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# THE CAUCHY PROBLEM FOR THE BENJAMIN–ONO EQUATION IN $L^2$ REVISITED

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Ionescu and Kenig proved that the Cauchy problem associated with the Benjamin–Ono equation is globally well posed in $L^2(\mathbb{R})$. In this paper we give a simpler proof of Ionescu and Kenig's result, which moreover provides stronger uniqueness results. In particular, we prove unconditional well-posedness in $H^s(\mathbb{R})$ for $s > \frac{1}{4}$. Note that our approach also permits us to simplify the proof of the global well-posedness in $L^2(\mathbb{T})$ and yields unconditional well-posedness in $H^{\frac{1}{2}}(\mathbb{T})$.

## 1. Introduction

The Benjamin–Ono equation is one of the fundamental equations describing the evolution of weakly nonlinear internal long waves. It has been derived by Benjamin [1967] as an approximate model for long-crested unidirectional waves at the interface of a two-layer system of incompressible inviscid fluids, one being infinitely deep. In nondimensional variables, the initial value problem (IVP) associated with the Benjamin–Ono equation (BO) is

$$\begin{cases} \partial_t u + \mathscr{H}\,\partial_x^2 u = u\,\partial_x u, \\ u(x,0) = u_0(x), \end{cases} \tag{1-1}$$

where $x \in \mathbb{R}$ or $\mathbb{T}$, $t \in \mathbb{R}$, $u$ is a real-valued function, and $\mathscr{H}$ is the Hilbert transform, defined on the line by

$$\mathscr{H} f(x) = \text{p.v.}\, \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(y)}{x-y}\, dy. \tag{1-2}$$

The Benjamin–Ono equation is, at least formally, completely integrable [Fokas and Ablowitz 1983] and thus possesses an infinite number of conservation laws. For example, the momentum and the energy, respectively given by

$$M(u) = \int u^2\, dx \quad \text{and} \quad E(u) = \frac{1}{2} \int \bigl|D_x^{\frac{1}{2}} u\bigr|^2\, dx + \frac{1}{6} \int u^3\, dx, \tag{1-3}$$

are conserved by the flow of (1-1).

The IVP associated with the Benjamin–Ono equation presents interesting mathematical difficulties and has been extensively studied in recent years. In the continuous case, well-posedness in $H^s(\mathbb{R})$ for $s > \frac{3}{2}$ was proved by Iório [1986] by using purely hyperbolic energy methods (see also [Abdelouhab et al. 1989] for global well-posedness in the same range of $s$). Then Ponce [1991] derived a local smoothing effect associated with the dispersive part of the equation, which, combined with compactness methods, enables

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us to reach $s = \frac{3}{2}$. This technique was refined by Koch and Tzvetkov [2003] and Kenig and Koenig [2003], who reached $s > \frac{5}{4}$ and $s > \frac{9}{8}$, respectively. On the other hand, Molinet, Saut, and Tzvetkov [Molinet et al. 2001] proved that the flow map associated with BO, when it exists, fails to be $C^2$ in any Sobolev space $H^s(\mathbb{R})$, $s \in \mathbb{R}$. This result is based on the fact that the dispersive smoothing effects of the linear part of BO are not strong enough to control the low-high frequency interactions appearing in the nonlinearity of (1-1). It was improved by Koch and Tzvetkov [2005], who showed that the flow map fails even to be uniformly continuous in $H^s(\mathbb{R})$ for $s > 0$ (see [Biagioni and Linares 2001] for the same result in the case $s < -\frac{1}{2}$). As the consequence of those results, one cannot solve the Cauchy problem for the Benjamin–Ono equation by a Picard iterative method implemented on the integral equation associated with (1-1) for initial data in the Sobolev space $H^s(\mathbb{R})$, $s \in \mathbb{R}$. In particular, the methods introduced by Bourgain [1993b] and Kenig, Ponce, and Vega [Kenig et al. 1993; 1996] for the Korteweg–de Vries equation do not apply directly to the Benjamin–Ono equation.

Therefore, the problem of obtaining well-posedness in less regular Sobolev spaces turns out to be far from trivial. Due to the conservations laws (1-3), $L^2(\mathbb{R})$ and $H^{\frac{1}{2}}(\mathbb{R})$ are two natural spaces where well-posedness is expected. In this direction, a decisive breakthrough was achieved by Tao [2004]. By combining a complex variant of the Cole–Hopf transform (which linearizes the Burgers equation) with Strichartz estimates, he proved well-posedness in $H^1(\mathbb{R})$. More precisely, to obtain estimates at the $H^1$-level, he introduced the new unknown

$$w = \partial_x P_{+hi}\big(e^{-\frac{i}{2}F}\big), \tag{1-4}$$

where $F$ is some spatial primitive of $u$ and $P_{+hi}$ denotes the projection on high positive frequencies. Then $w$ satisfies an equation of the form

$$\partial_t w - i\,\partial_x^2 w = -\partial_x P_{+hi}(\partial_x^{-1} w P_- \partial_x u) + \text{negligible terms}. \tag{1-5}$$

Observe that, thanks to the frequency projections, the nonlinear term appearing in the right-hand side of (1-5) does not exhibit any low-high frequency interaction terms. Finally, to invert this gauge transformation, one gets an equation of the form

$$u = 2ie^{\frac{i}{2}F} w + \text{negligible terms}. \tag{1-6}$$

Very recently, Burq and Planchon [2008] and Ionescu and Kenig [2007] were able to use Tao's ideas in the context of Bourgain's spaces to prove well-posedness for the Benjamin–Ono equation in $H^s(\mathbb{R})$ for $s > \frac{1}{4}$ and $s \geq 0$, respectively. The main difficulty arising here is that Bourgain's spaces do not enjoy an algebra property so that one is losing regularity when estimating $u$ in terms of $w$ via Equation (1-6). Burq and Planchon first paralinearized the equation and then used a localized version of the gauge transformation on the worst nonlinear term. On the other hand, Ionescu and Kenig decomposed the solution in two parts: the first one is the smooth solution of BO evolving from the low-frequency part of the initial data while the second one solves a dispersive system renormalized by a gauge transformation involving the first part. The authors were then able to solve the system via a fixed-point argument in a dyadic version of Bourgain's spaces (already used in the context of wave maps [Tataru 1998]) with a special structure in low frequencies. We observe that their result only ensures the uniqueness in the class of limits of smooth solutions while Burq and Planchon obtained a stronger uniqueness result. Indeed, by applying their approach to the equation satisfied by the difference of two solutions, they succeed in proving that the flow map associated with BO is Lipschitz in a weaker topology when the initial data belongs to $H^s(\mathbb{R})$, $s > \frac{1}{4}$.

In the periodic setting, Molinet [2007; 2008] proved well-posedness in $H^s(\mathbb{T})$ for $s \geq \frac{1}{2}$ and $s \geq 0$, successively. (This last result is proven to be sharp in [Molinet 2009].) Once again, these works combined Tao's gauge transformation with estimates in Bourgain's spaces. It should be pointed out that in the periodic case, one can assume that $u$ has mean value zero to define a primitive. Then it is easy to check by the mean-value theorem that the gauge transformation in (1-4) is Lipschitz from $L^2$ into $L^\infty$. This property, which is not true on the real line, is crucial to prove the uniqueness and the Lipschitz property of the flow map.

The aim of this paper is to give a simpler proof of Ionescu and Kenig's result, which also provides a stronger uniqueness result for the solutions at the $L^2$ level. It is worth noticing that to reach $L^2$ in [Ionescu and Kenig 2007] and [Molinet 2008], the authors replaced $u$ in (1-4) by the formula given in (1-6). The benefit of this substitution is that then $u$ no longer appears in (1-4). On the other hand, it introduces new technical difficulties in handling the multiplication by $e^{\mp iF/2}$ in Bourgain spaces. Here we are able to avoid this substitution, which will simplify the proof. Our main result is the following:

**Theorem 1.1.** *Let $s \geq 0$ be given.*

*Existence: For all $u_0 \in H^s(\mathbb{R})$ and all $T > 0$, there exists a solution*

$$u \in C([0,T]; H^s(\mathbb{R})) \cap X_T^{s-1,1} \cap L_T^4 W_x^{s,4} \tag{1-7}$$

*of (1-1) such that*

$$w = \partial_x P_{+hi}(e^{-\frac{i}{2}F[u]}) \in Y_T^s, \tag{1-8}$$

*where $F[u]$ is some primitive of $u$ defined in (3-2).*

*Uniqueness: This solution is unique in the following classes:*[1]

(i) $u \in L^\infty\big(]0,T[; L^2(\mathbb{R})\big) \cap L^4\big(]0,T[\times\mathbb{R}\big)$ *and* $w \in X_T^{0,\frac{1}{2}}$,

(ii) $u \in L^\infty\big(]0,T[; H^s(\mathbb{R})\big) \cap L_T^4 W_x^{s,4}$ *whenever* $s > 0$,

(iii) $u \in L^\infty\big(]0,T[; H^s(\mathbb{R})\big)$ *whenever* $s > \frac{1}{4}$.

*Moreover, $u \in C_b(\mathbb{R}; L^2(\mathbb{R}))$, and the flow-map data solution $u_0 \mapsto u$ is continuous from $H^s(\mathbb{R})$ into $C([0,T]; H^s(\mathbb{R}))$.*

Note that $H^s(\mathbb{R})$ above denotes the space of all real-valued functions with the usual norm, and $X_T^{s,b}$ and $Y_T^s$ are Bourgain spaces defined in Section 2B while the primitive $F[u]$ of $u$ is defined in Section 3A.

**Remark 1.2.** Since the function spaces in the uniqueness class (i) are reflexive and since $\partial_x P_{+hi}(e^{-\frac{i}{2}F[u_n]})$ converges to $\partial_x P_{+hi}(e^{-\frac{i}{2}F[u]})$ in $L^\infty(]-T,T[; L^2(\mathbb{R}))$ when $u_n$ converges to $u$ in $L^\infty(]-T,T[; L^2(\mathbb{R}))$, our result clearly implies the uniqueness in the class of $L^\infty(]-T,T[; L^2(\mathbb{R}))$-limits of smooth solutions.

[1] Note that according to the equation, the time derivative of a solution in these classes belongs to $L^\infty(-T,T; H^{-2})$, and thus such a solution has to belong to $C(-T,T; H^{-2})$.

**Remark 1.3.** For $s > 0$ we get a uniqueness class without any conditions on $w$ (see [Burq and Planchon 2008] for the case $s > \frac{1}{4}$).

**Remark 1.4.** According to (iii) we get unconditional well-posedness in $H^s(\mathbb{R})$ for $s > \frac{1}{4}$. Such a result was first proven, in a much less direct way, in [Burq and Planchon 2006] for $s \geq \frac{1}{2}$. It implies in particular the uniqueness of the (energy) weak solutions that belong to $L^\infty(\mathbb{R}; H^{\frac{1}{2}}(\mathbb{R}))$. These solutions are constructed by regularizing the equation and passing to the limit as the regularizing coefficient goes to 0 (taking into account some energy estimate for the regularizing equation related to the energy conservation of (1-1)).

Our proof also combines Tao's ideas with the use of Bourgain's spaces. Actually, it closely follows the strategy introduced by the first author in [Molinet 2007]. The main new ingredient is a bilinear estimate for the nonlinear term appearing in (1-5), which allows us to recover one derivative at the $L^2$ level. It is interesting to note that, at the $H^s$ level with $s > 0$, this estimate follows from the Cauchy–Schwarz method introduced by Kenig, Ponce, and Vega [Kenig et al. 1996] (see the Appendix for the use of this method in some region of integration). To reach $L^2$, one of the main difficulties is that we cannot substitute the Fourier transform of $u$ by its modulus in the bilinear estimate since we are not able to prove that $\mathcal{F}^{-1}(|\hat{u}|)$ belongs to $L^4_{x,t}$ but only that $u$ belongs to $L^4_{x,t}$. To overcome this difficulty we use a Littlewood–Paley decomposition of the functions and carefully divide the domain of integration into suitable disjoint subdomains.

To obtain our uniqueness result, following the same method as in the periodic setting, we derive a Lipschitz bound for the gauge transformation from some affine subspaces of $L^2(\mathbb{R})$ into $L^\infty(\mathbb{R})$. Recall that this is clearly not possible for general initial data since it would imply the uniform continuity of the flow map. The main idea is to notice that such a Lipschitz bound holds for solutions emanating from initial data having the same low frequency part, and this is sufficient for our purpose.

Let us point out some applications. First our uniqueness result allows us to simplify the proof of the continuity of the flow map associated with the Benjamin–Ono equation for the weak topology of $L^2(\mathbb{R})$. This result was recently proved by Cui and Kenig [2010].

It is also interesting to observe that the method of proof used here still works in the periodic setting, and thus, we reobtain the well-posedness result [Molinet 2008] in a simpler way. Moreover, as in the continuous case, we prove new uniqueness results (see Theorem 7.1). In particular, we get unconditional well-posedness in $H^s(\mathbb{T})$ as soon as $s \geq \frac{1}{2}$.

Finally, we believe that this technique may be useful for other nonlinear dispersive equations presenting the same kind of difficulties as the Benjamin–Ono equation. For example, consider the higher-order Benjamin–Ono equation

$$\partial_t v - b\mathcal{H}\,\partial_x^2 v + a\,\partial_x^3 v = cv\,\partial_x v - d\,\partial_x(v\mathcal{H}\,\partial_x v + \mathcal{H}(v\,\partial_x v)), \tag{1-9}$$

where $x, t \in \mathbb{R}$, $v$ is a real-valued function, $a \in \mathbb{R}$, and $b$, $c$, and $d$ are positive constants. The equation above corresponds to a second-order approximation model of the same phenomena described by the Benjamin–Ono equation. It was derived by Craig, Guyenne, and Kalisch [2005] using a Hamiltonian perturbation theory and possesses an energy at the $H^1$ level. As for the Benjamin–Ono equation, the flow map associated with (1-9) fails to be smooth in any Sobolev space $H^s(\mathbb{R})$, $s \in \mathbb{R}$ [Pilod 2008]. Recently, the Cauchy problem associated with (1-9) was proved to be well posed in $H^2(\mathbb{R})$ [Linares et al. 2011]. In a forthcoming paper, the authors will show that it is actually well posed in the energy space $H^1(\mathbb{R})$.

This paper is organized as follows: in the next section, we introduce the notations, define the function spaces, and recall some classical linear estimates. Section 3 is devoted to the key nonlinear estimates, which are used in Section 4 to prove the main part of Theorem 1.1 while the assertions (ii) and (iii) are proved in Section 5. In Section 6, we give a simple proof of the continuity of the flow map for the weak $L^2(\mathbb{R})$ topology whereas Section 7 is devoted to some comments and new results in the periodic case. Finally, in the Appendix we prove the bilinear estimate used in Section 5.

## 2. Notation, function spaces, and preliminary estimates

**2A. *Notation.*** For any positive numbers $a$ and $b$, the notation $a \lesssim b$ means that there exists a positive constant $c$ such that $a \le cb$. We also write $a \sim b$ when $a \lesssim b$ and $b \lesssim a$. Moreover, if $\alpha \in \mathbb{R}$, $\alpha_+$ and $\alpha_-$ will denote a number slightly greater and lesser than $\alpha$, respectively.

For $u = u(x,t) \in \mathcal{S}(\mathbb{R}^2)$, $\mathcal{F}u = \widehat{u}$ will denote its space-time Fourier transform whereas $\mathcal{F}_x u = (u)^{\wedge_x}$ and $\mathcal{F}_t u = (u)^{\wedge_t}$ will denote its Fourier transform in space and time, respectively. For $s \in \mathbb{R}$, we define the Bessel and Riesz potentials of order $-s$, $J_x^s$ and $D_x^s$, by

$$J_x^s u = \mathcal{F}_x^{-1}(1+|\xi|^2)^{\frac{s}{2}} \mathcal{F}_x u \quad \text{and} \quad D_x^s u = \mathcal{F}_x^{-1}(|\xi|^s \mathcal{F}_x u).$$

Throughout the paper, we fix a cutoff function $\eta$ such that

$$\eta \in C_0^\infty(\mathbb{R}), \quad 0 \le \eta \le 1, \quad \eta_{|_{[-1,1]}} = 1, \quad \operatorname{supp}(\eta) \subset [-2,2].$$

We define

$$\phi(\xi) := \eta(\xi) - \eta(2\xi) \quad \text{and} \quad \phi_{2^l}(\xi) := \phi(2^{-l}\xi).$$

Summations over capitalized variables such as $N$ are presumed to be dyadic with $N \ge 1$; i.e., these variables range over numbers of the form $2^n$, $n \in \mathbb{Z}_+$. Then we have

$$\sum_N \phi_N(\xi) = 1 - \eta(2\xi) \;\; \forall \xi \ne 0 \quad \text{and} \quad \operatorname{supp}(\phi_N) \subset \{\tfrac{1}{2}N \le |\xi| \le 2N\}.$$

Let us define the Littlewood–Paley multipliers by

$$P_N u = \mathcal{F}_x^{-1}(\phi_N \mathcal{F}_x u) \quad \text{and} \quad P_{\ge N} := \sum_{K \ge N} P_K.$$

We also define the operators $P_{hi}$, $P_{HI}$, $P_{lo}$, and $P_{LO}$ by

$$P_{hi} = \sum_{N \ge 2} P_N, \quad P_{HI} = \sum_{N \ge 8} P_N, \quad P_{lo} = 1 - P_{hi}, \quad \text{and} \quad P_{LO} = 1 - P_{HI}.$$

Let $P_+$ and $P_-$ denote the projections on the positive and the negative Fourier frequencies, respectively. Then

$$P_\pm u = \mathcal{F}_x^{-1}(\chi_{\mathbb{R}_\pm} \mathcal{F}_x u),$$

and we also define $P_{\pm hi} = P_\pm P_{hi}$, $P_{\pm HI} = P_\pm P_{HI}$, $P_{\pm lo} = P_\pm P_{lo}$, and $P_{\pm LO} = P_\pm P_{LO}$. Observe that $P_{hi}$, $P_{HI}$, $P_{lo}$, and $P_{LO}$ are bounded operators on $L^p(\mathbb{R})$ for $1 \le p \le \infty$ while $P_\pm$ is only bounded on $L^p(\mathbb{R})$ for $1 < p < \infty$. We also note that

$$\mathscr{H} = -i P_+ + i P_-.$$

Finally, we denote by $U(\cdot)$ the free group associated with the linearized Benjamin–Ono equation, which is to say,

$$\mathscr{F}_x(U(t)f)(\xi) = e^{-it|\xi|\xi}\mathscr{F}_x f(\xi).$$

**2B.** ***Function spaces.*** For $1 \le p \le \infty$, $L^p(\mathbb{R})$ is the usual Lebesgue space with the norm $\|\cdot\|_{L^p}$, and for $s \in \mathbb{R}$, the real-valued Sobolev spaces $H^s(\mathbb{R})$ and $W^{s,p}(\mathbb{R})$ denote the spaces of all real-valued functions with the usual norms

$$\|f\|_{H^s} = \|J^s u\|_{L^2} \quad \text{and} \quad \|f\|_{W^{s,p}} = \|J_x^s f\|_{L^p}.$$

For $1 < p < \infty$, we define the space $\tilde{L}^p$ as

$$\|f\|_{\tilde{L}^p} = \|P_{lo} f\|_{L^p} + \Big(\sum_N \|P_N f\|_{L^p}^2\Big)^{\frac{1}{2}}.$$

Observe that when $p \ge 2$, the Littlewood–Paley theorem on the square function and Minkowski's inequality imply that the injection $\tilde{L}^p \hookrightarrow L^p$ is continuous. Moreover, if $u = u(x,t)$ is a real-valued function defined for $x \in \mathbb{R}$ and $t$ in the time interval $[0,T]$ with $T > 0$, $B$ is one of the spaces defined above, and $1 \le p \le \infty$, we will define the mixed space-time spaces $L_T^p B_x$ and $L_t^p B_x$ by the norms

$$\|u\|_{L_T^p B_x} = \Big(\int_0^T \|u(\cdot,t)\|_B^p \, dt\Big)^{\frac{1}{p}} \quad \text{and} \quad \|u\|_{L_t^p B_x} = \Big(\int_{\mathbb{R}} \|u(\cdot,t)\|_B^p \, dt\Big)^{\frac{1}{p}},$$

respectively.

For $s, b \in \mathbb{R}$, we introduce the Bourgain spaces $X^{s,b}$ and $Z^{s,b}$ related to the Benjamin–Ono equation as the completion of the Schwartz space $\mathscr{S}(\mathbb{R}^2)$ under the norms

$$\|u\|_{X^{s,b}} = \Big(\int_{\mathbb{R}^2} \langle \tau + |\xi|\xi\rangle^{2b} \langle \xi\rangle^{2s} |\widehat{u}(\xi,\tau)|^2 \, d\xi \, d\tau\Big)^{\frac{1}{2}}, \tag{2-1}$$

$$\|u\|_{Z^{s,b}} = \bigg(\int_{\mathbb{R}} \Big(\int_{\mathbb{R}} \langle \tau + |\xi|\xi\rangle^{b} \langle \xi\rangle^{s} |\widehat{u}(\xi,\tau)| \, d\tau\Big)^2 d\xi\bigg)^{\frac{1}{2}}, \tag{2-2}$$

$$\|u\|_{\widetilde{Z}^{s,b}} = \|P_{lo} u\|_{Z^{s,b}} + \Big(\sum_N \|P_N u\|_{Z^{s,b}}^2\Big)^{\frac{1}{2}}, \tag{2-3}$$

$$\|u\|_{Y^s} = \|u\|_{X^{s,\frac{1}{2}}} + \|u\|_{\widetilde{Z}^{s,0}}, \tag{2-4}$$

where $\langle x\rangle := 1 + |x|$. We will also use the localized (in time) version of these spaces. Let $T > 0$ be a positive time and $\|\cdot\|_B = \|\cdot\|_{X^{s,b}}$, $\|\cdot\|_{\widetilde{Z}^{s,b}}$, or $\|\cdot\|_{Y^s}$. If $u : \mathbb{R} \times [0,T] \to \mathbb{C}$, then

$$\|u\|_{B_T} := \inf\{\, \|\tilde{u}\|_B \mid \tilde{u} : \mathbb{R} \times \mathbb{R} \to \mathbb{C},\ \tilde{u}|_{\mathbb{R}\times[0,T]} = u \,\}.$$

We recall that

$$Y_T^s \hookrightarrow Z_T^{s,0} \hookrightarrow C([0,T]; H^s(\mathbb{R})).$$

**2C.** ***Linear estimates.*** In this subsection, we recall some linear estimates in Bourgain's spaces that will be needed later. The first ones are well known (see [Ginibre et al. 1997], for example).

**Lemma 2.1** (homogeneous linear estimate). *Let $s \in \mathbb{R}$. Then*

$$\|\eta(t)\, U(t) f\|_{Y^s} \lesssim \|f\|_{H^s}. \tag{2-5}$$

**Lemma 2.2** (nonhomogeneous linear estimate). *Let $s \in \mathbb{R}$. Then, for any $0 < \delta < \frac{1}{2}$,*

$$\left\| \eta(t) \int_0^t U(t-t')\, g(t')\, dt' \right\|_{X^{s,\frac{1}{2}+\delta}} \lesssim \|g\|_{X^{s,-\frac{1}{2}+\delta}} \tag{2-6}$$

*and*

$$\left\| \eta(t) \int_0^t U(t-t')\, g(t')\, dt' \right\|_{Y^s} \lesssim \|g\|_{X^{s,-\frac{1}{2}}} + \|g\|_{\widetilde{Z}^{s,-1}}. \tag{2-7}$$

*Proof.* Lemmas 2.1 and 2.2 follow directly from the classical linear estimates for $X^{s,b}$ and $Z^{s,b}$ together with the fact that

$$\|u\|_{X^{s,b}} \sim \|P_{lo} u\|_{X^{s,b}} + \left( \sum_N \|P_N u\|_{X^{s,b}}^2 \right)^{\frac{1}{2}}. \qquad \square$$

**Lemma 2.3.** *For any $T > 0$, $s \in \mathbb{R}$ and for all $-\frac{1}{2} < b' \le b < \frac{1}{2}$,*

$$\|u\|_{X_T^{s,b'}} \lesssim T^{b-b'} \|u\|_{X_T^{s,b}}. \tag{2-8}$$

The following Bourgain–Strichartz estimates will also be useful:

**Lemma 2.4.** *It holds that*

$$\|u\|_{L^4_{x,t}} \lesssim \|u\|_{\widetilde{L}^4_{x,t}} \lesssim \|u\|_{X^{0,\frac{3}{8}}}, \tag{2-9}$$

*and for any $T > 0$ and $\frac{3}{8} \le b \le \frac{1}{2}$,*

$$\|u\|_{L^4_{x,T}} \lesssim T^{b-\frac{3}{8}} \|u\|_{X_T^{0,b}}. \tag{2-10}$$

*Proof.* Estimate (2-9) follows directly by applying the estimate

$$\|u\|_{L^4_{x,t}} \lesssim \|u\|_{X^{0,\frac{3}{8}}},$$

proved in the appendix of [Molinet 2007], to each dyadic block on the left-hand side of (2-9).

To prove (2-10), we choose an extension $\tilde{u} \in X^{0,b}$ of $u$ such that $\|\tilde{u}\|_{X^{0,b}} \le 2\|u\|_{X_T^{0,b}}$. Therefore, it follows from (2-8) and (2-9) that

$$\|u\|_{L^4_{x,T}} \le \|\tilde{u}\|_{L^4_{x,t}} \lesssim \|\tilde{u}\|_{X^{0,\frac{3}{8}}} \lesssim T^{b-\frac{3}{8}} \|u\|_{X_T^{0,b}}. \qquad \square$$

**2D. *Fractional Leibniz rules.*** First we state the classical fractional Leibniz rule estimate derived by Kenig, Ponce, and Vega (see Theorems A.8 and A.12 in [Kenig et al. 1993]).

**Proposition 2.5.** *Let $0<\alpha<1$, $p, p_1, p_2 \in (1,+\infty)$ with $\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p}$, and $\alpha_1,\alpha_2\in[0,\alpha]$ with $\alpha=\alpha_1+\alpha_2$. Then*

$$\|D_x^\alpha(fg)-fD_x^\alpha g-gD_x^\alpha f\|_{L^p}\lesssim\|D_x^{\alpha_1}g\|_{L^{p_1}}\,\|D_x^{\alpha_2}f\|_{L^{p_2}}. \tag{2-11}$$

*Moreover, for $\alpha_1=0$, the value $p_1=+\infty$ is allowed.*

The next estimate is a frequency-localized version of estimate (2-11) in the same spirit as Lemma 3.2 in [Tao 2004]. It allows sharing most of the fractional derivative in the first term on the right-hand side of (2-12).

**Lemma 2.6.** *Let $\alpha\ge 0$ and $1<q<\infty$. Then*

$$\|D_x^\alpha P_+(fP_-\partial_x g)\|_{L^q}\lesssim\|D_x^{\alpha_1}f\|_{L^{q_1}}\,\|D_x^{\alpha_2}g\|_{L^{q_2}} \tag{2-12}$$

*with $1<q_i<\infty$, $\frac{1}{q_1}+\frac{1}{q_2}=\frac{1}{q}$, and $\alpha_1\ge\alpha$, $\alpha_2\ge 0$, and $\alpha_1+\alpha_2=1+\alpha$.*

*Proof.* See Lemma 3.2 in [Molinet 2007]. □

Finally, we derive an estimate to handle the multiplication by a term of the form $e^{\pm\frac{i}{2}F}$, where $F$ is a real-valued function, in fractional Sobolev spaces.

**Lemma 2.7.** *Let $2\le q<\infty$ and $0\le\alpha\le\frac{1}{q}$. Consider $F_1$ and $F_2$, two real-valued functions such that $u_j=\partial_x F_j$ belongs to $L^2(\mathbb{R})$ for $j=1,2$. Then*

$$\big\|J_x^\alpha\big(e^{\pm\frac{i}{2}F_1}g\big)\big\|_{L^q}\lesssim(1+\|u_1\|_{L^2})\|J_x^\alpha g\|_{L^q}, \tag{2-13}$$

*and*

$$\big\|J_x^\alpha\big((e^{\pm\frac{i}{2}F_1}-e^{\pm\frac{i}{2}F_2})g\big)\big\|_{L^q}\lesssim\big(\|u_1-u_2\|_{L^2}+\|e^{\pm\frac{i}{2}F_1}-e^{\pm\frac{i}{2}F_2}\|_{L^\infty}(1+\|u_1\|_{L^2})\big)\|J_x^\alpha g\|_{L^q}. \tag{2-14}$$

*Proof.* In the case $\alpha=0$, we deduce from Hölder's inequality that

$$\big\|e^{\pm\frac{i}{2}F_1}g\big\|_{L^q}\le\|g\|_{L^q} \tag{2-15}$$

since $F_1$ is real-valued. Therefore, we assume that $0<\alpha\le\frac{1}{q}$, and it is enough to bound $\|D_x^\alpha(e^{\pm\frac{i}{2}F_1}g)\|_{L^q}$. First we observe that

$$\big\|D_x^\alpha\big(e^{\pm\frac{i}{2}F_1}g\big)\big\|_{L^q}\le\big\|D_x^\alpha\big(P_{lo}e^{\pm\frac{i}{2}F_1}g\big)\big\|_{L^q}+\big\|D_x^\alpha\big(P_{hi}e^{\pm\frac{i}{2}F_1}g\big)\big\|_{L^q}. \tag{2-16}$$

Estimate (2-11) and Bernstein's inequality imply that

$$\big\|D_x^\alpha\big(P_{lo}e^{\pm\frac{i}{2}F_1}g\big)\big\|_{L^q}\lesssim\big\|P_{lo}e^{\pm\frac{i}{2}F_1}\big\|_{L^\infty}\|D_x^\alpha g\|_{L^q}+\big\|D_x^\alpha P_{lo}e^{\pm\frac{i}{2}F_1}\big\|_{L^\infty}\|g\|_{L^q}\lesssim\|J_x^\alpha g\|_{L^q}. \tag{2-17}$$

On the other hand, by using estimate (2-11) again, we get that

$$\big\|D_x^\alpha\big(P_{hi}e^{\pm\frac{i}{2}F_1}g\big)\big\|_{L^q}\lesssim\big\|P_{hi}e^{\pm\frac{i}{2}F_1}\big\|_{L^\infty}\|D_x^\alpha g\|_{L^q}+\|g\|_{L^{q_1}}\,\big\|D_x^\alpha P_{hi}e^{\pm\frac{i}{2}F_1}\big\|_{L^{q_2}}$$

with $\frac{1}{q_1}=\frac{1}{q}-\alpha$ and $\frac{1}{q_2}=\alpha$, so $\frac{1}{q_1}+\frac{1}{q_2}=\frac{1}{q}$. Then it follows from the real-valuedness of $F_1$, the equality

$\partial_x F_1 = u_1$, and the Sobolev embedding that

$$\begin{aligned}\big\| D_x^{\alpha}\big(P_{hi} e^{\pm\frac{i}{2}F_1} g\big)\big\|_{L^q} &\lesssim \|D_x^{\alpha} g\|_{L^q} + \|J_x^{\alpha} g\|_{L^q}\, \big\| D_x^{\alpha+\frac12} P_{hi} e^{\pm\frac{i}{2}F_1}\big\|_{L^2} \\ &\lesssim \|J_x^{\alpha} g\|_{L^q}\big(1+\big\|\partial_x e^{\pm\frac{i}{2}F_1}\big\|_{L^2}\big) \\ &\lesssim \|J_x^{\alpha} g\|_{L^q}(1+\|u_1\|_{L^2}).\end{aligned} \tag{2-18}$$

The proof of estimate (2-13) is concluded gathering (2-15)–(2-18).

Estimate (2-14) can be obtained exactly in the same way, using that

$$\big\|\partial_x\big(e^{\pm\frac{i}{2}F_1} - e^{\pm\frac{i}{2}F_2}\big)\big\|_{L^2} \lesssim \|u_1-u_2\|_{L^2} + \big\|e^{\pm\frac{i}{2}F_1} - e^{\pm\frac{i}{2}F_2}\big\|_{L^\infty}\|u_1\|_{L^2}. \tag{2-19}$$

This completes the proof. □

## 3. A priori estimates in $H^s(\mathbb{R})$ for $s \geq 0$

In this section we will derive *a priori* estimates on a solution $u$ to (1-1) at the $H^s$-level for $s \geq 0$. First, following Tao [2004], we perform a nonlinear transformation on the equation to weaken the high-low frequency interaction in the nonlinearity. Furthermore, since we want to reach $L^2$, we will need to use Bourgain spaces. This requires a new bilinear estimate, which we derive in Section 3B.

**3A. *The gauge transformation.*** Let $u$ be a solution to the equation in (1-1). First we construct a spatial primitive $F = F[u]$ of $u$ (i.e., $\partial_x F = u$) that satisfies the equation

$$\partial_t F = -\mathcal{H}\,\partial_x^2 F + \tfrac12(\partial_x F)^2. \tag{3-1}$$

Note that these two properties defined $F$ up to a constant. In order to construct $F$ for $u$ with low regularity, we use the construction of Burq and Planchon [2008]. Consider $\psi \in C_0^\infty(\mathbb{R})$ such that $\int_{\mathbb{R}} \psi(y)\,dy = 1$ and define

$$F(x,t) = \int_{\mathbb{R}} \psi(y)\Big(\int_y^x u(z,t)\,dz\Big)\,dy + G(t) \tag{3-2}$$

as a mean of antiderivatives of $u$. Obviously, $\partial_x F = u$ and

$$\begin{aligned}\partial_t F(x,t) &= \int_{\mathbb{R}} \psi(y)\Big(\int_y^x \partial_t u(z,t)\,dz\Big)\,dy + G'(t) \\ &= \int_{\mathbb{R}} \psi(y)\Big(\int_y^x \big(-\mathcal{H}\,\partial_z^2 u(z,t) + \tfrac12\,\partial_z(u(z,t)^2)\big)\,dz\Big)\,dy + G'(t) \\ &= -\mathcal{H}\,\partial_x u(x,t) + \tfrac12 u(x,t)^2 + \int_{\mathbb{R}} \big(\mathcal{H}\psi'(y)\,u(y,t) - \psi(y)\tfrac12 u(y,t)^2\big)\,dy + G'(t).\end{aligned}$$

Therefore, we choose $G$ as

$$G(t) = \int_0^t \int_{\mathbb{R}} \big(-\mathcal{H}\psi'(y)\,u(y,s) + \psi(y)\tfrac12 u(y,s)^2\big)\,dy\,ds$$

to ensure that (3-1) is satisfied. Observe that this construction makes sense for $u \in L^2_{\mathrm{loc}}(\mathbb{R}^2)$. Next, we introduce the new unknown

$$W = P_{+hi}\big(e^{-\frac{i}{2}F}\big) \quad \text{and} \quad w = \partial_x W = -\tfrac{1}{2} i\, P_{+hi}\big(e^{-\frac{i}{2}F} u\big). \tag{3-3}$$

Then it follows from (3-1) and the identity $\mathscr{H} = -i(P_+ - P_-)$ that

$$\begin{aligned}\partial_t W + \mathscr{H}\partial_x^2 W = \partial_t W - i\,\partial_x^2 W &= -\tfrac{1}{2} i\, P_{+hi}\big(e^{-\frac{i}{2}F}(\partial_t F - i\,\partial_x^2 F - \tfrac{1}{2}(\partial_x F)^2)\big)\\ &= -P_{+hi}(W P_-\,\partial_x u) - P_{+hi}\big(P_{lo} e^{-\frac{i}{2}F} P_-\,\partial_x u\big)\end{aligned}$$

since the term $-P_{+hi}(P_{-hi} e^{-\frac{i}{2}F} P_-\,\partial_x u)$ cancels due to the frequency localization. Thus, it follows from differentiating that

$$\partial_t w - i\,\partial_x^2 w = -\partial_x P_{+hi}(W P_-\,\partial_x u) - \partial_x P_{+hi}\big(P_{lo} e^{-\frac{i}{2}F} P_-\,\partial_x u\big). \tag{3-4}$$

On the other hand, one can write $u$ as

$$u = F_x = e^{\frac{i}{2}F} e^{-\frac{i}{2}F} F_x = 2i e^{\frac{i}{2}F}\partial_x\big(e^{-\frac{i}{2}F}\big) = 2i e^{\frac{i}{2}F} w - e^{\frac{i}{2}F} P_{lo}\big(e^{-\frac{i}{2}F} u\big) - e^{\frac{i}{2}F} P_{-hi}\big(e^{-\frac{i}{2}F} u\big) \tag{3-5}$$

so that it follows from the frequency localization

$$P_{+HI} u = 2i\, P_{+HI}\big(e^{\frac{i}{2}F} w\big) - P_{+HI}\big(P_{+hi} e^{\frac{i}{2}F} P_{lo}(e^{-\frac{i}{2}F} u)\big) + 2i\, P_{+HI}\big(P_{+HI} e^{\frac{i}{2}F}\,\partial_x P_{-hi} e^{-\frac{i}{2}F}\big). \tag{3-6}$$

**Remark 3.1.** Note that the use of $P_{+HI}$ allows us to replace $e^{\frac{i}{2}F}$ by $P_{+hi} e^{\frac{i}{2}F}$ in the second term on the right-hand side of (3-6). This fact will be useful to obtain at least a quadratic term in $\|u\|_{L^\infty_T L^2_x}$ on the right-hand side of estimate (3-8) in Proposition 3.2.

Then we have the following *a priori* estimates for $u$ in terms of $w$:

**Proposition 3.2.** *Let* $0 \le s \le 1$, $0 < T \le 1$, $0 \le \theta \le 1$, *and $u$ be a solution to* (1-1) *in the time interval* $[0, T]$. *Then*

$$\|u\|_{X_T^{s-\theta,\theta}} \lesssim \|u\|_{L^\infty_T H^s_x} + \|u\|_{L^4_{T,x}}\,\|J^s_x u\|_{L^4_{T,x}}. \tag{3-7}$$

*Moreover, if* $0 \le s \le \frac{1}{4}$, *we have*

$$\|J^s_x u\|_{L^p_T L^q_x} \lesssim \|u_0\|_{L^2} + \big(1 + \|u\|_{L^\infty_T L^2_x}\big)\big(\|w\|_{Y^s_T} + \|u\|^2_{L^\infty_T L^2_x}\big) \tag{3-8}$$

*for* $(p, q) = (\infty, 2)$ *or* $(4, 4)$.

**Remark 3.3.** One can rewrite (3-8) in a convenient form for $s \ge \frac{1}{4}$; see [Molinet 2007].

*Proof.* We begin with the proof of estimate (3-7) and construct a suitable extension in time $\tilde{u}$ of $u$. First we consider $v(t) = U(-t)\,u(t)$ on the time interval $[0, T]$ and extend $v$ on $[-2, 2]$ by setting $\partial_t v = 0$ on $[-2, 2] \setminus [0, T]$. Then it is pretty clear that

$$\|\partial_t v\|_{L^2_{[-2,2]} H^r_x} = \|\partial_t v\|_{L^2_T H^r_x} \quad \text{and} \quad \|v\|_{L^2_{[-2,2]} H^r_x} \lesssim \|v\|_{L^\infty_T H^r_x}$$

for all $r \in \mathbb{R}$. Now we define $\tilde{u}(x, t) = \eta(t)\, U(t)\, v(t)$. Obviously,

$$\|\tilde{u}\|_{X^{s-1,1}} \lesssim \|\partial_t v\|_{L^2_{[-2,2]} H^{s-1}_x} + \|v\|_{L^2_{[-2,2]} H^{s-1}_x} \lesssim \|\partial_t v\|_{L^2_T H^{s-1}_x} + \|v\|_{L^\infty_T H^{s-1}_x} \tag{3-9}$$

and

$$\|\tilde{u}\|_{X^{s,0}} \lesssim \|v\|_{L^2_{[-2,2]}H^s_x} \lesssim \|v\|_{L^\infty_T H^s_x} = \|u\|_{L^\infty_T H^s_x}. \tag{3-10}$$

Interpolating between (3-9) and (3-10) and using the identity

$$\partial_t v = \mathscr{H}\,\partial_x^2 U(-t)\,u + U(-t)\,\partial_t u = U(-t)[\mathscr{H}\,\partial_x^2 u + \partial_t u],$$

we then deduce that

$$\|\tilde{u}\|_{X^{s-\theta,\theta}} \lesssim \|\partial_t u + \mathscr{H}\,\partial_x^2 u\|_{L^2_T H^{s-1}_x} + \|u\|_{L^\infty_T H^s_x} \tag{3-11}$$

for all $0 \le \theta \le 1$. Therefore, the fact that $u$ is a solution to (1-1) and the fractional Leibniz rule [Kenig et al. 1993] yield

$$\|\tilde{u}\|_{X^{s-\theta,\theta}} \lesssim \|u\|_{L^\infty_T H^s_x} + \|u\|_{L^4_{x,T}}\,\|J^s_x u\|_{L^4_{x,T}},$$

which concludes the proof of (3-7) since $\tilde{u}$ extends $u$ outside of $[0,T]$.

Next, we turn to the proof of (3-8). Let $0 \le T \le 1$, $0 \le s \le \frac{1}{4}$, $(p,q) = (\infty,2)$ or $(4,4)$, and $u$ be a smooth solution to the equation in (1-1). Since $u$ is real-valued, it that holds $P_- u = \overline{P_+ u}$ so that

$$\|J^s_x u\|_{L^p_T L^q_x} \lesssim \|P_{LO} u\|_{L^p_T L^q_x} + \|D^s_x P_{+HI} u\|_{L^p_T L^q_x}. \tag{3-12}$$

To estimate the second term on the right-hand side of (3-12), we use (3-6) to deduce that

$$\begin{aligned}\|D^s_x P_{+HI} u\|_{L^p_T L^q_x} \lesssim \big\|D^s_x P_{+HI}\big(e^{\frac{i}{2}F} w\big)\big\|_{L^p_T L^q_x} &+ \big\|D^s_x P_{+HI}\big(P_{+hi} e^{\frac{i}{2}F} P_{lo}(e^{-\frac{i}{2}F} u)\big)\big\|_{L^p_T L^q_x} \\ &+ \big\|D^s_x P_{+HI}\big(P_{+HI} e^{\frac{i}{2}F}\partial_x P_{-hi} e^{-\frac{i}{2}F}\big)\big\|_{L^p_T L^q_x} \\ =: I + II + III.&\end{aligned}$$

Estimates (2-10) and (2-13) yield

$$I \lesssim (1 + \|u\|_{L^\infty_T L^2_x})\|J^s_x w\|_{L^p_T L^q_x} \lesssim (1 + \|u\|_{L^\infty_T L^2_x})\|w\|_{Y^s_T}. \tag{3-13}$$

On the other hand, the fractional Leibniz rule (Proposition 2.5), Hölder's inequality in time, and the Sobolev embedding imply that

$$\begin{aligned} II &\lesssim \big\|D^s_x P_{+hi} e^{\frac{i}{2}F}\big\|_{L^p_T L^q_x}\,\big\|P_{+lo}\big(ue^{-\frac{i}{2}F}\big)\big\|_{L^\infty_{T,x}} + \big\|P_{+hi} e^{\frac{i}{2}F}\big\|_{L^\infty_{T,x}}\,\big\|D^s_x P_{+lo}\big(ue^{-\frac{i}{2}F}\big)\big\|_{L^p_T L^q_x} \\ &\lesssim \big\|\partial_x P_{+hi} e^{\frac{i}{2}F}\big\|_{L^p_T L^2_x}\,\big\|P_{+lo}\big(ue^{-\frac{i}{2}F}\big)\big\|_{L^\infty_T L^2_x} \lesssim T^{\frac{1}{p}}\|u\|^2_{L^\infty_T L^2_x}. \end{aligned} \tag{3-14}$$

Finally, estimate (2-12) with $\alpha_1 = \alpha_2 = (1+s)/2$ and $q_1 = q_2 = q$, Hölder's inequality in time, and the Sobolev embedding lead to

$$\begin{aligned} III &\lesssim \big\|D^{(1+s)/2}_x P_{+HI} e^{\frac{i}{2}F}\big\|_{L^{2p}_T L^{2q}_x}\,\big\|D^{(1+s)/2}_x P_{-hi} e^{-\frac{i}{2}F}\big\|_{L^{2p}_T L^{2q}_x} \\ &\lesssim T^{\frac{1}{p}}\big\|D^{1+\frac{s}{2}-\frac{1}{2q}}_x P_{+HI} e^{\frac{i}{2}F}\big\|_{L^\infty_T L^2_x}\,\big\|D^{1+\frac{s}{2}-\frac{1}{2q}}_x P_{-hi} e^{-\frac{i}{2}F}\big\|_{L^\infty_T L^2_x} \\ &\lesssim T^{\frac{1}{p}}\big\|\partial_x P_{+HI} e^{\frac{i}{2}F}\big\|_{L^\infty_T L^2_x}\,\big\|\partial_x P_{-hi} e^{-\frac{i}{2}F}\big\|_{L^\infty_T L^2_x} \lesssim T^{\frac{1}{p}}\|u\|^2_{L^\infty_T L^2_x}, \end{aligned} \tag{3-15}$$

since $0 \le s \le \frac{1}{q}$. Therefore, we deduce by gathering (3-13)–(3-15) that

$$\|D_x^s P_{+HI} u\|_{L^p_T L^q_x} \lesssim \big(1 + \|u\|_{L^\infty_T L^2_x}\big)\big(\|w\|_{Y^s_T} + T^{\frac{1}{p}} \|u\|^2_{L^\infty_T L^2_x}\big). \tag{3-16}$$

Next we turn to the first term on the right-hand side of (3-12) and consider the integral equation satisfied by $P_{LO}u$,

$$P_{LO}u = U(t) P_{LO} u_0 + \int_0^t U(t-\tau)\, P_{LO}\, \partial_x(u^2)(\tau)\, d\tau. \tag{3-17}$$

First observe that

$$\|P_{LO}u\|_{L^p_T L^q_x} \lesssim T^{\frac{1}{p}} \|P_{LO}u\|_{L^\infty_T L^2_x}.$$

Then we deduce from (3-17), using the fact that $U$ is a unitary group in $L^2$ and Bernstein's inequality, that

$$\begin{aligned} \|P_{LO}u\|_{L^p_T L^q_x} &\lesssim T^{\frac{1}{p}} \|u_0\|_{L^2_x} + T^{1+\frac{1}{p}} \|\partial_x P_{LO}(u^2)\|_{L^\infty_T L^2_x} \\ &\lesssim T^{\frac{1}{p}} \|u_0\|_{L^2_x} + T^{1+\frac{1}{p}} \|P_{LO}(u^2)\|_{L^\infty_T L^1_x} \\ &\lesssim \|u_0\|_{L^2_x} + \|u\|^2_{L^\infty_T L^2_x}, \end{aligned} \tag{3-18}$$

since $0 \le T \le 1$.

Thus, estimate (3-8) follows combining (3-12), (3-16), and (3-18). This concludes the proof of Proposition 3.2. □

**3B. *Bilinear estimates.*** The aim of this subsection is to derive the following estimate of $\|w\|_{Y^s_T}$:

**Proposition 3.4.** *Let $0 < T \le 1$, $0 \le s \le \frac{1}{2}$, and $u$ be a solution to (1-1) on the time interval $[0, T]$. Then*

$$\|w\|_{Y^s_T} \lesssim (1 + \|u_0\|_{L^2})\|u_0\|_{H^s} + \|u\|^2_{L^4_{x,T}} + \|w\|_{X^{s,\frac{1}{2}}_T}\big(\|u\|_{L^\infty_T L^2_x} + \|u\|_{L^4_{x,T}} + \|u\|_{X^{-1,1}_T}\big). \tag{3-19}$$

The main tools to prove Proposition 3.4 are the following crucial bilinear estimates:

**Proposition 3.5.** *For any $s \ge 0$, we have*

$$\big\|\partial_x P_{+hi}(\partial_x^{-1} w P_- \partial_x u)\big\|_{X^{s,-\frac{1}{2}}} \lesssim \|w\|_{X^{s,\frac{1}{2}}}\big(\|u\|_{L^2_{x,t}} + \|u\|_{L^4_{x,t}} + \|u\|_{X^{-1,1}}\big) \tag{3-20}$$

*and*

$$\big\|\partial_x P_{+hi}(\partial_x^{-1} w P_- \partial_x u)\big\|_{\widetilde{Z}^{s,-1}} \lesssim \|w\|_{X^{s,\frac{1}{2}}}\big(\|u\|_{L^2_{x,t}} + \|u\|_{L^4_{x,t}} + \|u\|_{X^{-1,1}}\big). \tag{3-21}$$

**Remark 3.6.** Note that $\partial_x^{-1} w$ is well defined since $w$ is localized in high frequencies.

*Proof.* We will only give the proof in the case of $s = 0$ since the case $s > 0$ can be deduced by using similar arguments. By duality, to prove (3-20) is equivalent to prove that

$$|I| \lesssim \|h\|_{L^2_{x,t}} \|w\|_{X^{0,\frac{1}{2}}}\big(\|u\|_{L^2_{x,t}} + \|u\|_{L^4_{x,t}} + \|u\|_{X^{-1,1}}\big), \tag{3-22}$$

where

$$I = \int_{\mathcal{D}} \frac{\xi}{\langle\sigma\rangle^{1/2}} \widehat{h}(\xi,\tau)\, \xi_1^{-1} \widehat{w}(\xi_1,\tau_1)\, \xi_2\, \widehat{u}(\xi_2,\tau_2)\, d\nu, \tag{3-23}$$

$$d\nu = d\xi\, d\xi_1\, d\tau\, d\tau_1, \quad \xi_2 = \xi - \xi_1, \quad \tau_2 = \tau - \tau_1, \quad \sigma_i = \tau_i + \xi_i|\xi_i|,\ i = 1, 2, \tag{3-24}$$

and

$$\mathcal{D} = \left\{ (\xi, \xi_1, \tau, \tau_1) \in \mathbb{R}^4 \;\middle|\; \xi \geq 1, \xi_1 \geq 1, \xi_2 \leq 0 \right\}. \tag{3-25}$$

Observe that we always have in $\mathcal{D}$ that

$$\xi_1 \geq \xi \geq 1 \quad \text{and} \quad \xi_1 \geq |\xi_2|. \tag{3-26}$$

In the case where $|\xi_2| \leq 1$, we have by using Hölder's inequality and estimate (2-9) that

$$|I| \lesssim \int_{\mathbb{R}^4} \frac{|\widehat{h}|}{\langle\sigma\rangle^{1/2}} |\widehat{w}(\xi_1, \tau_1)|\, |\widehat{u}(\xi_2, \tau_2)|\, d\nu \lesssim \left\| \left( \frac{|\widehat{h}|}{\langle\sigma\rangle^{1/2}} \right)^{\vee} \right\|_{L^4_{x,t}} \|(|\widehat{w}|)^{\vee}\|_{L^4_{x,t}} \|u\|_{L^2_{x,t}} \lesssim \|h\|_{L^2_{x,t}} \|w\|_{X^{\frac{3}{8}}} \|u\|_{L^2_{x,t}}.$$

From now on we will assume that $|\xi_2| \geq 1$ in $\mathcal{D}$.

By using a dyadic decomposition in space-frequency for the functions $h$, $w$, and $u$, one can rewrite $I$ as

$$I = \sum_{N,N_1,N_2} I_{N,N_1,N_2} \tag{3-27}$$

with

$$I_{N,N_1,N_2} := \int_{\mathcal{D}} \frac{\xi}{\langle\sigma\rangle^{1/2}} \widehat{P_N h}(\xi, \tau)\, \xi_1^{-1} \widehat{P_{N_1} w}(\xi_1, \tau_1)\, \xi_2 \widehat{P_{N_2} u}(\xi_2, \tau_2)\, d\nu$$

and the dyadic numbers $N$, $N_1$, and $N_2$ ranging from 1 to $+\infty$. Moreover, the resonance identity

$$\sigma_1 + \sigma_2 - \sigma = \xi_1^2 + (\xi - \xi_1)|\xi - \xi_1| - \xi^2 = -2\xi\xi_2 \tag{3-28}$$

holds in $\mathcal{D}$. Therefore, to calculate $I_{N,N_1,N_2}$, we split the integration domain $\mathcal{D}$ into the disjoint regions

$$\begin{aligned}
\mathcal{A}_{N,N_2} &= \left\{ (\xi, \xi_1, \tau, \tau_1) \in \mathcal{D} \;\middle|\; |\sigma| \geq \tfrac{1}{6} N N_2 \right\}, \\
\mathcal{B}_{N,N_2} &= \left\{ (\xi, \xi_1, \tau, \tau_1) \in \mathcal{D} \;\middle|\; |\sigma| < \tfrac{1}{6} N N_2,\ |\sigma_1| \geq \tfrac{1}{6} N N_2 \right\}, \\
\mathcal{C}_{N,N_2} &= \left\{ (\xi, \xi_1, \tau, \tau_1) \in \mathcal{D} \;\middle|\; |\sigma| < \tfrac{1}{6} N N_2,\ |\sigma_1| < \tfrac{1}{6} N N_2,\ |\sigma_2| \geq \tfrac{1}{6} N N_2 \right\},
\end{aligned}$$

and denote by $I^{\mathcal{A}_{N,N_2}}_{N,N_1,N_2}$, $I^{\mathcal{B}_{N,N_2}}_{N,N_1,N_2}$, and $I^{\mathcal{C}_{N,N_2}}_{N,N_1,N_2}$ the restriction of $I_{N,N_1,N_2}$ to each of these regions. Then it follows that

$$I_{N,N_1,N_2} = I^{\mathcal{A}_{N,N_2}}_{N,N_1,N_2} + I^{\mathcal{B}_{N,N_2}}_{N,N_1,N_2} + I^{\mathcal{C}_{N,N_2}}_{N,N_1,N_2},$$

and thus

$$|I| \leq |I_{\mathcal{A}}| + |I_{\mathcal{B}}| + |I_{\mathcal{C}}|, \tag{3-29}$$

where

$$I_{\mathcal{A}} := \sum_{N,N_1,N_2} I^{\mathcal{A}_{N,N_2}}_{N,N_1,N_2}, \quad I_{\mathcal{B}} := \sum_{N,N_1,N_2} I^{\mathcal{B}_{N,N_2}}_{N,N_1,N_2}, \quad I_{\mathcal{C}} := \sum_{N,N_1,N_2} I^{\mathcal{C}_{N,N_2}}_{N,N_1,N_2}.$$

Therefore, it suffices to bound $|I_{\mathcal{A}}|$, $|I_{\mathcal{B}}|$, and $|I_{\mathcal{C}}|$. Note that one of the two following cases holds:

(1) high-low interaction: $N_1 \sim N$ and $N_2 \le N_1$,

(2) high-high interaction: $N_1 \sim N_2$ and $N \le N_1$.

*Estimate for* $|I_{\mathcal{A}}|$. In the first case, we observe from the Cauchy–Schwarz inequality that

$$|I_{\mathcal{A}}| \sim \left| \int_{\mathbb{R}^2} \widehat{h} \sum_{N_1} \sum_{j=0}^{\frac{\ln N_1}{\ln 2}} \phi_{N_1} \xi \langle \sigma \rangle^{-\frac{1}{2}} \chi_{\{|\sigma| \ge \frac{1}{6} N_1{}^2 2^{-j}\}} \mathcal{F}\big(P_+(\partial_x^{-1} P_{N_1} w P_- \partial_x P_{2^{-j}N_1} u)\big)\, d\xi\, d\tau \right|$$

$$\lesssim \|\widehat{h}\|_{L^2_{\xi,\tau}} \left\| \sum_{N_1} \sum_{j \ge 0} N_1 (N_1^2 2^{-j})^{-\frac{1}{2}} \phi_{N_1} \left| \mathcal{F}\big(P_+(\partial_x^{-1} P_{N_1} w P_- \partial_x P_{2^{-j}N_1} u)\big) \right| \right\|_{L^2_{\xi,\tau}}.$$

Then the Plancherel identity and the triangular inequality imply that

$$|I_{\mathcal{A}}| \lesssim \|h\|_{L^2_{x,t}} \sum_{j \ge 0} \left( \sum_{N_1} 2^j \left\| P_{N_1}(\partial_x^{-1} P_{N_1} w P_- \partial_x P_{2^{-j}N_1} u) \right\|^2_{L^2_{x,t}} \right)^{\frac{1}{2}}.$$

By using the Hölder and Bernstein inequalities, we deduce that

$$\begin{aligned} |I_{\mathcal{A}}| &\lesssim \|h\|_{L^2_{x,t}} \sum_{j \ge 0} \left( \sum_{N_1} 2^{-j} \|P_{N_1} w\|^2_{L^4_{x,t}} \|P_{2^{-j}N_1} u\|^2_{L^4_{x,t}} \right)^{\frac{1}{2}} \\ &\lesssim \|h\|_{L^2_{x,t}} \left( \sum_{N} \|P_{N_1} w\|^2_{L^4_{x,t}} \right)^{\frac{1}{2}} \|u\|_{L^4_{x,t}}. \end{aligned} \tag{3-30}$$

In the second case, it follows using the same strategy as in the first case that

$$|I_{\mathcal{A}}| \lesssim \|h\|_{L^2_{x,t}} \sum_{j \ge 0} \left( \sum_{N_1} (2^{-j} N_1)^2 (2^{-j} N_1 N_1)^{-1} \left\| P_{2^{-j}N_1}(\partial_x^{-1} P_{N_1} w P_- \partial_x P_{N_1} u) \right\|^2_{L^2_{x,t}} \right)^{\frac{1}{2}},$$

which implies using the Hölder and Bernstein inequalities that

$$\begin{aligned} |I_{\mathcal{A}}| &\lesssim \|h\|_{L^2_{x,t}} \sum_{j \ge 0} \left( \sum_{N_1} 2^{-j} \|P_{N_1} w\|^2_{L^4_{x,t}} \|P_{N_1} u\|^2_{L^4_{x,t}} \right)^{\frac{1}{2}} \\ &\lesssim \|h\|_{L^2_{x,t}} \left( \sum_{N_1} \|P_{N_1} w\|^2_{L^4_{x,t}} \right)^{\frac{1}{2}} \|u\|_{L^4_{x,t}}. \end{aligned} \tag{3-31}$$

Therefore, we deduce by gathering (3-30)–(3-31) and using estimate (2-9) that

$$|I_{\mathcal{A}}| \le \|h\|_{L^2_{x,t}} \|w\|_{X^{0,\frac{3}{8}}} \|u\|_{L^4_{x,t}}. \tag{3-32}$$

*Estimate for* $|I_{\mathcal{B}}|$. By again using the triangular and the Cauchy–Schwarz inequalities, we have in the first case that

$$|I_{\mathcal{B}}| \le \|w\|_{X^{0,\frac{1}{2}}} \sum_{j \ge 0} \left( \sum_{N_1} N_1^{-2} (N_1 2^{-j} N_1)^{-1} \left\| P_{N_1}\Big( \partial_x P_{+hi} P_{N_1} \Big( \frac{\widehat{h}}{\langle \sigma \rangle^{1/2}} \Big)^{\vee} P_+ \partial_x P_{2^{-j}N_1} \tilde{u} \Big) \right\|^2_{L^2_{x,t}} \right)^{\frac{1}{2}},$$

where $\tilde{u}(x,t)=u(-x,-t)$. Thus, it follows from the Bernstein and Hölder inequalities that

$$
\begin{aligned}
|I_{\mathcal{B}}| &\lesssim \|w\|_{X^{0,\frac12}} \sum_{j\geq 0}\Bigl(\sum_{N_1} 2^{-j}\Bigl\|P_{N_1}\Bigl(\frac{\widehat{h}}{\langle\sigma\rangle^{1/2}}\Bigr)^{\vee}\Bigr\|^2_{L^4_{x,t}}\|P_{2^{-j}N_1}u\|^2_{L^4_{x,t}}\Bigr)^{\frac12}\\
&\lesssim \|w\|_{X^{0,\frac12}}\Bigl(\sum_{N_1}\Bigl\|P_{N_1}\Bigl(\frac{\widehat{h}}{\langle\sigma\rangle^{1/2}}\Bigr)^{\vee}\Bigr\|^2_{L^4_{x,t}}\Bigr)^{\frac12}\|u\|_{L^4_{x,t}}.
\end{aligned}
\tag{3-33}
$$

In the second case, we bound $|I_{\mathcal{B}}|$ by

$$
\bigl|I_{\mathcal{B}}\bigr| \leq \|w\|_{X^{0,\frac12}} \sum_{j\geq 0}\Bigl(\sum_{N_1} N_1^{-2}(2^{-j}N_1N_1)^{-1}\Bigl\|P_{N_1}\Bigl(\partial_x P_{+hi}P_{2^{-j}N_1}\Bigl(\frac{\widehat{h}}{\langle\sigma\rangle^{1/2}}\Bigr)^{\vee}P_+\,\partial_x P_{N_1}\tilde{u}\Bigr)\Bigr\|^2_{L^2_{x,t}}\Bigr)^{\frac12}
$$

so that

$$
\begin{aligned}
|I_{\mathcal{B}}| &\lesssim \|w\|_{X^{0,\frac12}} \sum_{j\geq 0}\Bigl(\sum_{N_1} 2^{-j}\Bigl\|P_{2^{-j}N_1}P_{+hi}\Bigl(\frac{\widehat{h}}{\langle\sigma\rangle^{1/2}}\Bigr)^{\vee}\Bigr\|^2_{L^4_{x,t}}\|P_{N_1}u\|^2_{L^4_{x,t}}\Bigr)^{\frac12}\\
&\lesssim \|w\|_{X^{0,\frac12}} \sum_{j\geq 0}2^{-\frac{j}{2}}\Bigl(\sum_{N_1}\Bigl\|P_{2^{-j}N_1}P_{+hi}\Bigl(\frac{\widehat{h}}{\langle\sigma\rangle^{1/2}}\Bigr)^{\vee}\Bigr\|^2_{L^4_{x,t}}\Bigr)^{\frac12}\|u\|_{L^4_{x,t}}\\
&\lesssim \|w\|_{X^{0,\frac12}}\Bigl(\sum_{N_1}\Bigl\|P_{N_1}\Bigl(\frac{\widehat{h}}{\langle\sigma\rangle^{1/2}}\Bigr)^{\vee}\Bigr\|^2_{L^4_{x,t}}\Bigr)^{\frac12}\|u\|_{L^4_{x,t}}.
\end{aligned}
\tag{3-34}
$$

In conclusion, we obtain by gathering (3-33)–(3-34) and using estimate (2-9) that

$$
|I_{\mathcal{B}}| \leq \|h\|_{L^2_{x,t}}\,\|w\|_{X^{0,\frac12}}\,\|u\|_{L^4_{x,t}}. \tag{3-35}
$$

*Estimate for* $|I_{\mathscr{C}}|$. First observe that

$$
|I_{\mathscr{C}}| \lesssim \int_{\widetilde{\mathscr{C}}} \frac{|\xi|}{\langle\sigma\rangle^{1/2}}|\widehat{h}(\xi,\tau)|\,|\xi_1|^{-1}|\widehat{w}(\xi_1,\tau_1)|\frac{|\xi_2|^2}{\langle\sigma_2\rangle}\frac{\langle\sigma_2\rangle}{|\xi_2|}|\widehat{u}(\xi_2,\tau_2)|\,d\nu, \tag{3-36}
$$

where

$$
\widetilde{\mathscr{C}} = \Bigl\{(\xi,\xi_1,\tau,\tau_1)\in\mathscr{D}\;\Big|\;(\xi,\xi_1,\tau,\tau_1)\in\bigcup_{N,N_2}\mathscr{C}_{N,N_2}\Bigr\}.
$$

Since $|\sigma_2|>|\sigma|$ and $|\sigma_2|>|\sigma_1|$ in $\widetilde{\mathscr{C}}$, it follows from (3-28) that $|\sigma_2|\gtrsim|\xi\xi_2|$. Then

$$
|\xi\xi_1^{-1}\xi_2^2\langle\sigma_2\rangle^{-1}|\lesssim 1 \tag{3-37}
$$

holds in $\widetilde{\mathscr{C}}$ so that, using Hölder's inequality and estimate (2-9), we deduce

$$
\begin{aligned}
|I_{\mathscr{C}}| &\lesssim \int_{\widetilde{\mathscr{C}}} \frac{|\widehat{h}(\xi,\tau)|}{\langle\sigma\rangle^{1/2}}|\widehat{w}(\xi_1,\tau_1)|\frac{\langle\sigma_2\rangle}{|\xi_2|}|\widehat{u}(\xi_2,\tau_2)|\,d\nu\\
&\lesssim \Bigl\|\Bigl(\frac{|\widehat{h}|}{\langle\sigma\rangle^{1/2}}\Bigr)^{\vee}\Bigr\|_{L^4_{x,t}}\|(|\widehat{w}|)^{\vee}\|_{L^4_{x,t}}\|u\|_{X^{-1,1}} \lesssim \|h\|_{L^2_{x,t}}\,\|w\|_{X^{0,\frac38}}\,\|u\|_{X^{-1,1}}.
\end{aligned}
\tag{3-38}
$$

Therefore, estimates (3-29), (3-32), (3-35), and (3-38) imply estimate (3-22), which concludes the proof of estimate (3-20).

To prove estimate (3-21), we also proceed by duality. Then it is sufficient to show that

$$|J| \lesssim \Big(\sum_N \|g_N\|^2_{L^2_\xi L^\infty_\tau}\Big)^{\frac12} \|w\|_{X^{0,\frac12}} \big(\|u\|_{L^2_{x,t}} + \|u\|_{L^4_{x,t}} + \|u\|_{X^{-1,1}}\big), \tag{3-39}$$

where

$$J = \sum_N \int_{\mathfrak{D}} \frac{\xi}{\langle\sigma\rangle} g_N(\xi,\tau)\, \phi_N(\xi)\, \xi_1^{-1} \widehat{w}(\xi_1,\tau_1)\, \xi_2\, \widehat{u}(\xi_2,\tau_2)\, d\nu,$$

and $d\nu$ and $\mathfrak{D}$ are defined in (3-24) and (3-25). As in the case of $I$, we can also assume that $|\xi_2| \ge 1$. By using dyadic decompositions as in (3-27), $J$ can be rewritten as

$$J = \sum_{N,N_1,N_2} J_{N,N_1,N_2},$$

where

$$J_{N,N_1,N_2} := \int_{\mathfrak{D}} \frac{\xi}{\langle\sigma\rangle} \phi_N(\xi) g_N(\xi,\tau)\, \xi_1^{-1} \widehat{P_{N_1} w}(\xi_1,\tau_1)\, \xi_2 \widehat{P_{N_2} u}(\xi_2,\tau_2)\, d\nu,$$

and the dyadic numbers $N$, $N_1$, and $N_2$ range from 1 to $+\infty$. Moreover, we will denote by $J^{\mathcal{A}_{N,N_2}}_{N,N_1,N_2}$, $J^{\mathcal{B}_{N,N_2}}_{N,N_1,N_2}$, and $J^{\mathcal{C}_{N,N_2}}_{N,N_1,N_2}$ the restriction of $J_{N,N_1,N_2}$ to the regions $\mathcal{A}_{N,N_2}$, $\mathcal{B}_{N,N_2}$, and $\mathcal{C}_{N,N_2}$ defined in (3-28). Then it follows that

$$|J| \le |J_{\mathcal{A}}| + |J_{\mathcal{B}}| + |J_{\mathcal{C}}|, \tag{3-40}$$

where

$$J_{\mathcal{A}} := \sum_{N,N_1,N_2} J^{\mathcal{A}_{N,N_2}}_{N,N_1,N_2}, \quad J_{\mathcal{B}} := \sum_{N,N_1,N_2} J^{\mathcal{B}_{N,N_2}}_{N,N_1,N_2}, \quad J_{\mathcal{C}} := \sum_{N,N_1,N_2} J^{\mathcal{C}_{N,N_2}}_{N,N_1,N_2}$$

so that it suffices to estimate $|J_{\mathcal{A}}|$, $|J_{\mathcal{B}}|$, and $|J_{\mathcal{C}}|$.

*Estimate for* $|J_{\mathcal{A}}|$. To estimate $|J_{\mathcal{A}}|$, we divide each region $\mathcal{A}_{N,N_2}$ into disjoint subregions

$$\mathcal{A}^q_{N,N_2} = \big\{ (\xi,\xi_1,\tau,\tau_1) \in \mathcal{A}_{N,N_2} \;\big|\; 2^{q-3} N N_2 \le |\sigma| < 2^{q-2} N N_2 \big\}$$

for $q \in \mathbb{Z}_+$. Thus, if $J^{\mathcal{A}^q_{N,N_2}}_{N,N_1,N_2}$ denotes the restriction of $J^{\mathcal{A}_{N,N_2}}_{N,N_1,N_2}$ to each of these regions, we have

$$J_{\mathcal{A}} = \sum_{q\ge 0} \sum_{N,N_1,N_2} J^{\mathcal{A}^q_{N,N_2}}_{N,N_1,N_2}.$$

In the case of high-low interactions, we deduce by using the Plancherel identity and the Cauchy–Schwarz and Minkowski inequalities that

$$|J_{\mathcal{A}}| \le \sum_{q\ge 0} \sum_{N_1} \sum_{N_2 \le N_1} \|g_{N_1} \chi_{\{|\sigma|\sim 2^q N_1 N_2\}}\|_{L^2_{\xi,\tau}} \times (2^q N_1 N_2)^{-1} N_1 \|\partial_x^{-1} P_{N_1} w P_- \partial_x P_{N_2} u\|_{L^2_{x,t}}.$$

Moreover, we get from Hölder's inequality

$$\|g_{N_1} \chi_{\{|\sigma|\sim 2^q N_1 N_2\}}\|_{L^2_{\xi,\tau}} \lesssim (2^q N N_2)^{\frac12} \|g_{N_1}\|_{L^2_\xi L^\infty_\tau}$$

so that the Cauchy–Schwarz inequality yields

$$\begin{aligned}|J_{\mathcal{A}}| &\lesssim \sum_{N_1}\sum_{N_2\le N_1}(N_2N_1^{-1})^{\frac12}\|g_{N_1}\|_{L^2_\xi L^\infty_\tau}\|P_{N_1}w\|_{L^4_{x,t}}\|P_{N_2}u\|_{L^4_{x,t}}\\ &\lesssim \|u\|_{L^4_{x,t}}\sum_{N_1}\|g_{N_1}\|_{L^2_\xi L^\infty_\tau}\|P_{N_1}w\|_{L^4_{x,t}} \lesssim \Big(\sum_{N_1}\|g_{N_1}\|^2_{L^2_\xi L^\infty_\tau}\Big)^{\frac12}\|w\|_{\tilde{L}^4_{x,t}}\|u\|_{L^4_{x,t}}. \end{aligned}\tag{3-41}$$

In the high-high interaction case, it follows from the Minkowski and Cauchy–Schwarz inequalities that

$$|J_{\mathcal{A}}| \le \sum_{q\ge0}\sum_{N_1}\sum_{N\le N_1}\|g_N\chi_{\{|\sigma|\sim 2^qNN_1\}}\|_{L^2_{\xi,\tau}}\times(2^qNN_1)^{-1}N\|\partial_x^{-1}P_{N_1}wP_-\partial_xP_{N_1}u\|_{L^2_{x,t}}.$$

Moreover, we deduce from Hölder's inequality that

$$\|g_N\chi_{\{|\sigma|\sim 2^qNN_1\}}\|_{L^2_{\xi,\tau}} \lesssim (2^qNN_1)^{\frac12}\|g_N\|_{L^2_\xi L^\infty_\tau}.$$

Then the Cauchy–Schwarz inequality implies that

$$\begin{aligned}|J_{\mathcal{A}}| &\lesssim \sum_{j\ge0}\sum_{N_1}(N_1^{-1}2^{-j}N_1)^{\frac12}\|g_{2^{-j}N_1}\|_{L^2_\xi L^\infty_\tau}\|P_{N_1}w\|_{L^4_{x,t}}\|P_{N_1}u\|_{L^4_{x,t}}\\ &\lesssim \sum_{j\ge0}2^{-\frac{j}{2}}\Big(\sum_{N_1}\|g_{2^{-j}N_1}\|^2_{L^2_\xi L^\infty_\tau}\Big)^{\frac12}\Big(\sum_{N_1}\|P_{N_1}w\|^2_{L^4_{x,t}}\Big)^{\frac12}\|u\|_{L^4_{x,t}}\\ &\lesssim \Big(\sum_{N_1}\|g_{N_1}\|^2_{L^2_\xi L^\infty_\tau}\Big)^{\frac12}\|w\|_{\tilde{L}^4_{x,t}}\|u\|_{L^4_{x,t}}.\end{aligned}\tag{3-42}$$

Then estimates (2-9), (3-41), and (3-42) yield

$$|J_{\mathcal{A}}| \lesssim \Big(\sum_{N}\|g_N\|^2_{L^2_\xi L^\infty_\tau}\Big)^{\frac12}\|w\|_{X^{0,\frac38}}\|u\|_{L^4_{x,t}}.\tag{3-43}$$

*Estimate for* $|J_{\mathcal{B}}|$ *and* $|J_{\mathcal{C}}|$. Arguing as in the proof of (3-20), it is deduced that

$$|J_{\mathcal{B}}|+|J_{\mathcal{C}}| \lesssim \Big(\Big\|\Big(\frac{g}{\langle\sigma\rangle}\Big)^\vee\Big\|_{\tilde{L}^4_{x,t}}+\Big\|\Big(\frac{|g|}{\langle\sigma\rangle}\Big)^\vee\Big\|_{\tilde{L}^4_{x,t}}\Big)\|w\|_{X^{0,\frac12}}\big(\|u\|_{L^4_{x,t}}+\|u\|_{X^{-1,1}}\big),$$

where $g=\sum_N \phi_N g_N$. Moreover, estimate (2-9) and Hölder's inequality imply

$$\Big\|\Big(\frac{g}{\langle\sigma\rangle}\Big)^\vee\Big\|_{\tilde{L}^4_{x,t}}+\Big\|\Big(\frac{|g|}{\langle\sigma\rangle}\Big)^\vee\Big\|_{\tilde{L}^4_{x,t}} \lesssim \Big\|\langle\sigma\rangle^{-\frac58}\sum_N\phi_Ng_N\Big\|_{L^2_{\xi,\tau}} \lesssim \Big(\sum_N\big\|\langle\sigma\rangle^{-\frac58}g_N\big\|^2_{L^2_{\xi,\tau}}\Big)^{\frac12} \lesssim \Big(\sum_N\|g_N\|^2_{L^2_\xi L^\infty_\tau}\Big)^{\frac12}$$

so that

$$|J_{\mathcal{B}}|+|J_{\mathcal{C}}| \lesssim \Big(\sum_N\|g_N\|^2_{L^2_\xi L^\infty_\tau}\Big)^{\frac12}\|w\|_{X^{0,\frac12}}\big(\|u\|_{L^4_{x,t}}+\|u\|_{X^{-1,1}}\big).\tag{3-44}$$

Finally (3-40), (3-43), and (3-44) imply (3-39), which concludes the proof of estimate (3-21). □

**Lemma 3.7.** *Let $0 < T \le 1$, $s \ge 0$, $u_1, u_2 \in L^\infty(\mathbb{R}; L^2(\mathbb{R})) \cap L^4(\mathbb{R}^2)$ be supported in the time interval $[-2T, 2T]$, and $F_1$, $F_2$ be some spatial primitives of $u_1$ and $u_2$, respectively. Then*

$$\left\| \partial_x P_{+hi}\left(P_{lo} e^{-\frac{i}{2}F_1} P_- \partial_x u_1\right)\right\|_{\tilde{Z}^{s,-1}} + \left\| \partial_x P_{+hi}\left(P_{lo} e^{-\frac{i}{2}F_1} P_- \partial_x u_1\right)\right\|_{X^{s,-\frac12}} \lesssim \|u_1\|^2_{L^4_{x,t}}, \tag{3-45}$$

*and*

$$\left\| \partial_x P_{+hi}\left(P_{lo}\left(e^{-\frac{i}{2}F_1} - e^{-\frac{i}{2}F_2}\right) P_- \partial_x u_2\right)\right\|_{\tilde{Z}^{s,-1}} + \left\| \partial_x P_{+hi}\left(P_{lo}\left(e^{-\frac{i}{2}F_1} - e^{-\frac{i}{2}F_2}\right) P_- \partial_x u_2\right)\right\|_{X^{s,-\frac12}} \\ \lesssim \left(\|u_1 - u_2\|_{L^\infty_t L^2_x} + \|e^{-\frac{i}{2}F_1} - e^{-\frac{i}{2}F_2}\|_{L^\infty_{x,t}} \|u_2\|_{L^\infty_t L^2_x}\right)\|u_2\|_{L^4_{x,t}}. \tag{3-46}$$

*Proof.* We deduce from the Cauchy–Schwarz inequality, the Sobolev embedding $\|f\|_{H_t^{-1/2+\varepsilon}} \lesssim \|f\|_{L_t^{1+\varepsilon'}}$ with $1+\varepsilon' = 1/(1-\varepsilon)$, and the Minkowski inequality that

$$\|f\|_{\tilde{Z}^{s,-1}} + \|f\|_{X^{s,-\frac12}} \lesssim \|f\|_{X^{s,-\frac12+\varepsilon}} = \left\| \left\| (J_x^s U(-t) f)^{\wedge_x}(\xi)\right\|_{H_t^{-\frac12+\varepsilon}}\right\|_{L^2_\xi} \\ \lesssim \left\| \left\| (J_x^s U(-t) f)^{\wedge_x}(\xi)\right\|_{L_t^{1+\varepsilon'}}\right\|_{L^2_\xi} \lesssim \|f\|_{L_t^{1+\varepsilon'} H^s_x}. \tag{3-47}$$

On the other hand, it follows from the frequency localization that

$$\partial_x P_{+hi}\left(P_{lo} e^{-\frac{i}{2}F} P_- \partial_x u\right) = \partial_x P_{+LO}\left(P_{lo} e^{-\frac{i}{2}F} P_{-LO}\, \partial_x u\right).$$

Therefore, by using (3-47), Bernstein's inequalities, and estimate (2-12), we can bound the left-hand side of (3-45) by

$$\left\| P_{+LO}\left(P_{lo} e^{-\frac{i}{2}F} P_{-LO}\, \partial_x u\right)\right\|_{L_t^{1+\varepsilon'} L^2_x} \lesssim T^\gamma \left\|\partial_x e^{-\frac{i}{2}F}\right\|_{L^4_{x,t}} \|u\|_{L^4_{x,t}} \tag{3-48}$$

with $\frac{1}{\gamma} = \frac12 - \varepsilon'$, which concludes the proof of estimate (3-45), recalling that $\partial_x F = u$ and $0 < T \le 1$. Estimate (3-46) can be proved exactly as above by recalling (2-19). □

*Proof of Proposition 3.4.* Let $0 \le s \le \frac12$, $0 < T \le 1$, and $\tilde{u}$ and $\tilde{w}$ be extensions of $u$ and $w$ such that $\|\tilde{u}\|_{X^{-1,1}} \le 2\|u\|_{X_T^{-1,1}}$ and $\|\tilde{w}\|_{X^{s,1/2}} \le 2\|w\|_{X_T^{s,1/2}}$. By the Duhamel principle, the integral formulation associated with (3-4) reads

$$w(t) = \eta(t)\, U(t)\, w(0) - \eta(t) \int_0^t U(t-t')\, \partial_x P_{+hi}\left(\eta_T \partial_x^{-1} \tilde{w} P_-(\eta_T\, \partial_x u)\right)(t')\, dt' \\ - \eta(t) \int_0^t U(t-t')\, \partial_x P_{+hi}\left(P_{lo}\left(\eta_T e^{-\frac{i}{2}\tilde{F}}\right) P_-(\eta_T \partial_x \tilde{u})\right)(t')\, dt'$$

for $0 < t \le T \le 1$. Therefore, we deduce gathering estimates (2-5), (2-7), (3-20), (3-21), and (3-45) that

$$\|w\|_{Y^s_T} \lesssim \|w(0)\|_{H^s} + \|u\|^2_{L^4_{x,T}} + \|w\|_{X_T^{s,\frac12}}\left(\|u\|_{L^\infty_T L^2_x} + \|u\|_{L^4_{x,T}} + \|u\|_{X_T^{-1,1}}\right).$$

This concludes the proof of estimate (3-19) since

$$\|w(0)\|_{H^s} \lesssim \left\| J^s_x\left(e^{-\frac{i}{2}F(\cdot,0)} u_0\right)\right\|_{L^2} \lesssim (1 + \|u_0\|_{L^2})\|u_0\|_{H^s} \tag{3-49}$$

follows from estimate (2-13) and the fact that $0 \le s \le \frac12$. □

## 4. Proof of Theorem 1.1

First, we can always assume that we deal with data having small $L^2(\mathbb{R})$-norm. Indeed, if $u$ is a solution to the IVP (1-1) on the time interval $[0, T]$, then for every $0 < \lambda < \infty$, $u_\lambda(x, t) = \lambda u(\lambda x, \lambda^2 t)$ is also a solution to the equation in (1-1) on the time interval $[0, \lambda^{-2}T]$ with initial data $u_{0,\lambda} = \lambda u_0(\lambda, \cdot)$. For $\varepsilon > 0$ let us denote by $B_\varepsilon$ the ball of $L^2(\mathbb{R})$ centered at the origin with radius $\varepsilon$. Since $\|u_\lambda(\cdot 0)\|_{L^2} = \lambda^{1/2}\|u_0\|_{L^2}$, we see that we can force $u_{0,\lambda}$ to belong to $B_\varepsilon$ by choosing $\lambda \sim \min(\varepsilon^2\|u_0\|_{L^2}^{-2}, 1)$. Therefore, the existence and uniqueness of a solution of (1-1) on the time interval $[0, 1]$ for small $L^2(\mathbb{R})$ initial data will ensure the existence of a unique solution $u$ to (1-1) for arbitrary large $L^2(\mathbb{R})$ initial data on the time interval $T \sim \lambda^2 \sim \min(\|u_0\|_{L^2}^{-4}, 1)$. Using the conservation of the $L^2(\mathbb{R})$-norm, this will lead to global well-posedness in $L^2(\mathbb{R})$.

**4A.** ***Uniform bound for small initial data.*** First we begin by deriving *a priori* estimates on smooth solutions associated with initial data $u_0 \in H^\infty(\mathbb{R})$ that are small in $L^2(\mathbb{R})$. It is known from the classical well-posedness theory [Iório 1986] that such initial data gives rise to a global solution $u \in C(\mathbb{R}; H^\infty(\mathbb{R}))$ to the Cauchy problem (1-1). Setting $0 < T \le 1$,

$$N_T^s(u) := \max\Big(\|u\|_{L_T^\infty H_x^s}, \|J_x^s u\|_{L_{x,T}^4}, \|w\|_{X_T^{s,\frac{1}{2}}}\Big), \tag{4-1}$$

and it follows from the smoothness of $u$ that $T \mapsto N_T^s(u)$ is continuous and nondecreasing on $\mathbb{R}_+^*$. Moreover, from (3-4), the linear estimate (2-7), (3-49), and (3-7), we infer that $\lim_{T\to 0+} N_T^s(u) \lesssim (1+\|u_0\|_{L^2})\|u_0\|_{H^s}$. On the other hand, combining (3-7)–(3-8) and (3-19) and the conservation of the $L^2$-norm, we infer that

$$N_T^0(u) \lesssim (1 + \|u_0\|_{L^2})\|u_0\|_{L^2} + \big(N_T^0(u)\big)^2 + \big(N_T^0(u)\big)^3.$$

By continuity, this ensures there exist $\varepsilon_0 > 0$ and $C_0 > 0$ such that $N_1^0(u) \le C_0\varepsilon$ given $\|u_0\|_{L^2} \le \varepsilon \le \varepsilon_0$. Finally, again using (3-7)–(3-8) and (3-19), this leads to $N_1^s(u) \lesssim \|u_0\|_{H^s}$ given $\|u_0\|_{L^2} \le \varepsilon \le \varepsilon_0$.

**4B.** ***Lipschitz bound for initial data having the same low-frequency part.*** To prove the uniqueness as well as the continuity of the solution, we will derive a Lipschitz bound on the solution map on some affine subspaces of $H^s(\mathbb{R})$ with values in $L_T^\infty H^s(\mathbb{R})$. We know from [Koch and Tzvetkov 2003] that such a Lipschitz bound does not exist in general in $H^s(\mathbb{R})$. Here we will restrict ourselves to solutions emanating from initial data having the same low-frequency part. This is clearly sufficient to get uniqueness, and it will turn out to be sufficient to get the continuity of the solution as well as the continuity of the flow map. Let $\varphi_1, \varphi_2 \in B_\varepsilon \cap H^s(\mathbb{R})$, $s \ge 0$, such that $P_{LO}\varphi_1 = P_{LO}\varphi_2$, and let $u_1$, $u_2$ be two solutions to (1-1) emanating from $\varphi_1$ and $\varphi_2$, respectively, that satisfy (7-1) on the time interval $[0, T]$, $0 < T < 1$. We also assume that the primitives $F_1 := F[u_1]$ and $F_2 := F[u_2]$ of $u_1$ and $u_2$, respectively, are such that the associated gauge functions $W_1$, $w_1$ and $W_2$, $w_2$, respectively, constructed in Section 3A, satisfy (7-2). Finally, we assume that

$$N_T^0(u_i) \le C_0\varepsilon \le C_0\varepsilon_0. \tag{4-2}$$

First, by construction, we observe that since $F(x)-F(y)=\int_x^y u(z)\,dz$,

$$P_{LO}\int_y^x u\,dz = P_{LO}(F(x)-F(y)) = P_{LO}F(x)-F(y)$$

holds. On the other hand, since $P_{LO}$ and $\partial_x$ do commute, we have $\partial_x P_{LO}F = P_{LO}u$ and, by integrating, $\int_y^x P_{LO}u\,dz = P_{LO}F(x)-P_{LO}F(y)$. Gathering these two identities, we get

$$\int_y^x P_{LO}u\,dz - P_{LO}\int_y^x u\,dz = F(y)-P_{LO}F(y) = P_{HI}F(y),$$

which leads to

$$P_{lo}\int_y^x u\,dz = P_{lo}\int_y^x P_{LO}u\,dz.$$

We thus infer that

$$\begin{aligned} P_{lo}(F_1-F_2)(x,0) &= \int_{\mathbb{R}}\psi(y)P_{lo}\int_y^x (u_1-u_2)(z,0)\,dz\,dy \\ &= \int_{\mathbb{R}}\psi(y)P_{lo}\int_y^x P_{LO}(\varphi_1(z)-\varphi_2(z))\,dz\,dy = 0. \end{aligned} \tag{4-3}$$

Then we set $v=u_1-u_2$, $Z=W_1-W_2$, and $z=w_1-w_2$. Obviously, $z$ satisfies

$$\begin{aligned} \partial_t z - i\,\partial_x^2 z = -\partial_x P_{+hi}(W_1P_-\partial_x v) - \partial_x P_{+hi}(ZP_-\partial_x u_2) \\ -\partial_x P_{+hi}\big(P_{lo}e^{-\frac{i}{2}F_1}P_-\partial_x v\big) - \partial_x P_{+hi}\big(P_{lo}\big(e^{-\frac{i}{2}F_1}-e^{-\frac{i}{2}F_2}\big)P_-\partial_x u_2\big). \end{aligned} \tag{4-4}$$

Thus, by gathering estimates (2-7), (3-20), (3-21), (3-45), and (3-46), we deduce that

$$\begin{aligned} \|z\|_{Y_1^s} \lesssim \|z(0)\|_{H^s} + \|w_1\|_{X_1^{s,\frac12}}\big(\|v\|_{X_1^{-1,1}} + \|v\|_{L^4_{x,1}} + \|v\|_{L_1^\infty L_x^2}\big) + \|v\|^2_{L^4_{x,1}} \\ + \|z\|_{X_1^{s,\frac12}}\big(\|u_2\|_{X_1^{-1,1}} + \|u_2\|_{L^4_{x,1}} + \|u_2\|_{L_1^\infty L_x^2}\big) + \big(\|v\|_{L_1^\infty L_x^2} + \big\|e^{-\frac{i}{2}F_1}-e^{-\frac{i}{2}F_2}\big\|_{L^\infty_{x,1}}\big)\|u_2\|_{L^4_{x,1}}, \end{aligned}$$

which, recalling (4-1) and (4-2), implies that

$$\|z\|_{Y_1^s} \lesssim \|z(0)\|_{H^s} + \varepsilon\big(\|v\|_{X_1^{-1,1}} + \|v\|_{L^4_{x,1}} + \|v\|_{L_1^\infty L_x^2}\big) + \varepsilon\big\|e^{-\frac{i}{2}F_1}-e^{-\frac{i}{2}F_2}\big\|_{L^\infty_{x,1}}, \tag{4-5}$$

where, by the mean-value theorem,

$$\begin{aligned} \|z(0)\|_{H^s} &\lesssim \|\varphi_1-\varphi_2\|_{H^s}\big(1+\|\varphi_1\|_{H^s}+\|\varphi_2\|_{L^2}\big) + \big\|e^{-\frac{i}{2}F_1(0)}-e^{-\frac{i}{2}F_2(0)}\big\|_{L^\infty}\|\varphi_1\|_{H^s}(1+\|\varphi_1\|_{L^2}) \\ &\lesssim \|\varphi_1-\varphi_2\|_{H^s} + \|F_1(0)-F_2(0)\|_{L^\infty}. \end{aligned} \tag{4-6}$$

On the other hand, the equation for $v=u_1-u_2$ reads

$$\partial_t v + \mathscr{H}\partial_x^2 v = \tfrac12\partial_x((u_1+u_2)v)$$

so that it is deduced from (3-11), (4-1), and the fractional Leibniz rule that

$$\|v\|_{X_1^{-1,1}} \lesssim \|\partial_t v + \mathscr{H}\partial_x^2 v\|_{L_1^2 H_x^{-1}} + \|v\|_{L_T^\infty L_x^2} \lesssim \varepsilon\|v\|_{L^4_{x,1}} + \|v\|_{L_1^\infty L_x^2}. \tag{4-7}$$

Next, proceeding as in (3-6), we infer that

$$\begin{aligned} P_{+HI}v &= 2i\,P_{+HI}\big(e^{\frac{i}{2}F_1}z\big) + 2i\,P_{+HI}\big((e^{\frac{i}{2}F_1}-e^{\frac{i}{2}F_2})w_2\big) \\ &\quad + 2i\,P_{+HI}\big(P_{+hi}e^{\frac{i}{2}F_1}\partial_x P_{+lo}\big(e^{-\frac{i}{2}F_1}-e^{-\frac{i}{2}F_2}\big)\big) \\ &\quad + 2i\,P_{+HI}\big(P_{+hi}\big(e^{\frac{i}{2}F_1}-e^{\frac{i}{2}F_2}\big)\,\partial_x P_{+lo}e^{-\frac{i}{2}F_2}\big) \\ &\quad + 2i\,P_{+HI}\big(P_{+HI}e^{\frac{i}{2}F_1}\partial_x P_-\big(e^{-\frac{i}{2}F_1}-e^{-\frac{i}{2}F_2}\big)\big) \\ &\quad + 2i\,P_{+HI}\big(P_{+HI}\big(e^{\frac{i}{2}F_1}-e^{\frac{i}{2}F_2}\big)\,\partial_x P_-e^{-\frac{i}{2}F_2}\big). \end{aligned}$$

Thus, we deduce using estimates (2-14) and (2-19) and arguing as in the proof of Proposition 3.2 that

$$\begin{aligned} \|J_x^s v\|_{L_1^pL_x^q} &\lesssim \big(\|u_1\|_{L_1^\infty L_x^2}+\|u_2\|_{L_1^\infty L_x^2}\big)\|v\|_{L_1^\infty L_x^2} + (1+\|u_1\|_{L_1^\infty L_x^2})\|z\|_{Y_1^s} \\ &\quad + \big(\|v\|_{L_1^\infty L_x^2} + \big\|e^{\frac{i}{2}F_1}-e^{\frac{i}{2}F_2}\big\|_{L^\infty_{x,1}}(1+\|u_1\|_{L_1^\infty L_x^2})\big)\|w_2\|_{Y_1^s} \\ &\quad + \|u_1\|_{L_1^\infty L_x^2}\big(\|v\|_{L_1^\infty L_x^2} + \big\|e^{-\frac{i}{2}F_1}-e^{-\frac{i}{2}F_2}\big\|_{L^\infty_{x,1}}\|u_1\|_{L_1^\infty L_x^2}\big) \\ &\quad + \|u_2\|_{L_1^\infty L_x^2}\big(\|v\|_{L_1^\infty L_x^2} + \big\|e^{\frac{i}{2}F_1}-e^{\frac{i}{2}F_2}\big\|_{L^\infty_{x,1}}\|u_1\|_{L_1^\infty L_x^2}\big) \end{aligned}$$

for $(p,q)=(\infty,2)$ or $(p,q)=(4,4)$, which, recalling (4-2), implies that

$$\|J_x^s v\|_{L_1^\infty L_x^2} + \|J_x^s v\|_{L^4_{x,1}} \lesssim \|z\|_{Y_1^s} + \varepsilon\big\|e^{-\frac{i}{2}F_1}-e^{-\frac{i}{2}F_2}\big\|_{L^\infty_{x,1}} + \varepsilon\big\|e^{\frac{i}{2}F_1}-e^{\frac{i}{2}F_2}\big\|_{L^\infty_{x,1}}. \tag{4-8}$$

Finally, we use the mean-value theorem to get the bound

$$\|e^{\pm\frac{i}{2}F_1}-e^{\pm\frac{i}{2}F_2}\|_{L^\infty_{x,1}} \lesssim \|F_1-F_2\|_{L^\infty_{x,1}}. \tag{4-9}$$

The following crucial lemma gives an estimate for the right-hand side of (4-9):

**Lemma 4.1.** *It holds that*

$$\|F_1(0)-F_2(0)\|_{L^\infty} \lesssim \|\varphi_1-\varphi_2\|_{L^2} \tag{4-10}$$

*and*

$$\|F_1-F_2\|_{L^\infty_{x,1}} \lesssim \|v\|_{L_1^\infty L_x^2}. \tag{4-11}$$

*Proof.* Equation (4-10) clearly follows from (4-3) together with Bernstein's inequality. To prove (4-11), we set $G=F_1-F_2$, $G_{lo}=P_{lo}G$, and $G_{hi}=P_{hi}G$. Then

$$\|G\|_{L^\infty_{x,1}} \le \|G_{lo}\|_{L^\infty_{x,1}} + \|G_{hi}\|_{L^\infty_{x,1}}. \tag{4-12}$$

Observe, from the Duhamel principle and (4-3), that $G_{lo}$ satisfies

$$G_{lo} = \frac{1}{2}\int_0^t U(t-\tau)\,P_{lo}((u_1+u_2)v)(\tau)\,d\tau.$$

Therefore, using Bernstein and Hölder's inequalities, it follows that

$$\|G_{lo}\|_{L^\infty_{x,1}} \lesssim \|(u_1+u_2)v\|_{L_1^\infty L_x^1} \lesssim \big(\|u_1\|_{L_1^\infty L_x^2}+\|u_2\|_{L_1^\infty L_x^2}\big)\|v\|_{L_1^\infty L_x^2}. \tag{4-13}$$

On the other hand, Bernstein's inequality ensures that

$$\|G_{hi}\|_{L^\infty_{x,1}} \lesssim \|\partial_x G_{hi}\|_{L^\infty_1 L^2_x} \lesssim \|v\|_{L^\infty_1 L^2_x} \tag{4-14}$$

since $\partial_x G = v$. The proof of Lemma 4.1 is concluded gathering (4-2) and (4-12)–(4-14). □

Finally, estimates (4-5)–(4-11) lead to

$$\|z\|_{Y^s_1} + \|v\|_{X^{s-1,1}_1} + \|v\|_{L^\infty_1 H^s_x} + \|J^s_x v\|_{L^4_{x,1}} \lesssim \|\varphi_1 - \varphi_2\|_{H^s} + \varepsilon\big(\|z\|_{Y^s_1} + \|v\|_{X^{s-1,1}_1} + \|v\|_{L^\infty_1 H^s_x} + \|J^s_x v\|_{L^4_{x,1}}\big).$$

Therefore, we conclude that there exists $0 < \varepsilon_1 \le \varepsilon_0$ such that

$$\|z\|_{Y^s_1} + \|v\|_{X^{s-1,1}_1} + \|v\|_{L^\infty_1 H^s_x} + \|J^s_x v\|_{L^4_{x,1}} \lesssim \|\varphi_1 - \varphi_2\|_{H^s}, \tag{4-15}$$

provided $u_1$ and $u_2$ satisfy (4-2) with $0 < \varepsilon \le \varepsilon_1$.

**4C. *Well-posedness.*** Let $u_0 \in B_{\varepsilon_1} \cap H^s(\mathbb{R})$, and consider the sequence of initial data $\{u_0^j\} \subset H^\infty(\mathbb{R})$, defined by

$$u_0^j = \mathcal{F}_x^{-1}\big(\chi_{|[-j,j]}\mathcal{F}_x u_0\big) \qquad \text{for all } j \ge 20. \tag{4-16}$$

Clearly $\{u_0^j\}$ converges to $u_0$ in $H^s(\mathbb{R})$. By the classical well-posedness theory, the associated sequence of solutions $\{u^j\}$ is a subset of $C([0,1]; H^\infty(\mathbb{R}))$, and according to Section 4A, it satisfies $N^s_1(u^j) \le C_0\varepsilon_1$. Moreover, since $P_{LO}u_0^j = P_{LO}u_0$ for all $j \ge 20$, it follows from the preceding subsection that

$$\|u^j - u^{j'}\|_{L^\infty_1 H^s_x} + \|u^j - u^{j'}\|_{L^4_1 W^{s,4}_x} + \|w^j - w^{j'}\|_{X_1^{0,\frac12}} \lesssim \|u_0^j - u_0^{j'}\|_{H^s_x}. \tag{4-17}$$

Therefore, the sequence $\{u^j\}$ converges strongly in $L^\infty_1 H^s(\mathbb{R}) \cap L^4_1 W^{s,4}_x$ to some function

$$u \in C\big([0,1]; H^s(\mathbb{R})\big),$$

and $\{w_j\}_{j\ge 4}$ converges strongly to some function $w$ in $X_1^{s,1/2}$. Thanks to these strong convergences, it is easy to check that $u$ is a solution to (1-1) emanating from $u_0$ and that $w = \partial_x P_{+hi}(e^{-\frac{i}{2}F[u]})$. Moreover, from the conservation of the $L^2(\mathbb{R})$-norm, $u \in C_b(\mathbb{R}; L^2(\mathbb{R})) \cap C(\mathbb{R}; H^s(\mathbb{R}))$.

Now let $\tilde u$ be another solution of (1-1) on $[0,T]$ emanating from $u_0$ belonging to the same class of regularity as $u$. By again using the scaling argument we can always assume that $\|\tilde u\|_{L^\infty_T L^2_x} + \|\tilde u\|_{L^4_{x,T}} \le C_0\varepsilon_1$. Moreover, setting $\tilde w := P_{+hi}(e^{-iF[\tilde u]})$, by the Lebesgue monotone convergence theorem, there exists $N > 0$ such that $\|P_{\ge N}\tilde w\|_{X_T^{0,1/2}} \le C_0\varepsilon_{1/2}$. On the other hand, using Lemmas 2.1–2.2, it is easy to check that

$$\begin{aligned}\big\|(1-P_{\ge N})\tilde w\big\|_{X_T^{0,\frac12}} &\lesssim \|u_0\|_{L^2} + NT^{\frac14}\|\tilde u\|_{L^4_{x,T}}\|\tilde w\|_{L^4_{x,T}} + \|\tilde u\|^2_{L^4_{x,T}} \\ &\lesssim \|u_0\|_{L^2} + NT^{\frac14}\|\tilde w\|_{X_T^{0,\frac12}}\|\tilde u\|_{L^4_{x,T}} + \|\tilde u\|^2_{L^4_{x,T}}.\end{aligned}$$

Therefore, for $T > 0$ small enough, we can require that $\tilde u$ satisfies the smallness condition (4-2) with $\varepsilon_1$, and thus by (4-15), $\tilde u \equiv u$ on $[0,T]$. This proves the uniqueness result for initial data belonging to $B_{\varepsilon_1}$.

Next we turn to the continuity of the flow map. Fix $u_0 \in B_{\varepsilon_1}$ and $\lambda > 0$ and consider the emanating solution $u \in C([0,1]; H^s(\mathbb{R}))$. We will prove that if $v_0 \in B_{\varepsilon_1}$ satisfies $\|u_0 - v_0\|_{H^s} \le \delta$, where $\delta$ will be fixed later, then the solution $v$ emanating from $v_0$ satisfies

$$\|u - v\|_{L^\infty_1 H^s_x} \le \lambda. \tag{4-18}$$

For $j \ge 1$, let $u_0^j$ and $v_0^j$ be constructed as in (4-16), and denote by $u^j$ and $v^j$ the solutions emanating from $u_0^j$ and $v_0^j$. Then it follows from the triangular inequality that

$$\|u - v\|_{L^\infty_1 H^s_x} \le \|u - u^j\|_{L^\infty_1 H^s_x} + \|u^j - v^j\|_{L^\infty_1 H^s_x} + \|v - v^j\|_{L^\infty_1 H^s_x}. \tag{4-19}$$

First, according to (4-17), we can choose $j_0$ large enough so that

$$\|u - u^{j_0}\|_{L^\infty_1 H^s_x} + \|v - v^{j_0}\|_{L^\infty_1 H^s_x} \le \tfrac{2}{3}\lambda.$$

Second, from the definition of $u_0^j$ and $v_0^j$ in (4-16), we infer that

$$\|u_0^j - v_0^j\|_{H^3} \le j^{3-s}\|u_0 - v_0\|_{H^s} \le j^{3-s}\delta.$$

Therefore, by using the continuity of the flow map for smooth initial data, we can choose $\delta > 0$ such that

$$\|u^{j_0} - v^{j_0}\|_{L^\infty_1 H^s_x} \le \frac{\lambda}{3}.$$

This concludes the proof of Theorem 1.1.

## 5. Improvement of the uniqueness result for $s > 0$

Now we prove that uniqueness holds for initial data $u_0 \in H^s(\mathbb{R})$, $s > 0$, in the class $u \in L^\infty_T H^s_x \cap L^4_T W^{s,4}_x$. The great interest of this result is that we no longer assume any condition on the gauge transform of $u$. Moreover, when $s > \frac{1}{4}$, the Sobolev embedding $L^\infty_T H^s_x \hookrightarrow L^4_T W^{0+,4}_x$ ensures that uniqueness holds in $L^\infty_T H^s_x$, and thus, the Benjamin–Ono equation is unconditionally well posed in $H^s(\mathbb{R})$ for $s > \frac{1}{4}$.

According to the uniqueness result (i) of Theorem 1.1, it suffices to prove that for any solution $u$ to (1-1) that belongs to $L^\infty_T H^s_x \cap L^4_T W^{s,4}_x$, the associated gauge function $w = \partial_x P_{hi}(e^{-\frac{i}{2}F[u]})$ belongs to $X^{0,\frac{1}{2}}_T$. The proof is based on the following bilinear estimate that is shown in the Appendix:

**Proposition 5.1.** *Let $s > 0$. Then there exist $0 < \delta < s/10$ and $\theta \in (\frac{1}{2}, 1)$, let us say $\theta = \frac{1}{2} + \delta$, such that*

$$\|P_{+hi}(W P_- \partial_x u)\|_{X^{\frac{1}{2},-\frac{1}{2}+2\delta}} \lesssim \|W\|_{X^{\frac{1}{2},\frac{1}{2}+\delta}}\big(\|J^s u\|_{L^2_{x,t}} + \|J^s u\|_{L^4_{x,t}} + \|u\|_{X^{s-\theta,\theta}}\big). \tag{5-1}$$

First note that by the same scaling argument as in Section 4C, for any given $\varepsilon > 0$, we can always assume that $\|J^s u\|_{L^\infty_T L^2_x} + \|J^s u\|_{L^4_{Tx}} \le \varepsilon$, and by (3-7) it follows that $\|u\|_{X^{s-\theta,\theta}_T} \lesssim \varepsilon$ for $0 \le \theta \le 1$.

Since $u \in L^\infty([0,T]; H^s(\mathbb{R})) \cap L^4_T W^{s,4}_x$ and satisfies (1-1), it follows that $u_t \in L^\infty([0,T]; H^{s-2}(\mathbb{R}))$. Therefore, $F := F[u] \in L^\infty([0,T]; H^{s+1}_{\mathrm{loc}})$, and $\partial_t F \in L^\infty([0,T]; H^{s-1}_{\mathrm{loc}})$. It ensures that

$$W := P_{hi}\big(e^{-\frac{i}{2}F}\big) \in L^\infty\big([0,T]; H^{s+1}(\mathbb{R})\big) \cap L^4_T W^{s+1,4}_x \hookrightarrow X^{1,0}_T, \tag{5-2}$$

$e^{-\frac{i}{2}F} \in L^\infty([0,T]; H^{s+1}_{\mathrm{loc}})$, and the following calculations are thus justified:

$$\begin{aligned}\partial_t W = \partial_t P_+\big(e^{-\frac{i}{2}F}\big) &= -\tfrac{1}{2} i P_{hi}(F_t e^{-\frac{i}{2}F}) \\ &= -\tfrac{1}{2} i P_{hi}\big(e^{-\frac{i}{2}F}(-\mathscr{H}F_{xx} + \tfrac{1}{2}F_x^2)\big),\\ \partial_{xx} W = \partial_{xx} P_{hi}\big(e^{-\frac{i}{2}F}\big) &= P_{hi}\big(e^{-\frac{i}{2}F}(-\tfrac{1}{4}F_x^2 - \tfrac{i}{2}F_{xx})\big).\end{aligned}$$

It follows that $W$ satisfies, at least in a distributional sense,

$$\begin{cases}\partial_t W - i\,\partial_x^2 W = -P_{+hi}(W P_- \partial_x u) - P_{+hi}\big(P_{lo} e^{-\frac{i}{2}F} P_-\,\partial_x u\big)\\ W(\,\cdot\,,0) = P_{+hi}\big(e^{-\frac{i}{2}F[u_0]}\big).\end{cases} \tag{5-3}$$

From (5-2) and Lemma 2.6, we thus deduce that $W \in X_T^{s,1}$ so that, by interpolation with (5-2), $W \in X_T^{\frac{1}{2},\frac{1}{2}+}$. But since $u$ is given in $L^\infty_T H^s_x \cap L^4_T W^{s,4}_x \cap X_T^{s-\theta,\theta}$, considering (2-6), the bilinear estimate (5-1), and (3-48), we infer that there exists only one solution to (5-3) in $X_T^{\frac{1}{2},\frac{1}{2}+}$. Hence, $w = \partial_x W$ belongs to $X_T^{-\frac{1}{2},\frac{1}{2}+}$ and is the unique solution to (3-4) in $X_T^{-\frac{1}{2},\frac{1}{2}+}$ emanating from the initial data $w_0 = \partial_x P_{hi}(e^{-\frac{i}{2}F[u_0]}) \in L^2(\mathbb{R})$. On the other hand, according to Proposition 3.4, one can construct a solution to (3-4) emanating from $w_0$ and belonging to $Y_T^s$ by using a Picard iterative scheme. Moreover, using (1-1) and Lemma 2.6 we can easily check that this solution belongs to $X_T^{-1,1}$ and thus by interpolation to $X_T^{s-,\frac{1}{2}+} \hookrightarrow X_T^{-\frac{1}{2},\frac{1}{2}+}$. This ensures that $w = \partial_x P_{hi}(e^{-iF/2})$ belongs to $Y_T^s \hookrightarrow X_T^{0,\frac{1}{2}}$, which concludes the proof.

## 6. Continuity of the flow map for the weak $L^2$-topology

In [Cui and Kenig 2010] it is proven that, for any $t \geq 0$, the flow map $u_0 \mapsto u(t)$ associated with the Benjamin–Ono equation is continuous from $L^2(\mathbb{R})$ equipped with the weak topology into itself. In this section, we explain how the uniqueness part of Theorem 1.1 enables us to simplify the proof of this result by following the approach developed in [Goubet and Molinet 2009].

Let $\{u_{0,n}\}_n \subset L^2(\mathbb{R})$ be a sequence of initial data that converges weakly to $u_0$ in $L^2(\mathbb{R})$, and let $u$ be the solution emanating from $u_0$ given by Theorem 1.1. From the Banach–Steinhaus theorem, we know that $\{u_{0,n}\}_n$ is bounded in $L^2(\mathbb{R})$, and from Theorem 1.1 we know that $\{u_{0,n}\}_n$ gives rise to a sequence $\{u_n\}_n$ of solutions to (1-1) bounded in $C([0,1]; L^2(\mathbb{R})) \cap L^4(]0,1[\times\mathbb{R})$ with an associated sequence of gauge functions $\{w_n\}_n$ bounded in $X_1^{0,\frac{1}{2}}$. Therefore, there exist $v \in L^\infty(]0,1[; L^2(\mathbb{R})) \cap X_1^{-1,1} \cap L^4(]0,1[\times\mathbb{R})$ and $z \in X_1^{0,\frac{1}{2}}$ such that, up to the extraction of a subsequence, $\{u_n\}_n$ converges to $v$ weakly in $L^4(]0,1[\times\mathbb{R})$ and weakly star in $L^\infty(]0,1[\times\mathbb{R})$, and $\{w_n\}_n$ converges to $z$ weakly in $X_1^{0,\frac{1}{2}}$. We now need some compactness on $\{u_n\}_n$ to ensure that $z$ is the gauge transform of $v$. In this direction, we first notice, since $\{w_n\}_n$ is bounded in $X_1^{0,\frac{1}{2}}$ and by using the Kato's smoothing effect injected in Bourgain's spaces framework, that $\{D_x^{1/4} w_n\}_n$ is bounded in $L^4_x L^2_1$. Let $\eta_R(\,\cdot\,) := \eta(\,\cdot\,/R)$. Using (3-6) and Lemma 2.6 we infer that

$$\begin{aligned}\|D_x^{\frac14}P_{+HI}u_n\|_{L^2(]0,1[\times]-R,R[)} &\lesssim \|D_x^{\frac14}P_{+HI}(e^{\frac{i}{2}F[u_n]}w_n\eta_R)\|_{L^2_{1,x}}+\|D_x^{\frac14}P_{+HI}(P_{+hi}e^{\frac{i}{2}F[u_n]}\,\partial_x P_{lo}e^{-\frac{i}{2}F[u_n]})\|_{L^2_{1,x}}\\ &\quad+\|D_x^{\frac14}P_{+HI}(P_{+HI}e^{\frac{i}{2}F[u_n]}\,\partial_x P_{-hi}e^{-\frac{i}{2}F[u_n]})\|_{L^2_{1,x}}\\ &\lesssim \|D_x^{\frac14}(w_n\eta_R)\|_{L^2_xL^2_1}+\|D_x^{\frac14}e^{iF[u_n]}\|_{L^8_{1,x}}\|w_n\|_{L^{\frac83}_{1,x}}+\|u_n\|^2_{L^4_{1,x}}.\end{aligned}$$

But clearly

$$\|D_x^{\frac14}(w_n\eta_R)\|_{L^2_xL^2_1}\lesssim C(R)\Big(\|D_x^{\frac14}w_n\|_{L^4_xL^2_1}+\|w_n\|_{L^2_{1,x}}\Big),$$

and by interpolation $\|D_x^{1/4}e^{iF[u_n]}\|_{L^8_{1,x}}\lesssim\|u_n\|^{3/4}_{L^2_{1,x}}$. Therefore, recalling that the $u_n$ are real-valued functions, it follows that $\{u_n\}_n$ is bounded in $L^2_1H^{1/4}(]-R,R[)$.

Since, according to Equation (1-1), $\{\partial_t u_n\}_n$ is bounded in $L^2_1H_x^{-2}$, Aubin–Lions compactness theorem and standard diagonal extraction arguments ensure that there exists an increasing sequence of integers $\{n_k\}_k$ such that $u_{n_k}\to v$ a.e. in $]0,1[\times\mathbb{R}$ and $u_{n_k}^2\rightharpoonup v^2$ in $L^2(]0,1[\times\mathbb{R})$. In view of our construction of the primitive $F[u_n]$ of $u_n$ (see Section 3A), it is then easy to check that $F[u_{n_k}]$ converges to the primitive $F[v]$ of $v$ a.e. in $]0,1[\times\mathbb{R}$. This ensures that $P_{+hi}(e^{-\frac{i}{2}F[u_{n_k}]})$ converges weakly to $P_{+hi}(e^{-\frac{i}{2}F[v]})$ in $L^2(]0,1[\times\mathbb{R})$, and thus, $z$ is the gauge transform of $v$. Passing to the limit in the equation, we conclude that $v$ satisfies (1-1) and belongs to the class of uniqueness of Theorem 1.1.

Moreover, setting $(\cdot,\cdot)$ for the $L^2_x$ scalar product, by (1-1) and the bounds above, it is easy to check that for any smooth space function $\phi$ with compact support, the family $\{t\mapsto(u_{n_k}(t),\phi)\}$ is uniformly equicontinuous on $[0,1]$. Ascoli's theorem then ensures that $(u_{n_k}(\cdot),\phi)$ converges to $(v(\cdot),\phi)$ uniformly on $[0,1]$, and thus, $v(0)=u_0$. By uniqueness, it follows that $v\equiv u$, which ensures that the whole sequence $\{u_n\}$ converges to $v$ in the sense above and not only a subsequence. Finally, from the above convergence result, it follows that $u_n(t)\rightharpoonup u(t)$ in $L^2_x$ for all $t\in[0,1]$. □

## 7. The periodic case

In this section, we explain how the bilinear estimate proved in Proposition 3.5 can lead to a great simplification of the global well-posedness result in $L^2(\mathbb{T})$ derived in [Molinet 2008] and to new uniqueness results in $H^s(\mathbb{T})$, where $\mathbb{T}=\mathbb{R}/2\pi\mathbb{Z}$. With the notations of [Molinet 2007], these new results lead to the following global well-posedness theorem:

**Theorem 7.1.** *Let $s\ge 0$ be given. For all $u_0\in H^s(\mathbb{T})$ and all $T>0$, there exists a solution*

$$u\in C\big([0,T];H^s(\mathbb{T})\big)\cap X_T^{s-1,1}\cap L^4_TW^{s,4}(\mathbb{T}) \tag{7-1}$$

*of (1-1) such that*

$$w=\partial_x P_{+hi}\big(e^{-\frac{i}{2}\partial_x^{-1}\tilde u}\big)\in Y^s_T, \tag{7-2}$$

*where*

$$\tilde u:=u\Big(t,x-t\fint u_0\Big)-\fint u_0 \quad \textit{and} \quad \widehat{\partial_x^{-1}}:=\frac{1}{i\xi},\ \xi\in\mathbb{Z}^*.$$

*This solution is unique in the following classes*:

(i) $u \in L^\infty\big(]0,T[;L^2(\mathbb{T})\big) \cap L^4\big(]0,T[\times\mathbb{T}\big)$ *and* $w \in X_T^{0,\frac12}$,

(ii) $u \in L^\infty\big(]0,T[;H^{\frac14}(\mathbb{T})\big) \cap L^4_T W^{\frac14,4}(\mathbb{T})$ *whenever* $s \geq \frac14$,

(iii) $u \in L^\infty\big(]0,T[;H^{\frac12}(\mathbb{T})\big)$ *whenever* $s \geq \frac12$.

*Moreover*, $u \in C_b(\mathbb{R};L^2(\mathbb{T}))$, *and the flow map data-solution* $u_0 \mapsto u$ *is continuous from* $H^s(\mathbb{T})$ *into* $C([0,T];H^s(\mathbb{T}))$.

*Sketch of the proof.* In the periodic case, following [Molinet 2007], the gauge transform is defined as follows: let $u$ be a smooth $2\pi$-periodic solution of BO with initial data $u_0$. In the sequel, we will assume that $u(t)$ has mean value zero for all time. Otherwise, we perform the change of unknowns

$$\tilde{u}(t,x) := u\Big(t, x - t\fint u_0\Big) - \fint u_0, \tag{7-3}$$

where $\fint u_0 := \frac{1}{2\pi}\int_{\mathbb{T}} u_0$ is the mean value of $u_0$. It is easy to see that $\tilde{u}$ satisfies BO with $u_0 - \fint u_0$ as initial data, and since $\fint \tilde{u}$ is preserved by the flow of BO, $\tilde{u}(t)$ has mean value zero for all time. We take for the primitive of $u$ the unique periodic, zero mean value primitive of $u$ defined by

$$\hat{F}(0) = 0 \quad \text{and} \quad \widehat{F}(\xi) = \frac{1}{i\xi}\hat{u}(\xi), \xi \in \mathbb{Z}^*.$$

The gauge transform is then defined by

$$W := P_+(e^{-iF/2}). \tag{7-4}$$

Since $F$ satisfies

$$F_t + \mathscr{H}F_{xx} = \tfrac12 F_x^2 - \frac12 \fint F_x^2 = \tfrac12 F_x^2 - \tfrac12 P_0(F_x^2),$$

we finally obtain that $w := W_x = -\frac12 i P_{+hi}(e^{-iF/2}F_x) = -\frac12 i P_+(e^{-iF/2}u)$ satisfies

$$\begin{aligned} w_t - i w_{xx} &= -\partial_x P_{hi}\big[e^{-iF/2}\big(P_-(F_{xx}) - \frac{i}{4}P_0(F_x^2)\big)\big] \\ &= -\partial_x P_{+hi}\big(W P_-(u_x)\big) + \frac{i}{4}P_0(F_x^2)w. \end{aligned} \tag{7-5}$$

Clearly the second term is harmless, and the first one has exactly the same structure as the one that we estimated in Proposition 3.5. Carefully following the proof of this proposition, it is not too hard to check that it also holds in the periodic case independent of the period $\lambda \geq 1$. Note in particular that (2-9) also holds with $L^4_{x,t}$ and $X^{0,\frac38}$ respectively replaced by $L^4_{t,\lambda}$ and $X_\lambda^{0,\frac38}$, $\lambda \geq 1$, where the subscript $\lambda$ denotes spaces of functions with space variable on the torus $\mathbb{R}/2\pi\lambda\mathbb{Z}$ (see [Bourgain 1993a] and also [Molinet 2007]). This leads to a great simplification of the proof the global well-posedness in $L^2(\mathbb{T})$ proved in [Molinet 2008].

Now to derive the new uniqueness result we proceed exactly as in Section 5 except that Proposition 5.1 does not hold on the torus. Actually, on the torus it should be replaced by the following:

**Proposition 7.2.** *For $s \geq \frac{1}{4}$ and all $\lambda \geq 1$, we have*

$$\|P_{+hi}(WP_-\,\partial_x u)\|_{X_\lambda^{s+\frac{1}{2},-\frac{1}{2}}} \lesssim \|W\|_{X_\lambda^{s+\frac{1}{2},\frac{1}{2}}}\big(\|J_x^s u\|_{L^2_{T,\lambda}} + \|J_x^s u\|_{L^4_{T,\lambda}} + \|u\|_{X_\lambda^{s-1,1}}\big). \tag{7-6}$$

Going back to the proof of the bilinear estimate, it is easy to be convinced that the above estimate works at the level $s = 0_+$ in the regions $\mathcal{A}$ and $\mathcal{B}$ (see the proof of Proposition 5.1), whereas in the region $\mathcal{C}$ we are clearly in trouble. Indeed, when $s = 0$, (3-37) must then be replaced by

$$\big|k^{\frac{1}{2}}k_1^{-\frac{1}{2}}k_2^2\langle\sigma_2\rangle^{-1}\big| \sim \big|k^{-\frac{1}{2}}k_1^{-\frac{1}{2}}k_2\big|,$$

which cannot be bounded when $|k_2| \gg k$. On the other hand, at the level $s = \frac{1}{4}$ it becomes

$$\big|k^{\frac{3}{4}}k_1^{-\frac{3}{4}}k_2^{\frac{7}{4}}\langle\sigma_2\rangle^{-1}\big| \sim \big|k^{-\frac{1}{4}}k_1^{-\frac{3}{4}}k_2^{\frac{3}{4}}\big| \lesssim k^{-\frac{1}{4}} \lesssim 1,$$

which yields the result.

With Proposition 7.2 in hand, exactly the same procedure as in Section 5 leads to the uniqueness result in the class $u \in L_T^\infty H^{\frac{1}{4}}(\mathbb{T}) \cap L_T^4 W^{\frac{1}{4},4}(\mathbb{T})$ and by Sobolev embedding to the uniqueness in the class $u \in L_T^\infty H^{\frac{1}{2}}(\mathbb{T})$, i.e., unconditional uniqueness in $H^{\frac{1}{2}}(\mathbb{T})$. As in the real line case, it proves the uniqueness of the (energy) weak solutions that belong to $L^\infty(\mathbb{R}; H^{\frac{1}{2}}(\mathbb{T}))$.

## Appendix

*Proof of Proposition 5.1.* We will need the following calculus lemma stated in [Ginibre et al. 1997].

**Lemma A.3.** *Let $0 < a_- \leq a_+$ such that $a_- + a_+ > \frac{1}{2}$. Then for all $\mu \in \mathbb{R}$,*

$$\int_{\mathbb{R}} \langle y\rangle^{-2a_-}\langle y-\mu\rangle^{-2a_+}\,dy \lesssim \langle\mu\rangle^{-s}, \tag{A-1}$$

*where $s = 2a_-$ if $a_+ > \frac{1}{2}$, $s = 2a_- - \varepsilon$ if $a_+ = \frac{1}{2}$, and $s = 2(a_+ + a_-) - 1$ if $a_+ < \frac{1}{2}$, and let $\varepsilon$ denote any small positive number.*

The proof of Proposition 5.1 closely follows the one of Proposition 3.5 except in the region $\sigma_2$-dominant where we use the approach developed in [Kenig et al. 1996]. Recalling the notation used in (3-24)–(3-25), we need to prove that

$$|K| \lesssim \|h\|_{L^2_{x,t}}\,\|f\|_{L^2_{x,t}}\big(\|u\|_{L^2_{x,t}} + \|u\|_{L^4_{x,t}} + \|u\|_{X^{-\theta,\theta}}\big), \tag{A-2}$$

where

$$K = \int_{\mathcal{D}} \frac{\langle\xi\rangle^{\frac{1}{2}}}{\langle\sigma\rangle^{\frac{1}{2}-2\delta}}\widehat{h}(\xi,\tau)\frac{\langle\xi_1\rangle^{-\frac{1}{2}}}{\langle\sigma_1\rangle^{\frac{1}{2}+\delta}}\widehat{f}(\xi_1,\tau_1)\,\xi_2\langle\xi_2\rangle^{-s}\widehat{u}(\xi_2,\tau_2)\,d\nu. \tag{A-3}$$

For the same reason as in the proof of Proposition 3.5, we can assume that $|\xi_2| \leq 1$. By using a Littlewood–Paley decomposition on $h$, $f$, and $u$, $K$ can be rewritten as

$$K = \sum_{N,N_1,N_2} K_{N,N_1,N_2}, \tag{A-4}$$

with

$$K_{N,N_1,N_2} := \int_{\mathcal{D}} \frac{\langle \xi \rangle^{\frac{1}{2}}}{\langle \sigma \rangle^{\frac{1}{2}-2\delta}} \widehat{P_N h}(\xi,\tau) \frac{\langle \xi_1 \rangle^{-\frac{1}{2}}}{\langle \sigma_1 \rangle^{\frac{1}{2}+\delta}} \widehat{P_{N_1} f}(\xi_1,\tau_1)\, \xi_2 \langle \xi_2 \rangle^{-s} \widehat{P_{N_2} u}(\xi_2,\tau_2)\, d\nu$$

and the dyadic numbers $N$, $N_1$, and $N_2$ ranging from 1 to $+\infty$. Moreover, we will denote by $K^{\mathcal{A}_{N,N_2}}_{N,N_1,N_2}$, $K^{\mathcal{B}_{N,N_2}}_{N,N_1,N_2}$, and $K^{\mathcal{C}_{N,N_2}}_{N,N_1,N_2}$ the restriction of $K_{N,N_1,N_2}$ to the regions $\mathcal{A}_{N,N_2}$, $\mathcal{B}_{N,N_2}$, and $\mathcal{C}_{N,N_2}$ defined in (3-28). Then it follows that

$$|K| \le |K_{\mathcal{A}}| + |K_{\mathcal{B}}| + |K_{\mathcal{C}}|, \tag{A-5}$$

where

$$K_{\mathcal{A}} := \sum_{N,N_1,N_2} J^{\mathcal{A}_{N,N_2}}_{N,N_1,N_2}, \quad K_{\mathcal{B}} := \sum_{N,N_1,N_2} K^{\mathcal{B}_{N,N_2}}_{N,N_1,N_2}, \quad \text{and} \quad K_{\mathcal{C}} := \sum_{N,N_1,N_2} J^{\mathcal{C}_{N,N_2}}_{N,N_1,N_2}$$

so that it suffices to estimate $|K_{\mathcal{A}}|$, $|K_{\mathcal{B}}|$, and $|K_{\mathcal{C}}|$. Recall that due to the structure of $\mathcal{D}$, one of the following case must hold:

(1) high-low interaction: $N_1 \sim N$ and $N_2 \le N_1$,

(2) high-high interaction: $N_1 \sim N_2$ and $N \le N_1$.

*Estimate for* $|K_{\mathcal{A}}|$. In the first case, it follows from the triangular inequality, Plancherel's identity, and Hölder's inequality that

$$\begin{aligned}
|K_{\mathcal{A}}| &\lesssim \|h\|_{L^2_{x,t}} \sum_{N_1} \sum_{N_2 \le N_1} \frac{N_1^{\frac{1}{2}}}{(N_1 N_2)^{\frac{1}{2}-2\delta}} \left\| P_{N_1}\Big( J_x^{-\frac{1}{2}} P_{N_1}\Big(\frac{\widehat{f}}{\langle \sigma_1 \rangle^{\frac{1}{2}+\delta}}\Big)^{\vee} P_- \,\partial_x J_x^{-s} P_{N_2} u\Big) \right\|_{L^2_{x,t}} \\
&\lesssim \|h\|_{L^2_{x,t}} \sum_{N_1} \sum_{N_2 \le N_1} \frac{N_2^{\frac{1}{2}-s+2\delta}}{(N_1)^{\frac{1}{2}-2\delta}} \left\| P_{N_1}\Big(\frac{\widehat{f}}{\langle \sigma_1 \rangle^{\frac{1}{2}+\delta}}\Big)^{\vee} \right\|_{L^4_{x,t}} \|P_{N_2} u\|_{L^4_{x,t}} \\
&\lesssim \|h\|_{L^2_{x,t}} \|u\|_{L^4_{x,t}} \sum_{N_1} N_1^{4\delta - s} \left\| P_{N_1}\Big(\frac{\widehat{f}}{\langle \sigma_1 \rangle^{\frac{1}{2}+\delta}}\Big)^{\vee} \right\|_{L^4_{x,t}}.
\end{aligned}$$

Then it is deduced from the Cauchy–Schwarz inequality in $N_1$ that

$$|K_{\mathcal{A}}| \lesssim \|h\|_{L^2_{x,t}} \bigg( \sum_{N_1} \left\| P_{N_1}\Big(\frac{\widehat{f}}{\langle \sigma_1 \rangle^{\frac{1}{2}+\delta}}\Big)^{\vee} \right\|^2_{L^4_{x,t}} \bigg)^{\frac{1}{2}} \|u\|_{L^4_{x,t}} \tag{A-6}$$

since $s > 10\delta$. On the other, estimate (A-6) also holds in the case of high-high interaction by arguing exactly as in (3-31) so that estimate (2-9) yields

$$|K_{\mathcal{A}}| \lesssim \|h\|_{L^2_{x,t}} \|f\|_{L^2_{x,t}} \|u\|_{L^4_{x,t}}. \tag{A-7}$$

*Estimate for* $|K_{\mathcal{B}}|$. The estimate

$$|K_{\mathcal{B}}| \lesssim \|h\|_{L^2_{x,t}} \|f\|_{L^2_{x,t}} \|u\|_{L^4_{x,t}} \tag{A-8}$$

follows by the same argument as in (A-6).

*Estimate for* $|K_{\mathscr{C}}|$. First observe that

$$|K_{\mathscr{C}}| \lesssim \int_{\widetilde{\mathscr{C}}} \frac{|\xi|^{\frac{1}{2}}}{\langle\sigma\rangle^{\frac{1}{2}-2\delta}} |\widehat{h}(\xi,\tau)| \frac{|\xi_1|^{-\frac{1}{2}}}{\langle\sigma_1\rangle^{\frac{1}{2}+\delta}} |\widehat{f}(\xi_1,\tau_1)| \frac{|\xi_2|^{(1+\theta-s)}}{\langle\sigma_2\rangle^{\theta}} \frac{\langle\sigma_2\rangle^{\theta}}{|\xi_2|^{\theta}} |\widehat{u}(\xi_2,\tau_2)|\, d\nu, \tag{A-9}$$

where

$$\widetilde{\mathscr{C}} = \left\{ (\xi,\xi_1,\tau,\tau_1) \in \mathscr{D} \;\middle|\; (\xi,\xi_1,\tau,\tau_1) \in \bigcup_{N,N_2} \mathscr{C}_{N,N_2} \right\}.$$

Since $|\sigma_2| > |\sigma|$ and $|\sigma_2| > |\sigma_1|$ in $\widetilde{\mathscr{C}}$, (3-28) implies that $|\sigma_2| \gtrsim |\xi\xi_2|$. Applying the Cauchy–Schwarz inequality twice, we deduce that

$$|K_{\mathscr{C}}| \lesssim \sup_{\xi_2,\tau_2} \big(L_{\widetilde{\mathscr{C}}}(\xi_2,\tau_2)\big)^{\frac{1}{2}} \|f\|_{L^2_{\xi,\tau}} \|g\|_{L^2_{\xi,\tau}} \|h\|_{L^2_{\xi,\tau}},$$

where

$$L_{\widetilde{\mathscr{C}}}(\xi_2,\tau_2) = \frac{|\xi_2|^{2+2(\theta-s)}}{\langle\sigma_2\rangle^{2\theta}} \int_{\mathscr{C}(\xi_2,\tau_2)} \frac{|\xi||\xi_1|^{-1}}{\langle\sigma\rangle^{1-4\delta}\langle\sigma_1\rangle^{1+2\delta}}\, d\xi_1\, d\tau_1$$

and

$$\widetilde{\mathscr{C}}(\xi_2,\tau_2) = \{ (\xi_1,\tau_1) \in \mathbb{R}^2 \mid (\xi,\xi_1,\tau,\tau_1) \in \mathscr{C} \}.$$

Thus, to prove that

$$|K_{\mathscr{C}}| \lesssim \|h\|_{L^2_{x,t}} \|f\|_{L^2_{x,t}} \|u\|_{X^{-\theta,\theta}}, \tag{A-10}$$

it is enough to prove that $L_{\widetilde{\mathscr{C}}}(\xi_2,\tau_2) \lesssim 1$ for all $(\xi_2,\tau_2) \in \mathbb{R}^2$. We deduce from (A-1) and (3-28) that

$$L_{\widetilde{\mathscr{C}}}(\xi_2,\tau_2) \lesssim \frac{|\xi_2|^{2+2(\theta-s)}}{\langle\sigma_2\rangle^{1+2\delta}} \int_{\xi_1} \frac{|\xi|\,|\xi_1|^{-1}}{\langle\sigma_2+2\xi\xi_2\rangle^{1-4\delta}}\, d\xi_1$$

since $\theta = 1+\delta$. To integrate with respect to $\xi_1$, we change variables

$$\mu_2 = \sigma_2 + 2\xi\xi_2 \quad \text{so that} \quad d\mu_2 = 2\xi_2\, d\xi_1 \quad \text{and} \quad |\mu_2| \le 4|\sigma_2|.$$

Moreover, (3-26) and (3-28) imply that

$$\frac{|\xi|\,|\xi_1|^{-1}|\xi_2|^{1+2(\theta-s)}}{|\xi_1|^2} \le |\xi\xi_2|^{\frac{1}{2}+\theta-s} \lesssim |\sigma_1|^{\frac{1}{2}+\theta-s}$$

in $\widetilde{\mathscr{C}}$. Then

$$L_{\widetilde{\mathscr{C}}}(\xi_2,\tau_2) \lesssim \frac{|\xi_2|^{1+2(\theta-s)}}{\langle\sigma_2\rangle^{1+2\delta}} \int_0^{4|\sigma_2|} \frac{|\xi|\,|\xi_1|^{-1}}{\langle\mu_2\rangle^{1-4\delta}}\, d\mu_2 \lesssim \frac{\langle\sigma_2\rangle^{\frac{1}{2}+\theta-s+4\delta}}{\langle\sigma_2\rangle^{1+2\delta}} \lesssim \langle\sigma_2\rangle^{3\delta-s} \lesssim 1$$

since $s - 3\delta > 0$.

Finally, we conclude the proof of Proposition 5.1 by gathering (A-2), (A-5), (A-7), (A-8), and (A-10).

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