x)\,dx\,dt < +\infty.$$

Furthermore, $\tilde{\rho}_2(t) \le \tilde{\rho}_2(t_0) + \sigma(t-t_0)$ for all $t\ge t_0$. Thus

$$\int_{t_0}^{+\infty}\int_{\mathbb{R}} \xi_1(\tilde{y}(t))(\tilde{z}_x^2+\tilde{z}^2)(t,x)\,dx\,dt < +\infty. \tag{7-18}$$

In addition, an easier calculation gives

$$\int_{t_0}^{+\infty}\int_{\mathbb{R}} a'(\varepsilon x)\xi_2(\tilde{y}(t))(\tilde{z}_x^2+\tilde{z}^2)(t,x)\,dx\,dt < +\infty. \tag{7-19}$$

From (7-18) and (7-19), we can perform a second integration with respect to x_0 in (7-16) to obtain

$$\int_{\mathbb{R}} \xi_2(\tilde{y}(t_0))\tilde{z}^2(t_0,x)\,dx \le K(\varepsilon),$$

uniformly for x_0 large. Since t_0 is arbitrary, this last estimate gives the conclusion. □

Lemma 7.5 (L^1-integrability and smallness). *Assume (7-3) holds. There exist $K, T_0 > 0$ large enough such that $u(t, \cdot + \tilde{\rho}_2(t)) \in L^1(\mathbb{R})$ for all $t \ge T_0$. Moreover,*

$$\left|\int_{\mathbb{R}} z(t)\right| \le \frac{1}{100}. \tag{7-20}$$

Finally, from the L^1 conservation law (1-9), we have $u(t) \in L^1(\mathbb{R})$ for all $t \in \mathbb{R}$ and

$$\int_{\mathbb{R}} u(t) = \int_{\mathbb{R}} Q. \tag{7-21}$$

Proof. Let $x_0 \ge \tilde{x}_0$ to be fixed below. If $|x| \ge x_0$ we have $2^{-1/(m-1)}Q_{c_\infty}(x) \le Ke^{-\sqrt{c_\infty}|x|}$. Since $\tilde{z}(t, x+\tilde{\rho}_2(t)) = u(t, x+\tilde{\rho}_2(t)) - 2^{-1/(m-1)}Q_{c_\infty}(x)$, by using Lemma 7.3 and the stability bound (6-2), in addition to a Gagliardo–Nirenberg type inequality, we then get, for all $x \le -x_0$,

$$|\tilde{z}(t, x+\tilde{\rho}_2(t))| \le K\|\tilde{z}(t, \cdot+\tilde{\rho}_2(t))\|_{L^2(y\ge x)}^{\frac12}\|\tilde{z}_y(t, \cdot+\tilde{\rho}_2(t))\|_{L^2(\mathbb{R})}^{\frac12} \le K\varepsilon^{1/4}e^{x/K},$$

For $x \in [-x_0, x_0]$ one has

$$\int_{[-x_0,x_0]} \tilde{z}(t, x+\tilde{\rho}_2(t)) \le Kx_0^{1/2}\|\tilde{z}(t, x+\tilde{\rho}_2(t))\|_{L^2(\mathbb{R})}^{1/2} \le Kx_0^{1/2}\varepsilon^{1/4}.$$

The case $x \ge x_0$ requires more care. From Lemma 7.4 and the Cauchy–Schwarz inequality, we have (for clarity we drop the dependence on $x+\tilde{\rho}_2(t)$)

$$\left|\int_{x\ge x_0} z(t)\right| \le \frac{K}{(x_0-\tilde{x}_0)^{1/2-}}\left(\int_{x\ge x_0}(1+(x-\tilde{x}_0)^2)z^2(t)\right)^{1/2} \le \frac{K}{x_0^{1/2-}},$$

for x_0 large enough, independent of ε. From these estimates we obtain the smallness condition (7-20).

That $u(t)$ is in $L^1(\mathbb{R})$ for all $t \in \mathbb{R}$ is a consequence of Proposition 2.1. It is clear that from this last fact (1-9) remains constant for all time and (7-21) holds. □

Conclusion of the proof. From (7-2) and Section A.6 we have

$$\lim_{t\to+\infty}\int_{\mathbb{R}} z(t) = \left(1 - \frac{(c^+)^{\theta-\frac14}}{2^{1/(m-1)}}\right)\int_{\mathbb{R}} Q = (1-\kappa_m)\int_{\mathbb{R}} Q \ne 0, \quad \text{with} \quad \kappa_m := \frac{c_\infty^{\frac{3-m}{2(m-1)}}}{2^{1/(m-1)}},$$

For $m = 3, 4$ it is easy to see that $1-\kappa_m > \frac{1}{10}$. For $m = 2$ we have $\kappa_2 = \frac{1}{2}c_\infty^{1/2}$; but from (4-19) we know that $c_\infty \le 2^{\frac{4}{3}}$. In any case, then, we have $1-\kappa_m > \frac{1}{10}$. Thus

$$\left| \lim_{t\to+\infty} \int_{\mathbb{R}} z(t) \right| \ge \frac{1}{10} \int_{\mathbb{R}} Q,$$

in contradiction with (7-20). This finishes the proof of (1-35). □

Appendix: Proofs of auxiliary results

A.1. ***Sketch of the proof of Theorem 3.1.*** We follow [Martel 2005], to which we refer the reader for all the details omitted here.

Let $(T_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ be an increasing sequence with $T_n \ge \frac{1}{2}T_\varepsilon$ for all n and $\lim_{n\to+\infty} T_n = +\infty$. For notational simplicity we denote by $\tilde{T}_n$ the sequence $(1-\lambda)T_n$. Consider the solution $u_n(t)$ of the Cauchy problem

$$\begin{cases} (u_n)_t + \big((u_n)_{xx} - \lambda u_n + a_\varepsilon u_n^m\big)_x = 0 \quad \text{in } \mathbb{R}_t \times \mathbb{R}_x, \\ u_n(-T_n) = Q(\cdot - \tilde{T}_n). \end{cases} \tag{A-1}$$

Thus u_n is a solution of (aKdV) that at time $t = -T_n$ corresponds to the soliton $Q(\cdot - \tilde{T}_n)$. Clearly, $Q(\cdot - \tilde{T}_n) \in H^s(\mathbb{R})$ for every $s \ge 0$; moreover, there exists a uniform constant $C = C(s) > 0$ such that

$$\|Q(\cdot - \tilde{T}_n)\|_{H^s(\mathbb{R})} \le C.$$

According to Propositions 2.1 and 2.2, u_n is locally well defined in time, and global for positive times in $H^1(\mathbb{R})$. Let I_n be its maximal interval of existence.

The next step is to establish uniform estimates starting from a fixed time $t = -\frac{1}{2}T_\varepsilon < 0$ so negative that the soliton is not influenced by the perturbation in the potential. That is the content of this proposition, proved in the next section:

Proposition A.1.1 (Uniform estimates in H^s for large times). *There exist constants $K, \gamma > 0$ and $\varepsilon_0 > 0$ such that, for all $0 < \varepsilon < \varepsilon_0$ and for all $n \in \mathbb{N}$ we have $[-T_n, -\frac{1}{2}T_\varepsilon] \subseteq I_n$ (so $u_n \in C([-T_n, -\frac{1}{2}T_\varepsilon], H^s(\mathbb{R}))$) and, for all $t \in [-T_n, -\frac{1}{2}T_\varepsilon]$,*

$$\|u_n(t) - Q(\cdot - (1-\lambda)t)\|_{H^s(\mathbb{R})} \le K\varepsilon^{-1}e^{\gamma\varepsilon t}. \tag{A-2}$$

In particular, there exists a constant $C_s > 0$ such that, for all $t \in [-T_n, -\frac{1}{2}T_\varepsilon]$,

$$\|u_n(t)\|_{H^s(\mathbb{R})} \le C_s. \tag{A-3}$$

Using this result we will obtain the existence of a *critical element* $u_{0,*} \in H^s(\mathbb{R})$, with good compact properties, nondispersive, and uniformly close to the desired soliton.

Indeed, consider the sequence $(u_n(-\frac{1}{2}T_\varepsilon))_{n\in\mathbb{N}} \subseteq H^s(\mathbb{R})$. A standard argument shows that, given any $\delta > 0$, there exist $\varepsilon_0 > 0$ and $K_0 > 0$ such that

$$\int_{|x|>K_0} u_n^2(-\tfrac{1}{2}T_\varepsilon) < \delta \quad \text{for all } 0 < \varepsilon < \varepsilon_0 \text{ and } n \in \mathbb{N}. \tag{A-4}$$

To use this in the proof of Theorem 3.1, note first that (A-3) implies that

$$\|u_n(-T_\varepsilon/2)\|_{H^1(\mathbb{R})} \leq C_0,$$

independently of n. Thus, up to a subsequence we may suppose that $u_n(-\frac{1}{2}T_\varepsilon) \rightharpoonup u_{*,0}$ in the $H^1(\mathbb{R})$ weak sense, and $u_n(-\frac{1}{2}T_\varepsilon) \to u_{*,0}$ in $L^2_{loc}(\mathbb{R})$, as $n \to +\infty$. In addition, from (A-4) we have strong convergence in $L^2(\mathbb{R})$. From interpolation and the bound (A-3) we have strong convergence in $H^s(\mathbb{R})$ for any $s \geq 1$.

Let $u_* = u_*(t)$ be the solution of (1-1) with initial data $u_*(-\frac{1}{2}T_\varepsilon) = u_{*,0}$. From Proposition 2.1 we have $u_* \in C(I, H^s(\mathbb{R}))$, where $-\frac{1}{2}T_\varepsilon \in I$, the corresponding maximal interval of existence. Thus, using the continuous dependence of u_n and u_*, we obtain $u_n(t) \to u_*(t)$ in $H^s(\mathbb{R})$ for every $t \leq -\frac{1}{2}T_\varepsilon \subseteq I$. Passing to the limit in (A-2) we obtain, for all $t \leq -\frac{1}{2}T_\varepsilon$,

$$\|u_*(t) - Q(\cdot - (1-\lambda)t)\|_{H^s(\mathbb{R})} \leq K\varepsilon^{-1}e^{\varepsilon\gamma t},$$

as desired. This completes the proof of the existence part of Theorem 3.1, assuming Proposition A.1.1.

A.1.2. *Uniform H^1 estimates.* Next we outline the proof of Proposition A.1.1. We consider only the H^1 case. The first step in the proof is the following bootstrap property:

Proposition A.1.3 (Uniform estimates with and without decay assumption). *Let $m = 2, 3, 4$ and $0 \leq \lambda \leq \lambda_0 < 1$. There exist constants $K, \gamma, \varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ the following is true.*

(1) *Suppose $m = 3$, or $m = 2, 4$ with $\lambda > 0$. Then there exists $\alpha_0 > 0$ such that, for all $0 < \alpha < \alpha_0$, if for some $-T_{n,*} \in [-T_n, -\frac{1}{2}T_\varepsilon]$ and for all $t \in [-T_n, -T_{n,*}]$ we have*

$$\|u_n(t) - Q(\cdot - (1-\lambda)t)\|_{H^1(\mathbb{R})} \leq 2\alpha, \tag{A-5}$$

then, for all $t \in [-T_n, -T_{n,*}]$,

$$\|u_n(t) - Q(\cdot - (1-\lambda)t)\|_{H^1(\mathbb{R})} \leq K\varepsilon^{-1}e^{\varepsilon\gamma t}. \tag{A-6}$$

(2) *Now suppose $m = 2, 4$ and $\lambda = 0$. Then (A-6) holds if for some $-T_{n,*} \in [-T_n, -\frac{1}{2}T_\varepsilon]$ and for all $t \in [-T_n, -T_{n,*}]$ one has*

$$\|u_n(t) - Q(\cdot - (1-\lambda)t)\|_{H^1(\mathbb{R})} \leq 2K\varepsilon^{-1}e^{\varepsilon\gamma t}. \tag{A-7}$$

Proof of Proposition A.1.1, assuming the validity of Proposition A.1.3. We assume item (1) of the proposition. The case in which we assume item (2) is similar. From (A-1) we have

$$\|u_n(-T_n) - Q(-(1-\lambda)T_n)\|_{H^1(\mathbb{R})} = 0,$$

so there exists $t_0 = t_0(n, \alpha) > 0$ such that (A-5) holds true for all $t \in [-T_n, -T_n + t_0]$. Now consider (we adopt the convention $T_{*,n} > 0$)

$$-\tilde{T}_{*,n} := \sup\bigl\{t \in [-T_n, -\tfrac{1}{2}T_\varepsilon] : \|u_n(t') - Q(\cdot t - (1-\lambda)t')\|_{H^1(\mathbb{R})} \leq 2\alpha \text{ for all } t' \in [-T_n, t]\bigr\}.$$

Assume, for a contradiction, that $-\tilde{T}_{*,n} < -\frac{1}{2}T_\varepsilon$. From Proposition A.1.3, we have

$$\left\| u_n(t') - Q(\cdot t - (1-\lambda)t') \right\|_{H^1(\mathbb{R})} \le K\varepsilon^{-1}e^{\gamma\varepsilon t} \le \alpha,$$

for ε small enough (recall that $t \le -\frac{1}{2}T_\varepsilon = -\frac{1}{2(1-\lambda)}\varepsilon^{-1-\frac{1}{100}}$). This contradicts the definition of $\tilde{T}_{*,n}$. □

We turn to the proof of Proposition A.1.3. The first step is to decompose the solution preserving a standard orthogonality condition. To obtain this, we suppose (without loss of generality, by taking $T_{n,*}$ even larger) that, for all $t \in [-T_n, -T_{n,*}]$,

$$\left\| u_n(t) - Q(\cdot - (1-\lambda)t - r_n(t)) \right\|_{H^1(\mathbb{R})} \le 2\alpha, \tag{A-8}$$

for all smooth $r_n = r_n(t)$ satisfying $r_n(-T_n) = 0$ and $|r_n'(t)| \le 1/t^2$. A posteriori we will prove that this condition can be improved and extended to any time $t \in [-T_n, -\frac{1}{2}T_\varepsilon]$.

In what follows we drop the index n in $-T_{*,n}$ and u_n, if no confusion can arise.

Lemma A.1.4 (Modulation). *There exist $K, \gamma, \varepsilon_0 > 0$ and a unique C^1 function $\rho_0 : [-T_n, -T_*] \to \mathbb{R}$ such that for all $0 < \varepsilon < \varepsilon_0$ the function z defined by*

$$z(t,x) := u(t,x) - R(t,x); \quad R(t,x) := Q(x - (1-\lambda)t - \rho_0(t)) \tag{A-9}$$

satisfies, for all $t \in [-T_n, -T_*]$,

$$\int_{\mathbb{R}} z(t,x)R_x(t,x)\,dx = 0, \quad \|z(t)\|_{H^1(\mathbb{R})} \le K\alpha, \quad \rho_0(-T_n) = 0. \tag{A-10}$$

Moreover, z satisfies the modified gKdV equation

$$z_t + \big(z_{xx} - \lambda z + a_\varepsilon((R+z)^m - R^m) + (1-a_\varepsilon)R^m\big)_x - \rho_0'(t)R_x = 0, \tag{A-11}$$

and

$$|\rho_0'(t)| \le K\big(e^{\varepsilon\gamma t} + \|z(t)\|_{H^1(\mathbb{R})} + \|z(t)\|^2_{L^2(\mathbb{R})}\big). \tag{A-12}$$

Proof of Lemma A.1.4. The proof of (A-10) is a standard consequence of the implicit function theorem, the definition of T_* $(= T_{*,n})$, and the definition of $u_n(-T_n)$ given in (A-1). Similarly, the proof of (A-11) follows after a simple computation.

Now we deal with (A-12). Taking the time derivative of (A-9) and using (A-11), we get

$$\begin{aligned} 0 &= \int_{\mathbb{R}} z_t R_x - (1-\lambda+\rho_0')\int_{\mathbb{R}} zR_{xx} \\ &= \int_{\mathbb{R}} \big(z_{xx} - z + a_\varepsilon((R+z)^m - R^m) + (1-a_\varepsilon)R^m\big)R_{xx} + \rho_0' \int_{\mathbb{R}} R_x(R_x + z_x). \end{aligned}$$

Note that

$$\int_{\mathbb{R}} R_x(R_x + z_x) = \int_{\mathbb{R}} Q'^2 + O(\|z(t)\|_{L^2(\mathbb{R})}).$$

On the other hand, from (1-13), (A-10), the uniform bound on $\rho_0'(t)$ in the definition of T_*, and the exponential decay of R, we have

$$\left|\int_{\mathbb{R}}(1-a_\varepsilon)R^m R_{xx}\right| \le Ke^{\varepsilon\gamma t}. \tag{A-13}$$

Indeed, first note that from (A-8), by integrating between $-T_n$ and t and using (A-10) we get

$$\rho_0(t) \le -\frac{1}{T_n}-\frac{1}{t} \le \frac{2}{T_\varepsilon} \le K\varepsilon^{1+\frac{1}{100}}.$$

Thus $t+\rho_0(t) \le t + K\varepsilon^{1+\frac{1}{100}} \le \frac{9}{10}t$. Therefore, by possibly redefining γ, we have from (1-13)

$$\begin{aligned}\left|\int_{\mathbb{R}}(1-a_\varepsilon)R^m R_{xx}\right| &\le K\int_{-\infty}^0 e^{\gamma\varepsilon x}e^{-(m+1)|x-(t+\rho_0(t))|}dx + Ke^{(m+1)(t+\rho_0(t))}\int_0^\infty e^{-(m+1)x}dx\\ &\le K\exp\bigl(\gamma\varepsilon(t+\rho_0(t))\bigr)+K\exp\bigl(\gamma(m+1)(t+\rho_0(t))\bigr)\le Ke^{\gamma\varepsilon t}.\end{aligned}$$

Finally,

$$\int_{\mathbb{R}} R_{xx}\bigl(z_{xx}-z+a_\varepsilon((R+z)^m-R^m)\bigr) = O\bigl(\|z(t)\|_{L^2(\mathbb{R})}+\|z(t)\|^2_{L^2(\mathbb{R})}\bigr).$$

Collecting the estimates above we obtain (A-12). □

A.1.5. *Almost conservation of mass and energy.* Recall that from the remark on page 586 that the modified mass defined in (2-8) satisfies

$$\tilde{M}[u](t) \le \tilde{M}[u](-T_n). \tag{A-14}$$

for all $-T_n \le t \le -\frac{1}{2}T_\varepsilon$. Moreover, in the case $m=2,4$ and $\lambda=0$, since (1-20) and (A-7) hold, there exist $K,\gamma>0$ such that

$$M[u](t) \le M[u](-T_n) + K\varepsilon e^{\gamma\varepsilon t}, \tag{A-15}$$

for ε small enough. By extending the definition of $\tilde{M}[u]$ to the latter case, we have almost conservation of mass, with exponential loss for all cases.

Similarly, in the region considered the soliton $R(t)$ is an almost solution of (1-15); in particular it must conserve mass $\tilde{M}$ (2-8) and the energy E_a (1-21), at least for large negative time. Indeed, an argument as in Lemma 6.3 (but easier) gives

$$E_a[R](-T_n)-E_a[R](t)+(1-\lambda)\bigl[\tilde{M}[R](-T_n)-\tilde{M}[R](t)\bigr] \le Ke^{\gamma\varepsilon t}. \tag{A-16}$$

for some constant $K>0$ and all time $t\in[-T_n,T_*]$

The next step is the use conservation of energy to provide control in the $R(t)$ direction (which is essential in order to obtain certain coercivity properties; see Lemma 2.3). Following Lemma 5.4, one has

$$\left|\int_{\mathbb{R}} Rz(t)\right| \le \frac{K}{1-\lambda}\bigl(e^{\gamma\varepsilon t}+\|z(t)\|^2_{L^2(\mathbb{R})}+e^{\gamma\varepsilon t}\|z(t)\|_{L^2(\mathbb{R})}\bigr). \tag{A-17}$$

for constants $K,\gamma>0$ independent of ε.

Now, consider the energy $E_a[u]$ and the mass $\tilde{M}[u]$ defined in (1-21) and (2-8). One has

$$E_a[u](t) + (1-\lambda)\tilde{M}[u](t) = E_a[R](t) + (1-\lambda)\tilde{M}[R](t) - \int_{\mathbb{R}} z(a_\varepsilon - 1)R^m + \mathcal{F}_0(t), \tag{A-18}$$

where $\mathcal{F}_0$ is the quadratic functional

$$\mathcal{F}_0(t) := \frac{1}{2}\int_{\mathbb{R}}(z_x^2 + \lambda z^2) + (1-\lambda)\tilde{M}[z] - \frac{1}{m+1}\int_{\mathbb{R}} a_\varepsilon\big((R+z)^{m+1} - R^{m+1} - (m+1)R^m z\big).$$

In addition, for any $t \in [-T_n, -T_*]$,

$$\left|\int_{\mathbb{R}} z(a_\varepsilon - 1)R^m\right| \le Ke^{\gamma\varepsilon t}\|z(t)\|_{L^2(\mathbb{R})}. \tag{A-19}$$

The proof of (A-18) is essentially an expansion of the energy-mass functional using the relation $u(t) = R(t) + z(t)$. The proof of (A-19) is similar to (A-13).

The functional $\mathcal{F}_0(t)$ just defined enjoys the following coercivity property: there exist $K, \lambda_0 > 0$ independent of ε such that for every $t \in [-T_n, -T_*]$

$$\mathcal{F}_0(t) \ge \lambda_0\|z(t)\|_{H^1(\mathbb{R})}^2 - \left|\int_{\mathbb{R}} R(t)z(t)\right|^2 - Ke^{\gamma\varepsilon t}\|z(t)\|_{L^2(\mathbb{R})}^2 - K\|z(t)\|_{L^2(\mathbb{R})}^3. \tag{A-20}$$

This bound is simply a consequence of the inequality $\lambda + (1-\lambda)a_\varepsilon^{1/m}(x) \ge 1$, (A-10) and Lemma 2.3.

A.1.6. *End of proof of Proposition A.1.3.* Now by using (A-18), (A-20), and the estimates (A-14), (A-15), and (A-17) we finally get (A-6). Indeed, note that

$$E_a[u](t) - E_a[u](-T_n) + (1-\lambda)[\tilde{M}[u](t) - \tilde{M}[u](-T_n)] \le Ke^{\varepsilon\gamma t}.$$

On the other hand, from (A-18) and (A-10),

$$\begin{aligned} E_a[u](t) - E_a[u](-T_n) + (1-\lambda)[\tilde{M}[u](t) - \tilde{M}[u](-T_n)]& \\ \ge \mathcal{F}_0(t) - Ke^{\gamma\varepsilon t} - Ke^{\gamma\varepsilon t}\|z(t)\|_{L^2(\mathbb{R})}&, \end{aligned}$$

since $z(-T_n) = 0$ and $\mathcal{F}_0(-T_n) = 0$. Finally, from (A-20) and (A-17) we get

$$\|z(t)\|_{H^1(\mathbb{R})} \le Ke^{\gamma\varepsilon t}.$$

Plugging this estimate into (A-12), we obtain that $|\rho_0'(t)| \le Ke^{\gamma\varepsilon t}$, and thus after integration we get the final uniform estimate (A-6) for the H^1-case. Note that we have also improved the estimate on $\rho_0'(t)$ assumed in (A-8). □

We now address the uniqueness part of Theorem 3.1. Recall that the solution u constructed above is in $C(\mathbb{R}, H^s(\mathbb{R}))$ for any $s \ge 1$, and satisfies the exponential decay condition (3-2). Moreover, every solution converging to a soliton satisfies this property:

Proposition A.1.7 (Exponential decay). *Let $m = 3$, or $m = 2, 4$ with $0 < \lambda \le \lambda_0$. Let $v = v(t)$ be a $C(\mathbb{R}, H^1(\mathbb{R}))$ solution of (1-1) satisfying*

$$\lim_{t \to -\infty}\|v(t) - Q(\cdot - (1-\lambda)t)\|_{H^1(\mathbb{R})} = 0.$$

Then there exist $K, \gamma, \varepsilon_0 > 0$ such that for every $t \le -T_\varepsilon$ we have

$$\|v(t) - Q(\cdot - (1-\lambda)t)\|_{H^1(\mathbb{R})} \le K\varepsilon^{-1}e^{\gamma\varepsilon t}.$$

Proof. Fix $\alpha > 0$ small. Let $\varepsilon_0 = \varepsilon_0(\alpha) > 0$ be small enough such that for all $\varepsilon \le \varepsilon_0$ and $t \le -T_\varepsilon$

$$\|v(t) - Q(\cdot - (1-\lambda)t)\|_{H^1(\mathbb{R})} \le \alpha.$$

Possibly choosing ε_0 even smaller, we can apply earlier arguments to the function $v(t)$ on the interval $(-\infty, -\frac{1}{2}T_\varepsilon]$ to obtain the desired result. We follow part (1) of Proposition A.1.3: Lemma A.1.4 holds for $z(t) := v(t) - Q(\cdot - (1-\lambda)t - \rho_0(t))$ and $t \le -\frac{1}{2}T_\varepsilon$, but now we have, by hypothesis,

$$\lim_{t\to-\infty} |\rho_0(t)| + \|z(t)\|_{H^1(\mathbb{R})} = 0;$$

and therefore $\lim_{t\to-\infty} \mathscr{F}_0(t) = 0$. (This can be made rigorous by taking a sequence $t_n \to -\infty$ large enough and such that $\|v(t_n) - Q(\cdot - (1-\lambda)t_n)\|_{H^1(\mathbb{R})} \le \frac{1}{n}$. With this choice one has $|\rho_{0,n}(t_n)| + \|z_n(t_n)\|_{H^1(\mathbb{R})} \to 0$, independently of ε. Rerunning as usual the proof in the interval $[t_n, t]$ and finally taking the limit $n \to +\infty$, we obtain the conclusion.) The rest of the proof is easy. □

Note that monotonicity of mass was a key ingredient in this proof. This property apparently does not hold when $\lambda = 0$ and $m = 2, 4$.

A.1.8. *Uniqueness of the solution.* Let $w(t) := v(t) - u(t)$. Then $w(t) \in H^1(\mathbb{R})$ and satisfies the equation

$$\begin{cases} w_t + (w_{xx} - \lambda w + a_\varepsilon[(u+w)^m - u^m])_x = 0 & \text{in } \mathbb{R}_t \times \mathbb{R}_x, \\ \|w(t)\|_{H^1(\mathbb{R})} \le K\varepsilon^{-1}e^{\gamma\varepsilon t} & \text{for all } t \le -\frac{1}{2}T_\varepsilon. \end{cases} \tag{A-21}$$

We must show that $w(t) \equiv 0$. Define the second-order functional

$$\mathscr{F}_0(t) := \frac{1}{2}\int_{\mathbb{R}} w_x^2 + \frac{1}{2}\int_{\mathbb{R}} w^2 - \frac{1}{m+1}\int_{\mathbb{R}} a_\varepsilon(x)[(u+w)^{m+1} - u^{m+1} - (m+1)u^m w].$$

Reasoning as in the proof of Lemma 5.6, it is easy to verify the following properties:

(1) Lower bound. There exists $K > 0$ such that for all $t \le -\frac{1}{2}T_\varepsilon$,

$$\mathscr{F}_0(t) \ge \frac{1}{2}\int_{\mathbb{R}} (w_x^2 + w^2 - mQ^{m-1}w^2)(t) - K\varepsilon^{-1}e^{\gamma\varepsilon t} \sup_{t'\le t} \|w(t')\|_{H^1(\mathbb{R})}^2.$$

(2) Upper bound. There exists $K, \gamma > 0$ such that

$$\mathscr{F}_0(t) \le K\varepsilon^{-2}e^{\gamma\varepsilon t} \sup_{t'\le t} \|w(t')\|_{H^1(\mathbb{R})}^2.$$

However, this functional is not coercive, so in order to obtain a satisfactory lower bound, one has to modify the function w in $(-\infty, -\frac{1}{2}T_\varepsilon]$ by setting

$$\tilde{w}(t) := w(t) + b(t)Q'(\cdot - t), \quad b(t) := \frac{\int_{\mathbb{R}} w(t)Q'(\cdot - t)}{\int_{\mathbb{R}} Q'^2},$$

This modified function enjoys several properties:

(1) Orthogonality to the Q' direction: $\displaystyle\int_{\mathbb{R}} \tilde{w}(t)Q'(\cdot - t) = 0.$

(2) Equivalence. There exists $C_1, C_2 > 0$ independent of ε such that

$$C_1\|w(t)\|_{H^1(\mathbb{R})} \le \|\tilde{w}(t)\|_{H^1(\mathbb{R})} + |b(t)| \le C_2\|w(t)\|_{H^1(\mathbb{R})}.$$

Moreover,

$$\frac{1}{2}\int_{\mathbb{R}}(w_x^2 + w^2 - mQ^{m-1}w^2)(t) = \frac{1}{2}\int_{\mathbb{R}}(\tilde{w}_x^2 + \tilde{w}^2 - mQ^{m-1}\tilde{w}^2)(t) + O(e^{-\varepsilon\gamma|t|}).$$

(3) Control in the Q direction:

$$\left|\int_{\mathbb{R}} \tilde{w}(t)Q(\cdot - t)\right| \le K\varepsilon^{-1}e^{\varepsilon\gamma t}\sup_{t'\le t}\|w(t')\|_{H^1(\mathbb{R})}.$$

This property is proved similarly to the proof of (6-15): We use the fact that variation in time of the above quantity is of quadratic order on $\tilde{w}$.

(4) Coercivity. There exists $\lambda > 0$ independent of t such that

$$\frac{1}{2}\int_{\mathbb{R}}(\tilde{w}_x^2 + \tilde{w}^2 - mQ^{m-1}\tilde{w}^2)(t) \ge \lambda\|\tilde{w}(t)\|^2_{H^1(\mathbb{R})} - K\left|\int_{\mathbb{R}} \tilde{w}(t)Q(\cdot - t)\right|^2.$$

(5) Sharp control. From the equivalence between w and $\tilde{w}$ and the coercivity property we obtain

$$\|\tilde{w}(t)\|_{H^1(\mathbb{R})} + \varepsilon|b(t)| \le K\varepsilon^{-2}e^{\varepsilon\gamma t/2}\sup_{t'\le t}\|w(t')\|_{H^1(\mathbb{R})}. \tag{A-22}$$

The bound on $b(t)$ is proved similarly to (6-14).

Finally, from (A-22) we obtain, for ε small enough and $t \le -\frac{1}{2}T_\varepsilon$,

$$\|w(t)\|_{H^1(\mathbb{R})} \le K\varepsilon^{-2}e^{\varepsilon\gamma t}\sup_{t'\le t}\|w(t')\|_{H^1(\mathbb{R})} < \tfrac{1}{2}\sup_{t'\le t}\|w(t')\|_{H^1(\mathbb{R})}.$$

This implies $w \equiv 0$, proving uniqueness. □

A.2. *Proof of Proposition 4.2.* The proof is similar to that of [Martel and Merle 2009, Proposition 2.2] (see also and [Martel and Merle 2007, Appendix]). We start by writing the error term $S[\tilde{u}]$ of (4-9) as

$$S[\tilde{u}] = I + II + III, \tag{A-23}$$

with

$$I := S[R], \tag{A-24}$$

$$II := w_t + (w_{xx} - \lambda w + m\, a_\varepsilon R^{m-1}w)_x, \tag{A-25}$$

$$III := \big(a_\varepsilon((R+w)^m - R^m - mR^{m-1}w)\big)_x. \tag{A-26}$$

Recall that $m = 2, 3, 4$.

Lemma A.2.1. *We have*

$$I = \varepsilon F_1(\varepsilon t; y) + \frac{\varepsilon^2 a''}{2\tilde{a}^m}(y^2 Q_c^m)_y + \varepsilon^3 f_I(\varepsilon t)F_c^I(y), \tag{A-27}$$

where

$$F_1(\varepsilon t;y) := \frac{c'}{\tilde a}\Lambda Q_c - \frac{\tilde a'}{\tilde a^2}(c-\lambda)Q_c + \frac{a'}{\tilde a^m}(yQ_c^m)_y \in \mathcal{Y},$$

and $|f_I(\varepsilon t)| \le K$, $F_c^I \in \mathcal{Y}$. *Finally, for every* $t\in[-T_\varepsilon,T_\varepsilon]$,

$$\|\varepsilon^3 f_I(\varepsilon t)F_c^I(y)\|_{H^2(\mathbb{R})} \le K\varepsilon^3. \tag{A-28}$$

Proof of Lemma A.2.1. Recall that $\tilde a := a^{\frac{1}{m-1}}$ and

$$R(t,x) = \frac{Q_{c(\varepsilon t)}(y)}{\tilde a(\varepsilon\rho(t))},\qquad y = x-\rho(t),\qquad \partial_t\rho(t) = c(\varepsilon t)-\lambda.$$

Thus

$$I = R_t + (R_{xx}-\lambda R + a_\varepsilon R^m)_x = \frac{\varepsilon c'}{\tilde a}\Lambda Q_c - \frac{c-\lambda}{\tilde a}Q_c' - \varepsilon\frac{\tilde a'(c-\lambda)}{\tilde a^2}Q_c + \frac{1}{\tilde a}Q_c^{(3)} - \frac{\lambda}{\tilde a}Q_c' + \frac{1}{\tilde a^m}(a(\varepsilon x)Q_c^m)_x.$$

A Taylor expansion gives

$$(a(\varepsilon x)Q_c^m)_x = a(\varepsilon\rho)(Q_c^m)_x + \varepsilon a'(\varepsilon\rho)(yQ_c^m)_x + \tfrac12\varepsilon^2 a''(\varepsilon\rho)(y^2Q_c^m)_x + O_{H^2(\mathbb{R})}(\varepsilon^3).$$

Therefore,

$$\begin{aligned}
I &= \frac{\varepsilon c'}{\tilde a}\Lambda Q_c - \frac{(c-\lambda)}{\tilde a}Q_c' - \frac{\varepsilon}{m-1}\frac{a'(c-\lambda)}{\tilde a^m}Q_c + \frac{1}{\tilde a}Q_c^{(3)} - \frac{\lambda}{\tilde a}Q_c' + \frac{1}{\tilde a}(Q_c^m)' + \frac{\varepsilon a'}{\tilde a^m}(yQ_c^m)_x \\
&\qquad + \frac{\varepsilon^2 a''}{2\tilde a^m}(y^2Q_c^m)_x + \varepsilon^3 f_I(\varepsilon t)F_c^I(y) \\
&= \frac{1}{\tilde a}(Q_c''-cQ_c+Q_c^m)' + \frac{\varepsilon c'}{\tilde a}\Lambda Q_c - \varepsilon\frac{\tilde a'}{\tilde a^2}(c-\lambda)Q_c + \frac{\varepsilon a'}{\tilde a^m}(yQ_c^m)_y + \frac{\varepsilon^2 a''}{2\tilde a^m}(y^2Q_c^m)_y + \varepsilon^3 f_I(\varepsilon t)F_c^I(y) \\
&= \varepsilon\Big(\frac{c'}{\tilde a}\Lambda Q_c - \frac{\tilde a'}{\tilde a^2}(c-\lambda)Q_c + \frac{a'}{\tilde a^m}(yQ_c^m)_y\Big) + \frac{\varepsilon^2 a''}{2\tilde a^m}(y^2Q_c^m)_y + \varepsilon^3 f_I(\varepsilon t)F_c^I(y).
\end{aligned}$$

Moreover $|f_I(\varepsilon t)| \le K$, $F_c^I(y)\in\mathcal{Y}$, and (A-28) is satisfied. □

Lemma A.2.2. *The quantity* II *is given by*

$$-\varepsilon(\mathcal{L}A_c)_y(\varepsilon t;y) + \varepsilon^2\big((A_c)_t + c'(\varepsilon t)\Lambda A_c\big)(\varepsilon t;y) + m\varepsilon^2\frac{a'(\varepsilon\rho)}{a(\varepsilon\rho)}\big(yQ_c^{m-1}(y)A_c(\varepsilon t;y)\big)_y + \varepsilon^3 F_c^{II}(\varepsilon t;y).$$

with $F_c^{II}(\varepsilon t;\cdot)\in\mathcal{Y}$, *uniformly in time. If, in addition, Property IP holds for* A_c, *then*

$$\|\varepsilon^3 F_c^{II}(\varepsilon t;y)\|_{H^2(\mathbb{R})} \le K\varepsilon^3 e^{-\gamma\varepsilon|t|}. \tag{A-29}$$

Proof. We compute

$$\begin{aligned}
II &= \varepsilon(A_c(\varepsilon t;y))_t + \varepsilon\Big[(A_c)_{yy}(\varepsilon t;y) - \lambda A_c(\varepsilon t;y) + \frac{a_\varepsilon}{a(\varepsilon\rho)}mQ_c^{m-1}(y)A_c(\varepsilon t;y)\Big]_x \\
&= -\varepsilon(\mathcal{L}A_c)_y(\varepsilon t;y) + \varepsilon^2(A_c)_t(\varepsilon t;y) + \varepsilon^2 c'(\varepsilon t)\Lambda A_c(\varepsilon t,y) \\
&\qquad + m\varepsilon^2\frac{a'(\varepsilon\rho)}{a(\varepsilon\rho)}(yQ_c^{m-1}(y)A_c(\varepsilon t;y))_y + \varepsilon^3 F_c^{II}(\varepsilon t;y),
\end{aligned}$$

where $F_c^{II}(\varepsilon t;y) = O\big(y^2Q_c^{m-1}(y)A_c(\varepsilon t;y)\big)_y \in\mathcal{Y}$. Now (A-29) follows from Property IP. □

Lemma A.2.3. *Suppose Property IP holds for A_c. Then we have*

$$III = \varepsilon^3 a'(\varepsilon x)[\varepsilon^{m-2} A_c^m(\varepsilon t; y) + \tilde{F}_c^{III}(\varepsilon t; y)] + \varepsilon^2 a_\varepsilon G_c^{III}(\varepsilon t; y),$$

with $\tilde{F}_c^{III}(\varepsilon t; \cdot), G_c^{III}(\varepsilon t; \cdot) \in \mathcal{Y}$, uniformly for every $t \in [-T_\varepsilon, T_\varepsilon]$. Moreover, we have the estimate

$$\|III\|_{H^2(\mathbb{R})} \leq K\varepsilon^2 e^{-\gamma\varepsilon|t|}, \tag{A-30}$$

for every $t \in [-T_\varepsilon, T_\varepsilon]$.

Proof. Define $\widehat{III} := a_\varepsilon\big((R+w)^m - R^m - mR^{m-1}w\big)$.

Suppose first that $m = 2$. Then $\widehat{III} = a_\varepsilon w^2 = \varepsilon^2 a_\varepsilon A_c^2$, and taking the derivative, $III = \varepsilon^3 a'(\varepsilon x) A_c^2 + \varepsilon^2 a_\varepsilon (A_c^2)'$. Here $(A_c^2)' \in \mathcal{Y}$ because of Property IP.

Now suppose $m = 3$. We have $\widehat{III} = \varepsilon^2 a_\varepsilon\big(3Q_c A_c^2 + \varepsilon A_c^3\big)$. From this we get

$$III = \varepsilon^3 a'(\varepsilon x)\big(3Q_c A_c^2 + \varepsilon A_c^3\big) + \varepsilon^2 a_\varepsilon\big(3(Q_c A_c^2)' + \varepsilon (A_c^3)'\big).$$

Finally, for the case $m = 4$,

$$\begin{aligned} III &= \big(a_\varepsilon \varepsilon^2 (6Q_c^2 A_c^2 + 4\varepsilon Q_c A_c^3 + \varepsilon^2 A_c^4)\big)_x \\ &= \varepsilon^3 a'(\varepsilon x)(6Q_c^2 A_c^2 + 4\varepsilon^2 Q_c A_c^3 + \varepsilon^2 A_c^4) + \varepsilon^2 a_\varepsilon\big(6(Q_c^2 A_c^2)' + 4\varepsilon (Q_c A_c^3)' + \varepsilon^2 (A_c^4)'\big). \end{aligned}$$

Thus (A-30) holds in each case, assuming Property IP. □

Now we collect the estimates from Lemmas A.2.1, A.2.2 and A.2.3. We finally get

$$S[\tilde{u}] = I + II + III = \varepsilon\big(F_1 - (\mathcal{L}A_c)_y\big)(\varepsilon t; y) + \varepsilon^2\big((A_c)_t + c'(\varepsilon t)\Lambda A_c\big)(\varepsilon t; y) + O(\varepsilon^2 e^{-\gamma\varepsilon|t|}),$$

provided Property IP holds for A_c.

A.3. ***End of proof of Proposition 4.7.*** In this section we will show that for all $t \in [-T_\varepsilon, T_\varepsilon]$ (cf. (4-29))

$$\|S[\tilde{u}](t)\|_{H^2(\mathbb{R})} \leq K\varepsilon^{\frac{3}{2}} e^{-\gamma\varepsilon|t|}, \tag{A-31}$$

where $\tilde{u}$ is the modified approximate solution defined in (4-26). We do this by writing a decomposition

$$S[\tilde{u}] = I + \widetilde{II} + \widetilde{III},$$

similar to that in Section A.2 (see (A-23)–(A-26)). Lemma A.2.1 applies, so the term I is given by (A-27) with no change. The term $\widetilde{III}$ can be written as

$$\widetilde{III} = \varepsilon^3 a'(\varepsilon x)\big(\varepsilon^{m-2}\eta_c^m A_c^m(\varepsilon t; y) + \tilde{F}_c^{III}(\varepsilon t; y)\big) + \varepsilon^2 a_\varepsilon\big(G_c^{III}(\varepsilon t; y) + \varepsilon^{m-1}(\eta_c^m)' A_c^m\big),$$

with $\tilde{F}_c^{III}(\varepsilon t; \cdot), G_c^{III}(\varepsilon t; \cdot) \in \mathcal{Y}$, uniformly for every $t \in [-T_\varepsilon, T_\varepsilon]$. This is proved exactly like Lemma A.2.3, the only novelty being the appearance of the term

$$\varepsilon^{m+1} a_\varepsilon (\eta_c^m)' A_c^m, \quad \text{with} \quad \|\varepsilon^{m+1} a_\varepsilon (\eta_c^m)' A_c^m\|_{H^2(\mathbb{R})} \leq K\varepsilon^{m+\frac{1}{2}} e^{-\gamma\varepsilon|t|}.$$

We thus get the estimate

$$\|\widetilde{III}\|_{H^2(\mathbb{R})} \leq K\varepsilon^2 e^{-\gamma\varepsilon|t|} \quad \text{for all } t \in [-T_\varepsilon, T_\varepsilon]. \tag{A-32}$$

Finally, for the term $\widetilde{II}$, we show that

$$\widetilde{II} = -\varepsilon\eta_c(y)(\mathscr{L}A_c)_y(\varepsilon t; y) + O_{H^2(\mathbb{R})}(\varepsilon^{\frac{3}{2}} e^{-\gamma\varepsilon|t|}). \tag{A-33}$$

This is done along the lines of the proof of Lemma A.2.2, as follows. We have

$$\begin{aligned}&(\varepsilon A_\#(\varepsilon t; y))_t \\ &\quad = -(c-\lambda)\varepsilon^2\eta_\varepsilon' A_c(\varepsilon t; y) - (c-\lambda)\varepsilon\eta_\varepsilon (A_c)_y(\varepsilon t; y) + \varepsilon^2\eta_\varepsilon (A_c)_t(\varepsilon t; y) + \varepsilon^2 c'(\varepsilon t)\eta_\varepsilon \Lambda A_c(\varepsilon t; y).\end{aligned}$$

We use Lemma 4.5 and (4-28) to estimate this expression, obtaining

$$(\varepsilon A_\#(\varepsilon t; y))_t = -(c-\lambda)\varepsilon\eta_\varepsilon(y)(A_c)_y(\varepsilon t; y) + O_{H^2(\mathbb{R})}(\varepsilon^{\frac{3}{2}} e^{-\gamma\varepsilon|t|}). \tag{A-34}$$

On the other hand,

$$\begin{aligned}
&\varepsilon\Big((A_\#)_{xx} - \lambda A_\# + \frac{a_\varepsilon}{a(\varepsilon\rho)} m Q_c^{m-1}(y) A_\#\Big)_x \\
&\quad = \varepsilon\Big(\eta_\varepsilon\Big((A_c)_{yy} - \lambda A_c + \frac{a_\varepsilon}{a(\varepsilon\rho)} m Q_c^{m-1}(y) A_c\Big) + 2\varepsilon\eta_\varepsilon'(A_c)_y + \varepsilon^2\eta_\varepsilon'' A_c\Big)_x \\
&\quad = \varepsilon\eta_\varepsilon\Big((A_c)_{yy} - \lambda A_c + \frac{a_\varepsilon}{a(\varepsilon\rho)} m Q_c^{m-1}(y) A_c\Big)_x \\
&\qquad\qquad + \varepsilon^2\big(3\eta_\varepsilon'(A_c)_{yy} - \lambda\eta_\varepsilon' A_c + a_\varepsilon m\eta_\varepsilon' Q_c^{m-1} A_c + 3\varepsilon\eta_\varepsilon''(A_c)_y + \varepsilon^2\eta_\varepsilon^{(3)} A_c\big) \\
&\quad = \varepsilon\eta_\varepsilon\big((A_c)_{yy} - \lambda A_c + m Q_c^{m-1}(y) A_c\big)_y + \varepsilon^2\eta_\varepsilon m\frac{a'(\varepsilon\rho)}{a(\varepsilon\rho)}(y Q_c^{m-1} A_c)_y \\
&\qquad + \varepsilon^2\big(3\eta_\varepsilon'(A_c)_{yy} - \lambda\eta_\varepsilon' A_c + a_\varepsilon m\eta_\varepsilon' Q_c^{m-1} A_c + 3\varepsilon\eta_\varepsilon''(A_c)_y + \varepsilon^2\eta_\varepsilon^{(3)} A_c\big) + O\big(\varepsilon^3\eta_\varepsilon(y^2 Q_c^{m-1} A_c)_y\big).
\end{aligned}$$

We now use Lemma 4.5 and Property IP to get the estimates

$$\begin{aligned}
m\varepsilon^2\left|\frac{a'(\varepsilon\rho)}{a(\varepsilon\rho)}\right| \big\|\eta_\varepsilon(y Q_c^{m-1} A_c)_y\big\|_{H^2(\mathbb{R})} &\le K\varepsilon^2 e^{-\gamma\varepsilon|t|}, \\
\big\|O(\varepsilon^3\eta_\varepsilon(y^2 Q_c^{m-1} A_c)_y)\big\|_{H^2(\mathbb{R})} &\le K\varepsilon^3, \\
\varepsilon^4\big\|\eta_c^{(3)} A_c\big\|_{H^2(\mathbb{R})} &\le \varepsilon^{\frac{7}{2}} e^{-\gamma\varepsilon|t|}, \\
\big\|\varepsilon^2\lambda\eta_\varepsilon' A_c\big\|_{H^2(\mathbb{R})} &\le K\lambda\varepsilon^{\frac{3}{2}} e^{-\gamma\varepsilon|t|}, \\
\varepsilon^2\big\|3\eta_\varepsilon'(A_c)_{yy} + a_\varepsilon m\eta_\varepsilon' Q_c^{m-1} A_c + 3\varepsilon\eta_\varepsilon''(A_c)_y\big\|_{H^2(\mathbb{R})} &\le K\varepsilon^2 e^{-\gamma\varepsilon|t|}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\varepsilon\Big((A_\#)_{xx} - \lambda A_\# + \frac{a_\varepsilon}{a(\varepsilon\rho)} m Q_c^{m-1}(y) A_\#\Big)_x \\
&\qquad = \varepsilon\eta_\varepsilon\big((A_c)_{yy} - \lambda A_c + m Q_c^{m-1}(y) A_c\big)_y + O_{H^2(\mathbb{R})}(\varepsilon^2 e^{-\gamma\varepsilon|t|} + \varepsilon^3).
\end{aligned} \tag{A-35}$$

Now (A-33) follows from (A-34) and (A-35).

We return to the global estimate on $S[\tilde{u}]$. From (A-27), (A-32), (A-33), and Lemma 4.5 we get

$$\begin{aligned} S[\tilde{u}] &= \varepsilon\big(F_1(\varepsilon t, y) - \eta_c(y)(\mathcal{L}A_c)_y\big)(\varepsilon t, y) + O_{H^2(\mathbb{R})}(\varepsilon^{\frac{3}{2}} e^{-\gamma\varepsilon|t|}) \\ &= \varepsilon(1-\eta_c(y))F_1(\varepsilon t; y) + O_{H^2(\mathbb{R})}(\varepsilon^{\frac{3}{2}} e^{-\gamma\varepsilon|t|}). \end{aligned}$$

The final conclusion of this appendix is a straightforward consequence of the following fact: For every $t \in [-T_\varepsilon, T_\varepsilon]$ we have

$$\|\varepsilon(1-\eta_c(y))F_1(\varepsilon t; y)\|_{H^2(\mathbb{R})} \le K\varepsilon e^{-\frac{1}{\varepsilon}-\gamma\varepsilon|t|} \ll K\varepsilon^{10}.$$

for ε small enough. Indeed, note that $\operatorname{supp}(1-\eta_c(\cdot)) \subseteq (-\infty, -\frac{1}{\varepsilon}]$. From (A-27),

$$|F_1(\varepsilon t; y)| \le K e^{-\gamma|y|-\gamma\varepsilon|t|}.$$

Now the desired estimate follows directly.

A.4. ***Proof of Lemma 6.4.*** Our proof of the virial inequality (6-26) follows closely that of [Martel and Merle 2005, Lemma 2]. Take $t \in [t_1, T^*]$ and set $y := x - \rho_2(t)$. We have

$$\partial_t \int_{\mathbb{R}} z^2 \psi_{A_0}(y) = 2\int_{\mathbb{R}} z z_t \psi_{A_0}(y) - \rho_2'(t)\int_{\mathbb{R}} z^2 \psi_{A_0}'(y).$$

Substituting the value of z_t given by (6-13), we can express the right-hand side as a sum of terms:

$$2\int_{\mathbb{R}} (z\psi_{A_0}(y))_x (z_{xx} - \lambda z + mQ_{c_2}^{m-1}(y)z) \tag{A-36}$$

$$-(c_2(t)-\lambda)\int_{\mathbb{R}} z^2\psi_{A_0}'(y) - 2(c_2(t)-\lambda-\rho_2')(t)\int_{\mathbb{R}} zQ_{c_2}'\psi_{A_0}(y) \tag{A-37}$$

$$2\int_{\mathbb{R}} (z\psi_{A_0}(y))_x[(R+z)^m - R^m - mR^{m-1}z] \tag{A-38}$$

$$-2c_2'(t)\int_{\mathbb{R}} z\Lambda Q_{c_2}\psi_{A_0}(y) + (c_2-\lambda-\rho_2')(t)\int_{\mathbb{R}} z^2\psi_{A_0}'(y) \tag{A-39}$$

$$\int_{\mathbb{R}} (z\psi_{A_0}(y))_x(a_\varepsilon - 2)(R+z)^m. \tag{A-40}$$

Following [Martel and Merle 2005] and using (6-14) and (6-15) it is easy to check that, for A_0 large enough and for some constants δ_0, ε_0 small,

$$|\text{(A-38)} + \text{(A-39)}| \le \frac{\delta_0}{100}\int_{\mathbb{R}} (z_x^2 + z^2)(t) e^{-\frac{1}{A_0}|y|}.$$

Estimating (A-36) and (A-37) is done as for B_1 and B_2 in [Martel and Merle 2005, Appendix B]. We get

$$\text{(A-36)} + \text{(A-37)} \le -\frac{\delta_0}{10}\int_{\mathbb{R}} (z_x^2 + z^2)(t) e^{-\frac{1}{A_0}|y|}.$$

Finally, (A-40) can be estimated as follows. From (6-11) and (6-12) we have for $t \ge t_1$

$$c_2(t) = c_\infty + O(\varepsilon^{1/2}), \quad \rho_2(t) = (c_\infty - \lambda)t + O(\varepsilon^{1/2}(t-t_1)),$$

and then

$$\tfrac{9}{10}c_\infty \le c_2(t) \le \tfrac{11}{10}c_\infty; \quad \rho_2(t) \ge \tfrac{9}{10}(c_\infty - \lambda)t. \tag{A-41}$$

On the other hand, we can write

$$\begin{aligned}\text{(A-40)} &= \int_{\mathbb{R}} (z\psi_{A_0})_x(a_\varepsilon - 2)[(R+z)^m - z^m] + \int_{\mathbb{R}} (z\psi_{A_0})_x(a_\varepsilon - 2)z^m \\ &= \int_{\mathbb{R}} (\psi_{A_0})_x(a_\varepsilon - 2)[(R+z)^m - z^m]z + \int_{\mathbb{R}} \psi_{A_0}(a_\varepsilon - 2)[(R+z)^m - z^m]z_x \\ &\qquad + \frac{m}{m+1}\int_{\mathbb{R}} (\psi_{A_0})_x(a_\varepsilon - 2)z^{m+1} - \frac{\varepsilon}{m+1}\int_{\mathbb{R}} \psi_{A_0} a'(\varepsilon x)z^{m+1}.\end{aligned}$$

Then, from (1-13), (6-25) and by using that $t \ge t_1 \ge \frac{1}{2}T_\varepsilon$, we get

$$\left|\int_{\mathbb{R}} (\psi_{A_0})_x(a_\varepsilon - 2)[(R+z)^m - z^m]z\right| \le KA_0 e^{-\varepsilon\rho_2(t)/A_0}\|z(t)\|_{H^1(\mathbb{R})} \le KA_0 e^{-\gamma\varepsilon t}\|z(t)\|_{H^1(\mathbb{R})}$$

for some $\gamma = \gamma(A_0, c_\infty, \lambda) > 0$ independent of ε and D_0. (See (A-13) for a similar computation.) Similarly,

$$\begin{aligned}\left|\int_{\mathbb{R}} \psi_{A_0}(a_\varepsilon - 2)\big((R+z)^m - z^m\big)z_x\right| &\le KA_0 e^{-\gamma\varepsilon t}\|z(t)\|_{H^1(\mathbb{R})}, \\ \left|\int_{\mathbb{R}} (\psi_{A_0})_x(a_\varepsilon - 2)z^{m+1}\right| &\le KA_0 e^{-\gamma\varepsilon t}\|z(t)\|^{m+1}_{H^1(\mathbb{R})}.\end{aligned}$$

Finally, from (6-24) and (A-41),

$$\left|\varepsilon\int_{\mathbb{R}} \psi_{A_0}(y)a'(\varepsilon x)z^{m+1}\right| \le KA_0 e^{-\gamma\varepsilon t}\|z(t)\|^{m+1}_{H^1(\mathbb{R})}.$$

In conclusion, (A-40) $= O(A_0 e^{-\gamma\varepsilon t}\|z(t)\|_{H^1(\mathbb{R})})$, for ε small enough.

From (A-41) we obtain the second term in (6-26). Collecting the estimates above we conclude the proof. □

A.5. *Proof of Lemma 6.8.* This is very similar to [Martel and Merle 2005, Lemma 3]. Recall that $\phi = \phi(\tilde{y}(x_0))$, with $\tilde{y}(x_0) = x - (\rho_2(t_0) + \sigma(t - t_0) + x_0)$. Therefore

$$\partial_t \int_{\mathbb{R}} u^2\phi = -\int_{\mathbb{R}} \Big(3u_x^2 + (\sigma+\lambda)u^2 - \frac{2ma_\varepsilon}{m+1}u^{m+1}\Big)\phi' + \int_{\mathbb{R}} u^2\phi^{(3)} - \frac{2\varepsilon}{m+1}\int_{\mathbb{R}} a'(\varepsilon x)u^{m+1}\phi,$$

and

$$\begin{aligned}\partial_t \int_{\mathbb{R}} \Big(u_x^2 - \frac{2a_\varepsilon(x)}{m+1}u^{m+1}\Big)\phi &= \int_{\mathbb{R}} \big(-(u_{xx} + a_\varepsilon u^m)^2 - 2u_{xx}^2 + 2ma_\varepsilon u_x^2 u^{m-1}\big)\phi' + \int_{\mathbb{R}} u_x^2\phi^{(3)} \\ &\quad -\sigma\int_{\mathbb{R}} \Big(u_x^2 - \frac{2a_\varepsilon}{m+1}u^{m+1}\Big)\phi' - \frac{\varepsilon}{m+1}\int_{\mathbb{R}} a_\varepsilon' u^{m+1}\phi'' - \frac{\varepsilon^2}{m+1}\int_{\mathbb{R}} a_\varepsilon'' u^{m+1}\phi';\end{aligned} \tag{A-42}$$

see for example [Martel and Merle 2005, Appendix C]. The conclusion follows from the arguments in the same reference, after we estimate the single new different term. In particular, we have

$$-\int_{\mathbb{R}} \Big(3u_x^2 + (\sigma+\lambda)u^2 - \frac{2ma_\varepsilon(x)}{m+1}u^{m+1}\Big)\phi' + \int_{\mathbb{R}} u^2\phi^{(3)} \le Ke^{-(t_0-t)/2K_0}e^{-x_0/K_0}. \tag{A-43}$$

Indeed, using that $1/K_0^2 \le \sigma/2$, we have (discarding the term with λ)

$$-\int_{\mathbb{R}}\Big(3u_x^2+\sigma u^2-\frac{2ma_\varepsilon(x)}{m+1}u^{m+1}\Big)\phi'+\int_{\mathbb{R}}u^2\psi^{(3)}\le-\int_{\mathbb{R}}\Big(3u_x^2+\frac{\sigma}{2}u^2-\frac{2ma_\varepsilon(x)}{m+1}u^{m+1}\Big)\phi'.$$

Now we estimate the nonlinear term. Let $R_0>0$, to be chosen later. Consider the region defined by $t\ge t_1$, $|x-\rho_2(t)|\ge R_0$. In this region we have, from the stability and Morrey's embedding,

$$|u(t,x)|\le\|u(t)-R(t)\|_{L^\infty(\mathbb{R})}+R(t,x)\le K\varepsilon^{1/2}+Ke^{-\gamma R_0},$$

with $\gamma>0$ a constant. Taking $0<\varepsilon\le\varepsilon_0$ sufficiently small and R_0 large enough, we have $|ma_\varepsilon(x)u^{m-1}|\le\sigma/4$ in the region considered. For the complementary region, $|x-\rho_2(t)|\le R_0$, we see from (6-11) and the hypothesis $\sigma<\frac{1}{2}(1-\lambda_0)$ that

$$|\tilde{y}(x_0)|\ge\big|\rho_2(t_0)-\rho_2(t)-\sigma(t_0-t)+x_0\big|-\big|x-\rho_2(t)\big|\ge\tfrac{1}{2}\sigma(t_0-t)+x_0-R_0. \tag{A-44}$$

Thus $|\phi'(\tilde{y})|\le Ke^{-\gamma(t_0-t)/K_0}e^{-x_0/K_0}$. Collecting the estimates above we obtain (A-43).

Now we claim that

$$\left|\frac{2\varepsilon}{m+1}\int_{\mathbb{R}}a'(\varepsilon x)u^{m+1}\phi\right|\le Ke^{-\varepsilon\gamma T_\varepsilon}e^{-\varepsilon\gamma(t_0-t)/K_0}e^{-\gamma\varepsilon x_0/K_0}. \tag{A-45}$$

Indeed, set $\tilde{x}(t):=\rho_2(t_0)+\sigma(t-t_0)+x_0$. Then from $\sigma<\frac{1}{2}(1-\lambda_0)$ and (6-11) we have

$$\begin{aligned}\tilde{x}(t)&=\rho_2(t_0)-\rho_2(t)-\sigma(t_0-t)+(x_0+\rho_2(t))\\&\ge\tfrac{1}{2}\sigma(t_0-t)+\rho_2(t_0)+x_0\ge\tfrac{1}{2}\sigma(t_0-t)+\tfrac{1}{2}T_\varepsilon+x_0,\end{aligned}$$

and thus for ε small,

$$\begin{aligned}\left|\frac{2\varepsilon}{m+1}\int_{\mathbb{R}}a'(\varepsilon x)u^{m+1}\phi\right|&\le K\varepsilon\int_{-\infty}^{\tilde{x}}e^{-\varepsilon\gamma|x|}e^{(x-\tilde{x})/K_0}dx+K\varepsilon\int_{\tilde{x}}^{\infty}e^{-\varepsilon\gamma x}\\&\le K\varepsilon e^{-\tilde{x}/K_0}+Ke^{-\varepsilon\gamma\tilde{x}}\le Ke^{-\gamma\varepsilon T_\varepsilon}e^{-\gamma\varepsilon(t_0-t)/K_0}e^{-\gamma\varepsilon x_0/K_0}.\end{aligned}$$

This last estimate proves (A-45). Integrating between t and t_0 we get (6-32).

Next, by following the same kind of calculations (see [Martel and Merle 2005]), we have

$$\partial_t\int_{\mathbb{R}}\Big(u_x^2+u^2-\frac{2a_\varepsilon(x)}{m+1}u^{m+1}\Big)\phi\le Ke^{-\gamma(t_0-t)/K_0}e^{-x_0/K_0}+Ke^{-\gamma\varepsilon T_\varepsilon}e^{-\gamma\varepsilon(t_0-t)/K_0}e^{-\gamma\varepsilon x_0/K_0}.$$

After integration we get (6-34).

Now we prove (6-33). The procedure is analogous to (6-32); the main differences are in (A-44) and (A-45). For the first case we have $\tilde{y}(-x_0)=x-\big(\rho_2(t_0)+\sigma(t-t_0)-x_0\big)$ satisfies

$$|\tilde{y}|\ge\big|\rho_2(t)-\rho_2(t_0)-\sigma(t-t_0)+x_0\big|-\big|x-\rho_2(t)\big|\ge\tfrac{1}{2}\sigma(t-t_0)+x_0-R.$$

From the hypothesis we have $\hat{x}(t) := \rho_2(t_0) + \sigma(t-t_0) - x_0 > t_1 \geq \frac{1}{2}T_\varepsilon$. Therefore (A-45) can be bounded as follows:

$$\left|\frac{2\varepsilon}{m+1}\int_{\mathbb{R}} a'(\varepsilon x)u^{m+1}\phi\right| \leq K\varepsilon\int_{-\infty}^{\hat{x}} e^{-\varepsilon\gamma|x|}e^{(x-\hat{x})/K_0}dx + K\varepsilon\int_{\hat{x}}^{\infty} e^{-\varepsilon\gamma x}$$
$$\leq K\varepsilon e^{-\hat{x}/K_0} + Ke^{-\varepsilon\gamma\hat{x}}k \leq Ke^{-\gamma\varepsilon\rho_2(t_0)}e^{-\gamma\varepsilon(t-t_0)/K_0}e^{\gamma\varepsilon x_0/K_0}.$$

Collecting the estimates above and integrating between t_0 and t, we obtain the conclusion. □

A.6. ***Some identities related to the soliton Q.*** The following identities can be found in [Martel and Merle 2007, Appendix C]. Recall that $Q_c := c^{\frac{1}{m-1}}Q(\sqrt{c}x)$ denotes the scaled soliton ($m > 1$). Recall also that $\theta = \frac{1}{m-1} - \frac{1}{4}$.

(1) *Energy*.

$$E_1[Q] = \frac{1}{2}(\lambda-\lambda_0)\int_{\mathbb{R}} Q^2 = (\lambda-\lambda_0)M[Q], \qquad \text{with } \lambda_0 = \frac{5-m}{m+3}.$$

(2) *Integrals*.

$$\int_{\mathbb{R}} Q_c = c^{\theta-\frac{1}{4}}\int_{\mathbb{R}} Q, \quad \int_{\mathbb{R}} Q_c^2 = c^{2\theta}\int_{\mathbb{R}} Q^2, \quad E_1[Q_c] = c^{2\theta+1}E_1[Q],$$
$$\int_{\mathbb{R}} Q_c^{m+1} = \frac{2(m+1)c^{2\theta+1}}{m+3}\int_{\mathbb{R}} Q^2, \qquad \int_{\mathbb{R}} \Lambda Q_c Q_c = \theta c^{2\theta-1}\int_{\mathbb{R}} Q^2.$$

Acknowledgments

. The author thanks Y. Martel and F. Merle for presenting this problem to him and for their continuing encouragement during the elaboration of this work. The author is also grateful of Gustavo Ponce for useful discussions.

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Received 22 Dec 2009. Revised 3 Jun 2010. Accepted 13 Jul 2010.

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Volume 4 No. 4 2011