ANALYSIS & PDE

Volume 5 No. 4 2012

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Analysis & PDE

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Analysis & PDE, at Mathematical Sciences Publishers, Department of Mathematics, University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFLOW™ from Mathematical Sciences Publishers.

PUBLISHED BY

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A NON-PROFIT CORPORATION

Typeset in LaTeX

ANALYSIS AND PDE
Vol. 5, No. 4, 2012
dx.doi.org/10.2140/apde.2012.5.705

ON THE GLOBAL WELL-POSEDNESS OF ENERGY-CRITICAL SCHRÖDINGER EQUATIONS IN CURVED SPACES

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In this paper we present a method to study global regularity properties of solutions of large-data critical Schrödinger equations on certain noncompact Riemannian manifolds. We rely on concentration compactness arguments and a global Morawetz inequality adapted to the geometry of the manifold (in other words we adapt the method of Kenig and Merle to the variable coefficient case), and a good understanding of the corresponding Euclidean problem (a theorem of Colliander, Keel, Staffilani, Takaoka and Tao).

As an application we prove global well-posedness and scattering in H^1 for the energy-critical defocusing initial-value problem

$$(i\partial_t + \Delta_{\mathbf{g}})u = u|u|^4, \quad u(0) = \phi,$$

on hyperbolic space $\mathbb{H}^3$.

1. Introduction

The goal of this paper is to present a somewhat general method to prove global well-posedness of critical[1] nonlinear Schrödinger initial-value problems of the form

$$(i\partial_t + \Delta_{\mathbf{g}})u = \mathcal{N}(u), \quad u(0) = \phi, \tag{1-1}$$

on certain noncompact Riemannian manifolds $(M, \mathbf{g})$. Here $\Delta_{\mathbf{g}} = g^{ij}(\partial_{ij} - \Gamma_{ij}^k \partial_k)$ is the (negative) Laplace–Beltrami operator of $(M, \mathbf{g})$. In Euclidean spaces, the subcritical theory of such nonlinear Schrödinger equations is well established; see for example the books [Cazenave 2003; Tao 2006] for many references. Many of the subcritical methods extend also to the study of critical equations with small

Ionescu was supported in part by a Packard Fellowship, and Staffilani by NSF Grant DMS 0602678. Pausader and Staffilani thank the MIT/France program during which this work was initiated.
MSC2000: 35Q55.
Keywords: global well-posedness, energy-critical defocusing NLS, nonlinear Schrödinger equation, induction on energy.

[1] Here critical refers to the fact that when $(M, \mathbf{g}) = (\mathbb{R}^3, \delta_{ij})$, the equation and the control (here the energy) are invariant under the rescaling $u(x,t) \to \lambda^{1/2} u(\lambda x, \lambda^2 t)$.

data. The case of large-data critical Schrödinger equations is more delicate, and was first considered in [Bourgain 1999] and [Grillakis 2000] for defocusing Schrödinger equations with pure power nonlinearities and spherically symmetric data. The spherical symmetry assumption was removed in dimension $d = 3$ in [Colliander et al. 2008]; global well-posedness was then extended to higher dimensions $d \geq 4$ in [Ryckman and Visan 2007; Visan 2007].

A key development in the theory of large-data critical dispersive problems was the article [Kenig and Merle 2006], on spherically symmetric solutions of the energy-critical focusing NLS in $\mathbb{R}^3$. The methods developed in this paper found applications in many other large-data critical dispersive problems, leading to complete solutions or partial results. We adapt this point of view in our variable coefficient setting as well.

To keep things as simple as possible on a technical level, in this paper we consider only the energy-critical defocusing Schrödinger equation

$$(i\partial_t + \Delta_{\boldsymbol{g}})u = u|u|^4 \tag{1-2}$$

in hyperbolic space $\mathbb{H}^3$. Suitable solutions of (1-2) on the time interval (T_1, T_2) satisfy mass and energy conservation, in the sense that the functions

$$E^0(u)(t) := \int_{\mathbb{H}^3} |u(t)|^2 \, d\mu, \quad E^1(u)(t) := \frac{1}{2}\int_{\mathbb{H}^3} |\nabla_{\boldsymbol{g}} u(t)|^2 \, d\mu + \frac{1}{6}\int_{\mathbb{H}^3} |u(t)|^6 \, d\mu \tag{1-3}$$

are constant on the interval (T_1, T_2). Our main theorem concerns global well-posedness and scattering in $H^1(\mathbb{H}^3)$ for the initial-value problem associated to (1-2).

Theorem 1.1. (a) (Global well-posedness.) *If $\phi \in H^1(\mathbb{H}^3)^2$ then there exists a unique global solution $u \in C(\mathbb{R} : H^1(\mathbb{H}^3))$ of the initial-value problem*

$$(i\partial_t + \Delta_{\boldsymbol{g}})u = u|u|^4, \quad u(0) = \phi. \tag{1-4}$$

In addition, the mapping $\phi \to u$ is a continuous mapping from $H^1(\mathbb{H}^3)$ to $C(\mathbb{R} : H^1(\mathbb{H}^3))$, and the quantities $E^0(u)$ and $E^1(u)$ defined in (1-3) *are conserved.*

(b) (Scattering.) *We have the bound*

$$\|u\|_{L^{10}(\mathbb{H}^3\times\mathbb{R})} \leq C(\|\phi\|_{H^1(\mathbb{H}^3)}). \tag{1-5}$$

As a consequence, there exist unique $u_\pm \in H^1(\mathbb{H}^3)$ such that

$$\|u(t) - e^{it\Delta_{\boldsymbol{g}}} u_\pm\|_{H^1(\mathbb{H}^3)} = 0 \text{ as } t \to \pm\infty. \tag{1-6}$$

It was observed by Banica [2007] that hyperbolic geometry cooperates well with the dispersive nature of Schrödinger equations, at least in the case of subcritical problems. In fact the long time dispersion of solutions is stronger in hyperbolic geometry than in Euclidean geometry. Intuitively, this is due to the fact that the volume of a ball of radius $R + 1$ in hyperbolic spaces is about twice as large as the volume

[2] Unlike in Euclidean spaces, in hyperbolic spaces $\mathbb{H}^d$ one has the uniform inequality $\int_{\mathbb{H}^d} |f|^2 \, d\mu \lesssim \int_{\mathbb{H}^d} |\nabla f|^2 \, d\mu$ for any $f \in C_0^\infty(\mathbb{H}^d)$. In other words $\dot{H}^1(\mathbb{H}^d) \hookrightarrow L^2(\mathbb{H}^d)$.

of a ball of radius R, if $R \geq 1$; therefore, as outgoing waves advance one unit in the geodesic direction they have about twice as much volume to disperse into. This heuristic can be made precise; see [Anker and Pierfelice 2009; Banica 2007; Banica et al. 2008; 2009; Banica and Duyckaerts 2007; Bouclet 2011; Christianson and Marzuola 2010; Ionescu and Staffilani 2009; Pierfelice 2008] for theorems concerning subcritical nonlinear Schrödinger equations in hyperbolic spaces (or other spaces that interpolate between Euclidean and hyperbolic spaces). The theorems proved in these papers are stronger than the corresponding theorems in Euclidean spaces, in the sense that one obtains better scattering and dispersive properties of the nonlinear solutions.

We remark, however, that the global geometry of the manifold cannot bring any improvements in the case of critical problems. To see this, consider only the case of data of the form

$$\phi_N(x) = N^{1/2}\psi(N\Psi^{-1}(x)), \tag{1-7}$$

where $\psi \in C_0^\infty(\mathbb{R}^3)$ and $\Psi : \mathbb{R}^3 \to \mathbb{H}^3$ is a suitable local system of coordinates. Assuming that ψ is fixed and letting $N \to \infty$, the functions $\phi_N \in C_0^\infty(\mathbb{H}^3)$ have uniformly bounded H^1 norm. For any $T \geq 0$ and ψ fixed, one can prove that the nonlinear solution of (1-4) corresponding to data ϕ_N is well approximated by

$$N^{1/2}v(N\Psi^{-1}(x), N^2t)$$

on the time interval $(-TN^{-2}, TN^{-2})$, for N sufficiently large (depending on T and ψ), where v is the solution on the time interval $(-T, T)$ of the Euclidean nonlinear Schrödinger equation

$$(i\partial_t + \Delta)v = v|v|^4, \quad v(0) = \psi. \tag{1-8}$$

See Section 4 for precise statements. In other words, the solution of the hyperbolic NLS (1-4) with data ϕ_N can be regular on the time interval $(-TN^{-2}, TN^{-2})$ only if the solution of the Euclidean NLS (1-8) is regular on the interval $(-T, T)$. This shows that understanding the Euclidean scale invariant problem is a prerequisite for understanding the problem on any other manifold. Fortunately, we are able to use the main theorem of Colliander et al. [2008] as a black box (see the proof of Lemma 4.2).

The previous heuristic shows that understanding the *scaling limit* problem (1-8) is part of understanding the full nonlinear evolution (1-4), at least if one is looking for uniform control on all solutions below a certain energy level. This approach was already used in the study of elliptic equations, first in the subcritical case (where the scaling limits are easier) by Gidas and Spruck [1981] and also in the H^1 critical setting, see for example Druet, Hebey and Robert [Druet et al. 2004], Hebey and Vaugon [1995], Schoen [1989] and (many) references therein. Note however that in the dispersive case, we have to contend with the fact that we are looking at perturbations of a linear operator $i\partial_t + \Delta_{\boldsymbol{g}}$ whose kernel is infinite dimensional.

Other critical dispersive models, such as large-data critical wave equations or the Klein–Gordon equation have also been studied extensively, both in the case of the Minkowski space and in other Lorentz manifolds. See, for example, [Bahouri and Gérard 1999; Bahouri and Shatah 1998; Burq et al. 2008; Burq and Planchon 2009; Grillakis 1990; 1992; Ibrahim and Majdoub 2003; Ibrahim et al. 2009; 2011; Kapitanski 1994; Kenig and Merle 2008; Killip et al. 2012; Laurent 2011; Shatah and Struwe 1993; 1994;

Struwe 1988; Tao 2006] for further discussion and references. In the case of the wave equation, passing to the variable coefficient setting is somewhat easier due the finite speed of propagation of solutions.

Nonlinear Schrödinger equations such as (1-1) have also been considered in the setting of compact Riemannian manifolds $(M, \boldsymbol{g})$; see [Bourgain 1993a; 1993b; Burq et al. 2004; 2005; Colliander et al. 2010; Gérard and Pierfelice 2010]. In this case the conclusions are generally weaker than in Euclidean spaces: there is no scattering to linear solutions, or some other type of asymptotic control of the nonlinear evolution as $t \to \infty$. We note however the recent result of Herr, Tataru and Tzvetkov [Herr et al. 2011] on the global well-posedness of the energy critical NLS with small initial data in $H^1(\mathbb{T}^3)$.

To simplify the exposition, we use some of the structure of hyperbolic spaces; in particular we exploit the existence of a large group of isometries that acts transitively on $\mathbb{H}^d$. However the main ingredients in the proof are more basic, and can probably be extended to more general settings[3]. These main ingredients are:

(1) a dispersive estimate such as (2-24), which gives a good large-data local well-posedness/stability theory (Propositions 3.1 and 3.2);

(2) a good Morawetz-type inequality (Proposition 3.3) to exploit the global defocusing character of the equation;

(3) a good understanding of the Euclidean problem, provided in this case by a result of Colliander, Keel, Staffilani, Takaoka and Tao [Colliander et al. 2008, Theorem 4.1];

(4) some uniform control of the geometry of the manifold at infinity.

The rest of the paper is organized as follows: in Section 2 we set up the notations, and record the main dispersive estimates on the linear Schrödinger flow on hyperbolic spaces. We prove also several lemmas that are used later.

In Section 3 we collect all the necessary ingredients described above, and outline the proof of the main theorem. The only component of the proof that is not known is Proposition 3.4 on the existence of a suitable minimal energy blow-up solution.

In Section 4 we consider nonlinear solutions of (1-4) corresponding to data that contract at a point, as in (1-7). Using the main theorem in [Colliander et al. 2008] we prove that such nonlinear solutions extend globally in time and satisfy suitable dispersive bounds.

In Section 5 we prove our main profile decomposition of H^1-bounded sequences of functions in hyperbolic spaces. This is the analogue of Keraani's theorem [2001] in Euclidean spaces. In hyperbolic spaces we have to distinguish between two types of profiles: Euclidean profiles which may contract at a point, after time and space translations, and hyperbolic profiles which live essentially at frequency[4] $N = 1$. Hyperbolic geometry guarantees that profiles of low frequency $N \ll 1$ can be treated as perturbations.

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[3]Two of the authors have applied a similar strategy to prove global regularity of the defocusing energy-critical NLS in other settings, such as $\mathbb{T}^3$ [Ionescu and Pausader 2012a] and $\mathbb{R} \times \mathbb{T}^3$ [Ionescu and Pausader 2012b], where other issues arise due to the presence of trapped geodesics or the lower power in the nonlinearity.

[4]Here we define the notion of frequency through the heat kernel, see (2-28).

Finally, in Section 6 we use our profile decomposition and orthogonality arguments to complete the proof of Proposition 3.4.

2. Preliminaries

In this subsection we review some aspects of the harmonic analysis and the geometry of hyperbolic spaces, and summarize our notations. For simplicity, we will use the conventions in [Bray 1994], but one should keep in mind that hyperbolic spaces are the simplest examples of symmetric spaces of the noncompact type, and most of the analysis on hyperbolic spaces can be generalized to this setting (see, for example, [Helgason 1994]).

Hyperbolic spaces: Riemannian structure and isometries. For integers $d \geq 2$ we consider the Minkowski space $\mathbb{R}^{d+1}$ with the standard Minkowski metric $-(dx^0)^2 + (dx^1)^2 + \ldots + (dx^d)^2$ and define the bilinear form on $\mathbb{R}^{d+1} \times \mathbb{R}^{d+1}$,

$$[x, y] = x^0 y^0 - x^1 y^1 - \cdots - x^d y^d.$$

Hyperbolic space $\mathbb{H}^d$ is defined as

$$\mathbb{H}^d = \{x \in \mathbb{R}^{d+1} : [x, x] = 1 \text{ and } x^0 > 0\}.$$

Let $\mathbf{0} = (1, 0, \ldots, 0)$ denote the origin of $\mathbb{H}^d$. The Minkowski metric on $\mathbb{R}^{d+1}$ induces a Riemannian metric $\boldsymbol{g}$ on $\mathbb{H}^d$, with covariant derivative $\boldsymbol{D}$ and induced measure $d\mu$.

We define $\mathbb{G} := \mathrm{SO}(d, 1) = \mathrm{SO}_e(d, 1)$ as the connected Lie group of $(d + 1) \times (d + 1)$ matrices that leave the form $[\cdot, \cdot]$ invariant. Clearly, $X \in \mathrm{SO}(d, 1)$ if and only if

$${}^{\mathrm{tr}}X \cdot I_{d,1} \cdot X = I_{d,1}, \quad \det X = 1, \quad X_{00} > 0,$$

where $I_{d,1}$ is the diagonal matrix $\mathrm{diag}[-1, 1, \ldots, 1]$ (since $[x, y] = -{}^t x \cdot I_{d,1} \cdot y$). Let $\mathbb{K} = \mathrm{SO}(d)$ denote the subgroup of $\mathrm{SO}(d, 1)$ that fixes the origin $\mathbf{0}$. Clearly, $\mathrm{SO}(d)$ is the compact rotation group acting on the variables $(x^1, \ldots, x^d)$. We define also the commutative subgroup $\mathbb{A}$ of $\mathbb{G}$,

$$\mathbb{A} := \left\{ a_s = \begin{bmatrix} \mathrm{ch}\, s & \mathrm{sh}\, s & 0 \\ \mathrm{sh}\, s & \mathrm{ch}\, s & 0 \\ 0 & 0 & I_{d-1} \end{bmatrix} : s \in \mathbb{R} \right\}, \tag{2-1}$$

and recall the Cartan decomposition

$$\mathbb{G} = \mathbb{K}\mathbb{A}_+\mathbb{K}, \quad \mathbb{A}_+ := \{a_s : s \in [0, \infty)\}. \tag{2-2}$$

The semisimple Lie group $\mathbb{G}$ acts transitively on $\mathbb{H}^d$ and hyperbolic space $\mathbb{H}^d$ can be identified with the homogeneous space $\mathbb{G}/\mathbb{K} = \mathrm{SO}(d, 1)/\,\mathrm{SO}(d)$. Moreover, for any $h \in \mathrm{SO}(d, 1)$ the mapping $L_h : \mathbb{H}^d \to \mathbb{H}^d$, $L_h(x) = h \cdot x$, defines an isometry of $\mathbb{H}^d$. Therefore, for any $h \in \mathbb{G}$, we define the isometries

$$\pi_h : L^2(\mathbb{H}^d) \to L^2(\mathbb{H}^d), \quad \pi_h(f)(x) = f(h^{-1} \cdot x). \tag{2-3}$$

We fix normalized coordinate charts which allow us to pass in a suitable way between functions defined on hyperbolic spaces and functions defined on Euclidean spaces. More precisely, for any $h \in \mathrm{SO}(d,1)$ we define the diffeomorphism

$$\Psi_h : \mathbb{R}^d \to \mathbb{H}^d, \quad \Psi_h(v^1,\dots,v^d) = h \cdot (\sqrt{1+|v|^2}, v^1,\dots,v^d). \tag{2-4}$$

Using these diffeomorphisms we define, for any $h \in \mathbb{G}$,

$$\widetilde{\pi}_h : C(\mathbb{R}^d) \to C(\mathbb{H}^d), \quad \widetilde{\pi}_h(f)(x) = f(\Psi_h^{-1}(x)). \tag{2-5}$$

We will use the diffeomorphism Ψ_I as a global coordinate chart on $\mathbb{H}^d$, where I is the identity element of $\mathbb{G}$. We record the integration formula

$$\int_{\mathbb{H}^d} f(x)\,d\mu(x) = \int_{\mathbb{R}^d} f(\Psi_I(v))(1+|v|^2)^{-1/2}\,dv \tag{2-6}$$

for any $f \in C_0(\mathbb{H}^d)$.

The Fourier transform on hyperbolic spaces. The Fourier transform (as defined by Helgason [1965] in the more general setting of symmetric spaces) takes suitable functions defined on $\mathbb{H}^d$ to functions defined on $\mathbb{R} \times \mathbb{S}^{d-1}$. For $\omega \in \mathbb{S}^{d-1}$ and $\lambda \in \mathbb{C}$, let $b(\omega) = (1,\omega) \in \mathbb{R}^{d+1}$ and

$$h_{\lambda,\omega} : \mathbb{H}^d \to \mathbb{C}, \quad h_{\lambda,\omega}(x) = [x, b(\omega)]^{i\lambda-\rho},$$

where

$$\rho = (d-1)/2.$$

It is known that

$$\Delta_{\boldsymbol{g}} h_{\lambda,\omega} = -(\lambda^2+\rho^2)h_{\lambda,\omega}, \tag{2-7}$$

where $\Delta_{\boldsymbol{g}}$ is the Laplace–Beltrami operator on $\mathbb{H}^d$. The Fourier transform of $f \in C_0(\mathbb{H}^d)$ is defined by the formula

$$\tilde{f}(\lambda,\omega) = \int_{\mathbb{H}^d} f(x)h_{\lambda,\omega}(x)\,d\mu = \int_{\mathbb{H}^d} f(x)[x,b(\omega)]^{i\lambda-\rho}\,d\mu. \tag{2-8}$$

This transformation admits a Fourier inversion formula: if $f \in C_0^\infty(\mathbb{H}^d)$ then

$$f(x) = \int_0^\infty \int_{\mathbb{S}^{d-1}} \tilde{f}(\lambda,\omega)[x,b(\omega)]^{-i\lambda-\rho}|\boldsymbol{c}(\lambda)|^{-2}\,d\lambda\,d\omega, \tag{2-9}$$

where, for a suitable constant C,

$$\boldsymbol{c}(\lambda) = C\frac{\Gamma(i\lambda)}{\Gamma(\rho+i\lambda)}$$

is the Harish-Chandra $\boldsymbol{c}$-function corresponding to $\mathbb{H}^d$, and the invariant measure of $\mathbb{S}^{d-1}$ is normalized to 1. It follows from (2-7) that

$$\widetilde{\Delta_{\boldsymbol{g}} f}(\lambda,\omega) = -(\lambda^2+\rho^2)\tilde{f}(\lambda,\omega). \tag{2-10}$$

We record also the nontrivial identity

$$\int_{\mathbb{S}^{d-1}} \tilde{f}(\lambda,\omega)[x,b(\omega)]^{-i\lambda-\rho}\,d\omega = \int_{\mathbb{S}^{d-1}} \tilde{f}(-\lambda,\omega)[x,b(\omega)]^{i\lambda-\rho}\,d\omega$$

for any $f \in C_0^\infty(\mathbb{H}^d)$, $\lambda \in \mathbb{C}$, and $x \in \mathbb{H}^d$.

According to the Plancherel theorem, the Fourier transform $f \to \tilde{f}$ extends to an isometry of $L^2(\mathbb{H}^d)$ onto $L^2(\mathbb{R}_+ \times \mathbb{S}^{d-1}, |\boldsymbol{c}(\lambda)|^{-2} d\lambda\, d\omega)$; moreover

$$\int_{\mathbb{H}^d} f_1(x)\overline{f_2(x)}\,d\mu = \frac{1}{2}\int_{\mathbb{R}\times\mathbb{S}^{d-1}} \tilde{f_1}(\lambda,\omega)\overline{\tilde{f_2}(\lambda,\omega)}|\boldsymbol{c}(\lambda)|^{-2}\,d\lambda\,d\omega, \tag{2-11}$$

for any $f_1, f_2 \in L^2(\mathbb{H}^d)$. As a consequence, any bounded multiplier $m : \mathbb{R}_+ \to \mathbb{C}$ defines a bounded operator T_m on $L^2(\mathbb{H}^d)$ by the formula

$$\widetilde{T_m(f)}(\lambda,\omega) = m(\lambda)\cdot \tilde{f}(\lambda,\omega). \tag{2-12}$$

The question of L^p boundedness of operators defined by multipliers as in (2-12) is more delicate if $p \neq 2$. A necessary condition for boundedness on $L^p(\mathbb{H}^d)$ of the operator T_m is that the multiplier m extends to an even analytic function in the interior of the region $\mathcal{T}_p = \{\lambda \in \mathbb{C} : |\Im\lambda| < |2/p-1|\rho\}$ [Clerc and Stein 1974]. Conversely, if $p \in (1,\infty)$ and $m : \mathcal{T}_p \to \mathbb{C}$ is an even analytic function which satisfies the symbol-type bounds

$$|\partial^\alpha m(\lambda)| \le C(1+|\lambda|)^{-\alpha} \quad \text{for any } \alpha \in [0, d+2]\cap\mathbb{Z} \text{ and } \lambda \in \mathcal{T}_p, \tag{2-13}$$

then T_m extends to a bounded operator on $L^p(\mathbb{H}^d)$ [Stanton and Tomas 1978].

As in Euclidean spaces, there is a connection between convolution operators in hyperbolic spaces and multiplication operators in the Fourier space. To state this connection precisely, we normalize first the Haar measures on $\mathbb{K}$ and $\mathbb{G}$ such that $\int_{\mathbb{K}} 1\,dk = 1$ and

$$\int_{\mathbb{G}} f(g\cdot\mathbf{0})\,dg = \int_{\mathbb{H}^d} f(x)\,d\mu$$

for any $f \in C_0(\mathbb{H}^d)$. Given two functions $f_1, f_2 \in C_0(\mathbb{G})$ we define the convolution

$$(f_1 * f_2)(h) = \int_{\mathbb{G}} f_1(g) f_2(g^{-1}h)\,dg. \tag{2-14}$$

A function $K : \mathbb{G} \to \mathbb{C}$ is called $\mathbb{K}$-biinvariant if

$$K(k_1 g k_2) = K(g) \quad \text{for any } k_1, k_2 \in \mathbb{K}. \tag{2-15}$$

Similarly, a function $K : \mathbb{H}^d \to \mathbb{C}$ is called $\mathbb{K}$-invariant (or radial) if

$$K(k\cdot x) = K(x) \quad \text{for any } k \in \mathbb{K} \text{ and } x \in \mathbb{H}^d. \tag{2-16}$$

If $f, K \in C_0(\mathbb{H}^d)$ and K is $\mathbb{K}$-invariant then we define (compare to (2-14))

$$(f * K)(x) = \int_{\mathbb{G}} f(g\cdot\mathbf{0})K(g^{-1}\cdot x)\,dg. \tag{2-17}$$

If K is $\mathbb{K}$-invariant then the Fourier transform formula (2-8) becomes

$$\widetilde{K}(\lambda,\omega)=\widetilde{K}(\lambda)=\int_{\mathbb{H}^d}K(x)\Phi_{-\lambda}(x)\,d\mu, \tag{2-18}$$

where

$$\Phi_\lambda(x)=\int_{\mathbb{S}^{d-1}}[x,b(\omega)]^{-i\lambda-\rho}\,d\omega \tag{2-19}$$

is the elementary spherical function. The Fourier inversion formula (2-9) becomes

$$K(x)=\int_0^\infty \widetilde{K}(\lambda)\Phi_\lambda(x)|\boldsymbol{c}(\lambda)|^{-2}\,d\lambda, \tag{2-20}$$

for any $\mathbb{K}$-invariant function $K\in C_0^\infty(\mathbb{H}^d)$. With the convolution defined as in (2-17), we have the important identity

$$\widetilde{(f*K)}(\lambda,\omega)=\tilde{f}(\lambda,\omega)\cdot\widetilde{K}(\lambda) \tag{2-21}$$

for any $f,K\in C_0(\mathbb{H}^d)$, provided that K is $\mathbb{K}$-invariant[5].

We define now the inhomogeneous Sobolev spaces on $\mathbb{H}^d$. There are two possible definitions: using the Riemannian structure $\boldsymbol{g}$ or using the Fourier transform. These two definitions agree. In view of (2-10), for $s\in\mathbb{C}$ we define the operator $(-\Delta)^{s/2}$ as given by the Fourier multiplier $\lambda\to(\lambda^2+\rho^2)^{s/2}$. For $p\in(1,\infty)$ and $s\in\mathbb{R}$ we define the Sobolev space $W^{p,s}(\mathbb{H}^d)$ as the closure of $C_0^\infty(\mathbb{H}^d)$ under the norm

$$\|f\|_{W^{p,s}(\mathbb{H}^d)}=\|(-\Delta)^{s/2}f\|_{L^p(\mathbb{H}^d)}.$$

For $s\in\mathbb{R}$ let $H^s=W^{2,s}$. This definition is equivalent to the usual definition of the Sobolev spaces on Riemannian manifolds (this is a consequence of the fact that the operator $(-\Delta_{\boldsymbol{g}})^{s/2}$ is bounded on $L^p(\mathbb{H}^d)$ for any $s\in\mathbb{C}$, $\Re s\le 0$, since its symbol satisfies the differential inequalities (2-13)). In particular, for $s=1$ and $p\in(1,\infty)$,

$$\|f\|_{W^{p,1}(\mathbb{H}^d)}=\|(-\Delta)^{1/2}f\|_{L^p(\mathbb{H}^d)}\approx_p\left(\int_{\mathbb{H}^d}|\nabla_{\boldsymbol{g}}f|^p\,d\mu\right)^{1/p}, \tag{2-22}$$

where

$$|\nabla_{\boldsymbol{g}}f|:=|\boldsymbol{D}^\alpha f\,\boldsymbol{D}_\alpha\bar{f}|^{1/2}.$$

We record also the Sobolev embedding theorem

$$W^{p,s}\hookrightarrow L^q\quad\text{ if }\ 1<p\le q<\infty\ \text{ and }\ s=d/p-d/q. \tag{2-23}$$

Dispersive estimates. Most of our perturbative analysis in the paper is based on the Strichartz estimates for the linear Schrödinger flow. For any $\phi\in H^s(\mathbb{H}^d)$, $s\in\mathbb{R}$, let $e^{it\Delta_{\boldsymbol{g}}}\phi\in C(\mathbb{R}:H^s(\mathbb{H}^d))$ denote the solution of the free Schrödinger evolution with data ϕ, i.e.,

$$\widetilde{e^{it\Delta_{\boldsymbol{g}}}\phi}(\lambda,\omega)=\widetilde{\phi}(\lambda,\omega)\cdot e^{-it(\lambda^2+\rho^2)}.$$

[5]Unlike in Euclidean Fourier analysis, there is no simple identity of this type without the assumption that K is $\mathbb{K}$-invariant.

The main inequality we need is the dispersive estimate[6] (see [Anker and Pierfelice 2009; Banica 2007; Banica et al. 2008; Ionescu and Staffilani 2009; Pierfelice 2008])

$$\|e^{it\Delta_g}\|_{L^p\to L^{p'}} \lesssim |t|^{-d(1/p-1/2)}, \quad p\in[2d/(d+2),2],\ p'=p/(p-1), \tag{2-24}$$

for any $t\in\mathbb{R}\setminus\{0\}$. The Strichartz estimates below then follow from a general theorem from [Keel and Tao 1998].

Proposition 2.1 (Strichartz estimates). *Assume that $d\geq 3$ and $I=(a,b)\subseteq\mathbb{R}$ is a bounded open interval.*

(i) *If $\phi\in L^2(\mathbb{H}^d)$ then*

$$\|e^{it\Delta_g}\phi\|_{(L^\infty_t L^2_x\cap L^2_t L^{2d/(d-2)}_x)(\mathbb{H}^d\times I)} \lesssim \|\phi\|_{L^2}. \tag{2-25}$$

(ii) *If $F\in(L^1_tL^2_x+L^2_tL^{2d/(d+2)}_x)(\mathbb{H}^d\times I)$ then*

$$\left\|\int_a^t e^{i(t-s)\Delta_g}F(s)\,ds\right\|_{(L^\infty_t L^2_x\cap L^2_t L^{2d/(d-2)}_x)(\mathbb{H}^d\times I)} \lesssim \|F\|_{(L^1_tL^2_x+L^2_tL^{2d/(d+2)}_x)(\mathbb{H}^d\times I)}. \tag{2-26}$$

To exploit these estimates in dimension $d=3$, for any interval $I\subseteq\mathbb{R}$ and $f\in C(I:H^{-1}(\mathbb{H}^3))$ we define

$$\begin{aligned}
\|f\|_{Z(I)} &:= \|f\|_{L^{10}_{t,x}(\mathbb{H}^3\times I)},\\
\|f\|_{S^k(I)} &:= \|(-\Delta)^{k/2}f\|_{(L^\infty_tL^2_x\cap L^2_tL^6_x)(\mathbb{H}^3\times I)}, \quad k\in[0,\infty),\\
\|f\|_{N^k(I)} &:= \|(-\Delta)^{k/2}f\|_{(L^1_tL^2_x+L^2_tL^{6/5}_x)(\mathbb{H}^3\times I)}, \quad k\in[0,\infty).
\end{aligned} \tag{2-27}$$

We use the S^1 norms to estimate solutions of linear and nonlinear Schrödinger equations. Nonlinearities are estimated using the N^1 norms. The L^{10} norm is the "scattering" norm, which controls the existence of strong solutions of the nonlinear Schrödinger equation, see Proposition 3.1 and Proposition 3.2 below.

Some lemmas. In this subsection we collect and prove several lemmas that will be used later in the paper. For $N>0$ we define the operator $P_N:L^2(\mathbb{H}^3)\to L^2(\mathbb{H}^3)$,

$$\begin{aligned}
&P_N := N^{-2}\Delta_g e^{N^{-2}\Delta_g},\\
&\widetilde{P_Nf}(\lambda,\omega) = -N^{-2}(\lambda^2+1)e^{-N^{-2}(\lambda^2+1)}\tilde{f}(\lambda,\omega).
\end{aligned} \tag{2-28}$$

One should think of P_N as a substitute for the usual Littlewood–Paley projection operator in Euclidean spaces that restricts to frequencies of size $\approx N$; this substitution is necessary in order to have a suitable L^p theory for these operators, since only real-analytic multipliers can define bounded operators on $L^p(\mathbb{H}^3)$ [Clerc and Stein 1974]. In view of the Fourier inversion formula we have

$$P_Nf(x) = \int_{\mathbb{H}^3} f(y)P_N(d(x,y))\,d\mu(y),$$

[6]In fact this estimate can be improved if $|t|\geq 1$, see [Ionescu and Staffilani 2009, Lemma 3.3]. This leads to better control of the longtime behavior of solutions of subcritical Schrödinger equations in hyperbolic spaces, compared to the behavior of solutions of the same equations in Euclidean spaces (see [Banica 2007; Banica et al. 2008; Ionescu and Staffilani 2009; Anker and Pierfelice 2009]).

where

$$|P_N(r)| \lesssim N^3(1+Nr)^{-5}e^{-4r}. \tag{2-29}$$

The estimates in the following lemma will be used in Section 5.

Lemma 2.2. (i) *Given $\epsilon \in (0,1]$ there is $R_\epsilon \geq 1$ such that for any $x \in \mathbb{H}^3$, $N \geq 1$, and $f \in H^1(\mathbb{H}^3)$,*

$$|P_N f(x)| \lesssim N^{1/2}(\|f \cdot \mathbf{1}_{B(x,R_\epsilon N^{-1})}\|_{L^6(\mathbb{H}^3)} + \epsilon \|f\|_{L^6(\mathbb{H}^3)})$$

where $B(x,r)$ denotes the ball $B(x,r) = \{y \in \mathbb{H}^3 : d(x,y) < r\}$.

(ii) *For any $f \in H^1(\mathbb{H}^3)$,*

$$\|f\|_{L^6(\mathbb{H}^3)} \lesssim \|\nabla f\|^{1/3}_{L^2(\mathbb{H}^3)} \sup_{\substack{N\geq 1\\ x\in\mathbb{H}^3}} \left[N^{-1/2}|P_N f(x)|\right]^{2/3}.$$

Proof. (i) The inequality follows directly from (2-29):

$$\begin{aligned}|P_N f(x)| &\lesssim \int_{B(x,R_\epsilon N^{-1})} |f(y)|\,|P_N(d(x,y))|\,d\mu(y) + \int_{{}^cB(x,R_\epsilon N^{-1})} |f(y)|\,|P_N(d(x,y))|\,d\mu(y)\\ &\lesssim \|f \cdot \mathbf{1}_{B(x,R_\epsilon N^{-1})}\|_{L^6(\mathbb{H}^3)} \cdot A_{N,0,6/5} + \|f\|_{L^6(\mathbb{H}^3)} \cdot A_{N,R_\epsilon,6/5},\end{aligned}$$

where, for $R \in [0,\infty)$, $N \in [1,\infty)$ and $p \in [1,2]$

$$\begin{aligned}A_{N,R,p} &:= \left(\int_{d(\mathbf{0},y)\geq RN^{-1}} |P_N(d(0,y))|^p\,d\mu(y)\right)^{1/p} \lesssim \left(\int_{RN^{-1}}^\infty |P_N(r)|^p(\operatorname{sh} r)^2\,dr\right)^{1/p}\\ &\lesssim N^3\left(\int_{RN^{-1}}^\infty (1+Nr)^{-5p}r^2\,dr\right)^{1/p} \lesssim N^{3-3/p}(1+R)^{-1}.\end{aligned}$$

The inequality follows if $R_\epsilon = 1/\epsilon$.

(ii) Such improved Sobolev embeddings in various settings have been used before, for example, in [Bahouri and Gérard 1999; Keraani 2001]. For any $f \in H^1(\mathbb{H}^3)$ we have the identity

$$f = c\int_{N=0}^\infty N^{-1}P_N(f)\,dN. \tag{2-30}$$

Thus, with $A := \sup_{N\geq 0}\|N^{-1/2}P_N f\|_{L^\infty(\mathbb{H}^3)}$

$$\begin{aligned}\int_{\mathbb{H}^3} |f|^6\,d\mu &\lesssim \int_{\mathbb{H}^3}\int_{0\leq N_1\leq\ldots\leq N_6} |P_{N_1}f|\cdot\ldots\cdot|P_{N_6}f|\,\frac{dN_1}{N_1}\ldots\frac{dN_6}{N_6}d\mu\\ &\lesssim A^4\int_{\mathbb{H}^3}\int_{0\leq N_5\leq N_6} N_5^2|P_{N_5}f||P_{N_6}f|\,\frac{dN_5}{N_5}\frac{dN_6}{N_6}d\mu\\ &\lesssim A^4\int_{\mathbb{H}^3}\int_0^\infty N|P_N f|^2\,dNd\mu,\end{aligned}$$

where the last inequality follows by Schur's lemma. The claim follows since

$$\int_{\mathbb{H}^3}\int_0^\infty N|P_N f|^2\,dNd\mu = c\|(-\Delta)^{1/2}f\|^2_{L^2(\mathbb{H}^3)},$$

as a consequence of the Plancherel theorem and the definition of the operators P_N, and, for any $N \in [0, 1)$,

$$\|N^{-1/2} P_N f\|_{L^\infty(\mathbb{H}^3)} \lesssim \|P_2 f\|_{L^\infty(\mathbb{H}^3)}. \tag{2-31}$$

□

We will also need the following technical estimate:

Lemma 2.3. *Assume $\psi \in H^1(\mathbb{H}^3)$ satisfies*

$$\|\psi\|_{H^1(\mathbb{H}^3)} \leq 1, \qquad \sup_{\substack{K\geq 1\\ t\in\mathbb{R}\\ x\in\mathbb{H}^3}} K^{-1/2}|P_K e^{it\Delta_g}\psi(x)| \leq \delta, \tag{2-32}$$

for some $\delta \in (0, 1]$. Then, for any $R > 0$ there is $C(R) \geq 1$ such that

$$N^{1/2}\|\nabla_{\boldsymbol{g}} e^{it\Delta_g}\psi\|_{L_t^5 L_x^{15/8}(B(x_0,RN^{-1})\times(t_0-R^2N^{-2},t_0+R^2N^{-2}))} \leq C(R)\delta^{1/20} \tag{2-33}$$

for any $N \geq 1$, any $t_0 \in \mathbb{R}$, and any $x_0 \in \mathbb{H}^3$.

Proof. We may assume $R = 1$, $x_0 = \mathbf{0}$, $t_0 = 0$. It follows from (2-32) that for any $K > 0$ and $t \in \mathbb{R}$

$$\|P_K e^{it\Delta_g}\psi\|_{L^\infty(\mathbb{H}^3)} \lesssim \delta K^{1/2}, \quad \|P_K e^{it\Delta_g}\psi\|_{L^6(\mathbb{H}^3)} \lesssim 1;$$

therefore, by interpolation,

$$\|P_K e^{it\Delta_g}\psi\|_{L^{12}(\mathbb{H}^3)} \lesssim \delta^{1/2}K^{1/4}.$$

Thus, for any $K > 0$ and $t \in \mathbb{R}$,

$$\|\nabla_{\boldsymbol{g}}(P_K e^{it\Delta_g}\psi)\|_{L^{12}(\mathbb{H}^3)} \lesssim \delta^{1/2}K^{1/4}(K+1),$$

which shows that, for any $K > 0$ and $N \geq 1$,

$$N^{1/2}\|\nabla_{\boldsymbol{g}}(P_K e^{it\Delta_g}\psi)\|_{L_t^5 L_x^{15/8}(B(\mathbf{0},N^{-1})\times(-N^{-2},N^{-2}))} \lesssim \delta^{1/2}K^{1/4}(K+1)N^{-5/4}. \tag{2-34}$$

We will prove below that, for any $N \geq 1$ and $K \geq N$,

$$\|\nabla_{\boldsymbol{g}}(P_K e^{it\Delta_g}\psi)\|_{L^2_{x,t}(B(\mathbf{0},N^{-1})\times(-N^{-2},N^{-2}))} \lesssim (NK)^{-1/2}. \tag{2-35}$$

Assuming this and using the energy estimate

$$\|\nabla_{\boldsymbol{g}}(P_K e^{it\Delta_g}\psi)\|_{L_t^\infty L_x^2(\mathbb{H}^3\times\mathbb{R})} \lesssim 1,$$

we have, by interpolation,

$$\|\nabla_{\boldsymbol{g}}(P_K e^{it\Delta_g}\psi)\|_{L_t^5 L_x^2(B(\mathbf{0},N^{-1})\times(-N^{-2},N^{-2}))} \lesssim (NK)^{-1/5}.$$

Therefore, for any $N \geq 1$ and $K \geq N$

$$N^{1/2}\|\nabla_{\boldsymbol{g}}(P_K e^{it\Delta_g}\psi)\|_{L_t^5 L_x^{15/8}(B(\mathbf{0},N^{-1})\times(-N^{-2},N^{-2}))} \lesssim N^{1/5}K^{-1/5}. \tag{2-36}$$

The desired bound (2-33) follows from (2-34), (2-36), and the identity (2-30).

It remains to prove the local smoothing bound (2-35). Many such estimates are known in more general settings; see, for example, [Doi 1996]. We provide below a simple self-contained proof specialized to our case. Assuming $N \geq 1$ fixed, we will construct a real-valued function $a = a_N \in C^\infty(\mathbb{H}^3)$ with the properties

$$\begin{aligned} |\boldsymbol{D}^\alpha a \boldsymbol{D}_\alpha a| &\lesssim 1 && \text{in } \mathbb{H}^3, \\ |\Delta_{\boldsymbol{g}}(\Delta_{\boldsymbol{g}} a)| &\lesssim N^3 && \text{in } \mathbb{H}^3, \\ X^\alpha X_\alpha \cdot N \mathbf{1}_{B(\mathbf{0},N^{-1})} &\lesssim X^\alpha X^\beta \boldsymbol{D}_\alpha \boldsymbol{D}_\beta a && \text{in } \mathbb{H}^3 \text{ for any vector-field } X \in T(\mathbb{H}^3). \end{aligned} \tag{2-37}$$

Assuming such a function is constructed, we define the Morawetz action

$$M_a(t) = 2\Im \int_{\mathbb{H}^3} \boldsymbol{D}^\alpha a(x) \cdot \bar{u}(x) \boldsymbol{D}_\alpha u(x)\, d\mu(x),$$

where $u := P_K e^{it\Delta_{\boldsymbol{g}}} \psi$. A formal computation (see [Ionescu and Staffilani 2009, Proposition 4.1] for a complete justification) shows that

$$\partial_t M_a(t) = 4\Re \int_{\mathbb{H}^3} \boldsymbol{D}^\alpha \boldsymbol{D}^\beta a \cdot \boldsymbol{D}_\alpha u \boldsymbol{D}_\beta \bar{u}\, d\mu - \int_{\mathbb{H}^3} \Delta_{\boldsymbol{g}}(\Delta_{\boldsymbol{g}} a) \cdot |u|^2\, d\mu.$$

Therefore, by integrating on the time interval $[-N^{-2}, N^{-2}]$ and using the first two properties in (2-37),

$$\begin{aligned} 4 \int_{-N^{-2}}^{N^{-2}} \int_{\mathbb{H}^3} &\Re(\boldsymbol{D}^\alpha \boldsymbol{D}^\beta a \cdot \boldsymbol{D}_\alpha u \boldsymbol{D}_\beta \bar{u})\, d\mu\, dt \\ &\leq 2 \sup_{|t| \leq N^{-2}} |M_a(t)| + \int_{-N^{-2}}^{N^{-2}} \int_{\mathbb{H}^3} |\Delta_{\boldsymbol{g}}(\Delta_{\boldsymbol{g}} a)| \cdot |u|^2\, d\mu\, dt \\ &\lesssim \sup_{|t| \leq N^{-2}} \|u(t)\|_{L^2(\mathbb{H}^3)} \|u(t)\|_{H^1(\mathbb{H}^3)} + N^3 \int_{-N^{-2}}^{N^{-2}} \|u(t)\|^2_{L^2(\mathbb{H}^3)}\, dt \lesssim K^{-1} + NK^{-2}. \end{aligned}$$

The desired bound (2-35) follows, in view of the inequality in the last line of (2-37) and the assumption $K \geq N$ since a is real valued.

Finally, it remains to construct a real-valued function $a \in C^\infty(\mathbb{H}^3)$ satisfying (2-37). We are looking for a function of the form

$$a(x) := \tilde{a}(\operatorname{ch} r(x)), \quad r = d(\mathbf{0}, x),\ \tilde{a} \in C^\infty([1,\infty)). \tag{2-38}$$

To prove the inequalities in (2-37) it is convenient to use coordinates induced by the Iwasawa decomposition of the group $\mathbb{G}$: we define the global diffeomorphism

$$\Phi : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{H}^3, \quad \Phi(v^1, v^2, s) = {}^{\mathrm{tr}}(\operatorname{ch} s + e^{-s}|v|^2/2, \operatorname{sh} s + e^{-s}|v|^2/2, e^{-s}v^1, e^{-s}v^2),$$

and fix the global orthonormal frame

$$e_3 := \partial_s, \quad e_1 := e^s \partial_{v^1}, \quad e_2 := e^s \partial_{v^2}.$$

With respect to this frame, the covariant derivatives are

$$\boldsymbol{D}_{e_\alpha} e_\beta = \delta_{\alpha\beta} e_3, \quad \boldsymbol{D}_{e_\alpha} e_3 = -e_\alpha, \quad \boldsymbol{D}_{e_3} e_\alpha = \boldsymbol{D}_{e_3} e_3 = 0 \qquad \text{for } \alpha, \beta = 1, 2.$$

See [Ionescu and Staffilani 2009, Section 2] for these calculations. In this system of coordinates we have

$$\operatorname{ch} r = \operatorname{ch} s + e^{-s}|v|^2/2. \tag{2-39}$$

Therefore, for a as in (2-38), we have

$$\boldsymbol{D}_3 a = (\operatorname{sh} s - e^{-s}|v|^2/2)\cdot \tilde{a}'(\operatorname{ch} r), \quad \boldsymbol{D}_1 a = v^1 \cdot \tilde{a}'(\operatorname{ch} r), \quad \boldsymbol{D}_2 a = v^2 \cdot \tilde{a}'(\operatorname{ch} r).$$

Using the formula

$$\boldsymbol{D}_\alpha \boldsymbol{D}_\beta a = e_\alpha(e_\beta(a)) - (\boldsymbol{D}_{e_\alpha} e_\beta)(a), \quad \alpha,\beta = 1,2,3,$$

we compute the Hessian:

$$\begin{aligned}
&\boldsymbol{D}_1\boldsymbol{D}_1 a = (v^1)^2\tilde{a}''(\operatorname{ch} r) + \operatorname{ch} r\,\tilde{a}'(\operatorname{ch} r), \quad \boldsymbol{D}_2\boldsymbol{D}_2 a = (v^2)^2\tilde{a}''(\operatorname{ch} r) + \operatorname{ch} r\,\tilde{a}'(\operatorname{ch} r),\\
&\boldsymbol{D}_1\boldsymbol{D}_2 a = \boldsymbol{D}_2\boldsymbol{D}_1 a = v^1v^2\tilde{a}''(\operatorname{ch} r), \qquad \boldsymbol{D}_3\boldsymbol{D}_3 f = (\operatorname{sh} s - e^{-s}|v|^2/2)^2\tilde{a}''(\operatorname{ch} r) + \operatorname{ch} r\,\tilde{a}'(\operatorname{ch} r),\\
&\boldsymbol{D}_1\boldsymbol{D}_3 a = \boldsymbol{D}_3\boldsymbol{D}_1 a = v^1(\operatorname{sh} s - e^{-s}|v|^2/2)\tilde{a}''(\operatorname{ch} r),\\
&\boldsymbol{D}_2\boldsymbol{D}_3 a = \boldsymbol{D}_3\boldsymbol{D}_2 a = v^2(\operatorname{sh} s - e^{-s}|v|^2/2)\tilde{a}''(\operatorname{ch} r).
\end{aligned}$$

Therefore, using again (2-39),

$$\boldsymbol{D}^\alpha a \boldsymbol{D}_\alpha a = (\operatorname{sh} r)^2(\tilde{a}'(\operatorname{ch} r))^2, \quad \Delta_{\boldsymbol{g}} a = ((\operatorname{ch} r)^2 - 1)\tilde{a}''(\operatorname{ch} r) + 3(\operatorname{ch} r)\tilde{a}'(\operatorname{ch} r), \tag{2-40}$$

and

$$X^\alpha X^\beta \boldsymbol{D}_\alpha \boldsymbol{D}_\beta a = \operatorname{ch} r\,\tilde{a}'(\operatorname{ch} r)|X|^2 + \tilde{a}''(\operatorname{ch} r)(X^1v^1 + X^2v^2 + X^3(\operatorname{sh} s - e^{-s}|v|^2/2))^2. \tag{2-41}$$

We fix now $\tilde{a}$ such that

$$\tilde{a}'(y) := (y^2 - 1 + N^{-2})^{-1/2}, \quad y \in [1,\infty).$$

The first identity in (2-37) follows easily from (2-40). To prove the second identity in (2-37), we use again (2-40) to derive

$$\Delta_{\boldsymbol{g}} a = b(\operatorname{ch} r), \quad \text{where } b(y) = 3y(y^2 - 1 + N^{-2})^{-1/2} - y(y^2-1)(y^2 - 1 + N^{-2})^{-3/2}.$$

Using (2-40) again, it follows that

$$|\Delta_{\boldsymbol{g}}(\Delta_{\boldsymbol{g}} a)| \lesssim y^2(y^2 - 1 + N^{-2})^{-3/2} \quad \text{where } y = \operatorname{ch} r,$$

which proves the second inequality in (2-37). Finally, using (2-41),

$$\begin{aligned}
X^\alpha X^\beta \boldsymbol{D}_\alpha \boldsymbol{D}_\beta a &\geq \operatorname{ch} r\,\tilde{a}'(\operatorname{ch} r)|X|^2 - ((\operatorname{ch} r)^2 - 1)|\tilde{a}''(\operatorname{ch} r)|\,|X|^2\\
&= N^{-2}\operatorname{ch} r((\operatorname{ch} r)^2 - 1 + N^{-2})^{-3/2}|X|^2,
\end{aligned}$$

which proves the last inequality in (2-37). This completes the proof of the lemma. □

3. Proof of the main theorem

In this section we outline the proof of Theorem 1.1. The main ingredients are a local well-posedness and stability theory for the initial-value problem, which in our case relies only on the Strichartz estimates in Proposition 2.1, a global Morawetz inequality, which exploits the defocusing nature of the problem, and a compactness argument, which depends on the Euclidean analogue of Theorem 1.1 proved in [Colliander et al. 2008].

We start with the local well-posedness theory. Let

$$\mathcal{P} = \{(I, u) : I \subseteq \mathbb{R} \text{ is an open interval and } u \in C(I : H^1(\mathbb{H}^3))\}$$

with the natural partial order

$$(I, u) \leq (I', u') \quad \text{if and only if} \quad I \subseteq I' \text{ and } u'(t) = u(t) \text{ for any } t \in I.$$

Proposition 3.1 (local well-posedness). *Assume $\phi \in H^1(\mathbb{H}^3)$. Then there is a unique maximal solution $(I, u) = (I(\phi), u(\phi)) \in \mathcal{P}$, $0 \in I$, of the initial-value problem*

$$(i\partial_t + \Delta_{\mathbf{g}})u = u|u|^4, \quad u(0) = \phi \tag{3-1}$$

on $\mathbb{H}^3 \times I$. The mass $E^0(u)$ and the energy $E^1(u)$ defined in (1-3) *are constant on I, and $\|u\|_{S^1(J)} < \infty$ for any compact interval $J \subseteq I$. In addition,*

$$\begin{aligned} \|u\|_{Z(I_+)} &= \infty \quad \textit{if } I_+ := I \cap [0, \infty) \textit{ is bounded}, \\ \|u\|_{Z(I_-)} &= \infty \quad \textit{if } I_- := I \cap (-\infty, 0] \textit{ is bounded}. \end{aligned} \tag{3-2}$$

In other words, local-in-time solutions of the equation exist and extend as strong solutions as long as their spacetime $L^{10}_{x,t}$ norm does not blow up. We complement this with a stability result.

Proposition 3.2 (stability). *Assume I is an open interval, $\rho \in [-1, 1]$, and $\tilde{u} \in C(I : H^1(\mathbb{H}^3))$ satisfies the approximate Schrödinger equation*

$$(i\partial_t + \Delta_{\mathbf{g}})\tilde{u} = \rho\tilde{u}|\tilde{u}|^4 + e \quad \text{on } \mathbb{H}^3 \times I.$$

Assume in addition that

$$\|\tilde{u}\|_{L^{10}_{t,x}(\mathbb{H}^3 \times I)} + \sup_{t \in I} \|\tilde{u}(t)\|_{H^1(\mathbb{H}^3)} \leq M, \tag{3-3}$$

for some $M \in [1, \infty)$. Assume $t_0 \in I$ and $u(t_0) \in H^1(\mathbb{H}^3)$ is such that the smallness condition

$$\|u(t_0) - \tilde{u}(t_0)\|_{H^1(\mathbb{H}^3)} + \|e\|_{N^1(I)} \leq \epsilon \tag{3-4}$$

holds for some $0 < \epsilon < \epsilon_1$, where $\epsilon_1 \leq 1$ is a small constant $\epsilon_1 = \epsilon_1(M) > 0$.

Then there exists a solution $u \in C(I : H^1(\mathbb{H}^3))$ of the Schrödinger equation

$$(i\partial_t + \Delta_{\mathbf{g}})u = \rho u|u|^4 \text{ on } \mathbb{H}^3 \times I,$$

and

$$\|u\|_{S^1(\mathbb{H}^3 \times I)} + \|\tilde{u}\|_{S^1(\mathbb{H}^3 \times I)} \leq C(M), \quad \|u - \tilde{u}\|_{S^1(\mathbb{H}^3 \times I)} \leq C(M)\epsilon. \tag{3-5}$$

Both Proposition 3.1 and Proposition 3.2 are standard consequences of the Strichartz estimates and Sobolev embedding theorem (2-23); see, for example, [Colliander et al. 2008, Section 3]. We will use Proposition 3.2 with $\rho = 0$ and with $\rho = 1$ to estimate linear and nonlinear solutions on hyperbolic spaces.

We next state the global Morawetz estimate:

Proposition 3.3 [Ionescu and Staffilani 2009, Proposition 4.1]. *Assume that $I \subseteq \mathbb{R}$ is an open interval, and $u \in C(I : H^1(\mathbb{H}^3))$ is a solution of the equation*

$$(i\partial_t + \Delta_{\mathbf{g}})u = u|u|^4 \text{ on } \mathbb{H}^3 \times I.$$

Then, for any $t_1, t_2 \in I$,

$$\|u\|^6_{L^6(\mathbb{H}^3\times[t_1,t_2])} \lesssim \sup_{t\in[t_1,t_2]} \|u(t)\|_{L^2(\mathbb{H}^3)} \|u(t)\|_{H^1(\mathbb{H}^3)}. \tag{3-6}$$

Next, recall the conserved energy $E^1(u)$ defined in (1-3). For any $E \in [0, \infty)$ let $S(E)$ be defined by

$$S(E) = \sup\{\|u\|_{Z(I)}, E^1(u) \le E\},$$

where the supremum is taken over all solutions $u \in C(I : H^1(\mathbb{H}^3))$ defined on an interval I and of energy less than E. We also define

$$E_{\max} = \sup\{E, S(E) < \infty\}.$$

Using Proposition 3.2 with $\tilde{u} \equiv 0$, $e \equiv 0$, $I = \mathbb{R}$, $M = 1$, $\epsilon \ll 1$, one checks that $E_{\max} > 0$. It follows from Proposition 3.1 that if u is a solution of (1-2) and $E(u) < E_{\max}$, then u can be extended to a globally defined solution which scatters.

If $E_{\max} = +\infty$, then Theorem 1.1 is proved, as a consequence of Propositions 3.1 and 3.2. If we assume that $E_{\max} < +\infty$, then, there exists a sequence of solutions satisfying the hypothesis of the following key proposition, to be proved later.

Proposition 3.4. *Let $u_k \in C((-T_k, T^k) : H^1(\mathbb{H}^3))$, $k = 1, 2, \ldots$, be a sequence of nonlinear solutions of the equation*

$$(i\partial_t + \Delta_{\mathbf{g}})u = u|u|^4,$$

defined on open intervals $(-T_k, T^k)$ such that $E(u_k) \to E_{\max}$. Let $t_k \in (-T_k, T^k)$ be a sequence of times with

$$\lim_{k\to\infty} \|u_k\|_{Z(-T_k,t_k)} = \lim_{k\to\infty} \|u_k\|_{Z(t_k,T^k)} = +\infty. \tag{3-7}$$

Then there exists $w_0 \in H^1(\mathbb{H}^3)$ and a sequence of isometries $h_k \in \mathbb{G}$ such that, up to passing to a subsequence, $u_k(t_k, h_k^{-1} \cdot x) \to w_0(x) \in H^1$ strongly.

Using these propositions we can now prove our main theorem.

Proof of Theorem 1.1. Assume for contradiction that $E_{\max} < +\infty$. Then, we first claim that there exists a solution $u \in C((-T_*, T^*) : H^1)$ of (1-2) such that

$$E(u) = E_{\max} \quad \text{and} \quad \|u\|_{Z(-T_*,0)} = \|u\|_{Z(0,T^*)} = +\infty. \tag{3-8}$$

Indeed, by hypothesis, there exists a sequence of solutions u_k defined on intervals $I_k = (-T_k, T^k)$ satisfying $E(u_k) \le E_{\max}$ and

$$\|u_k\|_{Z(I_k)} \to +\infty.$$

But this is exactly the hypothesis of Proposition 3.4, for suitable points $t_k \in (-T_k, T^k)$. Hence, up to a subsequence, we get that there exists a sequence of isometries $h_k \in \mathbb{G}$ such that $\pi_{h_k}(u_k(t_k)) \to w_0$ strongly in H^1. Now, let $u \in C((-T_*, T^*) : H^1(\mathbb{H}^3))$ be the maximal solution of (3-1) with initial data w_0, in the sense of Proposition 3.1. By the stability theory Proposition 3.2, if $\|u\|_{Z(0,T^*)} < +\infty$, then $T^* = +\infty$ and $\|u_k\|_{Z(t_k,+\infty)} \le C(\|u\|_{Z(0,+\infty)})$ which is impossible. Similarly, we see that $\|u\|_{Z(-T_*,0)} = +\infty$, which completes the proof of (3-8).

We now claim that the solution u obtained in the previous step can be extended to a global solution. Indeed, using Proposition 3.1, it suffices to see that there exists $\delta > 0$ such that, for all times $t \in (-T_*, T^*)$,

$$\|u\|_{Z((t-\delta,t+\delta)\cap(-T_*,T^*))} \le 1.$$

If this were not true, there would exist a sequence $\delta_k \to 0$ and a sequence of times $t_k \in (-T_* + \delta_k, T^* - \delta_k)$ such that

$$\|u\|_{Z(t_k-\delta_k,t_k+\delta_k)} \ge 1. \tag{3-9}$$

Applying Proposition 3.4 with $u_k = u$, we see that, up to a subsequence, $\pi_{h_k}(u_k(t_k)) \to w$ strongly in H^1 for some translations $h_k \in \mathbb{G}$. We consider z the maximal nonlinear solution with initial data w, then by the local theory Proposition 3.1, there exists $\delta > 0$ such that

$$\|z\|_{Z(-\delta,\delta)} \le \tfrac{1}{2}.$$

Proposition 3.2 gives that $\|u\|_{Z(t_k-\delta,t_k+\delta)} \le 1/2 + o_k(1)$, which again contradicts our hypothesis (3-9). In other words, we proved that if $E_{\max} < \infty$ then there is a global solution $u \in C(\mathbb{R} : H^1)$ of (1-2) such that

$$E(u) = E_{\max} \quad \text{and} \quad \|u\|_{Z(-\infty,0)} = \|u\|_{Z(0,\infty)} = +\infty.$$

We claim now that there exists $\delta > 0$ such that for all times,

$$\|u(t)\|_{L^6} \ge \delta. \tag{3-10}$$

Indeed, otherwise, we can find a sequence of times $t_k \in (0, \infty)$ such that $u(t_k) \to 0$ in L^6. Applying again Proposition 3.4 to this sequence, we see that, up to a subsequence, there exist $h_k \in \mathbb{G}$ such that $\pi_{h_k}(u(t_k)) \to w$ in H^1 with $w = 0$. But this contradicts conservation of energy.

But now we have a contradiction with the Morawetz estimate (3-6), which shows that $E_{\max} = +\infty$ as desired. □

Propositions 3.1 and 3.2 are standard consequences of the Strichartz estimates, while Proposition 3.3 was proved in [Ionescu and Staffilani 2009]. Therefore it only remains to prove Proposition 3.4. We collect the main ingredients in the next two sections and complete the proof of Proposition 3.4 in Section 6.

4. Euclidean approximations

In this section we prove precise estimates showing how to compare Euclidean and hyperbolic solutions of both linear and nonlinear Schrödinger equations. Since the global Euclidean geometry and the global hyperbolic geometry are quite different, such a comparison is meaningful only in the case of rescaled data that concentrate at a point.

We fix a spherically symmetric function $\eta \in C_0^\infty(\mathbb{R}^3)$ supported in the ball of radius 2 and equal to 1 in the ball of radius 1. Given $\phi \in \dot{H}^1(\mathbb{R}^3)$ and a real number $N \geq 1$ we define

$$\begin{aligned} Q_N\phi &\in C_0^\infty(\mathbb{R}^3), & (Q_N\phi)(x) &= \eta(x/N^{1/2}) \cdot (e^{\Delta/N}\phi)(x), \\ \phi_N &\in C_0^\infty(\mathbb{R}^3), & \phi_N(x) &= N^{1/2}(Q_N\phi)(Nx), \\ f_N &\in C_0^\infty(\mathbb{H}^3), & f_N(y) &= \phi_N(\Psi_I^{-1}(y)), \end{aligned} \tag{4-1}$$

where Ψ_I is defined in (2-4). Thus $Q_N\phi$ is a regularized, compactly supported[7] modification of the profile ϕ, ϕ_N is an $\dot{H}^1$-invariant rescaling of $Q_N\phi$, and f_N is the function obtained by transferring ϕ_N to a neighborhood of $\mathbf{0}$ in $\mathbb{H}^3$. We define also

$$E^1_{\mathbb{R}^3}(\phi) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla\phi|^2\,dx + \frac{1}{6}\int_{\mathbb{R}^3} |\phi|^6\,dx.$$

We will use the main theorem of [Colliander et al. 2008], in the following form.

Theorem 4.1. *Assume $\psi \in \dot{H}^1(\mathbb{R}^3)$. Then there is a unique global solution $v \in C(\mathbb{R} : \dot{H}^1(\mathbb{R}^3))$ of the initial-value problem*

$$(i\partial_t + \Delta)v = v|v|^4, \quad v(0) = \psi, \tag{4-2}$$

and

$$\big\| |\nabla v| \big\|_{L_t^\infty L_x^2 \cap L_t^2 L_x^6(\mathbb{R}^3\times\mathbb{R})} \leq \widetilde{C}(E^1_{\mathbb{R}^3}(\psi)). \tag{4-3}$$

This solution scatters in the sense that there exists $\psi^{\pm\infty} \in \dot{H}^1(\mathbb{R}^3)$ such that

$$\|v(t) - e^{it\Delta}\psi^{\pm\infty}\|_{\dot{H}^1(\mathbb{R}^3)} \to 0 \tag{4-4}$$

as $t \to \pm\infty$. If $\psi \in H^5(\mathbb{R}^3)$, then $v \in C(\mathbb{R} : H^5(\mathbb{R}^3))$ and $\sup_{t\in\mathbb{R}} \|v(t)\|_{H^5(\mathbb{R}^3)} \lesssim_{\|\psi\|_{H^5(\mathbb{R}^3)}} 1$.

The main result in this section is the following lemma:

Lemma 4.2. *Assume $\phi \in \dot{H}^1(\mathbb{R}^3)$, $T_0 \in (0,\infty)$, and $\rho \in \{0,1\}$ are given, and define f_N as in* (4-1).

(i) *There is $N_0 = N_0(\phi, T_0)$ sufficiently large such that for any $N \geq N_0$ there is a unique solution $U_N \in C((-T_0N^{-2}, T_0N^{-2}) : H^1(\mathbb{H}^3))$ of the initial-value problem*

$$(i\partial_t + \Delta_{\mathbf{g}})U_N = \rho U_N|U_N|^4, \quad U_N(0) = f_N. \tag{4-5}$$

[7] This modification is useful to avoid the contribution of ϕ coming from the Euclidean infinity, in a uniform way depending on the scale N.

Moreover, for any $N \geq N_0$,

$$\|U_N\|_{S^1(-T_0N^{-2},T_0N^{-2})} \lesssim_{E^1_{\mathbb{R}^3}(\phi)} 1. \tag{4-6}$$

(ii) *Assume $\varepsilon_1 \in (0,1]$ is sufficiently small (depending only on $E^1_{\mathbb{R}^3}(\phi)$), and let $\phi' \in H^5(\mathbb{R}^3)$ satisfy $\|\phi-\phi'\|_{\dot H^1(\mathbb{R}^3)} \leq \varepsilon_1$. Let $v' \in C(\mathbb{R} : H^5)$ denote the solution of the initial-value problem*

$$(i\partial_t+\Delta)v' = \rho v'|v'|^4, \quad v'(0)=\phi'.$$

For $R, N \geq 1$ we define

$$\begin{aligned}
v'_R(x,t) &= \eta(x/R)v'(x,t), && (x,t)\in\mathbb{R}^3\times(-T_0,T_0),\\
v'_{R,N}(x,t) &= N^{1/2}v'_R(Nx,N^2t), && (x,t)\in\mathbb{R}^3\times(-T_0N^{-2},T_0N^{-2}),\\
V_{R,N}(y,t) &= v'_{R,N}(\Psi_I^{-1}(y),t) && (y,t)\in\mathbb{H}^3\times(-T_0N^{-2},T_0N^{-2}).
\end{aligned} \tag{4-7}$$

Then there is $R_0 \geq 1$ (depending on T_0 and ϕ' and ε_1) such that, for any $R \geq R_0$,

$$\limsup_{N\to\infty} \|U_N - V_{R,N}\|_{S^1(-T_0N^{-2},T_0N^{-2})} \lesssim_{E^1_{\mathbb{R}^3}(\phi)} \varepsilon_1. \tag{4-8}$$

Proof. All of the constants in this proof are allowed to depend on $E^1_{\mathbb{R}^3}(\phi)$; for simplicity of notation we will not track this dependence explicitly. Using Theorem 4.1 we have

$$\|\nabla v'\|_{(L^\infty_t L^2_x \cap L^2_t L^6_x)(\mathbb{R}^3\times\mathbb{R})} \lesssim 1, \quad \sup_{t\in\mathbb{R}} \|v'(t)\|_{H^5(\mathbb{R}^3)} \lesssim_{\|\phi'\|_{H^5(\mathbb{R}^3)}} 1. \tag{4-9}$$

We will prove that for any R_0 sufficiently large there is N_0 such that $V_{R_0,N}$ is an almost-solution of (4-5), for any $N \geq N_0$. We will then apply Proposition 3.2 to upgrade this to an exact solution of the initial-value problem (4-5) and prove the lemma.

Let

$$\begin{aligned}
e_R(x,t) := \big((i\partial_t+\Delta)v'_R - \rho v'_R|v'_R|^4\big)(x,t) &= \rho\Big(\eta\Big(\frac{x}{R}\Big)-\eta\Big(\frac{x}{R}\Big)^5\Big)v'(x,t)|v'(x,t)|^4\\
&\quad + R^{-2}v'(x,t)(\Delta\eta)\Big(\frac{x}{R}\Big) + 2R^{-1}\sum_{j=1}^3 \partial_j v'(x,t)\partial_j\eta\Big(\frac{x}{R}\Big).
\end{aligned}$$

Since $|v'(x,t)| \lesssim_{\|\phi'\|_{H^5(\mathbb{R}^3)}} 1$, see (4-9), it follows that

$$\sum_{k=1}^3 |\partial_k e_R(x,t)| \lesssim_{\|\phi'\|_{H^3(\mathbb{R}^3)}} \mathbf{1}_{[R,2R]}(|x|)\cdot\Big(|v'(x,t)| + \sum_{k=1}^3 |\partial_k v'(x,t)| + \sum_{k,j=1}^3 |\partial_k\partial_j v'(x,t)|\Big).$$

Therefore

$$\lim_{R\to\infty} \big\||\nabla e_R|\big\|_{L^2_t L^2_x(\mathbb{R}^3\times(-T_0,T_0))} = 0. \tag{4-10}$$

Letting

$$e_{R,N}(x,t) := \big((i\partial_t+\Delta)v'_{R,N} - \rho v'_{R,N}|v'_{R,N}|^4\big)(x,t) = N^{5/2}e_R(Nx,N^2t),$$

it follows from (4-10) that there is $R_0 \geq 1$ such that, for any $R \geq R_0$ and $N \geq 1$,

$$\big\| |\nabla e_{R,N}| \big\|_{L_t^1 L_x^2(\mathbb{R}^3 \times (-T_0 N^{-2}, T_0 N^{-2}))} \leq \varepsilon_1. \tag{4-11}$$

With $V_{R,N}(y,t) = v'_{R,N}(\Psi_I^{-1}(y),t)$ as in (4-7), let

$$\begin{aligned} E_{R,N}(y,t) &:= \big((i\partial_t + \Delta_{\boldsymbol{g}})V_{R,N} - \rho V_{R,N}|V_{R,N}|^4\big)(y,t) \\ &= e_{R,N}(\Psi_I^{-1}(y),t) + \Delta_{g} V_{R,N}(y,t) - (\Delta v'_{R,N})(\Psi_I^{-1}(y),t). \end{aligned} \tag{4-12}$$

To estimate the difference in the formula above, let ∂_j, $j = 1,2,3$, denote the standard vector-fields on $\mathbb{R}^3$ and $\widetilde{\partial}_j := (\Psi_I)_*(\partial_j)$ and induced vector-fields on $\mathbb{H}^3$. Using the definition (2-4) we compute

$$\boldsymbol{g}_{ij}(y) := \boldsymbol{g}_y(\widetilde{\partial}_i, \widetilde{\partial}_j) = \delta_{ij} - \frac{v_i v_j}{1+|v|^2}, \quad y = \Psi_I(v).$$

Using the standard formula for the Laplace–Beltrami operator in local coordinates

$$\Delta_{\boldsymbol{g}} f = |\boldsymbol{g}|^{-1/2} \widetilde{\partial}_i (|\boldsymbol{g}|^{1/2} \boldsymbol{g}^{ij} \widetilde{\partial}_j f)$$

we derive the pointwise bound

$$\big|\widetilde{\nabla}^1\big(\Delta_{\boldsymbol{g}} f(y) - \Delta(f \circ \Psi_I)(\Psi_I^{-1}(y))\big)\big| \lesssim \sum_{k=1}^{3} |\Psi_I^{-1}(y)|^{k-1}\, |\widetilde{\nabla}^k f(y)|,$$

for any C^3 function $f : \mathbb{H}^3 \to \mathbb{C}$ supported in the ball of radius 1 around $\mathbf{0}$, where, by definition, for $k = 1,2,3$

$$|\widetilde{\nabla}^k h(y)| := \sum_{k_1+k_2+k_3=k} \big|\widetilde{\partial}_1^{k_1} \widetilde{\partial}_2^{k_2} \widetilde{\partial}_3^{k_3} h(y)\big|.$$

Therefore the identity (4-12) gives the pointwise bound

$$\begin{aligned} |\widetilde{\nabla}^1 E_{R,N}(y,t)| &\lesssim |\nabla e_{R,N}|(\Psi_I^{-1}(y),t) + \sum_{k=1}^{3} \sum_{k_1+k_2+k_3=k} |\Psi_I^{-1}(y)|^{k-1} \big|\partial_1^{k_1}\partial_2^{k_2}\partial_3^{k_3} v'_{R,N}(\Psi_I^{-1}(y),t)\big| \\ &\lesssim |\nabla e_{R,N}|(\Psi_I^{-1}(y),t) + R^3 N^{3/2} \sum_{k_1+k_2+k_3 \in \{1,2,3\}} \big|\partial_1^{k_1}\partial_2^{k_2}\partial_3^{k_3} v'_R(N(\Psi_I^{-1}(y),t)\big|. \end{aligned}$$

Using also (4-11), it follows that for any R_0 sufficiently large there is N_0 such that for any $N \geq N_0$

$$\big\| |\nabla_{\boldsymbol{g}} E_{R_0,N}| \big\|_{L_t^1 L_x^2(\mathbb{H}^3 \times (-T_0 N^{-2}, T_0 N^{-2}))} \leq 2\varepsilon_1. \tag{4-13}$$

To verify the hypothesis (3-3) of Proposition 3.2, we use (4-9) and the integral formula (2-6) to estimate, for N large enough,

$$\begin{aligned}
&\|V_{R_0,N}\|_{L^{10}_{x,t}(\mathbb{H}^3\times(-T_0N^{-2},T_0N^{-2}))}+\sup_{t\in(-T_0N^{-2},T_0N^{-2})}\|V_{R_0,N}(t)\|_{H^1(\mathbb{H}^3)}\\
&\quad\lesssim\|v'_{R_0,N}\|_{L^{10}_{x,t}(\mathbb{R}^3\times(-T_0N^{-2},T_0N^{-2}))}+\sup_{t\in(-T_0N^{-2},T_0N^{-2})}\|\nabla v'_{R_0,N}(t)\|_{L^2(\mathbb{R}^3)}\\
&\quad=\|v'_{R_0}\|_{L^{10}_{x,t}(\mathbb{R}^3\times(-T_0,T_0))}+\sup_{t\in(-T_0,T_0)}\|\nabla v'_{R_0}(t)\|_{L^2(\mathbb{R}^3)}\\
&\quad\lesssim 1.
\end{aligned}\tag{4-14}$$

Finally, to verify the inequality on the first term in (3-4) we estimate, for R_0, N large enough,

$$\begin{aligned}
\|f_N-V_{R_0,N}(0)\|_{H^1(\mathbb{H}^3)}&\lesssim\|\phi_N-v'_{R_0,N}(0)\|_{\dot H^1(\mathbb{R}^3)}=\|Q_N\phi-v'_{R_0}(0)\|_{\dot H^1(\mathbb{R}^3)}\\
&\le\|Q_N\phi-\phi\|_{\dot H^1(\mathbb{R}^3)}+\|\phi-\phi'\|_{\dot H^1(\mathbb{R}^3)}+\|\phi'-v'_{R_0}(0)\|_{\dot H^1}\le 3\varepsilon_1.
\end{aligned}\tag{4-15}$$

The conclusion of the lemma follows from Proposition 3.2, provided that ε_1 is fixed sufficiently small depending on $E^1_{\mathbb{R}^3}(\phi)$. □

As a consequence, we have:

Corollary 4.3. *Assume $\psi\in\dot H^1(\mathbb{R}^3)$, $\varepsilon>0$, $I\subseteq\mathbb{R}$ is an interval, and*

$$\big\||\nabla(e^{it\Delta}\psi)|\big\|_{L^p_tL^q_x(\mathbb{R}^3\times I)}\le\varepsilon,\tag{4-16}$$

where $2/p+3/q=3/2$, $q\in(2,6]$. For $N\ge1$ we define, as before,

$$(Q_N\psi)(x)=\eta(x/N^{1/2})\cdot(e^{\Delta/N}\psi)(x),\;\;\psi_N(x)=N^{1/2}(Q_N\psi)(Nx),\;\;\widetilde{\psi}_N(y)=\psi_N(\Psi_I^{-1}(y)).$$

Then there is $N_1=N_1(\psi,\varepsilon)$ such that, for any $N\ge N_1$,

$$\big\||\nabla_{\boldsymbol g}(e^{it\Delta_{\boldsymbol g}}\widetilde{\psi}_N)|\big\|_{L^p_tL^q_x(\mathbb{H}^3\times N^{-2}I)}\lesssim_q\varepsilon.\tag{4-17}$$

Proof. As before, the implicit constants may depend on $E^1_{\mathbb{R}^3}(\psi)$. We may assume that $\psi\in C_0^\infty(\mathbb{R}^3)$. Using the dispersive estimate (2-24), for any $t\ne0$,

$$\begin{aligned}
\|(-\Delta_{\boldsymbol g})^{1/2}(e^{it\Delta_{\boldsymbol g}}\widetilde{\psi}_N)\|_{L^q_x(\mathbb{H}^3)}&\lesssim|t|^{3/q-3/2}\|(-\Delta_{\boldsymbol g})^{1/2}\widetilde{\psi}_N\|_{L^{q'}_x(\mathbb{H}^3)}\lesssim|t|^{3/q-3/2}\big\||\nabla\psi_N|\big\|_{L^{q'}_x(\mathbb{R}^3)}\\
&\lesssim_\psi|t|^{3/q-3/2}N^{3/q-3/2}.
\end{aligned}$$

Thus, for $T_1>0$,

$$\big\||\nabla_{\boldsymbol g}(e^{it\Delta_{\boldsymbol g}}\widetilde{\psi}_N)|\big\|_{L^p_tL^q_x(\mathbb{H}^3\times[\mathbb{R}\setminus(-T_1N^{-2},T_1N^{-2})])}\lesssim_\psi T_1^{-1/p}.$$

Therefore we can fix $T_1=T_1(\psi,\varepsilon)$ such that, for any $N\ge1$,

$$\big\||\nabla_{\boldsymbol g}(e^{it\Delta_{\boldsymbol g}}\widetilde{\psi}_N)|\big\|_{L^p_tL^q_x(\mathbb{H}^3\times[\mathbb{R}\setminus(-T_1N^{-2},T_1N^{-2})])}\lesssim_q\varepsilon.$$

The desired bound on the remaining interval $N^{-2}I\cap(-T_1N^{-2},T_1N^{-2})$ follows from Lemma 4.2(ii) with $\rho=0$. □

5. Profile decomposition in hyperbolic spaces

In this section we show that given a bounded sequence of functions $f_k \in H^1(\mathbb{H}^3)$ we can construct certain *profiles* and express the functions f_k in terms of these profiles. In other words, we prove the analogue of Keraani's theorem [2001] in hyperbolic geometry.

Given $(f, t_0, h_0) \in L^2(\mathbb{H}^3) \times \mathbb{R} \times \mathbb{G}$ we define

$$\Pi_{t_0,h_0} f(x) = (e^{-it_0\Delta_g} f)(h_0^{-1}x) = (\pi_{h_0} e^{-it_0\Delta_g} f)(x). \tag{5-1}$$

As in Section 4 — see (4-1) — given $\phi \in \dot{H}^1(\mathbb{R}^3)$ and $N \geq 1$, we define

$$T_N\phi(x) := N^{1/2}\widetilde{\phi}(N\Psi_I^{-1}(x)), \quad \text{where } \widetilde{\phi}(y) := \eta(y/N^{1/2}) \cdot (e^{\Delta/N}\phi)(y), \tag{5-2}$$

and observe that

$$T_N : \dot{H}^1(\mathbb{R}^3) \to H^1(\mathbb{H}^3) \text{ is a bounded linear operator with } \|T_N\phi\|_{H^1(\mathbb{H}^3)} \lesssim \|\phi\|_{\dot{H}^1(\mathbb{R}^3)}. \tag{5-3}$$

Definition 5.1. (1) We define a *frame* to be a sequence $\mathcal{O}_k = (N_k, t_k, h_k) \in [1,\infty) \times \mathbb{R} \times \mathbb{G}$, $k = 1, 2, \dots,$ where $N_k \geq 1$ is a scale, $t_k \in \mathbb{R}$ is a time, and $h_k \in \mathbb{G}$ is a translation element. We also assume that either $N_k = 1$ for all k (in which case we call $\{\mathcal{O}_k\}_{k\geq 1}$ a hyperbolic frame) or that $N_k \nearrow \infty$ (in which case we call $\{\mathcal{O}_k\}_{k\geq 1}$ a Euclidean frame). Let $\mathcal{F}_e$ denote the set of Euclidean frames,

$$\mathcal{F}_e = \big\{\mathcal{O} = \{(N_k, t_k, h_k)\}_{k\geq 1} : N_k \in [1,\infty),\ t_k \in \mathbb{R},\ h_k \in \mathbb{G},\ N_k \nearrow \infty\big\},$$

and let $\mathcal{F}_h$ denote the set of hyperbolic frames,

$$\mathcal{F}_h = \big\{\widetilde{\mathcal{O}} = \{(1, t_k, h_k)\}_{k\geq 1} : t_k \in \mathbb{R},\ h_k \in \mathbb{G}\big\}.$$

(2) We say that two frames $\{(N_k, t_k, h_k)\}_{k\geq 1}$ and $\{(N'_k, t'_k, h'_k)\}_{k\geq 1}$ are *orthogonal* if

$$\lim_{k\to\infty} \big[|\ln(N_k/N_{k'})| + N_k^2|t_k - t'_k| + N_k d(h_k \cdot \mathbf{0}, h'_k \cdot \mathbf{0})\big] = +\infty. \tag{5-4}$$

Two frames that are not orthogonal are called *equivalent*.

(3) Given $\phi \in \dot{H}^1(\mathbb{R}^3)$ and a Euclidean frame $\mathcal{O} = \{\mathcal{O}_k\}_{k\geq 1} = \{(N_k, t_k, h_k)\}_{k\geq 1} \in \mathcal{F}_e$, we define *the Euclidean profile associated with* $(\phi, \mathcal{O})$ as the sequence $\widetilde{\phi}_{\mathcal{O}_k}$, where

$$\widetilde{\phi}_{\mathcal{O}_k} := \Pi_{t_k,h_k}(T_{N_k}\phi), \tag{5-5}$$

The operators Π and T are defined in (5-1) and (5-2).

(4) Given $\psi \in H^1(\mathbb{H}^3)$ and a hyperbolic frame $\widetilde{\mathcal{O}} = \{\widetilde{\mathcal{O}}_k\}_{k\geq 1} = \{(1, t_k, h_k)\}_{k\geq 1} \in \mathcal{F}_h$ we define *the hyperbolic profile associated with* $(\psi, \widetilde{\mathcal{O}})$ as the sequence $\widetilde{\psi}_{\widetilde{\mathcal{O}}_k}$, where

$$\widetilde{\psi}_{\widetilde{\mathcal{O}}_k} := \Pi_{t_k,h_k}\psi. \tag{5-6}$$

Definition 5.2. We say a sequence $(f_k)_k$ bounded in $H^1(\mathbb{H}^3)$ is *absent* from a frame $\mathcal{O}=\{(N_k,t_k,h_k)\}_k$ if its localization to $\mathcal{O}$ converges weakly to 0, i.e., if for all profiles $\widetilde{\phi}_{\mathcal{O}_k}$ associated to $\mathcal{O}$, we have

$$\lim_{k\to\infty}\langle f_k,\widetilde{\phi}_{\mathcal{O}_k}\rangle_{H^1\times H^1(\mathbb{H}^3)}=0. \tag{5-7}$$

Remark 5.3. (i) If $\mathcal{O}=(1,t_k,h_k)_k$ is a hyperbolic frame, this is equivalent to saying that

$$\Pi_{-t_k,h_k^{-1}}f_k\rightharpoonup 0$$

as $k\to\infty$ in $H^1(\mathbb{H}^3)$.

(ii) If $\mathcal{O}$ is a Euclidean frame, this is equivalent to saying that for all $R>0$

$$g_k^R(v)=\eta(v/R)N_k^{-1/2}\big(\Pi_{-t_k,h_k^{-1}}f_k\big)(\Psi_I(v/N_k))\rightharpoonup 0$$

as $k\to\infty$ in $\dot{H}^1(\mathbb{R}^3)$.

We prove first some basic properties of profiles associated to equivalent/orthogonal frames.

Lemma 5.4. (i) *Assume $\{\mathcal{O}_k\}_{k\geq 1}=\{(N_k,t_k,h_k)\}_{k\geq 1}$ and $\{\mathcal{O}'_k\}_{k\geq 1}=\{(N'_k,t'_k,h'_k)\}_{k\geq 1}$ are two equivalent Euclidean frames (or hyperbolic frames), and $\phi\in\dot{H}^1(\mathbb{R}^3)$ (or $\phi\in H^1(\mathbb{H}^3)$). Then there is $\phi'\in\dot{H}^1(\mathbb{R}^3)$ (or $\phi'\in H^1(\mathbb{H}^3)$) such that, up to a subsequence,*

$$\lim_{k\to\infty}\|\widetilde{\phi}_{\mathcal{O}_k}-\widetilde{\phi}'_{\mathcal{O}'_k}\|_{H^1(\mathbb{H}^3)}=0, \tag{5-8}$$

where $\widetilde{\phi}_{\mathcal{O}_k},\widetilde{\phi}'_{\mathcal{O}'_k}$ are as in Definition 5.1.

(ii) *Assume $\{\mathcal{O}_k\}_{k\geq 1}=\{(N_k,t_k,h_k)\}_{k\geq 1}$ and $\{\mathcal{O}'_k\}_{k\geq 1}=\{(N'_k,t'_k,h'_k)\}_{k\geq 1}$ are two orthogonal frames (either Euclidean or hyperbolic) and $\widetilde{\phi}_{\mathcal{O}_k},\widetilde{\psi}_{\mathcal{O}'_k}$ are associated profiles. Then*

$$\lim_{k\to\infty}\left|\int_{\mathbb{H}^3}\boldsymbol{D}^\alpha\widetilde{\phi}_{\mathcal{O}_k}\overline{\boldsymbol{D}_\alpha\widetilde{\psi}_{\mathcal{O}'_k}}\,d\mu\right|+\lim_{k\to\infty}\|\widetilde{\phi}_{\mathcal{O}_k}\widetilde{\psi}_{\mathcal{O}'_k}\|_{L^3(\mathbb{H}^3)}=0. \tag{5-9}$$

(iii) *If $\widetilde{\phi}_{\mathcal{O}_k}$ and $\widetilde{\psi}_{\mathcal{O}_k}$ are two Euclidean profiles associated to the same frame, then*

$$\begin{aligned}\lim_{k\to\infty}\langle\nabla_{\boldsymbol{g}}\widetilde{\phi}_{\mathcal{O}_k},\nabla_{\boldsymbol{g}}\widetilde{\psi}_{\mathcal{O}_k}\rangle_{L^2\times L^2(\mathbb{H}^3)}&=\lim_{k\to\infty}\int_{\mathbb{H}^3}\boldsymbol{D}^\alpha\widetilde{\phi}_{\mathcal{O}_k}\overline{\boldsymbol{D}_\alpha\widetilde{\psi}_{\mathcal{O}_k}}\,d\mu\\&=\int_{\mathbb{R}^3}\nabla\phi(x)\cdot\nabla\overline{\psi}(x)dx=\langle\nabla\phi,\nabla\psi\rangle_{L^2\times L^2(\mathbb{R}^3)}\end{aligned}$$

Proof. (i) The proof follows from the definitions if $\{\mathcal{O}_k\}_{k\geq 1},\{\mathcal{O}'_k\}_{k\geq 1}$ are hyperbolic frames: by passing to a subsequence we may assume $\lim_{k\to\infty}-t'_k+t_k=\bar{t}$ and $\lim_{k\to\infty}{h'_k}^{-1}h_k=\bar{h}$, and define

$$\phi':=\Pi_{\bar{t},\bar{h}}\phi.$$

To prove the claim if $\{\mathcal{O}_k\}_{k\geq 1},\{\mathcal{O}'_k\}_{k\geq 1}$ are equivalent Euclidean frames, we decompose first, using the Cartan decomposition (2-2)

$${h'_k}^{-1}h_k=m_ka_{s_k}n_k,\quad m_k,n_k\in\mathbb{K},\ s_k\in[0,\infty). \tag{5-10}$$

Therefore, using the compactness of the subgroup $\mathbb{K}$ and the definition (5-4), after passing to a subsequence, we may assume that

$$\lim_{k\to\infty} N_k/N_k' = \overline{N}, \quad \lim_{k\to\infty} N_k^2(t_k - t_k') = \bar{t}, \quad \lim_{k\to\infty} m_k = m, \quad \lim_{k\to\infty} n_k = n, \quad \lim_{k\to\infty} N_k s_k = \bar{s}. \tag{5-11}$$

We observe that for any $N \geq 1$, $\psi \in \dot{H}^1(\mathbb{R}^3)$, $t \in \mathbb{R}$, $g \in \mathbb{G}$, and $q \in \mathbb{K}$

$$\Pi_{t,gq}(T_N\psi) = \Pi_{t,g}(T_N\psi_q), \quad \text{where } \psi_q(x) = \psi(q^{-1}\cdot x).$$

Therefore, in (5-10) we may assume that

$$m_k = n_k = I, \quad {h_k'}^{-1} h_k = a_{s_k}.$$

With $\bar{x} = (\bar{s}, 0, 0)$, we define

$$\phi'(x) := \overline{N}^{1/2}(e^{-i\bar{t}\Delta}\phi)(\overline{N}x - \bar{x}), \quad \phi' \in \dot{H}^1(\mathbb{R}^3),$$

and define $\widetilde{\phi}'$, $\widetilde{\phi}'_{N_k'}$, and $\widetilde{\phi}'_{\mathbb{O}_k'}$ as in (5-5). The identity (5-8) is equivalent to

$$\lim_{k\to\infty} \|T_{N_k'}\phi' - \pi_{{h_k'}^{-1}h_k} e^{i(t_k'-t_k)\Delta_g}(T_{N_k}\phi)\|_{H^1(\mathbb{H}^3)} = 0. \tag{5-12}$$

To prove (5-12) we may assume that $\phi' \in C_0^\infty(\mathbb{R}^3)$, $\phi \in H^5(\mathbb{R}^3)$, and apply Lemma 4.2(ii) with $\rho = 0$. Let $v(x,t) = (e^{it\Delta}\phi)(x)$ and, for $R \geq 1$,

$$v_R(x,t) = \eta(x/R)v(x,t), \quad v_{R,N_k}(x,t) = N_k^{1/2} v_R(N_k x, N_k^2 t), \quad V_{R,N_k}(y,t) = v_{R,N_k}(\Psi_I^{-1}(y),t).$$

It follows from Lemma 4.2(ii) that for any $\varepsilon > 0$ sufficiently small there is R_0 sufficiently large such that, for any $R \geq R_0$,

$$\limsup_{k\to\infty} \left\| e^{i(t_k'-t_k)\Delta_g}(T_{N_k}\phi) - V_{R,N_k}(t_k'-t_k) \right\|_{H^1(\mathbb{H}^3)} \leq \varepsilon. \tag{5-13}$$

Therefore, to prove (5-12) it suffices to show that, for R large enough,

$$\limsup_{k\to\infty} \left\| \pi_{h_k^{-1}h_k'}(T_{N_k'}\phi') - V_{R,N_k}(t_k'-t_k) \right\|_{H^1(\mathbb{H}^3)} \lesssim \varepsilon,$$

which, after examining the definitions and recalling that $\phi' \in C_0^\infty(\mathbb{R}^3)$, is equivalent to

$$\limsup_{k\to\infty} \left\| {N_k'}^{1/2}\phi'(N_k'\Psi_I^{-1}({h_k'}^{-1}h_k\cdot y)) - N_k^{1/2} v_R(N_k\Psi_I^{-1}(y), N_k^2(t_k'-t_k)) \right\|_{H_y^1(\mathbb{H}^3)} \lesssim \varepsilon.$$

After changing variables $y = \Psi_I(x)$ this is equivalent to

$$\limsup_{k\to\infty} \left\| {N_k'}^{1/2}\phi'(N_k'\Psi_I^{-1}({h_k'}^{-1}h_k\cdot \Psi_I(x))) - N_k^{1/2} v_R(N_k x, N_k^2(t_k'-t_k)) \right\|_{\dot{H}_x^1(\mathbb{R}^3)} \lesssim \varepsilon.$$

Since, by definition, $\phi'(z) = \overline{N}^{1/2} v(\overline{N}z - \bar{x}, -\bar{t})$, this follows provided that

$$\lim_{k\to\infty} N_k\Psi_I^{-1}({h_k'}^{-1}h_k\cdot\Psi_I(x/N_k)) - x = \bar{x} \quad \text{for any } x \in \mathbb{R}^3.$$

This last claim follows by explicit computations using (5-11) and the definition (2-4).

(ii) It suffices to prove that one can extract a subsequence such that (5-9) holds. We analyze three cases:

Case 1: $\mathcal{O}, \mathcal{O}' \in \mathcal{F}_h$. We may assume that $\phi, \psi \in C_0^\infty(\mathbb{H}^3)$ and select a subsequence such that either

$$\lim_{k\to\infty} |t_k - t_k'| = \infty \tag{5-14}$$

or

$$\lim_{k\to\infty} t_k - t_k' = \bar{t} \in \mathbb{R}, \quad \lim_{k\to\infty} d(h_k \cdot \mathbf{0}, h_k' \cdot \mathbf{0}) = \infty. \tag{5-15}$$

Using (2-24) it follows that

$$\begin{aligned} \|\Pi_{t,h}\phi\|_{L^6(\mathbb{H}^3)} + \|\Pi_{t,h}(\Delta_{\boldsymbol{g}}\phi)\|_{L^6(\mathbb{H}^3)} &\lesssim_\phi (1+|t|)^{-1} \\ \|\Pi_{t,h}\psi\|_{L^6(\mathbb{H}^3)} + \|\Pi_{t,h}(\Delta_{\boldsymbol{g}}\psi)\|_{L^6(\mathbb{H}^3)} &\lesssim_\psi (1+|t|)^{-1}, \end{aligned}$$

for any $t \in \mathbb{R}$ and $h \in \mathbb{G}$. Thus

$$\|\widetilde{\phi}_{\mathcal{O}_k}\widetilde{\psi}_{\mathcal{O}_k'}\|_{L^3(\mathbb{H}^3)} \le \|\Pi_{t_k,h_k}\phi\|_{L^6(\mathbb{H}^3)}\,\|\Pi_{t_k',h_k'}\psi\|_{L^6(\mathbb{H}^3)} \lesssim_{\phi,\psi} (1+|t_k|)^{-1}(1+|t_k'|)^{-1}, \tag{5-16}$$

and

$$\begin{aligned} \left|\int_{\mathbb{H}^3} \boldsymbol{D}^\alpha\widetilde{\phi}_{\mathcal{O}_k}\overline{\boldsymbol{D}_\alpha\widetilde{\psi}_{\mathcal{O}_k'}}\,d\mu\right| = \left|\int_{\mathbb{H}^3}\Delta_{\boldsymbol{g}}\widetilde{\phi}_{\mathcal{O}_k}\cdot\overline{\widetilde{\psi}_{\mathcal{O}_k'}}\,d\mu\right| &= \left|\int_{\mathbb{H}^3}\pi_{h_k'^{-1}h_k}e^{-i(t_k-t_k')\Delta_g}(\Delta_{\boldsymbol{g}}\phi)\cdot\overline{\psi}\,d\mu\right| \\ &\lesssim \|\pi_{h_k'^{-1}h_k}e^{-i(t_k-t_k')\Delta_g}(\Delta_{\boldsymbol{g}}\phi)\|_{L^6(\mathbb{H}^3)}\|\psi\|_{L^{6/5}(\mathbb{H}^3)} \lesssim_{\phi,\psi} (1+|t_k-t_k'|)^{-1}. \end{aligned}$$

The claim (5-9) follows if the selected subsequence satisfies (5-14).

If the selected subsequence satisfies (5-15) then, as before,

$$\begin{aligned} &\left|\int_{\mathbb{H}^3}\boldsymbol{D}^\alpha\widetilde{\phi}_{\mathcal{O}_k}\overline{\boldsymbol{D}_\alpha\widetilde{\psi}_{\mathcal{O}_k'}}\,d\mu\right| \\ &\qquad = \left|\int_{\mathbb{H}^3}\pi_{h_k'^{-1}h_k}e^{-i(t_k-t_k')\Delta_g}\phi\cdot\overline{\Delta_{\boldsymbol{g}}\psi}\,d\mu\right| \\ &\qquad \lesssim \|\Delta_{\boldsymbol{g}}\psi\|_{L^2(\mathbb{H}^3)}\cdot\|e^{-i\bar{t}\Delta_g}\phi - e^{-i(t_k-t_k')\Delta_g}\phi\|_{L^2(\mathbb{H}^3)} + \int_{\mathbb{H}^3}|e^{-i\bar{t}\Delta_g}\phi|\cdot|\pi_{h_k^{-1}h_k'}\Delta_{\boldsymbol{g}}\psi|\,d\mu. \end{aligned}$$

The first limit in (5-9) follows. Using the bound (5-16), the second limit in (5-9) also follows, up to a subsequence, if $\limsup_{k\to\infty}|t_k| = \infty$. Otherwise, we may assume that $\lim_{k\to\infty} t_k = T$, $\lim_{k\to\infty} t_k' = T' = T - \bar{t}$ and estimate

$$\begin{aligned} \|\widetilde{\phi}_{\mathcal{O}_k}\widetilde{\psi}_{\mathcal{O}_k'}\|_{L^3(\mathbb{H}^3)} &= \|e^{-it_k\Delta_g}\pi_{h_k}\phi\cdot e^{-it_k'\Delta_g}\pi_{h_k'}\psi\|_{L^3(\mathbb{H}^3)} \\ &\lesssim_{\phi,\psi} \|e^{-it_k\Delta_g}\phi - e^{-iT\Delta_g}\phi\|_{L^6(\mathbb{H}^3)} \\ &\qquad + \|e^{-it_k'\Delta_g}\psi - e^{-iT'\Delta_g}\psi\|_{L^6(\mathbb{H}^3)} + \|e^{-iT\Delta_g}\phi\cdot\pi_{h_k^{-1}h_k'}(e^{-iT'\Delta_g}\psi)\|_{L^3(\mathbb{H}^3)}. \end{aligned}$$

The second limit in (5-9) follows in this case as well.

Case 2: $\mathcal{O} \in \mathcal{F}_h$, $\mathcal{O}' \in \mathcal{F}_e$. We may assume that $\phi \in C_0^\infty(\mathbb{H}^3)$ and $\psi \in C_0^\infty(\mathbb{R}^3)$. We estimate

$$\left|\int_{\mathbb{H}^3}\boldsymbol{D}^\alpha\widetilde{\phi}_{\mathcal{O}_k}\overline{\boldsymbol{D}_\alpha\widetilde{\psi}_{\mathcal{O}_k'}}\,d\mu\right| = \left|\int_{\mathbb{H}^3}\Pi_{t_k,h_k}(\Delta_g\phi)\cdot\overline{\Pi_{t_k',h_k'}(T_{N_k'}\psi)}\,d\mu\right| \lesssim_\phi \|T_{N_k'}\psi\|_{L^2(\mathbb{H}^3)} \lesssim_{\phi,\psi} N_k'^{-1}$$

and

$$\|\widetilde{\phi}_{\mathcal{O}_k}\widetilde{\psi}_{\mathcal{O}'_k}\|_{L^3(\mathbb{H}^3)} \le \|\Pi_{t_k,h_k}\phi\|_{L^\infty(\mathbb{H}^3)}\,\|\Pi_{t'_k,h'_k}(T_{N'_k}\psi)\|_{L^3(\mathbb{H}^3)}$$
$$\lesssim \|\Delta_{\boldsymbol{g}}\phi\|_{L^2(\mathbb{H}^3)}\,\|(-\Delta_{\boldsymbol{g}})^{1/4}(T_{N'_k}\psi)\|_{L^2(\mathbb{H}^3)} \lesssim_{\phi,\psi} {N'_k}^{-1/2}.$$

The limits in (5-9) follow.

Case 3: $\mathcal{O},\mathcal{O}'\in\mathcal{F}_e$. We may assume that $\phi,\psi\in C_0^\infty(\mathbb{R}^3)$ and select a subsequence such that either

$$\lim_{k\to\infty} N_k/N'_k = 0, \tag{5-17}$$

or

$$\lim_{k\to\infty} N_k/N'_k = \overline{N}\in(0,\infty), \qquad \lim_{k\to\infty} N_k^2|t_k-t'_k| = \infty, \tag{5-18}$$

or

$$\lim_{k\to\infty} N_k/N'_k = \overline{N}\in(0,\infty), \quad \lim_{k\to\infty} N_k^2(t_k-t'_k) = \bar{t}\in\mathbb{R}, \quad \lim_{k\to\infty} N_k d(h_k\cdot\mathbf{0},h'_k\cdot\mathbf{0}) = \infty. \tag{5-19}$$

Assuming (5-17) we estimate, as in Case 2,

$$\Big|\int_{\mathbb{H}^3}\boldsymbol{D}^\alpha\widetilde{\phi}_{\mathcal{O}_k}\overline{\boldsymbol{D}_\alpha\widetilde{\psi}_{\mathcal{O}'_k}}\,d\mu\Big| = \Big|\int_{\mathbb{H}^3}\Pi_{t_k,h_k}(\Delta_g(T_{N_k}\phi))\cdot\overline{\Pi_{t'_k,h'_k}(T_{N'_k}\psi)}\,d\mu\Big|$$
$$\lesssim \|\Delta_{\boldsymbol{g}}(T_{N_k}\phi)\|_{L^2(\mathbb{H}^3)}\|T_{N'_k}\psi\|_{L^2(\mathbb{H}^3)} \lesssim_{\phi,\psi} N_k {N'_k}^{-1}$$

and

$$\|\widetilde{\phi}_{\mathcal{O}_k}\widetilde{\psi}_{\mathcal{O}'_k}\|_{L^3(\mathbb{H}^3)} \le \|\Pi_{t_k,h_k}(T_{N_k}\phi)\|_{L^9(\mathbb{H}^3)}\cdot\|\Pi_{t'_k,h'_k}(T_{N'_k}\psi)\|_{L^{9/2}(\mathbb{H}^3)}$$
$$\lesssim \|(-\Delta_{\boldsymbol{g}})^{7/12}(T_{N_k}\phi)\|_{L^2(\mathbb{H}^3)}\cdot\|(-\Delta_{\boldsymbol{g}})^{5/12}(T_{N'_k}\psi)\|_{L^2(\mathbb{H}^3)} \lesssim_{\phi,\psi} N_k^{1/6}{N'_k}^{-1/6}.$$

The limits in (5-9) follow in this case.

To prove the limit (5-9) assuming (5-18), we estimate first, using (2-24),

$$\|\Pi_{t,h}(T_N f)\|_{L^6(\mathbb{H}^3)} \lesssim_f (1+N^2|t|)^{-1}, \tag{5-20}$$

for any $t\in\mathbb{R}$, $h\in\mathbb{G}$, $N\in[0,\infty)$, and $f\in C_0^\infty(\mathbb{R}^3)$. Thus

$$\|\widetilde{\phi}_{\mathcal{O}_k}\widetilde{\psi}_{\mathcal{O}'_k}\|_{L^3(\mathbb{H}^3)} \le \|\Pi_{t_k,h_k}(T_{N_k}\phi)\|_{L^6(\mathbb{H}^3)}\,\|\Pi_{t'_k,h'_k}(T_{N'_k}\psi)\|_{L^6(\mathbb{H}^3)}$$
$$\lesssim_{\phi,\psi} (1+N_k^2|t_k|)^{-1}(1+{N'_k}^2|t'_k|)^{-1},$$

and

$$\Big|\int_{\mathbb{H}^3}\boldsymbol{D}^\alpha\widetilde{\phi}_{\mathcal{O}_k}\overline{\boldsymbol{D}_\alpha\widetilde{\psi}_{\mathcal{O}'_k}}\,d\mu\Big| = \Big|\int_{\mathbb{H}^3}\widetilde{\phi}_{\mathcal{O}_k}\cdot\overline{\Delta_{\boldsymbol{g}}\widetilde{\psi}_{\mathcal{O}'_k}}\,d\mu\Big|$$
$$= \Big|\int_{\mathbb{H}^3}\pi_{{h'_k}^{-1}h_k}e^{-i(t_k-t'_k)\Delta_{\boldsymbol{g}}}(T_{N_k}\phi)\cdot\overline{\Delta_{\boldsymbol{g}}(T_{N'_k}\psi)}\,d\mu\Big|$$
$$\lesssim \|\pi_{{h'_k}^{-1}h_k}e^{-i(t_k-t'_k)\Delta_{\boldsymbol{g}}}(T_{N_k}\phi)\|_{L^6(\mathbb{H}^3)}\,\|\Delta_{\boldsymbol{g}}(T_{N'_k}\psi)\|_{L^{6/5}(\mathbb{H}^3)}$$
$$\lesssim_{\phi,\psi} (1+N_k^2|t_k-t'_k|)^{-1}.$$

The claim (5-9) follows if the selected subsequence verifies (5-18).

Finally, it remains to prove the limit (5-9) if the selected subsequence verifies (5-19). For this we will use the following claim: if $(g_k, M_k)_{k\geq 1} \in \mathbb{G}\times[1,\infty)$, $\lim_{k\to\infty} M_k = \infty$, $\lim_{k\to\infty} M_k d(g_k\cdot\mathbf{0},\mathbf{0}) = \infty$, and $f, g \in \dot{H}^1(\mathbb{R}^3)$ then

$$\lim_{k\to\infty}\left|\int_{\mathbb{H}^3} \pi_{g_k}(-\Delta_{\boldsymbol{g}})^{1/2}(T_{M_k} f)\cdot(-\Delta_{\boldsymbol{g}})^{1/2}(T_{M_k} g)\, d\mu\right| + \|\pi_{g_k}(T_{M_k} f)\cdot(T_{M_k} g)\|_{L^3(\mathbb{H}^3)} = 0. \quad (5\text{-}21)$$

Assuming this, we can complete the proof of (5-9). It follows from (5-12) that if $f \in \dot{H}^1(\mathbb{R}^3)$ and $\{s_k\}_{k\geq 1}$ is a sequence with the property that $\lim_{k\to\infty} N_k^2 s_k = \bar{s} \in \mathbb{R}$ then

$$\lim_{k\to\infty} \|e^{-is_k\Delta_g}(T_{N_k} f) - T_{N_k'} f'\|_{H^1(\mathbb{H}^3)} = 0, \quad (5\text{-}22)$$

where $f'(x) = \overline{N}^{1/2}(e^{-i\bar{s}\Delta} f)(\overline{N}x)$. We estimate

$$\begin{aligned}\left|\int_{\mathbb{H}^3} \boldsymbol{D}^\alpha \widetilde{\phi}_{\mathbb{O}_k} \overline{\boldsymbol{D}_\alpha \widetilde{\psi}_{\mathbb{O}_k'}}\, d\mu\right| &= \left|\int_{\mathbb{H}^3} (-\Delta_{\boldsymbol{g}})^{1/2} \pi_{h_k'^{-1} h_k} e^{-i(t_k - t_k')\Delta_g}(T_{N_k}\phi)\cdot \overline{(-\Delta_{\boldsymbol{g}})^{1/2}(T_{N_k'}\psi)}\, d\mu\right| \\ &\lesssim \left|\int_{\mathbb{H}^3} (-\Delta_{\boldsymbol{g}})^{1/2}\pi_{h_k'^{-1}h_k}(T_{N_k'}\phi')\cdot \overline{(-\Delta_{\boldsymbol{g}})^{1/2}(T_{N_k'}\psi)}\, d\mu\right| \\ &\quad + \|\psi\|_{\dot{H}^1(\mathbb{R}^3)}\cdot \left\|\pi_{h_k'^{-1}h_k} e^{-i(t_k-t_k')\Delta_g}(T_{N_k}\phi) - \pi_{h_k'^{-1}h_k}(T_{N_k'}\phi')\right\|_{H^1(\mathbb{H}^3)}.\end{aligned}$$

In view of (5-21) and (5-22), both terms in the expression above converge to 0 as $k\to\infty$, as desired. If $\lim_{k\to\infty} N_k^2|t_k| = \infty$ then, using (5-20), we estimate

$$\|\widetilde{\phi}_{\mathbb{O}_k}\widetilde{\psi}_{\mathbb{O}_k'}\|_{L^3(\mathbb{H}^3)} \leq \|\Pi_{t_k,h_k}(T_{N_k}\phi)\|_{L^6(\mathbb{H}^3)}\|\Pi_{t_k',h_k'}(T_{N_k'}\psi)\|_{L^6(\mathbb{H}^3)} \lesssim_{\phi,\psi} (1+N_k^2|t_k|)^{-1},$$

which converges to 0 as $k\to\infty$. Otherwise, up to a subsequence, we may assume that $\lim_{k\to\infty} N_k^2 t_k = T \in \mathbb{R}$, $\lim_{k\to\infty}$ and write

$$\|\widetilde{\phi}_{\mathbb{O}_k}\widetilde{\psi}_{\mathbb{O}_k'}\|_{L^3(\mathbb{H}^3)} = \left\|\pi_{h_k'^{-1}h_k} e^{-it_k\Delta_g}(T_{N_k}\phi)\cdot e^{-it_k'\Delta_g}(T_{N_k'}\psi)\right\|_{L^3(\mathbb{H}^3)}.$$

This converges to 0 as $k\to\infty$, using (5-21) and (5-22), as desired.

It remains to prove the claim (5-21). In view of the $\dot{H}^1(\mathbb{R}^3)\to H^1(\mathbb{H}^3)$ boundedness of the operators T_N, we may assume that $f, g \in C_0^\infty(\mathbb{R}^3)$ and replace $T_{M_k} f$ and $T_{M_k} g$ by $M_k^{1/2} f(M_k\Psi_I^{-1}(x))$ and $M_k^{1/2} g(M_k\Psi_I^{-1}(x))$ respectively, up to small errors. Then we notice that the supports of these functions become disjoint for k sufficiently large (due to the assumption $\lim_{k\to\infty} M_k d(g_k\cdot\mathbf{0},\mathbf{0}) = \infty$). The limit (5-21) follows.

(iii) By the boundedness of T_{N_k}, it suffices to consider the case when $\phi,\psi \in C_0^\infty(\mathbb{R}^3)$. In this case, we have

$$\left\|\nabla_{\boldsymbol{g}}\big(T_{N_k}\phi - N_k^{1/2}\phi(N_k\Psi_I^{-1}\cdot)\big)\right\|_{L^2(\mathbb{H}^3)} \to 0$$

as $k\to\infty$. Hence, by the unitarity of Π_{t_k,h_k}, it suffices to compute

$$\lim_{k\to\infty} N_k\big\langle\nabla_{\boldsymbol{g}}\big(\phi(N_k\Psi^{-1}\cdot)\big), \nabla_{\boldsymbol{g}}\big(\psi(N_k\Psi_I^{-1}\cdot)\big)\big\rangle_{L^2\times L^2(\mathbb{H}^3)} = \int_{\mathbb{R}^3}\nabla\phi(x)\cdot\nabla\overline{\psi}(x)dx,$$

which follows after a change of variables and use of the dominated convergence theorem. □

Our main result in this section is the following.

Proposition 5.5. *Assume that $(f_k)_{k\geq 1}$ is a bounded sequence in $H^1(\mathbb{H}^3)$. Then there are sequences of pairs $(\phi^\mu, \mathcal{O}^\mu) \in \dot{H}^1(\mathbb{R}^3) \times \mathcal{F}_e$ and $(\psi^\nu, \widetilde{\mathcal{O}}^\nu) \in H^1(\mathbb{H}^3) \times \mathcal{F}_h$, $\mu, \nu = 1, 2, \dots$, such that, up to a subsequence, for any $J \geq 1$,*

$$f_k = \sum_{1\leq\mu\leq J} \widetilde{\phi}^\mu_{\mathcal{O}^\mu_k} + \sum_{1\leq\nu\leq J} \widetilde{\psi}^\nu_{\widetilde{\mathcal{O}}^\nu_k} + r^J_k, \tag{5-23}$$

where $\widetilde{\phi}^\mu_{\mathcal{O}^\mu_k}$ and $\widetilde{\psi}^\nu_{\widetilde{\mathcal{O}}^\nu_k}$ are the associated profiles in Definition 5.1, and[8]

$$\lim_{J\to\infty} \limsup_{k\to\infty} \sup_{\substack{N\geq 1\\ t\in\mathbb{R}\\ x\in\mathbb{H}^3}} N^{-1/2} |P_N e^{it\Delta_{\mathbf{g}}} r^J_k|(x) = 0. \tag{5-24}$$

Moreover the frames $\{\mathcal{O}^\mu\}_{\mu\geq 1}$ and $\{\widetilde{\mathcal{O}}^\nu\}_{\nu\geq 1}$ are pairwise orthogonal. Finally, the decomposition is asymptotically orthogonal in the sense that

$$\lim_{J\to\infty} \limsup_{k\to\infty} \left| E^1(f_k) - \sum_{1\leq\mu\leq J} E^1(\widetilde{\phi}^\mu_{\mathcal{O}^\mu_k}) - \sum_{1\leq\nu\leq J} E^1(\widetilde{\psi}^\nu_{\widetilde{\mathcal{O}}^\nu_k}) - E^1(r^J_k) \right| = 0, \tag{5-25}$$

where E^1 is the energy defined in (1-3).

The profile decomposition in Proposition 5.5 is a consequence of the following finitary decomposition.

Lemma 5.6. *Let $(f_k)_{k\geq 1}$ be a bounded sequence of functions in $H^1(\mathbb{H}^3)$ and let $\delta \in (0, \delta_0]$ be sufficiently small. Up to passing to a subsequence, the sequence $(f_k)_{k\geq 1}$ can be decomposed into $2J + 1 = O(\delta^{-2})$ terms*

$$f_k = \sum_{1\leq\mu\leq J} \widetilde{\phi}^\mu_{\mathcal{O}^\mu_k} + \sum_{1\leq\nu\leq J} \widetilde{\psi}^\nu_{\widetilde{\mathcal{O}}^\nu_k} + r_k, \tag{5-26}$$

where $\widetilde{\phi}^\mu_{\mathcal{O}^\mu_k}$ and $\widetilde{\psi}^\nu_{\widetilde{\mathcal{O}}^\nu_k}$ are Euclidean and hyperbolic profiles, respectively, associated to the sequences $(\phi^\mu, \mathcal{O}^\mu) \in \dot{H}^1(\mathbb{R}^3) \times \mathcal{F}_e$ and $(\psi^\nu, \widetilde{\mathcal{O}}^\nu) \in H^1(\mathbb{H}^3) \times \mathcal{F}_h$ as in Definition 5.1.

Moreover the remainder r_k is absent from all the frames $\mathcal{O}^\mu$, $\widetilde{\mathcal{O}}^\nu$, $1 \leq \mu, \nu \leq J$ and

$$\limsup_{k\to\infty} \sup_{\substack{N\geq 1\\ t\in\mathbb{R}\\ x\in\mathbb{H}^3}} N^{-1/2} |e^{it\Delta_{\mathbf{g}}} P_N r_k|(x) \leq \delta. \tag{5-27}$$

In addition, the frames $\mathcal{O}^\mu$ and $\widetilde{\mathcal{O}}^\nu$ are pairwise orthogonal, and the decomposition is asymptotically orthogonal in the sense that

$$\|\nabla_{\mathbf{g}} f_k\|^2_{L^2} = \sum_{1\leq\mu\leq J} \|\nabla_{\mathbf{g}} \widetilde{\phi}^\mu_{\mathcal{O}^\mu_k}\|^2_{L^2} + \sum_{1\leq\nu\leq J} \|\nabla_{\mathbf{g}} \widetilde{\psi}^\nu_{\widetilde{\mathcal{O}}^\nu_k}\|^2_{L^2} + \|\nabla_{\mathbf{g}} r_k\|^2_{L^2} + o_k(1) \tag{5-28}$$

where $o_k(1) \to 0$ as $k \to \infty$.

[8] It is convenient to use the critical norm $\|N^{-1/2} P_N e^{it\Delta_{\mathbf{g}}} f\|_{L^\infty_{N,x,t}}$ to measure smallness of the remainder in (5-24), as it already selects the parameters of the frames. Other critical norms have been used as well; see, for example, [Keraani 2001] and [Laurent 2011]. In any case, by Sobolev and Strichartz estimates, one obtains full control of the Z norm of the remainders, see (6-1).

We show first how to prove Proposition 5.5 assuming the finitary decomposition of Lemma 5.6.

Proof of Proposition 5.5. We apply Lemma 5.6 repeatedly for $\delta = 2^{-l}$, $l = 1, 2, \dots$ and we obtain the result except for (5-25). To prove this, it suffices from (5-28) to prove the addition of the L^6-norms. But from Lemma 2.2 and (5-24), we see that

$$\limsup_{J\to\infty} \limsup_{k\to\infty} \|r_k^J\|_{L^6(\mathbb{H}^3)} = 0$$

so that

$$\limsup_{J\to\infty} \limsup_{k\to\infty} \Big(\big| \|f_k\|_{L^6}^6 - \|f_k - r_k^J\|_{L^6}^6 \big| + \|r_k^J\|_{L^6}^6 \Big) = 0. \tag{5-29}$$

Now, for fixed J, we see that

$$\begin{aligned}
&\Big| |f_k - r_k^J|^6 - \sum_{1\le\mu\le J} |\widetilde{\phi}^{\mu}_{\mathbb{O}^{\mu}_k}|^6 - \sum_{1\le\nu\le J} |\widetilde{\psi}^{\nu}_{\widetilde{\mathbb{O}}^{\nu}_k}|^6 \Big| \\
&\qquad \lesssim_J \sum_{1\le\alpha\neq\beta\le J} |\widetilde{\phi}^{\alpha}_{\mathbb{O}^{\alpha}_k}| \, |\widetilde{\phi}^{\beta}_{\mathbb{O}^{\beta}_k}|^5 + \sum_{1\le\alpha\neq\beta\le J} |\widetilde{\psi}^{\alpha}_{\widetilde{\mathbb{O}}^{\alpha}_k}| \, |\widetilde{\psi}^{\beta}_{\widetilde{\mathbb{O}}^{\beta}_k}|^5 + \sum_{1\le\mu,\nu\le J} \Big(|\widetilde{\phi}^{\mu}_{\mathbb{O}^{\mu}_k}| \, |\widetilde{\psi}^{\nu}_{\widetilde{\mathbb{O}}^{\nu}_k}|^5 + |\widetilde{\phi}^{\mu}_{\mathbb{O}^{\mu}_k}|^5 |\widetilde{\psi}^{\nu}_{\widetilde{\mathbb{O}}^{\nu}_k}| \Big)
\end{aligned}$$

so that

$$\Big| \|f_k - r_k^J\|_{L^6}^6 - \sum_{1\le\mu\le J} \|\widetilde{\phi}^{\mu}_{\mathbb{O}^{\mu}_k}\|_{L^6}^6 - \sum_{1\le\nu\le J} \|\widetilde{\psi}^{\nu}_{\widetilde{\mathbb{O}}^{\nu}_k}\|_{L^6}^6 \Big| \lesssim_J \sum_{\alpha,\beta} \|f_k^\alpha f_k^\beta\|_{L^3}$$

where the summation ranges over all pairs (f_k^α, f_k^β) of profiles such that $f_k^\alpha \neq f_k^\beta$ and where we have used the fact that the L^6 norm of each profile is bounded uniformly. From Lemma 5.4(ii), we see that this converges to 0 as $k \to \infty$. The identity (5-25) follows using also (5-29). □

Proof of Lemma 5.6. For $(g_k)_k$ a bounded sequence in $H^1(\mathbb{H}^3)$, we let

$$\delta((g_k)_k) = \limsup_{k\to\infty} \sup_{\substack{N\ge1\\ t\in\mathbb{R}\\ h\in\mathbb{G}}} N^{-\frac{1}{2}} \big| P_N\big(e^{it\Delta_{\mathbf{g}}} g_k(h\cdot\mathbf{0})\big)\big|. \tag{5-30}$$

If $\delta((f_k)_k) \le \delta$, then we let $J = 0$ and $f_k = r_k$ and Lemma 5.6 follows. Otherwise, we use inductively the following:

Claim. *Assume $(g_k)_k$ is a bounded sequence in $H^1(\mathbb{H}^3)$ which is absent from a family of frames $(\mathbb{O}^\alpha)_{\alpha\le A}$ and such that $\delta((g_k)_k) \ge \delta$. Then, after passing to a subsequence, there exists a new frame $\mathbb{O}'$ which is orthogonal to $\mathbb{O}^\alpha$ for all $\alpha \le A$ and a profile $\widetilde{\phi}_{\mathbb{O}'_k}$ of free energy*

$$\lim_{k\to\infty} \|\nabla_{\mathbf{g}} \widetilde{\phi}_{\mathbb{O}'_k}\|_{L^2} \gtrsim \delta \tag{5-31}$$

such that $g_k - \widetilde{\phi}_{\mathbb{O}'_k}$ is absent from the frames $\mathbb{O}'$ and $\mathbb{O}^\alpha$, $\alpha \le A$.

Once we have proved the claim, Lemma 5.6 follows by applying repeatedly the above procedure. Indeed, we let $(f_k^\alpha)_k$ be defined as follows: $(f_k^0)_k = (f_k)_k$ and if $\delta((f_k^\alpha)_k) \ge \delta$, then apply the above claim to $(f_k^\alpha)_k$ to get a new sequence

$$f_k^{\alpha+1} = f_k^\alpha - \widetilde{\phi}_{\mathcal{O}_k^{\alpha+1}}.$$

By induction, $(f_k^\alpha)_k$ is absent from all the frames $\mathcal{O}^\beta$, $\beta \le \alpha$. This procedure stops after a finite number ($O(\delta^{-2})$) of steps. Indeed, since $f_k^\alpha = f_k^{\alpha-1} - \widetilde{\phi}_{\mathcal{O}_k^\alpha}$ is absent from $\mathcal{O}_k^\alpha$, we get from (5-7) that

$$\begin{aligned}\|\nabla_{\boldsymbol{g}} f_k^{\alpha-1}\|_{L^2}^2 &= \|\nabla_{\boldsymbol{g}} f_k^\alpha\|_{L^2}^2 + \|\nabla_{\boldsymbol{g}} \widetilde{\phi}_{\mathcal{O}_k^\alpha}\|_{L^2}^2 + 2\langle f_k^\alpha, \widetilde{\phi}_{\mathcal{O}_k^\alpha}\rangle_{H^1\times H^1(\mathbb{H}^3)}\\ &= \|\nabla_{\boldsymbol{g}} f_k^\alpha\|_{L^2}^2 + \|\nabla_{\boldsymbol{g}} \widetilde{\phi}_{\mathcal{O}_k^\alpha}\|_{L^2}^2 + o_k(1)\end{aligned}$$

and therefore by induction,

$$\|\nabla_{\boldsymbol{g}} f_k\|_{L^2}^2 = \sum_{1\le\alpha\le A} \|\nabla_{\boldsymbol{g}} \widetilde{\phi}_{\mathcal{O}^\alpha}\|_{L^2}^2 + \|\nabla_{\boldsymbol{g}} f_k^A\|_{L^2}^2 + o_k(1).$$

Since each profile has a free energy $\gtrsim \delta$, this is a finite process and Lemma 5.6 follows.

Now we prove the claim. By hypothesis, there exists a sequence $\widetilde{\mathcal{O}}_k = (N_k, t_k, h_k)_k$ such that the $\limsup_{k\to\infty}$ in (5-30) is greater than $\delta/2$. If $\limsup_{k\to\infty} N_k = \infty$, then, up to passing to a subsequence, we may assume that $\{\widetilde{\mathcal{O}}_k\}_{k\ge1} = \mathcal{O}'$ is a Euclidean frame. Otherwise, up to passing to a subsequence, we may assume that $N_k \to N \ge 1$ and we let $\mathcal{O}' = \{(1, t_k, h_k)_k\}_{k\ge1}$ be a hyperbolic frame. In all cases, we get a frame $\mathcal{O}' = \{(M_k, t_k, h_k)_k\}_{k\ge1}$ such that

$$\delta/2 \le \lim_{k\to\infty} N_k^{-\frac12} \left|P_{N_k}(e^{it_k\Delta_{\boldsymbol{g}}})g_k\right|(h_k\cdot\mathbf{0}) = \lim_{k\to\infty} \left|\langle \Pi_{-t_k,h_k^{-1}} g_k, N_k^{-\frac12} P_{N_k}(\delta_{\mathbf{0}})\rangle_{L^2\times L^2(\mathbb{H}^3)}\right| \tag{5-32}$$

for some sequence N_k comparable to M_k.

Now, we claim that there exists a profile $\tilde{f}_{\mathcal{O}_k'}$ associated to the frame $\mathcal{O}'$ such that

$$\limsup_{k\to\infty} \|\nabla_{\boldsymbol{g}} \tilde{f}_{\mathcal{O}_k'}\|_{L^2} \lesssim 1$$

and

$$\Pi_{-t_k,h_k^{-1}} \tilde{f}_{\mathcal{O}_k'} - N_k^{-\frac52} e^{N_k^{-2}\Delta_{\boldsymbol{g}}}(\delta_{\mathbf{0}}) \to 0$$

strongly in $H^1(\mathbb{H}^3)$. Indeed, if $\mathcal{O}'$ is a hyperbolic frame, then $f := N^{-\frac52} e^{N^{-2}\Delta_{\boldsymbol{g}}}\delta_{\mathbf{0}}$. If $N_k \to \infty$, we let $f(x) := (4\pi)^{-\frac32} e^{-|x|^2/4} = e^{\Delta}\delta_0$. By the unitarity of Π it suffices to see that

$$\|N_k^{-\frac52} e^{N_k^{-2}\Delta_{\boldsymbol{g}}}\delta_{\mathbf{0}} - T_{N_k} f\|_{H^1(\mathbb{H}^3)} \to 0 \tag{5-33}$$

which follows by inspection of the explicit formula

$$(e^{z\Delta_{\boldsymbol{g}}}\delta_{\mathbf{0}})(P) = \frac{1}{(4\pi z)^{\frac32}} e^{-z} \frac{r}{\sinh r} e^{-\frac{r^2}{4z}}$$

for $r = d_{\boldsymbol{g}}(\mathbf{0}, P)$.

Since g_k is absent from the frames $\mathcal{O}^\alpha$, $\alpha \le A$, and we have a nonzero scalar product in (5-32), we see from the discussion after Definition 5.2 that $\mathcal{O}'$ is orthogonal to these frames.

Now, in the case $\mathcal{O}'$ is a hyperbolic frame, we let $\psi \in H^1(\mathbb{H}^3)$ be any weak limit of $\Pi_{-t_k,h_k^{-1}} g_k$. Then, passing to a subsequence, we may assume that for any $\varphi \in H^1(\mathbb{H}^3)$,

$$\langle \nabla_{\boldsymbol{g}}(\Pi_{-t_k,h_k^{-1}} g_k - \psi), \nabla_{\boldsymbol{g}} \varphi \rangle_{L^2\times L^2} = \langle \nabla_{\boldsymbol{g}}(g_k - \Pi_{t_k,h_k}\psi), \nabla_{\boldsymbol{g}} \Pi_{t_k,h_k}\varphi \rangle_{L^2\times L^2} \to 0,$$

so that $g'_k = g_k - \Pi_{t_k,h_k}\psi$ is absent from $\mathcal{O}'$. In particular, we see from (5-32) that

$$\begin{aligned}\delta/2 &\le \Big|\lim_{k\to\infty} \langle \Pi_{-t_k,h_k^{-1}} g_k, \Delta_{\boldsymbol{g}} N^{-\frac{5}{2}}(e^{N^{-2}\Delta_{\boldsymbol{g}}}\delta_{\mathbf{0}})\rangle_{L^2\times L^2}\Big| \\ &\le \big|\langle \psi, \Delta_{\boldsymbol{g}} N^{-\frac{5}{2}}(e^{N^{-2}\Delta_{\boldsymbol{g}}}\delta_{\mathbf{0}})\rangle_{L^2\times L^2}\big| \lesssim \|\nabla_{\boldsymbol{g}}\psi\|_{L^2(\mathbb{H}^3)}\end{aligned}$$

so that (5-31) holds. Finally, to prove that g'_k is also absent from the frames $\mathcal{O}^\alpha$, $1 \le \alpha \le A$ it suffices by hypothesis to prove this for $\widetilde{\psi}_{\mathcal{O}'_k}$, but this follows from Lemma 5.4(ii).

In the case $N_k \to \infty$, we first choose $R > 0$ and we define

$$\phi_k^R(v) = \eta(v/R) N_k^{-\frac{1}{2}} (\Pi_{-t_k,h_k^{-1}} g_k)(\Psi_I(v/N_k)), \tag{5-34}$$

where η is a smooth cut-off function as in (4-1). This sequence satisfies

$$\limsup_{k\to\infty} \|\nabla \phi_k^R\|_{L^2(\mathbb{R}^3)} \lesssim \limsup_{k\to\infty} \|\nabla_{\boldsymbol{g}} g_k\|_{L^2(\mathbb{H}^3)}$$

and therefore has a subsequence which is bounded in $\dot{H}^1(\mathbb{R}^3)$ uniformly in $R > 0$. Passing to a subsequence, we can find a weak limit $\phi^R \in \dot{H}^1(\mathbb{R}^3)$. Since the bound is uniform in $R > 0$, we can let $R \to \infty$ and find a weak limit ϕ such that

$$\phi^R \rightharpoonup \phi$$

in H^1_{loc} and $\phi \in \dot{H}^1(\mathbb{R}^3)$. Now, for $\varphi \in C_0^\infty(\mathbb{R}^3)$, we have

$$\big\| T_{N_k}\varphi - N_k^{\frac{1}{2}} \varphi(N_k \Psi_I^{-1}) \big\|_{H^1(\mathbb{H}^3)} \to 0$$

as $k \to \infty$ and with Lemma 5.4(iii), we compute that

$$\begin{aligned}\langle g_k, \Delta_{\boldsymbol{g}} \widetilde{\varphi}_{\mathcal{O}'_k} \rangle_{L^2\times L^2(\mathbb{H}^3)} &= \langle \Pi_{-t_k,h_k^{-1}} g_k, \Delta_{\boldsymbol{g}} T_{N_k}\varphi \rangle_{L^2\times L^2(\mathbb{H}^3)} \\ &= \big\langle \Pi_{-t_k,h_k^{-1}} g_k, \Delta_{\boldsymbol{g}} N_k^{\frac{1}{2}} \varphi(N_k \Psi_I^{-1}\cdot) \big\rangle_{L^2\times L^2(\mathbb{H}^3)} + o_k(1) \\ &= \langle \phi, \Delta\varphi \rangle_{L^2\times L^2(\mathbb{R}^3)} + o_k(1) \\ &= -\langle \widetilde{\phi}_{\mathcal{O}'_k}, \widetilde{\varphi}_{\mathcal{O}'_k} \rangle_{H^1\times H^1(\mathbb{H}^3)} + o_k(1).\end{aligned} \tag{5-35}$$

In particular, $g'_k = g_k - \widetilde{\phi}_{\mathcal{O}'_k}$ is absent from $\mathcal{O}'$ and from (5-32), we see that (5-31) holds. Finally, from Lemma 5.4(ii) again, g'_k is absent from all the previous frames.

This finishes the proof of the claim and hence the proof of the finitary statement.

6. Proof of Proposition 3.4

In this section, we first give the proof of Proposition 3.4 assuming a few lemmas that we prove at the end.

Proof of Proposition 3.4. Using the time translation symmetry, we may assume that $t_k = 0$ for all $k \geq 1$. We apply Proposition 5.5 to the sequence $(u_k(0))_k$ which is bounded in $H^1(\mathbb{H}^3)$ and we get sequences of pairs $(\phi^\mu, \mathcal{O}^\mu) \in \dot{H}^1(\mathbb{R}^3) \times \mathcal{F}_e$ and $(\psi^\nu, \widetilde{\mathcal{O}}^\nu) \in H^1(\mathbb{H}^3) \times \mathcal{F}_h$, $\mu, \nu = 1, 2, \dots$, such that the conclusion of Proposition 5.5 holds. Up to using Lemma 5.4(i), we may assume that for all μ, either $t_k^\mu = 0$ for all k or $(N_k^\mu)^2 |t_k^\mu| \to \infty$ and similarly, for all ν, either $t_k^\nu = 0$ for all k or $|t_k^\nu| \to \infty$.

Case I: all profiles are trivial, $\phi^\mu = 0$, $\psi^\nu = 0$ for all μ, ν. In this case, we get from Strichartz estimates, (5-24) and Lemma 2.2(ii) that $u_k(0) = r_k^J$ satisfies

$$\begin{aligned} \|e^{it\Delta_g}(u_k(0))\|_{Z(\mathbb{R})} &\lesssim \|e^{it\Delta_g}(u_k(0))\|_{L_t^6 L_x^{18}}^{\frac{3}{5}} \|e^{it\Delta_g}(u_k(0))\|_{L_t^\infty L_x^6}^{\frac{2}{5}} \\ &\lesssim \|\nabla u_k(0)\|_{L^2}^{\frac{11}{15}} \Big(\sup_{N \geq 1, t, x} N^{-\frac{1}{2}} |e^{it\Delta_g} P_N(u_k(0))|(x) \Big)^{\frac{4}{15}} \to 0 \end{aligned} \tag{6-1}$$

as $k \to \infty$. Applying Lemma 6.1, we see that

$$\|u_k\|_{Z(\mathbb{R})} \leq \|e^{it\Delta_g} u_k(0)\|_{L^{10}_{t,x}(\mathbb{H}^3 \times \mathbb{R})} + \|u_k - e^{it\Delta_g} u_k(0)\|_{S^1(\mathbb{R})} \to 0$$

as $k \to \infty$, which contradicts (3-7).

Now, for every linear profile $\widetilde{\phi}^\mu_{\mathcal{O}_k^\mu}$ (resp. $\widetilde{\psi}^\nu_{\widetilde{\mathcal{O}}_k^\nu}$), define the associated nonlinear profile $U^\mu_{e,k}$ (resp. $U^\nu_{h,k}$) as the maximal solution of (1-2) with initial data $U^\mu_{e,k}(0) = \widetilde{\phi}^\mu_{\mathcal{O}_k^\mu}$ (resp. $U^\nu_{h,k}(0) = \widetilde{\psi}^\nu_{\widetilde{\mathcal{O}}_k^\nu}$). We may write U_k^γ if we do not want to discriminate between Euclidean and hyperbolic profiles.

We can give a more precise description of each nonlinear profile.

(1) If $\mathcal{O}^\mu \in \mathcal{F}_e$ is a Euclidean frame, this is given in Lemma 6.2.

(2) If $t_k^\nu = 0$, letting (I^ν, W^ν) be the maximal solution of (1-2) with initial data $W^\nu(0) = \psi^\nu$, we see that for any interval $J \Subset I^\nu$,

$$\|U^\nu_{h,k}(t) - \pi_{h_k^\nu} W^\nu(t - t_k^\nu)\|_{S^1(J)} \to 0 \tag{6-2}$$

as $k \to \infty$ (indeed, this is identically 0 in this case).

(3) If $t_k^\nu \to +\infty$, then we define (I^ν, W^ν) to be the maximal solution of (1-2) satisfying[9]

$$\|W^\nu(t) - e^{it\Delta_g} \psi^\nu\|_{H^1(\mathbb{H}^3)} \to 0$$

as $t \to -\infty$. Then, applying Proposition 3.2, we see that on any interval $J = (-\infty, T) \Subset I^\nu$, we have (6-2). Using the time reversal symmetry $u(t, x) \to \bar{u}(-t, x)$, we obtain a similar description when $t_k^\nu \to -\infty$.

[9]Note that (I^ν, W^ν) exists by Strichartz estimates and Lemma 6.1.

Case IIa: there is only one Euclidean profile, i.e., there exists μ such that $u_k(0) = \widetilde{\phi}^{\mu}_{\mathbb{O}^{\mu}_k} + o_k(1)$ in $H^1(\mathbb{H}^3)$. Applying Lemma 6.2, we see that $U^{\mu}_{e,k}$ is global with uniformly bounded S^1-norm for k large enough. Then, using the stability Proposition 3.2 with $\tilde{u} = U^{\mu}_{e,k}$, we see that for all k large enough,

$$\|u_k\|_{Z(I)} \lesssim_{E_{\max}} 1$$

which contradicts (3-7).

Case IIb: there is only one hyperbolic profile, i.e., there is ν such that $u_k(0) = \widetilde{\psi}^{\nu}_{\widetilde{\mathbb{O}}^{\nu}_k} + o_k(1)$ in $H^1(\mathbb{H}^3)$. If $t^{\nu}_k \to +\infty$, then, using Strichartz estimates, we see that

$$\|\nabla_{\boldsymbol{g}} e^{it\Delta_{\boldsymbol{g}}} \Pi_{t^{\nu}_k, h^{\nu}_k} \psi^{\nu}\|_{L^{10}_t L^{\frac{30}{13}}_x(\mathbb{H}^3 \times (-\infty,0))} = \|\nabla_{\boldsymbol{g}} e^{it\Delta_{\boldsymbol{g}}} \psi^{\nu}\|_{L^{10}_t L^{\frac{30}{13}}_x(\mathbb{H}^3 \times (-\infty,-t^{\nu}_k))} \to 0$$

as $k \to \infty$, which implies that $\|e^{it\Delta_{\boldsymbol{g}}} u_k(0)\|_{Z(-\infty,0)} \to 0$ as $k \to \infty$. Using again Lemma 6.1, we see that, for k large enough, u_k is defined on $(-\infty, 0)$ and $\|u_k\|_{Z(-\infty,0)} \to 0$ as $k \to \infty$, which contradicts (3-7). Similarly, $t^{\nu}_k \to -\infty$ yields a contradiction. Finally, if $t^{\nu}_k = 0$, we get that

$$\pi_{(h^{\nu}_k)^{-1}} u_k(0) \to \psi^{\nu}$$

converges strongly in $H^1(\mathbb{H}^3)$, which is the desired conclusion of the proposition.

Case III: there exists μ or ν and $\eta > 0$ such that

$$2\eta < \limsup_{k\to\infty} E^1(\widetilde{\phi}^{\mu}_{\mathbb{O}^{\mu}_k}), \limsup_{k\to\infty} E^1(\widetilde{\psi}^{\nu}_{\widetilde{\mathbb{O}}^{\nu}_k}) < E_{\max} - 2\eta. \tag{6-3}$$

Taking k sufficiently large and maybe replacing η by $\eta/2$, we may assume that (6-3) holds for all k. In this case, we claim that, for J sufficiently large,

$$U^{\text{app}}_k = \sum_{1\le\mu\le J} U^{\mu}_{e,k} + \sum_{1\le\nu\le J} U^{\nu}_{h,k} + e^{it\Delta_{\boldsymbol{g}}} r^J_k = U^J_{\text{prof},k} + e^{it\Delta_{\boldsymbol{g}}} r^J_k$$

is a global approximate solution with bounded Z norm for all k sufficiently large.

First, by Lemma 6.2, all the Euclidean profiles are global. Using (5-25), we see that for all ν and all k sufficiently large, $E^1(U^{\nu}_{h,k}) < E_{\max} - \eta$. By (6-2), this implies that $E^1(W^{\nu}) < E_{\max} - \eta$ so that by the definition of $E_{\max}$, W^{ν} is global and by Proposition 3.2, $U^{\nu}_{h,k}$ is global for k large enough and

$$\|U^{\nu}_{h,k}(t) - \pi_{h_k} W^{\nu}(t - t^{\nu}_k)\|_{S^1(\mathbb{R})} \to 0 \tag{6-4}$$

as $k \to \infty$.

Now we claim that

$$\limsup_{k\to\infty} \|\nabla_{\boldsymbol{g}} U^{\text{app}}_k\|_{L^\infty_t L^2_x} \le 4E^{\frac{1}{2}}_{\max} \tag{6-5}$$

is bounded uniformly in J. Indeed, we first observe using (5-25) that

$$\begin{aligned} \|\nabla_{\boldsymbol{g}} U^{\text{app}}_k\|_{L^\infty_t L^2_x} &\le \|\nabla_{\boldsymbol{g}} U^J_{\text{prof},k}\|_{L^\infty_t L^2_x} + \|\nabla_{\boldsymbol{g}} r^J_k\|_{L^2_x} \\ &\le \|\nabla_{\boldsymbol{g}} U^J_{\text{prof},k}\|_{L^\infty_t L^2_x} + (2E_{\max})^{\frac{1}{2}}. \end{aligned}$$

Using Lemma 6.3, we get that for fixed t and J,

$$\begin{aligned}\|\nabla_{\mathbf{g}} U^J_{\mathrm{prof},k}(t)\|^2_{L^2_x} &\le \sum_{1\le\gamma\le 2J} \|\nabla_{\mathbf{g}} U^\gamma_k\|^2_{L^\infty_t L^2_x} + 2\sum_{\gamma\neq\gamma'} \langle \nabla_{\mathbf{g}} U^\gamma_k(t), \nabla_{\mathbf{g}} U^{\gamma'}_k(t)\rangle_{L^2\times L^2} \\ &\le 2\sum_{1\le\gamma\le 2J} E^1(U^\gamma_k) + o_k(1) \le 2E_{\max} + o_k(1),\end{aligned}$$

where $o_k(1)\to 0$ as $k\to\infty$ for fixed J.

We also have

$$\limsup_{k\to\infty} \|\nabla_{\mathbf{g}} U^{\mathrm{app}}_k\|_{L^{10}_t L^{\frac{30}{13}}_x} \lesssim_{E_{\max},\eta} 1 \tag{6-6}$$

is bounded uniformly in J. Indeed, from (6-3) and (5-25), we see that for all γ and all k sufficiently large (depending maybe on J), $E^1(U^\gamma_k) < E_{\max}-\eta$ and from the definition of $E_{\max}$, we conclude that

$$\sup_\gamma \|U^\gamma_k\|_{Z(\mathbb{R})} \lesssim_{E_{\max},\eta} 1.$$

Using Proposition 3.2, we see that this implies that

$$\sup_\gamma \|\nabla_{\mathbf{g}} U^\gamma_k\|_{L^{\frac{10}{3}}_{t,x}} \lesssim_{E_{\max},\eta} 1.$$

Besides, using Lemma 6.1, we obtain that

$$\|\nabla_{\mathbf{g}} U^\gamma_k\|^2_{L^{\frac{10}{3}}_{t,x}} \lesssim E^1(U^\gamma_k)$$

if $E^1(U^\gamma_k)\le\delta_0$ is sufficiently small. Hence there exists a constant $C=C(E_{\max},\eta)$ such that, for all γ, and all k large enough (depending on γ),

$$\begin{aligned}\|\nabla_{\mathbf{g}} U^\gamma_k\|^2_{L^{\frac{10}{3}}_{t,x}} &\le CE^1(U^\gamma_k) \lesssim_{E_{\max},\eta} 1, \\ \|U^\gamma_k\|^2_{L^{10}_{t,x}} \lesssim \|\nabla_{\mathbf{g}} U^\gamma_k\|^2_{L^{10}_t L^{\frac{30}{13}}_x} &\le CE^1(U^\gamma_k) \lesssim_{E_{\max},\eta} 1,\end{aligned} \tag{6-7}$$

the second inequality following from Hölder's inequality between the first and the trivial bound

$$\|\nabla_{\mathbf{g}} U^\gamma_k\|_{L^\infty_t L^2_x} \le 2E^1(U^\gamma_k).$$

Now, using (6-7) and Lemma 6.3, we see that

$$\begin{aligned}\left| \|\nabla_{\mathbf{g}} U^J_{\mathrm{prof},k}\|^{\frac{10}{3}}_{L^{\frac{10}{3}}_{t,x}} - \sum_{1\le\alpha\le 2J} \|\nabla_{\mathbf{g}} U^\alpha_k\|^{\frac{10}{3}}_{L^{\frac{10}{3}}_{t,x}} \right| &\le \sum_{1\le\alpha\neq\beta\le 2J} \|(\nabla_{\mathbf{g}} U^\alpha_k)^{\frac{7}{3}} \nabla_{\mathbf{g}} U^\beta_k\|_{L^1_{t,x}} \\ &\lesssim_{E_{\max},\eta} \sum_{1\le\alpha\neq\beta\le 2J} \|(\nabla_{\mathbf{g}} U^\alpha_k)\nabla_{\mathbf{g}} U^\beta_k\|_{L^{\frac{5}{3}}_{t,x}} \lesssim_{E_{\max},\eta} o_k(1).\end{aligned}$$

Consequently,

$$\begin{aligned}\|\nabla_{\mathbf{g}} U^J_{\mathrm{prof},k}\|^{\frac{10}{3}}_{L^{\frac{10}{3}}_{t,x}} &\le \sum_{1\le\alpha\le 2J} \|\nabla_{\mathbf{g}} U^\alpha_k\|^{\frac{10}{3}}_{L^{\frac{10}{3}}_{t,x}} + o_k(1) \\ &\lesssim_{E_{\max},\eta} C \sum_{1\le\alpha\le 2J} E^1(U^\alpha_k) + o_k(1) \lesssim_{E_{\max},\eta} 1\end{aligned}$$

and using Hölder's inequality and (6-5), we get (6-6).

Using (6-5) and (6-6) we can apply Proposition 3.2 to get $\delta > 0$ such that the conclusion of Proposition 3.2 holds.

Now, for $F(x) = |x|^4 x$, we have

$$e = \left(i\partial_t + \Delta_{\boldsymbol{g}}\right) U_k^{\mathrm{app}} - U_k^{\mathrm{app}} |U_k^{\mathrm{app}}|^4 = \sum_{1\le\alpha\le 2J} \left((i\partial_t + \Delta_{\boldsymbol{g}}) U_k^\alpha - F(U_k^\alpha)\right) + \sum_{1\le\alpha\le 2J} F(U_k^\alpha) - F(U_k^{\mathrm{app}}).$$

The first term is identically 0, while using Lemma 6.4, we see that taking J large enough, we can ensure that the second is smaller than δ given above in $L_t^2 H_x^{1,\frac{6}{5}}$-norm for all k large enough. Then, since $u_k(0) = U_k^{\mathrm{app}}(0)$, Sobolev's inequality and the conclusion of Proposition 3.2 imply that for all k large, and all interval J

$$\|u_k\|_{Z(J)} \lesssim \|u_k\|_{S^1(J)} \le \|u_k - U_k^{\mathrm{app}}\|_{S^1(J)} + \|U_k^{\mathrm{app}}\|_{S^1(\mathbb{R})} \lesssim_{E_{\max},\eta} 1$$

where we have used (6-6). Then, we see that u_k is global for all k large enough and that u_k has uniformly bounded Z-norm, which contradicts (3-7). This ends the proof.

Criterion for linear evolution.

Lemma 6.1. *For any $M > 0$, there exists $\delta > 0$ such that for any interval $J \subset \mathbb{R}$, if*

$$\|\nabla_{\boldsymbol{g}} \phi\|_{L^2(\mathbb{H}^3)} \le M \quad \text{and} \quad \|e^{it\Delta_{\boldsymbol{g}}}\phi\|_{Z(J)} \le \delta,$$

then for any $t_0 \in J$, the maximal solution (I, u) of (1-2) *satisfying $u(t_0) = e^{it_0\Delta_{\boldsymbol{g}}}\phi$ satisfies $J \subset I$ and*

$$\begin{aligned} \|u - e^{it\Delta_{\boldsymbol{g}}}\phi\|_{S^1(J)} &\le \delta^3, \\ \|u\|_{S^1(J)} &\le C(M,\delta). \end{aligned} \tag{6-8}$$

Besides, if $J = (-\infty, T)$, then there exists a unique maximal solution (I, u), $J \subset I$ of (1-2) *such that*

$$\lim_{t\to-\infty} \|\nabla_{\boldsymbol{g}}(u(t) - e^{it\Delta_{\boldsymbol{g}}}\phi)\|_{L^2(\mathbb{H}^3)} = 0 \tag{6-9}$$

and (6-8) *holds in this case too. The same statement holds in the Euclidean case when* $(\mathbb{H}^3, \boldsymbol{g})$ *is replaced by* $(\mathbb{R}^3, \delta_{ij})$.

Proof of Lemma 6.1. The first part is a direct consequence of Proposition 3.2. Indeed, let $v = e^{it\Delta_{\boldsymbol{g}}}\phi$. Then clearly (3-3) is satisfied while using Strichartz estimates,

$$\|\nabla_{\boldsymbol{g}} v|v|^4\|_{L_t^2 L_x^{\frac{6}{5}}(J\times\mathbb{H}^3)} \le \|v\|_{Z(J)}^4 \|\nabla_{\boldsymbol{g}} e^{it\Delta_{\boldsymbol{g}}}\phi\|_{L_t^{10} L_x^{\frac{30}{13}}(J\times\mathbb{H}^3)} \lesssim M\delta^4,$$

thus we get (3-4). Then we can apply Proposition 3.2 with $\rho = 1$ to conclude. The second claim is classical and follows from a fixed point argument. $\square$

Description of a Euclidean nonlinear profile. Let

$$\widetilde{\mathcal{F}}_e = \{(N_k, t_k, h_k)_k \in \mathcal{F}_e : t_k = 0 \text{ for all } k \text{ or } \lim_{k\to\infty} N_k^2|t_k| = \infty\},$$
$$\widetilde{\mathcal{F}}_h = \{\, (1,\, t_k,\, h_k)_k \in \mathcal{F}_h : t_k = 0 \text{ for all } k \text{ or } \lim_{k\to\infty} |t_k| = \infty\}.$$

Lemma 6.2. *Assume $\phi \in \dot{H}^1(\mathbb{R}^3)$ and $(N_k, t_k, h_k)_k \in \widetilde{\mathcal{F}}_e$. Let U_k be the solution of* (1-2) *such that $U_k(0) = \Pi_{t_k,h_k}(T_{N_k}\phi)$.*

(i) *For k large enough, $U_k \in C(\mathbb{R} : H^1)$ is globally defined, and*

$$\|U_k\|_{Z(\mathbb{R})} \le 2\tilde{C}(E^1_{\mathbb{R}^3}(\phi)). \tag{6-10}$$

(ii) *There exists a Euclidean solution $u \in C(\mathbb{R} : \dot{H}^1(\mathbb{R}^3))$ of*

$$(i\partial_t + \Delta)\, u = u|u|^4 \tag{6-11}$$

with scattering data $\phi^{\pm\infty}$ defined as in (4-4) *such that the following holds, up to a subsequence: for any $\varepsilon > 0$, there exists $T(\phi, \varepsilon)$ such that for all $T \ge T(\phi, \varepsilon)$ there exists $R(\phi, \varepsilon, T)$ such that for all $R \ge R(\phi, \varepsilon, T)$, we have*

$$\|U_k - \tilde{u}_k\|_{S^1(|t-t_k|\le TN_k^{-2})} \le \varepsilon, \tag{6-12}$$

for k large enough, where

$$(\pi_{h_k^{-1}}\tilde{u}_k)(t, x) = N_k^{1/2}\eta(N_k\Psi_I^{-1}(x)/R)u(N_k\Psi_I^{-1}(x), N_k^2(t - t_k)).$$

In addition, up to a subsequence,

$$\|U_k\|_{L_t^{10} H_x^{1,\frac{30}{13}} \cap L_t^{\frac{10}{3}} H_x^{1,\frac{10}{3}} (\mathbb{H}^3\times\{N_k^2|t-t_k|\ge T\})} \le \varepsilon \tag{6-13}$$

and for any $\pm(t - t_k) \ge TN_k^{-2}$,

$$\|\nabla_{\boldsymbol{g}}\big(U_k(t) - \Pi_{t_k-t,h_k} T_{N_k}\phi^{\pm\infty}\big)\|_{L^2} \le \varepsilon, \tag{6-14}$$

for k large enough (*depending on ϕ, ε, T, R*).

Proof. We may assume that $h_k = I$ for any k.

If $t_k = 0$ for any k then the lemma follows from Lemma 4.2 and Corollary 4.3: we let u be the nonlinear Euclidean solution of (6-11) with $u(0) = \phi$ and notice that for any $\delta > 0$ there is $T(\phi, \delta)$ such that

$$\|\nabla u\|_{L^{10/3}_{x,t}(\mathbb{R}^3\times\{|t|\ge T(\phi,\delta)\})} \le \delta.$$

The bound (6-12) follows for any fixed $T \ge T(\phi, \delta)$ from Lemma 4.2. Assuming δ is sufficiently small and T is sufficiently large (both depending on ϕ and ε), the bounds (6-13) and (6-14) then follow from Corollary 4.3 (which guarantees smallness of $\mathbf{1}_\pm(t)\cdot e^{it\Delta_g} U_k(\pm N_k^{-2} T(\phi, \delta))$ in $L_t^{10/3} H_x^{1,10/3}(\mathbb{H}^3 \times \mathbb{R})$) and Lemma 6.1.

Otherwise, if $\lim_{k\to\infty} N_k^2|t_k| = \infty$, we may assume by symmetry that $N_k^2 t_k \to +\infty$. Then we let u be the solution of (6-11) such that

$$\|\nabla(u(t) - e^{it\Delta}\phi)\|_{L^2(\mathbb{R}^3)} \to 0$$

as $t \to -\infty$ (thus $\phi^{-\infty} = \phi$). We let $\tilde{\phi} = u(0)$ and apply the conclusions of the lemma to the frame $(N_k, 0, h_k)_k \in \mathcal{F}_e$ and $V_k(s)$, the solution of (1-2) with initial data $V_k(0) = \pi_{h_k} T_{N_k}\tilde{\phi}$. In particular, we see from the fact that $N_k^2 t_k \to +\infty$ and (6-14) that

$$\|V_k(-t_k) - \Pi_{t_k,h_k} T_{N_k}\phi\|_{H^1(\mathbb{H}^3)} \to 0$$

as $k \to \infty$. Then, using Proposition 3.2, we see that

$$\|U_k - V_k(\cdot - t_k)\|_{S^1(\mathbb{R})} \to 0$$

as $k \to \infty$, and we can conclude by inspecting the behavior of V_k. This ends the proof. □

Noninteraction of nonlinear profiles.

Lemma 6.3. *Let $\widetilde{\phi}_{\mathcal{O}_k}$ and $\widetilde{\psi}_{\mathcal{O}'_k}$ be two profiles associated to orthogonal frames $\mathcal{O}$ and $\mathcal{O}'$ in $\widetilde{\mathcal{F}}_e \cup \widetilde{\mathcal{F}}_h$. Let U_k and U'_k be the solutions of the nonlinear equation* (1-2) *such that $U_k(0) = \widetilde{\phi}_{\mathcal{O}_k}$ and $U'_k(0) = \widetilde{\psi}_{\mathcal{O}'_k}$. Suppose also that $E^1(\widetilde{\phi}_{\mathcal{O}_k}) < E_{\max} - \eta$ (resp. $E^1(\widetilde{\psi}_{\mathcal{O}'_k}) < E_{\max} - \eta$) if $\mathcal{O} \in \mathcal{F}_h$ (resp. $\mathcal{O}' \in \mathcal{F}_h$). Then*

$$\sup_{T\in\mathbb{R}} \left|\langle \nabla_{\mathbf{g}} U_k(T), \nabla_{\mathbf{g}} U'_k(T)\rangle_{L^2\times L^2(\mathbb{H}^3)}\right| + \|U_k \nabla_{\mathbf{g}} U'_k\|_{L_t^{5/2} L_x^{15/8}(\mathbb{H}^3\times\mathbb{R})} + \|(\nabla_{\mathbf{g}} U_k)\nabla_{\mathbf{g}} U'_k\|_{L_{t,x}^{5/3}(\mathbb{H}^3\times\mathbb{R})} \to 0 \quad (6\text{-}15)$$

as $k \to \infty$.

Proof. It suffices to prove (6-15) up to extracting a subsequence, and fix $\varepsilon > 0$ sufficiently small.

We only provide the proof that the second norm in (6-15) decays; the other two claims are similar. Applying Lemma 6.2 if U_k is a profile associated to a Euclidean frame (respectively (6-4) if U_k is a profile associated to a hyperbolic frame), we see that

$$\|U_k\|_{S^1} + \|U'_k\|_{S^1} \le M < +\infty$$

and that there exist R and δ such that

$$\begin{gathered} \|\nabla_{\mathbf{g}} U_k\|_{L_t^{10} L_x^{30/13} \cap L_{x,t}^{10/3}((\mathbb{H}^3\times\mathbb{R})\setminus \mathcal{S}^R_{N_k,t_k,h_k})} + \|U_k\|_{L^{10}_{x,t}((\mathbb{H}^3\times\mathbb{R})\setminus \mathcal{S}^R_{N_k,t_k,h_k})} \le \varepsilon, \\ \sup_{S,h} \left[\|\nabla_{\mathbf{g}} U_k\|_{L_t^{10} L_x^{30/13} \cap L_{x,t}^{10/3}(\mathcal{S}^\delta_{N_k,S,h})} + \|U_k\|_{L^{10}_{x,t}(\mathcal{S}^\delta_{N_k,S,h})}\right] \le \varepsilon, \end{gathered} \quad (6\text{-}16)$$

where

$$\mathcal{S}^a_{N,T,h} := \left\{(x,t) \in \mathbb{H}^3 \times \mathbb{R} : d_{\mathbf{g}}(h^{-1}\cdot x, \mathbf{0}) \le aN^{-1} \text{ and } |t - T| \le a^2 N^{-2}\right\}. \quad (6\text{-}17)$$

A similar claim holds for U'_k with the same values of R, δ.

If $N_k/N'_k \to \infty$, then for k large enough we estimate

$$\begin{aligned}\|U_k \nabla_{\boldsymbol{g}} U'_k\|_{L^5_t L^{\frac{30}{16}}_x} &\le \|U_k \nabla_{\boldsymbol{g}} U'_k\|_{L^5_t L^{\frac{30}{16}}_x(\mathcal{S}^R_{N_k,t_k,h_k})} + \|U_k \nabla_{\boldsymbol{g}} U'_k\|_{L^5_t L^{\frac{30}{16}}_x((\mathbb{H}^3\times\mathbb{R})\setminus\mathcal{S}^R_{N_k,t_k,h_k})} \\ &\le \|U_k\|_{L^{10}_{t,x}} \|\nabla_{\boldsymbol{g}} U'_k\|_{L^{10}_t L^{\frac{30}{13}}(\mathcal{S}^\delta_{N'_k,t_k,h_k})} + \|U_k\|_{L^{10}_{t,x}((\mathbb{H}^3\times\mathbb{R})\setminus\mathcal{S}^R_{N_k,t_k,h_k})} \|\nabla_{\boldsymbol{g}} U'_k\|_{L^{10}_t L^{\frac{30}{13}}_x} \\ &\lesssim_M \varepsilon.\end{aligned}$$

The case when $N'_k/N_k \to \infty$ is similar.

Otherwise, we can assume that $C^{-1} \le N_k/N'_k \le C$ for all k, and then find k sufficiently large that $\mathcal{S}^R_{N_k,t_k,h_k} \cap \mathcal{S}^R_{N'_k,t'_k,h'_k} = \varnothing$. Using (6-16) it follows as before that

$$\|U_k \nabla_{\boldsymbol{g}} U'_k\|_{L^5_t L^{\frac{30}{16}}_x} \lesssim_M \varepsilon.$$

Hence, in all cases,

$$\limsup_{k\to\infty} \|U_k \nabla_{\boldsymbol{g}} U'_k\|_{L^5_t L^{\frac{15}{8}}_x} \lesssim_M \varepsilon.$$

The convergence to 0 of the second term in (6-15) follows. □

Control of the error term.

Lemma 6.4. *With the notations in the proof of Proposition 3.4*,

$$\lim_{J\to\infty} \limsup_{k\to\infty} \Big\| \nabla_{\boldsymbol{g}} \big(F(U^{\mathrm{app}}_k) - \sum_{1\le\alpha\le 2J} F(U^\alpha_k) \big) \Big\|_{L^2_t L^{\frac{6}{5}}_x} = 0. \tag{6-18}$$

Proof. Fix $\varepsilon_0 > 0$. For fixed J, we let

$$U^J_{\mathrm{prof},k} = \sum_{1\le\mu\le J} U^\mu_{e,k} + \sum_{1\le\nu\le J} U^\nu_{h,k} = \sum_{1\le\gamma\le 2J} U^\gamma_k$$

be the sum of the profiles. Then we separate

$$\begin{aligned}&\Big\| \nabla_{\boldsymbol{g}} \big(F(U^{\mathrm{app}}_k) - \sum_{1\le\alpha\le 2J} F(U^\alpha_k) \big) \Big\|_{L^2_t L^{\frac{6}{5}}_x} \\ &\qquad \le \Big\| \nabla_{\boldsymbol{g}} \big(F(U^{\mathrm{app}}_k) - F(U^J_{\mathrm{prof},k}) \big) \Big\|_{L^2_t L^{\frac{6}{5}}_x} + \Big\| \nabla_{\boldsymbol{g}} \big(F(U^J_{\mathrm{prof},k}) - \sum_{1\le\alpha\le 2J} F(U^\alpha_k) \big) \Big\|_{L^2_t L^{\frac{6}{5}}_x}.\end{aligned}$$

We first claim that, for fixed J,

$$\limsup_{k\to\infty} \Big\| \nabla_{\boldsymbol{g}} (F(U^J_{\mathrm{prof},k}) - \sum_{1\le\alpha\le 2J} F(U^\alpha_k)) \Big\|_{L^2_t L^{\frac{6}{5}}_x} = 0. \tag{6-19}$$

Indeed, using that

$$\Big| \nabla_{\boldsymbol{g}} \Big(F\Big(\sum_{1\le\alpha\le 2J} U^\alpha_k \Big) - \sum_{1\le\alpha\le 2J} F(U^\alpha_k) \Big) \Big| \lesssim \sum_{\alpha\ne\beta,\gamma} |U^\gamma_k|^3 |U^\alpha_k \nabla_{\boldsymbol{g}} U^\beta_k|,$$

we see that

$$\Big\|\nabla_{\boldsymbol{g}}\Big(F(U^J_{\mathrm{prof},k})-\sum_{1\le\alpha\le 2J}F(U^\alpha_k)\Big)\Big\|_{L^2_tL^{\frac{6}{5}}_x}\lesssim\sum_{\alpha\neq\beta,\gamma}\|U^\gamma_k\|^3_{L^{10}_{t,x}}\|U^\alpha_k\nabla_{\boldsymbol{g}}U^\beta_k\|_{L^5_tL^{\frac{15}{8}}_x}.$$

Therefore (6-19) follows from (6-15) since the sum is over a finite set and each profile is bounded in $L^{10}_{t,x}$ by (6-7).

Now we prove that, for any given $\varepsilon_0>0$,

$$\limsup_{J\to\infty}\limsup_{k\to\infty}\big\|\nabla_{\boldsymbol{g}}\big(F(U^{\mathrm{app}}_k)-F(U^J_{\mathrm{prof},k})\big)\big\|_{L^2_tL^{\frac{6}{5}}_x}\lesssim\varepsilon_0. \tag{6-20}$$

This would complete the proof of (6-18). We first remark that, from (6-6), $U^J_{\mathrm{prof},k}$ has bounded $L^{10}_tH^{1,\frac{30}{13}}_x$-norm, uniformly in J for k sufficiently large. We also let $j_0=j_0(\varepsilon_0)$ independent of J be such that[10]

$$\sup_{\alpha\ge j_0}\limsup_{k\to\infty}\|U^\alpha_k\|_{L^{10}_{t,x}}\lesssim\varepsilon_0. \tag{6-21}$$

Now we compute

$$\big\|\nabla_{\boldsymbol{g}}(F(U^J_{\mathrm{prof},k}+e^{it\Delta_{\boldsymbol{g}}}r^J_k)-F(U^J_{\mathrm{prof},k}))\big\|_{L^2_tL^{\frac{6}{5}}_x}\lesssim\sum_{j=1}^{5}\sum_{p=0}^{1}\big\|\nabla^p_{\boldsymbol{g}}(e^{it\Delta_{\boldsymbol{g}}}r^J_k)^j\nabla^{1-p}_{\boldsymbol{g}}(U^J_{\mathrm{prof},k})^{5-j}\big\|_{L^2_tL^{\frac{6}{5}}_x}.$$

Since both $U^J_{\mathrm{prof},k}$ and $e^{it\Delta_{\boldsymbol{g}}}r^J_k$ are bounded in $L^{10}_tH^{1,\frac{30}{13}}_x$ uniformly in J, if there is at least one term $e^{it\Delta_{\boldsymbol{g}}}r^J_k$ with no derivative, we can bound the norm in the expression above by

$$\big\|\nabla^p_{\boldsymbol{g}}(e^{it\Delta_{\boldsymbol{g}}}r^J_k)^j\nabla^{1-p}_{\boldsymbol{g}}(U^J_{\mathrm{prof},k})^{5-j}\big\|_{L^2_tL^{\frac{6}{5}}_x}\lesssim_{E_{\max},\eta}\|e^{it\Delta_{\boldsymbol{g}}}r^J_k\|_{L^{10}_{t,x}}$$

uniformly in J, so that taking the limit $k\to\infty$ and then $J\to\infty$, we get 0. Hence we need only consider the term

$$\Big\|(U^J_{\mathrm{prof},k})^4\nabla_{\boldsymbol{g}}(e^{it\Delta_{\boldsymbol{g}}}r^J_k)\Big\|_{L^2_tL^{\frac{6}{5}}_x}.$$

Expanding further $(U^J_{\mathrm{prof},k})^4$ and using Lemma 6.3 and (6-7), we see that

$$\begin{aligned}\limsup_{k\to\infty}\|(U^J_{\mathrm{prof},k})^4\nabla_{\boldsymbol{g}}(e^{it\Delta_{\boldsymbol{g}}}r^J_k)\|_{L^2_tL^{\frac{6}{5}}_x}&=\limsup_{k\to\infty}\sum_{1\le\alpha\le J}\|(U^\alpha_k)^4\nabla_{\boldsymbol{g}}(e^{it\Delta_{\boldsymbol{g}}}r^J_k)\|_{L^2_tL^{\frac{6}{5}}_x}\\&\lesssim\limsup_{k\to\infty}\sum_{1\le\alpha\le J}\|U^\alpha_k\|^3_{L^{10}_{t,x}}\|U^\alpha_k\nabla_{\boldsymbol{g}}(e^{it\Delta_{\boldsymbol{g}}}r^J_k)\|_{L^5_tL^{\frac{15}{8}}_x}\\&\lesssim_{E_{\max},\eta}\limsup_{k\to\infty}\sum_{1\le\alpha\le j_0}E^1(U^\alpha_k)\|U^\alpha_k\nabla_{\boldsymbol{g}}(e^{it\Delta_{\boldsymbol{g}}}r^J_k)\|_{L^5_tL^{\frac{15}{8}}_x}\\&\quad+\limsup_{k\to\infty}\sum_{j_0\le\alpha\le J}E^1(U^\alpha_k)\|U^\alpha_k\|_{L^{10}_{t,x}}\|\nabla_{\boldsymbol{g}}(e^{it\Delta_{\boldsymbol{g}}}r^J_k)\|_{L^{10}_tL^{\frac{30}{13}}_x}\end{aligned}$$

[10]The fact that j_0 exists follows from (5-25) and (6-7).

where j_0 is chosen in (6-21). Consequently, using the summation formula for the energies (5-25), we get

$$\limsup_{k\to\infty}\big\|(U^J_{\mathrm{prof},k})^4\nabla_{\boldsymbol g}(e^{it\Delta_g}r^J_k)\big\|_{L^2_tL^{\frac65}_t}\lesssim_{E_{\max},\eta}\varepsilon_0+\sup_{1\le\alpha\le j_0}\limsup_{k\to\infty}\big\|U^\alpha_k\nabla_{\boldsymbol g}(e^{it\Delta_g}r^J_k)\big\|_{L^5_tL^{\frac{15}{8}}_x}.$$

Finally, we obtain from Lemma 2.3 that for any profile U^α_k,

$$\lim_{J\to\infty}\limsup_{k\to\infty}\|U^\alpha_k\nabla_{\boldsymbol g}(e^{it\Delta_g}r^J_k)\|_{L^5_tL^{\frac{15}{8}}_x(\mathbb{H}^3\times\mathbb{R})}=0. \tag{6-22}$$

This would imply (6-20) and hence complete the proof of Lemma 6.4. To prove (6-22), fix $\varepsilon>0$. For U^α_k given, we consider the sets $\mathcal{S}^a_{N,T,h}$ as defined in (6-17). For R large enough we have, using (6-16),

$$\begin{aligned}\|U^\alpha_k\nabla_{\boldsymbol g}(e^{it\Delta_g}r^J_k)\|&_{L^5_tL^{\frac{15}{8}}_x((\mathbb{H}^3\times\mathbb{R})\setminus\mathcal{S}^R_{N_k,t_k,h_k})}\\&\le\|U^\alpha_k\|_{L^{10}_{x,t}((\mathbb{H}^3\times\mathbb{R})\setminus\mathcal{S}^R_{N_k,t_k,h_k})}\|\nabla_{\boldsymbol g}(e^{it\Delta_g}r^J_k)\|_{L^{10}_tL^{\frac{30}{13}}_x}\lesssim_{E_{\max},\eta}\varepsilon.\end{aligned}$$

Now in the case of a hyperbolic profile $U^\nu_{h,k}$, we know that W^ν as in (6-2) satisfies $W^\nu\in L^{10}_{x,t}(\mathbb{H}^3\times\mathbb{R})$. We choose $W^{\nu,\prime}\in C^\infty_c(\mathbb{H}^3\times\mathbb{R})$ such that

$$\|W^\nu-W^{\nu,\prime}\|_{L^{10}_{x,t}(\mathbb{H}^3\times\mathbb{R})}\le\varepsilon.$$

Using (6-4) we see that there exists a constant $C_{\nu,\varepsilon}$ such that

$$\begin{aligned}\|U^\nu_{h,k}\nabla_{\boldsymbol g}(e^{it\Delta_g}r^J_k)\|_{L^5_tL^{\frac{15}{8}}_x(\mathcal{S}^R_{N_k,t_k,h_k})}&\le\|(U^\nu_{h,k}-\pi_{h^\nu_k}W^{\nu,\prime}(\cdot-t^\nu_k))\nabla_{\boldsymbol g}(e^{it\Delta_g}r^J_k)\|_{L^5_tL^{\frac{15}{8}}_x(\mathcal{S}^R_{N_k,t_k,h_k})}\\&\quad+\|W^{\nu,\prime}\|_{L^\infty_{t,x}}\|\nabla_{\boldsymbol g}(e^{it\Delta_g}r^J_k)\|_{L^5_tL^{\frac{15}{8}}_x(\mathcal{S}^R_{N_k,t_k,h_k})}\\&\lesssim_{E_{\max},\eta}\varepsilon+C_{\nu,\varepsilon}\|\nabla_{\boldsymbol g}(e^{it\Delta_g}r^J_k)\|_{L^5_tL^{\frac{15}{8}}_x(\mathcal{S}^R_{N_k,t_k,h_k})}.\end{aligned}$$

In the case of a Euclidean profile, we choose $v\in C^\infty_c(\mathbb{R}^3\times\mathbb{R})$ such that

$$\|u-v\|_{L^{10}_{t,x}(\mathbb{R}^3\times\mathbb{R})}\le\varepsilon,$$

for u given in Lemma 6.2. Then, using (6-12), we estimate as before

$$\|U^\mu_{e,k}\nabla_{\boldsymbol g}(e^{it\Delta_g}r^J_k)\|_{L^5_tL^{\frac{15}{8}}_x(\mathcal{S}^R_{N_k,t_k,h_k})}\lesssim_{E_{\max},\eta}\varepsilon+C_{\mu,\varepsilon}(N^\mu_k)^{\frac12}\|\nabla_{\boldsymbol g}(e^{it\Delta_g}r^J_k)\|_{L^5_tL^{\frac{15}{8}}_x(\mathcal{S}^R_{N_k,t_k,h_k})}.$$

Therefore, we conclude that in all cases,

$$\|U^\alpha_k\nabla_{\boldsymbol g}(e^{it\Delta_g}r^J_k)\|_{L^5_tL^{\frac{15}{8}}_x(\mathcal{S}^R_{N_k,t_k,h_k})}\lesssim_{E_{\max},\eta}\varepsilon+C_{\alpha,\varepsilon}(N^\alpha_k)^{\frac12}\|\nabla_{\boldsymbol g}(e^{it\Delta_g}r^J_k)\|_{L^5_tL^{\frac{15}{8}}_x(\mathcal{S}^R_{N_k,t_k,h_k})}.$$

Finally we use Lemma 2.3 and (5-24) to conclude that

$$\lim_{J\to\infty}\limsup_{k\to\infty}\|U^\alpha_k\nabla_{\boldsymbol g}(e^{it\Delta_g}r^J_k)\|_{L^5_tL^{\frac{15}{8}}_x(\mathcal{S}^R_{N_k,t_k,h_k})}\lesssim_{E_{\max},\eta}\varepsilon.$$

Since ε was arbitrary, we obtain (6-22) and hence finish the proof. □

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Received 5 Aug 2010. Accepted 1 Apr 2011.

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ANALYSIS AND PDE
Vol. 5, No. 4, 2012
dx.doi.org/10.2140/apde.2012.5.747

GENERALIZED RICCI FLOW
I: HIGHER-DERIVATIVE ESTIMATES FOR COMPACT MANIFOLDS

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We consider a generalized Ricci flow with a given (not necessarily closed) three-form and establish higher-derivative estimates for compact manifolds. As an application, we prove the compactness theorem for this generalized Ricci flow. Similar results still hold for a more generalized Ricci flow.

1. Introduction

Throughout this paper manifolds always mean smooth and closed (compact and without boundary) manifolds. Let $\mathfrak{Met}(M)$ denote the space of smooth metrics on a manifold M, and $C^\infty(M)$ the set of all smooth functions on M. We denote by C the universal constants depending only on the dimension of M, which may take different values at different places.

An important and natural problem in differential geometry is to find a canonical metric on a given manifold. A classical example is the uniformization theorem (e.g., [Chow and Knopf 2004]), which says that every smooth surface admits a unique conformal metric of constant curvature. To generalize to higher dimensional manifolds, Hamilton [1982] introduced a system of equations

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij}, \tag{1-1}$$

now called the Ricci flow, an analogue of the heat equation for metrics.

There are two ways to understand the Ricci flow: one way comes from the two-dimensional sigma model (see [Bakas 2007]), while another comes from Perelman's energy functional [Perelman 2002] defined by

$$\mathscr{F}(g,f) = \int_M \left(R + |\nabla f|^2\right) e^{-f}\, dV_g, \quad (g,f) \in \mathfrak{Met}(M) \times C^\infty(M), \tag{1-2}$$

MSC2010: 53C44, 35K55.
Keywords: Ricci flow, Generalized Ricci flow, BBS derivative estimates, compactness theorems, energy functionals.

where R, ∇, and dV_g, is the scalar curvature, Levi-Civita connection, and volume form of g, respectively. He showed that the Ricci flow is the gradient flow of (1-2) and the functional $\mathscr{F}$ is monotonic along this gradient flow. Precisely, under the following system

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij}, \quad \frac{\partial f}{\partial t} = -R - \Delta f + |\nabla f|^2, \tag{1-3}$$

we have

$$\frac{d}{dt}\mathscr{F}(g, f) = 2\int_M \left|R_{ij} + \nabla_i\nabla_j f\right|^2 e^{-f}\, dV_g \geq 0. \tag{1-4}$$

Perelman's energy functional plays an essential role in determining the structures of singularities of the Ricci flow and then the proof of Poincaré conjecture and Thurston's generalization conjecture; for more details we refer readers to [Cao and Zhu 2006; Chow et al. 2006; 2007; 2008; 2010; Kleiner and Lott 2008; Morgan and Tian 2007; Perelman 2002].

Ricci flow coupled with a one-form or a two-form. If we consider the two-dimensional nonlinear sigma model [Bakas 2007; Oliynyk et al. 2006], then we obtain a generalized Ricci flow that is the Ricci flow coupled with the evolution equation for a two-form. This flow can be also obtained from the point of view of Perelman-type energy functional.

Denoting by $\mathscr{A}^p(M)$ the space of p-forms on M, we consider the energy functional

$$\mathscr{F}^{(1)} : \mathfrak{Met}(M) \times \mathscr{A}^2(M) \times C^\infty(M) \to \mathbb{R}$$

defined by

$$\mathscr{F}^{(1)}(g, B, f) = \int_M \left(R + |\nabla f|^2 - \tfrac{1}{12}|H|^2\right) e^{-f}\, dV_g, \tag{1-5}$$

where $H = dB$. As showed in [Oliynyk et al. 2006], the gradient flow of $\mathscr{F}^{(1)}$ satisfies

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij} - 2\nabla_i\nabla_j f + \tfrac{1}{2}H_i{}^{k\ell}H_{jk\ell}, \tag{1-6}$$

$$\frac{\partial B_{ij}}{\partial t} = 3\nabla_k H^k{}_{ij} - 3H^k{}_{ij}\nabla_k f, \tag{1-7}$$

$$\frac{\partial f}{\partial t} = -R - \Delta f + \tfrac{1}{4}|H|^2, \tag{1-8}$$

and under a family of diffeomorphisms the system (1-6)–(1-8) is equivalent to

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij} + \tfrac{1}{2}H_i{}^{k\ell}H_{jk\ell}, \tag{1-9}$$

$$\frac{\partial B_{ij}}{\partial t} = 3\nabla_k H^k{}_{ij}, \tag{1-10}$$

$$\frac{\partial f}{\partial t} = -R - \Delta f + |\nabla f|^2 + \tfrac{1}{4}|H|^2. \tag{1-11}$$

Using the adjoint operator d^*, Equation (1-10) can be written as

$$\frac{\partial B_{ij}}{\partial t} = -(d^* H)_{ij}, \tag{1-12}$$

and therefore (because of $H = dB$)

$$\frac{\partial H}{\partial t} = -dd^* H = \Delta_{\mathrm{HL}} H, \tag{1-13}$$

where $\Delta_{\mathrm{HL}} = -(dd^* + d^* d)$ denotes the Hodge–Laplace operator.

The flow (1-9)–(1-10) can be interpreted as the connection Ricci flow [Streets 2008]. If we replace $H = dB$ by $F = dA$, i.e., replace a two-form by a one-form, then the flow (1-6)–(1-7) or (1-9)–(1-10) is exactly the Ricci Yang–Mills flow studied by Streets [2007] and Young [2008].

Ricci flow coupled with a one-form and a two-form. There is another generalized Ricci flow which connects to Thurston's conjecture — roughly stating that a three-dimensional manifold with a given topology has a canonical decomposition into simple three-dimensional manifolds, each of which admits one, and only one, of eight homogeneous geometries: $\mathbb{S}^3$, the round three-sphere; $\mathbb{R}^3$, the Euclidean space; $\mathbb{H}^3$, the standard hyperbolic space; $\mathbb{S}^2 \times \mathbb{R}$; $\mathbb{H}^2 \times \mathbb{R}$; Nil, the three-dimensional nilpotent Heisenberg group; $\widetilde{\mathrm{SL}}(2, \mathbb{R})$; Sol, the three -dimensional solvable Lie group. The proof of Thurston's conjecture can be found in [Cao and Zhu 2006; Kleiner and Lott 2008; Morgan and Tian 2007; Perelman 2002].

To better understanding Thurston's conjecture, Gegenberg and Kunstatter [2004] proposed a generalized flow by considering the modified $3D$ stringy theory. This flow is the Ricci flow coupled with evolution equations for a one-form and a two-form. As in (1-5), we define an energy functional

$$\mathscr{F}^{(2)} : \mathfrak{Met}(M) \times \mathscr{A}^1(M) \times \mathscr{A}^2(M) \times C^\infty(M) \to \mathbb{R}$$

by

$$\mathscr{F}^{(2)}(g, A, B, f) = \int_M \left(R + |\nabla f|^2 - \tfrac{1}{12}|H|^2 - \tfrac{1}{2}|F|^2 \right) e^{-f}\, dV_g, \tag{1-14}$$

where $H = dB$, and $F = dA$. In [He et al. 2008], the authors showed that the gradient flow of $\mathscr{F}^{(2)}$ satisfies

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij} - 2\nabla_i \nabla_j f + \tfrac{1}{2} H_i{}^{k\ell} H_{jk\ell} + 2F_i{}^k F_{jk}, \tag{1-15}$$

$$\frac{\partial A_i}{\partial t} = 2\nabla_j F^j{}_i - 2F^j{}_i \nabla_j f, \tag{1-16}$$

$$\frac{\partial B_{ij}}{\partial t} = 3\nabla_k H^k{}_{ij} - 3H^k{}_{ij} \nabla_k f, \tag{1-17}$$

$$\frac{\partial f}{\partial t} = -R - \Delta f + \tfrac{1}{4}|H|^2 + |F|^2, \tag{1-18}$$

and under a family of diffeomorphisms the system (1-15)–(1-18) is equivalent to

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij} + \tfrac{1}{2} H_i{}^{k\ell} H_{jk\ell} + 2F_i{}^k F_{jk}, \tag{1-19}$$

$$\frac{\partial A_i}{\partial t} = 2\nabla_j F^j{}_i, \tag{1-20}$$

$$\frac{\partial B_{ij}}{\partial t} = 3\nabla_k H^k{}_{ij}, \tag{1-21}$$

$$\frac{\partial f}{\partial t} = -R - \Delta f + |\nabla f|^2 + \tfrac{1}{4}|H|^2 + |F|^2. \tag{1-22}$$

Using again the adjoint operator d^*, we have

$$\frac{\partial F}{\partial t} = \Delta_{\mathrm{HL}} F, \quad \frac{\partial H}{\partial t} = \Delta_{\mathrm{HL}} H. \tag{1-23}$$

The flow (1-19)–(1-21) clearly contains the Ricci flow, the flow (1-9)–(1-10) or the connection Ricci flow, and the Ricci Yang–Mills flow; we expect this flow can give another proof of the Poincaré conjecture and Thurston's generalization conjecture, with less analysis on singularities.

Main results. For convenience, we refer to GRF the generalized Ricci flow and RF(A, B) the Ricci flow coupled with a one-form A and a two-form B.

Let (M, g) denote an n-dimensional closed Riemannian manifold with a three-form $H = \{H_{ijk}\}$. In the first part of this paper we consider the following GRF on M:

$$\frac{\partial}{\partial t} g_{ij}(x,t) = -2R_{ij}(x,t) + \tfrac{1}{2} H_{ik\ell}(x,t) H_j{}^{k\ell}(x,t), \tag{1-24}$$

$$\frac{\partial}{\partial t} H(x,t) = \Delta_{\mathrm{HL},g(x,t)} H(x,t), \quad H(x,0) = H(x), \quad g(x,0) = g(x). \tag{1-25}$$

It is clearly from (1-9) and (1-13) that the gradient flow of the energy functional $\mathscr{F}^{(1)}$ is a special case of (1-24)–(1-25). The corresponding case that H is closed is called the refined generalized Ricci flow (RGRF):

$$\frac{\partial}{\partial t} g_{ij}(x,t) = -2R_{ij}(x,t) + \frac{1}{2} H_{ik\ell}(x,t) H_j{}^{k\ell}(x,t), \tag{1-26}$$

$$\frac{\partial}{\partial t} H(x,t) = -d d^*_{g(x,t)} H(x,t), \quad H(x,0) = H(x), \quad g(x,0) = g(x). \tag{1-27}$$

Here $d^*_{g(x,t)}$ is the dual operator of d with respect to the metric $g(x,t)$.

Lemma 1.1. *Under* RGRF, *$H(x,t)$ is closed if the initial value $H(x)$ is closed.*

Proof. Since the exterior derivative d is independent of the metric, we have

$$\frac{\partial}{\partial t} dH(x,t) = d \frac{\partial}{\partial t} H(x,t) = d\big(-d d^*_{g(x,t)} H(x,t)\big) = 0.$$

so $dH(x,t) = dH(x) = 0$. □

The closedness of H is very important and has physical interpretation [Bakas 2007; Oliynyk et al. 2006]. Streets [2008] considered the connection Ricci flow in which H is the geometric torsion of connection.

Proposition 1.2. *If $(g(x,t), H(x,t))$ is a solution of* RGRF *and the initial value $H(x)$ is closed, then it is also a solution of* GRF.

Proof. From Lemma 1.1 and the assumption we know that $H(x,t)$ are all closed. Hence

$$\Delta_{\mathrm{HL},g(x,t)} H(x,t) = -d d^*_{g(x,t)} H(x,t). \qquad \square$$

For GRF, a basic and natural question is the existence. The short-time existence for RGRF has been established in [He et al. 2008], where the authors have already showed the short-time existence for RF(A, B) obviously including RGRF. In this paper, we prove the short- time existence for RGF.

Theorem 1.3. *There is a unique solution to* GRF *for a short time. More precisely, let* $(M, g_{ij}(x))$ *be an* n*-dimensional closed Riemannian manifold with a three-form* $H = \{H_{ijk}\}$, *then there exists a constant* $T = T(n) > 0$ *depending only on* n *such that the evolution system*

$$\begin{aligned}\frac{\partial}{\partial t} g_{ij}(x,t) &= -2R_{ij}(x,t) + \tfrac{1}{2} g^{kp}(x,t) g^{\ell q}(x,t) H_{ik\ell}(x,t) H_{jpq}(x,t),\\ \frac{\partial}{\partial t} H(x,t) &= \Delta_{\mathrm{HL},g(x,t)} H(x,t), \quad H(x,0) = H(x), \quad g(x,0) = g(x),\end{aligned}$$

has a unique solution $(g_{ij}(x,t), H_{ijk}(x,t))$ *for a short time* $0 \le t \le T$.

After establishing the local existence, we are able to prove the higher derivatives estimates for GRF. Precisely, we have the following

Theorem 1.4. *Suppose that* $(g(x,t), H(x,t))$ *is a solution to* GRF *on a closed manifold* M^n *and* K *is an arbitrary given positive constant. Then for each* $\alpha > 0$ *and each integer* $m \ge 1$ *there exists a constant* C_m *depending on* $m, n, \max\{\alpha, 1\}$, *and* K *such that if*

$$|\mathrm{Rm}(x,t)|_{g(x,t)} \le K, \quad |H(x)|_{g(x)} \le K$$

for all $x \in M$ *and* $t \in [0, \alpha/K]$, *then*

$$|\nabla^{m-1} \mathrm{Rm}(x,t)|_{g(x,t)} + |\nabla^m H(x,t)|_{g(x,t)} \le \frac{C_m}{t^{m/2}} \tag{1-28}$$

for all $x \in M$ *and* $t \in (0, \alpha/K]$.

As an application, we can prove the compactness theorem for GRF.

Theorem 1.5 (compactness for GRF). *Let* $\{(M_k, g_k(t), H_k(t), O_k)\}_{k\in\mathbb{N}}$ *be a sequence of complete pointed solutions to* GRF *for* $t \in [\alpha, \omega) \ni 0$ *such that*:

(i) *There is a constant* $C_0 < \infty$ *independent of* k *such that*

$$\sup_{(x,t)\in M_k\times(\alpha,\omega)} \left|\mathrm{Rm}_{g_k(x,t)}\right|_{g_k(x,t)} \le C_0, \quad \sup_{x\in M_k} |H_k(x,\alpha)|_{g_k(x,\alpha)} \le C_0.$$

(ii) *There exists a constant* $\iota_0 > 0$ *satisfies*

$$\mathrm{inj}_{g_k(0)}(O_k) \ge \iota_0.$$

Then there exists a subsequence $\{j_k\}_{k\in\mathbb{N}}$ *such that*

$$(M_{j_k}, g_{j_k}(t), H_{j_k}(t), O_{j_k}) \to (M_\infty, g_\infty(t), H_\infty(t), O_\infty),$$

converges to a complete pointed solution $(M_\infty, g_\infty(t), H_\infty(t), O_\infty)$, $t \in [\alpha, \omega)$ *to* GRF *as* $k \to \infty$.

In the second part of this paper, we consider the Ricci flow coupled with a one-form and a two-form. This flow is the gradient flow of $\mathscr{F}^{(2)}$ and takes the form

$$\frac{\partial}{\partial t}g_{ij}(x,t) = -2R_{ij} + \tfrac{1}{2}H_i{}^{k\ell}(x,t)H_{jk\ell}(x,t) + 2F_i{}^k(x,t)F_{jk}(x,t), \tag{1-29}$$

$$\frac{\partial}{\partial t}A_i(x,t) = 2\nabla_j F^j{}_i(x,t), \quad A_i(x,0) = A_i(x), \quad g_{ij}(x,0) = g_{ij}(x), \tag{1-30}$$

$$\frac{\partial}{\partial t}B_{ij}(x,t) = 3\nabla_k H^k{}_{ij}(x,t), \quad B_{ij}(x,0) = B_{ij}(x). \tag{1-31}$$

Here $A = \{A_i\}$ and $B = \{B_{ij}\}$ is a one-form and a two-form on M, respectively, and $F = dA$, $H = dB$. For this flow, we can also prove the short-time existence, higher derivative estimates, and the compactness theorem.

The rest of this paper is organized as follows. In Section 2, we prove the short-time existence and uniqueness of the GRF for any given three-form H. In Section 3, we compute the evolution equations for the Levi-Civita connections, Riemann, Ricci, and scalar curvatures of a solution to the GRF. In Section 4, we establish higher derivative estimates for GRF, called *Bernstein–Bando–Shi (BBS) derivative estimates* (e.g., [Cao and Zhu 2006; Chow and Knopf 2004; Chow et al. 2007; 2008; 2010; Morgan and Tian 2007; Shi 1989]). In Section 5, we prove the compactness theorem for GRF by using BBS estimates. In Section 6, based on the work of [He et al. 2008], the similar results are established for RF(A, B).

2. Short-time existence of GRF

In this section we establish the short-time existence for GRF. Our method is standard: we use the DeTurck trick in Ricci flow to prove its short-time existence. We assume that M is an n-dimensional closed Riemannian manifold with metric

$$d\widetilde{s}^2 = \widetilde{g}_{ij}(x)\,dx^i\,dx^j \tag{2-1}$$

and with Riemannian curvature tensor $\{\widetilde{R}_{ijk\ell}\}$. We also assume that $\widetilde{H} = \{\widetilde{H}_{ijk}\}$ is a fixed three-form on M. In the following we put

$$h_{ij} := H_{ik\ell}H_j{}^{k\ell}. \tag{2-2}$$

Suppose the metrics

$$d\widehat{s}_t^2 = \tfrac{1}{2}\widehat{g}_{ij}(x,t)\,dx^i\,dx^j \tag{2-3}$$

are the solutions of[1]

$$\frac{\partial}{\partial t}\widehat{g}_{ij}(x,t) = -2\widehat{R}_{ij}(x,t) + \widehat{h}_{ij}(x,t), \quad \widehat{g}_{ij}(x,0) = \widetilde{g}_{ij}(x) \tag{2-4}$$

for a short time $0 \le t \le T$. Consider a family of smooth diffeomorphisms $\varphi_t : M \to M (0 \le t \le T)$ of M. Let

$$ds_t^2 := \varphi_t^* d\widehat{s}_t^2, \quad 0 \le t \le T \tag{2-5}$$

[1] In the following computations we don't need to use the evolution equation for $H(x,t)$, hence we only consider the evolution equation for metrics.

be the pull-back metrics of $d\hat{s}_t^2$. For coordinates system $x=\{x^1,\dots,x^n\}$ on M, let

$$ds_t^2 = g_{ij}(x,t)\,dx^i\,dx^j \tag{2-6}$$

and

$$y(x,t)=\varphi_t(x)=\{y^1(x,t),\dots,y^n(x,t)\}. \tag{2-7}$$

Then we have

$$g_{ij}(x,t)=\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\hat{g}_{\alpha\beta}(y,t). \tag{2-8}$$

By the assumption $\hat{g}_{\alpha\beta}(x,t)$ are the solutions of

$$\frac{\partial}{\partial t}\hat{g}_{\alpha\beta}(x,t)=-2\hat{R}_{\alpha\beta}(x,t)+\hat{h}_{\alpha\beta}(x,t),\quad \hat{g}_{\alpha\beta}(x,0)=\tilde{g}_{\alpha\beta}(x). \tag{2-9}$$

We use $R_{ij},\hat{R}_{ij},\tilde{R}_{ij}$; $\Gamma_{ij}^k,\hat{\Gamma}_{ij}^k,\tilde{\Gamma}_{ij}^k$; $\nabla,\hat{\nabla},\tilde{\nabla}$; $h_{ij},\hat{h}_{ij},\tilde{h}_{ij}$ to denote the Ricci curvatures, Christoffel symbols, covariant derivatives, and products of the three-form H with respect to $\tilde{g}_{ij},\hat{g}_{ij},g_{ij}$ respectively. Then

$$\frac{\partial}{\partial t}g_{ij}(x,t)=\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\left(\frac{\partial}{\partial t}\hat{g}_{\alpha\beta}(y,t)\right)+\frac{\partial}{\partial x^i}\left(\frac{\partial y^\alpha}{\partial t}\right)\frac{\partial y^\beta}{\partial x^j}\hat{g}_{\alpha\beta}(y,t)+\frac{\partial y^\alpha}{\partial x^i}\frac{\partial}{\partial x^j}\left(\frac{\partial y^\beta}{\partial t}\right)\hat{g}_{\alpha\beta}(y,t).$$

From (2-9) we have

$$\frac{\partial}{\partial t}\hat{g}_{\alpha\beta}(y,t)=-2\hat{R}_{\alpha\beta}(y,t)+\hat{h}_{\alpha\beta}(y,t)+\frac{\partial\hat{g}_{\alpha\beta}}{\partial y^\gamma}\frac{\partial y^\gamma}{\partial t},$$

and

$$\begin{aligned}\frac{\partial}{\partial t}g_{ij}(x,t)=&-2\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\hat{R}_{\alpha\beta}(y,t)+\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\hat{h}_{\alpha\beta}(y,t)\\&+\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial\hat{g}_{\alpha\beta}}{\partial y^\gamma}\frac{\partial y^\gamma}{\partial t}+\frac{\partial}{\partial x^i}\left(\frac{\partial y^\alpha}{\partial t}\right)\frac{\partial y^\beta}{\partial x^j}\hat{g}_{\alpha\beta}(y,t)+\frac{\partial y^\alpha}{\partial x^i}\frac{\partial}{\partial x^j}\left(\frac{\partial y^\beta}{\partial t}\right)\hat{g}_{\alpha\beta}(y,t).\end{aligned}$$

Since

$$R_{ij}(x,t)=\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\hat{R}_{\alpha\beta}(y,t),\quad h_{ij}(x,t)=\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\hat{h}_{\alpha\beta}(y,t),$$

using [Shi 1989, §2, (29)], we obtain

$$\frac{\partial}{\partial t}g_{ij}(x,t)=-2R_{ij}(x,t)+h_{ij}(x,t)+\nabla_i\left(\frac{\partial y^\alpha}{\partial t}\frac{\partial x^k}{\partial y^\alpha}g_{jk}\right)+\nabla_j\left(\frac{\partial y^\alpha}{\partial t}\frac{\partial x^k}{\partial y^\alpha}g_{ik}\right). \tag{2-10}$$

According to DeTurck trick, we define $y(x,t)=\varphi_t(x)$ by the equation

$$\frac{\partial y^\alpha}{\partial t}=\frac{\partial y^\alpha}{\partial x^k}g^{\beta\gamma}(\Gamma_{\beta\gamma}^k-\tilde{\Gamma}_{\beta\gamma}^k),\quad y^\alpha(x,0)=x^\alpha, \tag{2-11}$$

then (2-10) becomes

$$\frac{\partial}{\partial t}g_{ij}(x,t)=-2R_{ij}(x,t)+h_{ij}(x,t)+\nabla_iV_j+\nabla_jV_i,\quad g_{ij}(x,0)=\tilde{g}_{ij}(x), \tag{2-12}$$

where

$$V_i = g_{ik} g^{\beta\gamma} (\Gamma^k_{\beta\gamma} - \widetilde{\Gamma}^k_{\beta\gamma}). \tag{2-13}$$

Lemma 2.1. *The evolution equation* (2-12) *is a strictly parabolic system. Moreover*,

$$\begin{aligned}\frac{\partial}{\partial t} g_{ij} &= g^{\alpha\beta} \widetilde{\nabla}_\alpha \widetilde{\nabla}_\beta g_{ij} - g^{\alpha\beta} g_{ip} \widetilde{g}^{pq} \widetilde{R}_{j\alpha q\beta} - g^{\alpha\beta} g_{jp} \widetilde{g}^{pq} \widetilde{R}_{i\alpha q\beta} \\ &+ \tfrac{1}{2} g^{\alpha\beta} g^{pq} \big(\widetilde{\nabla}_i g_{p\alpha} \cdot \widetilde{\nabla}_j g_{q\beta} + 2 \widetilde{\nabla} g_{jp} \cdot \widetilde{\nabla}_q g_{i\beta} - 2 \widetilde{\nabla}_\alpha g_{jp} \cdot \widetilde{\nabla}_\beta g_{iq} - 2 \widetilde{\nabla}_j g_{p\alpha} \cdot \widetilde{\nabla}_\beta g_{iq} - 2 \widetilde{\nabla}_i g_{p\alpha} \cdot \widetilde{\nabla}_\beta g_{jq} \big) \\ &+ \tfrac{1}{2} g^{\alpha\beta} g^{pq} H_{i\alpha p} H_{j\beta q}.\end{aligned}$$

Proof. It is an immediate consequence of Lemma 2.1 of [Shi 1989]. □

Now we can prove the short-time existence of GRF.

Theorem 2.2. *There is a unique solution to* GRF *for a short time. More precisely*, *let* $(M, g_{ij}(x))$ *be an* n*-dimensional closed Riemannian manifold with a three-form* $H = \{H_{ijk}\}$, *then there exists a constant* $T = T(n) > 0$ *depending only on* n *such that the evolution system*

$$\begin{aligned}&\frac{\partial}{\partial t} g_{ij}(x,t) = -2R_{ij}(x,t) + \tfrac{1}{2} g^{kp}(x,t) g^{\ell q}(x,t) H_{ik\ell}(x,t) H_{jpq}(x,t), \\ &\frac{\partial}{\partial t} H(x,t) = \Delta_{\mathrm{HL}, g(x,t)} H(x,t), \quad H(x,0) = H(x), \quad g(x,0) = g(x),\end{aligned}$$

has a unique solution $(g_{ij}(x,t), H_{ijk}(x,t))$ *for a short time* $0 \le t \le T$.

Proof. We proved that the first evolution equation is strictly parabolic by Lemma 2.1. Form the Ricci identity, we have $\Delta_{\mathrm{HL}, g(x,t)} H = \Delta_{\mathrm{LB}, g(x,t)} H + \mathrm{Rm} * H$ which is also strictly parabolic. Hence from the standard theory of parabolic systems, the evolution system has a unique solution. □

3. Evolution of curvatures

The evolution equation for the Riemann curvature tensors to the usual Ricci flow (e.g., [Cao and Zhu 2006; Chow and Knopf 2004; Chow et al. 2007, 2008; 2010; Hamilton 1982; Morgan and Tian 2007; Shi 1989]) is given by

$$\frac{\partial}{\partial t} R_{ijk\ell} = \Delta R_{ijk\ell} + \psi_{ijk\ell}, \tag{3-1}$$

where

$$\psi_{ijk\ell} = 2(B_{ijk\ell} - B_{ij\ell k} - B_{i\ell jk} + B_{ikj\ell}) - g^{pq}(R_{pjk\ell} R_{q\ell} + R_{ipk\ell} R_{qj} + R_{ijp\ell} R_{qk} + R_{ijkp} R_{q\ell}),$$

and $B_{ijk\ell} = g^{pr} g^{qs} R_{piqj} R_{rks\ell}$. From this we can easily deduce the evolution equation for the Riemann curvature tensors to GRF.

Let $v_{ij}(x,t)$ be any symmetric 2-tensor, we consider the flow

$$\frac{\partial}{\partial t} g_{ij}(x,t) = v_{ij}(x,t). \tag{3-2}$$

Applying a formula in [Chow and Knopf 2004] to our case $v_{ij} := -2R_{ij} + \frac{1}{2}h_{ij}$ with $h_{ij} = H_{ik\ell}H_j{}^{k\ell}$, we obtain

$$\begin{aligned}\frac{\partial}{\partial t}R_{ijk\ell} &= -\tfrac{1}{2}\big(-2\nabla_i\nabla_k R_{j\ell} + \tfrac{1}{2}\nabla_i\nabla_k h_{j\ell} + 2\nabla_i\nabla_\ell R_{jk} - \tfrac{1}{2}\nabla_i\nabla_\ell h_{jk}\\ &\qquad\qquad + 2\nabla_j\nabla_k R_{i\ell} - \tfrac{1}{2}\nabla_j\nabla_k h_{i\ell} - 2\nabla_j\nabla_\ell R_{ik} + \tfrac{1}{2}\nabla_j\nabla_\ell h_{ik}\big)\\ &\quad + \tfrac{1}{2}g^{pq}\big[R_{ijkp}(-2R_{q\ell} + \tfrac{1}{2}h_{q\ell}) + R_{ijp\ell}(-2R_{qk} + \tfrac{1}{2}h_{qk})\big]\\ &= \nabla_i\nabla_k R_{j\ell} - \nabla_i\nabla_\ell R_{jk} - \nabla_j\nabla_k R_{i\ell} + \nabla_j\nabla_\ell R_{ik} - g^{pq}(R_{ijkp}R_{q\ell} + R_{ijp\ell}R_{qk})\\ &\quad + \tfrac{1}{4}\left(-\nabla_i\nabla_k h_{j\ell} + \nabla_i\nabla_\ell h_{jk} + \nabla_j\nabla_k h_{i\ell} - \nabla_j\nabla_\ell h_{ik}\right) + \tfrac{1}{4}g^{pq}\left(R_{ijkp}h_{q\ell} + R_{ijp\ell}h_{qk}\right)\\ &= \Delta R_{ijk\ell} + 2\left(B_{ijk\ell} - B_{ij\ell k} - B_{iljk} + B_{ikj\ell}\right)\\ &\quad - g^{pq}(R_{pjk\ell}R_{q\ell} + R_{ipk\ell}R_{qj} + R_{ijp\ell}R_{qk} + R_{ijkp}R_{q\ell})\\ &\quad + \tfrac{1}{4}\left(-\nabla_i\nabla_k h_{j\ell} + \nabla_i\nabla_\ell h_{jk} + \nabla_j\nabla_k h_{i\ell} - \nabla_j\nabla_\ell h_{ik}\right) + \tfrac{1}{4}g^{pq}\left(R_{ijkp}h_{q\ell} + R_{ijp\ell}h_{qk}\right).\end{aligned}$$

Proposition 3.1. *For* GRF *we have*

$$\begin{aligned}\frac{\partial}{\partial t}R_{ijk\ell} &= \Delta R_{ijk\ell} + 2\left(B_{ijk\ell} - B_{ij\ell k} - B_{i\ell jk} + B_{ikj\ell}\right)\\ &\quad - g^{pq}(R_{pjk\ell}R_{q\ell} + R_{ipk\ell}R_{qj} + R_{ijp\ell}R_{qk} + R_{ijkp}R_{q\ell})\\ &\quad + \tfrac{1}{4}(-\nabla_i\nabla_k h_{j\ell} + \nabla_i\nabla_\ell h_{jk} + \nabla_j\nabla_k h_{i\ell} - \nabla_j\nabla_\ell h_{ik}) + \tfrac{1}{4}g^{pq}\left(R_{ijkp}h_{q\ell} + R_{ijp\ell}h_{qk}\right).\end{aligned}$$

In particular:

Corollary 3.2. *For* GRF *we have*

$$\frac{\partial}{\partial t}\operatorname{Rm} = \Delta\operatorname{Rm} + \operatorname{Rm} * \operatorname{Rm} + H * H * \operatorname{Rm} + \sum_{i=0}^{2}\nabla^i H * \nabla^{2-i}H. \tag{3-3}$$

Proof. From Proposition 3.1, we obtain

$$\frac{\partial}{\partial t}\operatorname{Rm} = \Delta\operatorname{Rm} + \operatorname{Rm} * \operatorname{Rm} + \nabla^2 h + h * \operatorname{Rm}.$$

On the other hand, $h = H * H$ and

$$\nabla^2 h = \nabla(\nabla(H * H)) = \nabla(\nabla H * H) = \nabla^2 H * H + \nabla H * \nabla H.$$

Combining these terms, we obtain the result. □

Proposition 3.3. *For* GRF *we have*

$$\begin{aligned}\frac{\partial}{\partial t}R_{ik} &= \Delta R_{ik} + 2\langle R_{piqk}, R_{pq}\rangle - 2\langle R_{pi}, R_{pk}\rangle + \tfrac{1}{4}\big[\langle h_{\ell q}, R_{i\ell kq}\rangle + \langle R_{ip}, h_{kp}\rangle\big]\\ &\qquad + \tfrac{1}{4}\big[-\nabla_i\nabla_k|H|^2 + g^{j\ell}\nabla_i\nabla_\ell h_{jk} + g^{j\ell}\nabla_j\nabla_k h_{i\ell} - \Delta h_{ik}\big].\end{aligned}$$

Proof. Since

$$\frac{\partial}{\partial t}R_{ik} = g^{j\ell}\frac{\partial}{\partial t}R_{ijk\ell} + 2g^{jp}g^{\ell q}R_{ijk\ell}R_{pq}$$

and

$$g^{ij}h_{ij}=g^{ij}H_{ipq}H_j{}^{pq}=g^{ij}g^{pr}g^{qs}H_{ipq}H_{jrs}=|H|^2,$$

it follows that

$$\begin{aligned}g^{j\ell}[-\nabla_i\nabla_k h_{j\ell}+\nabla_i\nabla_\ell h_{jk}+\nabla_j\nabla_k h_{i\ell}-\nabla_j\nabla_\ell h_{ik}+g^{pq}h_{q\ell}R_{ijkp}+g^{pq}h_{qk}R_{ijp\ell}]\\ =-\nabla_i\nabla_k|H|^2+g^{j\ell}\nabla_i\nabla_\ell h_{jk}+g^{j\ell}\nabla_j\nabla_k h_{i\ell}-\Delta h_{ik}+g^{j\ell}g^{pq}h_{q\ell}R_{ijkp}+g^{pq}h_{qk}R_{ip}.\end{aligned}$$

From these identities, we get the result. □

As a consequence, we obtain the evolution equation for scalar curvature.

Proposition 3.4. *For* GRF *we have*

$$\frac{\partial}{\partial t}R=\Delta R+2\,|\mathrm{Ric}\,|^2-\tfrac{1}{2}\Delta|H|^2+\tfrac{1}{2}\langle h_{ij},R_{ij}\rangle+\tfrac{1}{2}g^{ik}g^{j\ell}\nabla_i\nabla_j h_{k\ell}.$$

Proof. From the usual evolution equation for scalar curvature under the Ricci flow, we have

$$\begin{aligned}\frac{\partial}{\partial t}R&=\Delta R+2\,|\mathrm{Ric}\,|^2+\tfrac{1}{4}g^{ik}[\langle h_{\ell q},R_{i\ell kq}\rangle+\langle R_{ip},h_{kp}\rangle]\\&\quad+\tfrac{1}{4}g^{ik}\big(-\nabla_i\nabla_k|H|^2+g^{j\ell}\nabla_i\nabla_\ell h_{jk}+g^{j\ell}\nabla_j\nabla_k h_{i\ell}-\Delta h_{ik}\big)\\&=\Delta R+2\,|\mathrm{Ric}\,|^2+\tfrac{1}{4}\langle h_{ij},R_{ij}\rangle+\tfrac{1}{4}\langle R_{ip},h_{ip}\rangle\\&\quad-\tfrac{1}{4}\Delta|H|^2+\tfrac{1}{4}g^{ik}g^{j\ell}\nabla_i\nabla_\ell h_{jk}+\tfrac{1}{4}g^{ik}g^{j\ell}\nabla_j\nabla_k h_{i\ell}-\tfrac{1}{4}\Delta|H|^2.\end{aligned}$$

Simplifying the terms, we obtain the required result. □

4. Derivative estimates

In this section we are going to prove BBS estimates. At first we review several basic identities of commutators $[\Delta,\nabla]$ and $[\partial/\partial t,\nabla]$. If $A=A(t)$ is a t-dependency tensor, and $\partial g_{ij}/\partial t=v_{ij}$, then applying the well-known formulas stated in [Chow and Knopf 2004] on GRF we have

$$\begin{aligned}\frac{\partial}{\partial t}\nabla\,\mathrm{Rm}&=\nabla\frac{\partial}{\partial t}\,\mathrm{Rm}+\mathrm{Rm}*\nabla(\mathrm{Rm}+H*H)\\&=\nabla(\Delta\,\mathrm{Rm}+\mathrm{Rm}*\mathrm{Rm}+H*H*\mathrm{Rm}+\nabla^2H*H+\nabla H*\nabla H)+\mathrm{Rm}*\nabla\,\mathrm{Rm}+H*\nabla H*\mathrm{Rm}\\&=\Delta(\nabla\,\mathrm{Rm})+\sum_{i+j=0}\nabla^i\,\mathrm{Rm}*\nabla^j\,\mathrm{Rm}+\sum_{i+j+k=0}\nabla^iH*\nabla^jH*\nabla^k\,\mathrm{Rm}+\sum_{i+j=0+2}\nabla^iH*\nabla^jH.\end{aligned}\tag{4-1}$$

More generally:

Proposition 4.1. *For* GRF *and any nonnegative integer ℓ we have*

$$\frac{\partial}{\partial t}\nabla^\ell\,\mathrm{Rm}=\Delta(\nabla^\ell\,\mathrm{Rm})+\sum_{i+j=\ell}\nabla^i\,\mathrm{Rm}*\nabla^j\,\mathrm{Rm}+\sum_{i+j+k=\ell}\nabla^iH*\nabla^jH*\nabla^k\,\mathrm{Rm}+\sum_{i+j=\ell+2}\nabla^iH*\nabla^jH.\tag{4-2}$$

Proof. For $\ell = 1$, this is (4-1). Suppose (4-2) holds for $1, \dots, \ell$. By induction on ℓ, for $\ell + 1$ we have

$$\begin{aligned}
&\frac{\partial}{\partial t}\nabla^{\ell+1}\operatorname{Rm}\\
&\quad= \frac{\partial}{\partial t}\nabla(\nabla^{\ell}\operatorname{Rm})\\
&\quad= \nabla\frac{\partial}{\partial t}(\nabla^{\ell}\operatorname{Rm}) + \nabla^{\ell}\operatorname{Rm} * \nabla(\operatorname{Rm} + H * H)\\
&\quad= \nabla\Bigg[\Delta(\nabla^{\ell}\operatorname{Rm}) + \sum_{i+j=\ell}\nabla^{i}\operatorname{Rm} * \nabla^{j}\operatorname{Rm} + \sum_{i+j+k=\ell}\nabla^{i}H * \nabla^{j}H * \nabla^{k}\operatorname{Rm} + \sum_{i+j=\ell+2}\nabla^{i}H * \nabla^{j}H\Bigg]\\
&\qquad\qquad + \nabla^{\ell}\operatorname{Rm} * \nabla\operatorname{Rm} + H * \nabla H * \nabla^{\ell}\operatorname{Rm}\\
&\quad= \Delta(\nabla^{\ell+1}\operatorname{Rm}) + \nabla\operatorname{Rm} * \nabla^{\ell}\operatorname{Rm} + \operatorname{Rm} * \nabla^{\ell+1}\operatorname{Rm}\\
&\qquad + \sum_{i+j=\ell}\big(\nabla^{i+1}\operatorname{Rm} * \nabla^{j}\operatorname{Rm} + \nabla^{i}\operatorname{Rm} * \nabla^{j+1}\operatorname{Rm}\big)\\
&\qquad + \sum_{i+j+k=\ell}\big(\nabla^{i+1}H * \nabla^{j}H * \nabla^{k}\operatorname{Rm} + \nabla^{i}H * \nabla^{j+1}H * \nabla^{k}\operatorname{Rm} + \nabla^{i}H * \nabla^{j}H * \nabla^{k+1}\operatorname{Rm}\big)\\
&\qquad + \sum_{i+j=\ell+2}(\nabla^{i+1}H * \nabla^{j}H + \nabla^{i}H * \nabla^{j+1}H) + H * \nabla H * \nabla^{l}\operatorname{Rm}.
\end{aligned}$$

Simplifying these terms, we obtain the required result. □

As an immediate consequence, we have an evolution inequality for $|\nabla^{l}\operatorname{Rm}|^2$.

Corollary 4.2. *For* GRF *and any nonnegative integer ℓ we have*

$$\begin{aligned}
\frac{\partial}{\partial t}|\nabla^{\ell}\operatorname{Rm}|^2 \le \Delta|\nabla^{l}\operatorname{Rm}|^2 - 2|\nabla^{\ell+1}\operatorname{Rm}|^2 + C\sum_{i+j=\ell}|\nabla^{i}\operatorname{Rm}|\cdot|\nabla^{j}\operatorname{Rm}|\cdot|\nabla^{\ell}\operatorname{Rm}|\\
+ C\sum_{i+j+k=\ell}|\nabla^{i}H|\cdot|\nabla^{j}H|\cdot|\nabla^{k}\operatorname{Rm}|\cdot|\nabla^{\ell}\operatorname{Rm}| + C\sum_{i+j=\ell+2}|\nabla^{i}H|\cdot|\nabla^{j}H|\cdot|\nabla^{\ell}\operatorname{Rm}|,
\end{aligned}\tag{4-3}$$

where C represents universal constants depending only on the dimension of M.

Next we derive the evolution equations for the covariant derivatives of H.

Proposition 4.3. *For* GRF *and any positive integer ℓ we have*

$$\frac{\partial}{\partial t}\nabla^{\ell}H = \Delta(\nabla^{\ell}H) + \sum_{i+j=\ell}\nabla^{i}H * \nabla^{j}\operatorname{Rm} + \sum_{i+j+k=\ell}\nabla^{i}H * \nabla^{j}H * \nabla^{k}H. \tag{4-4}$$

Proof. From the Bochner formula, the evolution equation for H can be rewritten as

$$\frac{\partial}{\partial t}H = \Delta H + \operatorname{Rm} * H. \tag{4-5}$$

For $\ell = 1$, we have

$$\begin{aligned}\frac{\partial}{\partial t}\nabla H &= \nabla\frac{\partial}{\partial t}H + H * \nabla(\mathrm{Rm} + H * H)\\ &= \nabla(\Delta H + \mathrm{Rm} * H) + H * \nabla\,\mathrm{Rm} + H * H * \nabla H\\ &= \nabla(\Delta H) + H * \nabla\,\mathrm{Rm} + \nabla H * \mathrm{Rm} + H * H * \nabla H\\ &= \Delta(\nabla H) + \nabla\,\mathrm{Rm} * H + \nabla H * \mathrm{Rm} + H * H * \nabla H.\end{aligned}$$

Using (4-2) and the same argument, we can prove the evolution equation for higher covariant derivatives. □

Similarly, we have an evolution inequality for $|\nabla^\ell H|^2$.

Corollary 4.4. *For* GRF *and for any positive integer l we have*

$$\frac{\partial}{\partial t}|\nabla^l H|^2 \leq \Delta|\nabla^\ell H|^2 - 2\,|\nabla^{\ell+1}H|^2 + C\sum_{i+j=\ell}|\nabla^i H|\cdot|\nabla^j\,\mathrm{Rm}\,|\cdot|\nabla^\ell H| + C\sum_{i+j+k=\ell}|\nabla^i H|\cdot|\nabla^j H|\cdot|\nabla^k H|\cdot|\nabla^l H|, \tag{4-6}$$

while

$$\frac{\partial}{\partial t}|H|^2 \leq \Delta|H|^2 - 2\,|\nabla H|^2 + C\cdot|\mathrm{Rm}\,|\cdot|H|^2. \tag{4-7}$$

Theorem 4.5. *Suppose that $(g(x,t), H(x,t))$ is a solution to* GRF *on a closed manifold M^n for a short time $0 \leq t \leq T$ and K_1, K_2 are arbitrary given nonnegative constants. Then there exists a constant C_n depending only on n such that if*

$$|\mathrm{Rm}(x,t)|_{g(x,t)} \leq K_1, \quad |H(x)|_{g(x)} \leq K_2$$

for all $x \in M$ and $t \in [0,T]$, then

$$|H(x,t)|_{g(x,t)} \leq K_2 e^{C_n K_1 t} \tag{4-8}$$

for all $x \in M$ and $t \in [0,T]$.

Proof. Since

$$\frac{\partial}{\partial t}|H|^2 \leq \Delta|H|^2 + C_n|\mathrm{Rm}\,|\cdot|H|^2 \leq \Delta|H|^2 + C_n K_1|H|^2,$$

using the maximum principle, we obtain $u(t) \leq u(0)e^{C_n K_1 t}$, where $u(t) = |H|^2$. □

The main result in this section is the following estimates for higher derivatives of Riemann curvature tensors and three-forms. Some special cases were proved in [Streets 2007; 2008; Young 2008].

Theorem 4.6. *Suppose that $(g(x,t), H(x,t))$ is a solution to* GRF *on a compact manifold M^n and K is an arbitrary given positive constant. Then for each $\alpha > 0$ and each integer $m \geq 1$ there exists a constant C_m depending on $m, n, \max\{\alpha, 1\}$, and K such that if*

$$|\mathrm{Rm}(x,t)|_{g(x,t)} \leq K, \quad |H(x)|_{g(x)} \leq K$$

for all $x \in M$ and $t \in [0, \alpha/K]$, then

$$|\nabla^{m-1} \mathrm{Rm}(x,t)|_{g(x,t)} + |\nabla^m H(x,t)|_{g(x,t)} \leq \frac{C_m}{t^{m/2}} \tag{4-9}$$

for all $x \in M$ and $t \in (0, \alpha/K]$.

Proof. In the following computations we always let C be any constants depending on $n, m, \max\{\alpha, 1\}$,and K, which may take different values at different places. From the evolution equations and Theorem 4.5, we have

$$\begin{aligned}
\frac{\partial}{\partial t}|\mathrm{Rm}|^2 &\leq \Delta|\mathrm{Rm}|^2 - 2|\nabla \mathrm{Rm}|^2 + C + C|\nabla^2 H| + C|\nabla H|^2, \\
\frac{\partial}{\partial t}|H|^2 &\leq \Delta|H|^2 - 2|\nabla H|^2 + C, \\
\frac{\partial}{\partial t}|\nabla H|^2 &\leq \Delta|\nabla H|^2 - 2|\nabla^2 H|^2 + C|\nabla \mathrm{Rm}|\cdot|\nabla H| + C|\nabla H|^2.
\end{aligned}$$

Consider the function $u = t|\nabla H|^2 + \gamma|H|^2 + t|\mathrm{Rm}|^2$. Directly computing, we obtain

$$\begin{aligned}
\frac{\partial}{\partial t}u &\leq \Delta u - 2t|\nabla^2 H|^2 + Ct|\nabla^2 H| + (C - 2\gamma)|\nabla H|^2 + C + C\gamma - 2t|\nabla \mathrm{Rm}|^2 + Ct\cdot|\nabla \mathrm{Rm}|\cdot|\nabla H| \\
&\leq \Delta u + 2(C - \gamma)\cdot|\nabla H|^2 + C(1 + \gamma).
\end{aligned}$$

If we choose $\gamma = C$, then $\frac{\partial}{\partial t}u \leq \Delta u + C$ which implies that $u \leq Ce^{Ct}$ since $u(0) \leq C$. With this estimate we are able to bound the first covariant derivative of Rm and the second covariant derivative of H. In order to control the term $|\nabla \mathrm{Rm}|^2$, we should use the evolution equations of $|H|^2$, $|\nabla H|^2$ and $|\nabla^2 H|^2$ to cancel with the bad terms, i.e., $|\nabla^2 \mathrm{Rm}|^2$, $|\nabla^2 H|^2$, and $|\nabla^3 H|^2$, in the evolution equation of $|\nabla \mathrm{Rm}|^2$:

$$\begin{aligned}
&\frac{\partial}{\partial t}|\nabla \mathrm{Rm}|^2 \\
&\quad\leq \Delta|\nabla \mathrm{Rm}|^2 - 2|\nabla^2 \mathrm{Rm}|^2 + C|\nabla \mathrm{Rm}|^2 + \frac{C}{t^{1/2}}|\nabla \mathrm{Rm}| + C\cdot|\nabla \mathrm{Rm}|\cdot|\nabla^3 H| + \frac{C}{t^{1/2}}|\nabla^2 H|\cdot|\nabla \mathrm{Rm}|,
\end{aligned}$$

$$\frac{\partial}{\partial t}|\nabla^2 H|^2 \leq \Delta|\nabla^2 H|^2 - 2|\nabla^3 H|^2 + C\cdot|\nabla^2 \mathrm{Rm}|\cdot|\nabla^2 H| + \frac{C}{t^{1/2}}|\nabla \mathrm{Rm}|\cdot|\nabla^2 H| + C|\nabla^2 H|^2 + \frac{C}{t}|\nabla^2 H|.$$

As above, we define

$$u := t^2(|\nabla^2 H|^2 + |\nabla \mathrm{Rm}|^2) + t\beta(|\nabla H|^2 + |\mathrm{Rm}|^2) + \gamma|H|^2,$$

and therefore, $\dfrac{\partial u}{\partial t} \leq \Delta u + C$. Motivated by cases for $m = 1$ and $m = 2$, for general m, we can define a function

$$u := t^m(|\nabla^m H|^2 + |\nabla^{m-1} \mathrm{Rm}|^2) + \sum_{i=1}^{m-1} \beta_i t^i (|\nabla^i H|^2 + |\nabla^{i-1} \mathrm{Rm}|^2) + \gamma|H|^2,$$

where β_i and γ are positive constants determined later. In the following, we always assume $m \geq 3$.

Suppose that $|\nabla^{i-1}\operatorname{Rm}|+|\nabla^i H|\le \frac{C_i}{t^{i/2}}$, for $i=1,2,\dots,m-1$. For such i, from Corollary 4.4, we have

$$\begin{aligned}
\frac{\partial}{\partial t}|\nabla^i H|^2 &\le \Delta|\nabla^i H|^2-2|\nabla^{i+1}H|^2+C\sum_{j=0}^{i}|\nabla^j H|\cdot|\nabla^{i-j}\operatorname{Rm}|\cdot|\nabla^i H|\\
&\qquad + C\sum_{j=0}^{i}\sum_{\ell=0}^{i-j}|\nabla^j H|\cdot|\nabla^{i-j-\ell}H|\cdot|\nabla^\ell H|\cdot|\nabla^i H|\\
&\le \Delta|\nabla^i H|^2-2|\nabla^{i+1}H|^2+C\cdot|\nabla^i H|\sum_{j=0}^{i}\frac{C_j}{t^{\frac{j}{2}}}\cdot\frac{C_{i-j+1}}{t^{\frac{i-j+1}{2}}}+C\cdot|\nabla^i H|\sum_{j=0}^{i}\sum_{\ell=0}^{i-j}\frac{C_j}{t^{\frac{j}{2}}}\cdot\frac{C_{i-j-\ell}}{t^{\frac{i-j-1}{2}}}\cdot\frac{C_\ell}{t^{\frac{l}{2}}}\\
&\le \Delta|\nabla^i H|^2-2|\nabla^{i+1}H|^2+\frac{C_i}{t^{\frac{i+1}{2}}}|\nabla^i H|+\frac{C_i}{t^{\frac{i}{2}}}|\nabla^i H|.
\end{aligned}$$

Similarly, from Corollary 4.2 we also have

$$\begin{aligned}
\frac{\partial}{\partial t}|\nabla^{i-1}\operatorname{Rm}|^2 &\le \Delta|\nabla^{i-1}\operatorname{Rm}|^2-2|\nabla^i\operatorname{Rm}|^2+C\sum_{j=0}^{i-1}|\nabla^j\operatorname{Rm}||\nabla^{i-1-j}\operatorname{Rm}||\nabla^{i-1}\operatorname{Rm}|\\
&\quad + C\sum_{j=0}^{i-1}\sum_{\ell=0}^{i-1-j}|\nabla^j H|\cdot|\nabla^{i-1-j-\ell}H|\cdot|\nabla^\ell\operatorname{Rm}|\cdot|\nabla^{i-1}\operatorname{Rm}|\\
&\quad + C\sum_{j=0}^{i+1}|\nabla^j H|\cdot|\nabla^{i+1-j}H|\cdot|\nabla^{i-1}\operatorname{Rm}|\\
&\le \Delta|\nabla^{i-1}\operatorname{Rm}|^2-2|\nabla^i\operatorname{Rm}|^2+C\cdot|\nabla^{i-1}\operatorname{Rm}|\sum_{j=0}^{i-1}\frac{C_{j+1}}{t^{\frac{j+1}{2}}}\cdot\frac{C_{i-j}}{t^{\frac{i-j}{2}}}\\
&\quad + C\cdot|\nabla^{i-1}\operatorname{Rm}|\sum_{j=0}^{i-1}\sum_{\ell=0}^{i-1-j}\frac{C_j}{t^{\frac{j}{2}}}\cdot\frac{C_{i-1-j-\ell}}{t^{\frac{i-1-j-\ell}{2}}}\cdot\frac{C_{\ell+1}}{t^{\frac{\ell+1}{2}}}\\
&\quad + C\cdot|\nabla^{i-1}\operatorname{Rm}|\sum_{j=1}^{i}\frac{C_j}{t^{\frac{j}{2}}}\cdot\frac{C_{i+1-j}}{t^{\frac{i+1-j}{2}}}+C\cdot|\nabla^{i+1}H|\cdot\frac{C_i}{t^{\frac{i}{2}}}\\
&\le \Delta|\nabla^{i-1}\operatorname{Rm}|^2-2|\nabla^i\operatorname{Rm}|^2+\frac{C_i}{t^{\frac{i+1}{2}}}\cdot|\nabla^{i-1}\operatorname{Rm}|+\frac{C_i}{t^{\frac{i}{2}}}|\nabla^{i+1}H|+\frac{C_i}{t^{\frac{i}{2}}}|\nabla^{i-1}\operatorname{Rm}|.
\end{aligned}$$

The evolution inequality for u is now given by

$$\begin{aligned}
\frac{\partial u}{\partial t} &\le mt^{m-1}(|\nabla^m H|^2+|\nabla^{m-1}\operatorname{Rm}|^2)+\sum_{i=1}^{m-1}i\beta_i t^{i-1}(|\nabla^i H|^2+|\nabla^{i-1}\operatorname{Rm}|^2)\\
&\quad + t^m\left(\frac{\partial}{\partial t}|\nabla^m H|^2+\frac{\partial}{\partial t}|\nabla^{m-1}\operatorname{Rm}|^2\right)+\sum_{i=1}^{m-1}\beta_i t^i\left(\frac{\partial}{\partial t}|\nabla^i H|^2+\frac{\partial}{\partial t}|\nabla^{i-1}\operatorname{Rm}|^2\right)+\gamma\cdot\frac{\partial}{\partial t}|H|^2.
\end{aligned}$$

It's easy to see that the second term is bounded by

$$\sum_{i=1}^{m-1} i\beta_i t^{i-1}\frac{C_i}{t^i} = \sum_{i=1}^{m-1} i\beta_i C_i t^{-1},$$

but this bound depends on t and approaches to infinity when t goes to zero. Hence we use the last second term to control this bad term. The evolution inequality for the third term is the combination of the inequalities

$$\begin{aligned}
&\frac{\partial}{\partial t}|\nabla^m H|^2\\
&\quad\le \Delta|\nabla^m H|^2 - 2\,|\nabla^{m+1}H|^2 + C\sum_{i=0}^{m}|\nabla^i H|\cdot|\nabla^{m-i}\operatorname{Rm}|\cdot|\nabla^m H|\\
&\qquad\qquad\qquad\qquad + C\sum_{i=0}^{m}\sum_{j=0}^{m-i}|\nabla^j H|\cdot|\nabla^{m-i-j}H|\cdot|\nabla^i H|\cdot|\nabla^m H|\\
&\quad\le \Delta|\nabla^m H|^2 - 2\,|\nabla^{m+1}H|^2 + C|\nabla^m H|^2 + C\cdot|\nabla^m\operatorname{Rm}|\cdot|\nabla^m H| + \frac{C_m}{t^{\frac{m+1}{2}}}|\nabla^m H| + \frac{C_m}{t^{\frac{m}{2}}}|\nabla^m H|
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial}{\partial t}|\nabla^{m-1}\operatorname{Rm}|^2 &\le \Delta|\nabla^{m-1}\operatorname{Rm}|^2 - 2\,|\nabla^m\operatorname{Rm}|^2 + C\sum_{i=0}^{m-1}|\nabla^i\operatorname{Rm}|\cdot|\nabla^{m-1-i}\operatorname{Rm}|\cdot|\nabla^{m-1}\operatorname{Rm}|\\
&\quad + C\sum_{i=0}^{m-1}\sum_{j=0}^{m-1-i}|\nabla^j H|\cdot|\nabla^{m-1-i-j}H|\cdot|\nabla^i\operatorname{Rm}|\cdot|\nabla^{m-1}\operatorname{Rm}|\\
&\quad + C\sum_{i=0}^{m+1}|\nabla^i H|\cdot|\nabla^{m+1-i}H|\cdot|\nabla^{m-1}\operatorname{Rm}|\\
&\le \Delta|\nabla^{m-1}\operatorname{Rm}|^2 - 2\,|\nabla^m\operatorname{Rm}|^2 + C|\nabla^{m-1}\operatorname{Rm}|^2 + \frac{C}{t^{\frac{1}{2}}}\cdot|\nabla^m H|\cdot|\nabla^{m-1}\operatorname{Rm}|\\
&\quad + C|\nabla^{m+1}H||\nabla^{m-1}\operatorname{Rm}| + \frac{C_m}{t^{\frac{m+1}{2}}}|\nabla^{m-1}\operatorname{Rm}| + \frac{C_m}{t^{\frac{m}{2}}}|\nabla^{m-1}\operatorname{Rm}|.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
\frac{\partial u}{\partial t} &\le m t^{m-1}(|\nabla^m H|^2 + |\nabla^{m-1}\operatorname{Rm}|^2) + \sum_{i=1}^{m-1} i\beta_i t^{i-1}(|\nabla^i H|^2 + |\nabla^{i-1}\operatorname{Rm}|^2)\\
&\quad + t^m\Bigg(\Delta|\nabla^m H|^2 - 2\,|\nabla^{m+1}H|^2 + \frac{C}{t^{\frac{m+1}{2}}}|\nabla^m H| + C|\nabla^m H|^2\\
&\qquad + C|\nabla^m\operatorname{Rm}|\cdot|\nabla^m H| + \Delta|\nabla^{m-1}\operatorname{Rm}|^2\\
&\qquad - 2\,|\nabla^m\operatorname{Rm}|^2 + \frac{C}{t^{\frac{m+1}{2}}}|\nabla^{m-1}\operatorname{Rm}| + C|\nabla^{m-1}\operatorname{Rm}|^2\\
&\qquad + \frac{C}{t^{1/2}}|\nabla^m H|\cdot|\nabla^{m-1}\operatorname{Rm}| + C|\nabla^{m+1}H|\cdot|\nabla^{m-1}\operatorname{Rm}|\Bigg)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{m-1}\beta_i t^i\bigg(\frac{C_i}{t^{\frac{i+1}{2}}}|\nabla^{i-1}\operatorname{Rm}|+\Delta|\nabla^i H|^2-2|\nabla^{i+1}H|^2\\
&\qquad\qquad+\Delta|\nabla^{i-1}\operatorname{Rm}|^2+\frac{C_i}{t^{\frac{i+1}{2}}}|\nabla^i H|+\frac{C_i}{t^{\frac{i}{2}}}|\nabla^{i+1}H|-2|\nabla^i\operatorname{Rm}|^2\bigg)\\
&+\gamma(\Delta|H|^2-2|\nabla H|^2+C)\\
\le{}&\Delta u-2t^m|\nabla^{m+1}H|^2+Ct^m|\nabla^{m+1}H|\cdot|\nabla^{m-1}\operatorname{Rm}|\\
&-2t^m|\nabla^m\operatorname{Rm}|^2+Ct^m|\nabla^m\operatorname{Rm}|\cdot|\nabla^m H|+\sum_{i=0}^{m-2}(i+1)\beta_{i+1}t^i(|\nabla^{i+1}H|^2+|\nabla^i\operatorname{Rm}|^2)\\
&-2\sum_{i=1}^{m-1}\beta_i t^i(|\nabla^{i+1}H|^2+|\nabla^i\operatorname{Rm}|^2)-2\gamma|\nabla H|^2+\gamma C\\
&+Ct^{m-1}|\nabla^m H|^2+Ct^{m-1}|\nabla^{m-1}\operatorname{Rm}|^2+jt^{\frac{m-1}{2}}|\nabla^m H|+Ct^{\frac{m-1}{2}}|\nabla^{m-1}\operatorname{Rm}|\\
&+Ct^{m-\frac{1}{2}}|\nabla^m H|\cdot|\nabla^{m-1}\operatorname{Rm}|+Ct^m|\nabla^{m+1}H|\cdot|\nabla^{m-1}\operatorname{Rm}|\\
&+\sum_{i=1}^{m-1}\beta_i C_i t^{\frac{i}{2}}|\nabla^{i+1}H|+\sum_{i=1}^{m-1}\beta_i C_i t^{\frac{i-1}{2}}(|\nabla^i H|^2+|\nabla^{i-1}\operatorname{Rm}|).
\end{aligned}
$$

Choosing

$$
(i+1)\beta_{i+1}=\beta_i,\quad \beta_i=\frac{A}{i!},\quad i\ge 0,
$$

where A is constant which is determined later, and noting that

$$
\sum_{i=1}^{m-1}\beta_i C_i t^{i/2}|\nabla^{i+1}H|\le\frac{1}{2}\sum_{i=1}^{m-1}\beta_i t^i|\nabla^{i+1}H|^2+\frac{1}{2}\sum_{i=1}^{m-1}\beta_i C_i^2
$$

and

$$
\begin{aligned}
&\sum_{i=1}^{m-1}\beta_i C_i t^{\frac{i-1}{2}}(|\nabla^i H|+|\nabla^{i-1}\operatorname{Rm}|)\\
&\qquad\le\beta_1 C_1(|\nabla H|+|\operatorname{Rm}|)+\sum_{i=1}^{m-2}\beta_{i+1}C_{i+1}t^{\frac{i}{2}}(|\nabla^{i+1}H|+|\nabla^i\operatorname{Rm}|)\\
&\qquad\le\beta_1 C_1(|\nabla H|+|\operatorname{Rm}|)+\sum_{i=1}^{m-2}\beta_{i+1}C_{i+1}\bigg(\frac{t^i|\nabla^{i+1}H|^2+t^i|\nabla^i\operatorname{Rm}|^2}{2\beta_{i+1}C_{i+1}/\beta_i}+\frac{\beta_{i+1}C_{i+1}}{\beta_i}\bigg)\\
&\qquad\le\beta_1 C_1(|\nabla H|+|\operatorname{Rm}|)+\frac{1}{2}\sum_{i=1}^{m-2}\beta_i t^i(|\nabla^{i+1}H|^2+|\nabla^i\operatorname{Rm}|^2)+\sum_{i=1}^{m-2}\frac{\beta_{i+1}^2C_{i+1}^2}{\beta_i},
\end{aligned}
$$

we obtain

$$\begin{aligned}
\frac{\partial}{\partial t}u \le{}& \Delta u - 2t^m|\nabla^{m+1}H|^2 + Ct^m|\nabla^{m+1}H|\cdot|\nabla^{m-1}\operatorname{Rm}| \\
&- 2t^m|\nabla^m \operatorname{Rm}|^2 + Ct^m|\nabla^m H|\cdot|\nabla^m \operatorname{Rm}| + Ct^{m-1}|\nabla^m H|^2 + Ct^{m-1}|\nabla^{m-1}\operatorname{Rm}|^2 \\
&+ Ct^{m-\frac{1}{2}}|\nabla^m H|\cdot|\nabla^{m-1}\operatorname{Rm}| + \beta_0(|\nabla H|^2 + |\operatorname{Rm}|^2) \\
&- \sum_{i=1}^{m-1}\beta_i t^i(|\nabla^{i+1}H|^2 + |\nabla^i \operatorname{Rm}|^2) + \sum_{i=1}^{m-2}\beta_i t^i(|\nabla^{i+1}H|^2 + |\nabla^i \operatorname{Rm}|^2) \\
&+ \tfrac{1}{2}\beta_{m-1}t^{m-1}|\nabla^m H|^2 + \beta_1 C_1|\nabla H| - 2\gamma|\nabla H|^2 + C + C\gamma \\
\le{}& \Delta u + Ct^{m-1}|\nabla^{m-1}\operatorname{Rm}|^2 + Ct^{m-1}|\nabla^m H|^2 Ct^{m-\frac{1}{2}}(|\nabla^m H|^2 + |\nabla^{m-1}\operatorname{Rm}|^2) + \beta_0|\nabla H|^2 \\
&+ \beta_1 C_1|\nabla H| - 2\gamma|\nabla H|^2 + C + C\gamma - \tfrac{1}{2}\beta_{m-1}t^{m-1}|\nabla^m H|^2 - \beta_{m-1}t^{m-1}|\nabla^m \operatorname{Rm}|^2 \\
\le{}& \Delta u + \tfrac{1}{2}(C\sqrt{t} + C - \beta_{m-1})t^{m-1}(|\nabla^{m-1}\operatorname{Rm}|^2 + |\nabla^m H|^2) \\
&+ (\beta_0 + \beta_1 C_1 - 2\gamma)|\nabla H|^2 + C + C\gamma + \beta_1 C_1.
\end{aligned}$$

When we chose A and γ sufficiently large, we obtain $\frac{\partial u}{\partial t} \le \Delta u + C$, which implies that $u(t) \le C$ since $u(0)$ is bounded. □

Finally we give an estimate that plays a crucial role in the next section.

Corollary 4.7. *Let $(g(x,t), H(x,t))$ be a solution of the generalized Ricci flow on a closed manifold M. If there are $\beta > 0$ and $K > 0$ such that*

$$|\operatorname{Rm}(x,t)|_{g(x,t)} \le K, \quad |H(x)|_{g(x)} \le K$$

for all $x \in M$ and $t \in [0,T]$, where $T > \beta/K$, then there exists for each $m \in \mathbb{N}$ a constant C_m depending on $m, n, \min\{\beta, 1\}$, and K such that

$$|\nabla^{m-1}\operatorname{Rm}(x,t)|_{g(x,t)} + |\nabla^m H(x,t)|_{g(x,t)} \le C_m K^{m/2}$$

for all $x \in M$ and $t \in [\min\{\beta,1\}/K, T]$.

Proof. The proof is the same as in [Chow et al. 2007]; we just copy it here. Let $\beta_1 := \min\{\beta, 1\}$. For any fixed point $t_0 \in [\beta_1/K, T]$ we set $T_0 := t_0 - \beta_1/K$. For $\bar{t} := t - T_0$ we let $(\bar{g}(\bar{t}), \bar{H}(\bar{t}))$ be the solution of the system

$$\frac{\partial \bar{g}}{\partial \bar{t}} = -2\overline{\operatorname{Ric}} + \tfrac{1}{2}\bar{h}, \quad \frac{\partial \bar{H}}{\partial \bar{t}} = \Delta_{\mathrm{HL},\bar{g}}\bar{H}, \quad \bar{g}(0) = g(T_0), \quad \bar{H}(0) = H(T_0).$$

The uniqueness of solution implies that $\bar{g}(\bar{t}) = g(\bar{t} + T_0) = g(t)$ for $\bar{t} \in [0, \beta_1/K]$. By the assumption we have

$$|\overline{\operatorname{Rm}}(x,\bar{t})|_{\bar{g}(x,\bar{t})} \le K, \quad |\bar{H}(x)|_{\bar{g}(x)} \le K$$

for all $x \in M$ and $\bar{t} \in [0, \beta_1/K]$. Applying Theorem 4.5 with $\alpha = \beta_1$, we have

$$|\bar{\nabla}^{m-1}\overline{\operatorname{Rm}}(x,\bar{t})|_{\bar{g}(x,\bar{t})} + |\bar{\nabla}^m H(x,\bar{t})|_{\bar{g}(x,\bar{t})} \le \frac{\bar{C}_m}{\bar{t}^{m/2}}$$

for all $x \in M$ and $\bar{t} \in (0, \beta_1/K]$. We have $\bar{t}^{m/2} \geq \beta_1^{m/2} 2^{-m/2} K^{-m/2}$ if $\bar{t} \in [\beta_1/2K, \beta_1/K]$. Taking $\bar{t} = \beta_1/K$, we obtain

$$|\nabla^{m-1}\operatorname{Rm}(x,t_0)|_{g(x,t_0)} + |\nabla^m H(x,t_0)|_{g(x,t_0)} \leq \frac{2^{m/2}\overline{C}_m K^{m/2}}{\beta_1^{m/2}}$$

for all $x \in M$. Since $t_0 \in [\beta/K, T]$ was arbitrary, the result follows. □

5. Compactness theorem

In this section we prove the compactness theorem for our generalized Ricci flow. We follow [Hamilton 1995] on the compactness theorem for the usual Ricci flow.

We review several definitions from [Chow et al. 2007]. Throughout this section, all Riemannian manifolds are smooth manifolds of dimensions n. The covariant derivative with respect to a metric g will be denoted by ${}^g\nabla$.

Definition 5.1. Let $K \subset M$ be a compact set and let $\{g_k\}_{k\in\mathbb{N}}$, g_∞, and g be Riemannian metrics on M. For $p \in \{0\} \cup \mathbb{N}$ we say that g_k *converges in C^p to g_∞ uniformly on K with respect to g* if for every $\epsilon > 0$ there exists $k_0 = k_0(\epsilon) > 0$ such that for $k \geq k_0$,

$$\|g_k - g_\infty\|_{C^p;K,g} := \sup_{0\leq\alpha\leq p}\sup_{x\in K} |{}^g\nabla^\alpha(g_k - g_\infty)(x)|_g < \epsilon. \tag{5-1}$$

Since we consider a compact set, the choice of background metric g does not change the convergence. Hence we may choose $g = g_\infty$.

Definition 5.2. Suppose $\{U_k\}_{k\in\mathbb{N}}$ is an exhaustion[2] of a smooth manifold M by open sets and g_k are Riemannian metrics on U_k. We say that (U_k, g_k) *converges in C^∞ to (M, g_∞) uniformly on compact sets in M* if for any compact set $K \subset M$ and any $p > 0$ there exists $k_0 = k_0(K, p)$ such that $\{g_k\}_{k\geq k_0}$ converges in C^p to g_∞ uniformly on K.

A *pointed Riemannian manifold* is a 3-tuple (M, g, O), where (M, g) is a Riemannian manifold and $O \in M$ is a basepoint. If the metric g is complete, the 3-tuple is called a *complete pointed Riemannian manifold.* We say $(M, g(t), H(t), O), t \in (\alpha, \omega)$, is a *pointed solution to the generalized Ricci flow* if $(M, g(t), H(t))$ is a solution to the generalized Ricci flow.

The so-called *Cheeger–Gromov convergence* in C^∞ is defined as follows:

Definition 5.3. A given sequence $\{(M_k, g_k, O_k)\}_{k\in\mathbb{N}}$ of complete pointed Riemannian manifolds *converges* to a complete pointed Riemannian manifold $(M_\infty, g_\infty, O_\infty)$ if there exist

(i) an exhaustion $\{U_k\}_{k\in\mathbb{N}}$ of M_∞ by open sets with $O_\infty \in U_k$, and

(ii) a sequence of diffeomorphisms $\Phi_k : M_\infty \ni U_k \to V_k := \Phi_k(U_k) \subset M_k$ with $\Phi_k(O_\infty) = O_k$

such that $(U_k, \Phi_k^*(g_k|_{V_k}))$ converges in C^∞ to (M_∞, g_∞) uniformly on compact sets in M_∞.

[2] If for any compact set $K \subset M$ there exists $k_0 \in \mathbb{N}$ such that $U_k \supset K$ for all $k \geq k_0$

The corresponding convergence for the generalized Ricci flow is similar to the convergence for the usual Ricci flow introduced by Hamilton [1995].

Definition 5.4. A given sequence $\{(M_k, g_k(t), H_k(t), O_k)\}_{k\in\mathbb{N}}$ of complete pointed solutions to the GRF *converges* to a complete pointed solution to the GRF

$$(M_\infty, g_\infty(t), H_\infty(t), O_\infty), \quad t \in (\alpha, \omega),$$

if there exist

(i) an exhaustion $\{U_k\}_{k\in\mathbb{N}}$ of M_∞ by open sets with $O_\infty \in U_k$,

(ii) a sequence of diffeomorphisms $\Phi_k : M_\infty \ni U_k \to V_k := \Phi_k(U_k) \subset M_k$ with $\Phi_k(O_\infty) = O_k$

such that $\big(U_k \times (\alpha, \omega), \Phi_k^*(g_k(t)|_{V_k}) + dt^2, \Phi_k^*(H_k(t)|_{V_k})\big)$ converges in C^∞ to

$$\big(M_\infty \times (\alpha, \omega), g_\infty(t) + dt^2, H_\infty(t)\big)$$

uniformly on compact sets in $M_\infty \times (\alpha, \omega)$. Here we denote by dt^2 the standard metric on (α, ω).

Let $\operatorname{inj}_g(O)$ be the injectivity radius of the metric g at the point O. The following compactness theorem is due to Cheeger and Gromov.

Theorem 5.5 (compactness for metrics). *Let $\{(M_k, g_k, O_k)\}_{k\in\mathbb{N}}$ be a sequence of complete pointed Riemannian manifolds satisfying these conditions*:

(i) *For all $p \geq 0$ and $k \in \mathbb{N}$, there is a sequence of constants $C_p < \infty$ independent of k such that*

$$|{}^{g_k}\nabla^p \operatorname{Rm}(g_k)|_{g_k} \leq C_p$$

on M_k.

(ii) *There exists some constant $\iota_0 > 0$ such that*

$$\operatorname{inj}_{g_k}(O_k) \geq \iota_0$$

for all $k \in \mathbb{N}$.

Then there exists a subsequence $\{j_k\}_{k\in\mathbb{N}}$ such that $\{(M_{j_k}, g_{j_k}, O_{j_k})\}_{k\in\mathbb{N}}$ converges to a complete pointed Riemannaian manifold $(M^n_\infty, g_\infty, O_\infty)$ as $k \to \infty$.

As a consequence of Theorem 5.5, we state our compactness theorem for GRF.

Theorem 5.6 (compactness for GRF). *Let $\{(M_k, g_k(t), H_k(t), O_k)\}_{k\in\mathbb{N}}$ be a sequence of complete pointed solutions to* GRF *for $t \in [\alpha, \omega) \ni 0$ satisfying these conditions*:

(i) *There is a constant $C_0 < \infty$ independent of k such that*

$$\sup_{(x,t)\in M_k\times(\alpha,\omega)} |\operatorname{Rm}(g_k(x,t))|_{g_k(x,t)} \leq C_0, \quad \sup_{x\in M_k} |H_k(x,\alpha)|_{g_k(x,\alpha)} \leq C_0.$$

(ii) *There exists a constant $\iota_0 > 0$ satisfying*

$$\operatorname{inj}_{g_k(0)}(O_k) \geq \iota_0.$$

Then there exists a subsequence $\{j_k\}_{k\in\mathbb{N}}$ *such that*

$$(M_{j_k}, g_{j_k}(t), H_{j_k}(t), O_{j_k}) \to (M_\infty, g_\infty(t), H_\infty(t), O_\infty),$$

converges to a complete pointed solution $(M_\infty, g_\infty(t), H_\infty(t), O_\infty)$, $t \in [\alpha, \omega)$, *to* GRF *as* $k \to \infty$.

To prove Theorem 5.6 we extend a lemma for Ricci flow to GRF. After establishing this lemma, the proof of Theorem 5.6 is similar to that of Theorem 3.10 in [Chow et al. 2007].

Lemma 5.7. *Let* (M, g) *be a Riemannian manifold with a background metric* g, *let* K *be a compact subset of* M, *and let* $(g_k(x,t), H_k(x,t))$ *be a collection of solutions to the generalized Ricci flow defined on neighborhoods of* $K \times [\beta, \psi]$, *where* $t_0 \in [\beta, \psi]$ *is a fixed time. Suppose that*:

(i) *The metrics* $g_k(x, t_0)$ *are all uniformly equivalent to* $g(x)$ *on* K, *i.e., for all* $V \in T_x M$, k, *and* $x \in K$,

$$C^{-1} g(x)(V,V) \le g_k(x,t_0)(V,V) \le C g(x)(V,V),$$

where $C < \infty$ *is a constant independent of* V, k, *and* x.

(ii) *The covariant derivatives of the metrics* $g_k(x,t_0)$ *with respect to the metric* $g(x)$ *are all uniformly bounded on* K, *i.e., for all* k *and* $p \ge 1$,

$$|{}^g\nabla^p g_k(x,t_0)|_{g(x)} + |{}^g\nabla^{p-1} H_k(x,t_0)|_{g(x)} \le C_p$$

where $C_p < \infty$ *is a sequence of constants independent of* k.

(iii) *The covariant derivatives of the curvature tensors* $\mathrm{Rm}(g_k(x,t))$ *and of the forms* $H_k(x,t)$ *are uniformly bounded with respect to the metric* $g_k(x,t)$ *on* $K \times [\beta, \psi]$, *i.e., for all* k *and* $p \ge 0$,

$$|{}^{g_k}\nabla^p \,\mathrm{Rm}(g_k(x,t))|_{g_k(x,t)} + |{}^{g_k}\nabla^p H_k(x,t)|_{g_k(x,t)} \le C'_p$$

where C'_p *is a sequence of constants independent of* k.

Then the metrics $g_k(x,t)$ *are uniformly equivalent to* $g(x)$ *on* $K \times [\beta, \psi]$, *i.e.,*

$$B(t,t_0)^{-1} g(x)(V,V) \le g_k(x,t)(V,V) \le B(t,t_0) g(x)(V,V),$$

where $B(t,t_0) = Ce^{C'_0|t-t_0|}$ (*here the constant* C'_0 *may not be equal to the previous one*), *and the time-derivatives and covariant derivatives of the metrics* $g_k(x,t)$ *with respect to the metric* $g(x)$ *are uniformly bounded on* $K \times [\beta, \psi]$, *i.e., for each* (p,q) *there is a constant* $\widetilde{C}_{p,q}$ *independent of* k *such that*

$$\left|\frac{\partial^q}{\partial t^q}{}^g\nabla^p g_k(x,t)\right|_{g(x)} + \left|\frac{\partial^q}{\partial t^q}{}^g\nabla^{p-1} H_k(x,t)\right|_{g(x)} \le \widetilde{C}_{p,q}$$

for all k.

Proof. We use [Chow et al. 2007, Lemma 3.13]: Suppose that the metrics g_1 and g_2 are equivalent, i.e., $C^{-1}g_1 \le g_2 \le Cg_1$. Then for any (p,q)-tensor T we have $|T|_{g_2} \le C^{(p+q)/2}|T|_{g_1}$. We denote by h

the tensor $h_{ij} := g^{kp}g^{lq}H_{ikl}H_{jpq}$. In the following we denote by C a constant depending only on n, β, and ψ, which may take different values at different places. For any tangent vector $V \in T_xM$ we have

$$\frac{\partial}{\partial t}g_k(x,t)(V,V) = -2\operatorname{Ric}(g_k(x,t))(V,V) + \tfrac{1}{2}h_k(x,t)(V,V),$$

and therefore

$$\begin{aligned}\left|\frac{\partial}{\partial t}\log g_k(x,t)(V,V)\right| &= \left|\frac{-2\operatorname{Ric}(g_k(x,t))(V,V)+\frac{1}{2}h_k(x,t)(V,V)}{g_k(x,t)(V,V)}\right|\\ &\leq C_0' + C|H_k(x,t)|^2_{g_k(x,t)}\\ &\leq C_0' + CC_0'^2 =: \overline{C},\end{aligned}$$

since

$$|\operatorname{Ric}(g_k(x,t))(V,V)| \leq C_0' g_k(x,t)(V,V), \quad |h_k(x,t)(V,V)| \leq C|H_k(x,t)|^2_{g_k(x,t)}g_k(x,t)(V,V).$$

Integrating on both sides, we have

$$\overline{C}|t_1-t_0| \geq \int_{t_0}^{t_1}\left|\frac{\partial}{\partial t}\log g_k(x,t)(V,V)\right|dt \geq \left|\int_{t_0}^{t_1}\frac{\partial}{\partial t}\log g_k(t)(V,V)\,dt\right| = \left|\log\frac{g_k(x,t_1)(V,V)}{g_k(x,t_0)(V,V)}\right|,$$

and hence we conclude that

$$e^{-\overline{C}|t_1-t_0|}g_k(x,t_0)(V,V) \leq g_k(x,t_1)(V,V) \leq e^{\overline{C}|t_1-t_0|}g_k(x,t_0)(V,V).$$

From the assumption (i), it immediately deduces from above that

$$C^{-1}e^{-\overline{C}|t_1-t_0|}g(x)(V,V) \leq g_k(x,t_1)(V,V) \leq Ce^{\overline{C}|t_1-t_0|}g(x)(V,V).$$

Since t_1 was arbitrary, the first part is proved. From the definition (or see [Chow et al. 2007, p. 134, (37)]), we have

$$(g_k)^{ec}({}^g\nabla_a(g_k)_{bc} + {}^g\nabla_b(g_k)_{ac} - {}^g\nabla_c(g_k)_{ab}) = 2({}^{g_k}\Gamma)^e_{ab} - 2({}^g\Gamma)^e_{ab}.$$

Thus $|{}^{g_k}\Gamma(x,t) - {}^g\Gamma(x)|_{g(x)} \leq C|{}^g\nabla g_k(x,t)|_{g_k(x)}$. On the other hand,

$${}^g\nabla_a(g_k)_{bc} = (g_k)_{eb}[({}^{g_k}\Gamma)^e_{ac} - ({}^g\Gamma)^e_{ac}] + (g_k)_{ec}[({}^{g_k}\Gamma)^e_{ab} - ({}^g\Gamma)^e_{ab}],$$

it follows that $|{}^g\nabla g_k(x,t)|_{g_k(x,t)} \leq C|{}^{g_k}\Gamma(x,t) - {}^g\Gamma(x)|_{g_k(x,t)}$ and therefore

$${}^g\nabla g_k \text{ is equivalent to } {}^{g_k}\Gamma - {}^g\Gamma = {}^{g_k}\nabla - {}^g\nabla. \tag{5-2}$$

The evolution equation for ${}^g\Gamma$ is

$$\begin{aligned}\frac{\partial}{\partial t}({}^{g_k}\Gamma)^c_{ab} = -(g_k)^{cd}[&({}^{g_k}\nabla)_a(\operatorname{Ric}(g_k))_{bd} + ({}^{g_k}\nabla)_b(\operatorname{Ric}(g_k))_{ad}\\ &-({}^{g_k}\nabla)_d(\operatorname{Ric}(g_k))_{ab}] + \tfrac{1}{4}(g_k)^{cd}\big[({}^{g_k}\nabla)_a(h_k)_{bd} + ({}^{g_k}\nabla)_b(h_k)_{ad} - ({}^{g_k}\nabla)_d(h_k)_{ab}\big].\end{aligned}$$

Since ${}^{g}\Gamma$ does not depend on t, it follows from the assumptions that

$$\begin{aligned}\left|\frac{\partial}{\partial t}({}^{g_k}\Gamma - {}^{g}\Gamma)\right|_{g_k} &\le C\left|{}^{g_k}\nabla(\mathrm{Ric}(g_k))\right|_{g_k} + C|{}^{g_k}\nabla(h_k)|_{g_k}\\ &\le CC_1' + C|{}^{g_k}\nabla H_k|_{g_k}\cdot|H_k|_{g_k} \le C_1'.\end{aligned}$$

Integrating on both sides,

$$C_1'|t_1 - t_0| \ge \left|\int_{t_0}^{t_1}\frac{\partial}{\partial t}(g^k\Gamma(t) - {}^{g}\Gamma)\,dt\right|_{g_k} \ge |{}^{g_k}\Gamma(t_1) - {}^{g}\Gamma|_{g_k} - |{}^{g_k}\Gamma(t_0) - {}^{g}\Gamma|_{g_k}.$$

Hence we obtain

$$\begin{aligned}|{}^{g_k}\Gamma(t) - {}^{g}\Gamma|_{g_k} &\le C_1'|t_1 - t_0| + |{}^{g_k}\Gamma(t_0) - {}^{g}\Gamma|_{g_k}\\ &\le C_1'|t_1 - t_0| + C|{}^{g}\nabla g_k(t_0)|_{g_k}\\ &\le C_1'|t - t_0| + C|{}^{g}\nabla g_k(t_0)|_{g}\\ &\le C_1'|t - t_0| + C_1.\end{aligned}$$

The equivalency of metrics tells us that

$$\begin{aligned}|{}^{g}\nabla g_k(t)|_g &\le B(t,t_0)^{3/2}|{}^{g}\nabla g_k(t)|_{g_k} \le B(t,t_0)^{3/2}\cdot C|{}^{g_k}\Gamma(t) - {}^{g}\Gamma|_{g_k}\\ &\le B(t,t_0)^{3/2}(C_1'|t - t_0| + C').\end{aligned}$$

Since $|t - t_0| \le \psi - \beta$, it follows that $|{}^{g}\nabla g_k(t)|_g \le \widetilde{C}_{1,0}$ for some constant $\widetilde{C}_{1,0}$. But g and g_k are equivalent, we have

$$|H_k(t)|_g \le C|H_k(t)|_{g_k} \le CC_1' = \widetilde{C}_{1,0}.$$

From the assumptions, we also have

$$\begin{aligned}|{}^{g}\nabla H_k|_g &\le |({}^{g}\nabla - {}^{g_k}\nabla)H_k + {}^{g_k}\nabla H_k|_g\\ &\le C|{}^{g}\nabla g_k|_g\cdot|H_k|_g + C|{}^{g_k}\nabla H_k|_{g_k}\\ &\le CC_1' + C\widetilde{C}_{1,0}\widetilde{C}_{1,0} := \widetilde{C}_{2,0}.\end{aligned}$$

Moreover,

$$\begin{aligned}\frac{\partial}{\partial t}{}^{g}\nabla H_k &= {}^{g}\nabla(\Delta_{g_k}H_k + \mathrm{Rm}(g_k) * H_k)\\ &= ({}^{g}\nabla - {}^{g_k}\nabla)\Delta_{g_k}H_k + {}^{g_k}\nabla\Delta_{g_k}H_k + {}^{g}\nabla\,\mathrm{Rm}(g_k) * H_k + \mathrm{Rm}(g_k) * {}^{g}\nabla H_k\end{aligned}$$

where Δ_{g_k} is the Laplace operator associated to g_k. Hence

$$\begin{aligned}&\left|\frac{\partial}{\partial t}{}^{g}\nabla H_k\right|_g\\ &\quad\le C|{}^{g}\nabla g_k|_g\cdot|\Delta_{g_k}H_k|_{g_k} + C|{}^{g_k}\nabla\Delta_{g_k}H_k|_g + C|{}^{g}\nabla\,\mathrm{Rm}(g_k)|_g\cdot|H_k|_g + C|\mathrm{Rm}(g_k)|_g\cdot|{}^{g}\nabla H_k|_g\\ &\quad\le \widetilde{C}_{2,1}.\end{aligned}$$

For higher derivatives we claim that

$$|{}^g\nabla^p \operatorname{Ric}(g_k)|_g \le C_p''|{}^g\nabla^p g_k|_g + C_p''', \quad |{}^g\nabla^p g_k|_g + |{}^g\nabla^{p-1} H_k|_g \le \widetilde{C}_{p,0}, \tag{5-3}$$

for all $p \ge 1$, where C_p'', C_p''', and $\widetilde{C}_{p,0}$ are constants independent of k. For $p = 1$, we have proved the second inequality, so we suffice to prove the first one with $p = 1$. Indeed,

$$\begin{aligned}|{}^g\nabla \operatorname{Ric}(g_k)|_g &\le C\,|({}^g\nabla - {}^{g_k}\nabla)\operatorname{Ric}(g_k) + {}^{g_k}\nabla \operatorname{Ric}(g_k)|_{g_k}\\ &\le C\,|{}^g\Gamma - {}^{g_k}\Gamma|_g \cdot |\operatorname{Ric}(g_k)|_{g_k} + C\,|{}^{g_k}\nabla \operatorname{Ric}(g_k)|_{g_k}\\ &\le C_1''|{}^g\nabla g_k|_g + C_1'''.\end{aligned}$$

Suppose the claim holds for all $p < N$ $(N \ge 2)$, we shall show that it also holds for $p = N$. From

$$\begin{aligned}|{}^g\nabla^N \operatorname{Ric}(g_k)|_g &= \left|\sum_{i=1}^{N} {}^g\nabla^{N-i}({}^g\nabla - {}^{g_k}\nabla){}^{g_k}\nabla^{i-1}\operatorname{Ric}(g_k) + {}^{g_k}\nabla^N \operatorname{Ric}(g_k)\right|_g\\ &\le \sum_{i=1}^{N}\left|{}^g\nabla^{N-i}({}^g\nabla - {}^{g_k}\nabla){}^{g_k}\nabla^{i-1}\operatorname{Ric}(g_k)\right|_g + |{}^{g_k}\nabla^N \operatorname{Ric}(g_k)|_g\end{aligned}$$

we estimate each term. For $i = 1$, by induction and the assumptions we have

$$\begin{aligned}&|{}^g\nabla^{N-1}({}^g\nabla - {}^{g_k}\nabla)\operatorname{Ric}(g_k)|_g\\ &\qquad\le C\,|{}^g\nabla^{N-1}({}^g\nabla g_k \cdot \operatorname{Ric}(g_k))|_g\\ &\qquad\le C\left|\sum_{j=0}^{N-1}\binom{N-1}{j}{}^g\nabla^{N-1-j}({}^g\nabla g_k)\cdot {}^g\nabla^j(\operatorname{Ric}(g_k))\right|_g\\ &\qquad\le C\sum_{j=0}^{N-1}\binom{N-1}{j}|{}^g\nabla^{N-j} g_k|_g \cdot |{}^g\nabla^j \operatorname{Ric}(g_k)|_g\\ &\qquad\le C\sum_{i=0}^{N-1}\binom{N-1}{j}(C_j''|{}^g\nabla^j g_k|_g + C_j''')|{}^g\nabla^{N-j} g_k|_g\\ &\qquad\le C\sum_{j=0}^{N-1}\binom{N-1}{j}(C_j''\widetilde{C}_{j,0} + C_j''')|{}^g\nabla^{N-j} g_k|_g\\ &\qquad= C(N-1)(C_0''\widetilde{C}_{j,0} + C_0''')|{}^g\nabla^N g_k|_g + C\sum_{j=1}^{N-1}\binom{N-1}{j}(C_j''\widetilde{C}_{j,0} + C_j''')\widetilde{C}_{N-j,0}\\ &\qquad\le C_N''|{}^g\nabla^N g_k|_g + C_N'''.\end{aligned}$$

For $i \geq 2$, we have

$$\begin{aligned}\left|{}^g\nabla^{N-i}({}^g\nabla - {}^{g_k}\nabla){}^{g_k}\nabla^{i-1}\operatorname{Ric}(g_k)\right|_g &\leq C\left|{}^g\nabla^{N-i}({}^g\nabla g_k \cdot {}^{g_k}\nabla^{i-1}\operatorname{Ric}(g_k))\right|_g \\ &\leq C\sum_{j=0}^{N-i}\binom{N-i}{j}|{}^g\nabla^{N-i-j+1}g_k|_g \cdot |{}^g\nabla^j \cdot {}^{g_k}\nabla^{i-1}\operatorname{Ric}(g_k)|_g.\end{aligned}$$

If $j = 0$, then

$$|{}^{g_k}\nabla^{i-1}\operatorname{Ric}(g_k)|_g \leq C''_{i-1}|{}^g\nabla^{i-1}g_k|_g + C'''_{i-1} \leq C''_{i-1}\widetilde{C}_{i-1,0} + C'''_{i-1}.$$

Suppose in the following that $j \geq 1$. Hence

$$\begin{aligned}\left|{}^g\nabla^j \cdot {}^{g_k}\nabla^{i-1}\operatorname{Ric}(g_k)\right|_g &= \left|(({}^g\nabla - {}^{g_k}\nabla) + {}^{g_k}\nabla)^j \cdot {}^{g_k}\nabla^{i-1}\operatorname{Ric}(g_k)\right|_g \\ &\leq C\sum_{l=0}^{j}\binom{j}{l}|{}^g\nabla^l g_k|_g \cdot |{}^{g_k}\nabla^{j-l+i-1}\operatorname{Ric}(g_k)|_g \\ &\leq C\sum_{l=0}^{j}\binom{j}{l}\widetilde{C}_{l,0}(C''_{j-l+i-1}\widetilde{C}_{j-l+i-1,0} + C'''_{j-l+i-1}),\end{aligned}$$

where we make use of (5-2) from first line to second line. Combining these inequalities, we get

$$|{}^g\nabla^N\operatorname{Ric}(g_k)|_g \leq C''_N|{}^g\nabla^N g_k|_g + C'''_N.$$

Similarly, we have

$$|{}^g\nabla^N h_k|_g \leq C''_N|{}^g\nabla^N g_k|_g + C'''_N.$$

Since $\frac{\partial}{\partial t}g_k = -2\operatorname{Ric}(g_k) + \frac{1}{2}h_k$, it follows that

$$\frac{\partial}{\partial t}{}^g\nabla^N g_k = {}^g\nabla^N(-2\operatorname{Ric}(g_k) + \tfrac{1}{2}h_k),$$

$$\begin{aligned}\frac{\partial}{\partial t}|{}^g\nabla^N g_k|_g^2 &\leq \left|\frac{\partial}{\partial t}{}^g\nabla^N g_k\right|_g^2 + |{}^g\nabla^N g_k|_g^2 \\ &\leq 8\,|{}^g\nabla^N\operatorname{Ric}(g_k)|_g^2 + \frac{1}{2}|{}^g\nabla^N h_k|_g^2 + |{}^g\nabla^N g_k|_g^2 \\ &\leq (1 + 18(C''_N)^2)|{}^g\nabla^N g_k|_g^2 + 18(C''_N)^2.\end{aligned}$$

Integrating the above inequality, we get $|{}^g\nabla g_k|_g \leq \widetilde{C}_{N,0}$ and therefore $|{}^g\nabla^N h_k|_g \leq \widetilde{C}_{N+1,0}$. We have proved lemma for $q = 0$. When $g \geq 1$, then

$$\frac{\partial^q}{\partial t^q}{}^g\nabla^p g_k(t) = g\nabla^p\frac{\partial^{q-1}}{\partial t^{q-1}}\big(-2\operatorname{Ric}(g_k(t)) + \tfrac{1}{2}h_k(t)\big).$$

Using the evolution equations for $\operatorname{Rm}(g_k(t))$ and $h_k(t)$, combining the induction to q and using the above method, we have

$$\left|\frac{\partial^q}{\partial t^q}{}^g\nabla^p g_k(t)\right|_g + \left|\frac{\partial^q}{\partial t^q}{}^g\nabla^{p-1}h_k(t)\right|_g \leq \widetilde{C}_{p,q}. \qquad \square$$

6. Generalization

In this section, we generalize the main results in Sections 4 and 5 to a kind of generalized Ricci flow for which local existence has been established [He et al. 2008].

Let $(M, g_{ij}(x))$ be an n-dimensional closed Riemannian manifold and let $A = \{A_i\}$ and $B = \{B_{ij}\}$ denote a one-form and a two-form respectively. Set $F = dA$ and $H = dB$. The authors in [He et al. 2008] proved that there exists a constant $T > 0$ such that the evolution equations

$$\begin{aligned}
&\frac{\partial}{\partial t} g_{ij}(x,t) = -2R_{ij}(x,t) + \tfrac{1}{2}h_{ij}(x,t) + 2f_{jk}(x,t), \quad g_{ij}(x,0) = g_{ij}(x),\\
&\frac{\partial}{\partial t} A_i(x,t) = -2\nabla_k F_i{}^k(x,t), \quad A_i(x,0) = A_i(x),\\
&\frac{\partial}{\partial t} B_{ij}(x,t) = 3\nabla_k H^k{}_{ij}(x,t), \quad B_{ij}(x,0) = B_{ij}(x)
\end{aligned}$$

has a unique smooth solution on $m \times [0,T)$, where $h_{ij} = H_{ikl}H_j{}^{kl}$ and $f_{ij} = F_i{}^k F_{jk}$. We call it RF(A, B). According to the definition of the adjoint operator d^*, we have

$$(d^*F)_i = 2\nabla_k F_i{}^k, \quad (d^*H)_{ij} = -3\nabla_k H^k{}_{ij}, \tag{6-1}$$

and hence

$$\frac{\partial}{\partial t} F(x,t) = -dd^*_{g(x,t)}F \;=\; \Delta_{\mathrm{HL},g(x,t)}F \;=\; \Delta F + \mathrm{Rm} * F, \tag{6-2}$$

$$\frac{\partial}{\partial t} H(x,t) = -dd^*_{g(x,t)}H \;=\; \Delta_{\mathrm{HL},g(x,t)}H \;=\; \Delta H + \mathrm{Rm} * H. \tag{6-3}$$

They also derived the evolution equations of curvatures:

$$\begin{aligned}
\frac{\partial}{\partial t} R_{ijk\ell} = \Delta R_{ijk\ell} &+ 2(B_{ijk\ell} - B_{ij\ell k} - B_{i\ell jk} + B_{ikj\ell})\\
&- g^{pq}(R_{pjk\ell}R_{qi} + R_{ipk\ell}R_{qj} + R_{ijp\ell}R_{qk} + R_{ijkp}R_{q\ell})\\
&+ \tfrac{1}{4}\big[\nabla_i\nabla_\ell(H_{kpq}H_j{}^{pq}) - \nabla_i\nabla_k(H_{jpq}H_\ell{}^{pq}) - \nabla_j\nabla_\ell(H_{kpq}H_i{}^{pq}) + \nabla_j\nabla_k(H_{ipq}H_\ell{}^{pq})\big]\\
&+ \tfrac{1}{4}g^{rs}(H_{kpq}H_r^{pq}R_{ijs\ell} + H_{rpq}H_\ell{}^{pq}R_{ijks})\\
&+ \nabla_i\nabla_\ell(F_k{}^pF_{jp}) - \nabla_i\nabla_k(F_j{}^pF_{\ell p}) - \nabla_j\nabla_\ell(F_k{}^pF_{ip}) + \nabla_j\nabla_k(F_i{}^pF_{\ell p})\\
&+ g^{rs}(F_k{}^pF_{rp}R_{ijs\ell} + F_r{}^pF_{\ell p}R_{ijks}).
\end{aligned}$$

Under our notation, it can be rewritten as

$$\begin{aligned}
\frac{\partial}{\partial t}\mathrm{Rm} = \Delta\,\mathrm{Rm} + \sum_{i+j=0}\nabla^i\,\mathrm{Rm} * \nabla^j\,\mathrm{Rm} + \sum_{i+j=0+2}\nabla^i H * \nabla^j H + \sum_{i+j=0+2}\nabla^i F * \nabla^j F&\\
+ \sum_{i+j+k=0}\nabla^i H * \nabla^j H * \nabla^k\,\mathrm{Rm} + \sum_{i+j+k=0} !!\nabla^i F * \nabla^j F * \nabla^k\,\mathrm{Rm}.&
\end{aligned} \tag{6-4}$$

As before, we have:

Proposition 6.1. *For* RF(A, B) *and any nonnegative integer* ℓ *we have*

$$\frac{\partial}{\partial t}\nabla^{\ell}\mathrm{Rm}=\Delta(\nabla^{l}\mathrm{Rm})+\sum_{i+j=\ell}\nabla^{i}\mathrm{Rm}*\nabla^{j}\mathrm{Rm}+\sum_{i+j=\ell+2}\nabla^{i}H*\nabla^{j}H+\sum_{i+j=\ell+2}\nabla^{i}F*\nabla^{j}F \\ +\sum_{i+j+k=\ell}\nabla^{i}H*\nabla^{j}H*\nabla^{k}\mathrm{Rm}+\sum_{i+j+k=\ell}\nabla^{i}F*\nabla^{j}F*\nabla^{k}\mathrm{Rm}. \tag{6-5}$$

In particular,

$$\frac{\partial}{\partial t}|\nabla^{l}\mathrm{Rm}|^{2}\le\Delta|\nabla^{l}\mathrm{Rm}|^{2}-2|\nabla^{\ell+1}\mathrm{Rm}|^{2}+C\sum_{i+j=\ell}|\nabla^{i}\mathrm{Rm}|\cdot|\nabla^{j}\mathrm{Rm}|\cdot|\nabla^{\ell}\mathrm{Rm}| \\ +C\sum_{i+j=\ell+2}|\nabla^{i}H|\cdot|\nabla^{j}H|\cdot|\nabla^{\ell}\mathrm{Rm}|+C\sum_{i+j=\ell+2}|\nabla^{i}F|\cdot|\nabla^{j}F|\cdot|\nabla^{\ell}\mathrm{Rm}| \\ +C\sum_{i+j+k=\ell}|\nabla^{i}H|\cdot|\nabla^{j}H|\cdot|\nabla^{k}\mathrm{Rm}|\cdot|\nabla^{\ell}\mathrm{Rm}|+C\sum_{i+j+k=\ell}|\nabla^{i}F|\cdot|\nabla^{j}F|\cdot|\nabla^{k}\mathrm{Rm}|\cdot|\nabla^{\ell}\mathrm{Rm}|.$$

Since $\frac{\partial}{\partial t}F=\Delta F+\mathrm{Rm}*F$ it follows that

$$\begin{aligned}\frac{\partial}{\partial t}\nabla F&=\nabla\frac{\partial}{\partial t}F+F*\nabla(\mathrm{Rm}+H*H+F*F)\\&=\nabla(\Delta F+\mathrm{Rm}*F)+F*\nabla\mathrm{Rm}+F*H*\nabla H+F*F*\nabla F\\&=\Delta(\nabla F)+\nabla\mathrm{Rm}*F+\mathrm{Rm}*\nabla F+F*H*\nabla H+F*F*\nabla F.\end{aligned}$$

It can be expressed as

$$\frac{\partial}{\partial t}\nabla F=\Delta(\nabla F)+\sum_{i+j=1}\nabla^{i}F*\nabla^{j}\mathrm{Rm} \\ +\sum_{i+j+k=1}\nabla^{i}F*\nabla^{j}F*\nabla^{k}F+\sum_{i=0}^{1-1}\sum_{j=0}^{1-i}\nabla^{i}F*\nabla^{j}H*\nabla^{1-i-j}H.$$

More generally, we can show:

Proposition 6.2. *For* RF(A, B) *and any positive integer* ℓ *we have*

$$\frac{\partial}{\partial t}\nabla^{\ell}F=\Delta(\nabla^{\ell}F)+\sum_{i+j=\ell}\nabla^{i}F*\nabla^{j}\mathrm{Rm} \\ +\sum_{i+j+k=\ell}\nabla^{i}F*\nabla^{j}F*\nabla^{k}F+\sum_{i=0}^{\ell-1}\sum_{j=0}^{\ell-i}\nabla^{i}F*\nabla^{j}H*\nabla^{\ell-i-j}H.$$

In particular,

$$\frac{\partial}{\partial t}|\nabla^{\ell}F|^{2}\le\Delta|\nabla^{\ell}F|^{2}-2|\nabla^{\ell+1}F|^{2}+C\sum_{i+j=\ell}|\nabla^{i}F|\cdot|\nabla^{j}\mathrm{Rm}|\cdot|\nabla^{\ell}F| \\ +C\sum_{i+j+k=\ell}|\nabla^{i}F|\cdot|\nabla^{j}F|\cdot|\nabla^{k}F|\cdot|\nabla^{l}F|+C\sum_{i=0}^{\ell-1}\sum_{j=0}^{\ell-i}|\nabla^{i}F|\cdot|\nabla^{j}H|\cdot|\nabla^{\ell-i-j}H|\cdot|\nabla^{\ell}F|.$$

Similarly, we obtain:

Proposition 6.3. *For* $\mathrm{RF}(A, B)$ *and any positive integer* l *we have*

$$\frac{\partial}{\partial t}\nabla^\ell H = \Delta(\nabla^\ell H) + \sum_{i+j=\ell} \nabla^i H * \nabla^j \mathrm{Rm} + \sum_{i+j+k=\ell} \nabla^i H * \nabla^j H * \nabla^k H + \sum_{i=0}^{\ell-1}\sum_{j=0}^{\ell-i} \nabla^i H * \nabla^j F * \nabla^{\ell-i-j} F.$$

In particular,

$$\frac{\partial}{\partial t}|\nabla^\ell H|^2 \le \Delta|\nabla^\ell H|^2 - 2\,|\nabla^{\ell+1} H|^2 + C \sum_{i+j=\ell} |\nabla^i H|\cdot|\nabla^j \mathrm{Rm}|\cdot|\nabla^\ell H| + C \sum_{i+j+k=\ell} |\nabla^i H|\cdot|\nabla^j H|\cdot|\nabla^k H|\cdot|\nabla^\ell H| + C\sum_{i=0}^{\ell-1}\sum_{j=0}^{\ell-i} |\nabla^i H|\cdot|\nabla^j F|\cdot|\nabla^{\ell-i-j} F|\cdot|\nabla^\ell H|.$$

From the evolution inequalities

$$\frac{\partial}{\partial t}|H|^2 \le \Delta|H|^2 - 2\,|\nabla H|^2 + C\cdot|\mathrm{Rm}|\cdot|H|^2,$$
$$\frac{\partial}{\partial t}|F|^2 \le \Delta|F|^2 - 2\,|\nabla F|^2 + C\cdot|\mathrm{Rm}|\cdot|F|^2,$$

the following theorem is obvious.

Theorem 6.4. *Suppose that* $(g(x,t), H(x,t), F(x,t))$ *is a solution to* $\mathrm{RF}(A, B)$ *on a compact manifold* M^n *for a short time* $0 \le t \le T$ *and* K_1, K_2, K_3 *are arbitrary given nonnegative constants. Then there exists a constant* C_n *depending only on* n *such that if*

$$|\mathrm{Rm}(x,t)|_{g(x,t)} \le K_1, \quad |H(x)|_{g(x)} \le K_2, \quad |F(x)|_{g(x)} \le K_3$$

for all $x \in M$ *and* $t \in [0, T]$, *then*

$$|H(x,t)|_{g(x,t)} \le K_2 e^{C_n K_1 t}, \quad |F(x,t)|_{g(x,t)} \le K_3 e^{C_n K_1 t}, \tag{6-6}$$

for all $x \in M$ *and* $t \in [0, T]$.

Parallel to Theorem 4.6, we can prove:

Theorem 6.5. *Suppose that* $(g(x,t), H(x,t), F(x,t))$ *is a solution to* $\mathrm{RF}(A, B)$ *on a compact manifold* M^n *and* K *is an arbitrary given positive constant. Then for each* $\alpha > 0$ *and each integer* $m \ge 1$ *there exists a constant* C_m *depending on* $m, n, \max\{\alpha, 1\}$, *and* K *such that if*

$$|\mathrm{Rm}(x,t)|_{g(x,t)} \le K, \quad |H(x)|_{g(x)} \le K, \quad |F(x)|_{g(x)} \le K$$

for all $x \in M$ *and* $t \in [0, \alpha/K]$, *then*

$$|\nabla^{m-1}\mathrm{Rm}(x,t)|_{g(x,t)} + |\nabla^m H(x,t)|_{g(x,t)} + |\nabla^m F(x,t)|_{g(x,t)} \le \frac{C_m}{t^{\frac{m}{2}}}, \tag{6-7}$$

for all $x \in M$ *and* $t \in (0, \alpha/K]$.

We can also establish the corresponding compactness theorem for $\mathrm{RF}(A, B)$. We omit the detail since the proof is close to the proof in Section 5. In the forthcoming paper, we will consider the BBS estimates for complete noncompact Riemannian manifolds.

Acknowledgment

The author thanks his advisor, Professor Shing-Tung Yau, for helpful discussions. The author expresses his gratitude to Professor Kefeng Liu for his interest in this work and for his numerous help in mathematics. He also thanks Valentino Tosatti and Jeff Streets for several useful conversations.

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Received 22 Sep 2010. Revised 4 Aug 2011. Accepted 27 Sep 2011.

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ANALYSIS AND PDE
Vol. 5, No. 4, 2012
dx.doi.org/10.2140/apde.2012.5.777

SMOOTH TYPE II BLOW-UP SOLUTIONS TO THE FOUR-DIMENSIONAL ENERGY-CRITICAL WAVE EQUATION

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We exhibit $\mathscr{C}^\infty$ type II blow-up solutions to the focusing energy-critical wave equation in dimension $N=4$. These solutions admit near blow-up time a decomposition

$$u(t,x)=\frac{1}{\lambda^{(N-2)/2}(t)}(Q+\varepsilon(t))\Big(\frac{x}{\lambda(t)}\Big),\quad \text{with } \|\varepsilon(t),\partial_t\varepsilon(t)\|_{\dot H^1\times L^2}\ll 1,$$

where Q is the extremizing profile of the Sobolev embedding $\dot H^1\to L^{2^*}$, and a blow-up speed

$$\lambda(t)=(T-t)e^{-\sqrt{|\log(T-t)|}(1+o(1))}\quad \text{as } t\to T.$$

1. Introduction

Setting of the problem. We deal in this paper with the energy-critical focusing wave equation

$$\begin{cases}\partial_{tt}u-\Delta u-f(u)=0 & \text{with } f(t)=t^{(N+2)/(N-2)},\\ (u,\partial_t u)_{|t=0}=(u_0,u_1), & (t,x)\in\mathbb{R}\times\mathbb{R}^N.\end{cases}\tag{1-1}$$

in dimension $N=4$. This is a special case of the nonlinear wave equation

$$\partial_{tt}u-\Delta u-f(u)=0,\tag{1-2}$$

which, since the pioneering [Jörgens 1961], has been the subject of a considerable amount of work. For the energy-critical nonlinearity $f(u)=\pm t^{(N+2)/(N-2)}$, the Cauchy problem is locally well posed in the energy space $\dot H^1\times L^2$ and the solution propagates regularity; see [Sogge 1995] and references therein. Recall that in this case, (1-2) admits a conserved energy

$$E(u(t))=E(u_0,u_1)=\frac12\int(\partial_t u)^2+\frac12\int|\nabla u|^2\mp\frac{N-2}{2N}\int u^{2N/(N-2)}$$

that is left invariant by the scaling symmetry of the flow,

$$u_\lambda(t,x)=\frac{1}{\lambda^{(N-2)/2}}u\Big(\frac{t}{\lambda},\frac{x}{\lambda}\Big).$$

Global existence in the defocusing case was proved by Struwe [1988] for radial data and Grillakis [1990] for general data. For focusing nonlinearities, a sharp threshold criterion of global existence and scattering

MSC2010: 35Q51.
Keywords: wave equation, blow-up.

or finite time blow-up is obtained by Kenig and Merle [2008] based on the soliton solution to (1-1),

$$Q(r)=\left(\frac{1}{1+r^2/(N(N-2))}\right)^{(N-2)/2}, \tag{1-3}$$

which is the extremizing profile of the Sobolev embedding $\dot{H}^1\to L^{2^*}$. Indeed, for initial data (u_0,u_1) such that $E(u_0,u_1)<E(Q,0)$, those with $\|\nabla u_0\|_{L^2}<\|\nabla Q\|_{L^2}$ have global solutions and scatter, while those with $\|\nabla u_0\|_{L^2}>\|\nabla Q\|_{L^2}$ lead to finite time blow-up.

Note that like in the works of Levine [1974] (see also [Strauss 1989]) and as is standard in a nonlinear dispersive setting, blow-up is derived through obstructive convexity arguments; see also [Karageorgis and Strauss 2007] for refined statements near the soliton Q. However, this approach gives very little insight into the description of the blow-up mechanism and the description of the flow even just near the ground state soliton Q is still only at its beginning.

On the energy-critical wave map problem. There is an important literature devoted to the construction of blow-up solutions for nonlinear wave equations; see [Alinhac 1995; Merle and Zaag 2003; 2008] for the study of the ODE-type of blow-up for subcritical nonlinearities. For energy-critical problems like (1-1), recent important progress has been made through the study of the two-dimensional energy-critical corotational wave map to the 2-sphere,

$$\partial_{tt}u-\partial_{rr}u-\frac{\partial_r u}{r}-\frac{k^2\sin 2u}{2r^2}=0, \tag{1-4}$$

where $k\in\mathbb{N}^*$ is the homotopy number. The ground state is given there by

$$Q(r)=2\tan^{-1}(r^k).$$

After the pioneering works of Christodoulou and Tahvildar-Zadeh [1993], Shatah and Tahvildar-Zadeh [1994] and Struwe [2003] and their detailed study of the concentration of energy scenario, the first explicit description of singularity formation for the $k=1$ case was derived by Krieger, Schlag and Tataru [2008] who constructed finite energy finite time blow-up solutions of the form

$$u(t,x)=(Q+\varepsilon)\left(t,\frac{x}{\lambda(t)}\right),\quad \text{with } \|\varepsilon(t),\partial_t\varepsilon(t)\|_{\dot{H}^1\times L^2}\ll 1, \tag{1-5}$$

with a blow-up speed given by

$$\lambda(t)=(T-t)^{\nu},$$

for any $\nu>\frac{3}{2}$; see also [Krieger et al. 2009a]. The spectacular feature of this result is that it exhibits arbitrarily slow blow-up regimes further and further from self-similarity which would correspond to the (forbidden; see [Struwe 2003]) self-similar law

$$\lambda(t)\sim T-t. \tag{1-6}$$

Numerics suggest that this blow-up scenario is nongeneric and corresponds to finite-codimensional manifolds [Bizoń et al. 2001]. After the pioneering work [Rodnianski and Sterbenz 2010] for large homotopy number $k\geq 4$, Raphaël and Rodnianski [2012] gave a complete description of *stable* blow-up dynamics that originate from smooth data for all homotopy numbers $k \geq 1$. The blow-up speed obeys in this regime a *universal law that depends in an essential way on the rate of convergence* of the ground state Q to its asymptotic value,

$$\pi - Q \sim \frac{1}{r^k} \quad \text{as } r \to \infty,$$

and indeed the stable blow-up regime corresponds to a decomposition (1-5) with blow-up speed

$$\lambda(t) \sim \begin{cases} c_k \dfrac{T-t}{|\log(T-t)|^{1/(2k-2)}} & \text{for } k \geq 2, \\ (T-t)e^{-\sqrt{|\log(T-t)|}} & \text{for } k = 1. \end{cases} \tag{1-7}$$

Note that this work draws an important analogy with another critical problem, the L^2 critical nonlinear Schrödinger equation, where a similar universality of the stable singularity formation near the ground state was proved in [Merle and Raphael 2003; 2004; 2005a; 2005b; 2006; Raphael 2005].

Statement of the result. For the power nonlinearity energy-critical problem (1-1), there has been recent progress towards the understanding of the flow near the solitary wave Q. Krieger and Schlag [2007] constructed in dimension $N = 3$ a codimension one manifold of initial data near Q that yield global solutions asymptotically converging to the soliton manifold. The strategy developed by Krieger et al. [2008] for the wave map problem has been adapted in [Krieger et al. 2009b] to show in dimension $N = 3$ the existence of finite energy finite time blow-up solutions of the form

$$u(t,x) = \frac{1}{\lambda^{(N-2)/2}(t)}(Q+\varepsilon)\left(t, \frac{x}{\lambda(t)}\right), \quad \text{with } \|\varepsilon(t), \partial_t\varepsilon(t)\|_{\dot{H}^1\times L^2} \ll 1,$$

and with a blow-up speed given by

$$\lambda(t) = (T-t)^{\nu}, \tag{1-8}$$

for any $\nu > \frac{3}{2}$. The quantization of the energy at blow-up for small type II blow-up solutions in dimension $N \in \{3, 5\}$ is proved in [Duyckaerts et al. 2011; 2012] in the radial and nonradial cases. In particular, for radial data, if $T < +\infty$ and

$$\sup_{t\in[0,T]} \left[|\nabla u(t)|_{L^2}^2 + \partial_t u|_{L^2}^2\right] \leq |\nabla Q|_{L^2}^2 + \alpha^*, \quad \alpha^* \ll 1,$$

then there exists a dilation parameter $\lambda(t) \to 0$ as $t \to T$ and asymptotic profiles $(u^*, v^*) \in H^1 \times L^2$ such that

$$\left(u(t,x) - \frac{1}{\lambda^{(N-2)/2}(t)} Q\left(\frac{x}{\lambda(t)}\right), \partial_t u(t)\right) \to (u^*, v^*) \quad \text{in } \dot{H}^1 \times L^2 \text{ as } t \to T;$$

see [Merle and Raphael 2005b] for related classification results for the L^2 critical (NLS).

These works however leave open the question of the existence of *smooth* type II blow-up solutions. We claim that such smooth type II blow-up solutions can be constructed in dimension $N = 4$ as the formal analogue of the singular dynamics exhibited by Raphaël and Rodnianski [2012] for the wave map problem in the least homotopy number class $k = 1$. The following theorem is the main result of this paper:

Theorem 1.1 (existence of smooth type II blow-up solutions in dimension $N=4$). *Let $N=4$. Then for all $\alpha^*>0$, there exist $\mathscr{C}^\infty$ initial data (u_0,u_1) with*

$$E(u_0,u_1)<E(Q,0)+\alpha^*$$

such that the corresponding solution to the energy-critical focusing wave equation (1-1) *blows up in finite time $T=T(u_0,u_1)<+\infty$ in a type II regime according to the following dynamics*: *there exist $(u^*,v^*)\in\dot H^1\times L^2$ such that*

$$\left(u(t,x)-\frac{1}{\lambda^{(N-2)/2}(t)}Q\left(\frac{x}{\lambda(t)}\right),\partial_t u(t)\right)\to(u^*,v^*)\quad \textit{in } \dot H^1\times L^2 \textit{ as } t\to T, \tag{1-9}$$

with a blow-up speed given by

$$\lambda(t)=(T-t)e^{-\sqrt{|\log(T-t)|}(1+o(1))}\quad \textit{as } t\to T. \tag{1-10}$$

Comments on the result. 1. On the smoothness of the initial data. An important feature of Theorem 1.1 is to exhibit a new blow-up speed which is valid for $\mathscr{C}^\infty$ solutions. Indeed, while the Krieger et al. [2009b] approach provides a continuum of blow-up speeds, the exact regularity of the obtained solutions is not known, which is an unpleasant consequence of their construction scheme. In fact, it is expected that $\mathscr{C}^\infty$ initial data should lead to *quantized* blow-up rates hence breaking the continuum of blow-up speeds (1-8), we refer to [van den Berg et al. 2003] for a related discussion in the context of the energy-critical harmonic heat flow. Hence we expect the blow-up rate (1-10) to correspond to the minimal type II blow-up speed of smooth solutions with small supercritical energy. Such a general lower bound on the blow-up rate in the spirit of the one obtained by Merle and Raphael [2006; Raphael 2005] for the L^2 critical NLS is an open problem. The construction of excited blow-up solutions with other speeds and $\mathscr{C}^\infty$ regularity also remains to be done. This problematic is related to the understanding of the structure of the flow near Q, which is still in its infancy.

2. On the codimension one manifold. The proof of Theorem 1.1 involves a detailed description of the set of initial data leading to the type II blow-up with speed (1-10). Indeed, given a small enough parameter $b_0>0$ and a suitable deformation Q_{b_0} of the soliton with

$$Q_{b_0}\to Q\quad \text{as } b_0\to 0$$

in some strong sense, we show that for any smooth and radially symmetric excess of energy

$$\|\eta_0,\eta_1\|_{H^2\times H^1}\lesssim\frac{b_0^2}{|\log(b_0)|},$$

we can find $d_+(b_0,\eta_0,\eta_1)\in\mathbb{R}$ such that the solution to (1-1) with initial data

$$u_0=Q_{b_0}+\eta_0+d_+\psi,\quad u_1=b_0\left(\frac{N-2}{2}Q_{b_0}+y\cdot\nabla Q_{b_0}\right)+\eta_1,$$

blows up in finite time in the regime described by Theorem 1.1. Here ψ is the bound state of the linearized operator close to Q and generates the unstable mode, we refer to Definition 3.4 and Proposition 3.5 for precise statements. Hence the set of blow-up solutions we construct lives on a codimension one manifold in the radial class in some weak sense. Following [Krieger and Schlag 2007; Krieger and Schlag 2009], the proof that this set is indeed a codimension one manifold relies on proving some Lipschitz regularity of the map $(b_0, \eta_0, \eta_1) \to d_+(b_0, \eta_0, \eta_1)$, and in particular some local uniqueness to begin with. The analysis in [Krieger and Schlag 2009] shows that this may be a delicate step in some cases. Our solution is constructed using a soft continuous topological argument of Brouwer-type coupled with suitable monotonicity properties in the spirit of Cote, Marte and Merle [2009]. In other related settings (see [Martel 2005; Raphaël and Szeftel 2011]) this strategy has proved to be quite powerful for eventually achieving strong uniqueness results. This interesting question in our setting will require additional efforts and needs to be addressed separately in detail.

3. Extension to higher dimensions. We focus on the case of dimension $N = 4$ for the sake of simplicity. Our main objective is to provide a robust framework to construct $\mathscr{C}^\infty$ type II blow-up solutions. However, following the heuristic developed in [Raphaël and Rodnianski 2012], the blow-up speed (1-10) corresponds to the $k = 1$ case in (1-7), and we similarly conjecture in dimension $N \geq 5$ the existence of type II finite time blow-up solutions close to Q with blow-up speed

$$\lambda(t) \sim c_N \frac{T-t}{|\log(T-t)|^{1/(N-4)}}.$$

Note from (1-3) that the higher the dimension, the fastest the decay of the ground state Q, and that this should help avoid some difficulties that occur only in low dimension like in [Raphaël and Rodnianski 2012] for large homotopy number $k \geq 4$. We expect the strategy developed in this paper to carry over to the cases $N = 5$ and 6, but the extension to large dimension will be confronted in particular with the difficulty of the lack of smoothness of the nonlinearity. Let us also insist on the fact that the case $N = 4$ is in many ways the more delicate one in terms of the strong coupling of the main part of the solution and the outgoing tail due to the slow decay of Q, which results in the somewhat pathological blow-up speed (1-10). This comment becomes even more dramatic in dimension $N = 3$, where we expect our analysis to be applicable to the construction of $\mathscr{C}^\infty$ type II blow-up solutions, but this seems to require a slightly different approach.

Strategy of the proof of Theorem 1.1.

Step 1: *Approximate self-similar solution.* Let D, Λ denote the differential operators in (1-18). Exact self-similar solutions to (1-1) of the form

$$u(t,x) = \frac{1}{\lambda^{(N-2)/2}(t)} Q_b\left(\frac{x}{\lambda(t)}\right), \quad \text{with } b = -\lambda_t,$$

where Q_b satisfies the self-similar equation

$$\Delta Q_b - b^2 D\Lambda Q_b + Q_b^3 = 0, \tag{1-11}$$

are known to develop a singularity on the light cone $y = (T-t)/\lambda(t) = 1/b$ leading to an unbounded Dirichlet energy $\|\nabla Q_b\|_{L^2} = +\infty$; see [Kavian and Weissler 1990]. We therefore assume $0 < b \ll 1$ and consider a one term expansion approximation

$$Q_b = Q + b^2 T_1,$$

which injected into (1-11) yields, at the order b^2,

$$HT_1 = -D\Lambda Q. \tag{1-12}$$

Here H is the linearized operator close to Q given by

$$H = -\Delta - \frac{N+2}{N-2} Q^{4/(N-2)}. \tag{1-13}$$

The spectral structure of H is well known in connection to the fact that Q is an extremizer of the Sobolev embedding $\dot{H}^1 \to L^{2^*}$, and in the radial sector H admits one nonpositive eigenvalue with well localized eigenvector ψ,

$$H\psi = -\zeta\psi, \quad \zeta > 0, \tag{1-14}$$

and a *resonance* at the boundary of the continuum spectrum generated by the scaling invariance of (1-1),

$$H(\Lambda Q) = 0, \quad \Lambda Q(r) \sim \frac{C}{r^{N-2}} \text{ as } r \to +\infty. \tag{1-15}$$

In order to solve (1-12), we first remove the leading-order growth in the exact solution $T_1 = \frac{1}{4}|y|^2 Q$ which is consequence of the flux computation

$$(D\Lambda Q, \Lambda Q) = \tfrac{1}{2} \lim_{y\to+\infty} y^4 |\Lambda Q|^2 > 0 \tag{1-16}$$

due to the slow decay of Q in dimension $N = 4$ from (1-3). For this, we solve

$$HT_1 = -D\Lambda Q + c_b \Lambda Q \mathbf{1}_{y\le 1/b}, \quad \text{with } c_b = \frac{(D\Lambda Q, \Lambda Q)}{\int_{y\le 1/b} |\Lambda Q|^2} \sim \frac{1}{2\,|\log b|} \text{ as } b \to 0.$$

The purpose of this construction is to yield after a suitable localization process an $o(b^2)$ approximate solution to the self-similar equation (1-11) whose dominant term near and past the light cone is still given by Q itself in the sense that

$$b^2|T_1| \ll Q \quad \text{for } y \ge 1/b.$$

This identifies Q as the leading-order radiation term.[1]

Step 2: *Bootstrap estimates.* We now roughly consider initial data of the form

$$u_0 = Q_{b_0} + d_+\psi + \eta_0, \quad u_1 = b_0 \Lambda Q_{b_0} + \eta_1, \quad \text{with } |d_+| + \|\eta_0, \eta_1\|_{H^2\times H^1} \ll b_0^2, \tag{1-17}$$

and introduce a modulated decomposition of the flow

$$u(t,x) = \frac{1}{\lambda^{(N-2)/2}(t)} (Q_{b(t)} + \varepsilon)\left(t, \frac{x}{\lambda(t)}\right), \quad b(t) = -\lambda_t.$$

[1] See [Raphaël and Rodnianski 2012] for a further discussion on this issue and the role played by the nonvanishing Pohozaev integration (1-16).

Here we face the major difference between the power nonlinearity wave equation (1-1) and the critical wave map problem (1-4), which is the presence of a negative eigenvalue in the first case (1-14) for the linearized operator H close to Q. This induces an instability in the modulation equations for b, λ that is absent in the wave map case, leading to stable blow-up dynamics. However, we claim that the ODE-type instability generated by (1-14) is the *only* instability mechanism.

The situation is conceptually similar to the one studied in [Cote et al. 2009] where multisolitary wave solutions are constructed in the supercritical regime despite the presence of exponentially growing modes for the linearized operator which are absent in the subcritical regime. We adapt a similar scheme of proof that does not rely on a fixed point argument to solve the problem from infinity in time,[2] but by directly following the flow for *any* initial data of the form (1-17). This reduces the full problem to a one-dimensional dynamical system for which a clever classical continuity argument yields the existence of $d_+(b_0, \eta_0, \eta_1)$ such that the unstable mode is extinct, see Section 5.

The key is hence to control the flow under the *a priori* control of the unstable mode, and here we adapt the technology developed in [Raphaël and Rodnianski 2012] which relies on *monotonicity properties* of the linearized Hamiltonian at the H^2 level of regularity. However, the analysis in [Raphaël and Rodnianski 2012] heavily relies on the existence of a decomposition of the Hamiltonian,

$$H = A^*A, \quad A = -\partial_y + V(y),$$

which is central to the proof of the main monotonicity property and is lost in our setting. This forces us to revisit the approach in several ways, and to rely in particular on fine algebraic properties of the flow[3] near Q and coercivity properties of suitable quadratic forms in the spirit of [Martel and Merle 2002; Merle and Raphael 2005a] (see Lemma 4.7) which remarkably turn out to be almost explicit thanks to the formula (1-3). We are eventually able to find $d_+(b_0, \eta_0, \eta_1)$ for which, to leading order,

$$b_s \sim -c_b b^2 \sim -\frac{b^2}{2\,|\log b|}, \quad b = -\lambda_t, \quad \frac{ds}{dt} = \frac{1}{\lambda}, \quad |d_+| + \|\partial_{yy}\varepsilon\|_{L^2} \ll b^2,$$

and whose reintegration in time yields finite time blow-up in the regime described by Theorem 1.1.

Notation. We define differential operators

$$\Lambda f = \frac{N-2}{2} f + y \cdot \nabla f \quad (\dot{H}^1 \text{ scaling}), \quad Df = \frac{N}{2} f + y \cdot \nabla f \quad (L^2 \text{ scaling}). \tag{1-18}$$

Denoting by

$$(f, g) = \int fg = \int_0^{+\infty} f(r)g(r)r^{N-1}dr$$

the $L^2(\mathbb{R}^N)$ radial inner product, we observe the integration by parts formulas

$$(Df, g) = -(f, Dg) \quad \text{and} \quad (\Lambda f, g) + (\Lambda g, f) = -2(f, g). \tag{1-19}$$

[2] After renormalization of the time.

[3] See in particular (4-23), (4-38).

Given f and $\lambda > 0$, we shall write

$$f_\lambda(t,r) = \frac{1}{\lambda^{(N-2)/2}} f\left(t, \frac{r}{\lambda}\right),$$

and the rescaled space variable will always be denoted by

$$y = \frac{r}{\lambda}.$$

We let χ be a smooth positive radial cut off function, $\chi(r) = 1$ for $r \leq 1$ and $\chi(r) = 0$ for $r \geq 2$. For a given parameter $B > 0$, we let

$$\chi_B(r) = \chi\left(\frac{r}{B}\right). \tag{1-20}$$

Given $b > 0$, we set

$$B_0 = \frac{2}{b}, \qquad B_1 = \frac{|\log b|}{b}. \tag{1-21}$$

To clarify the exposition we use the notation $a \lesssim b$ for when there exists a constant C with no relevant dependency on (a, b) such that $a \leq Cb$. In particular, we do not allow constants C to depend on the parameter M except in Appendix A.

2. Computation of the modified self-similar profile

This section is devoted to the construction of an approximate self-similar solution Q_b which describes the dominant part of the blow-up profile inside the backward light cone from the singular point $(0, T)$ and displays a slow decay at infinity which is eventually responsible for the modifications to the blow-up speed with respect to the self-similar law. The key to this construction is the fact that the structure of the linearized operator H close to Q is completely explicit in the radial sector thanks to the explicit formulas at hand for the elements of the kernel.

We introduce the direction

$$\Phi = D\Lambda Q, \tag{2-1}$$

which displays the cancellation

$$|\Phi(y)| \lesssim \frac{1}{1+y^4} \tag{2-2}$$

and the crucial nondegeneracy which follows from the Pohozaev integration by parts formula,

$$(\Phi, \Lambda Q) = \lim_{y\to+\infty} \left(\tfrac{1}{2} y^4 |\Lambda Q|^2\right) = 32 > 0. \tag{2-3}$$

Proposition 2.1 (approximate self-similar solution). *Let M denote a large enough constant. Then there exists $b^*(M) > 0$ small enough such that for all $0 < b < b^*(M)$, there exists a smooth radially symmetric profile T_1 satisfying the orthogonality condition*

$$(T_1, \chi_M \Phi) = 0 \tag{2-4}$$

such that

$$P_{B_1} = Q + \chi_{B_1} b^2 T_1 \tag{2-5}$$

is an approximate self-similar solution in the following sense. Let

$$\Psi_{B_1} = -\Delta P_{B_1} + b^2 D\Lambda P_{B_1} - f(P_{B_1}), \tag{2-6}$$

then for all $k \geq 0$, $0 \leq y \leq 1/b^2$,

$$\left|\frac{d^k T_1}{dy^k}(y)\right| \lesssim \frac{1}{1+y^k}\left[\frac{1+|\log(by)|}{|\log b|}\mathbf{1}_{2\leq y\leq \frac{B_0}{2}} + \frac{1}{b^2y^2|\log b|}\mathbf{1}_{y\geq \frac{B_0}{2}} + \frac{\log(M)+|\log(1+y)|}{1+y^2}\right], \tag{2-7}$$

$$\left|\frac{d^k}{dy^k}\frac{\partial P_{B_1}}{\partial b}\right| \lesssim \frac{b\mathbf{1}_{y\leq 2B_1}}{1+y^k}\left[\frac{1+|\log(by)|}{|\log b|}\mathbf{1}_{2\leq y\leq \frac{B_0}{2}} + \frac{1}{b^2y^2|\log b|}\mathbf{1}_{y\geq \frac{B_0}{2}} + \frac{\log(M)+|\log(1+y)|}{1+y^2}\right], \tag{2-8}$$

and, for all $k \geq 0$, $y \geq 0$,

$$\begin{aligned}\left|\frac{d^k}{dy^k}(\Psi_{B_1} - c_b b^2 \chi_{B_0/4}\Lambda Q)\right| &\\ \lesssim \frac{b^4}{1+y^k}&\left[\frac{1+|\log(by)|}{|\log b|}\mathbf{1}_{2\leq y\leq B_0/2} + \frac{1}{b^2y^2|\log b|}\mathbf{1}_{2B_1\geq y\geq B_0/2} + \frac{\log(M)+|\log(1+y)|}{1+y^2}\mathbf{1}_{y\leq 2B_1}\right]\\ &+ \frac{b^2}{(1+y^{4+k})}\mathbf{1}_{y\geq B_1/2}\end{aligned} \tag{2-9}$$

for some constant

$$c_b = \frac{1}{2\,|\log b|}\left(1 + O\left(\frac{1}{|\log b|}\right)\right). \tag{2-10}$$

Proof.

<u>Step 1:</u> *Inversion of H*. The first Green's function of H is given from scaling invariance by

$$\Lambda Q(y) = \frac{N-2}{2\big(1+y^2/(N(N-2))\big)^{N/2}}\left(1 - \frac{y^2}{N(N-2)}\right), \tag{2-11}$$

which admits the asymptotics

$$\forall\, k \geq 0, \quad \frac{d^k(\Lambda Q)}{dy^k}(y) = \begin{cases} O(1) & \text{as } y \to 0,\\ O(y^{-(N-2+k)}) & \text{as } y \to \infty.\end{cases} \tag{2-12}$$

Now let

$$\Gamma(y) = -\Lambda Q(y)\int_1^y \frac{ds}{s^{N-1}(\Lambda Q)^2(s)}$$

be another (singular at the origin[4]) element of the kernel of H, which can be found from the Wronskian relation

$$\Gamma'\Lambda Q - \Gamma(\Lambda Q)' = \frac{-1}{y^{N-1}}.$$

From this we easily find the asymptotics of $\Gamma^{(k)}$ for any integer k:

$$\frac{d^k\Gamma}{dy^k}(y) = \begin{cases} O(y^{-(N-2+k)}) & \text{as } y \to 0,\\ O(y^{-k}) & \text{as } y \to \infty.\end{cases} \tag{2-13}$$

[4]Note that Γ must be smooth at $y = \sqrt{N(N-2)}$, where ΛQ vanishes, because of the radial ODE $H\Gamma = 0$.

A smooth solution to $Hw = F$ is

$$w(y) = \Gamma(y)\int_0^y F(s)\Lambda Q(s)s^{N-1}ds - \Lambda Q(y)\int_0^y F(s)\Gamma(s)s^{N-1}ds. \tag{2-14}$$

We now look for a solution to the self-similar equation in the form $Q + b^2 T_1$. This yields

$$\begin{aligned}\Psi_b &= -\Delta Q_b + b^2 D\Lambda Q_b - f(Q_b)\\ &= b^2(HT_1 + D\Lambda Q) + b^4 D\Lambda T_1 - \left[f(Q+b^2T_1) - f(Q) - b^2 f'(Q)T_1\right].\end{aligned} \tag{2-15}$$

<u>Step 2:</u> *Computation of* T_1. Thanks to the anomalous decay (2-2), we choose T_1 to be a solution of

$$\begin{cases} HT_1 = F = -D\Lambda Q + c_b \chi_{B_0/4}\Lambda Q,\\ (T_1, \chi_M \Phi) = 0,\end{cases} \tag{2-16}$$

with c_b chosen such that

$$(F, \Lambda Q) = 0. \tag{2-17}$$

That is, from the Pohozaev integration by parts formula — see (1-21) and (2-3) —

$$\begin{aligned} c_b &= \frac{(D\Lambda Q, \Lambda Q)}{(\chi_{B_0/4}\Lambda Q, \Lambda Q)} = \frac{1}{2}\frac{\lim_{y\to+\infty} y^4|\Lambda Q(y)|^2}{\int \chi_{B_0/4}|\Lambda Q|^2}\\ &= \frac{1}{2\,|\log b|}\left(1 + O\left(\frac{1}{|\log b|}\right)\right) \quad \text{as } b \to 0.\end{aligned}$$

This yields (2-10). Following (2-14), we first consider

$$\tilde{T}_1(y) = \Gamma(y)\int_0^y F(s)\Lambda Q(s)s^3ds - \Lambda Q(y)\int_0^y F(s)\Gamma(s)s^3ds. \tag{2-18}$$

The smoothness of $\tilde{T}_1$ at the origin follows from (2-18) together with elliptic regularity from (2-16). We now examine the behavior of $\tilde{T}_1$ at large y.

We first observe that, from the orthogonality (2-17),

$$\tilde{T}_1(y) = -\left[\Gamma(y)\int_y^{+\infty} F(s)\Lambda Q(s)s^3ds + \Lambda Q(y)\int_0^y F(s)\Gamma(s)s^3ds\right].$$

Hence, from the degeneracy $|D\Lambda Q| = O(y^{-4})$, this yields that for $B_0/2 \le y \le 1/b^2$,

$$\begin{aligned}|\tilde{T}_1(y)| &\lesssim \int_y^{+\infty}\frac{s^3}{(1+s^4)(1+s^2)}ds + \frac{1}{y^2}\left[\int_0^y \frac{1+s^3}{1+s^4}ds + |c_b|\int_0^{B_0}\frac{s^3}{1+s^2}ds\right]\\ &\lesssim \frac{|\log(1+y)|}{1+y^2} + \frac{1}{b^2y^2|\log b|}.\end{aligned} \tag{2-19}$$

Similarly, for $1 \le y \le B_0/2$,

$$
\begin{aligned}
|\tilde{T}_1(y)| &= \left|\Gamma(y)\int_y^{+\infty} F(s)\Lambda Q(s)s^3\,ds + \Lambda Q(y)\int_0^y F(s)\Gamma(s)s^3\,ds\right| \\
&\lesssim \int_y^{+\infty}\frac{s^3}{(1+s^4)(1+s^2)}ds + |c_b|\int_y^{B_0}\frac{s^3}{(1+s^2)^2}ds + \frac{1}{1+y^2}\left[\int_0^y\frac{s^3}{1+s^4}ds + |c_b|\int_0^y\frac{s^3}{1+s^2}ds\right] \\
&\lesssim \frac{1+|\log(by)|}{|\log b|} + \frac{|\log(1+y)|}{1+y^2}. \qquad (2\text{-}20)
\end{aligned}
$$

We now choose, thanks to (2-3),

$$
T_1(y) = \tilde{T}_1(y) - c\Lambda Q \quad \text{with } c = \frac{(\tilde{T}_1, \chi_M\Phi)}{(\chi_M\Phi, \Lambda Q)},
$$

so that the orthogonality condition (2-4) is fulfilled. We note that the bounds (2-19) and (2-20) ensure that c remains bounded by $\log(M)$ uniformly in M and b, provided b is chosen sufficiently small with respect to M.

This yields (2-7) for $k=0$; the other cases follow similarly.

Step 3: *Estimate on Ψ_{B_1} and $\partial_b\Psi_{B_1}$.* We now cut off the slow decaying tail T_1 according to (2-5) and estimate the corresponding error to self-similarity Ψ_{B_1} given by (2-6).

We compute

$$
\begin{aligned}
\Psi_{B_1} = b^2\chi_{B_1}(HT_1 + D\Lambda Q) + b^2\big[-2\chi'_{B_1}T'_1 - T_1\Delta\chi_{B_1} + (1-\chi_{B_1})D\Lambda Q + b^2 D\Lambda(\chi_{B_1}T_1)\big] \\
- \big[f(Q + b^2\chi_{B_1}T_1) - f(Q) - \chi_{B_1}f'(Q)T_1\big].
\end{aligned}
$$

Outside the support of χ_{B_1} we have thus $\Psi_{B_1} = b^2 D\Lambda Q$. On the other hand, in dimension $N=4$, we have the Taylor expansion

$$
f(Q + b^2\chi_{B_1}T_1) - f(Q) - \chi_{B_1}f'(Q)T_1 = b^4\chi_{B_1}^2 T_1^2(y)\int_0^1 (1-\tau)(Q(y) + \tau b^2\chi_{B_1}T_1(y))\,d\tau.
$$

We thus estimate from (2-7), (2-15), (2-16) and the degeneracy (2-2) for $y \le 2B_1$ that

$$
\begin{aligned}
\big|\Psi_{B_1} - b^2 c_b\chi_{B_0/4}\Lambda Q\big| \lesssim b^2\mathbf{1}_{y\ge B_1/2}\left(\frac{T'_1}{1+y} + \frac{T_1}{1+y^2} + \frac{1}{1+y^4}\right) \\
+ b^4|D\Lambda(\chi_{B_1}T_1)| + b^4|T_1^2(y)|\int_0^1(1-\tau)|Q(y) + \tau b^2 T_1(y)|\,d\tau.
\end{aligned}
$$

(2-7) now yields (2-9) for $k=0$. Further derivatives are estimated similarly thanks to the smoothness of the nonlinearity. We emphasize here that, given $B>0$ large, we have $1/(1+y) \lesssim 1/B \lesssim 1/(1+y)$ on the support of χ'_B, so that differentiating χ_B acts as multiplication by $1/(1+y)$. Furthermore, we have $1/B_1 = o(b)$ so that we can always dominate $1/(1+y)$ by b on the support of χ'_{B_1}.

Finally, we compute $\partial_b P_{B_1}$ from (2-5).

To this end, we note that $\partial_b c_b = O\Big(\frac{1}{b|\log(b)|^2}\Big)$ when $b \to 0$, so that the source term for T_1 in (2-16) satisfies

$$\partial_b F = \left[O\left(\frac{1}{b\,|\log b|}\right)\chi_{B_0/4} + O\left(\frac{1}{b\,|\log b|}\right)\rho_{B_0/4}\right]\Lambda Q,$$

where $\rho(z) = z\chi'(z) \in \mathscr{C}^\infty_c(0,\infty)$ and we keep the convention for function dilation. Hence, the same arguments as for T_1 enable us to show first that $\partial_b \tilde{T}_1$, and then $\partial_b T_1$, satisfy the estimates

$$\left|\frac{d^k \partial_b T_1}{dy^k}(y)\right| \lesssim \frac{1}{b(1+y^k)}\left[\frac{1+|\log(by)|}{|\log b|}\mathbf{1}_{2\le y\le B_0/2} + \frac{1}{b^2y^2|\log b|}\mathbf{1}_{y\ge B_0/2} + \frac{1+|\log(1+y)|}{1+y^2}\right]. \tag{2-21}$$

Finally, we compute from (2-5) that

$$\partial_b P_{B_1} = 2b\chi_{B_1}T_1 + b^2\partial_b\log(B_1)\rho_{B_1}T_1 + b^2\chi_{B_1}\partial_b T_1. \tag{2-22}$$

This decomposition, together with (2-7) and the previous computation, yield (2-8), which concludes the proof of Proposition 2.1. □

3. Description of the trapped regime

We display in this section the regime which leads to the blow-up dynamics described by Theorem 1.1.

***Modulation of solutions to* (1-1).** Let us start by describing the set of solutions among which the finite time blow-up scenario described by Theorem 1.1 is likely to arise. We recall from (1-14) that ψ denotes the bound state of H with eigenvalue $-\zeta < 0$. The following lemma is a standard consequence of the implicit function theorem and the smoothness of the flow; see Appendix A.

Lemma 3.1 (modulation theory). *Let M be a large constant to be chosen later and $0 < b_0 < b_0^*(M)$ small enough. Let (η_0, η_1, d_+) satisfy the smallness condition*

$$|d_+| + \left\|\eta_0, \nabla\eta_0, \eta_1 + b_0(1-\chi_{B_1(b_0)})\Lambda Q, \nabla\eta_1\right\|_{\dot H^1\times\dot H^1\times L^2\times L^2} \lesssim \frac{b_0^2}{|\log b_0|}, \tag{3-1}$$

then there exists a time T_0 such that the unique solution $u \in \mathscr{C}^2([0,T_0]; L^2(\mathbb{R}^N)) \cap \mathscr{C}([0,T_0]; H^2(\mathbb{R}^N))$ to (1-1) *with initial data*

$$u_0 = P_{B_1(b_0)} + \eta_0 + d_+\psi, \quad u_1 = b_0\Lambda P_{B_1(b_0)} + \eta_1, \tag{3-2}$$

admits on $[0,T_0]$ a unique decomposition

$$u(t) = (P_{B_1(b(t))} + \varepsilon(t))_{\lambda(t)} \tag{3-3}$$

with $\lambda \in \mathscr{C}^2([0,T_0], \mathbb{R}_+^*)$ such that

$$(\varepsilon(t), \chi_M\Phi) = 0 \text{ and } b(t) = -\lambda_t \quad \text{for all } t \in [0,T_0], \tag{3-4}$$

and the following smallness condition is satisfied:

$$\|\nabla\varepsilon(t)\|_{L^2} \lesssim b_0|\log b_0|, \quad |b(t)-b_0| + |\lambda(t)-1| + \|\nabla^2\varepsilon(t)\|_{L^2} \lesssim \frac{b_0^2}{|\log b_0|} \quad \text{for all } t \in [0,T_0]. \tag{3-5}$$

Remark 3.2. Recall that the slow decay of Q and the choice of P_{B_1} induces an unbounded tail of ΛP_{B_1} in the energy norm, and more specifically $\|\Lambda Q\|_{L^2} = +\infty$, hence the need for the compensation in the norm of the time derivative in (3-1).

Decomposition of the flow and modulation equations. Considering initial data satisfying the assumption of the above lemma, we now write the evolution equation induced by (1-1) in terms of the decomposition (3-3). Let

$$u(t,r) = \frac{1}{[\lambda(t)]^{N/2-1}}\big(P_{B_1(b(t))}+\varepsilon\big)\left(t, \frac{r}{\lambda(t)}\right) = \big(P_{B_1(b(t))}\big)_{\lambda(t)} + w(t,r), \tag{3-6}$$

where $b = -\lambda_t$. Let us derive the equations for w and ε. Let

$$s(t) = \int_0^t \frac{d\tau}{\lambda(\tau)} \tag{3-7}$$

be the rescaled time. We shall make an intensive use of the rescaling formulas

$$u(t,r) = \frac{1}{\lambda^{N/2-1}} v(s,y), \quad y = \frac{r}{\lambda}, \quad \frac{ds}{dt} = \frac{1}{\lambda}, \tag{3-8}$$

$$\partial_t u = \frac{1}{\lambda}(\partial_s v + b\Lambda v)_\lambda, \tag{3-9}$$

$$\partial_{tt} u = \frac{1}{\lambda^2}\big[\partial_s^2 v + b(\partial_s v + 2\Lambda\partial_s v) + b^2 D\Lambda v + b_s \Lambda v\big]_\lambda. \tag{3-10}$$

In particular, we derive from (1-1) the equation for ε,

$$\partial_s^2\varepsilon + H_{B_1}\varepsilon = -\Psi_{B_1} - b_s\Lambda P_{B_1} - b(\partial_s P_{B_1} + 2\Lambda\partial_s P_{B_1}) - \partial_s^2 P_{B_1} - b(\partial_s\varepsilon + 2\Lambda\partial_s\varepsilon) - b_s\Lambda\varepsilon + N(\varepsilon), \tag{3-11}$$

where, implicitly, $B_1 = B_1(b(t))$ and H_{B_1} is the linear operator associated to the profile P_{B_1},

$$H_{B_1}\varepsilon = -\Delta\varepsilon + b^2 D\Lambda\varepsilon - f'(P_{B_1})\varepsilon, \tag{3-12}$$

and the nonlinearity

$$N(\varepsilon) = f(P_{B_1}+\varepsilon) - f(P_{B_1}) - f'(P_{B_1})\varepsilon. \tag{3-13}$$

Alternatively, the equation for w takes the form

$$\partial_t^2 w + \tilde{H}_{B_1} w = -\big[\partial_t^2 (P_{B_1})_\lambda - \Delta(P_{B_1})_\lambda - f((P_{B_1})_\lambda)\big] + N_\lambda(w),$$

with

$$\tilde{H}_{B_1} w = -\Delta w - f'((P_{B_1})_\lambda) w, \tag{3-14}$$

$$N_\lambda(w) = f((P_{B_1})_\lambda + w) - f((P_{B_1})_\lambda) - f'((P_{B_1})_\lambda) w. \tag{3-15}$$

We then expand using (3-9), (3-10), obtaining

$$\begin{aligned}\partial_t^2(P_{B_1})_\lambda-\Delta(P_{B_1})_\lambda-f((P_{B_1})_\lambda)&=\frac{1}{\lambda^2}\big[\partial_{ss}P_{B_1}+b(\partial_s P_{B_1}+2\Lambda\partial_s P_{B_1})+b_s\Lambda P_{B_1}+\Psi_{B_1}\big]_\lambda\\&=\frac{1}{\lambda^2}\big[b\Lambda\partial_s P_{B_1}+b_s\Lambda P_{B_1}+\Psi_{B_1}\big]_\lambda+\partial_t\bigg[\frac{1}{\lambda}(\partial_s P_{B_1})_\lambda\bigg],\end{aligned}$$

and rewrite the equation for w as

$$\partial_t^2 w+\tilde{H}_{B_1}w=-\frac{1}{\lambda^2}\big[b\Lambda\partial_s P_{B_1}+b_s\Lambda P_{B_1}+\Psi_{B_1}\big]_\lambda-\partial_t\bigg[\frac{1}{\lambda}(\partial_s P_{B_1})_\lambda\bigg]+N_\lambda(w). \tag{3-16}$$

For most of our arguments we prefer to view the linear operator H_{B_1} acting on w in (3-16) as a perturbation of the linear operator H_λ associated to Q_λ. Then

$$\begin{aligned}&\partial_t^2 w+H_\lambda w=F_{B_1} \qquad\qquad (3\text{-}17)\\&\quad=-\frac{1}{\lambda^2}\big[b\Lambda\partial_s P_{B_1}+b_s\Lambda P_{B_1}+\Psi_{B_1}\big]_\lambda-\partial_t\bigg[\frac{1}{\lambda}(\partial_s P_{B_1})_\lambda\bigg]-\big[f'(Q_\lambda)-f'((P_{B_1})_\lambda)\big]w+N_\lambda(w),\end{aligned}$$

with

$$H_\lambda w=-\Delta w+f'(Q_\lambda)w. \tag{3-18}$$

The set of bootstrap estimates. First we fix some notations. We introduce the energy $\mathscr{E}(t)$ associated to the Hamiltonian H_λ,

$$\mathscr{E}(t)=\lambda^2\int\big[(H_\lambda\partial_t w,\,\partial_t w)+(H_\lambda w)^2\big]. \tag{3-19}$$

Given the unstable eigenvalue $\zeta\in(0,\infty)$, we set

$$V_+=\left|\begin{array}{l}1\\ \sqrt{\zeta}\end{array}\right.,\qquad V_-=\left|\begin{array}{l}1\\ -\sqrt{\zeta}\end{array}\right., \tag{3-20}$$

and introduce the decomposition of the unstable direction,

$$\left|\begin{array}{l}(\varepsilon,\,\psi)\\ (\partial_s\varepsilon,\,\psi)\end{array}\right.=\tilde{a}_+(s)V_++\tilde{a}_-(s)V_-. \tag{3-21}$$

Let us write

$$\kappa_+(s)=\tilde{a}_+(s)+\frac{b_s}{2\sqrt{\zeta}}(\partial_b P_{B_1},\,\psi),\qquad \kappa_-(s)=\tilde{a}_-(s)-\frac{b_s}{2\sqrt{\zeta}}(\partial_b P_{B_1},\,\psi). \tag{3-22}$$

We note that the vectors V_+, V_- given by (3-20) yield an eigenbasis of

$$\begin{pmatrix}0&1\\ \zeta&0\end{pmatrix}$$

and hence correspond respectively to the unstable and stable mode of the two dimensional dynamical system

$$\frac{dY}{ds}=\begin{pmatrix}0&1\\ \zeta&0\end{pmatrix}Y,$$

which, to first order in b, is verified by the projection onto the unstable mode (ε, ψ); see (4-57). The deformation term $b_s(\partial_b P_{B_1}, \psi)$ in (3-22) is present to handle some possible time oscillations induced by the $\partial_s^2 P_{B_1}$ term in the right-hand side of (3-11), which cannot be estimated in absolute value but will be proved to be of lower order.

With these conventions, we may now parametrize the set of initial data described by Lemma 3.1 by $a_+ = \kappa_+(0)$, and then reformulate the initial smallness properties in terms of suitable initial bounds for ε; see Appendix A for the proof, which is standard.

Lemma 3.3 (initial parametrization of the unstable mode and initial bounds). *Let M and b_0 be given as in Lemma 3.1 and denote by $C(M)$ a sufficiently large constant. Then, given (η_0, η_1, a_+) satisfying*

$$|a_+| + \|\eta_0, \nabla\eta_0, \eta_1 + b_0(1-\chi_{B_1(b_0)})\Lambda Q, \nabla\eta_1\|_{\dot H^1\times\dot H^1\times L^2\times L^2} \le \frac{b_0^2}{|\log b_0|}, \tag{3-23}$$

there exists a unique d_+ with $|d_+| \lesssim b_0^2/|\log(b_0)|$ and $T_0 > 0$ such that the unique decomposition

$$u(t) = (P_{B_1(b(t))} + \varepsilon)_{\lambda(t)} = (P_{B_1(b(t))})_{\lambda(t)} + w(t)$$

of the unique smooth solution u to (1-1) *on* $[0, T_0]$ *with initial data* (3-2) *satisfies the initialization*

$$\kappa_+(0) = a_+ \tag{3-24}$$

and the following smallness conditions on $[0, T_0]$:

- *Smallness and positivity of b*:
$$0 < b(t) < 5b_0. \tag{3-25}$$
- *Pointwise bound on b_s*:
$$|b_s(t)|^2 \le C(M)\frac{[b(t)]^4}{|\log b(t)|^2}. \tag{3-26}$$
- *Smallness of the energy norm*:
$$\left\| \nabla w(t), \partial_t w(t) + \frac{b(t)}{\lambda(t)}\big((1-\chi_{B_1(b(t))})\Lambda Q\big)_{\lambda(t)} \right\|_{L^2\times L^2} \le \sqrt{b_0}. \tag{3-27}$$
- *Global $\dot H^2$ bound*:
$$|\mathscr{E}(t)| \le C(M)\frac{[b(t)]^4}{|\log b(t)|^2}. \tag{3-28}$$
- *A priori bound on the stable mode*:
$$|\kappa_-(t)| \le (C(M))^{1/8}\frac{[b(t)]^2}{|\log b(t)|}. \tag{3-29}$$
- *A priori bound of the unstable mode*:
$$|\kappa_+(t)| \le 2\frac{[b(t)]^2}{|\log b(t)|}. \tag{3-30}$$

We can now describe the bootstrap regime.

Definition 3.4 (exit time). Let $K(M)$ be a large constant. Given $a_+ \in [-b_0^2/|\log b_0|, b_0^2/|\log b_0|]$, we let $T(a_+)$ be the life time of the solution to (1-1) with initial data (3-2), and $T_1(a_+) > 0$ be the supremum of $T \in (0, T(a_+))$ such that for all $t \in [0, T]$, the following estimates hold:

- Smallness and positivity of b:
$$0 < b(t) < 5b_0. \tag{3-31}$$
- Pointwise bound on b_s:
$$|b_s|^2 \leq K(M)\frac{[b(t)]^4}{|\log b(t)|^2}. \tag{3-32}$$
- Smallness of the energy norm:
$$\left\| \nabla w(t), \partial_t w(t) + \frac{b(t)}{\lambda(t)}\big((1-\chi_{B_1(b(t))})\Lambda Q\big)_{\lambda(t)} \right\|_{L^2\times L^2} \leq \sqrt{b_0}. \tag{3-33}$$
- *Global* $\dot{H}^2$ bound:
$$|\mathscr{E}(t)| \leq K(M)\frac{[b(t)]^4}{|\log b(t)|^2}. \tag{3-34}$$
- *A priori* bound on the stable and unstable modes:
$$|\kappa_+(t)| \leq 2\frac{[b(t)]^2}{|\log b(t)|}, \quad |\kappa_-(t)| \leq (K(M))^{1/8}\frac{[b(t)]^2}{|\log b(t)|}. \tag{3-35}$$

The existence of blow-up solutions in the regime described by Theorem 1.1 now follows from the following proposition:

Proposition 3.5. *There exists* $a_+ \in [-b_0^2/|\log b_0|, b_0^2/|\log b_0|]$ *such that*
$$T_1(a_+) = T(a_+).$$
Then the corresponding solution to (1-1) *blows up in finite time in the regime described by Theorem 1.1.*

The proof of Proposition 3.5 relies on a monotonicity argument applied to the energy $\mathscr{E}$, which is the core of the analysis (see Proposition 4.6), and the strictly outgoing behavior of the unstable mode induced by the nontrivial eigenvalue $-\zeta < 0$ of H (see Lemma 4.10). The fact that the regime described by the bootstrap bounds (3-31)–(3-35) corresponds to a finite blow-up solution with a specific blow-up speed will then follow from the modulation equations and the sharp derivation of the blow speed as in [Raphaël and Rodnianski 2012].

4. Improved bounds

This section is devoted to the derivation of the main dynamical properties of the flow in the bootstrap regime described by Definition 3.4. The three main steps are first the derivation of a monotonicity property on $\mathscr{E}$, which allows us to improve the bounds (3-31)–(3-34) in $[0, T_1(a_+)]$, second the derivation of the dynamics of the eigenmode and the outgoing behavior of the unstable direction, and third the derivation of the sharp law for the parameter b, which allows to bootstrap its smallness (3-31) and will eventually allow us to derive the sharp blow-up speed.

Remark 4.1. Throughout the proof, we will introduce various constants $C(M), \delta(M) > 0$ *that do not depend* on the bootstrap constant $K(M)$. An important feature of all these constants is that, up to a smaller choice of $b^*(M)$ or a larger choice of $K(M)$, we assume that any product of the form $C(M) f(b)$, where $\lim_{b\to 0} f(b) = 0$, or that any ratio $\delta(M)/K(M)$ is small in the trapped regime. This will be used implicitly in this section.

Coercivity of $\mathscr{E}$**.** Let us start by showing that the linearized energy $\mathscr{E}$ yields a control of suitable weighted norms of (w, ε) in the regime $t \in [0, T_1(a_+)]$.

Lemma 4.2 (coercivity of $\mathscr{E}$). *There exists $M_0 \geq 1$ such that for all $M \geq M_0$, there exists*[5] $\delta(M) > 0$ *and* $C(M) < \infty$ *such that in the interval* $[0, T_1(a_+))$,

$$\mathscr{E} \geq \tfrac{1}{2}\lambda^2 \int (H_\lambda w)^2 + \delta(M)\lambda^2 \left[\int (\nabla \partial_t w)^2 + \int \frac{(\partial_r w)^2}{r^2}\right] - C(M)[K(M)]^{1/4} \frac{b^4}{|\log b|^2}. \tag{4-1}$$

Proof of Lemma 4.2. This is a consequence of the explicit distribution of the negative eigenvalues of H and the *a priori* bound on the unstable mode (3-35). Indeed, let $t \in [0, T_1(a_+))$, then first observe from (3-21), (3-22), (3-35) that

$$\begin{aligned} |(\varepsilon, \psi)|^2 + |(\partial_s \varepsilon, \psi)|^2 &\lesssim |\kappa_+|^2 + |\kappa_-|^2 + |b_s|^2 (\partial_b P_{B_1}, \psi)^2 \\ &\lesssim [K(M)]^{1/4} \frac{b^4}{|\log b|^2} + C(M) b^2 |b_s|^2 \lesssim [K(M)]^{1/4} b^4 / |\log b|^2, \end{aligned} \tag{4-2}$$

where we used the estimates of Proposition 2.1 and the fact that ψ is well localized. This yields

$$\begin{aligned} \frac{1}{\lambda^4}(w, \psi_\lambda)^2 + \frac{1}{\lambda^2}(\partial_t w, \psi_\lambda)^2 &= (\varepsilon, \psi)^2 + (\partial_s \varepsilon + b\Lambda\varepsilon, \psi)^2 \\ &\lesssim [K(M)]^{1/4} \frac{b^4}{|\log b|^2} + b^2 \left[\int \frac{\varepsilon^2}{y^4(1 + |\log(y)|)^2} + \int \frac{|\nabla \varepsilon|^2}{y^2}\right], \end{aligned} \tag{4-3}$$

and similarly, using the orthogonality condition (3-4),

$$\begin{aligned} \frac{1}{\lambda^4}(w, (\chi_M \Phi)_\lambda)^2 + \frac{1}{\lambda^2}(\partial_t w, (\chi_M \Phi)_\lambda)^2 &= (b\Lambda\varepsilon, \chi_M \Phi)^2 \\ &\lesssim b^2 M^C \left[\int \frac{\varepsilon^2}{y^4(1 + |\log(y)|)^2} + \int \frac{|\nabla \varepsilon|^2}{y^2}\right]. \end{aligned} \tag{4-4}$$

Applying Lemma C.3 yields

$$\lambda^2 \int |H_\lambda w|^2 = \int |H\varepsilon|^2 \geq \delta(M) \left[\int \frac{|\nabla \varepsilon|^2}{y^2} + \frac{\varepsilon^2}{y^4(1 + |\log(y)|)^2}\right].$$

[5] Recall Remark 4.1.

Introducing the rescaled version (C-13) of Lemma C.3, we then conclude that

$$\begin{aligned}\mathscr{E} \geq \frac{1}{2}\int \lambda^2(H_\lambda w)^2 + \delta_1(M)\left[\lambda^2\int(\nabla\partial_t w)^2 + \int\frac{|\nabla\varepsilon|^2}{y^2} + \int\frac{\varepsilon^2}{y^4(1+|\log(y)|)^2}\right]\\ - b^2M^C\left[\int\frac{\varepsilon^2}{y^4(1+|\log(y)|)^2} + \int\frac{|\nabla\varepsilon|^2}{y^2}\right] - C(M)[K(M)]^{1/4}\frac{b^4}{|\log b|^2}\\ \geq \frac{1}{2}\int\lambda^2(H_\lambda w)^2 + \delta(M)\lambda^2\left[\int(\nabla\partial_t w)^2 + \int\frac{(\partial_r w)^2}{r^2}\right] - C(M)[K(M)]^{1/4}\frac{b^4}{|\log b|^2},\end{aligned}$$

where we used the Hardy bound (C-3), and (4-1) is proved. □

Remark 4.3. Note that (4-1) together with the Hardy estimate (C-1), the coercivity estimate (C-9) and (4-4) yield the following weighted bound on ε which will be extensively used in the paper: Let

$$\eta(s,y) = \lambda^{(N-2)/2+1}\partial_t w(t,\lambda y) = \partial_s\varepsilon(s,y) + b\Lambda\varepsilon(s,y), \tag{4-5}$$

then

$$\int\frac{\varepsilon^2}{y^4(1+|\log y|^2)} + \int\frac{\eta^2}{y^2} + \int\frac{|\nabla\varepsilon|^2}{y^2} + \int|\nabla\eta|^2 \lesssim c(M)\left[|\mathscr{E}| + [K(M)]^{1/4}\frac{b^4}{|\log b|^2}\right], \tag{4-6}$$

$$\lesssim c(M)|\mathscr{E}| + \sqrt{K(M)}\frac{b^4}{|\log b|^2}. \tag{4-7}$$

First bound on b_s. We now derive a crude bound on b_s which appears as an *order-one* forcing term in the right-hand side of the equation (3-11) for ε. This bound is a simple consequence of the construction of the profile Q_b and the choice of the orthogonality condition (3-4).

Lemma 4.4 (rough pointwise bound on b_s). *We have the bound*[6]

$$\left(b_s + \frac{(\varepsilon, H\Phi)}{(\Lambda Q,\Phi)}\right)^2 \lesssim \frac{1}{M}|\mathscr{E}| + \sqrt{K(M)}\frac{b^4}{|\log b|^2}. \tag{4-8}$$

Remark 4.5. This is in contrast with [Raphaël and Rodnianski 2012], where the b_s term could be treated as degenerate with respect to ε thanks to a specific choice of orthogonality conditions and the factorization of the operator H in the wave map case. This difficulty in our case will be treated using a specific algebra generated by our choice of orthogonality condition (3-4) which gives the *right sign* to the leading-order terms involving b_s in the energy identity of Proposition 4.6; see (4-24), (4-38).

Proof of Lemma 4.4. Let us recall that the equation for ε in rescaled variables is given by (3-11)–(3-13). Observe also that from (1-19), the adjoint of H_B with respect to the $L^2(\mathbb{R}^N)$ inner product is

$$H^*_{B_1} = H_{B_1} + 2b^2D. \tag{4-9}$$

To compute b_s we take the scalar product of (3-11) with $\chi_M\Phi$. Using the orthogonality relations

$$(\partial_s^m\varepsilon, \chi_M\Phi) = (\partial_s^m(P_{B_1}-Q), \chi_M\Phi) = 0 \quad \text{for all } m\geq 0,$$

[6] Recall Remark 4.1.

we integrate by parts to get the algebraic identity

$$b_s\big[(\Lambda P_{B_1}, \chi_M\Phi) + 2b(\Lambda\partial_b P_{B_1}, \chi_M\Phi) + (\Lambda\varepsilon, \chi_M\Phi)\big] \\ = -(\Psi_{B_1}, \chi_M\Phi) - (\varepsilon, H^*_{B_1}(\chi_M\Phi)) + 2b(\partial_s\varepsilon, \Lambda(\chi_M\Phi)) + (N(\varepsilon), \chi_M\Phi). \tag{4-10}$$

We first derive from the estimates of Proposition 2.1 that

$$(\Psi_{B_1}, \chi_M\Phi)^2 \lesssim \frac{b^4}{|\log b|^2}. \tag{4-11}$$

Similarly, using (4-6) yields

$$(\partial_s\varepsilon, \Lambda(\chi_M\Phi))^2 \lesssim C(M)\left[c(M)|\mathscr{E}| + \sqrt{K(M)}\frac{b^4}{|\log b|^2}\right] \tag{4-12}$$

and

$$(\varepsilon, H^*_{B_1}(\chi_M\Phi)) = (\varepsilon, H\Phi) - (H\varepsilon, (1-\chi_M)\Phi) + O\left(M^C b^2\sqrt{c(M)|\mathscr{E}| + \sqrt{K(M)}b^4/|\log b|^2}\right).$$

We then use the improved decay (2-2) and (4-7) to estimate

$$(H\varepsilon, (1-\chi_M)\Phi)^2 \lesssim \left(\int_{y\geq M}\frac{|H\varepsilon|}{1+y^N}\right)^2 \lesssim \frac{|\mathscr{E}|}{M} + \sqrt{K(M)}\frac{b^4}{|\log b|^2}.$$

Thus

$$\left|(\varepsilon, H^*_{B_1}(\chi_M\Phi)) - (\varepsilon, H\Phi)\right|^2 \lesssim \frac{1}{M}|\mathscr{E}| + \sqrt{K(M)}\frac{b^4}{|\log b|^2}. \tag{4-13}$$

Similarly,

$$(\Lambda P_{B_1}, \chi_M\Phi) + 2b(\Lambda\partial_b P_{B_1}, \chi_M\Phi) + (\Lambda\varepsilon, \chi_M\Phi) \\ = (\Lambda Q, \Phi) + O\left(\frac{b}{|\log b|} + M^C\sqrt{|\mathscr{E}| + \sqrt{K(M)}b^4/|\log b|^2}\right) \\ = (\Lambda Q, \Phi) + O\left(\frac{b}{|\log b|}\right), \tag{4-14}$$

where we have used that in the trapped regime we have $\mathscr{E} \leq K(M)b^4/[\log(b)]^2$. Finally, on the support of χ_M and for $b < b_0^*(M)$ small enough, the term Q dominates in $Q_b = Q + b^2T_1$. Hence, for the nonlinear term, we have from the Sobolev inequality and (4-7) that

$$|(N(\varepsilon), \chi_M\Phi)| \lesssim \int\left(\frac{\varepsilon^2}{1+y^6} + \frac{\varepsilon^3}{1+y^4}\right) \lesssim \int\frac{|\varepsilon|^2}{(1+y^5)}\,[1 + \|y\varepsilon\|_{L^\infty}] \lesssim C(M)\left[\mathscr{E} + \sqrt{K(M)}\frac{b^4}{|\log b|^2}\right].$$

Injecting this, together with (4-11)–(4-14), into (4-10) yields (4-8).[7] □

[7] Recall Remark 4.1.

Global $\dot{H}^2$ bound. We derive in this section a monotonicity statement for the energy $\mathscr{E}$ that provides a global $\dot{H}^2$ estimate for the solution. The monotonicity statement involves suitable repulsive properties of the rescaled Hamiltonian H_λ in the focusing regime under the orthogonality condition (3-11) and the *a priori* control of the unstable mode (3-35), which themselves rely on the positivity of an explicit quadratic form; see Lemma 4.7.

Proposition 4.6 (H^2 control of the radiation). *In the trapped regime, there exists a function $\mathscr{F}$ satisfying*

$$\mathscr{F} \lesssim \frac{\mathscr{E}}{M} + \sqrt{K(M)}\frac{b^4}{|\log b|^2} \tag{4-15}$$

such that, for some $0<\alpha<1$ close enough to 1, we have

$$\frac{d}{dt}\left\{\frac{\mathscr{E}+\mathscr{F}}{\lambda^{2(1-\alpha)}}\right\} \le \frac{b}{\lambda^{3-2\alpha}}\left[\sqrt{K(M)}\frac{b^4}{|\log b|^2}\right]. \tag{4-16}$$

Proof.

Step 1: *Energy identity.* Let

$$\widetilde{V}(t,r) = \frac{N+2}{N-2}Q_\lambda^{4/(N-2)}(r) = \frac{1}{\lambda^2}V\left(\frac{r}{\lambda}\right), \quad V(y) = \frac{N+2}{N-2}Q^{4/(N-2)}(y).$$

We first have an algebraic energy identity that follows by integrating by parts from (3-17),

$$\frac{1}{2}\frac{d}{dt}\left\{\int(\partial_{tr}w)^2 - \int\widetilde{V}(\partial_t w)^2 + \int(H_\lambda w)^2\right\} = -\int\partial_t\widetilde{V}\left[\frac{(\partial_t w)^2}{2} + wH_\lambda w\right] + \int\partial_t w H_\lambda F_{B_1}. \tag{4-17}$$

We now use the w equation and integration by parts to compute

$$\begin{aligned} -\int\partial_t\widetilde{V}wH_\lambda w &= -\int\partial_t\widetilde{V}w(F_{B_1}-\partial_{tt}w) && (4\text{-}18)\\ &= \frac{d}{dt}\left\{\int\partial_t\widetilde{V}w\partial_t w\right\} - \int\partial_t\widetilde{V}wF_{B_1} - \int\partial_t\widetilde{V}(\partial_t w)^2 - \int\partial_{tt}\widetilde{V}w\partial_t w. && (4\text{-}19)\end{aligned}$$

We next pick $0<\alpha<1$ close enough to 1 and combine the above identities to get

$$\begin{aligned}\frac{1}{2\lambda^{2\alpha}}\frac{d}{dt}&\left\{\lambda^{2\alpha}\left[\int(\partial_{tr}w)^2 - \int\widetilde{V}(\partial_t w)^2 + \int(H_\lambda w)^2 - 2\int\partial_t\widetilde{V}w\partial_t w\right]\right\}\\ &= -R_1 + R_2 + \frac{2\alpha b}{\lambda}\int\partial_t\widetilde{V}w\partial_t w - \int\partial_{tt}\widetilde{V}w\partial_t w, \qquad (4\text{-}20)\end{aligned}$$

where R_1 collects the quadratic terms

$$\begin{aligned}R_1 &= \frac{\alpha b}{\lambda}\left[\int(\partial_{tr}w)^2 - \int\widetilde{V}(\partial_t w)^2 + \int(H_\lambda w)^2\right] + \frac{3}{2}\int\partial_t\widetilde{V}(\partial_t w)^2 - \frac{b_s}{\lambda^2}\int\partial_t\widetilde{V}(\Lambda Q)_\lambda w\\ &= \frac{b}{\lambda^3}\left[\alpha\int(\partial_y\eta)^2 - \alpha\int V\eta^2 + \alpha\int(H\varepsilon)^2 + \frac{3}{2}\int(2V+y\cdot\nabla V)\eta^2 - b_s\int\varepsilon(2V+y\cdot\nabla V)\Lambda Q\right] \qquad (4\text{-}21)\end{aligned}$$

and R_2 collects the nonlinear higher-order terms

$$R_2 = \int \partial_t w H_\lambda F_{B_1} - \int \partial_t \widetilde{V} w \left[F_{B_1} + \frac{b_s}{\lambda^2} (\Lambda Q)_\lambda \right]. \tag{4-22}$$

Step 2: *Derivation of the quadratic terms and treatment of the b_s term.* Let us now obtain a suitable *lower bound* for the quadratic term R_1. The main enemy is the b_s term which is of order one in ε and will be treated by using a specific algebra generated by the choice of the orthogonality condition (3-4).

Observe from $H(\Lambda Q) = 0$ that $(\Lambda Q/\lambda)_\lambda(y) = (1/\lambda)^{N/2}(\Lambda Q)(y/\lambda)$ satisfies

$$-\Delta(\Lambda Q/\lambda)_\lambda(y) - (1/\lambda)^2 V(y/\lambda)(\Lambda Q/\lambda)_\lambda(y) = 0.$$

Differentiating this relation at $\lambda = 1$ yields

$$H\Phi = H(D\Lambda Q) = (2V + y \cdot \nabla V)\Lambda Q.$$

We inject this into the modulation equation (4-8) to get

$$-b_s \int \varepsilon(2V + y \cdot \nabla V)\Lambda Q = b_s^2(\Phi, \Lambda Q) + |b_s| O\left(\frac{|\mathscr{E}|}{M} + \sqrt{K(M)} \frac{b^4}{|\log b|^2} \right)^{1/2}. \tag{4-23}$$

We thus conclude using the sign

$$(\Phi, \Lambda Q) > 0$$

and (4-8), (4-21) that

$$R_1 \geq \frac{b}{\lambda^3} \left[\alpha \int (\partial_y \eta)^2 + \int [(3-\alpha)V + \tfrac{3}{2} y \cdot \nabla V]\eta^2 + \alpha \int (H\varepsilon)^2 + c_1 (b_s)^2 + O\left(\frac{|\mathscr{E}|}{M} + \sqrt{K(M)} \frac{b^4}{|\log b|^2} \right) \right] \tag{4-24}$$

for some universal constant $c_1 > 0$ *independent of M*.

Step 3: *Coercivity of the quadratic form.* We now claim the following coercivity property of the quadratic form in η appearing on the right-hand side of (4-24) in the limit case $\alpha = 1$. The proof is given in Appendix B.

Lemma 4.7. *There exists a universal constant $c_0 > 0$ such that for all $\eta \in \dot{H}^1_{\text{rad}}$ we have*

$$\int (\partial_y \eta)^2 + \int \big[2V + \tfrac{3}{2} y \cdot \nabla V\big]\eta^2 \geq c_0 \int (\partial_y \eta)^2 - \frac{1}{c_0}\big[(\eta, \psi)^2 + (\eta, \Phi)^2\big].$$

From a simple continuity argument, there exists $0 < \alpha^* < 1$ such that given $0 < \alpha^* < \alpha \leq 1$, for all $\eta \in \dot{H}^1_{\text{rad}}$, we have

$$\alpha \int (\partial_y \eta)^2 + \int \big[(3-\alpha)V + \tfrac{3}{2} y \cdot \nabla V\big]\eta^2 \geq \frac{c_0}{2} \int (\partial_y \eta)^2 - \frac{2}{c_0}\big[(\eta, \psi)^2 + (\eta, \Phi)^2\big].$$

We now pick once and for all such an $\alpha < 1$ and control the negative directions.

Using (4-3) and (4-7) yields

$$(\eta, \psi)^2 \lesssim b|\mathscr{E}| + \sqrt{K(M)} \frac{b^4}{|\log b|^2}.$$

Similarly, we compute $(\eta, \Phi) = (\eta, \chi_M \Phi) + (\eta, (1-\chi_M)\Phi)$ for which (4-4) and (4-7) yield

$$(\eta, \chi_M \Phi)^2 \lesssim b|\mathscr{E}| + \sqrt{K(M)}\frac{b^4}{|\log b|^2},$$

and applying (C-1) we have

$$(\eta, (1-\chi_M)\Phi)^2 \leq \|y\eta\|_{L^\infty}^2 \left[\int_{y \geq M/2} \frac{|\Phi|}{y}\right]^2 \lesssim \frac{1}{M}\int |\partial_y \eta|^2.$$

This, together with (4-24), yields the lower bound on quadratic terms,

$$R_1 \geq \frac{b}{\lambda^3}\left[c_1((b_s)^2 + |\mathscr{E}|) + O\left(\sqrt{K(M)}\frac{b^4}{|\log b|^2}\right)\right] \tag{4-25}$$

for some universal constant $c_1 > 0$. Indeed, a straightforward integration by parts in (3-19) yields

$$\mathscr{E} \lesssim \int |\partial_y \eta|^2 + \int |H\varepsilon|^2.$$

Step 4: *Control of lower-order quadratic terms.* The lower-order quadratic terms in (4-20) are controlled similarly,

$$\begin{aligned}\left|\int \partial_t \widetilde{V} w \partial_t w\right| \lesssim \frac{b}{\lambda^2}\left[\int \frac{\varepsilon^2}{1+y^6} + \int \frac{\eta^2}{y^2}\right] &\lesssim \frac{1}{\lambda^2}\left(bC(M)|\mathscr{E}| + \sqrt{K(M)}\frac{b^4}{|\log b|^2}\right) \\ &\lesssim \frac{1}{\lambda^2}\left(\frac{|\mathscr{E}|}{M} + \sqrt{K(M)}\frac{b^4}{|\log b|^2}\right),\end{aligned} \tag{4-26}$$

and, with the help of (3-32),

$$\left|\int \partial_{tt} \widetilde{V} w \partial_t w\right| \lesssim \left(\frac{b^2}{\lambda^3} + \frac{|b_s|}{\lambda^3}\right)\left[\int \frac{\varepsilon^2}{1+y^6} + \int \frac{\eta^2}{y^2}\right] \lesssim \frac{b}{\lambda^3}\left(bC(M)|\mathscr{E}| + \sqrt{K(M)}\frac{b^4}{|\log b|^2}\right).$$

Remark 4.8. We note here that (4-26) is sufficient for the proof of our theorem. Indeed, the estimated term $\int \partial_t \widetilde{V} w \partial_t w$ has been integrated by parts with respect to time, so that it becomes a part of $\mathscr{F}$. Furthermore, we note that to compute (4-16), we multiply $\mathscr{F}$ by $\lambda^{2\alpha}$. Consequently, the commutator $b\alpha/\lambda \int \partial_t \widetilde{V} w \partial_t w$ appears on the right-hand side. However, (4-26) yields that, in the trapped regime, this supplementary term is controlled by $b/\lambda^3 \sqrt{K(M)} b^4/|\log b|^2$. Similar arguments will be repeated implicitly below for the terms that require an integration by parts with respect to time.

Step 5: *Rewriting the nonlinear R_2 terms.* It remains to control the nonlinear R_2 terms in (4-20) given by (4-22). According to (3-17), this term contains b_{ss}-types of terms which cannot be estimated in absolute value and require a further integration by parts in time. Let

$$F_{B_1} = F_1 - \partial_t F_2, \quad \text{with } F_2 = \frac{1}{\lambda}(\partial_s P_{B_1})_\lambda, \tag{4-27}$$

and write

$$R_2 = \int \partial_t w H_\lambda F_1 - \int \partial_t \widetilde{V} w\left[F_1 + \frac{b_s}{\lambda^2}(\Lambda Q)_\lambda\right] - \int \partial_t w H_\lambda \partial_t F_2 + \int \partial_t \widetilde{V} w \partial_t F_2.$$

We now integrate by parts in time to treat the F_2 term,

$$-\int \partial_t w H_\lambda \partial_t F_2 + \int \partial_t \widetilde{V} w \partial_t F_2 = -\frac{d}{dt}\left\{\int \partial_t w H_\lambda F_2 - \int \partial_t \widetilde{V} w F_2\right\} - \int (\partial_{tt}\widetilde{V} w + 2\partial_t \widetilde{V} \partial_t w) F_2 + \int \partial_{tt} w H_\lambda F_2.$$

The last term is rewritten using (3-17) and integration by parts,

$$\int \partial_{tt} w H_\lambda F_2 = \int [F_1 - \partial_t F_2 - H_\lambda w] H_\lambda F_2 = -\frac{1}{2}\frac{d}{dt}\left\{\int |\nabla F_2|^2 - \int \widetilde{V} F_2^2\right\} - \frac{1}{2}\int \partial_t \widetilde{V} F_2^2 + \int [F_1 - H_\lambda w] H_\lambda F_2.$$

Eventually we arrive at a manageable expression for R_2,

$$\begin{aligned} R_2 = -\frac{d}{dt}\left\{\int \partial_t w H_\lambda F_2 - \int \partial_t \widetilde{V} w F_2 + \frac{1}{2}\int |\nabla F_2|^2 - \frac{1}{2}\int \widetilde{V} F_2^2\right\} \\ - \int \partial_t \widetilde{V} w \left[F_1 + \frac{b_s}{\lambda^2}(\Lambda Q)_\lambda\right] + \int \partial_t w H_\lambda F_1 - \int (\partial_{tt}\widetilde{V} w + 2\partial_t \widetilde{V}\partial_t w) F_2 \\ - \frac{1}{2}\int \partial_t \widetilde{V} F_2^2 + \int [F_1 - H_\lambda w] H_\lambda F_2. \end{aligned} \tag{4-28}$$

We now aim at estimating all the terms in the right-hand side of (4-28). According to (3-17), we split F_1 into four terms

$$F_1 + \frac{b_s}{\lambda^2}(\Lambda Q)_\lambda = -\frac{1}{\lambda^2}\left[\Psi_{B_1} + F_{1,1} + F_{1,2} + N(\varepsilon)\right]_\lambda, \tag{4-29}$$

with

$$F_{1,1} = b\Lambda \partial_s P_{B_1} + b_s(\Lambda P_{B_1} - \Lambda Q), \qquad F_{1,2} = \left[f'(Q) - f'(P_{B_1})\right]\varepsilon. \tag{4-30}$$

Step 6: F_1 *terms.* These are the leading-order terms.

• Ψ_{B_1} *terms.* We first extract from (2-9) the rough bound

$$|\Psi_{B_1}| \lesssim \frac{b^2}{|\log b|(1+y^2)} + C(M) b^4 \mathbf{1}_{y \le 2B_1}, \tag{4-31}$$

which yields

$$\int \frac{1+|\log y|^2}{1+y^4}|\Psi_{B_1}|^2 \lesssim \frac{b^4}{|\log b|^2},$$

and thus, from (4-7),

$$\left|\int \partial_t \widetilde{V} w \frac{1}{\lambda^2}(\Psi_{B_1})_\lambda\right| \lesssim \frac{b}{\lambda^3}\int \frac{|\varepsilon||\Psi_{B_1}|}{(1+y^4)} \lesssim \frac{b}{\lambda^3}\frac{b^2}{|\log b|} C(M)\sqrt{|\mathscr{E}| + \sqrt{K(M)} b^4/|\log b|^2} \lesssim \frac{b}{\lambda^3}\sqrt{K(M)}\frac{b^4}{|\log b|^2}.$$

Next we use the fundamental cancellation $H(\Lambda Q)=0$ and (2-9) to estimate

$$|H\Psi_{B_1}|\lesssim\frac{b^4}{1+y^2}\left[\frac{1+|\log(by)|}{|\log b|}\mathbf{1}_{2\le y\le 2B_0}+\frac{1}{b^2y^2|\log b|}\mathbf{1}_{B_0/2\le y\le 2B_1}+\frac{\log(M)+|\log(1+y)|}{1+y^2}\mathbf{1}_{y\le 2B_1}\right]+\frac{b^2}{(1+y^4)|\log b|}\mathbf{1}_{y\ge B_1/2},$$

and thus get

$$\int(1+y^2)|H(\Psi_{B_1})|^2\lesssim\frac{b^6}{|\log b|^2}. \tag{4-32}$$

Hence

$$\left|\int\partial_t w H_\lambda\left(\frac{1}{\lambda^2}(\Psi_{B_1})_\lambda\right)\right|\lesssim\frac{b}{\lambda^3}\|\eta/y\|_{L^2}\left[\int\frac{1}{b^2}(1+y)^2|H(\Psi_{B_1})|^2\right]^{1/2}\lesssim\frac{b}{\lambda^3}\sqrt{K(M)}\frac{b^4}{|\log b|^2}.$$

• $F_{1,1}$ *terms.* From (2-7) and (2-8) we obtain

$$|F_{1,1}|\lesssim|b_s|b^2\left[\frac{1+|\log(by)|}{|\log b|}\mathbf{1}_{2\le y\le B_0/2}+\frac{1}{b^2y^2|\log b|}\mathbf{1}_{B_0/2\le y\le 2B_1}+\frac{\log(M)+|\log y|}{1+y^2}\right],$$

and, recalling that differentiation with respect to y acts as a multiplication by $1/(1+y)$,

$$\left|HF_{1,1}\right|\lesssim C(M)\frac{|b_s|b^2}{1+y^2}\left[\frac{1+|\log(by)|}{|\log b|}\mathbf{1}_{2\le y\le B_0/2}+\frac{1}{b^2y^2|\log b|}\mathbf{1}_{B_0/2\le y\le 2B_1}+\frac{\log(M)+|\log y|}{1+y^2}\right],$$

from which

$$\int(1+y^2)|H(F_{1,1})|^2\lesssim|b_s|^2\frac{b^2}{|\log b|^2},\qquad\int\frac{(1+|\log y|^2)}{(1+y^4)}|F_{1,1}|^2\lesssim|b_s|^2b^2. \tag{4-33}$$

Hence similar arguments as with the Ψ_{B_1} terms yield

$$\left|\int\partial_t\widetilde{V}wF_{1,1}\right|\lesssim\frac{b}{\lambda^3}b|b_s|C(M)\sqrt{|\mathscr{E}|+\sqrt{K(M)}b^4/|\log b|^2}\lesssim\frac{b}{\lambda^3}\sqrt{K(M)}\frac{b^4}{|\log b|^2},$$

and

$$\begin{aligned}\left|\int\partial_t wH_\lambda F_{1,1}\right|&\lesssim\frac{C(M)b}{\lambda^3}\frac{|b_s|}{|\log b|}\sqrt{|\mathscr{E}|+\sqrt{K(M)}b^4/|\log b|^2}\\&\lesssim\frac{b}{\lambda^3}\left[\frac{|b_s|^2}{|\log b|}+\frac{\mathscr{E}}{|\log b|}+\sqrt{K(M)}\frac{b^4}{|\log b|^2}\right]\lesssim\frac{b}{\lambda^3}\sqrt{K(M)}\frac{b^4}{|\log b|^2}.\end{aligned}$$

• $F_{1,2}$ *terms.* The explicit expansion of the cubic nonlinearity and the bound (2-7) yield

$$|F_{1,2}|\lesssim\frac{C(M)b^2}{1+y^2}|\varepsilon|\quad\text{and}\quad|\nabla F_{1,2}|\lesssim\frac{C(M)b^2}{1+y^3}|\varepsilon|+\frac{C(M)b^2}{1+y^2}|\nabla\varepsilon|, \tag{4-34}$$

from which

$$\frac{1}{\lambda^2}\left|\int\partial_t\widetilde{V}w(F_{1,2})_\lambda\right|\lesssim\frac{C(M)b^3}{\lambda^3}\int\frac{\varepsilon^2}{1+y^6}\lesssim\frac{b}{\lambda^3}\left(b|\mathscr{E}|+\sqrt{K(M)}\frac{b^4}{|\log b|^2}\right),$$

and, after integration by parts of the Laplacian term,

$$\frac{1}{\lambda^2}\left|\int \partial_t w H_\lambda (F_{1,2})_\lambda\right| \lesssim \frac{C(M)}{\lambda^3}\left[\int \frac{|\eta|}{1+y^4}\frac{b^2}{1+y^2}|\varepsilon| + \int |\nabla\eta|\left(\frac{b^2}{1+y^3}|\varepsilon| + \frac{b^2}{1+y^2}|\nabla\varepsilon|\right)\right]$$
$$\lesssim \frac{b}{\lambda^3}\left[\frac{|\mathscr{E}|}{M} + \sqrt{K(M)}\frac{b^4}{|\log b|^2}\right].$$

• *Nonlinear term* $N(\varepsilon)$*.* We expand the nonlinearity as

$$N(\varepsilon) = 3P_{B_1}\varepsilon^2 + \varepsilon^3.$$

This yields, using (3-27) and (C-1), the rough bound

$$|N(\varepsilon)| \lesssim \frac{\varepsilon^2}{1+y}.$$

In what follows, we will use the following bound on η, which follows from (4-6), (C-1):

$$\|y\eta\|_{L^\infty} \lesssim \|\nabla\eta\|_{L^2} \lesssim \left(c(M)|\mathscr{E}| + \sqrt{K(M)}\frac{b^4}{|\log b|^2}\right)^{1/2}.$$

We then estimate

$$\left|\frac{1}{\lambda^2}\int \partial_t \widetilde{V} w (N(\varepsilon))_\lambda\right| \lesssim \frac{b}{\lambda^3}\int \frac{|\varepsilon|^3}{1+y^5} \lesssim \frac{b}{\lambda^3}\|\nabla\varepsilon\|_{L^2}\left(c(M)|\mathscr{E}| + \sqrt{K(M)}\frac{b^4}{|\log b|^2}\right) \lesssim \frac{b}{\lambda^3}\sqrt{K(M)}\frac{b^4}{|\log b|^2}$$

for $b_0 < b^*(M)$ small enough. We split the second term into

$$\int \partial_t w H_\lambda\left(\frac{(N(\varepsilon))_\lambda}{\lambda^2}\right) = \int \nabla\partial_t w \cdot \nabla\left(\frac{(N(\varepsilon))_\lambda}{\lambda^2}\right) - \int \widetilde{V}\partial_t w\left(\frac{(N(\varepsilon))_\lambda}{\lambda^2}\right). \tag{4-35}$$

The second of these terms is estimated by brute force:

$$\left|\int \widetilde{V}\partial_t w\left(\frac{(N(\varepsilon))_\lambda}{\lambda^2}\right)\right| \lesssim \frac{1}{\lambda^3}\int \frac{|\eta|\,|\varepsilon|^2}{1+y^5} \lesssim \frac{1}{\lambda^3}\|y\eta\|_{L^\infty}\int \frac{|\varepsilon|^2}{1+y^6}$$
$$\lesssim \frac{1}{\lambda^3}\left(c(M)|\mathscr{E}| + \sqrt{K(M)}\frac{b^4}{|\log b|^2}\right)^{3/2} \lesssim \frac{b}{\lambda^3}\frac{b^4}{|\log b|^2}.$$

The first term in (4-35) is split into two parts:

$$\int \nabla\partial_t w \cdot \nabla\left(\frac{(N(\varepsilon))_\lambda}{\lambda^2}\right) = \int \nabla\partial_t w \cdot \left[\nabla(w^3) + 3(P_{B_1})_\lambda\nabla(w^2)\right] + \frac{3}{\lambda^3}\int \varepsilon^2\nabla\eta\cdot\nabla P_{B_1}.$$

The second term is integrated by parts in space and then estimated by brute force:

$$\begin{aligned}\left|\frac{3}{\lambda^3}\int \varepsilon^2\nabla\eta\cdot\nabla P_{B_1}\right| &= \frac{3}{\lambda^3}\left|\int \eta\left[\varepsilon^2\Delta P_{B_1}+2\varepsilon\nabla P_{B_1}\cdot\nabla\varepsilon\right]\right|\\ &\lesssim \frac{1}{\lambda^3}\int|\eta|\left[\frac{\varepsilon^2}{1+y^4}+\frac{|\varepsilon|\,|\nabla\varepsilon|}{1+y^3}\right]\\ &\lesssim \frac{1}{\lambda^3}\|y\eta\|_{L^\infty}\left[\int\frac{\varepsilon^2}{1+y^5}+\int\frac{|\nabla\varepsilon|^2}{y^2}\right]\\ &\lesssim \frac{1}{\lambda^3}\left(c(M)|\mathscr{E}|+\sqrt{K(M)}\frac{b^4}{|\log b|^2}\right)^{3/2}\lesssim \frac{b}{\lambda^3}\frac{b^4}{|\log b|^2}.\end{aligned}$$

The first term is more delicate and requires first a time integration by parts,

$$\begin{aligned}\int\nabla\partial_t w\cdot\left[\nabla(w^3)+3(P_{B_1})_\lambda\nabla(w^2)\right] &= \frac{d}{dt}\left\{\int|\nabla w|^2\left[\frac{3}{2}w^2+3(P_{B_1})_\lambda w\right]\right\}\\ &\quad -3\int w\partial_t w|\nabla w|^2-3\int|\nabla w|^2\left[w\partial_t(P_{B_1})_\lambda+(P_{B_1})_\lambda\partial_t w\right].\end{aligned}$$

We may now estimate all terms by brute force. First,

$$\left|\int|\nabla w|^2\left[\frac{3}{2}w^2+3(P_{B_1})_\lambda w\right]\right|\lesssim\frac{1}{\lambda^2}[\|y\varepsilon\|_{L^\infty}+\|yP_{B_1}\|_{L^\infty}]\|y\varepsilon\|_{L^\infty}\int\frac{|\nabla\varepsilon|^2}{y^2}\lesssim\frac{1}{\lambda^2}\frac{b^4}{|\log b|^2},$$

second,

$$\left|\int w\partial_t w|\nabla w|^2\right|\lesssim\frac{1}{\lambda^2}\|y\varepsilon\|_{L^\infty}\|y\eta\|_{L^\infty}\int\frac{|\nabla\varepsilon|^2}{y^2}\lesssim\left(c(M)|\mathscr{E}|+\sqrt{K(M)}\frac{b^4}{|\log b|^2}\right)^{3/2}\lesssim\frac{b}{\lambda^3}\frac{b^4}{|\log b|^2},$$

and third,

$$\begin{aligned}\left|\int|\nabla w|^2 w\partial_t(P_{B_1})_\lambda\right| &\lesssim \frac{\|yw\|_{L^\infty}}{\lambda^3}\int\frac{|\nabla w|^2}{y}\left[\frac{b}{1+y^2}+C(M)b|b_s|\mathbf{1}_{y\le B_1}\right]\\ &\lesssim \frac{b}{\lambda^3}|\nabla\varepsilon|_{L^2}\left(1+C(M)|b_s|\frac{|\log b|}{b}\right)\int\frac{|\nabla\varepsilon|^2}{y^2}\lesssim\frac{b}{\lambda^3}\frac{b^4}{|\log b|^2},\end{aligned}$$

where we used the rough bound extracted from (2-8), $|\partial_b P_{B_1}|\lesssim C(M)b\mathbf{1}_{y\le B_1}$. Finally,

$$\left|\int|\nabla w|^2(P_{B_1})_\lambda\partial_t w\right|\lesssim\frac{1}{\lambda^3}\|y\eta\|_{L^\infty}\int\frac{|\nabla\varepsilon|^2}{1+y^3}\lesssim\left(C(M)\mathscr{E}+\sqrt{K(M)}\frac{b^4}{|\log b|^2}\right)^{3/2}\lesssim\frac{b}{\lambda^3}\frac{b^4}{|\log b|^2},$$

for $b_0<b^*(M)$ small enough. The above chain of estimates together with Remark 4.8, achieves the control of the nonlinear term $N(\varepsilon)$.

Step 7: *F_2 terms.* We estimate from (2-8),

$$\int\left|\frac{\partial_b P_{B_1}}{(1+y)}\right|^2+\int\left|\nabla\partial_b P_{B_1}\right|^2\lesssim\frac{1}{|\log b|^2}\quad\text{and}\quad\int\frac{1}{1+y^3}\left|\partial_b P_{B_1}\right|^2\lesssim\frac{b}{|\log b|^2}. \tag{4-36}$$

Hence, first,

$$\begin{aligned}\left|\int \partial_t w H_\lambda F_2\right| &\lesssim \frac{|b_s|}{\lambda^2}\left[\int \frac{|\eta||\partial_b P_{B_1}|}{(1+y^4)}+\int |\nabla\eta||\nabla\partial_b P_{B_1}|\right]\\ &\lesssim \frac{1}{\lambda^2}\frac{|b_s|}{|\log b|}\sqrt{c(M)|\mathscr{E}|+\sqrt{K(M)}b^4/|\log b|^2}\\ &\lesssim \frac{1}{\lambda^2}\left[\frac{|\mathscr{E}|}{M}+\sqrt{K(M)}\frac{b^4}{|\log b|^2}\right],\end{aligned}$$

second,

$$\begin{aligned}\left|\int \partial_t \widetilde{V} w F_2\right| \lesssim \frac{|b_s|b}{\lambda^2}\int \frac{|\partial_b P_{B_1}||\varepsilon|}{(1+y^4)} &\lesssim \frac{|b_s|b}{\lambda^2|\log b|}\left[c(M)|\mathscr{E}|+\sqrt{K(M)}\frac{b^4}{|\log b|^2}\right]^{\frac{1}{2}}\\ &\lesssim \frac{1}{\lambda^2}\left[\frac{|\mathscr{E}|}{M}+\sqrt{K(M)}\frac{b^4}{|\log b|^2}\right],\end{aligned}$$

and third,

$$\int |\nabla F_2|^2+\left|\int V F_2^2\right| \lesssim \frac{|b_s|^2}{\lambda^2}\left[\int \frac{|\partial_b P_{B_1}|^2}{(1+y^4)}+\int |\nabla\partial_b P_{B_1}|^2\right] \lesssim \frac{1}{\lambda^2}\frac{(b_s)^2}{|\log b|^2} \lesssim \frac{1}{\lambda^2}\frac{b^4}{|\log b|^2}.$$

Similarly,

$$\begin{aligned}&\left|\int (\partial_{tt}\widetilde{V} w+2\partial_t\widetilde{V}\partial_t w)F_2\right|+\left|\int \partial_t \widetilde{V} F_2^2\right|\\ &\qquad\lesssim \frac{|b_s|}{\lambda^3}\left[\int \frac{((|b_s|+b^2)|\varepsilon|+b|\eta|)|\partial_b P_{B_1}|}{(1+y^4)}+|b_s|b\int \frac{|\partial_b P_{B_1}|^2}{1+y^4}\right]\\ &\qquad\lesssim \frac{|b_s|}{\lambda^3}\left[\frac{(|b_s|+b)}{|\log b|}\sqrt{c(M)|\mathscr{E}|+\sqrt{K(M)}b^4/|\log b|^2}+\frac{b^2}{|\log b|^2}|b_s|\right]\\ &\qquad\lesssim \frac{b}{\lambda^3}\left[\frac{|\mathscr{E}|}{M}+\sqrt{K(M)}\frac{b^4}{|\log b|^2}\right].\end{aligned}$$

Eventually, (4-32) and (4-33) ensure that

$$\int (1+y^2)|H(\Psi_{B_1}+F_{1,1})|^2 \lesssim \left[\frac{b^6}{|\log b|^2}+\frac{b^2|b_s|^2}{|\log b|^2}\right] \lesssim \frac{b^6}{|\log b|^2},$$

which together with (4-36) yields

$$\left|\int \frac{1}{\lambda^2}(\Psi_{B_1}+F_{1,1})_\lambda H_\lambda F_2\right| \lesssim \frac{1}{\lambda^3}\frac{b^3|b_s|}{|\log b|^2} \lesssim \frac{b}{\lambda^3}\frac{b^4}{|\log b|^2}.$$

We similarly estimate from (4-34), after integration by parts, that

$$\begin{aligned}\left|\int \frac{1}{\lambda^2}(F_{1,2})_\lambda H_\lambda F_2\right| &\lesssim \frac{|b_s|}{\lambda^3}\left[\int \frac{b^2|\varepsilon||\partial_b P_{B_1}|}{1+y^6}+\int |\nabla\partial_b P_{B_1}|\left(\frac{b^2|\varepsilon|}{1+y^3}+\frac{b^2|\nabla\varepsilon|}{1+y^2}\right)\right]\\ &\lesssim C(M)\frac{b^4}{\lambda^3|\log b|}\left(\int \frac{\varepsilon^2}{1+y^6}+\int \frac{|\nabla\varepsilon|^2}{1+y^4}\right)^{1/2} \lesssim \frac{b}{\lambda^3}\frac{b^4}{|\log b|^2}.\end{aligned}$$

For the nonlinear term, we extract from (2-8) the rough bound

$$|H(\partial_b P_{B_1})| \lesssim [C(M)+\log(b)]\frac{b}{1+y^2}\mathbf{1}_{y\le B_1},$$

which together with (C-1) ensures that

$$\begin{aligned}\left|\int \frac{1}{\lambda^2}(N(\varepsilon))_\lambda H_\lambda F_2\right| &\lesssim \frac{[C(M)+\log(b)]}{\lambda^3}|b_s|\int \frac{b}{1+y^2}\frac{\varepsilon^2}{1+y}\mathbf{1}_{y\le B_1}\\ &\lesssim C(M)\frac{|b_s|\,|\log b|^4}{\lambda^3}\int \frac{\varepsilon^2}{(1+y^4)|\log y|^2}\\ &\lesssim \frac{b}{\lambda^3}\sqrt{b}\left(c(M)|\mathscr{E}|+\sqrt{K(M)}\frac{b^4}{|\log b|^2}\right)\lesssim \frac{b}{\lambda^3}\frac{b^4}{|\log b|^2}.\end{aligned}$$

Step 10: *The remaining F_2 term has the right sign.* It remains to estimate the term

$$-\int H_\lambda w H_\lambda F_2$$

on the right-hand side of (4-28). Let us stress the fact that this term is *a priori* no better $O(\mathscr{E}/\lambda^3)$ due to the b_s contribution and the bound (4-8); recall Remark 4.5.

We now claim that the *main contribution has the right sign again*. Indeed, we first compute from the T_1 equation (2-16) that

$$HT_1 = -\Phi + c_b\chi_{B_0/4}\Lambda Q \quad \text{and} \quad H\partial_b T_1 = O\left(\frac{1}{b\,|\log b|}\frac{\mathbf{1}_{2\le y\le B_0/2}}{(1+y^2)}\right). \tag{4-37}$$

We then apply the decomposition (2-22),

$$H(\partial_b P_{B_1}) = H\big(2bT_1 + 2b(\chi_{B_1}-1)T_1 + b^2\partial_b\log(B_1)\rho_{B_1}T_1 + b^2\chi_{B_1}\partial_b T_1\big) = -2b\Phi + \Sigma,$$

and estimate using (2-8), (2-21), (4-37) that

$$|\Sigma| \lesssim \frac{b}{1+y^2}\left[\frac{1}{|\log b|}\mathbf{1}_{2\le y\le B_0/2} + \frac{1}{b^2y^2|\log b|}\mathbf{1}_{B_0/2\le y}\right].$$

In particular, $\int\Sigma^2 \lesssim b^2/|\log b|$, and thus using the modulation equation (4-8) gives

$$\begin{aligned}-\int H_\lambda w H_\lambda F_2 &= -\frac{b_s}{\lambda^3}\int (H\varepsilon)H(\partial_b P_{B_1})\\ &= -\frac{b_s}{\lambda^3}\int H\varepsilon\,(-2b\Phi+\Sigma)\\ &= 2\frac{b}{\lambda^3}b_s(\varepsilon, H\Phi) + \frac{b}{\lambda^3}O\left(\frac{|b_s|}{\sqrt{|\log b|}}\sqrt{|\mathscr{E}|+\sqrt{K(M)}b^4/|\log b|^2}\right)\\ &= 2\frac{b}{\lambda^3}\left[-\frac{(\varepsilon, H\Phi)}{(\Lambda Q, \Phi)} + O\left(\sqrt{|\mathscr{E}|/M+\sqrt{K(M)}b^4/|\log b|^2}\right)\right](\varepsilon, H\Phi) + \frac{b}{\lambda^3}O\left(\frac{b^4}{|\log b|^2}\right)\\ &= -\frac{2b}{\lambda^3}\frac{(\varepsilon, H\Phi)^2}{(\Lambda Q, \Phi)} + O\left(\frac{|\mathscr{E}|}{M}+\sqrt{K(M)}\frac{b^4}{|\log b|^2}\right) + \frac{b}{\lambda^3}O\left(\frac{b^4}{|\log b|^2}\right)\end{aligned} \tag{4-38}$$

$$\leq O\left(\frac{|\mathscr{E}|}{M}+\sqrt{K(M)}\frac{b^4}{|\log b|^2}\right). \tag{4-39}$$

Collecting all the above estimates yields (4-16) and concludes the proof of Proposition 4.6. □

Improved bound. We now claim that the *a priori* bound on the unstable direction (3-35), coupled with the monotonicity property of Proposition 4.6, implies the following:

Lemma 4.9 (improved bounds under the a priori control (3-35)). *In* $[0, T_1(a_+)]$ *we have*

$$\left\|\nabla w(t),\, \partial_t w(t)+\frac{b(t)}{\lambda(t)}\big((1-\chi_{B_1(b(t))})\Lambda Q\big)_{\lambda(t)}\right\|_{L^2\times L^2}\lesssim b_0|\log b_0|, \tag{4-40}$$

$$\frac{b^4(t)}{|\log b(t)|^2\lambda^{2(1-\alpha)}(t)}\geq\frac{b^4(0)}{|\log b(0)|^2\lambda^{2(1-\alpha)}(0)}, \tag{4-41}$$

$$|b_s|^2\leq\frac{K(M)}{2}\frac{b^4}{|\log b|^2}, \tag{4-42}$$

$$|\mathscr{E}(t)|\leq\frac{K(M)}{2}\frac{b^4}{(\log b)^2}. \tag{4-43}$$

Proof.

Step 1: *Energy bound.* The energy bound (4-40) is a consequence of the conservation of energy. Indeed, conservation of energy and the initial bounds of Lemma 3.1 ensure that

$$E(u,\partial_t u)=E(u_0,u_1)=E(Q)+O(b_0\sqrt{|\log b_0|}),$$

(see Appendix A) and thus give

$$E(Q)+O(b_0|\log b_0|)=\frac{1}{2}\int\big[\partial_t(P_{B_1})_\lambda+\partial_t w\big]^2+\frac{1}{2}\int\big|\nabla(P_{B_1})_\lambda+\nabla w\big|^2-\frac{1}{4}\int\big[(P_{B_1})_\lambda+w\big]^4. \tag{4-44}$$

We lower bound the first term by expanding

$$\begin{aligned}\partial_t(P_{B_1})_\lambda+\partial_t w&=\partial_t w+\frac{b}{\lambda}((1-\chi_{B_1})\Lambda Q)_\lambda+\frac{b}{\lambda}(\chi_{B_1}\Lambda Q)_\lambda+\frac{b^3}{\lambda}(\Lambda[\chi_{B_1}T_1])_\lambda+\frac{b_s}{\lambda}(\partial_b P_{B_1})_\lambda\\&=\partial_t w+\frac{b}{\lambda}((1-\chi_{B_1})\Lambda Q)_\lambda+\Sigma,\end{aligned}$$

with

$$\int\Sigma^2\lesssim b_0^2|\log b_0|,$$

where we used the bootstrap bounds (3-31) and (3-32). Finally,

$$\int\big[\partial_t(P_{B_1})_\lambda+\partial_t w\big]^2\geq\frac{1}{2}\int\left[\frac{b}{\lambda}((1-\chi_{B_1})\Lambda Q)_\lambda+\partial_t w\right]^2-O(b_0^2|\log b_0|). \tag{4-45}$$

We then expand the second term as

$$\begin{aligned}\frac{1}{2}\int\left[\nabla(P_{B_1})_\lambda+\nabla w\right]^2-\frac{1}{4}\int\left[(P_{B_1})_\lambda+w\right]^4\\ &=\frac{1}{2}\int\left[\nabla P_{B_1}+\nabla\varepsilon\right]^2-\frac{1}{4}\int\left[P_{B_1}+\varepsilon\right]^4\\ &=\frac{1}{2}\int\left|\nabla P_{B_1}\right|^2-\frac{1}{4}\int\left|P_{B_1}\right|^4-(\varepsilon,\Delta P_{B_1}+P_{B_1}^3)+\frac{1}{2}\left(\int|\nabla\varepsilon|^2-3\int P_{B_1}^2\varepsilon^2\right)-\frac{1}{4}\left(4P_{B_1}\varepsilon^3+\varepsilon^4\right).\end{aligned}$$

From the construction of P_{B_1},

$$\frac{1}{2}\int\left|\nabla P_{B_1}\right|^2-\frac{1}{4}\int\left|P_{B_1}\right|^4=E(Q)+O(b^2|\log b|). \tag{4-46}$$

The linear term is treated using (2-9), the improved decay (2-2) and (4-31). We get

$$\left|(\varepsilon,\Delta P_{B_1}+P_{B_1}^3)\right|=\left|(\varepsilon,b^2D\Lambda P_{B_1}-\Psi_{B_1})\right|\lesssim\|\varepsilon/y\|_{L^2}\|y(b^2D\Lambda P_{B_1}-\Psi_{B_1})\|_{L^2}\lesssim b|\nabla\varepsilon|_{L^2}. \tag{4-47}$$

We now rewrite the quadratic term as a small deformation of H and use the coercivity bound (C-8) to ensure that

$$\int|\nabla\varepsilon|^2-3\int P_{B_1}^2\varepsilon^2\geq c_0\int|\nabla\varepsilon|^2+Def, \tag{4-48}$$

with

$$Def:=3\int(Q^2-P_{B_1}^2)\varepsilon^2-\frac{(\varepsilon,\psi)^2}{c_0}.$$

Collecting (2-7) and (C-1), on the one hand, and (4-2) on the other hand, we compute

$$\left|\int(Q^2-P_{B_1}^2)\varepsilon^2\right|\leq\|y^2(Q^2-P_{B_1}^2)\|_{L^\infty}\|\nabla\varepsilon\|_{L^2}^2\lesssim b\|\nabla\varepsilon\|_{L^2}^2\quad\text{and}\quad|(\varepsilon,\psi)|^2\lesssim b^2|\log b|. \tag{4-49}$$

The nonlinear term is easily estimated by the Sobolev inequality:

$$\int\left|(3P_{B_1}+\varepsilon)\varepsilon^3\right|\leq\|yP_{B_1}\|_{L^\infty}\|y\varepsilon\|_{L^\infty}\|\nabla\varepsilon\|_{L^2}^2\lesssim\sqrt{b_0}\|\nabla\varepsilon\|_{L^2}^2. \tag{4-50}$$

Injecting (4-45), (4-47), (4-46), (4-49), (4-48), (4-50) into (4-44) now yields (4-40).

Step 2: *Lower bound on b.* We now turn to the proof of (4-41). First observe from the bootstrap estimate (3-32) that

$$|b_s|\leq\sqrt{K(M)}\frac{b^2}{|\log b|}\leq\frac{1-\alpha}{10}b^2. \tag{4-51}$$

This implies

$$\frac{d}{ds}\left(\frac{b^4}{(\log b)^2\lambda^{2(1-\alpha)}}\right)=\frac{4b^3}{\lambda^{2(1-\alpha)}(\log b)^2}\left[b_s\left(1-\frac{1}{2\log b}\right)+\frac{1-\alpha}{2}b^2\right]>0$$

and (4-41) follows.

Step 3: *Improved $\dot{H}^2$ bound.* We now turn to the proof of (4-43). We integrate (4-16) in time and conclude from (4-1) and (4-15) that

$$|\mathscr{E}(t)| \lesssim \left(\frac{\lambda(t)}{\lambda(0)}\right)^{2(1-\alpha)} |\mathscr{E}(0)| + (K(M))^{1/2}\left[\frac{b^4(t)}{|\log b(t)|^2} + [\lambda(t)]^{2(1-\alpha)}\int_0^t \frac{b(\tau)}{[\lambda(\tau)]^{3-2\alpha}}\frac{b^4(\tau)}{|\log b(\tau)|^2}d\tau\right]. \tag{4-52}$$

We then derive from (4-51) that

$$\begin{aligned}\int_0^t \frac{b(\tau)}{[\lambda(\tau)]^{3-2\alpha}}\frac{b^4(\tau)}{|\log b(\tau)|^2}\,d\tau &= -\int_0^t \frac{\lambda_t}{\lambda^{3-2\alpha}}\frac{b^4}{|\log b|^2}\,d\tau \\ &\le \frac{1}{2(1-\alpha)}\frac{b^4(t)}{\lambda^{2(1-\alpha)}(t)|\log b(t)|^2} - \frac{1}{2(1-\alpha)}\int_0^t \frac{b_s}{\lambda^{3-2\alpha}}\frac{b^3}{|\log b|^2}\left[1-\frac{2}{|\log b|^2}\right] \\ &\lesssim \frac{b^4(t)}{\lambda^{2(1-\alpha)}(t)|\log b(t)|^2} + \sqrt{K(M)}\int_0^t \frac{b(\tau)}{[\lambda(\tau)]^{3-2\alpha}}\frac{b^4(\tau)}{|\log b(\tau)|^2}\frac{1}{|\log b(\tau)|}d\tau,\end{aligned}$$

and hence obtain the bound

$$\lambda^{2(1-\alpha)}(t)\int_0^t \frac{b(\tau)}{[\lambda(\tau)]^{3-2\alpha}}\frac{b^4(\tau)}{|\log b(\tau)|^2}d\tau \lesssim \frac{b^4(t)}{|\log b(t)|^2}.$$

Injecting this into (4-52) and using the initial bounds (A-12), (A-17) and the monotonicity (4-41) yields

$$\mathscr{E}(t) \lesssim \left(\frac{\lambda(t)}{\lambda(0)}\right)^{2(1-\alpha)}\frac{b^4(0)}{|\log b(0)|^2} + (K(M))^{1/2}\frac{b^4(t)}{|\log b(t)|^2} \lesssim \sqrt{K(M)}\frac{b^4(t)}{|\log b(t)|^2} \tag{4-53}$$

and (4-43) follows. The bound (4-42) now follows from Lemma 4.4 and (4-53). This concludes the proof of Lemma 4.9. □

Dynamic of the unstable mode. We now focus onto the dynamic of the unstable mode. We recall the decomposition

$$Y(t) = \left|\begin{matrix}(\varepsilon, \psi) \\ (\partial_s\varepsilon, \psi)\end{matrix}\right. = \tilde{a}_+(t)V_+ + \tilde{a}_-(t)V_-, \tag{4-54}$$

and the variables given by (3-22),

$$\kappa_+(s) = \tilde{a}_+(s) + \frac{b_s}{2\sqrt{\zeta}}(\partial_b P_{B_1}, \psi), \quad \kappa_-(s) = \tilde{a}_-(s) - \frac{b_s}{2\sqrt{\zeta}}(\partial_b P_{B_1}, \psi).$$

Lemma 4.10 (control of the unstable mode). *For all $t \in [0, T_1(a_+)]$ we have*

$$|\kappa_-(t)| \le \tfrac{1}{2}(K(M))^{1/8}\frac{b^2}{|\log b|}, \tag{4-55}$$

and κ_+ is strictly outgoing,

$$\left|\frac{d\kappa_+}{ds} - \sqrt{\zeta}\kappa_+\right| \le \sqrt{b}\frac{b^2}{|\log b|}. \tag{4-56}$$

Proof. We compute the equation satisfied by the unstable direction (ε, ψ) by taking the inner product of (3-11) with the well localized direction ψ and get

$$\frac{d^2}{ds^2}(\varepsilon, \psi) - \zeta(\varepsilon, \psi) = E(\varepsilon) - (\partial_s^2 P_{B_1}, \psi), \tag{4-57}$$

with

$$\begin{aligned} E(\varepsilon) = -(\Psi_{B_1}, \psi) - b_s(\Lambda P_{B_1}, \psi) - b(\partial_s P_{B_1} + 2\Lambda\partial_s P_{B_1}, \psi) - b(\partial_s\varepsilon + 2\Lambda\partial_s\varepsilon, \psi) \\ - b_s(\Lambda\varepsilon, \psi) + (N(\varepsilon), \psi) + b^2(\Lambda\varepsilon, D\psi) + ((f'(P_{B_1}) - f'(Q))\varepsilon, \psi). \end{aligned} \tag{4-58}$$

Simple algebraic manipulations using (4-54) and (3-22) and the initial condition yield the equivalent system

$$\frac{d}{ds}\kappa_+ = \sqrt{\zeta}\kappa_+(s) + \frac{E_+(s)}{2\sqrt{\zeta}}, \quad \frac{d}{ds}\kappa_- = -\sqrt{\zeta}\kappa_-(s) - \frac{E_-(s)}{2\sqrt{\zeta}}\kappa_-(0), \tag{4-59}$$

with

$$E_+(s) = E(s) - \frac{b_s}{2}(\partial_b P_{B_1}, \psi), \quad E_-(s) = E(s) + \frac{b_s}{2}(\partial_b P_{B_1}, \psi). \tag{4-60}$$

We now have from the explicit formula (4-58) and (4-60), the exponential localization of ψ, the orthogonality

$$(\psi, \Lambda Q) = 0,$$

the estimates of Proposition 2.1 and the bootstrap estimate (3-32) the bound

$$\frac{1}{\sqrt{\zeta}}|E_\pm| \lesssim |b|\left(|b_s| + \sqrt{|\mathscr{E}|} + \sqrt{K(M)}\frac{b^2}{|\log b|}\right) \le \sqrt{b}\frac{b^2}{|\log b|}, \tag{4-61}$$

which together with (4-59) yields (4-56). Let then

$$\mathscr{G} = \kappa_-^2\frac{|\log b|^2}{b^4},$$

then from (4-59), (4-61), (3-32), we estimate that

$$\begin{aligned} \frac{d\mathscr{G}}{ds} &= 2\kappa_-\frac{d\kappa_-}{ds}\frac{|\log b|^2}{b^4} + \kappa_-^2 b_s\left[-\frac{4|\log b|^2}{b^5} + \frac{2\log b}{b^5}\right] \\ &= 2\frac{|\log b|^2}{b^4}\left[\kappa_-\left(-\sqrt{\zeta}\kappa_- - \frac{E_-}{\sqrt{\zeta}}\right)\right] + \kappa_-^2\frac{|\log b|^2}{b^4}O\left(\frac{|b_s|}{b}\right) \\ &\le -\frac{\sqrt{\zeta}}{2}\frac{|\log b|^2}{b^4}\kappa_-^2 + \frac{|\log b|^2}{b^4}\kappa_-\sqrt{b}\frac{b^2}{|\log b|} \lesssim -\frac{\sqrt{\zeta}}{2}\mathscr{G} + 1. \end{aligned}$$

We integrate this in time and get

$$\mathscr{G}(s) \le \mathscr{G}(0)e^{-\frac{\sqrt{\zeta}}{2}s} + \int_0^s e^{-\frac{(s-\sigma)}{2}\sqrt{\zeta}}d\sigma \lesssim 1,$$

where we used the initial inequality (A-18) yielding that $\mathscr{G}(0) \lesssim 1$. This concludes the proof of (4-55) and of Lemma 4.10. □

Derivation of the sharp law for b. We now turn to the derivation of the sharp law for b, which will yield the monotonicity statement on b needed to obtain the smallness bootstrap estimate (3-31), and will eventually lead to the derivation of the sharp blow-up speed (1-10).

Lemma 4.11 (sharp law for b). *Let*

$$\tilde{P}_{B_0} = \chi_{B_0/4} Q, \tag{4-62}$$

$$G(b) = b|\Lambda \tilde{P}_{B_0}|_{L^2}^2 + \int_0^b \tilde{b}(\partial_b \tilde{P}_{B_0}, \Lambda \tilde{P}_{B_0})\, d\tilde{b}, \tag{4-63}$$

$$\mathcal{J}(s) = (\partial_s \varepsilon, \Lambda \tilde{P}_{B_0}) + b(\varepsilon + 2\Lambda\varepsilon, \Lambda \tilde{P}_{B_0}) + b_s(\partial_b \tilde{P}_{B_0}, \Lambda \tilde{P}_{B_0}) - b_s\big(\partial_b(P_{B_1} - \tilde{P}_{B_0}), \Lambda \tilde{P}_{B_0}\big). \tag{4-64}$$

Then

$$G(b) = 64b\,|\log b| + O(b), \quad |\mathcal{J}| \lesssim K(M) b, \tag{4-65}$$

$$\left|\frac{d}{ds}\{G(b) + \mathcal{J}(s)\} + 32b^2\right| \lesssim K(M)\frac{b^2}{\sqrt{|\log b|}}. \tag{4-66}$$

Remark 4.12. Observe that (4-65) and (4-66) essentially yield a pointwise differential equation

$$b_s \sim -\frac{b^2}{2|\log b|},$$

which will allow us to derive the sharp scaling law via the relationship $-\lambda_s/\lambda = b$.

Proof of Lemma 4.11. The proof is inspired by the one in [Raphaël and Rodnianski 2012]. We multiply (3-11) by $\Lambda \tilde{P}_{B_0}$ and compute

$$\begin{aligned}(b_s \Lambda P_{B_1} &+ b(\partial_s P_{B_1} + 2\Lambda \partial_s P_{B_1}) + \partial_s^2 P_{B_1}, \Lambda \tilde{P}_{B_0}) \\ &= -(\Psi_{B_1}, \Lambda \tilde{P}_{B_0}) - (H_{B_1}\varepsilon, \Lambda \tilde{P}_{B_0}) - \big(\partial_s^2 \varepsilon + b(\partial_s \varepsilon + 2\Lambda \partial_s \varepsilon) + b_s \Lambda \varepsilon, \Lambda \tilde{P}_{B_0}\big) + (N(\varepsilon), \Lambda \tilde{P}_{B_0}).\end{aligned}$$

We further rewrite this as

$$\begin{aligned}&\big(b_s \Lambda \tilde{P}_{B_0} + b(\partial_s \tilde{P}_{B_0} + 2\Lambda \partial_s \tilde{P}_{B_0}) + \partial_s^2 \tilde{P}_{B_0}, \Lambda \tilde{P}_{B_0}\big) \\ &= -(\Psi_{B_1}, \Lambda \tilde{P}_{B_0}) - \big(b_s \Lambda (P_{B_1} - \tilde{P}_{B_0}) + b(\partial_s (P_{B_1} - \tilde{P}_{B_0}) + 2\Lambda \partial_s (P_{B_1} - \tilde{P}_{B_0})) + \partial_s^2 (P_{B_1} - \tilde{P}_{B_0}), \Lambda \tilde{P}_{B_0}\big) \\ &\qquad - (H_{B_1}\varepsilon, \Lambda \tilde{P}_{B_0}) - \big(\partial_s^2 \varepsilon + b(\partial_s \varepsilon + 2\Lambda \partial_s \varepsilon) + b_s \Lambda \varepsilon, \Lambda \tilde{P}_{B_0}\big) + (N(\varepsilon), \Lambda \tilde{P}_{B_0}).\end{aligned} \tag{4-67}$$

We now estimate all terms in this identity.

Step 1: *b terms.* An integration by parts in time allows us to rewrite the left-hand side of (4-67) as

$$\big(b_s \Lambda \tilde{P}_{B_0} + b(\partial_s \tilde{P}_{B_0} + 2\Lambda \partial_s \tilde{P}_{B_0}) + \partial_s^2 \tilde{P}_{B_0}, \Lambda \tilde{P}_{B_0}\big) = \frac{d}{ds}\big[G(b) + b_s(\partial_b \tilde{P}_{B_0}, \Lambda \tilde{P}_{B_0})\big] + |b_s|^2 \|\partial_b \tilde{P}_{B_0}\|_{L^2}^2, \tag{4-68}$$

with G given by (4-63). Observe from (3-32) the bound

$$|b_s|^2 \|\partial_b \tilde{P}_{B_0}\|_{L^2}^2 \lesssim \frac{|b_s|^2}{b^2} \lesssim (K(M))^2 \frac{b^2}{|\log b|^2} \lesssim \frac{b^2}{\sqrt{|\log b|}}.$$

We now turn to the key step in the derivation of the sharp b law which corresponds to the following outgoing flux computation:[8]

$$(\Psi_{B_1}, \Lambda\tilde{P}_{B_0}) = 32b^2\left(1+O\left(\frac{1}{|\log b|}\right)\right) \quad \text{as } b \to 0. \tag{4-69}$$

Indeed, we first estimate from (2-9) that

$$\left|(\Psi_{B_1} - c_b b^2 \chi_{B_0/4}\Lambda Q, \Lambda\tilde{P}_{B_0})\right| \lesssim b^4 \int_{y\le B_0/2}\left[\frac{1+|\log(by)|}{|\log b|(1+y^2)} + \frac{1+|\log(1+y)|}{(1+y^2)^2}\right] \lesssim \frac{b^2}{|\log b|}.$$

The remainder term is computed from (2-10) and the explicit formula for Q (1-3),

$$(c_b b^2\chi_{B_0/4}\Lambda Q, \Lambda\tilde{P}_{B_0}) = \frac{b^2}{2|\log b|}\left(1+O\left(\frac{1}{|\log b|}\right)\right)\left[\int_{y\le 1/2b}(\Lambda Q)^2 + O(1)\right] = 32b^2\left(1+O\left(\frac{1}{|\log b|}\right)\right),$$

and (4-69) follows.

We now estimate the lower-order terms in b that correspond to the second line of (4-67). One term is reintegrated by parts in time,

$$-(\partial_s^2(P_{B_1} - \tilde{P}_{B_0}), \Lambda\tilde{P}_{B_0}) = -\frac{d}{ds}\left\{b_s(\partial_b(P_{B_1} - \tilde{P}_{B_0}), \Lambda\tilde{P}_{B_0})\right\} + b_s^2\left(\partial_b(P_{B_1} - \tilde{P}_{B_0}), \partial_b\Lambda\tilde{P}_{B_0}\right).$$

The remaining terms are estimated in brute force using (2-8) and (3-32), which yield

$$\begin{aligned}\left|\left(b_s\Lambda(P_{B_1} - \tilde{P}_{B_0}) + b(\partial_s(P_{B_1} - \tilde{P}_{B_0}) + 2\Lambda\partial_s(P_{B_1} - \tilde{P}_{B_0})), \Lambda\tilde{P}_{B_0}\right)\right| + b_s^2\left|(\partial_b(P_{B_1} - \tilde{P}_{B_0}), \partial_b\Lambda\tilde{P}_{B_0})\right| \\ \lesssim |b_s| + \frac{|b_s|^2}{b^2} \lesssim K(M)\frac{b^2}{|\log b|}.\end{aligned}$$

<u>Step 2:</u> *ε terms.* We are left with estimating the third line on the right-hand side of (4-67). We first treat the linear term from (4-1), (4-7) and (3-34) and get

$$|(H_{B_1}\varepsilon, \Lambda\tilde{P}_{B_0})| \lesssim |(H\varepsilon, \Lambda\tilde{P}_{B_0})| + \int |\varepsilon||P_{B_1}^2 - Q^2||\Lambda\tilde{P}_{B_0}| + b^2|(D\Lambda\varepsilon, \Lambda\tilde{P}_{B_0})|. \tag{4-70}$$

On the one hand, (4-7) together with bootstrap estimates yields

$$\int |\varepsilon||P_{B_1}^2 - Q^2||\Lambda\tilde{P}_{B_0}| \lesssim b^2\int_{y\le B_0}\frac{|\varepsilon|}{(1+y^2)^2} \le b^{3/2}\left(\int\frac{|\varepsilon|^2}{(1+y)^5}\right)^{1/2} \lesssim \frac{b^2}{|\log(b)|}.$$

On the other hand, after integration by parts, we repeat the same arguments as before and apply (C-4).

[8] See again [Raphaël and Rodnianski 2012] for more details about the flux computation statement and its connection to the Pohozaev integration by parts formula.

This yields

$$\begin{aligned}
b^2\left|(D\Lambda\varepsilon,\,\Lambda\tilde P_{B_0})\right| &\le b^2\int_{y\le B_0}\frac{|\varepsilon|}{(1+y^4)}+b^2\int_{B_0/4\le y\le B_0/2}\frac{|\varepsilon|}{(1+y^2)}+b^2\int_{y\le B_0}|\nabla\varepsilon|\frac{y}{1+y^2}\\
&\lesssim b^{3/2}\left(\int\frac{|\varepsilon|^2}{(1+y^5)}\right)^{1/2}+\left(\int_{B_0/4\le y\le B_0/2}\frac{|\varepsilon|^2}{(1+y^4)}\right)^{1/2}+\left(\int_{y\le B_0}\frac{|\nabla\varepsilon|^2}{1+y^2}\right)^{1/2}\\
&\lesssim\sqrt{|\log(b)|}\left(c(M)|\mathscr{E}|+\sqrt{K(M)}\frac{b^4}{|\log b|^2}\right)^{1/2}\\
&\lesssim\sqrt{K(M)}\frac{b^2}{\sqrt{|\log(b)|}}.
\end{aligned}$$

Finally,

$$\begin{aligned}
|(H\varepsilon,\,\Lambda\tilde P_{B_0})| &\lesssim \|H\varepsilon\|_{L^2}\sqrt{|\log b|}+\sqrt{K(M)}\frac{b^2}{\sqrt{\log(b)}}\\
&\lesssim\sqrt{|\log b|}\sqrt{|\mathscr{E}|+\sqrt{K(M)}\,b^4/|\log b|^2}\lesssim\sqrt{K(M)}\frac{b^2}{\sqrt{|\log b|}}.
\end{aligned}$$

We further integrate by parts in time to obtain

$$\begin{aligned}
&\left(\partial_s^2\varepsilon+b(\partial_s\varepsilon+2\Lambda\partial_s\varepsilon)+b_s\Lambda\varepsilon,\,\Lambda\tilde P_{B_0}\right)\\
&\qquad=\frac{d}{ds}\left[(\partial_s\varepsilon,\,\Lambda\tilde P_{B_0})+b(\varepsilon+2\Lambda\varepsilon,\,\Lambda\tilde P_{B_0})\right]-b_s\left[(\partial_s\varepsilon+b\Lambda\varepsilon,\,\Lambda\partial_b\tilde P_{B_0})+(\varepsilon,\,\Phi_b)\right],
\end{aligned}$$

with

$$\Phi_b=-\Lambda\tilde P_{B_0}-\Lambda^2\tilde P_{B_0}-b\Lambda\partial_b\tilde P_{B_0}-b\Lambda^2\partial_b\tilde P_{B_0}.$$

We thus estimate from (4-1), (4-5), (4-7), (3-32) and (3-34) that

$$\begin{aligned}
|b_s||(\partial_s\varepsilon+b\Lambda\varepsilon,\,\Lambda\partial_b\tilde P_{B_0})+(\varepsilon,\,\Phi_b)| &\lesssim |b_s|\left[\int_{B_0/4\le y\le B_0}\frac{|\eta|}{y}+\int_{y\le B_0}\frac{|\varepsilon|}{1+y^2}\right]\\
&\lesssim\frac{|b_s||\log b|}{b^2}C(M)\sqrt{|\mathscr{E}|+\sqrt{K(M)}\,b^4/|\log b|^2}\lesssim K(M)\frac{b^2}{\sqrt{|\log b|}}.
\end{aligned}$$

The nonlinear term is estimated as before. Indeed, we have

$$\begin{aligned}
|(N(\varepsilon),\,\Lambda\tilde P_{B_0})| &\lesssim\int(|P_{B_1}|+|\varepsilon|)\varepsilon^2|\Lambda\tilde P_{B_0}|\\
&\lesssim\frac{1}{b^2}\|y(|P_{B_1}|+|\varepsilon|)\|_{L^\infty}\|(1+y^2)\Lambda\tilde P_{B_0}\|_{L^\infty}\int_0^{B_0}\frac{|\varepsilon|^2}{y(1+y^4)}\\
&\lesssim\frac{C(M)}{b^2}\left[\mathscr{E}+K(M)\frac{b^4}{|\log b|^2}\right]\lesssim K(M)\frac{b^2}{\sqrt{|\log b|}}.
\end{aligned}$$

Step 3: *Control of* $G(b)$ *and* $\mathscr{I}$. Injecting the estimates of Steps 1 and 2 into (4-67) yields (4-66). It remains to prove (4-65). The estimate for $G(b)$ is a straightforward consequence of the choice (4-62) and the explicit formula (1-3). It remains to control $\mathcal{I}$. We integrate by parts in space in (4-64) and get

$$\mathcal{I}(s) = (\partial_s\varepsilon + b\Lambda\varepsilon,\, \Lambda\tilde{P}_{B_0}) - b(\varepsilon,\, \Lambda\tilde{P}_{B_0} + \Lambda^2\tilde{P}_{B_0}) + b_s(\partial_b\tilde{P}_{B_0},\, \Lambda\tilde{P}_{B_0}) - b_s\big(\partial_b(P_{B_1} - \tilde{P}_{B_0}),\, \Lambda\tilde{P}_{B_0}\big).$$

The b terms are estimated as in Step 1,

$$|b_s|\big|(\partial_b\tilde{P}_{B_0},\, \Lambda\tilde{P}_{B_0}) - (\partial_b(P_{B_1} - \tilde{P}_{B_0}),\, \Lambda\tilde{P}_{B_0})\big| \lesssim \frac{|b_s|}{b} \lesssim b.$$

The linear term is estimated using (4-1), (4-5), (4-7), (3-32) and (3-34),

$$\begin{aligned}&\big|(\partial_s\varepsilon + b\Lambda\varepsilon,\, \Lambda\tilde{P}_{B_0}) - b(\varepsilon,\, \Lambda\tilde{P}_{B_0} + \Lambda^2\tilde{P}_{B_0})\big|\\ &\qquad\lesssim \int_{y\le B_0}\frac{|\eta|}{y^2} + b\int_{y\le B_0}\frac{|\varepsilon|}{y^2} \lesssim \frac{1}{b}\left(\int\frac{|\eta|^2}{y^2}\right)^{1/2} + \frac{|\log b|}{b^2}\left(\int_{y\le B_0}\frac{|\varepsilon|^2}{y^4(1+|\log y|^2)}\right)^{1/2} \lesssim K(M)b,\end{aligned}$$

and (4-65) is proved. This concludes the proof of Lemma 4.11. □

5. Sharp description of the singularity formation

We are now in position to conclude the proofs of Proposition 3.5 and Theorem 1.1 as simple consequences of the *a priori* bounds obtained in the previous section. The proofs rely on a topological argument that finishes the bootstrap argument, and then the sharp description of the blow-up dynamic is a consequence of the *a priori* bounds obtained on the solution and in particular the modulation equation (4-66).

Proof of Proposition 3.5. We argue by contradiction and assume that for all

$$a_+ \in \left[-\frac{b_0^2}{|\log b_0|},\, \frac{b_0^2}{|\log b_0|}\right], \quad T_1(a_+) < T(a_+).$$

In view of what Lemma 4.9 says about the bootstrap regime and the improved bounds of Lemmas 4.9 and 4.10, a simple continuity argument ensures that $T_1(a_+)$ is attained at the first time t where

$$|\kappa_+(t)| = \frac{|b(t)|^2}{2|\log(b(t))|}. \tag{5-1}$$

The fundamental fact used now is the outgoing behavior (4-56), which together with (5-1), ensures that

$$\left|\frac{d\kappa_+}{dt}(T_1(a_+))\right| > 0.$$

Thus from a standard argument,[9] the map

$$\left[-\frac{b_0^2}{|\log b_0|},\, \frac{b_0^2}{|\log b_0|}\right] \to \mathbb{R}*_+, \quad a_+ \mapsto T_1(a_+),$$

is continuous. We may thus consider the continuous map

$$\Phi : \left[-\frac{b_0^2}{|\log b_0|},\, \frac{b_0^2}{|\log b_0|}\right] \to \mathbb{R}, \quad a_+ \to \kappa_+(T_1(a_+))\frac{2|\log b(T_1(a_+))|}{b^2(T_1(a_+))}.$$

[9] See [Cote et al. 2009, Lemma 6] for a complete exposition.

On the one hand, (5-1) implies

$$\Phi\left(\left[-\frac{b_0^2}{|\log b_0|}, \frac{b_0^2}{|\log b_0|}\right]\right) \subset \{-1, 1\}.$$

On the other hand, the outgoing behavior (4-56) together with the initialization $\kappa_+(0) = a_+$ ensure that

$$\Phi\left(-\frac{b_0^2}{|\log b_0|}\right) = -1 \quad \text{and} \quad \Phi\left(\frac{b_0^2}{|\log b_0|}\right) = 1,$$

and a contradiction follows.[10] This concludes the proof of Proposition 3.5. □

Proof of Theorem 1.1.

Step 1: *Finite time blow-up and derivation of the blow-up speed.* Choose from Proposition 3.5 initial data with $T_1(a_+) = T(a_+)$. We first claim that u blows up in finite time,

$$T = T(a_+) < +\infty. \tag{5-2}$$

Indeed, from (4-41),

$$\lambda^{2(1-\alpha)} \lesssim b^3 \quad \text{and thus} \quad \lambda^{2/3} \lesssim \lambda^{2(1-\alpha)/3} \lesssim b = -\lambda_t.$$

Integrating this differential inequality yields

$$t \lesssim \lambda^{1/3}(0) - \lambda^{1/3}(t) \lesssim 1$$

and (5-2) follows. The $(\dot{H}^1 \cap \dot{H}^2) \times (L^2 \cap \dot{H}^1)$ bounds (3-33) and (3-34) on $(\varepsilon, \partial_t \varepsilon)$, and hence on $(u, \partial_t u)$ in the bootstrap regime, and standard H^2 local well posedness theory ensures that blow-up corresponds to

$$\lambda(t) \to 0 \quad \text{as } t \to T(a_+).$$

We now derive the blow-up speed by reintegrating the ODE (4-66) and briefly sketch the proof which follows as in [Raphaël and Rodnianski 2012].

First recall the standard scaling lower bound

$$\lambda(t) \leq C(u_0)(T - t),$$

which implies that the rescaled time is global,

$$s(t) = \int_0^t \frac{d\tau}{\lambda(\tau)} \to +\infty \quad \text{as } t \to T.$$

Let

$$\mathcal{J} = G + \mathcal{I}$$

so that from (4-65) we get

$$\mathcal{J} = 64b\,|\log b|\left(1 + O\left(\frac{1}{|\log b|}\right)\right) \quad \text{and} \quad b = \frac{\mathcal{J}}{64|\log \mathcal{J}|}\left(1 + O\left(\frac{1}{\sqrt{|\log \mathcal{J}|}}\right)\right), \tag{5-3}$$

[10] This topological argument is the one-dimensional version of Brouwer's fixed-point argument used in [Cote et al. 2009].

and $\mathcal{J}$ satisfies from (4-66) the ODE

$$\mathcal{J}_s + \frac{\mathcal{J}^2}{128|\log \mathcal{J}|^2}\left(1 + O\left(\frac{1}{\sqrt{|\log \mathcal{J}|}}\right)\right) = 0.$$

We multiply the above by $|\log \mathcal{J}|^2/\mathcal{J}^2$, integrate in time and obtain to leading order that

$$\mathcal{J} = \frac{128(\log s)^2}{s}\left(1 + O\left(\frac{1}{\sqrt{|\log s|}}\right)\right) \quad \text{that is,} \ -\frac{\lambda_s}{\lambda} = b = \frac{2\log s}{s}\left(1 + O\left(\frac{1}{\sqrt{|\log s|}}\right)\right),$$

where we used (5-3). Integrating this once more in time yields

$$-\log \lambda = (\log s)^2\left(1 + O\left(\frac{1}{\sqrt{|\log s|}}\right)\right)$$

and thus

$$b = -\lambda_t = \exp\left(-\sqrt{|\log \lambda|}\left(1 + O\left(\frac{1}{|\log \lambda|^{1/4}}\right)\right)\right).$$

Integrating this from t to T where $\lambda(T) = 0$ yields the asymptotic

$$\lambda(t) = (T - t)\exp\left(-\sqrt{|\log \lambda(t)|}\left(1 + O\left(\frac{1}{|\log \lambda(t)|^{1/4}}\right)\right)\right),$$

which yields (1-10).

Step 2: *Energy quantization.* It remains to prove (1-9), which can be derived exactly as in [Raphaël and Rodnianski 2012]; this is left to the reader. This concludes the proof of Theorem 1.1. □

Appendix A: Modulation theory

This appendix is devoted to the proof of Lemmas 3.1 and 3.3. The arguments are standard in the framework of modulation theory and we briefly sketch the main computations.

Proof of Lemma 3.1. First note that the bounds

$$\|\nabla(P_{B_1} - Q)\|_{L^2} + b\,\|\Lambda P_{B_1} - b(1-\chi_{B_1})\Lambda Q\|_{L^2} \lesssim b\,|\log b|$$

ensure that our initial data are of the form

$$u_0 = Q + \tilde{\eta}_0, \quad u_1 = \tilde{\eta}_1,$$

for a small excess of energy in the sense that

$$\|\nabla\tilde{\eta}_0, \tilde{\eta}_1\|_{L^2\times L^2} \lesssim b_0|\log b_0|, \quad \|\nabla^2\tilde{\eta}_0, \nabla\tilde{\eta}_1\|_{L^2\times L^2} \lesssim b_0. \tag{A-1}$$

Hence the continuity of the flow associated to (1-1) ensures the existence of a time $T_0 > 0$ (uniform in $\tilde{\eta}_0, \tilde{\eta}_1$) for which the solution u to (1-1) with initial data (u_0, u_1) satisfies on $[0, T_0]$ that

$$\sup_{[0,T_0]} \|\nabla(u - Q), \partial_t u\|_{L^2\times L^2} \lesssim b_0|\log b_0|. \tag{A-2}$$

Step 1: *Modulation near Q.* The nondegeneracy $(\Lambda Q, \Phi) \neq 0$ ensures[11] that u admits on $[0, T_0]$ a decomposition

$$u(t) = (Q + \tilde{\varepsilon}(t))_{\lambda(t)}, \tag{A-3}$$

with

$$(\tilde{\varepsilon}(t), \chi_M \Phi) = 0. \tag{A-4}$$

Moreover, $\lambda \in \mathscr{C}^2([0, T_0]; \mathbb{R}_+^*)$, and noting that $\tilde{\eta}_0$ satisfies

$$|(\tilde{\eta}_0, \chi_M \Phi)| \lesssim \frac{b_0^2}{|\log b_0|},$$

we obtain the bound

$$|\lambda(0) - 1| \lesssim \frac{b_0^2}{|\log b_0|}. \tag{A-5}$$

We then let $b(t) = -\lambda_t(t)$ on $[0, T_0]$.

Step 2: *Positivity of b.* Straightforward computations yield

$$\partial_t \tilde{\varepsilon}(t) = \left(\partial_t u - \frac{b(t)}{\lambda(t)} \Lambda u\right)_{1/\lambda(t)}.$$

Taking the scalar product with $\chi_M \Phi$, we obtain at the initial time

$$b(0) = \lambda(0) \frac{\big((u_1)_{1/\lambda(0)}, \chi_M \Phi\big)}{\big((\Lambda u_0)_{1/\lambda(0)}, \chi_M \Phi\big)}, \tag{A-6}$$

where (2-5) together with (A-5) imply

$$\big((u_1)_{1/\lambda(0)}, \chi_M \Phi\big) = b_0(\Lambda Q, \chi_M \Phi) + O\left(\frac{b_0^2}{|\log(b_0)|}\right), \tag{A-7}$$

$$\big((\Lambda u_0)_{1/\lambda(0)}, \chi_M \Phi\big) = (\Lambda Q, \chi_M \Phi) + O\big(b_0^2 |\log(b_0)|\big). \tag{A-8}$$

This yields the positivity of $b(0)$ and the positivity of $b(t)$ for small time, together with

$$b(t) = b_0 + O\left(\frac{b_0^2}{|\log(b_0)|}\right). \tag{A-9}$$

As $b > 0$, we may introduce the decomposition

$$u(t) = (Q + \tilde{\varepsilon})_{\lambda(t)} = (P_{B_1(b(t))} + \varepsilon)_{\lambda(t)}, \quad \text{where } \varepsilon(t) = \tilde{\varepsilon}(t) - (P_{B_1(b(t))} - Q). \tag{A-10}$$

Observe from (2-4) and (A-4) that

$$\forall\, t \in [0, T_0], \quad (\varepsilon(t), \chi_M \Phi) = 0. \tag{A-11}$$

The uniqueness of such a decomposition is guaranteed by the (local) uniqueness of $(\lambda, \tilde{\varepsilon})$.

[11] This is a direct consequence of the implicit function theorem and the smoothness of the flow (1-1).

Step 3: *Smallness of* ε. To complete the proof, we obtain smallness of ε in $\dot{H}^1$ and $\dot{H}^2$. To this end, we note that

$$\varepsilon(0) = (u_0)_{1/\lambda(0)} - P_{B_1(b(0))} = \left[\left(P_{B_1(b_0)}\right)_{1/\lambda(0)} - P_{B_1(b(0))}\right] + (\eta_0 + d_+\psi)_{1/\lambda(0)}.$$

Simple computations based on the estimates of Proposition 2.1 yield the expected result,

$$\|\nabla\varepsilon(0)\|_{L^2} \lesssim b_0|\log(b_0)| \quad \text{and} \quad \left\|\frac{\varepsilon(0)}{1+y^4}\right\|_{L^2} + \|\nabla^2\varepsilon(0)\|_{L^2} \lesssim \frac{b_0^2}{|\log(b_0)|}. \tag{A-12}$$

This concludes the proof. □

Proof of Lemma 3.3. The proof of this lemma is divided into two steps. First, given (η_0, η_1, d_+) satisfying the smallness condition (3-1) for small b_0, we prove that b, b_s and w satisfy (3-31)–(3-34). Then, we show that given (b_0, η_0, η_1), we can apply the inverse mapping theorem to $d_+ \mapsto \kappa_+(0)$ close to 0. The arguments used are standard and we refer to [Cote et al. 2009] for a detailed proof in a similar setting.

Step 1: *Smallness of initial modulation.* Given (η_0, η_1, d_+) satisfying the smallness condition (3-1), we can apply Lemma 3.1. This yields T_0 and b, ε, w such that (3-31) holds and

$$\|\nabla w(t)\|_{L^2} \lesssim b_0|\log(b_0)|, \quad \|\nabla^2 w(t)\|_{L^2} \lesssim \frac{b_0^2}{|\log(b_0)|^2}. \tag{A-13}$$

We emphasize that Lemma 3.1 implies in particular that $b_0/2 < b(0) < 2b_0$ for sufficiently small b_0.

As before, we focus now on bounds satisfied initially. We first compute $b_s(0)$ using (1-1) and the orthogonality condition (A-11). Recalling that $(\partial_b^k P_{B_1}, \chi_M\Phi) = (\partial_s^{k-1}\varepsilon, \chi_M\Phi) = 0$ for any integer k, we get, like for (4-10),

$$\begin{aligned} b_s[(\Lambda P_{B_1}, \chi_M\Phi) + 2b(\Lambda\partial_b P_{B_1}, \chi_M\Phi) + (\Lambda\varepsilon, \chi_M\Phi)] \qquad& \\ = -(\Psi_{B_1}, \chi_M\Phi) - (\varepsilon, H_{B_1}^*(\chi_M\Phi)) + b(\partial_s\varepsilon, \Lambda(\chi_M\Phi)) &+ (N(\varepsilon), \chi_M\Phi), \end{aligned}$$

where, denoting by *LHS* and *RHS* the two sides at initial time, we compute, for b_0 small enough with respect to M that

$$|RHS| \le C(M)\left(\frac{b_0^2}{|\log(b_0)|} + \|\partial_s\varepsilon\|_{L^2(y<M)}\right), \qquad \frac{|b_s(0)|}{2}(\Lambda Q, \chi_M\Phi) \le |LHS|. \tag{A-14}$$

At the same time, after time-differentiation, we obtain

$$\partial_s\varepsilon(0) = \lambda(0)\partial_t\varepsilon(0) = -b_s(0)\partial_b P_{B_1(b(0))} - b(0)\Lambda u_0 + \lambda(0)\left(b_0\Lambda P_{B_1(b_0)}\right)_{1/\lambda(0)}. \tag{A-15}$$

Observe now from (2-8) that

$$\left\|\partial_b P_{B_1(b_0)}\right\|_{L^2(y\le 2M)} \lesssim C(M)b_0 \le \sqrt{b_0},$$

which together with (A-5), (A-9) and (3-1) yields

$$\|\partial_s\varepsilon(0)\|_{L^2(y\le 2M)} = \lambda(0)\|\partial_t\varepsilon(0)\|_{L^2(y\le 2M)} \lesssim \frac{b_0^2}{|\log b_0|} + |b_s(0)|\sqrt{b_0}, \tag{A-16}$$

which together with (A-14) concludes the proof of the initial bound (3-26) on b_s.

Then we compute

$$\partial_t w(0) = u_1 - \left(\frac{b_s(0)}{\lambda(0)}\partial_b P_{B_1(b(0))} + \frac{b(0)}{\lambda(0)}\Lambda P_{B_1(b(0))}\right)_{\lambda(0)},$$

so that, introducing (A-15) and previous estimates on $b(0)$, we get

$$\left\|\partial_t w(0) + \frac{b(0)}{\lambda(0)}\big((1-\chi_{B_1(b(0))})\Lambda Q\big)_{\lambda(0)}\right\|_{L^2} \lesssim b_0|\ln(b_0)| \leq \sqrt{b_0},$$

and

$$\|\nabla\partial_t w(0)\|_{L^2} \lesssim \frac{b_0^2}{|\log b_0|}. \tag{A-17}$$

Together with (A-13), this yields (3-27) and (3-28).

Finally, straightforward computations yield

$$\kappa_- = \frac{1}{2}(\varepsilon, \psi) - \frac{1}{\zeta}(\partial_s\varepsilon, \psi) - \frac{b_s}{2\zeta}(\partial_b P_{B_1}, \psi).$$

Consequently, we apply (3-28), noting that $w(t) = (\varepsilon(t))_{\lambda(t)}$, and (A-15) because of the exponential decay of ψ to get

$$|\kappa_-(0)| \lesssim \frac{b_0^2}{|\log b_0|}. \tag{A-18}$$

Step 2: *Computation of* d_+. We now claim from an explicit computation that given a_+, the initialization (3-24) can be reformulated in the form

$$F(d_+) = a_+, \quad \text{with } \left.\frac{\partial F}{\partial d_+}\right|_{d_+=0} = \tfrac{1}{2}\|\psi\|_{L^2}^2 + O(b_0), \tag{A-19}$$

from which the implicit function theorem concludes the proof of Lemma 3.3.

Let us briefly justify (A-19). We want to study the mapping

$$\mathcal{V} \to \mathbb{R}^4, \quad d_+ \mapsto [b(t), b_s(t), (\varepsilon(0), \psi), (\partial_s\varepsilon(t), \psi)],$$

where $\mathcal{V}$ is a neighborhood of 0. To this end, it is necessary to study the dependencies of all initial parameters on d_+. For conciseness, we denote by d differentiation with respect to d_+ in what follows.

***Computation of* $(\boldsymbol{\lambda(0)}, \boldsymbol{\tilde{\varepsilon}(0)})$.** As a first step in modulation theory, we proved that $(\lambda(0), \tilde{\varepsilon}(0)) = \Phi(u_0)$, where Φ is a smooth mapping $\dot{H}^1(\mathbb{R}^N) \to \mathbb{R}\times\dot{H}^1(\mathbb{R}^N)$ defined on a neighborhood of Q. Due to the exponential decay of $\psi \in \mathscr{C}^\infty(\mathbb{R}^N)$ we thus have that $\lambda(0)$ is a smooth function of d_+ with differential $d\lambda(0) = d\lambda \in \mathbb{R}$. We have the same result for ε with differential $d\tilde{\varepsilon}(0) = d\tilde{\varepsilon} \in \dot{H}^1(\mathbb{R}^N)$. By definition, we have

$$\tilde{\varepsilon}(0) = u_0 - Q_{1/\lambda},$$

so that

$$d\tilde{\varepsilon} = \psi + \frac{d\lambda}{\lambda(0)}(\Lambda Q)_{1/\lambda(0)}.$$

Computation of $b(0)$. From (A-6), $b(0)$ is a $\mathscr{C}^1$ mapping with

$$db(0) = d\lambda \left[\frac{\big((u_1)_{1/\lambda(0)}, \chi_M \Phi\big)}{\big((\Lambda u_0)_{1/\lambda(0)}, \chi_M \Phi\big)} + \frac{\big((\Lambda^2 u_0)_{1/\lambda(0)}, \chi_M \Phi\big) - \big((\Lambda u_1)_{1/\lambda(0)}, \chi_M \Phi\big)}{\big((\Lambda u_0)_{1/\lambda(0)}, \chi_M \Phi\big)^2} \right] - \lambda(0) \frac{\big((u_1)_{1/\lambda(0)}, \chi_M \Phi\big)\big((\Lambda \psi)_{1/\lambda(0)}, \chi_M \Phi\big)}{\big((\Lambda u_0)_{1/\lambda(0)}, \chi_M \Phi\big)^2},$$

where (A-6) and (A-7) ensure that, for some $db \in \mathbb{R}$, we have

$$db(0) = db + O(b_0).$$

Computation of $\varepsilon(0)$. Next,

$$\varepsilon(0) = \tilde{\varepsilon}(0) - (P_{B_1(b(0))} - Q).$$

Consequently, $(\varepsilon(0), \psi)$ is also a smooth function of d_+ with derivative $dps_1(0)$ satisfying

$$dps_1(0) = (d\tilde{\varepsilon}, \psi) - db(0)(\partial_b P_{B_1(b(0))}, \psi).$$

Replacing $d\tilde{\varepsilon}$ by its values, and applying that $(\Lambda Q, \psi) = 0$ together with $|\lambda(0) - 1| \lesssim b_0^2/|\log(b_0)|$, we get

$$(d\tilde{\varepsilon}, \psi) = \|\psi\|_{L^2}^2 + O(b_0),$$

so that

$$dps_1(0) = \|\psi\|_{L^2}^2 + O(b_0).$$

Computation of $\partial_s \varepsilon(0) + b_s(0) \partial_b P_{B_1(b(0))}$. From (A-15),

$$\partial_s \varepsilon(0) = -b_s(0) \partial_b P_{B_1(b(0))} - b(0) \Lambda u_0 + \lambda(0) \left(b_0 \Lambda P_{B_1(b_0)}\right)_{1/\lambda(0)},$$

so that $(\partial_s \varepsilon(0) + b_s(0) \partial_b P_{B_1(b(0))}, \psi)$ is a smooth function of d_+ with derivative

$$dps_2(0) = -db(0)(\Lambda u_0, \psi) + d\lambda\Big(\Big[\big(b_0 \Lambda P_{B_1(b_0)}\big)_{1/\lambda(0)} + \big(b_0 \Lambda^2 P_{B_1(b_0)}\big)_{1/\lambda(0)}\Big], \psi\Big) - b(0)(\Lambda \psi, \psi),$$

where, for the same orthogonality reason $(\Lambda Q, \psi) = 0$, we have

$$(\Lambda u_0, \psi) = (\Lambda Q, \psi) + O(b_0) = O(b_0).$$

Consequently $dps_2(0) = O(b_0)$.

Conclusion. Finally, we have

$$\kappa_+(0) = \frac{1}{2}\left[(\varepsilon(0), \psi) + \frac{1}{\sqrt{\zeta}}(\partial_s \varepsilon(0) + b_s(0)\partial_b P_{B_1(b(0))}, \psi)\right],$$

and $\kappa_+(0) = a_+$ reduces to a simple one-dimensional equation $F(d_+) = a_+$, with F computed as combination of the above functions so that it is smooth in a neighborhood of 0. Moreover,

$$dF = \frac{1}{2}\left[dps_1(0) + \frac{1}{\sqrt{\zeta}} dps_2(0)\right] = \tfrac{1}{2}\|\psi\|_{L^2}^2 + O(b_0),$$

and (A-19) is proved. This concludes the proof of Lemma 3.3. □

Appendix B: Coercivity estimates

The aim of this section is to prove the coercivity properties of the quadratic form

$$B(\eta, \eta) = (\mathcal{B}v, v) = \int_{\mathbb{R}^4} |\partial_r \eta|^2 + \int_{\mathbb{R}^4} W\eta^2,$$

where

$$W(r) = 2V + \tfrac{3}{2}\, rV' = \frac{6}{(1+r^2/8)^2} - \frac{9}{4}\frac{r^2}{(1+r^2/8)^3}.$$

We use the elementary method developed in [Fibich et al. 2006]. The coercivity property of Lemma 4.7 is a consequence of the two following facts. First the index of B on

$$\dot{H}^1_r = \left\{ u \text{ radial} \;\middle|\; \int |\nabla u|^2 + \int \frac{u^2}{r^2} < +\infty \right\}$$

is at most 2. From standard Sturm–Liouville oscillation theorems, see Theorem XIII.8 [Reed and Simon 1978], this is equivalent to counting the number of zeroes of the solution to

$$\begin{cases} \mathcal{B}U = 0, \\ U(0) = 1, \quad U'(0) = 0, \end{cases} \tag{B-1}$$

on $(0, \infty)$, and this can be analytically reduced to counting the number of zeroes of a Bessel function. Then we need to show that the orthogonality conditions $(\eta, \psi) = (\eta, \Phi) = 0$ are enough to treat the two negative directions. Arguing exactly as in [Fibich et al. 2006] — see also [Marzuola and Simpson 2011] — this is equivalent to first inverting the operator $\mathcal{B}$ on $\dot{H}^1_{\text{rad}}$, and then showing that B restricted to $\text{Span}\{\mathcal{B}^{-1}\psi, \mathcal{B}^{-1}\Phi\}$ is negative definite, which is an elementary numerical check. We shall check these two facts below and refer to [Fibich et al. 2006] for the proofs that this implies the claimed coercivity property. The proofs there are given for exponentially decaying functions and potentials, but one checks easily that the decay of the potential $|W(r)| \sim 1/r^4$ at infinity and $|\Phi(r)| \sim 1/r^4$ are more than enough to have all proofs go through.

Computation of the index of B. We first show that the index of $\mathscr{B}$ on $\dot{H}^1_r$ is at most 2. We start by noting that $W(r) \geq \widehat{W}(r)$, where

$$\hat{W}(r) = -\frac{3}{2}\frac{r^2}{(1+r^2/8)^3}.$$

Hence, classical Sturm–Liouville theory ensures that U has less zeros than $\hat{U}$, the unique solution to

$$-\frac{1}{r^3}\frac{d}{dr}\left[r^3\frac{d}{dr}\widehat{U}\right] + \widehat{W}\widehat{U} = 0, \qquad \widehat{U}(0) = 1, \quad \widehat{U}'(0) = 0, \tag{B-2}$$

on $(0, \infty)$. Second, we look for $\widehat{U}$ of the form $\widehat{U}(r) = (2/r^2)\,\overline{U}(r^2/2)$, with $\overline{U}$ a sufficiently smooth function. Denoting by s the new variable $r^2/2$, straightforward calculations yield that $\overline{U}$ is a solution to

$$-\frac{d^2}{ds^2}\overline{U} + \overline{W}\,\overline{U} = 0, \qquad \overline{U}(0) = 0, \quad \overline{U}'(0) = 1, \tag{B-3}$$

on $(0, \infty)$, where

$$\overline{W}(s) = -\frac{3}{2}\frac{1}{(1+s/4)^3}.$$

Setting then $\overline{U}(s) = \sqrt{1+s/4}\,\widetilde{U}(1/\sqrt{1+s/4})$, we obtain that $\overline{U}$ is a solution to (B-3) if and only if $\widetilde{U}$ is a solution to

$$\tau^2\frac{d^2}{d\tau^2}\widetilde{U} + \tau\frac{d}{d\tau}\widetilde{U} + (96\tau^2-1)\widetilde{U} = 0, \qquad \widetilde{U}(1) = 0, \quad \widetilde{U}'(1) = -8,$$

on $(0, 1)$. Hence, $\widetilde{U}$ is a combination of Bessel functions: $\widetilde{U}(\tau) = C_1 J(1, 4\sqrt{6}\tau) + C_2 Y(1, 4\sqrt{6}\tau)$.

We compute (C_1, C_2) and draw the explicit combination with Maple (Figure 1). The computed solution $\widetilde{U}$ has two zeros on $(0, 1)$. Moreover, it diverges at 0 so that $\widetilde{U}(\tau) \sim K/\tau$ close to 0 with $K \neq 0$, As a

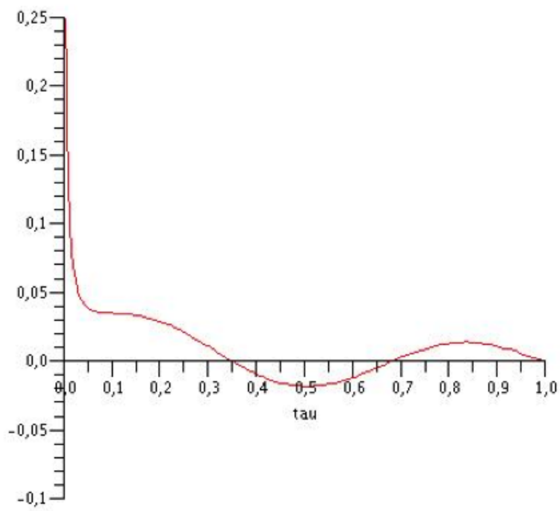

Figure 1. Solution to (B-3) computed by Maple.

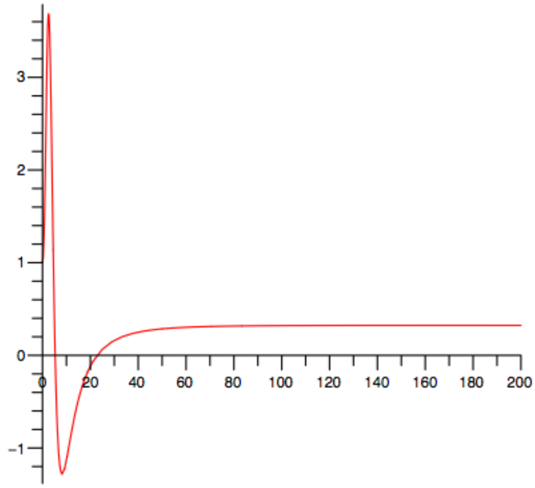

Figure 2. Solution to (B-1) computed by MAPLE.

consequence.

$$\hat{U}(r) \sim \tfrac{1}{4}K \neq 0 \quad \text{when } r \to \infty,$$

and thus the index of $-\Delta + \hat{W}$ on $\dot{H}^1_{\text{rad}}$ is exactly two. Hence the index of $\mathcal{B}$ is at most 2.

Choice of the orthogonality conditions. We now invert $\mathcal{B}$. We first check numerically that the solution U does not vanish at infinity, that is,

$$\lim_{r \to +\infty} U(r) > 0;$$

see Figure 2.

Hence U is not a resonance — note that if U had been a resonance, we could have removed the resonance by diminishing a bit the potential and getting a potential with index 2 and no resonance — and thus from standard ODE arguments [Fibich et al. 2006] there exists unique smooth solution in $\dot{H}^1{}_{\text{rad}}$ of

$$\mathcal{B}U = -\frac{1}{r^3}\frac{d}{dr}\left[r^3 \frac{d}{dr}U\right] + WU = \psi, \qquad U'(0) = 0, \tag{B-4}$$

on $(0, \infty)$, with $(1 + r^2)U \in L^\infty$, and

$$\mathcal{B}U = -\frac{1}{r^3}\frac{d}{dr}\left[r^3 \frac{d}{dr}U\right] + WU = \Phi, \qquad U'(0) = 0, \tag{B-5}$$

on $(0, \infty)$, with $(1 + r^2/\log r)U \in L^\infty$. We denote by $\mathcal{B}^{-1}\psi$ and $\mathcal{B}^{-1}\Phi$ the respective solutions to these systems. We recall the explicit formula

$$\Phi(r) = D\Lambda Q(r) = \frac{2 - 3r^2/4}{(1 + r^2/8)^3}.$$

In the remainder of this section we check numerically that the restriction of B to $\mathrm{Span}(\mathcal{B}^{-1}\psi, \mathcal{B}^{-1}\Phi)$ is negative definite, or equivalently:

Lemma. *The symmetric matrix* $\mathbb{B} = \begin{bmatrix} (\mathcal{B}^{-1}\psi, \psi) & (\mathcal{B}^{-1}\Phi, \psi) \\ (\mathcal{B}^{-1}\Phi, \psi) & (\mathcal{B}^{-1}\Phi, \Phi) \end{bmatrix}$ *satisfies*

$$(\mathcal{B}^{-1}\psi, \psi) < 0 \quad \textit{and} \quad \det \mathbb{B} > 0, \tag{B-6}$$

and is thus negative definite.

Numerical proof. We use standard MATLAB routines for the computation of solutions to (B-4) and (B-5). We note that we only fixed the initial value for $U'(0)$. The value $U(0)$ is left open in order to achieve the expected decay at infinity that characterizes the inverse. To obtain $\mathcal{B}^{-1}\psi$, we first compute ψ. We obtain that the corresponding eigenvalue is approximately $l = -0.5860808922$. Because ψ decays exponentially, we only need to obtain an approximation on a short time-range. We computed our solutions until $T_{\psi,\max} = 30$. We emphasize here that we use an explicit scheme. As a drawback, the accumulation of errors tends to make the numerical solution become negative when the exact solution is exponentially small. Hence, our scheme becomes unstable after time $\widetilde{T}_{\psi,\max} = 18$. Nevertheless, we extend our numerical solution by 0 after this time. This induces an exponentially small error. The pictures in Figure 3 illustrate this computation. On the left-hand side we draw the obtained solution. On the right-hand side, we draw $\psi_{\text{test}}(r) = \psi(r)\exp(\sqrt{-l}r)$. We observe here that our solution enters the exponential asymptotic regime before the instability comes into play.

The solution $\mathcal{B}^{-1}\psi$ is computed with the extension of ψ. Straightforward ODE analysis shows that the unique solution decaying fast at infinity behaves like $1/r^2$ asymptotically. The choice of $U(0)$ is made with respect to this criterion. Figure 4 illustrates that we obtained a solution with the suitable decay. As previously, on the left-hand side is a picture of the numerical solution. On the right-hand side we plot $\mathcal{B}^{-1}\psi_{\text{test}}(r) = r^2\mathcal{B}^{-1}\psi(r)$. In the latter computations, this solution is involved in scalar products

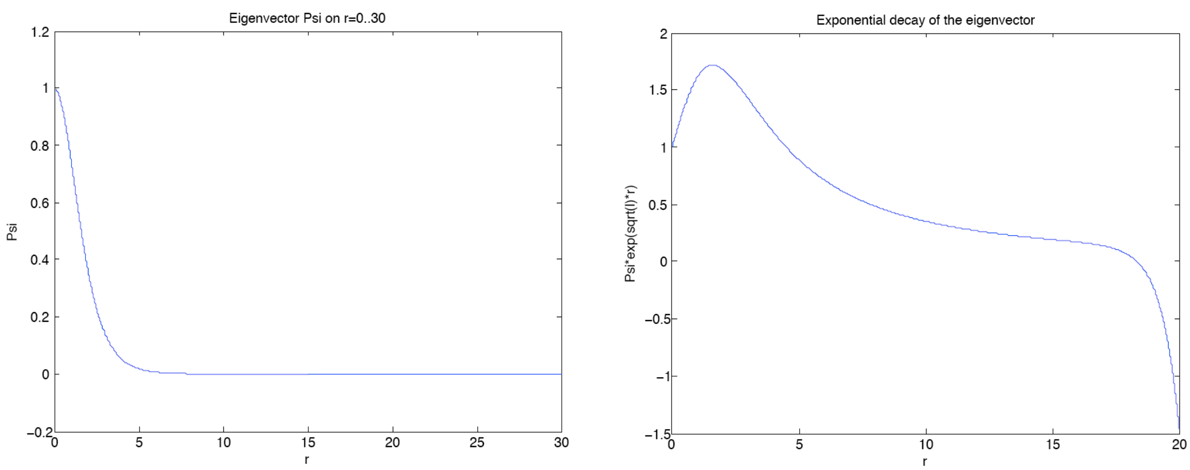

Figure 3. Numerical simulations for ψ.

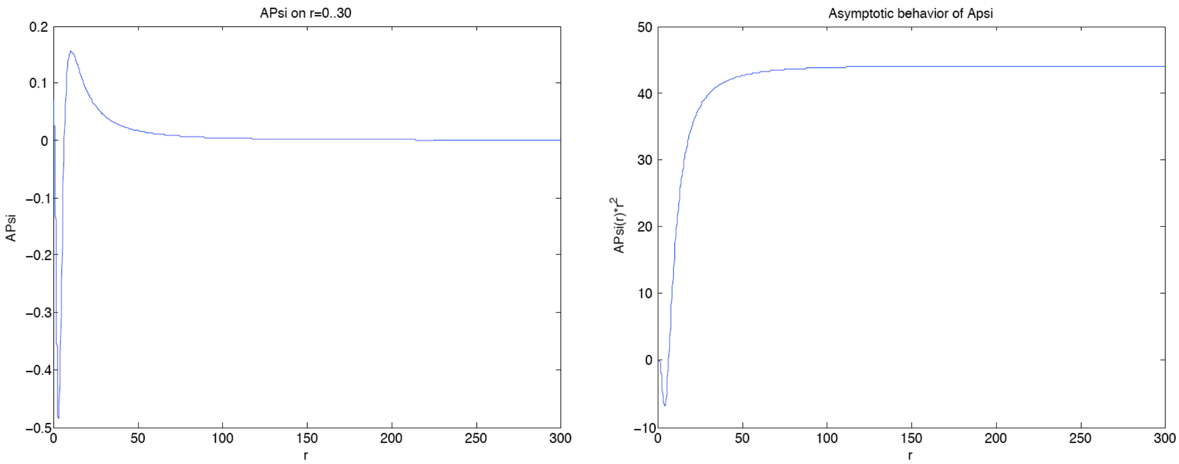

Figure 4. Numerical simulations for $\mathcal{B}^{-1}\psi$.

with ψ. Hence even if drawn until $T_{\max} = 300$, we only need a precise computation of this solution until $T_{\mathcal{B}^{-1}\psi,\max} = 18$.

The last solution $\mathcal{B}^{-1}\Phi$ is computed with the same method. In this second case, the expected decay of the solution is $\log(r)/r^2$. Figure 5 illustrates that we obtained a solution with the suitable decay. The picture on the right-hand side restricts to the time-interval $r = 0 \ldots 100$ because this is the significant region. In the latter computations, this solution is involved in integrals which converge slowly. Hence, we compute this solution until $T_{\mathcal{B}^{-1}\Phi,\max} = 1000$.

We now compute numerically the entries of the matrix $\mathbb{B}$. We first compute $(\mathcal{B}^{-1}\Phi, \psi) = (\mathcal{B}^{-1}\psi, \Phi)$. The exponential decay of ψ implies that we need to compute the first integral $(\mathcal{B}^{-1}\Phi, \psi)$ on a shorter time-interval. Hence, we prefer this computation to the second one. We compute the L^2 scalar products with a standard trapezoidal method. Changing the time-interval and the time-step, the computations are

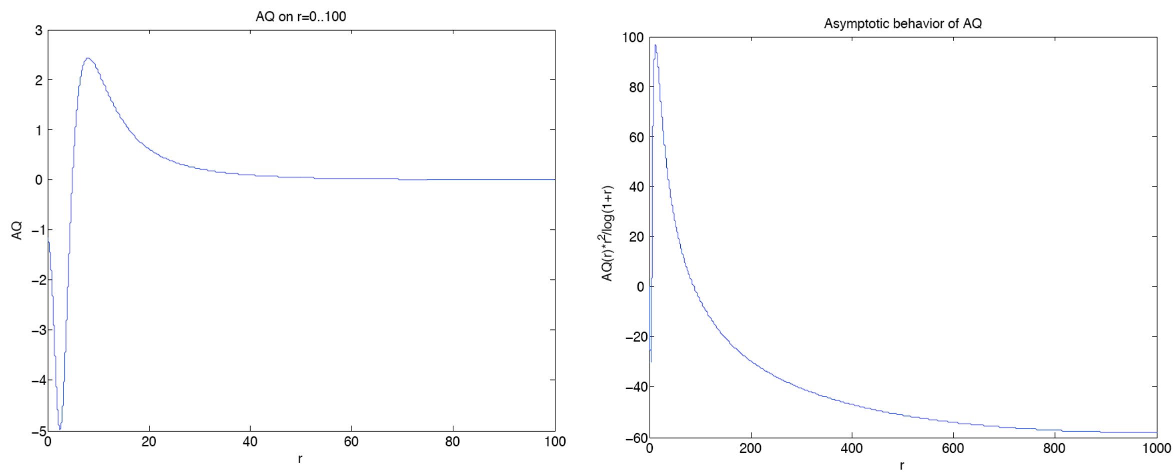

Figure 5. Numerical simulations for $\mathcal{B}^{-1}\Phi$.

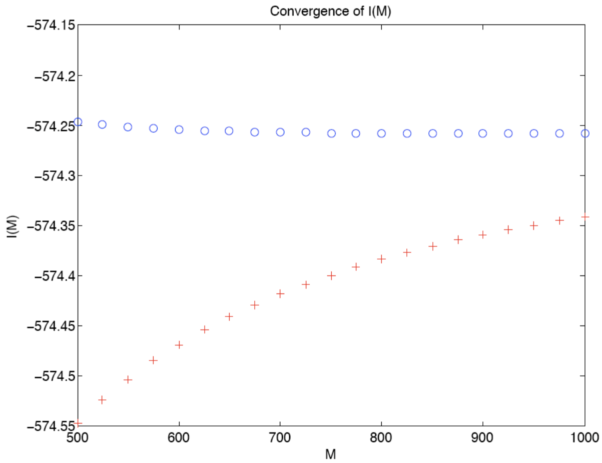

Figure 6. Computations for $(\mathcal{B}^{-1}\Phi, \Phi)$.

stable up to an error of 10^{-2}. We get the following approximations for the integrals involving ψ:

$$(\mathcal{B}^{-1}\psi, \psi) = -4.63 \pm 10^{-2} \quad \text{and} \quad (\mathcal{B}^{-1}\Phi, \psi) = 32.65 \pm 10^{-2}.$$

The last integral is a more involved computation. Indeed, standard real analysis implies that

$$I(M) := \int_0^M \mathcal{B}^{-1}\Phi Q(r)\Phi(r) r^3 \mathrm{d}r = (\mathcal{B}^{-1}\Phi, \Phi) + err(M),$$

with a remainder satisfying $err(M) = (K + o(1)) \ln(M)/M^2$ for some constant K. This remainder goes to 0 slowly; we see numerically that our computations have not converged even after integrating until $T_{\mathcal{B}^{-1}\Phi,\max} = 1000$ (see Figure 6, red crosses). To improve the rate of convergence we compute an approximation of coefficient K and subtract the estimated error term of our computations. This yields the blue circles in Figure 6. In this second computation we obtain a very good rate of convergence. Hence, we get the approximation $(\mathcal{B}^{-1}\Phi, \Phi) = -574.25 \pm 10^{-2}$, which leads to

$$\det(\mathbb{B}) = 1591 \pm 10,$$

concluding the numerical proof of the lemma. □

Appendix C: Some linear estimates

We start by recalling some obvious integration-by-part results:

Lemma C.1. *For any $N \geq 3$, there exists a constant C for which there holds, for any $v \in H^1_{\mathrm{rad}}(\mathbb{R}^N)$,*

$$\left[\int_{\mathbb{R}^N} \frac{|v(y)|^2}{|y|^2}\right]^{1/2} + \sup_{y \in \mathbb{R}^N} \left(|y|^{(N-2)/2}|v(y)|\right) \leq C \left[\int_{\mathbb{R}^n} |\nabla v(y)|^2\right]^{1/2}. \tag{C-1}$$

Looking for control on further derivatives, we prove a lemma.

Lemma C.2 (Hardy inequalities). *Let $N = 4$. Then for all $R > 2$ and $v \in H^2_{\mathrm{rad}}(\mathbb{R}^N)$, we have*

$$\int \frac{|\partial_y v|^2}{y^2} \lesssim \int (\Delta v)^2, \tag{C-2}$$

$$\int_{y\le R} \frac{|v|^2}{y^4(1+|\log y|)^2} \lesssim \int_{y\le R} \frac{|\partial_y v|^2}{y^2} + \int_{y\le 2} |v|^2, \tag{C-3}$$

$$\int_{R\le y\le 2R} \frac{|v|^2}{y^4} \lesssim \log R \int_{y\le R} \frac{|\partial_y v|^2}{y^2} + \int_{y\le 2} |v|^2. \tag{C-4}$$

Proof. Let v be smooth. (C-2) follows from the explicit formula after integration by parts,

$$\int (\Delta v)^2 = \int \left(\partial_{yy} v + \frac{N-1}{y}\partial_y v\right)^2 = \int (\partial_{yy} v)^2 + (N-1)\int \frac{|\partial_y v|^2}{y^2}.$$

To prove (C-3), let $a \in [1, 2]$ be such that

$$|v(a)|^2 \le \int_{1\le y\le 2} |v|^2. \tag{C-5}$$

Let $f(y) = -(1/y^3(1+\log(y)))\,\mathbf{e}_y$ so that $\nabla\cdot f = 1/(y^4(1+|\log y|)^2)$, and integrate by parts to get

$$\begin{aligned}\int_{a\le y\le R} \frac{|v|^2}{y^4(1+\log y)^2} &= \int_{a\le y\le R} |v|^2 \nabla\cdot f \\ &= -\left[\frac{|v|^2}{1+\log(y)}\right]_a^R + 2\int_{y\le R} \frac{v\partial_y v}{y^3(1+\log y)} \\ &\lesssim |v(a)|^2 + \left(\int_{y\le R} \frac{|v|^2}{y^4(1+|\log y|)^2}\right)^{1/2}\left(\int_{y\le R} \frac{|\partial_y v|^2}{y^2}\right)^{1/2}.\end{aligned} \tag{C-6}$$

similarly, using $\tilde f(y) = (1/y^3(1-\log(y)))\,\mathbf{e}_y$, we get

$$\begin{aligned}\int_{\varepsilon\le y\le a} \frac{|v|^2}{y^4(1-\log y)^2} &= \int_{a\le y\le R} |v|^2 \nabla\cdot \tilde f \\ &= \left[\frac{|v|^2}{1-\log(y)}\right]_\varepsilon^a + 2\int_{y\le a} v\partial_y v \frac{1}{y^3(1-\log y)} \\ &\lesssim |v(a)|^2 + \left(\int_{y\le R} \frac{|v|^2}{y^4(1+|\log y|)^2}\right)^{1/2}\left(\int_{y\le R} \frac{|\partial_y v|^2}{y^2}\right)^{1/2}.\end{aligned} \tag{C-7}$$

(C-5)–(C-7) now yield (C-3). The last inequality (C-4) is a straightforward variant of [Raphaël and Rodnianski 2012, Lemma B.1, (B.4)] and is left to the reader. □

Lemma C.3 (coercivity estimates with H). *Let ψ be the first eigenvector of H. Then there exist $c>0$ and $M_0 \geq 1$ such that for $M \geq M_0$, there exists $\delta(M)>0$ such that given $u \in H^1_{\mathrm{rad}}(\mathbb{R}^N)$, we have*

$$(Hu,u) \geq c\int (\partial_y u)^2 - \frac{1}{c}\big[(u,\psi)^2 + (u,\chi_M\Phi)^2\big], \tag{C-8}$$

$$\int (Hu)^2 \geq \delta(M)\left[\int \frac{(\partial_y u)^2}{y^2} + \int \frac{u^2}{y^4(1+|\log y|)^2}\right] - \frac{1}{\delta(M)}(u,\chi_M\Phi)^2. \tag{C-9}$$

Proof. (C-8) is a standard consequence of the coercivity of the linearized energy which admits exactly ψ as bound state and ΛQ as resonance at the origin, the good enough localization of Φ from (2-1) and the nondegeneracy from (2-2). The detailed proof is left to the reader.

To prove (C-9), we first observe the key subcoercivity property

$$\begin{aligned}\int (Hu)^2 = \int (\Delta u + Vu)^2 &= \int (\Delta u)^2 - 2\int V(\partial_y u)^2 + \int (\Delta V + V^2)u^2 \\ &\geq c\left[\int (\Delta u)^2 + \int \frac{u^2}{1+y^6}\right] - \frac{1}{c}\left[\int \frac{(\partial_y u)^2}{1+y^4} + \int \frac{u^2}{1+y^8}\right],\end{aligned} \tag{C-10}$$

where we used the asymptotic value

$$V(y) = \frac{N(N+2)(N-2)}{y^4}\left[1 + O\left(\frac{1}{y^2}\right)\right] \quad \text{as } y \to +\infty.$$

(C-9) now follows by contradiction. Let $M>0$ fixed and consider a sequence u_n such that

$$\int \frac{(\partial_y u_n)^2}{y^2} + \int \frac{u_n^2}{y^4(1+|\log y|)^2} = 1 \tag{C-11}$$

and

$$\int (Hu_n)^2 \leq \frac{1}{n}, \quad (u_n, \chi_M\Phi) = 0. \tag{C-12}$$

Then by semicontinuity of the norm, a subsequence of u_n weakly converges to a solution $u_\infty \in H^1_{\mathrm{loc}}$ of $Hu_\infty = 0$. The solution u_∞ is smooth away from the origin and hence the explicit integration of the ODE and the regularity assumption at the origin $u_\infty \in H^1_{\mathrm{loc}}$ imply that

$$u_\infty = \alpha\Lambda Q.$$

On one hand, the uniform bound (C-11) together with the local compactness of Sobolev embeddings ensure that, up to a subsequence,

$$\int \frac{(\partial_y u_n)^2}{1+y^4} + \int \frac{|u_n|^2}{1+y^8} \to \int \frac{(\partial_y u_\infty)^2}{1+y^4} + \int \frac{|u_\infty|^2}{1+y^8} \quad \text{and} \quad (u_n,\chi_M\Phi) \to (u_\infty,\chi_M\Phi),$$

thanks to the χ_M localization. We thus conclude that

$$\alpha(\Lambda Q, \chi_M\Phi) = (u_\infty, \chi_M\Phi) = 0$$

and thus $\alpha = 0$. On the other hand, the subcoercivity property (C-10), the Hardy control (C-2), (C-3) and

(C-11), (C-12) ensure that

$$\int \frac{(\partial_y u_n)^2}{1+y^4} + \int \frac{u_n^2}{1+y^8} \geq C > 0,$$

from which

$$\alpha^2 \left[\int \frac{(\partial_y \Lambda Q)^2}{1+y^4} + \int \frac{|\Lambda Q|^2}{1+y^8} \right] = \int \frac{(\partial_y u_\infty)^2}{1+y^4} + \int \frac{|u_\infty|^2}{1+y^8} \geq C > 0,$$

and thus $\alpha \neq 0$. A contradiction follows. This concludes the proof of (C-9) and of Lemma C.3. □

Straightforward computations show that the coercivity estimates with H can be adapted to any of the operators H_λ yielding, for any $\lambda > 0$ and $u \in H^1_{\mathrm{rad}}(\mathbb{R}^N)$,

$$(H_\lambda u, u) \geq c \int (\partial_y u)^2 - \frac{1}{c\lambda^4} \left[(u, (\psi)_\lambda)^2 + (u, (\chi_M \Phi)_\lambda)^2 \right] \tag{C-13}$$

for the same c and $\delta(M)$ as in Lemma C.3.

Acknowledgements

The authors thank Igor Rodnianski for stimulating discussions about this work. P. Raphaël is supported by the French ANR Jeune Chercheur SWAP.

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Received 8 Oct 2010. Accepted 13 May 2011.

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ANALYSIS AND PDE
Vol. 5, No. 4, 2012
dx.doi.org/10.2140/apde.2012.5.831

NONCONCENTRATION IN PARTIALLY RECTANGULAR BILLIARDS

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In specific types of partially rectangular billiards we estimate the mass of an eigenfunction of energy E in the region outside the rectangular set in the high-energy limit. We use the adiabatic ansatz to compare the Dirichlet energy form with a second quadratic form for which separation of variables applies. This allows us to use sharp one-dimensional control estimates and to derive the bound assuming that E is not resonating with the Dirichlet spectrum of the rectangular part.

1. Introduction

We study concentration and nonconcentration of eigenfunctions of the Laplace operator in stadium-like billiards. As predicted by the quantum/classical correspondence, such concentration is deeply linked with the classical underlying dynamics. In particular, the celebrated quantum ergodicity theorem roughly states that when the corresponding classical dynamics is ergodic then almost every sequence of eigenfunctions equidistributes in the high energy limit (see [Schnirelman 1974; Colin de Verdière 1985; Zelditch 1987] and [Gérard and Leichtnam 1993; Zelditch and Zworski 1996] in the billiard setting for a more precise statement). In strongly chaotic systems such as negatively curved manifolds, it is expected that every sequence of eigenfunctions equidistributes. This statement is the quantum unique ergodicity conjecture (Q.U.E.) and remains open in most cases despite several recent striking results (see for instance [Faure et al. 2003; Lindenstrauss 2006; Anantharaman 2008; Anantharaman and Nonnenmacher 2007]). On the other extreme, the Bunimovich stadium, although ergodic, is expected to violate Q.U.E. Indeed, it is expected that there exist bouncing ball modes, i.e., exceptional sequences of eigenfunctions concentrating on the cylinder of bouncing ball periodic orbits that sweep out the rectangular region (see [Bäcker et al. 1997] for instance). The existence of such bouncing ball modes is still open and only recently did Hassell prove that the generic Bunimovich stadium billiard indeed fails to be Q.U.E. (see [Hassell 2010]).

Our work is closely related to the search for bouncing ball modes but proceeds loosely speaking in the other direction. We actually aim at understanding how strong concentration of eigenfunctions in the rectangular part cannot be. We thus follow [Burq and Zworski 2005], where it is proved that even bouncing ball modes couldn't concentrate strictly inside the rectangular region. This was made precise by

Marzuola was supported in part by a Hausdorff Center Postdoctoral Fellowship at the University of Bonn, in part by an NSF Postdoctoral Fellowship at Columbia University. He also thanks the MATPYL program, which supported his coming to the University of Nantes, where this research began. Hillairet was partly supported by the ANR programs *NONaa* and *Methchaos*.
MSC2010: primary 35P20; secondary 35Q40, 58J51.
Keywords: eigenfunctions, billiards, nonconcentration.

Burq, Hassell and Wunsch in [Burq et al. 2007], where the following estimate was proved:

$$\|u\|_{L^2(W)} \geq E^{-1}\|u\|_{L^2(\Omega)},$$

in which $\|u\|_{L^2(W)}$ and $\|u\|_{L^2(\Omega)}$ denote the L^2 norm of the eigenfunction u in the wings and in the billiard, respectively.

Our main result for the Bunimovich stadium is the following:

Theorem 1. *Let Ω be a Bunimovich stadium with rectangular part $R := [-B_0, 0] \times [0, L_0]$. We set $W = \Omega \setminus R$ and denote by Σ the Dirichlet spectrum of R, i.e.,*

$$\Sigma = \left\{\frac{k^2\pi^2}{L_0^2} + \frac{l^2\pi^2}{B_0^2},\ k, l \in \mathbb{N}\right\}.$$

For any $\varepsilon \geq 0$ there exists E_0 and C such that if u is an eigenfunction of energy E such that $E > E_0$ and $\operatorname{dist}(E, \Sigma) > E^{-\varepsilon}$ *then the following estimates holds*:

$$\|u\|_{L^2(\Omega)} \leq CE^{\frac{5+8\varepsilon}{6}}\|u\|_{L^2(W)},$$

This bound improves on the Burq–Hassell–Wunsch bound provided that $\varepsilon < \frac{1}{8}$. It is natural that the smaller ε is the better the bound is. Indeed, the condition on the distance between E and Σ is comparable to a nonresonance condition and should imply heuristically that u must have some mass in the wing region. It is quite interesting to have a quantitative statement confirming this heuristics. We will actually give a more general statement concerning more general billiards (see Theorem 2). In particular we will consider billiards with smoother boundaries (see Section 2) disregarding the fact that these may not be ergodic. Here again we expect the bound to be better when the billiard becomes smoother and this statement is made quantitative in Theorem 2.

The method we propose relies on comparing the Dirichlet energy quadratic form with another quadratic form arising from the adiabatic ansatz presented in the numerical study of eigenfunctions by Bäcker, Schubert and Stifter [1997]. This adiabatic quadratic form has also appeared recently in [Hillairet and Judge 2009] in the study of the spectrum of the Laplacian on triangles. These two quadratic forms are close provided we do not enter too deeply into the wing region so that the nonconcentration estimate really takes place in a neighborhood of the rectangle that becomes smaller and smaller when the energy goes to infinity (see Sections 4.3.3 and 4.6.1). Since the new quadratic form may be addressed using separation of variables, we will show precise one-dimensional control estimates and then use them to prove our results. We have separated these one dimensional estimates in an appendix since they may be of independent interest. Finally, we remark that the method can be applied to quasimodes with some caution (see Remark 5.2) but there are no reasons to think that the bound we obtain is optimal.

2. The setting

Let L be a function defined on $[-B_0, B_1]$ with the following properties:

- For nonpositive x, $L(x) = L_0 > 0$.

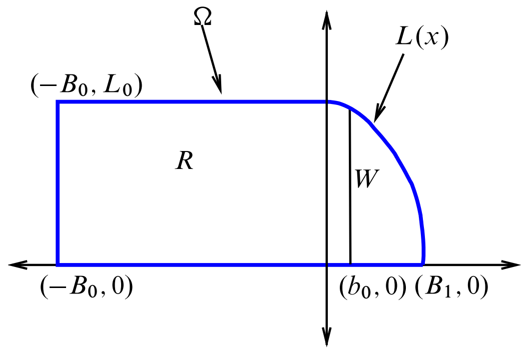

Figure 1. An example of a billiard Ω.

- On $(0, B_1)$, L is smooth, nonnegative and nonincreasing.
- When x goes to B_1, L' has a negative limit (either finite or $-\infty$).
- For small positive x, we have the asymptotic expansions

$$L(x) = L_0 - c_L x^\gamma + o(x^\gamma), \quad L'(x) = -c_L \gamma x^{\gamma-1} + o(x^{\gamma-1}) \tag{2-1}$$

for some positive c_L and $\gamma \geq \frac{3}{2}$.

The billiard Ω is then defined by

$$\Omega = \{(x, y) \mid -B_0 \leq x \leq B_1, \ 0 \leq y \leq L(x)\}.$$

See Figure 1 for an example of an applicable billiard. For any $b < B_1$, we will denote by $\Omega_b := \Omega \cap \{x \leq b\}$ and by $W_b := \Omega \cap \{0 \leq x \leq b\}$.

We study eigenfunctions of the positive Dirichlet Laplacian, Δ, on Ω. Namely, we study solutions u_E such that

$$\Delta u_E = -\left(\partial_x^2 + \partial_y^2\right) u_E = E u_E \quad \text{and} \quad u_E|_{\partial\Omega} = 0,$$

where $E > 0$.

We may formulate this equation using quadratic forms. We thus introduce q defined on $H^1(\Omega)$ by

$$q(u) = \int_\Omega |\nabla u|^2 dx\, dy.$$

The Euclidean Laplacian with Dirichlet boundary condition in Ω is the unique self-adjoint operator associated with q defined on $H_0^1(\Omega)$. We denote by q_b the restriction of q to $H^1(\Omega_b)$ and by Δ_b the Dirichlet Laplace operator on Ω_b. We will also denote by $\mathcal{D}_b$ the set of smooth functions with compact support in Ω_b.

3. Adiabatic approximation

Motivated by the well-known eigenvalue problem on a rectangular billiard and computational results in [Bäcker et al. 1997], we introduce a second family of quadratic forms a_b and compare it to q_b.

For any $b < B_1$ and any $u \in \mathcal{D}_b$, Fourier decomposition in y implies that

$$u(x, y) = \sum_k u_k(x) \sin\left(\frac{\pi k}{L(x)} y\right). \tag{3-1}$$

Since

$$\int_0^{L(x)} \left|\sin\left(k\pi \frac{y}{L(x)}\right)\right|^2 dy = \frac{L(x)}{2}$$

each Fourier coefficient u_k is given by

$$u_k(x) = \frac{2}{L(x)} \int_0^{L(x)} u(x, y) \sin\left(\frac{\pi k}{L(x)} y\right) dy.$$

For such u, we define

$$a_b(u) = \sum_{k\in\mathbb{N}} \int_{-B_0}^{b} \left(|u_k'(x)|^2 + \frac{k^2\pi^2}{L^2(x)} |u_k(x)|^2\right) \frac{L(x)}{2} dx,$$

$$N_b(u) = \sum_{k\in\mathbb{N}} \int_{-B_0}^{b} |u_k(x)|^2 \frac{L(x)}{2} dx.$$

Observe that for each fixed x, Plancherel's formula reads

$$\sum_{k\in\mathbb{N}} |u_k(x)|^2 \frac{L(x)}{2} = \int_0^{L(x)} |u(x, y)|^2 \, dy,$$

so that we get $N_b(u) = \|u\|^2_{L^2(\Omega_b)}$ by integration with respect to x.

Fixing some $0 < b_0 < B_1$, and using that L is uniformly bounded above and below on $[-B_0, b_0]$ we find a constant C such that for any $b \le b_0$ and $u \in L^2(\Omega_b)$:

$$C^{-1}\|u\|^2_{\Omega_b} \le \sum_{k=1}^{\infty} \|u_k\|^2_{L^2(-B_0,b)} \le C\|u\|^2_{\Omega_b}. \tag{3-2}$$

The quadratic form a_b appears as the direct sum of the following quadratic forms $a_{b,k}$ (that can be defined on the whole function space $H^1(-B_0, b)$):

$$a_{b,k}(u) := \int_{-B_0}^{b} \left(|u'|^2 + \frac{k^2\pi^2}{L^2(x)} |u|^2\right) \frac{L(x)}{2} dx. \tag{3-3}$$

Recall that, on an interval I, the standard H^1 norm is defined by

$$\|u\|_{H^1} := \left(\|u'\|^2_{L^2(I)} + \|u\|^2_{L^2(I)}\right)^{\frac{1}{2}}, \tag{3-4}$$

so that, for any k and $b < B_1$ and any $u \in \mathscr{C}_0^\infty(-B_0, b)$ we have

$$\min\left(L(b)\,,\ \frac{k^2\pi^2}{L_0}\right)\|u\|^2_{H^1} \le a_{b,k}(u) \le \max\left(L_0\,,\ \frac{k^2\pi^2}{L(b)}\right)\|u\|^2_{H^1}. \tag{3-5}$$

The norm $a^{\frac{1}{2}}_{b,k}$ thus defines on $H^1(-B_0, b)$ a norm that is equivalent to the standard H^1 norm.

3.1. ***Comparing a_b and q_b.*** To compare a_b and q_b, we introduce the following operators D and R defined on $\mathcal{D}_b$ by

$$\begin{aligned} Ru &= \frac{yL'(x)}{L(x)}\partial_y u, \\ Du &= \partial_x u + Ru. \end{aligned}$$

Using the Plancherel formula for each fixed x and then integrating, we obtain

$$a_b(u) = \int_{\Omega_b} |Du|^2 + |\partial_y u|^2 \, dx \, dy.$$

from which the following holds for any $u, v \in \mathcal{D}_b$:

$$\begin{aligned} a_b(u,v) - q_b(u,v) &= \langle Du, Dv\rangle - \langle \partial_x u, \partial_x v\rangle \\ &= \langle \partial_x u, Rv\rangle + \langle Ru, Dv\rangle, \qquad (3\text{-}6) \\ &= \langle \partial_x u, Rv\rangle + \langle Ru, \partial_x v\rangle + \langle Ru, Rv\rangle. \qquad (3\text{-}7) \end{aligned}$$

We thus obtain the following lemma.

Lemma 3.1. *Let δ be the function defined by*

$$\delta(b) = \sup_{(0,b]} |L'(x)| + \sup_{(0,b]} |L'(x)|^2.$$

Then for all $u, v \in \mathcal{D}_b$

$$|a_b(u,v) - q_b(u,v)| \le \delta(b) \cdot q_b^{\frac{1}{2}}(u) \cdot q_b^{\frac{1}{2}}(v).$$

Remark 3.1. The function δ is continuous on $(0, B_1)$ and $\delta(b) = O(b^{\gamma-1})$ when b goes to 0.

Proof. In (3-7), we use the Cauchy–Schwarz inequality, $\max(\|Du\|, \|\partial_y u\|) \le a_b^{\frac{1}{2}}(u)$, and the fact that $y/L(x)$ is uniformly bounded by 1 on Ω. ☐

The following corollary is then straightforward.

Corollary 3.2. *For any $0 < b < B_1$ and any $u \in H^1(\Omega)$, the linear functional Λ defined by $\Lambda(v) := a_b(u,v) - q_b(u,v)$ belongs to $H^{-1}(\Omega_b)$. Moreover*

$$\|\Lambda\|_{H^{-1}(\Omega_b)} \le \delta(b)\|u\|_{H^1(\Omega_b)}.$$

4. Nonconcentration

4.1. ***Preliminary reduction.*** Let u be an eigenfunction of q with eigenvalue E. And define the associated linear functional Λ using Corollary 3.2.

Integration by parts shows that for any $v \in H_0^1(\Omega_b)$ we have

$$q_b(u,v) = E \cdot \langle u, v\rangle_{L^2(\Omega)},$$

so that

$$a_b(u,v) - E \cdot N_b(u,v) \;=\; \Lambda(v). \tag{4-1}$$

We now deal with this equation using the adiabatic decomposition. We thus define Λ_k as the distribution over $\mathfrak{D}_b$ such that, for any $v \in \mathfrak{D}_b$,

$$\Lambda_k(v) := \Lambda\left(v(x)\sin\left(k\pi\frac{y}{L(x)}\right)\right). \tag{4-2}$$

Remark 4.1. From now on, u will always denote the eigenfunction that we are dealing with. We will denote by u_k the functions entering in the adiabatic decomposition of u, by Λ the linear functional associated with u and by Λ_k the one-dimensional linear functionals that are associated with Λ.

A straightforward computation yields, that for any $v \in \mathfrak{D}_b$ we have

$$a_{b,k}(u_k, v) - E \cdot \int_{-B_0}^{b} u_k(x)v(x)\frac{L(x)}{2}\,dx \;=\; \Lambda_k(v),$$

where $a_{b,k}$ is the quadratic form defined in (3-3).

An integration by parts then shows that, in the distributional sense in $(-B_0, b)$, we have

$$-\frac{1}{L}\frac{d}{dx}(Lu_k') + \left(\frac{k^2\pi^2}{L^2} - E\right)u_k \;=\; \tilde{\Lambda}_k, \tag{4-3}$$

where the linear functional $\tilde{\Lambda}_k$ is defined by

$$\tilde{\Lambda}_k(v) := \Lambda_k\left(\frac{2}{L}\cdot v\right). \tag{4-4}$$

Remark 4.2. Since L is not smooth, this definition of $\tilde{\Lambda}_k$ doesn't make sense as a distribution. However, in the next section, we will prove that Λ_k actually is in H^{-1} and, since multiplication by $2/L$ is a bounded operator from $H^1(-B_0, b)$ into itself, we thus get that $\tilde{\Lambda}_k$ is a perfectly legitimate element of H^{-1}. Moreover, for any b_0 there exists $C(b_0)$ such that for any $b \le b_0$, and $v \in \mathfrak{D}_b$, we have

$$\left\|\frac{2}{L}v\right\|_{H^1(-B_0,b)} \le C(b_0)\|v\|_{H^1(-B_0,b)}.$$

We denote by P_k the operator that is defined by

$$P_k(u) \;=\; -\frac{1}{L}\frac{d}{dx}(Lu') + \left(\frac{k^2\pi^2}{L^2} - E\right)u,$$

and we try to analyze the way a solution to equation (4-3) on $(-B_0, b)$ may be controlled by its behavior on $(0, b)$.

The strategy will depend upon whether k is large or not, but first we have to get a bound on Λ_k in some reasonable functional space of distributions.

4.2. ***Bounding Λ_k.*** In this section, we prove that each Λ_k is actually in $H^{-1}(-B_0, b)$ and provide a bound for its H^{-1} norm.

We first note that, using (3-4), for any $F \in H^{-1}(-B_0, b)$:

$$\|F\|_{H^{-1}(-B_0,b)} := \sup_{\phi\in\mathcal{D}_b} \frac{|F(\phi)|}{\|\phi\|_{H^1}} \leq \sup_{\phi\in\mathcal{D}_b} \frac{|F(\phi)|}{\|\phi'\|_{L^2}}. \tag{4-5}$$

Using (3-6) in the definition of Λ_k — see (4-2) — we obtain

$$\Lambda_k(v) = \left\langle \partial_x u, R\left(v(x)\sin\left(k\pi\frac{y}{L(x)}\right)\right)\right\rangle + \left\langle Ru, D\left(v(x)\sin\left(k\pi\frac{y}{L(x)}\right)\right)\right\rangle.$$

Denote by $A_k(v)$ the first term on the right and $B_k(v)$ the second term. By inspection, we have

$$A_k(v) := \frac{k\pi}{2}\int_0^b v(x)\frac{L'(x)}{L(x)}F_k(x)\,dx \quad\text{and}\quad B_k(v) := \frac{1}{2}\int_0^b v'(x)L'(x)G_k(x)\,dx,$$

where we have set

$$F_k(x) := \frac{2}{L(x)}\int_0^{L(x)} 1_W \cdot y\partial_x u(x,y)\cdot\cos\left(k\pi\frac{y}{L(x)}\right)dy, \tag{4-6}$$

$$G_k(x) := \frac{2}{L(x)}\int_0^{L(x)} 1_W \cdot y\partial_y u(x,y)\cdot\sin\left(k\pi\frac{y}{L(x)}\right)dy \tag{4-7}$$

Since $u \in H^1(\Omega)$, F_k and G_k are $L^2(0,b)$ and we can estimate the H^{-1} norm of Λ_k using them.

Lemma 4.1. *For any $b_0 < B_1$, and given Λ_k and F_k, G_k defined as above, there exists $C = C(\Omega_{b_0})$ such that*

$$\|\Lambda_k\|_{H^{-1}} \leq C(kb^\gamma\|F_k\|_{L^2(0,b)} + b^{\gamma-1}\|G_k\|_{L^2(0,b)}). \tag{4-8}$$

Proof. We estimate $A_k(v)$, using first an integration by parts

$$A_k(v) := -\frac{k\pi}{2}\int_{-B_0}^b v'(x)\left(\int_0^x \frac{L'(\xi)}{L(\xi)}F_k(\xi)\,d\xi\right)dx.$$

Using the Cauchy–Schwarz inequality and the fact that $L'(\xi) = O(\xi^{\gamma-1})$ we have

$$\left|\int_0^x \frac{L'(\xi)}{L(\xi)}F_k(\xi)\,d\xi\right| \leq Cx_+^{\gamma-\frac{1}{2}}\|F_k\|_{L^2(0,b)}.$$

Inserting into $A_k(v)$ and using the Cauchy–Schwarz inequality again we get

$$|A_k(v)| \leq C\cdot(kb^\gamma)\|F_k\|_{L^2(0,b)}\cdot\|v'\|_{L^2(-B_0,b)},$$

which gives the claimed bound using (4-5).

Next, the second term is estimated using directly the Cauchy–Schwarz estimate and the fact that $\sup_{[0,b]}|L'(x)| \leq Cb^{\gamma-1}$. We get

$$|B_k(v)| \leq C\cdot b^{\gamma-1}\|G_k\|_{L^2(0,b)}\cdot\|v'\|_{L^2(-B_0,b)}.$$

That gives the claimed bound using again (4-5). ☐

Define $F := 1_W \partial_x u$ and $G := 1_W \partial_y u$. By definition, $F_k(x)$ is the Fourier coefficient of the function $F(x, \cdot)$ with respect to the Fourier basis

$$\left(y \mapsto \cos\left(k\pi \frac{y}{L(x)}\right)\right)_{k \in \mathbb{N} \cup \{0\}}.$$

Using the Plancherel formula we get

$$\sum_{k \geq 1} F_k(x)^2 \frac{L(x)}{2} \leq \int_0^{L(x)} |F(x, y)|^2 dy.$$

For the same reason, but using this time the sin basis, we have

$$\sum_{k \geq 1} G_k(x)^2 \frac{L(x)}{2} = \int_0^{L(x)} |G(x, y)|^2 dy.$$

Integrating with respect to x and bounding y from above and $L(x)$ from below uniformly we get:

Lemma 4.2. *For any b_0 there exists C depending only on the billiard and b_0 such that, for any $b < b_0$,*

$$\sum_{k \geq 1} \|F_k\|^2_{L^2(0,b)} \leq C \|\partial_x u\|^2_{L^2(W_b)}, \tag{4-9}$$

$$\sum_{k \geq 1} \|G_k\|^2_{L^2(0,b)} \leq C \|\partial_y u\|^2_{L^2(W_b)}. \tag{4-10}$$

We now switch to the control estimate. We begin by dealing with the modes for which $\dfrac{k^2\pi^2}{L_0^2} - E \geq E$.

4.3. *Large modes.*

4.3.1. *A control estimate.* Equation (4-3) may be rewritten as

$$-u_k'' + \left(\frac{k^2\pi^2}{L^2(x)} - E\right) u_k = h_k, \tag{4-11}$$

where h_k is the element of H^{-1} defined by

$$h_k := \tilde{\Lambda}_k + \frac{L'}{L} u_k' \tag{4-12}$$

The H^{-1} norm of h_k is now estimated as follows:

Lemma 4.3. *There exists a constant $C := C(b_0)$ such that for any $b \leq b_0$ and any k with $\dfrac{k^2\pi^2}{L_0^2} - E \geq E$ the following estimate holds*:

$$\|h_k\|_{H^{-1}(-B_0,b)} \leq C(b_0)\big(k b^\gamma \|F_k\|_{L^2(0,b)} + b^{\gamma-1} \|G_k\|_{L^2(0,b)} + b^{\gamma-1} \|u_k\|_{L^2(0,b)}\big). \tag{4-13}$$

Proof. Using Remark 4.2, the norm of $\tilde{\Lambda}_k$ is uniformly controlled by the norm of Λ_k and the latter is estimated using Lemma 4.1. To estimate the H^{-1} norm of $(L'/L)u_k'$, we first set $v = (L'/L)u_k'$ and remark that

$$v = \left(\frac{L'}{L} u_k\right)' - \left(\frac{L''}{L} - \frac{(L')^2}{L^2}\right) u_k.$$

We choose a test function ϕ and estimate

$$I_1 = \int_{-B_0}^{b} \left(\frac{L'}{L} u_k\right)' \phi \, dx.$$

We perform an integration by parts, use that $L'(x)/L(x) \le C b^{\gamma-1} 1_{x>0}$, then apply the Cauchy–Schwarz inequality to get

$$\left| \int_{-B_0}^{b} \left(\frac{L'}{L} u_k\right)' \phi \, dx \right| = \left| \int_{-B_0}^{b} \frac{L'(x)}{L(x)} u_k(x) \phi'(x) \, dx \right| \le C b^{\gamma-1} \|u_k\|_{L^2(0,b)} \|\phi'\|_{L^2(-B_0,b)}.$$

We then estimate

$$I_2 = \left| \int_{-B_0}^{b} \left(\frac{L''(x)}{L(x)} - \frac{(L'(x))^2}{L^2(x)} \right) u(x) \phi(x) \, dx \right|.$$

We perform an integration by parts, use that

$$\left| \frac{L''(x)}{L(x)} - \frac{(L'(x))^2}{L^2(x)} \right| \le C x_+^{\gamma-2},$$

then twice apply the Cauchy–Schwarz inequality to get

$$I_2 \le C \int_{-B_0}^{b} \left(\int_0^x \xi_+^{\gamma-2} |u(\xi)| \, d\xi \right) |\phi'(x)| \, dx \le C b^{\gamma-1} \|u\|_{L^2(0,b)} \|\phi'\|_{L^2(0,b)}.$$

The claim follows using (4-5). □

The variational formulation of equation (4-11) is given by

$$\int_{-B_0}^{b} u_k' v' \, dx + \int_{-B_0}^{b} \left(\frac{k^2 \pi^2}{L^2(x)} - E \right) u_k v \, dx = h_k(v). \tag{4-14}$$

Since $k^2\pi^2/L_0^2 - E \ge E$, the left-hand side is a continuous quadratic form on $H_0^1(-B_0, b)$, so that, by Lax–Milgram theory, there is a unique v_k in $H_0^1(-B_0, b)$ satisfying (4-11) in the distributional sense.

The following lemma allows us to estimate the L^2 norm of this v_k.

Lemma 4.4. *There exists a constant C depending only on b_0 but not on $b < b_0$, k, or E such that, if $E \ge 1$ and $k^2\pi^2/L_0^2 - E \ge E$, the variational solution v_k in $H_0^1(-B_0, b)$ to equation* (4-11) *satisfies*

$$\|v_k\|_{L^2(-B_0,b)} \le C(b_0)\big(b^\gamma \|F_k\|_{L^2(0,b)} + E^{-\frac{1}{2}} b^{\gamma-1} \|G_k\|_{L^2(0,b)} + E^{-\frac{1}{2}} b^{\gamma-1} \|u_k\|_{L^2(0,b)}\big). \tag{4-15}$$

Proof. Since v_k is a variational solution, putting $v = v_k$ in (4-14) we get

$$\int_{-B_0}^{b} |v_k'(x)|^2 dx + \int_{-B_0}^{b} \left(\frac{k^2\pi^2}{L^2(x)} - E \right) |v_k(x)|^2 dx = h_k(v_k). \tag{4-16}$$

In the regime we are considering the second integral on the left is positive, so that we obtain

$$\int_{-B_0}^{b} |v_k'(x)|^2 dx \le |h_k(v_k)| \le \|h_k\|_{H^{-1}} \|v_k\|_{H^1}.$$

Since v_k is in $H_0^1(-B_0, b)$, Poincaré's inequality gives $c(b)$, a positive continuous function of b defined for $b > -B_0$ and satisfying

$$\int_{-B_0}^{b} |v_k'(x)|^2 dx \;\ge\; c(b)\|v_k\|_{H^1}^2 .$$

This gives a constant C depending only on b_0 such that, for any $0 < b < b_0$, we have

$$\|v_k\|_{H^1} \le C\|h_k\|_{H^{-1}} .$$

We now use (4-16) again to obtain

$$\left(\frac{k^2\pi^2}{L_0^2} - E\right)\int_{-B_0}^{b} |v_k(x)|^2 dx \;\le\; \|h_k\|_{H^{-1}}\|v_k\|_{H^1} \;\le\; C\|h_k\|_{H^{-1}}^2$$

with the preceding bound. Using the estimate (4-13) we obtain

$$\left(\frac{k^2\pi^2}{L_0^2} - E\right)^{\frac{1}{2}}\|v_k\|_{L^2(-B_0,b)} \;\le\; C\big(kb^{\gamma}\|F_k\|_{L^2(0,b)} + b^{\gamma-1}\|G_k\|_{L^2(0,b)} + b^{\gamma-1}\|u_k\|_{L^2(0,b)}\big).$$

We divide both sides by $(k^2\pi^2/L_0^2 - E)^{\frac{1}{2}}$. The coefficient in front of $b^{\gamma}\|F_k\|_{L^2(0,b)}$ is bounded by a constant that is uniform in k, using the fact that

$$\sup_{k^2\pi^2/L_0^2 - E \ge E} \frac{k^2}{k^2\pi^2/L_0^2 - E} = \sup_{Z \ge E} \frac{L_0^2}{\pi^2}\left(1 + \frac{E}{Z}\right) = \frac{L_0^2}{\pi^2}\left(1 + \frac{E}{E}\right).$$

For the two other terms, we use simply that $k^2\pi^2/L_0^2 - E \ge E$. This gives the lemma. □

We can now let $w_k = u_k - v_k$. By construction, w_k is a solution to the homogeneous equation

$$-w'' + \left(\frac{k^2\pi^2}{L^2(x)} - E\right)w = 0. \tag{4-17}$$

Moreover, since both u_k and v_k satisfy Dirichlet boundary condition at $-B_0$ we have that $w_k(-B_0) = 0$.

Since the "potential" part in equation (4-17) is bounded below by E, concentration properties of solutions may be obtained using convexity estimates.

Lemma 4.5. *For any $b \le b_0$, any solution w to* (4-17) *such that $w(-B_0) = 0$ satisfies*

$$b\int_{-B_0}^{b} |w|^2(x)\,dx \le (B_0 + b_0)\int_0^b |w|^2(x)\,dx.$$

Proof. Multiplying the equation by w we find

$$-w''w + \left(\frac{k^2\pi^2}{L^2(x)} - E\right)w^2 = 0.$$

It follows that $(w^2)'' \ge \beta^2 w^2$, for some positive β (here $\beta^2 = 2E$).

Since $w(-B_0) = 0$, using the maximum principle on $[-B_0, \xi]$, we obtain for all $-B_0 \le x \le \xi \le b_0$

$$w^2(x) \le w^2(\xi) \frac{\sinh(\beta(x + B_0))}{\sinh(\beta(\xi + B_0))}.$$

For any $t \in [0, 1]$, define $x(t) = -B_0 + t(B_0 + b)$ and $\xi(t) = tb$. Since for any t we have $-B_0 \le x(t) \le \xi(t) \le b_0$, we may integrate the preceding relation:

$$\int_0^1 w^2(x(t))\,dt \le \int_0^1 w^2(\xi(t)) \frac{\sinh(\beta(x(t) + B_0))}{\sinh(\beta(\xi(t) + B_0))}\,dt.$$

Since sinh is increasing the quotient of sinh is bounded above by 1 and we obtain

$$b \int_{-B_0}^{b} w^2(x)\,dx \le (B_0 + b) \int_0^b w^2(x)\,dx. \qquad \square$$

Putting these two lemmas together we obtain:

Proposition 4.6. *There exists a constant C depending only on b_0 such that for any $b \le b_0$, for any k and E such that $k^2\pi^2/L_0^2 - E \ge E$ and $E \ge 1$,*

$$\|u_k\|_{L^2(-B_0,b)} \le C\big(b^{\gamma-\frac{1}{2}}\|F_k\|_{L^2(0,b)} + E^{-\frac{1}{2}} b^{\gamma-\frac{3}{2}}\|G_k\|_{L^2(0,b)} + b^{-\frac{1}{2}}\|u_k\|_{L^2(0,b)}\big) \tag{4-18}$$

for $C = C(b_0)$.

Proof. According to Lemma 4.5 we have

$$\|w_k\|_{L^2(-B_0,b)} \le Cb^{-\frac{1}{2}}\|w_k\|_{L^2(0,b)},$$

where $w_k = u_k - v_k$ and v_k is the variational solution constructed above. Using the reverse triangle inequality, we obtain

$$\|u_k\|_{L^2(-B_0,b)} \le Cb^{-\frac{1}{2}}\|u_k\|_{L^2(0,b)} + (C + b_0^{\frac{1}{2}})b^{-\frac{1}{2}}\|v_k\|_{L^2(-B_0,b)}.$$

The claim will follow using estimate (4-15) of Lemma 4.4. Observe that the prefactor of $\|u_k\|_{L^2(0,b)}$ is at first (up to a constant prefactor)

$$b^{-\frac{1}{2}} + b^{-\frac{1}{2}}E^{-\frac{1}{2}}b^{\gamma-1}.$$

Since $E^{-\frac{1}{2}}b^{\gamma-1}$ is uniformly bounded we obtain the given estimate. $\square$

4.3.2. *Summing over k.* We will now sum the preceding estimates over k. We thus introduce

$$u_+(x, y) = \sum_{k^2\pi^2/L_0 - E \ge E} u_k(x) \sin\frac{k\pi y}{L(x)}$$

and prove the following proposition.

Proposition 4.7. *There exist b_0 and E_0 and a constant C depending only on E_0 and b_0 such that, if u is an eigenfunction with energy $E > E_0$ and $b < b_0$, then*

$$\|u_+\|^2_{L^2(R)} \le C\big(b^{2\gamma-1}\|\partial_x u\|^2_{W_b} + E^{-1}b^{2\gamma-3}\|\partial_y u\|^2_{L^2(W_b)} + b^{-1}\|u\|^2_{L^2(W_b)}\big).$$

Proof. We square estimate (4-18), sum over k, and use (3-2) and Lemma 4.2. □

Observe that the controlling term in the preceding estimate is supported in the wing region. However, compared to the usual bounds (as in [Burq et al. 2007]) there is a loss of derivatives since we need $\partial_x u$ and $\partial_y u$ in the wings.

Corollary 4.8. *Let b_0 and E_0 be fixed. There exists C depending on the billiard b_0 and E_0 but not on the eigenfunction nor on $b < b_0$ such that*

$$\|u_+\|^2_{L^2(R)} \le C\big((b^{2\gamma-1}E + b^{2\gamma-3})\|u\|^2_{L^2(\Omega)} + b^{-1}\|u\|^2_{L^2(W)}\big).$$

Proof. We bound $\|\partial_x u\|^2_{L^2(W_b)}$ and $\|\partial_y u\|^2_{L^2(W_b)}$ by $E\|u\|^2_{L^2(\Omega)}$ and use the fact that the norm over W_b is less than the norm over W. □

It remains to choose b in a clever way to obtain the desired bound.

4.3.3. *Optimizing b.* We will choose b to be of the form $M^{-1}E^{-\alpha}$ for some constants M and α to be chosen. As long as α is positive, there is some large E_0 such that for any $E \ge E_0$ then $b = ME^{-\alpha} < b_0$ so that we can use the preceding proposition.

We obtain

$$\|u_+\|^2_{L^2(R)} \le C\big((M^{1-2\gamma}E^{1-\alpha(2\gamma-1)} + E^{-\alpha(2\gamma-3)})\|u\|^2_{L^2(\Omega)} + ME^{\alpha}\|u\|^2_{L^2(W)}\big). \tag{4-19}$$

It remains to make good choices to obtain the following proposition.

Proposition 4.9. *There exists E_0 and C depending only on the billiard such that for any u eigenfunction with energy $E > E_0$ the following holds*:

$$\|u_+\|^2_{L^2(R)} \le \tfrac{1}{4}\|u\|^2_{L^2(\Omega)} + CE^{\frac{1}{2\gamma-1}}\|u\|^2_{L^2(W)} \tag{4-20}$$

Proof. We choose $\alpha := 1/(2\gamma-1)$ and M such that $CM^{1-2\gamma} = \frac{1}{8}$. For E large enough, $E^{-\alpha(2\gamma-3)}$ goes to zero. It is thus bounded by $1/(8C)$ for E large enough. Substituting in (4-19) we get (4-20). □

4.4. ***Small modes.*** We now consider modes for which $k^2\pi^2/L_0^2 - E \le E$, and this time we rewrite the equation $P_k(u_k) = \Lambda_k$ in the form

$$-u_k'' - z_k u_k = h_k, \tag{4-21}$$

in which we have set $z_k := E - k^2\pi^2/L_0^2$ and

$$h_k := \tilde{\Lambda}_k + \frac{L'}{L}u_k' + \frac{k^2}{\pi^2}\left(\frac{1}{L_0^2} - \frac{1}{L^2}\right)u_k.$$

4.4.1. *The control estimate.* Since $z_k \ge -E$ we can use the results of the Appendix to control the term $\|u_k\|_{L^2(-B_0,b)}$. To do so, we need to estimate the norm of h_k in $H^{-1}(-B_0, b)$.

Lemma 4.10. *There exists some constant C depending only on b_0 such that, for any $b \le b_0$ and any k such that $k^2\pi^2/L_0^2 - E \le E$, the following holds*:

$$\|h_k\|_{H^{-1}(-B_0,b)} \le C\big(kb^{\gamma}\|F_k\|_{L^2(0,b)} + b^{\gamma-1}\|G_k\|_{L^2(0,b)} + (b^{\gamma-1} + k^2b^{\gamma+1})\|u_k\|_{L^2(0,b)}\big). \tag{4-22}$$

Proof. From the definition,

$$h_k = \tilde{\Lambda}_k + \frac{L'}{L}u_k' + \frac{k^2}{\pi^2}\left(\frac{1}{L_0^2} - \frac{1}{L^2}\right)u_k$$

and estimate each term separately. The first term is estimated using Lemma 4.1 and Remark 4.2. The second is estimated as in the proof of Lemma 4.3. The same method applies to estimate the third term. We introduce

$$I_3 = \left|\int_{-B_0}^{b}\left(\frac{1}{L_0^2} - \frac{1}{L^2(x)}\right)u_k(x)\phi(x)\,dx\right|,$$

and observe that the quantity in parentheses is $O(x_+^\gamma)$. Integrating by parts and using the Cauchy–Schwarz inequality twice gives

$$I_3 \le Cb^{\gamma+1}\|\phi'\|_{L^2(0,b)}\|u_k\|_{L^2(0,b)}.$$

Using the definition of the H^{-1} norm (see (4-5)) and putting these estimates together yields the lemma. ☐

For any $E \in \mathbb{R}$, define

$$\nu(E) := \min\left\{\left|E - \frac{k^2\pi^2}{L_0^2} - \frac{l^2\pi^2}{B_0^2}\right|, (k,l) \in \mathbb{N}\times\mathbb{N}\right\}.$$

Remark 4.3. Taking $l = 1$ in the definition shows that, for E large, we have

$$\nu(E) < c\sqrt{E} \tag{4-23}$$

for some constant c.

Lemma 4.11. *For any $\beta > 0$, there exists some c such that the following holds. For any k such that $z_k = E - k^2\pi^2/L_0^2 \ge \beta^2$,*

$$|\sin(B_0\sqrt{z_k})| \ge c\cdot\frac{\nu(E)}{\sqrt{z_k}}.$$

Proof. First we use that there exists some c such that

$$|\sin x| \ge c\,\mathrm{dist}(x, \pi\mathbb{Z}) \quad \text{for all } x \in \mathbb{R}.$$

We denote by l_k the integer such that

$$\mathrm{dist}\left(\sqrt{z_k}, \frac{\pi}{B_0}\mathbb{Z}\right) = \left|\sqrt{z_k} - \frac{l_k\pi}{B_0}\right|,$$

so that we have

$$\left|\sin(B_0\sqrt{z_k})\right| \ge c\left|\sqrt{z_k} - \frac{l_k\pi}{B_0}\right| \ge c\frac{z_k - l_k^2\pi^2/B_0^2}{\sqrt{z_k} + l_k\pi/B_0} \ge c\frac{E - \dfrac{k^2\pi^2}{L_0^2} - \dfrac{l_k^2\pi^2}{B_0^2}}{\sqrt{z_k}},$$

where, for the last bound, we have used the Lemma 4.12 below.

The claim follows by the definition of $\nu(E)$. ☐

Lemma 4.12. *Fix $\alpha > 0$ and denote by l the (step-like) function on $[0,\infty)$ defined by*

$$|\lambda - l(\lambda)\alpha| = \operatorname{dist}(\lambda, \alpha\mathbb{Z})$$

Then there exists some C such that

$$\lambda + l(\lambda)\alpha \le C\lambda. \quad \textit{for all } \lambda \in [0,\infty).$$

Proof. Define f by

$$f(\lambda) = \frac{\lambda + l(\lambda)\alpha}{\lambda}.$$

Since l vanishes on $[0, \alpha/2]$, we have $f(\lambda) = 1$ on this interval. Next, f tends to the limit 2 when λ goes to infinity. Finally, on $[\alpha/2, M]$ we have

$$f(\lambda) = 1 + \frac{l(\lambda)}{\lambda}\alpha \le 1 + \frac{2M+1}{\alpha}.$$ □

Putting these estimates together, we get:

Proposition 4.13. *There exists b_0 and E_0 and a constant $C := C(b_0, E_0)$ such that the following holds. For any $E > E_0$, for any k such that $k^2\pi^2/L_0^2 - E \le E$ and for any $b < b_0$, we have the estimate*

$$\|u_k\|_{L^2(-B_0,b)} \le C\frac{E^{\frac{1}{2}}}{\nu(E)}\big(E^{\frac{1}{2}}b^{\gamma+\frac{1}{2}}\|F_k\|_{L^2(0,b)} + b^{\gamma-\frac{1}{2}}\|G_k\|_{L^2(0,b)} + (1 + Eb^{\gamma+2})b^{-\frac{1}{2}}\|u_k\|_{L^2(0,b)}\big). \tag{4-24}$$

Proof. For any k we let $z_k = E - k^2\pi^2/L_0^2$ and use the estimates of the appendix combined with the bound on h_k given by Lemma 4.10. For k such that z_k corresponds to estimates (A-10) and (A-12) of Theorem 3 we obtain

$$\begin{aligned}\|u_k\|_{L^2(-B_0,b)} &\le C\big(b^{\frac{1}{2}}\|h_k\|_{H^{-1}(-B_0,b)} + b^{-\frac{1}{2}}\|u_k\|_{L^2(0,b)}\big)\\ &\le C\big(kb^{\gamma+\frac{1}{2}}\|F_k\|_{L^2(0,b)} + b^{\gamma-\frac{1}{2}}\|G_k\|_{L^2(0,b)} + (b^\gamma + k^2b^{\gamma+2} + 1)b^{-\frac{1}{2}}\|u_k\|_{L^2(0,b)}\big).\end{aligned}$$

We now use that $k = O(E^{\frac{1}{2}})$ in the regime we are considering. We also remark that $b^\gamma + k^2b^{\gamma+2} + 1 = O(1 + Eb^{\gamma+2})$.

In the opposite case (for k such that z_k corresponds to estimate (A-11)), we have to add a global $|\sin(B_0\sqrt{z_k})|^{-1}$ prefactor. Using Lemma 4.11, we have

$$|\sin(B_0\sqrt{z_k})|^{-1} \le C\frac{\sqrt{z_k}}{\nu(E)} \le C\frac{E^{\frac{1}{2}}}{\nu(E)}.$$

We thus obtain that, for any k,

$$\begin{aligned}&\|u_k\|_{L^2(-B_0,b)}\\ &\quad\le C\cdot\max\left(1, \frac{E^{\frac{1}{2}}}{\nu(E)}\right)\big(E^{\frac{1}{2}}b^{\gamma+\frac{1}{2}}\|F_k\|_{L^2(0,b)} + b^{\gamma-\frac{1}{2}}\|G_k\|_{L^2(0,b)} + (1 + Eb^{\gamma+2})b^{-\frac{1}{2}}\|u\|_{L^2(0,b)}\big).\end{aligned}$$

Using (4-23), for large E we have $E^{1/2}/\nu(E)$ bounded from below, so that the claim follows. □

4.5. ***Summing over k.*** We use the estimates of the preceding sections to obtain a control on $\|u_-\|^2_{L^2(R)}$ in which we have set

$$u_-(x,y) = \sum_{k^2\pi^2/L_0 - E \le E} u_k(x)\sin\left(\frac{k\pi y}{L(x)}\right).$$

Proposition 4.14. *There exists b_0 and E_0 and a constant C depending only on E_0 and b_0 such that if u is an eigenfunction with energy $E > E_0$ and $b < b_0$, then*

$$\|u_-\|^2_{L^2(R)} \le C\frac{E}{\nu(E)^2}\big(Eb^{2\gamma+1}\|\partial_x u\|^2_{L^2(W)} + b^{2\gamma-1}\|\partial_y u\|^2_{L^2(W)} + (1+Eb^{\gamma+2})^2 b^{-1}\|u\|^2_{L^2(W)}\big).$$

Proof. We square (4-24) and sum with respect to k. The Lemma 4.2 controls $\sum\|F_k\|^2$ and $\sum\|G_k\|^2$. Plancherel formula takes care of $\sum\|u_k\|^2$. We also use as before that the norm over W_b is smaller than the norm over W. □

As for the large mode case, we get a corollary using the fact that $\|\partial_x u\|^2$ and $\|\partial_y u\|^2$ are bounded above by $E\|u\|^2_{L^2(\Omega)}$.

Corollary 4.15. *There exists b_0 and E_0 and a constant C depending only on E_0 and b_0 such that if u is an eigenfunction with energy $E > E_0$ and $b < b_0$, then*

$$\|u_-\|^2_{L^2(R)} \le C\left[\left(\frac{E^3}{\nu(E)^2}b^{2\gamma+1} + \frac{E^2}{\nu(E)^2}b^{2\gamma-1}\right)\|u\|^2_{L^2(\Omega)} + (1+Eb^{\gamma+2})^2\frac{Eb^{-1}}{\nu(E)^2}\|u\|^2_{L^2(W)}\right].$$

4.6. ***A nonresonance condition.*** We now want to make the previous estimates explicit with respect to E and b so that we can use a similar optimization procedure as for the large modes case. We thus impose some condition on $\nu(E)$. Namely, for any $\varepsilon \ge 0$, we introduce the set

$$\mathcal{X}_\varepsilon := \{E\in\mathbb{R} \mid \nu(E)\ge c_0E^{-\varepsilon}\} = \left\{E\in\mathbb{R} \;\middle|\; \left|E - \frac{k^2\pi^2}{L_0^2} - \frac{l^2\pi^2}{B_0^2}\right| \ge c_0E^{-\varepsilon} \text{ for } k,l\in\mathbb{N}\right\}.$$

In other words, the set $\mathcal{X}_\varepsilon$ consists in energies that are far from the Dirichlet spectrum of the rectangle $[-B_0,0]\times[0,L_0]$. It is natural to say that such energies are *not resonating* with the rectangle. The coefficient c_0 which is irrelevant when $\varepsilon > 0$ has been chosen in such a way that Weyl's law for the rectangle implies that $\mathcal{X}_0$ is not empty. Note however that, although expected, it is not clear that there actually are eigenvalues in $\mathcal{X}_0$, nor for that matter in $\mathcal{X}_\varepsilon$.

Once ε is fixed, the estimate of the Corollary 4.15 becomes

$$\|u_-\|^2_{L^2(R)} \le C\left[(b^{2\gamma+1}E^{3+2\varepsilon} + b^{2\gamma-1}E^{2+2\varepsilon})\|u\|^2_{L^2(\Omega)} + (1+Eb^{\gamma+2})^2b^{-1}E^{1+2\varepsilon}\|u\|^2_{L^2(W)}\right]. \tag{4-25}$$

4.6.1. *Optimizing b.* As before we let $b = ME^{-\alpha}$ for some positive α and try to optimize the bound.

Proposition 4.16. *Define α by*

$$\alpha = \max\left(\frac{3+2\varepsilon}{2\gamma+1}, \frac{2+2\varepsilon}{2\gamma-1}\right).$$

There exists E_0 and C such that for any u eigenfunction with energy E in $\mathfrak{X}_\varepsilon$ such that $E > E_0$, the following holds:

$$\|u_-\|^2_{L^2(R)} \leq \tfrac{1}{4}\|u\|^2_{L^2(\Omega)} + C \cdot E^{1+2\varepsilon+\alpha} \cdot \|u\|^2_{L^2(W)}. \tag{4-26}$$

Proof. With the given choice of α it is possible to choose M so that the prefactor of $\|u\|^2_{L^2(\Omega)}$ is $\frac{1}{4}$ for E large enough. The claim follows remarking that the definition of α implies

$$\alpha \geq \frac{3}{2\gamma+1} > \frac{1}{\gamma+2},$$

so that the prefactor $(1 + Eb^{\gamma+2})^2$ is uniformly bounded above. □

5. Nonconcentration estimate

We now put all the estimates together to obtain the following theorem.

Theorem 2. *Fix ε, and define ρ by*

$$\rho := \max\left(\frac{2+\gamma+2(\gamma+1)\varepsilon}{2\gamma+1}, \frac{1+2\gamma+4\gamma\varepsilon}{4\gamma-2}\right).$$

There exists E_0 and C such that any eigenfunction u of Ω with energy E in $\mathfrak{X}_\varepsilon$ such that $E > E_0$ satisfies:

$$\|u\|_{L^2(\Omega)} \leq C \cdot E^\rho \|u\|_{L^2(W)}.$$

Proof. We first remark that whatever the exponent α is we always have $1 + 2\varepsilon + \alpha \geq 1 > \frac{1}{2\gamma-1}$ so that the exponent for the small modes is always larger than the exponent for the large modes. Thus, adding the estimates from propositions 4.9 and 4.16, we obtain

$$\|u\|^2_{L^2(R)} \leq \tfrac{1}{2}\|u\|^2_{L^2(\Omega)} + CE^{1+2\varepsilon+\alpha}\|u\|^2_{L^2(W)}.$$

Since $\|u\|^2_{L^2(R)} = \|u\|^2_{L^2(\Omega)} - \|u\|^2_{L^2(W)}$ we get

$$\tfrac{1}{2}\|u\|^2_{L^2(\Omega)} \leq (1 + CE^{1+2\varepsilon+\alpha})\|u\|^2_{L^2(W)}$$

When E is large the constant 1 can be absorbed in the term with a power of E. The claim follows by computing $1 + 2\varepsilon + \alpha$ for both possible choices of α and taking square roots. □

We state as a corollary the corresponding statement for the Bunimovich billiard (see Theorem 1).

Corollary 5.1. *In the Bunimovich stadium, for any $\varepsilon \geq 0$ there exists E_0 and C such that if u is an eigenfunction of energy E in $\mathfrak{X}_\varepsilon$ such that $E > E_0$ then the following estimate holds*:

$$\|u\|_{L^2(\Omega)} \leq CE^{\frac{5+8\varepsilon}{6}}\|u\|_{L^2(W)}.$$

Proof. We let $\gamma = 2$, so $\alpha = \max\left(\frac{4+6\varepsilon}{5}, \frac{5+8\varepsilon}{6}\right)$. Since $\frac{4+6\varepsilon}{5} \leq \frac{5+8\varepsilon}{6}$ for any nonnegative ε, the proof is complete. □

Remark 5.1. The bounds in [Burq et al. 2007] gives a similar control with 1 as the exponent of E. Our bound thus gives a better estimate as long as $\varepsilon < \frac{1}{8}$. As it has been recalled in the introduction, it is quite natural that the nonresonance condition allows to get better bounds.

Remark 5.2. We could deal with quasimodes by adding an error term to Λ that is controlled by some negative power of E. There will be mainly two differences in the analysis. First the second term Λ will not have support away from the rectangle anymore and second, in the optimization process, we will have to take care of the new error term (which will possibly change the range of applicable exponents).

Remark 5.3. By adding the estimates in propositions 4.7 and 4.14, we get a different control estimate, where the control still is in the wings but now with a loss in derivatives. We haven't tried to optimize this bound.

Appendix: One-dimensional control estimates

The aim of this appendix is to provide a control estimate for the equation

$$-u'' - z \cdot u = h$$

on $[-B_0, b]$ of the form

$$\|u\|_{L^2(-B_0,0)} \leq C_1 \|h\|_{H^{-1}(-B_0,b)} + C_2 \|u\|_{L^2(0,b)},$$

in which we want an explicit dependence of the constants C_1 and C_2 on z and b. It is now standard (see [Burq and Zworski 2005]) that if b is fixed then we can choose C_1 and C_2 to be independent of z but what we need is an estimate when b goes to 0.

We first need a few preparatory lemmas.

Lemma A.2. *For any $\varepsilon > 0$, there exists a constant $C := C(\varepsilon)$ such that for any b, for any $h \in H^{-1}(-B_0, b)$ and any z such that $z \leq (1-\varepsilon)\pi^2/b^2$, there exists a solution $v_p \in H_0^1(0, b)$ to*

$$-v_p'' - z v_p = h,$$

in $\mathcal{D}'(0, b)$ and

$$\|v_p\|_{L^2(0,b)} \leq C b \|h\|_{H^{-1}(-B_0,b)}. \tag{A-1}$$

Proof. First we note that h, when restricted to $(0, b)$ also belongs to $H^{-1}(0, b)$ and that $\|h\|_{H^{-1}(0,b)} \leq \|h\|_{H^{-1}(-B_0,b)}$. The proof follows from a standard resolvent estimate since, on $(0, b)$, the bottom of the spectrum of the self-adjoint operator $v \mapsto -v''$ with Dirichlet boundary condition is π^2/b^2. We include it for the convenience of the reader. We decompose v_p in Fourier series:

$$v_p(x) = \sum_{k\geq 1} a_k \sin\Big(\frac{k\pi}{b} x\Big).$$

We have

$$h(x) = \sum_{k\geq 1} \Big(\frac{k^2\pi^2}{b^2} - z\Big) a_k \sin\Big(\frac{k\pi}{b} x\Big);$$

hence

$$\|h\|^2_{H^{-1}(0,b)} = \sum_{k\geq 1} \frac{\left(k^2\pi^2/b^2 - z\right)^2}{k^2\pi^2/b^2} |a_k^2|$$

or

$$\|h\|^2_{H^{-1}(0,b)} \geq \pi^2 b^{-2} \left(\inf_{k\geq 1} \frac{\left(1-\dfrac{zb^2}{k^2\pi^2}\right)^2}{\pi^2 k^2} \right) \|v_p\|^2_{H^1(0,b)} \geq c\pi^2 b^{-2} \|v_p\|^2_{L^2(0,b)}.$$

The claim follows since the inf is bounded away from zero in the regime we are considering. □

Lemma A.3. *Given $z \leq (1-\varepsilon)\pi^2/b^2$, let $w \in H^1_0(-B_0, b)$ be a solution to*

$$-w'' - zw = 0$$

in $\mathcal{D}'\left((-B_0, b)\setminus\{0\}\right)$. Then there exists a constant A such that $w = AG$, in which the function G is defined by

$$G(x) = \begin{cases} \dfrac{\sin(\sqrt{z}(x+B_0))}{\sqrt{z}} \dfrac{\sin(\sqrt{z}b)}{\sqrt{z}} & \text{if } x < 0, \\[2ex] \dfrac{\sin(\sqrt{z}(b-x))}{\sqrt{z}} \dfrac{\sin(\sqrt{z}B_0)}{\sqrt{z}} & \text{if } x > 0. \end{cases} \tag{A-2}$$

Proof. Let w be such a function then necessarily there exist two constants $A_\pm$ such that

$$w(x) = \begin{cases} A_- \dfrac{\sin(\sqrt{z}(x+B_0))}{\sqrt{z}} & \text{if } x < 0, \\[2ex] A_+ \dfrac{\sin(\sqrt{z}(b-x))}{\sqrt{z}} & \text{if } x > 0. \end{cases}$$

By assumption $w \in H^1$ and hence is continuous at 0, so

$$A_- \frac{\sin(\sqrt{z}B_0)}{\sqrt{z}} = A_+ \frac{\sin(\sqrt{z}b)}{\sqrt{z}}.$$

In the regime we are considering $\sin(\sqrt{z}b)/\sqrt{z} \neq 0$, hence we can divide by this expression and express A_- in terms of A_+. The claim follows. □

We finish these preparatory lemmas by establishing the control estimate for multiples of G.

Lemma A.4. (1) *For β such that $0 < \beta \leq \pi/B_0$, there exists $B_1 = B_1(\beta)$ and $C := C(\beta)$ such that, for any $z \leq \beta^2$ and any $b < B_1$, the following estimate holds*:

$$\|G\|_{L^2(-B_0,0)} \leq C b^{-\frac{1}{2}} \|G\|_{L^2(0,b)}. \tag{A-3}$$

(2) *For any $\beta, \varepsilon > 0$ there exists $B_1 := B_1(\beta)$ and $C := C(\beta, \varepsilon)$ such that, for any $b \leq B_1$ and $\beta^2 \leq z \leq (1-\varepsilon)\pi^2/b^2$, the following estimate holds*:

$$\|G\|_{L^2(-B_0,0)} \leq C \frac{b^{-\frac{1}{2}}}{\sin(\sqrt{z}B_0)} \|G\|_{L^2(0,b)}. \tag{A-4}$$

Proof. (1) We first assume that $z < -Z_0^2$ for some positive Z_0. We set $z = -\omega^2$ and compute

$$\int_{-B_0}^{0} |G(x)|^2\,dx = \frac{\sinh^2(\omega b)}{\omega^2}\int_{-B_0}^{0}\sinh^2(\omega(B_0+x))\,dx,$$
$$\int_{0}^{b} |G(x)|^2\,dx = \frac{\sinh^2(\omega B_0)}{\omega^2}\int_{0}^{b}\sinh^2(\omega(b-x))\,dx.$$

By a straightforward change of variables we get

$$\int_{-B_0}^{0} |G(x)|^2\,dx = \frac{\sinh^2(\omega b)}{\omega^3}\int_{0}^{\omega B_0}\sinh^2(\xi)\,d\xi,$$
$$\int_{0}^{b} |G(x)|^2\,dx = \frac{\sinh^2(\omega B_0)}{\omega^3}\int_{0}^{b}\sinh^2(\xi)\,d\xi.$$

We set $F(X) := \dfrac{\int_0^X \sinh^2(\xi)\,d\xi}{\sinh^2(X)}$, so that we finally obtain

$$\int_{-B_0}^{0} |G(x)|^2\,dx \;=\; \frac{F(\omega B_0)}{F(\omega b)}\cdot\int_{0}^{b}|G(x)|^2\,dx.$$

It is straightforward that $F(X)$ is positive, tends to 1 at infinity and that $F(X)/X$ tends to $\frac{1}{3}$ at 0. As a consequence, there exists some $C(Z_0)$ such that, for any $z < -Z_0^2$,

$$\int_{-B_0}^{0} |G(x)|^2\,dx \le C\max(1,(\omega b)^{-1})\int_{0}^{b}|G(x)|^2\,dx,$$

For $b < B_1$ and $\omega > Z_0$, we have $\max(1,(\omega b)^{-1}) \le \max(1,\omega^{-1})b^{-1} \le Cb^{-1}$ which gives the claim for this range of parameters.

We now assume that we have $-Z_0^2 < z < \beta^2$. We have

$$\begin{aligned}\int_{-B_0}^{0} |G(x)|^2\,dx &= \left|\frac{\sin(\sqrt{z}b)}{\sqrt{z}}\right|^2\int_{-B_0}^{0}\left|\frac{\sin(\sqrt{z}(x+B_0))}{\sqrt{z}}\right|^2 dx\\ &= b^2\left|\frac{\sin(\sqrt{z}b)}{\sqrt{z}b}\right|^2\int_{-B_0}^{0}\left|\frac{\sin(\sqrt{z}(x+B_0))}{\sqrt{z}(x+B_0)}\right|^2(x+B_0)^2\,dx \le Cb^2,\end{aligned}$$

where the constant C comes from the fact that the function $\sin(w)/w$ is continuous and its argument belongs to a fixed compact set. On the other hand, by a simple change or variables we have

$$\int_{0}^{b} |G(x)|^2\,dx = \left|\frac{\sin(\sqrt{z}B_0)}{\sqrt{z}}\right|^2\int_{0}^{b}\left|\frac{\sin(\sqrt{z}x)}{\sqrt{z}}\right|^2 dx \ge cB_0^2\frac{b^3}{3},$$

in which c is given by

$$c \;=\; \left|\frac{\sin(\sqrt{z}B_0)}{\sqrt{z}B_0}\right|^2\inf_{0\le x\le B_1}\left|\frac{\sin(\sqrt{z}x)}{\sqrt{z}x}\right|^2.$$

Using that $\sin(w)/w$ is continuous and does not vanish on $(-\infty, \pi)$ and choosing B_1 accordingly we obtain the first bound.

(2) We first use homogeneity and prove the bound for $\tilde{G} := zG$. We have

$$\int_{-B_0}^{0} \left|\tilde{G}(x)\right|^2 dx = \left|\sin(\sqrt{z}b)\right|^2 \int_{-B_0}^{0} \left|\sin(\sqrt{z}(x+B_0))\right|^2 \, dx \leq B_0 \left|\sin(X)\right|^2,$$

in which we have set $X := \sqrt{z}b$. On the other hand we have

$$\int_{0}^{b} \left|\tilde{G}(x)\right|^2 dx = \left|\sin(\sqrt{z}B_0)\right|^2 \int_{0}^{b} \left|\sin(\sqrt{z}x)\right|^2 \, dx = b \cdot \left|\sin(\sqrt{z}B_0)\right|^2 \cdot \frac{1}{X} \int_0^X |\sin(\xi)|^2 d\xi,$$

with the same X. Under the assumptions, X belongs to a compact subinterval of $[0, \pi)$. Since on this interval the function

$$X \mapsto \frac{1}{X|\sin(X)|^2} \int_0^X |\sin(\xi)|^2 d\xi$$

is continuous, the claim follows. □

Proposition A.5. *There exist β and $B_1 := B_1(\beta)$, such that if $b \leq B_1$ and $v \in H_0^1(-B_0, b)$ satisfies*

$$-v'' - zv = h,$$

with h that vanishes on $(-B_0, 0)$, then the following estimates hold:

(1) *If $z \leq \beta^2$, then*

$$\|v\|_{L^2(-B_0,0)} \leq C_1\big(b^{\frac{1}{2}}\|h\|_{H^{-1}(-B_0,b)} + b^{-\frac{1}{2}}\|v\|_{L^2(0,b)}\big), \tag{A-5}$$

(2) *If $\beta^2 \leq z \leq \dfrac{1}{b^2}$,*

$$\|v\|_{L^2(-B_0,0)} \leq C_1\left(\frac{b^{\frac{1}{2}}}{|\sin(B_0\sqrt{z})|}\|h\|_{H^{-1}(-B_0,b)} + \frac{b^{-\frac{1}{2}}}{|\sin(B_0\sqrt{z})|}\|v\|_{L^2(0,b)}\right), \tag{A-6}$$

(3) *If $\dfrac{1}{b^2} \leq z$ then*

$$\|v\|_{L^2(-B_0,0)} \leq C_3\big(b^{\frac{1}{2}}\|h\|_{H^{-1}(-B_0,b)} + b^{-\frac{1}{2}}\|v\|_{L^2(0,b)}\big). \tag{A-7}$$

Proof. In the first two cases, we have $z \leq \frac{1}{b^2} < \frac{\pi^2}{b^2}$. We may thus consider v_p as given by Lemma A.2 and define $\tilde{v}_p$ by extending v_p by 0 for negative x. Observe that $w := v - \tilde{v}_p$ is in $H_0^1(-B_0, b)$ and satisfies

$$-w'' - zw = 0$$

in $\mathfrak{D}((-B_0, b) \setminus \{0\})$ so that $v - \tilde{v}_p = AG$ for some A according to Lemma A.3. Using Lemma A.4 we obtain in the first case

$$\|v - \tilde{v}_p\|_{L^2(-B_0,0)} \leq Cb^{-\frac{1}{2}}\|v - \tilde{v}_p\|_{L^2(0,b)}.$$

We use the triangle inequality on the right-hand side and the fact that $\tilde{v}_p$ is 0 for negative x and coincide with v_p for positive v. We obtain

$$\|v\|_{L^2(-b_0,0)} \leq Cb^{-\frac{1}{2}}\left(\|v\|_{L^2(0,b)} + \|v_p\|_{L^2(0,b)}\right).$$

The claim then follows from the estimate on $\|v_p\|_{L^2(0,b)}$ in Lemma A.2. We prove the second case by following the same argument, inserting the corresponding bound for G.

The third case will follow the same lines but we will introduce a different particular solution v_p, following then even more closely the proof of [Burq and Zworski 2005]. We set $\lambda = \sqrt{z}$.

Denote by H the unique L^2 function on $(-B_0, b)$ that vanishes on $(-B_0, 0)$ and such that $H' = h$ in the distributional sense. The L^2 norm of H is related to the H^{-1} norm of h by the relation

$$\|H - \Big(\int_0^b H(y)\,dy\Big)\|_{L^2(-B_0,b)} = \|h\|_{H^{-1}(-B_0,b)}.$$

The Cauchy–Schwarz inequality then implies that

$$\|H\|_{L^2(-B_0,b)} \geq (1 + b^{\frac{1}{2}})^{-1}\|h\|_{H^{-1}(-B_0,b)}. \tag{A-8}$$

Set

$$v_p(x) = \int_{-B_0}^{x} \frac{\sin(\lambda(x-y))}{\lambda} H'(y)\,dy.$$

Then v_p satisfies

$$-v_p'' - \lambda^2 v_p = H'$$

in $\mathcal{D}'(-B_0, b)$ and $v_p(-B_0) = 0$ but the boundary condition need not be satisfied at b. We thus have

$$v(x) = v_p(x) - v_p(b)\frac{\sin(\lambda(x+B_0))}{\sin(\lambda(B_0+b))}.$$

The function $v - v_p$ is thus a multiple of $\sin(\lambda(x+B_0))$.

We have

$$\int_{-B_0}^{0} \big|\sin(\lambda(x+B_0))\big|^2\,dx \leq B_0$$

and

$$\int_0^b |\sin(\lambda(x+B_0))|^2\,dx \geq \frac{1}{2}\Big(b - \frac{1}{2\lambda}\Big).$$

Hence, in the regime under consideration we have

$$\|v - v_p\|_{L^2(-B_0,b)} \leq Cb^{-\frac{1}{2}}\|v - v_p\|_{L^2(0,b)}. \tag{A-9}$$

We perform an integration by parts in v_p and observe that the boundary contributions vanish because H vanishes near $-B_0$ and $\sin(\lambda(y-x))$ vanishes at $y = x$.

Finally, we obtain

$$v_p(x) = \int_{-B_0}^{x} \cos(\lambda(x-y))H(y)\,dy.$$

It follows that v_p is identically 0 on $(-B_0, 0)$ and that, on $(0, b)$, it satisfies

$$|v_p(x)| \leq \|H\|_{L^2(-B_0,b)}\sqrt{x}.$$

Squaring and integrating, we get

$$\|v_p\|_{L^2(0,b)} \le b\|H\|_{L^2(-B_0,b)}.$$

Using the triangle inequality in (A-9) and inserting this bound, the result follows for $b \le \frac{1}{2}$ using (A-8). □

In the paper, we will need to relax the condition that $v(b) = 0$. This can be done using a standard construction related to a commutator method. We will get the following

Theorem 3. *There exist β and four constants B_1, C_1, C_2, C_3 depending only on β such that the following holds. For any $b \le B_1$, for any function u in $H^1(-B_0, b)$ that satisfies*

$$-u'' - zu = h,$$

with $h \in H^{-1}(-B_0, b)$ and such that $u(-B_0) = 0$ and h vanishes on $(-B_0, 0)$. Then, the following estimates hold:

(1) *If $z \le \beta^2$, then*

$$\|u\|_{L^2(-B_0,0)} \le C_1\big(b^{\frac{1}{2}}\|h\|_{H^{-1}(-B_0,b)} + b^{-\frac{1}{2}}\|u\|_{L^2(0,b)}\big). \tag{A-10}$$

(2) *If $\beta^2 \le z \le \dfrac{1}{b^2}$, then*

$$\|u\|_{L^2(-B_0,0)} \le C_1\left(\frac{b^{\frac{1}{2}}}{|\sin(B_0\sqrt{z})|}\|h\|_{H^{-1}(-B_0,b)} + \frac{b^{-\frac{1}{2}}}{|\sin(B_0\sqrt{z})|}\|u\|_{L^2(0,b)}\right). \tag{A-11}$$

(3) *If $\dfrac{1}{b^2} \le z$, then*

$$\|u\|_{L^2(-B_0,0)} \le C_3\big(b^{\frac{1}{2}}\|h\|_{H^{-1}(-B_0,b)} + b^{-\frac{1}{2}}\|u\|_{L^2(0,b)}\big). \tag{A-12}$$

Proof. Define a smooth cutoff function ρ_1 such that $\rho_1(x)$ is identically 1 if $x \le \frac{1}{2}$ and identically 0 if $x \ge 1$ and let ρ_b be the function $x \mapsto \rho_1(x/b)$. Define $v := \rho_b u$ then $v \in H_0^1(-B_0, b)$ and satisfies

$$-v'' - zv = h + 2(\rho_b' u)' - \rho_b'' u.$$

The right-hand side vanishes on $(-B_0, 0)$ so that, in order to use Proposition A.5, we have to estimate its H^{-1} norm. The strategy is the same as in the proofs of Lemmas 4.3 and 4.10.

An integration by parts followed by the use of the Cauchy–Schwarz inequality gives

$$\left|\int_{-B_0}^{B_1} (\rho_b' u)'\phi\right| \le \|\rho' u\|_{L^2(0,b)}\|\phi'\|_{L^2} \le \frac{C}{b}\|u\|_{L^2(0,b)}\|\phi'\|_{L^2}.$$

Thus,

$$\|(\rho_b' u)'\|_{H^{-1}} \le \frac{C}{b}\|u\|_{L^2(0,b)}.$$

The third term can be estimated using the same method. Indeed,

$$\left|\int \rho_b'' u\phi\right| = \left|\int_0^b \left(\int_0^x \rho_b''(y)u(y)\,dy\right)\phi'(x)\,dx\right| \le \left\|\int_0^x \rho_b''(y)u(y)\,dy\right\|_{L^2(0,b)}\|\phi'\|_{L^2}.$$

Using again Cauchy–Schwarz inequality and the fact that $|\rho_b''(y)| \le Cb^{-2}$ we get

$$\left|\int_0^x \rho_b''(y)u(y)\,dy\right| \le Cb^{-2}\|u\|_{L^2(0,b)}\sqrt{x}.$$

We obtain

$$\left\|\int_0^x \rho_b''(y)u(y)\,dy\right\|_{L^2(0,b)} \le Cb^{-2}\|u\|_{L^2(0,b)}\|\sqrt{x}\|_{L^2(0,b)} \le Cb^{-1}\|u\|_{L^2(0,b)}.$$

It follows that

$$\|h + 2(\rho_b' u)' - \rho_b'' u\|_{H^{-1}(-B_0,b)} \le \|h\|_{H^{-1}(-B_0,b)} + Cb^{-1}\|u\|_{L^2(0,b)}.$$

We obtain the theorem by plugging this bound into the estimates of the Proposition A.5. ☐

Acknowledgement

The authors wish to thank Andrew Hassell and Alex Barnett for very helpful conversations during the preparation of this article.

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Received 30 Nov 2010. Revised 14 Jul 2011. Accepted 24 Oct 2011.

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ANALYSIS AND PDE
Vol. 5, No. 4, 2012
dx.doi.org/10.2140/apde.2012.5.855

GLOBAL WELL-POSEDNESS AND SCATTERING FOR THE DEFOCUSING QUINTIC NLS IN THREE DIMENSIONS

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We revisit the proof of global well-posedness and scattering for the defocusing energy-critical NLS in three space dimensions in light of recent developments. This result was obtained previously by Colliander, Keel, Staffilani, Takaoka, and Tao.

1. Introduction

The defocusing quintic nonlinear Schrödinger equation,

$$iu_t + \Delta u = |u|^4 u, \tag{1-1}$$

describes the evolution of a complex-valued function $u(t,x)$ of spacetime $\mathbb{R}_t \times \mathbb{R}^3_x$. This evolution conserves *energy*:

$$E(u(t)) := \int_{\mathbb{R}^3} \tfrac{1}{2}|\nabla u(t,x)|^2 + \tfrac{1}{6}|u(t,x)|^6 \, dx. \tag{1-2}$$

By Sobolev embedding, $u(0)$ has finite energy if and only if $u(0) \in \dot H^1_x(\mathbb{R}^3)$, which is the space of initial data that we consider. This is also a scale-invariant space; both the class of solutions to (1-1) and the energy are invariant under the scaling symmetry

$$u(t,x) \mapsto u^\lambda(t,x) := \lambda^{1/2} u(\lambda^2 t, \lambda x). \tag{1-3}$$

For this reason, the equation is termed *energy-critical*.

A function $u : I \times \mathbb{R}^3 \to \mathbb{C}$ on a nonempty time interval $I \ni 0$ is called a *strong solution* to (1-1) if it lies in the class $C^0_t \dot H^1_x(K \times \mathbb{R}^3) \cap L^{10}_{t,x}(K \times \mathbb{R}^3)$ for all compact $K \subset I$, and obeys the Duhamel formula

$$u(t) = e^{it\Delta}u(0) - i\int_0^t e^{i(t-s)\Delta}|u(s)|^4 u(s)\, ds, \tag{1-4}$$

for all $t \in I$. We say that u is a *maximal-lifespan solution* if the solution cannot be extended (in this class) to any strictly larger interval.

Our main result is a new proof of the following:

The first author was partially supported by NSF grant DMS-1001531. The second author was partially supported by the Sloan Foundation and NSF grant DMS-0901166. This work was completed while the second author was a Harrington Faculty Fellow at the University of Texas at Austin.
MSC2010: 35Q55.
Keywords: energy critical, nonlinear Schrödinger.

Theorem 1.1 (global well-posedness and scattering). *Let $u_0 \in \dot{H}_x^1(\mathbb{R}^3)$. Then there exists a unique global strong solution $u \in C_t^0 \dot{H}_x^1(\mathbb{R} \times \mathbb{R}^3)$ to* (1-1) *with initial data $u(0) = u_0$. Moreover, this solution satisfies*

$$\int_{\mathbb{R}} \int_{\mathbb{R}^3} |u(t,x)|^{10}\, dx\, dt \le C\big(\|u_0\|_{\dot{H}_x^1}\big). \tag{1-5}$$

Further, scattering occurs: (*i*) *there exist asymptotic states $u_\pm \in \dot{H}_x^1$ such that*

$$\big\|u(t) - e^{it\Delta}u_\pm\big\|_{\dot{H}_x^1} \to 0 \quad as \quad t \to \pm\infty \tag{1-6}$$

and (*ii*) *for any $u_+ \in \dot{H}_x^1$ (or $u_- \in \dot{H}_x^1$) there exists a unique global solution u to* (1-1) *such that* (1-6) *holds.*

Theorem 1.1 was proved by Colliander, Keel, Staffilani, Takaoka, and Tao in the ground-breaking paper [Colliander et al. 2008]. The key point is to prove the spacetime bound (1-5); scattering is an easy consequence of this. Note also that the solution described in Theorem 1.1 is in fact unique in the larger class of $C_t^0 \dot{H}_x^1$ functions obeying (1-4); this unconditional uniqueness statement is proved in [Colliander et al. 2008, §16] by adapting earlier work.

The paper [Colliander et al. 2008] advanced the induction on energy technique, introduced by Bourgain in [1999], and presaged many recent developments in dispersive PDE at critical regularity. The argument may be outlined as follows: (i) If a bound of the form (1-5) does not hold, then there must be a minimal almost-counterexample, that is, a minimal-energy solution with (prespecified) enormous spacetime norm. (ii) By virtue of its minimality, such a solution must have good tightness and equi-continuity properties. (iii) To be consistent with the interaction Morawetz identity such a solution must undergo a dramatic change of (spatial) scale in a short span of time. (iv) Such a rapid change is inconsistent with simultaneous conservation of mass and energy.

As just described, the argument appears to be by contradiction, but this is not the case. In fact, it is entirely quantitative, showing that in order to achieve such a large spacetime norm, the solution must have at least a certain amount of energy. The energy requirement diverges as the spacetime norm diverges and so yields an effective bound for the function C appearing in (1-5). This style of argument adapts also to other equations and dimensions; see, for example, [Nakanishi 1999; Ryckman and Vişan 2007; Tao 2005; Vişan 2007].

The downside to the induction on energy argument is its complexity. It is monolithic, as opposed to modular; the value of a small parameter introduced at the very beginning of the proof is not determined until the very end. In recent years, the induction on energy argument has been supplanted by a related contradiction argument that is completely modular and is much easier to understand; it is not quantitative.

The genesis of this new method comes from the discovery of Keraani [2006] that the estimates underlying the proof that minimal almost-counterexamples have good tightness/equicontinuity properties can be pushed further to show that failure of Theorem 1.1 guarantees the existence of a *minimal* counterexample. This insight was first applied to the well-posedness problem in an important paper of Kenig and Merle [2006], which considered the focusing equation with radial data in dimensions three, four, and five. Subsequent papers (by a wide array of authors) have greatly refined and expanded this methodology.

In this paper, we revisit the proof of Theorem 1.1 using this "minimal criminal" approach, which, we believe, results in significant expository simplification. We will also endeavor to convey that much of the original argument lives on, both in spirit and in the technical details, by explicit reference to [Colliander et al. 2008] as well as by maintaining their notations, as much as possible.

In some very striking recent work [Dodson 2012; 2011b; 2011a], Dodson has proved the analogue of Theorem 1.1 for the *mass-critical* nonlinear Schrödinger equation in arbitrary dimension. The most significant difference between [Colliander et al. 2008] and the argument presented here comes from the adaptation of some of his ideas (present already in the first paper [Dodson 2012]) to the problem (1-1). We postpone a fuller discussion of these matters until we have described some of the key steps in the proof.

Outline of the proof. We argue by contradiction. Simple contraction mapping arguments show that Theorem 1.1 holds for solutions with small energy; thus, if the theorem were not to hold there must be a transition energy above which the energy no longer controls the spacetime norm. The first step in the argument is to show that there is a minimal counterexample and that, by virtue of its minimality, this counterexample has good compactness properties.

Definition 1.2 (almost periodicity). A solution $u \in L_t^\infty \dot{H}_x^1(I \times \mathbb{R}^3)$ to (1-1) is said to be *almost periodic (modulo symmetries)* if there exist functions $N : I \to \mathbb{R}^+$, $x : I \to \mathbb{R}^3$, and $C : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $t \in I$ and $\eta > 0$,

$$\int_{|x-x(t)|\geq C(\eta)/N(t)} \left|\nabla u(t,x)\right|^2 dx + \int_{|\xi|\geq C(\eta)N(t)} |\xi|^2 \, |\hat{u}(t,\xi)|^2 \, d\xi \leq \eta. \tag{1-7}$$

We refer to the function $N(t)$ as the *frequency scale function* for the solution u, to $x(t)$ as the *spatial center function*, and to $C(\eta)$ as the *modulus of compactness*.

Remark 1.3. Together with boundedness in $\dot{H}_x^1$, the tightness plus equicontinuity statement (1-7) illustrates that almost periodicity is equivalent to the (co)compactness of the orbit modulo translation and dilation symmetries. In particular, from compactness we see that for each $\eta > 0$ there exists $c(\eta) > 0$ so that for all $t \in I$,

$$\int_{|x-x(t)|\leq c(\eta)/N(t)} \left|\nabla u(t,x)\right|^2 dx + \int_{|\xi|\leq c(\eta)N(t)} |\xi|^2 \, |\hat{u}(t,\xi)|^2 \, d\xi \leq \eta.$$

Similarly, compactness implies

$$\int_{\mathbb{R}^3} |\nabla u(t,x)|^2 \, dx \lesssim_u \int_{\mathbb{R}^3} |u(t,x)|^6 \, dx$$

uniformly for $t \in I$. This last observation plays the role of Proposition 4.8 in [Colliander et al. 2008].

With these preliminaries out of the way, we can now describe the first major milestone in the proof of Theorem 1.1:

Theorem 1.4 (reduction to almost periodic solutions [Kenig and Merle 2006; Killip and Vişan 2010]). *Suppose Theorem 1.1 fails. Then there exists a maximal-lifespan solution $u : I \times \mathbb{R}^3 \to \mathbb{C}$ to* (1-1) *which* *is almost periodic and blows up both forward and backward in time in the sense that for all* $t_0 \in I$,

$$\int_{t_0}^{\sup I} \int_{\mathbb{R}^3} |u(t,x)|^{10}\, dx\, dt = \int_{\inf I}^{t_0} \int_{\mathbb{R}^3} |u(t,x)|^{10}\, dx\, dt = \infty.$$

The theorem does not explicitly claim that u is a *minimal* counterexample; nevertheless, this is how it is constructed and, more importantly, how it is shown to be almost periodic. In [Colliander et al. 2008], the role of this theorem is played by Corollary 4.4 (equicontinuity) and Proposition 4.6 (tightness).

A précis of the proof of Theorem 1.4 can be found in [Kenig and Merle 2006], building on Keraani's method [2006]; for complete details see [Killip and Vişan 2010] or [Killip and Vişan 2008]. Just as for the results from [Colliander et al. 2008] mentioned above, the key ingredients in the proof are improved Strichartz inequalities, which show that concentration occurs, and perturbation theory, which shows that multiple simultaneous concentrations are inconsistent with minimality.

Continuity of the flow prevents rapid changes in the modulation parameters $x(t)$ and $N(t)$. In particular, from [Killip et al. 2009, Corollary 3.6] or [Killip and Vişan 2008, Lemma 5.18] we have

Lemma 1.5 (local constancy property). *Let* $u : I \times \mathbb{R}^3 \to \mathbb{C}$ *be a maximal-lifespan almost periodic solution to* (1-1). *Then there exists a small number* δ, *depending only on* u, *such that if* $t_0 \in I$ *then*

$$\big[t_0 - \delta N(t_0)^{-2},\, t_0 + \delta N(t_0)^{-2}\big] \subset I$$

and

$$N(t) \sim_u N(t_0) \quad \textit{whenever} \quad |t - t_0| \le \delta N(t_0)^{-2}.$$

We recall next a consequence of the local constancy property; see [Killip et al. 2009, Corollary 3.7; Killip and Vişan 2008, Corollary 5.19].

Corollary 1.6 ($N(t)$ at blowup). *Let* $u : I \times \mathbb{R}^3 \to \mathbb{C}$ *be a maximal-lifespan almost periodic solution to* (1-1). *If* T *is any finite endpoint of* I, *then* $N(t) \gtrsim_u |T - t|^{-1/2}$; *in particular*, $\lim_{t\to T} N(t) = \infty$.

Finally, we will need the following result linking the frequency scale function $N(t)$ of an almost periodic solution u and its Strichartz norms:

Lemma 1.7 (spacetime bounds). *Let* u *be an almost periodic solution to* (1-1) *on a time interval* I. *Then*

$$\int_I N(t)^2\, dt \lesssim_u \|\nabla u\|^q_{L_t^q L_x^r(I\times\mathbb{R}^3)} \lesssim_u 1 + \int_I N(t)^2\, dt \tag{1-8}$$

for all $\frac{2}{q} + \frac{3}{r} = \frac{3}{2}$ *with* $2 \le q < \infty$.

Proof. We recall that Lemma 5.21 in [Killip and Vişan 2008] shows that

$$\int_I N(t)^2\, dt \lesssim_u \int_I \int_{\mathbb{R}^3} |u(t,x)|^{10}\, dx\, dt \lesssim_u 1 + \int_I N(t)^2\, dt. \tag{1-9}$$

The second inequality in (1-8) follows from the second inequality above and an application of the Strichartz inequality. The first inequality follows by the same method used to prove the corresponding result in (1-9): The fact that $u \not\equiv 0$ ensures that $N(t)^{-2/q}\|\nabla u(t)\|_{L_x^r}$ never vanishes. Almost periodicity then implies that it is bounded away from zero and the inequality follows. $\square$

Let $u : I \times \mathbb{R}^3 \to \mathbb{C}$ be an almost periodic maximal-lifespan solution to (1-1). As a direct consequence of the preceding three results, we can tile the interval I with infinitely many *characteristic intervals* J_k, which have the following properties:

- $N(t) \equiv N_k$ is constant on each J_k.
- $|J_k| \sim_u N_k^{-2}$, uniformly in k.
- $\|\nabla u\|_{L_t^q L_x^r(J_k \times \mathbb{R}^3)} \sim_u 1$, for each $\frac{2}{q} + \frac{3}{r} = \frac{3}{2}$ with $2 \le q \le \infty$ and uniformly in k.

Note that the redefinition of $N(t)$ may necessitate a mild increase in the modulus of compactness. We may further assume that 0 marks a boundary between characteristic intervals, which we do, for expository reasons.

Returning to Theorem 1.4, a simple rescaling argument (see, for example, the proof of Theorem 3.3 in [Tao et al. 2007]) allows us to additionally assume that $N(t) \ge 1$ at least on half of the interval I, say, on $[0, T_{\max})$. Inspired by [Dodson 2012], we further subdivide into two cases dictated by the control given by the interaction Morawetz inequality. Putting everything together, we obtain

Theorem 1.8 (two special scenarios for blowup). *Suppose Theorem 1.1 failed. Then there exists an almost periodic solution* $u : [0, T_{\max}) \times \mathbb{R}^3 \to \mathbb{C}$, *such that*

$$\|u\|_{L^{10}_{t,x}([0,T_{\max})\times\mathbb{R}^4)} = +\infty$$

and $[0, T_{\max}) = \cup_k J_k$ *where* J_k *are characteristic intervals on which* $N(t) \equiv N_k \ge 1$. *Furthermore*,

$$\textit{either} \quad \int_0^{T_{\max}} N(t)^{-1}\,dt < \infty \quad \textit{or} \quad \int_0^{T_{\max}} N(t)^{-1}\,dt = \infty.$$

Thus, in order to prove Theorem 1.1 we just need to preclude the existence of the two types of almost periodic solution described in Theorem 1.8. By analogy with the trichotomies appearing in [Killip et al. 2009; Killip and Vişan 2010], we refer to the first type of solution as a *rapid low-to-high frequency cascade* and the second as a *quasisoliton*.

In each case, the key to showing that such solutions do not exist is a fundamentally nonlinear relation obeyed by the equation. In the cascade case, it is the conservation of mass; in the quasisoliton case, it is the interaction Morawetz identity (a monotonicity formula introduced in [Colliander et al. 2004]). Unfortunately, both of these relations have energy-subcritical scaling and so are not immediately applicable to $L_t^\infty \dot{H}_x^1$ solutions; additional control on the low frequencies is required. It is in how this control is achieved that we deviate most from [Colliander et al. 2008].

The argument in [Colliander et al. 2008] relies heavily on the interaction Morawetz identity. To cope with the noncritical scaling, a frequency localization is introduced. This produces error terms which are then controlled by means of a highly entangled bootstrap argument. Dodson's paper [2012] also uses a frequency-localized interaction Morawetz identity; however, the error terms are handled via spacetime estimates that are proved independently of this identity. Indeed, the proof of these estimates does not even rely on the defocusing nature of the nonlinearity.

In this paper, we adopt Dodson's strategy (see also [Vişan 2012]). The requisite estimates on the low-frequency part of the solution appear in Theorem 4.1. It seems to us that this theorem represents the limit of what can be achieved without the use of intrinsically nonlinear tools such as monotonicity formulae. The rationale for this assertion comes from consideration of the focusing equation and is discussed in Remark 4.3. Nevertheless, Theorem 4.1 does just suffice to treat the error terms in the frequency-localized interaction Morawetz identity (see Section 6), which is then used to preclude quasisolitons in Section 7.

The proof of Theorem 4.1 relies on a type of Strichartz estimate that we have not seen previously. This estimate, Proposition 3.1, has the flavor of a maximal function in that it controls the worst Littlewood–Paley piece at each moment of time. The necessity of considering a supremum over frequency projections (as opposed to a sum) is borne out by an examination of the ground-state solution to the focusing equation; see Remark 4.3. The proof of this proposition is adapted from the double Duhamel trick first introduced in [Colliander et al. 2008, §14]. The original application of this trick also appears here, namely, as Proposition 3.2.

The nonexistence of cascade solutions is proved in Section 5. The argument combines the following proposition and Theorem 4.1 to prove first that the mass is finite and then (to reach a contradiction) that it is zero. It is equally valid in the focusing case.

Proposition 1.9 (no-waste Duhamel formula, [Killip and Vişan 2008; Tao et al. 2008]). *Let $u : [0, T_{\max}) \times \mathbb{R}^3 \to \mathbb{C}$ be a solution as in Theorem 1.8. Then for all $t \in [0, T_{\max})$,*

$$u(t) = i \lim_{T \to T_{\max}} \int_t^T e^{i(t-s)\Delta} |u(s)|^4 u(s)\, ds,$$

where the limit is to be understood in the weak $\dot H^1_x$ topology.

2. Notation and useful lemmas

We use the notation $X \lesssim Y$ to indicate that there exists some constant C so that $X \leq CY$. Similarly, we write $X \sim Y$ if $X \lesssim Y \lesssim X$. We use subscripts to indicate the dependence of C on additional parameters. For example, $X \lesssim_u Y$ denotes the assertion that $X \leq C_u Y$ for some C_u depending on u.

We will make frequent use of the fractional differential/integral operators $|\nabla|^s$ together with the corresponding homogeneous Sobolev norms:

$$\|f\|_{\dot H^s_x} := \| |\nabla|^s f \|_{L^2_x} \quad \text{where} \quad \widehat{|\nabla|^s f}(\xi) := |\xi|^s \hat f(\xi).$$

We will also need some Littlewood–Paley theory. Specifically, let $\varphi(\xi)$ be a smooth bump supported in the ball $|\xi| \leq 2$ and equaling one on the ball $|\xi| \leq 1$. For each dyadic number $N \in 2^{\mathbb{Z}}$ we define the Littlewood–Paley operators

$$\widehat{P_{\leq N} f}(\xi) := \varphi(\xi/N) \hat f(\xi), \quad \widehat{P_{>N} f}(\xi) := (1 - \varphi(\xi/N)) \hat f(\xi), \quad \widehat{P_N f}(\xi) := (\varphi(\xi/N) - \varphi(2\xi/N)) \hat f(\xi).$$

Similarly, we can define $P_{<N}$, $P_{\geq N}$, and $P_{M<\cdot\leq N} := P_{\leq N} - P_{\leq M}$, whenever M and N are dyadic numbers. We will frequently write $f_{\leq N}$ for $P_{\leq N} f$ and similarly for the other operators.

The Littlewood–Paley operators commute with derivative operators, the free propagator, and complex conjugation. They are self-adjoint and bounded on every L_x^p and $\dot{H}_x^s$ space for $1 \le p \le \infty$ and $s \ge 0$. They also obey the following Sobolev and Bernstein estimates:

$$\| |\nabla|^{\pm s} P_N f \|_{L_x^p} \sim_s N^{\pm s} \| P_N f \|_{L_x^p}, \qquad \| P_N f \|_{L_x^q} \lesssim_s N^{\frac{3}{p}-\frac{3}{q}} \| P_N f \|_{L_x^p},$$

whenever $s \ge 0$ and $1 \le p \le q \le \infty$.

We will frequently denote the nonlinearity in (1-1) by $F(u)$, that is, $F(u) := |u|^4 u$. We will use the notation $\text{Ø}(X)$ to denote a quantity that resembles X, that is, a finite linear combination of terms that look like those in X, but possibly with some factors replaced by their complex conjugates and/or restricted to various frequencies. For example,

$$F(u+v) = \sum_{j=0}^{5} \text{Ø}(u^j v^{5-j}) \quad \text{and} \quad F(u) = F(u_{>N}) + \text{Ø}(u_{\le N} u^4) \text{ for any } N > 0.$$

We use $L_t^q L_x^r$ to denote the spacetime norm

$$\|u\|_{L_t^q L_x^r} := \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}^3} |u(t,x)|^r dx \right)^{q/r} dt \right)^{1/q},$$

with the usual modifications when q or r is infinity, or when the domain $\mathbb{R} \times \mathbb{R}^3$ is replaced by some smaller spacetime region. When $q = r$ we abbreviate $L_t^q L_x^r$ by $L_{t,x}^q$.

Let $e^{it\Delta}$ be the free Schrödinger propagator. In physical space this is given by the formula

$$e^{it\Delta} f(x) = \frac{1}{(4\pi i t)^{3/2}} \int_{\mathbb{R}^3} e^{i|x-y|^2/4t} f(y) dy.$$

In particular, the propagator obeys the *dispersive inequality*

$$\|e^{it\Delta} f\|_{L_x^\infty(\mathbb{R}^3)} \lesssim |t|^{-3/2} \|f\|_{L_x^1(\mathbb{R}^3)} \tag{2-1}$$

for all times $t \ne 0$. As a consequence of this dispersive estimate, one obtains the Strichartz estimates; see, for example, [Ginibre and Velo 1992; Keel and Tao 1998; Strichartz 1977]. The particular version we need is from [Colliander et al. 2008].

Lemma 2.1 (Strichartz inequality). *Let I be a compact time interval and let $u : I \times \mathbb{R}^3 \to \mathbb{C}$ be a solution to the forced Schrödinger equation*

$$i u_t + \Delta u = G$$

for some function G. Then we have

$$\left\{ \sum_{N \in 2^{\mathbb{Z}}} \|\nabla u_N\|^2_{L_t^q L_x^r (I \times \mathbb{R}^3)} \right\}^{1/2} \lesssim \|u(t_0)\|_{\dot{H}_x^1(\mathbb{R}^3)} + \|\nabla G\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'} (I \times \mathbb{R}^3)} \tag{2-2}$$

for any time $t_0 \in I$ and any exponents (q, r) and $(\tilde{q}, \tilde{r})$ obeying $\frac{2}{q} + \frac{3}{r} = \frac{2}{\tilde{q}} + \frac{3}{\tilde{r}} = \frac{3}{2}$ and $2 \le q, \tilde{q} \le \infty$. Here, as usual, p' denotes the dual exponent to p, that is, $1/p + 1/p' = 1$.

Elementary Littlewood–Paley theory shows that (2-2) implies

$$\|\nabla u\|_{L_t^q L_x^r(I\times\mathbb{R}^3)} \lesssim \|u(t_0)\|_{\dot H_x^1(\mathbb{R}^3)} + \|\nabla G\|_{L_t^{\tilde q'} L_x^{\tilde r'}(I\times\mathbb{R}^3)},$$

which corresponds to the usual Strichartz inequality; however, the Besov variant given above allows us to "Sobolev embed" into L_x^∞:

Lemma 2.2 (an endpoint estimate). *For any $u : I\times\mathbb{R}^3 \to \mathbb{R}$ we have*

$$\|u\|_{L_t^4 L_x^\infty(I\times\mathbb{R}^3)} \lesssim \|\nabla u\|_{L_t^\infty L_x^2}^{1/2} \Big\{ \sum_{N\in 2^{\mathbb{Z}}} \|\nabla u_N\|_{L_t^2 L_x^6(I\times\mathbb{R}^3)}^2 \Big\}^{1/4}.$$

In particular, for any frequency $N>0$,

$$\|u_{\le N}\|_{L_t^4 L_x^\infty(I\times\mathbb{R}^3)} \lesssim \|\nabla u_{\le N}\|_{L_t^\infty L_x^2}^{1/2} \Big\{ \sum_{M\le N} \|\nabla u_M\|_{L_t^2 L_x^6(I\times\mathbb{R}^3)}^2 \Big\}^{1/4}.$$

Proof. Using Bernstein's inequality we have,

$$\begin{aligned}
\|u\|_{L_t^4 L_x^\infty(I\times\mathbb{R}^3)}^4 &\lesssim \int_I \Big\{ \sum_{N\in 2^{\mathbb{Z}}} \|u_N(t)\|_{L_x^\infty} \Big\}^4 dt \\
&\lesssim \sum_{N_1\le N_2\le N_3\le N_4} \|u_{N_1}\|_{L_t^\infty L_x^\infty} \|u_{N_2}\|_{L_t^\infty L_x^\infty} \|u_{N_3}\|_{L_t^2 L_x^\infty} \|u_{N_4}\|_{L_t^2 L_x^\infty} \\
&\lesssim \sum_{N_1\le\cdots\le N_4} \Big[\tfrac{N_1 N_2}{N_3 N_4}\Big]^{\frac12} \|\nabla u_{N_1}\|_{L_t^\infty L_x^2} \|\nabla u_{N_2}\|_{L_t^\infty L_x^2} \|\nabla u_{N_3}\|_{L_t^2 L_x^6} \|\nabla u_{N_4}\|_{L_t^2 L_x^6} \\
&\lesssim \|\nabla u\|_{L_t^\infty L_x^2}^2 \sum_{N_3\le N_4} \Big[\tfrac{N_3}{N_4}\Big]^{\frac12} \|\nabla u_{N_3}\|_{L_t^2 L_x^6} \|\nabla u_{N_4}\|_{L_t^2 L_x^6}.
\end{aligned}$$

All spacetime norms above are over $I\times\mathbb{R}^3$. The claim now follows from Schur's test. □

3. Maximal Strichartz estimates

Proposition 3.1. *Let $(i\partial_t+\Delta)v = F+G$ on a compact interval $[0,T]$. Then for each $6<q\le\infty$,*

$$\Big\| M(t)^{\frac3q - 1} \|P_{M(t)} v(t)\|_{L_x^q} \Big\|_{L_t^2} \lesssim \big\| |\nabla|^{-\frac12} v \big\|_{L_t^\infty L_x^2} + \big\| |\nabla|^{-\frac12} G \big\|_{L_t^2 L_x^{6/5}} + \|F\|_{L_t^2 L_x^1}$$

uniformly for all functions $M : [0,T]\to 2^{\mathbb{Z}}$. All spacetime norms are over $[0,T]\times\mathbb{R}^3$.

It is not difficult to see that the conclusion is weaker than (and has the same scaling as) $|\nabla|^{-1/2}v \in L_t^2 L_x^6$. In fact, if $F\equiv 0$, this stronger result can be deduced immediately from the Strichartz inequality. However, this argument does not extend to give a proof of the proposition because $F\in L_x^1$ does not imply $|\nabla|^{-1/2}F \in L_x^{6/5}$. Indeed, the whole theory of the energy-critical NLS in three dimensions is dogged by the absence of endpoint estimates of this type.

The freedom of choosing an arbitrary function $M(t)$ makes this a maximal function estimate; at each time one can take the supremum over all choices of the parameter. Writing maximal functions in this way yields linear operators and so one may use the method of TT^*; this is an old idea dating at least to the work of Kolmogorov and Seliverstov in the 1920s (cf. [Zygmund 2002, Chapter XIII]). As we will see, the double Duhamel trick, which underlies the proof of Proposition 3.1, is a variant of the TT^* idea. Specifically, one takes the inner-product between two *different* representations of $v(t)$.

The double Duhamel trick was introduced in [Colliander et al. 2008, §14]. There it was used for a different purpose, namely, to obtain control over the mass on balls. This is then used to estimate error terms in the (localized) interaction Morawetz identity. We will also need this information and for exactly the same reasons; see (6-16). The following proposition captures the main thrust of [Colliander et al. 2008, §14]:

Proposition 3.2. *Let* $(i\partial_t + \Delta)v = F + G$ *on a compact interval* $[0, T]$ *and let*

$$[\mathcal{S}_R v](t, x) := \left((\pi R^2)^{-3/2} \int_{\mathbb{R}^3} |v(t, x + y)|^2 e^{-|y|^2/R^2}\, dy \right)^{1/2}. \tag{3-1}$$

Then for each $0 < R < \infty$ *and* $6 < q \le \infty$,

$$R^{\frac{1}{2} - \frac{3}{q}} \|\mathcal{S}_R v\|_{L^2_t L^q_x} \lesssim \|v\|_{L^\infty_t L^2_x} + \|G\|_{L^2_t L^{6/5}_x} + R^{-\frac{1}{2}} \|F\|_{L^2_t L^1_x}, \tag{3-2}$$

where all spacetime norms are over $[0, T] \times \mathbb{R}^3$.

We use the letter $\mathcal{S}$ for the operator appearing in (3-1) to signify both "smudging" and "square function". It is easy to see that the Gaussian smudging used here could be replaced by other methods without affecting the result; indeed, the analogous estimate in [Colliander et al. 2008] averages over balls. That paper also sets $q = 100$ and sums over a lattice rather than integrating in x. As $\mathcal{S}v$ is slowly varying, summation and integration yield comparable norms.

To control $\mathcal{S}_R v$ we need to estimate some complicated oscillatory (and nonoscillatory) integrals. By choosing a Gaussian weight, some of the integrals can be done both quickly and exactly; see the proof of Lemma 3.4. Before turning to that subject, we first show how the two propositions are interconnected. The proof of the next lemma also demonstrates how bounds on $\mathcal{S}_R$ can be used to deduce analogous results with other weights.

Lemma 3.3. *Fix* $6 < q \le \infty$. *Then*

$$\sup_{M>0} M^{\frac{3}{q}-1} \|f_M\|_{L^q_x} \lesssim \sup_{M>0} M^{\frac{3}{q}-1} \|\mathcal{S}_{M^{-1}}(f_M)\|_{L^q_x}. \tag{3-3}$$

Proof. Let $\tilde{P}_M = P_{M/2} + P_M + P_{2M}$ denote the fattened Littlewood–Paley projector. The basic relation $P_M = \tilde{P}_M P_M$ reduces our goal to showing that

$$\sup_{M>0} M^{\frac{3}{q}-1} \|\tilde{P}_M g\|_{L^q_x} \lesssim \sup_{M>0} M^{\frac{3}{q}-1} \|\mathcal{S}_{M^{-1}} g\|_{L^q_x} \tag{3-4}$$

for general functions $g : \mathbb{R}^3 \to \mathbb{C}$, say, $g = f_M$.

Recall that the convolution kernel for $\tilde{P}_M$ takes the form $M^3\psi(Mx)$ for some Schwartz function ψ. By virtue of its rapid decay, we can write

$$|\psi(x)| \le \int_0^\infty \pi^{-3/2} e^{-|x|^2/\lambda^2}\, d\mu(\lambda)$$

where μ is a positive measure with all moments finite. Indeed, since ψ is radial one can choose $d\mu(\lambda) = 20|\psi'(\lambda)|\, d\lambda$. Thus by the Cauchy–Schwarz inequality,

$$\big|[\tilde{P}_M g](x)\big|^2 \le \int_{\mathbb{R}^3} |g(x+y)|^2 M^3 |\psi(My)|\, dy \le \int_0^\infty \big|[\mathcal{S}_{\lambda M^{-1}} g](x)\big|^2 \lambda^3\, d\mu(\lambda).$$

Applying Minkowski's inequality in $L_x^{q/2}(\mathbb{R}^3)$ then easily yields (3-4); indeed, one can take the constant to be $[\int \lambda^{1+6/q}\, d\mu(\lambda)]^{1/2}$. $\square$

Lemma 3.4. *For fixed* $6 < q \le \infty$, *the integral kernel*

$$K_R(\tau, z; s, y; x) := (\pi R^2)^{-3/2} \langle \delta_z,\ e^{i\tau\Delta} e^{-|\cdot - x|^2/R^2} e^{is\Delta} \delta_y \rangle$$

obeys

$$\sup_{R>0} \int_0^\infty \int_0^\infty R^{2-\frac{6}{q}} \|K_R(\tau, z; s, y; x)\|_{L^\infty_{z,y} L_x^{q/2}} f(t+\tau) f(t-s)\, ds\, d\tau \lesssim \big|[\mathcal{M} f](t)\big|^2, \tag{3-5}$$

where $\mathcal{M}$ *denotes the Hardy–Littlewood maximal operator and* $f : \mathbb{R} \to [0, \infty)$.

Proof. From the exact formula for the propagator,

$$K_R(t, z; s, y; x) = \int_{\mathbb{R}^3} \frac{\exp\{i|z-x'|^2/4\tau - |x'-x|^2/R^2 + i|x'-y|^2/4s\}}{(4\pi i\tau)^{3/2} (4\pi is)^{3/2} (\pi R^2)^{3/2}}\, dx'. \tag{3-6}$$

Completing the square and doing the Gaussian integral yields

$$|K_R(t, z; s, y; x)| = (2\pi)^{-3} \big[16 s^2 \tau^2 + R^4 (s+\tau)^2\big]^{-3/4} \exp\Big\{-\frac{R^2 (s+\tau)^2 |x - x^*|^2}{16 s^2 \tau^2 + R^4 (s+\tau)^2}\Big\}$$

where $x^* = (sz + ty)/(s+t)$. One more Gaussian integral then yields

$$\|K_R(\dots)\|_{L_x^{q/2}} = (2\pi)^{-3} (2\pi/q)^{3/q} R^{-6/q} |s+\tau|^{-6/q} [16 s^2 \tau^2 + R^4 (s+\tau)^2]^{-3/4+3/q}.$$

Notice that there is no dependence on z or y. This is due to simultaneous translation and Galilean invariance. In this way, we deduce that

$$\text{LHS(3-5)} \lesssim \sup_{R>0} \int_0^\infty \int_0^\infty K_q^*(\alpha, \beta) f(t + R^2\alpha) f(t - R^2\beta)\, d\alpha\, d\beta, \tag{3-7}$$

where we have changed variables to $\alpha = R^{-2}\tau$ and $\beta = R^{-2}s$ and written

$$K_q^*(\alpha, \beta) := [\alpha + \beta]^{-6/q} \big[\alpha^2 \beta^2 + (\alpha+\beta)^2\big]^{-3/4+3/q}.$$

To finish the proof, we just need to show that K_q^* can be majorized by a convex combination of (L^1-normalized) characteristic functions of rectangles of the form $[0,\ell]\times[0,w]$. In fact, we can write it exactly as a positive linear combination of such rectangles:

$$K_q^*(\alpha,\beta)=\int_0^\infty\int_0^\infty \tfrac{1}{\ell}\chi_{[0,\ell]}(\alpha)\,\tfrac{1}{w}\chi_{[0,w]}(\beta)\,\rho(\ell,w)\,d\ell\,dw=\int_\alpha^\infty\int_\beta^\infty \frac{\rho(\ell,w)\,d\ell\,dw}{\ell w}$$

where $\rho(\ell,w):=\ell w\partial_\ell\partial_w K_q^*(\ell,w)\geq 0$. Thus, we just need to check that $\rho\in L^1$. With a little patience, one finds that $\rho(\ell,w)\lesssim_q K_q^*(\ell,w)$, which leaves us to integrate the latter over a quadrant. We use polar coordinates, $\ell+iw=re^{i\theta}$:

$$\begin{aligned}\int_0^\infty\int_0^\infty K_q^*(\ell,w)\,d\ell\,dw &\lesssim \int_0^\infty\int_0^{\pi/2} r^{-6/q}[r^4\sin^2(2\theta)+r^2]^{-3/4+3/q}\,r\,d\theta\,dr\\ &\lesssim \int_0^\infty r^{-1/2}(1+r)^{-3/2+6/q}\,dr\lesssim 1.\end{aligned}$$

Notice that convergence of the r integral relies on $q>6$. The estimate for the θ integral given above is only valid in the range $6<q<12$. When $q>12$, the correct form is $\int r^{-1/2}(1+r)^{-1}\,dr$ and when $q=12$, it is $\int r^{-1/2}\log(2+r)(1+r)^{-1}\,dr$. Nevertheless, both of these integrals are also finite. □

We now have all the necessary ingredients to complete the proofs of Propositions 3.1 and 3.2. We only provide the details for the former because the two arguments are so similar. Indeed, the proof of the latter essentially follows by choosing $M(t)\equiv R^{-1}$ and throwing away the Littlewood–Paley projector $P_{M(t)}$ in the argument we are about to present.

Proof of Proposition 3.1. In view of Lemma 3.3 we need to show that

$$\sup_{M>0} M^{\frac{3}{q}-1}\big\|\mathcal{G}_{M^{-1}}\big(P_M v(t)\big)\big\|_{L_x^q}\in L_t^2([0,T])$$

(with suitable bounds), where the supremum is taken pointwise in time.

As noted earlier, we will use the double Duhamel trick, which relies on playing two Duhamel formulae off against one another, one from each endpoint of $[0,T]$:

$$v(t)=e^{it\Delta}v(0)-i\int_0^t e^{i(t-s)\Delta}G(s)\,ds-i\int_0^t e^{i(t-s)\Delta}F(s)\,ds \tag{3-8}$$

$$=e^{-i(T-t)\Delta}v(T)+i\int_t^T e^{-i(\tau-t)\Delta}G(\tau)\,d\tau+i\int_t^T e^{-i(\tau-t)\Delta}F(\tau)\,d\tau. \tag{3-9}$$

The idea is to compute the L_x^2 norm of $P_M v(t)$ with respect to the Gaussian measure that defines $[\mathcal{G}_{M^{-1}}P_M v](t,x)$ by taking the inner product between these two representations. Actually, we deviate slightly from this idea because it is not clear how to estimate a pair of cross-terms. Our trick for avoiding this is the following simple fact about vectors in a Hilbert space:

$$v=a+b=c+d\quad\Longrightarrow\quad \|v\|^2\leq 3\|a\|^2+3\|c\|^2+2|\langle b,d\rangle|. \tag{3-10}$$

(The numbers are neither optimal nor important.) To prove this, write

$$\|v\|^2 = \langle a, v\rangle + \langle v, c\rangle - \langle a, c\rangle + \langle b, d\rangle$$

and then use the Cauchy–Schwarz inequality.

Let us invoke (3-10) with a and c representing (P_M applied to) the first two summands in (3-8) and (3-9), respectively, while b and d represent the summands which involve F. In this way, we obtain the pointwise statement

$$\begin{aligned}\big|[\mathcal{G}_{M^{-1}}(P_M v)](t,x)\big|^2 \lesssim \Big|\mathcal{G}_{M^{-1}}\Big(e^{it\Delta}v_M(0) - i\int_0^t e^{i(t-s)\Delta}G_M(s)\,ds\Big)(x)\Big|^2 \\ + \Big|\mathcal{G}_{M^{-1}}\Big(e^{-i(T-t)\Delta}v_M(T) + i\int_t^T e^{-i(\tau-t)\Delta}G_M(\tau)\,d\tau\Big)(x)\Big|^2 + h_M(t,x),\end{aligned}$$

where h_M is an abbreviation for

$$h_M(t,x) := \pi^{-3/2}M^3\Big|\Big\langle\int_t^T e^{-i(\tau-t)\Delta}F_M(\tau)\,d\tau,\, e^{-M^2|\cdot-x|^2}\int_0^t e^{i(t-s)\Delta}F_M(s)\,ds\Big\rangle\Big|$$

The contributions of the first two summands are easily estimated: For any function w, Young's and Bernstein's inequalities imply

$$M^{\frac{3}{q}-1}\big\|[\mathcal{G}_{M^{-1}}(P_M w)](t,x)\big\|_{L_x^q(\mathbb{R}^3)} \lesssim M^{-\frac12}\big\|P_M w(t)\big\|_{L_x^6(\mathbb{R}^3)} \lesssim \big\||\nabla|^{-\frac12}w(t)\big\|_{L_x^6(\mathbb{R}^3)}.$$

This can then be combined with Strichartz inequality, which shows

$$\Big\||\nabla|^{-\frac12}\Big(e^{it\Delta}v(0) - i\int_0^t e^{i(t-s)\Delta}G(s)\,ds\Big)\Big\|_{L_t^2L_x^6} \lesssim \big\||\nabla|^{-\frac12}v(0)\big\|_{L_x^2} + \big\||\nabla|^{-\frac12}G\big\|_{L_t^2L_x^{6/5}}$$

and similarly for the second summand.

The third summand, h_M, is the crux of the matter. Using the notation from Lemma 3.4 and changing variables, we have

$$h_M(t,x) = \Big|\int_0^{T-t}\int_0^t\iint \bar F_M(t+\tau',z)K_{M^{-1}}(\tau',z;s',y;x)F_M(t-s',y)\,dy\,dz\,ds'\,d\tau'\Big|.$$

Note also that by Bernstein's inequality and the maximal inequality, the function $f(t) := \|F(t)\|_{L_x^1}$ obeys

$$\|F_M(t)\|_{L_x^1} \lesssim f(t) \quad\text{and}\quad \|\mathcal{M}f\|_{L_t^2} \lesssim \|F\|_{L_t^2L_x^1}.$$

Thus using Lemma 3.4 (with f as just defined), we obtain

$$\Big\|\sup_{M>0} M^{\frac{6}{q}-2}\|h_M(t)\|_{L_x^{q/2}}\Big\|_{L_t^1} \lesssim \|F\|_{L_t^2L_x^1}^2.$$

Recalling that h_M appears in an upper bound on the *square* of the size of $P_M v$, the proposition follows. □

4. Long-time Strichartz estimates

The main result of this section is a long-time Strichartz estimate. As will be evident from the proof, the result is also valid for $L_t^\infty \dot H_x^1(\mathbb{R}^3)$ solutions to the focusing equation; see also Remark 4.3 at the end of this section.

Theorem 4.1 (long-time Strichartz estimate). *Let $u : (T_{\min}, T_{\max}) \times \mathbb{R}^3 \to \mathbb{C}$ be a maximal-lifespan almost periodic solution to* (1-1) *and $I \subset (T_{\min}, T_{\max})$ a time interval that is tiled by finitely many characteristic intervals J_k. Then for any fixed $6 < q < \infty$ and any frequency $N > 0$,*

$$A(N) := \Big\{ \sum_{M \le N} \|\nabla u_M\|^2_{L_t^2 L_x^6(I\times\mathbb{R}^3)} \Big\}^{1/2} \tag{4-1}$$

and

$$\tilde A_q(N) := N^{3/2} \Big\| \sup_{M \ge N} M^{\frac{3}{q}-1} \, \|u_M(t)\|_{L_x^q(\mathbb{R}^3)} \Big\|_{L_t^2(I)} \tag{4-2}$$

obey

$$A(N) + \tilde A_q(N) \lesssim_u 1 + N^{3/2} K^{1/2}, \tag{4-3}$$

where $K := \int_I N(t)^{-1}\,dt$. The implicit constant is independent of the interval I.

The proof of this theorem will occupy the remainder of this section. Throughout, we consider a single interval I and so the implicit dependence of $A(N)$, $\tilde A_q(N)$, and K on the interval should not cause confusion. Additionally, all spacetime norms will be on $I \times \mathbb{R}^3$, unless specified otherwise.

By Bernstein's inequality, $\tilde A_q(N)$ is monotone in q. Thus $q = \infty$ is also allowed.

The analogue of Theorem 4.1 in [Colliander et al. 2008] is Proposition 12.1. Our proof is very different and is inspired by Dodson's work [2012] on the mass-critical NLS (see also [Vişan 2012]). In [Colliander et al. 2008], this estimate is derived on the assumption that $u_{>N}$ obeys certain $L^4_{t,x}$ spacetime bounds. That the solution does admit these spacetime bounds is derived from the interaction Morawetz estimate, using the analogue of (4-3) to control certain error terms. This results in a tangled bootstrap argument across several sections of the paper. The argument that follows does not use the Morawetz identity, merely Strichartz and maximal Strichartz estimates, and so is equally valid in the focusing case. We also contend that it is simpler.

The attentive reader will discover that the implicit constant in (4-3) depends only on u through its $L_t^\infty \dot H_x^1$ norm and its modulus of compactness (cf. Definition 1.2). Indeed, the dependence on the latter can be traced to the following: Let $\eta > 0$ be a small parameter to be chosen later. Then, by Remark 1.3 and Sobolev embedding, there exists $c = c(\eta)$ such that

$$\|u_{\le cN(t)}\|_{L_t^\infty L_x^6} + \|\nabla u_{\le cN(t)}\|_{L_t^\infty L_x^2} \le \eta. \tag{4-4}$$

By elementary manipulations with the square function estimate and Lemma 2.2, respectively, we have

$$\|\nabla u_{\le N}\|_{L_t^2 L_x^6} \lesssim A(N), \quad \|u_{\le N}\|_{L_t^4 L_x^\infty} \lesssim A(N)^{1/2} \|\nabla u_{\le N}\|^{1/2}_{L_t^\infty L_x^2} \lesssim_u A(N)^{1/2}. \tag{4-5}$$

As noted earlier, the only reason for considering the Besov-type norm that appears in (4-1), rather than the simpler $L_t^2 L_x^6$ norm, is that it allows us to deduce these $L_t^4 L_x^\infty$ bounds.

By combining the Strichartz inequality (Lemma 2.1) with Lemma 1.7 we have

$$A(N)^2 \lesssim_u 1 + \int_I N(t)^2\, dt \lesssim_u \int_I N(t)^2\, dt. \tag{4-6}$$

Note that the second inequality relies on the fact that I contains at least one whole characteristic interval J_k. Similarly, using Proposition 3.1 and then Bernstein's inequality we find

$$\begin{aligned} \tilde{A}_q(N) &\lesssim N^{3/2}\Big\{ \big\| |\nabla|^{-1/2} u_{\geq N} \big\|_{L_t^\infty L_x^2} + \big\| |\nabla|^{-1/2} P_{\geq N} F(u) \big\|_{L_t^2 L_x^{6/5}} \Big\} \lesssim 1 + \|\nabla u\|_{L_t^2 L_x^6} \|u\|_{L_t^\infty L_x^6}^4 \\ &\lesssim_u \bigg(\int_I N(t)^2\, dt \bigg)^{1/2}. \end{aligned}$$

Thus

$$A(N) + \tilde{A}_q(N) \lesssim_u N^{3/2} K^{1/2} \quad \text{whenever} \quad N \geq \bigg(\frac{\int_I N(t)^2\, dt}{\int_I N(t)^{-1}\, dt} \bigg)^{1/3} \tag{4-7}$$

and so, in particular, when $N \geq N_{\max} := \sup_{t \in I} N(t)$. This is the base step for the inductive proof of Theorem 4.1. The passage to smaller values of N relies on the following:

Lemma 4.2 (recurrence relations for $A(N)$ and $\tilde{A}_q(N)$). *For η sufficiently small,*

$$A(N) \lesssim_u 1 + c^{-3/2} N^{3/2} K^{1/2} + \eta^2 \tilde{A}_q(2N) \tag{4-8}$$

$$\tilde{A}_q(N) \lesssim_u 1 + c^{-3/2} N^{3/2} K^{1/2} + \eta A(N) + \eta^2 \tilde{A}_q(2N), \tag{4-9}$$

uniformly in $N \in 2^{\mathbb{Z}}$. Here $c = c(\eta)$ as in (4-4).

Proof. The recurrence relations for $A(N)$ and $\tilde{A}_q(N)$ rely on Lemma 2.1 and Proposition 3.1, respectively. To estimate the contribution of the nonlinearity, we decompose $u(t) = u_{\leq cN(t)}(t) + u_{>cN(t)}(t)$ and then selectively $u = u_{\leq N} + u_{>N}$. Recalling that the Ø notation incorporates possible additional Littlewood–Paley projections, we may write

$$\begin{aligned} F(u) &= \text{Ø}\big(u_{>cN(t)}^2 u^3\big) + \text{Ø}\big(u_{\leq cN(t)}^2 u^3\big) \\ &= \text{Ø}\big(u_{>cN(t)}^2 u^3\big) + \text{Ø}\big(u_{\leq cN(t)}^2 u_{>N}^2 u\big) + \text{Ø}\big(u_{\leq cN(t)}^2 u_{\leq N}^2 u\big). \end{aligned} \tag{4-10}$$

Using this decomposition together with Lemma 2.1 and Bernstein's inequality, we obtain

$$\begin{aligned} A(N) &\lesssim \|\nabla u_{\leq N}\|_{L_t^\infty L_x^2} + \big\| \nabla P_{\leq N} \text{Ø}\big(u_{>cN(t)}^2 u^3\big) \big\|_{L_t^2 L_x^{6/5}} \\ &\qquad + \big\| \nabla P_{\leq N} \text{Ø}\big(u_{\leq cN(t)}^2 u_{>N}^2 u\big) \big\|_{L_t^2 L_x^{6/5}} + \big\| \nabla P_{\leq N} \text{Ø}\big(u_{\leq cN(t)}^2 u_{\leq N}^2 u\big) \big\|_{L_t^2 L_x^{6/5}} \\ &\lesssim_u 1 + N^{3/2} \|u_{>cN(t)}^2 u^3\|_{L_t^2 L_x^1} + N^{3/2} \|u_{\leq cN(t)}^2 u_{>N}^2 u\|_{L_t^2 L_x^1} + \big\| \nabla \text{Ø}\big(u_{\leq cN(t)}^2 u_{\leq N}^2 u\big) \big\|_{L_t^2 L_x^{6/5}}. \end{aligned} \tag{4-11}$$

Using instead Proposition 3.1 and Bernstein's inequality, we find

$$\begin{aligned}\tilde{A}_q(N) &\lesssim N^{3/2}\Big\{ \big\| |\nabla|^{-1/2} u_{\geq N} \big\|_{L_t^\infty L_x^2} + \|u^2_{>cN(t)} u^3\|_{L_t^2 L_x^1} + \|u^2_{\leq cN(t)} u^2_{>N} u\|_{L_t^2 L_x^1} \\ &\qquad\qquad + \big\| |\nabla|^{-1/2} P_{\geq N} \emptyset\big(u^2_{\leq cN(t)} u^2_{\leq N} u\big) \big\|_{L_t^2 L_x^{6/5}} \Big\} \\ &\lesssim_u 1 + N^{3/2} \|u^2_{>cN(t)} u^3\|_{L_t^2 L_x^1} + N^{3/2} \|u^2_{\leq cN(t)} u^2_{>N} u\|_{L_t^2 L_x^1} + \big\| \nabla \emptyset\big(u^2_{\leq cN(t)} u^2_{\leq N} u\big) \big\|_{L_t^2 L_x^{6/5}}. \end{aligned} \tag{4-12}$$

Therefore, to obtain the desired recurrence relations it remains to estimate the (identical) last three terms on the right-hand sides of (4-11) and (4-12). We will consider these terms individually, working from left to right.

To treat the first term, we decompose the time interval I into characteristic subintervals J_k where $N(t) \equiv N_k$. On each of these subintervals, we apply Hölder's inequality, Sobolev embedding, Bernstein's inequality, and Lemma 1.7 to obtain

$$\begin{aligned}\|u^2_{>cN(t)} u^3\|_{L_t^2 L_x^1(J_k \times \mathbb{R}^3)} &\lesssim \|u_{>cN_k}\|^2_{L^4_{t,x}(J_k \times \mathbb{R}^3)} \|u\|^3_{L_t^\infty L_x^6} \\ &\lesssim_u c^{-3/2} N_k^{-3/2} \|\nabla u_{>cN_k}\|^2_{L_t^4 L_x^3(J_k \times \mathbb{R}^3)} \lesssim_u c^{-3/2} N_k^{-3/2}.\end{aligned}$$

Squaring and summing the estimates above over the subintervals J_k, we find

$$N^{3/2}\, \|u^2_{>cN(t)} u^3\|_{L_t^2 L_x^1} \lesssim_u c^{-3/2} N^{3/2} K^{1/2}, \tag{4-13}$$

which is the origin of this term on the right-hand sides of (4-8) and (4-9).

To estimate the second term, we begin with a preliminary computation: Using Bernstein's inequality and Schur's test (for the last step), we estimate

$$\begin{aligned}\big\| \emptyset\big(u^2_{>N} u\big) \big\|_{L_t^2 L_x^{3/2}} &\lesssim \Big\| \sum_{\substack{M_1 \geq M_2 \geq M_3 \\ M_2 > N}} \|u_{M_1}(t)\|_{L_x^2} \|u_{M_2}(t)\|_{L_x^q} \|u_{M_3}(t)\|_{L_x^{\frac{6q}{q-6}}} \Big\|_{L_t^2} \\ &\lesssim \Big\| \sup_{M>N} \|M^{\frac{3}{q}-1} u_M(t)\|_{L_x^q} \sum_{M_1 \geq M_3} \big(\tfrac{M_3}{M_1}\big)^{3/q} \|\nabla u_{M_1}(t)\|_{L_x^2} \|\nabla u_{M_3}(t)\|_{L_x^2} \Big\|_{L_t^2} \\ &\lesssim_u N^{-3/2} \tilde{A}_q(2N).\end{aligned} \tag{4-14}$$

Using this, Hölder, and (4-4), we find

$$N^{3/2} \|u^2_{\leq cN(t)} u^2_{>N} u\|_{L_t^2 L_x^1} \lesssim N^{3/2} \|u_{\leq cN(t)}\|^2_{L_t^\infty L_x^6} \|\emptyset\big(u^2_{>N} u\big)\|_{L_t^2 L_x^{3/2}} \lesssim_u \eta^2 \tilde{A}_q(2N). \tag{4-15}$$

This is the origin of the last term on the right-hand sides of (4-8) and (4-9).

Finally, to estimate the contribution coming from the last term in (4-11) and (4-12), we distribute the gradient, use Hölder's inequality, and then (4-4) and (4-5):

$$\begin{aligned}\big\| \nabla \emptyset\big(u^2_{\leq cN(t)} u^2_{\leq N} u\big) \big\|_{L_t^2 L_x^{6/5}} &\lesssim \|\nabla u_{\leq N}\|_{L_t^2 L_x^6} \|u_{\leq cN(t)}\|_{L_t^\infty L_x^6} \|u\|^3_{L_t^\infty L_x^6} \\ &\qquad + \|\nabla u\|_{L_t^\infty L_x^2} \|u_{\leq cN(t)}\|_{L_t^\infty L_x^6} \|u_{\leq N}\|^2_{L_t^4 L_x^\infty} \|u\|_{L_t^\infty L_x^6} \\ &\lesssim_u \eta A(N).\end{aligned} \tag{4-16}$$

As $A(N)$ is known to be finite (cf. (4-6)), this can be brought to the other side of (4-8); naturally, this requires η to be sufficiently small depending on u and certain absolute constants, but not on I.

Collecting estimates (4-13) through (4-15) and choosing η sufficiently small, this completes the proof of the lemma. □

We now have all the ingredients needed to complete the proof of Theorem 4.1.

Proof of Theorem 4.1. With the base step (4-7) and Lemma 4.2 in place, Theorem 4.1 follows from a straightforward induction argument, provided η is chosen sufficiently small depending on u. □

Remark 4.3. In the introduction it was asserted that the long-time Strichartz estimates in Theorem 4.1 are essentially best possible in the focusing case. We now elaborate that point. For the energy-critical equation, the principal difficulty is to obtain control over the low frequencies, because all known conservation laws (with the exception of energy) and monotonicity formulae are energy-subcritical. If (by some miracle) we knew our putative minimal counterexample u belonged to $L_t^\infty L_x^2$, the whole argument could be brought to a swift conclusion, even in the focusing case (cf. [Killip and Vişan 2010]). Thus any potential improvement of Theorem 4.1 should be judged by whether it gives better control on the low frequencies.

It is well-known that

$$W(x) = \left(1 + \tfrac{1}{3}|x|^2\right)^{-1/2} \quad \text{obeys} \quad \Delta W + W^5 = 0 \tag{4-17}$$

and so is a static solution of the focusing energy-critical NLS. In particular, it is almost periodic with parameters $N(t) \equiv 1$ and $x(t) \equiv 0$.

As $\int W(x)^5\,dx = 4\pi\sqrt{3}$, we can read off from (4-17) that

$$\hat{W}(\xi) = 4\pi\sqrt{3}|\xi|^{-2} + O\big(|\xi|^\varepsilon\big) \quad \text{as} \quad \xi \to 0 \tag{4-18}$$

and so deduce $\|W_M\|_{L^q} \sim M^{1-3/q}$ for M small and $6 \le q \le \infty$. This shows that the supremum is essential in (4-2); we cannot expect the bound (4-3) for the sum of the Littlewood–Paley pieces. It also shows that the $L_t^2 L_x^6$ norm of $\nabla W_{\le N}$ on long time intervals decays no faster than the $N^{3/2}$ rate proved for $A(N)$.

5. Impossibility of rapid frequency cascades

In this section, we show that the first type of almost periodic solution described in Theorem 1.8 (for which $\int_0^{T_{\max}} N(t)^{-1}\,dt < \infty$) cannot exist. We will show that its existence is inconsistent with the conservation of mass, $M(u) := \int_{\mathbb{R}^3} |u(t,x)|^2\,dx$. The argument does not utilize the defocusing nature of the equation beyond the fact that the solution belongs to $L_t^\infty \dot{H}_x^1$.

Lemma 5.1 (finite mass). *Let* $u : [0, T_{\max}) \times \mathbb{R}^3 \to \mathbb{C}$ *be an almost periodic solution to* (1-1) *with* $\|u\|_{L^{10}_{t,x}([0,T_{\max})\times\mathbb{R}^3)} = +\infty$ *and*

$$K := \int_0^{T_{\max}} N(t)^{-1}\,dt < \infty. \tag{5-1}$$

(Note $T_{\max} = \infty$ is allowed.) *Then* $u \in L_t^\infty L_x^2$; *indeed, for all* $0 < N < 1$,

$$\|u_{N\le\cdot\le 1}\|_{L_t^\infty L_x^2([0,T_{\max})\times\mathbb{R}^3)} + \frac{1}{N}\Big\{\sum_{M<N}\|\nabla u_M\|^2_{L_t^2 L_x^6([0,T_{\max})\times\mathbb{R}^3)}\Big\}^{1/2} \lesssim_u 1. \tag{5-2}$$

Proof. The key point is to prove (5-2); finiteness of the mass follows easily from this. Indeed, letting $N \to 0$ in (5-2) to control the low frequencies and using $\nabla u \in L_t^\infty L_x^2$ and Bernstein for the high frequencies, we obtain

$$\|u\|_{L_t^\infty L_x^2} \le \|u_{\le 1}\|_{L_t^\infty L_x^2} + \|u_{>1}\|_{L_t^\infty L_x^2} \lesssim_u 1. \tag{5-3}$$

In the inequality above and for the remainder of the proof all spacetime norms are over $[0, T_{\max}) \times \mathbb{R}^3$.

As K is finite, the conclusion (4-3) of Theorem 4.1 extends (by exhaustion) to the time interval $[0, T_{\max})$. Observe that the second summand in (5-2) is $N^{-1}A(N/2)$, in the notation of that theorem.

We will estimate the left-hand side of (5-2) by a small multiple of itself plus a constant. For this statement to be meaningful, we need the left-hand side of (5-2) to be finite. This follows easily from Theorem 4.1 and Bernstein's inequality:

$$\mathrm{LHS}(5\text{-}2) \lesssim N^{-1}\|\nabla u\|_{L_t^\infty L_x^2} + N^{-1}A(N/2) \lesssim_u N^{-1}(1+N^3K)^{1/2} < \infty. \tag{5-4}$$

The origin of the small constant lies with the almost periodicity of the solution. Indeed, by Remark 1.3 and Sobolev embedding, for $\eta > 0$ (a small parameter to be chosen later) there exists $c = c(\eta)$ such that

$$\|u_{\le cN(t)}\|_{L_t^\infty L_x^6} + \|\nabla u_{\le cN(t)}\|_{L_t^\infty L_x^2} \le \eta. \tag{5-5}$$

To continue, fix $0 < N < 1$. Using the Duhamel formula from Proposition 1.9 together with the Strichartz inequality we obtain

$$\mathrm{LHS}(5\text{-}2) \lesssim \tfrac{1}{N}\|\nabla P_{<N}F(u)\|_{L_t^2 L_x^{6/5}} + \|P_{N\le\cdot\le 1}F(u)\|_{L_t^2 L_x^{6/5}}. \tag{5-6}$$

To estimate the nonlinearity, we decompose $u(t) = u_{\le cN(t)}(t) + u_{>cN(t)}(t)$ and then $u = u_{<N} + u_{N\le\cdot\le 1} + u_{>1}$. As the $\emptyset$ notation incorporates possible additional Littlewood–Paley projections, we may write

$$\begin{aligned} F(u) &= \emptyset\big(u^2_{>cN(t)}u^3\big) + \emptyset\big(u_{\le cN(t)}u^2_{<N}u^2\big) + \emptyset\big(u_{\le cN(t)}u^2_{\le 1}u_{N\le\cdot\le 1}u\big) \\ &\quad + \emptyset\big(u_{\le cN(t)}u^2_{>1}u^2\big). \end{aligned} \tag{5-7}$$

Next, we estimate the contributions of each of these terms to (5-6), working from left to right.

Using Bernstein's inequality and (4-13), we bound the contribution of the first term as follows:

$$\begin{aligned} \tfrac{1}{N}\big\|\nabla P_{<N}\emptyset\big(u^2_{>cN(t)}u^3\big)\big\|_{L_t^2 L_x^{6/5}} &+ \big\|P_{N\le\cdot\le 1}\emptyset\big(u^2_{>cN(t)}u^3\big)\big\|_{L_t^2 L_x^{6/5}} \\ &\lesssim (N^{1/2}+1)\|\emptyset(u^2_{>cN(t)}u^3)\|_{L_t^2 L_x^1} \\ &\lesssim_u c^{-3/2}K^{1/2}. \end{aligned}$$

To estimate the contribution of the second term in (5-7) to (5-6), we use Bernstein's inequality on the second summand and distribute the gradient, followed by Hölder's inequality, (4-5), and (5-5):

$$\begin{aligned}
&\tfrac{1}{N}\big\|\nabla P_{<N}\mathcal{O}(u_{\le cN(t)}u_{<N}^2u^2)\big\|_{L_t^2L_x^{6/5}}+\big\|P_{N\le\cdot\le 1}\mathcal{O}(u_{\le cN(t)}u_{<N}^2u^2)\big\|_{L_t^2L_x^{6/5}}\\
&\quad\lesssim \tfrac{1}{N}\|\nabla u_{\le cN(t)}\|_{L_t^\infty L_x^2}\|u_{<N}\|_{L_t^4L_x^\infty}^2\|u\|_{L_t^\infty L_x^6}^2+\tfrac{1}{N}\|u_{\le cN(t)}\|_{L_t^\infty L_x^6}\|\nabla u_{<N}\|_{L_t^2L_x^6}\|u\|_{L_t^\infty L_x^6}^3\\
&\qquad\qquad+\tfrac{1}{N}\|u_{\le cN(t)}\|_{L_t^\infty L_x^6}\|u_{<N}\|_{L_t^4L_x^\infty}^2\|\nabla u\|_{L_t^\infty L_x^2}\|u\|_{L_t^\infty L_x^6}\\
&\quad\lesssim_u \eta\,\mathrm{LHS}(5\text{-}2).
\end{aligned}$$

Using Bernstein's inequality, Theorem 4.1, (4-5), and (5-5), we estimate the contribution of the third term in (5-7) as follows:

$$\begin{aligned}
&\tfrac{1}{N}\big\|\nabla P_{<N}\mathcal{O}\big(u_{\le cN(t)}u_{\le 1}^2u_{N\le\cdot\le 1}u\big)\big\|_{L_t^2L_x^{6/5}}+\big\|P_{N\le\cdot\le 1}\mathcal{O}\big(u_{\le cN(t)}u_{\le 1}^2u_{N\le\cdot\le 1}u\big)\big\|_{L_t^2L_x^{6/5}}\\
&\quad\lesssim \|u_{\le cN(t)}\|_{L_t^\infty L_x^6}\|u_{\le 1}\|_{L_t^4L_x^\infty}^2\|u_{N\le\cdot\le 1}\|_{L_t^\infty L_x^2}\|u\|_{L_t^\infty L_x^6}\\
&\quad\lesssim_u \eta(1+K^{1/2})\,\mathrm{LHS}(5\text{-}2).
\end{aligned}$$

Finally, to estimate the contribution to (5-6) of the last term in (5-7) we use Bernstein's inequality, Theorem 4.1, (4-14), and (5-5):

$$\begin{aligned}
&\tfrac{1}{N}\big\|\nabla P_{\le N/2}\mathcal{O}\big(u_{\le cN(t)}u_{>1}^2u^2\big)\big\|_{L_t^2L_x^{6/5}}+\big\|P_{N\le\cdot\le 1}\mathcal{O}\big(u_{\le cN(t)}u_{>1}^2u^2\big)\big\|_{L_t^2L_x^{6/5}}\\
&\quad\lesssim (N^{1/2}+1)\big\|\mathcal{O}\big(u_{\le cN(t)}u_{>1}^2u^2\big)\big\|_{L_t^2L_x^1}\\
&\quad\lesssim \|u_{\le cN(t)}\|_{L_t^\infty L_x^6}\|\mathcal{O}(u_{>1}^2u)\|_{L_t^2L_x^{3/2}}\|u\|_{L_t^\infty L_x^6}\\
&\quad\lesssim_u \eta(1+K^{1/2}).
\end{aligned}$$

Collecting all the estimates above, (5-6) implies

$$\mathrm{LHS}(5\text{-}2)\lesssim_u \eta(1+K^{1/2})\,\mathrm{LHS}(5\text{-}2)+1+c^{-3/2}K^{1/2}.$$

Recalling (5-1) and (5-4) and taking η small enough depending on u and K yields (5-2). ☐

We are now ready to prove the main result of this section:

Theorem 5.2 (no rapid frequency-cascades). *There are no almost periodic solutions $u:[0,T_{\max})\times\mathbb{R}^3\to\mathbb{C}$ to* (1-1) *with $\|u\|_{L_{t,x}^{10}([0,T_{\max})\times\mathbb{R}^3)}=+\infty$ and*

$$\int_0^{T_{\max}} N(t)^{-1}\,dt<\infty. \tag{5-8}$$

Proof. We argue by contradiction. Let u be such a solution. By Corollary 1.6,

$$\lim_{t\to T_{\max}} N(t)=\infty, \tag{5-9}$$

when $T_{\max}$ is finite; this is also true when $T_{\max}$ is infinite by virtue of (5-8).

We will prove that the existence of such a solution u is inconsistent with the conservation of mass. In Lemma 5.1 we found that the mass is finite; to derive the desired contradiction we will prove that the mass is not only finite, but zero!

We first show that the mass at low frequencies is small. To do this, we use the Duhamel formula from Proposition 1.9 together with the Strichartz inequality, followed by Bernstein's inequality:

$$\|u_{\le N}\|_{L_t^\infty L_x^2} \lesssim \|P_{\le N} F(u)\|_{L_t^2 L_x^{6/5}} \lesssim N^{1/2} \|F(u)\|_{L_t^2 L_x^1}.$$

In the display above and for the remainder of the proof all spacetime norms are over $[0, T_{\max}) \times \mathbb{R}^3$.

To estimate the nonlinearity we decompose it as follows:

$$F(u) = \emptyset(u_{\le 1}^3 u^2) + \emptyset(u_{>1}^3 u^2).$$

By Theorem 4.1, (4-5), (5-8), Bernstein, and finiteness of the mass,

$$\|\emptyset(u_{\le 1}^3 u^2)\|_{L_t^2 L_x^1} \lesssim \|u_{\le 1}\|_{L_t^4 L_x^\infty}^2 \|u_{\le 1}\|_{L_{t,x}^\infty} \|u\|_{L_t^\infty L_x^2}^2 \lesssim_u 1,$$

while by Theorem 4.1, (4-14), and (5-8),

$$\|\emptyset(u_{>1}^3 u^2)\|_{L_t^2 L_x^1} \lesssim \|u\|_{L_t^\infty L_x^6}^2 \|u_{>1}^3\|_{L_t^2 L_x^{3/2}} \lesssim_u 1.$$

Thus,

$$\|u_{\le N}\|_{L_t^\infty L_x^2} \lesssim_u N^{1/2}.$$

By comparison, control over the mass at middle and high frequencies can be obtained with just Bernstein's inequality and the fact that for any $\eta > 0$ there exists $c = c(u, \eta) > 0$ so that

$$\|\nabla u_{\le cN(t)}(t)\|_{L_x^2} \le \eta,$$

which was noted in Remark 1.3. Altogether, we have that for any $t \in [0, T_{\max})$,

$$\begin{aligned}
\|u(t)\|_{L_x^2} &\lesssim \|u_{\le N}(t)\|_{L_x^2} + \|P_{>N} u_{\le cN(t)}(t)\|_{L_x^2} + \|u_{>cN(t)}(t)\|_{L_x^2} \\
&\lesssim_u N^{1/2} + N^{-1} \|\nabla u_{\le cN(t)}(t)\|_{L_x^2} + c^{-1} N(t)^{-1} \|\nabla u\|_{L_t^\infty L_x^2} \\
&\lesssim_u N^{1/2} + N^{-1}\eta + c^{-1} N(t)^{-1}.
\end{aligned}$$

Using (5-9), we can make the right-hand side here as small as we wish. (Choose N small, then η small, and then t close to $T_{\max}$.) Because mass is conserved under the flow, this allows us to conclude that $\|u\|_{L_t^\infty L_x^2} = 0$ and thus $u \equiv 0$ in contradiction to the hypothesis $\|u\|_{L_{t,x}^{10}([0,T_{\max})\times\mathbb{R}^3)} = +\infty$. □

6. The frequency-localized interaction Morawetz inequality

In this section, we prove a spacetime bound on the high-frequency portion of the solution:

Theorem 6.1 (a frequency-localized interaction Morawetz estimate). *Suppose $u : [0, T_{\max}) \times \mathbb{R}^3 \to \mathbb{C}$ is an almost periodic solution to* (1-1) *such that $N(t) \ge 1$ and let $I \subset [0, T_{\max})$ be a union of contiguous* *characteristic intervals J_k. Fix $0 < \eta_0 \le 1$. For $N > 0$ sufficiently small (depending on η_0 but not on I),*

$$\int_I \int_{\mathbb{R}^3} |u_{>N}(t,x)|^4 \, dx \, dt \lesssim_u \eta_0 \big(N^{-3} + K \big), \tag{6-1}$$

where $K := \int_I N(t)^{-1} \, dt$. Importantly, the implicit constant in the inequality above does not depend on η_0 or the interval I.

Unlike Theorem 4.1, the argument does not rely solely on estimates for the linear propagator and is not indifferent to the sign of the nonlinearity. Instead, we use a special monotonicity formula associated with (1-1), namely, the interaction Morawetz identity. This is a modification of the traditional Morawetz identity (cf. [Lin and Strauss 1978; Morawetz 1975]) introduced in [Colliander et al. 2004]. We begin with a general form of the identity:

Proposition 6.2. *Suppose $i\partial_t \phi = -\Delta\phi + |\phi|^4\phi + \mathcal{F}$ and let*

$$M(t) := 2 \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |\phi(y)|^2 a_k(x-y) \operatorname{Im}\{\phi_k(x)\bar\phi(x)\} \, dx \, dy, \tag{6-2}$$

for some weight $a : \mathbb{R}^d \to \mathbb{R}$. Then

$$\partial_t M(t) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \Big\{ \tfrac{4}{3} a_{kk}(x-y) |\phi(x)|^6 |\phi(y)|^2 \tag{6-3}$$

$$+ 2 a_k(x-y) |\phi(y)|^2 \operatorname{Re}\big[\phi_k(x)\bar{\mathcal{F}}(x) - \mathcal{F}_k(x)\bar\phi(x)\big] \tag{6-4}$$

$$+ 4 a_k(x-y) (\operatorname{Im} \mathcal{F}(y)\bar\phi(y)) (\operatorname{Im} \phi_k(x)\bar\phi(x)) \tag{6-5}$$

$$+ 4 a_{jk}(x-y) \big[|\phi(y)|^2 \bar\phi_j(x)\phi_k(x) - (\operatorname{Im}\bar\phi(y)\phi_j(y))(\operatorname{Im}\bar\phi(x)\phi_k(x)) \big] \tag{6-6}$$

$$- a_{jjkk}(x-y) \, |\phi(y)|^2 |\phi(x)|^2 \Big\} \, dx \, dy. \tag{6-7}$$

Subscripts denote spatial derivatives and repeated indices are summed.

The significance of this identity to our problem is best seen by choosing $a(x) = |x|$ and ϕ to be a solution to (1-1). In this case, $\mathcal{F} = 0$ and the fundamental theorem of calculus yields

$$8\pi \int_I \int_{\mathbb{R}^3} |\phi(t,x)|^4 \, dx \, dt \le 2 \|M(t)\|_{L_t^\infty(I)} \le 4 \|\phi\|^3_{L_t^\infty L_x^2(I\times\mathbb{R}^3)} \|\phi\|_{L_t^\infty \dot H_x^1(I\times\mathbb{R}^3)}.$$

The left-hand side originates from (6-7); the terms (6-6) and (6-3) are both positive.

Unfortunately for us, a minimal blowup solution need not have finite L_x^2 norm at any time. Thus it is necessary to localize the identity to high frequencies, that is, choose $\phi = u_{>N}$. Naturally, this produces myriad error terms; nevertheless, in spatial dimensions four and higher they can be controlled (cf. [Ryckman and Vişan 2007; Vişan 2007; 2012]). In the three-dimensional case under consideration here, there is one error term (originating from (6-5)) that cannot be satisfactorily controlled. (See also Remark 6.9 at the end of this section.) This was observed already in [Colliander et al. 2008] and as there, our solution is to truncate the function a. This truncation ruins the convexity properties of a that made some of the terms in Proposition 6.2 positive, thus creating more error terms to control.

For reasons we will explain in due course, it is important to perform the cutoff of a in a very careful fashion. We choose a to be a smooth spherically symmetric function, which we regard interchangeably as a function of $x \in \mathbb{R}^3$ or $r = |x|$. We specify it further in terms of its radial derivative:

$$a(0) = 0, \quad a_r \geq 0, \quad a_{rr} \leq 0, \quad a_r = \begin{cases} 1 & \text{if } r \leq R, \\ 1 - J^{-1}\log(r/R) & \text{if } eR \leq r \leq e^{J-J_0}R, \\ 0 & \text{if } e^J R \leq r, \end{cases} \tag{6-8}$$

where $J_0 \geq 1$, $J \geq 2J_0$, and R are parameters that will be determined in due course. It is not difficult to see that one may fill in the regions where a_r is not yet defined so that the function obeys

$$|\partial_r^k a_r| \lesssim_k J^{-1} r^{-k} \qquad \text{for each } k \geq 1, \tag{6-9}$$

uniformly in r and in the choice of parameters.

When $|x| \leq R$, we see that $a(x) = |x|$, while a is a constant when $|x| \geq e^J R$. The key point about the transition between these two regimes is that

$$\frac{2}{r} a_r \geq \frac{2J_0}{Jr} \quad \text{but} \quad |a_{rr}| \leq \frac{1}{Jr} \tag{6-10}$$

when $eR \leq r \leq e^{J-J_0}R$. Thus the Laplacian $a_{kk} = a_{rr} + \frac{2}{r}a_r$ is dominated by the first derivative term and so remains coercive at these radii. (This also appears implicitly in [Colliander et al. 2008, §11] and is the key point behind the "averaging over R" argument there.)

As noted above, we will be applying Proposition 6.2 with

$$\phi = u_{\mathrm{hi}} := u_{>N}, \quad \text{and so} \quad \mathcal{F} = P_{\mathrm{hi}} F(u) - F(u_{\mathrm{hi}}). \tag{6-11}$$

(We will also write $u_{\mathrm{lo}} := u_{\leq N}$.) Here N is an additional parameter that will be chosen small (depending on η_0 and u). We require that N, R, and J are related via

$$e^J RN = 1. \tag{6-12}$$

Actually, it is merely essential that $e^J RN \leq 1$, but choosing equality makes the exposition simpler. Our first restriction on these parameters is that N is small enough and R is large enough so that given $\eta = \eta(\eta_0, u)$,

$$\int_{\mathbb{R}^3} |\nabla u_{\mathrm{lo}}(t,x)|^2 \, dx + \int_{\mathbb{R}^3} |N u_{\mathrm{hi}}(t,x)|^2 \, dx + \int_{|x-x(t)|>\frac{R}{2}} |\nabla u_{\mathrm{hi}}(t,x)|^2 \, dx < \eta^2 \tag{6-13}$$

uniformly for $0 \leq t < T_{\max}$. The possibility of doing this follows immediately from the fact that u is almost periodic modulo symmetries and $N(t) \geq 1$.

Before moving on to estimating the terms in Proposition 6.2, we pause to review the tools at our disposal. Besides using the norm $\|u_{\mathrm{hi}}\|_{L^4_{t,x}}$ to estimate itself, we will also make recourse to Theorem 4.1 and Proposition 3.2. For ease of reference, we record these results in the forms we will use:

Corollary 6.3 (a priori bounds). *For all* $\frac{2}{q}+\frac{3}{r}=\frac{3}{2}$ *with* $2\le q\le\infty$ *and any* $s<1-\frac{3}{q}$,

$$\|\nabla u_{\mathrm{lo}}\|_{L_t^qL_x^r}+\big\|N^{1-s}|\nabla|^s u_{\mathrm{hi}}\big\|_{L_t^qL_x^r}\lesssim_u \big(1+N^3K\big)^{1/q}. \tag{6-14}$$

Under the hypothesis (6-13),

$$\|u_{\mathrm{lo}}\|_{L_t^4L_x^\infty}\lesssim_u \eta^{1/2}(1+N^3K)^{1/4}. \tag{6-15}$$

Furthermore, for any $\rho\le Re^J=N^{-1}$,

$$\int_I \sup_{x\in\mathbb{R}^3}\int_{|x-y|\le\rho}|u_{\mathrm{hi}}(t,y)|^2\,dy\,dt \;\lesssim_u\; \rho\big(K+N^{-3}\big). \tag{6-16}$$

Proof. Recall that Theorem 4.1 implies

$$A(M):=\Big\{\sum_{M'\le M}\|\nabla u_{M'}\|^2_{L_t^2L_x^6(I\times\mathbb{R}^3)}\Big\}^{1/2}\lesssim_u (1+M^3K)^{1/2}$$

uniformly in M. Setting $M=N$ yields all the estimates on u_{lo} stated in the corollary. More explicitly, the $q=2$ case of (6-14) as well as (6-15) follow from this statement and (4-5). The other values of q can then be deduced by interpolation with the (conserved) energy.

Similarly, to estimate u_{hi} we write

$$M^{1-s}\||\nabla|^s u_M\|_{L_t^qL_x^r}\lesssim\|\nabla u_M\|_{L_t^qL_x^r}\lesssim A(M)^{2/q}\|\nabla u\|^{(q-2)/q}_{L_t^\infty L_x^2}\lesssim_u (1+M^3K)^{1/q},$$

multiply through by M^{s-1}, and sum over $M\ge N$. Notice that the condition $\frac{3}{q}+s<1$ guarantees the convergence of this sum.

Claim (6-16) will follow by combining Proposition 3.2 and Theorem 4.1. First we write $(i\partial_t+\Delta)u_{>N}=F+G$ with $F=P_{>N}\mathcal{O}(u^2_{>N}u^3)$ and $G=P_{>N}\mathcal{O}(u^4_{\le N}u)$ and then estimate these as follows: By Theorem 4.1 and (4-14),

$$\|F\|_{L_t^2L_x^1}\lesssim\|u\|^2_{L_t^\infty L_x^6}\|\mathcal{O}(u^2_{>N}u)\|_{L_t^2L_x^{3/2}}\lesssim_u N^{-3/2}+K^{1/2},$$

while by Bernstein, Theorem 4.1, and (4-5),

$$\begin{aligned}\|G\|_{L_t^2L_x^{6/5}}&\lesssim N^{-1}\|\nabla\mathcal{O}(u^4_{\le N}u)\|_{L_t^2L_x^{6/5}}\lesssim N^{-1}\|\nabla u\|_{L_t^\infty L_x^2}\|u\|^2_{L_t^\infty L_x^6}\|u_{\le N}\|^2_{L_t^4L_x^\infty}\\&\lesssim_u N^{-1}+N^{1/2}K^{1/2}.\end{aligned}$$

Putting these together with Proposition 3.2 yields

$$\rho^{1/2}\big\|\mathcal{S}u_{>N}\big\|_{L_t^2L_x^\infty(I\times\mathbb{R}^3)}\lesssim_u N^{-1}+\big(N^{1/2}+\rho^{-1/2}\big)\big(K+N^{-3}\big)^{1/2}.$$

Noting from (3-1) that, modulo a factor of $\rho^{-3/2}$, $\mathcal{S}u_{>N}(t,x)$ controls the L_x^2 norm on the ball around x, and recalling the restriction on ρ, we deduce the claim. □

We now will analyze the individual terms in Proposition 6.2, beginning with the most important one:

Lemma 6.4 (mass-mass interactions).

$$8\pi \|u_{\mathrm{hi}}\|^4_{L^4_{t,x}(I\times\mathbb{R}^3)} - \int_I\iint -a_{jjkk}(x-y)\,|u_{\mathrm{hi}}(y)|^2|u_{\mathrm{hi}}(x)|^2\,dx\,dy\,dt \lesssim_u \frac{\eta^2 e^{2J}}{J}(K+N^{-3}).$$

Proof. In three dimensions, $\Delta|x| = 2|x|^{-1}$ and $-(4\pi|x|)^{-1}$ is the fundamental solution of Laplace's equation. In this way, we are left to estimate the error terms originating from the truncation of a at radii $|x-y|\geq R$. Combining (6-9) and (6-16) yields

$$\int_I\iint_{|x-y|\geq R}\bigl|a_{jjkk}(x-y)\bigr|\,|u_{\mathrm{hi}}(y)|^2|u_{\mathrm{hi}}(x)|^2\,dx\,dy\,dt \lesssim_u J^{-1}\|u_{\mathrm{hi}}\|^2_{L^\infty_t L^2_x}\sum_{j=0}^{J}(Re^j)^{-3}(Re^j)(K+N^{-3}).$$

To obtain the lemma, we simply invoke (6-13) as well as (6-12). □

The second most important term originates from (6-3). Its importance stems from the fact that it contains additional coercivity that we will use to estimate other error terms below.

Lemma 6.5. *We estimate* (6-3) *in two pieces*:

$$B_I := \int_I\iint_{|x-y|\leq e^{J-J_0}R}\tfrac{4}{3}a_{kk}(x-y)|u_{\mathrm{hi}}(x)|^6|u_{\mathrm{hi}}(y)|^2\,dx\,dy\,dt \geq 0, \tag{6-17}$$

as $a_{kk}\geq 0$ *there*, *and on the complementary region*,

$$\int_I\iint_{|x-y|\geq e^{J-J_0}R}|a_{kk}(x-y)||u_{\mathrm{hi}}(x)|^6|u_{\mathrm{hi}}(y)|^2\,dx\,dy\,dt \lesssim \tfrac{J_0^2}{J}\bigl(K+N^{-3}\bigr). \tag{6-18}$$

Proof. That $a_{kk}\geq 0$ and hence $B_I\geq 0$ is immediate from (6-10). Further, by construction, $|a_{kk}|\lesssim J_0(Jr)^{-1}$ when $r\geq e^{J-J_0}R$. In this way, we see that (6-18) relies only on controlling

$$\int_I\iint_{e^{J-J_0}R\leq|x-y|\leq e^JR}\frac{J_0|u_{\mathrm{hi}}(x)|^6|u_{\mathrm{hi}}(y)|^2}{J|x-y|}\,dx\,dy\,dt,$$

which by (6-16) is

$$\lesssim_u J_0\|u_{\mathrm{hi}}\|^6_{L^\infty_t L^6_x}\sum_{j=J-J_0}^{J}(Je^jR)^{-1}\cdot(e^jR)\bigl(K+N^{-3}\bigr) \lesssim_u \tfrac{J_0^2}{J}\bigl(K+N^{-3}\bigr),$$

as needed. □

Now we come to the most dangerous-looking term, (6-6). Satisfactory control relies on the full strength of (6-10).

Lemma 6.6. *Let*

$$\Phi_{jk}(x,y) := |u_{\mathrm{hi}}(y)|^2\partial_j\overline{u_{\mathrm{hi}}}(x)\partial_k u_{\mathrm{hi}}(x) - (\operatorname{Im}\overline{u_{\mathrm{hi}}}(y)\partial_j u_{\mathrm{hi}}(y))(\operatorname{Im}\overline{u_{\mathrm{hi}}}(x)\partial_k u_{\mathrm{hi}}(x)).$$

Then

$$-\int_I\iint 4a_{jk}(x-y)\Phi_{jk}(x,y)dx\,dy\,dt \lesssim_u \Bigl(\eta^2+\frac{J_0}{J}\Bigr)(K+N^{-3}) + \tfrac{1}{J_0}B_I.$$

(*For the* B_I *notation*, *refer to* (6-17).)

Proof. As $a_{jk}(x-y)$ is invariant under $x \leftrightarrow y$, we may replace Φ by the matrix

$$\tfrac{1}{2}\Phi_{jk}(x,y) + \tfrac{1}{2}\Phi_{jk}(y,x),$$

which is Hermitian-symmetric. Moreover, for each x, y this matrix defines a positive semidefinite quadratic form on $\mathbb{R}^3$. To see this, notice that for any vector $e \in \mathbb{R}^3$ and any function ϕ,

$$\begin{aligned}\bigl|e_k e_j(\operatorname{Im}\bar{\phi}(y)\phi_j(y))(\operatorname{Im}\bar{\phi}(x)\phi_k(x))\bigr| &\le |\phi(y)|\,|e\cdot\nabla\phi(y)|\,|\phi(x)|\,|e\cdot\nabla\phi(x)|\\ &\le \tfrac{1}{2}|\phi(x)|^2|e\cdot\nabla\phi(y)|^2 + \tfrac{1}{2}|\phi(y)|^2|e\cdot\nabla\phi(x)|^2.\end{aligned}$$

As a_{jk} is a real symmetric matrix (for any x and y), its eigenvectors are real. Thus, wherever a_{jk} is positive semidefinite (i.e., a is convex), the integrand has a favorable sign. In general, the eigenvalues of the Hessian of a spherically symmetric function are a_{rr} and $r^{-1}a_r$ with the latter having multiplicity two (ambient dimension minus one). In our case $a_r \ge 0$ and $|a_{rr}| \lesssim J^{-1}r^{-1}$. Therefore, we are left to estimate

$$\int_I \iint_{R<|x-y|<e^J R} \frac{|\nabla u_{\mathrm{hi}}(x)|^2|u_{\mathrm{hi}}(y)|^2}{J|x-y|}\,dx\,dy\,dt. \tag{6-19}$$

To do this, we break the integral into two regions: $|x-x(t)| > R/2$ and $|x-x(t)| \le R/2$. In the former case, we use (6-13) and (6-16) to obtain the bound

$$\lesssim_u \|\nabla u_{\mathrm{hi}}\|^2_{L_t^\infty L_x^2(|x-x(t)|>R/2)} \sum_{j=0}^{J}(Je^jR)^{-1}\cdot(e^jR)\bigl(K+N^{-3}\bigr) \lesssim_u \eta^2\bigl(K+N^{-3}\bigr).$$

When $|x-x(t)| \le R/2$, we further subdivide into two regions. When additionally $|x-y| \ge Re^{J-J_0}$, we estimate in much the same manner as above to obtain the bound

$$\lesssim_u \|\nabla u\|^2_{L_t^\infty L_x^2} \sum_{j=J-J_0}^{J}(Je^jR)^{-1}\cdot(e^jR)\bigl(K+N^{-3}\bigr) \lesssim_u \tfrac{J_0}{J}\bigl(K+N^{-3}\bigr).$$

This leaves us to consider the integral (6-19) over the region where $|x-x(t)| \le R/2$ and $|x-y| < Re^{J-J_0}$. Here we use the fact that by the almost periodicity of u (cf. also Remark 1.3 and (6-13)),

$$\int_{\mathbb{R}^3} |\nabla u_{\mathrm{hi}}(t,x)|^2\,dx \lesssim_u \int_{\mathbb{R}^3} |u_{\mathrm{hi}}(t,x)|^6\,dx \lesssim_u \int_{|x-x(t)|\le R/2} |u_{\mathrm{hi}}(t,x)|^6\,dx,$$

uniformly for $t \in [0, T_{\max})$. We also observe from (6-10) that $J_0(Jr)^{-1} \le a_{kk}$; recall $J_0 \ge 1$. Therefore, the remaining integral is $\lesssim_u \frac{1}{J_0}B_I$. □

The terms appearing in (6-4) are referred to as momentum bracket terms on account of the notation

$$\{\mathcal{F},\phi\}_p := \operatorname{Re}(\mathcal{F}\nabla\bar{\phi} - \phi\nabla\bar{\mathcal{F}}). \tag{6-20}$$

Note that applying Proposition 6.2 with $\phi = u_{>N}$ gives $\mathcal{F} = P_{\mathrm{hi}}F(u) - F(u_{\mathrm{hi}})$. These error terms are comparatively easy to control:

Lemma 6.7 (Momentum bracket terms). *For any* $\varepsilon \in (0, 1]$,

$$\int_I \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |u_{\mathrm{hi}}(t, y)|^2 \nabla a(x-y) \cdot \{\mathcal{F}, \phi\}_p \, dx \, dy \, dt \lesssim_u \varepsilon B_I + \eta \|u_{\mathrm{hi}}\|_{L^4_{t,x}}^4 + \big(\varepsilon^{-1}\eta + \varepsilon \tfrac{J_0^2}{J}\big)(N^{-3} + K). \tag{6-21}$$

Proof. We begin by expanding the momentum bracket into several terms. First, we note that $\{F(\phi), \phi\}_p = -\frac{2}{3}\nabla|\phi|^6$ and so

$$\{\mathcal{F}, u_{\mathrm{hi}}\}_p = -\tfrac{2}{3}\nabla\big(|u|^6 - |u_{\mathrm{lo}}|^6 - |u_{\mathrm{hi}}|^6\big) - \{F(u) - F(u_{\mathrm{lo}}), u_{\mathrm{lo}}\}_p - \{P_{\mathrm{lo}} F(u), u_{\mathrm{hi}}\}_p.$$

Then, using $\{f, g\}_p = \nabla(fg) + \text{Ø}(f \nabla g)$, we obtain

$$\{\mathcal{F}, u_{\mathrm{hi}}\}_p = \nabla \sum_{j=1}^{5} O(u_{\mathrm{hi}}^j u_{\mathrm{lo}}^{6-j}) + \text{Ø}(u^2 u_{\mathrm{hi}} u_{\mathrm{lo}}^2 \nabla u_{\mathrm{lo}}) + \text{Ø}(u^3 u_{\mathrm{hi}}^2 \nabla u_{\mathrm{lo}}) + \nabla \text{Ø}(u_{\mathrm{hi}} P_{\mathrm{lo}} F(u)) + \text{Ø}(u_{\mathrm{hi}} \nabla P_{\mathrm{lo}} F(u)). \tag{6-22}$$

We will treat each of these terms in succession. The presence of the gradient in front of a term is a signal that we will integrate by parts in (6-21) before estimating its contribution.

We begin with the first term in (6-22). Integrating by parts and using

$$\sum_{j=1}^{5} |u_{\mathrm{hi}}|^j |u_{\mathrm{lo}}|^{6-j} \lesssim \varepsilon |u_{\mathrm{hi}}|^6 + \varepsilon^{-1} |u_{\mathrm{lo}}|^2 |u_{\mathrm{hi}}| \big[|u_{\mathrm{hi}}| + |u_{\mathrm{lo}}|\big]^3,$$

we find that we need to obtain satisfactory estimates for

$$\varepsilon \int_I \iint |a_{kk}(x-y)| |u_{\mathrm{hi}}(t, y)|^2 |u_{\mathrm{hi}}(t, x)|^6 \, dx \, dy \, dt, \tag{6-23}$$

which follow already from Lemma 6.5, and for

$$\int_I \iint \frac{|u_{\mathrm{hi}}(t, y)|^2 |u_{\mathrm{lo}}(t, x)|^2 |u_{\mathrm{hi}}(t, x)| [|u_{\mathrm{hi}}(t, x)| + |u_{\mathrm{lo}}(t, x)|]^3}{\varepsilon |x-y|} \, dx \, dy \, dt. \tag{6-24}$$

(To obtain this compact form, we use the fact that $|a_{kk}(x-y)| \lesssim |x-y|^{-1}$.) To bound this second integral, we use the Hölder and Hardy–Littlewood–Sobolev inequalities, as well as Corollary 6.3 and (6-13):

$$\begin{aligned} \text{(6-24)} &\lesssim \varepsilon^{-1} \big\| |x|^{-1} * |u_{\mathrm{hi}}|^2 \big\|_{L^4_t L^6_x} \|u_{\mathrm{hi}}\|_{L^4_t L^3_x} \|u_{\mathrm{lo}}\|_{L^4_t L^\infty_x}^2 \|u\|_{L^\infty_t L^6_x}^3 \\ &\lesssim \varepsilon^{-1} \|u_{\mathrm{hi}}\|_{L^\infty_t L^2_x} \|u_{\mathrm{hi}}\|_{L^4_t L^3_x}^2 \|u_{\mathrm{lo}}\|_{L^4_t L^\infty_x}^2 \|u_{\mathrm{lo}}\|_{L^\infty_t L^6_x}^3 \\ &\lesssim_u \varepsilon^{-1} \eta (N^{-3} + K). \end{aligned}$$

We now move on to estimating the contribution of the second term in (6-22). This is easily estimated using Corollary 6.3:

$$\|\text{Ø}(u^2 u_{\mathrm{hi}} u_{\mathrm{lo}}^2 \nabla u_{\mathrm{lo}})\|_{L^1_{t,x}} \lesssim \|u_{\mathrm{hi}}\|_{L^\infty_t L^2_x} \|\nabla u_{\mathrm{lo}}\|_{L^2_t L^6_x} \|u_{\mathrm{lo}}\|_{L^4_t L^\infty_x}^2 \|u\|_{L^\infty_t L^6_x}^2 \lesssim_u \eta N^{-1}(1 + N^3 K).$$

This takes the desired form when multiplied by

$$\int_{\mathbb{R}^3} |u_{\mathrm{hi}}(t,y)|^2\,dy \lesssim_u \eta^2 N^{-2}. \tag{6-25}$$

To control the third term in (6-22), we use Bernstein together with Corollary 6.3:

$$\begin{aligned}\|\emptyset(u^3 u_{\mathrm{hi}}^2 \nabla u_{\mathrm{lo}})\|_{L^1_{t,x}} &\lesssim \|\nabla u_{\mathrm{lo}}\|_{L^2_t L^\infty_x} \|u_{\mathrm{hi}}\|^2_{L^4_{t,x}} \|u\|^3_{L^\infty_t L^6_x}\\ &\lesssim_u N^{1/2}\|\nabla u_{\mathrm{lo}}\|_{L^2_t L^6_x}\|u_{\mathrm{hi}}\|^2_{L^4_{t,x}}\\ &\lesssim_u N^2\|u_{\mathrm{hi}}\|^4_{L^4_{t,x}} + N^{-1}(1+N^3K).\end{aligned}$$

Next, we estimate the contribution from the fourth term in (6-22), which, after integration by parts, this takes the form

$$-\int_I\iint |u_{\mathrm{hi}}(t,y)|^2 a_{kk}(x-y)\emptyset\big(u_{\mathrm{hi}} P_{\mathrm{lo}} F(u)\big)(t,x)\,dx\,dy\,dt.$$

To continue, we write $u_{\mathrm{hi}}(t,x) = \operatorname{div}(\nabla\Delta^{-1}u_{\mathrm{hi}}(t,x))$ and integrate by parts once more. This breaks the contribution into two parts; after applying Hölder's inequality and the Mikhlin multiplier theorem, the total contribution is bounded by

$$\big\| |x|^{-1} * |u_{\mathrm{hi}}|^2\big\|_{L^4_t L^{12}_x} \| |\nabla|^{-1} u_{\mathrm{hi}}\|_{L^2_t L^6_x} \|\nabla P_{\mathrm{lo}} F(u)\|_{L^4_t L^{4/3}_x} \tag{6-26}$$

$$+ \big\| |x|^{-2} * |u_{\mathrm{hi}}|^2\big\|_{L^4_t L^{12/5}_x} \| |\nabla|^{-1} u_{\mathrm{hi}}\|_{L^2_t L^6_x} \| P_{\mathrm{lo}} F(u)\|_{L^4_t L^{12/5}_x}. \tag{6-27}$$

Applying the Hardy–Littlewood–Sobolev inequality to the first factor in each term and using Sobolev embedding on the very last factor, yields

$$(6\text{-}26) + (6\text{-}27) \lesssim \big\| |u_{\mathrm{hi}}|^2\big\|_{L^4_t L^{4/3}_x} \big\| |\nabla|^{-1} u_{\mathrm{hi}}\big\|_{L^2_t L^6_x} \|\nabla P_{\mathrm{lo}} F(u)\|_{L^4_t L^{4/3}_x}. \tag{6-28}$$

To estimate $\nabla P_{\mathrm{lo}} F(u)$, we decompose $F(u) = F(u_{\mathrm{lo}}) + \emptyset(u_{\mathrm{hi}} u^4)$. Using Hölder, Bernstein, and Corollary 6.3, we obtain

$$\|\nabla P_{\mathrm{lo}} F(u_{\mathrm{lo}})\|_{L^4_t L^{4/3}_x} \lesssim N^{3/4}\|\nabla F(u_{\mathrm{lo}})\|_{L^4_t L^1_x} \lesssim N^{3/4}\|\nabla u_{\mathrm{lo}}\|_{L^4_t L^3_x}\|u_{\mathrm{lo}}\|^4_{L^\infty_t L^6_x} \lesssim_u N^{3/4}(1+N^3K)^{1/4},$$

$$\|\nabla P_{\mathrm{lo}} \emptyset(u_{\mathrm{hi}} u^4)\|_{L^4_t L^{4/3}_x} \lesssim N^{3/2}\|u_{\mathrm{hi}} u^4\|_{L^4_t L^{12/11}_x} \lesssim N^{3/2}\|u_{\mathrm{hi}}\|_{L^4_{t,x}}\|u\|^4_{L^\infty_t L^6_x} \lesssim_u N^{3/2}\|u_{\mathrm{hi}}\|_{L^4_{t,x}},$$

Putting these together with Corollary 6.3 and (6-13) yields

$$\begin{aligned}(6\text{-}28) &\lesssim \|u_{\mathrm{hi}}\|_{L^\infty_t L^2_x}\|u_{\mathrm{hi}}\|_{L^4_{t,x}}\| |\nabla|^{-1}u_{\mathrm{hi}}\|_{L^2_t L^6_x}\big(N^{3/4}(1+N^3K)^{1/4} + N^{3/2}\|u_{\mathrm{hi}}\|_{L^4_{t,x}}\big)\\ &\lesssim_u \eta N^{-1}\|u_{\mathrm{hi}}\|_{L^4_{t,x}} N^{-2}(1+N^3K)^{1/2}\big(N^{3/4}(1+N^3K)^{1/4} + N^{3/2}\|u_{\mathrm{hi}}\|_{L^4_{t,x}}\big)\\ &\lesssim_u \eta\left(\|u_{\mathrm{hi}}\|^4_{L^4_{t,x}} + (N^{-3}+K)\right)\end{aligned}$$

For the fifth (and last) term in (6-22), we again write $u_{\mathrm{hi}} = \operatorname{div}(\nabla\Delta^{-1}u_{\mathrm{hi}})$. After integrating by parts once, the contribution splits into two pieces, one of which is controlled by (6-26) and another which we bound by

$$\big\|(\nabla a) * |u_{\mathrm{hi}}|^2\big\|_{L^\infty_t L^\infty_x}\big\| |\nabla|^{-1}u_{\mathrm{hi}}\big\|_{L^2_t L^6_x}\|\Delta P_{\mathrm{lo}} F(u)\|_{L^2_t L^{6/5}_x}. \tag{6-29}$$

We now decompose $F(u) = F(u_{\rm lo}) + \emptyset(u_{\rm hi}u_{\rm lo}^2 u^2) + \emptyset(u_{\rm hi}^2 u^3)$. Using the Hölder and Bernstein inequalities, we deduce

$$\|\Delta P_{\rm lo} F(u_{\rm lo})\|_{L_t^2 L_x^{6/5}} \lesssim N\|\nabla u_{\rm lo}\|_{L_t^2 L_x^6} \|u_{\rm lo}\|_{L_t^\infty L_x^6}^4 \lesssim_u N(1+N^3K)^{1/2},$$
$$\|\Delta P_{\rm lo}\emptyset(u_{\rm hi}u_{\rm lo}^2u^2)\|_{L_t^2 L_x^{6/5}} \lesssim N^2\|u_{\rm hi}\|_{L_t^\infty L_x^2}\|u_{\rm lo}\|_{L_t^4 L_x^\infty}^2 \|u\|_{L_t^\infty L_x^6}^2 \lesssim_u N(1+N^3K)^{1/2},$$

and

$$\|\Delta P_{\rm lo}\emptyset(u_{\rm hi}^2u^3)\|_{L_t^2 L_x^{6/5}} \lesssim N^{5/2}\|u_{\rm hi}\|_{L_{t,x}^4}^2 \|u\|_{L_t^\infty L_x^6}^3 \lesssim_u N^{5/2}\|u_{\rm hi}\|_{L_{t,x}^4}^2.$$

Putting it all together we find

$$\begin{aligned}(6\text{-}29) &\lesssim \|u_{\rm hi}\|_{L_t^\infty L_x^2}^2 N^{-2}(1+N^3K)^{1/2}\big(N(1+N^3K)^{1/2} + N^{5/2}\|u_{\rm hi}\|_{L_{t,x}^4}^2\big)\\ &\lesssim_u \eta^2\big(\|u_{\rm hi}\|_{L_{t,x}^4}^4 + (N^{-3}+K)\big).\end{aligned}$$

With the last term estimated satisfactorily, the proof of Lemma 6.7 is now complete. □

Looking back to Proposition 6.2, we are left with just one term in $\partial_t M(t)$ to estimate, namely, (6-5). As in [Colliander et al. 2008], we call this the mass (Poisson) bracket term and use the notation

$$\{\mathcal{F}, \phi\}_m := \operatorname{Im}(\mathcal{F}\bar\phi).$$

Notice that $\{|\phi|^4\phi, \phi\}_m = 0$ for any function ϕ.

Lemma 6.8 (mass bracket terms). *For any $\varepsilon > 0$,*

$$\left|\operatorname{Im}\int_I\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\{\mathcal{F}, u_{\rm hi}\}_m(t,y)\nabla a(x-y)\cdot\nabla u_{\rm hi}(t,x)\overline{u_{\rm hi}(t,x)}\,dx\,dy\,dt\right| \lesssim \eta^{1/4}\big(\|u_{\rm hi}\|_{L_{t,x}^4}^4 + N^{-3} + K\big). \tag{6-30}$$

Proof. Exploiting the cancellation noted above and

$$F(u) - F(u_{\rm hi}) - F(u_{\rm lo}) = \emptyset\big(u_{\rm lo}u_{\rm hi}u^3\big),$$

we write

$$\begin{aligned}\{\mathcal{F}, u_{\rm hi}\}_m &= \{P_{\rm hi}F(u) - F(u_{\rm hi}), u_{\rm hi}\}_m\\ &= \{P_{\rm hi}[F(u) - F(u_{\rm hi}) - F(u_{\rm lo})], u_{\rm hi}\}_m - \{P_{\rm lo}F(u_{\rm hi}), u_{\rm hi}\}_m + \{P_{\rm hi}F(u_{\rm lo}), u_{\rm hi}\}_m\\ &= \emptyset\big(u_{\rm lo}u_{\rm hi}^2u^3\big) - \{P_{\rm lo}F(u_{\rm hi}), u_{\rm hi}\}_m + \{P_{\rm hi}F(u_{\rm lo}), u_{\rm hi}\}_m.\end{aligned} \tag{6-31}$$

We will treat their contributions in reverse order (right to left) since this corresponds to increasing complexity.

The contribution of the third term is easily seen to be bounded by

$$\begin{aligned}\|u_{\rm hi}\nabla u_{\rm hi}\|_{L_t^\infty L_x^1}\|u_{\rm hi}P_{\rm hi}F(u_{\rm lo})\|_{L_{t,x}^1} &\lesssim \|\nabla u_{\rm hi}\|_{L_t^\infty L_x^2}\|u_{\rm hi}\|_{L_t^\infty L_x^2}^2 N^{-1}\|\nabla F(u_{\rm lo})\|_{L_t^1 L_x^2}\\ &\lesssim_u \eta^2 N^{-3}\|\nabla u_{\rm lo}\|_{L_t^2 L_x^6}\|u_{\rm lo}\|_{L_t^4 L_x^\infty}^2\|u_{\rm lo}\|_{L_t^\infty L_x^6}^2\\ &\lesssim_u \eta^2(N^{-3}+K).\end{aligned}$$

For the second term in (6-31) we write $u_{\mathrm{hi}} = \operatorname{div}(\nabla\Delta^{-1}u_{\mathrm{hi}})$ and integrate by parts. This yields two contributions to LHS(6-30), which we bound as follows:

$$\begin{aligned}\|u_{\mathrm{hi}}\nabla u_{\mathrm{hi}}\|_{L_t^\infty L_x^1}\big\||\nabla|^{-1}u_{\mathrm{hi}}\big\|_{L_t^2L_x^6}\big\|\nabla P_{\mathrm{lo}}F(u_{\mathrm{hi}})\big\|_{L_t^2L_x^{6/5}} &\lesssim_u \|u_{\mathrm{hi}}\|_{L_t^\infty L_x^2}N^{-2}(1+N^3K)^{1/2}N^{3/2}\|F(u_{\mathrm{hi}})\|_{L_t^2L_x^1}\\ &\lesssim_u \eta(N^{-3}+K)^{1/2}\|u_{\mathrm{hi}}\|_{L_{t,x}^4}^2\|u_{\mathrm{hi}}\|_{L_t^\infty L_x^6}^3\\ &\lesssim_u \eta\big(\|u_{\mathrm{hi}}\|_{L_{t,x}^4}^4+N^{-3}+K\big)\end{aligned}$$

and

$$\begin{aligned}&\big\||x|^{-1}*|u_{\mathrm{hi}}\nabla u_{\mathrm{hi}}|\big\|_{L_t^4L_x^{12}}\big\||\nabla|^{-1}u_{\mathrm{hi}}\big\|_{L_t^2L_x^6}\big\|P_{\mathrm{lo}}F(u_{\mathrm{hi}})\big\|_{L_t^4L_x^{4/3}}\\ &\qquad\lesssim \|\nabla u_{\mathrm{hi}}\|_{L_t^\infty L_x^2}\|u_{\mathrm{hi}}\|_{L_{t,x}^4}N^{-2}(1+N^3K)^{1/2}N^{3/4}\|F(u_{\mathrm{hi}})\|_{L_t^4L_x^1}\\ &\qquad\lesssim_u \|u_{\mathrm{hi}}\|_{L_{t,x}^4}N^{1/4}(N^{-3}+K)^{1/2}\|u_{\mathrm{hi}}\|_{L_{t,x}^4}\|u_{\mathrm{hi}}\|_{L_t^\infty L_x^6}^{15/4}\|u_{\mathrm{hi}}\|_{L_t^\infty L_x^2}^{1/4}\\ &\qquad\lesssim_u \eta^{1/4}\big(\|u_{\mathrm{hi}}\|_{L_{t,x}^4}^4+N^{-3}+K\big).\end{aligned}$$

We now move to the first term in (6-31). This term, or more precisely, the term $\mathcal{O}(u_{\mathrm{lo}}u_{\mathrm{hi}}^5)$ contained therein, is the reason we needed to introduce the spatial truncation on a. Using $Re^J = N^{-1}$, we estimate this term via

$$\begin{aligned}\|\nabla u_{\mathrm{hi}}\|_{L_t^\infty L_x^2}\|u_{\mathrm{hi}}\|_{L_{t,x}^4}\|\nabla a\|_{L_t^\infty L_x^4}\|u_{\mathrm{hi}}\|_{L_{t,x}^4}^2\|u_{\mathrm{lo}}\|_{L_t^4L_x^\infty}\|u\|_{L_t^\infty L_x^6}^3 &\lesssim_u \|u_{\mathrm{hi}}\|_{L_{t,x}^4}^3(e^JR)^{3/4}\eta^{1/2}(1+N^3K)^{1/4}\\ &\lesssim_u \eta^{1/2}\big(\|u_{\mathrm{hi}}\|_{L_{t,x}^4}^4+N^{-3}+K\big).\end{aligned}$$

This completes the control of the mass bracket terms. □

Proof of Theorem 6.1. From Hölder's inequality, we see that when $\phi = u_{\mathrm{hi}}$ and a is as above, the interaction Morawetz quantity defined in (6-2) obeys

$$\sup_{t\in I}|M(t)| \le 2\|u_{\mathrm{hi}}\|_{L_t^\infty L_x^2(I\times\mathbb{R}^3)}^3\|\nabla u_{\mathrm{hi}}\|_{L_t^\infty L_x^2(I\times\mathbb{R}^3)} \lesssim_u \eta^3N^{-3},$$

provided, of course, that N is small enough so that (6-13) holds. Applying the fundamental theorem of calculus to the identity in Proposition 6.2 and putting together all the lemmas in this section, we reach the conclusion that

$$8\pi\|u_{\mathrm{hi}}\|_{L_{t,x}^4(I\times\mathbb{R}^3)}^4 + B_I \lesssim_u \Big(\varepsilon+\frac{1}{J_0}\Big)B_I + \eta^{\frac14}\|u_{\mathrm{hi}}\|_{L_{t,x}^4(I\times\mathbb{R}^3)}^4 + \Big(\eta^{\frac14}+\frac{\eta}{\varepsilon}+\frac{J_0^2}{J}+\eta^2\frac{e^{2J}}{J}\Big)(N^{-3}+K).$$

We remind the reader that this estimate is uniform in $\varepsilon, \eta \in (0, 1]$, but was derived under several overarching hypotheses: (6-13), $NRe^J = 1$, and $J \ge 2J_0 \ge 2$.

We now choose our parameters as follows: First ε and J_0^{-1} are made small enough so that the B_I term on the RHS can be absorbed by that on the LHS. Next η and J^{-1} are chosen small enough both to handle the $L_{t,x}^4$ on the RHS and to ensure that the prefactor in front of $(N^{-3}+K)$ is smaller than η_0. We now choose R and N^{-1} large enough so that (6-13) holds and then further increase N^{-1} or R so as to ensure $NRe^J = 1$.

To fully justify bringing the two terms across the inequality, we need to verify that they are indeed finite. This is easily done:

$$\|u_{\mathrm{hi}}\|_{L^4_{t,x}}^4 \lesssim \||\nabla|^{1/4} u_{\geq N}\|_{L^4_t L^3_x}^4 \lesssim N^{-3} \|\nabla u_{\mathrm{hi}}\|_{L^4_t L^3_x}^4 \lesssim_u N^{-3} + N^{-3} \int_I N(t)^2 \, dt,$$

by Sobolev embedding, Bernstein, and Lemma 1.7. Similarly,

$$\begin{aligned} B_I &\lesssim \big\| |x|^{-1} * |u_{\mathrm{hi}}|^2 \big\|_{L^4_t L^{12}_x} \|u_{\mathrm{hi}}\|_{L^\infty_t L^2_x}^{5/4} \|u_{\mathrm{hi}}\|_{L^{19/3}_t L^{114/7}_x}^{19/4} \\ &\lesssim \|u_{\mathrm{hi}}\|_{L^4_t L^4_x} \|u_{\mathrm{hi}}\|_{L^\infty_t L^2_x}^{9/4} \|\nabla u_{\mathrm{hi}}\|_{L^{19/3}_t L^{38/15}_x}^{19/4} \lesssim_u N^{-3} + N^{-3} \int_I N(t)^2 \, dt, \end{aligned}$$

by also using the Hardy–Littlewood–Sobolev inequality. □

Remark 6.9. As noted in the course of the proof, the necessity of truncating $a(x)$ stems from our inability to estimate one term. It would be possible to give a much simpler proof if we could show (a priori) that

$$\|u_{\mathrm{hi}}^5 u_{\mathrm{lo}}\|_{L^1_{t,x}} \lesssim_u N^{-2} + \eta N K, \tag{6-32}$$

for N sufficiently small. We will now describe what appears to be an intrinsic obstacle to doing this.

With current technology, proving (6-32) without using the interaction Morawetz identity seems to require proving that it also holds for almost periodic solutions of the *focusing* equation; however, the static solution W described in Remark 4.3 shows (6-32) does not hold in that setting. From (4-18) and simple arguments,

$$\begin{aligned} \lim_{N\to 0} N^{-1} \int_{\mathbb{R}^3} \big[W_{>N}(x)\big]^5 W_{\leq N}(x)\, dx &= \lim_{N\to 0} N^{-1} \int_{\mathbb{R}^3} W(x)^5 W_{\leq N}(x)\, dx \\ &= \lim_{N\to 0} N^{-1} \int_{\mathbb{R}^3} |\xi|^2 |\hat{W}(\xi)|^2 \varphi(\xi/N)\, d\xi \sim 1. \end{aligned}$$

As $N(t) \equiv 1$, it follows that $K = |I|$ and so $\|W_{\mathrm{hi}}^5 W_{\mathrm{lo}}\|_{L^1_{t,x}} \gtrsim NK$ for N small.

7. Impossibility of quasisolitons

In this section, we show that the second type of almost periodic solution described in Theorem 1.8, namely, those with $\int_0^{T_{\max}} N(t)^{-1}\, dt = \infty$, cannot exist. This is because their existence is inconsistent with the interaction Morawetz estimate obtained in the last section.

Theorem 7.1 (no quasisolitons). *There are no almost periodic solutions $u : [0, T_{\max}) \times \mathbb{R}^3 \to \mathbb{C}$ to* (1-1) *with $N(t) \equiv N_k \geq 1$ on each characteristic interval $J_k \subset [0, T_{\max})$ which satisfy $\|u\|_{L^{10}_{t,x}([0,T_{\max})\times\mathbb{R}^3)} = +\infty$ and*

$$\int_0^{T_{\max}} N(t)^{-1}\, dt = \infty. \tag{7-1}$$

Proof. We argue by contradiction and assume there exists such a solution u.

First we observe that there exists $C(u) > 0$ such that

$$N(t) \int_{|x-x(t)|\leq C(u)/N(t)} |u(t,x)|^4 \, dx \geq 1/C(u) \tag{7-2}$$

uniformly for $t \in [0, T_{\max})$. That this is true for a single time t follows from the fact that $u(t)$ is not identically zero. To upgrade this to a statement uniform in time, we use the fact that u is almost periodic. More precisely, we note that the left-hand side of (7-2) is both scale- and translation-invariant and that the map $u(t) \mapsto \mathrm{LHS}(7\text{-}2)$ is continuous on L_x^6 and hence also on $\dot{H}_x^1$.

Moreover, by Hölder's inequality,

$$N(t) \int_{|x-x(t)| \le C(u)/N(t)} |u_{\le N}(t,x)|^4 \, dx \lesssim_u \|u_{\le N}(t)\|_{L_x^6}^4 \quad \text{for any } N > 0,$$

uniformly for $t \in [0, T_{\max})$. Combining this with (7-2) and Theorem 6.1 shows that for each $\eta_0 > 0$ there exists some $N = N(\eta_0)$ sufficiently small so that

$$\int_I N(t)^{-1} \, dt \lesssim_u \eta_0 N^{-3} + \eta_0 \int_I N(t)^{-1} \, dt$$

uniformly for time intervals $I \subset [0, T_{\max})$ that are a union of characteristic subintervals J_k. In particular, we may choose η_0 small enough to defeat the implicit constant in this inequality and so deduce that

$$\int_0^{T_{\max}} N(t)^{-1} \, dt = \lim_{T \nearrow T_{\max}} \int_0^T N(t)^{-1} \, dt \lesssim_u 1,$$

which contradicts (7-1). □

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Received 5 Mar 2011. Revised 12 May 2011. Accepted 11 Jun 2011.

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ANALYSIS & PDE

Volume 5 No. 4 2012

2157-5045(2012)5:4;1-G