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PERIODIC SOLUTIONS OF NONLINEAR SCHRÖDINGER EQUATIONS: A PARADIFFERENTIAL APPROACH

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This paper is devoted to the construction of periodic solutions of nonlinear Schrödinger equations on the torus, for a large set of frequencies. Usual proofs of such results rely on the use of Nash–Moser methods. Our approach avoids this, exploiting the possibility of reducing, through paradifferential conjugation, the equation under study to an equivalent form for which periodic solutions may be constructed by a classical iteration scheme.

Introduction

This paper is devoted to the existence of families of periodic solutions of Hamiltonian nonlinear Schrödinger equations on the torus $\mathbb{T}^d$. Our goal is to show that such results may be proved without using Nash–Moser methods, replacing them by a technically simpler conjugation idea.

We consider equations of type

$$(-i\partial_t - \Delta + \mu)u = \epsilon \frac{\partial F}{\partial \bar{u}}(\omega t, x, u, \bar{u}, \epsilon) + \epsilon f(\omega t, x),$$

where $t \in \mathbb{R}$, $x \in \mathbb{T}^d$, F is a smooth function, vanishing at order 3 at $(u, \bar{u}) = 0$, f is a smooth function on $\mathbb{R} \times \mathbb{T}^d$, 2π-periodic in time, ω a frequency parameter, μ a real parameter and $\epsilon > 0$ a small number. One wants to show that for ϵ small and ω in a Cantor set whose complement has small measure, the equation has time periodic solutions.

Let us recall known results for that type of problems. The first periodic solutions for nonlinear wave or Schrödinger equations were constructed in [Kuksin 1993; Wayne 1990], which deal with one space dimension, with x staying in a compact interval, and imposing on the extremities of this interval convenient boundary conditions. Later on, Craig and Wayne [1993; 1994] treated the same problem for time-periodic solutions defined on $\mathbb{R} \times \mathbb{S}^1$. Periodic solutions of nonlinear wave equations in higher space dimensions (on $\mathbb{R} \times \mathbb{T}^d$, $d \geq 2$) were obtained in [Bourgain 1994]. These results concern nonlinearities which are analytic. More recently, some work has been devoted to the same problem when the nonlinearity is a smooth function: Berti and Bolle [2010] have proved in this setting existence of time-periodic solutions for the nonlinear wave equation on $\mathbb{R} \times \mathbb{T}^d$. We refer also to the paper of Berti, Bolle and Procesi [Berti et al. 2010], where the case of equations on Zoll manifolds is treated. Very recently, Berti and Procesi

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[2011] have studied the same problem, for wave or Schrödinger equations, on a homogeneous space. We refer also to [Craig 2000; Kuksin 2000] for more references.

The proofs of all these results rely on the use of the Nash–Moser theorem, to overcome unavoidable losses of derivatives coming from the small divisors appearing when inverting the linear part of the equation. Our goal here is to show that one may construct periodic solutions of nonlinear Schrödinger equations (for large sets of frequencies), using just a standard iterative scheme instead of the quadratic scheme of the Nash–Moser method. This approach allows one to separate on the one hand the treatment of losses of derivatives coming from small divisors, and on the other hand the question of convergence of the sequence of approximations, while in a Nash–Moser scheme, both problems have to be treated at the same time. The basic idea is inspired by our work in [Delort 2010] concerning linear Schrödinger equations with smooth time dependent potential. It is shown in that paper that a linear equation of type $(i\partial_t - \Delta + V(t,x))u = 0$ may be reduced by conjugation to an equation of type $(i\partial_t - \Delta + V_{\mathrm{D}})v = Rv$, where R is a smoothing operator and V_{D} a block diagonal operator of order zero. We aim at applying a similar method when the linear potential V is replaced by a nonlinear one, so that, in the reduced equation, the block-diagonal operator V_{D} depends on v itself, and R sends essentially H^s to H^{2s-a} (where a is a fixed constant, and H^s the Sobolev scale). It is pretty clear that such a reduced equation will be solvable by a standard iterative scheme, even if the inversion of $i\partial_t - \Delta + V_{\mathrm{D}}$ loses derivatives because of small divisors, since such losses are recovered by the smoothing properties of R on the right side.

Before describing the different sections of the paper, let us give some more references and add some comments. There are actually a few results concerning existence of periodic solutions which do not appeal to Nash–Moser theorem. Bambusi and Paleari [2001; 2002] constructed such solutions without making use of Nash–Moser or KAM methods, but only for a family of frequency parameters of measure zero (instead of a set of parameters whose *complement* has small measure). Related results, concerning the case of rational frequencies, may be found in [Berti 2007, Chapter 5]. Recently, Gentile and Procesi [2009] found, for analytic nonlinearities, an alternative approach to Nash–Moser using expansions in terms of Lindsted series.

Let us also mention that we restrict in this paper to one of the many variants that may be considered when constructing periodic solutions. Most of the known results we cited so far concern the case of periodic solutions of the nonlinear equation, whose frequency is close to the frequency of a periodic solution of the linear equation obtained for $\epsilon = 0$. The problem may be written, using a Liapunov–Schmidt decomposition, as a coupling between a non-resonant equation (the (P) equation) and a resonant one (the (Q) equation). In most works, the resonant equation is a finite-dimensional equation, while (P) is infinite-dimensional. One uses Nash–Moser to solve (P), getting a solution depending on finitely many parameters. Plugging this solution in (Q), one gets for these finitely many parameters an equation in closed form, that may be solved using implicit functions-like theorems. Actually, Berti and Bolle [2006] have shown that such a strategy may be also adapted to the case when (Q) is completely resonant, i.e., infinite-dimensional.

Since our objective here is to show that one may avoid the use of Nash–Moser theorems, we limited ourselves to the forced oscillations equation written at the beginning of the introduction, which corresponds to a (P) equation for which there is no associated (Q) equation. Berti and Bolle [2010] have studied similar forced oscillations for the wave equation. It is very likely that our method could be adapted to recover as well known results for resonant periodic Schrödinger equations, even if one would have to write a detailed proof. In the same way, since the results in [Delort 2010] concerning the Schrödinger equation hold not only on $\mathbb{T}^d$, but also on Zoll manifolds or on some surfaces of revolution, we conjecture that the analogue of the main theorem of this paper extends to this setting, or even to the case of a product of several Zoll manifolds.

Organization of the paper. Section 1 states the main theorem and introduces notation.

Section 2 is devoted to the paralinearization of the equation. After defining convenient classes of paradifferential operators, we perform a first reduction, localizing the unknown of the problem close to the characteristic variety of the linear Schrödinger operator. This is done using the standard implicit function theorem. Next, we paralinearize the equation, reducing it to

$$(-i\omega\partial_t - \Delta + V)v = R(v)v + \epsilon f$$

where V is a paradifferential operator of order zero, depending on v, and $R(v)$ is a smoothing operator (Actually, we shall have to consider a system in $(v, \bar{v})$ instead of a scalar equation). A consequence of the fact that our starting equation is Hamiltonian will be that V is self-adjoint.

Section 3 is the heart of the paper. We construct a paradifferential conjugation of the preceding equation to transform it into

$$(-i\omega\partial_t - \Delta + V_{\mathrm{D}}(w))w = R(w)w + \epsilon f$$

where $R(w)$ is still a smoothing operator, and V_{D} is block diagonal relatively to an orthogonal decomposition of $L^2(\mathbb{T}^d)$ in a sum of finite-dimensional subspaces introduced in [Bourgain 1999].

Section 4 is devoted to the construction of the solution to the block diagonal equation by a standard iteration scheme. We first show that on each block $-i\omega\partial_t - \Delta + V_{\mathrm{D}}(w)$ is invertible for ω outside a convenient small subset. This is done by the usual argument, exploiting that the ω-derivative of the eigenvalues of $-i\omega\partial_t - \Delta$ is large. In order that the set of excluded parameters remain small, we have to allow small divisors when inverting $-i\omega\partial_t - \Delta + V_{\mathrm{D}}(w)$. As the right-hand side of the equation involves a smoothing operator $R(w)$, we may compensate the losses of derivatives coming from such small divisors, and construct a sequence of approximations of the solution.

Let us conclude this introduction with a few words concerning the limitations of our method. First, it does not seem that it could be adapted to find periodic solutions of nonlinear wave equations, as the construction of Section 3 relies on a specific separation property for the eigenvalues of $-\Delta$ on $\mathbb{T}^d$. On the other hand, it might be applied to equations where one has a nice separation of eigenvalues, like KdV or one-dimensional water wave equations with surface tension. Second, we do not know if our method could be modified to construct quasi-periodic solutions. Recall that such solutions have been obtained for the equation set on an interval [Kuksin 1993; Kuksin 2000; Kuksin and Pöschel 1996]. The case of solutions on $\mathbb{S}^1$ has been treated in [Bourgain 1994]. In higher dimensions, Bourgain [1998] constructed such periodic solutions on $\mathbb{T}^2$. The case of general $\mathbb{T}^d$ has been treated in [Bourgain 2005;

Eliasson and Kuksin 2010]. One of the difficulties of the quasi-periodic case versus the periodic one lies in the fact that, even close to the characteristic variety, time frequencies might be much larger than space frequencies. In our proof below, the fact that these frequencies are of the same magnitude plays an important role. We do not know whether the multiscale methods of Bourgain, Eliasson, and Kuksin could be combined to the arguments we use in the periodic case to construct quasi-periodic solutions without making appeal to a Newton scheme.

1. Periodic solutions of semi-linear Schrödinger equations

1.1. ***Statement of the main theorem.*** Let $\mathbb{T}^d$ $(d \geq 1)$ be the standard torus, $\mathbb{S}^1$ the unit circle. Consider a C^∞ function

$$\begin{aligned} F : (t, x, u, \bar{u}, \epsilon) &\longrightarrow F(t, x, u, \bar{u}, \epsilon) \\ \mathbb{R}\times\mathbb{T}^d \times \mathbb{C}^2 \times [0, 1] &\rightarrow \mathbb{R} \end{aligned} \tag{1.1.1}$$

which is 2π-periodic in t, and satisfies $\partial^\alpha_{u,\bar{u}} F(t, x, 0, 0, \epsilon) \equiv 0$ for $|\alpha| \leq 2$. We study the equation

$$(D_t - \Delta + \mu)u = \epsilon \frac{\partial F}{\partial \bar{u}}(\omega t, x, u, \bar{u}, \epsilon) + \epsilon f(\omega t, x) \tag{1.1.2}$$

where Δ is the Laplace operator on $\mathbb{T}^d$, $D_t = \frac{1}{i}\frac{\partial}{\partial t}$, $\epsilon \in [0, 1]$, $\mu \in \mathbb{R}$, $\omega \in \mathbb{R}^*_+$, f is a smooth function on $\mathbb{R} \times \mathbb{T}^d$, 2π-periodic in t, with values in $\mathbb{C}$, and where we look for $\frac{2\pi}{\omega}$-periodic solutions of the equation when ϵ is small. Changing t to t/ω, we have to find solutions on $\mathbb{S}^1 \times \mathbb{T}^d$ to the equivalent equation

$$(\omega D_t - \Delta + \mu)u = \epsilon \frac{\partial F}{\partial \bar{u}}(t, x, u, \bar{u}, \epsilon) + \epsilon f(t, x) \tag{1.1.3}$$

for small enough ϵ and for ω outside a subset of small measure. To fix ideas, we shall take ω inside a fixed compact subinterval of $]0, +\infty[$, say $\omega \in [1, 2]$.

Let us define the Sobolev space in which we shall look for solutions. If $u \in \mathcal{D}'(\mathbb{S}^1 \times \mathbb{T}^d)$, we set for $(j, n) \in \mathbb{Z} \times \mathbb{Z}^d$

$$\hat{u}(j, n) = \frac{1}{(2\pi)^{\frac{d+1}{2}}} \int_{\mathbb{S}^1\times\mathbb{T}^d} e^{-itj-in\cdot x} u(t, x)\, dt\, dx.$$

For $s \in \mathbb{R}$, define $\tilde{\mathcal{H}}^s(\mathbb{S}^1 \times \mathbb{T}^d; \mathbb{C})$ to be the space of those $u \in \mathcal{D}'(\mathbb{S}^1 \times \mathbb{T}^d)$ such that

$$\|u\|^2_{\tilde{\mathcal{H}}^s} \stackrel{\text{def}}{=} \sum_{j\in\mathbb{Z}} \sum_{n\in\mathbb{Z}^d} (1 + |j| + |n|^2)^s |\hat{u}(j, n)|^2 < +\infty. \tag{1.1.4}$$

We shall use the similar notation $\tilde{\mathcal{H}}^s(\mathbb{S}^1 \times \mathbb{T}^d; \mathbb{C}^2)$, $\tilde{\mathcal{H}}^s(\mathbb{S}^1 \times \mathbb{T}^d; \mathbb{R}^2)$ for $\mathbb{C}^2$ or $\mathbb{R}^2$-valued functions.

Let us state our main theorem.

Theorem 1.1.1. *Let $\mu \in \mathbb{R} - \mathbb{Z}_-$. There are $s_0 > 0, \zeta > 0$ and for any $s \geq s_0$, any $q_0 > 0$, there are constants $\delta_0 \in]0, 1]$, $B > 0$ and for any $f \in \tilde{\mathcal{H}}^{s+\zeta}(\mathbb{S}^1 \times \mathbb{T}^d; \mathbb{C})$ with $\|f\|_{\tilde{\mathcal{H}}^{s+\zeta}} \leq q_0$, there is a subset $\mathbb{O} \subset [1, 2] \times]0, 1]$ such that*:

- *For any* $\delta \in]0, \delta_0]$ *and* $\epsilon \in [0, \delta^2]$

$$\text{meas}\{\omega \in [1,2] : (\omega, \epsilon) \in \mathbb{O}\} \leq B\delta. \tag{1.1.5}$$

- *For any* $\delta \in]0, \delta_0]$, *any* $\epsilon \in [0, \delta^2]$, *and any* $\omega \in [1,2]$ *such that* $(\omega, \epsilon) \notin \mathbb{O}$, (1.1.3) *has a solution* $u \in \widetilde{\mathcal{H}}^s(\mathbb{S}^1 \times \mathbb{T}^d; \mathbb{C})$ *satisfying* $\|u\|_{\widetilde{\mathcal{H}}^s} \leq B\epsilon\delta^{-1}$.

Remark. As mentioned in the introduction, this theorem is a version, for Schrödinger equations, of [Berti and Bolle 2010, Theorem 1.1], which concerns wave equations. Our point will be to give a proof that does not appeal to Nash–Moser methods.

1.2. ***Spaces of functions and notations.*** For $n \in \mathbb{Z}^d$, $u \in \mathcal{D}'(\mathbb{T}^d)$, we denote by Π_n the spectral projector

$$\Pi_n u = \hat{u}(n)\frac{e^{in\cdot x}}{(2\pi)^{d/2}} = \int_{\mathbb{T}^d} e^{-in\cdot x}u(x)\,\frac{dx}{(2\pi)^{d/2}}\frac{e^{in\cdot x}}{(2\pi)^{d/2}}. \tag{1.2.1}$$

When $u(t,x)$ is in $\mathcal{D}'(\mathbb{S}^1 \times \mathbb{T}^d)$, we use the same notation, considering t as a parameter. We shall make use of the following "separation property" result attributed by Bourgain to Granville and Spencer (see [Bourgain 1999, Lemma 8.1]; for the proof see also [Bourgain 2005, Lemma 19.10]).

Lemma 1.2.1. *For any* $\beta \in]0, \frac{1}{10}[$, *there are* $\rho \in]0, \beta[$, $\theta > 0$ *and a partition* $(\Omega_\alpha)_{\alpha\in\mathcal{A}}$ *of* $\mathbb{Z}^d$ *such that*

$$\begin{aligned} |n-n'| + \big||n|^2 - |n'|^2\big| &< \theta + |n|^\beta \quad \textit{for all } \alpha \in \mathcal{A},\ n \in \Omega_\alpha,\ n' \in \Omega_\alpha, \\ |n-n'| + \big||n|^2 - |n'|^2\big| &> |n|^\rho \quad \textit{for all } \alpha, \alpha' \in \mathcal{A}\ (\alpha \neq \alpha'),\ n \in \Omega_\alpha,\ n' \in \Omega_{\alpha'}. \end{aligned} \tag{1.2.2}$$

For each $\alpha \in \mathcal{A}$, we choose some $n(\alpha) \in \Omega_\alpha$. There is a constant $\Theta_0 > 0$ such that, if we set $\langle n\rangle = (1+|n|^2)^{1/2}$ for $n \in \mathbb{Z}^d$, then

$$\Theta_0^{-1}\langle n(\alpha)\rangle \leq \langle n\rangle \leq \Theta_0\langle n(\alpha)\rangle \tag{1.2.3}$$

for any $\alpha \in \mathcal{A}$, any $n \in \Omega_\alpha$. It also follows from (1.2.2) that, for some uniform constant $\Theta_1 > 0$,

$$\#\Omega_\alpha \leq \Theta_1\langle n(\alpha)\rangle^{\beta d}. \tag{1.2.4}$$

For any $\alpha \in \mathcal{A}$, we set

$$\widetilde{\Pi}_\alpha = \sum_{n\in\Omega_\alpha} \Pi_n. \tag{1.2.5}$$

We define a closed subspace $\mathcal{H}^s(\mathbb{S}^1 \times \mathbb{T}^d; \mathbb{C})$ of $\widetilde{\mathcal{H}}^s(\mathbb{S}^1 \times \mathbb{T}^d; \mathbb{C})$ by

$$\begin{aligned} \mathcal{H}^s(\mathbb{S}^1 \times \mathbb{T}^d; \mathbb{C}) = \bigcap_{\alpha\in\mathcal{A}} \big\{u \in \widetilde{\mathcal{H}}^s(\mathbb{S}^1 \times \mathbb{T}^d; \mathbb{C}) : \hat{u}(j,n) = 0 \text{ for all } n \in \Omega_\alpha \text{ and all } j \text{ such that} \\ |j| > K_0\langle n(\alpha)\rangle^2 \text{ or } |j| < K_0^{-1}\langle n(\alpha)\rangle^2\big\}, \end{aligned} \tag{1.2.6}$$

where $K_0 = K_0(\mu)$ will be chosen later on.

In other words, non vanishing modes (j,n) of an element u of $\mathcal{H}^s(\mathbb{S}^1 \times \mathbb{T}^d; \mathbb{C})$ have to satisfy $K_0^{-1}\langle n(\alpha)\rangle^2 \leq |j| \leq K_0\langle n(\alpha)\rangle^2$ if $n \in \Omega_\alpha$. This shows that the restriction to $\mathcal{H}^s$ of the $\widetilde{\mathcal{H}}^s$-norm given by (1.1.4) is equivalent to the square root of

$$\sum_{j\in\mathbb{Z}}\sum_{n\in\mathbb{Z}^d}\langle n\rangle^{2s}|\hat{u}(j,n)|^2 \tag{1.2.7}$$

and to the square root of

$$\sum_{\alpha\in\mathcal{A}}\langle n(\alpha)\rangle^{2s}\|\widetilde{\Pi}_\alpha u\|^2_{L^2(\mathbb{S}^1\times\mathbb{T}^d,\mathbb{C})}. \tag{1.2.8}$$

We use similar notation for spaces $\mathcal{H}^s(\mathbb{S}^1\times\mathbb{T}^d;\mathbb{C}^2)$, $\mathcal{H}^s(\mathbb{S}^1\times\mathbb{T}^d;\mathbb{R}^2)$, and so on.

2. Paralinearization of the equation

The goal of this section is to rewrite (1.1.3) as a paradifferential equation in the sense of [Bony 1981], on spaces of form (1.2.6). We first define the classes of operators we shall use.

2.1. *Spaces of operators.* We fix from now on some real number $\sigma_0>\frac{d}{2}+1$. If $s\in\mathbb{R}, q>0$, we denote by $B_q(\mathcal{H}^s)$ the open ball with center 0, radius q in $\mathcal{H}^s(\mathbb{S}^1\times\mathbb{T}^d;\mathbb{C})$, $\mathcal{H}^s(\mathbb{S}^1\times\mathbb{T}^d;\mathbb{C}^2)$,...

Definition 2.1.1. Let $m\in\mathbb{R}$, $q>0$, $N\in\mathbb{N}$, $\sigma\in\mathbb{R}$, $\sigma\geq\sigma_0+2N+d+1$. One denotes by $\Psi^m(N,\sigma,q)$ the space of maps $U\to a(U)$ defined on the open ball of center 0, radius q in $\mathcal{H}^\sigma(\mathbb{S}^1\times\mathbb{T}^d;\mathbb{C}^2)$, with values in the space of linear maps from $C^\infty(\mathbb{S}^1\times\mathbb{T}^d;\mathbb{C})$ to $\mathcal{D}'(\mathbb{S}^1\times\mathbb{T}^d;\mathbb{C})$, such that, for any $n,n'\in\mathbb{Z}^d$, the map $U\to\Pi_n a(U)\Pi_{n'}$ is smooth with values in $\mathcal{L}(\mathcal{H}^0(\mathbb{S}^1\times\mathbb{T}^d;\mathbb{C}))$ and satisfies for any $M\in\mathbb{N}$ with $d+1\leq M\leq\sigma-\sigma_0-2N$, any $U\in B_q(\mathcal{H}^\sigma)$, any $j\in\mathbb{N}$, any $W_1,\dots,W_j\in\mathcal{H}^\sigma(\mathbb{S}^1\times\mathbb{T}^d;\mathbb{C}^2)$, any $n,n'\in\mathbb{Z}^d$,

$$\begin{aligned}&\|\Pi_n(\partial_U^j a(U)\cdot(W_1,\dots,W_j))\Pi_{n'}\|_{\mathcal{L}(\mathcal{H}^0)}\\&\qquad\leq C(1+|n|+|n'|)^m\langle n-n'\rangle^{-M}\mathbb{1}_{|n-n'|\leq\frac{1}{10}(|n|+|n'|)}\prod_{\ell=1}^{j}\|W_\ell\|_{\mathcal{H}^{\sigma_0+2N+M}}.\end{aligned} \tag{2.1.1}$$

Remarks. • In (2.1.1), the decay $\langle n-n'\rangle^{-M}$ reflects the available x-smoothness of the symbol of a pseudo-differential or paradifferential operator. This smoothness is controlled by the upper bound $\sigma-\sigma_0-2N$ that we assume for M. The cut-off $|n-n'|\leq\frac{1}{10}(|n|+|n'|)$ means that we are considering paradifferential operators. The integer N measures some loss of smoothness, relatively to the index σ, that will appear in some expansions of operators.

• Definition 2.1.1 implies that if $a\in\Psi^m(N,\sigma,q)$, then $\partial_t[a(U)]$ belongs to $\Psi^m(N+1,\sigma,q)$. Actually, $\partial_t a(U)=\partial_U a(U)\cdot\partial_t U$, so (2.1.1) allows us to estimate

$$\left\|\Pi_n(\partial_U^j(\partial_t[a(U)])\cdot(W_1,\dots,W_j))\Pi_{n'}\right\|_{\mathcal{L}(\mathcal{H}^0)}$$

from $\|\partial_t U\|_{\mathcal{H}^{\sigma_0+2N+M}}\prod_{\ell=1}^j\|W_\ell\|_{\mathcal{H}^{\sigma_0+2N+M}}$, and by the definition (1.2.6) of $\mathcal{H}^s$,

$$\|\partial_t U\|_{\mathcal{H}^{\sigma_0+2N+M}}\leq K_0\|U\|_{\mathcal{H}^{\sigma_0+2(N+1)+M}}\leq K_0\|U\|_{\mathcal{H}^\sigma}$$

if we assume $M\leq\sigma-2(N+1)-\sigma_0$.

The definition implies boundedness properties for the operators.

Lemma 2.1.2. *Let σ, m, N, q be as in the definition. Assume that $\sigma \geq \sigma_0 + 2N + d + 1$. Then for any $U \in B_q(\mathcal{H}^\sigma)$, for any $s \in \mathbb{R}$, $a(U)$ is a bounded operator from $\mathcal{H}^s(\mathbb{S}^1 \times \mathbb{T}^d; \mathbb{C})$ to $\mathcal{H}^{s-m}(\mathbb{S}^1 \times \mathbb{T}^d; \mathbb{C})$. Moreover, $U \to a(U)$ is a smooth map from $B_q(\mathcal{H}^\sigma)$ to the space $\mathcal{L}(\mathcal{H}^s, \mathcal{H}^{s-m})$, and for any $j \in \mathbb{N}$, there is $C > 0$, such that, for any $U \in B_q(\mathcal{H}^\sigma)$ and any $W_1, \dots, W_j \in \mathcal{H}^\sigma(\mathbb{S}^1 \times \mathbb{T}^d; \mathbb{C})$,*

$$\left\| \partial_U^j a(U) \cdot (W_1, \dots, W_j) \right\|_{\mathcal{L}(\mathcal{H}^s, \mathcal{H}^{s-m})} \leq C \prod_{\ell=1}^{j} \| W_\ell \|_{\mathcal{H}^{\sigma_0 + 2N + d + 1}}. \tag{2.1.2}$$

Proof. One has just to apply (2.1.1) with $M = d+1$ and use the fact that, by (1.2.7), $\|v\|^2_{\mathcal{H}^s}$ is equivalent to $\sum_{n \in \mathbb{Z}^d} \langle n \rangle^{2s} \|\Pi_n v\|^2_{L^2}$. □

We define a class of smoothing operators as well.

Definition 2.1.3. Let $\sigma \in \mathbb{R}$, $N \in \mathbb{N}$, and $\nu \in \mathbb{N}$, with $\sigma \geq \sigma_0 + 2N + d + 1$, $q > 0$, $r \in \mathbb{R}_+$. One denotes by $\mathcal{R}^r_\nu(N, \sigma, q)$ the space of smooth maps $U \to R(U)$ defined on $B_q(\mathcal{H}^\sigma)$, with values in $\mathcal{L}(\mathcal{H}^s(\mathbb{S}^1 \times \mathbb{T}^d; \mathbb{C}), \mathcal{H}^{s+r}(\mathbb{S}^1 \times \mathbb{T}^d; \mathbb{C}))$ for any $s \geq \sigma_0 + \nu$, such that there is for any j, any $s \geq \sigma_0 + \nu$, a constant $C > 0$ with

$$\left\| \partial_U^j R(U) \cdot (W_1, \dots, W_j) \right\|_{\mathcal{L}(\mathcal{H}^s, \mathcal{H}^{s+r})} \leq C \prod_{\ell=1}^{j} \| W_\ell \|_{\mathcal{H}^\sigma} \tag{2.1.3}$$

for any $U \in B_q(\mathcal{H}^\sigma)$, $W_1, \dots, W_j \in \mathcal{H}^\sigma$.

Remark. Lemma 2.1.2 shows that if $r \geq 0$ and $\sigma \geq \sigma_0 + 2N + d + 1$, the space $\Psi^{-r}(N, \sigma, q)$ is contained in $\mathcal{R}^r_0(N, \sigma, q)$.

Proposition 2.1.4. (i) *Let $\sigma \geq \sigma_0 + 2N + d + 1$, $a \in \Psi^m(N, \sigma, q)$. Then $a^* \in \Psi^m(N, \sigma, q)$.*

(ii) *Let $m_1, m_2 \in \mathbb{R}$. Assume $\sigma \geq \sigma_0 + 2N + d + 1 + (m_1 + m_2)_+$. Set*

$$r = \sigma - \sigma_0 - 2N - (d+1) - (m_1 + m_2) \geq 0. \tag{2.1.4}$$

If $a \in \Psi^{m_1}(N, \sigma, q)$ and $b \in \Psi^{m_2}(N, \sigma, q)$, there are $c \in \Psi^{m_1+m_2}(N, \sigma, q)$ and $R \in \mathcal{R}^r_0(N, \sigma, q)$ such that

$$a(U) \circ b(U) = c(U) + R(U). \tag{2.1.5}$$

Proof. Part (i) follows immediately from the definition. For (ii), define

$$c(U) = \sum_n \sum_{n'} \Pi_n [a(U) \circ b(U)] \Pi_{n'} \mathbb{1}_{|n-n'| \leq \frac{1}{10}(|n|+|n'|)}.$$

To check that (2.1.1) is satisfied by c when $j = 0$ we write

$$\|\Pi_n c(U) \Pi_{n'}\|_{\mathcal{L}(\mathcal{H}^0)} \leq \sum_k \|\Pi_n a(U) \Pi_k\|_{\mathcal{L}(\mathcal{H}^0)} \|\Pi_k b(U) \Pi_{n'}\|_{\mathcal{L}(\mathcal{H}^0)}$$

for n, n' with $|n-n'| \leq \frac{1}{10}(|n|+|n'|)$. Applying (2.1.1) to a, b with $d+1 \leq M \leq \sigma-\sigma_0-2N$, we get the bound

$$C(1+|n|+|n'|)^{m_1+m_2}\sum_k \langle n-k\rangle^{-M}\langle k-n'\rangle^{-M} \leq C(1+|n|+|n'|)^{m_1+m_2}\langle n-n'\rangle^{-M}.$$

One estimates $\partial_U^j c(U)$ in the same way.

The remainder $R(U)=a(U)\circ b(U)-c(U)$ will satisfy by definition of c:

$$\|\Pi_n R(U)\Pi_{n'}\|_{\mathscr{L}(\mathscr{H}^0)} \leq \sum_k \|\Pi_n a(U)\Pi_k\|_{\mathscr{L}(\mathscr{H}^0)}\,\|\Pi_k b(U)\Pi_{n'}\|_{\mathscr{L}(\mathscr{H}^0)}\mathbb{1}_{|n-n'|>\frac{1}{10}(|n|+|n'|)},$$

and so will be bounded using (2.1.1) for a, b by

$$C(1+|n|+|n'|)^{m_1+m_2}\sum_k \langle n-k\rangle^{-M}\langle k-n'\rangle^{-M}\mathbb{1}_{|k-n|\leq\frac{1}{10}(|n|+|k|)}\,\mathbb{1}_{|k-n'|\leq\frac{1}{10}(|n'|+|k|)}\,\mathbb{1}_{|n-n'|>\frac{1}{10}(|n|+|n'|)}$$

for any M between $d+1$ and $\sigma-\sigma_0-2N$. Since on the summation, either $|n-k| \geq \frac{1}{2}|n-n'|$ or $|n'-k| \geq \frac{1}{2}|n-n'|$, and $|n-n'| \leq \frac{1}{2}(|n|+|n'|)$, we get the bound

$$\|\Pi_n R(U)\Pi_{n'}\|_{\mathscr{L}(\mathscr{H}^0)} \leq C(1+|n|+|n'|)^{m_1+m_2-M}\mathbb{1}_{|n-n'|\leq\frac{1}{2}(|n|+|n'|)}$$

for any M between $d+1$ and $\sigma-\sigma_0-2N$. Reasoning as in the proof of Lemma 2.1.2, we obtain that $R(U)$ sends $\mathscr{H}^s$ to $\mathscr{H}^{s+r}$ for any s and r given by (2.1.4). The estimates of $\partial_U^j R(U)\cdot(W_1,\ldots,W_j)$ are obtained in the same way. □

In the rest of this paper, we shall use several variants of these classes. We shall denote by $\Psi_{\mathbb{R}}^m(N,\sigma,q)$ the subspace of $\Psi^m(N,\sigma,q)$ made of those operators $a(U)$ sending real-valued functions to real-valued functions, i.e., satisfying $\overline{a(U)}=a(U)$. We define $\mathscr{R}_{\nu,\mathbb{R}}^r(N,\sigma,q)$ from $\mathscr{R}_\nu^r(N,\sigma,q)$ analogously. We denote by

$$\Psi^m(N,\sigma,q)\otimes\mathscr{M}_2(\mathbb{R}) \quad \text{and} \quad \mathscr{R}_\nu^r(N,\sigma,q)\otimes\mathscr{M}_2(\mathbb{R})$$

the space of 2×2 matrices with entries in $\Psi^m(N,\sigma,q)$ and in $\mathscr{R}_\nu^r(N,\sigma,q)$ respectively. We use similar notation for $\Psi_{\mathbb{R}}^m(N,\sigma,q)$ and $\mathscr{R}_{\nu,\mathbb{R}}^r(N,\sigma,q)$.

Finally, we shall consider operators $a(U,\omega,\epsilon)$, $R(U,\omega,\epsilon)$ depending on (ω,ϵ) staying in a bounded domain of $\mathbb{R}^2$. We say $a(U,\omega,\epsilon)$ is C^1 in (ω,ϵ) if $(\omega,\epsilon)\to\Pi_n a(U,\omega,\epsilon)\Pi_{n'}$ is C^1 in (ω,ϵ) with values in $\mathscr{L}(\mathscr{H}^0)$ and if (2.1.1) is satisfied also by $\partial_\omega a, \partial_\epsilon a$. Likewise, $R(U,\omega,\epsilon)$ is C^1 if $(\omega,\epsilon)\to R(U,\omega,\epsilon)$ is C^1 in (ω,ϵ) with values in $\mathscr{L}(\mathscr{H}^s,\mathscr{H}^{s+r})$ and if (2.1.3) is satisfied by $\partial_\omega R, \partial_\epsilon R$.

2.2. ***Equivalent formulation of the equation.*** The goal of this subsection is to reduce (1.1.3) to an equivalent equation for a new unknown belonging to the space $\mathscr{H}^s$ defined by (1.2.6) instead of $\widetilde{\mathscr{H}}^s$. Recall that we fixed some $\sigma_0 > \frac{d}{2}+1$.

For $\sigma\in\mathbb{R}$, we consider the space $\mathscr{H}^\sigma(\mathbb{S}^1\times\mathbb{T}^d;\mathbb{R}^2)\subset\widetilde{\mathscr{H}}^\sigma(\mathbb{S}^1\times\mathbb{T}^d;\mathbb{R}^2)$ and denote by $\mathscr{F}^\sigma(\mathbb{S}^1\times\mathbb{T}^d;\mathbb{R}^2)$ the orthogonal complement of the first space in the second one.

Definition 2.2.1. Let $\sigma \geq \sigma_0$. Denote by $\mathcal{H}_1^\sigma, \mathcal{H}_2^\sigma$ any of the preceding spaces. Let X be an open subset of $\mathcal{H}_1^\sigma$, $k \in \mathbb{Z}$. One denotes by $\Phi^{\infty,k}(X, \mathcal{H}_2^{\sigma-k})$ the space of C^∞ maps $G : X \to \mathcal{H}_2^{\sigma-k}$ such that, for any $s \geq \sigma$ and $u \in X \cap \mathcal{H}_1^s$:

- $G(u) \in \mathcal{H}_2^{s-k}$.
- The linear map $DG(u) \in \mathcal{L}(\mathcal{H}_1^\sigma, \mathcal{H}_2^{\sigma-k})$ extends as an element of $\mathcal{L}(\mathcal{H}_1^{\sigma'}, \mathcal{H}_2^{\sigma'-k})$ for any $\sigma' \in [-s, s]$. Moreover, $v \to DG(v)$ is smooth from $X \cap \mathcal{H}_1^s$ to the preceding space.
- The bilinear map $D^2G(u) \in \mathcal{L}_2(\mathcal{H}_1^\sigma \times \mathcal{H}_1^\sigma; \mathcal{H}_2^{\sigma-k})$ extends as an element of $\mathcal{L}_2(\mathcal{H}_1^{\sigma_1} \times \mathcal{H}_1^{\sigma_2}; \mathcal{H}_2^{-\sigma_3-k})$ for any triple $\{\sigma_1, \sigma_2, \sigma_3\} = \{\sigma', -\sigma', \max(\sigma_0, \sigma')\}$ with $\sigma' \in [0, s]$. Moreover, $v \to D^2G(v)$ is smooth from $X \cap \mathcal{H}_1^s$ to the preceding space.

Let us give an example of an element of $\Phi^{\infty,0}(\widetilde{\mathcal{H}}^\sigma, \widetilde{\mathcal{H}}^\sigma)$. Consider $F : \mathbb{S}^1 \times \mathbb{T}^d \times \mathbb{R}^2 \to \mathbb{R}^2$, a smooth function satisfying $F(t, x, 0) \equiv 0$, $\partial_u F(t, x, 0) \equiv 0$. By Lemma A.1 of the appendix, for $\sigma > \frac{d}{2} + 1$ and $u \in \widetilde{\mathcal{H}}^\sigma(\mathbb{S}^1 \times \mathbb{T}^d; \mathbb{R}^2)$, we have $F(\,\cdot\,, u) \in \widetilde{\mathcal{H}}^\sigma(\mathbb{S}^1 \times \mathbb{T}^d; \mathbb{R}^2)$, and by Corollary A.2, $u \to F(\,\cdot\,, u)$ is smooth. If we define $G(u) = F(\,\cdot\,, u)$, then $DG(u) \cdot h = \partial_u F(\,\cdot\,, u)h$, which, by Lemma A.3, extends as a linear map from $\widetilde{\mathcal{H}}^{\sigma'}$ to itself for any $\sigma' \in [-s, s]$, when $u \in \widetilde{\mathcal{H}}^s$ and $s > \frac{d}{2} + 1$. In the same way, $D^2G(u) \cdot (h_1, h_2) = \partial_u^2 F(\,\cdot\,, u) \cdot (h_1, h_2)$ extends from $\widetilde{\mathcal{H}}^{\sigma_1} \times \widetilde{\mathcal{H}}^{\sigma_2}$ to $\widetilde{\mathcal{H}}^{-\sigma_3}$ for $\sigma_1, \sigma_2, \sigma_3$ as in the statement of the definition, by Lemma A.3.

Definition 2.2.2. Let $\sigma \geq \sigma_0$, X an open subset of $\mathcal{H}_1^\sigma$, $k \in \mathbb{Z}$. One denotes by $C^{\infty,k}(X; \mathbb{R})$ the space of C^1 functions $\Phi : X \to \mathbb{R}$, such that for any $s \geq \sigma$ and $u \in X \cap \mathcal{H}_1^s$, we have $\nabla\Phi(u) \in \mathcal{H}_1^{s-k}$ and $u \to \nabla\Phi(u)$ belongs to $\Phi^{\infty,k}(X, \mathcal{H}_1^{\sigma-k})$.

If $F : \mathbb{S}^1 \times \mathbb{T}^d \times \mathbb{R}^2 \to \mathbb{R}$ is a smooth function, with $F(t, x, 0) \equiv 0$, $\partial_u F(t, x, 0) \equiv 0$, $\partial_u^2 F(t, x, 0) \equiv 0$, and if $\Phi(u) = \int_{\mathbb{S}^1 \times \mathbb{T}^d} F(t, x, u(t, x))\, dt\, dx$, then $\nabla\Phi(u) = \partial_u F(\,\cdot\,, u) \in \widetilde{\mathcal{H}}^s$ if $u \in \widetilde{\mathcal{H}}^s$ and $s > \frac{d}{2} + 1$ (see Lemma A.1), and the example following Definition 2.2.1 shows that $\Phi \in C^{\infty,0}(\widetilde{\mathcal{H}}^\sigma, \mathbb{R})$ for $\sigma \geq \sigma_0$.

Remark. In the sequel we shall have to consider elements $G(u, \omega, \epsilon)$, $\Phi(u, \omega, \epsilon)$ of the preceding spaces depending on the real parameters (ω, ϵ). We shall say that G, Φ are C^1 in (ω, ϵ) if the conditions of Definition 2.2.1 (resp. Definition 2.2.2) are satisfied by $G, \partial_\omega G, \partial_\epsilon G$ (resp. $\Phi, \partial_\omega \Phi, \partial_\epsilon \Phi$).

Lemma 2.2.3. *Let $\sigma \geq \sigma_0$, $k \in \mathbb{N}$, X an open subset of $\mathcal{H}_1^\sigma$, $G \in \Phi^{\infty,-k}(X, \mathcal{H}_2^{\sigma+k})$, Y an open subset of $\mathcal{H}_2^{\sigma+k}$ containing $G(X)$, $\Phi \in C^{\infty,k}(Y, \mathbb{R})$. Then $\Phi \circ G \in C^{\infty,0}(X, \mathbb{R})$.*

Proof. The assumption on G implies that for $v \in X \cap \mathcal{H}_1^s$, $s \geq \sigma$ and for σ' with $|\sigma'| \leq s$,

$$DG(v) \in \mathcal{L}(\mathcal{H}_1^{\sigma'}, \mathcal{H}_2^{\sigma'+k}) \subset \mathcal{L}(\mathcal{H}_1^{\sigma'}, \mathcal{H}_2^{\sigma'}). \tag{2.2.1}$$

Moreover, since $\nabla\Phi \in \Phi^{\infty,k}(Y, \mathcal{H}_2^\sigma)$, we have $G(v) \in Y \cap \mathcal{H}_2^{s+k}$ for $v \in X \cap \mathcal{H}_1^s$, so $\nabla\Phi(G(v)) \in \mathcal{H}_2^s$ and for any σ'' with $|\sigma''| \leq s + k$, $(D(\nabla\Phi))(G(v))$ is in $\mathcal{L}(\mathcal{H}_2^{\sigma''}, \mathcal{H}_2^{\sigma''-k})$. In particular, for any σ' with $|\sigma'| \leq s$,

$$D(\nabla\Phi)(G(v)) \in \mathcal{L}(\mathcal{H}_2^{\sigma'+k}, \mathcal{H}_2^{\sigma'}). \tag{2.2.2}$$

We deduce from (2.2.1) that $\nabla(\Phi \circ G)(v) = {}^tDG(v) \cdot (\nabla\Phi)(G(v))$ belongs to $\mathcal{H}_1^s$ when $v \in X \cap \mathcal{H}_1^s$. Let us check that $\nabla(\Phi \circ G)$ belongs to $\Phi^{\infty,0}(X, \mathcal{H}_1^\sigma)$. If $u \in X \cap \mathcal{H}_1^s$ $(s \geq \sigma)$ and $h \in \mathcal{H}_1^{\sigma'}$ with $\sigma' \in [-s, s]$, we write

$$D[\nabla(\Phi\circ G)(v)]\cdot h = {}^tDG(v)\cdot\big((D\nabla\Phi)(G(v))\cdot DG(v)\cdot h\big)+(D({}^tDG)(v)\cdot h)\cdot\nabla\Phi(G(v)). \quad (2.2.3)$$

By (2.2.1) and (2.2.2), the first term on the right belongs to $\mathcal{H}_1^{\sigma'}$. To check that the last term in (2.2.3) belongs to the same space, we integrate it against $h'\in\mathcal{H}_1^{-\sigma'}$. We get

$$\int[(D({}^tDG)(v)\cdot h)\cdot\nabla\Phi(G(v))]h'\,dt\,dx = \int(\nabla\Phi)(G(v))D^2G(v)\cdot(h,h')\,dt\,dx. \quad (2.2.4)$$

By Definition 2.2.1,

$$D^2G(v)\cdot(h,h')\in\mathcal{H}_2^{-\max(\sigma_0,\sigma')+k}\subset\mathcal{H}_2^{-\max(\sigma_0,\sigma')}.$$

Since $\nabla\Phi(G(v))\in\mathcal{H}_2^s\subset\mathcal{H}_2^{\max(\sigma_0,\sigma')}$, this shows that the right side of (2.2.4) defines a continuous linear form in $h'\in\mathcal{H}_1^{-\sigma'}$.

We now study $D^2[\nabla(\Phi\circ G)(v)]\cdot(h_1,h_2)$, with $(h_1,h_2)\in\mathcal{H}_1^{\sigma_1}\times\mathcal{H}_1^{\sigma_2}$. To prove that

$$D^2[\nabla(\Phi\circ G)(v)]\cdot(h_1,h_2)\in\mathcal{H}_1^{-\sigma_3},$$

we compute, for $h_3\in\mathcal{H}_1^{\sigma_3}$,

$$D^2\int\nabla(\Phi\circ G)(v)h_3\,dt\,dx = D^2\int[(\nabla\Phi)(G(v))][DG(v)\cdot h_3]\,dt\,dx.$$

We get the following contributions (up to symmetries) for the action on $(h_1,h_2)\in\mathcal{H}_1^{\sigma_1}\times\mathcal{H}_1^{\sigma_2}$:

$$\int[(\nabla\Phi)(G(v))][D^3G(v)\cdot(h_1,h_2,h_3)]\,dt\,dx, \quad (2.2.5a)$$

$$\int[D((\nabla\Phi)(G(v)))\cdot h_1][D^2G(v)\cdot(h_2,h_3)]\,dt\,dx, \quad (2.2.5b)$$

$$\int[(D\nabla\Phi)(G(v))\cdot D^2G(v)\cdot(h_1,h_2)][DG(v)\cdot h_3]\,dt\,dx, \quad (2.2.5c)$$

$$\int[(D^2\nabla\Phi)(G(v))\cdot(DG(v)\cdot h_1,DG(v)\cdot h_2)][DG(v)\cdot h_3]\,dt\,dx. \quad (2.2.5d)$$

In (2.2.5a), we may assume for instance $h_1\in\mathcal{H}_1^{\sigma'}$, $h_2\in\mathcal{H}_1^{-\sigma'}$, $h_3\in\mathcal{H}_1^{\max(\sigma_0,\sigma')}$. Since $u\to D^2G(u)$ is C^1 on $X\cap\mathcal{H}_1^{\max(\sigma_0,\sigma')}$ with values in $\mathcal{L}_2(\mathcal{H}_1^{\sigma'}\times\mathcal{H}_1^{-\sigma'};\mathcal{H}_2^{-\max(\sigma_0,\sigma')+k})$, the second factor in the integrand belongs to $\mathcal{H}_2^{-\max(\sigma_0,\sigma')+k}$, so may be integrated against $\nabla\Phi(G(v))\in\mathcal{H}_2^s\subset\mathcal{H}_2^{\max(\sigma_0,\sigma')}$ for $s\geq\sigma'\geq 0$ and $s\geq\sigma$.

In (2.2.5b), $D^2G(v)\cdot(h_2,h_3)\in\mathcal{H}_2^{-\sigma_1+k}$. On the other hand $D((\nabla\Phi)(G(v)))\cdot h_1$ is in $\mathcal{H}_2^{\sigma_1}$ by (2.2.1) and (2.2.2), which allows one to integrate the product of the two factors.

In (2.2.5c), $DG(v)\cdot h_3$ lies in $\mathcal{H}_2^{\sigma_3+k}$. The other factor is given by the action of $(D\nabla\Phi)(G(v))$ on $D^2G(v)\cdot(h_1,h_2)\in\mathcal{H}_2^{-\sigma_3+k}$, whence again the wanted duality in the integral, using (2.2.2).

Finally, in (2.2.5d), we integrate $DG(v)\cdot h_3\in\mathcal{H}_2^{\sigma_3+k}$ against the action of $(D^2\nabla\Phi)(G(v))$ on a couple belonging to $\mathcal{H}_2^{\sigma_1+k}\times\mathcal{H}_2^{\sigma_2+k}\subset\mathcal{H}_2^{\sigma_1}\times\mathcal{H}_2^{\sigma_2}$. Since this vector is in $\mathcal{H}_2^{-\sigma_3-k}$ by definition of $C^{\infty,k}(Y,\mathbb{R})$, we get the conclusion. □

Let us write an equivalent form of (1.1.3) using the classes of functions above. Since the Hamiltonian F in (1.1.2) is real-valued, we may write (1.1.3) as a 2×2 system

$$\begin{aligned}(\omega D_t - \Delta + \mu)u &= \epsilon f(t,x) + \epsilon \frac{\partial F}{\partial \bar{u}}(t,x,u,\bar{u},\epsilon),\\ (-\omega D_t - \Delta + \mu)\bar{u} &= \epsilon \bar{f}(t,x) + \epsilon \frac{\partial F}{\partial u}(t,x,u,\bar{u},\epsilon).\end{aligned} \tag{2.2.6}$$

We identify $u = v_1 + iv_2$ with $v = \left[\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right]$ and $f = f_1 + if_2$ with $f = \left[\begin{smallmatrix} f_1 \\ f_2 \end{smallmatrix}\right]$. If we set

$$\nabla F(v) = \begin{bmatrix} \partial F/\partial v_1 \\ \partial F/\partial v_2 \end{bmatrix}$$

and

$$L_\omega = \begin{bmatrix} \Delta - \mu & -\omega\partial_t \\ \omega\partial_t & \Delta - \mu \end{bmatrix}, \tag{2.2.7}$$

Equation (2.2.6) is equivalent to

$$L_\omega v = -\epsilon f - \epsilon \nabla_v F(t,x,v). \tag{2.2.8}$$

Define, for $v \in \widetilde{\mathscr{H}}^s(\mathbb{S}^1 \times \mathbb{T}^d; \mathbb{R}^2)$,

$$\Phi_1(v, f, \omega, \epsilon) = \frac{1}{2}\int_{\mathbb{S}^1 \times \mathbb{T}^d} (L_\omega v) v \, dt \, dx + \epsilon \int_{\mathbb{S}^1 \times \mathbb{T}^d} f(t,x) v(t,x) \, dt \, dx \tag{2.2.9}$$

and

$$\Phi_2(v, \epsilon) = \int_{\mathbb{S}^1 \times \mathbb{T}^d} F(t,x,v(t,x),\epsilon) \, dt \, dx. \tag{2.2.10}$$

Then $\nabla\Phi_1(v) = L_\omega v + \epsilon f$, so $\Phi_1 \in C^{\infty,2}(\widetilde{\mathscr{H}}^\sigma \times \widetilde{\mathscr{H}}^\sigma, \mathbb{R})$ if $\sigma \geq \sigma_0$, since, by the definition of $\widetilde{\mathscr{H}}^\sigma(\mathbb{S}^1 \times \mathbb{T}^d; \mathbb{R}^2)$, L_ω is bounded from $\widetilde{\mathscr{H}}^\sigma$ to $\widetilde{\mathscr{H}}^{\sigma-2}$. By the statement following Definition 2.2.2, $\Phi_2 \in C^{\infty,0}(\widetilde{\mathscr{H}}^\sigma, \mathbb{R})$ $(\sigma \geq \sigma_0)$. Moreover (2.2.8) may be written

$$\nabla_v[\Phi_1(v, f, \omega, \epsilon) + \epsilon\Phi_2(v, \epsilon)] = 0. \tag{2.2.11}$$

Using the notation introduced at the bottom of page 646, we decompose any $v \in \widetilde{\mathscr{H}}^s(\mathbb{S}^1 \times \mathbb{T}^d; \mathbb{R}^2)$ as $v = v' + v''$ on the decomposition

$$\widetilde{\mathscr{H}}^s(\mathbb{S}^1 \times \mathbb{T}^d; \mathbb{R}^2) = \mathscr{H}^s(\mathbb{S}^1 \times \mathbb{T}^d; \mathbb{R}^2) \oplus \mathscr{F}^s(\mathbb{S}^1 \times \mathbb{T}^d; \mathbb{R}^2).$$

We denote for $q > 0$ by $B_q(\widetilde{\mathscr{H}}^s)$, $B_q(\mathscr{H}^s)$, $B_q(\mathscr{F}^s)$ the ball of center 0 and radius q in these spaces. By (1.2.6), if $v \in \mathscr{F}^s(\mathbb{S}^1 \times \mathbb{T}^d; \mathbb{R}^2)$, $(j,n) \in \mathbb{Z} \times \Omega_\alpha \subset \mathbb{Z} \times \mathbb{Z}^d$ and $\hat{v}(j,n) \neq 0$, then $|j| > K_0\langle n(\alpha)\rangle^2$ or $|j| < K_0^{-1}\langle n(\alpha)\rangle^2$. Moreover, since $\mu \in \mathbb{R} - \mathbb{Z}_-$, $\big||n|^2 + \mu\big| \geq c(\mu)\langle n(\alpha)\rangle^2$ when $n \in \Omega_\alpha$, for some constant $c(\mu) > 0$. If we fix K_0 large enough, and use that ω stays in $[1,2]$, we conclude that the eigenvalues of L_ω satisfy the bounds

$$\big|\omega j + |n|^2 + \mu\big| \geq c(|j| + \langle n(\alpha)\rangle^2) \quad \text{for } j \in \mathbb{Z},\ n \in \Omega_\alpha,\ \alpha \in \mathscr{A}.$$

This shows that the restriction of L_ω to $\mathscr{F}^{s+2}$ is an invertible operator from $\mathscr{F}^{s+2}$ to $\mathscr{F}^s$ (uniformly in $\omega \in [1,2]$).

Let us reduce (2.2.11) to an equation on the space $\mathscr{H}^s(\mathbb{S}^1 \times \mathbb{T}^d; \mathbb{R}^2)$.

Proposition 2.2.4. *Let $\sigma \geq \sigma_0, q > 0$, $f' \in B_q(\mathscr{H}^\sigma)$. There are $\gamma_0 \in]0,1]$,*

- *an element $(v', f'') \to \psi_2(v', f'', \omega, \epsilon)$ of $C^{\infty,0}(W_q; \mathbb{R})$, where $W_q = B_q(\mathscr{H}^\sigma(\mathbb{S}^1 \times \mathbb{T}^d; \mathbb{R}^2)) \times B_q(\mathscr{F}^\sigma(\mathbb{S}^1 \times \mathbb{T}^d; \mathbb{R}^2))$, with C^1 dependence in $(\omega, \epsilon) \in [1,2] \times [0, \gamma_0]$, and*
- *an element $(v', f'') \to G(v', f'', \omega, \epsilon)$ of $\Phi^{\infty,-2}(W_q, \mathscr{F}^{\sigma+2})$, with C^1 dependence in (ω, ϵ),*

such that, for any given subset $A \subset [1,2] \times [0, \gamma_0]$, the following two conditions are equivalent:

(i) *The function $v = (v', G(v', f'', \omega, \epsilon))$ satisfies for any $(\omega, \epsilon) \in A$*

$$L_\omega v + \epsilon f + \epsilon \nabla_v \Phi_2(v, \epsilon) = 0, \tag{2.2.12}$$

where $f = f' + f''$.

(ii) *The function v' satisfies for any $(\omega, \epsilon) \in A$*

$$L_\omega v' + \epsilon f' + \epsilon \nabla_{v'} \psi_2(v', f'', \omega, \epsilon) = 0. \tag{2.2.13}$$

Proof. Write (2.2.12) as

$$L_\omega v' + \epsilon f' + \epsilon \nabla_{v'} \Phi_2(v', v'', \epsilon) = 0, \tag{2.2.14a}$$

$$L_\omega v'' + \epsilon f'' + \epsilon \nabla_{v''} \Phi_2(v', v'', \epsilon) = 0. \tag{2.2.14b}$$

We are looking for a solution of the second equation under the form $v'' = -\epsilon L_\omega^{-1} f'' + \epsilon w''$. The new unknown w'' satisfies

$$w'' = -L_\omega^{-1} \nabla_{v''} \Phi_2(v', -\epsilon L_\omega^{-1} f'' + \epsilon w'', \epsilon). \tag{2.2.15}$$

Let $q_0 > 0$ be such that $\|L_\omega^{-1} \nabla_{v''} \Phi_2(v', h, \epsilon)\|_{\mathscr{F}^{\sigma+2}} \leq q_0/2$ for any $(v', h) \in B_q(\mathscr{H}^\sigma) \times B_q(\mathscr{F}^\sigma)$, any $\epsilon \in [0,1]$, any $\omega \in [1,2]$. The fixed point theorem with parameters shows that there is $\gamma_0 \in]0,1]$ such that for any $(v', f'') \in W_q$, any $\epsilon \in [0, \gamma_0]$, Equation (2.2.15) has a unique solution $w'' \in B_{q_0}(\mathscr{F}^{\sigma+2})$. We denote this solution by $G(v', f'', \omega, \epsilon)$. This is a smooth function of $(v', f'') \in W_q$, with C^1 dependence in (ω, ϵ). If moreover $(v', f'') \in \widetilde{\mathscr{H}}^s$ for some $s \geq \sigma$, it follows from (2.2.15) that $w'' \in \mathscr{F}^{s+2}$ (using that L_ω^{-1} gains two derivatives in the $\mathscr{F}^s$ scale). Let us show that G belongs to $\Phi^{\infty,-2}(W_q, \mathscr{F}^{\sigma+2})$. By the definition of G

$$\begin{aligned} D_{v'} G(v', f'', \omega, \epsilon) &= -L_\omega^{-1}(\mathrm{Id} - \epsilon M''(v', f'', \omega, \epsilon) L_\omega^{-1})^{-1} M'(v', f'', \omega, \epsilon), \\ D_{f''} G(v', f'', \omega, \epsilon) &= \epsilon L_\omega^{-1}(\mathrm{Id} - \epsilon M''(v', f'', \omega, \epsilon) L_\omega^{-1})^{-1} M''(v', f'', \omega, \epsilon) L_\omega^{-1}, \end{aligned} \tag{2.2.16}$$

with

$$\begin{aligned} M'(v', f'', \omega, \epsilon) &= (D_{v'} \nabla_{v''} \Phi_2)(v', -\epsilon L_\omega^{-1} f'' + \epsilon G, \epsilon), \\ M''(v', f'', \omega, \epsilon) &= -(D_{v''} \nabla_{v''} \Phi_2)(v', -\epsilon L_\omega^{-1} f'' + \epsilon G, \epsilon). \end{aligned} \tag{2.2.17}$$

Since $\Phi_2 \in C^{\infty,0}(W_q, \mathbb{R})$, when $(v', f'') \in W_q \cap \widetilde{\mathscr{H}}^s$ for some $s \geq \sigma$, one can extend $M''(v', f'', \omega, \epsilon)$ into an element of $\mathscr{L}(\mathscr{F}^{\sigma'}, \mathscr{F}^{\sigma'})$ for any $\sigma' \in [-s, s]$; similarly, $M'(v', f'', \omega, \epsilon)$ extends as an element of $\mathscr{L}(\mathscr{H}^{\sigma'}, \mathscr{F}^{\sigma'})$. We choose γ_0 small enough that for $\epsilon \in [0, \gamma_0]$, $\epsilon \| M''(v', f'', \omega, \epsilon) L_\omega^{-1} \|_{\mathscr{L}(\mathscr{F}^\sigma, \mathscr{F}^\sigma)}$ is smaller than $\frac{1}{2}$. Let us check that G satisfies the first condition in Definition 2.1.1. We may write the first equation in (2.2.16) as

$$\begin{aligned} &D_{v'} G(v', f'', \omega, \epsilon) \\ &\quad = -\sum_{k=0}^{2N-1} L_\omega^{-1} (\epsilon M'' L_\omega^{-1})^k M' - L_\omega^{-1} (\epsilon M'' L_\omega^{-1})^N (\mathrm{Id} - \epsilon M'' L_\omega^{-1})^{-1} (\epsilon M'' L_\omega^{-1})^N M', \end{aligned} \tag{2.2.18}$$

and a similar formula holds for $D_{f''} G$. If N is chosen large enough relatively to s, and $\sigma' \in [-s, s]$, $(\epsilon M'' L_\omega^{-1})^N M'$ sends $\mathscr{H}^{\sigma'}$ to $\mathscr{F}^\sigma$, over which $(\mathrm{Id} - \epsilon M'' L_\omega^{-1})^{-1}$ is bounded. Consequently, the last contribution in (2.2.18) is in $\mathscr{F}^{s+2} \subset \mathscr{F}^{\sigma'+2}$. The sum on the right side being bounded from $\mathscr{H}^{\sigma'}$ to $\mathscr{F}^{\sigma'+2}$ for any $\sigma' \in [-s, s]$, we get the same property for $D_{v'} G$. We argue in the same way for $D_{f''} G$. To check the second condition in Definition 2.1.1, we compute from (2.2.16), for $(h_1, h_2) \in \mathscr{H}^{\sigma_1} \times \mathscr{H}^{\sigma_2}$

$$\begin{aligned} D_{v'}^2 G(v', f'', \omega, \epsilon) \cdot (h_1, h_2) &= -L_\omega^{-1} (\mathrm{Id} - \epsilon M'' L_\omega^{-1})^{-1} [(D_{v'} M' \cdot h_1) \cdot h_2] \\ &\quad - L_\omega^{-1} (\mathrm{Id} - \epsilon M'' L_\omega^{-1})^{-1} (\epsilon D_{v'} M'' L_\omega^{-1} \cdot h_1) (\mathrm{Id} - \epsilon M'' L_\omega^{-1})^{-1} M' \cdot h_2. \end{aligned}$$

If $\{\sigma_1, \sigma_2, \sigma_3\} = \{\sigma', -\sigma', \max(\sigma_0, \sigma')\}$, the assumption on Φ_2 implies that $D_{v'} M'$ sends $\mathscr{H}^{\sigma_1} \times \mathscr{H}^{\sigma_2}$ to $\mathscr{F}^{-\sigma_3}$, and $D_{v'} M''$ sends $\mathscr{H}^{\sigma_1} \times \mathscr{F}^{\sigma_2}$ to $\mathscr{F}^{-\sigma_3}$. Using expansions as in (2.2.18), we conclude that if $(h_1, h_2) \in \mathscr{H}^{\sigma_1} \times \mathscr{H}^{\sigma_2}$, $D_{v'}^2 G(v', f'', \omega, \epsilon) \cdot (h_1, h_2) \in \mathscr{F}^{-\sigma_3+2}$. One studies $D_{v'} D_{f''} G$ and $D_{f''}^2 G$ in the same way. It is clear that $DG, D^2 G$ are smooth in $(v', f'') \in W_q \cap \widetilde{\mathscr{H}}^s$ and have a C^1 dependence in (ω, ϵ); hence $G \in \Phi^{\infty,-2}(W_q, \mathscr{F}^{\sigma+2})$.

Let us obtain the equivalent form (2.2.13) of (2.2.12) or (2.2.11). By (2.2.9), (2.2.10)

$$\begin{aligned} \Phi_1(v', v'', \omega, \epsilon) + \epsilon \Phi_2(v', v'', \epsilon) &= \frac{1}{2} \int (L_\omega v') v' \, dt \, dx + \epsilon \int f' v' \, dt \, dx \\ &\quad + \frac{1}{2} \int (L_\omega v'') v'' \, dt \, dx + \epsilon \int f'' v'' \, dt \, dx + \epsilon \Phi_2(v', v'', \epsilon). \end{aligned}$$

We plug into this expression the solution $v'' = -\epsilon L_\omega^{-1} f'' + \epsilon G(v', f'', \omega, \epsilon)$ of (2.2.14b). We get after simplification the function

$$\Psi(v', f'', \omega, \epsilon) = \frac{1}{2} \int (L_\omega v') v' \, dt \, dx + \epsilon \int f' v' \, dt \, dx - \frac{\epsilon^2}{2} \int (L_\omega^{-1} f'') f'' \, dt \, dx + \epsilon \psi_2(v', f'', \omega, \epsilon),$$

where

$$\psi_2(v', f'', \omega, \epsilon) = \frac{\epsilon}{2} \int G(L_\omega G) \, dt \, dx + \Phi_2(v', -\epsilon L_\omega^{-1} f'' + \epsilon G, \epsilon). \tag{2.2.19}$$

The integral in (2.2.19) is the composition of the function defined on $\mathscr{F}^\sigma$ by $w'' \to \int w''(L_\omega w'') \, dt \, dx$, which is an element of $C^{\infty,2}(\mathscr{F}^\sigma, \mathbb{R})$, with the map

$$\begin{aligned} (v', f'') &\to G(v', f'', \omega, \epsilon), \\ \widetilde{\mathscr{H}}^\sigma &\to \mathscr{F}^{\sigma+2}, \end{aligned}$$

which is an element of $\Phi^{\infty,-2}(W_q, \mathscr{F}^{\sigma+2})$. By Lemma 2.2.3, we conclude that $\psi_2 \in C^{\infty,0}(W_q, \mathbb{R})$.

Since G is defined as the critical point (up to an affine change of variables) of the map

$$v'' \to (\Phi_1 + \epsilon\Phi_2)(v', v'', \omega, \epsilon),$$

and since Ψ is the corresponding critical value, v' solves (2.2.14a) if and only of $\nabla_{v'}\Psi(v', f'', \omega, \epsilon) = 0$. This gives (2.2.13). $\square$

We finish this subsection with a lemma that will be useful in the sequel. Let X be an open subset of $\mathcal{H}^{\sigma_0}(\mathbb{S}^1 \times \mathbb{T}^d; \mathbb{R}^2)$, ψ an element of $C^{\infty,0}(X; \mathbb{R})$. For $v \in X \cap \mathcal{H}^{+\infty}$, $w_1, w_2 \in \mathcal{H}^{+\infty}$, we set

$$L(v; w_1, w_2) = D^2\psi(v) \cdot (w_1, w_2). \tag{2.2.20}$$

This is a continuous bilinear form in $(w_1, w_2) \in \mathcal{H}^0 \times \mathcal{H}^0$, by the definition of $C^{\infty,0}(X; \mathbb{R})$. By the Riesz theorem, we write it

$$L(v; w_1, w_2) = \int_{\mathbb{S}^1 \times \mathbb{T}^d} (W(v)w_1)w_2 \, dt \, dx$$

for some symmetric $\mathcal{H}^0$-bounded operator $W(v)$. Since Definition 2.2.2 implies that $v \to D^2\psi(v)$ is a smooth map on X with values in the space of continuous bilinear forms on $\mathcal{H}^0 \times \mathcal{H}^0$, we know that $v \to W(v)$ is smooth with values in $\mathcal{L}(\mathcal{H}^0, \mathcal{H}^0)$. Thus we may write, for $j = 1, \dots, d$,

$$\begin{aligned} L(v; \partial_{x_j} w_1, w_2) + L(v; w_1, \partial_{x_j} w_2) &= -\int_{\mathbb{S}^1 \times \mathbb{T}^d} ((\partial_{x_j} W(v))w_1)w_2 \, dt \, dx \\ &= -(\partial_v L)(v; w_1, w_2) \cdot (\partial_{x_j} v), \end{aligned} \tag{2.2.21}$$

for any $v \in X \cap \mathcal{H}^{+\infty}$ and $w_1, w_2 \in \mathcal{H}^{+\infty}$.

We denote by $\mathbb{C}[X_\alpha; \alpha \in \mathbb{N}^d]$ the space of polynomials in indeterminates X_α, indexed by elements of $\mathbb{N}^d$. If $X_{\alpha_1}^{k_1} \cdots X_{\alpha_\ell}^{k_\ell}$ is a monomial, its weight will be defined as $k_1|\alpha_1| + \cdots + k_\ell|\alpha_\ell|$. The weight of any polynomial is then defined in the natural way.

Lemma 2.2.5. *For any $N \in \mathbb{N}$ and $\ell \in \mathbb{N}$, there is a polynomial $Q_N^\ell \in \mathbb{C}[X_\alpha; \alpha \in \mathbb{N}^d]$, of weight less or equal to N, and for any $q > 0$ a constant $C > 0$ such that, for any $v \in B_q(\mathcal{H}^{\sigma_0}) \cap \mathcal{H}^{+\infty} \cap X$, any $h_1, \dots, h_\ell$ in $\mathcal{H}^{+\infty}$, any $n, n' \in \mathbb{Z}^d$*

$$\begin{aligned} &\| \Pi_n \partial_v^\ell W(v) \cdot (h_1, \dots, h_\ell) \Pi_{n'} \|_{\mathcal{L}(\mathcal{H}^0)} \\ &\qquad \le C \langle n - n' \rangle^{-N} \sum_{N_0 + \cdots + N_\ell = N} Q_{N_0}^\ell ((\|\partial^\alpha v\|_{\mathcal{H}^{\sigma_0}})_\alpha) \prod_{\ell'=1}^{\ell} \|h_{\ell'}\|_{\mathcal{H}^{\sigma_0 + N_{\ell'}}}. \end{aligned} \tag{2.2.22}$$

Proof. Since ${}^t\Pi_n = \Pi_{-n}$, we may write, for any $w_1, w_2 \in \mathcal{H}^{+\infty}$,

$$\begin{aligned} (n_j - n'_j) \int (\Pi_n W(v) \Pi_{n'} w_1) w_2 \, dt \, dx &= (n_j - n'_j) L(v; \Pi_{n'} w_1, \Pi_{-n} w_2) \\ &= i[L(v; \partial_{x_j} \Pi_{n'} w_1, \Pi_{-n} w_2) + L(v; \Pi_{n'} w_1, \partial_{x_j} \Pi_{-n} w_2)] \\ &= -i(\partial_v L)(v; \Pi_{n'} w_1, \Pi_{-n} w_2) \cdot (\partial_{x_j} v), \end{aligned}$$

by (2.2.21). Iterating the computation, we get for

$$\langle n-n'\rangle^N \left|\int (\Pi_n W(v)\Pi_{n'}w_1)w_2\,dt\,dx\right|$$

an estimate in terms of quantities

$$\left|(\partial_v^p L)(v;\Pi_{n'}w_1,\Pi_{-n}w_2)\cdot(\partial^{\alpha_1}v,\dots,\partial^{\alpha_p}v)\right|,$$

with $|\alpha_1|+\cdots+|\alpha_p|\le N$. By the properties of L, this is bounded from above by

$$C\|\Pi_{n'}w_1\|_{L^2}\|\Pi_{-n}w_2\|_{L^2}\prod_{p'=1}^{p}\|\partial^{\alpha_{p'}}v\|_{\mathcal{H}^{\sigma_0}}$$

when v stays in a fixed $\mathcal{H}^{\sigma_0}$-ball. This implies (2.2.22) for $\ell=0$. The proof for general ℓ is similar, up to notation. □

2.3. *Reduction to a paradifferential equation.* We want to construct, under the conditions of the statement of Theorem 1.1.1, periodic solutions to (2.2.6). We have rewritten this equation under the real form (2.2.8) (or (2.2.11)). By Proposition 2.2.4, if we find a periodic solution v' for (2.2.13), we get a periodic solution v for (2.2.12), which is a rewriting of (2.2.11). We are thus reduced to finding a solution $v'\in\mathcal{H}^\sigma(\mathbb{S}^1\times\mathbb{T}^d;\mathbb{R}^2)$ to (2.2.13). Since the force term $f=f'+f''$ will be fixed, we no longer write the f'' dependence in the function ψ_2 defined in Proposition 2.2.4. Moreover, since, in the rest of the paper, we will study only the equivalent formulation (2.2.13) of our initial problem, we drop the primes; that is, we study

$$L_\omega v+\epsilon f+\epsilon\nabla_v\psi_2(v,\omega,\epsilon)=0, \tag{2.3.1}$$

where $v\in B_q(\mathcal{H}^\sigma(\mathbb{S}^1\times\mathbb{T}^d;\mathbb{R}^2))$, $f\in\mathcal{H}^s(\mathbb{S}^1\times\mathbb{T}^d;\mathbb{R}^2)$, ψ_2 is in $C^{\infty,0}(B_q(\mathcal{H}^\sigma),\mathbb{R})$ for some $\sigma\in[\sigma_0,s]$, $q>0$ and for $\epsilon\in[0,\gamma_0]$, with $\gamma_0\in\,]0,1]$ small enough. We shall use the equivalent norms (1.2.7) and (1.2.8) on the spaces we consider.

Our objective in this subsection is to rewrite the nonlinearity in (2.3.1) using paradifferential operators.

Proposition 2.3.1. *Let $q>0$, $\sigma\ge\sigma_0+d+1$ be given. Set*

$$r=\sigma-\sigma_0-d-1. \tag{2.3.2}$$

There is a symmetric element $\widetilde{V}\in\Psi^0_{\mathbb{R}}(0,\sigma,q)\otimes\mathcal{M}_2(\mathbb{R})$ and an element $\widetilde{R}\in\mathcal{R}^r_{0,\mathbb{R}}(0,\sigma,q)\otimes\mathcal{M}_2(\mathbb{R})$, with C^1 dependence in (ω,ϵ), such that, for any $v\in B_q(\mathcal{H}^\sigma)$, $\epsilon\in[0,\gamma_0]$, and $\omega\in[1,2]$,

$$\nabla_v\psi_2(v,\omega,\epsilon)=\widetilde{V}(v,\omega,\epsilon)v+\widetilde{R}(v,\omega,\epsilon)v. \tag{2.3.3}$$

Let us comment about the interest of this decomposition of $\nabla_v\psi_2$. It allows us to express the nonlinearity in (2.3.1) as the sum of a remainder and of the action of the paradifferential potential $\widetilde{V}(v,\omega,\epsilon)$ on v. In that way, the main contribution to the nonlinearity is expressed in terms of a class of operators enjoying a nice calculus. This will be exploited below to perform a block diagonalization.

We introduce some notation for the proof. For $p \in \mathbb{N}$, $v \in \mathcal{H}^0(\mathbb{S}^1 \times \mathbb{T}^d; \mathbb{R}^2)$, we set

$$\begin{aligned} \Delta_0 v &= \Pi_0 v, & \Delta_p v &= \sum_{\substack{n\in\mathbb{Z}^d \\ 2^{p-1}\le |n| < 2^p}} \Pi_n v & &\text{for } p \ge 1, \\ S_0 v &= 0, & S_p v &= \sum_{p'=0}^{p-1} \Delta_{p'} v = \sum_{\substack{n\in\mathbb{Z}^d \\ |n| < 2^{p-1}}} \Pi_n v & &\text{for } p \ge 1. \end{aligned} \tag{2.3.4}$$

We also consider the frequency cut-offs defined for $n, n' \in \mathbb{Z}^d$ by

$$S(n,n') = \sum_{|n''| \le 2(1+\min(|n|,|n'|))} \Pi_{n''}. \tag{2.3.5}$$

Lemma 2.3.2. *Let $\sigma \ge \sigma_0 + d + 1$, $q > 0$. There is a map $(v, \omega, \epsilon) \to W(v, \omega, \epsilon)$ defined for $v \in B_q(\mathcal{H}^\sigma)$, $\epsilon \in [0, \gamma_0]$, $\omega \in [1, 2]$, with values in the space of bounded symmetric operators on $\mathcal{H}^0(\mathbb{S}^1 \times \mathbb{T}^d; \mathbb{R}^2)$, which is C^∞ in v and has C^1 dependence in (ω, ϵ), such that for any (v, ω, ϵ)*

$$\psi_2(v, \omega, \epsilon) = \int_{\mathbb{S}^1\times\mathbb{T}^d} [W(v, \omega, \epsilon)v] v \, dt \, dx \tag{2.3.6}$$

and such that the following estimate holds: For $(\ell, N) \in \mathbb{N}\times\mathbb{N}$, there are polynomials $Q_N^\ell \in \mathbb{C}[X_\alpha; \alpha \in \mathbb{N}]$, of weight at most N, and there is for any $M \in \mathbb{N}$ and $\ell \in \mathbb{N}$ a constant C, depending only on ℓ, q, M, such that for any $v \in B_q(\mathcal{H}^\sigma)$, any $\epsilon \in [0, \gamma_0]$, any $\omega \in [1, 2]$, any $(a_0, a_1) \in \mathbb{N}^2$ with $a_0 + a_1 \le 1$, any $(h_1, \dots, h_\ell) \in (\mathcal{H}^\sigma)^\ell$, and any $n, n' \in \mathbb{Z}^d$,

$$\begin{aligned} &\left\| \Pi_n \partial_\omega^{a_0} \partial_\epsilon^{a_1} D_v^\ell W(v, \omega, \epsilon) \cdot (h_1, \dots, h_\ell) \Pi_{n'} \right\|_{\mathcal{L}(\mathcal{H}^0)} \\ &\qquad \le C \langle n - n' \rangle^{-M} \sum_{N_0 + \dots + N_\ell = M} Q_{N_0}^\ell \big((\| \partial^\alpha S(n,n') v \|_{\mathcal{H}^{\sigma_0}})_\alpha \big) \prod_{\ell'=1}^{\ell} \| S(n,n') h_{\ell'} \|_{\mathcal{H}^{\sigma_0 + N_{\ell'}}}. \end{aligned} \tag{2.3.7}$$

Proof. We do not write ω, ϵ, which play the role of parameters. Since ψ_2 vanishes at order 3 at $v = 0$, and $S_p v \to v$ in $\mathcal{H}^\sigma$ when $p \to +\infty$, we write

$$\psi_2(v) = \sum_{p_1=0}^{+\infty} (\psi_2(S_{p_1+1} v) - \psi_2(S_{p_1} v)) = \sum_{p_1=0}^{+\infty} \int_0^1 (\partial\psi_2)(S_{p_1} v + \tau_1 \Delta_{p_1} v) \, d\tau_1 \cdot \Delta_{p_1} v.$$

Repeating the process, we get

$$\psi_2(v) = \sum_{p_1=0}^{+\infty} \sum_{p_2=0}^{+\infty} \int_0^1 \int_0^1 (\partial^2 \psi_2)(\Omega_{p_1,p_2}(\tau_1, \tau_2) v) \, d\tau_2 \cdot (\Delta_{p_2}(S_{p_1} + \tau_1 \Delta_{p_1}) v, \Delta_{p_1} v) \, d\tau_1,$$

where $\Omega_{p_1,p_2}(\tau_1,\tau_2) = \prod_{\ell=1}^2 (S_{p_\ell} + \tau_\ell \Delta_{p_\ell})$. By the discussion before Lemma 2.2.5, there is a symmetric operator $\widetilde{W}(v)$ satisfying (2.2.22), such that

$$\partial^2 \psi_2(v) \cdot (w_1, w_2) = \int [\widetilde{W}(v) w_1] w_2 \, dt \, dx.$$

We set

$$
\begin{aligned}
W(v) = \frac{1}{2} \sum_{p_1} \sum_{p_2} \int_0^1 \int_0^1 \Delta_{p_1}[\widetilde{W}(\Omega_{p_1,p_2}(\tau_1,\tau_2)v)\Delta_{p_2}(S_{p_1} + \tau_1 \Delta_{p_1})]\, d\tau_1 d\tau_2 \\
+ \frac{1}{2} \sum_{p_1} \sum_{p_2} \int_0^1 \int_0^1 \Delta_{p_2}(S_{p_1} + \tau_1 \Delta_{p_1})[\widetilde{W}(\Omega_{p_1,p_2}(\tau_1,\tau_2)v)\Delta_{p_1}]\, d\tau_1 d\tau_2. \quad (2.3.8)
\end{aligned}
$$

This is a symmetric operator. We apply (2.2.22) to $\widetilde{W}$. Because of the cut-offs in the argument of $\widetilde{W}$ in (2.3.8), we may write $\Pi_n W(v)\Pi_{n'} = \Pi_n W(S(n,n')v)\Pi_{n'}$. Consequently, (2.2.22) implies (2.3.7). Note that since $\sigma \geq \sigma_0 + d + 1$, we may take some integer $M > d$, such that $\sigma_0 + M \leq \sigma$, so that for $v, h_{\ell'}$ in $\mathcal{H}^\sigma$, the right side of (2.3.7) is bounded from above by $C\langle n - n'\rangle^{-M}$. This shows that $W(v)$ is indeed bounded on $\mathcal{H}^0$. □

Proof of Proposition 2.3.1. Let h_1 be in $\mathcal{H}^{+\infty}(\mathbb{S}^1 \times \mathbb{T}^d; \mathbb{R}^2)$ and write

$$
D\psi_2(v,\omega,\epsilon)\cdot h_1 = 2\int_{\mathbb{S}^1\times\mathbb{T}^d} (W(v,\omega,\epsilon)v)h_1\, dt\, dx + \int_{\mathbb{S}^1\times\mathbb{T}^d} ((DW(v,\omega,\epsilon)\cdot h_1)v)v\, dt\, dx. \quad (2.3.9)
$$

Define

$$
\widetilde{V} = 2\sum_{n,n'} \mathbb{1}_{|n-n'|\leq \frac{1}{10}(|n|+|n'|)} \Pi_n W(v,\omega,\epsilon)\Pi_{n'}.
$$

In (2.3.7), we can bound $\|\partial^\alpha S(n,n')v\|_{\mathcal{H}^{\sigma_0}}$ by $C\|v\|_{\mathcal{H}^\sigma}$ when $|\alpha| \leq M \leq \sigma - \sigma_0$, and we can control $\|S(n,n')h_{\ell'}\|_{\mathcal{H}^{\sigma_0+N_{\ell'}}}$ by $C\|h_{\ell'}\|_{\mathcal{H}^{\sigma_0+M}}$. We obtain that $\widetilde{V}$ satisfies (2.1.1), and is thus an element of $\Psi^0(0,\sigma,q)$. We show that the remaining terms in (2.3.9) give contributions to the last term in (2.3.3). Set

$$
R_1(v,\omega,\epsilon) = 2\sum_n \sum_{n'} \Pi_n W(v,\omega,\epsilon)\Pi_{n'} \mathbb{1}_{|n-n'|> \frac{1}{10}(|n|+|n'|)}.
$$

We estimate

$$
\left\| \Pi_n \partial_\omega^{a_0} \partial_\epsilon^{a_1} \partial_v^\ell R_1(v,\omega,\epsilon)\cdot(h_1,\dots,h_\ell)\Pi_{n'} \right\|_{\mathcal{L}(\mathcal{H}^0)} \quad (2.3.10)
$$

using (2.3.7) with $M > \sigma - \sigma_0$. Since $\|S(n,n')w\|_{\mathcal{H}^{\sigma_0+\beta}} \leq C(1 + \inf(|n|,|n'|))^{(\beta+\sigma_0-\sigma)_+}\|w\|_{\mathcal{H}^\sigma}$, we get for (2.3.10) the upper bound

$$
C(1 + |n| + |n'|)^{-M}(1 + \inf(|n|,|n'|))^{M+\sigma_0-\sigma} \prod_{\ell'=1}^{\ell} \|h_{\ell'}\|_{\mathcal{H}^\sigma}.
$$

Taking M large enough, we deduce the boundedness of $R_1(v,\omega,\epsilon)$ and of its derivatives from $\mathcal{H}^s$ to $\mathcal{H}^{s+(\sigma-\sigma_0-d-1)}$, for any $s \geq \sigma_0$; thus $R_1 \in \mathcal{R}^r_{0,\mathbb{R}}(0,\sigma,q)$.

We treat next the last contribution to (2.3.9), defining an operator $R_2(v,\omega,\epsilon)$ by

$$
\int [(DW(v,\omega,\epsilon)\cdot h)v]w\, dt\, dx = \int [R_2(v,\omega,\epsilon)w]h\, dt\, dx \quad (2.3.11)
$$

for any $h, w \in \mathcal{H}^{+\infty}$. On the left side, we decompose the last v as $\sum_{n'} \Pi_{n'} v$ and w as $\sum_n \Pi_n w$. We bound the modulus of (2.3.11) by

$$\sum_n \sum_{n'} \left\| \Pi_n DW(v, \omega, \epsilon) \cdot h \Pi_{n'} \right\|_{\mathcal{L}(\mathcal{H}^0)} \|\Pi_{n'} v\|_{\mathcal{H}^0} \|\Pi_n w\|_{\mathcal{H}^0}. \tag{2.3.12}$$

To show that $R_2(v, \omega, \epsilon)$ is bounded from $\mathcal{H}^s$ to $\mathcal{H}^{s+r}$, we bound $\|\Pi_n w\|_{\mathcal{H}^0} \leq c_n \langle n \rangle^{-s} \|w\|_{\mathcal{H}^s}$, for a ℓ^2-sequence $(c_n)_n$ and take $h \in \mathcal{H}^{-s-r}$. We use (2.3.7) with $\ell = 1$. We have the bound

$$Q^1_{N_0}\big((\|\partial^\alpha S(n, n')v\|_{\mathcal{H}^{\sigma_0}})_\alpha\big) \|S(n, n')h\|_{\mathcal{H}^{\sigma_0 + N_1}} \leq C(1 + \inf(|n|, |n'|))^{M+s+r+\sigma_0} \|h\|_{\mathcal{H}^{-s-r}}$$

since v is bounded in $\mathcal{H}^\sigma$. Consequently, the general term of (2.3.12) is smaller than

$$C \langle n - n' \rangle^{-M} (1 + \inf(|n|, |n'|))^{M+s+r+\sigma_0} \langle n \rangle^{-s} c_n \|w\|_{\mathcal{H}^s} \|h\|_{\mathcal{H}^{-s-r}} \langle n' \rangle^{-\sigma} c'_{n'} \|v\|_{\mathcal{H}^\sigma} \tag{2.3.13}$$

for some ℓ^2-sequence $(c'_{n'})_{n'}$. Taking $M = d + 1$, and using the value (2.3.2) of r and $s \geq 0$, $\sigma \geq 0$, one checks that the sum in n, n' of (2.3.13) converges. This shows the boundedness of $R_2(v, \omega, \epsilon)$ from $\mathcal{H}^s$ to $\mathcal{H}^{s+r}$. One treats in the same way $\partial_\omega^{a_0} \partial_\epsilon^{a_1} \partial_v^\ell R_2(v, \omega, \epsilon)$. Consequently $R_2 \in \mathcal{R}^r_{0,\mathbb{R}}(0, \sigma, q)$. This concludes the proof of the proposition. ☐

Let us conclude this section writing in complex coordinates the equation we are interested in. By Proposition 2.3.1, Equation (2.3.1) may be written

$$L_\omega v + \epsilon f + \epsilon \widetilde{V}(v, \omega, \epsilon)v + \epsilon \widetilde{R}(v, \omega, \epsilon)v = 0. \tag{2.3.14}$$

We write $v = \left[\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right] \in \mathbb{R}^2$ and set $u = v_1 + i v_2$, $U = \left[\begin{smallmatrix} u \\ \bar{u} \end{smallmatrix}\right]$, $I' = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$.

Corollary 2.3.3. *Let $q > 0$, $\sigma \geq \sigma_0 + d + 1$, r given by* (2.3.2). *There is an element $V(U, \omega, \epsilon)$ in $\Psi^0(0, \sigma, q) \otimes \mathcal{M}_2(\mathbb{R})$ with $V(U, \omega, \epsilon)^* = V(U, \omega, \epsilon)$, there is $R(U, \omega, \epsilon)$ in $\mathcal{R}^r_0(0, \sigma, q) \otimes \mathcal{M}_2(\mathbb{R})$ such that* (2.3.14) *is equivalent to*

$$[(\omega I' D_t + (-\Delta + \mu)I) + \epsilon V(U, \omega, \epsilon)]U = \epsilon R(U, \omega, \epsilon)U + \epsilon f \tag{2.3.15}$$

(*where*, *abusing notation*, *we write f for* $\left[\begin{smallmatrix} f_1 + if_2 \\ f_1 - if_2 \end{smallmatrix}\right]$).

Proof. Write $\widetilde{V}(v, \omega, \epsilon) = (\widetilde{V}_{i,j}(v, \omega, \epsilon))_{1 \leq i,j \leq 2}$, $\widetilde{R}(v, \omega, \epsilon) = (\widetilde{R}_{i,j}(v, \omega, \epsilon))_{1 \leq i,j \leq 2}$ and note that (2.3.14) implies

$$\begin{aligned}(\omega D_t &- \Delta + \mu)u \\ &= \epsilon(f_1 + if_2) - \epsilon V_{11}(U, \omega, \epsilon)u - \epsilon V_{12}(U, \omega, \epsilon)\bar{u} + \epsilon R_{11}(U, \omega, \epsilon)u + \epsilon R_{12}(U, \omega, \epsilon)\bar{u},\end{aligned} \tag{2.3.16}$$

where we have set

$$\begin{aligned} V_{11} &= -\tfrac{1}{2}[\widetilde{V}_{11} + \widetilde{V}_{22} + i(\widetilde{V}_{21} - \widetilde{V}_{12})], \quad V_{12} = -\tfrac{1}{2}[\widetilde{V}_{11} - \widetilde{V}_{22} + i(\widetilde{V}_{21} + \widetilde{V}_{12})], \\ R_{11} &= \tfrac{1}{2}[\widetilde{R}_{11} + \widetilde{R}_{22} + i(\widetilde{R}_{21} - \widetilde{R}_{12})], \quad R_{12} = \tfrac{1}{2}[\widetilde{R}_{11} - \widetilde{R}_{22} + i(\widetilde{R}_{21} + \widetilde{R}_{12})]. \end{aligned} \tag{2.3.17}$$

We define $V_{21} = \overline{V}_{12}$, $V_{22} = \overline{V}_{11}$, $R_{21} = \overline{R}_{12}$, $R_{22} = \overline{R}_{11}$, $V = (V_{ij})_{1 \leq i,j \leq 2}$, $R = (R_{ij})_{1 \leq i,j \leq 2}$. Since ${}^t\widetilde{V} = \widetilde{V}$ and $\overline{\widetilde{V}} = \widetilde{V}$, we see that $V^* = V$ and (2.3.16), (2.3.17) imply (2.3.15). This concludes the proof. ☐

3. Diagonalization of the problem

The goal of this section is to deduce from (2.3.15) a new equation where, up to remainders, $V(U,\omega,\epsilon)$ will be replaced by a block diagonal operator relatively to the decomposition $\mathcal{H}^0 = \bigoplus_\alpha \operatorname{Range}(\widetilde{\Pi}_\alpha)$ coming from (1.2.5). This is the key point that will allow us to avoid using Nash–Moser methods in the construction of the solution performed in Section 4.

3.1. *Spaces of diagonal and non diagonal operators.*

Definition 3.1.1. Let $\sigma \in \mathbb{R},\ N \in \mathbb{N},\ \sigma \geq \sigma_0 + d + 1 + 2N,\ m \in \mathbb{R},\ q > 0$.

(i) One denotes by $\Sigma^m(N,\sigma,q)$ the space $\Psi^m(N,\sigma,q) \otimes \mathcal{M}_2(\mathbb{R})$. Abusing notation, we also write $\mathcal{R}^r_\nu(N,\sigma,q)$ for $\mathcal{R}^r_\nu(N,\sigma,q) \otimes \mathcal{M}_2(\mathbb{R})$.

(ii) One denotes by $\Sigma^m_{\mathrm{D}}(N,\sigma,q)$ the subspace of $\Sigma^m(N,\sigma,q)$ consisting of elements $A(U,\omega,\epsilon) = (A_{ij}(U,\omega,\epsilon))_{1\leq i,j\leq 2}$ such that $A_{12} = A_{21} = 0$ and, for any $\alpha, \alpha' \in \mathcal{A}$ with $\alpha \neq \alpha'$,

$$\widetilde{\Pi}_\alpha A_{11}(U,\omega,\epsilon)\widetilde{\Pi}_{\alpha'} \equiv 0, \quad \widetilde{\Pi}_\alpha A_{22}(U,\omega,\epsilon)\widetilde{\Pi}_{\alpha'} \equiv 0. \tag{3.1.1}$$

(iii) One denotes by $\Sigma^m_{\mathrm{ND}}(N,\sigma,q)$ the subspace of $\Sigma^m(N,\sigma,q)$ made up of elements $A(U,\omega,\epsilon)$ such that, for any $\alpha \in \mathcal{A}$,

$$\widetilde{\Pi}_\alpha A_{11}(U,\omega,\epsilon)\widetilde{\Pi}_{\alpha} \equiv 0, \quad \widetilde{\Pi}_\alpha A_{22}(U,\omega,\epsilon)\widetilde{\Pi}_{\alpha} \equiv 0. \tag{3.1.2}$$

Clearly, we get a direct sum decomposition $\Sigma^m(N,\sigma,q) = \Sigma^m_{\mathrm{D}}(N,\sigma,q) \oplus \Sigma^m_{\mathrm{ND}}(N,\sigma,q)$.

Definition 3.1.2. Let $\rho \in \,]0,1]$.

(i) $\mathcal{L}^m_\rho(N,\sigma,q)$ denotes the subspace of $\Sigma^{m-\rho}(N,\sigma,q)$ consisting of elements $(A_{ij}(U,\omega,\epsilon))_{1\leq i,j\leq 2}$ that satisfy

$$A_{11}, A_{22} \in \Psi^{m-\rho}(N,\sigma,q), \quad A_{12}, A_{21} \in \Psi^{m-2}(N,\sigma,q). \tag{3.1.3}$$

(ii) $\mathcal{L}'^{\,m}_\rho(N,\sigma,q)$ denotes the subspace of $\Sigma^{m-\rho}(N,\sigma,q)$ consisting of elements $(A_{ij}(U,\omega,\epsilon))_{1\leq i,j\leq 2}$ that satisfy (3.1.3) and

$$A^*_{11} = -A_{11}, \quad A^*_{22} = -A_{22}, \quad A^*_{12} = A_{21}. \tag{3.1.4}$$

Remark. It follows from the definition and from Proposition 2.1.4(ii) that, if $A \in \mathcal{L}^{m_1}_\rho(N,\sigma,q)$, $B \in \mathcal{L}^{m_2}_\rho(N,\sigma,q)$ with $\sigma \geq \sigma_0 + 2N + d + 1 + (m_1 + m_2 - 2\rho)_+$, then AB is the sum of an element of $\mathcal{L}^{m_1+m_2-\rho}_\rho(N,\sigma,q)$ and an element of $\mathcal{R}^r_0(N,\sigma,q)$ with

$$r = \sigma - \sigma_0 - (d+1) - m_1 - m_2 + 2\rho - 2N.$$

Proposition 3.1.3. *Let $A(U,\omega,\epsilon)$ be a self-adjoint element of $\Sigma^m_{\mathrm{ND}}(N,\sigma,q)$. There exist $B(U,\omega,\epsilon)$ in $\mathcal{L}'^{\,m}_\rho(N,\sigma,q)$ and $R(U,\omega,\epsilon)$ in $\mathcal{R}^{r(\sigma,N)-m}_0(N,\sigma,q)$, with $r(\sigma,N) = \rho(\sigma - \sigma_0 - 2N - d - 1)$, such that*

$$B(U,\omega,\epsilon)^*(\Delta - \mu) + (\Delta - \mu)B(U,\omega,\epsilon) = A(U,\omega,\epsilon) + R(U,\omega,\epsilon) \tag{3.1.5}$$

(*where ρ is given by Lemma 1.2.1, for a given $\beta \in \,]0, \frac{1}{10}[$*). *Moreover $[\Delta, B]$ is in $\Sigma^m(N,\sigma,q)$.*

Proof. By assumption, we may write

$$A(U,\omega,\epsilon)=\begin{bmatrix} a(U,\omega,\epsilon) & b(U,\omega,\epsilon)\\ b(U,\omega,\epsilon)^* & c(U,\omega,\epsilon)\end{bmatrix},$$

with $a^*=a,\ c^*=c$, and $\widetilde{\Pi}_\alpha a\widetilde{\Pi}_{\alpha'}=0=\widetilde{\Pi}_\alpha c\widetilde{\Pi}_{\alpha'}$ if $\alpha,\alpha'\in\mathcal{A}$ with $\alpha\neq\alpha'$. Write $a=a'+a''$, with

$$a'=\sum_{n,n'}\mathbb{1}_{|n-n'|\leq c(|n|+|n'|)^\rho}\Pi_n a\Pi_{n'},\quad a''=\sum_{n,n'}\mathbb{1}_{|n-n'|>c(|n|+|n'|)^\rho}\Pi_n a\Pi_{n'},$$

where c is a small positive constant. Applying (2.1.1) with $M=\sigma-\sigma_0-2N-d-1$, we get

$$\begin{aligned}&\|\Pi_n\partial_U^j a''(U)(W_1,\dots,W_j)\Pi_{n'}\|_{\mathcal{L}(\mathcal{H}^0)}\\ &\qquad\leq C(1+|n|+|n'|)^{m-r(\sigma,N)}\langle n-n'\rangle^{-d-1}\mathbb{1}_{|n-n'|\leq\frac{1}{10}(|n|+|n'|)}\prod_{\ell=1}^{j}\|W_\ell\|_{\mathcal{H}^\sigma},\end{aligned}$$

which implies a bound of type (2.1.3) for any $s\geq\sigma_0$, with r replaced by $r(\sigma,N)-m$. Consequently, a'' gives a contribution to R in (3.1.5) and, changing notation, we may assume that $a=a'$. We do the same for the c-contribution, so that we reduce ourselves to a,c verifying that

$$\Pi_n a\Pi_{n'}=0\quad\text{and}\quad\Pi_n c\Pi_{n'}=0\quad\text{if } |n-n'|>c(|n|+|n'|)^\rho. \tag{3.1.6}$$

We look for

$$B(U,\omega,\epsilon)=\begin{bmatrix} a_1(U,\omega,\epsilon) & b_1(U,\omega,\epsilon)\\ b_1(U,\omega,\epsilon)^* & c_1(U,\omega,\epsilon)\end{bmatrix},$$

for some a_1,b_1,c_1 satisfying $a_1^*=-a_1,\ c_1^*=-c_1$ such that $A(U,\omega,\epsilon)$ equals the left side of (3.1.5). The latter may be written as

$$\begin{bmatrix} [\Delta,a_1] & (\Delta-\mu)b_1+b_1(\Delta-\mu)\\ b_1^*(\Delta-\mu)+(\Delta-\mu)b_1^* & [\Delta,c_1]\end{bmatrix}. \tag{3.1.7}$$

Consequently, we have to solve the equations

$$[\Delta,a_1]=a,\quad(\Delta-\mu)b_1+b_1(\Delta-\mu)=b,\quad[\Delta,c_1]=c. \tag{3.1.8}$$

The first of these is equivalent to

$$(|n'|^2-|n|^2)\Pi_n a_1\Pi_{n'}=\Pi_n a\Pi_{n'}\text{ for any } n,n'\in\mathbb{Z}^d. \tag{3.1.9}$$

Since $A\in\Sigma^m_{\mathrm{ND}}(N,\sigma,q)$, Definition 3.1.1(ii) implies that the right side in (3.1.9) vanishes if n,n' belong to a same Ω_α of the partition of Lemma 1.2.1. Consequently, we may define

$$a_1(U,\omega,\epsilon)=\sum_{\substack{\alpha,\alpha'\in\mathcal{A}\\ \alpha\neq\alpha'}}\sum_{n\in\Omega_\alpha}\sum_{n'\in\Omega_{,\alpha'}}(|n'|^2-|n|^2)^{-1}\Pi_n a(U,\omega,\epsilon)\Pi_{n'}. \tag{3.1.10}$$

If we use the second lower bound in (1.2.2), Definition 2.1.1, and (3.1.6) with a small enough $c>0$, we see that a_1 satisfies (2.1.1) with m replaced by $m-\rho$. Thus $a_1\in\Psi^{m-\rho}(N,\sigma,q)$, and by (3.1.10) and the fact that $a^*=a$, we get $a_1^*=-a_1$. The last equation (3.1.8) is solved in the same way.

We are left with finding $b_1(U,\omega,\epsilon)$. The equation giving it is equivalent to

$$-(|n|^2+|n'|^2+2\mu)\Pi_n b_1 \Pi_{n'} = \Pi_n b \Pi_{n'}. \tag{3.1.11}$$

Since $\mu \notin \mathbb{Z}_-$ by assumption, we may always define b_1 by division. Coming back to Definition 2.1.1, we see that we get an element of $\Psi^{m-2}(N,\sigma,q)$, which is moreover self-adjoint. This concludes the proof since (3.1.7) shows that by construction $[\Delta,a_1],[\Delta,c_1]$ belong to $\Psi^m(N,\sigma,q)$, and since $\Delta b_1, b_1\Delta$ and their adjoints are in $\Psi^m(N,\sigma,q)$. □

3.2. *Diagonalization theorem.* The main result of this subsection is the following one, which gives a reduction for the left side of (2.3.15).

Proposition 3.2.1. *Let r be a given positive number and fix an integer N such that $(N+1)\rho \geq r+2$. Let $\sigma \in \mathbb{R}$ satisfy*

$$\sigma \geq \sigma_0 + 2(N+1) + d + 1 + r/\rho. \tag{3.2.1}$$

Let $q>0$ be given. One may find elements $Q_j(U,\omega,\epsilon)$ in $\mathcal{L}_\rho^{-j\rho}(j,\sigma,q)$, $0\leq j\leq N$, elements $V_{\mathrm{D},j}(U,\omega,\epsilon)$ in $\Sigma_{\mathrm{D}}^{-j\rho}(j,\sigma,q)$, $0\leq j\leq N-1$, and an element $R_1(U,\omega,\epsilon)$ in $\mathcal{R}_2^r(N+1,\sigma,q)$, with C^1 dependence in (ω,ϵ), such that if one denotes

$$Q(U,\omega,\epsilon)=\sum_{j=0}^{N} Q_j(U,\omega,\epsilon),\ V_{\mathrm{D}}(U,\omega,\epsilon)=\sum_{j=0}^{N-1} V_{\mathrm{D},j}(U,\omega,\epsilon),\ I'=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}, \tag{3.2.2}$$

one gets, for any $U \in B_q(\mathcal{H}^\sigma(\mathbb{S}^1\times\mathbb{T}^d;\mathbb{C}^2))$,

$$\begin{aligned}(\mathrm{Id}+\epsilon Q(U,\omega,\epsilon))^*(\omega I' D_t + (-\Delta+\mu)I+\epsilon V(U,\omega,\epsilon))(\mathrm{Id}+\epsilon Q(U,\omega,\epsilon))\\ = \omega I' D_t + (-\Delta+\mu)I + \epsilon V_{\mathrm{D}}(U,\omega,\epsilon) - \epsilon R_1(U,\omega,\epsilon).\end{aligned} \tag{3.2.3}$$

We shall prove Proposition 3.2.1 by constructing recursively Q_j, $0\leq j\leq N$ so that Q_j may be written $Q_j = Q_j' + Q_j''$ with

$$\begin{aligned}&Q_j' \in \mathcal{L}_\rho'^{-j\rho}(j,\sigma,q), && [\Delta,Q_j'] \in \Sigma^{-j\rho}(j,\sigma,q), && j=0,\dots,N,\\ &Q_j'' \in \mathcal{L}_\rho^{-(j+1)\rho}(j,\sigma,q), && [\Delta,Q_j''] \in \Sigma^{-(j+1)\rho}(j,\sigma,q), && j=0,\dots,N-1,\\ &Q_N''=0.\end{aligned} \tag{3.2.4}$$

We compute first the left side of (3.2.3).

Proposition 3.2.2. *Let r,σ,N satisfy $(N+1)\rho \geq r+2$ and $\sigma \geq \sigma_0+2(N+1)+d+1+r$. Let $Q(U,\omega,\epsilon)=\sum_{j=0}^N Q_j(U,\omega,\epsilon)$ be given, with $Q_j = Q_j'+Q_j''$ satisfying* (3.2.4).

- *There are elements*

$$S_j(U,\omega,\epsilon) \in \mathcal{L}_\rho^{-(j+1)\rho}(j,\sigma,q), \quad 0\leq j\leq N-1, \tag{3.2.5}$$

with $[\Delta,S_j] \in \Sigma^{-(j+1)\rho}(j,\sigma,q)$, and S_j depending only on Q_ℓ' $(0\leq \ell \leq j)$ and Q_ℓ'' $(0\leq\ell\leq j-1)$.

- *There are elements*

$$V_j(U,\omega,\epsilon) \in \Sigma^{-j\rho}(j,\sigma,q), \quad 0 \le j \le N$$

with $(V_j)^* = V_j$, V_j *depending only on* Q_ℓ $(\ell \le j-1)$.

- *There is an element* $R \in \mathcal{R}_2^r(N+1,\sigma,q)$ *such that, if we set*

$$V^N(U,\omega,\epsilon) = \sum_{j=0}^{N} V_j(U,\omega,\epsilon), \quad S^N(U,\omega,\epsilon) = \sum_{j=0}^{N-1} S_j(U,\omega,\epsilon),$$

$$Q' = \sum_{j=0}^{N} Q'_j, \quad Q'' = \sum_{j=0}^{N} Q''_j, \quad \widetilde{L}_\omega = \omega I' D_t + (-\Delta+\mu)I,$$

then

$$\begin{aligned}(\mathrm{Id}+\epsilon Q)^*[\widetilde{L}_\omega+\epsilon V](\mathrm{Id}+\epsilon Q) = \widetilde{L}_\omega + \epsilon V^N + \epsilon[(S^N)^*\widetilde{L}_\omega + \widetilde{L}_\omega(S^N)]\\ +\epsilon[Q'^*(-\Delta+\mu)+(-\Delta+\mu)Q'] + \epsilon[Q''^*\widetilde{L}_\omega + \widetilde{L}_\omega Q''] + \epsilon R. \quad (3.2.6)\end{aligned}$$

Before starting the proof, we compute some commutators.

Lemma 3.2.3. (i) *One can find* $A_j \in \Sigma^{-j\rho}(j-1,\sigma,q)$ $(1 \le j \le N)$ *depending only on* Q_ℓ $(\ell \le j-1)$ *and satisfying* $A_j^* = A_j$, *one can find* $B_j \in \mathcal{L}_\rho^{-(j+1)\rho}(j,\sigma,q)$ $(0 \le j \le N-1)$ *depending only on* Q'_ℓ $(\ell \le j)$ *and* Q''_ℓ $(\ell \le j-1)$ *and satisfying* $[\Delta, B_j] \in \Sigma^{-(j+1)\rho}(j,\sigma,q)$, *and one can find* $R \in \mathcal{R}_2^r(N+1,\sigma,q)$, *such that, if one sets* $A = \sum_{j=1}^{N} A_j$, $B = \sum_{j=0}^{N-1} B_j$, *then*

$$[Q^*,\widetilde{L}_\omega]Q + Q^*[\widetilde{L}_\omega, Q] = A + B^*\widetilde{L}_\omega + \widetilde{L}_\omega B + R. \quad (3.2.7)$$

(ii) *One can find* A_j *as above* $(1 \le j \le N)$, *one can find* $B_j \in \mathcal{L}_\rho^{-(j+1)\rho}(j,\sigma,q)$ $(0 \le j \le N-1)$, *satisfying* $[\Delta, B_j] \in \Sigma^{-(j+1)\rho}(j,\sigma,q)$ *and depending only on* Q'_ℓ $(\ell \le j)$ *and* Q''_ℓ $(\ell \le j-1)$, *and one can find* $R \in \mathcal{R}_2^r(N+1,\sigma,q)$ *such that, with the same notation as in* (*i*),

$$Q^*\widetilde{L}_\omega Q = A + B^*\widetilde{L}_\omega + \widetilde{L}_\omega B + R. \quad (3.2.8)$$

Proof. (i) Write

$$\begin{aligned}[\widetilde{L}_\omega, Q] &= -[\Delta, Q] + \omega[I'D_t, Q] = -[\Delta, Q] + \omega I'[D_t, Q] + \omega[I', Q]D_t\\ &= -[\Delta, Q] + \omega I'[D_t, Q] + [I', Q]I'(\Delta-\mu) + [I', Q]I'\widetilde{L}_\omega.\end{aligned}$$

The left side of (3.2.7) may be written

$$\begin{aligned}-Q^*[\Delta,Q]+\omega Q^*I'[D_t,Q]+Q^*[I',Q]I'(\Delta-\mu)-[Q^*,\Delta]Q+\omega[Q^*,D_t]I'Q+(\Delta-\mu)I'[Q^*,I']Q\\ +Q^*[I',Q]I'\widetilde{L}_\omega + \widetilde{L}_\omega I'[Q^*,I']Q. \quad (3.2.9)\end{aligned}$$

Denote by $\tilde{A}$ the first line of (3.2.9). Then $\tilde{A}$ is self-adjoint and may be written as $\sum_{j=1}^{2N+2} \tilde{A}_j$, where $\tilde{A}_j$ is the sum of the following terms:

$$\sum_{\substack{j_1+j_2=j-1\\0\le j_1,j_2\le N}} \big(-[Q_{j_1}^*,\Delta]Q_{j_2}-Q_{j_2}^*[\Delta,Q_{j_1}]\big) \qquad (j\ge 1), \tag{3.2.10}$$

$$\omega\sum_{\substack{j_1+j_2=j-2\\0\le j_1,j_2\le N}} \big(Q_{j_1}^* I'[D_t,Q_{j_2}]+[Q_{j_2}^*,D_t]I'Q_{j_1})\big) \qquad (j\ge 2), \tag{3.2.11}$$

$$\sum_{\substack{j_1+j_2=j-1\\0\le j_1,j_2\le N}} \big(Q_{j_1}^*[I',Q_{j_2}]I'(\Delta-\mu)+(\Delta-\mu)I'[Q_{j_2}^*,I']Q_{j_1}\big) \quad (j\ge 1). \tag{3.2.12}$$

Let us check that we may write $\tilde{A}_j = A_j + R_{1,j}$ with A_j in $\Sigma^{-j\rho}(\min(N+1,j-1),\sigma,q)$ and $R_{1,j}$ in $\mathcal{R}_0^r(\min(N+1,j-1),\sigma,q)$. Since $\mathcal{L}_\rho^{-j_\ell\rho}(j_\ell,\sigma,q)\subset\Sigma^{-(j_\ell+1)\rho}(j_\ell,\sigma,q)$, it follows from (3.2.4) and from (ii) of Proposition 2.1.4 that the general term in (3.2.10) may be written as a contribution to A_j plus a remainder belonging to $\mathcal{R}_0^{r_1}(\min(N,j-1),\sigma,q)$ with

$$r_1=\sigma-\sigma_0-2N-(d+1)+(j_1+j_2+1)\rho\ge r.$$

Moreover these contributions depend only on Q_ℓ $(\ell\le j-1)$.

Consider the general term of (3.2.11). The second remark following Definition 2.1.1 implies that $[D_t,Q_{j_2}]\in\Sigma^{-(j_2+1)\rho}(j_2+1,\sigma,q)$. Consequently, using again (ii) of Proposition 2.1.4, we may write (3.2.11) as a contribution to A_j, plus a remainder belonging to $\mathcal{R}_0^{r_1}(\min(N+1,j-1),\sigma,q)$, depending only on Q_ℓ $(\ell\le j-2)$.

Finally, consider (3.2.12). If $C=(C_{ij}(U,\omega,\epsilon))_{1\le i,j\le 2}$ is an element of $\mathcal{L}_\rho^m(N,\sigma,q)$, it follows from (3.1.3) that $[I',C]=\left[\begin{smallmatrix}0 & 2c_{12}\\ -2c_{21} & 0\end{smallmatrix}\right]$ belongs to $\Sigma^{m-2}(N,\sigma,q)$. Hence, the first term in the sum (3.2.12) is given by the composition of an element in $\Sigma^{-(j_1+1)\rho}(j_1,\sigma,q)$ and of an element in $\Sigma^{-j_2\rho}(j_2,\sigma,q)$. By applying Proposition 2.1.4 once more, we may write this as a contribution to A_j plus a remainder in $\mathcal{R}_0^r(\min(N,j-1),\sigma,q)$, depending only on Q_ℓ $(\ell\le j-1)$. The second term in the argument of the sum (3.2.12) is treated in the same way. This shows that the sum of the first two lines in (3.2.9) contributes to $A+R$ on the right side of (3.2.7), since for $j\ge N+1$, A_j is in $\Sigma^{-(N+1)\rho}(N+1,\sigma,q)$, hence in $\mathcal{R}_0^r(N+1,\sigma,q)$ by the inequality $(N+1)\rho\ge r$ and the remark after the statement of Definition 2.1.3. Let us show that the last line in (3.2.9) contributes to $B^*\widetilde{L}_\omega+\widetilde{L}_\omega B+R$ in (3.2.7). We have seen above that the fact that $Q_j'\in\mathcal{L}_\rho^{-j\rho}(j,\sigma,q)$ implies $[Q_j',I']=\left[\begin{smallmatrix}0 & e_1\\ e_2 & 0\end{smallmatrix}\right]$ with $e_\ell\in\Psi^{-j\rho-2}(j,\sigma,q)$; similarly, $Q_j''\in\mathcal{L}_\rho^{-(j+1)\rho}(j,\sigma,q)$ implies $[Q_j'',I']=\left[\begin{smallmatrix}0 & e_1\\ e_2 & 0\end{smallmatrix}\right]$ with $e_\ell\in\Psi^{-(j+1)\rho-2}(j,\sigma,q)$). We set

$$\tilde{B}_j=\sum_{\substack{j_1+j_2=j\\0\le j_1,j_2\le N}} I'[Q_{j_1}'^*,I']Q_{j_2}'+\sum_{\substack{j_1+j_2=j-1\\0\le j_1,j_2\le N}} I'([Q_{j_1}'^*,I']Q_{j_2}''+[Q_{j_1}''^*,I']Q_{j_2}')+\sum_{\substack{j_1+j_2=j-2\\0\le j_1,j_2\le N}} I'[Q_{j_1}''^*,I']Q_{j_2}''.$$

Applying Proposition 2.1.4, we again have a decomposition $\tilde{B}_j=B_j+R_j$, where B_j belongs to the class $\mathcal{L}_\rho^{-(j+1)\rho}(\min(N,j),\sigma,q)$ (actually, B_j is in $\Sigma^{-(j+1)\rho-2}(\min(N,j),\sigma,q)$) and R_j belongs to

$\mathcal{R}_0^{r+2}(\min(j,N),\sigma,q)$ because of (3.2.1). Moreover, B_j depends only on Q'_ℓ $(\ell \leq j)$ and Q''_ℓ $(\ell \leq j-1)$, and by construction, $[\Delta, B_j] \in \Sigma^{-(j+1)\rho}(\min(N,j),\sigma,q)$. For $j \leq N-1$, we get contributions to B and R in (3.2.8), noting that $R_j \widetilde{L}_\omega, \widetilde{L}_\omega R_j$ are in $\mathcal{R}_2^r(N,\sigma,q)$. For $j \geq N$, B_j as well as R_j contribute to the remainder in (3.2.7) since $(N+1)\rho \geq r$. This concludes the proof of (i).

(ii) We write

$$Q^* \widetilde{L}_\omega Q = \tfrac{1}{2}[Q^* Q \widetilde{L}_\omega + \widetilde{L}_\omega Q^* Q] + \frac{1}{2}[Q^*[\widetilde{L}_\omega, Q] + [Q^*, \widetilde{L}_\omega]Q].$$

By (i), the last term may be written as a contribution to the right side of (3.2.8). Let us write the first term on the right side under the form $B^* \widetilde{L}_\omega + \widetilde{L}_\omega B + R$. We write $Q^* Q$ as the sum in j of

$$\sum_{\substack{j_1+j_2=j \\ 0 \leq j_1, j_2 \leq N}} Q'^*_{j_1} Q'_{j_2} + \sum_{\substack{j_1+j_2=j-1 \\ 0 \leq j_1, j_2 \leq N}} (Q'^*_{j_1} Q''_{j_2} + Q''^*_{j_1} Q'_{j_2}) + \sum_{\substack{j_1+j_2=j-2 \\ 0 \leq j_1, j_2 \leq N}} Q''^*_{j_1} Q''_{j_2}.$$

By (3.2.4) and the remark following Definition 3.1.2, this expression may be written as $B_j + R_j$, where $B_j \in \mathcal{L}_\rho^{-(j+1)\rho}(\min(N,j),\sigma,q)$ depends only on Q'_ℓ $(\ell \leq j)$ and Q''_ℓ $(\ell \leq j-1)$, $[B_j, \Delta]$ belongs to $\Sigma^{-(j+1)\rho}(\min(N,j),\sigma,q)$, and R_j belongs to $\mathcal{R}_0^{r_2}(\min(N,j),\sigma,q)$, with

$$r_2 = \sigma - \sigma_0 - (d+1) + (j+2)\rho - 2\min(j,N) \geq r+2.$$

We obtain contributions to the right side of (3.2.8) when $j \leq N-1$, and to the remainder R when $j \geq N$ since $(N+1)\rho \geq r+2$. This concludes the proof. $\square$

Proof of Proposition 3.2.2. We write the left side of (3.2.6) as

$$\begin{aligned}\widetilde{L}_\omega + \epsilon V(U,\omega,\epsilon) + \epsilon[Q'^*(-\Delta+\mu) + (-\Delta+\mu)Q'] + \epsilon[Q''^* \widetilde{L}_\omega + \widetilde{L}_\omega Q''] \\ + \epsilon[Q'^* I' \omega D_t + \omega I' D_t Q'] + \epsilon^2 Q^* \widetilde{L}_\omega Q + \epsilon^2[Q^* V + VQ] + \epsilon^3 Q^* VQ. \quad (3.2.13)\end{aligned}$$

The term V in (3.2.13) contributes to the V_0 component of V^N on the right side of (3.2.6). The first two brackets in (3.2.13) give rise to the last two in (3.2.6). To study the contribution of $Q^* \widetilde{L}_\omega Q$, we use (3.2.8). The B_j component of B on the right side of (3.2.8) contributes to the S_j component of S^N in (3.2.6). Let us study the third bracket in (3.2.13). By (3.2.4) and Definition 3.1.2, we may write $Q'_{j-1} = \left[\begin{smallmatrix} a & b \\ b^* & c \end{smallmatrix}\right]$ with $a, c \in \Psi^{-j\rho}(j-1,\sigma,q)$, $b \in \Psi^{-(j-1)\rho-2}(j-1,\sigma,q)$, $a^* = -a$, and $c^* = -c$. This implies that

$$Q'^*_{j-1} I' D_t + I' D_t Q'_{j-1} = \begin{bmatrix} [D_t, a] & [D_t, b] \\ -[D_t, b^*] & -[D_t, c] \end{bmatrix}$$

is a self-adjoint operator belonging to $\Sigma^{-j\rho}(j,\sigma,q)$, $1 \leq j \leq N$, by the second remark at the bottom of page 644. We thus get a contribution to V_j in (3.2.6).

Finally, let us check that the last two terms in (3.2.13) may be written as contributions to V^N and to R on the right side of (3.2.6). Actually, we may write $Q^* V + VQ + \epsilon Q^* VQ$ as the sum in j of

$$\begin{aligned} & Q'^*_{j-1} V + V Q'_{j-1} + Q''^*_{j-2} V + V Q''_{j-2} \\ & + \epsilon \sum_{j_1+j_2=j-2} Q'^*_{j_1} V Q'_{j_2} + \epsilon \sum_{j_1+j_2=j-3} (Q''^*_{j_1} V Q'_{j_2} + Q'^*_{j_1} V Q''_{j_2}) + \epsilon \sum_{j_1+j_2=j-4} Q''^*_{j_1} V Q''_{j_2}. \quad (3.2.14)\end{aligned}$$

Using that $Q'_j \in \Sigma^{-(j+1)\rho}(j,\sigma,q)$, $Q''_j \in \Sigma^{-(j+2)\rho}(j,\sigma,q)$, $V \in \Sigma^0(0,\sigma,q)$, we write (3.2.14) as $V_j + R_j$, where V_j depends only on Q'_ℓ ($\ell \le j-1$) and Q''_ℓ ($\ell \le j-2$) and is in $\Sigma^{-j\rho}(\min(N, j-1),\sigma,q)$ and $R_j \in \mathcal{R}^r_0(N,\sigma,q)$. This concludes the proof. $\square$

Proof of Proposition 3.2.1. Let us construct recursively Q'_j ($0 \le j \le N$) and Q''_j ($0 \le j \le N-1$) so that the right side of (3.2.6) may be written as the right side of (3.2.3). Assume that $Q_0, \dots, Q_{j-1}$ have been already determined in such a way that the right side of (3.2.6) may be written

$$\widetilde{L}_\omega + \epsilon \sum_{j'=0}^{j-1} V_{\mathrm{D},j'} + \epsilon \sum_{j'=j}^{N-1} [S^*_{j'} \widetilde{L}_\omega + \widetilde{L}_\omega S_{j'}] + \epsilon \sum_{j'=j}^{N} [Q'^*_{j'}(-\Delta+\mu) + (-\Delta+\mu) Q'_{j'}] \\ + \epsilon \sum_{j'=j}^{N-1} [Q''^*_{j'} \widetilde{L}_\omega + \widetilde{L}_\omega Q''_{j'}] + \epsilon \sum_{j'=j}^{N} V_{j'} + \epsilon R. \quad (3.2.15)$$

Write $V_j = \left[\begin{smallmatrix} a & b \\ b^* & c \end{smallmatrix}\right]$ with $a, b, c \in \Psi^{-j\rho}(j,\sigma,q)$, $a^* = a$, $c^* = c$, and define

$$V_{\mathrm{D},j} = \sum_{\alpha \in \mathcal{A}} \widetilde{\Pi}_\alpha \left[\begin{smallmatrix} a & 0 \\ 0 & c \end{smallmatrix}\right] \widetilde{\Pi}_\alpha, \; V_{\mathrm{ND},j} = V_j - V_{\mathrm{D},j}.$$

Then $V_{\mathrm{D},j} \in \Sigma^{-j\rho}_{\mathrm{D}}(j,\sigma,q)$, $(V_{\mathrm{D},j})^* = V_{\mathrm{D},j}$ and $V_{\mathrm{ND},j}$ is in $\Sigma^{-j\rho}_{\mathrm{ND}}(j,\sigma,q)$, $(V_{\mathrm{ND},j})^* = V_{\mathrm{ND},j}$. Moreover $V_{\mathrm{ND},j}$ depends only on Q_ℓ ($\ell \le j-1$). We apply Proposition 3.1.3 to find $Q'_j \in \mathcal{L}'^{-j\rho}_\rho(j,\sigma,q)$ and $R_j \in \mathcal{R}^{r(\sigma,j)+j\rho}_0(j,\sigma,q)$ such that $Q'^*_j(-\Delta+\mu) + (-\Delta+\mu)Q'_j = V_{\mathrm{ND},j} + R_j$ and $[\Delta, Q'_j]$ is in $\Sigma^{-j\rho}(j,\sigma,q)$. The assumption (3.2.1) on σ shows that R_j contributes to R_1 in (3.2.3). Moreover condition (3.2.4) is satisfied by Q'_j, so that we have eliminated the j-th component in the fourth and sixth terms of (3.2.15). To eliminate the j-th component of the third and fifth terms, we set $Q''_j = -S_j$, $j \le N-1$, $Q''_N = 0$. Then condition (3.2.4) is satisfied by Q''_j, and the definition is consistent since S_j depends only on Q'_ℓ ($\ell \le j$) and Q''_ℓ ($\ell \le j-1$). This concludes the proof. $\square$

4. Iterative scheme

This section will be devoted to the proof of Theorem 1.1.1. We shall construct a solution to (2.3.15) — which is equivalent to (1.1.3) — writing this equation under an equivalent form involving the right side of (3.2.3). The first subsection will be devoted to the study of the restriction of the operator $\widetilde{L}_\omega + \epsilon V_{\mathrm{D}}(U,\omega,\epsilon)$ to the range of one of the projectors $\widetilde{\Pi}_\alpha$. We shall show that, for (ω,ϵ) outside a subset of small measure, this restriction is invertible. As usual in these problems, the inverse we construct loses derivatives. This will not cause much trouble, since Proposition 3.2.1 allows us to write the equation essentially under the form $(\widetilde{L}_\omega + \epsilon V_{\mathrm{D}}(U,\omega,\epsilon))W = \epsilon R_1(U,\omega,\epsilon)W$ for a new unknown W. Since R_1 is smoothing, it gains enough derivatives to compensate the losses coming from $(\widetilde{L}_\omega + \epsilon V_{\mathrm{D}})^{-1}$. Because of that, we may construct the solution using a standard iterative scheme.

4.1. ***Lower bounds for eigenvalues.*** Let $\gamma_0 \in]0,1]$, $\sigma \in \mathbb{R}$, $N \in \mathbb{N}$, $\zeta \in \mathbb{R}_+$ such that

$$\sigma \ge \sigma_0 + \frac{\zeta}{\rho} + 2(N+1) + d + 1.$$

We denote by $\mathscr{E}^\sigma(\zeta)$ the space of functions from $\mathbb{S}^1 \times \mathbb{T}^d \times [1,2] \times [0,\gamma_0]$ to $\mathbb{C}^2$,

$$(t,x,\omega,\epsilon) \longrightarrow U(t,x,\omega,\epsilon), \tag{4.1.1}$$

which are continuous functions of ω with values in $\mathscr{H}^\sigma(\mathbb{S}^1 \times \mathbb{T}^d; \mathbb{C}^2)$ and C^1 functions of ω with values in $\mathscr{H}^{\sigma-\zeta-2}(\mathbb{S}^1 \times \mathbb{T}^d; \mathbb{C}^2)$, uniformly in $\epsilon \in [0,\gamma_0]$. We set

$$\|U\|_{\mathscr{E}^\sigma(\zeta)} = \sup_{(\omega,\epsilon)\in[1,2]\times[0,\gamma_0]} \|U(\cdot,\omega,\epsilon)\|_{\mathscr{H}^\sigma} + \sup_{(\omega,\epsilon)\in[1,2]\times[0,\gamma_0]} \|\partial_\omega U(\cdot,\omega,\epsilon)\|_{\mathscr{H}^{\sigma-\zeta-2}}. \tag{4.1.2}$$

If $\widetilde{\Pi}_\alpha$ is the projector of $\mathscr{H}^0$ given by (1.2.5), we set $F_\alpha = \mathrm{Range}(\widetilde{\Pi}_\alpha)$, $D_\alpha = \dim F_\alpha$. By (1.2.4) and (1.2.6), $D_\alpha \le C_1 \langle n(\alpha)\rangle^{\beta d+2}$ for some $C_1 > 0$. We define for $U \in \mathscr{E}^\sigma(\zeta)$, $\omega \in [1,2]$, $\epsilon \in [0,\gamma_0]$

$$A_\alpha(\omega; U,\epsilon) = \widetilde{\Pi}_\alpha(\widetilde{L}_\omega + \epsilon V_{\mathrm{D}}(U,\omega,\epsilon))\widetilde{\Pi}_\alpha. \tag{4.1.3}$$

This is a self-adjoint operator on F_α, with C^1 dependence in ω, since it follows from the expression (3.2.2) of V_{D}, condition (2.1.1) in the definition of $\Psi^m(N,\sigma,q)$, the fact that $\partial_\omega U \in \mathscr{H}^{\sigma-\zeta-2}$, and the assumption made on σ, that $\omega \to \widetilde{\Pi}_\alpha V_{\mathrm{D}}(U(t,x,\omega,\epsilon),\omega,\epsilon)\widetilde{\Pi}_\alpha$ is C^1. The main result of this subsection is the following:

Proposition 4.1.1. *For any $\mu \in \mathbb{R} - \mathbb{Z}_-$ and $q > 0$, there are $\gamma_0 \in]0,1]$, $C_0 > 0$, $\mathscr{A}_0 \subset \mathscr{A}$ a finite subset, and for any $U \in \mathscr{E}^\sigma(\zeta)$ with $\|U\|_{\mathscr{E}^\sigma(\zeta)} < q$, any $\epsilon \in [0,\gamma_0]$, any $\alpha \in \mathscr{A}$, the eigenvalues of A_α form a finite family of C^1 real valued functions of ω, depending on (U,ϵ),*

$$\omega \to \lambda_\ell^\alpha(\omega; U,\epsilon),\ 1 \le \ell \le D_\alpha \tag{4.1.4}$$

satisfying the following properties:

(i) *For any $\alpha \in \mathscr{A}$, any $U, U' \in \mathscr{H}^\sigma$ with $\|U\|_{\mathscr{H}^\sigma} < q$, $\|U'\|_{\mathscr{H}^\sigma} < q$, any $\ell \in \{1,\dots,D_\alpha\}$, any $\epsilon \in [0,\gamma_0]$, and any $\omega \in [1,2]$, there is $\ell' \in \{1,\dots,D_\alpha\}$ such that*

$$|\lambda_\ell^\alpha(\omega; U,\epsilon) - \lambda_{\ell'}^\alpha(\omega; U',\epsilon)| \le C_0 \epsilon \|U - U'\|_{\mathscr{H}^\sigma}. \tag{4.1.5}$$

(ii) *For any $a \in \mathscr{A} - \mathscr{A}_0$, any $U \in \mathscr{E}^\sigma(\zeta)$ with $\|U\|_{\mathscr{E}^\sigma(\zeta)} < q$, any $\epsilon \in [0,\gamma_0]$, and any $\ell \in \{1,\dots,D_\alpha\}$, either*

$$C_0^{-1}\langle n(\alpha)\rangle^2 \le \frac{\partial \lambda_\ell^\alpha}{\partial\omega}(\omega; U,\epsilon) \le C_0 \langle n(\alpha)\rangle^2 \qquad \text{for any } \omega \text{ in } [1,2] \tag{4.1.6}$$

or

$$-C_0\langle n(\alpha)\rangle^2 \le \frac{\partial \lambda_\ell^\alpha}{\partial\omega}(\omega; U,\epsilon) \le -C_0^{-1} \langle n(\alpha)\rangle^2 \quad \text{for any } \omega \text{ in } [1,2]. \tag{4.1.7}$$

(iii) *For $\delta \in]0,1]$, $\epsilon \in [0,\gamma_0]$, $\alpha \in \mathscr{A}$, and $U \in \mathscr{E}^\sigma(\zeta)$ with $\|U\|_{\mathscr{E}^\sigma(\zeta)} < q$, set*

$$I(\alpha,U,\epsilon,\delta) = \{\omega \in [1,2] : \forall \ell \in \{1,\dots,D_\alpha\}, |\lambda_\ell^\alpha(\omega; U,\epsilon)| \ge \delta \langle n(\alpha)\rangle^{-\zeta}\}. \tag{4.1.8}$$

Then there is a constant E_0, depending only on the dimension, such that for any $\omega \in I(\alpha,U,\epsilon,\delta)$, $A_\alpha(\omega; U,\epsilon)$ is invertible and

$$\|A_\alpha(\omega; U,\epsilon)^{-1}\|_{\mathscr{L}(\mathscr{H}^0)} \le E_0 \delta^{-1} \langle n(\alpha)\rangle^\zeta, \quad \|\partial_\omega A_\alpha(\omega; U,\epsilon)^{-1}\|_{\mathscr{L}(\mathscr{H}^0)} \le E_0 \delta^{-2} \langle n(\alpha)\rangle^{2\zeta+2}. \tag{4.1.9}$$

Proof. The proof of this result is quite classical, and may be found in the references given in the introduction. For completeness, we give it in detail.

(i) By construction, A_α is a self-adjoint operator, acting on a space of finite dimension D_α. Moreover, A_α is a C^1 function of ω if $U \in \mathscr{E}^\sigma(\zeta)$. By a theorem of Rellich (see [Kato 1976, Theorem 6.8], for instance), we know that we may index the eigenvalues of that matrix so that they are C^1 functions of ω: $\lambda_\ell^\alpha(\omega; U, \epsilon)$, for $1 \le \ell \le D_\alpha$. Moreover, if B and B' are self-adjoint matrices of the same dimension, for any eigenvalue $\lambda_\ell(B)$ of B there is an eigenvalue $\lambda_{\ell'}(B')$ of B' such that $|\lambda_\ell(B)-\lambda_{\ell'}(B')| \le \|B-B'\|$. Combining this with the fact that $U \to A_\alpha(\omega; U, \epsilon)$ is lipschitz with values in $\mathscr{L}(\mathscr{H}^0)$, with lipschitz constant $C\epsilon$, we get (4.1.5).

(ii) Set

$$\Lambda^0_\pm(\alpha) = \{\pm j\omega + |n|^2 + \mu : j \in \mathbb{N}, n \in \Omega_\alpha, K_0^{-1}\langle n(\alpha)\rangle^2 \le j \le K_0\langle n(\alpha)\rangle^2\},$$

so that the spectrum of $\widetilde{\Pi}_\alpha \widetilde{L}_\omega \widetilde{\Pi}_\alpha$ is $\Lambda^0_+(\alpha) \cup \Lambda^0_-(\alpha)$. The difference between an eigenvalue in $\Lambda^0_+(\alpha)$, parametrized by (j, n), and an eigenvalue in $\Lambda^0_-(\alpha)$, parametrized by (j', n') $(j > 0, j' < 0)$ is bounded from below by

$$\omega(j - j') + |n|^2 - |n'|^2 \ge 2K_0^{-1}\langle n(\alpha)\rangle^2 - \theta - C\langle n(\alpha)\rangle^\beta,$$

by the first estimate (1.2.2), for some $C > 0, \beta \in]0, \frac{1}{10}[$. If we take the subset $\mathscr{A}_0$ large enough, we get that when $\alpha \in \mathscr{A} - \mathscr{A}_0$, the difference between two such eigenvalues is bounded from below by $K_0^{-1}\langle n(\alpha)\rangle^2$. Consequently, if $0 \le \epsilon < \gamma_0$ small enough, the spectrum of A_α may be split in two subsets $\Lambda_+(\alpha) \cup \Lambda_-(\alpha)$ whose distance is bounded from below by $\frac{1}{2}K_0^{-1}\langle n(\alpha)\rangle^2$. Let Γ be a contour in the complex plane turning once around $\Lambda^0_+(\alpha)$, of length $O(\langle n(\alpha)\rangle^2)$, such that the distance between Γ and the spectrum of $\widetilde{L}^\alpha_\omega = \widetilde{\Pi}_\alpha \widetilde{L}_\omega \widetilde{\Pi}_\alpha$ is bounded from below by $c\langle n(\alpha)\rangle^2$, and such that $\Lambda^0_-(\alpha)$ is outside Γ. If γ_0 is small enough, this contour satisfies the same conditions with $\Lambda^0_\pm(\alpha)$ replaced by $\Lambda_\pm(\alpha)$ and $\widetilde{L}^\alpha_\omega$ replaced by A_α. The spectral projectors $\widetilde{\Pi}^+_\alpha(\omega)$ and $\widetilde{\Pi}^{+,0}_\alpha$ associated to the eigenvalues $\Lambda_+(\alpha)$ and $\Lambda^0_+(\alpha)$ of A_α and $\widetilde{L}^\alpha_\omega$, respectively, are given by

$$\widetilde{\Pi}^+_\alpha(\omega) = \frac{1}{2i\pi}\int_\Gamma (\zeta \mathrm{Id} - A_\alpha)^{-1}\, d\zeta, \quad \widetilde{\Pi}^{+,0}_\alpha = \frac{1}{2i\pi}\int_\Gamma (\zeta \mathrm{Id} - \widetilde{L}^\alpha_\omega)^{-1}\, d\zeta. \tag{4.1.10}$$

Note that the second projector is just the orthogonal projector on

$$\mathrm{Vect}\,\{e^{i(jt+n\cdot x)} : n \in \Omega_\alpha, K_0^{-1}\langle n(\alpha)\rangle^2 \le j \le K_0\langle n(\alpha)\rangle^2\},$$

so it is independent of ω. Write

$$\widetilde{\Pi}^+_\alpha(\omega) - \widetilde{\Pi}^{+,0}_\alpha = \frac{1}{2i\pi}\int_\Gamma (\zeta \mathrm{Id} - A_\alpha)^{-1}(A_\alpha - \widetilde{L}^\alpha_\omega)(\zeta \mathrm{Id} - \widetilde{L}^\alpha_\omega)^{-1}\, d\zeta. \tag{4.1.11}$$

Using (4.1.3) and the definition of $\widetilde{L}^\alpha_\omega$ we get

$$\|A_\alpha - \widetilde{L}^\alpha_\omega\|_{\mathscr{L}(F_\alpha)} + \|\partial_\omega(A_\alpha - \widetilde{L}^\alpha_\omega)\|_{\mathscr{L}(F_\alpha)} \le C\epsilon, \quad \|\partial_\omega A_\alpha\|_{\mathscr{L}(F_\alpha)} + \|\partial_\omega \widetilde{L}^\alpha_\omega\|_{\mathscr{L}(F_\alpha)} \le C\langle n(\alpha)\rangle^2.$$

Consequently (4.1.11) implies

$$\begin{aligned}\|\widetilde{\Pi}_\alpha^+(\omega)-\widetilde{\Pi}_\alpha^{+,0}\|_{\mathscr{L}(F_\alpha)}&\leq C\epsilon\langle n(\alpha)\rangle^{-2},\\ \|\partial_\omega\widetilde{\Pi}_\alpha^+(\omega)\|_{\mathscr{L}(F_\alpha)}=\|\partial_\omega(\widetilde{\Pi}_\alpha^+(\omega)-\widetilde{\Pi}_\alpha^{+,0})\|_{\mathscr{L}(F_\alpha)}&\leq C\epsilon\langle n(\alpha)\rangle^{-2}.\end{aligned}$$

Writing

$$\begin{aligned}\widetilde{\Pi}_\alpha^+(\omega)A_\alpha\widetilde{\Pi}_\alpha^+(\omega)=(\widetilde{\Pi}_\alpha^+(\omega)-\widetilde{\Pi}_\alpha^{+,0})A_\alpha\widetilde{\Pi}_\alpha^+(\omega)+\widetilde{\Pi}_\alpha^{+,0}(A_\alpha-\widetilde{L}_\omega^\alpha)\widetilde{\Pi}_\alpha^+(\omega)\\ +\widetilde{\Pi}_\alpha^{+,0}\widetilde{L}_\omega^\alpha(\widetilde{\Pi}_\alpha^+(\omega)-\widetilde{\Pi}_\alpha^{+,0})+\widetilde{\Pi}_\alpha^{+,0}\widetilde{L}_\omega^\alpha\widetilde{\Pi}_\alpha^{+,0}\end{aligned}$$

we obtain

$$\|\partial_\omega[\widetilde{\Pi}_\alpha^+(\omega)A_\alpha\widetilde{\Pi}_\alpha^+(\omega)-\widetilde{\Pi}_\alpha^{+,0}\widetilde{L}_\omega^\alpha\widetilde{\Pi}_\alpha^{+,0}]\|_{\mathscr{L}(F_\alpha)}\leq C\epsilon. \tag{4.1.12}$$

Let I be an interval contained in $[1,2]$ over which one of the eigenvalues $\lambda_\ell^\alpha(\omega;U,\epsilon)$ of the matrix $\widetilde{\Pi}_\alpha^+(\omega)A_\alpha(\omega;U,\epsilon)\widetilde{\Pi}_\alpha^+(\omega)$ has constant multiplicity m, and denote by $P(\omega)$ the associated spectral projector. Then $P(\omega)$ is C^1 in $\omega\in I$ and satisfies $P(\omega)^2=P(\omega)$, whence $P(\omega)P'(\omega)P(\omega)=0$. We get therefore for

$$\lambda_\ell^\alpha(\omega;U,\epsilon)=\frac{1}{m}\operatorname{tr}[P(\omega)\widetilde{\Pi}_\alpha^+(\omega)A_\alpha(\omega;U,\epsilon)\widetilde{\Pi}_\alpha^+(\omega)P(\omega)]$$

the equality

$$\partial_\omega\lambda_\ell^\alpha(\omega;U,\epsilon)=\frac{1}{m}\operatorname{tr}[P(\omega)\partial_\omega(\widetilde{\Pi}_\alpha^+(\omega)A_\alpha(\omega;U,\epsilon)\widetilde{\Pi}_\alpha^+(\omega))P(\omega)].$$

By (4.1.12), we obtain

$$\partial_\omega\lambda_\ell^\alpha(\omega;U,\epsilon)=\frac{1}{m}\operatorname{tr}[P(\omega)\partial_\omega(\widetilde{\Pi}_\alpha^{+,0}\widetilde{L}_\omega^\alpha\widetilde{\Pi}_\alpha^{+,0})P(\omega)]+O(\epsilon). \tag{4.1.13}$$

Since $\widetilde{\Pi}_\alpha^{+,0}\widetilde{L}_\omega^\alpha\widetilde{\Pi}_\alpha^{+,0}$ is by definition of $\widetilde{L}_\omega^\alpha$ a diagonal matrix with entries $j\omega+|n|^2+\mu$, $n\in\Omega_\alpha$, $K_0^{-1}\langle n(\alpha)\rangle^2\leq j\leq K_0\langle n(\alpha)\rangle^2$, we see that (4.1.13) stays between $K_0^{-1}\langle n(\alpha)\rangle^2-C\epsilon$ and $K_0\langle n(\alpha)\rangle^2+C\epsilon$. This implies (4.1.6) if $\epsilon\in[0,\gamma_0]$ with γ_0 small enough. The case of eigenvalues corresponding to $\Lambda_-(\alpha)$ is treated in a similar way, and gives (4.1.7).

(iii) The first estimate in (4.1.9) follows from the fact that the eigenvalues $\lambda_\ell^\alpha(\omega;U,\epsilon)$ of A_α satisfy the lower bound given by the definition of (4.1.8). The second estimate is a consequence of the first one and of the fact that $\|\partial_\omega A_\alpha(\omega;U,\epsilon)\|_{\mathscr{L}(\mathscr{H}^0)}\leq C\langle n(\alpha)\rangle^2$ by definition of A_α. This concludes the proof. ☐

4.2. ***Iterative scheme.*** This subsection will be devoted to the proof of Theorem 1.1.1, constructing the solution as the limit of an iterative scheme. We fix indices $s,\sigma,N,\zeta,r,\delta$ satisfying the inequalities:

$$\begin{aligned}&\sigma\geq\sigma_0+2(N+1)+d+1+r/\rho,\quad r=\zeta,\\ &(N+1)\rho\geq r+2,\quad s\geq\sigma+\zeta+2,\quad \delta\in]0,\delta_0],\end{aligned} \tag{4.2.1}$$

where $\delta_0>0$ will be chosen small enough. We also assume that the parameter μ is in $\mathbb{R}-\mathbb{Z}_-$. We shall solve (2.3.15) when its force term f is given in $\mathscr{H}^{s+\zeta}(\mathbb{S}^1\times\mathbb{T}^d;\mathbb{C}^2)$. To achieve this goal, the main task will be to construct a sequence $(G_k,\mathbb{O}_k,\psi_k,U_k,W_k)$, $k\geq 0$, where $G_k,\mathbb{O}_k$ will be subsets of

$[1,2]\times[0,\delta^2]$, ψ_k will be a real valued function defined on $[1,2]\times[0,\delta^2]$, U_k, W_k will be functions of $(t,x,\omega,\epsilon)\in\mathbb{S}^1\times\mathbb{T}^d\times[1,2]\times[0,\delta^2]$ with values in $\mathbb{C}^2$. At order $k=0$, we define

$$\begin{aligned} &U_0 = W_0 = 0,\\ &\mathcal{O}_0 = \big\{(\omega,\epsilon)\in[1,2]\times[0,\gamma_0] : \exists\alpha\in\mathcal{A}_0, \exists\ell\in\{1,\dots,D_\alpha\} \text{ with } |\lambda_\alpha^\ell(\omega;0,\epsilon)|<2\delta\big\}, \end{aligned}\tag{4.2.2}$$

using the notation of Proposition 4.1.1. For any $\epsilon\in[0,\gamma_0]$, we denote by $\mathcal{O}_{0,\epsilon}$ the ϵ-section of $\mathcal{O}_0$ and set

$$G_0 = \left\{(\omega,\epsilon)\in[1,2]\times[0,\gamma_0] : d(\omega,\mathbb{R}-\mathcal{O}_{0,\epsilon}) \geq \frac{\delta}{8C_0'}\right\},$$

where $C_0'>0$ is a constant such that $|\partial_\omega\lambda_\ell^\alpha(\omega;0,\epsilon)|\leq C_0'$ for any $\alpha\in\mathcal{A}_0$, any $\ell\in\{1,\dots,D_\alpha\}$, and any $(\omega,\epsilon)\in[1,2]\times[0,\gamma_0]$. Then $\mathcal{O}_0$ is an open subset of $[1,2]\times[0,\gamma_0]$ and for any $\epsilon\in[0,\gamma_0]$, $G_{0,\epsilon}$ is a closed subset of $[1,2]$, contained in the open subset $\mathcal{O}_{0,\epsilon}$. By Urysohn's lemma, we may for each fixed ϵ construct a C^1 function $\omega\to\psi_0(\omega,\epsilon)$, compactly supported in $\mathcal{O}_{0,\epsilon}$, equal to one on $G_{0,\epsilon}$, such that for any ω and ϵ with $0\leq\psi_0(\omega,\epsilon)\leq 1$, we have $|\partial_\omega\psi_0(\omega,\epsilon)|\leq C_1\delta^{-1}$ for some uniform constant C_1 depending only on C_0'.

We set

$$\widetilde{S}_k = \sum_{\substack{\alpha\in\mathcal{A}\\ \langle n(\alpha)\rangle<2^k}} \widetilde{\Pi}_\alpha, \quad k\geq 1. \tag{4.2.3}$$

Proposition 4.2.1. *There are $\delta_0\in\,]0,\sqrt{\gamma_0}]$, positive constants C_1, B_1, B_2 and, for any $k\geq 1$ and $\delta\in\,]0,\delta_0]$, a 5-uple $(G_k,\mathcal{O}_k,\psi_k,U_k,W_k)$ satisfying the following conditions*:

$$\begin{aligned} \bullet\qquad &\mathcal{O}_k = \left\{(\omega,\epsilon)\in[1,2]\times[0,\delta^2] : \begin{matrix}\exists\alpha\in\mathcal{A}-\mathcal{A}_0 \text{ and } \ell\in\{1,\dots,D_\alpha\} \text{ such that}\\ 2^{k-1}\leq\langle n(\alpha)\rangle<2^k,\ |\lambda_\alpha^\ell(\omega;U_{k-1},\epsilon)|<2\delta 2^{-k\zeta}\end{matrix}\right\},\\ &G_k = \left\{(\omega,\epsilon)\in[1,2]\times[0,\delta^2] : d(\omega,\mathbb{R}-\mathcal{O}_{k,\epsilon})\geq\frac{\delta}{8C_0}2^{-k(\zeta+2)}\right\}, \end{aligned}\tag{4.2.4}$$

where C_0 is the constant in (4.1.6), (4.1.7);

- *$\psi_k : [1,2]\times[0,\delta^2]\to[0,1]$ is supported in $\mathcal{O}_k$, equal to 1 on G_k,*

$$C^1 \text{ in } \omega, \text{ and satisfies } |\partial_\omega\psi_k(\omega,\epsilon)|\leq\frac{C_1}{\delta}2^{k(\zeta+2)} \text{ for all } (\omega,\epsilon); \tag{4.2.5}$$

- *for any $\epsilon\in[0,\delta^2]$, the function $(t,x,\omega)\to W_k(t,x,\omega,\epsilon)$ is a continuous function of ω with values in $\mathcal{H}^s(\mathbb{S}^1\times\mathbb{T}^d;\mathbb{C}^2)$, which is a C^1 function of ω with values in $\mathcal{H}^{s-\zeta-2}(\mathbb{S}^1\times\mathbb{T}^d;\mathbb{C}^2)$ satisfying*

$$\|W_k(\cdot,\omega,\epsilon)\|_{\mathcal{H}^s} + \delta\|\partial_\omega W_k(\cdot,\omega,\epsilon)\|_{\mathcal{H}^{s-\zeta-2}} \leq B_1\frac{\epsilon}{\delta}; \tag{4.2.6}$$

moreover, for any $(\omega,\epsilon)\in[1,2]\times[0,\delta^2]-\bigcup_{k'=0}^k\mathcal{O}_{k'}$, W_k satisfies

$$\begin{aligned}(\widetilde{L}_\omega+\epsilon V_{\mathrm{D}}(U_{k-1},\omega,\epsilon))W_k &= \epsilon\widetilde{S}_k(\mathrm{Id}+\epsilon Q(U_{k-1},\omega,\epsilon))^*R(U_{k-1},\omega,\epsilon)U_{k-1}\\ &\quad+\epsilon\widetilde{S}_k[R_1(U_{k-1},\omega,\epsilon)W_{k-1}]+\epsilon\widetilde{S}_k(\mathrm{Id}+\epsilon Q(U_{k-1},\omega,\epsilon))^*f, \end{aligned}\tag{4.2.7}$$

where R is defined by (2.3.15) *and Q, V_{D}, R_1 are defined in* (3.2.2), (3.2.3);

• *the function U_k is defined from W_k by*

$$U_k(t,x,\omega,\epsilon) = (\mathrm{Id} + \epsilon Q(U_{k-1},\omega,\epsilon))W_k \tag{4.2.8}$$

and it satisfies

$$\|U_k - U_{k-1}\|_{\mathscr{H}^\sigma} \le 2B_2\frac{\epsilon}{\delta}2^{-k\zeta}, \quad \|U_k(\,\cdot\,,\omega,\epsilon)\|_{\mathscr{H}^s} + \delta\|\partial_\omega U_k(\,\cdot\,,\omega,\epsilon)\|_{\mathscr{H}^{s-\zeta-2}} \le B_2\frac{\epsilon}{\delta}; \tag{4.2.9}$$

moreover,

$$\|W_k - W_{k-1}\|_{\mathscr{H}^\sigma} \le B_2\frac{\epsilon}{\delta}2^{-k\zeta}. \tag{4.2.10}$$

Remark. Since we assume $\epsilon \le \delta^2$, the second inequality in (4.2.9) implies, with the notation introduced in (4.1.2), the uniform bound

$$\|U_k\|_{\mathscr{E}^\sigma(\zeta)} < q \tag{4.2.11}$$

for some q.

Let us write the equation for U_k following from (4.2.8) and (4.2.7). Because of the uniform estimate (4.2.11) for U_{k-1}, if $0 \le \delta \le \delta_0$ with δ_0 small enough, $(\mathrm{Id} + \epsilon Q(U_{k-1},\omega,\epsilon))^*$ is invertible for any $(\omega,\epsilon) \in [1,2]\times[0,\delta^2]$. If we write

$$(\widetilde{L}_\omega + \epsilon V(U_{k-1},\omega,\epsilon))U_k = (\widetilde{L}_\omega + \epsilon V(U_{k-1},\omega,\epsilon))(\mathrm{Id} + \epsilon Q(U_{k-1},\omega,\epsilon))W_k$$

and use (3.2.3) multiplied on the left by $(\mathrm{Id} + \epsilon Q(U_{k-1},\omega,\epsilon)^*)^{-1}$ and (4.2.7), we get

$$\begin{aligned}(\widetilde{L}_\omega + \epsilon V(U_{k-1},\omega,\epsilon))U_k = \epsilon(\mathrm{Id} + \epsilon Q(U_{k-1},\omega,\epsilon)^*)^{-1}[\widetilde{S}_k(\mathrm{Id} + \epsilon Q(U_{k-1},\omega,\epsilon)^*)R(U_{k-1},\omega,\epsilon)U_{k-1} \\ + \widetilde{S}_k R_1(U_{k-1},\omega,\epsilon)W_{k-1} + \widetilde{S}_k(\mathrm{Id} + \epsilon Q(U_{k-1},\omega,\epsilon)^*)f - R_1(U_{k-1},\omega,\epsilon)W_k]\end{aligned} \tag{4.2.12}$$

for any $(\omega,\epsilon) \in [1,2]\times[0,\delta^2] - \bigcup_{k'=0}^{k}\mathbb{O}_{k'}$, $\delta \in [0,\delta_0]$.

Proof of Proposition 4.2.1. We assume that $(G_k, \mathbb{O}_k, \psi_k, U_k, W_k)$ have been constructed satisfying (4.2.4) to (4.2.9), and shall construct these data at rank $k+1$, if δ_0 is small enough and the constants C_1, B_1, B_2 are large enough.

The sets $\mathbb{O}_{k+1}, G_{k+1}$ are defined by (4.2.4) at rank $k+1$ as soon as U_k is given. Then for fixed ϵ, $G_{k+1,\epsilon}$ is a compact subset of the open set $\mathbb{O}_{k+1,\epsilon}$, whose distance to the complement of $\mathbb{O}_{k+1,\epsilon}$ is bounded from below by $\frac{\delta}{8C_0}2^{-(k+1)(\zeta+2)}$. We may construct by Urysohn's lemma a function ψ_{k+1} satisfying (4.2.5) at rank $k+1$. Let us construct W_{k+1} for $(\omega,\epsilon) \in [1,2]\times[0,\delta^2] - \bigcup_{k'=0}^{k+1} G_{k'}$. Since $V_{\mathrm{D}}(U_k,\omega,\epsilon)$ is by construction a block-diagonal operator, we may write (4.2.7) at rank $k+1$ as the system of equations

$$\begin{aligned}(\widetilde{L}_\omega + \epsilon V_{\mathrm{D}}(U_k,\omega,\epsilon))\widetilde{\Pi}_\alpha W_{k+1} = \epsilon\widetilde{\Pi}_\alpha\widetilde{S}_{k+1}(\mathrm{Id} + \epsilon Q(U_k,\omega,\epsilon)^*)R(U_k,\omega,\epsilon)U_k \\ + \epsilon\widetilde{\Pi}_\alpha\widetilde{S}_{k+1}R_1(U_k,\omega,\epsilon)W_k + \epsilon\widetilde{\Pi}_\alpha\widetilde{S}_{k+1}(\mathrm{Id} + \epsilon Q(U_k,\omega,\epsilon)^*)f\end{aligned} \tag{4.2.13}$$

for any $\alpha \in \mathscr{A}$. If $\langle n(\alpha)\rangle \ge 2^{k+1}$, the right side of (4.2.13) vanishes by definition of $\widetilde{S}_{k+1}$, so that we may set in this case $\widetilde{\Pi}_\alpha W_{k+1} = 0$ by definition. Let us solve (4.2.13) for those α satisfying $\langle n(\alpha)\rangle < 2^{k+1}$. We shall apply Proposition 4.1.1, using the following lemma:

Lemma 4.2.2. *There is $\delta_0 \in]0,1]$, depending only on the constants B_1, B_2, such that for any $k \geq 0$, any $k' \in \{1, \dots, k+1\}$, any $\delta \in [0, \delta_0]$, any $\epsilon \in [0, \delta^2]$, and any $\alpha \in \mathcal{A} - \mathcal{A}_0$ with $2^{k'} \leq \langle n(\alpha) \rangle < 2^{k'+1}$,*

$$[1,2] - G_{k',\epsilon} \subset I(\alpha, U_k, \epsilon, \delta), \tag{4.2.14}$$

where $I(\cdot)$ is defined by (4.1.8). *The same conclusion holds when $k' = 0$, $\alpha \in \mathcal{A}_0$.*

Proof. Consider first the case $k' \neq 0$. Let $\omega \in [1,2] - \mathcal{O}_{k',\epsilon}$. Take $\ell \in \{1, \dots, D_\alpha\}$. By (i) of Proposition 4.1.1 applied to $(U, U') = (U_k, U_{k'-1})$, there is $\ell' \in \{1, \dots, D_\alpha\}$ such that

$$|\lambda_\ell^\alpha(\omega; U_k, \epsilon)| \geq |\lambda_{\ell'}^\alpha(\omega; U_{k'-1}, \epsilon)| - C_0 \epsilon \|U_k - U_{k'-1}\|_{\mathcal{H}^\sigma} \geq 2\delta 2^{-k'\zeta} - 2C_0 B_2 \frac{\epsilon^2}{\delta} \frac{2^{-k'\zeta}}{1 - 2^{-\zeta}}, \tag{4.2.15}$$

where the second lower bound follows from the definition (4.2.4) of $\mathcal{O}_{k'}$ and from (4.2.9). Since $\epsilon \leq \delta^2$, we obtain the lower bound

$$|\lambda_\ell^\alpha(\omega; U_k, \epsilon)| \geq \frac{3}{2} \delta 2^{-k'\zeta} \tag{4.2.16}$$

if $\omega \in [1,2] - \mathcal{O}_{k',\epsilon}$ and $\delta \in [0, \delta_0]$ with δ_0 small enough. If $\omega \in \mathcal{O}_{k',\epsilon} - G_{k',\epsilon}$, we take $\tilde{\omega} \in [1,2] - \mathcal{O}_{k',\epsilon}$ with $|\omega - \tilde{\omega}| < \frac{\delta}{8C_0} 2^{-k'(\zeta+2)}$. By (4.1.6), (4.1.7), we know that for any $U \in \mathscr{E}^\sigma(\zeta)$ with $\|U\|_{\mathscr{E}^\sigma(\zeta)} < q$, any $\alpha \in \mathcal{A} - \mathcal{A}_0$, any $\ell \in \{1, \dots, D_\alpha\}$,

$$\sup_{\omega' \in [1,2]} |\partial_\omega \lambda_\ell^\alpha(\omega'; U, \epsilon)| \leq C_0 \langle n(\alpha) \rangle^2.$$

Enlarging C_0, we may assume that this inequality is also valid when $\alpha \in \mathcal{A}_0$. By condition (4.2.11), we may apply it when $U = U_k$. Using (4.2.16), we get since $2^{2k'} \leq \langle n(\alpha) \rangle^2 < 2^{2(k'+1)}$

$$|\lambda_\ell^\alpha(\omega; U_k, \epsilon)| \geq |\lambda_\ell^\alpha(\tilde{\omega}; U_k, \epsilon)| - C_0 \langle n(\alpha) \rangle^2 |\omega - \tilde{\omega}| \geq \delta 2^{-k'\zeta} \geq \delta \langle n(\alpha) \rangle^{-\zeta}.$$

When $k' = 0$, we argue in the same way, taking in (4.2.15) $U_{k'-1} = 0$. This shows that ω belongs to $I(\alpha, U_k, \epsilon, \delta)$. □

To solve (4.2.13), we shall need, in addition to the preceding lemma, estimates for its right side. Set

$$\begin{aligned} H_{k+1}(U_k, W_k) = {} & \widetilde{S}_{k+1}(\mathrm{Id} + \epsilon Q(U_k, \omega, \epsilon)^*) R(U_k, \omega, \epsilon) U_k \\ & + \widetilde{S}_{k+1} R_1(U_k, \omega, \epsilon) W_k + \widetilde{S}_{k+1}(\mathrm{Id} + \epsilon Q(U_k, \omega, \epsilon)^*) f. \end{aligned} \tag{4.2.17}$$

Lemma 4.2.3. *There is a constant $C > 0$, depending on q in* (4.2.11) *but independent of k, such that for any $\omega \in [1,2]$, any $\epsilon \in [0, \delta^2]$, and any $\delta \in]0, \delta_0]$,*

$$\|H_{k+1}(U_k, W_k)\|_{\mathcal{H}^{s+\zeta}} \leq C[\|U_k(\cdot, \omega, \epsilon)\|_{\mathcal{H}^s} + \|W_k(\cdot, \omega, \epsilon)\|_{\mathcal{H}^s}] + (1 + C\epsilon)\|f\|_{\mathcal{H}^{s+\zeta}}, \tag{4.2.18}$$

$$\begin{aligned} \|\partial_\omega H_{k+1}(U_k, W_k)\|_{\mathcal{H}^{s-2}} \leq {} & C[\|U_k(\cdot, \omega, \epsilon)\|_{\mathcal{H}^s} + \|\partial_\omega U_k(\cdot, \omega, \epsilon)\|_{\mathcal{H}^{s-\zeta-2}} \\ & + \|W_k(\cdot, \omega, \epsilon)\|_{\mathcal{H}^s} + \|\partial_\omega W_k(\cdot, \omega, \epsilon)\|_{\mathcal{H}^{s-\zeta-2}} + \epsilon \|f\|_{\mathcal{H}^{s-2}}], \end{aligned} \tag{4.2.19}$$

$$\begin{aligned} \|H_{k+1}(U_k, W_k) - H_k(U_{k-1}, W_{k-1})\|_{\mathcal{H}^{\sigma+\zeta}} \leq {} & C[\|U_k - U_{k-1}\|_{\mathcal{H}^\sigma} + \|W_k - W_{k-1}\|_{\mathcal{H}^\sigma}] \\ & + 2^{-k\zeta}[C(\|U_k\|_{\mathcal{H}^{\sigma+\zeta}} + \|W_k\|_{\mathcal{H}^{\sigma+\zeta}}) + (1 + C\epsilon)\|f\|_{\mathcal{H}^{\sigma+2\zeta}}]. \end{aligned} \tag{4.2.20}$$

Proof. The operators R and R_1 belong to $\mathcal{R}_2^r(N+1,\sigma,q)$ with $r=\zeta$. By Definition 2.1.3, and because of the assumption (4.2.1) on the indices, they are bounded from $\mathcal{H}^s$ to $\mathcal{H}^{s+\zeta}$. Moreover, $Q(U_k,\omega,\epsilon)^*$ is in $\Psi^0(N,\sigma,q)\otimes\mathcal{M}_2(\mathbb{R})$, so is bounded on any $\mathcal{H}^s$-space by Lemma 2.1.2. This gives (4.2.18).

To obtain (4.2.19), one has to study the boundedness properties of

$$\partial_\omega[Q(U_k,\omega,\epsilon)] = \partial_U Q(\,\cdot\,,\omega,\epsilon)\cdot(\partial_\omega U_k) + \partial_\omega Q(U_k,\omega,\epsilon), \tag{4.2.21a}$$

$$\partial_\omega[R(U_k,\omega,\epsilon)] = \partial_U R(\,\cdot\,,\omega,\epsilon)\cdot(\partial_\omega U_k) + \partial_\omega R(U_k,\omega,\epsilon), \tag{4.2.21b}$$

$$\partial_\omega[R_1(U_k,\omega,\epsilon)] = \partial_U R_1(\,\cdot\,,\omega,\epsilon)\cdot(\partial_\omega U_k) + \partial_\omega R_1(U_k,\omega,\epsilon). \tag{4.2.21c}$$

By (2.1.2), the inequalities in (4.2.1), and the fact that, by (4.2.11), $\partial_\omega U_k$ is uniformly bounded in $\mathcal{H}^{s-\zeta-2}\subset\mathcal{H}^\sigma$, we see that the operator in (4.2.21a) is bounded on any space $\mathcal{H}^{s'}$. By (2.1.3), and the assumption $s\geq\sigma+\zeta+2$ in (4.2.1), we see in the same way that (4.2.21b) and (4.2.21c) give bounded operators from $\mathcal{H}^{s-\zeta-2}$ to $\mathcal{H}^{s-2}$ and from $\mathcal{H}^s$ to $\mathcal{H}^{s+\zeta}$. This gives estimate (4.2.19).

To prove (4.2.20), let us write the difference $H_{k+1}(U_k,W_k)-H_k(U_{k-1},W_{k-1})$ from the quantities

$$\left.\begin{aligned} (\widetilde{S}_{k+1}-\widetilde{S}_k)(\mathrm{Id}+\epsilon Q(U_k,\omega,\epsilon)^*)R(U_k,\omega,\epsilon)U_k,\\ (\widetilde{S}_{k+1}-\widetilde{S}_k)R_1(U_k,\omega,\epsilon)W_k,\\ (\widetilde{S}_{k+1}-\widetilde{S}_k)(\mathrm{Id}+\epsilon Q(U_k,\omega,\epsilon)^*)f, \end{aligned}\right\} \tag{4.2.22}$$

$$\left.\begin{aligned} \epsilon\widetilde{S}_k[Q(U_k,\omega,\epsilon)^*-Q(U_{k-1},\omega,\epsilon)^*]R(U_k,\omega,\epsilon)U_k,\\ \widetilde{S}_k(\mathrm{Id}+\epsilon Q(U_{k-1},\omega,\epsilon)^*)[R(U_k,\omega,\epsilon)-R(U_{k-1},\omega,\epsilon)]U_k,\\ \widetilde{S}_k[R_1(U_k,\omega,\epsilon)-R_1(U_{k-1},\omega,\epsilon)]W_k,\\ \epsilon\widetilde{S}_k[Q(U_k,\omega,\epsilon)^*-Q(U_{k-1},\omega,\epsilon)^*]f, \end{aligned}\right\} \tag{4.2.23}$$

$$\left.\begin{aligned} \widetilde{S}_k(\mathrm{Id}+\epsilon Q(U_{k-1},\omega,\epsilon)^*)R(U_{k-1},\omega,\epsilon)(U_k-U_{k-1}),\\ \widetilde{S}_kR_1(U_k,\omega,\epsilon)(W_k-W_{k-1}). \end{aligned}\right\} \tag{4.2.24}$$

By (4.2.6) and (4.2.9), U_k and W_k stay in a bounded subset of $\mathcal{H}^\sigma$ and R, R_1 act from $H^{\sigma+\zeta}$ to $H^{\sigma+2\zeta}$. Using the cut-off $\widetilde{S}_{k+1}-\widetilde{S}_k$, we see that the $\mathcal{H}^{\sigma+\zeta}$ norm of (4.2.22) is bounded from above by the last term in the right side of (4.2.20).

By (2.1.3), the $\mathcal{L}(\mathcal{H}^\sigma,\mathcal{H}^{\sigma+\zeta})$ operator norm of $R(U_k,\omega,\epsilon)-R(U_{k-1},\omega,\epsilon)$ and of $R_1(U_k,\omega,\epsilon)-R_1(U_{k-1},\omega,\epsilon)$ is bounded from above by $C\|U_k-U_{k-1}\|_{\mathcal{H}^\sigma}$. By (2.1.2), the $\mathcal{L}(\mathcal{H}^{\sigma+\zeta},\mathcal{H}^{\sigma+\zeta})$-norm of $Q(U_k,\omega,\epsilon)^*-Q(U_{k-1},\omega,\epsilon)^*$ is bounded by the same quantity. This shows that the $\mathcal{H}^{\sigma+\zeta}$ norm of (4.2.23) is bounded from above by the right side of (4.2.20).

Finally, (4.2.24) is trivially estimated. This concludes the proof. $\square$

We continue with the proof of Proposition 4.2.1. We have seen that $\widetilde{\Pi}_\alpha W_{k+1}$ is a solution to (4.2.13). Let $k'\in\{1,\dots,k+1\}$ and $\alpha\in\mathcal{A}-\mathcal{A}_0$ such that $2^{k'}\leq\langle n(\alpha)\rangle<2^{k'+1}$, or $k'=0,\alpha\in\mathcal{A}_0$. Let $\omega\in[1,2]-G_{k',\epsilon}$. By Lemma 4.2.2 and Proposition 4.1.1, the operator $A_\alpha(\omega;U_k,\epsilon)$ is invertible, ant its inverse satisfies estimates (4.1.9). For such ω, we may write (4.2.13) as

$$\widetilde{\Pi}_\alpha W_{k+1} = \epsilon A_\alpha(\omega;U_k,\epsilon)^{-1}\widetilde{\Pi}_\alpha H_{k+1}(U_k,W_k). \tag{4.2.25}$$

Applying (4.1.9) we obtain, for any $k' \in \{1, \dots, k+1\}$, any $\alpha \in \mathcal{A} - \mathcal{A}_0$ with $2^{k'} \le \langle n(\alpha)\rangle < 2^{k'+1}$, and any $(\omega, \epsilon) \in [1,2] \times [0, \delta^2] - G_{k'}$ (and also any $\alpha \in \mathcal{A}_0$ and $(\omega, \epsilon) \in [1,2] \times [0, \delta^2] - G_0$), the bound

$$\|\widetilde{\Pi}_\alpha W_{k+1}(\,\cdot\,, \omega, \epsilon)\|_{\mathcal{H}^s} \le E_0 \frac{\epsilon}{\delta} \|\widetilde{\Pi}_\alpha H_{k+1}(U_k, W_k)(\,\cdot\,, \omega, \epsilon)\|_{\mathcal{H}^{s+\zeta}}. \tag{4.2.26}$$

In the same way, one gets the estimate

$$\begin{aligned}\|\widetilde{\Pi}_\alpha \partial_\omega W_{k+1}(\,\cdot\,, \omega, \epsilon)\|_{\mathcal{H}^{s-\zeta-2}} \\ \le E_0 \frac{\epsilon}{\delta} \|\widetilde{\Pi}_\alpha \partial_\omega H_{k+1}(U_k, W_k)(\,\cdot\,, \omega, \epsilon)\|_{\mathcal{H}^{s-2}} + E_0 \frac{\epsilon}{\delta^2} \|\widetilde{\Pi}_\alpha H_{k+1}(U_k, W_k)(\,\cdot\,, \omega, \epsilon)\|_{\mathcal{H}^{s+\zeta}}.\end{aligned} \tag{4.2.27}$$

We define $W_{k+1}(t, x, \omega, \epsilon)$ for any value of (ω, ϵ) in $[1,2] \times [0, \delta^2]$ from (4.2.25) by setting

$$\begin{aligned}W_{k+1}(t, x, \omega, \epsilon) = \sum_{k'=1}^{k+1} \sum_{\substack{\alpha \in \mathcal{A} - \mathcal{A}_0 \\ 2^{k'} \le \langle n(\alpha)\rangle < 2^{k'+1}}} (1 - \psi_{k'}(\omega, \epsilon)) \widetilde{\Pi}_\alpha W_{k+1}(t, x, \omega, \epsilon) \\ + \sum_{\alpha \in \mathcal{A}_0} (1 - \psi_0)(\omega, \epsilon) \widetilde{\Pi}_\alpha W_{k+1}(t, x, \omega, \epsilon).\end{aligned} \tag{4.2.28}$$

Note that the right side is well defined since (4.2.25) determines $\widetilde{\Pi}_\alpha W_{k+1}(\,\cdot\,, \omega, \epsilon)$ on the support of $1 - \psi_{k'}$ when (α, k') satisfy the conditions in the summation.

We combine (4.2.28), (4.2.26) and (4.2.18). Taking into account (4.2.6) and (4.2.9), we get

$$\|W_{k+1}(\,\cdot\,, \omega, \epsilon)\|_{\mathcal{H}^s} \le E_0 \frac{\epsilon}{\delta} \Big(C(B_1 + B_2) \frac{\epsilon}{\delta} + \|f\|_{\mathcal{H}^{s+\zeta}} (1 + C\epsilon) \Big). \tag{4.2.29}$$

To bound the ∂_ω-derivative, we use that by (4.2.5)

$$\|\partial_\omega \psi_{k'} \widetilde{\Pi}_\alpha W_{k+1}\|_{\mathcal{H}^{s-\zeta-2}} \le \frac{C_1}{\delta} \|\widetilde{\Pi}_\alpha W_{k+1}\|_{\mathcal{H}^s}$$

when $2^{k'} \le \langle n(\alpha)\rangle < 2^{k'+1}$, $\alpha \in \mathcal{A} - \mathcal{A}_0$ if $k' \ne 0$, and when $\alpha \in \mathcal{A}_0$ if $k' = 0$. We apply this inequality together with (4.2.28), (4.2.27), (4.2.18), (4.2.19) and the uniform bounds (4.2.6), (4.2.9), to get

$$\begin{aligned}\|\partial_\omega W_{k+1}(\,\cdot\,, \omega, \epsilon)\|_{\mathcal{H}^{s-\zeta-2}} \le E_0 \frac{\epsilon}{\delta} \Big(C(B_1 + B_2) \frac{\epsilon}{\delta^2} + C\epsilon \|f\|_{\mathcal{H}^{s-2}} \Big) \\ + E_0 \frac{\epsilon}{\delta^2} \Big(C(B_1 + B_2) \frac{\epsilon}{\delta} + (1 + C\epsilon)\|f\|_{\mathcal{H}^{s+\zeta}} \Big) + E_0 C_1 \frac{\epsilon}{\delta^2} \Big(C(B_1 + B_2) \frac{\epsilon}{\delta} + (1 + C\epsilon)\|f\|_{\mathcal{H}^{s+\zeta}} \Big).\end{aligned} \tag{4.2.30}$$

In (4.2.29) and (4.2.30), C depends on the *a priori* bound given by (4.2.11), while E_0, C_1 are uniform constants. Consequently, if we take B_1 large enough relatively to $\|f\|_{\mathcal{H}^{s+\zeta}}$, E_0, C_1 and then $\epsilon \le \delta^2 \le \delta_0^2$, with δ_0 small enough, we deduce from (4.2.29) and (4.2.30) that (4.2.6) holds at rank $k+1$. The second estimate in (4.2.9) at rank $k+1$ follows, with for instance $B_2 = 2B_1$, if δ_0 is small enough. We are left with establishing the first estimate in (4.2.9) at rank $k+1$ and (4.2.10).

First let us bound $W_{k+1} - W_k$. By (4.2.25), for any $k' \in \{1, \dots, k\}$, any $(\omega, \epsilon) \in [1,2] \times [0, \delta^2] - G_{k'}$, $\alpha \in \mathcal{A} - \mathcal{A}_0$, and any $2^{k'} \le \langle n(\alpha)\rangle < 2^{k'+1}$ (or any $(\omega, \epsilon) \in [1,2] \times [0, \delta^2] - G_0$ and $\alpha \in \mathcal{A}_0$), we have

$$\begin{aligned}(\widetilde{L}_\omega + \epsilon V_{\mathrm{D}}(U_k, \omega, \epsilon)) \widetilde{\Pi}_\alpha W_{k+1} = \epsilon \widetilde{\Pi}_\alpha H_{k+1}(U_k, W_k), \\ (\widetilde{L}_\omega + \epsilon V_{\mathrm{D}}(U_{k-1}, \omega, \epsilon)) \widetilde{\Pi}_\alpha W_k = \epsilon \widetilde{\Pi}_\alpha H_k(U_{k-1}, W_{k-1}),\end{aligned}$$

whence the equation

$$
\begin{aligned}
(\widetilde{L}_\omega + \epsilon V_{\mathrm{D}}(U_k,\omega,\epsilon))\widetilde{\Pi}_\alpha(W_{k+1}-W_k) &= \epsilon\widetilde{\Pi}_\alpha[V_{\mathrm{D}}(U_{k-1},\omega,\epsilon)-V_{\mathrm{D}}(U_k,\omega,\epsilon)]W_k \\
&\quad + \epsilon\widetilde{\Pi}_\alpha[H_{k+1}(U_k,W_k)-H_k(U_{k-1},W_{k-1})]. \qquad (4.2.31)
\end{aligned}
$$

We make act $\mathcal{A}_\alpha(\omega;U_k,\epsilon)^{-1}$ on both sides as in (4.2.25). Applying inequality (4.1.9) we get

$$
\begin{aligned}
\|\widetilde{\Pi}_\alpha(W_{k+1}-W_k)\|_{\mathcal{H}^\sigma} \le \frac{E_0\epsilon}{\delta}\big[&\|\widetilde{\Pi}_\alpha[V_{\mathrm{D}}(U_{k-1},\omega,\epsilon)-V_{\mathrm{D}}(U_k,\omega,\epsilon)]W_k\|_{\mathcal{H}^{\sigma+\zeta}} \\
&+ \|\widetilde{\Pi}_\alpha[H_{k+1}(U_k,W_k)-H_k(U_{k-1},W_{k-1})]\|_{\mathcal{H}^{\sigma+\zeta}}\big]. \qquad (4.2.32)
\end{aligned}
$$

This estimate holds outside $G_{k'}$ when $k'\neq 0$, $\alpha\in\mathcal{A}-\mathcal{A}_0$, $2^{k'}\le\langle n(\alpha)\rangle<2^{k'+1}$, and outside G_0 when $\alpha\in\mathcal{A}_0$. By (4.2.28), we may write

$$
\begin{aligned}
(W_{k+1}-W_k)(t,x,\omega,\epsilon) &= \sum_{\alpha\in\mathcal{A}_0}(1-\psi_0)\widetilde{\Pi}_\alpha(W_{k+1}-W_k) \\
&+ \sum_{k'=1}^{k}\sum_{\substack{\alpha\in\mathcal{A}-\mathcal{A}_0\\ 2^{k'}\le\langle n(\alpha)\rangle<2^{k'+1}}}(1-\psi_{k'})\widetilde{\Pi}_\alpha(W_{k+1}-W_k) + \sum_{\substack{\alpha\in\mathcal{A}-\mathcal{A}_0\\ 2^{k+1}\le\langle n(\alpha)\rangle<2^{k+2}}}(1-\psi_{k+1})\widetilde{\Pi}_\alpha W_{k+1}. \qquad (4.2.33)
\end{aligned}
$$

The $\mathcal{H}^\sigma$ norm of the last term is bounded by $C_2 2^{-k(s-\sigma)}\|W_{k+1}\|_{\mathcal{H}^s}\le C_2B_1\frac{\epsilon}{\delta}2^{-k(s-\sigma)}$ by (4.2.6), for some universal constant C_2. The $\mathcal{H}^\sigma$-norm of the k'-sum in (4.2.33) may be estimated using (4.2.32), (4.2.20) and the bound

$$
\|(V_{\mathrm{D}}(U_{k-1},\omega,\epsilon)-V_{\mathrm{D}}(U_k,\omega,\epsilon))W_k\|_{\mathcal{H}^{\sigma+\zeta}}\le C\|U_k-U_{k-1}\|_{\mathcal{H}^\sigma}\|W_k\|_{\mathcal{H}^s}
$$

which follows from (2.1.2), and where we used $s\ge\sigma+\zeta$. Using the induction hypothesis (4.2.9), (4.2.10), we get

$$
\begin{aligned}
&\|W_{k+1}-W_k\|_{\mathcal{H}^\sigma} \\
&\le E_0\frac{\epsilon}{\delta}\Big(CB_1\frac{\epsilon}{\delta}2B_2\frac{\epsilon}{\delta}2^{-k\zeta}+3CB_2\frac{\epsilon}{\delta}2^{-k\zeta}+C2^{-k\zeta}(B_1+B_2)\frac{\epsilon}{\delta}+(1+C\epsilon)\|f\|_{\mathcal{H}^{\sigma+2\zeta}}2^{-k\zeta}\Big) \\
&\qquad + C_2B_1\frac{\epsilon}{\delta}2^{-k(s-\sigma)}. \qquad (4.2.34)
\end{aligned}
$$

Since $s\ge\sigma+\zeta$, we may take B_1 large enough relatively to E_0, $\|f\|_{\mathcal{H}^{s+\zeta}}$, and B_2 large enough relatively to C_2, B_1, and $\epsilon/\delta\le\delta\le\delta_0$ small enough, so that (4.2.34) is smaller than $B_2(\epsilon/\delta)2^{-(k+1)\zeta}$, whence (4.2.10) at rank $k+1$. Writing

$$
U_{k+1}-U_k = (\mathrm{Id}+\epsilon Q(U_k,\omega,\epsilon))(W_{k+1}-W_k)+\epsilon(Q(U_k,\omega,\epsilon)-Q(U_{k-1},\omega,\epsilon))W_k,
$$

we deduce from that the first inequality in (4.2.9) at rank $k+1$, for small enough ϵ. This concludes the proof of the proposition. □

Proof of Theorem 1.1.1. By (4.2.9), the series $\sum(U_k-U_{k-1})$ converges in $\mathcal{H}^\sigma(\mathbb{S}^1\times\mathbb{T}^d;\mathbb{C}^2)$ and its sum U satisfies $U\in\mathcal{H}^s(\mathbb{S}^1\times\mathbb{T}^d;\mathbb{C}^2)$ with

$$
\|U(\,\cdot\,,\omega,\epsilon)\|_{\mathcal{H}^s}+\delta\|\partial_\omega U(\,\cdot\,,\omega,\epsilon)\|_{\mathcal{H}^{s-\zeta-2}}\le B_2\frac{\epsilon}{\delta}.
$$

We have to check that U gives a solution to our problem outside a set of parameters of small measure. Let

$$(\omega,\epsilon)\in[1,2]\times[0,\delta^2]-\bigcup_{k'=0}^{\infty}\mathbb{O}_{k'}$$

and $\delta\in\,]0,\delta_0]$. Then (4.2.12) is satisfied for any k. We make $k\to+\infty$. Since we have uniform $\mathcal{H}^s$ bounds for U_k, W_k and $\mathcal{H}^\sigma$ convergence for these quantities, the limit U satisfies

$$(\widetilde{L}_\omega+\epsilon V(U,\omega,\epsilon))U=\epsilon R(U,\omega,\epsilon)U+\epsilon f,$$

which is (2.3.15). We have seen that this equation is equivalent to (2.3.14), which is, by Proposition 2.3.1, the same as (2.2.13). Since Proposition 2.2.4 shows that, up to a change of notation, this equation is equivalent to the formulation (2.2.6) of (1.1.3), we obtain a solution satisfying the requirements of Theorem 1.1.1. We still have to check that (1.1.5) holds with $\mathbb{O}=\bigcup_{k'=0}^{\infty}\mathbb{O}_{k'}$. According to (4.2.2), the set $\mathbb{O}_0$ is included in the set of those (ω,ϵ) such that there are (j,n) in a given finite subset of $\mathbb{Z}^2$ such that $\left|j\omega+|n|^2+\mu\right|<2\delta$. The ω-measure of this set is $O(\delta)$, $\delta\to0$ (Note that since $\mu\notin\mathbb{Z}_-$, we may always assume $j\neq0$). For $k'>0$, $\mathbb{O}_{k'}$ is the union for $\alpha\in\mathcal{A}-\mathcal{A}_0$ with $2^{k'-1}\le\langle n(\alpha)\rangle<2^{k'}$ and $\ell\in\{1,\dots,D_\alpha\}$ of the set of those (ω,ϵ) satisfying

$$|\lambda_\alpha^\ell(\omega;U_{k'-1},\epsilon)|<2\delta2^{-k'\zeta}.$$

By (4.1.6), (4.1.7) the ω-measure of each of these sets in bounded by $C\langle n(\alpha)\rangle^{-2}\delta2^{-k'\zeta}\le C2^{-(k'+2)\zeta}\delta$. Since $D_\alpha\le C_12^{k'(\beta d+2)}$ by (1.2.4), (1.2.6), we obtain for the measure of the ϵ-section of $\mathbb{O}$ the bound

$$C\sum_{k'=0}^{+\infty}2^{-(k'+2)\zeta+k'(\beta d+2)+k'd}\delta.$$

If we take $\zeta>(\beta+1)d+2$, we obtain the wanted $O(\delta)$ bound. This concludes the proof. □

Appendix

We gather here some elementary results used throughout the paper.

Lemma A.1. *Let $s>\frac{d}{2}+1$. Then $\widetilde{\mathcal{H}}^s(\mathbb{S}^1\times\mathbb{T}^d;\mathbb{C})\subset L^\infty$. Moreover, if F is a smooth function on $\mathbb{S}^1\times\mathbb{T}^d\times\mathbb{C}$, satisfying $F(t,x,0)\equiv0$, there is some continuous function $\tau\to C(\tau)$ such that, for any $u\in\widetilde{\mathcal{H}}^s$, we have $F(\cdot,u)\in\widetilde{\mathcal{H}}^s$ with the estimate*

$$\|F(\cdot,u)\|_{\widetilde{\mathcal{H}}^s}\le C(\|u\|_{L^\infty})\|u\|_{\widetilde{\mathcal{H}}^s}.$$

Proof. Let $\varphi\in C_0^\infty(]0,+\infty[)$, $\varphi\ge0$, $\varphi\equiv1$ on $[1,2]$ be such that $\sum_{\ell=-\infty}^{+\infty}\varphi(2^{-\ell}\lambda)\equiv1$ for $\lambda\in\mathbb{R}_+^*$, and define $\psi(\lambda)=\sum_{-\infty}^{0}\varphi(2^{-\ell}\lambda)$. Consider, for $(j,n)\in\mathbb{Z}\times\mathbb{Z}^d$,

$$\begin{aligned}\Phi_k(j,n)&=\varphi(2^{-2k}(j^2+|n|^4)^{1/2}),\ k\ge1,\\ \Phi_0(j,n)&=\psi((j^2+|n|^4)^{1/2}).\end{aligned}\tag{A.1}$$

Define for $u \in \widetilde{\mathscr{H}}^0$ and $k \in \mathbb{N}$,

$$\Delta_k u = \sum_{j,n} \Phi_k(j,n)\hat{u}(j,n)\frac{e^{i(tj+k\cdot n)}}{(2\pi)^{(d+1)/2}}, \quad K_k(t,x) = \frac{1}{(2\pi)^{d+1}}\sum_{j,n}\Phi_k(j,n)e^{i(tj+k\cdot n)}. \tag{A.2}$$

Then, for any $N \in \mathbb{N}$,

$$|K_k(t,x)| \le C_N 2^{2k(1+d/2)}(1+2^{2k}|e^{it}-1|+2^k|e^{ix}-1|)^{-N} \tag{A.3}$$

and $u \in \widetilde{\mathscr{H}}^s$ if and only if $(2^{ks}\|\Delta_k u\|_{L^2})_k$ is in ℓ^2.

The first statement of the lemma follows from the inequality $\|\Delta_k u\|_{L^\infty} \le C2^{k(1+d/2)}\|\Delta_k u\|_{L^2}$, which is a consequence of (A.3) (for the kernel corresponding to an enlarged Φ_k). To get the second statement, we consider first the case of a function F that does not depend on (t,x). We set $S_k = \sum_{k'\le k-1}\Delta_{k'}$ when $k \ge 1$, $S_0 = 0$ and write

$$F(u) = \sum_{k=0}^{+\infty}(F(S_{k+1}u) - F(S_k u)) = \sum_{k=0}^{+\infty} m_k(u)\Delta_k u$$

where $m_k(u) = \int_0^1 F'(S_k u + \tau\Delta_k u)\,d\tau$. It follows from the definition of S_k that this operator is given by a convolution kernel obeying the same estimates as in (A.3). Consequently, for any $(\alpha,\beta) \in \mathbb{N}\times\mathbb{N}^d$,

$$\|\partial_t^\alpha\partial_x^\beta m_k(u)\|_{L^\infty} \le C2^{2k\alpha+k|\beta|} \tag{A.4}$$

with a constant depending only on $\|u\|_{L^\infty}$. One writes for some $N_0 \in \mathbb{N}$ to be chosen

$$\Delta_j[F(u)] = \sum_{k=0}^{j-1-N_0}\Delta_j[m_k(u)\Delta_k u] + \sum_{k=j-N_0}^{+\infty}\Delta_j[m_k(u)\Delta_k u]. \tag{A.5}$$

The L^2-norm of the second sum is bounded by $Cc_j 2^{-js}\|u\|_{\widetilde{\mathscr{H}}^s}$ for some sequence $(c_j)_j$ in the unit ball of ℓ^2, and some C depending only on $\|u\|_{L^\infty}$. If N_0 is fixed large enough, because of the support properties of the Fourier transforms,

$$\Delta_j[m_k(u)\Delta_k u] = \Delta_j\big[[(\mathrm{Id} - S_{j-N_0})m_k(u)]\Delta_k u\big]$$

when $k \le j-1-N_0$. We estimate the L^2-norm of this quantity by

$$\|(\mathrm{Id} - S_{j-N_0})m_k\|_{L^\infty}\|\Delta_k u\|_{L^2} \tag{A.6}$$

and use that, for any N, we have $\|(\mathrm{Id}-S_{j-N_0})m_k\|_{L^\infty} \le C_N 2^{-4jN}\|P^N m_k\|_{L^\infty}$ where $P = \partial_t^2 + \Delta^2 + 1$. It follows from (A.4) that (A.6) is bounded from above by $C_N 2^{-4(j-k)N}\|\Delta_k u\|_{L^2}$, from which we deduce that the L^2-norm of the first sum in (A.5) is also smaller than $C2^{-js}c_j\|u\|_{\widetilde{\mathscr{H}}^s}$. This concludes the proof for functions F independent of (t,x). In the general case, we note that since u is bounded, we may always assume that F is compactly supported, and we write

$$F(t,x,u) = \frac{1}{2\pi}\int_{\mathbb{R}} F_1(u,\theta)b(t,x,\theta)\,d\theta,$$

where $F_1(u,\theta)=e^{iu\theta}-1$ and $b(t,x,\theta)$ is the Fourier transform of $u\to F(t,x,u)$. Then it follows from the preceding proof that $F_1(u,\theta)$ is in $\widetilde{\mathcal{H}}^s$ with a bound $\|F_1(u,\theta)\|_{\widetilde{\mathcal{H}}^s}\le C\langle\theta\rangle^{N(s)}$, for some exponent $N(s)$. Moreover, for any N, $\|b(\cdot,\theta)\|_{\widetilde{\mathcal{H}}^s}\le C_N\langle\theta\rangle^{-N}$. We get the conclusion by superposition. □

Corollary A.2. *Let $F:\mathbb{S}^1\times\mathbb{T}^d\times\mathbb{C}\to\mathbb{C}$ be a smooth function with $F(t,x,0)\equiv 0$. Then $u\to F(\cdot,u)$ is a smooth map from $\widetilde{\mathcal{H}}^\sigma$ to itself, for any $\sigma>\frac{d}{2}+1$.*

Proof. We write

$$F(t,x,u+h)-F(t,x,u)-\partial_u F(t,x,u)h=\int_0^1\int_0^1 (D^2F)(t,x,u+\tau_1\tau_2 h)\tau_1\cdot h^2\,d\tau_1 d\tau_2$$

and we apply the lemma to $D^2F(t,x,u)-D^2F(t,x,0)$. □

Lemma A.3.
- *Let $s>\frac{d}{2}+1$. If $u\in\widetilde{\mathcal{H}}^s$ and $v\in\widetilde{\mathcal{H}}^{\sigma'}$ for some $\sigma'\in[-s,s]$, then $uv\in\widetilde{\mathcal{H}}^{\sigma'}$.*
- *For any $\sigma\in\mathbb{R}$ and $\sigma_0>\frac{d}{2}+1$, $\widetilde{\mathcal{H}}^\sigma\cdot\widetilde{\mathcal{H}}^{-\sigma}\subset\widetilde{\mathcal{H}}^{-\max(\sigma,\sigma_0)}$.*

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STANDING RING BLOWUP SOLUTIONS FOR CUBIC NONLINEAR SCHRÖDINGER EQUATIONS

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For all dimensions $N \geq 3$ we prove there exist solutions to the focusing cubic nonlinear Schrödinger equations that blow up on a set of codimension two. The blowup set is identified both as the site of L^2 concentration and by a bounded supercritical norm outside any neighborhood of the set. In all cases, the global H^1 norm grows at the log-log rate.

1. Introduction

Consider the cubic focusing nonlinear Schrödinger equation in dimension $N \geq 3$:

$$\begin{cases} iu_t + \Delta u + u\,|u|^2 = 0, \\ u(0,x) = u_0 : \mathbb{R}^N \to \mathbb{C}. \end{cases} \tag{1-1}$$

This is a canonical model equation arising in physics and engineering [Sulem and Sulem 1999]. This equation, and other closely related equations, have been the subject of many recent mathematical studies.

Equation (1-1) is locally wellposed for data

$$u_0 \in H^s(\mathbb{R}^N)$$

for any $s \in \left(\frac{N}{2} - 1, \frac{N}{2}\right]$ or integer $s > \frac{N}{2}$; see [Cazenave 2003]. In these cases we have the classic blowup alternative: either $T_{\max} = +\infty$ or $\|u(t)\|_{H^s} \to \infty$ as $t \to T_{\max}$. Higher regularity persists under local-in-time dynamics and the maximal time $T_{\max} > 0$ for which u belongs to $C([0, T_{\max}), H^s)$ is the same for all $s > \frac{N}{2} - 1$. Evolution under (1-1) preserves

$$\int_{\mathbb{R}^3} |u(t,x)|^2\, dx = \int |u_0|^2\, dx = M[u_0] \qquad \text{(mass)}, \tag{1-2}$$

$$\int |\nabla_x u(t,x)|^2\, dx - \frac{1}{2}\int |u(t,x)|^4\, dx = E[u(t,x)] = E[u_0] \qquad \text{(energy)}, \tag{1-3}$$

$$\mathrm{Im}\left(\int \bar{u}(t,x)\nabla u(t,x)\, dx\right) = \mathrm{Im}\left(\int \overline{u_0}\nabla u_0\, dx\right) \qquad \text{(momentum)}. \tag{1-4}$$

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There are corresponding symmetries. If $u(t,x)$ satisfies (1-1), so do the following:

$$\begin{aligned} &u(t, x+x_0) && \forall\, x_0 \in \mathbb{R}^N && \text{(spatial translation invariance)} \\ &u(t+t_0, x) && \forall\, t_0 \in \mathbb{R} && \text{(time translation invariance)} \\ &u(t,x)e^{i\gamma_0} && \forall\, \gamma_0 \in \mathbb{R} && \text{(phase invariance)} \\ &u(t, x-\beta_0 t)e^{i\frac{\beta_0}{2}\cdot(x-\frac{\beta_0}{2}t)} && \forall\, \beta_0 \in \mathbb{R}^N && \text{(Galilean invariance)} \\ &\lambda_0 u(\lambda_0^2 t, \lambda_0 x) && \forall\, \lambda_0 > 0 && \text{(scaling invariance)} \end{aligned}$$

Scaling invariance leaves the $\dot{H}^{\frac{N}{2}-1}(\mathbb{R}^N)$ norm of data unchanged and for this reason Equation (1-1) is deemed $H^{\frac{N}{2}-1}$*-critical*. Local wellposedness for $s > \frac{N}{2} - 1$ and the scaling symmetry prove that all solutions that blow up in finite time $T_{\max} < +\infty$ must obey the *scaling lower bounds*

$$\|u(t)\|_{H^s} \gtrsim \frac{1}{(T_{\max} - t)^{\frac{s}{2} - \frac{N-2}{4}}}.$$

Equation (1-1) has standing wave solutions. The ansatz $u(t,x) = e^{it}W(x)$ leads to the elliptic PDE

$$\begin{cases} \Delta W - W + W\,|W|^2 = 0, \\ W(|x|) > 0 \quad \text{for } x \in \mathbb{R}^N. \end{cases} \tag{1-5}$$

The unique positive radial solution[1] to (1-5) is the *ground-state* solution of (1-1). We reserve the notation Q for the ground-state solution of the *two-dimensional* problem,

$$\begin{cases} \Delta_{\mathbb{R}^2} Q - Q + Q\,|Q|^2 = 0, \\ Q(|y|) > 0 \quad \text{for } y \in \mathbb{R}^2. \end{cases} \tag{1-6}$$

Classification of dynamics. In the case $N = 2$, if $M[u_0] < M[Q]$ solutions to (1-1) exist for all time [Weinstein 1983] and scatter [Killip et al. 2009]. Negative-energy data $u_0 \in H^1$ lead to blowup in finite time if it is radially symmetric or has finite variance, $u_0 \in \Sigma = H^1 \cap \{f : |x|\, f(x) \in L^2\}$; see [Ogawa and Tsutsumi 1991]. By adjusting the quadratic phase of negative-energy data, one can produce examples of blowup solutions with arbitrary energy [Cazenave 2003, Remark 6.5.9].[2] At the threshold $M[u_0] = M[Q]$ there is, up to symmetries, a unique explicit blowup solution [Merle 1993].

In the cases $N = 3$ and $N = 4$ the situation is more complicated. Assume that

$$M[u_0]^{4-N} E[u_0]^{N-2} < M[W]^{4-N} E[W]^{N-2}. \tag{1-7}$$

The following classification is independent of time:

$$\begin{aligned} &\|u(t)\|_{L^2}^{4-N} \|\nabla u(t)\|_{L^2}^{N-2} < \|W\|_{L^2}^{4-N} \|\nabla W\|_{L^2}^{N-2}, && \text{with global existence and scattering, or} \\ &\|u(t)\|_{L^2}^{4-N} \|\nabla u(t)\|_{L^2}^{N-2} > \|W\|_{L^2}^{4-N} \|\nabla W\|_{L^2}^{N-2}, && \text{with unbounded } H^1 \text{ norm growth.} \end{aligned}$$

[1] The classic proof in the case $N = 3$ is in [Coffman 1972]. For other dimensions, see [Weinstein 1983; Berestycki and Lions 1983; Kwong 1989]. For a concise overview of these results, see [Tao 2006, Appendix B].

[2] These arguments also apply to the other energy-subcritical case, $N = 3$. Further sufficient conditions for blowup based on the virial identity in $N = 3$ are known; see [Holmer et al. 2010].

This was shown in [Kenig and Merle 2006] in the case $N = 4$ with radial symmetry and in [Duyckaerts et al. 2008; Holmer and Roudenko 2008; 2010] in the case $N = 3$. Again, the data at the threshold of (1-7) may be classified in terms of three special solutions, up to symmetries; see [Duyckaerts and Merle 2009; Duyckaerts and Roudenko 2010]. One of these solutions blows up in finite time, but the exact dynamic is unknown.

In all cases, the description of behavior above the threshold is an ongoing challenge.

In the case of $N = 2$, Merle and Raphaël have completely described the dynamic of an open class of data with $M[Q] < M[u_0] < M[Q] + \delta$. This open class includes all data with negative energy and is thought to describe the generic behavior. See Theorem 1.1, below, for references. Their work follows the earlier simulation [Landman et al. 1988] and construction [Perelman 2001] of solutions in H^1 that blow up at the rate of the scaling lower bound with a log-log correction.

Recently, in the case of $N = 3$, data with $M[W]\,E[W] < M[u_0]\,E[u_0] < M[W]\,E[W] + \delta$ has been shown in [Nakanishi and Schlag 2010] to satisfy one of nine scenarios involving scattering, finite-time blowup, or trapping in the neighborhood of a manifold of solitons.

Known blowup regimes. There remain, in all cases, very few examples of explicit blowup regimes. We have already alluded to Merle and Raphaël's results in the case $N = 2$:

Theorem 1.1 (log-log blowup of L^2-critical NLS [Merle and Raphaël 2003; 2004; 2005a; 2006; Raphaël 2005]). *Consider the focusing L^2-critical NLS in dimension* $1 \le d \le 5$,

$$i u_t + \Delta u + u\,|u|^{\frac{4}{d}} = 0.$$

There exists an open set of data in $H^1(\mathbb{R}^d)$, with mass a little larger than the groundstate, that blow up at the log-log rate:

$$\|u(t)\|_{H^1} \approx \left(\frac{\log|\log(T_{\max}-t)|}{T_{\max}-t}\right)^{\frac{1}{2}}. \tag{1-8}$$

These solutions concentrate exactly the groundstate profile in L^2 at a point. That is, the remainder of the solution has a strong limit in L^2 as $t \to T_{\max}$. Moreover, data in the same range of mass that do not belong to the log-log class give solutions that either exist for all time, or blow up at the rate $\|u(t)\|_{H^1} \gtrsim (T_{\max}-t)^{-1}$.

Theorem 1.1 gives a precise understanding of the stable blowup regime of the one-dimensional quintic NLS. To prove the next theorem, Raphaël demonstrated a reduction of the two-dimensional quintic problem (which is $H^{\frac{1}{2}}$-critical) to this one-dimensional log-log regime.

Theorem 1.2 (standing ring blowups for quintic NLS in 2D [Raphaël 2006]). *There exists an open set of radially symmetric data in $H^1(\mathbb{R}^2)$ for which the corresponding solution to $i u_t + \Delta u + u\,|u|^4 = 0$ exhibits blowup at the log-log rate*, (1-8), *and concentration in L^2 at a ring of fixed radius.*

The argument was extended to reduce the energy critical and supercritical quintic equations to the same one-dimensional log-log regime:

Theorem 1.3 (codimension-one blowups for quintic NLS [Raphaël and Szeftel 2009]). *For all* $N \geq 3$, *there exists an open set of radially symmetric data in* $H^N(\mathbb{R}^N)$ *for which the corresponding solution to* $iu_t + \Delta u + u\,|u|^4 = 0$ *exhibits blowup at the log-log rate and concentration in* L^2 *at a fixed radius.*

Our aim is to implement this approach for the cubic problem, and, under cylindrical symmetry, reduce the N-dimensional problem (1-1) to the two-dimensional problem, which we understand by Theorem 1.1. The following result and our main result, Theorem 1.6, were developed simultaneously.[3]

Theorem 1.4 (standing ring blowups for cubic NLS in 3D [Holmer and Roudenko 2011]). *There exists an open set of cylindrically symmetric data in* $H^1(\mathbb{R}^3)$ *for which the corresponding solution to* (1-1) *exhibits blowup at the log-log rate and concentration in* L^2 *at a ring of fixed radius.*

The stability proven in Theorem 1.4 is at the level of H^1 regularity, which allows for a different approach and a larger class of data than our main result in the case $N = 3$. The new techniques of Theorem 1.4 are not directly applicable for the energy critical and supercritical cases, $N > 3$.

It is anticipated that the L^2-supercritical NLS will demonstrate further unique blowup behavior not described by Theorems 1.2, 1.3, 1.4 or our Theorem 1.6. In particular, due to the asymptotic analysis of Fibich, Gavish, and Wang [Fibich et al. 2007], all supercritical problems with subquintic nonlinearities are expected to admit radially symmetric blowup solutions that focus onto a shell whose radius collapses to zero. In the case of cubic nonlinearity and dimension N, the shell is expected to have radius $\approx (T_{\max} - t)^{N/(N-1)^2}$. In the case $N = 3$, the H^1 norm is expected to grow at the scaling lower bound. See also [Holmer and Roudenko 2007]. The existence and uniqueness of these radial blowup solutions remain important problems.

Notation 1.5. We use $f \lesssim g$, $f \gtrsim g$ and $f \approx g$ to denote that there exist constants $C_1, C_2 > 0$ such that $f \leq C_1 g$, $f \geq C_2 g$, and $C_2 g \leq f \leq C_1 g$, respectively. The notation $f \sim g$ is used in more casual discussion to say that f and g are of the same order. We will use $\delta(\alpha)$ to denote any function of α with the property $\delta(\alpha) \to 0$ as $\alpha \to 0$. The exact form of δ will depend on the context. Frequently, we use the operator

$$\Lambda = 1 + y \cdot \nabla_y, \quad \text{where } y \text{ is a two-dimensional variable.}$$

For $f, g \in L^2(\mathbb{R}^2)$ we have $(\Lambda f, g) = -(f, \Lambda g)$.

Statement of result. For all $N \geq 3$ we introduce cylindrical coordinates $x = (r, z, \theta) \in [0, \infty) \times \mathbb{R} \times S^{N-2}$ for $x \in \mathbb{R}^N$. We refer to functions that are symmetric with respect to θ as cylindrically symmetric, and we let $H^s_{\mathrm{cyl}}(\mathbb{R}^N)$ denote the cylindrically symmetric subset of H^s.

Theorem 1.6 (main result). *For all* $N \geq 3$, *there exists a set of cylindrically symmetric data* $u_0 \in \mathcal{P}$, *open in* $H^N_{\mathrm{cyl}}(\mathbb{R}^N)$, *for which the corresponding solution* $u(t)$ *of* (1-1) *has maximum* (*forward*) *lifetime* $0 < T_{\max} < +\infty$ *and exhibits the following properties*:

[3] After Theorem 1.2, the idea of considering other $H^{\frac{1}{2}}$-critical problems was first suggested to the author's thesis advisor by Justin Holmer and Svetlana Roudenko in private conversation.

Concentration. *There exist parameters* $\lambda(t) > 0$, $r(t) > 0$, $z(t) \in \mathbb{R}^{N-2}$, *and* $\gamma(t) \in \mathbb{R}$, *with convergence*

$$(r(t), z(t)) \longrightarrow (r_{\max}, z_{\max}) \quad \textit{as } t \to T_{\max}, \ \textit{with } r_{\max} \sim 1, \tag{1-9}$$

such that there is the following strong convergence in $L^2(\mathbb{R}^N)$:

$$u(t, r, z, \theta) - \frac{1}{\lambda(t)} Q\left(\frac{(r, z) - (r(t), z(t))}{\lambda(t)}\right) e^{-i\gamma(t)} \longrightarrow u^*(r, z, \theta) \quad \textit{as } t \to T_{\max}. \tag{1-10}$$

Persistent regularity away from singular ring. *For any* $R > 0$,

$$u^* \in H^{\frac{N}{2}-\frac{1}{2}}\left(|(r, z) - (r_{\max}, z_{\max})| > R\right). \tag{1-11}$$

Log-log blowup rate. *The solution leaves* H^1 *at the log-log rate*:

$$\frac{\left(\dfrac{\log|\log T_{\max} - t|}{T_{\max} - t}\right)^{\frac{1}{2}}}{\|u(t)\|_{H^1(\mathbb{R}^3)}} \longrightarrow \frac{\sqrt{2\pi}}{\|Q\|_{L^2(\mathbb{R}^2)}} \quad \textit{as } t \to T_{\max}. \tag{1-12}$$

Moreover, the higher-order norm behaves appropriately:

$$\frac{\|u(t)\|_{H^N}}{\|u(t)\|_{H^1}^N \log\|u(t)\|_{H^1}} \longrightarrow 0 \quad \textit{as } t \to T_{\max}. \tag{1-13}$$

Remark 1.7 (nature of u^*). For the L^2-critical problem, Theorem 1.1, it is known that the residual profile u^* is not in H^1 [Merle and Raphaël 2005b]. Indeed, (1-11) fails for $R = 0$. See also Remark 5.1.

Brief heuristic. In cylindrical coordinates the Laplacian is written as

$$\Delta_x = \partial_r^2 + \partial_z^2 + (N-2)\frac{\partial_r}{r}. \tag{1-14}$$

Suppose that a solution to (1-1) is cylindrically symmetric and concentrated near the ring $(r, z) \sim (r_0, z_0)$. Then for an appropriately small $\lambda_0 > 0$ we may write

$$u(t, x) = \frac{1}{\lambda_0} v\left(\frac{t}{\lambda_0^2}, \frac{(r, z) - (r_0, z_0)}{\lambda_0}\right), \tag{1-15}$$

where the function v is supported on the half-plane $(r, z) \in [-r_0/\lambda_0, \infty) \times \mathbb{R}$. Neglect that our parameters may vary in time. After changing coordinates, v satisfies

$$i\partial_s v + \Delta_y v + (N-2)\frac{\lambda_0}{r}\partial_{y_1} v + v|v|^2 = 0, \quad \text{where } s = \frac{t}{\lambda_0^2}, \ y = \frac{(r, z) - (r_0, z_0)}{\lambda_0}. \tag{1-16}$$

For a solution $u(t, x)$ tightly concentrated near (r_0, z_0), we might choose $\lambda_0 \ll 1$ as the width of the window of concentration. Then, $(N-2)(\lambda_0/r)\partial_{y_1} v$ can be taken as a lower-order correction, and the evolution of v is essentially that of the two-dimensional cubic NLS. If $v(s, y)$ falls within the robust log-log blowup dynamic, we would expect the concentration near (r_0, z_0) to increase, and for the lower-order correction in (1-16) to become less relevant.

We can identify our main challenge: to ensure persistence of sufficient decay in the original variables near $r=0$ such that conditions there mimic those at infinity during a log-log blowup of two-dimensional cubic NLS.

2. Setting of the bootstrap

In this section we identify data concentrated near the set $(r,z)\sim(1,0)$, according to properties we will later show persist. Our subsequent arguments are based on the two-dimensional L^2-critical log-log blowup dynamic, which has been comprehensively investigated in [Merle and Raphaël 2003; 2004; 2005a; 2005b; 2006; Raphaël 2005]. This work stems from those detailed studies.

Definition 2.1 (fundamental properties of almost self-similar profiles). For all $b>0$ sufficiently small, there exists a solution $\widetilde{Q}_b\in H^1(\mathbb{R}^2)$ of

$$\Delta\widetilde{Q}_b-\widetilde{Q}_b+ib\Lambda\widetilde{Q}_b+\widetilde{Q}_b\,|\widetilde{Q}_b|^2=-\Psi_b$$

that is supported on the ball of radius $2/|b|$ and converges to Q in $C^3(\mathbb{R}^2)$ as $b\to 0$. The profiles $\widetilde{Q}_b$ have mass of the order of b^2 larger than Q, and energy of the order $e^{-C/b}$. The truncation error Ψ_b acts as the source of the linear radiation,

$$\Delta\zeta_b-\zeta_b+ib\Lambda\zeta_b=\Psi_b.$$

The radiation ζ_b is not in L^2, with the precise decay rate $\Gamma_b=\lim_{|y|\to+\infty}|y|\,|\zeta_b|^2$. It is known that $\Gamma_b\sim e^{-\pi/b}$, and it is this decay property linked to the central profile $\widetilde{Q}_b$ that is responsible for the log-log rate of the two-dimensional L^2-critical problem. For our analysis, we will truncate ζ_b near $|y|\sim e^{+a/b}$ for a small fixed parameter a. See page 694 for details.

Lemma 2.2 (smoothness of $\widetilde{Q}_b$). *The almost self-similar profiles $\widetilde{Q}_b$ are smooth. For any $s\geq 3$,*

$$\limsup_{b\to 0}\|\widetilde{Q}_b\|_{C^s(\mathbb{R}^2)}<+\infty\quad\textit{and}\quad\limsup_{b\to 0}\|\widetilde{Q}_b\|_{H^s(\mathbb{R}^2)}<+\infty.\tag{2-17}$$

Geometric decomposition. In place of $(r,z,\theta)\in\mathbb{R}^N$ we change coordinates to the rescaled half-plane:

$$y=\frac{(r,z)-(r_0,z_0)}{\lambda_0}\in[-r_0/\lambda_0,+\infty)\times\mathbb{R}.\tag{2-18}$$

The fixed parameters r_0, z_0, λ_0 will later be replaced by $r(t)$, $z(t)$, and $\lambda(t)$. This will be clear from the context. Note the measure due to cylindrical symmetry, $dx=\lambda_0\mu_{\lambda_0,r_0}(y)\,dy$ is given by

$$\mu_{\lambda_0,r_0}(y)=|S^{N-2}|(\lambda_0 y_1+r_0)^{N-2}\mathbb{1}_{y_1\geq -r_0/\lambda_0}.\tag{2-19}$$

We will shortly hypothesize parameters of the decomposition in such a way that the support of both $\widetilde{Q}_b$ and $\tilde{\zeta}_b$ are well away from the boundary of domain (2-18). For convenience we will omit the constant factor $|S^{N-2}|$ and approximate $\mu(y)\sim 1$ on this region; see (2-55). Integrals in y can then be seen as taken over all of $\mathbb{R}^2$, and regular integration by parts applies. Any integral that cannot be localized in this way will be treated separately, and very carefully.

To begin, we modulate suitable cylindrically symmetric data as if it were two-dimensional:

Lemma 2.3 (existence of geometric decomposition at a fixed time [Raphaël 2006, Lemma 2]). *Suppose that $v \in H^1_{\mathrm{cyl}}(\mathbb{R}^N)$ may be written in the form*

$$v(r,z,\theta) = \frac{1}{\lambda_v}(\tilde{Q}_{b_v} + \epsilon_v)\left(\frac{(r,z)-(r_v,z_v)}{\lambda}\right) e^{-i\gamma_v} \tag{2-20}$$

for some parameters $\lambda_v, b_v, r_v > 0$ and $\gamma_v, z_v \in \mathbb{R}$ such that

$$\int \left|\nabla_y \epsilon_v\right|^2 \mu_{\lambda_v,r_v}(y)\,dy + \int_{|y|\le 10/b_v} |\epsilon_v|^2 e^{-|y|}\,dy < \Gamma_{b_v}^{\frac{1}{2}}, \tag{2-21}$$

$$|(r_v,z_v)-(1,0)| < \tfrac{1}{3} \quad \text{and} \quad 10\lambda_v < b_v < \alpha^*. \tag{2-22}$$

Then there are nearby parameters $\lambda_0, b_0, r_0 > 0$ and $\gamma_0, z_0 \in \mathbb{R}$ with

$$|b_0 - b_v| + \left|\frac{\lambda_0}{\lambda_v} - 1\right| + \frac{|(r_0,z_0)-(r_v,z_v)|}{\lambda_v} \le \Gamma_{b_0}^{\frac{1}{5}}, \tag{2-23}$$

such that the corresponding ϵ_0,

$$\epsilon_0(y) = \lambda_0\, v\,(\lambda_0 y + (r_0,z_0))\, e^{i\gamma_0} - \tilde{Q}_{b_0}, \tag{2-24}$$

satisfies the two-dimensional orthogonality conditions[4]

$$\mathrm{Re}(\epsilon_0, |y|^2 \tilde{Q}_{b_0}) = \mathrm{Re}(\epsilon_0, y\tilde{Q}_{b_0}) = \mathrm{Im}(\epsilon_0, \Lambda^2 \tilde{Q}_{b_0}) = \mathrm{Im}(\epsilon_0, \Lambda \tilde{Q}_{b_0}) = 0. \tag{2-25}$$

We now identify a neighborhood of the singular set, the complement of which is contiguous and includes both the origin and infinity. In the case of $N = 3$, the singular set is a ring and the neighborhood a toroid. Define two smooth cutoff functions,

$$\chi(r,z,\theta) = \begin{cases} 1 & \text{for } |(r,z)-(1,0)| \ge \frac{2}{3}, \\ 0 & \text{for } |(r,z)-(1,0)| \le \frac{1}{3}, \end{cases} \qquad \chi_0(r,z,\theta) = \begin{cases} 1 & \text{for } |(r,z)-(1,0)| \ge \frac{1}{7}, \\ 0 & \text{for } |(r,z)-(1,0)| \le \frac{1}{8}. \end{cases} \tag{2-26}$$

In Section 4 we will define a further series of cutoff functions ψ and φ, supported on bounded regions where $\chi_0 \equiv 1$. We now describe the initial data for our bootstrap procedure.

Definition 2.4 (description of initial data $\mathcal{P}$). For $\alpha^* > 0$ a constant to be determined, let $\mathcal{P}(\alpha^*)$ be the set of cylindrically symmetric $u_0 \in H^N_{\mathrm{cyl}}(\mathbb{R}^N)$ that can be written in the form

$$\begin{aligned} u_0(r,z) &= \frac{1}{\lambda_0}(\tilde{Q}_{b_0} + \epsilon_0)\left(\frac{(r,z)-(r_0,z_0)}{\lambda_0}\right) e^{-i\gamma_0} \\ &= \frac{1}{\lambda_0}(\tilde{Q}_{b_0})\left(\frac{(r,z)-(r_0,z_0)}{\lambda_0}\right) e^{-i\gamma_0} + \tilde{u}_0(r,z), \end{aligned} \tag{2-27}$$

in a way that satisfies the following two sets of conditions:

[4]The decomposition of [Merle and Raphaël 2003] used slightly different orthogonality conditions. Equation (2-25) is the decomposition introduced [Merle and Raphaël 2004, Lemma 6], which leads to a better estimate on the phase parameter than was achieved in the former paper.

Singularity of a log-log nature:

C1.1. *Radial profile is focused near a singular ring:*

$$|(r_0, z_0) - (1, 0)| < \alpha^*. \tag{2-28}$$

C1.2. *Radial profile is close to Q near the singular ring:* The profiles $\widetilde{Q}_b$ have nearly the mass of Q and account for nearly all mass globally:

$$0 < b_0 + \|\tilde{u}_0\|_{L^2(\mathbb{R}^N)} < \alpha^*; \tag{2-29}$$

$\epsilon_0(y)$ satisfies both the orthogonality conditions

$$\mathrm{Re}(\epsilon_0, |y|^2\, \widetilde{Q}_{b_0}) = \mathrm{Re}(\epsilon_0, y\widetilde{Q}_{b_0}) = \mathrm{Im}(\epsilon_0, \Lambda^2 \widetilde{Q}_{b_0}) = \mathrm{Im}(\epsilon_0, \Lambda \widetilde{Q}_{b_0}) = 0 \tag{2-30}$$

and the smallness condition

$$\int \left|\nabla_y \epsilon_0(y)\right|^2 \mu_{\lambda_0, r_0}(y)\, dy + \int_{|y| \le \frac{10}{b_0}} |\epsilon_0(y)|^2 e^{-|y|}\, dy < \Gamma_{b_0}^{\frac{6}{7}}. \tag{2-31}$$

C1.3. *Conformal and scaling parameters are consistent with log-log blowup speed:*

$$e^{-e^{\frac{2\pi}{b_0}}} < \lambda_0 < e^{-e^{\frac{\pi}{2}\frac{1}{b_0}}}. \tag{2-32}$$

C1.4. *Energy and localized momentum are normalized:*

$$\lambda_0^2\, |E_0| + \lambda_0 \left| \mathrm{Im}\left(\int \nabla_x \psi^{(x)} \cdot \nabla_x u_0 \bar{u}_0 \right) \right| < \Gamma_{b_0}^{10}, \tag{2-33}$$

where $\psi^{(x)}$ is a cylindrically symmetric smooth cutoff function given by

$$\psi^{(x)}(r, z, \theta) = \begin{cases} r + z & \text{for } |(r, z) - (1, 0)| \le \frac{1}{2}, \\ 0 & \text{for } |(r, z) - (1, 0)| \ge \frac{3}{4}. \end{cases} \tag{2-34}$$

Regularity away from the singularity:

C2.1. *Scaling-consistent $\dot{H}^N$ norm:*

$$\|u_0\|_{H^N(\mathbb{R}^N)} < \frac{C_{\widetilde{Q}}}{\lambda_0^N}, \tag{2-35}$$

where $C_{\widetilde{Q}}$ is a universal constant due to Lemma 2.2.

C2.2. *Strong hierarchy of regularity away from the singular ring:*

$$\|\chi_0 u_0\|_{H^{N-\kappa}(\mathbb{R}^N)} < \frac{1}{\lambda_0^{N-2\kappa}}, \tag{2-36}$$

for each half-integer $\frac{1}{2} \le \kappa \le \frac{N}{2}$.

C2.3. *Vanishing lower-order norms away from the singular ring:*

$$\|\chi_0 u_0\|_{H^{\frac{N}{2}-\frac{1}{2}}{}_1(\mathbb{R}^N)} < (\alpha^*)^{\frac{1}{2}}. \tag{2-37}$$

Lemma 2.3 guarantees that $\mathcal{P}(\alpha^*)$ is open in $H^1_{\mathrm{cyl}} \cap H^N_{\mathrm{cyl}}$. See the Appendix for a proof that $\mathcal{P}(\alpha^*)$ is nonempty.

For the remainder of this paper, fix an arbitrary $u_0 \in \mathcal{P}(\alpha^*)$. Let $u(t)$ denote the evolution under (1-1), with maximum (forward) lifetime $T_{\max} > 0$, possibly infinite.

Continuous evolution in $H^N(\mathbb{R}^N)$ implies the same in $H^1(\mathbb{R}^N)$, so by Lemma 2.3 there is some $T_{\mathrm{geo}} \in (0, T_{\max}]$ — which may be assumed maximal — for which the geometric decomposition of Lemma 2.3 can be applied on $[0, T_{\mathrm{geo}})$. There exist unique continuous functions $\lambda(t), b(t), r(t) : [0, T_{\mathrm{geo}}) \to (0, \infty)$ and $\gamma(t), z(t) : [0, T_{\mathrm{geo}}) \to \mathbb{R}$, with the expected initial values, where

$$\begin{aligned} u(t, r, z, \theta) &= \frac{1}{\lambda(t)} \big(\tilde{Q}_{b(t)} + \epsilon(t)\big) \left(\frac{(r, z) - (r(t), z(t))}{\lambda} \right) e^{-i\gamma(t)} \\ &= \frac{1}{\lambda(t)} \big(\tilde{Q}_{b(t)}\big) \left(\frac{(r, z) - (r(t), z(t))}{\lambda} \right) e^{-i\gamma(t)} + \tilde{u}(t, r, z, \theta), \end{aligned} \tag{2-38}$$

such that $\epsilon(t, y)$ satisfies the two-dimensional orthogonality conditions

$$\operatorname{Re}\big(\epsilon(t), |y|^2 \tilde{Q}_{b(t)}\big) = 0, \tag{2-39}$$

$$\operatorname{Re}\big(\epsilon(t), y \tilde{Q}_{b(t)}\big) = 0, \tag{2-40}$$

$$\operatorname{Im}\big(\epsilon(t), \Lambda^2 \tilde{Q}_{b(t)}\big) = 0, \tag{2-41}$$

$$\operatorname{Im}\big(\epsilon(t), \Lambda \tilde{Q}_{b(t)}\big) = 0. \tag{2-42}$$

We may now define the rescaled time,

$$s(t) = \int_0^t \frac{1}{\lambda^2(\tau)} d\tau + s_0, \quad \text{where } s_0 = e^{\frac{3\pi}{4b_0}}. \tag{2-43}$$

The choice of s_0 will prove convenient in the proof of Lemma 3.15 (page 699).

Also set $s_1 = s(T_{\mathrm{hyp}})$, for T_{hyp} as in Definition 2.6 below.

Notation 2.5 (fixed parameters). To aid the reader, we provide a brief summary of the various parameters that will be introduced, in the order one might ultimately determine them:

- η and a are parameters that determine the cutoff shape of $\tilde{Q}_b$ and $\tilde{\zeta}_b$; see (3-80) and (3-104). The value of $a > 0$ is assumed sufficiently small for the proof of Lemma 3.22, relative to some universal constant. Before that, the proof of Lemma 3.19 is conditioned on the choice of $\eta < a/C_0$, for another universal constant $C_0 > 0$; see (3-196). These choices affect the class of initial data $\mathcal{P}$, both by setting the profiles $\tilde{Q}_b$ and by forcing an upper bound on the value of α^*.
- σ_1, σ_2 and σ_3: parameters in the statements of Lemma 4.1, Lemma 4.3, and Corollary 4.4. Their value is chosen (repeatedly) according to circumstance.
- σ_4: an arbitrary universal constant, $0 < \sigma_4 \ll 1$, used in the proof of Lemma 4.8.
- σ_5: defined for Lemma 4.8. Its value depends on σ_4, and is uniform over all $m > 0$ small enough.
- m': existence of $m' < m$ with particular properties in a key assertion of Proposition 2.8. Some particular value $m' \in (m - \sigma_5/2, m)$ is chosen for the proof of Lemma 4.10.

- σ_6: parameter in the statement of Lemma 4.11. Its value is fixed for the proof of Lemma 4.13.
- σ_7: defined for Lemma 4.17. Its value is uniform over all $m > 0$ small enough.
- m: a fixed constant $m > 0$ that features in the bootstrap hypotheses of Definition 2.6. For the purpose of various proofs in Section 4, m will be assumed sufficiently small. The exact value of m may be determined apriori, and will affect the class of initial data $\mathcal{P}$ by forcing an upper bound on the value of α^*.
- α^*: A fixed positive constant to be determined last. For the purpose of various proofs throughout this paper, α^* will be assumed sufficiently small.

The following bootstrap hypotheses are possible due to our choice of data in $\mathcal{P}$.

Definition 2.6 (time $T_{\text{hyp}} > 0$ and bootstrap hypotheses). Let $0 < T_{\text{hyp}} \leq T_{\max}$ be the maximum time such that for all $t \in [0, T_{\text{hyp}})$ the following two sets of conditions hold:

Singularity remains of a log-log nature:

H1.1. *Profile remains focused near a singular ring:*

$$\left|(r(t), z(t)) - (1, 0)\right| < (\alpha^*)^{\frac{1}{2}}. \tag{2-44}$$

H1.2. *Profile remains close to Q near the singular ring:*

$$0 < b(t) + \|\tilde{u}(t)\|_{L^2(\mathbb{R}^N)} < (\alpha^*)^{\frac{1}{10}}, \tag{2-45}$$

$$\int \left|\nabla_y \epsilon(t)\right|^2 \mu_{\lambda(t), r(t)}(y)\, dy + \int_{|y| \leq \frac{10}{b(t)}} |\epsilon(t)|^2 e^{-|y|}\, dy \leq \Gamma_{b(t)}^{\frac{3}{4}}. \tag{2-46}$$

H1.3. *Conformal and scaling parameters remain consistent with log-log blowup speed:*

$$\frac{\pi}{10} \frac{1}{\log s} < b(s) < \frac{10\pi}{\log s}, \qquad e^{-e^{\frac{10\pi}{b(s)}}} < \lambda(s) < e^{-e^{\frac{\pi}{10} \frac{1}{b(s)}}}. \tag{2-47}$$

H1.4. *Energy and localized momentum remain normalized:*

$$\lambda^2(t)\, |E_0| + \lambda(t) \left| \operatorname{Im} \left(\int \nabla \psi^{(x)} \cdot \nabla u(t) \bar{u}(t) \right) \right| < \Gamma_{b(t)}^2. \tag{2-48}$$

H1.5. *Norm growths are almost monotonic:*

$$\lambda(s_b) \leq 3\lambda(s_a) \quad \text{for all } \; s_a \leq s_b \in [s_0, s_1]. \tag{2-49}$$

Regularity away from the singularity persists:

H2.1. *Growth of $\dot{H}^N$ is near scaling:*

$$\|u(t)\|_{H^N(\mathbb{R}^N)} < \frac{e^{+\frac{m}{b(t)}}}{\lambda^N(t)}. \tag{2-50}$$

H2.2. *Strong hierarchy of regularity away from the singular ring persists:*

$$\|\chi u(t)\|_{H^{N-\kappa}} < \frac{e^{+(1+\kappa)\frac{m}{b(t)}}}{\lambda^{N-2\kappa}(t)}, \tag{2-51}$$

for each half-integer $\frac{1}{2} \leq \kappa < \frac{N}{2}$, and

$$\|\chi u(t)\|_{H^{\frac{N}{2}}} < e^{+\frac{2m+\pi}{b(t)}}. \tag{2-52}$$

H2.3. *Lower-order norms away from the singular ring remain bounded:*

$$\|\chi u(t)\|_{H^{\frac{N}{2}-\frac{1}{2}}} < (\alpha^*)^{\frac{1}{10}}. \tag{2-53}$$

An important consequence of H1.2, H1.3, and the forthcoming estimate on Γ_b, (3-103), is that

$$\lambda(t) < e^{-e^{\frac{\pi}{10b(t)}}} < \Gamma_{b(t)}^{10}. \tag{2-54}$$

Therefore as a consequence of H1.1 and the definition of A in (3-104) below,

$$\tfrac{2}{3} \leq \mu(y) \leq \tfrac{3}{2} \quad \text{for all } \ |y| \leq 5A(t). \tag{2-55}$$

The region $|y| \leq 5A(t)$ is exceptionally wide, encompassing the support of both the central profile $\widetilde{Q}_b$ and the associated radiation $\tilde{\zeta}_b$.

Remark 2.7 (geometric decomposition is well defined). Hypotheses H1.1–H1.5 easily satisfy the conditions of Lemma 2.3, ensuring that $T_{\text{hyp}} \leq T_{\text{geo}}$ and the unique geometric decomposition (2-38) is available.

Proposition 2.8 (bootstrap conclusion). *For $\alpha^* > 0$ sufficiently small, hypotheses* (2-44)–(2-53) *are not sharp. There exists $m' < m$ such that, for all $t \in [0, T_{\text{hyp}})$:*

I1.1.
$$\left|(r(t), z(t)) - (1,0)\right| < (\alpha^*)^{\frac{2}{3}}. \tag{2-56}$$

I1.2.
$$0 < b(t) + \|\tilde{u}(t)\|_{L^2(\mathbb{R}^N)} < (\alpha^*)^{\frac{1}{5}}, \tag{2-57}$$

$$\int \left|\nabla_y \epsilon(t)\right|^2 \mu_{\lambda(t),r(t)}(y)\, dy + \int_{|y| \leq \frac{10}{b(t)}} |\epsilon(t)|^2 e^{-|y|}\, dy \leq \Gamma_{b(t)}^{\frac{4}{5}}. \tag{2-58}$$

I1.3.
$$\frac{\pi}{5}\frac{1}{\log s} < b(s) < \frac{5\pi}{\log s}, \qquad e^{-e^{\frac{5\pi}{b(t)}}} < \lambda(t) < e^{-e^{\frac{\pi}{5}\frac{1}{b(t)}}}. \tag{2-59}$$

I1.4.
$$\lambda^2(t)\,|E_0| + \lambda(t)\left|\mathrm{Im}\left(\int \nabla\psi^{(x)} \cdot \nabla u(t)\bar{u}(t)\right)\right| < \Gamma_{b(t)}^4. \tag{2-60}$$

I1.5.
$$\lambda(s_b) \leq 2\lambda(s_a) \quad \text{for all } \ s_a \leq s_b \in [s_0, s_1]. \tag{2-61}$$

I2.1.
$$\|u(t)\|_{H^N(\mathbb{R}^N)} < \frac{e^{+\frac{m'}{b(t)}}}{\lambda^N(t)}. \tag{2-62}$$

I2.2.
$$\|\chi u(t)\|_{H^{N-\kappa}} < \frac{e^{+(1+\kappa)\frac{m'}{b(t)}}}{\lambda^{N-2\kappa}(t)} \tag{2-63}$$

for each half-integer $\frac{1}{2} \le \kappa < \frac{N}{2}$, *and*

$$\|\chi u(t)\|_{H^{\frac{N}{2}}} < e^{+\frac{2m'+\pi}{b(t)}}. \tag{2-64}$$

I2.3.
$$\|\chi u(t)\|_{H^{\frac{N}{2}-\frac{1}{2}}} < (\alpha^*)^{\frac{1}{5}}. \tag{2-65}$$

As a consequence, $T_{\text{hyp}} = T_{\max}$.

Strategy of proof: the log-log argument. We will establish statements I1.1–I1.5 in Section 3 using the arguments of [Merle and Raphaël 2003; 2006]. Here we identify the main challenge in maintaining the log-log dynamics. As with all modulation arguments, we seek to reduce the question of blowup to a finite-dimensional ODE dynamic for the parameters. This is only possible due to the algebraic structure associated with Q. Recall the operator $\Lambda = 1 + y \cdot \nabla_y$, which one might recognize from either the argument $E(Q) = 0$:

$$(0, \Lambda(Q)) = \big(\Delta Q - Q + Q\,|Q|^2\,, \Lambda(Q)\big) = -2E(Q), \tag{2-66}$$

or from the Pohozaev identity:

$$(0, \Lambda(v)) = \operatorname{Re}\big(i\,v_s + \Delta_y v + v\,|v|^2\,, \Lambda(v)\big) = -\frac{1}{2}\frac{d}{ds}\operatorname{Im}\int v\, y\cdot\nabla\bar{v}\,dy - 2E(v), \tag{2-67}$$

which is also a consequence of formally calculating the virial identity's term

$$\frac{d^2}{d^2 s}\int |y|^2\,|v|^2\,dy.$$

Substitution of (2-38) into (1-1) will produce an equation for ϵ. Ignoring the distinction between Q and $\widetilde{Q}_b$, the terms linear in ϵ are $i\,\partial_s\epsilon + L(\epsilon)$, where L is the linearized propagator near Q. As a matrix on real and imaginary parts,

$$L(\epsilon) = \begin{bmatrix} 0 & L_- \\ -L_+ & 0 \end{bmatrix}\begin{bmatrix} \epsilon_{\text{re}} \\ i\,\epsilon_{\text{im}} \end{bmatrix} \quad \text{with} \quad \begin{cases} L_+ = -\Delta + 1 - 3Q^2, \\ L_- = -\Delta + 1 - Q^2. \end{cases} \tag{2-68}$$

Weinstein [1985] noted that

$$L_-(|y|^2\,Q) = -2\Lambda Q, \quad L_-(yQ) = -2\nabla Q, \quad \text{and} \quad L_+(\Lambda Q) = -2Q. \tag{2-69}$$

These algebraic properties are the inspiration for the orthogonality conditions, so that, by taking appropriate inner products of the ϵ-equation, linear terms cancel. For example, the imaginary part of the inner product with $|y|^2\,Q$ has no linear terms due to conditions (2-39) and (2-42). The imaginary part of the inner product with yQ is controlled by the momentum.

The most fruitful calculation is when we take the real part of an inner product of the ϵ-equation with ΛQ. This is of course a localized version of (2-67). We substitute conservation of energy to eliminate the linear term, $2\,\mathrm{Re}(\epsilon, Q)$, which is due to the third identity of (2-69). The remaining terms quadratic in ϵ form the following,

$$H(\epsilon,\epsilon)=\begin{bmatrix}\mathscr{L}_{\mathrm{re}} & 0\\ 0 & \mathscr{L}_{\mathrm{im}}\end{bmatrix}\begin{bmatrix}\epsilon_{\mathrm{re}}\\ i\,\epsilon_{\mathrm{im}}\end{bmatrix}\cdot\begin{bmatrix}\epsilon_{\mathrm{re}}\\ -i\,\epsilon_{\mathrm{im}}\end{bmatrix}\quad\text{with}\quad\begin{cases}\mathscr{L}_{\mathrm{re}}=-\Delta+3Qy\cdot\nabla Q,\\ \mathscr{L}_{\mathrm{im}}=-\Delta+Qy\cdot\nabla Q.\end{cases}\tag{2-70}$$

The operator $H(\epsilon,\epsilon)$ is the derivative with respect to scaling of the conserved energy of the linear flow. It has coercivity properties that mirror the stability of Q:

Proposition 2.9 (spectral property). *There exists a universal constant $\delta_0>0$ such that, for any $v\in H^1$,*

$$H(v,v)\geq\delta_0\left(\int_{y\in\mathbb{R}^2}\left|\nabla_y v\right|^2+\int_{y\in\mathbb{R}^2}|v^2|e^{-|y|}\right)-\frac{1}{\delta_0}\times$$
$$\big((\mathrm{Re}(v,Q))^2+(\mathrm{Re}(v,\Lambda Q))^2+(\mathrm{Re}(v,yQ))^2+(\mathrm{Im}(v,\Lambda Q))^2+(\mathrm{Im}(v,\Lambda^2Q))^2+(\mathrm{Im}(v,\nabla Q))^2\big).\tag{2-71}$$

The two-dimensional spectral property as stated here has a numerical proof [Fibich et al. 2006].[5] Assuming we can ensure H is coercive, the goal is to prove the *local virial identity*

$$b_s\geq\delta_1\,|||\epsilon|||-\Gamma_b^{1-C\eta},\tag{2-72}$$

where we have defined

$$|||\epsilon|||=\int\left|\nabla_y\epsilon\right|^2\mu\,dy+\int_{|y|\leq\frac{10}{b}}|\epsilon|^2\,e^{-|y|}\,dy.\tag{2-73}$$

To prove (2-72) using the spectral property requires that we control the contribution from all other terms of the conservation of energy. In particular, we must establish *nonlocal* control:

$$\int_{\mathbb{R}^2}|\epsilon(y)|^4\,\mu(y)\ll\int_{\mathbb{R}^2}\left|\nabla_y\epsilon\right|^2\mu(y).\tag{2-74}$$

This is our main challenge.

The local virial identity (2-72) is a satisfactory control for ϵ at times where $b_s<0$. However, our argument is based on approximating the central profile of the solution; therefore we cannot expect monotinicity in our modulation parameters. Including the radiation $\tilde{\zeta}$ to better approximate the central profile, repeating the local virial calculation, and taking into account the mass flux leaving the support of the radiation, Merle and Raphaël [2006] discovered a Lyapunov functional. It is remarkable that we can approximate the Lyapunov functional very precisely in terms of a positive multiple of a norm of ϵ. The functional is then used to bridge the control of ϵ between times where $b_s<0$. The approximation here is achieved through the conservation of energy, and involves (2-74) a second time.

Regarding (2-74), change variables to get

$$\int_{\mathbb{R}^2}|\epsilon(y)|^4\,\mu(y)=\lambda^2\int_{\mathbb{R}^N}|\tilde{u}|^4=\lambda^2\int_{\mathbb{R}^N}|\chi\tilde{u}|^4+\lambda^2\int_{\mathbb{R}^N}(1-\chi^4)\,|\tilde{u}|^4.$$

[5] The numerical proof is given for the L^2-critical nonlinearities of Theorem 1.1 in dimensions $d=2,3,4$, and in $d=5$ with a slight change of orthogonality conditions. In dimension $d=1$ the proof is explicit [Merle and Raphaël 2005a, Proposition 2].

Since the support of χ includes the origin, we must apply N-dimensional Sobolev to that term:

$$\|\chi\tilde{u}\|_{L^4(\mathbb{R}^N)}^4 \lesssim \|\chi u\|_{H^{\frac{N}{2}-1}(\mathbb{R}^N)}^2 \|\chi u\|_{H^1(\mathbb{R}^N)}^2 .$$

Change variables again and observe that to achieve (2-74) requires at least that $\|\chi u\|_{H^{\frac{N}{2}-1}(\mathbb{R}^N)} \ll 1$.

Strategy of proof: persistence of regularity. Once we have established the log-log nature of our blowup, we expect powers of $1/\lambda$ to be as integrable in time as powers of

$$\sqrt{\frac{\log|\log(T_{\max}-t)|}{T_{\max}-t}}.$$

Indeed, as noted in [Raphaël and Szeftel 2009],

$$\int_0^t \frac{d\tau}{\lambda^\mu(\tau)} \le C(\delta)\frac{1}{\lambda^{\mu-2+\delta}(t)}, \tag{2-75}$$

for any $\delta > 0$ and $\nu > 2$. *Our argument cannot allow any loss of scaling.* At all times we must be cautious to account for all factors of $e^{\frac{1}{b}}$. We prove that

$$\int_0^t \frac{e^{\frac{\sigma^*}{b(\tau)}}}{\lambda^\mu(\tau)}\,d\tau \le C(\sigma^*,\sigma,\nu)\frac{e^{\frac{\sigma^*+\sigma}{b(t)}}}{\lambda^{\mu-2}(t)}, \tag{2-76}$$

for any $\sigma^* < \sigma^* + \sigma$ of either sign and $\mu > 2$. The arguments of Section 4, to establish statements I2.1–I2.3, proceed in three stages.

Control of $\|u\|_{H^N}$. We explicitly calculate $\frac{d}{dt}\|\nabla^N u\|_{L^2}^2$ and seek to estimate the resulting error terms separately in two regions of space. First, away from the singularity, on the truly N-dimensional region that includes the origin, the estimates are simpler, due to hypotheses H2.2 and H2.3. Second, on a neighborhood of the singular set, things are more delicate, and we split the solution into the rescaled almost self-similar profile and $\tilde{u}$, defined in (2-38). Since $\widetilde{Q}_b$ is smooth, the higher-order norms scale exactly with $1/\lambda$. In particular,

$$\left\|\frac{1}{\lambda}\widetilde{Q}_b(y)\right\|_{H^N(\mathbb{R}^N)} \le \frac{C(\widetilde{Q}_b)}{\lambda^N(t)}, \tag{2-77}$$

where the constant is uniform for all b sufficiently small; see Lemma 2.2. Note that (2-77) is better than H2.1. For terms in $\tilde{u}$, the H^1 norm is better than $1/\lambda$, due to H1.3. By assuming $m > 0$ is small enough, we use this superior H^1 control to offset the logarithmic loss due to our use of H2.1. We prove that

$$\left|\frac{d}{dt}\|u\|_{H^N}^2\right| \lesssim \frac{1}{\lambda^{2N+2}} + \frac{e^{-\frac{\sigma_5}{b}}}{\lambda^2}\|u\|_{H^N}^2 .$$

To prove I2.1, we integrate carefully with (2-76).

Remark 2.10. The exact scaling of the smooth central profile was not needed by Raphaël and Szeftel [2009] to control a higher order norm. With the Strauss radial embedding, those authors prove an estimate analogous to $\frac{d}{dt}\|u\|_{H^N}^2 \lesssim \|u\|_{H^N}^{2-\delta}\|u\|_{H^1}^{N\delta+2}$, which is compatible with (2-75) and a hypothesis of the form $\|u\|_{H^N} \lesssim 1/\lambda^{N+C(\delta)}$.

Initial regularity improvement. Let ψ^A be a smooth cutoff function that covers the support of $\nabla\chi$ — this includes the boundary of a neighborhood of the singular set that acts as an interface between the singular dynamics and the truly N-dimensional dynamics. We hope for any control of $\|\psi^A u\|_{H^\nu}$ that is better than an interpolation of H2.1. Calculate $\frac{d}{dt}\|\psi^A u\|_{H^\nu}^2$ directly from (1-1) and integrate in time. The result is effectively Kato's smoothing effect and a Strichartz estimate,

$$\|\psi^A u\|_{L_t^\infty H^\nu}^2 \lesssim \|\psi^B u\|_{L_t^2 H^{\nu+\frac{1}{2}}}^2 + \int_0^t \left| \int D^\nu(\psi^A u\,|u|^2)\, D^\nu(\psi^A \bar{u}) \right|, \tag{2-78}$$

where ψ^B is some other cutoff function with slightly larger support.

Due to (2-76), we see that the term in $H^{\nu+\frac{1}{2}}$ is in fact of the order $1/\lambda^{2(\nu-\frac{1}{2})}$. This is exactly the sort of control we want, but the nonlinear term of (2-78) is uncooperative.

To estimate the nonlinear term of (2-78) we prove a modified Brezis–Gallouët estimate that does not break scaling too badly, the proof of which requires hypothesis H2.1 to be scaling consistent up to a sufficiently small power of $e^{\frac{1}{b}}$. See Remark 4.15. This delicacy is not required in the radial case [Raphaël 2006; Raphaël and Szeftel 2009] as Strauss's radial embedding is already scaling consistent. In place of a Brezis–Gallouët estimate, Holmer and Roudenko [2011] use an elegant microlocal estimate to smooth the nonlocal part of the nonlinearity.

Iterated smoothing. The next stage is to prove I2.2 and I2.3 hold on the support of $\nabla\chi$. We iterate the argument of (2-78), in half-integer steps, beginning with $\nu = N - \frac{1}{2}$, and introducing a new cutoff with smaller support each time. Due to the initial regularity improvement, it is possible to handle the nonlinear term of (2-78) systematically, and at the same order as the term in $H^{\nu+\frac{1}{2}}$. Due to integration (2-76), at each stage we may smooth (almost) a half-derivative farther, relative to scaling, than was proved in the previous stage. After N iterates, we find that $\|\psi^C u\|_{H^{\frac{N}{2}}}$ is (almost) order-zero in $\frac{1}{\lambda}$. The final iterate proves $\|\psi^D u\|_{H^{\frac{N}{2}-\frac{1}{2}}}$ is constant.

To complete the proof of I2.2 and I2.3, we repeat the iteration scheme for χu. The combination of hypotheses H2.2 and H2.3 with the results of the first iteration make the second iteration substantially simpler.

3. Proof of log-log singular behavior

In this section we will prove that properties I1.1–I1.5 are a consequence of hypotheses H1.1–H1.5 and the bound

$$\|\chi u(t)\|_{H^{\frac{N}{2}-1}} < (\alpha^*)^{\frac{1}{10}}, \tag{3-79}$$

which is a particular consequence of H2.3.

Almost self-similar profiles. The parameter $\eta > 0$ about to be used is universal, sufficiently small, and will be determined later on (see after (3-183) and after (3-196)). For $b \neq 0$, let

$$R_b = \frac{2}{b}\sqrt{1-\eta} \quad \text{and} \quad R_b^- = R_b\sqrt{1-\eta}, \tag{3-80}$$

and let ϕ_b denote a radially symmetric cutoff function with

$$\phi_b(y) = \begin{cases} 1 & \text{for } |y| \le R_b^-, \\ 0 & \text{for } |y| \ge R_b, \end{cases} \quad \text{and} \quad |\nabla\phi_b|_{L^\infty} + |\Delta\phi_b|_{L^\infty} \to 0 \text{ as } |b| \to 0. \tag{3-81}$$

The following result was originally shown in [Merle and Raphaël 2003, Proposition 1]. The refined cutoff, with parameter η, is introduced in [Merle and Raphaël 2004, Propositions 8 and 9].

Proposition 3.1 (localized self-similar profiles). *For all $\eta > 0$ sufficiently small there exists positive $b^*(\eta)$ and $\delta(\eta)$ such that for all $|b| < b^*(\eta)$ there exists a unique radial solution Q_b to,*

$$\begin{cases} \Delta Q_b - Q_b + ib\Lambda Q_b + Q_b\,|Q_b|^2 = 0, \\ P_b = Q_b e^{i\frac{b|y|^2}{4}} > 0 \textit{ for } y \in [0, R_b), \\ |Q_b(0) - Q(0)| < \delta(\eta), \quad Q_b(R_b) = 0. \end{cases} \tag{3-82}$$

The truncation to $|y| < \frac{2}{b}$, $\widetilde{Q}_b(y) = Q_b(y)\phi_b(y)$, satisfies

$$\Delta\widetilde{Q}_b - \widetilde{Q}_b + ib\Lambda\widetilde{Q}_b + \widetilde{Q}_b\,|\widetilde{Q}_b|^2 = -\Psi_b, \tag{3-83}$$

with the explicit error term

$$-\Psi_b = Q_b\Delta\phi_b + 2\nabla\phi_b\cdot\nabla Q_b + ibQ_b\,y\cdot\nabla\phi_b + (\phi_b^3 - \phi_b)Q_b\,|Q_b|^2. \tag{3-84}$$

Moreover, $\widetilde{Q}_b$ satisfies the following properties:

- *Uniform closeness to the ground state*:

$$\left\| e^{C|y|}(\widetilde{Q}_b - Q)\right\|_{C^3} \to 0 \quad \textit{as } b \to 0. \tag{3-85}$$

- *Derivative with respect to b*:

$$\left\| e^{C|y|}\left(\frac{\partial}{\partial b}\widetilde{Q}_b + i\frac{|y|^2}{4}Q\right)\right\|_{C^2} \to 0 \quad \textit{as } b \to 0. \tag{3-86}$$

- *Supercritical mass*:

$$\frac{d}{d(b^2)}\left(\int \left|\widetilde{Q}_b\right|^2\right)\bigg|_{b^2=0} = d_0 \quad \textit{with } 0 < d_0 < +\infty. \tag{3-87}$$

As a consequence of (3-85), *for any polynomial $P(y)$ and $k = 0, 1$,*

$$\left|P(y)\nabla^k\Psi_b\right|_{L^\infty} \le e^{-\frac{C(P)}{|b|}}. \tag{3-88}$$

In particular, energy and momentum are degenerate:

$$\left|E(\widetilde{Q}_b)\right| \le e^{-(1-C\eta)\frac{\pi}{|b|}} \quad \textit{and} \quad \operatorname{Im}\left(\int \nabla_y\widetilde{Q}_b\,\overline{\widetilde{Q}_b}\right) = 0. \tag{3-89}$$

The linearized Schrödinger operator near $\widetilde{Q}_b$ is, $M\left[\begin{smallmatrix} v \\ iw \end{smallmatrix}\right] = M_+(v,w) + iM_-(v,w)$, with,

$$M_+(v,w) = -\Delta_y v + v - \left(\frac{\widetilde{Q}_b^2}{|\widetilde{Q}_b|^2} + 2\right)|\widetilde{Q}_b|^2 v - \operatorname{Im}(\widetilde{Q}_b^2)\, w, \tag{3-90}$$

$$M_-(v,w) = -\Delta_y w + w - \left(2 - \frac{\widetilde{Q}_b^2}{|\widetilde{Q}_b|^2}\right)|\widetilde{Q}_b|^2 w - \operatorname{Im}(\widetilde{Q}_b^2)\, v. \tag{3-91}$$

As with L from (2-68), there is an associated bilinear operator

$$H_b(\epsilon,\epsilon) = H(\epsilon,\epsilon) + \widetilde{H}_b(\epsilon,\epsilon), \tag{3-92}$$

where $H(\epsilon,\epsilon)$ is the usual form (2-70) associated with L. The correction term may be written as

$$\widetilde{H}_b(\epsilon,\epsilon) = \int V_{11}{\epsilon_{\mathrm{re}}}^2 + \int V_{12}\epsilon_{\mathrm{re}}\epsilon_{\mathrm{im}} + \int V_{22}{\epsilon_{\mathrm{im}}}^2, \tag{3-93}$$

for well-localized potentials built on $\widetilde{Q}_b$, Q and $y\cdot\nabla$; see [Merle and Raphaël 2004, Appendix C]. Due to proximity with Q — see (3-85) — there is universal constant C with

$$\left\| e^{C|y|} V_{ij} \right\|_{L^\infty} \to 0 \quad \text{as } b \to 0. \tag{3-94}$$

The following variation of H is of a different nature. Set

$$\widetilde{H}(\epsilon,\epsilon) = H(\epsilon,\epsilon) - \frac{1}{\|\Lambda Q\|_{L^2}^2}(\epsilon_{\mathrm{re}}, L_+\Lambda^2 Q)\,(\epsilon_{\mathrm{re}}, \Lambda Q)\,, \tag{3-95}$$

which simply alters the definition of $\mathscr{L}_+$ given in (2-70). The following is a consequence of (2-69) and the spectral property:

Lemma 3.2 (alternative spectral property [Merle and Raphaël 2004, page 616]). *There exists a universal positive constant $\tilde{\delta}_0 < \delta_0$ such that, for all $\epsilon \in H^1$,*

$$\widetilde{H}(\epsilon,\epsilon) \geq \tilde{\delta}_0 \left(\int_{y\in\mathbb{R}^2} |\nabla_y \epsilon|^2 + \int_{y\in\mathbb{R}^2} |\epsilon^2| e^{-|y|}\right) - \frac{1}{\delta_0} \times$$
$$\big((\operatorname{Re}(\epsilon,Q))^2 + (\operatorname{Re}(\epsilon,|y|^2 Q))^2 + (\operatorname{Re}(\epsilon, yQ))^2 + (\operatorname{Im}(\epsilon,\Lambda Q))^2 + (\operatorname{Im}(\epsilon,\Lambda^2 Q))^2 + (\operatorname{Im}(\epsilon,\nabla Q))^2\big). \tag{3-96}$$

In Lemma 3.17 and Remark 3.18 we will find that the study of linear radiation gives an accurate description of mass ejection from the singular regime. Here is a background result:

Lemma 3.3 (linear radiation [Merle and Raphaël 2004, Lemma 15]). *There are universal constants $C > 0$ and $\eta^* > 0$ such that for all $0 < \eta < \eta^*$ there is $b^*(\eta) > 0$ such that for all $0 < b < b^*(\eta)$ there exists a unique radial solution ζ_b to*

$$\begin{cases} \Delta\zeta_b - \zeta_b + ib\Lambda\zeta_b = \Psi_b, \\ \int |\nabla\zeta_b|^2 < +\infty, \end{cases} \tag{3-97}$$

where Ψ_b is the truncation error given by (3-83); *moreover, the solution satisfies the following properties,* *where we have set*

$$\Gamma_b = \lim_{|y|\to+\infty} |y|\,|\zeta_b(y)|^2 : \tag{3-98}$$

- *Decay past the support of* Ψ_b:

$$\big\| |y|\,|\zeta_b| + |y|^2\,|\nabla\zeta_b| \big\|_{L^\infty(|y|\geq R_b)} \leq \Gamma_b^{\frac{1}{2}-C\eta} < +\infty. \tag{3-99}$$

- *Smallness in* $\dot{H}^1$:

$$\int \left|\nabla_y \zeta_b\right|^2 \leq \Gamma_b^{1-C\eta}. \tag{3-100}$$

- *Derivative with respect to* b:

$$\left\| \frac{\partial \zeta_b}{\partial b} \right\|_{C^1} \leq \Gamma_b^{\frac{1}{2}-C\eta}. \tag{3-101}$$

- *Stronger decay for larger* $|y|$:

$$\big\| |y|^2\,|\nabla\zeta_b| \big\|_{L^\infty(|y|\geq R_b^2)} \leq C\frac{\Gamma_b^{\frac{1}{2}}}{|b|}, \tag{3-102}$$

$$e^{-(1+C\eta)\frac{\pi}{b}} \leq \tfrac{4}{5}\Gamma_b \leq \big\| |y|^2\,|\zeta_b|^2 \big\|_{L^\infty(|y|\geq R_b^2)} \leq e^{-(1-C\eta)\frac{\pi}{b}}. \tag{3-103}$$

(As an estimate on Γ_b, (3-103) will be indispensable.)

The small universal parameter $a > 0$ in the next equation will be introduced later, via (3-165), and determined on page 706, in the proof of Lemma 3.22. It influences the choice of η. We set

$$A(t) = e^{a\frac{\pi}{b(t)}}, \qquad \text{so that} \qquad \Gamma_b^{-\frac{a}{2}} \leq A \leq \Gamma_b^{-\frac{3a}{2}}, \tag{3-104}$$

and we let ϕ_A denote a radially symmetric cutoff function with

$$\phi_A(y) = \begin{cases} 1 & \text{for } |y| \leq A, \\ 0 & \text{for } |y| \geq 2A. \end{cases} \tag{3-105}$$

The truncated radiation $\tilde{\zeta}_b(y) = \phi_A(y)\zeta_b$ satisfies

$$\Delta\tilde{\zeta}_b - \tilde{\zeta}_b + ib\Lambda\tilde{\zeta}_b = \Psi_b + F, \tag{3-106}$$

where the error term F is explicit:

$$F = \zeta_b\Delta\phi_A + 2\nabla\phi_A\cdot\nabla\zeta_b + ib\zeta_b\, y\cdot\nabla\phi_A. \tag{3-107}$$

In particular, by (3-102) and (3-103),

$$|F|_{L^\infty} + |y\cdot\nabla F|_{L^\infty} \leq C\frac{\Gamma_b^{\frac{1}{2}}}{A}. \tag{3-108}$$

Remark 3.4. For smaller values of η the central profiles $\tilde{Q}_b$ approximate the mass of the singular region more closely — see (3-80) — at the cost that estimates (3-85)–(3-89) are only known for ever smaller values of b. When η is larger, to compensate for the imperfection of our central profile we require more of the radiative tail to get an accurate picture of mass transport, requiring a larger choice of a. See [Merle and Raphaël 2006, page 53] for similar remarks on the optimality in choice of $A(t)$.

Estimates directly due to geometric decomposition. The next lemma explains our choice of norm for ϵ.

Lemma 3.5 (weighted and local L^2 estimates). *For any $\kappa > 0$ and for all $v \in H^1(\mathbb{R}^2)$,*

$$\int_{y\in\mathbb{R}^2} |v(y)|^2 e^{-\kappa|y|} \le C(\kappa)\left(\int |\nabla v(y)|^2 + \int_{|y|\le 1} |v(y)|^2 e^{-|y|}\right), \tag{3-109}$$

$$\int_{|y|\le\kappa} |v(y)|^2 \le C\,\kappa^2 \log\kappa \left(\int |\nabla v(y)|^2 + \int_{|y|\le 1} |v(y)|^2 e^{-|y|}\right). \tag{3-110}$$

Equation (3-110) is found in [Merle and Raphaël 2006, (4.11)]. While the original proof of (3-109) in [Merle and Raphaël 2004, Lemma 5] has a flaw, the methods of [Merle and Raphaël 2006] give an alternate proof.

Remark 3.6 (nonconcern for μ). In practice, we apply these lemmas and the interaction estimates below only on regions within $\{|y| \lesssim A(t)\}$. That is, (2-55) always applies and we may choose to include measure $\mu(y)$ as appropriate.

Lemma 3.7 (estimates on interaction terms [Merle and Raphaël 2003, Section 5.3(C)]). *Let $s \in [s_0, s_1)$, and recall from* (2-73) *that $\|\!|\epsilon|\!\|$ stands for $\int \left|\nabla_y \epsilon\right|^2 \mu\, dy + \int_{|y|\le \frac{10}{b}} |\epsilon|^2 e^{-|y|}\, dy$.*

- *Estimate of first-order terms*:

$$\left|\left(\epsilon(y), P(y)\frac{d^k}{dy^k}\widetilde{Q}_b(y)\right)\right| \le C(P)\|\!|\epsilon|\!\|^{\frac{1}{2}}, \tag{3-111}$$

where $P(y)$ is any polynomial and $0 \le k \le 3$.

- *Estimate of second-order terms*:

$$\left|\left(R(\epsilon), P(y)\frac{d^k}{dy^k}\widetilde{Q}_b(y)\right)\right| \le C(P)\|\!|\epsilon|\!\|, \tag{3-112}$$

where $P(y)$ is any polynomial, $0 \le k \le 3$, and $R(\epsilon)$ is the terms of $(\epsilon + \widetilde{Q}_b)|\epsilon + \widetilde{Q}_b|^2$ formally quadratic in ϵ — see Equation (3-131).

- *Estimate of localized higher-order terms*:

$$\int \left|J(\epsilon) - |\epsilon|^4\right| \mu(y)\, dy \le \delta(\alpha^*)\|\!|\epsilon|\!\|, \tag{3-113}$$

where $J(\epsilon) - |\epsilon|^4 = 4\,\mathrm{Re}\big(\epsilon\,|\epsilon|^2, \widetilde{Q}_b\big)$ is the term of $|\epsilon + \widetilde{Q}_b|^4$ formally cubic in ϵ and localized to the support of $\widetilde{Q}_b$. Similarly,

$$\big(\widetilde{R}(\epsilon), \Lambda\widetilde{Q}_b\big) \le \delta(\alpha^*)\|\!|\epsilon|\!\|, \tag{3-114}$$

where $\widetilde{R}(\epsilon) = \epsilon\,|\epsilon|^2$ is the term of $(\epsilon + \widetilde{Q}_b)|\epsilon + \widetilde{Q}_b|^2$ formally cubic in ϵ.

The following estimate is our first nontrivial departure from the L^2-critical argument.

Lemma 3.8 (complete estimate on $J(\epsilon)$). *For all $s \in [s_0, s_1)$,*

$$\int |\epsilon(y)|^4 \, \mu(y) \, dy \leq \delta(\alpha^*) \|\!|\epsilon|\!\|. \tag{3-115}$$

With (3-113), *this gives a complete estimate for $J(\epsilon)$.*

Proof. Partition the support of ϵ into two- and three- dimensional regions:

$$\int |\epsilon(y)|^4 \, \mu(y) \, dy = \int (1-\chi^4) \, |\epsilon(y)|^4 \, \mu(y) \, dy + \int \left|\chi\left(\lambda y + (r,z)(s)\right) \epsilon(y)\right|^4 \mu(y) \, dy. \tag{3-116}$$

The first term on the right is supported away from $r = 0$, and due to H1.1 the support of $1 - \chi^4$ is approximately $\left\{|y| < \frac{2}{3}\frac{1}{\lambda}\right\}$, so that $\frac{1}{3} \lesssim \mu(y) \lesssim \frac{5}{3}$. We estimate this term by two-dimensional Sobolev embedding and the small mass assumption H1.2. Regarding the second term, the support of χ^4 excludes the support of $\widetilde{Q}_b$ by the same reasons. Changing variables, we obtain

$$\int |\chi(x(y)) \, \epsilon(y)|^4 \, \mu(y) \, dy = \lambda^2 \int_{x \in \mathbb{R}^N} |\chi(x) u(x)|^4 \, dx. \tag{3-117}$$

By the N-dimensional Sobolev embedding, $\dot{H}^{\frac{N}{4}} \hookrightarrow L^4(\mathbb{R}^N)$, and interpolation,

$$\begin{aligned} \lambda^2 \int_{x \in \mathbb{R}^N} |\chi(x) u(x)|^4 \, dx &\lesssim \|\chi u\|^2_{\dot{H}^{\frac{N}{2}-1}} \lambda^2 \, \|\chi u\|^2_{\dot{H}^1(\mathbb{R}^N)} \\ &\lesssim \|\chi u\|^2_{H^{\frac{N}{2}-1}} \left(\int \left|\nabla_y \epsilon\right|^2 \mu \, dy \right). \end{aligned} \tag{3-118}$$

To complete the proof, we use the assumed control H2.3 for the first and only time. ☐

Lemma 3.9 (estimates due to conservation laws). *For all $s \in [s_0, s_1)$ the following are true*:

(a) *Due to conservation of mass*:

$$b^2 + \int |\tilde{u}|^2 \lesssim (\alpha^*)^{\frac{1}{2}}. \tag{3-119}$$

(b) *Due to conservation of energy*:

$$2\,\mathrm{Re}(\epsilon, \widetilde{Q}_b) - \int |\nabla \epsilon|^2 \, \mu(y) \, dy + 3 \int_{|y| \leq \frac{10}{b}} Q^2 {\epsilon_{re}}^2 + \int_{|y| \leq \frac{10}{b}} Q^2 {\epsilon_{\mathrm{im}}}^2 \leq \Gamma_b^{1 - C\eta} + \delta(\alpha^*) \|\!|\epsilon|\!\|. \tag{3-120}$$

(c) *Due to localized momentum* (2-48):

$$\left|\mathrm{Im}(\epsilon, \nabla \widetilde{Q})\right| \leq \Gamma_b^2 + \delta(\alpha^*) \|\!|\epsilon|\!\|^{\frac{1}{2}}. \tag{3-121}$$

In particular, (3-121) *also holds for* $\left|(\epsilon_{\mathrm{im}}, \mathrm{Re}(\nabla \widetilde{Q}_b))\right|$, *by Hölder and* (3-85).

Proof. (a) Conservation of mass gives $\int_{\mathbb{R}^N} |u(t)|^2\,dx = \int |u_0|^2$. From the geometric decomposition, expand and change some variables, obtaining

$$\int \left|\widetilde{Q}_b(y)\right|^2 \mu(y)\,dy + 2\operatorname{Re}\left(\int \epsilon \overline{\widetilde{Q}}_b \mu(y)\,dy\right) + \int |\tilde{u}(t)|^2 = \int |u_0|^2. \tag{3-122}$$

Expand the measure μ. Due to the bound on λ in (2-54), together with hypotheses H1.1 and H1.2 and the supercritical mass of $\widetilde{Q}_b$,

$$\begin{aligned}\int \left|\widetilde{Q}_b\right|^2 \mu(y)\,dy - \int Q^2 = \left(\int \left|\widetilde{Q}_b\right|^2 - \int Q^2\right) + (r^{N-2}(t)-1)\int \left|\widetilde{Q}_b\right|^2 + \int \mathcal{O}(\lambda y_1)\left|\widetilde{Q}_b\right|^2 dy \\ \gtrsim b^2 - \sqrt{\alpha^*}.\end{aligned} \tag{3-123}$$

Due to the smallness of b_0 and the small mass of ϵ_0 (see C1.2), we have $\left|\int_{\mathbb{R}^N} |u_0|^2 - \int_{\mathbb{R}^2} Q^2\right| \lesssim C\alpha^*$. Due to local support and hypothesis H1.2, $\left|\int \epsilon \overline{\widetilde{Q}_b}\mu\right| \lesssim \alpha^*$.

(b) Conservation of energy gives $\int_{\mathbb{R}^N} |\nabla u(t)|^2\,dx - \frac{1}{2}\int |u|^4 = 2E_0$. From the geometric decomposition,

$$2\lambda^2 E_0 = \int \left|\nabla_y(\widetilde{Q}_b + \epsilon)\right|^2 \mu(y)\,dy - \frac{1}{2}\int |\widetilde{Q}_b + \epsilon|^4 \mu(y)\,dy \tag{3-124}$$

Partially expand the measure μ:

$$\begin{aligned}\int \left|\nabla_y(\widetilde{Q}_b + \epsilon)\right|^2 \mu(y)\,dy = r^{N-2}(t)\int \left|\nabla_y \widetilde{Q}_b\right|^2 + \int \mathcal{O}(\lambda y_1)\left(\left|\nabla_y \widetilde{Q}_b\right|^2 + 2\operatorname{Re}(\epsilon \overline{\widetilde{Q}_b})\right) dy \\ + 2^{N-2} r(t)\operatorname{Re}\left(\int \nabla_y \epsilon \cdot \nabla_y \overline{\widetilde{Q}_b}\right) + \int |\nabla_y \epsilon|^2 \mu(y)\,dy.\end{aligned} \tag{3-125}$$

Due to the support of $\widetilde{Q}_b$, the second line is of order λ, and thus inconsequential. Via a similar approach,

$$\begin{aligned}&-\frac{1}{2}\int \left|\widetilde{Q}_b + \epsilon\right|^4 \mu(y)\,dy \\ &\quad = -r^{N-2}(t)\left(\frac{1}{2}\int \left|\widetilde{Q}_b\right|^4 + 2\operatorname{Re}\left(\int \epsilon \overline{\widetilde{Q}_b}\left|\widetilde{Q}_b\right|^2\right) + \int |\epsilon|^2 \left|\widetilde{Q}_b\right|^2 + \operatorname{Re}\left(\int \epsilon^2 \overline{\widetilde{Q}_b}^2\right)\right) \\ &\qquad + \lambda\,\mathcal{O}\left(\left|\widetilde{Q}_b\right|^2\right) - \frac{1}{2}\int J(\epsilon)\mu(y)\,dy.\end{aligned} \tag{3-126}$$

Now proceed as in the L^2-critical argument. Integrate $\int \nabla_y \epsilon \cdot \nabla_y \overline{\widetilde{Q}_b}$ by parts and substitute the equation for $\widetilde{Q}_b$ (3-83); this cancels the term of (3-126) linear in ϵ. Recall the bound for Ψ_b (3-88), the degenerate energy of $\widetilde{Q}_b$ (3-89), proximity to Q (3-85), that $r(t) \sim 1$, and the non-trivial estimate on J, Equation (3-115).

(c) Our starting point is (2-48). In cylindrical coordinates, $\nabla_x f \cdot \nabla_x g = \partial_r f \partial_r g + \partial_z f \partial_z f$. For this proof we denote r by x_1 and z by x_2. Fix either $j = 1$ or $j = 2$. From the geometric decomposition,

$$\lambda \operatorname{Im}\left(\int_{\mathbb{R}^N} \partial_{x_j}\psi^{(x)} \partial_{x_j} u\,\bar{u}\,dx\right) = \operatorname{Im}\left(\int \partial_{x_j}\psi^{(x)} \partial_{y_j}\left(\widetilde{Q}_b + \epsilon\right)\overline{\left(\widetilde{Q}_b + \epsilon\right)}\mu(y)\,dy\right). \tag{3-127}$$

Directly from definition (2-34), we have $\partial_{x_j}\psi^{(x)} = 1$ on the support of $\widetilde{Q}_b$. Expand the measure μ as $r^{N-2}(t) + \mathbb{O}(\lambda y_1)$. Integrate by parts the interaction term in $r^{N-2}(t)\partial_{y_j}\epsilon\overline{\widetilde{Q}_b}$. With the degenerate momentum of $\widetilde{Q}_b$ — see (3-89) — we have

$$2r^{N-2}(t)\,\mathrm{Im}\big(\epsilon, \partial_{y_j}\widetilde{Q}_b\big) = \mathrm{Im}\left(\int \mathbb{O}(\lambda y_1)\big(\partial_{y_j}\epsilon\overline{\widetilde{Q}_b} + \partial_{y_j}\widetilde{Q}_b\bar{\epsilon} + \partial_{y_j}\widetilde{Q}_b\overline{\widetilde{Q}_b}\big)\,dy\right) + \mathrm{Im}\left(\int \partial_{x_j}\psi^{(x)}\partial_{y_j}\epsilon\bar{\epsilon}\mu(y)\,dy\right) - \lambda(t)\,\mathrm{Im}\left(\int_{\mathbb{R}^N}\partial_{x_j}\psi^{(x)}\partial_{x_j}u\,\bar{u}\,dx\right). \quad (3\text{-}128)$$

The first term on the right is of order λ, and thus negligible. For the next term we apply Hölder and the small mass assumption H1.2. The final term is controlled by H1.4. □

Remark 3.10 (role of momentum conservation). The estimate analogous to (3-121) in the L^2-critical context is proven with the conservation of momentum in place of H1.4; see [Merle and Raphaël 2003, Appendix A]. As might be expected, the proof of I1.4 will resemble the proof of momentum conservation. See (3-149).

Definition 3.11 (NLS reformulated for ϵ). For $s \in [s_0, s_1)$, $y \in [-r(t)/\lambda(t), +\infty) \times \mathbb{R}$, and a suitable boundary condition at $y_1 = -r(t)/\lambda(t)$, the function ϵ satisfies

$$ib_s\frac{\partial\widetilde{Q}_b}{\partial b} + i\epsilon_s - M(\epsilon) + \frac{N-2}{r(y_1)}\lambda\partial_{y_1}\epsilon + ib\Lambda\epsilon = i\left(\frac{\lambda_s}{\lambda} + b\right)\Lambda\widetilde{Q}_b + \tilde{\gamma}_s\widetilde{Q}_b + i\frac{(r_s, z_s)}{\lambda}\cdot\nabla_y\widetilde{Q}_b + i\left(\frac{\lambda_s}{\lambda} + b\right)\Lambda\epsilon + \tilde{\gamma}_s\epsilon + i\frac{(r_s, z_s)}{\lambda}\cdot\nabla_y\epsilon + \Psi_b - R(\epsilon), \quad (3\text{-}129)$$

where we have introduced the new variable

$$\tilde{\gamma}(s) = -s - \gamma(s). \quad (3\text{-}130)$$

Note the single new term due to cylindrical symmetry. As already mentioned, the term $R(\epsilon)$ corresponds to those terms formally quadratic in ϵ:

$$R(\epsilon) = \big(\epsilon + \widetilde{Q}_b\big)\big|\epsilon + \widetilde{Q}_b\big|^2 - \widetilde{Q}_b\big|\widetilde{Q}_b\big|^2 - 2\big|\widetilde{Q}_b\big|^2\epsilon - \big(2\widetilde{Q}_b^2 - \mathrm{Re}(\widetilde{Q}_b^2)\big)\bar{\epsilon}. \quad (3\text{-}131)$$

Lemma 3.12 (estimates due to orthogonality conditions). *For all $s \in [s_0, s_1)$,*

$$\left|\frac{\lambda_s}{\lambda} + b\right| + |b_s| \lesssim \Gamma_b^{1-C\eta} + \|\!|\epsilon|\!\| \quad (3\text{-}132)$$

and

$$\left|\tilde{\gamma}_s - \frac{1}{|\Lambda Q|_{L^2}^2}\Big(\epsilon_{re}, L_+(\Lambda^2 Q)\Big)\right| + \left|\frac{r_s}{\lambda}\right| + \left|\frac{z_s}{\lambda}\right| \le \Gamma_b^{1-C\eta} + \delta(\alpha^*)\|\!|\epsilon|\!\|^{\frac{1}{2}}. \quad (3\text{-}133)$$

Estimates (3-132) and (3-133) are a direct result of orthogonality conditions (2-39), (2-40), (2-41) and (2-42) by taking the respective inner products with ϵ; see (3-129). Due to (2-54), terms resulting from $\frac{N-2}{r(y_1)}\lambda\partial_{y_1}\epsilon$ are inconsequential. The estimates due to energy and momentum, (3-120) and (3-121), are involved in the estimates of $|b_s|$ and $|r_s/\lambda|+|z_s/\lambda|$ respectively. Otherwise, all calculations are localized to the support of $\widetilde{Q}_b$ and are identical to the L^2-critical argument. See [Merle and Raphaël 2004, Appendix C] or [Raphaël 2005, Appendix A] for the complete calculations.

Lemma 3.13 (local virial identity). *For all $s\in[s_0,s_1)$,*

$$b_s \ge \delta_1 \|\!|\epsilon\|\!| - \Gamma_b^{1-C\eta}, \tag{3-134}$$

where $\delta_1>0$ is a universal constant and $\|\!|\epsilon\|\!| = \int |\nabla_y\epsilon|^2\,\mu\,dy + \int_{|y|\le\frac{10}{b}} |\epsilon|^2 e^{-|y|}\,dy$ as in (2-73).

Brief proof. Begin with the method used to prove preliminary estimate (3-132). Take the real part of the inner product of ϵ in (3-129) with $\Lambda\widetilde{Q}_b$. Recognize that $\partial_s \operatorname{Im}(\epsilon,\Lambda\widetilde{Q}_b)=0$ due to orthogonality condition (2-42). An adapted version of the algebraic property $L_+(\Lambda Q)=-2Q$ is applied [Merle and Raphaël 2004, equation (101)]. After recognizing the equation of $\widetilde{Q}_b$, injecting the conservation of energy cancels the remaining terms linear in ϵ. The resulting terms quadratic in ϵ are the bilinear operator $H_b(\epsilon,\epsilon)$ of (3-92). The remaining terms cubic in ϵ (due to the original inner product) were estimated as part of Lemma 3.7. See [Merle and Raphaël 2004, Appendix C] for the complete calculation. Controlling the auxiliary terms of the conservation of energy with (3-120) we have

$$\begin{aligned}-b_s \operatorname{Im}\left(\frac{\partial}{\partial b}\widetilde{Q}_b,\Lambda\widetilde{Q}_b\right) \gtrsim\ & H_b(\epsilon,\epsilon) + b_s \operatorname{Im}\left(\epsilon,\Lambda\frac{\partial}{\partial b}\widetilde{Q}_b\right) - \left(\frac{\lambda_s}{\lambda}+b\right)\operatorname{Im}\big(\epsilon,\Lambda^2\widetilde{Q}_b\big)\\ &- \tilde{\gamma}_s \operatorname{Re}\left(\epsilon,\Lambda\widetilde{Q}_b\right) - \frac{(r_s,z_s)}{\lambda}\cdot\operatorname{Im}\big(\epsilon,\nabla\widetilde{Q}_b\big) - \Gamma_b^{1-C\eta} - \delta(\alpha^*)\,\|\!|\epsilon\|\!|. \end{aligned}\tag{3-135}$$

Recall that $\partial_b\widetilde{Q}_b \approx -\frac{1}{4}i\,|y|^2\,Q$, make the correction (3-94) for $\widetilde{H}_b$, and apply the preliminary estimates (3-132) and (3-133). With the proximity to Q we can write

$$b_s\,\tfrac{1}{4}\,\|yQ\|_{L^2}^2 \gtrsim H(\epsilon,\epsilon) - \tilde{\gamma}_s(\epsilon_{\mathrm{re}},\Lambda Q) - \Gamma_b^{1-C\eta} - \delta(\alpha^*)\,\|\!|\epsilon\|\!|. \tag{3-136}$$

Identify the alternate form Equation (3-95) of $\widetilde{H}$, apply the preliminary estimate for $\tilde{\gamma}_s$, Equation (3-133), and apply the adapted version of the spectral property, Lemma 3.2. □

Remark 3.14 (progress in proving Proposition 2.8). We have already proven the first half of I1.2 as the preliminary estimate (3-119). The local virial identity with preliminary estimate (3-132) produce a closed expression for λ and b, which we treat with simple arguments to prove the following lemma. In particular, (3-138) implies the first lower bound of I1.3. Following similar methods, we will then prove the second upper bound of I1.3, I1.4, I1.5, and I1.1.

Lemma 3.15 (upper bound on blowup rate). *For all $s\in[s_0,s_1)$,*

$$b(s) \ge \frac{3\pi}{4\log s} \tag{3-137}$$

and

$$\lambda(s) \le \sqrt{\lambda_0}\,e^{-\frac{\pi}{3}\frac{s}{\log s}}. \tag{3-138}$$

Proof. Inject hypothesis H1.2 into the local virial identity (3-134) and carefully integrate in time. From $b > 0$ and the bound on Γ_b (3-103), we have

$$\partial_s e^{+\frac{3\pi}{4b}} = -\frac{b_s}{b^2}\frac{3\pi}{4}e^{+\frac{3\pi}{4b}} \leq 1, \quad \text{which implies} \quad e^{+\frac{3\pi}{4b}} \leq s - s_0 + e^{+\frac{3\pi}{4b_0}}. \tag{3-139}$$

Now (3-137) follows from our clever choice of s_0 in (2-43).

Next we view the preliminary estimate (3-132) and hypothesis H1.2 as the approximate dynamics of λ:

$$\left|\frac{\lambda_s}{\lambda} + b\right| + |b_s| < \Gamma_b^{\frac{1}{2}}. \tag{3-140}$$

In particular, as $b > 0$ is small, $-\frac{\lambda_s}{\lambda} \geq \frac{2b}{3}$, which we integrate with (3-137) to get

$$-\log\lambda \geq -\log\lambda_0 + \int_{s_0}^{s} \frac{\pi}{2\log\sigma}\,d\sigma. \tag{3-141}$$

Assume s_0 is sufficiently large through the choice of data (2-29) with α^* sufficiently small, then,

$$\int_{s_0}^{s} \frac{\pi}{2\log\sigma}d\sigma \geq \frac{\pi}{3}\left(\frac{s}{\log s} - \frac{s_0}{\log s_0}\right). \tag{3-142}$$

From the choice of data C1.3, and (2-43), $-\log\lambda_0 \geq e^{\frac{\pi}{2b_0}} = s_0^{\frac{3}{2}}$. Thus

$$-\log\lambda \geq -\tfrac{1}{2}\log\lambda_0 + \frac{\pi}{3}\frac{s}{\log s},$$

and we have proved (3-138). □

A simple change of variables in (3-138) and the choice of data (2-29) and (2-32) yield a corollary:

$$T_{\text{hyp}} = \int_{s_0}^{s_1} \lambda^2(\sigma)\,d\sigma \leq \lambda_0 \int_2^{+\infty} e^{-\frac{2\pi}{3}\frac{s}{\log s}}\,ds < \alpha^*. \tag{3-143}$$

Proof of second upper bound in I1.3. As a direct consequence of (3-138), again assuming $s_0 > 0$ sufficiently large,

$$-\log(s\lambda(s)) \geq \frac{\pi}{3}\frac{s}{\log s} - \log s \geq \frac{s}{\log s}. \tag{3-144}$$

Taking the logarithm and applying (3-137),

$$\log\left|-\log\left(s\lambda(s)\right)\right| \geq \log\frac{s}{\log s} \geq \frac{4}{15}\log s \geq \frac{\pi}{5b(s)}, \tag{3-145}$$

which leads successively to $s\lambda(s) \leq e^{-e^{\frac{\pi}{5b}}}$ and $\lambda \leq e^{-e^{\frac{\pi}{5b}}}$, the second upper bound of I1.3. □

Proof of I1.4. Recall the approximate dynamic (3-140), which was due to the preliminary estimate (3-132) and the hypothesized control on ϵ. As a consequence, for $s \in [s_0, s_1)$,

$$\frac{d}{ds}\left(\lambda^2 e^{\frac{5\pi}{b}}\right) = 2\lambda^2 e^{\frac{5\pi}{b}}\left(\frac{\lambda_s}{\lambda} + b - b - \frac{5\pi b_s}{2b^2}\right) \leq -\lambda^2 b e^{5\pi} b < 0, \tag{3-146}$$

which implies

$$\lambda^2(t)e^{\frac{5\pi}{b(t)}} \le \lambda_0^2 e^{\frac{5\pi}{b_0}}. \tag{3-147}$$

Then, with the estimate (3-103) on Γ_b, the choice of data (2-33), and the estimate on Γ_b again, we obtain the energy-normalization part of I1.4:

$$\lambda^2(t)\,|E_0| < \Gamma_{b(t)}^4\, e^{\frac{5\pi}{b_0}}\lambda_0^2\,|E_0| < \Gamma_{b(t)}^4\, e^{\frac{5\pi}{b_0}}\,\Gamma_{b_0}^{10} \ll \Gamma_{b(t)}^4, \tag{3-148}$$

Regarding the localized momentum, calculate directly from (1-1) that,

$$\begin{aligned}\frac{d}{dt}\operatorname{Im}&\left(\int \nabla\psi^{(x)}\cdot\nabla u\bar{u}\right)\\ &= \operatorname{Re}\left(\int \partial_{x_j}\partial_{x_k}\psi^{(x)}\partial_{x_k}u\partial_{x_j}\bar{u}\right) - \frac{1}{2}\int \Delta\psi^{(x)}\,|u|^4 - \frac{1}{4}\int \Delta^2\psi^{(x)}\,|u|^2. \end{aligned}\tag{3-149}$$

This is a special case of the general Morawetz calculation; see, for instance, [Tao 2006, equation (3.36)]. Recall from definition (2-34) that the support of $\psi^{(x)}$ is well away from $r=0$. Apply the two-dimensional Sobolev embedding $H^{\frac{1}{2}} \hookrightarrow L^4$ to estimate

$$\left|\frac{d}{dt}\operatorname{Im}\left(\int \nabla\psi^{(x)}\cdot\nabla u\bar{u}\right)\right| \le C(\psi^{(x)})\,\|u(t)\|_{H^1}^2 \lesssim \frac{1}{\lambda^2}, \tag{3-150}$$

where the final inequality is due to hypothesized control on ϵ and the small excess mass H1.2. Note that $\int_0^t \frac{d\tau}{\lambda^2(\tau)} = \int_{s_0}^s d\sigma \le s$, so we have proven

$$\lambda(t)\left|\operatorname{Im}\left(\nabla\psi^{(x)}\cdot\nabla u(t)\bar{u}(t)\right)\right| \le \lambda(t)\left|\operatorname{Im}\left(\nabla\psi^{(x)}\cdot\nabla u_0\bar{u}_0\right)\right| + C\lambda(t)s(t).$$

From the estimate (3-103) on Γ_b and (3-145) from the previous proof, we have $C\lambda(t)s(t) \le C\Gamma_{b(t)}^{10} \ll \Gamma_b^4$. Using virtually the same calculation that gave us (3-146)–(3-148) we obtain, for $s \in [s_0, s_1)$,

$$\frac{d}{ds}\left(\lambda e^{\frac{6\pi}{b}}\right) \le -\tfrac{1}{2}\lambda b e^{\frac{6\pi}{b}} < 0, \tag{3-151}$$

and hence

$$\lambda(t)e^{\frac{6\pi}{b(t)}} \le \lambda_0 e^{\frac{6\pi}{b_0}}.$$

By the estimate on Γ_b and choice of data (2-33), we obtain the localized-momentum part of I1.4:

$$\lambda(t)\left|\operatorname{Im}\left(\nabla\psi^{(x)}\cdot\nabla u_0\bar{u}_0\right)\right| \le \Gamma_{b(t)}^5\, e^{\frac{6\pi}{b_0}}\,\Gamma_{b_0}^{10} \ll \Gamma_{b(t)}^4. \qquad \square$$

Proof of I1.5. We follow the argument found in the proof of [Raphaël 2005, Lemma 7]. Fix some $s_2 \le s_3 \in [s_0, s_1)$. Substitute the local virial identity (3-134) into the preliminary estimate (3-132) to control the norm of ϵ. With a crude bound for Γ_b,

$$\left|\frac{\lambda_s}{\lambda} + b\right| \le C(b_s + b^2), \tag{3-152}$$

From hypothesis H1.2, $0 < b^2 < \delta(\alpha^*)b$ where $\delta(\alpha^*) \to 0$ as $\alpha^* \to 0$. Then,

$$-\log\frac{\lambda(s_2)}{\lambda(s_3)} = \int_{s_2}^{s_3}\Big(\frac{\lambda_s}{\lambda}+b\Big) - \int_{s_2}^{s_3} b \le \delta(\alpha^*) - \frac{1}{2}\int_{s_2}^{s_3} b \le \delta(\alpha^*). \tag{3-153}$$

In particular, we may assume that α^* is such that $\delta(\alpha^*) < \log 2$, which proves I1.5. □

Proof of H1.1. The preliminary estimate (3-133) can be crudely simplified to

$$\left|\frac{r_s}{\lambda}\right| + \left|\frac{z_s}{\lambda}\right| \le 1. \tag{3-154}$$

Then we have for all $s \in [s_0, s_1)$

$$|r(s)-r_0| + |z(s)-z_0| \le \int_{s_0}^{s} |r_s| + |z_s| \le \int_{s_0}^{s} \lambda(\sigma)\,d\sigma \le \sqrt{\lambda_0}\int_2^{+\infty} e^{-\frac{\pi}{3}\frac{\sigma}{\log\sigma}}\,d\sigma < \alpha^*, \tag{3-155}$$

where we applied (3-138), the choice of data (2-32) and the smallness of b_0 (2-29). With our choice of r_0,z_0 (2-28), this proves I1.1. □

3.1. ***Lyapunov functional.*** To begin this section, we repeat the calculation of the local virial identity, this time including the linear radiation $\tilde{\zeta}_b$ as part of the central profile. That is, we write

$$\tilde{\epsilon} = \epsilon - \tilde{\zeta}_b \implies u(t,x) = \frac{1}{\lambda(t)}\big(\widetilde{Q}_{b(t)} + \tilde{\zeta}_{b(t)} + \tilde{\epsilon}(t)\big)\left(\frac{(r,z)-(r(t),z(t))}{\lambda}\right)e^{-i\gamma(t)}, \tag{3-156}$$

where the parameters of the geometric decomposition are unchanged. The equation for $\tilde{\epsilon}$ may then be written analogously to (3-129), with a new linearized evolution operator analogous to M, (3-90).

Lemma 3.16 (radiative virial identity [Merle and Raphaël 2006]). *For all $s \in [s_0, s_1)$,*

$$\partial_s f_1 \ge \delta_2 \|\!|\tilde{\epsilon}|\!\| + \Gamma_b - \frac{1}{\delta_2}\int_{A\le|y|\le 2A} |\epsilon|^2\,dy, \tag{3-157}$$

where $\|\!|\tilde{\epsilon}|\!\| = \int \left|\nabla_y\tilde{\epsilon}\right|^2 \mu(y)\,dy + \int_{|y|\le\frac{10}{b}} |\tilde{\epsilon}|^2 e^{-|y|}\,dy$ (cf. (2-73)), $\delta_2, c > 0$ *are universal constants, and*

$$f_1(s) = \frac{b}{4}\left|y\widetilde{Q}_b\right|_{L^2}^2 + \frac{1}{2}\operatorname{Im}\left(\int y\cdot\nabla\tilde{\zeta}_b\overline{\tilde{\zeta}_b}\right) + \operatorname{Im}(\epsilon, \Lambda\tilde{\zeta}_b). \tag{3-158}$$

Compared with the local virial identity, the radiative virial identity is useless to control ϵ in $\dot{H}^1$ due to the presence of mass term $\int_{A\le|y|\le 2A}|\epsilon|^2$. See (3-110) for further discouragement. Nevertheless, we will link this term to the ejection of mass from the singularity, through the radiation, into the dispersive regime (Lemma 3.17). Then, we will show this mass ejection is more or less uninterrupted by demonstrating the Lyapunov functional (Lemma 3.19). Finally, through conservation of energy we will prove precise bounds on the Lyapunov functional in terms of the excess mass at the singularity and $|\epsilon|_{\dot{H}^1}$ (Lemma 3.20). These bounds will allow us to bridge between times where $b_s \le 0$ (times where the local virial identity is useful) to control ϵ pointwise in time (Lemma 3.22).

Let ϕ_∞ be a smooth radial cutoff function on $\mathbb{R}^2$ satisfying

$$\phi_\infty(y) = \begin{cases} 0 & \text{for } |y| \le \frac{1}{2}, \\ 1 & \text{for } |y| \ge 3, \end{cases} \tag{3-159}$$

$$\tfrac{1}{4} \le \phi_\infty' \le \tfrac{1}{2} \quad \text{for } 1 \le |y| \le 2, \qquad 0 \le \phi_\infty' \quad \text{for all } y. \tag{3-160}$$

The following lemma is proved on page 707:

Lemma 3.17 (mass ejection from singular and radiative regimes).

$$\partial_s \left(\frac{1}{r^{N-2}(t)} \int \phi_\infty\left(\frac{y}{A}\right) |\epsilon|^2 \, \mu(y) \, dy \right) \ge \frac{b}{400} \int_{A \le |y| \le 2A} |\epsilon|^2 \, dy - \Gamma_b^{\frac{a}{2}} \int \left|\nabla_y \epsilon\right|^2 \mu(y) \, dy - \Gamma_b^2.$$

Remark 3.18 (interpretation of Lemma 3.17). Assume for the sake of heuristics that $\epsilon \approx \zeta_b$ in the region $|y| \sim A$. With the definition of Γ_b in (3-98) and the control on ϵ afforded by hypothesis H1.2, the lemma's inequality suggests continuous ejection of mass from the region $|y| < A/2$, regardless of whether that region is growing or contracting.

Lemma 3.19 (Lyapunov functional). *For all $s \in [s_0, s_1)$,*

$$\partial_s \mathcal{J} \le -Cb \left(\Gamma_b + \|\tilde{\epsilon}\| + \int_{A \le |y| \le 2A} |\epsilon|^2 \right), \tag{3-161}$$

where $C > 0$ is a universal constant and

$$\mathcal{J}(s) = \int |\widetilde{Q}_b|^2 - \int |Q|^2 + 2\,\mathrm{Re}(\epsilon, \widetilde{Q}_b) + \frac{1}{r^{N-2}(s)} \int \left(1 - \phi_\infty\left(\frac{y}{A}\right)\right) |\epsilon|^2 \, \mu(y) \, dy$$
$$- \frac{\delta_2}{800} \left(b \tilde{f}_1(b) - \int_0^b \tilde{f}_1(v) \, dv + b \,\mathrm{Im}(\epsilon, \Lambda \tilde{\zeta}_b) \right), \tag{3-162}$$

$\tilde{f}_1$ being is the principal part of f_1 from (3-158):

$$\tilde{f}_1(b) = \frac{b}{4} \left|y \widetilde{Q}_b\right|_{L^2}^2 + \tfrac{1}{2} \,\mathrm{Im}\left(\int y \cdot \nabla \tilde{\zeta}_b \bar{\tilde{\zeta}}_b \right). \tag{3-163}$$

The proof, which we defer until page 708, involves the radiative virial estimate (3-157), the mass dispersion estimate in Lemma 3.17, and conservation of mass.

Now let us discuss what $\mathcal{J}$ is.

Lemma 3.20 (estimates on Lyapunov functional). *For all $s \in [s_0, s_1)$ we have the crude estimate*

$$|\mathcal{J} - d_0 b^2| < \delta_3 b^2, \tag{3-164}$$

where $0 < \delta_3 \ll 1$ is a universal constant and $d_0 b^2$ is the approximate excess mass of the profile $\widetilde{Q}_b$ (see (3-87)). *There also holds a more refined estimate*:

$$-\Gamma_b^{1-Ca} + \frac{1}{C} \|\epsilon\| \le \mathcal{J}(s) - f_2(b(s)) \le \Gamma_b^{1-Ca} + CA^2 \|\epsilon\|, \tag{3-165}$$

where f_2 *is the principal part of* $\mathcal{J}$ *concerned with the mass of the profile*:

$$f_2(b)=\int \left|\widetilde{Q}_b\right|^2-\int |Q|^2-\frac{\delta_2}{800}\left(b\tilde{f}_1(b)-\int_0^b \tilde{f}_1(v)\,dv\right). \tag{3-166}$$

Proof. To prove (3-164) we will approximate each term of (3-162). To estimate the term in $|\epsilon|^2$, recall from (3-159) the support of ϕ_∞, and derive the consequence for $\mu(y)$ similar to Equation (2-55). One obtains

$$\int \left(1-\phi_\infty\left(\frac{y}{A}\right)\right)|\epsilon|^2\,\mu(y)\,dy \lesssim \int_{|y|\le 3A}|\epsilon|^2 \lesssim A^2\log A\|\!|\epsilon|\!\| \le \Gamma_b^{\frac{1}{2}}, \tag{3-167}$$

where the second inequality is due to Lemma 3.5 and the final inequality is from the definition (3-104) of A and the hypothesized control of ϵ. Estimate $(\epsilon,\widetilde{Q}_b)$ by the same control, and the terms in $\tilde{\zeta}_b$ by (3-100). Equation (3-164) then follows from (3-87) by noting that the constant δ_2 due to the radiative virial identity (3-157) can be assumed small with respect to universal constant d_0, so that $0<(\partial f_2/\partial b^2)\big|_{b^2=0}<\infty$.

Next we prove the refined estimate. Note that

$$\mathcal{J}(s)-f_2(b(s))=2\operatorname{Re}(\epsilon,\widetilde{Q}_b)+\frac{1}{r^{N-2}(t)}\int (1-\phi_\infty)\,|\epsilon|^2\,\mu(y)-\frac{\delta_2}{800}b\operatorname{Im}(\epsilon,\Lambda\tilde{\zeta}_b). \tag{3-168}$$

By the bounds for $\tilde{\zeta}_b$ in Lemma 3.5 and the choice of A, we have

$$\left|\operatorname{Im}(\epsilon,\Lambda\tilde{\zeta}_b)\right| \le \Gamma_b^{\frac{1}{2}-C\eta}\left(\int_{|y|\le A}|\epsilon|^2\right)^{\frac{1}{2}} \lesssim \Gamma_b^{\frac{1}{2}-C\eta}A(\log A)^{\frac{1}{2}}\|\!|\epsilon|\!\|^{\frac{1}{2}} \lesssim \Gamma_b^{1-Ca}+\|\!|\epsilon|\!\|. \tag{3-169}$$

Since b is small, the contribution of (3-169) is a factor of α^* smaller than the desired bound. Similar terms will be omitted for the remainder of the proof.

Regarding the two other terms in (3-168), the term linear in ϵ we recognize from the conservation of energy (3-120). Indeed, the upper bound for (3-168) follows from (3-120) with (3-109) and

$$\int (1-\phi_\infty)\,|\epsilon|^2\,\mu(y)\,dy \lesssim A^2\log A\,\|\!|\epsilon|\!\|, \tag{3-170}$$

which is due to (3-110).

To establish a lower bound for (3-168) we will need the following lemma, whose proof is based on a spectral result due to [Martel and Merle 2001], with additional properties proven in [Mariş 2002] and [McLeod 1993]. See [Merle and Raphaël 2006, Lemma 8] for that spectral property, and Appendix D of the same reference for a proof of the lemma.

Lemma 3.21 (elliptic estimate for L). *Recall the linearized Schrödinger operator* L *from* (2-68). *There exists a universal constant* $\delta_4>0$ *such that, for all* $v\in H^1(\mathbb{R}^2)$,

$$\begin{aligned}&\operatorname{Re}(L(v),v)-\int \phi_\infty\,|v|^2\\ &\quad\ge \delta_4\left(\int |\nabla v|^2+\int |v|^2e^{-|y|}\right)-\frac{1}{\delta_4}\left(\operatorname{Re}(v,Q)+\operatorname{Re}(v,|y|^2Q)+\operatorname{Re}(v,yQ)+\operatorname{Im}(v,\Lambda^2Q)\right)^2.\end{aligned} \tag{3-171}$$

Introduce a new radially symmetric cutoff function, analogous to ϕ_A (3-105) but with larger support, such that $(1-\phi_B(y))(1-\phi_\infty(y/A)) = 0$:

$$\phi_B(y) = \begin{cases} 1 & \text{for } |y| \le 3A, \\ 0 & \text{for } |y| \ge 4A. \end{cases} \tag{3-172}$$

By (2-55), we can rewrite the principal part of the conservation of energy estimate (3-120) as

$$2\operatorname{Re}(\epsilon, \widetilde{Q}_b) \approx \int (1-\phi_B^2)\,|\nabla\epsilon|^2\,\mu(y)\,dy + \int \phi_B^2\,|\nabla\epsilon|^2\;dy - 3\int Q^2(\phi_B\epsilon_{\text{re}})^2 - \int Q^2(\phi_B\epsilon_{\text{im}})^2, \tag{3-173}$$

where we used the exponential spatial decay of Q and the lower bound for Γ_b (3-98) to control the excess in $Q^2\epsilon^2$ on $|y| > \frac{10}{b}$. With integration by parts,

$$\int \phi_B^2\,|\nabla\epsilon|^2\;dy = \int |\nabla(\phi_B\epsilon)|^2\;dy + \int \Delta\phi_B\,\phi_B\,|\epsilon|^2\;dy. \tag{3-174}$$

The principal part of (3-168) is then

$$\begin{aligned} 2\operatorname{Re}(\epsilon, \widetilde{Q}_b) + \frac{1}{r^{N-2}(t)}\int (1-\phi_\infty)\,|\epsilon|^2\,\mu(y)\,dy &\approx \int (1-\phi_B^2)\,|\nabla\epsilon|^2\,\mu(y)\,dy \\ +\biggl(\operatorname{Re}(L(\phi_B\epsilon),\phi_B\epsilon) - \int \phi_\infty\,|\phi_B\epsilon|^2\biggr) + \int \Delta\phi_B\,\phi_B\,|\epsilon|^2 &+ \int (1-\phi_\infty)\biggl(\frac{\mu}{r^{N-2}(t)} - \phi_B^2\biggr)|\epsilon|^2. \end{aligned} \tag{3-175}$$

The final term can be neglected, since $(1-\phi_\infty)(\mu/r^{N-2}(t) - \phi_B^2)$ is of order λy_1, and supported on $|y| < 4A$. The lower bound for (3-168) then follows from Lemma 3.21, an integration by parts, and the straightforward comparison,

$$\int \phi_B^2\,|\nabla\epsilon|^2 + \int |\phi_B\epsilon|^2\,e^{-|y|} \gtrsim \int \phi_B^2\,|\nabla\epsilon|^2\,\mu(y) + \int_{|y|\le\frac{10}{b}} |\epsilon|^2\,e^{-|y|}, \tag{3-176}$$

again due to the support of ϕ_B and the bound on λ. This completes the proof of (3-165). □

Lemma 3.22 (lower bound on blowup rate). *For all $s \in [s_0, s_1)$,*

$$b(s) \le \frac{4\pi}{3\log s} \tag{3-177}$$

and

$$\int_{s_0}^{s} (\Gamma_{b(\sigma)} + |||\epsilon|||)\,d\sigma \le C\alpha^*, \tag{3-178}$$

where $C > 0$ is a universal constant and

$$|||\epsilon||| \le \Gamma_b^{\frac{4}{5}}. \tag{3-179}$$

(This is (2-58), the remaining part of I1.2.)

Note that (3-177) is the first upper bound of I1.3. The only estimate still required in order to establish Proposition 2.8 follows as a corollary.

Proof of the second lower bound I1.3. Recall from (3-140) the approximate dynamics of λ. Since $b > 0$ is small, we have $-\frac{\lambda_s}{\lambda} \le 3b$, which we integrate with (3-177),

$$-\log\lambda(s) \le -\log\lambda_0 + 4\pi\int_{s_0}^{s}\frac{1}{\log\sigma}\,d\sigma \le -\log\lambda_0 + 4\pi(s-s_0). \tag{3-180}$$

Use (3-177) again, and recall the definition of s_0 (2-43) and choice of data (2-32), to obtain

$$\lambda(s) \ge \lambda_0 e^{4\pi s_0}\,e^{-4\pi e^{\frac{4\pi}{3b(s)}}} > e^{-e^{\frac{5\pi}{b(s)}}}. \qquad \square$$

Proof of Lemma 3.22. First, in view of the crude estimate (3-164), we may divide the Lyapunov inequality (3-161) by $\sqrt{\mathcal{J}}$ and integrate in time, leaving

$$\int_{s_0}^{s}\big(\Gamma_{b(\sigma)} + |\!|\!|\epsilon|\!|\!|\big)\,d\sigma \le C\big(\sqrt{\mathcal{J}(s_0)} - \sqrt{\mathcal{J}(s)}\big) \le Cb_0. \tag{3-181}$$

The choice of data (2-29) then proves (3-178). Alternately, we may view the crude estimate (3-164) and the Lyapunov inequality (3-161) as giving a differential inequality for $\mathcal{J}$:

$$\partial_s e^{+\frac{5\pi}{4}\sqrt{\frac{d_0}{\mathcal{J}}}} \gtrsim \frac{b}{\mathcal{J}}\,\Gamma_b e^{\frac{5\pi}{4}\sqrt{\frac{d_0}{\mathcal{J}}}} \ge 1, \tag{3-182}$$

which implies

$$e^{+\frac{5\pi}{4}\sqrt{\frac{d_0}{\mathcal{J}(s)}}} \ge e^{+\frac{5\pi}{4}\sqrt{\frac{d_0}{\mathcal{J}(s_0)}}} + s - s_0. \tag{3-183}$$

Here we applied the bound (3-103) on Γ_b, for which it is essential that $\frac{5}{4} > 1 + C\eta$; see Remark 4.15. By the crude estimate (3-164) and the definition of s_0 in (2-43), we have

$$e^{+\frac{5\pi}{4}\sqrt{\frac{d_0}{\mathcal{J}(s_0)}}} > e^{\frac{\pi}{b_0}} > s_0, \tag{3-184}$$

which, again with estimate (3-164), proves (3-177) from (3-183).

It remains to establish the pointwise control of ϵ. Fix $s \in [s_0, s_1)$.

1. If $\partial_s b(s) \le 0$, then (3-179) follows from the local virial identity, Lemma 3.13.

2. If $\partial_s b(s) > 0$, there exists a largest interval (s_+, s), with $s_0 \le s_+$, on which $\partial_s b > 0$. This implies $b(s_+) < b(s)$ and either

$$s_+ = s_0 \quad\text{or}\quad \partial_s b(s_+) = 0.$$

In the first case we use the choice of small ϵ_0 and in the second the local virial identity, to obtain in either case

$$\int\big|\nabla_y\epsilon(s_+,y)\big|^2\mu(y)\,dy + \int_{|y|\le\frac{10}{b(s_+)}}|\epsilon(s_+,y)|^2 e^{-|y|}\,dy \le \Gamma_{b(s_+)}^{\frac{6}{7}}.$$

From the upper bound of the refined estimate (3-165), and assuming $a > 0$ is sufficiently small,

$$\mathcal{J}(s_+) - f_2(b(s_+)) \le \Gamma_{b(s_+)}^{\frac{5}{6}} < \Gamma_{b(s)}^{\frac{5}{6}}. \tag{3-185}$$

Since $\mathcal{J}$ is non-increasing, and from the lower bound of refined estimate (3-165), we get

$$\begin{aligned}\Gamma_{b(s)}^{\frac{5}{6}} &\geq \mathcal{J}(s) - f_2(b(s_+)) \\ &\gtrsim \left(\int \left|\nabla_y \epsilon(s,y)\right|^2 \mu(y)\,dy + \int_{|y|\leq \frac{10}{b(s)}} |\epsilon(s,y)|^2 e^{-|y|}\,dy \right) - \Gamma_{b(s)}^{1-Ca} + \big(f_2(b(s)) - f_2(b(s_+))\big).\end{aligned} \tag{3-186}$$

As noted in the proof of the crude estimate (3-164), we may assume the constant δ_2 of (3-157) is small enough relative to d_0 so that $0 < \partial f_2/\partial_{b^2}\big|_{b^2=0} < \infty$, proving that $\big(f_2(b(s)) - f_2(b(s_+))\big) > 0$. Assuming $a > 0$ is sufficiently small, this proves (3-179). □

Proof of Lemma 3.17. Directly from (1-1) we obtain

$$\begin{aligned}\tfrac{1}{2}\partial_s \left(\int \phi_\infty \left(\frac{(r,z) - (r(t), z(t))}{\lambda A} \right) |u|^2\,dx \right) &= \frac{1}{\lambda A} \operatorname{Im} \left(\int \nabla_x \phi_\infty \left(\frac{y}{A}\right) \cdot \nabla_x u\, \bar{u}\,dx \right) \\ &\quad - \frac{1}{2\lambda^2 A} \int \left(\left(\frac{\lambda_s}{\lambda} + \frac{A_s}{A} \right) y + \frac{\partial_s(r,z)}{\lambda} \right) \cdot \nabla_x \phi_\infty \left(\frac{y}{A}\right) |u|^2\,dx.\end{aligned} \tag{3-187}$$

By the choice of A in (3-104) and the properties of ϕ_∞ in (3-159), the support of $\widetilde{Q}_b$ and $\phi_\infty(y/A)$ are disjoint. With the geometric decomposition and a change of variables we can rewrite (3-187) in terms of $|\epsilon|^2$:

$$\begin{aligned}&\frac{1}{2}\frac{d}{ds} \int \phi_\infty \left(\frac{y}{A}\right) |\epsilon|^2 \mu(y)\,dy \\ &\quad = \frac{1}{A} \operatorname{Im} \left(\int \nabla_x \phi_\infty \left(\frac{y}{A}\right) \cdot \nabla_y \epsilon\, \bar{\epsilon} \mu(y)\,dy \right) + \frac{b}{2} \int \frac{y}{A} \cdot \nabla_x \phi_\infty \left(\frac{y}{A}\right) |\epsilon|^2 \mu(y)\,dy \\ &\qquad - \frac{1}{2A} \int \left(\left(\frac{\lambda_s}{\lambda} + b + \frac{A_s}{A} \right) y + \frac{\partial_s(r,z)}{\lambda} \right) \cdot \nabla_x \phi_\infty \left(\frac{y}{A}\right) |\epsilon|^2 \mu(y)\,dy.\end{aligned} \tag{3-188}$$

By Cauchy–Schwarz, the definition of A in (3-104) and the lower bound on Γ_b in (3-103),

$$\begin{aligned}&\left| \frac{1}{A} \operatorname{Im} \left(\int \nabla_x \phi_\infty \left(\frac{y}{A}\right) \cdot \nabla_y \epsilon\, \bar{\epsilon} \mu(y)\,dy \right) \right| \\ &\qquad \leq \frac{1}{A} \left(\int |\nabla \epsilon|^2 \mu(y)\,dy \right)^{\frac{1}{2}} \left(\int \left| \nabla_x \phi_\infty \left(\frac{y}{A}\right) \right| |\epsilon|^2 \mu(y)\,dy \right)^{\frac{1}{2}} \\ &\qquad \leq \frac{1}{2} \Gamma_b^{\frac{a}{2}} \int |\nabla \epsilon|^2 \mu(y)\,dy + \frac{b}{40} \int \left| \nabla_x \phi_\infty \left(\frac{y}{A}\right) \right| |\epsilon|^2 \mu(y)\,dy.\end{aligned} \tag{3-189}$$

The factor $\frac{b}{40}$ is arbitrary by assuming b is sufficiently small. The following term is the principal part of (3-188). From Equation (3-160) we know the support of ϕ'_∞ and that $\phi'_\infty \geq 0$, so

$$\frac{b}{2} \int \frac{y}{A} \cdot \nabla_x \phi_\infty \left(\frac{y}{A}\right) |\epsilon|^2 \mu(y)\,dy \geq \frac{b}{5} \int \left| \nabla_x \phi_\infty \left(\frac{y}{A}\right) \right| |\epsilon|^2 \mu(y)\,dy. \tag{3-190}$$

Regarding the last line of (3-188), apply preliminary estimates (3-132) and (3-133), the support of ϕ'_∞, and the definition of A to estimate

$$\frac{1}{2A}\left|\left(\frac{\lambda_s}{\lambda}+b+\frac{A_s}{A}\right)y+\frac{\partial_s(r,z)}{\lambda}\right|\leq\frac{b}{40}. \tag{3-191}$$

Due to the bounds for $\phi'_\infty(y/A)$ on $A\leq|y|\leq 2A$, and lower bounds for μ similar to (2-55), we have

$$\int\left|\nabla_x\phi_\infty\left(\frac{y}{A}\right)\right||\epsilon|^2\,\mu(y)\,dy\geq\frac{1}{6}\int_{A\leq|y|\leq 2A}|\epsilon|^2\,dy. \tag{3-192}$$

From (3-188) we have proven,

$$\frac{d}{ds}\int\phi_\infty\left(\frac{y}{A}\right)|\epsilon|^2\,\mu(y)\,dy\geq\frac{b}{20}\int_{A\leq|y|\leq 2A}|\epsilon|^2\,dy-\Gamma_b^{\frac{a}{2}}\int|\nabla\epsilon|^2\,\mu(y)\,dy. \tag{3-193}$$

Finally note that by the preliminary estimate (3-133), the fact that $r(t)\sim 1$ from H1.1, a change of variables, and the log-log relationship (2-54), we have the easy estimate

$$\left|\frac{r_s}{r^{N-1}(t)}\int\phi_\infty\left(\frac{y}{A}\right)|\epsilon|^2\,\mu(y)\,dy\right|\ll\lambda\int|\tilde{u}|^2\ll\Gamma_b^2. \tag{3-194}$$

This completes the proof of Lemma 3.17. □

Proof of Lemma 3.19. We multiply the radiative virial identity (3-157) by $\dfrac{\delta_2 b}{800}$ and sum with the mass ejection estimate in Lemma 3.17 to cancel the bad sign of $\int_{A\leq|y|\leq 2A}|\epsilon|^2$:

$$\begin{aligned}\partial_s&\left(\frac{1}{r^{N-2}(t)}\int\phi_\infty\left(\frac{y}{A}\right)|\epsilon|^2\,\mu(y)\,dy\right)+\frac{\delta_2 b}{800}\partial_s f_1\\&\geq\frac{\delta_2^2 b}{800}|||\tilde{\epsilon}|||+\frac{b}{800}\int_{A\leq|y|\leq 2A}|\epsilon|^2\,dy+\frac{\delta_2 b}{1000}\Gamma_b-\Gamma_b^{\frac{a}{2}}\int\left|\nabla_y\epsilon\right|^2\mu(y)\,dy.\end{aligned} \tag{3-195}$$

The final term of (3-195) has the bad sign. Recall from (3-156) that $\epsilon=\tilde{\epsilon}+\tilde{\zeta}_b$, and from (3-100) that $\tilde{\zeta}_b$ is small in $\dot{H}^1$, on the support of which we can estimate μ, so that

$$\begin{aligned}\Gamma_b^{\frac{a}{2}}\int\left|\nabla_y\epsilon\right|^2\mu(y)\,dy&\lesssim\Gamma_b^{\frac{a}{2}}\left(\Gamma_b^{1-C\eta}+\int|\nabla\tilde{\epsilon}|^2\,\mu(y)\,dy\right)\\&\leq\Gamma_b^{1+\frac{a}{4}}+\Gamma_b^{\frac{a}{2}}\int|\nabla\tilde{\epsilon}|^2\,\mu(y)\,dy,\end{aligned} \tag{3-196}$$

where for the second inequality we require $a>4C\eta$; see Remark 3.4. To rewrite $\dfrac{\delta_2 b}{800}\partial_s f_1$, note that

$$b\partial_s f_1=\partial_s\left(b\tilde{f}_1(b)-\int_0^b\tilde{f}_1(v)\,dv+b\,\mathrm{Im}(\epsilon,\Lambda\tilde{\zeta}_b)\right)-\partial_s b\,\mathrm{Im}(\epsilon,\Lambda\tilde{\zeta}_b), \tag{3-197}$$

where $\tilde{f}_1$ is the principal part of f_1; see (3-163) and (3-158), respectively. Estimate the final term of (3-197) with a combination of the preliminary estimate (3-132), Hölder, Lemma 3.5, and H1.2.

Equation (3-195) is transformed into

$$\partial_s\left(\frac{1}{r^{N-2}(t)}\int \phi_\infty\left(\frac{y}{A}\right)|\epsilon|^2\,\mu(y)\,dy+\frac{\delta_2}{800}\left(b\tilde{f}_1(b)-\int_0^b \tilde{f}_1(v)\,dv+b\,\mathrm{Im}\big(\epsilon,\Lambda\tilde{\zeta}_b\big)\right)\right)$$
$$\geq\frac{\delta_2^2 b}{800}\left(|\!|\!|\tilde{\epsilon}|\!|\!|+\int_{A\leq|y|\leq 2A}|\epsilon|^2\,dy\right)+\frac{\delta_2 b}{2000}\Gamma_b. \quad (3\text{-}198)$$

To identify the left-hand side of (3-198) with $-\partial_s\mathcal{J}$, inject the conservation of mass, $\int_{\mathbb{R}^N}|u(t)|^2=\int|u_0|^2$. As we did for Equation (3-122), rewrite $u(t)$ with the geometric decomposition, expand the product, change variables, expand the measure μ, divide by $r^{N-2}(t)$, and take the derivative ∂_s:

$$\partial_s\left(\frac{1}{r^{N-2}(t)}\int \phi_\infty\left(\frac{y}{A}\right)|\epsilon|^2\,\mu(y)\,dy\right)=-\partial_s\left(\int|\tilde{Q}_b|^2-\int|Q|^2+2\,\mathrm{Re}\big(\epsilon,\tilde{Q}_b\big)\right)$$
$$-\,\partial_s\left(\frac{1}{r^{N-2}(t)}\int\mathcal{O}(\lambda y_1)\left(\big|\tilde{Q}_b\big|^2+2\,\mathrm{Re}\big(\epsilon\overline{\tilde{Q}_b}\big)\right)\right)-\frac{\partial_s r}{r^{N-1}(t)}\int|u_0|^2. \quad (3\text{-}199)$$

Through a combination of the preliminary estimates (3-132) and (3-133), the ϵ-equation (3-129), and the log-log rate (2-54), we obtain

$$\left|-\partial_s\left(\frac{1}{r^{N-2}(t)}\int\mathcal{O}(\lambda y_1)\left(\big|\tilde{Q}_b\big|^2+2\,\mathrm{Re}\big(\epsilon\overline{\tilde{Q}_b}\big)\right)\right)\right|\lesssim\lambda<\Gamma_b^2.$$

Likewise,

$$\left|\frac{\partial_s r}{r^2(t)}\right|\int|u_0|^2\lesssim\lambda\int|u_0|^2<\Gamma_b^2.$$

Inserting (3-199) into (3-198) completes the proof of Lemma 3.19. □

4. Proof of global behavior

In this section we prove that the properties I2.1–I2.3 follow from hypotheses H1.1–H2.3. The following properties of the singular dynamic proven in Section 3 will be used: the specific log-log rate, the geometric decomposition and resulting control on b_s, and the integrability of $\|\tilde{u}\|_{L_t^2 H_x^1}$.

Growth of $\|u\|_{H^N}$. It is left until Section 5 to show that $1/\lambda$ follows the log-log rate (1-12). Here, we use the log-log rate in the form H1.3, and the control of b_s, to prove directly that $\lambda^{-1}(t)$ has the same integrability in time as

$$\sqrt{\frac{\log|\log(T-t)|}{T-t}}.$$

Lemma 4.1 (Integrability due to log-log rate[Raphaël and Szeftel 2009, (51)]). *Let $0\leq\mu<2$ and $\sigma_1\in\mathbb{R}$. Then,*

$$\int_0^t\frac{e^{\frac{\sigma_1}{b(\tau)}}}{\lambda^\mu(\tau)}\,d\tau\lesssim C(\mu,\sigma_1,\alpha^*), \quad (4\text{-}200)$$

where for fixed μ and σ_1, $C(\mu,\sigma_1,\alpha^*)$ decays much faster than $e^{-\frac{1}{\alpha^*}}$ as $\alpha^*\to 0$.

Proof. The log-log rate, H1.3, gives $e^{\frac{\pi}{10b}} < |\log\lambda|$ and $\frac{\pi}{10b} > \frac{1}{100}\log s$; hence

$$\frac{1}{\lambda} > e^{+e^{\frac{\pi}{10b}}} > e^{s^{\frac{1}{100}}}.$$

By a change of variables and the almost-monotony of λ, H1.5,

$$\int_0^t \frac{e^{\frac{\sigma_1}{b(\tau)}}}{\lambda^{\mu}(\tau)}\,d\tau < \int_{s_0}^s \frac{|\log\lambda|^{\frac{10\sigma_1}{\pi}}}{\lambda^{\mu-2}(\tau')}\,d\tau' \lesssim \frac{|\log\lambda(t)|^{\frac{10\sigma_1}{\pi}}}{\lambda^{\mu-2}(t)}\,(s(t)-s_0) \lesssim e^{(\mu-2)s^{\frac{1}{100}}(t)}.$$

Finally, to prove the behavior of $C(\mu,\sigma_1,\alpha^*)$, recall that $s(t) \ge s_0 = e^{\frac{3\pi}{4b_0}}$ and $b_0 < \alpha^*$. □

Remark 4.2 (Lemma 4.1 for $\mu \ge 2$). From the log-log rate, H1.3, $s(t) - s_0 \lesssim e^{\frac{10}{\pi}\frac{1}{b(t)}}$, so by the same proof,

$$\int_0^t \frac{1}{\lambda^{\mu}} \lesssim \frac{e^{\frac{10}{\pi}\frac{1}{b(t)}}}{\lambda^{\mu-2}}. \tag{4-201}$$

This is the primary integrability tool of [Raphaël and Szeftel 2009]. The following improvement will be crucial.

Lemma 4.3 (refined integrability due to control of b_s). *Let $\mu > 2$ and σ^* be arbitrary and assume $\alpha^* > 0$ is sufficiently small. Then for any $\sigma_2 > 0$ and all $t \in [0, T_{\text{hyp}})$,*

$$\int_0^t \frac{e^{-\frac{\sigma^*}{b(\tau)}}}{\lambda^{\mu}(\tau)}\,d\tau \le C(\mu,\sigma_2,\alpha^*)\,\frac{e^{-\frac{\sigma^*}{b(t)}}e^{+\frac{\sigma_2}{b(t)}}}{\lambda^{\mu-2}(t)}, \tag{4-202}$$

where, for fixed μ and σ_2, $C(\mu,\sigma_2,\alpha^*) \to 0$ as $\alpha^* \to 0$.

Proof. To begin, we prove the case $\sigma^* = 0$. By direct calculation,

$$\frac{d}{ds}\left(\frac{1}{b}\frac{1}{\lambda^{\mu-2}}\right) = \frac{1}{\lambda^{\mu-2}}\left((\mu-2) - \frac{b_s}{b^2} - (\mu-2)\frac{\frac{\lambda_s}{\lambda}+b}{b}\right).$$

For α^* sufficiently small relative to μ, from H1.2 and the control of b_s, (3-132),

$$\frac{1}{\lambda^{\mu}} \le C(\mu)\frac{1}{\lambda^2}\frac{d}{ds}\left(\frac{1}{b}\frac{1}{\lambda^{\mu-2}}\right) = C(\mu)\frac{d}{dt}\left(\frac{1}{b}\frac{1}{\lambda^{\mu-2}}\right). \tag{4-203}$$

After integration, we estimate

$$C(\mu)\frac{1}{b}\frac{1}{\lambda^{\mu-2}} \le C(\mu,\sigma_2,\alpha^*)\frac{e^{+\frac{\sigma_2}{b}}}{\lambda^{\mu-2}}.$$

For those cases where $\sigma^* \ne 0$, integrate by parts:

$$\int_0^t \frac{e^{-\frac{\sigma^*}{b(\tau)}}}{\lambda^{\mu}(\tau)}d\tau = e^{-\frac{\sigma^*}{b(\tau)}}\int_0^{\tau}\frac{1}{\lambda^{\mu}(\tau')}d\tau'\bigg|_0^t - \int_0^t \sigma^*\left(\frac{b_\tau}{b^2}e^{-\frac{\sigma^*}{b(\tau)}}\int_0^{\tau}\frac{1}{\lambda^{\mu}(\tau')}d\tau'\right)d\tau. \tag{4-204}$$

Apply the previous case to the first term on the right. For the second term, make the change of variable $b_\tau = b_{s(\tau)}/\lambda^2(\tau)$ and apply the previous case for some $\sigma_2 \ll \frac{1}{2}$. Use (3-132) to approximate b_s, and we have bounded the second term by a small multiple of the left-hand side. □

Lemma 4.3 is not true for $\mu = 2$. As a substitute, we prove a corollary of the integrated Lyapunov inequality, (3-178).

Corollary 4.4. *Let $\sigma_3 \geq 0$. For all $t \in [0, T_{\mathrm{hyp}})$,*

$$\int_0^t e^{\frac{\sigma_3}{b(\tau)}} \left(\|\tilde{u}(\tau)\|_{H^1}^2 + \frac{\Gamma_{b(\tau)}}{\lambda^2(\tau)} \right) d\tau \lesssim C(\alpha^*) e^{\frac{\sigma_3}{b(t)}}. \tag{4-205}$$

Proof. By change of variables and integration by parts,

$$\begin{aligned}\int_0^t & e^{\frac{\sigma_3}{b(\tau)}} \left(\|\tilde{u}(\tau)\|_{H^1}^2 + \frac{\Gamma_{b(\tau)}}{\lambda^2(\tau)} \right) d\tau \\ &= e^{\frac{\sigma_3}{b(\sigma)}} \int_0^\sigma \lambda^2(\sigma') \left\|\tilde{u}(\sigma')\right\|_{H_x^1}^2 + \Gamma_{b(\sigma')}\, d\sigma' \Big|_{s_0}^{s(t)} + \sigma_3 \int_{s_0}^{s(t)} \frac{b_s}{b^2} e^{\frac{\sigma_3}{b(\sigma)}} \left(\int_0^\sigma \lambda^2 \|\tilde{u}\|_{H_x^1}^2 + \Gamma_b \right) d\sigma.\end{aligned}$$

Then observe the control on b_s in (3-132) and the estimate (3-178). □

Remark 4.5 (optimality of (4-205)). Corollary 4.4 is the best possible integrability of $e^{\frac{\delta}{b}}/\lambda^2$ for constant δ. As a heuristic, assume that $\lambda \sim \sqrt{T-t}$ and $e^{\frac{1}{b}} \sim |\log \lambda| \sim |\log(T-t)|$, motivated by the log-log rate H1.3. The integral $\int^T \frac{|\log(T-t)|^\delta}{T-t}\, dt$ is only finite for values of δ sufficiently negative. In our case, the maximum threshold for δ is given dynamically by (3-178).

Next, we translate hypotheses H2.1–H2.3 into a gain of derivative during particular N-dimensional Sobolev embeddings. Consider a smooth cutoff function with support on $\chi^{-1}(\{1\})$:

$$\tilde{\chi}(r, z, \theta) = \begin{cases} 1 & \text{for } |(r,z) - (1,0)| \geq \frac{3}{4}, \\ 0 & \text{for } |(r,z) - (1,0)| \leq \frac{2}{3}. \end{cases} \tag{4-206}$$

Lemma 4.6 (consequences of bootstrap hypotheses). *Let $v = \tilde{\chi} u$. Then,*

$$\int \left|\nabla^{N-1} v\right|^2 |v|^2 \leq C\left(\tilde{\chi}, \alpha^*\right) \|v\|_{H^N}^2, \tag{4-207}$$

where $C(\tilde{\chi}, \alpha^*) \to 0$ as $\alpha^* \to 0$. Furthermore, suppose that

$$\begin{aligned} N - 1 \geq l_1 \geq l_2 \geq l_3 \geq 0 \quad &\text{with} \quad l_1 + l_2 + l_3 = N, \\ k_1 \geq k_2 \geq k_3 \geq 0 \quad &\text{with} \quad k_1 + k_2 + k_3 = N - 2. \end{aligned}$$

Then

$$\begin{aligned}\int \left|\nabla^N v\right| \left|\nabla^{l_1} v\right| \left|\nabla^{l_2} v\right| \left|\nabla^{l_3} v\right| + \int \left|\nabla^{k_1} v\right| \left|\nabla^{k_2} v\right| \left|\nabla^{k_3} v\right| \left|\nabla^{l_1} v\right| \left|\nabla^{l_2} v\right| \left|\nabla^{l_3} v\right| \\ + \int \left|\nabla^{N-1} v\right|^2 \left(|\nabla v|^2 + |v|^4\right) \leq C(\tilde{\chi}) \frac{1}{\lambda^{2N+1}}.\end{aligned} \tag{4-208}$$

Proof. For (4-207), apply the N-dimensional Sobolev embeddings $H^1 \hookrightarrow L^{\frac{2N}{N-2}}$ and $H^{\frac{N}{2}-1} \hookrightarrow L^N$:

$$\int \left|\nabla^{N-1} v\right|^2 |v|^2 \leq \|\nabla^{N-2} v\|_{L^{\frac{2N}{N-2}}}^2 \|v\|_{L^N}^2 \lesssim \|v\|_{H^N}^2 \|v\|_{H^{\frac{N}{2}-1}}^2.$$

Then recall hypothesis H2.3.

We consider in turn the three integrals in (4-208), applying Hölder and N-dimensional Sobolev embeddings in each case. For the first,

$$\int |\nabla^N v||\nabla^{l_1} v||\nabla^{l_2} v||\nabla^{l_3} v| \lesssim \|v\|_{\dot H^N} \prod_{j=1,2,3} \|\nabla^{l_j} v\|_{L^{\frac{2N}{l_j}}} \lesssim \|v\|_{\dot H^N} \prod_{j=1,2,3} \|v\|_{H^{\frac{N}{2}+\frac{l_j}{2}+\delta}}, \tag{4-209}$$

where $\frac{1}{2} \gg \delta > 0$ is only necessary if $l_3 = 0$. Apply hypotheses H2.1 and H2.2, interpolating if $\delta \neq 0$. The resulting bound is of the order $1/\lambda^{2N}$.

To deal with the second integral in (4-208), choose $r_j = 2N \dfrac{N-2}{N-2} \dfrac{1}{k_j}$ and $q_j = 2N \dfrac{N}{N+1} \dfrac{1}{l_j}$, so

$$\sum \frac{1}{r_j} = \frac{N-1}{2N} \quad \text{and} \quad \sum \frac{1}{q_j} = \frac{N+1}{2N}.$$

Then

$$\begin{aligned}\int |\nabla^{k_1} v||\nabla^{k_2} v||\nabla^{k_3} v||\nabla^{l_1} v||\nabla^{l_2} v||\nabla^{l_3} v| &\lesssim \prod_{j=1,2,3} \|\nabla^{k_j} v\|_{L^{r_j}} \|\nabla^{l_j} v\|_{L^{q_j}} \\ &\lesssim \prod_{j=1,2,3} \|v\|_{H^{\frac{N}{2}+\frac{k_j}{2}\frac{N-3}{N-2}+\delta}} \|v\|_{H^{\frac{N}{2}+\frac{l_j}{2}\frac{N-1}{N}+\delta}},\end{aligned} \tag{4-210}$$

where, again, $\frac{1}{2} \gg \delta > 0$ is only necessary if k_2, k_3 or $l_3 = 0$. Apply hypotheses H2.1–H2.3, interpolating where necessary. The resulting bound is of the order $1/\lambda^{2N-4}$.

For the third integral, we write

$$\begin{aligned}\int |\nabla^{N-1} v|^2 (|\nabla v|^2 + |v|^4) &\lesssim \|\nabla^{N-1} v\|^2_{L^{\frac{2N}{N-2}}} (\|\nabla v\|^2_{L^N} + \|v\|^4_{L^{2N}}) \\ &\lesssim \|v\|^2_{H^N} (\|v\|^2_{H^{\frac{N}{2}}} + \|v\|^4_{H^{\frac{N}{2}-1}}).\end{aligned} \tag{4-211}$$

Apply hypotheses H2.1 and H2.3. The resulting bound is of the order $1/\lambda^{2N}$.

Finally, use hypothesis H1.3 to estimate the neglected factors of $e^{\frac{1}{b}}$ by a single factor of $1/\lambda$. □

Near the singular ring, and in particular on the support of $\nabla\chi$, we do not have the luxury of bootstrap hypotheses. However, under cylindrical symmetry this region is essentially two-dimensional. Indeed, two-dimensional type Sobolev embeddings may be applied to functions supported on this region, as we remark in the next paragraph. Coupled to the geometric decomposition, these embeddings will achieve precisely the weakest usable bounds.

Remark 4.7 (comparison of $H^\nu(\mathbb{R}^2)$ and $H^\nu(\mathbb{R}^N)$). Consider $\Omega_N \subset \{0 < R < r < 2R < \infty\} \subset \mathbb{R}^N$, a fixed cylindrical symmetric compact domain away from the origin, as is, in particular, the support of $\nabla\chi$. Let $\Omega_2 \subset \mathbb{R}^2$ denote the obvious projection, and let f denote any cylindrically symmetric function supported on Ω_N. For $\nu \geq 0$ we claim that

$$\|f\|_{H^\nu(\mathbb{R}^2)} \approx_{R,N,\nu} \|f\|_{H^\nu(\mathbb{R}^N)} \quad \text{whenever } f \in H_0^{\lceil \nu \rceil}(\Omega_N).$$

The canonical linear mapping $T : f(x \in \mathbb{R}^N) \to f((r,z) \in \mathbb{R}^2)$ is seen, by explicit computation, to be continuous as a map $L^2_0(\Omega_N) \to L^2_0(\Omega_2)$ or $H^{\lceil \nu \rceil}_0(\Omega_N) \to H^{\lceil \nu \rceil}_0(\Omega_2)$. Moreover, the mapping is compact (see [Lions 1982]), so the same map between the interpolation spaces $H^{\nu}_0(\Omega_N)$, $H^{\nu}_0(\Omega_2)$ of the interpolation pairs $L^2(\Omega_N), H^{\lceil \nu \rceil}(\Omega_N)$ and $L^2(\Omega_2), H^{\lceil \nu \rceil}(\Omega_2)$ is also compact [Persson 1964].

Lemma 4.8 (two-dimensional version of Lemma 4.6). *Let $v = (1-\tilde{\chi})u$. There exists $\sigma_5 > 0$ such that*

$$\int \left|\nabla^{N-1} v\right|^2 |v|^2 \le C(\tilde{\chi}, \widetilde{Q}_b)\left(\frac{1}{\lambda^{2N}} + e^{-\frac{\sigma_5}{b}} \|u\|^2_{H^N}\right), \tag{4-212}$$

and that

$$\int \left|\nabla^N v\right|\left|\nabla^{l_1} v\right|\left|\nabla^{l_2} v\right|\left|\nabla^{l_3} v\right| + \int \left|\nabla^{k_1} v\right|\left|\nabla^{k_2} v\right|\left|\nabla^{k_3} v\right|\left|\nabla^{l_1} v\right|\left|\nabla^{l_2} v\right|\left|\nabla^{l_3} v\right| \\ + \int \left|\nabla^{N-1} v\right|^2\left(\left|\nabla v\right|^2 + |v|^4\right) \le C(\tilde{\chi}, \widetilde{Q}_b)\left(\frac{1}{\lambda^{2N+2}} + \frac{e^{-\frac{\sigma_5}{b}}}{\lambda^2} \|u\|^2_{H^N}\right), \tag{4-213}$$

where k_j and l_j are as in Lemma 4.6. The value of $\sigma_5 > 0$ is uniform over all $m > 0$ sufficiently small.

Proof. Due to the concentrated support of $\widetilde{Q}_b$ — see (2-55) — we have

$$(1-\tilde{\chi})\, u(r,z,\theta) = \frac{1}{\lambda}\widetilde{Q}_b\left(\frac{(r,z)-(r_0,z_0)}{\lambda}\right) e^{-i\gamma} + (1-\tilde{\chi})\tilde{u}(r,z), \tag{4-214}$$

which we denote by $W + w$. Due to Lemma 2.2, the various norms of W are explicit. For example, $\left\|\nabla^N W\right\|_{L^\infty} \le C(\widetilde{Q}_b)/\lambda^{N+1}$, where the constant is uniform over all b sufficiently small. To prove (4-212) and (4-213), we substitute $v = W + w$ and consider two cases: all factors are W, or, at least one factor is w. The first case is explicit and trivial. In the second case we will extract a factor that is a power of $\|w\|_{H^1}$. Assuming $m > 0$ is sufficiently small, H1.2 will then yield the factor of $e^{-\sigma_5/b}$. Throughout this proof, we preserve the correct multiplicity of $1/\lambda$ and $\|u\|_{H^N}$ by avoiding the Sobolev embedding into L^∞.

Make the substitution $v = W + w$. To prove (4-212) we need to show the same bound for,

$$\int \left|\nabla^{N-1} w\right|\left|\nabla^{N-1} v\right| |v|^2 + \int \left|\nabla^{N-1} v\right|^2 |v|\, |w|. \tag{4-215}$$

In the first case, apply the two-dimensional embedding $H^{\frac{1}{2}} \hookrightarrow L^4$, and interpolate:

$$\begin{aligned} \int \left|\nabla^{N-1} w\right|\left|\nabla^{N-1} v\right| |v|^2 &\le \left\|\nabla^{N-1} w\right\|_{L^4}\left\|\nabla^{N-1} v\right\|_{L^4} \|v\|^2_{L^4} \\ &\lesssim \|w\|_{H^{N-\frac{1}{2}}} \|v\|_{H^{N-\frac{1}{2}}} \|v\|^2_{H^{\frac{1}{2}}} \\ &\lesssim \left(\|w\|^{1-\frac{1}{2(N-1)}}_{H^N} \|w\|^{\frac{1}{2(N-1)}}_{H^1}\right) \|v\|_{H^{N-\frac{1}{2}}} \|v\|^2_{H^{\frac{1}{2}}}. \end{aligned} \tag{4-216}$$

Interpolate the norms in v between $\|u\|_{L^2}$ and $\|u\|_{H^N}$. The factor of $\|w\|_{H^1}$ provides a factor of $e^{-\frac{\sigma_5}{b}}$ for some $\sigma_5 > 0$, assuming that the constant $m > 0$ of hypothesis H2.1 is sufficiently small. For the second term of (4-215) follow the same strategy, except use the interpolation $\|w\|_{H^{\frac{1}{2}}} \lesssim \|w\|_{H^1}^{\frac{1}{2}} \|w\|_{L^2}^{\frac{1}{2}}$. This completes the proof of (4-212).

Now consider the three left-hand side terms of (4-213) in turn. In each case make the substitution $v = W + w$ and assume at least one factor is w.

1. We need to show the same bound for

$$\int |\nabla^N w||\nabla^{l_a} W||\nabla^{l_b} W||\nabla^{l_c} W| + \int |\nabla^N v||\nabla^{l_a} v||\nabla^{l_b} v||\nabla^{l_c} w|, \tag{4-217}$$

where $N-1 \geq l_a, l_b, l_c \geq 0$, with $l_a + l_b + l_c = N$, is some permutation of l_1, l_2, l_3. Integrate the first term of (4-217) by parts, use Hölder and interpolate:

$$\|\nabla^{N-1} w\|_{L^2} \|\nabla^{l_a} W \nabla^{l_b} W \nabla^{l_c} W\|_{H^1} \lesssim \|w\|_{H^N}^{1-\frac{1}{N-1}} \|w\|_{H^1}^{\frac{1}{N-1}} \|\nabla^{l_a} W \nabla^{l_b} W \nabla^{l_c} W\|_{H^1}.$$

The norms of W have explicit scaling-consistent bounds of the order $(1/\lambda)^{(2+l_a+l_b+l_c)}$. Again, $\|w\|_{H^1}$ provides a factor of $e^{-\frac{\sigma_5}{b}}$ for some $\sigma_5 > 0$ and the resulting bound is of the order $1/\lambda^{2N+1}$.

The remaining term of (4-217) is more difficult. Choose $q_a, q_b, q_c > 0$ such that

$$\sum_{j=a,b,c} \frac{1}{q_j} = \frac{1}{2} \quad \text{with } \frac{1}{q_j} < \frac{l_j}{2} \text{ if } l_j \neq 0 \quad \text{and} \quad \frac{1}{q_j} < \sigma_4 \text{ if } l_j = 0, \tag{4-218}$$

where $0 < \sigma_4 \ll 1$ is an arbitrary universal constant. Apply Hölder and two-dimensional Sobolev embeddings

$$\begin{aligned}\int |\nabla^N v||\nabla^{l_a} v||\nabla^{l_b} v||\nabla^{l_c} w| &\leq \|v\|_{H^N} \|\nabla^{l_a} v\|_{L^{q_a}} \|\nabla^{l_b} v\|_{L^{q_b}} \|\nabla^{l_c} w\|_{L^{q_c}} \\ &\lesssim \|v\|_{H^N} \prod_{j=a,b} \|v\|_{H^{2(\frac{1}{2}-\frac{1}{q_j})+l_j}} \|w\|_{H^{2(\frac{1}{2}-\frac{1}{q_c})+l_c}}.\end{aligned} \tag{4-219}$$

Due to the choice in (4-218), the final three norms of (4-219) may be interpolated strictly between H^N and H^1, or strictly between H^1 and L^2, if $l_j = 0$. We are guaranteed a factor in $\|w\|_{H^1}$,

$$\text{(4-219)} \lesssim \begin{cases} \|u\|_{H^N}^2 \|u\|_{H^1}^{2-C(l_c)} \|w\|_{H^1}^{C(l_c)} & \text{if all the } l_j \text{ are nonzero,} \\ \|u\|_{H^N}^{2+C(\sigma_4)} \|u\|_{H^1}^{2-NC(\sigma_4)-C(l_c)} \|w\|_{H^1}^{C(l_c)} & \text{if some } l_j \text{ is zero.} \end{cases} \tag{4-220}$$

For $m > 0$ sufficiently small (relative to σ_4), there is a spare factor of $e^{-\frac{\sigma_5}{b}}$, for some $\sigma_5 > 0$. This proves the bound for the first term on the left in (4-213).

2. Choose $r_1, r_2, r_3 > 0$ and $q_1, q_2, q_3 > 0$ according to the rules of (4-218). Then,

$$\int |\nabla^{k_1} v||\nabla^{k_2} v||\nabla^{k_3} v||\nabla^{l_1} v||\nabla^{l_2} v||\nabla^{l_3} v| \lesssim \prod_{j=1,2,3} \|\nabla^{k_j} v\|_{L^{r_j}} \|\nabla^{l_j} v\|_{L^{q_j}}. \tag{4-221}$$

Recall that at least one factor of v in (4-221) is in fact w. Continue with two-dimensional Sobolev embeddings and interpolation as we did at Equation (4-219). This proves the bound for the second term on the left in (4-213).

3. Apply Hölder and two-dimensional Sobolev to get

$$\begin{aligned}\int \left|\nabla^{N-1} w\right|\left|\nabla^{N-1} v\right||\nabla v|^2 &\leq \left\|\nabla^{N-1} w\right\|_{L^4}\left\|\nabla^{N-1} v\right\|_{L^4}\|\nabla v\|_{L^4}^2 \\ &\lesssim \|w\|_{H^{N-\frac{1}{2}}}\|v\|_{H^{N-\frac{1}{2}}}\|v\|_{H^{\frac{3}{2}}}^2, \end{aligned} \tag{4-222}$$

$$\begin{aligned}\int \left|\nabla^{N-1} w\right|\left|\nabla^{N-1} v\right||v|^4 &\leq \left\|\nabla^{N-1} w\right\|_{L^4}\left\|\nabla^{N-1} v\right\|_{L^4}\|v\|_{L^8}^4 \\ &\lesssim \|w\|_{H^{N-\frac{1}{2}}}\|v\|_{H^{N-\frac{1}{2}}}\|v\|_{H^{\frac{3}{4}}}^4. \end{aligned} \tag{4-223}$$

The bound for the third term on the left-hand side of (4-213) follows from interpolation. □

Lemma 4.9 (H^N energy identity). *Denote the N-th order energy by*

$$E_N(u)=\int\left|\nabla^N u\right|^2-\left(2\int\left|\nabla^{N-1} u\right|^2|u|^2+\operatorname{Re}\int(\nabla^{N-1}\bar{u})^2 u^2\right). \tag{4-224}$$

Then

$$\begin{aligned}\frac{1}{C}\left|\frac{d}{dt} E_N(u)\right| \leq \int|\nabla^N u||\nabla^{l_1} u||\nabla^{l_2} u||\nabla^{l_3} u|+\int|\nabla^{k_1} u||\nabla^{k_2} u||\nabla^{k_3} u||\nabla^{l_1} u||\nabla^{l_2} u||\nabla^{l_3} u| \\ +\int|\nabla^{N-1} u|^2\left(|\nabla u|^2+|u|^4\right), \end{aligned} \tag{4-225}$$

where the right-hand side is implicitly summed over $N-1 \geq l_1 \geq l_2 \geq l_3 \geq 0$ with $l_1+l_2+l_3=N$ and $k_1 \geq k_2 \geq k_3 \geq 0$ with $k_1+k_2+k_3=N-2$.

Proof. We refer to the right-hand side of (4-225) as error terms of type I, II, and III respectively. By direct calculation,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\int\left|\nabla^N u\right|^2\right) &= -\operatorname{Im}\int \nabla^N(\Delta u+u|u|^2)\,\nabla^N\bar{u} \\ &= -2\operatorname{Im}\int \nabla(\nabla^{N-1}u\,|u|^2)\,\nabla^N\bar{u}-\operatorname{Im}\int\nabla(\nabla^{N-1}\bar{u}u^2)\,\nabla^N\bar{u} \\ &\qquad + \text{terms of the form } \int \nabla^{N-2}(\nabla u\nabla u\, u)\,\nabla^N\bar{u}. \end{aligned} \tag{4-226}$$

The final terms here are errors of type I. Regarding the first term on the right in (4-226),

$$\begin{aligned}-2\operatorname{Im}\int\nabla(\nabla^{N-1}u\,|u|^2)\nabla^N\bar{u} &= 2\operatorname{Im}\int\nabla^{N-1}u\,|u|^2\,\nabla^{N-1}\Delta\bar{u} \\ &= \int\frac{d}{dt}\left(\left|\nabla^{N-1}u\right|^2\right)|u|^2-2\operatorname{Im}\int\nabla^{N-1}u\,|u|^2\,\nabla^{N-1}(\bar{u}\,|u|^2). \end{aligned} \tag{4-227}$$

Recognize the last term of (4-227) as error of type II and III. Regarding the other term,

$$\int\frac{d}{dt}\left(\left|\nabla^{N-1}u\right|^2\right)|u|^2=\frac{d}{dt}\left(\int\left|\nabla^{N-1}u\right|^2|u|^2\right)+2\operatorname{Im}\int\left|\nabla^{N-1}u\right|^2\left(\Delta u+u\,|u|^2\right)\bar{u}. \tag{4-228}$$

After integration by parts, we recognize the final term of (4-228) as being of type I and III. We have shown that

$$-2\,\mathrm{Im}\int \nabla\big(\nabla^{N-1}u\,|u|^2\big)\,\nabla^N\bar{u} = \frac{1}{2}\frac{d}{dt}\left(\int \left|\nabla^{N-1}u\right|^2|u|^2\right),$$

up to error terms. It is virtually the same calculation to show that,

$$-\,\mathrm{Im}\int \nabla\big(\nabla^{N-1}\bar{u}u^2\big)\nabla^N\bar{u} = \frac{1}{2}\frac{d}{dt}\left(\mathrm{Re}\int (\nabla^{N-1}\bar{u})^2u^2\right),$$

also up to error terms of type I, II, and III. This completes the proof of (4-225). □

Now we simply combine the previous three Lemmas. Equations (4-207) and (4-212) prove that $E_N \approx \|u\|_{H^N}$. Equations (4-208) and (4-213) control dE_N/dt. Integrate the bound on $dE_N/dt\,E_N$ using Lemma 4.3, with $\sigma_2 < \min(\sigma_5, 2m)$. Choose $m' > 0$ to be any value, $m - \sigma_5/2 < m' < m$. Assuming α^* is sufficiently small (depending on the choice of m'), we have proved statement I2.1:

Lemma 4.10 (controlled growth of H^N). *For all* $t \in [0, T_{\mathrm{hyp}})$,

$$\|u(t)\|_{H^N(\mathbb{R}^N)} < \frac{e^{\frac{m'}{b(t)}}}{\lambda^N(t)}. \tag{4-229}$$

Behavior away from both infinity and the singularity. In this section we concentrate on the interface between the singular set and the truly N-dimensional region that contains the origin. On this interface, away from $r = 0$, the dynamics remains essentially two-dimensional and L^2-critical

Lemma 4.11 (Two-dimensional endpoint Sobolev control away from the singularity). *If* $\sigma_6 > 0$ *and* φ *is a smooth cutoff function compactly supported away from both the singular set and the origin*,

$$\|\varphi u(t)\|_{L^\infty(\mathbb{R}^N)} \le C(\sigma_6,\varphi)e^{+\frac{\sigma_5}{b(t)}}\left(\|\tilde{u}(t)\|_{H^1(\mathbb{R}^N)} + \frac{\Gamma_{b(t)}^{\frac{1}{2}}}{\lambda(t)}\right). \tag{4-230}$$

(This is a two-dimensional type of estimate due to the support of φ.)

The key feature of Lemma 4.11 is that it lets us avoid the Sobolev embedding $H^{1+\epsilon}(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$. At the order of the blowup parameter λ, Equation (4-230) is consistent with scaling. The analogue in the case of radial symmetry, given in [Raphaël 2006; Raphaël and Szeftel 2009], is Strauss's radial embedding.

Proof of Lemma 4.11. We adapt an argument of Brezis and Gallouët. Our estimate is for a fixed time $t \in [0, T_{\mathrm{hyp}})$. Choose

$$R = \|\tilde{u}(t)\|_{H^1} + \frac{\Gamma_{b(t)}^{\frac{1}{2}}}{\lambda(t)} \gg 0.$$

Consider $v = \varphi u$ as a compactly supported function of two variables, and partition phase space as follows:

$$|v| \le \|\hat{v}\|_{L^1(\mathbb{R}^2)} = \int_{|\xi|\le R} |\hat{v}(\xi)|\,d\xi + \int_{|\xi|>R} |\hat{v}(\xi)|\,d\xi$$

Let $\langle\xi\rangle$ denote $\sqrt{1+|\xi|^2}$ and rewrite the low frequencies:

$$\int_{|\xi|\le R} |\hat{v}|\, d\xi = \int_{|\xi|\le R} \big(\langle\xi\rangle^{\frac{1}{2}} |\hat{v}|^{\frac{1}{2}}\big)\big(|\hat{v}|^{\frac{1}{2}}\big)\big(\langle\xi\rangle^{-\frac{1}{2}}\big)\, d\xi \le \|v\|_{H^1}^{\frac{1}{2}} \|v\|_{L^2}^{\frac{1}{2}} \left(\int_{|\xi|\le R} \frac{1}{\langle\xi\rangle}\, d\xi\right)^{\frac{1}{2}}, \tag{4-231}$$

Estimate the final integral of (4-231) as $\int_{|\xi|\le R} \frac{1}{\langle\xi\rangle}\, d\xi \le \int_0^R \frac{1}{\rho}\rho\, d\rho = R$. Apply a similar argument for high frequencies, with parameter $\nu(\sigma_6, m) > 1$ to be determined:

$$\begin{aligned}\int_{|\xi|>R} |\hat{v}|\, d\xi &= \int_{|\xi|>R} \big(\langle\xi\rangle^{\nu} |\hat{v}|\big) \frac{1}{\langle\xi\rangle^{\nu}}\, d\xi \lesssim \|v\|_{H^\nu} \left(\int_R^{+\infty} \frac{1}{\langle\rho\rangle^{2\nu}} \rho\, d\rho\right)^{\frac{1}{2}} \\ &\le \frac{1}{2(\nu-1)} \|v\|_{H^\nu}\ \frac{1}{R^{\nu-1}} \lesssim \frac{1}{2(\nu-1)} \big(\|v\|_{H^1}^{2-\nu} R^{\nu-1}\big)\left(\frac{\|v\|_{H^2}^{\nu-1}}{R^{2(\nu-1)}}\right).\end{aligned} \tag{4-232}$$

Due to hypothesis H2.1 and the Γ_b-estimate (3-103), the final term of (4-232) is bounded by $e^{+\frac{\sigma_6}{b(t)}}$ for any choice of $\nu > 1$ sufficiently small. □

Definition 4.12 (cutoffs to cover $\operatorname{Supp} \nabla\chi$). Fix $N+4$ smooth cylindrically symmetric cutoff functions $\psi^{(0)}$, $\psi^{(\frac{1}{2})}$, $\psi^{(1)}$, $\varphi^{(N-\frac{1}{2})}$, $\varphi^{(N-1)}, \dots, \varphi^{(\frac{N}{2})}$, $\varphi^{(\frac{N}{2}-\frac{1}{2})}$ with the following properties:

1. *They cover the support of* $\nabla\chi$: Each function is 1 on $\big\{\frac{1}{3} < |(r,z)-(1,0)| < \frac{2}{3}\big\}$.
2. *The tails do not overlap:* The support of each cutoff is contained where the previous cutoff is 1.
3. *They are supported away from both the singularity and the origin:* The largest support, that of $\psi^{(0)}$, is contained in $\big\{\frac{1}{7} < |(r,z)-(1,0)| < \frac{6}{7}\big\}$.

Lemma 4.13 (annular $H^{\frac{1}{2}}$ control: the crucial first step). *For all $t \in [0, T_{\mathrm{hyp}})$,*

$$\big\|\psi^{(\frac{1}{2})} u\big\|_{H^{\frac{1}{2}}} \lesssim \frac{1}{\lambda^{C(\alpha^*)}(t)}, \tag{4-233}$$

where $C(\alpha^*) \to 0$ as $\alpha^* \to 0$.

This is the first proof that any behavior better than scaling extends beyond the support of hypotheses H2.2 and H2.3.

Remark 4.14 (analogue in [Raphaël 2006; Raphaël and Szeftel 2009]). In radial cases, one proves Lemma 4.13 for H^ν, $\nu < \frac{1}{2}$. The subsequent $H^{\frac{1}{2}}$ bound — see, for example, [Raphaël 2006, Lemma 10] — should be seen as comparable to the forthcoming Lemma 4.17.

Proof of Lemma 4.13. By direct calculation,

$$\frac{1}{2}\frac{d}{dt}\big\|\psi^{(\frac{1}{2})} u\big\|_{\dot{H}^{\frac{1}{2}}}^2 = \operatorname{Im}\left(\int D^{\frac{1}{2}}\big(u\Delta\psi^{(\frac{1}{2})} + 2\nabla\psi^{(\frac{1}{2})}\cdot\nabla u - \psi^{(\frac{1}{2})} u\,|u|^2\big)\, D^{\frac{1}{2}}\big(\psi^{(\frac{1}{2})}\bar{u}\big)\right). \tag{4-234}$$

Estimate the first and second terms on the right in (4-234):

$$\begin{gathered}\big\|D^{\frac{1}{2}}\big(u\Delta\psi^{(\frac{1}{2})}\big)\big\|_{L^2}\big\|\psi^{(\frac{1}{2})}u\big\|_{H^{\frac{1}{2}}} \le C\big(\psi^{(\frac{1}{2})}\big)\big\|\psi^{(0)}u\big\|_{H^{\frac{1}{2}}}\big\|\psi^{(\frac{1}{2})}u\big\|_{H^{\frac{1}{2}}}, \\ \big\|\nabla\psi^{(\frac{1}{2})}\cdot\nabla u\big\|_{L^2}\big\|\psi^{(\frac{1}{2})}u\big\|_{H^1} \le C\big(\psi^{(\frac{1}{2})}\big)\big\|\psi^{(0)}u\big\|_{H^1}^2.\end{gathered}$$

The nonlinear term of (4-234) does not enjoy any real-valued cancellations, as the operator D does not have an exact Leibniz property. Apply standard commutation estimates:

$$\begin{aligned}\big\|D^{\frac{1}{2}}\big(\psi^{(\frac{1}{2})}u\,|u|^2\big)\big\|_{L^2} &\lesssim \big\|\psi^{(\frac{1}{2})}u\big\|_{H^{\frac{1}{2}}}\big\|\psi^{(0)}u\big\|^2_{L^\infty(\mathbb{R}^2)} + \big\|\psi^{(\frac{1}{2})}u\big\|_{L^4}\big\|\psi^{(0)}u\big\|_{W^{\frac{1}{2},4}}\big\|\psi^{(0)}u\big\|_{L^\infty(\mathbb{R}^2)}\\ &\lesssim \big\|\psi^{(\frac{1}{2})}u\big\|_{H^{\frac{1}{2}}}\big(\big\|\psi^{(0)}u\big\|^2_{L^\infty(\mathbb{R}^2)} + \big\|\psi^{(0)}u\big\|_{H^1}\big\|\psi^{(0)}u\big\|_{L^\infty(\mathbb{R}^2)}\big).\end{aligned} \tag{4-235}$$

From support away from the singularity, $\psi^{(0)}u = \psi^{(0)}\tilde{u}$, and we may apply the endpoint estimate of Lemma 4.11. Denote $\big\|\psi^{(\frac{1}{2})}u(t)\big\|_{\dot{H}^{\frac{1}{2}}}$ by f. We have the simple ODE

$$\frac{1}{2}\frac{d}{dt}(f^2) \le C\big(\psi^{(\frac{1}{2})}\big)\big(f\,\|\tilde{u}(t)\|^{\frac{1}{2}}_{H^1}\,\|\tilde{u}(t)\|^{\frac{1}{2}}_{L^2} + \|\tilde{u}(t)\|^2_{H^1}\big) + C\big(\sigma_6, \psi^{(0)}\big)\,f^2 e^{+\frac{\sigma_6}{b(t)}}\left(\|\tilde{u}(t)\|^2_{H^1} + \frac{\Gamma_{b(t)}}{\lambda^2(t)}\right).$$

The final term is dominant. After integration, by Corollary 4.4,

$$\big\|\psi^{(\frac{1}{2})}u(t)\big\|_{H^{\frac{1}{2}}} \lesssim e^{\left(C(\alpha^*)\,C(\sigma_6,\psi^{(0)})e^{+\frac{\sigma_6}{b(t)}}\right)}. \tag{4-236}$$

To complete the proof, choose $\sigma_6 = \frac{\pi}{10}$ and recall the log-log rate H1.3. □

Remark 4.15 (justification for Lemma 4.11). The open nature of hypothesis H1.3 is an essential feature of any modulation argument. It is for this reason that we must be free to choose σ_6. The standard Brezis–Gallouët estimate, $\|v\|_{L^\infty(\mathbb{R}^2)} \lesssim \|v\|_{H^1}\sqrt{\log(\|v\|_{H^2})}$, would not suffice to prove Lemma 4.13.

We now reformulate the calculation of (4-234) for repeated application.

Lemma 4.16 (standard Gronwall argument). *Let ψ^A be supported where $\psi^B \equiv 1$, let I be any subinterval of* $[0, T_{\text{hyp}})$, *and let* $\nu \ge 0$. *Then*

$$\big\|\psi^A u\big\|_{L^\infty_I H^\nu} \le C(\psi^A)\big(\big\|\psi^B u_0\big\|_{H^\nu} + |I| + \big\|\psi^B u\big\|_{L^2_I H^{\nu+\frac{1}{2}}} + \big\|\psi^A u\,|u|^2\big\|_{L^1_I H^\nu}\big). \tag{4-237}$$

Lemma 4.17 (annular H^1 control: propagation of Lemma 4.13). *There exists* $\sigma_7 > 0$, *universal for all* $m > 0$ *sufficiently small*, *such that for*, *all* $t \in [0, T_{\text{hyp}})$,

$$\big\|\psi^{(1)}u(t)\big\|_{H^1} < C(\alpha^*)\frac{e^{-\frac{\sigma_7}{b(t)}}}{\lambda^{\frac{1}{2}}(t)}, \tag{4-238}$$

where $C(\alpha^*) \to 0$ *as* $\alpha^* \to 0$.

Proof. Apply (4-237) for $\nu = 1$, $I = [0, t < T_{\text{hyp}}]$, $\psi^A = \psi^{(1)}$, and $\psi^B = \psi^{(\frac{1}{2})}$. Note that $\psi^{(1)}u = \psi^{(1)}\tilde{u}$. Through interpolation and hypotheses H1.2 and H2.1,

$$\big\|\psi^{(1)}u\big\|_{L^2_I H^{1+\frac{1}{2}}} \lesssim \left(\int \|\tilde{u}\|^{2-\frac{1}{N-1}}_{H^1}\,\|\tilde{u}\|^{\frac{1}{N-1}}_{H^N}\right)^{\frac{1}{2}} \lesssim \left(\int e^{-(\frac{1}{4}-\frac{m}{N-1})\frac{1}{b}}\frac{1}{\lambda^N}\right)^{\frac{1}{2}}. \tag{4-239}$$

Assuming $m > 0$ is sufficiently small, apply integrability Lemma 4.3 for $\sigma_2 > 0$, also sufficiently small. Regarding the final term of (4-237), apply Hölder, two-dimensional Sobolev embedding, and interpolate:

$$\begin{aligned}\|\psi^{(1)}u\,|u|^2\|_{H^1} &\lesssim \|\nabla(\psi^{(\frac12)}u)\,(\psi^{(\frac12)}u)^2\|_{L^2} \lesssim \|\nabla(\psi^{(\frac12)}u)\|_{L^4}\|\psi^{(\frac12)}u\|_{L^8}^2 \lesssim \|\psi^{(\frac12)}u\|_{H^{\frac32}}\|\psi^{(\frac12)}u\|_{H^{\frac34}}^2\\ &\lesssim \|u\|_{H^N}^{\frac32\frac{1}{N-\frac12}}\|\psi^{(\frac12)}u\|_{H^{\frac12}}^{3-\frac32\frac{1}{N-\frac12}} \lesssim \frac{1}{\lambda^{\frac32\frac{N}{N-\frac12}+C(\alpha^*)}(t)},\end{aligned}\tag{4-240}$$

where the final inequality is due to hypothesis H2.1 and Lemma 4.13. The final exponent is less than 2 for $\alpha^*>0$ sufficiently small and $N\geq 3$. Apply Lemma 4.1. □

Remark 4.18 (scheme for the remainder of Section 4). The proof of Lemma 4.17 may be repeated, with a shrunken cutoff and $H^{\frac32}$ in place of H^1. However, due to the new version of Equation (4-239), iteration to higher norms will not yield more than the same $\frac12$-derivative improvement over scaling.

Instead, we switch direction. Starting with I2.1, at each stage the previous iterate will give progressively better control on the equivalent of (4-239). Lemma 4.17 will be used to help control the equivalent of Equation (4-240).

Lemma 4.19 (Moser-type product estimate). *Let $v\in H^{\nu+\frac12}(\mathbb{R}^d)$ for some $\nu\geq\frac{d}{2}-\frac12$, not necessarily an integer. Then,*

$$\|v^3\|_{H^\nu}\lesssim\|v\|_{H^{\nu+\frac12}}\|v\|_{H^{\frac{d}{2}}}^2.\tag{4-241}$$

Lemma 4.20 (I2.2 and I2.3 on the support of $\nabla\chi$). *For any $t\in[0,T_{\text{hyp}})$ and any half-integer $\frac12\leq\kappa<\frac{N}{2}$, we have*

$$\|\varphi^{(N-\kappa)}u\|_{H^{N-\kappa}}<C(\alpha^*)\frac{e^{(1+\kappa)\frac{m'}{b(t)}}}{\lambda^{N-2\kappa}},\tag{4-242}$$

$$\|\varphi^{(\frac{N}{2})}u(t)\|_{H^{\frac{N}{2}}}<C(\alpha^*)e^{+\frac{2m'+\pi}{b(t)}},\tag{4-243}$$

$$\|\varphi^{(\frac{N}{2}-\frac12)}u\|_{H^{\frac{N}{2}-\frac12}}<C(\alpha^*)(\alpha^*)^{\frac15},\tag{4-244}$$

where in each case $C(\alpha^*)\to 0$ as $\alpha^*\to 0$.

Proof. We prove (4-242) by induction in κ. The base case $\kappa=0$ is Lemma 4.10. Hypothesize that (4-242) holds for $\kappa-\frac12$, some $\kappa\geq\frac12$. Set $\nu=N-\kappa$ and apply the standard Gronwall argument for $I=[0,t<T_{\text{hyp}}]$, $\psi^A=\phi^{(\nu)}$ and $\psi^B=\phi^{(\nu+\frac12)}$,

$$\|\varphi^{(\nu)}u\|_{H^\nu}\lesssim\|\chi_0u_0\|_{H^\nu}+\|\varphi^{(\nu+\frac12)}u\|_{L_t^2H^{\nu+\frac12}}+\|\varphi^{(\nu)}u\,|u|^2\|_{L_t^1H^\nu}.\tag{4-245}$$

Apply our induction hypothesis to the second term on the right in (4-245):

$$\|\varphi^{(\nu+\frac12)}u\|_{L_t^2H^{\nu+\frac12}}\lesssim\left(\int_I\left(\frac{e^{(1+(\kappa-\frac12))\frac{m'}{b(\tau)}}}{\lambda^{N-2(\kappa-\frac12)}(\tau)}\right)^2d\tau\right)^{\frac12}\lesssim\frac{e^{(1+\kappa)\frac{m'}{b(t)}}}{\lambda^{N-2\kappa}(t)}\big(e^{\frac{\sigma_2-m'}{b(t)}}\big)^{\frac12}.\tag{4-246}$$

where, since $\kappa<\frac{N}{2}$, we applied Lemma 4.3 for some $\sigma_2<m'$. Examine the final term of (4-245). Note that, $\varphi^{(\nu)}u=\varphi^{(\nu)}\big(\varphi^{(\nu+\frac12)}u\big)$. Recall Remark 4.7, apply Lemma 4.19 in the two-dimensional case, and

inject both the induction hypothesis and the H^1 control of Lemma 4.17,

$$\left\|\varphi^{(\nu)}u\,|u|^2\right\|_{H^\nu} \lesssim \left\|\varphi^{(\nu+\frac{1}{2})}u\right\|_{H^{\nu+\frac{1}{2}}}\left\|\varphi^{(\nu+\frac{1}{2})}u\right\|_{H^1}^2$$
$$\lesssim \frac{e^{+\frac{(1+(\kappa-\frac{1}{2}))m'}{b}}}{\lambda^{N-2(\kappa-\frac{1}{2})}}\frac{e^{-\frac{2\sigma_7}{b}}}{\lambda} = \frac{e^{\frac{(\text{neg})}{b}}}{\lambda^2}\frac{1}{\lambda^{N-2\kappa}}, \tag{4-247}$$

where we made the assumption that $m > 0$ is sufficiently small relative to σ_7. Finally, apply Lemma 4.3 for some σ_2 less than the negative exponent. This completes the proof of (4-242).

To prove (4-243) let $\kappa = \frac{N}{2}$. We proceed exactly as above, using (4-242) in place of the induction hypothesis, and applying Corollary 4.4 in place of Lemma 4.3.

To prove (4-244), let $\kappa = \frac{N}{2} + \frac{1}{2}$. We proceed exactly as above using (4-243) in place of the induction hypothesis, and applying Lemma 4.1 in place of Lemma 4.3. □

Improved behavior at infinity. With Lemma 4.20 covering the support of $\nabla\chi$, we prove the corresponding result for χ by similar methods. Note the argument is now in three dimensions.

Proof of I2.2 and I2.3. We revisit the proof of the standard Gronwall argument. Let $I = [0, t < T_{\text{hyp}}]$, $\nu \geq 0$, and set $v = \chi u$. With (1-1), we get

$$i v_t + \Delta v + v\,|v|^2 = u\Delta\chi + 2\nabla\chi\cdot\nabla u + (\chi^2-1)\chi u\,|u|^2\,. \tag{4-248}$$

Note that the terms on the right-hand side of (4-248) are localized to the support of $\nabla\chi$, a region of two-dimensional character where $\varphi^{(\frac{N}{2}-\frac{1}{2})} \equiv 1$. By direct calculation,

$$\frac{1}{2}\left\|\chi u\right\|_{L_I^\infty H^\nu} \leq \left\|\chi u_0\right\|_{H^\nu} + \left\|\chi u\,|\chi u|^2\right\|_{L_I^1 H^\nu}$$
$$+ C(\chi)\big(\left\|\varphi^{(1)}u_0\right\|_{H^\nu} + |I| + \left\|\varphi^{(1)}u\right\|_{L_I^2 H^{\nu+\frac{1}{2}}} + \left\|\varphi^{(1)}u|\varphi^{(1)}u|^2\right\|_{L_I^1 H^\nu}\big). \tag{4-249}$$

Consider $\nu = N-\kappa$ for some $\kappa \in \left[\frac{1}{2}, \frac{N}{2}+\frac{1}{2}\right]$. Due to Definition 4.12, all the conclusions of Lemma 4.20 apply to $\varphi^{(\frac{N}{2}-\frac{1}{2})}u$, which we use in place of an induction hypothesis to control the second line of (4-249), exactly as we did Equation (4-245). These terms will give the largest contribution.

Finally, examine the term nonlinear in χu. Apply the Moser-type estimate of Lemma 4.19, interpolate, and inject hypotheses H2.2,

$$\left\|\chi u\,|\chi u|^2\right\|_{L_I^1 H^{N-\kappa}} \lesssim \left\|\left\|\chi u\right\|_{H^{N-(\kappa-\frac{1}{2})}}\left\|\chi u\right\|_{H^{\frac{N}{2}}}^2\right\|_{L_I^1}$$
$$\lesssim \begin{cases} \displaystyle\int_I \frac{e^{\left(1+(\kappa-\frac{1}{2})\right)\frac{m}{b(\tau)}}}{\lambda^{N-2(\kappa-\frac{1}{2})}(\tau)} e^{2\left(\frac{2m+\pi}{b(\tau)}\right)}\,d\tau & \text{for } \kappa \leq \frac{N}{2}, \\ \displaystyle\int_I e^{3\left(\frac{2m+\pi}{b(\tau)}\right)}\,d\tau & \text{for } \kappa = \frac{N}{2}+\frac{1}{2}. \end{cases} \tag{4-250}$$

Apply Lemma 4.1 for $\kappa \geq \frac{N}{2}$, Corollary 4.4 for $\kappa = \frac{N}{2} - \frac{1}{2}$, and Lemma 4.3 for $\frac{N}{2} - 1 \geq \kappa \geq \frac{1}{2}$. Note that the result of Equation (4-250) is an entire order better than necessary. □

This completes all the deferred proofs necessary to establish Proposition 2.8.

5. Proof of Theorem 1.6

Proof of norm growth (1-12), (1-13). From Proposition 2.8 we have that $T_{\mathrm{hyp}} = T_{\max}$, and from (3-143) we have blowup in finite time. By the failure of local wellposedness we have that $\lambda(t) \to 0$ as $t \to T_{\max}$. Recall the approximate dynamics of λ, Equation (3-140), which with the control on b implies in particular that $|\lambda_s/\lambda| < 1$ on $[s_0, s_{\max})$, which easily integrates to

$$|\log \lambda(s)| \lesssim 1 + s. \tag{5-251}$$

Therefore $s_{\max} = +\infty$. By direct calculation and a change of variable,

$$-\partial_t\big(\lambda^2 \log|\log\lambda|\big) = -\frac{\lambda_s}{\lambda}\log|\log\lambda|\left(2 + \frac{1}{|\log\lambda|\log|\log\lambda|}\right).$$

The approximate dynamics (3-140) gives

$$\frac{b}{2} \le -\frac{\lambda_s}{\lambda} \le 2b,$$

so with the log-log rate H1.3 we have proven that, for some universal constant $C > 0$ and all $t \in [0, T_{\max})$,

$$\frac{1}{C} \le -\partial_t\big(\lambda^2 \log|\log\lambda|\big) \le C. \tag{5-252}$$

For all $t \in [0, T_{\max})$, integrate Equation (5-252). Since λ is very small we can estimate

$$\frac{1}{C}\left(\frac{T_{\max} - t}{\log|\log(T_{\max} - t)|}\right)^{\frac{1}{2}} \le \lambda(t) \le C\left(\frac{T_{\max} - t}{\log|\log(T_{\max} - t)|}\right)^{\frac{1}{2}}. \tag{5-253}$$

We do not prove the exact value of the constant in (1-12); see [Merle and Raphaël 2006, Proposition 6]. Finally, we conclude that (1-13) follows from the log-log relationship H1.3, higher-order norm control H2.1, and from $m > 0$ small. As an aside, recall that $ds/dt = 1/\lambda^2$, so with (5-253) one would conclude

$$\frac{1}{C}\,|\log(T_{\max} - t)| \le s(t) \le C\,|\log(T_{\max} - t)|\,. \tag{5-254}$$

Then from the explicit lower and upper bounds for b in (3-137) and (3-177) we obtain

$$\frac{1}{C\log|\log(T_{\max} - t)|} \le b(t) \le \frac{C}{\log|\log(T_{\max} - t)|}. \qquad \square$$

Proof of stable locus of concentration, (1-9). The preliminary estimate (3-133) implies in particular that

$$\left|\frac{\partial_s(r,z)}{\lambda}\right| < 1 \tag{5-255}$$

on $[s_0, s_1)$. Then by a change of variable, (5-253) and the bound (3-143) on $T_{\max}$,

$$\int_0^{T_{\max}} |\partial_t(r,z)|\,dt < \int_0^{T_{\max}} \frac{1}{\lambda(t)}\,dt < \delta(\alpha^*). \tag{5-256}$$

Equation (1-9) follows from choice of initial data C1.1. $\square$

Proof of regularity away from singular ring, (1-11). Given $R > 0$, define χ_R to be a suitable modification of χ (2-26), equal to 1 for $|(r,z)-(r_{\max},z_{\max})| > R$. Choose some $t(R) \in [0, T_{\max})$ such that

$$A(t)\lambda(t) + |(r(t),z(t)) - (r_{\max},z_{\max})| \ll R \quad \text{for all } t \in [t(R), T_{\max}), \tag{5-257}$$

and hence $\chi_R u = \chi_R \tilde{u}$ for all $t \in [t(R), T_{\max})$. Let $t_3 \in (t(R), T_{\max}]$ be the largest value such that

$$\left\| \chi_R u(t) \right\|_{H^1} < 2 \left\| \chi_R u(t(R)) \right\|_{H^1} \quad \text{for all } t \in [t(R), t_3). \tag{5-258}$$

This choice of $t_3 > t(R)$ is possible since $u(t)$ is strongly continuous in H^1 at time $t(R) < T_{\max}$. With interpolation, (5-258) replaces the bootstrap hypotheses H2.2 and H2.3. Repeating the arguments of Section 4 proves that $t_3 = T_{\max}$ and

$$\|\tilde{u}(t)\|_{H^1(|(r,z)-(r_{\max},z_{\max})|>R)} < C(R) \quad \text{for all } t \in [0, T_{\max}). \tag{5-259}$$

This yields (1-11). □

Proof of mass concentration, (1-10). Let $R > 0$. To begin we will prove there exists a residual profile in L^2 away from the singular ring:

$$\tilde{u}(t) \to u^* \quad \text{in } L^2_x\big(|(r,z)-(r_{\max},z_{\max})| \geq R\big) \quad \text{as } t \to T_{\max}. \tag{5-260}$$

Then to establish (1-10) we will prove

$$u^* \in L^2(\mathbb{R}^N) \quad \text{and} \quad \int |u^*|^2 = \lim_{t \to T_{\max}} \int |\tilde{u}(t)|^2. \tag{5-261}$$

Let $\epsilon_0 > 0$ be arbitrary. Due to (3-178), we may choose $t(R) < T_{\max}$ such that

$$T_{\max} - t(R) < \frac{\epsilon_0}{1 + C(R/4)} \quad \text{and} \quad \int_{t(R)}^{T_{\max}} \int |\nabla \tilde{u}|^2 \, dx \, dt < \epsilon_0, \tag{5-262}$$

where $C(R/4)$ is the constant from Equation (5-259). We may assume that, for $t \in [t(R), T_{\max})$, $u(t) = \tilde{u}$ on $\big\{|(r,z)-(r_{\max},z_{\max})| > R/4\big\}$. Let $\tau > 0$ be a parameter to be fixed later, and define

$$v^\tau(t,x) = u(t+\tau,x) - u(t,x). \tag{5-263}$$

Since $t(R) < T_{\max}$, $u(t)$ is strongly continuous in L^2 at time $t(R)$. Thus, there exists τ_0 such that

$$\int \left| v^\tau(t(R)) \right|^2 \, dx < \epsilon_0 \quad \text{for all } \tau \in [0, \tau_0]. \tag{5-264}$$

Denote a smooth cutoff function ϕ_R analogous to ϕ_∞ (see (3-159) and (3-160)):

$$\phi_R(r,z) = \phi_\infty^4 \left(\frac{(r,z) - (r_{\max},z_{\max})}{R} \right). \tag{5-265}$$

By direct calculation,

$$\tfrac{1}{2}\partial_t \left(\int \phi_R |v^\tau|^2 \right) = \operatorname{Im}\left(\int \nabla\phi_R \cdot \nabla v^\tau \overline{v^\tau} \, dx \right) + \operatorname{Im}\left(\int \phi_R v^\tau \overline{\big(u\,|u|^2(t+\tau) - u\,|u|^2(t)\big)} \, dx \right). \tag{5-266}$$

Regarding the first term on the right in (5-266), from Hölder and our choice of $t(R)$ we have

$$\int_{t(R)}^{T_{\max}} \left| \operatorname{Im}\left(\int \nabla \phi_R \cdot \nabla v^\tau \overline{v^\tau}\, dx \right) dt \right| \le C \left(\int_{t(R)}^{T_{\max}} 1^2\, dt \right)^{\frac{1}{2}} \epsilon_0^{\frac{1}{2}} < C\epsilon_0. \tag{5-267}$$

Regarding the second term on the right in (5-266), by homogeneity,

$$\left| \phi_R v^\tau \overline{\left(u\,|u|^2\,(t+\tau) - u\,|u|^2\,(t) \right)} \right| \le C \left(\left| \phi_R^{\frac{1}{4}} u(t+\tau) \right|^4 + \left| \phi_R^{\frac{1}{4}} u(t) \right|^4 \right). \tag{5-268}$$

Then, as we did in proving estimate (3-115), apply the Sobolev embedding $H^{\frac{N}{4}} \hookrightarrow L^4(\mathbb{R}^N)$ and interpolate, obtaining

$$\int \left| \phi_R^{\frac{1}{4}} u \right|^4 \le C \left\| \phi_R^{\frac{1}{4}} u \right\|_{H^{\frac{N}{2}-1}}^2 \left\| \phi_R^{\frac{1}{4}} u \right\|_{H^1}^2.$$

By (5-259), the uniform control of $H^{\frac{N}{2}-1}$, and our choice of $t(R)$,

$$\int_{t(R)}^{T_{\max}} \left| \phi_R v^\tau \overline{\left(u\,|u|^2\,(t+\tau) - u\,|u|^2\,(t) \right)} \right| dt \le C\epsilon_0. \tag{5-269}$$

Through the integration of (5-266) we have proved that

$$\int \phi_R \left| v^\tau(t) \right|^2 dx < C\epsilon_0 \quad \text{for all } \tau \in [0, \tau_0] \text{ and } t \in [t(R), T_{\max} - \tau). \tag{5-270}$$

This shows that $\tilde{u}$ is Cauchy, which proves (5-260).

We turn to (5-261). Denote the thickness of the toroidal support of the singular profile and radiation by

$$R(t) = A(t)\lambda(t). \tag{5-271}$$

Recall the definition of $A(t)$ in (3-104). By the log-log rate H1.3, we have $A(t) \approx |\log(T_{\max} - t)|^C$; in particular, $R(t) \to 0$ with the bound $R(t) \le (T_{\max} - t)^{\frac{1}{2}-\delta}$. Now consider

$$\phi_{R(t),\tau} = \phi_\infty^4 \left(\frac{(r,z) - (r(\tau), z(\tau))}{R(t)} \right),$$

a family of time-variable cutoffs similar to $\phi_{R(t)}$. For fixed time $t < T_{\max}$ we calculate directly that

$$\begin{aligned} &\frac{1}{2}\partial_\tau \left(\int \phi_{R(t),\tau} \,|u(\tau)|^2\, dx \right) \\ &= \frac{1}{R(t)} \operatorname{Im}\left(\int \nabla_x \phi_{R(t),\tau} \cdot \nabla_x u(\tau) \overline{u(\tau)}\, dx \right) - \frac{1}{2R(t)} \int \partial_\tau (r(\tau), z(\tau)) \cdot \nabla_x \phi_{R(t),\tau} \,|u(\tau)|^2\, dx, \end{aligned} \tag{5-272}$$

where we use $\nabla_x \phi_{R(t),\tau}$ to denote

$$\nabla_y \phi_\infty^4(y) \Big|_{y = \frac{(r,z) - (r(\tau), z(\tau))}{R(t)}}.$$

Regarding the first term on the right in (5-272),

$$\left|\frac{1}{R(t)}\operatorname{Im}\left(\int \nabla_x\phi_{R(t),\tau}\cdot\nabla_x u(\tau)\overline{u(\tau)}\,dx\right)\right| \lesssim \frac{1}{R(t)}\,\|u(\tau)\|_{H^1} \lesssim \frac{1}{A(t)\lambda(t)\lambda(\tau)}.$$

Regarding the last term of (5-272), apply the preliminary estimate (3-133),

$$\left|\frac{1}{2R(t)}\int \partial_\tau(r(\tau),z(\tau))\cdot\nabla_x\phi_{R(t),\tau}\,|u(\tau)|^2\,dx\right| \lesssim \frac{1}{A(t)\lambda(t)\lambda(\tau)}\int |u_0|^2.$$

Integrate (5-272) in τ, and apply the bounds for A and λ to obtain

$$\begin{aligned}
&\left|\int \phi_{R(t),T_{\max}}\,|u^*|^2\,dx - \int \phi_{R(t),t}\,|u(t)|^2\,dx\right| \\
&\qquad\le C\frac{1}{A(t)\lambda(t)}\int_t^{T_{\max}}\frac{1}{\lambda(\tau)}\,d\tau \\
&\qquad\le \frac{C}{|\log(T_{\max}-t)|^C}\left(\frac{\log|\log(T_{\max}-t)|}{T_{\max}-t}\right)^{\frac{1}{2}}\int_t^{T_{\max}}\left(\frac{\log|\log(T_{\max}-\tau)|}{T_{\max}-\tau}\right)^{\frac{1}{2}}d\tau \qquad (5\text{-}273)\\
&\qquad\le \frac{1}{|\log(T_{\max}-t)|^{\frac{C}{2}}}.
\end{aligned}$$

The final inequality relied upon $T_{\max}-t < T_{\max} < \alpha^*$, Equation (3-143), both to approximate the integral and then to approximate $C\log|\log(T_{\max}-t)| < |\log(T_{\max}-t)|^{\frac{C}{2}}$. Taking the limit $t\to T_{\max}$ we see that

$$\int |u^*|^2 = \lim_{t\to T_{\max}}\int \phi_{R(t),t}\,|u(t)|^2. \qquad (5\text{-}274)$$

From the definition of $A(t)$ and (3-110) we can bound $\int(1-\phi_{R(t)})\,|\tilde u(t)|^2$ to prove that

$$\lim_{t\to T_{\max}}\int \phi_{R(t),t}\,|u(t)|^2 = \lim_{t\to T_{\max}}\int \phi_{R(t),t}\,|\tilde u(t)|^2 = \lim_{t\to T_{\max}}\int |\tilde u(t)|^2;$$

this shows that the limit in (5-274) exists and establishes (5-261). This completes the proof of (1-10). □

Remark 5.1 (consistency with $u^*\notin H^1$). By repeating the proof of I2.3, we expect that following the proof of (5-260) it could be shown that $\tilde u(t)\to u^*$ in $H^1\big(|(r,z)-(r_{\max},z_{\max})|\ge R\big)$. Nevertheless, an attempt to prove a version of (5-261) in H^1 will fail. Indeed, the last term in (5-272) would require a bound for $|\nabla u(\tau)|$ on the support of $\nabla\phi$, with nothing to take the role of mass conservation.

Appendix

Proof that $\mathcal{P}$ is nonempty. Choose $r_0=1,\ z_0=0,\ b_0>0$ small enough to satisfy (2-29), and λ_0 in the range of C1.3. Fix some smooth real-valued radially symmetric function $f(y)$, with support in $|y|\le 2$ and such that $\|f\|_{H^N(\mathbb{R}^2)}\le 1,\ (f,Q)=1$ and, for any $\nu\in\mathbb{C}$ to be determined, such that $\epsilon_0(y)=\nu f(y)$ satisfies the orthogonality conditions (2-30). One can explicitly calculate such an f from $\widetilde{Q}_{b_0}$. With $\gamma_0=0$, we now find $\nu=\nu(b_0)$ to satisfy C1.4 and the small-mass requirement of C1.2.

By the choice of λ_0, we have $|(r,z)-(1,0)|<\frac{1}{3}$ on the support of $\widetilde{Q}_{b_0}(y)$, which includes the support of $f(y)$. After a change of variables, we will expand $\mu_{\lambda_0,1}(y)$ as $(1+\lambda_0 y_1)^{N-2}$ so that

$$\begin{aligned}\lambda_0^2|E_0| &= \left|\frac{1}{2}\int \left|\nabla_y(\widetilde{Q}_{b_0}+\nu f)\right|^2 \mu_{\lambda_0,1}(y)\,dy - \frac{1}{4}\int \left|\widetilde{Q}_{b_0}+\nu f\right|^4 \mu_{\lambda_0,1}(y)\,dy\right| \\ &\lesssim \left|\frac{1}{2}\int \left|\nabla_y(\widetilde{Q}_{b_0}+\nu f)\right|^2 dy - \frac{1}{4}\int \left|\widetilde{Q}_{b_0}+\nu f\right|^4 dy\right| + \mathcal{O}(\lambda_0), \end{aligned} \tag{5-275}$$

which is a small correction from the two-dimensional energy. Directly from (1-6) we get

$$\frac{d}{dw}\left(\frac{1}{2}\int \left|\nabla_y(Q+wf)\right|^2 dy - \frac{1}{4}\int |Q+wf|^4\, dy\right)\bigg|_{w=0} = -\operatorname{Re}(f,Q) = -1, \tag{5-276}$$

so the left-hand side does not depend on the imaginary component of ν. By the degenerate energy of $\widetilde{Q}_{b_0}$ we may choose the real part of ν of the order $|\nu|\le \Gamma_{b_0}^{1-C\eta}$ such that $E_0=0$. Note the choice $\nu=0$ is impossible as the energy of $\widetilde{Q}_{b_0}$ alone is too large to satisfy C1.4.

Next we show the momentum requirement of C1.4 is satisfied. Again from the choice of λ_0, the support of $\widetilde{Q}_{b_0}+\nu f$ lies well within $|(r,z)-(1,0)|\le\frac{1}{2}$, a region where $\nabla_x\psi^{(x)}$ is constant; see (2-34). With the radial symmetry of $\widetilde{Q}_b$ and f we have

$$\begin{aligned}\lambda_0 \operatorname{Im}\left(\int \nabla_x\psi^{(x)}\cdot\nabla_x u_0\bar{u}_0\right) &= (1,1)\cdot\operatorname{Im}\left(\int \nabla_y\big(\widetilde{Q}_{b_0}+\nu f\big)\big(\overline{\widetilde{Q}_{b_0}+\nu f}\big)\mu_{\lambda_0,1}(y)\,dy\right) \\ &= 2\operatorname{Im}\int \nu f\,\overline{\widetilde{Q}_{b_0}}\,dy + \mathcal{O}(\lambda_0), \end{aligned}\tag{5-277}$$

and there is a $\mathcal{O}(\lambda_0)$ choice of the imaginary part of ν such that (5-277) is zero. Finally, we note that C1.4 is satisfied,

$$\|\tilde{u}_0\|_{L^2(\mathbb{R}^N)} = |\nu|\left(\int |f(y)|^2\,\mu_{\lambda_0,1}(y)\,dy\right)^{\frac{1}{2}} < \alpha^*. \tag{5-278}$$

The requirements C2.2 and C2.3 are automatic from the support of f. The constant C in C2.1 is due to Lemma 2.2 and the choice of ν. □

Relationship with the classic virial argument. For data $u_0\in H^1$ with finite variance, due to the classic virial identity, a sufficient condition for blowup is

$$\left[\operatorname{Im}\left(\int x\nabla u_0\overline{u_0}\right)\right]^2 > 2\,\|xu_0\|_{L^2}^2\, E(u_0). \tag{5-279}$$

We remark that there exists $u_0\in\mathcal{P}$ for which condition (5-279) fails.

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NON-WEYL RESONANCE ASYMPTOTICS FOR QUANTUM GRAPHS

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We consider the resonances of a quantum graph $\mathscr{G}$ that consists of a compact part with one or more infinite leads attached to it. We discuss the leading term of the asymptotics of the number of resonances of $\mathscr{G}$ in a disc of a large radius. We call $\mathscr{G}$ a *Weyl graph* if the coefficient in front of this leading term coincides with the volume of the compact part of $\mathscr{G}$. We give an explicit topological criterion for a graph to be Weyl. In the final section we analyze a particular example in some detail to explain how the transition from the Weyl to the non-Weyl case occurs.

1. Introduction

Quantum graphs. Let $\mathscr{G}_0$ be a finite compact metric graph. That is, $\mathscr{G}_0$ has finitely many edges and each edge is equipped with coordinates (denoted x) that identify this edge with a bounded interval of the real line. We choose some subset of vertices of $\mathscr{G}_0$, to be called *external vertices*, and attach one or more copies of $[0, \infty)$, to be called *leads*, to each external vertex; the point 0 in a lead is thus identified with the relevant external vertex. We call the thus extended graph $\mathscr{G}$. We assume that $\mathscr{G}$ has no "tadpoles", i.e., no edge starts and ends at the same vertex; this can always be achieved by introducing additional vertices, if necessary. In order to distinguish the edges of $\mathscr{G}_0$ from the leads, we will call the former the *internal edges* of $\mathscr{G}$.

In $L^2(\mathscr{G})$ we consider the self-adjoint operator $H = -d^2/dx^2$ with the continuity condition and the Kirchhoff boundary condition at each vertex of $\mathscr{G}$; see Section 2 for the precise definitions. The metric graph $\mathscr{G}$ equipped with the self-adjoint operator H in $L^2(\mathscr{G})$ is called the *quantum graph*. We refer to the surveys [Kuchment 2004; 2008] for a general exposition of quantum graph theory. Important earlier work on resonances of quantum graphs has been carried out by Kottos and Smilansky [2003] and Kostrykin and Schrader [1999] (see also [Kostrykin and Schrader 2006; Kostrykin et al. 2007]), but their results have little overlap with ours. For more recent progress see [Exner and Lipovský 2010; Davies et al. 2010].

If the set of leads is nonempty, it is easy to show by standard techniques (see [Ong 2006, Lemma 1], for example) that the spectrum of H is $[0, \infty)$. The operator H may have embedded eigenvalues.

Resonances of H. The "classical" definition of resonances is this:

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Definition 1.1. We will say that $k \in \mathbb{C}$, $k \neq 0$, is a *resonance* of H (or, by a slight abuse of terminology, a resonance of $\mathcal{G}$) if there exists a *resonance eigenfunction* $f \in L^2_{\mathrm{loc}}(\mathcal{G})$, $f \not\equiv 0$, which satisfies the equation

$$-f''(x) = k^2 f(x), \quad x \in \mathcal{G}, \tag{1-1}$$

on each edge and lead of $\mathcal{G}$, is continuous on $\mathcal{G}$, satisfies the Kirchhoff boundary condition at each vertex of $\mathcal{G}$ and the *radiation condition*

$$f(x) = f(0)e^{ikx}$$

on each lead of $\mathcal{G}$. We denote the set of all resonances of H by $\mathcal{R}$.

Any solution to (1-1) on a lead $\ell = [0, \infty)$ satisfies

$$f(x) = \gamma_\ell e^{ikx} + \gamma'_\ell e^{-ikx};$$

Definition 1.1 requires that there exists a nonzero solution with all coefficients γ'_ℓ vanishing. It is easy to see that all resonances must satisfy $\operatorname{Im} k \leq 0$; indeed, if k_0 with $\operatorname{Im} k_0 > 0$ is a resonance then the corresponding resonance eigenfunction is in $L^2(\mathcal{G})$, so k_0^2 is an eigenvalue of H, which is impossible since $k_0^2 \notin [0, \infty)$. As we will only be interested in the asymptotics of the number of resonances in large disks, we exclude the case $k = 0$ from further consideration. In the absence of leads, the spectrum of H consists of nonnegative eigenvalues and $k \neq 0$ is a resonance if and only if $k \in \mathbb{R}$ and k^2 is an eigenvalue of H.

It is well known (see [Exner and Lipovský 2007; 2010], for example) that this "classical" definition of a resonance coincides with the definition via exterior complex scaling (see [Aguilar and Combes 1971; Simon 1973; Sjöstrand and Zworski 1991]). In the complex scaling approach, the resonances of H are identified with the eigenvalues of an auxiliary nonselfadjoint operator $H(i\theta)$, $\theta \in (0, \pi)$. The *algebraic multiplicity* of a resonance is then defined as the algebraic multiplicity of the corresponding eigenvalue of $H(i\theta)$. We discuss this in more detail in Section 2, where we show that the multiplicity is independent of θ. In particular, we show (in Theorem 2.3) that any $k \in \mathbb{R}$, $k \neq 0$, is a resonance if and only if k^2 is an eigenvalue of H and in this case the corresponding multiplicities coincide.

We define the *resonance counting function* by

$$N(R) = \#\{k : k \in \mathcal{R},\ |k| \leq R\}, \quad R > 0,$$

with the convention that each resonance is counted with its algebraic multiplicity taken into account. Note that the set $\mathcal{R}$ of resonances is invariant under the symmetry $k \to -\overline{k}$, so this method of counting yields, roughly speaking, twice as many resonances as one would obtain if one imposed an additional condition $\operatorname{Re}(k) \geq 0$. In particular, in the absence of leads, $N(R)$ equals twice the number of eigenvalues $\lambda \neq 0$ of H (counting multiplicities) with $\lambda \leq R^2$.

Main result. This paper is concerned with the asymptotics of the resonance counting function $N(R)$ as $R \to \infty$. We say that $\mathcal{G}$ is a *Weyl graph* if

$$N(R) = \frac{2}{\pi} \operatorname{vol}(\mathcal{G}_0) R + o(R), \quad \text{as } R \to \infty, \tag{1-2}$$

where $\mathrm{vol}(\mathcal{G}_0)$ is the sum of the lengths of the edges of $\mathcal{G}_0$. If there are no leads then H has pure point spectrum, resonances are identified with eigenvalues of H and Weyl's law (1-2) may be proved by Dirichlet–Neumann bracketing. Thus, every compact quantum graph is Weyl in our sense. As we show below, in the presence of leads this may not be the case.

We call an external vertex v of $\mathcal{G}$ *balanced* if the number of leads attached to v equals the number of internal edges attached to v. If v is not balanced, we call it *unbalanced*. Our main result is this:

Theorem 1.2. *One has*

$$N(R) = \frac{2}{\pi} W R + O(1), \quad \text{as } R \to \infty, \tag{1-3}$$

where the coefficient W *satisfies* $0 \le W \le \mathrm{vol}(\mathcal{G}_0)$. *One has* $W = \mathrm{vol}(\mathcal{G}_0)$ *if and only if every external vertex of* $\mathcal{G}$ *is unbalanced.*

This theorem shows, in particular, that as the graph becomes larger and more complex the failure of Weyl's law becomes increasingly likely in an obvious sense.

Discussion. The simplest example of a graph $\mathcal{G}$ with a balanced external vertex occurs when exactly one lead ℓ and exactly one internal edge e meet at a vertex. In this case, one can merge e and ℓ into a new lead; this will not affect the resonances of $\mathcal{G}$ but will reduce $\mathrm{vol}\,\mathcal{G}_0$. This already shows that $\mathcal{G}$ cannot be Weyl. Section 6 discusses the another simple example.

Our proof of Theorem 1.2 consists of two steps. The first step is to identify the set $\mathcal{R}$ of resonances with the set of zeros of $\det A(k)$, where $A(k)$ is a certain analytic matrix-valued function. This identification is straightforward, but it has a subtle aspect: this is to show that the algebraic multiplicity of a resonance coincides with the order of the zero of $\det A(k)$. This is done in Sections 4 and 5 by employing a range of rather standard techniques of spectral theory, including a resolvent identity which involves the Dirichlet-to-Neumann map.

The function $\det A(k)$ turns out to be an exponential polynomial. By a classical result (Theorem 3.2), the asymptotics of the zeros of an exponential polynomial can be explicitly expressed in terms of the coefficients of this polynomial. Thus, the second step of our proof is a direct and completely elementary analysis of the matrix $A(k)$ which allows us to relate the required information about the coefficients of the polynomial $\det A(k)$ to the question of whether the external vertices of $\mathcal{G}$ are balanced. This is done in Section 3.

Resonance asymptotics of Weyl type have been established for compactly supported potentials on the real line, a class of superexponentially decaying potentials on the real line, compactly supported potentials on cylinders and Laplace operators on surfaces with finite volume hyperbolic cusps in [Zworski 1987; Froese 1997; Christiansen 2004; Parnovski 1995] respectively. The proofs rely upon theorems about the zeros of certain classes of entire functions. Likewise, our analysis uses a simple classical result (Theorem 3.2) about zeros of exponential polynomials.

The situation with resonance asymptotics for potential and obstacle scattering in Euclidean space in dimensions greater than one and in hyperbolic space is more complicated and still not fully understood; the current state of knowledge is described in [Stefanov 2006; Borthwick 2010]. Here we remark only that generically the resonance asymptotics in the multidimensional case is not given by the Weyl formula. We hope that Theorem 1.2 can provide some insight to the multidimensional case.

One may approach the resonances of quantum graphs by studying the scattering matrix. A detailed account of resonance scattering for quantum graphs from the physics perspective and some associated numerical calculations can be found in [Kottos and Smilansky 2003]. The graphs considered in that reference have no balanced external vertices, so the non-Weyl phenomenon does not occur there. Resonances for quantum graphs have also been discussed in [Exner and Lipovský 2010]. Our paper has little technical content in common with either of those articles, in spite of their common themes.

After this paper was written the main results were extended in [Davies et al. 2010] to graphs with general self-adjoint boundary conditions at the vertices; the results there emphasise the exceptional nature of non-Weyl resonance asymptotics.

Example. In Section 6 we consider the resonances of a particularly simple quantum graph which can be described as a circle with two leads attached to it. Theorem 1.2 says that if the leads are attached at different points on the circle, the corresponding quantum graph is Weyl, and if they are attached at the same point, we have a non-Weyl graph. When the two points where the leads are attached move closer to each other and eventually coalesce, one observes the transition from the Weyl to the non-Weyl case. We study this transition in much detail. We show that as the two external vertices get closer, "half" of the resonances move off to infinity. In the course of this analysis, we also obtain bounds on the positions of individual resonances for this model.

The same example was recently considered by Exner and Lipovský [Exner and Lipovský 2010] subject to general boundary conditions that include the Kirchhoff's boundary condition case as a singular limit. Although some of their results are broadly similar to ours, none of our theorems may be found in [Exner and Lipovský 2010].

2. Resonances via complex scaling

Here we introduce the necessary notation, recall the definition of resonances via the complex scaling procedure and show that the resonances on the real axis coincide with the eigenvalues of H.

Notation. Let E^{int} be the set of all internal edges of $\mathcal{G}$ (i.e., the set of all edges of $\mathcal{G}_0$) and let E^{ext} be the set of all leads; we also denote $E = E^{\mathrm{int}} \cup E^{\mathrm{ext}}$. The term "edge" without an adjective will refer to any element of E. For $e \in E^{\mathrm{int}}$, we denote by $\rho(e)$ the length of e; i.e., an edge $e \in E^{\mathrm{int}}$ is identified with the interval $[0, \rho(e)]$.

Let V be the set of all vertices of $\mathcal{G}$, let V^{ext} be the set of all external vertices, and let $V^{\mathrm{int}} = V \setminus V^{\mathrm{ext}}$; the elements of V^{int} will be called *internal vertices*. The degree of a vertex v is denoted by $d(v)$. The number of leads attached to an external vertex v is denoted by $q(v)$; we also set $q(v) = 0$ for $v \in V^{\mathrm{int}}$.

If an edge or a lead e is attached to a vertex v, we write $v \in e$. If two vertices u, v are connected by one or more edges, we write $u \sim v$.

We denote by $\mathcal{G}_\infty$ the graph $\mathcal{G}$ with all the internal edges and vertices removed. We let χ_0 and χ_∞ be the characteristic functions of $\mathcal{G}_0$ and $\mathcal{G}_\infty$.

Let $f:\mathscr{G}\to\mathbb{C}$ be a function such that the restriction of f onto every edge is continuously differentiable. Then for $v\in V$, we denote by $N_v f$ the sum of the outgoing derivatives of f at v over all edges attached to v. If v is an external vertex, we denote by $N_v^{\mathrm{int}} f$ (resp. $N_v^{\mathrm{ext}} f$) the sum of all outgoing derivatives of f at v over all internal edges (resp. leads) attached to v.

Let $\widetilde{C}(\mathscr{G})$ be the class of functions $f:\mathscr{G}\to\mathbb{C}$ which are continuous on $\mathscr{G}\setminus V^{\mathrm{ext}}$ and such that for each external vertex v the function $f(x)$ approaches a limiting value (to be denoted by $D_v^{\mathrm{int}} f$) as x approaches v along any internal edge and $f(x)$ approaches another limiting value (to be denoted by $D_v^{\mathrm{ext}} f$) as x approaches v along any lead.

For any finite set A, we denote by $|A|$ the number of elements of A. We will use the identity

$$\sum_{v\in V} d(v) = 2|E^{\mathrm{int}}| + |E^{\mathrm{ext}}|. \tag{2-1}$$

Finally, we use the notation $\mathbb{C}_+ = \{z\in\mathbb{C} : \operatorname{Im} z > 0\}$.

The operator $H(\varkappa)$. The domain of the self-adjoint operator H consists of all continuous functions $f:\mathscr{G}\to\mathbb{C}$ such that the restriction of f onto any $e\in E$ lies in the Sobolev space $W_2^2(e)$, and f satisfies the Kirchhoff boundary condition $N_v f = 0$ on every vertex v of $\mathscr{G}$.

For $\varkappa\in\mathbb{R}$, let $U(\varkappa): L^2(\mathscr{G})\to L^2(\mathscr{G})$ be the unitary operator which acts as identity on $L^2(\mathscr{G}_0)$ and as a dilation on all leads $\ell=[0,\infty)$:

$$(U(\varkappa)f)(x) = e^{\varkappa/2} f(e^{\varkappa}x), \quad x\in\ell. \tag{2-2}$$

Note that $U(\varkappa)^* = U(-\varkappa)$ for any $\varkappa\in\mathbb{R}$. Consider the operator

$$H(\varkappa) = U(\varkappa) H U(-\varkappa). \tag{2-3}$$

It admits an analytic continuation to $\varkappa\in\mathbb{C}$, which we describe below.

Definition 2.1. For $\varkappa\in\mathbb{C}$, the operator $H(\varkappa)$ in $L^2(\mathscr{G})$ acts according to the formula

$$(H(\varkappa)f)(x) = \begin{cases} -f''(x) & \text{if } x\in\mathscr{G}_0, \\ -e^{-2\varkappa} f''(x) & \text{if } x\in\mathscr{G}_\infty. \end{cases} \tag{2-4}$$

The domain of $H(\varkappa)$ is defined to be the set of all $f:\mathscr{G}\to\mathbb{C}$ which satisfy the following conditions:

(i) The restriction of f onto any $e\in E$ lies in the Sobolev space $W_2^2(e)$.

(ii) $f\in\widetilde{C}(\mathscr{G})$.

(iii) f satisfies the condition $N_v f = 0$ at every internal vertex v.

(iv) For any $v\in V^{\mathrm{ext}}$, one has

$$D_v^{\mathrm{int}} f - e^{-\varkappa/2} D_v^{\mathrm{ext}} f = 0, \tag{2-5}$$

$$N_v^{\mathrm{int}} f + e^{-3\varkappa/2} N_v^{\mathrm{ext}} f = 0. \tag{2-6}$$

In particular, $H(0)$ is the operator called H so far. For complex $\varkappa$, the operator $H(\varkappa)$ is in general nonselfadjoint. A standard straightforward computation shows that for any $\varkappa \in \mathbb{C}$ the operator $H(\varkappa)$ is closed and

$$H(\varkappa)^* = H(\bar{\varkappa}). \tag{2-7}$$

Resonances via complex scaling. The following theorem is standard in the method of complex scaling; see [Aguilar and Combes 1971; Simon 1973; Sjöstrand and Zworski 1991; Exner and Lipovský 2007]:

Theorem 2.2. *The family of operators $H(\varkappa)$, $\varkappa \in \mathbb{C}$, is analytic in the sense of Kato* (see, for example, [Reed and Simon 1978, Section XII.2]), *and*

$$H(\varkappa + \varkappa_0) = U(\varkappa_0)H(\varkappa)U(-\varkappa_0) \quad \textit{for all } \varkappa \in \mathbb{C} \textit{ and all } \varkappa_0 \in \mathbb{R}. \tag{2-8}$$

The essential spectrum of $H(\varkappa)$ coincides with the half-line $e^{-2\varkappa}[0,\infty)$. Let $\theta \in (0,\pi)$; then the sector $0 < \arg\lambda < 2\pi - 2\theta$, $\lambda \neq 0$, contains no eigenvalues of $H(i\theta)$, and any $\lambda \neq 0$ in the sector $2\pi - 2\theta < \arg\lambda \leq 2\pi$ is an eigenvalue of $H(i\theta)$ if and only if $\lambda = k^2$ with $k \in \mathfrak{R}$.

For completeness, we give the proof in Section 5.

As $\theta \in (0,\pi)$ increases monotonically, the essential spectrum $e^{-2i\theta}[0,\infty)$ of $H(i\theta)$ rotates clockwise, uncovering more and more eigenvalues λ. These eigenvalues are identified with the resonances k of H via $\lambda = k^2$. If $\lambda \neq 0$ is an eigenvalue of $H(i\theta)$, $\theta \in (0,\pi)$, $2\pi - 2\theta < \arg\lambda \leq 2\pi$, Kato's theory of analytic perturbations implies that the eigenvalue and associated Riesz spectral projection depend analytically on θ. Noting (2-8) and using analytic continuation it follows that the algebraic multiplicity of λ is independent of θ. It is easy to see directly that the geometric multiplicity of λ is also independent of θ. The *algebraic* (resp. *geometric*) *multiplicity* of a resonance k is defined as the algebraic (resp. geometric) multiplicity of the corresponding eigenvalue $\lambda = k^2$ of $H(i\theta)$.

Resonances on the real line. The geometric multiplicities of resonances will not play any role in our analysis. However, we note that for the Schrödinger operator on the real line, resonances can have arbitrary large algebraic multiplicity [Korotyaev 2005], while their geometric multiplicity is always equal to one. This gives an example of resonances with distinct algebraic and geometric multiplicities. It would be interesting to see if one can have distinct algebraic and geometric multiplicities of resonances for quantum graphs in the situation we are discussing. We have nothing to say about this except for the case of the resonances on the real line:

Theorem 2.3. (i) *If $k \in \mathbb{R}$, $k \neq 0$, is a resonance of H then the algebraic and geometric multiplicities of k coincide.*

(ii) *Any $k \in \mathbb{R}$, $k \neq 0$, is a resonance of H if and only if k^2 is an eigenvalue of H and the multiplicity of the resonance k coincides with the multiplicity of the eigenvalue k^2.*

Proof. **1.** Let $\lambda > 0$ be an eigenvalue of H with the eigenfunction f. If $\ell = [0,\infty)$ is a lead, then $f(x) = \gamma_\ell e^{ikx} + \gamma'_\ell e^{-ikx}$, $x \in \ell$, where $k^2 = \lambda$. Since $f \in L^2(\ell)$, we conclude that $\gamma_\ell = \gamma'_\ell = 0$ and so

$f \equiv 0$ on all leads. It follows that $f \in \operatorname{Dom} H(i\theta)$ for all θ and $H(i\theta)f = \lambda f$. This argument proves that

$$\dim \operatorname{Ker}(H(i\theta) - \lambda I) \geq \dim \operatorname{Ker}(H - \lambda I). \tag{2-9}$$

2. Let $f \in \operatorname{Ker}(H(i\theta) - \lambda I)$, $\lambda > 0$, $\theta \in (0, \pi)$. Let us prove that f vanishes identically on all leads. Let $\lambda = k^2$, $k > 0$. On every lead, we have

$$f(x) = f(0) \exp(ie^{i\theta} kx). \tag{2-10}$$

Consider the difference

$$\omega(f) = \int_{\mathscr{G}_0} |f'(x)|^2 dx - \lambda \int_{\mathscr{G}_0} |f(x)|^2 dx = \int_{\mathscr{G}_0} |f'(x)|^2 dx + \int_{\mathscr{G}_0} f''(x) \overline{f(x)} dx. \tag{2-11}$$

Integrating by parts, we get

$$\omega(f) = - \sum_{v \in V^{\text{ext}}} (N_v^{\text{int}} f) \overline{D_v^{\text{int}} f}.$$

Using the boundary condition (2-5) and formula (2-10), we obtain

$$\omega(f) = ik \sum_{v \in V^{\text{ext}}} |D_v^{\text{ext}} f|^2 q(v).$$

By the definition (2-11) of $\omega(f)$, we have $\operatorname{Im} \omega(f) = 0$. This yields that $|D_v^{\text{ext}} f| = 0$ on all external vertices v. By (2-10), it follows that f vanishes identically on all leads.

3. By combining the previous step of the proof with (2-5) and (2-6) we obtain $D_v^{\text{int}} f = N_v^{\text{int}} f = 0$. It follows that for any $f \in \operatorname{Ker}(H(i\theta) - \lambda I)$, $\lambda > 0$, $\theta \in (0, \pi)$, we have $f \in \operatorname{Dom} H$ and $Hf = \lambda f$. This argument also proves that

$$\dim \operatorname{Ker}(H - \lambda I) \geq \dim \operatorname{Ker}(H(i\theta) - \lambda I). \tag{2-12}$$

4. It remains to prove that if $\lambda > 0$ is an eigenvalue of $H(i\theta)$, $\theta \in (0, \pi)$, then its algebraic and geometric multiplicities coincide. Suppose this is not the case. Then there exist nonzero elements $f, g \in \operatorname{Dom} H(i\theta)$ such that $H(i\theta)g = \lambda g$ and $(H(i\theta) - \lambda I)f = g$.

By step 2 of the proof, g vanishes on all leads. It follows that on all leads the function f satisfies (2-10). Next, since $g(x) = -f''(x) - \lambda f(x)$ on $\mathscr{G}_0$, we have

$$0 < \int_{\mathscr{G}_0} |g(x)|^2 dx = - \int_{\mathscr{G}_0} (f''(x) + \lambda f(x)) \overline{g(x)}\, dx. \tag{2-13}$$

Integrating by parts, we get

$$\begin{aligned} -\int_{\mathscr{G}_0} &(f''(x) + \lambda f(x)) \overline{g(x)}\, dx \\ &= -\int_{\mathscr{G}_0} f(x) \big(\overline{g''(x)} + \lambda \overline{g(x)} \big)\, dx + \sum_{v \in V^{\text{ext}}} (N_v^{\text{int}} f)(\overline{D_v^{\text{int}} g}) - \sum_{v \in V^{\text{ext}}} (D_v^{\text{int}} f)(\overline{N_v^{\text{int}} g}). \end{aligned} \tag{2-14}$$

Consider the three terms in the right-hand side of (2-14). The first term vanishes since $H(i\theta)g = \lambda g$. Next, since $g \equiv 0$ on $\mathcal{G}_\infty$, we have $D_v^{\text{ext}} g = N_v^{\text{ext}} g = 0$ for any $v \in V^{\text{ext}}$. By the boundary conditions (2-5) and (2-6) for g it follows that $D_v^{\text{int}} g = N_v^{\text{int}} g = 0$. Thus, the second and third terms in the right-hand side of (2-14) also vanish. This contradicts (2-13). ☐

3. Proof of Theorem 1.2

Here we describe the resonances as zeros of $\det A(k)$, where $A(k)$ is certain entire matrix-valued function. Using this characterisation, we prove our main result.

Definition of $A(k)$. Fix $k \in \mathbb{C}_+$. Let $\mathcal{L}(k)$ denote the space of all solutions $f \in L^2(\mathcal{G})$ to $-f'' = k^2 f$ on $\mathcal{G}$ without any boundary conditions. The restriction of $f \in \mathcal{L}(k)$ to any internal edge e has the form $f_e(x) = \alpha_e e^{ikx} + \beta_e e^{-ikx}$, and the restriction of f to any lead ℓ has the form $f_\ell(x) = \gamma_\ell e^{ikx}$. Thus, $\dim \mathcal{L}(k) = 2|E^{\text{int}}| + |E^{\text{ext}}|$.

Let us describe in detail the set of all conditions on $f \in \mathcal{L}(k)$ required to ensure that f is a resonance eigenfunction. If f_e denotes the restriction of f to an edge e, then we can write the continuity conditions at the vertex v as

$$f_e(v) = \zeta_v \quad \text{for all } e \ni v, \tag{3-1}$$

where $\zeta_v \in \mathbb{C}$ is an auxiliary variable. We also have the condition

$$N_v f = 0, \quad v \in V. \tag{3-2}$$

Writing down conditions (3-1), (3-2) for every vertex $v \in V$, we obtain

$$N = \sum_{v \in V} d(v) + |V| = 2|E^{\text{int}}| + |E^{\text{ext}}| + |V|$$

conditions. Our variables are ζ_v, α_e, β_e, γ_ℓ; altogether we have

$$|V| + \dim \mathcal{L}(k) = |V| + 2|E^{\text{int}}| + |E^{\text{ext}}| = N$$

variables. Let ζ, α, β, γ be the sequences of coordinates ζ_v, α_e, β_e, γ_ℓ of length $|V|$, $|E^{\text{int}}|$, $|E^{\text{int}}|$, $|E^{\text{ext}}|$ respectively, and let $\nu = (\zeta, \alpha, \beta, \gamma)^\top \in \mathbb{C}^N$. We may write the constraints (3-1), (3-2) in the form $A\nu = 0$, where A is an $N \times N$ matrix. Each row of A relates to one of the constraints, and each constraint is of the form

$$y \cdot \zeta + a \cdot \alpha + b \cdot \beta + g \cdot \gamma = 0. \tag{3-3}$$

If the constraint is of the form (3-2), then $y = 0$ and a, b, g all contain a multiplicative factor ik which we eliminate before proceeding. The coefficient a_e is 0, ± 1, or $\pm e^{ik\rho(e)}$, and the coefficient b_e is 0, ± 1, or $\pm e^{-ik\rho(e)}$. The coefficient g_ℓ is 0 or 1, and the coefficient y_v is 0 or -1.

We have not specified the order of the rows or columns of $A(k)$. However, the object of importance in the sequel is the set of zeros of $\det A(k)$, and the choice of the order of rows or columns of $A(k)$ will not affect this set.

Example. As an example, let us display the matrix $A(k)$ for a graph which consists of two vertices v_1 and v_2, two edges e_1 and e_2 of length ρ_1 and ρ_2 and a lead attached at v_1. In this case we have, with $z_j = e^{ik\rho_j}$,

$$A(k) = \begin{pmatrix} 0 & 0 & z_1 & z_2 & -z_1^{-1} & -z_2^{-1} & 0 \\ 0 & 0 & 1 & 1 & -1 & -1 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & z_1 & 0 & z_1^{-1} & 0 & 0 \\ 0 & -1 & 0 & z_2 & 0 & z_2^{-1} & 0 \end{pmatrix}. \tag{3-4}$$

***Resonances as zeros of* det $A(k)$.** Although $A(k)$ was defined for $k \in \mathbb{C}_+$, we see that all elements of $A(k)$ are entire functions of $k \in \mathbb{C}$. Thus, we will consider $A(k)$ as an entire matrix-valued function of k.

In Sections 4 and 5 we prove:

Theorem 3.1. *Any $k_0 \neq 0$ is a resonance of H if and only if* $\det A(k_0) = 0$. *In this case, the algebraic multiplicity of the resonance k_0 coincides with the order of k_0 as a zero of* $\det A(k)$.

The first part of this theorem is obvious: by the construction of the matrix A, we have $\det A(k_0) = 0$ iff there exists a nonzero resonance eigenfunction $f \in \mathcal{L}(k_0)$. The part concerning multiplicity is less obvious. Unfortunately, we were not able to find a completely elementary proof of this part. The proof we give in Sections 4–5 involves a standard set of techniques from the spectral theory of quantum graphs: a resolvent identity involving the Dirichlet-to-Neumann map and a certain trace formula.

By Theorem 3.1, the question reduces to counting the total multiplicity of zeros of the entire function $\det A(k)$ in large discs. As is clear from the structure of the matrix $A(k)$, its determinant is an exponential polynomial, i.e., a linear combination of the terms of the type $e^{i\sigma k}$, $\sigma \in \mathbb{R}$. Thus, we need to discuss the zeros of exponential polynomials.

Zeros of exponential polynomials. Exponential polynomials are entire functions $F(k)$, $k \in \mathbb{C}$, of the form

$$F(k) = \sum_{r=1}^{n} a_r \mathrm{e}^{i\sigma_r k}, \tag{3-5}$$

where $a_r, \sigma_r \in \mathbb{C}$ are constants. The study of the zeros of such polynomials has a long history; see, e.g., [Langer 1931] and references therein. For more recent literature see [Moreno 1973]. Some of these results were rediscovered in [Davies 2003; Davies and Incani 2010; Incani 2009], where they were used to analyze the spectra of nonselfadjoint systems of ODEs and directed finite graphs. The asymptotic distribution of the zeros of F depends heavily on the location of the extreme points of the convex hull of the set $\cup_{r=1}^{n}\{\sigma_r\}$.

We are only interested in the case in which σ_r are distinct real numbers. We set $\sigma^- = \min\{\sigma_1, \dots, \sigma_n\}$ and $\sigma^+ = \max\{\sigma_1, \dots, \sigma_n\}$. For $R > 0$ we denote the number of zeros of F (counting their orders) in the disc $\{k \in \mathbb{C} : |k| < R\}$ by $N(R; F)$. The following classical statement is from [Langer 1931, Theorem 3].

Theorem 3.2. *Let F be a function of the form* (3-5), *where a_r are nonzero complex numbers and σ_r are distinct real numbers. Then there exists a constant $K < \infty$ such that all the zeros of F lie within a strip of the form $\{k : |\operatorname{Im}(z)| \le K\}$. The counting function $N(R; F)$ satisfies*

$$N(R;F) = \frac{\sigma^+ - \sigma^-}{\pi} R + O(1) \quad \text{as } R \to +\infty.$$

Estimate for $N(R;F)$. Here we prove the first part of the main Theorem 1.2. Let $F(k) = \det A(k)$. From the structure of $A(k)$ it is clear that $F(k)$ is given by (3-5) where a_r, σ_r are real coefficients. By Theorem 3.2, it suffices to prove that in the representation (3-5) we have

$$\sigma^+ \le \operatorname{vol}(\mathscr{G}_0), \quad \sigma^- \ge -\operatorname{vol}(\mathscr{G}_0). \tag{3-6}$$

In order to prove (3-6), let us discuss the entries of $A(k)$ in detail. For simplicity of notation we will not draw attention in our equations to the fact that all of the matrices below depend on k.

The matrix A has some constant terms and some terms that are exponential in k. The term $e^{ik\rho(e)}$ can only appear in the column associated with the variable α_e and the term $e^{-ik\rho(e)}$ can only appear in the column associated with the variable β_e. The columns associated with the variables ζ and γ contain only constant terms. Since the determinant is formed from the products of entries of A where every column contributes one entry to each product, we see that the maximum possible value for the coefficient σ_r in (3-5) is attained when every column corresponding to the variable α_e contributes the term $e^{ik\rho(e)}$ and every column corresponding to β_e contributes a constant term to the product. The maximal value of σ_r thus attained will be exactly $\sum_{e\in E^{\text{int}}} \rho(e) = \operatorname{vol}\mathscr{G}_0$. This proves the first inequality in (3-6). The second one is proven in the same way by considering the minimal possible value for σ_r.

Of course, the coefficients $a^\pm$ of the terms $e^{\pm ik\operatorname{vol}(\mathscr{G}_0)}$ in the representation (3-5) for $\det A$ may well happen to be zero. Theorem 1.2 will be proven if we show that these coefficients do not vanish if and only if every external vertex of $\mathscr{G}$ is unbalanced. In what follows, for an exponential polynomial F with the representation (3-5) we denote by $a^\pm(F)$ the coefficient a_r of the term $e^{i\sigma_r k}$, $\sigma_r = \pm\operatorname{vol}(\mathscr{G}_0)$.

Invariance of resonances with respect to a change of orientation. Before proceeding with the proof, we need to discuss a minor technical point. Our definition of the matrix $A(k)$ assumes that a certain orientation of all internal edges of $\mathscr{G}$ is fixed. Suppose we have changed the parametrization of an internal edge e by reversing its orientation. In other words, suppose that instead of the variable $x \in [0, \rho(e)]$ we decided to use the variable $x' = \rho(e) - x$. We claim that this change will not affect the zeros of $\det A(k)$.

Indeed, let $A'(k)$ be the matrix corresponding to the new parametrization. The matrix $A'(k)$ corresponds to the parametrization of solutions $f \in \mathscr{L}(k)$ on e by $f(x) = \alpha'_e e^{ikx'} + \beta'_e e^{-ikx'}$ instead of $\alpha_e e^{ikx} + \beta_e e^{-ikx}$. We have

$$\begin{pmatrix} \alpha'_e \\ \beta'_e \end{pmatrix} = \begin{pmatrix} 0 & e^{-ik\rho(e)} \\ e^{ik\rho(e)} & 0 \end{pmatrix} \begin{pmatrix} \alpha_e \\ \beta_e \end{pmatrix}, \quad \det \begin{pmatrix} 0 & e^{-ik\rho(e)} \\ e^{ik\rho(e)} & 0 \end{pmatrix} = -1,$$

and thus $\det A'(k) = -\det A(k)$.

Proof of Theorem 1.2: the balanced case. Assume that a particular external vertex v of $\mathscr{G}$ is balanced. Below we prove that the coefficient $a^+(\det A)$ vanishes.

Let us reorder the rows and columns of A by reference to the vertex v. We assume that q internal edges and q leads are attached to v, $q \geq 2$. (The case $q = 1$ is trivial because one may then merge the lead with the edge to which it is connected.) Using the invariance of resonances with respect to a change of orientation (page 738), we can choose an orientation of these internal edges so that they all end at v (i.e., v is identified with the point $\rho(e)$ of the intervals $[0, \rho(e)]$). Let the first $2q$ rows of A be those relating to the conditions (3-1) for the vertex v and let the $(2q+1)$-st row be the one relating to the condition (3-2) for the vertex v. The ordering of the remaining rows does not matter. Let the first $2q$ columns be related to the variables $\gamma_1, \dots, \gamma_q, \alpha_1, \dots, \alpha_q$ and let the $(2q+1)$-st column be related to the variable ζ_v; see the definition of the matrix $A(k)$ in Section 3. The ordering of the remaining columns does not matter.

We write A in the block form

$$A = \begin{pmatrix} B & C \\ D & E \end{pmatrix} \tag{3-7}$$

where B is a $(2q + 1) \times (2q + 1)$ matrix. For example, in the case $q = 2$ we have

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & z_1 & 0 & -1 \\ 0 & 0 & 0 & z_2 & -1 \\ 1 & 1 & -z_1 & -z_2 & 0 \end{pmatrix}, \tag{3-8}$$

where $z_r = e^{ik\rho(e_r)}$.

The determinant is the sum of the products of entries of A where every column contributes one entry to each product. In order for the product to be of the type $a_+ e^{ik \operatorname{vol}(\mathscr{G}_0)}$, each column corresponding to a variable α_e must contribute the entry $e^{ik\rho(e)}$. Thus, the constant entries of the columns corresponding to the variables α_e are irrelevant to our question and can be replaced by zeros; this will not affect the value of $a^+(\det A)$. Noticing that the columns of D corresponding to the variables $\gamma_1, \dots, \gamma_q$ and ζ_v are all zeros, we conclude that

$$a^+(\det A) = a^+(\det A_0), \quad \text{where } A_0 = \begin{pmatrix} B & C \\ 0 & E \end{pmatrix}.$$

By a general matrix identity, $\det A_0 = \det B \det E$. Finally, a simple row reduction shows that $\det B = 0$; this is easy to see in the case of (3-8). Thus, the coefficient $a^+(\det A)$ vanishes. By (3-6), it follows that $\sigma^+ < \operatorname{vol} \mathscr{G}_0$, as claimed.

We note (although this is not needed for our proof) that $\sigma^- = -\operatorname{vol} \mathscr{G}_0$ both in the balanced and in the unbalanced case; this will be clear from the next part of the proof.

Proof of Theorem 1.2: the unbalanced case. Assume that all external vertices are unbalanced. We will prove that

$$\sigma^+ = \operatorname{vol}(\mathscr{G}_0), \quad \sigma^- = -\operatorname{vol}(\mathscr{G}_0). \tag{3-9}$$

The proof uses the same reduction as (3-7), but the details are somewhat more complicated, since now we have to consider *all* external vertices.

We label the external vertices by $v_1, \dots, v_m$, where $m = |V^{\text{ext}}|$. For $r = 1, 2, \dots, m$, let $\mathcal{G}_r$ denote the graph obtained from $\mathcal{G}_0$ by adding all the leads of $\mathcal{G}$ that have ends in the set $\{v_1, \dots, v_r\}$, so that $\mathcal{G}_m = \mathcal{G}$. Let A_r denote the constraint matrix A corresponding to the graph $\mathcal{G}_r$ and let $a_r^{\pm} = a^{\pm}(\det A_r)$.

By the previous reasoning, the graph $\mathcal{G}_r$ is Weyl if and only if $a_r^+ \neq 0$ and $a_r^- \neq 0$. Our claim (3-9) follows inductively from the following statements:

1. The graph $\mathcal{G}_0$ is Weyl.
2. The coefficient a_r^- is nonzero for all r.
3. For all r, if $a_{r-1}^+ \neq 0$ then $a_r^+ \neq 0$.

Item 1 holds because the operator H on $\mathcal{G}_0$ has discrete spectrum and no other resonances. The eigenvalues obey the Weyl law by a standard variational argument using Dirichlet–Neumann bracketing.

Let us prove item 3. We reorder the rows and columns of A_r with reference to v_r as in the balanced case (see previous page). We assume that p internal edges $e_1, \dots, e_p$ and q leads $\ell_1, \dots, \ell_q$ are attached to v_r, and $q \neq p$. The first $q + p + 1$ columns of A_r are those relating to the variables $\gamma_1, \dots, \gamma_q$ (associated with $\ell_1, \dots, \ell_q$), $\alpha_1, \dots, \alpha_p$ (associated with $e_1, \dots, e_p$), and ζ_r. The first $q + p + 1$ rows of A_r are those relating to the conditions (3-1) and (3-2) for the vertex v_r. As in the balanced case, this allows us to write

$$A_r = \begin{pmatrix} B_r & C_r \\ D_r & E_r \end{pmatrix} \tag{3-10}$$

where B_r is a $(q+p+1) \times (q+p+1)$ matrix. Writing the matrix A_{r-1} in the same way with reference to *the same vertex* v_r, we obtain

$$A_{r-1} = \begin{pmatrix} \widetilde{B}_{r-1} & \widetilde{C}_{r-1} \\ \widetilde{D}_{r-1} & E_r \end{pmatrix}, \tag{3-11}$$

where $\widetilde{B}_{r-1}$ is a $(p+1) \times (p+1)$ matrix. In other words, $\widetilde{B}_{r-1}$, $\widetilde{C}_{r-1}$, $\widetilde{D}_{r-1}$ are the matrices B_r, C_r, D_r with relevant q rows and q columns deleted. The deleted columns correspond to the variables $\gamma_1, \dots, \gamma_q$, and the deleted rows correspond to the conditions (3-1) associated with the leads $\ell_1, \dots, \ell_q$. Note that the matrix E_r is the same in (3-10) and (3-11).

Next, just as in the argument used in the balanced case, we notice that

$$a_r^+ = a^+(\det B_r \det E_r) \quad \text{and} \quad a_{r-1}^+ = a^+(\det \widetilde{B}_{r-1} \det E_r).$$

Finally, by a simple row reduction we obtain

$$\det B_r = (q-p) z_1 \dots z_p, \tag{3-12}$$

$$\det \widetilde{B}_{r-1} = (-p) z_1 \dots z_p, \tag{3-13}$$

where $z_j = e^{ik\rho(e_j)}$. It follows that a_r^+ and a_{r-1}^+ differ by a nonzero coefficient $(p-q)/p$. This proves item 3.

Let us prove item 2. Here the argument follows that of the proof of item 3, only instead of keeping track of the coefficient of $e^{ik\,\mathrm{vol}(\mathscr{G}_0)}$ we need to keep track of the coefficient of $e^{-ik\,\mathrm{vol}(\mathscr{G}_0)}$, and instead of the variables $\alpha_1, \dots, \alpha_p$ we consider the variables $\beta_1, \dots, \beta_p$. Instead of the coefficient $(q-p)$ in (3-12) we get $(q+p)$, which never vanishes (even if v_r is balanced). This proves our claim.

4. A resolvent identity and its consequences

To complete the proof of Theorem 1.2, it remains to provide the proof of Theorem 3.1. Theorem 4.2 below provides an explicit formula for the difference of the resolvents of $H(\varkappa)$ and an auxiliary operator $H_D(\varkappa)$; this formula is given in terms of the Dirichlet-to-Neumann map. This leads immediately to the trace formula (4-13), which is the key to our proof of Theorem 3.1 in Section 5. The formulae obtained in this section are "complex-scaled" versions of resolvent identities well known in the theory of boundary value problems; see, for example, [Gesztesy et al. 2009; 2007]

Dirichlet-to-Neumann map. Throughout this section, we assume that the parameter $k \in \mathbb{C}_+$ is fixed. Let $\mathscr{L}(k)$ be as defined on page 736 and let $\mathscr{M}(k) = \mathscr{L}(k) \cap C(\mathscr{G})$. Each $f \in \mathscr{M}(k)$ determines a vector $\zeta \in \mathbb{C}^{|V|}$ by restriction to V. Conversely, every $\zeta \in \mathbb{C}^{|V|}$ arises from a function $f \in \mathscr{M}(k)$; this can be seen by comparing $\dim \mathscr{L}(k)$ with the number of constraints imposed by writing

$$f(v) = \zeta_v, \quad v \in V.$$

Finally, the assumption $k \in \mathbb{C}_+$ implies that only one function $f \in \mathscr{M}(k)$ corresponds to each set of values $\zeta \in \mathbb{C}^{|V|}$ (otherwise we would have a complex eigenvalue of the operator with Dirichlet boundary conditions on all vertices). This shows that we may define the Dirichlet-to-Neumann map

$$\Lambda(k) : \mathbb{C}^{|V|} \to \mathbb{C}^{|V|}$$

by

$$(\Lambda(k)\zeta)_v = N_v f,$$

where f corresponds to ζ as described above and N_v was defined in Section 2. This map is a well known tool in the spectral theory of boundary value problems and has also been used in quantum graph theory [Ong 2006; Kuchment 2005].

The functions φ_v and formulae for Λ. Given $v \in V$, let φ_v be the function in $\mathscr{M}(k)$ that satisfies

$$\varphi_v(u) = \delta_{uv} \quad \text{for all } u, v \in V.$$

The functions φ_v are given by the following explicit expressions. Let $v \in e$, $e \in E^{\mathrm{int}}$ and identify e with $[0, \rho]$ where v corresponds to the point 0. Then

$$\varphi_v(x) = \frac{\sin k(\rho - x)}{\sin k\rho}, \quad x \in [0, \rho] = e. \tag{4-1}$$

In the same way, if $e \in E^{\mathrm{ext}}$ and v is identified with the point 0, then

$$\varphi_v(x) = e^{ikx}, \quad x \in [0, \infty) = e. \tag{4-2}$$

If the dependence on k needs to be emphasized, we will write $\varphi_v(x;k)$ instead of $\varphi_v(x)$.

Lemma 4.1. *If $k \in \mathbb{C}_+$ then the map $\Lambda(k)$ is invertible. Its matrix entries are given by*

$$\Lambda_{uv} = 0 \qquad \text{if } u \neq v,\ u \not\sim v; \tag{4-3}$$

$$\Lambda_{uv} = \sum_{\substack{e \in E^{\text{int}} \\ u,v \in e}} \frac{k}{\sin k\rho(e)} \qquad \text{if } u \neq v,\ u \sim v; \tag{4-4}$$

$$\Lambda_{vv} = ikq(v) - k \sum_{\substack{e \in E^{\text{int}} \\ v \in e}} \cot k\rho(e) \quad \text{for any } v \in V; \tag{4-5}$$

where $q(v)$ was defined in Section 2.

Proof. If $\Lambda(k)\zeta = 0$, then the corresponding function $f \in \mathcal{M}(k) \subset L^2(\mathcal{G})$ satisfies the Kirchhoff boundary condition at every vertex, which implies that $f \in \operatorname{Dom} H$ and $Hf = k^2 f$. Since $\operatorname{Spec}(H) = [0,\infty)$ and $\operatorname{Im} k > 0$, this implies that $f = 0$. Therefore $\Lambda(k)$ is invertible.

By the definition of φ_v, we have

$$\Lambda_{uv} = N_u \varphi_v.$$

The formulae for the matrix entries are obtained by combining this with (4-1) and (4-2). □

It follows from Lemma 4.1 that $\Lambda(k)$ can be extended to a meromorphic function of $k \in \mathbb{C}$ whose poles are all on the real axis, and that for any $u, v \in V$ one has

$$\Lambda_{uv}(k) = \Lambda_{vu}(k) \quad \text{and} \quad \overline{\Lambda_{uv}(k)} = \Lambda_{uv}(-\bar{k}), \quad k \in \mathbb{C}. \tag{4-6}$$

In the calculations below the expressions Λ^{-1}_{uv} will denote the matrix entries of $(\Lambda(k))^{-1}$.

The complex-scaled version of φ_v. We will need a version of the functions φ_v pertaining to the complex-scaled operator $H(\varkappa)$. Let $k \in \mathbb{C}_+$ and $\varkappa \in \mathbb{C}$ be such that $ke^{\varkappa} \in \mathbb{C}_+$. Given $v \in V$, we define $\varphi_v^{\varkappa}$ by

$$\varphi_v^{\varkappa}(x;k) = \begin{cases} \varphi_v(x;k) & \text{if } x \in \mathcal{G}_0; \\ \varphi_v(0;k)e^{\varkappa/2}\exp(ike^{\varkappa}x) & \text{if } x \in \ell = [0,\infty),\ \ell \in E^{\text{ext}}. \end{cases}$$

Clearly, $\varphi_v^{\varkappa}$ is a solution to the equation $H(\varkappa)\varphi_v^{\varkappa} = k^2\varphi_v^{\varkappa}$ on every edge of $\mathcal{G}$. It is also straightforward to see that $\varphi_v^{\varkappa} \in \widetilde{C}(\mathcal{G})$ and $\varphi_v^{\varkappa}$ satisfies the boundary condition (2-5) on every external vertex. For $f \in \widetilde{C}(\mathcal{G})$, let us denote

$$N_v^{\varkappa} f = \begin{cases} N_v f & \text{if } v \in V^{\text{int}}, \\ N_v^{\text{int}} f + e^{-3\varkappa/2} N_v^{\text{ext}} f & \text{if } v \in V^{\text{ext}}. \end{cases}$$

It is straightforward to see that

$$\Lambda_{uv} = N_u^{\varkappa}\varphi_v^{\varkappa} \quad \text{for all } u, v \in V, \tag{4-7}$$

where the left-hand side depends on k but not on $\varkappa$. Moreover,

$$\overline{\varphi_v^{\varkappa}(x;k)} = \varphi_v^{\bar{\varkappa}}(x;-\bar{k}). \tag{4-8}$$

The resolvent identity. Let H_D be the self-adjoint operator in $L^2(\mathcal{G})$ defined by $H_D f = -f''$ with a Dirichlet boundary condition at every vertex of $\mathcal{G}$. Given $\varkappa \in \mathbb{C}$, we define the "complex-scaled" version of H_D as follows; $H_D(\varkappa)$ is the operator acting in $L^2(\mathcal{G})$ defined by

$$(H_D(\varkappa)f)(x) = \begin{cases} -f''(x) & \text{if } x \in \mathcal{G}_0, \\ -e^{-2\varkappa} f''(x) & \text{if } x \in \mathcal{G}_\infty, \end{cases}$$

with a Dirichlet boundary condition at every vertex of $\mathcal{G}$. Of course, $H_D(\varkappa)$ splits into an orthogonal sum of operators acting on $L^2(e)$ for all $e \in E$. We see immediately that in addition to its essential spectrum $e^{-2\varkappa}[0,\infty)$, the operator $H_D(\varkappa)$ has a discrete set of positive eigenvalues with finite multiplicities.

We set

$$R_D^\varkappa(k) = (H_D(\varkappa) - k^2 I)^{-1}, \quad R^\varkappa(k) = (H(\varkappa) - k^2 I)^{-1},$$

whenever the inverse operators exist. We denote by $R^\varkappa(k; x, y)$, where $x, y \in \mathcal{G}$, the integral kernel of the resolvent $R^\varkappa(k)$; we define $R_D^\varkappa(k; x, y)$ from $R_D^\varkappa(k)$ analogously.

The fact that $H_D(\varkappa)$ and $H(\varkappa)$ coincide except for different boundary conditions at each of the $|V|$ vertices indicates that the difference of the two resolvents should have rank $|V|$. Our next theorem makes this explicit. Formulae of this type are well known in the theory of boundary value problems; see [Gesztesy et al. 2009; 2007], for example. In the context of graphs, similar considerations have been used in [Kostrykin and Schrader 1999; 2006; Kostrykin et al. 2007; Ong 2006].

Theorem 4.2. *For any $k \in \mathbb{C}_+$ and any $\varkappa \in \mathbb{C}$, such that $ke^\varkappa \in \mathbb{C}_+$, we have*

$$R^\varkappa(k; x, y) - R_D^\varkappa(k; x, y) = -\sum_{u,v \in V} \Lambda_{uv}^{-1}(k) \varphi_v^\varkappa(x; k) \varphi_u^\varkappa(y; k), \tag{4-9}$$

for any $x, y \in \mathcal{G}$.

Proof. **1.** Let $\widetilde{R}^\varkappa(k)$ be the operator in $L^2(\mathcal{G})$ with the integral kernel given by

$$\widetilde{R}^\varkappa(k; x, y) = R_D^\varkappa(k; x, y) - \sum_{u,v \in V} \Lambda_{uv}^{-1}(k) \varphi_v^\varkappa(x; k) \varphi_u^\varkappa(y; k).$$

We need to prove that $\widetilde{R}^\varkappa(k)$ is a bounded operator, that it maps $L^2(\mathcal{G})$ into $\operatorname{Dom} H(\varkappa)$ and that the identities

$$(H(\varkappa) - k^2 I) \widetilde{R}^\varkappa(k) = I \tag{4-10}$$

$$\widetilde{R}^\varkappa(k)(H(\varkappa) - k^2 I) = I \tag{4-11}$$

hold true. First note that since $\varphi_v^\varkappa$ decays exponentially on all leads, the boundedness of $\widetilde{R}^\varkappa(k)$ is obvious. Next, using (4-6), (4-8) one obtains $\widetilde{R}^\varkappa(k)^* = \widetilde{R}^{\overline{\varkappa}}(-\overline{k})$. From here and (2-7) by taking adjoints we see that (4-11) is equivalent to

$$(H(\overline{\varkappa}) - (-\overline{k})^2) \widetilde{R}^{\overline{\varkappa}}(-\overline{k}) = I$$

which is (4-10) with $-\overline{k}$, $\overline{\varkappa}$ instead of k, $\varkappa$. We note that $k \in \mathbb{C}_+$, $ke^\varkappa \in \mathbb{C}_+$ if and only if $-\overline{k} \in \mathbb{C}_+$, $-\overline{k}e^{\overline{\varkappa}} \in \mathbb{C}_+$. Thus, (4-11) follows from (4-10).

2. It suffices to prove that for a dense set of elements $f \in L^2(\mathcal{G})$, the inclusion $\widetilde{R}^\varkappa(k)f \in \operatorname{Dom} H(\varkappa)$ and the identity

$$(H(\varkappa) - k^2 I)\widetilde{R}^\varkappa(k)f = f \tag{4-12}$$

hold true. Let f be from the dense set of all continuous functions compactly supported on $\mathcal{G}$ and vanishing near all vertices of $\mathcal{G}$. Let us check that the function $g = \widetilde{R}^\varkappa(k)f$ belongs to $\operatorname{Dom} H(\varkappa)$. It is clear that the restriction of g onto any edge e of $\mathcal{G}$ belongs to the Sobolev space $W_2^2(e)$. Thus, it suffices to check that g belongs to $\widetilde{C}(\mathcal{G})$ and satisfies the boundary conditions (2-5) and (2-6).

Denote $g_0 = R_D^\varkappa(k)f$. Since $g_0 \in \operatorname{Dom} H_D(\varkappa)$, g_0 vanishes on all vertices. Therefore g_0 lies in $\widetilde{C}(\mathcal{G})$ and satisfies (2-5) at every external vertex v. As mentioned on page 742, the functions $\varphi_v^\varkappa$ also belong to $\widetilde{C}(\mathcal{G})$ and satisfy (2-5) at every external vertex v. Thus, g also has these properties.

Our next task is to prove that the boundary condition (2-6) is satisfied for the function g. Suppose that f is supported on a single edge, which we identify with $[0, \rho]$. Then the integral kernel of $R_D^\varkappa(k)$ can be explicitly calculated, which gives

$$g_0'(0) = \int_0^\rho \frac{\sin k(\rho - x)}{\sin k\rho} f(x)\, dx.$$

Similarly, if f is supported on a lead $[0, \infty)$, then a direct calculation shows that

$$g_0'(0) = e^{2\varkappa} \int_0^\infty \exp(ike^\varkappa x) f(x)\, dx.$$

Combining this, we see that for any $w \in V^{\text{ext}}$ we have

$$N_w^\varkappa g_0 = \int_{\mathcal{G}} f(x)\varphi_w^\varkappa(x)\, dx.$$

Using the last identity and (4-7), for any $w \in V^{\text{ext}}$ we get:

$$N_w^\varkappa g = \int_{\mathcal{G}} f(x)\varphi_w^\varkappa(x)\, dx - \sum_{u,v \in V} \Lambda_{uv}^{-1}\Lambda_{wv} \int_{\mathcal{G}} f(x)\varphi_u^\varkappa(x)\, dx = 0,$$

and so the boundary condition (2-6) is satisfied for g. Thus, $g \in \operatorname{Dom} H(\varkappa)$, as required.

3. It remains to note that the identity (4-12) follows from the fact that $R_D^\varkappa$ is the resolvent of $H_D(\varkappa)$ and the fact that $\varphi_v^\varkappa$ satisfies the equation $H(\varkappa)\varphi_v^\varkappa = k^2\varphi_v^\varkappa$ on every edge and lead of $\mathcal{G}$. □

A trace formula. The trace formula (4-13) below results by calculating the traces of both sides of (4-9). Since the right-hand side of (4-9) is a finite rank operator, the trace is well defined; the fact that the value of (4-13) does not depend on $\varkappa$ can be proved by complex scaling, but the direct proof is almost as easy.

The identity (4-13) below can be rephrased by saying that the (modified) perturbation determinant of the pair of operators $H(\varkappa)$, $H_D(\varkappa)$ equals $\det \Lambda(k)$. Statements of this nature (for $\varkappa = 0$) are well known in the theory of boundary value problems; see e.g. [Carron 2002] and references therein. The key to our proof of Theorem 3.1 will be (4-13) and Lemma 5.1, in which $\det A(k)$ and $\det \Lambda(k)$ are related.

Theorem 4.3. *For any $k \in \mathbb{C}_+$ and any $\varkappa \in \mathbb{C}$, such that $ke^{\varkappa} \in \mathbb{C}_+$, we have*

$$\operatorname{Tr}(R^{\varkappa}(k) - R^{\varkappa}_D(k)) = -\frac{\frac{d}{dk}\det \Lambda(k)}{2k \det \Lambda(k)}. \tag{4-13}$$

In particular, the left-hand side is independent of $\varkappa$.

Proof. **1.** Theorem 4.2 yields

$$\operatorname{Tr}(R^{\varkappa}(k) - R^{\varkappa}_D(k)) = -\sum_{u,v\in V} \Lambda^{-1}_{uv}(k)\sigma^{\varkappa}_{uv}(k), \tag{4-14}$$

where

$$\sigma^{\varkappa}_{uv}(k) = \int_{\mathscr{G}} \varphi^{\varkappa}_u(x;k)\varphi^{\varkappa}_v(x;k)dx. \tag{4-15}$$

We next compute the coefficients σ_{uv} explicitly. If $v \neq u$ and $v \not\sim u$ then $\operatorname{supp}\varphi^{\varkappa}_v \cap \operatorname{supp}\varphi^{\varkappa}_u = \varnothing$ and so $\sigma_{uv} = 0$. If $v \neq u$ and $v \sim u$ then by (4-1)

$$\sigma_{uv} = \sum_{\substack{e\in E^{\mathrm{int}}\\ u,v\in e}} \int_0^{\rho} \frac{\sin kx}{\sin k\rho(e)} \frac{\sin k(\rho(e)-x)}{\sin k\rho(e)}\, dx = \frac{1}{2k} \sum_{\substack{e\in E^{\mathrm{int}}\\ u,v\in e}} \frac{\sin k\rho(e) - k\rho(e)\cos k\rho(e)}{(\sin k\rho(e))^2},$$

and finally,

$$\begin{aligned}\sigma_{vv} &= \sum_{\substack{e\in E^{\mathrm{int}}\\ v\in e}} \int_0^{\rho(e)} \left(\frac{\sin kx}{\sin k\rho(e)}\right)^2 dx + q(v)\int_0^{\infty} (e^{\varkappa/2}\exp(ike^{\varkappa}x))^2 dx\\ &= \frac{1}{2k}\sum_{\substack{e\in E^{\mathrm{int}}\\ v\in e}} \frac{k\rho(e) - \cos k\rho(e)\sin k\rho(e)}{(\sin k\rho(e))^2} + \frac{i}{2k}q(v).\end{aligned}$$

2. Noting that σ_{uv} depend on k but not on $\varkappa$, a direct calculation using (4-3)–(4-5) yields

$$\frac{1}{2k}\frac{d}{dk}\Lambda_{uv}(k) = \sigma_{uv}(k).$$

It follows that

$$\operatorname{Tr}(R^{\varkappa}(k) - R^{\varkappa}_D(k)) = -\sum_{u,v\in V} \Lambda^{-1}_{uv}(k)\frac{1}{2k}\frac{d}{dk}\Lambda_{uv}(k) = -\frac{1}{2k}\operatorname{Tr}\left(\Lambda^{-1}(k)\frac{d}{dk}\Lambda(k)\right) = -\frac{\frac{d}{dk}\det\Lambda(k)}{2k\det\Lambda(k)},$$

as required. □

5. Proof of Theorems 3.1 and 2.2

***Calculation of* det $A(k)$.** Given $k \in \mathbb{C}$, we define

$$\delta(k) = \prod_{e\in E^{\mathrm{int}}} (k \sin k\rho(e)). \tag{5-1}$$

Let $A(k)$ be the matrix defined on page 736.

Lemma 5.1. *For any $k \in \mathbb{C}_+$, we have the identity*

$$\det A(k) = \pm \frac{2^{|E^{\mathrm{int}}|} i^{|E^{\mathrm{int}}|-|V|}}{k^{|E^{\mathrm{int}}|+|V|}} \delta(k) \det \Lambda(k), \tag{5-2}$$

where the sign $\pm$ depends on the ordering of the rows and columns of the matrix $A(k)$.

Proof. **1.** Let us order the rows and the columns of $A(k)$ in such a way that the first $|V|$ rows correspond to the conditions $N_v(u) = 0$, and the first $|V|$ columns correspond to the variables ζ. Then $A(k)$ can be written in the block form as

$$A = \begin{pmatrix} 0 & M \\ -N & P \end{pmatrix} \tag{5-3}$$

where 0 is the $|V| \times |V|$ zero matrix and P is a $(2|E^{\mathrm{int}}|+|E^{\mathrm{ext}}|)\times(2|E^{\mathrm{int}}|+|E^{\mathrm{ext}}|)$ matrix. The elements of N are 0 or 1, the elements of M are 0, ± 1, $\pm e^{\pm ik\rho}$, and the elements of P are 0, ± 1, or $e^{\pm ik\rho}$. For example, the matrix (3-4) is written in this form.

2. Let us reorder the rows of P in such a way that any two constraints associated with the continuity conditions at the two endpoints of the same edge follow one another. Let us also reorder the columns of P such that each variable β_e follows the corresponding variable α_e. For example, the block P of the matrix (3-4) after such reordering will be

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ z_1 & z_1^{-1} & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & z_2 & z_2^{-1} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

In general, after this reordering, P assumes a block-diagonal structure with blocks either of size 2×2 with elements

$$\begin{pmatrix} 1 & 1 \\ e^{ik\rho} & e^{-ik\rho} \end{pmatrix}$$

or of size 1×1 with the element 1. From here it follows that

$$\det P = \pm \prod_{e \in E^{\mathrm{int}}} (2i \sin(k\rho(e))) = \pm (2i)^{|E^{\mathrm{int}}|} k^{-|E^{\mathrm{int}}|} \delta(k). \tag{5-4}$$

In particular, since $k \in \mathbb{C}_+$, the matrix P is invertible.

3. By applying the Schur complement method to (5-3) one obtains

$$\det A = \det P \det(MP^{-1}N). \tag{5-5}$$

Let us prove that

$$ikMP^{-1}N = \Lambda(k). \tag{5-6}$$

Let $\zeta \in \mathbb{C}^{|V|}$ and let $a = P^{-1}N\zeta$. The vector a represents a set of parameters α, β, γ. Let $f \in \mathcal{L}(k)$ be the solution with this set of parameters. The equation $Pa = N\zeta$ implies that the solution f is continuous on $\mathscr{G}$ and satisfies $f(v) = \zeta_v$ for any vertex v. Next, the coordinates of the vector $ikMP^{-1}N\zeta = ikMa$ are given by

$$ik(Ma)_v = N_v f.$$

This shows that $ikMa = \Lambda(k)\zeta$, as required.

4. By combining (5-4)–(5-6) one obtains

$$\det A(k) = \det P(k) \det(M(k)P^{-1}(k)N(k)) = \pm(2i)^{|E^{\mathrm{int}}|} k^{-|E^{\mathrm{int}}|} \delta(k) \det((ik)^{-1}\Lambda(k)),$$

which yields (5-2) immediately. □

Proof of Theorem 3.1. **1.** Let $k \in \mathbb{C}_+$ and let χ_0 and χ_∞ be defined as in Section 2. Clearly, $\chi_0 R_D(k)\chi_0$ is an orthogonal sum of resolvents of the operators $-d^2/dx^2$ on the intervals $(0, \rho(e))$, $e \in E^{\mathrm{int}}$, with Dirichlet boundary conditions. For each such operator we have that $(-d^2/dx^2 - k^2)^{-1}$ is trace class and

$$\begin{aligned}\mathrm{Tr}(-d^2/dx^2 - k^2)^{-1} &= \sum_{n=0}^{\infty} \Big((\pi n/\rho)^2 - k^2\Big)^{-1} = -\frac{1}{2k^2} - \frac{1}{2k} \sum_{n=-\infty}^{\infty} \frac{1}{k - \pi n/\rho} \\ &= -\frac{1}{2k^2} - \frac{\rho}{2k}\cot(k\rho) = -\frac{\frac{d}{dk}(k\sin(k\rho))}{2k(k\sin(k\rho))}.\end{aligned}$$

Summing over all edges, a direct calculation shows that $\chi_0 R_D(k)\chi_0$ is a trace class operator and

$$\mathrm{Tr}(\chi_0 R_D(k)\chi_0) = -\frac{\frac{d}{dk}\delta(k)}{2k\delta(k)}. \tag{5-7}$$

2. Let $k \in \mathbb{C}_+$, $ke^{\varkappa} \in \mathbb{C}_+$. It is easy to see that the resolvent $R_D^{\varkappa}(k)$ commutes with χ_0, χ_∞ and that

$$\chi_0 R_D^{\varkappa}(k)\chi_0 = \chi_0 R_D(k)\chi_0.$$

Therefore we have

$$R^{\varkappa}(k) - \chi_\infty R_D^{\varkappa}(k)\chi_\infty = R^{\varkappa}(k) - R_D^{\varkappa}(k) + \chi_0 R_D(k)\chi_0. \tag{5-8}$$

By combining Theorem 4.3 and (5-8), we obtain

$$\mathrm{Tr}(R^{\varkappa}(k) - \chi_\infty R_D^{\varkappa}(k)\chi_\infty) = -\frac{\frac{d}{dk}\det\Lambda(k)}{2k\det\Lambda(k)} - \frac{\frac{d}{dk}\delta(k)}{2k\delta(k)} = -\frac{\frac{d}{dk}(\delta(k)\det\Lambda(k))}{2k\delta(k)\det\Lambda(k)}. \tag{5-9}$$

Using Lemma 5.1, we then obtain

$$\mathrm{Tr}(R^{\varkappa}(k) - \chi_\infty R_D^{\varkappa}(k)\chi_\infty) = \frac{|E^{\mathrm{int}}| + |V|}{2k^2} - \frac{\frac{d}{dk}\det A(k)}{2k\det A(k)}, \tag{5-10}$$

for all $k \in \mathbb{C}_+$ and $ke^{\varkappa} \in \mathbb{C}_+$.

3. The right-hand side of (5-10) is a single-valued meromorphic function of $k \in \mathbb{C}$. Let $\tau^{\varkappa}(k)$ be the left-hand side of (5-10). For each fixed $\varkappa \in \mathbb{C}$, the function $\tau^{\varkappa}(k)$ is meromorphic in $\mathbb{C}$ with the cut along the line determined by the condition $k^2 \in \sigma_{\mathrm{ess}}(H(\varkappa)) = e^{-2\varkappa}[0, \infty)$. In other words, $\tau^{\varkappa}$ is meromorphic and single-valued in each of the two half-planes $\operatorname{Im} ke^{\varkappa} > 0$ and $\operatorname{Im} ke^{\varkappa} < 0$. By the uniqueness of analytic continuation, for each $\varkappa$ the identity (5-10) extends to all k such that $\operatorname{Im} ke^{\varkappa} > 0$.

4. Let $k_0 \in \mathfrak{R}$ with the algebraic multiplicity $m(k_0) \geq 1$ and let $\theta \in (0, \pi)$ with $-\theta < \arg k_0 \leq 0$. Then $\operatorname{Im} k_0 e^{i\theta} > 0$ and so the identity (5-10) with $\varkappa = i\theta$ holds for all k near k_0. If γ is a sufficiently small circle with centre at k_0, then the multiplicity $m(k_0)$ equals the rank, or equivalently the trace, of the Riesz spectral projection

$$P^{\theta}(k_0) = -\frac{1}{2\pi i}\int_{\gamma} R^{i\theta}(k) 2k\, dk. \tag{5-11}$$

Next, since the operator $H_D(i\theta)$ restricted to $L^2(\mathcal{G}_\infty)$ has no eigenvalues, the operator valued function $\chi_\infty R_D^{i\theta}(k)\chi_\infty$ is analytic for $\operatorname{Im} ke^{i\theta} \neq 0$. It follows that

$$-\frac{1}{2\pi i}\int_{\gamma} \chi_\infty R_D^{i\theta}(k)\chi_\infty 2k\, dk = 0.$$

By taking the trace of the difference of the last two equations and using (5-10) we obtain

$$m(k_0) = -\frac{1}{2\pi i}\int_{\gamma} \operatorname{Tr}(R^{i\theta}(k) - \chi_\infty R_D^{i\theta}(k)\chi_\infty) 2k\, dk = \frac{1}{2\pi i}\int_{\gamma} \frac{\frac{d}{dk}\det A(k)}{\det A(k)}\, dk.$$

Therefore $m(k_0)$ equals the order of the zero of $\det A(k)$ at $k = k_0$, as required. □

Proof of Theorem 2.2. This theorem is well known but we give its proof for completeness.

1. First note that by Theorem 4.2, the difference of the resolvents of $H(\varkappa)$ and $H_D(\varkappa)$ is a finite rank operator. By Weyl's theorem on the invariance of the essential spectrum under a relatively compact perturbation we obtain

$$\sigma_{\mathrm{ess}}(H(\varkappa)) = \sigma_{\mathrm{ess}}(H_D(\varkappa)) = e^{-2\varkappa}[0, \infty).$$

2. The fact that the family $H(\varkappa)$ is analytic in the sense of Kato follows again from Theorem 4.2, since $H_D(\varkappa)$ is analytic in the sense of Kato and each of the functions $\varphi_v^{\varkappa}$ is analytic in $\varkappa$.

3. The identity (2-8) can be checked by a direct calculation.

4. Let $k \in \mathfrak{R}$ and let f be the corresponding eigenfunction. For any $\theta \in (0, \pi)$ with $-\theta < \arg k \leq 0$, let f_θ be the function defined *formally* by $f_\theta = U(i\theta) f$. More precisely, we set $f_\theta = f$ on $\mathcal{G}_0$ and

$$f_\theta(x) = f(0) e^{i\theta/2} \exp(i k e^{i\theta} x) \tag{5-12}$$

for x on any lead $\ell = [0, \infty)$. By the choice of θ, we have $\operatorname{Im} ke^{i\theta} > 0$ and so $f_\theta \in L^2(\mathcal{G})$. A straightforward inspection shows that $f_\theta \in \operatorname{Dom} H(i\theta)$ and $H(i\theta) f_\theta = k^2 f_\theta$.

5. Conversely, let $\lambda \notin e^{-2i\theta}[0, \infty)$ be an eigenvalue of $H(i\theta)$ for $\theta \in (0, \pi)$. Write $\lambda = k^2$ with $\operatorname{Im} ke^{i\theta} > 0$. Then, for the corresponding eigenfunction g of $H(i\theta)$ we have $g(x) = g(0)\exp(ike^{i\theta}x)$ on any lead of $\mathcal{G}$. A direct inspection shows that $g = f_\theta$ in the same sense as (5-12), where f is a resonance eigenfunction. Thus, $k \in \mathfrak{R}$ and in particular, $\operatorname{Im} k \leq 0$. It follows that $2\pi - 2\theta < \arg k^2 \leq 2\pi$. □

6. An example

Here we consider resonances of a particular simple graph $\mathscr{G}(c)$, where $c \in [0, 1]$ is a certain geometric parameter. The graph $\mathscr{G}(c)$ was also considered in [Exner and Lipovský 2010, Section 4], but with different boundary conditions at the vertices. The graph $\mathscr{G}(c)$ is Weyl for $c < 1$ and non-Weyl for $c = 1$. This section has two goals. The first one is to discuss the transition between the Weyl and the non-Weyl cases in order to throw new light on the failure of the Weyl law. Our second goal is to obtain rigorous bounds on the locations of individual resonances of $\mathscr{G}(c)$, which was not addressed by Exner and Lipovský.

Definition of $\mathscr{G}(c)$. Given $c \in [0, 1)$, we consider the graph $\mathscr{G}_0(c)$ which consists of two vertices v_1 and v_2 and two edges $e_1 = [0, \rho_1]$, $\rho_1 = (1-c)\pi$, and $e_2 = [0, \rho_2]$, $\rho_2 = (1+c)\pi$. The vertex v_2 is identified with the point 0 of e_1 and with the point 0 of e_2, and the vertex v_1 is identified with the point ρ_1 of e_1 and with the point ρ_2 of e_2. Thus, the graph $\mathscr{G}_0(c)$ is simply a circle with the circumference $\operatorname{vol} \mathscr{G}_0(c) = 2\pi$ for all c. We attach a lead ℓ_1 at v_1 and a lead ℓ_2 at v_2 and denote the thus extended graph by $\mathscr{G}(c)$. Geometrically, $\mathscr{G}(c)$ is a circle with two leads attached to it. Finally, for $c = 1$, let $\mathscr{G}(c)$ be the circle of length 2π with two leads attached at the same point.

We will denote by $H(c)$ the operator $-d^2/dx^2$ acting in $L^2(\mathscr{G}(c))$ subject to the usual continuity and Kirchhoff boundary conditions at the vertices v_1 and v_2. By Theorem 1.2, the graph $\mathscr{G}(c)$ is Weyl if and only if $c < 1$. At the same time, the graph $\mathscr{G}(1)$ can be regarded as the limit of $\mathscr{G}(c)$ as $c \to 1$ in an obvious geometric sense, so we need to explain what happens to resonances as $c \to 1$. As we will see, roughly speaking, a half of the resonances of $H(c)$ move off to infinity as $c \to 1$. We will obtain bounds on the curves along which the resonances move as c increases from 0 to 1.

The matrix $A(k, c)$ ***for*** $\mathscr{G}(c)$. Let us display the constraints (3-3) corresponding to the graph $\mathscr{G}(c)$; the matrix $A(k, c)$ will be built up of the rows corresponding to these constraints. We denote $z_j = e^{ik\rho_j/2}$, $j = 1, 2$. The constraints corresponding to the vertex v_1 are

$$\alpha_1 z_1^2 + \beta_1 z_1^{-2} - \zeta_1 = 0, \tag{R_1}$$

$$\alpha_2 z_2^2 + \beta_2 z_2^{-2} - \zeta_1 = 0, \tag{R_2}$$

$$\gamma_1 - \zeta_1 = 0, \tag{R_3}$$

$$-\alpha_1 z_1^2 + \beta_1 z_1^{-2} - \alpha_2 z_2^2 + \beta_2 z_2^{-2} + \gamma_1 = 0. \tag{R_4}$$

The first three lines are the continuity conditions, and the last is the requirement that the sum of the outgoing derivatives vanishes. Similarly, the constraints corresponding to the vertex v_2 are

$$\alpha_1 + \beta_1 - \zeta_2 = 0, \tag{R_5}$$

$$\alpha_2 + \beta_2 - \zeta_2 = 0, \tag{R_6}$$

$$\gamma_2 - \zeta_2 = 0, \tag{R_7}$$

$$\alpha_1 - \beta_1 + \alpha_2 - \beta_2 + \gamma_2 = 0. \tag{R_8}$$

We list these constraints in the order R_1, R_5, R_2, R_6, R_3, R_7, R_4, R_8, and order the variables as α_1, $\beta_1, \alpha_2, \beta_2, \gamma_1, \gamma_2, \zeta_1, \zeta_2$. This leads to the matrix

$$A(k,c)=\begin{pmatrix} z_1^2 & z_1^{-2} & 0 & 0 & 0 & 0 & -1 & 0\\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1\\ 0 & 0 & z_2^2 & z_2^{-2} & 0 & 0 & -1 & 0\\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & -1\\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1\\ -z_1^2 & z_1^{-2} & -z_2^2 & z_2^{-2} & 1 & 0 & 0 & 0\\ 1 & -1 & 1 & -1 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

***Calculation of* $\det A(k,c)$.** The graph $\mathcal{G}(c)$ has a reflection symmetry with respect to the midpoints of e_1 and e_2. This allows to decompose the space $\mathcal{L}(k)$ into the direct sum of the subspaces corresponding to even and odd functions with respect to this symmetry. We use this decomposition to represent the matrix $A(k,c)$ in a block-diagonal form where the blocks correspond to the even and odd solutions. More precisely, let

$$T_1=\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1\\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}, \qquad T_2=\begin{pmatrix} z_1^{-1} & 0 & 0 & 0 & z_1^{-1} & 0 & 0 & 0\\ z_1 & 0 & 0 & 0 & -z_1 & 0 & 0 & 0\\ 0 & z_2^{-1} & 0 & 0 & 0 & z_2^{-1} & 0 & 0\\ 0 & z_2 & 0 & 0 & 0 & -z_2 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

A straightforward calculation shows that $\det T_1 = \det T_2 = 16$. Next, let $\widetilde{A}(k,c) = T_1 A(k,c) T_2$; the reader is invited to check that the matrix $\widetilde{A}(k)$ can be written as

$$\widetilde{A} = 2\begin{pmatrix} \widetilde{A}_{\text{even}} & 0\\ 0 & \widetilde{A}_{\text{odd}} \end{pmatrix},$$

with blocks

$$\widetilde{A}_{\text{even}}=\begin{pmatrix} 2C_1 & 0 & 0 & -1\\ 0 & 2C_2 & 0 & -1\\ 0 & 0 & 1 & -1\\ -2iS_1 & -2iS_2 & 1 & 0 \end{pmatrix}, \quad \widetilde{A}_{\text{odd}}=\begin{pmatrix} 2iS_1 & 0 & 0 & -1\\ 0 & 2iS_2 & 0 & -1\\ 0 & 0 & 1 & -1\\ -2C_1 & -2C_2 & 1 & 0 \end{pmatrix},$$

where we have used the notation $C_j = \cos(k\rho_j/2)$, $S_j = \sin(k\rho_j/2)$, $j = 1, 2$. Straightforward calculations of $\det(\widetilde{A}_{\text{even}})$ and $\det(\widetilde{A}_{\text{odd}})$ now yield

Theorem 6.1. *For all $k \in \mathbb{C}$ and all $c \in [0, 1)$ one has*

$$\det A(k,c) = 4F_{\text{even}}(k,c)F_{\text{odd}}(k,c),$$

where

$$F_{\text{even}}(k,c) = i\cos(kc\pi) + i\cos(k\pi) + 2\sin(k\pi),$$
$$F_{\text{odd}}(k,c) = i\cos(kc\pi) - i\cos(k\pi) - 2\sin(k\pi).$$

We will call the zeros of $F_{\text{even}}(\cdot,c)$ (resp. of $F_{\text{odd}}(\cdot,c)$) the even (resp. odd) resonances. It is not difficult to check that the resonance eigenfunctions which correspond to even/odd resonances are even/odd with respect to the symmetry of the graph $\mathscr{G}(c)$. By Theorem 2.3, the real even/odd resonances are actually eigenvalues of $H(c)$ and therefore we will call them even/odd eigenvalues.

Finally, it is not difficult to check that the resonances of $H(1)$ are given, as expected, by the zeros of $\det A(k,1)$. In fact, in this case we have $F_{\text{odd}}(k,1) = -2\sin(k\pi)$ and

$$F_{\text{even}}(k,1) = 2ie^{-ik\pi} \neq 0 \quad \text{for all } k \in \mathbb{C}. \tag{6-1}$$

Thus, the resonances of $H(1)$ coincide with the solutions to $\sin(k\pi) = 0$, i.e., they are given by $k \in \mathbb{Z}$. By Theorem 2.3, these resonances (for $k \neq 0$) coincide with the eigenvalues of $H(1)$ and all of them have multiplicity one. This shows that for $c = 1$ we have the asymptotics (1-3) with $W = \pi = \frac{1}{2}\operatorname{vol}\mathscr{G}_0$.

Locating the odd resonances.

Theorem 6.2. (i) *For any $c \in [0,1]$, any $n \in \mathbb{Z}$ and any $y \geq 0$ one has $F_{\text{odd}}(n+\frac{1}{2}-iy,c) \neq 0$.*

(ii) *For any $c \in [0,1]$ and any $k = x - iy$ with $y > |x|/\sqrt{3}$ one has $F_{\text{odd}}(k,c) \neq 0$.*

Proof. (i) By an explicit calculation,

$$F_{\text{odd}}(n+\tfrac{1}{2}-iy,c) = i\cos((n+\tfrac{1}{2}-iy)\pi c) + (-1)^n\sinh(y\pi) - 2(-1)^n\cosh(y\pi) = A + B,$$

where

$$\begin{aligned}|A| &= |\cos((n+\tfrac{1}{2}-iy)\pi c)|\\ &= |\cos((n+\tfrac{1}{2})\pi c)\cosh(y\pi c) + i\sin((n+\tfrac{1}{2})\pi c)\sinh(y\pi c)|\\ &\leq \cosh(y\pi c) \leq \cosh(y\pi)\end{aligned}$$

and

$$|B| = 2\cosh(y\pi) - \sinh(y\pi) = \cosh(y\pi) + \mathrm{e}^{-y\pi}.$$

We deduce that

$$|F_{\text{odd}}(n+\tfrac{1}{2}-iy,c)| \;\geq\; |B| - |A| \;\geq\; \mathrm{e}^{-y\pi} \;>\; 0.$$

(ii) We start by observing that $|F_{\text{odd}}(k,c)| \geq 2A - B$ where

$$A = |\sin(k\pi)|, \quad B = |\cos(k\pi) - \cos(k\pi c)| = \left|\int_c^1 k\pi\sin(k\pi s)\,ds\right|.$$

If $u \in \mathbb{R}$ and $v \geq 0$ then

$$\sin(u - iv) = \sin(u)\cosh(v) - i\cos(u)\sinh(v).$$

Therefore

$$\sinh(v) \le |\sin(u - iv)| \le \cosh(v).$$

We deduce that $A \ge \sinh(y\pi)$ and

$$B \le \int_c^1 |k|\pi \cosh(y\pi s)\, ds = \frac{|k|}{y}(\sinh(y\pi) - \sinh(y\pi s)) \le \frac{|k|}{y}\sinh(y\pi).$$

These bounds imply that $2A - B > 0$ if $2y > |k|$, which yields the theorem immediately. ☐

It follows that all odd resonances are located in the rectangles

$$\Pi_n^{\text{odd}} = \left\{ x - iy : |x - n| < \tfrac{1}{2},\ 0 \le y \le \frac{2|n|+1}{2\sqrt{3}} \right\}, \quad n \in \mathbb{Z}.$$

The following statement, in combination with Rouche's theorem, shows that each of the rectangles Π_n^{odd} contains exactly one odd resonance of algebraic multiplicity one for all $c \in [0, 1]$.

Theorem 6.3. *If $c = 0$ there is a resonance of algebraic multiplicity one at $k = n - i\log(3)/\pi$ for every odd $n \in \mathbb{Z}$ and an eigenvalue of multiplicity one at $k = n$ for every nonzero even $n \in \mathbb{Z}$. There is also a resonance of algebraic multiplicity one at $k = 0$. No other odd resonances or eigenvalues exist if $c = 0$.*

The proof follows from the explicit formula

$$F_{\text{odd}}(k, 0) = \frac{i}{2}(e^{ik\pi} + 3)(1 - e^{-ik\pi}).$$

By the implicit function theorem, we obtain that each of the zeros of $F_{\text{odd}}(\cdot; c)$ is a real analytic function of $c \in [0, 1]$ with values in Π_n^{odd}. The set of all odd resonances for all such c is therefore the union of a sequence of bounded real analytic curves.

It is interesting to note that each of these resonance curves intersects the real axis, thereby (by Theorem 2.3) giving rise to embedded eigenvalues. This happens at rational values of c. More precisely, a direct computation shows that $F_{\text{odd}}(k, c) = 0$ for $k \in \mathbb{R}$ if and only if

$$k = m + n \quad \text{and} \quad c = \frac{m - n}{m + n} \quad \text{for some } m, n \in \mathbb{N}.$$

Figure 1 plots a typical odd resonance curve as c increases from 0 to 1. It starts at $7 - i\log(3)/\pi$, when $c = 0$. The curve then passes through 7 when $c = \frac{1}{7}, \frac{3}{7}, \frac{5}{7}, 1$.

Locating the even resonances.

Theorem 6.4. (i) *For any $c \in [0, 1]$, any $n \in \mathbb{Z}$ and any $y \ge 0$ one has $F_{\text{even}}(n + \frac{1}{2} - iy, c) \ne 0$.*

(ii) *For any $c \in [0, 1)$ and any $k = x - iy$ with $y > \dfrac{\log 3}{\pi(1 - |c|)}$, one has $F_{\text{odd}}(k, c) \ne 0$.*

Proof. (i) We have

$$F_{\text{even}}(n + \tfrac{1}{2} - iy, c) = A - B,$$

where A, B are as in the proof of Theorem 6.2(i). The rest of the proof is the same as in Theorem 6.2(i).

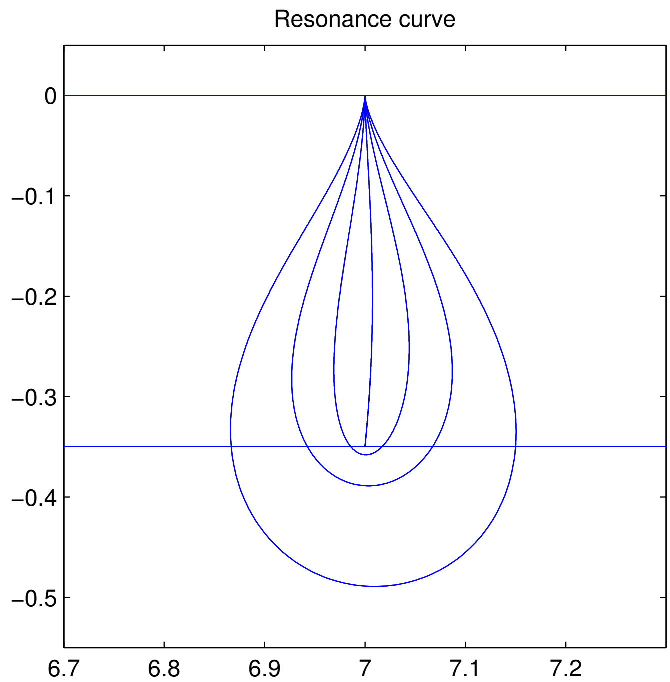

Figure 1. The odd resonance curve in Π_7^{odd}.

(ii) For any $k = x - iy$ we have

$$\begin{aligned}\tfrac{1}{2}\mathrm{e}^{y\pi|c|} + \tfrac{1}{2} \geq \cosh(y\pi c) &\geq |\cos(x\pi c)\cosh(y\pi c) + i\,\sin(x\pi c)\cosh(y\pi c)| \\ &\geq |\cos(x\pi c)\cosh(y\pi c) + i\,\sin(x\pi c)\sinh(y\pi c)| = |\cos(k\pi c)| \end{aligned} \tag{6-2}$$

and

$$|i\,\cos(k\pi) + 2\sin(k\pi)| \geq \tfrac{1}{2}|\mathrm{e}^{ik\pi}| - \tfrac{3}{2}|\mathrm{e}^{-ik\pi}| = \tfrac{1}{2}\mathrm{e}^{y\pi} - \tfrac{3}{2}\mathrm{e}^{-y\pi}. \tag{6-3}$$

Now suppose $F_{\mathrm{even}}(k, c) = 0$; then $\cos(k\pi c) = -i\,\cos(k\pi) - 2\sin(k\pi)$ and therefore, combining (6-2) and (6-3), we obtain

$$\mathrm{e}^{y\pi} \leq \mathrm{e}^{y\pi|c|} + 1 + 3\mathrm{e}^{-y\pi}.$$

If $y \geq \log(3)/\pi$ or equivalently $\mathrm{e}^{y\pi} \geq 3$ then

$$\mathrm{e}^{y\pi} \leq \mathrm{e}^{y\pi|c|} + 2 \leq \mathrm{e}^{y\pi|c|} + \tfrac{2}{3}\mathrm{e}^{y\pi}.$$

A simple manipulation then yields that $y \leq \dfrac{\log 3}{\pi(1-|c|)}$, and the required result follows. □

It follows that for $c \in [0, 1)$ the even resonances are located in the rectangles

$$\Pi_n^{\mathrm{even}}(c) = \left\{x + iy : |x - n| < \tfrac{1}{2},\ 0 \leq y \leq \frac{\log 3}{\pi(1-|c|)}\right\}.$$

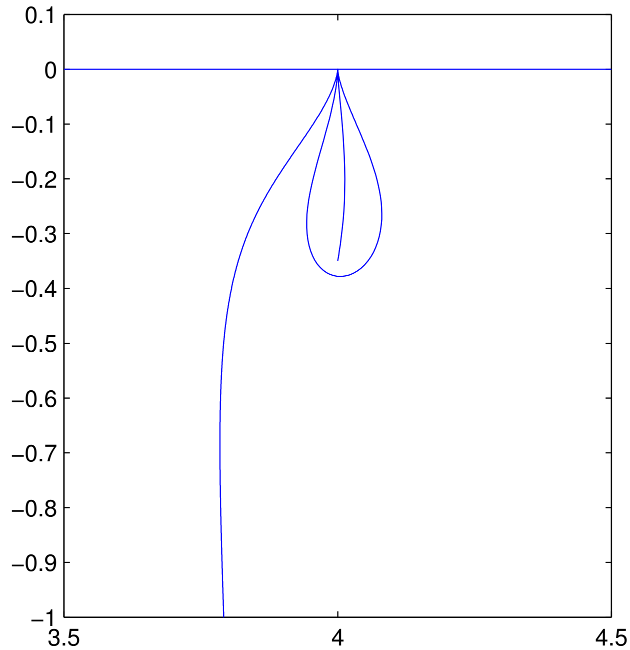

Figure 2. The even resonance curve in Π_4^{even}.

Just as in the odd case, the following statement shows that for each $n \in \mathbb{Z}$ and $c \in [0, 1)$, the rectangle $\Pi_n^{\text{even}}(c)$ contains exactly one resonance.

Theorem 6.5. *If $c = 0$ there is an even resonance of the algebraic multiplicity one at $k = n - i \log(3)/\pi$ for every even $n \in \mathbb{Z}$ and an even eigenvalue of multiplicity one at $k = n$ for every nonzero odd $n \in \mathbb{Z}$. There are no other even resonances.*

The proof follows from the explicit formula

$$F_{\text{even}}(k, 0) = -\tfrac{1}{2} i (e^{ik\pi} - 3)(1 + e^{-ik\pi}).$$

Just as in the odd case, we obtain that the resonances are given by branches of real analytic functions of $c \in [0, 1)$ with values in $\Pi_n^{\text{even}}(c)$. However, in contrast with the odd case, the height of the rectangles $\Pi_n^{\text{even}}(c)$ is not uniformly bounded in c. Moreover:

Theorem 6.6. *Let $n \in \mathbb{Z}$ and let $k_n = k_n(c)$ be the unique solution to $F_{\text{even}}(k, c) = 0$ with $k_n(c) \in \Pi_n^{\text{even}}(c)$. Then $\operatorname{Im} k_n(c) \to -\infty$ as $c \to 1$.*

Proof. Suppose that the conclusion of the theorem is false. Then there exists a sequence $c_m \to 1$ such that $\operatorname{Im} k_n(c_m)$ is bounded. By passing to a subsequence we can assume that $k_n(c_m) \to k_n^\infty \in \mathbb{C}$ as $m \to \infty$. This would imply that $F_{\text{even}}(k_n^\infty, 1) = 0$ by the joint continuity of the function F_{even}. This is impossible by (6-1). $\square$

Therefore, all even resonances move off to infinity and this explains the failure of the Weyl law for $c = 1$. Formal calculations and numerical analysis suggest that the rate of divergence of $\operatorname{Im} k_n(c)$ as $c \to 1$ is logarithmic.

As in the odd case, the even resonance curves intersect the real axis for some rational values of k. A direct computation shows that $F_{\text{even}}(k, c) = 0$ for $k \in \mathbb{R}$ if and only if

$$k = m + n - 1 \quad \text{and} \quad c = \frac{m-n}{m+n-1} \quad \text{for some } n, m \in \mathbb{N}.$$

Figure 2 plots a typical even resonance curve as c increases from 0 to 1. It starts at $4 - i\log(3)/\pi$ when $c = 0$. The curve then passes through 4 when $c = \frac{1}{5}, \frac{3}{5}$ and diverges to ∞ as $c \to 1$.

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IMPROVED LOWER BOUNDS FOR GINZBURG–LANDAU ENERGIES VIA MASS DISPLACEMENT

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We prove some improved estimates for the Ginzburg–Landau energy (with or without a magnetic field) in two dimensions, relating the asymptotic energy of an arbitrary configuration to its vortices and their degrees, with possibly unbounded numbers of vortices. The method is based on a localization of the "ball construction method" combined with a mass displacement idea which allows to compensate for negative errors in the ball construction estimates by energy "displaced" from close by. Under good conditions, our main estimate allows to get a lower bound on the energy which includes a finite order "renormalized energy" of vortex interaction, up to the best possible precision, i.e., with only a $o(1)$ error per vortex, and is complemented by local compactness results on the vortices. Besides being used crucially in a forthcoming paper, our result can serve to provide lower bounds for weighted Ginzburg–Landau energies.

Introduction

We are interested in proving lower bounds and compactness results for Ginzburg–Landau type energies of the form

$$G_\varepsilon(u, A) = \frac{1}{2}\int_{\Omega_\varepsilon} |\nabla_A u|^2 + (\operatorname{curl} A)^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2}$$

where ε is a small parameter, u is a complex-valued function called the *order parameter*, A is $\mathbb{R}^2$-valued and is the vector potential of the magnetic field $h := \operatorname{curl} A$, and $\nabla_A = \nabla - iA$. Here the domain of integration Ω_ε is a smooth bounded domain in $\mathbb{R}^2$, which can depend on ε. We are interested in particular in the case where Ω_ε gets large as $\varepsilon \to 0$. Note that one may set $A \equiv 0$ to recover the simpler Ginzburg–Landau energy

$$E_\varepsilon(u) = \frac{1}{2}\int_{\Omega_\varepsilon} |\nabla u|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2}$$

without a magnetic field. Our results apply to this energy functional by making this trivial choice of A.

The Ginzburg–Landau energy is a famous model for superconductivity. In this model the order-parameter u often has quantized vortices, which are the zeroes of u with nonzero topological degree. Obtaining ansatz-free lower bounds for G_ε in terms of the vortices of u has proven to be crucial in studying the asymptotics of minimizers of G_ε, in particular via Γ-convergence methods.

The first study establishing lower bounds for Ginzburg–Landau was the work of Bethuel, Brezis, and Hélein [Bethuel et al. 1994] for solutions to the Ginzburg–Landau equations without magnetic field

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with energy E_ε bounded by $C|\log\varepsilon|$. Such an energy bound ensures that the total number of vortices remains bounded as $\varepsilon\to 0$. This was later improved and extended in two different directions in [Han and Shafrir 1995] and [Almeida and Bethuel 1998] for arbitrary configurations, still with a number of vortices that remains bounded. The main limitation of such estimates is that the error terms blow up as the number of vortices gets large. Then, Jerrard [1999] and Sandier [1998] introduced the "ball construction method", which provides lower bounds in terms of vortices for arbitrary configurations, allowing unbounded numbers of vortices and much larger energies. This is crucial for many applications, since energy minimizers of the functional with applied magnetic field do not always satisfy a $C|\log\varepsilon|$ bound on their energy. Subsequent refinements of the ball construction method were given (see for example [Sandier and Serfaty 2007, Chapter 4] for a recent result). The lower bound provided by the ball construction method also provides a crucial compactness result on the vorticity (roughly the sum of Dirac masses at the vortex centers, weighted by their degrees). These are the so-called "Jacobian estimates"; see [Jerrard and Soner 2002] and [Sandier and Serfaty 2007, Chapter 6]. They say roughly that the vorticity is controlled by $|\log\varepsilon|^{-1}$ times the energy. For other subsequent works refining those results in a slightly different direction, see also [Sandier and Serfaty 2004; Jerrard and Spirn 2008; Serfaty and Tice 2008].

In a way our objective here can be seen as obtaining next order terms (order 1 as opposed to order $|\log\varepsilon|$) in such estimates, both energy estimates and compactness results.

For a given (u, A), let us define the energy density

$$e_\varepsilon(u,A)=\frac{1}{2}\left(|\nabla_A u|^2+(\operatorname{curl} A)^2+\frac{(1-|u|^2)^2}{2\varepsilon^2}\right).$$

If (u, A) is clear from the context and defined on a set E, we will often use the abbreviation $e_\varepsilon(E)$ for $\int_E e_\varepsilon(u,A)$, and e_ε for the density $e_\varepsilon(u,A)$. We then introduce the measure

$$f_\varepsilon := e_\varepsilon-\pi|\log\varepsilon|\sum_B d_B\delta_{a_B},$$

where the a_B are the centers of the vortex balls constructed via Jerrard's and Sandier's ball construction, the d_B are the degrees of the balls and δ is the Dirac mass. Calculating $\int f_\varepsilon$ corresponds to subtracting off the cost of all vortices from the total energy: what remains should then correspond to the interaction energy between the vortices, which we can call "renormalized energy" by analogy with [Bethuel et al. 1994]. In order to obtain next order estimates of the energy G_ε, we show here lower bounds on the energy $\int f_\varepsilon$, as well as coerciveness properties of f_ε, which say, roughly, that f_ε, or in other words, the renormalized energy, suffices to control the vorticity. (This is again to be compared with the previous ball construction and Jacobian estimate, where the vorticity is controlled by $e_\varepsilon/|\log\varepsilon|$).

The motivation for this is our joint paper [Sandier and Serfaty 2010], where we establish a next-order Γ-convergence result for the Ginzburg–Landau energy with applied magnetic field, and derive a limiting interaction energy between points in the plane, thus making the link to the question of the famous Abrikosov lattice (the Abrikosov lattice is a triangular lattice of vortices in superconductors observed in experiments and predicted by Abrikosov). More precisely, we show there an asymptotic expansion for the minimal energy of the form

$$\min G_\varepsilon = G_\varepsilon^N + N \min W + o(N)$$

where $N \gg 1$ is the optimal number of vortices (determined by the intensity of the applied field), G_ε^N is a constant of order N^2 (the leading order estimate) and W is a renormalized energy governing the pattern formed by the vortices after blow-up at the scale $\sqrt{N}$. Moreover, we show that the patterns formed by the vortices of minimizers after this blow-up minimize W (almost surely, in some sense). We prove in addition that among lattice configurations (of fixed volume), W is uniquely minimized by the triangular lattice. The natural conjecture is that this lattice is also a minimizer among all point configurations, and if this were proved, it would completely justify the emergence of the Abrikosov triangular lattice.

To achieve this, with an error only $o(N)$, we needed lower bounds on the cost of vortices with a precision $o(1)$ per vortex (with still a possibly infinite number of vortices), which is finer than was available in the literature. We also needed to control the (local number of) vortices by the renormalized energy. In fact the energy density we end up having to analyze in [Sandier and Serfaty 2010] is exactly f_ε, and we need to be able to control the vortices through it.

The other problem we need to overcome in that paper is that f_ε is obviously not positive or even bounded below, and this prevents from applying standard lower semicontinuity ideas, and the abstract scheme for Γ-convergence of 2-scale energies which we introduce there. This reflects the fact that the energy e_ε is not exactly where the vortices are, as we will explain below. The remedy which we implement here, is that we can "deform" f_ε into an energy density g_ε which is bounded below and enjoys nice coerciveness properties. To accomplish this we show that we can transport the positive mass in f_ε into the support of the negative mass in f_ε, with mass traveling at most at fixed finite distances (say distance 1), and so that the result of the operation, g_ε, is bounded below. This is done by using the following rather elementary transport lemma:

Lemma 3.1. *Assume f is a finite Radon measure on a compact set A, that Ω is open and that for any positive Lipschitz function ξ in $\mathrm{Lip}_\Omega(A)$, i.e., vanishing on $\Omega \setminus A$,*

$$\int \xi \, df \geq -C_0 |\nabla \xi|_{L^\infty(A)}.$$

Then there exists a Radon measure g on A such that $0 \leq g \leq f_+$ and such that

$$\|f - g\|_{\mathrm{Lip}_\Omega(A)^*} \leq C_0.$$

Thus what is needed is a control on the negative part of f_ε, which will be provided by the ball construction lower bounds and additional improvements of it.

The norm $\|f_\varepsilon - g_\varepsilon\|_{\mathrm{Lip}_\Omega(\Omega)^*}$ will measure how far mass has been displaced in the process. This control appears in Theorem 1.1 below and more particularly Corollary 1.2. Since $\int g_\varepsilon$ will be close to $\int f_\varepsilon$, it also can be seen as a renormalized energy. Since g_ε is bounded below, we can then hope that it enjoys nice coerciveness properties, we can in fact obtain the desired compactness results which allow to control the vorticity locally by g_ε. This will be the object of Theorem 1.5 below.

Finally, let us point out that our results can in principle serve to obtain lower bounds for weighted Ginzburg–Landau energies, see Remark 1.7.

We now describe briefly the method that we use, which allows us to control the negative part of f_ε.

The best vortex ball construction lower bound on e_ε available (such as that in [Sandier and Serfaty 2007, Chapter 4]) is of the following type: given $(u_\varepsilon, A_\varepsilon)$ and any (small) number r, there exists a family of disjoint closed balls $\mathcal{B}$ covering all the zeros of u_ε, the sum of the radii of the balls being bounded above by r, and such that

$$\int_{\bigcup_{B\in\mathcal{B}} B} e_\varepsilon(u_\varepsilon, A_\varepsilon) \geq \pi D\left(\log\frac{r}{\varepsilon D} - C\right), \tag{0-1}$$

where $D = \sum_{B\in\mathcal{B}} |d_B|$ with $d_B = \deg(u_\varepsilon, \partial B)$ if $B \subset \Omega$ and 0 otherwise. We shall reprove here in Proposition 2.1 a version of this result using Jerrard's ball construction.

This above estimate says that a vortex of degree d costs an energy at least $\simeq \pi|d||\log\varepsilon|$, but this is only really true when the vortex is well isolated from other vortices and from the boundary, and if there are not too many of them locally, as the factor r/D in the logarithm above somewhat reflects: an ideal lower bound would be

$$e_\varepsilon(B) \geq \pi|d_B|\left(\log\frac{r}{\varepsilon} - C\right),$$

and compared to this, the lower bound above contains a negative error $-\pi D\log D$ which tends to $-\infty$ if the total number of vortices becomes large when $\varepsilon \to 0$. In truth, this ideal lower bound cannot hold in general as can be seen in the case of n vortices of degree 1 all positioned regularly near the boundary of the domain, a case where (0-1) is optimal.

Moreover the energy density e_ε is not localized exactly where the vortices are: vortices can be viewed as points, while their energy is spread over annular regions around these points. The ball construction lower bounds such as (0-1) capture well the energy which lies very near the vortices, but some energy is missing from it, in particular when vortices accumulate locally around a point. The missing energy in that case can be recovered by the method of "lower bounds on annuli" which we introduced in [Sandier and Serfaty 2003] and used again in [Sandier and Serfaty 2007, Chapter 9]. It is based on the following: Let $B(x_0, r_1)\backslash B(x_0, r_0)$ be an annulus that contains no zeros of u. Roughly speaking we have

$$e_\varepsilon\big(B(x_0, r_1)\backslash B(x_0, r_0)\big) \geq \pi D^2 \log\frac{r_1}{r_0},$$

where $D = \deg(u, \partial B(x_0, r_1)) = \deg(u, \partial B(x_0, r_0))$. In other words, if a fixed size ball in the domain contains some large degree D of vorticity, then there is an energy of order D^2 lying not in that ball, but in a thick enough annulus around that ball. This energy of order D^2 should suffice to "neutralize" the error term $-\pi D\log D$ found above through the ball construction. However, it lies at a certain (finite) distance from the center of the vortices. The main technique is then to combine in a systematic way the ball construction lower bounds and the "lower bounds on annuli", in order to recover enough energy.

Let us finally emphasize a technical difficulty. Since we want a local control on the vortices, the lower bound (0-1) provided by the ball construction is not quite sufficient because it cannot be localized in general, i.e., we cannot deduce a bound for $\int_B e_\varepsilon$ for each $B \in \mathcal{B}$. It is only possible to do so when a matching upper bound on the total in (0-1) is known. See Proposition 2.1 for more details.

The idea for remedying this difficulty is to "localize" the construction, splitting the domain into pieces on which one expects to have a bounded vorticity, then apply the ball construction on each piece, and paste together the constructions and lower bounds obtained this way, whose error terms will now be bounded below by a constant. However, this is not completely easy: one needs to localize the construction and still get a global covering of the vortices by balls while preserving the disjointness of the balls. In applications, trying to split the domain into pieces where the vorticity is expected to be bounded leads us to splitting the domain into very small (as $\varepsilon \to 0$) pieces. Equivalently after rescaling one can consider very large domains cut into bounded size pieces. In other words, in order to be able to treat the case where the vortex density becomes large, we need to be able to treat the case of unbounded domains as $\varepsilon \to 0$.

This is precisely what we do in this paper: we consider possibly large domains. This way we may in practice rescale our domains as much as needed until the local density of vortices remains bounded as $\varepsilon \to 0$. We consider vortex ball constructions obtained over coverings of Ω_ε by domains of fixed size, and we work at pasting together these lower bounds while combining them with the method of lower bounds on annuli, as explained above, and finally retrieving "finite numbers of vortices" estimates (of the type in [Bethuel et al. 1994]) which bound from below the energy f_ε or g_ε by the exact renormalized energy, up to only $o(1)$ errors.

1. Statement of the main results

In this paper we will deal with families $(u_\varepsilon, A_\varepsilon)_\varepsilon$ defined on domains $\{\Omega_\varepsilon\}_\varepsilon$ in $\mathbb{R}^2$ which become large as $\varepsilon \to 0$. The example we have in mind is $\Omega_\varepsilon = \lambda_\varepsilon \Omega$ where Ω is a fixed bounded smooth domain and $\lambda_\varepsilon \to +\infty$ as $\varepsilon \to 0$, but we don't need to make any particular hypothesis on $\{\Omega_\varepsilon\}_\varepsilon$, which could even be a fixed bounded domain.

Next we introduce some notation.

For $E \subset \mathbb{R}^2$ we let

$$\widehat{E} = \{x \in \Omega_\varepsilon, \operatorname{dist}(x, E) \le 1\}.$$

We also define, for any real-valued or vector-valued function f in Ω_ε,

$$\hat{f}(x) = \sup\{|f(y)|, y \in B(x, 1) \cap \Omega_\varepsilon\}.$$

Note that both $\hat{f}$ and $\widehat{E}$ depend on ε, but the value of ε will be clear from the context. The choice of 1 in the definitions is arbitrary but constrains the choice of other constants below.

In all the paper, f_+ and f_- will denote the positive and negative parts of a function or measure, both being positive functions or measures, and $\|f\|$ is the total variation of f. If f and g are two measures then $f \le g$ means that $g - f$ is a nonnegative measure.

Given a family $\{(u_\varepsilon, A_\varepsilon)\}_\varepsilon$, where $u_\varepsilon : \Omega_\varepsilon \to \mathbb{C}$ and $A_\varepsilon : \Omega_\varepsilon \to \mathbb{R}^2$ we define the *currents* and *vorticities* to be

$$j_\varepsilon = (iu_\varepsilon, \nabla_{A_\varepsilon} u_\varepsilon), \qquad \mu_\varepsilon = \operatorname{curl} j_\varepsilon + h_\varepsilon,$$

where $(a, b) = \frac{1}{2}(a\bar{b} + \bar{a}b)$ and $h_\varepsilon = \operatorname{curl} A_\varepsilon$ is the *induced magnetic field.*

We denote by $\mathrm{Lip}_\Omega(A)$ the set of Lipschitz functions on A which are 0 on $\Omega \setminus A$, and let $\|f\|_{\mathrm{Lip}_\Omega(A)^*} = \sup \int \xi \, df$, the supremum being taken over functions $\xi \in \mathrm{Lip}_\Omega(A)$ such that $|\nabla \xi|_{L^\infty(A)} \le 1$.

We say a family $\{f_\alpha\}_\alpha$ is subordinate to a cover $\{A_\alpha\}_\alpha$ if $\mathrm{Supp}(f_\alpha) \subset A_\alpha$ for every α.

Despite the slightly confusing notation, the covering A_α will have nothing to do with the magnetic gauge A_ε. Also, the densities f_α and g_α, as well as n_α and ν_α will implicitly depend on ε, and should be really $f_{\varepsilon,\alpha}$ and $g_{\varepsilon,\alpha}$, etc, but for simplicity we do not indicate this dependence.

Theorem 1.1. *Let $\{\Omega_\varepsilon\}_{\varepsilon>0}$ be a family of bounded open sets in $\mathbb{R}^2$. Assume that $\{(u_\varepsilon, A_\varepsilon)\}_\varepsilon$, where $(u_\varepsilon, A_\varepsilon)$ is defined over Ω_ε, satisfies for some $0 < \beta < 1$ small enough*

$$G_\varepsilon(u_\varepsilon, A_\varepsilon) \le \varepsilon^{-\beta}. \tag{1-1}$$

Then the following holds, for ε small enough:

(1) (vortices). *There exists a measure ν_ε, depending only on u_ε (and not on A_ε) of the form $2\pi \sum_i d_i \delta_{a_i}$ for some points $a_i \in \Omega_\varepsilon$ and some integers d_i such that, C denoting a generic constant independent of ε,*

$$\|\mu_\varepsilon - \nu_\varepsilon\|_{(C_0^{0,1}(\Omega_\varepsilon))^*} \le C\sqrt{\varepsilon}G_\varepsilon(u_\varepsilon, A_\varepsilon), \tag{1-2}$$

and for any measurable set E

$$|\nu_\varepsilon|(E) \le C\frac{e_\varepsilon(\widehat{E})}{|\log \varepsilon|}.$$

(2) (covering). *There exists a cover $\{A_\alpha\}_\alpha$ of Ω_ε by open sets with diameter and overlap number bounded by a universal constant, and measures $\{f_\alpha\}_\alpha$, $\{\nu_\alpha\}_\alpha$ subordinate to this cover such that, letting $f_\varepsilon := e_\varepsilon - \frac{1}{2}|\log \varepsilon|\nu_\varepsilon$,*

$$f_\varepsilon \ge \sum_\alpha f_\alpha, \quad \nu_\varepsilon = \sum_\alpha \nu_\alpha, \quad \nu_{\alpha_1} \perp \nu_{\alpha_2} \text{ for } \alpha_1 \ne \alpha_2.$$

(3) (energy transport). *Letting $n_\alpha := \|\nu_\alpha\|/2\pi$, for each α the following holds*: *If* $\mathrm{dist}(A_\alpha, {\Omega_\varepsilon}^c) > \varepsilon$ *there exists a measure $g_\alpha \ge -C$ such that either*

$$\|f_\alpha - g_\alpha\|_{\mathrm{Lip}_\Omega(A_\alpha)^*} \le Cn_\alpha(1 + \beta|\log \varepsilon|) \quad \text{and} \quad g_\alpha(A_\alpha) \ge cn_\alpha|\log \varepsilon|, \tag{1-3}$$

or

$$\|f_\alpha - g_\alpha\|_{\mathrm{Lip}_\Omega(A_\alpha)^*} \le Cn_\alpha(1 + \log n_\alpha) \quad \text{and} \quad g_\alpha(A_\alpha) \ge c{n_\alpha}^2 - Cn_\alpha, \tag{1-4}$$

where and $c, C > 0$ are universal positive constants.

If $\mathrm{dist}(A_\alpha, {\Omega_\varepsilon}^c) \le \varepsilon$, *there exists $g_\alpha \ge 0$ such that for any function ξ*

$$\int \xi \, d(f_\alpha - g_\alpha) \le Cn_\alpha\big(|\nabla\xi|_{L^\infty(A_\alpha)} + \beta|\log \varepsilon|\,|\xi|_{L^\infty(A_\alpha)}\big). \tag{1-5}$$

(4) (properties of g_ε). *Letting* $g_\varepsilon = f_\varepsilon + \sum_\alpha (g_\alpha - f_\alpha)$, *we have*

$$-C \le g_\varepsilon \le e_\varepsilon + \tfrac{1}{2}|\log \varepsilon|(\nu_\varepsilon)_-, \tag{1-6}$$

and for any measurable set $E \subset \Omega_\varepsilon$,

$$(g_\varepsilon)_-(E) \le C\frac{e_\varepsilon(\widehat{E})}{|\log \varepsilon|}, \quad (g_\varepsilon)_+(E) \le C e_\varepsilon(\widehat{E}). \tag{1-7}$$

Moreover, assuming $|u_\varepsilon| \le 1$ *in* Ω_ε *and that* $E + B(0, C) \subset \Omega_\varepsilon$, *for some* $C > 0$ *large enough, then for every* $p < 2$,

$$\int_E |j_\varepsilon|^p \le C_p\big((g_\varepsilon)_+(E + B(0, C)) + |E|\big). \tag{1-8}$$

The third item admits, or rather implies the following form, from which the covering $\{A_\alpha\}_\alpha$ is hidden.

Corollary 1.2. *Under the hypothesis above and using the same notation, for every* $0 < \eta \le 1$ *we have if* $\varepsilon > 0$ *is small enough*: *First, for every Lipschitz function* ξ *vanishing on* $\partial\Omega_\varepsilon$

$$\int_{\Omega_\varepsilon} \xi \, d(g_\varepsilon - f_\varepsilon) \le C \int_{\Omega_\varepsilon} \widehat{\nabla\xi} \left[d|\nu_\varepsilon| + (\beta + \eta)\, d(g_\varepsilon)_+ + \frac{|\log \eta|^2}{\eta}\, dx \right] + C\beta \int_{\widehat{\partial\Omega_\varepsilon}} \hat{\xi} e_\varepsilon. \tag{1-9}$$

Second, if $d(E, \partial\Omega_\varepsilon) > C$ *then*

$$|\nu_\varepsilon|(E) \le C \left(\eta (g_\varepsilon)_+(\widehat{E}) + \frac{1}{\eta}|\widehat{E}| + \frac{e_\varepsilon(\widehat{E} \cap \widehat{\partial\Omega_\varepsilon})}{|\log \varepsilon|} \right). \tag{1-10}$$

The point in introducing the extra parameter η is that we want to be able to use only a small η-fraction of the "remaining" energy g_ε to control the error $f_\varepsilon - g_\varepsilon$ between the original energy and the displaced one. This corollary is obtained by simply summing the relations (1-3)–(1-5) and controlling n_α and $n_\alpha \log n_\alpha$ by a small fraction of ${n_\alpha}^2$ through the elementary relations

$$x \log x \le \eta x^2 + C\frac{\log^2 \eta}{\eta} \qquad 2x \le \eta x^2 + \frac{1}{\eta}$$

and then controlling ${n_\alpha}^2$ by $g_\alpha(A_\alpha)$ via (1-3) or (1-4).

Remark 1.3. If we let $\eta = 1$ and if E and the support of ξ are at distance at least 1 from $\partial\Omega$, then (1-9) and (1-10) reduce to

$$\int \xi \, d(f_\varepsilon - g_\varepsilon) \le C \int_{\Omega_\varepsilon} \widehat{\nabla\xi} \left[d(g_\varepsilon)_+ + d|\nu_\varepsilon| \right] \tag{1-11}$$

and

$$|\nu_\varepsilon|(E) \le C \left((g_\varepsilon)_+(\widehat{E}) + |\widehat{E}| \right).$$

If one takes $\xi = \chi_R$ to be a positive cut-off function supported in $B(0, R)$ and $\equiv 1$ in $B(0, R-1)$ then the right-hand side in (1-11) scales like a boundary term (i.e., like R) as R gets large, while the left-hand side scales like an interior term.

Remark 1.4. Assume we have proved Theorem 1.1 and Corollary 1.2. Given $\{(u_\varepsilon, A_\varepsilon)\}_\varepsilon$ and $\{\Omega_\varepsilon\}_\varepsilon$ satisfying the hypothesis, we may consider for some fixed $\sigma > 0$ the rescaled quantities $\tilde\varepsilon = \varepsilon/\sigma$, $\tilde x = x/\sigma$ and let

$$\tilde u_\varepsilon(\tilde x) = u_\varepsilon(x), \quad \widetilde{A}_\varepsilon(\tilde x) = \sigma A_\varepsilon(x), \quad \widetilde{\Omega}_\varepsilon = \Omega_\varepsilon/\sigma.$$

Then, letting $h = \operatorname{curl} A$ and $\tilde h = \operatorname{curl} \widetilde{A}$, we have

$$e_\varepsilon^\sigma(u, A) := \sigma^2\left(\frac{1}{2}|\nabla_A u|^2 + \frac{\sigma^2}{2}h^2 + \frac{1}{4\varepsilon^2}(1-|u|^2)^2\right) = \tfrac{1}{2}|\nabla_{\widetilde{A}}\tilde u|^2 + \tfrac{1}{2}\tilde h^2 + \frac{1}{4\tilde\varepsilon^2}(1-|\tilde u|^2)^2.$$

We may then apply the theorem to the tilded quantities, yielding a measure $\tilde g_\varepsilon$. Then if we let $g_\varepsilon(x) = \tilde g_\varepsilon(\tilde x)$, the measure g_ε will satisfy the properties stated in Theorem 1.1 and Corollary 1.2, with e_ε replaced by e_ε^σ (and with a different C) provided we modify the definition of $\widehat{E}$ to

$$\widehat{E} = \{x \mid \operatorname{dist}(\tilde x, \widetilde{E}) < 1\} = \{x \mid \operatorname{dist}(x, E) < \sigma\},$$

(note that we can keep the original definition provided $\sigma \le 1$).

Then we may add to both e_ε and g_ε the quantity $\left(\frac{1}{2} - \frac{1}{2}\sigma^2\right){h_\varepsilon}^2$ and obtain in this manner a new g_ε satisfying the listed properties and — for the particular choice $\sigma^2 = \frac{1}{2}$ — the lower bound

$$g_\varepsilon \ge \frac{{h_\varepsilon}^2}{4} - C. \tag{1-12}$$

We will then usually assume when applying Theorem 1.1 that this lower bound holds as well as the other conclusions of the theorem.

The next result shows how g_ε has the desired coerciveness properties, and behaves like the renormalized energy. Indeed, under the assumption that the family $\{g_\varepsilon\}_\varepsilon$ is bounded on compact sets (recall that the domains become increasingly large as $\varepsilon \to 0$) we have compactness results for the vorticities and currents, and lower bounds on $\int g_\varepsilon$ (hence $\int f_\varepsilon$ via (1-9)) in terms of the renormalized energy W.

Before stating that result, we introduce some additional notation. We denote by $\{\boldsymbol{U}_R\}_{R>0}$ a family of sets in $\mathbb{R}^2$ such that, for some constant $C > 0$ independent of R,

$$\boldsymbol{U}_R + B(0,1) \subset \boldsymbol{U}_{R+C} \quad \text{and} \quad \boldsymbol{U}_{R+1} \subset \boldsymbol{U}_R + B(0, C). \tag{1-13}$$

For example, $\{\boldsymbol{U}_R\}_{R>0}$ can be the family $\{B_R\}_{R>0}$ of balls centered at 0 of radius R.

Then we use the notation $\chi_{\boldsymbol{U}_R}$ for cutoff functions satisfying, for some C independent of R,

$$|\nabla\chi_{\boldsymbol{U}_R}| \le C, \quad \operatorname{Supp}(\chi_{\boldsymbol{U}_R}) \subset \boldsymbol{U}_R, \quad \chi_{\boldsymbol{U}_R}(x) = 1 \text{ if } \operatorname{dist}(x, \boldsymbol{U}_R^c) \ge 1. \tag{1-14}$$

Finally, given a vector field $j : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\operatorname{curl} j = 2\pi\sum_{p\in\Lambda}\delta_p + h$ with Λ, where h is in L^2_{loc} and Λ a discrete set, we define the *renormalized energy* of j by

$$W(j) = \limsup_{R\to\infty} \frac{W(j, \chi_{B_R})}{|B_R|},$$

where for any χ

$$W(j,\chi)=\liminf_{\eta\to 0}\left(\frac{1}{2}\int_{\mathbb{R}^2\setminus\bigcup_{p\in\Lambda}B(p,\eta)}\chi|j|^2+\pi\log\eta\sum_{p\in\Lambda}\chi(p)\right). \tag{1-15}$$

Various results on W, in particular on its minimizers, are proved in [Sandier and Serfaty 2010]. Note in particular that if we assume $\operatorname{div} j=0$, then the lim inf in (1-15) is in fact a limit, because in this case $j=\nabla^\perp H$ with $\Delta H=2\pi\delta_p+h$ in a neighborhood of p, and thus $H=\log|\cdot-p|+f$ with $f\in H^1$ in this neighborhood.

Theorem 1.5. *Let the hypothesis of Theorem 1.1 hold, and assume $|u_\varepsilon|\le 1$ in Ω_ε.*

(1) *Assume that* $\operatorname{dist}(0,\partial\Omega_\varepsilon)\to+\infty$ *as $\varepsilon\to 0$ and that, for any $R>0$,*

$$\limsup_{\varepsilon\to 0} g_\varepsilon(\boldsymbol{U_R})\,dx<+\infty, \tag{1-16}$$

where $\{\boldsymbol{U_R}\}_{\boldsymbol{R}}$ satisfies (1-13). *Then, up to extraction of a subsequence, the vorticities $\{\mu_\varepsilon\}_\varepsilon$ converge in $W^{-1,p}_{\mathrm{loc}}(\mathbb{R}^2)$ to a measure ν of the form $2\pi\sum_{p\in\Lambda}\delta_p$, where Λ is a discrete subset of $\mathbb{R}^2$, the currents $\{j_\varepsilon\}_\varepsilon$ converge weakly in $L^p_{\mathrm{loc}}(\mathbb{R}^2,\mathbb{R}^2)$ for any $p<2$ to j, and the induced fields $\{h_\varepsilon\}_\varepsilon$ converge weakly in $L^2_{\mathrm{loc}}(\mathbb{R}^2)$ to h which are such that*

$$\operatorname{curl} j=\nu-h \quad \text{in } \mathbb{R}^2.$$

(2) *If we replace the assumption* (1-16) *by the stronger assumption*

$$\limsup_{\varepsilon\to 0} g_\varepsilon(\boldsymbol{U_R})<CR^2, \tag{1-17}$$

where C is independent of R, then the limit j of the currents satisfies, for any $p<2$,

$$\limsup_{R\to+\infty}\fint_{\boldsymbol{U_R}}|j|^p\,dx<+\infty. \tag{1-18}$$

Moreover for every family $\chi_{\boldsymbol{U_R}}$ satisfying (1-14) *we have*

$$\liminf_{\varepsilon\to 0}\int_{\mathbb{R}^2}\frac{\chi_{\boldsymbol{U_R}}}{|\boldsymbol{U_R}|}\,dg_\varepsilon\ge\left(\frac{W(j,\chi_{\boldsymbol{U_R}})}{|\boldsymbol{U_R}|}+\frac{1}{2}\fint_{\boldsymbol{U_R}}h^2+\frac{\gamma}{2\pi}\fint_{\boldsymbol{U_R}}h\right)+o_{\boldsymbol{R}}(1), \tag{1-19}$$

where γ is a constant defined below and $o_{\boldsymbol{R}}(1)$ is a function tending to 0 as $R\to+\infty$.

Remark 1.6. The constant γ in (1-19) was introduced in [Bethuel et al. 1994] and may be defined by

$$\gamma=\lim_{R\to\infty}\left(\frac{1}{2}\int_{B_R}|\nabla u_0|^2+\frac{(1-|u_0|^2)^2}{2}-\pi\log R\right),$$

where $u_0(r,\theta)=f(r)e^{i\theta}$ is the unique (up to translation and rotation) radially symmetric degree-one vortex. See [Bethuel et al. 1994; Mironescu 1996].

Remark 1.7. Lower bounds immediately follow from this theorem. Indeed f_ε is the energy density minus the energetic cost of a vortex, and $f_\varepsilon - g_\varepsilon$ is controlled by Theorem 1.1; see also Remark 1.3. This, combined with the lower bound (1-19) shows that in good cases the averages over large balls of f_ε are bounded below by W plus explicit constants, which proves a sharp lower bound for the energy with a $o(1)$ order error, à la [Bethuel et al. 1994].

The bound (1-9) may also be interpreted as a lower bound for the Ginzburg–Landau energy with weight. Assuming a fixed domain Ω and $G_\varepsilon(u_\varepsilon, A_\varepsilon) < C|\log \varepsilon|$ for instance, and that $\mu_\varepsilon \to 2\pi \sum_{i=1}^n \delta_{a_i}$, where $a_i \in \Omega$, then by blowing up by a factor independent of ε we may assume the points are at distance 2, say, from the boundary and then if ξ is a fixed positive weight we may multiply it by a cutoff $0 \le \chi \le 1$ equal to zero on $\widehat{\partial\Omega}$ and equal to 1 at each a_i. Then (1-9) becomes

$$\int_\Omega \xi e_\varepsilon \ge \pi |\log \varepsilon| \sum_{i=1}^n \xi(a_i) + \int \chi\xi \, dg_\varepsilon - C \int \widehat{\nabla(\chi\xi)} \left(d|\nu_\varepsilon| + (\beta+\eta)\, d(g_\varepsilon)_+ + \frac{|\log \eta|^2}{\eta}\, dx \right).$$

Typically, there will be an upper bound for the energy which implies that $(g_\varepsilon)_+(\Omega) < C$ and since also $g_\varepsilon \ge -C$, the integrals on the right-hand side may be bounded below by a constant independent of ε.

The paper is organized as follows: In Section 2 we state without proof the result on lower bounds via Jerrard's ball construction (the proof is postponed to Section 5) which we adapt for our purposes, and explain how we use it on a covering of Ω_ε by a collection U_α of balls of finite size. In Section 3, we present the tool used to transport the negative part of f_ε to absorb it into the positive part, and deduce Theorem 1.1. In Section 4, we prove Theorem 1.5. Finally in Section 5, we prove the ball-construction lower bound.

2. Use of the ball construction and coverings of the domain

The first step consists in performing a ball construction in Ω_ε in order to obtain lower bounds. This follows essentially the method of [Jerrard 1999], the difficulty being that we are not allowed more than an error of order one per vortex. This is hopeless if the total number of vortices diverges when $\varepsilon \to 0$, hence we need to localize the construction in pieces of Ω_ε small enough for the number of vortices in each piece to remain bounded as $\varepsilon \to 0$.

The ball construction lower bound. We start by stating the result of Jerrard's ball construction in a version adapted to our situation, in particular including the magnetic field. The proof is postponed to Section 5. In all what follows, if $\mathcal{B}$ is a collection of balls, $r(\mathcal{B})$ denotes the sum of the radii of the balls in the collection. In all the sequel we will sometimes abuse notation by writing $\mathcal{B}$ for $\bigcup_{B\in\mathcal{B}} B$, identifying the collection of balls with the set it covers.

Proposition 2.1. *There exist $\varepsilon_0, C > 0$ such that if $U \subset \mathbb{R}^2$, $\varepsilon \in (0, \varepsilon_0)$, and $(u_\varepsilon, A_\varepsilon)$ defined on U are such that $G_\varepsilon(u_\varepsilon, A_\varepsilon) \le \varepsilon^{-\beta}$, where $\beta \in (0,1)$, the following holds. For every $r \in (C\varepsilon^{1-\beta}, \frac{1}{2})$, there exists a collection of disjoint closed balls $\mathcal{B}$ depending only on u_ε (and not on A_ε) such that, letting*

$U_\varepsilon = \{x \mid d(x, U^c) > \varepsilon\}$, we have

(1) $\{x \in U_\varepsilon \mid |u_\varepsilon(x)| < \frac{1}{2}\} \subset \mathcal{B}$,

(2) $r(\mathcal{B}) \le r$, *and*

(3) *for any* $2 \le \overline{C} \le (r/\varepsilon)^{1/2}$, *either* $e_\varepsilon(\mathcal{B} \cap U) \ge \overline{C} \log \frac{r}{\varepsilon}$ *or*

$$e_\varepsilon(B) \ge \pi |d_B| \left(\log \frac{r}{\varepsilon \overline{C}} - C \right) \quad \textit{for all } B \in \mathcal{B} \textit{ such that } B \subset U_\varepsilon,$$

where $d_B = \deg(u_\varepsilon, \partial B)$.

A natural choice of $\overline{C}$ above is πD, where $D = \sum_{B \in \mathcal{B}} |d_B|$ and we have let $d_B = 0$ if $B \not\subset U_\varepsilon$. With this choice we find in all cases

$$e_\varepsilon(\mathcal{B} \cap U) \ge \pi D \left(\log \frac{r}{\varepsilon D} - C \right)$$

i.e., we recover the same lower bound as in [Sandier and Serfaty 2007, Theorem 4.1], mentioned in the introduction as (0-1). The reason why we don't simply use that theorem directly is that we need to keep the dichotomy above, and thus a lower bound localized in each ball.

Localizing the ball construction. For any $\varepsilon > 0$ we construct an open cover $\{U_\alpha\}_\alpha$ of Ω_ε as follows: We consider the collection $\mathcal{B}$ of balls of radius ℓ_0 — where $\ell_0 \in (0, \frac{1}{8})$ is to be chosen below, small enough but independent of ε — centered at the points of $\ell_0 \mathbb{Z}^2$. The cover consists of the open sets $\Omega_\varepsilon \cap B$, for $B \in \mathcal{B}$.

This cover depends on ε, but the maximal number of *neighbors* of a given α — defined as the indices β such that $U_\alpha \cap U_\beta \ne \varnothing$ — is bounded independently of ε by an integer we denote by m (in fact $m = 9$). Note that m also bounds the *overlap number* of the cover, that is, the maximal number of U_α's to which a given x can belong. There is also $\ell > 0$ independent of ε which is a Lebesgue number of the cover, i.e., such that for every $x \in \Omega_\varepsilon$, there exists α such that $B(x, \ell) \cap \Omega_\varepsilon \subset U_\alpha$ or, equivalently, $\operatorname{dist}(x, \Omega_\varepsilon \cap U_\alpha^c) \ge \ell$.

Assuming $\beta < \frac{1}{4}$, and applying Proposition 2.1 to $(u_\varepsilon, A_\varepsilon)$ in U_α for every α we obtain, since $\sqrt{\varepsilon} > C\varepsilon^{1-\beta}$ if ε is small enough, a collection $\mathcal{B}_\varepsilon^{\alpha, r}$ for every $\sqrt{\varepsilon} \le r \le \frac{1}{2}$.

If ρ is chosen small enough depending on ℓ and m only, thus less than a universal constant, we may extract from $\bigcup_\alpha \mathcal{B}_\varepsilon^{\alpha,\rho}$ a subcollection $\mathcal{B}_\varepsilon$ such that any two balls B, B' in $\mathcal{B}_\varepsilon$ satisfy $\Omega_\varepsilon \cap B \cap B' = \varnothing$. We will say $\mathcal{B}_\varepsilon$ is *disjoint in* Ω_ε:

Proposition 2.2. *Assume $\rho \le \ell/(8m)$. Then, writing in short $\mathcal{B}_\varepsilon^\alpha$ instead of $\mathcal{B}_\varepsilon^{\alpha,\rho}$, there exists a subcollection of $\bigcup_\alpha \mathcal{B}_\varepsilon^\alpha$ — call it $\mathcal{B}_\varepsilon$ — which is disjoint in Ω_ε and such that*

$$\{|u_\varepsilon| \le \tfrac{1}{2}\} \cap \{x \mid \operatorname{dist}(x, {\Omega_\varepsilon}^c) > \varepsilon\} \subset \bigcup_{B \in \mathcal{B}_\varepsilon} B. \tag{2-1}$$

Moreover, for every $B \in \mathcal{B}_\varepsilon \cap \mathcal{B}_\varepsilon^\alpha$ we have $B \cap \Omega_\varepsilon = B \cap U_\alpha$ and

$$\operatorname{dist}(B, {\Omega_\varepsilon}^c) > \varepsilon \iff \operatorname{dist}(B, {U_\alpha}^c) > \varepsilon.$$

Proof. Assume $C = \Omega_\varepsilon \cap (B_1 \cup \cdots \cup B_k)$ is a connected component of $\Omega_\varepsilon \cap \big(\bigcup_\alpha \mathcal{B}_\varepsilon^\alpha\big)$. Reordering if necessary, we may assume that $B_i \cap (B_1 \cup \cdots \cup B_{i-1}) \neq \varnothing$ for every $1 \le i \le k$. There exists $x \in \Omega_\varepsilon \cap B_1$ and α such that $\mathrm{dist}(x, \Omega_\varepsilon \cap U_\alpha^c) \ge \ell$. Then $\mathrm{dist}(B_1, \Omega_\varepsilon \cap U_\alpha^c) > 3\ell/4$. Assume

$$\mathrm{dist}(B_1 \cup \cdots \cup B_{i-1}, \Omega_\varepsilon \cap U_\alpha^c) \ge \frac{3\ell}{4}.$$

Then $\mathrm{dist}(B_i, \Omega_\varepsilon \cap U_\alpha^c) > \ell/2$ hence for every $1 \le j \le i$ the ball B_j belongs to $\mathcal{B}_\varepsilon^\beta$, where β is a neighbor of α. It follows that $r_1 + \cdots + r_i \le m\rho \le \ell/8$, where r_i is the radius of B_i, and we deduce that $B_1 \cup \cdots \cup B_i \subset B(x, \ell/4)$ and then

$$\mathrm{dist}(B_1 \cup \cdots \cup B_i, \Omega_\varepsilon \cap U_\alpha^c) \ge \frac{3\ell}{4}.$$

We have thus proved by induction that $C \subset U_\alpha$ and even that $\mathrm{dist}(C, \Omega_\varepsilon \cap U_\alpha^c) \ge 3\ell/4$ for every i.

We delete from $\{B_1, \dots, B_k\}$ the balls which do not belong to $\mathcal{B}_\varepsilon^\alpha$ and call C' the union of the remaining balls. If y belongs to

$$C \cap \{|u_\varepsilon| \le \tfrac{1}{2}\} \cap \{x \mid \mathrm{dist}(x, {\Omega_\varepsilon}^c) > \varepsilon\}$$

then, since $\mathrm{dist}(C, \Omega_\varepsilon \cap U_\alpha^c) \ge 3\ell/4$ and $\mathrm{dist}(y, {\Omega_\varepsilon}^c) > \varepsilon$, provided $\varepsilon < 3\ell/4$ we have that $\mathrm{dist}(y, {U_\alpha}^c) > \varepsilon$ hence y belongs to some ball $B \in \mathcal{B}_\varepsilon^\alpha$ (since $\mathcal{B}_\varepsilon^\alpha$ covers the set $\{|u_\varepsilon| \le \frac{1}{2}\} \cap \{\mathrm{dist}(x, U_\alpha^c) > \varepsilon\}$), thus $y \in C'$. The balls in C' are disjoint in Ω_ε since they belong to the collection $\mathcal{B}_\varepsilon^\alpha$ which is itself disjoint in Ω_ε.

Performing this operation on each connected component of $\Omega_\varepsilon \cap \big(\bigcup_\alpha \mathcal{B}_\varepsilon^\alpha\big)$ we thus obtain a collection $\mathcal{B}_\varepsilon$ which covers $\{|u_\varepsilon| \le \frac{1}{2}\} \cap \{x \mid \mathrm{dist}(x, {\Omega_\varepsilon}^c) > \varepsilon\}$ and is disjoint in Ω_ε. Moreover, if $B \in \mathcal{B}_\varepsilon \cap \mathcal{B}_\varepsilon^\alpha$ then $\mathrm{dist}(B, \Omega_\varepsilon \cap U_\alpha^c) \ge 3\ell/4$ hence $B \cap \Omega_\varepsilon = B \cap U_\alpha$ and

$$\mathrm{dist}(B, {\Omega_\varepsilon}^c) > \varepsilon \iff \mathrm{dist}(B, {U_\alpha}^c) > \varepsilon. \qquad \square$$

The value of ρ will be fixed smaller than $\ell/8m$ and independent of ε, as specified below. Proposition 2.2 provides us for any $\varepsilon > 0$ small enough with collections of balls $\mathcal{B}_\varepsilon$ and $\mathcal{B}_\varepsilon^\alpha$.

Definition 2.3. For any $\sqrt{\varepsilon} \le r \le \rho$ and any $B \in \mathcal{B}_\varepsilon^\alpha$, we let $\mathcal{B}_\varepsilon^{B,r}$ be the collection of balls in $\mathcal{B}_\varepsilon^{\alpha,r}$ which are included in B. Then we let

$$\mathcal{B}_\varepsilon^r = \bigcup_{B \in \mathcal{B}_\varepsilon} \mathcal{B}_\varepsilon^{B,r}.$$

It is disjoint in Ω_ε and covers the set $\{|u_\varepsilon| \le \frac{1}{2}\} \cap \{x \mid \mathrm{dist}(x, {\Omega_\varepsilon}^c) > \varepsilon\}$ and of course if $B \in \mathcal{B}_\varepsilon^r \cap \mathcal{B}_\varepsilon^{\alpha,r}$, then $B \cap \Omega_\varepsilon = B \cap U_\alpha$ and

$$\mathrm{dist}(B, {\Omega_\varepsilon}^c) > \varepsilon \iff \mathrm{dist}(B, {U_\alpha}^c) > \varepsilon.$$

In other words, the disjoint collection $\mathcal{B}_\varepsilon$ permits us to construct disjoint collections of smaller radius by discarding from $\mathcal{B}_\varepsilon^{\alpha,r}$ those balls which are inside a ball discarded from $\mathcal{B}_\varepsilon^{\alpha,\rho}$. The collection $\mathcal{B}_\varepsilon^{\sqrt{\varepsilon}}$ should be seen as the collection of "small balls" and $\mathcal{B}_\varepsilon$ (obtained from $\mathcal{B}_\varepsilon^{\alpha,\rho}$) as the collection of "large balls". We will sometimes also use the collection of the intermediate size balls $\mathcal{B}_\varepsilon^r$ with $\sqrt{\varepsilon} \le r \le \rho$.

Finally we let

$$\nu_\varepsilon = \sum_{\substack{B\in\mathcal{B}_\varepsilon^{\sqrt{\varepsilon}}\\ \mathrm{dist}(B,\Omega_\varepsilon{}^c)>\varepsilon}} 2\pi d_B\delta_{a_B}, \quad |\nu_\varepsilon| = \sum_{\substack{B\in\mathcal{B}_\varepsilon^{\sqrt{\varepsilon}}\\ \mathrm{dist}(B,\Omega_\varepsilon{}^c)>\varepsilon}} 2\pi |d_B|\delta_{a_B}, \tag{2-2}$$

where a_B is the center of B, and d_B denotes the winding number of $u_\varepsilon/|u_\varepsilon|$ restricted to ∂B. This is the ν_ε given by the conclusion of the theorem. Note that since the balls only depend on u_ε (and not on A_ε), ν_ε satisfies the same. If B is any ball which does not cross the boundary of balls in $\mathcal{B}_\varepsilon^{\sqrt{\varepsilon}}$ and $\mathrm{dist}(B,\Omega_\varepsilon{}^c) > \varepsilon$ then $\nu_\varepsilon(B) = 2\pi d_B$. From the Jacobian estimate (see [Jerrard and Soner 2002] or [Sandier and Serfaty 2007, Theorem 6.1]) we have that (1-2) is satisfied.

Lemma 2.4. *There exists $\varepsilon_0 > 0$ such that if $\beta < \frac14$ in* (1-1) *and $\varepsilon < \varepsilon_0$ then*

$$|\nu_\varepsilon|(E) \le 16\,\frac{e_\varepsilon(\Omega_\varepsilon\cap\widehat{E})}{|\log\varepsilon|}$$

for any measurable set E, so that choosing $E = \Omega_\varepsilon$ and taking logarithms,

$$\log\|\nu_\varepsilon\| \le \beta|\log\varepsilon| + C, \tag{2-3}$$

where $\|\cdot\|$ denotes the total variation of a measure.

Proof. We use the properties of $\mathcal{B}_\varepsilon^{\alpha,\sqrt{\varepsilon}}$. Letting $\overline{C} = (\sqrt{\varepsilon}/\varepsilon)^{1/2} = \varepsilon^{-1/4}$, it is impossible when ε is small enough that $e_\varepsilon(\Omega_\varepsilon\cap\mathcal{B}_\varepsilon^{\alpha,\sqrt{\varepsilon}}) \ge \overline{C}\log(\sqrt{\varepsilon}/\varepsilon)$ since we assumed that $e_\varepsilon(\Omega_\varepsilon) \le \varepsilon^{-\beta}$. Thus Proposition 2.1 implies that, for every $B\in\mathcal{B}_\varepsilon^{\alpha,\sqrt{\varepsilon}}$ such that $\mathrm{dist}(B,U_\alpha{}^c) > \varepsilon$,

$$e_\varepsilon(B) \ge \pi|d_B|(\log\varepsilon^{-1/4} - C) \ge \frac{\pi}{8}|d_B||\log\varepsilon|,$$

if ε is small enough. If, moreover, $B\in\mathcal{B}_\varepsilon^{\sqrt{\varepsilon}}$, then Definition 2.3 implies that $\mathrm{dist}(B,U_\alpha{}^c) > \varepsilon$ if and only if $\mathrm{dist}(B,\Omega_\varepsilon{}^c) > \varepsilon$. Hence, using (2-2) and the fact that balls in $\mathcal{B}_\varepsilon^{\sqrt{\varepsilon}}$ have radius smaller than $\frac12$ if ε is small enough, we obtain for any set E

$$|\nu_\varepsilon|(E) \le \sum|\nu_\varepsilon|(B) \le 16\frac{e_\varepsilon(\Omega_\varepsilon\cap\widehat{E})}{|\log\varepsilon|},$$

where the sum is over all B intersecting E and satisfying $B\in\mathcal{B}_\varepsilon^{\sqrt{\varepsilon}}$ and $\mathrm{dist}(B,\Omega_\varepsilon{}^c) > \varepsilon$. □

Definition 2.5. For any α, let ν_α denote the restriction of ν_ε to the balls in $\mathcal{B}_\varepsilon\cap\mathcal{B}_\varepsilon^\alpha$ and $n_\alpha = \|\nu_\alpha\|/2\pi$, so that

$$\nu_\varepsilon = \sum_\alpha \nu_\alpha, \quad n_\alpha = \sum_{B\in\mathcal{B}_\varepsilon\cap\mathcal{B}_\varepsilon^\alpha}\frac{|\nu_\varepsilon|(B)}{2\pi}, \quad \|\nu_\varepsilon\| = 2\pi\sum_\alpha n_\alpha.$$

We also define

$$\overline{C}_\alpha = \begin{cases} \max\Big(Mn_\alpha, \dfrac{3e_\alpha}{|\log\varepsilon|}\Big) & \text{if } n_\alpha \ne 0,\\ 2 & \text{otherwise,}\end{cases} \tag{2-4}$$

where M is a large universal constant to be chosen later and $e_\alpha = \sum_{B\in\mathcal{B}_\varepsilon^\alpha} e_\varepsilon(B\cap U_\alpha)$.

Note that n_α is the sum of the absolute values of the degrees of the small balls included in the large balls of $\mathcal{B}_\varepsilon^\alpha$.

Proposition 2.6. *There exist $\varepsilon_0, C_0 > 0$ such that if $\beta < \frac{1}{4}$ in* (1-1) *and $\varepsilon < \varepsilon_0$, $\varepsilon^{1/2} < r < \rho$ then $2 \le \overline{C}_\alpha \le (r/\varepsilon)^{1/2}$ and for any $B \in \mathcal{B}_\varepsilon^r \cap \mathcal{B}_\varepsilon^{\alpha,r}$ such that* $\operatorname{dist}(B, {\Omega_\varepsilon}^c) > \varepsilon$ *we have*

$$e_\varepsilon(B) \ge 2\pi |d_B| \Lambda_\varepsilon^{\alpha,r}, \quad \text{where} \quad \Lambda_\varepsilon^{\alpha,r} = \frac{1}{2}\left(\log \frac{r}{\varepsilon \overline{C}_\alpha} - C_0\right). \tag{2-5}$$

Moreover, $0 \le \Lambda_\varepsilon^{\alpha,r} \le \frac{1}{2}|\log \varepsilon|$ *and*

$$0 \le \tfrac{1}{2}|\log \varepsilon| - \Lambda_\varepsilon^{\alpha,r} \le \tfrac{1}{2}(\beta|\log \varepsilon| + |\log r| + C_0). \tag{2-6}$$

Proof. From the definition (2-4), from (1-1) and Lemma 2.4 we have for ε small enough that $2 \le \overline{C}_\alpha \le \varepsilon^{-\beta}$. It follows that if $\varepsilon^{1/2} < r < 1$ then $2 \le \overline{C}_\alpha \le (r/\varepsilon)^{1/2}$, since $\beta < \frac{1}{4}$. Also, from the definition of $\overline{C}_\alpha$ it is impossible that $e_\varepsilon(\mathcal{B}_\varepsilon^{\alpha,r} \cap U_\alpha) \ge \overline{C}_\alpha \log(r/\varepsilon)$ since for $\sqrt{\varepsilon} \le r \le \rho$ we have $\overline{C}_\alpha \ge 3e_\varepsilon(\mathcal{B}_\varepsilon^{\alpha,r})/|\log \varepsilon|$.

Then from Proposition 2.1, letting $\overline{C} = \overline{C}_\alpha$, we deduce (2-5) for any $B \in \mathcal{B}_\varepsilon^{\alpha,r}$ with $\operatorname{dist}(B, {U_\alpha}^c) > \varepsilon$, which is equivalent to $\operatorname{dist}(B, {\Omega_\varepsilon}^c > \varepsilon)$ if $B \in \mathcal{B}_\varepsilon^r \cap \mathcal{B}_\varepsilon^{\alpha,r}$.

Finally, $r/(\varepsilon \overline{C}_\alpha) \ge \varepsilon^{-1/4}$ using $\overline{C}_\alpha \le (r/\varepsilon)^{1/2}$ and $r \ge \sqrt{\varepsilon}$, which easily implies that $\Lambda_\varepsilon^{\alpha,r} > 0$ if ε is small enough, and $\Lambda_\varepsilon^{\alpha,r} \le \frac{1}{2}|\log \varepsilon|$ is clear from the definition. The last inequality in (2-6) then follows from $\frac{1}{2}|\log \varepsilon| - \Lambda_\varepsilon^{\alpha,r} = \frac{1}{2}\big(\log(\overline{C}_\alpha/r) + C_0\big)$, since $\overline{C}_\alpha \le \varepsilon^{-\beta}$. □

3. Mass transport

We proceed to study the displacement of the negative part of

$$f_\varepsilon = e_\varepsilon - \tfrac{1}{2}|\log \varepsilon| \nu_\varepsilon.$$

Abstract lemmas. For the displacements we will use two lemmas. The first one was already stated in the introduction and uses optimal transportation for the 1-Wasserstein distance (or minimal connection cost).

Lemma 3.1. *Assume f is a finite Radon measure on a compact set A, that Ω is open, and that for any positive Lipschitz function ξ in* $\operatorname{Lip}_\Omega(A)$, *i.e., vanishing on* $\Omega \setminus A$,

$$\int \xi \, df \ge -C_0 |\nabla \xi|_{L^\infty(A)}.$$

Then there exists a Radon measure g on A such that $0 \le g \le f_+$ and such that

$$\|f - g\|_{\operatorname{Lip}_\Omega(A)^*} \le C_0.$$

Proof. The proof uses convex analysis. Let $X = C(A)$ denotes the space of continuous functions and for $\xi \in X$ let

$$\varphi(\xi) = \int \xi_+ \, df_+ \quad \text{and} \quad \psi(\xi) = \begin{cases} +\infty & \text{if } |\nabla \xi|_{L^\infty(A)} > 1 \text{ or } \xi \notin \operatorname{Lip}_\Omega(A), \\ -\int \xi df & \text{otherwise.} \end{cases}$$

Then ψ is lower semicontinuous because $\{\xi \in \mathrm{Lip}_\Omega(A) \mid |\nabla\xi|_{L^\infty} \le 1\}$ is closed under uniform convergence, and φ is continuous. Both functions are convex, and finite for $\xi = 0$. Then the theorem of Fenchel and Rockafellar (see for instance [Ekeland and Témam 1999]) yields

$$\inf_X (\varphi + \psi) = \max_{\mu\in X^*} \big(-\varphi^*(-\mu) - \psi^*(\mu)\big),$$

where X^* is the dual of X (i.e., the space of Radon measures on A) and

$$\varphi^*(\mu) = \sup_{\xi\in X} \int \xi\, d\mu - \int \xi_+\, df_+ = \begin{cases} 0 & \text{if } 0 \le \mu \le f_+, \\ +\infty & \text{otherwise,} \end{cases}$$
$$\psi^*(\mu) = \sup_{\substack{\xi\in\mathrm{Lip}_\Omega \\ |\nabla\xi|_\infty\le 1}} \int \xi\, d\mu + \int \xi\, df = \|\mu + f\|_{\mathrm{Lip}^*_\Omega}.$$

We deduce that

$$\inf_{\substack{\xi\in\mathrm{Lip}_\Omega \\ |\nabla\xi|_{L^\infty}\le 1}} \int \xi_+\, df_+ - \int \xi\, df = \max_{0\le -\mu\le f_+} \big(-\|\mu + f\|_{\mathrm{Lip}^*_\Omega}\big)$$

and then the existence of a Radon measure g such that $-g$ maximizes the right-hand side, i.e., such that $0 \le g \le f_+$ and

$$-\|f - g\|_{\mathrm{Lip}^*_\Omega} = \inf_{\substack{\xi\in\mathrm{Lip}_\Omega \\ |\nabla\xi|_{L^\infty}\le 1}} \int \xi_+\, df_+ - \int \xi\, df.$$

But

$$\begin{aligned}
\inf_{\substack{\xi\in\mathrm{Lip}_\Omega \\ |\nabla\xi|_{L^\infty}\le 1}} \int \xi_+\, df_+ - \int \xi\, df &= -\sup_{\substack{\xi\in\mathrm{Lip}_\Omega \\ |\nabla\xi|_{L^\infty}\le 1}} \left(\int \xi\, df - \int \xi_+\, df_+\right) \\
&= -\sup_{\substack{\xi\in\mathrm{Lip}_\Omega \\ |\nabla\xi|_{L^\infty}\le 1}} \left(\int \xi_+\, d(f - f_+) - \int \xi_-\, df\right) \\
&= -\sup_{\substack{\xi\in\mathrm{Lip}_\Omega \\ |\nabla\xi|_{L^\infty}\le 1}} \left(-\int \xi_-\, df\right) = \inf_{\substack{\xi\in\mathrm{Lip}_\Omega \\ |\nabla\xi|_{L^\infty}\le 1}} \int \xi_-\, df.
\end{aligned}$$

The assumption of the lemma implies that this last right-hand side is at least $-C_0$; therefore

$$\|f - g\|_{\mathrm{Lip}_\Omega(A)^*} \le C_0. \qquad \square$$

Lemma 3.2. *Assume f is a finite Radon measure supported in Ω and such that $f(\Omega) \ge 0$. Then there exists $0 \le g \le f_+$ such that for any Lipschitz function ξ*

$$\int_\Omega \xi\, d(f - g) \le 2\,\mathrm{diam}(\Omega)|\nabla\xi|_{L^\infty(\Omega)} f_-(\Omega).$$

Proof. This follows from the previous lemma but can be proved directly by letting

$$g = f_+ \left(1 - \frac{f_-(\Omega)}{f_+(\Omega)}\right)$$

(assuming f is nonzero; otherwise $g = 0$ is the answer). Then g is positive because $f(\Omega) \geq 0$ implies $f_-(\Omega) \leq f_+(\Omega)$ and

$$\int \xi \, d(f - g) = \int \xi \, d\left(f_+ \frac{f_-(\Omega)}{f_+(\Omega)} - f_-\right) = \int (\xi - \bar{\xi}) \, d\left(f_+ \frac{f_-(\Omega)}{f_+(\Omega)} - f_-\right),$$

where $\bar{\xi}$ is the average of ξ over Ω, and the right-hand side is clearly bounded above by

$$2 \operatorname{diam}(\Omega) |\nabla \xi|_\infty f_-(\Omega). \qquad \square$$

Mass displacement in the balls.

Definition 3.3. For $B \in \mathcal{B}_\varepsilon \cap \mathcal{B}_\varepsilon^\alpha$, we let

$$f_\varepsilon^B = (e_\varepsilon - \Lambda_\varepsilon^\alpha \nu_\varepsilon) \mathbf{1}_{B \cap \Omega_\varepsilon}.$$

where $\Lambda_\varepsilon^{\alpha,r}$ is defined in (2-5) and we have set $\Lambda_\varepsilon^\alpha = \Lambda_\varepsilon^{\alpha,\rho}$.

This corresponds to the excess energy in the balls, i.e., the energy remaining after subtracting off the expected value from the ball construction. There is a difference of order $|\nu_\varepsilon|(B) \log \overline{C}_\alpha$ between $f_\varepsilon(B)$ and $f_\varepsilon^B(B)$, which will be dealt with later.

Proposition 3.4. *There exists $\varepsilon_0, C > 0$ such that for any $\varepsilon < \varepsilon_0$, and any $B \in \mathcal{B}_\varepsilon \cap \mathcal{B}_\varepsilon^\alpha$, there exists a positive measure g_ε^B defined in $B \cap \Omega_\varepsilon$ and such that*

$$g_\varepsilon^B \leq e_\varepsilon + \Lambda_\varepsilon^\alpha (\nu_\varepsilon)_- \quad \textit{and} \quad \int_{B \cap \Omega_\varepsilon} \xi \, d(f_\varepsilon^B - g_\varepsilon^B) \leq C |\nabla \xi|_{L^\infty(B \cap \Omega_\varepsilon)} |\nu_\varepsilon|(B), \tag{3-1}$$

for any Lipschitz function ξ vanishing on $\Omega_\varepsilon \setminus B$.

Proof. To prove the existence of g_ε^B, in view of Lemma 3.1 and since $(f_\varepsilon^B)_+ = e_\varepsilon + \Lambda_\varepsilon^\alpha (\nu_\varepsilon)_-$ on B it suffices to prove that for any *positive* function ξ defined on B and vanishing on $B \setminus \Omega_\varepsilon$ we have

$$\int \xi \, d f_\varepsilon^B \geq -C |\nabla \xi|_{L^\infty(B)} |\nu_\varepsilon|(B). \tag{3-2}$$

We turn to the proof of (3-2). Let $B \in \mathcal{B}_\varepsilon \cap \mathcal{B}_\varepsilon^\alpha$ and ξ be as above. Then

$$\int \xi \, d f_\varepsilon^B = \int_0^{+\infty} f_\varepsilon^B \, (E_t \cap B) \, dt, \tag{3-3}$$

where we have set $E_t = \{x \in B \mid \xi(x) \geq t\}$ and $f_\varepsilon^B(A) = \int_A f_\varepsilon^B$.

We will divide the integral (3-3) into $\int_0^{t_\varepsilon} + \int_{t_\varepsilon}^{+\infty}$, with $t_\varepsilon = \varepsilon |\nabla \xi|_{L^\infty}$. The first integral is straightforward to bound from below. Indeed, $(f_\varepsilon^B)_-(B) \leq C |\log \varepsilon| \, |\nu_\varepsilon|(B)$; hence

$$\int_0^{t_\varepsilon} f_\varepsilon^B \, (E_t) \, dt \geq -C \varepsilon |\log \varepsilon| \, |\nabla \xi|_{L^\infty} |\nu_\varepsilon|(B) \geq -C |\nabla \xi|_{L^\infty} |\nu_\varepsilon|(B). \tag{3-4}$$

On the other hand, if $t > t_\varepsilon$ — and this motivated our choice of t_ε — then since $\xi = 0$ in $B \setminus \Omega_\varepsilon$ we have $\operatorname{dist}(E_t, {\Omega_\varepsilon}^c) > \varepsilon$. So let $t > t_\varepsilon$, and let $a \in E_t$ be a point in the support of ν_ε. For any $r \in [\sqrt{\varepsilon}, \rho]$, there exists a ball $B_{a,r} \in \mathcal{B}_\varepsilon^r$ containing a. Since $\{\mathcal{B}_\varepsilon^r\}$ is monotonic with respect to r, $B_{a,r} \subset B$. Put

$$r(a,t) = \sup\{r \in [\sqrt{\varepsilon}, \rho),\, B_{a,r} \subset E_t\}$$

if the set on the right is nonempty, and $r(a,t) = 0$ otherwise. Then let

$$B_a^t = B_{a,r(a,t)}.$$

If $0 < r(a,t) < \rho$ then $r(a,t)$ bounds from above the distance of a to the complement of E_t. In particular,

$$\xi(a) - t \le r(a,t)|\nabla\xi|_{L^\infty}. \tag{3-5}$$

Indeed for any $r(a,t) < s < \rho$ we have $B_{a,s} \subset B$ and $B_{a,s} \cap (E_t)^c \neq \varnothing$; hence there exists $b \in B_{a,s} \cap \partial E_t$. Then $\xi(a) - \xi(b) \le s|\nabla\xi|_{L^\infty}$ and since $\partial E_t \subset \{\xi = t\}$ we deduce $\xi(a) - t \le s|\nabla\xi|_{L^\infty}$, proving (3-5) by making s tend to $r(a,t)$ from above.

A second fact is that if $r(a,t) = 0$, then $\overline{B_{a,\sqrt{\varepsilon}}}$ intersects $B \setminus E_t$, and as above we deduce

$$\xi(a) - t \le \sqrt{\varepsilon}\,|\nabla\xi|_{L^\infty(B)}. \tag{3-6}$$

The third fact is that the collection $\{B_a^t\}_a$, where a ranges over E_t and the a's for which $r(a,t) = 0$ have been excluded, is disjoint. Indeed take $a, b \in E_t$ and assume that $r(a,t) \ge r(b,t)$. Then, since $\mathcal{B}_{r(a,t)}$ is disjoint, the balls $B_{a,r(a,t)}$ and $B_{b,r(a,t)}$ are either equal or disjoint. If they are disjoint we note that $r(a,t) \ge r(b,t)$ implies that $B_{b,r(b,t)} \subset B_{b,r(a,t)}$ and therefore $B_b^t = B_{b,r(b,t)}$ and $B_a^t = B_{a,r(a,t)}$ are disjoint. If they are equal, then $B_{b,r(a,t)} \subset E_t$ and therefore $r(b,t) \ge r(a,t)$, which implies $r(b,t) = r(a,t)$ and then $B_b^t = B_a^t$.

Now, for any $B' \in \{B_a^t\}_a$ we have $B' \subset E_t$ and $\operatorname{dist}(E_t, {\Omega_\varepsilon}^c) > \varepsilon$, hence $\operatorname{dist}(B', {\Omega_\varepsilon}^c) > \varepsilon$. Now let r be the common value of $r(a,t)$ for all $a \in B'$ in the support of ν_ε. From Proposition 2.6, we have

$$e_\varepsilon(B') \ge |\nu_\varepsilon(B')| \left(\Lambda_\varepsilon^\alpha - \frac{1}{2}\log\frac{\rho}{r}\right)_+,$$

since $\Lambda_\varepsilon^{\alpha,r} = \Lambda_\varepsilon^{\alpha,\rho} - \frac{1}{2}\log(\rho/r)$. We can rewrite this as

$$e_\varepsilon(B') \ge \left| \sum_{a \in B' \cap \operatorname{Supp}\nu_\varepsilon} \nu_\varepsilon(a) \left(\Lambda_\varepsilon^\alpha - \frac{1}{2}\log\frac{\rho}{r(a,t)}\right)_+ \right|,$$

and summing over $B' \in \{B_a^t\}_a$ we deduce

$$e_\varepsilon(E_t \cap B) \ge \left| \sum_{a \in \mathcal{P}_t} \left(\Lambda_\varepsilon^\alpha - \frac{1}{2}\log\frac{\rho}{r(a,t)}\right)_+ \nu_\varepsilon(a) \right|,$$

where $\mathcal{P}_t$ is the set of points in $E_t \cap \operatorname{Supp}\nu_\varepsilon$ such that $r(a,t) > 0$. We will let $\mathcal{Q}_t$ be the set of points in $E_t \cap \operatorname{Supp}\nu_\varepsilon$ such that $r(a,t) = 0$.

Since $\nu_\varepsilon(E_t) = \nu_\varepsilon(\mathcal{P}_t) + \nu_\varepsilon(\mathcal{Q}_t)$, subtracting from $\Lambda_\varepsilon^\alpha \nu_\varepsilon(E_t)$ the above we find

$$f_\varepsilon^B(E_t) \geq -\sum_{a\in\mathcal{Q}_t} |\nu_\varepsilon|(a)\Lambda_\varepsilon^\alpha - \frac{1}{2}\sum_{a\in\mathcal{P}_t} |\nu_\varepsilon|(a) \log \frac{\rho}{r(a,t)}.$$

From (3-6), a given $a \in \operatorname{Supp}\nu_\varepsilon \cap B$ can belong to $\mathcal{Q}_t$ only if $|t-\xi(a)| \leq \sqrt{\varepsilon}\,|\nabla\xi|_{L^\infty}$. Therefore integrating the above with respect to t yields, using the fact that $t \leq \xi(a)$ if $a \in E_t$, that

$$\int_{t_\varepsilon}^\infty f_\varepsilon^B(E_t)\,dt \geq -\sum_{a\in\operatorname{Supp}\nu_\varepsilon\cap B} |\nu_\varepsilon|(a)\left(\int_{\xi(a)-\sqrt{\varepsilon}|\nabla\xi|_{L^\infty}}^{\xi(a)+\sqrt{\varepsilon}|\nabla\xi|_{L^\infty}} \Lambda_\varepsilon^\alpha\,dt + \frac{1}{2}\int_0^{\xi(a)} \left(\log\frac{\rho}{r(a,t)}\right)_+ dt\right);$$

hence

$$\int_{t_\varepsilon}^\infty f_\varepsilon^B(E_t)\,dt \geq -2\Lambda_\varepsilon^\alpha\sqrt{\varepsilon}|\nabla\xi|_{L^\infty}|\nu_\varepsilon|(B) - \frac{1}{2}\sum_{a\in\operatorname{Supp}\nu_\varepsilon\cap B} |\nu_\varepsilon|(a)\int_0^{\xi(a)} \left(\log\frac{\rho}{r(a,t)}\right)_+ dt.$$

We now note that — since $\Lambda_\varepsilon^\alpha \leq \frac{1}{2}|\log\varepsilon|$ — $\sqrt{\varepsilon}\Lambda_\varepsilon^\alpha$ is bounded independently of $\varepsilon \leq 1$ and, using the inequality (3-5), we get

$$\int_0^{\xi(a)} \left(\log\frac{\rho}{r(a,t)}\right)_+ dt \leq \int_0^{\xi(a)} \left(\log\frac{\rho|\nabla\xi|_{L^\infty}}{\xi(a)-t}\right)_+ dt = \int_{\xi(a)-\rho|\nabla\xi|_{L^\infty}}^{\xi(a)} \log\frac{\rho|\nabla\xi|_{L^\infty}}{\xi(a)-t}\,dt,$$

and the rightmost integral is equal, via the change of variables $u = \dfrac{\xi(a)-t}{\rho|\nabla\xi|_{L^\infty}}$, to $\rho|\nabla\xi|_{L^\infty}$. Therefore

$$\int_{t_\varepsilon}^{+\infty} f_\varepsilon^B(E_t)\,dt \geq -C|\nu_\varepsilon|(B)|\nabla\xi|_{L^\infty}.$$

In view of (3-3), adding (3-4) yields the result. □

Remark 3.5. In the proof of (3-2), the final radius ρ may be replaced by any $r \in (\sqrt{\varepsilon}, \rho)$. This yields the following result: Assume that $r \in (\sqrt{\varepsilon}, \rho)$ and that $B \in \mathcal{B}_\varepsilon^r$ is included in some ball in $\mathcal{B}_\varepsilon \cap \mathcal{B}_\varepsilon^\alpha$. Then, for any positive function ξ vanishing on $B \setminus \Omega_\varepsilon$,

$$\int_B (e_\varepsilon - \Lambda_\varepsilon^{\alpha,r}\nu_\varepsilon)\xi \geq -C|\nabla\xi|_{L^\infty(B)}|\nu_\varepsilon|(B). \tag{3-7}$$

We record the following lower bounds:

Proposition 3.6. *For ε small enough and $B \in \mathcal{B}_\varepsilon \cap \mathcal{B}_\varepsilon^\alpha$:*

$$e_\varepsilon(\Omega_\varepsilon \cap B) \geq \left(\tfrac{1}{8}|\log\varepsilon| - C\right)|\nu_\varepsilon|(B). \tag{3-8}$$

For ε small enough and $B \in \mathcal{B}_\varepsilon \cap \mathcal{B}_\varepsilon^\alpha$ such that $\operatorname{dist}(B, \Omega_\varepsilon^c) > \varepsilon$, *we have*

$$g_\varepsilon^B(\Omega_\varepsilon \cap B) \geq \left(\tfrac{1}{8}|\log\varepsilon| - C\right)|\nu_\varepsilon|(B) - \tfrac{1}{2}|\log\varepsilon|\,|\nu_\varepsilon(B)|. \tag{3-9}$$

If in addition $d_B < 0$, then

$$g_\varepsilon^B(\Omega_\varepsilon \cap B) - \left(\tfrac{1}{2}|\log\varepsilon| - \Lambda_\varepsilon^\alpha\right)\nu_\varepsilon(B) \geq \left(\tfrac{1}{8}|\log\varepsilon| - C\right)|\nu_\varepsilon|(B). \tag{3-10}$$

The meaning of this lower bound is that $e_\varepsilon(B)$ is not only bounded below by $\Lambda_\varepsilon^\alpha|\nu_\varepsilon(B)|$, which to leading order is $\frac{1}{2}|\log\varepsilon||\nu_\varepsilon(B)|$ — this is the positivity of g_ε^B in the proposition above — but also by some constant times $|\log\varepsilon||\nu_\varepsilon|(B)$, even though the constant is no longer guaranteed to be the (optimal) value $\frac{1}{2}$. This information is valuable in the case where $|\nu_\varepsilon(B)|$ is much smaller than $|\nu_\varepsilon|(B)$. The precise value of the constants is unimportant.

Proof. As we noticed, $\overline{C}_\alpha < (\sqrt{\varepsilon}/\varepsilon)^{1/2}$ implies $\sqrt{\varepsilon}/(\varepsilon\overline{C}_\alpha) \geq \varepsilon^{-1/4}$. Thus, using Proposition 2.6,

$$e_\varepsilon(B\cap\Omega_\varepsilon) \geq \sum e_\varepsilon(B') \geq \sum \pi|d_{B'}|(\log\varepsilon^{-1/4} - C) = |\nu_\varepsilon|(B)\big(\tfrac{1}{8}|\log\varepsilon| - \tfrac{1}{2}C\big),$$

where the sums are over $B' \in \mathscr{B}_\varepsilon^{\sqrt{\varepsilon}}$ such that $B' \subset B$ and $\operatorname{dist}(B', {\Omega_\varepsilon}^c) > \varepsilon$. This proves the first assertion. Secondly, note that from (3-1), if $\operatorname{dist}(B, \Omega_\varepsilon^c) > \varepsilon$, choosing ξ compactly supported in Ω_ε such that $\xi = 1$ in B, we have

$$f_\varepsilon^B(B\cap\Omega_\varepsilon) = g_\varepsilon^B(B\cap\Omega_\varepsilon).$$

Since $\Lambda_\varepsilon^\alpha \leq \frac{1}{2}|\log\varepsilon|$ we deduce (3-9) in view of

$$g_\varepsilon^B(B\cap\Omega_\varepsilon) = f_\varepsilon^B(B\cap\Omega_\varepsilon) \geq |\nu_\varepsilon|(B)\big(\tfrac{1}{8}|\log\varepsilon| - C\big) - \tfrac{1}{2}|\log\varepsilon||\nu_\varepsilon(B)|.$$

For the last assertion, since $\nu_\varepsilon(B) = 2\pi d_B < 0$, we write

$$g_\varepsilon^B(B\cap\Omega_\varepsilon) - \big(\tfrac{1}{2}|\log\varepsilon| - \Lambda_\varepsilon^\alpha\big)\nu_\varepsilon(B) = e_\varepsilon(B\cap\Omega_\varepsilon) - \tfrac{1}{2}|\log\varepsilon|\nu_\varepsilon(B) \geq e_\varepsilon(B\cap\Omega_\varepsilon),$$

and this is bounded below using (3-8). □

Mass displacement of the remainder. Proposition 3.4 will allow us to replace f_ε^B by the positive g_ε^B, and we have

$$f_\varepsilon - \sum_{B\in\mathscr{B}_\varepsilon} f_\varepsilon^B = e_\varepsilon \mathbf{1}_{\mathscr{B}_\varepsilon^c} + \sum_\alpha \big(\tfrac{1}{2}|\log\varepsilon| - \Lambda_\varepsilon^\alpha\big)\nu_\alpha. \tag{3-11}$$

We now proceed to absorb the negative part of $f_\varepsilon - \sum f_\varepsilon^B$, which is $\big(\frac{1}{2}|\log\varepsilon| - \Lambda_\varepsilon^\alpha\big)(\nu_\alpha)_+$. This will be easy if $\overline{C}_\alpha = 3e_\alpha/|\log\varepsilon|$; and if not, in view of (2-5), we have

$$0 \leq \tfrac{1}{2}|\log\varepsilon| - \Lambda_\varepsilon^\alpha \leq \tfrac{1}{2}\log n_\alpha + C,$$

which allows to bound the mass of the negative part by $C\sum_\alpha n_\alpha(\log n_\alpha + 1)$. Following the method in [Sandier and Serfaty 2003] (see also [Sandier and Serfaty 2007, Chapter 9]), this will be balanced by a lower bound by $c[n_\alpha]^2$ for the energy on annuli surrounding U_α.

Recall that $U_\alpha = B(x_\alpha, \ell_0)\cap\Omega_\varepsilon$. We set

$$r_0 = \ell_0, \quad r_1 = 3\ell_0, \quad A_\alpha = B(x_\alpha, r_1).$$

Choosing ℓ_0 small enough, we can require that

$$\operatorname{diam}(A_\alpha) < 1 \quad\text{and}\quad \Big(A_\alpha \cap {\Omega_\varepsilon}^c \neq \varnothing \implies A_\alpha \subset \big\{x \mid \operatorname{dist}(x, \partial\Omega_\varepsilon) < \tfrac{1}{2}\big\}\Big).$$

We will denote below by m' a bound, uniform in ε, for the overlap number of the $\{A_\alpha\}_\alpha$.

Now we choose ρ such that $|T_\varepsilon^\alpha| \ge \ell_0$ for any $\varepsilon > 0$, where

$$T_\varepsilon^\alpha = \big\{ t \in (r_0, r_1) \mid \{|x - x_\alpha| = t\} \cap \mathcal{B}_\varepsilon = \varnothing \big\}.$$

Indeed, the number of U_β's that intersect $B(x_\alpha, r_1)$ is bounded by a certain number N, independent of ε and α. Choosing $\rho = \ell_0/N$, the sum of the radii of balls in $\bigcup_\beta \mathcal{B}_\varepsilon^\beta$ which intersect $B(x_\alpha, r_1)$ is bounded above by ℓ_0, hence $|T_\varepsilon^\alpha| \ge (r_1 - r_0) - \ell_0 = \ell_0$.

Lower bounds on annuli. For any α let

$$(\tilde g_\varepsilon^\alpha)_+ = \frac{1}{4m'} \Big(e_\varepsilon \mathbf{1}_{\mathcal{B}_\varepsilon^c} + \sum_{B \in \mathcal{B}_\varepsilon} g_\varepsilon^B \Big) \mathbf{1}_{A_\alpha}, \quad (\tilde g_\varepsilon^\alpha)_- = \big(\tfrac12 |\log \varepsilon| - \Lambda_\varepsilon^\alpha\big)(\nu_\varepsilon)_+ \mathbf{1}_{\mathcal{B}_\varepsilon \cap \mathcal{B}_\varepsilon^\alpha}, \tag{3-12}$$

and $\tilde g_\varepsilon^\alpha = (\tilde g_\varepsilon^\alpha)_+ - (\tilde g_\varepsilon^\alpha)_-$. We have

$$\tilde g_\varepsilon - \sum_\alpha \tilde g_\varepsilon^\alpha \ge \frac34 \Big(e_\varepsilon \mathbf{1}_{\mathcal{B}_\varepsilon^c} + \sum_{B \in \mathcal{B}_\varepsilon} g_\varepsilon^B \Big) + \sum_\alpha \big(\tfrac12 |\log \varepsilon| - \Lambda_\varepsilon^\alpha\big)(\nu_\varepsilon)_- \mathbf{1}_{\mathcal{B}_\varepsilon \cap \mathcal{B}_\varepsilon^\alpha}.$$

In particular,

$$(\tilde g_\varepsilon^\alpha)_+(A_\alpha) \le \frac{1}{3m'} \Big(\tilde g_\varepsilon - \sum_\beta \tilde g_\varepsilon^\beta \Big)(A_\alpha).$$

Proposition 3.7. *There exist $\varepsilon_0, C, c > 0$ such that if $\beta < \frac14$ in* (1-1), *then for any $\varepsilon < \varepsilon_0$ and any index α*

$$(\tilde g_\varepsilon^\alpha)_-(A_\alpha) \le \pi n_\alpha \left(\beta |\log \varepsilon| + C\right). \tag{3-13}$$

If moreover $\operatorname{dist}(A_\alpha, \Omega_\varepsilon{}^c) > \varepsilon$ *then at least one of the following is true*:

$$(\tilde g_\varepsilon^\alpha)_-(A_\alpha) \le \pi n_\alpha \left(\beta |\log \varepsilon| + C\right), \quad (\tilde g_\varepsilon^\alpha)_+(A_\alpha) \ge c n_\alpha |\log \varepsilon| \tag{3-14}$$

or

$$(\tilde g_\varepsilon^\alpha)_-(A_\alpha) \le \pi n_\alpha \left(\log n_\alpha + C\right), \quad (\tilde g_\varepsilon^\alpha)_+(A_\alpha) \ge c {n_\alpha}^2. \tag{3-15}$$

Proof. The bound (3-13) follows from (3-12), (2-6). Now assume $\operatorname{dist}(A_\alpha, \Omega_\varepsilon{}^c) > \varepsilon$.

First, if $n_\alpha = 0$ then $(\tilde g_\varepsilon^\alpha)_- = 0$, $(\tilde g_\varepsilon^\alpha)_+ \ge 0$; hence (3-14) is true.

Second, if $3e_\alpha / |\log \varepsilon| \ge M n_\alpha$ then, since for $B \subset A_\alpha$, we have $g_\varepsilon^B(B) = f_\varepsilon{}^B(B) = e_\varepsilon(B) - \Lambda_\varepsilon^\alpha \nu_\varepsilon(B)$ and $\Lambda_\varepsilon^\alpha \le \frac12 |\log \varepsilon|$ it follows that

$$\begin{aligned}(\tilde g_\varepsilon^\alpha)_+(A_\alpha) &\ge \frac{1}{4m'} \int_{A_\alpha} e_\varepsilon - \frac{1}{4m'} \Lambda_\varepsilon^\alpha \sum_{B \in \mathcal{B}_\varepsilon \cap A_\alpha} |d_B| \ge \frac{1}{4m'} \int_{U_\alpha} e_\varepsilon - \frac{1}{4m'} \Lambda_\varepsilon^\alpha \sum_{B \in \mathcal{B}_\varepsilon \cap A_\alpha} |d_B| \\ &\ge \frac{M}{12m'} n_\alpha |\log \varepsilon| - \pi n_\alpha |\log \varepsilon| \ge \left(\frac{M}{12m'} - \pi \right) n_\alpha |\log \varepsilon|.\end{aligned}$$

Together with (3-13), this implies (3-14) if M was chosen strictly greater than $12m'\pi$. The last case is that where $\overline{C}_\alpha = M n_\alpha$. Then $\frac12 |\log \varepsilon| - \Lambda_\varepsilon^\alpha = \frac12 \log n_\alpha + C$ and therefore, using (2-3),

$$(\tilde g_\varepsilon^\alpha)_-(A_\alpha) \le 2\pi n_\alpha \big(\tfrac12 \log n_\alpha + C\big) \le n_\alpha (\pi \beta |\log \varepsilon| + C). \tag{3-16}$$

We define

$$D_0^+ = \sum_{\substack{B\in\mathcal{B}_\varepsilon,\, d_B>0\\ B\subset B(x_\alpha,r_0)}} d_B, \qquad D_1^- = \sum_{\substack{B\in\mathcal{B}_\varepsilon,\, d_B<0\\ B\subset B(x_\alpha,r_1)}} |d_B|,$$

and again we distinguish several cases.

First from (3-16) we will have proven (3-14) if we prove that

$$(\tilde{g}_\varepsilon^\alpha)_+(A_\alpha) \geq c n_\alpha |\log\varepsilon|, \tag{3-17}$$

for some $c > 0$. This inequality holds in the following two cases.

First case: $D_1^- > n_\alpha/20$. This means there is a significant proportion of balls with negative degrees. For each such negative ball we have from (3-10), and since $|\nu_\varepsilon|(B) \geq |\nu_\varepsilon(B)|$,

$$g_\varepsilon^B(B) \geq g_\varepsilon^B(B) - \big(\tfrac12|\log\varepsilon| - \Lambda_\varepsilon^\alpha\big)\nu_\varepsilon(B) \geq \big(\tfrac18|\log\varepsilon| - C\big)2\pi|d_B|.$$

This implies that

$$(\tilde{g}_\varepsilon^\alpha)_+(A_\alpha) \geq \frac{1}{4m'}\big(\tfrac18|\log\varepsilon| - C\big)2\pi D_1^-;$$

hence (3-17) is satisfied when $D_1^- > n_\alpha/20$.

Second case: $D_0^+ \leq n_\alpha/10$ *and* $D_1^- \leq n_\alpha/20$. Then for each $B \in \mathcal{B}_\varepsilon \cap \mathcal{B}_\varepsilon^\alpha$, Proposition 3.6 yields

$$g_\varepsilon^B(B) \geq \begin{cases} \big(\tfrac18|\log\varepsilon| - C\big)|\nu_\varepsilon|(B) - \tfrac12|\log\varepsilon|\,|\nu_\varepsilon(B)| & \text{if } |d_B| > 0\\ \big(\tfrac18|\log\varepsilon| - C\big)|\nu_\varepsilon|(B) & \text{if } |d_B| < 0.\end{cases}$$

Summing with respect to B we find, since $B \in \mathcal{B}_\varepsilon \cap \mathcal{B}_\varepsilon^\alpha$ implies $B \subset B(x_\alpha, r_0)$, that

$$(\tilde{g}_\varepsilon^\alpha)_+(A_\alpha) \geq \frac{1}{4m'}\big(\tfrac18|\log\varepsilon| - C\big)n_\alpha - \frac{1}{4m'}D_0^+\tfrac12|\log\varepsilon|,$$

which again yields (3-17) when $D_0^+ \leq n_\alpha/10$.

We are left with the complementary case, when $D_0^+ > n_\alpha/10$ and $D_1^- \leq n_\alpha/20$. In this case (3-17) and then (3-14) do not necessarily hold. We need to prove (3-15) instead, which in view of (3-16) reduces to proving

$$(\tilde{g}_\varepsilon^\alpha)_+(A_\alpha) \geq c {n_\alpha}^2.$$

For this we really need to use the lower bounds on annuli of the type first introduced in [Sandier and Serfaty 2003]. We set

$$\mathcal{C}_\varepsilon^\alpha = B(x_\alpha, r_1) \setminus (B(x_\alpha, r_0) \cup \mathcal{B}_\varepsilon)\,.$$

For any $t \in T_\varepsilon^\alpha$ we let $B_t = B(x_\alpha, t)$ and $\gamma_t = \partial B_t$; recall that γ_t does not intersect $\mathcal{B}_\varepsilon$. If $t \in T_\varepsilon^\alpha$ then $|u_\varepsilon| \geq \frac12$ on γ_t because of (2-1) and the fact that $\operatorname{dist}(A_\alpha, {\Omega_\varepsilon}^c) > \varepsilon$.

It follows (see for instance [Sandier and Serfaty 2007, Lemma 4.4], or (5-4) below) that for some constant $c > 0$ we have

$$\int_{\gamma_t}\left(\frac{1}{2}|\nabla_A u|^2 + \frac{(1-|u|^2)^2}{4\varepsilon^2}\right) + \frac{1}{2}\int_{B_t}(\operatorname{curl} A)^2 \ge c\frac{|d_\varepsilon^t|^2}{t}, \tag{3-18}$$

where d_ε^t is the degree of $u_\varepsilon/|u_\varepsilon|$ on γ_t. Integrating (3-18) with respect to $t \in T_\varepsilon^\alpha$, which has measure less than 1, the left-hand side will be bounded above by $e_\varepsilon(A_\alpha)$. In view of the lower bound $d_\varepsilon^t \ge (D_0^+ - D_1^-)$, which is valid for any $t \in T_\varepsilon^\alpha$, since $|T_\varepsilon^\alpha| \ge \ell_0$, and from the assumption on D_0^+ and D_1^- we deduce that

$$e_\varepsilon(A_\alpha \setminus \mathcal{B}_\varepsilon) \ge c(D_0^+ - D_1^-)^2 \ge c{n_\alpha}^2.$$

Then, since $(\tilde g_\varepsilon^\alpha)_+ = \dfrac{1}{4m'}e_\varepsilon$ on $(\mathcal{B}_\varepsilon)^c$ we deduce $(\tilde g_\varepsilon^\alpha)_+(A_\alpha) \ge c{n_\alpha}^2$ and (3-15) is proved. □

Proof of Theorem 1.1 and Corollary 1.2. (1) The estimate (1-2) was already mentioned after the definition (2-2) of ν_ε, and the bound $|\nu_\varepsilon|(E) \le Ce_\varepsilon(\widehat{E})/|\log\varepsilon|$ was proved in Lemma 2.4.

(2) We define

$$f_\alpha = \sum_{B\in\mathcal{B}_\varepsilon\cap\mathcal{B}_\varepsilon^\alpha}(f_\varepsilon^B - g_\varepsilon^B) + (\tilde g_\varepsilon^\alpha)_+ - (\tilde g_\varepsilon^\alpha)_-.$$

Then clearly f_α is supported in A_α. Moreover, using the fact (see (3-11)) that

$$f_\varepsilon - \sum_{B\in\mathcal{B}_\varepsilon} f_\varepsilon^B = e_\varepsilon \mathbf{1}_{\mathcal{B}_\varepsilon^c} - \sum_\alpha\left(\tfrac{1}{2}|\log\varepsilon| - \Lambda_\varepsilon^\alpha\right)\nu_\alpha$$

and since $\sum_\alpha \mathbf{1}_{A_\alpha} \le m'$ we easily obtain

$$f_\varepsilon - \sum_\alpha f_\alpha = \sum_\alpha\left(\tfrac{1}{2}|\log\varepsilon| - \Lambda_\varepsilon^\alpha\right)(\nu_\alpha)_- + \left(e_\varepsilon\mathbf{1}_{\mathcal{B}_\varepsilon^c} + \sum_{B\in\mathcal{B}_\varepsilon} g_\varepsilon^B\right)\left(1 - \frac{1}{4m'}\sum_\alpha \mathbf{1}_{A_\alpha}\right). \tag{3-19}$$

Since $\sum_\alpha \mathbf{1}_{A_\alpha} \le m'$ we find

$$f_\varepsilon - \sum_\alpha f_\alpha \ge \sum_\alpha\left(\tfrac{1}{2}|\log\varepsilon| - \Lambda_\varepsilon^\alpha\right)(\nu_\alpha)_- + \frac{3}{4}\left(e_\varepsilon\mathbf{1}_{\mathcal{B}_\varepsilon^c} + \sum_{B\in\mathcal{B}_\varepsilon} g_\varepsilon^B\right) \ge 0. \tag{3-20}$$

(3) We define g_α. In the case $\operatorname{dist}(A_\alpha, {\Omega_\varepsilon}^c) \le \varepsilon$ we let $g_\alpha = (\tilde g_\varepsilon^\alpha)_+$. Then

$$\int \xi\, d(f_\alpha - g_\alpha) = \sum_{B\in\mathcal{B}_\varepsilon\cap\mathcal{B}_\varepsilon^\alpha}\int \xi\, d(f_\varepsilon^B - g_\varepsilon^B) - \int \xi\, d(\tilde g_\varepsilon^\alpha)_-.$$

This implies (1-5), summing (3-1) over $B \in \mathcal{B}_\varepsilon \cap \mathcal{B}_\varepsilon^\alpha$ and using (3-13).

In the case $\operatorname{dist}(A_\alpha, {\Omega_\varepsilon}^c) > \varepsilon$ we let

$$c_\alpha = \left(\frac{\tilde g_\varepsilon^\alpha(A_\alpha)}{|A_\alpha|}\right)_-.$$

We deduce easily from (3-14), (3-15) and if β is small enough that $c_\alpha \le C$ and applying Lemma 3.2 in A_α to $\tilde g_\varepsilon^\alpha + c_\alpha$ we obtain φ_α defined on A_α and such that $0 \le \varphi_\alpha \le (\tilde g_\varepsilon^\alpha)_+ + c_\alpha$ and, for any Lipschitz function ξ,

$$\int_{A_\alpha} \xi \, d\left(\tilde{g}_\varepsilon^\alpha - g_\alpha\right) \le C|\nabla\xi|_{L^\infty(A_\alpha)}(\tilde{g}_\varepsilon^\alpha)_-(A_\alpha), \quad \text{where } g_\alpha := \varphi_\alpha - c_\alpha.$$

Moreover $-C \le -c_\alpha \le g_\alpha \le (\tilde{g}_\varepsilon^\alpha)_+$.

Then

$$\begin{aligned}\int_{A_\alpha} \xi \, d(f_\alpha - g_\alpha) &= \int_{A_\alpha} \xi \, d(f_\alpha - \tilde{g}_\varepsilon^\alpha) + \int_{A_\alpha} \xi \, d(\tilde{g}_\varepsilon^\alpha - g_\alpha) \\ &= \sum_{B\in\mathcal{B}_\varepsilon\cap\mathcal{B}_\varepsilon^\alpha} \int \xi \, d(f_\varepsilon^B - g_\varepsilon^B) + \int_{A_\alpha} \xi \, d(\tilde{g}_\varepsilon^\alpha - g_\alpha) \\ &\le C|\nabla\xi|_{L^\infty(A_\alpha)}\big(n_\alpha + (\tilde{g}_\varepsilon^\alpha)_-(A_\alpha)\big),\end{aligned} \tag{3-21}$$

where we have used (3-1) to bound the integral involving $f_\varepsilon^B - g_\varepsilon^B$. Moreover, $g_\alpha(A_\alpha) = \tilde{g}_\varepsilon^\alpha(A_\alpha)$.

If (3-14) holds, then (1-3) follows immediately from (3-21) when $\pi\beta < c/2$, with c the constant in (3-14). If (3-15) holds we deduce (1-4) from (3-21) by noting that $c{n_\alpha}^2 - Cn_\alpha(\log n_\alpha + 1) \ge \frac{c}{2}{n_\alpha}^2 - C'n_\alpha$ if C' is chosen large enough depending on c, C.

(4) To prove (1-8), we adapt an argument from [Struwe 1994].

First, $g_\varepsilon - \sum_\alpha g_\alpha = f_\varepsilon - \sum_\alpha f_\alpha$ thus from (3-20) and since $\sum_\alpha g_\alpha \ge -C$ we find

$$g_\varepsilon \ge \frac{3}{4}\left(e_\varepsilon \mathbf{1}_{\mathcal{B}_\varepsilon^c} + \sum_{B\in\mathcal{B}_\varepsilon} g_\varepsilon^B\right) - C. \tag{3-22}$$

Then, assuming $U_\alpha \subset \Omega_\varepsilon$, denote by $\mathcal{B}_\varepsilon^{r,\alpha}$ the set of balls in $\mathcal{B}_\varepsilon^r$ which are included in some ball belonging to $\mathcal{B}_\varepsilon^\alpha \cap \mathcal{B}_\varepsilon$, so that $\nu_\alpha(\mathcal{B}_\varepsilon) = \nu_\varepsilon(\mathcal{B}_\varepsilon^\alpha \cap \mathcal{B}_\varepsilon) = \nu_\varepsilon(\mathcal{B}_\varepsilon^{r,\alpha})$. Applying Remark 3.5 for some $r \in (\sqrt{\varepsilon}, \rho)$ with $\xi = 1$ and summing (3-7) over $B \in \mathcal{B}_\varepsilon^{r,\alpha}$ we find $e_\varepsilon(\mathcal{B}_\varepsilon^{r,\alpha}) \ge \Lambda_\varepsilon^{\alpha,r}\nu_\varepsilon(\mathcal{B}_\varepsilon^{r,\alpha})$ and then

$$\begin{aligned}e_\varepsilon(\mathcal{B}_\varepsilon \cap \mathcal{B}_\varepsilon^\alpha \setminus \mathcal{B}_\varepsilon^{r,\alpha}) &\le e_\varepsilon(\mathcal{B}_\varepsilon \cap \mathcal{B}_\varepsilon^\alpha) - \Lambda_\varepsilon^\alpha \nu_\alpha(\mathcal{B}_\varepsilon) + \left(\Lambda_\varepsilon^\alpha - \Lambda_\varepsilon^{\alpha,r}\right)\nu_\alpha(\mathcal{B}_\varepsilon) \\ &= \sum_{B\in\mathcal{B}_\varepsilon\cap\mathcal{B}_\varepsilon^\alpha} g_\varepsilon^B(B) + \tfrac{1}{2}\log\frac{1}{r}\nu_\alpha(\mathcal{B}_\varepsilon),\end{aligned}$$

where we have used that $f_\varepsilon^B(B) = g_\varepsilon^B(B)$. It follows using (3-22) that

$$e_\varepsilon(\mathcal{B}_\varepsilon \cap \mathcal{B}_\varepsilon^\alpha \setminus \mathcal{B}_\varepsilon^{r,\alpha}) \le C\left((g_\varepsilon)_+(U_\alpha) + n_\alpha \log\frac{1}{r} + 1\right). \tag{3-23}$$

Then comes the argument in [Struwe 1994]: For any integer k, let $r_k = 2^{-k}\rho$, and let $\mathcal{C}_k$ be the intersection of $\mathcal{B}_\varepsilon^{r_k} \setminus \mathcal{B}_\varepsilon^{r_{k+1}}$ and $\mathcal{B}_\varepsilon^\alpha$. Then $|\mathcal{C}_k| \le C2^{-2k}\rho^2$, since $\rho 2^{-k}$ bounds the total radius of the balls in $\mathcal{B}_\varepsilon^{r_k} \cap \mathcal{B}_\varepsilon^\alpha$. Moreover $j_\varepsilon = (iu_\varepsilon, \nabla u_\varepsilon - iA_\varepsilon)$ and thus assuming $|u_\varepsilon| \le 1$ we have $|j_\varepsilon|^2 \le 2e_\varepsilon$. Then using Hölder's inequality in $\mathcal{C}_k$ and (3-23) we find for $p < 2$

$$\begin{aligned}\int_{\mathcal{C}_k} |j_\varepsilon|^p &\le |\mathcal{C}_k|^{1-p/2}\big(e_\varepsilon(\mathcal{C}_k)\big)^{p/2} \le |\mathcal{C}_k|^{1-p/2}\big(e_\varepsilon(\mathcal{B}_\varepsilon \cap \mathcal{B}_\varepsilon^\alpha \setminus \mathcal{B}_\varepsilon^{r_{k+1}})\big)^{p/2} \\ &\le C_p 2^{-(2-p)k}\big(e_\varepsilon(\mathcal{B}_\varepsilon \cap \mathcal{B}_\varepsilon^\alpha \setminus \mathcal{B}_\varepsilon^{r_{k+1}})\big)^{p/2} \le C_p 2^{-(2-p)k}\big((g_\varepsilon)_+(U_\alpha) + kn_\alpha\log 2 + 1\big)^{p/2}.\end{aligned}$$

Using (1-10) we find

$$\int_{\mathscr{C}_k} |j_\varepsilon|^p \le C_p 2^{-(2-p)k}(1+k\log 2)^{p/2}\big((g_\varepsilon)_+(U_\alpha)+1\big)^{p/2}.$$

Summing these inequalities for k ranging from 0 to the largest integer K such that $r_K \ge \sqrt{\varepsilon}$ — so that in particular $r_K \le 2\sqrt{\varepsilon}$ — we find

$$\int_{\mathscr{B}_\varepsilon\cap\mathscr{B}_\varepsilon^\alpha\setminus\mathscr{B}_\varepsilon^{2\sqrt{\varepsilon}}} |j_\varepsilon|^p \le C_p\big((g_\varepsilon)_+(U_\alpha)+1\big)^{p/2},$$

where C_p is a constant times the sum of the convergent series $\sum_k 2^{-(2-p)k}(1+k\log 2-\log\rho)^{p/2}$. To this inequality we add

$$\int_{\mathscr{B}_\varepsilon^\alpha\cap\mathscr{B}_\varepsilon^{2\sqrt{\varepsilon}}} |j_\varepsilon|^p \le C\varepsilon^{1-p/2}e_\varepsilon(U_\alpha)^{p/2},$$

which follows from Hölder's inequality after estimating $\big|\mathscr{B}_\varepsilon^{2\sqrt{\varepsilon}}\cap\mathscr{B}_\varepsilon^\alpha\big|$ by $C\varepsilon$, as above. But since $e_\varepsilon = f_\varepsilon + \frac{1}{2}|\log\varepsilon|\nu_\varepsilon$ we may write using (1-9), (1-10),

$$e_\varepsilon(U_\alpha) \le C(g_\varepsilon)_+(\widehat{U}_\alpha) + C|\nu_\varepsilon|(\widehat{U}_\alpha)(1+|\log\varepsilon|) \le C|\log\varepsilon|\,\big((g_\varepsilon)_+(U_\alpha+B(0,2))+1\big). \tag{3-24}$$

Thus

$$\int_{\mathscr{B}_\varepsilon^\alpha\cap\mathscr{B}_\varepsilon^{2\sqrt{\varepsilon}}} |j_\varepsilon|^p \le C\varepsilon^{1-\frac{p}{2}}|\log\varepsilon|^{p/2}\big((g_\varepsilon)_+(U_\alpha+B(0,2))^{p/2}+1\big) \le C\big((g_\varepsilon)_+(U_\alpha+B(0,2))^{p/2}+1\big).$$

We also add

$$\int_{U_\alpha\setminus\mathscr{B}_\varepsilon} |j_\varepsilon|^p \le C\big((g_\varepsilon)_+(U_\alpha)+1\big)$$

which follows from (3-22). Finally we obtain

$$\int_{U_\alpha} |j_\varepsilon|^p \le C_p\big((g_\varepsilon)_+(U_\alpha+B(0,2))+1\big).$$

Summing with respect to the α's such that $E\cap U_\alpha \ne \varnothing$, this proves (1-8) and concludes the proof of Theorem 1.1. ∎

Proof of Corollary 1.2. Note that

$$\int \xi\, d(f_\varepsilon - g_\varepsilon) = \sum_\alpha \int \xi\, d(f_\alpha - g_\alpha).$$

Three types of indices occur.

First we consider indices α such that $\operatorname{dist}(A_\alpha, {\Omega_\varepsilon}^c) > \varepsilon$ and (1-3) holds. Since

$$g_\alpha \le g_\varepsilon - \sum_{\beta\ne\alpha} g_\beta \le g_\varepsilon + C, \tag{3-25}$$

we deduce from (1-3) that if $n_\alpha \ge 1$ and ε is small enough, $g_\varepsilon(A_\alpha) \ge cn_\alpha|\log\varepsilon|$ and then using (1-3) again that

$$\int \xi \, d(f_\alpha - g_\alpha) \le C|\nabla\xi|_{L^\infty(A_\alpha)} \left(n_\alpha + \beta(g_\varepsilon)_+(A_\alpha)\right). \tag{3-26}$$

If $n_\alpha = 0$ the same inequality holds since from (1-3) the left-hand side is zero.

Second we consider indices α such that $\operatorname{dist}(A_\alpha, \Omega_\varepsilon{}^c) > \varepsilon$ and (1-4) holds. We note that if C is large enough then $x \log x \le \eta x^2 + C \log^2 \eta/\eta$ holds for every $x > 0$ and $\eta \le 1$, for instance by distinguishing the cases $\eta > (\log x)/x$ and $\eta \le (\log x)/x$. We use this and (3-25), together with (1-4) to find that if $n_\alpha \ge 1$ then

$$\int \xi \, d(f_\alpha - g_\alpha) \le C|\nabla\xi|_{L^\infty(A_\alpha)} \left(n_\alpha + \eta(g_\varepsilon)_+(A_\alpha) + \frac{\log^2 \eta}{\eta}\right). \tag{3-27}$$

Again the inequality is true if $n_\alpha = 0$ since from (1-4) the left-hand side is zero in this case.

Finally we consider indices α such that $\operatorname{dist}(A_\alpha, \Omega_\varepsilon{}^c) \le \varepsilon$. In this case, noting that from Lemma 2.4 we have $n_\alpha |\log \varepsilon| \le C e_\varepsilon(A_\alpha)$, we rewrite (1-5) as

$$\int \xi \, d(f_\alpha - g_\alpha) \le C \left(|\nabla\xi|_{L^\infty(A_\alpha)} n_\alpha + \beta|\xi|_{L^\infty(A_\alpha)} e_\varepsilon(A_\alpha)\right). \tag{3-28}$$

To conclude we sum either (3-26), (3-27) or (3-28) according to the type of index α, noting that since $\operatorname{diam}(A_\alpha) \le 1$, we have $|f|_{L^\infty(A_\alpha)} \le \hat{f}$ on A_α for any function f. Since the overlap number of the A_α's is bounded by a universal constant, we deduce (1-9).

We prove (1-10). We start by proving that when $\operatorname{dist}(A_\alpha, \Omega_\varepsilon{}^c) > \varepsilon$ we have

$$\min\left(n_\alpha{}^2, n_\alpha |\log \varepsilon|\right) \le C\left((g_\varepsilon)_+(A_\alpha) + 1\right). \tag{3-29}$$

If $n_\alpha = 0$ this is trivial, if not then it follows from either (1-3) or (1-4) using (3-25).

Assume α is such that $\operatorname{dist}(A_\alpha, \Omega_\varepsilon{}^c) > \varepsilon$, then since $2x \le \eta x^2 + 1/\eta$ and since $x \le \eta x |\log \varepsilon|$ is trivially true if $1/|\log \varepsilon| < \eta$, we deduce from (3-29) that

$$n_\alpha \le C\left(\eta(g_\varepsilon)_+(A_\alpha) + 1/\eta\right). \tag{3-30}$$

On the other hand Lemma 2.4 implies that for any α

$$n_\alpha \le C \frac{e_\varepsilon(A_\alpha \cap \Omega_\varepsilon)}{|\log \varepsilon|}. \tag{3-31}$$

Summing (3-30) or (3-31) according to whether $\operatorname{dist}(A_\alpha, \Omega_\varepsilon{}^c)$ is $> \varepsilon$ or $\le \varepsilon$ we deduce (1-10). □

4. Proof of Theorem 1.5

Convergence. We study the consequences of the hypothesis

$$M_R := \limsup_{\varepsilon \to 0} \int_{U_R} g_\varepsilon(x) \, dx < +\infty \quad \text{for all } R > 0. \tag{4-1}$$

and prove that it implies the convergence of the vorticities and currents in the appropriate sense.

Note that we assume $\operatorname{dist}(0, \partial\Omega_\varepsilon) \to +\infty$ so that for every R, $\mathbf{U}_R \subset \Omega_\varepsilon$ for ε small enough. From (1-13) there exists $C > 0$ such that for any R large enough

$$B_{R/C} \subset \boldsymbol{U_R} \subset B_{CR}, \quad \frac{1}{C} \le \frac{|\boldsymbol{U_R}|}{R^2} \le C.$$

We now gather several easy consequences of Theorem 1.1 and (4-1).

Proposition 4.1. *Assume* (4-1) *holds, and let g_ε be as in Theorem 1.1. Then for any R and ε small enough depending on R we have*

$$\sum_{\alpha \mid A_\alpha \subset \boldsymbol{U_R}} \min({n_\alpha}^2, n_\alpha |\log \varepsilon|) \le C(M_{\boldsymbol{R}+C} + R^2), \tag{4-2}$$

$$|\nu_\varepsilon|(\boldsymbol{U_R}) \le C(M_{\boldsymbol{R}+C} + R^2), \tag{4-3}$$

$$\int (f_\varepsilon - g_\varepsilon)\chi_{\boldsymbol{U_R}} \le C \sum_{\alpha \mid A_\alpha \subset \boldsymbol{U}_{R+C} \setminus \boldsymbol{U}_{R-C}} n_\alpha(\log n_\alpha + 1) \le C(M_{\boldsymbol{R}+C} + R^2), \tag{4-4}$$

where $\{\chi_{\boldsymbol{U_R}}\}_{\boldsymbol{R}}$ are any functions satisfying (1-14).

For any $1 \le p < 2$ there exists $C_p > 0$ such that for any $R > 0$, and ε small enough

$$\int_{\boldsymbol{U_R}} |j_\varepsilon|^p \le C_p(M_{\boldsymbol{R}+C} + R^2). \tag{4-5}$$

Up to extraction of a subsequence, $\{j_\varepsilon\}_\varepsilon$ converges weakly in $L^p_{\mathrm{loc}}(\mathbb{R}^2)$, $p < 2$ to some $j : \mathbb{R}^2 \to \mathbb{R}^2$; $\{\nu_\varepsilon\}_\varepsilon$ converges in the weak sense of measures to a measure ν on $\mathbb{R}^2$ of the form $2\pi \sum_{p\in\Lambda} d_p\delta_p$, where Λ is a discrete set and $d_p \in \mathbb{Z}$; $\{\mu_\varepsilon\}_\varepsilon$ converges to the same ν in $W^{-1,p}_{\mathrm{loc}}(\mathbb{R}^2)$ for any $p < 2$; and $\{h_\varepsilon\}$ converges weakly in $L^2_{\mathrm{loc}}(\mathbb{R}^2)$ to h. Moreover,

$$\operatorname{curl} j = \nu - h. \tag{4-6}$$

Proof. Assertions (4-2), (4-3) and (4-5) are direct consequences of (3-29), (1-10) and (1-8), respectively.

We prove (4-4). As a consequence of (4-1), for every $R > 0$, if $\varepsilon > 0$ is small enough and $A_\alpha \subset \boldsymbol{U_R}$ then (1-4) holds. Indeed if (1-3) is true with $n_\alpha \ge 1$ (note that if $n_\alpha = 0$ then (1-3) and (1-4) are identical) then $g_\varepsilon(A_\alpha) \ge c|\log \varepsilon| - C$, using (3-25), which contradicts (4-1) if ε is small enough.

Then we use (1-3) with $\xi = \chi_{\boldsymbol{U_R}}$. Since $\chi_{\boldsymbol{U_R}}$ is supported in $\boldsymbol{U}_{R+C}$ and since $\operatorname{dist}(\boldsymbol{U}_{R+C}, \partial\Omega_\varepsilon) \to +\infty$ we have, if ε is small enough and $A_\alpha \cap \boldsymbol{U}_{R+C} \ne \varnothing$, that $\operatorname{dist}(A_\alpha, \partial\Omega_\varepsilon) > \varepsilon$. Then summing (1-3) over all such α we find

$$\int \chi_{\boldsymbol{U_R}}\, d(f_\varepsilon - g_\varepsilon) \le C \sum_{\alpha \mid A_\alpha \subset \boldsymbol{U}_{R+C} \setminus \boldsymbol{U}_{R-C}} n_\alpha\, (\log n_\alpha + 1)\,,$$

which is the first inequality in (4-4). The second one then easily follows from (3-29).

We now turn to the convergence results. The weak local convergence of j_ε follows from a bound for $\int_{\boldsymbol{U_R}} |j_\varepsilon|^p$ valid for any ε small enough, depending on R, which is implied by (4-1) and (4-5). From (4-3), $\{\nu_\varepsilon\}_\varepsilon$ is bounded on any compact subset of $\mathbb{R}^2$, hence converges (up to extraction) to a measure ν, which by (2-2) has to be of the form $2\pi \sum_{p\in\Lambda} d_p\delta_p$ where Λ is a discrete set and $d_p \in \mathbb{Z}$ for every $p \in \Lambda$ (we will prove below that $d_p = 1$).

The weak local convergence of h_ε follows from (1-12) combined with the bound (4-1).

The convergence of $\{\mu_\varepsilon\}_\varepsilon$ in $W^{-1,p}_{\mathrm{loc}}$ uses the Jacobian estimate (see [Jerrard and Soner 2002] or [Sandier and Serfaty 2007, Theorem 6.2]) from which we deduce that for any $R > 0$ and any $\gamma \in (0, 1)$, and since $r(\mathcal{B}_\varepsilon^{\sqrt{\varepsilon}} \cap B_R) \le C\sqrt{\varepsilon}$,

$$\|\mu_\varepsilon - \nu_\varepsilon\|_{(C_0^{0,\gamma}(B_R))^*} \le C(\sqrt{\varepsilon})^\gamma\,(e_\varepsilon(B_R) + 1)\,, \tag{4-7}$$

where C depends on R but not on ε.

But $\{\nu_\varepsilon\}_\varepsilon$ is bounded in B_R as measures, hence in $(C_0^{0,\gamma})^*$, and arguing again as in (3-24),

$$e_\varepsilon(B_R) \le (g_\varepsilon)_+(B_{R+1}) + \tfrac{1}{2}|\log\varepsilon|\,|\nu_\varepsilon|(B_{R+C}) \le C|\log\varepsilon|.$$

Therefore the right-hand side in (4-7) tends to 0 as $\varepsilon \to 0$ and $\{\mu_\varepsilon\}_\varepsilon$ is bounded in $(C_0^{0,\gamma}(B_R))^*$. We deduce that $\mu_\varepsilon \to \nu$ in $W^{-1,p}_{\mathrm{loc}}$ by noting that for any $1 < p < 2$ there exists $0 < \gamma < 1$ such that $W_0^{1,p'}(B_R) \hookrightarrow C_0^{0,\gamma}$ with compact imbedding — where $1/p + 1/p' = 1$ — which implies by duality that $(C_0^{0,\gamma})^* \hookrightarrow W_0^{-1,p}$ with compact imbedding.

Finally (4-6) is obtained by passing to the limit in $\mu_\varepsilon = \operatorname{curl} j_\varepsilon + \operatorname{curl} A_\varepsilon$ since by Remark 1.4 we may assume (up to extraction) that $\operatorname{curl} A_\varepsilon \to h$ weakly locally in L^2 as $\varepsilon \to 0$. □

Remark 4.2. From the above results, it is easy to deduce (1-18) under the stronger assumption (1-17). In this case we have $M_R \le CR^2$ and therefore (4-3), (4-5) and Remark 1.4 imply that

$$|\nu_\varepsilon|(U_R) \le CR^2, \qquad \int_{U_R} |j_\varepsilon|^p \le CR^2, \qquad \int_{U_R} |h_\varepsilon|^2 \le CR^2 \tag{4-8}$$

which in turn implies (1-18).

Lower bound by the renormalized energy. We turn to the proof of the remaining statement in Theorem 1.5, namely that ν is of the form $2\pi\sum_{p\in\Lambda}\delta_p$ (we already know it is of the form $2\pi\sum_{p\in\Lambda} d_p\delta_p$, where the d_p's are nonzero integers) and that under assumption (1-17) the lower bound (1-19) holds. Both are related to a lower bound of $\int \chi_R g_\varepsilon$ by the renormalized energy, where $\chi_R := \chi_{U_R}$. This reproduces more or less arguments present in [Bethuel et al. 1994] and [Bethuel and Rivière 1995]. Throughout this subsection we assume that (1-17) holds, and begin by bounding from below the integral of $(e_\varepsilon - \frac{1}{2}|\log\varepsilon|\,\nu_\varepsilon)\chi_R$.

Choose $R > 0$. From (4-3) we have that $|\nu_\varepsilon|$ is bounded independently of ε on the support of χ_R, thus a subsequence of $\{|\nu_\varepsilon|\mathbf{1}_{\operatorname{Supp}\chi_R}\}_\varepsilon$ converges to a positive measure $\tilde\nu$ of the form $2\pi\sum_{i=1}^k k_i\delta_{a_i}$, where k_i is a positive integer for every i (the a_i's are a subset of Λ).

From the weak convergence of j_ε to j in L^p_{loc} and using the inequality $|\nabla_{A_\varepsilon} u_\varepsilon| \geq |j_\varepsilon|$ (following from the assumption $|u_\varepsilon| \leq 1$) we have for any $r > 0$

$$\liminf_{\varepsilon\to 0} \int_{\mathbb{R}^2 \setminus \bigcup_{p\in\Lambda} B(p,r)} \chi_R |\nabla_{A_\varepsilon} u_\varepsilon|^2 \geq \liminf_{\varepsilon\to 0} \int_{\mathbb{R}^2 \setminus \bigcup_{p\in\Lambda} B(p,r)} \chi_R |j_\varepsilon|^2$$
$$\geq \int_{\mathbb{R}^2 \setminus \bigcup_{p\in\Lambda} B(p,r)} \chi_R |j|^2. \tag{4-9}$$

Indeed either the left-hand side is equal to $+\infty$ and the statement is true, or there is weak L^2 convergence of the currents on the complement of $\bigcup_p B(p,r)$ and (4-9) follows by weak lower semicontinuity of the integrand. Similarly, by weak convergence of h_ε to h we have

$$\liminf_{\varepsilon\to 0} \int_{\mathbb{R}^2 \setminus \bigcup_{p\in\Lambda} B(p,r)} \chi_R {h_\varepsilon}^2 \geq \int_{\mathbb{R}^2 \setminus \bigcup_{p\in\Lambda} B(p,r)} \chi_R h^2. \tag{4-10}$$

Then consider any $\eta \in (0,1)$ small enough so that the balls $B(a_i, 2\eta)$ are disjoint. Note that since the limit of $|\nu_\varepsilon|$ on the support of χ_R is a sum of Dirac masses concentrated at the points $\{a_i\}_i$ we have for ε small enough

$$|\nu_\varepsilon|(\operatorname{Supp} \chi_R \setminus \bigcup_i B(a_i,\eta)) = 0, \quad \nu_\varepsilon(B(a_i,\eta)) = 2\pi d_i,$$

where $2\pi d_i = \nu(a_i)$.

We use two distinct lower bounds for the integral of $\chi_R(e_\varepsilon - \frac{1}{2}|\log\varepsilon|\,\nu_\varepsilon)$ on balls. We distinguish the set I of indices such that $B(a_i, 2\eta) \subset \{\chi_R = 1\}$ and the remaining indices J. Note that if $i \in J$ then $B(a_i, 2\eta)$ intersects the set where $\chi_R \neq 1$ and the support of χ_R, thus $B(a_i, 2\eta) \subset \boldsymbol{U}_{R+C} \setminus \boldsymbol{U}_{R-C}$ for some $C > 0$ independent of $R > 0$, $\eta \in (0,1)$ and i.

In the case $i \in I$ we use

$$\int_{B(a_i,\eta)} e_\varepsilon \geq \pi |d_i| \log\frac{\eta}{\varepsilon} + C_{|d_i|} + o_{\eta,\varepsilon}(1), \tag{4-11}$$

where C_d is a constant depending only on d such that $C_1 = \gamma$, (where γ is defined after Theorem 1), where $C_0 = 0$, and where

$$\lim_{\eta\to 0} \limsup_{\varepsilon\to 0} o_{\eta,\varepsilon}(1) = 0.$$

We postpone the proof of this well-known statement. It is very similar to analogous ones found in [Bethuel et al. 1994] or [Bethuel and Rivière 1995]. Then we deduce from (4-11) that for any $i \in I$ and letting $C_{d_i} = +\infty$ if $d_i < 0$,

$$\liminf_{\varepsilon\to 0} \int_{B(a_i,\eta)} \left(e_\varepsilon - \tfrac{1}{2}|\log\varepsilon|\, d\nu_\varepsilon\right) \geq \pi d_i \log\eta \; + C_{d_i} + o_\eta(1), \tag{4-12}$$

where $\lim_{\eta\to 0} o_\eta(1) = 0$.

In the case $i \in J$ we have to introduce the weight χ_R that is no longer constant on the ball. Then we resort to Remark 3.5. Consider the family of balls $\mathscr{C}_\varepsilon$ consisting of the balls B in $\mathscr{B}_\varepsilon^{\eta/2}$ which intersect the support of χ_{R+1}, and such that $|\nu_\varepsilon|(B) \neq \varnothing$. For any $B \in \mathscr{C}_\varepsilon$, since $|\nu_\varepsilon|(B) \neq 0$ and

$|\nu_\varepsilon| \to 2\pi \sum_i k_i \delta_{a_i}$, and since $r(B) \le \eta/2$, we have for ε small enough depending on R that there is some index i for which $B \subset B(a_i, \eta)$. Let $\mathscr{C}_\varepsilon^i$ denote the balls included in $B(a_i, \eta)$ and partition $\mathscr{C}_\varepsilon^i$ as $\bigcup_\alpha \mathscr{C}_\varepsilon^{i,\alpha}$ and $\mathscr{C}_\varepsilon$ as $\bigcup_\alpha \mathscr{C}_\varepsilon^\alpha$ where the superscript α corresponds to the balls which are included in a ball $B \in \mathscr{B}_\varepsilon^\alpha$ (we assume $\eta/2 < \rho$).

From (3-7), for every $B \in \mathscr{C}_\varepsilon^{i,\alpha}$

$$\int_B \chi_R\big(e_\varepsilon - \Lambda_\varepsilon^{\alpha,\eta/2}\, d\nu_\varepsilon\big) \ge -C|\nabla\chi_R|_\infty |\nu_\varepsilon|(B) \ge -C|\nu_\varepsilon|(B). \tag{4-13}$$

Now we note that since (1-17) holds, then for ε small enough $\overline{C}_\alpha = M n_\alpha$, for otherwise we would have $e_\varepsilon(\mathscr{B}_\varepsilon^\alpha) \ge (M/3) n_\alpha |\log\varepsilon|$ and then

$$\sum_{B\in\mathscr{B}_\varepsilon^\alpha} g_\varepsilon^B(B) = \sum_{B\in\mathscr{B}_\varepsilon^\alpha} f_\varepsilon^B(B) = \sum_{B\in\mathscr{B}_\varepsilon^\alpha} (e_\varepsilon - \Lambda_\varepsilon^\alpha \nu_\varepsilon)(B) \ge \Big(\frac{M}{3} - \pi\Big) n_\alpha |\log\varepsilon| \quad \xrightarrow{\varepsilon\to 0} \quad +\infty,$$

if we choose $M > 3\pi$ and since $n_\alpha \ge 1$. This is a contradiction with (1-16) since $g_\varepsilon \ge \sum_B g_\varepsilon^B - C$ by (1-6), proving that $\overline{C}_\alpha = M n_\alpha$.

Then we have from (2-5) that

$$\Lambda_\varepsilon^{\alpha,\eta/2} - \tfrac12 |\log\varepsilon| = \tfrac12 \log\eta + \Delta, \quad \text{where } |\Delta| \le C\,(\log n_\alpha + 1)$$

and

$$\left| \int_B \big(\chi_R - \chi_R(a_i)\big)\, d\nu_\varepsilon \right| \le C\eta |\nu_\varepsilon|(B).$$

Hence with (4-13)

$$\begin{aligned}\int_B \chi_R\big(e_\varepsilon - \tfrac12 |\log\varepsilon|\, d\nu_\varepsilon\big) &= \int_B \chi_R\big(e_\varepsilon - \Lambda_\varepsilon^{\alpha,\eta/2}\, d\nu_\varepsilon\big) + \big(\frac12 \log\eta + \Delta\big) \int_B \chi_R\, d\nu_\varepsilon \\ &\ge -C|\nu_\varepsilon|(B) + \frac{\log\eta}{2}\chi_R(a_i)\nu_\varepsilon(B) - \frac{\eta}{2}\log\eta |\nu_\varepsilon|(B) - |\Delta|\,|\nu_\varepsilon|(B) \\ &\ge \frac{\log\eta}{2}\nu_\varepsilon(B)\chi_R(a_i) - C|\nu_\varepsilon|(B)\big(1 + \log n_\alpha\big).\end{aligned}$$

Summing over $B \in \mathscr{C}_\varepsilon^{i,\alpha}$ and then over α and $i \in J$ we find, since

$$\sum_{B\in\mathscr{C}_\varepsilon^i} \nu_\varepsilon(B) = \nu_\varepsilon(B(a_i,\eta)) \to \nu(B(a_i,\eta)) = 2\pi d_i,$$

that

$$\liminf_{\varepsilon\to 0} \int_{\bigcup_{i\in J} B(a_i,\eta)} \chi_R\big(e_\varepsilon - \tfrac12 |\log\varepsilon|\, d\nu_\varepsilon\big) \ge \pi \sum_{i\in J} d_i \chi_R(a_i) \log\eta - C\Delta(R),$$

where

$$\Delta(R) = \limsup_{\varepsilon\to 0} \sum_{\alpha \,|\, U_\alpha \subset U_{R+C} \setminus U_{R-C}} n_\alpha\,(\log n_\alpha + 1)\,.$$

Summing (4-12) over $i \in I$ and adding the above and (4-9)–(4-10), we deduce

$$\liminf_{\varepsilon\to 0}\int \chi_R(e_\varepsilon - \tfrac{1}{2}|\log\varepsilon|\,d\nu_\varepsilon) \ge \frac{1}{2}\int_{\mathbb{R}^2\setminus\bigcup_{p\in\Lambda}B(p,\eta)}\chi_R(|j|^2+h^2) \\ + \sum_{i\in I}\chi_R(a_i)\left(\pi d_i\log\eta + C_{d_i}\right) + \sum_{i\in J}\chi_R(a_i)\pi d_i\log\eta - C\Delta(R) - o_\eta(1). \quad (4\text{-}14)$$

We will now take the limit $\eta\to 0$ on the right-hand side. For that we use a Hodge decomposition of j in $B(a_i,\eta_0)$, writing $j=-\nabla^\perp H+\nabla K$, with $H=0$ on $\partial B(a_i,\eta_0)$. Then since $-\Delta H=\nu-h=2\pi d_i\delta_{a_i}-1$ we have $H(x)=d_i\log|x-a_i|+F$, where F is in H^2 in the neighborhood of a_i, in particular $H\in W^{1,p}$ for any $p<2$, and since $j\in L^p$, this implies that $K\in W^{1,p}$ also. Then an easy computation shows that

$$\lim_{\eta\to 0}\frac{1}{2}\int_{B(a_i,\eta_0)\setminus B(a_i,\eta)}\chi_R|\nabla^\perp H|^2+\pi(\log\eta)d_i{}^2\chi_R(a_i)$$

exists and is finite, while

$$\int_{B(a_i,\eta_0)\setminus B(a_i,\eta)}\chi_R|j|^2 \ge \int_{B(a_i,\eta_0)\setminus B(a_i,\eta)}\chi_R\left(|\nabla^\perp H|^2+\nabla^\perp H\cdot\nabla K\right).$$

Decomposing H and integrating by parts we have, writing $C_{i,\eta}=B(a_i,\eta_0)\setminus B(a_i,\eta)$,

$$\int_{C_{i,\eta}}\nabla^\perp H\cdot(\chi_R\nabla K)=\int_{C_{i,\eta}}\nabla^\perp F\cdot(\chi_R\nabla K)-d_i\int_{C_{i,\eta}}K\nabla^\perp\log\cdot\nabla\chi_R,$$

and this remains bounded as $\eta\to 0$, using the regularity of χ_R, F, and the boundedness of H, K, log in $W^{1,p}$. We may then deduce that

$$\liminf_{\eta\to 0}\frac{1}{2}\int_{B(a_i,\eta_0)\setminus B(a_i,\eta)}\chi_R|j|^2+\pi(\log\eta)d_i{}^2\chi_R(p)$$

is not equal to $-\infty$.

As a consequence, writing $d_i=d_i{}^2-(d_i{}^2-d_i)$ in the right-hand side of (4-14), and this right-hand side being bounded above independently of η, we have that $\sum_i(d_i{}^2-d_i)\chi_R(a_i)\log\frac{1}{\eta}$ is bounded above as $\eta\to 0$. Thus we have $d_i\in\{0,1\}$ for any i such that $\chi_R(a_i)\neq 0$ and then $d_i=1$ since d_i was assumed to be nonzero. In view of this, (4-14) can be rewritten as

$$\liminf_{\varepsilon\to 0}\int \chi_R(e_\varepsilon - \tfrac{1}{2}|\log\varepsilon|\,d\nu_\varepsilon) \ge \frac{1}{2}\int_{\mathbb{R}^2\setminus\bigcup_{p\in\Lambda}B(p,\eta)}\chi_R(|j|^2+h^2) \\ + \sum_{p\in\Lambda}\chi_R(p)\,(\pi\log\eta+\gamma)-C\Delta(R)-o_\eta(1),$$

where we recall that $\gamma=C_1$ and we have absorbed $C_1\sum_{i\in J}\chi_R(a_i)$ in $C\Delta(R)$.

Letting $\eta\to 0$ we thus find (see (1-15))

$$\liminf_{\varepsilon\to 0}\int \chi_R(e_\varepsilon - \tfrac{1}{2}|\log\varepsilon|\,d\nu_\varepsilon) \ge W(j,\chi_R)+\frac{1}{2}\int\chi_R h^2+\sum_{p\in\Lambda}\chi_R(p)\gamma-C\Delta(R).$$

From (4-4) we may replace $e_\varepsilon - \frac{1}{2}|\log\varepsilon|\nu_\varepsilon$ by g_ε, with an error term which may be absorbed in $C\Delta(R)$ hence

$$\liminf_{\varepsilon\to 0}\int \chi_R\, dg_\varepsilon \geq W(j,\chi_R)+\frac{1}{2}\int \chi_R h^2 + \sum_{p\in\Lambda}\chi_R(p)\gamma - C\Delta(R). \tag{4-15}$$

Now, under hypothesis (1-17) and using (4-2), we have $\limsup_{\varepsilon\to 0} \sum_{\alpha\,|\,A_\alpha\subset \boldsymbol{U}_R} {n_\alpha}^2 \leq CR^2$, and thus

$$\limsup_{R\to\infty}\limsup_{\varepsilon\to 0}\frac{1}{R^2}\sum_{\alpha|A_\alpha\subset \boldsymbol{U}_{R+C}\setminus \boldsymbol{U}_{R-C}} n_\varepsilon^\alpha|\log n_\varepsilon^\alpha| = 0.$$

Indeed, using Hölder's inequality, and bounding the number of α's involved in the above sum by CR, we find

$$\sum_{\alpha|A_\alpha\subset \boldsymbol{U}_{R+C}\setminus \boldsymbol{U}_{R-C}} [n_\varepsilon^\alpha]^{3/2} \leq (CR)^{1/4}\Bigg(\sum_{\alpha\,|\,A_\alpha\subset \boldsymbol{U}_{R+C}} {n_\alpha}^2\Bigg)^{3/4} \leq CR^{1/4+3/2}.$$

It follows, since $U_\alpha\subset A_\alpha$, that

$$\limsup_{R\to+\infty}\frac{\Delta(R)}{R^2}=0 \tag{4-16}$$

and in particular $\nu(\boldsymbol{U}_{R+C}\setminus \boldsymbol{U}_{R-C}) = o(R^2)$. Then we write, using $\nu = \operatorname{curl} j + h$,

$$\sum_{p\in\Lambda}\chi_R(p) = \frac{1}{2\pi}\int \chi_R\, d\nu = \frac{1}{2\pi}\int\chi_R h - \frac{1}{2\pi}\int \nabla^\perp\chi_R\cdot j.$$

Let $E_R = \{0<\chi_R<1\}$. Then since $E_R\subset \boldsymbol{U}_{R+C}\setminus \boldsymbol{U}_{R-C}$ we have $|E_R|\leq CR$ and using (4-8) together with Hölder's inequality we find

$$\int_{E_R}\chi_R h \leq |E_R|^{1/2}\left(\int_{E_R} h^2\right)^{1/2}\leq CR^{3/2},$$

and a similar bound for $\int\nabla^\perp\chi_R\cdot j$ using (4-8) again, since it is equal to $\int_{E_R}\nabla^\perp\chi_R\cdot j$. Therefore

$$\sum_{p\in\Lambda}\chi_R(p) = \frac{1}{2\pi}\int_{\{\chi_R=1\}} h + o(R^2) = \frac{1}{2\pi}\int_{\boldsymbol{U}_R} h + o(R^2),$$

the second equality being proved again with the help of (4-8) and Hölder's inequality. Together with (4-16) and (4-15), this proves (1-19).

There remains to prove (4-11). For this it is convenient to blow-up $B(a_i,\eta)$ to the unit ball B_1. Then (4-11) becomes

$$\frac{1}{2}\int_{B_1}\left(|\nabla_B v|^2 + \left|\frac{\operatorname{curl} B}{\eta}\right|^2 + \frac{(1-|v|^2)^2}{2\varepsilon'^2}\right) \geq \pi|d_i|\log\frac{1}{\varepsilon'} + C_{d_i} + o_{\eta,\varepsilon}(1), \tag{4-17}$$

where $v(x) = u_\varepsilon(\eta x)$, $B(x) = \eta A_\varepsilon(\eta x)$ and $\eta\varepsilon' = \varepsilon$, so that ε' tends to 0 with ε. Note that (v, B) depends on ε but we omit this in the notation for the rest of the proof.

Since $\operatorname{curl} A_\varepsilon \to h$ weakly in L^2_{loc}, it follows that $\|\operatorname{curl} B\|_{L^2(B_1)} \le 2\eta \|\operatorname{curl} A_\varepsilon\|_{L^2(B_\eta)} \le C\eta$. Then, choosing to work in the gauge $\operatorname{div} B = 0$, $B\cdot\tau =$ constant on ∂B_1, we have $\|B\|_{H^1(B_1)} \le C\eta$. Since $j(u_\varepsilon, A_\varepsilon)$ is bounded in $L^p_{\mathrm{loc}}(\mathbb{R}^2)$ for any $p < 2$, we deduce immediately that $\|j(v,B)\|_{L^p(B_1)} \le C\eta^{1-2/p}$. But by Sobolev embedding, $\|B\|_{L^q(B_1)} = O(\eta)$ for any $q > 1$ hence the integral of $B\cdot j(v,B)$ on B_1 is $o_\eta(1)$. Then, since

$$|\nabla_B v|^2 = |\nabla v|^2 - 2B\cdot j(v,B) + |B|^2|v|^2,$$

(4-17) will follow if we show that

$$\frac{1}{2}\int_{B_1}\left(|\nabla v|^2 + \frac{(1-|v|^2)^2}{2\varepsilon'^2}\right) \ge \pi|d_i|\log\frac{1}{\varepsilon'} + C_{d_i} + o_{\eta,\varepsilon}(1). \tag{4-18}$$

To prove (4-18) we modify B in order for the current to be divergence-free: As before we use the Hodge decomposition $j(v) := (iv, \nabla v) = -\nabla^\perp H + \nabla K$ with $H = 0$ on ∂B_1, and let $\tilde{v} = ve^{-iK}$. Then denoting $e(v)$ the integrand in (4-18) we have

$$e(\tilde{v}) = e(v) - \nabla K\cdot j(v) + \frac{|v|^2}{2}|\nabla K|^2.$$

We replace $j(v) = -\nabla^\perp H + \nabla K$ and note that, integrating by parts, $\nabla K\cdot\nabla^\perp H$ integrates to 0 on B_1. Therefore

$$\int_{B_1} e(\tilde{v}) = \int_{B_1}\left(e(v) + \left(\frac{|v|^2}{2} - 1\right)|\nabla K|^2\right) \le \int_{B_1} e(v).$$

Thus if we show the lower bound (4-18) for $\tilde{v}$, then we are done. For this we may assume, without loss of generality, that the upper bound

$$\frac{1}{2}\int_{B_1}\left(|\nabla \tilde{v}|^2 + \frac{(1-|\tilde{v}|^2)^2}{2\varepsilon'^2}\right) \le \pi|d_i|\log\frac{1}{\varepsilon'} + C_{d_i} \tag{4-19}$$

holds.

The advantage is that now we have

$$j(\tilde{v}) = -\nabla^\perp H + (1-|v|^2)\nabla K.$$

But $\lim_{\varepsilon\to 0}(1-|v|^2) = 0$ in $L^q(B_1)$ for any $q > 1$, being bounded in L^∞ and tending to 0 in L^2. Moreover, we have seen that $\|j(v,B)\|_{L^p(B_1)} \le C\eta^{1-2/p}$, and that $B = O(\eta)$ in every L^p, so

$$j(v,B) - j(v) = |v|^2 B = O(\eta) \tag{4-20}$$

and therefore $j(v) = O(\eta^{1-2/p})$ in L^p, which implies that H and K are $O(\eta^{1-2/p})$ in $W^{1,p}$. It follows from the above that

$$j(\tilde{v}) + \nabla^\perp H = o_{\eta,\varepsilon}(1). \tag{4-21}$$

in $L^p(B_1)$, for every $p < 2$.

Moreover, since $\operatorname{curl} j(u_\varepsilon, A_\varepsilon) + h_\varepsilon \to 2\pi d_i \delta_{a_i}$ in $W^{-1,p}$ as $\varepsilon \to 0$, we have that $\operatorname{curl} j(v, B) + \eta \operatorname{curl} B \to 2\pi d_i \delta_0$. Hence using (4-20) we deduce $-\Delta H = \operatorname{curl} j(v) \to 2\pi d_i \delta_0 + o_\eta(1)$ as $\varepsilon \to 0$ in $W^{-1,p}$. Since $H = 0$ on ∂B_1 we then have

$$H(x) = -2\pi d_i \log|x| + o_\eta(1) \tag{4-22}$$

in $W^{1,p}$.

From (4-21), (4-22) we may find radii $\{r_\varepsilon\}_\varepsilon$ such that

$$\text{(i)} \lim_{\varepsilon\to 0} r_\varepsilon = 1, \quad \text{(ii)} \|j(\tilde v) + \nabla^\perp H\|_{L^p(\partial B_{r_\varepsilon})} = o_{\eta,\varepsilon}(1), \quad \text{(iii)} \|H + 2\pi d_i \log\|_{W^{1,p}(\partial B_{r_\varepsilon})} = o_\eta(1).$$

We may further require that $\rho := |\tilde v| \to 1$ uniformly as $\varepsilon \to 0$ on $\partial B_{r_\varepsilon}$. Indeed from (4-19) we have

$$\frac{1}{2}\int_{B_1} |\nabla\rho|^2 + \frac{1}{2\varepsilon'^2}(1-\rho^2)^2 \le C \log\varepsilon'$$

thus a mean value argument easily implies that r_ε may be chosen such that

$$\frac{1}{2}\int_{\partial B_{r_\varepsilon}} |\nabla\rho|^2 + \frac{1}{2\varepsilon'^2}(1-\rho^2)^2 \le C(\log\varepsilon')^2.$$

This in turn implies using (5-1) that $\|\rho - 1\|_{L^\infty(\partial B_{r_\varepsilon})} \to 0$ as $\varepsilon \to 0$.

Then, writing $\tilde v = \rho e^{i\varphi}$, we have $j(\tilde v) = \rho^2 \nabla\varphi$, and the above implies that

$$\tilde v = (1+\tilde\rho) e^{i(\theta_0 + d_i\theta + \tilde\varphi)} \quad \text{for some } \theta_0 \in \mathbb{R},$$

where

$$\|\tilde\varphi\|_{W^{1,p}(\partial B_1)} = o_{\eta,\varepsilon}(1) \quad \text{and} \quad \|\tilde\rho\|_{L^\infty(\partial B_1)} = o_\varepsilon(1).$$

Without going into further detail (see [Bethuel et al. 1994, Chapter VIII], for instance), this implies that

$$\frac{1}{2}\int_{B_1}\left(|\nabla\tilde v|^2 + \frac{(1-|\tilde v|^2)^2}{2\varepsilon'^2}\right) \ge \min\left\{\frac{1}{2}\int_{B_1}\left(|\nabla u|^2 + \frac{1}{2\varepsilon'^2}(1-|u|^2)^2\right) \,\middle|\, u = e^{id_i\theta} \text{ on } \partial B_1\right\} + o_{\eta,\varepsilon}(1).$$

From [Bethuel et al. 1994], the right-hand side is precisely equal to $\pi|d_i|\log(1/\varepsilon') + C_{|d_i|} + o_\varepsilon(1)$, where the constant C_d is equal to γ if $d = 1$. Thus we have proved (4-18), and then (4-11).

5. Proof of Proposition 2.1

The proof of Proposition 2.1 is based on the ball construction of R. Jerrard [1999], hence we will only emphasize the points which need some modification, mostly to take into account the presence of the magnetic potential A the way we do in [Sandier and Serfaty 2007]. We will denote by c, C, respectively, a small and a large generic universal constant. We will number the constants we need to keep track of. Throughout this section U is a bounded domain in $\mathbb{R}^2$ and (u, A) are defined on U.

The first ingredient is a lower bound for the energy of $|u|$ on a circle [Jerrard 1999, Lemma 2.3]. It is valid for any $\varepsilon > 0$.

Lemma 5.1. *Assuming $2r \geq \varepsilon > 0$ and x are such that the closed ball $B(x,r) \subset U$, we have*

$$\frac{1}{2}\int_{\partial B(x,r)} \left|\nabla|u|\right|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2} \geq c_0\frac{(1-m)^2}{\varepsilon}, \tag{5-1}$$

where $m = \min_{\partial B(x,r)} |u|$.

In contrast to [Jerrard 1999] and because we wish to work with constants independent of U we introduce

$$U_\varepsilon = \{x \in U \mid \operatorname{dist}(x, U^c) > \varepsilon\}.$$

Then $u : U \to \mathbb{C}$ being given we introduce, following Jerrard, $S = \{x \in U_\varepsilon \mid |u| \leq \frac{1}{2}\}$. Assuming u is continuous the connected components of S which are included in U_ε are compact, and $u/|u|$ has a well defined degree, or winding number on their boundary. Then we let

$$S_E = \text{union of the components of } S \text{ with nonzero boundary degree.}$$

Still following Jerrard, for any compact $K \subset U$ such that $\partial K \cap S_E = \varnothing$ we let

$$\deg_E(u, \partial K) = \sum_{S_i \text{ component of } S_E} \deg(u, \partial S_i).$$

This degree is defined even if $|u|$ vanishes on ∂K, provided the points where it vanishes are not in S_E.

The previous lemma implies (see [Jerrard 1999, Proposition 3.3]):

Lemma 5.2. *There exists a collection of disjoint closed balls $B_1, \dots, B_k$ of radii $r_1, \dots, r_k$ such that for all i we have $r_i \geq \varepsilon$ and $e_\varepsilon(U \cap B_i) \geq c_1 r_i/\varepsilon$, and that*

$$S_E \cap U_\varepsilon \subset \bigcup_{i=1}^{k} B_i.$$

Proof. We only sketch the proof. If $x \in S_E$ then either $\partial B_r(x)$ intersects $\{|u| \leq \frac{1}{2}\}$ for every $\varepsilon/2 \leq r \leq \varepsilon$, in which case Lemma 5.1 implies that $e_\varepsilon(U \cap B(x,\varepsilon)) \geq c$, or there exists $\varepsilon/2 \leq r \leq \varepsilon$ such that $|u| > \frac{1}{2}$ on $\partial B_r(x)$, and then the connected component of x in S_E, which has nonzero degree, is included in $B(x,r)$. The nonzero degree implies again (see [Jerrard 1999]) that $e_\varepsilon(U \cap B(x,\varepsilon)) \geq c$. We thus have a cover of S_E by balls that satisfy $e_\varepsilon(B) \geq c r(B)/\varepsilon$.

From Besicovitch's lemma, there exists a disjoint subcollection $\{B_k\}_k$ such that $\{\widetilde{B}_k\}_k$ covers S_E, where $\widetilde{B}_k = CB_k$ with C a universal constant. These balls still satisfy $e_\varepsilon(B) \geq c r(B)/\varepsilon$, though with a smaller constant. Then, grouping the balls which intersect in larger ones as in [Jerrard 1999] (see also [Sandier and Serfaty 2000]) we can obtain a disjoint cover of S_E with the same property. The condition $r_i \geq \varepsilon$ is trivially verified since the balls we started with had radius ε. Note also that the balls obtained here only depend on S_E, hence on u. ∎

Still following Jerrard, we have:

Proposition 5.3. *Choose $c_2 \in (0, c_1)$ small enough and let*

$$\lambda_\varepsilon(x) = \min\left(\frac{c_2}{\varepsilon}, \frac{\pi}{x}\frac{1}{1+\frac{x}{2}+\frac{\pi\varepsilon}{c_0 x}}\right).$$

Then, assuming that $B_r \subset U_\varepsilon$, that $\partial B_r \cap S_E = \varnothing$ and that $\varepsilon \le r \le |d|/2$, where $d = \deg_E(u, \partial B_r)$ is assumed to be different from 0, *we have*

$$\frac{1}{2}\int_{\partial B_r} |\nabla_A u|^2 + \frac{1}{2}\int_{B_r} |\mathrm{curl}\, A|^2 + \frac{1}{4\varepsilon^2}\int_{\partial B_r} \left(1-|u|^2\right)^2 \ge \lambda_\varepsilon\left(\frac{r}{|d|}\right). \tag{5-2}$$

Moreover, the primitive function $\Lambda_\varepsilon(x) = \int_0^x \lambda_\varepsilon$ is increasing, $s \mapsto \Lambda_\varepsilon(s)/s$ is decreasing,

$$\lim_{s\searrow 0}\frac{\Lambda_\varepsilon(s)}{s} = \frac{\min(c_0, c_2)}{\varepsilon} < \frac{c_1}{\varepsilon}, \quad \frac{\Lambda_\varepsilon(\varepsilon)}{\varepsilon} \ge \frac{c_3}{\varepsilon}$$

and finally, for any $\varepsilon \le s \le \frac{1}{2}$, and for some $C_0 > 0$,

$$\Lambda_\varepsilon(s) \ge \pi \log\frac{s}{\varepsilon} - C_0. \tag{5-3}$$

Proof. First, in the case where ∂B_r intersects $\{|u| \le \frac{1}{2}\}$ we deduce from (5-1) that (5-2) is satisfied with $c_2 = c_0/4$.

When on the contrary $|u| > \frac{1}{2}$ on ∂B_r we have $\deg_E(u, \partial B_r) = \deg(u, \partial B_r)$. Then we bound from below $\frac{1}{2}\int_{\partial B_r} |u|^2|\nabla\varphi - A|^2$, where $u = |u|e^{i\varphi}$ as follows: Still denoting $m = \min_{\partial B_r}|u|$, using the Cauchy–Schwarz inequality we have

$$\frac{1}{2}\int_{\partial B_r} |u|^2|\nabla\varphi - A|^2 \ge \frac{m^2}{2}\frac{1}{2\pi r}\left(\int_{\partial B_r}\frac{\partial\varphi}{\partial\tau} - A\cdot\tau\right)^2 = \frac{m^2}{4\pi r}(2\pi d - X)^2$$

where we write $X := \int_{B_r} \mathrm{curl}\, A = \int_{\partial B_r} A\cdot\tau$. On the other hand, by Cauchy–Schwarz again

$$\frac{1}{2}\int_{B_r} |\mathrm{curl}\, A|^2 \ge \frac{1}{2\pi r^2}\left(\int_{B_r}\mathrm{curl}\, A\right)^2 = \frac{X^2}{2\pi r^2}$$

Adding the two relations we obtain

$$\frac{1}{2}\int_{\partial B_r} |u|^2|\nabla\varphi - A|^2 + \frac{1}{2}\int_{B_r} |\mathrm{curl}\, A|^2 \ge \frac{1}{2\pi r}\left(\frac{m^2}{2}(2\pi d - X)^2 + \frac{1}{r}X^2\right).$$

Minimizing the right-hand side with respect to X yields

$$\frac{1}{2}\int_{\partial B_r} |u|^2|\nabla\varphi - A|^2 + \frac{1}{2}\int_{B_r} |\mathrm{curl}\, A|^2 \ge \frac{\pi d^2}{r}\frac{m^2}{1+\frac{m^2 r}{2}}. \tag{5-4}$$

Adding (5-1) we deduce for $r \ge \varepsilon$ that

$$e_\varepsilon(\partial B_r) \ge \frac{\pi|d|}{r}\frac{|d|}{\frac{1}{m^2}+\frac{r}{2}} + c_0\frac{(1-m)^2}{\varepsilon}. \tag{5-5}$$

If $|d|>1$, then either $m^2<2/3$ and we find $e_\varepsilon>c/\varepsilon$ for a well chosen $c>0$ or $m^2\ge 2/3$ and, since $r/2<|d|/4$, we have $m^{-2}+r/2\le 3/2+|d|/4\le|d|$ implying $e_\varepsilon\ge\pi|d|/r$. Thus, if $|d|>1$, (5-2) is satisfied. If $|d|=1$ then minimizing the right-hand side of (5-5) with respect to m yields

$$e_\varepsilon(\partial B_r)\ge\frac{\pi}{r}\frac{1}{1+\frac{r}{2}+\frac{\pi\varepsilon}{c_0 r}},$$

so that in every case we have $e_\varepsilon(\partial B_r)\ge\lambda_\varepsilon(r/|d|)$, if c_2 is chosen small enough.

We now turn to the properties of Λ_ε. Since λ_ε is positive, decreasing, then Λ_ε is increasing and $\Lambda_\varepsilon(s)/s$ is decreasing. It is clear that as $s\to 0$, we have $\lambda_\varepsilon(s)\sim\min(c_0,c_2)/\varepsilon\sim\Lambda_\varepsilon(s)/s$. Moreover, if $x>c\varepsilon$, with $c=\pi/c_2$, then

$$\lambda_\varepsilon(x)=\frac{\pi}{x}\frac{1}{1+\frac{x}{2}+\frac{\pi\varepsilon}{c_0 x}};$$

hence, if $s\ge c\varepsilon$,

$$\Lambda_\varepsilon(s)\ge\int_{c\varepsilon}^{s}\frac{\pi}{x}\frac{1}{1+\frac{x}{2}+\frac{\pi\varepsilon}{c_0 x}}\,dx\ge\int_{c\varepsilon}^{s}\frac{\pi}{x}\left(1-\frac{x}{2}-\frac{\pi\varepsilon}{c_0 x}\right)dx\ge\pi\log\frac{s}{\varepsilon}-C_0,$$

for some constant C_0. If $s<c\varepsilon$ then the inequality remains true if C_0 is chosen large enough, since $\Lambda_\varepsilon(s)\ge 0$.

Finally, $\Lambda_\varepsilon(\varepsilon)\ge\varepsilon\lambda_\varepsilon(\varepsilon)\ge c_3$, if $c_3>0$ is chosen small enough. □

From there, the ball construction procedure (growing and merging of balls) from [Jerrard 1999] (or see [Sandier and Serfaty 2000, Proposition 3.1]) allows one to deduce this:

Proposition 5.4. *For any $0<s<\frac{1}{2}$ there exists a family of disjoint closed balls $\mathcal{B}(s)$ (depending only on u_ε) such that:*

(1) *The family of balls is monotonic; that is, if $s<t$, we have $\mathcal{B}(s)\subset\mathcal{B}(t)$. Moreover, denoting by $r(B)$ the radius of B, the function $s\to\sum_{B\in\mathcal{B}(s)}r(B)$ is continuous.*

(2) *For any s we have $S_E\subset\mathcal{B}(s)$.*

(3) *For any $B\in\mathcal{B}(s)$,*
$$e_\varepsilon(U\cap B)\ge r(B)\frac{\Lambda_\varepsilon(s)}{s}.$$

(4) *If $B\in\mathcal{B}(s)$ and $B\subset U_\varepsilon$ then, letting $d_B=\deg_E(u_\varepsilon,\partial B)$, we have $r(B)\ge s|d_B|$.*

Proof. We let $\mathcal{B}(s_0)$ be the family of balls given by Lemma 5.2, where we choose s_0 small enough so that items 3 and 4 are satisfied (item 2 obviously is). We let $\mathcal{B}(s)=\mathcal{B}(s_0)$ for every $s\le s_0$. For $s\ge s_0$ we apply the method of growing and merging of [Jerrard 1999] which we sketch briefly: It consists in continuously increasing the parameter s and at the same time making those balls included in U_ε such that $r(B)=s|d_B|$ grow so that the equality remains satisfied. When balls touch, the parameter s is stopped and the balls are merged into a larger ball with radius the sum of the radii of the merged balls, and this is repeated if the resulting family is still not disjoint. This does not change the total radius and when it is done — that is, when the family is disjoint again — the increasing of s is resumed, and the process is repeated This yields a family of disjoint closed balls which is monotonic, such that $s \to \sum_{B\in\mathcal{B}(s)} r(B)$ is continuous and such that $r(B) \geq s|d_B|$ for every ball included in U_ε. Obviously $S_E \cap U_\varepsilon \subset \mathcal{B}(s)$ for every s. Also the growing and merging process depends only on the initial balls and the degrees of u_ε, hence on u_ε.

The lower bound $e_\varepsilon(U \cap B) \geq r(B)\Lambda_\varepsilon(s)/s$ is true initially and is preserved through the merging process, it is also preserved through the growing process as long as (5-2) remains valid, i.e., $r(B) < |d_B|/2$ for every $B \subset U_\varepsilon$ such that $d_B \neq 0$. This results from the properties of Λ_ε, as detailed in [Jerrard 1999]. Then for the process to stop, there must be a ball B for which $r(B) = s|d_B|$, i.e., a growing ball, with $r(B) \geq |d_B|/2$, hence we must have $s \geq \frac{1}{2}$. □

Proof of Proposition 2.1. We first construct a family $\mathcal{B}'(s)$ containing S_E instead of $\{x \in U_\varepsilon \mid |u| \leq \frac{1}{2}\}$ but satisfying items (2) and (3) in the conclusion of the proposition.

Under the hypotheses, Proposition 5.4 applies, and yields for every $0 < s < \frac{1}{2}$ a family of balls $\mathcal{B}'(s)$ satisfying the four items stated. Choosing s_0 small enough we have $\Lambda(s_0)/s_0 \geq c/\varepsilon$. Hence, letting r_0 denote the total radius of the balls in $\mathcal{B}'(s_0)$,

$$\varepsilon^{-\beta} \geq G_\varepsilon(u, A) \geq \frac{c r_0}{\varepsilon}$$

and therefore $r_0 \leq C\varepsilon^{1-\beta}$.

Let $r \in (C\varepsilon^{1-\beta}, \frac{1}{2})$, and let r_1 be the total radius of the balls in $\mathcal{B}'(\frac{1}{2})$. If $r > r_1$ then $\mathcal{B}'(\frac{1}{2})$ satisfies item (2) trivially and moreover for any $B \in \mathcal{B}'(\frac{1}{2})$ we have from Proposition 5.4 and using (5-3) that

$$e_\varepsilon(B) \geq |d_B|\Lambda_\varepsilon(\tfrac{1}{2}) \geq \pi|d_B|\left(\log\frac{1}{2\varepsilon} - C\right) \geq \pi|d_B|\left(\log\frac{r}{\overline{C}_\alpha\varepsilon} - C'\right),$$

for any $r \leq \frac{1}{2}$ and any $\overline{C}_\alpha \geq 2$, proving item (3) in this case.

If $r < r_1$ then there exists $s \in (s_0, \frac{1}{2})$ such that $\mathcal{B}' := \mathcal{B}'(s)$ satisfies $r(\mathcal{B}') = r$. Then item 2 of the proposition is satisfied for this collection. Let us check item 3.

Assume then $e_\varepsilon(\mathcal{B}') \leq \overline{C}\log(r/\varepsilon)$, with $2 \leq \overline{C} \leq (r/\varepsilon)^{1/2}$. We show by contradiction that if M is chosen large enough, then

$$s \geq \frac{r}{M\overline{C}}.$$

Since $e_\varepsilon(\mathcal{B}') \geq r\Lambda_\varepsilon(s)/s$ and since $\Lambda_\varepsilon(s)/s$ is decreasing, if $s < r/(M\overline{C})$ and $r/(M\overline{C}) \leq \frac{1}{2}$ then

$$\overline{C}\log\frac{r}{\varepsilon} \geq M\overline{C}\Lambda_\varepsilon\left(\frac{r}{M\overline{C}}\right) \geq \pi M\overline{C}\log\left(\frac{r}{\varepsilon M\overline{C}}\right) - C_0 M\overline{C}.$$

It follows that

$$(1 - \pi M)\log\frac{r}{\varepsilon} + \pi M\log\overline{C} + \pi M\log M - C_0 M \geq 0,$$

which yields a contradiction for $M = 3/\pi$ and $r \geq C\varepsilon$, with C large enough, recalling that $\overline{C} \leq (r/\varepsilon)^{1/2}$. Therefore $s \geq \pi r/(3\overline{C})$ and then for every $B \in \mathcal{B}'$ such that $B \subset U_\varepsilon$ we have

$$e_\varepsilon(B) \geq r(B)\frac{\Lambda_\varepsilon(s)}{s} \geq |d_B|\Lambda_\varepsilon(s) \geq |d_B|\Lambda_\varepsilon\left(\frac{\pi r}{3\overline{C}}\right),$$

which in view of (5-3) yields, for all $B \in \mathcal{B}'$ such that $B \subset U_\varepsilon$,

$$e_\varepsilon(B) \geq \pi |d_B| \left(\log \frac{r}{\varepsilon \overline{C}} - C \right),$$

if C is chosen large enough.

It remains to modify $\mathcal{B}'(s)$ so that $S := \{x \in U_\varepsilon \mid |u| \leq \frac{1}{2}\} \subset \mathcal{B}(r)$. First we note that a well known application of the coarea formula yields rather easily (see [Sandier and Serfaty 2007, Proposition 4.8]) that S can be covered by a collection of disjoint closed balls $\mathcal{C}$ such that $r(\mathcal{C}) \leq C\varepsilon G_\varepsilon \leq C\varepsilon^{1-\beta}$. Then for every s we do the merging of the balls in $\mathcal{C} \cup \mathcal{B}'(s)$ as in the proof of Proposition 5.4 to obtain $\mathcal{B}(s)$. If we chose s such that $r(\mathcal{B}'(s)) = r/2$ with $C\varepsilon^{1-\beta} < r < 1$ and C large enough, then $r(\mathcal{B}(s)) \leq r$ since $r(\mathcal{C}) \leq C\varepsilon^{1-\beta}$. Moreover, if $B \in \mathcal{B}(s)$ is such that $B \subset U_\varepsilon$ then $\deg(u, \partial B)$ is the sum of $\deg_E(u, \partial B')$ for $B' \in \mathcal{B}'(s)$ and $B' \subset B$. Then, if $e_\varepsilon(B) \leq \overline{C} \log(r/2\varepsilon)$ the same bound holds for the B''s and summing the above lower bounds we find

$$e_\varepsilon(B) \geq \pi |d_B| \left(\log \frac{r}{2\varepsilon \overline{C}} - C \right).$$

Changing the constant C we can get rid of the factor 2 and $\mathcal{B}(s)$ has all the desired properties. □

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