Analysis & PDE

Volume 5 No. 1 2012

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mathematical sciences publishers

ANALYSIS AND PDE
Vol. 5, No. 1, 2012
dx.doi.org/10.2140/apde.2012.5.1

# A CHARACTERIZATION OF TWO WEIGHT NORM INEQUALITIES FOR MAXIMAL SINGULAR INTEGRALS WITH ONE DOUBLING MEASURE

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Let $\sigma$ and $\omega$ be positive Borel measures on $\mathbb{R}$ with $\sigma$ doubling. Suppose first that $1 < p \leq 2$. We characterize boundedness of certain maximal truncations of the Hilbert transform $T_\natural$ from $L^p(\sigma)$ to $L^p(\omega)$ in terms of the strengthened $A_p$ condition

$$\Big(\int_{\mathbb{R}} s_Q(x)^p\, d\omega(x)\Big)^{1/p}\Big(\int_{\mathbb{R}} s_Q(x)^{p'}\, d\sigma(x)\Big)^{1/p'} \leq C|Q|,$$

where $s_Q(x) = |Q|/(|Q| + |x - x_Q|)$, and two testing conditions. The first applies to a restricted class of functions and is a strong-type testing condition,

$$\int_Q T_\natural(\chi_E\sigma)(x)^p\, d\omega(x) \leq C_1 \int_Q d\sigma(x) \quad \text{for all } E \subset Q,$$

and the second is a weak-type or dual interval testing condition,

$$\int_Q T_\natural(\chi_Q f\sigma)(x)\, d\omega(x) \leq C_2\Big(\int_Q |f(x)|^p\, d\sigma(x)\Big)^{1/p}\Big(\int_Q d\omega(x)\Big)^{1/p'}$$

for all intervals $Q$ in $\mathbb{R}$ and all functions $f \in L^p(\sigma)$. In the case $p > 2$ the same result holds if we include an additional necessary condition, the Poisson condition

$$\int_{\mathbb{R}}\Big(\sum_{r=1}^{\infty}|I_r|_\sigma|I_r|^{p'-1}\sum_{\ell=0}^{\infty}\frac{2^{-\ell}}{|(I_r)^{(\ell)}|}\chi_{(I_r)^{(\ell)}}(y)\Big)^p\, d\omega(y) \leq C\sum_{r=1}^{\infty}|I_r|_\sigma|I_r|^{p'},$$

for all pairwise disjoint decompositions $Q = \bigcup_{r=1}^{\infty} I_r$ of the dyadic interval $Q$ into dyadic intervals $I_r$. We prove that analogues of these conditions are sufficient for boundedness of certain maximal singular integrals in $\mathbb{R}^n$ when $\sigma$ is doubling and $1 < p < \infty$. Finally, we characterize the weak-type two weight inequality for certain maximal singular integrals $T_\natural$ in $\mathbb{R}^n$ when $1 < p < \infty$, without the doubling assumption on $\sigma$, in terms of analogues of the second testing condition and the $A_p$ condition.

## 1. Introduction

Sawyer [1984; 1982; 1988] characterized two weight inequalities for maximal functions and other positive operators, in terms of the obviously necessary conditions that the operators be uniformly bounded on a restricted class of functions, namely indicators of intervals and cubes. Thus, these characterizations have a form reminiscent of the $T1$ theorem of David and Journé.

Lacey is supported in part by the NSF, through grant DMS-0456538. Sawyer is supported in part by NSERC. Uriarte-Tuero is supported in part by the NSF, through grant DMS-0901524.
*MSC2000:* 42B20.
*Keywords:* two weight, singular integral, maximal function, maximal truncation.

Corresponding results for even the Hilbert transform have only recently been obtained [Nazarov et al. 2010; Lacey et al. 2011] and even then only for $p = 2$; evidently these are much harder to obtain. We comment in more detail on prior results below, including the innovative work of Nazarov, Treil and Volberg [1999; 2008; 2010; 2003].

Our focus is on providing characterizations of the boundedness of certain maximal truncations of a fixed operator of singular integral type. The singular integrals will be of the usual type, for example the Hilbert transform or paraproducts. Only size and smoothness conditions on the kernel are assumed; see (1-9). The characterizations are in terms of certain obviously necessary conditions, in which the class of functions being tested is simplified. For such examples, we prove unconditional characterizations of both strong-type and weak-type two weight inequalities for certain maximal truncations of the Hilbert transform, but with the additional assumption that $\sigma$ is *doubling* for the strong-type inequality. A major point of our characterizations is that they hold for *all* $1 < p < \infty$. The methods in [Lacey et al. 2011] and those of Nazarov, Treil and Volberg apply only to the case $p = 2$, where the orthogonality of measure-adapted Haar bases prove critical. The doubling hypothesis on $\sigma$ may not be needed in our theorems, but is required by the use of Calderón–Zygmund decompositions in our method.

As the precise statements of our general results are somewhat complicated, we illustrate them with an important case here. Let

$$Tf(x) = \lim_{\varepsilon \to 0} \int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)} \frac{1}{y} f(x-y)\,dy$$

denote the Hilbert transform, let

$$T_\flat f(x) = \sup_{0<\varepsilon<\infty} \left| \int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)} \frac{1}{y} f(x-y)\,dy \right|$$

denote the usual maximal singular integral associated with $T$, and finally let

$$T_\natural f(x) = \sup_{\substack{0<\varepsilon_1,\varepsilon_2<\infty \\ 1/4<\varepsilon_2/\varepsilon_1<4}} \left| \int_{\mathbb{R}\setminus(-\varepsilon_1,\varepsilon_2)} \frac{1}{y} f(x-y)\,dy \right|$$

denote the new *strongly* (or *noncentered*) maximal singular integral associated with $T$ that is defined more precisely below. Suppose $\sigma$ and $\omega$ are two locally finite positive Borel measures on $\mathbb{R}$ that have no point masses in common. Then we have the following weak and strong-type characterizations, which we emphasize hold for *all* $1 < p < \infty$.

- The operator $T_\flat$ is *weak* type $(p, p)$ with respect to $(\sigma, \omega)$, that is,

$$\|T_\flat(f\sigma)\|_{L^{p,\infty}(\omega)} \le C\|f\|_{L^p(\sigma)} \tag{1-1}$$

  for all $f$ bounded with compact support *if and only if* the two weight $A_p$ condition

$$\frac{1}{|Q|}\int_Q d\omega \Big(\frac{1}{|Q|}\int_Q d\sigma\Big)^{p-1} \le C,$$

holds for all intervals $Q$ and the dual $T_\flat$ interval testing condition

$$\int_Q T_\flat(\chi_Q f\sigma)\,d\omega \le C\Big(\int_Q |f|^p\,d\sigma\Big)^{1/p}\Big(\int_Q d\omega\Big)^{1/p'}, \tag{1-2}$$

holds for all intervals $Q$ and $f \in L^p_Q(\sigma)$ (part 4 of Theorem 1.8). The same is true for $T_\natural$. It is easy to see that (1-2) is equivalent to the more familiar dual interval testing condition

$$\int_Q |L^*(\chi_Q\omega)|^{p'}\,d\sigma \le C\int_Q d\omega, \tag{1-3}$$

for all intervals $Q$ and linearizations $L$ of the maximal singular integral $T_\flat$ (see (2-10)).

- Suppose in addition that $\sigma$ is doubling and $1 < p < \infty$. Then the operator $T_\natural$ is strong-type $(p, p)$ with respect to $(\sigma, \omega)$, that is,

  $$\|T_\natural(f\sigma)\|_{L^p(\omega)} \le C\|f\|_{L^p(\sigma)}$$

  for all $f$ bounded with compact support *if and only if* these four conditions hold: (1) the strengthened $A_p$ condition

  $$\Big(\int_Q s_Q(x)^p\,d\omega(x)\Big)^{1/p}\Big(\int_I s_Q(x)^{p'}\,d\sigma(x)\Big)^{1/p'} \le C|Q|,$$

  where $s_Q(x) = \frac{|Q|}{|Q|+|x-x_Q|}$, holds for all intervals $Q$; (2) the dual $T_\natural$ interval testing condition

  $$\int_Q T_\natural(\chi_Q f\sigma)\,d\omega \le C\Big(\int_Q |f|^p\,d\sigma\Big)^{1/p}\Big(\int_Q d\omega\Big)^{1/p'},$$

  holds for all intervals $Q$ and $f \in L^p_Q(\sigma)$; (3) the forward $T_\natural$ testing condition

  $$\int_Q T_\natural(\chi_E\sigma)^p\,d\omega \le C\int_Q d\sigma, \tag{1-4}$$

  holds for all intervals $Q$ and all compact subsets $E$ of $Q$; and (4) the Poisson condition

  $$\int_{\mathbb{R}}\Big(\sum_{r=1}^{\infty}|I_r|_\sigma|I_r|^{p'-1}\sum_{\ell=0}^{\infty}\frac{2^{-\ell}}{|(I_r)^{(\ell)}|}\chi_{(I_r)^{(\ell)}}(y)\Big)^p\,d\omega(y) \le C\sum_{r=1}^{\infty}|I_r|_\sigma|I_r|^{p'},$$

  for all pairwise disjoint decompositions $Q = \bigcup_{r=1}^{\infty} I_r$ of the dyadic interval $Q$ into dyadic intervals $I_r$ for any fixed dyadic grid. In the case $1 < p \le 2$, only the first three conditions are needed (Theorem 1.10). Note that in (1-4) we are required to test over all compact subsets $E$ of $Q$ on the left side, but retain the upper bound over the (larger) cube $Q$ on the right side.

As these results indicate, the imposition of the weight $\sigma$ on both sides of (1-1) is a standard part of weighted theory, and is in general necessary for the testing conditions to be sufficient. Compare to the characterization of the two weight maximal function inequalities in Theorem 1.2 below.

**Problem 1.1.** In (1-4), our testing condition is more complicated than one would like, in that one must test over all compact $E \subset Q$ in (1-4). There is a corresponding feature of (1-2), seen after one unwinds the definition of the linearization $L^*$. We do not know if these testing conditions can be further simplified. The form of these testing conditions is dictated by our use of what we call the "maximum principle"; see Lemma 2.6.

We now recall the two weight inequalities for the maximal function as they are central to the new results of this paper. Define the maximal function

$$\mathcal{M}\nu(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |\nu| \quad \text{for } x \in \mathbb{R},$$

where the supremum is taken over all cubes $Q$ (by which we mean cubes with sides parallel to the coordinate axes) containing $x$.

**Theorem 1.2** (maximal function inequalities). *Suppose that $\sigma$ and $\omega$ are positive locally finite Borel measures on $\mathbb{R}^n$, and that $1 < p < \infty$. The maximal operator $\mathcal{M}$ satisfies the two weight norm inequality* [Sawyer 1982]

$$\|\mathcal{M}(f\sigma)\|_{L^p(\omega)} \le C\|f\|_{L^p(\sigma)} \quad \textit{for } f \in L^p(\sigma), \tag{1-5}$$

if and only if *for all cubes $Q \subset \mathbb{R}^n$*,

$$\int_Q \mathcal{M}(\chi_Q \sigma)(x)^p \, d\omega(x) \le C_1 \int_Q d\sigma(x). \tag{1-6}$$

*The maximal operator $\mathcal{M}$ satisfies the* weak-type *two weight norm inequality* [Muckenhoupt 1972]

$$\|\mathcal{M}(f\sigma)\|_{L^{p,\infty}(\omega)} \equiv \sup_{\lambda > 0} \lambda |\{\mathcal{M}(f\sigma) > \lambda\}|_\omega^{1/p} \le C\|f\|_{L^p(\sigma)} \quad \textit{for } f \in L^p(\sigma), \tag{1-7}$$

*if and only if the two weight $A_p$ condition holds for all cubes $Q \subset \mathbb{R}^n$:*

$$\Bigl(\frac{1}{|Q|}\int_Q d\omega\Bigr)^{1/p}\Bigl(\frac{1}{|Q|}\int_Q d\sigma\Bigr)^{1/p'} \le C_2. \tag{1-8}$$

The necessary and sufficient condition (1-6) for the strong-type inequality (1-5) states that one need only test the strong-type inequality for functions of the form $\chi_Q \sigma$. Not only that, but the full $L^p(\omega)$ norm of $\mathcal{M}(\chi_Q \sigma)$ need not be evaluated. There is a corresponding weak-type interpretation of the $A_p$ condition (1-8). Finally, the proofs given in [Sawyer 1982] and [Muckenhoupt 1972] for absolutely continuous weights carry over without difficulty for the locally finite measures considered here.

**1.3.** ***Two weight inequalities for singular integrals.*** Let us set notation for our theorems. Consider a kernel function $K(x, y)$ defined on $\mathbb{R}^n \times \mathbb{R}^n$ satisfying the size and smoothness conditions

$$\begin{aligned} &|K(x, y)| \le C|x - y|^{-n}, \\ &|K(x, y) - K(x', y)| \le C\delta\Bigl(\frac{|x - x'|}{|x - y|}\Bigr)|x - y|^{-n}, \qquad \frac{|x - x'|}{|x - y|} \le \frac{1}{2}, \end{aligned} \tag{1-9}$$

where $\delta$ is a Dini modulus of continuity, that is, a nondecreasing function on $[0, 1]$ with $\delta(0) = 0$ and $\int_0^1 \delta(s)s^{-1}\,ds < \infty$.

Next we describe the truncations we consider. Let $\zeta, \eta$ be fixed smooth functions on the real line satisfying

$$\begin{gathered}
\zeta(t) = 0 \quad \text{for } t \leq \tfrac{1}{2} \quad \text{and} \quad \zeta(t) = 1 \quad \text{for } t \geq 1, \\
\eta(t) = 0 \quad \text{for } t \geq 2 \quad \text{and} \quad \eta(t) = 1 \quad \text{for } t \leq 1, \\
\zeta \text{ is nondecreasing and } \eta \text{ is nonincreasing.}
\end{gathered}$$

Given $0 < \varepsilon < R < \infty$, set $\zeta_\varepsilon(t) = \zeta(t/\varepsilon)$ and $\eta_R(t) = \eta(t/R)$ and define the smoothly truncated operator $T_{\varepsilon,R}$ on $L^1_{\text{loc}}(\mathbb{R}^n)$ by the absolutely convergent integrals

$$T_{\varepsilon,R} f(x) = \int K(x, y)\zeta_\varepsilon(|x - y|)\eta_R(|x - y|) f(y)\,dy \quad \text{for } f \in L^1_{\text{loc}}(\mathbb{R}^n).$$

Define the *maximal* singular integral operator $T_\flat$ on $L^1_{\text{loc}}(\mathbb{R}^n)$ by

$$T_\flat f(x) = \sup_{0<\varepsilon<R<\infty} |T_{\varepsilon,R} f(x)| \quad \text{for } x \in \mathbb{R}^n.$$

We also define a corresponding *new* notion of *strongly maximal* singular integral operator $T_\natural$ as follows. In dimension $n = 1$, we set

$$T_\natural f(x) = \sup_{\substack{0<\varepsilon_i<R<\infty \\ 1/4\leq\varepsilon_1/\varepsilon_2\leq 4}} |T_{\boldsymbol{\varepsilon},R} f(x)| \quad \text{for } x \in \mathbb{R},$$

where $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2)$ and

$$T_{\boldsymbol{\varepsilon},R} f(x) = \int K(x, y)\{\zeta_{\varepsilon_1}(x - y) + \zeta_{\varepsilon_2}(y - x)\}\eta_R(|x - y|) f(y)\,dy.$$

Thus the local singularity has been removed by a *noncentered* smooth cutoff — $\varepsilon_1$ to the left of $x$ and $\varepsilon_2$ to the right of $x$, but with controlled eccentricity $\varepsilon_1/\varepsilon_2$. There is a similar definition of $T_\natural f$ in higher dimensions involving in place of $\zeta_\varepsilon(|x - y|)$, a product of smooth cutoffs,

$$\zeta_{\boldsymbol{\varepsilon}}(x - y) \equiv 1 - \prod_{k=1}^{n} (1 - \{\zeta_{\varepsilon_{2k-1}}(x_k - y_k) + \zeta_{\varepsilon_{2k}}(y_k - x_k)\}),$$

satisfying $1/4 \leq \varepsilon_{2k-1}/\varepsilon_{2k} \leq 4$ for $1 \leq k \leq n$. The advantage of this larger operator $T_\natural$ is that in many cases boundedness of $T_\natural$ (or collections thereof) implies boundedness of the maximal operator $\mathcal{M}$. Our method of proving boundedness of $T_\flat$ and $T_\natural$ requires boundedness of the maximal operator $\mathcal{M}$ anyway, and as a result we can in some cases give necessary and sufficient conditions for strong boundedness of $T_\natural$. As for weak-type boundedness, we can in many more cases give necessary and sufficient conditions for weak boundedness of the usual truncations $T_\flat$.

**Definition 1.4.** We say that $T$ is a *standard singular integral operator with kernel $K$* if $T$ is a bounded linear operator on $L^q(\mathbb{R}^n)$ for some fixed $1 < q < \infty$, that is

$$\|Tf\|_{L^q(\mathbb{R}^n)} \le C\|f\|_{L^q(\mathbb{R}^n)} \quad \text{for } f \in L^q(\mathbb{R}^n), \tag{1-10}$$

if $K(x, y)$ is defined on $\mathbb{R}^n \times \mathbb{R}^n$ and satisfies both (1-9) and the Hörmander condition,

$$\int_{B(y,2\varepsilon)^c} |K(x, y) - K(x, y')|\, dx \le C \quad \text{for } y' \in B(y, \varepsilon),\ \varepsilon > 0, \tag{1-11}$$

and finally if $T$ and $K$ are related by

$$Tf(x) = \int K(x, y) f(y)\, dy \quad \text{for a.e.-} x \notin \operatorname{supp} f, \tag{1-12}$$

whenever $f \in L^q(\mathbb{R}^n)$ has compact support in $\mathbb{R}^n$. We call a kernel $K(x, y)$ *standard* if it satisfies (1-9) and (1-11).

For standard singular integral operators, we have this classical result. (See the appendix on truncation of singular integrals on [Stein 1993, page 30] for the case $R = \infty$; the case $R < \infty$ is similar.)

**Theorem 1.5.** *Suppose that $T$ is a standard singular integral operator. Then the map $f \to T_\flat f$ is of weak type $(1, 1)$, and bounded on $L^p(\mathbb{R})$ for $1 < p < \infty$. There exist sequences $\varepsilon_j \to 0$ and $R_j \to \infty$ such that for $f \in L^p(\mathbb{R})$ with $1 \le p < \infty$,*

$$\lim_{j\to\infty} T_{\varepsilon_j, R_j} f(x) \equiv T_{0,\infty} f(x)$$

*exists for a.e. $x \in \mathbb{R}$. Moreover, there is a bounded measurable function $a(x)$ (depending on the sequences) satisfying*

$$Tf(x) = T_{0,\infty} f(x) + a(x) f(x) \quad \textit{for } x \in \mathbb{R}^n.$$

We state a conjecture, so that the overarching goals of this subject are clear.

**Conjecture 1.6.** *Suppose that $\sigma$ and $\omega$ are positive Borel measures on $\mathbb{R}^n$, let $1 < p < \infty$, and suppose $T$ is a standard singular integral operator on $\mathbb{R}^n$. Then the following two statements are equivalent:*

$$\int |T(f\sigma)|^p \omega \le C \int |f|^p \sigma \quad \textit{for } f \in C_0^\infty,$$

$$\left.\begin{aligned} \Big(\frac{1}{|Q|}\int_Q d\omega\Big)^{1/p}\Big(\frac{1}{|Q|}\int_Q d\sigma\Big)^{1/p'} &\le C, \\ \int_Q |T\chi_Q\sigma|^p &\le C' \int_Q \sigma, \\ \int_Q |T^*\chi_Q\omega|^{p'}\sigma &\le C'' \int_Q \omega, \end{aligned}\right\} \quad \textit{for all cubes } Q.$$

**Remark 1.7.** The first of the three testing conditions above is the two-weight $A_p$ condition. We expect that this condition can be strengthened to a "Poisson two-weight $A_p$ condition". See [Nazarov et al. 2010; Volberg 2003].

The most important instances of this conjecture occur when $T$ is one of a few canonical singular integral operators, such as the Hilbert transform, the Beurling transform, or the Riesz transforms. This question occurs in different instances, such as the Sarason conjecture concerning the composition of Hankel operators, or the semicommutator of Toeplitz operators [Cruz-Uribe et al. 2007; Zheng 1996], mathematical physics [Peherstorfer et al. 2007], as well as perturbation theory of some self-adjoint operators. See references in [Volberg 2003].

To date, this has only been verified for positive operators, such as Poisson integrals and fractional integral operators [Sawyer 1984; 1982; 1988]. Recently the authors have used the methods of Nazarov, Treil and Volberg to prove a special case of the conjecture for the Hilbert transform when $p = 2$ and an energy hypothesis is assumed [Lacey et al. 2011]. Earlier in [2010] Nazarov, Treil and Volberg used a stronger pivotal condition in place of the energy hypothesis, but neither of these conditions are necessary [Lacey et al. 2011]. The two weight Helson–Szegö theorem was proved many years earlier by Cotlar and Sadosky [1979; 1983]; thus the $L^2$ case for the Hilbert transform is completely settled.

Nazarov, Treil and Volberg [1999; 2010] have characterized those weights for which the class of Haar multipliers is bounded when $p = 2$. They also have a result for an important special class of singular integral operators, the "well-localized" operators of [2008]. Citing the specific result here would carry us too far afield, but this class includes the important Haar shift examples, such as the one found by S. Petermichl [2000], and generalized in [2002]. Consequently, characterizations are given in [Volberg 2003] and [Nazarov et al. 2010] for the Hilbert transform and Riesz transforms in weighted $L^2$ spaces under various additional hypotheses. In particular they obtain an analogue of the case $p = 2$ of the strong-type theorem below. Our results can be reformulated in the context there, a theme we do not pursue further here.

We now characterize the weak-type two weight norm inequality for both maximal singular integrals and strongly maximal singular integrals.

**Theorem 1.8** (maximal singular integral weak-type inequalities). *Suppose that $\sigma$ and $\omega$ are positive locally finite Borel measures on $\mathbb{R}^n$, let $1 < p < \infty$, and let $T_\flat$ and $T_\natural$ be the maximal singular integral operators as above with kernel $K(x, y)$ satisfying* (1-9).

(1) *Suppose that the maximal operator $\mathcal{M}$ satisfies* (1-7). *Then $T_\natural$ satisfies the weak-type two weight norm inequality*

$$\|T_\natural(f\sigma)\|_{L^{p,\infty}(\omega)} \le C\|f\|_{L^p(\sigma)} \quad \textit{for } f \in L^p(\sigma), \tag{1-13}$$

if and only if

$$\int_Q T_\natural(\chi_Q f\sigma)(x)\,d\omega(x) \le C_2\Big(\int_Q |f(x)|^p\,d\sigma(x)\Big)^{1/p}\Big(\int_Q d\omega(x)\Big)^{1/p'}, \tag{1-14}$$

*for all cubes $Q \subset \mathbb{R}^n$ and all functions $f \in L^p(\sigma)$.*

(2) *The same characterization as above holds for $T_\flat$ in place of $T_\natural$ everywhere.*

(3) *Suppose that $\sigma$ and $\omega$ are absolutely continuous with respect to Lebesgue measure, that the maximal operator $\mathcal{M}$ satisfies* (1-7), *and that $T$ is a standard singular integral operator with kernel $K$ as*

*above. If* (1-13) *holds for* $T_\natural$ *or* $T_\flat$, *then it also holds for* $T$:

$$\|T(f\sigma)\|_{L^{p,\infty}(\omega)} \le C\|f\|_{L^p(\sigma)} \quad \text{for } f \in L^p(\sigma),\ f\sigma \in L^\infty \text{ with } \operatorname{supp} f\sigma \text{ compact}. \tag{1-15}$$

(4) *Suppose* $c > 0$ *and that* $\{K_j\}_{j=1}^J$ *is a collection of* standard *kernels such that for* each *unit vector* $\boldsymbol{u}$ *there is* $j$ *satisfying*

$$|K_j(x, x+t\boldsymbol{u})| \ge ct^{-n} \quad \text{for } t \in \mathbb{R}. \tag{1-16}$$

*Suppose also that* $\sigma$ *and* $\omega$ *have no common point masses, that is,* $\sigma(\{x\}) \cdot \omega(\{x\}) = 0$ *for all* $x \in \mathbb{R}^n$. *Then*

$$\|(T_j)_\flat(f\sigma)\|_{L^{p,\infty}(\omega)} \le C\|f\|_{L^p(\sigma)} \quad \text{for } f \in L^p(\sigma),\ \text{with } 1 \le j \le J,$$

*if and only if the two weight* $A_p$ *condition* (1-8) *holds and*

$$\int_Q (T_j)_\flat(\chi_Q f\sigma)(x)\, d\omega(x) \le C_2 \Big(\int_Q |f(x)|^p\, d\sigma(x)\Big)^{1/p} \Big(\int_Q d\omega(x)\Big)^{1/p'}, \\ f \in L^p(\sigma),\ \text{cubes } Q \subset \mathbb{R}^n,\ 1 \le j \le J.$$

While in (1)–(3), we assume that the maximal function inequality holds, in point (4), we obtain an *unconditional* characterization of the weak-type inequality for a large class of families of (centered) maximal singular integral operators $T_\flat$. This class includes the individual maximal Hilbert transform in one dimension, the individual maximal Beurling transform in two dimensions, and the families of maximal Riesz transforms in higher dimensions; see Lemma 2.11.

Note that in (1) above, there is only size and smoothness assumptions placed on the kernel, so that it could for instance be a degenerate fractional integral operator, and therefore unbounded on $L^2(dx)$. But, the characterization still has content in this case, if $\omega$ and $\sigma$ are not of full dimension.

In (3), we deduce a two weight inequality for standard singular integrals $T$ without truncations when the measures are absolutely continuous. The proof of this is easy. From (1-13) and the pointwise inequality $T_{0,\infty} f\sigma(x) \le T_\flat f\sigma(x) \le T_\natural f\sigma(x)$, we obtain that for any limiting operator $T_{0,\infty}$ the map $f \to T_{0,\infty} f\sigma$ is bounded from $L^p(\sigma)$ to $L^{p,\infty}(\omega)$. By (1-7) $f \to \mathcal{M} f\sigma$ is bounded; hence $f \to f\sigma$ is bounded, and so Theorem 1.5 shows that $f \to Tf\sigma = T_{0,\infty} f\sigma + af\sigma$ is also bounded, provided we initially restrict attention to functions $f$ for which $f\sigma$ is bounded with compact support.

The characterizing condition (1-14) is a weak-type condition, with the restriction that one only needs to test the weak-type condition for functions supported on a given cube, and test the weak-type norm over that given cube. It also has an interpretation as a dual inequality $\int_Q |L^*(\chi_Q\omega)|^{p'}\, d\sigma \le C_2 \int_Q d\omega$, which we return to below; see (2-10) and (2-11).

We now consider the two weight norm inequality for a strongly maximal singular integral $T_\natural$, but assuming that the measure $\sigma$ is doubling.

**Theorem 1.9** (maximal singular integral strong-type inequalities). *Suppose that* $\sigma$ *and* $\omega$ *are positive locally finite Borel measures on* $\mathbb{R}^n$ *with* $\sigma$ *doubling, let* $1 < p < \infty$, *and let* $T_\flat$ *and* $T_\natural$ *be the maximal singular integral operators as above with kernel* $K(x, y)$ *satisfying* (1-9).

(1) *Suppose that the maximal operator* $\mathcal{M}$ *satisfies* (1-5) *and also the "dual" inequality*

$$\|\mathcal{M}(g\omega)\|_{L^{p'}(\sigma)} \le C\|g\|_{L^{p'}(\omega)} \quad \text{for } g \in L^{p'}(\omega). \tag{1-17}$$

*Then* $T_\natural$ *satisfies the two weight norm inequality*

$$\int_{\mathbb{R}^n} T_\natural(f\sigma)(x)^p \, d\omega(x) \le C \int_{\mathbb{R}^n} |f(x)|^p \, d\sigma(x), \tag{1-18}$$

*for all* $f \in L^p(\sigma)$ *that are bounded with compact support in* $\mathbb{R}^n$, *if and only if both the dual cube testing condition* (1-14) *and the condition*

$$\int_Q T_\natural(\chi_Q g\sigma)(x)^p \, d\omega(x) \le C_1 \int_Q d\sigma(x), \tag{1-19}$$

*holds for all cubes* $Q \subset \mathbb{R}^n$ *and all functions* $|g| \le 1$.

(2) *The same characterization as above holds for* $T_\flat$ *in place of* $T_\natural$ *everywhere. In fact*

$$|T_\natural f\sigma(x) - T_\flat f\sigma(x)| \le C\mathcal{M}(f\sigma)(x).$$

(3) *Suppose that* $\sigma$ *and* $\omega$ *are absolutely continuous with respect to Lebesgue measure, that the maximal operator* $\mathcal{M}$ *satisfies* (1-5), *and that* $T$ *is a standard singular integral operator. If* (1-18) *holds for* $T_\natural$ *or* $T_\flat$, *then it also holds for* $T$:

$$\int_{\mathbb{R}^n} |T(f\sigma)(x)|^p \, d\omega(x) \le C \int_{\mathbb{R}^n} |f(x)|^p \, d\sigma(x) \quad \text{for } f \in L^p(\sigma),\ f\sigma \in L^\infty, \text{ with } \operatorname{supp}(f\sigma) \text{ compact}.$$

(4) *Suppose that* $\{K_j\}_{j=1}^n$ *is a collection of standard kernels satisfying for some* $c > 0$,

$$\pm \operatorname{Re} K_j(x, y) \ge \frac{c}{|x-y|^n} \quad \text{for } \pm(y_j - x_j) \ge \tfrac{1}{4}|x-y|, \tag{1-20}$$

*where* $x = (x_j)_{1\le j\le n}$. *If both* $\omega$ *and* $\sigma$ *are doubling, then* (1-18) *holds for* $(T_j)_\natural$ *and* $(T_j^*)_\natural$ *for all* $1 \le j \le n$ *if and only if both* (1-19) *and* (1-14) *hold for* $(T_j)_\natural$ *and* $(T_j^*)_\natural$ *for all* $1 \le j \le n$.

Note that the second condition (1-19) is a stronger condition than we would like: it is the $L^p$ inequality, applied to functions *bounded by* 1 and supported on a cube $Q$, but with the $L^p(\sigma)$ norm of $\mathbf{1}_Q$ on the right side. It is easy to see that the bounded function $g$ in (1-19) can be replaced by $\chi_E$ for every compact subset $E$ of $Q$. Indeed if $L$ ranges over all linearizations of $T_\natural$, then with

$$g_{h,Q,L} = L^*(\chi_Q h\omega)/|L^*(\chi_Q h\omega)|$$

we have

$$\begin{aligned}
\sup_{|g|\le 1}\int_Q T_\natural(\chi_Q g\sigma)^p\omega &= \sup_{|g|\le 1}\sup_L \sup_{\|h\|_{L^{p'}(\omega)}\le 1}\Big|\int_Q L(\chi_Q g\sigma)h\omega\Big| \\
&= \sup_L \sup_{\|h\|_{L^{p'}(\omega)}\le 1}\sup_{|g|\le 1}\Big|\int_Q L^*(\chi_Q h\omega)g\sigma\Big| \\
&= \sup_L \sup_{\|h\|_{L^{p'}(\omega)}\le 1}\int_Q L^*(\chi_Q h\omega)g_{h,Q,L}\sigma \\
&= \sup_{\|h\|_{L^{p'}(\omega)}\le 1}\sup_L \int_Q L(\chi_Q g_{h,Q,L})h\omega\sigma \\
&\le \sup_{\|h\|_{L^{p'}(\omega)}\le 1}\sup_L \int_Q T_\natural(\chi_Q g_{h,Q,L}\sigma)^p\omega.
\end{aligned}$$

Since $g_{h,Q,L}$ takes on only the values $\pm 1$, it is easy to see that we can take $g=\chi_E$. Point (3) is again easy, just as in the previous weak-type theorem.

And in (4), we note that the truncations, in the way that we formulate them, dominate the maximal function, so that our assumption on $\mathcal{M}$ in (1)–(3) is not unreasonable. The main result of [Nazarov et al. 2010] assumes $p=2$ and that $T$ is the Hilbert transform, and makes similar kinds of assumptions. In fact it is essentially the same as our result in the case $p=2$, but without doubling on $\sigma$ and only for $T$ and not $T_\flat$ or $T_\natural$. Finally, we observe that by our definition of the truncation $T_\natural$, we obtain in point (4) a characterization for doubling measures of the strong-type inequality for appropriate families of standard singular integrals and their adjoints, including the Hilbert and Riesz transforms; see Lemma 2.12.

We don't know if the bounded function $g$ in condition (1-19) can be replaced by the constant function 1.

We now give a characterization of the strong-type weighted norm inequality for the *individual* strongly maximal Hilbert transform $T_\natural$ when $1<p<\infty$ and the measure $\sigma$ is *doubling*. If $p>2$ we use an extra necessary condition (see (1-24)) that involves a "dyadic" Poisson function $\sum_{\ell=0}^{\infty}(2^{-\ell}/|I^{(\ell)}|)\chi_{I^{(\ell)}}(y)$, where $I$ is a dyadic interval and $I^{(\ell)}$ denotes its $\ell$-th ancestor in the dyadic grid, that is, the unique dyadic interval containing $I$ with $|I^{(\ell)}|=2^\ell|I|$. This condition is a variant of the pivotal condition of Nazarov, Treil and Volberg in [2010]; when $1<p\le 2$ it is a consequence of the $A_p$ condition (1-8).

**Theorem 1.10.** *Suppose that $\sigma$ and $\omega$ are positive locally finite Borel measures on $\mathbb{R}$ with $\sigma$* doubling, *let $1<p<\infty$, and let $T_\natural$ be the strongly maximal Hilbert transform. Then $T_\natural$ is* strong *type $(p,p)$ with respect to $(\sigma,\omega)$, that is,*

$$\|T_\natural(f\sigma)\|_{L^p(\omega)}\le C\|f\|_{L^p(\sigma)},$$

*for all $f$ bounded with compact support if and only if the following four conditions hold. In the case $1<p\le 2$, the fourth condition* (1-24) *is implied by the $A_p$ condition* (1-8), *and so in this case we only need the first three conditions below*:

(1) *The dual $T_\natural$ interval testing condition*

$$\int_Q T_\natural(\chi_Q f\sigma)\,d\omega \le C\Big(\int_Q |f|^p\,d\sigma\Big)^{1/p}\Big(\int_Q d\omega\Big)^{1/p'} \tag{1-21}$$

*holds for all intervals $Q$ and $f \in L^p_Q(\sigma)$.*

(2) *The forward $T_\natural$ testing condition*

$$\int_Q T_\natural(\chi_E\sigma)^p\,d\omega \le C\int_Q d\sigma \tag{1-22}$$

*holds for all intervals $Q$ and all compact subsets $E$ of $Q$.*

(3) *The strengthened $A_p$ condition*

$$\Big(\int_{\mathbb{R}}\Big(\frac{|Q|}{|Q|+|x-x_Q|}\Big)^p\,d\omega(x)\Big)^{1/p}\Big(\int_{\mathbb{R}}\Big(\frac{|Q|}{|Q|+|x-x_Q|}\Big)^{p'}\,d\sigma(x)\Big)^{1/p'} \le C|Q| \tag{1-23}$$

*holds for all intervals $Q$.*

(4) *The Poisson condition*

$$\int_{\mathbb{R}}\Big(\sum_{r=1}^{\infty}|I_r|_\sigma|I_r|^{p'-1}\sum_{\ell=0}^{\infty}\frac{2^{-\ell}}{|(I_r)^{(\ell)}|}\chi_{(I_r)^{(\ell)}}(y)\Big)^p\,d\omega(y) \le C\sum_{r=1}^{\infty}|I_r|_\sigma|I_r|^{p'} \tag{1-24}$$

*holds for all pairwise disjoint decompositions $Q=\bigcup_{r=1}^{\infty} I_r$ of the dyadic interval $Q$ into dyadic intervals $I_r$, for any fixed dyadic grid.*

**Remark 1.11.** The strengthened $A_p$ condition (1-23) can be replaced with the weaker “half” condition where the first factor on the left is replaced by $(\int_Q d\omega)^{1/p}$. We do not know if the first three conditions suffice when $p>2$.

## 2. Overview of the proofs and general principles

If $Q$ is a cube, then $\ell(Q)$ is its side length, $|Q|$ is its Lebesgue measure and for a positive Borel measure $\nu$, $|Q|_\nu=\int_Q d\nu$ is its $\nu$-measure.

**2.1.** ***Calderón–Zygmund decompositions.*** Our starting place is the argument in [Sawyer 1988] used to prove a two weight norm inequality for fractional integral operators on Euclidean space. Of course the fractional integral is a positive operator with a monotone kernel, properties we do not have in the current setting.

A central tool arises from the observation that for any positive Borel measure $\mu$, one has the boundedness of a maximal function associated with $\mu$. Define the dyadic $\mu$-maximal operator $\mathcal{M}^{dy}_\mu$ by

$$\mathcal{M}^{dy}_\mu f(x) = \sup_{\substack{Q\in\mathcal{D}\\ x\in Q}}\frac{1}{|Q|_\mu}\int_Q |f|\mu, \tag{2-1}$$

with the supremum taken over all dyadic cubes $Q \in \mathcal{D}$ containing $x$. It is immediate to check that $\mathcal{M}_\mu^{dy}$ satisfies the weak-type (1, 1) inequality, and the $L^\infty(\mu)$ bound is obvious. Hence we have

$$\int (\mathcal{M}_\mu^{dy} f)^p \mu \leq C \int f^p \mu \quad \text{for } f \geq 0 \text{ on } \mathbb{R}^n. \tag{2-2}$$

This observation places certain Calderón–Zygmund decompositions at our disposal. Exploitation of this brings in the testing condition (1-19) involving the bounded function $g$ on a cube $Q$, and indeed, $g$ turns out to be the "good" function in a Calderón–Zygmund decomposition of $f$ on $Q$. The associated "bad" function requires the dual testing condition (1-14) as well.

**2.2.** ***Edge effects of dyadic grids.*** Our operators are not dyadic operators, nor — in contrast to the fractional integral operators — can they be easily obtained from dyadic operators. This leads to the necessity of considering for instance triples of dyadic cubes, which are not dyadic.

Also, dyadic grids distinguish points by for instance making some points on the boundary of many cubes. As our measures are arbitrary, they could conspire to assign extra mass to some of these points. To address this point, Nazarov, Treil and Volberg [2010; 2003; 1997] use a random shift of the grid.

A random approach would likely work for us as well, though the argument would be different from those in the cited papers above. Instead, we will use a nonrandom technique of shifted dyadic grid from [Muscalu et al. 2002], which goes back to P. Jones and J. Garnett. Define a *shifted dyadic grid* to be the collection of cubes

$$\mathcal{D}^\alpha = \left\{2^j(k + [0, 1)^n + (-1)^j \alpha) : j \in \mathbb{Z}, k \in \mathbb{Z}^n\right\}, \quad \text{where } \alpha \in \{0, \tfrac{1}{3}, \tfrac{2}{3}\}^n. \tag{2-3}$$

The basic properties of these collections are these: In the first place, each $\mathcal{D}^\alpha$ is a grid, that is, for $Q, Q' \in \mathcal{D}^\alpha$ we have $Q \cap Q' \in \{\varnothing, Q, Q'\}$ and $Q$ is a union of $2^n$ elements of $\mathcal{D}^\alpha$ of equal volume. In the second place (and this is the novel property for us), for any cube $Q \subset \mathbb{R}^n$ there is a choice of some $\alpha$ and some $Q' \in \mathcal{D}_\alpha$ such that $Q \subset (9/10)Q'$ and $|Q'| \leq C|Q|$.

We define the analogues of the dyadic maximal operator in (2-1), namely

$$\mathcal{M}_\mu^\alpha f(x) = \sup_{\substack{Q \in \mathcal{D}^\alpha \\ x \in Q}} \frac{1}{|Q|_\mu} \int_Q |f| \mu. \tag{2-4}$$

These operators clearly satisfy (2-2). Shifted dyadic grids will return in Section 4.5.

**2.3.** ***A maximum principle.*** A second central tool is a "maximum principle" (or good $\lambda$ inequality) that will permit one to localize large values of a singular integral, provided the maximal function is bounded. It is convenient for us to describe this in conjunction with another fundamental tool of this paper, a family of Whitney decompositions.

We begin with the Whitney decompositions. Fix a finite measure $\nu$ with compact support on $\mathbb{R}^n$ and for $k \in \mathbb{Z}$, let

$$\Omega_k = \{x \in \mathbb{R}^n : T_\natural \nu(x) > 2^k\}. \tag{2-5}$$

Note that $\Omega_k \neq \mathbb{R}^n$ has compact closure for such $\nu$. Fix an integer $N \geq 3$. We can choose $R_W \geq 3$ sufficiently large, depending only on the dimension and $N$, such that there is a collection of cubes $\{Q_j^k\}_j$ that satisfy the following properties:

$$
\begin{array}{rl}
\text{(disjoint cover)} & \Omega_k = \bigcup_j Q_j^k \ \text{ and } \ Q_j^k \cap Q_i^k = \varnothing \text{ if } i \neq j, \\
\text{(Whitney condition)} & R_W Q_j^k \subset \Omega_k \ \text{ and } \ 3R_W Q_j^k \cap \Omega_k^c \neq \varnothing \ \text{ for all } k,\, j, \\
\text{(bounded overlap)} & \sum_j \chi_{NQ_j^k} \leq C\chi_{\Omega_k} \ \text{ for all } k, \\
\text{(crowd control)} & \#\{Q_s^k : Q_s^k \cap NQ_j^k \neq \varnothing\} \leq C \ \text{ for all } k,\, j, \\
\text{(nested property)} & Q_j^k \subsetneqq Q_i^\ell \ \text{ implies } \ k > \ell.
\end{array}
\tag{2-6}
$$

Indeed, one should choose the $\{Q_j^k\}_j$ satisfying the Whitney condition, and then show that the other properties hold. The different combinatorial properties above are fundamental to the proof. And alternate Whitney decompositions are constructed in Section 4.9.1 below.

**Remark 2.4.** Our use of the Whitney decomposition and the maximum principle are derived from the two weight fractional integral argument of Sawyer; see [1988, Section 2]. In particular, the properties above are as Sawyer's, aside from the crowd control property above, which is $N = 3$ there.

**Remark 2.5.** In our notation for the Whitney cubes, the superscript indicates a "height" and the subscript an arbitrary enumeration of the cubes. We will use super- and subscripts below in this manner consistently throughout the paper. It is important to note that a fixed cube $Q$ can arise in *many* Whitney decompositions: There are integers $K_-(Q) \leq K_+(Q)$ with $Q = Q_{j(k)}^k$ for some choice of $j(k)$ for all $K_-(Q) \leq k \leq K_+(Q)$. (The last point follows from the nested property.) There is no *a priori* upper bound on $K_+(Q) - K_-(Q)$.

**Lemma 2.6** (maximum principle). *Let $\nu$ be a finite (signed) measure with compact support. For any cube $Q_j^k$ as above, we have the pointwise inequality*

$$
\sup_{x \in Q_j^k} T_\natural(\chi_{(3Q_j^k)^c}\nu)(x) \leq 2^k + C\boldsymbol{P}(Q_j^k, \nu) \leq 2^k + CM(Q_j^k, \nu),
\tag{2-7}
$$

*where $\boldsymbol{P}(Q, \nu)$ and $M(Q, \nu)$ are defined by*

$$
\begin{aligned}
\boldsymbol{P}(Q, \nu) &\equiv \frac{1}{|Q|}\int_Q d|\nu| + \sum_{\ell=0}^{\infty} \frac{\delta(2^{-\ell})}{|2^{\ell+1}Q|}\int_{2^{\ell+1}Q \setminus 2^\ell Q} d|\nu|, \\
M(Q, \nu) &\equiv \sup_{Q' \supset Q} \frac{1}{|Q'|}\int_{Q'} d|\nu|.
\end{aligned}
\tag{2-8}
$$

The bound in terms of $\boldsymbol{P}(Q, \nu)$ should be regarded as one in terms of a modified Poisson integral. It is both slightly sharper than that of $M(Q, \nu)$, and a linear expression in $|\nu|$, a fact will be used in the proof of the strong-type estimates.

*Proof.* To see this, take $x \in Q_j^k$ and note that for each $\eta > 0$ there is $\boldsymbol{\varepsilon}$ with $\ell(Q_j^k) < \max_{1\le j\le n} \varepsilon_j < R < \infty$ and $\theta \in [0, 2\pi)$ such that

$$\begin{aligned} T_\natural(\chi_{(3Q_j^k)^c}\nu)(x) &\le (1+\eta)\Big|\int_{(3Q_j^k)^c} K(x, y)\zeta_\varepsilon(x-y)\eta_R(x-y)d\nu(y)\Big| \\ &= (1+\eta)e^{i\theta}T_{\boldsymbol{\varepsilon},R}(\chi_{(3Q_j^k)^c}\nu)(x). \end{aligned}$$

For convenience we take $\eta = 0$ in the sequel. By the Whitney condition in (2-6), there is a point $z \in 3R_W Q_j^k \cap \Omega_k^c$ and it now follows that (remember that $\ell(Q_j^k) < \max_{1\le j\le n} \varepsilon_j$)

$$\begin{aligned} &|T_{\boldsymbol{\varepsilon},R}(\chi_{(3Q_j^k)^c}\nu)(x) - T_{\boldsymbol{\varepsilon},R}\nu(z)| \\ &\quad\le C\frac{1}{|6R_W Q_j^k|}\int_{6R_W Q_j^k} d|\nu| + |T_{\varepsilon,R}(\chi_{(6R_W Q_j^k)^c}\nu)(x) - T_{\boldsymbol{\varepsilon},R}(\chi_{(6R_W Q_j^k)^c}\nu)(z)| \\ &\quad= C\frac{1}{|6R_W Q_j^k|}\int_{6R_W Q_j^k} d|\nu| \\ &\qquad\quad + \int_{(6R_W Q_j^k)^c} |K(x, y)\zeta_\varepsilon(x-y)\eta_R(x-y) - K(z, y)\zeta_{\boldsymbol{\varepsilon}}(z-y)\eta_R(z-y)|d|\nu|(y) \\ &\quad\le C\frac{1}{|6R_W Q_j^k|}\int_{6R_W Q_j^k} d|\nu| + C\int_{(6R_W Q_j^k)^c} \delta\Big(\frac{|x-z|}{|x-y|}\Big)\frac{1}{|x-y|^n}d|\nu|(y) \\ &\quad\le C\boldsymbol{P}(Q_j^k, \nu). \end{aligned}$$

Thus

$$T_\natural(\chi_{(3Q_j^k)^c}\nu)(x) \le |T_\natural\nu(z)| + C\boldsymbol{P}(Q_j^k, \nu) \le 2^k + C\boldsymbol{P}(Q_j^k, \nu),$$

which yields (2-7) since $\boldsymbol{P}(Q, \nu) \le CM(Q, \nu)$. □

**2.7. *Linearizations.*** We now make comments on the linearizations of our maximal singular integral operators. We would like, at different points, to treat $T_\natural$ as a linear operator, which of course it is not. Nevertheless $T_\natural$ is a pointwise supremum of the linear truncation operators $T_{\varepsilon,R}$, and as such, the supremum can be linearized with measurable selection of the parameters $\boldsymbol{\varepsilon}$ and $R$, as was just done in the previous proof. We make this a definition.

**Definition 2.8.** We say that $L$ is a linearization of $T_\natural$ if there are measurable functions $\boldsymbol{\varepsilon}(x) \in (0, \infty)^n$ and $R(x) \in (0, \infty)$ with $1/4 \le \varepsilon_i/\varepsilon_j \le 4$, $\max_{1\le i\le n} \varepsilon_i < R(x) < \infty$ and $\theta(x) \in [0, 2\pi)$ such that

$$Lf(x) = e^{i\theta(x)}T_{\boldsymbol{\varepsilon}(x),R(x)}f(x) \quad \text{for } x \in \mathbb{R}^n. \tag{2-9}$$

For fixed $f$ and $\delta > 0$, we can always choose a linearization $L$ so that $T_\natural f(x) \le (1+\delta)Lf(x)$ for all $x$. In a typical application of this lemma, one takes $\delta$ to be one.

Note that condition (1-19) is obtained from inequality (1-18) by testing over $f$ of the form $f = \chi_Q g$ with $|g| \le 1$, and then restricting integration on the left to $Q$. By passing to linearizations $L$, we can "dualize" (1-14) to the testing conditions

$$\int_Q |L^*(\chi_Q \omega)(x)|^{p'} \, d\sigma(x) \le C_2 \int_Q d\omega(x), \tag{2-10}$$

or equivalently (note that in (1-19) the presence of $g$ makes a difference, but not here),

$$\int_Q |L^*(\chi_Q g\omega)(x)|^{p'} \, d\sigma(x) \le C_2 \int_Q d\omega(x) \quad \text{for } |g| \le 1, \tag{2-11}$$

with the requirement that these inequalities hold *uniformly* in all linearizations $L$ of $T_\natural$.

While the smooth truncation operators $T_{\boldsymbol{\varepsilon},R}$ are essentially self-adjoint, the dual of a linearization $L$ is generally complicated. Nevertheless, the dual $L^*$ does satisfy one important property, which plays a crucial role in the proof of Theorem 1.9, the $L^p$-norm inequalities.

**Lemma 2.9.** *$L^*\mu$ is $\delta$-Hölder continuous (where $\delta$ is the Dini modulus of continuity of the kernel $K$) with constant $C\boldsymbol{P}(Q,\mu)$ on any cube $Q$ satisfying $\int_{3Q} d|\mu| = 0$, that is,*

$$|L^*\mu(y) - L^*\mu(y')| \le C\boldsymbol{P}(Q,\mu)\delta\Big(\frac{|y-y'|}{\ell(Q)}\Big) \quad \textit{for } y, y' \in Q. \tag{2-12}$$

*Here, recall the definition* (2-8) *and that $\boldsymbol{P}(Q,\mu) \le CM(Q,\mu)$.*

*Proof.* Suppose $L$ is as in (2-9). Then for any finite measure $\nu$,

$$L\nu(x) = e^{i\theta(x)} \int \zeta_{\boldsymbol{\varepsilon}(x)}(x-y)\eta_{R(x)}(x-y)K(x,y)d\nu(y).$$

Fubini's theorem shows that the dual operator $L^*$ is given on a finite measure $\mu$ by

$$L^*\mu(y) = \int \zeta_{\boldsymbol{\varepsilon}(x)}(x-y)\eta_{R(x)}(x-y)K(x,y)e^{i\theta(x)}d\mu(x). \tag{2-13}$$

For $y, y' \in Q$ and $|\mu|(3Q) = 0$, we thus have

$$\begin{aligned} L^*\mu(y) - L^*\mu(y') = \int \{(\zeta_{\boldsymbol{\varepsilon}(x)}\eta_{R(x)})(x-y) - (\zeta_{\boldsymbol{\varepsilon}(x)}\eta_{R(x)})(x-y')\}K(x,y)e^{i\theta(x)}d\mu(x) \\ + \int (\zeta_{\boldsymbol{\varepsilon}(x)}\eta_{R(x)})(x-y')(K(x,y) - K(x,y'))e^{i\theta(x)}d\mu(x), \end{aligned}$$

from which (2-12) follows easily if we split the two integrals in $x$ over dyadic annuli centered at the center of $Q$. □

**2.10.** ***Control of maximal functions.*** Next we record the facts that $T$ and $T_\natural$ control $\mathcal{M}$ for many (sets of) standard singular integrals $T$, including the Hilbert transform, the Beurling transform and the sets of Riesz transforms in higher dimensions.

**Lemma 2.11.** *Suppose that $\sigma$ and $\omega$ have no point masses in common, and that $\{K_j\}_{j=1}^J$ is a collection of* standard *kernels satisfying* (1-9) *and* (1-16). *If the corresponding operators $T_j$ given by* (1-12) *satisfy*

$$\|\chi_E T_j(f\sigma)\|_{L^{p,\infty}(\omega)} \le C\|f\|_{L^p(\sigma)} \quad \textit{where} \quad E = \mathbb{R}^n \setminus \operatorname{supp} f,$$

*for* $1 \le j \le J$, *then the two weight* $A_p$ *condition* (1-8) *holds*, *and hence also the weak-type two weight inequality* (1-7).

*Proof.* Part of the "one weight" argument of [Stein and Shakarchi 2005, page 21] yields the *asymmetric* two weight $A_p$ condition

$$|Q|_\omega |Q'|_\sigma^{p-1} \le C|Q|^p, \tag{2-14}$$

where $Q$ and $Q'$ are cubes of equal side length $r$ and distance approximately $C_0 r$ apart for some fixed large positive constant $C_0$ (for this argument we choose the unit vector $\boldsymbol{u}$ in (1-16) to point in the direction from the center of $Q$ to the center of $Q'$, and then with $j$ as in (1-16), $C_0$ is chosen large enough by (1-9) that (1-16) holds for all unit vectors $\boldsymbol{u}$ pointing from a point in $Q$ to a point in $Q'$). In the one weight case treated in [Stein and Shakarchi 2005], it is easy to obtain from this (even for a *single* direction $\boldsymbol{u}$) the usual (symmetric) $A_p$ condition (1-8). Here we will instead use our assumption that $\sigma$ and $\omega$ have no point masses in common for this purpose.

So fix an open dyadic cube $Q_0$ in $\mathbb{R}^n$, say with side length 1, let $\mathsf{Q}_0 = Q_0 \times Q_0$ and set

$$\Omega = \{\mathsf{Q} = Q \times Q' \text{ dyadic} : \mathsf{Q} \subset \mathsf{Q}_0 \text{ and (2-14) holds for } Q \text{ and } Q'\}.$$

Note that with $\mathsf{Q} = Q \times Q'$, inequality (2-14) can be written

$$\mathcal{A}_p(\omega, \sigma; \mathsf{Q}) \le C|\mathsf{Q}|^{p/2}, \tag{2-15}$$

where

$$\mathcal{A}_p(\omega, \sigma; \mathsf{Q}) = |Q|_\omega |Q'|_\sigma^{p-1}.$$

Here $\mathcal{A}_2(\omega, \sigma; \mathsf{Q}) = |\mathsf{Q}|_{\omega\times\sigma}$, where $\omega \times \sigma$ denotes product measure on $\mathbb{R}^n \times \mathbb{R}^n$. For $1 < p < \infty$ we easily see that if $\mathsf{Q}_0 = \bigcup_\alpha \mathsf{Q}_\alpha$ is a pairwise disjoint union of cubes $\mathsf{Q}_\alpha$, then the Lebesgue measures satisfy

$$\sum_\alpha |\mathsf{Q}_\alpha|^{p/2} \le C|Q_0 \times Q_0|^{p/2} = C|Q_0|^p.$$

Suppose first that $1 < p \le 2$. Divide $\mathsf{Q}_0$ into $2^n \times 2^n = 4^n$ congruent subcubes $\mathsf{Q}_0^1, \dots, \mathsf{Q}_0^{4^n}$ of side length $\frac{1}{2}$, and set aside those $\mathsf{Q}_0^j \in \Omega$ (those for which (2-14) holds) into a collection of *stopping cubes* $\Gamma$. Continue to divide the remaining $\mathsf{Q}_0^j$ into $4^n$ congruent subcubes $\mathsf{Q}_0^{j,1}, \dots, \mathsf{Q}_0^{j,4^n}$ of side length $\frac{1}{4}$, and again, set aside those $\mathsf{Q}_0^{j,i} \in \Omega$ into $\Gamma$, and continue subdividing those that remain. We continue with such subdivisions for $N$ generations so that all the cubes *not* set aside into $\Gamma$ have side length $2^{-N}$. The important property these cubes have is that they all lie within distance $r2^{-N}$ of the diagonal $\mathcal{D} = \{(x, x) : (x, x) \in \mathsf{Q}_0\}$ in $\mathsf{Q}_0 = Q_0 \times Q_0$ since (2-14) holds for all pairs of cubes $Q$ and $Q'$ of equal side length $r$ having distance approximately $C_0 r$ apart. Enumerate the cubes in $\Gamma$ as $\{\mathsf{Q}_\alpha\}_\alpha$ and those remaining that are not in $\Gamma$ as $\{\mathsf{P}_\beta\}_\beta$. Thus we have the pairwise disjoint decomposition

$$\mathsf{Q}_0 = \Big(\bigcup_\alpha \mathsf{Q}_\alpha\Big) \cup \Big(\bigcup_\beta \mathsf{Q}_\beta\Big).$$

In the case $p=2$, the countable additivity of the product measure $\omega\times\sigma$ shows that

$$\mathscr{A}_2(\omega,\sigma;\mathsf{Q}_0)=\sum_\alpha \mathscr{A}_2(\omega,\sigma;\mathsf{Q}_\alpha)+\sum_\beta \mathscr{A}_2(\omega,\sigma;\mathsf{P}_\beta).$$

For the more general case $1<p\le 2$, note that at each division described above, we have using $0<p-1\le 1$

$$\mathscr{A}_p(\omega,\sigma;\mathsf{Q}_0)=\Big(\sum_{i=1}^{2^n}|\mathsf{Q}_0^j|_\omega\Big)\Big(\sum_{i=1}^{2^n}|\mathsf{Q}_0^j|_\sigma\Big)^{p-1}\le\Big(\sum_{i=1}^{2^n}|\mathsf{Q}_0^j|_\omega\Big)\Big(\sum_{i=1}^{2^n}|\mathsf{Q}_0^j|_\sigma^{p-1}\Big)=\sum_{j=1}^{4^n}\mathscr{A}_p(\omega,\sigma;\mathsf{Q}_0^j),$$

$$\mathscr{A}_p(\omega,\sigma;\mathsf{Q}_0^j)\le\sum_{i=1}^{4^n}\mathscr{A}_p(\omega,\sigma;\mathsf{Q}_0^{j,i})\quad\text{for }\mathsf{Q}_0^j\notin\Gamma,$$

and so on. It follows that

$$\begin{aligned}\mathscr{A}_p(\omega,\sigma;\mathsf{Q}_0)&\le\sum_\alpha\mathscr{A}_p(\omega,\sigma;\mathsf{Q}_\alpha)+\sum_\beta\mathscr{A}_p(\omega,\sigma;\mathsf{P}_\beta)\\&\le C\sum_\alpha|\mathsf{Q}_\alpha|^{p/2}+\sum_\beta\mathscr{A}_p(\omega,\sigma;\mathsf{P}_\beta)\le C|Q_0|^p+\sum_\beta\mathscr{A}_p(\omega,\sigma;\mathsf{P}_\beta).\end{aligned}$$

Since $\omega$ and $\sigma$ have no point masses in common, it is not hard to show, using that the side length of $\mathsf{P}_\beta=P_\beta\times P'_\beta$ is $2^{-N}$ and $\operatorname{dist}(\mathsf{P}_\beta,\mathfrak{D})\le C2^{-N}$, that we have the limit

$$\sum_\beta\mathscr{A}_p(\omega,\sigma;\mathsf{P}_\beta)\to 0\quad\text{as }N\to\infty.$$

Indeed, if $\sigma$ has no point masses at all, then

$$\begin{aligned}\sum_\beta\mathscr{A}_p(\omega,\sigma;\mathsf{P}_\beta)&=\sum_\beta|P_\beta|_\omega|P'_\beta|_\sigma^{p-1}\\&\le\Big(\sum_\beta|P_\beta|_\omega\Big)\sup_\beta|P'_\beta|_\sigma^{p-1}\le C|Q_0|_\omega\sup_\beta|P'_\beta|_\sigma^{p-1}\to 0\quad\text{as }N\to\infty.\end{aligned}$$

If $\sigma$ contains a point mass $c\delta_x$, then

$$\sum_{\beta:x\in P'_\beta}\mathscr{A}_p(\omega,\sigma;\mathsf{P}_\beta)\le\Big(\sum_{\beta:x\in P'_\beta}|P_\beta|_\omega\Big)\sup_{\beta:x\in P'_\beta}|P'_\beta|_\sigma^{p-1}\le C\Big(\sum_{\beta:x\in P'_\beta}|P_\beta|_\omega\Big)\to 0\quad\text{as }N\to\infty$$

since $\omega$ has no point mass at $x$. The argument in the general case is technical, but involves no new ideas, and we leave it to the reader. We thus conclude that

$$\mathscr{A}_p(\omega,\sigma;\mathsf{Q}_0)\le C|Q_0|^p,$$

which is (1-8). The case $2\le p<\infty$ is proved in the same way using that (2-14) can be written

$$\mathscr{A}_{p'}(\sigma,\omega;\mathsf{Q}_\alpha)\le C'|\mathsf{Q}_\alpha|^{p'/2}.$$

□

**Lemma 2.12.** *If* $\{T_j\}_{j=1}^n$ *satisfies* (1-20), *then*

$$\mathcal{M}\nu(x) \leq C \sum_{j=1}^{n} (T_j)_\natural \nu(x) \quad \textit{for } x \in \mathbb{R}^n, \textit{ with } \nu \geq 0 \textit{ a finite measure with compact support.}$$

*Proof.* We prove the case $n = 1$, the general case being similar. Then with $T = T_1$ and $r > 0$ we have

$$\begin{aligned}\operatorname{Re}(T_{r,r/4,100r}\nu(x) - T_{r,4r,100r}\nu(x)) &= \int (\zeta_{r/4}(y-x) - \zeta_{4r}(y-x)) \operatorname{Re} K(x,y) d\nu(y) \\ &\geq \frac{c}{r} \int_{[x+r/2, x+2r]} d\nu(y).\end{aligned}$$

Thus

$$T_\natural \nu(x) \geq \max\{|T_{r,r/4,100r}\nu(x)|, |T_{r,4r,100r}\nu(x)|\} \geq \frac{c}{r} \int_{[x+r/2,x+2r]} d\nu(y),$$

and similarly

$$T_\natural \nu(x) \geq \frac{c}{r} \int_{[x-2r,x-r/2]} d\nu(y).$$

It follows that

$$\begin{aligned}\mathcal{M}\nu(x) &\leq \sup_{r>0} \frac{1}{4r} \int_{[x-2r,x+2r]} d\nu(y) \\ &= \sup_{r>0} \sum_{k=0}^{\infty} 2^{-k} \frac{1}{2^{2-k} r} \int_{[x-2^{1-k}r, x-2^{-1-k}r] \cup [x+2^{-1-k}r, x+2^{1-k}r]} d\nu(y) \leq C T_\natural \nu(x). \qquad \square\end{aligned}$$

Finally, we will use the following covering lemma of Besicovitch type for multiples of dyadic cubes (the case of triples of dyadic cubes arises in (4-50) below).

**Lemma 2.13.** *Let $M$ be an odd positive integer, and suppose that $\Phi$ is a collection of cubes $P$ with bounded diameters and having the form $P = MQ$, where $Q$ is dyadic* (*a product of clopen dyadic intervals*). *If $\Phi^*$ is the collection of* maximal *cubes in $\Phi$, that is, $P^* \in \Phi^*$ provided there is no strictly larger $P$ in $\Phi$ that contains $P^*$, then the cubes in $\Phi^*$ have finite overlap at most $M^n$.*

*Proof.* Let $Q_0 = [0, 1)^n$ and assign labels $1, 2, 3, \ldots, M^n$ to the dyadic subcubes of side length one of $MQ_0$. We say that the subcube labeled $k$ is of type $k$, and we extend this definition by translation and dilation to the subcubes of $MQ$ having side length that of $Q$. Now we simply observe that if $\{P_i^*\}_i$ is a set of cubes in $\Phi^*$ containing the point $x$, then for a given $k$, there is at most one $P_i^*$ that contains $x$ in its subcube of type $k$. The reason is that if $P_j^*$ is another such cube and $\ell(P_j^*) \leq \ell(P_i^*)$, we must have $P_j^* \subset P_i^*$ (draw a picture in the plane for example). $\square$

**2.14.** ***Preliminary precaution.*** Given a positive locally finite Borel measure $\mu$ on $\mathbb{R}^n$, there exists a rotation such that all boundaries of rotated dyadic cubes have $\mu$-measure zero (see [Mateu et al. 2000] where they actually prove a stronger assertion when $\mu$ has no point masses, but our conclusion is obvious for a sum of point mass measures). We will assume that such a rotation has been made so that all boundaries of rotated dyadic cubes have $(\omega+\sigma)$-measure zero, where $\omega$ and $\sigma$ are the positive Borel measures appearing in the theorems above (of course $\sigma$ doubling implies that $\sigma$ cannot contain any point masses, but this argument works as well for general $\sigma$ as in the weak type theorem). While this assumption is not essential for the proof, it relieves the reader of having to consider the possibility that boundaries of dyadic cubes have positive measure at each step of the argument below.

Recall also (see for example [Rudin 1987, Theorem 2.18]) that any positive locally finite Borel measure on $\mathbb{R}^n$ is both inner and outer regular.

## 3. The proof of Theorem 1.8: Weak-type inequalities

We begin with the necessity of condition (1-14):

$$\begin{aligned}\int_Q T_\natural(\chi_Q f\sigma)\omega &= \int_0^\infty \min\bigl\{|Q|_\omega, |\{T_\natural(\chi_Q f\sigma)>\lambda\}|_\omega\bigr\}d\lambda \\ &\le \Bigl(\int_0^A+\int_A^\infty\Bigr)\min\Bigl\{|Q|_\omega, C\lambda^{-p}\int|f|^p\,d\sigma\Bigr\}d\lambda \\ &\le A|Q|_\omega + CA^{1-p}\int|f|^p\,d\sigma = (C+1)|Q|_\omega^{1/p'}\Bigl(\int|f|^p\,d\sigma\Bigr)^{1/p},\end{aligned}$$

if we choose $A=(\int|f|^p\,d\sigma/|Q|_\omega)^{1/p}$.

Now we turn to proving (1-13), assuming both (1-14) and (1-7), and moreover that $f$ is bounded with compact support. We will prove the quantitative estimate

$$\|T_\natural f\sigma\|_{L^{p,\infty}(\omega)} \le C\{\mathfrak{A}+\mathfrak{T}_*\}\|f\|_{L^p(\sigma)}, \tag{3-1}$$

$$\mathfrak{A} = \sup_Q \sup_{\|f\|_{L^p(\sigma)}=1} \sup_{\lambda>0} \lambda|\{\mathcal{M}(f\sigma)>\lambda\}|_\omega^{1/p}, \tag{3-2}$$

$$\mathfrak{T}_* = \sup_{\|f\|_{L^p(\sigma)}=1} \sup_Q |Q|_\omega^{-1/p'}\int_Q T_\natural(\chi_Q f\sigma)(x)\,d\omega(x). \tag{3-3}$$

We should emphasize that the term (3-2) is comparable to the two weight $A_p$ condition (1-8).

Standard considerations [Sawyer 1984, Section 2] show that it suffices to prove the following good-$\lambda$ inequality: There is a positive constant $C$ such that for $\beta>0$ sufficiently small, and provided

$$\sup_{0<\lambda<\Lambda} \lambda^p|\{x\in\mathbb{R}^n : T_\natural f\sigma(x)>\lambda\}|_\omega < \infty \quad \text{for } \Lambda<\infty, \tag{3-4}$$

we have this inequality:

$$\begin{aligned}&|\{x\in\mathbb{R}^n : T_\natural f\sigma(x)>2\lambda \text{ and } \mathcal{M}f\sigma(x)\le\beta\lambda\}|_\omega \\ &\qquad\le C\beta\mathfrak{T}_*^p|\{x\in\mathbb{R}^n : T_\natural f\sigma(x)>\lambda\}|_\omega + C\beta^{-p}\lambda^{-p}\int|f|^p\,d\sigma.\end{aligned} \tag{3-5}$$

Our presumption (3-4) holds due to the $A_p$ condition (1-8) and the fact that

$$\{x\in\mathbb{R}^n : T_\natural f\sigma(x)>\lambda\}\subset B(0,c\lambda^{-1/n}) \quad \text{for } \lambda>0 \text{ small},$$

Hence it is enough to prove (3-5).

To prove (3-5) we choose $\lambda = 2^k$ and apply the decomposition in (2-6). In this argument, we can take $k$ to be fixed, so that we can suppress its appearance as a superscript in this section. (When we come to $L^p$ estimates, we will not have this luxury.)

Define

$$E_j = \{x \in Q_j : T_\natural f\sigma(x) > 2\lambda \text{ and } \mathcal{M} f\sigma(x) \le \beta\lambda\}.$$

Then for $x \in E_j$, we can apply Lemma 2.6 to deduce

$$T_\natural(\chi_{(3Q_j)^c} f\sigma)(x) \le (1 + C\beta)\lambda. \tag{3-6}$$

If we take $\beta > 0$ so small that $1 + C\beta \le \frac{3}{2}$, then (3-6) implies that for $x \in E_j$

$$2\lambda < T_\natural f\sigma(x) \le T_\natural \chi_{3Q_j} f\sigma(x) + T_\natural \chi_{(3Q_j)^c} f\sigma(x) \le T_\natural \chi_{3Q_j^k} f\sigma(x) + \tfrac{3}{2}\lambda.$$

Integrating this inequality with respect to $\omega$ over $E_j$ we obtain

$$\lambda |E_j|_\omega \le 2 \int_{E_j} (T_\natural \chi_{3Q_j} f\sigma)\omega. \tag{3-7}$$

The disjoint cover condition in (2-6) shows that the sets $E_j$ are disjoint, and this suggests we should sum their $\omega$-measures. We split this sum into two parts, according to the size of $|E_j|_\omega / |3Q_j|_\omega$. The left side of (3-5) satisfies

$$\sum_j |E_j|_\omega \le \beta \sum_{j:|E_j|_\omega \le \beta |3Q_j|_\omega} |3Q_j|_\omega + \beta^{-p} \sum_{j:|E_j|_\omega > \beta |3Q_j|_\omega} |E_j|_\omega \Big( \frac{2}{\lambda} \frac{1}{|3Q_j|_\omega} \int_{E_j} (T_\natural \chi_{3Q_j} f\sigma)\omega \Big)^p.$$

Call the added pieces of this $I$ and $II$. Now

$$I \le \beta \sum_j |3Q_j^k|_\omega \le C\beta |\Omega|_\omega,$$

by the finite overlap condition in (2-6). From (1-14) with $Q = 3Q_j$ we have

$$\begin{aligned} II &\le \Big(\frac{2}{\beta\lambda}\Big)^p \sum_j |E_j|_\omega \Big( \frac{1}{|3Q_j|_\omega} \int_{E_j^k} (T_\natural \chi_{3Q_j} f\sigma)\omega \Big)^p \\ &\le C \Big(\frac{2}{\beta\lambda}\Big)^p \mathfrak{T}_*^p \sum_j |E_j|_\omega \frac{1}{|3Q_j|_\omega^p} |3Q_j|_\omega^{p-1} \int_{3Q_j} |f|^p \, d\sigma \\ &\le C \Big(\frac{2}{\beta\lambda}\Big)^p \mathfrak{T}_*^p \int \Big( \sum_j \chi_{3Q_j^k} \Big) |f|^p \, d\sigma \le C \Big(\frac{2}{\beta\lambda}\Big)^p \mathfrak{T}_*^p \int |f|^p \, d\sigma, \end{aligned}$$

by the finite overlap condition in (2-6) again. This completes the proof of the good-$\lambda$ inequality (3-5).

The proof of assertion 2 regarding $T_\flat$ is similar. Assertion 3 was discussed earlier and assertion 4 follows readily from assertion 2 and Lemma 2.11. □

## 4. The proof of Theorem 1.9: Strong-type inequalities

Since conditions (1-19) and (1-14) are obviously necessary for (1-18), we turn to proving the weighted inequality (1-18) for the strongly maximal singular integral $T_\natural$.

**4.1.** ***The quantitative estimate.*** In particular, we will prove

$$\|T_\natural f\sigma\|_{L^p(\omega)} \le C\big(\mathfrak{M}+\gamma^2\mathfrak{M}_*+\gamma^2\mathfrak{T}+\mathfrak{T}_*\big)\|f\|_{L^p(\sigma)}, \tag{4-1}$$

$$\mathfrak{M} = \sup_{\|f\|_{L^p(\sigma)}=1} \|\mathcal{M}(f\sigma)\|_{L^p(\omega)}, \tag{4-2}$$

$$\mathfrak{M}_* = \sup_{\|g\|_{L^{p'}(\omega)}=1} \|\mathcal{M}(g\omega)\|_{L^{p'}(\sigma)}, \tag{4-3}$$

$$\mathfrak{T} = \sup_{Q}\sup_{\|f\|_{L^\infty}\le 1} |Q|_\sigma^{-1/p}\|\chi_Q T_\natural(\chi_Q f\sigma)\|_{L^p(\omega)}, \tag{4-4}$$

$$\mathfrak{T}_* = \sup_{\|f\|_{L^p(\sigma)}=1}\sup_{Q}|Q|_\omega^{-1/p'}\int_Q T_\natural(\chi_Q f\sigma)(x)\,d\omega(x), \tag{4-5}$$

where $\gamma \ge 2$ is a doubling constant for the measure $\sigma$; see (4-19) below. Note that $\gamma$ appears only in conjunction with $\mathfrak{T}$ and $\mathfrak{M}_*$. The norm estimates on the maximal function (4-2) and (4-3) are equivalent to the testing conditions in (1-6) and its dual formulation. The term $\mathfrak{T}_*$ also appeared in (3-3).

**4.2.** ***The initial construction.*** We suppose that both (1-19) and (1-14) hold, that is, (4-4) and (4-5) are finite, and that $f$ is bounded with compact support on $\mathbb{R}^n$. Moreover, in the case (1-20) holds, we see that (1-19) (the finiteness of (4-4)) implies (1-6) by Lemma 2.12, and so by Theorem 1.2 we may also assume that the maximal operator $\mathcal{M}$ satisfies the two weight norm inequality (1-5). It now follows that $\int(T_\natural f\sigma)^p\omega < \infty$ for $f$ bounded with compact support. Indeed, $T_\natural f\sigma \le C\mathcal{M}f\sigma$ far away from the support of $f$, while $T_\natural f\sigma$ is controlled by the finiteness of the testing condition (4-4) near the support of $f$.

Let $\{Q_j^k\}$ be the cubes as in (2-5) and (2-6), with the measure $\nu$ that appears in there being $\nu = f\sigma$. We will use Lemma 2.6 with this choice of $\nu$ as well. Now define an "exceptional set" associated to $Q_j^k$ to be

$$E_j^k = Q_j^k \cap (\Omega_{k+1}\setminus\Omega_{k+2}).$$

See Figure 4.1. One might anticipate the definition of the exceptional set to be more simply $Q_j^k\cap\Omega_{k+1}$. We are guided to this choice by the work on fractional integrals [Sawyer 1988]. And indeed, the choice of exceptional set above enters in a decisive way in the analysis of the bad function at the end of the proof.

We estimate the left side of (1-18) in terms of this family of dyadic cubes $\{Q_j^k\}_{k,j}$ by

$$\begin{aligned}\int (T_\natural f\sigma)^p\omega(dx) &\le \sum_{k\in\mathbb{Z}}(2^{k+2})^p|\Omega_{k+1}\setminus\Omega_{k+2}|_\omega \\ &\le \sum_{k,j}(2^{k+2})^p|E_j^k|_\omega.\end{aligned} \tag{4-6}$$

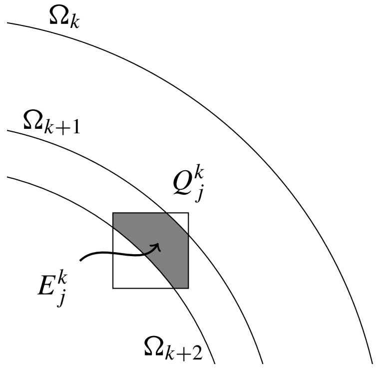


**Figure 4.1.** The set $E_j^k(Q)$.

Choose a linearization $L$ of $T_\natural$ as in (2-9) so that (recall $R(x)$ is the upper limit of truncation)

$$R(x) \le \tfrac{1}{2}\ell(Q_j^k) \quad \text{for } x \in E_j^k, \tag{4-7}$$

$$\text{and} \quad T_\natural(\chi_{3Q_j^k} f\sigma)(x) \le 2L(\chi_{3Q_j^k} f\sigma)(x) + C\frac{1}{|3Q_j^k|}\int_{3Q_j^k} |f|\sigma \quad \text{for } x \in E_j^k.$$

For $x \in E_j^k$, the maximum principle (2-7) yields

$$T_\natural \chi_{3Q_j^k} f\sigma(x) \ge T_\natural f\sigma(x) - T_\natural \chi_{(3Q_j^k)^c} f\sigma(x) > 2^{k+1} - 2^k - C\boldsymbol{P}(Q_j^k, f\sigma) = 2^k - C\boldsymbol{P}(Q_j^k, f\sigma).$$

From (4-7) we conclude that

$$L\chi_{3Q_j^k} f\sigma(x) \ge 2^{k-1} - C\boldsymbol{P}(Q_j^k, f\sigma).$$

Thus either $2^k \le 4\inf_{E_j^k} L\chi_{3Q_j^k} f\sigma$ or $2^k \le 4C\boldsymbol{P}(Q_j^k, f\sigma) \le 4CM(Q_j^k, f\sigma)$. So we obtain either

$$|E_j^k|_\omega \le C2^{-k}\int_{E_j^k} (L\chi_{3Q_j^k} f\sigma)\omega(dx), \tag{4-8}$$

or

$$|E_j^k|_\omega \le C2^{-pk}|E_j^k|_\omega M(Q_j^k, f\sigma)^p \le C2^{-pk}\int_{E_j^k} (\mathcal{M} f\sigma)^p\omega(dx). \tag{4-9}$$

Now consider the following decomposition of the set of indices $(k, j)$:

$$\begin{aligned} \mathbb{E} &= \{(k, j) : |E_j^k|_\omega \le \beta|NQ_j^k|_\omega\}, \\ \mathbb{F} &= \{(k, j) : \text{(4-9) holds}\}, \\ \mathbb{G} &= \{(k, j) : |E_j^k|_\omega > \beta|NQ_j^k|_\omega \text{ and (4-8) holds}\}, \end{aligned} \tag{4-10}$$

where $0 < \beta < 1$ will be chosen sufficiently small at the end of the argument. (It will be of the order of $c^p$ for a small constant $c$.) By the "bounded overlap" condition of (2-6), we have

$$\sum_j \chi_{NQ_j^k} \le C \quad \text{for } k \in \mathbb{Z}. \tag{4-11}$$

We then have the corresponding decomposition:

$$\begin{aligned}
\int (T_\natural f\sigma)^p \omega &\le \Big(\sum_{(k,j)\in\mathbb{E}} + \sum_{(k,j)\in\mathbb{F}} + \sum_{(k,j)\in\mathbb{G}}\Big)(2^{k+2})^p |E_j^k|_\omega \qquad (4\text{-}12)\\
&\le \beta \sum_{(k,j)\in\mathbb{E}} (2^{k+2})^p |NQ_j^k|_\omega + C \sum_{(k,j)\in\mathbb{F}} \int_{E_j^k} (\mathcal{M} f\sigma)^p \omega \\
&\qquad + C \sum_{(k,j)\in\mathbb{G}} |E_j^k|_\omega \Big(\frac{1}{\beta |NQ_j^k|_\omega} \int_{E_j^k} (L\chi_{3Q_j^k} f\sigma)\omega\Big)^p \\
&= J(1) + J(2) + J(3) \\
&\le C_0\Big(\beta \int (T_\natural f\sigma)^p \omega + \beta^{-p} \int |f|^p \sigma\Big), \qquad (4\text{-}13)
\end{aligned}$$

where $C_0 \le C(\mathfrak{M} + \gamma^2\mathfrak{M}_* + \gamma^2\mathfrak{T} + \mathfrak{T}_*)^p$. The last line is the claim that we take up in the remainder of the proof. Once it is proved, note that if we take $0 < C_0\beta < \frac{1}{2}$ and use the fact that $\int (T_\natural f\sigma)^p\omega < \infty$ for $f$ bounded with compact support, we have proved assertion (1) of Theorem 1.9, and in particular (4-1).

The proof of the strong-type inequality requires a complicated series of decompositions of the dominating sums, which are illustrated for the reader's convenience as a schematic tree in Figure 4.2.

**4.3.** ***Two easy estimates.*** Note that the first term $J(1)$ in (4-12) satisfies

$$J(1) = \beta \sum_{(k,j)\in\mathbb{E}} (2^{k+2})^p |NQ_j^k|_\omega \le C\beta \int (T_\natural f\sigma)^p \omega,$$

by the finite overlap condition (4-11). The second term $J(2)$ is dominated by

$$C \sum_{(k,j)\in\mathbb{F}} \int_{E_j^k} (\mathcal{M} f\sigma)^p \omega \le C\mathfrak{M}^p \|f\|_{L^p(\sigma)}^p,$$

by our assumption (1-5). It is useful to note that this is the *only* time in the proof that we use the maximal function inequality (1-5) — from now on we use the *dual* maximal function inequality (1-17).

**Remark 4.4.** In the arguments below we can use [Sawyer 1988, Theorem 2] to replace the dual maximal function assumption $\mathfrak{M}_* < \infty$ with two assumptions, namely a "Poisson two weight $A_p$ condition" and the analogue of the dual pivotal condition of Nazarov, Treil and Volberg [2010]. The Poisson two weight $A_p$ condition is in fact necessary for the two weight inequality, but the pivotal conditions are *not* necessary for the Hilbert transform two weight inequality [Lacey et al. 2011]. On the other hand, the assumption $\mathfrak{M} < \infty$ cannot be weakened here, reflecting that our method requires the maximum principle in Lemma 2.6.

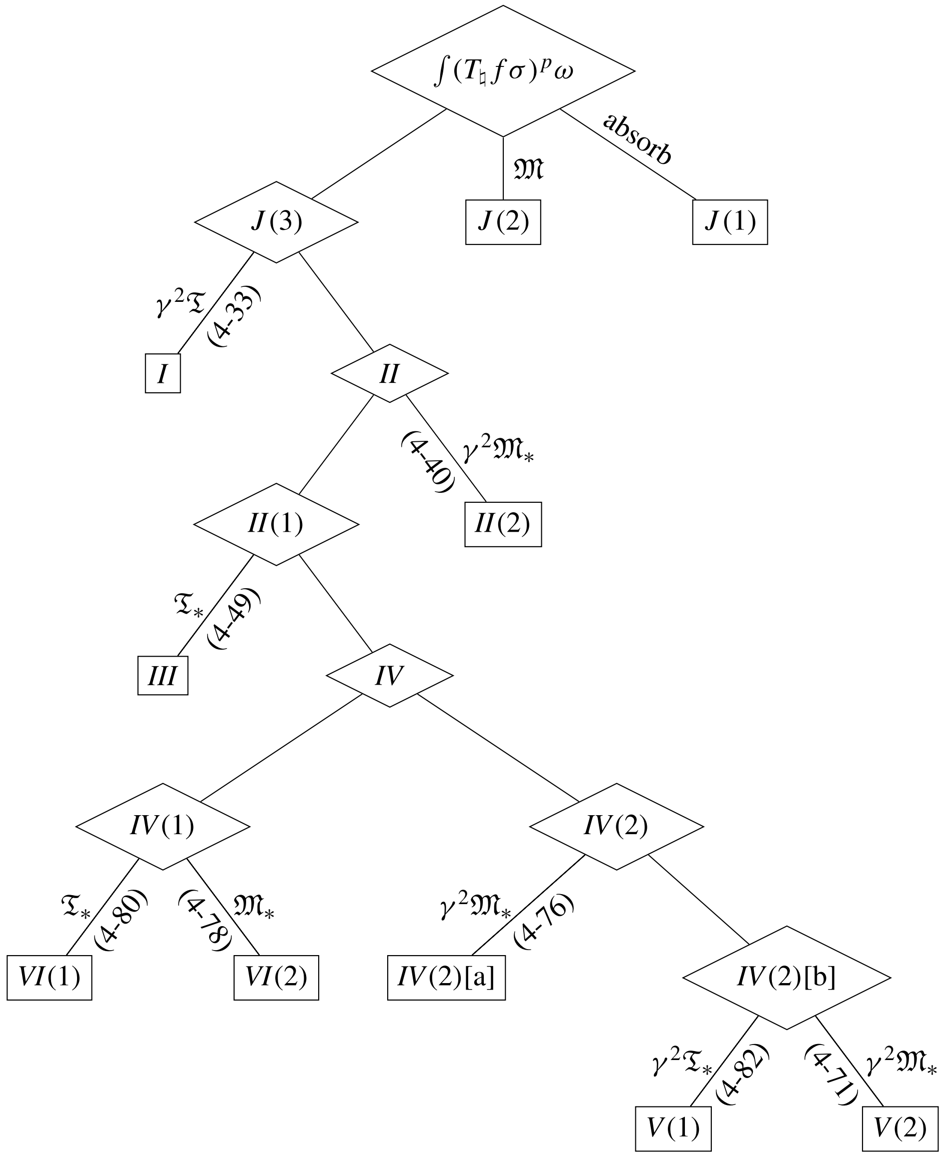


**Figure 4.2.** This is a schematic tree of how the integral $\int (T_\natural f\sigma)^p \omega$ has been, and will continue to be, decomposed. We have suppressed superscripts, subscripts and sums in the tree. Terms in diamonds are further decomposed, while terms in rectangles are final estimates. The edges leading into rectangles are labeled by $\mathfrak{M}$, $\mathfrak{M}_*$, $\mathfrak{I}$ or $\mathfrak{I}_*$ whose finiteness is used to control that term. Those terms controlled by the doubling constant $\gamma$ are also indicated. Equation references are to where the final estimates on the term is obtained. The word "absorb" leading into $J(1)$ indicates that this term is a small multiple of $\int (T_\natural f\sigma)^p \omega$ and can be absorbed into the left-hand side of the inequality. As most of the terms involve the maximal theorem (Equation (2-2)), we do not indicate its use in the schematic tree.

It is the third term $J(3)$ that is the most involved; see Figure 4.2. The remainder of the proof is taken up with the proof of

$$\sum_{(k,j)\in\mathbb{G}} R_j^k \Big| \int_{E_j^k} (L\chi_{3Q_j^k} f\sigma)\omega\Big|^p \le C\{\gamma^{2p}\mathfrak{M}_*^p + \gamma^{2p}\mathfrak{T}^p + \mathfrak{T}_*^p\}\|f\|_{L^p\sigma(dx)}^p, \tag{4-14}$$

where

$$R_j^k = \frac{|E_j^k|_\omega}{|NQ_j^k|_\omega^p}. \tag{4-15}$$

Once this is done, the proof of (4-12) is complete, and the proof of assertion (1) is finished.

**4.5.** ***The Calderón–Zygmund decompositions.*** To carry out this proof, we make Calderón–Zygmund decompositions relative to the measure $\sigma$. These decompositions will be done at *all heights simultaneously*. We will use the shifted dyadic grids; see (2-3). Suppose that $\gamma \ge 2$ is a doubling constant for the measure $\sigma$:

$$|3Q|_\sigma \le \gamma |Q|_\sigma \quad \text{for all cubes } Q. \tag{4-16}$$

For $\alpha \in \{0, \frac{1}{3}, \frac{2}{3}\}^n$, let

$$\mathcal{M}_\sigma^\alpha f(x) = \sup_{x\in Q\in\mathcal{D}^\alpha} \frac{1}{|Q|_\sigma}\int_Q |f|\,d\sigma,$$
$$\Gamma_t^\alpha = \{x\in\mathbb{R} : \mathcal{M}_\sigma^\alpha f(x) > \gamma^t\} = \bigcup_s G_s^{\alpha,t}, \tag{4-17}$$

where $\{G_s^{\alpha,t}\}_{(t,s)\in\mathbb{L}^\alpha}$ are the maximal $\mathcal{D}^\alpha$ cubes in $\Gamma_t^\alpha$, and $\mathbb{L}^\alpha$ is the set of pairs we use to label the cubes. This implies that we have the nested property: If $G_s^{\alpha,t} \subsetneqq G_{s'}^{\alpha,t'}$ then $t > t'$. Moreover, if $t > t'$ there is some $s'$ with $G_s^{\alpha,t} \subset G_{s'}^{\alpha,t'}$. These are the cubes used to make a Calderón–Zygmund decomposition at height $\gamma^t$ for the grid $\mathcal{D}^\alpha$ with respect to the measure $\sigma$. We will refer to the cubes $\{G_s^{\alpha,t}\}_{(t,s)\in\mathbb{L}^\alpha}$ as *principal cubes*.

Of course we have from the maximal inequality in (2-2)

$$\sum_{(t,s)\in\mathbb{L}^\alpha} \gamma^{pt}|G_s^{\alpha,t}|_\sigma \le C\|f\|_{L^p(\sigma)}^p. \tag{4-18}$$

The point of these next several definitions is to associate to each dyadic cube $Q$, a good shifted dyadic grid, and an appropriate height, at which we will build our Calderón–Zygmund decomposition.

We now use a consequence of the doubling condition (4-16) for the measure $\sigma$, that

$$|P(G)|_\sigma \le \gamma |G|_\sigma \quad \text{for } G\in\mathcal{D}^\alpha. \tag{4-19}$$

The average $|G_s^{\alpha,t}|_\sigma^{-1}\int_{G_s^{\alpha,t}}|f|\,d\sigma$ is thus at most $\gamma^{t+1}$ by (4-19) and the maximality of the cubes in (4-17):

$$\gamma^t < \frac{1}{|G_s^{\alpha,t}|_\sigma}\int_{G_s^{\alpha,t1}}|f|\,d\sigma \le \frac{|P(G_s^{\alpha,t})|_\sigma}{|G_s^{\alpha,t}|_\sigma}\frac{1}{|P(G_s^{\alpha,t})|_\sigma}\int_{P(G_s^{\alpha,t})}|f|\,d\sigma \le \gamma\gamma^t = \gamma^{t+1}. \tag{4-20}$$

**Select a shifted grid:** Let $\vec{\alpha} : \mathcal{D} \longrightarrow \{0, \frac{1}{3}, \frac{2}{3}\}^n$ be a map such that for $Q \in \mathcal{D}$, there is a $\hat{Q} \in \mathcal{D}^{\vec{\alpha}(Q)}$ such that $3Q \subset \hat{Q}$ and $|\hat{Q}| \le C|Q|$. Here, $C$ is an appropriate constant depending only on dimension. Thus, $\vec{\alpha}(Q)$ picks a "good" shifted dyadic grid for $Q$. Moreover we will assume that $\hat{Q}$ is the smallest such cube. Note that we are discarding the extra requirement that $3Q \subset \frac{9}{10}\hat{Q}$ since this property will not be used. Also we have

$$\hat{Q} \subset MQ, \tag{4-21}$$

for some positive dimensional constant $M$. The cubes $\hat{Q}_j^k$ will play a critical role below. See Figure 4.3

**Select a principal cube:** Define $\mathcal{A}(Q)$ to be the smallest cube from the collection $\{G_s^{\vec{\alpha}(Q),t} \mid (t, s) \in \mathbb{L}^\alpha\}$ that contains $3Q$; such $\mathcal{A}(Q)$ is uniquely determined by $Q$ and the choice of function $\vec{\alpha}$. Define

$$\mathbb{H}_s^{\alpha,t} = \{(k, j) : \mathcal{A}(Q_j^k) = G_s^{\alpha,t}\} \quad \text{for } (s, t) \in \mathbb{L}^\alpha. \tag{4-22}$$

This is an important definition for us. The combinatorial structure this places on the corresponding cubes is essential for this proof to work. Note that $3Q_j^k \subset \hat{Q}_j^k \subset \mathcal{A}(Q_j^k)$.

**Parents:** For any of the shifted dyadic grids $\mathcal{D}^\alpha$, a $Q \in \mathcal{D}^\alpha$ has a unique parent denoted as $P(Q)$, the smallest member of $\mathcal{D}^\alpha$ that strictly contains $Q$. We suppress the dependence upon $\alpha$ here.

**Indices:** Let

$$\mathcal{K}_s^{\alpha,t} = \{r \mid G_r^{\alpha,t+1} \subset G_s^{\alpha,t}\}. \tag{4-23}$$

We use a calligraphic font $\mathcal{K}$ for sets of indices related to the grid $\{G_s^{\alpha,t}\}$, and a blackboard font $\mathbb{H}$ for sets of indices related to the grid $\{Q_j^k\}$.

**The good and bad functions:** Let $A_{G_r^{\alpha,t+1}} = |G_r^{\alpha,t+1}|_\sigma^{-1} \int_{G_r^{\alpha,t+1}} f\sigma$ be the $\sigma$-average of $f$ on $G_r^{\alpha,t+1}$. Define functions $g_s^{\alpha,t}$ and $h_s^{\alpha,t}$ satisfying $f = g_s^{\alpha,t} + h_s^{\alpha,t}$ on $G_s^{\alpha,t}$ by

$$g_s^{\alpha,t}(x) = \begin{cases} A_{G_r^{\alpha,t+1}} & \text{for } x \in G_r^{\alpha,t+1} \text{ with } r \in \mathcal{K}_s^{\alpha,t}, \\ f(x) & \text{for } x \in G_s^{\alpha,t} \setminus \bigcup\{G_r^{\alpha,t+1} : r \in \mathcal{K}_s^{\alpha,t}\}, \end{cases} \tag{4-24}$$

$$h_s^{\alpha,t}(x) = \begin{cases} f(x) - A_{G_r^{\alpha,t+1}} & \text{for } x \in G_r^{\alpha,t+1} \text{ with } r \in \mathcal{K}_s^{\alpha,t}, \\ 0 & \text{for } x \in G_s^{\alpha,t} \setminus \bigcup\{G_r^{\alpha,t+1} : r \in \mathcal{K}_s^{\alpha,t}\}. \end{cases} \tag{4-25}$$

We extend both $g_s^{\alpha,t}$ and $h_s^{\alpha,t}$ to all of $\mathbb{R}^n$ by defining them to vanish outside $G_s^{\alpha,t}$.

Now $|A_{G_r^{\alpha,t+1}}| \le \gamma^{t+1}$ by (4-20). Thus Lebesgue's differentiation theorem shows that (any of the standard proofs can be adapted to the dyadic setting for positive locally finite Borel measures on $\mathbb{R}^n$)

$$|g_s^{\alpha,t}(x)| \le \gamma^{t+1} < \frac{\gamma}{|G_s^{\alpha,t}|_\sigma} \int_{G_s^{\alpha,t}} |f|\sigma \quad \text{for } \sigma\text{-a.e. } x \in G_s^{\alpha,t} \text{ and } (t, s) \in \mathbb{L}^\alpha. \tag{4-26}$$

That is, $g_s^{\alpha,t}$ is the "good" function and $h_s^{\alpha,t}$ is the "bad" function.

We can now refine the final sum on the left side of (4-14) according to the decomposition of $\mathcal{M}_\sigma^\alpha f$. We carry this out in three steps. In the first step, we fix an $\alpha \in \{0, \frac{1}{3}, \frac{2}{3}\}^n$, and for the remainder of the proof, we only consider $Q_j^k$ for which $\vec{\alpha}(Q_j^k)=\alpha$. Namely, we will modify the important definition of $\mathbb{G}$ in (4-10) to

$$\mathbb{G}^\alpha=\bigl\{(k,j):\vec{\alpha}(Q_j^k)=\alpha,\ |E_j^k|_\omega>\beta|NQ_j^k|_\omega \text{ and (4-8) holds}\bigr\}, \tag{4-27}$$

In the second step, we partition the indices $(k,j)$ into the sets $\mathbb{H}_s^{\alpha,t}$ in (4-22) for $(t,s)\in\mathbb{L}^\alpha$. In the third step, for $(k,j)\in\mathbb{H}_s^{\alpha,t}$, we split $f$ into the corresponding good and bad parts, yielding the decomposition

$$\sum_{(k,j)\in\mathbb{G}^\alpha} R_j^k\Bigl|\int_{E_j^k}(L\chi_{3Q_j^k}f\sigma)\omega\Bigr|^p\le C(I+II), \tag{4-28}$$

$$I=\sum_{(t,s)\in\mathbb{L}^\alpha} I_s^t,\qquad II=\sum_{(t,s)\in\mathbb{L}^\alpha} II_s^t, \tag{4-29}$$

$$I_s^t=\sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}} R_j^k\Bigl|\int_{E_j^k}(L\chi_{3Q_j^k}g_s^{\alpha,t}\sigma)\omega\Bigr|^p, \tag{4-30}$$

$$II_s^t=\sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}} R_j^k\Bigl|\int_{E_j^k}(L\chi_{3Q_j^k}h_s^{\alpha,t}\sigma)\omega\Bigr|^p, \tag{4-31}$$

$$\mathbb{I}_s^{\alpha,t}=\mathbb{G}\cap\mathbb{H}_s^{\alpha,t}. \tag{4-32}$$

Recall the definition of $R_j^k$ in (4-15). In the definitions of $I$, $I_s^t$ and $II$, $II_s^t$, we will suppress the dependence on $\alpha\in\{0,\frac{1}{3},\frac{2}{3}\}^n$. The same will be done for the subsequent decompositions of the (difficult) term $II$, although we usually retain the superscript $\alpha$ in the quantities arising in the estimates. In particular, the combinatorial properties of the cubes associated with $\mathbb{I}_s^{\alpha,t}$ are essential to completing this proof.

Term $I$ requires only the forward testing condition (1-19) and the maximal theorem (2-2), while term $II$ requires only the dual testing condition (1-14), along with the dual maximal function inequality (1-17) and the maximal theorem (2-2). The reader is again directed to Figure 4.2 for a map of the various decompositions of the terms and the conditions used to control them.

**4.6.** ***The analysis of the good function.*** We claim that

$$I\le C\gamma^{2p}\mathfrak{T}^p\|f\|_{L^p(\sigma)}^p. \tag{4-33}$$

*Proof.* We use boundedness of the "good" function $g_s^{\alpha,t}$, as defined in (4-24), the testing condition (1-19) for $T_\natural$ (see also (4-4)), and finally the universal maximal function bound (2-2) with $\mu=\omega$. Here are the details. For $x\in E_j^k$, (4-7) implies that $L\chi_{3Q_j^k}g_s^{\alpha,t}\sigma(x)=Lg_s^{\alpha,t}\sigma(x)$ and so

$$\begin{aligned}
I=\sum_{(t,s)\in\mathbb{L}^\alpha} I_s^t&=C\sum_{(t,s)\in\mathbb{L}^\alpha}\ \sum_{(k,j)\in\mathbb{G}^\alpha\cap\mathbb{H}_s^{\alpha,t}} R_j^k\Bigl|\int_{E_j^k}(Lg_s^{\alpha,t}\sigma)\omega\Bigr|^p\\
&\le C\sum_{(t,s)\in\mathbb{L}^\alpha}\int|\mathcal{M}_\omega^{dy}(\chi_{G_s^{\alpha,t}}Lg_s^{\alpha,t}\sigma)|^p\omega\le C\sum_{(t,s)\in\mathbb{L}^\alpha}\int_{G_s^{\alpha,t}}|Lg_s^{\alpha,t}\sigma|^p\omega\\
&\le C\gamma^{2p}\sum_{(t,s)\in\mathbb{L}^\alpha}\gamma^{pt}\int_{G_s^{\alpha,t}}\Bigl(T_\natural\frac{g_s^{\alpha,t}}{\gamma^{t+2}}\sigma\Bigr)^p\omega\le C\gamma^{2p}\mathfrak{T}^p\sum_{(t,s)\in\mathbb{L}^\alpha}\gamma^{pt}|G_s^{\alpha,t}|_\sigma,
\end{aligned}$$

where we have used (4-26) and (1-19) with $g = g_s^{\alpha,t}/\gamma^{t+2}$ in the final inequality. This last sum is controlled by (4-18), and completes the proof of the claim. $\square$

**4.7.** ***The analysis of the bad function: Part 1.*** It remains to estimate term $II$, as in (4-31), but this is in fact the harder term. Recall the definition of $\mathcal{H}_s^{\alpha,t}$ in (4-23). We now write

$$h_s^{\alpha,t} = \sum_{r\in\mathcal{H}_s^{\alpha,t}} (f - A_{G_r^{\alpha,t+1}})\chi_{G_r^{\alpha,t+1}} \equiv \sum_{r\in\mathcal{H}_s^{\alpha,t}} b_r, \tag{4-34}$$

where the "bad" functions $b_r$ are supported in the cube $G_r^{\alpha,t+1}$ and have $\sigma$-mean zero, $\int_{G_r^{\alpha,t+1}} b_r\sigma = 0$. To take advantage of this, we will pass to the dual $L^*$ below.

But first we must address the fact that the triples of the $\mathcal{D}^\alpha$ cubes $G_r^{\alpha,t+1}$ do not form a grid. Fix $(t,s)\in\mathbb{L}^\alpha$ and let

$$\mathfrak{C}_s^{\alpha,t} = \{3G_r^{\alpha,t+1} : r\in\mathcal{H}_s^{\alpha,t}\} \tag{4-35}$$

be the collection of triples of the $\mathcal{D}^\alpha$ cubes $G_r^{\alpha,t+1}$ with $r\in\mathcal{H}_s^{\alpha,t}$. We select the *maximal* triples

$$\{3G_{r_\ell}^{t+1}\}_{\ell\in\mathcal{L}_s^{\alpha,t}} \equiv \{T_\ell\}_{\ell\in\mathcal{L}_s^{\alpha,t}} \tag{4-36}$$

from the collection $\mathfrak{C}_s^{\alpha,t}$, and assign to each $r\in\mathcal{H}_s^{\alpha,t}$, the maximal triple $T_\ell = T_{\ell(r)}$ containing $3G_r^{\alpha,t+1}$ with least $\ell$. Note that $T_{\ell(r)}$ extends outside $G_s^{\alpha,t}$ if $G_r^{\alpha,t+1}$ and $G_s^{\alpha,t}$ share a face. By Lemma 2.13 applied to $\mathcal{D}^\alpha$ the maximal triples $\{T_\ell\}_{\ell\in\mathcal{L}_s^{\alpha,t}}$ have finite overlap $3^n$, and this will prove crucial in (4-49), (4-82) and (4-50) below.

We will pass to the dual of the linearization.

$$\int_{E_j^k} (Lh_s^{\alpha,t}\sigma)\omega = \sum_{r\in\mathcal{H}_s^{\alpha,t}} \int_{E_j^k} (Lb_r\sigma)\omega = \sum_{r\in\mathcal{H}_s^{\alpha,t}} \int_{G_r^{\alpha,t+1}\cap 3Q_j^k} (L^*\chi_{E_j^k}\omega)b_r\sigma \tag{4-37}$$

Note that (4-7) implies $L^*\nu$ is supported in $3Q_j^k$ if $\nu$ is supported in $E_j^k$, explaining the range of integration above. Continuing, we have for fixed $(k,j)\in\mathbb{I}_s^{\alpha,t}$,

$$|(4\text{-}37)| \le \Big|\sum_{r\in\mathcal{H}_s^{\alpha,t}} \int_{G_r^{\alpha,t+1}\cap 3Q_j^k} (L^*\chi_{E_j^k\cap T_{\ell(r)}}\omega)b_r\sigma\Big| + C\sum_{r\in\mathcal{H}_s^{\alpha,t}} \boldsymbol{P}(G_r^{\alpha,t+1}, \chi_{E_j^k\setminus 3G_r^{\alpha,t+1}}\omega)\int_{G_r^{\alpha,t+1}} |f|\sigma. \tag{4-38}$$

To see the inequality above, note that for $r\in\mathcal{H}_s^{\alpha,t}$ we are splitting the set $E_j^k$ into $E_j^k\cap T_{\ell(r)}$ and $E_j^k\setminus T_{\ell(r)}$. On the latter set, the hypotheses of Lemma 2.9 are in force, namely the set $E_j^k\setminus T_{\ell(r)}$ does not intersect $3G_r^{\alpha,t+1}$, whence we have an estimate on the $\delta$-Hölder modulus of continuity of $L^*\chi_{E_j^k}\setminus T_{\ell(r)}\omega$. Combine this with the fact that $b_r$ has $\sigma$-mean zero on $G_r^{\alpha,t+1}$ to derive the estimate below, in which $y_r^{t+1}$ is the center of the cube $G_r^{\alpha,t+1}$.

$$\begin{aligned}\Big|\int_{G_r^{\alpha,t+1}} (L^*\chi_{E_j^k\setminus T_{\ell(r)}}\omega)b_r\sigma\Big| &= \Big|\int_{G_r^{\alpha,t+1}} (L^*\chi_{E_j^k\setminus T_{\ell(r)}}\omega(y) - L^*\chi_{E_j^k\setminus T_{\ell(r)}}\omega(y_r^{t+1}))(b_r\sigma)\Big| \\ &\le \int_{G_r^{\alpha,t+1}\cap 3Q_j^k} C\boldsymbol{P}(G_r^{\alpha,t+1}, \chi_{E_j^k\setminus T_{\ell(r)}}\omega)\delta\Big(\frac{|y-y_r^{t+1}|}{\ell(G_r^{\alpha,t+1})}\Big)|b_r(y)|\,d\sigma(y)\end{aligned}$$

$$\leq C\,\boldsymbol{P}(G_r^{\alpha,t+1},\,\chi_{E_j^k\setminus 3G_r^{\alpha,t+1}}\omega)\int_{G_r^{\alpha,t+1}}|f|\,d\sigma.$$

We have after application of (4-38),

$$II_s^t=\sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}}R_j^k\Big(\int_{E_j^k}(Lh_s^{\alpha,t}\sigma)\omega\Big)^p\leq II_s^t(1)+II_s^t(2),$$

where

$$II_s^t(1)=\sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}}R_j^k\Big|\sum_{r\in\mathcal{H}_s^{\alpha,t}}\int_{G_r^{\alpha,t+1}}(L^*\chi_{E_j^k\cap T_{\ell(r)}}\omega)b_r\sigma\Big|^p,\tag{4-39}$$

$$II_s^t(2)=\sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}}R_j^k\Big(\sum_{r\in\mathcal{H}_s^{\alpha,t}}\boldsymbol{P}(G_r^{\alpha,t+1},\,\chi_{E_j^k}\omega)\int_{G_r^{\alpha,t+1}}|f|\sigma\Big)^p.$$

Note that we may further restrict the integration in (4-39) to $G_r^{\alpha,t+1}\cap 3Q_j^k$ since $L^*\chi_{E_j^k}\cap T_{\ell(r)}\omega$ is supported in $3Q_j^k$.

**4.7.1.** *Analysis of $II(2)$.* Recalling the definition of $\mathfrak{M}_*$ in (4-3), we claim that

$$\sum_{(t,s)\in\mathbb{L}^\alpha}II_s^t(2)\leq C\gamma^{2p}\mathfrak{M}_*^p\int|f|^p\sigma.\tag{4-40}$$

*Proof.* We begin by defining a linear operator by

$$\boldsymbol{P}_j^k(\mu)\equiv\sum_{r\in\mathcal{H}_s^{\alpha,t}}\boldsymbol{P}(G_r^{\alpha,t+1},\,\chi_{E_j^k}\mu)\chi_{G_r^{\alpha,t+1}}.\tag{4-41}$$

In this notation, we have for $(k,\,j)\in\mathbb{I}_s^{\alpha,t}$ (see (4-22) and (4-31)),

$$\begin{aligned}\sum_{r\in\mathcal{H}_s^{\alpha,t}}\boldsymbol{P}\big(G_r^{\alpha,t+1},\,\chi_{E_j^k}\omega(dx)\big)\int_{G_r^{\alpha,t+1}}|f|\sigma&=\sum_{r\in\mathcal{H}_s^{\alpha,t}}\boldsymbol{P}\big(G_r^{\alpha,t+1},\,\chi_{E_j^k}\omega\big)\int_{G_r^{\alpha,t+1}}\sigma\Big(\frac{1}{|G_r^{\alpha,t+1}|_\sigma}\int_{G_r^{\alpha,t+1}}|f|\sigma\Big)\\&\leq\gamma^{t+2}\int_{G_s^{\alpha,t}}\boldsymbol{P}_j^k(\omega)\sigma=\gamma^{t+2}\int_{E_j^k}(\boldsymbol{P}_j^k)^*(\chi_{G_s^{\alpha,t}}\sigma)\omega.\end{aligned}$$

By assumption, the maximal function $\mathcal{M}(\omega\cdot)$ maps $L^{p'}(\omega)$ to $L^{p'}(\sigma)$, and we now note a particular consequence of this. In the definition (4-41) we were careful to insert $\chi_{E_j^k}$ on the right hand side. These sets are pairwise disjoint, whence we have the inequality below for measures $\mu$.

$$\begin{aligned}\sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}}\boldsymbol{P}_j^k(\mu)(x)&\leq\sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}}\sum_{r\in\mathcal{H}_s^{\alpha,t}}\sum_{\ell=0}^\infty\frac{\delta(2^{-\ell})}{|2^\ell G_r^{\alpha,t+1}|}\Big(\int_{2^\ell G_r^{\alpha,t+1}}\chi_{E_j^k}\mu\Big)\chi_{G_r^{\alpha,t+1}}(x)\\&\leq\sum_{\ell=0}^\infty\sum_{r\in\mathcal{H}_s^{\alpha,t}}\frac{\delta(2^{-\ell})}{|2^\ell G_r^{\alpha,t+1}|}\Big(\int_{2^\ell G_r^{\alpha,t+1}\cap G_s^{\alpha,t}}\mu\Big)\chi_{G_r^{\alpha,t+1}}(x)\leq C\chi_{G_s^{\alpha,t}}\mathcal{M}(\chi_{G_s^{\alpha,t}}\mu)(x).\end{aligned}\tag{4-42}$$

Thus the inequality

$$\Big\|\chi_{G_s^{\alpha,t}}\sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}}\boldsymbol{P}_j^k(|g|\omega)\Big\|_{L^{p'}(\sigma)}\le C\mathfrak{M}_*\|\chi_{G_s^{\alpha,t}}g\|_{L^{p'}(\omega)} \tag{4-43}$$

follows immediately. By duality we then have

$$\Big\|\chi_{G_s^{\alpha,t}}\sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}}(\boldsymbol{P}_j^k)^*(|h|\sigma)\Big\|_{L^{p}(w)}\le C\mathfrak{M}_*\|\chi_{G_s^{\alpha,t}}h\|_{L^{p}(\sigma)}. \tag{4-44}$$

Note that it was the linearity that we wanted in (4-41), so that we could appeal to the dual maximal function assumption.

We thus obtain

$$II_s^t(2)\le\gamma^{p(t+2)}\sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}}R_j^k\Big(\int_{Q_j^k}(\boldsymbol{P}_j^k)^*(\chi_{G_s^{\alpha,t}}\sigma)\,d\omega\Big)^p.$$

Summing in $(t,s)$ and using $(\boldsymbol{P}_j^k)^*\le\sum_{(\ell,i)\in\mathbb{I}_s^{\alpha,t}}(\boldsymbol{P}_i^\ell)^*$ for $(k,j)\in\mathbb{I}_s^{\alpha,t}$, we obtain

$$\begin{aligned}\sum_{(t,s)\in\mathbb{L}^\alpha}II_s^t(2)&\le C\gamma^{2p}\sum_{(t,s)\in\mathbb{L}^\alpha}\gamma^{pt}\sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}}R_j^k\Big(\int_{Q_j^k}(\boldsymbol{P}_j^k)^*(\chi_{G_s^{\alpha,t}}\sigma)\,d\omega\Big)^p &\text{(4-45)}\\ &=C\gamma^{2p}\sum_{(t,s)\in\mathbb{L}^\alpha}\gamma^{pt}\sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}}|E_j^k|_\omega\Big(\frac{1}{|NQ_j^k|_w}\int_{Q_j^k}(\boldsymbol{P}_j^k)^*(\chi_{G_s^{\alpha,t}}\sigma)\omega\Big)^p\\ &\le C\gamma^{2p}\sum_{(t,s)\in\mathbb{L}^\alpha}\gamma^{pt}\int\Big(\mathcal{M}_\omega^{dy}(\chi_{G_s^{\alpha,t}}\sum_{(\ell,i)\in\mathbb{I}_s^{\alpha,t}}(\boldsymbol{P}_i^\ell)^*(\chi_{G_s^{\alpha,t}}\sigma))\Big)^p\omega &\text{(4-46)}\\ &\le C\gamma^{2p}\sum_{(t,s)\in\mathbb{L}^\alpha}\gamma^{pt}\int_{G_s^{\alpha,t}}\Big(\sum_{(\ell,i)\in\mathbb{I}_s^{\alpha,t}}(\boldsymbol{P}_i^\ell)^*(\chi_{G_s^{\alpha,t}}\sigma)\Big)^p\omega &\text{(4-47)}\\ &\le C\gamma^{2p}\mathfrak{M}_*^p\sum_{(t,s)\in\mathbb{L}^\alpha}\gamma^{pt}|G_s^{\alpha,t}|_\sigma,\end{aligned}$$

which is bounded by $C\gamma^{2p}\mathfrak{M}_*^p\int|f|^p\sigma$. In the last line we are applying (4-44) with $h\equiv1$. □

**4.7.2.** *Decomposition of II*(1). We note that the term $II_s^t(1)$ is dominated by $II_s^t(1)\le III_s^t+IV_s^t$, where

$$\begin{aligned}III_s^t&=\sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}}R_j^k\Big|\sum_{r\in\mathcal{K}_s^{\alpha,t}}\int_{G_r^{\alpha,t+1}\setminus\Omega_{k+2}}(L^*\chi_{E_j^k\cap T_{\ell(r)}}\omega)b_r\sigma\Big|^p,\\ IV_s^t&=\sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}}R_j^k\Big|\sum_{r\in\mathcal{K}_s^{\alpha,t}}\int_{G_r^{\alpha,t+1}\cap\Omega_{k+2}}(L^*\chi_{E_j^k\cap T_{\ell(r)}}\omega)b_r\sigma\Big|^p. &\text{(4-48)}\end{aligned}$$

The term $III_s^t$ includes that part of $b_r$ supported on $G_r^{\alpha,t+1}\setminus\Omega_{k+2}$, and the term $IV_s^t$ includes that part of $b_r$ supported on $G_r^{\alpha,t+1}\cap\Omega_{k+2}$, which is the more delicate case.

**Remark 4.8.** The key difference between the terms $III_s^t$ and $IV_s^t$ is the range of integration: $G_r^{\alpha,t+1}\setminus\Omega_{k+2}$ for $III_s^t$ and $G_r^{\alpha,t+1}\cap\Omega_{k+2}$ for $IV_s^t$. Just as for the fractional integral case, it is the latter case that is harder, requiring combinatorial facts, which we come to at the end of the argument. An additional fact that we return to in different forms is that the set $G_r^{\alpha,t+1} \cap \Omega_{k+2}$ can be further decomposed using Whitney decompositions of $\Omega_{k+2}$ in the grid $\mathcal{D}^\alpha$.

Recall the definition of $\mathfrak{T}_*$ in (4-5). We claim

$$\sum_{(t,s)\in\mathbb{L}^\alpha} III_s^t \le C\mathfrak{T}_*^p \int |f|^p \sigma. \tag{4-49}$$

*Proof.* Let $\widetilde{E}_j^k = 3Q_j^k \setminus \Omega_{k+2}$ (note that $\widetilde{E}_j^k$ is much larger than $E_j^k$). We will use the definition of $R_j^k$ in (4-15), and the fact that

$$\sum_{\ell\in\mathcal{L}_s^{\alpha,t}} \chi_{T_\ell} \le 3^n \tag{4-50}$$

provided $N \ge 9$. We will apply the form (2-11) of (1-14) with $g = \chi_{E_j^k \cap T_\ell}$ — also see (4-5) — and with

$$Q \equiv T_\ell \cap \hat{Q}_j^k \quad \text{and} \quad Q \equiv T_\ell$$

in the cases $T_\ell \cap \hat{Q}_j^k$ is a cube and is not a cube, respectively (the latter is possible since $T_\ell$ is the *triple* of a $\mathcal{D}^\alpha$-cube). In each case we claim that

$$Q \subset T_\ell \cap 3\hat{Q}_j^k.$$

Indeed, recall that $\hat{Q}_j^k$ is the cube in the shifted grid $\mathcal{D}^\alpha$ that is selected by $Q_j^k$ as in the definition "Select a shifted grid" above and satisfies $3\hat{Q}_j^k \subset MQ_j^k \subset NQ_j^k$, where $N$ is as in Remark 2.4, by choosing $R_W$ sufficiently large in (2-6). Now $T_\ell$ is a triple of a cube in the grid $\mathcal{D}^\alpha$ and $\hat{Q}_j^k$ is a cube in $\mathcal{D}^\alpha$. Thus if $T_\ell \cap \hat{Q}_j^k$ is *not* a cube, then we must have $T_\ell \subset 3\widetilde{Q}_j^k$ and this proves the claim. We then have

$$\begin{aligned}
III_s^t &\le \sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}} R_j^k \Big( \sum_{\ell\in\mathcal{L}_s^{\alpha,t}} \sum_{r\in\mathcal{H}_s^{\alpha,t}:\ell=\ell(r)} \int_{G_r^{\alpha,t+1}\cap\widetilde{E}_j^k} |L^*\chi_{E_j^k\cap T_{\ell(r)}}\omega|^{p'}\sigma \Big)^{p-1} \int_{\widetilde{E}_j^k} |h_s^{\alpha,t}|^p\sigma \\
&\le \sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}} R_j^k \Big( \sum_{\ell\in\mathcal{L}_s^{\alpha,t}} \int_{T_\ell\cap 3\hat{Q}_j^k} |L^*\chi_{E_j^k\cap T_\ell}\omega|^{p'}\sigma \Big)^{p-1} \int_{\widetilde{E}_j^k} |h_s^{\alpha,t}|^p\sigma \\
&\le \mathfrak{T}_*^p \sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}} R_j^k \Big( \sum_{\ell\in\mathcal{L}_s^{\alpha,t}} |T_\ell\cap 3\hat{Q}_j^k|_\omega \Big)^{p-1} \int_{\widetilde{E}_j^k} |h_s^{\alpha,t}|^p\sigma \\
&\le \mathfrak{T}_*^p \sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}} \frac{|E_j^k|_\omega}{|NQ_j^k|_\omega} |NQ_j^k|_\omega^{p-1} \int_{\widetilde{E}_j^k} |h_s^{\alpha,t}|^p\sigma \\
&\le C\mathfrak{T}_*^p \sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}} \int_{\widetilde{E}_j^k} |h_s^{\alpha,t}|^p\sigma \le C\mathfrak{T}_*^p \sum_{(k,j)\in\mathbb{G}^\alpha\cap\mathbb{H}_s^{\alpha,t}} \int_{\widetilde{E}_j^k} (|f|^p + |\mathcal{M}_\sigma^\alpha f|^p)\sigma.
\end{aligned}$$

Using

$$\sum_{(t,s)\in\mathbb{L}^\alpha} \sum_{(k,j)\in\mathbb{G}^\alpha\cap\mathbb{H}_s^{\alpha,t}} \chi_{\widetilde{E}_j^k} = \sum_{\text{all } k,j} \chi_{\widetilde{E}_j^k} \le C, \tag{4-51}$$

we thus obtain (4-49). □

**4.9.** ***The analysis of the bad function: Part 2.*** This is the most intricate and final case. We will prove

$$\sum_{(t,s)\in\mathbb{L}^\alpha} IV_s^t \le C(\gamma^{2p}\mathfrak{T}^p + \mathfrak{T}_*^p + \gamma^{2p}\mathfrak{M}_*^p)\int |f|^p\sigma, \tag{4-52}$$

where $\mathfrak{T}$, $\mathfrak{T}_*$ and $\mathfrak{M}_*$ are defined in (4-4), (4-5) and (4-3), respectively. The estimates (4-33), (4-40), (4-49), (4-52) prove (4-12), and so complete the proof of assertion 1 of the strong-type characterization in Theorem 1.9. Assertions 2 and 3 of Theorem 1.9 follow as in the weak-type Theorem 1.8. Finally, to prove assertion 4 we note that Lemma 2.12 and condition (1-19) imply (1-6), which by Theorem 1.2 yields (1-5).

**4.9.1.** *Whitney decompositions with shifted grids.* We now use the shifted grid $\mathcal{D}^\alpha$ in place of the dyadic grid $\mathcal{D}$ to form a Whitney decomposition of $\Omega_k$ in the spirit of (2-6). However, in order to fit the $\mathcal{D}^\alpha$-cubes $\hat{Q}_j^k$ defined above in "Select a shifted grid", it will be necessary to use a smaller constant than the constant $R_W$ already used for the Whitney decomposition of $\Omega_k$ into $\mathcal{D}$-cubes. Recall the dimensional constant $M$ defined in (4-21): it satisfies $\hat{Q}\subset MQ$. Define the new constant

$$R'_W = \frac{R_W}{M}.$$

We now use the decomposition of $\Omega_k$ in (2-6), but with $\mathcal{D}$ replaced by $\mathcal{D}^\alpha$ and with $R_W$ replaced by $R'_W$. We have thus decomposed

$$\Omega_k = \bigsqcup_m B_m^k$$

into a Whitney decomposition of pairwise disjoint cubes $B_m^k$ in $\mathcal{D}^\alpha$ satisfying

$$\begin{gathered} R'_W B_m^k \subset \Omega_k, \\ 3R'_W B_m^k \cap \Omega_k^c \neq \varnothing, \end{gathered} \tag{4-53}$$

and the following analogue of the nested property in (2-6):

$$B_j^k \subsetneqq B_i^\ell \quad \text{implies} \quad k > \ell. \tag{4-54}$$

Now we introduce yet another construction. For every pair $(k, j)$ let $\widetilde{Q}_j^k$ be the unique $\mathcal{D}^\alpha$-cube $B_m^k$ containing $\hat{Q}_j^k$. Note that such a cube $\widetilde{Q}_j^k = B_m^k$ exists since $\hat{Q}_j^k \subset MQ_j^k$ by (4-21) and $R_W Q_j^k \subset \Omega_k$ by (2-6) implies that $R'_W \hat{Q}_j^k \subset \Omega_k$. Of course the cube $\widetilde{Q}_j^k = B_m^k$ satisfies

$$R'_W \widetilde{Q}_j^k \subset \Omega_k. \tag{4-55}$$

Moreover, we can arrange to have

$$3\widetilde{Q}_j^k \subset NQ_j^k, \tag{4-56}$$

where $N$ is as in Remark 2.4, by choosing $R_W$ sufficiently large in (2-6). See Figure 4.3.

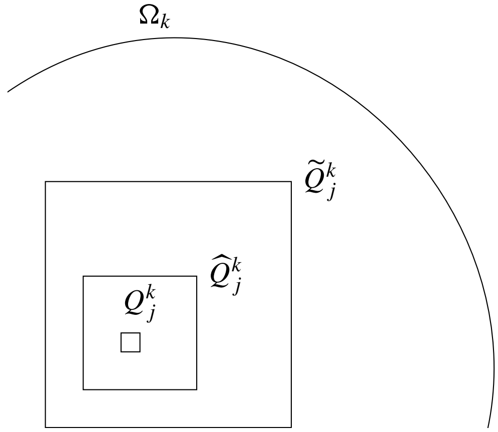


**Figure 4.3.** The relative positions of the cubes $Q_j^k$, $\widehat{Q}_j^k$, and $\widetilde{Q}_j^k$ inside a set $\Omega_k$.

We will use this decomposition for the set $\Omega_{k+2} = \sqcup_m B_m^{k+2}$ in our arguments below. The corresponding cubes $\widetilde{Q}_i^{k+2}$ that arise as certain of the $B_m^{k+2}$ satisfy the conditions

$$3Q_i^{k+2} \subset \widehat{Q}_i^{k+2} \subset \widetilde{Q}_i^{k+2} \subset 3\widetilde{Q}_i^{k+2} \subset N Q_i^{k+2} \subset \Omega_{k+2}. \tag{4-57}$$

Note that the set of indices $m$ arising in the decomposition of $\Omega_{k+2}$ into $\mathcal{D}^\alpha$ cubes $B_m^{k+2}$ is *not* the same as the set of indices $i$ arising in the decomposition of $\Omega_{k+2}$ into $\mathcal{D}$ cubes $Q_i^{k+2}$, but this should not cause confusion. So we will usually write $B_i^{k+2}$ with dummy index $i$ unless it is important to distinguish the cubes $B_i^k + 2$ from the cubes $Q_i^{k+2}$. This distinction will be important in the proof of the "bounded occurrence of cubes" property in Section 4.14.7 below.

Now use $\Omega_{k+2} = \bigcup B_i^{k+2}$ to split the term $IV_s^t$ in (4-48) into two pieces as follows:

$$\begin{aligned} IV_s^t \le & \sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}} R_j^k \Big| \sum_{r\in\mathcal{K}_s^{\alpha,t}} \sum_{i\in\mathcal{I}_s^t} \int_{G_r^{\alpha,t+1}\cap B_i^{k+2}} (L^* \chi_{E_j^k \cap T_{\ell(r)}} \omega) b_r \sigma \Big|^p \\ & + \sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}} R_j^k \Big| \sum_{r\in\mathcal{K}_s^{\alpha,t}} \sum_{i\in\mathcal{J}_s^t} \int_{G_r^{\alpha,t+1}\cap B_i^{k+2}} (L^* \chi_{E_j^k \cap T_{\ell(r)}} \omega) b_r \sigma \Big|^p \\ = & IV_s^t(1) + IV_s^t(2), \end{aligned} \tag{4-58}$$

where

$$\mathcal{I}_s^t = \{i : \mathsf{A}_i^{k+2} > \gamma^{t+2}\} \quad \text{and} \quad \mathcal{J}_s^t = \{i : \mathsf{A}_i^{k+2} \le \gamma^{t+2}\}, \tag{4-59}$$

and where

$$\mathsf{A}_i^{k+2} = \frac{1}{|B_i^{k+2}|_\sigma} \int_{B_i^{k+2}} |f| \, d\sigma \tag{4-60}$$

denotes the $\sigma$-average of $|f|$ on the cube $B_i^{k+2}$. Thus $IV(1)$ corresponds to the case where the averages are "big" and $IV(2)$ where the averages are "small". The analysis of $IV_s^t(1)$ in (4-58) is the hard case, taken up later.

**4.9.2.** *A first combinatorial argument.*

**Lemma 4.10** (bounded occurrence of cubes). *A given cube $B \in \mathcal{D}^\alpha$ can occur only a bounded number of times as $B_i^{k+2}$, where*

$$B_i^{k+2} \subset \widetilde{Q}_j^k \quad \text{with } (k, j) \in \mathbb{G}^\alpha.$$

*Specifically, let $(k_1, j_1), \ldots, (k_M, j_M) \in \mathbb{G}^\alpha$, as defined in* (4-27), *be such that $B = B_{i_\sigma}^{k_\sigma+2}$ for some $i_\sigma$ and $B \subset \widetilde{Q}_{j_\sigma}^{k_\sigma}$ for $1 \le \sigma \le M$. It follows that $M \le C\beta^{-1}$, where $\beta$ is the small constant chosen in the definition of $\mathbb{G}^\alpha$. The constant $C$ here depends only on dimension.*

The Whitney structure (see (2-6)) is decisive here, as well as the fact that $|E_j^k|_\omega \ge \beta |NQ_j^k|_\omega$ for $(k, j) \in \mathbb{G}^\alpha$. For this proof it will be useful to use $m$ to index the cubes $B_m^k + 2$ and to use $i$ to index the cubes $Q_i^{k+2}$. The following lemma captures the main essence of the Whitney structure, and will be applied to cubes $B_m^{k+2}$ satisfying (4-53) and cubes $Q_i^{k+2}$ satisfying (2-6).

**Lemma 4.11.** *Suppose that $Q$ is a member of the Whitney decomposition of $\Omega$ with respect to the grid $\mathcal{D}$ and with Whitney constant $R_W$. Suppose also that a cube $B$ is a member of a Whitney decomposition of the same open set $\Omega$ but with respect to the grid $\mathcal{D}^\alpha$ and with Whitney constant $R'_W$. If $N < \frac{1}{2} R_W$ and $B \subset NQ$, then the side lengths of $Q$ and $B$ are comparable*:

$$\ell(Q) \approx \ell(B).$$

*Proof of Lemma 4.11.* Since $N < \frac{1}{2} R_W$ and $Q$ is a Whitney cube we have

$$\ell(Q) \approx \operatorname{dist}(Q, \partial\Omega) \approx \sup_{x \in NQ} \operatorname{dist}(x, \partial\Omega) \approx \inf_{x \in NQ} \operatorname{dist}(x, \partial\Omega).$$

Then since $B \subset NQ$ and $B$ is a Whitney cube (for the other decomposition) we have

$$\ell(Q) \approx \operatorname{dist}(B, \partial\Omega) \approx \ell(B). \qquad \square$$

*Proof of Lemma 4.10.* So suppose that $(k_1, j_1), \ldots, (k_M, j_M) \in \mathbb{G}^\alpha$ and $B = B_{i_\sigma}^{k_\sigma+2} \subset \widetilde{Q}_{j_\sigma}^{k_\sigma}$ for $1 \le \sigma \le M$, with the pairs of indices $(k_\sigma, j_\sigma)$ being distinct. Observe that the finite overlap property in (2-6) applies to the cubes $\widetilde{Q}_{j_\sigma}^{k_\sigma}$ in the Whitney decomposition (4-53) of $\Omega_{k_\sigma}$ with grid $\mathcal{D}^\alpha$ and Whitney constant $R'_W$. Thus for fixed $k$, the number of $(k_\sigma, j_\sigma)$ with $k_\sigma = k$ is bounded by the finite overlap constant since $B$ is inside each $\widetilde{Q}_{j_\sigma}^{k_\sigma}$. This gives us the observation that a single integer $k$ can occur only a *bounded* number $C_b$ of times among the $k_1, \ldots, k_M$.

After a relabeling, we can assume that all the $k_\sigma$ for $1 \le \sigma \le M'$ are distinct, listed in increasing order, and that the number $M'$ of $k_\sigma$ satisfies $M \le C_b M'$. The nested property of (2-6) assures us that $B$ is an element of the Whitney decomposition (4-53) of $\Omega_k$ for *all* $k_1 \le k \le k_{M'}$.

**Remark 4.12.** Note that the $k_\sigma$ are not necessarily consecutive since we require that $(k_\sigma, j_\sigma) \in \mathbb{G}^\alpha$. Nevertheless, the cube $B$ *does* occur among the $B_i^{k+2}$ for any $k$ that lies between $k_\sigma$ and $k_{\sigma+1}$. These latter occurrences of $B$ may be unbounded, but we are only concerned with bounding those for which $(k_\sigma, j_\sigma) \in \mathbb{G}^\alpha$, and it is these occurrences that our argument is treating.

Thus for $3 \le \sigma \le M'$, we have $k_1 \le k_\sigma - 2 \le k_{M'}$, and it follows from Remark 4.12 that the cube $B$ is a member of the Whitney decomposition (4-53) of the open set $\Omega_{k_\sigma}$ with grid $\mathcal{D}^\alpha$ and Whitney constant $R'_W$. But we also have that $Q^{k_\sigma}_{j_\sigma}$ is a member of the Whitney decomposition (2-6) of $\Omega_{k_\sigma}$ with grid $\mathcal{D}$ and Whitney constant $R_W$. Thus Lemma 4.11 gives us the equivalence of side lengths $\ell(Q^{k_\sigma}_{j_\sigma}) \approx \ell(B)$. Combining this with the containment $NQ^{k_\sigma}_{j_\sigma} \supset B$, we see that the number of possible locations for the cubes $Q^{k_\sigma}_{j_\sigma} \in \mathcal{D}$ is bounded by a constant $C'_b$ depending only on dimension.

Apply the pigeonhole principle to the possible locations of the $Q^{k_\sigma}_{j_\sigma}$. After a relabeling, we can argue under the assumption that all $Q^{k_\sigma}_{j_\sigma}$ equal the same cube $Q'$ for all choices of $1 \le \sigma \le M''$, where $M' \le C'_b M''$. Now comes the crux of the argument where the condition that the indices $(k_\sigma, j_\sigma)$ lie in $\mathbb{G}^\alpha$, as given in (4-27), proves critical. In particular we have $|E^{k_\sigma}_{j_\sigma}|_\omega \ge \beta |NQ'|_\omega$ where $N$ is as in Remark 2.4. The $k_\sigma$ are distinct, and the sets $E^{k_\sigma}_{j_\sigma} \subset Q'$ are pairwise disjoint; hence

$$M''\beta|NQ'|_\omega \le \sum_{\sigma=1}^{M''} |E^{k_\sigma}_{j_\sigma}|_\omega \le |Q'|_\omega \quad \text{implies } M'' \le \beta^{-1}.$$

Thus $M \le C_b C'_b \beta^{-1}$ and our proof of the claim is complete. □

**4.12.1.** *Replace bad functions by averages.* The first task in the analysis of the terms $IV^t_s(1)$ and $IV^t_s(2)$ will be to replace part of the "bad functions" $b_r$ by their averages over $B^k_i + 2$, or more exactly the averages $\mathsf{A}^{k+2}_i$. We again appeal to the Hölder continuity of $L^*\chi_{E^k_j \cap T_\ell}\omega$. By construction, $3B^{k+2}_i$ does not meet $E^k_j$, so that Lemma 2.9 applies. If $B^{k+2}_i \subset G^{\alpha,t+1}_r$ for some $r$, then there is a constant $c^{k+2}_i$ satisfying $|c^{k+2}_i| \le 1$ such that

$$\begin{aligned}\Big| \int_{B^{k+2}_i} (L^*\chi_{E^k_j \cap T_{\ell(r)}}\omega) b_r \sigma - \Big(c^{k+2}_i \int_{B^{k+2}_i} (L^*\chi_{E^k_j \cap T_{\ell(r)}}\omega)\sigma\Big)(|A^{\alpha,t+1}_r| + \mathsf{A}^{k+2}_i)\Big| \\ \le C\boldsymbol{P}(B^{k+2}_i, \chi_{E^k_j \cap T_{\ell(r)}}\omega) \int_{B^{k+2}_i} |b_r|\sigma. \quad (4\text{-}61)\end{aligned}$$

Indeed, if $z^{k+2}_i$ is the center of the cube $B^{k+2}_i$, we have

$$\begin{aligned}&\int_{B^{k+2}_i} (L^*\chi_{E^k_j \cap T_{\ell(r)}}\omega) b_r \sigma \\ &\quad = L^*(\chi_{E^k_j \cap T_{\ell(r)}}\omega)(z^{k+2}_i) \int_{B^{k+2}_i} b_r\sigma + O\Big(\boldsymbol{P}(B^{k+2}_i, \chi_{E^k_j \cap T_{\ell(r)}}\omega) \int_{B^{k+2}_i} |b_r|\sigma\Big) \\ &\quad = \Big(\int_{B^{k+2}_i} (L^*\chi_{E^k_j \cap T_{\ell(r)}}\omega)\sigma\Big) \frac{1}{|B^{k+2}_i|_\sigma} \int_{B^{k+2}_i} b_r\sigma + O\Big(\boldsymbol{P}(B^{k+2}_i, \chi_{E^k_j \cap T_{\ell(r)}}\omega) \int_{B^{k+2}_i} |b_r|\sigma\Big).\end{aligned}$$

Now, the functions $b_r$ are given in (4-34), and by construction, we note that

$$\frac{1}{|B^{k+2}_i|_\sigma} \Big| \int_{B^{k+2}_i} b_r\sigma \Big| \le \Big| \frac{1}{|G^{\alpha,t+1}_r|_\sigma} \int_{G^{\alpha,t+1}_r} f\sigma \Big| + \frac{1}{|B^{k+2}_i|_\sigma} \int_{B^{k+2}_i} |f|\sigma = |A^{\alpha,t+1}_r| + \mathsf{A}^{k+2}_i.$$

So with

$$c^{k+2}_i = \frac{1}{|A^{\alpha,t+1}_r| + \mathsf{A}^{k+2}_i} \frac{1}{|B^{k+2}_i|_\sigma} \int_{B^{k+2}_i} b_r\sigma,$$

we have $|c_i^{k+2}| \le 1$ and

$$\int_{B_i^{k+2}} (L^* \chi_{E_j^k \cap T_{\ell(r)}} \omega) b_r \sigma = \Big( c_i^{k+2} \int_{B_i^{k+2}} (L^* \chi_{E_j^k \cap T_{\ell(r)}} \omega)\sigma \Big) (|A_r^{\alpha,t+1}| + \mathsf{A}_i^{k+2}) + O\Big( \boldsymbol{P}(B_i^{k+2}, \chi_{E_j^k \cap T_{\ell(r)}} \omega) \int_{B_i^{k+2}} |b_r|\sigma \Big).$$

In the special case where $B_i^{k+2}$ is equal to $G_r^{\alpha,t+1}$, we have $\int_{B_i^{k+2}} b_r\sigma = \int b_r\sigma = 0$ and the proof above shows that

$$\Big| \int_{G_r^{\alpha,t+1}} (L^* \chi_{E_j^k \cap T_{\ell(r)}} \omega) b_r \sigma \Big| \le C \boldsymbol{P}(G_r^{\alpha,t+1}, \chi_{E_j^k \cap T_{\ell(r)}} \omega) \int_{G_r^{\alpha,t+1}} |f|\sigma \ , \tag{4-62}$$

since $\int_{G_r^{\alpha,t+1}} |b_r|\sigma = \int_{G_r^{\alpha,t+1}} |f - A_r^{\alpha,t+1}|\sigma \le 2\int_{G_r^{\alpha,t+1}} |f|\sigma$.

Our next task is to organize the sum over the cubes $B_i^{k+2}$ relative to the cubes $G_r^{\alpha,t+1}$. This is needed because the cubes $B_i^{k+2}$ are *not* pairwise disjoint in $k$, and we thank Tuomas Hytonen for bringing this point to our attention. The cube $B_i^{k+2}$ must intersect $\bigcup_{r \in \mathcal{H}_s^{\alpha,t}} G_r^{\alpha,t+1}$ since otherwise

$$\int_{G_r^{\alpha,t+1} \cap B_i^{k+2}} (L^* \chi_{E_j^k \cap T_{\ell(r)}} \omega) b_r \sigma = 0 \quad \text{for } r \in \mathcal{H}_s^{\alpha,t}.$$

Thus $B_i^{k+2}$ satisfies exactly one of the following two cases which we indicate by writing $i \in \text{Case(a)}$ or $i \in \text{Case(b)}$

Case(a) $B_i^{k+2}$ *strictly* contains at least one of the cubes $G_r^{\alpha,t+1}$ for $r \in \mathcal{H}_s^{\alpha,t}$.

Case(b) $B_i^{k+2} \subset G_r^{\alpha,t+1}$ for some $r \in \mathcal{H}_s^{\alpha,t}$.

Note that the cubes $B_i^{k+2}$ with $i \in \mathcal{I}_s^t$ can only satisfy Case(b), while the cubes $B_i^{k+2}$ with $i \in \mathcal{J}_s^t$ can satisfy either of the two cases above. However, we have the following claim.

**Claim 4.13.** *For each fixed $r \in \mathcal{H}_s^{\alpha,t}$, we have*

$$\sum_{(k+2,i,j) \text{ admissible}} \chi_{B_i^{k+2}} \le C,$$

*where the sum is taken over all* admissible *index* triples $(k+2, i, j)$, *that is, those for which the cube $B_i^{k+2}$ arises in term $IV_s^t$ with both $B_i^{k+2} \subset G_r^{\alpha,t+1}$ and $B_i^{k+2} \subset \widetilde{Q}_j^k$.*

But we first establish a containment that will be useful later as well. Recall that $\Omega_{k+2}$ decomposes as a pairwise disjoint union of cubes $B_i^{k+2}$, and thus we have

$$\int_{G_r^{\alpha,t+1} \cap \Omega_{k+2}} (L^* \chi_{E_j^k \cap T_{\ell(r)}} \omega) b_r \sigma = \sum_{i : B_i^{k+2} \cap \widetilde{Q}_j^k \ne \varnothing} \int_{B_i^{k+2}} (L^* \chi_{E_j^k \cap T_{\ell(r)}} \omega) b_r \sigma,$$

since the support of $L^* \chi_{E_j^k \cap T_{\ell(r)}} \omega$ is contained in $2Q_j^k \subset \hat{Q}_j^k \subset \widetilde{Q}_j^k$ by (4-7). Since both $B_i^{k+2}$ and $\widetilde{Q}_j^k$ lie in the grid $\mathcal{D}^\alpha$ and have nonempty intersection, one of these cubes is contained in the other. Now $B_i^{k+2}$

cannot *strictly* contain $\widetilde{Q}_j^k$ since $\widetilde{Q}_j^k = B_\ell^k$ for some $\ell$ and the cubes $\{B_j^k\}_{k,j}$ satisfy the nested property (4-54). It follows that we must have

$$B_i^{k+2} \subset \widetilde{Q}_j^k \quad \text{whenever } B_i^{k+2} \cap \widetilde{Q}_j^k \neq \varnothing. \tag{4-63}$$

Now we return to Claim 4.13, and note that for a fixed index pair $(k+2, i)$, the bounded overlap condition in (2-6) shows that there are only a bounded number of indices $j$ such that $B_i^{k+2} \subset \widetilde{Q}_j^k \subset N Q_j^k$ — see (4-56). We record this observation here:

$$\#\{j : B_i^{k+2} \subset \widetilde{Q}_j^k\} \leq C \quad \text{for each pair } (k+2, i). \tag{4-64}$$

Thus Claim 4.13 is reduced to this one:

**Claim 4.14.** $\sum\{\chi_{B_i^{k+2}} : B_i^{k+2} \subset G_r^{\alpha,t+1}$ *for some* $(k, j) \in \mathbb{I}_s^{\alpha,t}$ *with* $B_i^{k+2} \subset \widetilde{Q}_j^k\} \leq C$ *for each* $r \in \mathcal{K}_s^{\alpha,t}$.

As is the case with similar assertions in this argument, a central obstacle is that a given cube $B$ can arise in many different ways as a $B_i^{k+2}$.

*Proof of Claim 4.14.* We will appeal to the "bounded occurrence of cubes" in Section 4.9.2 above. This principle relies upon the definition of $\mathbb{G}^\alpha$ in (4-27), and applies in this setting due to the definition of $\mathbb{I}_s^{\alpha,t}$ in (4-28). We also appeal to the following fact:

$$G_r^{\alpha,t+1} \subset \widetilde{Q}_j^k \quad \text{whenever } B_i^{k+2} \subset G_r^{\alpha,t+1} \cap \widetilde{Q}_j^k \text{ with } (k, j) \in \mathbb{I}_s^{\alpha,t}. \tag{4-65}$$

To see (4-65), we note that both of the cubes $G_r^{\alpha,t+1}$ and $\widetilde{Q}_j^k$ lie in the grid $\mathcal{D}^\alpha$ and have nonempty intersection (they contain $B_i^{k+2}$), so that one of these cubes must be contained in the other. However, if $\widetilde{Q}_j^k \subset G_r^{\alpha,t+1}$, then $3Q_j^k \subset \hat{Q}_j^k \subset \widetilde{Q}_j^k$ implies $\mathcal{A}(Q_j^k) \subset G_r^{\alpha,t+1}$, which contradicts $(k, j) \in \mathbb{I}_s^{\alpha,t}$. Therefore we must have $G_r^{\alpha,t+1} \subset \widetilde{Q}_j^k$ as asserted in (4-65).

So to see that Claim 4.14 holds, suppose that $\mathcal{A}(Q_j^k) = G_s^{\alpha,t}$ and $B_i^{k+2} \subset G_r^{\alpha,t+1}$ with an associated cube $\widetilde{Q}_j^k$ as in (4-65). Then by (4-65) and (4-57) the side length $\ell(Q_j^k)$ of $Q_j^k$ satisfies

$$\ell(Q_j^k) = \frac{1}{N}\ell(NQ_j^k) \geq \frac{1}{N}\ell(\widetilde{Q}_j^k) \geq \frac{1}{N}\ell(G_r^{\alpha,t+1}). \tag{4-66}$$

Also, if $B_\ell^k$ is *any* Whitney cube at level $k$ that is contained in $G_r^{\alpha,t+1}$, then by (4-65) and (4-57) we have

$$B_\ell^k \subset G_r^{\alpha,t+1} \subset \widetilde{Q}_j^k \subset NQ_j^k,$$

so that Lemma 4.11 shows that $B_\ell^k$ and $Q_j^k$ have comparable side lengths:

$$\ell(B_\ell^k) \approx \ell(Q_j^k). \tag{4-67}$$

Moreover, if $B_{\ell'}^{k'}$ is any Whitney cube at level $k' < k$ that is contained in $G_r^{\alpha,t+1}$, then there is some Whitney cube $B_\ell^k$ at level $k$ such that $B_\ell^k \subset B_{\ell'}^{k'}$. Thus we have the containments $B_\ell^k \subset B_{\ell'}^{k'} \subset NQ_j^k$, and it follows from (4-67) that

$$\ell(B_{\ell'}^{k'}) \approx \ell(Q_j^k). \tag{4-68}$$

Now momentarily *fix* $k_0$ such that there is a cube $B_i^{k_0+2}$ satisfying the conditions in Claim 4.14. Then *all* of the cubes $B_\ell^{k+2}$ that arise in Claim 4.14 with $k \le k_0 - 2$ satisfy

$$\ell(B_\ell^{k+2}) \approx \ell(Q_j^{k_0}) \ge \frac{1}{N}\ell(G_r^{\alpha,t+1}).$$

Thus all of the cubes $B_\ell^{k+2}$ with $k \le k_0$, except perhaps those with $k \in \{k_0-1, k_0\}$, have side lengths bounded below by $c\ \ell(G_r^{\alpha,t+1})$, which bounds the number of possible locations for these cubes by a dimensional constant. However, those cubes $B_i^{k_0+1}$ at level $k_0+1$ are pairwise disjoint, as are those cubes $B_i^{k_0+2}$ at level $k_0+2$. Consequently, we can apply the "bounded occurrence of cubes" to show that the sum in Claim 4.14, when restricted to $k \le k_0$, is bounded by a constant $C$ independent of $k_0$. Since $k_0$ is arbitrary, this completes the proof of Claim 4.14. $\square$

As a result of Claim 4.14, for those $i$ in either $\mathcal{I}_s^t$ or $\mathcal{J}_s^t$ that satisfy Case(b), we will be able to apply below the Poisson argument used to estimate term $II_s^t(2)$ in (4-40) above.

We now further split the sum over $i \in \mathcal{J}_s^t$ in term $IV_s^t(2)$ into two sums according to Case(a) and Case(b) above:

$$\begin{aligned} IV_s^t(2) \le& \sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}} R_j^k \Big| \sum_{r\in\mathcal{H}_s^{\alpha,t}} \sum_{\substack{i\in\mathcal{J}_s^t \\ i\in\text{Case(a)}}} \int_{G_r^{\alpha,t+1}\cap B_i^{k+2}} (L^*\chi_{E_j^k\cap T_{\ell(r)}}\omega) b_r\sigma \Big|^p \\ &+ \sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}} R_j^k \Big| \sum_{r\in\mathcal{H}_s^{\alpha,t}} \sum_{\substack{i\in\mathcal{J}_s^t \\ i\in\text{Case(b)}}} \int_{G_r^{\alpha,t+1}\cap B_i^{k+2}} (L^*\chi_{E_j^k\cap T_{\ell(r)}}\omega) b_r\sigma \Big|^p \\ \equiv& IV_s^t(2)[\text{a}] + IV_s^t(2)[\text{b}]. \end{aligned} \tag{4-69}$$

We apply the definition of Case(b) and (4-61), to decompose $IV_s^t(2)[\text{b}]$ as follows:

$$\begin{aligned} IV_s^t(2)[\text{b}] =& \sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}} R_j^k \Big| \sum_{r\in\mathcal{H}_s^{\alpha,t}} \sum_{\substack{i\in\mathcal{J}_s^t \\ B_i^{k+2}\subset G_r^{\alpha,t+1}}} \int_{B_i^{k+2}} (L^*\chi_{E_j^k\cap T_{\ell(r)}}\omega) b_r\sigma \Big|^p \\ \le& \sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}} R_j^k \Big| \sum_{r\in\mathcal{H}_s^{\alpha,t}} \sum_{\substack{i\in\mathcal{J}_s^t \\ B_i^{k+2}\subset G_r^{\alpha,t+1}}} \Big(\int_{B_i^{k+2}} (L^*\chi_{E_j^k\cap T_{\ell(r)}}\omega)\sigma\Big) \times c_i^{k+2}(|A_r^{\alpha,t+1}| + \mathsf{A}_i^{k+2}) \Big|^p \\ &+ \sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}} R_j^k \Big| \sum_{r\in\mathcal{H}_s^{\alpha,t}} \sum_{\substack{i\in\mathcal{J}_s^t \\ B_i^{k+2}\subset G_r^{\alpha,t+1}}} \boldsymbol{P}(B_i^{k+2}, \chi_{E_j^k\cap T_{\ell(r)}}\omega) \int_{B_i^{k+2}} |b_r|\sigma \Big|^p \\ =& V_s^t(1) + V_s^t(2). \end{aligned} \tag{4-70}$$

**4.14.1.** *The bound for* $V(2)$. We claim that

$$\sum_{(t,s)\in\mathbb{L}^\alpha} V_s^t(2) \le C\gamma^{2p}\mathfrak{M}_*^p \|f\|_{L^p(\sigma)}^p. \tag{4-71}$$

Here, $\mathfrak{M}_*$ is defined in (4-3), and $V_s^t(2)$ is defined in (4-70).

*Proof.* The estimate for term $V_s^t(2)$ is similar to that of $II_s^t(2)$ above (see (4-40)), except that this time we use Claim 4.13 to handle a complication arising from the extra sum in the cubes $B_i^{k+2}$. We define

$$\boldsymbol{P}_j^k(\mu) \equiv \sum_{\ell} \sum_{\substack{r \in \mathcal{K}_s^{\alpha,t} \\ \ell(r)=\ell}} \sum_{\substack{i \in \mathcal{J}_s^t \\ B_i^{k+2} \subset G_r^{\alpha,t+1}}} \boldsymbol{P}(B_i^{k+2}, \chi_{E_j^k \cap T_\ell} \mu) \chi_{B_i^{k+2}}. \tag{4-72}$$

We observe that by Claim 4.14 the sum of these operators satisfies

$$\sum_{(k,j) \in \mathbb{I}_s^{\alpha,t}} \boldsymbol{P}_j^k(\mu) \le C \chi_{G_s^{\alpha,t}} \mathcal{M}(\chi_{G_s^{\alpha,t}} \mu), \tag{4-73}$$

and hence the analogue of (4-44) holds with $\boldsymbol{P}_j^k$ defined as above:

$$\Big\| \chi_{G_s^{\alpha,t}} \sum_{(k,j) \in \mathbb{I}_s^{\alpha,t}} (\boldsymbol{P}_j^k)^*(|h|\sigma) \Big\|_{L^p(w)} \le C \mathfrak{M}_* \| \chi_{G_s^{\alpha,t}} h \|_{L^p(\sigma)}. \tag{4-74}$$

For our use below, we note that this conclusion holds independent of the assumption, imposed in (4-72), that $i \in \mathcal{J}_s^t$.

With this notation, the summands in the definition of $V_s^t(2)$, as given in (4-70), are

$$\begin{aligned}
\sum_{\ell} \sum_{\substack{r \in \mathcal{K}_s^{\alpha,t} \\ \ell(r)=\ell}} \sum_{\substack{i \in \mathcal{J}_s^t \\ B_i^{k+2} \subset G_r^{\alpha,t+1}}} & \boldsymbol{P}(B_i^{k+2}, \chi_{E_j^k \cap T_\ell} \omega) \Big( \int_{B_i^{k+2}} \sigma \Big) \Big( \frac{1}{|B_i^{k+2}|_\sigma} \int_{B_i^{k+2}} |f| \sigma \Big) \\
&\le \gamma^{t+2} \sum_{\ell} \int \sum_{\substack{r \in \mathcal{K}_s^{\alpha,t} \\ \ell(r)=\ell}} \sum_{\substack{i \in \mathcal{J}_s^t \\ B_i^{k+2} \subset G_r^{\alpha,t+1}}} \boldsymbol{P}(B_i^{k+2}, \chi_{E_j^k \cap T_\ell} \omega) \chi_{B_i^{k+2}} \sigma \quad (\text{since } i \in \mathcal{J}_s^t) \\
&\le \gamma^{t+2} \int_{G_s^{\alpha,t}} \boldsymbol{P}_j^k(\omega) \sigma = \gamma^{t+2} \int_{E_j^k} (\boldsymbol{P}_j^k)^*(\chi_{G_s^{\alpha,t}} \sigma) \omega.
\end{aligned} \tag{4-75}$$

We then have from (4-70) and (4-75) by the argument for term $II_s^t(2)$,

$$\begin{aligned}
\sum_{(t,s) \in \mathbb{L}^\alpha} V_s^t(2) &\le C \gamma^{2p} \sum_{(t,s) \in \mathbb{L}^\alpha} \gamma^{pt} \sum_{(k,j) \in \mathbb{I}_s^{\alpha,t}} R_j^k \Big| \int_{Q_j^k} (\boldsymbol{P}_j^k)^*(\chi_{G_s^{\alpha,t}} \sigma)\, \omega \Big|^p \\
&\le C \gamma^{2p} \sum_{(t,s) \in \mathbb{L}^\alpha} \gamma^{pt} \int \Big| \mathcal{M}_\omega \Big( \chi_{G_s^{\alpha,t}} \sum_{(\ell,i) \in \mathbb{I}_s^{\alpha,t}} (\boldsymbol{P}_i^\ell)^*(\chi_{G_s^{\alpha,t}} \sigma) \Big) \Big|^p \omega \\
&\le C \gamma^{2p} \sum_{(t,s) \in \mathbb{L}^\alpha} \gamma^{pt} \int_{G_s^{\alpha,t}} \Big( \sum_{(\ell,i) \in \mathbb{I}_s^{\alpha,t}} (\boldsymbol{P}_i^\ell)^*(\chi_{G_s^{\alpha,t}} \sigma) \Big)^p \omega \\
&\le C \gamma^{2p} \mathfrak{M}_*^p \sum_{(t,s) \in \mathbb{L}^\alpha} \gamma^{pt} \sum_{\ell} |G_s^{\alpha,t}|_\sigma \le C \gamma^{2p} \mathfrak{M}_*^p \int |f|^p \, \sigma.
\end{aligned}$$

In last lines we are using the boundedness (1-17) of the maximal operator. □

We will use the same method to treat term $V(1)$ and term $VI(1)$ below, and we postpone the argument for now.

**4.14.2.** *The bound for* $IV(2)[\mathrm{a}]$. We turn to the term defined in (4-69). In Case(a) the cubes $B_i^{k+2}$ satisfy

$$G_r^{\alpha,t+1} \subset B_i^{k+2} \quad \text{whenever } G_r^{\alpha,t+1} \cap B_i^{k+2} \neq \varnothing.$$

and so recalling that $i \in \mathcal{I}_s^t$ and $i \in \text{Case(a)}$, we obtain from (4-62) that

$$\begin{aligned} IV_s^t(2)[\mathrm{a}] &= \sum_{(k,j)\in \mathbb{I}_s^{\alpha,t}} R_j^k \Big| \sum_{\substack{i\in\mathcal{I}_s^t \\ i\in\text{Case(a)}}} \sum_{r:G_r^{\alpha,t+1}\subset B_i^{k+2}} \int_{G_r^{\alpha,t+1}} (L^*\chi_{E_j^k}\omega) b_r \sigma \Big|^p \\ &\le C \sum_{(k,j)\in \mathbb{I}_s^{\alpha,t}} R_j^k \Big| \sum_{i\in\text{Case(a)}} \sum_{r:G_r^{\alpha,t+1}\subset B_i^{k+2}} \boldsymbol{P}(G_r^{\alpha,t+1}, \chi_{E_j^k}\omega) \int_{G_r^{\alpha,t+1}} |f|\sigma \Big|^p \\ &\le C\gamma^{p(t+2)} \sum_{(k,j)\in \mathbb{I}_s^{\alpha,t}} R_j^k \Big| \sum_{r:G_r^{\alpha,t+1}\subset 3Q_j^k} \boldsymbol{P}(G_r^{\alpha,t+1}, \chi_{E_j^k}\omega) |G_r^{\alpha,t+1}|_\sigma \Big|^p. \end{aligned}$$

But this last sum is identical to the estimate for the term $II_s^t(2)$ used in (4-45) above. The estimate there thus gives

$$\sum_{(t,s)\in\mathbb{L}^\alpha} IV_s^t(2)[a] \le C\gamma^{2p}\mathfrak{M}_*^p \sum_{(t,s)\in\mathbb{L}^\alpha} \gamma^{pt} |G_s^{\alpha,t}|_\sigma \le C\gamma^{2p}\mathfrak{M}_*^p \int |f|^p\sigma, \tag{4-76}$$

which is the desired estimate.

**4.14.3.** *The decomposition of* $IV(1)$. This term is the first term on the right hand side of (4-58). Recall that for $i \in \mathcal{I}_s^t$ we have $i \in \text{Case(b)}$ and so $B_i^{k+2} \subset G_r^{\alpha,t+1} \subset T_{\ell(r)}$ for some $r \in \mathcal{K}_s^{\alpha,t}$. From (4-63) we also have $B_i^{k+2} \subset \widetilde{Q}_j^k$. To estimate $IV_s^t(1)$ in (4-58), we again apply (4-61) to be able to write

$$\begin{aligned} IV_s^t(1) &\le C \sum_{(k,j)\in \mathbb{I}_s^{\alpha,t}} R_j^k \Big( \sum_\ell \sum_{\substack{i\in\mathcal{I}_s^t \\ B_i^{k+2}\subset T_\ell\cap\widetilde{Q}_j^k}} \Big( \int_{B_i^{k+2}} |L^*\chi_{E_j^k\cap T_\ell}\omega|\sigma \Big) \mathsf{A}_i^{k+2} \Big)^p \\ &\quad + C \sum_{(k,j)\in \mathbb{I}_s^{\alpha,t}} R_j^k \Big( \sum_\ell \sum_{\substack{i\in\mathcal{I}_s^t \\ B_i^{k+2}\subset T_\ell\cap\widetilde{Q}_j^k}} \boldsymbol{P}(B_i^{k+2}, \chi_{E_j^k\cap T_\ell}\omega) \int_{B_i^{k+2}} |f|\sigma \Big)^p \qquad (4\text{-}77) \\ &= VI_s^t(1) + VI_s^t(2). \end{aligned}$$

We can dominate the averages on $B_i^{k+2}$ of the bad function $b_r$ by $\mathsf{A}_i^{k+2} + |A_r^{\alpha,t+1}| \le 2\mathsf{A}_i^{k+2}$, since in this case $i \in \mathcal{I}_s^t$ (see (4-59)), and this implies that the average of $|b_r| = |f - A_r^{\alpha,t+1}|$ over the cube $B_i^{k+2}$ is dominated by

$$\mathsf{A}_i^{k+2} + |A_r^{\alpha,t+1}| \le \mathsf{A}_i^{k+2} + \gamma^{t+2} < 2\mathsf{A}_i^{k+2}.$$

**4.14.4.** *The bound for $VI(2)$.* We claim that

$$VI_s^t(2) \le C\mathfrak{M}_*^p \sum_{k,i}^{s,t,\mathscr{I}} |B_i^{k+2}|_\sigma (\mathsf{A}_i^{k+2})^p. \tag{4-78}$$

Here, the sum on the right is over all pairs of integers $k, i \in \mathscr{I}_s^t$ such that $B_i^{k+2} \subset T_\ell \cap \widetilde{Q}_j^k$ for some $\ell$, $j$ with $(k, j) \in \mathbb{I}_s^{\alpha,t}$. (Below, we will need a similar sum, with the condition $i \in \mathscr{I}_s^t$ replaced by $i \in \mathscr{J}_s^t$ and $i \in \text{Case(b)}$.) This is a provisional bound, one that requires additional combinatorial arguments in Section 4.14.7.

*Proof.* The term $VI_s^t(2)$ can be handled the same way as the term $V_s^t(2)$ (see (4-71)), with these two changes. First, in the definition of $\mathbf{P}_j^k$, we replace $\mathscr{J}_s^t$ by $\mathscr{I}_s^t$, and second, we use the function

$$h = \sum_{k,i}^{s,t,\mathscr{I}} \mathsf{A}_i^{k+2} \chi_{B_i^{k+2}}$$

in (4-74). That argument then obtains

$$\Big\| \chi_{G_s^{\alpha,t}} \sum_{k,j} (\boldsymbol{P}_j^k)^* (\chi_{G_s^{\alpha,t}} h\sigma) \Big\|_{L^p(\omega)}^p \le C\mathfrak{M}_*^p \sum_{k,i}^{s,t,\mathscr{I}} |B_i^{k+2}|_\sigma (\mathsf{A}_i^{k+2})^p. \tag{4-79}$$

Here we are using the bounded overlap of the cubes $B_i^{k+2}$ given in Claim 4.13, along with the fact recorded in (4-64) that for fixed $(k+2, i)$, only a bounded number of $j$ satisfy $B_i^{k+2} \subset \widetilde{Q}_j^k$. Claim 4.13 applies in this setting, as we are in a subcase of the analysis of $IV$. We then use the universal maximal function bound (2-2).

$$\begin{aligned}
VI_s^t(2) &= \sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}} R_j^k \Big( \sum_\ell \sum_{\substack{i\in\mathscr{I}_s^t \\ B_i^{k+2}\subset T_\ell\cap\widetilde{Q}_j^k}} \boldsymbol{P}(B_i^{k+2}, \chi_{E_j^k\cap T_\ell}\omega) \Big( \int_{B_i^{k+2}} \sigma \Big) \mathsf{A}_i^{k+2} \Big)^p \\
&= C \sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}} R_j^k \Big| \int_{Q_j^k} (\boldsymbol{P}_j^k)^*(h\sigma)\omega \Big|^p \\
&\le C \int \Big( \mathcal{M}_\omega \Big( \chi_{G_s^{\alpha,t}} \sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}} (\boldsymbol{P}_j^k)^* (\chi_{G_s^{\alpha,t}} h\sigma) \Big) \Big)^p \omega \\
&\le C \int \Big( \chi_{G_s^{\alpha,t}} \sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}} (\boldsymbol{P}_j^k)^* (\chi_{G_s^{\alpha,t}} h\sigma) \Big)^p \omega.
\end{aligned}$$

In view of (4-79), this completes the proof of the provisional estimate (4-78). □

**4.14.5.** *The bound for $VI(1)$.* Recall the definition of $VI(1)$ from (4-77), and also from (4-63) the fact that $B_i^{k+2} \subset \widetilde{Q}_j^k$ whenever $B_i^{k+2} \cap \widetilde{Q}_j^k \neq \varnothing$. We claim that

$$VI_s^t(1) \le C\mathfrak{T}_*^p \sum_{k,i}^{s,t,\mathscr{I}} |B_i^{k+2}|_\sigma (\mathsf{A}_i^{k+2})^p. \tag{4-80}$$

The notation here is as in (4-78), but since $i \in \mathcal{I}_s^t$ implies $i$ belongs to Case(b), the sum over the right is over $k, i \in \mathcal{I}_s^t$ such that $B_i^{k+2} \subset G_r^{\alpha,t+1} \subset T_{\ell(r)} \cap \widetilde{Q}_j^k$, for some integers $j, r$, with $(k, j) \in \mathbb{I}_s^{\alpha,t}$. As with (4-78), this is a provisional estimate.

*Proof.* We first estimate the sum in $i$ inside term $VI_s^t(1)$. Recall that the sum in $i$ is over those $i$ such that $B_i^{k+2} \subset G_r^{\alpha,t+1} \subset T_\ell$ for some $r$ with $\ell = \ell(r)$, and where $\{T_\ell\}_\ell$ is the set of maximal cubes in the collection $\{3G_r^{\alpha,t+1} : r \in \mathcal{K}_s^{\alpha,t}\}$. See the discussion at (4-35), and (4-50). We will write $\ell(i) = \ell(r)$ when $B_i^{k+2} \subset G_r^{\alpha,t+1}$. It is also important to note that the sum in $i$ deriving from term $IV_s^t$ is also restricted to those $i$ such that $B_i^{k+2} \subset \widetilde{Q}_j^k$ by (4-63), so that altogether, $B_i^{k+2} \subset T_\ell \cap \widetilde{Q}_j^k$. We have

$$\begin{aligned}
&\Big|\sum_i \Big(\int_{B_i^{k+2}} |L^* \chi_{E_j^k \cap T_{\ell(i)}} \omega| \sigma\Big) \mathsf{A}_i^{k+2}\Big|^p \\
&\qquad \le \sum_i |B_i^{k+2}|_\sigma (\mathsf{A}_i^{k+2})^p \Big(\sum_i |B_i^{k+2}|_\sigma^{1-p'} \Big(\int_{B_i^{k+2}} |L^* \chi_{E_j^k \cap T_{\ell(i)}} \omega| \sigma\Big)^{p'}\Big)^{p-1} \\
&\qquad \le \sum_i |B_i^{k+2}|_\sigma (\mathsf{A}_i^{k+2})^p \Big(\sum_i \int_{B_i^{k+2}} |L^* \chi_{E_j^k \cap T_{\ell(i)}} \omega|^{p'} \sigma\Big)^{p-1} \\
&\qquad \le C \sum_i |B_i^{k+2}|_\sigma (\mathsf{A}_i^{k+2})^p \Big(\sum_\ell \sum_{i:\ell(i)=\ell} \int_{B_i^{k+2}} |L^* \chi_{E_j^k \cap T_{\ell(i)}} \omega|^{p'} \sigma\Big)^{p-1}.
\end{aligned}$$

Now we will apply the form (2-11) of (1-14) with $g = \chi_{E_j^k \cap T_\ell}$ and $Q$ chosen to be either $T_\ell$ or $\widetilde{Q}_j^k$ depending on the relative positions of $T_\ell$ and $\widetilde{Q}_j^k$. Since $T_\ell$ is a triple of a cube in the grid $\mathcal{D}^\alpha$ and $\widetilde{Q}_j^k$ is a cube in the grid $\mathcal{D}^\alpha$, we must have either

$$\widetilde{Q}_j^k \subset T_\ell \quad \text{or} \quad T_\ell \subset 3\widetilde{Q}_j^k.$$

If $\widetilde{Q}_j^k \subset T_\ell$ we choose $Q$ in (2-11) to be $\widetilde{Q}_j^k$ and note that by bounded overlap of Whitney cubes, there are only a bounded number of such cases. If on the other hand $T_\ell \subset 3\widetilde{Q}_j^k$, then we choose $Q$ to be $T_\ell$ and note that the cubes $T_\ell$ have bounded overlap. This gives

$$\sum_\ell \sum_{i:\ell(i)=\ell} \int_{B_i^{k+2}} |L^* \chi_{E_j^k \cap T_{\ell(i)}} \omega|^{p'} \sigma \lesssim \mathfrak{T}_*^p |3\widetilde{Q}_j^k|_\omega,$$

and hence

$$\Big|\sum_i \Big(\int_{B_i^{k+2}} |L^* \chi_{E_j^k \cap T_{\ell(i)}} \omega| \sigma\Big) \mathsf{A}_i^{k+2}\Big|^p \le C \mathfrak{T}_*^p \sum_i |B_i^{k+2}|_\sigma (\mathsf{A}_i^{k+2})^p |N Q_j^k|_\omega^{p-1},$$

since $3\widetilde{Q}_j^k \subset N Q_j^k$ by (4-56). With this we obtain

$$\begin{aligned}
VI_s^t(1) &\le C \mathfrak{T}_*^p \sum_{(k,j)\in \mathbb{I}_s^{\alpha,t}} R_j^k \sum_{i \in \mathcal{I}_s^t} |B_i^{k+2}|_\sigma (\mathsf{A}_i^{k+2})^p |N Q_j^k|_\omega^{p-1} \qquad (4\text{-}81)\\
&\le C \mathfrak{T}_*^p \, {\sum_{k,i}}^{*\,s,t,\mathcal{I}} |B_i^{k+2}|_\sigma (\mathsf{A}_i^{k+2})^p,
\end{aligned}$$

where we are using $R_j^k |NQ_j^k|_\omega^{p-1} \le 1$ and (4-64) in the final line. $\square$

**4.14.6.** *The bound for* $V(1)$. We will use the same method as in the estimate for term $VI(1)$ above to obtain

$$\sum_{(t,s)\in\mathbb{L}^\alpha} V_s^t(1) \le C\mathfrak{T}_*^p \gamma^{2p} \|f\|_{L^p(\sigma)}^p. \tag{4-82}$$

Recall from (4-70) that $V_s^t(1)$ is given by

$$\sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}} R_j^k \Big| \sum_{r\in\mathcal{K}_s^{\alpha,t}} \sum_{\substack{i\in\mathcal{J}_s^t\\ B_i^{k+2}\subset G_r^{\alpha,t+1}}} \Big(\int_{B_i^{k+2}} (L^*\chi_{E_j^k\cap T_{\ell(r)}}\omega)\sigma\Big) c_i^{k+2} (|A_r^{\alpha,t+1}| + \mathsf{A}_i^{k+2})\Big|^p.$$

The main difference here, as opposed to the previous estimate, is that $i\in\mathcal{J}_s^t$ rather than in $\mathcal{I}_s^t$; see (4-59). As a result, we have the estimate

$$|A_r^{\alpha,t+1}| + \mathsf{A}_i^{k+2} \lesssim \gamma^{t+2}, \tag{4-83}$$

instead of $|A_r^{\alpha,t+1}| + \mathsf{A}_i^{k+2} \lesssim \mathsf{A}_i^{k+2}$, which holds when $i\in\mathcal{I}_s^t$.

*Proof of* (4-82). We follow the argument leading up to and including (4-81) in the estimate for term $VI(1)$ above, but using instead (4-83). The result is as below, where we are using the notation of (4-78), with the condition $i\in\mathcal{I}_s^t$ replaced by $i\in\mathcal{J}_s^t$ and $i\in Case(b)$, and so we use an asterisk and $\mathcal{J}$ in the notation below.

$$V_s^t(1) \le C\mathfrak{T}_*^p \, {\sum_{k,i}}^{*\,s,t,\mathcal{J}} |B_i^{k+2}|_\sigma (\gamma^{t+2})^p.$$

Now we collect those cubes $B_i^{k+2}$ that lie in a given cube $G_r^{\alpha,t+1}$ and write the right hand side above as a constant times

$$\mathfrak{T}_*^p \gamma^{(t+2)p} \sum_{r\in\mathcal{K}_s^{\alpha,t}} {\sum_{k,i}}^{*\,s,t,\mathcal{J}} |B_i^{k+2}|_\sigma := \mathfrak{T}_*^p \gamma^{(t+2)p} \sum_{r\in\mathcal{K}_s^{\alpha,t}} \mathcal{S}_{s,r}^{\alpha,t}.$$

By Claim 4.13, which applies as we are in a subcase of $IV$, we have $\mathcal{S}_{s,r}^{\alpha,t} \le C|G_r^{\alpha,t+1}|_\sigma$, and it follows that

$$V_s^t(1) \le C\mathfrak{T}_*^p \gamma^{(t+2)p} \sum_{r\in\mathcal{K}_s^{\alpha,t}} |G_r^{\alpha,t+1}|_\sigma \le C\mathfrak{T}_*^p \gamma^{(t+2)p} |G_s^{\alpha,t}|_\sigma,$$

and hence from (4-18) that

$$\sum_{(t,s)\in\mathbb{L}^\alpha} V_s^t(1) \le C\mathfrak{T}_*^p \gamma^{2p} \sum_{(t,s)\in\mathbb{L}^\alpha} \gamma^{tp} |G_s^{\alpha,t}|_\sigma \le C\mathfrak{T}_*^p \gamma^{2p} \|f\|_{L^p(\sigma)}^p. \qquad \square$$

**4.14.7.** *The final combinatorial arguments.* Our final estimate in the proof of (4-52) is to dominate by $C\int|f|^p\,d\sigma$ the sum of the right hand sides of (4-78) and (4-80) over $(t,s)\in\mathbb{L}^\alpha$, namely

$$\sum_{(t,s)\in\mathbb{L}^\alpha} {\sum_{k,i}}^{s,t,\mathcal{J}} \le C\int |f|^p\,d\sigma. \tag{4-84}$$

The proof of (4-84) will require combinatorial facts related to the principal cubes, and the definition of the collection $\mathbb{G}^\alpha$ in (4-27). Also essential is the implementation of the shifted dyadic grids. We now detail the arguments.

**Definition 4.15.** We say that a cube $B_i^{k+2}$ satisfying the defining condition in $VI_s^t(1)$, namely

there is $(k,j) \in \mathbb{I}_s^{\alpha,t} = \mathbb{G}^\alpha \cap \mathbb{H}_s^{\alpha,t}$ such that
$B_i^{k+2} \subset \widetilde{Q}_j^k$ and $B_i^{k+2} \subset$ some $G_r^{\alpha,t+1} \subset G_s^{\alpha,t}$ satisfying $\mathsf{A}_i^{k+2} > \gamma^{t+2}$,

is a *final type* cube for the pair $(t,s) \in \mathbb{L}^\alpha$ generated from $Q_j^k$.

The collection $\mathscr{F}$ of cubes $B_i^{k+2}$ such that $B_i^{k+2}$ is a *final type* cube generated from some $Q_j^k$ with $(k,j) \in \mathbb{I}_s^{\alpha,t}$ for some pair $(t,s) \in \mathbb{L}^\alpha$ satisfies the following three properties:

**Property 1.** *$\mathscr{F}$ is a nested grid in the sense that given any two distinct cubes in $\mathscr{F}$, either one is strictly contained in the other, or they are disjoint (ignoring boundaries).*

**Property 2.** *If $B_i^{k+2}$ and $B_{i'}^{k'+2}$ are two distinct cubes in $\mathscr{F}$ with $B_{i'}^{k'+2} \subsetneqq B_i^{k+2}$, and $k$ and $k'$ have the same parity, then*

$$\mathsf{A}_{i'}^{k'+2} > \gamma \mathsf{A}_i^{k+2}.$$

**Property 3.** *A given cube $B_i^{k+2}$ can occur at most a bounded number of times in the grid $\mathscr{F}$.*

*Proof of Properties 1, 2 and 3.* Property 1 is obvious from the properties of the dyadic shifted grid $\mathscr{D}^\alpha$. Property 3 follows from the "bounded occurrence of cubes" noted above. So we turn to Property 2. It is this property that prompted the use of the shifted dyadic grids.

Indeed, since $B_{i'}^{k'+2} \subsetneqq B_i^{k+2}$, it follows from the nested property (4-54) that $k' > k$. By Definition 4.15 there are cubes

$$Q_{j'}^{k'} \quad \text{and} \quad Q_j^k \quad \text{satisfying} \quad B_{i'}^{k'+2} \subset \widetilde{Q}_{j'}^{k'} \quad \text{and} \quad B_i^{k+2} \subset \widetilde{Q}_j^k,$$

and also cubes $G_{s'}^{\alpha,t'} \subset G_s^{\alpha,t}$ such that $(k',j') \in \mathbb{I}_{s'}^{\alpha,t'}$ and $(k,j) \in \mathbb{I}_s^{\alpha,t}$ with $(t',s'),(t,s) \in \mathbb{L}^\alpha$, so that in particular,

$$\widetilde{Q}_{j'}^{k'} \subset G_{s'}^{\alpha,t'} \quad \text{and} \quad \widetilde{Q}_j^k \subset G_s^{\alpha,t}.$$

Now $k' \geq k+2$ and in the extreme case where $k' = k+2$, it follows that the $\mathscr{D}^\alpha$-cube $\widetilde{Q}_{j'}^{k'}$ is one of the cubes $B_\ell^{k+2}$, so in fact it must be $B_i^{k+2}$ since $B_{i'}^{k'+2} \subset B_i^{k+2}$. Thus we have

$$B_{i'}^{k'+2} \subset \widetilde{Q}_{j'}^{k'} = B_i^{k+2}.$$

In the general case $k' \geq k+2$ we have instead

$$B_{i'}^{k'+2} \subset \widetilde{Q}_{j'}^{k'} \subset B_i^{k+2}.$$

Now $\mathsf{A}_i^{k+2} > \gamma^{t+2}$ by Definition 4.15, and so there is $t_0 \geq t+2$ determined by the condition

$$\gamma^{t_0} < \mathsf{A}_i^{k+2} \leq \gamma^{t_0+1}, \tag{4-85}$$

and also $s_0$ such that

$$B_i^{k+2} \subset G_{s_0}^{\alpha,t_0} \subset G_s^{\alpha,t},$$

where the label $(t_0, s_0)$ need not be principal. Combining inclusions we have

$$\widetilde{Q}_{j'}^{k'} \subset B_i^{k+2} \subset G_{s_0}^{\alpha,t_0},$$

and since $(k', j') \in \mathbb{I}_{s'}^{\alpha,t'}$, we obtain $G_{s'}^{\alpha,t'} \subset G_{s_0}^{\alpha,t_0}$. Since $(t', s') \in \mathbb{L}^\alpha$ is a principal label, we have the key property that

$$t' \geq t_0. \tag{4-86}$$

Indeed, if $G_{s'}^{\alpha,t'} = G_{s_0}^{\alpha,t_0}$ then (4-86) holds because $(t', s') \in \mathbb{L}^\alpha$ is a principal label, and otherwise the maximality of $G_{s'}^{\alpha,t'}$ shows that

$$\gamma^{t_0} < \frac{1}{|G_{s_0}^{\alpha,t_0}|_\sigma} \int_{G_{s_0}^{\alpha,t_0}} |f| \, d\sigma \leq \gamma^{t'+1} \quad \text{that is,} \quad t_0 < t'+1.$$

Thus using (4-86) and (4-85) we obtain Property 2:

$$\mathsf{A}_{i'}^{k'+2} > \gamma^{t'+2} \geq \gamma^{t_0+2} \geq \gamma \mathsf{A}_i^{k+2}. \qquad \square$$

*Proof of* (4-84). Now for $Q = B_i^{k+2} \in \mathcal{F}$ set

$$\mathsf{A}(Q) = \frac{1}{|Q|_\sigma} \int_Q |f|\sigma = \mathsf{A}_i^{k+2} = \frac{1}{|B_i^{k+2}|_\sigma} \int_{B_i^{k+2}} |f|\sigma.$$

With the three properties above we can now prove (4-84) as follows. Recall that in term $IV(1)$ we have $i \in \mathcal{I}_s^t$ which implies $B_i^{k+2}$ satisfies Case(b). In the display below by $\sum_i^*$ we mean the sum over $i$ such that $B_i^{k+2}$ is contained in some $G_r^{\alpha,t+1} \subset G_s^{\alpha,t}$, and also in some $\widetilde{Q}_j^k$ with $(k, j) \in \mathbb{I}_s^{\alpha,t}$, and satisfying $\mathsf{A}_i^{k+2} > 2^{t+2}$. The left side of (4-84) is dominated by

$$\begin{aligned}
\sum_{(t,s)\in\mathbb{L}^\alpha} \sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}} \sum_i^* |B_i^{k+2}|_\sigma (\mathsf{A}_i^{k+2})^p &= \sum_{Q\in\mathcal{F}} |Q|_\sigma \mathsf{A}(Q)^p = \sum_{Q\in\mathcal{F}} |Q|_\sigma \Big( \frac{1}{|Q|_\sigma} \int_Q |f|\sigma \Big)^p \\
&= \int_{\mathbb{R}^n} \sum_{Q\in\mathcal{F}} \chi_Q(x) \Big( \frac{1}{|Q|_\sigma} \int_Q |f|\sigma \Big)^p \, d\sigma(x) \\
&\leq C \int_{\mathbb{R}^n} \sup_{x\in Q: Q\in\mathcal{F}} \Big( \frac{1}{|Q|_\sigma} \int_Q |f|\sigma \Big)^p \, d\sigma(x) \\
&\leq C \int_{\mathbb{R}^n} \mathcal{M}_\sigma^\alpha f(x)^p \sigma(dx) \leq C \int_{\mathbb{R}^n} |f(x)|^p \, d\sigma(x),
\end{aligned}$$

where the second to last line follows since for fixed $x \in \mathbb{R}^n$, the sum

$$\sum_{Q\in\mathcal{F}} \chi_Q(x) \Big( \frac{1}{|Q|_\sigma} \int_Q |f|\sigma \Big)^p$$

is supergeometric by Properties 1, 2 and 3 above, that is, for any two distinct cubes $Q$ and $Q'$ in $\mathcal{F}$ each containing $x$, the ratio of the corresponding values is bounded away from 1, more precisely,

$$\frac{\left(\frac{1}{|Q|_\sigma}\int_Q |f|\sigma\right)^p}{\left(\frac{1}{|Q'|_\sigma}\int_{Q'} |f|\sigma\right)^p} \notin [\gamma^{-p}, \gamma^p) \quad \text{for } \gamma \geq 2.$$

This completes the proof of (4-84). □

## 5. The proof of Theorem 1.10 on the strongly maximal Hilbert transform

To prove Theorem 1.10 we first show that in the proof of Theorem 1.9 above, we can replace the use of the dual maximal function inequality (1-17) with the dual weighted Poisson inequality (5-5) defined below. After that we will show that in the case of standard kernels satisfying (1-9) with $\delta(s) = s$ in dimension $n = 1$, the dual weighted Poisson inequality (5-5) is implied by the *half-strengthened* $A_p$ condition

$$\left(\int_{\mathbb{R}}\left(\frac{|Q|}{|Q|+|x-x_Q|}\right)^{p'} d\sigma(x)\right)^{1/p'}\left(\int_Q d\omega(x)\right)^{1/p} \leq \mathcal{A}_p(\omega,\sigma)|Q|, \tag{5-1}$$

for all intervals $Q$, together with the dual pivotal condition (5-2) of Nazarov, Treil and Volberg [2010], namely that

$$\sum_{r=1}^{\infty} |Q_r|_\sigma \mathrm{P}(Q_r, \chi_{Q_0}\omega)^{p'} \leq \mathfrak{C}_*^{p'}|Q_0|_\omega, \tag{5-2}$$

holds for all decompositions of an interval $Q_0$ into a union of pairwise disjoint intervals $Q_0 = \bigcup_{r=1}^{\infty} Q_r$. We will assume $1 < p \leq 2$ for this latter implication. Finally, for $p > 2$, we show that (5-5) is implied by (5-1), (5-2) and the Poisson condition (1-24).

It follows from work in [Nazarov et al. 2010] and [Lacey et al. 2011] that the strengthened $A_2$ condition (5-16) is necessary for the two weight inequality for the Hilbert transform, and also from [Lacey et al. 2011] that the dual pivotal condition (5-2) is necessary for the dual testing condition

$$\int_Q T(\chi_Q\omega)^2\, d\sigma \leq C\int_Q d\omega,$$

for $T$ when $p = 2$ and $\sigma$ is doubling. We show below that these results extend to $1 < p < \infty$. A slightly weaker result was known earlier from work of Nazarov, Treil and Volberg — namely that the pivotal conditions are necessary for the Hilbert transform $H$ when *both* of the weights $\omega$ and $\sigma$ are doubling and $p = 2$. However, [Lacey et al. 2011] gives an example that shows that (5-2) is *not* in general necessary for boundedness of the Hilbert transform $T$ when $p = 2$.

Finally, we show below that when $\sigma$ is doubling, the dual weighted Poisson inequality (5-5) is implied by the two weight inequality for the Hilbert transform. Since the Poisson condition (1-24) is a special case of the inequality dual to (5-5), we obtain the necessity of (1-24) for the two weight inequality for the Hilbert transform.

**5.1.** ***The Poisson inequalities.*** We begin working in $\mathbb{R}^n$ with $1 < p < \infty$. Recall the definition of the Poisson integral $\mathsf{P}(Q, \nu)$ of a measure $\nu$ relative to a cube $Q$, given by

$$\mathsf{P}(Q, \nu) \equiv \sum_{\ell=0}^{\infty} \frac{\delta(2^{-\ell})}{|2^\ell Q|} \int_{2^\ell Q} d|\nu|. \tag{5-3}$$

We will consider here only the standard Poisson integral with $\delta(s) = s$ in (5-3), and so we also suppose that $\delta(s) = s$ in (1-9) above. We now fix a cube $Q_0$ and a collection of pairwise disjoint subcubes $\{Q_r\}_{r=1}^{\infty}$. Corresponding to these cubes we define a positive linear operator

$$\mathbb{P}\nu(x) = \sum_{r=1}^{\infty} \mathsf{P}(Q_r, \nu)\chi_{Q_r}(x). \tag{5-4}$$

We wish to obtain *sufficient* conditions for the following "dual" weighted Poisson inequality,

$$\int_{\mathbb{R}^n} \mathbb{P}(f\omega)(x)^{p'}\, d\sigma(x) \le C \int_{\mathbb{R}^n} f^{p'}\, d\omega(x) \quad \text{for } f \ge 0. \tag{5-5}$$

uniformly in $Q_0$ and pairwise disjoint subcubes $\{Q_r\}_{r=1}^{\infty}$. As we will see below, this inequality is necessary for the two weight Hilbert transform inequality when $\sigma$ is doubling.

The reason for wanting the dual Poisson inequality (5-5) is that in Theorem 1.9 above, we can replace the assumption (1-17) on dual boundedness of the maximal operator $\mathcal{M}$ by the dual Poisson inequality (5-5). Indeed, this will be revealed by simple modifications of the proof of Theorem 1.9 above. In fact (5-5) can replace (1-17) in estimating term $II_s^t(2)$, as well as in the similar estimates for terms $V_s^t(2)$ and $VI_s^t(2)$. We turn now to the proofs of these assertions before addressing the question of sufficient conditions for the dual Poisson inequality (5-5).

**5.1.1.** *Sufficiency of the dual Poisson inequality.* We begin by demonstrating that the term $II_s^t(2)$ in (4-40) can be handled using the "dual" Poisson inequality (5-5) in place of the maximal inequality (1-17). We are working here in $\mathbb{R}^n$ with $1 < p < \infty$. In fact we claim that

$$\sum_{(t,s)\in\mathbb{L}^\alpha} II_s^t(2) \le C\gamma^{2p}\mathfrak{P}_*^p \int |f|^p \sigma, \tag{5-6}$$

where $\mathfrak{P}_*$ is the norm of the dual Poisson inequality (5-5) if we take $Q_0$ and its collection of pairwise disjoint subcubes $\{Q_r\}_{r=1}^{\infty}$ to be $G_s^{\alpha,t}$ and $\{G_r^{\alpha,t+1}\}_{r\in\mathcal{K}_s^{\alpha,t}}$. Now the maximal inequality (1-17) was used in the proof of (4-40) only in establishing (4-43), which says

$$\Big\| \chi_{G_s^{\alpha,t}} \sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}} \boldsymbol{P}_j^k(|g|\omega) \Big\|_{L^{p'}(\sigma)} \le C\mathfrak{M}_* \|\chi_{G_s^{\alpha,t}} g\|_{L^{p'}(\omega)},$$

where

$$\boldsymbol{P}_j^k(\mu) \equiv \sum_{r\in\mathcal{K}_s^{\alpha,t}} \boldsymbol{P}(G_r^{\alpha,t+1}, \chi_{E_j^k}\mu)\chi_{G_r^{\alpha,t+1}}.$$

We now note that

$$\begin{aligned}\sum_{(k,j)\in \mathbb{I}_s^{\alpha,t}} \boldsymbol{P}_j^k(|g|\omega) &= \sum_{(k,j)\in \mathbb{I}_s^{\alpha,t}} \sum_{r\in \mathcal{K}_s^{\alpha,t}} \boldsymbol{P}(G_r^{\alpha,t+1}, \chi_{E_j^k}|g|\omega)\chi_{G_r^{\alpha,t+1}} \\ &\le \sum_{r\in \mathcal{K}_s^{\alpha,t}} \boldsymbol{P}(G_r^{\alpha,t+1}, \chi_{G_s^{\alpha,t}}|g|\omega)\chi_{G_r^{\alpha,t+1}} = \mathbb{P}(\chi_{G_s^{\alpha,t}}|g|\omega)(x),\end{aligned}$$

which proves

$$\Big\|\chi_{G_s^{\alpha,t}} \sum_{k,j} \boldsymbol{P}_j^k(|g|\omega)\Big\|_{L^{p'}(\sigma)} \le C\mathfrak{P}_*\|\chi_{G_s^{\alpha,t}} g\|_{L^{p'}(\omega)},$$

which yields (5-6) as before.

The terms $V(2)$ and $VI(2)$ are handled similarly. Indeed, Claim 4.14 yields the following analogue of (4-73):

$$\sum_{(k,j)\in \mathbb{I}_s^{\alpha,t}} \boldsymbol{P}_j^k(\mu) \le C\chi_{G_s^{\alpha,t}}\mathbb{P}(\chi_{G_s^{\alpha,t}}\mu),$$

from which the arguments above yield both (4-71) and (4-78) with $\mathfrak{M}_*$ replaced by $\mathfrak{P}_*$.

**5.1.2.** *Sufficient conditions for Poisson inequalities.* We continue to work in $\mathbb{R}^n$ with $1 < p < \infty$. We note that (5-5) can be rewritten

$$\sum_{r=1}^{\infty} |Q_r|_\sigma \mathsf{P}(Q_r, f\omega)^{p'} \le C\int_{\mathbb{R}^n} f^{p'}\, d\omega \quad \text{for } f \ge 0,$$

and this latter inequality can then be expressed in terms of the Poisson operator $\mathbb{P}_+$ in the upper half space $\mathbb{R}_+^{n+1}$ given by

$$\mathbb{P}_+(f\omega)(x,t) = \int_{\mathbb{R}^n} P_t(x-y)f(y)\, d\omega(y).$$

Indeed, let $Z_r = (x_{Q_r}, \ell(Q_r))$ be the point in $\mathbb{R}_+^{n+1}$ that lies above the center $x_{Q_r}$ of $Q_r$ at a height equal to the side length $\ell(Q_r)$ of $Q_r$. Define an atomic measure $ds$ in $\mathbb{R}_+^{n+1}$ by

$$ds(x,t) = \sum_{r=1}^{\infty} |Q_r|_\sigma \delta_{Z_r}(x,t). \tag{5-7}$$

Then (5-5) is equivalent to the inequality (this is where we use $\delta(s) = s$),

$$\int_{\mathbb{R}_+^{n+1}} \mathbb{P}_+(f\omega)(x,t)^{p'}\, ds(x,t) \le C\int_{\mathbb{R}^n} f^{p'}\, d\omega(x) \quad \text{for } f \ge 0. \tag{5-8}$$

We can use [Sawyer 1988, Theorem 2] to characterize this latter inequality in terms of testing conditions over $\mathbb{P}_+$ and its dual $\mathbb{P}_+^*$ given by

$$\mathbb{P}_+^*(gw)(x,t) = \int_{\mathbb{R}_+^{n+1}} P_t(y-x)g(x,t)dw(x,t).$$

Let $\hat{Q}$ denote the cube in $\mathbb{R}_+^{n+1}$ with $Q$ as a face. Then [ibid., Theorem 2] yields the following.

**Theorem 5.2.** *The Poisson inequality* (5-5) *holds for given data* $Q_0$ *and* $\{Q_r\}_{r=1}^{\infty}$ if and only if *the measure* $s$ *in* (5-7) *satisfies*

$$\int_{\mathbb{R}_+^{n+1}} \mathbb{P}_+(\chi_Q\omega)^{p'}\,ds \le C\int_Q d\omega \qquad \text{for all cubes } Q\in\mathcal{D},$$
$$\int_{\mathbb{R}^n} \mathbb{P}_+^*(t^{p'-1}\chi_{\hat{Q}}\,ds)^p\,d\omega \le C\int_{\hat{Q}} t^{p'}\,ds \quad \text{for all cubes } Q\in\mathcal{D}.$$

Note that

$$\int_{\mathbb{R}_+^{n+1}} \mathbb{P}_+(\chi_Q\omega)^{p'}\,ds \approx \sum_{r=1}^{\infty}|Q_r|_\sigma \mathrm{P}(Q_r,\chi_Q\omega)^{p'}.$$

**Claim 5.3.** *Let* $n=1$ *and suppose that* $\sigma$ *is doubling. First assume that* $1<p<\infty$. *Then for the special measure* $s$ *in* (5-7), *inequality* (5-8) *follows from the dual pivotal condition* (5-2), *the Poisson condition* (1-24), *and the half-strengthened* $A_p$ *condition* (5-1). *Now assume that* $1<p\le 2$. *Then for the special measure* $s$ *in* (5-7), *inequality* (5-8) *follows from* (5-2) *and* (5-1) *without* (1-24).

With Claim 5.3 proved, the discussion above yields the following result.

**Theorem 5.4.** *Let* $n=1$ *and suppose that* $\sigma$ *is doubling. First assume that* $1<p<\infty$. *Then the dual Poisson inequality* (5-5) *holds uniformly in* $Q_0$ *and* $\{Q_r\}_{r=1}^{\infty}$ *satisfying* $\bigcup_{r=1}^{\infty} Q_r\subset Q_0$ *provided the half-strengthened* $A_p$ *condition* (5-1), *the dual pivotal condition* (5-2), *and the Poisson condition* (1-24) *all hold. Now assume that* $1<p\le 2$. *Then* (5-5) *holds uniformly in* $Q_0$ *and* $\{Q_r\}_{r=1}^{\infty}$ *satisfying* $\bigcup_{r=1}^{\infty} Q_r\subset Q_0$ *provided* (5-1) *and* (5-2) *both hold.*

**Remark 5.5.** We do not know if Claim 5.3 and Theorem 5.4 hold without the assumption that $\sigma$ is doubling, nor do we know if the Poisson condition (1-24) is implied by (5-1) and (5-2) when $p>2$.

We work exclusively in dimension $n=1$ from now on.

**5.5.1.** *Proof of Claim 5.3.* Instead of applying Theorem 5.2 directly, we first reduce matters to proving that certain $\mathcal{D}^\alpha$-dyadic analogues hold of the two conditions in Theorem 5.2. For $\alpha\in\{0,\frac{1}{3},\frac{2}{3}\}$ we use the following atomic measures $ds_\alpha$ on $\mathbb{R}_+^2$, along with the following $\mathcal{D}^\alpha$-dyadic analogues of the Poisson operators $\mathbb{P}$ and $\mathbb{P}_+$ (with $\delta(s)=s$),

$$\begin{aligned}\mathbb{P}_\alpha^{dy}\nu(x) &= \sum_{r=1}^{\infty}\mathrm{P}_\alpha^{dy}(I_r^\alpha,\nu)\chi_{I_r^\alpha}(x), \quad \mathbb{P}_{+,\alpha}^{dy}\nu(x,t) = \sum_{\substack{Q\in\mathcal{D}^\alpha\\ x\in Q \text{ and } \ell(Q)\ge t}} \frac{t}{\ell(Q)}\frac{1}{|Q|}\int_Q d\nu,\\ ds_\alpha(x,t) &= \sum_{r=1}^{\infty}|I_r^\alpha|_\sigma\delta_{Z_r^\alpha}(x,t),\end{aligned} \tag{5-9}$$

where

(1) the interval $I_r^\alpha$ is chosen to be a *maximal* $\mathcal{D}^\alpha$-interval contained in $Q_r$ with *maximum* length (there can be at most two such intervals, in which case we choose the leftmost one),

(2) the $\mathcal{D}^\alpha$-Poisson integral $\mathsf{P}_\alpha^{dy}(Q,\nu)$ is given by

$$\mathsf{P}_\alpha^{dy}(Q,\nu)\equiv\sum_{\ell=0}^{\infty}\frac{2^{-\ell}}{|Q^{(\ell)}|}\int_{Q^{(\ell)}}d\nu\quad\text{for } Q\in\mathcal{D}^\alpha,$$

where $Q^{(\ell)}$ denotes the $\ell$-th dyadic parent of $Q$ in $\mathcal{D}^\alpha$, and

(3) the point $Z_r^\alpha=(x_{I_r^\alpha},\ell(I_r^\alpha))$ in $\mathbb{R}_+^2$ lies above the center $x_{I_r^\alpha}$ of $I_r^\alpha$ at a height equal to the side length $\ell(I_r^\alpha)$ of $I_r^\alpha$.

We will use the following dyadic analogue of Theorem 5.2, whose proof is the obvious dyadic analogue of the proof of Theorem 5.2 as given in [Sawyer 1988].

**Theorem 5.6.** *The $\mathcal{D}^\alpha$-Poisson inequality*

$$\int_{\mathbb{R}_+^2}\mathbb{P}_{+,\alpha}^{dy}(f\omega)^{p'}\,ds_\alpha\le C\int_Q f^{p'}\,d\omega\quad\text{for } f\ge 0,$$

*holds if and only if*

$$\begin{aligned}\int_{\mathbb{R}_+^2}\mathbb{P}_{+,\alpha}^{dy}(\chi_Q\omega)^{p'}\,ds_\alpha&\le C\int_Q d\omega &&\text{for all intervals } Q\in\mathcal{D}^\alpha,\\ \int_{\mathbb{R}}(\mathbb{P}_{+,\alpha}^{dy})^*(t^{p'-1}\chi_{\hat{Q}}\,ds_\alpha)^p\,d\omega&\le C\int_{\hat{Q}}t^{p'}\,ds_\alpha &&\text{for all intervals } Q\in\mathcal{D}^\alpha.\end{aligned}\tag{5-10}$$

We claim that for any positive measure $\nu$, the set of shifted dyadic grids $\{\mathcal{D}^\alpha\}_{\alpha\in\{0,1/3,2/3\}}$ satisfies

$$\mathsf{P}(Q_r,\nu)=\sum_{\ell=0}^{\infty}\frac{2^{-\ell}}{|2^\ell Q_r|}\int_{2^\ell Q_r}d\nu\approx\sum_{\alpha\in\{0,1/3,2/3\}}\sum_{\ell=0}^{\infty}\frac{2^{-\ell}}{|(I_r^\alpha)^{(\ell)}|}\int_{(I_r^\alpha)^{(\ell)}}d\nu=\sum_{\alpha\in\{0,1/3,2/3\}}\mathsf{P}_\alpha^{dy}(I_r^\alpha,\nu)$$

for all $r$. Indeed, for each interval $2^\ell Q_r$, there is $\alpha\in\{0,1/3,2/3\}$ and an interval $Q\in\mathcal{D}^\alpha$ containing $2^\ell Q_r$ whose length is comparable to that of $2^\ell Q_r$. Thus $Q=(I_r^\alpha)^{(\ell+c)}$ for some universal positive integer $c$. Now

$$\mathbb{P}_+(\nu)(x_{Q_r},\ell(Q_r))=\int_{\mathbb{R}}P_{\ell(Q_r)}(x_{Q_r}-y)d\nu(y)\approx\sum_{\ell=0}^{\infty}2^{-\ell}\frac{1}{|2^\ell Q_r|}\int_{2^\ell Q_r}d\nu=\mathsf{P}(Q_r,\nu).$$

Since $\sigma$ is *doubling* and $I_r^\alpha$ is a maximal $\mathcal{D}^\alpha$-interval in $Q_r$ with maximum length, we have $|Q_r|_\sigma\lesssim|I_r^\alpha|_\sigma$ and

$$\begin{aligned}\int_{\mathbb{R}_+^{n+1}}\mathbb{P}_+\nu(x,t)^p\,ds&=\sum_{r=1}^{\infty}|Q_r|_\sigma\mathbb{P}_+\nu(x_{Q_r},\ell(Q_r))^p\approx\sum_{r=1}^{\infty}|Q_r|_\sigma\mathsf{P}(Q_r,\nu)^p\\&\approx\sum_{\alpha\in\{0,1/3,2/3\}}\sum_{r=1}^{\infty}|I_r^\alpha|_\sigma\mathsf{P}_\alpha^{dy}(I_r^\alpha,\nu)^p=\sum_{\alpha\in\{0,1/3,2/3\}}\int_{\mathbb{R}_+^2}\mathbb{P}_{+,\alpha}^{dy}\nu(x,t)^p\,ds_\alpha.\end{aligned}$$

This together with Theorem 5.6 reduces the proof of Claim 5.3 to showing that (5-10) holds for all $\alpha\in\{0,1/3,2/3\}$.

Now the definition of $s_\alpha$ in (5-9) shows that the left side of the first line in (5-10) is

$$\int_{\mathbb{R}^2_+} \mathbb{P}^{dy}_{+,\alpha}(\chi_Q\omega)^{p'}\, ds_\alpha = \sum_{r=1}^{\infty} |I^\alpha_r|_\sigma \mathsf{P}^{dy}_\alpha (I^\alpha_r, \chi_Q\omega)^{p'}.$$

Recall that $I^\alpha_r, Q \in \mathcal{D}^\alpha$. Now if $Q \subset I^\alpha_r$ for some $r$, then the sum above consists of just one term that satisfies

$$|I^\alpha_r|_\sigma \mathsf{P}^{dy}_\alpha (I^\alpha_r, \chi_Q\omega)^{p'} \le C\frac{|I^\alpha_r|_\sigma |Q|^{p'-1}_\omega}{|I^\alpha_r|^{p'}}|Q|_\omega \le C\mathcal{A}_p(\omega,\sigma)^{p'}|Q|_\omega.$$

Otherwise we have

$$\begin{aligned}\int_{\mathbb{R}^2_+} \mathbb{P}^{dy}_{+,\alpha}(\chi_Q\omega)^{p'}\, ds_\alpha &\lesssim \sum_{I^\alpha_r \subset Q} |I^\alpha_r|_\sigma \mathsf{P}^{dy}_\alpha (I^\alpha_r, \chi_Q\omega)^{p'} + \sum_{I^\alpha_r \cap Q = \varnothing} |I^\alpha_r|_\sigma \mathsf{P}^{dy}_\alpha (I^\alpha_r, \chi_Q\omega)^{p'} \\ &\le \mathfrak{C}^{p'}_* \int_Q d\omega + \sum_{I^\alpha_r \cap Q = \varnothing} |I^\alpha_r|_\sigma \Big(\sum_{\ell=0}^{\infty} \frac{2^{-\ell}}{|(I^\alpha_r)^{(\ell)}|}\int_{Q\cap (I^\alpha_r)^{(\ell)}} d\omega\Big)^{p'},\end{aligned}$$

where the local term has been estimated by the dual pivotal condition (5-2) applied to $Q$.

Now if $I^\alpha_r \subset Q^{(m)} \setminus Q^{(m-1)}$, then $Q \cap Q^{(\ell)}_r \ne \varnothing$ only if $Q^{(m)} \subset (I^\alpha_r)^{(\ell)}$. Thus the second term on the right can be estimated by

$$\begin{aligned}&\sum_{m=1}^{\infty} \sum_{I^\alpha_r \subset Q^{(m)}\setminus Q^{(m-1)}} |I^\alpha_r|_\sigma \Big(\sum_{\ell=0}^{\infty} \frac{2^{-\ell}}{|(I^\alpha_r)^{(\ell)}|}\int_{Q\cap (I^\alpha_r)^{(\ell)}} d\omega\Big)^{p'} \\ &\quad\le \sum_{m=1}^{\infty} \sum_{I^\alpha_r \subset Q^{(m)}\setminus Q^{(m-1)}} |I^\alpha_r|_\sigma \sum_{\ell=0}^{\infty} 2^{-\ell}\bigg(\frac{\int_{Q\cap (I^\alpha_r)^{(\ell)}} d\omega}{|(I^\alpha_r)^{(\ell)}|}\bigg)^{p'} \\ &\quad\le C\sum_{m=1}^{\infty} \sum_{I^\alpha_r \subset Q^{(m)}\setminus Q^{(m-1)}} |I^\alpha_r|_\sigma \sum_{\ell=0}^{\infty} 2^{-\ell}\bigg(\frac{\int_Q d\omega}{|Q^{(m)}|}\bigg)^{p'} \\ &\quad\le \Big(\sum_{m=1}^{\infty}\frac{|Q^{(m)}|_\sigma}{|Q^{(m)}|^{p'}}\Big)|Q|^{p'-1}_\omega \int_Q d\omega \\ &\quad= \Big(\frac{1}{|Q|^{p'}}\Big(\int s^{dy}_{Q,\alpha}(x)^{p'}\, d\sigma(x)\Big)|Q|^{p'-1}_\omega\Big)\int_Q d\omega \le C\mathcal{A}_p(\omega,\sigma)^{p'}\int_Q d\omega,\end{aligned}$$

where we have used

$$s^{dy}_{Q,\alpha}(x) \equiv \sum_{m=0}^{\infty} \frac{|Q|}{|Q^{(m)}|}\chi_{Q^{(m)}}(x) \lesssim s_Q(x),$$

and the half-strengthened $A_p$ condition (5-1) in the final inequality.

Now we turn to showing that the second line in (5-10) holds using only the $A_p$ condition (1-8). First we compute the dual operator $(\mathbb{P}^{dy}_{+,\alpha})^*$. Since the kernel of $\mathbb{P}^{dy}_{+,\alpha}$ is

$$\mathbb{P}^{dy}_{+,\alpha}[(x,t),y] \equiv \sum_{I\in\mathcal{D}^\alpha : \ell(I)\ge t} \chi_I(x)\frac{t}{\ell(I)}\frac{1}{|I|}\chi_I(y),$$

we have for any positive measure $\mu(x,t)$ on the upper half space $\mathbb{R}^2_+$,

$$(\mathbb{P}^{dy}_{+,\alpha})^*\mu(y)=\int_{\mathbb{R}^2_+}\Big(\sum_{I\in\mathcal{D}^\alpha:\ell(I)\geq t}\chi_I(x)\frac{t}{\ell(I)}\frac{1}{|I|}\chi_I(y)\Big)d\mu(x,t)=\sum_{I\in\mathcal{D}^\alpha:y\in I}\frac{1}{|I|}\int_{\hat{I}}\frac{t}{\ell(I)}d\mu(x,t).$$

Using the third line in (5-9) we compute that

$$\int_{\hat{Q}}t^{p'}\,ds_\alpha=\sum_{I^\alpha_r\subset Q}|I^\alpha_r|_\sigma|I^\alpha_r|^{p'},$$

and

$$\begin{aligned}(\mathbb{P}^{dy}_{+,\alpha})^*(t^{p'-1}\chi_{\hat{Q}}\,ds_\alpha)(y)&=\sum_{I\in\mathcal{D}^\alpha:y\in I}\frac{1}{|I|}\int_{\hat{I}\cap\hat{Q}}\frac{t}{\ell(I)}t^{p'-1}\,ds_\alpha(x,t)\\&=\sum_{I^\alpha_r\subset Q}|I^\alpha_r|_\sigma|I^\alpha_r|^{p'-1}\sum_{\ell=0}^\infty\frac{2^{-\ell}}{|(I^\alpha_r)^{(\ell)}|}\chi_{(I^\alpha_r)^{(\ell)}}(y).\end{aligned}$$

Thus we must prove

$$\int_{\mathbb{R}}\Big(\sum_{I^\alpha_r\subset Q}|I^\alpha_r|_\sigma|I^\alpha_r|^{p'-1}\sum_{\ell=0}^\infty\frac{2^{-\ell}}{|(I^\alpha_r)^{(\ell)}|}\chi_{(I^\alpha_r)^{(\ell)}}(y)\Big)^p\,d\omega(y)\leq C\mathcal{A}_p(\omega,\sigma)^p\sum_{I^\alpha_r\subset Q}|I^\alpha_r|_\sigma|I^\alpha_r|^{p'};\tag{5-11}$$

but this is the Poisson condition (1-24) in Theorem 1.10 for the shifted dyadic grid $\mathcal{D}^\alpha$. This completes the proof of the first assertion in Claim 5.3 regarding the case $1<p<\infty$. We now assume that $1<p\leq 2$ for the remainder of the proof.

To obtain (5-11) it suffices to show that for each $\ell\geq 0$

$$\int_{\mathbb{R}}\Big(\sum_{I^\alpha_r\subset Q}|I^\alpha_r|_\sigma|I^\alpha_r|^{p'-2}2^{-2\ell}\chi_{(I^\alpha_r)^{(\ell)}}(y)\Big)^p\,d\omega(y)\leq C2^{-p\ell}\mathcal{A}_p(\omega,\sigma)^p\sum_{I^\alpha_r\subset Q}|I^\alpha_r|_\sigma|I^\alpha_r|^{p'}.\tag{5-12}$$

Indeed, with this in hand, Minkowski's inequality yields

$$\begin{aligned}\|(\mathbb{P}^{dy}_{+,\alpha})^*(t\chi_{\hat{Q}}\,ds_\alpha)\|_{L^p(\omega)}&=\Big\|\sum_{\ell=0}^\infty\sum_{I^\alpha_r\subset Q}|I^\alpha_r|_\sigma|I^\alpha_r|^{p'-2}2^{-2\ell}\chi_{(I^\alpha_r)^{(\ell)}}\Big\|_{L^p(\omega)}\\&\leq\sum_{\ell=0}^\infty\Big\|\sum_{I^\alpha_r\subset Q}|I^\alpha_r|_\sigma|I^\alpha_r|^{p'-2}2^{-2\ell}\chi_{(I^\alpha_r)^{(\ell)}}\Big\|_{L^p(\omega)}\\&\leq C\sum_{\ell=0}^\infty 2^{-\ell}\mathcal{A}_p(\omega,\sigma)\Big(\sum_{I^\alpha_r\subset Q}|I^\alpha_r|_\sigma|I^\alpha_r|^{p'}\Big)^{1/p},\end{aligned}\tag{5-13}$$

as required.

Note that for $a>0$ and $p>1$,

$$h(x)\equiv(a+x)^p-a^p-p(a+x)^{p-1}x,$$

is decreasing on $[0,\infty)$ since $h'(x)=-p(p-1)(a+x)^{p-2}x<0$ for $x>0$. Since $h(0)=0$ we have $h(x)\le 0$ for $x\ge 0$, that is,

$$(a+x)^p-a^p\le p(a+x)^{p-1}x \quad \text{for } a,x>0 \text{ and } p>1. \tag{5-14}$$

Now fix an interval $Q$ in (5-12) and arrange the intervals $I_r^\alpha$ that are contained in $Q$ into a sequence $\{I_r^\alpha\}_{r=1}^N$ in which the lengths $|I_r^\alpha|$ are increasing (we may suppose without loss of generality that $N$ is finite). Recall we are now assuming $1<p\le 2$. Integrate by parts and apply (5-14) to estimate the left side of (5-12) by

$$\begin{aligned}
&2^{-2p\ell}\int_{\mathbb{R}}\Big(\sum_{r=1}^{N}|I_r^\alpha|_\sigma|I_r^\alpha|^{p'-2}\chi_{(I_r^\alpha)^{(\ell)}}(y)\Big)^p\,d\omega(y)\\
&\quad=2^{-2p\ell}\int_{\mathbb{R}}\sum_{n=1}^{N}\Big(\Big(\sum_{r=1}^{n}|I_r^\alpha|_\sigma|I_r^\alpha|^{p'-2}\chi_{(I_r^\alpha)^{(\ell)}}(y)\Big)^p-\Big(\sum_{r=1}^{n-1}|I_r^\alpha|_\sigma|I_r^\alpha|^{p'-2}\chi_{(I_r^\alpha)^{(\ell)}}(y)\Big)^p\Big)\,d\omega(y)\\
&\quad\le 2^{-2p\ell}\int_{\mathbb{R}}\sum_{n=1}^{N}\Big(p\Big(\sum_{r=1}^{n}|I_r^\alpha|_\sigma|I_r^\alpha|^{p'-2}\chi_{(I_r^\alpha)^{(\ell)}}(y)\Big)^{p-1}|I_n^\alpha|_\sigma|I_n^\alpha|^{p'-2}\chi_{(I_n^\alpha)^{(\ell)}}(y)\Big)\,d\omega(y)\\
&\quad\le 2^{-2p\ell}p\sum_{n=1}^{N}\int_{\mathbb{R}}\Big(\Big(\sum_{r=1}^{n}|I_r^\alpha|_\sigma\chi_{(I_r^\alpha)^{(\ell)}}(y)\Big)^{p-1}|I_n^\alpha|_\sigma|I_n^\alpha|^{p'-2}|I_n^\alpha|^{(p'-2)(p-1)}\chi_{(I_n^\alpha)^{(\ell)}}(y)\Big)\,d\omega(y),
\end{aligned}$$

where we have used (5-14) with

$$a=\sum_{r=1}^{n-1}|I_r^\alpha|_\sigma|I_r^\alpha|^{p'-2}\chi_{(I_r^\alpha)^{(\ell)}}(y) \quad\text{and}\quad x=|I_n^\alpha|_\sigma|I_n^\alpha|^{p'-2}\chi_{(I_n^\alpha)^{(\ell)}}(y),$$

and then used $|I_r^\alpha|^{p'-2}\le|I_n^\alpha|^{p'-2}$ for $1\le r\le n$, which follows from $|I_r^\alpha|\le|I_n^\alpha|$ and $p'\ge 2$. If $(I_r^\alpha)^{(\ell)}\cap(I_n^\alpha)^{(\ell)}\neq\varnothing$ and $1\le r\le n$, then $I_r^\alpha\subset(I_n^\alpha)^{(\ell)}$ and so

$$\begin{aligned}
&\int_{\mathbb{R}}\Big(\sum_{I_r^\alpha\subset Q}|I_r^\alpha|_\sigma|I_r^\alpha|^{p'-2}2^{-2\ell}\chi_{(I_r^\alpha)^{(\ell)}}(y)\Big)^p\,d\omega(y)\\
&\qquad\le 2^{-2p\ell}p\sum_{n=1}^{N}|I_n^\alpha|_\sigma|I_n^\alpha|^{p'p-2p}\int_{\mathbb{R}}\Big(\sum^{n}_{\substack{1\le r\le n\\ I_r^\alpha\subset(I_n^\alpha)^{(\ell)}}}|I_r^\alpha|_\sigma\Big)^{p-1}\chi_{(I_n^\alpha)^{(\ell)}}(y)\,d\omega(y)\\
&\qquad\le 2^{-2p\ell}p\sum_{n=1}^{N}|I_n^\alpha|_\sigma|I_n^\alpha|^{p'p-2p}|(I_n^\alpha)^{(\ell)}|_\sigma^{p-1}|(I_n^\alpha)^{(\ell)}|_\omega\\
&\qquad\le 2^{-2p\ell}p\mathcal{A}_p(\omega,\sigma)^p\sum_{n=1}^{N}|I_n^\alpha|_\sigma|I_n^\alpha|^{p'p-2p}|(I_n^\alpha)^{(\ell)}|^p\\
&\qquad=2^{-p\ell}p\mathcal{A}_p(\omega,\sigma)^p\sum_{n=1}^{N}|I_n^\alpha|_\sigma|I_n^\alpha|^{p'}=2^{-p\ell}p\mathcal{A}_p(\omega,\sigma)^p\sum_{I_r^\alpha\subset Q}|I_n^\alpha|_\sigma|I_n^\alpha|^{p'}.
\end{aligned}$$

Thus we have proved (5-12) for $p \in (1, 2]$, which completes the proof of (5-10). This finishes the proof of Claim 5.3, and hence also that of Theorem 5.4.

**5.7.** ***Necessity of the conditions.*** Here we consider the two weight Hilbert transform inequality for $1 < p < \infty$. We show the necessity of the strengthened $A_p$ condition for general weights, as well as the necessity of the dual pivotal condition for the dual testing condition, and the dual Poisson inequality for the dual Hilbert transform inequality, when $\sigma$ is doubling.

**5.7.1.** *The strengthened $A_p$ condition.* Here we derive a necessary condition for the weighted inequality (1-18) but with the Hilbert transform $T$ in place of $T_\natural$, that is,

$$\int_{\mathbb{R}\setminus \operatorname{supp} f} T(f\sigma)(x)^p \, d\omega(x) \le C \int_{\mathbb{R}^n} |f(x)|^p \, d\sigma(x). \tag{5-15}$$

The condition,

$$\left(\int_{\mathbb{R}} \left(\frac{|Q|}{|Q|+|x-x_Q|}\right)^p d\omega(x)\right)^{1/p} \left(\int_{\mathbb{R}} \left(\frac{|Q|}{|Q|+|x-x_Q|}\right)^{p'} d\sigma(x)\right)^{1/p'} \le C|Q| \tag{5-16}$$

for all intervals $Q$, is stronger than the two weight $A_p$ condition (1-8), and we call it the *strengthened* $A_p$ condition.

Preliminary results in this direction were obtained by Muckenhoupt and Wheeden, and in the setting of fractional integrals by Gabidzashvili and Kokilashvili, and here we follow the argument proving [Sawyer and Wheeden 1992, (1.9)], where “two-tailed” inequalities of the type (5-16) originated in the fractional integral setting. A somewhat different approach to this for the conjugate operator in the disk when $p = 2$ uses conformal invariance and appears in [Nazarov et al. 2010], and provides the first instance of a strengthened $A_2$ condition being proved necessary for a two weight inequality for a singular integral.

Fix an interval $Q$ and for $a \in \mathbb{R}$ and $r > 0$, let

$$s_Q(x) = \frac{|Q|}{|Q|+|x-x_Q|} \quad \text{and} \quad f_{a,r}(y) = \chi_{(a-r,a)}(y) s_Q(y)^{p'-1},$$

where $x_Q$ is the center of the interval $Q$. For convenience we assume that neither $\omega$ nor $\sigma$ have any point masses — see [Lacey et al. 2011] for the modifications necessary when point masses are present. For $y < x$ we have

$$|Q|(x-y) = |Q|(x-x_Q) + |Q|(x_Q - y) \le (|Q|+|x-x_Q|)(|Q|+|x_Q-y|),$$

and so

$$\frac{1}{x-y} \ge |Q|^{-1} s_Q(x) s_Q(y) \quad \text{for } y < x.$$

Thus for $x > a$ we obtain that

$$H(f_{a,r}\sigma)(x) = \int_{a-r}^{a} \frac{1}{x-y} s_Q(y)^{p'-1} \, d\sigma(y) \ge |Q|^{-1} s_Q(x) \int_{a-r}^{a} s_Q(y)^{p'} \, d\sigma(y),$$

and hence by (5-15) for the Hilbert transform $H$,

$$|Q|^{-p}\int_a^\infty s_Q(x)^p\Big(\int_{a-r}^a s_Q(y)^{p'}\,d\sigma(y)\Big)^p\,d\omega(x) \\ \le \int |H(f_{a,r}\sigma)(x)|^p\,d\omega(x) \le C\int |f_{a,r}(y)|^p\,d\sigma(y) = C\int_{a-r}^a s_Q(y)^{p'}\,d\sigma(y).$$

From this we obtain

$$|Q|^{-p}\Big(\int_a^\infty s_Q(x)^p\,d\omega(x)\Big)\Big(\int_{a-r}^a s_Q(y)^{p'}\,d\sigma(y)\Big)^{p-1} \le C,$$

and upon letting $r\to\infty$ and taking $p$-th roots, we get

$$\Big(\int_a^\infty s_Q(x)^p\,d\omega(x)\Big)^{1/p}\Big(\int_{-\infty}^a s_Q(y)^{p'}\,d\sigma(y)\Big)^{1/p'} \le C|Q|.$$

Similarly we have

$$\Big(\int_{-\infty}^a s_Q(x)^p\,d\omega(x)\Big)^{1/p}\Big(\int_a^\infty s_Q(y)^{p'}\,d\sigma(y)\Big)^{1/p'} \le C|Q|.$$

Now we choose $a$ so that

$$\int_{-\infty}^a s_Q(y)^{p'}\,d\sigma(y) = \int_a^\infty s_Q(y)^{p'}\,d\sigma(y) = \frac{1}{2}\int s_Q(y)^{p'}\,d\sigma(y),$$

and conclude that

$$\begin{aligned}
&\Big(\int s_Q(x)^p\,d\omega(x)\Big)^{1/p}\Big(\int s_Q(y)^{p'}d\sigma(y)\Big)^{1/p'} \\
&\le \Big(\int_{-\infty}^a s_Q(x)^p\,d\omega(x)\Big)^{1/p}\Big(\int s_Q(y)^{p'}d\sigma(y)\Big)^{1/p'} + \Big(\int_a^\infty s_Q(x)^p\,d\omega(x)\Big)^{1/p}\Big(\int s_Q(y)^{p'}d\sigma(y)\Big)^{1/p'} \\
&\le 2^{1/p'}\Big(\int_{-\infty}^a s_Q(x)^p\,d\omega(x)\Big)^{1/p}\Big(\int_a^\infty s_Q(y)^{p'}d\sigma(y)\Big)^{1/p'} \\
&\qquad + 2^{1/p'}\Big(\int_a^\infty s_Q(x)^p\,d\omega(x)\Big)^{1/p}\Big(\int_{-\infty}^a s_Q(y)^{p'}d\sigma(y)\Big)^{1/p'} \\
&\le 2^{1+1/p'}C|Q|.
\end{aligned}$$

**5.7.2.** *Necessity of the dual pivotal condition and the dual Poisson inequality for a doubling measure.* Here we show first that if $\sigma$ is a *doubling* measure, then the dual pivotal condition (5-2) with $\delta(s)=s$ is implied by the $A_p$ condition (1-8) and the dual testing condition for the Hilbert transform $H$, that is,

$$\int_I |H(\chi_I\omega)(x)|^{p'}\,d\sigma(x) \le C_{\omega,\sigma,p}|I|_\omega \quad \text{for all intervals } I. \tag{5-17}$$

After this we show that the dual Poisson inequality (5-5) is implied by the $A_p$ condition (1-8) and the dual Hilbert transform inequality,

$$\int_I |H(\chi_I g\omega)(x)|^{p'}\,d\sigma(x) \le C_{\omega,\sigma,p}\int_I g(x)^{p'}\,d\omega(x) \quad \text{for all } g\ge 0 \text{ and intervals } I. \tag{5-18}$$

**Lemma 5.8.** *Suppose that $\sigma$ is doubling and $T = H$ is the Hilbert transform. Then the dual pivotal condition* (5-2) *is implied by the $A_p$ condition* (1-8) *and the dual testing condition* (5-17).

*Proof.* We begin by proving that for any interval $I$ and any positive measure $\nu$ supported in $\mathbb{R} \setminus I$, we have

$$\mathbb{P}(I;\nu) \le \frac{1}{|I|}\int_I d\nu + 2|I| \inf_{x,y\in I} \frac{H(\chi_{I^c}\nu)(x) - H(\chi_{I^c}\nu)(y)}{x-y}, \tag{5-19}$$

where we here redefine

$$\mathbb{P}(I;\nu) \equiv \frac{1}{|I|}\int_I d\nu + \frac{|I|}{2}\int_{\mathbb{R}\setminus I} \frac{1}{|z-z_I|^2}d\nu(z), \tag{5-20}$$

with $z_I$ the center of $I$. Note that this definition of $\mathbb{P}(I;\nu)$ is comparable to that in (5-3) with $\delta(s) = s$. Note also that $H(\chi_{I^c}\nu)$ is defined by (5-15) on $I$, and increasing on $I$ when $\nu$ is positive, so that the infimum in (5-19) is nonnegative.

To see (5-19), we suppose without loss of generality that $I = (-a, a)$, and a calculation then shows that for $-a \le x < y \le a$,

$$\begin{aligned} &H(\chi_{I^c}\nu)(y) - H(\chi_{I^c}\nu)(x) \\ &\quad = \int_{\mathbb{R}\setminus I}\Big(\frac{1}{z-y} - \frac{1}{z-x}\Big)d\nu(z) = (y-x)\int_{\mathbb{R}\setminus I}\frac{1}{(z-y)(z-x)}d\nu(z) \ge \tfrac14 (y-x)\int_{\mathbb{R}\setminus I}\frac{1}{z^2}d\nu(z), \end{aligned}$$

since $((z-y)(z-x))^{-1}$ is positive and satisfies

$$\frac{1}{(z-y)(z-x)} \ge \frac{1}{4z^2}$$

on each interval $(-\infty, -a)$ and $(a, \infty)$ in $\mathbb{R} \setminus I$ when $-a \le x < y \le a$. Thus we have from (5-20)

$$\mathbb{P}(I;\nu) = \frac{1}{|I|}\int_I d\nu + \frac{|I|}{2}\int_{\mathbb{R}\setminus I}\frac{1}{z^2}d\nu(z) \le \frac{1}{|I|}\int_I d\nu + 2|I| \inf_{x,y\in I}\frac{H(\chi_{I^c}\nu)(y) - H(\chi_{I^c}\nu)(x)}{y-x}.$$

Now we return to the dual pivotal condition (5-2), and let $C_{\omega,\sigma,p}$ be the best constant in the dual testing condition (5-17) for $H$. Let $Q_0 = \bigcup_{r=1}^{\infty} Q_r$ be a pairwise disjoint decomposition of $Q_0$ and consider $\varepsilon, \delta > 0$, which will be chosen at the end of the proof (we will take $\delta = \frac12$ and $\varepsilon > 0$ very small). For each interval $Q_r$, let $\alpha_r \in Q_r$ minimize $|H(\chi_{Q_r^c}\omega)|$ on $Q_r$, that is,

$$|H(\chi_{Q_r^c}\omega)(\alpha_r)| = \min_{x\in I}|H(\chi_{Q_r^c}\omega)(x)|,$$

and set

$$J_{r,\varepsilon} \equiv (\alpha_r - \varepsilon|Q_r|, \alpha_r + \varepsilon|Q_r|) \cap Q_r.$$

Now for each interval $Q_r$, consider the following three mutually exclusive and exhaustive cases:

Case 1: $\displaystyle \frac{1}{|Q_r|}\int_{Q_r} d\omega > \frac{|Q_r|}{4}\int_{\mathbb{R}\setminus Q_r}\frac{1}{|z-z_{Q_r}|^2}\,d\omega(z),$

Case 2: $\displaystyle \frac{1}{|Q_r|}\int_{Q_r} d\omega \le \frac{|Q_r|}{4}\int_{\mathbb{R}\setminus Q_r}\frac{1}{|z-z_{Q_r}|^2}\,d\omega(z) \quad \text{and} \quad |Q_r \setminus J_{r,\varepsilon}|_\sigma \ge \delta|Q_r|_\sigma,$

Case 3: $\dfrac{1}{|Q_r|}\displaystyle\int_{Q_r} d\omega \le \frac{|Q_r|}{4}\int_{\mathbb{R}\setminus Q_r} \frac{1}{|z-z_{Q_r}|^2}\,d\omega(z)$ and $|J_{r,\varepsilon}|_\sigma > (1-\delta)|Q_r|_\sigma$.

If $Q_r$ is a Case 1 interval we have $\mathbb{P}(Q_r, \chi_{Q_0}\omega) \le 3|Q_r|^{-1}\int_{Q_r} d\omega$ and so

$$\sum_{Q_r \text{ satisfies Case 1}} |Q_r|_\sigma \mathbb{P}(Q_r, \chi_{Q_0}\omega)^{p'} \le 3^p \sum_{r=1}^{\infty} |Q_r|_\sigma \Big(\frac{1}{|Q_r|}\int_{Q_r} d\omega\Big)^{p'}$$
$$\le C_p \sum_{r=1}^{\infty} \frac{|Q_r|_\sigma |Q_r|_\omega^{p'-1}}{|Q_r|^{p'}} \int_{Q_r} d\omega \le C_p \|(\omega,\sigma)\|_{A_p}^{p'} \int_{Q_0} d\omega.$$

If $Q_r$ is a Case 2 or Case 3 interval we have from (5-19) with $\nu = \chi_{Q_0}\omega$ that for all $x \in Q_r \setminus J_{r,\varepsilon}$,

$$\mathbb{P}(Q_r; \chi_{Q_0}\omega) \le 6|Q_r| \frac{H(\chi_{Q_0\cap Q_r^c}\omega)(x) - H(\chi_{Q_0\cap Q_r^c}\omega)(\alpha_r)}{x-\alpha_r}$$
$$\le 6|Q_r| \frac{1}{\varepsilon|Q_r|}\big(|H(\chi_{Q_0\cap Q_r^c}\omega)(x)| + |H(\chi_{Q_0\cap Q_r^c}\omega)(\alpha_r)|\big) \le \frac{12}{\varepsilon}|H(\chi_{Q_0\cap Q_r^c}\omega)(x)|.$$

If now $Q_r$ is a Case 2 interval, we also have $|Q_r|_\sigma \le \delta^{-1}|Q_r \setminus J_{r,\varepsilon}|_\sigma$ and so

$$\sum_{Q_r \text{ satisfies Case 2}} |Q_r|_\sigma \mathbb{P}(Q_r, \chi_{Q_0}\omega)^{p'}$$
$$\le \frac{1}{\delta} \sum_{Q_r \text{ satisfies Case 2}} |Q_r \setminus J_{r,\varepsilon}|_\sigma \mathbb{P}(Q_r, \chi_{Q_0}\omega)^{p'}$$
$$\le \frac{1}{\delta}\sum_{r=1}^{\infty}\Big(\frac{12}{\varepsilon}\Big)^{p'} \int_{Q_r\setminus J_{r,\varepsilon}} |H(\chi_{Q_0\cap Q_r^c}\omega)(x)|^{p'}\,d\sigma(x)$$
$$\le C_{\varepsilon,\delta,p} \sum_{r=1}^{\infty} \int_{Q_r\setminus J_{r,\varepsilon}} \big(|H(\chi_{Q_0}\omega)(x)|^{p'} + |H(\chi_{Q_r}\omega)(x)|^{p'}\big)\,d\sigma(x)$$
$$\le C_{\varepsilon,\delta,p}\Big(\int_{Q_0} |H(\chi_{Q_0}\omega)(x)|^{p'}\,d\sigma(x) + \sum_{r=1}^{\infty}\int_{Q_r} |H(\chi_{Q_r}\omega)(x)|^{p'}\,d\sigma(x)\Big)$$
$$\le C_{\varepsilon,\delta,p}\Big(C|Q_0|_\omega + \sum_{r=1}^{\infty} C|Q_r|_\omega\Big) = C_{\varepsilon,\delta,p}|Q_0|_\omega,$$

where the final inequality follows from (5-17) with $I = Q_0$ and then $I = Q_r$.

Now we use our assumption that $\sigma$ is doubling. There are $C, \eta > 0$ such that

$$|J|_\sigma \le C\Big(\frac{|J|}{|Q|}\Big)^\eta |Q|_\sigma$$

whenever $J$ is a subinterval of an interval $Q$. If $Q_r$ is a Case 3 interval we have both

$$\frac{|J_{r,\varepsilon}|}{|Q_r|} \le 2\varepsilon \quad \text{and} \quad |J_{r,\varepsilon}|_\sigma > (1-\delta)|Q_r|_\sigma,$$

which altogether yields

$$(1-\delta)|Q_r|_\sigma < |J_{r,\varepsilon}|_\sigma \le C\Big(\frac{|J_{r,\varepsilon}|}{|Q_r|}\Big)^\eta |Q_r|_\sigma \le C(2\varepsilon)^\eta |Q_r|_\sigma,$$

which is a contradiction if $\delta = 1/2$ and $\varepsilon > 0$ is chosen sufficiently small, so that $\varepsilon < 1/2(1/(2C))^{1/\eta}$. With this choice, there are no Case 3 intervals, and so we are done. □

**Lemma 5.9.** *Suppose that $\sigma$ is doubling and $T = H$ is the Hilbert transform. Then the dual Poisson inequality* (5-5) *is implied by the $A_p$ condition* (1-8) *and the dual Hilbert transform inequality* (5-18).

*Proof.* The proof is virtually identical to that of Lemma 5.8 but with $d\nu = \chi_{Q_0} g\, d\omega$ in place of $\chi_{Q_0}\, d\omega$ where $g \ge 0$. Indeed, if $Q_r$ is a Case 1 interval we then have $\mathbb{P}(Q_r, \chi_{Q_0} g\omega) \le 3|Q_r|^{-1}\int_{Q_r} g\, d\omega$ and so

$$\begin{aligned}\sum_{Q_r \text{ satisfies Case 1}} |Q_r|_\sigma \mathbb{P}(Q_r, \chi_{Q_0} g\omega)^{p'} &\le 3^p \sum_{r=1}^\infty |Q_r|_\sigma \Big(\frac{1}{|Q_r|}\int_{Q_r} g\, d\omega\Big)^{p'} \\ &\le C_p \sum_{r=1}^\infty \frac{|Q_r|_\sigma |Q_r|_\omega^{p'-1}}{|Q_r|^{p'}} \int_{Q_r} g^{p'}\, d\omega \le C_p \|(\omega,\sigma)\|_{A_p}^{p'} \int_{Q_0} g^{p'}\, d\omega.\end{aligned}$$

If $Q_r$ is a Case 2 interval, then $|Q_r|_\sigma \le \delta^{-1}|Q_r \setminus J_{r,\varepsilon}|_\sigma$ and

$$\begin{aligned}&\sum_{Q_r \text{ satisfies Case 2}} |Q_r|_\sigma \mathbb{P}(Q_r, \chi_{Q_0} g\omega)^{p'} \\ &\quad\le \frac{1}{\delta} \sum_{Q_r \text{ satisfies Case 2}} |Q_r \setminus J_{r,\varepsilon}|_\sigma \mathbb{P}(Q_r, \chi_{Q_0} g\omega)^{p'} \\ &\quad\le \frac{1}{\delta}\sum_{r=1}^\infty \Big(\frac{12}{\varepsilon}\Big)^{p'} \int_{Q_r \setminus J_{r,\varepsilon}} |H(\chi_{Q_0 \cap Q_r^c} g\omega)(x)|^{p'}\, d\sigma(x) \\ &\quad\le C_{\varepsilon,\delta,p} \sum_{r=1}^\infty \int_{Q_r\setminus J_{r,\varepsilon}} (|H(\chi_{Q_0} g\omega)(x)|^{p'} + |H(\chi_{Q_r} g\omega)(x)|^{p'})\, d\sigma(x) \\ &\quad\le C_{\varepsilon,\delta,p}\Big(\int_{Q_0} |H(\chi_{Q_0} g\omega)(x)|^{p'}\, d\sigma(x) + \sum_{r=1}^\infty \int_{Q_r} |H(\chi_{Q_r} g\omega)(x)|^{p'}\, d\sigma(x)\Big) \\ &\quad\le C_{\varepsilon,\delta,p}\Big(C\int_{Q_0} g^{p'}\, d\omega + \sum_{r=1}^\infty C\int_{Q_r} g^{p'}\, d\omega\Big) = C_{\varepsilon,\delta,p}\int_{Q_0} g^{p'}\, d\omega,\end{aligned}$$

upon using (5-18) with $Q_0$ and $Q_r$, which is (5-5). As before, Case 3 intervals don't exist if $\sigma$ is doubling and $\varepsilon > 0$ is sufficiently small. □

*Proof of Theorem 1.10.* Theorem 5.4 shows that the dual Poisson inequality (5-5) holds uniformly in $Q_0$ and pairwise disjoint $\{Q_r\}_{r=1}^\infty$ satisfying $\bigcup_{r=1}^\infty Q_r \subset Q_0$, provided both the half-strengthened $A_p$ condition (5-1) and the dual pivotal condition (5-2) hold when $1 < p \le 2$ — and provided (5-1), (5-2) and the Poisson condition (1-24) hold when $p > 2$. Since $\sigma$ is doubling, Lemma 5.8 shows that the dual pivotal condition (5-2) follows from the dual testing condition (1-21) — and Lemma 5.9 shows that the dual Poisson inequality (5-5), and hence also the Poisson condition (1-24), follows from the dual Hilbert transform inequality (5-18). Thus Theorem 1.10 now follows from the claim proved in Section 5.1.1 that (5-5) can be substituted for (1-17) in the proof of Theorem 1.9. □

## Acknowledgment

We began this work during research stays at the Fields Institute in Toronto, Canada, and continued at the Centre de Recerca Matemàtica in Barcelona, Spain. We thank these institutions for their generous hospitality. In addition, this paper has been substantially improved by the careful attention of the referee, for which we are particularly grateful.

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Received 7 Oct 2009. Revised 2 Feb 2011. Accepted 2 Mar 2011.

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Typeset in LaTeX

# Analysis & PDE

Volume 5 No. 1 2012



2157-5045(2012)5:1;1-J