ANALYSIS & PDE

Volume 5 No. 3 2012

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mathematical sciences publishers

ANALYSIS AND PDE
Vol. 5, No. 3, 2012
dx.doi.org/10.2140/apde.2012.5.475

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BLOW-UP SOLUTIONS ON A SPHERE FOR THE 3D QUINTIC NLS IN THE ENERGY SPACE

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We prove that if $u(t)$ is a log-log blow-up solution, of the type studied by Merle and Raphaël, to the L^2 critical focusing NLS equation $i\partial_t u + \Delta u + |u|^{4/d}u = 0$ with initial data $u_0 \in H^1(\mathbb{R}^d)$ in the cases $d = 1, 2$, then $u(t)$ remains bounded in H^1 away from the blow-up point. This is obtained without assuming that the initial data u_0 has any regularity beyond $H^1(\mathbb{R}^d)$. As an application of the $d = 1$ result, we construct an open subset of initial data in the radial energy space $H^1_{\text{rad}}(\mathbb{R}^3)$ with corresponding solutions that blow up on a sphere at positive radius for the 3D quintic ($\dot{H}^1$-critical) focusing NLS equation $i\partial_t u + \Delta u + |u|^4 u = 0$. This improves the results of Raphaël and Szeftel [2009], where an open subset in $H^3_{\text{rad}}(\mathbb{R}^3)$ is obtained. The method of proof can be summarized as follows: On the whole space, high frequencies above the blow-up scale are controlled by the bilinear Strichartz estimates. On the other hand, outside the blow-up core, low frequencies are controlled by finite speed of propagation.

1. Introduction

Consider the L^2 critical focusing nonlinear Schrödinger equation (NLS)

$$i\partial_t u + \Delta u + |u|^{4/d}u = 0, \tag{1-1}$$

where $u = u(x,t) \in \mathbb{C}$ and $x \in \mathbb{R}^d$, in dimensions $d = 1$ and $d = 2$. It is locally well-posed in $H^1(\mathbb{R}^d)$ and its solutions satisfy conservation of mass $M(u)$, momentum $P(u)$, and energy $E(u)$:

$$M(u) = \|u\|^2_{L^2}, \quad P(u) = \operatorname{Im}\int \bar{u}\,\nabla u\,dx, \quad E(u) = \tfrac{1}{2}\|\nabla u\|^2_{L^2} - \frac{1}{4/d+2}\|u\|^{4/d+2}_{L^{4/d+2}}; \tag{1-2}$$

see [Tao 2006, Chapter 3] and [Cazenave 2003, Chapter 4] for exposition and references. The Galilean identity (see [Tao 2006, Exercise 2.5]) transforms any solution to one with zero momentum, so there is no loss in considering only solutions $u(t)$ such that $P(u) = 0$.

The unique (up to translation) minimal mass H^1 solution of

$$-Q + \Delta Q + |Q|^{4/d}Q = 0, \quad \text{with } Q = Q(x), \tag{1-3}$$

is called the *ground state*. It is smooth, radial, real-valued and positive, and exponentially decaying; see [Tao 2006, Appendix B]. In the case $d = 1$, we have explicitly

$$Q(x) = 3^{1/4}\operatorname{sech}^{1/2}(x). \tag{1-4}$$

MSC2000: 35Q55.
Keywords: blow-up, nonlinear Schrödinger equation.

Weinstein [1982] proved that solutions to (1-1) with $M(u) < M(Q)$ necessarily satisfy $E(u) > 0$ and remain bounded in H^1 globally in time (that is, they do not blow up in finite time).

Building upon the earlier heuristic and numerical result of Landman, Papanicolaou, Sulem and Sulem [Landman et al. 1988] and the first analytical result of Perelman [2001], Merle and Raphaël in a series of papers (see [Merle and Raphaël 2005] and references therein) studied H^1 solutions to (1-1) such that

$$E(u) < 0, \quad P(u) = 0, \quad M(Q) < M(u) < M(Q) + \alpha^* \tag{1-5}$$

for some small absolute constant $\alpha^* > 0$. They showed that any such solution blows up in finite time at the *log-log rate* — more precisely, they proved that there exists a *threshold time* $T_0(u_0) > 0$ and a *blow-up time* $T(u_0) > T_0(u_0)$ such that

$$\|\nabla u(t)\|_{L^2_x} \sim \left(\frac{\log|\log(T-t)|}{T-t}\right)^{1/2} \quad \text{for } T_0 \le t < T, \tag{1-6}$$

where the implicit constant in (1-6) is universal. Also, with scale parameter $\lambda(t) = \|\nabla Q\|_{L^2}/\|\nabla u(t)\|_{L^2}$, there exist parameters of position $x(t) \in \mathbb{R}^d$ and phase $\gamma(t) \in \mathbb{R}$ such that if we define the *blow-up core*

$$u_{\text{core}}(x,t) = \frac{e^{i\gamma(t)}}{\lambda(t)^{d/2}} Q\left(\frac{x - x(t)}{\lambda(t)}\right), \tag{1-7}$$

and *remainder* $\tilde{u} = u - u_{\text{core}}$, then $\|\tilde{u}\|_{L^2} \le \alpha_*$ and

$$\|\nabla \tilde{u}(t)\|_{L^2} \lesssim \left(\frac{1}{|\log(T-t)|^C (T-t)}\right)^{1/2} \tag{1-8}$$

for some $C > 1$. There is, in addition, a well-defined *blow-up point* $x_0 := \lim_{t \nearrow T} x(t)$. We refer to the region of space $\{x \in \mathbb{R}^d \mid |x - x_0| > R\}$, for any fixed $R > 0$, as the *external* region. While the Merle–Raphaël analysis accurately describes the activity of the solution in the blow-up core, the only information it directly yields about the external region is the bound (1-8).

However, it is a consequence of the analysis in [Raphaël 2006] that in the case $d = 1$, H^1 solutions in the class (1-5) have bounded $H^{1/2}$ norm in the external region all the way up to the blow-up time T. In [Holmer and Roudenko 2011], we extended this result to the case $d = 2$. Raphaël and Szeftel [2009] established for $d = 1$ that solutions with regularity H^N for $N \ge 3$ satisfying (1-5) remain bounded in the $H^{(N-1)/2}$ norm in the external region, and Zwiers [2011] extended this result to the case $d = 2$. These results leave open the possibility that there is a loss of roughly half the regularity in passing from the initial data to the solution in the external region at blow-up time. The first main result of this paper is that such a loss *does not occur*. Specifically, we prove that H^1 solutions in the class (1-5) remain bounded in the H^1 norm in the external region all the way up to the blow-up time, resolving an open problem posed in [Raphaël and Szeftel 2009, Comment 1 on page 976].

Theorem 1.1. *Consider dimension $d = 1$ or $d = 2$. Suppose that $u(t)$ is an H^1 solution to* (1-1) *in the Merle–Raphaël class* (1-5) (*no higher regularity is assumed*). *Let $T > 0$ be the blow-up time and $x_0 \in \mathbb{R}^d$*

the blow-up point. Then for any $R > 0$,

$$\|\nabla u(t)\|_{L^\infty_{[0,T]} L^2_{|x-x_0|\ge R}} \le C, \quad \textit{where } C \textit{ depends on } R,\ T_0(u_0),\ \textit{and } \|\nabla u_0\|_{L^2}.^{1}$$

We remark that H^1, the *energy space*, is a natural space in which to study the equation (1-1) since the conservation laws (1-2) are defined and Lyapunov–Hamiltonian type methods, such as those used by Merle and Raphaël in their blow-up theory, naturally yield coercivity on H^1 quantities.

The retention of regularity in the external region has applications to the construction of new blow-up solutions, with special geometry, for L^2 supercritical NLS equations. Using their partial regularity methods, Raphaël [2006] and Raphaël and Szeftel [2009] constructed spherically symmetric finite-time blow-up solutions to the quintic NLS

$$i\partial_t u + \Delta u + |u|^4 u = 0 \tag{1-9}$$

in dimension $d \ge 2$ that contract toward a sphere $|x| = r_0 \sim 1$ following the one-dimensional quintic blow-up dynamics (1-6)(1-7) in the radial variable near $r = r_0$. Specifically, they showed there exists an open subset of initial data in some radial function class with corresponding solutions adhering to the blow-up dynamics described above. In [Raphaël 2006], for $d = 2$, an open subset of initial data in the radial energy space $H^1_{\text{rad}}(\mathbb{R}^2)$ was obtained. For $d = 3$, in which case (1-9) is $\dot{H}^1$ critical, Raphaël and Szeftel [2009] obtained an open subset of initial data in a comparably "thin" subset $H^3_{\text{rad}}(\mathbb{R}^3)$ of the radial energy space $H^1_{\text{rad}}(\mathbb{R}^3)$.

As an application of the techniques used to prove Theorem 1.1, we prove, for $d = 3$, the existence of an open subset of initial data in the full radial energy space $H^1_{\text{rad}}(\mathbb{R}^3)$. For the statement, take Q to be the solution to (1-3) in the case $d = 1$, explicitly given by (1-4). The following theorem follows the motif of the $d = 3$ case of [Raphaël and Szeftel 2009, Theorem 1] except that $\mathcal{P}$, the initial data, is an open subset of $H^1_{\text{rad}}(\mathbb{R}^3)$ rather than $H^3_{\text{rad}}(\mathbb{R}^3)$.

Theorem 1.2. *There exists an open subset* $\mathcal{P} \subset H^1_{\text{rad}}(\mathbb{R}^3)$ *such that the following holds true. Let* $u_0 \in \mathcal{P}$ *and let* $u(t)$ *denote the corresponding solution to* (1-9) *in the case* $d = 3$. *Then there exist a blow-up time* $0 < T < +\infty$ *and parameters of scale* $\lambda(t) > 0$, *radial position* $r(t) > 0$, *and phase* $\gamma(t) \in \mathbb{R}$ *such that if we take*

$$u_{\text{core}}(t, r) := \frac{1}{\lambda(t)^{1/2}} Q\left(\frac{r - r(t)}{\lambda(t)}\right) e^{i\gamma(t)}$$

and the remainder $\tilde{u}(t) := u(t) - u_{\text{core}}(t)$, *then the following hold*:

(1) *The remainder converges in* L^2: $\tilde{u}(t) \to u^*$ *in* $L^2(\mathbb{R}^3)$ *as* $t \nearrow T$.

(2) *The position of the singular sphere converges*: $r(t) \to r_0 > 0$ *as* $t \nearrow T$.

[1] We did not see in the Merle–Raphaël papers the threshold time $T_0(u_0)$ or the blow-up time $T(u_0)$ estimated quantitatively in terms of properties of the initial data ($\|\nabla u_0\|_{L^2}$, $E(u_0)$, etc.). If such dependence could be quantified, then the constant C in Theorem 1.1 could be quantified.

(3) *The solution contracts toward the sphere at the log-log rate*:

$$\lambda(t)\left(\frac{\log|\log(T-t)|}{T-t}\right)^{1/2} \to \frac{\sqrt{2\pi}}{\|Q\|_{L^2}} \quad \textit{as } t \nearrow T.$$

(4) *The solution remains H^1-small away from the singular sphere*: *For each* $R > 0$,

$$\|u(t)\|_{H^1_{|r-r(T)|\geq R}(\mathbb{R}^3)} \leq \epsilon.$$

The 3D quintic NLS equation (1-9) is energy-critical, and the global well-posedness and scattering problem is one of several critical regularity problems that has received a lot of attention in the last decade [Bourgain 1999; Colliander et al. 2008; Kenig and Merle 2006]. The global well-posedness for small data in $\dot{H}^1$ is classical and follows from the Strichartz estimates. Our Theorem 1.2 takes a large, but special "prefabricated" approximate blow-up solution, and installs it near radius $r = 1$ on top of a small global H^1 background. The main difficulty, of course, is showing that the two different components — the blow-up portion on the one hand, and the evolution of the small $\dot{H}^1$ background on the other — have limited interaction and can effectively evolve separately. Thus, it is not surprising that the techniques to prove Theorem 1.1 are relevant to this analysis.

We now outline the method used to prove Theorem 1.1. We start with a given blow-up solution $u(t)$ in the Merle–Raphaël class, and by scaling and shifting this solution, it suffices to assume that the blow-up point is $x_0 = 0$ and the blow-up time is $T = 1$, and moreover, (1-6) holds over times $0 \leq t < 1$. Since (1-1) is L^2 critical, the size of the L^2 norm is highly relevant. By mass conservation, we know that $\|P_N u(t)\|_{L^2_x} \lesssim 1$ for all N and all $0 \leq t < 1$, where P_N denotes the Littlewood–Paley frequency projection. However, (1-6) shows that for $N \gg (1-t)^{-(1+\delta)/2}$, we have $\|P_N u(t)\|_{L^2_x} \lesssim N^{-1}(1-t)^{-(1+\delta)/2}$, which is a better estimate for these large frequencies N. In Section 3, we show that this smallness of high frequencies reinforces itself and ultimately proves that for $N \gg (1-t)^{-(1+\delta)/2}$, the solution is H^1 bounded. This is achieved using dispersive estimates typically employed in local well-posedness arguments — the Strichartz and Bourgain's bilinear Strichartz estimates — after the equation has been restricted to high frequencies. We note that this improvement of regularity at high frequencies is proved *globally in space*.

For the Schrödinger equation, frequencies of size N propagate at speed N, and thus, travel a distance $O(1)$ over a time N^{-1}. Therefore, at time $t < 1$, a component of the solution in the blow-up core at frequency N will effectively only make it out of the blow-up core and into the external region before the blow-up time, provided $N \gtrsim (1-t)^{-1}$. Thus, we expect that the blow-up action, which is taking place at frequency $\sim (1-t)^{-1/2}\log|\log(1-t)| \ll (1-t)^{-1}$, will not be able to exit the blow-up core before blow-up time. This is the philosophy behind the analysis in Section 4. Recall that in Section 3, we have controlled the solution at frequencies above $(1-t)^{-(1+\delta)/2}$. In Section 4, we apply a spatial localization to the external region, and then look to control the remaining low frequencies, i.e., those frequencies below $(1-t)^{-(1+\delta)/2}$. We examine the equation solved by $P_{\leq (1-t)^{-3/4}}\psi u(t)$, where ψ is a spatial restriction to the external region. In estimating the inhomogeneous terms, we can make use of the frequency restriction to exchange α-spatial derivatives for a time factor $(1-t)^{-3\alpha/4}$. This enables us to prove a low-frequency recurrence: The H^s size of the solution in the external region is bounded by the $H^{s-1/8}$ size of the solution in a slightly larger external region. Iteration gives the H^1 boundedness.

The structure of the paper is as follows. Preliminaries on the Strichartz and bilinear Strichartz estimates appear in Section 2. The proof of Theorem 1.1 is carried out in Sections Section 3 and 4. The proof of Theorem 1.2 is carried out in Section 5.

2. Standard estimates

All of the estimates outlined in this section are now classical and well known. Let P_N, $P_{\le N}$, and $P_{\ge N}$ denote the Littlewood–Paley frequency projections.

We say that (q, p) is an *admissible* pair if $2 \le p \le \infty$ and

$$\frac{2}{q} + \frac{d}{p} = \frac{d}{2},$$

excluding the case $d = 2$, $q = 2$, and $p = \infty$.

Lemma 2.1 (Strichartz estimate). *If (q, p) is an admissible pair, then*

$$\|e^{it\Delta}\phi\|_{L_t^q L_x^p} \lesssim \|\phi\|_{L_x^2}.$$

Proof. See [Strichartz 1977] and [Keel and Tao 1998]. □

Lemma 2.2 (Bourgain bilinear Strichartz estimate). *Suppose that $N_1 \ll N_2$. Then*

$$\|P_{N_1} e^{it\Delta}\phi_1 P_{N_2} e^{it\Delta}\phi_2\|_{L_t^2 L_x^2} \lesssim \left(\frac{N_1^{d-1}}{N_2}\right)^{1/2} \|\phi_1\|_{L_x^2}\|\phi_2\|_{L_x^2}, \tag{2-1}$$

$$\|P_{N_1} e^{it\Delta}\phi_1 \overline{P_{N_2} e^{it\Delta}\phi_2}\|_{L_t^2 L_x^2} \lesssim \left(\frac{N_1^{d-1}}{N_2}\right)^{1/2} \|\phi_1\|_{L_x^2}\|\phi_2\|_{L_x^2}. \tag{2-2}$$

Proof. For the 2D estimate (2-1), see [Bourgain 1998, Lemma 111]; the 1D case appears in [Colliander et al. 2001, Lemma 7.1]; another nice proof is given in [Koch and Tataru 2007, Proposition 3.5], the other dimensions are analogous. We review the 1D proof to show that the second estimate (2-2) holds as well.

Denote $u = e^{it\Delta}(P_{N_1}\phi_1)$ and $v = e^{\pm it\Delta}(P_{N_2}\phi_2)$. Then in the 1D case,

$$\widehat{uv}(\xi, \tau) = \int_{\xi_1+\xi_2=\xi} \widehat{P_{N_1}\phi_1}(\xi_1)\widehat{P_{N_2}\phi_2}(\xi_2)\delta(\tau - (\xi_1^2 \pm \xi_2^2))\, d\xi_1 \tag{2-3}$$

$$= \frac{1}{|g'_{\xi_1}(\xi_1, \xi_2)|}\widehat{P_{N_1}\phi_1}\,\widehat{P_{N_2}\phi_2}|_{(\xi_1,\xi_2)}, \tag{2-4}$$

where $g(\xi_1, \xi_2) = \tau - (\xi_1^2 \pm \xi_2^2)$, thus, $|g'_{\xi_1}| = 2|\xi_1 \pm \xi_2|$. To estimate the $L^2_{\xi,\tau}$ norm of uv, we square the expression above and integrate in τ and ξ. Changing variables (τ, ξ) to (ξ_1, ξ_2) with $\tau = \xi_1^2 \pm \xi_2^2$ and $\xi = \xi_1 + \xi_2$, we obtain $d\tau d\xi = J\, d\xi_1 d\xi_2$ with the Jacobian $J = 2|\xi_1 \pm \xi_2|$, which is of size N_2 (note that $\pm$ does not matter here, since $N_2 \gg N_1$). Bringing the square inside, we get

$$\|uv\|_{L_x^2}^2 \lesssim \int_{|\xi_1|\sim N_1, |\xi_2|\sim N_2} |\widehat{\phi_1}(\xi_1)|^2 |\widehat{\phi_2}(\xi_2)|^2 \frac{d\xi_1\, d\xi_2}{|\xi_1 \pm \xi_2|} \lesssim \frac{1}{N_2}\|\phi_1\|_{L_x^2}^2\|\phi_2\|_{L_x^2}^2.$$ □

Now we introduce the Fourier restriction norms. For $\tilde{u} \in \mathcal{S}(\mathbb{R}^{1+d})$,

$$\|\tilde{u}\|_{X_{s,b}} = \big\| \langle D_t \rangle^b \langle D_x \rangle^s e^{-it\Delta} \tilde{u}(\,\cdot\,, t) \big\|_{L_t^2 L_x^2} = \left(\int_\xi \int_\tau |\widehat{\tilde{u}}(\xi, \tau)|^2 \langle \xi \rangle^{2s} \langle \tau + |\xi|^2 \rangle^{2b} \, d\xi \, d\tau \right)^{1/2}.$$

If $I \subset \mathbb{R}$ is an open subinterval and $u \in \mathcal{D}'(I \times \mathbb{R}^d)$, define

$$\|u\|_{X_{s,b}(I)} = \inf_{\tilde{u}} \|\tilde{u}\|_{X_{s,b}},$$

where the infimum is taken over all distributions $\tilde{u} \in \mathcal{S}'(\mathbb{R}^{1+d})$ such that $\tilde{u}|_I = u$.

Lemma 2.3. *If θ is a function such that* $\operatorname{supp} \theta \subset I$, *then for all* $0 < b < 1$,

$$\|\theta u\|_{X_{s,b}} \lesssim (\|\theta\|_{L^\infty} + \|D_t^{\max(1/2,b)} \theta\|_{L^2}) \|u\|_{X_{s,b}(I)}. \tag{2-5}$$

If $0 \le b < \frac{1}{2}$ and χ_I is the (sharp) characteristic function of the time interval I, then

$$\|\chi_I u\|_{X_{s,b}} \sim \|u\|_{X_{s,b}(I)}. \tag{2-6}$$

Proof. It suffices to take $s = 0$. The inequality (2-5) follows from the fractional Leibniz rule. To address (2-6), we note that Jerison and Kenig [1995] prove that $\|\chi_{(0,+\infty)} f\|_{H_t^b} \lesssim \|f\|_{H_t^b}$ for $-\frac{1}{2} < b < \frac{1}{2}$. Consequently, $\|\chi_I f\|_{H_t^b} \lesssim \|f\|_{H_t^b}$ for any time interval I. Let $\tilde{u}$ be an extension of u (meaning $\tilde{u}|_I = u$) so that $\|\tilde{u}\|_{X_{0,b}} \le 2\|u\|_{X_{0,b}(I)}$. Then

$$\begin{aligned} \|\chi_I u\|_{X_{0,b}} &= \|\langle D_t \rangle^b e^{-it\Delta} \chi_I \tilde{u}\|_{L_t^2 L_x^2} \\ &= \big\| \|\chi_I e^{-it\Delta} \tilde{u}\|_{H_t^b} \big\|_{L_x^2} \lesssim \big\| \|e^{-it\Delta} \tilde{u}\|_{H_t^b} \big\|_{L_x^2} \\ &= \|\tilde{u}\|_{X_{0,b}} \le 2\|u\|_{X_{0,b}(I)}. \end{aligned}$$

On the other hand, the inequality $\|u\|_{X_{0,b}(I)} \lesssim \|\chi_I u\|_{X_{0,b}}$ is trivial, since $\chi_I u$ is an extension of $u|_I$. ☐

Lemma 2.4. *If $i\partial_t u + \Delta u = f$ on a time interval $I = (a_1, a_2)$ with $|I| = O(1)$, then*

(1) *For $\frac{1}{2} < b \le 1$, taking $I' = (a_1 - \omega, a_2 + \omega)$, $0 < \omega \le 1$, we have*

$$\|u(t) - e^{i(t-a_1)\Delta} u(a_1)\|_{X_{0,b}(I)} \lesssim \omega^{1/2-b} \|f\|_{X_{0,b-1}(I')}. \tag{2-7}$$

(2) *For $0 \le b < \frac{1}{2}$,*

$$\|u(t) - e^{i(t-a_1)\Delta} u(a_1)\|_{X_{0,b}(I)} \lesssim \|f\|_{L_I^1 L_x^2}. \tag{2-8}$$

Moreover, for all b,

$$\|e^{i(t-a_1)\Delta} \phi\|_{X_{0,b}(I)} \lesssim \|\phi\|_{L_x^2}.$$

Proof. Without loss, we take $a_1 = 0$. First we consider (2-7). Since, for $t \in I$,

$$e^{-it\Delta} u(\,\cdot\,, t) = u(0) - i\theta(t) \int_0^t e^{-it'\Delta} \theta(t') f(\,\cdot\,, t') \, dt',$$

where θ is a cutoff function such that $\theta(t)=1$ on I and $\operatorname{supp}\theta\subset I'$, the estimate reduces to the space-independent estimate

$$\Big\|\theta(t)\int_0^t h(t')\,dt'\Big\|_{H_t^b}\lesssim\|h\|_{H_t^{b-1}}\quad\text{for } \tfrac12<b\le1 \tag{2-9}$$

by (2-5). Now we prove estimate (2-9). Divide $h=P_{\le1}h+P_{\ge1}h$ and use that

$$\int_0^t P_{\ge1}h(t')=\frac12\int(\operatorname{sgn}(t-t')+\operatorname{sgn}(t'))P_{\ge1}h(t')\,dt'$$

to obtain the decomposition

$$\theta(t)\int_0^t h(t')\,dt'=H_1(t)+H_2(t)+H_3(t),$$

where

$$\begin{aligned}H_1(t)&=\theta(t)\int_0^t P_{\le1}h(t')\,dt',\\ H_2(t)&=\tfrac12\theta(t)[\operatorname{sgn}*P_{\ge1}h](t)\,dt',\\ H_3(t)&=\tfrac12\theta(t)\int_{-\infty}^{+\infty}\operatorname{sgn}(t')P_{\ge1}h(t')\,dt'.\end{aligned}$$

We begin by addressing term H_1. By Sobolev embedding (recall $\frac12<b\le1$) and the $L^p\to L^p$ boundedness of the Hilbert transform for $1<p<\infty$,

$$\|H_1\|_{H_t^b}\lesssim\|H_1\|_{L_t^2}+\|\partial_tH_1\|_{L_t^{2/(3-2b)}}.$$

Using that $|I|=O(1)$ and $\|P_{\le1}h\|_{L_t^\infty}\lesssim\|h\|_{H_t^{b-1}}$, we thus conclude

$$\|H_1\|_{H_t^b}\lesssim\big(\|\theta\|_{L_t^2}+\|\theta\|_{L_t^{2/(3-2b)}}+\|\theta'\|_{L_t^{2/3-2b}}\big)\|h\|_{H_t^{b-1}}.$$

Next we address the term H_2. By the fractional Leibniz rule,

$$\|H_2\|_{H_t^b}\lesssim\|\langle D_t\rangle^b\theta\|_{L_t^2}\|\operatorname{sgn}*P_{\ge1}h\|_{L_t^\infty}+\|\theta\|_{L_t^\infty}\|\langle D_t\rangle^b(\operatorname{sgn}*P_{\ge1}h)\|_{L_t^2}.$$

However,

$$\|\operatorname{sgn}*P_{\ge1}h\|_{L_t^\infty}\lesssim\|\langle\tau\rangle^{-1}\hat h(\tau)\|_{L_\tau^1}\lesssim\|h\|_{H_t^{b-1}}.$$

On the other hand,

$$\|\langle D_t\rangle^b\operatorname{sgn}*P_{\ge1}h\|_{L_t^2}\lesssim\|\langle\tau\rangle^b\langle\tau\rangle^{-1}\hat h(\tau)\|_{L_\tau^2}\lesssim\|h\|_{H_t^{b-1}}.$$

Consequently,

$$\|H_2\|_{H_t^b}\lesssim(\|\langle D_t\rangle^b\theta\|_{L_t^2}+\|\theta\|_{L_t^\infty})\|h\|_{H_t^{b-1}}.$$

For term H_3, we have

$$\|H_3\|_{H_t^b}\lesssim\|\theta\|_{H_t^b}\Big\|\int_{-\infty}^{+\infty}\operatorname{sgn}(t')P_{\ge1}h(t')\,dt'\Big\|_{L_t^\infty}.$$

However, the second term is handled via Parseval's identity

$$\int_{t'} \operatorname{sgn}(t') P_{\geq 1} h(t')\, dt' = \int_{|\tau|\geq 1} \tau^{-1} \hat{h}(\tau)\, d\tau,$$

from which the appropriate bounds follow again by Cauchy–Schwarz. Collecting our estimates for H_1, H_2, and H_3, we have

$$\left\| \theta(t) \int_0^t h(t')\, dt' \right\|_{H_t^b} \lesssim C_\theta \|h\|_{H_t^{b-1}},$$

where

$$C_\theta = \|\theta\|_{L_t^2} + \|\theta'\|_{L_t^{2/(3-2b)}} + \|\langle D_t\rangle^b \theta\|_{L_t^2} + \|\theta\|_{L_t^{2/(3-2b)}} + \|\theta\|_{L_t^\infty} \lesssim \omega^{1/2-b}.$$

This completes the proof of (2-7). Next, we prove (2-8). We have

$$e^{-it\Delta} u(\,\cdot\,, t) = u(0) - i \int_0^t e^{-it'\Delta} f(\,\cdot\,, t')\, dt',$$

and thus, (2-8) reduces, by (2-6), to

$$\left\| \chi_I \int_0^t g(t')\, dt' \right\|_{H_t^b} \lesssim \|g\|_{L_I^1}, \quad \text{for } 0 \leq b < \tfrac{1}{2}. \tag{2-10}$$

To prove (2-10), note that

$$\chi_I(t) \int_0^t g(t')\, dt' = \chi_I(t) [\chi_I * (g\chi_I)](t).$$

Hence,

$$\|\chi_I \int_0^t g(t')\, dt'\|_{H_t^b} \lesssim \|\langle D\rangle^b \chi_I\|_{L_t^2} \|g\|_{L_I^1}.$$

The Fourier transform of χ_I is smooth and decays like $|\tau|^{-1}$ as $|\tau| \to \infty$, and hence, $\|\langle D\rangle^b \chi_I\|_{L_t^2} < \infty$ for $0 \leq b < \frac{1}{2}$. □

Lemma 2.5 (Strichartz estimate). *If (q, r) is an admissible pair, then we have the embedding*

$$\|u\|_{L_I^q L_x^p} \lesssim \|u\|_{X_{0,1/2+\delta}(I)}.$$

Proof. We reproduce the well-known argument. Replace u by an extension to $t \in \mathbb{R}$ such that $\|u\|_{X_{0,1/2+\delta}} \leq 2\|u\|_{X_{0,1/2+\delta}(I)}$. Write

$$u(x, t) = \int_\xi \int_\tau e^{it\tau} e^{ix\cdot\xi} \hat{u}(\xi, \tau)\, d\tau\, d\xi.$$

Change variables $\tau \mapsto \tau - |\xi|^2$ and apply Fubini to obtain

$$u(x, t) = \int_\tau e^{it\tau} \int_\xi e^{-it|\xi|^2} e^{ix\cdot\xi} \hat{u}(\xi, \tau - |\xi|^2)\, d\xi\, d\tau.$$

Define $f_\tau(x)$ by $\hat{f}_\tau(\xi) = \hat{u}(\xi, \tau - |\xi|^2)$. Then the above reads

$$u(x, t) = \int_\tau e^{it\tau} e^{it\Delta} f_\tau(x)\, d\tau,$$

and hence,

$$|u(x,t)| \le \int_\tau |e^{it\Delta} f_\tau(x)|\, d\tau.$$

Apply the Strichartz norm, the Minkowski integral inequality, appeal to Lemma 2.1, and invoke Plancherel to obtain

$$\|u\|_{L_I^q L_x^p} \lesssim \int_\tau \|\hat{f}_\tau(\xi)\|_{L_\xi^2}\, d\tau.$$

The argument is completed using Cauchy–Schwarz in τ (note that we need $b > \frac{1}{2}$, since $\int_{\mathbb{R}} \langle \tau \rangle^{-2b}\, d\tau$ has to be finite). □

Lemma 2.6 (Bourgain bilinear Strichartz estimate). *Let $N_1 \ll N_2$. Then*

$$\|P_{N_1} u_1 P_{N_2} u_2\|_{L_I^2 L_x^2} \lesssim \left(\frac{N_1^{d-1}}{N_2}\right)^{1/2} \|u_1\|_{X_{0,1/2+\delta}(I)} \|u_2\|_{X_{0,1/2+\delta}(I)},$$

$$\|P_{N_1} u_1 \overline{P_{N_2} u_2}\|_{L_I^2 L_x^2} \lesssim \left(\frac{N_1^{d-1}}{N_2}\right)^{1/2} \|u_1\|_{X_{0,1/2+\delta}(I)} \|u_2\|_{X_{0,1/2+\delta}(I)}.$$

Proof. We reproduce the well-known argument. As in the proof of Lemma 2.5, taking $f_{j,\tau}(x)$ defined by $\hat{f}_{j,\tau}(\xi) = \hat{u}_1(\xi, \tau - |\xi|^2)$, we have

$$u_j(x,t) = \int_\tau e^{it\tau} e^{it\Delta} f_{j,\tau}(x)\, d\tau.$$

Plug these into the expression $\|P_{N_1} u_1 \; P_{N_2} u_2\|_{L_t^2 L_x^2}$, and then estimate using Lemma 2.2. □

We need to take $b = \frac{1}{2} - \delta$ in some places. In those situations, we use this:

Lemma 2.7 (interpolated Strichartz). *Take $d = 1$ or $d = 2$ and suppose that $0 \le b < \frac{1}{2}$ and $2 \le p \le \infty$ and $2 < q \le \infty$ satisfy*

$$\frac{2}{q} + \frac{d}{p} > \frac{d}{2} + (1 - 2b), \tag{2-11}$$

$$\frac{2}{q} - \frac{1}{p} \le \frac{1}{2} \qquad \text{in the case } d = 1 \text{ only} \tag{2-12}$$

(*see Figure 1*). *Then*

$$\|u\|_{L_I^q L_x^p} \lesssim \|u\|_{X_{0,b}(I)}. \tag{2-13}$$

with implicit constant dependent upon the size of the gap from equality in (2-11).

Proof. Let

$$\alpha := \frac{1}{2}\left(\frac{2}{q} + \frac{d}{p} - \frac{d}{2} - (1 - 2b)\right) > 0. \tag{2-14}$$

Using $0 \le \theta \le 1$ as an interpolation parameter, we aim to deduce (2-13) by interpolation between

$$\|u\|_{L_t^{\tilde{q}} L_x^{\tilde{p}}} \lesssim \|u\|_{X_{0,b/(2(b-\alpha))}}, \tag{2-15}$$

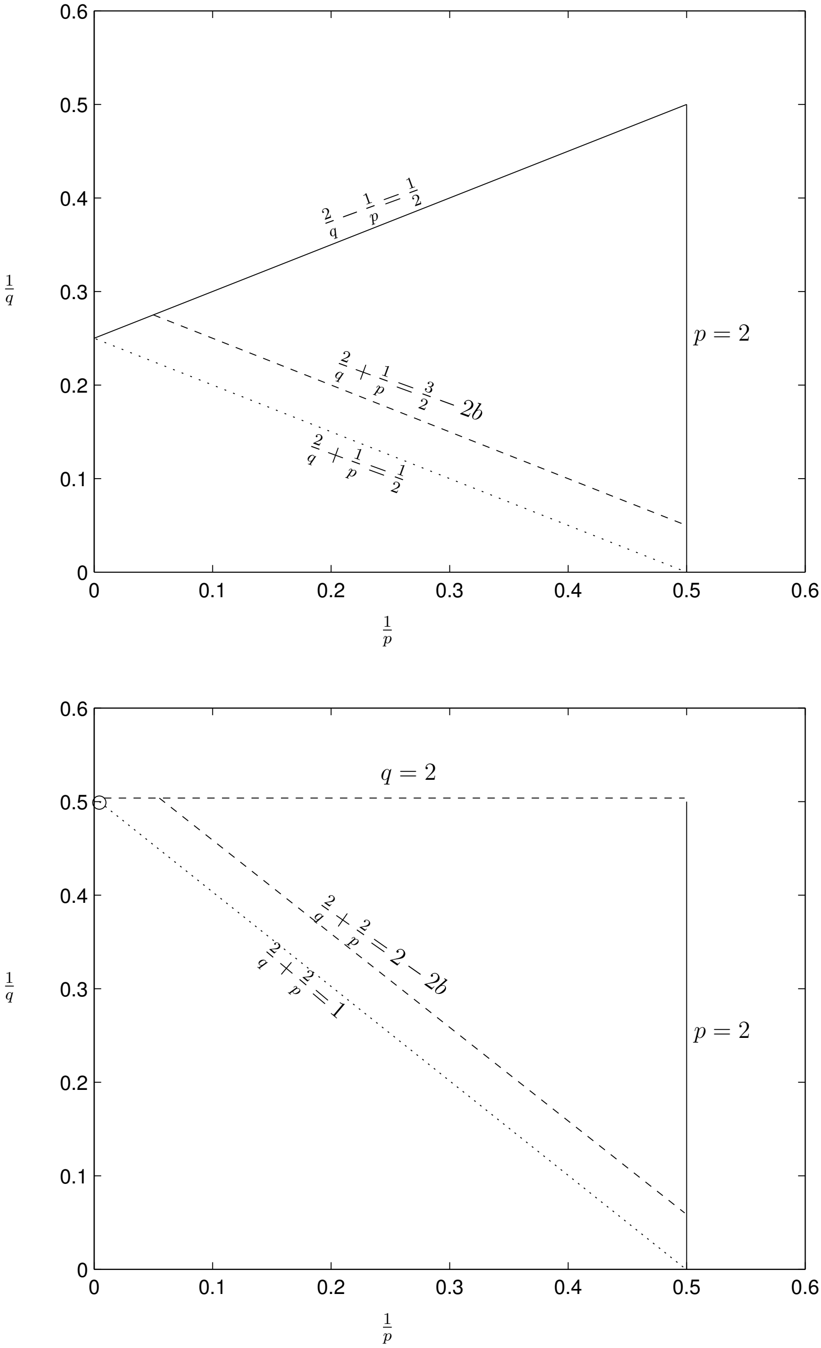

Figure 1. The enclosed triangular region gives the values of $(1/q, 1/p)$ meeting the hypotheses of Lemma 2.7. The top frame is the case $d = 1$ and the bottom frame is the case $d = 2$. The proof of Lemma 2.7 involves interpolating between a point on the line $2/q + d/p = d/2$ and the point $(1/2, 1/2)$.

with weight θ, for some Strichartz admissible pair $(\tilde{q}, \tilde{p})$, and the trivial estimate (equality, in fact)

$$\|u\|_{L_t^2 L_x^2} \lesssim \|u\|_{X_{0,0}}, \tag{2-16}$$

with weight $1-\theta$. The interpolation conditions read

$$\frac{1}{q} = \frac{\theta}{\tilde{q}} + \frac{1-\theta}{2} \quad \text{and} \quad \frac{1}{p} = \frac{\theta}{\tilde{p}} + \frac{1-\theta}{2}. \tag{2-17}$$

Multiplying the first of these relations by 2 and adding d times the second, and using the Strichartz admissibility condition for $(\tilde{q}, \tilde{p})$, we obtain

$$\frac{2}{q} + \frac{d}{p} = \frac{d}{2} + (1-\theta).$$

Combining this relation with (2-14), we get $\theta = 2b - 2\alpha$. We can then solve for $\tilde{q}$ and $\tilde{p}$ using (2-17). □

Lemma 2.8 (interpolated bilinear Strichartz). *Let $d = 1$ or $d = 2$ and $N_1 \ll N_2$. Then*

$$\|P_{N_1}u_1 \; P_{N_2}u_2\|_{L_I^2 L_x^2} \lesssim \frac{N_1^{(d-1)/2}}{N_2^{1/2-\delta'}} \|u_1\|_{X_{0,1/2-\delta}(I)} \|u_2\|_{X_{0,1/2-\delta}(I)}.$$

Proof. First, observe that

$$\|P_{N_1}u_1 \; P_{N_2}u_2\|_{L_I^2 L_x^2} \lesssim \|u_1\|_{L_I^4 L_x^4} \|u_2\|_{L_I^4 L_x^4}. \tag{2-18}$$

In the case $d = 1$, $L_I^4 L_x^4$ interpolates between $L_I^6 L_x^6$ and $L_I^2 L_x^2$, and thus $\|u_j\|_{L_I^4 L_x^4} \lesssim \|u_j\|_{X_{0,3/8+\delta}(I)}$ by Lemma 2.7. We conclude that

$$\|P_{N_1}u_1 \; P_{N_2}u_2\|_{L_I^2 L_x^2} \lesssim \|u_1\|_{X_{0,3/8+\delta}(I)} \|u_2\|_{X_{0,3/8+\delta}(I)}.$$

Interpolating this with the result of Lemma 2.6 completes the proof in the case $d = 1$.

In the case $d = 2$, we still begin with (2-18). Fix $\epsilon > 0$ small. By Sobolev embedding,

$$\|P_{N_j}u_j\|_{L_I^4 L_x^4} \lesssim N_j^{\epsilon} \|P_{N_j}u_j\|_{L_I^4 L_x^{4/(1+2\epsilon)}}.$$

By Lemma 2.7, we have

$$\|P_{N_j}u_j\|_{L_I^4 L_x^{4/(1+2\epsilon)}} \lesssim \|u_j\|_{X_{0,b}}$$

for any $b > \frac{1}{2}(1-\epsilon)$. Plugging into (2-18), we obtain

$$\|P_{N_1}u_1 \; P_{N_2}u_2\|_{L_I^2 L_x^2} \lesssim N_2^{2\epsilon} \|u_1\|_{X_{0,b}} \|u_2\|_{X_{0,b}} \quad \text{for any } b > \tfrac{1}{2}(1-\epsilon).$$

Interpolating this with the result of Lemma 2.6 completes the proof in the case $d = 2$. □

Remark 2.9. After this section we will adopt new notation: Instead of $X_{s,1/2+\delta}$ we will simply write $X_{s,1/2+}$. If an expression has two different Bourgain spaces, it will mean that the delta's will be different. Similarly, if an expression involves δ in the estimate on the right side, it will mean that this δ will be different from the one that would be chosen for spaces such as $X_{s,1/2+}$ or L^{p-}.

The following is a simple consequence of the pseudodifferential calculus; see [Stein 1993, Theorem1 on page 234 and Theorem 2 on page 237]; see also [Evans and Zworski 2003].

Lemma 2.10. *Suppose that ϕ is a smooth function on $\mathbb{R}$ such that $\|\partial_x^\alpha \phi\|_{L^\infty} \le c_\alpha$ for all $\alpha \ge 0$. Then*

$$\|P_{\ge N}(\phi g) - \phi P_{\ge N} g\|_{L^2} \lesssim N^{-1} \|g\|_{L^2} \quad \text{for } N \ge 1.$$

Proof. Let $\chi(\xi)$ be a smooth function that is 1 for $|\xi| \ge 1$ and is 0 for $|\xi| \le \frac{1}{2}$. $P_{\ge N}$ is a pseudodifferential operator with symbol $\chi(N^{-1}\xi)$ and M_ϕ, the operator of multiplication by ϕ, is a pseudodifferential operator with symbol $\phi(x)$. The commutator $[P_N, M_\phi]$ has symbol with top-order asymptotic term $N^{-1}\chi'(N^{-1}\xi)\phi'(x)$. The result then follows from the $L^2 \to L^2$ boundedness of 0-order operators. □

3. Additional high-frequency regularity

In this section, we begin the proof of Theorem 1.1 by showing improved regularity at high frequencies, above the blow-up scale, *with no restriction in space* — this appears as Proposition 3.4 below. In Section 4 below, we will complete the proof of Theorem 1.1 by appealing to a finite-speed of propagation argument for lower frequencies *after we have restricted in space* to outside the blow-up core.

Consider a solution $u(t)$ to (1-1) in the Merle–Raphaël class (1-5); let $T_0 > 0$ be the threshold time, $T > T_0$ the blow-up time and x_0 the blow-up point, as described in the introduction. Our analysis focuses on the time interval $[T_0, T)$ on which the log-log asymptotics (1-6) kick in. Apply a space-time (rescaling) shift, in which $x = x_0$ is sent to $x = 0$ and the time interval $[T_0, T)$ is sent to $[0, 1)$, to obtain a transformed solution that we henceforth still denote by $u(t)$. Now the blow-up time is $T = 1$, the blow-up point is $x = 0$, and (1-6) becomes[2]

$$\|\nabla u(t)\|_{L_x^2} \sim \left(\frac{\log|\log(1-t)|}{1-t}\right)^{1/2}, \tag{3-1}$$

which is now valid for all $0 \le t < 1$. Note that now, however, the time $t = 0$ "initial data", which we henceforth denote u_0, does not correspond to the original initial data u_0 in Theorem 1.1. We remark that the estimate (1-8) on the remainder $\tilde{u}(t)$ becomes

$$\|\nabla \tilde{u}(t)\|_{L_x^2} \lesssim \frac{1}{(1-t)^{1/2}|\log(1-t)|}. \tag{3-2}$$

In our analysis, the norm $L_I^\infty L_x^2$ for an interval $I = [0, T']$, $T' < T$, will be replaced by the norm $X_{0,1/2+}(I)$. While we have, from Lemma 2.5, the bound

$$\|u\|_{L_I^\infty L_x^2} \lesssim \|u\|_{X_{0,1/2+}(I)},$$

the reverse bound does not in general hold. Nevertheless, (3-1) indicates that the solution is blowing up close to the scale rate $(1-t)^{-1/2}$. Thus, the local theory combined with (3-1) implies a bound on $\|u\|_{X_{1,1/2+}(I)}$, where $\log|\log(1-T')|$ is weakened to $(1-T')^{-\delta}$.

[2] The rescaling is the following. If we take $u(x,t)$ in the original frame (for $T_0 \le t < T$), and let

$$u(x,t) = \mu^{d/2} v(\mu(x-x_0), \mu^2(t-T_0))$$

with $\mu = (T-T_0)^{-1/2}$, then $v(y,s)$ is defined in the modified frame (for $0 \le s < 1$). Moreover, we have $\|\nabla v(s)\|_{L_x^2} \sim (\log|\log \mu^{-2}(1-s)|)^{1/2}(1-s)^{-1/2}$, so now the implicit constant of comparability in (3-1) depends on $T - T_0$.

Lemma 3.1. *For* $I = [0, T']$ *with* $T' < T$, *for* $0 < s \le 1$, *we have*

$$\|u\|_{X_{s,\frac{1}{2}+}(I)} \le c_s (1-T')^{-s(1+\delta)/2}, \quad \text{with } c_s \nearrow +\infty \text{ as } s \searrow 0.$$

The fact that c_s diverges as $s \searrow 0$ results from the fact that (1-1) is L^2-critical, and thus, the local theory estimates break down at $s = 0$. At the technical level, some slack is needed in applying the Strichartz and bilinear Strichartz estimates; hence, we need to take $b = 1/2 - \delta$ in place of $b = 1/2 + \delta'$.

Proof. We just carry out the argument for $s = 1$. Let $\lambda(t) = \|\nabla u(t)\|_{L^2}^{-1}$. Let s_k be the increasing sequence of times[3] such that $\lambda(s_k) = 2^{-k}$, so that $\|\nabla u(t)\|_{L^2}$ doubles over $[s_k, s_{k+1}]$. From (3-1), we compute that $s_k = 1 - 2^{-2k} \log k$. Note that $s_{k+1} - s_k \approx 2^{-2k} \log k$. Hence, we can rescale the cutoff solution $u(t)$ on the time interval $[s_k, s_{k+1}]$ to a solution u' on the time interval $[0, \log k]$ so that $\|u'\|_{L^\infty_{[0,\log k]} H^1_x} \sim 1$. We invoke the local theory over $\sim \log k$ time intervals J each of unit size to obtain $\|u'\|_{X_{1,1/2+}(J)} \sim 1$, which are square summed to obtain $\|u'\|_{X_{1,1/2+}(0,\log k)} \sim (\log k)^{1/2}$. Returning to the original frame of reference, we conclude that

$$\|u\|_{X_{1,1/2+}(s_k,s_{k+1})} \lesssim 2^{k(1+\delta)},$$

where a δ-loss is incurred in part from the $(\log k)^{1/2}$ factor but also from the $b = \frac{1}{2} + \delta$ weight in the X norm. Thus,

$$\|u\|_{X_{1,1/2+}(0,s_K)} = \left(\sum_{k=1}^{K-1} 2^{2k(1+\delta)} \right)^{1/2} \sim 2^{K(1+\delta)}. \qquad \square$$

Now suppose that $u(t)$ satisfies (3-1). Let $t_k = 1 - 2^{-k}$ and $I_k = [0, t_k]$. Then from (3-1) and mass conservation, we have

$$\|P_{\ge N} u(t)\|_{L^\infty_{I_k} L^2_x} \lesssim \begin{cases} 2^{k(1+\delta)/2} N^{-1} & \text{for } N \ge 2^{k(1+\delta)/2}, \\ 1 & \text{for } N \le 2^{k(1+\delta)/2}. \end{cases} \tag{3-3}$$

To refine (3-3), we will work with local-theory estimates and thus use the analogous bound on the Bourgain norm $X_{0,1/2+}(I_k)$. From Lemma 3.1 we obtain

$$\|P_{\ge N} u\|_{X_{0,1/2+}(I_k)} \lesssim N^{-s} \|P_{\ge N} u\|_{X_{s,1/2+}(I_k)} \le c_s N^{-s} 2^{ks(1+\delta)/2}. \tag{3-4}$$

We obtain from (3-4) that

$$\|P_{\ge N} u\|_{X_{0,1/2+}(I_k)} \lesssim \begin{cases} 2^{k(1+\delta)/2} N^{-1} & \text{for } N \ge 2^{k(1+\delta)/2}, \\ 2^{k\delta'} & \text{for } N \le 2^{k(1+\delta)/2}. \end{cases} \tag{3-5}$$

The next step is to run local-theory estimates to improve (3-5) at *high* frequencies. Frequencies $N \lesssim 2^k \sim (1-t_k)^{-1}$ on I_k effectively do not make it out of the blow-up core before blow-up time due to the finite speed of propagation for such frequencies.[4] Hence, these *low* frequencies can be controlled by spatial location, which we address in Section 4. On the other hand, (3-5) shows that the solution at

[3] One of the conclusions of the Merle–Raphaël analysis is the almost monotonicity of the scale parameter $\lambda(t) = \|\nabla u(t)\|_{L^2}^{-1}$: $\lambda(t_2) < 2\lambda(t_1)$ for all $t_2 \ge t_1$.

[4] Recall that for the Schrödinger equation, frequencies of size N propagate at speed N and thus travel a distance $O(1)$ in time N^{-1}.

frequencies $N \gtrsim 2^{k(1+\delta)/2}$ is small. Thus, for these *high* frequencies, dispersive estimates might be able, upon iteration, to show that the solution is even smaller at these high frequencies.

To chose an intermediate dividing point between the high frequencies that are capable of exiting the blow-up core before blow-up time ($N \gtrsim 2^k$) and the frequency scale at which the blow-up is taking place ($N \sim 2^{k/2}(\log k)^{1/2}$), we consider frequencies $\geq 2^{3k/4}$ to be *high* frequencies and frequencies $\leq 2^{3k/4}$ to be *low* frequencies. The goal of this section is Proposition 3.4 below, which shows that the high frequencies are bounded in H^1. In Section 4 below, we will localize in space to the external region and then control the low frequencies.

We first address the dimension $d = 1$ case.

Lemma 3.2 (high frequency recurrence in one dimension). *Take $d = 1$. Let $t_k = 1 - 2^{-k}$ and $I_k = [0, t_k]$. Let $u(t)$ be a solution such that (3-1) holds, and define*

$$\alpha(k, N) = \|P_{\geq N} u\|_{X_{0,1/2+}(I_k)}. \tag{3-6}$$

Then there exists an absolute constant $0 < \mu \ll 1$ such that for $N \geq 2^{k(1+\delta)/2}$,

$$\|P_{\geq N}(u - e^{it\partial_x^2} u_0)\|_{X_{0,1/2+}(I_k)} \lesssim 2^{k(1+\delta)/2} N^{-1+\delta} \alpha(k+1, \mu N) + 2^{k\delta} \alpha(k+1, \mu N)^2. \tag{3-7}$$

In particular, by Lemma 2.4,

$$\alpha(k, N) \lesssim \|P_{\geq N} u_0\|_{L_x^2} + 2^{k(1+\delta)/2} N^{-1+\delta} \alpha(k+1, \mu N) + 2^{k\delta} \alpha(k+1, \mu N)^2. \tag{3-8}$$

Proof. By (2-7) of Lemma 2.4 with $\omega = 2^{-k-1}$ and $I = I_k$,

$$\|P_{\geq N}(u - e^{it\partial_x^2} u_0)\|_{X_{0,1/2+}(I_k)} \lesssim 2^{k\delta} \|P_{\geq N}(|u|^4 u)\|_{X_{0,-1/2+}(I_{k+1})}.$$

In the rest of the proof, we estimate the right side of the estimate above, and we will just write I_k instead of I_{k+1} for convenience. By duality,

$$\|P_{\geq N}(|u|^4 u)\|_{X_{0,-1/2+}(I_k)} = \sup_{\|w\|_{X_{0,1/2-}(I_k)}=1} \int_{I_k} \int_{x \in \mathbb{R}} P_{\geq N}(|u|^4 u)\, w\, dx\, dt.$$

Fix w with $\|w\|_{X_{0,1/2-}(I_k)} = 1$ and let

$$J := \int_{I_k} \int_{x \in \mathbb{R}} P_{\geq N}(|u|^4 u)\, w\, dx\, dt.$$

Then J can be decomposed into a finite sum of terms J_α, each of the form (we have dropped complex conjugates, since they are unimportant in the analysis)

$$J_\alpha := \int_0^{t_k} \int_{x \in \mathbb{R}} P_{\geq N}(u_1 u_2 u_3 u_4 u_5)\, w\, dx\, dt$$

such that each term (after a relabeling of the u_j for $1 \leq j \leq 5$) falls into exactly one of the following two categories.[5]

[5] Indeed, decompose each u_j as $u_j = u_{j,\text{lo}} + u_{j,\text{med}} + u_{j,\text{hi}}$, where $u_{j,\text{lo}} = P_{\leq N/160} u_j$, $u_{j,\text{med}} = P_{N/160 \leq \cdot \leq N/20}$, and $u_{j,\text{hi}} = P_{\geq N/20} u_j$. Then in the expansion of $u_1 u_2 u_3 u_4 u_5$, at least one term must be "hi"; without loss take this to be u_5.

Note that w is frequency supported in $|\xi| \gtrsim N$.

Case 1 (exactly one high). Each u_j for $1 \le j \le 4$ is frequency supported in $|\xi| \le \mu N$ and u_5 is frequency supported in $|\xi| \ge 8\mu N$. In this case, we estimate as

$$|J_\alpha| \le \|u_1\|_{L^\infty_{I_k} L^\infty_x} \|u_2\|_{L^\infty_{I_k} L^\infty_x} \|u_3 u_5\|_{L^2_{I_k} L^2_x} \|u_4 w\|_{L^2_{I_k} L^2_x}. \tag{3-9}$$

For $j = 1, 2$, Gagliardo–Nirenberg and (3-1) implies

$$\|u_j\|_{L^\infty_{I_k} L^\infty_x} \lesssim \|u_j\|^{1/2}_{L^\infty_{I_k} L^2_x} \|\partial_x u_j\|^{1/2}_{L^\infty_{I_k} L^2_x} \lesssim 2^{k(1+\delta)/4}. \tag{3-10}$$

The bilinear Strichartz estimate (Lemma 2.6) yields

$$\|u_3 u_5\|_{L^2_{I_k} L^2_x} \lesssim N^{-1/2} \|u_3\|_{X_{0,1/2+}(I_k)} \|u_5\|_{X_{0,1/2+}(I_k)} \lesssim N^{-1/2} 2^{k\delta} \alpha(k, \mu N). \tag{3-11}$$

The interpolated bilinear Strichartz estimate (Lemma 2.8) yields

$$\|u_4 w\|_{L^2_{I_k} L^2_x} \lesssim N^{-1/2+\delta} \|u_4\|_{X_{0,1/2+}(I_k)} \|w\|_{X_{0,1/2-}(I_k)} \lesssim N^{-1/2+\delta} 2^{k\delta}. \tag{3-12}$$

Substituting (3-10), (3-11), and (3-12) into (3-9), we obtain

$$|J_\alpha| \lesssim 2^{k(1+\delta)/2} N^{-1+\delta} \alpha(k, \mu N).$$

Case 2 (at least two high). Both u_4 and u_5 are frequency supported in $|\xi| \ge \mu N$ (no restrictions on u_j for $1 \le j \le 3$). Then we estimate as

$$|J_\alpha| \le \|u_1\|_{L^6_{I_k} L^{6+\delta}_x} \|u_2\|_{L^6_{I_k} L^6_x} \|u_3\|_{L^6_{I_k} L^6_x} \|u_4\|_{L^6_{I_k} L^6_x} \|u_5\|_{L^6_{I_k} L^6_x} \|w\|_{L^6_{I_k} L^{6-\delta'}_x}. \tag{3-13}$$

For $2 \le j \le 3$ we invoke the Strichartz estimate (Lemma 2.5) and (3-5) to obtain

$$\|u_j\|_{L^6_{I_k} L^6_x} \lesssim \|u_j\|_{X_{0,1/2+}(I_k)} \le 2^{k\delta}. \tag{3-14}$$

For $4 \le j \le 5$ we invoke the Strichartz estimate (Lemma 2.5) and (3-6) to obtain

$$\|u_j\|_{L^6_{I_k} L^6_x} \lesssim \|u_j\|_{X_{0,1/2+}} \le \alpha(k, \mu N). \tag{3-15}$$

For $j = 1$, by Sobolev embedding, the Strichartz estimate (Lemma 2.5), and (3-5),

$$\|u_1\|_{L^6_{I_k} L^{6+}_x} \lesssim \|D^\delta_x u_1\|_{L^6_{I_k} L^6_x} \lesssim \|u_1\|_{X_{\delta,1/2+}(I_k)} \lesssim 2^{k\delta}. \tag{3-16}$$

By the interpolated Strichartz estimate (Lemma 2.7), we have

$$\|w\|_{L^6_t L^{6-}_x} \lesssim \|w\|_{X_{0,1/2-}(I_k)} = 1. \tag{3-17}$$

Using (3-14)–(3-17) in (3-13),

$$|J_\alpha| \lesssim 2^{k\delta} \alpha(k, \mu N)^2.$$ □

In the 2D case, we will just go ahead and assume that $N \ge 2^{3k/4}$ to reduce confusion with deltas.

Case 1 corresponds to $u_{1,\mathrm{lo}} u_{2,\mathrm{lo}} u_{3,\mathrm{lo}} u_{4,\mathrm{lo}} u_{5,\mathrm{hi}}$ and Case 2 corresponds to everything else (at least one u_j for $1 \le j \le 4$ must be "med" or "hi". Hence, we can take $\mu = 1/160$.

Lemma 3.3 (high frequency recurrence, 2D). *Take $d=2$. Let $t_k = 1-2^{-k}$ and $I_k=[0,t_k]$. Let $u(t)$ be a solution such that (3-1) holds and define*

$$\alpha(k,N) := \|P_{\geq N} u\|_{X_{0,1/2+}(I_k)}. \tag{3-18}$$

Then there exists an absolute constant $0<\mu \ll 1$ such that for $N \gtrsim 2^{3k/4}$,

$$\|P_{\geq N}(u - e^{it\Delta}u_0)\|_{X_{0,1/2+}(I_k)} \lesssim 2^{k\delta} N^{-1/6+\delta} \alpha(k+1, \mu N). \tag{3-19}$$

In particular, by Lemma 2.4,

$$\alpha(k,N) \lesssim \|P_{\geq N} u\|_{L^2_x} + 2^{k\delta} N^{-1/6+\delta} \alpha(k+1,\mu N). \tag{3-20}$$

Proof. By Lemma 2.4 (2-7) with $I = I_k$ and $\omega = 2^{-k-1}$,

$$\|P_{\geq N}(u - e^{it\Delta}u_0)\|_{X_{0,1/2+}(I_k)} \lesssim 2^{k\delta} \|P_{\geq N}(|u|^2 u)\|_{X_{0,-1/2+}(I_{k+1})}.$$

In the remainder of the proof, we estimate the right side, and for convenience take I_{k+1} to be I_k. By duality,

$$\|P_{\geq N}(|u|^2u)\|_{X_{0,-1/2+}(I_k)} = \sup_{\|w\|_{X_{0,1/2-}(I_k)}=1} \int_{I_k}\int_{x\in\mathbb{R}} P_{\geq N}(|u|^2 u)\, w\, dx\, dt.$$

Fix w with $\|w\|_{X_{0,1/2-}(I_k)} = 1$ and let

$$J := \int_{I_k}\int_{x\in\mathbb{R}} P_{\geq N}(|u|^2u)\, w\, dx\, dt.$$

Then J can be decomposed into a finite sum of terms J_α, each of the form (we have dropped complex conjugates, since they are unimportant in the analysis)

$$J_\alpha := \int_0^{t_k}\int_{x\in\mathbb{R}} P_{\geq N}(u_1u_2u_3)\, w\, dx\, dt$$

such that each term (after a relabeling of the u_j for $1 \leq j \leq 3$) falls into exactly one of the following two categories.[6] Note that w is frequency supported in $|\xi| \gtrsim N$.

Case 1′ (exactly one high). Both u_1 and u_2 are frequency supported in $|\xi| \leq N^{5/6}$ and u_3 is frequency supported in $|\xi| \geq N/12$. In this case, we estimate as

$$|J_\alpha| \lesssim \|u_1 w\|_{L^2_{I_k}L^2_x} \|u_2u_3\|_{L^2_{I_k}L^2_x}.$$

By the interpolated bilinear Strichartz estimate (Lemma 2.8),

$$\|u_1w\|_{L^2_{I_k}L^2_x} \lesssim (N^{5/6})^{1/2} N^{-1/2+\delta} \|u_1\|_{X_{0,1/2-}(I_k)} \|w\|_{X_{0,1/2-}(I_k)} \lesssim N^{-1/12+\delta} 2^{k\delta},$$

[6] Indeed, decompose $u_j = u_{j,\text{lo}} + u_{j,\text{med}} + u_{j,\text{hi}}$, where $u_{j,\text{lo}} = P_{\leq N^{5/6}} u_j$, $u_{j,\text{med}} = P_{N^{5/6} \leq \cdot \leq N/12}$, and $u_{j,\text{hi}} = P_{\geq N/12} u_j$. Then at least one term must be "hi"; take it to be u_3. Case 1′ corresponds to $u_{1,\text{lo}}u_{2,\text{lo}}u_{3,\text{hi}}$ and Case 2′ corresponds to all other possibilities. Hence, we can take $\mu = 1/12$.

and by Lemma 2.6 directly,

$$\|u_2u_3\|_{L^2_{I_k}L^2_x} \lesssim (N^{5/6})^{1/2}N^{-1/2+\delta}\|u_2\|_{X_{0,1/2+}(I_k)}\|u_3\|_{X_{0,1/2+}(I_k)} \lesssim N^{-1/12+\delta}2^{k\delta}\alpha(k,\mu N).$$

Combining yields

$$|J_\alpha| \lesssim N^{-1/6+\delta}2^{k\delta}\alpha(k,\mu N).$$

Case 2′ (at least two high). Here we suppose that u_2 is frequency supported in $|\xi| \geq N^{5/6}$ and u_3 is frequency supported in $|\xi| \geq \mu N$; we make no assumptions about u_1. Then we estimate as

$$|J_\alpha| \lesssim \|u_1\|_{L^4_{I_k}L^{4+\delta}_x}\|u_2\|_{L^4_{I_k}L^4_x}\|u_3\|_{L^4_{I_k}L^4_x}\|w\|_{L^4_{I_k}L^{4-\delta}_x}.$$

For u_1, we use Sobolev embedding and (3-5) to obtain

$$\|u_1\|_{L^4_{I_k}L^{4+\delta}_x} \lesssim \|D^\delta_x u_1\|_{L^4_{I_k}L^4_x} \lesssim \|u_1\|_{X_{\delta,\frac{1}{2}+}(I_k)} \lesssim 2^{k\delta}.$$

Since $N \gtrsim 2^{3k/4}$, we have $N^{5/6} \gtrsim 2^{5k/8} \gg 2^{k(1+\delta)/2}$, and thus by Lemma 2.5 and (3-5),

$$\begin{aligned}\|u_2\|_{L^4_{I_k}L^4_x} \lesssim 2^{k(1+\delta)/2}N^{-5/6} &\lesssim (2^{k(1+\delta)}N^{-2/3})N^{-1/6}\\ &\lesssim 2^{k\alpha}N^{-1/6}, \quad \text{since } N \gtrsim 2^{3k/4}.\end{aligned}$$

For u_3, we use Lemma 2.5 and (3-18) to obtain

$$\|u_3\|_{L^4_{I_k}L^4_x} \lesssim \alpha(k,\mu N).$$

Combining, we obtain (changing deltas)

$$|J_\alpha| \lesssim 2^{k\delta}N^{-1/6}\alpha(k,\mu N).$$ □

The main result of this section is the following. It states that high frequencies (those strictly above $2^{3k/4}$) are H^1 bounded on I_k. Moreover, if we subtract the linear flow, we obtain $H^{4/3-\delta}$ boundedness for frequencies above $2^{3k/4}$ in the case $d = 1$ and $H^{7/6-\delta}$ boundedness for frequencies above $2^{3k/4}$ in the case $d = 2$.[7]

Proposition 3.4. *Let $t_k = 1 - 2^{-k}$, $I_k = [0, t_k]$, and let $u(t)$ be a solution to (1-1) such that (3-1) holds. Then we have*

$$\|P_{\geq 2^{3k/4}}u(t)\|_{L^\infty_{I_k}H^1_x} \lesssim \|P_{\geq 2^{3k/4}}u(t)\|_{X_{1,1/2+}(I_k)} \lesssim 1.$$

Moreover, we have the following regularity above H^1 *after the linear flow of the initial data is removed: For any $0 \leq s \leq \frac{4}{3} - \delta$ in the case $d = 1$ and for any $0 \leq s \leq \frac{7}{6} - \delta$ in the case $d = 2$, we have*

$$\|P_{\geq 2^{3k/4}}(u(t) - e^{it\Delta}u_0)\|_{L^\infty_{I_k}H^s_x} \lesssim \|P_{\geq 2^{3k/4}}(u(t) - e^{it\Delta}u_0)\|_{X_{s,1/2+\delta}(I_k)} \lesssim 1. \tag{3-21}$$

[7] In fact, the threshold $\geq 2^{3k/4}$, to obtain H^1 boundedness (but not (3-21)), can be replaced by $2^{k(1+\delta)/2}$ for any $\delta > 0$; in the $d = 1$ case, one can appeal to Lemma 3.2 with a strictly smaller choice of δ in order to obtain a nontrivial gain upon each application of Lemma 3.2. The number of applications of Lemma 3.2 is still finite number but δ-dependent. In the 2D case, Lemma 3.3 would first need to be rewritten. We have stated the proposition with threshold $\geq 2^{3k/4}$ because this is all that is needed in Section 4, and it allows us to avoid confusion with multiple small parameters.

Proof. We carry out the $d=1$ case in full, which is a consequence of Lemma 3.2. The $d=2$ case follows from Lemma 3.3 in a similar way.

By (3-5), we start with the knowledge that $\alpha(k,N) \lesssim 2^{k(1+\delta)/2}N^{-1}$ for $N \geq 2^{k(1+\delta)/2}$. Note

$$\|P_{\geq N}u_0\|_{L_x^2} \lesssim N^{-1}\|\nabla u_0\|_{L_x^2} \lesssim N^{-1}.$$

By (3-8) in Lemma 3.2,

$$\alpha(k,N) \lesssim N^{-1} + 2^{k(1+\delta)/2}N^{-1+\delta}\alpha(k+1,\mu N). \tag{3-22}$$

Application of (3-22) m times gives

$$\alpha(k,N) \lesssim N^{-1}\Bigl(\sum_{j=0}^{m-1}(2^{k(1+\delta)/2}N^{-1+\delta})^j\Bigr) + (2^{k(1+\delta)/2}N^{-1+\delta})^m\alpha(k+m,\mu^m N).$$

Since $N \geq 2^{3k/4}$, we have $2^{k/2}N^{-1} \lesssim N^{-1/3}$. Taking $m=7$ we obtain $\alpha(k,N) \lesssim N^{-1}$. Substituting this into (3-7) of Lemma 3.2, we obtain

$$\|P_{\geq N}(u(t) - e^{it\partial_x^2}u_0)\|_{X_{0,1/2+}(I_k)} \lesssim 2^{k(1+\delta)/2}N^{-2+\delta} \lesssim N^{-4/3+\delta}. \qquad \square$$

4. Finite speed of propagation

Recall that the main result of the last section was Proposition 3.4, which showed that the solution at frequencies $\geq 2^{3k/4}$ is H^1 bounded on I_k. This was achieved without applying any restriction in space. In this section, we apply a spatial restriction to $|x| \geq R$ (outside the blow-up core), and study the low frequencies $\leq 2^{3k/4}$ on I_k. Since frequencies of size N propagate at speed N, and thus travel a distance $O(1)$ over a time N^{-1}, we expect that frequencies of size $\lesssim 2^k$ involved in the blow-up dynamics will be incapable of exiting the blow-up core $|x| \leq R$ before blow-up time.

Since $I_k = [0,t_k]$ and $t_k = 1-2^{-k}$, restricting to frequencies $\leq 2^{3k/4}$ on I_k for each k is effectively equivalent to inserting a time-dependent spatial frequency projection $P_{\leq(1-t)^{-3/4}}$. The main technical Lemma 4.3 below shows that, for $0 < r_1 < r_2 < \infty$, the H^s size of the solution in the external region $|x| \geq r_2$ is bounded by the $H^{s-1/8}$ size of the solution in the slightly larger external region $|x| \geq r_1$. This lemma is proved by studying the equation solved by $P_{\leq(1-t)^{-3/4}}\psi u$, where ψ is a spatial cutoff. In estimating the inhomogeneous terms of this equation, we use that the presence of the $P_{\leq(1-t)^{-3/4}}$ projection enables an exchange of α spatial derivatives for a factor of $(1-t)^{-3\alpha/4}$. This is the manner in which finite speed of propagation is implemented. Lemma 4.3 is the main recurrence device for proving Proposition 4.4, giving the H^1 boundedness of the solution in the external region, completing the proof of Theorem 1.1.

Before getting to Lemma 4.3, we begin by using the method of Raphaël [2006], based on the use of local smoothing and (3-2), to achieve a small gain of regularity.[8]

[8] In the $d=1$ case, we obtain a gain of $2/5$ derivatives in this first step, but in fact the proof could be rewritten to achieve a gain of $s < 1/2$ derivatives. The reason $s = 1/2$ derivatives cannot be achieved in one step is the failure of the $H^{1/2} \hookrightarrow L^\infty$ embedding needed to estimate the nonlinear term. One could achieve $1/2$ derivatives by running the same argument twice, but

Lemma 4.1 (a little regularity, $d = 1$ case). *Suppose $d = 1$. Suppose that $u(t)$ solving (1-1) with H^1 initial data satisfies (3-1). Fix $R > 0$. Then*

$$\|\langle D_x\rangle^{2/5}\psi_R u\|_{L^\infty_{[0,1)}L^2_x} \lesssim 1,$$

where $\psi_R(x) = \psi(x/R)$ and $\psi(x)$ is a smooth cutoff with $\psi(x) = 1$ for $|x| \ge 1/2$ and $\psi(x) = 0$ for $|x| \le 1/4$.

Proof. Let $w = \psi_R u$ and $q = \psi_{R/2}u$. Then w solves the equation

$$i\partial_t w + \partial_x^2 w = -|q|^4 w + 2\partial_x(\psi_R' u) - \psi_R'' u = F_1 + F_2 + F_3.$$

Apply $\langle D_x\rangle^{2/5}$, and estimate with $I = [T_1, 1)$ using the (dual) local smoothing estimate for the F_2 term:

$$\|\langle D_x\rangle^{2/5} w\|_{L^\infty_I L^2_x} \lesssim \|\langle D_x\rangle^{2/5} w(T_1)\|_{L^2_x} + \|\langle D_x\rangle^{2/5} F_1\|_{L^1_I L^2_x} + \|\langle D_x\rangle^{2/5}\langle D_x\rangle^{-1/2} F_2\|_{L^2_I L^2_x} + \|\langle D_x\rangle^{2/5} F_3\|_{L^1_I L^2_x}.$$

We begin by estimating term F_1. By the fractional Leibniz rule,

$$\begin{aligned}\|D_x^{2/5} F_1\|_{L^1_I L^2_x} &\lesssim \||q|^4\|_{L^1_I L^\infty_x}\|D_x^{2/5} w\|_{L^\infty_I L^2_x} + \|D_x^{2/5}|q|^4\|_{L^1_I L^{5/2}_x}\|w\|_{L^\infty_I L^{10}_x}.\\ &\lesssim \big(\||q|^4\|_{L^1_I L^\infty_x} + \|D_x^{2/5}|q|^4\|_{L^1_I L^{5/2}_x}\big)\|D_x^{2/5} w\|_{L^\infty_I L^2_x}.\end{aligned}$$

By Sobolev/Gagliardo–Nirenberg embedding and (3-2),

$$\||q|^4\|_{L^\infty_x} + \|D_x^{2/5}|q|^4\|_{L^{5/2}_x} \lesssim \|q\|_{L^2_x}^2\|\partial_x q\|_{L^2_x}^2 \lesssim (1-t)^{-1}(\log(1-t)^{-1})^{-2}.$$

Applying the L^1_I time norm, we obtain a bound by $(\log(1-T_1)^{-1})^{-1}$. Hence,

$$\|\langle D_x\rangle^{2/5} F_1\|_{L^1_I L^2_x} \lesssim (\log(1-T_1)^{-1})^{-1}\|\langle D_x\rangle^{2/5} w\|_{L^\infty_I L^2_x}.$$

Next, we address term F_2. We have

$$\|\langle D_x\rangle^{2/5}\langle D_x\rangle^{-1/2} F_2\|_{L^2_I L^2_x} \lesssim \|\langle D_x\rangle^{9/10} q\|_{L^2_I L^2_x} \lesssim \|q\|_{L^\infty_I L^2_x}^{1/10}\|\|\langle\partial_x\rangle q\|_{L^2_x}^{9/10}\|_{L^2_I}.$$

From (3-2), we have $\|\partial_x q\|_{L^2_x} \lesssim (T-t)^{-1/2}|\log(1-t)|^{-1}$ and hence

$$\|\langle D_x\rangle^{2/5}\langle D_x\rangle^{-1/2} F_2\|_{L^2_I L^2_x} \lesssim (1-T_1)^{1/10}.$$

Term F_3 is comparatively straightforward. Indeed, we obtain

$$\|\langle D_x\rangle^{2/5} F_3\|_{L^1_I L^2_x} \lesssim \|u\|_{L^\infty_I L^2_x}^{3/5}\|\|\langle\partial_x\rangle\psi_2 u\|_{L^2_x}^{2/5}\|_{L^1_I} \lesssim (1-T_1)^{4/5}.$$

Collecting the estimates above, we obtain

$$\|\langle D_x\rangle^{2/5} w\|_{L^\infty_I L^2_x} \lesssim \|\langle D_x\rangle^{2/5} w(T_1)\|_{L^2_x} + (\log(1-T_1)^{-1})^{-1}\|\langle D_x\rangle^{2/5} w\|_{L^\infty_I L^2_x} + (1-T_1)^{1/10}.$$

---

this is unnecessary since we only need a small gain of $s > 0$ to complete the proof of our main new Lemma 4.3/Proposition 4.4 below, which enables us to reach the full $s = 1$ gain. One cannot achieve a gain of $s > 1/2$ by the method employed in the proof of Lemma 4.1 alone due to the term $\partial_x(\psi_R' u)$.

By taking T_1 sufficiently close to 1 so that $(\log(1-T_1)^{-1})^{-1}$ beats out the (absolute) implicit constants furnished by the estimates, we obtain

$$\|\langle D_x\rangle^{2/5} w\|_{L_I^\infty L_x^2} \lesssim \|\langle D_x\rangle^{2/5} w(T_1)\|_{L_x^2} + (1-T_1)^{1/10}. \qquad \square$$

Lemma 4.2 (a little regularity, $d = 2$ case). *Suppose $d = 2$. Suppose that $u(t)$ solving (1-1) with H^1 initial data satisfies (3-1). Fix $R > 0$. Then*

$$\|\langle D_x\rangle^{1/2} \psi_R u\|_{L^\infty_{[0,1)} L_x^2} \lesssim 1,$$

where $\psi_R(x) = \psi(x/R)$ and $\psi(x)$ is a smooth cutoff with $\psi(x) = 1$ for $|x| \ge \frac{1}{2}$ and $\psi(x) = 0$ for $|x| \le \frac{1}{4}$.

Proof. Let $w = \psi_R u$ and $q = \psi_{R/2} u$, and take $\tilde{\psi} = \nabla_x \psi_R$ and $\tilde{\tilde{\psi}} = \Delta_x \psi_R$. Then w solves the equation

$$i\partial_t w + \Delta w = -|q|^2 w + 2\nabla_x \cdot (\tilde{\psi}\, u) - \tilde{\tilde{\psi}}\, u = F_1 + F_2 + F_3.$$

Apply $\langle D_x\rangle^{1/2}$, and estimate with $I = [T_1, 1)$ using the (dual) local smoothing estimate for the term F_2:

$$\|\langle D_x\rangle^{1/2} w\|_{L_I^\infty L_x^2} + \|\langle D_x\rangle^{1/2} w\|_{L_I^4 L_x^4} \lesssim \|\langle D_x\rangle^{1/2} w_0\|_{L_x^2} + \|\langle D_x\rangle^{1/2} F_1\|_{L_I^{4/3} L_x^{4/3}} + \|F_2\|_{L_I^2 L_x^2} + \|\langle D_x\rangle^{1/2} F_3\|_{L_I^1 L_x^2}.$$

Before we begin treating term F_1, let us note that by (3-2), $\|\nabla q\|_{L_x^2} \lesssim (1-t)^{-1/2} (\log(1-t)^{-1})^{-1}$ and hence $\|\nabla q\|_{L_I^2 L_x^2} \lesssim (\log(1-T_1)^{-1})^{-1/2}$. By the fractional Leibniz rule and Sobolev/Gagliardo–Nirenberg embedding,

$$\|D_x^{1/2} |q|^2\|_{L_x^2} \lesssim \|D_x^{1/2} q\|_{L_x^4} \|q\|_{L_x^4} \lesssim \|q\|_{L_x^2}^{1/2} \|\nabla q\|_{L_x^2}^{3/2}.$$

Hence,

$$\|D_x^{1/2} |q|^2\|_{L_I^{4/3} L_x^2} \lesssim \|q\|_{L_I^\infty L_x^2}^{1/2} \|\nabla q\|_{L_I^2 L_x^2}^{3/2} \lesssim (\log(1-T_1)^{-1})^{-3/4}. \tag{4-1}$$

Also, we have

$$\|q\|_{L_x^4} \lesssim \|D_x^{1/2} q\|_{L_x^2} \lesssim \|q\|_{L_x^2}^{1/2} \|\nabla q\|_{L_x^2}^{1/2},$$

and hence

$$\|q\|_{L_I^4 L_x^4}^2 \lesssim \|q\|_{L_I^\infty L_x^2} \|\nabla q\|_{L_I^2 L_x^2} \lesssim (\log(1-T_1)^{-1})^{-1/2}. \tag{4-2}$$

Now we proceed with the estimates for term F_1. By the fractional Leibniz rule (in x),

$$\|\langle D_x\rangle^{1/2} F_1\|_{L_I^{4/3} L_x^{4/3}} \lesssim \|\langle D_x\rangle^{1/2} |q|^2\|_{L_I^{4/3} L_x^2} \|w\|_{L_I^\infty L_x^4} + \||q|^2\|_{L_I^2 L_x^2} \|\langle D_x\rangle^{1/2} w\|_{L_I^4 L_x^4}.$$

By (4-1) and (4-2), we obtain

$$\|\langle D_x\rangle^{1/2} F_1\|_{L_I^{4/3} L_x^{4/3}} \lesssim (\log(1-T_1)^{-1})^{-1/2} (\|\langle D_x\rangle^{1/2} w\|_{L_I^\infty L_x^2} + \|\langle D_x\rangle^{1/2} w\|_{L_I^4 L_x^4}).$$

Next, we treat the F_2 term. Again since $\|\nabla q\|_{L_x^2} \lesssim (1-t)^{-1/2} (\log(1-t)^{-1})^{-1}$,

$$\|F_2\|_{L_I^2 L_x^2} \lesssim (\log(1-T_1)^{-1})^{-1}.$$

The F_3 term is comparatively straightforward.

Collecting the estimates above, we have

$$\begin{aligned}\|\langle D_x\rangle^{1/2}w\|_{L_I^\infty L_x^2}+\|\langle D_x\rangle^{1/2}w\|_{L_I^4L_x^4} &\lesssim \|\langle D_x\rangle^{1/2}w(T_1)\|_{L_x^2}+(\log(1-T_1)^{-1})^{-1}\\ &\quad+(\log(1-T_1)^{-1})^{-1/2}(\|\langle D_x\rangle^{1/2}w\|_{L_I^\infty L_x^2}+\|\langle D_x\rangle^{1/2}w\|_{L_I^4L_x^4}).\end{aligned}$$

By taking T_1 sufficiently close to 1, we obtain

$$\|\langle D_x\rangle^{1/2}w\|_{L_I^\infty L_x^2}\lesssim \|\langle D_x\rangle^{1/2}w(T_1)\|_{L_x^2}+(\log(1-T_1)^{-1})^{-1}. \qquad \square$$

Lemma 4.3 (low frequency recurrence). *Let $d=1$ or $d=2$, $0<R\le r_1<r_2$ and $\frac{1}{8}\le s\le 1$. Let $\psi_1(x)$ and $\psi_2(x)$ be smooth radial cutoff functions such that*

$$\psi_1(x)=\begin{cases}0 & \text{on } |x|\le r_1,\\ 1 & \text{on } |x|\ge \frac12(r_1+r_2)\end{cases} \quad \text{and} \quad \psi_2(x)=\begin{cases}0 & \text{on } |x|\le \frac12(r_1+r_2),\\ 1 & \text{on } |x|\ge r_2.\end{cases}$$

Then

$$\|D_x^s\psi_2u\|_{L^\infty_{[0,1)}L_x^2}\lesssim 1+\|\langle D_x\rangle^{s-1/8}\psi_1u\|_{L^\infty_{[0,1)}L_x^2}.$$

Proof. Let $\chi(\rho)$ be a smooth function such that $\chi(\rho)=1$ for $|\rho|\le 1$ for $\chi(\rho)=0$ for $|\rho|\ge 2$. Let $P_-=P_{\le(T-t)^{-3/4}}$ be the time-dependent multiplier operator defined by $\widehat{Pf}(\xi)=\chi((T-t)^{3/4}|\xi|)\hat f(\xi)$ (where the Fourier transform is in space only). Note that the Fourier support of P at time $t_k=1-2^{-k}$ is $\lesssim 2^{3k/4}$. We further have that

$$\partial_t P_-f=\tfrac34 i(1-t)^{-1/4}QD_xf+P\partial_t f,$$

where $Q=Q_{(1-t)^{-3/4}}$ is the time-dependent multiplier

$$\widehat{Qf}(\xi)=\chi'((1-t)^{3/4}|\xi|)\hat f(\xi).$$

Note that the Fourier support of Q at time $t_k=1-2^{-k}$ is $\sim 2^{3k/4}$. Note also that if $g=g(x)$ is any function, then

$$\|PD_x^\alpha g\|_{L_x^2}\le(1-t)^{-3\alpha/4}\|g\|_{L_x^2}. \tag{4-3}$$

Let $w=P_-\psi_2u$. Taking $\widetilde\psi_2=\nabla_x\psi_2$ and $\widetilde{\widetilde\psi}_2=\Delta_x\psi_2$, we have

$$\begin{aligned}i\partial_t w+\Delta w&=-i(1-t)^{-1/4}Q\cdot\nabla_x\, w-P_-\psi_2|u|^{4/d}u+2P_-\nabla_x\cdot[\widetilde\psi_2u]-P_-\widetilde{\widetilde\psi}_2u\\ &=F_1+F_2+F_3+F_4.\end{aligned}$$

By the energy method,

$$\|D_x^sw\|^2_{L^\infty_{[0,1)}L_x^2}\lesssim\|D_x^sw(0)\|^2_{L_x^2}+\int_0^1|\langle D_x^sF_1(s),D_x^sw(s)\rangle_{L_x^2}|\,ds+10\sum_{j=2}^4\|D_x^sF_j\|^2_{L^1_{[0,1)}L_x^2}.$$

For term F_1, we argue as follows. Let $\tilde Q$ be a projection onto frequencies of size $(1-t)^{-3/4}$. Then

$$\int_0^1|\langle D_x^sF_1(s),D_x^sw(s)\rangle_{L_x^2}|\,ds\lesssim\int_0^1(1-s)^{-1/4}\|D_x^{1/2+s}\tilde Q\psi_2u(s)\|^2_{L_x^2}\,ds.$$

Applying (4-3) with $\alpha = \frac{1}{2}$, we can control the above by

$$\int_0^1 (1-s)^{-1} \|D_x^s \tilde{Q}\psi_2 u(s)\|_{L_x^2}^2 \, ds.$$

Dividing the time interval $[0, 1) = \bigcup_{k=1}^{\infty} [t_k, t_{k+1})$, we bound the above by

$$\sum_{k=1}^{+\infty} 2^k \int_{t_k}^{t_{k+1}} \|D_x^s P_{2^{3k/4}} \psi_2 u(s)\|_{L_x^2}^2 \, ds \lesssim \sum_{k=1}^{+\infty} \|D_x^s P_{2^{3k/4}} \psi_2 u(s)\|_{L^\infty_{[t_k,t_{k+1})} L_x^2}^2,$$

where $P_{2^{3k/4}}$ is the projection onto frequencies of size $\sim 2^{3k/4}$ (and not $\lesssim 2^{3k/4}$). However, writing $u(t) = e^{it\Delta}u_0 + (u(t) - e^{it\Delta}u_0)$, the above is controlled by (taking $s = 1$, the worst case)

$$\sum_{k=1}^{\infty} \|\nabla_x P_{2^{3k/4}} u_0\|_{L_x^2}^2 + \sum_{k=1}^{+\infty} \|\nabla_x P_{2^{3k/4}} (u(t) - e^{it\Delta}u_0)\|_{L_x^2}^2.$$

By (3-21) of Proposition 3.4,

$$\|\nabla_x u_0\|_{L_x^2}^2 + \sum_{k=1}^{+\infty} 2^{-k/8} \lesssim 1.$$

In conclusion, for term F_1 we obtain

$$\int_0^1 |\langle D_x^s F_1(s), D_x^s w(s)\rangle_{L_x^2}| \, ds \lesssim 1.$$

We next address term F_2. Insert $\psi_2 \psi_1^{4/d+1} = \psi_2$, then apply (4-3) with $\alpha = s$ to obtain (in the worst case $s = 1$),

$$\|D_x^s F_2\|_{L^1_{[0,1)} L_x^2} \lesssim \|(1-t)^{-3/4} \psi_2 |u|^{4/d} u\|_{L^1_{[0,1)} L_x^2} \lesssim \|(1-t)^{-3/4} \|\psi_1 u\|_{L_x^{2(4/d+1)}}^{4/d+1} \|_{L^1_{[0,1)}}.$$

We consider the cases $d = 1$ and $d = 2$ separately. When $d = 1$,

$$\|\psi_1 u\|_{L_x^{10}} \lesssim \|D_x^{2/5} \psi_1 u\|_{L_x^2} \lesssim 1,$$

by Lemma 4.1. Consequently,

$$\|D_x^s F_2\|_{L^1_{[0,1)} L_x^2} \lesssim \|(1-t)^{-3/4}\|_{L^1_{[0,1)}} \lesssim 1.$$

On the other hand, when $d = 2$, we have

$$\|\psi_1 u\|_{L_x^6} \lesssim \|D_x^{2/3} \psi_1 u\|_{L_x^2} \lesssim \|D_x^{1/2} \psi_1 u\|_{L_x^2}^{2/3} \|\nabla_x \psi_1 u\|_{L_x^2}^{1/3} \lesssim (1-t)^{-1/6}$$

by Lemma 4.2 and (3-2). Consequently,

$$\|D_x^s F_2\|_{L^1_{[0,1)} L_x^2} \lesssim \|(1-t)^{-3/4} (1-t)^{-1/6}\|_{L^1_{[0,1)}} \lesssim 1.$$

Next, we address term F_3. By (4-3) with $\alpha = 9/8$,

$$\|D_x^s F_3\|_{L^1_{[0,1)} L_x^2} \lesssim \|(1-t)^{-27/32}\|_{L^1_{[0,1)}} \|D_x^{s-1/8} (\tilde{\psi}_2 u)\|_{L^\infty_{[0,1)} L_x^2}.$$

Since $\|(1-t)^{-27/32}\|_{L^1_{[0,1)}} \sim 1$ and the support of $\tilde{\psi}_2$ is contained in the set where $\psi_1 = 1$, we have

$$\|D_x^s F_3\|_{L^1_{[0,1)} L^2_x} \lesssim \|\langle D_x \rangle^{s-1/8} \psi_1 u\|_{L^\infty_{[0,1)} L^2_x}.$$

Finally, we consider F_4. We have

$$\|D_x^s F_4\|_{L^1_{[0,1)} L^2_x} \lesssim \|\langle \nabla_x \rangle P_- \psi_1 u\|_{L^1_{[0,1)} L^2_x} \lesssim \|(1-t)^{-3/4}\|_{L^1_{[T_1,1)}} \|u\|_{L^\infty_{[0,1)} L^2_x} \lesssim 1$$

by (4-3) with $\alpha = 1$. □

Proposition 4.4. *Suppose that $u(t)$ solving (1-1) with H^1 initial data satisfies (3-1). Fix $R > 0$. Then*

$$\|u\|_{L^\infty_{[0,1)} H^1_{|x| \geq R}} \lesssim 1.$$

Proof. Iterate Lemma 4.3 eight times on successively larger external regions. □

Proposition 4.4 completes the proof of Theorem 1.1.

5. Application to 3D standing sphere blow-up

We now outline the proof of Theorem 1.2 utilizing the techniques of Section 3 and 4. Theorem 1.2 pertains to radial solutions of (1-9). We define the initial data set $\mathcal{P}$ as in[9] Raphaël and Szeftel [2009, Definition 1, page 980–1], except that condition (v) is replaced by $\|\tilde{u}_0\|_{H^1(|r-1| \geq 1/10)} \leq \epsilon^5$. The goal then becomes to complete the proof of the bootstrap Proposition 1 on page 982, where the "improved regularity estimates" (35)–(37) are effectively replaced with

$$\|u(t)\|_{L^\infty_{[0,t_1]} H^1_{|x| \leq 1/2}} \leq \epsilon.$$

Let us formulate a more precise statement:

Proposition 5.1 (partial bootstrap argument). *Let Q be the 1D ground state given by (1-4), and let $\epsilon > 0$, $T > 0$ be fixed with $T \leq \epsilon^{200}$. Suppose that $u(t)$ is a radial 3D solution to*

$$i\partial_t u + \Delta u + |u|^4 u = 0$$

on an interval $[0, T'] \subset [0, T)$ such that the following "bootstrap inputs" hold:

(1) *There exist parameters $\lambda(t) > 0$, $\gamma(t) \in \mathbb{R}$, and $|r(t) - 1| \leq 1/10$, such that if we define*

$$\tilde{u}(r, t) = u(r, t) - \frac{1}{\lambda(t)^{1/2}} Q\left(\frac{r - r(t)}{\lambda(t)}\right), \tag{5-1}$$

then, for $0 \leq t \leq T'$,

$$\|\nabla u(t)\|_{L^2_x} = \lambda(t)^{-1} \sim \left(\frac{\log|\log(T-t)|}{T-t}\right)^{1/2}, \tag{5-2}$$

and

$$\|\nabla \tilde{u}(t)\|_{L^2_x} \lesssim \frac{1}{|\log(T-t)|^{1+}(T-t)^{1/2}}. \tag{5-3}$$

[9] We are considering the case dimension $d = 3$ (in their notation $N = 3$).

(2) *Interior Strichartz control*: $\|\langle\nabla\rangle u(t)\|_{L^5_{[0,T']}L^{30/11}_{|x|\le 1/2}} \le \epsilon$.

(3) *Initial data remainder control*: $\|\langle\nabla\rangle\tilde{u}_0\|_{L^2_x} \le \epsilon^5$.

Then we have the following "bootstrap output":

$$\|\langle\nabla\rangle u(t)\|_{L^\infty_{[0,T']}L^2_{|x|\le 1/2}} + \|\langle\nabla\rangle u(t)\|_{L^5_{[0,T']}L^{30/11}_{|x|\le 1/2}} \lesssim \epsilon^5. \tag{5-4}$$

The goal of this section is to prove Proposition 5.1, which shows that the bootstrap input (2) is reinforced. Proposition 5.1 is, however, an incomplete bootstrap and by itself does not establish Theorem 1.2. The analysis which uses (5-4) to reinforce the bootstrap assumption (1) is rather elaborate but will be omitted here as it follows the arguments in [Raphaël 2006] and [Raphaël and Szeftel 2009]. Moreover, these papers demonstrate how the assertions in Theorem 1.2 follow.

The proof of Proposition 5.1 follows the methods developed in Section 3–4 used to prove Theorem 1.1. We do not, however, rescale the solution so that $T = 1$ as was done in Section 3.

Remark 5.2. Let us list some notational conventions for the rest of the section. We take $t_k = T - 2^{-k}$ and denote $I_k = [0, t_k]$. Let $v(r,t) = ru(r,t)$, and consider v as a 1D function in r extended to $r < 0$ as an odd function. Note that v solves

$$i\partial_t v + \partial_r^2 v = -r^{-4}|v|^4 v.$$

The frequency projection P_N will always refer to the 1D frequency projection in the r-variable. The Bourgain norm $\|v\|_{X_{s,b}}$ refers to the 1D norm in the r-variable.

Let $\lambda_0 = \lambda(0)$ and take $k_0 \in \mathbb{N}$ such that $2^{-k_0/2}(\log k_0)^{-1/2} \sim \lambda_0$. We then have $T \sim 2^{-k_0}$. The assumption $T \le \epsilon^{40}$ equates to $2^{-k_0/8} \le \epsilon^5$. Note that $\lambda(t_k) = 2^{-k/2}(\log k)^{-1/2}$.

Lemma 5.3 (smallness of initial data). *Under the assumption (3) in Proposition 5.1 on the initial data, and with $v_0 = ru_0$, we have*

$$\|P_{\ge 2^{3k_0/4}}\partial_r v_0\|_{L^2_r} + \|\partial_r v_0\|_{L^2_{r\le 1/2}} \lesssim \epsilon^5.$$

Proof. Let $\tilde{v}_0 = r\tilde{u}_0$. Since $\partial_r\tilde{v}_0 = \tilde{u}_0 + r\partial_r\tilde{u}_0$, we have by Hardy's inequality

$$\|\partial_r\tilde{v}_0\|_{L^2_r} \lesssim \||x|^{-1}\tilde{u}_0\|_{L^2_x} + \|\nabla\tilde{u}_0\|_{L^2_x} \lesssim \|\nabla\tilde{u}_0\|_{L^2_x} \lesssim \epsilon^5.$$

Recalling the definition of $\tilde{u}_0 = \tilde{u}(0)$ in (5-1) (with $t = 0$), we have

$$v_0 = \frac{r}{\lambda_0^{1/2}} Q\left(\frac{r - r_0}{\lambda_0}\right) + \tilde{v}_0.$$

The result then follows from the exponential localization and smoothness of Q. □

Lemma 5.4 (radial Strichartz). *Suppose that $u(t)$ is a 3D radial solution to*

$$i\partial_t u + \Delta u = f.$$

Let $v(r,t) = ru(r,t)$ and $g(r,t) = rf(r,t)$ and consider v as a 1D function in r (extended to be odd), so that

$$i\partial_t v + \partial_r^2 v = g.$$

Then for (q,r) *and* $(\tilde{q},\tilde{r})$ *satisfying the* 3D *admissibility condition*,

$$\|r^{2/p-1}v\|_{L_t^q L_r^p} \lesssim \|v_0\|_{L_r^2} + \|r^{2/p'-1}g\|_{L_t^{\tilde{q}'} L_r^{\tilde{p}'}}.$$

Proof. The left side is equivalent to $\|\nabla u\|_{L_t^q L_x^p}$ and the right side is equivalent to $\|u_0\|_{L_x^2} + \|f\|_{L_t^{\tilde{q}'} L_x^{\tilde{p}}}$, so it is just a restatement of the 3D Strichartz estimates. □

Lemma 5.5 (3D to 1D conversion). *Suppose that* $u(x)$ *is a* 3D *radial function, and write* $u(r) = u(x)$. *Let* $v(r) = ru(r)$. *Then for* $1 < p < 3$, *we have*

$$\|r^{2/p-1}\partial_r v\|_{L_r^p} \lesssim \|\nabla_x u\|_{L_x^p}. \tag{5-5}$$

Also for $\frac{3}{2} < p < +\infty$, *we have*

$$\|\nabla_x u\|_{L_x^p} \lesssim \|r^{2/p-1}\partial_r v\|_{L_r^p}. \tag{5-6}$$

Consequently, for 3D *admissible pairs* (q, p) *such that* $2 \le p < 3$, *we have*

$$\|\nabla u\|_{L_t^q L_x^p} \sim \|r^{2/p-1}\partial_r v\|_{L_t^q L_r^p}. \tag{5-7}$$

We remark that $q = 5$ and $p = \frac{30}{11}$ falls within the range of validity for (5-7).

Proof. The proof of (5-5) and (5-6) is a standard application of the Hardy inequality.

First, we prove (5-5). Using $v = ru$,

$$r^{2/p-1}\partial_r v = r^{2/p}\partial_r u + r^{2/p-1}u,$$

and thus,

$$\|r^{2/p-1}\partial_r v\|_{L_r^p} \le \|r^{2/p}\partial_r u\|_{L_r^p} + \|r^{2/p-1}u\|_{L_r^p}.$$

We have, for $r > 0$,

$$u(r) = -(u(+\infty) - u(r)) = \int_{s=1}^{+\infty} \frac{d}{ds}(u(sr))\,ds = \int_{s=1}^{+\infty} u'(sr)r\,ds.$$

By the Minkowski integral inequality,

$$\|r^{2/p-1}u\|_{L_r^p} \le \int_{s=1}^{+\infty} \|u'(sr)r^{2/p}\|_{L_{r>0}^p}\,ds.$$

Changing variable $r \mapsto s^{-1}r$, we obtain that the right-hand side is bounded by

$$\biggl(\int_{s=1}^{+\infty} s^{-3/p}\,ds\biggr)\|r^{2/p}u'\|_{L_{r>0}^p}$$

and the s integral is finite provided $p < 3$.

Next, we prove (5-6). We have

$$r^{2/p}\partial_r u = r^{2/p}\partial_r(r^{-1}v) = -r^{2/p-2}v + r^{2/p-1}\partial_r v,$$

and hence,

$$\|r^{2/p}\partial_r u\|_{L_r^p} \le \|r^{2/p-2}v\|_{L_r^p} + \|r^{2/p-1}\partial_r v\|_{L_r^p}.$$

We have

$$v(r)=v(r)-v(0)=\int_{s=0}^{1}\frac{d}{ds}(v(sr))\,ds=\int_{s=0}^{1}v'(sr)r\,ds.$$

By the Minkowski integral inequality,

$$\|r^{2/p-2}v\|_{L_r^p}\le\int_{s=0}^{1}\|v'(sr)r^{2/p-1}\|_{L_r^p}\,ds.$$

Changing variable $r\mapsto s^{-1}r$ in the right side, we obtain

$$\|r^{2/p-2}v\|_{L_r^p}\le\Big(\int_{s=0}^{1}s^{-3/p+1}ds\Big)\|v'(r)r^{2/p-1}\|_{L_r^p}$$

and the s integral is finite provided $p>\frac{3}{2}$. □

The replacement for Lemma 3.1 is Lemma 5.6 below. The difference is that in Lemma 5.6, we only use $b<\frac{1}{2}$ when working at $\dot H^1$ regularity.

Lemma 5.6. *Suppose that the assumptions of Proposition 5.1 and Remark 5.2 hold. Then for* $\frac{1}{2}-\delta\le b<\frac{1}{2}$,

$$\|\partial_r v\|_{X_{0,b}(I_k)}\lesssim 2^{kb}(\log k)^{b+1/2}=(T-t)^{-b}(\log|\log(T-t)|)^{b+1/2}. \tag{5-8}$$

Also, for $\frac{1}{2}-\delta<b<\frac{1}{2}+\delta$,

$$\|v\|_{X_{0,b}(I_k)}\lesssim_\delta 2^{k\delta}=(T-t)^{-\delta}. \tag{5-9}$$

Proof. We will only carry out the proof of (5-8), which stems from (5-2).[10] The proof of (5-9) is similar, and stems from the bound on $\|u(t)\|_{H^\delta}$ obtained from interpolation between (5-2) and mass conservation.

In the proof below, T has no relation to the T representing blow-up time in the rest of the article.

Let $\lambda=\lambda(t_k)=2^{-k/2}(\log k)^{-1/2}$. Let $r=\lambda R$, $x=\lambda X$, and $t=\lambda^2T+t_k$. Define the functions

$$V(R,T)=\lambda^{1/2}v(\lambda R,\lambda^2T+t_k)=\lambda^{1/2}v(r,t),$$
$$U(X,T)=\lambda^{1/2}u(\lambda X,\lambda^2T+t_k)=\lambda^{1/2}u(x,t).$$

Note that the identity $v(r)=ru(r)$ corresponds to $V(R)=\lambda RU(R)$.

We study $V(R,T)$ on $T\in[0,\log k]$, which corresponds to $t\in[t_k,t_{k+1}]$. We have $\|V\|_{L_R^2}=\|v\|_{L_r^2}\sim O(1)$ (by mass conservation) and $\|\partial_R V\|_{L_R^2}=\lambda\|\partial_r v\|_{L_r^2}$. Hence, $\|\partial_R V\|_{L^\infty_{[0,\log k]}L_R^2}=O(1)$. The equation satisfied by V is

$$i\partial_T V+\partial_R^2V=-\lambda^{-4}R^{-4}|V|^4V.$$

Let $J=[a,b]$ be a unit-sized time interval in $[0,\log k]$. Then by Lemma 2.4,

$$\|\partial_R V\|_{X_{0,b}(J)}\lesssim\|\partial_R V(a)\|_{L^2}+\|\partial_R(\lambda^{-4}R^{-4}|V|^4V)\|_{L_J^1L_R^2}.$$

[10]The need to take $b<1/2$ comes from Lemma 2.4, (2-7) versus (2-8); when working at $\dot H^1$ regularity near the origin, we cannot suffer any loss of derivatives. The fact that $\|\partial_r v\|_{X_{0,b}(I_k)}$ for $b<1/2$ is only a $\dot H^1$ subcritical quantity is of no harm as the only application of (5-8) in the subsequent arguments is to control the solution for $r\ge 1/2$, where the equation is effectively L^2 critical.

Let $\chi_1(r) = 1$ for $r \le \frac{1}{4}$ and $\operatorname{supp}\chi_1 \subset B(0, \frac{3}{8})$. Let $\chi_2 = 1 - \chi_1$. Let $g_1 = \partial_R(\lambda^{-4}R^{-4}\chi_1(\lambda R)|V|^4 V)$ and $g_2 = \partial_R(\lambda^{-4}R^{-4}\chi_2(\lambda R)|V|^4 V)$, so that the above becomes

$$\|\partial_R V\|_{X_{0,b}(J)} \lesssim \|\partial_R V(a)\|_{L^2} + \|g_1\|_{L^1_J L^2_R} + \|g_2\|_{L^1_J L^2_R}. \tag{5-10}$$

We begin with estimating $\|g_2\|_{L^1_J L^2_R}$. We have

$$\|g_2\|_{L^1_J L^2_R} \lesssim \|V^5\|_{L^1_J L^2_R} + \|V^4(\partial_R V)\|_{L^1_J L^2_R}. \tag{5-11}$$

We now treat the first term in (5-11). Of course, $\|V^5\|_{L^1_J L^2_R} = \|V\|^5_{L^5_J L^{10}_R}$. By Sobolev embedding $\|V\|_{L^{10}_R} \lesssim \|D_R^{2/5} V\|_{L^2_R}$ and by Hölder,

$$\begin{aligned}\|V\|_{L^5_J L^{10}_R} \lesssim |J|^{1/10}\|D_R^{2/5}V\|_{L^{10}_J L^2_R} &\lesssim |J|^{1/10}(\|V\|_{L^{10}_J L^2_R} + \|\partial_R V\|_{L^{10}_J L^2_R})\\ &\le |J|^{1/10}(|J|^{1/10}\,\|V\|_{L^\infty_J L^2_R} + \|\partial_R V\|_{L^{10}_J L^2_R}).\end{aligned}$$

Using that $\|V\|_{L^\infty_J L^2_R} \sim 1$, that $|J| \sim 1$ and Lemma 2.7, provided $\frac{2}{5} < b < \frac{1}{2}$, we have

$$\|V\|_{L^5_J L^{10}_R} \lesssim |J|^{1/10}(1 + \|\partial_R V\|_{X_{0,b}}). \tag{5-12}$$

We now treat the second term in (5-11), similarly estimating the term $\|V\|_{L^{10}_R}$. We have

$$\begin{aligned}\|V^4\partial_R V\|_{L^1_J L^2_R} &\lesssim |J|^{7/20}\|V\|^4_{L^{10}_J L^{10}_R}\|\partial_R V\|_{L^4_J L^{10}_R}\\ &\lesssim |J|^{7/20}(1 + \|\partial_R V\|_{L^{10}_J L^2_R})^4\|\partial_R V\|_{L^4_J L^{10}_R}.\end{aligned}$$

Appealing to Lemma 2.7, provided $\frac{9}{20} < b < \frac{1}{2}$, we obtain

$$\|V^4\partial_R V\|_{L^1_J L^2_R} \lesssim |J|^{7/20}(1 + \|\partial_R V\|_{X_{0,b}})^5. \tag{5-13}$$

Combining (5-12) and (5-13), we have

$$\|g_2\|_{L^1_J L^2_R} \lesssim |J|^{7/20}(1 + \|\partial_R V\|_{X_{0,b}})^5. \tag{5-14}$$

Next we estimate $\|g_1\|_{L^1_J L^2_R}$. By rescaling,

$$\|g_1\|_{L^1_J L^2_R} = \lambda\|\partial_r(\chi_1 r^{-4}|v|^4 v)\|_{L^1_{[t_k,t_{k+1}]}L^2_r}.$$

Let $w = \tilde{\chi}_1 u$, where $\tilde{\chi}_1 = 1$ on $\operatorname{supp}\chi_1$ but $\operatorname{supp}\tilde{\chi}_1 \subset B(0, \frac{1}{2})$. Replacing $u = r^{-1}v$, we obtain $\partial_r(r\chi_1 u^5) = \partial_r(r\chi_1 w^5)$, and hence,

$$\|g_1\|_{L^2_R} \lesssim \lambda(\|w\|^5_{L^{10}_r} + \|r w^4\partial_r w\|_{L^2_r}) \lesssim \lambda(\||x|^{-1/5}w\|^5_{L^{10}_x} + \|w^4\nabla w\|_{L^2_x}). \tag{5-15}$$

By Hardy's inequality and 3D Sobolev embedding,

$$\||x|^{-1/5}w\|_{L^{10}_x} \lesssim \|D_x^{1/5}w\|_{L^{10}_x} \lesssim \|\nabla w\|_{L^{30/11}_x}.$$

By Hölder's inequality and 3D Sobolev embedding,

$$\|w^4\nabla w\|_{L^2_x} \le \|w\|^4_{L^{30}_x}\|\nabla w\|_{L^{30/11}_x} \lesssim \|\nabla w\|^5_{L^{30/11}_x}.$$

Returning to (5-15) and invoking (2) of Proposition 5.1,

$$\|g_1\|_{L^1_{I_k} L^2_r} \lesssim \lambda \|\nabla w\|^5_{L^5_{I_k} L^{30/11}_x} \lesssim \lambda \epsilon^5. \tag{5-16}$$

By putting (5-14) and (5-16) into (5-10), we obtain

$$\|\partial_R V\|_{X_{0,b}(J)} \lesssim \|\partial_R V(a)\|_{L^2} + |J|^{7/20}(1 + \|\partial_R V\|_{X_{0,b}(J)})^5 + \lambda \epsilon^5.$$

From this, we conclude that we can take $|J|$ sufficiently small (but still "unit-sized"[11]) so that it follows that

$$\|\partial_R V\|_{X_{0,b}(J)} \le O(1).$$

Square summing over unit-sized intervals J filling $[0, \log k]$,

$$\|\partial_R V\|_{X_{0,b}([0,\log k])} \lesssim (\log k)^{1/2}.$$

This estimate scales back to

$$\|\partial_r v\|_{X_{0,b}([t_k,t_{k+1}])} \lesssim (\log k)^{1/2} \lambda(t_k)^{-2b} = 2^{kb} (\log k)^{b+1/2}.$$

Now square sum over k from $k = 0$ to $k = K$ to obtain a bound of $2^{Kb}(\log K)^{b+1/2}$ over the time interval I_K, which is the claimed estimate (5-8). □

The analogue of Lemma 3.2 will be Lemma 5.7 below. We note that as a consequence of Lemma 5.6, the hypothesis of Lemma 5.7 below is satisfied with $\alpha(k, N) = 2^{-k/2} N^{-1}$.

Lemma 5.7 (high-frequency recurrence). *Let the assumptions of Proposition 5.1 and Remark 5.2 hold, and let*[12]

$$\beta(k, N) := \|P_{\ge N} \partial_r v\|_{X_{0,1/2-}(I_k)}.$$

Then there exists an absolute constant $0 < \mu \ll 1$ *such that for* $N \ge 2^{k(1+\delta)/2}$, *we have*

$$\beta(k, N) + \|r^{2/p-1} P_{\ge N} \partial_r v\|_{L^q_{I_k} L^p_r} \lesssim \|P_{\ge N} \partial_r v_0\|_{L^2_r} + 2^{k(1+\delta)/2} N^{-1+\delta} \beta(k, \mu N) + N^{-1+\delta} 2^{k\delta} \beta(k, \mu N)^2 + 2^{-k\delta} + \epsilon^5 \tag{5-17}$$

for all 3D *admissible* (q, p).

Proof. Note that v solves

$$i \partial_t v + \partial_r^2 v = -r|u|^4 u = -r^{-4}|v|^4 v.$$

Let $\chi_1(r)$ be a smooth function such that $\chi_1(r) = 1$ for $|r| \le \frac{1}{4}$ and χ_1 is supported in $|r| \le \frac{3}{8}$. Let $\chi_2 = 1 - \chi_1$. Apply $P_{\ge N} \partial_r$ to obtain

$$(i \partial_t + \partial_r^2) P_{\ge N} \partial_r v = g_1 + g_2,$$

[11] Meaning: with size independent of any small parameters like ϵ or λ

[12] Note the inclusion of one derivative in the definition of β, in contrast to the choice of definition for α in Proposition 3.4.

where

$$g_j(r) = -P_{\geq N}\partial_r(\chi_j r^{-4}|v|^4 v) \quad \text{for } j = 1, 2.$$

Then by Lemma 2.4[13] and Lemma 5.4,

$$\|P_{\geq N}\partial_r v\|_{X_{0,1/2-}(I_k)} + \|r^{2/p-1}P_{\geq N}\partial_r v\|_{L^q_{I_k}L^p_r} \lesssim \|P_{\geq N}\partial_r v_0\|_{L^2_r} + \|g_1\|_{L^1_{I_k}L^2_r} + \|g_2\|_{L^1_{I_k}L^2_r}.$$

The term $\|g_2\|_{L^1_t L^2_r}$ is controlled in a manner similar to the analysis in the proof of Lemma 3.2. For this term, $\chi_2 r^{-4}$ and $\partial_r(\chi_2 r^{-4})$ are smooth bounded functions, with all derivatives bounded. By Lemma 2.10,

$$\|g_2\|_{L^2_r} \lesssim \|P_{\geq N}\langle\partial_r\rangle v^5\|_{L^2_r} + N^{-1}\|\langle\partial_r\rangle v^5\|_{L^2_r}. \tag{5-18}$$

By an analysis similar to the proof of Lemma 3.2, utilizing the bounds in Lemma 5.6, we obtain

$$\|P_{\geq N}\langle\partial_r\rangle v^5\|_{L^1_{I_k}L^2_r} \lesssim 2^{k(1+\delta)/2}N^{-1+\delta}\beta(k,\mu N) + N^{-1+\delta}2^{k\delta}\beta(k,\mu N)^2. \tag{5-19}$$

Also by the Strichartz estimates, as in the proof of Lemma 5.6 above,

$$\|\langle\partial_r\rangle v^5\|_{L^1_{I_k}L^2_r} \lesssim \|D^\delta v\|^4_{X_{0,b}}\|\partial_R v\|_{X_{0,b}} \lesssim 2^{k(1+\delta)/2}. \tag{5-20}$$

Inserting (5-19) and (5-20) into (5-18), we obtain

$$\|g_2\|_{L^1_{I_k}L^2_r} \lesssim 2^{k(1+\delta)/2}N^{-1+\delta}\beta(k,\mu N) + N^{-1+\delta}2^{k\delta}\beta(k,\mu N)^2 + N^{-1}2^{k(1+\delta)/2}. \tag{5-21}$$

The last term, $N^{-1}2^{k(1+\delta)/2}$, gives the contribution $2^{-k\delta}$ in (5-17) due to the restriction $N \geq 2^{k(1+\delta)/2}$ (different deltas).

Next we address $\|g_1\|_{L^1_{I_k}L^2_r}$. We estimate away $P_{\geq N}$ by

$$\|g_1\|_{L^1_{I_k}L^2_r} \lesssim \|\tilde g_1\|_{L^1_{I_k}L^2_r}, \tag{5-22}$$

where (ignoring complex conjugates)

$$\tilde g_1 = \partial_r(r^{-4}\chi_1 v^5).$$

Let $w = \tilde\chi_1 u$, where $\tilde\chi_1 = 1$ on $\operatorname{supp}\chi_1$ but $\operatorname{supp}\tilde\chi_1 \subset B(0, \frac12)$. Replacing $u = r^{-1}v$, we obtain $\tilde g_1 = \partial_r(r\chi_1 u^5) = \partial_r(r\chi_1 w^5)$, and hence,

$$\|\tilde g_1\|_{L^2_r} \lesssim \|w\|^5_{L^{10}_r} + \|rw^4\partial_r w\|_{L^2_r} \lesssim \||x|^{-1/5}w\|^5_{L^{10}_x} + \|w^4\nabla w\|_{L^2_x}.$$

By Hardy's inequality and 3D Sobolev embedding,

$$\||x|^{-1/5}w\|_{L^{10}_x} \lesssim \|D^{1/5}_x w\|_{L^{10}_x} \lesssim \|\nabla w\|_{L^{30/11}_x}.$$

By Hölder's inequality and 3D Sobolev embedding,

$$\|w^4\nabla w\|_{L^2_x} \leq \|w\|^4_{L^{30}_x}\|\nabla w\|_{L^{30/11}_x} \lesssim \|\nabla w\|^5_{L^{30/11}_x}.$$

[13] We were able to obtain the $L^1_{I_k}L^2_r$ right side (without δ loss), because we took $b < 1/2$ in the Bourgain norm.

Hence, $\|\tilde{g}_1\|_{L^2_r} \lesssim \|\nabla w\|^5_{L_x^{30/11}}$. Returning to (5-22) and invoking (2) of Proposition 5.1,

$$\|g_1\|_{L^1_{I_k}L^2_r} \lesssim \|\nabla w\|^5_{L^5_{I_k}L_x^{30/11}} \lesssim \epsilon^5. \qquad \square$$

The analogue of Proposition 3.4 is this:

Proposition 5.8 (high-frequency control). *Let the assumptions of Proposition 5.1 and Remark 5.2 hold. Then for any 3D Strichartz admissible pair (q, p), we have*

$$\|P_{\geq 2^{3k/4}}\partial_r v\|_{X_{0,1/2-}(I_k)} + \|r^{2/p-1}P_{\geq 2^{3k/4}}\partial_r v\|_{L^q_{I_k}L^p_r} \lesssim \epsilon^5.$$

Proof. Several applications of Lemma 5.7, just as Proposition 3.4 is deduced from Lemma 3.2. $\square$

Due to the $\dot{H}^1$ criticality of the problem, we do not have improved regularity of $v(t) - e^{it\partial_r^2}v_0$ as was the case in Proposition 3.4. As a substitute, we can use the methods of Lemma 5.7 to obtain the following lemma:

Lemma 5.9 (additional high-frequency control). *Suppose that the assumptions of Proposition 5.1 and Remark 5.2 hold. Then*

$$\left(\sum_{k=k_0}^{+\infty} \|P_{2^{3k/4}}\partial_r v\|^2_{L^\infty_{[t_{k-1},t_k]}L^2_r}\right)^{1/2} \lesssim \epsilon^5. \tag{5-23}$$

Proof. It suffices to prove the estimate with the sum terminating at $k = K$, provided we obtain a bound independent of K. For each k in $k_0 \leq k \leq K$, write the integral equation on I_k. For $t \in [t_{k-1}, t_k]$

$$v(t) = e^{it\partial_r^2}v_0 - i\int_0^t e^{i(t-t')\partial_r^2}(r^{-4}|v|^4 v(t'))\,dt'.$$

Apply $P_{2^{3k/4}}\partial_r$ to obtain

$$P_{2^{3k/4}}\partial_r v(t) = P_{2^{3k/4}}e^{it\partial_r^2}\partial_r v_0 - i\int_0^t e^{i(t-t')\partial_r^2}P_{2^{3k/4}}\partial_r(r^{-4}|v|^4 v(t'))\,dt'.$$

Estimate

$$\|P_{2^{3k/4}}\partial_r v\|_{L^\infty_{[t_{k-1},t_k]}L^2_r} \leq \|P_{2^{3k/4}}\partial_r v_0\|_{L^2_r} + \|P_{2^{3k/4}}\partial_r(r^{-4}|v|^4 v)\|_{L^1_{I_k}L^2_r}.$$

By the inequality $(a+b)^2 \leq 2a^2 + 2b^2$, this implies

$$\|P_{2^{3k/4}}\partial_r v\|^2_{L^\infty_{[t_{k-1},t_k]}L^2_r} \lesssim \|P_{2^{3k/4}}\partial_r v_0\|^2_{L^2_r} + \|P_{2^{3k/4}}\partial_r(r^{-4}|v|^4 v)\|^2_{L^1_{I_k}L^2_r}.$$

Let $\chi_1(r)$ be a smooth function such that $\chi_1(r) = 1$ for $|r| \leq \frac{1}{4}$ and χ_1 is supported in $|r| \leq \frac{3}{8}$. Let $\chi_2 = 1 - \chi_1$. Let $g_j = P_{2^{3k/4}}\partial_r(\chi_j r^{-4}|v|^4 v)$ for $j = 1, 2$.

Recall that in the proof of Lemma 5.7, we showed that

$$\|P_{\geq N}\partial_r \chi_2 r^{-4}|v|^4 v\|_{L^1_{I_k}L^2_r} \lesssim 2^{k(1+\delta)/2}N^{-1+\delta}\beta(k,\mu N) + N^{-1+\delta}2^{k\delta}\beta(k,\mu N)^2 + N^{-1}2^{k(1+\delta)/2},$$

and Proposition 5.8 showed that $\beta(k, 2^{3k/4}) \lesssim 1$. Combining gives $\|g_2\|_{L^1_{I_k} L^2_r} \lesssim 2^{-k/8}$, and hence,

$$\left(\sum_{k=k_0}^{K} \|g_2\|^2_{L^1_{I_k} L^2_r}\right)^{1/2} \lesssim 2^{-k_0/8} \le \epsilon^5.$$

Now we address g_1. Let $w = \tilde{\chi}_1 u$. For each k, lengthen I_k to $I := I_K$ to obtain

$$\sum_{k=k_0}^{K} \|g_1\|^2_{L^1_{I_k} L^2_r} \lesssim \|P_{2^{3k/4}} \partial_r (r^{-4} \chi_1 |w|^4 w)\|^2_{\ell^2_k L^1_I L^2_r}.$$

By the Minkowski inequality, for any space-time function F, we have

$$\|P_{2^{3k/4}} F\|_{\ell^2_k L^1_I L^2_r} \le \|P_{2^{3k/4}} F\|_{L^1_I \ell^2_k L^2_r} \lesssim \|F\|_{L^1_I L^2_r}.$$

Hence,

$$\sum_{k=k_0}^{K} \|g_1\|^2_{L^1_{I_k} L^2_r} \lesssim \|\partial_r (\chi_1 r^{-4} |w|^4 w)\|^2_{L^1_I L^2_r}.$$

At this point we proceed as in Lemma 5.7 to obtain a bound by ϵ^5. □

Now we begin to insert spatial cutoffs away from the blow-up core and obtain the missing low frequency bounds. The first step is to obtain a little regularity above L^2, since it is needed in the proof of Lemma 5.11.

Lemma 5.10 (small regularity gain). *Suppose that the assumptions of Proposition 5.1 and Remark 5.2 hold. Let $\psi_{3/4}(r)$ be a smooth function such that $\psi_{3/4}(r) = 1$ for $|r| \le \frac{3}{4}$ and $\psi_{3/4}(r) = 0$ for $|r| \ge \frac{7}{8}$. Then*

$$\|\langle D_r \rangle^{3/7} \psi_{3/4} v\|_{L^\infty_{[0,T)} L^2_r} \lesssim \epsilon^5.$$

Proof. Taking $\psi = \psi_{3/4}$, let $w = \psi v$. Then

$$\begin{aligned} i\partial_t w + \partial_r^2 w &= \psi(i\partial_t + \partial_r^2) v + 2\partial_r(\psi' v) - \psi'' v \\ &= -r^{-4} \psi |v|^4 v + 2\partial_r(\psi' v) - \psi'' v = F_1 + F_2 + F_3. \end{aligned}$$

Local smoothing and energy estimates provide the estimate

$$\|D_r^{3/7} w\|_{L^\infty_{[0,T)} L^2_r} \lesssim \|D_r^{3/7} w_0\|_{L^2_r} + \|D_r^{3/7} F_1\|_{L^1_{[0,T)} L^2_r} + \|D_r^{-1/2} D_r^{3/7} F_2\|_{L^2_{[0,T)} L^2_r} + \|D_r^{3/7} F_3\|_{L^1_{[0,T)} L^2_r}. \tag{5-24}$$

We begin with the F_1 estimate. Let $\tilde{\psi}$ be a smooth function such that

$$\tilde{\psi}(r) = \begin{cases} 0 & \text{if } r \le \frac{1}{4}, \\ 1 & \text{if } \frac{1}{2} \le r \le \frac{7}{8}, \\ 0 & \text{if } r \ge \frac{7}{8}. \end{cases}$$

Let $q = r^{-1} \tilde{\psi} v$. By writing $1 = (1 - \tilde{\psi}^4) + \tilde{\psi}^4$, we obtain

$$F_1 = -(1 - \tilde{\psi}^4) \psi r^{-4} |v|^4 v - |q|^4 w.$$

Note that $(1-\tilde{\psi}^4)\psi$ is supported in $|r| \le \frac{1}{2}$ and $\tilde{\psi}^4\psi$ is supported in $\frac{1}{4} \le |r| \le \frac{15}{16}$.

For the term $(1-\tilde{\psi}^4)\psi r^{-4}|v|^4 v$, we appeal to the bootstrap hypothesis (2) in the same way we did in the proof of Lemma 5.7 to obtain a bound by ϵ^5. As for the term $|q|^4 w$, by the fractional Leibniz rule,

$$\|D_r^{3/7}(|q|^4 w)\|_{L^1_{[0,T)}L^2_r} \lesssim \|D_r^{3/7}|q|^4\|_{L^1_{[0,T)}L_r^{7/3}}\|w\|_{L^\infty_{[0,T)}L_r^{14}} + \||q|^4\|_{L^1_{[0,T)}L_r^\infty}\|D_r^{3/7}w\|_{L^\infty_{[0,T)}L^2_r}.$$

By Sobolev embedding and Gagliardo–Nirenberg,

$$\|D_r^{3/7}|q|^4\|_{L_r^{7/3}} + \||q|^4\|_{L_r^\infty} \lesssim \|q\|^2_{L^2_r}\|\partial_r q\|^2_{L^2_r} \quad \text{and} \quad \|w\|_{L_r^{14}} \lesssim \|D_r^{3/7}w\|_{L^2_r}.$$

Hence,

$$\|D_r^{3/7}(|q|^4 w)\|_{L^1_{[0,T)}L^2_r} \lesssim \|q\|^2_{L^\infty_{[0,T)}L^2_r}\|\partial_r q\|^2_{L^2_{[0,T)}L^2_r}\|D_r^{3/7}w\|_{L^\infty_{[0,T)}L^2_r}.$$

By (5-3), $\|\partial_r q\|_{L^2_{[0,T)}L^2_r} \lesssim (|\log T|)^{-1} \lesssim (\log \epsilon^{-1})^{-1}$. Consequently, we obtain

$$\|D_r^{3/7}F_1\|_{L^1_{[0,T)}L^2_r} \lesssim \epsilon^5 + (\log\epsilon^{-1})^{-1}\|D_r^{3/7}w\|_{L^\infty_{[0,T)}L^2_r}.$$

As for F_2, we start by bounding

$$\|D_r^{-1/2}D_r^{3/7}F_2\|_{L^2_{[0,T)}L^2_r} \lesssim \|D_r^{13/14}(\psi' v)\|_{L^2_{[0,T)}L^2_r}.$$

On the support of ψ', we have $v = rq$. Noting that on the support of ψ' we have $r \sim 1$ and using the interpolation, we get

$$\|D_r^{13/14}(\psi' rq)\|_{L^2_r} \lesssim \|q\|_{L^2_r} + \|q\|^{1/14}_{L^2_r}\|\partial_r q\|^{13/14}_{L^2_r}.$$

By (5-3),

$$\|\|\partial_r q\|^{13/14}_{L^2_r}\|_{L^2_{[0,T)}} \lesssim T^{1/28} \lesssim \epsilon^5.$$

Consequently,

$$\|D_r^{-1/2}D_r^{3/7}F_2\|_{L^2_{[0,T)}L^2_r} \lesssim T^{1/2} + T^{1/28} \lesssim \epsilon^5.$$

Finally, for the term F_3, we estimate

$$\|D_r^{3/7}F_3\|_{L^1_{[0,T)}L^2_r} \lesssim \|q\|_{L^1_{[0,T)}L^2_r} + \|\partial_r q\|_{L^1_{[0,T)}L^2_r} \lesssim T + T^{1/2} \lesssim \epsilon^5.$$

Collecting the above estimates and inserting into (5-24), we obtain

$$\|D_r^{3/7}w\|_{L^2_{[0,T)}L^2_r} \lesssim \|D_r^{3/7}w_0\|_{L^2_r} + (\log\epsilon^{-1})^{-1}\|D_r^{3/7}w\|_{L^\infty_{[0,T)}L^2_r} + \epsilon^5,$$

and the result follows (by bootstrap assumption (3), $\|D_r^{3/7}w_0\|_{L^2_r} \lesssim \epsilon^5$). □

We will need to apply the following lemma eight times in the proof of Proposition 5.12 below. As in Section 4, the use of the frequency projection $P_{\lesssim (T-t)^{-3/4}}$ and the process of exchanging derivatives for time factors via (5-25) is essentially an appeal to the finite speed of propagation for low frequencies.

Lemma 5.11 (low frequency recurrence). *Let the assumptions of Proposition 5.1 and Remark 5.2 hold. Let $\frac{5}{8} < r_1 < r_2 < \frac{3}{4}$ and $\frac{1}{8} \le s \le 1$. Let $\psi_1(r)$ and $\psi_2(r)$ be smooth cutoff functions such that*

$$\psi_1(r) = \begin{cases} 1 & \text{on } |r| \le r_1, \\ 0 & \text{on } |r| \ge \frac{1}{2}(r_1+r_2) \end{cases} \quad \text{and} \quad \psi_2(r) = \begin{cases} 1 & \text{on } |r| \le \frac{1}{2}(r_1+r_2), \\ 0 & \text{on } |r| \ge r_2. \end{cases}$$

Then

$$\|D_r^s(\psi_1 v)\|_{L^\infty_{[0,T)}L^2_r} \lesssim \|D_r^{s-1/8}(\psi_2 v)\|_{L^\infty_{[0,T)}L^2_r} + \epsilon^5.$$

Proof. Let $\chi(\xi) = 1$ for $|\xi| \le 1$ and $\chi(\xi) = 0$ for $|\xi| \ge 2$ be a smooth function. Let $P = P_{\le (T-t)^{-3/4}}$ be the time-dependent multiplier operator defined by $\widehat{Pf}(\xi) = \chi((T-t)^{3/4}\xi)\hat{f}(\xi)$ (where Fourier transform is in space only). Note that the Fourier support of P at time $T - t = 2^{-k}$ is $\lesssim 2^{3k/4}$. We further have that

$$\partial_t Pf = \tfrac{3}{4} i (T-t)^{-1/4} Q \partial_r f + P \partial_t f,$$

where $Q = Q_{(T-t)^{-3/4}}$ is the time-dependent multiplier

$$\widehat{Qh}(\xi) = \chi'((T-t)^{3/4}\xi)\,\widehat{h}(\xi).$$

Note that the Fourier support of Q at time $t = T - 2^{-k}$ is $\sim 2^{3k/4}$. Note also that if $g = g(r)$ is any function, then

$$\|PD_r^\alpha g\|_{L^2_r} \le (T-t)^{-3\alpha/4}\|g\|_{L^2_r}. \tag{5-25}$$

Let $\tilde{\psi}$ be a smooth function such that

$$\tilde{\psi}(r) = \begin{cases} 0 & \text{if } |r| \le \frac{1}{4}, \\ 1 & \text{if } \frac{1}{2} \le |r| \le \frac{1}{2}(r_1+r_2), \\ 0 & \text{if } |r| \ge r_2. \end{cases}$$

Let $w = P_{\le (T-t)^{-3/4}} D_r^s(\psi_1 v)$. By Proposition 5.8, it suffices to show that

$$\|w\|_{L^\infty_{[0,T)}L^2_r} \lesssim \|D_r^{s-1/8}(\psi_2 v)\|_{L^\infty_{[0,T)}L^2_r} + \epsilon^5.$$

Note that w solves

$$\begin{aligned} i\partial_t w + \partial_r^2 w &= -\tfrac{3}{4}(T-t)^{-1/4} Q \partial_r D_r^s(\psi_1 v) - PD_r^s(\psi_1 r^{-4}|v|^4 v) + 2P\partial_r D_r^s(\psi_1' v) - PD_r^s(\psi_1'' v) \\ &= F_1 + F_2 + F_3 + F_4. \end{aligned}$$

By the energy method, we obtain

$$\|w\|^2_{L^\infty_t L^2_r} \le \|w_0\|^2_{L^2_r} + \int_0^T |\langle F_1, w\rangle_{L^2_r}| + 10\sum_{j=2}^4 \|F_j\|^2_{L^1_{[0,T)}L^2_r}.$$

We estimate F_1 using Lemma 5.9 as follows.[14] Let $\tilde{Q}$ be a projection onto frequencies of size $\sim (T-t)^{-3/4}$ (importantly, *not* $\lesssim (T-t)^{-3/4}$). Then

$$\int_0^T |\langle F_1, w\rangle_{L_r^2}| \lesssim \int_0^T (T-t)^{-1/4} \|\tilde{Q} D_r^{1/2+s}(\psi_1 v)\|_{L_r^2}^2.$$

It suffices to take $s=1$, the worst case. The presence of $\tilde{Q}$ allows for the exchange $D_r^{1/2} \sim (T-t)^{-3/8}$, which gives

$$\int_0^T |\langle F_1, w\rangle_{L_r^2}| \lesssim \int_0^T (T-t)^{-1} \|\tilde{Q}\partial_r(\psi_1 v)\|_{L_r^2}^2.$$

By decomposing $[0,T) = \bigcup_{k=k_0}^{\infty}[t_k, t_{k+1}]$, and using that $(T-t)^{-1} = 2^k$ on $[t_k, t_{k+1}]$, we have

$$\int_0^T (T-t)^{-1} \|\tilde{Q}\partial_r(\psi_1 v)\|_{L_r^2}^2 = \sum_{k=k_0}^{\infty} \int_{[t_k,t_{k+1}]} 2^k \|P_{2^{3k/4}}\partial_r(\psi_1 v)\|_{L_r^2}^2.$$

Since $|[t_k, t_{k+1}]| = 2^{-k}$, the above is controlled by $\sum_{k=k_0}^{\infty} \|P_{2^{3k/4}}\partial_r(\psi_1 v)\|_{L^\infty_{[t_k,t_{k+1}]}L_r^2}^2$, the square root of which is bounded by ϵ^5 (by Lemma 5.9).

For the nonlinear term F_2, by writing $1 = 1 - \tilde{\psi}^4 + \tilde{\psi}^4$, we have

$$F_2 = -PD_r^s(r^{-4}(1-\tilde{\psi}^4)\psi_1|v|^4 v) - PD_r^s(r^{-4}\tilde{\psi}^4\psi_1|v|^4 v) = F_{21} + F_{22}.$$

The support of $(1-\tilde{\psi}^4)\psi_1$ is contained in $|r| \le \frac{1}{2}$, and we can use the bootstrap hypothesis (2) to obtain

$$\|F_{21}\|_{L^1_{[0,T)}L_r^2} \lesssim \epsilon^5,$$

as was done in the proof of Lemma 5.7 (for any $s \le 1$). For F_{22}, taking $\tilde{v} = \psi_2 v$ and noting that $\psi_1\psi_2 = \psi_1$, we have $F_{22} = PD_r^s(r^{-4}\tilde{\psi}^4\psi_1|\tilde{v}|^4\tilde{v})$. By (5-25) with $\alpha = \frac{1}{8}$,

$$\|F_{22}\|_{L^1_{[0,T)}L_r^2} \le \Big\|(T-t)^{-3/32}\|D_r^{s-1/8}(r^{-4}\tilde{\psi}^4\psi_1|\tilde{v}|^4\tilde{v})\|_{L_r^2}\Big\|_{L^1_{[0,T)}}.$$

Since $\tilde{\psi}$ is supported in $\frac{1}{4} \le |r| \le r_2$, the function $\tilde{\psi}^4\psi_1 r^{-4}$ is smooth and compactly supported. By the fractional Leibniz rule,

$$\|D_r^{s-1/8}(r^{-4}\tilde{\psi}^4\psi_1|\tilde{v}|^4\tilde{v})\|_{L_r^2} \lesssim \|\tilde{v}\|_{L_r^\infty}^4 \|\langle D_r\rangle^{s-1/8}\tilde{v}\|_{L_r^2} \lesssim \|D_r^{3/7}\tilde{v}\|_{L_r^2}^{7/2}\,\|\partial_r\tilde{v}\|_{L_r^2}^{1/2}\,\|\langle D_r\rangle^{s-\frac{1}{8}}\tilde{v}\|_{L_r^2}.$$

Using the bound $\|\partial_r\tilde{v}\|_{L_r^2} \le (T-t)^{-1/2}$ from (5-3) and the bound on $\|D_r^{3/7}\tilde{v}\|_{L^\infty_{[0,T)}L_r^2}$ from Lemma 5.10, we obtain

$$\|F_{22}\|_{L^1_{[0,T)}L_r^2} \lesssim \|(T-t)^{-3/32}(T-t)^{-1/4}\|_{L^1_{[0,T)}}\|\langle D_r\rangle^{s-1/8}\tilde{v}\|_{L^\infty_{[0,T)}L_r^2} \lesssim \epsilon^5\|\langle D_r\rangle^{s-1/8}\tilde{v}\|_{L^\infty_{[0,T)}L_r^2}.$$

To bound F_3, we use (5-25) with $\alpha = \frac{9}{8}$ to obtain

$$\|F_3\|_{L^1_{[0,T)}L_r^2} \lesssim \|(T-t)^{-27/32}\|_{L^1_{[0,T)}}\|D_r^{s-1/8}\tilde{v}\|_{L^\infty_{[0,T)}L_r^2}.$$

[14] It seems that the energy method is needed here, since it furnishes $\int_0^T |\langle F_1, w\rangle_{L_r^2}|$; we cannot see a way to estimate $\|F_1\|_{L^1_{[0,T)}L_r^2}$. Indeed, by pursuing the method here, one ends up with a bound $\|F_1\|_{L^1_{[0,T)}L_r^2} \lesssim \sum_{k=k_0}^{\infty}\|P_{2^{3k/4}}\psi_1 v\|_{L_r^2}$, which is not controlled by Lemma 5.9, since it is not a *square* sum.

The F_4 term is more straightforward than F_3, since there is one fewer derivative. □

The H^1 control will complete part of the bootstrap estimate (5-4) in Proposition 5.1:

Proposition 5.12 (H^1 control). *Suppose that the assumptions of Proposition 5.1 and Remark 5.2 hold. Then*

$$\|\partial_r v\|_{L^\infty_{[0,T)} L^2_{|r|\le 5/8}} \lesssim \epsilon^5.$$

Proof. Let $r_k = \frac{5}{8} + \frac{1}{64}(k-1)$. Apply Lemma 5.11 on $[r_k, r_{k+1}]$ for $k = 1, \dots, 8$ to obtain collectively by Lemma 5.10 that

$$\|\partial_r v\|_{L^\infty_{[0,T)} L^2_{|r|\le 5/8}} \lesssim \epsilon^5 + \|v\|_{L^2_{|r|\le 3/4}} \le \epsilon^5. \qquad \square$$

Proposition 5.13 (local smoothing control). *Let the assumptions of Proposition 5.1 and Remark 5.2 hold. Let $\psi_{9/16}$ be a smooth function such that $\psi_{9/16}(r) = 1$ for $|r| \le \frac{9}{16}$ and $\psi_{9/16}(r) = 0$ for $|r| \ge \frac{5}{8}$. Then*

$$\|D_r^{3/2}(\psi_{9/16} v)\|_{L^2_{[0,T)} L^2_r} \lesssim \epsilon^5.$$

Proof. Let $\chi(\xi) = 1$ for $|\xi| \le 1$ and $\chi(\xi) = 0$ for $|\xi| \ge 2$ be a smooth function. Let $\chi_- = \chi$ and $\chi_+ = 1 - \chi$. Let P_- be the Fourier multiplier with symbol $\chi_-((T-t)^{3/4}\xi)$ and P_+ be the Fourier multiplier with symbol $\chi_+((T-t)^{3/4}\xi)$. Then $I = P_- + P_+$ for each t, and P_- projects onto frequencies $\lesssim (T-t)^{-3/4}$, while P_+ projects onto frequencies $\gtrsim (T-t)^{-3/4}$. Letting Q be the Fourier multiplier with symbol $\frac{3}{4}\chi'((T-t)^{3/4}\xi)$, we have $\partial_t P_\pm f = \pm i(T-t)^{-1/4} Q \partial_r f + P \partial_t f$. Note that Q has Fourier support in $|\xi| \sim (T-t)^{-3/4}$.

First, we can discard low frequencies. From Proposition 5.12 and (5-25) with $\alpha = \frac{1}{2}$,

$$\|D_r^{3/2} P_- \psi_{9/16} v\|_{L^2_{[0,T)} L^2_r} \lesssim \|(T-t)^{-3/8} \partial_r \psi_{9/16} v\|_{L^2_{[0,T)} L^2_r} \lesssim T^{1/8} \|\partial_r \psi_{9/16} v\|_{L^\infty_{[0,T)} L^2_r} \lesssim \epsilon^5.$$

For the high-frequency portion, $D_r^{3/2} P_+ \psi_{9/16} v$, we first need to dispose of the spatial cutoff. We have

$$D_r^{3/2} P_+ \psi_{9/16} = \psi_{9/16} D_r^{3/2} P_+ + [D_r^{3/2} P_+, \psi_{9/16}].$$

The leading order term in the symbol of the commutator $[D_r^{3/2} P_+, \psi_{9/16}]$, by the pseudodifferential calculus, is $\xi^{1/2}\chi_+(\xi(T-t)^{3/4})\psi'(r) + \xi^{3/2}(T-t)^{3/4}\chi_+'(\xi(T-t)^{3/4})\psi'(r)$. Hence, we obtain the bound

$$\|[D_r^{3/2} P_+, \psi_{9/16}]\langle D_r\rangle^{-1/2}\|_{L^2_r \to L^2_r} \lesssim 1,$$

independently of t. Thus, $\|[D_r^{3/2} P_+, \psi_{9/16}]v\|_{L^2_{[0,T)} L^2_r}$ is easily bounded by Proposition 5.12.

It remains to show that $\|\psi_{9/16} D_r^{3/2} P_+ v\|_{L^2_{[0,T)} L^2_r} \lesssim \epsilon^5$, the estimate for the high-frequency portion with no spatial cutoff to the right of the frequency cut-off. To obtain local smoothing via the energy method, we need to introduce the pseudodifferential operator A of order 0 with symbol $\exp(-(\operatorname{sgn}\xi)(\tan^{-1} r))$, where $\operatorname{sgn}\xi$ is a smoothed signum function. Note that by the sharp Gärding inequality, A is positive. The key property of A is

$$\partial_r^2 A f = A \partial_r^2 f - 2i(1+r^2)^{-1} D_r A f + B f,$$

where B is an order 0 pseudodifferential operator. The first-order term $i(1+r^2)^{-1}D_r Af$ will generate the local smoothing estimate.

Let $w = AP_+v$. By the sharp Gärding inequality,

$$\|\psi_{9/16}D_r^{3/2}P_+v\|_{L^2_{[0,T)}L^2_r} \lesssim \|(1+r^2)^{-1/2}D_r^{3/2}w\|_{L^2_{[0,T)}L^2_r}$$

and it suffices to prove that $\|(1+r^2)^{-1/2}D_r^{3/2}w\|_{L^2_{[0,T)}L^2_r} \lesssim \epsilon^5$. The equation satisfied by w is

$$i\partial_t w + \partial_r^2 w + 2i(1+r^2)^{-1}D_r w = (T-t)^{-1/4}AQ\partial_r v - AP_+r^{-4}|v|^4v + Bv = F_1+F_2+F_3,$$

where B is a order 0 operator (satisfying bounds independent of t). By applying ∂_r and pairing this equation with $\partial_r w$ (energy method), we obtain, upon time integration,

$$\|\partial_r w\|^2_{L^\infty_{[0,T)}L^2_r} + \|(1+r^2)^{-1/2}D_r^{3/2}w\|^2_{L^2_{[0,T)}L^2_r} \lesssim \int_0^T |\langle \partial_r F_1, w\rangle| + 10\|\partial_r F_2\|^2_{L^1_{[0,T)}L^2_r} + 10\|\partial_r F_3\|^2_{L^1_{[0,T)}L^2_r}.$$

The F_3 term is easily controlled using Proposition 5.12.

The F_1 term is controlled as in the proof of Lemma 5.11 (a similar first term). For the F_2 term, let ψ be a smooth function such that $\psi(r) = 1$ for $|r| \le \frac{1}{4}$ and $\psi(r) = 0$ for $|r| \le \frac{1}{2}$. Writing $1 = \psi^5 + (1-\psi^5)$, we have

$$F_2 = AP_+\psi^5 r^{-4}|v|^4v + AP_+(1-\psi^5)r^{-4}|v|^4v = F_{21}+F_{22}.$$

We estimate $\|\partial_r F_{21}\|_{L^1_{[0,T)}L^2_r}$ as we did in the proof of Lemma 5.7. For the term F_{22}, take $\psi_+ = (1-\psi^5)r^{-4}$, and note that ψ_+ is smooth and well localized. In the proof of Lemma 5.7 (see (5-18) and (5-21)), we showed that

$$\|P_{\ge N}\partial_r\psi_+|v|^4v\|_{L^1_{I_k}L^2_r} \lesssim 2^{k(1+\delta)/2}N^\delta\beta(k,\mu N) + N^{-1+\delta}2^{k\delta}\beta(k,\mu N)^2 + N^{-1}2^{k(1+\delta)/2}.$$

Furthermore, Proposition 5.8 showed that $\beta(k, 2^{3k/4}) \lesssim 1$. Combining with the above gives

$$\|P_{\ge 2^{3k/4}}\partial_r\psi_+|v|^4v\|_{L^1_{I_k}L^2_r} \lesssim 2^{-k/8}.$$

Thus,

$$\|\partial_r F_{22}\|_{L^1_{[0,T)}L^2_r} \lesssim \sum_{k=k_0}^\infty \|P_{\ge 2^{3k/4}}\partial_r\psi_+|v|^4v\|_{L^1_{I_k}L^2_r} \lesssim \sum_{k=k_0}^\infty \|P_{\ge 2^{3k/4}}\partial_r\psi_+|v|^4v\|_{L^1_{I_k}L^2_r} \lesssim 2^{-k_0/8} \lesssim \epsilon^5. \qquad \square$$

Proposition 5.14 (Strichartz control). *Suppose that the assumptions of Proposition 5.1 and Remark 5.2 hold. Then*

$$\|r^{2/p-1}\partial_r v\|_{L^q_{[0,T)}L^p_{|r|\le 1/2}} \lesssim \epsilon^5.$$

Proof. Let ψ be a smooth function such that $\psi(r) = 1$ for $|r| \le \frac{1}{2}$ and $\psi(r) = 0$ for $|r| \ge \frac{9}{16}$. Let $w = \psi v$. Then w solves

$$i\partial_t w + \partial_r^2 w = -\psi r^{-4}|v|^4v + 2\partial_r(\psi' v) - \psi'' v = F_1+F_2+F_3.$$

By the Strichartz estimate and dual local smoothing estimate, we obtain

$$\|r^{2/p-1}\partial_r w\|_{L^q_{[0,T)}L^p_r} \lesssim \|\partial_r w_0\|_{L^2_r} + \|\partial_r F_1\|_{L^1_{[0,T)}L^2_r} + \|D_r^{-1/2}\partial_r F_2\|_{L^2_{[0,T)}L^2_r} + \|\partial_r F_3\|_{L^1_{[0,T)}L^2_r}.$$

Let $\tilde{\psi}$ be a smooth function such that $\tilde{\psi}(r) = 1$ for $|r| \le \frac{1}{4}$ and $\tilde{\psi}(r) = 0$ for $|r| \ge \frac{1}{2}$. By writing $1 = \tilde{\psi}^5 + (1 - \tilde{\psi}^5)$, we have

$$F_1 = -\psi\tilde{\psi}^5 r^{-4}|v|^4 v - \psi(1-\tilde{\psi}^5)r^{-4}|v|^4 v = F_{11} + F_{12}.$$

Since the support of $\psi\tilde{\psi}^5$ is contained in $|r| \le \frac{1}{2}$, we can estimate the term $\|\partial_r F_{11}\|_{L^1_{[0,T)}L^2_r}$ by ϵ^5 using bootstrap assumption (2) as in the proof of Lemma 5.7. Since $(1-\tilde{\psi}^5)\psi r^{-4}$ is a bounded and smooth function,

$$\|\partial_r F_{12}\|_{L^1_{[0,T)}L^2_r} \lesssim \|\langle\partial_r\rangle v^5\|_{L^1_{[0,T)}L^2_{|r|\le 5/8}} \lesssim T\|\langle\partial_r\rangle v\|^5_{L^\infty_{[0,T)}L^2_{|r|\le 5/8}} \lesssim \epsilon^5.$$

Also, by Proposition 5.13,

$$\|D_r^{1/2} F_2\|_{L^2_{[0,T)}L^2_r} \lesssim \|\langle D_r\rangle^{3/2}\psi_{9/16} v\|_{L^2_{[0,T)}L^2_r} \lesssim \epsilon^5.$$

Finally,

$$\|\partial_r F_3\|_{L^1_{[0,T)}L^2_r} \lesssim T\|\langle\partial_r\rangle v\|_{L^\infty_{[0,T)}L^2_{|r|\le 5/8}} \lesssim \epsilon^5$$

by Proposition 5.12. Collecting the estimates above, we obtain the claimed bound. □

This completes the proof of Proposition 5.1 (via Lemma 5.5).

Acknowledgments

Holmer is partially supported by a Sloan fellowship and NSF grant DMS-0901582. Roudenko is partially supported by NSF grant DMS-0808081. Holmer thanks Mike Christ and Daniel Tataru for patient mentorship, in work related to the paper [Koch and Tataru 2007], on the use of the bilinear Strichartz estimates.

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Received 23 Jul 2010. Revised 10 Jan 2011. Accepted 21 Feb 2011.

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Analysis & PDE, at Mathematical Sciences Publishers, Department of Mathematics, University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFLOW™ from Mathematical Sciences Publishers.

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Typeset in LaTeX

ANALYSIS & PDE

Volume 5 No. 3 2012