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GENERALIZED RICCI FLOW I: HIGHER-DERIVATIVE ESTIMATES FOR COMPACT MANIFOLDS

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We consider a generalized Ricci flow with a given (not necessarily closed) three-form and establish higher-derivative estimates for compact manifolds. As an application, we prove the compactness theorem for this generalized Ricci flow. Similar results still hold for a more generalized Ricci flow.

1. Introduction

Throughout this paper manifolds always mean smooth and closed (compact and without boundary) manifolds. Let $\mathfrak{Met}(M)$ denote the space of smooth metrics on a manifold M, and $C^\infty(M)$ the set of all smooth functions on M. We denote by C the universal constants depending only on the dimension of M, which may take different values at different places.

An important and natural problem in differential geometry is to find a canonical metric on a given manifold. A classical example is the uniformization theorem (e.g., [Chow and Knopf 2004]), which says that every smooth surface admits a unique conformal metric of constant curvature. To generalize to higher dimensional manifolds, Hamilton [1982] introduced a system of equations

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij}, \tag{1-1}$$

now called the Ricci flow, an analogue of the heat equation for metrics.

There are two ways to understand the Ricci flow: one way comes from the two-dimensional sigma model (see [Bakas 2007]), while another comes from Perelman's energy functional [Perelman 2002] defined by

$$\mathcal{F}(g, f) = \int_M \left(R + |\nabla f|^2\right) e^{-f} dV_g, \quad (g, f) \in \mathfrak{Met}(M) \times C^\infty(M), \tag{1-2}$$

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where R, ∇, and dV_g, is the scalar curvature, Levi-Civita connection, and volume form of g, respectively. He showed that the Ricci flow is the gradient flow of (1-2) and the functional $\mathcal{F}$ is monotonic along this gradient flow. Precisely, under the following system

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij}, \quad \frac{\partial f}{\partial t} = -R - \Delta f + |\nabla f|^2, \tag{1-3}$$

we have

$$\frac{d}{dt}\mathcal{F}(g, f) = 2\int_M \left|R_{ij} + \nabla_i \nabla_j f\right|^2 e^{-f} dV_g \geq 0. \tag{1-4}$$

Perelman's energy functional plays an essential role in determining the structures of singularities of the Ricci flow and then the proof of Poincaré conjecture and Thurston's generalization conjecture; for more details we refer readers to [Cao and Zhu 2006; Chow et al. 2006; 2007; 2008; 2010; Kleiner and Lott 2008; Morgan and Tian 2007; Perelman 2002].

Ricci flow coupled with a one-form or a two-form. If we consider the two-dimensional nonlinear sigma model [Bakas 2007; Oliynyk et al. 2006], then we obtain a generalized Ricci flow that is the Ricci flow coupled with the evolution equation for a two-form. This flow can be also obtained from the point of view of Perelman-type energy functional.

Denoting by $\mathcal{A}^p(M)$ the space of p-forms on M, we consider the energy functional

$$\mathcal{F}^{(1)} : \mathfrak{Met}(M) \times \mathcal{A}^2(M) \times C^\infty(M) \to \mathbb{R}$$

defined by

$$\mathcal{F}^{(1)}(g, B, f) = \int_M \left(R + |\nabla f|^2 - \tfrac{1}{12}|H|^2\right) e^{-f} dV_g, \tag{1-5}$$

where $H = dB$. As showed in [Oliynyk et al. 2006], the gradient flow of $\mathcal{F}^{(1)}$ satisfies

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij} - 2\nabla_i \nabla_j f + \tfrac{1}{2} H_i{}^{k\ell} H_{jk\ell}, \tag{1-6}$$

$$\frac{\partial B_{ij}}{\partial t} = 3\nabla_k H^k{}_{ij} - 3H^k{}_{ij} \nabla_k f, \tag{1-7}$$

$$\frac{\partial f}{\partial t} = -R - \Delta f + \tfrac{1}{4}|H|^2, \tag{1-8}$$

and under a family of diffeomorphisms the system (1-6)–(1-8) is equivalent to

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij} + \tfrac{1}{2} H_i{}^{k\ell} H_{jk\ell}, \tag{1-9}$$

$$\frac{\partial B_{ij}}{\partial t} = 3\nabla_k H^k{}_{ij}, \tag{1-10}$$

$$\frac{\partial f}{\partial t} = -R - \Delta f + |\nabla f|^2 + \tfrac{1}{4}|H|^2. \tag{1-11}$$

Using the adjoint operator d^*, Equation (1-10) can be written as

$$\frac{\partial B_{ij}}{\partial t} = -(d^* H)_{ij}, \tag{1-12}$$

and therefore (because of $H = dB$)

$$\frac{\partial H}{\partial t} = -dd^*H = \Delta_{\mathrm{HL}} H, \tag{1-13}$$

where $\Delta_{\mathrm{HL}} = -(dd^* + d^*d)$ denotes the Hodge–Laplace operator.

The flow (1-9)–(1-10) can be interpreted as the connection Ricci flow [Streets 2008]. If we replace $H = dB$ by $F = dA$, i.e., replace a two-form by a one-form, then the flow (1-6)–(1-7) or (1-9)–(1-10) is exactly the Ricci Yang–Mills flow studied by Streets [2007] and Young [2008].

Ricci flow coupled with a one-form and a two-form. There is another generalized Ricci flow which connects to Thurston's conjecture — roughly stating that a three-dimensional manifold with a given topology has a canonical decomposition into simple three-dimensional manifolds, each of which admits one, and only one, of eight homogeneous geometries: $\mathbb{S}^3$, the round three-sphere; $\mathbb{R}^3$, the Euclidean space; $\mathbb{H}^3$, the standard hyperbolic space; $\mathbb{S}^2 \times \mathbb{R}$; $\mathbb{H}^2 \times \mathbb{R}$; Nil, the three-dimensional nilpotent Heisenberg group; $\widetilde{\mathrm{SL}}(2, \mathbb{R})$; Sol, the three -dimensional solvable Lie group. The proof of Thurston's conjecture can be found in [Cao and Zhu 2006; Kleiner and Lott 2008; Morgan and Tian 2007; Perelman 2002].

To better understanding Thurston's conjecture, Gegenberg and Kunstatter [2004] proposed a generalized flow by considering the modified $3D$ stringy theory. This flow is the Ricci flow coupled with evolution equations for a one-form and a two-form. As in (1-5), we define an energy functional

$$\mathcal{F}^{(2)} : \mathfrak{Met}(M) \times \mathcal{A}^1(M) \times \mathcal{A}^2(M) \times C^\infty(M) \to \mathbb{R}$$

by

$$\mathcal{F}^{(2)}(g, A, B, f) = \int_M \left(R + |\nabla f|^2 - \tfrac{1}{12}|H|^2 - \tfrac{1}{2}|F|^2\right) e^{-f} dV_g, \tag{1-14}$$

where $H = dB$, and $F = dA$. In [He et al. 2008], the authors showed that the gradient flow of $\mathcal{F}^{(2)}$ satisfies

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij} - 2\nabla_i \nabla_j f + \tfrac{1}{2} H_i{}^{k\ell} H_{jk\ell} + 2F_i{}^k F_{jk}, \tag{1-15}$$

$$\frac{\partial A_i}{\partial t} = 2\nabla_j F^j{}_i - 2F^j{}_i \nabla_j f, \tag{1-16}$$

$$\frac{\partial B_{ij}}{\partial t} = 3\nabla_k H^k{}_{ij} - 3H^k{}_{ij} \nabla_k f, \tag{1-17}$$

$$\frac{\partial f}{\partial t} = -R - \Delta f + \tfrac{1}{4}|H|^2 + |F|^2, \tag{1-18}$$

and under a family of diffeomorphisms the system (1-15)–(1-18) is equivalent to

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij} + \tfrac{1}{2} H_i{}^{k\ell} H_{jk\ell} + 2F_i{}^k F_{jk}, \tag{1-19}$$

$$\frac{\partial A_i}{\partial t} = 2\nabla_j F^j{}_i, \tag{1-20}$$

$$\frac{\partial B_{ij}}{\partial t} = 3\nabla_k H^k{}_{ij}, \tag{1-21}$$

$$\frac{\partial f}{\partial t} = -R - \Delta f + |\nabla f|^2 + \tfrac{1}{4}|H|^2 + |F|^2. \tag{1-22}$$

Using again the adjoint operator d^*, we have

$$\frac{\partial F}{\partial t} = \Delta_{\mathrm{HL}} F, \quad \frac{\partial H}{\partial t} = \Delta_{\mathrm{HL}} H. \tag{1-23}$$

The flow (1-19)–(1-21) clearly contains the Ricci flow, the flow (1-9)–(1-10) or the connection Ricci flow, and the Ricci Yang–Mills flow; we expect this flow can give another proof of the Poincaré conjecture and Thurston's generalization conjecture, with less analysis on singularities.

Main results. For convenience, we refer to GRF the generalized Ricci flow and RF(A, B) the Ricci flow coupled with a one-form A and a two-form B.

Let (M, g) denote an n-dimensional closed Riemannian manifold with a three-form $H = \{H_{ijk}\}$. In the first part of this paper we consider the following GRF on M:

$$\frac{\partial}{\partial t} g_{ij}(x,t) = -2R_{ij}(x,t) + \tfrac{1}{2} H_{ik\ell}(x,t) H_j{}^{k\ell}(x,t), \tag{1-24}$$

$$\frac{\partial}{\partial t} H(x,t) = \Delta_{\mathrm{HL},g(x,t)} H(x,t), \quad H(x,0) = H(x), \quad g(x,0) = g(x). \tag{1-25}$$

It is clearly from (1-9) and (1-13) that the gradient flow of the energy functional $\mathcal{F}^{(1)}$ is a special case of (1-24)–(1-25). The corresponding case that H is closed is called the refined generalized Ricci flow (RGRF):

$$\frac{\partial}{\partial t} g_{ij}(x,t) = -2R_{ij}(x,t) + \frac{1}{2} H_{ik\ell}(x,t) H_j{}^{k\ell}(x,t), \tag{1-26}$$

$$\frac{\partial}{\partial t} H(x,t) = -d d^*_{g(x,t)} H(x,t), \quad H(x,0) = H(x), \quad g(x,0) = g(x). \tag{1-27}$$

Here $d^*_{g(x,t)}$ is the dual operator of d with respect to the metric $g(x,t)$.

Lemma 1.1. *Under* RGRF, $H(x,t)$ *is closed if the initial value* $H(x)$ *is closed.*

Proof. Since the exterior derivative d is independent of the metric, we have

$$\frac{\partial}{\partial t} dH(x,t) = d \frac{\partial}{\partial t} H(x,t) = d\big(-d d^*_{g(x,t)} H(x,t)\big) = 0.$$

so $dH(x,t) = dH(x) = 0$. □

The closedness of H is very important and has physical interpretation [Bakas 2007; Oliynyk et al. 2006]. Streets [2008] considered the connection Ricci flow in which H is the geometric torsion of connection.

Proposition 1.2. *If* $(g(x,t), H(x,t))$ *is a solution of* RGRF *and the initial value* $H(x)$ *is closed, then it is also a solution of* GRF.

Proof. From Lemma 1.1 and the assumption we know that $H(x,t)$ are all closed. Hence

$$\Delta_{\mathrm{HL},g(x,t)} H(x,t) = -d d^*_{g(x,t)} H(x,t).$$ □

For GRF, a basic and natural question is the existence. The short-time existence for RGRF has been established in [He et al. 2008], where the authors have already showed the short-time existence for RF(A, B) obviously including RGRF. In this paper, we prove the short- time existence for RGF.

Theorem 1.3. *There is a unique solution to* GRF *for a short time. More precisely, let* $(M, g_{ij}(x))$ *be an* n*-dimensional closed Riemannian manifold with a three-form* $H = \{H_{ijk}\}$*, then there exists a constant* $T = T(n) > 0$ *depending only on* n *such that the evolution system*

$$\frac{\partial}{\partial t} g_{ij}(x,t) = -2R_{ij}(x,t) + \tfrac{1}{2} g^{kp}(x,t) g^{\ell q}(x,t) H_{ik\ell}(x,t) H_{jpq}(x,t),$$
$$\frac{\partial}{\partial t} H(x,t) = \Delta_{\mathrm{HL},g(x,t)} H(x,t), \quad H(x,0) = H(x), \quad g(x,0) = g(x),$$

has a unique solution $(g_{ij}(x,t), H_{ijk}(x,t))$ *for a short time* $0 \le t \le T$.

After establishing the local existence, we are able to prove the higher derivatives estimates for GRF. Precisely, we have the following

Theorem 1.4. *Suppose that* $(g(x,t), H(x,t))$ *is a solution to* GRF *on a closed manifold* M^n *and* K *is an arbitrary given positive constant. Then for each* $\alpha > 0$ *and each integer* $m \ge 1$ *there exists a constant* C_m *depending on* $m, n, \max\{\alpha, 1\}$*, and* K *such that if*

$$|\mathrm{Rm}(x,t)|_{g(x,t)} \le K, \quad |H(x)|_{g(x)} \le K$$

for all $x \in M$ *and* $t \in [0, \alpha/K]$*, then*

$$|\nabla^{m-1} \mathrm{Rm}(x,t)|_{g(x,t)} + |\nabla^m H(x,t)|_{g(x,t)} \le \frac{C_m}{t^{m/2}} \tag{1-28}$$

for all $x \in M$ *and* $t \in (0, \alpha/K]$.

As an application, we can prove the compactness theorem for GRF.

Theorem 1.5 (compactness for GRF). *Let* $\{(M_k, g_k(t), H_k(t), O_k)\}_{k\in\mathbb{N}}$ *be a sequence of complete pointed solutions to* GRF *for* $t \in [\alpha, \omega) \ni 0$ *such that*:

(i) *There is a constant* $C_0 < \infty$ *independent of* k *such that*

$$\sup_{(x,t)\in M_k \times (\alpha,\omega)} \left|\mathrm{Rm}_{g_k(x,t)}\right|_{g_k(x,t)} \le C_0, \quad \sup_{x\in M_k} |H_k(x,\alpha)|_{g_k(x,\alpha)} \le C_0.$$

(ii) *There exists a constant* $\iota_0 > 0$ *satisfies*

$$\mathrm{inj}_{g_k(0)}(O_k) \ge \iota_0.$$

Then there exists a subsequence $\{j_k\}_{k\in\mathbb{N}}$ *such that*

$$(M_{j_k}, g_{j_k}(t), H_{j_k}(t), O_{j_k}) \to (M_\infty, g_\infty(t), H_\infty(t), O_\infty),$$

converges to a complete pointed solution $(M_\infty, g_\infty(t), H_\infty(t), O_\infty)$, $t \in [\alpha, \omega)$ *to* GRF *as* $k \to \infty$.

In the second part of this paper, we consider the Ricci flow coupled with a one-form and a two-form. This flow is the gradient flow of $\mathcal{F}^{(2)}$ and takes the form

$$\frac{\partial}{\partial t}g_{ij}(x,t) = -2R_{ij} + \tfrac{1}{2}H_i{}^{k\ell}(x,t)H_{jk\ell}(x,t) + 2F_i{}^k(x,t)F_{jk}(x,t), \tag{1-29}$$

$$\frac{\partial}{\partial t}A_i(x,t) = 2\nabla_j F^j{}_i(x,t), \quad A_i(x,0) = A_i(x), \quad g_{ij}(x,0) = g_{ij}(x), \tag{1-30}$$

$$\frac{\partial}{\partial t}B_{ij}(x,t) = 3\nabla_k H^k{}_{ij}(x,t), \quad B_{ij}(x,0) = B_{ij}(x). \tag{1-31}$$

Here $A = \{A_i\}$ and $B = \{B_{ij}\}$ is a one-form and a two-form on M, respectively, and $F = dA$, $H = dB$. For this flow, we can also prove the short-time existence, higher derivative estimates, and the compactness theorem.

The rest of this paper is organized as follows. In Section 2, we prove the short-time existence and uniqueness of the GRF for any given three-form H. In Section 3, we compute the evolution equations for the Levi-Civita connections, Riemann, Ricci, and scalar curvatures of a solution to the GRF. In Section 4, we establish higher derivative estimates for GRF, called *Bernstein–Bando–Shi (BBS) derivative estimates* (e.g., [Cao and Zhu 2006; Chow and Knopf 2004; Chow et al. 2007; 2008; 2010; Morgan and Tian 2007; Shi 1989]). In Section 5, we prove the compactness theorem for GRF by using BBS estimates. In Section 6, based on the work of [He et al. 2008], the similar results are established for RF(A, B).

2. Short-time existence of GRF

In this section we establish the short-time existence for GRF. Our method is standard: we use the DeTurck trick in Ricci flow to prove its short-time existence. We assume that M is an n-dimensional closed Riemannian manifold with metric

$$d\tilde{s}^2 = \tilde{g}_{ij}(x)\,dx^i\,dx^j \tag{2-1}$$

and with Riemannian curvature tensor $\{\tilde{R}_{ijk\ell}\}$. We also assume that $\tilde{H} = \{\tilde{H}_{ijk}\}$ is a fixed three-form on M. In the following we put

$$h_{ij} := H_{ik\ell}H_j{}^{k\ell}. \tag{2-2}$$

Suppose the metrics

$$d\hat{s}_t^2 = \tfrac{1}{2}\hat{g}_{ij}(x,t)\,dx^i\,dx^j \tag{2-3}$$

are the solutions of[1]

$$\frac{\partial}{\partial t}\hat{g}_{ij}(x,t) = -2\hat{R}_{ij}(x,t) + \hat{h}_{ij}(x,t), \quad \hat{g}_{ij}(x,0) = \tilde{g}_{ij}(x) \tag{2-4}$$

for a short time $0 \le t \le T$. Consider a family of smooth diffeomorphisms $\varphi_t : M \to M (0 \le t \le T)$ of M. Let

$$ds_t^2 := \varphi_t^* d\hat{s}_t^2, \quad 0 \le t \le T \tag{2-5}$$

[1] In the following computations we don't need to use the evolution equation for $H(x,t)$, hence we only consider the evolution equation for metrics.

be the pull-back metrics of $d\hat{s}_t^2$. For coordinates system $x = \{x^1, \dots, x^n\}$ on M, let

$$ds_t^2 = g_{ij}(x,t)\, dx^i\, dx^j \tag{2-6}$$

and

$$y(x,t) = \varphi_t(x) = \{y^1(x,t), \dots, y^n(x,t)\}. \tag{2-7}$$

Then we have

$$g_{ij}(x,t) = \frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\hat{g}_{\alpha\beta}(y,t). \tag{2-8}$$

By the assumption $\hat{g}_{\alpha\beta}(x,t)$ are the solutions of

$$\frac{\partial}{\partial t}\hat{g}_{\alpha\beta}(x,t) = -2\hat{R}_{\alpha\beta}(x,t) + \hat{h}_{\alpha\beta}(x,t), \quad \hat{g}_{\alpha\beta}(x,0) = \tilde{g}_{\alpha\beta}(x). \tag{2-9}$$

We use $R_{ij}, \hat{R}_{ij}, \tilde{R}_{ij}$; $\Gamma_{ij}^k, \hat{\Gamma}_{ij}^k, \tilde{\Gamma}_{ij}^k$; $\nabla, \hat{\nabla}, \tilde{\nabla}$; $h_{ij}, \hat{h}_{ij}, \tilde{h}_{ij}$ to denote the Ricci curvatures, Christoffel symbols, covariant derivatives, and products of the three-form H with respect to $\tilde{g}_{ij}, \hat{g}_{ij}, g_{ij}$ respectively. Then

$$\frac{\partial}{\partial t}g_{ij}(x,t) = \frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\left(\frac{\partial}{\partial t}\hat{g}_{\alpha\beta}(y,t)\right) + \frac{\partial}{\partial x^i}\left(\frac{\partial y^\alpha}{\partial t}\right)\frac{\partial y^\beta}{\partial x^j}\hat{g}_{\alpha\beta}(y,t) + \frac{\partial y^\alpha}{\partial x^i}\frac{\partial}{\partial x^j}\left(\frac{\partial y^\beta}{\partial t}\right)\hat{g}_{\alpha\beta}(y,t).$$

From (2-9) we have

$$\frac{\partial}{\partial t}\hat{g}_{\alpha\beta}(y,t) = -2\hat{R}_{\alpha\beta}(y,t) + \hat{h}_{\alpha\beta}(y,t) + \frac{\partial \hat{g}_{\alpha\beta}}{\partial y^\gamma}\frac{\partial y^\gamma}{\partial t},$$

and

$$\begin{aligned}\frac{\partial}{\partial t}g_{ij}(x,t) = {} & -2\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\hat{R}_{\alpha\beta}(y,t) + \frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\hat{h}_{\alpha\beta}(y,t) \\ & + \frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial \hat{g}_{\alpha\beta}}{\partial y^\gamma}\frac{\partial y^\gamma}{\partial t} + \frac{\partial}{\partial x^i}\left(\frac{\partial y^\alpha}{\partial t}\right)\frac{\partial y^\beta}{\partial x^j}\hat{g}_{\alpha\beta}(y,t) + \frac{\partial y^\alpha}{\partial x^i}\frac{\partial}{\partial x^j}\left(\frac{\partial y^\beta}{\partial t}\right)\hat{g}_{\alpha\beta}(y,t).\end{aligned}$$

Since

$$R_{ij}(x,t) = \frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\hat{R}_{\alpha\beta}(y,t), \quad h_{ij}(x,t) = \frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\hat{h}_{\alpha\beta}(y,t),$$

using [Shi 1989, §2, (29)], we obtain

$$\frac{\partial}{\partial t}g_{ij}(x,t) = -2R_{ij}(x,t) + h_{ij}(x,t) + \nabla_i\left(\frac{\partial y^\alpha}{\partial t}\frac{\partial x^k}{\partial y^\alpha}g_{jk}\right) + \nabla_j\left(\frac{\partial y^\alpha}{\partial t}\frac{\partial x^k}{\partial y^\alpha}g_{ik}\right). \tag{2-10}$$

According to DeTurck trick, we define $y(x,t) = \varphi_t(x)$ by the equation

$$\frac{\partial y^\alpha}{\partial t} = \frac{\partial y^\alpha}{\partial x^k}g^{\beta\gamma}(\Gamma_{\beta\gamma}^k - \tilde{\Gamma}_{\beta\gamma}^k), \quad y^\alpha(x,0) = x^\alpha, \tag{2-11}$$

then (2-10) becomes

$$\frac{\partial}{\partial t}g_{ij}(x,t) = -2R_{ij}(x,t) + h_{ij}(x,t) + \nabla_i V_j + \nabla_j V_i, \quad g_{ij}(x,0) = \tilde{g}_{ij}(x), \tag{2-12}$$

where

$$V_i = g_{ik} g^{\beta\gamma} (\Gamma^k_{\beta\gamma} - \widetilde{\Gamma}^k_{\beta\gamma}). \tag{2-13}$$

Lemma 2.1. *The evolution equation* (2-12) *is a strictly parabolic system. Moreover*,

$$\begin{aligned}\frac{\partial}{\partial t} g_{ij} &= g^{\alpha\beta} \widetilde{\nabla}_\alpha \widetilde{\nabla}_\beta g_{ij} - g^{\alpha\beta} g_{ip} \widetilde{g}^{pq} \widetilde{R}_{j\alpha q\beta} - g^{\alpha\beta} g_{jp} \widetilde{g}^{pq} \widetilde{R}_{i\alpha q\beta} \\ &\quad + \tfrac{1}{2} g^{\alpha\beta} g^{pq} \big(\widetilde{\nabla}_i g_{p\alpha} \cdot \widetilde{\nabla}_j g_{q\beta} + 2 \widetilde{\nabla} g_{jp} \cdot \widetilde{\nabla}_q g_{i\beta} - 2 \widetilde{\nabla}_\alpha g_{jp} \cdot \widetilde{\nabla}_\beta g_{iq} - 2 \widetilde{\nabla}_j g_{p\alpha} \cdot \widetilde{\nabla}_\beta g_{iq} - 2 \widetilde{\nabla}_i g_{p\alpha} \cdot \widetilde{\nabla}_\beta g_{jq} \big) \\ &\quad + \tfrac{1}{2} g^{\alpha\beta} g^{pq} H_{i\alpha p} H_{j\beta q}.\end{aligned}$$

Proof. It is an immediate consequence of Lemma 2.1 of [Shi 1989]. □

Now we can prove the short-time existence of GRF.

Theorem 2.2. *There is a unique solution to* GRF *for a short time. More precisely, let* $(M, g_{ij}(x))$ *be an* n*-dimensional closed Riemannian manifold with a three-form* $H = \{H_{ijk}\}$*, then there exists a constant* $T = T(n) > 0$ *depending only on* n *such that the evolution system*

$$\begin{aligned}\frac{\partial}{\partial t} g_{ij}(x,t) &= -2R_{ij}(x,t) + \tfrac{1}{2} g^{kp}(x,t) g^{\ell q}(x,t) H_{ik\ell}(x,t) H_{jpq}(x,t), \\ \frac{\partial}{\partial t} H(x,t) &= \Delta_{\mathrm{HL}, g(x,t)} H(x,t), \quad H(x,0) = H(x), \quad g(x,0) = g(x),\end{aligned}$$

has a unique solution $(g_{ij}(x,t), H_{ijk}(x,t))$ *for a short time* $0 \le t \le T$.

Proof. We proved that the first evolution equation is strictly parabolic by Lemma 2.1. Form the Ricci identity, we have $\Delta_{\mathrm{HL}, g(x,t)} H = \Delta_{\mathrm{LB}, g(x,t)} H + \mathrm{Rm} * H$ which is also strictly parabolic. Hence from the standard theory of parabolic systems, the evolution system has a unique solution. □

3. Evolution of curvatures

The evolution equation for the Riemann curvature tensors to the usual Ricci flow (e.g., [Cao and Zhu 2006; Chow and Knopf 2004; Chow et al. 2007, 2008; 2010; Hamilton 1982; Morgan and Tian 2007; Shi 1989]) is given by

$$\frac{\partial}{\partial t} R_{ijk\ell} = \Delta R_{ijk\ell} + \psi_{ijk\ell}, \tag{3-1}$$

where

$$\psi_{ijk\ell} = 2(B_{ijk\ell} - B_{ij\ell k} - B_{i\ell jk} + B_{ikj\ell}) - g^{pq}(R_{pjk\ell} R_{q\ell} + R_{ipk\ell} R_{qj} + R_{ijp\ell} R_{qk} + R_{ijkp} R_{q\ell}),$$

and $B_{ijk\ell} = g^{pr} g^{qs} R_{piqj} R_{rks\ell}$. From this we can easily deduce the evolution equation for the Riemann curvature tensors to GRF.

Let $v_{ij}(x,t)$ be any symmetric 2-tensor, we consider the flow

$$\frac{\partial}{\partial t} g_{ij}(x,t) = v_{ij}(x,t). \tag{3-2}$$

Applying a formula in [Chow and Knopf 2004] to our case $v_{ij} := -2R_{ij} + \frac{1}{2}h_{ij}$ with $h_{ij} = H_{ik\ell}H_j{}^{k\ell}$, we obtain

$$\begin{aligned}\frac{\partial}{\partial t}R_{ijk\ell} &= -\tfrac{1}{2}\big(-2\nabla_i\nabla_k R_{j\ell} + \tfrac{1}{2}\nabla_i\nabla_k h_{j\ell} + 2\nabla_i\nabla_\ell R_{jk} - \tfrac{1}{2}\nabla_i\nabla_\ell h_{jk} \\ &\qquad + 2\nabla_j\nabla_k R_{i\ell} - \tfrac{1}{2}\nabla_j\nabla_k h_{i\ell} - 2\nabla_j\nabla_\ell R_{ik} + \tfrac{1}{2}\nabla_j\nabla_\ell h_{ik}\big) \\ &\quad + \tfrac{1}{2}g^{pq}\big[R_{ijkp}(-2R_{q\ell} + \tfrac{1}{2}h_{q\ell}) + R_{ijp\ell}(-2R_{qk} + \tfrac{1}{2}h_{qk})\big] \\ &= \nabla_i\nabla_k R_{j\ell} - \nabla_i\nabla_\ell R_{jk} - \nabla_j\nabla_k R_{i\ell} + \nabla_j\nabla_\ell R_{ik} - g^{pq}(R_{ijkp}R_{q\ell} + R_{ijp\ell}R_{qk}) \\ &\quad + \tfrac{1}{4}\left(-\nabla_i\nabla_k h_{j\ell} + \nabla_i\nabla_\ell h_{jk} + \nabla_j\nabla_k h_{i\ell} - \nabla_j\nabla_\ell h_{ik}\right) + \tfrac{1}{4}g^{pq}\left(R_{ijkp}h_{q\ell} + R_{ijp\ell}h_{qk}\right) \\ &= \Delta R_{ijk\ell} + 2\left(B_{ijk\ell} - B_{ij\ell k} - B_{ilj k} + B_{ikj\ell}\right) \\ &\quad - g^{pq}(R_{pjk\ell}R_{q\ell} + R_{ipk\ell}R_{qj} + R_{ijp\ell}R_{qk} + R_{ijkp}R_{q\ell}) \\ &\quad + \tfrac{1}{4}\left(-\nabla_i\nabla_k h_{j\ell} + \nabla_i\nabla_\ell h_{jk} + \nabla_j\nabla_k h_{i\ell} - \nabla_j\nabla_\ell h_{ik}\right) + \tfrac{1}{4}g^{pq}\left(R_{ijkp}h_{q\ell} + R_{ijp\ell}h_{qk}\right).\end{aligned}$$

Proposition 3.1. *For* GRF *we have*

$$\begin{aligned}\frac{\partial}{\partial t}R_{ijk\ell} &= \Delta R_{ijk\ell} + 2\left(B_{ijk\ell} - B_{ij\ell k} - B_{i\ell jk} + B_{ikj\ell}\right) \\ &\quad - g^{pq}(R_{pjk\ell}R_{q\ell} + R_{ipk\ell}R_{qj} + R_{ijp\ell}R_{qk} + R_{ijkp}R_{q\ell}) \\ &\quad + \tfrac{1}{4}(-\nabla_i\nabla_k h_{j\ell} + \nabla_i\nabla_\ell h_{jk} + \nabla_j\nabla_k h_{i\ell} - \nabla_j\nabla_\ell h_{ik}) + \tfrac{1}{4}g^{pq}\left(R_{ijkp}h_{q\ell} + R_{ijp\ell}h_{qk}\right).\end{aligned}$$

In particular:

Corollary 3.2. *For* GRF *we have*

$$\frac{\partial}{\partial t}\operatorname{Rm} = \Delta\operatorname{Rm} + \operatorname{Rm} * \operatorname{Rm} + H * H * \operatorname{Rm} + \sum_{i=0}^{2}\nabla^i H * \nabla^{2-i}H. \tag{3-3}$$

Proof. From Proposition 3.1, we obtain

$$\frac{\partial}{\partial t}\operatorname{Rm} = \Delta\operatorname{Rm} + \operatorname{Rm} * \operatorname{Rm} + \nabla^2 h + h * \operatorname{Rm}.$$

On the other hand, $h = H * H$ and

$$\nabla^2 h = \nabla(\nabla(H * H)) = \nabla(\nabla H * H) = \nabla^2 H * H + \nabla H * \nabla H.$$

Combining these terms, we obtain the result. □

Proposition 3.3. *For* GRF *we have*

$$\begin{aligned}\frac{\partial}{\partial t}R_{ik} &= \Delta R_{ik} + 2\langle R_{piqk}, R_{pq}\rangle - 2\langle R_{pi}, R_{pk}\rangle + \tfrac{1}{4}\big[\langle h_{\ell q}, R_{i\ell kq}\rangle + \langle R_{ip}, h_{kp}\rangle\big] \\ &\qquad + \tfrac{1}{4}\big[-\nabla_i\nabla_k|H|^2 + g^{j\ell}\nabla_i\nabla_\ell h_{jk} + g^{j\ell}\nabla_j\nabla_k h_{i\ell} - \Delta h_{ik}\big].\end{aligned}$$

Proof. Since

$$\frac{\partial}{\partial t}R_{ik} = g^{j\ell}\frac{\partial}{\partial t}R_{ijk\ell} + 2g^{jp}g^{\ell q}R_{ijk\ell}R_{pq}$$

and

$$g^{ij}h_{ij} = g^{ij}H_{ipq}H_j{}^{pq} = g^{ij}g^{pr}g^{qs}H_{ipq}H_{jrs} = |H|^2,$$

it follows that

$$\begin{aligned} &g^{j\ell}[-\nabla_i\nabla_k h_{j\ell} + \nabla_i\nabla_\ell h_{jk} + \nabla_j\nabla_k h_{i\ell} - \nabla_j\nabla_\ell h_{ik} + g^{pq}h_{q\ell}R_{ijkp} + g^{pq}h_{qk}R_{ijp\ell}] \\ &\quad = -\nabla_i\nabla_k|H|^2 + g^{j\ell}\nabla_i\nabla_\ell h_{jk} + g^{j\ell}\nabla_j\nabla_k h_{i\ell} - \Delta h_{ik} + g^{j\ell}g^{pq}h_{q\ell}R_{ijkp} + g^{pq}h_{qk}R_{ip}. \end{aligned}$$

From these identities, we get the result. □

As a consequence, we obtain the evolution equation for scalar curvature.

Proposition 3.4. *For* GRF *we have*

$$\frac{\partial}{\partial t}R = \Delta R + 2\,|\mathrm{Ric}\,|^2 - \tfrac{1}{2}\Delta|H|^2 + \tfrac{1}{2}\langle h_{ij}, R_{ij}\rangle + \tfrac{1}{2}g^{ik}g^{j\ell}\nabla_i\nabla_j h_{k\ell}.$$

Proof. From the usual evolution equation for scalar curvature under the Ricci flow, we have

$$\begin{aligned} \frac{\partial}{\partial t}R &= \Delta R + 2\,|\mathrm{Ric}\,|^2 + \tfrac{1}{4}g^{ik}[\langle h_{\ell q}, R_{i\ell kq}\rangle + \langle R_{ip}, h_{kp}\rangle] \\ &\quad + \tfrac{1}{4}g^{ik}\big(-\nabla_i\nabla_k|H|^2 + g^{j\ell}\nabla_i\nabla_\ell h_{jk} + g^{j\ell}\nabla_j\nabla_k h_{i\ell} - \Delta h_{ik}\big) \\ &= \Delta R + 2\,|\mathrm{Ric}\,|^2 + \tfrac{1}{4}\langle h_{ij}, R_{ij}\rangle + \tfrac{1}{4}\langle R_{ip}, h_{ip}\rangle \\ &\quad - \tfrac{1}{4}\Delta|H|^2 + \tfrac{1}{4}g^{ik}g^{j\ell}\nabla_i\nabla_\ell h_{jk} + \tfrac{1}{4}g^{ik}g^{j\ell}\nabla_j\nabla_k h_{i\ell} - \tfrac{1}{4}\Delta|H|^2. \end{aligned}$$

Simplifying the terms, we obtain the required result. □

4. Derivative estimates

In this section we are going to prove BBS estimates. At first we review several basic identities of commutators $[\Delta, \nabla]$ and $[\partial/\partial t, \nabla]$. If $A = A(t)$ is a t-dependency tensor, and $\partial g_{ij}/\partial t = v_{ij}$, then applying the well-known formulas stated in [Chow and Knopf 2004] on GRF we have

$$\begin{aligned} \frac{\partial}{\partial t}\nabla\,\mathrm{Rm} &= \nabla\frac{\partial}{\partial t}\,\mathrm{Rm} + \mathrm{Rm}*\nabla(\mathrm{Rm} + H*H) \\ &= \nabla(\Delta\,\mathrm{Rm} + \mathrm{Rm}*\mathrm{Rm} + H*H*\mathrm{Rm} + \nabla^2 H*H + \nabla H*\nabla H) + \mathrm{Rm}*\nabla\,\mathrm{Rm} + H*\nabla H*\mathrm{Rm} \\ &= \Delta(\nabla\,\mathrm{Rm}) + \sum_{i+j=0}\nabla^i\,\mathrm{Rm}*\nabla^j\,\mathrm{Rm} + \sum_{i+j+k=0}\nabla^i H*\nabla^j H*\nabla^k\,\mathrm{Rm} + \sum_{i+j=0+2}\nabla^i H*\nabla^j H. \end{aligned} \tag{4-1}$$

More generally:

Proposition 4.1. *For* GRF *and any nonnegative integer ℓ we have*

$$\frac{\partial}{\partial t}\nabla^\ell\,\mathrm{Rm} = \Delta(\nabla^\ell\,\mathrm{Rm}) + \sum_{i+j=\ell}\nabla^i\,\mathrm{Rm}*\nabla^j\,\mathrm{Rm} + \sum_{i+j+k=\ell}\nabla^i H*\nabla^j H*\nabla^k\,\mathrm{Rm} + \sum_{i+j=\ell+2}\nabla^i H*\nabla^j H. \tag{4-2}$$

Proof. For $\ell = 1$, this is (4-1). Suppose (4-2) holds for $1, \dots, \ell$. By induction on ℓ, for $\ell + 1$ we have

$$\begin{aligned}
&\frac{\partial}{\partial t}\nabla^{\ell+1}\operatorname{Rm} \\
&\quad = \frac{\partial}{\partial t}\nabla(\nabla^{\ell}\operatorname{Rm}) \\
&\quad = \nabla\frac{\partial}{\partial t}(\nabla^{\ell}\operatorname{Rm}) + \nabla^{\ell}\operatorname{Rm} * \nabla(\operatorname{Rm} + H * H) \\
&\quad = \nabla\left[\Delta(\nabla^{\ell}\operatorname{Rm}) + \sum_{i+j=\ell}\nabla^{i}\operatorname{Rm} * \nabla^{j}\operatorname{Rm} + \sum_{i+j+k=\ell}\nabla^{i}H * \nabla^{j}H * \nabla^{k}\operatorname{Rm} + \sum_{i+j=\ell+2}\nabla^{i}H * \nabla^{j}H\right] \\
&\qquad\qquad + \nabla^{\ell}\operatorname{Rm} * \nabla\operatorname{Rm} + H * \nabla H * \nabla^{\ell}\operatorname{Rm} \\
&\quad = \Delta(\nabla^{\ell+1}\operatorname{Rm}) + \nabla\operatorname{Rm} * \nabla^{\ell}\operatorname{Rm} + \operatorname{Rm} * \nabla^{\ell+1}\operatorname{Rm} \\
&\qquad + \sum_{i+j=\ell}\left(\nabla^{i+1}\operatorname{Rm} * \nabla^{j}\operatorname{Rm} + \nabla^{i}\operatorname{Rm} * \nabla^{j+1}\operatorname{Rm}\right) \\
&\qquad + \sum_{i+j+k=\ell}\left(\nabla^{i+1}H * \nabla^{j}H * \nabla^{k}\operatorname{Rm} + \nabla^{i}H * \nabla^{j+1}H * \nabla^{k}\operatorname{Rm} + \nabla^{i}H * \nabla^{j}H * \nabla^{k+1}\operatorname{Rm}\right) \\
&\qquad + \sum_{i+j=\ell+2}(\nabla^{i+1}H * \nabla^{j}H + \nabla^{i}H * \nabla^{j+1}H) + H * \nabla H * \nabla^{l}\operatorname{Rm}.
\end{aligned}$$

Simplifying these terms, we obtain the required result. □

As an immediate consequence, we have an evolution inequality for $|\nabla^{l}\operatorname{Rm}|^2$.

Corollary 4.2. *For* GRF *and any nonnegative integer ℓ we have*

$$\begin{aligned}
\frac{\partial}{\partial t}|\nabla^{\ell}\operatorname{Rm}|^2 \le \Delta|\nabla^{l}\operatorname{Rm}|^2 - 2|\nabla^{\ell+1}\operatorname{Rm}|^2 + C\sum_{i+j=\ell}|\nabla^{i}\operatorname{Rm}|\cdot|\nabla^{j}\operatorname{Rm}|\cdot|\nabla^{\ell}\operatorname{Rm}| \\
+ C\sum_{i+j+k=\ell}|\nabla^{i}H|\cdot|\nabla^{j}H|\cdot|\nabla^{k}\operatorname{Rm}|\cdot|\nabla^{\ell}\operatorname{Rm}| + C\sum_{i+j=\ell+2}|\nabla^{i}H|\cdot|\nabla^{j}H|\cdot|\nabla^{\ell}\operatorname{Rm}|, \quad (4\text{-}3)
\end{aligned}$$

where C represents universal constants depending only on the dimension of M.

Next we derive the evolution equations for the covariant derivatives of H.

Proposition 4.3. *For* GRF *and any positive integer ℓ we have*

$$\frac{\partial}{\partial t}\nabla^{\ell}H = \Delta(\nabla^{\ell}H) + \sum_{i+j=\ell}\nabla^{i}H * \nabla^{j}\operatorname{Rm} + \sum_{i+j+k=\ell}\nabla^{i}H * \nabla^{j}H * \nabla^{k}H. \quad (4\text{-}4)$$

Proof. From the Bochner formula, the evolution equation for H can be rewritten as

$$\frac{\partial}{\partial t}H = \Delta H + \operatorname{Rm} * H. \quad (4\text{-}5)$$

For $\ell = 1$, we have

$$\begin{aligned}\frac{\partial}{\partial t}\nabla H &= \nabla\frac{\partial}{\partial t}H + H * \nabla(\mathrm{Rm} + H * H)\\ &= \nabla(\Delta H + \mathrm{Rm} * H) + H * \nabla\,\mathrm{Rm} + H * H * \nabla H\\ &= \nabla(\Delta H) + H * \nabla\,\mathrm{Rm} + \nabla H * \mathrm{Rm} + H * H * \nabla H\\ &= \Delta(\nabla H) + \nabla\,\mathrm{Rm} * H + \nabla H * \mathrm{Rm} + H * H * \nabla H.\end{aligned}$$

Using (4-2) and the same argument, we can prove the evolution equation for higher covariant derivatives. □

Similarly, we have an evolution inequality for $|\nabla^\ell H|^2$.

Corollary 4.4. *For* GRF *and for any positive integer l we have*

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla^l H|^2 \le{}& \Delta|\nabla^\ell H|^2 - 2\,|\nabla^{\ell+1} H|^2\\ &+ C\sum_{i+j=\ell}|\nabla^i H|\cdot|\nabla^j\,\mathrm{Rm}\,|\cdot|\nabla^\ell H| + C\sum_{i+j+k=\ell}|\nabla^i H|\cdot|\nabla^j H|\cdot|\nabla^k H|\cdot|\nabla^l H|, \quad (4\text{-}6)\end{aligned}$$

while

$$\frac{\partial}{\partial t}|H|^2 \le \Delta|H|^2 - 2\,|\nabla H|^2 + C\cdot|\mathrm{Rm}\,|\cdot|H|^2. \quad (4\text{-}7)$$

Theorem 4.5. *Suppose that $(g(x,t), H(x,t))$ is a solution to* GRF *on a closed manifold M^n for a short time $0 \le t \le T$ and K_1, K_2 are arbitrary given nonnegative constants. Then there exists a constant C_n depending only on n such that if*

$$|\mathrm{Rm}(x,t)|_{g(x,t)} \le K_1, \quad |H(x)|_{g(x)} \le K_2$$

for all $x \in M$ and $t \in [0,T]$, then

$$|H(x,t)|_{g(x,t)} \le K_2 e^{C_n K_1 t} \quad (4\text{-}8)$$

for all $x \in M$ and $t \in [0,T]$.

Proof. Since

$$\frac{\partial}{\partial t}|H|^2 \le \Delta|H|^2 + C_n|\mathrm{Rm}\,|\cdot|H|^2 \le \Delta|H|^2 + C_n K_1|H|^2,$$

using the maximum principle, we obtain $u(t) \le u(0)e^{C_n K_1 t}$, where $u(t) = |H|^2$. □

The main result in this section is the following estimates for higher derivatives of Riemann curvature tensors and three-forms. Some special cases were proved in [Streets 2007; 2008; Young 2008].

Theorem 4.6. *Suppose that $(g(x,t), H(x,t))$ is a solution to* GRF *on a compact manifold M^n and K is an arbitrary given positive constant. Then for each $\alpha > 0$ and each integer $m \ge 1$ there exists a constant C_m depending on $m, n, \max\{\alpha, 1\}$, and K such that if*

$$|\mathrm{Rm}(x,t)|_{g(x,t)} \le K, \quad |H(x)|_{g(x)} \le K$$

for all $x \in M$ *and* $t \in [0, \alpha/K]$, *then*

$$|\nabla^{m-1} \operatorname{Rm}(x,t)|_{g(x,t)} + |\nabla^m H(x,t)|_{g(x,t)} \le \frac{C_m}{t^{m/2}} \tag{4-9}$$

for all $x \in M$ *and* $t \in (0, \alpha/K]$.

Proof. In the following computations we always let C be any constants depending on $n, m, \max\{\alpha, 1\}$,and K, which may take different values at different places. From the evolution equations and Theorem 4.5, we have

$$\begin{aligned}
\frac{\partial}{\partial t}|\operatorname{Rm}|^2 &\le \Delta|\operatorname{Rm}|^2 - 2\,|\nabla \operatorname{Rm}|^2 + C + C|\nabla^2 H| + C|\nabla H|^2,\\
\frac{\partial}{\partial t}|H|^2 &\le \Delta|H|^2 - 2\,|\nabla H|^2 + C,\\
\frac{\partial}{\partial t}|\nabla H|^2 &\le \Delta|\nabla H|^2 - 2\,|\nabla^2 H|^2 + C|\nabla \operatorname{Rm}| \cdot |\nabla H| + C|\nabla H|^2.
\end{aligned}$$

Consider the function $u = t|\nabla H|^2 + \gamma|H|^2 + t|\operatorname{Rm}|^2$. Directly computing, we obtain

$$\begin{aligned}
\frac{\partial}{\partial t}u &\le \Delta u - 2t|\nabla^2 H|^2 + Ct|\nabla^2 H| + (C - 2\gamma)|\nabla H|^2 + C + C\gamma - 2t|\nabla \operatorname{Rm}|^2 + Ct \cdot |\nabla \operatorname{Rm}| \cdot |\nabla H|\\
&\le \Delta u + 2(C - \gamma) \cdot |\nabla H|^2 + C(1 + \gamma).
\end{aligned}$$

If we choose $\gamma = C$, then $\frac{\partial}{\partial t}u \le \Delta u + C$ which implies that $u \le Ce^{Ct}$ since $u(0) \le C$. With this estimate we are able to bound the first covariant derivative of Rm and the second covariant derivative of H. In order to control the term $|\nabla \operatorname{Rm}|^2$, we should use the evolution equations of $|H|^2$, $|\nabla H|^2$ and $|\nabla^2 H|^2$ to cancel with the bad terms, i.e., $|\nabla^2 \operatorname{Rm}|^2$, $|\nabla^2 H|^2$, and $|\nabla^3 H|^2$, in the evolution equation of $|\nabla \operatorname{Rm}|^2$:

$$\begin{aligned}
&\frac{\partial}{\partial t}|\nabla \operatorname{Rm}|^2\\
&\quad \le \Delta|\nabla \operatorname{Rm}|^2 - 2\,|\nabla^2 \operatorname{Rm}|^2 + C|\nabla \operatorname{Rm}|^2 + \frac{C}{t^{1/2}}|\nabla \operatorname{Rm}| + C \cdot |\nabla \operatorname{Rm}| \cdot |\nabla^3 H| + \frac{C}{t^{1/2}}|\nabla^2 H| \cdot |\nabla \operatorname{Rm}|,\\
&\frac{\partial}{\partial t}|\nabla^2 H|^2 \le \Delta|\nabla^2 H|^2 - 2\,|\nabla^3 H|^2 + C \cdot |\nabla^2 \operatorname{Rm}| \cdot |\nabla^2 H| + \frac{C}{t^{1/2}}|\nabla \operatorname{Rm}| \cdot |\nabla^2 H| + C|\nabla^2 H|^2 + \frac{C}{t}|\nabla^2 H|.
\end{aligned}$$

As above, we define

$$u := t^2(|\nabla^2 H|^2 + |\nabla \operatorname{Rm}|^2) + t\beta(|\nabla H|^2 + |\operatorname{Rm}|^2) + \gamma|H|^2,$$

and therefore, $\frac{\partial u}{\partial t} \le \Delta u + C$. Motivated by cases for $m = 1$ and $m = 2$, for general m, we can define a function

$$u := t^m(|\nabla^m H|^2 + |\nabla^{m-1} \operatorname{Rm}|^2) + \sum_{i=1}^{m-1} \beta_i t^i(|\nabla^i H|^2 + |\nabla^{i-1} \operatorname{Rm}|^2) + \gamma|H|^2,$$

where β_i and γ are positive constants determined later. In the following, we always assume $m \ge 3$.

Suppose that $|\nabla^{i-1}\operatorname{Rm}|+|\nabla^i H|\le \dfrac{C_i}{t^{i/2}}$, for $i=1,2,\dots,m-1$. For such i, from Corollary 4.4, we have

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla^i H|^2 &\le \Delta|\nabla^i H|^2-2\,|\nabla^{i+1}H|^2+C\sum_{j=0}^{i}|\nabla^j H|\cdot|\nabla^{i-j}\operatorname{Rm}|\cdot|\nabla^i H|\\ &\qquad +C\sum_{j=0}^{i}\sum_{\ell=0}^{i-j}|\nabla^j H|\cdot|\nabla^{i-j-\ell}H|\cdot|\nabla^\ell H|\cdot|\nabla^i H|\\ &\le \Delta|\nabla^i H|^2-2\,|\nabla^{i+1}H|^2+C\cdot|\nabla^i H|\sum_{j=0}^{i}\frac{C_j}{t^{\frac{j}{2}}}\cdot\frac{C_{i-j+1}}{t^{\frac{i-j+1}{2}}}+C\cdot|\nabla^i H|\sum_{j=0}^{i}\sum_{\ell=0}^{i-j}\frac{C_j}{t^{\frac{j}{2}}}\cdot\frac{C_{i-j-\ell}}{t^{\frac{i-j-1}{2}}}\cdot\frac{C_\ell}{t^{\frac{l}{2}}}\\ &\le \Delta|\nabla^i H|^2-2\,|\nabla^{i+1}H|^2+\frac{C_i}{t^{\frac{i+1}{2}}}|\nabla^i H|+\frac{C_i}{t^{\frac{i}{2}}}|\nabla^i H|.\end{aligned}$$

Similarly, from Corollary 4.2 we also have

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla^{i-1}\operatorname{Rm}|^2 &\le \Delta|\nabla^{i-1}\operatorname{Rm}|^2-2\,|\nabla^i\operatorname{Rm}|^2+C\sum_{j=0}^{i-1}|\nabla^j\operatorname{Rm}||\nabla^{i-1-j}\operatorname{Rm}||\nabla^{i-1}\operatorname{Rm}|\\ &\quad +C\sum_{j=0}^{i-1}\sum_{\ell=0}^{i-1-j}|\nabla^j H|\cdot|\nabla^{i-1-j-\ell}H|\cdot|\nabla^\ell\operatorname{Rm}|\cdot|\nabla^{i-1}\operatorname{Rm}|\\ &\quad +C\sum_{j=0}^{i+1}|\nabla^j H|\cdot|\nabla^{i+1-j}H|\cdot|\nabla^{i-1}\operatorname{Rm}|\\ &\le \Delta|\nabla^{i-1}\operatorname{Rm}|^2-2\,|\nabla^i\operatorname{Rm}|^2+C\cdot|\nabla^{i-1}\operatorname{Rm}|\sum_{j=0}^{i-1}\frac{C_{j+1}}{t^{\frac{j+1}{2}}}\cdot\frac{C_{i-j}}{t^{\frac{i-j}{2}}}\\ &\quad +C\cdot|\nabla^{i-1}\operatorname{Rm}|\sum_{j=0}^{i-1}\sum_{\ell=0}^{i-1-j}\frac{C_j}{t^{\frac{j}{2}}}\cdot\frac{C_{i-1-j-\ell}}{t^{\frac{i-1-j-\ell}{2}}}\cdot\frac{C_{\ell+1}}{t^{\frac{\ell+1}{2}}}\\ &\quad +C\cdot|\nabla^{i-1}\operatorname{Rm}|\sum_{j=1}^{i}\frac{C_j}{t^{\frac{j}{2}}}\cdot\frac{C_{i+1-j}}{t^{\frac{i+1-j}{2}}}+C\cdot|\nabla^{i+1}H|\cdot\frac{C_i}{t^{\frac{i}{2}}}\\ &\le \Delta|\nabla^{i-1}\operatorname{Rm}|^2-2\,|\nabla^i\operatorname{Rm}|^2+\frac{C_i}{t^{\frac{i+1}{2}}}\cdot|\nabla^{i-1}\operatorname{Rm}|+\frac{C_i}{t^{\frac{i}{2}}}|\nabla^{i+1}H|+\frac{C_i}{t^{\frac{i}{2}}}|\nabla^{i-1}\operatorname{Rm}|.\end{aligned}$$

The evolution inequality for u is now given by

$$\begin{aligned}\frac{\partial u}{\partial t} &\le mt^{m-1}(|\nabla^m H|^2+|\nabla^{m-1}\operatorname{Rm}|^2)+\sum_{i=1}^{m-1}i\beta_i t^{i-1}(|\nabla^i H|^2+|\nabla^{i-1}\operatorname{Rm}|^2)\\ &\quad +t^m\left(\frac{\partial}{\partial t}|\nabla^m H|^2+\frac{\partial}{\partial t}|\nabla^{m-1}\operatorname{Rm}|^2\right)+\sum_{i=1}^{m-1}\beta_i t^i\left(\frac{\partial}{\partial t}|\nabla^i H|^2+\frac{\partial}{\partial t}|\nabla^{i-1}\operatorname{Rm}|^2\right)+\gamma\cdot\frac{\partial}{\partial t}|H|^2.\end{aligned}$$

It's easy to see that the second term is bounded by

$$\sum_{i=1}^{m-1} i\beta_i t^{i-1}\frac{C_i}{t^i} = \sum_{i=1}^{m-1} i\beta_i C_i t^{-1},$$

but this bound depends on t and approaches to infinity when t goes to zero. Hence we use the last second term to control this bad term. The evolution inequality for the third term is the combination of the inequalities

$$\begin{aligned}
&\frac{\partial}{\partial t}|\nabla^m H|^2 \\
&\quad\le \Delta|\nabla^m H|^2 - 2\,|\nabla^{m+1}H|^2 + C\sum_{i=0}^{m}|\nabla^i H|\cdot|\nabla^{m-i}\,\mathrm{Rm}\,|\cdot|\nabla^m H| \\
&\qquad\qquad + C\sum_{i=0}^{m}\sum_{j=0}^{m-i}|\nabla^j H|\cdot|\nabla^{m-i-j}H|\cdot|\nabla^i H|\cdot|\nabla^m H| \\
&\quad\le \Delta|\nabla^m H|^2 - 2\,|\nabla^{m+1}H|^2 + C|\nabla^m H|^2 + C\cdot|\nabla^m\,\mathrm{Rm}\,|\cdot|\nabla^m H| + \frac{C_m}{t^{\frac{m+1}{2}}}|\nabla^m H| + \frac{C_m}{t^{\frac{m}{2}}}|\nabla^m H|
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial}{\partial t}|\nabla^{m-1}\,\mathrm{Rm}\,|^2 &\le \Delta|\nabla^{m-1}\,\mathrm{Rm}\,|^2 - 2\,|\nabla^m\,\mathrm{Rm}\,|^2 + C\sum_{i=0}^{m-1}|\nabla^i\,\mathrm{Rm}\,|\cdot|\nabla^{m-1-i}\,\mathrm{Rm}\,|\cdot|\nabla^{m-1}\,\mathrm{Rm}\,| \\
&\quad + C\sum_{i=0}^{m-1}\sum_{j=0}^{m-1-i}|\nabla^j H|\cdot|\nabla^{m-1-i-j}H|\cdot|\nabla^i\,\mathrm{Rm}\,|\cdot|\nabla^{m-1}\,\mathrm{Rm}\,| \\
&\quad + C\sum_{i=0}^{m+1}|\nabla^i H|\cdot|\nabla^{m+1-i}H|\cdot|\nabla^{m-1}\,\mathrm{Rm}\,| \\
&\le \Delta|\nabla^{m-1}\,\mathrm{Rm}\,|^2 - 2\,|\nabla^m\,\mathrm{Rm}\,|^2 + C|\nabla^{m-1}\,\mathrm{Rm}\,|^2 + \frac{C}{t^{\frac{1}{2}}}\cdot|\nabla^m H|\cdot|\nabla^{m-1}\,\mathrm{Rm}\,| \\
&\quad + C|\nabla^{m+1}H||\nabla^{m-1}\,\mathrm{Rm}\,| + \frac{C_m}{t^{\frac{m+1}{2}}}|\nabla^{m-1}\,\mathrm{Rm}\,| + \frac{C_m}{t^{\frac{m}{2}}}|\nabla^{m-1}\,\mathrm{Rm}\,|.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
\frac{\partial u}{\partial t} &\le mt^{m-1}(|\nabla^m H|^2 + |\nabla^{m-1}\,\mathrm{Rm}\,|^2) + \sum_{i=1}^{m-1} i\beta_i t^{i-1}(|\nabla^i H|^2 + |\nabla^{i-1}\,\mathrm{Rm}\,|^2) \\
&\quad + t^m\Bigg(\Delta|\nabla^m H|^2 - 2\,|\nabla^{m+1}H|^2 + \frac{C}{t^{\frac{m+1}{2}}}|\nabla^m H| + C|\nabla^m H|^2 \\
&\qquad + C|\nabla^m\,\mathrm{Rm}\,|\cdot|\nabla^m H| + \Delta|\nabla^{m-1}\,\mathrm{Rm}\,|^2 \\
&\qquad - 2\,|\nabla^m\,\mathrm{Rm}\,|^2 + \frac{C}{t^{\frac{m+1}{2}}}|\nabla^{m-1}\,\mathrm{Rm}\,| + C|\nabla^{m-1}\,\mathrm{Rm}\,|^2 \\
&\qquad + \frac{C}{t^{1/2}}|\nabla^m H|\cdot|\nabla^{m-1}\,\mathrm{Rm}\,| + C|\nabla^{m+1}H|\cdot|\nabla^{m-1}\,\mathrm{Rm}\,|\Bigg)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{m-1}\beta_i t^i\Big(\frac{C_i}{t^{\frac{i+1}{2}}}|\nabla^{i-1}\operatorname{Rm}|+\Delta|\nabla^i H|^2-2|\nabla^{i+1}H|^2\\
&\qquad\qquad+\Delta|\nabla^{i-1}\operatorname{Rm}|^2+\frac{C_i}{t^{\frac{i+1}{2}}}|\nabla^i H|+\frac{C_i}{t^{\frac{i}{2}}}|\nabla^{i+1}H|-2|\nabla^i\operatorname{Rm}|^2\Big)\\
&+\gamma(\Delta|H|^2-2|\nabla H|^2+C)\\
\le\ &\Delta u-2t^m|\nabla^{m+1}H|^2+Ct^m|\nabla^{m+1}H|\cdot|\nabla^{m-1}\operatorname{Rm}|\\
&-2t^m|\nabla^m\operatorname{Rm}|^2+Ct^m|\nabla^m\operatorname{Rm}|\cdot|\nabla^m H|+\sum_{i=0}^{m-2}(i+1)\beta_{i+1}t^i(|\nabla^{i+1}H|^2+|\nabla^i\operatorname{Rm}|^2)\\
&-2\sum_{i=1}^{m-1}\beta_i t^i(|\nabla^{i+1}H|^2+|\nabla^i\operatorname{Rm}|^2)-2\gamma|\nabla H|^2+\gamma C\\
&+Ct^{m-1}|\nabla^m H|^2+Ct^{m-1}|\nabla^{m-1}\operatorname{Rm}|^2+jt^{\frac{m-1}{2}}|\nabla^m H|+Ct^{\frac{m-1}{2}}|\nabla^{m-1}\operatorname{Rm}|\\
&+Ct^{m-\frac{1}{2}}|\nabla^m H|\cdot|\nabla^{m-1}\operatorname{Rm}|+Ct^m|\nabla^{m+1}H|\cdot|\nabla^{m-1}\operatorname{Rm}|\\
&+\sum_{i=1}^{m-1}\beta_i C_i t^{\frac{i}{2}}|\nabla^{i+1}H|+\sum_{i=1}^{m-1}\beta_i C_i t^{\frac{i-1}{2}}(|\nabla^i H|^2+|\nabla^{i-1}\operatorname{Rm}|).
\end{aligned}
$$

Choosing

$$(i+1)\beta_{i+1}=\beta_i,\quad \beta_i=\frac{A}{i!},\quad i\ge 0,$$

where A is constant which is determined later, and noting that

$$\sum_{i=1}^{m-1}\beta_i C_i t^{i/2}|\nabla^{i+1}H|\le\frac{1}{2}\sum_{i=1}^{m-1}\beta_i t^i|\nabla^{i+1}H|^2+\frac{1}{2}\sum_{i=1}^{m-1}\beta_i C_i^2$$

and

$$
\begin{aligned}
&\sum_{i=1}^{m-1}\beta_i C_i t^{\frac{i-1}{2}}(|\nabla^i H|+|\nabla^{i-1}\operatorname{Rm}|)\\
&\qquad\le\beta_1 C_1(|\nabla H|+|\operatorname{Rm}|)+\sum_{i=1}^{m-2}\beta_{i+1}C_{i+1}t^{\frac{i}{2}}(|\nabla^{i+1}H|+|\nabla^i\operatorname{Rm}|)\\
&\qquad\le\beta_1 C_1(|\nabla H|+|\operatorname{Rm}|)+\sum_{i=1}^{m-2}\beta_{i+1}C_{i+1}\Big(\frac{t^i|\nabla^{i+1}H|^2+t^i|\nabla^i\operatorname{Rm}|^2}{2\beta_{i+1}C_{i+1}/\beta_i}+\frac{\beta_{i+1}C_{i+1}}{\beta_i}\Big)\\
&\qquad\le\beta_1 C_1(|\nabla H|+|\operatorname{Rm}|)+\frac{1}{2}\sum_{i=1}^{m-2}\beta_i t^i(|\nabla^{i+1}H|^2+|\nabla^i\operatorname{Rm}|^2)+\sum_{i=1}^{m-2}\frac{\beta_{i+1}^2C_{i+1}^2}{\beta_i},
\end{aligned}
$$

we obtain

$$\begin{aligned}\frac{\partial}{\partial t}u &\le \Delta u - 2t^m|\nabla^{m+1}H|^2 + Ct^m|\nabla^{m+1}H|\cdot|\nabla^{m-1}\operatorname{Rm}| \\ &\quad - 2t^m|\nabla^m \operatorname{Rm}|^2 + Ct^m|\nabla^m H|\cdot|\nabla^m \operatorname{Rm}| + Ct^{m-1}|\nabla^m H|^2 + Ct^{m-1}|\nabla^{m-1}\operatorname{Rm}|^2 \\ &\quad + Ct^{m-\frac{1}{2}}|\nabla^m H|\cdot|\nabla^{m-1}\operatorname{Rm}| + \beta_0(|\nabla H|^2 + |\operatorname{Rm}|^2) \\ &\quad - \sum_{i=1}^{m-1}\beta_i t^i(|\nabla^{i+1}H|^2 + |\nabla^i \operatorname{Rm}|^2) + \sum_{i=1}^{m-2}\beta_i t^i(|\nabla^{i+1}H|^2 + |\nabla^i \operatorname{Rm}|^2) \\ &\quad + \tfrac{1}{2}\beta_{m-1}t^{m-1}|\nabla^m H|^2 + \beta_1 C_1|\nabla H| - 2\gamma|\nabla H|^2 + C + C\gamma \\ &\le \Delta u + Ct^{m-1}|\nabla^{m-1}\operatorname{Rm}|^2 + Ct^{m-1}|\nabla^m H|^2 Ct^{m-\frac{1}{2}}(|\nabla^m H|^2 + |\nabla^{m-1}\operatorname{Rm}|^2) + \beta_0|\nabla H|^2 \\ &\quad + \beta_1 C_1|\nabla H| - 2\gamma|\nabla H|^2 + C + C\gamma - \tfrac{1}{2}\beta_{m-1}t^{m-1}|\nabla^m H|^2 - \beta_{m-1}t^{m-1}|\nabla^m \operatorname{Rm}|^2 \\ &\le \Delta u + \tfrac{1}{2}(C\sqrt{t} + C - \beta_{m-1})t^{m-1}(|\nabla^{m-1}\operatorname{Rm}|^2 + |\nabla^m H|^2) \\ &\quad + (\beta_0 + \beta_1 C_1 - 2\gamma)|\nabla H|^2 + C + C\gamma + \beta_1 C_1.\end{aligned}$$

When we chose A and γ sufficiently large, we obtain $\dfrac{\partial u}{\partial t} \le \Delta u + C$, which implies that $u(t) \le C$ since $u(0)$ is bounded. □

Finally we give an estimate that plays a crucial role in the next section.

Corollary 4.7. *Let $(g(x,t), H(x,t))$ be a solution of the generalized Ricci flow on a closed manifold M. If there are $\beta > 0$ and $K > 0$ such that*

$$|\operatorname{Rm}(x,t)|_{g(x,t)} \le K, \quad |H(x)|_{g(x)} \le K$$

for all $x \in M$ and $t \in [0,T]$, where $T > \beta/K$, then there exists for each $m \in \mathbb{N}$ a constant C_m depending on $m, n, \min\{\beta, 1\}$, and K such that

$$|\nabla^{m-1}\operatorname{Rm}(x,t)|_{g(x,t)} + |\nabla^m H(x,t)|_{g(x,t)} \le C_m K^{m/2}$$

for all $x \in M$ and $t \in [\min\{\beta,1\}/K, T]$.

Proof. The proof is the same as in [Chow et al. 2007]; we just copy it here. Let $\beta_1 := \min\{\beta, 1\}$. For any fixed point $t_0 \in [\beta_1/K, T]$ we set $T_0 := t_0 - \beta_1/K$. For $\bar{t} := t - T_0$ we let $(\bar{g}(\bar{t}), \bar{H}(\bar{t}))$ be the solution of the system

$$\frac{\partial \bar{g}}{\partial \bar{t}} = -2\overline{\operatorname{Ric}} + \tfrac{1}{2}\bar{h}, \quad \frac{\partial \bar{H}}{\partial \bar{t}} = \Delta_{\mathrm{HL},\bar{g}}\overline{H}, \quad \bar{g}(0) = g(T_0), \quad \overline{H}(0) = H(T_0).$$

The uniqueness of solution implies that $\bar{g}(\bar{t}) = g(\bar{t} + T_0) = g(t)$ for $\bar{t} \in [0, \beta_1/K]$. By the assumption we have

$$|\overline{\operatorname{Rm}}(x,\bar{t})|_{\bar{g}(x,\bar{t})} \le K, \quad |\overline{H}(x)|_{\bar{g}(x)} \le K$$

for all $x \in M$ and $\bar{t} \in [0, \beta_1/K]$. Applying Theorem 4.5 with $\alpha = \beta_1$, we have

$$|\overline{\nabla}^{m-1}\overline{\operatorname{Rm}}(x,\bar{t})|_{\bar{g}(x,\bar{t})} + |\overline{\nabla}^m H(x,\bar{t})|_{\bar{g}(x,\bar{t})} \le \frac{\overline{C}_m}{\bar{t}^{m/2}}$$

for all $x \in M$ and $\bar{t} \in (0, \beta_1/K]$. We have $\bar{t}^{m/2} \geq \beta_1^{m/2} 2^{-m/2} K^{-m/2}$ if $\bar{t} \in [\beta_1/2K, \beta_1/K]$. Taking $\bar{t} = \beta_1/K$, we obtain

$$|\nabla^{m-1} \operatorname{Rm}(x, t_0)|_{g(x,t_0)} + |\nabla^m H(x, t_0)|_{g(x,t_0)} \leq \frac{2^{m/2} \overline{C}_m K^{m/2}}{\beta_1^{m/2}}$$

for all $x \in M$. Since $t_0 \in [\beta/K, T]$ was arbitrary, the result follows. □

5. Compactness theorem

In this section we prove the compactness theorem for our generalized Ricci flow. We follow [Hamilton 1995] on the compactness theorem for the usual Ricci flow.

We review several definitions from [Chow et al. 2007]. Throughout this section, all Riemannian manifolds are smooth manifolds of dimensions n. The covariant derivative with respect to a metric g will be denoted by ${}^g\nabla$.

Definition 5.1. Let $K \subset M$ be a compact set and let $\{g_k\}_{k\in\mathbb{N}}$, g_∞, and g be Riemannian metrics on M. For $p \in \{0\} \cup \mathbb{N}$ we say that g_k *converges in* C^p *to* g_∞ *uniformly on* K *with respect to* g if for every $\epsilon > 0$ there exists $k_0 = k_0(\epsilon) > 0$ such that for $k \geq k_0$,

$$\|g_k - g_\infty\|_{C^p;K,g} := \sup_{0\leq\alpha\leq p} \sup_{x\in K} |{}^g\nabla^\alpha (g_k - g_\infty)(x)|_g < \epsilon. \tag{5-1}$$

Since we consider a compact set, the choice of background metric g does not change the convergence. Hence we may choose $g = g_\infty$.

Definition 5.2. Suppose $\{U_k\}_{k\in\mathbb{N}}$ is an exhaustion[2] of a smooth manifold M by open sets and g_k are Riemannian metrics on U_k. We say that (U_k, g_k) *converges in* C^∞ *to* (M, g_∞) *uniformly on compact sets in* M if for any compact set $K \subset M$ and any $p > 0$ there exists $k_0 = k_0(K, p)$ such that $\{g_k\}_{k\geq k_0}$ converges in C^p to g_∞ uniformly on K.

A *pointed Riemannian manifold* is a 3-tuple (M, g, O), where (M, g) is a Riemannian manifold and $O \in M$ is a basepoint. If the metric g is complete, the 3-tuple is called a *complete pointed Riemannian manifold*. We say $(M, g(t), H(t), O), t \in (\alpha, \omega)$, is a *pointed solution to the generalized Ricci flow* if $(M, g(t), H(t))$ is a solution to the generalized Ricci flow.

The so-called *Cheeger–Gromov convergence* in C^∞ is defined as follows:

Definition 5.3. A given sequence $\{(M_k, g_k, O_k)\}_{k\in\mathbb{N}}$ of complete pointed Riemannian manifolds *converges* to a complete pointed Riemannian manifold $(M_\infty, g_\infty, O_\infty)$ if there exist

(i) an exhaustion $\{U_k\}_{k\in\mathbb{N}}$ of M_∞ by open sets with $O_\infty \in U_k$, and

(ii) a sequence of diffeomorphisms $\Phi_k : M_\infty \supset U_k \to V_k := \Phi_k(U_k) \subset M_k$ with $\Phi_k(O_\infty) = O_k$

such that $(U_k, \Phi_k^*(g_k|_{V_k}))$ converges in C^∞ to (M_∞, g_∞) uniformly on compact sets in M_∞.

[2] If for any compact set $K \subset M$ there exists $k_0 \in \mathbb{N}$ such that $U_k \supset K$ for all $k \geq k_0$

The corresponding convergence for the generalized Ricci flow is similar to the convergence for the usual Ricci flow introduced by Hamilton [1995].

Definition 5.4. A given sequence $\{(M_k, g_k(t), H_k(t), O_k)\}_{k\in\mathbb{N}}$ of complete pointed solutions to the GRF *converges* to a complete pointed solution to the GRF

$$(M_\infty, g_\infty(t), H_\infty(t), O_\infty), \quad t \in (\alpha, \omega),$$

if there exist

(i) an exhaustion $\{U_k\}_{k\in\mathbb{N}}$ of M_∞ by open sets with $O_\infty \in U_k$,

(ii) a sequence of diffeomorphisms $\Phi_k : M_\infty \supset U_k \to V_k := \Phi_k(U_k) \subset M_k$ with $\Phi_k(O_\infty) = O_k$

such that $\big(U_k \times (\alpha, \omega), \Phi_k^*(g_k(t)|_{V_k}) + dt^2, \Phi_k^*(H_k(t)|_{V_k})\big)$ converges in C^∞ to

$$\big(M_\infty \times (\alpha, \omega), g_\infty(t) + dt^2, H_\infty(t)\big)$$

uniformly on compact sets in $M_\infty \times (\alpha, \omega)$. Here we denote by dt^2 the standard metric on (α, ω).

Let $\mathrm{inj}_g(O)$ be the injectivity radius of the metric g at the point O. The following compactness theorem is due to Cheeger and Gromov.

Theorem 5.5 (compactness for metrics). *Let $\{(M_k, g_k, O_k)\}_{k\in\mathbb{N}}$ be a sequence of complete pointed Riemannian manifolds satisfying these conditions*:

(i) *For all $p \geq 0$ and $k \in \mathbb{N}$, there is a sequence of constants $C_p < \infty$ independent of k such that*

$$|^{g_k}\nabla^p \mathrm{Rm}(g_k)|_{g_k} \leq C_p$$

on M_k.

(ii) *There exists some constant $\iota_0 > 0$ such that*

$$\mathrm{inj}_{g_k}(O_k) \geq \iota_0$$

for all $k \in \mathbb{N}$.

Then there exists a subsequence $\{j_k\}_{k\in\mathbb{N}}$ such that $\{(M_{j_k}, g_{j_k}, O_{j_k})\}_{k\in\mathbb{N}}$ converges to a complete pointed Riemannaian manifold $(M_\infty^n, g_\infty, O_\infty)$ as $k \to \infty$.

As a consequence of Theorem 5.5, we state our compactness theorem for GRF.

Theorem 5.6 (compactness for GRF). *Let $\{(M_k, g_k(t), H_k(t), O_k)\}_{k\in\mathbb{N}}$ be a sequence of complete pointed solutions to* GRF *for $t \in [\alpha, \omega) \ni 0$ satisfying these conditions*:

(i) *There is a constant $C_0 < \infty$ independent of k such that*

$$\sup_{(x,t)\in M_k\times(\alpha,\omega)} |\mathrm{Rm}(g_k(x,t))|_{g_k(x,t)} \leq C_0, \quad \sup_{x\in M_k} |H_k(x,\alpha)|_{g_k(x,\alpha)} \leq C_0.$$

(ii) *There exists a constant $\iota_0 > 0$ satisfying*

$$\mathrm{inj}_{g_k(0)}(O_k) \geq \iota_0.$$

Then there exists a subsequence $\{j_k\}_{k\in\mathbb{N}}$ *such that*

$$(M_{j_k}, g_{j_k}(t), H_{j_k}(t), O_{j_k}) \to (M_\infty, g_\infty(t), H_\infty(t), O_\infty),$$

converges to a complete pointed solution $(M_\infty, g_\infty(t), H_\infty(t), O_\infty)$, $t \in [\alpha, \omega)$, *to* GRF *as* $k \to \infty$.

To prove Theorem 5.6 we extend a lemma for Ricci flow to GRF. After establishing this lemma, the proof of Theorem 5.6 is similar to that of Theorem 3.10 in [Chow et al. 2007].

Lemma 5.7. *Let* (M, g) *be a Riemannian manifold with a background metric* g, *let* K *be a compact subset of* M, *and let* $(g_k(x,t), H_k(x,t))$ *be a collection of solutions to the generalized Ricci flow defined on neighborhoods of* $K \times [\beta, \psi]$, *where* $t_0 \in [\beta, \psi]$ *is a fixed time. Suppose that*:

(i) *The metrics* $g_k(x, t_0)$ *are all uniformly equivalent to* $g(x)$ *on* K, *i.e., for all* $V \in T_x M$, k, *and* $x \in K$,

$$C^{-1} g(x)(V,V) \le g_k(x,t_0)(V,V) \le C g(x)(V,V),$$

where $C < \infty$ *is a constant independent of* V, k, *and* x.

(ii) *The covariant derivatives of the metrics* $g_k(x, t_0)$ *with respect to the metric* $g(x)$ *are all uniformly bounded on* K, *i.e., for all* k *and* $p \ge 1$,

$$|{}^g\nabla^p g_k(x,t_0)|_{g(x)} + |{}^g\nabla^{p-1} H_k(x,t_0)|_{g(x)} \le C_p$$

where $C_p < \infty$ *is a sequence of constants independent of* k.

(iii) *The covariant derivatives of the curvature tensors* $\mathrm{Rm}(g_k(x,t))$ *and of the forms* $H_k(x,t)$ *are uniformly bounded with respect to the metric* $g_k(x,t)$ *on* $K \times [\beta, \psi]$, *i.e., for all* k *and* $p \ge 0$,

$$|{}^{g_k}\nabla^p \mathrm{Rm}(g_k(x,t))|_{g_k(x,t)} + |{}^{g_k}\nabla^p H_k(x,t)|_{g_k(x,t)} \le C'_p$$

where C'_p *is a sequence of constants independent of* k.

Then the metrics $g_k(x,t)$ *are uniformly equivalent to* $g(x)$ *on* $K \times [\beta, \psi]$, *i.e.*,

$$B(t,t_0)^{-1} g(x)(V,V) \le g_k(x,t)(V,V) \le B(t,t_0) g(x)(V,V),$$

where $B(t,t_0) = Ce^{C'_0|t-t_0|}$ (*here the constant* C'_0 *may not be equal to the previous one*), *and the time-derivatives and covariant derivatives of the metrics* $g_k(x,t)$ *with respect to the metric* $g(x)$ *are uniformly bounded on* $K \times [\beta, \psi]$, *i.e., for each* (p,q) *there is a constant* $\widetilde{C}_{p,q}$ *independent of* k *such that*

$$\left|\frac{\partial^q}{\partial t^q}{}^g\nabla^p g_k(x,t)\right|_{g(x)} + \left|\frac{\partial^q}{\partial t^q}{}^g\nabla^{p-1} H_k(x,t)\right|_{g(x)} \le \widetilde{C}_{p,q}$$

for all k.

Proof. We use [Chow et al. 2007, Lemma 3.13]: Suppose that the metrics g_1 and g_2 are equivalent, i.e., $C^{-1} g_1 \le g_2 \le C g_1$. Then for any (p,q)-tensor T we have $|T|_{g_2} \le C^{(p+q)/2} |T|_{g_1}$. We denote by h

the tensor $h_{ij} := g^{kp} g^{lq} H_{ikl} H_{jpq}$. In the following we denote by C a constant depending only on n, β, and ψ, which may take different values at different places. For any tangent vector $V \in T_x M$ we have

$$\frac{\partial}{\partial t} g_k(x,t)(V,V) = -2\,\mathrm{Ric}(g_k(x,t))(V,V) + \tfrac{1}{2} h_k(x,t)(V,V),$$

and therefore

$$\begin{aligned}\left|\frac{\partial}{\partial t} \log g_k(x,t)(V,V)\right| &= \left|\frac{-2\,\mathrm{Ric}(g_k(x,t))(V,V) + \frac{1}{2} h_k(x,t)(V,V)}{g_k(x,t)(V,V)}\right| \\ &\le C_0' + C|H_k(x,t)|^2_{g_k(x,t)} \\ &\le C_0' + C C_0'^2 =: \overline{C},\end{aligned}$$

since

$$|\mathrm{Ric}(g_k(x,t))(V,V)| \le C_0' g_k(x,t)(V,V), \quad |h_k(x,t)(V,V)| \le C|H_k(x,t)|^2_{g_k(x,t)} g_k(x,t)(V,V).$$

Integrating on both sides, we have

$$\overline{C}|t_1 - t_0| \ge \int_{t_0}^{t_1} \left|\frac{\partial}{\partial t} \log g_k(x,t)(V,V)\right| dt \ge \left|\int_{t_0}^{t_1} \frac{\partial}{\partial t} \log g_k(t)(V,V)\, dt\right| = \left|\log \frac{g_k(x,t_1)(V,V)}{g_k(x,t_0)(V,V)}\right|,$$

and hence we conclude that

$$e^{-\overline{C}|t_1-t_0|} g_k(x,t_0)(V,V) \le g_k(x,t_1)(V,V) \le e^{\overline{C}|t_1-t_0|} g_k(x,t_0)(V,V).$$

From the assumption (i), it immediately deduces from above that

$$C^{-1} e^{-\overline{C}|t_1-t_0|} g(x)(V,V) \le g_k(x,t_1)(V,V) \le C e^{\overline{C}|t_1-t_0|} g(x)(V,V).$$

Since t_1 was arbitrary, the first part is proved. From the definition (or see [Chow et al. 2007, p. 134, (37)]), we have

$$(g_k)^{ec}({}^g\nabla_a (g_k)_{bc} + {}^g\nabla_b (g_k)_{ac} - {}^g\nabla_c (g_k)_{ab}) = 2({}^{g_k}\Gamma)^e_{ab} - 2({}^g\Gamma)^e_{ab}.$$

Thus $|{}^{g_k}\Gamma(x,t) - {}^g\Gamma(x)|_{g(x)} \le C|{}^g\nabla g_k(x,t)|_{g_k(x)}$. On the other hand,

$${}^g\nabla_a (g_k)_{bc} = (g_k)_{eb}[({}^{g_k}\Gamma)^e_{ac} - ({}^g\Gamma)^e_{ac}] + (g_k)_{ec}[({}^{g_k}\Gamma)^e_{ab} - ({}^g\Gamma)^e_{ab}],$$

it follows that $|{}^g\nabla g_k(x,t)|_{g_k(x,t)} \le C|{}^{g_k}\Gamma(x,t) - {}^g\Gamma(x)|_{g_k(x,t)}$ and therefore

$${}^g\nabla g_k \text{ is equivalent to } {}^{g_k}\Gamma - {}^g\Gamma = {}^{g_k}\nabla - {}^g\nabla. \tag{5-2}$$

The evolution equation for ${}^g\Gamma$ is

$$\begin{aligned}\frac{\partial}{\partial t}({}^{g_k}\Gamma)^c_{ab} = &-(g_k)^{cd}[({}^{g_k}\nabla)_a(\mathrm{Ric}(g_k))_{bd} + ({}^{g_k}\nabla)_b(\mathrm{Ric}(g_k))_{ad} \\ &-({}^{g_k}\nabla)_d(\mathrm{Ric}(g_k))_{ab}] + \tfrac{1}{4}(g_k)^{cd}\big[({}^{g_k}\nabla)_a(h_k)_{bd} + ({}^{g_k}\nabla)_b(h_k)_{ad} - ({}^{g_k}\nabla)_d(h_k)_{ab}\big].\end{aligned}$$

Since ${}^g\Gamma$ does not depend on t, it follows from the assumptions that

$$\begin{aligned}\left|\frac{\partial}{\partial t}({}^{g_k}\Gamma - {}^g\Gamma)\right|_{g_k} &\leq C\left|{}^{g_k}\nabla(\mathrm{Ric}(g_k))\right|_{g_k} + C|{}^{g_k}\nabla(h_k)|_{g_k}\\ &\leq CC_1' + C|{}^{g_k}\nabla H_k|_{g_k}\cdot|H_k|_{g_k} \leq C_1'.\end{aligned}$$

Integrating on both sides,

$$C_1'|t_1 - t_0| \geq \left|\int_{t_0}^{t_1}\frac{\partial}{\partial t}(g^k\Gamma(t) - {}^g\Gamma)\,dt\right|_{g_k} \geq |{}^{g_k}\Gamma(t_1) - {}^g\Gamma|_{g_k} - |{}^{g_k}\Gamma(t_0) - {}^g\Gamma|_{g_k}.$$

Hence we obtain

$$\begin{aligned}|{}^{g_k}\Gamma(t) - {}^g\Gamma|_{g_k} &\leq C_1'|t_1 - t_0| + |{}^{g_k}\Gamma(t_0) - {}^g\Gamma|_{g_k}\\ &\leq C_1'|t_1 - t_0| + C|{}^g\nabla g_k(t_0)|_{g_k}\\ &\leq C_1'|t - t_0| + C|{}^g\nabla g_k(t_0)|_g\\ &\leq C_1'|t - t_0| + C_1.\end{aligned}$$

The equivalency of metrics tells us that

$$\begin{aligned}|{}^g\nabla g_k(t)|_g &\leq B(t,t_0)^{3/2}|{}^g\nabla g_k(t)|_{g_k} \leq B(t,t_0)^{3/2}\cdot C|{}^{g_k}\Gamma(t) - {}^g\Gamma|_{g_k}\\ &\leq B(t,t_0)^{3/2}(C_1'|t - t_0| + C').\end{aligned}$$

Since $|t - t_0| \leq \psi - \beta$, it follows that $|{}^g\nabla g_k(t)|_g \leq \widetilde{C}_{1,0}$ for some constant $\widetilde{C}_{1,0}$. But g and g_k are equivalent, we have

$$|H_k(t)|_g \leq C|H_k(t)|_{g_k} \leq CC_1' = \widetilde{C}_{1,0}.$$

From the assumptions, we also have

$$\begin{aligned}|{}^g\nabla H_k|_g &\leq |({}^g\nabla - {}^{g_k}\nabla)H_k + {}^{g_k}\nabla H_k|_g\\ &\leq C|{}^g\nabla g_k|_g\cdot|H_k|_g + C|{}^{g_k}\nabla H_k|_{g_k}\\ &\leq CC_1' + C\widetilde{C}_{1,0}\widetilde{C}_{1,0} := \widetilde{C}_{2,0}.\end{aligned}$$

Moreover,

$$\begin{aligned}\frac{\partial}{\partial t}{}^g\nabla H_k &= {}^g\nabla(\Delta_{g_k}H_k + \mathrm{Rm}(g_k) * H_k)\\ &= ({}^g\nabla - {}^{g_k}\nabla)\Delta_{g_k}H_k + {}^{g_k}\nabla\Delta_{g_k}H_k + {}^g\nabla\,\mathrm{Rm}(g_k) * H_k + \mathrm{Rm}(g_k) * {}^g\nabla H_k\end{aligned}$$

where Δ_{g_k} is the Laplace operator associated to g_k. Hence

$$\begin{aligned}&\left|\frac{\partial}{\partial t}{}^g\nabla H_k\right|_g\\ &\leq C|{}^g\nabla g_k|_g\cdot|\Delta_{g_k}H_k|_{g_k} + C|{}^{g_k}\nabla\Delta_{g_k}H_k|_g + C|{}^g\nabla\,\mathrm{Rm}(g_k)|_g\cdot|H_k|_g + C|\mathrm{Rm}(g_k)|_g\cdot|{}^g\nabla H_k|_g\\ &\leq \widetilde{C}_{2,1}.\end{aligned}$$

For higher derivatives we claim that

$$|{}^g\nabla^p \operatorname{Ric}(g_k)|_g \le C_p''|{}^g\nabla^p g_k|_g + C_p''', \quad |{}^g\nabla^p g_k|_g + |{}^g\nabla^{p-1} H_k|_g \le \widetilde{C}_{p,0}, \tag{5-3}$$

for all $p \ge 1$, where C_p'', C_p''', and $\widetilde{C}_{p,0}$ are constants independent of k. For $p = 1$, we have proved the second inequality, so we suffice to prove the first one with $p = 1$. Indeed,

$$\begin{aligned}
|{}^g\nabla \operatorname{Ric}(g_k)|_g &\le C|({}^g\nabla - {}^{g_k}\nabla)\operatorname{Ric}(g_k) + {}^{g_k}\nabla \operatorname{Ric}(g_k)|_{g_k} \\
&\le C|{}^g\Gamma - {}^{g_k}\Gamma|_g \cdot |\operatorname{Ric}(g_k)|_{g_k} + C|{}^{g_k}\nabla \operatorname{Ric}(g_k)|_{g_k} \\
&\le C_1''|{}^g\nabla g_k|_g + C_1'''.
\end{aligned}$$

Suppose the claim holds for all $p < N$ $(N \ge 2)$, we shall show that it also holds for $p = N$. From

$$\begin{aligned}
|{}^g\nabla^N \operatorname{Ric}(g_k)|_g &= \left|\sum_{i=1}^N {}^g\nabla^{N-i}({}^g\nabla - {}^{g_k}\nabla)^{g_k}\nabla^{i-1} \operatorname{Ric}(g_k) + {}^{g_k}\nabla^N \operatorname{Ric}(g_k)\right|_g \\
&\le \sum_{i=1}^N \left|{}^g\nabla^{N-i}({}^g\nabla - {}^{g_k}\nabla)^{g_k}\nabla^{i-1} \operatorname{Ric}(g_k)\right|_g + |{}^{g_k}\nabla^N \operatorname{Ric}(g_k)|_g
\end{aligned}$$

we estimate each term. For $i = 1$, by induction and the assumptions we have

$$\begin{aligned}
&|{}^g\nabla^{N-1}({}^g\nabla - {}^{g_k}\nabla)\operatorname{Ric}(g_k)|_g \\
&\qquad \le C|{}^g\nabla^{N-1}({}^g\nabla g_k \cdot \operatorname{Ric}(g_k))|_g \\
&\qquad \le C\left|\sum_{j=0}^{N-1}\binom{N-1}{j}{}^g\nabla^{N-1-j}({}^g\nabla g_k)\cdot {}^g\nabla^j(\operatorname{Ric}(g_k))\right|_g \\
&\qquad \le C\sum_{j=0}^{N-1}\binom{N-1}{j}|{}^g\nabla^{N-j} g_k|_g \cdot |{}^g\nabla^j \operatorname{Ric}(g_k)|_g \\
&\qquad \le C\sum_{i=0}^{N-1}\binom{N-1}{j}(C_j''|{}^g\nabla^j g_k|_g + C_j''')|{}^g\nabla^{N-j} g_k|_g \\
&\qquad \le C\sum_{j=0}^{N-1}\binom{N-1}{j}(C_j''\widetilde{C}_{j,0} + C_j''')|{}^g\nabla^{N-j} g_k|_g \\
&\qquad = C(N-1)(C_0''\widetilde{C}_{j,0} + C_0''')|{}^g\nabla^N g_k|_g + C\sum_{j=1}^{N-1}\binom{N-1}{j}(C_j''\widetilde{C}_{j,0} + C_j''')\widetilde{C}_{N-j,0} \\
&\qquad \le C_N''|{}^g\nabla^N g_k|_g + C_N'''.
\end{aligned}$$

For $i \geq 2$, we have

$$\begin{aligned}\left|{}^g\nabla^{N-i}({}^g\nabla - {}^{g_k}\nabla)^{g_k}\nabla^{i-1}\operatorname{Ric}(g_k)\right|_g &\leq C\left|{}^g\nabla^{N-i}({}^g\nabla g_k \cdot {}^{g_k}\nabla^{i-1}\operatorname{Ric}(g_k))\right|_g \\ &\leq C\sum_{j=0}^{N-i}\binom{N-i}{j}|{}^g\nabla^{N-i-j+1}g_k|_g \cdot |{}^g\nabla^j \cdot {}^{g_k}\nabla^{i-1}\operatorname{Ric}(g_k)|_g.\end{aligned}$$

If $j = 0$, then

$$|{}^{g_k}\nabla^{i-1}\operatorname{Ric}(g_k)|_g \leq C''_{i-1}|{}^g\nabla^{i-1}g_k|_g + C'''_{i-1} \leq C''_{i-1}\widetilde{C}_{i-1,0} + C'''_{i-1}.$$

Suppose in the following that $j \geq 1$. Hence

$$\begin{aligned}\left|{}^g\nabla^j \cdot {}^{g_k}\nabla^{i-1}\operatorname{Ric}(g_k)\right|_g &= \left|(({}^g\nabla - {}^{g_k}\nabla) + {}^{g_k}\nabla)^j \cdot {}^{g_k}\nabla^{i-1}\operatorname{Ric}(g_k)\right|_g \\ &\leq C\sum_{l=0}^{j}\binom{j}{l}|{}^g\nabla^l g_k|_g \cdot |{}^{g_k}\nabla^{j-l+i-1}\operatorname{Ric}(g_k)|_g \\ &\leq C\sum_{l=0}^{j}\binom{j}{l}\widetilde{C}_{l,0}(C''_{j-l+i-1}\widetilde{C}_{j-l+i-1,0} + C'''_{j-l+i-1}),\end{aligned}$$

where we make use of (5-2) from first line to second line. Combining these inequalities, we get

$$|{}^g\nabla^N\operatorname{Ric}(g_k)|_g \leq C''_N|{}^g\nabla^N g_k|_g + C'''_N.$$

Similarly, we have

$$|{}^g\nabla^N h_k|_g \leq C''_N|{}^g\nabla^N g_k|_g + C'''_N.$$

Since $\dfrac{\partial}{\partial t}g_k = -2\operatorname{Ric}(g_k) + \frac{1}{2}h_k$, it follows that

$$\begin{aligned}\frac{\partial}{\partial t}{}^g\nabla^N g_k &= {}^g\nabla^N(-2\operatorname{Ric}(g_k) + \tfrac{1}{2}h_k), \\ \frac{\partial}{\partial t}|{}^g\nabla^N g_k|_g^2 &\leq \left|\frac{\partial}{\partial t}{}^g\nabla^N g_k\right|_g^2 + |{}^g\nabla^N g_k|_g^2 \\ &\leq 8\,|{}^g\nabla^N\operatorname{Ric}(g_k)|_g^2 + \frac{1}{2}|{}^g\nabla^N h_k|_g^2 + |{}^g\nabla^N g_k|_g^2 \\ &\leq (1 + 18(C''_N)^2)|{}^g\nabla^N g_k|_g^2 + 18(C''_N)^2.\end{aligned}$$

Integrating the above inequality, we get $|{}^g\nabla g_k|_g \leq \widetilde{C}_{N,0}$ and therefore $|{}^g\nabla^N h_k|_g \leq \widetilde{C}_{N+1,0}$. We have proved lemma for $q = 0$. When $g \geq 1$, then

$$\frac{\partial^q}{\partial t^q}{}^g\nabla^p g_k(t) = {}^g\nabla^p\frac{\partial^{q-1}}{\partial t^{q-1}}\big(-2\operatorname{Ric}(g_k(t)) + \tfrac{1}{2}h_k(t)\big).$$

Using the evolution equations for $\operatorname{Rm}(g_k(t))$ and $h_k(t)$, combining the induction to q and using the above method, we have

$$\left|\frac{\partial^q}{\partial t^q}{}^g\nabla^p g_k(t)\right|_g + \left|\frac{\partial^q}{\partial t^q}{}^g\nabla^{p-1}h_k(t)\right|_g \leq \widetilde{C}_{p,q}. \qquad \square$$

6. Generalization

In this section, we generalize the main results in Sections 4 and 5 to a kind of generalized Ricci flow for which local existence has been established [He et al. 2008].

Let $(M, g_{ij}(x))$ be an n-dimensional closed Riemannian manifold and let $A = \{A_i\}$ and $B = \{B_{ij}\}$ denote a one-form and a two-form respectively. Set $F = dA$ and $H = dB$. The authors in [He et al. 2008] proved that there exists a constant $T > 0$ such that the evolution equations

$$\begin{aligned}
&\frac{\partial}{\partial t} g_{ij}(x,t) = -2R_{ij}(x,t) + \tfrac{1}{2} h_{ij}(x,t) + 2f_{jk}(x,t), \quad g_{ij}(x,0) = g_{ij}(x),\\
&\frac{\partial}{\partial t} A_i(x,t) = -2\nabla_k F_i{}^k(x,t), \quad A_i(x,0) = A_i(x),\\
&\frac{\partial}{\partial t} B_{ij}(x,t) = 3\nabla_k H^k{}_{ij}(x,t), \quad B_{ij}(x,0) = B_{ij}(x)
\end{aligned}$$

has a unique smooth solution on $m \times [0, T)$, where $h_{ij} = H_{ikl} H_j{}^{kl}$ and $f_{ij} = F_i{}^k F_{jk}$. We call it RF(A, B). According to the definition of the adjoint operator d^*, we have

$$(d^* F)_i = 2\nabla_k F_i{}^k, \quad (d^* H)_{ij} = -3\nabla_k H^k{}_{ij}, \tag{6-1}$$

and hence

$$\frac{\partial}{\partial t} F(x,t) = -d\, d^*_{g(x,t)} F \;=\; \Delta_{\mathrm{HL},g(x,t)} F \;=\; \Delta F + \mathrm{Rm} * F, \tag{6-2}$$

$$\frac{\partial}{\partial t} H(x,t) = -d\, d^*_{g(x,t)} H \;=\; \Delta_{\mathrm{HL},g(x,t)} H \;=\; \Delta H + \mathrm{Rm} * H. \tag{6-3}$$

They also derived the evolution equations of curvatures:

$$\begin{aligned}
\frac{\partial}{\partial t} R_{ijk\ell} &= \Delta R_{ijk\ell} + 2(B_{ijk\ell} - B_{ij\ell k} - B_{i\ell jk} + B_{ikj\ell})\\
&\quad - g^{pq}(R_{pjk\ell} R_{qi} + R_{ipk\ell} R_{qj} + R_{ijp\ell} R_{qk} + R_{ijkp} R_{q\ell})\\
&\quad + \tfrac{1}{4}\big[\nabla_i\nabla_\ell(H_{kpq} H_j{}^{pq}) - \nabla_i\nabla_k(H_{jpq} H_\ell{}^{pq}) - \nabla_j\nabla_\ell(H_{kpq} H_i{}^{pq}) + \nabla_j\nabla_k(H_{ipq} H_\ell{}^{pq})\big]\\
&\quad + \tfrac{1}{4} g^{rs}(H_{kpq} H_r^{pq} R_{ijs\ell} + H_{rpq} H_\ell{}^{pq} R_{ijks})\\
&\quad + \nabla_i\nabla_\ell(F_k{}^p F_{jp}) - \nabla_i\nabla_k(F_j{}^p F_{\ell p}) - \nabla_j\nabla_\ell(F_k{}^p F_{ip}) + \nabla_j\nabla_k(F_i{}^p F_{\ell p})\\
&\quad + g^{rs}(F_k{}^p F_{rp} R_{ijs\ell} + F_r{}^p F_{\ell p} R_{ijks}).
\end{aligned}$$

Under our notation, it can be rewritten as

$$\begin{aligned}
\frac{\partial}{\partial t}\mathrm{Rm} = \Delta\,\mathrm{Rm} + \sum_{i+j=0} \nabla^i \mathrm{Rm} * \nabla^j \mathrm{Rm} + \sum_{i+j=0+2} \nabla^i H * \nabla^j H + \sum_{i+j=0+2} \nabla^i F * \nabla^j F\\
+ \sum_{i+j+k=0} \nabla^i H * \nabla^j H * \nabla^k \mathrm{Rm} + \sum_{i+j+k=0} !!\nabla^i F * \nabla^j F * \nabla^k \mathrm{Rm}\,.
\end{aligned} \tag{6-4}$$

As before, we have:

Proposition 6.1. *For* $\mathrm{RF}(A, B)$ *and any nonnegative integer* ℓ *we have*

$$\frac{\partial}{\partial t}\nabla^\ell \mathrm{Rm} = \Delta(\nabla^l \mathrm{Rm}) + \sum_{i+j=\ell} \nabla^i \mathrm{Rm} * \nabla^j \mathrm{Rm} + \sum_{i+j=\ell+2} \nabla^i H * \nabla^j H + \sum_{i+j=\ell+2} \nabla^i F * \nabla^j F + \sum_{i+j+k=\ell} \nabla^i H * \nabla^j H * \nabla^k \mathrm{Rm} + \sum_{i+j+k=\ell} \nabla^i F * \nabla^j F * \nabla^k \mathrm{Rm}. \quad (6\text{-}5)$$

In particular,

$$\frac{\partial}{\partial t}|\nabla^l \mathrm{Rm}|^2 \le \Delta|\nabla^l \mathrm{Rm}|^2 - 2|\nabla^{\ell+1}\mathrm{Rm}|^2 + C\sum_{i+j=\ell} |\nabla^i \mathrm{Rm}|\cdot|\nabla^j \mathrm{Rm}|\cdot|\nabla^\ell \mathrm{Rm}| + C\sum_{i+j=\ell+2} |\nabla^i H|\cdot|\nabla^j H|\cdot|\nabla^\ell \mathrm{Rm}| + C\sum_{i+j=\ell+2} |\nabla^i F|\cdot|\nabla^j F|\cdot|\nabla^\ell \mathrm{Rm}| + C\sum_{i+j+k=\ell} |\nabla^i H|\cdot|\nabla^j H|\cdot|\nabla^k \mathrm{Rm}|\cdot|\nabla^\ell \mathrm{Rm}| + C\sum_{i+j+k=\ell} |\nabla^i F|\cdot|\nabla^j F|\cdot|\nabla^k \mathrm{Rm}|\cdot|\nabla^\ell \mathrm{Rm}|.$$

Since $\frac{\partial}{\partial t}F = \Delta F + \mathrm{Rm} * F$ it follows that

$$\begin{aligned}\frac{\partial}{\partial t}\nabla F &= \nabla\frac{\partial}{\partial t}F + F * \nabla(\mathrm{Rm} + H * H + F * F)\\ &= \nabla(\Delta F + \mathrm{Rm} * F) + F * \nabla \mathrm{Rm} + F * H * \nabla H + F * F * \nabla F\\ &= \Delta(\nabla F) + \nabla \mathrm{Rm} * F + \mathrm{Rm} * \nabla F + F * H * \nabla H + F * F * \nabla F.\end{aligned}$$

It can be expressed as

$$\frac{\partial}{\partial t}\nabla F = \Delta(\nabla F) + \sum_{i+j=1} \nabla^i F * \nabla^j \mathrm{Rm} + \sum_{i+j+k=1} \nabla^i F * \nabla^j F * \nabla^k F + \sum_{i=0}^{1-1}\sum_{j=0}^{1-i} \nabla^i F * \nabla^j H * \nabla^{1-i-j} H.$$

More generally, we can show:

Proposition 6.2. *For* $\mathrm{RF}(A, B)$ *and any positive integer* ℓ *we have*

$$\frac{\partial}{\partial t}\nabla^\ell F = \Delta(\nabla^\ell F) + \sum_{i+j=\ell} \nabla^i F * \nabla^j \mathrm{Rm} + \sum_{i+j+k=\ell} \nabla^i F * \nabla^j F * \nabla^k F + \sum_{i=0}^{\ell-1}\sum_{j=0}^{\ell-i} \nabla^i F * \nabla^j H * \nabla^{\ell-i-j} H.$$

In particular,

$$\frac{\partial}{\partial t}|\nabla^\ell F|^2 \le \Delta|\nabla^\ell F|^2 - 2|\nabla^{\ell+1}F|^2 + C\sum_{i+j=\ell} |\nabla^i F|\cdot|\nabla^j \mathrm{Rm}|\cdot|\nabla^\ell F| + C\sum_{i+j+k=\ell} |\nabla^i F|\cdot|\nabla^j F|\cdot|\nabla^k F|\cdot|\nabla^l F| + C\sum_{i=0}^{\ell-1}\sum_{j=0}^{\ell-i} |\nabla^i F|\cdot|\nabla^j H|\cdot|\nabla^{\ell-i-j} H|\cdot|\nabla^\ell F|.$$

Similarly, we obtain:

Proposition 6.3. *For* $\mathrm{RF}(A, B)$ *and any positive integer* l *we have*

$$\frac{\partial}{\partial t}\nabla^\ell H = \Delta(\nabla^\ell H) + \sum_{i+j=\ell} \nabla^i H * \nabla^j \mathrm{Rm} + \sum_{i+j+k=\ell} \nabla^i H * \nabla^j H * \nabla^k H + \sum_{i=0}^{\ell-1}\sum_{j=0}^{\ell-i} \nabla^i H * \nabla^j F * \nabla^{\ell-i-j} F.$$

In particular,

$$\frac{\partial}{\partial t}|\nabla^\ell H|^2 \le \Delta|\nabla^\ell H|^2 - 2\,|\nabla^{\ell+1} H|^2 + C\sum_{i+j=\ell} |\nabla^i H|\cdot|\nabla^j \mathrm{Rm}\,|\cdot|\nabla^\ell H| + C\sum_{i+j+k=\ell} |\nabla^i H|\cdot|\nabla^j H|\cdot|\nabla^k H|\cdot|\nabla^\ell H| + C\sum_{i=0}^{\ell-1}\sum_{j=0}^{\ell-i} |\nabla^i H|\cdot|\nabla^j F|\cdot|\nabla^{\ell-i-j} F|\cdot|\nabla^\ell H|.$$

From the evolution inequalities

$$\frac{\partial}{\partial t}|H|^2 \le \Delta|H|^2 - 2\,|\nabla H|^2 + C\cdot|\mathrm{Rm}\,|\cdot|H|^2,$$
$$\frac{\partial}{\partial t}|F|^2 \le \Delta|F|^2 - 2\,|\nabla F|^2 + C\cdot|\mathrm{Rm}\,|\cdot|F|^2,$$

the following theorem is obvious.

Theorem 6.4. *Suppose that* $(g(x,t), H(x,t), F(x,t))$ *is a solution to* $\mathrm{RF}(A, B)$ *on a compact manifold* M^n *for a short time* $0 \le t \le T$ *and* K_1, K_2, K_3 *are arbitrary given nonnegative constants. Then there exists a constant* C_n *depending only on* n *such that if*

$$|\mathrm{Rm}(x,t)|_{g(x,t)} \le K_1, \quad |H(x)|_{g(x)} \le K_2, \quad |F(x)|_{g(x)} \le K_3$$

for all $x \in M$ *and* $t \in [0, T]$, *then*

$$|H(x,t)|_{g(x,t)} \le K_2 e^{C_n K_1 t}, \quad |F(x,t)|_{g(x,t)} \le K_3 e^{C_n K_1 t}, \tag{6-6}$$

for all $x \in M$ *and* $t \in [0, T]$.

Parallel to Theorem 4.6, we can prove:

Theorem 6.5. *Suppose that* $(g(x,t), H(x,t), F(x,t))$ *is a solution to* $\mathrm{RF}(A, B)$ *on a compact manifold* M^n *and* K *is an arbitrary given positive constant. Then for each* $\alpha > 0$ *and each integer* $m \ge 1$ *there exists a constant* C_m *depending on* $m, n, \max\{\alpha, 1\}$, *and* K *such that if*

$$|\mathrm{Rm}(x,t)|_{g(x,t)} \le K, \quad |H(x)|_{g(x)} \le K, \quad |F(x)|_{g(x)} \le K$$

for all $x \in M$ *and* $t \in [0, \alpha/K]$, *then*

$$|\nabla^{m-1}\mathrm{Rm}(x,t)|_{g(x,t)} + |\nabla^m H(x,t)|_{g(x,t)} + |\nabla^m F(x,t)|_{g(x,t)} \le \frac{C_m}{t^{\frac{m}{2}}}, \tag{6-7}$$

for all $x \in M$ *and* $t \in (0, \alpha/K]$.

We can also establish the corresponding compactness theorem for RF(A, B). We omit the detail since the proof is close to the proof in Section 5. In the forthcoming paper, we will consider the BBS estimates for complete noncompact Riemannian manifolds.

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